Liapunov-Floquet Transformation: Computation and Applications to Periodic Systems

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Liapunov-Floquet Transformation: Computation and Applications to Periodic Systems

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In this paper, a new analysis technique in the study of dynamical systems with periodically varying parameters is presented. The method is based on the fact that all linear periodic systems can be replaced by similar linear time-invariant systems through a suitable periodic transformation known as the Liapunov-Floquet (L-F) transformation. A general technique for the computation of the L-F transformation matrices is suggested. In this procedure, the state vector and the periodic matrix of the linear system equations are expanded in terms of the shifted Chebyshev polynomials over the principal period. Such an expansion reduces the original differential problem to a set of linear algebraic equations from which the state transition matrix (STM) can be constructed over the period in closed form. Application of Floquet theory and eigenanalysis to the resulting STM yields the L-F transformation matrix in a form suitable for algebraic manipulations. The utility of the L-F transformation in obtaining solutions of both linear and nonlinear dynamical systems with periodic coefficients is demonstrated. It is shown that the application of L-F transformation to free and harmonically forced linear periodic systems directly provides the conditions for internal and combination resonances and external resonances, respectively. The application of L-F transformation to quasilinear periodic systems provides a dynamically similar quasilinear systems whose linear parts are time-invariant and the solutions of such systems can be obtained through an application of the time-dependent normal form theory. *These solutions can be transformed back to the original coordinates using the inverse L-F transformation. Two dynamical systems, namely, a commutative system and a Mathieu type equation are considered to demonstrate the effectiveness of the method. It is shown that the present technique is virtually free from the small parameter restriction unlike averaging and perturbation procedures and* can be used even for those systems for which the generating solutions do not exist in the classical sense. *The results obtained from the proposed technique are compared with those obtained via the perturbation method and numerical solutions computed using a Runge-Kutta type algorithm.*

1 Introduction

The study of systems governed by a set of ordinary differential equations with periodic coefficients is of great importance in diverse branches of science and engineering. The stability problem associated with these equations involves analysis of a set of linear ordinary differential equations with periodic coefficients. The same mathematical problem also arises in the study of nonlinear autonomous systems in the event when the stability of a particular periodic solution needs to be investigated. Besides the stability issues, the linear control problems associated with rotating systems also lead to the same type of equations.

Hill's method (Yakubovich and Starzhinski, 1975), perturbation techniques (Nayfeh and Mook, 1979) and Floquet theory (Coddington and Levinson, 1955) are some of the most commonly used mathematical methods in the analysis of linear periodic systems. It is well-known that the Hill's approach is only suitable for determining the stability boundaries for such systems. The perturbation and averaging methods have their own limitations due to the fact that they can only be applied to systems where the periodic coefficients can be expressed in terms of a small parameter. Also, they are suitable for relatively smaller systems. Therefore, Floquet analysis coupled with a numerical integration code has served as the main tool in various applications (Friedmann et al., 1977). Most commonly a fourth or higher-order Runge-Kutta type numerical code is employed in a "single pass" scheme for an efficient computation of the state transition matrix (Peters et al., 1971; Friedmann et al., 1977). Recently, Sinha and his associates (Sinha and Wu, 1991a; Sinha et al., 1993; Joseph et al., 1993) have developed a new technique for the stability and control analysis of linear periodic systems through an application of the Chebyshev polynomials in conjunction with the Floquet theory.

Although the linearized equations play an important role in the stability analyses, they fail to provide answers to many questions associated with the nonlinear systems. For a better understanding, one must investigate the nonlinear equations of the perturbed motion. For obvious reasons solutions of such equations have not been treated extensively in the literature. Averaging (Sanders and Verhulst, 1985) and perturbation techniques (Nayfeh and Mook, 1979) have been found applicable to such systems under rather restrictive conditions. In both methods it is tacitly assumed that a generating solution exists and the periodic terms as well as the nonlinearities can be expressed in terms of a suitable small parameter. On the other hand, one can always apply standard numerical techniques such as shooting methods (Seydel, 1981) and provide strategies for

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calculating branch points, etc. But such methods are certainly not free from numerical instabilities and require vast amount of computations in order to obtain global information. An alternate method is to construct the Poincaré map by which the original nonautonomous differential equation is replaced by a set of difference equations which do not explicitly depend on time. However, in order to obtain the corresponding difference equations, one must construct an exact, an approximate or a numerical solution of a system of nonlinear time-varying equations. Since most of the time one must settle for an approximate or a numerical representation of the point mapping, recent studies (Hsu, 1987; Guttalu and Flashner, 1990) have suggested the use of a combination of Runge-Kutta and perturbation technique or a purely numerical scheme for obtaining a truncated version of the Poincaré map. It should be noted that such mappings can be computed to an arbitrary degree of precision and play an important role in global analysis. However, the computational problems associated with such techniques once again cannot be ignored.

Another school of thought in dealing with such systems has been to use the Liapunov-Floquet Theorem so that the linear periodic systems can be transformed to a new set of similar equations which are totally time-invariant. However, to this date there are no methods available to compute this transformation matrix for general periodic systems as indicated by Verhulst (1990) and Bellman (1970a). For certain special classes of linear systems, called the commutative systems, it is possible to obtain the Liapunov-Floquet transformation in a closed form, as shown in the literature (Wu, 1978; Lukes, 1982). In order to determine such a transformation for a general periodic system one must compute the state transition matrix (STM) as an explicit function of time. Recently Sinha and his coauthors (Sinha and Wu, 1991a; Sinha et al., 1993; Wu and Sinha, 1994) have been successful in obtaining the state transition matrices of general linear periodic systems in numerical as well as symbolic forms using the shifted Chebyshev polynomials of the first kind. These studies have shown that the STMs can be computed to a desired accuracy by including the appropriate number of terms in Chebyshev expansion. The method is computationally efficient when compared with the standard numerical schemes and suitable for relatively large problems (see Sinha and Wu, 1991a and Wu and Sinha, 1994, where systems up to dimensions 20 × 20 have been analyzed). Further, while analyzing a 16 × 16 periodic system, Joseph et al. (1993) have studied the convergence of the exponents of the monodromy matrix (which implies the convergence of the elements of the STM) as a function of the number of terms in Chebyshev expansion. Sinha and Wu (1991) also presented an error bound analysis for this technique. The extension to parametrically excited nonlinear problems has been undertaken by Sinha et al. (1993). The objective of this paper is to exploit the Chebyshev expansion technique for the computation of the L-F transformation matrix associated with a general periodic system and to demonstrate its application in the analyses of linear and quasilinear dynamical systems. A perturbation solution is also included following the approach suggested by Yakubovich and Starzhinski (1975). It is demonstrated that for a parametrically excited free and forced linear system, the conditions for resonance can be explicitly obtained from the transformed equations. As an example, the responses of a free and forced Mathieu equation under resonance conditions are computed and compared with numerical solutions. In the second part, it is shown that the original quasilinear periodic system can be transformed via the L-F transformation to a dynamically similar form in which the linear part is time-invariant. The analysis of the transformed equations has been carried out through the use of *time-dependent normal form theory.* The solutions thus obtained can be mapped back to the original coordinates by applying the inverse L-F transformation and compared with the numerical results obtained by a Runge-Kutta type algorithm. A nonlinear Mathieu type equation is chosen to demonstrate the procedure. The proposed technique is applicable to a wide class of problems including the situations where the generating solutions do not exist and/or the parameter multiplying the linear periodic terms are no longer small. It is also shown that in many cases it is possible to obtain approximate analytical solutions which compare extremely well with the numerical solutions. The authors believe that the solutions of this nature for a quasilinear periodic systems have been presented for the very first time.

2 Dynamical Equations of Time-Periodic Systems

In many engineering applications time-periodic dynamic systems appear naturally. Such systems may be represented by differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{t})\mathbf{x} + \mathbf{e}_2(\mathbf{x}, \mathbf{t}) + \mathbf{e}_3(\mathbf{x}, \mathbf{t}) + \ldots + \mathbf{e}_k(\mathbf{x}, \mathbf{t}) + \mathbf{O}(|\mathbf{x}|^{k+1}, \mathbf{t}) + \mathbf{F}(\Omega, \mathbf{t}) \quad (1)$$

where the $n \times n$ matrix $\mathbf{A}(\mathbf{t})$ and the $n \times l$ nonlinear terms $\mathbf{e}_k(\mathbf{x}, \mathbf{t})$ are T-periodic functions of t. The forcing vector $\mathbf{F}(\Omega, \mathbf{t})$ is a periodic function of period $2\pi/\Omega$. It is to be noted that the nonlinear terms $\mathbf{e}_k(\cdot)$ in Eq. (1) represent homogeneous monomials in $\mathbf{x}_i$ of order k. For later use, we rewrite the above equation as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{t})\mathbf{x} + \mathbf{E}(\mathbf{x}, \mathbf{t}) + \mathbf{F}(\Omega, \mathbf{t}) \quad (2)$$

where $\mathbf{E}(\cdot)$ is appropriately defined in terms of $\mathbf{e}_k(\cdot)$'s. The linear homogeneous part of Eq. (2) is, of course, given by

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{t})\mathbf{x}; \quad \mathbf{A}(\mathbf{t}) = \mathbf{A}(\mathbf{t} + \mathbf{T}) \quad (3)$$

The stability and response of Eq. (3) can be discussed using the well-known Floquet theory. Using the transformation (Liapunov, 1896)

$$\mathbf{x}(\mathbf{t}) = \mathbf{L}(\mathbf{t})\mathbf{z}(\mathbf{t}); \quad \mathbf{L}(\mathbf{t}) = \mathbf{L}(\mathbf{t} + \mathbf{T}) \quad (4)$$

Eq. (3) can be transformed to a time-invariant system

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \quad (5)$$

where $\mathbf{C}$ is a constant matrix, and $\mathbf{L}(\mathbf{t})$ is known as the Liapunov-Floquet (L-F) transformation matrix. Application of this transformation to Eq. (2) results in

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{L}^{-1}(\mathbf{t})\mathbf{E}(\mathbf{z}, \mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{F}(\Omega, \mathbf{t}) \quad (6)$$

In general, matrices $\mathbf{C}$ and $\mathbf{L}$ are complex. For a real representation, one may use the 2T periodic real L-F transformation matrix $\mathbf{Q}(\mathbf{t})$ such that $\mathbf{x}(\mathbf{t}) = \mathbf{Q}(\mathbf{t})\mathbf{z}(\mathbf{t})$ yields

$$\dot{\mathbf{z}} = \mathbf{R}\mathbf{z} + \mathbf{Q}^{-1}(\mathbf{t})\mathbf{E}(\mathbf{z}, \mathbf{t}) + \mathbf{Q}^{-1}(\mathbf{t})\mathbf{F}(\Omega, \mathbf{t}) \quad (7)$$

It is evident that the internal resonant conditions for Eq. (3) are characterized by the eigenvalues of matrix $\mathbf{C}$ (or $\mathbf{R}$). Similarly, the external resonance conditions of the linear forced problem (i.e., when $\mathbf{E}(\mathbf{x}, \mathbf{t}) \equiv 0$ in Eq. (2)) can be easily obtained from Eq. (6). Also the nonlinear free vibration of Eq. (2) (i.e., when $\mathbf{F}(\Omega, \mathbf{t}) \equiv 0$) can be analyzed using the *theory of time-dependent normal forms.* But first, the computation of L-F transformations is discussed.

3 Computation of the Liapunov-Floquet Transformation Matrix

3.1 Computation of L-F Transformation Matrix for a Commutative System. For a very special class of periodic systems called the commutative systems, it is possible to obtain the L-F transformation matrices in closed forms. The main properties of such systems are described below.

The periodic matrix $\mathbf{A}(t)$ in Eq. (3) is called commutative if there exists a matrix $\mathbf{B}(t)$ such that

$$d\mathbf{B}(t)/dt = \mathbf{A}(t) \tag{8}$$

satisfying the relation $\mathbf{A}(t)\mathbf{B}(t) - \mathbf{B}(t)\mathbf{A}(t) = 0$. Such a $\mathbf{B}(t)$ is called a commuting antiderivative of $\mathbf{A}(t)$. The differential Eq. (3) is called commutative if the coefficient matrix $\mathbf{A}(t)$ is commutative.

Let $\mathbf{B}(t)$ be a commutative antiderivative of the T-periodic matrix $\mathbf{A}(t)$ appearing in Eq. (3) such that $\mathbf{B}(T)\mathbf{B}(0) - \mathbf{B}(0)\mathbf{B}(T) = 0$. Then the state transition matrix $\Phi(t)$ of system (3) has the factorization

$$\Phi(t) = e^{\mathbf{B}(t)} = e^{\mathbf{B}_T(t)}e^{\mathbf{C}t} = \mathbf{L}(t)e^{\mathbf{C}t} \tag{9}$$

where $\mathbf{B}_T(t) = \mathbf{B}(t) - \mathbf{C}t$ is the T-periodic matrix and $\mathbf{C}$ is the average matrix given by

$$\mathbf{C} = \frac{1}{T}\int_0^T \mathbf{A}(\zeta)d\zeta \tag{10}$$

Since $\dot{\mathbf{B}}(t) = \mathbf{A}(t)$,

$$\mathbf{B}(t) = \mathbf{B}(0) + \int_0^t \mathbf{A}(\zeta)d\zeta \tag{11}$$

Therefore, in this case the L-F matrix $\mathbf{L}(t) \equiv e^{\mathbf{B}_T(t)}$ and one can see that it can be computed in a closed form.

As an example, consider

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{12}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{13}$$

In this case there exists a commuting antiderivative for the matrix $\mathbf{A}(t)$ given by

$$\mathbf{B}(t) = \int_0^t \mathbf{A}(\zeta)d\zeta = \begin{bmatrix} \sin t & 1 - \cos t \\ \cos t - 1 & \sin t \end{bmatrix} \tag{14}$$

The state transition matrix can be calculated as

$$\Phi(t) = e^{\sin t - \sin \tau} \times \begin{bmatrix} \cos(\cos t - \cos\tau) & -\sin(\cos t - \cos\tau) \\ \sin(\cos t - \cos\tau) & \cos(\cos t - \cos\tau) \end{bmatrix} \tag{15}$$

where τ is a normalization constant. In order to normalize the principal period T to unity, we set $t = 2\pi t'$ and $\tau = 2\pi$, yielding

$$\Phi(t') = e^{[(1/2\pi)\sin(2\pi t')]} \begin{bmatrix} \cos\left(\frac{1}{2\pi}[\cos(2\pi t') - 1]\right) & -\sin\left(\frac{1}{2\pi}[\cos(2\pi t') - 1]\right) \\ \sin\left(\frac{1}{2\pi}[\cos(2\pi t') - 1]\right) & \cos\left(\frac{1}{2\pi}[\cos(2\pi t') - 1]\right) \end{bmatrix} \tag{16}$$

Evaluating the STM at the end of the principal period one obtains

$$\Phi(1) = e^{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

This is a unique situation where the STM itself is periodic with initial condition $\Phi(0) = \Phi(\mathbf{T}) = \mathbf{I}$. As a result we find that the constant matrix $\mathbf{C}$ is in fact real over the principal period $\mathbf{T}$, i.e.,

$$\mathbf{C} = \begin{bmatrix} \ln 1 & 0 \\ 0 & \ln 1 \end{bmatrix} \tag{18}$$

Thus

$$e^{\mathbf{C}t} = \begin{bmatrix} e^{t\ln 1} & 0 \\ 0 & e^{t\ln 1} \end{bmatrix} = \mathbf{I} \tag{19}$$

The L-F transformation matrix then reduces down identically equal to the state transition matrix, i.e.,

$$\mathbf{L}(t) = \Phi(t) \tag{20}$$

3.2 Computation of L-F Transformation Matrix via Chebyshev Polynomials. Following the works of Sinha and Wu (1991a), Sinha et al. (1993) and Sinha and Juneja (1991b), the solution vector $\mathbf{x}(t)$ and the periodic matrix $\mathbf{A}(t)$ in Eq. (3) are expanded in terms of the shifted Chebyshev polynomials in the interval $[0, T]$ as

$$x_i(t) \approx \sum_{r=0}^{m-1} b_r^i s_r^*(t) \equiv s^{*T}(t)\mathbf{b}^i, \quad i = 1, 2 \ldots, n \tag{21}$$

$$\mathbf{A}(t) \approx \sum_{r=0}^{m-1} d_r^{ij} s_r^*(t) \equiv s^{*T}(t)\mathbf{d}^{ij}, \quad i, j = 1, 2, \ldots, n \tag{22}$$

where

$$\mathbf{b}^i = \{b_0^i b_1^i \ldots b_{m-1}^i\}^T, \quad \mathbf{d}^{ij} = \{d_0^{ij} d_1^{ij} \ldots d_{m-1}^{ij}\}^T, \quad \text{and}$$

$$s^{*T}(t) = \{s_0^*(t) s_1^*(t) \ldots s_{m-1}^*(t)\}.$$

Here b_r^i are unknown expansion coefficients of $x_i(t)$, d_r^{ij} are known expansion coefficients of $A_{ij}(t)$ and $s_r^*(t)$ are the shifted Chebyshev polynomials of the first kind. $\{\cdot\}^T$ represents the transpose of the quantity $\{\cdot\}$. For convenience in algebraic manipulation an $(n \times nm)$ Chebyshev polynomial matrix is defined as

$$\hat{\mathbf{S}}(t) = \mathbf{I} \otimes s^{*T}(t) \tag{23}$$

where $\otimes$ represents the Kronecker product (Bellman, 1970b) (also see Appendix-A), and $\mathbf{I}$ is an $(n \times n)$ identity matrix. Using the representations given by Eqs. (21), (22) and (23), $\mathbf{x}(t)$ and $\mathbf{A}(t)$ can be rewritten as

$$\mathbf{x}(t) = \hat{\mathbf{S}}(t)\bar{\mathbf{b}}; \quad \mathbf{A}(t) = \hat{\mathbf{S}}(t)\mathbf{D} \quad \text{and}$$
$$\mathbf{A}(t)\mathbf{x}(t) = \hat{\mathbf{S}}(t)\bar{\mathbf{Q}}\bar{\mathbf{b}} \tag{24}$$

where $\bar{\mathbf{b}} = \{\mathbf{b}^1\mathbf{b}^2\mathbf{b}^3 \ldots \mathbf{b}^n\}^T$ is an $(nm \times 1)$ vector, $\mathbf{D} = [\mathbf{d}^{i1}\mathbf{d}^{i2}\mathbf{d}^{i3} \ldots \mathbf{d}^{ij}]$, $i, j = 1, 2, 3, \ldots n$ is an $(nm \times n)$ matrix and $\bar{\mathbf{Q}}$ is an $(nm \times nm)$ *product operation matrix* given in the Appendix-A (for details see Sinha and Wu, 1991a).

The integral form of Eq. (3) is

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{A}(\xi)\mathbf{x}(\xi)d\xi \tag{25}$$

where ξ represents a dummy variable. Substituting Eq. (24) in

(25) and following the approach of Sinha and Wu (1991a), one can obtain a set of linear algebraic equations of the form

$$[\mathbf{I} - \hat{\mathbf{Z}}]\bar{\mathbf{b}} = \mathbf{x}(0) \qquad (26)$$

where $\hat{\mathbf{Z}}$ is an $(nm \times nm)$ constant matrix defined in Appendix-A by Eq. (A2) and $\bar{\mathbf{b}}$ is the vector of unknown Chebyshev coefficients. Once the $\mathbf{b}^i$ are obtained from Eq. (26), the solution for $\mathbf{x}(\mathbf{t})$ is given by Eq. (21) which simply represents a super convergent power series in t (see Fox and Parker, 1968).

In order to compute the L-F transformation matrix, $\mathbf{L}(\mathbf{t})$, one needs to find the state transition matrix $\mathbf{\Phi}(\mathbf{t})$ associated with the linear system given by Eq. (3). This requires a set of solutions of Eq. (3) with n initial conditions: $\mathbf{x}_i(0) = (1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. It is to be noted that all $\bar{\mathbf{b}}_i$'s corresponding to the above set of initial conditions can be determined simultaneously by defining the right hand side of Eq. (26) in the matrix form. Then the STM is given by

$$\mathbf{\Phi}(\mathbf{t}) = \hat{\mathbf{S}}(\mathbf{t})\bar{\mathbf{B}} \qquad (27)$$

where $\bar{\mathbf{B}} = [\bar{\mathbf{b}}_1\bar{\mathbf{b}}_2\bar{\mathbf{b}}_3 \ldots \bar{\mathbf{b}}_n]$ and $\mathbf{\Phi}(0) = \mathbf{I}$. It has to be noted that this STM is valid only for $0 \le \mathbf{t} \le T$ since the shifted Chebyshev polynomials of the first kind are defined over the interval $[0, T]$. When $\mathbf{t} > T$, the STM can be evaluated using the formula (Coddington and Levinson, 1955)

$$\mathbf{\Phi}(\mathbf{t}) = [\mathbf{\Phi}(\psi)][\mathbf{\Phi}(\mathbf{T})]^n \qquad (28)$$

where $t = nT + \psi$, $\psi \in [0, T]$, *and* $n = 1, 2, 3, \ldots$. Once $\mathbf{\Phi}(\mathbf{t})$ is known, the T-periodic complex matrix $\mathbf{L}(\mathbf{t})$ or the 2T-periodic real matrix $\mathbf{Q}(\mathbf{t})$ can be computed in the following way (Arrowsmith and Place, 1990). Since $\mathbf{\Phi}(0) = \mathbf{I}$, $\mathbf{L}(0) = \mathbf{L}(\mathbf{T}) = \mathbf{I}$, the Floquet Transition Matrix (FTM), $\mathbf{\Phi}(\mathbf{T})$ can be written as

$$\mathbf{\Phi}(\mathbf{T}) = \mathbf{e}^{\mathbf{CT}} \qquad (29)$$

where $\mathbf{C}$ is a $n \times n$ constant complex matrix. By performing an eigen-analysis on $\mathbf{\Phi}(\mathbf{T})$, the matrix $\mathbf{C}$ can be computed easily. Then the T-periodic L-F transformation matrix is

$$\mathbf{L}(\mathbf{t}) = \mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-\mathbf{Ct}} \qquad (30)$$

In order to evaluate the 2T-periodic real L-F transformation matrix $\mathbf{Q}(\mathbf{t})$, first we note that (c.f. Coddington and Levinson, 1955)

$$\mathbf{\Phi}(2\mathbf{T}) = \mathbf{\Phi}^2(\mathbf{T}) = \mathbf{e}^{\mathbf{CT}}\mathbf{e}^{\mathbf{C^*T}} = \mathbf{e}^{2\mathbf{RT}} \qquad (31)$$

where $\mathbf{C}^*$ is the conjugate matrix of $\mathbf{C}$, the $n \times n$ constant real matrix $\mathbf{R} = \{\mathbf{C} + \mathbf{C}^*\}/2$ and the 2-T periodic L-F matrix $\mathbf{Q}(\mathbf{t})$ can be represented as

$$\mathbf{Q}(\mathbf{t}) = \mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-\mathbf{Rt}}; \quad 0 \le \mathbf{t} \le \mathbf{T}$$
$$\mathbf{Q}(\tau + \mathbf{T}) = \mathbf{\Phi}(\tau)\mathbf{Q}(\mathbf{T})\mathbf{e}^{-\mathbf{R}\tau};$$
$$\mathbf{T} \le (\mathbf{T} + \tau) \le 2\mathbf{T}; \quad 0 \le \tau \le \mathbf{T} \qquad (32)$$

It should be noted that $\mathbf{Q}(\mathbf{t}) = \mathbf{Q}(\mathbf{t} + 2\mathbf{T})$. If one is interested in finding $\mathbf{L}^{-1}(\mathbf{t})$ or $\mathbf{Q}^{-1}(\mathbf{t})$, then there are two avenues. $\mathbf{L}(\mathbf{t})$ and $\mathbf{Q}(\mathbf{t})$ can possibly be inverted through a symbolic software like MACSYMA/MATHEMATICA/MAPLE or one can first find the STM $\mathbf{\Psi}(\mathbf{t})$ of the adjoint system [corresponding to Eq. (3)]

$$\dot{\eta} = -\mathbf{A}^{\mathrm{T}}(\mathbf{t})\eta \qquad (33)$$

and use the following relationship (c.f. Yakubovich and Starzhinski, 1975),

$$\mathbf{\Phi}^{-1}(\mathbf{t}) = \mathbf{\Psi}^{\mathrm{T}}(\mathbf{t}) \qquad (34)$$

The computation of $\mathbf{\Phi}^{-1}(\mathbf{t})$ is essential in determining $\mathbf{L}^{-1}(\mathbf{t})$ or $\mathbf{Q}^{-1}(\mathbf{t})$. For example, the inverse T-periodic L-F transformation matrix $\mathbf{L}^{-1}(\mathbf{t})$ can be evaluated utilizing the properties of the adjoint system as shown below.

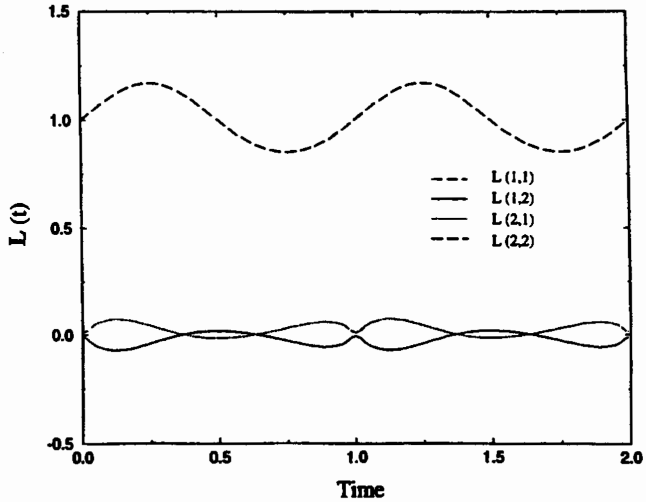

Fig. 1 Liapunov-Floquet transformation matrix 9 term expansion on the commutative system

$$\mathbf{L}^{-1}(\mathbf{t}) = [\mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-\mathbf{Ct}}]^{-1} = \mathbf{e}^{\mathbf{Ct}}\mathbf{\Phi}^{-1}(\mathbf{t}) = \mathbf{e}^{\mathbf{Ct}}\mathbf{\Psi}^{\mathrm{T}}(\mathbf{t}) \quad (35)$$

It should be observed that the accuracy of $\mathbf{L}(\mathbf{t})$ or $\mathbf{L}^{-1}(\mathbf{t})$ is dependent upon the number of Chebyshev terms used in the computation of the STM. Depending upon the size of the system, an optimum number of terms can always be selected (see Wu and Sinha, 1994 and Joseph et al., 1993) by incorporating a desired tolerance parameter in the computational scheme.

Since the elements $\mathbf{L}_{ij}(\mathbf{t})$ or $\mathbf{Q}_{ij}(\mathbf{t})$ are periodic with period $\mathbf{T}$, they have the truncated Fourier representation

$$\mathbf{L}_{ij}(\mathbf{t}) \approx \sum_{n=-q}^{q} \mathbf{c}_n \exp(\mathrm{i}2\pi \mathrm{nt}/\mathbf{T}), \quad \mathrm{i} = \sqrt{-1} \qquad (36)$$

or

$$\mathbf{Q}_{ij}(\mathbf{t}) \approx \frac{\mathbf{a}_0}{2} + \sum_{n=1}^{q} \mathbf{a}_n \cos(\pi \mathrm{nt}/\mathbf{T}) + \sum_{n=1}^{q} \mathbf{b}_n \sin(\pi \mathrm{nt}/\mathbf{T}) \quad (37)$$

Since the complex matrix $\mathbf{L}(\mathbf{t})$ (or the real matrix $\mathbf{Q}(\mathbf{t})$) can be computed as a function of $\mathbf{t}$, all algebraic manipulations involving this matrix can be done in symbolic form. $\mathbf{L}_{ij}^{-1}(\mathbf{t})$ and $\mathbf{Q}_{ij}^{-1}(\mathbf{t})$ have similar Fourier representations.

Since we have an exact L-F transformation $\mathbf{L}(\mathbf{t})$ for the commutative system discussed in Section 3.1, it seems natural to compare the exact result with the result obtained by the Chebyshev expansion technique. After some algebraic manipulations, we find that the $\mathbf{I} - \hat{\mathbf{Z}}$ matrix, appearing in Eq. (26) takes the form

$$\mathbf{I} - \hat{\mathbf{Z}} = \begin{bmatrix} \mathbf{I} - \mathbf{G}^{\mathrm{T}}\dot{\mathbf{C}} & \mathbf{G}^{\mathrm{T}}\dot{\mathbf{S}} \\ \mathbf{G}^{\mathrm{T}}\dot{\mathbf{S}} & \mathbf{I} - \mathbf{G}^{\mathrm{T}}\dot{\mathbf{C}} \end{bmatrix} \qquad (38)$$

where $\mathbf{C}$ and $\mathbf{S}$ are the coefficient matrices of expansions of *sine* and *cosine* functions, respectively in the form of Eq. (21) (see Sinha et al., 1993) and $\mathbf{G}$ is as given in Appendix-A. The STM and the L-F transformation matrix were then computed as explicit functions of time for a various number of terms in the Chebyshev expansion. Since theoretically $\mathbf{L}(\mathbf{T}) \equiv \mathbf{I}$, $\|\mathbf{L}(\mathbf{T}) - \mathbf{I}\| < \epsilon$ was used as the error control criterion, where ϵ is a small positive number. As representative samples, the computed L-F transformation matrix is plotted in Figs. 1 and 2 over the first two periods for 9 term and 16 term expansions, respectively. As seen from Fig. 2, the 16 term solution and the exact result are identical.

4 Some General Results and Application to the Mathieu Equation

4.1 Computation of L-F Transformation Matrix. In this section we propose to study the general free and harmonically forced problems and present specific results for the Mathieu equation of the form

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} 0 & 1 \\ -(\alpha^2 + \beta \cos \omega t) & 0 \end{bmatrix} \mathbf{x}(\mathbf{t}) + \begin{Bmatrix} 0 \\ \mathbf{F}(\Omega, \mathbf{t}) \end{Bmatrix} \quad (39)$$

The L-F transformation matrix is computed for Eq. (39) using the proposed Chebyshev expansion scheme. Following the steps outlined in Section 3.2, it can be shown that the matrix $\mathbf{I} - \hat{\mathbf{Z}}$, appearing in Eq. (26) takes the form

$$[\mathbf{I} - \hat{\mathbf{Z}}] = \begin{bmatrix} \mathbf{I} & -\mathbf{G}^{\mathbf{T}} \\ -\mathbf{G}^{\mathbf{T}}(\alpha^2 + \beta\bar{\mathbf{Q}})(2\pi/\omega)^2 & \mathbf{I} \end{bmatrix} \quad (40)$$

After the coefficient vector $\bar{\mathbf{b}}$ is determined, the STM $\mathbf{\Phi}(\mathbf{t})$ is computed using the representation given in Eq. (27). The L-F transformation matrix $\mathbf{L}(\mathbf{t})$ or $\mathbf{Q}(\mathbf{t})$ is then computed from Eq. (30) or (32). A convergence study of the elements of $\mathbf{Q}(\mathbf{t})$ as a function of the number of Chebyshev expansion terms has been presented in Table 1. The Table indicates the values of the elements sampled at $T/4$, $3T/4$ and T where the Period $T = 2$ Secs. Of course, at $t = 2$, $\mathbf{Q}_{11} + \mathbf{Q}_{22} \rightarrow 2$ and $\mathbf{Q}_{12} + \mathbf{Q}_{21} \rightarrow 0$ as the number of terms increase.

At this point, it should be pointed out that for small β, the L-F transformation matrix can also be obtained by the perturbation method as shown by Yakubovich and Starzhinski (1975). For this procedure to be applicable to Eq. (3), the matrix $\mathbf{A}(\mathbf{t})$ should be of the form $\mathbf{A}(\mathbf{t}) \equiv \mathbf{A}_0 + \beta\bar{\mathbf{A}}_1(\mathbf{t})$. According to this procedure the STM, $\mathbf{\Phi}(\mathbf{t})$ of Eq. (3) may be expressed as

$$\mathbf{\Phi}(\mathbf{t}) = \mathbf{L}(\mathbf{t}, \beta)e^{\mathbf{C}(\beta)\mathbf{t}}$$

where

$$\mathbf{L}(\mathbf{t}, \beta) = \mathbf{I} + \beta\mathbf{L}_1(\mathbf{t}) + \beta^2\mathbf{L}_2(\mathbf{t}) + \ldots,$$
$$\mathbf{C}(\beta) = \mathbf{C}_0 + \beta\mathbf{C}_1 + \beta^2\mathbf{C}_2 + \ldots, \quad (41)$$

$\mathbf{C}(0) = \mathbf{C}_0$ and $\mathbf{L}(\mathbf{t} + \mathbf{T}, \beta) \equiv \mathbf{L}(\mathbf{t}, \beta)$. Substituting Eq. (41) in Eq. (3) yields

$$\mathbf{dL}/\mathbf{dt} = \mathbf{A}(\mathbf{t})\mathbf{L}(\mathbf{t}, \beta) - \mathbf{L}(\mathbf{t}, \beta)\mathbf{C}(\beta) \quad (42)$$

Comparing the coefficients of like powers of β, a sequence of matrix differential equations can be obtained. However, in the

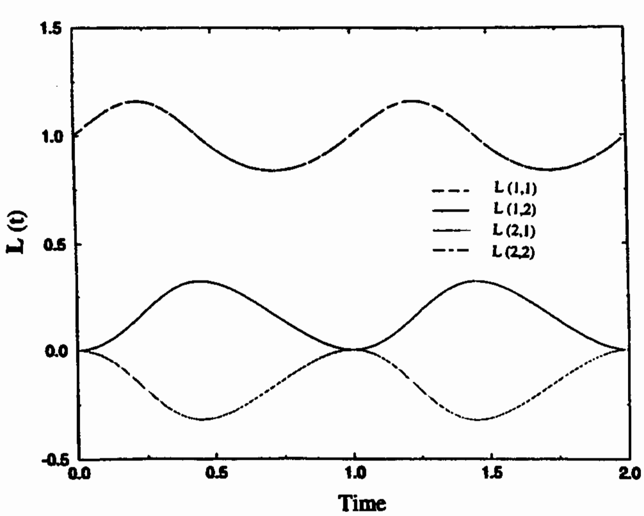

Fig. 2 Liapunov-Floquet transformation matrix 16 term expansion and the exact solution for the commutative system

Table 1 Convergence of L-F transformation elements $\alpha^2 = 0.5$, $\beta = 4.0$, $\omega = 2.0$

Time (Secs)	No. of Chebyshev Coefficients	$\sum Q_{ij}(t), i=j$	$\sum Q_{ij}(t), i \neq j$
0.5	6.0	4.95885d-02	-5.93414
	8.0	2.73162d-03	-5.15794
	10.0	-2.54377d-03	-5.16673
	12.0	2.88849d-04	-5.22464
	14.0	1.94931d-05	-5.22910
	16.0	-8.53952d-06	-5.22871
	18.0	8.34913d-07	-5.22859
	20.0	1.84261d-08	-5.22858
1.5	6.0	-4.95885d-02	5.93414
	8.0	-2.73162d-03	5.15794
	10.0	2.54377d-03	5.16672
	12.0	-2.88850d-04	5.22464
	14.0	-1.94936d-05	5.22910
	16.0	8.53984d-06	5.22871
	18.0	-8.35125d-07	5.22859
	20.0	-1.85315d-08	5.22858
2.0	6.0	1.86490	1.30012
	8.0	1.99586	0.12003
	10.0	2.00941	-8.42177d-02
	12.0	1.99881	8.57687d-03
	14.0	1.99991	9.09347d-04
	16.0	2.00004	-3.12005d-04
	18.0	2.00000	2.69006d-05
	20.0	2.00000	1.28317d-06

following, differential equations corresponding to the powers of β and β^2 only are shown as

$$\mathbf{A}_0 = \mathbf{C}_0$$
$$\mathbf{dL}_1/\mathbf{dt} = \mathbf{C}_0\mathbf{L}_1(\mathbf{t}) - \mathbf{L}_1(\mathbf{t})\mathbf{C}_0 + \bar{\mathbf{A}}_1(\mathbf{t}) - \mathbf{C}_1$$
$$\mathbf{dL}_2/\mathbf{dt} = \mathbf{C}_0\mathbf{L}_2(\mathbf{t}) - \mathbf{L}_2(\mathbf{t})\mathbf{C}_0 + \bar{\mathbf{A}}_1(\mathbf{t})\mathbf{L}_1(\mathbf{t}) - \mathbf{L}_1(\mathbf{t})\mathbf{C}_1 - \mathbf{C}_2 \quad (43)$$

The solution of the above set of matrix differential equations is anything but simple. However, for the special choice of $\mathbf{C}_0 = 0$ or $c\mathbf{I}$ where c is an arbitrary constant, a solution can be obtained without much difficulty.

To illustrate this approach, the homogeneous part of Eq. (39) is considered and is rewritten after normalizing with $\omega t = 2\pi \mathbf{t}'$ such that $\omega = 2$ as

$$\dot{\mathbf{x}}(\mathbf{t}') = \left\{ \begin{bmatrix} 0 & 1 \\ -\alpha^2\pi^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\beta\pi^2 \cos 2\pi \mathbf{t}' & 0 \end{bmatrix} \right\} \mathbf{x}(\mathbf{t}') \quad (44)$$

where β is assumed to be small. In the following, the $\mathbf{L}$ and $\mathbf{C}$ matrices are computed for the case when the parameter $\alpha^2 \equiv 0$. The other singular and nonsingular cases have been dealt with elaborately by Yakubovich and Starzhinski (1975). For this case, the constituting matrices up to the order of β and β^2 can be computed using the formulae

$$\mathbf{C}_1 = \frac{1}{\mathbf{T}} \int_0^{\mathbf{T}} \mathbf{A}(\mathbf{t})\mathbf{dt}; \quad \mathbf{L}_1(\mathbf{t}) = \int_0^{\mathbf{t}} [\mathbf{A}(\mathbf{t}_1) - \mathbf{C}_1]\mathbf{dt}_1$$
$$\mathbf{C}_2 = \frac{1}{\mathbf{T}} \int_0^{\mathbf{T}} [\mathbf{A}(\mathbf{t})\mathbf{L}_1(\mathbf{t}) - \mathbf{L}_1(\mathbf{t})\mathbf{C}_1]\mathbf{dt}$$
$$\mathbf{L}_2(\mathbf{t}) = \int_0^{\mathbf{t}} [[\mathbf{A}(\mathbf{t}_1)\mathbf{L}_1(\mathbf{t}_1) - \mathbf{L}_1(\mathbf{t}_1)\mathbf{C}_1] - \mathbf{C}_2]\mathbf{dt}_1 \quad (45)$$

Therefore, the L-F transformation matrix and the corresponding exponent matrix, $\mathbf{C}(\beta)$ can be obtained using Eq. (41) in terms of the small parameter β as

$$\mathbf{L}(\mathbf{t}, \beta) = \begin{bmatrix} 1 + \frac{\beta\pi^2}{4}(\cos 2\pi\mathbf{t}' - 1) & 0 \\ -\sin\frac{2\pi\mathbf{t}'}{2} & 1 - \frac{\beta\pi^2}{4}(\cos 2\pi\mathbf{t}' - 1) \end{bmatrix} \quad (46)$$

and

$$\mathbf{C}(\beta) = \begin{bmatrix} 0 & \beta\pi^2 \\ 0 & 0 \end{bmatrix}$$

For $\beta = 0.001$, the exponent matrix, $\mathbf{C}(\beta)$ is computed using the Chebyshev expansion and the perturbation methods and is recorded below

$$\mathbf{C}_{\text{pert}} = \begin{bmatrix} 0 & 0.0099 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C}_{\text{Cheb}} = \begin{bmatrix} 0 & 1.0005 \\ 0 & 0 \end{bmatrix} \quad (47)$$

With $\beta = 0.01$, they take the forms

$$\mathbf{C}_{\text{pert}} = \begin{bmatrix} 0 & 0.0988 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C}_{\text{Cheb}} = \begin{bmatrix} 0 & 1.0050 \\ -0.0001 & 0 \end{bmatrix} \quad (48)$$

It can be readily observed from the exponent matrix $\mathbf{C}_{\text{Cheb}}$ that even in such a small range of β, the character of the system changed from two zero characteristic exponents to two purely imaginary characteristic exponents. However, the exponent matrix obtained by perturbation method $\mathbf{C}_{\text{pert}}$ always yields two zero eigenvalues for all values of β and hence fails to provide the true system characteristics. Therefore, the perturbation procedure (at least the first order) is not suitable for this problem.

The important advantage of the L-F transformation matrix is that it allows one to study the time-periodic systems in the time-invariant space. Once the T periodic Liapunov-Floquet transformation matrix, $\mathbf{L}(\mathbf{t})$ is computed, Eq. (39) takes the form

$$\dot{\mathbf{z}}(\mathbf{t}) = \mathbf{C}\mathbf{z}(\mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{F}(\Omega, \mathbf{t}) \quad (49)$$

by applying Eq. (4), $\mathbf{x}(\mathbf{t}) = \mathbf{L}(\mathbf{t})\mathbf{z}(\mathbf{t})$ where $\mathbf{C}$ is a complex matrix. The solution of the above equation is given by

$$\mathbf{z}(\mathbf{t}) = \mathbf{e}^{\mathbf{C}\mathbf{t}}\mathbf{Z}(0) + \int_0^{\mathbf{t}} \mathbf{e}^{\mathbf{C}(\mathbf{t}-\mathbf{s})}\{\mathbf{L}^{-1}(\mathbf{s})\mathbf{F}(\Omega, \mathbf{s})\}\mathbf{ds} \quad (50)$$

where $\mathbf{Z}(0) = \{\mathbf{z}(0), \dot{\mathbf{z}}(0)\}^{\mathbf{T}}$. It should be noted that the above solution is valid for arbitrary $\mathbf{F}(\Omega, \mathbf{t})$ and can certainly be used to find response due to stochastic excitations as well. However, in this study $\mathbf{F}(\Omega, \mathbf{t})$ is restricted to a periodic function.

The homogeneous solution of Eq. (39) as obtained from Eq. (50) are plotted in Figs. 3 and 4 against Runge-Kutta numerical solutions for various parameter combinations. In both cases 16 term Chebyshev solutions are virtually identical to the Runge-Kutta results while the 9 term solutions also yield good approximations to the numerically integrated trajectories. For the stable case in Fig. 3, β is small and the solution obtained by the averaging method is quite accurate. However, such is not the case for the unstable solution shown in Fig. 4 because β is no longer small.

4.2 Stability and External Resonance Conditions. The stability conditions for the periodic Eq. (3) can be easily determined from the homogeneous part of Eq. (49) through a discussion of the eigenvalues of the time-invariant matrix $\mathbf{C}$. For the special case of Mathieu Eq. (39), it is well known that for sufficiently small β, the internal resonance condition is given by $\omega = 2\alpha$. This corresponds to the case when the $\mathbf{C}$ matrix (2 $\times$ 2) has a repeated zero eigenvalue. Similar situations arise in higher order systems.

In the following, it is shown that the proposed technique can be effectively applied to obtain the conditions for external resonances for a general $n \times n$ linear periodic system of the form given by Eq. (49). Since the elements of $\mathbf{L}^{-1}(\mathbf{t})$ are periodic functions of time with frequency ω, and $\mathbf{F}(\Omega, \mathbf{t})$ can be expanded in a Fourier series, $\mathbf{L}^{-1}(\mathbf{t})\mathbf{F}(\Omega, \mathbf{t})$ can be represented as

$$\mathbf{L}^{-1}(\mathbf{t})\mathbf{F}(\Omega, \mathbf{t}) = \sum_{\mu=-s}^{s}\sum_{m=-r}^{r} \mathbf{a}_{\mu m}\mathbf{e}^{\mathbf{i}(\mu\omega \pm m\Omega)\mathbf{t}}; \quad \mathbf{i} = \sqrt{-1} \quad (51)$$

where $\mathbf{a}_{\mu m}$ are arbitrary complex numbers.

At this point the resonance conditions are apparent if one looks at the eigenvalues of matrix $\mathbf{C}$. In general, the eigenvalues

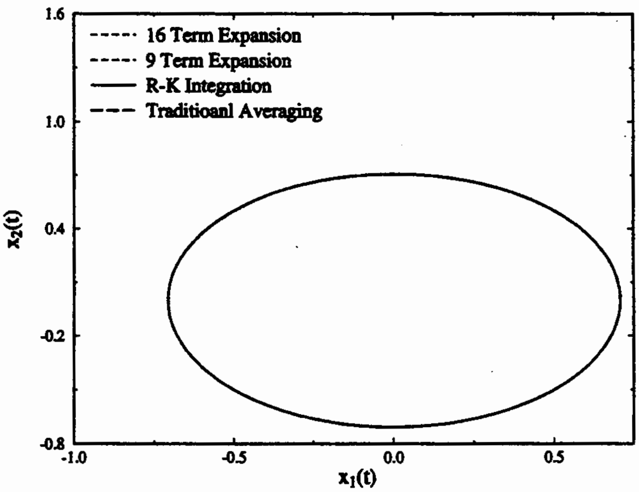

Fig. 3 Linear homogeneous Mathieu equation $\alpha^2 = 1/(4\pi^2)$, $\beta = 0.01\alpha^2$, $\omega = 1$, $IC = \{0.5, 0.5\}^T$

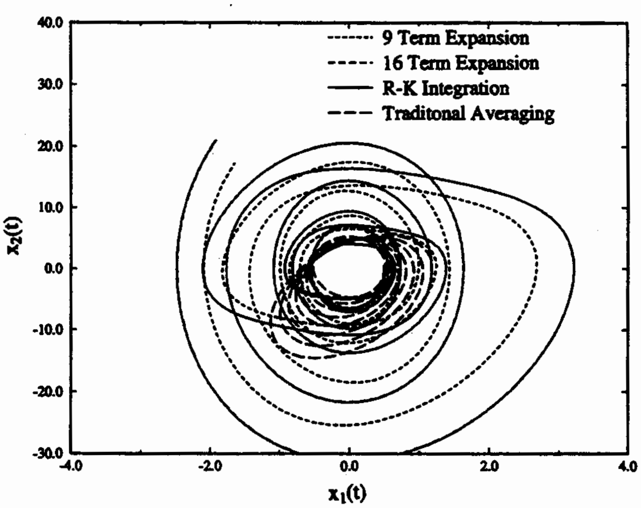

Fig. 4 Linear homogeneous Mathieu equation $\alpha^2 = 2.5$, $\beta = 2.0$, $\omega = 1$, $IC = \{0.5, 0.5\}^T$

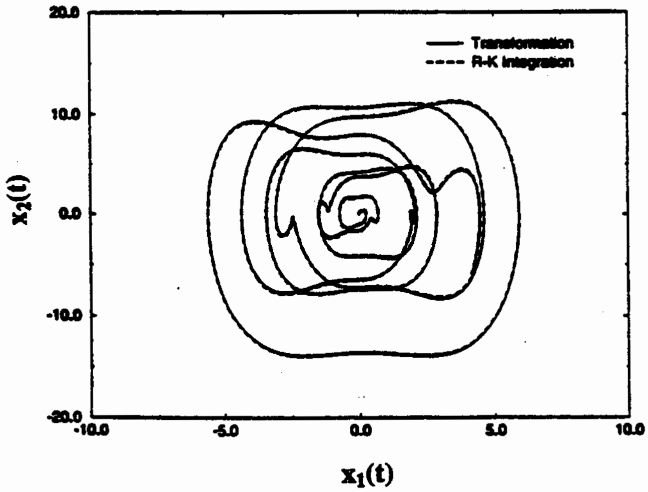

Fig. 5 Mathieu equation $n = m = 1$, $\mu = 0$ resonance $\alpha^2 = 4.0$, $\beta = 7.468$, $\omega = 2\pi$, $\Omega = 2.3257297$, $t \rightarrow 20$Secs

are complex, that is, of the form $\alpha_n \pm i\omega_n$. Therefore the resonance conditions are given by

$$\pm\omega_n = \mu\omega \pm m\Omega;\ m, \mu = 0, 1, 2, \ldots,\ n = 1, 2, \ldots \tag{52}$$

As an example, consider the case of Mathieu equation such that the roots of the matrix $\mathbf{C}$ has a pair of purely imaginary roots and if $\mathbf{F}(\Omega, \mathbf{t}) \equiv \mathbf{F} \cos \Omega\mathbf{t}$, the sequence of forcing frequencies Ω which give rise to resonances can be obtained from Eq. (52) by setting $n = m = 1$ and $\mu = 0, 1, 2, \ldots$. This case is particularly interesting because it corresponds to the bifurcation point if one is dealing with a nonlinear problem where the linear part consists of a Mathieu type system. The results for one case of purely imaginary roots are plotted for the first three resonant forcing frequencies in Figs. 5–6. In each case the solution to the original equation increases without bound, however, it is noted that the amplitude of the harmonic corresponding to $\mu = 0$ increases most rapidly. The amplitudes of the higher harmonics also increase but at much slower rates.

5 Analysis of Parametrically Excited Nonlinear Systems

In the following, the analysis of quasilinear Eq. (2) with $\mathbf{F}(\Omega, \mathbf{t}) \equiv 0$ is considered. First, the 2T L-F transformation matrix $\mathbf{Q}(\mathbf{t})$ is used such that

$$\mathbf{x}(\mathbf{t}) = \mathbf{Q}(\mathbf{t})\mathbf{z}(\mathbf{t}) \tag{53}$$

transforms Eq. (2) to a vector field where the linear part is time-invariant. The resulting equation is of the form

$$\dot{\mathbf{z}} = \mathbf{R}\mathbf{z} + \mathbf{Q}^{-1}\mathbf{E}(\mathbf{Q}\mathbf{z}, \mathbf{t}) \tag{54}$$

where $\mathbf{R}$ is an $n \times n$ real constant matrix. The form of Eq. (54) is amenable to direct application of the method of *time dependent normal forms* (TDNF) for equations with periodic coefficients as shown by Arnold (1988).

Equation (54) in its Jordan canonical form can be written as

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{w}_2(\mathbf{y}, \mathbf{t}) + \ldots + \mathbf{w}_k(\mathbf{y}, \mathbf{t}) + \ldots \mathbf{O}(|\mathbf{y}|^{k+1}, \mathbf{t}) \tag{55}$$

where $\mathbf{J}$ is the Jordan form of matrix $\mathbf{R}$ and $\mathbf{w}_k$'s are 2T-periodic functions and contain homogeneous monomials of $\mathbf{y}_i$ of order $\mathbf{k}$. The aim of the normal form is to construct a sequence of transformations which successively remove the nonlinear terms $\mathbf{w}_r(\mathbf{y}, \mathbf{t})$ for $\mathbf{r} = 2, \ldots k$, where k is the highest order considered. The transformation is of the form

$$\mathbf{y} = \mathbf{v} + \mathbf{h}_r(\mathbf{v}, \mathbf{t}) \tag{56}$$

where $\mathbf{h}_r(\mathbf{v}, \mathbf{t})$ is a formal power series in $\mathbf{v}$ of degree $\mathbf{r}$ with periodic coefficients having the principal period 2T. Applying the above change of coordinates, and following the procedure given in Arrowsmith and Place (1990) the solvability condition for the given degree of nonlinearity can be obtained as

$$\mathbf{h}_{r,j,\sigma} = \mathbf{a}_{r,j,\sigma}/(i\sigma\rho + \mathbf{m}.\mathbf{\Lambda} - \lambda_j);\quad \sigma = \pi/\mathrm{T} \tag{57}$$

where $\mathbf{h}_{r,j,\sigma}$, $\mathbf{a}_{r,j,\sigma}$ are the Fourier coefficients of expansions of $\mathbf{h}_r(\mathbf{v}, \mathbf{t})$ and $\mathbf{w}_r(\mathbf{y}, \mathbf{t})$, respectively. $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of matrix $\mathbf{J}$. It is clear that when

$$(i\sigma\rho + \mathbf{m}.\mathbf{\Lambda} - \lambda_j) \neq 0\ \forall\sigma \tag{58}$$

Eq. (55) can be reduced to a linear form. Otherwise, the corresponding resonant terms will stay in the reduced equation so that Eq. (55) takes the simplest nonlinear form as

$$\dot{\mathbf{v}} = \mathbf{J}\mathbf{v} + \sum_{r=2}^{k} \mathbf{w}_r(\mathbf{v}, \mathbf{t}) + \ldots \mathbf{O}(|\mathbf{v}|^{k+1}, \mathbf{t}) \tag{59}$$

It is important to note that the $\mathbf{w}_r(\mathbf{v}, \mathbf{t})$ contains only a finite number q of Fourier harmonics. In order to obtain an approximate solution of Eq. (55), the variations of the periodic coefficients of nonlinear terms can be neglected in comparison with

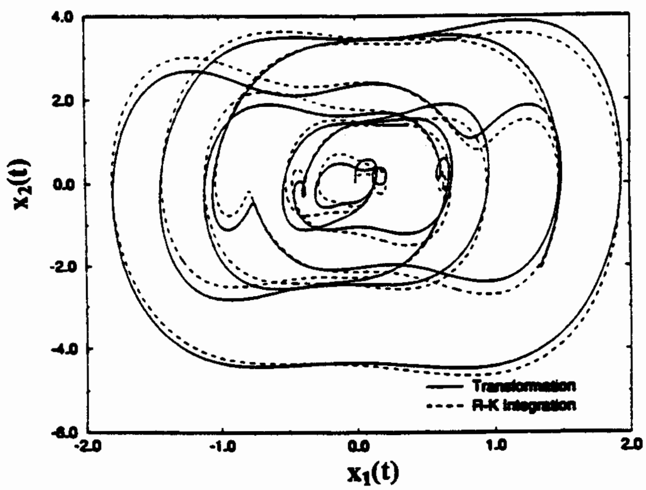

Fig. 6(a) Mathieu equation $n = m = 1$, $\mu = 1$ resonance $\alpha^2 = 4.0$, $\beta = 7.468$, $\omega = 2\pi$, $\Omega = 3.9574556$, $t \rightarrow 20$Secs

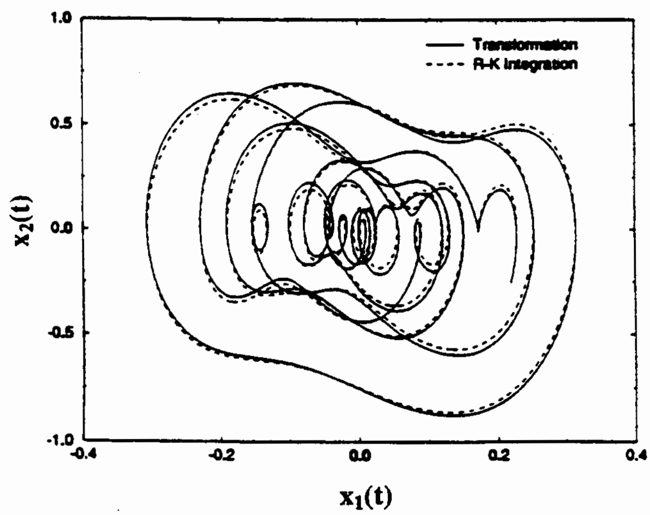

Fig. 6(b) Mathieu equation $n = m = 1$, $\mu = -1$ resonance $\alpha^2 = 4.0$, $\beta = 7.468$, $\omega = 2\pi$, $\Omega = 8.6089150$, $t \rightarrow 20$Secs their predominant means (Rosenblat and Cohen, 1980). This approximation results in an equation which is completely autonomous and the solution can be obtained via the *time-independent normal form theory* (TINF).

To demonstrate the applicability and effectiveness of the suggested approaches, a nonlinear Mathieu equation is considered. The parameters of the equation are selected in such a way that there exists no generating solution to apply first order averaging. The results of the proposed technique based on L-F transformation and normal forms provide reasonably good solutions even for moderately large parameters multiplying the nonlinear terms whereas the traditional averaging procedure is applicable only when the parameters multiplying the linear periodic terms as well as the nonlinear terms are both small.

Example: Nonlinear Mathieu's Equation

Consider the Mathieu Equation with cubic nonlinearity of the form

$$\ddot{x} + \delta\dot{x} + (\alpha^2 + \beta\cos\omega t)x + \epsilon x^3 = 0 \qquad (60)$$

where δ, α, β, ω and ϵ are the parameters of the system. Applications of the real L-F transformation $\mathbf{Q}(t)$, the Jordan canonical transformation, and the near-identity transformation to Eq. (60) in the state space form yields

$$\begin{Bmatrix}\dot{y}_1\\ \dot{y}_2\end{Bmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} + \epsilon\begin{Bmatrix}f_{11}(t,\tau)y_1^3 + f_{12}(t,\tau)y_1^2y_2 + f_{13}(t,\tau)y_1y_2^2 + f_{14}(t,\tau)y_2^3\\ f_{21}(t,\tau)y_1^3 + f_{22}(t,\tau)y_1^2y_2 + f_{23}(t,\tau)y_1y_2^2 + f_{24}(t,\tau)y_2^3\end{Bmatrix} \qquad (61)$$

where $\tau = 2T$, $\Lambda = \{\lambda_1, \lambda_2\}$ are the eigenvalues of $\mathbf{R}$ and the periodic coefficients $f_{ij}(t, \tau)$; $i, j = 1, \ldots, 4$ are expressed as

$$f_{ij}(t,\tau) = a_0^{ij} + \sum_{n-1}^{l} a_n^{ij}\cos\left(\frac{2\pi nt}{\tau}\right) + b_n^{ij}\sin\left(\frac{2\pi nt}{\tau}\right) \qquad (62)$$

After experimenting with various sets of system parameters, it was observed that $l = 15$ to 18 provided accurate representations of functions $f_{ij}(t, \tau)$. It is also consistent with the number of Fourier terms taken in the representation of the L-F transformation $\mathbf{Q}(t)$.

In order to obtain a solution of Eq. (61) using TDNF, consider a near-identity nonlinear transformation

$$\begin{aligned} y_1 &= u + g_{11}(t,\tau)u^3 + g_{12}(t,\tau)u^2v + g_{13}(t,\tau)uv^2 + g_{14}(t,\tau)v^3\\ y_2 &= v + g_{21}(t,\tau)u^3 + g_{22}(t,\tau)u^2v + g_{23}(t,\tau)uv^2 + g_{24}(t,\tau)v^3 \end{aligned} \qquad (63)$$

where the periodic coefficients $g_{ij}(t, \tau)$; $i, j = 1, \ldots, 4$ are once again of the form given by Eq. (62) but with unknown constants $\bar{a}_n$ and $\bar{b}_n$. Substituting Eq. (63) in Eq. (61) and solving the resulting homological equation, the unknown constants $\bar{a}_n$ and $\bar{b}_n$ can be evaluated. In situations when there is no resonance, the Fourier series assumed for $g_{ij}(t, \tau)$ and its derivative are found to be convergent (Arnold, 1988). Therefore, the solution of the nonlinear Mathieu's equation in the original coordinates can be obtained by substituting back all the intermediate transformations. Even when some of the nonlinear terms remain due to resonance, the resulting equation can still be used to provide many useful conclusions about the stability and dynamical behavior of the system. Such procedures are described in the literature (Bruno, 1989; Hale and Kocak, 1991).

Equation without Generating Solution ($\alpha^2 = 0$). Note that when $\alpha^2 = 0$ the fundamental frequency ω_0 of the autonomous part of Eq. (60) is also zero and there is no generating solution in the classical sense. Consequently, the averaging or perturbation method cannot be applied to this case.

Parameter Set 1: $\alpha^2 = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\epsilon = 0.3$;

For this set, the solutions based on TDNF and TINF theory have been found to predict the dynamic behavior quite precisely in comparison with the numerical solution as shown in Fig. 7.

Parameter Set 2: $\alpha^2 = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\epsilon = 3.0$;

This set shows that the behavior of the system is well predicted by the suggested TDNF and TINF techniques when compared with the numerical solution even though the nonlinearity parameter ϵ is 10 times larger than the value used in set 1. The comparison is shown in Fig. 8. An approximate analytical solution obtained using the L-F transformation and the time-dependent normal form theory pertaining to this parameter set is provided in Appendix B.

6 Conclusions

In this paper, a general technique for the computation of L-F transformation matrix in terms of the shifted Chebyshev polynomials is presented. As shown in literature, (Sinha and Wu, 1991a; Sinha et al., 1993; Joseph et al., 1993) this Chebyshev expansion technique is numerically efficient and can be successfully used for the computation of the state transition matrices for large scale systems. As shown in Section 3.2, the L-F transformation matrices can be evaluated as a by-product of these computations. On the basis of the convergence study reported by Joseph et al. (1993) and our experience with the various case studies reported here, a 12 to 18 term Chebyshev polynomial representations of the L-F transformation matrices provide excellent accuracy. This transformation converts the linear part of the quasilinear periodic dynamical equation to a time-invariant form regardless of the magnitude of the parameters associated with the periodic matrix. For all the linear cases considered in this study, it is evident that increasing the number of terms in the STM expansion increases the accuracy of the L-F transformation and the solution. As seen from Table 1, uniform convergence has been observed as the number of terms were increased from 6 to 20. For most single and double preci-

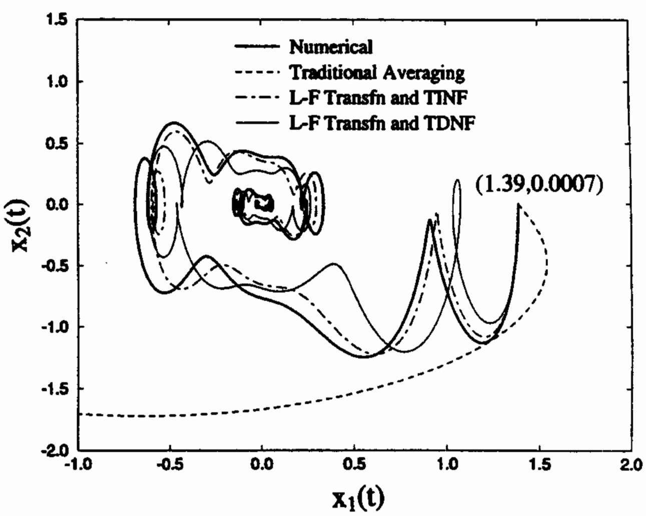

Fig. 7 Comparison of solution of Mathieu's equation with cubic nonlinearity $\alpha^2 = 0$, $\beta = 4.0$, $\delta = 0.4243$, $\epsilon = 0.3$

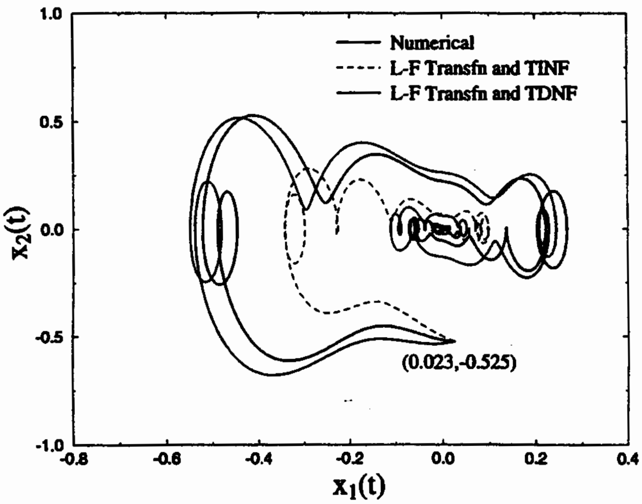

Fig. 8 Comparison of solution of Mathieu's equation with cubic nonlinearity $\alpha^2 = 0$, $\beta = 4.0$, $\delta = 0.4243$, $\epsilon = 3.0$

sion calculations, the 16 term expansion is found to be more than sufficient. Similar observations have been made for larger systems as well (see Pandiyan and Sinha, 1993).

The successful computational procedure for the L-F transformation has resulted in new methods for the analysis of general linear and quasilinear dynamic systems with periodically varying coefficients. Since the linear part can be made time-invariant, the solutions of transformed systems can be obtained using the standard time-invariant methods and converted back to the original coordinates via the inverse L-F transformation. In case of quasilinear periodic systems, the resulting dynamical equations can be analyzed using the *time-dependent normal form theory.*

The symbolic inversion of L-F transformation matrices is a computationally intensive task and therefore it is recommended that the adjoint equation of (3) be utilized to compute the inverse L-F transformation matrix, as suggested in this study. Applying the transformation to externally excited systems gives an accurate representation of the resonance conditions. The Mathieu equation example shows how easily and explicitly all resonance conditions can be determined through the proposed approach. It is observed that the solutions obtained via *time-dependent normal forms* show a slower behavior to the numerical solutions. Nevertheless, they eventually depict the correct behavior of the system. The reason for such a slow behavior can be attributed to different pointing directions of the averaged drift of such methods and the projection of the true motion (Arnold, 1988).

To conclude, the authors would like to state that it is for the first time the computation and application of Liapunov-Floquet transformation has been demonstrated in the analysis of periodic dynamical systems. It has been shown that the proposed analysis techniques are virtually free from the small parameter limitations and can provide approximate analytical solutions in most cases through the application of a symbolic software. In addition, the question of the existence of the so called "generating solution" does not arise, which is certainly not the case with the classical methods such as averaging, perturbation, etc. The controllers for periodically varying systems can be effectively designed using the time-invariant methods through the application of the Liapunov-Floquet transformation. Such studies have been reported elsewhere (Sinha and Joseph, 1993).

Acknowledgments

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APPENDIX A

In Eq. (3) let

$$\mathbf{A}(\mathbf{t}) = \mathbf{A}_0 + \bar{\mathbf{A}}(\mathbf{t}) \tag{A1}$$

where $\mathbf{A}_0$ and $\bar{\mathbf{A}}(\mathbf{t})$ are $n \times n$ matrices.

The $\hat{\mathbf{Z}}$ matrix appearing in Eq. (26) can be written as

$$\hat{\mathbf{Z}}_{nm\times nm} = [\mathbf{A}_0 \otimes \mathbf{G}^{\mathbf{T}} + \mathbf{C}_{\mathbf{A}} \otimes \mathbf{G}^{\mathbf{T}}\bar{\mathbf{Q}}] \tag{A2}$$

where $\mathbf{C}_A$ is the coefficient matrix of $\bar{\mathbf{A}}(\mathbf{t})$ and from reference by Sinha and Wu, (1991a), $\mathbf{G}^T$ the $(m \times m)$ integration operational matrix and $\bar{\mathbf{Q}}(\mathbf{d}_j)$ the $m \times m$ product operational matrix are given as

$$\mathbf{G}^{\mathbf{T}} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \cdot\ \cdot & \cdot & 0 \\ -1/8 & 0 & 1/8 & 0 & \cdot\ \cdot & \cdot & 0 \\ -1/6 & -1/4 & 0 & 1/12 & \cdot\ \cdot & \cdot & 0 \\ 1/16 & 0 & -1/8 & 0 & \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \dfrac{1}{4(m-1)} \\ \dfrac{(-1)^m}{2m(m-2)} & \cdot & \cdot & \cdot & \cdot\ \cdot & \dfrac{-1}{4(m-2)} & 0 \end{bmatrix} \tag{A3}$$

$$\bar{\mathbf{Q}}(\mathbf{d}_j) = \begin{bmatrix} \mathbf{d}_0 & \mathbf{d}_1/2 & \mathbf{d}_2/2 & \cdot\ \cdot & \mathbf{d}_{m-1} \\ \mathbf{d}_1 & \mathbf{d}_0 + \mathbf{d}_2/2 & (\mathbf{d}_1 + \mathbf{d}_3)/2 & \cdot\ \cdot & (\mathbf{d}_{m-2} + \mathbf{d}_m)/2 \\ \mathbf{d}_2 & (\mathbf{d}_1 + \mathbf{d}_3)/2 & \mathbf{d}_0 + \mathbf{d}_4/2 & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \mathbf{d}_{m-1} & (\mathbf{d}_{m-2} + \mathbf{d}_m)/2 & \cdot & \cdot\ \cdot & \mathbf{d}_0 + \mathbf{d}_{2m-2}/2 \end{bmatrix} \tag{A4}$$

Kronecker Product. Consider a 2×2 square matrix $\mathbf{A}$ and an $n \times m$ matrix $\mathbf{B}$. The Kronecker product of the two matrices is defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{a}_{11}\mathbf{B} & \mathbf{a}_{12}\mathbf{B} \\ \mathbf{a}_{21}\mathbf{B} & \mathbf{a}_{22}\mathbf{B} \end{bmatrix}$$

The resulting matrix is of size $2n \times 2m$.

APPENDIX B

Approximate analytical solution of the nonlinear Mathieu equation for the parameter set 2. The symbols used are $\gamma = \pi t$, $\xi = \mathbf{u}_0$ and $\varkappa = \mathbf{v}_0$ where $\mathbf{u}_0$ and $\mathbf{v}_0$ are the initial conditions in the normal form coordinates u and v, respectively. (a, b) represents complex number a + ib, $\mathbf{i} = \sqrt{(-1)}$.

$$\begin{aligned} \mathbf{x}_1 = {} & \mathbf{e}^{(-0.2122,0.8193)t}\{(0.2291, 0.7064) + (0.0248, 0.0764)\cos(2\gamma) + (0.0003, 0.0010)\cos(4\gamma) \\ & + (0.0198, -0.0064)\sin(2\gamma) + (0.0004, -0.0001)\sin(4\gamma)\}\xi + \mathbf{e}^{(-0.2122,-0.8193)t}\{(-0.2954, -0.7821) \\ & + (-0.0319, -0.0846)\cos(2\gamma) + (-0.0004, -0.0011)\cos(4\gamma) + (0.0220, -0.0083)\sin(2\gamma) \\ & + (0.0004, -0.0002)\sin(4\gamma)\}\varkappa + \mathbf{e}^{(-0.6365,0.8193)t}\{(-0.0071, -0.0050) + (0.0005, -0.0025)\cos(2\gamma) \\ & + (0.00001, -0.00008)\cos(4\gamma) + (0, 0.00001)\cos(2\gamma)\cos(4\gamma) + (0.0008, -0.00002)\cos(2\gamma)\sin(2\gamma) \\ & + 0.0001\cos(4\gamma)\sin(2\gamma) - 0.00005\cos(2\gamma)\sin(4\gamma) + (0., -0.00001)\sin(2\gamma)\sin(4\gamma) - 0.00004\sin(6\gamma) \\ & + (0.0001, -0.0002)\cos^2(2\gamma) - 0.00001\cos(6\gamma) + (-0.0078, -0.0003)\sin(2\gamma) + (0.0001, -0.0002)\sin^2(2\gamma) \\ & - 0.0007\sin(4\gamma)\}\xi^2\varkappa + \mathbf{e}^{(-0.6365,-0.8193)t}\{(-0.0022, 0.0083) + (-0.0002, 0.0009)\cos(2\gamma) \\ & + (0.00004, 0.00001)\cos(4\gamma) + (-0.0003, 0.00007)\sin(2\gamma) - 0.00002\sin(2\gamma)\sin(4\gamma)\}\xi\varkappa^2 \\ & + \mathbf{e}^{(-0.6365,2.4579)t}\{(0.3836, -0.1547) + (0.0971, -0.0100)\cos(2\gamma) + (0.0061, 0.0007)\cos^2(2\gamma) \\ & + (0.0024, -0.0002)\cos(4\gamma) + 0.0003\cos(2\gamma)\cos(4\gamma) + 0.00003\cos(6\gamma) + (0.0003, -0.0682)\sin(2\gamma) \\ & + (0.0006, -0.0010)\cos(2\gamma)\sin(2\gamma) + 0.00003\cos(4\gamma)\sin(2\gamma) + (0.0066, 0.0003)\sin^2(2\gamma) \\ & + (-0.0002, -0.0012)\sin(4\gamma) - 0.00001\cos(2\gamma)\sin(4\gamma) + 0.0004\sin(2\gamma)\sin(4\gamma) + (-0.00005, -0.00001)\sin(6\gamma)\}\xi^3 \end{aligned}$$

$$
\begin{aligned}
\mathbf{x}_2 = {} & e^{(-0.2122,0.8193)t}\{(-0.6273, 0.0378) + (0.0568, -0.0363)\cos(2\gamma) + (0.0039, -0.0015)\cos(4\gamma) \\
& + (0.00005, -0.00002)\cos(6\gamma) + (-0.1546, -0.4623)\sin(2\gamma) + (-0.0041, -0.0124)\sin(4\gamma) \\
& + (-0.00004, -0.0001)\sin(6\gamma)\}\xi + e^{(-0.2122,-0.8193)t}\{(-0.5781, 0.4080) + (0.0754, -0.0080)\cos(2\gamma) \\
& + (0.0045, -0.0014)\cos(4\gamma) + (0.1892, 0.5151)\sin(2\gamma) + (0.00005, 0.0001)\sin(6\gamma) \\
& + (0.0051, 0.0138)\sin(4\gamma) + (0.00005, -0.00002)\cos(6\gamma)\}\varkappa + e^{(-0.6365,0.8193)t}\{(-0.0485, -0.00008)\cos(2\gamma) \\
& + (-0.0043, 0.00004)\cos(4\gamma) + 0.00003\cos^2(4\gamma) + (0.0036, -0.0027) - 0.00005\cos(6\gamma) \\
& + (0.0022, 0.0097)\sin(2\gamma) + (-0.0006, 0.0006)\cos(2\gamma)\sin(2\gamma) + (0.0050, 0.0004)\sin^2(2\gamma) \\
& + (0.0003, 0.0004)\sin(4\gamma) - 0.00002\cos(2\gamma)\sin(4\gamma) + (0.0006, 0.00001)\sin(2\gamma)\sin(4\gamma) \\
& + 0.00001\sin^2(4\gamma) + 0.0002\sin(6\gamma) - 0.00002\cos(2\gamma)\sin(6\gamma) + 0.00003\sin(2\gamma)\sin(6\gamma) \\
& + 0.00078\cos(2\gamma)\cos(4\gamma) - 0.00001\cos(4\gamma)\sin(2\gamma) + (0.0051, 0.0002)\cos^2(2\gamma)\}\xi^2\varkappa \\
& + e^{(-0.6365,-0.8193)t}\{(0.0082, -0.0035) + (-0.0008, 0.00002)\cos(2\gamma) + (-0.00006, -0.00002)\cos(4\gamma) \\
& + (0.0013, -0.0055)\sin(2\gamma) + 0.00003\cos(4\gamma)\sin(2\gamma) + (0.0005, -0.0002)\sin(4\gamma) + 0.00005\sin(6\gamma) \\
& - 0.00003\cos(2\gamma)\sin(2\gamma) - 0.00005\cos(2\gamma)\sin(4\gamma)\}\xi\varkappa^2 + e^{(-0.6365,2.4579)*t}\{(0.1350, 1.0413) \\
& + (-0.0365, -0.1839)\cos(2\gamma) + (-0.0021, 0.0082)\cos^2(2\gamma) + (-0.0026, -0.0084)\cos(4\gamma) \\
& + (-0.00004, 0.0007)\cos(2\gamma)\cos(4\gamma) + (0., 0.00001)\cos^2(4\gamma) + (-0.00003, -0.0001)\cos(6\gamma) \\
& + (-0.4434, 0.1072)\sin(2\gamma) + (-0.0161, -0.0032)\cos(2\gamma)\sin(2\gamma) + (0.0001, 0.00006)\cos(4\gamma)\sin(2\gamma) \\
& + (-0.0064, 0.0221)\sin^2(2\gamma) + (-0.0144, 0.0030)\sin(4\gamma) + (-0.0002, -0.0001)\cos(2\gamma)\sin(4\gamma) \\
& + 0.00002\cos(4\gamma)\sin(4\gamma) + (-0.0002, 0.0009)\sin(2\gamma)\sin(4\gamma) + (-0.0003, 0.00003)\sin(6\gamma) \\
& + 0.00002\cos(2\gamma)\sin(6\gamma) + 0.00003\sin(2\gamma)\sin(6\gamma)\}\xi^3 + e^{(-0.6365,-2.4579)t}\{0.00003\cos(2\gamma) + (0., -0.00001)\sin(2\gamma)\}\varkappa^3
\end{aligned}
$$