Collection of papers related to Kovacic algorithm for solving differential equations

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Contents

These are collection of papers related to Kovacic algorithm for solving differential equations. As I was studying the algorithm, I thought it will be good to have any related paper I find on the subject in one place for easy access. The HTML version has only the outlines. Only the PDF version has the actual papers shown.

CHAPTER 1

THE PAPERS

1.1 An Algorithm for Solving Second Order Linear Homogeneous Differential Equations (1985 version)

This is the original paper by JERALD J. KOVACIC.

J. Symbolic Computation (1986) **2**, 3–43

An Algorithm for Solving Second Order Linear Homogeneous Differential Equations

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In this paper we present an algorithm for finding a "closed-form" solution of the differential equation $y''+ay'+by$, where a and b are rational functions of a complex variable x, provided a "closed-form" solution exists. The algorithm is so arranged that if no solution is found, then no solution can exist.

1. Introduction

In this paper we present an algorithm for finding a "closed-form" solution of the differential equation $y''+ay'+by$, where a and b are rational functions of a complex variable x, provided a "closed-form" solution exists. The algorithm is so arranged that if no solution is found, then no solution can exist.

The first section makes precise what is meant by "closed-form" and shows that there are four possible cases. The first three cases are discussed in sections 3, 4 and 5 respectively. The last case is the case in which the given equation has no "closed-form" solution. It holds precisely when the first three cases fail.

In the second section we present conditions that are necessary for each of the three cases. Although this material could have been omitted, it seems desirable to know in advance which cases are possible.

The algorithm in cases 1 and 2 is quite simple and can usually be carried out by hand, provided the given equation is relatively simple. However, the algorithm in case 3 involves quite extensive computations. It can be programmed on a computer for a specific differential equation with no difficulty. In fact, the author has worked through several examples using only a programmable calculator. Only in one example was a computer necessary, and this was because intermediate numbers grew to 20 decimal digits, more than the calculator could handle. Fortunately, the necessary conditions for case 3 are quite strong so this case can often be eliminated from consideration.

The algorithm does require that the partial fraction expansion of the coefficients of the differential equation be known, thus one needs to factor a polynomial in one variable over the complex numbers into linear factors. Once the partial fraction expansions are known, only linear algebra is required.

Using the MACSYMA computer algebra system, see, for example, Pavelle & Wang (1985), Bob Caviness and David Saunders of Rensselear Polytechnic Institute programmed the entire algorithm (see Saunders (1981)). Meanwhile, the algorithm has

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been implemented also in the MAPLE computer algebra system, see, for example, Char *et al.* (1985), by Carolyn Smith (1984).

This paper is arranged so that the algorithm may be studied independently of the proofs. In section 1, parts 1 and 2 are necessary to understand the algorithm, parts 3 and 4 are devoted to proofs. In the other sections, part 1 describes the algorithm, part 2 contains examples, and the remaining parts contain proofs.

Since the first appearance of this paper as a technical report, a number of papers have appeared on the same problem: Baldassarri (1980), Baldassarri & Dwork (1979), Singer (1979, 1981, 1985).

Special thanks are due to Bob Caviness and David Saunders of RPI for their encouragement and assistance during the preparation of this paper.

1. The Four Cases

In the first part of this section we define precisely what we mean by "closed-form" solution. In the second part we state the four possible cases that can occur. These cases are treated individually in the latter sections. The third part is devoted to a brief description of the Galois theory of differential equations. This theory is used in the proofs of the theorems of the present chapter and those of sections 4 and 5. Part 4 contains a proof of the theorem stated in part 2.

1.1. LIOUVILLIAN EXTENSIONS

The goal of this paper is to find "closed-form" solutions of differential equations. By a "closed-form" solution we mean, roughly, one that can be written down by a first-year calculus student. Such a solution may involve esponentials, indefinite integrals and solutions of polynomial equations. (As we are considering functions of a complex variable, we need not explicitly mention trigonometric functions, they can be written in terms of exponentials. Note that logarithms are indefinite integrals and hence are allowed.) A more precise definition involves the notion of Liouvillian field.

DEFINITION. Let $\mathbf{F}$ be a differential field of functions of a complex variable x that contains $\mathbb{C}(x)$. (*Thus* $\mathbf{F}$ is a field and the derivation operator $'$ $(=d/dx)$ carries $\mathbf{F}$ into itself). $\mathbf{F}$ is said to be *Liouvillian* if there is a tower of differential fields

$$\mathbb{C}(x) = \mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \cdots \subseteq \mathbf{F}_n = \mathbf{F}$$

such that, for each $i = 1, \ldots, n$,

either $\mathbf{F}_i = \mathbf{F}_{i-1}(\alpha)$ where $\alpha'/\alpha \in \mathbf{F}_{i-1}$
($\mathbf{F}_i$ is generated by an exponential of an integral over $\mathbf{F}_{i-1}$)

or $\mathbf{F}_i = \mathbf{F}_{i-1}(\alpha)$ where $\alpha' \in \mathbf{F}_{i-1}$
($\mathbf{F}_i$ is generated by an integral over $\mathbf{F}_{i-1}$)

or $\mathbf{F}_i$ is finite algebraic over $\mathbf{F}_{i-1}$.

A function is said to be *Liouvillian* if it is contained in some Liouvillian differential field. Suppose that η is a (non-zero) Liouvillian solution of the differential equation

$y''+ay'+by$, where $a, b\in\mathbb{C}(x)$. It follows that every solution of this differential equation is Liouvillian. Indeed, the method of reduction of order produces a second solution, namely $\eta\int(e^{-\int a}/\eta^2)$. This second solution is evidently Liouvillian and the two solutions are linearly independent. Thus any solution, being a linear combination of these two, is Liouvillian.

We may use a well-known change of variable to eliminate the term involving y' from the differential equation. Set $z=e^{\frac{1}{2}\int a}y$. Then $z''+(b-\frac{1}{4}a^2-\frac{1}{2}a')z=0$. This new equation still has coefficients in $\mathbb{C}(x)$ and evidently y is Liouvillian if and only if z is Liouvillian. Thus no generality is lost by assuming that the term involving y' is missing from the differential equation.

1.2 THE FOUR CASES

In the remainder of this paper we shall consider the equation

$$y''=ry,\quad r\in\mathbb{C}(x).$$

We shall refer to this equation as "the DE". To avoid triviality, we assume that $r\notin\mathbb{C}$. By a solution of the DE is always meant a non-zero solution.

THEOREM. *There are precisely four cases that can occur.*

Case 1. *The DE has a solution of the form* $e^{\int\omega}$ *where* $\omega\in\mathbb{C}(x)$.

Case 2. *The DE has a solution of the form* $e^{\int\omega}$ *where* ω *is algebraic over* $\mathbb{C}(x)$ *of degree* 2, *and case* 1 *does not hold.*

Case 3. *All solutions of the DE are algebraic over* $\mathbb{C}(x)$ *and cases* 1 *and* 2 *do not hold.*

Case 4. *The DE has no Liouvillian solution.*

It is evident that these cases are mutually exclusive, the theorem states that they are exhaustive. The proof of this theorem will be presented in part 1.4.

1.3. THE DIFFERENTIAL GALOIS GROUP

Here we present a brief summary of the Picard–Vessiot theory of differential equations (see Kaplansky (1957), or Chapter 6 of Kolchin (1973)), which is tailored specifically to the DE $y''=ry$.

Suppose that η,ζ is a fundamental system of solutions of the DE (where η,ζ are functions of a complex variable x). Form the differential extension field $\mathbf{G}$ of $\mathbb{C}(z)$ generated by η,ζ, thus

$$\mathbf{G}=\mathbb{C}(x)\langle\eta,\zeta\rangle=\mathbb{C}(x)(\eta,\eta',\zeta,\zeta').$$

Then the Galois group of $\mathbf{G}$ over $\mathbb{C}(x)$, denoted by $G(\mathbf{G}/\mathbb{C}(x))$, is the group of all differential automorphisms of $\mathbf{G}$ that leave $\mathbb{C}(x)$ invariant. (An automorphism σ is differential if $\sigma(a')=(\sigma a)'$ for every $a\in\mathbf{G}$.) We refer the reader to the references cited above for a proof that the Fundamental Theorem of Galois Theory holds in this context.

There is an isomorphism of $G(\mathbf{G}/\mathbb{C}(x))$ with a subgroup of $GL(2)$, the group of invertible 2×2 matrices with coefficients in $\mathbb{C}$. Let $\sigma\in G(\mathbf{G}/\mathbb{C}(x))$. Then

$$(\sigma\eta)''=\sigma(\eta'')=\sigma(r\eta)=r\sigma\eta.$$

Hence, $\sigma\eta$ is also a solution of the DE and so is a linear combination $\sigma\eta = a_\sigma\eta + c_\sigma\zeta$, a_σ, $c_\sigma \in \mathbb{C}$, of η, ζ. Similarly, $\sigma\zeta = b_\sigma\eta + d_\sigma\zeta$ for some $b_\sigma, d_\sigma \in \mathbb{C}$.

$$c : \sigma \to \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

is immediately seen to be an injective group homomorphism.

This representation $c : G(\mathbf{G}/\mathbb{C}(x)) \to GL(2)$ certainly does depend on the choice of fundamental system η, ζ. If η_1, ζ_1 is another fundamental system, then there is a matrix $X \in GL(2)$ such that $(\eta_1, \zeta_1) = (\eta, \zeta)X$. Therefore,

$$\mathbf{G} = \mathbb{C}(x)\langle\eta, \zeta\rangle = \mathbb{C}(x)\langle\eta_1, \zeta_1\rangle \quad \text{and} \quad c_1(\sigma) = X^{-1}c(\sigma)X.$$

The representation $G(\mathbf{G}/\mathbb{C}(x)) \to GL(2)$ is determined by the DE only up to conjugation. By abuse of language, we allow ourselves to speak of any one of these conjugate groups as the Galois group of the DE. If a fundamental system η, ζ is fixed, then we refer to $c(G(\mathbf{G}/\mathbb{C}(x))) \subseteq GL(2)$ as the Galois group of the DE relative to η, ζ.

Fix a fundamental system η, ζ of solutions of the DE and let $G \subseteq GL(2)$ be the Galois group relative to η, ζ. Let $W = \eta\zeta' - \eta'\zeta$ be the Wronskian of η, ζ. A simple computation, using the DE, shows that $W' = 0$, so W is a (non-zero) constant and is left fixed by any element of $G(\mathbf{G}/\mathbb{C}(x))$. Let $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, then, using the notation above,

$$\begin{aligned} W = \sigma W &= (a_\sigma\eta + c_\sigma\zeta)(b_\sigma\eta' + d_\sigma\zeta') - (a_\sigma\eta' + c_\sigma\zeta')(b_\sigma\eta + d_\sigma\zeta) \\ &= (a_\sigma d_\sigma - b_\sigma c_\sigma)W = \det c(\sigma) \cdot W. \end{aligned}$$

Thus $G \subseteq SL(2)$, the group of 2×2 matrices with determinant 1.

Recall that a subgroup G of $GL(2)$ is an algebraic group if there exist a finite number of polynomials

$$P_1, \ldots, P_r \in \mathbb{C}[X_1, X_2, X_3, X_4] \quad \text{such that} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

if and only if

$$P_1(a, b, c, d) = \cdots = P_r(a, b, c, d) = 0.$$

One of the principal facts in the Picard–Vessiot theory is that the Galois group of a differential equation is an algebraic group. For a proof in all generality, see the references cited above. Here we sketch a proof in the special case that we are considering.

Let Y, Z, Y_1, Z_1 be indeterminates over $\mathbb{C}(x)$ and consider the substitution homomorphism

$$\mathbb{C}[x, Y, Z, Y_1, Z_1] \to \mathbb{C}[x, \eta, \zeta, \eta', \zeta'].$$

The kernel of this mapping is a prime ideal $\mathbf{p}$. Any element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2)$ induces an automorphism of $\mathbb{C}[x, Y, Z, Y_1, Z_1]$ over $\mathbb{C}[x]$ by the formula

$$(Y, Z, Y_1, Z_1) \to (aY + cZ, bY + dZ, aY_1 + cZ_1, bY_1 + dZ_1).$$

Moreover, $A \in G$ if and only if $\mathbf{p}$ is carried into itself. The ideal $\mathbf{p}$ is finitely generated, say $\mathbf{p} = (q_1, \ldots, q_s)$, where $q_1, \ldots, q_s$ are linearly independent over $\mathbb{C}$. Let n be the maximum of the degrees of $q_1, \ldots, q_s$ in x, Y, Z, Y_1, Z_1 and let V be the vector space over $\mathbb{C}$ of all polynomials in $\mathbb{C}[x, Y, Z, Y_1, Z_1]$ of degree n or less. Evidently the action of $SL(2)$ on

$\mathbb{C}[x, Y, Z, Y_1, Z_1]$ restricts to V. If $q_1, \ldots, q_s, q_{s+1}, \ldots, q_t$ is a basis of V, then there exist polynomials $P_{ij} \in \mathbb{C}[X_1, X_2, X_3, X_4]$ such that the result of the action of A on q_i is

$$\sum_{j=1}^{t} P_{ij}(a, b, c, d)q_j.$$

It follows that $A \in G$ if and only if $P_{ij}(a, b, c, d) = 0$ for $i = 1, \ldots, s$, $j = s+1, \ldots, t$. Therefore G is an algebraic group.

1.4. PROOF

In this section we shall prove the theorem that was stated in 1.2. We shall use several facts about algebraic groups. Suitable references are Borel (1956), Kaplansky (1957), and Chapter 5 of Kolchin (1973). The following result is contained in Kaplansky (1957, p. 31).

LEMMA. *Let G be an algebraic subgroup of $SL(2)$. Then one of four cases can occur.*

Case 1. *G is triangulisable.*

Case 2. *G is conjugate to a subgroup of*

$$D^{\dagger} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\},$$

and case 1 *does not hold.*

Case 3. *G is finite and cases* 1 *and* 2 *do not hold.*

Case 4. $G = SL(2)$.

Proof. Denote the component of the identity of G by G°. First we note that any two-dimensional Lie algebra is solvable, hence either $\dim G = 3$ (in which case $G = SL(2)$) or else G° is solvable. In the latter case, G° is triangulisable by the Lie–Kolchin Theorem. Assume that G° is triangular.

If G° is not diagonalisable, then G° contains a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$ (since an algebraic group contains the unipotent and semi-simple parts of all of its elements). Since G° is normal in G, any matrix in G conjugates $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ into a triangular matrix. A direct computation shows that only triangular matrices have this property. Thus G itself is triangular. This is case 1.

Assume next that G° is diagonal and infinite, so G° contains a non-scalar diagonal matrix A. Because G° is normal in G, any element of G conjugates A into a diagonal matrix. A direct computation shows that any matrix with this property must be contained in $D^{\dagger}$. Therefore either G is diagonal, this being case 1, or else G is contained in $D^{\dagger}$, this being case 2.

Finally we observe that if G° is finite (and therefore $G^{\circ} = \{1\}$), then G must also be finite. This is case 3. This proves the lemma.

We shall now prove the theorem of section 2.

Let η, ζ be a fundamental system of solutions of the DE and let G be the Galois group relative to η, ζ. Set $\mathbf{G} = \mathbb{C}(x)\langle \eta, \zeta \rangle$.

Case 1. G is triangulisable. We may assume that G is triangular. Then, for every

$\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, $\sigma\eta = c_\sigma \eta$, where $c_\sigma \in \mathbb{C}$, $c_\sigma \neq 0$. Therefore $\sigma\omega = \omega$, where $\omega = \eta'/\eta$, which implies that $\omega \in \mathbb{C}(x)$.

Case 2. G is conjugate to be a subgroup of $D^\dagger$. We may assume that G is a subgroup of $D^\dagger$. If $\omega = \eta'/\eta$ and $\phi = \zeta'/\zeta$, then, for every $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, either $\sigma\omega = \omega$, $\sigma\phi = \phi$ or $\sigma\omega = \phi$, $\sigma\phi = \omega$. Thus ω is quadratic over $\mathbb{C}(x)$.

Case 3. G is finite. In this case $\mathbf{G}$ has only a finite number of differential automorphisms $\sigma_1, \ldots, \sigma_n$. Since the elementary symmetric function of $\sigma_1\eta, \ldots, \sigma_n\eta$ are invariant under $G(\mathbf{G}/\mathbb{C}(x))$, η is algebraic over $\mathbb{C}(x)$. Similarly, ζ is algebraic over $\mathbb{C}(x)$. Because every solution of the DE is contained in $\mathbf{G}$, every solution of the DE is algebraic.

Case 4. $G = SL(2)$. Suppose that the DE had a Liouvillian solution. Then, as pointed out in 1.1, every solution of the DE is Liouvillian. Thus $\mathbf{G}$ is contained in a Liouvillian field. It follows that G° is solvable (Kolchin, 1973, p. 415). Since $G^\circ = SL(2)$ is not solvable, the DE can have no Liouvillian solution.

This proves the theorem.

2. Necessary Conditions

In this section we discuss some easy conditions that are necessary for cases 1, 2, or 3 to hold. These conditions give a sufficient condition for case 4 to hold (namely when the necessary conditions for cases 1, 2, and 3 fail). Throughout, we shall consider the DE $y'' = ry$, $r \in \mathbb{C}(x)$.

2.1. THE NECESSARY CONDITIONS

Since r is a rational function, we may speak of the poles of r, by which we shall always mean the poles in the finite complex plane $\mathbb{C}$. If $r = s/t$, with $s, t \in \mathbb{C}[x]$, relatively prime, then the poles of r are the zeros of t and the order of the pole is the multiplicity of the zero of t. By the order of r at ∞ we shall mean the order of ∞ as a zero of r, thus the order of r at ∞ is $\deg t - \deg s$.

THEOREM. *The following conditions are necessary for the respective cases to hold.*

Case 1. *Every pole of r must have even order or else have order* 1. *The order of r at* ∞ *must be even or else be greater than* 2.

Case 2. *r must have at least one pole that either has odd order greater than* 2 *or else has order* 2.

Case 3. *The order of a pole of r cannot exceed* 2 *and the order of r at* ∞ *must be at least* 2. *If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j},$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$, *for each i*, $\sum_j \beta_j = 0$, *and if*

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1+4\gamma} \in \mathbb{Q}$.

2.2. EXAMPLES

Airey's Equation $y''=xy$ has no Liouvillian solution (i.e. case 4 holds). This is clear because the necessary conditions for cases 1, 2, and 3 all fail. More generally, $y''=Py$, where $P\in\mathbb{C}[x]$ has odd degree, has no Liouvillian solution.

For Bessel's Equation

$$y''=\frac{4(n^2-x^2)-1}{4x^2}y, \quad n\in\mathbb{C}$$

(in self-adjoint form), only cases 1, 2, and 4 are possible.

For Weber's Equation

$$y''=(\tfrac{1}{4}x^2-\tfrac{1}{2}-n)y, \quad n\in\mathbb{C},$$

only cases 1 and 4 are possible.

2.3. PROOF

In this section we prove the theorem of Section 1.

Case 1. In this case the DE has a solution of the form $\eta=e^{\int\omega}$ where $\omega\in\mathbb{C}(x)$. Since $\eta''=r\eta$, it follows that $\omega'+\omega^2=r$ (the Riccatti Equation). Both ω and r have Laurent series expansions about any point c of the complex plane, for ease of notation we take $c=0$. Say

$$\omega=bx^\mu+\cdots, \quad \mu\in\mathbb{Z}, \quad b\neq 0$$
$$r=\alpha x^\nu+\cdots, \quad \nu\in\mathbb{Z}, \quad \alpha\neq 0.$$

(The dots represent terms involving x raised to powers higher than that shown.) Using the Riccatti Equation, we find that

$$\mu bx^{\mu-1}+\cdots+b^2x^{2\mu}+\cdots=\alpha x^\nu+\cdots.$$

As we need to show that every pole of r either has order 1 or else has even order, we may assume that $\nu\leqslant -3$. Since $\alpha\neq 0$, $-3\geqslant\nu\geqslant\min(\mu-1, 2\mu)$. It follows that $\mu<-1$ and $2\mu<\mu-1$. Since $b^2\neq 0$, $2\mu=\nu$, which implies that ν is even. For use in section 3.3, we remark that if r has a pole of order $2\mu\geqslant 4$ at c, then ω must have a pole of order μ at c.

Now consider the Laurent series expansions of r and ω at ∞.

$$\omega=bx^\mu+\cdots, \quad \mu\in\mathbb{Z}, \quad b\neq 0$$
$$r=\alpha x^\nu+\cdots, \quad \nu\in\mathbb{Z}, \quad a\neq 0.$$

(The dots represent terms involving x raised to a power lower than that shown. The order of r at ∞ is $-\nu$.) As we need to show that either the order of r at ∞ is $\geqslant 3$ or else is even, we may assume that $\nu\geqslant -1$. Using the Riccatti Equation, we have

$$\mu bx^{\mu-1}+\cdots+b^2x^{2\mu}+\cdots=\alpha x^\nu+\cdots. .$$

Just as above, $-1\leqslant\nu\leqslant\max(\mu-1, 2\mu)$, $\mu>-1$, $2\mu>\mu-1$. Since $b^2\neq 0$, $2\mu=\nu$, so ν is even. For use in section 3,3, we remark that if r has a pole of order $2\mu\geqslant 0$ at ∞, then ω has a pole of order μ at ∞.

This verifies the necessary conditions for case 1.

Case 2. We analyse this case by considering the differential Galois group that must obtain. By section 1.4 the group must be conjugate to a subgroup G of $D^\dagger$, which is not triangulisable (otherwise case 1 would hold). Let η, ζ be a fundamental system of solutions of the DE relative to the group G. For every $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, either $\sigma\eta = c_\sigma \eta$, $\sigma\zeta = c_\sigma^{-1}\zeta$ or $\sigma\eta = -c_\sigma^{-1}\zeta$, $\sigma\zeta = c_\sigma \eta$. Evidently $\eta^2\zeta^2$ is an invariant of $G(\mathbf{G}/\mathbb{C}(x))$ and therefore $\eta^2\zeta^2 \in \mathbb{C}(x)$. Moreover, $\eta\zeta \notin \mathbb{C}(x)$, for otherwise G would be a subgroup of the diagonal group, which is case 1.

Writing

$$\eta^2\zeta^2 = \prod (x-c_i)^{e_i} \quad (e_i \in \mathbb{Z}),$$

we have that at least one exponent e_i is odd. Without loss of generality we may assume that

$$\eta^2\zeta^2 = x^e \prod (x-c_i)^{e_i}$$

and that e is odd. Let

$$\theta = (\eta\zeta)'/(\eta\zeta) = \tfrac{1}{2}(\eta^2\zeta^2)'/(\eta^2\zeta^2) = \tfrac{1}{2}ex^{-1} + \cdots,$$

where the dots represent terms involving x to non-negative powers. Since $\eta'' = r\eta$ and $\zeta'' = r\zeta$,

$$\theta'' + 3\theta'\theta + \theta^3 = 4r\theta + 2r'.$$

Let $r = \alpha x^\nu + \cdots$ be the Laurent series expansion of r at 0, where $\alpha \neq 0$ and $\nu \in \mathbf{Z}$. From the equation above we obtain

$$(e - \tfrac{3}{4}e^2 + \tfrac{1}{8}e^3)x^{-3} + \cdots = 2\alpha(e+\nu)x^{-1} + \cdots.$$

If $\nu > -2$, then $0 = 8e - 6e^2 + e^3 = e(e-2)(e-4)$. This contradicts the fact that e is odd. Therefore $\nu \leqslant -2$. If $\nu < -2$, then $e + \nu = 0$, so ν is odd.

This verifies the necessary conditions for case 2.

Case 3. In this case the DE has a solution η that is algebraic over $\mathbb{C}(x)$. η has a Puiseaux series expansion about any point c in the complex plane, for ease of notation we take $c = 0$. Then $\eta = ax^\mu + \cdots$, where $a \in \mathbb{C}$, $a \neq 0$, $\mu \in \mathbb{Q}$. Since $r \in \mathbb{C}(x)$, $r = \alpha x^\nu + \cdots$, where $\alpha \neq 0$ and $\nu \in \mathbb{Z}$. The DE implies that

$$\mu(\mu-1)ax^{\mu-2} + \cdots = \alpha a x^{\nu+\mu} + \cdots.$$

It follows that $\nu \geqslant -2$, i.e. r has no pole of order greater than 2. If $\nu = -2$, then $\mu(\mu-1) = \alpha$. Because $\mu \in \mathbb{Q}$, we must have $\sqrt{1+4\alpha} \in Q$.

So far we have shown that the partial fraction expansion of r has the form

$$r = \sum_i \frac{\alpha_i}{(x-x_i)^2} + \sum_j \frac{\beta_j}{x-d_j} + P,$$

where $P \in \mathbb{C}[x]$ and $\sqrt{1+4\alpha_i} \in \mathbb{Q}$ for each i.

Next, we consider the series expansions about ∞,

$$\eta = ax^\mu + \cdots, \quad r = \gamma x^\nu + \cdots,$$

where the dots represent lower powers of x than those shown. From the DE we obtain

$$\mu(\mu-1)ax^{\mu-2} + \cdots = \nu\gamma a x^{\nu+\mu} + \cdots.$$

Just as above, we obtain $\nu \leqslant -2$ and therefore $P = 0$. But

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j}$$
$$= (\sum_j \beta_j)x^{-1} + \gamma x^{-2} + \cdots,$$

where $\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j$. Therefore $\sum_j \beta_j = 0$ and $\mu(\mu-1) = \gamma$. Since $\mu \in \mathbb{Q}$, $\sqrt{1+4\gamma} \in \mathbb{Q}$.

This completes the proof of the theorem stated in section 2.1.

3. The Algorithm for Case 1

The first part of this section is devoted to a description of the algorithm. It is somewhat complicated to describe in full generality, yet, as the examples in part 2 show, it is often quite easy to apply. The third part is devoted to a proof that the algorithm is correct.

3.1. DESCRIPTION OF THE ALGORITHM

The goal of this algorithm is to find a solution of the DE of the form $\eta = Pe^{\int \omega}$, where $P \in \mathbb{C}[x]$ and $\omega \in \mathbb{C}(x)$. Since η may be written as $\eta = e^{\int (P'/P+\omega)}$, this is of the form described in section 1.2. The first step on the algorithm consists of determining "parts" of the partial fraction expansion of ω. In the second step we put these "parts" together to form a candidate for ω. The maximum number of candidates possible is $2^{\rho+1}$ where ρ is the number of poles of r. If there are no candidates, then case 1 cannot hold. The third and last step is applied to each candidate for ω and consists of searching for a suitable polynomial P. If one is found, then we have the desired solution of the DE. If, for each candidate for ω, we fail to find a suitable P, then case 1 cannot hold.

We assume that the necessary condition of section 2.1 for case 1 holds, and we denote by Γ the set of poles of r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$ we define a rational function $[\sqrt{r}]_c$ and two complex numbers α_c^+, α_c^- as described below.

(c_1) If $c \in \Gamma$ and c is a pole of order 1, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^+ = \alpha_c^- = 1.$$

(c_2) If $c \in \Gamma$ and c is a pole of order 2, then

$$[\sqrt{r}]_c = 0.$$

Let b be the coefficient of $1/(x-c)^2$ in the partial fraction expansion for r. Then

$$\alpha_c^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

(c_3) If $c \in \Gamma$ and c is a pole of order $2\nu \geqslant 4$ (necessarily even by the conditions of section 2.1), then $[\sqrt{r}]_c$ is the sum of terms involving $1/(x-c)^i$ for $2 \leqslant i \leqslant \nu$ in the Laurent series expansion of $\sqrt{r}$ at c. There are two possibilities for $[\sqrt{r}]_c$, one being the negative of the other, either one may be chosen. Thus

$$[\sqrt{r}]_c = \frac{a}{(x-c)^\nu} + \cdots + \frac{d}{(x-c)^2}.$$

In practice, one would not form the Laurent series for $\sqrt{r}$, but rather would determine $[\sqrt{r}]_c$ by using undetermined coefficients. Let b be the coefficient of $1/(x-c)^{\nu+1}$ in r minus the coefficient of $1/(x-c)^{\nu+1}$ in $([\sqrt{r}]_c)$. Then

$$\alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a}+\nu\right).$$

(∞_1) If the order of r at ∞ is >2, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2) If the order of r at ∞ is 2, then

$$[\sqrt{r}]_\infty = 0.$$

Let b be the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞. (If $r = s/t$, where $s, t \in \mathbb{C}[x]$ are relatively prime, then b is the leading coefficient of s divided by the leading coefficient of t.) Then

$$\alpha_\infty^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

(∞_3) If the order of r at ∞ is $-2\nu \leqslant 0$ (necessarily even by the conditions of section 2.1), then $[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leqslant i \leqslant \nu$ in the Laurent series for $\sqrt{r}$ at ∞. (Either one of the two possibilities may be chosen.) Thus

$$[\sqrt{r}]_\infty = ax^\nu + \cdots + d.$$

Let b be the coefficient of $x^{\nu-1}$ in r minus the coefficient of $x^{\nu-1}$ in $([\sqrt{r}]_\infty)^2$. Then

$$\alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a}-\nu\right).$$

Step 2. For each family $s = (s(c))_{c\in\Gamma\cup\{\infty\}}$, where $s(c)$ is $+$ or $-$, let

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}.$$

If d is a non-negative integer, then

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

is a candidate for ω. If d is not a non-negative integer, then the family s may be removed from consideration.

Step 3. This step should be applied to each of the families retained from Step 2, until success is achieved or the supply of families has been exhausted. In the latter event, case 1 cannot hold.

For each family, search for a monic polynomial P of degree d (as defined in Step 2) that satisfies the differential equation

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

This is conveniently done by using undetermined coefficients and is a simple problem in linear algebra, which may or may not have a solution. If such a polynomial exists, then

$\eta = Pe^{\int\omega}$ is a solution of the DE. If no such polynomial is found for any family retained from Step 2, then Case 1 cannot hold.

3.2. EXAMPLES

Example 1. Consider the DE $y'' = ry$ where

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x}$$
$$= x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}.$$

Since r has a single pole (at 0) and the order there is 4, the necessary conditions of section 2.1 for case 2 do not hold. Evidently the necessary conditions for case 3 also do not hold. We apply the algorithm for case 1 to this DE.

The order of r at the pole 0 is $2\nu = 4$. Therefore $[\sqrt{r}]_0 = a/x^2$, and $a^2 = 1$. We choose $a = 1$, so $[\sqrt{r}]_0 = 1/x^2$. $b = -5 - 0 = -5$, and therefore $\alpha_0^{\pm} = \frac{1}{2}(\pm(-5/1) + 2)$, which gives $\alpha_0^+ = -3/2$ and $\alpha_0^- = 7/2$.

At ∞, $\nu = 1$, and $[\sqrt{r}]_\infty = ax + d$. Comparing r and $[\sqrt{r}]_\infty^2 = a^2x^2 + 2adx + d^2$ we see that $a^2 = 1$ and $2ad = -2$. We choose $a = 1$, $d = -1$. Thus $[\sqrt{r}]_\infty = x - 1$. $b = 3 - 1 = 2$, and $\alpha^{+\infty} = 1/2$, $\alpha_\infty^- = -3/2$.

There are four families to consider.

$$\begin{array}{lll} s(0) = +, & s(\infty) = +, & d = 1/2 - (-3/2) \quad = 2 \\ s(0) = +, & s(\infty) = -, & d = -3/2 - (-3/2) = 0 \\ s(0) = -, & s(\infty) = +, & d = 1/2 - 7/2 \qquad = -3 \\ s(0) = -, & s(\infty) = -, & d = -3/2 - 7/2 \quad = -5. \end{array}$$

Only the first two remain for consideration.

We shall treat the second family first, since $d = 0$ in that case. The candidate for ω is

$$\omega = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} - [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} - x + 1.$$

We now search for a monic polynomial P of degree 0 such that

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

Since $P = 1$, the existence of P is a question of whether or not $\omega' + \omega^2 - r = 0$. But the coefficient of $1/x$ in $\omega' + \omega^2 - r$ is -6. Thus no such polynomial P can exist.

The only remaining family is the first family. The candidate for ω is

$$\omega = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} + [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} + x - 1.$$

We now search for a monic polynomial P of degree 2 that satisfies the linear differential equation given above. Writing $P = x^2 + ax + b$, we easily determine that $a = 0$, $b = -1$ provides a solution.

Therefore a solution of the DE is given by

$$\eta = Pe^{\int\omega} = (x^2 - 1)e^{\int(1/x^2 - 3/(2x) + x - 1)}$$
$$= x^{-3/2}(x^2 - 1)e^{-1/x + x^2/2 - x}.$$

Example 2. In this example we begin the discussion of Bessel's Equation

$$y'' = \left(\frac{4n^2-1}{4x^2} - 1\right) y, \quad n \in \mathbb{C}.$$

The necessary conditions of section 2.1 imply that case 3 cannot hold. Here we consider case 1, case 2 is worked out in section 3.2.

The only pole of r is at $c=0$ and the order there is 2. Thus

$$[\sqrt{r}]_0 = 0, \qquad b = (4n^2-1)/4, \qquad \alpha_0^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b} = \tfrac{1}{2} \pm n.$$

At ∞, r has order 0 and $[\sqrt{r}]_\infty = i$. Evidently $b=0$ so $\alpha_\infty^{\pm}=0$.

There are four families to consider.

$$\begin{array}{lll} s(0)=+, & s(\infty)=+, & d=-1/2-n \\ s(0)=+, & s(\infty)=-, & d=-1/2-n \\ s(0)=-, & s(\infty)=+, & d=-1/2+n \\ s(0)=-, & s(\infty)=-, & d=-1/2+n. \end{array}$$

A necessary condition that case 1 holds is that $-1/2 \pm n$ be a non-negative integer, i.e. that n be half an odd integer. We claim that this condition is also sufficient.

If n is negative, and half an odd integer, then $m = -1/2 - n \in N$. This corresponds to the first family, in which case $\omega = -m/x + i$. We need to find a polynomial P of degree m such that

$$\begin{aligned} 0 &= P'' + 2\omega P' + (\omega' + \omega^2 - r)P \\ &= P'' + 2\left(-\frac{m}{x} + i\right)P' - \frac{2im}{x}P. \end{aligned}$$

It is straightforward to verify that

$$P = \sum_{j=0}^{m} \frac{1}{(-2i)^{m-j}} \frac{(2m-j)!}{j!\,(m-j)!} x^j$$

is the desired polynomial. A solution to Bessel's Equation is given by $\eta = x^{-m} P e^{ix}$.

If n is positive, then $m = -1/2 + n$ is a non-negative integer. This corresponds to the third family. In this case $\omega = -m/x + i$, and we are back to the case considered above.

Example 3. In this example we treat the general situation where r is a polynomial of degree 2. We may write $r = (ax+d)^2 + b$ for some $a, b, d \in \mathbb{C}$ (a and d are determined by r only up to sign, we choose either of the two possibilities). We claim that the DE has a Liouvillian solution if and only if b/a is an odd integer.

The necessary condition of section 2.1 implies that only cases 1 and 4 are possible. We consider case 1.

Evidently $[\sqrt{r}]_\infty = ax + d$ and $\alpha_\infty^{\pm} = \tfrac{1}{2}(\pm(b/a) - 1)$. There are no poles. Thus d equals α_∞^+ or α_∞^-, so one of these two numbers must be a non-negative integer for case 1 to hold. It follows that b/a must be an odd integer, which is the necessity part of our claim.

For sufficiency, we may assume that $b/a = 2n+1$ is positive, since a may be replaced by $-a$. Case 1 will hold provided that there is a monic polynomial P of degree n such that

$$\begin{aligned} 0 &= P'' + 2\omega P' + (\omega' + \omega^2 - r)P \\ &= P'' + 2(ax+d)P' - 2naP. \end{aligned}$$

If we write

$$P=\sum_{i=0}^{n} P_i x^i$$

and substitute, we obtain a system of linear equations in $P_0, \ldots, P_{n-1}$ ($P_n = 1$) that has a solution, namely

$$P_i = \frac{(2n+1)(i+1)}{n-i} P_{i+1} + \frac{(i+2)(i+1)}{2a(n-i)} P_{i+2} \quad (i = n-1, \ldots, 0)$$

where $P_{n+1} = 0$ and $P_n = 1$.

A special case of this example is Weber's Equation

$$y'' = (\tfrac{1}{4}x^2 - \tfrac{1}{2} - n)y, \quad n \in \mathbb{C}.$$

Here $a = -1/2$, $b = -1/2 - n$, $d = 0$. Thus $b/a = 2n+1$ is an odd integer if and only if n is an integer.

3.3. PROOF

In case 1, the DE has a solution of the form $\eta = e^{\int \theta}$, with $\theta \in \mathbb{C}(x)$. Since $\eta'' = r\eta$, we have

$$\theta' + \theta^2 = r \quad \text{(Riccatti Equation)}.$$

We shall determine the partial fraction expansion of θ using the Laurent series expansion of r and this Riccatti Equation.

For $c \in \mathbb{C}$, we denote the "component at c" of the partial fraction expansion of θ by

$$[\theta]_c + \frac{\alpha}{x-c} = \sum_{i=2}^{\nu} \frac{a_i}{(x-c)^i} + \frac{\alpha}{x-c}.$$

In order to simplify the notation, we assume that $c = 0$ and drop the subscript "0". We shall also need to consider the Laurent series expansion of θ about 0

$$\theta = [\theta] + \frac{\alpha}{x} + \bar{\theta}.$$

Here $\bar{\theta} = * + *x + \cdots$, where the * denotes a complex number whose particular value is irrelevant to our discussion.

We assume that the necessary conditions for case 1 (see section 2.1) are satisfied, in particular we assume that the poles of r are either of even order or else of order 1. We split our proof into parts, depending on the nature of r at 0. This parallels the division of Step 1 of the algorithm.

(c_1) Suppose that 0 is a pole of r of order 1, so $r = */x + \cdots$. The Riccatti equation becomes

$$-\frac{\nu a_\nu}{x^{\nu+1}} + \cdots + \frac{a_\nu^2}{x^\nu} + \cdots = \frac{*}{x} + \cdots.$$

Since $a_\nu^2 \neq 0$, $\nu \leqslant 1$ and $[\theta] = 0$.

Substituting $\theta = \alpha/x + \bar{\theta}$ into the Riccatti Equation, we have

$$-\frac{\alpha}{x^2} + \bar{\theta}' + \frac{\alpha^2}{x^2} + \frac{2\alpha}{x}\bar{\theta} + \bar{\theta}^2 = \frac{*}{x} + \cdots.$$

Therefore $-\alpha+\alpha^2=0$, so $\alpha=0$ or $\alpha=1$. Were $\alpha=0$, the left-hand side of this equation would have 0 as an ordinary point; however, the right-hand side has a pole at 0. We conclude that $\alpha=1$ and the component of the partial fraction expansion of θ at 0 is (in the notation of the algorithm)

$$\frac{\alpha^\pm}{x}, \quad \text{where } \alpha^\pm = 1.$$

(c_2) Suppose that r has a pole at 0 of order 2, say

$$r = \frac{b}{x^2} + \frac{*}{x} + \cdots.$$

As in (c_1), $[\theta]=0$ and $-\alpha+\alpha^2=b$. Thus the component of the partial fraction expansion of θ at 0 is

$$\frac{\alpha^\pm}{x}, \quad \text{where } \alpha^\pm = \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1+4b}.$$

(c_3) Suppose that r has a pole at 0 of order $2\mu \geqslant 4$. In section 2.3, we showed that $\nu=\mu$. Recall from section 3.1 that

$$[\sqrt{r}] = \frac{a}{x^\nu} + \cdots + \frac{*}{x^2},$$

where we have dropped the subscript "0".

Let $\bar{r}=\sqrt{r}-[\sqrt{r}]$. Then $r=[\sqrt{r}]^2+2\bar{r}[\sqrt{r}]+\bar{r}^2$. From the Riccati Equation we obtain the following formula

$$(\&) \qquad \begin{aligned} ([\theta]-[\sqrt{r}])\cdot([\theta]+[\sqrt{r}]) \\ = -[\theta]' + \frac{\alpha}{x^2} - \bar{\theta}' - \frac{2\alpha}{x}[\theta] - 2\bar{\theta}[\theta] \\ - \frac{\alpha^2}{x^2} - \frac{2\alpha}{x}\bar{\theta} - \bar{\theta}^2 + 2\bar{r}[\sqrt{r}] + \bar{r}^2. \end{aligned}$$

An examination of the right-hand side of this equation determines that it is free of terms involving $1/x^i$ for $i=\nu+2,\ldots,2$ (since $\nu \geqslant 1$). This implies that the left-hand side is 0. Indeed, since

$$([\theta]-[\sqrt{r}])+([\theta]+[\sqrt{r}]) = 2[\theta],$$

at least one of the factors involves $1/x^\nu$. Were the other factor non-zero, it would involve $1/x^i$ for some $i \geqslant 2$. The product would then involve $1/x^{\nu+i}$ for some $i \geqslant 2$, which is absurd. Hence $[\theta]=\pm[\sqrt{r}]$.

The coefficient of $1/x^{\nu+1}$ in the right-hand side of (&) is $\pm\nu a \mp 2\alpha a + b$, where b is the coefficient of $1/x^{\nu+1}$ in $2\bar{r}[\sqrt{r}]+\bar{r}^2 = r-[\sqrt{r}]^2$. Therefore $\alpha^\pm = \frac{1}{2}(\pm b/a+\nu)$. We have shown that if 0 is a pole of r of order $2\nu \geqslant 4$, then the component of the partial fraction expansion of θ at 0 is

$$\pm[\sqrt{r}] + \frac{\alpha^\pm}{x}, \quad \text{where} \quad \alpha^\pm = \frac{1}{2}\left(\pm\frac{b}{a}+\nu\right).$$

(c_4) Finally, we must consider what happens when 0 is an ordinary point of r. As in (c_1), $[\theta]=0$ and $-\alpha+\alpha^2=0$. Contrary to the situation in (c_1), however, we cannot conclude that $\alpha \neq 0$. Hence the component of the partial fraction expansion of r at 0 is either 0 or $1/x$.

We collect together what we have proven so far. Let Γ be the set of poles of r. Then

$$\theta=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+\sum_{i=1}^{d}\frac{1}{x-d_i}+R,$$

where $R\in\mathbb{C}[x]$, $s(c)=+$ or $-$, and $[\sqrt{r}]_c$, $\alpha_c^{s(c)}$ are as in the statement of the algorithm.

Next we consider the Laurent series about ∞. Suppose that

$$\theta=R+\frac{\alpha_\infty}{x}+\cdots.$$

(∞_1) If r has order $v>2$ at ∞, then

$$r=\frac{*}{x^v}+\frac{*}{x^{v+1}}+\cdots.$$

The Riccati Equation implies that $R=0$ and $-\alpha_\infty+\alpha_\infty^2=0$, so $\alpha_\infty=0$ or 1.

(∞_2) If r has order 2 at ∞, then

$$r=\frac{b}{x^2}+\frac{*}{x^3}+\cdots.$$

The Riccati Equation implies that $R=0$ *and* $-\alpha_\infty+\alpha_\infty^2=b$, hence

$$\alpha_\infty=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1+4b}.$$

(∞_3) In the other cases, the order of r at ∞ must be even, by the necessary conditions of section 2. Following an argument similar to that used in (c_3) we find that

$$R=\pm[\sqrt{r}]_\infty,\qquad \alpha_\infty=\frac{1}{2}\left(\pm\frac{b}{a}-v\right),$$

where $-2v$ is the order of r at ∞, a is the leading coefficient of $[\sqrt{r}]_\infty$ and b is the coefficient of $1/x^{v-1}$ in $r-[\sqrt{r}]_\infty^2$.

We now know that the partial fraction expansion of θ has the form

$$\theta=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty+\sum_{i=1}^{d}\frac{1}{x-d_i}.$$

Moreover, the coefficient of $1/x$ in the Laurent series expansion of θ at ∞ is $\alpha_\infty^{s(\infty)}$. Thus

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}\in\mathbb{N}.$$

Let

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty,$$

and

$$P=\prod_{i=1}^{d}(x-d_i).$$

Then $\theta=\omega+P'/P$. Again, using the Riccati Equation, we obtain

$$P''+2\omega P'+(\omega'+\omega^2-r)P=0.$$

The converse, namely that if P is a solution of this equation, then θ satisfies the Riccati Equation, is a simple verification. It follows that if P is a solution of this equation, then $\eta=Pe^{\int\omega}$ is a solution of the DE $y''=ry$.

This proves that the algorithm for case 1 is correct.

4. The Algorithm for Case 2

Following the pattern of section 3, we shall describe the algorithm in section 4.1, give examples in section 4.2 and the proof in section 4.3. The algorithm and its proof assume that case 1 is known to fail.

4.1. DESCRIPTION OF THE ALGORITHM

Just as for case 1, we first collect data for each pole c of r and also for ∞. The form of the data is a set E_c (or E_∞) consisting of from one to three integers. Next we consider families of elements of these sets, perhaps discarding some and retaining others. If no families are retained, case 2 cannot hold. For each family retained we search for a monic polynomial that satisfies a certain linear differential equation. If no such polynomial exists for any family, then case 2 cannot hold. If such a polynomial does exist, then a solution to the DE has been found.

Let Γ be the set of poles of r.

Step 1. For each $c \in \Gamma$ we define E_c as follows.

(c_1) If c is a pole of r of order 1, then $E_c = \{4\}$.

(c_2) If c is a pole of r of order 2 and if b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \{2 + k\sqrt{1+4b} \mid k = 0, \pm 2\} \cap \mathbb{Z}.$$

(c_3) If c is a pole of r of order $\nu > 2$, then $E_c = \{\nu\}$.

(∞_1) If r has order >2 at ∞, then $E_\infty = \{0, 2, 4\}$.

(∞_2) If r has order 2 at ∞ and b is the coefficient of x^{-2} in the Laurent series expansion of r at ∞, then

$$E_\infty = \{2 + k\sqrt{1+4b} \mid k = 0, \pm 2\} \cap \mathbb{Z}.$$

(∞_3) If the order of r at ∞ is $\nu < 2$, then $E_\infty = \{\nu\}$.

Step 2. We consider all families $(e_c)_{c \in \Gamma \cup \{\infty\}}$ with $e_c \in E_c$. Those families all of whose coordinates are even may be discarded. Let

$$d = \tfrac{1}{2}\Big(e_\infty - \sum_{c \in \Gamma} e_c\Big).$$

If d is a non-negative integer, the family should be retained, otherwise the family is discarded. If no families remain under consideration, case 2 cannot hold.

Step 3. For each family retained from Step 2, we form the rational function

$$\theta = \tfrac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x-c}.$$

Next we search for a monic polynomial P of degree d (as defined in Step 2) such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

If no such polynomial is found for any family retained from Step 2, then case 2 cannot hold.

Suppose that such a polynomial is found. Let $\phi = \theta + P'/P$ and let ω be a solution of the equation

$$\omega^2 + \phi\omega + (\tfrac{1}{2}\phi' + \tfrac{1}{2}\phi^2 - r) = 0.$$

Then $\eta = e^{\int \omega}$ is a solution of the DE $y'' = ry$.

4.2. EXAMPLES

Example 1. Consider the DE $y'' = ry$ where

$$r = \frac{1}{x} - \frac{3}{16x^2}.$$

The necessary conditions of section 2 show that cases 1 and 3 cannot hold. (The order of r at ∞ is 1.) The only pole of r is at 0 and the order there is 2. The coefficient of $1/x^2$ in the partial fraction expansion of r is $b = -3/16$. Since $2\sqrt{1+4b} = 1$ is an integer, $E_0 = \{1, 2, 3\}$. The order of r at ∞ is 1 and $E_\infty = \{1\}$.

We have three families to consider.

$$\begin{aligned} &e_0 = 2, \quad && e = 1, \quad && d = -1/2 \\ &e_0 = 3, && e = 1, && d = -1 \\ &e_0 = 1, && e = 1, && d = 0. \end{aligned}$$

Only the third family need remain in consideration. For this family, $\theta = 1/2x$ and we need to find a monic polynomial P, of degree 0, such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

Evidently P must be 1, so the existence of P is a question of whether or not $\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'$ is zero. That expression does happen to be 0, so $P = 1$ is the desired polynomial.

Next we form

$$\phi = \theta + P'/P = \frac{1}{2x}.$$

The equation for ω is

$$0 = \omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi - r\right) = \omega^2 - \frac{1}{2x}\omega + \frac{1}{16x^2} - \frac{1}{x}.$$

The roots are

$$\omega = \frac{1}{4x} \pm \frac{1}{\sqrt{x}}.$$

It follows that

$$\eta = e^{\int \omega} = e^{\int (1/(4x) + 1/\sqrt{x})} = x^{1/4} e^{2\sqrt{x}}$$

is a solution of the DE. (And $x^{1/4} e^{-2\sqrt{x}}$ is also a solution.)

Example 2. In this example we finish consideration of Bessel's Equation

$$y'' = \left(\frac{4n^2 - 1}{4x^2} - 1\right) y, \quad n \in \mathbb{C},$$

that was started in section 3.2. In that section we observed that case 3 cannot hold and that case 1 holds if and only if n is half an odd integer. Here we treat case 2 and make the assumption that n is not half an odd integer.

The only pole of r is at 0 and the order there is 2. Since

$$2\sqrt{1+4b} = 2\sqrt{1 + 4(4n^2 - 1)/4} = 4n,$$

either $E_0 = \{2\}$ or $E_0 = \{2, 2+4n, 2-4n\}$, depending on whether $4n$ is an integer or not. If $4n$ is not an integer, then there is only one case to consider.

$$e_0 = 2, \qquad e_\infty = 0, \qquad d = -1.$$

Thus if $4n$ is not an integer, case 2 cannot hold. If $4n$ is an integer, there are three cases to consider.

$$\begin{aligned} e_0 &= 2, & e_\infty &= 0, & d &= -1 \\ e_0 &= 2+4n, & e_\infty &= 0, & d &= -1-2n \\ e_0 &= 2-4n, & e_\infty &= 0, & d &= -1+2n. \end{aligned}$$

In order that d be a non-negative integer, it is necessary that n be half an integer. Since n is not half an odd integer, n must be half an even integer, that is n is an integer. But, for such n, both e_0 and e_∞ are even. Hence all families are discarded and case 2 cannot hold.

In this example, and in Example 2 of section 3.2, we have shown that Bessel's Equation has a Liouvillian solution if and only if n is half an odd integer.

4.3. PROOF

For the proof of the algorithm for case 2 we shall rely heavily on the differential Galois group of the DE. In case 2, this group is (conjugate to) a subgroup of

$$D^\dagger = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C},\ c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C},\ c \neq 0\right\}.$$

Moreover, we may assume that case 1 does not hold, so the differential Galois group is not triangulisable. Let η, ζ be a fundamental system of solutions of the DE corresponding to the subgroup of $D^\dagger$. For any differential automorphism σ of $\mathbb{C}(x)\langle\eta,\zeta\rangle$ over $\mathbb{C}(x)$, either $\sigma\eta = c\eta$, $\sigma\zeta = c^{-1}\zeta$ or $\sigma\eta = -c^{-1}\zeta$, $\sigma\zeta = c\eta$, for some $c \in \mathbb{C}$, $c \neq 0$. Evidently $\sigma(\eta^2\zeta^2) = \eta^2\zeta^2$, therefore $\eta^2\zeta^2 \in \mathbb{C}(x)$. Moreover, $\eta\zeta \notin \mathbb{C}(x)$ since case 1 does not hold.

We write

$$\eta^2\zeta^2 = g \prod_{c \in \Gamma} (x-c)^{e_c} \prod_{i=1}^{m} (x-d_i)^{f_i},$$

where Γ is the set of poles of r and the exponents e_c, f_i are integers. Our goal is to determine these exponents.

Let

$$\phi = (\eta\zeta)'/(\eta\zeta) = \tfrac{1}{2}(\eta^2\zeta^2)'/(\eta^2\zeta^2) = \tfrac{1}{2}\sum_{c \in \Gamma} \frac{e_c}{x-c} + \tfrac{1}{2}\sum_{i=1}^{m} \frac{f_i}{x-d_i}.$$

Because $\phi = \eta'/\eta + \zeta'/\zeta$, it follows that

(*) $$\phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r'.$$

We first determine e_c for $c \in \Gamma$. In order to simplify the notation, we assume that $c = 0$.

(c_1) Suppose that 0 is a pole of r of order 1. The Laurent series expansions of r and ϕ at 0 are of the form

$$\begin{aligned} r &= \alpha x^{-1} + \cdots \quad (\alpha \neq 0) \\ \phi &= \tfrac{1}{2}ex^{-1} + f + \cdots \quad (e \in \mathbb{Z}, f \in \mathbb{C}). \end{aligned}$$

Substituting these series into the equation (*) and retaining all those terms that involve x^{-3} and x^{-2}, we obtain the following.

$$ex^{-3} + \cdots - \tfrac{3}{4}e^2x^{-3} - \tfrac{3}{2}efx^{-2} + \cdots + \tfrac{1}{8}e^3x^{-3} + \tfrac{3}{4}e^2fx^{-2} + \cdots = 2\alpha ex^{-2} + \cdots - \alpha x^{-2} + \cdots.$$

Therefore $e-\frac{3}{4}e^2+\frac{1}{8}e^3=0$, so $e=0,2,4$. Also $-\frac{3}{2}ef+\frac{3}{4}e^2f=2\alpha e-\alpha$. Because $\alpha\neq 0$, $e\neq 0, 2$. Hence, e must be 4.

(c_2) Suppose that 0 is a pole of r of order 2 and that b is the coefficient of $1/x^2$ in the Laurent series for r. That is

$$r = bx^{-2}+\cdots,\qquad \phi=\tfrac{1}{2}ex^{-1}+\cdots.$$

Equating the coefficients of x^{-3} on the two sides of equation (*), we obtain

$$e-\tfrac{3}{4}e^2+\tfrac{1}{8}e^3 = 2eb-4b.$$

The roots of this equation are $e=2$, $e=2\pm 2\sqrt{1+4b}$. Of course, the latter two roots may be discarded in the case that they are non-integral.

(c_3) Finally we consider the possibility that 0 is a pole of r of order $\nu>2$. Then $r=x^{-\nu}+\cdots$ and $\phi=\frac{1}{2}ex^{-1}+\cdots$. Equating the coefficients of $x^{-\nu-1}$ in (*) we obtain $0=2\alpha e-2\alpha\nu$, hence $e=\nu$.

In determining the exponents f_i we may use the calculation of (c_1) above if we replace α by 0 (since d_i must be an ordinary point of r). We find that $f_i=0$, 2, or 4. We cannot exclude the possibility that $f_i=2$, but we can, of course, exclude the possibility $f_i=0$.

We have shown so far that

$$\eta^2\zeta^2=\prod_{c\in\Gamma}(x-c)^{e_c}P^2,$$

where $e_c\in E_c$ (as defined in section 4.1) and $P\in\mathbb{C}[x]$.

$$\text{Set}\quad \theta=\tfrac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c},\qquad \text{so}\quad \phi=\theta+P'/P.$$

The next step in our proof is to determine the degree d of P, which we do by examining the Laurent series expansion of ϕ at ∞ and using equation (*).

$$\phi=\tfrac{1}{2}e_\infty x^{-1}+\cdots,\qquad e_\infty=\sum_{c\in\Gamma}e_c+2d.$$

(∞_1) Suppose that the order of r at ∞ is 2. As in (c_1) we find that $e_\infty=0$, 2 or 4.

(∞_2) Suppose that the order of r at ∞ is 2 and that b is the coefficient of x^{-2} in the Laurent series expansion of r at ∞. Then, as in (c_2), $e_\infty=2$, $2\pm 2\sqrt{1+4b}$ and e_∞ is integral.

(∞_3) Suppose that the order of r at ∞ is $\nu<2$. As in (c_3), it follows that $e_\infty=\nu$.

Note that at least one of the e_c ($c\in\Gamma$) is odd, since $\eta\zeta\notin\mathbb{C}(x)$.

Using equation (*) and the equation $\phi=\theta+P'/P$, we obtain

$$P'''+3\theta P''+(3\theta^2+3\theta'-4r)P'+(\theta''+3\theta\theta'+\theta^3-4r\theta-2r')P=0.$$

This is a linear homogeneous differential equation for P, so there is a polynomial solution if and only if there is a monic polynomial which is a solution.

Now let ω be a solution of the equation

(**) $$\omega^2-\phi\omega+\tfrac{1}{2}\phi'+\tfrac{1}{2}\phi^2-r=0.$$

To complete the proof we need to show that $\eta=e^{\int\omega}$ is a solution of the DE $y^*=ry$.

From (**) we obtain (by differentiation)

$$(2\omega-\phi)\omega'=\phi'\omega-\tfrac{1}{2}\phi''-\phi\phi'+r'.$$

The factor $(2\omega-\phi)$ cannot be zero. Indeed, if $\phi=2\omega$, then $\omega'+\omega^2-r=0$ (from (**)) so $\eta=e^{\int\omega}$ is a solution of the DE. But $\omega=\frac{1}{2}\phi\in\mathbb{C}(x)$. This is case 1, which was assumed to fail. Using (**) and (*) we have

$$2(2\omega-\phi)(\omega'+\omega^2-r)=-\phi''-3\phi\phi'-\phi^3+4r\phi+2r'=0.$$

Thus $\omega'+\omega^2=r$ so $\eta=e^{\int\omega}$ is a solution of the DE.

This completes the proof that the algorithm for case 2 is correct.

5. The Algorithm for Case 3

Following the pattern established in the previous two sections, we describe the algorithm in section 5.1 and give examples in section 5.2. The proof of the algorithm requires a knowledge of the finite subgroups of $SL(2)$ and their invariants, which is provided in section 5.3. The proof of the algorithm is presented in section 5.4.

In case 3, the DE has only algebraic solutions and we assume that cases 1 and 2 are known to fail. (It is possible for the DE to have only algebraic solutions and for cases 1 or 2 to apply. For example, case 1 gives the solution $\eta=x^{1/4}$ to the DE $y''=-(3/16x^2)y$, then reduction of order gives $\zeta=x^{3/4}$ as a second solution.)

5.1. DESCRIPTION OF THE ALGORITHM

Let η be a solution of the DE $y''=ry$ and set $\omega=\eta'/\eta$. Then, as we shall show in section 5.4, ω is algebraic over $\mathbb{C}(x)$ of degree 4, 6 or 12. It is the minimal polynomial for ω that we shall determine. We are unable to determine the minimal equation for η (which would be of degree 24, 48 or 120).

There are two possible methods for finding the minimal equation for ω. We could find a polynomial of degree 12 and then factor it. We shall prove that if ω is any solution of the 12th degree polynomial found by our method, then $\eta=e^{\int\omega}$ is a solution of the DE, hence any one of the irreducible factors may be used. This is the most direct method; however, the factorisation can be a formidable problem, even with the assistance of a computer. We illustrate this by example, in section 5.2. The alternative is to first attempt to find a 4th degree equation for ω, then a 6th degree equation and finally a 12th degree equation. The advantage is that if an equation is found, then it is guaranteed to be irreducible.

In our description of the algorithm, we shall combine the various possibilities, denoting by n the degree of the equation being sought. As before, we denote by Γ the set of poles of r. Recall that, by the necessary conditions of section 2, r cannot have a pole of order >2.

Step 1. For each $c\in\Gamma\cup\{\infty\}$ we define a set E_c of integers as follows.

(c_1) If c is a pole of r of order 1, then $E_c=\{12\}$.

(c_2) If c is a pole of r of order 2 and if α is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c=\left\{6+\frac{12k}{n}\sqrt{1+4\alpha}\,|\,k=0,\pm1,\pm2,\ldots,\pm\frac{n}{2}\right\}\cap Z.$$

(∞) If the Laurent series for r at ∞ is

$$r=\gamma x^{-2}+\cdots\quad(\gamma\in\mathbb{C},\text{ possibly }0),$$

then

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4\gamma} \mid k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap Z.$$

Step 2. We consider all families $(e_c)_{c \in \Gamma \cup \{\infty\}}$ such that $e_c \in E_c$. For each such family, define

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right).$$

If d is a non-negative integer, the family is retained, otherwise the family is discarded. If no families are retained, then ω cannot satisfy a polynomial equation of degree n with coefficients in $\mathbb{C}(x)$.

Step 3. For each family retained from step 2, form the rational function

$$\theta = \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x-c}.$$

Also define

$$S = \prod_{c \in \Gamma} (x-c).$$

Next search for a monic polynomial $P \in \mathbb{C}[x]$ of degree d (as defined in step 2) such that when we define polynomials $P_n, P_{n-1}, \ldots, P_{-1}$ recursively by the formulas below, then $P_{-1} = 0$ (identically).

$$P_n = -P$$

$$P_{i-1} = -SP_i' + ((n-i)S' - S\theta)P_i - (n-i)(i+1)S^2 r P_{i+1} \qquad (i = n, n-1, \ldots, 0).$$

This may be conveniently done by using undetermined coefficients for P. If no such polynomial P is found for any family retained from step 2, then ω cannot satisfy a polynomial equation of degree n with coefficients in $\mathbb{C}(x)$.

Assume that a family and its associated polynomial P has been found. Let ω be a solution of the equation

$$\sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!}\,\omega^i = 0.$$

Then $\eta = e^{\int \omega}$ is a solution of the DE.

5.2. EXAMPLES

Example 1. Our first example illustrates the alternative technique mentioned at the beginning of the last section, namely to bypass the search for equations of degrees 4 and 6 for ω and proceed directly to the search for an equation of degree 12.

We consider the hypergeometric equation $y'' = ry$ where

$$r = -\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}.$$

The necessary conditions of section 2 show that all four cases are possible.

Applying the algorithm for case 1, we find that

$$\alpha_0^+ = 3/4, \qquad \alpha_0^- = 1/4$$
$$\alpha_1^+ = 2/3, \qquad \alpha_1^- = 1/3$$
$$\alpha_\infty^+ = 2/3, \qquad \alpha_\infty^- = 1/3,$$

and $d = \alpha_\infty^\pm - \alpha_0^\pm - \alpha_1^\pm$ can never be a non-negative integer. Case 1 fails.

Applying the algorithm for case 2, we find that

$$E_0 = \{2, 3, 1\}$$
$$E_1 = \{2\}$$
$$E_\infty = \{2\},$$

and $d = e_\infty - e_0 - e_1$ can never be a non-negative integer. Case 2 fails.

We apply the algorithm for case 3, searching for an equation of degree 12 for ω, thus $n = 12$ in the algorithm.

At $c = 0$, $\alpha = -3/16$ and $\sqrt{1+4\alpha} = 1/2$ (or $-1/2$). Hence $E_0 = \{3, 4, 5, 6, 7, 8, 9\}$. At $c = 1$, $\alpha = -2/9$ and $\sqrt{1+4\alpha} = 1/3$. So $E_1 = \{4, 5, 6, 7, 8\}$. At ∞, $\gamma = -2/9$ and $E_\infty = \{4, 5, 6, 7, 8\}$.

Following the instructions of step 2, we now form the expression $d = e_\infty - e_0 - e_1$ for every choice of $e_\infty \in E_\infty$, $e_0 \in E_0$, $e_1 \in E_1$. We discard those families for which d is a negative integer. Only four possibilities remain.

$$\begin{array}{llll} e_\infty = 7, & e_0 = 3, & e_1 = 4, & d = 0 \\ e_\infty = 8, & e_0 = 3, & e_1 = 4, & d = 1 \\ e_\infty = 8, & e_0 = 3, & e_1 = 5, & d = 0 \\ e_\infty = 8, & e_0 = 4, & e_1 = 4, & d = 0. \end{array}$$

We now consider the first possibility, following step 3. We set $\theta = 3/x + 4/(x-1)$, $S = x^2 - x$, and search for a monic polynomial P of degree 1 that satisfies the conditions given in step 3. Of course, $P = 1$.

The computations are far too complicated to be accurately done by hand; however, they are easily programmed into a computer. Since P_i is always a polynomial $(i = 12, \ldots, -1)$ whose degree is easily predicted (in this example $\deg P_i = 12 - i$) arrays of coefficients may be manipulated to carry through the computations. In order to avoid roundoff error, we computed $12^{12-i}P_i$ using 33 digit integer arithmetic. The results follow.

$$P_{12} = -1$$
$$P_{11} = 7x - 3$$
$$P_{10} = (1/12)(-536x^2 + 459x - 99)$$
$$P_9 = (3!/(3\cdot 12^2))(18544x^3 - 23799x^2 + 10260x - 1485)$$
$$P_8 = (4!/(16\cdot 12^2))(-127488x^4 + 217972x^3 - 140879x^2 + 40770x - 4455)$$
$$P_7 = (5!/(2\cdot 12^3))(174080x^5 - 371748x^4 + 320305x^3 - 138975x^2 + 30375x - 2673)$$
$$P_6 = (6!/12^5)(-8257536x^6 + 21145136x^5 - 22757500x^4 + 13168377x^3 - 4318083x^2 + 760347x - 56133)$$
$$P_5 = (7!/(2\cdot 12^5))(7929856x^7 - 23673984x^6 + 30564708x^5 - 22107287x^4 + 9668646x^3 - 2555280x^2 + 377622x - 24057)$$
$$P_4 = (8!/(16\cdot 12^6))(-26421152x^8 + 900984832x^7 - 1356734768x^6 + 1177673400x^5 - 644082327x^4 + 227124972x^3 - 50398362x^2 + 6429780x - 360855)$$
$$P_3 = (9!/(3\cdot 12^8))(174483046x^9 - 6688997376x^8 + 11509039440x^7 - 11656902184x^6 + 7654170465x^5 - 3376695033x^4 + 1000183626x^3 - 191681802x^2 + 21552885x - 1082565)$$

$$P_2 = (10!/(2\cdot 12^9))(-2281701376x^{10}+9713634848x^9-18799438080x^8 +21766009616x^7-16683774768x^6+8840413683x^5 -3277319535x^4+838780110x^3-141739470x^2+14270175x -649539)$$

$$P_1 = (11!/12^{10})(1342177280x^{11}-6282018816x^{10}+13507531776x^9-17598922384x^8 +15426848952x^7-9546427017x^6+4252638672x^5-1362816657x^4 +307684656x^3-46576539x^2+4251528x-177147)$$

$$P_0 = (12!/12^{12})(-8589934592x^{12}+43838865408x^{11}-103681720320x^{10} +150145637824x^9-148170380976x^8+104901110964x^7 -54596424249x^6-21032969490x^5-5948563455x^4 +1203654816x^3-165278151x^2+13817466x-531441)$$

$$P_{-1} = 0$$

Therefore $\eta = e^{\int \omega}$ is a solution of the DE, where ω is a solution of the equation

$$\sum_{i=0}^{12} \frac{(x^2-x)^i P_i}{(12-i)!}\,\omega^i = 0.$$

Professors Caviness and Saunders of Rensselaer Polytechnic Institute kindly offered to attempt a factorisation of this polynomial for ω. They used the exceedingly powerful system for algebraic manipulation called MACSYMA at MIT. The program took less than 5 minutes to write but took 3 minutes of CPU time to execute. The result is that the polynomial above is the cube of the following polynomial.

$$(x^2-x)^4\omega^4-(1/3)(x^2-x)^3(7x-3)\omega 3+(1/24)(x^2-x)^2(48x^2-41x+9)\omega^2 -(1/432)(x^2-x)(320x^3-409x^2+180x-27)\omega +(1/20736)(2048x^4-3484x^3+2313x^2-702x+81)$$

Example 2. In this example we consider the DE $y'' = ry$, where

$$r = -\frac{5x+27}{36(x-1)^2}.$$

The necessary conditions of section 2 show that all four cases are possible.

Note that the partial fraction expansion of r has the form

$$r = -\frac{2}{9(x+1)^2}+\cdots-\frac{2}{9(x-1)^2}+\cdots$$

and the Laurant series for r about ∞ is

$$r = -\frac{5}{36x^2}+\cdots.$$

Applying the algorithm for case 1 we find that

$$\alpha_{-1}^{\pm} = 2/3, \qquad \alpha_{-1}^{-} = 1/3$$
$$\alpha_1^{+} = 2/3, \qquad \alpha_1^{-} = 1/3$$
$$\alpha_\infty^{+} = 5/6, \qquad \alpha_\infty^{-} = 1/6.$$

For no choice of signs is $d = \alpha_\infty^{\pm}-\alpha_{-1}^{\pm}-\alpha_1^{\pm}$ a non-negative integer, thus case 1 cannot hold.

Applying the algorithm for case 2 we find that $E_{-1}=E_1=E_\infty=\{2\}$, and case 2 does not hold.

We now apply the algorithm for case 3, attempting to find an equation of degree 4 over $\mathbb{C}(x)$ that is satisfied by ω.

From step 1 we have that

$$E_{-1}=\{4,4,6,7,8\},\quad E_1=\{4,5,6,7,8\}\quad\text{and}\quad E_\infty=\{2,4,6,8,10\}.$$

There are four families with the property that $d=\frac{1}{3}(e_\infty-e_{-1}-e_1)$ is a non-negative integer, namely

$$\begin{aligned}
&e_\infty=8, && e_{-1}=4, && e_1=4, && d=0,\\
&e_\infty=10, && e_{-1}=4, && e_1=6, && d=0,\\
&e_\infty=10, && e_{-1}=5, && e_1=5, && d=0,\\
&e_\infty=10, && e_{-1}=6, && e_1=4, && d=0.
\end{aligned}$$

The first possibility gives

$$\theta=\frac{1}{3}\left(\frac{4}{x+1}+\frac{4}{x-1}\right)=\frac{8x}{3(x^2-1)}.$$

Setting $S=x^2-1$, we have $S\theta=\frac{8}{3}x$, $S^2r=-\frac{1}{36}(5x^2+27)$. We then have

$$\begin{aligned}
P_4&=-1\\
P_3&=(8/3)x\\
P_2&=-(1/3)(15x^2+1)\\
P_1&=(1/9)(50x^3+14x)\\
P_0&=-(1/54)(125x^4+134x^2-3)\\
P_{-1}&=0.
\end{aligned}$$

Let ω be a solution of the equation

$$S\omega^4=\tfrac{8}{3}xS\omega^3-\tfrac{1}{6}(15x^2+1)S\omega^2+\tfrac{1}{27}(25x^3+7x)S\omega-\tfrac{1}{1296}(125x^4+134x^2-3).$$

If we make the substitution $6S\omega=z+4x$, the equation simplifies to

$$z^4=6(x^2-1)z^2-8x(x^2-1)z+3(x^2-1)^2.$$

Then

$$\eta=e^{\int\omega}=(x^2-1)^{1/3}\exp\left(\int(z/(x^2-1))\,dx\right)$$

is a solution of the DE.

5.3. FINITE SUBGROUPS OF SL(2)

In this section we determine the finite subgroups of $SL(2)$, up to conjugation, and their invariants. This work is classical, being found in the work of Klein, Jordan and others. For the sake of completeness we sketch the results here in the form needed in the subsequent section.

THEOREM 1. *Let G be a finite subgroup of SL(2). Then either*

(*i*) *G is conjugate to a subgroup of the group*

$$D^{\dagger} = D \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot D,$$

where D is the diagonal group, or

(*ii*) *the order of G is* 24 (*the "tetrahedral" case*), *or*

(*iii*) *the order of G is* 48 (*the "octahedral" case*), *or*

(*iv*) *the order of G is* 120 (*the "icosahedral" case*).

In the last three cases G contains the scalar matrix -1.

The geometric names were used by Klein; however, our proof will be entirely algebraic.

Proof. We assume that G is not conjugate to a subgroup of $D^{\dagger}$. Let H be the set of scalar matrices in G, thus $H=\{1\}$ or $\{1, -1\}$, so the order of H is 1 or 2. For any $x \in G-H$ (i.e. $x \in G$ and $x \notin H$) we denote by $Z(x)$ the centraliser of x in G and by $N(x)$ the normaliser of $Z(x)$ in G.

Let $x \in G-H$. Since x is of finite order, x is diagonalisable. (The Jordan form of a non-diagonalisable matrix in $SL(2)$ must be $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.) Since the centraliser in $SL(2)$ of a diagonal non-scalar matrix is D (by direct computation) $Z(x)$ must be the intersection of G and a conjugate of D. Hence $Z(x)=Z(y)$ if and only if $y \in Z(x)$. Using this fact and the fact that $Z(gxg^{-1}) = gZ(x)g^{-1}$ we may conclude that (for arbitrary $x, y, g, g' \in G$) either

$$gZ(x)g^{-1} \cap g'Z(y)g'^{-1} = H \quad \text{or} \quad gZ(x)g^{-1} = g'Z(y)g'^{-1}$$

and in the latter case $y \in g'^{-1}gZ(x)g^{-1}g'$. In addition $gZ(x)g^{-1} = g'Z(x)g'^{-1}$ if and only if $g'^{-1}g \in N(x)$. Therefore we may write G as a disjoint union

$$G = \bigcup_{i=1}^{s} \bigcup (gZ(x_i)g^{-1} - H) \cup H \quad \text{(disjoint)},$$

where the inner union is taken over all cosets $gN(x_i)$ in $G/N(x_i)$, s is some natural number and $x_1, \ldots, x_s \in G-H$.

The group $N(x_i)$ is easy to describe since x_i is diagonalisable. First note that the only matrices in $SL(2)$ that conjugate a diagonal non-scalar matrix into a diagonal matrix are the matrices in $D^{\dagger}$ (by direct computation). It follows that $N(x_i)$ is the intersection of G and a conjugate of $D^{\dagger}$, in particular the index of $Z(x_i)$ in $N(x_i)$, $[N(x_i):Z(x_i)]$, is either 1 or 2.

Let $M = \operatorname{ord}(G/H)$ and $e_i = \operatorname{ord}(Z(x_i)/H)$. The representation of G as a disjoint union gives the following formulas.

$$M \cdot \operatorname{ord} H = \sum_{i=1}^{s} [G:N(x_i)](e_i \cdot \operatorname{ord} H - \operatorname{ord} H) + \operatorname{ord} H,$$

or

$$M = \sum_{i=1}^{s} \frac{M}{[N(x_i):Z(x_i)] \cdot e_i}(e_i - 1) + 1,$$

or

(#)
$$\frac{1}{M} = \sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]}\left(\frac{1}{e_i} - 1\right) + 1.$$

Certainly $s \neq 0$ since $G \neq H$. If $s = 1$, then

$$1/M \geqslant 1/([N(x_1):Z(x_1)]e_1) = 1/\text{ord}\,(N(x_1)/H), \text{ so } G = N(x_1).$$

This contradicts the fact that G is not conjugate to a subgroup of $D^\dagger$.

Since $e_i \geqslant 2$ $(i = 1, \ldots, s)$ we have

$$0 < \frac{1}{M} \leqslant 1 - \tfrac{1}{2} \sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]}$$

so

$$\sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]} < 2.$$

Because

$$[N(x_i):Z(x_i)] = 1 \text{ or } 2,$$

there are only three solutions of this inequality.

$$s = 2, \quad [N(x_1):Z(x_1)] = 1, \quad [N(x_2):Z(x_2)] = 2 \quad ,$$
$$s = 2, \quad [N(x_1):Z(x_1)] = [N(x_2):Z(x_2)] = 2,$$
$$s = 3, \quad [N(x_1):Z(x_1)] = [N(x_2):Z(x_2)] = [N(x_3):Z(x_3)] = 2.$$

For all solutions $[N(x_2):Z(x_2)] = 2$. Thus G contains a conjugate of a matrix in $D^\dagger - D$, i.e. the conjugate of a matrix of the form $\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}$. The square of such a matrix is -1. Hence ord $H = 2$.

The first solution gives $1/M = 1/e_1 + 1/(2e_2) - 1/2$, so $e_1 = 3$, $e_2 = 2$ and $M = 12$, so ord $G = 24$. (The point being that $M > 2e_2$, since G is not conjugate to a subgroup of D, and therefore $e_1 \geqslant 3$.)

The second solution gives $1/M = 1/(2e_1) + 1/(2e_2)$, which is impossible since $M > 2e_2$.

The third solution gives

$$\frac{2}{M} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1.$$

Assuming that $e_1 \leqslant e_2 \leqslant e_3$ we find that $e_1 < 3$ so $e_1 = 2$ and

$$\frac{2}{M} = \frac{1}{e_2} + \frac{1}{e_3} - \frac{1}{2}.$$

Also $e_2 = 3$ since $M > 2e_3$. The solutions are

$$\begin{aligned} e_1 = 2, \quad e_2 = 3, \quad & e_3 = 3, \quad M = 12, \quad \text{ord}\, G = 24, \\ & e_3 = 4, \quad M = 24, \quad \text{ord}\, G = 48, \\ & e_3 = 5, \quad M = 60, \quad \text{ord}\, G = 120. \end{aligned}$$

This proves the theorem.

In the following sequence of theorems we shall explicitly determine the three "geometric" groups. To that end we need the following lemma.

LEMMA. *Let G be a finite subgroup of $SL(2, C)$ that is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then G/H has no normal cyclic subgroup.*

PROOF. If xH is a generator of a normal cyclic subgroup of G/H then the group generated by x and $-x$ is diagonalisable. Since this group would be normal in G, G would be conjugate to a subgroup of $D^\dagger$.

THEOREM 2. *Let G be a subgroup of $SL(2, C)$ of order 24 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H=\{1, -1\}$. Then G/H is isomorphic to A_4, the alternating group on 4 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \phi\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$

where ξ is a primitive 6th root of 1 and $3\phi=2\xi-1$.

PROOF. Since ord G/H is 12, and because of the previous lemma, G/H has 4 Sylow 3-groups, and G/H acts by conjugation on the set of these Sylow 3-groups. This action induces a homomorphism $G/H \to S_4$ (the symmetric group on 4 letters). The subgroup of the image consisting of those permutations that leave a particular Sylow 3-group fixed must have index 4 since G/H acts transitively. Therefore the order of the image is divisible by 4. It follows that the order of the kernel is 1, 2 or 3. By the previous lemma, the order of the kernel must be 1, so G/H is isomorphic to a subgroup of S_4. Now consider the composite homomorphism $G/H \to S_4 \to \{1, -1\}$, with the last arrow being given by $\sigma \to \text{signum}(\sigma)$. By the previous lemma, G/H cannot have a normal subgroup of order 6 (since a subgroup of order 6 contains a unique subgroup of order 3 which would be normal in G/H). Therefore the composite homomorphism has trivial image and G/H is isomorphic to A_4.

Let $\tau: G \to A_4$ be a homomorphism with kernel H. Let $A \in \tau^{-1}(123)$. We may conjugate G so that A is a diagonal matrix. Thus

$$A=\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

Since $\tau A^3=(1)$, $A^3 \in H$. However, $\tau A \neq (1)$ and $\tau A^2 \neq (1)$, thus $A \notin H$ and $A^2 \notin H$. Replacing A by $-A$, if necessary, we may assume that ξ is a primitive 6th root of 1.

Let $B \in \tau^{-1}(12)(34)$. Since $\tau(AB) \neq \tau(BA)$, B cannot be a diagonal matrix, i.e. not both B_{12} and B_{21} are zero. In fact neither is zero because if one were zero and the other non-zero, then B would have infinite order.

We may conjugate G by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ without affecting A. If we choose $c=B_{21}$ and $d=2B_{12}$, then B has the form

$$B=\begin{pmatrix} \phi & \psi \\ 2\psi & -\chi \end{pmatrix}.$$

Now $\tau B^2=(1)$ so $B^2 \in H$. A direct computation shows that $\chi=\phi$.

Next we observe that $\tau(BA^2)=\tau(AB)^2$ so $BA^2=\pm(AB)^2$. We perform the computation and discover that $\phi(\xi^2-1)=\pm\xi^4$ (using the fact that $\psi \neq 0$). Replacing B by $-B$, if necessary, we may assume that $\phi(\xi^2-1)=\xi^4$, hence $3\phi=2\xi-1$ (using the relation $\xi^2=\xi-1$).

Next we use the fact that $\det B=1$ to obtain the formula $\phi^2+2\psi^2=-1$, or $3\psi=\pm(2\xi-1)$. If necessary, we conjugate G by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $3\psi=2\xi-1=3\phi$. This proves the theorem.

The group of this theorem is called the tetrahedral group.

THEOREM 3. *Let G be a subgroup of $SL(2)$ of order 48 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H=\{1,-1\}$. Then G/H is isomorphic to $\mathbf{S}_4$, the symmetric group on 4 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \phi\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where ξ is a primitive 8th root of 1 and $2\phi=\xi(\xi^2+1)$.

PROOF. Since ord $G/H=24$, and because of the previous lemma, G/H has 4 Sylow 3-groups. The action of G/H on the set of Sylow 3-groups (via conjugation) induces a homomorphism $G/H\to\mathbf{S}_4$. The image contains a subgroup of index 4, namely the subgroup of permutations leaving a particular Sylow 3-group fixed, since G/H acts transitively on the set of Sylow 3-groups. Hence the order of the image is divisible by 4, so the order of the kernel is 1, 2, 3 or 6. Were the order of the kernel 6, then the kernel would contain a unique subgroup of order 3 which would be normal in G. This contradicts the lemma. Indeed, the lemma implies that ord ker $=1$, so G/H is isomorphic to $\mathbf{S}_4$.

Let $\tau: G\to\mathbf{S}_4$ be a homomorphism with kernel H and let $A\in\tau^{-1}(1234)$. We may conjugate G so that A is a diagonal matrix

$$A=\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

Since $\tau A^4=(1)$, $\xi^4=\pm 1$. However, were $\xi^4=1$, then $\xi^2=\pm 1$ and $A^2\in H$. But $\tau A^2\neq(1)$. Hence ξ is a primitive 8th root of 1.

Let $B\in\tau^{-1}(12)$. Since $\tau(AB)\neq\tau(BA)$, B cannot be a diagonal matrix, thus not both B_{12} and B_{21} are zero. In fact, neither is zero since B has finite order. We may conjugate G, without disturbing A, by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c^2=B_{21}$ and $d^2=B_{12}$. Then B has the form

$$B=\begin{pmatrix} \phi & \psi \\ \psi & -\chi \end{pmatrix}.$$

Using the fact $\tau B^2=(1)$, i.e. $B^2\in H$, we obtain easily that $\chi=\phi$.

Because $\tau(BA^3)=\tau(AB)^2$, $BA^3=\pm(AB)^2$. Making this computation, and using the fact that $\psi\neq 0$, we find that $\phi(\xi^2-1)=\pm\xi$, or $2\phi=\pm\xi(\xi^2+1)$. Replacing B by $-B$, if necessary, we may assume that $2\phi=\xi(\xi^2+1)$. Then $2\phi^2=-1$. Now we use the fact that $1=\det B=-\phi^2-\psi^2$ to conclude that $2\psi^2=-1$. Conjugate G, if necessary, by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $\psi=\phi$.

Because $\tau A, \tau B$ generate $\mathbf{S}_4$ and the group generated by A, B contains H, we can conclude that A, B generate G. This proves the theorem.

The group of this theorem is called the octahedral group.

THEOREM 4. *Let G be a subgroup of $SL(2)$ of order 120 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H=\{1,-1\}$. Then G/H is isomorphic to $\mathbf{A}_5$, the alternating group on 5 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix},$$

where ξ is a primitive 10th root of 1, $5\phi=3\xi^3-\xi^2+4\xi-2$, and $5\psi=\xi^3+3\xi 2-2\xi+1$.

PROOF. The proof that G/H is isomorphic to $\mathbf{A}_5$ may be found in Burnside (1955, 127, p. 161–2).

Let $\tau: G \to \mathbf{A}_5$ be a homomorphism with kernel H and let $A \in \tau^{-1}(12345)$. We may conjugate G so that A is a diagonal matrix

$A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Since $\tau A^5 = (1)$, $\xi^5 = \pm 1$. Replacing A with $-A$, if necessary, we may assume that $\xi^5 = -1$. Evidently ξ is a primitive 10th root of 1.

Let $B \in \tau^{-1}(12)(34)$. As in the proof of Theorem 3, we may assume that B has the form

$$B = \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}.$$

Because $\tau(A^4B) = \tau(BA)^2$, $A^4B = \pm(BA)^2$. Making this computation we find that $\phi(1+\xi^3) = \pm\xi^4$, or $5\phi = \pm(3\xi^3 - \xi^2 + 4\xi - 2)$. Replacing B by $-B$, if necessary, we may assume that the plus sign obtains. Now we use the fact that $1 = \det B$ to conclude that $5\psi = \pm(\xi^3 + 3\xi^2 - 2\xi + 1)$. Conjugate G by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, if necessary, so that the plus sign obtains.

Note that $\tau A, \tau B$ generate $\mathbf{A}_5$. (This group generated by τA and τB contains an element of order 5, an element of order 2 and an element of order 3. Thus the order of this group is divisible by 30. Since $\mathbf{A}_5$ is simple, this group must be $\mathbf{A}_5$.) Also the group generated by A, B contains H. Therefore A, B generate G. This proves the theorem.

The group described in this theorem is called the icosahedral group.

For use in the next section, we also need to know the invariants of the three "geometric" groups.

THEOREM 5. *Let G be the Galois group of the DE $y'' = ry$ and let η, ζ be a fundamental system of solutions relative to the group G.*

(i) *If G is the tetrahedral group, then $(\eta^4 + 8\eta\zeta^3)^3 \in \mathbb{C}(x)$.*
(ii) *If G is the octahedral group, then $(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbb{C}(x)$.*
(iii) *If G is the icosahedral group, then $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11} \in \mathbb{C}(x)$.*

PROOF. (i) Consider the tetrahedral group, using the notation of Theorem 2. Recall that $\xi^3 = -1$, $\xi^2 = \xi - 1$ and $3\phi = 2\xi - 1$.

$\eta^4 + 8\eta\zeta^3$ is carried into $\xi^4(\eta^4 + 8\eta\zeta^3)$ by the matrix $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. The matrix $\phi\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ carries

$$\eta^4 + 8\eta\zeta^3 = \eta \cdot (\eta + 2\zeta) \cdot (\eta + 2\xi^2\zeta) \cdot (\eta - 2\xi\zeta)$$

into

$$\begin{aligned} &\phi(\eta+2\zeta) \cdot 3\phi\eta \cdot \phi(2\xi-1)(\eta-2\xi\zeta) \cdot \phi(1-2\xi)(\eta+2\xi^2\zeta) \\ &\quad = -3 \cdot \phi^4 \cdot (2\xi-1)^2 \cdot (\eta 4 + 8\eta\zeta^3) \\ &\quad = -3 \cdot (-1/3)^2 \cdot (-3) \cdot (\eta^4 + 8\eta\zeta^3) = \eta^4 + 8\eta\zeta^3. \end{aligned}$$

Thus $(\eta^4 + 8\eta\zeta^3)^3$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

(ii) Consider the octahedral group, using the notation of Theorem 3. Recall that $\xi^4 = -1$ and $2\phi = \xi(\xi^2 + 1)$.

$\eta^5\zeta-\eta\zeta^5$ is carried into $\xi^4(\eta^5\zeta-\eta\zeta^5)$ by the matrix $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$. The matrix $\phi\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ carries

$$\eta^5\zeta-\eta\zeta^5 = \eta\cdot\zeta\cdot(\eta+\zeta)\cdot(\eta-\zeta)\cdot(\eta+\xi^2\zeta)\cdot(\eta-\xi^2\zeta)$$

into

$$\begin{aligned}\phi(\eta+\zeta)\cdot(\eta-\zeta)\cdot 2\phi\eta\cdot 2\phi\zeta\cdot\phi(1+\xi^2)(\eta-\xi^2\zeta)\cdot\phi(1-\xi^2)(\eta+\xi^2\zeta)\\ = 4\cdot\phi^6\cdot(1-\xi^4)\cdot(\eta^5\zeta-\eta\zeta^5)\\ = 8\cdot(-1/2)^3\cdot(\eta^5\zeta-\eta\zeta^5) = -(\eta^5\zeta-\eta\zeta^5).\end{aligned}$$

Thus $(\eta^5\zeta-\eta\zeta^5)^2$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

(iii) Consider the icosahedral group and use the notation of Theorem 4. First we collect some easily derivable formulas.

$$\begin{aligned}\xi^5 = -1,\quad \xi^4 = \xi^3-\xi^2+\xi-1,\\ 5\phi^2 = \xi^3-\xi^2-3,\quad 5\psi^2 = -\xi^3+\xi^2-2,\\ 5\phi\psi = 2\xi^3-2\xi^2-1 = 5(\phi^2-\psi^2).\end{aligned}$$

The matrix $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ carries $\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$ into itself. The matrix $\begin{pmatrix}\phi & \psi\\ \psi & -\phi\end{pmatrix}$ carries

$$\begin{aligned}\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11} = \eta\zeta\cdot(\eta^2-\eta\zeta-\zeta^2)\cdot(\eta^2+\xi^3\eta\zeta+\xi\zeta^2)\cdot\\ (\eta^2-\xi^2\eta\zeta-\xi^4\zeta^2)\cdot(\eta^2+\xi\eta\zeta-\xi^2\zeta^2)\cdot(\eta^2-\xi^4\eta\zeta+\xi^3\zeta^2)\end{aligned}$$

into

$$\begin{aligned}\phi\psi(\eta^2-\eta\zeta-\zeta^2)\cdot 5\phi\psi\eta\zeta\cdot(-\xi)(\eta^2-\xi^2\eta\zeta-\xi^4\zeta^2)\cdot(\xi^4)(\eta^2+\xi^3\eta\zeta+\xi\zeta^2)\cdot\\ (-1)(\eta^2+\xi\eta\zeta-\xi^2\zeta^2)\cdot(-1)(\eta^2-\xi^4\eta\zeta+\xi^3\zeta^2)\\ = 5\cdot(\phi\psi)^2\cdot(\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}) = \eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}.\end{aligned}$$

Thus $\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

This proves the theorem.

5.4. PROOF OF THE ALGORITHM

We must prove the validity of four separate algorithms. We must show that the algorithms for finding a 4th, 6th and 12th degree equation for ω are correct for the tetrahedral, octahedral and icosahedral groups and that the equation obtained is irreducible, and finally that the algorithm for finding a 12th degree equation is all-inclusive (although the equation obtained need not be irreducible). In so far as is possible, we carry out the proofs simultaneously.

We begin by showing that the equations obtained for ω in the tetrahedral, octahedral and icosahedral cases are minimal. Throughout we assume that the Galois group G of the DE $y''=ry$ is the tetrahedral, octahedral or icosahedral group. We also fix a fundamental system of solutions η, ζ of the DE relative to the group G and set $\omega=\eta'/\eta$.

THEOREM 1. *Let η_1 be any solution of the DE and let $\omega_1=\eta_1'/\eta_1$.*

(*i*) *If G is the tetrahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 4 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 4.$$

(*ii*) *If G is the octahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 6 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 6.$$

(*iii*) *If G is the icosahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 12 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 12.$$

PROOF. Since ω is left fixed by the group G_1 generated by $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$, where ξ is a primitive 6th, 8th or 10th root of 1 in the tetrahedral, octahedral or icosahedral cases, the degree of ω over $\mathbb{C}(x)$ is $\leqslant [G:G_1] = 4$, 6 or 12. The reverse inequality is proven more generally, as indicated in the statement of the theorem.

Let G_1 be the subgroup of G that fixes η_1. Complete η_1 to a fundamental system of solutions η_1, ζ_1 of the DE and conjugate G to XGX^{-1} so that XGX^{-1} is the Galois group of the DE relative to η_1, ζ_1. Then XG_1X^{-1} consists of matrices of the form $\begin{pmatrix}c & d\\ 0 & c^{-1}\end{pmatrix}$. Since XG_1X^{-1} is finite, $d=0$ and $c^m=1$, where m is the order of G_1. Evidently XG_1X^{-1} is a subgroup of the cyclic group

$$\left\{\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\middle| c^m = 1\right\}$$

and therefore is cyclic. Hence G_1/H (where H is the centre of G) is isomorphic to a cyclic subgroup of $\mathbf{A}_4$ in the tetrahedral case, of $\mathbf{S}_4$ in the octahedral case, and of $\mathbf{A}_5$ in the icosahedral case. So ord $G_1/H \leqslant 3, 4, 5$ or ord $G_1 \leqslant 6, 8, 10$. *Thus*

$$\deg_{\mathbb{C}(x)}\omega_1 = [G:G_1] \geqslant 4, 6, 12.$$

This proves the theorem.

Throughout the remainder of this section we shall be considering a certain differential equation written recursively, namely

$$a_n = -1$$

$$(\#)_n \qquad a_{i-1} = -a_i' - za_i - (n-i)(i+1)ra_{i+1} \quad (i = n, \ldots, 0)$$

By a solution of $(\#)_n$ is meant a function z such that when $a_n, \ldots, a_{-1}$ are defined as above, then a_{-1} is (identically) zero.

THEOREM 2. *Let z be a solution of* $(\#)_n$, *and let* ω *be any solution of*

$$\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!}\,\omega^i.$$

Then $\eta = e^{\int\omega}$ *is a solution of the DE* $y'' = ry$.

Proof. Let

$$A = -w^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!}\,w^i = \sum_{i=0}^{n} \frac{a_i}{(n-i)!}\,w^i \quad (a_n = -1),$$

where w is an indeterminate. We claim that

$$\frac{\partial^{k+1}A}{\partial w^{k+1}}(w^2 - r) = \frac{\partial^{k+1}A}{\partial w^k\,\partial x} + [(n-2k)w + z]\frac{\partial^k A}{\partial w^k} + k(n-k+1)\frac{\partial^{k-1}A}{\partial w^{k-1}} \quad (k = 0, 1, \ldots).$$

For $k=0$, we have

$$\frac{\partial A}{\partial w}(w^2-r) = \left(\sum_{i=1}^{n} \frac{ia_i}{(n-i)!} w^{i-1}\right)(w^2-r)$$

$$= na_n w^{n+1} + \sum_{i=0}^{n-1} \frac{ia_i}{(n-i)!} w^{i+1} - \sum_{i=0}^{n-1} \frac{(i+1)ra_{i+1}}{(n-1-i)!} w^i$$

$$= nwA - \sum_{i=0}^{n-1} \frac{(n-i)a_i}{(n-i)!} w^{i+1} - \sum_{i=0}^{n-1} \frac{(n-i)(i+1)ra_{i+1}}{(n-i)!} w^i$$

$$= nwA + a_{n-1}A - \sum_{i=0}^{n-1} \frac{a_{n-1}a_i}{(n-i)!} w^i - \sum_{i=0}^{n} \frac{a_{i-1}}{(n-i)!} w^i$$

$$- \sum_{i=0}^{n} \frac{(n-i)(i+1)ra_{i+1}}{(n-i)!} w^i$$

$$= (nw+z)A - \sum_{i=0}^{n} \frac{1}{(n-i)!}[za_i + {}^{a}{}_{i-1} + (n-i)(i+1)ra_{i+1}]w^i$$

$$= (nw+z)A + \sum_{i=0}^{n} \frac{a_i'}{(n-i)!} w^i = (nw+z)A + \frac{\partial A}{\partial x}.$$

Our claim now follows by induction.

To show that $\eta = e^{\int \omega}$ is a solution of the DE is equivalent to showing that $\omega' + \omega^2 = r$. We assume that $\omega' + \omega^2 - r \neq 0$ and force a contradiction.

Since $A(\omega) = 0$, we have

$$\frac{\partial A}{\partial w}(\omega)\omega' + \frac{\partial A}{\partial x}(\omega) = 0.$$

Therefore

$$\frac{\partial A}{\partial w}(\omega)(\omega' + \omega^2 - r) = -\frac{\partial A}{\partial x}(\omega) + (n\omega + z)A(\omega) + \frac{\partial A}{\partial x}(\omega) = 0.$$

Hence

$$\frac{\partial A}{\partial w}(\omega) = 0.$$

Assuming that

$$\frac{\partial^{k-1} A}{\partial w^{k-1}}(\omega) = \frac{\partial^k A}{\partial w^k}(\omega) = 0,$$

we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^k A}{\partial w^k}(\omega)\right) = \frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega)\omega' + \frac{\partial^{k+1} A}{\partial w^k x}(\omega).$$

Thus

$$\frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega)(\omega' + \omega^2 - r)$$

$$= -\frac{\partial^{k+1} A}{\partial w^k\, \partial x}(\omega) + \frac{\partial^{k+1} A}{\partial w^k\, \partial x}(\omega) + [(n-2k)\omega + z]\frac{\partial^k A}{\partial w^k}(\omega) + k(n-k+1)\frac{\partial^{k-1} A}{\partial w^{k-1}}(\omega)$$

$$= 0,$$

so

$$\frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega) = 0.$$

The desired contradiction follows from the fact that

$$\frac{\partial^n A}{\partial w^n}(\omega) = -n! \neq 0.$$

This proves the theorem.

THEOREM 3.

(i) *Suppose that* $(\#)_4$ *has a solution* $z \in \mathbb{C}(x)$. *Then the polynomial*

$$w^4 - \sum_{i=0}^{3} \frac{a_i}{(4-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

(ii) *Suppose that* $(\#)_6$ *has a solution* $z \in \mathbb{C}(x)$. *Then the polynomial*

$$w^6 - \sum_{i=0}^{5} \frac{a_i}{(6-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

(iii) *Suppose that* $(\#)_{12}$ *has a solution* $z \in \mathbb{C}(x)$ *and that* $(\#)_4$ *and* $(\#)_6$ *do not have solutions in* $\mathbb{C}(x)$. *Then the polynomial*

$$w^{12} - \sum_{i=0}^{11} \frac{a_i}{(12-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

PROOF. By Theorems 1 and 2, any root of the polynomial

$$w^n - \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i \quad (a_i \in \mathbb{C}(x))$$

must have degree 4, 6 or 12 over $\mathbb{C}(x)$. Statement (i) of the present theorem is clear. Statement (ii) follows from the fact that if a sextic is reducible, then one of the factors has degree $\leqslant 3$. To prove (iii) it suffices to show that if $\deg_{\mathbb{C}(x)} \omega = n$, then $(\#)_n$ has a solution $z \in \mathbb{C}(x)$.

Let $A \in \mathbb{C}(x)[w]$ be the minimal polynomial for ω over $\mathbb{C}(x)$. Let $\deg_w A = n$ and write

$$A = -w^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i = \sum_{i=0}^{n} \frac{a_i}{(n-i)!} w^i \quad (a_n = -1).$$

Consider the polynomial

$$B = \frac{\partial A}{\partial w}(r - w^2) + \frac{\partial A}{\partial x} + (nw + z)A,$$

where

$$z = a_{n-1} \in \mathbb{C}(x).$$

The coefficient of w^{n+1} in B is

$$-na_n + na_n = 0,$$

and the coefficient of w^n in B is

$$-(n-1)a_{n-1} + a_n' + na_{n-1} + za_n = a_{n-1} - z = 0,$$

since $a_n = -1$ and $a_{n-1} = z$. Therefore $\deg_w B < n$. But

$$\begin{aligned} B(\omega) &= \frac{\partial A}{\partial w}(\omega)(r-\omega^2) + \frac{\partial A}{\partial x}(\omega) + (n\omega + z)A(\omega) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(A(\omega)) + (n\omega+z)A(\omega) \\ &= 0. \end{aligned}$$

Therefore $B = 0$. The coefficient of w^i in B is

$$\begin{aligned} 0 &= (i+1)\frac{a_{i+1}}{(n-1-i)!}r - (i-1)\frac{a_{i-1}}{(n+1-i)!} + \frac{a_i'}{(n-i)!} + n\frac{a_{i-1}}{(n+1-i)!} + z\frac{a_i}{(n-i)!} \\ &= \frac{1}{(n-i)!}[(n-i)(i+1)ra_{i+1} + a_{i-1} + a_i' + za_i], \end{aligned}$$

where $a_{-1} = 0$. These are precisely the equations of $(\#)_n$. This proves the theorem.

For any function b we denote by $l\delta b = b'/b$ the "logarithmic derivative" of b.

THEOREM 4. *Let F be any form (homogeneous polynomial) of degree n in solutions of the DE. Then $z = l\delta F$ is a solution of $(\#)_n$.*

PROOF. First we prove that if F_1 and F_2 are functions such that $l\delta F_1$ and $l\delta F_2$ are solutions of $(\#)_n$, then $l\delta(c_1F_1 + c_2F_2)$ is a solution of $(\#)_n$ for any $c_1, c_2 \in \mathbb{C}$. Let a_i^1, a_i^2, a_i^3 denote the sequences determined by $(\#)_n$ for $z = l\delta F_1, l\delta F_2, l\delta(c_1F_1 + c_1F_2)$ respectively.

We claim that

$$(c_1F_1 + c_2F_2)a_i^3 = c_1F_1a_i^1 + c_2F_2a_i^2.$$

This is clear for $i = n$. Also

$$\begin{aligned} (c_1F_1 + c_2F_2)a_{i-1}^3 &= (c_1F_1 + c_2F_2)[-a_i^{3\prime} - l\delta(c_1F_1 + c_2F_2)a_i^3 - (n-i)(i+1)ra_{i+1}^3] \\ &= -[(c_1F_1 + c_2F_2)a_i^3]' - (n-i)(i+1)r(c_1F_1 + c_2F_2)a_{i+1}^3 \\ &= -[c_1F_1a_i^1 + c_2F_2a_i^2]' - (n-i)(i+1)[c_1F_1a_{i+1}^1 + c_2F_2a_{i+1}^2] \\ &= c_1F_1a_{i-1}^1 + c_2F_2a_{i-1}^2 \quad (i = n, \ldots, 0). \end{aligned}$$

Therefore

$$(c_1F_1 + c_2F_2)a_{-1}^3 = c_1F_1a_{-1}^1 + c_2F_2a_{-1}^2 = 0,$$

which verifies our assertion.

To prove the theorem, we may assume that

$$F = \prod_{i=1}^{n} \eta_i,$$

where $\eta_1, \ldots, \eta_n$ are solutions of the DE.

Let $\omega_i = \eta_i'/\eta_i$ and denote by σ_{mk} the kth symmetric function of $\omega_1, \ldots, \omega_m$. Thus $\sigma_{mk} = 0$ for $k < 0$ or $k > m$, $\sigma_{m0} = 1$ and

$$\sigma_{mk} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} \omega_{i_1} \cdots \omega_{i_k}$$

for $1 \leqslant k \leqslant m$. First we claim that

$$\sigma_{mk}' = (m+1-k)r\sigma_{m,k-1} - \sigma_{m1}\sigma_{mk} + (k+1)\sigma_{m,k+1}.$$

This formula is easily checked for $m=1$ and, for $m>1$,

$$\begin{aligned}\sigma'_{mk} &= (\sigma_{m-1,k}+\sigma_{m-1,k-1}\omega_m)' \\ &= (m-k)r\sigma_{m-1,k-1}-\sigma_{m-1,1}\sigma_{m-1,k}+(k+1)\sigma_{m-1,k+1} \\ &\quad +[(m+1-k)r\sigma_{m-1,k-2}-\sigma_{m-1,1}\sigma_{m-1,k-1}+k\sigma_{m-1,k}]\omega_m \\ &\quad +\sigma_{m-1,k-1}(r-\omega_m^2) \\ &= (m+1-k)r(\sigma_{m-1,k-1}+\sigma_{m-1,k-2}\omega_m)-(\sigma_{m-1,1}+\omega_m)(\sigma_{m-1,k}+\sigma_{m-1,k-1}\omega_m) \\ &\quad +(k+1)(\sigma_{m-1,k+1}+\sigma_{m-1,k}\omega_m) \\ &= (m+1-k)r\sigma_{m,k-1}-\sigma_{m1}\sigma_{mk}+(k+1)\sigma_{m,k+1},\end{aligned}$$

which completes the induction.

Next we use induction on i to prove that

$$a_i = (-1)^{n-i+1}(n-i)!\,\sigma_{n,n-i}.$$

Evidently

$$a_{n-1} = z = l\delta F = \sum_{i=1}^{n} \omega_i = \sigma_{n1}.$$

Using $(\#)_n$, we have

$$\begin{aligned}a_{i-1} &= -a'_i - za_i-(n-i)(i+1)ra_{i+1} \\ &= (-1)^{n-i}(n-i)!\,\sigma'_{n,n-i}+\sigma_{n1}(-1)^{n-i}(n-i)!\,\sigma_{n,n-i} \\ &\quad -(n-i)(i+1)r(-1)^{n-i}(n-1-i)!\,\sigma_{n,n-1-i} \\ &= (-1)^{n-i}(n-i)!\,[\sigma'_{n,n-i}+\sigma_{n1}\sigma_{n,n-i}-(i+1)r\sigma_{n,n-1-i}] \\ &= (-1)^{n-i}(n-i)!\,(n-i+1)\sigma_{n,n-i+1} \\ &= (-1)^{n-i}(n-i+1)!\,\sigma_{n,n-i+1}.\end{aligned}$$

Hence

$$a_{-1} = (-1)^n(n+1)!\,\sigma_{n,n+1} = 0.$$

This completes the proof of the theorem.

THEOREM 5.

(i) *If G is the tetrahedral group, then $(\#)_4$ has a solution $z=l\delta u$, where $u^3\in\mathbb{C}(x)$.*

(ii) *If G is the octahedral group, then $(\#)_6$ has a solution $z=l\delta u$, where $u^2\in\mathbb{C}(x)$.*

(iii) *If G is either the tetrahedral group, the octahedral group or the icosahedral group, then $(\#)_{12}$ has a solution $z=l\delta u$, where $u\in\mathbb{C}(x)$.*

PROOF. This theorem is a corollary of Theorem 3 of the present section and Theorem 3 of the previous section. For part (i) we may take $u=\eta^4+8\eta\zeta^3$, for part (ii) we may take $u=\eta^5\zeta-\eta\zeta^5$ and for part (iii) we may take $u=(\eta^4+8\eta\zeta^3)^3$, $(\eta^5\zeta-\eta\zeta^5)^2$ or $\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$.

We shall write

$$u^{12/n} = \prod_{c\in\mathbb{C}} (x-c)^{e_c}\in\mathbb{C}(x),$$

where $n=4$, 6 or 12 and $e_c\in Z$. Our next step in the proof is to determine the various possibilities for e_c, as stated in step 1 of the algorithm. For ease of notation, we shall assume that $c=0$. To this end we shall use the Laurent series for

$$z = l\delta u = \frac{n}{12}\,l\delta(u^{12/n}),$$

namely

$$z = \frac{n}{12} e x^{-1} + \cdots \quad (e = e_0 \in \mathbb{Z}, \text{ possibly } 0)$$

and for r, namely

$$r = \alpha x^{-2} + \beta x^{-1} + \cdots \quad (\alpha, \beta \in \mathbb{C}, \text{ possibly } 0).$$

(Note that, by the necessary conditions of section 2, r can have no pole of order exceeding 2.)

First we consider the possibility that $\alpha = 0$ and $\beta \neq 0$, corresponding to (c_1) of Step 1 of the algorithm.

THEOREM 6. *If* $\alpha = 0$ *and* $\beta \neq 0$, *then* $e = 12$.

PROOF. We write

$$z = \frac{n}{12} e x^{-1} + f + \cdots,$$

and treat e and f as indeterminates. Then

$$a_i = A_i x^{i-n} + B_i x^{i-n+1} + C_i f x^{i-n+1} + \cdots,$$

where A_i, B_i, C_i are polynomials in e with coefficients in $\mathbb{C}$. Using $(\#)_n$ we find that

$$A_n = -1, \qquad B_n = C_n = 0,$$

$$A_{i-1} = \left(n - i - \frac{n}{12} e\right) A_i,$$

$$B_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) B_i - (n-i)(i+1)\beta A_{i+1},$$

$$C_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) C_i - A_i,$$

for $i = n, \ldots, 0$.

We leave to the reader the verification that the solution to these equations is given by

$$A_i = -\prod_{j=0}^{n-i-1} \left(j - \frac{n}{12} e\right)$$

$$B_i = \beta \sum_{j=0}^{n-i-2} (j+1)(n-j) \prod_{\substack{k=0 \\ k \neq j}}^{n-i-2} \left(k - \frac{n}{12} e\right),$$

$$C_i = (n-i) \prod_{j=0}^{n-i-2} \left(j - \frac{n}{12} e\right) \quad (i = n, \ldots, 0)$$

because

$$0 = a_{-1} = A_{-1} x^{-n-1} + B_{-1} x^{-n} + C_{-1} f x^{-n} + \cdots,$$

$$0 = A_{-1} = -\prod_{j=0}^{n} \left(j - \frac{n}{12} e\right)$$

and

$$0 = B_{-1} + C_{-1} f$$

$$= \beta \sum_{j=0}^{n-1} (j+1)(n-j) \prod_{\substack{k=0 \\ k \neq j}}^{n-1} \left(k - \frac{n}{12} e\right) + f(n+1) \prod_{k=0}^{n-1} \left(k - \frac{n}{12} e\right).$$

The first equation implies that

$$e = \frac{12}{n} l$$

for some $l = 0, \ldots, n$. Suppose that $l \neq n$. Then the second equation gives

$$C = \beta(l+1)(n-l) \prod_{\substack{k=0 \\ k \neq l}}^{n-1} (k-l),$$

which implies that $\beta = 0$. This contradiction shows that $l = n$ and therefore $e = 12$. This proves the theorem.

Next we consider the possibility that $\alpha \neq 0$. This corresponds to (c_2) of Step 1 of the algorithm. As above we write $a_i = A_i x^{i-n} + \cdots$.

LEMMA. *A_i is a polynomial in e with coefficients in $\mathbb{Q}[\alpha]$ whose degree is $n-i$ and whose leading coefficient is $-(-(n/12))^{n-i}$.*

PROOF. Using $(\#)_n$ we have

$$A_n = -1,$$

$$A_{i-1} = \left(n - i - \frac{n}{12} e\right) A_i - (n-i)(i+1)\alpha A_{i+1}.$$

The lemma is immediate from these formulas.

The author did not succeed in finding a closed-form solution of these equations, thus we shall use an indirect argument.

Assume that $\alpha \neq -1/4$. Then the DE $y'' = ry$ has Puisseaux series solutions of the form

$$\eta_1 = x^{\mu_1} + \cdots, \qquad \mu_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha},$$

$$\eta_2 = x^{\mu_1} + \cdots, \qquad \mu_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4a}.$$

By Theorem 4, $l\delta(\eta_1^i \eta_2^{n-i})$ is a solution of $(\#)_n$ for every $i = 0, \ldots, n$. Since

$$l\delta(\eta_1^i \eta_2^{n-i}) = (i\mu_1 + (n-i)\mu_2)x^{-1} + \cdots$$

$$= \left(\frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha}\right) x^{-1} + \cdots,$$

the polynomial A_{-1} must vanish for

$$\frac{12}{n} e = \frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha} \quad (i = 0, \ldots, n).$$

THEOREM 7.

(*i*) *Assume that G is the tetrahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 3, \pm 6$.*

(*ii*) *Assume that G is the octahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 2, \pm 4, \pm 6$.*

(*iii*) *Assume that G is either the tetrahedral group, the octahedral group or the icosahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 1, \ldots, \pm 6$.*

PROOF. (i) In this case $n=4$. If $\alpha \neq 1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{4} \left(\frac{e}{3} - 2 + (2-i)\sqrt{1+4\alpha}\right).$$

Thus

$$e = 6 + k\sqrt{1+4\alpha}, \qquad k = 0, \pm 3, \pm 6.$$

If $\alpha = -1/4$, we compute directly, using the recurrence relations given above.

$$\begin{aligned}
A_4 &= -1 \\
A_3 &= \tfrac{1}{3}e \\
A_2 &= -\tfrac{1}{9}(e^2 - 3e + 9) \\
A_1 &= \tfrac{1}{27}(e^3 - 9e^2 + \tfrac{81}{2}e - 54) \\
A_0 &= -\tfrac{1}{81}(e^4 - 18e^3 + 135e^2 - 459e + \tfrac{1215}{2}) \\
A_{-1} &= \tfrac{1}{243}(e^5 - 30e^4 + 360e^3 - 2160e^2 + 6480e - 7776) \\
&= \tfrac{1}{243}(e-6)^5.
\end{aligned}$$

(ii) In this case $n=6$. *If* $\alpha \neq -1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{6} \left(\frac{e}{2} - 3 + (3-i)\sqrt{1+4\alpha}\right).$$

Thus

$$e = 6 + k\sqrt{1+4\alpha}, \qquad k = 0, \pm 2, \pm 4, \pm 6.$$

If $\alpha = -1/4$, we compute directly.

$$\begin{aligned}
A_6 &= -1 \\
A_5 &= \tfrac{1}{2}e \\
A_4 &= -\tfrac{1}{4}(e^2 - 2e + 6) \\
A_3 &= \tfrac{1}{8}(e^3 - 6e^2 + 24e - 24) \\
A_2 &= -\tfrac{1}{16}(e^4 - 12e^3 + 72e^2 - 192e + 216) \\
A_1 &= \tfrac{1}{32}(e^5 - 20e^4 + 180e^3 - 840e^2 + 2040e - 2016) \\
A_0 &= -\tfrac{1}{64}(e^6 - 30e^5 + 390e^4 - 2760e^3 + 11160e^2 - 24336e + 22320) \\
A_{-1} &= \tfrac{1}{128}(e^7 - 42e^6 + 756e^5 - 7560e^4 + 45360e^3 - 163296e^2 + 326592e - 279936) \\
&= \tfrac{1}{128}(e-6)^7.
\end{aligned}$$

(iii) In this case $n=12$. If $\alpha \neq -1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{12} (e - 6 + (6-i)\sqrt{1+4\alpha}).$$

Thus

$$e = 6 + k\sqrt{1+4\alpha}, \quad k = 0, \pm 1, \ldots, \pm 6.$$

If $\alpha = -1/4$, we compute directly. Using a programmable calculator we obtained the following.

$$
\begin{aligned}
A_{12} &= -1 \\
A_{11} &= e \\
A_{10} &= -e^2+e-3 \\
A_9 &= e^3-3e^2+\tfrac{21}{2}e-6 \\
A_8 &= -e^4+6e^3-27e^2+45e-\tfrac{81}{2} \\
A_7 &= e^5-10e^4+60e^3-180e^2+315e-216 \\
A_6 &= -e^6+15e^5-120e^4+540e^3-1485e^2+2241e-1485 \\
A_5 &= e^7-21e^6+\tfrac{441}{2}e^5-1365e^4+5355e^3-13041e^2+\tfrac{36477}{2}e-11178 \\
A_4 &= -e^8+28e^7-378e^6+3066e^5-16170e^4+56196e^3-125118e^2 \\
&\quad +162378e-\tfrac{187677}{2} \\
A_3 &= e^9-36e^8+612e^7-6300e^6+42903e^5-199206e^4+628236e^3 \\
&\quad -1293732e^2+\tfrac{3150495}{2}e-862488 \\
A_2 &= -e^{10}+45e^9-945e^8+12060e^7-103005e^6+612927e^5-2566620e^4 \\
&\quad -7453620e^3-\tfrac{28689795}{2}e2+\tfrac{33002235}{2}e-\tfrac{17213877}{2} \\
A_1 &= e^{11}-55e^{10}+\tfrac{2805}{2}e^9-21780e^8+228195e^7-1690227e^6 \\
&\quad +\tfrac{18035325}{2}e^5-34613865e^4+\tfrac{187185735}{2}e^3-\tfrac{339306165}{2}e^2 \\
&\quad +\tfrac{741729879}{4}e-92538045 \\
A_0 &= -e^{12}+66e^{11}-2013e^{10}+37455e^9-\tfrac{945945}{2}e^8 \\
&\quad -28176687e^6+137179251e^5-\tfrac{976923585}{2}e^4+1240169535e^3 \\
&\quad -\tfrac{4261026627}{2}e^2+\tfrac{4446102717}{2}e-\tfrac{4261026627}{4} \\
A_{-1} &= e^{13}-78e^{12}+2808e^{11}-61776e^{10}+926640e^9-10007712e^8 \\
&\quad +80061696e^7-480370176e^6+2161665792e^5-7205552640e^4 \\
&\quad +17293326336e^3-28298170368e^2+28298170368e-13060694016 \\
&= (e-6)^{13}.
\end{aligned}
$$

This proves the theorem.

Finally we consider what happens if $\alpha = \beta = 0$, i.e. at an ordinary point of r. Using the previous theorem, we have that $(n/12)e$ is an integer.

Let Γ denote the set of poles of r. We have proven the following.

(i) In the tetrahedral case, $(\#)_4$ has a solution $z = l\delta u$, where

$$u^3 = P^3 \prod_{c\in\Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_e is an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k = 0, \pm 3, \pm 6$.

(ii) In the octahedral case, $(\#)_6$ has a solution $z = l\delta u$, where

$$u^2 = P^2 \prod_{c\in\Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_c in an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k=0, \pm 2, \pm 4, \pm 6$.

(iii) In either the tetrahedral case, the octahedral case or the icosahedral case, $(\#)_{12}$ has a solution $z = l\delta u$, where

$$u = P \prod_{c \in \Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_c is an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k=0, \pm 1, \ldots, \pm 6$.

Let $d = \deg P$. Then the Laurent series for z at ∞ has the form

$$z = \frac{n}{12}\left(\frac{12}{n} d + \sum_{c \in \Gamma} e_c\right) x^{-1} + \cdots$$

and the Laurent series for r at ∞ has the form

$$r = \gamma x^{-2} + \cdots.$$

(By the necessary conditions of section 2, the order of r at ∞ is at least 2.)

If we let

$$e_\infty = \frac{12}{n} d + \sum_{c \in \Gamma} e_c,$$

then, by a theorem analogous to Theorem 7, e_∞ satisfies the same conditions as does each e_c. Also

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

must be a non-negative integer. This is a restatement of step 2 of the algorithm.

We shall complete the proof of the algorithm by showing that the recursive relations of step 3 are identical with $(\#)_n$.

Let

$$\theta = \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x-c} \quad \text{and} \quad S = \prod_{c \in \Gamma} (x-c).$$

Then $z = l\delta u = P'/P + \theta$. Also set $P_i = S^{n-i} P a_i$. Using $(\#)_n$, we have

$$\begin{aligned}
P_n &= -P \\
P_{i-1} &= S^{n-i+1} P a_{i-1} \\
&= S^{n-i+1} P(-a_i' - z a_i - (n-i)(i+1) r a_{i+1}) \\
&= -S(S^{n-i} P a_i)' + (n-i) S^{n-i} S' P a_i + S^{n-i+1} P' a_i \\
&\quad - S(P' + P\theta)(S^{n-i} a_i) - (n-i)(i+1) S^2 r (S^{n-i-1} P a_{i+1}) \\
&= -S P_i' + ((n-i) - S\theta) P_i - (n-i)(i+1) S^2 r P_{i+1}.
\end{aligned}$$

This is precisely the equation of step 2 of the algorithm.

Finally, the equation

$$\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} \omega^i$$

may be rewritten as

$$0 = -S^n P \omega^n + \sum_{i=0}^{n-1} \frac{S^n P a_i}{(n-i)!} \omega^i = \sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!} \omega^i.$$

This completes the proof of the algorithm.

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1.2 An Algorithm for Solving Second Order Linear Homogeneous Differential Equations (2005 version)

By JERALD J. KOVACIC.

An algorithm for solving second order linear homogeneous differential equations

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Abstract

The Galois group tells us a lot about a linear homogeneous differential equation - specifically whether or not it has "closed-form" solutions. Using it, we have been able to develop an algorithm for finding "closed-form" solutions.

First we will compute the Galois group of some very simple equations. We then will solve a more complicated one, using the techniques of the algorithm. This example illustrates how the algorithm was discovered and the kinds of calculations used by it.

1 Introduction

Every student of calculus wants a formula to solve differential equations. Of course that is impossible, at least if we want "closed-form" solutions. The situation is analogous to that of polynomial equations. We'd like to have a formula for solutions in terms of radicals, and we know from the Galois theory that we can't.

In fact, the Galois group tells us a great deal about the kinds of solutions that an equation has. That is, after all, the point of the Galois theory. Therefore one wants to compute the Galois group of an equation.

That was my point of view when I started working on [6]. I wanted to find some criteria to determine the Galois group of a differential equation. I wanted them to be explicit, and easy - at that time pencil and paper was the accepted form of symbolic computation.

What came out, to my surprise, was an explicit algorithm to either find a "simple" solution or to prove that none exist.

There are no new ideas in the algorithm. It is simply brute-force calculation. And the hardest parts were worked out in the 19th century.

According to Felix Ulmer and Jacques-Arthur Weil [13], algorithms for finding rational solutions are in Liouville [7] (1833).

Ulmer and Weil also claim that algorithms for finding algebraic solutions are in Fuchs [2] (1878) and Pèpin [9] (1881). But Michael Singer [11] claims that these authors did not give a complete decision procedure, and that Baldassarri and Dwork [1] were the first to have done so.

An algorithm for finding Liouvillian solutions requires the Picard-Vessiot (Galois) theory, and so awaited the work of Kolchin in this century.

The original algorithm has some severe implementation difficulties. But recent work as eliminated most of the problems. Most of the recent work done to improve the algorithm is due to Abramov, Bronstein, Singer, Ulmer, and Weil.

There is a web version for 2^{nd} and 3^{rd} order equations:

`http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html`

A very nice, and complete, description of the history of the algorithm and further developments is Michael Singer's survey article in [5, Direct and inverse problems in differential Galois theory, p. 527–554]. The bibliography has 168 items!

For expositions of Picard-Vessiot (differential Galois) theory see: Kaplansky [3], Kolchin [4], Magid [8], and van der Put-Singer [14].

2 The DE

We consider a second order linear homogeneous ordinary differential equation

$$z'' + az' + bz = 0$$

where $a, b \in \mathcal{F} = \mathbb{C}(x)$ (and $x' = 1$). There is a change of variables that "normalizes" the equation. Let

$$y = e^{\frac{1}{2}\int a} z$$

then

$$y'' = (b - \tfrac{1}{4}a^2 - \tfrac{1}{2}a')y\,.$$

If we can find y we can find z, at least up to the problem of integrating $\frac{1}{2}\int a$. But that is a "easier" problem and we consider it "solved". In fact, there is an algorithm, called the *Risch algorithm*, for integration.

Throughout the remainder of this talk, we consider only

$$y'' = ry$$

where $r \in \mathcal{F} = \mathbb{C}(x)$. We call this *the DE.*

There are several types of solutions that we are particularly interested in.

Definition 2.1. Let η be a solution of the DE.

1. η is *algebraic* if it the solution of a polynomial equation over $\mathcal{F}$
2. η is *primitive* if $\eta' \in \mathcal{F}$, that is $\eta = \int f$ for some $f \in \mathcal{F}$,

3. η is *exponential* if $\eta'/\eta \in \mathcal{F}$, that is $\eta = e^{\int f}$.

The third should really be called "exponential of a primitive".

Definition 2.2. A solution η of the differential equation is said to be *Liouvillian* if there is a tower of differential fields

$$\mathcal{F} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset ... \subset \mathcal{G}_m = \mathcal{G},$$

with $\eta \in \mathcal{G}$ and for each $i = 1, ..., m$, $\mathcal{G}_i = \mathcal{G}_{i-1}(\eta_i)$ with η_i either algebraic, primitive, or exponential over $\mathcal{G}_{i-1}$.

A Liouvillian solution is built up by integration and exponentiation. So we get log's, the trig functions, but not things like the Bessel functions. These are "closed-form" solutions familiar to a first year calculus student.

This is a little more generous than "elementary" functions (logs and exp's only) in that we allow arbitrary indefinite integration.

Proposition 2.3. *If $y'' = ry$ has one Liouvillian solution, then every solution is Liouvillian.*

Use reduction of order: set y $= \eta z$ where η is a Liouvillian solution. One finds that $\eta z'' + 2\eta' z' = 0$. Therefore $z = \int \frac{1}{\eta^2}$ and $y = \eta \int \frac{1}{\eta^2}$ is another Liovillian solution.

3 Picard-Vessiot (differential Galois) theory

We learned in college that the DE has a "fundamental system of solutions" η,ζ. This means that η and ζ are functions that satisfy the equation and are linearly independent over constants ($\mathbb{C}$). In addition, every solution is a linear combination over $\mathbb{C}$ of η and ζ.

We also learned that linear independence is equivalent to the Wronskian

$$W = \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix}$$

having non-zero determinant

$$\det W \neq 0$$

Observe that

$$(\det W)' = (\eta\zeta' - \eta'\zeta)' = \eta'\zeta' + \eta\zeta'' - \eta''\zeta - \eta'\zeta' = \eta r\zeta - r\eta\zeta = 0\,.$$

Thus $\det W \in \mathbb{C}$.

Consider the differential field ($\mathcal{F} = \mathbb{C}(x)$)

$$\mathcal{G} = \mathcal{F}\langle \eta, \zeta \rangle = \mathcal{F}(\eta, \zeta, \eta', \zeta')\,.$$

Definition 3.1. The group of all differential automorphisms of $\mathcal{G}$ that leave $\mathcal{F}$ invariant (element-wise) is called the *differential Galois group* of the $\mathcal{G}$ over $\mathcal{F}$ and is denoted by

$$G(\mathcal{G}/\mathcal{F})\,.$$

If $\sigma \in G(\mathcal{G}/\mathcal{F})$, is a differential automorphism over $\mathcal{F}$ then $\sigma\eta$ and $\sigma\zeta$ are solutions of the DE. Therefore

$$\sigma \begin{pmatrix} \eta & \zeta \end{pmatrix} = \begin{pmatrix} \eta & \zeta \end{pmatrix} \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\,,$$

where

$$c(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

is an invertible (since σ is an automorphism) matrix of constants, i.e.

$$c(\sigma) \in \mathrm{GL}_{\mathbb{C}}(2) = \mathrm{GL}(2)\,.$$

But we can do better. Since $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathbb{C}$, we have

$$\sigma \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix} = \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix} \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\,.$$

i.e.

$$\sigma W = W c(\sigma)$$

Taking determinants we have

$$\sigma(\det W) = \det W \det c(\sigma)$$

But

$$\sigma(\det W) = \det W$$

since $\det W \in \mathbb{C}$. Therefore $\det c(\sigma) = 1$, i.e.

$$c(\sigma) \in \mathrm{SL}(2)\,.$$

Theorem 3.2. *The mapping*

$$c\colon G(\mathcal{G}/\mathcal{F}) \longrightarrow \mathrm{SL}(2) = \mathrm{SL}_{\mathbb{C}}(2)$$

is an injective homomorphism whose image is an algebraic subgroup of $\mathrm{SL}(2)$.

There is a "fundamental theorem of Galois theory", i.e. a bijection between algebraic subgroups of $G(\mathcal{G}/\mathcal{F})$ and intermediate differential fields $\mathcal{F} \subset \mathcal{E} \subset \mathcal{G}$.

Theorem 3.3. *If* η_1, ζ_1 *is another fundamental set of solutions of the DE then the image of* c_1 *in* $\mathrm{SL}(2)$ *is conjugate to the image of* c.

I.e. There is an element $X \in \mathrm{SL}(2)$ such that

$$c_1(G(\mathcal{G}/\mathcal{F})) = X\, c(G(\mathcal{G}/\mathcal{F}))\, X^{-1}\,.$$

4 Example 1

Consider $y'' = y$. Then e^x, e^{-x} is a fundamental system of solutions and

$$\mathcal{G} = \mathcal{F}\langle e^x, e^{-x}\rangle = \mathcal{F}(e^x)\,.$$

Because $(e^x)' = e^x$, we must have

$$(\sigma e^x)' = \sigma e^x$$

for every $\sigma \in G(\mathcal{G}/\mathcal{F})$. This implies that

$$\sigma e^x = d_\sigma e^x \qquad \text{and} \qquad \sigma e^{-x} = d_\sigma^{-1} e^{-x}$$

for some constant $d_\sigma \in \mathbb{C}$ Therefore

$$\sigma \begin{pmatrix} \eta & \zeta \end{pmatrix} = \begin{pmatrix} \eta & \zeta \end{pmatrix} \begin{pmatrix} d_\sigma & 0 \\ 0 & d_\sigma^{-1} \end{pmatrix}$$

i.e.

$$G(\mathcal{G}/\mathcal{F}) \approx \left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \middle| \, d \in \mathbb{C} \right\} \subset \mathrm{SL}(2)\,.$$

5 Example 2

Consider

$$y'' = -\frac{1}{4x^2} y\,.$$

One solution is $\eta = \sqrt{x}$. We get the other one by reduction of order, so a fundamental system of solutions is

$$\eta = \sqrt{x}, \qquad \zeta = \sqrt{x} \log x\,.$$

Now we can compute the Galois group. Let $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then

$$\sigma\eta = \pm\eta\,.$$

$\log x$ is a solution of $y' = 1/x$ and every solution of that equation is of the form $\log x + c$ for some constant c. Therefore

$$\sigma\zeta = \pm\sqrt{x}(\log x + c_\sigma) = c_\sigma \eta \pm \zeta$$

Thus

$$G(\mathcal{G}/\mathcal{F}) \approx \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| \, c \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix} \middle| \, c \in \mathbb{C} \right\}$$

6 The four cases

Theorem 6.1. *There are precisely four cases that can occur.*

Case 1. *G is triangulisable, i.e. G is conjugate to a subgroup of*

$$\left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \middle| c, d \in \mathbb{C}, c \neq 0 \right\}$$

Case 2. *G is conjugate to a subgroup of*

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\}$$

Case 3. *G is a finite group: the tetrahedral group, the octahedral group or the icosahedral group.*

Case 4. *$G = \mathrm{SL}(2)$.*

7 Case 1

Suppose that

$$G \subset \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \middle| c, d \in \mathbb{C}, c \neq 0 \right\}$$

and η, ζ is a fundamental system of solutions relative to G. For every $\sigma \in G(\mathcal{G}/\mathcal{F})$,

$$\sigma \begin{pmatrix} \eta & \zeta \end{pmatrix} = \begin{pmatrix} \eta & \zeta \end{pmatrix} \begin{pmatrix} c_\sigma & d_\sigma \\ 0 & c_\sigma^{-1} \end{pmatrix}$$

so

$$\sigma\eta = c_\sigma \eta\,.$$

We say that η is a *semi-invariant*.

Let $\theta = \eta'/\eta$ then

$$\sigma\theta = \frac{\sigma\eta'}{\sigma\eta} = \frac{c_\sigma\eta'}{c_\sigma\eta} = \theta \qquad \Longrightarrow \qquad \theta \in \mathcal{F} = \mathbb{C}(x)\,.$$

We say that θ is an *invariant*.

θ satisfies the *Riccati equation*

$$\theta' + \theta^2 = \frac{\eta\eta'' - \eta'\eta'}{\eta^2} + \left(\frac{\eta'}{\eta}\right)^2 = \frac{\eta\eta''}{\eta^2} = r\,.$$

i.e. the Riccati equation has a rational solution.

8 Case 2

Suppose that

$$G \subset \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\}$$

Then either

$$\begin{array}{lll} \sigma\eta = c_\sigma \eta & \text{and} \quad \sigma\zeta = c_\sigma^{-1}\zeta, & \text{or} \\ \sigma\eta = -c_\sigma^{-1}\zeta & \text{and} \quad \sigma\zeta = c_\sigma \eta & \end{array}$$

Therefore

$$\sigma(\eta\zeta) = \pm\eta\zeta$$

so $(\eta\zeta)^2$ is an invariant and is in $\mathbb{C}(x)$. We write

$$(\eta\zeta)^2 = a\prod_i (x - c_i)^{e_i}$$

for some $e_i \in \mathbb{Z}$, $a, c_i \in \mathbb{C}$.

Let

$$\phi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{1}{2}\frac{((\eta\zeta)^2)'}{(\eta\zeta)^2} = \frac{1}{2}a\sum_i \frac{e_i}{x - c_i}$$

One computes that

$$\phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r'\,.$$

This is the Riccati equation associated to the third order linear homogeneous differential equation satisfied by $\eta\zeta$. In this case it has a solution of a very special sort.

9 Case 3

For the tetrahedral group,

$$\xi = (\eta^4 + 8\eta\zeta^3)$$

then ξ^3 is an invariant (and therefore is in $\mathbb{C}(x)$) and

$$\phi = \frac{\xi'}{\xi} = \frac{1}{3}\frac{(\xi^3)'}{\xi^3}$$

satisfies a 4$^{\text{th}}$ order Riccati equation.

For the octahedral group,

$$\xi = \eta^5\zeta - \eta\zeta^5$$

and ξ^2 is an invariant.

$$\phi = \frac{\xi'}{\xi} = \frac{1}{2}\frac{(\xi^2)'}{\xi^2}$$

satisfies a 6$^{\text{th}}$ order Riccati equation.

For the icosahedral group

$$\phi = \eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}$$

is invariant and satisfies a 12$^{\text{th}}$ order Riccati equation.

10 Case 4

This case is the easiest. The DE does not have Liouvillian solution.

11 Another example

Consider

$$y'' = \left(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\right) y = ry$$

We are going to try for Case 1, so we look for a rational solution θ of the Ricatti equation

$$\theta' + \theta^2 = r$$

Since $\theta \in \mathbb{C}(x)$, it has a partial fraction decomposition

$$\begin{aligned}\theta = & \frac{a_n}{x^n} + \cdots + \frac{a_1}{x} \\ & + \frac{b_{1m_1}}{(x-c_1)^{m_1}} + \cdots + \frac{b_{11}}{x-c_1} \\ & + \quad \cdots \\ & + \frac{b_{rm_d}}{(x-c_d)^{m_d}} + \cdots + \frac{b_{d1}}{x-c_d} \\ & + f_0 + \cdots + f_e x^e\end{aligned}$$

I separate out the pole $x = 0$ because it is a pole of r. The others $(c_1, \ldots, c_d)$ are not. It turns out that the Riccati equation can have singularities that are not present in the original equation.

It's easier to use Laurent series. Look first at 0:

$$\theta = \frac{a}{x^n} + \cdots + \frac{b}{x} + \cdots .$$

From the Riccati equation we get

$$-\frac{na}{x^{n+1}} + \cdots + \frac{a^2}{x^{2n}} + \cdots = r = \frac{1}{x^4} + \cdots$$

It immediately follows that

$$n = 2 \qquad \text{and} \qquad a = \pm 1$$

Using the Riccati equation again, we get

$$-\frac{2a}{x^3} - \frac{b}{x^2} + \cdots + \frac{a^2}{x^4} + \frac{2ab}{x^3} + \cdots = \frac{1}{x^4} - \frac{5}{x^3} + \cdots$$

therefore

$$-2a + 2ab = -5$$

So we have the possibilities:

$$a = \quad 1 \qquad b = -\frac{3}{2} \qquad\qquad \theta = \quad \frac{1}{x^2} - \frac{3/2}{x} + \cdots$$

$$a = -1 \qquad b = \quad \frac{7}{2} \qquad\qquad \theta = -\frac{1}{x^2} + \frac{7/2}{x} + \cdots$$

Now let's try some other point:

$$\theta = \frac{a}{x - c^n} + \cdots$$

From the Riccati equation we get

$$-\frac{na}{(x-c)^{n+1}} + \cdots + \frac{a^2}{(x-c)^2} + \cdots = 0 + \cdots$$

so

$$n = 1 \qquad \text{and} \qquad a = 1$$

So far

$$\theta = \quad \frac{1}{x^2} - \frac{3/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_i} + f_0 + \cdots + f_e x^e$$

or

$$\theta = -\frac{1}{x^2} + \frac{7/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_i} + f_0 + \cdots + f_e x^e$$

Unfortunately we do do not know, yet, what d is or what the c_i are (not to mention the polynomial part).

Next we look at ∞. Write

$$\theta = ax^n + \cdots + bx + cx^{-1} + \cdots$$

Then

$$nax^{n-1} + \cdots + a^2x^{2n} + \cdots = r = x^2 + \cdots$$

Therefore

$$n = 1 \qquad \text{and} \qquad a = \pm 1$$

So

$$\theta = ax + b + \frac{c}{x} + \cdot$$

and, from the Riccati equation,

$$a + \cdots + a^2x^2 + 2abx + 2ac + b^2 + \cdots = x^2 - 2x + 3 + \cdots$$

Comparing coefficients we get

$$a = 1 \qquad b = -1 \qquad c = \frac{1}{2} \qquad \theta = x - 1 + \frac{1/2}{x} + \cdots$$

$$a = -1 \qquad b = 1 \qquad c = -\frac{3}{2} \qquad \theta = -x + 1 - \frac{3/2}{x} + \cdots$$

From our analysis of the finite poles we had two possibilities for θ. The first was

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_d} + f_0 + \cdots + f_e x^e$$

$$= f_e x^e + \cdots + f_0 + \frac{d - 3/2}{x} + \frac{?}{x^2} + \cdots$$

Comparing with the first case above we have

$$e = 1, \quad f_e = 1, \quad f_0 = -1, \quad d - 3/2 = 1/2, \quad d = 2$$

Comparing with the second case we have

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d - 3/2 = -3/2, \quad d = 0$$

But we had a second possibility for theta:

$$\theta = -\frac{1}{x^2} + \frac{7/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_d} + f_0 + \cdots + f_e x^e$$

$$= f_e x^e + \cdots f_0 + \frac{d + 7/2}{x} + \frac{?}{x^2} + \cdots$$

Comparing with the equations we got at ∞ we have

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d + 7/2 = -3/2, \quad d = -3$$

which is impossible.

The last case is

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d + 7/2 = -3/2, \quad d = -5$$

which is also impossible.

Let's try for $d = 0$. In that case

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + 1 - x$$

We try this in the Riccati equation and get

$$\theta' + \theta^2 = x^2 - 2x + 3 - \frac{5}{x} + \frac{23/4}{x^2} - \frac{5}{x^3} + \frac{1}{x^4} \neq r$$

This is *not* the right answer! So θ does not give a solution.

On to the case

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + \frac{1}{x - c_1} + \frac{1}{x - c_2} - 1 + x$$

We do not know what c_1 and c_2 are.

Let

$$\omega = \frac{1}{x^2} - \frac{3/2}{x} - 1 + x$$

and

$$P = (x - c_1)(x - c_2)$$

Then

$$\eta = e^{\int \theta} = P e^{\int \omega}$$

is supposed to be a solution of the original DE ($y'' = ry$). This gives

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0$$

or

$$P'' + \left(\frac{2}{x^2} - \frac{3}{x} - 2 + 2x\right) P' + (\frac{4}{x} - 4)P = 0$$

Substituting $P = x^2 + ax + b$ one easily finds that

$$P = x^2 - 1 = (x-1)(x+1)$$

So $c_1 = 1$, $c_2 = -1$.

The solution to the original DE

$$y'' = ry$$

is

$$\eta = Pe^{\int \omega} = (x^2-1)e^{\int \frac{1}{x^2} - \frac{3}{2x} - 1 + x} = x^{-3/2}(x^2-1)e^{-1/x + -x + x^2/2}$$

12 Higher order

First of all, there really are only two cases: either the equation has a Liouvillian solution or it doesn't. And if it does, the Lie-Kolchin theorem tells us that the DE will have a Liouvillian solution if and only if the connected component of the identity of G, denoted by G^o, is triangulizable:

$$G^o = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0) \cdots & c_{nn} & \end{pmatrix}$$

For every $\sigma \in G^o$,

$$\sigma\eta = c_{11}\eta$$

so

$$\sigma\frac{\eta'}{\eta} = \frac{\eta'}{\eta}$$

Because G^o has finite index in G,

$$\frac{\eta'}{\eta}$$

is algebraic over $\mathbb{C}(x)$. The degree d is the index of G^o in G. Then the symmetric functions in η, ζ of degree d are invariants. These are solutions of a Riccati equation of order at most d.

Unfortunately, the index of G^o in G may be arbitrarily large. However we do have:

Theorem 12.1. *If $G \subset \mathrm{SL}(n)$ has a non-trivial triangularizable subgroup (not necessarily G^o, but always G^0), then the index is no greater than a computable number $I(n)$.*

$I(n)$ tends to be rather large, for example

$$I(2) = 384,064\,.$$

The following was proven by Michael Singer [10] and [11].

Theorem 12.2. *Given a linear homogeneous differential equation of order n*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0\,,$$

there is an algorithm that either finds a Liouvillian solution or proves that it has none.

13 The Galois group

The algorithm actually tells us something about the Galois group of the differential equation. In case 1, for example, the group is reducible (triangularizable). We can break the cases into various subcases and refine the algorithm to determine which subcase the equation belongs to.

For example, in case 1 could have $d = 0$. In this case the group is diagonal

$$G = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},$$

and $\eta\zeta$ is an invariant. If c is an n-th root of unity, then η^n is an invariant.

Singer and Ulmer [12] actually calculate the Galois group.

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1.3 An Implementation of Kovacic's Algorithm for Solving Second Order Linear Homogeneous Differential Equations

By B. David Saunders

An Implementation of Kovacic's Algorithm for Solving Second Order Linear Homogeneous Differential Equations

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1. Introduction.

Kovacic [3] has given an algorithm for the closed form solution of differential equations of the form $ay'' + by' + cy = 0$, where a, b, and c are rational functions with complex coefficients of the independent variable x. The algorithm provides a Liouvillian solution (i.e. one that can be expressed in terms of integrals, exponentials and algebraic functions) or reports that no such solution exists.

In this note a version of Kovacic's algorithm is described. This version has been implemented in MACSYMA and tested successfully on examples in Boyce and DiPrima [1], Kamke [2], and Kovacic [3]. Modifications to the algorithm have been made to minimize the amount of code needed and to avoid the complete factorization of a polynomial called for. In Section 2 these issues are discussed and in Section 3 the author's current version of the algorithm is described.

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2. Issues.

Some notation: For any function u of x we write u' for du/ux, and C denotes the complex numbers, C[x] the polynomials over C, and C(x) the rational functions over C.

By a standard change of variables,

(1) $au'' + bu' + cu = 0, \quad a \neq 0,$
$a, b, c \in C(x)$

is equivalent to

(2) $y'' = ry, \quad r = (2b'a-2ba'+b^2-4ac)/4a^2$

in that (1) has a Liouvillian solution if and only if (2) has a Liouvillian solution ($u = \exp(\int -b/2a\, dx)\cdot y$).

Using the Galois theory of differential fields, Kovacic shows that equation (2) has a Liouvillian solution if and only if it has a solution of the form $y = e^{\int\omega\, dx}$ where ω is algebraic of degree 1, 2, 4, 6, or 12 over C(x). In turn, equation (2) has a solution of the form $y = e^{\int\omega\, dx}$ if and only if the Ricatti equation

(3) $\omega' + \omega^2 = r$

has a solution ω which is algebraic of degree n = 1, 2, 4, 6, or 12 over C(x). The algorithm then consists in finding such ω. This is done largely by analysis of the singularities of r. If no such ω exists it can be asserted that equation (1) has no Liouvillian solution whatsoever.

The goal of the algorithm then is to determine the coefficients of the minimal polynomial of ω, $m_\omega(z) = z^n + \sum_{i=0}^{n-1} m_i z^i$,

$m_i \in C(x)$. Kovacic determines for each possible degree n in turn candidate polynomials m_ω and conditions on them in such a way that if the conditions are met then ω is a solution to the Ricatti equation and if the conditions are not met for any of the candidate m_ω then no ω of degree n is a solution.

Computationally, three features of the algorithm deserve special attention.

1. Candidates and conditions must be generated and the condidates tested against the conditions for each possible degree separately, (though degrees 4, 6, 12 can be treated simultaneously). In the version described in Section 3, we have taken advantage of similarities in the candidate generation process to minimize the amount of code.
2. It is assumed that the poles of r are known (analysis of singularities dominates the process of determining the candidate solutions). This implies a complete splitting of the denominator of r, which can be difficult in itself and requires also that computations be carried out in an algebraic extension field (of the constant field generated by the coefficients of r) over which the denominator splits. But the solution - if found - generally has coefficients in a simpler extension involving only certain square roots. The version of the algorithm described here avoids explicit realization of the individual odd order poles but does require determination of the even order poles. There is reason to think that one could also avoid splitting the even order square free factors of the denominator, but this has not been achieved as yet. It should be remarked that this matter only begins to cause computational concern when the denominator of r has a factor t^k, k even, with t irreducible of degree 2 or greater, a rare occurrence.
3. The number of candidates which must be checked is in the worst case $(n + 1)^{k+1}$ where k is the number of even order poles of r. Thus the algorithm requires exponential time, though in practice this is unlikely to be of much concern in that k is generally small and aspects of the production of candidates dominate the computing time. For example $k = 1$ seems to be the upper bound for textbook problems (cf. [1], [2]).

3. The Algorithm.

Kovacic describes his algorithm by treating three cases. Case 1 (n=1): search for a rational function ω satisfying the Ricatti equation $\omega' + \omega^2 = r$, case 2 (n=2): search for a solution ω quadratic over $C(x)$, and case 3 (n=4, 6, or 12): search for a higher degree function. In each case one first computes a number d and rational function θ, then a polynomial P of degree d whose coefficients are determined by d linear equations expressed in terms of θ. Each case has 3 steps, (1) determination of parts for d and θ (by analysis of the poles of r), (2) assembly of trial d's and θ's from the parts, (3) determination from each trial d and θ a system of linear equations for the coefficients of P. If any of these systems is solvable the desired function ω may be expressed in terms of the corresponding θ and P. Basically, the coefficient of z^{n-1} in the minimal polynomial $m_\omega(z)$ of ω is $-P'/P - \theta$ and the other coefficients are expressible in terms of P, θ, and the input r via a recursive formula. For example if n=1 we have $\omega = P'/P + \theta$ (then $e^{\int \omega\, dx} = Pe^{\int \theta\, dx}$). If n=2, ω satisfies $\omega^2 - \phi\omega + \phi'/2 + \phi^2/2 - r = 0$, where $\phi = P'/P + \theta$.

The algorithm outlined below unifies step 1 and step 2 across the three cases. Thus only one description (piece of code) for these steps (with n as a parameter) is needed. Step 3 remains in 3 parts, one for each case, and is as described by Kovacic. Hence step 3 is not developed in detail below. Step 3 can be given a unified treatment also, and it may be best to do this. However, such a unification has not been implemented as yet.

Kovacic's algorithm (modified)

[Given $r \in C(x)$, solve $\omega' + \omega^2 = r$ for ω.]

Step 0: [Preliminaries.]

(a) Let $r = s/t$, with $\gcd(s,t) = 1$, $s, t \in C[x]$. Compute the square free factorization of t:

$$t = t_1 t_2^2 t_3^3 \cdots t_m^m.$$

Let $o(\infty) = \deg(t) - \deg(s)$ [order of ∞ as a zero of r].

(b) [Necessary conditions.]
Form a set L of possible degrees over C(x) of a solution ω. [L is a subset of {1, 2, 4, 6, 12}.]

$1 \in L$ only if
$t_i = 1$ for all odd $i \geq 3$
and $o(\infty)$ is even or $o(\infty) > 2$.

$2 \in L$ only if
$t_2 \neq 1$ or $t_i \neq 1$ for some odd $i \geq 3$.

$4, 6, 12 \in L$ only if
$t_i = 1$ for all $i > 2$
(i.e., $m \leq 2$) and $o(\infty) \geq 2$.

Step 1: [Form parts for d and θ.]

(a) [Fixed parts.]
$d_{fix} = \frac{1}{4}(\min(o(\infty), 2) - \deg(t) - 3\deg(t_1))$.
$\theta_{fix} = \frac{1}{4}(t'/t + 3t_1'/t_1)$.

(b) [Poles of order 2.]
Find the roots $c_1, \ldots, c_{k_2}$ of t_2.
For i = 1 to k_2 let $d_i = \sqrt{1 + 4b}$, $\theta_i = d_i/(x - c_i)$.

(c) [High order poles.]
If $1 \in L$ then find the roots $c_{k_2+1}, \ldots, c_k$ of $t_4, t_6, \ldots, t_m$.
For $i = k_2 + 1$ to k let $d_i = b/a$, $\theta_i = 2[\sqrt{r}]_{c_i} + d_i/(x\ c_i)$.

[Here a, b, $[\sqrt{r}]_c$ are as described by Kovacic [3, pg. 21, 34, 45]. They are computed from the Laurent series expansion of r at c. However the parts d_i and θ_i are different. This reflects the computation of d_{fix} and θ_{fix} above and the different scheme for assembling parts in step 2. Also, parts d_0 and θ_0 corresponding to the 'zero' at infinity are computed as in (b) or (c) if $o(\infty) = 2$ or $o(\infty) < 2$ respectively (but replace $d_0/(x - c_0)$ by 0).]

Step 2: [Form trial d's, θ's.]

For n in L (taken in increasing order) do:
  If n = 1 then m = k else $m = k_2$.
  For all sequences $s = (s_0, s_1, \ldots, s_m)$ where each $s_i \in \{-\frac{1}{2}n, -\frac{1}{2}n+1, \ldots, \frac{1}{2}n\}$, [Start with $s = (-\frac{1}{2}n, \ldots, -\frac{1}{2}n)$, view s as a m-digit number base n + 1 which is incremented by 1 at each pass until $s = (\frac{1}{2}n, \ldots, \frac{1}{2}n)$] do:
    Let $d_s = n \cdot d_{fix} - \sum_{i=0}^{m} s_i d_i$.
    If d_s is an integer ≥ 0 then
      let $\theta_s = n \cdot \theta_{fix} + \sum_{i=0}^{m} s_i \theta_i$ and
        apply step $3_n(d_s, \theta_s)$.
  If step 3_n is successful then stop - solution is found.
  [It is expressed in terms of θ_s and a polynomial P of degree d_s found in step 3_n by solving a system of linear equations.]
  End 'for all sequences s do'.
End 'for n in L do'.

[If this point is reached, all possibilities for a solution have been tried, so...]

Stop - no solution exists.

The d_s and θ_s computed in step 2 are Kovacic's d's and θ's, except that in case 1 he uses 'ω' rather than 'θ' (see [3, pg. 23, 35, 46]). Thus this version may be verified by calculations to show that the same expressions are indeed obtained, and then reference to Kovacic's proof of the algorithm.

It is a tribute to the expressive power of MACSYMA that the code for this algorithm is about the same length as Kovacic's (mathematician's) description and contains little in the way of extra 'programmer's' details. One exception to this is the vector s used to keep track of the trial d's and θ's.

We conclude with some sample computations using the above algorithm.

Example 1.

Differential Equation: $y'' = (x^2 + 3)y$

Step 0: $r = x^2 + 3$, $L = \{1\}$,
$t = 1$ [no poles except at ∞], $o(\infty) = -2$.

Step 1:
$d_{fix} = -\frac{1}{2}$, $\theta_{fix} = 0$,
$d_0 = -3$, $\theta_0 = 2x$ [for the pole at ∞].

Step 2:
$s = (-\frac{1}{2})$, $d_s = -2$ [reject],
$s = (\frac{1}{2})$, $d_s = 1$, $\theta_s = x$.

Step 3: Successful with $P = x$.
$\omega = P'/P + \theta = 1/x + x$ solves the Ricatti equation.
$y = e^{\int\omega} = xe^{\frac{1}{2}x^2}$ solves the D.E.

Example 2.

Differential Equation : $y'' = (\frac{1}{x} - \frac{3}{16x^2})y$

Step 0: $r = (x - 3/16)/x^2$,
$t = x^2$, $t_2 = x$ [one pole at 0 of order 2],
$o(\infty) = 1$, $L = \{2\}$.

Step 1:
$d_{fix} = -\frac{1}{4}$, $\theta_{fix} = 1/2x$,
$d_1 = \frac{1}{4}$, $\theta_1 = 1/2x$ [for the pole $c_1=0$].

Step 2:
$s = (-1)$, $d_s = 0$, $\theta_s = 1/2x$.
[the values $s = (0)$ and $s = (1)$, had they been tried first, would have been rejected since in those cases d_s is not a non-negative integer.]

Step 3: Successful with $P = 1$.
$P'/P + \theta = 1/2x$.
ω satisfies $\omega^2 - (1/2x)\omega + (1 - 16x)/16x^2 = 0$.
$\omega = 1/4x + x^{-\frac{1}{2}}$ solves the Ricatti equation.
$y - e^{\int\omega\, dx} = x^{\frac{1}{4}} e^{2x^{\frac{1}{2}}}$ solves the D.E.

Remark: No d_0, θ_0 are computed when $1 \notin L$.

Example 3.

Differential Equation: $x^2 y'' = 2y$

Step 0: $r = 2/x^2$
$t = x^2$, $t_2 = x$ [one pole at 0 of order 2]
$o(\infty) = 2$, $L = \{1, 2, 4, 6, 12\}$.

Step 1:
$d_{fix} = 0$, $\theta_{fix} = 1/2x$,
$d_0 = 3$, $\theta_0 = 0$ [for the pole at ∞],
$d_1 = 3$, $\theta_1 = 3/x$ [pole at $c_1 = 0$].

Step 2:
$s = (-\frac{1}{2}, -\frac{1}{2})$, $d_s = 3$, $\theta_s = -1/x$.

Step 3: Successful with $P = x^3$.
$\omega = P'/P + \theta = 2/x$ solves the Ricatti equation.
$y = e^{\int\omega} = x^2$ solves the D.E.

References

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1.4 A DISCUSSION AND IMPLEMENTATION OF KOVACICS ALGORITHM FOR ORDINARY DIFFERENTIAL EQUATIONS

By Carolyn J. Smith.

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Preface

This essay was written in partial fulfillment of the requirements for the degree of Master of Mathematics in Computer Science at the University of Waterloo.

Thanks go to all the members of the Maple group for their assistance during the implementation of the algorithm.

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CHAPTER 1

Introduction

1.1. Overview

The following essay is a discussion of an algorithm by J. Kovacic [3] to solve certain second order ordinary differential equations, and of an implementation of this algorithm in Maple.

In the remainder of this chapter, the exact form of the problem to be solved is given. In chapter 2, some of the theory that is intrinsic to the algorithm is developed and the general form of the algorithm is specified.

In chapter 3, Kovacic's algorithm is presented and proved correct. In chapter 4, a variant of the algorithm developed by B. D. Saunders for implementation in Macsyma [5] is discussed and some corrections made. The corrected algorithm is also verified to correspond to Kovacic's algorithm.

In chapter 5, the implementation of the algorithm is discussed and some problems with Maple as the implementation system are mentioned. Some recommendations for future work on Maple and on the implementation of the algorithm are made. Source code is presented in appendix A. Some examples of the use of the algorithm and of the implementation are presented in appendix B.

1.2. Purpose

The equation to be solved is assumed of the following form:

$$a \cdot z'' + b \cdot z' + c \cdot z = 0 \tag{1.2a}$$

where a, b and c are rational functions of a (complex) variable x with coefficients in the field of complex numbers $\mathbf{C}$, $a \neq 0$ and z is a (complex) function of x.

The goal of the algorithm is to find one Liouvillian solution of the equation. A solution is Liouvillian if it is an element of a Liouvillian field, where a Liouvillian field is defined as follows:

Definition: Let F be a differential field of functions of a complex variable x, that contains $\mathbf{C}(x)$, i.e. F is a field of characteristic zero with a differentiation operator $'$ with the following two properties: $(a+b)' = a'+b'$ and $(ab)' = ab'+a'b$, for all a and b in F. (The characteristic of a field is the least integer $q > 0$ for which $qa = 0$ for all a in the field. If no such q exists, then the characteristic of the field is zero.) F is Liouvillian if there exists a tower of differential fields

$$\mathbf{C}(x) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F$$

such that for each $i = 1,\ldots,n$

$$F_i = F_{i-1}(\alpha) \quad \text{where } \frac{\alpha'}{\alpha} \in F_{i-1}$$

(i.e. F_i is generated by an exponential of an integral over F_{i-1})

or

$$F_i = F_{i-1}(\alpha) \quad \text{where } \alpha' \in F_{i-1}$$

(i.e. F_i is generated by an integral over F_{i-1})

or

F_i is finite algebraic over F_{i-1}.

(i.e. $F_i = F_{i-1}(\alpha)$ and α satisfies a polynomial of the form $a_0+a_1\alpha+ \cdots +a_n\alpha^n=0$ where the a_i are in F_{i-1} and are not all zero)

The algorithm need only find one Liouvillian solution of the equation because a second solution may be found by the method of reduction of order as follows. The second solution is assumed of the form $z_2 = v \cdot z_1$ where z_1 is the known first solution and v is some function to be determined. Using the differential equation (1.2a), one obtains a first order equation for v which can be solved to give the second solution as

$$z_2 = z_1 \cdot \int \frac{e^{-\int \frac{b}{a} dx}}{z_1^2} dx$$

If the first solution, z_1, is Liouvillian it is clear that the second, z_2, is also Liouvillian and hence all solutions of (1.2a) are Liouvillian (since all solutions are a linear combination of z_1 and z_2).

In order to reduce the original differential equation to a simpler form, the following transformation is made:

$$y(x) = z(x) \cdot e^{-\int \frac{b}{2a} dx}$$

Then (1.2a) becomes

$$y'' - \left(\frac{b^2 + 2ab' - 2ba' - 4ac}{4a^2} \right) y = 0$$

or

$$y'' = ry \qquad (1.2b)$$

where

$$r = \frac{b^2 + 2ab' - 2ba' - 4ac}{4a^2}$$

Clearly, if the solutions to (1.2a) are Liouvillian, then so are the solutions to (1.2b).

In what follows, the equation to be solved will be assumed of the form (1.2b). The implementation of the algorithm accepts equations of the form (1.2a), makes the transformation, solves the transformed equation, then transforms the solutions using the inverse transformation

$$z(x) = y(x) \cdot e^{\int \frac{b}{2a} dx}$$

CHAPTER 2

Some Preliminaries

2.1. The Four Cases

The following theorem by Kovacic [3] determines the form that the algorithm will take.

Theorem: For the differential equation $y'' = ry$, $r \in \mathbf{C}(x)$, there are four cases that can occur.

(1) The d.e. has a solution of the form $\eta = e^{\int \omega}$ where $\omega \in \mathbf{C}(x)$.

(2) The d.e. has a solution of the form $\eta = e^{\int \omega}$ where ω is algebraic of degree 2 over $\mathbf{C}(x)$ and case (1) does not hold.

(3) All solutions of the d.e. are algebraic over $\mathbf{C}(x)$ and cases (1) and (2) do not hold. The solutions are of the form $\eta = e^{\int \omega}$ where ω is algebraic of degree 4, 6 or 12 over $\mathbf{C}(x)$.

(4) The d.e. has no Liouvillian solutions.

The remainder of this section will cover the proof of this theorem and the background necessary to understand it.

Let η, ζ be any two independent solutions of the d.e. $y'' = ry$. Define $\overline{G}$ to be the differential extension field of $\mathbf{C}(x)$ generated by η and ζ, i.e. $\overline{G} = \mathbf{C}(x)(\eta, \eta', \zeta, \zeta')$. (Higher derivatives of η and ζ are not necessary since $\eta'' = r\eta \in \overline{G}$, $\eta''' = r'\eta + r\eta' \in \overline{G}$, etc.)

Now, the *Galois group of the differential equation* is the Galois group of $\overline{G}$ over $\mathbf{C}(x)$, and is denoted $G = G(\overline{G}/\mathbf{C}(x))$. G is the group of all differential automorphisms of $\overline{G}$ leaving $\mathbf{C}(x)$ elementwise fixed.

Recall that an automorphism of a group H is an isomorphism from H to itself. A differential automorphism is an automorphism that commutes with $'$ (the differentiation operator).

This means that G is the group of all automorphisms $\sigma : \overline{G} \to \overline{G}$ such that $\sigma(a') = (\sigma a)'$ for all $a \in \overline{G}$, and $\sigma f = f$ for all $f \in \mathbf{C}(x)$.

The Galois group, G, is isomorphic to a subgroup of $GL(2,\mathbf{C})$, the group of all 2×2 invertible matrices with complex coefficients, i.e. each $\sigma \in G$ corresponds to a matrix $\begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix}$ where a_σ, b_σ, c_σ, $d_\sigma \in \mathbf{C}$. This correspondence occurs as follows. Because η and ζ are solutions of $y'' = ry$, and because any $\sigma \in G$ is a differential automorphism, then

$$(\sigma(\eta))'' = \sigma(\eta'') = \sigma(r\eta) = \sigma r \cdot \sigma\eta = r\sigma\eta$$

and hence $\sigma\eta$ must be a solution of the d.e. too. Further, $\sigma\eta$ must be a linear combination of η and ζ, since every solution of the d.e. is a linear combination of any two independent solutions of the d.e. We may then write

$$\sigma\eta = a_\sigma \cdot \eta + b_\sigma \cdot \zeta \quad a_\sigma, b_\sigma \in \mathbf{C}$$

Following the same arguments

$$\sigma\zeta = c_\sigma \cdot \eta + d_\sigma \cdot \zeta \quad c_\sigma, d_\sigma \in \mathbf{C}$$

Combining these two results we have

$$\begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} a_\sigma\eta + b_\sigma\zeta \\ c_\sigma\eta + d_\sigma\zeta \end{bmatrix}$$

and σ clearly corresponds to the matrix $\begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix}$.

Using the Wronskian of η and ζ, we can show that G is isomorphic to a subgroup of $SL(2,\mathbf{C})$ ($\subset GL(2,\mathbf{C})$), the group of 2×2 invertible matrices with determinant 1. The Wronskian, W, of η and ζ is by definition $W = \eta\zeta' - \eta'\zeta$. Take the derivative of W and get

$$W' = \eta'\zeta' + \eta\zeta'' - \eta'\zeta' - \eta''\zeta = \eta\zeta'' - \eta''\zeta = \eta r\zeta - r\eta\zeta = 0$$

Hence W must be a constant and so for any $\sigma \in G$, $\sigma W = W$ (because $W \in \mathbf{C}(x)$ and σ, by definition, leaves $\mathbf{C}(x)$ fixed). This implies

$$\sigma W = \sigma(\eta\zeta' - \eta'\zeta) = \sigma\eta(\sigma\zeta)' - (\sigma\eta)'\sigma\zeta$$

$$= (a_\sigma\eta + b_\sigma\zeta)(c_\sigma\eta' + d_\sigma\zeta') - (a_\sigma\eta' + b_\sigma\zeta')(c_\sigma\eta + d_\sigma\zeta)$$

$$= (a_\sigma d_\sigma - b_\sigma c_\sigma)(\eta\zeta' - \eta'\zeta) = (a_\sigma d_\sigma - b_\sigma c_\sigma)W$$

and thus $a_\sigma d_\sigma - b_\sigma c_\sigma = \det \sigma = 1$.

We will now state two facts without proof. The Galois group of the d.e., G, (relative to η and ζ), is (isomorphic to) an algebraic subgroup of $SL(2,\mathbf{C})$. (This is a fundamental fact from Picard-Vessiot theory. A proof may be found in [3].) Recall that any subgroup K of $GL(2,\mathbf{C})$ is said to be an *algebraic group* if there exist a finite number of polynomials $P_1, \ldots, P_n$, where each $P_i \in \mathbf{C}[X_1,X_2,X_3,X_4]$, such that a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an element of K iff $P_1(a,b,c,d) = \cdots = P_n(a,b,c,d) = 0$.

Further, for any algebraic subgroup of $SL(2,\mathbf{C})$ the following lemma holds. (See [2] for the proof of this lemma.)

Lemma: If G is an algebraic subgroup of $SL(2,\mathbf{C})$ then one of the following four cases can occur.

(1) G is triangulisable, i.e. there exists an $x \in G$ such that for every $g \in G$, xgx^{-1} is a triangular matrix. We can assume that xgx^{-1} is a lower triangular matrix and hence of the form $\begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix}$ with a and $b \in \mathbf{C}$. (Recall that G is a subgroup of $SL(2,\mathbf{C})$ and hence the determinant of xgx^{-1} must be 1.)

(2) G is conjugate to a subgroup of D^+ where

$$D^+ = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mid c \in \mathbf{C}, c \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \mid c \in \mathbf{C}, c \neq 0 \right\}$$

and case (1) does not hold, i.e. there exists an $x \in G$ such that for every $g \in G$, xgx^{-1} is either a diagonal matrix or a skew-diagonal matrix but there is no $x \in G$ such that all xgx^{-1} are triangular (this includes strictly diagonal too). (Note that the determinant of each xgx^{-1} is 1 since the determinant of each matrix in D^+ is 1.)

(3) G is finite and cases (1) and (2) do not hold.

(4) $G = SL(2,\mathbf{C})$, i.e. G is the infinite group of all 2×2 matrices with determinant 1.

What do we have now? We know that G, the Galois group of the d.e., is (isomorphic to) an algebraic subgroup of $SL(2,\mathbf{C})$. We also know that any algebraic subgroup of $SL(2,\mathbf{C})$ satisfies the above lemma. We can now apply the lemma to the

Galois group of the d.e. and see what relevance it has to the solutions of the d.e.

In case (1), G is triangulisable. Assume $x \in G$ has been found and every matrix conjugated to a lower triangular matrix. (This is equivalent to changing the basis of the vector space or picking two different independent solutions $\bar{\eta}$ and $\bar{\zeta}$.) Then every $\sigma \in G$ is of the form $\begin{bmatrix} a_\sigma & 0 \\ c_\sigma & a_\sigma^{-1} \end{bmatrix}$ $a_\sigma, c_\sigma \in \mathbf{C}$ and maps η to $\sigma\eta = a_\sigma\eta$.

Now if we set $\omega = \dfrac{\eta'}{\eta}$ (or equivalently, $\eta = e^{\int \omega}$), then

$$\sigma\omega = \sigma\left(\frac{\eta'}{\eta}\right) = \frac{(\sigma\eta)'}{\sigma\eta} = \frac{a_\sigma\eta'}{a_\sigma\eta} = \frac{\eta'}{\eta} = \omega$$

and hence $\omega \in \mathbf{C}(x)$. This is case (1) of the original theorem; the d.e. has a solution of the form $\eta = e^{\int \omega}$ where $\omega \in \mathbf{C}(x)$.

In case (2), G is conjugate to a subgroup of D^+. Assume we have conjugated G and every $\sigma \in G$ is of the form $\begin{bmatrix} a_\sigma & 0 \\ 0 & a_\sigma^{-1} \end{bmatrix}$ or $\begin{bmatrix} 0 & b_\sigma \\ -b_\sigma^{-1} & 0 \end{bmatrix}$ so that either $\sigma\eta = a_\sigma\eta$, $\sigma\zeta = a_\sigma^{-1}\zeta$, or $\sigma\eta = b_\sigma\zeta$, $\sigma\zeta = -b_\sigma^{-1}\eta$. (Note that in either case, $\sigma(\eta^2\zeta^2) = \eta^2\zeta^2$, so that $\eta^2\zeta^2 \in \mathbf{C}(x)$.) If we set $\omega = \dfrac{\eta'}{\eta}$ $(\eta = e^{\int \omega})$ and $\phi = \dfrac{\zeta'}{\zeta}$, then either $\sigma\omega = \omega$ and $\sigma\phi = \phi$, or $\sigma\omega = \phi$ and $\sigma\phi = \omega$. Minimally both cases are handled by $\sigma^2\omega = \omega$ or $\sigma^2\omega - \omega = 0$ so ω satisfies a polynomial of degree 2 over $\mathbf{C}(x)$, hence is algebraic of degree 2 over $\mathbf{C}(x)$. This is case (2) of the original theorem; the d.e. has a solution of the form $\eta = e^{\int \omega}$ where ω is algebraic of degree 2 over $\mathbf{C}(x)$.

In case (3), G is a finite group, i.e. there are only a finite number of automorphisms, $\sigma_1, \sigma_2, \ldots, \sigma_n$. Look at any elementary symmetric function of the functions $\sigma_1\eta$, $\sigma_2\eta$, $\cdots$ $\sigma_n\eta$, e.g. $\sum\sigma_i\eta = \sigma_1\eta + \sigma_2\eta + \cdots + \sigma_n\eta$. For any $\sigma_j \in G$, $\sigma_j\left(\sum\sigma_i\eta\right) = \sum\sigma_i\eta$ because $\sigma_j \cdot \sigma_i \in G$ for all σ_i (because G is a group and hence is closed). Hence, $\sum\sigma_i\eta = f(x) \in \mathbf{C}(x)$ and η satisfies $\sigma_1\eta + \cdots + \sigma_n\eta - f(x) = 0$ and is algebraic over $\mathbf{C}(x)$. A similar argument holds for ζ so that since η and ζ are algebraic over $\mathbf{C}(x)$, all solutions of the d.e. are algebraic over $\mathbf{C}(x)$.

To be more specific about the nature of G in case (3), we will state the following theorem without proof. (See Kovacic [3] for details.)

Theorem: If K is a finite subgroup of $SL(2,\mathbf{C})$ then either

(1) K is conjugate to a subgroup of D^+.

(2) The order of K is 24.

(3) The order of K is 48.

(4) The order of K is 120.

Clearly, the first case of this theorem is a subcase of case (2). This means that for case (3), G has order 24, 48 or 120 only, and hence the order of η over $\mathbf{C}(x)$ is 24, 48 or 120 respectively.

In each of these cases, the following functions of η and ζ are known to be in $\mathbf{C}(x)$; if G has order 24 then $(\eta^4 + 8\eta\zeta^3)^2 \in \mathbf{C}(x)$, if G has order 48 then $(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbf{C}(x)$, and if G has order 120 then $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11} \in C(x)$. (See [3] for proofs of these statements.)

In case (4), $G = SL(2,\mathbf{C})$. We want to show that in this case, the d.e. has no Liouvillian solution. We will assume the contrary and force a contradiction.

Assume the d.e. has one Liouvillian solution. Then a second solution, obtained by the method of reduction of order, must also be Liouvillian and hence all solutions of the d.e. must be Liouvillian (because all solutions are a linear combination of the two independent solutions). Clearly, $\overline{G} = \mathbf{C}(x)(\eta,\eta',\zeta,\zeta')$ must be contained in a Liouvillian field. This implies that the component of the identity of G, G°, must be solvable.

The component of the identity of any group is the largest connected subgroup of the group containing the identity. Recall that a point set is connected if any two points in the set can be joined by a segmental arc all of whose points belong to the point set.

A group H is said to be solvable (in the Galois theory sense) if $H = H_0 \supset H_1 \supset \cdots \supset H_m = \{e\}$ where each H_{i+1} is normal in H_i, each factor group H_i/H_{i+1} is abelian and e is the identity element of H.

If $G = SL(2,\mathbf{C})$, then $G^\circ = SL(2,\mathbf{C})$ and hence $SL(2,\mathbf{C})$ must be solvable. But $SL(2,\mathbf{C})$ is not solvable and a contradiction has been shown. Hence, the original assumption must be false and the d.e. has no Liouvillian solutions. This is case (4) of the original theorem.

2.2. Form of the Algorithm

At this point, we have that for the d.e. $y'' = ry$, the solution is of the form $e^{\int \omega}$ with either $\omega \in \mathbf{C}$, ω algebraic of degree 2 over $\mathbf{C}(x)$, or ω algebraic of degree 4, 6 or

12 over $\mathbf{C}(x)$, or there is no closed-form solution. The algorithm to solve the given d.e. now takes the following form.

```
apply sub-algorithm to find a solution of form η = e^∫ω where ω ∈ C(x)
if success then
      RETURN(solution);
fi;
apply sub-algorithm to find a solution of form η = e^∫ω where ω is algebraic of
degree 2 over C(x)
if success then
      RETURN(solution);
fi;
apply sub-algorithm to find a solution of form η = e^∫ω where ω is algebraic of
degree 4, 6 or 12 over C(x)
if success then
      RETURN(solution);
fi;
FAIL();
```

Each of the three sub-algorithms must be such that if a solution of the required form exists, then it will always be found. Only then can one guarantee that if all three sub-algorithms fail, then there is no Liouvillian solution.

2.3. Necessary Conditions

In order to reduce the work involved in solving the d.e., Kovacic has determined some conditions on the function r that must be true in order for each of the first three cases to be possible, i.e. for a Liouvillian solution to exist. If we can determine using these conditions that a case is not possible, then the sub-algorithm for that case need not be attempted.

These conditions are necessary but not sufficient; i.e. if the conditions for a given case do not hold then the corresponding sub-algorithm need not be tried as it will certainly fail, but if the conditions do hold then the sub-algorithm may or may not succeed.

In order to understand the necessary conditions and their development, some facts from complex analysis are needed.

Recall that any analytic function, f, of a complex variable z, can be expanded about any point a in the complex plane in a Laurent series as follows.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

The analytic part of the given expansion is $a_0 + a_1(z-a) + \cdots$; the principal part is $\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$. By definition, a is a *pole of $f(z)$ of order n* if the last term of the principal part of the Laurent series expansion of f about a is $\frac{a_{-n}}{(z-a)^n}$. Equivalently, if f is a rational function, a is a pole of $f(z)$ of order n if it is a root of the denominator of f of multiplicity n.

Further, the *order of f at ∞* is defined to be the order of ∞ as a zero of $f(z)$, i.e. the order of 0 as a pole of $f(\frac{1}{z})$. Equivalently, if f is a rational function, then the order of f at ∞ is the degree of the denominator minus the degree of the numerator.

The following theorem regarding the necessary conditions for the three cases may now be stated.

Theorem: For the d.e. $y'' = ry$ the following conditions are necessary for the respective cases to hold, i.e. for a Liouvillian solution of the specified form to exist.

(1) Every pole of r has order 1 or even order. The order of r at ∞ is even or greater than 2.

(2) r has at least one pole of either order 2 or odd order greater than 2.

(3) No pole of r has order greater than 2. The order of r at ∞ is at least 2. If the partial fraction expansion of r is $r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j}$ then $\sqrt{1+4\alpha_i} \in \mathbf{Q}$ for each i, $\sum_j \beta_j = 0$, and $\sqrt{1+4\gamma} \in \mathbf{Q}$ where $\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j$.

The following is a summary of the proof of this theorem. It uses several ideas that are explained in more detail in the proof of the algorithm itself (see section 3.2.).

In case (1), the d.e. has a solution of the form $\eta = e^{\int \omega}$, where $\omega \in \mathbf{C}(x)$. This implies that

$$\omega' + \omega^2 = r \tag{2.3a}$$

(See section 3.2. for a proof of this statement.) Since both r and $\omega \in \mathbf{C}(x)$, they can be expanded in Laurent series about any point c in the complex plane as follows.

$$\omega = b(x-c)^{\mu} + \text{higher powers of } x-c, \quad \mu \in \mathbf{Z},\ b \neq 0 \tag{2.3b}$$

$$r = \alpha(x-c)^{\nu} + \text{higher powers of } x-c, \quad \nu \in \mathbf{Z},\ \alpha \neq 0 \tag{2.3c}$$

Substituting (2.3b) and (2.3c) into (2.3a) we get

$$\mu b(x-c)^{\mu-1} + \cdots + b^2(x-c)^{2\mu} + \cdots = \alpha(x-c)^{\nu} + \cdots$$

We want to demonstrate that if c is a pole of r (i.e. $\nu < 0$), then its order is either 1 or even. If we assume ν is not -1 or -2, we can show it must be even. Assume $\nu \leq -3$, then matching coefficients of the lowest power of $x-c$ above gives $\nu \geq \min(\mu-1, 2\mu)$. With $\nu \leq -3$, this implies that $\mu < -1$ and $2\mu < \mu-1$. Because $b^2 \neq 0$ (by assumption), then $\nu = 2\mu$, i.e. ν is even as required.

This also demonstrates that if r has a pole of order $\nu = 2\mu \geq 4$ at c, then ω has a pole of order μ at c. This fact will be used in the proof of the algorithm in section 3.2.1.

The proof of the conditions on the order of r at ∞ is exactly analogous and is done by expanding r and ω at ∞.

In case (2), the d.e. has a solution of the form $\eta = e^{\int \omega}$, ω algebraic over $\mathbf{C}(x)$ of degree 2. The Galois group of the d.e., G, is conjugate to a subgroup of D^+ so that for every $\sigma \in G$ either $\sigma\eta = a_\sigma \eta$, $\sigma\zeta = a_\sigma^{-1}\zeta$ or $\sigma\eta = b_\sigma\zeta$, $\sigma\zeta = -b_\sigma^{-1}\eta$. In either case, $\sigma(\eta^2\zeta^2) = \eta^2\zeta^2$ so $\eta^2\zeta^2 \in \mathbf{C}(x)$. Also $\eta\zeta \notin \mathbf{C}(x)$ because if it were, we would have $\sigma(\eta\zeta) = \eta\zeta = a_\sigma\eta \cdot a_\sigma^{-1}\zeta$ and G would be a diagonal matrix with a_σ and a_σ^{-1} on the diagonal (i.e. the case $\sigma\eta = b_\sigma\zeta$, $\sigma\zeta = -b_\sigma^{-1}\eta$ could not occur).

Hence we can write $\eta^2\zeta^2$ as $\prod(x-c_i)^{e_i}$, $e_i \in \mathbf{Z}$, where at least one of the e_i must be odd. Assume $\eta^2\zeta^2 = (x-c)^e\prod(x-c_i)^{e_i}$ with e odd. Let $\phi = \dfrac{(\eta\zeta)'}{\eta\zeta} = \dfrac{\frac{1}{2}(\eta^2\zeta^2)'}{(\eta^2\zeta^2)}$. Because $\eta'' = r\eta$ and $\zeta'' = r\zeta$, then

$$\phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r' \tag{2.3d}$$

Expand both r and ϕ in Laurent series about c. Then

$$\phi = \frac{\frac{1}{2}e}{x-c} + \text{polynomial in } x-c \tag{2.3e}$$

$$r = \alpha(x-c)^{\nu} + \text{higher powers of } x-c \tag{2.3f}$$

and substitute (2.3e) and (2.3f) into (2.3d) to get

$$\frac{e}{(x-c)^3} + \cdots + \frac{-\frac{3}{4}e^2}{(x-c)^3} + \cdots + \frac{\frac{1}{8}e^3}{(x-c)^3} + \cdots = 2\alpha(e+\nu)(x-c)^{\nu-1} + \cdots$$

If $\nu > -2$, then $e - \frac{3}{4}e^2 + \frac{1}{8}e^3 = 0$ and $e = 0, 2, 4$. But e must be odd, so $\nu \leq -2$. If $\nu < -2$, then $2\alpha(e+\nu) = 0$ and $e = -\nu$ so that ν is odd. Hence either $\nu = -2$ or $\nu < -2$ and odd, i.e. r has a pole of either order 2 or odd order > 2.

In case (3), η is algebraic over $\mathbf{C}(x)$ so it can be expanded in a Puiseaux series (Laurent series with fractional exponents) about any point c in the complex plane. Also, η is a solution of the d.e. so

$$\eta'' = r\eta \tag{2.3g}$$

Expand η and r about c

$$\eta = a(x-c)^{\mu} + \text{higher powers of } x-c, \quad a \in \mathbf{C}, \quad a \neq 0, \quad \mu \in \mathbf{Q} \tag{2.3h}$$

$$r = \alpha(x-c)^{\nu} + \text{higher powers of } x-c, \quad \alpha \neq 0, \quad \nu \in \mathbf{Z} \tag{2.3i}$$

and substitute (2.3h) and (2.3i) into (2.3g) to get

$$a\mu(\mu-1)(x-c)^{\mu-2} + \cdots = \alpha a(x-c)^{\mu+\nu} + \cdots$$

The lowest order term on the right, being composed of the product of the lowest order terms of η and r, cannot be zero, so $\mu + \nu \geq \mu - 2$ and $\nu \geq -2$, i.e. the poles of r have order 1 or 2.

If $\nu = -2$, then matching coefficients of $(x-c)^{\mu-2}$ on both sides gives $\alpha = \mu(\mu-1)$ or $\mu = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\alpha}$. Because $\mu \in \mathbf{Q}$, by assumption, then $\sqrt{1+4\alpha} \in \mathbf{Q}$ and the partial fraction expansion of r must be

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j} + \text{polynomial}$$

with $\sqrt{1+4\alpha_i} \in \mathbf{Q}$ for each i.

The remainder of the conditions are obtained in an exactly analogous manner by expanding r and η about ∞ and substituting into $\eta'' = r\eta$.

CHAPTER 3

Kovacic's Algorithm and Proof

3.1. Kovacic's Algorithm

In his original paper, Kovacic describes and proves the sub-algorithms for each of the three cases separately. We will exploit the similarities in the three algorithms and describe them together. This will make it easier to follow the unification Saunders does for his variant of the algorithm. Proofs of the three sub-algorithms will be done separately in section 3.2.

The goal of the algorithm will be to determine the minimal polynomial for ω. Since $\eta = e^{\int \omega}$ in all cases, this determines a solution of the d.e. (In some cases, we may not be able to obtain an explicit expression for ω.)

We will first determine a function $\phi = \phi(\eta, \zeta)$. Then, in each case, the minimal polynomial for ω is written in terms of ϕ (and r). The form of ϕ as a function of η and ζ will be determined by the invariant of the Galois group of the d.e. in each case, where the *invariant of the Galois group of the differential equation* is defined to be that function of η and ζ that is kept invariant by all σ in the group and hence is in $C(x)$. Recall that in case (1), $\frac{\eta'}{\eta}$ is invariant, in case (2) $\eta^2\zeta^2$ is invariant, and in case (3) $(\eta^4 + 8\eta\zeta^3)^2$, $(\eta^5\zeta - \eta\zeta^5)^2$ and $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}$ are invariant.

In all cases, ϕ will be written as $\phi = \theta + \frac{P'}{P}$. The main component of θ is a sum $\sum_{c \in \Gamma} \frac{e_c}{x-c}$, where Γ is the set of poles c of r. A major part of the work of the algorithm will be determining possible values for the e_c's.

The function P will be a polynomial whose roots are *ordinary points of r* (i.e. not poles). We will not determine the roots of P (i.e. the poles of $\frac{P'}{P}$) explicitly; rather we will determine a possible degree d of P in terms of the e_c's and e_∞ (determined from the expansion of r at ∞). Then, we can determine the coefficients of P using an equation relating P, θ and r.

We will need to consider all combinations of possibilities for the e_c's and e_∞ to get a solution for P. If a solution can be found for P, we will have found the proper combination of e_c's and e_∞ and will have determined θ exactly (since it is a function of the e_c's) and hence ϕ and ω.

The algorithm is divided into 3 steps. As a preliminary stage, the set Γ of the poles of r is computed. Also, the degree of r at infinity is computed; it will be required in the computation of e_∞.

The quantity n will be the degree of ω over $\mathbf{C}(x)$ in each case; for case (1) $n = 1$, for case (2) $n = 2$ and for case (3) $n = 4$, 6 or 12.

Step (1)

For each c in Γ define a set E_c of possible values of e_c as follows:

(a) If c is a pole of r of order 1 then

case (1) – $E_c = \{1\}$

case (2) – $E_c = \{4\}$

case (3) – $E_c = \{12\}$

(b) If c is a pole of r of order 2 and b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction expansion of r then

case (1) – $E_c = \{\frac{1}{2} + k\sqrt{1+4b} \mid k = \pm\frac{1}{2}\}$

case (2) – $E_c = \{2 + 2k\sqrt{1+4b} \mid k = 0, \pm 1\} \cap \mathbf{Z}$

case (3) – $E_c = \{6 + \frac{12}{n}k\sqrt{1+4b} \mid k = 0, \pm 1, \ldots, \pm\frac{n}{2}\} \cap \mathbf{Z}$

(c) If c is a pole of r of order $\nu > 2$ then

case (1) – $E_c = \{\frac{1}{4}\nu + k\frac{b}{a} \mid k = \pm\frac{1}{2}\}$

where $[\sqrt{r}]_c$ is the sum of terms involving $\frac{1}{(x-c)^i}$ for $i = 2, \ldots, \frac{\nu}{2}$ in the Laurent series expansion of $\sqrt{r}$ at c, a is the coefficient of $\frac{1}{(x-c)^{\frac{\nu}{2}}}$ in $[\sqrt{r}]_c$, and b is the coefficient of $\frac{1}{(x-c)^{\frac{\nu}{2}+1}}$ in $r - ([\sqrt{r}]_c)^2$

case (2) – $E_c = \{\nu\}$

case (3) – $E_c = \{\}$ (since there are no poles of order > 2 in case (3))

Also define a set E_∞ as follows.

(a) If the order of r at $\infty > 2$ then

case (1) – $E_\infty = \{\frac{1}{2} + k \mid k = \pm\frac{1}{2}\}$

case (2) – $E_\infty = \{2 + 2k \mid k = 0, \pm 1\}$

case (3) – $E_\infty = \{6 + \frac{12}{n}k \mid k = 0, \pm 1, \ldots, \pm \frac{n}{2}\}$

(b) If the order of r at $\infty = 2$ and b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ then

case(1) – $E_\infty = \{\frac{1}{2} + k\sqrt{1+4b} \mid k = \pm\frac{1}{2}\}$

case(2) – $E_\infty = \{2 + 2k\sqrt{1+4b} \mid k = 0, \pm 1\} \cap \mathbf{Z}$

case(3) – $E_\infty = \{6 + \frac{12}{n}k\sqrt{1+4b} \mid k = 0, \pm 1, \ldots, \pm\frac{n}{2}\} \cap \mathbf{Z}$

(c) If the order of r at $\infty = \nu < 2$ then

case (1) – $E_\infty = \{\frac{1}{4}\nu + k\frac{b}{a} \mid k = \pm\frac{1}{2}\}$

where $[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $i = \frac{-\nu}{2}, \ldots, 0$ in the Laurent series expansion of $\sqrt{r}$ at ∞, a is the coefficient of $x^{\frac{-\nu}{2}}$ in $[\sqrt{r}]_\infty$, and b is the coefficient of $x^{\frac{-\nu}{2}-1}$ in $r - ([\sqrt{r}]_\infty)^2$

case (2) – $E_\infty = \{\nu\}$

case (3) – $E_\infty = \{\}$ (since the order of r at ∞ is ≥ 2 in case(3))

Step (2)

Consider all possible tuples $(e_{c_1}, e_{c_2}, \ldots, e_{c_n}, e_\infty)$, where the c_i are the distinct elements of Γ and each e_{c_i} and e_∞ is an element of the corresponding set E_{c_i} and E_∞ respectively. (For case (2), we may discard a tuple if all of its coordinates are even.)

Form the quantity d as follows:

case (1) – $d = e_\infty - \sum_{c\in\Gamma} e_c$

case (2) – $d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$

case (3) – $d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$

If d is a non-negative integer, retain the tuple for step (3); otherwise, discard the tuple.

Step (3)

For each tuple retained from step (2), form the rational function θ as follows:

case (1) – $\theta = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c\right) + s(\infty)[\sqrt{r}]_\infty$

where $[\sqrt{r}]_c$ is computed only for poles c of order > 2 and $s(c)$ is the sign of k in the corresponding e_c in the tuple, and $[\sqrt{r}]_\infty$ is computed only if the order of r at ∞ is < 2 and $s(\infty)$ is the sign of k in the corresponding e_∞ in the tuple.

case (2) – $\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$

case (3) – $\theta = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}$

Now search for a polynomial P of degree d defined by the following equations for $n+2$ polynomials P_i, $i = n, n-1, \ldots, 0, -1$. (These are somewhat different from those given in Kovacic's paper; the rationale for them is given in section 3.2.3.)

cases (1), (2) and (3) – $P_n = -P$

$$P_{i-1} = -P_i' - \theta P_i - (n-i)(i+1)rP_{i+1} \qquad i = n, n-1, \ldots, 0$$

and

$$P_{-1} = 0 \text{ (identically)}$$

In each case, P is computed by constructing the polynomial of degree d with undetermined coefficients, substituting into the above equations and solving the final equation $P_{-1} = 0$ for the undetermined coefficients. If the polynomial P exists, then compute ω as follows. (Again, this is slightly different from Kovacic's paper; see section 3.2.3.)

cases (1), (2) and (3) – ω is a solution of $\sum_{i=0}^{n}\frac{P_i}{(n-i)!}\omega^i = 0$

(Note that it may be impossible to obtain an explicit solution for ω.)

Then, $\eta = e^{\int\omega}$ is a solution of the d.e. If no polynomial P exists for any tuple retained from step (2), then the case cannot hold.

3.2. Proof of Kovacic's Algorithm

In the proofs that follow, we will use the following fact several times. The d.e. $y'' = ry$ has a solution of the form $\eta = e^{\int\omega}$ iff ω satisfies the Riccati equation $\omega' + \omega^2 = r$.

The first half of the proof is as follows. Because η is a solution to $y'' = ry$, $\eta'' = r\eta$. Since $\eta = e^{\int\omega}$, $\eta' = \omega e^{\int\omega}$ and $\eta'' = \omega' e^{\int\omega} + \omega^2 e^{\int\omega}$. Thus, $\omega'\eta + \omega^2\eta = r\eta$ and dividing through by $\eta = e^{\int\omega}$ (since it is not zero), we have $\omega' + \omega^2 = r$.

In the converse case, define $\eta = e^{\int\omega}$, i.e. $\omega = \frac{\eta'}{\eta}$. Then, $\omega' = \frac{\eta''}{\eta} - \frac{(\eta')^2}{\eta^2}$ and $\omega^2 = \frac{(\eta')^2}{\eta^2}$. Since $\omega' + \omega^2 = r$, by assumption,

$$\frac{\eta''}{\eta} - \frac{(\eta')^2}{\eta^2} + \frac{(\eta')^2}{\eta^2} = \frac{\eta''}{\eta} = r,$$

proving $\eta'' = r\eta$ and that η is a solution of $y'' = ry$.

3.2.1. Proof of Algorithm for Case (1)

In case (1), we are searching for a solution to the d.e. $y'' = ry$ of the form $\eta = e^{\int \omega}$, $\omega \in \mathbf{C}(x)$. Recall from section 3.1 that we are looking for a function $\phi = \theta + \frac{P'}{P}$. In this case, $\omega = \phi$.

Since $\omega \in \mathbf{C}(x)$, it can be expanded in a Laurent series about any point in the complex plane. The algorithm proceeds by determining the partial fraction expansion of ω and is proved using the Laurent series expansion of r and the Riccati equation

$$\omega' + \omega^2 = r \tag{3.2.1a}$$

We will write the Laurent series expansion of ω about a pole c of r as

$$\omega = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i} + \frac{e_c}{x-c} + \sum_{j=0}^{\infty} b_i (x-c)^i$$

Further, we will not need to determine the a_i's and b_i's explicitly, so we will set $[\omega]_c = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i}$ and $\overline{\omega}_c = \sum_{j=0}^{\infty} b_i (x-c)^i$. Then

$$\omega = [\omega]_c + \frac{e_c}{x-c} + \overline{\omega}_c = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i} + \frac{e_c}{x-c} + \sum_{j=0}^{\infty} b_i (x-c)^i \tag{3.2.1b}$$

We will call $[\omega]_c + \frac{e_c}{x-c}$, the "component at c" of the expansion of ω.

The major task of the algorithm is to determine parts of ω, i.e. the e_c and $[\omega]_c$ and the polynomial remainder part $\overline{\omega}_c$.

Now we know that the poles of r are of either order 1, order 2 or even order $\geq$ 4, from the necessary conditions for case (1).

Suppose c is a pole of r of order 1. Then

$$r = \frac{\#}{x-c} + \text{polynomial in } x-c \tag{3.2.1c}$$

(We will use # as a placeholder, to denote a complex constant whose value is unknown and unimportant.) Substitute (3.2.1b) and (3.2.1c) into the Riccati equation (3.2.1a) and get

$$\frac{-\mu a_\mu}{(x-c)^{\mu+1}} + \cdots + \frac{a_\mu^2}{(x-c)^{2\mu}} + \cdots = \frac{\#}{x-c} + \cdots$$

Since, by assumption, $a_\mu \neq 0$, matching coefficients of $\frac{1}{x-c}$ on both sides gives $\min(\mu+1,2\mu) = 1 \rightarrow \mu \leq 0$ and hence $[\omega]_c = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i} = 0$ (because μ is supposed to be ≥ 2) and $\omega = \frac{e_c}{x-c} + \overline{\omega}_c$.

Use this expression and the Riccati equation (3.2.1a) again and get

$$\frac{-e_c}{(x-c)^2} + \overline{\omega}_c{}' + \frac{e_c{}^2}{(x-c)^2} + \frac{2e_c\overline{\omega}_c}{x-c} + \overline{\omega}_c{}^2 = \frac{\#}{x-c} + \cdots$$

Matching coefficients of $\frac{1}{(x-c)^2}$ on both sides gives $-e_c + e_c{}^2 = 0$, i.e. e_c is either 0 or 1. The solution $e_c = 0$ can be eliminated since in that case, the right hand side of the above equation has a pole at c and the left hand side does not.

Hence, if c is a pole of r of order 1, then the component at c of ω is

$$\frac{\mathbf{e_c}}{\mathbf{x-c}} \qquad \mathbf{e_c = 1}$$

Now suppose that c is a pole of r of order 2. Then

$$r = \frac{b}{(x-c)^2} + \frac{\#}{x-c} + \cdots \tag{3.2.1d}$$

Substitute (3.2.1b) and (3.2.1d) into the Riccati equation (3.2.1a) (as before) and get

$$\frac{-\mu a_\mu}{(x-c)^{\mu+1}} + \cdots + \frac{a_\mu^2}{(x-c)^{2\mu}} + \cdots = \frac{b}{(x-c)^2} + \frac{\#}{x-c} + \cdots$$

As before, match coefficients of $\frac{1}{(x-c)^2}$ on both sides and get $\min(\mu+1,2\mu) = 2 \rightarrow \mu \leq 1$ and again $[\omega]_c = 0$ (because μ should be ≥ 2) and $\omega = \frac{e_c}{x-c} + \overline{\omega}_c$.

Now use this expression and the Riccati equation (3.2.1a) again and get

$$\frac{-e_c}{(x-c)^2} + (\overline{\omega}_c)' + \frac{e_c{}^2}{(x-c)^2} + \frac{2e_c\overline{\omega}_c}{x-c} + \overline{\omega}_c{}^2 = \frac{b}{(x-c)^2} + \frac{\#}{x-c} + \cdots$$

Matching coefficients of $\frac{1}{(x-c)^2}$ on both sides gives $-e_c + e_c{}^2 = b$, i.e. two possibilities for e_c, $e_c = \frac{1}{2} + \frac{1}{2}\sqrt{1+4b}$ or $e_c = \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}$.

Hence, if c is a pole of r of order 2, then the component at c of ω is

$$\frac{e_c}{x-c} \qquad e_c = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$$

Now suppose that c is a pole of r of order $\nu = 2\tau \geq 4$. From the proof of the necessary conditions for case (1) (see section 2.3.) we have that ω must have a pole of order $\frac{\nu}{2}$ at c, i.e. $[\omega]_c = \sum_{i=2}^{\frac{\nu}{2}} \frac{a_i}{(x-c)^i}$.

As defined in the statement of the algorithm (section 3.1)

$$[\sqrt{r}]_c = \frac{a}{(x-c)^{\frac{\nu}{2}}} + \cdots + \frac{\#}{(x-c)^2} \tag{3.2.1e}$$

If we now define $\bar{r}_c = \sqrt{r} - [\sqrt{r}]_c$ then $r = (\bar{r}_c + [\sqrt{r}]_c)^2 = \bar{r}_c^{\,2} + 2\bar{r}_c[\sqrt{r}]_c + ([\sqrt{r}]_c)^2$ and

$$r - ([\sqrt{r}]_c)^2 = \bar{r}_c^{\,2} + 2\bar{r}_c[\sqrt{r}]_c \tag{3.2.1f}$$

Using (3.2.1e) and (3.2.1f) and the Riccati equation (3.2.1a), we can show (after several lines of not very interesting or important algebra) that

$$([\omega]_c + [\sqrt{r}]_c) \cdot ([\omega]_c - [\sqrt{r}]_c)$$

$$= -[\omega]_c{}' + \frac{e_c}{(x-c)^2} - \bar{\omega}_c{}' + r - ([\sqrt{r}]_c)^2 - \frac{2e_c[\omega]_c}{x-c} - \frac{e_c^{\,2}}{(x-c)^2} - \frac{2e_c\bar{\omega}_c}{x-c} - 2\bar{\omega}_c[\omega]_c - \bar{\omega}_c^{\,2}$$

The left hand side of this equation has only terms involving $\frac{1}{(x-c)^i}$ for $i = 4, \ldots, \nu$. The right hand side has terms involving $\frac{1}{(x-c)^i}$ for $i = 1, \ldots, \frac{\nu}{2}+1$ and polynomials in $x-c$. Because there are no terms with $\frac{1}{(x-c)^i}$ for $i = \frac{\nu}{2}+2, \ldots, \nu$ on the right hand side, the left hand side must be equal to zero and hence either $[\omega]_c = [\sqrt{r}]_c$ or $[\omega]_c = -[\sqrt{r}]_c$, and $\omega = \pm[\sqrt{r}]_c + \frac{e_c}{x-c} + \bar{\omega}_c$.

Use this expression and the Riccati equation (3.2.1a) again and (after several more lines of not very interesting algebra) get

$$\frac{\pm a\cdot\frac{\nu}{2}}{(x-c)^{\frac{\nu}{2}+1}} + \cdots + \frac{e_c}{(x-c)^2} - \overline{\omega}_c{}' + \frac{b}{(x-c)^{\frac{\nu}{2}+1}} + \cdots$$

$$\frac{\mp 2ae_c}{(x-c)^{\frac{\nu}{2}+1}} - \frac{{e_c}^2}{(x-c)^2} - \frac{2e_c\overline{\omega}_c}{x-c} \mp \frac{2\overline{\omega}_c a}{(x-c)^{\frac{\nu}{2}}} + \cdots = 0$$

Matching coefficients of $\frac{1}{(x-c)^{\frac{\nu}{2}+1}}$ on both sides gives $\pm a\cdot\frac{\nu}{2} + b \mp 2ae_c = 0$ and $e_c = \frac{1}{2}(\frac{\nu}{2} + \frac{b}{a})$ or $e_c = \frac{1}{2}(\frac{\nu}{2} - \frac{b}{a})$.

Hence, if c is a pole of r of even order $\nu \geq 4$, then the component at c of the partial fraction expansion of ω is

$$\frac{\mathbf{e_c}}{\mathbf{x-c}} + [\sqrt{\mathbf{r}}]_{\mathbf{c}} \qquad \mathbf{e_c} = \frac{1}{2}(\frac{\nu}{2} + \frac{\mathbf{b}}{\mathbf{a}})$$

or

$$\frac{\mathbf{e_c}}{\mathbf{x-c}} - [\sqrt{\mathbf{r}}]_{\mathbf{c}} \qquad \mathbf{e_c} = \frac{1}{2}(\frac{\nu}{2} - \frac{\mathbf{b}}{\mathbf{a}})$$

Now, look at g, an ordinary point of r, i.e. not a pole, so that r is a polynomial in $x-g$. Expanding ω about g and using the Riccati equation (3.2.1a) and arguments similar to the first case, we can show that $\omega = \frac{f}{x-g}$ + polynomial in $x-g$ where f is either 0 or 1.

Collecting what we have so far, if Γ is the set of poles of r, then

$$\omega = [\omega]_c + \frac{e_c}{x-c} + \overline{\omega}_c = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} \pm [\sqrt{r}]_c\right) + \sum_{i=1}^{d}\frac{1}{x-g_i} + R$$

where $[\sqrt{r}]_c = 0$ if c is not a pole of order ≥ 4 and R is a polynomial in $\mathbf{C}[x]$.

We now determine the polynomial part R using the expansion of ω about ∞, namely

$$\omega = R + \frac{e_\infty}{x} + \text{lower powers of } x \tag{3.2.1g}$$

Using arguments analogous to the previous cases we obtain $e_\infty = 0, 1,\ R = 0$ if $o(\infty) > 2$, $e_\infty = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$, $R = 0$ if $o(\infty) = 2$ and $e_\infty = \frac{1}{2}(\frac{\nu}{2} \pm \frac{b}{a})$, $R = \pm[\sqrt{r}]_\infty$ if $o(\infty) = \nu \leq 0$.

Hence,

$$\omega = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c\right) + s(\infty)[\sqrt{r}]_\infty + \sum_{i=1}^{d}\frac{1}{x-g_i} \qquad \textbf{(3.2.1h)}$$

where $s(c)$ is + or - according to the sign in the corresponding e_c, $s(\infty)$ is + or - according to the sign in e_∞, $[\sqrt{r}]_c = 0$ if c is not a pole of r of order ≥ 4 and $[\sqrt{r}]_\infty = 0$ if $o(\infty) \geq 2$. By expanding (3.2.1h) about ∞ and setting it equal to (3.2.1g), we obtain the equation $e_\infty = \sum_{c\in\Gamma} e_c + \sum_{i-1}^{d} 1$ and hence an expression for d in terms of the e_c's and e_∞, namely $d = e_\infty - \sum_{c\in\Gamma} e_c$.

If we now set $P = \prod_{i=1}^{d}(x-g_i)$ (note that d is the degree of P) so that $\frac{P'}{P} = \sum_{i=1}^{d}\frac{1}{x-g_i}$ and if $\theta = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} \pm [\sqrt{r}]_c\right) \pm [\sqrt{r}]_\infty$, we have

$$\omega = \phi = \theta + \frac{P'}{P} \qquad (3.2.1i)$$

All of θ is known; we require a method of determining P.

Using the Riccati equation (3.2.1a) again, and $\omega = \theta + \frac{P'}{P}$, we obtain

$$\omega' = \theta' + \frac{P\cdot P'' - P'^2}{P^2}, \qquad \omega^2 = \theta^2 + \frac{2\theta P'}{P} + \frac{P'^2}{P^2}$$

$$P'' + 2\theta P' + P(\theta' + \theta^2 - r) = 0 \qquad (3.2.1j)$$

We have that if ω satisfies the Riccati equation $\omega' + \omega^2 = r$ then P satisfies (3.2.1j). We can verify that if P satisfies (3.2.1j), then ω satisfies the Riccati equation and hence $\eta = e^{\int\omega}$ satisfies the d.e.

$$\omega' + \omega^2 = \theta' + \frac{P'' - P'^2}{P^2} + \theta^2 + \frac{2\theta P'}{P} + \frac{P'^2}{P^2} = \frac{P'' + 2\theta P' + P(\theta' + \theta^2)}{P} = \frac{Pr}{P} = r$$

This completes the proof of the correctness of the algorithm for case (1).

3.2.2. Proof of Algorithm for Case (2)

In case (2), we are searching for a solution to the d.e. $y'' = ry$ of the form $\eta = e^{\int\omega}$ where ω is algebraic of degree 2 over $\mathbf{C}(x)$. The Galois group of the d.e. is conjugate to a subgroup of

$$D^+ = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix} \mid c \in \mathbf{C}(x),\ c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0\end{pmatrix} \mid c \in \mathbf{C},\ c \neq 0\right\}$$

and $\eta^2\zeta^2$ is an invariant of the group. Hence, $\eta^2\zeta^2 \in \mathbf{C}(x)$ and $\eta\zeta \notin \mathbf{C}(x)$ (or else we would have case (1)). Therefore we can write

$$\eta^2\zeta^2 = \text{constant}\cdot \prod_{c\in\Gamma}(x-c)^{e_c}\prod_{i=1}^{m}(x-g_i)^{f_i}$$

and

$$\phi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{1}{2}\frac{(\eta^2\zeta^2)'}{\eta^2\zeta^2} = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{1}{2}\sum_{i=1}^{m}\frac{f_i}{x-g_i} \tag{3.2.2a}$$

The task of the algorithm will be to determine the e_c and f_i. (We do not need to determine the g_i explicitly.) Once ϕ is determined, there is a quadratic equation depending on ϕ that determines ω and hence the solution.

Because η and ζ are solutions to the d.e., i.e. $\eta'' = r\eta$ and $\zeta'' = r\zeta$,

$$\phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r' \tag{3.2.2b}$$

This has given us a relationship between ϕ (and the e_c and f_i), and r, a known function.

We can now determine the e_c by looking at the poles of r and the Laurent series expansion of r and ϕ about these poles.

Suppose c is a pole of r of order 1. Then

$$r = \frac{\alpha}{x-c} + \text{polynomial in } x-c \tag{3.2.2c}$$

and

$$\phi = \frac{\frac{1}{2}e_c}{x-c} + k + \text{polynomial in } x-c, \quad k \in \mathbf{C} \tag{3.2.2d}$$

Substituting (3.2.2c) and (3.2.2d) into (3.2.2b), we obtain

$$\frac{e_c}{(x-c)^3} + \cdots + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \cdots + \frac{-\frac{3}{2}e_ck}{(x-c)^2} + \cdots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} + \frac{\frac{3}{4}e_c^2k}{(x-c)^2} + \cdots$$
$$= \frac{2\alpha e_c}{(x-c)^2} + \cdots + \frac{-2\alpha}{(x-c)^2} + \cdots$$

Matching coefficients of $\frac{1}{(x-c)^3}$ on both sides gives $e_c - \frac{3}{4}e_c^2 + \frac{1}{8}e_c^3 = 0$ $\rightarrow e_c = 0, 2, 4$. Matching coefficients of $\frac{1}{(x-c)^2}$ on both sides gives $-\frac{3}{2}e_ck + \frac{3}{4}e_c^2k = 2\alpha e_c - 2\alpha$. Because $\alpha \neq 0$, $e_c \neq 0, 2$.

Hence, if c is a pole of r of order 1, then

$$\mathbf{e_c = 4}$$

Now suppose c is a pole of r of order 2. Then

$$r = \frac{b}{(x-c)^2} + \frac{\#}{x-c} + \text{polynomial in } x-c \qquad (3.2.2e)$$

$$\phi = \frac{\frac{1}{2}e_c}{x-c} + \text{polynomial in } x-c \qquad (3.2.2f)$$

Substituting (3.2.2e) and (3.2.2f) into (3.2.2b) we get

$$\frac{e_c}{(x-c)^3} + \cdots + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \cdots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} = \frac{2be_c}{(x-c)^3} + \cdots + \frac{-4b}{(x-c)^3} + \cdots$$

Matching coefficients of $\frac{1}{(x-c)^3}$ on both sides gives $e_c - \frac{3}{4}e_c^2 + \frac{1}{8}e_c^3 = 2be_c - 4b$ or three possibilities for e_c, $e_c = 2,\ 2 \pm 2\sqrt{1+4b}$. Since e_c is assumed an integer, non-integral solutions for e_c may be discarded.

Hence, if c is a pole of r of order 2 then

$$\mathbf{e_c = 2,\ 2 \pm 2\sqrt{1+4b} \in Z}$$

Now suppose that c is a pole of r of order $\nu > 2$. Then

$$r = \frac{\alpha}{(x-c)^\nu} + \text{higher powers of } x-c \qquad (3.2.2g)$$

$$\phi = \frac{\frac{1}{2}e_c}{x-c} + \text{polynomial in } x-c \qquad (3.2.2h)$$

Substitute (3.2.2g) and (3.2.2h) into (3.2.2b) and get

$$\frac{e_c}{(x-c)^3} + \cdots + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \cdots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} + \cdots = \frac{2\alpha e_c}{(x-c)^{\nu+1}} + \cdots + \frac{-2\alpha\nu}{(x-c)^{\nu+1}}$$

Since $\nu > 2$, $\nu+1 > 3$ and $2\alpha e_c - 2\alpha\nu = 0 \rightarrow e_c = \nu$.

Hence if c is a pole of r of order $\nu > 2$ then

$$\mathbf{e_c = \nu}$$

Now look at the g_i which are poles of ϕ but ordinary points of r. Then

$$r = \text{polynomial in } x-g_i \qquad (3.2.2i)$$

and

$$\phi = \frac{\frac{1}{2}f_i}{x-g_i} + k + \text{polynomial in } x-g_i, \quad k \in \mathbf{C} \tag{3.2.2j}$$

Substitute (3.2.2i) and (3.2.2j) into (3.2.2b) and get

$$\frac{f_i}{(x-g_i)^3} + \cdots + \frac{-\frac{3}{4}f_i^2}{(x-g_i)^3} + \frac{-\frac{3}{2}f_i g}{(x-g_i)^2} + \cdots + \frac{\frac{1}{8}f_i^3}{(x-g_i)^3} + \frac{\frac{3}{4}f_i^2 g}{(x-g_i)^2} + \cdots$$

$$= \frac{\#}{x-g_i} + \cdots$$

Since there are no terms in $\frac{1}{(x-g_i)^3}$ on the right hand side, $f_i - \frac{3}{4}f_i^2 + \frac{1}{8}f_i^3 = 0$ $\rightarrow f_i = 0, 2, 4$; hence all the f_i in ϕ are even.

Collecting what we have so far, $\eta^2\zeta^2 = \text{constant}\cdot\prod_{c\in\Gamma}(x-c)^{e_c}\cdot P^2$ where $P \in \mathbf{C}[x]$ and $P^2 = \prod_{i=1}^{m}(x-g_i)^{f_i}$.

We can now use the expansion of ϕ about ∞, namely

$$\phi = \frac{\frac{1}{2}e_\infty}{x} + \text{lower powers of } x \tag{3.2.2k}$$

and arguments exactly analogous to the previous cases, to obtain $e_\infty = 0, 2, 4$ if $o(\infty) > 2$, $e_\infty = 2, 2 \pm 2\sqrt{1+4b}$ if $o(\infty) = 2$, and $e_\infty = \nu$ if $o(\infty) = \nu < 2$.

By expanding (3.2.2a) about ∞, setting it equal to (3.2.2k) and extracting the coefficient of $\frac{1}{x}$ on both sides, we can obtain the following equation, $\frac{1}{2}e_\infty = \frac{1}{2}\sum_{c\in\Gamma}e_c + \frac{1}{2}\sum_{i=1}^{m}f_i$. If d is the degree of P, then $2d = \sum_{i=1}^{m}f_i$ so that $d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma}e_c\right)$ (an expression in terms of the e_c's and e_∞).

If we now let $\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$, then $\phi = \theta + \frac{P'}{P}$. Use this expression and (3.2.2b) and obtain

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0 \tag{3.2.2l}$$

We still don't have ω, the objective of the algorithm. Kovacic introduces the following equation, algebraic in ω,

$$\omega^2 - \phi\omega + \frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r = 0 \tag{3.2.2m}$$

We can verify that if ω is a solution of this equation and case (2) holds, then ω

satisfies $\omega' + \omega^2 = r$ and hence $\eta = e^{\int \omega}$ satisfies the d.e. $y'' = ry$.

If we differentiate (3.2.2m) we get

$$(2\omega - \phi)\omega' = \phi'\omega - \frac{1}{2}\phi'' - \phi\phi' + r'$$

From (3.2.2m), we have that $\omega^2 - r = \phi\omega - \frac{1}{2}\phi' - \frac{1}{2}\phi^2$ so that

$$(2\omega - \phi)(\omega' + \omega^2 - r) = -\frac{1}{2}(\phi'' + 3\phi\phi' + \phi^3 - 4r\phi - 2r') = -\frac{1}{2} \cdot (3.2.2b) = 0$$

so that either $2\omega - \phi = 0$ or $\omega' + \omega^2 - r = 0$. Now $2\omega - \phi$ cannot be zero, since in that case $\omega = \frac{1}{2}\phi \in \mathbf{C}(x)$ and that is covered in case (1), assumed to fail for case (2).

Hence, $\omega' + \omega^2 = r$ and $\eta = e^{\int \omega}$ is a solution of the d.e. This proves the correctness of the algorithm for case (2).

3.2.3. Proof of Algorithm for Case (3)

In case (3), we are searching for an algebraic solution, η, of the d.e. $y'' = ry$ and as in previous cases, we determine it by computing the minimal polynomial for $\omega = \frac{\eta'}{\eta}$ ($\eta = e^{\int \omega}$). As stated in section 2.1., the order of the Galois group of the d.e. in this case is 24, 48 or 120; the following theorem says that the degree of the corresponding ω over $\mathbf{C}(x)$ is then 4, 6 or 12 respectively. (See Kovacic [3] for details of the proof.)

Theorem: If $\omega = \frac{\eta'}{\eta}$ where η and ζ are solutions of the d.e. and G is the Galois group of the d.e. relative to η and ζ, then if G has order 24, 48 or 120, ω has degree 4, 6 or 12 respectively over $\mathbf{C}(x)$. Also, for any other $\omega_1 = \frac{\eta_1'}{\eta_1}$ where η_1 is also a solution of the d.e., the degree of ω_1 over $\mathbf{C}(x)$ is greater than or equal to 4, 6 or 12 respectively, i.e. the ω obtained are minimal.

The algorithm for this case can be carried out in one of two ways: either find a 12th degree polynomial for ω, factor it into irreducible factors and use any of the factors for ω; or try for a 4th degree polynomial for ω, then a 6th degree polynomial for ω, and finally a 12th degree polynomial for ω.

In the implementation, the second alternative was chosen since factoring a 12th degree polynomial is difficult in Maple (if not impossible if it has algebraic extensions). This is also what is done in Saunders' algorithm for the same reasons.

It will be noted in the relevant places in the proof of this case of the algorithm where the algorithms for cases (1) and (2) can be derived from this case. It turns out that the three cases are more similar than Kovacic's paper originally leads us to believe.

The backbone of the algorithm (and the algorithms for cases (1) and (2)) is an n-th order ordinary differential equation for a function ϕ defined in terms of ϕ and r by $a_{-1} = 0$, where a_{-1} is defined by the following equations, denoted $(\#)_n$.

$$\left.\begin{array}{ll} a_n = -1 & \\ a_{i-1} = -a_i' - \phi a_i - (n-i)(i+1)ra_{i+1} & i = n, \ldots, 0 \end{array}\right\} \qquad (\#)_n$$

(Note that in the formula for a_{n-1} there is no a_{n+1} term since $n-i = 0$ when $i = n$.)

In all three cases, we will construct ϕ as $\phi = \theta + \frac{P'}{P}$, where n is the degree of ω over $\mathbf{C}(x)$ The function θ is constructed as a function of the poles of r and P is then defined in terms of θ and r by the equation $P_{-1} = 0$ where $P_i = P \cdot a_i$.

In case (3), $\theta = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}$; compare this with $\theta = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c\right) + s(\infty)[\sqrt{r}]_\infty$ in case (1) and $\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$ in case (2).

The following three theorems by Kovacic are used in the proof of the algorithm for case (3) but they apply equally well to the algorithms for cases (1) and (2).

Theorem(1): If ϕ is a solution of $a_{-1} = 0$ where a_i is defined by the equations denoted $(\#)_n$, and ω is a solution of the equation

$$\omega^n = \sum_{i=0}^{n-1}\frac{a_i}{(n-i)!}\omega^i$$

then $\eta = e^{\int\omega}$ is a solution of the d.e. $y'' = ry$.

Proof:

We first define the polynomial $A(u)$ in terms of the a_i as

$$A(u) = -u^n + \sum_{i=0}^{n-1}\frac{a_i}{(n-i)!}u^i$$

or

$$A(u) = \sum_{i=0}^{n} \frac{a_i}{(n-i)!} u^i \qquad a_n = -1$$

Kovacic claims that

$$(u^2 - r)\frac{\partial^{k+1}A(u)}{\partial u^{k+1}} = \frac{\partial^{k+1}A(u)}{\partial u^i \partial x} + [(n-2k)u + \phi]\frac{\partial^k A(u)}{\partial u^k} + k(n-k+1)\frac{\partial^{k-1}A(u)}{\partial u^{k-1}} \quad (3.2.3a)$$

for all integer $k \geq 0$. (See [3] for a proof by induction on n of this claim.) From the definition of $A(u)$ and the assumed definition of ω, $A(\omega) = 0$. We will show that $A(\omega) = 0$ implies that $\omega' + \omega^2 = r$ (equivalent to $\eta = e^{\int \omega}$ is a solution of $y'' = ry$), by assuming the contrary and forcing a contradiction.

Because $A(\omega)$ is a constant, $\frac{dA(\omega)}{dx} = 0$. Then

$$\frac{dA(\omega)}{dx} = \frac{\partial A(\omega)}{\partial \omega}\frac{\partial \omega}{\partial x} + \frac{\partial A(\omega)}{\partial x} = 0$$

$$\omega' \frac{\partial A(\omega)}{\partial \omega} + \frac{\partial A(\omega)}{\partial x} = 0$$

$$(\omega' + \omega^2 - r)\frac{\partial A(\omega)}{\partial \omega} = -\frac{\partial A(\omega)}{\partial x} + (\omega^2 - r)\frac{\partial A(\omega)}{\partial \omega}$$

From (3.2.3a) with $k = 0$,

$$(\omega' + \omega^2 - r)\frac{\partial A(\omega)}{\partial \omega} = -\frac{\partial A(\omega)}{\partial x} + (n\omega + \phi)A(\omega) + \frac{\partial A(\omega)}{\partial x} = 0$$

(since $A(\omega) = 0$). Because $\omega' + \omega^2 - r \neq 0$ by assumption, $\frac{\partial A(\omega)}{\partial \omega}$ must be zero. Hence, we have that $\frac{\partial^l A(\omega)}{\partial \omega^l} = 0$ for $l = 0$ and $l = 1$. Now we can use induction to prove $\frac{\partial^{k+1}A(\omega)}{\partial \omega^{k+1}} = 0$. Assume it is true for arbitrary $l = k-1$ and $l = k$, i.e. $\frac{\partial^{k-1}A(\omega)}{\partial \omega^{k-1}} = \frac{\partial^k A(\omega)}{\partial \omega^k} = 0$. If $\frac{\partial^k A(\omega)}{\partial \omega^k} = 0$ then

$$\frac{d}{dx}\left(\frac{\partial^k A(\omega)}{\partial \omega^k}\right) = 0$$

$$\frac{\partial^{k+1}A(\omega)}{\partial \omega^{k+1}}\frac{\partial \omega}{\partial x} + \frac{\partial^{k+1}A(\omega)}{\partial \omega^k \partial x} = 0$$

$$(\omega' + \omega^2 - r)\frac{\partial^{k+1}A(\omega)}{\partial \omega^{k+1}} = -\frac{\partial^{k+1}A(\omega)}{\partial \omega^k \partial x} + (\omega^2 - r)\frac{\partial^{k+1}A(\omega)}{\partial \omega^{k+1}}$$

From (3.2.3a) we have

$$(\omega' + \omega^2 - r)\frac{\partial^{k+1}A(\omega)}{\partial\omega^{k+1}}$$

$$= -\frac{\partial^{k+1}A(\omega)}{\partial\omega^k\partial x} + \frac{\partial^{k+1}A(\omega)}{\partial\omega^k\partial x} + [(n-2k)\omega + \phi]\frac{\partial^k A(\omega)}{\partial\omega^k} + k(n-k+1)\frac{\partial^{k-1}A(\omega)}{\partial\omega^{k-1}}$$

Since $\frac{\partial^k A(\omega)}{\partial\omega^k} = \frac{\partial^{k-1}A(\omega)}{\partial\omega^{k-1}} = 0$ (by assumption), then $(\omega' + \omega^2 - r)\frac{\partial^{k+1}A(\omega)}{\partial\omega^{k+1}} = 0$, and since $\omega' + \omega^2 \neq r$ (by assumption), then $\frac{\partial^{k+1}A(\omega)}{\partial\omega^{k+1}} = 0$.

The desired contradiction then falls out since

$$\frac{\partial^n A(\omega)}{\partial\omega^n} = \frac{\partial^n}{\partial\omega^n}\left(\sum_{i=0}^{n}\frac{a_i}{(n-i)!}\omega^i\right) = \sum_{i=0}^{n}\left(\frac{a_i}{(n-i)!}\frac{\partial(\omega^i)}{\partial\omega^n}\right) = \frac{\partial^n(a_n\omega^n)}{\partial\omega^n} = -n! \neq 0$$

and hence $A(\omega) = 0$ implies $\omega' + \omega^2 = r$ and $\eta = e^{\int\omega}$ is a solution of $y'' = ry$.

Thus if ω satisfies $\omega^n = \sum_{i=0}^{n-1}\frac{a_i}{(n-i)!}\omega^i$ where the a_i correspond to a solution ϕ of $(\#)_n$, then $e^{\int\omega}$ is a solution of $y'' = ry$. This completes the proof of the theorem. □

This theorem implies that if we can construct ϕ and then calculate the corresponding a_i using $(\#)_n$, we can determine an equation for ω and hence a solution of the d.e.

The following theorem says that the equation for ω obtained using the a_i corresponding to ϕ is the minimal polynomial for ω. Further, ϕ is proved to be a function in $\mathbf{C}(x)$.

Theorem(2): If the degree of ω over $\mathbf{C}(x)$ is n, then ϕ is a solution of $a_{-1} = 0$ where a_i is defined by $(\#)_n$, and ϕ is a rational function of x with coefficients in $\mathbf{C}$, i.e. $\phi \in \mathbf{C}(x)$.

Proof:

Let $A(u)$ be a polynomial with coefficients in $\mathbf{C}(x)$, and let $A(u)$ be the minimal polynomial for ω. Let deg $A(u) = n$ so that the degree of ω over $\mathbf{C}(x)$ is n, then $A(u)$ can be of the form

$$A(u) = -u^n + \sum_{i=0}^{n-1}\frac{a_i}{(n-i)!}u^i = \sum_{i=0}^{n}\frac{a_i}{(n-i)!}u^i \qquad a_n = -1$$

Consider the following polynomial $B(u)$

$$B(u) = (r-u^2)\frac{\partial A(u)}{\partial u} + \frac{\partial A(u)}{\partial x} + (nu + \phi)A(u)$$

where $\phi = a_{n-1}$ and $\phi \in C(x)$. We will show that ϕ satisfies $a_{-1} = 0$ where the a_i are defined by $(\#)_n$, by determining the coefficients of powers of u in $B(u)$.

The u^{n+1} term in $B(u)$ comes from $-u^2\frac{\partial A(u)}{\partial u}$ and $nuA(u)$ and is

$$-u^2 \cdot \frac{na_nu^{n-1}}{0!} + nu \cdot \frac{a_nu^n}{0!} = -na_nu^{n+1} + na_nu^{n+1} = 0$$

The u^n term comes from $-u^2\frac{\partial A(u)}{\partial u}$, $\frac{\partial A(u)}{\partial x}$, $nuA(u)$ and $\phi A(u)$ and is

$$-u^2 \cdot \frac{(n-1)a_{n-1}u^{n-2}}{1!} + \frac{a_n'u^n}{0!} + nu \cdot \frac{a_{n-1}u^{n-1}}{1!} + \phi \cdot \frac{a_nu^n}{0!}$$

$$= -(n-1)a_{n-1}u^n + a_n'u^n + na_{n-1}u^n + \phi a_nu^n$$

$$= (a_{n-1} + a_n' + \phi a_n)u^n = 0$$

since $a_n = -1$, $a_n' = 0$ and since $\phi = a_{n-1}$, $\phi a_n = -a_{n-1}$. Hence, there are no u^{n+1} or u^n terms in $B(u)$ and the degree of $B(u)$ is less than n.

Now $B(\omega) = 0$ as follows

$$B(\omega) = (r-\omega^2)\frac{\partial A(\omega)}{\partial \omega} + \frac{\partial A(\omega)}{\partial x} + (n\omega + \phi)A(\omega)$$

$$= \omega'\frac{\partial A(\omega)}{\partial \omega} + \frac{\partial A(\omega)}{\partial x} + (n\omega + \phi)A(\omega) = \frac{dA(\omega)}{dx} + (n\omega + \phi)A(\omega) = 0$$

since $A(\omega) = 0$ by definition because it is the minimal polynomial for ω. Hence, the coefficients of ω^i in $B(\omega)$ for $i < n$ must all be zero. The ω^i term in $B(\omega)$ is

$$r \cdot \frac{(i+1)a_{i+1}}{(n-i-1)!}\omega^i - \omega^2 \cdot \frac{(i-1)a_{i-1}}{(n-i+1)!}\omega^{i-2} + \frac{a_i'}{(n-i)!}\omega^i + n\omega \cdot \frac{a_{i-1}}{(n-i+1)!}\omega^{i-1} + \phi \cdot \frac{a_i}{(n-i)!}\omega_i$$

$$= \left[\frac{r(i+1)a_{i+1}}{(n-i-1)!} - \frac{(i-1)a_{i-1}}{(n-i+1)!} + \frac{na_{i-1}}{(n-i+1)!} + \frac{\phi a_i}{(n-i)!}\right]\omega^i$$

$$= \frac{1}{(n-i)!}\left[(n-i)(i+1)ra_{i+1} - \frac{(i-1)}{(n-i+1)}a_{i-1} + \frac{na_{i-1}}{(n-i+1)} + a_i' + \phi a_i\right]\omega^i$$

$$= \frac{1}{(n-i)!}\left[(n-i)(i+1)ra_{i+1} + a_{i-1} + a_i' + \phi a_i\right]\omega^i$$

So $(n-i)(i+1)ra_{i+1} + a_{i-1} + a_i' + \phi a_i = 0$ for all $i = 0, \ldots, n$ where $a_{-1} = 0$. These are exactly the equations defining $(\#)_n$ so ϕ is a solution of $a_{-1} = 0$ as required. This completes the proof of the theorem. □

What do we have now? We have that $\phi \in \mathbf{C}(x)$ is a solution of $a_{-1} = 0$ defined by $(\#)_n$ iff ω satisfying $\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i$ is a solution of $\omega' + \omega^2 = r$ and ω is algebraic of degree n over $\mathbf{C}(x)$. The algorithms for all three cases use this fact; implicitly in cases (1) and (2), and explicitly in case (3). We progressively try $n = 1$, $n = 2$, $n = 4$, $n = 6$, $n = 12$. At each step we attempt to find a ϕ in $\mathbf{C}(x)$ satisfying $(\#)_n$. If that is possible then the minimal polynomial for ω is $-\omega^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} \omega^i$. If it is not possible, we proceed to the next value of n. If no value of n produces a ϕ in $\mathbf{C}(x)$ then there is no solution to the d.e. $y'' = ry$.

The following theorem provides a way of building the function ϕ.

Theorem(3): If u is any homogeneous polynomial of degree n in solutions η and ζ of the d.e. then $\phi = \frac{u'}{u}$ is a solution of $(\#)_n$. (Note: for example, $\eta^2\zeta^3 + 3\eta\zeta^4$ is a homogeneous polynomial of degree 5 in solutions of the d.e.)

See [3] for the details of the proof.

We require that ϕ be in $\mathbf{C}(x)$ so that $\phi = \frac{u'}{u} = \frac{\frac{1}{m}(u^m)'}{u^m}$ must be in $\mathbf{C}(x)$. If $\frac{u'}{u}$ is in $\mathbf{C}(x)$ or u^m, $m \geq 1$, is in $\mathbf{C}(x)$ then this requirement is satisfied. The functions u are written in terms of the invariants of the Galois group of the d.e. for each case. (See section 2.2 for the derivation of the invariants.) Recall that the invariant of the Galois group of the d.e. is a function of η and ζ left fixed by all σ in the group, i.e. it is a function in $\mathbf{C}(x)$.

Table 1 - Galois group invariants

n	u	invariant	m
1	η	u'/u	1
2	$\eta\zeta$	u^2	2
4	$\eta^4+8\eta\zeta^3$	u^3	3
6	$\eta^5\zeta-\eta\zeta^5$	u^2	2
12	$\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$	u	1

We can then write the invariant in terms of the poles of r and certain exponents e_c, and a polynomial part. Recall that in case (1),

$$\phi = \frac{u'}{u} = \omega = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c\right) + s(\infty)[\sqrt{r}]_\infty + \frac{P'}{P}$$

and in case (2),

$$\phi = \frac{\frac{1}{2}(u^2)'}{u^2} = \frac{\frac{1}{2}(\eta^2\zeta^2)'}{\eta^2\zeta^2} = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{P'}{P}$$

For case (3), we combine the cases $n = 4,\ 6,\ 12$ by writing

$$u^{\frac{12}{n}} = \prod_{c\in\Gamma}(x-c)^{e_c}\prod_{i=1}^{m}(x-g_i)^{f_i}$$

$$\phi = \frac{\frac{n}{12}(u^{\frac{12}{n}})'}{u^{\frac{12}{n}}} = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{P'}{P}$$

The e_c's are determined in a manner analogous to the process described for cases (1) and (2), and so will not be covered again here. The derivation is slightly more complicated here because of the parameter n, but basically the same.

The results are as follows. If c is a pole of r of order 1, then $e_c = 12$. If c is a pole of r of order 2, then $e_c = 6 + k\sqrt{1+4b}$, where k is one of $0, \pm 3, \pm 6$ if $n = 4$, k is one of $0, \pm 2, \pm 4, \pm 6$ if $n = 6$, and k is one of $0, \pm 1, \ldots, \pm 6$ if $n = 12$.

Similarly, if the order of r at ∞ is greater than 2, then $e_\infty = 6 + k$ and if the order of r at ∞ is 2, then $e_\infty = 6 + k\sqrt{1+4b}$, where $k = 0, \pm 3, \pm 6$ if $n = 4$, $k = 0, \pm 2, \pm 4, \pm 6$ if $n = 6$, and $k = 0, \pm 1, \ldots, \pm 6$ if $n = 12$.

It can also be shown that $\frac{n}{12}f_i$ is an integer for all i so that for $n = 4$, $u^3 = P^3\prod_{c\in\Gamma}(x-c)^{e_c}$; for $n = 6$, $u^2 = P^2\prod_{c\in\Gamma}(x-c)^{e_c}$; and for $n = 12$, $u = P\prod_{c\in\Gamma}(x-c)^{e_c}$, where the degree of P, $d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma}e_c\right)$, is a non-negative integer.

Compare the above results for case (3) those obtained in cases (1) and (2). The order of the invariant m is defined to be the least integer such that $\phi = \dfrac{\frac{1}{m}(u^m)'}{u^m} \in \mathbf{C}(x)$.

Table 2 - Formulas for e_c

n	m	e_c (c order 1)	e_c (c order 2)	k
1	1	1	$\frac{1}{2}+k\sqrt{1+4b}$	$-\frac{1}{2}, \frac{1}{2}$
2	2	4	$2(1+k\sqrt{1+4b})$	-1, 0, 1
4	3	12	$3(2+k\sqrt{1+4b})$	-2, -1, 0, 1, 2
6	2	12	$2(3+k\sqrt{1+4b})$	-3 , ... , 3
12	1	12	$6+k\sqrt{1+4b}$	-6 , ... , 6

The value calculated for e_c if c is a pole of r of order 1 is $n \cdot m$; the value calculated for e_c if c is a pole of r of order 2 is $m(\frac{n}{2} + k\sqrt{1+4b})$ where $k = -\frac{n}{2}, \ldots, \frac{n}{2}$.

So for case (3) we have

$$\phi = \frac{u'}{u} = \frac{\frac{n}{12}(u^{\frac{12}{n}})'}{u^{\frac{12}{n}}} = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{P'}{P}$$

If we set $\theta = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}$, then $\phi = \theta + \frac{P'}{P}$ as expected.

We now set $P_i = P \cdot a_i$ and demonstrate that the recursive relations for P are correct. Kovacic actually sets $P_i = S^{n-i} \cdot P \cdot a_i$, where $S = \prod_{c\in\Gamma}(x-c)$, with no justification at all. This is completely unnecessary and obscures the similarities of the three cases.

If $P_i = P \cdot a_i$, then $P_n = P \cdot a_n = -P$. Also

$$P_{i-1} = P \cdot a_{i-1} = P \cdot (-a_i' - \phi a_i - (n-i)(i+1)ra_{i+1})$$

$$= -Pa_i' - P\phi a_i - (n-i)(i+1)Pra_{i+1} = -Pa_i' - P\left(\theta + \frac{P'}{P}\right)a_i - (n-i)(i+1)rP_{i+1}$$

$$= -Pa_i' - P\theta a_i - P'a_i - (n-i)(i+1)rP_{i+1} = -P_i' - \theta P_i - (n-i)(i+1)rP_{i+1}$$

Because $a_{-1} = 0$, then $P_{-1} = 0$.

Hence P is defined by the following equations

$$P_n = -P$$

$$P_{i-1} = -P_i' - \theta P_i - (n-i)(i+1)rP_{i+1}$$

and

$$P_{-1} = 0 \quad \text{(identically)}$$

Rewriting the equation for ω in terms of P_i gives

$$0 = -\omega^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!}\omega^i = -P\omega^n + \sum_{i=0}^{n-1} \frac{Pa_i}{(n-i)!}\omega^i = a_n P\omega^n + \sum_{i=0}^{n-1} \frac{P_i}{(n-i)!}\omega^i = \sum_{i=0}^{n} \frac{P_i}{(n-i)!}\omega^i$$

$$\text{i.e.} \quad \sum_{i=0}^{n} \frac{P_i}{(n-i)!}\omega^i = 0$$

We can now verify that these two sets of equations for P and ω are the same as those produced in cases (1) and (2) as follows.

In case (1), $n = 1$.

$$P_1 = -P$$

$$P_0 = -P_1' - \theta P_1 - (1-1)(1+1)rP_2$$

$$= P' + \theta P$$

$$P_{-1} = -P_0' - \theta P_0 - (1-0)(0+1)rP_1$$

$$= -(P' + \theta P)' - \theta(P' + \theta P) - r(-P) = -P'' - \theta' P - \theta P' - \theta P' - \theta^2 P + rP$$

$$= -P'' - 2\theta P' - (\theta' + \theta^2 - r)P$$

because $P_{-1} = a_{-1} \cdot P = 0$ (because $a_{-1} = 0$) then

$$P'' + 2\theta P' + (\theta' + \theta^2 - r)P = 0 \qquad (3.2.3b)$$

Also $\sum_{i=0}^{1} \frac{P_i}{(1-i)!}\omega^i = 0$ so that

$$\frac{P_0}{(1-0)!}\omega^0 + \frac{P_1}{(1-1)!}\omega^1 = 0$$

$$P_0 + P_1\omega = 0$$

$$P' + \theta P - P\omega = 0$$

$$\omega = \frac{P' + \theta P}{P} = \theta + \frac{P'}{P} \tag{3.2.3c}$$

Note that (3.2.3b) and (3.2.3c) are the same equations as those produced for case (1), namely (3.2.1j) and (3.2.1i) respectively.

In case (2), $n = 2$.

$$P_2 = -P$$

$$P_1 = -P_2' - \theta P_2 - (2-2)(2+1)rP_3$$

$$= P' + \theta P$$

$$P_0 = -P_1' - \theta P_1 - (2-1)(1+1)rP_2$$

$$= -(P' + \theta P)' - \theta(P' + \theta P) - 2r(-P)$$

$$= -P'' - \theta' P - \theta P' - \theta P' - \theta^2 P + 2rP$$

$$= -P'' - 2\theta P' - (\theta' + \theta^2 - 2r)P$$

$$P_{-1} = -P_0' - \theta P_0 - (2-0)(0+1)rP_1$$

$$= -(-P'' - 2\theta P' - (\theta' + \theta^2 - 2r)P)' - \theta(-P'' - 2\theta P' - (\theta' + \theta^2 - 2r)P) - 2r(P' + \theta P)$$

$$= P''' + 2\theta' P' + 2\theta P'' + (\theta' + \theta^2 - 2r)P' + (\theta'' + 2\theta\theta' - 2r')P + \theta P'' + 2\theta^2 P'$$

$$+ \theta\theta' P + \theta^3 P - 2r\theta P - 2rP' - 2r\theta P$$

$$= P''' + 3\theta P'' + (3\theta' + 3\theta^2 - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P$$

Because P_{-1} must be zero then

$$P''' + 3\theta P'' + (3\theta' + 3\theta^2 - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0 \tag{3.2.3d}$$

Also $\sum_{i=0}^{2} \frac{P_i}{(2-i)!}\omega^i = 0$, so that

$$\frac{P_0}{(2-0)!}\omega^0 + \frac{P_1}{(2-1)!}\omega^1 + \frac{P_2}{(2-2)!}\omega^2 = 0$$

$$\tfrac{1}{2}P_0 + P_1\omega + P_2\omega^2 = 0$$

$$\tfrac{1}{2}(-P'' - 2\theta P' - (\theta' + \theta^2 - 2r)P) + (P' + \theta P)\omega + (-P)\omega^2 = 0$$

$$\omega^2 + \left(\frac{P' + \theta P}{-P}\right)\omega + \frac{1}{2}\left(\frac{-P'' - 2\theta P' - (\theta' + \theta^2 - 2r)P}{-P}\right) = 0$$

$$\omega^2 - \left(\theta + \frac{P'}{P}\right)\omega + \frac{1}{2}\left(\frac{P''}{P} + \frac{2\theta P'}{P} + (\theta' + \theta^2 - 2r)\right) = 0$$

Since $\phi = \theta + \frac{P'}{P}$, $\phi' = \theta' + \frac{P''}{P} - \left(\frac{P'}{P}\right)^2$ then

$$\omega^2 - \phi\omega + \frac{1}{2}(\theta' + \theta^2 - 2r) = 0 \qquad (3.2.3e)$$

Note that (3.2.3d) and (3.2.3e) are the same equations as those produced for case (2), namely (3.2.2l) and (3.2.2m) respectively.

Noticing that the equations for P and ω could be unified for all three cases was of benefit in implementing the algorithm. Saunders' [5] had unified only steps (1) and (2) of the algorithm and implemented step (3) separately for each of the three cases. In the implementation for Maple, step (3) could also be implemented as a single procedure.

CHAPTER 4

Saunders' Algorithm

4.1. Saunders' Modifications to Kovacic's Algorithm

Saunders [5] has produced a modified version of Kovacic's algorithm where most of the algorithm has been unified to avoid implementing each of the cases separately. In his algorithm, only the final portion of step (3) remained to be implemented separately for each case. By noting the similarities of the three cases, it was possible to unify this step as well.

Saunders does not prove that his version of Kovacic's algorithm does in fact correctly implement Kovacic's algorithm and it is certainly not obvious that the two are the same algorithm. In fact, in the course of comparing the two, several bugs were discovered in Saunders' algorithm. A corrected version is presented here and verified correct by comparison with Kovacic's algorithm as given in section 3.1.

Saunders' algorithm is noteworthy in that he has unified the computation of d (the degree of P) and θ in steps (1) and (2) of Kovacic's algorithm. (Recall that we are computing a function $\phi = \theta + \frac{P'}{P}$, then obtaining the minimal polynomial for ω in terms of ϕ and r.) We have now unified the computation of P and ω in step (3) of Kovacic's algorithm; see section 3.2.3 for details.

Saunders' unification is carried out by computing "parts" of d and θ. Recall that in Kovacic's algorithm, $d = \text{constant} \cdot \left(e_\infty - \sum_{c \in \Gamma} e_c\right)$ for all three cases, and that the e_c's and e_∞ were of the form $\text{expression}_1 + k \cdot \text{expression}_2$, where $k = -\frac{n}{2}, \ldots, \frac{n}{2}$. Here, we will compute the sum of all the expression_1's in d as e_{fix}, and each expression_2 as e_i, with $i = 0$ for e_∞.

A similar process is carried out for θ. Recall that in Kovacic's algorithm, the main component of θ was the sum $\sum_{c \in \Gamma} \frac{e_c}{x-c}$.

In a preliminary step of the algorithm, we normalise $r = \frac{s}{t}$ where $\gcd(s,t) = 1$ and s and t are polynomials in $\mathbf{C}[x]$. We then perform a square-free decomposition on t and obtain $t = t_1 \cdot t_2^2 \cdot t_3^3 \cdots t_m^m$. We will not have to determine all the poles of r

as in Kovacic's algorithm, only the poles with even order. We also determine the order of r at ∞, $o(\infty) = \deg t - \deg s$.

Next, we determine which of the three cases are possible (equivalently, what degrees of ω over $\mathbf{C}(x)$ are possible) by checking the necessary conditions and creating a list L of possible degrees as follows.

$1 \in L$	if $t_i = 1$ for all odd $i \geq 2$ and $o(\infty)$ is even or > 2
$2 \in L$	if $t_2 \neq 1$ or $t_i \neq 1$ for some odd $i \geq 3$
$4,\ 6,\ 12 \in L$	if $t_i = 1$ for all $i > 2$ and $o(\infty) \geq 2$

Then, the algorithm is as follows.

Step (1)

Form parts of d and θ.

(a) $e_{fix} = \frac{1}{4}\min(o(\infty),2) - \frac{1}{4}\deg t - \frac{3}{4}\deg t_1$

$$\theta_{fix} = \frac{\frac{1}{4}t'}{t} + \frac{\frac{3}{4}t_1'}{t_1}$$

(b) find the poles $c_1, \ldots, c_{k_2}$ of r of order 2 (i.e. the roots of t_2)

for i from 1 to k_2 do

b_i = the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r

$e_i = \sqrt{1+4b_i}$

$\theta_i = \frac{e_i}{x-c_i}$

od

(c) if $1 \in L$ then find the poles $c_{k_2+1}, \ldots, c_k$ of order $4, 6, 8, \ldots, m$ (i.e. the roots of $t_4, t_6, \ldots, t_m$)

for i from k_2+1 to k do

$[\sqrt{r}]_{c_i}$ = the sum of terms involving $\frac{1}{(x-c_i)^k}$, for $k = 2, \ldots, \frac{\nu}{2}$ and ν the order of the pole c_i, in the Laurent series expansion of $\sqrt{r}$ at c_i

a_i = the coefficient of $(x-c)^{\frac{-\nu}{2}}$ in $[\sqrt{r}]_{c_i}$

b_i = the coefficient of $(x-c_i)^{-\frac{\nu}{2}+1}$ in $r - ([\sqrt{r}]_{c_i})^2$

$e_i = \frac{b_i}{a_i}$

$\theta_i = 2[\sqrt{r}]_{c_i} + \frac{e_i}{x-c_i}$

od

(d) if $o(\infty) > 2$ then

$e_0 = 1$

$\theta_0 = 0$

elsif $o(\infty) = 2$ then

$b_0 =$ the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞

$e_0 = \sqrt{1+4b_0}$

$\theta_0 = 0$

else

if $1 \in L$ then

$[\sqrt{r}]_\infty =$ the sum of terms involving x^i, for $i = \frac{-\nu}{2}, \ldots, 0$ and $\nu = o(\infty)$, in the Laurent series expansion of $\sqrt{r}$ at ∞

$a_0 =$ the coefficient of $x^{\frac{-\nu}{2}}$ in $[\sqrt{r}]_\infty$

$b_0 =$ the coefficient of $x^{\frac{-\nu}{2}+1}$ in $r - ([\sqrt{r}]_\infty)^2$

$e_0 = \frac{b_0}{a_0}$

$\theta_0 = 2[\sqrt{r}]_\infty$

else

$e_0 = 0$

$\theta_0 = 0$

fi

fi

Step (2)

Form the trial d's and θ's.

for each n in L (in increasing order) do

if $n = 1$ then $m = k$ else $m = k_2$ fi

if $n = 2$ and $o(\infty) < 2$ then

$e_0 = 0$

$\theta_0 = 0$

fi

for all sequences $s = (s_0, \ldots, s_m)$ where $s_i \in \{-\frac{n}{2}, -\frac{n}{2}+1, \ldots, \frac{n}{2}\}$ do

$d = n \cdot e_{fix} + s_0 e_0 - \sum_{i=1}^{m} s_i e_i$

if d is an integer ≥ 0 then

$\theta = n \cdot \theta_{fix} + \sum_{i=0}^{m} s_i \theta_i$

apply step (3) to d and θ

if successful then

RETURN(solution)

fi

fi

od

od

FAIL(); (no solution exists)

Step (3)

Find the polynomial P (if possible) and ω.

form P in terms of undetermined coefficients a_i

$P = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$

generate the recursive relations P_i

$P_n = -P$

for i from n by -1 to 0 do

$P_{i-1} = -P_i' - \theta P_i - (n-i)(i+1) r P_{i+1}$

od

solve $P_{-1} = 0$ for the a_i

if a solution exists then

generate the minimal polynomial for ω

minpoly = 0

for i from 0 to n do

$$\text{minpoly} = \text{minpoly} + \frac{P_i}{(n-i)!} \cdot \omega^i$$

od

solve minpoly for ω

RETURN(solution as $e^{\int \omega}$)

e.se

not successful

fi

The following corrections were made to Saunders' original algorithm. The expression for d in step (2) was $d = n \cdot e_{fix} - \sum_{i=0}^{m} s_i e_i$. Unless one multiplied the expressions for e_0 by -1, this expression was incorrect. The corrected expression is more desirable simply because it corresponds more closely to the expression for d in Kovacic's algorithm. (See step (2) in section 3.1.)

Also, it was stated that e_0 and θ_0 were only computed if $o(\infty)$ was ≤ 2, and were never computed if 1 was not in the list L, i.e. if case (1) was not possible. This will be seen to be untrue in the following section.

4.2. Proof of Saunders' (Corrected) Algorithm

In order to verify this algorithm, we will require several identities. If $t = t_1 \cdot t_2^2 \cdot t_3^3 \cdots t_m^m$, with the t_i square-free and pair-wise relatively prime, then

$$\deg t = \deg t_1 + 2 \cdot \deg t_2 + 3 \cdot \deg t_3 + \cdots + m \cdot \deg t_m$$

and

$$\deg t_k = \text{the number of poles of } r \text{ of order } k$$

Also,

$$\frac{t'}{t} = \frac{t_1'}{t_1} + \frac{2t_2'}{t_2} + \cdots + \frac{m \cdot t_m'}{t_m}$$

and

$$\frac{t_k'}{t_k} = \sum_{j=1}^{\deg t_k} \frac{1}{x - a_j}$$

where the a_j , $j = 1, \ldots, \deg t_k$ are the roots of t_k, i.e. the poles of r of order k.

Now, we will verify the corrected version of Saunders' algorithm for each of the three cases of Kovacic's algorithm, i.e. $n = 1$, $n = 2$ and $n = 4$, 6 or 12. In all cases, the verification consists of proving that the d's and θ's computed agree for the two algorithms. The d's and θ's computed by Saunders' algorithm will be denoted d_s and θ_s; those computed by Kovacic's algorithm will be denoted by d_k and θ_k.

Case (1)

$n = 1$

$$d_s = 1 \cdot e_{fix} + s_0 e_0 - \sum_{i=1}^{k} s_i e_i$$

$$= \frac{1}{4}\min(o(\infty),2) - \frac{1}{4}\deg t - \frac{3}{4}\deg t_1 + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i - \sum_{i=k_2+1}^{k} s_i e_i$$

$$= \frac{1}{4}\min(o(\infty),2) - \deg t_1 - \frac{1}{2}\deg t_2 - \frac{1}{4}\sum_{\substack{\text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \nu\cdot\deg t_\nu + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i - \sum_{i=k_2+1}^{k} s_i e_i$$

$$= \frac{1}{4}\min(o(\infty),2) + s_0 e_0 - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order 1}}} 1 - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order 2}}} \frac{1}{2} - \sum_{i=1}^{k_2} s_i e_i - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \frac{1}{4}\nu - \sum_{i=k_2+1}^{k} s_i e_i$$

$$= \frac{1}{4}\min(o(\infty),2) + s_0 e_0 - \sum_{\substack{\text{poles of} \\ \text{order 1}}} 1 - \sum_{\substack{\text{poles of} \\ \text{order 2}}} \left(\frac{1}{2} + s_i\sqrt{1+4b}\right) - \sum_{\substack{\text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{1}{4}\nu + s_i\frac{b}{a}\right)$$

Compare this formula with the formula for d_k.

$$d_k = e_\infty - \sum_{c\epsilon\Gamma} e_c$$

$$= e_\infty - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order 1}}} 1 - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order 2}}} \left(\frac{1}{2} + k\sqrt{1+4b}\right) - \sum_{\substack{c\epsilon\Gamma \\ \text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{1}{4}\nu + k\frac{b}{a}\right) \quad \text{where } k = \pm\frac{1}{2}$$

The three sums are clearly equal in the two formulas since $s_i = -\frac{n}{2}, \ldots, \frac{n}{2} = \pm\frac{1}{2}$ when $n = 1$. It remains to show that $\frac{1}{4}\min(o(\infty),2) + s_0 e_0$ corresponds to e_∞. If $o(\infty) > 2$ then

$$\frac{1}{4}\min(o(\infty),2) + s_0 e_0 = \frac{1}{4}\cdot 2 + \pm\frac{1}{2}\cdot 1 = \frac{1}{2} \pm \frac{1}{2} = e_\infty$$

and if $o(\infty) = 2$ then

$$\frac{1}{4}\min(o(\infty),2) + s_0 e_0 = \frac{1}{4}\cdot 2 + \pm\frac{1}{2}\cdot\sqrt{1+4b} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b} = e_\infty$$

and if $o(\infty) < 2$ then

$$\frac{1}{4}\min(o(\infty),2) + s_0 e_0 = \frac{1}{4}\cdot\nu + \pm\frac{1}{2}\cdot\frac{b}{a} = \frac{1}{4}\nu \pm \frac{1}{2}\frac{b}{a} = e_\infty$$

so $d_s = d_k$ for case (1).

Now

$$\theta_s = 1\cdot\theta_{fix} + \sum_{i=0}^{k} s_i\theta_i$$

$$= \frac{1}{4}\frac{t'}{t} + \frac{\frac{3}{4}t_1'}{t_1} + s_0\theta_0 + \sum_{i=1}^{k_2} s_i\theta_i + \sum_{i=k_2+1}^{k} s_i\theta_i$$

$$= \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} \frac{1}{x-c_i} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} \frac{\frac{1}{2}}{x-c_i} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \frac{\frac{1}{4}\nu}{x-c_i} + s_0\theta_0 + \sum_{i=1}^{k_2} s_i\theta_i + \sum_{i=k_2+1}^{k} s_i\theta_i$$

$$= \sum_{\substack{\text{poles of} \\ \text{order 1}}} \frac{1}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order 2}}} \frac{\frac{1}{2}+s_ie_i}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{\frac{1}{4}\nu+s_ie_i}{x-c_i} + 2s_i[\sqrt{r}]_{c_i} \right) + s_0\theta_0$$

$$= \sum_{\substack{\text{poles of} \\ \text{order 1}}} \frac{1}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order 2}}} \frac{\frac{1}{2}+s_i\sqrt{1+4b}}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{\frac{1}{4}\nu+s_i\frac{b}{a}}{x-c_i} + 2s_i[\sqrt{r}]_{c_i} \right) + s_0\theta_0$$

Since $s_i = -\frac{n}{2}, \ldots, \frac{n}{2} = \pm\frac{1}{2}$

$$\theta_s = \sum_{\substack{\text{poles of} \\ \text{order 1}}} \frac{1}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order 2}}} \frac{\frac{1}{2}+s_i\sqrt{1+4b}}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{\frac{1}{4}\nu+s_i\frac{b}{a}}{x-c_i} + \text{sign}(s_i)\cdot[\sqrt{r}]_{c_i} \right) + s_0\theta_0$$

Recall that the formula for θ_k is

$$\theta_k = \sum_{c\in\Gamma} \left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c \right) + s(\infty)[\sqrt{r}]_\infty$$

$$= \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} \frac{1}{x-c} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} \frac{\frac{1}{2} + k\sqrt{1+4b}}{x-c} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2 \\ \nu \text{ even}}} \left(\frac{\frac{1}{4}\nu+k\frac{b}{a}}{x-c} + s(c)[\sqrt{r}]_c \right) + s(\infty)[\sqrt{r}]_\infty$$

Clearly, the three sums are equal in the two formulas. It remains to show that $s_0\theta_0$ corresponds to $s(\infty)[\sqrt{r}]_\infty$. If $o(\infty) \geq 2$ then

$$s_0\theta_0 = \pm\frac{1}{2}\cdot 0 = 0$$

and if $o(\infty) < 2$ then

$$s_0\theta_0 = \pm\frac{1}{2}\cdot 2[\sqrt{r}]_\infty = s(\infty)[\sqrt{r}]_\infty$$

so $\theta_s = \theta_k$ for case (1) and Saunders' algorithm is verified correct for $n = 1$.

Case (2)

$n = 2$

$$d_s = 2\cdot e_{fix} + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

$$= \frac{1}{2}\min(o(\infty),2) - \frac{1}{2}\deg t - \frac{3}{2}\deg t_1 + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

$$= \frac{1}{2}\min(o(\infty),2) - 2\deg t_1 - \deg t_2 - \frac{1}{2}\sum_{\nu>2} \nu \deg t_\nu + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

$$= \frac{1}{2}\min(o(\infty),2) + s_0 e_0 - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} 2 - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} 1 - \sum_{i=1}^{k_2} s_i e_i - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2}} \frac{1}{2}\nu$$

$$= \frac{1}{2}\min(o(\infty),2) + s_0 e_0 - \sum_{\substack{\text{poles of} \\ \text{order 1}}} 2 - \sum_{\substack{\text{poles of} \\ \text{order 2}}} 1+s_i\sqrt{1+4b} - \sum_{\substack{\text{poles of} \\ \text{order } \nu>2}} \frac{1}{2}\nu$$

$$= \frac{1}{2}\left(\min(o(\infty),2) + 2s_0 e_0 - \sum_{\substack{\text{poles of} \\ \text{order 1}}} 4 - \sum_{\substack{\text{poles of} \\ \text{order 2}}} 2+2s_i\sqrt{1+4b} - \sum_{\substack{\text{poles of} \\ \text{order } \nu>2}} \nu\right)$$

Recall that the formula for d_k is

$$d_k = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

$$= \frac{1}{2}\left(e_\infty - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} 4 - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} 2+2k\sqrt{1+4b} - \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2}} \nu\right)$$

The three sums are clearly equal in the two formulas since $s_i = -\frac{n}{2}, \ldots, \frac{n}{2} = -1,0,1 = 0,\pm 1$ when $n = 2$. It remains to show that $\min(o(\infty),2) + 2s_0 e_0$ corresponds to e_∞. If $o(\infty) > 2$ then

$$\min(o(\infty),2) + 2s_0 e_0 = 2 + 2\cdot s_0\cdot 1 = 2 + 2s_0 = e_\infty$$

and if $o(\infty) = 2$ then

$$\min(o(\infty),2) + 2s_0 e_0 = 2 + 2s_0\sqrt{1+4b} = e_\infty$$

and if $o(\infty) = \nu < 2$ then

$$\min(o(\infty),2) + 2s_0 e_0 = \nu + 2\cdot s_0\cdot 0 = \nu = e_\infty$$

so that $d_s = d_k$ for case (2).

Now

$$\theta_s = 2\cdot\theta_{fix} + \sum_{i=0}^{k_2} s_i\theta_i$$

$$= \frac{\frac{1}{2}t'}{t} + \frac{\frac{3}{2}t_1'}{t_1} + s_0\theta_0 + \sum_{i=1}^{k_2} s_i\theta_i$$

$$= \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} \frac{2}{x-c_i} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} \frac{1}{x-c_i} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2}} \frac{\frac{1}{2}\nu}{x-c_i} + s_0\theta_0 + \sum_{i=1}^{k_2} \frac{s_ie_i}{x-c_i}$$

$$= \sum_{\substack{\text{poles of} \\ \text{order 1}}} \frac{2}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order 2}}} \frac{1+s_i\sqrt{1+4b}}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order } \nu>2}} \frac{\frac{1}{2}\nu}{x-c_i} + s_0\theta_0$$

$$= \frac{1}{2}\left(\sum_{\substack{\text{poles of} \\ \text{order 1}}} \frac{4}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order 2}}} \frac{2+2s_i\sqrt{1+4b}}{x-c_i} + \sum_{\substack{\text{poles of} \\ \text{order } \nu>2}} \frac{\nu}{x-c_i} + 2s_0\theta_0 \right)$$

Recall that the formula for θ_k is

$$\theta_k = \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c}$$

$$= \frac{1}{2}\left(\sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 1}}} \frac{4}{x-c} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order 2}}} \frac{2+2k\sqrt{1+4b}}{x-c} + \sum_{\substack{c\in\Gamma \\ \text{poles of} \\ \text{order } \nu>2}} \frac{\nu}{x-c} \right)$$

Clearly the three sums are equal in the two formulas. Since $\theta_0 = 0$ in this case regardless of the value of $o(\infty)$, we have $\theta_s = \theta_k$ for case (2) and Saunders' algorithm is verified correct for $n = 2$.

Case (3)
$n = 4,6,12$

$$d_s = n\cdot e_{fix} + s_0e_0 - \sum_{i=1}^{k_2} s_ie_i$$

$$= \frac{n}{4}\min(o(\infty),2) - \frac{n}{4}\deg t - \frac{3n}{4}\deg t_1 + s_0e_0 - \sum_{i=1}^{k_2} s_ie_i$$

$$= \frac{n}{4}\min(o(\infty),2) - n\deg t_1 - \frac{n}{2}\deg t_2 + s_0e_0 - \sum_{i=1}^{k_2} s_ie_i$$

$$= \frac{n}{4}\min(o(\infty),2) + s_0e_0 - \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 1}}} n - \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 2}}} \frac{n}{2} - \sum_{i=1}^{k_2} s_ie_i$$

$$= \frac{n}{12}\left(3\cdot\min(o(\infty),2) + \frac{12}{n}s_0e_0 - \sum_{\substack{\text{poles of}\\ \text{order 1}}} 12 - \sum_{\substack{\text{poles of}\\ \text{order 2}}} 6+\frac{12}{n}s_i\sqrt{1+4b}\right)$$

Compare this formula with the formula for d_k.

$$d_k = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

$$= \frac{n}{12}\left(e_\infty - \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 1}}} 12 - \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 2}}} 6+\frac{12}{n}k\sqrt{1+4b}\right)$$

The three sums are clearly equal in the two formulas since $s_i = -\frac{n}{2}, \ldots, \frac{n}{2} = 0, \pm 1, \pm 2, \ldots, \pm\frac{n}{2}$ when $n = 4$, 6 or 12. It remains to show that $3\cdot\min(o(\infty),2) + \frac{12}{n}s_0e_0$ corresponds to e_∞. If $o(\infty) > 2$ then

$$3\cdot\min(o(\infty),2) + \frac{12}{n}s_0e_0 = 3\cdot 2 + \frac{12}{n}\cdot s_0\cdot 1 = 6 + \frac{12}{n}s_0 = e_\infty$$

and if $o(\infty) = 2$ then

$$3\cdot\min(o(\infty),2) + \frac{12}{n}s_0e_0 = 3\cdot 2 + \frac{12}{n}\cdot s_0\sqrt{1+4b} = 6 + \frac{12}{n}s_0\sqrt{1+4b} = e_\infty$$

and $o(\infty) < 2$ does not occur in case (3).

Now

$$\theta_s = n\cdot\theta_{fix} + \sum_{i=0}^{k_2} s_i\theta_i$$

$$= \frac{\frac{n}{4}t'}{t} + \frac{\frac{3n}{4}t_1'}{t_1} + s_0\theta_0 + \sum_{i=0}^{k_2} s_i\theta_i$$

$$= \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 1}}} \frac{n}{x-c_i} + \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 2}}} \frac{\frac{n}{2}}{x-c_i} + s_0\theta_0 + \sum_{i=0}^{k_2} s_i\theta_i$$

$$= \sum_{\substack{\text{poles of}\\ \text{order 1}}} \frac{n}{x-c_i} + \sum_{\substack{\text{poles of}\\ \text{order 2}}} \frac{\frac{n}{2}+s_ie_i}{x-c_i} + s_0\theta_0$$

$$= \sum_{\substack{\text{poles of}\\ \text{order 1}}} \frac{n}{x-c_i} + \sum_{\substack{\text{poles of}\\ \text{order 2}}} \frac{\frac{n}{2}+s_i\sqrt{1+4b}}{x-c_i} + s_0\theta_0$$

$$= \frac{n}{12}\left(\sum_{\substack{\text{poles of}\\ \text{order 1}}} \frac{12}{x-c_i} + \sum_{\substack{\text{poles of}\\ \text{order 2}}} \frac{6+\frac{12}{n}s_i\sqrt{1+4b}}{x-c_i} + \frac{12}{n}s_0\theta_0 \right)$$

Recall that the formula for θ_k is

$$\theta_k = \frac{n}{12}\sum_{c\in\Gamma} \frac{e_c}{x-c}$$

$$= \frac{n}{12}\left(\sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 1}}} \frac{12}{x-c} + \sum_{\substack{c\in\Gamma\\ \text{poles of}\\ \text{order 2}}} \frac{6+\frac{12}{n}k\sqrt{1+4b}}{x-c} \right)$$

Clearly, the sums are equal in the two formulas. Since $\theta_0 = 0$ for any value of $o(\infty)$, then $\theta_s = \theta_k$ for case (3) and Saunders' algorithm is verified correct for $n = 4,6,12$.

Hence, the corrected version of Saunders' algorithm is verified correct as presented.

CHAPTER 5

Implementation in Maple

5.1. Details of the Implementation

The implementation in Maple follows Saunders' variant of the algorithm fairly closely. Several subsections of the implementation will be discussed further here.

The square-free decomposition required in the preliminary step of Saunders' algorithm is done using Yun's algorithm (c) as follows. If P is a primitive polynomial, its square-free decomposition $P_1 \cdot P_2^2 \cdot P_3^3 \cdots P_m^m$ is computed by:

$$G \leftarrow \gcd\left(P, \frac{dP}{dx}\right)$$

$$C_1 \leftarrow \frac{P}{G}$$

$$D_1 \leftarrow \frac{dP}{dx}/G - \frac{dC_1}{dx}$$

for i from 1 while $C_1 \neq 1$ do

$$P_i \leftarrow \gcd(C_i, D_i)$$

$$C_{i+1} \leftarrow \frac{C_i}{P_i}$$

$$D_{i+1} \leftarrow \frac{D_i}{P_i} - \frac{dC_{i+1}}{dx}$$

od

A proof that this algorithm does calculate the square-free decomposition of P may be found in [7].

The procedure to find the roots of the parts $t_2, t_4, \ldots, t_m$ was developed to circumvent intrinsic limitations in Maple's *solve* routine. (See section 5.2 for further details.) First, Maple's *solve* is called to determine the roots. If *solve* finds all the roots (the number of roots should be the degree of the input polynomial), then *radsimp* is called to simplify them. Otherwise, *factor* is called in an attempt to reduce the polynomial to factors that can be handled by *solve*. If *factor* cannot produce any factors, the differential equation cannot be solved and the routine fails.

In step 1 of Saunders' algorithm, the calculation of the d_i and of $[\sqrt{r}]_{c_i}$ requires computation of coefficients of the Laurent series expansion of a rational function at its poles. The function is never explicitly expanded in a Laurent series; instead the method of undetermined coefficients is used. This also requires long division of the rational function to reduce $\frac{s}{t}$ to $squo + \frac{srem}{t}$ where $\deg(srem) < \deg(t)$. The algorithm used for division is algorithm D by Knuth [4].

Then, if a is a root of t with multiplicity m, $t(x) = (x-a)^m \cdot g(x)$ and f is a polynomial in x

$$\frac{srem}{t} = \frac{srem}{(x-a)^m \cdot g} = \frac{A_m}{(x-a)^m} + \frac{A_{m-1}}{(x-a)^{m-1}} + \cdots + \frac{A_1}{x-a} + \frac{f}{g}$$

where

$$A_i = \frac{1}{(m-i)!} \left. \frac{d^{m-i}}{dx^{m-i}} \left(\frac{srem}{g} \right) \right|_{x=a}$$

This algorithm is from the CRC Standard Mathematical Tables [6].

The calculation of $[\sqrt{r}]_{c_i}$ also uses this algorithm for undetermined coefficients. If c_i is a pole of order $\nu = 2\tau \geq 4$, then $[\sqrt{r}]_{c_i}$ is the sum of terms involving $\frac{1}{(x-c_i)^i}$ for $i = 2,3, \ldots, \tau$ in the Laurent series expansion of $\sqrt{r}$ at c_i. So

$$[\sqrt{r}]_{c_i} = \frac{a_\tau}{(x-c_i)^\tau} + \frac{a_{\tau-1}}{(x-c_i)^{\tau-1}} + \cdots + \frac{a_2}{(x-c_i)^2}$$

and $([\sqrt{r}]_{c_i})^2$ agrees with $(\sqrt{r})^2 = r$ for powers of $\frac{1}{(x-c_i)^k}$, $k = \tau+2, \ldots, 2\tau$. We exploit this fact by squaring $[\sqrt{r}]_{c_i}$ and obtaining expressions equal to the coefficients of $\frac{1}{(x-c_i)^k}$ in r, then using the undetermined coefficients algorithm on r.

$$([\sqrt{r}]_{c_i})^2 = \left(\frac{a_\tau}{(x-c_i)^\tau} + \frac{a_{\tau-1}}{(x-c_i)^{\tau-1}} + \cdots + \frac{a_2}{(x-c_i)^2} \right) = \sum_{i=4}^{\nu} \left[\frac{1}{(x-c_i)^i} \cdot \left(\sum_{j=i-\tau}^{\tau} a_j \cdot a_{i-j} \right) \right]$$

We first solve for a_τ by noting

$$a_\tau^2 = \text{coefficient of } \frac{1}{(x-c_i)^{2\tau}} \text{ in } r$$

Then each succeeding coefficient, $a_{\tau-1}, a_{\tau-2}, \ldots, a_2$ can be solved for using the equation $\sum_{j=i-\tau}^{\tau} a_j \cdot a_{i-j} =$ coefficient of $\frac{1}{(x-c_i)^i}$ in r and substitution of previously computed a's.

The calculation of $[\sqrt{r}]_\infty$ is very similar. If the order of r at ∞ is $\nu = -2\tau$, then $[\sqrt{r}]_\infty$ is the sum of terms involving x_i, $i = 0, \ldots, \tau$ in the Laurent series expansion of $\sqrt{r}$ at ∞, i.e.

$$[\sqrt{r}]_\infty = a_\tau x^\tau + \cdots + a_0$$

and $([\sqrt{r}]_\infty)^2$ agrees with r for powers of x^k, $k = 0, \ldots, 2\tau$. A formula can be derived for the coefficients as before and matched with coefficients of powers of x in *squo*.

In step (2) of Saunders' algorithm, it is necessary to generate all possible sequences $(s_0, s_1, \ldots, s_m)$ where each $s_i \in \{0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\}$. The quantities s_i are implemented as a one-dimensional table of rational numbers and the entire vector treated as a $m+1$ digit number base $n+1$ in order to generate all the combinations. If the vector s is initialised to

$$\begin{matrix} s_m & s_{m-1} & \cdots & s_0 \\ -\frac{n}{2} & -\frac{n}{2} & & -\frac{n}{2} \end{matrix}$$

then repeatedly adding 1 until s is

$$\begin{matrix} s_m & s_{m-1} & \cdots & s_0 \\ \frac{n}{2} & \frac{n}{2} & & \frac{n}{2} \end{matrix}$$

will produce all the combinations required.

As in Saunders' implementation, step 3 of the algorithm is implemented in three separate procedures for $n = 1$, $n = 2$, and $n = 4, 6, 12$. The three routines are very similar. Each generates a polynomial p of degree d as $x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ and then generates an expression as a function of p and the given θ that must be zero. By computing the numerator of the generated expression (using *normal* and *numerator*), then extracting the coefficients of each power of x, a set of expressions is found, each of which must be zero. These equations are linear functions of the unknown coefficients a_i and can be solved using a linear equation solver (courtesy of M. B. Monagan). This routine is noteworthy in that it can handle the case where the system of equations is overdetermined.

5.2. Limitations of the Implementation

While the algorithm as stated claims to solve any d.e. of the form $ay'' + by' + cy = 0$ where $a, b, c \in \mathbf{C}(x)$, the implemented algorithm will handle only a subset of these equations for a number of reasons.

In the preliminary step of the implemented algorithm, it is necessary to compute r as $\frac{s}{t}$ where $\gcd(s,t) = 1$. As long as $r \in \mathbf{Q}(x)$, Maple's *normal* will correctly simplify r to the form required. At present, however, it will not accept constants such as $\sqrt{2}$. In this case, calling radsimp will correctly normalise but radsimp will not correctly handle any more complicated constants, such as exp(2) for example.

It is clear that more powerful normalisation facilities are needed in Maple to handle the non-rational coefficients.

In the first step of the implemented algorithm, it is necessary to find the roots of a polynomial in $\mathbf{C}[x]$. In reality, the polynomial must be in $\mathbf{Q}[x]$ for several reasons. First, the roots computed by solve are simplified using radsimp which can handle rationals and rationals to rational powers, and no other constants. Then, if solve is unable to find the roots of the polynomial, Maple's factor package is called and it can only factor over the integers (actually over the Gaussian integers if (-1)**(1/2) is represented as I).

The use of the factor package could be a limitation in and of itself in that it is very slow and relatively untested. Fortunately, for most of the examples tested (and hopefully most of the examples to be tried in the future), the polynomial we require the roots of is of low degree, generally not more than degree 4 and can be adequately handled by solve and radsimp.

Another problem with Maple's solve function manifested itself when implementing step 3 of the algorithm. In that stage, a polynomial is constructed with undetermined coefficients and an expression depending on that polynomial is set to zero. The unknown coefficients are determined by expanding the expression and extracting the coefficients of each power of the dependent variable. This often results in an over-determined system of equations. Maple's solve routine will only handle a square system of equations, i.e. n equations in n unknowns. However, M. B. Monagan was able to provide a linear equation solver that handles both under- and over- determined systems of equations as well as square systems. This routine (or a variant of it) should be provided in the Maple library.

Two problems arise in the final stages of the implemented algorithm. In step 3 in all 3 cases, the solution is returned as $e^{\int \text{something}}$. Then in the course of transforming the equation from the form $y'' = ry$ back to the input form $az'' + bz' + cz = 0$, another $e^{\int \text{something}}$ is computed. In the course of testing, the limitations of Maple's integrator were clearly demonstrated at this point. It is a well-known fact that much more work is needed on this function and I will not belabour the point.

In the cases where the integral could be computed, the second problem became evident. If the solution to the d.e. is a polynomial, it will often be computed as $e^{\log(polynomial)}$ and since Maple had no knowledge of the relationship between exp and log, it would remain in that form in the output solution. This form could hardly be considered simplified and it seemed clear that any user of the package would not appreciate such a solution. Fortunately, B. Char produced a simplifier for expressions with exp's and log's. This system of routines knows the basic rules of exponentials and logarithms, namely $e^a \cdot e^b = c^{a+b}$ and $\log x + \log y = \log xy$, and also knows that exp and log are related by $\exp(\log(\cdots)) = \cdots$. (It does not yet know that $\log(\exp(\cdots)) = \cdots$; hopefully, this can also be implemented.)

5.3. Some Further Observations

The implemented algorithm was tested extensively using the equations in Kamke [1]. During this testing another problem was noted, namely that the algorithm will not handle parameterised equations, i.e. if the input equation is $ay'' + by' + cy = 0$, then a, b and c must be polynomials in x and no other parameters. In step (2), a decision must be made as to whether the quantity d is greater than or equal to zero and is an integer. If the expression for d is parameterised and the question was simply, is $d \geq 0$, we could proceed by following the two cases, $d < 0$ and $d \geq 0$, through the remainder of the code and outputting the conditions on d along with the solutions computed in each case. However, the question, is d an integer, does not lend itself to handling a finite number of cases. A true constant must be determined for d.

In step (3) of the algorithm for case (3) only, an equation for ω is computed of degree 4, 6 or 12. It may be impossible to produce an explicit expression for one of the solutions of this equation if it is of degree 6 or 12. In that case, we return an unevaluated integral, $e^{\int \omega}$, in the solution and a condition on ω, namely the equation we were unable to solve.

These two cases demonstrate a clear need for Maple (or any other system) to be able to handle and simplify expressions with side relations. The system must have certain basic side relations built in, e.g. $i^2 + 1 = 0$, $\sin^2 x + \cos^2 x = 1$, etc. The user must also be able to add side relations to the knowledge of the system, whether directly or indirectly, e.g. via conditions on a solution to a differential equation.

In the preliminary step of the algorithm, it is necessary to perform a square-free factorisation of a polynomial. Square-free factorisation is also required by Maple's radsimp routine, (undoubtedly) by the factor package, and by the Risch integration package (as yet, not in the Maple library). It is probably also used by other routines as well. It is recommended that a good square-free factorisation routine (for both univariate and multivariate polynomials) be added to the Maple library. The four routines mentioned above all use a slightly different square-free factorisation routine, though they may all be using the same algorithm.

The determination of the coefficients of the partial fraction expansion of a r in step (1) of the algorithm requires a routine to do polynomial long division. Maple's divide routine divides only if the input polynomials divide exactly. It is suggested that a library routine to do quotient-remainder division might be useful to a number of packages.

APPENDIX A

Source Code

```
read `/u/csmith/cs787/radsimp.m`;
read `/u/csmith/essay/lsolve.m`;
read `/u/bwchar/mathware/symbolic/maple/functioncall`;
read `/u/bwchar/mathware/symbolic/maple/lncontract`;
read `/u/bwchar/mathware/symbolic/maple/scanmap`;

#
#
#--> osolve:  Solve second order ordinary differential equations
#
#    Calling sequence:  osolve(in_ode,dep,indep)
#
#    Purpose:  Solves second order ordinary differential equations
#              of the form a * y'' + b * y' + c * y = 0 using
#              Kovacic's algorithm for second order linear
#              homogeneous equations
#
#    Input:  in_ode - either an equation or an expression assumed
#                     equal to zero representing the differential
#                     equation to be solved
#
#            dep - the dependent variable to be solved for, given
#                  as an undefined function e.g. y(x)
#
#            indep - the independent variable
#
#    Output:  function value -- a set of two independent solutions
#                               of the o.d.e. if they can be found
#
#    Functions required:  odeorder,kovode
#
#

osolve := proc(in_ode,dep,indep)

  local ode,a,b,c,d,i;

  if type(in_ode,`=`) then
    ode := expand(op(1,in_ode)-op(2,in_ode));
```

```
else
  ode := expand(in_ode);
fi;

if odeorder(ode,dep,indep) <> 2 then
  ERROR(‘not a second order o.d.e‘);
fi;

a := 0; b := 0; c := 0; d := 0;

if type(ode,‘+‘) then
  for i from 1 to nops(ode) do
    op(i,ode);
    if has(“,diff(diff(dep,indep),indep)) then
      a := a + “ / diff(diff(dep,indep),indep);
    elif has(“,diff(dep,indep)) then
      b := b + “ / diff(dep,indep);
    elif has(“,dep) then
      c := c + “ / dep;
    else
      d := d + “;
    fi;
  od;
else
  if has(ode,diff(diff(dep,indep),indep)) then
    a := ode / diff(diff(dep,indep),indep);
  elif has(ode,diff(dep,indep)) then
    b := ode / diff(dep,indep);
  elif has(ode,dep) then
    c := ode / dep;
  else
    d := ode;
  fi;
fi;

if (indets(a) + indets(b) + indets(c)) - {indep} <> {} then
  ERROR(‘invalid coefficients‘);
fi;
```

```
  if d <> 0 then
    print('WARNING: non-homogeneous, trying homogeneous case');
  fi;

  kovode(a,b,c,indep);

end;
```

```
#
#
#--> odeorder:  Determines the order of an ordinary differential equation
#
#    Calling sequence:  odeorder(ode,dep,indep)
#
#    Purpose:  Determines the order of an o.d.e., i.e. the highest
#              derivative of the dependent variable with respect to
#              the independent variable.
#
#    Input:  ode - an equation representing the o.d.e.
#
#            dep - the dependent variable, e.g. y(x)
#
#            indep - the independent variable, e.g. x
#
#    Output:  function value -- integer order of the o.d.e., -1 if the
#                               input equation is not an o.d.e.
#
#    Functions required:
#
#

odeorder := proc(ode,dep,indep)

  if type(ode,function) and (op(0,ode) = 'diff') then
    if op(2,ode) <> indep then
      RETURN(-1);
    elif op(1,ode) = dep then
      RETURN(1);
    else
      odeorder(op(1,ode),dep,indep);
      if " = -1 then
        RETURN(-1);
      else
        RETURN("+1);
      fi;
    fi;
  elif type(ode,`+`) or type(ode,`*`) then
```

```
    map(odeorder,{op(ode)},dep,indep);
    RETURN(max(op(")));
  elif type(ode,'**') then
    RETURN(odeorder(op(1,ode),dep,indep));
  else
    RETURN(-1);
  fi;

end;
```

```
#
#
#--> kovode:  Kovacic's algorithm for second order o.d.e.'s
#
#     Calling sequence:  kovode(fa,fb,fc,var)
#
#     Purpose:  Solve a second order linear homoegeneous differential
#               equation of the form fa * y'' + fb * y' + fc * y = 0
#
#     Input:  fa,fb,fc - coefficients in the differential equation,
#                    must be in Q(x)
#
#     Output:  function value -- set of two independent solutions of
#                         the o.d.e. and possibly an equation
#                         that is a condition on the solutions
#
#     Functions required:  normal,numerator,rdivide,sqfr,undetcoeff,
#                     roots,radsimp,solve,step3n1,step3n2,step3n4,
#                     esimp,int
#
#

kovode := proc(fa,fb,fc,var)

  local s,t,squo,srem,tcont,sdec,m,ord_inf,listl,oddti,i,j,k,l,
      d,theta,dfix,thetafix,ds,thetas,t1,t2,trest,rlist2,rlisthigher,
      k1,k2,soln,soln1,soln2,ac,rt,nu,vtemp,n,alls,sprod;

# transform the equation to the form z'' = (s/t)*z

  s := 2*diff(fb,var)*fa - 2*fb*diff(fa,var) + fb*fb - 4*fa*fc;
  t := 4*fa*fa;

# step 0 - preliminaries

  normal(s/t);
  s := numerator(",'t');

  rdivide(s,t,var,'squo','srem');
```

```
sdec := sqfr(t,var,'tcont');
m := nops(sdec);
t := tcont;
for i from 1 to m do
  t := t * op(i,sdec)**i;
od;
if m > 0 then
  t1 := op(1,sdec);
else
  t1 := 1;
fi;
if m > 1 then
  t2 := op(2,sdec);
else
  t2 := 1;
fi;

ord_inf := degree(t,var) - degree(s,var);

listl := [];

oddti := true;
for i from 3 by 2 to m do
  if op(i,sdec) <> 1 then
    oddti := false;
    break;
  fi;
od;

if oddti and (type(ord_inf/2,integer) or (ord_inf > 2)) then
  listl := [op(listl),1];
fi;

if not oddti or (t2 <> 1) then
  listl := [op(listl),2];
fi;

if (m <= 2) and (ord_inf >= 2) then
  listl := [op(listl),4,6,12];
```

```
  fi;

  if nops(listl) = 0 then
    FAIL();
  fi;

# step 1 - form parts for d and theta

  dfix := (min(ord_inf,2) - degree(t,var) - 3 * degree(t1,var)) / 4;
  thetafix := normal((diff(t,var) / t + 3 * diff(t1,var) / t1) / 4);

  rlist2 := roots(t2,var);

  t2 := t2 / lcoeff(t2);

  k2 := nops(rlist2);

  for i from 1 to k2 do
    trest := t / t2**2;
    rt := op(i,rlist2);
    for j from 1 to i-1 do
      trest := trest * (var - op(j,rlist2))**2;
    od;
    for j from i+1 to k2 do
      trest := trest * (var - op(j,rlist2))**2;
    od;
    undetcoeff(srem,trest,var,rt,2,2);
    d[i] := radsimp((1+4*")**(1/2));
    theta[i] := radsimp(d[i]/(var-rt));
  od;

  k1 := k2;
  if member(1,listl) then
    for i from 4 by 2 to m do
      op(i,sdec);
      rlisthigher := roots(",var);
      sdec := [op(1..i-1,sdec),""/lcoeff(""),op(i+1..m,sdec)];
      nu := i/2;
      for j from 1 to nops(rlisthigher) do
```

```
      k1 := k1 + 1;
      rt := op(j,rlisthigher);
      trest := t / op(i,sdec)**i;
      for l from 1 to j-1 do
        trest := trest * (var-op(l,rlisthigher))**i;
      od;
      for l from j+1 to nops(rlisthigher) do
        trest := trest * (var-op(l,rlisthigher))**i;
      od;
      undetcoeff(srem,trest,var,rt,2*nu,2*nu);
      ac[nu] := radsimp("**(1/2));
      for k from nu-1 by -1 to 2 do
        ac[k] := vtemp;
        0;
        for l from nu by -1 to k do
          " + ac[l] * ac[nu+k-l];
        od;
        ac[k] := solve("=undetcoeff(srem,trest,var,rt,2*nu,k+nu),vtemp);
      od;
      0;
      for k from 2 to nu-1 do
        " + ac[k] * ac[nu+1-k];
      od;
      d[k1] := (undetcoeff(srem,trest,var,rt,2*nu,nu+1) - ")/ac[nu];
      0;
      for k from 2 to nu do
        " + ac[k] / (var - rt)**k;
      od;
      theta[k1] := 2 * " + d[k1] / (var - rt);
    od;
  od;
fi;

if ord_inf > 2 then
  d[0] := 1;
  theta[0] := 0;
elif ord_inf = 2 then
  lcoeff(s) / lcoeff(t);
  d[0] := radsimp((1+4*")**(1/2));
```

```
    theta[0] := 0;
  elif member(1,listl) then
    nu := (-ord_inf) / 2;
    ac[nu] := radsimp(coeff(squo,var,2*nu)**(1/2));
    for i from nu-1 by -1 to 0 do
      ac[i] := vtemp;
      0;
      for j from i to nu do
        " + ac[j] * ac[i+nu-j];
      od;
      ac[i] := solve("=coeff(squo,var,i+nu),vtemp);
    od;
    0;
    for l from 0 to nu-1 do
      " + ac[l] * ac[nu-l-1];
    od;
    if nu = 0 then
      coeff(srem,var,degree(t,var)-1)/lcoeff(t) - ";
    else
      coeff(squo,var,nu-1) - ";
    fi;
    if " = 0 then
      d[0] := 0;
    else
      d[0] := " / ac[nu];
    fi;
    0;
    for l from 0 to nu do
      " + ac[l] * var**l;
    od;
    theta[0] := 2*";
  else
    d[0] := 0;
    theta[0] := 0;
  fi;

# step 2 - form trial d's and theta's

  for i from 1 to nops(listl) do
```

```
n := op(i,listl);
if n = 1 then
  m := k1;
else
  m := k2;
fi;

if (n = 2) and (ord_inf < 2) then
  d[0] := 0;
  theta[0] := 0;
fi;

for j from 0 to m do
  sq[j] := -1/2 * n;
od;

alls := false;
while not alls do

  sq[0] * d[0];
  for l from 1 to m do
    " - sq[l] * d[l];
  od;
  ds := radsimp(n * dfix + ");

  if type(ds,integer) and ds >= 0 then
    0;
    for l from 0 to m do
      " + sq[l] * theta[l];
    od;
    thetas := radsimp(n * thetafix + ");

# step 3 - determine polynomial P if possible and hence omega and solution

    soln := step3(n,ds,thetas,s/t,var);

    if op(1,[soln]) <> `@FAIL` then
      fb/fa;
      soln1 := exp(int(-1/2 * ",var)) * op(1,[soln]);
```

```
            soln2 := soln1 *
                  int(esimp(exp(-int("",var))/(soln1*soln1)),var);
            if nops([soln]) = 1 then
              RETURN([esimp(soln1),esimp(soln2)]);
            else
              RETURN([esimp(soln1),esimp(soln2)],op(2,[soln]));
            fi;
          fi;
        fi;

        for j from m by -1 to 0 do
          if sq[j] = (1/2 * n) then
            sq[j] := -1/2 * n;
          else
            sq[j] := sq[j] + 1;
            break;
          fi;
        od;

        if j < 0 then
          alls := true;
        fi;
      od;

    od;

    FAIL();

  end;
```

```
#
#
#--> step3:  Step 3 of Kovacic's algorithm
#
#     Calling sequence:  step3(n,d,theta,rhs,var)
#
#     Purpose:  Perform step 3 of Kovacic's algorithm
#
#     Input:  n -- degree of omega over C(x)
#
#             d -- degree of the polynomial to be constructed
#
#             theta -- trial function theta
#
#             rhs -- right hand side of the o.d.e z'' = r*z
#
#             var -- independant variable of the o.d.e.
#
#     Output:  function value -- solution of the o.d.e. z'' = r*z
#                              namely exp(int(omega))
#
#     Functions required:  radsimp,numerator,Lsolve,int
#
#

step3 := proc(n,d,theta,rhs,var)

  local p,listv,i,a,pr,sete,soln,trial,w;

  p := var**d;
  listv := [];
  for i from d-1 by -1 to 0 do
    a.i := evaln(a.i);
    p := p + a.i * var**i;
    listv := [op(listv),a.i];
  od;

  pr[n] := -p;
  for i from n by -1 to 0 do
```

```
  pr[i-1] := normal(-diff(pr[i],var) - theta * pr[i]
           - (n-i) * (i+1) * rhs * pr[i+1]);
od;

trial := expand(numerator(radsimp(pr[-1])));

if trial <> 0 then
  sete := {};
  for i from ldegree(trial,var) to degree(trial,var) do
    coeff(trial,var,i);
    if " <> 0 then
      sete := sete + {"};
    fi;
  od;

  soln := Lsolve(sete,listv);
  if op(1,[soln]) = [] then
    RETURN(`@FAIL`);
  fi;

  for i from d-1 by -1 to 0 do
    a.i := op(2,op(d-i,soln));
  od;
fi;

trial := 0;
for i from 0 to n do
  trial := trial + pr[i] * w**i / (n-i)!;
od;

[solve(trial,w)];
if " = [] then
  RETURN(exp(int(`@W`(var),var)),subs(w=`@W`,trial)=0);
fi;

w := radsimp(op(1,"));

exp(int(w,var));
```

```
  RETURN(");

end;
```

```
#
#
#--> roots:  Find the roots of a polynomial
#
#     Calling sequence:  roots(poly,var)
#
#     Purpose:  Find the roots of a given polynomial in Z[var]
#
#     Input:  poly -- a univariate polynomial in var with integer
#                     coefficients
#
#             var -- the indeterminate of the polynomial
#
#     Output:  function value -- a list of the radically simplified
#                                roots of the polynomial, an ERROR
#                                if we could not find degree(poly,var)
#                                roots, i.e. all of them
#
#     Functions required:  solve,radsimp,factor
#
#

roots := proc(poly,var)

  local newp,rlist,i;

  if degree(poly,var) = 0 then
    RETURN([]);
  fi;

  soln := [solve(poly,var)];

  if nops(soln) = degree(poly,var) then
    RETURN(map(radsimp,soln));
  fi;

  newp := factor(poly);

  if newp = poly then
```

```
    ERROR(`unable to find roots of the demoninator`);
  fi;

  rlist := [];
  for i from 1 to nops(newp) do
    roots(op(i,newp),var);
    rlist := [op(rlist),op(")];
  od;

  RETURN(rlist);

end;
```

```
#
#
#--> sqfr:  Perform a square-free factorization of a polynomial
#
#      Calling sequence:  sqfr(poly,var,'cont')
#
#      Purpose:  Do a square-free factorization of a univariate polynomial
#
#      Input:  poly -- a univariate polynomial with integer coefficients
#
#              var -- the indeterminate of the polynomial
#
#      Output:  function value -- a list of the form [t1,t2,t3,...,tm]
#                                 where poly = t1 * t2**2 * t3**3 *
#                                 ... * tm**m
#
#              'cont' -- (call-by-name) if the third argument is
#                        present the integer content of poly is assigned
#                        to the name 'cont'
#
#      Functions required:  primpart,gcd
#
#      Reference:  Yun's paper "On Square-Free Decomposition Algorithms"
#
#

sqfr := proc(poly,var,cont)

  local i,signp,pp,tc,tlist1,c,d;

  signp := sign(poly);
  expand(poly) / signp;

  if nargs > 2 then
    pp := primpart(",{var},'tc');
    cont := tc * signp;
  else
    pp := primpart(",{var});
  fi;
```

```
  tlist1 := [];

  expand(diff(pp,var));
  gcd(pp,",'c','d');

  for i from 1 while c <> 1 do
    expand(d - diff(c,var));
    gcd(c,",'c','d');
    tlist1 := [op(tlist1),"];
  od;

  tlist1;

end;
```

```
#
#
#--> undetcoeff:  Determine the coefficient of a factor in a partial
#               fraction expansion
#
#     Calling sequence:  undetcoeff(num,rden,var,root,m,ex);
#
#     Purpose:  Determine the coefficient of 1 / (var - root)**ex
#              in the partial fraction expansion of
#              num / (rden * (var - root)**m)
#
#     Input:  num -- polynomial with rational coefficients as above
#
#            rden -- polynomial with rational coefficients as above
#
#            var -- indeterminate of quotient
#
#            root -- root of the denominator of quotient
#
#            m -- multiplicity of root in denominator
#
#            ex -- particular coefficient required
#
#     Output:  function value -- coefficient as described above
#
#     Functions required:  normal, rsubs (only until 3.1 is released)
#
#     Reference:  CRC Standard Mathematical Tables
#
#

undetcoeff := proc(num,rden,var,root,m,ex)

  local k,p,i;

  k := m - ex;
  p := num / rden;

  for i from 1 to k do
```

```
    p := diff(p,var);
  od;

  p := normal(p);

  RETURN(rsubs(p,var=root)/(k!));

end;
```

```
#
#
#--> rdivide:  Divide one polynomial by another and return quotient and
#           remainder
#
#     Calling sequence:  rdivide(a,b,var,'quo','rem')
#
#     Purpose:  Divides a polynomial "a" by a polynomial "b" and produces
#            the quotient and remainder polynomials
#
#     Input:  a -- univariate polynomial with rational coefficients
#
#           b -- univariate polynomial with rational coefficients
#
#           var -- indeterminate of the two polynomials
#
#
#     Output:  function value -- none (of any relevance)
#
#           'quo' -- (call-by-name) the quotient when a is divided
#                 by b
#
#           'rem' -- (call-by-name) the remainder when a is divided
#                 by b
#
#     Functions required:
#
#     Reference:  Knuth, Volume 1, Algorithm D
#
#

rdivide := proc(a,b,var,quo,rem)

  local m,n,exa,exb,i,j,u,v,q;

  m := degree(a,var);
  n := degree(b,var);
  exa := expand(a);
  exb := expand(b);
```

```
  for i from 0 to m do
    u[i] := coeff(exa,var,i);
  od;
  for i from m+1 to n-1 do
    u[i] := 0;
  od;

  for i from 0 to n do
    v[i] := coeff(exb,var,i);
  od;

  for i from m-n by -1 to 0 do
    q[i] := u[n+i] / v[n];
    for j from n+i-1 by -1 to i do
      u[j] := u[j] - q[i] * v[j-i];
    od;
  od;

  0;
  for i from 0 to n-1 do
    " + u[i] * var**i;
  od;
  rem := ";

  0;
  for i from 0 to m-n do
    " + q[i] * var**i;
  od;
  quo := ";

end;
```

```
#
#
#--> esimp:  Simplify expressions with exponentials and logarithms
#
#     Calling sequence:  esimp(expr)
#
#     Purpose:  Simplify an expression with exp's and log's (ln's)
#               using the standard rules for exp's and log' and
#               the rule exp(log(...)) = ...
#
#     Input:  expr -- expression with exp's and log'
#
#     Output:  function value -- simplified version of expr
#
#     Functions required:  scanmap,expcontract,lncontract,explnsimp
#
#

esimp := proc(expr)

  scanmap(expr,[expcontract,lncontract,explnsimp]);
  RETURN(");

end;

save '/u/csmith/essay/kovode.m';
quit;
```

APPENDIX B

Examples and Tests

An Example of the Use of Kovacic's Algorithm

We will consider the differential equation

$$x^2z'' - 2z = 0$$

Making the transformation

$$y = z \cdot e^{-\int \frac{b}{2a}dx} = z$$

the equation becomes

$$y'' = \frac{2}{x^2}y$$

so that

$$r = \frac{2}{x^2}$$

There is only one pole of r, $c = 0$, and it has order 2, so that $\Gamma = \{2\}$. The order of r at ∞ is degree(x^2) $-$ degree(2) $= 2$.

Checking the necessary conditions (section 2.3), we find that all three cases are possible so we must try the sub-algorithms for all three cases.

First, the algorithm for case (1). The pole $c = 0$ is of order 2 so $E_0 = \{\frac{1}{2} + k\sqrt{1+4b}\}$, $k = \pm\frac{1}{2}$, where $b = 2$ (the coefficient of $\frac{1}{x^2}$in the partial fraction expansion of r at 0), i.e. $E_0 = \{\frac{1}{2} + \frac{1}{2}\sqrt{1+4\cdot 2}, \frac{1}{2} - \frac{1}{2}\sqrt{1+4\cdot 2}\} = \{2, -1\}$.

The order of r at ∞ is 2 so $E_\infty = \{\frac{1}{2} + k\sqrt{1+4b}\}$, $k = \pm\frac{1}{2}$, where $b = 2$ (the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞), i.e. $E_{in} = \{\frac{1}{2} + \frac{1}{2}\sqrt{1+4\cdot 2}, \frac{1}{2} - \frac{1}{2}\sqrt{1+4\cdot 2}\} = \{2, -1\}$.

There are four possible tuples to consider; $(e_0, e_\infty) = (2,2)$, $(2,-1)$, $(-1,2)$ and $(-1,-1)$. Since $d = e_\infty - \sum_{c\in\Gamma} e_c$ and $\theta = \sum_{c\in\Gamma} \frac{e_c}{x-c}$, the possible values for d and θ are

e_0	e_∞	d	θ
2	2	0	$\frac{2}{x}$
2	-1	-3	$\frac{2}{x}$
-1	2	3	$\frac{-1}{x}$
-1	-1	0	$\frac{-1}{x}$

We can eliminate the second tuple since in that case d is not a non-negative integer.

We now search for a monic polynomial of degree d satisfying $P'' + 2\theta P' + (\theta' + \theta^2 - r)P = 0$. For the first tuple, $d = 0$ so P must be 1. We check whether this satisfies the required equation.

$$1'' + 2\cdot\frac{2}{x}\cdot 1' + \left(\left(\frac{2}{x}\right)' + \left(\frac{2}{x}\right)^2 - \frac{2}{x^2}\right) = \frac{-2}{x^2} + \frac{4}{x^2} - \frac{2}{x^2} = 0$$

So we have the correct P and θ.

Now

$$\omega = \theta + \frac{P'}{P} = \frac{2}{x} + \frac{1'}{1} = \frac{2}{x}$$

and the solution is

$$\eta = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$$

We can transform this back using the inverse transformation

$$z = y \cdot e^{\int \frac{b}{2a} dx} = y$$

and get $z_1 = x^2$. Then by the method of reduction of order, the second solution is

$$z_2 = z_1 \cdot \int \frac{e^{-\int \frac{b}{a} dx}}{z_1^2} dx = x^2 \cdot \int \frac{1}{\left(x^2\right)^2} dx = x^2 \cdot \int \frac{1}{x^4} dx = x^2 \cdot \frac{-1/3}{x^3} = \frac{-1}{3x}$$

An Example of the Use of the Maple Implementation

The following is a listing of a Maple session using the implementation of Kovacic's algorithm.

```
Script started on Fri Aug 12 21:04:31 1983
Warning: no access to tty; thus no job control in this shell...
% maple <testrun
    |\^/|
._|\|   |/|_.
 \  MAPLE  /   Version 3.0 --- May 1983
 <____ ____>
      |
# read '/u/csmith/essay/kovode.m';
words used 1399
# prettyprint := 0;
words used 34125
prettyprint := 0
# eq1 := diff(y(x),x,x) + y(x) = 0;
eq1 := diff(diff(y(x),x),x)+y(x)=0
# osolve(eq1,y(x),x);
words used 36194
    .
    .
    .
words used 132824
[exp((-1)*I*x),1/2/I*exp(I*x)]
# eq2 := diff(y(x),x,x) + 4*x*diff(y(x),x) + (4*x**2+2)*y(x) = 0;
eq2 := diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x**2+2)*y(x)=0
# osolve(eq2,y(x),x);
words used 134826
    .
    .
    .
words used 147080
[exp((-1)*x**2),exp((-1)*x**2)*x]
# eq4 := x**2*diff(y(x),x,x) - 2*x*diff(y(x),x) + (x**2+2)*y(x) = 0;
eq4 := x**2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x**2+2)*y(x)=0
```

```
# osolve(eq4,y(x),x);
words used 149144
        .
        .
        .
words used 206842
[x*exp((-1)*I*x),1/2*x/I*exp(I*x)]
# eq5 := (x-2)**2*diff(y(x),x,x) - (x-2)*diff(y(x),x) - 3*y(x) = 0;
eq5 := (x+(-2))**2*diff(diff(y(x),x),x)-(x+(-2))*diff(y(x),x)-3*y(x)=0
# osolve(eq5,y(x),x);
words used 208881
        .
        .
        .
words used 279114
[(x+(-2))**(-3/2)*(x**2-4*x+4)**(1/4),(-8*x+6*x**2-2*x**3+1/4*x**4)*
(x+(-2))**(-3/2)*(x**2-4*x+4)**(1/4)]
# map(radsimp,");
words used 281126
        .
        .
        .
words used 418931
[(x+(-2))**(-1),(-8*x+6*x**2-2*x**3+1/4*x**4)/(x+(-2))]
# quit;
Final 'words used'=420608,  storage=1047028
%
script done on Fri Aug 12 21:12:15 1983
```

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1.5 Analysis and object oriented implementation of the Kovacic algorithm

By Nasser M. Abbasi

Analysis and object oriented implementation of the Kovacic algorithm

Nasser M. Abbasi*

Abstract

This paper gives a detailed overview and a number of worked out examples illustrating the Kovacic [1] algorithm for solving second order linear differential equation $A(x)y'' + B(x)y' + C(x)y = 0$ where A, B, C are rational functions with complex coefficients in the independent variable x. All three cases of the algorithm were implemented in a software package based on an object oriented design and complete source code listing given in the appendix with usage examples. Implementation used the Maple computer algebra language.[1]. This package was then used to analyze the distribution of Kovacic algorithm cases on 3000 differential equations.

*https://12000.org/

[1] The complete Kovacic package in one mpl file accompany the arXiv version of this paper

1 Introduction

Kovacic [1] gave an algorithm for finding a closed form Liouvillian[2] solution to any linear second order differential equation $Ay'' + By' + Cy = 0$ if such a solution exists. Smith [2] gave an implementation based on a modified version of Kovacic algorithm by Saunders [3].

The current implementation is based on the original paper by Kovacic and uses the new object oriented features in Maple. The accompanied software package have been tested on 3000 differential equations with each solution verified using Maple's odetest. The test suite is included as a separate module. The Appendix describes how to use the software.

The Kovacic algorithm finds one (basis) solution of $Ay'' + By' + Cy = 0$. The second basis solution is found using reduction of order. The general solution is a linear combination of the two basis solutions found.

The algorithm starts by writing the input ode $Ay'' + By' + Cy = 0$ as

$$y'' + ay' + by = 0 \tag{1}$$

Where $a = \frac{B}{A}, b = \frac{C}{A}$. The substitution

$$z = ye^{\frac{1}{2}\int a\,dx} \tag{2}$$

is then applied to (1) which transforms it to a second order ode in the new dependent variable $z(x)$ without the first derivative

$$z'' = rz \tag{3}$$

r in the above is given by

$$r = \frac{1}{4}a^2 + \frac{1}{2}a' - b \tag{4}$$

It is ode (3) which is solved by the algorithm and not (1). Equation (3) will be called the DE from now on.

If a solution $z(x)$ to the DE is found, then the first basis solution to the original ode is obtained using the transformation (2) in reverse

$$y = ze^{-\frac{1}{2}\int a\,dx}$$

The second solution is found using reduction of order.

[2]Wikipedia defines Liouvillian function as function of one variable which is the composition of a finite number of arithmetic operations $(+, -, \times, \div)$, exponentials, constants, solutions of algebraic equations (a generalization of nth roots), and antiderivatives. Kovacic in his original paper says "Such a solution may involve exponentials, indefinite integrals and solutions of polynomial equations. (As we are considering functions of a complex variable, we need not explicitly mention trigonometric functions, they can be written in terms of exponentials. Note that logarithms are indefinite integrals and hence are allowed."

These are the four possible cases to consider.

1. DE has solution $z = e^{\int \omega dx}$ with $\omega \in \mathbb{C}(x)$.

2. DE has solution $z = e^{\int \omega dx}$ with ω polynomial over $\mathbb{C}(x)$ of degree 2.

3. Solutions of DE are algebraic over $\mathbb{C}(x)$.

4. DE has no Liouvillian solution.

Before describing how the algorithm works, there are necessary (but not sufficient) conditions that determine which case the DE satisfies. Only those cases that meet the necessary conditions will be attempted.

The following are the necessary conditions for each case. To check each case, let $r = \frac{s}{t}$ where $\gcd(s,t) = 1$. The order of r at ∞ (from now on referred to as $\mathcal{O}(\infty)$) is defined as $\deg(t) - \deg(s)$. The poles of r and the order of each pole need to be determined.

Knowing the order of the poles of r and $\mathcal{O}(\infty)$ is all what is needed to determine the necessary conditions for each case. These conditions are the following

1. Case 1. Either no pole exists, or if a pole exists, the order must be either one or even. If $\mathcal{O}(\infty)$ is less 3, then it must be even otherwise it can be even or odd.

2. Case 2. r must have at least one pole either of order 2 or odd order greater than 2. There are no conditions on $\mathcal{O}(\infty)$.

3. Case 3. r must have a pole either of order 1 or 2. No other order is allowed. $\mathcal{O}(\infty)$ must be at least 2.

If the conditions of a case are not satisfied then the case will be attempted as the algorithm guarantees that there will be no Liouvillian solution. However if the conditions are satisfied, this does not necessarily mean a solution exists. As an example $y'' = 1/x^6 y$ satisfies only case one, but running the algorithm on case one shows that there is no Liouvillian solution.

The following table summarizes the above conditions for each case.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole of order 2 or pole of odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. The following are examples of pole orders which are allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no conditions
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 1: Necessary conditions for each Kovacic case

Some observations: In case one, no odd order pole is allowed except for order 1. Case one is the only case that could have no pole in r, which is the same as a pole of order zero. Case two and three require at least one pole. For case three, only poles of order 1 or 2 are allowed. If $\mathcal{O}(\infty)$ is zero, then only possibility is either case one or two. For case one, if $\mathcal{O}(\infty)$ is negative, then it must be even.

The above table also shows that when r has only one pole of order 2 and $\mathcal{O}(\infty)$ equals 2 or higher then all three cases are possible. Also, if r has two poles one of order 1 and the other of order 2 and $\mathcal{O}(\infty)$ equals 2 or higher then all three cases are possible.

These are the only two possibilities where all three cases have the same necessary conditions.

2 Description of algorithm for each case

2.1 Case one

2.1.1 step 1

Assuming that the necessary conditions for case one are satisfied and $z'' = rz, r = \frac{s}{t}$. Let Γ be the set of all poles of r. For each pole c in this set, three quantities are calculated: Rational function $\left[\sqrt{r}\right]_c$ and two complex numbers α_c^+, α_c^-.

How this is done depends on the order of the pole as described below. If the set Γ is empty (when there are no poles), then this part is skipped.

1. If the pole c has order 1 then

$$
\begin{aligned}
\left[\sqrt{r}\right]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

2. If the pole c is of order 2 then

$$
\begin{aligned}
\left[\sqrt{r}\right]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\
\alpha_c^- &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}
\end{aligned}
$$

   Where b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction decomposition of r.

3. If the pole is of order $\{4,6,8,\dots\}$ (poles must be all even from the conditions of case one), then the computation is more involved. Let $2v$ be the order of the pole. Hence if the pole was order 4, then $v = 2$. Let $\left[\sqrt{r}\right]_c$ be the sum of terms involving $\frac{1}{(x-c)^i}$ for $2 \le i \le v$ in the Laurent series expansion of $\sqrt{r}$ (not r) at c. Therefore

$$
\begin{aligned}
\left[\sqrt{r}\right]_c &= \sum_{i=2}^{v} \frac{a_i}{(x-c)^i} \\
&= \frac{a_2}{(x-c)^2} + \frac{a_3}{(x-c)^3} + \cdots + \frac{a_v}{(x-c)^v} \qquad (1)
\end{aligned}
$$

α_c^+, α_c^- are found using

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a_v} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a_v} + v\right)$$

Where in the above a_v is the coefficient of the term $\frac{a_v}{(x-c)^v}$ in (1) and b is the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r itself (found from the partial fraction decomposition), minus the coefficient of same term in the Laurent series expansion of $\sqrt{r}$ at c.

The coefficients in the Laurent series can be obtained as follows. Given $r(x)$ with a pole of finite order N at $x = c$, then its Laurent series expansion at c is given by the sum of the analytic part and the principal part of the of the Laurent series. The coefficients b_n are contained in the principal part of the series.

$$\begin{aligned} r(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n + \sum_{n=1}^{N} \frac{b_n}{(x-c)^n} \\ &= \sum_{n=0}^{\infty} a_n (x-c)^n + \frac{b_1}{(x-c)} + \frac{b_2}{(x-c)^2} + \frac{b_3}{(x-c)^3} + \cdots + \frac{b_N}{(x-c)^N} \end{aligned} \tag{2}$$

To obtain b_1 (which is the residue of $r(x)$ at c), both sides of the above are multiplied by $(x-c)^N$ which gives

$$(x-c)^N r(x) = \sum_{n=0}^{\infty} a_n (x-c)^{n+N} + b_1 (x-c)^{N-1} + b_2 (x-c)^{N-2} + \cdots + b_N \tag{3}$$

Differentiating both sides of (3) $(N-1)$ times w.r.t. x gives

$$\frac{d^{N-1}}{dx^{(N-1)}}\left((x-c)^N f(x)\right) = \sum_{n=0}^{\infty} \frac{d^{N-1}}{dx^{(N-1)}}\left(a_n (x-c)^{n+N}\right) + b_1 (N-1)!$$

Evaluating the above at $x = c$ gives

$$b_1 = \frac{\lim_{x\to c} \frac{d^{N-1}}{dx^{(N-1)}}\left((x-c)^N r(x)\right)}{(N-1)!}$$

To find the next coefficient b_2, both sides of (3) are differentiated $(N-2)$ times

$$\frac{d^{N-2}}{dx^{(N-2)}}\left((x-c)^N r(x)\right) = \sum_{n=0}^{\infty} \frac{d^{N-2}}{dx^{(N-2)}}\left(a_n (x-c)^{n+N}\right) + b_1 (N-1)! (x-c) + b_2 (N-2)!$$

Evaluating the above at $x = c$ gives

$$b_2 = \frac{\lim_{x\to c} \frac{d^{N-2}}{dx^{(N-2)}}\left((x-c)^N r(x)\right)}{(N-2)!}$$

The above is repeated to find $b_3, b_4, \cdots, b_N$. The general formula for find coefficient b_n is therefore

$$b_n = \frac{\lim_{x\to c} \frac{d^{N-n}}{dx^{(N-n)}}\left((x-c)^N r(x)\right)}{(N-n)!} \tag{4}$$

For the special case of the last term b_N the above simplifies to

$$b_N = \lim_{x\to c}(x-c)^N r(x) \tag{5}$$

The above is implemented in the function `laurent_coeff()` in the Kovacic class.

This completes finding all the quantities $\{[\sqrt{r}]_c, \alpha_c^+, \alpha_c^+\}$ for each pole in the set Γ for case one.

The next step calculates the following three quantities for $\mathcal{O}(\infty)$.

1. If $\mathcal{O}(\infty) \le 0$, which must be even, then let $-2v = \mathcal{O}(\infty)$ and $[\sqrt{r}]_\infty$ is then the sum of all terms x^i for for $0 \le i \le v$ in the Laurent series expansion of $\sqrt{r}$ at ∞.

$$[\sqrt{r}]_\infty = \sum_{i=0}^{v} a_i x^v = a_0 + a_1 x + a_2 x^2 \cdots + a_v x^v \tag{6}$$

The coefficients a_i are found by setting $x = \frac{1}{y}$ in r and then finding the Laurent series of $\left[\sqrt{r(y)}\right]$ expanded around $y = zero$. The process for finding the coefficient is the same one used as described earlier where now the limit is taken as y approaches zero from the right. This gives all the terms of (6). This is implemented in the function `laurent_coeff()` in the Kovacic class.

The corresponding $\{\alpha_\infty^+, \alpha_\infty^-\}$ are given by

$$\alpha_\infty^+ = \frac{1}{2}\left(\frac{b}{a_v} - v\right)$$
$$\alpha_\infty^- = \frac{1}{2}\left(-\frac{b}{a_v} - v\right)$$

Where a_v is coefficient of x^v in (6) and b is the coefficient of x^{v-1} in r itself (found using long division) minus the coefficient of x^{v-1} in $\left([\sqrt{r}]_\infty\right)^2$.

2. If $\mathcal{O}(\infty) = 2$ then $[\sqrt{r}]_\infty = 0$. The corresponding $\{\alpha_\infty^+, \alpha_\infty^-\}$ are given by

$$\alpha_\infty^+ = \frac{1}{2} + \frac{1}{2}\sqrt{1+4b}$$
$$\alpha_\infty^- = \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}$$

Here $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. lcoef(s) is the leading coefficient of s and similarly, lcoef(t) is the leading coefficient of t.

3. If $\mathcal{O}(\infty) > 2$ then

$$[\sqrt{r}]_\infty = 0$$
$$\alpha_\infty^+ = 0$$
$$\alpha_\infty^- = 1$$

2.1.2 step 2

Using quantities calculated in step 1, the algorithm now searches for a non-negative integer d using

$$d = \alpha_\infty^\pm - \sum_{c\in\Gamma} \alpha_c^\pm$$

If non-negative d is found, a candidate ω_d is calculated using

$$\omega_d = \sum_{c\in\Gamma}\left((\pm)\left[\sqrt{r}\right]_c + \frac{\alpha_c^\pm}{x-c}\right) + (\pm)\left[\sqrt{r}\right]_\infty$$

If no non-negative integer d could be found, then no Liouvillian solution exists using this case. Case two or three are tried next if these are available.

2.1.3 step 3

In this step the algorithm finds polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + x^d$ of degree d. This is done by solving for the coefficients a_i from

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{7}$$

Where ω is from the second step above and r is from $z'' = rz$.

For an example, if $d = 2$, then $p(x) = x^2 + a_1x + a_0$ is substituted in (3) and a_0, a_1 are solved for. If solution exists, then the solution to $z'' = rz$ will be

$$z = p(x)e^{\int \omega dx}$$

If the degree $d = 1$ then $p(x) = x + a_0$ and the same process is applied. If the degree $d = 0$, then $p(x) = 1$.

The first basis solution to the original ode is now be found from

$$y_1 = ze^{-\frac{1}{2}\int a\,dx}$$

And the second basis solution using reduction of order formula is

$$y_2 = y_1 \int \frac{e^{-\int a\,dx}}{y_1^2}\,dx$$

Hence the general solution to the original ode is

$$y(x) = c_1y_1 + c_2y_2$$

This completes the full algorithm for case 1. The part that needs most care is in finding $\{[\sqrt{r}]_c, \alpha_c^\pm, [\sqrt{r}]_\infty, \alpha_\infty^\pm\}$. Once these are calculated, the rest of the algorithm is much more direct.

2.1.4 Algorithm flow chart for case one

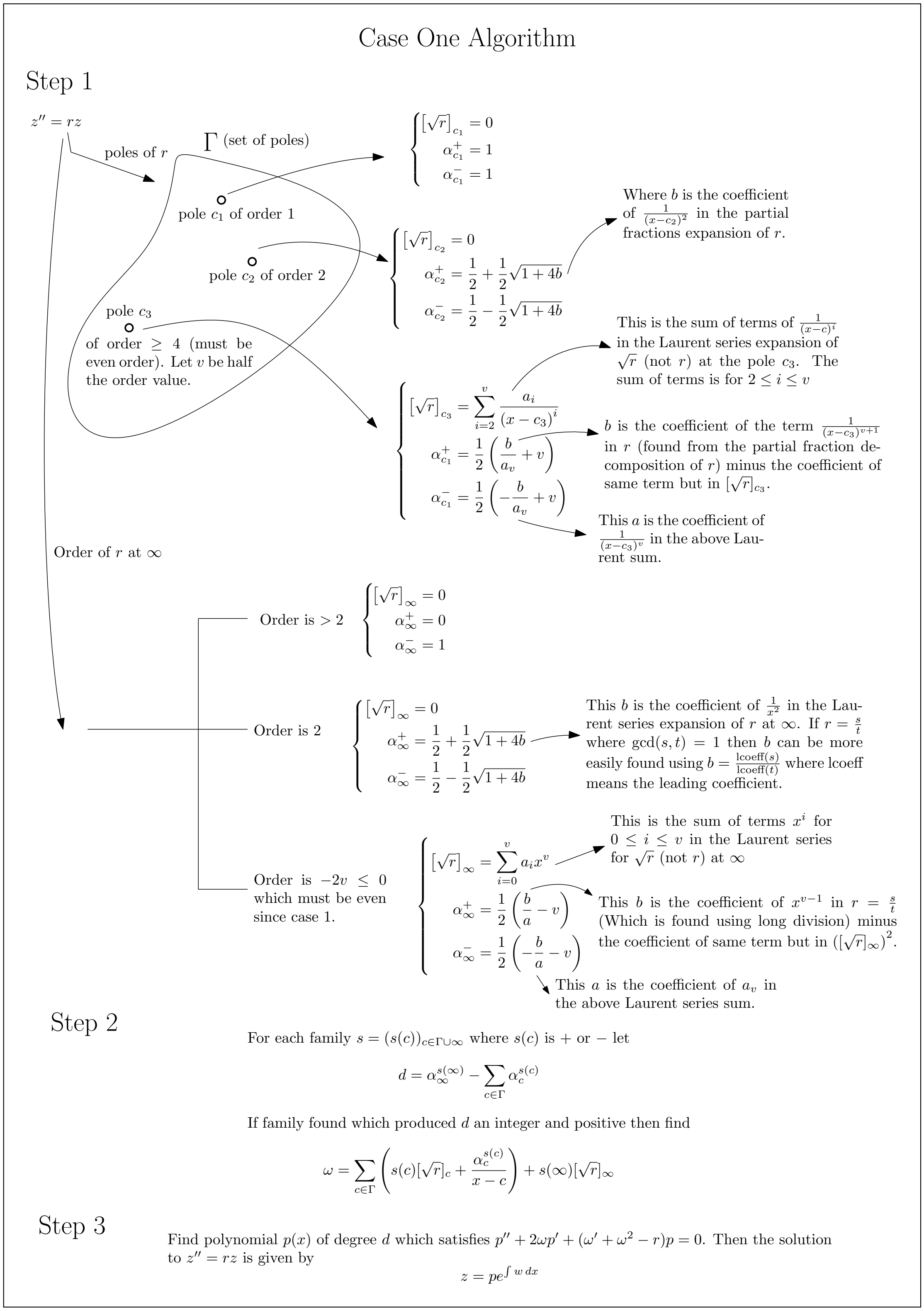


Figure 1: Case 1 Kovacic algorithm

2.2 Case two

2.2.1 step 1

Assuming that the necessary conditions for case two are satisfied and $z'' = rz, r = \frac{s}{t}$. Let Γ be the set of all poles of r. For each pole c in this set, E_c is found as follows

1. If the pole c has order 1 then $E_c = \{4\}$.
2. If the pole c is of order 2 then $E_c = \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}$ where b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction decomposition of r. In the above set E_c, only integer values are kept.
3. If the pole c is of order $v > 2$ then $E_c = \{v\}$

The next step is to determine E_∞.

1. If $\mathcal{O}(\infty) > 2$ then $E_\infty = \{0, 2, 4\}$
2. If $\mathcal{O}(\infty) = 2$ then $E_\infty = \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}$ where $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. lcoef(s) is the leading coefficient of s and similarly lcoef(t) is the leading coefficient of t. In the above set E_∞ only integer values are kept.
3. If $\mathcal{O}(\infty) < 2$ then $E_\infty = \mathcal{O}(\infty)$.

2.2.2 step 2

Using quantities calculated in step 1, the algorithm now searches for a non-negative integer d using

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above $e_c \in E_c$, $e_\infty \in E_\infty$ found in step 1. If non-negative d is found, then

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$$

If no non-negative integer d could be found, then no Liouvillian solution exists using this case. Case three is tried next if it is available.

2.2.3 step 3

In this step the algorithm determines a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + x^d$ of degree d. This is done by solving for the coefficients a_i from

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1}$$

Where θ was found in step 2 and r is from $z'' = rz$. If $p(x)$ can be found that satisfies (1) then

$$\phi = \theta + \frac{p'}{p} \tag{2}$$

ω is then solved for from

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0 \tag{3}$$

If solution ω to (3) can be found, then the solution to $z'' = rz$ is given by

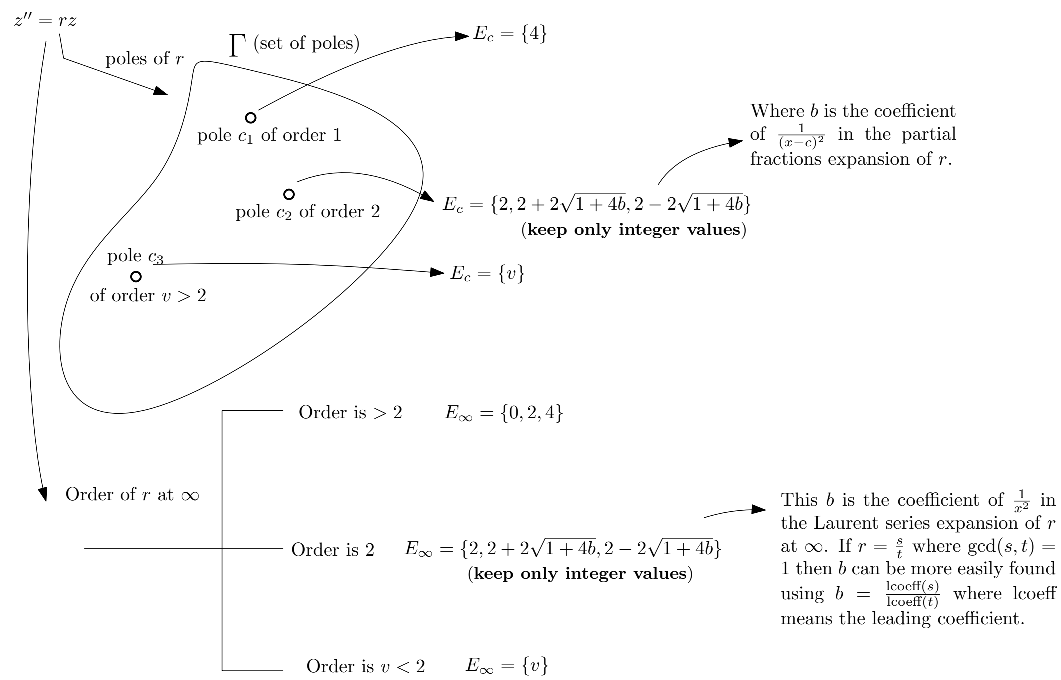
$$z = e^{\int \omega \, dx}$$

This completes the full algorithm for case two. The general solution to the original ode is now determined as outlined at the end of case one above.

2.2.4 Algorithm flow chart for case two

Case Two Algorithm

Step 1



Step 2

For each family $(e_c)_{c\in\Gamma\cup\infty}$ with $e_c \in E_c$ let

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

If family found which produced d an integer and positive then find

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$$

Step 3

Find polynomial $p(x)$ of degree d which satisfies

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0$$

Let

$$\phi = \theta + \frac{p'}{p}$$

The find solution ω for the equation

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

If solution can be found then

$$z = e^{\int w\,dx}$$

Is the solution to $z'' = rz$

Figure 2: Case 2 Kovacic algorithm

2.3 Case three

2.3.1 step 1

Assuming the necessary conditions for case three are satisfied and $z'' = rz, r = \frac{s}{t}$. Let Γ be the set of all poles of r. Recall that case three can have either a pole of order 1 or order 2 only. For each pole c in this set, E_c is found as follows

1. If the pole c has order 1 then $E_c = \{12\}$.
2. If the pole c is of order 2 then

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b}\right\} \qquad \text{for} \quad k = -\frac{n}{2}\cdots\frac{n}{2} \tag{1}$$

   Where k is incremented by 1 each time, and n is any of $\{4,6,12\}$ and b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction decomposition of r. In the above set E_c, only integer values are kept. For an example, when $n = 4$ then $k = \{-2,-1,0,1,2\}$ and $E_c = \{6-6\sqrt{1+4b}, 6-3\sqrt{1+4b}, 6, 6+3\sqrt{1+4b}, 6+6\sqrt{1+4b}\}$ and similarly for $n = 6$ and $n = 12$.

The next step determines E_∞. This is found using same formula as (1) but b is calculated differently using $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. lcoef(s) is the leading coefficient of s and lcoef(t) is the leading coefficient of t.

2.3.2 step 2

Using quantities calculated in step 1, the algorithm now searches for a non-negative integer d using

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above $e_c \in E_c$, $e_\infty \in E_\infty$ n is any of $\{4,6,12\}$ values. If non-negative d is found, then

$$\theta = \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}$$

The sum above is over all families of $\{e_\infty, e_c\}$ which generated the non-negative integer d. Next define

$$S = \prod_{c\in\Gamma}(x-c)$$

The product above is over families of $\{e_\infty, e_c\}$ which generated the non-negative integer d. If no non-negative integer d is found, then no Liouvillian solution exists.

2.3.3 step 3

In this step the algorithm determines a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + x^d$ of degree d. Define set of polynomials $\{P_n, P_{n-1}, \cdots, P_{-1}$ where

$$
\begin{aligned}
P_n &= -p(x) \\
P_{i-1} &= -SP_i' + \big((n-i)S' - S\theta\big)P_i - (n-i)(i+1)S^2 r P_{i+1} \qquad i = n \cdots 0
\end{aligned}
$$

The last polynomial $P_{-1}(x)$ is used to solve for the coefficients a_i using

$$P_{-1}(x) = 0 \tag{2}$$

In Maple this is done using the `solve` command with the `identity` option. If it is possible to find coefficients a_i such that (2) is satisfied, then define the equation

$$\sum_{i=0}^{n} \frac{S^i P_i(x)}{(n-i)!} \omega^i = 0$$

ω is solved for from the above equation. If solution ω is found then the solution to $z'' = rz$ will be

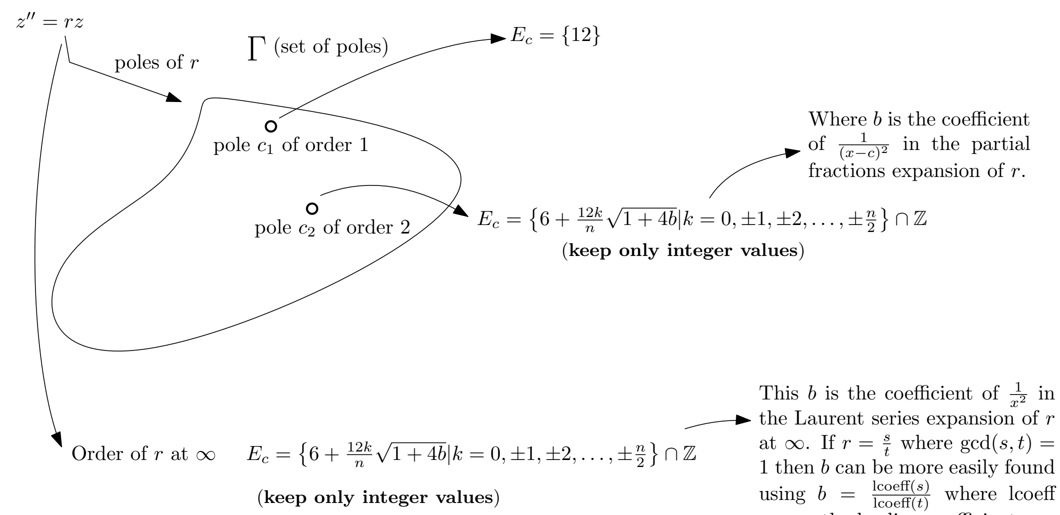
$$z = e^{\int \omega \, dx}$$

This completes the full algorithm for case three. The general solution to the original ode can now be determined as outlined at the end of case one above.

2.3.4 Algorithm flow chart for case three

Case 3 Algorithm

Step 1



Step 2

For each family $(e_c)_{c \in \Gamma \cup \infty}$ with $e_c \in E_c$ let

$$d = \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

If family found which produced d an integer and positive then find

$$\theta = \frac{n}{12}\sum_{c \in \Gamma} \frac{e_c}{x - c}$$

And

$$S = \prod_{c \in \Gamma}(x - c)$$

Step 3

Let polynomial $p(x)$ of degree d with coefficients a_i.
The following set of equations are set up in order to determine $p(x)$.

$$P_n = -p(x)$$
$$P_{i-1} = -Sp_i' + ((n-i)S' - S\theta)\, P_i - (n-i)(i+1)S^2 r P_{i+1} \qquad i = n \cdots 0$$

Where n above is either 4,6 or 12. The last polynomial $P_{-1}(x)$ is used to solve for the coefficients a_i using

$$P_{-1}(x) = 0$$

The equation for ω is found using

$$\sum_{i=0}^{n} \frac{S^i P_i(x)}{(n-i)!}\omega^i = 0$$

If solution ω can be found then

$$z = e^{\int w \, dx}$$

Is the solution to $z'' = rz$

Figure 3: Case 3 Kovacic algorithm

2.4 Statistics and discussion of results obtained using Kovacic algorithm

This gives summary of results obtained using testsuite of 3000 differential equations, all of which were selected as linear with rational coefficients as functions of x that can be solved using this algorithm.

The ode's used in the testsuite were collected by the author and stored in sql database. These were collected from a number of standard textbooks and other references such as "Differential Equations. E. Kamke. 3th edition. Chelsea." and "Ordinary Differential Equations And Their Solutions. Murphy, George Moseley. Dover. 2011".

All the ode's were successfully solved using the Kovacic algorithm as implemented here and each solution was verified using Maple odetest.

The following diagram shows the percentage of ode's solved using each case.

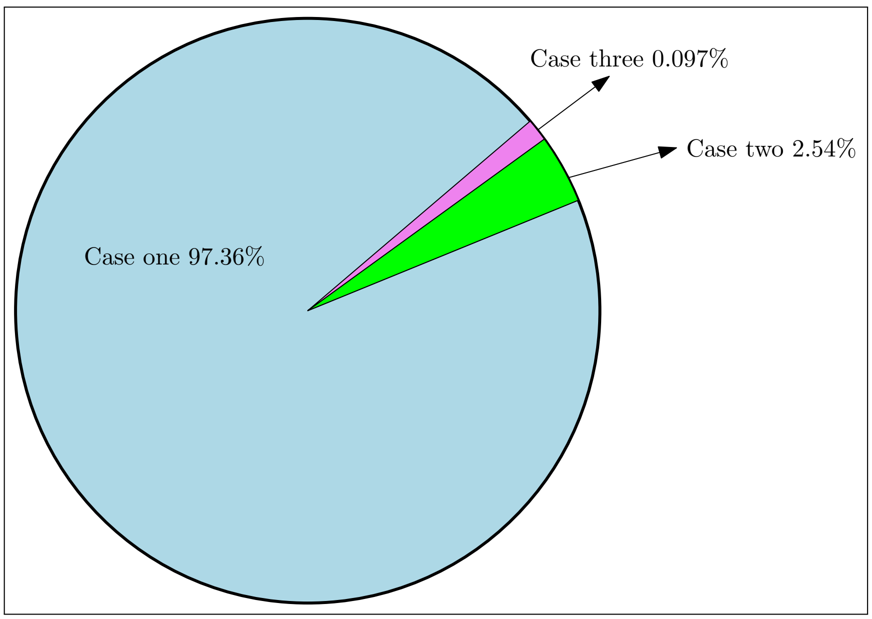


Figure 4: Kovacic cases distributions

Case 3 was required for solving only 3 odes. It used $n = 4$ for all 3 ode's. $n = 6$ and $n = 12$ were not reached or required to try. Recall that n for case 3 is the degree of the polynomial in ω used to solve for in order to find the z solution from $z = e^{\int \omega dx}$.

This result shows that case 1 and 2 combined is all what is needed to solve 99.9% of ode's used in practice. Larger collection of ode's than the 3000 used could produce different results, but the overall trend is that case 3 is rarely needed in practice and within case 3, $n = 6$ and $n = 12$ are even less likely to be required.

When forcing the algorithm to use case 3 and only use $n = 12$, this resulted in a very long computation time on some ode's. For an example, using ode $y'' + xy' + y = 0$ which satisfies all three cases, and asking the solver to use case 3 and $n = 12$, it was found that it required $p(x)$ of degree $d = 24$ in order to find ω of degree 12 that can be solved. The total number of trials in step 3 of case three to find such solution was found to be 2367. This took over 30 minutes to complete.

In comparison, the same ode was solved using case one in less than one second giving the same solution on the same computer.

The testsuite also calculates the distribution of cases which has its necessary conditions satisfied for each ode. Recall that having the necessary conditions for a case satisfied does not mean a solution would be found using that case. The following bar chart shows the percentages of the 3000 ode's that satisfied the necessary conditions each case. This chart shows that many ode's satisfy the conditions for more than one case at the same time.

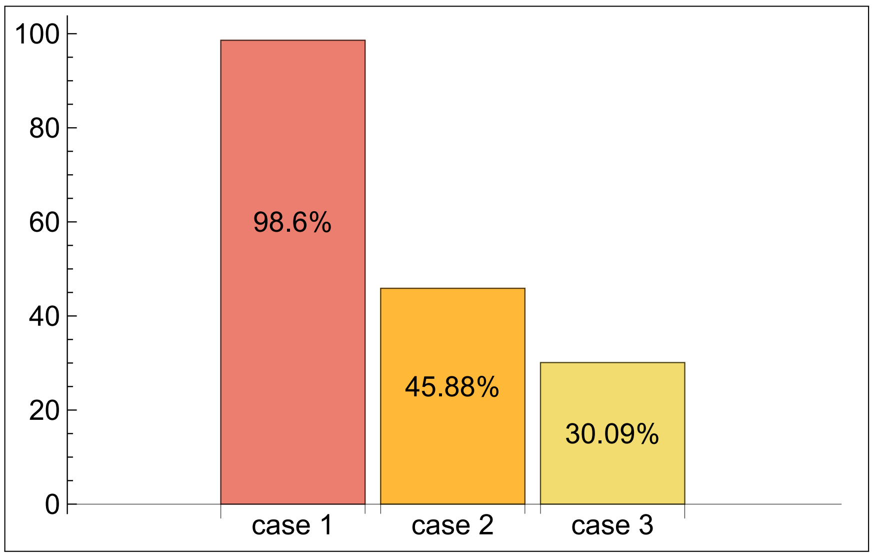


Figure 5: Percentage of ode's that satisfy each Kovacic case necessary conditions

3 Worked example for each case

3.1 case one

3.1.1 Example 1

Given the ode

$$(2x+1)y'' - 2y' - (2x+3)y = 0$$

Converting it $y'' + ay' + by = 0$ gives

$$y'' - \frac{2}{2x+1}y' - \frac{2x+3}{2x+1}y = 0$$

Where $a = -\frac{2}{2x+1}, b = -\frac{2x+3}{2x+1}$. Applying the transformation $z = ye^{\frac{1}{2}\int a\,dx}$ gives $z'' = rz$ where $r = \frac{1}{4}a^2 + \frac{1}{2}a' - b$. This results in

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2+8x+6}{(2x+1)^2} = \frac{s}{t} \end{aligned}$$

There is one pole at $x = -\frac{1}{2}$, hence $\Gamma = \{-\frac{1}{2}\}$. The order is 2 and $\mathcal{O}(\infty) = \deg(t) - \deg(s) = 0$. Table 1 shows that the necessary conditions for case one and two are both satisfied. This is solved first using case one. Since the order of the pole is 2, then

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_c^- &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned} \tag{1}$$

Where b is the coefficient of $\frac{1}{(x-c)^2} = \frac{1}{(x+\frac{1}{2})^2}$ in the partial fraction decomposition of r which is $r = 1 + \frac{3}{4}\frac{1}{(x+\frac{1}{2})^2} + \frac{1}{x+\frac{1}{2}}$. Therefore $b = \frac{3}{4}$ and the above becomes

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = -\frac{1}{2} \end{aligned}$$

Since $\mathcal{O}(\infty) = 0$ then $v = 0$ and $\left[\sqrt{r}\right]_\infty$ is the sum of all terms x^i for $0 \le i \le v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is found as follows. Since $\sqrt{r} = \sqrt{\frac{4x^2+8x+6}{(2x+1)^2}}$ then setting $x = \frac{1}{y}$ gives $\sqrt{r(y)} = \sqrt{\frac{4\left(\frac{1}{y}\right)^2+8\frac{1}{y}+6}{\left(2\frac{1}{y}+1\right)^2}}$ and since $v = 0$ then the constant term is $\lim_{y\to 0}\sqrt{r(y)} = 1$. Therefore $\left[\sqrt{r(x)}\right]_\infty = 1$. Hence $a = 1$.

b is the coefficient of $x^{v-1} = \frac{1}{x}$ in r minus the coefficient of $\frac{1}{x}$ in $\left(\left[\sqrt{r}\right]_\infty\right)^2 = 1$ which is zero since there is no term $\frac{1}{x}$. Because $v = 0$, long division is used is used find the coefficient of $\frac{1}{x}$ in r.

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2+8x+6}{4x^2+4x+1} \\
&= Q + \frac{R}{t} \\
&= 1 + \frac{4x+5}{4x^2+4x+1}
\end{aligned}
$$

The coefficient of $\frac{1}{x}$ in r is the leading coefficient in R minus the leading coefficient in t which gives $\frac{4}{4} = 1$. Therefore $b = 1 - 0 = 1$. This results in

$$
\begin{aligned}
\left[\sqrt{r}\right]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a_v} - v\right) &= \frac{1}{2}\left(\frac{1}{1} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a_v} - v\right) &= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &= -\frac{1}{2}
\end{aligned}
$$

The above completes step 1 for case one. Step 2 searches for a non-negative integer d using

$$
d = \alpha_\infty^\pm - \sum_{c\in\Gamma} \alpha_c^\pm \tag{2}
$$

Where the above is carried over all possible combinations resulting in the following 4 possibilities (in this example, there is only one pole, hence the sum contains only one term) and the number of possible combinations is therefore $2^2 = 4$

$$
\begin{aligned}
d &= \alpha_\infty^+ - \alpha_c^+ = \frac{1}{2} - \left(\frac{3}{2}\right) &&= -1 \\
d &= \alpha_\infty^+ - \alpha_c^- = \frac{1}{2} - \left(-\frac{1}{2}\right) &&= 1 \\
d &= \alpha_\infty^- - \alpha_c^+ = -\frac{1}{2} - \left(\frac{3}{2}\right) &&= -2 \\
d &= \alpha_\infty^- - \alpha_c^- = -\frac{1}{2} - \left(-\frac{1}{2}\right) &&= 0
\end{aligned}
$$

The above shows there are two possible d values to use. $d = 1$ or $d = 0$. Each is tried until one produces a solution or all fail to do so. For each valid d found an ω is found using

$$
\omega = \sum_c \left((\pm)\left[\sqrt{r}\right]_c + \frac{\alpha_c^\pm}{x-c} \right) + (\pm)\left[\sqrt{r}\right]_\infty
$$

But $\left[\sqrt{r}\right]_c = 0$ in this example, hence the above simplifies to

$$\omega = \sum_c \left(\frac{\alpha_c^{\pm}}{x-c}\right) + (\pm)\left[\sqrt{r}\right]_\infty$$

Since there is one pole, then the candidate ω to try are the following

$$\begin{aligned}
\omega &= \frac{\alpha_c^{+}}{x-c} + (+1)\left[\sqrt{r}\right]_\infty \\
\omega &= \frac{\alpha_c^{+}}{x-c} + (-1)\left[\sqrt{r}\right]_\infty \\
\omega &= \frac{\alpha_c^{-}}{x-c} + (+1)\left[\sqrt{r}\right]_\infty \\
\omega &= \frac{\alpha_c^{-}}{x-c} + (-1)\left[\sqrt{r}\right]_\infty
\end{aligned}$$

Substituting the known values found in step 1 into the above gives

$$\begin{aligned}
\omega &= \frac{\frac{3}{2}}{x+\frac{1}{2}} + (+1)(1) = \frac{6+2x}{2x+3} \\
\omega &= \frac{\frac{3}{2}}{x+\frac{1}{2}} + (-1)(1) = -\frac{2x}{2x+3} \\
\omega &= \frac{-\frac{1}{2}}{x+\frac{1}{2}} + (+1)(1) = \frac{2x}{2x+1} \\
\omega &= \frac{-\frac{1}{2}}{x+\frac{1}{2}} + (-1)(1) = -\frac{2(1+x)}{2x+1}
\end{aligned}$$

So there are two possible d values to try, and for each, there are 4 possible $w(x)$, which gives 8 possible tries. This completes step 2. For each trial, step 3 is now invoked.

Starting with $d=0$ and using $\omega = \frac{2x}{2x+1}$, and since the degree is $d=0$ then $p(x)=1$. This polynomial is now checked to see if it satisfies

$$\begin{aligned}
p'' + 2\omega p' + (\omega' + \omega^2 - r)p &= 0 \qquad (3) \\
(\omega' + \omega^2 - r)p &= 0 \\
\frac{d}{dx}\left(\frac{2x}{2x+1}\right) + \left(\frac{2x}{2x+1}\right)^2 - \frac{4x^2+8x+6}{(2x+1)^2} &= 0 \\
-\frac{4}{2x+1} &= 0
\end{aligned}$$

Since the left side is not identically zero, then this candidate ω has failed. Carrying out this process for the other 3 possible ω values shows that non are satisfied as well. $d=1$ is now tried. This implies the polynomial is $p(x) = a_0 + x$. The coefficient a_0 needs to be determined such that $p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0$ is satisfied. Starting with $\omega = \frac{2x}{2x+1}$ gives

$$\begin{aligned}
p'' + 2\omega p' + (\omega' + \omega^2 - r)p &= 0 \\
2\omega p' + (\omega' + \omega^2 - r)p &= 0
\end{aligned}$$

Substituting $p = a_0 + x$ and $\omega = \frac{2x}{2x+1}$ and $r = \frac{4x^2+8x+6}{(2x+1)^2}$ into the above and simplifying gives $-\frac{4a_0}{2x+1} = 0$. This implies that (3) can be satisfied for $a_0 = 0$. Therefore the polynomial of degree one is found and given by

$$p(x) = x$$

Therefore the solution to $z'' = rz$ is

$$\begin{aligned} z &= pe^{\int \omega \, dx} \\ &= xe^{\int \frac{2x}{2x+1} \, dx} \\ &= xe^{x - \frac{\ln(2x+1)}{2}} \\ &= \frac{xe^x}{\sqrt{2x+1}} \end{aligned}$$

Given this solution for $z(x)$, the first basis solution of the original ode in y is found using the inverse of the original transformation used to generate the z ode which is $z = ye^{\frac{1}{2}\int a \, dx}$, therefore

$$\begin{aligned} y_1 &= ze^{-\frac{1}{2}\int a \, dx} \\ &= \frac{xe^x}{\sqrt{2x+1}} e^{-\frac{1}{2}\int -\frac{2}{2x+1} \, dx} \\ &= \frac{xe^x}{\sqrt{2x+1}} e^{\frac{\ln(2x+1}{2}} \\ &= \frac{xe^x}{\sqrt{2x+1}} \sqrt{2x+1} \\ &= xe^x \end{aligned}$$

The second basis solution is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -a \, dx} \, dx}{y_1^2} \, dx \\ &= xe^x \int \frac{e^{\int -\frac{-2}{2x+1} \, dx} \, dx}{(xe^x)^2} \, dx \\ &= xe^x \int \frac{e^{\ln(2x+1)}}{(xe^x)^2} \, dx \\ &= -e^{-x} \end{aligned}$$

Therefore the general solution to the original ode $(2x+1)y'' - 2y' - (2x+3)y = 0$ is

$$\begin{aligned} y(x) &= c_1 y_1 + x_2 y_2 \\ y(x) &= c_1 xe^x - c_2 e^{-x} \end{aligned}$$

This completes the solution.

3.1.2 Example 2

Given the ode

$$x^2\left(x^2-2x+1\right)y''-x(3+x)y'+(4+x)y=0$$

Converting it $y''+ay'+by=0$ gives

$$y''-\frac{x(3+x)}{x^2\left(x^2-2x+1\right)}y'+\frac{4+x}{x^2\left(x^2-2x+1\right)}y=0$$

Where $a=-\frac{x(3+x)}{x^2(x^2-2x+1)}, b=\frac{4+x}{x^2(x^2-2x+1)}$. Applying the transformation $z=ye^{\frac{1}{2}\int a\,dx}$ gives $z''=rz$ where $r=\frac{1}{4}a^2+\frac{1}{2}a'-b$. This results in

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{7x^2+10x-1}{4x^2(x-1)^4} \end{aligned}$$

There is one pole at $x=0$ of order 2 and pole at $x=1$ of order 4, hence $\Gamma=\{0,1\}$, and $\mathcal{O}(\infty)=\deg(t)-\deg(s)=6-2=4$. Table 1 shows that the necessary conditions for case one and two are both satisfied. For the pole at 0 and since its order is 2 then

$$\begin{aligned} \left[\sqrt{r}\right]_0 &= 0 \qquad (1)\\ \alpha_0^+ &= \frac{1}{2}+\frac{1}{2}\sqrt{1+4b} \\ \alpha_0^- &= \frac{1}{2}-\frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-0)^2}=\frac{1}{x^2}$ in the partial fraction decomposition of r which is

$$r=\frac{4}{(x-1)^4}-\frac{2}{(x-1)^3}-\frac{1}{4x^2}-\frac{3}{2(x-1)}+\frac{3}{2x}+\frac{7}{4(x-1)^2} \qquad (2)$$

The above shows that $b=-\frac{1}{4}$. Equation (1) becomes becomes

$$\begin{aligned} \left[\sqrt{r}\right]_0 &= 0 \\ \alpha_0^+ &= \frac{1}{2}+\frac{1}{2}\sqrt{1-4\frac{1}{4}}=\frac{1}{2} \\ \alpha_0^- &= \frac{1}{2}-\frac{1}{2}\sqrt{1-4\frac{1}{4}}=\frac{1}{2} \end{aligned}$$

For the second pole at $x=1$, since its order is 4, then $2v=4$ or $v=2$. Therefore the corresponding $\left[\sqrt{r}\right]_1$ is the sum of all terms involving $\frac{1}{(x-1)^i}$ for $2\le i\le v$ or $2\le i\le 2$ in the Laurent series expansion of $\sqrt{r}$ (not r) around this pole. This results in

$$\begin{aligned} [\sqrt{r}]_1 &= \sum_{i=2}^{2}\frac{a_i}{(x-1)^i} \\ &= \frac{a_2}{(x-1)^2} \qquad (3) \end{aligned}$$

a_2 is found using

$$
\begin{aligned}
a_2 &= \lim_{x\to 1}(x-1)^2\sqrt{r}\\
&= \lim_{x\to 1}(x-1)^2\sqrt{\frac{7x^2+10x-1}{4x^2(x-1)^4}}\\
&= 2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
[\sqrt{r}]_1 &= \frac{2}{(x-1)^2}\\
\alpha_1^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) &= \frac{1}{2}\left(\frac{b}{2}+2\right)\\
\alpha_1^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &= \frac{1}{2}\left(-\frac{b}{2}+2\right)
\end{aligned}
\tag{4}
$$

What remains is to determine b. This is the coefficient of $\frac{1}{(x-c)^{v+1}} = \frac{1}{(x-1)^3}$ in the partial fraction decomposition of r which from (2) is -2 minus the coefficient of same term in $[\sqrt{r}]_1$ which from (3) is zero. Therefore $b = -2 - 0 = -2$. (4) now becomes

$$
\begin{aligned}
[\sqrt{r}]_1 &= \frac{2}{(x-1)^2}\\
\alpha_1^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) &= \frac{1}{2}\left(\frac{-2}{2}+2\right) &= \frac{1}{2}\\
\alpha_1^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &= \frac{1}{2}\left(-\frac{-2}{2}+2\right) &= \frac{3}{2}
\end{aligned}
\tag{5}
$$

The above completes finding $[\sqrt{r}]_c, \alpha_c^+, \alpha_c^+$ for all poles in the set Γ.

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= 0\\
\alpha_\infty^- &= 1
\end{aligned}
$$

This completes the first step of the solution. The following tables summarizes the findings so far

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Table 2: First step, case one. Γ set information

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Table 3: First step, case one. $\mathcal{O}(\infty)$ information

This completes step 1 for case one. Step 2 searches for non-negative integer d using

$$d = \alpha_\infty^\pm - \sum_{c\in\Gamma} \alpha_c^\pm$$

Where the above is carried over all possible combinations resulting in the following 8 possibilities (in this example, there are two poles, hence the sum contains two term) and the number of possible combinations is therefore $2^3 = 8$

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_0^+ + \alpha_1^+\right) = 0 - \left(\frac{1}{2} + \frac{1}{2}\right) = -1\\
d &= \alpha_\infty^+ - \left(\alpha_0^+ + \alpha_1^-\right) = 0 - \left(\frac{1}{2} + \frac{3}{2}\right) = -2\\
d &= \alpha_\infty^+ - \left(\alpha_0^- + \alpha_1^+\right) = 0 - \left(\frac{1}{2} + \frac{1}{2}\right) = -1\\
d &= \alpha_\infty^+ - \left(\alpha_0^- + \alpha_1^-\right) = 0 - \left(\frac{1}{2} + \frac{3}{2}\right) = -2\\
d &= \alpha_\infty^- - \left(\alpha_0^+ + \alpha_1^+\right) = 1 - \left(\frac{1}{2} + \frac{1}{2}\right) = 0\\
d &= \alpha_\infty^- - \left(\alpha_0^+ + \alpha_1^-\right) = 1 - \left(\frac{1}{2} + \frac{3}{2}\right) = -1\\
d &= \alpha_\infty^- - \left(\alpha_0^- + \alpha_1^+\right) = 1 - \left(\frac{1}{2} + \frac{1}{2}\right) = 0\\
d &= \alpha_\infty^- - \left(\alpha_0^- + \alpha_1^-\right) = 1 - \left(\frac{1}{2} + \frac{3}{2}\right) = -1
\end{aligned}$$

There is only one possible $d = 0$ values to use. Candidate ω are now found using

$$\omega = \sum_c \left((\pm) \left[\sqrt{r}\right]_c + \frac{\alpha_c^\pm}{x-c} \right) + (\pm) \left[\sqrt{r}\right]_\infty$$

Which gives

$$
\begin{aligned}
\omega_1 &= \left((+)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^+}{x}\right) + \left((+)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^+}{x-1}\right) + (+)\left[\sqrt{r}\right]_\infty \\
\omega_2 &= \left((+)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^+}{x}\right) + \left((-)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^-}{x-1}\right) + (+)\left[\sqrt{r}\right]_\infty \\
\omega_3 &= \left((-)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^-}{x}\right) + \left((+)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^+}{x-1}\right) + (+)\left[\sqrt{r}\right]_\infty \\
\omega_4 &= \left((-)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^-}{x}\right) + \left((-)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^-}{x-1}\right) + (+)\left[\sqrt{r}\right]_\infty \\
\omega_5 &= \left((+)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^+}{x}\right) + \left((+)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^+}{x-1}\right) + (-)\left[\sqrt{r}\right]_\infty \\
\omega_6 &= \left((+)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^+}{x}\right) + \left((-)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^-}{x-1}\right) + (-)\left[\sqrt{r}\right]_\infty \\
\omega_7 &= \left((-)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^-}{x}\right) + \left((+)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^+}{x-1}\right) + (-)\left[\sqrt{r}\right]_\infty \\
\omega_8 &= \left((-)\left[\sqrt{r}\right]_0 + \frac{\alpha_0^-}{x}\right) + \left((-)\left[\sqrt{r}\right]_1 + \frac{\alpha_1^-}{x-1}\right) + (-)\left[\sqrt{r}\right]_\infty
\end{aligned}
$$

Substituting values found in step 1 into the above gives

$$
\begin{aligned}
\omega_1 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((+)\frac{2}{(x-1)^2} + \frac{\frac{1}{2}}{x-1}\right) = \frac{2x^2+x+1}{2x(x-1)^2} \\
\omega_2 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((-)\frac{2}{(x-1)^2} + \frac{\frac{3}{2}}{x-1}\right) = \frac{4x^2-9x+1}{2x(x-1)^2} \\
\omega_3 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((+)\frac{2}{(x-1)^2} + \frac{\frac{1}{2}}{x-1}\right) = \frac{2x^2+x+1}{2x(x-1)^2} \\
\omega_4 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((-)\frac{2}{(x-1)^2} + \frac{\frac{3}{2}}{x-1}\right) = \frac{4x^2-9x+1}{2x(x-1)^2} \\
\omega_5 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((+)\frac{2}{(x-1)^2} + \frac{\frac{1}{2}}{x-1}\right) = \frac{2x^2+x+1}{2x(x-1)^2} \\
\omega_6 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((-)\frac{2}{(x-1)^2} + \frac{\frac{3}{2}}{x-1}\right) = \frac{4x^2-9x+1}{2x(x-1)^2} \\
\omega_7 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((+)\frac{2}{(x-1)^2} + \frac{\frac{1}{2}}{x-1}\right) = \frac{2x^2+x+1}{2x(x-1)^2} \\
\omega_8 &= \left(\frac{\frac{1}{2}}{x}\right) + \left((-)\frac{2}{(x-1)^2} + \frac{\frac{3}{2}}{x-1}\right) = \frac{4x^2-9x+1}{2x(x-1)^2}
\end{aligned}
$$

Which shows there are only two different ω to try, these are ω_1, ω_2. This complete step 2. For each trial, step 3 is now invoked. Starting with $d = 0$ and $\omega = \omega_1 = \frac{2x^2+x+1}{2x(x-1)^2}$. Since the degree $d=0$ then $p(x)=1$. This polynomial needs to satisfy

$$\begin{aligned} p''+2\omega p'+(\omega'+\omega^2-r)p &= 0 \\ (\omega'+\omega^2-r)p &= 0 \\ \frac{d}{dx}\left(\frac{2x^2+x+1}{2x(x-1)^2}\right)+\left(\frac{2x^2+x+1}{2x(x-1)^2}\right)^2-\frac{7x^2+10x-1}{4x^2(x-1)^4} &= 0 \\ 0 &= 0 \end{aligned} \tag{6}$$

Because the equation is satisfied, the polynomial $p(x)=1$ can be used. The solution to $z''=rz$ is now found from

$$\begin{aligned} z &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{2x^2+x+1}{2x(x-1)^2} dx} \\ &= e^{\frac{\ln(x-1)}{2}-\frac{2}{x-1}+\frac{\ln(x)}{2}} \\ &= \sqrt{x-1}\,\sqrt{x}\,\mathrm{e}^{-\frac{2}{x-1}} \end{aligned}$$

Given this solution for $z(x)$, the first basis solution of the original ode in y is found using the inverse of the original transformation used to generate the z ode which is $z=ye^{\frac{1}{2}\int a\,dx}$, therefore

$$\begin{aligned} y_1 &= ze^{-\frac{1}{2}\int a\,dx} \\ &= \sqrt{x-1}\sqrt{x}\mathrm{e}^{-\frac{2}{x-1}}\mathrm{e}^{-\frac{1}{2}\int -\frac{x(3+x)}{x^2(x^2-2x+1)} dx} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{x-1}\mathrm{e}^{-\frac{4}{x-1}}$$

The second solution y_2 to the original ode is found using reduction of order as was done in the first example.

3.1.3 Example 3

This ode is a standard second order representing the oscillating harmonics ode with constant coefficients and does not require Kovacic algorithm to solve it as it can be readily solved using standard method by finding the roots of the characteristic equation. It is included here in order to illustrate the Kovacic algorithm.

$$\begin{aligned} y''+y'+y &= 0 \\ Ay''+By'+Cy &= 0 \end{aligned}$$

Converting it to $z''=rz$ as shown before gives

$$\begin{aligned} z'' &= \frac{s}{t}z \\ &= \frac{-3}{4}z \end{aligned}$$

Hence $r = \frac{-3}{4}$. There are no poles therefore $\Gamma = \{\}$, and $\mathcal{O}(\infty) = \deg(t) - \deg(s) = 0$. Table 1 shows that the necessary conditions for case one are only satisfied. Since the set Γ is empty, then only the quantities related to $\mathcal{O}(\infty)$ need to be calculated. The order of r at ∞ is $O_r(\infty) = 0$ therefore $v = 0$. r has no x in it, hence the Laurent series of $\sqrt{r}$ at ∞ is itself

$$\sqrt{r} = \frac{i\sqrt{3}}{2}$$

Therefore

$$a = \frac{i\sqrt{3}}{2}$$

And since r is constant then $b = 0$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{i\sqrt{3}}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{3}}{2}} - 0\right) &&= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{3}}{2}} - 0\right) &&= 0
\end{aligned}$$

This completes step 1 for case one. Step 2 searches for non-negative integer d using

$$d = \alpha_\infty^\pm - \sum_{c \in \Gamma} \alpha_c^\pm$$

Since there are no poles then

$$\begin{aligned}
d &= \alpha_\infty^- \\
&= 0
\end{aligned}$$

Since d is non-negative integer integer it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above reduces to

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{i\sqrt{3}}{2}\right) \\
&= -\frac{i\sqrt{3}}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1}$$

Since $d = 0$ then let $p(x) = 1$. Substituting this in the above gives

$$\begin{aligned}(0) + 2\left(-\frac{i\sqrt{3}}{2}\right)(0) + \left((0) + \left(-\frac{i\sqrt{3}}{2}\right)^2 - \left(-\frac{3}{4}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{i\sqrt{3}}{2} dx} \\ &= \mathrm{e}^{-\frac{i\sqrt{3}x}{2}}\end{aligned}$$

The first solution to the original ode in y is now found from (using $A = 1, B = 1$)

$$\begin{aligned}y_1 &= z e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z e^{-\int \frac{1}{2} dx} \\ &= z e^{-\frac{x}{2}} \\ &= \mathrm{e}^{-\frac{i\sqrt{3}x}{2}}\left(\mathrm{e}^{-\frac{x}{2}}\right) \\ &= \mathrm{e}^{-\frac{x\left(1+\mathrm{I}\sqrt{3}\right)}{2}}\end{aligned}$$

The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \, dx \\ &= y_1 \int \frac{e^{-\int dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\ &= \left(\mathrm{e}^{-\frac{x\left(1+\mathrm{I}\sqrt{3}\right)}{2}}\right)\left(-\frac{i\sqrt{3}\,\mathrm{e}^{i\sqrt{3}x}}{3}\right) \\ &= -\frac{\mathrm{I}}{3}\mathrm{e}^{\frac{x\left(\mathrm{I}\sqrt{3}-1\right)}{2}}\sqrt{3}\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x\left(1+i\sqrt{3}\right)}{2}}\right) + c_2\left(-\frac{i\mathrm{e}^{\frac{x\left(i\sqrt{3}-1\right)}{2}}\sqrt{3}}{3}\right)\end{aligned}$$

Using Euler's formula the above can be simplified to the standard looking solution

$$y(x) = e^{-\frac{x}{2}}\left(C_1 \sin\left(\frac{\sqrt{3}\,x}{2}\right) + C_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

3.2 case two

3.2.1 Example 1

Given the ode

$$2x^2y'' - xy' + (1+x)y = 0$$

Converting it $y'' + ay' + by = 0$ gives

$$y'' - \frac{1}{2x}y' + \frac{1+x}{2x^2}y = 0$$

Where $a = -\frac{1}{2x}, b = \frac{1+x}{2x^2}$. Applying the transformation $z = ye^{\frac{1}{2}\int a\,dx}$ gives $z'' = rz$ and $r = \frac{1}{4}a^2 + \frac{1}{2}a' - b$. Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-3-8x}{16x^2} \end{aligned}$$

There is one pole at $x = 0$ of order 2 and $\mathcal{O}(\infty) = \deg(t) - \deg(s) = 2 - 1 = 1$. Table 1 shows that necessary conditions for only case two are satisfied. Since pole is order 2, the set $E_0 = \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}$ where b is the coefficient of $\frac{1}{(x)^2}$ in the partial fraction decomposition of r given by

$$r = -\frac{3}{16x^2} - \frac{1}{2x}$$

Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_0 &= \left\{2, 2+2\sqrt{1-4\frac{3}{16}}, 2-2\sqrt{1-4\frac{3}{16}}\right\} \\ &= \{2, 3, 1\} \end{aligned}$$

Since $\mathcal{O}(\infty) = 1 < 2$ then $E_\infty = \mathcal{O}(\infty) = \{1\}$. This completes step 1 of case two. Step 2 is used to determine a non-negative integer d. Using $e_\infty = 1$ gives

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

There is only one pole, so the sum contains only one term. There are 3 possible combinations to try, using either $e_0 = 2$, $e_0 = 3$ or $e_0 = 1$. Therefore

$$\begin{aligned} d &= \frac{1}{2}(e_\infty - (e_0)) = \frac{1}{2}(1-2) = -\frac{1}{2} \\ &= \frac{1}{2}(e_\infty - (e_0)) = \frac{1}{2}(1-3) = -1 \\ &= \frac{1}{2}(e_\infty - (e_0)) = \frac{1}{2}(1-1) = 0 \end{aligned}$$

The above shows that only the family $\{e_\infty = 1, e_0 = 1\}$ generated non-negative $d = 0$. θ is now found. In the following sum, only e_c retained from the above are used. In this example, this will be $e_0 = 1$ since it is the member of E_0 which generated non-negative integer. If there were more than one e_i found, then each would be tried at time.

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{e_0}{x-0}\right) \\ &= \frac{1}{2x} \end{aligned}$$

This completes step 2. Step 3 finds polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + x^d$ of degree d. Since $d = 0$ then $p(x) = 1$. This polynomial has to satisfy the following

$$p''' + 3\theta p''\left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0$$

Substituting $p = 1, \theta = \frac{1}{2x}$ into the above and simplifying gives

$$0 = 0$$

Since $p(x) = 1$ is verified, then

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Next, ω solution is found using

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above gives

$$w^2 - \frac{w}{2x} + \frac{1+8x}{16x^2} = 0$$

Solving for ω gives two roots, either one can be used. Using

$$\omega = \frac{1 + 2\sqrt{2}\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\sqrt{-x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is now found from

$$\begin{aligned} y_1 &= z e^{\int -\frac{1}{2} a \, dx} \\ &= z e^{-\int \frac{1}{2} \frac{-x}{2x^2} dx} \\ &= z e^{\frac{\ln(x)}{4}} \\ &= \left(x^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\sqrt{-x}} \right) x^{\frac{1}{4}} \\ &= \sqrt{x}\, \mathrm{e}^{\sqrt{2}\sqrt{-x}} \end{aligned}$$

The second solution y_2 to the original ode can be found using reduction of order.

3.2.2 Example 2

This is an ode in which the necessary conditions for all three cases are satisfied, but solved using case two to illustrate the algorithm.

$$(1-x)x^2 y'' + (5x-4)xy' + (6-9x)y = 0$$

Converting it $y'' + ay' + by = 0$ gives

$$y'' + \frac{5x-4}{(1-x)x} y' + \frac{6-9x}{(1-x)x^2} y = 0$$

Where $a = \frac{5x-4}{(1-x)x}, b = \frac{6-9x}{(1-x)x^2}$. Applying the transformation $z = ye^{\frac{1}{2}\int a\,dx}$ gives $z'' = rz$ where $r = \frac{1}{4}a^2 + \frac{1}{2}a' - b$. Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4-x}{4x(x-1)^2} \end{aligned}$$

There is a pole at $x = 0$ of order 1 and a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are satisfied. Since there is a pole of order 2 then the necessary conditions for case two are also satisfied. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are also satisfied. Any one of the three cases algorithm could be used to solve this, but here case two will be used for illustration.

The pole of order 1 at $x = 0$ gives $E_0 = \{4\}$ and the pole of order 2 at $x = 1$ gives $E_1 = \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}$ where b is the coefficient of $\frac{1}{(x-1)^2}$ in the partial fraction decomposition of r. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-1)^2} - \frac{1}{x-1} + \frac{1}{x}$$

The above shows that $b = \frac{3}{4}$, therefore $E_1 = \{-2, 2, 6\}$.

$\mathcal{O}(\infty) = 2$ therefore $E_\infty = \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}$ where $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. This gives $b = -\frac{1}{4}$, hence

$$
\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \left\{2, 2+2\sqrt{1-4\frac{1}{4}}, 2-2\sqrt{1-4-\frac{1}{4}}\right\} \\
&= \{2, 2, 2\} \\
&= \{2\}
\end{aligned}
$$

The following table summarizes step 1 results.

pole c location	pole order	E_c
0	1	$\{4\}$
1	2	$\{-2, 2, 6\}$

Table 4: First step, case two. E_c set information

Order of r at ∞	E_∞
2	$\{2\}$

Table 5: First step, case two. $\mathcal{O}(\infty)$ information

The above completes step 1 for case two. Step 2 searches for a non-negative integer d using

$$
d = \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)
$$

Where in the above $e_c \in E_c$, $e_\infty \in E_\infty$ were found in the first step. The following are the possible combinations to use

$$
\begin{aligned}
d &= \frac{1}{2}(2-(4-2)) = 0 \\
&= \frac{1}{2}(2-(4+2)) = -2 \\
&= \frac{1}{2}(2-(4+6)) = -4
\end{aligned}
$$

The above shows that $E_c = \{e_0 = 4, e_1 = -2\}$ are the family of values to use and all other values are discarded.

The following rational function θ is determined using

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{4}{(x-(0))}+\frac{-2}{(x-(1))}\right) \\
&= \frac{2}{x}-\frac{1}{x-1}
\end{aligned}
$$

The algorithm now searches for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0
$$

Since $d=0$, then $p(x)=1$. Substituting the values found in step 2 in the above equation and simplifying gives

$$
0=0
$$

Hence $p(x)=1$ can be used. Let

$$
\begin{aligned}
\phi &= \theta+\frac{p'}{p} \\
&= \frac{2}{x}-\frac{1}{x-1}
\end{aligned}
$$

And ω be the solution of

$$
\omega^2-\phi\omega+\left(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r\right)=0
$$

Substituting the values for ϕ and r into the above equation gives

$$
w^2-\left(\frac{2}{x}-\frac{1}{x-1}\right)w+\frac{(x-2)^2}{4(x-1)^2x^2}=0
$$

Solving for ω gives two roots, either one can be used. Using

$$
\omega=\frac{x-2}{2(x-1)x}
$$

Therefore to $z''=rz$ is

$$
\begin{aligned}
z &= e^{\int \omega\,dx} \\
&= e^{\int \frac{x-2}{2(x-1)x}dx} \\
&= \frac{x}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is now found from

$$
\begin{aligned}
y_1 &= z e^{-\int \frac{1}{2} a \, dx} \\
&= z e^{-\int \frac{1}{2} \frac{5x^2-4x}{-x^3+x^2} \, dx} \\
&= z e^{2\ln(x)+\frac{\ln(x-1)}{2}} \\
&= z \left(x^2 \sqrt{x-1}\right) \\
&= \frac{x}{\sqrt{x-1}} \left(x^2 \sqrt{x-1}\right) \\
&= x^3
\end{aligned}
$$

The second solution y_2 to the original ode is found using reduction of order.

3.3 case three

3.3.1 Example 1

This is the same ode used in second example above for case two as the necessary conditions for case three are also satisfied.

$$
(1-x)x^2 y'' + (5x-4)xy' + (6-9x)y = 0
$$

As shown earlier, this ode is transformed to $z'' = rz$ where

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4-x}{4x(x-1)^2}
\end{aligned}
$$

There is a pole at $x=0$ of order 1 and a pole at $x=1$ of order 2. For the pole of order 1 at $x=0$, $E_0 = \{12\}$. For the pole of order 2 at $x=1$

$$
E_1 = \left\{ 6 + \frac{12k}{n} \sqrt{1+4b} \right\} \qquad \text{for} \quad k = -\frac{n}{2} \cdots \frac{n}{2} \tag{1}
$$

Where k is incremented by 1 each time, and n is any of $\{4,6,12\}$ and b is the coefficient of $\frac{1}{(x-1)^2}$ in the partial fraction decomposition of r which is

$$
r = \frac{3}{4(x-1)^2} - \frac{1}{x-1} + \frac{1}{x}
$$

The above shows that $b = \frac{3}{4}$. Starting with $n=4$ (if this n produces no solution then $n = 6, 12$ will be tried also). Equation (1) now becomes

$$
E_1 = \left\{ 6 + \frac{12k}{4} \sqrt{1+4\left(\frac{3}{4}\right)} \right\} \qquad \text{for} \quad k = -2 \cdots 2
$$

Which simplifies to

$$\begin{aligned} E_1 &= \{6+6k\} \qquad \text{for} \quad k=-2\cdots 2 \\ &= \{-6,0,6,12,18\} \end{aligned}$$

E_∞ is found using (1) but with different b. In this case b is given by $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. lcoef(s) is the leading coefficient of s and lcoef(t) is the leading coefficient of t. Since $r = \frac{-x+4}{4x^3-8x^2+4x}$ then $b = -\frac{1}{4}$. Equation (1) becomes

$$E_\infty = \left\{6 + \frac{12k}{4}\sqrt{1-4\left(\frac{1}{4}\right)}\right\} \qquad \text{for} \quad k=-2\cdots 2$$

This simplifies to

$$E_\infty = \{6\}$$

The following table summarizes step 1 results using $n=4$.

pole c location	pole order	E_c
0	1	$\{12\}$
1	2	$\{-6,0,6,12,18\}$

Table 6: First step, case three using $n=4$. E_c set information

Order of r at ∞	E_∞
2	$\{6\}$

Table 7: First step, case three using $n=4$. $\mathcal{O}(\infty)$ information

The next step is to determine a non negative integer d using

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier.

This results in the following values for d using $n=4$ and $e_\infty = 6$.

$$\begin{aligned} d &= \frac{1}{3}(6-(12-6)) &&= 0 \\ &= \frac{1}{3}(6-(12+0)) &&= -2 \\ &= \frac{1}{3}(6-(12+6)) &&= -4 \\ &= \frac{1}{3}(6-(12+12)) &&= -6 \\ &= \frac{1}{3}(6-(12+18)) &&= -8 \end{aligned}$$

Therefore only the first case using $e_0 = 12, e_1 = -6$ generated non-negative integer d. The following rational function is now formed

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{4}{12}\left(\frac{12}{(x-(0))}+\frac{-6}{(x-(1))}\right) \\ &= \frac{2x-4}{(x-1)x} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= (x-0)(x-1) \\ &= x(x-1) \end{aligned}$$

This completes the step 2 of the algorithm.

Since the degree $d=0$, then $p(x)=1$. Now $P_i(x)$ polynomials are generated using

$$\begin{aligned} P_n &= -p(x) \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned}$$

These generate the following set

$$\begin{aligned} P_4 &= -1 \\ P_3 &= 2x-4 \\ P_2 &= -3(x-2)^2 \\ P_1 &= 3(x-2)^3 \\ P_0 &= -\frac{3(x-2)^4}{2} \\ P_{-1} &= 0 \end{aligned}$$

There is nothing to solve for from the last equation $P_{-1} = 0$ as $p(x) = 1$ is already known because the degree d was zero and hence there are no coefficients a_i to solve for.

Next ω is determined as the solution to the following equation using $n = 4$.

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0 \\
\frac{P_0}{4!} + \frac{SP_1}{3!}\omega + \frac{S^2P_2}{2!}\omega^2 + \frac{S^3P_3}{1!}\omega^3 + + \frac{S^4P_4}{0!}\omega^4 &= 0 \\
-\frac{1}{16}\left(2\omega x^2 - 2x\omega - x + 2\right)^4 &= 0
\end{aligned}
$$

Solving the above and using any one of the roots results in

$$
\omega = \frac{1}{2x(x-1)}(x-2)
$$

The above ω is used to find a solution to $z'' = rz$ from

$$
\begin{aligned}
z &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{x-2}{2x(x-1)}dx} \\
&= \frac{x}{\sqrt{x-1}}
\end{aligned}
$$

Therefore one solution to the original ode in y is

$$
\begin{aligned}
y &= ze^{\int -\frac{1}{2}a\,dx} \\
&= ze^{-\int \frac{1}{2}\frac{5x^2-4x}{-x^3+x^2}dx} \\
&= ze^{2\ln(x)+\frac{\ln(x-1)}{2}} \\
&= \left(\frac{x}{\sqrt{x-1}}\right)\left(x^2\sqrt{x-1}\right)
\end{aligned}
$$

The second solution to the original ode is found using reduction of order. This completes the solution using case 3 for degree $n = 4$ of ω.

3.3.2 Example 2

The ode is

$$
x^2(1+x)y'' + x(2x+1)y'' - (4+6x)y = 0
$$

Converting it $y'' + ay' + by = 0$ gives

$$
y'' + \frac{2x+1}{x(1+x)}y' - \frac{4+6x}{x^2(1+x)}y = 0
$$

Where $a = \frac{2x+1}{x(1+x)}, b = -\frac{4+6x}{x^2(1+x)}$. Applying the transformation $z = ye^{\frac{1}{2}\int a\,dx}$ results in $z'' = rz$ where $r = \frac{1}{4}a^2 + \frac{1}{2}a' - b$ where

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{24x^2 + 40x + 15}{4(x(x+1))^2}
\end{aligned}
$$

There is a pole at $x = 0$ of order 2 and a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are satisfied. Since there is a pole of order 2 then necessary conditions for case two are also satisfied. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are also satisfied. This is now solved as case three for illustration.

Starting with $n = 4$, and for the pole of order 2 at $x = -1$

$$
E_{-1} = \left\{6 + \frac{12k}{n}\sqrt{1+4b}\right\} \qquad \text{for} \quad k = -\frac{n}{2} \cdots \frac{n}{2}
$$

This simplifies to

$$
E_{-1} = \left\{6 + 3k\sqrt{1+4b}\right\} \qquad \text{for} \quad k = -2 \cdots 2 \tag{1}
$$

b is the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given by

$$
r = -\frac{1}{4(1+x)^2} + \frac{15}{4x^2} - \frac{5}{2(1+x)} + \frac{5}{2x}
$$

The above shows that $b = -\frac{1}{4}$. Equation (1) becomes

$$
\begin{aligned}
E_{-1} &= \left\{6 + 3k\sqrt{1 - 4\left(\frac{1}{4}\right)}\right\} \qquad \text{for} \quad k = -2 \cdots 2 \\
&= \{6\}
\end{aligned}
$$

For the pole at $x = 0$ of order 2, b is the coefficient of $\frac{1}{x^2}$ in the above partial fractions decomposition of r. This shows that $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
E_0 &= \left\{6 + 3k\sqrt{1 + 4\left(\frac{15}{4}\right)}\right\} \qquad \text{for} \quad k = -2 \cdots 2 \\
&= \{-18, -6, 6, 18, 30\}
\end{aligned}
$$

E_∞ is found using equation (1) but with different b. In this case b is given by $b = \frac{\text{lcoef}(s)}{\text{lcoeff}(t)}$ where $r = \frac{s}{t}$. lcoef(s) is the leading coefficient of s and lcoef(t) is the leading coefficient of t. Since $r = \frac{24x^2+40x+15}{4x^4+8x^3+4x^2}$ then $b = 6$. Equation (1) becomes

$$
\begin{aligned}
E_\infty &= \left\{6 + 3k\sqrt{1+4(6)}\right\} \qquad \text{for} \quad k = -2 \cdots 2 \\
&= \{-24, -9, 6, 21, 36\}
\end{aligned}
$$

The following table summarizes step 1 results using $n = 4$.

pole c location	pole order	E_c
-1	2	$\{6\}$
0	2	$\{-18, -6, 6, 18, 30\}$

Table 8: First step, case three using $n = 4$. E_c set information

Order of r at ∞	E_∞
2	$\{-24, -9, 6, 21, 36\}$

Table 9: First step, case three using $n = 4$. $\mathcal{O}(\infty)$ information

The next step is to determine a non negative integer d using

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier.

This results in the following values for d using $n = 4$.

$$\begin{aligned}
d &= \frac{1}{3}(-24-(6-18)) &&= -4 \\
&= \frac{1}{3}(-24-(6+6)) &&= -12 \\
&= \frac{1}{3}(-24-(6-6)) &&= -8 \\
&= \frac{1}{3}(-24-(6+18)) &&= -16 \\
&= \frac{1}{3}(-24-(6+30)) &&= -20 \\
&= \frac{1}{3}(-9-(6-18)) &&= 1 \\
&= \frac{1}{3}(-9-(6+6)) &&= -7 \\
&= \frac{1}{3}(-9-(6-6)) &&= -3 \\
&= \frac{1}{3}(-9-(6+18)) &&= -11 \\
&= \frac{1}{3}(-9-(6+30)) &&= -15 \\
&= \frac{1}{3}(6-(6-18)) &&= 6 \\
&= \frac{1}{3}(6-(6+6)) &&= -2 \\
&= \frac{1}{3}(6-(6-6)) &&= 2 \\
&= \frac{1}{3}(6-(6+18)) &&= -6 \\
&= \frac{1}{3}(6-(6+30)) &&= -10 \\
&= \frac{1}{3}(21-(6-18)) &&= 11 \\
&= \frac{1}{3}(21-(6+6)) &&= 3 \\
&= \frac{1}{3}(21-(6-6)) &&= 7 \\
&= \frac{1}{3}(21-(6+18)) &&= -1 \\
&= \frac{1}{3}(21-(6+30)) &&= -5 \\
&= \frac{1}{3}(36-(6-18)) &&= 16 \\
&= \frac{1}{3}(36-(6+6)) &&= 8 \\
&= \frac{1}{3}(36-(6-6)) &&= 12 \\
&= \frac{1}{3}(36-(6+18)) &&= 4 \\
&= \frac{1}{3}(36-(6+30)) &&= 0
\end{aligned}$$

From the above, the following families all produce non-negative d

$$
\begin{array}{llll}
e_\infty = -9 & e_{-1} = 6 & e_0 = -18 & d = 1 \\
e_\infty = 6 & e_{-1} = 6 & e_0 = -18 & d = 6 \\
e_\infty = 6 & e_{-1} = 6 & e_0 = -6 & d = 2 \\
e_\infty = 21 & e_{-1} = 6 & e_0 = -18 & d = 11 \\
e_\infty = 21 & e_{-1} = 6 & e_0 = 6 & d = 3 \\
e_\infty = 21 & e_{-1} = 6 & e_0 = -6 & d = 7 \\
e_\infty = 36 & e_{-1} = 6 & e_0 = -18 & d = 16 \\
e_\infty = 36 & e_{-1} = 6 & e_0 = 6 & d = 8 \\
e_\infty = 36 & e_{-1} = 6 & e_0 = -6 & d = 12 \\
e_\infty = 36 & e_{-1} = 6 & e_0 = 18 & d = 4 \\
e_\infty = 36 & e_{-1} = 6 & e_0 = 30 & d = 0
\end{array}
$$

Starting with the smallest $d = 0$ as that is the least computationally expensive with its corresponding $e_{-1} = 6, e_0 = 30, e_\infty = 36$, the following rational function is now formed

$$
\begin{aligned}
\theta &= \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x - c} \\
&= \frac{4}{12} \left(\frac{6}{x+1} + \frac{30}{x} \right) \\
&= \frac{12x + 10}{x(1+x)}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c \in \Gamma} (x - c) \\
&= (x+1)x
\end{aligned}
$$

This completes the step 2 of the algorithm.

Polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then

$$
p(x) = 1
$$

The $P_i(x)$ polynomials are generated using

$$
\begin{aligned}
P_n &= -p(x) \\
P_{i-1} &= -S p_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned}
$$

The above results in the following set

$$
\begin{aligned}
P_4 &= -1 \\
P_3 &= 12x + 10 \\
P_2 &= -3(6x+5)^2 \\
P_1 &= 3(6x+5)^3 \\
P_0 &= -\frac{3(6x+5)^4}{2} \\
P_{-1} &= 0
\end{aligned}
$$

There is coefficient for $p(x)$ to solve for from the last equation $P_{-1} = 0$ as $p(x) = 1$ is already known because the degree d is zero. ω is now determined as the solution to the following equation, using $n = 4$

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0 \\
\frac{P_0}{4!} + \frac{SP_1}{3!}\omega + \frac{S^2P_2}{2!}\omega^2 + \frac{S^3P_3}{1!}\omega^3 + + \frac{S^4P_4}{0!}\omega^4 &= 0 \\
-\frac{1}{16}\left(2\omega x^2 + 2x\omega - 6x - 5\right)^4 &= 0
\end{aligned}
$$

Solving the above and using any one of the roots gives

$$
\omega = \frac{x-2}{2x(x-1)}
$$

This ω is used to find a solution to $z'' = rz$ from

$$
\begin{aligned}
z &= e^{\int \omega dx} \\
&= \mathrm{e}^{\int \frac{6x+5}{2x(1+x)} dx} \\
&= x^{\frac{5}{2}}\sqrt{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y &= z e^{\int -\frac{1}{2} a \, dx} \\
&= z e^{-\int \frac{1}{2} \frac{2x^2+x}{x^2(1+x)} dx} \\
&= z e^{-\frac{\ln(x(1+x))}{2}} \\
&= \left(x^{\frac{5}{2}}\sqrt{1+x}\right)\left(\frac{1}{\sqrt{x(1+x)}}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x^2
$$

The second solution to the original ode is found using reduction of order.

4 Conclusion

Detailed description of the Kovacic algorithm with worked out examples were given. All three cases of the Kovacic algorithm were implemented using object oriented design in Maple. The software was then used to analyze over 3000 differential equations. The results showed that case one and two combined provided coverage for 99.9% of the ode's with 97.36% of the ode's solved using case one algorithm and 2.54% solved using case 2 algorithm with only 0.1% requiring case 3. Not a single ode was found that required the use of case three with $n = 6$ or $n = 12$.

One restriction found on the use of the algorithm is that it requires an ode with its coefficients being numerical and not symbolic. This is because the algorithm has to decide in step 2 if d (the degree of polynomial $p(x)$) is non-negative integer or not in order to continue to step 3. If some of the ode coefficients were symbolic, it will not be able to decide on this (without additional assumptions provided). Therefore this algorithm works best with ode's having its coefficients given with numerical values.

5 Appendix

5.1 Instructions and examples using the Kovacic package

The Kovacic class is included in the file KOV.mpl and the Kovacic testsuite module is in the file kovacic_tester.mpl. These two files accompany the arXiv version of this paper.

To use these, download these two files to some directory at your computer. For example, on windows, assuming the files were downloaded to c:/my_folder/, then now start Maple and type

```
read "c:/my_folder/KOV.mpl"
read "c:/my_folder/kovacic_tester.mpl"
```

The above will load the kovacic_class and the testsuite module. Once the above is successfully completed, then to solve an ode the command is

```
ode  := diff(y(x),x$2)+diff(y(x),x)+y(x)=0;
o    := Object(kovacic_class,ode,y(x)); #create the object
sol  := o:-dsolve();
```

The above command will automatically try all the cases that have been detected one by one until a solution is found. If no solution is found, it returns FAIL. To verify the solution, the command is

```
if sol<>FAIL then
   odetest(sol,ode);
fi;
```

Which returns 0 if the solution is correct.

A note on the type of ode's supported: it is recommenced to use only ode's with numeric coefficients and not symbolic coefficients. This is because the Kovacic algorithm needs to decide if the degree d of the polynomial $p(x)$ is non-negative or not. If some of the coefficients are purely symbolic, then it can fail to decide this. An example of this is given in the original Kovacic paper as example 2 on page 14, which is to solve the Bessel ode $y'' = \left(\frac{4n^2-1}{4x^2} - 1\right) y$. This will now return FAIL since the algorithm can not decide if d is non-negative integer without knowing any assumptions or having numerical value for n. Replacing n by any half odd integer, then it can solve it as follows

```
ode :=diff(diff(y(x),x),x)=((4*n^2-1)/(4*x^2)-1)*y(x);
n   :=-3/2;
o   := Object(kovacic_class,ode,y(x));
sol := o:-dsolve();
```

To solve an ode using specific case number, say case 2, the command is

```
ode   := ...;
o     := Object(kovacic_class,ode,y(x));
sol   := o:-dsolve_case(2);
```

If the ode happened to satisfy cases 1 and 2 for an example, then the above command will only use case 2 to solve it and will skip case 1. If the command o:-dsolve() was used instead, then the ode will be solved using case 1 instead as that is the first one tried. Case 2 will only be tried is no solution is found using case 1.

The object created above, named as "o", has additional public methods that can be invoked. The following is description of all public methods available.

- o:-get_y_ode() This returns the original ode.
- o:-get_z_ode() This returns the ode solved by Kovacic algorithm which is $z'' = rz$.
- o:-get_r() This returns r only.
- o:-get_poles() This returns list of the poles of r. It has the format

  `[ [pole location,pole order],[pole location,pole order], ...]`

  If there are no poles, then the empty list [] is returned.
- o:-get_order_at_infinity() This returns the order of r at infinity.
- o:-get_possible_cases() This returns list of possible Kovacic cases detected which can be [1], [2], [1,2], [1,2,3]. If no Kovacic cases are found, then the empty list [] is returned.
- o:-get_case_used() This returns the actual case number used if solution to the ode was successful. This can be 1,2 or 3. If no solution is found after trying all cases whose conditions were satisfied, then -1 is returned.
- o:-get_n_case_3() This return n used when case 3 was used to solve the ode. This can be 4,6 or 12. If case 3 was not used, or no solution is found, then -1 is returned.

To run the full testsuite of 3000 ode's that comes with the package, the command is

```
kovacic_tester:-unit_test_main_api();
                   "Test ", 6735, " PASSED "
                   "Test ", 6736, " PASSED "
                   "Test ", 6737, " PASSED "
                             .
                             .
                   "Test ", 7579, " PASSED "
                   "Test ", 7580, " PASSED "
                   "Test ", 7581, " PASSED "
```

To run testsuite using specific cases only, the commands are

```
kovacic_tester:-unit_test_case_1();
kovacic_tester:-unit_test_case_2();
kovacic_tester:-unit_test_case_3();
```

5.2 Source code

```
#------------------------------------------------------------------
#FILE NAME  :   KOV.mpl
#
# Copyright (C) 2022, Nasser M. Abbasi. All rights reserved
# email: nma@12000.org
#
# Free software to use and modify in anyway as long as the above
# copyright notice remains attached in the file
#
# Change history
#-----------------
# Oct 27, 2022.     Initial version
#
# Note that the latest version and any updates can alaways be obtained
# from the author web site at
# http://12000.org/my_notes/my_paper_on_kovacic/paper.htm
#
# Any problems found in the software please report so I can correct.
# This implementation was done using Maple 2022.2 on windows 10.
#
#------------------------------------------------------------------

#------------------------------------------------------------------
# An Object oriented implementation of the Kovacic algortithm
# using Maple 2022.
#
# based on original Kovacic paper description of the algorithm.
#
# This file contains two modules. The first is called kovacic_class
# used to create kovacic object. The second is kovacic_tester module
# used for unit testing the kovacic_class module
#
# To use this file just do
#
#    read "KOV.mpl"
#    ode := diff(y(x),x$2)=2/x^2*y(x)
#    o   := Object(kovacic_class,ode,y(x));
#    sol := o:-dsolve();
#
# Make sure to set the currentdir() in Maple correctly in order to find
# where you downloaded the file "KOV.mpl" to.
#------------------------------------------------------------------

#------------------------------------------------------------------
# This class is used to solve an ode using Kovacic algorithm.
#
#  Please see documantion section in the paper for additional
#  information on using this class.
#------------------------------------------------------------------

kovacic_class :=module()
```

```
option object;

#class to hold one entry in the gamma set for case 1
local case_one_gamma_entry := module()
    option object;
    export pole_location := 0;
    export pole_order    := 0;
    export sqrt_r        := 0;
    export alpha_plus    := 0;
    export alpha_minus   := 0;
    export b:=0;
end module;

#class to hold O_infinity information for case 1
local case_one_O_inf := module()
    option object;
    export sqrt_r_inf      := 0;
    export alpha_plus_inf  := 0;
    export alpha_minus_inf := 0;
    export a := 0;
    export b := 0;
end module;

#class to hold one entry in the gamma set for case 2,3
local case_2_and_3_gamma_entry:=module()
   option object;

   export pole_location := 0;
   export pole_order    := 0;
   export Ec::set       := {};
   export b := 0;
end module;

#-------------------------------------------------------------------
#  PRIVATE variables for the kovacic class
#  only methods inside this class can access these
#-------------------------------------------------------------------

local original_ode;

#coefficients of original linear ode A y''+ B y' + C y =0
local A,B,C;

#original ode dependent and independent variables
local y::symbol,x::symbol;

local modified_ode;  #this is the z''=r*z    ode;
local z::symbol; #dependent variable for the r_ode z''(x)=r*z(x)
local r,s,t;     #where r=s/t;

local O_inf; #order of r at infinity. degree(s)-degree(t)

# poles or r. format is [ [pole,order],[pole,order],...]
# if no poles, for example r=x, then list is empty []
```

```
# this means pole order zero is not in the list, since no pole.
local poles_list::list := [];

#contains all possible cases detected. Hence [1] or [1,2] or [1,2,3]
#or remains empty [] for case 4.
local list_of_possible_cases::list := [];

local case_used_to_solve::integer:=-1; #this will have case used: 1,2 or 3

#this will have n=4,6 or 12 used for case 3 only
local n_used_for_case_3::integer:=-1;

#-----------------------------------------------------------------------
#  CONSTRUCTOR
#
#  the input is  the ode itself as first argument, and the
#  dependent variable, y(x) for example, as second argument.
#-----------------------------------------------------------------------
export ModuleCopy::static:= proc( _self,
            proto::kovacic_class,
            ode::`=`,func::function(name) ,$)
    local A,B,C;
    local x,y,r,z::nothing,s,t;

    x,y,A,B,C := parse_and_validate_ode(ode,func);

    _self:-original_ode := ode;

    #force r to be relatively prime ratio of 2 polynomials
    r := normal( (2*diff(B,x)*A-2*B*diff(A,x)+B^2-4*A*C)/(4*A^2));
    if not type(r,'ratpoly'(anything,x)) then
        ERROR("r= ", r ," is not polynomial or rational function in ",x);
    fi;

    s := numer(r);
    t := denom(r);

    #save all findings into the object private data
    #so it can be used later
    _self:-modified_ode := diff(z(x),x$2) = r*z(x);
    _self:-O_inf := degree(t,x)-degree(s,x);

    _self:-r  := r;
    _self:-x  := x;
    _self:-y  := y;
    _self:-z  := z;
    _self:-s  := s;
    _self:-t  := t;
    _self:-C  := C;
    _self:-B  := B;
    _self:-A  := A;

    #determine all poles and order
    _self:-generate_poles_and_order_list();
```

```

    #determine all possible kovacic cases
    _self:-find_possible_cases();

    return NULL;
end proc;

#-------------------------------------------------------------------
# This module private function is called by constructor to parse and
# validate the ode
#-------------------------------------------------------------------
local parse_and_validate_ode:=proc(ode::`=`,func::function(name),$)
    local x,y,A,B,C,L::list;
    local item,dep_variables_found;

    if nops(func)<>1 then
        ERROR("dependent variable ",func," has more than one argument");
    fi;

    y := op(0,func);
    x := op(1,func);

    #basic verification
    if not has(ode,y) then ERROR("Supplied ",ode," has no ",y); fi;
    if not has(ode,x) then ERROR("Supplied ", ode," has no ",x); fi;
    if not has(ode,func) then ERROR("Supplied ",ode," has no ",func); fi;

    #check it is second order
    if PDEtools:-difforder(ode)<>2 then
        ERROR("Only second order ode's can be used in Kocacic algorithm. ");
    fi;

    #check all dependent variables in ode which is y, match given func y(x)
    try
        dep_variables_found := PDEtools:-Library:-GetDepVars([y],ode);
    catch:
        ERROR(lastexception);
    end try;

    #o over dep_variables_found and check the
    #independent variable is same as x i.e. ode can be y'(z)+y(z)=0 but
    #function is y(x).
    for item in dep_variables_found do
        if not type(item,function) then
            ERROR("Parse error. Expected ",func," found ",item," in", ode);
        else
            if op(1,item) <> x then
                ERROR("Parse error. Expected ",func," found ",item," in",ode);
            fi;
            if nops(item)<>1 then
                ERROR("Parse error. One argument allowed in ",
                      func," found ", item," in " , ode);
            fi;
        fi;
```

```
    od;

    #now go over all indents in ode and check that y shows as y(x)
    #and not as just y as the PDEtools:-Library:-GetDepVars([_self:-y],ode)
    #code above does not detect this. i.e. it does not check  y'(x)+y=0
    if numelems(indets(ode,identical(y))) > 0 then
       ERROR("Parsing error, Can not have ",y," with no argument inside ",ode);
    fi;

    #check ode is linear in y(x)
    if not has(DEtools:-odeadvisor(ode,func,['linear']),'_linear') then
        ERROR("Only linear ode's can be used in Kocacic algorithm. ");
    fi;

    #extract coefficients of the ode
    L:=DEtools:-convertAlg(ode,func); #this only works on linear ode's

    if L[2]<>0 then ERROR("Not homogeneous ode"); fi;

    #Finished parsing the ode.  A y''+B y' + C y = 0
    C  := L[1,1];
    B  := L[1,2];
    A  := L[1,3];

    if not type(A,'ratpoly'(anything,x)) or  not type(B,'ratpoly'(anything,x))
        or not type(C,'ratpoly'(anything,x)) then
        error "ode coefficients are not rational functions of ",x;
    fi;

    return x,y,A,B,C;

end proc;

#--------------------------------------------------------------------
# main API.  Called to solve the ode using Kovacic algorithm.
# returns the solution in the form of y(x)=.... or if no solution
# is found returns FAIL.
# see user guide how to use.
#--------------------------------------------------------------------
export dsolve::static:=proc(_self,$)
    local sol:=FAIL, current_case_number::posint;

    if nops(_self:-list_of_possible_cases) = 0 then
        return FAIL;
    fi;

    #keep trying all possible cases, starting from case 1 to case 3.
    #until one case succeed or all are tried

    for current_case_number in _self:-list_of_possible_cases do
        sol := _self:-dsolve_case(current_case_number);
        if sol<>FAIL then
            _self:-case_used_to_solve:=current_case_number;
            return sol;
```

```
        fi;
    od;

    return FAIL;

end proc;

#-------------------------------------------------------------------
#  Called to solve the ode using a specific case number
#  made public to allow user to solve using specific case
#-------------------------------------------------------------------
export dsolve_case::static:=proc(_self,case_number::posint,$)
    local sol;

    if case_number>3 then ERROR("Only case number 1,2,3 are allowed"); fi;

    if nops(_self:-list_of_possible_cases)=0 then
        ERROR("No possible cases detected for this ode");
    fi;

    if not member(case_number,_self:-list_of_possible_cases) then
        ERROR("Case ", case_number,
           " not one of possible cases ", _self:-list_of_possible_cases);
    fi;

    if case_number = 1 then
       sol:= _self:-solve_case_1();
    elif case_number = 2 then
       sol:= _self:-solve_case_2();
    else
       sol:= _self:-solve_case_3();
    fi;

    _self:-case_used_to_solve := case_number;
    return sol;

end proc;
#-------------------------------------------------------------------
# returns back the z''(x) = r(x) z(x) ode, which is the one
# actually solved by Kovacic algorithm
#-------------------------------------------------------------------
export get_z_ode::static:=proc(_self,$)
    local x := _self:-x;
    local r := _self:-r;
    local z := _self:-z;

    if _self:-s = 0 then
         return diff(z(x),x$2) = 0;
    else
         return diff(z(x),x$2) = numer(r)%/denom(r) * z(x);
    fi;
end proc;

#-------------------------------------------------------------------
```

```
# returns back the r term in the z''(x) = r(x) z(x) ode
#--------------------------------------------------------------------
export get_r::static:=proc(_self,$)
    if _self:-s = 0 then
        return 0;
    else
        return numer(_self:-r)%/denom(_self:-r);
    fi;
end proc;

#--------------------------------------------------------------------
#  returns s, where r = s/t and z'' = r z(t)
#--------------------------------------------------------------------
export get_s::static:=proc(_self,$)
    return _self:-s;
end proc;

#--------------------------------------------------------------------
#  returns t, where r = s/t and z'' = r z(t)
#--------------------------------------------------------------------
export get_t::static:=proc(_self,$)
    return _self:-t;
end proc;

#--------------------------------------------------------------------
# Returns back the original ode ( A y''+ B y' + C y = 0 )
#--------------------------------------------------------------------
export get_y_ode::static:=proc(_self,$)
    return _self:-original_ode;
end proc;

#--------------------------------------------------------------------
#  returns list of poles of r, where  z''(x) = r z(x)
#  The list has format [ [pole location,order], ...]
#--------------------------------------------------------------------
export get_poles::static:=proc(_self,$)
    return _self:-poles_list;
end proc;

#--------------------------------------------------------------------
#  returns O_infinity of r, where  z''(x) = r z(x)
#--------------------------------------------------------------------
export get_order_at_infinity::static:=proc(_self,$)
    return _self:-O_inf;
end proc;

#--------------------------------------------------------------------
#  returns list of possible kovacic cases possible.
#--------------------------------------------------------------------
export get_possible_cases::static:=proc(_self,$)::list;
    return _self:-list_of_possible_cases;
end proc;

#--------------------------------------------------------------------
```

```
#  returns actual case number used when solving ode.
#  can be 1,2 or 3
#  if no cases applicable, then -1 is returned
#----------------------------------------------------------------
export get_case_used::static:=proc(_self,$)::integer;
    return _self:-case_used_to_solve;
end proc;

#----------------------------------------------------------------
#  returns n used for case 3. Either 4,6, or 12.
#  if not case 3 used, then -1 is returned.
#----------------------------------------------------------------
export get_n_case_3::static:=proc(_self,$)::integer;
    return _self:-n_used_for_case_3;
end proc;


#----------------------------------------------------------------
# All functions below are private
#----------------------------------------------------------------

#----------------------------------------------------------------
#This proc find all possible kovacic cases. These can be 1,2 or 3.
#if none of these found, then empty list is returned, which is
#case 4 in the paper. For example, if case 1 is only possible,
#then [1] is returned. if case 1 and 2 are possible, then [1,2]
#is returned, if all three cases possible, then [1,2,3] is returned.
#If no cases possible then [] returned.
#----------------------------------------------------------------
local find_possible_cases::static:=proc(_self,$)
    local L::list := [];
    local poles_order := convert(_self:-poles_list[..,2],set);
    local T::set;

    #check for case 1
    T := select(Z-> Z>2,poles_order);
    if  nops( select(Z->type(Z,odd),T) )=0  and
        (_self:-O_inf<0 and type(_self:-O_inf,even)) or
            _self:-O_inf=0 or _self:-O_inf>1 then
        L:= [1];
    fi;

    if nops(poles_order)>0 then #must have at least one pole for 2,3 cases

        if nops(poles_order)<>0 then #can not have pole order 0, i.e. no poles

            T:=select(Z-> Z>2,poles_order);

            # r must have at least one pole that is either odd order greater
            #than 2 or else has order 2

            if  nops( select(Z->type(Z,odd),T) )<>0
                      or member(2,poles_order) then
                L:= [ op(L),2 ];
```

```
            fi;

            #check for case 3

            if not member(0,poles_order)
                   and nops(select(Z-> Z>2,poles_order))=0
                   and _self:-O_inf>=2 then

               L:= [ op(L),3 ];

            fi;
        fi;
    fi;

    _self:-list_of_possible_cases := L;

    return NULL;
end proc;

#--------------------------------------------------------------------
#called to find all poles of r and the order
#of each pole. Uses Maple's sqrfree.
#--------------------------------------------------------------------
local generate_poles_and_order_list::static:=proc(_self,$)
    local L::list := [];
    local sol::list;
    local current_sol;
    local poles_list::list;
    local current_pole;
    local r := _self:-r, x := _self:-x;

    poles_list := sqrfree(denom(r),x);
    poles_list := poles_list[2,..]; #we do not need the overall factor

    if nops(poles_list) = 0 then
        _self:-poles_list:=[];
    else
        for current_pole in poles_list do

            sol := solve(current_pole[1]=0,[x]);
            sol := ListTools:-Flatten(sol);

            for current_sol in sol do
                L := [op(L), [rhs(current_sol) ,current_pole[2] ] ];
            od;

        od;

        _self:-poles_list := L;
    fi;

    return NULL;

end proc:
```

```
#-------------------------------------------------------------------
#
#       C A S E     O N E      I M P L E M E N T A T I O N
#
#  returns ode solution using case 1, or FAIL is no solution exist
#-------------------------------------------------------------------
local solve_case_1::static:=proc(_self,$)
    local O_infinity_set::kovacic_class:-case_one_O_inf;
    local gamma_set::set(kovacic_class:-case_one_gamma_entry):={};

    gamma_set, O_infinity_set := _self:-case_1_step_1();
    return _self:-case_1_step_2(gamma_set, O_infinity_set);
end proc;

#-------------------------------------------------------------------
# called from _self:-solve_case_1()
#-------------------------------------------------------------------
local case_1_step_1::static:=proc(_self,$)::
          set(kovacic_class:-case_one_gamma_entry),
          kovacic_class:-case_one_O_inf;

    local current_pole;
    local e::kovacic_class:-case_one_gamma_entry;
    local o::kovacic_class:-case_one_O_inf;
    local b,a,v,i::integer;
    local b_coeff_in_r,b_coeff_in_r_inf_square;
    local x := _self:-x, r := _self:-r;
    local N::integer;
    local laurent_c;
    local current_term;
    local b_from_r,b_from_laurent_series;

    #this contains all information generated for each pole of r
    #this is what is called the set GAMMA in the paper and
    #in the diagram of algorithm above.
    local gamma_set::set(kovacic_class:-case_one_gamma_entry) := {};

    if nops(_self:-poles_list) = 0 then
        gamma_set := {};
    else
        for current_pole in _self:-poles_list do
            e := Object(kovacic_class:-case_one_gamma_entry);

            e:-pole_location  := current_pole[1];
            e:-pole_order     := current_pole[2];

            if e:-pole_order =1 then

               e:-sqrt_r      := 0;
               e:-alpha_plus  := 1;
               e:-alpha_minus := 1;
               gamma_set      := { op(gamma_set), e };

            elif e:-pole_order = 2 then
```

```
                e:-sqrt_r := 0;
                e:-b      := _self:-b_partial_fraction(r,x,e:-pole_location,2);
                e:-alpha_plus  := 1/2+1/2*sqrt(1+4*e:-b);
                e:-alpha_minus := 1/2-1/2*sqrt(1+4*e:-b);;
                gamma_set := { op(gamma_set), e };

            else

                v         := e:-pole_order/2;
                e:-sqrt_r := 0;

                for N from 2 to v  do
                    laurent_c := _self:-laurent_coeff(sqrt(r),x,
                                                      e:-pole_location,v,N);
                    current_term := laurent_c/(x-e:-pole_location)^N;
                    e:-sqrt_r    := e:-sqrt_r + current_term;
                    if N = v then
                        a := laurent_c;
                    fi;
                od;

                b_from_r :=_self:-b_partial_fraction(r,x,e:-pole_location,v+1);
                b_from_laurent_series :=_self:-laurent_coeff(sqrt(r),x,
                                                    e:-pole_location,v,v+1);

                e:-b := b_from_r - b_from_laurent_series;

                e:-alpha_plus  := 1/2*((e:-b)/a  + v);
                e:-alpha_minus := 1/2*(-(e:-b)/a + v);
                gamma_set      := { op(gamma_set), e };

            fi;
        od;
    fi;

    o := Object(case_one_O_inf);

    if _self:-O_inf > 2 then

        o:-sqrt_r_inf      := 0;
        o:-alpha_plus_inf  := 0;
        o:-alpha_minus_inf := 1;

    elif _self:-O_inf=2 then

        o:-sqrt_r_inf :=0;
        o:-b          := lcoeff(_self:-s) / lcoeff(_self:-t);
        b             := radsimp((1+4*o:-b)^(1/2));
        o:-alpha_plus_inf  := 1/2 + 1/2*b;
        o:-alpha_minus_inf := 1/2 - 1/2*b;

    else #order at infinity -2*v<= 0 which must be even

```

```
        v     := (-_self:-O_inf) / 2;
        o:-sqrt_r_inf :=0;

        for i from 0 to v do

            laurent_c     := _self:-laurent_coeff(sqrt(r),x,infinity,v,i);
            o:-sqrt_r_inf := o:-sqrt_r_inf + laurent_c*x^i;
            if i = v then
               o:-a := laurent_c;
            fi;

        od;

        b_coeff_in_r_inf_square := _self:-laurent_coeff(
                                            o:-sqrt_r_inf^2,x,0,v,v-1);

        b_coeff_in_r       := _self:-get_coefficient_of_r(v-1);
        o:-b               := b_coeff_in_r - b_coeff_in_r_inf_square;
        o:-alpha_plus_inf  := 1/2*(  (o:-b)/(o:-a) - v);
        o:-alpha_minus_inf := 1/2*( -(o:-b)/(o:-a) - v);
    fi;

    return gamma_set,o;

end proc;

#---------------------------------------------------------------------
# Finds b coefficient in r for the case one only when v<=0
# using long division
#---------------------------------------------------------------------
local get_coefficient_of_r::static:=proc(_self,the_degree,$)
    local r := _self:-r;
    local x := _self:-x;
    local c,R,t;

    try
        c := coeff(r,x,the_degree);
    catch:
        R := rem(numer(r),denom(r),x);
        t := denom(r);
        c := lcoeff(R)/lcoeff(t);
    end try;

    return c;
end proc;

#---------------------------------------------------------------------
# called from _self:-solve_case_1()
# This determines the set of d non-negative integers and
# corresponding w for each d.
#---------------------------------------------------------------------
local case_1_step_2::static:=proc(_self,
           gamma_set::set(kovacic_class:-case_one_gamma_entry),
           O_infinity::kovacic_class:-case_one_O_inf,$)
```

```

    local d,N,K,number_of_poles::integer,current_alpha_infinity;
    local item::list;
    local the_sign::integer,sign_list::list;
    local w;
    local r_solution,y_solution;
    local x := _self:-x;

    #this will contains all data found for any nonnegative d. Each
    #entry will be a list of this form
    #       [d, w]

    local good_d_found := Array(1..0);
    local B::Matrix; #this will contain good_d_found but as matrix

    number_of_poles := nops(gamma_set);
    sign_list := combinat:-permute([1$number_of_poles,-1$number_of_poles],
                                number_of_poles);

    for K,current_alpha_infinity in
            [O_infinity:-alpha_plus_inf,O_infinity:-alpha_minus_inf] do

        for item in sign_list do

            d := 0;

            if number_of_poles>0 then

                for N,the_sign in item do
                    if the_sign = -1 then
                        d := d-gamma_set[N]:-alpha_minus;
                    else
                        d := d-gamma_set[N]:-alpha_plus;
                    fi;
                od;

            fi;

            d := simplify(d + current_alpha_infinity);

            if type(d,'integer') and d >= 0 then
                w := 0;

                for N,the_sign in item do

                    if number_of_poles>0 then

                        if the_sign = -1 then
                            w := w +(-1)*gamma_set[N]:-sqrt_r +
                                    (gamma_set[N]:-alpha_minus)/
                                          (x - gamma_set[N]:-pole_location);
                        else
                            w := w + gamma_set[N]:-sqrt_r  +
                                    (  gamma_set[N]:-alpha_plus)/
```

```
                                          (x  - gamma_set[N]:-pole_location);
                        fi;

                    fi;

                od;

                #this to get the sign in according to paper. First + then -
                if K = 1 then
                   w := w + O_infinity:-sqrt_r_inf;
                else
                   w := w - O_infinity:-sqrt_r_inf;
                fi;

                good_d_found ,= [ d, w];
            fi;
        od;
    od;

    if numelems(good_d_found) = 0 then return FAIL; fi;

    #now convert the array to Matrix, and sort on d, so we
    #start will the smallest
    #degree d, which is the first column, as that will be most efficient.
    #convert to set first, to remove any possible duplicat entries
    #then convert to Matrix

    convert(good_d_found,set);
    B := convert(convert(%,list),Matrix);
    B := B[sort(B[.., 1], 'output'= 'permutation')];

    for N from 1 to LinearAlgebra:-RowDimension(B) do

        r_solution := _self:-case_1_step_3(B[N,1], B[N,2]); #(d,w)

        if r_solution <> FAIL then
            y_solution := _self:-build_y_solution_from_r_solution(r_solution);
            return y_solution;
        fi;

    od;

    return FAIL;

end proc;

#-----------------------------------------------------------------------
# called from _self:-case_1_step_2()
#-----------------------------------------------------------------------
local case_1_step_3::static:=proc(_self,d::nonnegative,w,$)
    local x := _self:-x;
    local r := _self:-r;
    local p;
    local i::integer;
```

```
    local a::nothing;
    local eq;
    local coeff_sol;
    local tmp,W;
    local final_result;

    p := x^d;

    for i from d-1 by -1 to 0 do
        p := p + a[i] * x^i;
    od;

    #using original kovacis method. Not Smith.
    eq := simplify( diff(p,x$2)+2*w*diff(p,x)+(diff(w,x)+w^2-r)*p) = 0;

    if d = 0 then
        if not evalb(eq) then return FAIL; fi;
    else
        #solve for coefficients
        try
           coeff_sol := timelimit(30,solve(
                                 identity(eq,x), [seq(a[i],i=0..d-1)]));
        catch:
            return FAIL;
        end try;

        if nops(coeff_sol) = 0 then return FAIL; fi;

        tmp := map(evalb,coeff_sol[1]);
        if has(tmp,true) then  return FAIL; fi;

        p := eval(p,coeff_sol[1]); #to force a[i] solutions to update
    fi;

    W := diff(p,x)/p + w;
    W := radsimp(W);

    tmp := diff(W,x)+W^2;

    if evalb( tmp = r) or is(tmp = r) or simplify(tmp-r)=0 then #can be used
        try
            tmp := int(w, x);
        catch:
            return FAIL;
        end try;

        if has(tmp,int) then return FAIL; fi;

        final_result := simplify(p*exp(tmp));
        if has(final_result,signum) or has(final_result,csgn) then
            final_result := p*exp(tmp);
        fi;

        return final_result;
```

```
    else
        return FAIL;
    fi;

end proc;

#-------------------------------------------------------------------
#
#  C A S E   T W O   I M P L E M E N T A T I O N
#
#  returns ode solution using case 2, or FAIL is no solution exist
#-------------------------------------------------------------------
local solve_case_2::static:=proc(_self,$)

    local E_inf::set;
    local gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry):={};

    gamma_set, E_inf := _self:-case_2_step_1();
    return _self:-case_2_step_2(gamma_set, E_inf );

end proc;

#-------------------------------------------------------------------
#
#-------------------------------------------------------------------
local case_2_step_1::static:=proc(_self,$)::
        set(kovacic_class:-case_2_and_3_gamma_entry),set;

    local current_pole;
    local e::kovacic_class:-case_2_and_3_gamma_entry;
    local E_inf::set;
    local b;
    local x := _self:-x;
    local r := _self:-r;

    #this contains all information generated for each pole of r
    #this is what is called the set GAMMA in the paper and in the diagram of
    #algorithm above.
    local gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry) := {};

    for current_pole in _self:-poles_list do

        e := Object(kovacic_class:-case_2_and_3_gamma_entry);
        e:-pole_location  := current_pole[1];
        e:-pole_order     := current_pole[2];

        if e:-pole_order = 1 then

           e:-Ec     := {4};
           gamma_set := { op(gamma_set), e };

        elif e:-pole_order = 2 then

            e:-b      := _self:-b_partial_fraction(r,x,e:-pole_location,2);
```

```
            e:-Ec     := {2,2+2*sqrt(1+4* e:-b),2-2*sqrt(1+4* e:-b)};
            e:-Ec     := select(z->type(z,integer),e:-Ec);
            gamma_set := { op(gamma_set), e };

        else

           e:-Ec     := {e:-pole_order};
           gamma_set := { op(gamma_set), e };

        fi;
    od;

    if _self:-O_inf>2 then

        E_inf := {0,2,4};

    elif _self:-O_inf=2 then

        b     := lcoeff(_self:-s) / lcoeff(_self:-t);
        E_inf := {2,2+2*sqrt(1+4*b),2-2*sqrt(1+4*b)};
        E_inf := select(z->type(z,integer),E_inf);

    else #order at infinity v< 2

        E_inf := {_self:-O_inf};

    fi;

    return gamma_set,E_inf;
end proc;

#--------------------------------------------------------------------
# called from _self:-solve_case_2()
# This determines the set of d non-negative integers and
# corresponding w for each d.
#--------------------------------------------------------------------
local case_2_step_2::static:=proc(_self,
           gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry),
           E_inf::set,
           $)

    local L::list := [];
    local item,current_E_inf;
    local d,N,ee,theta;
    local x := _self:-x;
    local r_solution,y_solution;

    #this will contains all data found for any nonnegative d. Each
    #entry will be a list of this form
    #       [d, theta]
    #so if we obtain say 3 values of d that are nonnegative,
    #there will be 3 such lists in this array

    local good_d_found := Array(1..0);
```

```
    local B::Matrix; #this will contain good_d_found as matrix

    for item in gamma_set do
        L := [ op(L), convert(item:-Ec,list) ];
    od;

    #now find all possible tuples
    if nops(L)>1 then
       L := kovacic_class:-cartProdSeq(op(L));
    else
       L := L[1];
    fi;

    for current_E_inf in E_inf do
        for item in L do

            d := 1/2*( current_E_inf - add(item));

            if type(d,'integer') and d >= 0 then
                theta :=0;

                for N,ee in item do
                    theta := theta + ee/(x-gamma_set[N]:-pole_location);
                od;

                theta          := 1/2*theta;
                good_d_found ,= [ d, theta];
            fi;

        od;
    od;

    if numelems(good_d_found) = 0 then return FAIL; fi;

    #now convert the array to Matrix, and sort on d, so
    #we start will the smallest degree d, which is the first column, as
    #that will be most efficient.
    #convert to set first, to remove any possible duplicat entries
    #then convert to Matrix

    convert(good_d_found,set);
    B := convert(convert(%,list),Matrix);
    B := B[sort(B[.., 1], 'output'= 'permutation')];

    for N from 1 to LinearAlgebra:-RowDimension(B) do

        r_solution := _self:-case_2_step_3(B[N,1], B[N,2]);  #(d,theta)

        if r_solution <> FAIL then
            y_solution := _self:-build_y_solution_from_r_solution(r_solution);
            return y_solution;
        fi;

    od;
```

```

    return FAIL;

end proc;

#----------------------------------------------------------------
# called from _self:-case_2_step_2()
#----------------------------------------------------------------
local case_2_step_3::static:=proc(_self,d::nonnegative,theta,$)

    local p, i;
    local a::nothing;
    local tmp;
    local coeff_sol := [];
    local eq;
    local phi;
    local sol_w;
    local current_w;
    local r:=_self:-r;
    local w::nothing;
    local x := _self:-x;
    local sol;

    p := x^d;

    for i from d-1 by -1 to 0 do
        p := p + a[i] * x^i;
    od;

    eq:= simplify(diff(p,x$3) + 3*theta*diff(p,x$2) +
            (3*theta^2+3*diff(theta,x) -4*r) * diff(p,x)
              + ( diff(theta,x$2)+3*theta*diff(theta,x)+theta^3 -
                 4*r*theta - 2*diff(r,x))*p) = 0;

    if d=0 then
        if not evalb(eq) then return FAIL; fi;
    else
        #solve for polynomial coefficients
        try
            coeff_sol:= timelimit(30,solve(
                                   identity(eq,x), [seq(a[i],i=0..d-1)]));
        catch:
            return FAIL;
        end try;

        if nops(coeff_sol) = 0 then return FAIL; fi;

        tmp := map(evalb,coeff_sol[1]);
        if has(tmp,true) then return FAIL; fi;

        p := eval(p,coeff_sol[1]); #to force a[i] solutions to update
    fi;

    phi := theta + diff(p,x)/p;
```

```
    eq  := w^2 - phi*w +  simplify(1/2*diff(phi,x)+1/2*phi^2-r) = 0;

    try
        sol_w := timelimit(30,solve(eq,[w])); #changed to []
        sol_w := ListTools:-Flatten(sol_w);
    catch:
        return FAIL;
    end try;

    if nops(sol_w) = 0 then return FAIL; fi;

    for current_w in sol_w do

        current_w := radsimp(rhs(current_w));

        #verify w before using it. Added 1/12/2022 4 PM
        tmp := diff(current_w,x)+current_w^2;
        if evalb( tmp= r) or is(tmp = r) or simplify(tmp-r)=0 then
            try
               sol := timelimit(30,int(current_w, x));
            catch:
               return FAIL;
            end try;

            if has(sol,int) then return FAIL; fi;

            return simplify(exp(sol));
        fi;

    od;

end proc;

#---------------------------------------------------------------------
#
#  C A S E   T H R E E   I M P L E M E N T A T I O N
#
#  returns ode solution using case 3, or FAIL is no solution exist
#---------------------------------------------------------------------

local solve_case_3::static:=proc(_self,$)

    local E_inf::set;
    local gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry):={};
    local sol;

    #these are possible degress of w to try until one works or none works
    #local w_degree::list := [4,6,12];
    local w_degree::list := [4,6,12];
    local n::posint;

    for n in w_degree do
        gamma_set, E_inf := _self:-case_3_step_1(n);
        sol              := _self:-case_3_step_2(gamma_set, E_inf, n );
```

```

        if sol<>FAIL then return sol; fi;
    od;

    return FAIL;

end proc;

#-------------------------------------------------------------------
# First step in case 3
#-------------------------------------------------------------------
local case_3_step_1::static:=proc(_self,
           n::posint,
           $)::set(kovacic_class:-case_2_and_3_gamma_entry),set;


    local current_pole;
    local e::kovacic_class:-case_2_and_3_gamma_entry;
    local E_inf::set;
    local b,k;
    local x := _self:-x;
    local r := _self:-r;

    #this contains all information generated for each pole of r
    #this is what is called the set GAMMA in the paper and in the diagram of
    #algorithm above.
    local gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry):={};

    for current_pole in _self:-poles_list do

        e := Object(kovacic_class:-case_2_and_3_gamma_entry);

        e:-pole_location := current_pole[1];
        e:-pole_order    := current_pole[2];

        if e:-pole_order = 1 then

           e:-Ec      := {12};
           gamma_set := { op(gamma_set), e };

        elif e:-pole_order = 2 then

            e:-b      := _self:-b_partial_fraction(r,x,e:-pole_location,2);
            e:-Ec     := {seq( 6+12*k/n*sqrt(1+4*(e:-b)),k=-n/2..n/2,1)};
            e:-Ec     := select(z->type(z,integer),e:-Ec);
            gamma_set := { op(gamma_set), e };

        else
           ERROR("Internal error. Case 3 can only have poles of order 1 or 2");
        fi;
    od;

    #same formula, but different b
    b     := lcoeff(_self:-s) / lcoeff(_self:-t);
```

```
    E_inf := {seq( 6+12*k/n*sqrt(1+4*b),k=-n/2..n/2,1)};
    E_inf := select(z->type(z,integer),E_inf);

    return gamma_set,E_inf;

end proc;

#-------------------------------------------------------------------
# Second step in case 3
#
# called from _self:-solve_case_3()
# This determines the set of d non-negative integers and
# corresponding w for each d.
#-------------------------------------------------------------------
local case_3_step_2::static:=proc(
        _self,
        gamma_set::set(kovacic_class:-case_2_and_3_gamma_entry),
        E_inf::set,
        n::posint,$)

    local L::list := [];
    local item,current_E_inf;
    local d,ee,theta,N;
    local x := _self:-x;
    local r_solution,y_solution;
    local S;
    local current_iteration::integer;
    local tmp;

    #this will contains all data found for any nonnegative d. Each
    #entry will be a list of this form
    #       [d, theta, S]
    #so if we obtain say 3 values of d that are nonnegative
    local good_d_found := Array(1..0);
    local B::Matrix; #this will contain good_d_found as matrix

    #DEBUG();
    for item in gamma_set do
        L := [ op(L), convert(item:-Ec,list) ];
    od;

    #now find all possible tuples
    if nops(L)>1 then
       L := kovacic_class:-cartProdSeq(op(L));
    else
       L := L[1];
    fi;

    current_iteration:=0;
    for current_E_inf in E_inf do
        for item in L do
            current_iteration := current_iteration+1;

            d := n/12*( current_E_inf - add(item));
```

```
            if type(d,'integer') and d >= 0 then
                theta :=0;
                for N,ee in item do
                    theta := theta + ee/(x-gamma_set[N]:-pole_location);
                od;
                theta := n/12*theta;
                theta := simplify(theta);
                S := mul( (x-gamma_set[N]:-pole_location), N=1..nops(item));
                tmp := simplify(S) assuming real;
                if not has(tmp,csgn) and not has(tmp,signum) then
                   S:= tmp;
                fi;

                good_d_found ,= [ d, theta, S];
            fi;
        od;
    od;

    #now convert the array to Matrix, and sort on d, so we start
    # with the smallest
    #degree d, which is the first column, as that will be most efficient.

    if numelems(good_d_found) = 0 then return FAIL; fi;

    #convert to set first, to remove any possible duplicat entries
    #then convert to Matrix
    convert(good_d_found,set);
    B := convert(convert(%,list),Matrix);
    B := B[sort(B[.., 1], 'output'= 'permutation')];

    for N from 1 to LinearAlgebra:-RowDimension(B) do
        #(d,theta,S,n)
        r_solution := _self:-case_3_step_3(B[N,1], B[N,2], B[N,3],n);

        if r_solution <> FAIL then
            _self:-n_used_for_case_3 := n;
            y_solution := _self:-build_y_solution_from_r_solution(r_solution);
            return y_solution;
        fi;
    od;

    return FAIL;
end proc;

#-----------------------------------------------------------------
# Third and final step in case 3.
# called from _self:-case_3_step_2(). Returns solution or FAIL is no
# solution found.
#-----------------------------------------------------------------
local case_3_step_3::static:=proc(
                  _self,
                  d::nonnegative,
                  theta,
```

```
                 S,
                 n::posint,
                 $)

    local p, i,P_minus_1;
    local a::nothing;
    local final_result,tmp;
    local coeff_sol := [];
    local sol_w;
    local current_w;
    local r := _self:-r;
    local x := _self:-x;
    local omega::symbol;
    local P := Array(-1..n);
    local result_of_simplify;
    local result_of_is_check;
    local omega_equation;

    #this makes p(x). For example if d=3, then the result will be
    #         p(x) = x^3 + a(2) x^2 + a(1) x  + a(0)
    #where the number of unknowns to determine is always the same
    #as the degree, in this case a(0),a(1),a(2)

    p := x^d; #this will be 1 if degree is zero

    for i from d-1 by -1 to 0 do
        p := p + a[i] * x^i;
    od;

    #build the P_n polynomials based on p(x) above
    P[n] := -p;

    for i from n by -1 to 0 do

        if i=n then
            P[i-1] := -S * diff(P[i], x) - S*theta*P[i];
            P[i-1] := simplify( P[i-1]);
        else
            P[i-1] := -S * diff(P[i], x) +
                  ( (n-i)*diff(S,x) - S*theta)*P[i] - (n-i)*(i+1)*S^2*r*P[i+1];
            P[i-1] := simplify( P[i-1]);
        fi;

    od;

    #solve P[-1] for p(x) if needed.
    P_minus_1 := expand(numer(radsimp(P[-1])));

    if P_minus_1 <>0 and evalb(p<>1) then
        if not hastype(P_minus_1,'indexed')  then #it must be indexed now
           ERROR("Internal error. Please report. This should not happen");
        fi;

        try
```

```
            coeff_sol := timelimit(30,solve(identity(P_minus_1,x), [seq(a[i],i=0..d-1)]));
        catch:
            return FAIL;
        end try;

        if nops(coeff_sol)=0 then #unable to solve
           return FAIL;
        fi;

        map(evalb,coeff_sol[1]); #check all solved for
        if has(%,true) then return FAIL; fi;
    fi;

    #build the equation for omega
    omega_equation := 0;

    for i from 0 to n do
        omega_equation := omega_equation + S^i*P[i]/(n-i)! * omega^i ;
    od;

    omega_equation := simplify(omega_equation);

    if nops(coeff_sol)<>0 then
        #to force a[i] solutions to update to solved coefficients
        omega_equation := eval(omega_equation,coeff_sol[1]);
    fi;

    try
        sol_w := timelimit(30, solve(omega_equation=0, [omega]));
        sol_w := ListTools:-Flatten(sol_w);
    catch:
        return FAIL;
    end try;

    if nops(sol_w) = 0 then return FAIL; fi;

    #go over each w solution and use one that works. verify before using

    for current_w in sol_w do

        current_w := rhs(current_w);

        if not has(current_w, RootOf) then
            try
                current_w := timelimit(30,simplify(current_w));
            catch:
                NULL;
            end try;

            if is_w_verified(current_w,x,r) then
                final_result := _self:-simplify_final_result(current_w,x);
                if final_result<>FAIL then
                   return final_result;
                fi;
```

```
            fi;
        else #no Rootof, try to resolve
            try
                current_w := timelimit(30,[allvalues(current_w)]);
            catch:
                return FAIL;
            end try;

            current_w := current_w[1]; #just use any root. Pick first

            if not has(current_w, RootOf) then
                try
                    current_w := timelimit(30,simplify(current_w));
                catch:
                    NULL;
                end try;

                if is_w_verified(current_w,x,r) then
                    final_result := _self:-simplify_final_result(current_w,x);
                    if final_result<>FAIL then
                       return final_result;
                    fi;

                fi;
            fi;
        fi;
    od;

    return FAIL;

end proc;


#-------------------------------------------------------------------
# Called from case 3, step 3 to verify w
#-------------------------------------------------------------------
local is_w_verified:=proc(current_w,x,r)::truefalse;
    local tmp;
    local result_of_is_check::truefalse;
    local result_of_simplify;

    tmp := diff(current_w,x)+current_w^2;

    if evalb( tmp=r) then return true; fi;

    try
        result_of_simplify := timelimit(30,simplify(tmp-r));
        if evalb(result_of_simplify=0) then return true; fi;
    catch:
        NULL;
    end try;

    try
        result_of_is_check := timelimit(30,is(tmp = r));
```

```
        if result_of_is_check then return true; fi;
    catch:
        NULL;
    end try;


    return false;
end proc;

#-------------------------------------------------------------------
# Called from case 3, step 3 to simplify final result
#-------------------------------------------------------------------

local simplify_final_result::static:=proc(_self,omega,x,$)
    local final_result;
    local integral_result;

    try
        integral_result := timelimit(30,int(omega, x));
    catch:
        return FAIL;
    end try;

    if has(integral_result,int) or has(integral_result,Int) then
        return exp(Int(omega, x));
    fi;

    try
        final_result := timelimit(30,simplify(exp(integral_result))) assuming real;
        if has(final_result,signum) or has(final_result,csgn)
            or has(final_result,abs) then
            final_result := exp(integral_result);
        fi;
    catch:
        final_result:= exp(integral_result);
    end try;

    return final_result;
end proc;


#-------------------------------------------------------------------
# External proc helper
# provided thanks to Joseph Riel
#-------------------------------------------------------------------
local cartProdSeq:= proc(L::seq(list))
    local Seq::nothing,i::nothing,j;
    option `Copyright (C) 2007, Joseph Riel. All rights reserved.`;
        eval([subs(Seq= seq, foldl(Seq, [cat(i, 1..nargs)],
                                 seq(cat(i,j)= L[j], j= nargs..1, -1)))])
end proc:

#-------------------------------------------------------------------
# Checks for special math function.
```

```
# This function was provided thanks to Carl Love.
# modified to allow some special functions.
#---------------------------------------------------------------------
local has_special_math_function:= subs(
    _F= {(op@FunctionAdvisor)~(FunctionAdvisor("class_members", "quiet"), "quiet")[]}
        minus (   {FunctionAdvisor("elementary", "quiet")[]}
                  union {erf,erfc, erfi, Im,Re,signum,max,argument} ),
    proc(e::algebraic, x::{name, set(name)}:= {})
        hastype(e, And(specfunc(_F), dependent(x)))
    end proc
):

#---------------------------------------------------------------------
# called if step 3 is successful. Build y solution from r solution
#---------------------------------------------------------------------
local build_y_solution_from_r_solution::static:=proc(_self,r_solution,$)

    local y1,y2;
    local int_B_over_A;
    local A :=_self:-A, B:=_self:-B;
    local x :=_self:-x, y:= _self:-y;
    local tmp;

    if B = 0 then
       y1 := r_solution;
       int_B_over_A := 0;
    else
        try
            int_B_over_A := timelimit(60,int(-B/A,x));
            if has(int_B_over_A,int) or has(int_B_over_A,RootOf) then
                int_B_over_A := Int(-B/A, x);
                y1 :=r_solution*exp(1/2*int_B_over_A);
            else
                y1 := simplify(r_solution*exp(1/2*int_B_over_A));
                if has(y1,signum) or has(y1,csgn) then
                    y1 := r_solution*exp(1/2*int_B_over_A);
                fi;
            fi;
        catch:
            int_B_over_A := Int(-B/A,x);
            y1 := r_solution*exp(1/2*int_B_over_A);
        end try;
    fi;

    try
        y2 := timelimit(30, int( simplify(exp(int_B_over_A))/y1^2,x));

        if kovacic_class:-has_special_math_function(y2,x) then
            y2 := y1 * Int( exp(int_B_over_A)/y1^2,x)
        else
            y2 := y1 * y2;
        fi;
        tmp := simplify(y2);
        if has(tmp,signum) or has(tmp,csgn) then
```

```
            y2 := y1 * y2;
        else
            y2 := tmp;
        fi;
    catch:
        y2 := y1 * Int( exp(int_B_over_A)/y1^2,x)
    end try;

    return y(x) = _C1*y1 + _C2*y2;

end proc;

#--------------------------------------------------------------------
# Helper private function called to obtain b from the partial fraction
# decomposition of r needed by the differenent case implementations
#--------------------------------------------------------------------
local b_partial_fraction::static:=proc(_self,
        r,
        x::symbol,
        pole_location,
        the_power::posint,$)

    local r_partial_fraction,T::nothing;

    r_partial_fraction := allvalues(convert(r,'fullparfrac',x));

    #adding dummy T so that select below always works
    r_partial_fraction := T+r_partial_fraction;
    select(z->hastype(z,
            anything/(anything*identical(x - pole_location)^the_power)),
            r_partial_fraction);

    return coeff(%,1/(x-pole_location)^the_power);

end proc;

#--------------------------------------------------------------------
# Helper private function called to return Laurent series coefficient
# of the 1/(x-c)^n  term. where n=1,2,3,.... for the function f(x)
# expandid about pole of order m
# if c=infinity, then x is replaced by 1/y and expansion is around 0
# for infinity, n=0,1,2,...
#--------------------------------------------------------------------
local laurent_coeff::static:=proc(_self,
        f,
        x::symbol,
        c,
        m::integer,
        n::integer,$)

    local the_coeff,y::nothing,fy;

    if c = infinity then
        fy := eval(f,x=1/y);
```

```
        if m-n>0 then
           the_coeff := limit( diff( y^m * fy,y$(m-n)),y=0,right)/(m-n)!;
        elif m-n=0 then
            the_coeff := limit( y^m * fy,y=0,right)/(m-n)!;
        else
           the_coeff := 0;
        fi;
    else
        if m-n>0 then
            the_coeff := limit( diff( (x-c)^m * f,x$(m-n)),x=c,right)/(m-n)! ;
        elif m-n=0 then
           the_coeff := limit( (x-c)^m * f,x=c,right)/(m-n)!;
        else
           the_coeff := 0;
        fi;
    fi;

    the_coeff := eval(the_coeff,[csgn=1,signum=1]);

    return the_coeff;

end proc;

end module;
```

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1.6 Kovacic's Algorithm and Its Application to Some Families of Special Functions

By Anne Duval, and Michele Loday-Richaud.

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Kovacic's Algorithm and Its Application to Some Families of Special Functions

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Abstract. We apply the Kovacic algorithm to some families of special functions, mainly the hypergeometric one and that of Heun, in order to discuss the existence of closed-form solutions. We begin by giving a slightly modified version of the Kovacic algorithm and a sketch proof.

Keywords: Second order linear differential equation, Liouvillian or closed-form solution, Riccati equation, Differential galois group, Hypergeometric equation, Heun's equation, Confluent equation

This article deals with the classical question of deciding whether or not a linear ordinary differential equation with rational coefficients has *liouvillian* solutions and computing them if any. Liouvillian (or closed-form) solutions are those which can be built up from $\mathbf{C}(x)$ by algebraic operations and taking exponentials or primitives.

A decision procedure to solve this problem for arbitrary order equations is presented in [20]. Although this solves the problem in principle, this procedure is not an algorithm that can be readily implemented (for recent work concerning improvements for equations of order three, see [24], [21]). In contrast, the case of equations of order two was solved by Kovacic [12] in a completely effective manner. His algorithm is implemented several times: in MACSYMA (Caviness and Saunders [16], Pavelle and Wang [14]), in MAPLE (Smith [23], Char [4]) or in SCRATCHPAD (Sénéchaud and Siebert [18]). All these implementations are conceived for equations having their coefficients in $\mathbf{C}(x)$ (actually $\mathbf{Q}(x)$), which is the natural environment of the original algorithm.

In this article we apply a slightly modified version of the algorithm to *families* of second order equations, that is equations in which the coefficients depend upon some external parameters. The examples treated are mainly hypergeometric and Heun's equations. By the way we recover some of Kimura's results ([10]) and also obtain relations between generalized hypergeometric functions (Proposition 10). As far as we know, our results concerning Heun's equations are new. The study of examples is the aim of the third section of the paper. Our version of the algorithm is detailed in the first section and a sketch of the proof is presented in the second section: we follow mainly Kovacic's arguments with some modifications pointed out in our paper.

This new version is being implemented in SCRATCHPAD by Fouché [6] who plans to use the resources of computer algebra to handle the parameters. It could be a stage in an automatic calculus of the differential galois group.

This article grew out of a series of lectures by the authors as well as C. Mitschi and F. Richard-Jung given in J.-P. Ramis Seminar in 1986–87. We also would like to thank the referees for some valuable suggestions.

1 Description of the Algorithm

1.1 The Method

Let us recall a few basic facts from differential algebra. We consider $\mathbf{C}(x)$ equipped with the usual derivation $\frac{d}{dx}$ as a *differential field.* A *differential extension* of $\mathbf{C}(x)$ is a differential overfield, the derivation of which extends $\frac{d}{dx}$.

Let

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{1}$$

be a differential equation of order two with polynomial coefficients. If $\{\eta, \zeta\}$ is a fundamental set of solutions of this equation in some differential extension of $\mathbf{C}(x)$, with the same field of constants, the field $K = \mathbf{C}(x)(\eta, \eta', \zeta, \zeta')$ is a *Picard–Vessiot* extension of $\mathbf{C}(x)$ associated to Eq. (1).

A field automorphism of K is called:

- a *differential automorphism* if it commutes with the derivation of K
- a $\mathbf{C}(x)$-*automorphism* if it is a differential automorphism leaving $\mathbf{C}(x)$ pointwise invariant.

The group G of all $\mathbf{C}(x)$-automorphisms of K is called the *differential galois group* of Eq. (1).

It is known (Kolchin [11]) that the Picard–Vessiot extension is unique up to $\mathbf{C}(x)$-differential isomorphism and the G may be viewed as an *algebraic* subgroup of $GL(2, \mathbf{C})$ by associating to $\sigma \in G$ the matrix

$$g_\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that $\sigma(\eta) = \alpha\eta + \gamma\zeta$ and $\sigma(\zeta) = \beta\eta + \delta\zeta$.

We are now able to give a precise definition of a *liouvillian* extension or solution.

Definition 1. *A differential field extension L of $\mathbf{C}(x)$ is called* liouvillian *if there is a tower of fields $\mathbf{C}(x) = L_0 \subset L_1 \subset \cdots \subset L_m = L$, such that for $i = 0, \ldots, m-1$, $L_{i+1} = L_i(\eta_i)$, where one of the following conditions holds*:

- η_i *is algebraic over* L_i, (algebraic extension)
- $\eta_i' \in L_i$, (extension by a primitive)
- $\frac{\eta_i'}{\eta_i} \in L_i$, (extension by the exponential of a primitive).

A solution of Eq. (1) is called liouvillian *if it belongs to some liouvillian extension of* $\mathbf{C}(x)$.

As the order of the equation is two, the "variation of the constant" method proves that either *no* solution of Eq. (1) is liouvillian or *all* solutions are liouvillian.

Replacing in Eq. (1) the function y by the *simultaneously liouvillian function* $y \exp \int \frac{b}{2a}$ we only have to look at equations in the *reduced* form

$$y'' - r(x)y = 0 \tag{2}$$

where $r \in \mathbf{C}(x)$. Note that if $a \neq 0$,

$$r = \frac{2b'a - 2ba' + b^2 - 4ac}{4a^2}.$$

It is a classical fact (Kaplansky [9]) that the differential galois group of Eq. (2) is a subgroup of $SL(2, \mathbf{C})$. This is generally not the case for the differential galois group of (1); indeed the previous transformation does not preserve the galois group.

Proposition 1 (Kovacic). *The equation* $y'' - ry = 0$ *has liouvillian solutions if and only if its differential galois group is a* proper *algebraic subgroup of* $SL(2, \mathbf{C})$.

The logarithmic derivative ω of a solution of Eq. (2) satisfies the Riccati equation

$$\omega' + \omega^2 = r \tag{3}$$

By Lie-Kolchin's theorem, it is known that Eq. (2) has liouvillian solutions if and only if Eq. (3) has an *algebraic* solution, the degree of which belongs to

$$L_{\max} = \{1, 2, 4, 6, 12\}.$$

It is now possible to sketch the main steps of the algorithm.

1. By examining the nature of the poles of r, one can define a subset L of $L_{\max}$ containing the possible values for the particular equation under consideration. This is achieved in what is called **First step** in the algorithm. There, one finds our improved "necessary conditions".

2. Exploring increasing values of n in L, one searches for a minimal polynomial of degree n

$$M(\omega) = \omega^n + a_{n-1}\omega^{n-1} + \cdots + a_0 \tag{4}$$

with $a_i \in \mathbf{C}(x)$, which could be satisfied by a solution ω of the Riccati Eq. (3).

The differential galois group of Eq. (2) acts on the solutions of Eq. (3) and it acts algebraically (i.e. as an ordinary Galois group) on its algebraic solutions. Thus if a root of the irreducible polynomial (4) is a solution of Eq. (3) then *any* root of this polynomial is also a solution of Eq. (3). By this property and since the Riccati equation is of order one, one gets $a_{n-2}, \ldots, a_0$ as explicit rational functions of a_{n-1} and one just has to find a_{n-1}.

3. One looks for a_{n-1} of the form

$$a_{n-1} = \theta + \frac{P'}{P}$$

where:

- θ is related to the poles of r, that is the *fixed* singularities of the Riccati Eq. (3). For each n there is a *finite* set of possible values for θ and they can be computed from the *exponents* of the Eq. (2) in its singular points (including ∞ if necessary). These possible values for θ are listed in the **Third step** of the algorithm.
- P is a *polynomial* related to the *moving* singularities of the Riccati Eq. (3), the degree of which is a known function of the pair (n, θ). The coefficients of P have to satisfy a *linear*, generally *overdetermined* system. When for at least one pair (r, θ) such a polynomial P can be found, the polynomial (4) corresponding to the smallest such n is certainly irreducible.

The final answer takes the form:

- either the differential Eq. (2) has no liouvillian solution,
- or this equation has only liouvillian solutions. The algorithm produces an *irreducible* polynomial with rational coefficients. To any root ω of this polynomial is associated the solution $\eta = \exp \int \omega$ of Eq. (2).

When one liouvillian solution is found, another one, linearly independent, is $\eta \int \frac{1}{\eta^2}$. This second solution is not always detected by the algorithm; moreover the various solutions given by the algorithm are not necessarily linearly independent.

1.2 The Algorithm

Notations. Let $L_{\max} = \{1, 2, 4, 6, 12\}$ and let h be the function defined on $L_{\max}$ by

$$h(1) = 1,\ h(2) = 2,\ h(4) = 3,\ h(6) = 2,\ h(12) = 1.$$

Input: A rational function

$$r(x) = \frac{s(x)}{t(x)}.$$

The polynomials $s, t \in \mathbf{C}[x]$ are supposed to be relatively prime, t being unitary. The differential equation under consideration is

$$y'' - ry = 0.$$

First step: *The set L of possible degrees of algebraic solutions of the Riccati equation.*

1. Make a full linear factorisation of $t(x)$ over $\mathbf{C}$.
If $t(x) = 1$ then $m = 0$, else according to this factorisation set

$$t(x) = t_1(x) t_2(x)^2 \cdots t_m(x)^m$$

where the t_i's are unitary pairwise relatively prime polynomials with no multiple root and $t_m \neq 1$.
Denote by Γ' the set of complex roots of t and let $\Gamma = \Gamma' \cup \{\infty\}$.
Introduce the *order* $o(c)$ of an element $c \in \Gamma$ by

$$o(c) = \begin{cases} i & \text{when } c \text{ is a root of } t_i, \\ \max(0, 4 + \deg s - \deg t) & \text{when } c = \infty. \end{cases}$$

Let $m^+ = \max(m, o(\infty))$.
For $0 \leqq i \leqq m^+$, denote by $\Gamma_i = \{c \in \Gamma \mid o(c) = i\}$ the subset of all elements of order i.

2. If $m^+ \geqq 2$ then $\gamma_2 = \operatorname{card} \Gamma_2$ else $\gamma_2 = 0$. Compute

$$\gamma = \gamma_2 + \operatorname{card}\Bigg(\bigcup_{\substack{\text{odd } k \\ 3 \leqq k \leqq m^+}} \Gamma_k \Bigg).$$

3. Define the subset L' of $L_{\max}$ by the following rules:

$$\begin{aligned} 1 \in L' &\Leftrightarrow \gamma = \gamma_2 \\ 2 \in L' &\Leftrightarrow \gamma \geqq 2 \\ 4, 6, 12 \in L' &\Leftrightarrow m^+ \leqq 2. \end{aligned}$$

4. For each $c \in \Gamma_1 \cup \Gamma_2$, compute the numbers α_c and β_c defined by

$$r(x) = \begin{cases} \dfrac{\alpha_c}{(x-c)^2} + \dfrac{\beta_c}{x-c} + O(1) & \text{when } c \in \mathbf{C}, \\ \dfrac{\alpha_\infty}{x^2} + \dfrac{\beta_\infty}{x^3} + O\left(\dfrac{1}{x^4}\right) & \text{when } c = \infty. \end{cases}$$

5. If $4 \notin L'$ then $L = L'$ else

$$\text{if } \begin{cases} \forall c \in \Gamma, \quad \sqrt{1 + 4\alpha_c} \in \mathbf{Q}, \\ \displaystyle\sum_{c \in \Gamma_1 \cup \Gamma_2} \beta_c = 0 \\ 1 + 4\Bigg(\displaystyle\sum_{c \in \Gamma_2} \alpha_c + \sum_{c \in \Gamma_1 \cup \Gamma_2} \beta_c \Bigg) \in \mathbf{Q} \end{cases} \quad \text{then } L = L',$$

else $L = L' \setminus \{4, 6, 12\}$.

6. If $L = \varnothing$ then go to OUTPUT2
else assign n the smallest value in L.

Second step: *Exponents at the singular points*

1. If $\infty \in \Gamma_0$ then $E_\infty = h(n)\{0, 1, \ldots, n\}$.
2. For each $c \in \Gamma_1$, define the set E_c by $E_c = \{nh(n)\}$.
3. When $n = 1$, for each $c \in \Gamma_2$, define the set E_c by

$$E_c = \left\{ \tfrac{1}{2}\left(1 \pm \sqrt{1 + 4\alpha_c} \right) \right\}.$$

4. When $n \geqq 2$, for each $c \in \Gamma_2$, define the set E_c by

$$E_c = \left\{ \frac{nh(n)}{2}\left(1 - \sqrt{1 + 4\alpha_c} \right) + h(n) j \sqrt{1 + 4\alpha_c} \,\middle|\, j = 0, \ldots, n \right\} \cap \mathbf{Z}.$$

If at least one set E_c is empty then go to CONTINUE.

5. When $n=1$, for each $c\in\Gamma_{2\nu}$ with $\nu\geqq 2$, compute one of the two "squareroots" $[\sqrt{r}]_c$ of r, defined up to sign by the following conditions:
   - for $c\in\mathbf{C}$,

$$[\sqrt{r}]_c=\frac{\alpha_c}{(x-c)^\nu}+\sum_{i=\nu-1}^{2}\frac{\lambda_{i,c}}{(x-c)^i}$$

$$r-[\sqrt{r}]_c^2=\frac{\beta_c}{(x-c)^{\nu+1}}+O\left(\frac{1}{(x-c)^\nu}\right)$$

   - for $c=\infty$,

$$[\sqrt{r}]_\infty=\alpha_\infty x^{\nu-2}+\sum_{i=\nu-3}^{0}\lambda_{i,\infty}x^i$$

$$r-[\sqrt{r}]_\infty^2=-\beta_\infty x^{\nu-3}+O(x^{\nu-4}).$$

   Now define the set E_c by

$$E_c=\left\{\frac{1}{2}\left(\nu+\varepsilon\frac{\beta_c}{\alpha_c}\right)\middle|\varepsilon=\pm 1\right\}$$

   and a "sign" function on E_c by

$$\operatorname{sign}\left(\frac{1}{2}\left(\nu+\varepsilon\frac{\beta_c}{\alpha_c}\right)\right)=\begin{cases}\varepsilon & \text{if } \beta_c\neq 0,\\ +1 & \text{if } \beta_c=0.\end{cases}$$

6. When $n=2$, for each $c\in\Gamma_\nu$ with $\nu\geqq 3$, define the set E_c by $E_c=\{\nu\}$.

Third step: *Possible degrees for P and possible values for θ*

1. For each family $\underline{e}=(e_c)_{c\in\Gamma}$ of elements e_c in E_c compute

$$d(\underline{e})=n-\frac{1}{h(n)}\sum_{c\in\Gamma}e_c.$$

2. Select all families $\underline{e}$ satisfying the two following conditions:
   (a) $d(\underline{e})\in\mathbf{N}$,
   (b) when $n=2$ or 4, at least two e_c are odd and moreover, when $n=4$ at least two e_c are multiple of 3.
   If no family is selected then go to CONTINUE.
3. For each selected family $\underline{e}$ set

$$\theta=\frac{1}{h(n)}\sum_{c\in\Gamma}\frac{e_c}{x-c}+\delta_n^1\sum_{c\in\bigcup_{\nu\geqq 2}\Gamma_{2\nu}}\operatorname{sign}(e_c)[\sqrt{r}]_c$$

   where δ_n^1 is the Kronecker symbol.

Fourth step: *Tentative computation of P*

For each $(\underline{e},\theta)$ try to find a polynomial P of degree $d=d(\underline{e})$ satisfying the linear system in the coefficients of P:

$$(*)_n \quad \begin{cases} P_n = -P \\ P_{i-1} = -P_i' - \theta P_i - (i+1)(n-i)rP_{i+1} \quad \text{for} \quad n \geqq i \geqq 0 \\ P_{-1} = 0 \end{cases}$$

(P_i' denotes the derivative of P_i with respect to x).

Output: If a pair (θ, P) is found then

OUTPUT1: The differential equation $y'' - ry = 0$ admits the liouvillian solution $\eta = \exp \int \omega$, where ω is any solution of the irreducible algebraic equation

$$\sum_{i=0}^{n} \frac{P_i}{(n-i)!} \omega^i = 0$$

else

CONTINUE: If n is not the greatest element in L
then assign n the next value in L and go to **Second step**
else

OUTPUT2: The differential equation $y'' - ry = 0$ has no liouvillian solution.

1.3 Comments

The Kovacic algorithm has been conceived for equations with rational coefficients. The various articles or reports on its implementation listed some of the practical problems encountered. We have not tried to solve these problems as our purpose was to handle families of equations, and from this point of view things are partially better. Specifically the problem of factoring the denominator of r, which is generally not an easy task, is avoided here while we deal with *normalized* equations in which the poles of r are a priori given.

In his original article, Kovacic already mentioned that cases $n = 1$ and $n = 2$ can generally be carried out by hand or with the help of a calculator. The situation is not so simple with parameters as, for instance, the degree of the polynomial P may depend on these parameters and the answer in no longer a matter of linear algebra (see the last example). Anyway we have been able to give a complete answer in most of the cases as long as only the case $n = 1$ or $n = 2$ was under consideration.

When $n = 1$, the final step is to decide if the second order linear differential equation for P equivalent to $(*)_1$ has a polynomial solution. In the two families studied (the hypergeometric and the Heun's) this equation belongs to the same family. Thus, it could be of interest to first solve this problem, which is a particular case of the general problem of finding rational solutions [20] Lemma 3.1 or algebraic solutions [25], see also [3].

In case it is reducible, the differential equation may or not have a basis of solutions of the form $\eta = \exp \int \omega$ ($\omega \in \mathbf{C}(x)$) and it is easy to complete the case $n = 1$ in the algorithm in order to compute all such solutions. In the example of the confluent hypergeometric equation we observe an exact correspondence between this fact and the nature of the differential galois group (see Sect. 3.1.2). Exploring this connection could be of interest.

2 Justification of the Algorithm

2.1 Some Results From Galois Differential Theory

From the well known classification of algebraic subgroups of $SL(2,\mathbf{C})$ (see Kovacic [12] and Kaplansky [9] Theorem 6.4 and 4.12) one proves:

Theorem 1. *The differential galois group G of the equation*

$$y'' - r(x)y = 0 \tag{2}$$

is an algebraic subgroup of $SL(2,\mathbf{C})$ *of one of the following forms:*

i) G *is* triangularisable *and then Eq.* (2) *is reducible and has a solution of the form* $\exp\int\omega$ *where* $\omega\in\mathbf{C}(x)$. (case $n=1$)

ii) G *is* imprimitive *and then Eq.* (2) *has a solution of the form* $\exp\int\omega$ *where* ω *is algebraic of degree 2 over* $\mathbf{C}(x)$. (case $n=2$)

iii) G *is* primitive and finite, *and then Eq.* (2) *has an algebraic solution of the form* $\exp\int\omega$ *where* ω *is algebraic of degree 4, 6 or 12 over* $\mathbf{C}(x)$. (cases $n=4,6,12$)

iv) $G=SL(2,\mathbf{C})$ *and then Eq.* (2) *has no liouvillian solution.*

This formulation was suggested to us by one of the referees. A similar statement for equations of order 3 can be found in a recent paper by Singer and Ulmer [21]. The already mentioned value for $L_{\max}$ is a consequence of this theorem.

Summarizing the different results which can be found in Kovacic [12] one has:

Proposition 2. *Suppose that the Riccati Eq.* (3) *has an algebraic solution* ω *of minimal order* $n\geqq 2$ *and let* $\eta=\exp\int\omega$. *Then there exists a solution* ζ *of the differential Eq.* (2) *and a polynomial* $U_n(X,Y)\in\mathbf{C}[X,Y]$ *such that*

1. $\deg U_n = n$
2. $u_n(x)=U_n(\eta,\zeta)^{h(n)}\in\mathbf{C}(x)$, *where h is the function already defined on* $L_{\max}$ *by* $h(1)=1, h(2)=2, h(4)=3, h(6)=2, h(12)=1$. *More explicitely one has:*

$$U_2(X,Y)=XY$$
$$U_4(X,Y)=X^4+8XY^3$$
$$U_6(X,Y)=X^5Y-XY^5$$
$$U_{12}(X,Y)=X^{11}Y-11X^6Y^6-XY^{11}.$$

Note that these polynomials are *semi-invariant* polynomials under the action of G as a subgroup of $SL(2,\mathbf{C})$.

2.2 Algebraic Solutions of the Riccati Equation

In this section we look for an algebraic solution of the Riccati Eq. (3) by computing its minimal polynomial, if any.

Let $A(w,\underline{a})$ be a unitary polynomial of degree n:

$$A(w,\underline{a}) = w^n - \sum_{i=0}^{n-1}\frac{a_i(x)}{(n-i)!}w^i \tag{5}$$

with coefficients $\underline{a}=(a_{n-1},\dots,a_1,a_0)$ in some differential extension of $\mathbf{C}(x)$.

The polynomial A is the expected minimal polynomial (4) if it is irreducible, it has its coefficients in $\mathbf{C}(x)$ and if any root of A is a solution of the Riccati Eq. (3).

Proposition 3. *Each root of the polynomial* (5) *is a solution of the Riccati Eq.* (3) *if and only if* $\underline{a}$ *satisfies the condition*

$$(\#)_n \quad \begin{cases} (n) & a_n = -1 \\ \vdots & \\ (i-1) & a_{i-1} = -a_i' - a_{n-1}a_i - (i+1)(n-i)a_{i+1}r \quad \textit{for} \quad n \geqq i \geqq 0 \\ \vdots & \\ (-1\text{ bis}) & a_{-1} = 0. \end{cases}$$

Proof. Kovacic [12] proved the "if" part (Theorem 2) and the "only if" part (Theorem 3) in the case when $n = 4, 6$ or 12 and A is the expected minimal polynomial. However we give a complete proof.

Let $\dfrac{\partial A}{\partial w}$ and $\dfrac{\partial A}{\partial x}$ denote the derivatives of the function $A(w, \underline{a}(x))$ with respect to the independent variables w and x. Let $A_1(w, \underline{a})$ be the polynomial of degree $n+1$ in w given by

$$A_1(w, \underline{a}) = \frac{\partial A}{\partial w} \cdot (r - w^2) + \frac{\partial A}{\partial x}.$$

The condition $(\#)_n$ says that A_1 is a multiple of A, more precisely that

$$A_1(w, \underline{a}) = -(nw + a_{n-1})A(w, \underline{a}). \tag{6}$$

To prove the "only if" part, we therefore have to show that if ω is a root of order at least p of A then it is a root of order at least p of A_1. Let us suppose then that

$$\frac{\partial^k A}{\partial w^k}(\omega, \underline{a}) = 0, \quad 0 \leqq k \leqq p-1. \tag{$[k]$}$$

By differentiating the relation [0] with respect to x one gets

$$\frac{\partial A}{\partial w}(\omega, \underline{a})\omega' + \frac{\partial A}{\partial x}(\omega, \underline{a}) = 0.$$

Taking into account the Riccati Eq. (3) this means that $A_1(\omega, \underline{a}) = 0$, which is the expected result when $p = 1$.

In the general case, the k'th derivative $(k = 0, 1, \ldots, p-1)$ with respect to w of the polynomial A_1 takes the form

$$\frac{\partial^k A_1}{\partial w^k} = \frac{\partial^{k+1} A}{\partial w^{k+1}} \cdot (r - w^2) - 2kw \frac{\partial^k A}{\partial w^k} - k(k-1)\frac{\partial^{k-1} A}{\partial w^{k-1}} + \frac{\partial^{k+1} A}{\partial w^k \partial x}. \tag{7}$$

On the other hand by differentiating the relation $[k]$ with respect to x $(k = 0, 1, \ldots, p-1)$ one has the relation

$$\frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega, \underline{a})\omega' + \frac{\partial^{k+1} A}{\partial w^k \partial x}(\omega, \underline{a}) = 0.$$

Taking into account the Riccati Eq. (3), this gives $\frac{\partial^k A_1}{\partial w^k}(\omega, \underline{a}) = 0$ for $0 \leqq k \leqq p-1$. This ends the proof of the "only if" part.

Conversely suppose now that the relation (6) is satisfied. Let p be the order of ω as a root of $A(w, \underline{a})$ (this is a well defined natural integer as A is not the null polynomial). By differentiating $p-1$ times the relation (6) and using the formulas (7) one gets

$$\frac{\partial^p A}{\partial w^p}(\omega, \underline{a})\cdot(r - \omega^2) + \frac{\partial^p A}{\partial w^{p-1}\partial x}(\omega, \underline{a}) = 0,$$

since $\frac{\partial^k A}{\partial w^k}(\omega, \underline{a}) = 0$ for $k = 0, \ldots, p-1$.

On the other hand, by differentiating the relation $\frac{\partial^{p-1} A}{\partial w^{p-1}}(\omega, \underline{a}) = 0$ with respect to x one gets

$$\frac{\partial^p A}{\partial w^p}(\omega, \underline{a})\omega' + \frac{\partial^p A}{\partial w^{p-1}\partial x}(\omega, \underline{a}) = 0.$$

By combining these two formulas and since $\frac{\partial^p A}{\partial w^p}(\omega, \underline{a}) \neq 0$, one deduces

$$\omega' = r - \omega^2. \quad \square$$

Proposition 4. *A solution of* $(\#)_n$ *is rational* ($\underline{a} \in (\mathbf{C}(x))^n$) *if and only if* $a_{n-1} \in \mathbf{C}(x)$.

Proof. The equation $(n-1)$ in $(\#)_n$ is trivial. By descending induction on i, for $n-1 \geqq i \geqq 0$, the equation $(i-1)$ gives a_{i-1} as a rational function of a_{n-1} and of its derivatives. $\square$

Remarks

1. In the system $(\#)_n$ one can eliminate the a_i for $i = 0, \ldots, n-2$. So this system is equivalent to a single non linear differential equation of order n for a_{n-1}. For $n = 1$ this equation is of course the Riccati Eq. (3) itself.
2. If n is the smallest element of $L_{\max}$ such that $(\#)_n$ has a rational solution then the corresponding polynomial $A(w, \underline{a})$ is *irreducible* in $\mathbf{C}(x)[w]$ (see Kovacic [12]).
3. The proof of Proposition 3 relies on the fact that the Riccati equation is of order 1. There is no known analogue for this proposition in higher orders so an efficient algorithm to compute all the coefficients of a minimal polynomial is much more difficult to obtain (see Singer–Ulmer [21]).

Proposition 5. *Let* $F \in \mathbf{C}[X, Y]$ *be a homogeneous polynomial of (total) degree* n *and* y_1, y_2 *two solutions of the differential Eq. (2). There exists a solution* $\underline{a}$ *of* $(\#)_n$ *such that* a_{n-1} *is the logarithmic derivative of* $\varphi(x) = F(y_1, y_2)$.

Proof. By factoring F in linear terms one sees that it suffices to prove the proposition when $\varphi(x) = \prod_{i=1}^{n} \eta_i$ where the η_i are (possibly equal) solutions of the Eq. (2).

Setting $\omega_i = \frac{\eta_i'}{\eta_i}$, one has $\frac{\varphi'}{\varphi} = \sum_{i=1}^{n} \omega_i$ where ω_i $(i = 1, \ldots, n)$ is a solution of the Riccati Eq. (3).

Let $\underline{a}$ be defined by the relation

$$A(w, \underline{a}) = \prod_{i=1}^{n} (w - \omega_i).$$

According to Proposition 3, $\underline{a}$ is a solution of $(\#)_n$ and obviously

$$a_{n-1} = \sum_{i=1}^{n} \omega_i = \frac{\varphi'}{\varphi}. \quad \square$$

Corollary 1. *If the Riccati equation has an algebraic solution of degree* $n \geqq 2$, *the system* $(\#)_n$ *has a* rational *solution.*

Proof. In the previous proposition take $F = U_n$, $y_1 = \eta$, $y_2 = \zeta$ as defined in Proposition 2. The logarithmic derivative $\frac{\varphi'}{\varphi}$ belongs to $\mathbf{C}(x)$ and the result follows from Propositions 4 and 5. $\square$

This section can be summarized in the

Theorem 2. *If the differential equation* $y'' - ry = 0$ *has liouvillian solutions then*

1. *for some* m *in* $L_{\max}$, *the system* $(\#)_m$ *has a rational solution uniquely determined from* a_{m-1},
2. *if* n *is the smallest such* m, *then the polynomial (4) associated to a rational solution of* $(\#)_n$ *is the minimal polynomial of an algebraic solution of the Riccati equation,*
3. *if* $n \geqq 2$ *there is a rational solution such that* a_{n-1} *is the logarithmic derivative of* $u_n(x)$.

Remark. The system $(\#)_n$ can be put in the form $(*)_n$ used in the algorithm by setting $a_{n-1} = \theta + \frac{P'}{P}$ and $a_i = \frac{P_i}{P}$ for $i = -1, 0, \ldots, n-2$.

2.3 The Proof

2.3.1 Conditions on the Poles of r. In [12] Kovacic deduces a set of necessary conditions for cases i) to iii) to occur. These conditions concern the poles of r. In order to let ∞ play the same rôle as any point in $\mathbf{C}$, we make the change of function and variable defined by

$$Y(x) = xy\left(\frac{1}{x}\right). \tag{8}$$

The Eq. (2) transforms to equation

$$Y'' - RY = 0 \quad \text{with} \quad R(x) = \frac{1}{x^4} r\left(\frac{1}{x}\right) \tag{9}$$

which is still in reduced form.

The order of ∞ in Eq. (2) is then defined as the order of 0 in (9):

$$o(\infty) = 4 + \deg s - \deg t \quad \text{if} \quad r = \frac{s}{t}.$$

We call the point ∞ a *pole* of r if $o(\infty) > 0$.

Using the same arguments as Kovacic [12] for every point in $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$, we are led to classify the liouvillian equations according to the following list, giving in each case a set of necessary conditions (N.C.):

Case 1: the Eq. (2) has a solution η such that $\frac{\eta'}{\eta} \in \mathbf{C}(x)$.

N.C.: *the order of each pole of r (in* $\mathbf{P}^1(\mathbf{C})$*) is 1 or is even.*

Case 2: the Eq. (2) has a solution η such that $\frac{\eta'}{\eta}$ is quadratic over $\mathbf{C}(x)$.

N.C.: *r has at least two poles with order either 2 or odd and greater than 2.*

Case 3: the Eq. (2) has a solution η such that $\frac{\eta'}{\eta}$ is algebraic over $\mathbf{C}(x)$ of degree 4, 6 or 12.

N.C.: *the order of each pole of r is less or equal to 2. Moreover in this case the arithmetic conditions listed in* **First step 5** *of the algorithm must be fulfilled.*

Remarks

1. The condition obtained in **Case 2** is stronger than the one given by Kovacic as we consider ∞ as any other point in $\mathbf{P}^1(\mathbf{C})$.
2. The arithmetic conditions of **Case 3** are straightened out in the work of Singer and Ulmer [21]; they also give a reference to an old similar result by Fuchs.

As a consequence of all these conditions the set $L_{\max}$ can be reduced to the subset L.

2.3.2 The Exponents when $n \geqq 2$. Throughout the next two sections we suppose that the Riccati equation has an algebraic solution of degree $n \geqq 2$.

For some finite set $\mathscr{C}$ such that $\Gamma' \subset \mathscr{C} \subset \mathbf{C}$, the rational function $u_n(x)$ (Proposition 2) may be written

$$u_n(x) = \lambda \prod_{c \in \mathscr{C}} (x - c)^{e_c}$$

with $\lambda \in \mathbf{C}$ and $e_c \in \mathbf{Z}$. This formula is not unique as we accept $e_c = 0$.

Using Proposition 5 and its corollary, one deduces that the system $(\#)_n$ has the solution

$$a_{n-1} = \frac{1}{h(n)} \sum_{c \in \mathscr{C}} \frac{e_c}{x - c}.$$

By examining the system $(\#)_n$ one gets conditions on the exponents e_c.

Theorem 3. *With the previous notations, for all* $c \in \mathbf{C}$ *the following assertions are true:*

i) *If c is not a pole of r then* $e_c \in h(n)\{0, 1, \ldots, n\}$.

ii) *If c is a simple pole of r then* $e_c = nh(n)$.

iii) *If c is a double pole of r then e_c, which must be an integer, is of the form*:

$$\frac{nh(n)}{2}\left(1-\sqrt{1+4\alpha_c}\right)+h(n)j\sqrt{1+4\alpha_c}$$

with $j\in\{0,1,\ldots,n\}$ *and* α_c *defined as in* **First step 4** *of the algorithm.*

iv) *If* $n=2$ *and* c *is a pole of order* $\nu\geqq 3$ *of* r, *then* $e_c=\nu$.

Proof. Without loss of generality one may suppose $c=0$. We write e in place of e_0 and α for α_c.

Let us assume firstly that $n\geqq 2$ and that r has the form $r=\frac{\alpha}{x^2}+\frac{\beta}{x}+O(1)$. As a_{n-1} has only simple poles, one can define e and f by the condition $a_{n-1}=\frac{e}{h(n)x}+f+O(1)$.

If the a_i are defined recursively by $(\#)_n$ it is clear by induction that the valuation of a_i satisfies $\mathrm{val}(a_i)\geqq -(n-i)$. Let A_i and B_i be defined by

$$a_i(x)=\frac{A_i}{x^{n-i}}+\frac{B_i}{x^{n-i-1}}+O\left(\frac{1}{x^{n-i-2}}\right).$$

From $(\#)_n$ one deduces the induction formulas for $0\leqq i\leqq n$:

$$A_{i-1}=\left(n-i-\frac{e}{h(n)}\right)A_i-(n-i)(i+1)\alpha A_{i+1}$$

starting with $A_{n+1}=0$, $A_n=-1$ and with the extra condition $A_{-1}=0$.

- In case i) or ii) $\alpha=0$ and one gets easily

$$A_i=\frac{(-1)^{n-i+1}}{(n-1)!}\prod_{j=0}^{n-i+1}\left(\frac{e}{h(n)}-j\right).$$

The condition $A_{-1}=0$ implies that $\frac{e}{h(n)}$ has to be a natural integer less or equal to n. This proves case i).

- In case ii) one looks now at the induction formula satisfied by B_i, $0\leqq i\leqq n$:

$$B_{i-1}=\left(n-i-1-\frac{e}{h(n)}\right)B_i-fA_i-(i+1)(n-i)\beta A_{i+1}$$

starting with $B_n=0$ and with the extra condition $B_{-1}=0$. Then one proves that if the integer $l=\frac{e}{h(n)}$ satisfies $l<n$ then $\beta=0$, in contradiction with the order of 0 as a pole of r:

One has $A_i=0$ for $n\leqq i<n-l-1$ and $A_{n-l}\neq 0$. So from $B_{-1}=0$ one deduces $B_i=0$ for $0\leqq i\leqq n-l-2$. In particular

$$0=B_{n-l-2}=0\times B_{n-l-1}-\beta(n-l)(l+1)A_{n-l}$$

which implies $\beta=0$.

- In case iii) $\alpha\neq 0$ and the induction formula for A_i is not so easy to solve. It is convenient to replace A_i by $D_i = \dfrac{A_i}{(n-i)!}$ which satisfies for $0 \leqq i \leqq n$

$$(n-i+1)D_{i-1} = \left(n - i - \frac{e}{h(n)}\right)D_i - \alpha(i+1)D_{i+1} \tag{10}$$

with $D_{n+1} = 0$, $D_n = -1$, $D_{-1} = 0$.

We remark that, due to the factor $(i+1)$ in the last term, if $(D_i)_{-1 \leqq i \leqq n+1}$ satisfies these conditions then the sequence $(D_i)_{i \in Z}$ obtained by setting $D_i = 0$ for $i \leqq -1$ and $i \geqq n+1$ also satisfies the induction formula (10). Accordingly its generating function $\varphi(w) = \sum_{i \in Z} D_i w^i$ satisfies the differential equation

$$(w^2 - w - \alpha)\varphi'(w) + \left(\frac{e}{h(n)} - n(w-1)\right)\varphi(w) = 0.$$

Thus

$$\frac{\varphi'(w)}{\varphi(w)} = \frac{-\dfrac{e}{h(n)} + n(w-1)}{w^2 - w - \alpha}$$

which, if $\alpha \neq -\frac{1}{4}$, admits the partial fraction decomposition

$$\sum_{j=1,2} \frac{\dfrac{e}{h(n)} + n(\mu_j - 1)}{(2\mu_j - 1)(w - \mu_j)}$$

with $\mu_j = \frac{1}{2}\left(1 + (-1)^j \sqrt{1 + 4\alpha}\right)$.

But $\varphi(w)$ must be a *polynomial* of degree n and this implies some conditions on the partial fraction decomposition of its logarithmic derivative. Assertion iii) follows from these conditions.

Let us now suppose that $n = 2$ and that 0 a pole of order $v > 2$ of r. If we write $r = \dfrac{\alpha}{x^v} + o\left(\dfrac{1}{x^v}\right)$ with $\alpha \neq 0$, one deduces from $(\#)_2$:

$$a_2 = -1$$

$$a_1 = \frac{e}{2x} + o\left(\frac{1}{x}\right)$$

$$a_0 = \frac{2\alpha}{x^v} + o\left(\frac{1}{x^v}\right)$$

$$a_{-1} = \frac{2\alpha(v-e)}{x^{v+1}} + o\left(\frac{1}{x^{v+1}}\right).$$

The result now follows from the condition $a_{-1} = 0$. □

Corollary 2. *The statements of Theorem 3 are valid for* $c = \infty$ *with*

$$e_\infty = nh(n) - \sum_{c \in \mathscr{C}} e_c.$$

Proof. The function $\tilde{u}_n(x)$ associated to the equation obtained by the transformation (8) satisfies

$$\tilde{u}_n(x) = x^{nh(n)} u_n\left(\frac{1}{x}\right)$$

in which 0 has the given order e_∞. □

2.3.3 End of the Proof when $n \geqq 2$. One deduces from Theorem 3i) that if liouvillian solutions of the corresponding type do exist, there is a *polynomial P* such that

$$\prod_{c \in \mathscr{C} \setminus \Gamma'} (x-c)^{e_c} = P(x)^{h(n)}.$$

Lemma 1. *The polynomial P just mentioned has degree*

$$d = n - \frac{1}{h(n)} \sum_{c \in \Gamma} e_c.$$

Proof. From the definition of P one has

$$\begin{aligned} dh(n) &= \sum_{c \in \mathscr{C} \setminus \Gamma'} e_c = \sum_{c \in \mathscr{C}} e_c - \sum_{c \in \Gamma'} e_c \\ &= nh(n) - e_\infty - \sum_{c \in \Gamma'} e_c \\ &= nh(n) - \sum_{c \in \Gamma} e_c. \quad \square \end{aligned}$$

Let θ be defined by

$$\theta = \frac{1}{h(n)} \sum_{c \in \Gamma'} \frac{e_c}{x-c}$$

so that $a_{n-1} = \theta + \frac{P'}{P}$. Using this decomposition of a_{n-1} the system $(\#)_n$ takes the form $(*)_n$ given explicitly in the algorithm.

2.3.4 Sketch of the Proof when $n = 1$. This time we have to look directly for a rational solution ω of the Riccati equation $\omega' + \omega^2 = r$ and, contrary to what happened for a_{n-1} when $n \geqq 2$, this solution could have poles of arbitrary order.

Proposition 6 (Kovacic). *If the Riccati Eq. (3) has the rational solution ω then any pole $c \in \mathbf{C}$ of ω satisfies*

i) *if c is not a pole of r or is a pole of order 1 or 2 of r then c is a simple pole of ω. Its residue e_c is given by the formulas of Theorem 3 with $n = 1$ if one sets $h(1) = 1$,*

ii) *If c is a pole of r of even order 2ν with $\nu \geqq 2$, then c is a pole of order ν of ω and*

$$\omega = \theta_c + \frac{e_c}{x-c} + O(1)$$

where $\theta_c = \sum_{i=\nu}^{2} \frac{\lambda_{i,c}}{(x-c)^i}$ *is defined up to sign by the condition*

$$r - \theta_c^2 = O\left(\frac{1}{(x-c)^\nu}\right).$$

For each choice of sign the possible values of e_c *are given in the algorithm* (**Second step 5.**)

Suitable modifications of these results lead to the value of e_∞ and eventually of $[\sqrt{r}]_\infty$. This time the exponent e_c for the solution $\omega = \frac{y'}{y}$ at a finite point c is defined as $e_c = \operatorname{Res}(\omega, c)$. At ∞ we set $e_\infty = \operatorname{Res}(\Omega, 0)$ where Ω is the logarithmic derivative of $Y(x) = xy\left(\frac{1}{x}\right)$ (transformation (8)). Therefore we get the relation

$$\operatorname{Res}(\Omega, 0) = 1 + \operatorname{Res}(\omega, \infty).$$

As previously the Riccati equation admits a rational solution if and only if there is a *polynomial* P such that

$$\sum_{c \in \mathscr{C} \setminus \Gamma} \frac{e_c}{x-c} = \frac{P'}{P}.$$

If such a polynomial exists its degree is given by the formula of Lemma 1 and it satisfies the following differential equation equivalent to $(*)_1$:

$$P'' + 2\theta P' + (\theta' + \theta^2 - r)P = 0 \tag{11}$$

where θ has the value given in **Third step 3** of the algorithm.

Remark. The third order differential equation for P equivalent to $(\#)_2$ is (Kovacic [12])

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0 \tag{12}$$

which can be seen to have $\{y_1^2, y_1 y_2, y_2^2\}$ as a fundamental set of solutions where $\{y_1, y_2\}$ is a fundamental set of solutions of the second order equation

$$y'' + \theta y' + (\tfrac{1}{2}\theta' + \tfrac{1}{4}\theta^2 - r)y = 0 \tag{13}$$

(the corresponding statement for $n > 2$ seems not to be true).

If D denotes the second order equation, we will denote by $D^{\circledS^2}$ the third order Eq. (12). More generally one may define $D^{\circledS^n}$ for $n \in \mathbf{N}^*$. Some properties of these operators were studied by Singer (see [20] and also [21]).

3 Examples

3.1 The Hypergeometric Family

The *hypergeometric fuchsian equation* normalized in such a way that its three singular points are located at $0, 1$ and ∞ is

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0 \tag{14}$$

where a, b and c are complex parameters.

The reduced form is more conveniently written with the parameters λ, μ and ν defined by $\lambda = 1 - c$, $\mu = c - a - b$ and $\nu = a - b$.

The change of function $y \mapsto x^{(\lambda-1)/2}(1-x)^{(\mu-1)/2} y$ puts the equation in reduced form with

$$r(x) = \frac{\lambda^2 - 1}{4x^2} + \frac{\mu^2 - 1}{4(1-x)^2} + \frac{\lambda^2 + \mu^2 - \nu^2 - 1}{4x(1-x)}.$$

The *confluent hypergeometric equation* obtained from the previous one by making "1 tend to ∞" has two classical forms:

- Kummer's form

$$xy'' + (c-x)y' - ay = 0 \tag{15}$$

- Whittaker's form

$$y'' - \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2}\right)y = 0 \tag{16}$$

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$.

Note that the second form is reduced and that the liouvillian change of function $y \mapsto x^{-(1/2)-\mu} e^{x/2} y$ transforms the first equation into the second one.

By double confluence, that is making "0 and 1 tend to ∞", one gets the *parabolic cylinder equation*

$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 0. \tag{17}$$

The following form of this equation is studied by Rehm [15]:

$$y'' - (a^2x^2 + 2abx + c)y = 0, \quad a \neq 0. \tag{18}$$

The change of variable $x \mapsto \sqrt{\frac{2}{a}}(ax + b)$ and the relation $\nu = \frac{b^2 - c}{2a} - \frac{1}{2}$ lead from one equation to the other. We will use Rehm's form in order to compare his results with ours.

Applying the general principle that the more confluent the equation, the easier it is to handle, we start with this last case.

3.1.1 The Parabolic Cylinder Equation. The only singular point of the Eq. (18) $y'' - (a^2x^2 + 2abx + c)y = 0$ is located at ∞ and, as we suppose $a \neq 0$, it has order 6.

The only possible value for n is 1.

One may take $[\sqrt{r}]_\infty = ax + b$ and then

$$E_\infty = \left\{ \frac{1}{2}\left(3 + \varepsilon\frac{b^2 - c}{a}\right) \middle| \varepsilon = \pm 1 \right\}.$$

Using Lemma 1 one gets the *necessary condition* (see Rehm [15]):

$$\boxed{\frac{b^2-c}{a} \text{ is an odd integer}}$$

With the other form of this equation this condition reads $\boxed{\nu \in \mathbf{Z}}$.

We suppose now this condition satisfied and, as the Eq. (18) remains unchanged by $(a,b) \mapsto (-a,-b)$, one may suppose that this integer is positive, namely that $\frac{b^2-c}{a} = 2d+1$.

One has to find a polynomial P of degree d such that

$$P'' - 2(ax+b)P' + 2daP = 0.$$

Under the previous condition such a polynomial always exists (an explicit formula is given in [15]), so that the condition is also *sufficient*. Only one solution (up to scalar multiplication) is detected by the algorithm.

With the help of Kovacic's algorithm, it is very easy to see that the equation $y'' + (bx+c)y = 0$ (a form of Airy equation when $b \neq 0$) has liouvillian solutions only if $b = 0$, as otherwise $o(\infty) = 5$ so that $\gamma_2 = 0$ and $\gamma = 1$, giving $L = \varnothing$.

3.1.2 The Confluent Hypergeometric Equation. We begin by using the Whittaker form (16). One has to distinguish three cases, according to the order of r at 0.

1. If $4\mu^2 - 1 = \kappa = 0$, the equation admits trivially the two liouvillian solutions $e^{-x/2}$ and $e^{x/2}$. They both can be detected by the algorithm.

2. If $4\mu^2 - 1 = 0$, $\kappa \neq 0$, one has $L = \{1\}$ and an easy computation gives for the possible degree of the polynomial $d = \pm\kappa - 1$. Thus we get the necessary condition

$$\boxed{\kappa \in \mathbf{Z}^*}.$$

If this condition is satisfied and if $\varepsilon = \pm 1$ is (uniquely) defined in such a way that $\varepsilon\kappa \in -\mathbf{N}^*$, we have to look for a polynomial of degree $d = -\varepsilon\kappa - 1$ satisfying the (confluent hypergeometric) equation

$$xP'' + (2+\varepsilon x)P' - \varepsilon dP = 0.$$

It is well known that the Laguerre polynomial

$$L_d^{(1)}(-\varepsilon x) = \sum_{m=0}^{d} \varepsilon^m \binom{d+1}{d-m} \frac{x^m}{m!}$$

is such a solution; thus the given necessary condition is also sufficient. The liouvillian solution of (16) given by the algorithm is then

$$xL_d^{(1)}(-\varepsilon x)e^{\varepsilon x/2}.$$

3. If $4\mu^2 - 1 \neq 0$, then once more $L = \{1\}$ and, if it exists, d takes the form $d = -\frac{1}{2} + \varepsilon\mu + \varepsilon'\kappa$. Thus we get the necessary condition

$$\boxed{\tfrac{1}{2} + \kappa + \mu \in \mathbf{Z} \quad \text{or} \quad \tfrac{1}{2} - \kappa + \mu \in \mathbf{Z}}$$

which allows one or two possibilities for $d \in \mathbf{N}$. The equation for P is then

$$xP'' + (1 + 2\varepsilon\mu + \varepsilon' x)P' - \varepsilon' dP = 0$$

which admits the Laguerre polynomial solution

$$L_d^{2\varepsilon\mu}(-\varepsilon' x) = \sum_{m=0}^{d} (\varepsilon')^m \binom{d+2\varepsilon\mu}{d-m} \frac{x^m}{m!}.$$

The corresponding liouvillian solution of (16) is

$$L_d^{2\varepsilon\mu}(-\varepsilon' x) x^{1/2+\varepsilon\mu} e^{\varepsilon' x/2}.$$

Translating these results for the Kummer Eq. (15), we discuss the number of solutions possibly given by the algorithm in the next proposition. Using the relations between the parameters of the two forms of the equation, we have $4\mu^2 - 1 = c(c-2)$, $\kappa = \frac{c}{2} - a$, $\frac{1}{2} + \kappa + \mu = c - a$, and $\frac{1}{2} - \kappa + \mu = a$.

Proposition 7. *The confluent hypergeometric equation $xy'' + (c-x)y' - ay = 0$ has liouvillian solutions if and only if*

$$\begin{cases} c \in \{0,2) \quad \textit{and} \quad a \in \mathbf{Z} \\ \quad \textit{or} \\ c \notin \{0,2\} \quad \textit{and} \quad \begin{cases} a \in \mathbf{Z} \\ \textit{or} \\ c - a \in \mathbf{Z} \end{cases} \end{cases}$$

Moreover

- *if $c = 0$ or $c = 2$ and $a = \frac{c}{2}$, two linearly independent solutions are detected*
- *if $c = 0$* or *$c = 2$, $a \in \mathbf{Z}$, $a \neq \frac{c}{2}$, one solution is detected*
- *if $c(c-2) \neq 0$, $a \in \mathbf{Z}$ or $c - a \in \mathbf{Z}$, $k \leqq 2$ solutions are detected where*

$$k = \mathrm{card}(\mathbf{N} \cap \{a-c, -a, a-1, c-a-1\}).$$

But when $k = 2$ these solutions are linearly independent if and only if

$$c \leqq a \leqq 0 \quad \textit{or} \quad 1 \leqq a \leqq c-1.$$

These results may be parallelled with the computation of the galois group of the Kummer equation performed by Martinet and Ramis [13]: the same values of the parameters are the relevant ones. In this table k is the number of linearly independent solutions given by the algorithm.

k	Differential galois group
0	$GL(2, \mathbf{C})$
1	$\mathbf{C}^* \ltimes \mathbf{C}$
2	$\mathbf{C}^*$

3.1.3 The Fuchsian Hypergeometric Equation. The problem of finding liouvillian hypergeometric equations is solved by Kimura [10]. Working without the help of a computer we recover only part of his results but we also obtain formulas connecting some usual or higher order hypergeometric functions.

According to the order of 0, 1 and ∞ in r there are five cases to consider. Up to a homography leaving the set $\{0, 1, \infty\}$ fixed, these cases are:

C1 $\lambda^2 = \mu^2 = \nu^2 = 1$, in which case r has no pole;

C2 $\lambda^2 = \mu^2 = 1$, $\nu^2 \neq 1$, in which case r has two simple poles (0 and 1) and a double pole (∞);

C3 $\lambda^2 = \mu^2 \neq 1$, $\nu^2 = 1$, in which case r has two double poles (0 and 1);

C4 $\lambda^2 \neq 1, \mu^2 \neq 1, \lambda^2 \neq \mu^2, \nu^2 = 1$, in which case r has two double poles (0 and 1) and a simple pole (∞);

C5 $\lambda^2 \neq 1$, $\mu^2 \neq 1$, $\nu^2 \neq 1$, in which case r has three double poles.

The case **C1** is trivial. In the other cases there is a double pole; thus m^+ is always 2 and then $1 \in L$. Actually $L = L_{\max}$ except in case **C2** where $2 \notin L$.

Let us try $\boxed{n = 1}$.

The exponent sets at 0 and 1 are

$$E_0 = \{\tfrac{1}{2}(1 + \varepsilon_0 \lambda) | \varepsilon_0 = \pm 1\}$$
$$E_1 = \{\tfrac{1}{2}(1 + \varepsilon_1 \mu) | \varepsilon_1 = \pm 1\}$$

except in case **C2** where $E_0 = E_1 = \{1\}$.

The exponent set E_∞ is

- $E_\infty = \{\tfrac{1}{2}(1 + \varepsilon_\infty \nu) | \varepsilon_\infty = \pm 1\}$ in cases **C2** and **C5**
- $E_\infty = \{0, 1\}$ in case **C3**
- $E_\infty = \{1\}$ in case **C4**.

The following table shows the degree of a possible P and the condition expressing that one such number is a positive integer.

	d	Necessary Condition
C2	$-\frac{1}{2}(3 + \varepsilon_\infty \nu)$	$\nu \in 1 + 2\mathbf{Z} \backslash \{\pm 1\}$
C3	$-e_\infty - \frac{1}{2}(\varepsilon_0 + \varepsilon_1)\lambda$	$-$
C4	$-1 - \frac{1}{2}(\varepsilon_0 \lambda + \varepsilon_1 \mu)$	$\lambda \pm \mu \in 2\mathbf{Z}$
C5	$-\frac{1}{2}(1 + \varepsilon_0 \lambda + \varepsilon_1 \mu + \varepsilon_\infty \nu)$	$\lambda \pm \mu \pm \nu \in 2\mathbf{Z} + 1$

In case **C3** one easily finds the liouvillian solution

$$y(x) = x^{(1 + \varepsilon_0 \lambda)/2}(1 - x)^{(1 - \varepsilon_0 \lambda)/2}.$$

In the three other cases the conditions found for d all reduce to the unique one

$$\boxed{\lambda \pm \mu \pm \nu \text{ is an odd integer}} \qquad (\star)$$

Let us suppose that this condition is fulfilled. By computing in each case the value of θ one finds the following differential equation for P:

case **C2** $x(1 - x)P'' + 2(1 - 2x)P' + d(d + 3)P = 0$

case **C4** $x(1 - x)P'' + (1 + \varepsilon_0 \lambda + 2dx)P' - d(d + 1)P = 0$

case **C5** $x(1 - x)P'' + (1 + \varepsilon_0 \lambda + (2d + \varepsilon_\infty \nu - 1)x)P' - d(d + \varepsilon_\infty \nu)P = 0.$

One may note that the equation as well as the degree d for case **C4** is the particular case $\varepsilon_\infty \nu = 1$ of case **C5**.

All these differential equations are hypergeometric and do have polynomial solutions of the suitable degree d, namely

case **C2**: ${}_2F_1\left(\begin{matrix} -d, d+3 \\ 2 \end{matrix}\middle| x\right)$ which is up to a constant factor the *Gegenbauer* polynomial $C_d^{3/2}(1-2x)$ or the *Jacobi* polynomial $P_d^{(1,1)}(1-2x)$

cases **C4** or **C5**: ${}_2F_1\left(\begin{matrix} -d, -d-\varepsilon_\infty \nu \\ 1+\varepsilon_0\lambda \end{matrix}\middle| x\right)$ which is up to a constant factor the *Jacobi* polynomial $P_d^{(\varepsilon_0\lambda, \varepsilon_1\mu)}(1-2x)$.

These results are summarized in the next proposition which is part A of theorem I in Kimura [10]. In 1894, Beke [2] proved the same result in terms of the parameters a, b, c, namely that equation (14) is reducible if and only if one of the numbers a, b, $c-a$, $c-b$ is a negative integer (we thank the referee who mentioned this fact to us).

Proposition 8. *If the three parameters* λ, μ, ν *of the hypergeometric differential equation*

$$y'' - \left[\frac{\lambda^2-1}{4x^2} + \frac{\mu^2-1}{4(1-x)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4x(1-x)}\right]y = 0$$

are such that $\pm\lambda\pm\mu\pm\nu$ *is an odd integer then the equation has liouvillian solutions.*

If the condition ($\star$) is not fulfilled, we try $\boxed{n=2}$ in cases **C3**–**C5**.

The arithmetic necessary conditions for the exponents show that among the three integers e_c exactly one has to be even. Up to a convenient homography let us suppose that e_∞ is even and that e_0 and e_1 are odd. We get the corresponding forms of the parameters listed below:

	Parameters
C3	$\lambda = \frac{1}{2} + l, l \in \mathbf{Z}$
C4	$\lambda = \frac{1}{2} + l, \mu = \frac{1}{2} + m, l \neq m, l \neq -m-1, l+m$ even
C5	$\lambda = \frac{1}{2} + l, \mu = \frac{1}{2} + m$, if $\nu \in \mathbf{Z}$, $l+m+\nu$ odd

In cases **C4** and **C5** one can prove that among the possible values for d there is always the one given by the following table:

d	$l \geqq 0, m \geqq 0$	$l \geqq 0, m \leqq -1$	$l \leqq -1, m \geqq 0$	$l \leqq -1, m \leqq -1$
C4	$l+m-1$	$l-m-2$	$m-l-1$	$-l-m-3$
C5	$l+m$	$l+m-1$	$m-l-1$	$-l-m-2$

We may suppose $l \geqq 0$ and $m \geqq 0$ as these conditions are achieved by proper transformations on the parameters of the type $(l, m) \mapsto (l, -m-1)$.

Under these conditions we will study each case separately.

Let us first prove two lemmas.

Lemma 2. *Let $d \in \mathbf{N}$ and $e \in \frac{1}{2} - \mathbf{N}$; if $e \neq -\frac{d}{2}$ the differential equation*

$$4x(1-x)y'' + 4(dx+e)y' - d(d+2)y = 0$$

has a solution the square of which is a polynomial of degree d.

Proof. Near 0 this equation admits the two solutions $y_1(x) = {}_2F_1\left(\begin{matrix} -d/2, -1-d/2 \\ e \end{matrix}\middle| x\right)$ and $y_2(x) = x^{1-e}{}_2F_1\left(\begin{matrix} 1-e-d/2, -e-d/2 \\ 2-e \end{matrix}\middle| x\right)$. Let us recall that if a or b is a negative integer then ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| x\right)$ is a polynomial. Moreover if a and b are both negative integers, the degree of this polynomial is $\min(-a, -b)$. Therefore if d is even, y_1 is a polynomial of degree $\frac{d}{2}$.

If d is odd, we set $d = 2\delta - 1$ so that

$$y_2(x) = x^{l+1/2}{}_2F_1\left(\begin{matrix} 1+l-\delta, l-\delta \\ l+\frac{3}{2} \end{matrix}\middle| x\right)$$

and y_2^2 is a polynomial of degree d if $0 \leqq l \leqq \delta - 1$. The differential equation also admits the solution (near 1)

$$y_3(x) = (1-x)^{2\delta+1/2-l}{}_2F_1\left(\begin{matrix} \delta+1-l, \delta-l \\ \frac{3}{2}+2\delta-l \end{matrix}\middle| 1-x\right)$$

and if $\delta + 1 \leqq l \leqq 2\delta$, y_3^2 is a polynomial of degree d.

The proof is complete as the condition $e \neq -\frac{d}{2}$ is equivalent to $l \neq \delta$. □

Lemma 3. *Let $d \in \mathbf{N}$, $e \in \frac{1}{2} - \mathbf{N}$ and $v \in \mathbf{C}$, if $\frac{1}{2} - d \leqq e \leqq \frac{1}{2}$ and if $v \notin \mathbf{Z}$ or $|v| > d$ then the hypergeometric differential equation*

$$4x(1-x)y'' + 4((d-1)x+e)y' + (v^2 - d^2)y = 0$$

has two linearly independent solutions the product of which is a polynomial of degree d.

Proof. Let us set $e = \frac{1}{2} - l$ and let D be the given differential equation. The two functions

$$\begin{cases} y_1(x) = x^{(1/2)(d+v)}{}_2F_1\left(\begin{matrix} -\frac{1}{2}(d+v), \frac{1}{2}(1-d-v)+l \\ 1-v \end{matrix}\middle| \frac{1}{x}\right) \\ \text{and} \\ y_2(x) = x^{(1/2)(d-v)}{}_2F_1\left(\begin{matrix} \frac{1}{2}(v-d), \frac{1}{2}(1+v-d)+l \\ 1+v \end{matrix}\middle| \frac{1}{x}\right) \end{cases}$$

form a fundamental set of solutions of D near ∞. Let $z(x)$ be their product. By using the general formula giving the product of two hypergeometric functions one has (see Bailey [1])

$$z(x)=\sum_{j\geqq 0} A_j\, {}_4F_3\left(\begin{matrix} -j,-\frac{1}{2}(d-v),\frac{1}{2}(1+v-d)+l,v-j \\ 1+v,1+\frac{1}{2}(d+v)-j,\frac{1}{2}(1+d+v)-l-j\end{matrix}\middle|\, 1\right)x^{d-j}$$

with

$$A_j=\frac{\langle -\frac{1}{2}(d+v)\rangle_j\langle\frac{1}{2}(1+v-d)+l\rangle_j}{\langle 1-v\rangle_j\, j!}$$

where we write for $a\in C$ and $j\in\mathbf{N}$, $\langle a\rangle_j=a(a+1)\cdots(a+j-1)$ if $j>0$ and $\langle a\rangle_0=1$.

For $j=d+1$ the term ${}_4F_3$ can be written

$${}_3F_2\left(\begin{matrix} -d-1,\frac{1}{2}(1+v-d)+l,v-d-1 \\ 1+v,\frac{1}{2}(v-d-1)-l\end{matrix}\middle|\, 1\right)$$

which, according to Dixon's formula (see [22]), vanishes if and only if $0\leqq l\leqq d$.

Therefore $z(x)$ has the Laurent series expansion

$$z(x)=\sum_{j\leqq d}\lambda_j x^j$$

with $\lambda_d=1$ and $\lambda_{-1}=0$.

On the other hand $z(x)$ is a solution of the third order differential equation $D^{\circledS^2}$ which can be written in terms of the Euler operator $\vartheta=\dfrac{d}{dx}$

$$D_3=(1-x)^2\vartheta^3+3(1-x)(dx-\tfrac{1}{2}l)\vartheta^3+\Phi(x)\vartheta+(v^2-d^2)(dx-l)x$$

with $\Phi(x)=(3d^2-v^2)x^2+(v^2-d^2-4ld-l-d-\frac{1}{2})x+\frac{1}{2}+2l^2+2l$.

One deduces that the coefficients of $z(x)$ satisfy for $j\in\mathbf{Z}$, $j\leqq d$ an induction formula of the form

$$A(j)\lambda_{j-1}+B(j)\lambda_j+(j+1)(2l-j)(l-j-\tfrac{1}{2})\lambda_{j+1}=0 \qquad [j]$$

where $A(j)$ and $B(j)$ are polynomials in j and one has set $\lambda_{d+1}=0$.

As $A(j)=(d-j+1)(v^2-(d-j+1)^2)$, this coefficient is non zero for $0\leqq j\leqq d$ if the conditions of the lemma are fulfilled, so that the sequence $(\lambda_j)_{j\leqq d}$ is uniquely defined. The relation $[-1]$ shows that, as $\lambda_{-1}=0$, $\lambda_{-2}=0$ too, and then recursively $\lambda_j=0$ for $j<0$. In other words $z(x)$ is a *polynomial* of degree d. □

The study of the three cases can now be achieved.

1. If the condition for case **C3** is satisfied one can see that a solution with $n=1$ has already been obtained and, as the differential equation for $n=2$ is $D^{\circledS^2}$ if D is the equation for $n=1$, in general we get nothing new.
2. If the conditions for **C4** are satisfied we set $e=\frac{1}{2}-l$ and $d=l+m-1$ which is an odd integer. With these notations

$$r=\frac{e(e-2)}{4x^2}+\frac{(d+e)(d+e+2)}{4(1-x)^2}+\frac{e^2+(d+e)^2+2d}{4x(1-x)}$$

and

$$\theta=\frac{e}{x}+\frac{d+e}{1-x}.$$

Notice that $e \neq -\frac{d}{2}$ as otherwise $l=m$. Thus, according to Lemma 2 if $D=4y''+4\theta y'+(2\theta'+\theta^2-4r)y$ the equation $D^{\circledS^2}$, which is equivalent to $(*)_2$, has a polynomial solution of degree d. One corresponding liouvillian solution is then given by the algorithm. As this solution is the square of a solution of D, only one solution is obtained at this step.

3. If now the conditions for case **C5** are satisfied, the same arguments with $d=l+m$ and the use of Lemma 3 lead to the same conclusion when ν is not an integer or if $|\nu|>l+m$. This time the algorithm gives two linearly independent solutions.

Finally if ν is an integer such that $\nu+l+m$ is odd and $|\nu|\leqq l+m$, another possible value for the degree is $d=l+m+\varepsilon_\infty \nu$.

Setting $a=-\frac{1}{2}(l+m+\nu)$, $b=-\frac{1}{2}(l+m-\nu)$, $c=\frac{1}{2}-l$ and writing down r and θ in terms of these coefficients one sees that the differential equation $4y''+4\theta y'+(2\theta'+\theta^2-4r)y=0$ admits the two solutions:

$$\begin{cases} y_1(x)=x^{1-c}{}_2F_1\left(\begin{matrix}1+a-c,1+b-c\\ 2-c\end{matrix}\middle| x\right)\\ y_2(x)=x^{c-a-b}{}_2F_1\left(\begin{matrix}c-a,c-b\\ c+1-a-b\end{matrix}\middle| 1-x\right).\end{cases}$$

Among the four integers $1+a-c$, $1+b-c$, $c-a$, $c-b$ at least one is negative so that y_1^2 or y_2^2 is a *polynomial* of degree $l+m\pm\nu$.

We may summarize these results in the following proposition which is consistent with Kimura's results:

Proposition 9. *If among the three parameters λ, μ, ν of the hypergeometric differential equation*

$$y''-\left[\frac{\lambda^2-1}{4x^2}+\frac{\mu^2-1}{4(1-x)^2}+\frac{\lambda^2+\mu^2-\nu^2-1}{4x(1-x)}\right]y=0$$

at least two belong to $\frac{1}{2}+\mathbf{Z}$ then the equation has liouvillian solutions.

Remark. In case **C4** as well as in case **C5** the third order equation $D^{\circledS^2}$ may happen to be a generalized hypergeometric equation in x or in $1-x$. This leads to some identities listed in the next proposition.

Proposition 10. *Let l, d and p be integers such that $l\leqq d$ and let $\nu\in C\setminus Z$, then*
Identity 1:

$${}_3F_2\left(\begin{matrix}-2d+1,-2d,-2d-1\\ -4d,-2d+\frac{1}{2}\end{matrix}\middle| x\right)=(1-x)\,{}_2F_1\left(\begin{matrix}-d+1,-d\\ \frac{3}{2}\end{matrix}\middle| 1-x\right)^2$$

Identity 2:

$${}_4F_3\left(\begin{matrix}-d-p+1,\frac{1}{2}(\nu-d),\frac{1}{2}(\nu+1-d)+l,\nu-d-p-l\\ 1+\nu,\frac{1}{2}(\nu-d)-p,\frac{1}{2}(\nu+1-d)-l-p\end{matrix}\middle| 1\right)=0$$

Identity 3:

$$\frac{1^2 3^2 5^2 \cdots (2d-1)^2}{(v^2-1^2)(v^2-2^2)\cdots(v^2-d^2)} {}_2F_1\left(\begin{matrix} -\frac{1}{2}(v+d), \frac{1}{2}(d+1-v) \\ 1-v \end{matrix}\middle| \frac{1}{x}\right).$$

$${}_2F_1\left(\begin{matrix} \frac{1}{2}(v-d), \frac{1}{2}(d+1+v) \\ 1+v \end{matrix}\middle| \frac{1}{x}\right) = {}_3F_2\left(\begin{matrix} -d, -d-v, -d+v \\ -2d, -d+\frac{1}{2} \end{matrix}\middle| x\right)$$

Identity 4:

$$x^d {}_2F_1\left(\begin{matrix} -\frac{1}{2}(v+d), \frac{1}{2}(1-d-v)+l \\ 1-v \end{matrix}\middle| \frac{1}{x}\right) {}_2F_1\left(\begin{matrix} \frac{1}{2}(v-d), \frac{1}{2}(1+v-d)+l \\ 1+v \end{matrix}\middle| \frac{1}{x}\right)$$

$$= (x-1)^d {}_2F_1\left(\begin{matrix} -\frac{1}{2}(v+d), \frac{1}{2}(1+d-v)-l \\ 1-v \end{matrix}\middle| \frac{1}{1-x}\right) {}_2F_1\left(\begin{matrix} \frac{1}{2}(v-d), \frac{1}{2}(1+v+d)-l \\ 1+v \end{matrix}\middle| \frac{1}{1-x}\right)$$

Proof

1. The second formula expresses that negative powers of x in $z(x)$ have a zero coefficient.
2. The first and the third formulas come from the fact that the third order differential equation $D^{\circledS^2}$ (or the one deduced from the change of variable $x \mapsto 1-x$) is itself a generalized hypergeometric equation if $l = d+1$ in case **C4** and if $l = d$ in case **C5**.
3. Taking into account the dependence of D_3 on d and l, one sees that the change of variable $x \mapsto 1-x$ transforms $D_3(d, l)$ in $D_3(d, d-l)$. The result follows. □

Working "by hand" we have got very few results for $n \geqq 4$ cases. We will not mention them here since using Singer and Ulmer's paper [21] one will certainly get much better conditions. In these cases the solutions if liouvillian are algebraic and listed in Schwarz [17].

3.2 The Heun Family

The Heun equation is the generic differential equation with four regular singular points located at 0, 1, c and ∞. In its reduced form r has the value

$$r(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-c} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-c)^2} \tag{19}$$

where

$$\begin{cases} A = -\frac{\alpha\beta}{2} - \frac{\alpha\gamma}{2c} + \frac{\delta\eta h}{c} & B = \frac{\alpha\beta}{2} - \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(h-1)}{c-1} \\ C = \frac{\alpha\gamma}{2c} + \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(c-h)}{c(c-1)} \cdot & D = \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right) \\ E = \frac{\beta}{2}\left(\frac{\beta}{2}-1\right) & F = \frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right) \\ \text{with } \alpha+\beta+\gamma-\delta-\eta = 1 \end{cases}$$

The confluent scheme for this equation giving the following list of reduced forms can be found in [5]

- If $c \to \infty$ one gets the *confluent* Heun equation for which

$$r(x) = \frac{\alpha^2}{4} + \frac{2\eta - 1}{2x} + \frac{2(\delta + \eta) - 1}{2(1-x)} + \frac{\beta^2 - 1}{4x^2} + \frac{\gamma^2 - 1}{4(1-x)^2}$$

- If $\left.\begin{matrix} c \\ 1 \end{matrix}\right\} \to \infty$ one gets the *biconfluent* Heun equation for which

$$r(x) = x^2 + \beta x + \frac{\beta^2}{4} - \gamma + \frac{\delta}{2x} + \frac{\alpha^2 - 1}{4x^2}$$

- if $\left\{\begin{matrix} c \to \infty \\ 1 \to 0 \end{matrix}\right.$ one gets the *double confluent* Heun equation for which

$$r(x) = \frac{\alpha^2}{4} - \frac{\gamma}{x} - \frac{\delta}{x^2} - \frac{\beta}{x^3} + \frac{\alpha^2}{4x^4}$$

- If $\left.\begin{matrix} c \\ 1 \\ 0 \end{matrix}\right\} \to \infty$ one gets the *triconfluent* Heun equation for which

$$r(x) = \frac{9x^4}{4} + \frac{3}{2}\gamma x^2 - \beta x + \frac{\gamma^2}{4} - \alpha.$$

Working without the help of a computer we have been able to find a complete answer for the tri-, double and biconfluent forms and a partial answer for the confluent equation, namely the case $n = 1$. Cases $n \geqq 4$ may occur only for the Heun equation.

3.2.1 The Triconfluent Heun Equation. The only singular point is ∞ which has order $o(\infty) = 8$. The only possible value for n is 1. One may take $[\sqrt{r}]_\infty = \frac{3}{2}x^2 + \frac{\gamma}{2}$ and easily get the two values for a possible degree $d = -1 \pm \frac{\beta}{3}$ which gives the necessary condition

$$\boxed{\beta \in 3\mathbf{Z}^*}$$

If $\beta = 3\varepsilon(d+1)$ with $\varepsilon = \pm 1$ and $d \in \mathbf{N}$, one has to find a polynomial of degree d satisfying the differential equation

$$y'' - \varepsilon(3x^2 + \gamma)y' + (3\varepsilon d x + \alpha)y = 0.$$

It is not difficult to see that the linear system for the $d + 1$ coefficients of such a polynomial is homogeneous and has $d + 1$ equations, so that it admits a non trivial solution if and only if the following determinant $Q_{d+1}(\alpha, \gamma)$ is zero:

$$\begin{vmatrix} \alpha & -\gamma & 2.1 & 0 & \cdots & & & 0 \\ 3d & \alpha & -2\gamma & 3.2 & \cdots & & & 0 \\ 0 & 3(d-1) & \alpha & -3\gamma & \cdots & & & 0 \\ 0 & 0 & 3(d-2) & \alpha & \cdots & & & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & \cdots & & \cdots & 3.3 & \alpha & -(d-1)\gamma & d(d-1) \\ 0 & \cdots & & \cdots & 0 & 3.2 & \alpha & -d\gamma \\ 0 & \cdots & & \cdots & 0 & 0 & 3.1 & \alpha \end{vmatrix}$$

In conclusion we have

Proposition 11. *A triconfluent Heun equation has liouvillian solutions if and only if* $\beta \in 3\mathbf{Z}^*$ *and the determinant* $Q_{d+1}(\alpha, \gamma) = 0$.

3.2.2 The Double Confluent Heun Equation. The following lemma is useful in most of the cases we have been able to study completely.

Lemma 4. *Let* $a, b, u, v, w, \xi \in \mathbf{C}$ *and* $d \in \mathbf{N}$. *The differential equation*

$$(bx + ax^2)y'' + (u + vx - \xi x^2)y' + (w + d\xi x)y = 0$$

is satisfied by a polynomial of degree d if and only if $\Pi_{d+1}(a, b, u, v, \xi, w) = 0$ *where* Π_{d+1} *is the determinant*

$$\begin{vmatrix} w & u & 0 & 0 & 0 & \cdots & 0 \\ d\xi & w+1(v+0a) & 2(u+b) & 0 & 0 & \cdots & 0 \\ 0 & (d-1)\xi & w+2(v+1a) & 3(u+2b) & 0 & \cdots & 0 \\ 0 & 0 & (d-2)\xi & w+3(v+2a) & 4(u+3b) & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & \cdots & & \cdots & 2\xi & w+(d-1)(v+(d-2)a) & d(u+(d-1)b) \\ 0 & \cdots & & \cdots & 0 & \xi & w+d(v+(d-1)a) \end{vmatrix}$$

Proof. If $P(x) = \sum_{k=0}^{d} \lambda_k x^k$ is a solution the λ_k satisfy the induction formula

$$\xi(d-k+1)\lambda_{k-1} + [w + kv + k(k-1)a]\lambda_k + (k+1)(u+kb)\lambda_{k+1} = 0$$

for all $k \in \mathbf{Z}$ with $\lambda_k = 0$ if $k < 0$ or $k \geqq d+1$ and $\lambda_d \neq 0$. In other words the vector $(\lambda_0, \lambda_1, \ldots, \lambda_d)$ has to be a non trivial solution of a linear system, the determinant of which is $\Pi_{d+1}(a, b, u, \xi, w)$.

Notice that this condition can be seen as a polynomial in w of degree $d+1$ and that it is always satisfied when $w = u = 0$. □

In the double confluent Heun equation the change of variable $x \mapsto \frac{1}{x}$ permutes β and γ so that, apart from the trivial case $\alpha = \beta = \gamma = \delta = 0$, there are five cases to consider:

C1: $\alpha \neq 0$
C2: $\alpha = 0, \beta \neq 0$ and $\gamma \neq 0$
C3: $\alpha = \beta = 0, \gamma \neq 0$ and $\delta \neq 0$
C4: $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$
C5: $\alpha = \beta = \delta = 0$ and $\gamma \neq 0$.

Examination of the set L of possible values for n shows that **C5** cannot be liouvillian. On the contrary **C4** is the always liouvillian elementary equation $x^2y'' - \delta y = 0$. In each of the three remaining cases, only one value for n is possible, namely $n = 1$ in **C1** and n = 2 otherwise. A complete answer can be obtained, as shown in the next proposition.

Proposition12. *The double confluent Heun equation has liouvillian solutions if and only if* one *of the following conditions is satisfied:*

1. $\alpha = \beta = \gamma = 0$
2. $\alpha = 0 = \beta\gamma$ *and there exists* $m \in Z$ *such that* $\delta = \dfrac{(3+2m)(1-2m)}{16}$
3. $\alpha \neq 0$, $\pm\beta \pm \gamma \in \alpha\mathbf{Z}^*$ and if $\varepsilon_0, \varepsilon_\infty \in \{\pm 1\}$ *are such that* $d = \dfrac{\varepsilon_0\beta - \varepsilon_\infty\gamma}{\alpha} \in \mathbf{N}$ then

$$\Pi_{d+1}\left(1, 0, \alpha\varepsilon_0, 2\left(1 - \varepsilon_0\,\beta/\alpha\right), -\varepsilon_\infty\alpha, w\right) = 0$$

where $w = \delta + \dfrac{1}{2}\varepsilon_0\varepsilon_\infty\alpha^2 - \varepsilon_0\dfrac{\beta}{\alpha} + \dfrac{\beta^2}{\alpha^2}.$

Proof. In case **C1** the value for a possible degree d appears to be $d = -1 + \dfrac{\pm\beta\pm\gamma}{\alpha}$, giving the necessary condition in the third possibility. If this condition is fulfilled, there is at least one and at most two possible choices ε_0 and $\varepsilon_\infty \in \{\pm 1\}$ such that $\varepsilon_0\beta - \varepsilon_\infty\gamma = \alpha(d+1)$ for $d \in \mathbf{N}$. One has then to look for a polynomial of degree d satisfying the differential equation

$$x^2y'' + \left[\varepsilon_0\alpha + 2\left(1 - \varepsilon_0\,\beta/\alpha\right)x + \varepsilon_\infty\alpha x^2\right]y' + (w - \varepsilon_\infty\alpha d x)y = 0$$

where w has the value given in the proposition. Thus Lemma 4 ends the third case.

In case **C2** one finds -1 for d so that this case is never liouvillian.

In case **C3** the value for d is the one given in the second possibility. It is obtained by using the fact that at least two exponents have to be odd. One deduces $\theta = \dfrac{1-2d}{2x}$ and the third order differential equation equivalent to $(\#)_2$ is

$$x^2y''' + 3x(1-2d)y'' + (4\gamma x - d(1-2d))y' - 4d\gamma y = 0.$$

An easy computation shows that this equation always admits a polynomial solution of the suitable degree.

The first possibility has already been mentioned. □

3.2.3 The Biconfluent Heun Equation. According to the value of $o(\infty)$ there are three cases to consider:

C1: $\alpha \neq \pm 1$
C2: $\alpha = \pm 1$ and $\delta \neq 0$
C3: $\alpha = \pm 1$ and $\delta = 0$.

In all cases the only possible value for n is $n = 1$. The results are summarized in the proposition below.

Proposition 13. *The biconfluent Heun equation has liouvillian solutions if and only if* one *of the following conditions is fulfilled:*

1. $\alpha^2 = 1$, $\delta = 0$ *and* $\gamma \in 1 + 2\mathbf{Z}$
2. $\alpha^2 = 1$, $\delta \neq 0$, $\gamma \in 1 + 2\mathbf{Z}^*$ *with* $|\gamma| \geqq 3$ *and, if* $\varepsilon = \operatorname{sign} \gamma$, *then*

$$\Pi_{(|\gamma|-1)/2}\left(0, 1, 2, \varepsilon\beta, -2\varepsilon, \varepsilon\beta - \frac{\delta}{2}\right) = 0$$

3. $\alpha \neq \pm 1$, $\pm \alpha \pm \gamma \in 2\mathbf{Z}^*$ *and, if* $\varepsilon_0, \varepsilon_\infty \in \{\pm 1\}$ *are such that* $\varepsilon_\infty \gamma - \varepsilon_0 \alpha = 2d^* \in 2\mathbf{N}^*$ *then*

$$\Pi_{d^*}(0, 1, 1 + \alpha\varepsilon_0, \varepsilon_\infty\beta, -2\varepsilon_\infty, \tfrac{1}{2}(\varepsilon_\infty\beta(1 + \varepsilon_0\alpha) - \delta)) = 0.$$

Moreover, in case of liouvillian solutions, at least one of them has the form $\exp \int w$ *where* w *is a rational function.*

Proof. In case **C1** d is given by $d = -1 + \frac{1}{2}(\varepsilon_\infty\gamma - \varepsilon_0\alpha)$, and the differential equation is

$$xy'' + (1 + \varepsilon_0\alpha + \varepsilon_\infty\beta x + 2\varepsilon_\infty x^2)y' + (w - 2\varepsilon_\infty dx)y = 0$$

with $w = \frac{1}{2}(\varepsilon_\infty\beta(1 + \varepsilon_0\alpha) - \delta)$.

According to Lemma 4 this equation has a solution which is a polynomial of degree d if and only if $\Pi_{d+1}(0, 1, 1 + \varepsilon_0\alpha, \varepsilon_\infty\beta, -2\varepsilon_\infty, w) = 0$.

The two other cases can be deduced from this one:

- **C2** by replacing $\varepsilon_0\alpha$ by 1,
- **C3** by replacing $\varepsilon_0\alpha$ by -1 and δ by 0 but this time, as $w = u = 0$ in Lemma 4 there is no more condition. □

Remark. The case **C3** is the particular case of the parabolic cylinder equation (in Rehm's form) corresponding to $a = 1$, $b = \frac{\beta}{2}$ and $c = \frac{\beta^2}{4} - \gamma$.

3.2.4 The Confluent Heun Equation. We note that if $\alpha = \delta = 0$ the confluent Heun equation is the hypergeometric equation with parameters $\lambda = \beta$, $\mu = \gamma$ and $\nu^2 = \beta^2 + \gamma^2 + 1 - 4\eta$. So we will only take care of the cases where $o(\infty) = 4$ (namely $\alpha \neq 0$) or $o(\infty) = 3$ (namely $\alpha = 0$ and $\delta \neq 0$). In this last cases one sees that $L = \varnothing$ unless 0 and 1 are both double poles of r. Taking into account the symmetrical rôles played by 0 and 1 there are seven cases to consider. The following table lists them, indicates the corresponding set L and gives for each case the necessary condition obtained for $n = 1$.

	Conditions	L	Nec. Cond. when $n = 1$
C1	$\alpha \neq 0, \beta^2 \neq 1, \gamma^2 \neq 1$	$\{1,2\}$	$\pm\beta \pm \gamma \pm 2(\delta/\alpha) \in 2\mathbf{Z}^*$
C2	$\alpha \neq 0, \gamma^2 \neq 1, \beta^2 = 1, \eta \neq \frac{1}{2}$	$\{1\}$	$\pm\gamma \pm 2(\delta/\alpha) \in 1 + 2\mathbf{Z} \backslash \{\pm 1\}$
C3	$\alpha \neq 0, \gamma^2 \neq 1, \beta^2 = 1, \eta = \frac{1}{2}$	$\{1\}$	$\pm\gamma \pm 2(\delta/\alpha) \in 1 + 2\mathbf{Z}$
C4	$\alpha \neq 0, \beta^2 = \gamma^2 = 1, \eta \neq \frac{1}{2}, \delta + \eta \neq \frac{1}{2}$	$\{1\}$	$\delta/\alpha \in \mathbf{Z} \backslash \{-1, 0, 1\}$
C5	$\alpha \neq 0, \beta^2 = \gamma^2 = 1, \eta = \frac{1}{2}, \delta \neq 0$	$\{1\}$	$\delta/\alpha \in \mathbf{Z}^*$
C6	$\alpha \neq 0, \beta^2 = \gamma^2 = 1, \eta = \frac{1}{2}, \delta = 0$	$\{1\}$	–
C7	$\alpha = 0, \beta^2 \neq 1, \gamma^2 \neq 1$	$\{2\}$	–

Case **C6** corresponds to the clearly Liouvillian equation $4y'' - \alpha^2 y = 0$.

We only give a complete answer for the value $n = 1$ as shown in the next proposition.

Proposition 14. *A non hypergeometric confluent Heun equation admits a solution of the form* $\exp \int \omega$ *with* $\omega \in \mathbf{C}(x)$ *if and only if*

$$\boxed{\alpha \neq 0, \ \pm\beta \pm \gamma \pm 2\tfrac{\delta}{\alpha} \in 2\mathbf{Z}^*}$$

and one *of the following conditions is satisfied*

1. $\beta^2 = 1$ *and* $\eta = \frac{1}{2}$;
2. $\gamma^2 = 1$ *and* $\delta + \eta = \frac{1}{2}$;
3. $\beta^2 \neq 1, \gamma^2 \neq 1$ *and*

$$\Pi_d(-2, 2, 2(1 - \varepsilon_0\beta), 2(\varepsilon_0(\alpha + \beta) + \varepsilon_1\gamma - 2), -2\varepsilon_\infty\alpha, w) = 0$$

with $w = 1 - 2\eta - (1 - \varepsilon_0\beta)(\varepsilon_\infty\alpha + 1 - \varepsilon_1\gamma)$, *for at least one choice of signs* ε_i *such that* $2d := \varepsilon_0\beta + \varepsilon_1\gamma + 2\varepsilon_\infty\delta/\alpha \in 2\mathbf{N}^*$;

4. $\beta^2 \neq 1, \gamma^2 = 1, \delta + \eta \neq \frac{1}{2}$ *and*

$$\Pi_d(-2, 2, 2(1 - \varepsilon_0\beta), 2(\varepsilon_0\beta - \varepsilon_\infty\alpha - 3), -2\varepsilon_\infty\alpha, w) = 0$$

with $w = 1 - 2\eta - (1 - \varepsilon_0\beta)(\varepsilon_\infty\alpha + 2)$ *for at least one choice of signs* ε_0 *and* ε_∞ *such that* $2d + 1 := \varepsilon_0\beta + 2\varepsilon_\infty\delta/\alpha \in 3 + 2\mathbf{N}$;

5. $\beta^2 = 1, \gamma^2 \neq 1, \eta \neq \frac{1}{2}$ *and*

$$\Pi_d(-2, 2, 4, 2(\varepsilon_1\gamma - \varepsilon_\infty\alpha - 3), -2\varepsilon_\infty\alpha, w) = 0$$

with $w = 1 - 2\eta - 2(\varepsilon_\infty\alpha + 1 - \varepsilon_1\gamma)$ *for at least one choice of signs* ε_1 *and* ε_∞ *such that* $2d + 1 := \varepsilon_1\gamma + 2\varepsilon_\infty\delta/\alpha \in 3 + 2\mathbf{N}$;

6. $\beta^2 = \gamma^2 = 1, \eta \neq \frac{1}{2}, \delta + \eta \neq \frac{1}{2}$ *and*

$$\Pi_d(-2, 2, 4, -2(\varepsilon\alpha + 4), -2\varepsilon\alpha, 1 - 2\eta - 2(\varepsilon\alpha + 2)) = 0$$

for at least one choice of the sign ε *such that* $d + 1 := \varepsilon\delta/\alpha \in 2 + \mathbf{N}$.

Proof. In case **C1**, if $\varepsilon_0\beta + \varepsilon_1\gamma + 2\varepsilon_\infty\dfrac{\delta}{\alpha} = 2(d+1)$ the differential equation to be considered is

$$2x(1-x)y'' + 2(\varepsilon_\infty\alpha x^2 - (\varepsilon_\infty\alpha + 2 - \varepsilon_0\beta - \varepsilon_1\gamma)x + 1 - \varepsilon_0\beta)y' + (w - 2\varepsilon_\infty\alpha d x)y = 0$$

with $w = 1 - 2\eta - (1 - \varepsilon_0\beta)(1 - \varepsilon_1\gamma + \varepsilon_\infty\alpha)$. The result follows once more from Lemma 4. Note that the polynomial solutions of this equation has been studied by Hautot who proves that, if there is one, it is a linear combination of Laguerre polynomials (see[7]).

The other cases can be regarded as particular cases of **C1** in the following way:

C2: replace $\varepsilon_0\beta$ by 1.

C3. replace $\varepsilon_0\beta$ by 1 and η by $\frac{1}{2}$. In this case the differential equation may be simplified and there is no additional condition

C4: replace $\varepsilon_0\beta$ and ε_1 by -1

C5: replace $\varepsilon_0\beta$ by 1, $\varepsilon_1\gamma$ by -1 and η by $\frac{1}{2}$. As in case **C3** there is no new condition. □

Concerning the case $n=2$ we only mention the two arithmetic necessary conditions

- in case **C1:** β and $\gamma \in \frac{1}{2} + \mathbf{Z}$
- in case **C7:** β or $\gamma \in \frac{1}{2} + \mathbf{Z}$.

3.2.5 The Heun Equation. A complete study is certainly very difficult as, this time, finite galois groups are possible. We just mention that the value $n=1$ leads to the necessary condition

$$\boxed{\pm\alpha \pm \beta \pm \gamma \pm (\delta - \eta) \in 1 + 2\mathbf{Z}}$$

and the differential equation to look for has the form

$$x(x-1)(x-c)y'' + (ax^2 + bx + c)y' + (ex + f)y = 0.$$

The polynomial solutions of this type of equations are also studied by Hautot in [8].

3.3 The Equation $y'' - (\alpha x^p + bx^q)y = 0$

This equation is studied by Setoyanagi in [19].

We assume that $p, q \in N$ with $p > q$ and that $a, b \in \mathbf{C}^*$.

It is convenient to replace p and q by the parameters s and σ defined by

$$\begin{cases} p - q = \sigma \\ p = 2(s\sigma - 1) \end{cases}$$

and submitted to the conditions $\sigma \in \mathbf{N}^*$ and $2(s\sigma - 1) \in \mathbf{N}^*$.

The equation has one singular point located at ∞, with order $o(\infty) = p + 4$. The set L is $\{1\}$ if p is even and $\varnothing$ if p is odd. Thus we obtain a first necessary condition

$$\boxed{s\sigma - 1 \in \mathbf{N}^*}$$

which, from now on, is supposed satisfied.

Expanding

$$r(x)^{1/2} = a^{1/2} x^{s\sigma - 1}\left(1 + \frac{b}{a}\frac{1}{x^\sigma}\right)^{1/2}$$

in powers of $\frac{1}{x}$ one finds

$$[\sqrt{r}]_\infty = a^{1/2} x^{s\sigma-1} \sum_{j=0}^{s'-1} \binom{1/2}{j} \left(\frac{b}{a}\right)^j \frac{1}{x^{j\sigma}}$$

with $s'-1 = \left\lfloor \frac{s\sigma - 1}{\sigma} \right\rfloor$ where $\lfloor \cdot \rfloor$ denotes the lower integer part.

As $r - [\sqrt{r}]_\infty^2 = 2a\binom{\frac{1}{2}}{s'}\left(\frac{b}{a}\right)^{s'} x^{2(s\sigma-1)-s'\sigma}(1+o(1))$ one gets with the previous notations of the algorithm

$$a_\infty = a^{1/2}$$

$$\beta_\infty = \begin{cases} 0 & \text{if } s \notin \mathbf{N}^* \\ -2\binom{\frac{1}{2}}{s}\left(\frac{b}{a}\right)^{s-1} b & \text{if } s \in \mathbf{N}^* \end{cases}$$

$$d = \begin{cases} -\frac{1}{2}(s\sigma - 1) & \text{if } s \notin \mathbf{N}^* \\ -\frac{1}{2}(s\sigma - 1) \pm a^{1/2}\binom{\frac{1}{2}}{s}\left(\frac{b}{a}\right)^{s} & \text{if } s \in \mathbf{N}^* \end{cases}$$

The necessary condition $d \in \mathbf{N}$ implies $s \in \mathbf{N}^*$ and thus $s' = s$. If we choose the square root $a^{1/2}$ which induces a $+$ in the formula, we may summarize the various hypothesis and necessary conditions in

$$\boxed{\begin{array}{c} \sigma \in \mathbf{N}^*, \quad s \in \mathbf{N}^*, \quad s\sigma \neq 1 \\ d := -\frac{1}{2}(s\sigma - 1) + a^{1/2}\binom{\frac{1}{2}}{s}\left(\frac{b}{a}\right)^{s} \in \mathbf{N} \end{array}}$$

With these conditions one has to look for a polynomial P of degree d satisfying the differential equation

$$P'' + 2\theta P' + (\theta' + \theta^2 - a x^{2s\sigma-2} - b x^{\sigma(2s-1)-2})P = 0 \tag{20}$$

where

$$\theta = a^{1/2}\left(\frac{b}{a}\right)^s \sum_{j=1}^{s} \binom{\frac{1}{2}}{s-j}\left(\frac{a}{b}\right)^j x^{j\sigma-1}.$$

Substituting the polynomial $P(x) = \sum_{k=0}^{d} \lambda_k x^k$ in the equation gives

$$(k + s\sigma)(k + s\sigma - 1)\lambda_{k+s\sigma} + \sum_{j=1}^{s-1} A_j(k)\lambda_{k+j\sigma} - 2a^{1/2}(d-k)\lambda_k = 0$$

with $A_j(k) = (2k + (s+j)\sigma - 1)L_j + (1 - \delta_j^{s-1})a\left(\frac{b}{a}\right)^{s+j} M_{j,s}$

where $M_{j,s} = \sum_{i=1}^{s-j-1} \binom{\frac{1}{2}}{s-i}\binom{\frac{1}{2}}{i+j}$, $\left(\text{when } s=1, \sum_1^0 = 0\right)$ and $L_j = a^{1/2}\binom{\frac{1}{2}}{j}\left(\frac{b}{a}\right)^j$.

These relations have to be valid for all $k \in \mathbf{Z}$ with the extra conditions $\lambda_d \neq 0$, say $\lambda_d = 1$, and $\lambda_k = 0$ for $k < 0$ or $k \geqq d+1$. The coefficient of λ_k is zero if and only if $k = d$. On the other hand the coefficient of $\lambda_{k+s\sigma}$ vanishes if and only if $k + s\sigma = 0$ or 1. Denoting by $d = \delta\sigma + t$, $0 \leqq t < \sigma$ the euclidian division of d by σ, we get the new necessary condition

$$\boxed{t = 0 \quad \text{or} \quad 1}$$

By examining the induction formulas above, one can remark that a solution P, if any, takes the form $P(x) = x^t Q(x^\sigma)$, for some polynomial Q of degree δ. The differential equation for Q is, if $u = x^\sigma$,

$$\sigma u Q''(u) + B(u)Q'(u) + C(u)Q(u) = 0 \tag{21}$$

with

$$B(u) = \sigma - 1 + 2t + 2\sum_{j=0}^{s-1} L_j u^{s-j}$$

$$C(u) = \sum_{j=1}^{s-1}\left[\left(s - j + \frac{2t-1}{\sigma}\right)L_j + \frac{1}{\sigma}M_{j,s}a\left(\frac{b}{a}\right)^{s+j}\right]u^{s-j-1} - 2a^{1/2}\delta u^{s-1}.$$

We have been able to discuss completely the problem of existence of a polynomial solution with proper degree in the two cases $s = 1$ and $s = 2$. Paradoxally we have not been able yet to solve the cases $s \geqq 3$, where the problem is however overdetermined by $s - 1$ equations, except for small values of d where, as expected, no liouvillian solution is found.

Case $s = 1$. Going back to the initial parameters, it means $p = 2p'$ and $q = p' - 1$, $p' \in \mathbf{N}^*$. The differential Eq. (21) is the confluent hypergeometric equation

$$uQ''(u) + \left(1 + \frac{2t-1}{p'+1} + \frac{2a^{1/2}}{p'+1}u\right)Q'(u) - \frac{2a^{1/2}}{p'+1}\delta Q(u) = 0$$

which admits the Laguerre polynomial $L_\delta^{(2t-1)/(p'+1)}\left(-\frac{2a^{1/2}}{p'+1}u\right)$ as a unique (up to constant multiple) polynomial solution.

Thus in this case, the algorithm gives the liouvillian solution

$$\begin{aligned} y(x) &= x^t L_\delta^{(2t-1)/(p'+1)}\left(-\frac{2a^{1/2}}{p'+1}u\right)\exp\left[\frac{a^{1/2}}{p'+1}x^{p'+1}\right] \\ &= x^t\, {}_1F_1\left(\begin{matrix} -\delta \\ \frac{2a^{1/2}}{p'+1}(p'+2t) \end{matrix} \,\middle|\, -\frac{2a^{1/2}}{p'+1}x^{p'+1}\right). \end{aligned}$$

Case $s = 2$. The following lemma gives the answer.

Lemma 5. *Let $\sigma, \delta, d \in \mathbf{N}^*$ such that $d = \sigma\delta + t$ with $t < \sigma$ and $t \in \{0, 1\}$. In order that there exists a sequence $(\mu_k)_{k\in\mathbf{Z}}$ satisfying the induction formulas (E_k):*

$$2k\sigma\mu_k = (2d - (2k-3)\sigma - 1)\mu_{k-1} - \frac{(d-(k-2)\sigma)(d-(k-2)\sigma-1)}{2d+2\sigma-1}\mu_{k-2}$$

with $\mu_0 = 1$ *and* $\mu_k = 0$ *for* $k < 0$ *or* $k \geqq \delta + 1$, *it is necessary and sufficient that* $\sigma = 1$.

Proof. As the coefficients are rational numbers, if a solution exists, the μ_k are rational numbers. Let us assume first that $\sigma > 1$. One easily proves by ascending induction that for $1 \leqq k \leqq \delta$

$$\mu_k > \frac{d-(k-1)\sigma}{2d+2\sigma-1}\mu_{k-1} > 0.$$

Hence for $k = \delta$ we get

$$\mu_\delta > \frac{\sigma+t}{2d+2\sigma-1}\mu_{\delta-1} > 0.$$

One deduces from equation $(E_{\delta+1})$

$$\mu_{\delta+1} = 0 = (2d+2\sigma-1)(\sigma+2t-1)\mu_\delta - (\sigma+t)(\sigma+t-1)\mu_{\delta-1}$$

and therefore

$$\mu_\delta \leqq \frac{\sigma+t}{2d+2\sigma-1}\mu_{\delta-1}$$

which contradicts the previous inequality.

If now $\sigma = 1$ and thus $d = \delta$ the relations (E_{d+1}) and (E_{d+2}) are

$$2(d+2)\mu_{d+2} = -2\mu_{d+1}$$

$$\mu_{d+1} = 0$$

and the μ_k, $0 \leqq k \leqq d$ have to satisfy a linear homogeneous system of d equations which obviously has a solution. This ends the proof of the lemma. □

Now suppose $P(x) = \sum_{k=0}^{d} \lambda_k x^k$ is a solution of Eq. (20) with $s = 2$; then if we set $\mu_k = \left(\frac{2a}{b}\right)^k \lambda_{d-k\sigma}$, the μ_k have to satisfy the recurrence relations and the limit conditions of Lemma 5. Hence the corresponding equation has liouvillian solutions if and only if $\sigma = 1$. In that case $p = 2$ and $q = 1$ so the equation is a parabolic cylinder one (in the form studied by Rehm).

The results above are summarized in the

Theorem 4. *Let* $a, b \in \mathbf{C}^*$ *and* $p, q \in \mathbf{N}$ *such that* $p > q$.

- *The differential equation*

$$y'' - (ax^p + bx^q)y = 0$$

has no liouvillian solution unless there exist

1. *two positive integers* $s, \sigma \in \mathbf{N}^*$ *such that* $\begin{cases} p = 2\sigma s - 2 \\ q = p - \sigma \end{cases}$
2. *a non negative integer* $d \cong 0$ *or* 1 *modulo* σ *such that*

$$\boxed{4\binom{\frac{1}{2}}{s}^2 b^{2s} = a^{2s-1}(2d+\sigma s-1)^2}$$

- *The differential equation*

$$y'' - (ax^{2q+2} + bx^q)y = 0$$

has liouvillian solutions if and only if there exists a non negative integer $d \cong 0$ *or* 1 *modulo* $q+2$ *such that*

$$\boxed{b^2 = a(2d + q + 1)^2}$$

- *The differential equation*

$$y'' - (ax^{4q+2} + bx^{3q+1})y = 0$$

has liouvillian solutions if and only if

$$\boxed{q = 0 \quad \textit{and} \quad \frac{b^2}{4a^{3/2}} \quad \textit{is an odd integer}} \; .$$

Remarks.

1. In Setoyanagi's paper [19] the same results are obtained as long as they concern cases $s = 1$ and $s = 2$. In particular an analogue of Lemma 5 is proved.

2. The Setoyanagi equation with $p = 4$, $q \leqq 2$ and $a = \frac{9}{4}$ is a triconfluent Heun equation with

$$\begin{aligned}
&\text{if } q = 2, \quad \gamma = \tfrac{2}{3}b, \quad \alpha = \tfrac{1}{4}\gamma^2 \quad \text{and} \quad \beta = 0 \\
&\text{if } q = 1, \quad \alpha = \gamma = 0 \quad \text{and} \quad \beta = -b \\
&\text{if } q = 0, \quad \beta = \gamma = 0 \quad \text{and} \quad \alpha = -b.
\end{aligned}$$

According to the previous results liouvillian solutions are possible only in the second case, namely for $q = 1$. Comparing the conditions obtained in this section and in Sect. 3.2.1 one can prove the relation, with the notations of Sect. 3.2.1,

$$Q_{d+1}(0,0) \neq 0 \quad \Leftrightarrow \quad d = 3\delta + 2$$

(a direct proof is easy).

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1.7 Note on Kovacic's Algorithm

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Note on Kovacic's Algorithm[†]

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Algorithms exist to find Liouvillian solutions of second order homogeneous linear differential equations (Kovacic, 1986, Singer and Ulmer, 1993b). In this paper, we show how, by carefully combining the techniques of those algorithms, one can find the Liouvillian solutions of an irreducible second order linear differential equation by computing only rational solutions of some associated linear differential equations. The result is an easy-to-implement simplified version of the Kovacic algorithm, based as much as possible on the computation of rational solutions of linear differential equations.

1. Differential Galois Theory

The material presented in this section is well known and has been included to make the exposition self contained. We refer to Kaplansky (1976), Kolchin (1948), Singer (1990) for further details about this section.

1.1. Introduction

A *differential field* (k, δ) is a field k together with a derivation δ on k. We also write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \ldots$ for $\delta(y), \delta^2(y), \ldots$. The field of constants $\{c \in k \mid c' = 0\}$ is denoted $\mathcal{C}$. Unless otherwise stated, a differential equation $L(y) = 0$ over k always means an ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k).$$

In the following we will look at solutions of $L(y) = 0$ in a differential field extension of k. A *differential field extension* of (k, δ) is a differential field (K, Δ) such that K is a field extension of k and Δ is an extension of the derivation δ of k to a derivation on K. The differential Galois group $\mathcal{G}(K/k)$ of a differential field extension K of k is the set of k-automorphisms of K which commute with the derivation of K. There is a unique way to extend the derivation of k to an algebraic extension of k making any algebraic extension of k into a differential extension.

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Definition 1.1. *A differential field extension* (K, Δ) *of* (k, δ) *is called a* Liouvillian extension *if there is a tower of fields*

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K,$$

where K_{i+1} *is a simple field extension* $K_i(\eta_i)$ *of* K_i*, such that one of the following holds:*

(i) η_i *is algebraic over* K_i*, or*
(ii) $\delta(\eta_i) \in K_i$ *(extension by an integral), or*
(iii) $\delta(\eta_i)/\eta_i \in K_i$ *(extension by the exponential of an integral).*

A solution of $L(y) = 0$ which is contained in:

(i) k, the coefficient field, will be called a *rational* solution,
(ii) an algebraic extension of k will be called an *algebraic* solution,
(iii) a Liouvillian extension of k will be called a *Liouvillian* solution

A solution z of $L(y) = 0$ is called *exponential*† if z'/z is in the coefficient field k. In the following we will have to compute rational and exponential solutions of $L(y) = 0$. For this reason we always assume that k is a differential field over which such solutions can be computed (e.g. $(\mathbb{C}(x), \frac{d}{dx})$). The computation of an exponential solution is usually much more difficult than the computation of a rational solution.

For $k = \mathbb{C}(x)$ and a differential equation $L(y) = 0$ with coefficients in k, an algorithm to compute

(i) rational solutions is given in Liouville (1833). More recent algorithms for more general coefficient fields are presented in Bronstein (1992), Singer (1991);
(ii) algebraic solutions of a second order equation $L(y) = 0$ is given in Fuchs (1878) and in Pépin (1881). The study of the third order case is started in Jordan (1878), a general algorithm was given by Boulanger and Singer, cf. Singer (1979);
(iii) Liouvillian solution of a second order equation is given in Kovacic (1986). A general procedure for equations of arbitrary order is presented in Singer (1981). The third order case is treated in Singer and Ulmer (1993b).

Definition 1.2. *Let* $L(y) = 0$ *be a homogeneous linear differential equation of order* n *with coefficients in* k*. A differential field extension* K *of* k *is called a* Picard–Vessiot extension (PVE) *of* k *for* $L(y) = 0$ *if*

(i) $K = k\langle y_1, \ldots, y_n \rangle$*, the* differential *field extension of* k *generated by* $y_1, \ldots, y_n$ *where* $\{y_1, \ldots, y_n\}$ *is a fundamental set of solutions of* $L(y) = 0$*.*
(ii) K *and* k *have the same field of constants.*

A PVE extension plays the role of a splitting field for $L(y) = 0$. A PVE exists and is unique up to differential isomorphisms if the field of constants of k is algebraically closed of characteristic 0 (Kaplansky, 1976, p. 21 and Kolchin, 1948). In the sequel we will always assume that the coefficient field is algebraically closed of characteristic 0. By definition the differential Galois group $\mathcal{G}(L)$ of $L(y) = 0$ is the differential Galois group of K/k, where

† Note that the exponential solutions of $L(y) = 0$ do not form a ring.

K is a PVE of k for $L(y) = 0$. If we choose a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of the equation $L(y) = 0$, then for each $\sigma \in \mathcal{G}(L)$ we get $\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j$, where $c_{ij} \in \mathcal{C}$. This gives a faithful representation of $\mathcal{G}(L)$ as a subgroup of $\mathrm{GL}(n, \mathcal{C})$. Different choices of bases $\{y_1, y_2, \ldots, y_n\}$ give equivalent representations. In the sequel we always consider this equivalence class of representations as *the* representation (module) of $\mathcal{G}(L)$. In fact, $\mathcal{G}(L)$ is a linear algebraic subgroup of $\mathrm{GL}(n, \mathcal{C})$ (Kolchin, 1948; Kovacic, 1986). We can limit our considerations to differential equations with $\mathcal{G}(L) \subseteq \mathrm{SL}(n, \mathcal{C})$:

Theorem 1.1. (Kaplansky, p. 41) *The differential Galois group of a differential equation of the form*

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k) \tag{1.1}$$

is a unimodular group (i.e. $\mathcal{G}(L) \subseteq \mathrm{SL}(n, \mathcal{C})$) if and only if $\exists W \in k$, such that $W'/W = a_{n-1}$.

In particular for a differential equation of the form

$$L(y) = y^{(n)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0 \tag{1.2}$$

we have $\mathcal{G}(L) \subseteq \mathrm{SL}(n, \mathcal{C})$. Using the variable transformation $y = z \cdot e^{\left(-\frac{\int a_{n-1}}{n}\right)}$ it is always possible to transform a given differential equation $L(y)$ into an equation $\widetilde{L}(y)$ of the form (1.2) without altering the Liouvillian character of the solutions. This transformation is always performed in Kovacic (1986). The algorithm presented in this paper works independently of this particular form and avoids unnecessary transformations.

1.2. properties of the differential Galois group

Properties of the equation $L(y) = 0$ are reflected by properties of the group $\mathcal{G}(L)$. To the equation (1) we associate a linear differential operator:

$$p(\delta) = a_n \delta^n + a_{n-1} \delta^{(n-1)} + \cdots + a_0.$$

The set of differential operators forms a ring $k[\delta]$ where multiplication is defined by $\delta a = a\delta + \delta(a)$. The ring $k[\delta]$ is a right and left euclidian ring in which a right (resp. left) least common multiple of differential operators can be computed (Ore, 1933). The factorization of differential operators in $k[\delta]$ is not unique but, as shown in Kolchin (1948), Singer (1990), or Singer (1996), we have:

Theorem 1.2. *The linear differential equation $L(y)$*

(i) *factors as a linear differential operator, if and only if $\mathcal{G}(L) \subseteq \mathrm{GL}(n, \mathcal{C})$ is a reducible linear group.*
(ii) *is the least common left multiple of irreducible operators if and only if $\mathcal{G}(L) \subseteq \mathrm{GL}(n, \mathcal{C})$ is a completely reducible linear group.*

Another property of $L(y) = 0$ that can be characterized by a property of $\mathcal{G}(L)$ is the solvability in terms of Liouvillian solutions. Note that if a second order equation has a Liouvillian solution, then another Liouvillian solution can be found using the d'Alembert reduction method. Thus a second order equation has either no Liouvillian solutions or only Liouvillian solutions.

Theorem 1.3. (Kolchin, 1948) *A differential equation $L(y) = 0$ with coefficients in k has only Liouvillian solutions over k if and only if the component of the identity $\mathcal{G}(L)^\circ$ of $\mathcal{G}(L)$ in the Zariski topology is solvable. In this case $L(y) = 0$ has a solution whose logarithmic derivative is algebraic over k.*

If $\mathcal{G}(L)^\circ$ is solvable, then it can be put simultaneously in triangular form (Lie–Kolchin Theorem, Kolchin, 1948) and thus has a common eigenvector z. In particular z'/z is in the fixed field of $\mathcal{G}(L)^\circ$ and thus, using the Galois correspondence, algebraic over k of degree at most $[\mathcal{G}(L) : \mathcal{G}(L)^\circ] < \infty$. In Singer (1981), it is shown that the algebraic degree of the logarithmic derivative z_1'/z_1 of *a particular* solution z_1 can be bounded independently of the equation $L(y) = 0$ (Singer, 1981; Ulmer, 1992). To compute the coefficients of the minimal polynomial of $u_1 = z_1'/z_1$ one notes that all conjugates u_i of u_1 under $\mathcal{G}(L)$ are also logarithmic derivatives of solutions z_i, the minimal polynomial $P(u)$ of u_1 can be written as

$$P(u) = \prod_{i=1}^{m}\left(u - \frac{\delta(z_i)}{z_i}\right) \tag{1.3}$$

$$= u^m - \frac{\delta\left(\prod_{i=1}^m z_i\right)}{\prod_{i=1}^m z_i}u^{m-1} + \cdots + (-1)^m \prod_{i=1}^{m}\frac{\delta(z_i)}{z_i}. \tag{1.4}$$

In particular, the coefficient of u^{m-1} is the negative logarithmic derivative of a product of m solutions of $L(y) = 0$. It is possible (Singer, 1979) to construct a differential equation whose solutions are the products of length m of solutions of $L(y) = 0$:

Definition 1.3. *Let $L(y) = 0$ be a homogeneous linear differential equation of order n and let $\{y_1, \ldots, y_n\}$ be a fundamental system of solutions. The differential equation $L^{\circledS m}(y)$ whose solution space is generated by the monomials of degree m in $y_1, \cdots, y_n$ is called the* mth symmetric power† *of $L(y) = 0$.*

To construct the equation $L^{\circledS m}(y)$ one starts with $Y = \prod_{i=1}^m z_i$, where z_i are arbitrary solutions of $L(y) = 0$. Taking derivatives of Y and replacing derivatives of order $\geq m$ of the z_i on the right-hand side by lower order derivatives using $L(y) = 0$ gives a linear differential equation for Y of order at most $\binom{n+m-1}{n-1}$ (Singer and Ulmer, 1993a). The group $\mathcal{G}(L)$ operates on the solutions space of $L^{\circledS m}(y)$ in a natural way which gives another representation of $\mathcal{G}(L)$.

From (1.4) we get that the coefficient of u^{m-1} in the minimal polynomial $P(u)$ is the negative logarithmic derivative of an exponential solution of $L^{\circledS m}(y)$.

Example. Let $L(y) = y'' + \dfrac{3}{16x^2}y$ and $k = \mathbb{C}(x)$. This equation has a solution whose logarithmic derivative is a solution of

$$P(u) = u^2 - \frac{1}{x}u + \frac{3}{16x^2}.$$

† One of the referees proposed the following equivalent definition. The differential equation corresponds to a differential module M together with a cyclic element e such that $Le = 0$ and L has minimal order with respect to this property. Let $\mathcal{S}^m(M)$ denote the mth symmetric power of M (Lang, 1984, p. 586). The minimal equation $L^{\circledS m}(y)$ of the element $e \otimes \cdots \otimes e \in \mathcal{S}^m(M)$ is called the *mth symmetric power* of L. Note that $e \otimes \cdots \otimes e$ is not always cyclic.

The coefficient of u is the negative logarithmic derivative of the solution $y = x$ of

$$L^{\circledS 2}(y) = y''' + \frac{3}{4x^2}y' - \frac{3}{4x^3}y = 0.$$

In this case the exponential solution is even rational. ⋄

In general the order of $L^{\circledS m}(y)$ can be less than $\binom{n+m-1}{n-1}$. For second order equations, the order is always $m+1$ (Singer and Ulmer, 1993a, Lemma 3.5) and the solution space of $L^{\circledS m}(y)$ is isomorphic to the mth symmetric power $\mathcal{S}^m(V)$ (Lang, 1984, p. 586) of the solution space V of $L(y) = 0$. In particular the character χ_m of the representation of $\mathcal{G}(L)$ on the solution space of $L^{\circledS m}(y)$ is the symmetrization of the character χ of the representation of $\mathcal{G}(L)$ on the solution space of $L(y) = 0$. For finite groups one can compute χ_m from χ (Singer and Ulmer, 1993a).

Definition 1.4. (see, e.g., Sturmfels, 1993) *Let V be a $\mathcal{C}$-vector space, call $\{y_1, \ldots, y_n\}$ a basis for V, and let $G \subseteq \mathrm{GL}(V)$ be a linear group. Define an action of $g \in G$ on $\mathcal{C}[y_1, \ldots, y_n]$ by $g \cdot (p(y_1, \ldots, y_n)) = p\,(g(y_1), \ldots, g(y_n))$. A polynomial with the property that*

$$\forall g \in G, \quad g\,(p(y_1, \ldots, y_n)) = \psi_p(g) \cdot (p(y_1, \ldots, y_n))\,, \ \textit{with } \psi_p(g) \in \mathcal{C}$$

is called a semi-invariant *of G. If $\forall g \in G$ we have $\psi_p(g) = 1$, then $p(y_1, \ldots, y_n)$ is called an* invariant *of G.*

Clearly, ψ_p must be a character of degree one. In the above definition the $y_1, \ldots y_n$ are independent variables. If we evaluate a polynomial $p(y_1, \ldots, y_n)$ by replacing the variables by the elements of a fundamental set of solutions of $L(y) = 0$, we get a function of the PVE associated to $L(y) = 0$. By differential Galois theory, since an invariant I of degree m of $\mathcal{G}(L)$ is left fixed by $\mathcal{G}(L)$, it must evaluate to a rational solution of $L^{\circledS m}(y) = 0$. In this paper we will identify the invariants with this rational solution and by *computing an invariant* we always mean *computing the corresponding rational solution*. Similarly a semi-invariant of degree m evaluates to an exponential solution of $L^{\circledS m}(y) = 0$ and thus, if it is not 0, to a right factor of order one of $L^{\circledS m}(y)$.

If $L(y) = 0$ is a second order equation, then any semi-invariant S of degree m of $\mathcal{G}(L)$ is a non-trivial exponential solution of $L^{\circledS m}(y) = 0$. To this semi-invariant corresponds a character of degree 1 in the decomposition of χ_m (the character of the representation of $\mathcal{G}(L)$ on the solution space of $L^{\circledS m}(y)$). For finite groups, the existence of a non-trivial semi-invariant of degree m can be deduced from the existence of a character of degree 1 in the decomposition of χ_m into irreducible characters.

Using this terminology, we see from (1.4) that the coefficient of u^{m-1} in $P(u)$ is a semi-invariant of degree m of $\mathcal{G}(L)$. In Section 2, we will show that to any semi-invariant of $\mathcal{G}(L)$ corresponds a unique polynomial $P(u)$ whose irreducible *factors* are all minimal polynomials of logarithmic derivatives of some solutions of $L(y) = 0$.

Example. Let $L(y) = y'' + \frac{3}{16x^2}y$ and $k = \mathbb{C}(x)$. we choose the two exponential solutions

$$y_1 = e^{\int \frac{1}{4x}} = x^{\frac{1}{4}}, \qquad y_2 = e^{\int \frac{3}{4x}} = x^{\frac{3}{4}}$$

as a basis of the solution space of $L(y) = 0$. A PVE of k for $L(y) = 0$ is the algebraic extension $\mathbb{C}(x)(x^{\frac{1}{4}})$ and $\mathcal{G}(L)$ is cyclic of order 4. The group $\mathcal{G}(L)$ is an abelian group and has four characters of degree one: the trivial character $\mathbf{1}$, a character $\psi_{1,1}$ of order 2 (i.e. $(\psi_{1,1})^2 = \mathbf{1}$) and two characters $\psi_{1,2}$ and $\psi_{1,3}$ of order 4. In the basis $\{y_1, y_2\}$, the group $\mathcal{G}(L)$ is generated by:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

From the above form we get that $\chi = \psi_{1,2} + \psi_{1,3}$ and thus that $\mathcal{G}(L)$ has two linearly independent semi-invariants $S_{1,1} = x^{\frac{1}{4}}$ and $S_{1,2} = x^{\frac{3}{4}}$ of degree one corresponding to the characters $\psi_{1,2}$ and $\psi_{1,3}$. To the logarithmic derivatives of $S_{1,1}$ and $S_{1,2}$ correspond two minimal polynomials $\left(u - \frac{1}{4x}\right)$ and $\left(u - \frac{3}{4x}\right)$ of logarithmic derivatives $u_i = y_i'/y_i$ of solutions of $L(y) = 0$.
A basis of the solution space of $L^{\circledS 2}(y) = 0$ (cf. previous Example) is given by:

$$(y_1)^2 = x^{\frac{1}{2}}, \qquad y_1 y_2 = x, \qquad (y_2)^2 = x \cdot x^{\frac{1}{2}}.$$

In the basis $\{(y_1)^2, y_1 y_2, (y_2)^2\}$, the group $\mathcal{G}\left(L^{\circledS 2}\right)$ is generated by:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From the above form we get that $\chi_2 = \mathbf{1} + 2\psi_{1,1}$ and thus that $\mathcal{G}(L)$ has an invariant $I_2 = y_1 y_2 = x$ of degree 2 and two linearly independent semi-invariants $S_{2,1} = y_1^2 = x^{\frac{1}{2}}$ and $S_{2,2} = y_2^2 = x \cdot x^{\frac{1}{2}}$ of degree 2 corresponding both to the character $\psi_{1,1}$. To the logarithmic derivative $\frac{1}{x}$ of I_2 corresponds the polynomial $\left(u^2 - \frac{1}{x}u + \frac{3}{16x^2}\right)$. This polynomial is not irreducible, but is the product of the above polynomials of degree one corresponding to $\psi_{1,2}$ and $\psi_{1,3}$. We will show in this paper that this factorization corresponds to the factorization $I_2 = S_{1,1} \cdot S_{1,2}$. ◇

Since exponential solutions (semi-invariants) are usually more difficult to compute than rational solutions (invariants), we want to compute whenever possible the minimal polynomials corresponding to rational solutions (invariants) and, if necessary, factor the corresponding polynomial $P(u)$. In particular we will show that for *irreducible* second order equations this will always be possible.

1.3. SECOND ORDER EQUATION

Let $L(y) = y'' + a_1 y' + a_0 y$ be a second order equation with coefficients in k and unimodular Galois group $\mathcal{G}(L) \subset \mathrm{SL}(2, \mathcal{C})$. The logarithmic derivatives of the solutions are precisely the solutions of the associated Riccati equation $\mathrm{Ri}(u) := u' + a_0 + a_1 u + u^2 = 0$. The possible groups $\mathcal{G}(L)$ are the linear algebraic subgroups of $\mathrm{SL}(2, \mathcal{C})$ which can be classified, up to conjugacy, as follows (Kovacic, 1986):

(i) The reducible but non-reductive groups, where a non-trivial $\mathcal{G}(L)$-invariant subspace has no complementary $\mathcal{G}(L)$-invariant subspace.
(ii) The diagonal linear algebraic subgroups of $\mathrm{SL}(2, \mathcal{C})$.
(iii) The imprimitive subgroups of $\mathrm{SL}(2, \mathcal{C})$ which are up to conjugacy:

 (a) The finite groups $D_n^{\mathrm{SL}_2}$ of order $4n$ (central extensions of the dihedral groups D_n) and generated by:

$$\begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(b) The infinite group:

$$D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \right\} \qquad \text{where } a \in \mathcal{C}^*.$$

(iv) The primitive finite subgroups of $\mathrm{SL}(2, \mathcal{C})$ which are isomorphic to either the tetrahedral, the octahedral or the icosahedral group; we denote them respectively $A_4^{\mathrm{SL}_2}$, $S_4^{\mathrm{SL}_2}$ and $A_5^{\mathrm{SL}_2}$. A definition for these groups is given in Kovacic (1986) or Singer and Ulmer (1993b).

(v) The group $\mathrm{SL}(2, \mathcal{C})$.

In order to bound the degree of an algebraic solution of $\mathrm{Ri}(u) = 0$, we compute a maximal subgroup H_z having a common eigenvector z, i.e. a reducible subgroup of $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$. The group H_z is the stabilizer of z'/z and thus, if the index $[\mathcal{G}(L) : H_z]$ is finite, the minimal polynomial of z'/z will be of degree $[\mathcal{G}(L) : H_z]$.

Lemma 1.5. *Let H be a finite reducible subgroup of* $\mathrm{SL}(2, \mathcal{C})$ *which is not contained in the center* $Z(\mathrm{SL}(2, \mathcal{C}))$ *of* $\mathrm{SL}(2, \mathcal{C})$. *Then H is cyclic and there exists up to multiples a unique basis in which H is a diagonal subgroup of* $\mathrm{SL}(2, \mathcal{C})$.

Proof. Since H is finite, Maschke's theorem shows that any invariant subspace has a complementary invariant subspace. Thus, we can put the elements of H simultaneously in diagonal form. Since $H \subset \mathrm{SL}(2, \mathcal{C})$ the diagonal entries will be given by characters χ and χ^{-1}. Therefore the map $h \in H \mapsto \chi(h)$ is an isomorphism of H onto a finite (and therefore cyclic) subgroup of $\mathcal{C}$. The result now follows from the linear independence of characters (Lang, 1984). □

Lemma 1.6. *Let $L(y) = 0$ be an irreducible second order equation over k whose differential Galois group $\mathcal{G}(L)$ is a finite unimodular group. Let $Z(\mathcal{G}(L))$ be the center of $\mathcal{G}(L)$. Then, the number of irreducible minimal polynomials of degree $m < [\mathcal{G}(L) : Z(\mathcal{G}(L))]$ of algebraic solutions of the Riccati equation* $\mathrm{Ri}(u) = 0$ *is equal to $2/m$ times the number of maximal cyclic subgroups (i.e. not contained in a larger cyclic subgroup) of index m of $\mathcal{G}(L)$. In particular, this number is always finite. All other solutions of the Riccati equation are algebraic of degree* $[\mathcal{G}(L) : Z(\mathcal{G}(L))]$.

Proof. Let w be an algebraic solution of $\mathrm{Ri}(u)$. The degree m of the minimum polynomial of w equals the index $[\mathcal{G}(L) : H_1]$ of the stabilizer $H_1 = \mathrm{Stab}_{\mathcal{G}(L)}(w)$ of w in $\mathcal{G}(L)$. Note that $\mathrm{Stab}_{\mathcal{G}(L)}(w)$ always contains $Z(\mathcal{G}(L))$. If $m < [\mathcal{G}(L) : Z(\mathcal{G}(L))]$ then, by the above Lemma, H_1 is a non-central cyclic group having up to multiples a unique basis $\{y_1, y_2\}$ in which it is a diagonal group.

Denote z_1 the solution of $L(y) = 0$ such that $z_1'/z_1 = w$. Then z_1 spans an H_1-invariant subspace, which by Maschke's Theorem has a complementary subspace spanned by some solution z_2. Since H_1 is also diagonal in the basis $\{z_1, z_2\}$, z_1 must be a multiple of y_1 or y_2, say y_1. The cyclic group H_1 cannot be contained in a larger cyclic subgroup of $\mathcal{G}(L)$: from Lemma 1.5 such a group would also be diagonal in the (up to multiples unique) basis $\{y_1, y_2\}$ and thus would be contained in H_1, the stabilizer of $w = y_1'/y_1$. In particular H_1 is also the stabilizer of y_2'/y_2 which must be algebraic of the same degree as y_1'/y_1.

It follows that the stabilizer of any algebraic solution of degree $m < [\mathcal{G}(L) : Z(\mathcal{G}(L))]$ of $R_i(u) = 0$ is a maximal cyclic subgroup, and each maximal cyclic subgroup of index $m < [\mathcal{G}(L) : Z(\mathcal{G}(L))]$ is the stabilizer of exactly two algebraic solutions of degree m of $R_i(u) = 0$. If there are N maximal cyclic subgroups of index $m < [\mathcal{G}(L) : Z(\mathcal{G}(L))]$, there are exactly $2N$ solutions of $R_i(u) = 0$ which are algebraic of degree m, and we must have exactly $2N/m$ minimum polynomials of degree m for these solutions. □

Using for example the group theory system CAYLEY one gets:

COROLLARY 1.7. *Let $L(y) = 0$ be a second order equation over k. For the possible minimal polynomials of the algebraic solutions of the Riccati equation we get:*

- *If $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$ (quaternion group), there are exactly three minimal polynomials of degree 2 and all the others are of degree 4.*
- *If $\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2}$ (tetrahedral group), there are exactly two minimal polynomials of degree 4, one of degree 6, and all the others are of degree 12.*
- *If $\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2}$ (octahedral group), there is exactly one minimal polynomial of degree 6, one of degree 8, one of degree 12, and all the others are of degree 24.*
- *If $\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2}$ (icosahedral group), there is exactly one minimal polynomial of degree 12, one of degree 20, one of degree 30, and all the others are of degree 60.*

This gives a partial proof of the following theorem which is the basis of the Kovacic algorithm:

THEOREM 1.4. (KOVACIC, 1986) *Let $L(y) = 0$ be a second order linear differential equation with $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$.*

(i) *$\mathcal{G}(L)$ is a reducible linear group if and only if the differential operator associated to $L(y)$ factors. In this case $L(y) = 0$ has an exponential solution.*
(ii) *If the previous case does not hold, then $\mathcal{G}(L)$ is an imprimitive linear group if and only if $L(y) = 0$ has a solution whose logarithmic derivative is algebraic of degree 2.*
(iii) *If the previous cases do not hold, then $\mathcal{G}(L)$ is a primitive finite linear group if and only if $L(y) = 0$ has a solution whose logarithmic derivative is algebraic of degree 4, 6 or 12.*
(iv) *If the previous cases do not hold, then $\mathcal{G}(L) = \mathrm{SL}(2, \mathcal{C})$ and $L(y) = 0$ has no Liouvillian solution.*

In the above result only the minimal degrees of an algebraic logarithmic derivative is mentioned. In this paper, in order to use invariants instead of semi-invariants, we will consider also other solutions, whose minimal polynomial is of higher degree.

2. Algebraic Solutions of the Riccati and Semi-invariants

Let $L(y) = y'' + a_1 y' + a_0 y$ be a second order equation with coefficients in k, and $\mathrm{Ri}(u) = u' + a_0 + a_1 u + u^2 = 0$ be the associated Riccati equation. We saw in Section 1.2 that, in order to compute a Liouvillian solution of $L(y) = 0$, one can compute the minimal polynomial $P(u) = u^m + b_{m-1}u^{m-1} + \cdots + b_0$ of an algebraic solution of $\mathrm{Ri}(u) = 0$. The main reason for the efficiency of the Kovacic algorithm is the fact that, for $k = \mathbb{C}(x)$ and

$a_1 = 0$, the coefficients of $P(u)$ are given by a linear recurrence from the knowledge of b_{m-1} (Kovacic, 1986; Duval and Loday-Richaud, 1992). In this section we give a proof of this fact without assuming that $\mathcal{G}(L)$ is unimodular or that $k = \mathbb{C}(x)$. The proof also applies to reducible polynomials, which will be fundamental to our approach.

A differential extension $k\{u\}$ of k by a differential variable u is obtained by adjoining to k a variable u and new variables u_i for the ith derivative of u. A derivation Δ on $k\{u\}$ is defined by $\Delta(a) = \delta(a)$ for $a \in k$ and $\Delta(u) = u_1$ which we also denote by u', $\Delta^2(u) = u_2, \ldots$. Note that one can also consider u as a usual variable and that we have $k[u] \subset k\{u\}$ which will result in some abuse of notation in what follows. Also note that Δ is not a derivation on $k[u]$ and that in the following we will consider simultaneously different derivations among which some are derivations on $k[u]$ and some are not.

Definition 2.1. *Let $P \in k[u]$ and D be a derivation on $k[u]$. A polynomial P is called* special *for D if P divides $D(P)$ in $k[u]$.*

The special polynomials exist in wider contexts (Weil, 1994, and references therein).

Lemma 2.2. *If $P_1, P_2 \in k[u]$ are special for a derivation D on $k[u]$, then P_1P_2 is special for D. Conversely, if P is special for D, then all its factors are special for D.*

Proof. If $D(P_1) = Q_1P_1$ and $D(P_2) = Q_2P_2$ with $Q_1, Q_2 \in k[u]$, then $D(P_1P_2) = D(P_1)P_2 + P_1D(P_2) = (Q_1 + Q_2)P_1P_2$.

Conversely, suppose that $D(P) = Q \cdot P$ with $Q \in k[u]$. If $P = P_1^n P_2$ where P_1 is prime and P_1 and P_2 are relatively prime, then $D(P) = QP_1^nP_2 = nP_1^{n-1}D(P_1)P_2 + P_1^nD(P_2)$. Since P_1^n divides both sides and P_1 is prime with P_2, P_1 must divide $D(P_1)$. Similarly P_2 must divides $D(P_2)$. By induction it follows that all irreducible factors of P are special. □

Using the following two derivations on $k[u]$:

$$\partial_k\left(\sum_{i=0}^{m} b_i u^i\right) = \sum_{i=0}^{m} \delta(b_i)u^i$$
$$\frac{\partial}{\partial u}\left(\sum_{i=0}^{m} b_i u^i\right) = \sum_{i=0}^{m} i b_i u^{i-1}.$$

We define a derivation $\mathcal{D}_{L,k}$ on $k[u]$ by:

$$\mathcal{D}_{L,k}\left(P(u)\right) = \partial_k\left(P(u)\right) - (a_0 + a_1u + u^2)\frac{\partial}{\partial u}\left(P(u)\right).$$

The derivative of a polynomial $P(u) = \sum_{i=0}^{m} b_iu^i \in k[u] \subset k\{u\}$ by Δ can now be written:

$$\begin{aligned}\Delta\left(P(u)\right) &= \partial_k\left(P(u)\right) + u'\frac{\partial P}{\partial u}(u)\\ &= \partial_k\left(P(u)\right) - (a_0 + a_1u + u^2)\frac{\partial P}{\partial u}(u) + \left(u' + u^2 + a_0 + a_1u\right)\frac{\partial P}{\partial u}(u)\\ &= \mathcal{D}_{L,k}\left(P(u)\right) + \mathrm{Ri}(u)\cdot\frac{\partial P}{\partial u}(u).\end{aligned}$$

Lemma 2.3. *If K is a differential field extension of k and $P(u) \in k[u]$ is special for $\mathcal{D}_{L,K}$, then $P(u)$ is special for $\mathcal{D}_{L,k}$*

Proof. Since $P \in k[u]$ and $\delta(k) \subset k$, we have that $\mathcal{D}_{L,K}(P) = \mathcal{D}_{L,k}(P)$ is in $k[u]$. If P divides $\mathcal{D}_{L,k}(P)$ over $K[u]$, then, by the uniqueness of the euclidian division, P divides $\mathcal{D}_{L,k}(P)$ over $k[u]$. □

Lemma 2.4. (Weil, 1994) *All zeroes of $P \in k[u]$ are solutions of the Riccati equation if and only if P is special for $\mathcal{D}_{L,k}$.*

Proof. Suppose that P is special and pick any irreducible factor P_1 which must again be special (Lemma 2.2). Since P_1 divides $\mathcal{D}_{L,k}(P_1)$, we have that $P_1(v) = 0$ implies $\mathcal{D}_{L,k}(P_1)(v) = 0$. Since P_1 is prime, it can not divide $\frac{\partial}{\partial u}(P_1)$. From

$$\Delta\left(P_1(u)\right) = \mathcal{D}_{L,k}\left(P_1\right)(u) + \mathrm{Ri}(u) \cdot \left(\frac{\partial}{\partial u} P_1\right)(u), \tag{2.1}$$

we finally get that if $P_1(v) = 0$, then $\mathrm{Ri}(v) = 0$. Since any zero of P is a zero of an irreducible factor, the result follows.

Conversely, suppose that all zeroes of $P(u)$ are zeroes of $\mathrm{Ri}(u)$. Pick an irreducible factor $P_1(u)$ of $P(u)$; then, reasoning as above, we get from (2.1) that, since $\mathrm{Ri}(u) = 0$, all zeroes of P_1 are zeroes of $\mathcal{D}_{L,k}(P_1)$ and thus that P_1 is special. Since all irreducible factors of $P(u)$ are special for $\mathcal{D}_{L,k}$, $P(u)$ is special for $\mathcal{D}_{L,k}$ (Lemma 2.2). □

Remark. This result also follows from Corrollary 1.6 and Lemma 1.10 of Bronstein (1990). ◇

A polynomial $P(u) = u^m + b_{m-1}u^{m-1} + \cdots + b_0$ is special if and only if $\mathcal{D}_{L,k}(P(u))$ is divisible by $P(u)$. Performing the division and setting the remainder equal to 0 gives the following system $(\sharp)_m$ for the coefficients b_i:

$$(\sharp)_m : \quad \begin{cases} b_m = 1 \\ b_{i-1} = \frac{-b_i' + b_{m-1}b_i + a_1(i-m)b_i + a_0(i+1)b_{i+1}}{m-i+1}, & m-1 \geq i \geq 0. \\ b_{-1} = 0 \end{cases}$$

Note that $P(u)$ is special if and only its coefficients b_i satisfy the above system. The last equation $b_{-1} = 0$ plays a central role in Kovacic (1986) but is not used in our proofs. From the form of the system we see that the coefficients b_i are all determined from the knowledge of the coefficient b_{m-1}. A special polynomial is thus uniquely determined by its degree m and by its coefficient b_{m-1}: we may say that such a b_{m-1} solves the system $(\sharp)_m$. Note that, if $P_1 = u^m + b_{m-1}u^{m-1} + \cdots$ and $P_2 = u^n + \beta_{n-1}u^{n-1} + \cdots$ are special, then $P_1P_2 = u^{m+n} + (b_{m-1} + \beta_{n-1})u^{m+n-1} + \cdots$ is also special and so $b_{m-1} + \beta_{n-1}$ solves the system $(\sharp)_{n+m}$. Our next step is to characterize the elements b_{m-1} of k which solve the system $(\sharp)_m$ and thus give a special polynomial of degree m.

Theorem 2.1. *Let $L(y) = y'' + a_1y' + a_0y$ be a second order equation with $a_i \in k$, then all zeroes of $P(u) = u^m + \sum_{i=0}^{m-1} b_iu^i$ with $b_i \in k$ are solutions of the Riccati equation $\mathrm{Ri}(u) = 0$ if and only if*

(1) *the coefficient b_{m-1} is the negative logarithmic derivative of an exponential solution*

(over k) of $L^{\circledS m}(y) = 0$*, i.e.* b_{m-1} *is the negative logarithmic derivative of a semi-invariant of* $\mathcal{G}(L) \subseteq \mathrm{GL}(2, \mathcal{C})$.

(2) *for* $i < m-1$ *the coefficients* b_i *of* P *are determined from* b_{m-1} *by the system* $(\sharp)_m$.

Proof. Suppose that $P(u) \in k[u]$ is special. From Lemma 2.4 we get that all zeroes of $P(u) = 0$ are solutions of $\mathrm{Ri}(u) = 0$. From relation (1.4) we get that b_{m-1} is the negative logarithmic derivative of an exponential solution of $L^{\circledS m}(y) = 0$.

We now show that any exponential solution z of a $L^{\circledS m}(y) = 0$ yields a special polynomial of degree m. Consider the polynomial $P(u) = u^m + \sum_{i=0}^{m-1} b_i u^i$, where $b_{m-1} = -z'/z$ and where the other coefficients $b_{m-2}, \ldots, b_0$ are given according to the recurrence $(\sharp)_m$. Since $b_{m-1} \in k$, all b_i will also be in k and thus $P(u) \in k[u]$. Let y_1, y_2 be a fundamental system of solutions of $L(y) = 0$ and (K, Δ) be a PVE of (k, δ) for $L(y) = 0$. Since z is a semi-invariant of degree m of $\mathcal{G}(L)$, it can be written as a homogeneous form $z = F(y_1, y_2)$ of degree m in y_1, y_2 over $\mathcal{C}$. As $\mathcal{C}$ is algebraically closed, $F(y_1, y_2)$ can be factored over K as a product of m linear forms: $F(y_1, y_2) = \prod_{i=1}^{m}(\beta_i y_1 - \alpha_i y_2)$ with $\beta_i, \alpha_i \in \mathcal{C}$. We note that $u_i = \Delta(\beta_i y_1 - \alpha_i y_2)/(\beta_i y_1 - \alpha_i y_2)$ is a solution of $\mathrm{Ri}(u) = 0$. Thus all zeros of the polynomial $Q(u) = \prod_{i=1}^{m}(u - u_i) \in K[u]$ are solutions of $\mathrm{Ri}(u) = 0$. The polynomial $Q(u) = 0$ must be special for $\mathcal{D}_{L,K}$ (Lemma 2.4) and its coefficients must satisfy $(\sharp)_m$. In particular, since $z'/z = -b_{m-1} = \sum_{i=1}^{m} u_i$, the coefficients of $Q(u) = 0$ are in k. Since $P(u)$ and $Q(u)$ are of the same degree and are both constructed from z'/z and $(\sharp)_m$, we have $P(u) = Q(u)$. The polynomial $P(u) = Q(u)$ is special for $\mathcal{D}_{L,K}$ and has coefficients in k. By Lemma 2.3, $P(u)$ is also special for $\mathcal{D}_{L,k}$. From Lemma 2.4 we get that all roots of $P(u) = 0$ are solutions of $\mathrm{Ri}(u) = 0$.□

This gives a bijection between monic polynomials of degree m over k whose roots are solutions of the Riccati equation and exponential solutions of $L^{\circledS m}(y) = 0$, i.e. semi-invariants of degree m of $\mathcal{G}(L)$. In particular, if z_1 and z_2 are two semi-invariants, then the special polynomial associated with the product $z_1 z_2$ is the product of the special polynomials associated with z_1 and z_2 respectively. In the sequel, we will use this remark without further mention.

For higher order linear differential equations, the minimum polynomial of an algebraic solution of the Riccati equation is no longer special, and the bijection does not exist any more.

3. The Algorithm

In this section, we will always assume that $L(y)$ is a second order equation with $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$. The previous section shows that there is a bijection between exponential solutions of $L^{\circledS m}(y) = 0$ and polynomials of degree m whose zeroes are solutions of the Riccati. We now propose an algorithm where rational solutions of $L^{\circledS m}(y) = 0$ are used as much as possible instead of exponential solutions.

The proposed algorithm can be outlined as follows:

(i) Test if $L(y)$ has a non-trivial rational (and thus Liouvillian) solution.

(ii) Test if $L^{\circledS 2}$ has a non-trivial rational solution. If it is the case, then $\mathcal{G}(L)$ is a reducible subgroup of $\mathrm{SL}(2, \mathcal{C})$.

(a) If the space of rational solutions of $L^{\circledS 2}$ is of dimension 3, then $\mathcal{G}(L) = \{id, -id\}$

and any special polynomial $P(u)$ of degree 2 associated to a non-trivial rational solution of $L^{ⓢ2}$ is reducible. A factor of $P(u)$ gives a Liouvillian solution.

(b) If the previous case does not hold, then $\mathcal{G}(L)$ is a completely reducible group if an only if the special polynomial $P(u)$ of degree 2 associated to a non-trivial rational solution of $L^{ⓢ2}$ factors but is not a square. The two factors of $P(u)$ give two exponential solutions.

(c) If the above cases do not hold, then the special polynomial $P(u)$ of degree 2 associated to a non-trivial rational solution of $L^{ⓢ2}$ is either a square or is irreducible. In both cases a Liouvillian solution is found.

(iii) Test if $L(y) = 0$ has a non-trivial exponential (and thus Liouvillian) solution. Such a solution must then be unique and gives a unique right factor of order one of $L(y)$.

(iv) Test if $L^{ⓢ4}$ has non-trivial rational solutions. The special polynomial $P(u)$ associated to an arbitrary non-trivial rational solution of $L^{ⓢ4}$ is either the square of an irreducible special polynomial of order 2 or is irreducible.

(v) Test for increasing $m \in \{6, 8, 12\}$ if $L^{ⓢm}$ has a non-trivial rational solution. The corresponding special polynomial will be irreducible.

(vi) Conclude that $L(y) = 0$ has no Liouvillian solution.

The steps have to be performed in the given order and the algorithm terminates as soon as a solution is found in one of the cases. The third step is the only one where instead of some rational solution one has to compute an exponential solution of $L(y)$ (which is, however, known to be unique in this case). We note that it is not difficult to test if a special polynomial $P(u)$, known to be either irreducible or a square, is a square. This is the case if and only if $Q(u) = \gcd\left(P(u), \frac{d}{du}P(u)\right)$ is not constant in u, in which case, under the given assumption, $\left(Q(u)\right)^2 = P(u)$.

In the remainder of this section we prove that the proposed algorithm is correct and compute examples in each case.

3.1. The reducible case

Proposition 4.2 of Singer and Ulmer (1993a) describes the reducible Galois groups, in particular if L has a rational solution. The next lemma complements this proposition.

Lemma 3.1. *Let $L(y)$ be a second order equation with $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$ having no non-trivial rational solutions. If $L^{ⓢ2}(y) = 0$ has a non-trivial rational solution, then $\mathcal{G}(L)$ is a reducible subgroup of $\mathrm{SL}(2, \mathcal{C})$.*

(i) *If the space of rational solutions of $L^{ⓢ2}$ is of dimension 3, then $\mathcal{G}(L) = \{id, -id\}$ and any special polynomial $P(u)$ associated to a non-trivial rational solution of $L^{ⓢ2}$ factors.*

(ii) *If the previous case does not hold, then $\mathcal{G}(L)$ is a completely reducible group if and only if the special polynomial $P(u)$ associated to a non-trivial rational solution of $L^{ⓢ2}$ factors but is not a square. The two factors of $P(u)$ give two exponential solutions which are linearly independent over $\mathcal{C}$.*

(iii) *If the above cases do not hold, then the special polynomial $P(u)$ associated to a non-trivial rational solution of $L^{ⓢ2}$ is either a square or is irreducible. In both cases a Liouvillian solution is found.*

PROOF. We first note that if $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ is irreducible (i.e. primitive or imprimitive), then $L^{ⓢ2}$ has no non-trivial rational solution because there is no invariant of degree 2 in those cases (cf. proofs of Lemmas 3.2 and 3.3). Thus, if $L^{ⓢ2}(y) = 0$ has a non-trivial rational solution, then $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ is reducible.

Assume that $\mathcal{G}(L)$ is completely reducible. For a basis denoted $\{y_1, y_2\}$ all elements g of $\mathcal{G}(L)$ must be of the form

$$g = \begin{pmatrix} a_g & 0 \\ 0 & a_g^{-1} \end{pmatrix}.$$

In particular y_1 and y_2 are semi-invariants and $y_1 y_2$ is an invariant of $\mathcal{G}(L)$.

(i) If $\mathcal{G}(L)$ has another linearly independent invariant of degree two, say $F(y_1, y_2) = \alpha(y_1)^2 + \beta(y_2)^2$, then, for $g \in \mathcal{G}(L)$, we have $g \cdot F(y_1, y_2) = a_g^2 \alpha(y_1)^2 + a_g^{-2}\beta(y_2)^2$. Thus $\forall g \in \mathcal{G}(L), a_g^2 = 1$ and we get $\mathcal{G}(L) = \{id, -id\}$. In this case any homogeneous form of degree 2 is invariant and $L^{ⓢ2}(y) = 0$ has a rational solution space of dimension 3. Any solution of $L(y) = 0$ is an exponential solution and thus any polynomial $P(u)$ factors into two linear polynomials.

(ii) If $\mathcal{G}(L)$ has no other linearly independent invariant of degree two, then any rational solution of $L^{ⓢ2}(y) = 0$ is a multiple of $y_1 y_2$ and factors. The polynomial $P(u)$ associated to a non-trivial rational solution of $L^{ⓢ2}(y) = 0$ will be the product of the distinct minimal polynomials associated to the semi-invariants y_1 and y_2. In particular, $P(u)$ is not a square.

Suppose that $P(u)$ factors but is not a square, then each factor is a special polynomial of order one corresponding to a different logarithmic derivative z_1'/z_1 and z_2'/z_2. The corresponding solutions z_1 and z_2 must be linearly independent over $\mathcal{C}$. In the basis $\{z_1, z_2\}$, the group $\mathcal{G}(L)$ is diagonal and thus completely reducible.

The only cases left are those where $\mathcal{G}(L) \neq \{id, -id\}$ and $P(u)$ is a square or is irreducible over $k[u]$. By the above, $\mathcal{G}(L)$ is reducible, but cannot be completely reducible. □

Remark. The fact that factorization of differential operators is easier in the completely reducible case was used by Singer (1996). ◇

An example of a completely reducible group is the example given in Section 1.2 which we now summarize:

Example. Let $L(y) = y'' + \frac{3}{16x^2}y$. This equation has no non-trivial rational solution, and the equation $L^{ⓢ2}(y) = 0$ has a one-dimensional space of rational solutions generated by x. Thus $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ is a reducible group and $L(y) = 0$ factors. Since the rational solution space of $L^{ⓢ2}(y) = 0$ is not of dimension 3, we have $\mathcal{G}(L) \neq \{id, -id\}$. The special polynomial obtained from the logarithmic derivative $1/x$ of x is

$$u^2 - \frac{1}{x}u + \frac{3}{16x^2}$$

which factors into $\left(u - \frac{1}{4x}\right)\left(u - \frac{3}{4x}\right)$. Since $P(u)$ is not a square, $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ is a completely reducible group. From the factorization of $P(u)$, we get the following two Liouvillian solutions of $L(y) = 0$:

$$y_1 = e^{\int \frac{1}{4x}}, \qquad y_2 = e^{\int \frac{3}{4x}}.$$

Viewed as an operator, L is the least common left multiple of $\delta - \frac{1}{4x}$ and $\delta - \frac{3}{4x}$. ◇

In the following example, we deal with a reducible but not completely reducible linear group:

Example. Consider $L(y) = y'' + \left(\frac{3}{16x^2} + \frac{1}{4(x-1)^2} - \frac{1}{4x(x-1)}\right)y$. The equation $L^{\circledS 2}(y) = 0$ has no non-trivial rational solution and thus $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ has no invariant of degree 2. In this case the exponential solution $e^{\int \frac{3x-1}{4x(x-1)}}$ is a semi-invariant of degree one, but there exists no other linearly independent semi-invariant of degree one. We thus get a unique polynomial $P(u) = u - \frac{3x-1}{4x(x-1)}$ of degree one. The group $\mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ is reducible but not completely reducible.
We note that even if no invariant of degree two exists, there could exist other invariants of higher degree. In this example $L^{\circledS 4}$ has a one-dimensional rational solution space generated by $x(x-1)^2$.
The example shows that, even if no invariant of degree 2 exists, the equation $L(y)$ could be reducible, and that in order to proceed in the algorithm, one must look for exponential solutions of $L(y) = 0$ at this stage. ◇

3.2. THE IMPRIMITIVE CASE

In this case we show that the computation of a Liouvillian solution of a second order equation $L(y) = 0$ is reduced to the computation of a rational solution of $L^{\circledS 4}(y) = 0$ and that the special polynomial associated to the logarithmic derivative is either a square or irreducible. In this section we need to assume that $L(y) = 0$ is an irreducible equation.

LEMMA 3.2. *Let $L(y) = 0$ be an irreducible second order equation over K whose Galois group $\mathcal{G}(L)$ is unimodular. Then $\mathcal{G}(L)$ is an imprimitive subgroup of* $\mathrm{SL}(2,\mathcal{C})$ *if and only if $L^{\circledS 4}$ has a rational solution q. The special polynomial obtained from the logarithmic derivative of q is then*

(i) *The square of a unique special polynomial of degree* 2 *if $L^{\circledS 4}$ has a one-dimensional rational solution space.*
(ii) *Either the square of a special polynomial of degree* 2 *or is irreducible if $L^{\circledS 4}$ has a two-dimensional rational solution space, in which case $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$.*

PROOF. Denote $\{y_1, y_2\}$ a basis in which all $g \in \mathcal{G}(L) \subseteq \mathrm{SL}(2,\mathcal{C})$ are simultaneously in the form

$$\begin{pmatrix} a_g & 0 \\ 0 & a_g^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -a_g \\ a_g^{-1} & 0 \end{pmatrix}$$

(cf. Section 1.3). Since $\forall g \in \mathcal{G}(L)$ we have $g(y_1 y_2) = \pm y_1 y_2$, we get that $y_1 y_2$ is a semi-invariant of degree 2 and that $y_1^2 y_2^2$ is an invariant of degree 4 of $\mathcal{G}(L)$. Since $L^{\circledS 4}(y) = 0$ has no rational solution if $\mathcal{G}(L)$ is a primitive subgroup of $\mathrm{SL}(2,\mathcal{C})$ (cf. character decompositions of the finite primitive groups in the next subsection and Springer, 1973, for $\mathrm{SL}(2,\mathcal{C})$), we get the first assertion (this result is also proven in Singer and Ulmer, 1993a, Theorem 4.1 or Kovacic, 1986, p. 20).

If the space of rational solutions of $L^{\circledS 4}$ is one dimensional then, up to a constant multiple, this rational solution is the square of $y_1 y_2$. Thus, the (unique) special polynomial corresponding to the (unique) logarithmic derivative of a rational solution $y_1^2y_2^2$ of $L^{ⓢ4}$ will be the square of the special polynomial associated with the semi-invariant y_1y_2. Note that the special polynomial associated with y_1y_2 must be irreducible, because $\mathcal{G}(L)$ is irreducible and thus has no semi-invariants of degree 1. Since for second order equations there is a bijection between rational solutions and invariants, we now look at the ring of invariants to see if the $\mathcal{C}$-subspace of invariants of degree 4 is of dimension 1. As shown in Springer (1973, p. 95), the ring of invariants of $D_n^{\mathrm{SL}_2}$ is generated by:

$$I_1 = y_1^2y_2^2, I_2 = y_1^{2n} + (-1)^n y_2^{2n}, I_3 = y_1y_2(y_1^{2n} - (-1)^n y_2^{2n}).$$

The group D_∞ has up to scalar multiples only one invariant $y_1^2y_2^2$ of degree 4. To see this one looks at the diagonal subgroup and, as in the proof of Lemma 3.1, shows that this diagonal subgroup would be of order at most 4 making D_∞ finite, a contradiction. Thus the group $D_2^{\mathrm{SL}_2}$ is the only imprimitive group for which the space of rational solutions of $L^{ⓢ4}$ is of dimension 2 and not 1.

The group $D_2^{\mathrm{SL}_2}$ has 5 irreducible characters, the trivial one denoted $\mathbf{1}$, 3 characters $\zeta_{1,1}$, $\zeta_{1,2}$, $\zeta_{1,3}$ of degree one and one character ζ_2 of degree two. The non-trivial characters of degree one have the property that the product $\zeta_{1,i}\zeta_{1,j}$ is $\mathbf{1}$ for $i = j$ and different from $\mathbf{1}$ otherwise. If a second order equation $L(y) = 0$ has Galois group $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$, then the corresponding character of $\mathcal{G}(L)$ will be ζ_2. The character χ_m of $\mathcal{G}(L^{ⓢm})$ can be computed according to the formula given in Singer and Ulmer (1993a, p. 15):

$$\chi_2 = \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3}, \qquad \chi_3 = 2\zeta_2, \qquad \chi_4 = 2\cdot\mathbf{1} + \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3}$$

this shows that there are three semi-invariants S_i associated to the characters $\zeta_{1,i}$ ($i \in \{1,2,3\}$) whose squares are rational. The products S_1S_2, S_1S_3 and S_2S_3 are not invariants (i.e. do not correspond to a rational solution) since the products of the associated characters are not the trivial character. Thus a rational solution of $L^{ⓢ4}$ is either the square of a semi-invariant S_i of order 2 (in which case the associated special polynomial will be a square), or it is not the product of semi-invariants and the associated special polynomial is irreducible. □

Example. Consider the irreducible equation

$$L(y) = y'' - \frac{2}{2x-1}y' + \frac{\left(27x^4 - 54x^3 + 5x^2 + 22x + 27\right)(2x-1)^2}{144x^2(x-1)^2(x^2-x-1)^2}y = 0.$$

It is unimodular because $\frac{2}{2x-1}$ is the logarithmic derivative of $2x-1$. The equation $L^{ⓢ4}(y) = 0$ has a one-dimensional space of rational solutions generated by $x(x-1)(x^2-x-1)^2$. The special polynomial that is associated with the logarithmic derivative $\frac{(2x-1)(3x^2-3x-1)}{(x^2-x-1)(x-1)x}$ is:

$$\begin{aligned} u^4 &- \frac{(2x-1)(3x^2-3x-1)}{(x^2-x-1)(x-1)x}u^3 \\ &+ \frac{(2x-1)^2(243x^4-486x^3+77x^2+166x+27)}{72x^2(x-1)^2(x^2-x-1)^2}u^2 \\ &- \frac{(81x^4-162x^3+23x^2+58x+9)(2x-1)^3(3x^2-3x-1)}{144x^3(x-1)^3(x^2-x-1)^3}u \end{aligned}$$

$$+\frac{(81x^4-162x^3+23x^2+58x+9)^2(2x-1)^4}{20736x^4(x-1)^4(x^2-x-1)^4}$$

which is the square of:

$$u^2-\frac{(2x-1)\left(3x^2-3x-1\right)}{2x(x-1)(x^2-x-1)}u+\frac{(81x^4-162x^3+23x^2+58x+9)(2x-1)^2}{144x^2(x-1)^2(x^2-x-1)^2}.$$

Since $L^{\circledS 6}$ also has a rational solution $x^2(x-1)^2(x^2-x-1)^2$, we get from the above proof that $\mathcal{G}(L)$ is $D_3^{\mathrm{SL}_2}$ $\diamond$

The next example has a Galois group $\mathcal{G}(L)\cong D_2^{\mathrm{SL}_2}$

Example. Consider the irreducible equation

$$L(y)=y''-\frac{2}{2x-1}y'+\frac{3(2x-1)^2\left(x^4-2x^3+x+1\right)}{16x^2(x-1)^2\left(x^2-x-1\right)^2}y.$$

The fourth symmetric power has a two-dimensional rational solution space generated by $J_0=x(x-1)\left(x^2-x-1\right)$ and $J_1=x(x-1)\left(x^2-x+1\right)\left(x^2-x-1\right)$. Thus, $\mathcal{G}(L)$ is the quaternion group and we get the following two special polynomials:

$$\begin{aligned}u^4&-\frac{(2x-1)(2x^2-2x-1)}{x(x-1)(x^2-x-1)}u^3+\frac{(2x-1)^2(11x^4-22x^3+11x+3)}{8x^2(x-1)^2(x^2-x-1)^2}u^2\\&-\frac{(2x-1)^3(2x^2-2x-1)(3x^4-6x^3+3x+1)}{16x^3(x-1)^3(x^2-x-1)^3}u\\&+\frac{(3x^4-6x^3+3x+1)^2(2x-1)^4}{256x^4(x-1)^4(x^2-x-1)^4}\end{aligned}$$

and

$$\begin{aligned}u^4&-\frac{(2x-1)(3x^4-6x^3+3x^2-1)}{x(x-1)(x^2-x+1)(x^2-x-1)}u^3\\&+\frac{3(2x-1)^2(9x^6-27x^5+19x^4+7x^3-8x^2+1)}{8x^2(x-1)^2(x^2-x-1)^2(x^2-x+1)}u^2\\&-\frac{(2x-1)^3(27x^8-108x^7+117x^6+27x^5-86x^4+x^3+21x^2+x-1)}{16x^3(x-1)^3(x^2-x-1)^3(x^2-x+1)}u\\&+\frac{(2x-1)^4(81x^{10}-405x^9+621x^8-54x^7-572x^6+204x^5+231x^4-55x^3-48x^2-3x+1)}{256x^4(x-1)^4(x^2-x-1)^4(x^2-x+1)}.\end{aligned}$$

From the theorem, we know that each polynomial is either a square or is irreducible. In this example, the first polynomial is a square and the second is irreducible. $\diamond$

3.3. THE PRIMITIVE CASE

The following shows that for the primitive case it is always possible to look only for rational solutions of symmetric powers. However the algebraic solution of the Riccati found this way will not be of lowest algebraic degree for $A_4^{\mathrm{SL}_2}$ and $S_4^{\mathrm{SL}_2}$.

LEMMA 3.3. *Let $L(y)=0$ be a second order equation whose differential Galois group $\mathcal{G}(L)$ is a finite primitive subgroup of* $\mathrm{SL}(2,\mathcal{C})$.

- *If $\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2}$, then the unique special polynomial obtained from the logarithmic derivative of a non-trivial rational solution of $L^{ⓢ 6}$ is irreducible.*
- *If $\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2}$, then the unique special polynomial obtained from the logarithmic derivative of a non-trivial rational solution of $L^{ⓢ 8}$ is irreducible. Also the unique special polynomial obtained from the logarithmic derivative of a non-trivial rational solution of $L^{ⓢ 12}$ is the square of a unique special polynomial of degree* 6.
- *If $\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2}$, then the unique special polynomial obtained from the logarithmic derivative of a non-trivial rational solution of $L^{ⓢ 12}$ is irreducible.*

In all cases, it is the special polynomial of lowest order that one can construct using rational *solutions of symmetric powers of $L(y)$.*

Proof. The (abstract) group $A_4^{\mathrm{SL}_2}$ has seven irreducible characters, the trivial one denoted $\mathbf{1}$, two characters $\zeta_{1,1}$ and $\zeta_{1,2}$ of degree 1, two characters $\zeta_{2,1}$ and $\zeta_{2,2}$ of degree 2 (where the trace of an element of order 3 is different from one and thus the representation is not in $\mathrm{SL}(2, \mathcal{C})$), another character ζ_2 of degree two (corresponding to a representation in $\mathrm{SL}(2, \mathcal{C})$) and a character ζ_3 of degree 3. If a second order equation $L(y) = 0$ has Galois group $\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2}$, then the corresponding character of $\mathcal{G}(L)$ will be $\chi = \zeta_2$. The character χ_m of $\mathcal{G}(L^{ⓢ m})$ can be computed according to the formula given in Singer and Ulmer (1993a, p. 15):

$$\chi_2 = \zeta_3 \qquad \chi_4 = \zeta_{1,1} + \zeta_{1,2} + \zeta_3 \qquad \chi_6 = \mathbf{1} + 2\zeta_3$$
$$\chi_3 = \zeta_{2,1} + \zeta_{2,2} \qquad \chi_5 = \zeta_{2,1} + \zeta_{2,2} + \zeta_2.$$

Since there are no semi-invariants of degree 2 or 3, the unique special polynomial obtained from the logarithmic derivative of a rational solution of $L^{ⓢ 6}$ cannot be the product of special polynomials of lower order.

The proof in the other cases are similar and can be deduced from the decompositions that follow:

- The (abstract) group $S_4^{\mathrm{SL}_2}$ has eight irreducible characters, the trivial one $\mathbf{1}$, another character $\zeta_{1,1}$ of degree 1, one character ζ_2 of degree 2 which is not faithful, two (conjugate) characters $\zeta_{2,0}$ and $\zeta_{2,1}$ of degree 2 (corresponding to representations in $\mathrm{SL}(2, \mathcal{C})$), two characters $\zeta_{3,1}$ and $\zeta_{3,2}$ of degree 3 and a character ζ_4 of degree 4. For $\zeta_{2,i}$ we set $j \equiv i + 1 \pmod 2$ and get:

  $$\chi_2 = \zeta_{3,1} \qquad \chi_5 = \zeta_{2,j} + \zeta_4 \qquad \chi_8 = \mathbf{1} + \zeta_2 + \zeta_{3,1} + \zeta_{3,2}$$
  $$\chi_3 = \zeta_4 \qquad \chi_6 = \zeta_{1,1} + \zeta_{3,1} + \zeta_{3,2} \qquad \chi_{12} = \mathbf{1} + \zeta_{1,1} + \zeta_2 + \zeta_{3,1} + 2\zeta_{3,2}$$
  $$\chi_4 = \zeta_2 + \zeta_{3,2} \qquad \chi_7 = \zeta_{2,i} + \zeta_{2,j} + \zeta_4.$$

  In the above case we note that the character χ_{12} as a unique trivial summand and thus that $L^{ⓢ 12}(y) = 0$ has a one-dimensional rational solution space and thus that (up to multiples) there is a unique invariant of degree 12. But this invariant must be the square of the semi-invariant of degree 6 since the one-dimensional character $\zeta_{1,1}$ is of order 2. The special polynomial associated to the invariant of degree 12 must be the square of the unique special polynomial of degree 6.
- The (abstract) group $A_5^{\mathrm{SL}_2}$ has nine irreducible characters, the trivial one $\mathbf{1}$, two (conjugate) characters $\zeta_{2,0}$ and $\zeta_{2,1}$ of degree 2 (corresponding to two representations in $\mathrm{SL}(2, \mathcal{C})$), two characters $\zeta_{3,1}$ and $\zeta_{3,2}$ of degree 3, two characters $\zeta_{4,1}$ and $\zeta_{4,2}$ of degree 4, a character ζ_5 of degree 5 and a character ζ_6 of degree 6. For

$\zeta_{2,i}$ we set $j \equiv i+1 \pmod 2$ and get:

$$\begin{array}{lll} \chi_2 = \zeta_{3,i} & \chi_6 = \zeta_{3,j} + \zeta_{4,2} & \chi_{10} = \zeta_{3,1} + \zeta_{3,2} + \zeta_5 \\ \chi_3 = \zeta_{4,1} & \chi_7 = \zeta_{2,j} + \zeta_6 & \chi_{11} = \zeta_{2,i} + \zeta_{4,1} + \zeta_6 \\ \chi_4 = \zeta_5 & \chi_8 = \zeta_{4,2} + \zeta_5 & \chi_{12} = \mathbf{1} + \zeta_{3,i} + \zeta_{4,2} + \zeta_5. \\ \chi_5 = \zeta_6 & \chi_9 = \zeta_{4,1} + \zeta_6. & \end{array}$$

□

Example. Consider the irreducible equation

$$L(y) = y'' - \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y.$$

This equation is studied in Kovacic (1986, p. 23), where a minimal polynomial of degree 4 of an algebraic solution of the Riccati equation is given. This minimal polynomial corresponds to an exponential solution of $L^{ⓢ4}$ which is not rational, but which is the cube root of a rational function. The same equation is also studied in Singer and Ulmer (1993b, p. 68) where the minimal polynomial of a solution (not of a logarithmic derivative) is computed.

Using our approach, since $L^{ⓢ4}$ has no rational solution we know that $\mathcal{G}(L)$ is a primitive subgroup of $\mathrm{SL}(2, \mathcal{C})$. Since $L^{ⓢ6}$ has a rational solution $x^2(x-1)^2$, we get that $\mathcal{G}(L)$ is the tetrahedral group and that the special polynomial associated with the logarithmic derivative $\frac{4x-2}{x^2-x}$ will be irreducible. This gives the following minimal polynomial for an algebraic solution of the Riccati:

$$\begin{aligned} &u^6 - 2\frac{(2x-1)}{x(x-1)}u^5 + \frac{5(64x^2-63x+15)}{48x^2(x-1)^2}u^4 \\ &\quad - \frac{5(512x^3-745x^2+351x-54)}{432x^3(x-1)^3}u^3 \\ &\quad + \frac{5(4096x^4-7840x^3+5485x^2-1674x+189)}{6912x^4(x-1)^4}u^2 \\ &\quad - \frac{(3645x-16254x^2+35781x^3-38720x^4+16384x^5-324)}{20736x^5(x-1)}u \\ &\quad + \frac{-29889x+169209x^2-506331x^3+842008x^4+262144x^6-735232x^5+2187}{2985984x^6(x-1)^6}. \end{aligned}$$

◇

4. Rationality Problem

In order to use differential Galois theory and in particular the existence of a PVE for $L(y) = 0$, we needed to assume that the field of constants of the coefficient field is algebraically closed of characteristic 0. This implies that even if the coefficients of $L(y) = 0$ belong to $\mathbb{Q}(x)$, the coefficient of a special polynomial could be in $\overline{\mathbb{Q}}(x)$ but not in $\mathbb{Q}(x)$. The question of which algebraic extension of the constant field is needed to represent a special polynomial is studied in Hendriks and van der Put (1993,1995) and Ulmer (1994). The following result is trivial but useful, since it connects the approach used in this paper to the rationality problem:

Lemma 4.1. *Let $L(y) = 0$ be a linear differential equations whose coefficients belong to a differential field $k_0 \subseteq \mathbb{C}(x) = k$. If a special polynomial $P(u)$ is obtained from an invariant of degree m corresponding to a solution in k_0 of $L^{ⓢm}(y) = 0$, then the coefficients of $P(u)$ are in k_0, i.e. no algebraic extension is needed to represent the coefficients of this particular special polynomial $P(u)$.*

To see how to use this result we note that:

(i) The coefficient of any symmetric power $L^{ⓢm}(y)$ of $L(y)$ are obtained by solving a linear system over k_0 and thus also belong to k_0.
(ii) An invariant of degree m is a rational solution of $L^{ⓢm}(y) = 0$. By Theorem 9.1 of Bronstein (1992), there exists a basis of the rational solution space of $L^{ⓢm}(y) = 0$ in k_0 which can be computed without extending the constant field†.
(iii) If the invariant and thus b_{m-1} is in k_0, then all other coefficients of $P(u)$ obtained by the recurrence $(\sharp)_m$ will also be in k_0.

In what follows we assume (e.g. using the algorithm given in Bronstein, 1992, Theorem 9.1) that all computed invariants from now on are in k_0, the smallest field containing the coefficients. Thus, if a special polynomial can be computed using an invariant of some degree (i.e. a rational solution of some symmetric power), then this special polynomial also has coefficients in k_0. Our results imply that this is possible in all cases except for the non-reductive subgroups $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$. For reducible non-reductive groups, there is a unique exponential solution, and so the result of Hendriks and van der Put (1995) quoted above shows that no extension of the constant field is needed to express this solution‡. Thus, one can always find (at least) *one* special polynomial without increasing the constant field.

4.1. the reducible case

If we are in a non-completely reducible case then, as seen just above, there a unique exponential solution and its logarithmic derivative lies in k_0.

In Section 3.1, we showed that the Galois group $\mathcal{G}(L) \neq \{id, -id\}$ is reducible and completely reducible if and only if it has an invariant of degree 2 such that the corresponding special polynomial factors but is not a square. In that case, an algebraic extension of degree 2 of the constant field may be needed to factor the special polynomial, as shown in this example:

Example. Consider $L(y) = y'' + \frac{7}{16x^2}y$ whose coefficients belong to $k_0 = \mathbb{Q}(x) \subset \overline{\mathbb{Q}}(x) = k$. A rational solutions of $L^{ⓢ2}$ is x and we get the special polynomial

$$u^2 - \frac{1}{x}u + \frac{7}{16x^2}.$$

† If $L(y) = 0$ has coefficients in $k_0 = \mathcal{C}_0(x)$ and V is the $\mathcal{C}_0$-space of solutions of $L(y) = 0$ in k_0, then $W = \overline{\mathcal{C}}_0 \otimes_{\mathcal{C}_0} V$ is the $\overline{\mathcal{C}_0}$-space of solutions of $L(y) = 0$ in $\overline{\mathcal{C}_0}k_0$. In particular, a $\mathcal{C}_0$-basis of V will be a $\overline{\mathcal{C}_0}$-basis of W.

‡ However, it is not certain yet that an extension of the constant field will not be needed during the computational process that provides this unique exponential solution.

This special polynomial is irreducible over $\mathbb{Q}(x)$, but factors over $\mathbb{Q}(\sqrt{-3})(x)$ into

$$\left(u - \frac{2-\sqrt{-3}}{4x}\right)\left(u - \frac{2+\sqrt{-3}}{4x}\right).$$

We get the following two Liouvillian solutions of $L(y) = 0$:

$$y_1 = e^{\int\left(\frac{2-\sqrt{-3}}{4x}\right)}, \qquad y_2 = e^{\int\left(\frac{2+\sqrt{-3}}{4x}\right)}.$$

◇

4.2. THE IRREDUCIBLE CASE

For irreducible equations $L(y) = 0$ we showed how to construct an irreducible special polynomial using an invariant. So, in this case, no algebraic extension of the coefficient field is needed to represent a solution. But, for the quaternion and the tetrahedral groups, the special polynomial proposed is not of minimal degree. To construct the special polynomial of minimal degree, an algebraic extension of k_0 is sometimes necessary. In fact, there are exactly two cases when one may need to augment the constant field; we now detail them.

4.3. THE GROUP OF QUATERNIONS

If $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$ (the group of quaternions), we saw that there are three irreducible special polynomials of degree 2 and all the other irreducible ones of degree 4. With our approach, one can also find the polynomials of degree 2. The idea, explained through the following example, is to choose the correct linear combination of invariants in order to guarantee that the corresponding special polynomial is a square.

Example. Consider the equation $y'' + \frac{27x}{8(x^3-2)^2}y = 0$ (from Hendriks and van der Put, 1995). Applying our algorithm, we find that $\mathcal{G}(L)$ has no invariant of degree less than 4 and that $L^{\circledS 4}(y) = 0$ has a basis of rational solutions given by $J_1 = \left(x^3 - 2\right)$ and $J_2 = x(-2 + x^3)$. Thus, $\mathcal{G}(L)$ is the quaternion group and we get the following two special polynomials $P_1(u)$ and $P_2(u)$:

$$u^4 - 3\frac{x^2}{x^3-2}u^3 + \frac{3x(4x^3+1)}{4(x^3-2)^2}u^2 - \frac{8x^6+13x^3-4}{8(x^3-2)^3}u + \frac{27x^2(-1+2x^3)}{64(x^3-2)^4}$$

and

$$u^4 - 2\frac{(-1+2x^3)}{(x^3-2)x}u^3 + \frac{3x(8x^3-7)}{4(x^3-2)^2}u^2 - \frac{16x^6-19x^3+1}{4(x^3-2)^3}u + \frac{(4x^3-3x-2)(16x^6+12x^4-16x^3+9x^2-6x+4)}{64(x^3-2)^4 x}.$$

A simple gcd computation shows that none of these is a square, so they both provide Liouvillian solutions. We now wish to compute the special polynomials of minimal degree 2 using a linear combination $J_\lambda = J_0 + \lambda J_1$ and construct the special polynomial $P_\lambda(u)$ associated with J_λ. The results of Section 3.2 show that there are exactly three values of λ such that P_λ is a square (and it is irreducible otherwise). Call R_u the resultant in u of $P_\lambda(u)$ and $\frac{\partial}{\partial u}P_\lambda(u)$; then, we must have $R_u(x, \lambda) = 0$ for all x. So, we compute the gcd of all coefficients in x and obtain $\left(2\lambda^3+1\right)^2$ (in fact, the resultant was $-115\,964\,116\,992(x^3-2)^{22}(1+2\lambda^3)^2(\lambda x+1))$. Call α a solution of $2\alpha^3+1=0$. Then, P_α is necessarily a square. Actually, we have $P_\alpha = Q_\alpha^2$, where $Q_\alpha(u)$ is:

$$u^2 - \frac{(2x^2+x\alpha^2-\alpha)}{(x^2+2x\alpha^2-2\alpha)(x-2\alpha^2)}u + \frac{(4x^3-3\alpha x-2)(x+\alpha^2)}{4(x^2+2x\alpha^2-2\alpha)^2(x-2\alpha^2)^2}.$$

Note that there are 3 conjugate solutions of $2\alpha^3+1=0$ and thus we have three minimum polynomials of degree 2 given by the above relation. The above process can be applied to any equation with a quaternion Galois group. ⋄

4.4. The Tetrahedral Group

In the finite primitive cases, Kovacic (1986) already mentioned that one could get the minimum special polynomials by factoring special polynomials obtained from invariants of degree 12. In the tetrahedral case, there is a 2-dimensional space of invariants. Taking the same notation as in the proof of Lemma 3.3, one can see this from:

$$\chi_{12} = 2\cdot\mathbf{1} + \zeta_{1,1} + \zeta_{1,2} + 3\zeta_3.$$

Among those invariants of degree 12, two must be the cube of the two semi-invariants of degree 4, since the corresponding linear characters $\zeta_{1,1}$ and $\zeta_{1,2}$ are of order 3.

One can proceed like for the group of quaternions and look for the linear combinations of the two invariants of degree 12 whose corresponding special polynomials are the cubes of one of the two special polynomials of degree 4. The linear combination may require a quadratic extension of the field of constants of k_0.

5. Conclusion

We do not claim that the algorithm presented here is always better/faster than the Kovacic algorithm. However we feel that the formulation via rational solutions simplifies the presentation and makes the algorithm easier to implement.

The algorithm presented here is not limited to the case $k=\mathbb{C}(x)$ and holds for any second order equation with unimodular Galois group (i.e. the special form $p(x)y''-q(x)y(x)=0$ used in Kovacic (1986) is not always needed). The fact that we reduce almost everything to the computation of rational solutions of some auxiliary linear differential equations allows us to work with complicated singularities without having to factor polynomials (Bronstein, 1992).

It turns out that an implementation of our approach treats easily examples with several complicated singularities and finite group (see our examples pages 193 and 198) which the known implementations of the Kovacic algorithm could hardly solve; in practice, the only case that remains difficult is the non-reductive case where the Riccati equation has a unique rational solution.

The necessary conditions used in the Kovacic algorithm can also be used in our approach to distinguish between the different case. Similar necessary conditions (even stronger in some cases) to those given in Kovacic (1986) which do not assume the special form $p(x)y''-q(x)y(x)=0$ are given in Singer and Ulmer (1994). Necessary conditions for the group of quaternions are given in Ulmer (1994).

In the case of a finite group, an alternative to our approach is to use the algorithm of

Singer and Ulmer (1993b) to compute the minimum polynomial of an algebraic solution of $L(y) = 0$ instead of its logarithmic derivative.

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1.8 Necessary Conditions for Liouvillian Solutions of (Third Order) Linear Differential Equations

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Necessary Conditions for Liouvillian Solutions of (Third Order) Linear Differential Equations*

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Abstract. In this paper we show how group theoretic information can be used to derive a set of necessary conditions on the coefficients of $L(y)$ for $L(y) = 0$ to have a liouvillian solution. The method is used to derive (and improve in one case) the necessary conditions of the Kovacic algorithm and to derive an explicit set of necessary conditions for third order differential equations.

Keywords: Linear differential equations, Liouvillian solutions, Differential Galois group, Monodromy group, Exponents

1. Introduction

In our previous work [20], [21], we have shown how group theoretic techniques can be used to develop effective algorithms to calculate Galois groups of second and third order homogeneous linear differential equations and to decide questions about the algebraic nature of the solution of such equations (e.g., solvability in terms of liouvillian functions or in terms of linear differential equations of lower order). In [12], Kovacic gave an algorithm to decide if a second order homogeneous linear differential equation has liouvillian solutions. In the process of doing this, Kovacic derived a very strong set of necessary conditions for the existence of such solutions. In this paper, we show how one can derive these conditions from general

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* A weaker version of these results were announced in *Liouvillian Solutions of Third Order Linear Differential Equations: New Bounds and Necessary Conditions*, Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation, ACM Press

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group theoretic properties. We also derive similar conditions for the third order case.

We shall assume that the reader is familiar with the elementary concepts of differential Galois theory and elementary properties of linear operators. The necessary facts are reviewed in [20]. We shall need results from [21], but these will be stated in full for the convenience of the reader.

The rest of the paper is organized as follows. In section 2, we give some elementary results concerning exponents and monodromy groups. Section 3 contains the necessary conditions. The last section deals with linear differential equations having 3 singular points and examples.

We wish to thank the referees for pointing out a mistake in the original proof of Theorem 3.5 and for suggestions that allowed us to sharpen our original version of the necessary conditions for case 2.

2. Exponents and Monodromy Groups

In this section we review some facts about formal and analytic solutions of linear differential equations. There is a well developed general theory of formal solutions at any point (of both systems and single equations) as well as an asymptotic theory (cf. [1, 5, 14]). We have restricted ourselves to the most basic facts in order to give an elementary and direct exposition of the results that we need.

In what follows we will frequently wish to consider solutions of a linear differential equation that can be expressed as a series in (not necessarily integral) powers of x. This motivates the following discussion.

Let

$$L(y) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$$

be a linear differential equation with coefficients in the field $\mathbb{C}((x))$ of formal Laurent series and let $a_i(x) = \sum_{j \geqq i_0} a_{i,j}x^j$ where $a_{i,i_0} \neq 0$. If $L(y) = 0$ has a solution of the form $y = x^\rho \sum_{j \geqq 0} c_j x^j$, $c_0 \neq 0$, then formally substituting this expression into $L(y) = 0$ and examining the coefficient of the smallest power of x, one sees that ρ will satisfy the equation

$$P(\rho) = \sum a_{k,k_0}(\rho)_k = 0$$

where $(\rho)_j = \rho(\rho - 1)\cdots(\rho - (j-1))$ and the sum is over all k with $k_0 - k = \min_{0 \leqq i \leqq n}\{i_0 - i\}$. $P(\rho)$ is called the *indicial equation of* $L(y)$ (*at* 0). We shall refer to the roots of the indicial polynomial as *exponents of* $L(y)$ (*at* 0).

If $L(y)$ has coefficients in $\mathbb{C}(x)$, then for any $c \in \mathbb{C}$ we can expand these coefficients in powers of $(x - c)$. Analogously, we can define the indicial equation at c and exponents at c. Via the transformation $x = 1/t$, $\frac{d}{dx} = -t^2\frac{d}{dt}$ we can also define the indicial equation and exponents at infinity by considering the point $t = 0$ of the transformed equation. We say that $x = c$ is an *ordinary point* if, for $0 \leqq i \leqq n$, $a_i(x)/a_n(x)$ is analytic at $x = c$ (for $c = \infty$ this property is related to the point $t = 0$ of the transformed equation). The following lemma contains facts that are easily verified by computation (cf. [5], Ch. 4, 5; [14], Ch. IV, V).

Lemma 2.1. *Let* $L(y) = 0$ *have coefficients in* $\mathbb{C}(x)$ *and let* $c \in \mathbb{C}$.

1. If $L(y)=0$ has a formal solution of the form

$$y=(x-c)^{\rho}\sum_{i\geqq 0} c_i(x-c)^{r_i},\quad r_i\in\mathbb{R},\quad 0=r_0<r_1<\ldots,c_0\neq 0,$$

then ρ is an exponent of $L(y)$ at c.

2. The degree of $P(\rho)$ is at most n.

3. If c is an ordinary point then the exponents at c are $\{0,\ldots,n-1\}$.

The converse of part 1 of the above lemma can be false in the case when log terms occur in formal solutions. For example, $L(y)=x^2y''-(3-2x)xy'+(3-6x)y$ has exponents $\{1, 3\}$ at 0 and at this point it has a fundamental set of solutions of the form $\{x^3, -4x^3\ln x+x\phi(x)\}$, where $\phi(x)$ is a power series in x with $\phi(0)\neq 0$. No linear combination of these two solutions can be of the form $x\psi(x)$ where $\psi(x)$ is a power series with $\psi(0)\neq 0$.

If a point c is not an ordinary point, it is called a *singular point*. If, in addition, the indicial polynomial at this point has degree n, we say that the point is a *regular singular point*. An obviously equivalent way of saving this is that the order of the pole of $a_{n-i}(x)/a_n(x)$ at c is at most i. A linear differential equation with only regular singular points (including ∞) is called a *fuchsian linear differential equation.* We gather some facts about regular singular points in the following lemma, but first give the following definition:

Definition 2.1. *Let $\{y_1,\ldots,y_n\}$ be a fundamental set of solutions of the linear differential equation $L(y)=0$. The m^{th} symmetric power $L^{Ⓢm}(y)$ of $L(y)$ is the differential equation of smallest order, such that the solution space of $L^{Ⓢm}(y)=0$ is spanned by $\{y_1^m, y_1^{m-1}y_2,\ldots,y_n^m\}$.*

Note that this definition is independent of the choice of fundamental solutions and that $L^{Ⓢm}(y)$ is the linear differential equation of smallest order satisfied by all homogeneous forms $F(y_1,\ldots,y_n)$ of degree m with constant coefficients in solutions $y_1,\ldots,y_n$ of $L(y)=0$. See [17] for a further discussion of this concept. An algorithm to construct the equation $L^{Ⓢm}(y)$ is given in [17] and [20], Section 3.2.2.

Lemma 2.2. *Let $L(y)=0$ be of order n and have coefficients in $\mathbb{C}(x)$ and let $c\in\mathbb{C}$.*

1. If c is a regular singular point and ρ is a root of multiplicity r of the indicial equation, then $L(y)=0$ has r independent solutions of the form

$$y_i=(x-c)^{\rho}\left(\sum_{j=0}^{i}\phi_{i,j}(x)(\log(x-c))^j\right)\tag{1}$$

where $i=0,\ldots,r-1$, the $\phi_{i,j}(x)$ are analytic at c and for each i there is some j such that $\phi_{i,j}(c)\neq 0$. Furthermore, if $\rho+N$ is not an exponent of $L(y)=0$ for any positive $N\in\mathbb{Z}$, then $L(y)=0$ has a solution of the form $(x-c)^{\rho}\varphi(x)$, where *$\varphi(x)$ is analytic at c and $\varphi(c)\neq 0$.*

2. c is a regular singular point if and only if in any open angular sector Ω at c, all solutions of $L(y)=0$ analytic in Ω, satisfy $\lim_{x\to c}(x-c)^N y=0$ for some $N\geqq 0$.

3. If all solutions of $L(y)=0$ are algebraic over $\mathbb{C}(x)$, then $L(y)=0$ is a fuchsian equation and at any singular point c there are n distinct *rational exponents. Furthermore, at any point there is a fundamental set of solutions of the form $y_i=(x-c)^{a_i}\phi_i(x)$ where $\phi_i(x)$ is analytic at c, $\phi_i(c)\neq 0$, and $a_1,\ldots,a_n$ are the exponents.*

4. *Let $L(y)=0$ be a fuchsian equation and $a_1,\ldots,a_n$ exponents at some point c. Then the m^{th} symmetric power $L^{\circledS m}(y)$ is fuchsian and the exponents at c are in the set $\{(\sum_{j=1}^{m} a_{i_j})+t \mid i_j\in\{1,\ldots,n\};\ t\in\mathbb{Z};\ t\geqq 0\}$.*

Proof. 1 and 2 are proved in [5], Chap. 5, Sects. 1–5 and [14] Chap. V, Sects. 16, 17. Lemma 2.2.2 is usually referred to as Fuchs' Criterion (cf. [5], p. 124 and [14], pp. 65–68). Lemma 2.2.2 implies that if all solutions are algebraic, the equation must be fuchsian. To prove 3., let $c=0$ and let $y_1,\ldots,y_n$ be solutions of $L(y)=0$ as described in 1. Since these are algebraic cover $\mathbb{C}(x)$, they are also algebraic over $\mathbb{C}((x))$. Let k be the differential field $\mathbb{C}((x))(x^{\rho_1},\ldots,x^{\rho_n})$, where $\rho_1,\ldots,\rho_n$ are the exponents at 0. We claim that for each solution y_i of the above form, we must have $\phi_{i,j}=0$ for $j=1,\ldots,i$. If not, then $\log x$ would be algebraic over k. The Kolchin–Ostrowski Theorem ([11]) implies that $\log x$ would be in $\mathbb{C}((x))$. Since there is no element y in $\mathbb{C}((x))$ such that $y'/y=1/x$, we have a contradiction. Therefore each y_i is of the form $x^{\rho_i}\phi_i(x)$, for the $\phi_i(x)$, $\phi_i(0)\neq 0$ in $\mathbb{C}((x))$. Since each y_i is algebraic over $\mathbb{C}((x))$, we have that each ρ_i is rational. We now claim that all the exponents of $L(y)$ are distinct. Consider the set of bases $B=(y_1,\ldots,y_n)$ where each $y_i=x^{\rho_i}\phi_i(x)$, $\phi_i(0)\neq 0$ where $\rho_1\leqq\cdots\leqq\rho_n$ and define an order on this set by letting $B<\tilde{B}$ if $(\rho_1,\ldots,\rho_n)<(\tilde{\rho}_1,\ldots,\tilde{\rho}_n)$ in the lexicographical order. Let $\hat{B}$ be a maximal element of this set, then for this set the $\hat{\rho}_i$ are distinct. This will show that there are n distinct exponents. Assume not and let j be the smallest index so that all $\hat{\rho}_i$ with $i>j$ are distinct. We then have that $\hat{\rho}_j=\hat{\rho}_{j+1}$. Therefore, there is a constant d such that $z=y_j-dy_{j+1}=x^{\rho}\psi(x)$ for some $\rho>\hat{\rho}_j$. Replacing y_{j+1} by z gives a new basis which (after we perhaps rearrange the elements) is larger in the ordering. This contradicts the maximality of $\hat{B}$.

To prove 4, consider a basis of the solution space of $L(y)=0$ of the form $y_i=(x-c)^{a_i}(\sum_{j=0}^{k_i}\phi_{i,j}(x)(\log(x-c))^j)$ where the $\phi_{i,j}(x)$ are analytic at c, some $\phi_{i,j}(c)\neq 0$, and $a_1,\ldots,a_n$ are the exponents of $L(y)=0$ at c. Note that $L^{\circledS m}(y)$ has a basis of solutions of the form $y_1^{i_1}y_2^{i_2}\cdots y_n^{i_n}$, where $i_1+\cdots+i_n=m$. The second assertion of the Lemma implies that c is at worst a regular singular point of $L^{\circledS m}(y)$. Let ρ be an exponent of $L^{\circledS m}(y)$ at c. From 1 we get that $L^{\circledS m}(y)=0$ has a solution of the form $y=(x-c)^{\rho}(\sum_{r=0}^{h}\psi_r(x)(\log(x-c))^r)$, where $\psi_r(x)$ is analytic at c, and some $\psi_r(c)\neq 0$. Therefore there exist constants c_I such that

$$\begin{aligned} y &= \sum_{I=(i_1,\ldots,i_n)} c_I y_1^{i_1}\cdot\cdots\cdot y_n^{i_n} \\ &= \sum_{I=(i_1,\ldots,i_n)} c_I (x-c)^{a_1 i_1+\cdots+a_n i_n}\left(\sum_{j=0}^{k}\varphi_{I,j}(x)(\log(x-c))^j\right) \end{aligned}$$

where the sum is over all $I=(i_1,\ldots,i_n)$ with $i_1+\cdots+i_n=m$, the $\varphi_{I,j}(x)$ are analytic at c and for some (I,j), $\varphi_{I,j}(c)\neq 0$. Since the $\varphi_{I,j}(x)$ contain only nonnegative powers of x, comparing the lowest power of $\log(x)$ whose coefficient is not zero and taking into account possible cancelation, yields the result. ■

For non fuchsian equations very little can be said regarding the relationship between the exponents of a linear differential equation and those of its symmetric power. The following example shows that one can have a linear differential equation with no exponents at 0 while the symmetric square has an exponent at 0. Let

$$L(y)=y''-\frac{2}{x(x-2)}y'+\frac{2x^2-3x+2}{x^4(x-2)}y.$$

This equation has a basis of solutions $\{x\exp(1/x), \exp(-1/x)\}$ and no exponents at 0. The second symmetric power is

$$y''' - \frac{6}{x(x-2)}y'' + \frac{4(3x^3-6x^2+8x-4)}{x^4(x-2)^2}y' - \frac{4(3x^3-6x^2+8x-4)}{x^5(x-2)^2}y$$

and this has 1 as an exponent since x is a solution.

We do note that the theory of formal solutions at a point does allow one to give apriori bounds on the exponents of the symmetric power even in the non-fuchsian case (cf. [5], Chap. 5, Sect. 2 and [1, 19]). This involves calculating formal exponents, a task that is more difficult than calculating exponents defined above. Since this will not be needed, we have stated Lemma 2.2.4 just for the fuchsian case.

Consider the class of linear differential equation with coefficients in $\mathbb{C}(x)$. For these equations we can use analytic considerations to define a group called the *monodromy group* that is a subgroup of the differential Galois group $\mathcal{G}(L)$ of $L(y)=0$ over $\mathbb{C}(x)$ (see e.g. [10, 20]). Let $c_1, \ldots, c_m$ can be singular points of $L(y)=0$ (including infinity if it is a singular point) and let c_0 be an ordinary point of the equation. We consider these points as lying on the Riemann Sphere S^2. Let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of $L(y)=0$ analytic at c_0 and let γ be a closed path in $S^2 - \{c_1, \ldots, c_m\}$ that begins and ends at c_0. One can analytically continue $\{y_1, \ldots, y_n\}$ along γ and get new fundamental solutions $\{\tilde{y}_1, \ldots, \tilde{y}_n\}$ analytic at c_0. These two sets must be related via $(\tilde{y}_1, \ldots, \tilde{y}_n)^T = M_\gamma(y_1, \ldots, y_n)^T$ where $M_\gamma \in GL(n, \mathbb{C})$. One can show that M_γ depends only on the homotopy class of γ and that the map $\gamma \mapsto M_\gamma$ defines a group homomorphism from $\pi_1(S^2 - \{c_1, \ldots, c_m\})$ to $GL(n, \mathbb{C})$. The image of this map depends on the choice of c_0 and $\{y_1, \ldots, y_n\}$ but is unique up to conjugacy and is called the *monodromy group of* $L(y)$. In general the image of this group will be a proper subgroup of $\mathcal{G}(L)$, but when $L(y)$ is fuchsian, the Zariski closure of this group will be the full differential Galois group $\mathcal{G}(L)$ (cf. [23]). In particular if $\mathcal{G}(L)$ is finite (i.e., all solutions of $L(y)=0$ are algebraic) then the map is surjective and the monodromy and Galois groups coincide. The fact that the monodromy group is a subgroup of the differential Galois group allows us to prove the following:

Lemma 2.3. *Let $L(y)=0$ be a linear differential equation with coefficients in $\mathbb{C}(x)$, assume all solutions of $L(y)=0$ are algebraic over $\mathbb{C}(x)$. If a is an exponent at any point c, then $Na \in \mathbb{Z}$ for some integer N that is the order of an element of $\mathcal{G}(L)$.*

Proof. Lemma 2.2 implies that there is a fundamental set of solutions of the form $y_i = (x-c)^{a_i}\phi_i(x)$ where $\phi_i(x)$ is analytic at $c, \phi_i(c) \neq 0$, and $a_1, \ldots, a_n$ are the exponents. If we continue analytically along a small loop around c, we get the solutions $\tilde{y}_i = e^{(2\pi\sqrt{-1})\alpha_i}(x-c)^{a_i}\phi_i(x)$. Therefore the diagonal matrix $g = \operatorname{daig}(e^{(2\pi\sqrt{-1})a_1}, \ldots, e^{(2\pi\sqrt{-1})a_n})$ is conjugate to an element of the differential Galois group. Since $g^N = 1$, we have $Na_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. ∎

The relationship between the exponents of $L(y)$ at c and the eigenvalues of the monodromy matrix M_γ corresponding to a simple loop γ around c, containing no other singular point, is given by the following:

Lemma 2.4. *Let $L(y)=0$ be a linear differential equation with coefficients in $\mathbb{C}(x)$, c a regular singular point of $L(y)=0$ and $a_1, \ldots, a_n$ the exponents at c. The eigenvalues of the monodromy matrix M_γ, corresponding to a loop γ around c are $e^{2\pi\sqrt{-1}a_1}, \ldots, e^{2\pi\sqrt{-1}a_n}$.*

Proof. For convenience let $c = 0$. Separate the exponents into different sets $S_1, \ldots, S_r$, such that the elements of each set differ by integers, and elements of different sets differ by non-integers. Let $m_j = |S_j|$ and $n = m_1 + \cdots + m_r$. For each set S_j, we denote by a_j an element with the property that for any integer $m \geqq 1$, $a_j + m \notin S_j$. From Lemma 2.2 we get that there exists a solution of the form $x^{a_j}\phi_j(x)$, where $\phi_j(x)$ is analytic at 0 and $\phi_j(0) \neq 0$. Therefore $\lambda_j = e^{2\pi\sqrt{-1}a_j}$ is an eigenvalue of the monodromy matrix. We claim that each λ_j is an eigenvalue of multiplicity at least m_j. Since $n = m_1 + \cdots + m_r$, this will imply that λ_j is of multiplicity m_j. Since for any $a \in S_j$, $\lambda_j = e^{2\pi\sqrt{-1}a}$, we will have established the conclusion of the Lemma. To prove the claim, let $a_1, \ldots, a_t$ be the *distinct* elements of S_j and assume that $a_i \in S_j$ is a root of multiplicity s_i of the indicial polynomial. For each a_i there are s_i independent solutions of the form (1). If $y = x^{a_i}(\sum_{j=0}^{h} \phi_j(x)(\log(x))^j)$ with $0 \leqq h < s_i$ is one of these solutions, then

$$
\begin{aligned}
(M_\gamma - e^{2\pi\sqrt{-1}a_i} Id)(y) &= e^{2\pi\sqrt{-1}a_i} x^{a_i} \left(\sum_{j=0}^{h} \phi_j(x)(\log(x) + 2\pi\sqrt{-1})^j \right) \\
&\quad - e^{2\pi\sqrt{-1}a_i} x^{a_i} \left(\sum_{j=0}^{h} \phi_j(x)(\log(x))^j \right) \\
&= x^{a_i} \left(\sum_{j=0}^{h-1} \psi_j(x)(\log(x))^j \right),
\end{aligned}
$$

where the $\psi_j(x)$ are analytic at 0. This shows that $(M_\gamma - e^{2\pi\sqrt{-1}a_i} Id)^{s_i - 1}(y) = x^{a_i}\psi(x)$, where $\psi(x)$ is analytic at 0, and thus we get $(M_\gamma - e^{2\pi\sqrt{-1}a_i} Id)^{s_i}(y) = 0$. Note that for distinct a_i we get independent solutions of the form (1). Therefore the generalized eigenspace corresponding to $\lambda_j = e^{2\pi\sqrt{-1}a_j}$ has dimension at least $s_1 + \cdots + s_t = m_j$. ∎

3. Necessary Conditions

In this section we show how the results of [21] can be used to derive strong necessary conditions for a third order linear differential equation with coefficients in $\mathbb{C}(x)$ to have a solution liouvillian over $\mathbb{C}(x)$ (see e.g. [10, 11, 12, 18, 20, 21] for definitions). We will also derive the necessary conditions of the Kovacic algorithm using our approach, and improve them.

In practice, one is given a linear differential equation $L(y) = 0$ whose coefficients lie in $F(x)$, where F is a finitely generated extension of $\mathbb{Q}$. In order to do our calculations, we need to assume that one has algorithms to perform the field operations and to factor polynomials over F. When this is the case, it is known that the same is true for the algebraic closure $\bar{F}$ if F ([7]). Although we give necessary conditions for certain behavior over $\mathbb{C}(x)$, it is clear from our procedures, that all calculations can be done in $\bar{F}(x)$. We have not addressed the issue of finding the most efficient way to calculate in this field. In particular, we assume that all polynomials over the field F can be factored into linear factors whenever needed (for example, to consider individually each singular point of the differential equation). Reducing the need to factor (and work in algebraic extension) would increase the efficiency of our procedures. Techniques for attacking this problem are discussed in [2, 3].

We shall continue using the philosophy of [20] and [21], that is, to distinguish 3 different cases (for second order differential equations these correspond to the first 3 cases of the Kovacic algorithm). If a third order equation $L(y) = 0$ has a liouvillian solution, then one of the following holds (cf. [20]):

1. The differential Galois group is a reducible subgroup of $SL(3, \mathbb{C})$ (cf. case 1 in the Kovacic algorithm [12]). In this case the computation of the liouvillian solutions of $L(y)$ can be reduced by factorisation to the computation of the liouvillian solutions of some second order linear differential equation over $\mathbb{C}(x)$ (cf. [21], Sect. 2).

2. The differential Galois group is an imprimitive subgroup of $SL(3, \mathbb{C})$ (cf. case 2 in the Kovacic algorithm). In this case all solutions of the third order equation $L(y) = 0$ are Liouvillian, and $L(y) = 0$ has a fundamental system of solutions $\{y_1, y_2, y_3\}$ whose logarithmic derivatives y_i'/y_i are algebraic of degree 3 over $\mathbb{C}(x)$ (cf. [21], Sect. 3).

3. The differential Galois group is a finite primitive subgroup of $SL(3, \mathbb{C})$ (cf. case 3 in the Kovacic algorithm). In this case all solutions of $L(y) = 0$ are algebraic over $\mathbb{C}(x)$ (cf. [21], Sect. 4).

If none of the above cases holds, then the differential Galois group is an infinite primitive subgroup of $SL(3, \mathbb{C})$ and $L(y) = 0$ has no liouvillian solutions. Case 1 must be tested first, since this case is assumed to not hold in the other case. (Note that the above classification can be used for differential equations of arbitrary prime order.)

In the following, we will assume that the differential Galois group of $L(y) = 0$ is unimodular (i.e. $\mathcal{G}(L) \subseteq SL(3, \mathbb{C})$), or equivalently (cf. [10], Chap. VI, Sect. 24) that $L(y) = 0$ is of the form

$$L(y) = y''' + Ay'' + By' + Cy = 0, \quad A, B, C \in \mathbb{C}(x),$$

where A is of the form $\sum_i \frac{n_i}{x - a_i}$, with $n_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{C}$. If $\mathcal{G}(L)$ is not unimodular, we can always transform the equation by a change of variable to an equation where $A = 0$ (cf. [10], Chap. VI, Sect. 24). The new differential equation has liouvillian solutions if and only if the original differential equation does.

3.1. Reducible Case

In this section we give conditions on the coefficients of $L(y) = 0$ for this case to occur.

3.1.1. *First step.* We start by showing that for a given differential equation $L(y) = 0$ of arbitrary order n and coefficients in $\mathbb{C}(x)$, all solutions of the form $P(x)\prod_i(x - c_i)^{a_i}$, where $P(x) \in \mathbb{C}[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i, can be computed by only finding polynomial solutions of some n-th order linear differential equations with coefficients in $\mathbb{C}(x)$.

We first note the following necessary condition:

Lemma 3.1. (cf. [16], Sect. 178) *If a linear differential equation $L(y) = 0$ of degree n has a solution of the form $P(x)\prod_i(x - c_i)^{a_i}$, where $P(x) \in \mathbb{C}[x]$, $c_i \neq \infty$ are the singular points of $L(y) = 0$, a_i are exponents at c_i and $P(x)$ is a polynomial, then there is an* *exponent e_∞ at ∞ such that the sum $(\sum_i a_i) + e_\infty$ is a non-positive integer. In particular, if $L(y)=0$ has no singularities other than ∞ and has such a solution, then it must have a non-positive integer exponent at ∞.*

Proof. Each a_i is an exponent at the singular point c_i. Expanding $P(x)\prod_i (x-c_i)^{a_i}$ in increasing powers of x^{-1}, we see that $-deg(P) - \sum_i a_i$ is the exponent of the leading term. Therefore this must be an exponent e_∞ at infinity. ■

Lemma 3.2. *Let $c_i, a_i \in \mathbb{C}$ and y an m-times differentiable function, then*

$$\left(y\prod_i (x-c_i)^{a_i}\right)^{(m)} = \left(\sum_{j=0}^{m} (q_j(x) y^{(j)})\right)\prod_i (x-c_i)^{a_i},$$

where $q_j(x) \in \mathbb{C}(x)$.

Proof. The result is true for $m=0$. For $1 \leqq m$ we get:

$$\left(y\prod_i (x-c_i)^{a_i}\right)^{(m)} = \left(y'\prod_i (x-c_i)^{a_i}\right)^{(m-1)} + \left(y\sum_i \left(a_i (x-c_i)^{a_i-1}\prod_{j\neq i}(x-c_j)^{a_j}\right)\right)^{(m-1)}$$

$$= \left(y'\prod_i (x-c_i)^{a_i}\right)^{(m-1)} + \left(\left(y\sum_i \frac{a_i}{x-c_i}\right)\prod_i (x-c_i)^{a_i}\right)^{(m-1)}.$$

Now the result follows by induction by using y' and $y\sum_i \frac{a_i}{x-c_i}$ instead of y. ■

For a given equation $L(y)=0$ of degree n, there are only a finite number of singularities and at each singularity c_i, there are at most n possible exponents a_i. If $L(y)=0$ has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y)=0$, $P(x) \in \mathbb{C}(x)$ and a_i are exponents of $L(y)=0$ at c_i, then there are at most a finite number of possibilities for $\prod_i (x-c_i)^{a_i}$. For each possible term $\prod_i (x-c_i)^{a_i}$, we consider the differential equation:

$$\tilde{L}(y) := \frac{L(y\prod_i (x-c_i)^{a_i})}{\prod_i (x-c_i)^{a_i}} = 0.$$

From Lemma 3.2 we get that the coefficients of $\tilde{L}(y)$ belong to $\mathbb{C}(x)$. If $L(y)=0$ has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, then $\tilde{L}(y)=0$ has a solution $P(x) \in \mathbb{C}[x]$. We thus have to compute a basis $\{p_1(x), \ldots, p_k(x)\}$ $(k \leqq n)$ of the polynomial solutions of $\tilde{L}(y)=0$ to get a basis of the solutions of the form $P(x)\prod_i (x-c_i)^{a_i}$.

3.1.2. *Necessary Conditions for Case 1.* If the differential Galois group of a differential equation $L(y)=0$ of order n is a reducible subgroup of $GL(n, \mathbb{C})$, then $L(y)$ factors as a differential operator and algorithms performing such a factorisation are known (cf. [20], Sect. 3.2.1). For a third order equation, this leads either to a right factor of order one or to a left factor of order one. Since a left factor of order one leads to a right factor of order one of the adjoint differential operator $L^*(y)$ (cf. [14], p. 38), testing reducibility leads to the computation of a right factor of order one of $L(y)$ or $L^*(y)$. If an equation has a right factor of order one, then it has a solution y whose logarithmic derivative $u = y'/y$ is rational. If $L(y)=0$ is of fuchsian type, then any solution whose logarithmic derivative is rational must be of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y)=0$ and a_i are exponents of $L(y)=0$ at c_i (cf. [16], Sect. 178). Using Lemma 3.1 and the fact that the adjoint of a differential equation of fuchsian type is also of fuchsian type (This follows from the facts that the solutions of $L^*(y)=0$ are contained in a Picard–Vessiot extension K associated with $L(y)=0$ (cf. [5], p. 101, Ex. 19; [14], p. 43, Ex. 12 and p. 38) and any element of K must have the growth properties described in Lemma 2.2.2) we get:

Corollary 3.3. *Let $L(y)=0$ be a third order differential equation which is of fuchsian type. If $L(y)=0$ is reducible, then for either $L(y)=0$ or $L^*(y)=0$ at each finite singular point c_i there are exponents a_i such that for some exponent e_∞ at ∞, the sum $(\sum_i a_i)+e_\infty$ is a non-positive integer.*

We now consider the non fuchsian case. The riccati equation associated with $L(y)=y'''+Ay''+\mathrm{B}y'+Cy=0$ is

$$R(u)=u''+3uu'+Au'+u^3+Au^2+Bu+C=0.$$

and the adjoint $L^*(y)=y'''-Ay''+(B-2A')y'+(-C+B'-A'')y$ of $L(y)$ has riccati equation:

$$R^*(u)=u''+3uu'-Au'+u^3-Au^2+(B-2A')u+(-C+B'-A'')=0.$$

In our computations we will usually assume that the finite singularity is at 0. For the Laurent series at 0 or ∞ we introduce the following notation:

$$\begin{aligned} A&=\alpha x^a+\cdots \quad \text{(higher order terms)}\\ B&=\beta x^b+\cdots \quad \text{(higher order terms)}\\ C&=\gamma x^c+\cdots \quad \text{(higher order terms)}\\ A&=\alpha_\infty x^{a_\infty}+\cdots \quad \text{(lower order terms)}\\ B&=\beta_\infty x^{b_\infty}+\cdots \quad \text{(lower order terms)}\\ C&=\gamma_\infty x^{c_\infty}+\cdots \quad \text{(lower order terms)}\end{aligned}$$

For $A=A_1/A_2$ where $A_i\in\mathbb{C}[x]$, a_∞ denotes $deg_x(A_1)-deg_x(A_2)$.

If a Puiseux or Laurent series of a solution y of $L(y)=0$ or a solution u of $R(u)=0$ exists at the point 0 or ∞, we denote them by:

$$\begin{aligned} y&=\rho x^r+\cdots \quad \text{(higher order terms)}\\ u&=\eta x^h+\cdots \quad \text{(higher order terms)}\\ y&=\rho_\infty x^{r_\infty}+\cdots \quad \text{(lower order terms)}\\ u&=\eta_\infty x^{h_\infty}+\cdots \quad \text{(lower order terms)}\end{aligned}$$

Necessary conditions for case 1. *Let $L(y)=y'''+Ay''+By'+Cy$ be a third order linear differential equation with coefficients in $\mathbb{C}(x)$ such that $L(y)=0$ and the adjoint $L^*(y)=0$ have no solutions of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x)\in\mathbb{C}[x]$, $c_i\neq\infty$ are singular points of $L(y)=0$ and a_i are exponents of $L(y)=0$ at c_i. If $L(y)$ is reducible over $\mathbb{C}(x)$, then one of the following holds:*

1. $C=0$ *and*

$$L(y)=\left(\frac{d^2}{dx^2}+A\frac{d}{dx}+B\right)\left(\left(\frac{d}{dx}\right)(y)\right)$$

2. $C\neq 0$ *and at some finite singular point of $L(y)=0$ the coefficients A, B and C have exponents a, b and c such that one of the following holds:*

(a) $A\neq 0$, $a\leqq -2$, $3a\leqq c$ *and (if* $B\neq 0$ *then* $2a\leqq b$*).*

(*b*) $B \neq 0$, $b \leqq -4$, $b \in 2\mathbb{Z}$, $b \leqq \frac{2}{3}c$ *and* (*if* $A \neq 0$ *then* $b \leqq 2a$).
(*c*) $c \leqq -6$, $c \in 3\mathbb{Z}$, (*if* $A \neq 0$ *then* $c \leqq 3a$) *and* (*if* $B \neq 0$ *then* $c \leqq \frac{3}{2}b$).
(*d*) $AB \neq 0$, $a < -2$, $b < -4$, $b - a < -1$, $2a < b$ *and* $2b - a \leqq c$.
(*e*) $A \neq 0$, $a < -2$, $c < -6$, $c - a < -4$, $(c - a) \in 2\mathbb{Z}$, $3a < c$ *and* (*if* $B \neq 0$ *then* $a + c \leqq 2b$).
(*f*) $B \neq 0$, $c < -6$, $b < -4$, $c \leqq b - 2$, $3b < 2c$ *and* (*if* $A \neq 0$ *then* $2b < a + c$).

3. $C \neq 0$ *and for* A, B *and* C, *one of the following holds*:
(*a*) $A \neq 0$, $a_\infty \geqq 0$, $3a_\infty \geqq c_\infty$ *and* (*if* $B \neq 0$ *then* $2a_\infty \geqq b_\infty$).
(*b*) $B \neq 0$, $0 \leqq b_\infty$, $b_\infty \in 2\mathbb{Z}$, $b_\infty \geqq \frac{2}{3}c_\infty$ *and* (*if* $A \neq 0$ *then* $b_\infty \geqq 2a$).
(*c*) $0 \leqq c_\infty \in 3\mathbb{Z}$, (*if* $A \neq 0$ *then* $c_\infty \geqq 3a_\infty$) *and* (*if* $B \neq 0$ *then* $c_\infty \geqq \frac{3}{2}b_\infty$).
(*d*) $AB \neq 0$, $0 < a_\infty$, $a_\infty \leqq b_\infty$ *and* $2b_\infty - a_\infty \geqq c_\infty$.
(*e*) $A \neq 0, 0 \leqq c_\infty - a_\infty, c_\infty - a_\infty \in 2\mathbb{Z}, 0 < c_\infty < 3a_\infty$ *and* (*if* $B \neq 0$ *then* $a_\infty + c_\infty \geqq 2b_\infty$).
(*f*) $B \neq 0$, $0 < b_\infty \leqq c_\infty$, $3b_\infty > 2c_\infty$ *and* (*if* $A \neq 0$ *then* $2b_\infty > a_\infty + c_\infty$).

Example: We use the above necessary conditions to show that for the following equation (which is the second symmetric power of the Airy equation)

$$L(y) = \frac{d^3y}{dx^3} - 4x\frac{dy}{dx} - 2y,$$

case 1 (of a reducible group $\mathscr{G}(L) \subseteq SL(3,\mathbb{C})$) cannot occur. The point ∞ is the only singular point of $L(y)$, and the only exponent at ∞ is 1/2. From Lemma 3.1 we get that $L(y) = 0$ has no solution of the form $P(x)\prod_i(x - c_i)a^i$. The first condition above cannot hold, since the coefficient of y is not 0. Since $L(y) = 0$ has no finite singular point, the above second condition does not hold. Finally, for $A = 0$, $b_\infty = 1$ and $c_\infty = 0$ the third condition above does not hold. Thus, since $L(y)$ is selfadjoint, $L(y) = 0$ is an irreducible equation. ■

Proof of the Necessary conditions for case 1: We first note that if $C = 0$, then $L(y) = 0$ has a right factor of order one of the form stated. We assume from now on that $C \neq 0$. If $L(y) = 0$ is a reducible third order differential equation, then either $L(y)$ or $L^*(y)$ has a right factor of order 1 ([20]).

1. First assume $L(y) = L_1(L_2(y))$, where $L_2(y)$ is of order 1, then $R(u)$ has a solution $u \in \mathbb{C}(x)$.
   (a) If u has a pole of order bigger than 1 then u must have a pole of order > 1 at some singularity of $L(y) = 0$ which we assume to be 0. We have $u = \eta x^h + \cdots$ (higher order terms) and $h < -1$. Plugging u into $R(u) = 0$ we get:

$$(\eta h(h-1)x^{h-2} + \cdots) + (3\eta^2 h x^{2h-1} + \cdots) + (\eta\alpha h x^{a+h-1} + \cdots)$$
$$+ (\eta^3 x^{3h} + \cdots) + (\eta^2 \alpha x^{2h+a} + \cdots) + (\beta\eta x^{b+h} + \cdots) + (\gamma x^c + \cdots) = 0$$

   Since $h < -1$ we get $h - 2 > 3h$, $2h - 1 > 3h$ and $a + h - 1 > a + 2h$. For the lowest term to cancel one of the following must hold:
      i. If $3h$ is the lowest exponent, then:
         A. if $3h = 2h + a$, in which case $A \neq 0$, then $a \leqq -2$, $3a \leqq c$ and (if $B \neq 0$ then $2a \leqq b$).
         B. if $3h = h + b$, in which case $B \neq 0$, then $b \leqq -4$, $b \in 2\mathbb{Z}$, $b \leqq \frac{2}{3}c$, and (if $A \neq 0$ then $b \leqq 2a$).
         C. if $3h = c$, then $c \leqq -6$, $c \in 3\mathbb{Z}$, (if $A \neq 0$ then $c \leqq 3a$) and (if $B \neq 0$ then $c \leqq \frac{3}{2}b$).

ii. If $a+2h$ is the lowest exponent but not $3h$, then:
   A. if $a+2h=h+b$, in which case $AB\neq 0$, then $a<-2$, $b<-4$, $b-a<-1$, $2a<b$ and $2b-a\leqq c$.
   B. if $a+2h=c$, in which case $A\neq 0$, then $a<-2$, $c<-6$, $c-a<-4$, $(c-a)\in 2\mathbb{Z}$, $3a<c$ and (if $B\neq 0$ then $a+c\leqq 2b$).
iii. If $a+2h$ and $3h$ are not the lowest exponents, then $B\neq 0$, $c<-6$, $b<-4$, $c\leqq b-2$, $3b<2c$ and (if $A\neq 0$ then $2b<a+c$).

(b) If u has no pole of order bigger than 1, then u is of the form

$$u=\sum\frac{\gamma_i}{x-c_i}+Q(x),$$

where $Q(x)\in\mathbb{C}[x]$. If $Q(x)=0$, then $L(y)=0$ has a solution of the form $P(x)\prod_i(x-c_i)^{\gamma_i}$, where $P(x)\in\mathbb{C}[x]$, $c_i\neq\infty$ are singular points of $L(y)=0$ and γ_i are exponents of $L(y)=0$ at c_i. Since we assume that this is not the case, we get $h_\infty\geqq 0$. Plugging u into $R(u)=0$ we get an equation similar to the one in 1(a) above (the only difference is that we have added the subscript ∞).

Since $h_\infty\geqq 0$ we get $h_\infty-2<3h_\infty$, $2h_\infty-1<3h_\infty$, $a_\infty+h_\infty-1<a_\infty+2h_\infty$. For the highest term to cancel one of the following must hold:

i. If $3h_\infty$ is the highest exponent, then:
   A. if $3h_\infty=a_\infty+2h_\infty$, in which case $A\neq 0$, then $a_\infty\geqq 0$, $3a_\infty\geqq c_\infty$ and (if $B\neq 0$ then $2a_\infty\geqq b_\infty$).
   B. if $3h_\infty=b_\infty+h_\infty$, in which case $B\neq 0$, then $0\leqq b_\infty$, $b_\infty\in 2\mathbb{Z}$, $b_\infty\geqq\frac{2}{3}c_\infty$ and (if $A\neq 0$ then $b_\infty\geqq 2a_\infty$).
   C. if $3h_\infty=c_\infty$, then $0\leqq c_\infty\in 3\mathbb{Z}$, (if $A\neq 0$ then $c_\infty\geqq 3a_\infty$) and (if $B\neq 0$ then $c_\infty\geqq\frac{3}{2}b_\infty$).
ii. If $a_\infty+2h_\infty$ is the highest exponent but not $3h_\infty$, then:
   A. if $a_\infty+2h_\infty=b_\infty+h_\infty$, in which case $AB\neq 0$, then $0<a_\infty$, $a_\infty\leqq b_\infty$ and $2b_\infty-a_\infty\geqq c_\infty$.
   B. if $a_\infty+2h_\infty=c_\infty$, in which case $A\neq 0$, then $0\leqq c_\infty-a_\infty$, $c_\infty-a_\infty\in 2\mathbb{Z}$, $0<c_\infty<3a_\infty$ and (if $B\neq 0$ then $a_\infty+c_\infty\geqq 2b_\infty$).
iii. If $b_\infty+h_\infty$ is the highest exponents, but not $3h_\infty$ or $a_\infty+2h_\infty$, then $B\neq 0$, $0<b_\infty\leqq c_\infty$, $3b_\infty>2c_\infty$ and (if $A\neq 0$ then $2b_\infty>a_\infty+c_\infty$).

2. Now assume $L^*(y)$ has a right factor of order one. Note that $L^*(y)$ has the same singular points as $L(y)=0$. When one expands the coefficients appearing in $R^*(u)$ and applies the above arguments, one gets the same conditions on a, b and c (resp. a_∞, b_∞ and c_∞). ■

3.2. Case of an Imprimitive Unimodular Differential Galois Group

If a third order linear differential equation $L(y)=0$ with coefficients in $\mathbb{C}(x)$ has a differential Galois group which is an imprimitive subgroup of $SL(3,\mathbb{C})$, then all the solutions of $L(y)=0$ are liouvillian. In fact in this case $L(y)=0$ has a solution z such that $u=z'/z$ is algebraic over $\mathbb{C}(x)$ of degree at most 3 (Theorem 3.3, [21]). The minimal polynomial of u can be computed by the method described in Sect. 3.2 of [21]. We now derive a necessary condition for this case to hold.

Since an imprimitive subgroup of prime degree (e.g. 3) is a monomial group (cf. [24], Definition 3.2), we will now derive necessary conditions for the differential

Galois group to be monomial. For third (or just prime) order equations, these are necessary conditions for the imprimitive case.

Necessary condition for case 2. *Let $L(y)=0$ be an irreducible linear differential equation of order n over $\mathbb{C}(x)$ with monomial differential Galois group $\mathcal{G}(L)\subseteq SL(n,\mathbb{C})$. The n-th symmetric power $L^{\circledS n}(y)=0$ of $L(y)=0$ must have a non trivial solution of the form $P(x)\prod_i(x-c_i)^{\alpha_i}$, where $P(x)\in\mathbb{C}[x]$, $c_i\neq\infty$ are singular points of $L^{\circledS n}(y)=0$ with exponents $\alpha_i\in\frac{1}{2}\mathbb{Z}$. In particular, any singularity of $L^{\circledS n}(y)=0$ must have an exponent of the form $b/2$, where $b\in\mathbb{Z}$.*

If, in addition, $L(y)=0$ is of fuchsian type, then at each of the m singular points a_i of $L(y)=0$ on the Riemann Sphere there must be exponents $\alpha_{i,1},\alpha_{i,2},\ldots,\alpha_{i,n}$ of $L(y)=0$ at a_i such that

- *$(\sum_{j=1}^{n}\alpha_{i,j})\in\frac{1}{2}\mathbb{Z}$, and*
- *$(\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{i,j})\in\mathbb{Z}$ and is non positive.*

Note that the elements $\alpha_{i,1},\alpha_{i,2},\ldots,\alpha_{i,n}$ are not necessarily distinct. Furthermore, note that the above conditions imply that $L(y)=0$ has a solution whose square is a rational function. For third order equations, the condition that $L^{\circledS 3}(y)=0$ has a solution whose square is rational is in fact necessary and sufficient (cf. [20], Theorem 4.6).

Example: We now use the above necessary conditions to show that, for the second symmetric power of the Airy equation

$$L(y)=\frac{d^3}{dx^3}-4x\frac{dy}{dx}-2y,$$

the case 2 (of an imprimitive group $\mathcal{G}(L)\subseteq SL(3,\mathbb{C})$) cannot hold. Since $L(y)=0$ is not of fuchsian type, we have to compute $L^{\circledS 3}(y)$ which gives:

$$L^{\circledS 3}(y)=\frac{d^7y}{dx^7}-56x\frac{d^5y}{dx^5}-140\frac{d^4y}{dx^4}+784x^2\frac{d^3y}{dx^3}+2352x\frac{d^2y}{dx^2}$$
$$-4(456x^3-295)\frac{dy}{dx}-3456x^2y.$$

The only possible exponent of $L^{\circledS 3}(y)$ at the unique singular point ∞ is $3/2$, which does not rule out the possibility of an imprimitive group $\mathcal{G}(L)\subseteq SL(3,\mathbb{C})$. We thus have to test the stronger condition that if $\mathcal{G}(L)\subseteq SL(3,\mathbb{C})$ is an imprimitive group, then $L^{\circledS 3}(y)=0$ must have a solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x)\in\mathbb{C}[x]$, $c_i\neq\infty$ are singular points of $L(y)=0$ and $a_i\in\frac{1}{2}\mathbb{Z}$. Since ∞ is the only singular point and $3/2$ is the only exponent, Lemma 3.1 implies that $L^{\circledS 3}(y)=0$ has no non-zero solution of this form. Thus case 2 (in which case we always have a liouvillian solution) does not hold. ∎

Proof of the Necessary conditions for case 2: The "non fuchsian" part of our necessary conditions follows from:

Theorem 3.4 (cf. Proposition 3.6, [20]). *If an irreducible linear differential equation $L(y)=0$ of order n with coefficients in $\mathbb{C}(x)$ has a monomial differential Galois group $\mathcal{G}(L)\subseteq SL(n,\mathbb{C})$, then the n-th symmetric power $L^{\circledS n}(y)=0$ of $L(y)=0$ has a solution which is the square root of an element of $\mathbb{C}(x)$. At any singularity, $L^{\circledS n}(y)=0$ must have an exponent of the form $a/2$, where $a\in\mathbb{Z}$.*

When $L(y)$ is fuchsian we can do better. If for some solutions y_i of $L(y)=0$ we have $(y_1 y_2 \cdots y_n)^2$ is a rational function, then $(y_1 y_2 \cdots y_n)$ must be of the form

$$P(x)\prod(x-a_i)^{b_i/2},$$

where $P(x)\in\mathbb{C}[x]$, a_i are singular points of $L(y)=0$ and $b_i/2=e_i$ with $b_i\in\mathbb{Z}$ is an exponent of $L^{\circledS n}(y)=0$ at a_i (the apparent singularities may also contribute to $P(x)$). Thus, for some exponent e_∞ of $L^{\circledS n}(y)=0$ at ∞ we have that $-deg(P)-\sum_i e_i = e_\infty$. Thus $e_\infty+\sum_i e_i$ is a non-positive integer. Using Lemma 2.2 we can express the exponents of $L^{\circledS n}(y)=0$ at a_i in terms of the exponents α_k of $L(y)=0$ at a_i. We get that there exists non negative integers $m_1,\ldots,m_n$ with $\sum_{j=1}^n m_j = n$ such that

$$\left(\sum_{j=1}^{n} m_j\alpha_{\infty,j}\right)+t_\infty+\sum_i\left(\left(\sum_{j=1}^{n} m_j\alpha_{i,j}\right)+t_i\right)$$

(where t_j are positive integer) is non positive and in $\mathbb{Z}$. In particular

$$\left(\sum_{j=1}^{n} m_j\alpha_{\infty,j}\right)+\sum_i\left(\sum_{j=1}^{n} m_j\alpha_{i,j}\right),$$

is non positive and in $\mathbb{Z}$. Since $e_i=\sum_{j=1}^n m_j\alpha_{i,j}+t_i$ and $e_i=b_i/2$ with $b_i\in\mathbb{Z}$, we get that $\sum_{j=1}^n m_j\alpha_{i,j}\in\frac{1}{2}\mathbb{Z}$. This proves the fuchsian part of the Theorem. ■

For a second order differential equation $L(y)=y''-ry$ ($r\in\mathbb{C}(x)$), one gets the condition of the Kovacic algorithm by looking at the riccati equation of the third order linear differential equation $L^{\circledS 2}(y)=0$ which is (cf. [21], p. 10 and [6]):

$$\theta''+3\theta'\theta+\theta^3=4r\theta+2r'$$

For a third order differential equation $L(y)=0$ the order of $L^{\circledS 3}(y)$, which is less than or equal to 10, is not known in advance (from [20], Lemma 3.5 we get that the order an be 7, 9 or 10). This makes a general discussion as in [12] difficult.

3.3. Case of a Finite Primitive Unimodular Differential Galois Group

There are, up to isomorphism, only 8 primitive finite subgroups of $SL(3,\mathbb{C})$. Following [20] Sect. 2.2, we denote them $A_6^{SL_3}$, A_5, $A_5\times C_3$, G_{168}. $G_{168}\times C_3$, $H_{216}^{SL_3}$, $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$. We note that the last 3 groups are solvable, G_{168} is the simple group of 168 elements and $A_6^{SL_3}$ is a central extension of A_6. By Jordan's Theorem, such a finite list exists for any degree.

The order of a one dimensional character ζ is the smallest integer i, such that $(\zeta)^i$ is the trivial character. If G is a group and V a G-module, then we denote the m^{th} symmetric power of V, which is also a G-module, by $\mathscr{S}^m(V)$ (cf. [13], p. 586).

Our necessary conditions in this case are based on the following Theorem:

Theorem 3.5. *Let $L(y)=0$ be a differential equation of degree n whose differential Galois group is a finite subgroup G of $SL(n,\mathbb{C})$. We denote by V the solution space of $L(y)=0$. If $\mathscr{S}^m(V)$ has a G-summand of dimension 1 whose character has order i, then, at each point c including ∞, there exists positive integers $m_1,\ldots,m_n$, with $\sum_{j=1}^n m_j=m$, such that $i(\sum_{j=0}^n m_je_j)\in\mathbb{Z}$, where $e_1,\ldots,e_n$ are the exponents of $L(y)=0$ at c.*

Proof. There is a natural map Φ_m of $\mathscr{S}^m(V)$ into the Picard–Vessiot extension for $L(y)=0$ given by sending $z_1\circledS\cdots\circledS z_m$ to $z_1\cdot\cdots\cdot z_m$. Since a finite group is completely reducible, the solution space of $L^{\circledS m}(y)=0$ is G-isomorphic to a direct summand of $\mathscr{S}^m(V)$ (cf. [20] Lemma 3.5). Let $\{y_1,\ldots,y_n\}$ be a fundamental set of solutions of $L(y)=0$. Since G is a finite group and $\mathscr{S}^m(V)$ has a one dimensional summand, there is a homogeneous polynomial $P(Y_1,\ldots,Y_n)$ of degree m whose (possibly trivial) image $P(y_1,\ldots,y_n)$ under Φ_m is a semi-invariant of G and such that $P(y_1,\ldots,y_n)^i$ is rational (cf. [21], Proposition 1.6).

If $P(y_1,\ldots,y_n)\neq 0$, then the exponents of $P(y_1,\ldots,y_n)$ at any point c are of the form $(\sum_{j=0}^n m_j e_j)+h$, where h and m_j are positive integers and $\sum_{j=1}^n m_j=m$ (cf. Lemma 2.2.4). Since $(P(y_1,\ldots,y_n))^i$ is rational, we have $i((\sum_{j=0}^n m_j e_j)+h)\in\mathbb{Z}$, thus $i(\sum_{j=0}^n m_j e_j)\in\mathbb{Z}$.

Now assume $P(y_1,\ldots,y_n)=0$ and let ζ be the character of the one dimensional summand of $\mathscr{S}^m(V)$. In this case we will show that there is a linear differential equation $\tilde{L}(y)=0$ with the following properties:

i. $L(y)$ and $\tilde{L}(y)$ have the same differential Galois group G,
ii. The solution space $\tilde{V}$ of $\tilde{L}(y)=0$ is G-isomorphic to V,
iii. For some fundamental set of solutions $\{\tilde{y}_1,\ldots,\tilde{y}_n\}$ of $\tilde{L}(y)=0$, $P(\tilde{y}_1,\ldots,\tilde{y}_n)\neq 0$ and $P(\tilde{y}_1,\ldots,\tilde{y}_n)$ generates a one dimensional invariant subspace of the solution space of $\tilde{L}^{\circledS m}(y)=0$ corresponding to the character ζ,
iv. For any point p on the Riemann Sphere, the sets of exponents $\mathscr{E}_p$ of $L(y)$ and $\tilde{\mathscr{E}}_p$ of $\tilde{L}(y)$ are the same mod $\mathbb{Z}$.

Assuming we have constructed $\tilde{L}(y)$, we can finish the proof. We apply the argument of the preceding paragraph to the equation $\tilde{L}(y)$ and conclude that, at each point c there exist positive integers $m_1,\ldots,m_n$ with $\sum_{j=1}^n m_j=m$, such that $i(\sum_{j=0}^n m_j\tilde{e}_j)\in\mathbb{Z}$, where $\tilde{e}_1,\ldots,\tilde{e}_n$ are the exponents of $\tilde{L}(y)=0$ at c. Since the exponents of $\tilde{L}(y)$ and $L(y)$ coincide mod $\mathbb{Z}$, we achieve the conclusion of the theorem.

To construct $\tilde{L}(h)$ we proceed as follows. Let $R_1,\ldots,R_n$ be new variables and consider the substitution

$$\begin{pmatrix} Y_1\\ \vdots\\ Y_n\end{pmatrix}=\begin{pmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)}\\ \vdots & \vdots & \vdots & \vdots\\ y_n & y_n' & \cdots & y_n^{(n-1)}\end{pmatrix}\begin{pmatrix} R_1\\ \vdots\\ R_n\end{pmatrix}$$

Under this substitution, the polynomial $P(Y_1,\ldots,Y_n)$ becomes $\hat{P}(R_1,\ldots,R_n)$, a polynomial with coefficients in K, the Picard–Vessiot extension associated with $L(y)$. Since the matrix $(y_i^{(j)})$ is nonsingular, we see that $\hat{P}(R_1,\ldots,R_n)$ is a non-zero polynomial. Let $\hat{Q}(R_1,\ldots,R_n)$ be the image of the differential polynomial $det(Wr(Y_1,\ldots,Y_n))$ (where Wr is the wronskian matrix) under this substitution. One can select the $r_i\in\mathbb{C}(x)$ so that $\hat{P}(r_1,\ldots,r_n)\cdot\hat{Q}(r_1,\ldots,r_n)\neq 0$ (see [15], p. 35). Let

$$\begin{pmatrix} \tilde{y}_1\\ \vdots\\ \tilde{y}_n\end{pmatrix}=\begin{pmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)}\\ \vdots & \vdots & \vdots & \vdots\\ y_n & y_n' & \cdots & y_n^{(n-1)}\end{pmatrix}\begin{pmatrix} r_1\\ \vdots\\ r_n\end{pmatrix}$$

We have that $\tilde{y}_1,\ldots,\tilde{y}_n$ belong to K and, by construction, these are linearly independent over $\mathbb{C}$. Applying elements of the Galois group G to $\tilde{y}_1,\ldots,\tilde{y}_n$, one sees that the $\mathbb{C}$-span $\tilde{V}$ of these elements is left invariant by G. Furthermore, these elements form a fundamental set of solutions of the linear differential equation $\tilde{L}(y)=det(Wr(y,\tilde{y}_1,\ldots,\tilde{y}_n))/det(Wr(\tilde{y}_1,\ldots,\tilde{y}_n))$ whose coefficients are left fixed by G and so lie in $\mathbb{C}(x)$.

Note that the matrix $S = (Wr(y_1, \ldots, y_n)).(Wr(\tilde{y}_1, \ldots, \tilde{y}_n))^{-1}$ is left invariant by the differential Galois group and so has entries in $\mathbb{C}(x)$. Comparing first rows of $(Wr(y_1, \ldots, y_n))$ and $S \cdot Wr(\tilde{y}_1, \ldots, \tilde{y}_n)$ we have that $y_i = \sum_{j=0}^{n-1} s_j \tilde{y}_i^{(j)}$ with $s_j \in \mathbb{C}(x)$. The map $y \mapsto \sum_{j=0}^{n-1} r_j y_i^{(j)}$ (resp. $\tilde{y} \mapsto \sum_{j=0}^{n-1} s_j \tilde{y}_i^{(j)}$) will take solutions of $L(y) = 0$ to solutions of $\tilde{L}(y) = 0$ (resp. solutions of $\tilde{L}(y) = 0$ to solutions of $L(y) = 0$). Therefore, the extension $\mathbb{C}(x)\langle \tilde{y}_1, \ldots, \tilde{y}_n \rangle$ coincides with K. In particular, $L(y)$ and $\tilde{L}(y)$ have the same differential Galois group, so property i. holds. To verify that property ii. holds, one observes that the matrix of any $\sigma \in G$ is the same with respect to the bases $\{y_1, \ldots, y_n\}$ and $\{\tilde{y}_1, \ldots, \tilde{y}_n\}$. Property iii. follows by construction. To verify property iv., let $p = 0$ and let $y = x^\rho \sum_n a_n x^n$ be a solution of $L(y) = 0$. We then have that $\tilde{y} = \sum_{j=0}^{n-1} r_j y^{(j)}$ will be a solution of $\tilde{L}(y)$ and furthermore it will be of the form $\tilde{y} = x^\rho \sum_n b_n x^n$ (the r_i are at worst meromorphic at 0). Therefore, if ρ is an exponent of $L(y) = 0$ at 0, then $\rho + N$ will be an exponent of $\tilde{L}(y)$ at 0. Therefore elements of $\mathscr{E}_0$ are congruent to elements of $\tilde{\mathscr{E}}_0$ mod $\mathbb{Z}$. Reversing the roles of $L(y)$ and $\tilde{L}(y)$ and arguing similarly, one establishes the last property. ■

We now state the necessary conditions for third order equations in this case:

Necessary conditions for case 3. *Let $L(y)$ be an irreducible third order linear differential equation with Galois group a finite primitive group $\mathscr{G}(L) \subset SL(3, \mathbb{C})$. Then $L(y) = 0$ must be a differential equation of fuchsian type having 3 distinct rational exponents at any singularity. Furthermore, if c is any singularity of $L(y) = 0$ and e_1, e_2, e_3 are the exponents at c, then one of the following holds:*

1. *If $\mathscr{G}(L) \cong A_6^{SL_3}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$ and*
   (b) *there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 6$ and $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
2. *If $\mathscr{G}(L) \cong A_5$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 5\}$, and*
   (b) *there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 2$ and $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
3. *If $\mathscr{G}(L) \cong A_5 \times C_3$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 5, 6, 15\}$, and*
   (b) *there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 2$ and $3(\sum_{i=1}^3 n_i e_i) \in \mathbb{Z}$.*
4. *If $\mathscr{G}(L) \cong G_{168}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 7\}$, and*
   (b) *there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 4$ and $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
5. *$\mathscr{G}(L) \cong G_{168} \times C_3$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 7, 12, 21\}$, and*

(*b*) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 4$ *and* $3(\sum_{i=1}^{3} n_i e_i) \in \mathbb{Z}$.

6. $\mathscr{G}(L) \cong H_{216}^{SL_3}$, *then*
   (*a*) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 9, 12, 18\}$, *and*
   (*b*) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 6$ *and* $3(\sum_{i=1}^{3} n_i e_i) \in \mathbb{Z}$.
7. $\mathscr{G}(L) \cong H_{72}^{SL_3}$, *then*
   (*a*) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 12\}$, *and*
   (*b*) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 6$ *and* $\sum_{i=1}^{3} n_i e_i \in \mathbb{Z}$.
8. $\mathscr{G}(L) \cong F_{36}^{SL_3}$, *then*
   (*a*) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 12\}$, *and*
   (*b*) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 3$ *and* $4(\sum_{i=1}^{3} n_i e_i) \in \mathbb{Z}$.

In order to exclude this case it is enough to show that none of the conditions on the exponents is satisfied.

Example: The second symmetric power of the Airy equation

$$L(y) = \frac{d^3y}{dx^3} - 4x\frac{dy}{dx} - 2y$$

has an unimodular differential Galois group and is not of fuchsian type. Thus none of the necessary conditions 1, 2 and 3 hold and case 1, 2 and 3 can not occur for $L(y) = 0$. This proves that $L(y) = 0$ has no liouvillian solutions. ■

Proof of Necessary conditions for case 3: The conditions that $L(y) = 0$ is of fuchsian type and that all exponents are rational are a consequence of the result that any solution of $L(y) = 0$ has to be algebraic in this case (cf. Lemma 2.2.3).

From Lemma 2.3 we get that all exponents are of the form a/m, where m is the order of an element of $\mathscr{G}(L)$. The possible set given in the necessary conditions is just the set of orders of elements of $\mathscr{G}(L)$.

For each irreducible three dimensional character χ of a finite primitive group G, using the recurrence relation given in [20] Sect. 2.3 we can compute the character χ_m of $\mathscr{S}^m(V)$. The following are the decompositions of the χ_m into irreducible characters of G in which a one dimensional summand appears for the first time (For example, in 7., χ_6 is the sum of the trivial character 1, 3 non trivial different one dimensional characters $\zeta_{1,1}$, $\zeta_{1,2}$ and $\zeta_{1,3}$, and 3 times the same irreducible eight dimensional character ζ_8):

1. For $\mathscr{G}(L) \cong A_6^{SL_3}$ we get $\chi_6 = 1 + \zeta_{5,1} + \zeta_{5,2} + \zeta_8 + \zeta_9$.
2. For $\mathscr{G}(L) \cong A_5$ we get $\chi_2 = 1 + \zeta_5$
3. For $\mathscr{G}(L) \cong A_5 \times C_3$ we get $\chi_2 = \zeta_1 + \zeta_5$, and ζ_1 is of order 3.
4. For $\mathscr{G}(L) \cong G_{168}$ we get $\chi_4 = 1 + \zeta_6 + \zeta_8$
5. For $\mathscr{G}(L) \cong G_{168} \times C_3$ we get $\chi_4 = \zeta_1 + \zeta_6 + \zeta_8$ and ζ_1 is of order 3.
6. For $\mathscr{G}(L) \cong H_{216}^{SL_3}$ we get $\chi_6 = \zeta_1 + \zeta_3 + \zeta_{8,1} + 2\zeta_{8,2}$, and ζ_1 is or order 3.

7. For $\mathscr{G}(L) \cong H_{72}^{SL_3}$ we get $\chi_6 = 1 + \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3} + 3\zeta_8$, and $\zeta_{1,i}$ is of order 2.
8. For $\mathscr{G}(L) \cong F_{36}^{SL_3}$ we get $\chi_3 = \zeta_{1,1} + \zeta_{1,2} + \zeta_{4,1} + \zeta_{4,2}$, and $\zeta_{1,i}$ is of order 4.

Theorem 3.5 now gives the result. ■

We now show how our approach can be used to get and to improve the necessary conditions of the Kovacic algorithm is this case (case 3 in [12]). The finite primitive subgroups of $SL(2, \mathbb{C})$ are the tetrahedral, octahedral and icosahedral group, which are denoted resp. $A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$ (see e.g. [20] Sect. 2.2)

Theorem 3.6. *Let $L(y)$ be an irreducible second order linear differential equation whose differential Galois group is a finite primitive group $\mathscr{G}(L) \subseteq SL(2, \mathbb{C})$. Then $L(y) = 0$ must be a differential equation of fuchsian type having 2 distinct rational exponents at any singularity. Furthermore, if c is any singularity of $L(y) = 0$ and e_1, e_2 are the exponents at c, then one of the following holds:*

1. *$\mathscr{G}(L) \cong A_5^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2) \in \{1, 2, 3, 4, 5, 6, 10\}$, and*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 12$ and $n_1 e_1 + n_2 e_2 \in \mathbb{Z}$.*
2. *$\mathscr{G}(L) \cong S_4^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2) \in \{1, 2, 3, 4, 6, 8\}$,*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 6$ and $2(n_1 e_1 + n_2 e_2) \in \mathbb{Z}$.*
3. *$\mathscr{G}(L) \cong A_4^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2) \in \{1, 2, 3, 4, 6\}$.*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 4$ and $3(n_1 e_1 + n_2 e_2) \in \mathbb{Z}$.*

Proof. The proof is similar to the proof of the Necessary Conditions 3. For any irreducible character χ of degree 2 of $A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$, we have to find the decomposition of the character χ_m of $\mathscr{S}^m(V)$ in which a one dimensional summand appears for the first time. Using the same notation as in proof of the Necessary Conditions 3 we get:

1. For $A_4^{SL_2}$ we get $\chi_4 = \zeta_{1,1} + \zeta_{1,2} + \zeta_3$, where $\zeta_{1,i}$ is of degree 1 and order 3.
2. For $S_4^{SL_2}$ we get $\chi_6 = \zeta_1 + 2\zeta_3$, where ζ_1 is of degree 1 and order 2.
3. For $A_5^{SL_2}$ we get $\chi_{12} = 1 + \zeta_3 + \zeta_4 + \zeta_5$.

We conclude as in Necessary Condition 3. ■

The above improvement of the necessary conditions given in [12] (see also [6]), which impose divisibility conditions on the denominator of the exponents, can also be found in a paper of L. Fuchs (cf. [8]). We note that for the above result one does not need to compute the semi invariants, but only the order of a one dimensional character.

3.4. Solvability in Terms of Lower Order Equations

If a second order equation with unimodular differential Galois group has no non-zero liouvillian solutions, then its differential Galois group is $SL(2, \mathbb{C})$ (cf. [12, 20]). If a third order equation $L(y)=0$ with unimodular differential Galois group has no non-zero liouvillian solutions, then its differential Galois group $\mathscr{G}(L)$ is either $SL(3,\mathbb{C})$ or is congugate to $\rho_2(SL(2,\mathbb{C}))$ or $\rho_2\,(SL(2,\mathbb{C}))\times C_3$, where

$$\rho_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

and C_3 is the center of $SL(3,\mathbb{C})$ which is cyclic of order 3 (cf. [20]; Note that $\rho_2(SL(2,\mathbb{C}))\cong PGL(2,\mathbb{C})$). If $\mathscr{G}(L)$ is not $SL(3,\mathbb{C})$, then $L(y)=0$ is solvable in terms of second order equations (cf. [18]).

If λ_1 and λ_2 are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the eigenvalues of $\rho_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $\{\lambda_1^2, 1, \lambda_2^2\}$ (since $\lambda_1\lambda_2=1$). This observation yields:

Lemma 3.7. *Let $L(y)=0$ be a fuchsian third order linear differential equation whose differential Galois group is conjugate to a subgroup of $\rho_2(SL(2,\mathbb{C}))\times C_3$. Then at each singular point, some exponent is of the form $\frac{n}{3}$ for some $n\in\mathbb{Z}$.*

Proof. The monodromy matrix at any point is an element of the differential Galois group. Since the elements of C_3 commute with the elements of $\rho_2(SL(2,\mathbb{C}))$ one sees that the eigenvalues of a monodromy matrix are of the form $\{\omega\lambda_1^2,\omega,\omega\lambda_2^2\}$, where $\lambda_1\lambda_2=1$ and $\omega^3=1$. Since $L(y)=0$ is fuchsian, Lemma 2.4 implies that these must be of the form $\{e^{2\pi\sqrt{-1}a_1}, e^{2\pi\sqrt{-1}a_2}, e^{2\pi\sqrt{-1}a_3}\}$. So $a_2=\frac{n}{3}$ with $n\in\mathbb{Z}$. ■

In the fuchsian case, the above yields necessary conditions useful in testing solvability in terms of second order equation and in computing differential Galois groups. This will be used in the next section.

4. Differential Equations with three Singular Points

In the previous section we derived necessary conditions on the exponents of a differential equation $L(y)=0$ with coefficients in $\mathbb{C}(x)$ to have a liouvillian solution. By assuming that we have only three singular points, we will be able to sharpen the necessary conditions for the finite primitive groups.

Theorem 4.1. *Let $L(y)=0$ be a third order linear differential equation over $\mathbb{C}(x)$ having three singular points s_1, s_2 and s_3 and whose differential Galois group $\mathscr{G}(L)$ is a finite primitive subgroup of $SL(3,\mathbb{C})$. Then for each possible group, all lists $[lcm_1, lcm_2, lcm_3]$ of possible least common multiples lcm_i of the denominators $d_{i,1}$, $d_{i,2}$, $d_{i,3}$ of the rational exponents $\frac{n_{i,1}}{d_{i,1}}, \frac{n_{i,2}}{d_{i,2}}, \frac{n_{i,3}}{d_{i,3}}$, where $(n_{i,j}, d_{i,j})=1$, at s_i can be derived.*

1. *If $\mathscr{G}(L)\cong A_6^{SL_3}$, we get:* [2,4,15], [2,5,12], [2,5,15], [2,12,15], [2,15,15], [3,3,4], [3,3,5], [3,3,12], [3,3,15], [3,4,5], [3,4,15], [3,5,5], [3,5,12], [3,5,15], [3,12,15], [3,15,15], [4,4,4], [4,4,5], [4,4,15], [4,5,5], [4,5,6], [4,5,12], [4,5,15], [4,6,15], [4,12,12], [4,12,15], [4,15,15], [5,5,5], [5,5,6], [5,5,12], [5,5,15], [5,6,12], [5,6,15], [5,12,12], [5,12,15], [5,15,15], [6,12,15], [6,15,15], [12,12,12], [12,12,15], [12,15,15] *or* [15,15,15].

2. *If* $\mathscr{G}(L) \cong A_5$, *we get*: [2, 3, 5], [2, 5, 3], [2, 5, 5], [3, 3, 5], [3, 5, 5] *or* [5, 5, 5].
3. *If* $\mathscr{G}(L) \cong A_5 \times C_3$, *we get*: [2, 3, 15], [2, 15, 15], [3, 3, 5], [3, 3, 15], [3, 5, 6], [3, 5, 15], [3, 6, 15], [3, 15, 15], [5, 6, 15], [5, 15, 15], [6, 15, 15] *or* [15, 15, 15].
4. *If* $\mathscr{G}(L) \cong G_{168}$, *we get*: [2, 3, 7], [2, 4, 7], [2, 7, 7], [3, 3, 4], [3, 3, 7], [3, 4, 4], [3, 4, 7], [3, 7, 7], [4, 4, 4], [4, 4, 7], [4, 7, 7] *or* [7, 7, 7].
5. *If* $\mathscr{G}(L) \cong G_{168} \times C_3$, *we get*: [2, 3, 21], [2, 12, 21], [2, 21, 21], [3, 3, 4], [3, 3, 7], [3, 3, 12], [3, 3, 21], [3, 4, 12], [3, 4, 21], [3, 6, 7], [3, 6, 21], [3, 7, 12], [3, 7, 21], [3, 12, 12], [3, 12, 21], [3, 21, 21], [4, 6, 21], [4, 12, 12], [4, 12, 21], [4, 21, 21], [6, 7, 12], [6, 7, 21], [6, 12, 21], [6, 21, 21], [7, 12, 12], [7, 12, 21], [7, 21, 21], [12, 12, 12], [12, 12, 21], [12, 21, 21] *or* [21, 21, 21].
6. *If* $\mathscr{G}(L) \cong H_{216}^{SL_3}$, *we get*: [3, 3, 4], [3, 3, 12], [3, 3, 18], [3, 4, 9], [3, 4, 18], [3, 9, 12], [3, 9, 18], [3, 12, 18], [4, 9, 18], [4, 18, 18], [9, 12, 18], [12, 18, 18] *or* [18, 18, 18].
7. *If* $\mathscr{G}(L) \cong H_{72}^{SL_3}$, *we get:* [4, 4, 4], [4, 4, 12], [4, 12, 12] *or* [12, 12, 12].
8. *If* $\mathscr{G}(L) \cong F_{36}^{SL_3}$, *we get*: [2, 4, 12], [2, 12, 12], [3, 4, 4], [3, 4, 12], [3, 12, 12], [4, 4, 6], [4, 6, 12] *or* [6, 12, 12].

Proof. We use the monodromy group of $L(y) = 0$, which is introduced in the first section. To the singular points at s_1, s_2 and s_3 correspond the matrices M_{s_1}, M_{s_2} and M_{s_3} of the monodromy group. The product $M_{s_1}M_{s_2}M_{s_3}$ corresponds to the zero path on the punctured Riemann Sphere and thus must be the identity. Since M_{s_1}, M_{s_2} and M_{s_3} generate the monodromy group and $M_{s_1}M_{s_2} = M_{s_3}^{-1}$, we get that the group $\mathscr{G}(L)$ is generated by two elements. Using the group theory system CAYLEY (see [4]) we can compute all possible sets of two generators for each finite primitive subgroup of $SL(3, \mathbb{C})$ and compute their order and the order of their product. From the proof of Lemma 2.3 we get that the order of M_{s_i} is the least common multiple of the denominator of the exponents at s_i. This leads to the possibilities listed above. ■

We note that the above possibilities often allow one to distinguish the groups A_5 from $A_5 \times C_3$ and G_{168} from $G_{168} \times C_3$.

Example: The following equation due to Hurwitz (see [9]):

$$H(y) = x^2(x-1)^2y''' + (7x-4)x(x-1)y'' + \left(\frac{72}{7}(x^2-x) - \frac{20}{9}(x-1) + \frac{3}{4}x\right)y' + \left(\frac{792}{7^3}(x-1) + \frac{5}{8} + \frac{2}{63}\right)y$$

is of fuchsian type and has a unimodular differential Galois group. The exponents of $H(y) = 0$ at the regular singular points 0, 1 and ∞ are respectively $\{0, -\frac{1}{3}, -\frac{2}{3}\}$, $\{\frac{1}{2}, 0, -\frac{1}{2}\}$ and $\{\frac{11}{7}, \frac{9}{7}, \frac{8}{7}\}$.

We now use the necessary conditions to test the possible structure of the differential Galois group $\mathscr{G}(H) \subseteq SL(3, \mathbb{C})$ of $H(y) = 0$:

1. *Reducibility*: Since $H(y)$ is fuchsian, we can use Corollary 3.3. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $H(y) = 0$ has no right factor of order one. The adjoint $H^*(y) = 0$ of

$H(y)=0$ is:

$$y''' + \left(\frac{-7x+4}{x(x-1)}\right)y'' + \left(\frac{\frac{170}{7}x^2 - \frac{6995}{252}x + \frac{92}{9}}{x^2(x-1)^2}\right)y'$$
$$+ \left(\frac{\frac{12650}{343}x^3 - \frac{1561655}{24696}x^2 + \frac{1143403}{24696}x + \frac{112}{9}}{x^3(x-1)^3}\right)y$$

The exponents of $H^*(y)=0$ at 0, 1 and ∞ are respectively $\{\frac{8}{3}, 2, \frac{7}{3}\}$, $\{\frac{5}{2}, 2, \frac{3}{2}\}$ and $\{-\frac{22}{7}, -\frac{23}{7}, -\frac{25}{7}\}$. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $H^*(y)=0$ has no right factor of order one. This shows, that the differential equation $H(y)=0$ is irreducible.

2. *Imprimitivity*: We note that at ∞, the only sum of three (possibly repeated) exponents which is in $\frac{1}{2}\mathbb{Z}$ is the sum of the three exponents $\frac{11}{7}, \frac{9}{7}$ and $\frac{8}{7}$ whose sums is 4. At 0 the possible sums of three exponents which are in $\frac{1}{2}\mathbb{Z}$ are $0, -1$ and -2. Since the sums of three exponents at 1 is $\geqq -\frac{3}{2}$, we get that no sum of the form prescribed in the Necessary Conditions 2 will be a non positive element of $\mathbb{Z}$. Thus the differential Galois group $\mathscr{G}(H)$ of this equation can not be an imprimitive group.

3. *Finite primitive groups*: Since the list of least common multiples of the denominators of the exponents at the singularities is $[2, 3, 7]$, we get from the above theorem, that if $\mathscr{G}(H)$ is a finite primitive group, then $\mathscr{G}(H)$ is isomorphic to G_{168} (note that the Necessary Conditions 3 would lead to the two possibilities $\mathscr{G}(H) \cong G_{168}$ and $\mathscr{G}(H) \cong G_{168} \times C_3$).

4. *Infinite primitive groups*: From Lemma 3.7 it follows that the differential Galois group of $H(y)=0$ cannot be a subgroup of $\rho_2(SL(2,\mathbb{C})) \times C_3$, since at ∞ no exponent is one third of an integer.

Therefore our necessary conditions show that the group $\mathscr{G}(H)$ is isomorphic to either G_{168} or $SL(3,\mathbb{C})$. In [22] we use the results of [20] and the fact that $H^{\circledS 4}(y)=0$ is only of order 14 to deduce that $\mathscr{G}(H)$ is not isomorphic to $SL(3,\mathbb{C})$ and thus that $\mathscr{G}(H) \cong G_{168}$. ■

Example: The general third order linear differential equation $L_p(y)=0$ of fuchsian type having respectively exponents $\{0, \frac{1}{6}, \frac{5}{6}\}$, $\{\frac{1}{12}, \frac{1}{3}, \frac{7}{12}\}$ and $\{\frac{1}{12}, \frac{1}{6}, \frac{3}{4}\}$ at $0, 1$, and ∞ is of the form:

$$L_p(y) = y''' + \left(\frac{2}{x-1} + \frac{2}{x}\right)y'' + \left(\frac{\frac{13}{48}}{(x-1)^2} + \frac{\frac{43}{24}}{x-1} + \frac{\frac{5}{36}}{x^2} + \frac{\frac{-43}{24}}{x}\right)y'$$
$$+ \left(\frac{\frac{-7}{432}}{(x-1)^3} + \frac{p+\frac{23}{864}}{(x-1)^2} + \frac{-2p-\frac{23}{864}}{x-1} + \frac{p}{x^2} + \frac{2p+\frac{23}{864}}{x}\right)y$$

where p is an arbitrary parameter.

We now use the necessary conditions to test the possible structure of the differential Galois group $\mathscr{G}(L_p) \subseteq SL(3,\mathbb{C})$ of $L_p(y)=0$:

1. *Reducibility*: Since $L_p(y)$ is fuchsian, we can use Corollary 3.3. Since all exponents of $L_p(y)=0$ are non-negative and at 1 and ∞ in fact positive, no sum of the prescribed form can be a negative integer. Thus $L_p(y)=0$ has no right factor of

order one. The adjoint $L_p^*(y)=0$ of $L_p(y)=0$ is:

$$y''' + \left(\frac{-2}{x-1} + \frac{-2}{x}\right)y'' + \left(\frac{\frac{205}{48}}{(x-1)^2} + \frac{\frac{43}{24}}{x-1} + \frac{\frac{149}{36}}{x^2} + \frac{\frac{-43}{24}}{x}\right)y'$$
$$+ \left(\frac{\frac{-1955}{432}}{(x-1)^3} + \frac{-p-\frac{1571}{864}}{(x-1)^2} + \frac{2p+\frac{23}{864}}{x-1} + \frac{\frac{-67}{18}}{x^3} + \frac{-p+\frac{43}{24}}{x^2} + \frac{-2p-\frac{23}{864}}{x}\right)y$$

The exponents of $L_p^*(y)=0$ at 0, 1 and ∞ are respectively $\{\frac{11}{6}, 2, \frac{7}{6}\}$, $\{\frac{5}{3}, \frac{17}{12}, \frac{23}{12}\}$ and $\{\frac{-25}{12}, \frac{-11}{4}, \frac{-13}{6}\}$. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $L_p^*(y)=0$ has no right factor of order one. This shows, that for any value of p the differential equation $L_p(y)=0$ is irreducible.

2. *Imprimitivity*: According to the Necessary Conditions 2 the differential Galois group $\mathscr{G}(L_p)$ of this equation can not be an imprimitive group, since, because all exponents are non-negative and at 1 positive, the sum of (possibly repeated) three exponents at each singular point will never be non positive.

3. *Finite primitive groups*: Since the list of least common multiples of the denominators of the exponents at the singularities is [6, 12, 12], we get from the above theorem, that if $\mathscr{G}(L_p)$ is a finite primitive group, then $\mathscr{G}(L_p)$ is isomorphic to $F_{36}^{SL_3}$ (note that the Necessary Conditions 3 would only exclude the cases $\mathscr{G}(L_p) \cong A_5$, $\mathscr{G}(L_p) \cong A_5 \times C_3$ and $\mathscr{G}(L_p) \cong G_{168}$).

4. *Infinite primitive groups*: From Lemma 3.7 it follows that the differential Galois group of $L_p(y)=0$ cannot be a subgroup of $\rho_2(SL(2,\mathbb{C})) \times C_3$, since at ∞ no exponent is one third of an integer.

Therefore our necessary conditions show that for any p, the group $\mathscr{G}(L_p)$ is isomorphic to either $F_{36}^{SL_3}$ or $SL(3,\mathbb{C})$. ■

For second order differential equations with three singular points, the method of the above Theorem always leads to at most one possible finite primitive group, as the conditions found are exclusive:

Theorem 4.2. *Let $L(y)=0$ be a second order linear differential equation over $\mathbb{C}(x)$ having three singular points s_1, s_2 and s_3 and whose differential Galois group $\mathscr{G}(L)$ is a finite primitive subgroup of $SL(3,\mathbb{C})$. Then for each possible group, all lists $[lcm_1, lcm_2, lcm_3]$ of possible least common multiples lcm_i of the denominators $d_{i,1}, d_{i,2}$ of the rational exponents $\frac{n_{i,1}}{d_{i,1}}, \frac{n_{i,2}}{d_{i,2}}$, where $(n_{i,j}, d_{i,j})=1$, at s_i can be derived.*

1. *If $\mathscr{G}(L) \cong A_5^{SL_2}$, we get*: [3, 3, 10], [3, 4, 5], [3, 4, 10], [3, 5, 5], [3, 5, 6], [3, 5, 10], [3, 10, 10], [4, 5, 5], [4, 5, 6], [4, 5, 10], [4, 6, 10], [4, 10, 10], [5, 5, 6], [5, 5, 10], [5, 6, 10], [6, 6, 10], [6, 10, 10] *or* [10, 10, 10].
2. *If $G(L) \cong S_4^{SL_2}$, we get*: [3, 4, 8], [3, 8, 8], [4, 6, 8] *or* [6, 8, 8].
3. *If $\mathscr{G}(L) \cong A_4^{SL_2}$, we get*: [3, 3, 4], [3, 3, 6], [3, 4, 6], [4, 6, 6] *or* [6, 6, 6].

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1.9 How to Solve Linear Differential Equations: An Outline

By FELIX ULMER

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How to Solve Linear Differential Equations: An Outline[1]

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Abstract—There are several definitions of closed form solutions to linear differential equations. In this paper, we look for the so-called Liouvillian solutions. Through examples, we give an overview of how the differential Galois theory leads to algorithms to find the Liouvillian solutions. We will outline the general ideas and results, but will give examples instead of proofs.

1. INTRODUCTION

For a detailed introduction to the differential Galois theory see [5, 7, 8, 9]. The recent work on the algorithms discussed here can be found in [3, 4, 13, 14].

A differential field k is a field together with a derivation δ. In this paper, we will always consider $k = \mathbb{C}(x)$ and $\delta = \frac{d}{dx}$. As we will see later, it is important that the field $\mathcal{C} = \{a \in k | \delta(a) = 0\}$ of constants be algebraically closed of characteristic 0. A differential field extension K of k is a field K containing k together with a derivation Δ that coincides with δ on k.

Example: An algebraic extension of $k = \mathbb{C}(x)$ is always a differential field extension. The field $k(y)$, obtained by adjoining a root of $y^3 + xy - 1 = 0$, is a differential field with the derivation

$$y' = \frac{-y}{3y^2 + x} = \frac{-6x}{4x^3 + 27} + \frac{2x^2}{4x^3 + 27}y - \frac{9}{4x^3 + 27}y^2$$

obtained by differentiating the above equation.

By analogy to the Galois theory of polynomials, one looks for solutions of a linear differential equation defined in some differential field extension.

Example: The equation

$$L_1(y) = \frac{d^2}{dx^2}y(x) + \frac{6x^2}{4x^3 + 27}\frac{d}{dx}y(x) - \frac{2x}{4x^3 + 27}y(x) = 0$$

defined over $k = \mathbb{C}(x)$ has a solution in an algebraic extension $k(y)$. Indeed the roots of $y^3 + xy - 1 = 0$ are solutions of this equation. ◇

The computation of the algebraic solutions of a linear differential equation $L(y) = 0$ over the field of rational functions was a problem of great interest at the end of the last century. P. Pepin, H. Schwarz, L. Fuchs, F. Klein, C. Jordan, and others worked on this problem and solved it for second-order equations.

However, algebraic solutions form a small class and it is difficult to decide if there is an algebraic solution. It is thus more convenient to look for solutions in a larger class of differential field extensions, where one also allows the symbols $\int$ and $e^{\int}$.

Definition 1. A differential field extension (K, Δ) of (k, δ) is a Liouvillian extension if there is a tower of fields

$$k = K_0 \subset K_1 \subset \ldots \subset K_m = K,$$

where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

- η_i is algebraic over K_i, or
- $\delta(\eta_i) \in K_i$ (extension by an integral), or
- $\delta(\eta_i)/\eta_i \in K_i$ (extension by the exponential of an integral).

A function contained in a Liouvillian extension of k is called a *Liouvillian function* over k.

The Liouvillian functions form a large class containing all algebraic functions

Example: There are many ways to write an algebraic function as a Liouvillian function. The solution returned by Maple V. 5 for the above equation $L_1(y) = 0$ is

$$\frac{e^{-1/3\operatorname{arctanh}(1/9\sqrt{12x^3+81}) + 1/2\ln(x) + 1/4\ln(4x^3+27)}}{\sqrt[4]{4x^3+27}}.$$

It is not at all obvious that this solution is algebraic and can be written in terms of the roots of $y^3 + xy - 1 = 0$. Deciding whether a Liouvillian solution is algebraic is a difficult problem. ◇

The goal of this paper is to give the ideas behind the modern algorithms for computing Liouvillian solu-

[1] This article was submitted by the author in English.

tions. We will give the results for second order equations and only point to the problems when generalizing them to third-order equations. We assume the reader to be familiar with the basics of the Galois and representation theories and linear algebraic groups. The analogy with the classical Galois theory is

(1) There is a differential Galois theory and the differential Galois group *measures* if the equation is solvable.

(2) There is an infinite set of possible Galois groups, but only a finite number of types (cf. Section 3).

(3) One can separate types using computations in the base fields (like the Galois group is a subgroup of A_n if the discriminant is a square).

Drawing this analogy and putting it into an algorithm is the goal of this paper. Many of the tools needed are available in Maple, and we will point them out as we go.

2. DIFFERENTIAL GALOIS THEORY

For the rest of the paper we fix the notation for a differential ground field (k, δ) and an nth order linear differential equation over k

$$L(y) = \sum_{i=0} a_i \delta^i(y) = 0 \quad (a_i \in k).$$

Under our assumption of an algebraic closed field of constants of characteristic 0, one can show that a *unique differential splitting field* K for $L(y) = 0$, called the Picard–Vessiot extension (PVE), exists (up to isomorphisms):

(i) $K = k\langle y_1, y_2, \ldots, y_n\rangle$ is the *differential* field generated by k and $y_1, y_2, \ldots, y_n$, where $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of $L(y) = 0$.

(ii) K and k have the same field of constants.

Since the solution space V of $L(y) = 0$ is a vector space over the constants, there is always a technical point in controlling the constants. Note that the PVE extension is obtained by adjoining a generating set $\{y_1, \ldots, y_n\}$ of the solution space, as well as all the first $n - 1$ derivatives of each y_i to k. Thus, the extension K/k is of a transcendance degree at most n^2 over k.

Example: The PVE extension of the Airy equation $L(y) = y'' - xy = 0$ is of transcendance degree 3 over $k = \mathbb{C}(x)$. There is one relation between y_1, y_1', y_2, y_2'.

Definition 2. The differential Galois group of $L(y) = 0$ is a subgroup of the Galois group of the PVE extension K/k:

$$\mathcal{G}(L) = \{\sigma \in Aut(K/k) | \sigma\delta = \delta\sigma.$$

Example: The solution space of $L_1(y) = 0$ is spanned by two solutions of $y^3 + xy - 1 = 0$. Since the classical Galois group of the polynomial is the symmetric group S_3, the differential Galois group $\mathcal{G}(L_1)$ is also S_3 (for algebraic extensions, the differential and the classical Galois groups coincide). ◇

We will need one property, called normality, of $\mathcal{G}(L)$: an element $z \in K$ that is not moved by any group element; i.e., $\forall \sigma \in \mathcal{G}(L)$, $\sigma(z) = z$, must be in k.

The compatibility with the derivation insures that a solution of $L(y) = 0$ is sent to another solution under the differential Galois group. In particular, y_i is sent to $\sum_{j=1}^n \alpha_{i,j} y_j$, which gives a faithful representation ρ: $\mathcal{G}(L) \longrightarrow GL(n, \mathcal{C})$. In the following, we will frequently identify $\mathcal{G}(L)$ with its image $\rho(\mathcal{G}(L))$. One can show that $\mathcal{G}(L)$ is a linear algebraic subgroup of $GL(n, \mathcal{C})$.

Example: Since $\mathcal{G}(L_1) \cong S_3$, we get that, for an appropriate change of the basis $\{y_1, y_2\}$ of the solution space, $\rho(\mathcal{G}(L_1))$ is generated by

$$\sigma_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\xi^2 + \xi + 1 = 0$. It is an example of a monomial group, i.e., groups whose elements in a certain basis all have only one nonzero entry in each row and column. ◇

The solvability in terms of algebraic of Liouvillian solutions is measured by the differential Galois group:

Theorem 1 ([?]). A differential equation $L(y) = 0$ with coefficients in k has

(i) only solutions that are algebraic over k if and only if $\mathcal{G}(L)$ is a finite group,

(ii) only Liouvillian solutions over k if and only if the component of the identity $\mathcal{G}(L)^0$ of $\mathcal{G}(L)$ in the Zariski topology is solvable.

Unfortunately, the only known way to compute $\mathcal{G}(L)$ is to first solve the equation $L(y) = 0$ [10]. So the knowledge of $\mathcal{G}(L)$ is not available in the algorithms. However, it is possible to list the possible types of Galois groups, as we will see. The algorithms then simply work their way through the list. In order to keep this list small, we will always assume that the equation $L(y) = 0$ is irreducible; i.e. over k the equation cannot be decomposed as a composition of differential operators $L_1(L_2(y))$. Factorization is always the first step in the algorithms. Since factorization algorithms over $\mathbb{C}(x)$ exist and are implemented in Maple, this is not a restriction (cf. [2]).

3. THE POSSIBLE GALOIS GROUPS

Assuming that the $L(y)$ is irreducible and has a unimodular Galois group, the possible Galois groups are (see [11, 12] for a precise description)

(1) for second-order equations:

(a) The imprimitive subgroups of $SL(2, \mathcal{C})$: groups whose elements in a certain basis all have only one nonzero entry in each row and each column.

(b) The finite primitive groups: there are three such groups of order 120, 48, and 24.

(c) The infinite primitive groups: $SL(2, \mathcal{C})$.

(2) for third-order equations:

(a) The imprimitive subgroups of $SL(3, \mathscr{C})$: groups whose elements in a certain basis all have only one nonzero entry in each row and each column.

(b) The finite primitive groups: there are eight such groups of order 1080, 648, 504, 168, 216, 180, 108, and 60.

(c) The infinite primitive groups: $SL(3, \mathscr{C})$, $PGL(2, \mathscr{C})$.

A similar result holds for any order.

Kolchin's theorem shows that a second-order equation with $\mathscr{G}(L)$, an irreducible subgroup of $SL(2, \mathscr{C})$, has a Liouvillian solution if and only if $\mathscr{G}(L)$ is not $SL(2, \mathscr{C})$ and a third-order equation with $\mathscr{G}(L)$, an irreducible subgroup of $SL(3, \mathscr{C})$, has a Liouvillian solution if and only if $\mathscr{G}(L)$ is neither $SL(3, \mathscr{C})$ nor $PGL(2, \mathscr{C})$. Our goal will be to distinguish the different possibilities using only the given equation $L(y)$. So we are looking for information about $\mathscr{G}(L)$ that can be obtained from $L(y)$.

We extend the action of $\mathscr{G}(L)$ on the solution space V generated by $\{y_1, \ldots, y_n\}$ to the polynomial ring $\mathscr{C}[Y_1, \ldots, Y_n]$ via

$$\forall \sigma \in \mathscr{G}(L), f(Y_1, \ldots, Y_n) \in \mathscr{C}[Y_1, \ldots, Y_n],$$

$$\sigma f(Y_1, \ldots, Y_n) = f(\sigma Y_1, \ldots, \sigma Y_n),$$

and where the action on the Y_i's is the same as the action on the y_i's. We used capitals for the unknown Y_i because, unlike the solutions y_i, there cannot be any polynomial relations between the Y_i's. This gives us a representation of $\mathscr{G}(L)$ on the space $Sym^m(V)$ of homogeneous polynomials of degree m in $\mathscr{C}[Y_1, \ldots, Y_n]$ whose basis is $\{Y_1^m, Y_1^{m-1}, \ldots, Y_n^m\}$.

Definition 3. A homogeneous polynomial $f(Y_1, \ldots, Y_n) \in Sym^m(V)$ is called an invariant of the differential Galois group $\mathscr{G}(L)$ if for all $g \in \mathscr{G}(L)$ the action of g on $f(Y_1, \ldots, Y_n)$ is trivial, i.e., $gf(Y_1, \ldots, Y_n) = f(Y_1, \ldots, Y_n)$.

Since the homogeneous components of an invariant are also invariants, we will always consider homogeneous invariants.

There is an obvious homomorphism ϕ: $\mathscr{C}[Y_1, \ldots, Y_n] \longrightarrow \mathscr{C}[y_1, \ldots, y_n] \subset K$ given by $f(Y_1, \ldots, Y_n) \mapsto f(y_1, \ldots, y_n)$. For $n = 2$, the morphism ϕ is an isomorphism; but for $n \geq 3$, this may no longer be true. From the normality of PVE extensions we get that invariants are mapped to elements of k under this evaluation morphism. This will allow us to distinguish groups via computations in the ground field (like the discriminant is a square in the classical Galois theory).

Example: Knowing that $\mathscr{G}(L_1) \cong S_3$ in the basis $\{y_1, y_2\}$ of the solution space is generated by

$$\sigma_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that the action of those generators of $\mathscr{G}(L_1)$ on $z_1 = y_1y_1$, $z_2 = y_1y_2$, $z_3 = y_3y_3$ is given by

$$\tilde{\sigma}_1 = \begin{pmatrix} \xi^{-1} & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{pmatrix}, \quad \tilde{\sigma}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In particular, y_1y_2 is invariant. Thus, we must have that $y_1y_2 \in \mathbb{C}(x)$. ◇

But unlike in the above example, we usually do not know the differential Galois group but have to start from the equation $L(y) = 0$.

4. COMPUTING INVARIANTS

In order to obtain the invariants we will

(1) construct an equation whose solution space in $Sym^m(V)$;

(2) find the solutions of this equation in $k = \mathbb{C}(x)$.

We start with the computation of rational solutions, i.e., solutions in k.

For $k = \mathbb{C}(x)$ the computation of the solution in k of $L(y) = 0$ can be done using linear algebra. We write the equation as an equation over $\mathbb{C}[x]$

$$L(y) = \sum_{i=0}^{n} p_i(x) y^{(i)} \quad p_i(x) \in C[x]$$

and look for a rational solution of the form

$$\frac{n(x)}{d(x)} = \frac{n(x)}{\prod_{j=1}^{s} (x - c_i)^{-\alpha_i}} = p(x) \prod_{j=1}^{s} (x - s_i)^{\alpha_i}.$$

The $s_i \in \mathbb{C}$ are singular points of $L(y)$ and must be roots of $p_n(x)$; and the α_i are exponents at s_i and must be roots of the indicial equation at s_i, which is of order n. Taking the smallest negative integer N_i solution of the indicial equation at s_i (if none exists, there are no rational solutions), we get a differential equation for $n(x)$ by setting

$$L_n(n(x)) = L\left(n(x) \prod_{j=1}^{s} (x - s_i)^{N_i}\right)$$

$$= \sum_{i=0}^{n} p_i(x) \left(\frac{n(x)}{\prod_{j=1}^{s} (x - c_i)^{N_i}} \right)^{(i)} = \sum_{i=0}^{n} \tilde{p}_i(x)(n(x))^{(i)}.$$

The degree of a polynomial solution of $L_n(y) = 0$ is the negative of a negative integer solution of the indicial equation at ∞ of $L_n(y) = 0$. Plugging a polynomial of a maximal degree with unknown coefficients into the equation gives a linear system for these coefficients. For a complete algorithm see [1].

Example: In Maple, there is a command **ratsols** to find rational solutions. In the example $L_2(y) = 0$ computed below, we also computed the exponents at the singularities: 0, 1, 2 at the three roots of $4x^3 + 27$ and -1, 1/2, 2 at ∞. From the exponent, we see that a rational solution must be polynomial of degree 1:

```
with(DEtools):
L2:=symmetric_power(eq1, 2, z(x));
```

$$L2 := -4\frac{z(x)}{4x^3+27} + 4\frac{x\left(\frac{d}{dx}z(x)\right)}{4x^3+27} + 18\frac{x^2\left(\frac{d^2}{dx^2}z(x)\right)}{4x^3+27} + \left(\frac{d^3}{dx^3}z(x)\right)$$

```
gen_exp(L2, z(x), T, x = RootOf(4*x^3+27));
   [[0, 1, T = x - RootOf(4_Z^3 + 27)],
        [1/2, T = x - RootOf(4_Z^3 + 27)]]
gen_exp(L2, z(x), T, x = infinity);
                       [[-1, 2, T = 1/x], [1/2, T = 1/x]]
ratsols(L2, z(x));
                                 [x]
```

In order to compute the invariants via their values, we have to construct a differential equation $L^{ⓢ^m}(y)$ whose solution space is spanned by $\{y_1^m, y^{m-1}y_2, \ldots, y_n^m\}$. For that, we proceed as follows:

(1) Put $Y = z_1z_2\ldots z_m$ where z_i are arbitrary solutions of $L(y) = 0$.

(2) Take derivatives of Y and use the rule

$$z_i^{(n)} = \frac{-1}{a_n}\left(\sum_{i=1}^{n-1} a_i z_i^{(i)}\right).$$

This expresses $Y^{(i)}$ as a linear combination over k of expressions $z_1^{(i_1)} z_2^{(i_2)} \ldots z_m^{(i_m)}$ where $i_j < n$.

(3) Because of the symmetry, there are at most $N = \binom{n+m-1}{n-1}$ different expressions of this type, and $\{1, Y, Y', \ldots, Y^N\}$ must be linearly dependent over k. This gives a linear differential equation noted $L^{ⓢ^m}(y)$ for $Y = z_1z_2\ldots z_m$ in order at most $\binom{n+m-1}{n-1}$.

Example: We have $L_1^{ⓢ^2}(y) = L_2(y)$, as can be checked in Maple using

```
> with(Detools):
> L2:=symmetric_power(L1, 2, y(x));
```

So we know that $\mathcal{G}(L_1)$ has an invariant of degree 2 whose value is the rational function x.

In order to obtain the invariant corresponding to this rational solution, we compute a basis of series solutions $s_1(x)$, $s_2(x)$ of $L_1(y) = 0$ at some regular point, say $x = 0$. Then we look for a linear combination $\sum_{i_1+i_2=2} a_{i_1,i_2} s_1(x)^{i_1} s_2(x)^{i_2}$ over $\mathcal{C}$ which equals the rational solution x:

```
> with(DEtools):
> s1:=convert(format_sol(L1,z(x),x,terms=6)[1][1],polynom);
            s1 := -1 - 1/81x^3
> s2:=convert(format_sol(L1,z(x),x,terms=6)[1][2],polynom);
            s2 := x - 1/81x^4
> p:=rem(x-(a1*s1*s1+a2*s1*s2+a3*s2*s2),x^6,x);
        p := 2/81a3x^5 - 2/81a1x^3 - a3x^2 + (1 + a2)x - a1
> eq:={coeffs(collect(p,x),x)};
        eq := {-2/81a1, -a3, 2/81a3, 1 + a2, -a1}
> solve(eq,{a1,a2,a3});
            {a1 = 0, a3 = 0, a2 = -1}
```

Thus, for the following basis of solutions of $L_1(y) = 0$ at $x = 0$

$$s_1(x) = -1 - 1/81x^3 + O(x^6),$$

$$s_2(x) = x - 1/81x^4 + O(x^6)$$

we get that x is the value of the invariant $-y_2y_1$. ◇

This method will always work for order 2 equations and if ϕ is an isomorphism. To handle the general case and get efficient algorithms, one uses first-order systems (cf. [4]).

5. INVARIANTS AND LIOUVILLIAN SOLUTIONS

In the sequel we assume that $\mathcal{G}(L)$ is unimodular, i.e., a subgroup of $SL(n, \mathcal{C})$. This can be achieved (cf. [5]) by performing the variable transformation

$$y = z\exp\left(-\frac{\int a_{n-1}}{n}\right).$$

This is not necessary to compute Liouvillian solutions (cf. Example $L_1(y) = 0$), but it guarantees that the algorithms find all Liouvillian solutions.

The following result shows that Liouvillian solutions are directly linked to the existence of particular invariants:

Theorem 2. [13] *If $\mathcal{G}(L)$ is an irreducible subgroup of $SL(n, \mathcal{C})$, then the linear differential equation $L(y) = 0$ has a Liouvillian solution if and only if $\mathcal{G}(L)$ has a homogeneous invariant that factors into linear forms.*

Since a homogeneous polynomial in two variables always factors into linear forms, we get

Corollary 1. [6, 14] *A second-order linear differential equation $L(y) = 0$ with $\mathcal{G}(L)$ an irreducible subgroup of $SL(2, \mathcal{C})$ has a Liouvillian solution if and only if $\mathcal{G}(L)$ has an invariant.*

The corollary shows why second-order examples are much easier to handle. Note that even if the group is not unimodular, an invariant will give us a Liouviilian solution. But in the nonunimodular case, we have to consider so called semi-invariants to get all Liouvillian solutions.

Example: Since the Galois group $\mathcal{G}(L_1)$ of the second order equation $L_1(y) = 0$ ha is an invariant of degree 2, we get that $L_1(y) = 0$ has a Liouvillian solution.

In order to turn our procedure into an algorithm, one needs to bound the degree of the minimal invariant that factors into linear forms. This can be done using the classification of the groups

Theorem 3. [6, 11, 14] *A second-order linear differential equation $L(y) = 0$ with $\mathcal{G}(L)$ an irreducible subgroup of $SL(2, \mathcal{C})$ has a Liouvillian solution if and only if $\mathcal{G}(L)$ has a homogeneous invariant of degree 2, 4, 6, 8, or 12.*

A similar result holds for any order. For third-order equations, one has to go up to degree 36 [11].

Once an invariant (that factors into linear forms) is found, it is always possible to construct a polynomial

$$P(u) = u^m + b_{m-1}u^{m-1} + \ldots + b_1u + b_0$$

having as zero the logarithmic derivative $u = z'/z$ of a (special) Liouvillian solution z of $L(y)$. This gives us a Liouvillian solution of the form $z = e^{\int u}$. For a second-order equation $L(y) = y'' + a_1y' + a_0y$, the coefficients b_i of $P(u)$ are obtained from the value $I(x)$ of an invariant by the following recursion (cf. [6, 14]):

$$\begin{cases} b_m = 1, \\ b_{m-1} = -\dfrac{I(x)'}{I(x)}, \\ b_{i-1} = \dfrac{-b_i' + b_{m-1}b_i + a_1(i-m)b_i + a_0(i+1)b_{i+1}}{m-i+1}, \\ m-1 > i \geq 0. \end{cases}$$

The polynomial $p(u)$ obtained will always have the root z'/z, but it is not always irreducible and one has to factor it. But any factor will also be a valid $p(u)$ (cf. [14]).

Example: For the equation $L_1(y) = 0$, we found an invariant of degree 2 whose value was x. Thus, we start the recursion with $b_2 = 1$ and $b_1 = -1/x$ and get the polynomial:

$$p(u) = u^2 - 1/xu + \frac{x}{4x^3 + 27}.$$

The two roots u_i of $p(u) = 0$ give us two solutions

$$z_1 = e^{\int 1/2\frac{4x^3 + 27 + 3\sqrt{12x^3 + 81}}{27x + 4x^4}}, \quad z_2 = e^{\int 1/2\frac{4x^3 + 27 - 3\sqrt{12x^3 + 81}}{27x + 4x^4}},$$

which can be expressed by Maple (compare with the first page) as

$$z_1 = e^{-1/3\operatorname{arctanh}(1/9\sqrt{12x^3 + 81}) + 1/2\ln(x)},$$

$$z_2 = e^{1/3\operatorname{arctanh}(1/9\sqrt{12x^3 + 81}) + 1/2\ln(x)}$$

However, in fact, not only $u_i = z_i'/z_i$ but in this case also, z_i is algebraic (which is not obvious from the above description). In fact, the $z_i = e^{\int u_i}$ are solutions of the following minimal polynomial (computed by J.A. Weil)

$$-\frac{1}{729}Y^6 - \frac{64}{729}Y^3 - \frac{1}{64}Y^3\alpha + x^3,$$

where α is a root of $-63844352 + 5971968x + 531441x^2$. Solving this equation gives us, for example, the following solution of $L_1(y) = 0$:

$$3/2\sqrt[3]{-12\sqrt{3}+4\sqrt{4x^3+27}}.$$

It is possible to solve a second-order equation interactively on the web:

http://www-lmc.imag.fr/-bronstei/kovacic-demo.html

The results for third-order equations are more tricky to explain, even if all the results remain valid. The reader is referred to the papers [3, 4, 13].

6. CONCLUSION

A similar approach works for linear differential equations of any order. Currently, implementations exist only for $n = 2$ and $n = 3$. The bibliography is by no means complete. Only a few recent publications directly linked to the problem under consideration have been quoted, and the reader should go to those papers to find more literature on the subject.

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1.10 Solving Second Order Linear Differential Equations with Kleins Theorem

By Mark van Hoeij, Jacques-Arthur Weil and Thomas Cluzeau.

Solving Second Order Linear Differential Equations with Klein's Theorem

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This talk is dedicated to the memory of Manuel Bronstein

Liouvillian Solutions of Second Order Linear ODE's: The Problem

$$y'' + a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad a_i(x) \in k$$

k is a differential field, e.g $C(x)$, $C(x, \exp(x))$

A solution y is called

1. Rational: if $y \in k$
2. Exponential: if $y'/y \in k$
3. Liouvillian: if y can be presented by any combination of: algebraic extensions, arithmetic operations, exp( ), and $\int$

The problem is to compute Liouvillian solutions.

Liouvillian Solutions of Second Order Linear ODE's: Algorithms

1. Kovacic, 1977, 1986
   - If a Liouvillian solution exists then $\exists$ solution of the form $y = \exp\left(\int \omega\right)$ with ω algebraic. Minimal polynomial of ω is computed using *semi-invariants* and a recursive formula.
2. Ulmer & Weil, 1996
   - Compute minpoly ω from *invariants* (easier to implement).
3. Fakler, 1997, computes algebraic solutions y in nicer form:
   - Gives the minpoly of y instead of minpoly of ω.
4. Klein (1877) ... Berkenbosch, van Hoeij, and Weil (2002)
   - Write Liouvillian solutions as hypergeometric functions composed with a function (called the *pullback*) in k.
   - Formulas for pullback given in B.H.W. using *semi-invariants*.

$$L(y) := y'' + a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad a_i(x) \in k$$

Write Liouvillian Solutions as $H(f) \cdot \exp\left(\int v\right)$ where H is a Hypergeometric function from Klein's table and $f, v \in k$.

Advantage: A more compact representation of the solutions.

Sketch of the Approach:

- Invariants $\Longrightarrow$ The differential Galois group $G(L)$ and v.
- $G(L)$ and Klein's table $\Longrightarrow$ H.
- a_0, a_1, v and a pre-computed formula $\Longrightarrow$ f.

Our contribution: Formulas to compute v and f using invariants.

- Makes it easy to implement: need only rational sols of linear ODE's (and exponential sols of L itself, if L is reducible).
- Available in Maple 10 and in Bernina.

Solutions and Differential Galois Groups

$$L(y) := y'' + a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad a_i(x) \in k$$

May assume $a_1 = 0$.

To L is associated a diff. Galois group $G(L)$.

$G(L)$ is a group of 2×2 matrices, acts on sol. space.

- Discriminate between groups via computing semi-invariants (Kovacic, Singer-Ulmer) or invariants (Ulmer-Weil)
- Invariants are found by finding *rational solutions* (in k) of an auxilliary operator: the *symmetric power* $Sym^m(L)$.

Invariants and the UW-Kovacic algorithm

$$L(y) := y'' + a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad a_i(x) \in k$$

Assume L is irreducible (no exponential sols).

Projective group $PG(L)$:= $G(L)$ mod center.

We compute $PG(L)$ (and later also v and f) from invariants:

1. If $\exists$ invariant(s) of degree 4: group is D_n or D_∞.
   ($n = 2$ is a special case, there we will compute v and f from the invariant of degree 6).
2. else, if $\exists$ invariant of degree 6: group is A_4
3. else, if $\exists$ invariant of degree 8: group is S_4
4. else, if $\exists$ invariant of degree 12: group is A_5
5. else: group is PSL_2 (no Liouvillian solutions).

Pullbacks

Definition

Let $L \in C(z)\left[\frac{d}{dz}\right]$ and $\mathcal{L} \in k\left[\partial\right]$ be differential operators.

1. $\mathcal{L}$ is a *proper pullback* of L by $f \in k$ if the change of variable $z \mapsto f$ changes L into $\mathcal{L}$. Then:

   Solutions $y(z)$ of $L \iff$ Solutions $y(f)$ of $\mathcal{L}$.

2. $\mathcal{L}$ is a *(weak) pullback* of L by $f \in k$ if $\exists v \in k$ such that we can transform L into $\mathcal{L}$ by doing a
   - change of variable: $z \mapsto f$, followed by
   - scaling: multiplying all solutions by $\exp(\int v)$.

   Then:

   Solutions $y(z)$ of $L \iff$ Solutions $y(f) \cdot \exp(\int v)$ of $\mathcal{L}$.

Klein's pullback theorem

To each $G \in \{D_n, A_4, S_4, A_5\}$, one associates a Standard Equation (we scaled them in such a way that the invariant has value 1)

$$St_{D_2} = \partial^2 + \frac{4}{3}\frac{z}{(z^2-1)}\partial - \frac{5}{144}\frac{z^2+3}{(z^2-1)^2} \quad (1)$$

$$St_{A_4} = \partial^2 + \frac{2(3z^2-1)}{3z(z^2-1)}\partial + \frac{5}{144z^2(z^2-1)} \quad (2)$$

$$\ldots \quad (3)$$

Theorem (Klein)

Let L be a second order irreducible linear differential operator over k with projective differential Galois group $PG(L)$. If $PG(L)$ is finite then L is a (weak) pullback of $St_{PG(L)}$.

This means: can write solutions of L as $H_{PG(L)}(f)\exp\left(\int v\right)$ where $H_G(z)$ = Hypergeometric sols of St_G.

The Algorithm: Example of the A_4 Group

Suppose for example the input of our algorithm is a differential operator L with group $PG(L) = A_4$. How would the algorithm determine $PG(L)$, the pullback f, and the solutions of L?

Group..

1. L irreducible. No invariants of degree $1, 2, 4$ and an invariant of degree 6 with value I_6. So the projective group is A_4.

Scaling..v

2. Divide solutions of L by $I_6^{1/6} \Longrightarrow$ new operator L_S that must be a *proper pullback* of St_{A_4} (because both operators have invariant value 1, and $y(z) \mapsto y(f)$ sends 1 to 1).

Pullback..f

3. Write $L_S = \partial^2 + a_1\partial + a_0$. Compute $g := 2a_1 + \frac{a_0'}{a_0}$, and the pullback mapping is $f = \pm\sqrt{1 + \frac{64}{5}\frac{a_0}{g^2}}$ it is rational!

   Solutions: $H_{A_4}(f) \cdot I_6^{1/6}$ for any solution H_{A_4} of St_{A_4}

How the pullback formula was found

For $G = A_4$ the *pullback formula* on the previous page was $f = \pm\sqrt{1 + \frac{64}{5}\frac{a_0}{g^2}}$ where $g = 2a_1 + \frac{a_0'}{a_0}$.

Our algorithm contains a pullback formula for each group G. These formulas were found as follows:

- Take a standard equation for G from Klein's table.
- Key idea: Scale it so that the invariant has value 1. Doing this to all operators reduces weak pullpacks to proper pullbacks!
- Change of variable $z \mapsto F$. One obtains a differential operator $\partial^2 + a_1\partial + a_0$ where $a_1, a_0 \in C(F, F', F'')$.
- Use differential elimination to express F in terms of a_1, a_0.
- For A_4 we got $F = \pm\sqrt{1 + \frac{64}{5}\frac{a_0}{g^2}}$ where $g = 2a_1 + \frac{a_0'}{a_0}$.
- For S_4 we got $F = \frac{-7}{144}\frac{g^2}{a_0}$.
- For other groups: see paper.

Example: group A_4 concretely

$$L(y) :=$$
$$y'' - \frac{1}{144}\,\frac{404\,(e^x)^2 x - 27\,x^2 + 108\,x^3 + 54\,x^4 + 9\,x^6 - 36\,x^5 + 216\,(e^x)^4 + \cdots}{(x-e^x)^2 (x+e^x)^2 (x-1)^2}\,y = 0$$

Group

1. No invariants of degree 1 or 2 or 4
2. Invariant of degree 6, value $I_6 = \frac{\left(x^2 - e^{2x}\right)^2}{e^x (x-1)^3}$: $Sym^6(L)(I_6) = 0$
   $\Longrightarrow PG(L) = A_4$

Normalize

3. Rescale operator L: Get L_S such that $Sym^6(L_S)(1) = 0$.
   L_S is a proper pullback of St_{A_4} because $Sym^6(St_{A_4})(1) = 0$.

Pullback

4. Apply *pullback formula* to coeffs of L_S gives pullback $f = \frac{e^x}{x}$
5. Solutions are

$$\frac{\left(x^2 - e^{2x}\right)^{2/3}}{\sqrt{x-1}} \left(C_1 \frac{{}_2F_1\left([\frac{7}{24}, \frac{19}{24}], [\frac{3}{4}], \frac{e^{2x}}{x^2}\right)}{e^{\frac{x}{4}}\, x^{\frac{7}{12}}} + C_2 \frac{e^{\frac{x}{4}}\; {}_2F_1\left([\frac{13}{24}, \frac{25}{24}], [\frac{5}{4}], \frac{e^{2x}}{x^2}\right)}{x^{\frac{13}{12}}} \right)$$

Example: group A_5

$L(y) := 48x(x-1)(75x-139)y'' + (2520x^2 - 47712x/5 + 3336)y' + (36001/75 - 19x)y = 0.$

1. $PG(L)$ equals A_5 in this example.
2. Both the standard Kovacic algorithm and our pullback method need to compute the invariant of degree 12.
3. However, the pullback method produces much smaller solutions:
   1. Solutions in Maple 9.5 (standard Kovacic): 236789 bytes.
   2. Solutions in Maple 10 (using pullback): 1360 bytes.
4. The old output is very large is because it contains an algebraic function represented by its minimal polynomial, and every coefficient of this polynomial is a large rational function.
5. In contrast, the output from the pullback method contains only one large rational function, namely f (which has degree 31 in this example).

Conclusion

- Keys to the algorithm are:
  1. We choose standard equations with invariant value 1.
  2. Given an equation we want to solve, we compute its invariant, and then scale it so that it too has value 1.
  3. This reduces a weak pullback to a proper pullback,
  4. which allows us to find a formula for the pullback.
- Easy to implement (one can simply add the pullback formulas to existing Kovacic implementations).
- Slightly faster than Kovacic due to smaller output size.
- $\exists$ extensions to order 3 by Berkenbosch (no algo but good)
- Other works on special functions using special forms (e.g Cheb-Terrab 2004) or essential singularities (e.g Bronstein and Lafaille 2002): get non-Liouvillian functions.

Thank you for your attention.

1.11 Liouvillian solutions for second order linear differential equations with polynomial coefficients

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SPECIAL SECTION: GEOMETRY IN ALGEBRA AND ALGEBRA IN GEOMETRY

Liouvillian solutions for second order linear differential equations with polynomial coefficients

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Abstract
In this paper we present an algebraic study concerning the general second order linear differential equation with polynomial coefficients. By means of Kovacic's algorithm and asymptotic iteration method we find a degree independent algebraic description of the spectral set: the subset, in the parameter space, of Liouville integrable differential equations. For each fixed degree, we prove that the spectral set is a countable union of non accumulating algebraic varieties. This algebraic description of the spectral set allow us to bound the number of eigenvalues for algebraically quasi-solvable potentials in the Schrödinger equation.

Keywords Anharmonic oscillators · Asymptotic iteration method · Kovacic algorithm · Liouvillian solutions · Parameter space · Quasi-solvable model · Schrödinger equation · Spectral varieties

Mathematics Subject Classification Primary 34M15 · Secondary 81Q35

1 Introduction

Let us consider the family of second order linear differential equations,

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$$u'' + P(x)u' + Q(x)u = 0, \tag{1}$$

with polynomial coefficients of bounded degree. This family is parameterized by the coefficients of P and Q and therefore endowed of an structure of affine algebraic variety. We are interested in characterizing the moduli of Liouville integrable differentential equations in (1) and describing how the Liouvillian solutions of those integrable equations depend on the coefficients. From a result of Singer [17], we expect that this moduli to be enumerable union of constructible set corresponding to possible choices of local exponents at infinity of Liouvillian solutions.

With this purpose we explore the application of Kovacic's algorithm (see [10, 12]) to the family (1). Some steps of the algorithm, dealing with polynomial solutions of auxiliar equations, are very sensitive to changes of the parameters. However, the Asymptotic Iteration Method (see [7]) allows us to describe the algebraic conditions on the parameters giving rise to the existence of Liouvillian solutions.

The structure of the paper is as follows. Section 2 is devoted to the definitions of parameter space $\mathbb{P}_{2n}$, spectral set $\mathbb{L}_{2n}$, spectral varieties $\mathbb{L}_{2n,d}$ and the statement of our first main result, Theorem 2.3. Section 3 is devoted to the definition of polynomial-hyperexponential solutions, the reduction of the parameter space through D'Alembert transformation, and Kovacic's algorithm. The analysis of equation $y'' = (x^{2n} + \mu x^{n-1})y$ allows us to prove the non-emptyness of the spectral varieties $\mathbb{L}_{2n,kn}$ and $\mathbb{L}_{2n,kn+1}$ (Corolary 3.6). Section 4 contains the results of this paper related to Asymptotic Iteration Method. We find a sequence of differential polynomials $\Delta_d(a, b)$ in two variables that codify the equations of the spectral varieties $\mathbb{L}_{2n,d}$ inpendently of n (Theorem 4.3). The proof of Theorem 2.3 is included at the end of the section. Section 5 is devoted to the Liovullian solutions of Schrödinger equations with polynomial potentials. We proof that the number of the values of the energy parameter allowing a Liouvillian eigenfunction is bounded by *the arithmetic condition* which is a simple function of the coefficients of the potential (Theorem 5.2).

2 Parameter space

Let us consider Eq. (1) with polynomial coefficients $P(x) = \sum_{j=1}^{n} p_j x^j \in \mathbb{C}[x]_{\leq n}$ and $Q(x) = \sum_{j=1}^{2n} q_j x^j \in \mathbb{C}[x]_{\leq 2n}$. We also take into account a non-degeneracy condition $p_n^2 - q_{2n} \neq 0$, which implies that the equation can be reduced to trace free form with a polynomial coefficient of degree $2n$. Thus, the *parameter space* corresponding to the family of Eq. (1) is,

$$\mathbb{P}_{2n} = (\mathbb{C}[x]_{\leq n} \times \in \mathbb{C}[x]_{\leq 2n}) - \{p_n^2 - q_{2n} = 0\},$$

that we consider an an affine algebraic variety of dimension $3n + 2$ with affine coordinates $p_0, \dots, p_n, q_0, \dots, q_{2n}$. Our purpose is to describe algebraically the *spectral set* $\mathbb{L}_{2n} \subseteq \mathbb{P}_{2n}$. That is, the set of equations in the family (1) admitting a Liouvillian solution. An important class of Liouvillian functions, specifically relevant for the integrability of Eq. (1) is the following.

Definition 2.1 A polynomial-hyperexponential function of polynomial degree d and exponential degree k is a function of the form

$$u(x) = P_d(x)e^{\int A_k(x)dx}, \tag{2}$$

with $P_d(x)$ and $A_k(x)$ polynomials of degree d and k respectively.

Definition 2.2 The spectral subvariety $\mathbb{L}_{2n,d}$ is the subset of $\mathbb{L}_{2n}$ corresponding to equations in the family (1) having a polynomial-hyperexponentional solution of polynomial degree d.

Theorem 2.3 *Let $\mathbb{L}_{2n} \subset \mathbb{P}_{2n}$ be the set of equations in family (1) having a Liouvillian solution, and $\mathbb{L}_{2n,d}$ be the set of equations in family (1) having a polynomial-hyperexponential solution of polynomial degree d . The following statements hold:*

(a) *For any fixed $n \in \mathbb{N}$ there is an infinite set of values of d such that $\mathbb{L}_{2n,d}$ is not empty.*
(b) *If not empty, $\mathbb{L}_{2n,d}$ is an algebraic variety of codimension $\leq n$ in $\mathbb{P}_{2n}$.*
(c) *For any $d \neq k$ the algebraic varieties $\mathbb{L}_{2n,d}$ and $\mathbb{L}_{2n,k}$ are disjoint in $\mathbb{P}_{2n}$.*
(d) *Any compact subset of $\mathbb{P}_{2n}$ intersects only a finite number of algebraic varieties of the family $\{\mathbb{L}_{2n,d}\}_{d\in\mathbb{N}}$.*

Furthermore,

$$\mathbb{L}_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}_{2n,d}.$$

Therefore we conclude that $\mathbb{L}_{2n}$ is a singular analytic submanifold of $\mathbb{P}_{2n}$ consisting in the enumerable union of pairwise disjoint algebraic varieties of codimension $\leq n$ in $\mathbb{P}_{2n}$.

In what follows we will deal with the proof of Theorem 2.3 and the calculation of the equations of the spectral subvarieties $\mathbb{L}_{2n,d}$ in suitable coordinates.

3 Liouvillian solutions

3.1 Reduction of the parameter space

As it is well known, Eq. (1) can be reduced to trace free form

$$y'' = R(x)y \tag{3}$$

by means of D'Alembert transform $u = \exp\left(-\frac{1}{2}\int P(x)dx\right)y$, where $R(x) = \frac{P(x)^2}{4} + \frac{P'(x)}{2} - Q(x)$. Note that the degree of $R(x)$ is not greater than $\max\{\deg(Q(x)), 2\deg(P(x))\}$. Note that the family of equations of the form (3) with

$R(x)$ of fixed degree $2n$ are parameterized by the space $\mathbb{K}_{2n} = \mathbb{C}[x]_{2n}$ of polynomials of degree $2n$ that we see as an affine algebraic variety of dimension $2n+1$, parameterized by the coefficients of $R(x)$ and thus isomorphic to $\mathbb{C}^* \times \mathbb{C}^{2n}$. Note that the family (3) is included in (1), where $R(x) \in \mathbb{K}_{2n}$ corresponds to $(0, -R(x)) \in \mathbb{P}_{2n}$. The D'Alembert transformation is a polynomial map in the coefficients of $P(x)$ and $Q(x)$ and it can be seen as a retract,

$$\mathrm{dal}_{2n} : \mathbb{P}_{2n} \to \mathbb{K}_{2n} \subset \mathbb{P}_{2n}, \quad (P(x), Q(x)) \mapsto R(x) = \frac{P(x)^2}{4} + \frac{P'(x)}{2} - Q(x) \mapsto (0, -R(x))$$

of the natural inclusion $\mathbb{K}_{2n} \subset \mathbb{P}_{2n}$. Taking into account that the ratio between u and y is the exponential of a polynomial, we obtain that $(P(x), Q(x)) \in \mathbb{L}_{2n,d}$ if and only if $R(x) \in \mathbb{L}_{2n,d} \cap \mathbb{K}_{2n}$. Therefore, the analysis of polynomial-hyperexponential solutions of a given polynomial degree can be restricted to the trace free family $\mathbb{K}_{2n}$.

Let us write $R(x) = \sum_{j=0}^{2n} r_j x^j$. Equation (3) can be reduced to the case of monic polynomial coefficient by the change of variables $x \mapsto \sqrt[2n+2]{\frac{1}{r_{2n}}}x$ which lead us to the equation

$$y'' = \left(x^{2n} + \sum_{j=0}^{2n-1} \frac{r_j}{\sqrt[2n+2]{\frac{1}{r_{2n}}}} x^j \right) y, a_k \in \mathbb{C}^*, a_i \in \mathbb{C}. \tag{4}$$

For the next step, let us consider $\mathbb{M}_{2n} \subset \mathbb{K}_{2n}$ the family of Eq. (3) with monic polynomial coefficient. It is an an algebraic variety isomorphic to $\mathbb{C}^{2n}$. Since the $(2n+2)$-th root of r_{2n} is an algebraic multivalued function of r_{2n}, any equation in $\mathbb{K}_{2n}$ has $2n+2$ different equivalent reductions in $\mathbb{M}_{2n}$. This can be seen as an algebraic correspondence $\mathcal{C}_{2n} \subset \mathbb{K}_{2n} \times \mathbb{M}_{2n}$. This algebraic correspondence is a $(2n+2)$-fold covering space of $\mathbb{K}_{2n}$ by the first projection, π_1 and the $(2n+2)$ monic reductions of the equation of coefficient $R(x)$ are given by $\pi_2(\pi_1^{-1}(\{R(x)\}))$. Note that $R(x)$ is in $\mathbb{L}_{2n,d}$ if and only if so are any of its monic reductions. Therefore, if suffices to focus our analysis to equations in the family $\mathbb{M}_{2n}$.

3.2 Kovacic's algorithm and adapted coordinates in $\mathbb{M}_{2n}$

From now on let $\mathbb{L}'_{2n} = \mathbb{L}_{2n} \cap \mathbb{M}_{2n}$ be the *reduced spectral set* consisting of equations in $\mathbb{M}_{2n}$ having a Liouvillian solution, and let $\mathbb{L}'_{2n,d} = \mathbb{L}_{2n,d} \cap \mathbb{M}_{2n}$ be the *reduced spectral variety* consisting of equations in $\mathbb{M}_{2n}$ having a polynomial-hyperexponential solution of polynomial degree d.

Note that, since D'Alembert reduction does not affect the polynomial degree of polynomial-hyperexponential solutions then a differential equation in the family (1) has a polynomial-hyperexponential solution of polynomial degree d if and only if so has any of its monic D'Alembert reductions. Therefore, if $\mathbb{L}_{2n,d}$ is a subvariety of $\mathbb{P}_{2n}$ then $\mathrm{codim}(\mathbb{L}_{2n,d}, \mathbb{P}_{2n}) = \mathrm{codim}(\mathbb{L}'_{2n,d}, \mathbb{M}_{2n})$.

Here we will analyze the existence of Liovillian solutions of equations in the family $\mathbb{M}_{2n}$. This is done in terms of some known theoretical results obtained by application of Kovacic's algorithm [12]. A first step is to introduce a system of coordinates in $\mathbb{M}_{2n}$ that fits our analysis of Eq. (3) better than the coefficients of $R(x)$. The following Lemma that can be traced back to [15, p. 474], allows to decompose the monic polynomial $R(x)$ in a suitable form for the application of the algorithm.

Lemma 3.1 *Every monic polynomial M(x) of even degree 2n can be written in one only way completing squares, that is,*

$$M(x) = A(x)^2 + B(x), \tag{5}$$

with $A(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$ *is a monic polynomial of degree n and* $B(x) = \sum_{j=0}^{n-1} b_j x^j$ *is a polynomial of degree at most* $n-1$.

According to the proof given in [1, Lemma 2.4, p. 275] it also clear that the decomposition map $\mathbb{M}_{2n} \to \mathbb{C}^{2n}$, $R(x) \mapsto (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ where $R(x) = A(x)^2 + B(x)$ is a regular invertible polynomial map. Therefore, we may consider the coefficients of $A(x)$ and $B(x)$ as a system of regular coordinates is $\mathbb{M}_{2n}$. The following results gives us precise information about the sets $\mathbb{L}'_{2n}$ and $\mathbb{L}'_{2n,d}$.

Theorem 3.2 [1, Theorem 2.5, pp. 276] *Let us consider the differential equation,*

$$y'' = M(x)y, \tag{6}$$

with $M(x) \in \mathbb{C}[x]$ *a monic polynomial of degree* $k > 0$. *Then its differential Galois Group G with coefficients in* $\mathbb{C}(x)$ *falls in one of the following cases:*

1. $G = SL_2(\mathbb{C})$ *(non-abelian, non-solvable, connected group).*
2. $G = \mathbb{C}^* \ltimes \mathbb{C}$ *(non-abelian, solvable, connected group).*

Furthermore, the second case is given if and only if the following conditions holds:

3. *M(x) has even degree* $k = 2n$,
4. *Writing* $M(x) = A(x)^2 + B(x)$ *as in Lemma 3.1, the quantity* $\pm b_{n-1} - n$ *is a non-negative even integer* $2d$, $d \in \mathbb{Z}_{\geq 0}$.
5. *There exist a monic polynomial* P_d *of degree d satisfying at least one of the following differential equations,*

$$P_d'' + 2AP_d' - (B - A')P_d = 0, \tag{7}$$

$$P_d'' - 2AP_d' - (B + A')P_d = 0. \tag{8}$$

In such case, Liouvillian solutions are given by

$$y_1 = P_d e^{\int A dx}, y_2 = y_1 \int \frac{e^{-2\int A dx}}{P_d^2}, or \tag{9}$$

$$y_1 = P_d e^{-\int A dx}, y_2 = y_1 \int \frac{e^{2\int A dx}}{P_d^2}. \tag{10}$$

A careful read of Theorem 3.2 gives us the following.

Corollary 3.3 *The sets $\mathbb{L}'_{2n}$ and $\mathbb{L}'_{2n,d}$ in $\mathbb{M}_{2n}$ satisfy the following.*

1. $\mathbb{L}'_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}'_{2n,d}$.
2. $\mathbb{L}_{2n,d}$ *is contained in the hypersurface of* $\mathbb{M}_{2n}$ *of equation* $b_{n-1}^2 - (n+2d)^2 = 0$.

Therefore, the sets $\mathbb{L}_{2n}$ and $\mathbb{L}_{2n,d}$ in $\mathbb{P}_{2n}$ satisfy $\mathbb{L}_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}_{2n,d}$.

Proof

1. It is a consequence of the dichotomy of the Galois group. In case the group is not $SL_2(\mathbb{C})$ it leads to a polynomial-hyperexponential solution.
2. It is a direct consequence of point 2 in the second part of Theorem 3.2, The last statement of the corollary is a consequence of the point 1. and the fact the the reductions process from $\mathbb{P}_{2n}$ to $\mathbb{M}_{2n}$ preserves polynomial-hyperexponential solutions. □

3.3 Canonical equation

The following example:

$$y'' = (x^{2n} + \mu x^{n-1})y, \quad \mu \in \mathbb{C}. \tag{11}$$

that we refer to as *canonical equation* gives us some information about the non emptiness of the sets $\mathrm{L}_{2n,d}$ for large d. Due to theorem 3.2, if (11) has a Liouvillian solution, the parameter μ in the canonical coefficient $x^{2n} + \mu x^{n-1}$ is forced to be a discrete parameter that can be $\mu = 2d + n$ or either $\mu = -2d - n$, where d is a non-negative integer, which lead us to deal with two different equations,

$$y'' = (x^{2n} + (2d+n)x^{n-1})y, or \tag{12}$$

$$y'' = (x^{2n} - (2d+n)x^{n-1})y. \tag{13}$$

Proposition 3.4 *The differential equation (12) is integrable in the liouvillian sense if and only if, $d = (n+1)k$ or $d = (n+1)k + 1$ where k is a non-negative integer.*

Proof The differential equation (12), is transformed into the Whittaker differential equation,

$$\mathcal{W}'' = \left(\frac{1}{4} - \frac{\frac{-2d-n}{2n+2}}{z} + \frac{4(\frac{1}{2n+2})^2 - 1}{4z^2}\right)\mathcal{W}, \tag{14}$$

through the change of variables $z = \frac{2}{n+1}x^{n+1}$, $y = z^{-\frac{n}{2n+2}}\mathcal{W}$. Applying Martinet-Ramis theorem, see [13], we have that

$$\pm\frac{-2d-n}{2n+2} \pm \frac{1}{2n+2} = \frac{1}{2} + k, k \in \mathbb{Z}_{\geq 0},$$

which left only two posibilities, $d = (n+1)k$ or $d = (n+1)k + 1$. □

It is easy to see that the change of variables made in above proof also transform the Eq. (13) into a Whittaker equation. Nevertheless this new equation will have parameters $\kappa = \frac{2d+n}{2n+2}$ and $\mu = \frac{1}{2n+2}$. So via Martinet–Ramis theorem we can enunciate the following result analogous to the previous proposition.

Proposition 3.5 *The differential equation (13) is integrable in the liouvillian sense if and only if, $d = (n+1)k$ or $d = (n+1)k + 1$ where k is a non-negative integer.*

Moreover, the solutions to the Eq. (11) can be explicitly written as

$$\begin{aligned} y_{d,n}(x) &= P_{d,n}(x)e^{\frac{x^{n+1}}{n+1}}, if \mu = 2d+n, or \\ y_{d,n}(x) &= P_{d,n}(x)e^{-\frac{x^{n+1}}{n+1}}, if \mu = -(2d+n), \end{aligned} \tag{15}$$

where the polynomials $P_{d,n}$ can be find by a Frobenius-like method. Having said that, it is a tedious process. In any case, for $d = (n+1)k$ we have that

$$P_{d,n}(x) = x^d + \sum_{j=n+1}^{d} \zeta_j x^{d-j}, where \zeta_j = 0 for j \neq (n+1)m. \tag{16}$$

On the other hand, for $d = (n+1)k + 1$

$$P_{d,n}(x) = x^d + \sum_{j=n+2}^{d} \zeta_j x^{d+1-j}, where \zeta_j = 0 for j \neq (n+1)m + 1. \tag{17}$$

ζ_j	$j = (n+1)m,\ j = (n+1)m+1$
$\mu = 2d+2$	$\prod_{r=1}^{m} -\frac{(d+2-r(n+1))(d+3-r(n+1))}{-2(d+2-r(n+1)-n)-2d}$
$\mu = -2d-2$	$\prod_{r=1}^{m} -\frac{(d+2-r(n+1))(d+3-r(n+1))}{2(d+2-r(n+1)-n)+2d}$

Corollary 3.6 *For any pair (n, d) of degrees with $d \equiv 0$ or $d \equiv 1 \bmod (n+1)$ there exist a monic polynomial M(x) of degree 2n such that the equation*

$$y'' = M(x)y \tag{18}$$

has a polynomial-hyperexponential solution of exponential degree $n+1$ and polynomial degree d; therefore $\mathbb{L}_{2n,d}$ is non-empty.

4 Analysis of auxiliary equations

We refer to Eqs. (7) and (8) as *auxiliary equations* for Eq. (6). As it is stated in Theorem 3.2 the existence of a Liouvillian solution of Eq. (6) depends of the existence of a polynomial solution of the auxiliary equations. In what follows we will show that conditions for the existence of a polynomial solution P_d of given degree is algebraic in the coefficients of $A(x)$ and $B(x)$, and therefore in the coefficients of $M(x)$.

4.1 Asymptotic iteration method

The asymptotic iteration method or AIM was introduced by Ciftci et al in [7] as a tool to solve homogeneous differential equations of the form

$$y'' = \ell_0 y' + r_0 y \tag{19}$$

where ℓ_0 and r_0 are smooth functions defined on a real interval. Nevertheless, the method is purely differential algebraic, so we can extent the result to differential rings of characteristic zero. By derivation of Eq. (19) we obtain a sequence of differential equations,

$$y^{(j+2)} = \ell_j y' + r_j y \tag{20}$$

where the sequences $\{\ell_j\}_{j\in\mathbb{N}}$ and $\{r_j\}_{j\in\mathbb{N}}$ are defined by the recurrence,

$$\ell_{j+1} = \ell_j' + r_j + \ell_0\ell_j, \quad r_{j+1} = r_j' + r_0\ell_j. \tag{21}$$

and the sequence of obstructions,

$$\delta_j = r_j\ell_{j-1} - \ell_j r_{j-1}.$$

We say that the AIM *stabilizes* at $p > 0$ if $\delta_p = 0$. The following statement is a differential algebraic translation of [16, Theorem 1].

Theorem 4.1 *Let ℓ_0 and r_0 be elements of a differential field R of characteristic zero. If there exist $p > 0$ such that*

$$\frac{r_p}{\ell_p} = \frac{r_{p-1}}{\ell_{p-1}} := \alpha, \tag{22}$$

then differential equation (19) *has general solution,*

$$y = u^{-1}(c_2 + c_1\beta), \quad c_1, c_2 \text{ arbitrary constants,} \tag{23}$$

in the extension $R\langle \alpha, u, u^{-1}, v, \beta\rangle$ *where* u, v, β *are solutions of* $u' = \alpha u$, $v' = \ell_0 v$ *y* $\beta' = u^2 v$*, respectively.*

Proof By derivation of Eq. (19) we obtain,

$$y^{(p+2)} = \ell_p y' + r_p y,$$

and from there

$$\log(y^{(p+1)})' = \frac{\ell_p\left(y' + \frac{r_p}{\ell_p}y\right)}{\ell_{p-1}\left(y' + \frac{r_{p-1}}{\ell_{p-1}}y\right)}.$$

If condition (22) is satisfied, then we have

$$(y^{(p+1)})' = \frac{\ell_p}{\ell_{p-1}} y^{(p+1)}.$$

On the other hand, from the recurrence, we have,

$$\frac{\ell_p}{\ell_{p-1}} = \log(\ell_{p-1})' + \alpha + \ell_0,$$

and replacing into the above equation we obtain,

$$(y^{(p+1)})' = (\log(\ell_{p-1})' + \alpha + \ell_0)y^{(p+1)}.$$

We have that $y^{(p+1)} = c_1\ell_{p-1}uv$ is a general solution for this equation and finally we obtain

$$y' + \alpha y = c_1 uv$$

that yields the general solution of the statement. □

The AIM method tests whether the auxiliary equations have polynomial solution. The following statement is a differential algebraic translation of [16, Theorem 2]. There is no difference in the proof, so we refer the reader to the original text.

Theorem 4.2 *Let* ℓ_0, r_0 *be elements in a differential field* R *of characteristic zero that contains* $\mathbb{C}[x]$.

(i) *If* (19) *has a polynomial solution of degree p, then* $\delta_p = 0$
(ii) If $\ell_p\ell_{p-1} \neq 0$ *and* $\delta_p = 0$*, then the differential equation* (19) *has a polynomial solution of degree at most p.*

4.2 Liouvillian solutions by means of AIM

Let us proceed to the AIM of auxiliary equations (7) and (8). For Eq. (7) we should start with $\ell_0^+ = -2A(x)$ and $r_0^+ = B(x) - A'(x)$. By the recurrence law (21) we have a sequence:

$$\begin{bmatrix} \ell_{p+1}^+ \\ r_{p+1}^+ \end{bmatrix} = \begin{bmatrix} \ell_p^+ \\ r_p^+ \end{bmatrix}' + \begin{bmatrix} -2A(x) & 1 \\ B(x) - A'(x) & 0 \end{bmatrix} \begin{bmatrix} \ell_p^+ \\ r_p^+ \end{bmatrix}$$

A condition for the existence of a polynomial solution of degree at most p of (7) is the vanishing of the polynomial $\delta_p^+ = r_p^+ \ell_{p-1}^+ - r_{p-1}^+ \ell_p^+$. We proceed analogously with Eq. (8) obtaining sequences of polynomials r_p^-, ℓ_p^- and δ_p^-.

In order to model this process, let us consider $\mathbb{Q}\{a, b\}$ the ring of differential polynomials in two differential variables a, b. We may consider the following $\mathbb{Q}$ -linear differential operator in the space of 2 by 2 matrices (Table 1).

$$\varphi : \mathrm{Mat}_{2\times 2}(\mathbb{Q}\{a, b\}) \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}\{a, b\}), \quad C \mapsto \varphi(C) = C' + \begin{bmatrix} -2a & 1 \\ b - a' & 0 \end{bmatrix} C$$

We consider the iterations of this map. If we give to the differential variables a, b the values of the polynomials $A(x)$ and $B(x)$ we obtain:

$$\left(\varphi^{p+1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) (A(x), B(x)) = \begin{bmatrix} \ell_p & \ell_{p-1} \\ r_p & r_{p-1} \end{bmatrix}.$$

Let us define the sequence of universal differential polynomials,

$$\Delta_p = -\det \left(\varphi^{p+1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in \mathbb{Q}\{a, b\}.$$

As we will see this sequence $\{\Delta_d\}_{d\in\mathbb{N}}$ of differential polynomials governs the Liouvillian integrability of Eq. (6) for any even degree $2n$ of $M(x)$.

Table 1 First values of the universal differential polynomials Δ_n

Δ_0	$b - a'$
Δ_1	$2a\left(a'' - b'\right) + 4ba' - 3(a')^2 - b^2$
Δ_2	$-9b^2 a' - 15(a')^3 - 2(-3a''b' + 2(a^2(a^{(3)} - b'') + (a'')^2) + (b')^2)$
	$+b(-a^{(3)} + 2a(3b' - 5a'') + 23(a')^2 + b'') + a'(3a^{(3)} - 2a(7b' - 9a'') - 3b'') + b^3$
Δ_3	$2(2a'' + 2a(b - 4a') + 4a^3 - b')(a^{(4)} - 4a^2(b' - a'') + b(10a'' - 4b') + 10a'b' + 8a^3(b - a')$
	$+2a(-a^{(3)} - 14ba' + 12(a')^2 + b'' + 2b^2) - 16a'a'' - b^{(3)}) - (-5a^{(3)} + a(26a'' - 10b')$
	$+12a^2(b - 5a') - 10ba' + 21(a')^2 + 16a^4 + 3b'' + b^2)(-a^{(3)} - 2a(b' - a'') + 4a^2(b - a')$
	$-6b(a') + 5a'^2 + b'' + b^2)$

Theorem 4.3 *Equation* (6) *with* $M(x) = A(x)^2 + B(x)$ *has a polynomial-hyperexponential solution of polynomial degree d if and only if,*

$$b_{n-1}^2 = (n+2d)^2 \quad \text{and} \quad \Delta_d(A(x), B(x))\Delta_d(-A(x), B(x)) = 0.$$

Therefore $\mathbb{L}'_{2n,d}$ *is an algebraic subvariety of* $\mathbb{M}_{2n}$ *contained in the union of the irreducible hypersurfaces of equations:*

$$b_{n-1} = 2d + n, \quad -b_{n-1} = 2d + n.$$

Proof Note that, by definition of the sequence Δ_d we have $\delta_d^+ = \Delta_d(A(x), B(x))$. Analogously, the application of the AIM to Eq. (8) produces an obstruction, δ_d^-. Note that, because of the symmetry between Eqs. (7) and (8) $\delta_d^- = \Delta_d(-A(x), B(x))$.

We need only to check that $\ell_{d-1}^{\pm}\ell_d^{\pm} \neq 0$ for the auxiliar equations. This comes easily from the fact that $\ell_0^{\pm} = \pm 2A(x)$ is of bigger degree than $r_0 = B \pm A'$. Note that $\Delta_d(A(x), B(x))\Delta_d(-A(x), B(x))$ is a polynomial in $x, a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$. Its coefficients as a polynomial in x are the algebraic equations of the restricted spectral variety $\mathbb{L}'_{2n,d}$ in $\mathbb{M}_{2n}$. □

Example 4.4 As a first example of AIM applications, let us consider an equation on $\mathbb{M}_2$

$$y'' = ((x + a_0)^2 + b_0)y. \tag{24}$$

An elementary traslation as $x \mapsto x + a_0$ reduces the determination of $\mathbb{L}'_2$ structure to an analysis of liouvillian-integrability conditions for quantum harmonic oscillator

$$y'' = (x^2 + b_0)y. \tag{25}$$

These conditions are $b_0^2 = (2d+1)^2$ and $\Delta_d(x, b_0)\Delta_d(-x, b_0) = 0$. It is easy to verify that $\Delta_d(x, b_0) = \Delta_d(-x, b_0) = 2^{d+1}\prod_{k=0}^{d} d - k$. Therefore,

$$\mathbb{L}'_{2,d} \;:\; (b_0 + 2d + 1)(b_0 - 2d - 1) = 0 \tag{26}$$

Let us note that for a given equation $\mathbb{M}_{2n}$, conditions $b_{n+1} = 2d + n$ and $b_{n+1} = -2d - n$ are mutually exclusive. In the first case, auxiliary equation (7) may have a polynomial solution but not (8). The opposite occurs in the second case. Therefore, we decompose the spectral variety $\mathbb{L}'_{2n,d}$ as the disjoint union of two components $\mathbb{L}'_{2n,d} = \mathbb{L}^+_{2n,d} \cup \mathbb{L}^-_{2n,d}$. The first component $\mathbb{L}^+_{2n,d}$ correspond to equations whose auxiliary equation (7) has a polynomial solution of degree d and the second component $\mathbb{L}^-_{2n,d}$ correspond to equations whose auxiliary equation (8) has a polynomial solution of degree d.

Definition 4.5 $\mathbb{L}'_{2n,d} = \mathbb{L}^+_{2n,d} \cup \mathbb{L}^-_{2n,d}$, where

$$\mathbb{L}^+_{2n,d} = \begin{cases} b_{n-1} = 2d + n \\ \Delta_d(A(x), B(x)) = 0, \end{cases}$$

and

$$\mathbb{L}^{-}_{2n,d} = \begin{cases} b_{n-1} = -2d - n \\ \Delta_d(-A(x), B(x)) = 0. \end{cases}$$

As in the previous Example 4.4 it is always possible to get rid of the coefficient a_{n-1} by means of a translation in the x axis. Therefore is convenient to consider the sets,

$$V^{\pm}_{2n,d} = \{a_{n-1} = 0\} \cap \mathbb{L}^{\pm}_{2n,d}.$$

whose equations are easier to describe. For instance, in $\mathbb{M}_4$ and $\mathbb{M}_6$ we restrict our analysis to equations of the forms:

$$y'' = ((x^2 + a_0)^2 + b_1 x + b_0)y. \tag{27}$$

and

$$y'' = ((x^3 + a_1 x + a_0)^2 + b_2 x^2 + b_1 x + b_0)y \tag{28}$$

respectively. The following calculations of the equations of $V^{+}_{2n,d}$ for $n = 2, 3$ and small values of d, in Tables 2 and 3 is performed by means of the universal differential polynomials Δ_d.

4.3 Codimension of the spectral variety

As the degree in x of the polynomials $\Delta_p(A(x), B(x))$ grows quickly with p and the degree of $M(x) = A(x)^2 + B(x)$ it seems that the sets $\mathbb{L}_{2n,p}$ are smaller as p grows. However, a direct analysis of the auxiliary equations allows us to bound the codimension of the spectral varieties $\mathbb{L}_{2n,d}$ in $\mathbb{M}_{2n}$. As we have seen before the algebraic

Table 2 Algebraic equations of restricted spectral varieties $V^{+}_{4,d} = \mathbb{L}^{+}_{4,d} \cap \{a_1 = 0\}$ for small values of d

$V^{+}_{4,0}$	$\begin{cases} b_1 = 2 \\ b_0 = 0 \end{cases}$
$V^{+}_{4,1}$	$\begin{cases} b_1 = 4 \\ b_0^2 + 4a_0 = 0 \end{cases}$
$V^{+}_{4,2}$	$\begin{cases} b_1 = 6 \\ b_0^3 + 16a_0 b_0 - 16 = 0 \end{cases}$
$V^{+}_{4,3}$	$\begin{cases} b_1 = 8 \\ b_0^4 + 40a_0 b_0^2 - 96b_0 + 144a_0^2 = 0 \end{cases}$
$V^{+}_{4,4}$	$\begin{cases} b_1 = 10 \\ b_0^5 + 80a_0 b_0^3 - 336b_0^2 + 1024a_0^2 b_0 - 3072a_0 = 0 \end{cases}$
$V^{+}_{4,5}$	$\begin{cases} b_1 = 12 \\ b_0^6 - 140a_0 b_0^4 + 896b_0^3 - 4144a_0^2 b_0^2 + 28160a_0 b_0 - 14400a_0^3 - 25600 = 0 \end{cases}$
$V^{+}_{4,6}$	$\begin{cases} b_1 = 14 \\ b_0^7 + 224a_0 b_0^5 - 2016b_0^4 + 12544a_0^2 b_0^3 - 142848a_0 b_0^2 + 147456a_0^3 b_0 + 288000b_0 - 884736a_0^2 = 0 \end{cases}$

Table 3 Algebraic equations of restricted spectral varieties $V^+_{6,d} = \mathbb{L}^+_{6,d} \cap \{a_1 = 0\}$ for small values of d

$V^+_{6,0}$	$\begin{cases} b_2 = 3 \\ b_1 = 0 \\ b_0 - a_1 = 0 \end{cases}$
$V^+_{6,1}$	$\begin{cases} b_2 = 5 \\ 2a_1b_1 - 8a_0 - 2b_0b_1 = 0 \\ -6a_1 - b_1^2 + 2b_0 = 0 \\ 4a_1b_0 - 2a_0b_1 - 3a_1^2 - b_0^2 = 0 \end{cases}$
$V^+_{6,2}$	$\begin{cases} b_2 = 7 \\ 23a_1^2b_0 - 9a_1b_0^2 - 14a_0a_1b_1 + 6a_0b_0b_1 - 15a_1^3 - 24a_1 + 32a_0^2 + b_0^3 - 2b_1^2 + 8b_0 = 0 \\ 9a_1^2b_1 - 12a_1b_0b_1 + 6a_0b_1^2 + 24a_0b_0 - 24a_0a_1 + 3b_0^2b_1 - 12b_1 = 0 \\ -3a_1b_1^2 + 36a_1b_0 + 24a_0b_1 - 30a_1^2 - 6b_0^2 + 3b_0b_1^2 - 48 = 0 \\ 22a_1b_1 - 32a_0 + b_1^3 - 6b_0b_1 = 0 \end{cases}$
$V^+_{6,3}$	$\begin{cases} b_2 = 9 \\ 176a_1^3b_0 - 86a_1^2b_0^2 - 116a_0a_1^2b_1 + 16a_1b_0^3 - 20a_1b_1^2 + 264a_1b_0 + 80a_0a_1b_0b_1 \\ \quad - 12a_0^2b_1^2 - 144a_0^2b_0 - 12a_0b_0^2b_1 + 120a_0b_1 - 105a_1^4 - 372a_1^2 + 432a_0^2a_1 \\ \quad - b_0^4 - 36b_0^2 + 8b_0b_1^2 - 288 = 0 \\ 60a_1^3b_1 - 92a_1^2b_0b_1 + 56a_0a_1b_1^2 + 192a_0a_1b_0 + 36a_1b_0^2b_1 + 96a_1b_1 - 48a_0b_0^2 \\ \quad - 24a_0b_0b_1^2 - 288a_0^2b_1 - 144a_0a_1^2 + 576a_0 + 8b_1^3 - 4b_0^3b_1 = 0 \\ \quad - 18a_1^2b_1^2 + 372a_1^2b_0 - 132a_1b_0^2 + 24a_1b_0b_1^2 - 72a_0a_1b_1 - 12a_0b_1^3 - 72a_0b_0b_1 \\ \quad - 252a_1^3 - 288a_1 + 12b_0^3 - 6b_0^2b_1^2 + 48b_1^2 + 288b_0 = 0 \\ 4a_1b_1^3 - 48a_0b_1^2 + 136a_1^2b_1 - 160a_1b_0b_1 + 288a_0b_0 - 864a_0a_1 - 4b_0b_1^3 + 24b_0^2b_1 \\ \quad + 192b_1 = 0 \\ -52a_1b_1^2 + 144a_0b_1 + 120a_1b_0 - 252a_1^2 - b_1^4 + 12b_0b_1^2 - 12b_0^2 - 288 = 0 \end{cases}$

equations for $\mathbb{L}_{2n,0}$ are well expressed by the obstruction $\Delta_0(a, b) = b - a'$, so henceforth we will consider $d > 0$.

Proposition 4.6 *If* $\mathbb{L}'_{2n,d}$ *is not empty, then* $\operatorname{codim}(\mathbb{L}'_{2n,d}, \mathbb{M}_{2n}) \leq n$.

Proof Now, let us suppose that $P_d = \sum_{k=0}^{d} p_k x^k$ is a solution to one of the following algebraic equations

$$P_d'' \pm 2AP_d' - (B \mp A')P_d = 0 \tag{29}$$

where $A = x^n + \sum_{k=1}^{n} a_{n-k}x^{n-k}$ and $B = \sum_{k=1}^{n} b_{n-k}x^{n-k}$. Hence the coefficients of the polynomial

$$\sum_{k=2}^{d} k(k-1)p_k x^{k-2} \pm \left(2x^n + \sum_{k=1}^{n} 2a_{n-k}x^{n-k}\right)\left(\sum_{k=1}^{d} kp_k x^{k-1}\right)$$
$$-\left(\sum_{k=1}^{n} b_{n-k}x^{n-k} \mp nx^{n-1} + \sum_{k=1}^{n-1}(n-k)a_{n-k}x^{n-1-k}\right)\left(\sum_{k=0}^{d} p_k x^k\right) = 0 \tag{30}$$

in $\mathbb{C}[x]$ vanish. This give place to a system of equations which are sufficient conditions for the existence of P_d,

$$\begin{bmatrix} a_1-b_0 & 2a_2 & 2 & 0 & \cdots & 0 & 0 \\ 2a_2-b_1 & 3a_1-b_0 & 4a_0 & 6 & \cdots & 0 & 0 \\ & & & & \ddots & * & * \\ 0 & 0 & 0 & 0 & \cdots & 2(d-1)+n-b_{n-1} & (2d+n-1)a_{n-1}-b_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2d+n-b_{n-1} \end{bmatrix}\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{bmatrix} = 0. \tag{31}$$

We will denote the coefficient matrix of the system (31) by $M^{\pm}_{d,n}(A,B)$. Note this matrix has size $(d+n)\times(d+1)$ and it also has the property

$$M^{\pm}_{d,n}(A,B) = \left[\begin{array}{ccc|c} & M^{\pm}_{d-1,n}(A,B) & & \begin{matrix} 0 \\ \vdots \\ * \end{matrix} \\ \hline 0 & \cdots & 0 & 2d+n\pm b_{n-1} \end{array}\right]. \tag{32}$$

Remark 4.7 As there is no solution P of degree less than d, then $\operatorname{rank}(M^{\pm}_{d-1,n}(A,B)) = d$.

In order to determinate the codimension of $\mathbb{L}'_{2n,d}$ around a point (A_0,B_0) we shall choose a suitable $d\times d$ submatrix D of $M^{\pm}_{d-1,n}(A_0,B_0)$ such that its determinant is different from zero. In addition, the vanishing of the determinants of the matrices set by adding one of the remaining n rows of $M^{\pm}_{d-1,n}(A_0,B_0)$ to D, generates n conditional equations which guarantees the existence of a non-trivial solution to (31). □

4.4 An example: case $n = 3$

As an useful example in order to illustrate further computes, specially for looking accurate spectral values on Schrödinger type problems, let us assume that $A(x) = x^3 + a_{3,1}x^2 + a_{3,2}x + a_{3,3}$ and $B(x) = b_{2,0}x^2 + b_{2,1}x + b_{2,2}$. So, the analysis on previous Sect. 4.3 for case $n = 3$ can be summarized with the following proposition.

Proposition 4.8 *A necessary condition for equation*

$$y'' - (2x^3 + 2a_{3,1}x^2 + 2a_{3,2}x + 2a_{3,3})y' - ((b_{2,0}+3)x^2 + (2a_{3,1}+b_{2,1})x + a_{3,2} + b_{2,2})y = 0 \tag{33}$$

in order to have a polynomial solution of degree d is $b_{2,0}+3 = -2d$ for $d = 0, 1, 2, \ldots$

On the other hand, sufficient conditions are coded by the solutions of the linear system associated to the matrix

$$M_d^-(A,B)=\begin{bmatrix}\alpha_0 & \beta_0 & \gamma_0 & & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & & & \eta_d & \zeta_d & \alpha_d \\ & & & & 0 & \eta_{d+1} & \zeta_{d+1}\end{bmatrix} \quad (34)$$

where

$$\begin{aligned}\alpha_k &= -a_{3,2}(2k+1)-b_{2,2}, k=0,1,2,\dots\\ \beta_k &= -2a_{3,3}(k+1), \qquad k=0,1,2,\dots\\ \gamma_k &= (k+2)(k+1), \qquad k=0,1,2,\dots\\ \zeta_k &= -2a_{3,1}k-b_{2,1}, \qquad k=1,2,3\dots\\ \eta_k &= -2k-b_{2,0}+1, \qquad k=2,3,4\dots\end{aligned}$$

It creates a set of at most two polynomial equations in the variables $a_{3,0}$, $a_{3,1}$, $a_{3,2}$, $a_{3,3}$, $a_{2,0}$, $a_{2,1}$, $a_{2,2}$ which guarantees likewise a non-trivial solution to the associated system to $M_d^-(A,B)$ and a polynomial solution to Eq. (33).

Proof This is a restriction of the analysis developed on Sect. 4.3 to $n=3$. □

Generically we can suppose that our $d\times d$ principal minor is different from zero, so the equations given by Proposition 4.8 are the determinants

$$\Delta^1_{d,3}(-A,B)=\begin{vmatrix}\alpha_0 & \beta_0 & \gamma_0 & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & \eta_d & \zeta_d & \alpha_d & \end{vmatrix}=0 \quad (35)$$

and

$$\Delta^2_{d,3}(-A,B)=\begin{vmatrix}\alpha_0 & \beta_0 & \gamma_0 & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & 0 & \eta_{d+1} & \zeta_{d+1} & \end{vmatrix}=0 \quad (36)$$

Several detailed examples of this equations can be found on [8, 19].

4.5 Proof of Theorem 2.3

We can now state the proof, which follows easily from the other results. Statement (a) is a direct consequence of Proposition 3.5. Statement (b) is a consequence of Theorems 4.3 and Proposition 4.6. Note that, from d'Alembert reduction, the codimension of $\mathbb{L}'_{2n,d}$ in $\mathbb{M}_{2n}$ coincide with that of $\mathbb{L}_{2n,d}$ in $\mathbb{P}_{2n}$. Statement (c) and (d) are also clear, as $\mathbb{L}'_{2n,d}$ is contained in the union of hyperplanes of equations $b_{n+1} = 2d + n$ and $b_{n-1} = -2d - n$. □

5 Schrödinger equation

Let us summarize briefly the known results about explicit solutions for the one dimensional stationary Schrödinger equation. We start mentioning that Natanzon in 1971, see [14], introduced *exactly solvable potentials*, which today are known as *Natanzon potentials*. The seminal work of Natanzon inspired further researchers about exactly solvable potentials, although in the sense of Natanzon exactly solvable potentials also include potentials in where Schrödinger equations have eigenfunctions of hypergeometric type, not necessarily Liouvillian functions. The exactly solvable potentials, also known as solvable potentials, we extended to Schrödinger equations with explicit eigenfunctions. In this sense, solvable potentials are related to Schrödinger equations with eigenfunctions belonging to the set of special functions (Airy, Bessel, Error, Ei, Hypergeometric, Whittaker, Heun), not necessarily Liouvillian! Moreover, in case of Coulomb and 3D harmonic oscillator potentials correspond to Schrödinger equations which are transformed into Whittaker differential equations, Martinet-Ramis in [13] established the necessary and sufficient conditions to determine the obtaining of Liouvillian solutions of the Whittaker differential equations. Recently Combot in [9] developed another method to obtain exactly solvable potentials, in the sense of Natanzon, involving *rigid functions* in the sense of Katz.

To avoid confusion between explicit and Liouvillian solutions it was introduced the concept of *algebraic spectrum* in [2]. Also known as Liouvillian spectral set it is the set of eigenvalues for which the Schrödinger equation has Liouvillian eigenfunctions, see also [3, 4]. In some scenarios it is known that bounded eigenfunctions of Schrödinger operator are necessarily Liouvillian, see [6]. Potentials with infinite countable algebraic spectrum are called *algebraically solvable potentials* and those with finite algebraic spectrum *algebraically quasi-solvable potentials*, for complete details see [4, §3.1, pp. 316] and see also [2, 3].

On the other hand, Turbiner in 1988, see [18], following the same philosophy of Natanzon, introduced *quasi-solvable potentials*. The seminal paper of Turbiner leaded to the seminal paper of Bender and Dunne in 1996, see [5], in where they obtain a family of orthogonal polynomials in the energy values of the Schrödinger equation with sextic anharmonic potentials, see also [11] for the study of more general sextic anharmonic oscillators. Due to Schrödinger equation with quartic anharmonic oscillator potential falls in *triconfluent Heun equation*, see [10], it is in some sense a generalized Natanzon potential (exactly solvable) although there no exist Liouvillian eigenfunctions. In a similar way for algebraically solvable potentials, in [4, §3.1, pp. 316] also was introduced the concept of *algebraically quasi-solvable potential* as those finite non empty algebraic spectrum, see also [2, 3]. Examples of algebraically solvable potentials and algebraically quasi-solvable potentials (quartic and sextic oscillators) were presented in [2–4] using [1, Theorem 2.5, pp. 276], which corresponds to the application of Kovacic algorithm for reduced second order linear differential equation with polynomial coefficients.

Let us consider the one dimensional stationary Schrödinger equation

$$\psi'' = (\lambda - U(x))\psi \tag{37}$$

with a polynomial potential $U(x)$. It is clear that the potential $U(x)$ is *algebraically quasi-exactly solvable* if there are some values of λ for wich equation (37) has a Liouvillian solution. This is equivalent to say that the line,

$$\{\lambda - U(x) \ : \ \lambda \in \mathbb{C}\} \subseteq \mathbb{M}_{2n}$$

paremeterized by λ, intersects the spectral set $\mathbb{L}_{2n}$.

As it is well know, and we examined in Example 4.4, any quadratic potential is quasi-exactly solvable (and more over, exactly solvable). It is also clear that any quasi-exactly solvable potential is of even degree. Let us assume from now on that $U(x)$ is of degree $2n \geq 4$.

We consider the decomposition $-U(x) = A(x)^2 + B(x)$ as in Theorem 3.2. We define the *arithmetic condition* of $U(x)$ as the complex number,

$$d = \frac{|b_{n-1}| - n}{2}$$

where b_{n-1} is the coefficient of x^{n-1} the polynomial $B(x)$ appearing in the unique decomposition $-U(x) = A(x)^2 + B(x)$. Note that a necessary condition for $U(x)$ to be quasi exactly solvable is its arithmetic condition to be a non-negative integer. In such case the intersection between the line:

$$\{\lambda - U(x) \ : \ \lambda \in \mathbb{C}\} \subseteq \mathbb{M}_{2n}$$

and $\mathbb{L}_{2n}$ is confined to the spectral variety $\mathbb{L}_{2n,d}$.

Let us consider the universal sequence of differential polynomials $\Delta_d \in \mathbb{Q}\{a, b\}$ as in Theorem 4.3. The following lemma allows us to bound the number of admissible values of energy (for which the Schrödinger equation admits a Liouvillian solution) of any quasi-exactly solvable polynomial potential. Let us make clear that by the degree of a differential polynomial Δ_d in the variable b we mean its ordinary degree: that is we consider $a, a', a'', \ldots, b, b', b'', \ldots$ as an infinite set of independent variables.

Lemma 5.1 *The degree of Δ_d in the variable b is at most $d + 1$.*

Proof Let us recall the differential polynomials ℓ_d and r_d appearing in the definition of Δ_d. Let us prove first:

(a) The degree of ℓ_d in the variable b is small or equal to $\frac{d+1}{2}$.
(b) The degree of r_d in the variable b is small or equal to $\frac{d+2}{2}$.

The degree of $\ell_0 = -2a$ in the variable b is 0 an the degree of $r_0 = b - a'$ in the variable b is 1. Therefore (a) and (b) hold for $d = 0$. Now, from the recurrence law (21) we have that the degree in b of ℓ_{j+1} is at most that of r_j and that the degree in b of r_{j+1} is at most a unit bigger that the degree of ℓ_j. This proves (a) and (b). The degree of δ_d is at most the maximum between the sum of the degrees of ℓ_d and r_{d-1} and the sum of the degrees of ℓ_{d-1} and r_d; which is at most $d + 1$. □

Theorem 5.2 *Let $U(x)$ be an algebraically quasi-solvable polynomial potential, and let d be its arithmetic condition. The number of values of the energy parameter λ such that Eq.* (37) *has a Liouvillian solution is at most $d + 1$.*

Proof Generically, we may consider that $U(x)$ has no independent term. Then the condition on λ for the existence of a Liouvillian solution is the vanishing of $\Delta_d(A(x), B(x) + \lambda)$ which is a polynomial in x of λ. The number of values of λ for which this polynomial vanish can not be greater than its degree in λ. Clearly, the degree in λ of $\Delta_d(A(x), B(x) + \lambda)$ can not exceed the degree in b of $\Delta_d(a, b)$ which is bounded by $d + 1$ by Lemma 5.1. □

Example 5.3 In order to illustrate the procedures developed here let us consider the non-singular Turbiner potential

Table 4 Spectral system of Schrödinger equation associated to (38)

d	$\mathbf{C}_d$
1	$\begin{cases} J = 1 \\ \lambda = 0 \end{cases}$
3	$\begin{cases} J = 2 \\ \lambda^2 - 24 = 0 \end{cases}$
5	$\begin{cases} J = 3 \\ \lambda^3 - 128\lambda = 0 \end{cases}$
7	$\begin{cases} J = 4 \\ \lambda^4 - 400\lambda^2 + 12096 = 0 \end{cases}$
9	$\begin{cases} J = 5 \\ -\lambda^5 + 960\lambda^3 - 129024\lambda = 0 \end{cases}$
11	$\begin{cases} J = 6 \\ \lambda^6 - 1960\lambda^4 + 729280\lambda^2 - 26611200 = 0 \end{cases}$
13	$\begin{cases} J = 7 \\ -\lambda^7 + 3584\lambda^5 - 2934784\lambda^3 + 438829056\lambda = 0 \end{cases}$

$$U(x) = x^6 - (4J + 1)x^2 \tag{38}$$

where J is a non-negative integer. This potential has been studied in several papers, including [5]. Let $\mathbf{C}_d \subset \mathbb{L}'_{6,d}$ be the set consisting of all possible values for J and λ with polynomial hyperexponential solutions of polynomial degree d. In virtue of Theorem 4.3 it is a subvariety of $V(2J - d - 1)$. So, d shall only take non-negative odd values (Table 4).

On the other hand, we can easily compute the equations of $\mathbf{C}_d$ through the universal differential polynomial $\Delta_d(x^3, -(4J + 1)x^2 - \lambda)$ for the auxiliary equation

$$P''_d - 2x^3 P'_d - (3x^2 - (4J + 1)x^2 - \lambda)P_d = 0. \tag{39}$$

For the case $d = 1$ we get the following equations

$$\begin{cases} 2J - 2 = 0 \\ -\lambda^2 = 0 \\ 2(-4J - 1)\lambda + 12\lambda = 0 \\ -(-4J - 1)^2 - 8(-4J - 1) - 15 = 0. \end{cases} \tag{40}$$

Taking into account above consideration we compute the first seven equations for $\mathbf{C}_d$

6 Final remarks

In this paper we developed a technique to obtain Liouvillian solutions for parameterized second order linear differential equations with polynomial coefficients. In particular case, we study the set of possible values of energy to get Liouvillian solutions of Schrödinger equations with anharmonic potentials. We adapted asymptotic iteration method, Kovacic's Algorithm and previous results provided in [1–4] in terms of algebraic varieties extending slightly the known results about polynomial quasi-solvable potentials.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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1.12 Liouvillian Solutions of Linear Differential Equations with Liouvillian Coefficients

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Liouvillian Solutions of Linear Differential Equations with Liouvillian Coefficients

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Let $L(y)=b$ be a linear differential equation with coefficients in a differential field K. We discuss the problem of deciding if such an equation has a non-zero solution in K and give a decision procedure in case K is an elementary extension of the field of rational functions or is an algebraic extension of a transcendental liouvillian extension of the field of rational functions. We show how one can use this result to give a procedure to find a basis for the space of solutions, liouvillian over K, of $L(y)=0$ where K is such a field and $L(y)$ has coefficients in K.

1. Introduction

In this paper the following two questions will be considered. Let K be a differential field and let $a_{n-1}, \ldots, a_0, b \in K$. Let $L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0 y$.

QUESTION 1. When does $L(y)=b$ have non-zero solutions in K and how can one find all such solutions?

QUESTION 2. When does $L(y)=0$ have a non-zero solution y such that $y'/y \in K$ and how does one find all such solutions?

An algorithm is presented to answer these questions when K is an elementary extension of $C(x)$ or K is an algebraic extension of a purely transcendental liouvillian extension of $C(x)$, where C is a computable algebraically closed field of characteristic zero. We will discuss why these are important questions and how they are related to each other. Before starting, let us recall some definitions. A field K is said to be a *differential field* with derivation $D: K \to K$ if D satisfies $D(a+b)=D(a)+D(b)$ and $D(ab)=(Da)b+a(Db)$ for all $a, b \in K$. The set $C(K)=\{c \mid Dc=0\}$ is a subfield called the *field of constants* of K. We will usually denote the derivation by $'$, i.e. $a'=Da$. A good example to keep in mind is the field of rational functions $\mathbb{C}(x)$ with derivation d/dx ($\mathbb{C}$ denotes the complex numbers). All fields in this paper, without further mention, are of characteristic zero. We say K is a *liouvillian extension* of k if there is a tower of fields $k=K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that for each $i=1, \ldots, n$, $K_i = K_{i-1}(t_i)$ where either (a) $t_i' \in K_{i-1}$ or (b) $t_i'/t_i \in K_{i-1}$ or (c) t_i is algebraic over K_{i-1}. For example $\mathbb{C}(x, e^{x^2}, e^{\int e^{x^2}})$ is a liouvillian extension of $\mathbb{C}(x)$. We say K is an *elementary extension* of k if there is a tower of fields $k=K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that for each $i=1, \ldots, n$, $K_i = K_{i-1}(t_i)$ where either (a) for some $u_i \neq 0$ in K_{i-1}, $t_i' = u_i'/u_i$ or (b) for some u_i in K_{i-1}, $t_i'/t_i = u_i'$ or (c) t_i is algebraic over K_{i-1}. For example, $\mathbb{C}(x, \log x, e^{(\log x)^2})$ is an elementary extension

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of $\mathbb{C}(x)$. The example following the definition of liouvillian extension is not an elementary extension of $\mathbb{C}(x)$ since $\int e^{x^2}$ lies in no elementary extension of $\mathbb{C}(x)$ (Rosenlicht, 1972). We say that w is *liouvillian* (*elementary*) over k if w belongs to a liouvillian (elementary) extension of k.

Algorithms to answer questions 1 and 2 would be useful in solving two other problems. First of all, an answer to question 1 would have a bearing on the Risch Algorithm. In a series of papers (Risch, 1968; 1969; 1970), Risch gave a procedure to answer the following question: Given α in an elementary extension K of $C(x)$ (C a finitely generated extension of the rational numbers $\mathbb{Q}$ and $C(K) = C$), decide if $\int \alpha$ lies in an elementary extension of K. Liouville's Theorem (Rosenlicht, 1972) states that if α has an anti-derivative in an elementary extension of K, then $\alpha = v_0' + \sum c_i(v_i'/v_i)$ where $v_0 \in K$, $v_1, \ldots, v_n \in \bar{C}K$ and $c_i \in \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Risch's algorithm gives a procedure to decide if such elements exist. As a corollary of Liouville's Theorem, one can show that if α is of the form $f\,e^g$ with f and g in K, then α has an elementary anti-derivative if and only if $y' + g'y = f$ has a solution y in K (i.e. if and only if there is a y in K such that $(y\,e^g)' = f\,e^g$). In general, Risch's Algorithm forces one to deal, again and again, with this same question: given f and g in an elementary extension K of $C(x)$, decide if $y' + g'y = f$ has a solution in K. When K is a purely transcendental extension of $C(x)$, one may write $K = E(t)$ with $t' \in E$ or $t'/t \in E$ and t transcendental over E. Letting

$$y = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}(t)}{(p_i(t))^j} + h(t)$$

be the partial fraction decomposition of y, one can plug this expression into $y' + g'y = f$. Equating powers and using the uniqueness of partial fraction decompositions, one can find a finite number of candidates for the p_is and bound the degree of h. This allows one to find all possible solutions y. (In fact there are now improvements on this idea. Rothstein (1976) showed how one can use "Hermite Reduction" to postpone, as much as possible, the need to factor polynomials.) When K is not a purely transcendental extension of $C(x)$, but involves algebraics in the tower, things are more complicated. In the purely transcendental case, partial fractions gives us a global normal form that captures all the necessary local information (e.g. the factors of the denominators and the powers to which they appear). When algebraics occur, one does not have this normal form. If $K = E(t, \gamma)$ with γ algebraic of degree n over $E(t)$, one may write $y = b_0 + b_1\gamma + \cdots + b_{n-1}\gamma^{n-1}$ with the $b_i \in E(t)$. To find the b_i, one is forced to work with puiseux expansions (a local normal form) at each place of the function field $E(t, \gamma)$. Although Risch showed that this approach does yield an algorithm, it is much more complex than the purely transcendental case (Bronstein (1990) has made significant improvements in the Risch algorithm and can avoid puiseux expansions in many situations, but he is still forced to consider them in certain cases). One would like to reduce the question of deciding if $y' + g'y = f$ has a solution in $E(t, \gamma)$ to a similar question in $E(t)$, where one could apply partial fraction techniques and a suitable induction hypothesis. In section 3, we shall see that we can reduce the problem of solving such an equation in an algebraic extension of a field to solving linear differential equations (more than one and possibly of order greater than one) in that field. We are then forced to answer question 1 for that field.

The second place these questions arise is in the general problem of finding liouvillian solutions of linear differential equations with liouvillian coefficients. In Singer (1981) it was shown that given a homogeneous linear differential equation $L(y) = 0$ with coefficients in F, a finite algebraic extension of $\mathbb{Q}(x)$, one can find in a finite number of steps, a basis for the vector space of liouvillian solutions of $L(y)=0$. I would like to extend this result to find, given a homogeneous linear differential equation with coefficients in a liouvillian extension K of $\mathbb{Q}(x)$, a basis for the liouvillian solutions of $L(y)=0$. One can show that to solve this problem, it is sufficient to find one non-zero liouvillian solution. An inductive procedure would then allow one to find all such solutions (see Lemma 2.5(iii) below). To see how problem 2 fits into this, I will outline the procedure to decide if a given $L(y)=0$ with coefficients in K has a non-zero liouvillian solution. It is known (Singer, 1981) that if $L(y)=0$ has a non-zero liouvillian solution, then there is a solution y such that $u=y'/y$ is algebraic over K of degree bounded by an integer N that depends only on the order of $L(y)$. Furthermore there are effective estimates for N. Therefore, for some $m\leq N$, u satisfies an irreducible equation of the form $f(u)=u^m+a_{m-1}u^{m-1}+\cdots+a_0=0$ with the $a_i\in K$. We must now find the possible $a_i\in K$ and test to see if, for such a choice of a_i, $e^{\int u}$ satisfies $L(y)=0$. For example, let us try to determine the possible a_{m-1}. If $u=u_1,\ldots,u_m$ are the roots of $f(u)=0$ and $y_1=e^{\int u_1}$ satisfies $L(y)=0$, then for $i=2,\ldots,m$, $y_i=e^{\int u_i}$ also satisfies $L(y)=0$. We have

$$a_{m-1}=-(u_1+\cdots+u_m)=-\left(\frac{y_1'}{y_1}+\cdots+\frac{y_m'}{y_m}\right)=-\left(\frac{(y_1\cdots y_m)'}{y_1\cdots y_m}\right).$$

One can show that the product $y=y_1\cdots y_m$ satisfies a homogeneous linear differential equation $L_m(y)=0$ and that $y'/y\in K$. Finding all such solutions is just problem 2 above. Theorem 4.2 below states that for certain liouvillian extensions K, we can fill in the details of the above argument and give a procedure to find a basis for the vector space of all solutions of $L(y)=0$ that are liouvillian over K.

Finally, we note that it appears that to answer one of these two questions we need to be able to answer the other. The rest of the paper is organized as follows. Section 2 is devoted to showing how one can algorithmically reduce question 2 to question 1. Section 3 contains procedures to answer question 1 in certain cases. Section 4 contains some final comments and open problems. The results of this paper were announced in Singer (1989).

2. Reducing Question 2 to Question 1

In this section we shall consider fields of the form $E(t)$, where either $t'\in E$, $t'/t\in E$ or t is algebraic over E and where E satisfies certain hypotheses. We shall show that for these fields, if we can answer question 1 algorithmically then we can answer question 2 algorithmically. This is made precise in Proposition 2.1, but first we need some definitions. We call a differential field K a *computable differential field* if the field operations and the derivation are recursive functions and if we can effectively factor polynomials over K. We say that we can *effectively solve homogeneous linear differential equations over* K if for any homogeneous linear differential equation $L(y)=0$ with coefficients in K, we can effectively find a basis for the vector space of all $y\in K$ such that $L(y)=0$. We say that we can *effectively find all exponential solutions of homogeneous linear differential equations over* K if for any homogeneous linear differential equation $L(y)=0$ with coefficients in K, we can effectively find $u_1,\ldots,u_m$ in K such that if $L(e^{\int u})=0$ for some $u\in K$, then $e^{\int u}/e^{\int u_i}\in K$ for some i.

The main result of this section is:

PROPOSITION 2.1. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$, an algebraically closed field, and assume that either $t'\in E$, $t'/t\in E$, or t is algebraic over E.*

Assume that we can effectively solve homogeneous linear differential equations over $E(t)$ and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can effectively find all exponential solutions of homogeneous linear differential equations over $E(t)$.

We will deal with each of the three cases for t separately in the following propositions and lemmas. We start by defining and reviewing some facts about the Riccati equation. If u is a differential variable and $y = e^{\int u}$, formal differentiation yields $y^{(i)} = P_i(u, u', \ldots, u^{(i-1)})\, e^{\int u}$, where the P_i are polynomials with integer coefficients satisfying $P_0 = 1$ and $P_i = P'_{i-1} + uP_{i-1}$. If $L(y) = y^{(n)} + A_{n-1}y^{(n-1)} + \cdots + A_0 y = 0$ is a linear differential equation, then $y = e^{\int u}$ satisfies $L(y) = 0$ if and only if u satisfies $R(u) = P_n(u, \ldots, u^{(n-1)}) + A_{n-1}P_{n-1}(u, \ldots, u^{(n-2)}) + \cdots + A_0 = 0$. This latter equation is called *the Riccati equation associated with* $L(y) = 0$. We will need the following technical lemma.

LEMMA 2.2. *Let $E(t)$ be a differential field with t transcendental over E and either $t' \in E$ or $t'/t \in E$. Let $p(t)$ be an irreducible polynomial in $E[t]$ where $p \neq t$ if $t'/t \in E$.*

(i) *Let $u \in E(t)$ have p-adic expansion of the form $u = u_\gamma/p^\gamma$ + higher order terms, where $\gamma > 0$, $u_\gamma \neq 0$, and $\deg_t u_\gamma < \deg_t p$. If $\gamma > 1$ then for $i \geq 1$, $P_i(u, \ldots, u^{(i-1)}) = v_{i\gamma}/p^{i\gamma}$ + higher order terms, where $v_{i\gamma} \equiv (u_\gamma)^i \bmod p$. If $\gamma = 1$ then for $i \geq 1$, $P_i(u, \ldots, u^{(i-1)}) = v_i/p^i$ + higher order terms where $v_i \equiv \prod_{j=0}^{i-1}(u_1 - jp') \bmod p$.*

(ii) *Assume that $t' \in E$ and that $u \in E(t)$ has $(1/t)$-adic expansion of the form $u = u_\gamma t^\gamma$ + higher powers of $1/t$, $u_\gamma \neq 0$. If $\gamma > 0$ then the $(1/t)$-adic expansion of $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i t^{i\gamma}$ + higher powers of $1/t$. If $\gamma = 0$, then the $(1/t)$-adic expansion of $P_i(u, \ldots, u^{(i-1)}) = P_i(u_0, \ldots, u_0^{(i-1)})$ + higher powers of $1/t$.*

(iii) *Assume that $t'/t \in E$ and that $u \in E(t)$. If u has t-adic expansion of the form $u = u_\gamma/t^\gamma$ + higher powers of t, $\gamma > 0$, $u_\gamma \neq 0$, then $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i/t^{i\gamma}$ + higher powers of t. If u has $(1/t)$-adic expansion $u = u_\gamma t^\gamma$ + higher powers of $1/t$, $u_\gamma \neq 0$, then $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i t^{i\gamma}$ + higher powers of $1/t$ if $\gamma > 0$ and $P(u, \ldots, u^{(i-1)}) = \gamma_i(u_0, \ldots, u_0^{(i-1)})$ + higher powers of $1/t$ if $\gamma = 0$.*

PROOF. We proceed in all cases by induction.

(i): Note that for p as above, p does not divide p'. First assume that $\gamma > 1$. If $i = 1$, $P_1 = u$, so $v_{1\gamma} = u_\gamma$. If $i > 0$, then

$$P_{i+1} = P'_i + uP_i = \left(\frac{-i\gamma v_{i\gamma}p'}{p^{i\gamma+1}} + \cdots\right) + \left(\frac{u_\gamma}{p^\gamma} + \cdots\right)\left(\frac{v_{i\gamma}}{p^{i\gamma}} + \cdots\right)$$

$$= \left(\frac{u_\gamma v_{i\gamma}}{p^{(i+1)\gamma}} + \cdots\right) \qquad \text{since } (i+1)\gamma > i\gamma + 1$$

$$= \frac{v_{(i+1)\gamma}}{p^{(i+1)\gamma}} + \cdots \qquad \text{where } v_{(i+1)\gamma} \equiv (u_\gamma)^{i+1} \bmod p.$$

Now assume that $\gamma = 1$. If $i = 1$ then the result is obvious. For $i > 0$,

$$P_{i+1} = P'_i + uP_i = \left(\frac{-iv_ip'}{p^{i+1}} + \cdots\right) + \left(\frac{u_1}{p} + \cdots\right)\left(\frac{v_i}{p^i} + \cdots\right)$$

$$= \frac{v_i(u_1 - ip')}{p^{i+1}} + \cdots$$

$$= \frac{v_{i+1}}{p^{i+1}} + \cdots \qquad \text{where } v_{i+1} \equiv \prod_{j=0}^{i}(u_1 - jp') \bmod p.$$

(ii) and (iii): The proofs are similar to (i), proceeding by induction and comparing leading terms.

PROPOSITION 2.3. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$ and assume that either $t' \in E$ or $t'/t \in E$ and that t is transcendental over E. Furthermore, assume that we can effectively solve homogeneous linear differential equations over $E(t)$ and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can decide if a homogeneous linear differential equation $L(y) = 0$ with coefficients in $E(t)$ has a solution $e^{\int u}$ with $u \in E(t)$.*

PROOF. Assume that $t' \in E$. We wish to decide if there is a u in $E(t)$ such that $R(u) = 0$ where $R(u)$ is the Riccati equation associated with $L(y) = 0$. We shall try and determine the possible partial fraction decomposition for such a u. Let $p(t)$ be a monic irreducible polynomial in $E[t]$ and let $u = u_\gamma/p^\gamma + u_{\gamma-1}/p^{\gamma-1} + \cdots$, where $\deg_t u_i < \deg_t p$ and $\gamma > 1$. I claim that one can find γ and u_γ up to some finite set of choices. The following method is very similar to the Newton polygon process used to expand algebraic functions. Let $L(y) = y^{(n)} + A_{n-1}y^{(n-1)} + \cdots + A_0 y$ and $A_i = a_{i\alpha_i}/p^{\alpha_i} + \cdots$. The leading powers in $R(u) = P_n + A_{n-1}P_{n-1} + \cdots + A_0$ must cancel. The leading term of A_iP_i is $(a_{i\alpha_i}v_{i\gamma})/p^{\alpha_i + i\gamma}$ (using the notation of Lemma 2.2). Therefore for some i, j, $i \neq j$, we have $\alpha_i + i\gamma = \alpha_j + j\gamma$ or $\gamma = \alpha_i - \alpha_j/(i-j)$. Fix a value of γ and a corresponding j such that $\alpha_k + k\gamma \leq \alpha_j + j\gamma$ for all other k (of course we only consider such γ that are integers > 1). Summing over all h such that $\alpha_h + h\gamma = \alpha_j + j\gamma$ we have $\sum a_{h\alpha_h}v_{h\gamma} = 0$. Lemma 2.2 implies that $\sum a_{h\alpha_h}u_\gamma^h = 0 \bmod p$. Since $\deg_t u_\gamma < \deg_t p$, this latter equation determines u_γ up to some finite set (to find u_γ we factor $\sum a_{h\alpha_h}Y^h$ in $(E(t)/p)[Y]$ and consider the linear factors). We now alter our original $L(y)$. Let $\tilde{L}(y) = L(y\, e^{\int(u_\gamma/p^\gamma)})/e^{\int(u_\gamma/p^\gamma)}$. We now look for solutions of $\tilde{L}(y) = 0$ of the form $e^{\int \tilde{u}}$ with $\tilde{u} \in E$ and $\tilde{u} = \tilde{u}_\delta/p^\delta + \cdots$ with δ an integer. We proceed now as above, except we only consider those δ with $\delta < \gamma$. Note that if $u = u_\delta/p^\delta + \cdots$ satisfies $R(u) = 0$ with $\delta > 1$, then p must occur in the denominator of some A_i. Therefore, we continue until we can assume that u is of the form $\sum u_{j1}/p_j + s$, where $s \in E[t]$. Some of the p_j occur in denominators of the A_i and some do not. Let $p = p_j$ occur in the denominator of some A_i and let $u_1 = u_{j1}$. We then look for cancellation as before. Fixing a value of i and summing over all h such that $\alpha_h + h = \alpha_i + i$, we have that $\sum a_{h\alpha_h}v_h = 0$. We have that $v_h \equiv \prod_{j=0}^{h-1}(u_1 - jp') \bmod p$ by Lemma 2.2, so u_1 will satisfy $\sum a_{h\alpha_h}(\prod_{j=0}^{h-1}(u_1 - jp')) \equiv 0 \bmod p$. This equation is a non-zero polynomial in u_1, and u_1 is assumed to have degree less than the degree of p, so we can determine u_1 up to some finite set of choices, as before. We can alter $L(y)$ as before and assume that u is of the form $u = \sum u_{j1}/p_j + s$, where this sum is over all p_j that do not occur in the denominator of some A_i. For such a p_j (which we again refer to as p), the leading term in the p-adic expansion of $R(u)$ is v_n/p^n (by Lemma 2.2), so $v_n = 0$ and so $\prod_{j=0}^{n-1}(u_i - jp') \equiv 0 \bmod p$. Therefore $u_1 = jp'$ for some j, $1 \leq j \leq n-1$. This allows us to assume that u is of the form $u = \sum (n_jp_j')/p_j + s$ where the n_j are integers and s and the p_j are polynomials not yet determined. We now proceed to determine $s = s_mt^m + \cdots + s_0$. First assume that $m > 1$. Expanding u in decreasing powers of t, we have $u = s_mt^m +$ smaller powers of t. Lemma 2.2(ii) implies that $P_i(u) = s_m^it^{im} +$ lower powers of t. Writing $A_i = a_it^{\alpha_i} +$ lower powers of t, we see that for cancellation to occur in $R(u)$ we must have $\alpha_i + im = \alpha_j + jm$ for some $i \neq j$. Therefore m can be determined up to some finite set of possibilities by considering the possible integer $\alpha_i - \alpha_j/(j-i)$. We fix such a value of m and a j such that $\alpha_k + km \leq \alpha_j + jm$ for all other k. Summing over all h such that $\alpha_h + hm = \alpha_j + jm$, we have $\sum a_hs_m^h = 0$.

Therefore s_m is determined up to a finite set of possibilities. We can again alter $L(y)$ until we are in a position to assume that $u=u_0+\sum(n_ip_i')/p_i$. Looking for cancellation in $R(u)=0$, we have, by Lemma 2.2(ii), that $\sum a_iP_i(u_0,\ldots,u_0^{(i-1)})=0$, where the summation is over all i with $\alpha_i=\max_j(\alpha_j)$. Therefore $e^{\int u_0}$ satisfies $\hat{L}(y)=0$, where $\hat{L}(y)=\sum a_iy^{(i)}$, the summation being over all i with $\alpha_i=\max_j(\alpha_j)$. Since we can effectively find all exponential solutions of homogeneous linear differential equations over E, we can find a finite set $\{v_0,\ldots,v_r\}$ such that $e^{\int u_0}/e^{\int v_i}=r_i\in E(t)$ for some i. For each i, we form $L_i(y)=L(y\,e^{\int v_i})/e^{\int v_i}$. We then have that $y=r_i\exp\int(\sum(n_ip_i')/p_i)=r_i\prod p_i^{n_i}$ will satisfy some $L_i(y)$. Since we can effectively solve homogeneous linear differential equations over $E(t)$, we can find such a solution, and so reconstruct an exponential solution of our original differential equation.

We now deal with the case when $t'/t\in E$. We again try to determine the possible partial fraction expansions for solutions of $R(u)=0$. Let p be a monic irreducible polynomial in $E[t]$ and *assume* $p\neq t$. If p occurs in the denominator of u to a power larger than 1, then p must occur in the denominator of some A_i. For these p, we can proceed with the reduction used above. We can therefore assume that $u=\sum u_{j1}/p_j+s$, where the p_j are monic irreducible polynomials, $p_j\neq t$ and $s=s_m/t^m+\cdots+s_0+\cdots+s_Mt^M$. We can eliminate those p_j that appear in the denominator of some A_i as before and so assume that the p_j that appear do not occur in the denominator of any A_i. Fix some p_i, say p. The leading term in $R(u)=P_n(u)+A_{n-1}P_{n-1}(u)+\cdots+A_0$ is (using the notation Lemma 2.2(i)) v_n/p^n, where $v_n\equiv\prod_{j=0}^{n-1}(u_1-jp')\bmod p$ and u_1 is the leading coefficient in the p-adic expansion of u. Since $v_n=0$, we must have $u_1\equiv jp'\bmod p$ for some j, $1\leq j\leq n-1$. Since p is monic and the degree of p is the same as the degree of p' (say N), we have that $u_1=jp'-Nj\zeta p$ where $\zeta=t'/t$. Therefore $u=\sum((n_ip_i'+m_i\zeta p_i)/p_i)+s$, where the n_i and m_i are integers and $\deg_t(n_ip_i'+m_i\zeta p_i)<\deg_t p_i$. We now will determine the coefficients in $s=s_m/t^m+\cdots+s_Mt^M$. If $A_i=a_{i\alpha_i}/t^{\alpha_i}+$higher powers of t, then the leading term in the t-adic expansion of A_iP_i is $(a_{i\alpha_i}s_m^i)/t^{\alpha_i+mi}$. To get cancellation in $R(u)=0$, we must have two such terms being equal. This determines m up to some finite set of choices. Selecting an m and a j such that $km+\alpha_k\leq jm+\alpha_j$ for all other k and summing over all h such that $hm+\alpha_h=jm+\alpha_j$, we have $\sum a_{h\alpha_h}s_m^h=0$. Therefore s_m is determined up to some finite set of possibilities. We can determine s_M in a similar way. We can alter $L(y)$ as before until we are in a position to assume that $u=u_0+\sum(n_ip_i'+m_i\zeta p_i)/p_i$. Looking for cancellation in the $(1/t)$-adic expansion of $R(u)$, we have by Lemma 2.2(iii) that $\sum a_{h\alpha_h}P_h(u_0,\ldots,u_0^{(i-1)})=0$ where $A_i=a_{i\alpha_i}t^{\alpha_i}+$higher powers of $1/t$ and the summation is over all h such that $\alpha_h=\max_j(\alpha_j)$. Therefore $e^{\int u_0}$ satisfies $\hat{L}(y)=0$ where $\hat{L}(y)=\sum a_{i\alpha_i}y^{(i)}$, the summation being as before. Since we can effectively find all exponential solutions of homogeneous linear differential equations over E, we can find a finite set $\{v_0,\ldots,v_r\}$ such that $e^{\int u_0}/e^{\int v_i}=w_i\in E(t)$ for some i. For each i, $y_i=y\,e^{-\int v_i}=w_i\exp(\int\sum(n_ip_i'+m_i\zeta p_i)/p_i)=w_it^{(\sum m_i)}\prod p_i^{n_i}\in E(t)$. y_i also satisfies the linear differential equation $L_i(y)=L(y\,e^{\int v_i})/e^{\int v_i}=0$. Since we can effectively solve homogeneous linear differential equations over $E(t)$, we can decide if this equation has a non-zero solution in $E(t)$. If not, then $L(y)=0$ has no solution of the desired form and if so, we can reconstruct a solution of the desired form.

Examples are now given to illustrate Proposition 2.3.

EXAMPLE 2.3.1. Let $E=\mathbb{Q}(x)$ and $t=\log x$. We shall consider the differential equation

$$L(y)=y''-\frac{1}{x(\log x+1)}y'-(\log x+1)^2y=0$$

and decide if it has solutions of the form $e^{\int u}$ with $u \in E(t)$. We shall assume that the hypotheses of the theorem are satisfied by E (this will be shown later). The associated Riccati equation is

$$R(u) = (u' + u^2) - \frac{1}{x(\log x + 1)} u - (\log x + 1)^2 = 0.$$

Assume that u is a solution of $R(u) = 0$ in $E(t) = \mathbb{Q}(x, \log x)$. If $p(t) \neq t+1$ is irreducible in $E[t]$, then as we have noted above the order of u at $p(t)$ is bigger than or equal to -1. At $t+1 = \log x + 1$, we may write

$$u = \frac{u_\gamma}{(\log x + 1)^\gamma} + \frac{u_{\gamma-1}}{(\log x + 1)^{\gamma-1}} + \cdots.$$

Substituting this expression in $R(u)$ and comparing leading terms, one sees that if $\gamma > 1$, then the leading term in $R(u)$ is $u_\gamma (\log x + 1)^{2\gamma}$. If $\gamma = 1$, then the leading term (after some cancellation) is $u_1^2 (\log x + 1)^2$. This means that u cannot have a pole at $\log x + 1$. We therefore have that $u = \sum p_i'/p_i + s$ where the p_i are irreducible polynomials in $E[t]$, not equal to $t+1$ and s is a polynomial in $E[t]$. We now proceed to determine $s(t) = s_m t^m + \cdots + s_0$. Plugging into $R(u)$ and comparing terms we see that $m = 1$ and $s_1 = \pm 1$ and so $s(t) = \pm t + s_0 = \pm \log x + s_0$. We therefore alter $L(y)$ in two ways. Let

$$\begin{aligned} L_1(y) &= L(y\, e^{-\int \log x})/e^{-\int \log x} \\ &= y'' + \frac{-2x \log^2 x - 2x \log x - 1}{x \log x + x} y' + \frac{-2x \log^2 x - 3x \log x - x - 1}{x \log x + x} y. \end{aligned}$$

Let

$$\begin{aligned} L_2(y) &= L(y\, e^{\int \log x})/e^{\int \log x} \\ &= y'' + \frac{2x \log^2 x + 2x \log x - 1}{x \log x + x} y' + \frac{-2x \log^2 x - 3x \log x - x + 1}{x \log x + x} y. \end{aligned}$$

To determine the possible s_0 we consider L_1 and L_2 separately. In both cases we are looking for solutions of this equation of the form $y = e^{\int s_0 + (\sum p_i'/p_i)}$ with s_0 in E. For L_1, if we expand the coefficients in decreasing powers of $\log x$, we get

$$L_1(y) = y'' + (2 \log x + \cdots) y' + (-2 \log x + \cdots) y = 0.$$

$e^{\int s_0}$ will satisfy $\hat{L}_1(y) = 2y' - 2y = 0$. By the hypotheses, we can find exponential solutions of this latter equation over $E = \mathbb{Q}(x)$. In fact, e^x is the only such solution, i.e. the only possibility for s_0 is 1. We now modify $L_1(y)$ and form

$$\begin{aligned} \bar{L}_1(y) &= L_1(y\, e^x)/e^x \\ &= y'' + \frac{2x \log^2 x + 4x \log x + 2x - 1}{x \log x + x} y'. \end{aligned}$$

We are looking for solutions of this latter equation of the form $r(\exp(\int (\sum p_i'/p_i)))$ with r in $E(t)$, that is, solutions in $E(t)$. A partial fractions argument shows that the only such solutions are constants. This implies that our original equation has a solution of the form $e^{\int (\log x + 1)} = e^{x \log x}$. Repeating this procedure for $L_2(y)$ would yield a solution of our original equation of the form $e^{\int (-\log x - 1)} = e^{-x \log x}$.

EXAMPLE 2.3.2. Let $E = \mathbb{Q}(x)$ and $t = e^x$. We shall consider the differential equation

$$L(y) = y'' + (-2\,e^x - 1)y' + e^{2x}y = 0.$$

The associated Riccati equation is

$$R(u) = (u' + u^2) + (-2\,e^x - 1)u + e^{2x} = 0.$$

One easily shows that all solutions in $E(t)$ of $R(u) = 0$ must be of the form $u = s + \sum (n_i p_i' + m_i p_i)/p_i$ where p_i are irreducible in $E[t]$ and not equal to t, and $s = s_m/t^m + \cdots + s_M t^M$. One easily sees that $m = 0$. Substituting u in $R(u)$ and expanding in powers of t, we have

$$M s_M t^M + \cdots + s_M^2 t^{2M} + \cdots - 2M s_M t^{M+1} + \cdots + t^2 = 0.$$

Therefore $M = 1$ and $s_M = 1$. Therefore $u = t + s_0 + \sum (n_i p_i' + m_i p_i)/p_i$. We alter the equation $L(y) = 0$ to get $L_1(y) = L(y\,e^{\int e^x})/e^{\int e^x} = y'' - y'$. We are looking for solutions of the form $e^{\int s_0 + \sum (n_i p_i' + m_i p_i)/p_i}$. We find that $s_0 = 1$ or 0. We now form the equations $L_{11}(y) = L_1(y\,e^{\int 1})/e^{\int 1} = y'' + y'$ and $L_{12}(y) = L_1(y\,e^{\int 0})/e^{\int 0} = y'' - y'$ and look for solutions of these equations that lie in $E(t)$. These have solutions $e^{-x}, 1$ and $e^x, 1$ respectively. Therefore the original equation $L(y)$ has solutions e^{e^x} and $e^{x + e^x}$.

Note that in these last two examples we have found all exponential solutions of $L(y) = 0$, not just a single one. The algorithm described in Proposition 2.3 can be modified to do this, but we would rather do this task in the following

LEMMA 2.4. *Let K be a computable differential field.*

(i) *Assume that for any homogeneous linear differential equation $L(y) = 0$ with coefficients in K we can decide if there exists a $u \in K$ such that $L(e^{\int u}) = 0$ and if so find such an element. Then we can effectively find all exponential solutions of homogeneous linear differential equations over K.*

(ii) *Assume that we can effectively solve homogeneous linear differential equations over K and that we can find all exponential solutions of homogeneous linear differential equations over K. If $L(y) = 0$ is a homogeneous linear differential equation with coefficients in K then one can find u_i, $1 \le i \le r$ and v_{ij}, $1 \le i \le r$, $1 \le j \le n_j$, such that if $u \in K$ and $L(e^{\int u}) = 0$ then there exists an i, $1 \le i \le r$ and constants c_{ij} such that $e^{\int u} = (\sum_j c_{ij} u_{ij})\,e^{\int u_i}$.*

PROOF. (i) We proceed by induction on the order of the linear differential equation. Let $L(y) = 0$ be a homogeneous linear differential equation of order n with coefficients in K. Decide if there exists a $u \in K$ such that $L(e^{\int u}) = 0$. If no such element exists, we are done. Otherwise find such an element. Let $L_1(y) = L(y\,e^{\int u})/e^{\int u}$. $L_1(y)$ has no term of order zero, so we may write $L_1(y) = \tilde{L}(y')$, where $\tilde{L}(y)$ has order $n - 1$. By induction we can find $u_1, \ldots, u_r$ in K such that if v is in K and $\tilde{L}(e^{\int v}) = 0$, then $e^{\int v}/e^{\int u_i}$ is in K. Let $w \in K$ satisfy $L(e^{\int w}) = 0$. We then have $0 = \tilde{L}((e^{\int w - u})') = \tilde{L}(e^{\int w - u + (w' - u')/(w - u)})$. Therefore $e^{\int w - u}/e^{\int u_i} \in K$ or $(e^{\int w - u})' = 0$. We can conclude that if $e^{\int w}$ satisfies $L(y) = 0$, then either $e^{\int w}/e^{\int u_i + u} \in K$ or $e^{\int w}/e^{\int u} \in K$.

(ii) Let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in K. We can find $u_1, \ldots, u_r$ such that if $u \in K$ and $L(e^{\int u}) = 0$, the $e^{\int u}/e^{\int u_i} \in K$. For each i, form $L_i(y) = L(y\,e^{\int u_i})/e^{\int u_i}$ and find a basis $\{u_{ij}\}$ for the vector space of solutions in K of $L_i(y) = 0$. This choice of u_i and u_{ij} satisfies the conclusion of the lemma.

EXAMPLE 2.4.1. We consider the same equation as in Example 2.3.2, $L(y)=y''+(-2\,e^x-1)y'+e^{2x}y=0$. $e^{\int u}$ is a solution of this equation where $u=e^x$, so we form $L_1(y)=L(y\,e^{e^x})/e^{e^x}=y''-y'$. Therefore $\tilde{L}(y)=y'-y$. This latter equation has solution $e^{\int u}$ where $u=x$. Therefore if $w\in K=\mathbb{Q}(x,e^x)$ and $L(e^{\int w})=0$ then either $e^{\int w}/e^{e^x}\in K$ or $e^{\int w}/e^{e^x+x}\in K$.

LEMMA 2.5. *Let K be a computable differential field with an algebraically closed field of constants and $L(y)=0$ a homogeneous linear differential equation with coefficients that lie in a finitely generated algebraic extension E of K. Assume that one can effectively find all solutions of homogeneous linear differential equations over K and effectively find all exponential solutions of homogeneous linear differential equations over K. Then*

(i) *One can decide if there exists an element u algebraic over K such that $L(e^{\int u})=0$ and if so find a minimal polynomial of u over K.*

(ii) *One can find an algebraic extension F of E and elements u_i, u_{ij} in F such that if u is an element in F and $L(e^{\int u})=0$, then there exist an i and constants c_{ij} such that $e^{\int u}=(\sum_j c_{ij}u_{ij})\,e^{\int u_i}$.*

(iii) *One can find elements $y_1,\ldots,y_r$, liouvillian over K, that span the space of all solutions of $L(y)=0$ that are liouvillian over K.*

PROOF. (i) This follows from the techniques and results of Singer (1981). For the convenience of the reader we outline the proof here. We know from Theorem 2.4 of Singer (1981) that if $L(y)=0$ has a solution of the prescribed form then it has one where u is algebraic over E of degree bounded by an integer N that depends only on the order n of L. Furthermore, there is a recursive function $I(n)$ such that $N\le I(n)$. Therefore u will satisfy a polynomial equation over F of degree at most $I(n)[E:F]$. Fix an integer $m\le I(n)[E:F]$. We wish to decide if there exist $a_{m-1},\ldots,a_0$ in K such that if $f(u)=u^m+a_{m-1}u^{m-1}+\cdots+a_0=0$, then $L(e^{\int u})=0$. We shall first determine the possible a_{m-1} that can occur. We may assume that E is a normal extension of F and let $G=\{\sigma_1,\ldots,\sigma_t\}$ be the galois group of E over K. For each $\sigma_i\in G$, let $L_i(y)=0$ be the homogeneous linear differential equation obtained by applying σ_i to the coefficients of $L(y)=0$. Let $\tilde{L}(y)=0$ be the homogeneous linear differential equation whose solution space is $\{y_1+\cdots+y_m \mid L_i(y_i)=0,\ i=1,\ldots,m\}$ (see Lemma 3.8 of Singer (1981)). Assuming that f is irreducible, we have that $\tilde{L}(e^{\int u_i})=0$ for all roots u_i of $f(u)=0$. Let $y_i=e^{\int u_i}$. We see that

$$
\begin{aligned}
a_{m-1}&=-(u_1+\cdots+u_m)\\
&=-\left(\frac{y_1'}{y_1}+\cdots+\frac{y_m'}{y_m}\right)\\
&=-\left(\frac{(y_1\cdots y_m)'}{y_1\cdots y_m}\right).
\end{aligned}
$$

Let $L_1(y)=0$ be the homogeneous linear differential equation with coefficients in E, satisfied by $z_m=y_1\cdots y_m$. This can be calculated from $\tilde{L}(y)$ using Lemma 3.8 of Singer (1981). Since $z_m'/z_m\in K$, $\hat{L}_1(z_m)=0$ where $\hat{L}_1(y)=\sum_{i=1}^{m}(L_1(y))^{\sigma}$. $\hat{L}_1(y)$ has coefficients in K, so by Lemma 2.4(ii), we can find v_i and v_{ij} in K such that $z_m=(\sum c_{ij}v_{ij})\,e^{\int v_i}$ for some constants c_{ij}. Therefore, for some i, $a_{m-1}=v_i+(\sum c_{ij}v_{ij})'/(\sum c_{ij}v_{ij})$. We can conclude that we can construct a finite number of rational functions $R_{m-1,i}(c_{i,j})$ with coefficients in K such that for some choice of constants c_{ij} and i, $a_{m-1}=R_{m-1,i}(c_{i,j})$. To compute a_{m-2}, note that

$$a_{m-2}=\sum_{1\le i,j\le m} u_i u_j$$
$$=\sum_{1\le i,j\le m}\frac{y_i' y_j'}{y_i y_j}$$
$$=c(\prod y_j)^{-1}\sum y_{i_1}' y_{i_2}' y_{i_3}\cdots y_{i_m}$$

where this latter sum is taken over all permutations of $(1,\ldots,m)$. Let $P(Y_1,\ldots,Y_m)=\sum Y_{i_1}' Y_{i_2}' Y_{i_3}\cdots Y_{i_m}$. We can construct a linear differential equation $L_2(y)$ with coefficients in E, such that for any solutions $y_1,\ldots,y_m$ of $L(y)=0$, $L_2(P(y_1,\ldots,y_m))=0$. Note that $\prod y_j=\sum_j c_{ij}v_{ij}\,e^{\int v_i}$ for some i and constants c_{ij} (as above). Therefore, for some i, we have that $(\prod y_j)a_{m-2}=\sum c_{ij}v_{ij}\,e^{\int v_i}a_{m-2}=P(y_1,\ldots,y_m)$ and so $e^{-\int v_i}P(y_1,\ldots,y_m)$ is in K. Let $L_{2,i}(y)=L_2(e^{\int v_i}y)/e^{\int v_i}$ and let $\hat{L}_{2,i}(y)=\sum_{i=1}^{t}(L_{2,i}(y))^{\sigma_i}$. Since $L_{2,i}(e^{\int v_i}P(y_1,\ldots,y_n))=0$, and $e^{-\int v_i}P(y_1,\ldots,y_m)\in K$, we have $\hat{L}_{2,i}(e^{-\int v_i}P(y_1,\ldots,y_m))=0$. By assumption, we can find $\{w_{ij}\}$ in K such that for each i, $\{w_{i,j}\}$ forms a basis of the vector space of solutions of $\hat{L}_{2,i}(y)=0$ in K. Therefore, for some i and constants c_{ij}, d_{ij}, $a_{m-2}=c(\prod y_i)^{-1}P(y_1,\ldots,y_n)=(\sum_j c_{ij}v_{ij}\,e^{\int v_i})^{-1}P(y_1,\ldots,y_n)=(\sum_j c_{ij}v_{ij})^{-1}(\sum_j d_{ij}w_{ij})$. We denote this rational function as $R_{2,i}$. In a similar way we can find, for the other a_{n-1}, expressions $R_{h,i}$ that are rational functions of known quantities with unknown constant coefficients. For all possible choices of $i=(i_0,\ldots,i_{m-1})$ we form

$$f_i(u)=u^m+R_{m-1,i_{m-1}}u^{m-1}+\cdots+R_{0,i_0}.$$

We wish to determine if there is a choice of constants $c_{ij}, d_{ij},\ldots$ such that any solution of $f_i(u)=0$ is a solution of $R(u)=0$, where $R(u)=0$ is the Riccati equation associated with $L(y)=0$. If we reduce $R(u)$ with respect to $f_i(u)$ as in Ritt (1966, p. 6), we get a remainder $H_i(u)$ that must vanish identically. This forces a collection of polynomials (with coefficients in E) in the $c_{ij}, d_{ij},\ldots$ to vanish identically. Since we are looking for constant solutions, there is an equivalent set of polynomials with constant coefficients. We can then decide if there exist constants that satisfy these polynomial equations. If such a set of constants do not exist then $L(y)=0$ does not have a solution of the desired form. If such a set does exist, then we factor $f_i(u)$ to find a minimal polynomial for u.

(ii) We proceed by induction on the order of $L(y)$. If the order is 1, then $L(y)=y'+ay$, for some $a\in K$. We then let $F=E$ and note that for any u in F such that $L(e^{\int u})=0$, we have $e^{\int u}=c\,e^{\int -a}$, for some constant c. Now assume that $L(y)$ has order $n>1$. By part (i) of this lemma, we can decide if there is a u algebraic over K such that $L(e^{\int u})=0$. Let $L_1(y)=L(y\,e^{\int u})/e^{\int u}$. $L_1(y)$ has no term of order zero so we may write $L_1(y)=\tilde{L}(y')$, where $\tilde{L}(y)$ has order $n-1$ and coefficients in $E(u)$. By induction, there exists an algebraic extension F of $E(u)$ and elements v_i in F such that if v is in F and $\tilde{L}(e^{\int v})=0$, then $e^{\int v}/e^{\int v_i}\in F$. If $w\in F$ and $L(e^{\int w})=0$, then $\tilde{L}((e^{\int w-u})')=0$, so $(w-u)\,e^{\int w-u}/e^{\int v_i}\in F$ or $e^{\int w}=c\,e^{\int u}$ for some constant c. Therefore $e^{\int w}/e^{\int v_i+u}\in F$ or $e^{\int w}/e^{\int u}\in F$. Let $u_1=v_1+u,\ldots,u_r=v_r+u$, $u_{r+1}=u$ and $L_i(y)=L(y\,e^{\int u_i})/e^{\int u_i}$ for $i=1,\ldots,r+1$. Each L_i has coefficients in F, an explicitly given algebraic extension of K. In Proposition 3.1 we shall see that we can effectively solve homogeneous linear differential equations in F. Therefore, we can find u_{ij} such that, for each i, $\{u_{ij}\}$ forms a basis for the set of solutions of $L_i(y)=0$ in F. We have then found the desired u_i and u_{ij}.

(iii) We again proceed by induction on the order of $L(y)$. When $L(y)$ has order 1, then $L(y)=y'+ay$ for some a in K. Therefore $y_1=e^{-\int a}$ will satisfy the conclusion of the lemma. For $n>1$, Theorem 2.4 of Singer (1981) implies that if $L(y)=0$ has a solution liouvillian over K, then there exists an element u algebraic over K such that $L(e^{\int u})=0$. Part (i) of this lemma lets us decide if this is the case and if so, find such an element. For such a u, let $L_1(y)=L(y\,e^{\int u})/e^{\int u}$. Since L_1 has no term of order 0, we may write $L_1(y)=\tilde{L}(y')$, where $\tilde{L}(y)$ has order $n-1$. By induction, we can find $z_1,\dots,z_r$ that span the space of solutions of $\tilde{L}(y)=0$ liouvillian over $K(u)$. $e^{\int u}$, $e^{\int u}\int z_1,\dots,e^{\int u}\int z_r$ then span the solutions of $L(y)=0$ liouvillian over K.

EXAMPLE 2.5.1. Let $K=\mathbb{Q}(x,\log x)$ and consider the linear differential equation

$$L(y)=y''+\frac{4x\log x+2x}{4x^2\log x}y'-\frac{1}{4x^2\log x}y=0.$$

We will find all liouvillian solutions of this equation. We start by looking for all solutions of the form $e^{\int u}$ where u is algebraic over K of degree at most 2. In general, we would have to decide if there is such a solution with u algebraic over K of degree bounded by some computable function of the order of L, but in this case we will see that the number 2 is enough. u satisfies an equation of the form $f(u)=u^2+au+b=0$ with $a,b\in K$. We will furthermore assume that $f(u)$ is irreducible. We then have that $a=-(y_1'/y_1+y_2'/y_2)$ where y_1 and y_2 are solutions of $L(y)=0$. We now construct a linear differential equation $L_2(y)$ satisfied by all elements y_1y_2 where y_1 and y_2 are solutions of $L(y)=0$. An algorithm for this is given in Singer (1981). We have

$$L_2(y)=y'''+\frac{6x\log x+3x}{2x^2\log x}y''+\frac{2\log x+1}{2x^2\log x}y'=0.$$

We need to find all solutions y of this latter equation such that $y'/y\in K$. An algorithm for this is given in Lemma 2.3 and Lemma 2.4. We find that the only such solutions are constants. This implies that $a=0$. To determine b, we note that $b=(y_1'y_2')/y_1y_2$, so $y_1y_2b=y_1'y_2'$. Since y_1y_2 must be a constant, $y_1'y_2'$ must be in K. We again construct a linear differential equation $L_3(y)=0$ satisfied by all expressions of the form $y_1'y_2'$. We find

$$\begin{aligned}L_3(y)=y'''&+\frac{18x^2\log^2 x+9x^2\log x}{2x^3\log^2 x}y''+\frac{38\log^2 x+43\log x+6x}{2x^3\log^2 x}y'\\&+\frac{16\log^2 x+32\log x+10}{2x^3\log^2 x}y\\&=0.\end{aligned}$$

We must find all solutions of this latter equation in K. An algorithm for this is given in Proposition 3.10. We find that the only such solutions are constant multiples of $1/(4x^2\log x)$. Therefore $f(u)$ must be of the form $u^2+c/(4x^2\log x)$ for some constant c. If $f(u)=0$, then $u^2=-c/(4x^2\log x)$ and $u'=-\frac{1}{2}[8x\log x+4x/(4x^2\log x)]\cdot u$. Substituting these expressions in

$$R(u)=u^2+u'+\frac{4x\log x+2x}{4x^2\log x}u-\frac{1}{4x^2\log x}=0$$

we see that $c=-1$. Therefore $f(u)=u^2-1/(4x^2\log x)$ so $L(y)$ has solutions of the form $y=e^w$ where $w=\pm(\log x)^{1/2}$. These two solutions form a basis for the space of all solutions of $L(y)=0$.

Proposition 2.6. *Let $E \subset E(t)$ be computable differential fields and assume that t is algebraic over E and that $C(E) = C(E(t))$ is algebraically closed. Assume that we can effectively solve homogeneous linear differential equations over E and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can decide if a homogeneous linear differential equation $L(y) = 0$ with coefficients in $E(t)$ has a solution $e^{\int u}$ with $u \in E(t)$.*

Proof. Let F, u_i, u_{ij} be as in Lemma 2.5(ii) where $E(t) \subset F$. If $L(e^{\int u}) = 0$ for some $u \in E(t)$, then there exists an i and constants c_{ij} such that $u = u_i + (\sum c_{ij} u_{ij})'/(\sum c_{ij} u_{ij})$. Therefore we need to decide if there exist constants c_{ij} such that $u_i + (\sum c_{ij} u_{ij})'/(\sum c_{ij} u_{ij}) \in E(t)$. If we write this in terms of a basis of F over $E(t)$, this is equivalent to a system of polynomials in the c_{ij}, with coefficients in $E(t)$ vanishing. There is an equivalent polynomial system with constant coefficients and we can decide if this has a solution in the subfield of constants.

Proof of Proposition 2.1. This follows immediately from Propositions 2.3 and 2.6 and Lemma 2.4(i).

3. Question 1

In this section we discuss the problem of answering question 1 for fields of the form $E(t)$ where E satisfies a suitable hypothesis and either $t'/t \in E$, $t' \in E$ or t is algebraic over E. We actually deal with a slightly more general question related to the following definition. Let K be a differential field. We say that we can *effectively solve parameterized linear differential equations over* K if given $a_{n-1}, \ldots, a_0, b_m, \ldots, b_0$ in K, one can effectively find $h_1, \ldots, h_r$ in K and a system $\mathscr{L}$ in $m+r$ variables with coefficients in $C(K)$ such that $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = c_1 b_1 + \cdots + c_m b_m$ for $y \in K$ and c_i in $C(K)$ if and only if $y = y_1 h_1 + \cdots + y_r h_r$ where the $y_i \in C(K)$ and $c_1, \ldots, c_m, y_1, \ldots, y_r$ satisfy $\mathscr{L}$. Obviously, if K is computable and we can effectively solve parameterized linear differential equations over K, then we can effectively solve homogeneous linear differential equations over K. Propositions 3.1 and 3.4 can be proved if both the hypotheses and conclusions regarding solving parameterized linear differential equations are replaced by the weaker statement that we can effectively solve homogeneous linear differential equations. In Proposition 3.9, we need the stronger statement to make the induction work. We prove these stronger statements with the hope that they will be more useful in applications.

We first deal with the field $E(t)$ where t is algebraic over E. Let $E[D]$ be the ring of differential operators with coefficients in E. This is the set of expressions of the form $a_n D^n + \cdots + a_0$ where multiplication corresponds to composition of these operators. In general, this is not a commutative ring, since $Da = D(a) + aD$. It is known that this ring has a right and left division algorithm (Poole, 1960, p. 31), so we can row and column reduce any matrix with coefficients in $E[D]$ (Poole 1960, p. 39).

Proposition 3.1. *Let E be a computable differential field and t an element algebraic over E. If we can effectively solve parameterized linear differential equations over E then we can effectively solve parameterized linear differential equations over $E(t)$.*

Proof. Let $1, t, \ldots, t^N$ form a vector space basis of $E(t)$ over E and let $y = y_0 + y_1 t + \cdots + y_N t^N$ where $y_0, \ldots, y_N$ are new variables. Using the fact that t' may be explicitly written as an element of $E(t)$, we may then write

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = c_1 b_1 + \cdots + c_m b_m$$

as

$$L_0(y_0, \ldots, y_N) + L_1(y_0, \ldots, y_N)t + \cdots + L_N(y_0, \ldots, y_N)t^N$$
$$= B_0(c_1, \ldots, c_m) + B_1(c_1, \ldots, c_m)t + \cdots + B_N(c_1, \ldots, c_m)t^N$$

where the L_i are linear differential equations in the y_j with coefficients in E and the B_i are linear polynomials in the c_j with coefficients in K. We can write this latter expression in matrix form $AY = B$ where A is an $N+1 \times N+1$ matrix with entries in $E[D]$, $Y = (y_0, \ldots, y_N)^T$ and $B = (B_0, \ldots, B_N)^T$. Using row and column reduction, we can find matrices U and V with entries in $E[D]$ such that U has a left inverse, V has a right inverse and $UAV = C$ where

$$C = \begin{bmatrix} \tilde{L}_0 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{L}_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdots & \tilde{L}_N \end{bmatrix}$$

and the $\tilde{L}_i$ are in $E[D]$. Y is a solution of $AY = B$ if and only if $W = V^{-1}Y$ is a solution of $CW = UB$. Solving this latter system is equivalent to solving $N+1$ equations $\tilde{L}_i(w_i) = \sum c_j \tilde{b}_{ij}$, where the $\tilde{b}_j$ are in E. Since we can effectively solve parameterized linear differential equations in E we can find appropriate h_{ij} in E and systems of linear equations $\mathscr{L}_i$. Using these we can construct elements h_j in $E(t)$ and a system $\mathscr{L}$ of linear equations satisfying the conditions for $L(y) = \sum c_i b_i$ in the definition of effectively solving parameterized linear differential equations.

An example illustrating the above proposition is given in Davenport & Singer (1986, p. 242). We now turn to fields of the form $E(t)$ where $t'/t \in E$ or $t' \in E$.

LEMMA 3.2. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$, t transcendental over E and either $t'/t \in E$ or $t' \in E$. Assume*:

(i) *we can effectively solve parameterized linear differential equations over E,*

(ii) *if $t'/t \in E$ and $A_n, \ldots, A_0$, $B_m, \ldots, B_1$ are in $E[t, t^{-1}]$, we can effectively find an integer M such that if $Y = y_\gamma/t^\gamma + \cdots + y_0 + \cdots + y_\delta t^\delta$ with $y_i \in E$, $y_\delta y_\gamma \neq 0$, satisfies $A_n Y^{(n)} + \cdots + A_0 Y = c_m B_m + \cdots + c_1 B_1$ for some $c_i \in C(E)$, then $\gamma \leq M$ and $\delta \leq M$.*

(iii) *if $t' \in E$ and $A_n, \ldots, A_0$, $B_m, \ldots, B_1 \in E[t]$, we can effectively find an integer M such that if $Y = y_0 + \cdots + y_\gamma t^\gamma$ with $y_i \in E$, $y_\gamma \neq 0$, satisfies $A_n Y^{(n)} + \cdots + A_0 Y = c_m B_m + \cdots + c_1 B_1$ for some c_i in $C(E)$, then $\gamma \leq M$.*

Then we can effectively solve parameterized linear differential equations over $E(t)$.

PROOF. We first consider the case where $t'/t \in E$. Let

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = c_1 b_1 + \cdots + c_m b_m \tag{1}$$

with the $a_i, b_i \in E(t)$. Let p be a monic irreducible polynomial in $E[t]$, $p \neq t$, and let

$$y = \frac{y_\alpha}{p^\alpha} + \cdots$$

$$a_i = \frac{a_{i\alpha_i}}{p^{\alpha_i}} + \cdots$$

$$b_i = \frac{b_{i\beta_i}}{p^{\beta_i}} + \cdots$$

be the p-adic expansions of these elements (for convenience we define $a_n = 1$ so $a_{n0} = 1$). Differentiating, we see that

$$y^{(j)} = \frac{u_j}{p^{\alpha+j}} + \cdots$$

where $u_j \equiv \pm\alpha(\alpha+1)\cdots(\alpha+j-1)y_\alpha(p')^j \bmod p$. Note that p' and p are relatively prime so that $u_j \neq 0$. If $\alpha > 0$, then some $\alpha_i > 0$ or some $\beta_i > 0$. Therefore only $p \neq t$ that occur to negative powers in the partial fraction decomposition of a solution of (1) have this property. We shall first try to bound α for such a p. In order for cancellation to occur in (1), we must have that either $\max_i(\alpha + i + \alpha_i) \leq \max_i \beta_i$, in which case we can bound α or $\gamma = \max_i(\alpha + i + \alpha_i) > \max_i \beta_i$. In this latter case we must have $\sum a_{i\alpha_i} u_i \equiv 0 \bmod p$, where the sum is over all i such that $\gamma = \alpha + i + \alpha_i$. This latter equation can be rewritten as $\sum a_{i\alpha_i}(\pm\alpha(\alpha+1)\cdots(\alpha+i-1)y_\alpha(p')^i) \equiv 0 \bmod p$. We can divide by y_α and get $\sum a_{i\alpha_i}(\pm\alpha(\alpha+1)\cdots(\alpha+i-1)(p')^i) \equiv 0 \bmod p$. Since p' and p are relatively prime and, for each i, $a_{i\alpha_i}$ and p are relatively prime, this latter equation gives a non-zero polynomial that α must satisfy. α is therefore determined up to some finite set of choices and so we can effectively find a bound α^*. Set $y = Y/p_1^{\alpha_1^*}\cdots p_k^{\alpha_k^*}$, where the p_j are those monic irreducible polynomials ($\neq t$) appearing in the denominators of some a_i or b_i and the α_j^* are the bounds calculated above. Substitute this into $L(y) = c_1 b_1 + \cdots + c_m b_m$ and clear denominators to get

$$A_n Y^{(n)} + A_{n-1} Y^{(n-1)} + \cdots + A_0 Y = c_m B_m + \cdots + c_m B_m, \tag{2}$$

where

$$Y = y_\gamma / t^\gamma + \cdots + y_0 + \cdots + y_\delta t^\delta$$

with the y_i and the $a_{i\alpha_i}$ in E and $A_n, \ldots, A_0, B_1, \ldots, B_m$ in $E[t, t^{-1}]$. By our hypotheses, we can find an M such that $\delta \leq M$ and $\gamma \leq M$.

We now wish to determine the y_j. Substituting our expression for Y into (2) and writing this in terms of powers of t, we have

$$\begin{aligned} L_{N_1}(y_\gamma, \ldots, y_\delta)t^{-N_1} + \cdots + L_{N_2}(y_\gamma, \ldots, y_\delta)t^{N_2} \\ = C_{N_3}(c_1, \ldots, c_m)t^{-N_3} + \cdots + C_{N_4}(c_1, \ldots, c_m)t^{N_4} \end{aligned}$$

for some $N_1 \leq N_2$ and $N_3 \leq N_4$ integers, where the L_i are linear differential equations in the y_j with coefficients in E and the C_j are linear in the c_i with coefficients in E. If $N_3 > N_1$, we set $C_{N_3} = \cdots = C_{N_1+1} = 0$ and get a system of linear equations $\mathcal{L}_1$ for the c_i. We similarly can get a system of linear equations $\mathcal{L}_2$ if $N_2 > N_4$. For $N_1 \leq i \leq N_2$, we have the equations $L_i(y_\alpha, \ldots, y_\gamma) = C_i(c_1, \ldots, c_m)$. This system can be written as $AY = B$, where A is an $[N_2 + N_1 + 1] \times [N_2 + N_1 + 1]$ matrix with coefficients in $E[D]$, $Y = (y_\gamma, \ldots, y_\delta)^T$ and $B = (C_\gamma, \ldots, C_\delta)^T$. We can find (as in Proposition 3.1) an equivalent diagonal system $CW = UB$ and apply the hypotheses of this proposition to find linear systems $\mathcal{L}_i$ in the c_j and appropriate h_{ij}. Transforming these back to our system $AY = B$

and then substituting into $y=y_\gamma t^{-\gamma}+\cdots+y_\delta t^\delta$ gives us the appropriate h_i for the conclusion of this proposition. We may take $\mathscr{L}=\mathscr{L}_1\cup\mathscr{L}_2\cup(\cup\mathscr{L}_i)$.

The proof when $t'\in E$ follows in a similar manner and will be omitted.

LEMMA 3.3. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$, t transcendental over E and $t'/t\in E$. Assume:*

(i) *we can effectively find all exponential solutions of homogeneous linear differential equations over E, and*

(ii) *for any u in E, we can decide if $y'+uy$ has a non-zero solution in $E(t)$ and find such a solution.*

Then given any $A_n,\ldots,A_0$, $B_m,\ldots,B_1$ in $E[t,t^{-1}]$, we can effectively find an M such that if $Y=y_\gamma/t^\gamma+\cdots+y_\delta t^\delta$ with $y_i\in E$, $y_\gamma y_\delta\neq 0$, satisfies $A_nY^{(n)}+\cdots+A_0Y=c_mB_m+\cdots+c_1B_1$ for some $c_i\in C(E)$, then $\gamma\leq M$ and $\delta\leq M$.

PROOF. We first show how to bound γ. Let

$$A_i=\frac{a_{i\alpha_i}}{t^{\alpha_i}}+\cdots+a_{i\beta_i}t^{\beta_i}$$

$$c_1B_1+\cdots+c_mB_m=\frac{b_\mu}{t^\mu}+\cdots+b_\nu t^\nu$$

with the a_{ij} in E and the b_i linear in the c_j with coefficients in E. Note that (if $\gamma>0$) we have

$$Y^{(i)}=\frac{u_i}{t^\gamma}+\cdots\qquad\text{where } u_i=\left(\frac{y_\gamma}{t^\gamma}\right)^{(i)}t^\gamma\in E.$$

Furthermore, $u_i\neq 0$, since otherwise t would be algebraic over E. Substituting the above expression for Y into

$$A_nY^{(n)}+\cdots+A_0Y=c_mB_m+\cdots+c_1B_1 \tag{3}$$

and equating coefficients, we see that $\bar\alpha=\max_i(\gamma+\alpha_i)<\max_i\beta_i$, in which case γ can be bounded, or $\bar\alpha>\max_i\beta_i$. In this latter case, the leading term on the left hand side of (3) is $\sum a_{i\alpha_i}u_i/t^{\gamma+\alpha_i}$ where the summation is over all i such that $\gamma+\alpha_i=\bar\alpha$. We then will have

$$0=t^{-\gamma}\sum a_{i\alpha_i}u_i=\sum a_{i\alpha_i}\left(\frac{y_\gamma}{t^\gamma}\right)^{(i)}.$$

Therefore, $Z=y_\gamma/t^\gamma$ is a solution of $L(Z)=\sum a_{i\alpha}Z^{(i)}=0$. By our assumptions we can find u_j and u_{ij} in E such that for some j, $y_\gamma t^{-\gamma}=\sum d_iu_{ij}\,e^{\int u_i}$ for some constants d_i. This implies that for some j, $y'-u_jy=0$ has a solution in $E(t)$. Finding all such solutions allows us to bound γ. We can bound δ in a similar way.

PROPOSITION 3.4. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$, t transcendental over E and $t'/t\in E$. Assume that we can effectively find all exponential solutions of homogeneous linear differential equations over E and that for any u in E decide if $y'+uy=0$ has a non-zero solution in $E(t)$ and find all such a solution if it exists. Then we can effectively solve parameterized linear differential equations over $E(t)$.*

PROOF. Immediate from Lemma 3.2 and Lemma 3.3.

EXAMPLE 3.4.1. Let $E=\mathbb{Q}$ and $t=e^x$. Consider the linear differential equation

$$L(y)=y''+\frac{-24e^x-25}{4e^x+5}y'+\frac{20e^x}{4e^x+5}y=0.$$

We wish to find all solutions of this equation in $\mathbb{Q}(e^x)$. Using p-adic expansions for $p\neq t$, one can easily show that any solution must be of the form $y_\gamma/t^\gamma+\cdots+y_\delta t^\delta$. We therefore clear denominators in the above differential equation and consider

$$(4t+5)y''+(-24t-25)y'+20ty=0. \tag{4}$$

Comparing highest powers of t, we see that $y_\delta t^\delta$ satisfies $4y''-24y'+20y=0$. This latter equation has solutions e^{5x} and e^x that are exponential over $E=\mathbb{Q}$. Both of these are in $\mathbb{Q}(e^x)$. Therefore $\delta\le 5$. Comparing lowest powers of t, we see that y_γ/t^γ satisfies $5y''-25y'+20y=0$. This latter equation has solutions e^{4x} and e^x in $\mathbb{Q}(e^x)$. Since $\gamma\ge 0$, we conclude that either $\gamma=0$ or $y_\gamma=0$. Therefore $y=y_5t^5+\cdots+y_0$ for some y_i constants. If we substitute this expression in (4) we get the following

$$-12y_4t^5+(-20y_4-16y_3)t^4+(-30y_3-12y_2)t^3+(-30y_2)t^2+(20y_0-20y_1)t=0.$$

Equating powers of t to 0 and solving gives us that $y_2=y_3=y_4=0$ and $y_0=y_1$. Therefore, solutions of (4) in $E(t)$ are of the form $c_1e^{5x}+c_2(e^x+1)$ where c_1 and c_2 are arbitrary constants.

A few words need to be said about the assumption in the previous proposition that for $u\in E$ we can decide if $y'+uy=0$ has a solution in $E(t)$. A priori, this is stronger than the assumption that we can decide effectively find all exponential solutions or all solutions of homogeneous linear differential equations over E. Since $t'/t\in E$, it is known (Rosenlicht, 1976, Theorem 2) that any solution in $E(t)$ of $y'+uy=0$ must be of the form y_nt^n for some integer n. y_n will then satisfy $y_n'+(u+n(t'/t))y_n=0$. We are therefore asking to decide if there is some integer n such that this latter equation has a non-zero solution in E. Similar problems come up in the Risch algorithm for integration in finite terms (we are asking if $\int u=\log y_n+n\log t$ for some y_n and integer n). We do not know how to reduce this question to the assumptions that we can effectively find all exponential solutions or effectively solve homogeneous linear differential equations. The following lemma shows that there are classes of fields for which this hypothesis is true.

LEMMA 3.5. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$ and assume t is transcendental over E with $t'/t\in E$ or $t'\in E$.*

(i) *If E is an elementary extension of $C(x)$, $x'=1$, and $u\in E$, then one can decide if $y'+uy=0$ has a non-zero solution in $E(t)$ and find such a solution.*

(ii) *If E is a purely transcendental liouvillian extension of $C(x)$, $x'=1$, and $u\in E$, then one can decide if $y'+uy=0$ has a non-zero solution in $E(t)$ and find such a solution.*

PROOF. In this proof we shall rely heavily on the results of Rothstein & Caviness (1979) and the appendix of Singer *et al.* (1985). If $t'\in E$, then the Corollary to Theorem 1 of Rosenlicht (1976) implies that any solution u of $y'+uy=0$ in $E(t)$ is actually in E. If E is an elementary extension of $C(x)$, the result follows from Risch (1968). If E is a purely transcendental liouvillian extension of $C(x)$, the result follows from Theorem $A1(b)$ of Singer *et al.* (1985) and the fact that we can effectively embed such an extension in a log-explicit extension. We now assume that $t'/t\in E$ and let $t'/t=v$.

(i) Assume that E is an elementary extension of $C(x)$. We can use the Risch Algorithm (Risch, 1968) to decide if v has an elementary anti-derivative. If it does, then we can find $v_1, \ldots, v_r$ in E such that $E(\int v) \subset E(\log v_1, \ldots, \log v_r)$. Since, for each i, $E_i = E(\log v_1, \ldots, \log v_i)$ is an elementary extension of E, we can inductively decide if $\log v_{i+1}$ is algebraic over E_i (and so in E_i) or transcendental over E_i. Therefore we can assume that $E_r = E(\log v_1, \ldots, \log v_r)$ is a computable differential field. The corollary to Theorem 1 of Rosenlicht (1976) implies that t is transcendental over E_r. $E_r(t)$ is a generalized log-explicit extension of C and we can write $E_r(t) = C(t_1, \ldots, t_n)$ as in (Rothstein & Caviness, 1979, Theorem 3.1). It is enough to decide, for a given u in E, if $y' + uy = 0$ has a solution in $E_r(t)$, since the corollary to Theorem 1 of Rosenlicht (1976) implies that such a solution will lie in $E(t)$. To decide if $y' + uy$ has a solution in $E_r(t)$, we use Corollary 3.2 of Rothstein & Caviness (1979). According to this result, if such a solution existed then

$$u = c + \sum_{i \in \mathscr{L}} r_i t_i + \sum_{i \in \mathscr{E}} r_i a_i$$

where c is a constant, $\mathscr{L} = \{i \mid t_i' = a_i'/a_i$, for some $a_i \in C(t_1, \ldots, t_{i-1})\}$, and $\mathscr{E} = \{i \mid t_i'/t_i = a_i'$ for some $a_i \in C(t_1, \ldots, t_{i-1})\}$. Writing this last equation as $u' = \sum r_i t_i' + \sum r_i a_i'$, and expanding in terms of a $\mathbb{Q}$-basis of $C(t_1, \ldots, t_n)$, we can find a rational solution $\{r_i\}$ if one exists. If such a solution exists, then $y = e^{\int u} = d \prod_{i \in L} a_i^{r_i} \prod_{i \in E} t_i^{r_i}$, for some constant d. This means that for some integer N (that can be determined from the r_i) $(y/(d^{1/N}))^N \in E_r(t)$. $E_r(t)$ is a computable field, so to determine if $y \in E$, we need only factor $Y^N - (\prod_{i \in L} a_i^{r_i} \prod_{i \in E} t^{r_i})^N$ over $E_r(t)$.

If $\int v$ is not elementary over E, then $E(\int v, t)$ is a log explicit extension of C and we can proceed as above.

(ii) Either $\int v$ is in E or it is transcendental over E. Lemma 3.4 of Rothstein & Caviness (1979) and Theorem A1 of Singer *et al.* (1985) imply that one can effectively embed $E(t)$ into a regular (i.e. purely transcendental) log-explicit extension F of C. Furthermore F will be of the form $E(t_1, \ldots, t_n)$, with the t_i' in E. The corollary to Theorem 1 of Rosenlicht (1976) implies that t is transcendental over F. Given u in E it is enough to decide if $y' + uy = 0$ has a solution in $F(t)$, since the corollary to Theorem 1 of Rosenlicht (1976) will imply this solution lies in E. Therefore, let us assume that E is a regular log-explicit extension of C. Theorem A1(b) now allows us to decide if $y' + uy = 0$ has a solution in $E(t)$ and find such a solution if it does.

We will now prove a result similar to Lemma 3.3 for fields of the form $E(t)$ with $t' \in E$. This lemma will describe an algorithm to find a certain integer M that bounds the degree of solutions in $E[t]$ of linear differential equations. To show the algorithm is correct, we need to consider more general extensions of E and we will prove two simple lemmas about these extensions.

Let $E \subset E(t)$ be countable differential fields, $C(E) = C(E(t))$, t transcendental over E and $t' \in E$. Since $C(E)$ is countable we may assume that $C(E) \subset \mathbb{C}$. Let $F = \mathbb{C} \otimes_{C(E)} E$. We first note that t is transcendental over F. If not, then $t^n + a_{n-1} t^{n-1} + \cdots a_0 = 0$ for some $a_i \in F$. Differentiating this equation, we have $nt^{n-1}t' + a_{n-1}' t^{n-1} + \cdots = 0$ so $t' = (1/n) a_{n-1}'$. Therefore there exists a $u \in F$ such that $u' = t'$. Let $\{\gamma_i\} \subset \mathbb{C}$ be an E-basis of F and write $u = \sum \gamma_i u_i$ for some $u_i \in E$. We then have $\sum \gamma_i u_i' = t' \in E$. Therefore for some i, $\gamma_i = 1$ and $t' = u_i'$. This implies that in $E(t)$, $(u - t)' = 0$ so $t \in E$, a contradiction.

We now consider the field $K=F((t^{-1}))$, the field of formal Laurent series in t^{-1} with coefficients in F. We can extend the derivation on F to K by defining

$$\left(\sum_{i\le n_0} a_i t^i\right)' = a'_{n_0} t^{n_0} + \sum_{i\le n_0} (i a_i t' + a'_{i-1}) t^{i-1}.$$

Let $\kappa \in \mathbb{C}-\mathbb{Q}$ and define an extension $K(u)$ of K where u is transcendental over K and $u'/u=\kappa t'/t$. We then have

LEMMA 3.6. (i) $C(F)=C(K)$. (ii) $C(K)=C(K(u))$.

PROOF. (i) Let $(\sum_{i\le n_0} a_i t^i)'=0$. First assume that $n_0\ne 0$. We then have $a'_{n_0}=n_0 a_{n_0} t' + a'_{n_0-1}=0$. Therefore $t'=(a_{n_0-1}/n_0 a_{n_0})'$, so $t-(a_{n_0-1}/n_0 a_{n_0})\in C(F)$ contradicting the fact that t is transcendental over F. If $n_0=0$, let $n_1<n_0$ be the largest integer such that $a_{n_1}\ne 0$. We then have $a'_{n_0}=a'_{n_1}=0$ and $n_1 a_{n_1} t' + a'_{n_1-1}=0$ and we get a similar contradiction as above.

(ii) If $C(K)$ is properly contained in $C(K(t))$ then there exists an integer n and a $v\in K$ such that $v'/v=n\kappa t'/t$ (Risch, 1969). If we write $v=a_{n_0}t^{n_0}+a_{n_0-1}t^{n_0-1}+\cdots$, then

$$\frac{v'}{v}=\frac{a'_{n_0}t^{n_0}+(n_0 a_{n_0} t' + a'_{n_0-1})t^{n_0-1}+\cdots}{a_{n_0}t^{n_0}+\cdots}=n\kappa\frac{t'}{t}.$$

Therefore, $a'_{n_0}=0$ and $(n_0 a_{n_0} t' + a'_{n_0-1})/a_{n_0}=n\kappa t'$. This implies that $(n\kappa-n_0)t'=(a_{n_0-1}/a_{n_0})'$. Since $t\notin K$, we must have $n\kappa-n_0=0$, contradicting the fact that $\kappa\notin\mathbb{Q}$.

We need one more lemma before we can prove that the algorithm described in Lemma 3.8 terminates.

LEMMA 3.7. *Let $K\subset F$ be differential fields and assume that we can solve parameterized linear differential equations over K. Let $A_0,\ldots,A_n$, $B_1,\ldots,B_m\in K$ and let $\mathcal{L}$ be a set of homogeneous linear equations with coefficients in $C(K)$ and $z_1,\ldots,z_r$ be elements of K such that $A_n y^n+\cdots+A_0 y=c_m B_m+\cdots+c_1 B_1$ for $y\in K$, $c_i\in C(K)$ if and only if $y=\sum h_i z_i$ for some $h_i\in C(K)$ and $c_1,\ldots,c_m$, $h_1,\ldots,h_r$ satisfy $\mathcal{L}$. Then for $y\in K\cdot C(F)$ and $c_i\in C(F)$, we have $A_n y^{(n)}+\cdots+A_0 y=c_m B_m+\cdots+c_1 B_1$ if and only if $y=\sum h_i z_i$ for some $h_i\in C(F)$ and $c_1,\ldots,c_m$, $h_1,\ldots,h_r$ satisfy $\mathcal{L}$.*

PROOF. The proof follows by expanding $y\in K\cdot C(F)$ in a K-basis and noting that all equations (both differential and algebraic) involved are linear.

LEMMA 3.8. *Let $E\subset E(t)$ be computable differentiable fields with $C(E)=C(E(T))$, t transcendental over E and $t'\in E$. Assume that we can effectively solve parameterized linear differential equations over E. Let $A_n,\ldots,A_0$, $B_m,\ldots,B_1\in E[t]$. Then we can effectively find an integer M such that if $Y=y_0+\cdots+y_\gamma t^\gamma$, $y_\gamma\ne 0$, is a solution of*

$$A_n Y^{(n)}+\cdots+A_0 Y=c_m B_m+\cdots+c_1 B_1 \tag{5}$$

for some $c_i\in C(E)$ then $\gamma<M$.

PROOF. We shall describe a procedure that successively attempts to compute $y_\gamma, y_{\gamma-1},\ldots$. We shall then show that for some i, in the process of computing $y_{\gamma-i}$, we shall find a bound for γ. This bound will be independent of the c_is. At present we have no way of giving an a priori estimate for the i such that the computation of $y_{\gamma-i}$ gives us the bound for γ.

Let

$$A_i = a_{i\alpha} t^{\alpha} + \cdots + a_{i0}$$

$$B_i = b_{i\beta} t^{\beta} + \cdots + b_{i0}$$

where some $a_{i\alpha} \neq 0$ and some $b_{i\beta} \neq 0$. We replace Y in (5) by $Y = y_\gamma t^\gamma + \cdots + y_0$ and equate powers of t. We first consider the highest power of t, that is $t^{\gamma+\alpha}$. There are two cases: either $\gamma + \alpha \leq \beta$ or $\gamma + \alpha > \beta$ and

$$L_\gamma(y_\gamma) = \sum_{i=0}^{n} a_{i\alpha} y_\gamma^{(i)} = 0.$$

By our hypotheses, we can find $z_{\gamma 1}, \ldots, z_{\gamma r_\gamma}$ in E such that any solution y_γ of $L_\gamma(y_\gamma) = 0$ in E is of the form $y_\gamma = \sum_i c_{\gamma i} z_{\gamma i}$ for some $c_{\gamma i}$ in $C(E)$. If there are no non-zero solutions of $L_\gamma(y_\gamma) = 0$, we stop and have $\gamma \leq \beta - \alpha$. Otherwise, we now replace y_γ in (5) by $\sum c_{\gamma i} z_{\gamma i}$ (where the $c_{\gamma i}$ are indeterminants) and consider the coefficients of $t^{\gamma+\alpha-1}$. Either $\gamma + \alpha - 1 \leq \beta$ or $\gamma + \alpha - 1 > \beta$ and the coefficient of $t^{\gamma+\alpha-1}$ is

$$L_{\gamma-1}(y_{\gamma-1}) = \sum a_{i\alpha} y_{\gamma-1}^{(i)} - \left(\sum c_{\gamma j} e_{\gamma j} + \sum \gamma c_{\gamma j} f_{\gamma j}\right) = 0$$

where the $e_{\gamma j}$ and $f_{\gamma j}$ are known elements of E. By our hypotheses we can find $z_{\gamma-1,1}, \ldots, z_{\gamma-1,r_{\gamma-1}}$ in E and a linear system $\mathcal{L}_{\gamma-1}$ in $r_{\gamma-1} + r_\gamma$ variables with coefficients in $C(E)$ such that $y_{\gamma-1} = \sum c_{\gamma-1,i} z_{\gamma-1,i}$ is a solution of $L_{\gamma-1}(y_{\gamma-1}) = 0$ for some choice of $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}, \gamma$ if and only if $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}, \gamma c_{\gamma,1}, \ldots, \gamma c_{\gamma,r_\gamma})$ satisfies $\mathcal{L}_{\gamma-1}$. We can replace $\mathcal{L}_{\gamma-1}$ with a linear system $\mathcal{L}^*_{\gamma-1}$ having coefficients in $C[\gamma]$ such that $y_{\gamma-1} = \sum c_{\gamma-1,i} z_{\gamma-1,i}$ is a solution of $L_{\gamma-1}(y_{\gamma-1}) = 0$ for some choice of $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}$ if and only if $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ satisfies $\mathcal{L}^*_{\gamma-1}$. Using elimination theory, we can effectively find systems $\mathcal{S}_1^{(1)}, \ldots, \mathcal{S}_{n_1}^{(1)}$ where each $\mathcal{S}_i^{(1)} = \{f_{i,1}^{(1)} = 0, \ldots, f_{i,m_i}^{(1)} = 0, g_i^{(1)} \neq 0\}$ where $f_{i,j}^{(1)}, g_i^{(1)} \in C(E)[\gamma]$ such that for γ in some algebraically closed extension field k of $C(E)$, γ satisfies some $\mathcal{S}_i^{(1)}$ if and only if $\mathcal{L}^*_{\gamma-1}$ has a solution $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ with $(c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}) \neq (0, \ldots, 0)$. We shall deal with two cases:

Case 1. Each $\mathcal{S}_i^{(1)}$ has only a finite number of solutions γ. In this case we can bound γ by $\gamma \leq \max(\beta - \alpha$, integer solutions of the $\mathcal{S}_i^{(1)})$.

Case 2. Some $\mathcal{S}_i^{(1)}$ has an infinite number of solutions. In this case, such an $\mathcal{S}_i^{(1)}$ is of the form $\{0 = 0, g_i^{(1)} \neq 0\}$. When this happens we continue and attempt to calculate $y_{\gamma-2}$ in the following way.

We now replace $y_{\gamma-1}$ by $\sum c_{\gamma-1,j} z_{\gamma-1,j}$ in (5), where the $c_{\gamma-1,j}$ are undetermined coefficients and consider the coefficient of $t^{\gamma-2}$. This will be of the form

$$L_{\gamma-2}(y_{\gamma-2}) = \sum a_{i\alpha} y_{\gamma-2}^{(i)} - M_{\gamma-2}(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1) c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j})$$

where $M_{\gamma-2}$ is a linear form in the $c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1) c_{\gamma j}, \ldots$ with known coefficients from E. By our hypotheses, we can find $z_{\gamma-2,1}, \ldots, z_{\gamma-2,r_{\gamma-2}}$ in E and a system of linear equations $\mathcal{L}_{\gamma-2}$ with coefficients in $C(E)$ such that $y_{\gamma-2} = \sum c_{\gamma-2,i} z_{\gamma-2,i}$ is a solution of $L_{\gamma-2}(y_{\gamma-2}) = 0$ for some choice of $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1) c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j})$ if and only if $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1) c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j}, c_{\gamma-2,j})$ satisfies $\mathcal{L}_{\gamma-2}$. As before, we can absorb γ into the coefficients and produce a system of homogeneous linear equations $\mathcal{L}^*_{\gamma-2}$ with coefficients in $C(E)[\gamma]$ such that $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1) c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j}, c_{\gamma-2,j})$ is a solution of $\mathcal{L}_{\gamma-2}$ if and only if $(c_{\gamma,j}, c_{\gamma-1,j}, c_{\gamma-2,j})$ is a solution of $\mathcal{L}^*_{\gamma-2}$. Again there exist systems $S_1^{(2)}, \ldots, S_{n_2}^{(2)}$ where each $S_i^{(2)} = \{f_{i,1}^{(2)} = 0, \ldots, f_{i,m_i}^{(2)} = 0, g_i \neq 0\}$ with $f_{i,j}^{(2)}, g_i^{(2)} \in C(E)[\gamma]$, such that for any γ in some algebraically closed extension k of $C(E)$, $\mathcal{L}^*_{\gamma-2} \cup \mathcal{L}^*_{\gamma-1}$ has a solution

$(c_{\gamma,j}, c_{\gamma-1,j}, c_{\gamma-2,j})$ in k with the first r_γ coordinates not identically zero if and only if γ satisfies $\mathscr{S}_i^{(2)}$ for some i. We again have two cases:

Case 1. Each $\mathscr{S}_i^{(2)}$ has only a finite number of solutions. In this case we can bound γ by $\gamma \le \max(\beta-\alpha-1$, integer solutions of the $\mathscr{S}_i^{(2)})$.

Case 2. Some $\mathscr{S}_i^{(2)}$ has an infinite number of solutions. In this case such an $\mathscr{S}_i^{(2)}$ is of the form $\{0=0, g_i^{(2)} \ne 0\}$.

If we encounter case 2, we continue this process, otherwise we stop. Assume that we do not encounter case 1 before the kth repetition of the process. We have at this point found $z_{\gamma,1}, \ldots, z_{\gamma,r_\gamma}, \ldots, z_{\gamma-k+1,1}, \ldots, z_{\gamma-k+1,r_{\gamma-k+1}}$ and systems of linear equations $\mathscr{L}^*_{\gamma-1}, \ldots, \mathscr{L}^*_{\gamma-k+1}$, with coefficients in $C(E)[\gamma]$ such that for some c_i in $C(E)$ if $y = y_\gamma t^\gamma + \cdots$ is a solution of (5) with $y_\gamma \ne 0$ and $\gamma > \beta - \alpha + k - 1$, then there exist $c_{i,\gamma-j} \in C(E)$, $1 \le i \le r_{\gamma-j}$, $0 \le j \le k-1$ such that $y_{\gamma-j} = \sum c_{i,\gamma-j} z_{i,\gamma-j}$ and $\{c_{i,\gamma-j}\}$ satisfy $\mathscr{L}^*_{\gamma-1} \cup \cdots \cup \mathscr{L}^*_{\gamma-k+1}$. Furthermore, there are systems $\mathscr{S}_1^{(k-1)}, \ldots, \mathscr{S}_{n_{k-1}}^{(k-1)}$ such that $\mathscr{L}^*_{\gamma-1} \cup \cdots \cup \mathscr{L}^*_{\gamma-k+1}$ has a solution with $c_{\gamma,1} \cdots c_{\gamma,r_\gamma}$ not all zero if and only if γ satisfies some $\mathscr{S}_i^{(k-1)}$. We can continue if and only if some $\mathscr{S}_i^{(k-1)}$ is of the form $\{0=0, g_i^{(k-1)} \ne 0\}$. We shall show that for some k, we have that no $\mathscr{S}_i^{(k-1)}$ is of this form. This will show that the algorithm terminates.

We argue by contradiction, so assume the process continues indefinitely. We now think of $C(E)$ as being embedded in $\mathbb{C}$ and fix some $\kappa \in \mathbb{C}$ transcendental over $C(E)$ (note that $C(E)$ is countable and so this can be done). For each k, We are assuming that there is an $\mathscr{S}_i^{(k-1)}$ of the form $\{0=0, g_i^{(k-1)} \ne 0\}$. Clearly κ satisfies $\mathscr{S}_i^{(k-1)}$. Therefore, for this κ we can solve $\mathscr{L}^*_{\gamma-1} \cup \cdots \cup \mathscr{L}^*_{\gamma-k+1}$ in $\mathbb{C}$ with non-zero $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}$. Note that for fixed k the set V_k of $(c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ in $\mathbb{C}^{r_\gamma}$ such that $\mathscr{L}^*_{\gamma-1} \cup \cdots \cup \mathscr{L}^*_{\gamma-k+1}$ has a solution is a vector space. Notice that $V_k \supset V_{k+1}$ and $V_k \ne 0$. Therefore, for some k, we have $V_k = V_{k+1} = \cdots \ne 0$. This implies (using Lemma 3.7) that there exist $c_{i,\kappa-j} \in \mathbb{C}$, $1 \le i \le r_{\kappa-j}$, $0 \le r_{\kappa-j} < \infty$ such that

$$w_\kappa = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{r_{\kappa-j}} c_{i,\kappa-j} z_{i,\kappa-j} \right) t^{\kappa-j}$$

is a solution of $A_n y^{(n)} + \cdots + A_0 y = 0$ with $c_{\kappa,1}, \ldots, c_{\kappa,r_\kappa}$ not all zero. We can repeat the above argument for $\gamma = \kappa - 1, \ldots, \gamma = \kappa - n$ and produce $n+1$ solutions $w_\kappa, \ldots, w_{\kappa-n}$ in $E((t^{-1}))(t^\kappa)$ of $A_n y^{(n)} + \cdots + A_0 y = 0$. Note that by looking at leading terms, we can see that these solutions are linearly independent over $C(E)$ and therefore (by Lemma 3.6) over $C(E((t^{-1}))(t^\kappa)$. Since a homogeneous linear differential equation can have at most n solutions linearly independent over the constants (Kaplansky, 1957), this yields a contradiction. Therefore the process described above terminates.

Proposition 3.9. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$, t transcendental over E and $t' \in E$. Assume that we can effectively solve parameterized linear differential equations over E. Then we can effectively solve parameterized linear differential equations over $E(t)$.*

Proof. Immediate from Lemma 3.2 and Lemma 3.9.

Example 3.9.1. Let $E = \mathbb{Q}(x)$ and $t = \log x$. Let

$$L(y) = (x^2 \log^2 x) y'' + (x \log^2 x - 3x \log x) y' + 3y = 0$$

We will look for solutions y of $L(y) = 0$ in $E(t) = \mathbb{Q}(x, \log x)$. Considering y as a rational function of t, we see that the only possible irreducible factor of the denominator is $t=\log x$. If we expand y in powers of $\log x$ and write $y=y_\alpha/(\log x)^\alpha+\cdots$, we see that the leading coefficient in $L(y)$ is $y_\alpha[\alpha(\alpha+1)-3(-\alpha)+3]$. Since this must equal zero, we have that $(\alpha+3)(\alpha+1)=0$. This means that any solution of $L(y)=0$ in $E(t)$ is actually in $E[t]$. We let $y=y_\gamma t^\gamma+y_{\gamma-1}t^{\gamma-1}+\cdots$ and substitute into $L(y)=0$. Calculating the coefficients of powers of t, we get the following:

l	Coefficient of t^l
$\gamma+2$	$L_\gamma(y_\gamma)=x^2y''_\gamma+xy'_\gamma$
$\gamma+1$	$L_{\gamma-1}(y_{\gamma-1})=x^2y''_{\gamma-1}+xy'_{\gamma-1}+(2\gamma x-3x)y'_\gamma$
γ	$L_{\gamma-2}(y_{\gamma-2})=x^2y''_{\gamma-2}+xy'_{\gamma-2}+(2\gamma x-5x)y'_{\gamma-1}+(\gamma^2-4\gamma+3)y_\gamma$

It is easy to see that $L_\gamma(y_\gamma)=0$ has only constant solutions in E. Replacing y_γ by $c_{\gamma,1}\cdot 1$ in $L_{\gamma-1}(y_{\gamma-2})$ yields the equation $x^2y''_{\gamma-1}+xy'_{\gamma-1}=0$ for $y_{\gamma-1}$. This new equation has only constant solutions in E and places no restrictions on γ. We let $y_{\gamma-1}=c_{\gamma-1,1}\cdot 1$ and substitute in the expression $L_{\gamma-2}(y_{\gamma-2})$. We obtain

$$x^2y''_{\gamma-2}+xy'_{\gamma-2}+(\gamma^2-4\gamma+3)c_{\gamma,1}=0.$$

Since $c_{\gamma,1}\neq 0$, this latter equation has a solution in E if and only if $\gamma^2-4\gamma+3=0$. This implies that $\gamma\leq 3$. Therefore $y=y_3t^3+y_2t^2+y_1t+y_0$. Substituting this expression into $L(y)=0$ and calculating the coefficients of powers of t, we find:

l	Coefficient of t^l
5	$L_3(y_3)=x^2y''_3+xy'_3$
4	$L_2(y_2)=x^2y''_2+xy'_2+3xy'_3$
3	$L_1(y_1)=x^2y''_1+xy'_1+xy'_2$
2	$L_0(y_0)=x^2y''_0+xy'_0-xy'_1-4y_2$
1	$-3xy'_0$
0	$3y_0$

Successively setting these expressions equal to zero and finding solutions in E yields that y_3 and y_1 are arbitrary constants and y_2 and y_0 are 0. Therefore all solutions of $L(y)=0$ in $\mathbb{Q}(x,\log x)$ are of the form $c_1(\log x)^3+c_2\log x$.

4. Final Comments

Using the results of the last two sections, we can answer questions 1 and 2 for certain classes of fields.

THEOREM 4.1. *Let C be an algebraically closed computable field and assume that either*:

(i) *K is an elementary extension of $C(x)$ with $x'=1$ and $C(K)=C$, or*

(ii) *K is an algebraic extension of a purely transcendental liouvillian extension of C with $C(K)=C$.*

Then one can effectively find exponential solutions of homogeneous linear differential equations over K and effectively solve parameterized linear differential equations over K.

PROOF. It is easy to see that one can find exponential solutions of homogeneous linear differential equations and effectively solve parameterized linear differential equations over C. Using Propositions 2.1, 3.1, 3.4, 3.9 and Lemma 3.5, one can prove this theorem by induction on the number of elements used to define the tower leading to K.

As a consequence of this and Lemma 2.5(ii), one can generalize the results of Singer (1981) in the following way:

THEOREM 4.2. *Let C and K be as in Theorem* 4.1. *If $L(y)=0$ is a homogeneous linear differential equation with coefficients in K, then one can find a basis for the space of solutions of $L(y)=0$ liouvillian over K.*

There remain several open problems and directions for further research.

(a) The algorithms presented above are certainly not very efficient. Efficiency could certainly be improved by using (where possible) Hermite reduction techniques (cf. Bronstein, 1990). We also have sometimes assumed that the field of constants is algebraically closed. For actual computations one has a finitely generated field and one is forced to compute the necessary algebraic extension. Work needs to be done efficiently to find minimal algebraic extensions that are sufficient and also incorporate the D^5 method (Della Dora *et al.*, 1985; Dicrescenzo & Duval, 1989).

(b) There should be a more direct algorithm to solve the problem stated in Proposition 2.6. In particular, one should not have to first decide if there exists a u algebraic over $E(t)$ such that $L(e^{\int u})=0$ in order to decide if there is a u in $E(t)$ satisfying this property. A procedure just working in $E(t)$ would be preferable and would possibly avoid the need to assume that the field of constants is algebraically closed.

(c) We do not have a priori bounds on how many cycles are required in the procedure presented in Lemma 3.8. Is there a simple function $f(n)$ (where n is the order of the differential equation) such that the algorithm terminates after $f(n)$ steps?

(d) We would like to extend Theorems 4.1 and 4.2 to other classes of fields, in particular liouvillian extensions of C (not just purely transcendental liouvillian extensions). At present this would require extending Lemma 3.5 to such fields. This seems to be related to the problem of parameterized integration in finite terms mentioned in Davenport & Singer (1986).

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1.13 A rationality result for Kovacic's algorithm

By Marius van der Put, Peter A. Hendriks

A rationality result for Kovacic's algorithm

Marius van der Put Peter A. Hendriks

1 Statement of the result

Consider the second order linear differential equation $y'' = ry$, with $r \in \mathbf{Q}(x)$. Let $\mathbf{Q}^{cl}$ denote the algebraic closure of the field of rational numbers $\mathbf{Q}$ and let G denote the differential Galois group over $\mathbf{Q}^{cl}(x)$ of this equation. Then $G \subset Sl(2, \mathbf{Q}^{cl})$. For any solution $y \neq 0$ of the equation the element $u = \frac{y'}{y}$ satisfies the Riccati equation $u' + u^2 = r$. The main result in [Kov86] is the following theorem.

Theorem 1.1 *(See [Kov86])*

1. *If there is no algebraic solution over $\mathbf{Q}^{cl}(x)$ of the Riccati equation then $G = Sl(2, \mathbf{Q}^{cl})$.*
2. *If there is an algebraic solution of the Riccati equation then the minimal degree n of such an equation can be $1, 2, 4, 6, 12$ and*
   (a) *if $n = 1$ then $G \subset \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \mid c \in \mathbf{Q}^{cl} \setminus \{0\}, d \in \mathbf{Q}^{cl} \right\}$.*
   (b) *if $n = 2$ then $G = D_\infty$ or $G = D_m$ with $m \geq 3$.*
   (c) *if $n = 4$ then G is the tetrahedral group.*
   (d) *if $n = 6$ then G is the octahedral group.*
   (e) *if $n = 12$ then G is the icosahedral group.*

The list of conjugacy classes of algebraic subgroups of $Sl(2, \mathbf{Q}^{cl})$ appearing in above theorem is well known to be exhaustive. The Kovacic's algorithm for the calculation of an algebraic solution u uses algebraic extensions of $\mathbf{Q}$ of arbitrary degree. In [B92] a "rational" version of the Kovacic algorithm is indicated for $n = 2$.

In this paper we want to proof the following rationality result.

Theorem 1.2 *Suppose that the Riccati equation $u' + u^2 = r$ has a solution, which is algebraic over $\mathbf{Q}^{cl}(x)$. Then there exists an algebraic solution u of minimal degree n of the Riccati equation such that the coefficients of the minimum polynomial of u over $\mathbf{Q}^{cl}(x)$ lie in a field $K(x)$ with $[K : \mathbf{Q}] \leq 2$. Moreover, only in the cases: $n = 1$ and G is the multiplicative group $\mathbf{G}_m$ or a finite cyclic group of order > 2 or $n = 4$ and G the tetrahedral group, a field extension K of degree 2 of $\mathbf{Q}$ can be needed.*

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2 The proof

Again let $r \in \mathbf{Q}(x)$ and consider the differential equation $y'' = ry$. Suppose that $\alpha \in \mathbf{Q}$ is a regular point of the differential equation. Then there exists two independent solutions $y_0, y_1 \in \mathbf{Q}[[x - \alpha]]$ of this equation. Let $F = \mathbf{Q}^{cl}(x, y_0, y_1, y_0', y_1')$. The differential field $F \subset \mathbf{Q}^{cl}((x-\alpha))$ is a Picard-Vessiot extension of $\mathbf{Q}^{cl}(x)$ associated with the equation $y'' = ry$. By $DGal(E/F)$ of an extension of differential fields we denote the group of F-linear automorphisms of the field E commuting with the differentiation.

Lemma 2.1 *The sequence*

$$1 \to DGal(F/\mathbf{Q}^{cl}(x)) \xrightarrow{i} DGal(F/\mathbf{Q}(x))$$

$$\xrightarrow{\rho} Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to 1$$

is split exact.

Proof. We give a description of a splitting homomorphism $\tilde{s} : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to DGal(F/\mathbf{Q}(x))$. Let $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ and $\sum_{i=k}^{\infty} a_i(x-\alpha)^i \in \mathbf{Q}^{cl}((x-\alpha))$. Then we define the group homomorphism

$$s : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to DGal(\mathbf{Q}^{cl}((x-\alpha))/\mathbf{Q}(x))$$

$$\text{by } (s\sigma). \sum_{i=k}^{\infty} a_i(x-\alpha)^i = \sum_{i=k}^{\infty} \sigma(a_i)(x-\alpha)^i$$

It is easily seen that $s\sigma$ is a $\mathbf{Q}(x)$-linear automorphism of $\mathbf{Q}^{cl}((x-\alpha))$ and that $s\sigma$ and $\frac{d}{dx}$ commute. Further for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ we have $(s\sigma)(F) = F$, because $(s\sigma)\mathbf{Q}^{cl}(x) = \mathbf{Q}^{cl}(x)$ and $(s\sigma)(y_i^{(j)}) = y_i^{(j)}$ for $i, j = 0, 1$. We define $\tilde{s}\sigma$ to be the restriction of $s\sigma$ to F. It is clear that $\rho \circ \tilde{s} = id$. Hence $\tilde{s} : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to$

$DGal(F/\mathbf{Q}(x))$ is a splitting homomorphism. The exactness of the sequence is now obvious.

Proof of theorem 1.2. The proof is given case by case.

1. Suppose $n = 1$. There are three possibilities.

   (a) If G contains the additive group $\mathbf{G}_a$ then the Riccati equation has only one solution $u \in \mathbf{Q}^{cl}(x)$. If $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)u$ is also a solution of the Riccati equation. Hence $(\tilde{s}\sigma)u = u$ and $u \in \mathbf{Q}(x)$.

   (b) If G is the multiplicative group $\mathbf{G}_m$ or a finite cyclic group of order greater than 2 then there are exactly two solutions $u_1, u_2 \in \mathbf{Q}^{cl}(x)$. Hence for all $\sigma \in Gal(\mathbf{Q}^{cl/}/\mathbf{Q})$ we have $(\tilde{s}\sigma)\{u_1, u_2\} = \{u_1, u_2\}$. We conclude that $u_1, u_2 \in K(x)$, where K is a field such that $[K : \mathbf{Q}] \leq 2$.

   (c) If G is a cyclic group of order equal to 1 or 2 then the Riccati equation has infinitely many solutions $u \in \mathbf{Q}^{cl}(x)$. Let $u_0 = \frac{y_0'}{y_0}$. Then $u_0 \in \mathbf{Q}^{cl}(x) \cap \mathbf{Q}((x - \alpha)) = \mathbf{Q}(x)$ and u_0 satisfies the Riccati equation.

2. Suppose $n = 2$. Now $G = D_\infty$ or $G = D_m$ with $m \geq 3$. It is not difficult to verify that there are exactly two solutions u_1 and u_2 of the Riccati equation, which are quadratic over $\mathbf{Q}^{cl}(x)$. These solutions have the same minimum polynomial over $\mathbf{Q}^{cl}(x)$. If $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, u_2\} = \{u_1, u_2\}$. Hence the coefficients of the minimum polynomial over $\mathbf{Q}^{cl}(x)$ are fixed under $\tilde{s}\sigma$ and so the coefficients are already in $\mathbf{Q}(x)$.

3. Suppose $n = 4$. Now G is the tetrahedral group and $\#G = 24$. If u is a solution of the Riccati equation then $H_u = \{\sigma \in G \mid \sigma(u) = u\}$ is a cyclic subgroup of G and u is algebraic of degree $[G : H_u]$ over $\mathbf{Q}^{cl}(x)$. Conversely if $H \subset G$ is a cyclic subgroup then there are solutions u of the Riccati equation algebraic of degree $[G : H]$ over $\mathbf{Q}^{cl}(x)$, which are fixed under the action of H. Moreover, if $\#H \geq 3$ then there are exactly two solutions of the Riccatti equation algebraic of degree $[G : H]$ over $\mathbf{Q}^{cl}(x)$, which are fixed under the action of H. In the tetrahedral case $n = 4$ and there are four cyclic subgroups of G of order 6. Hence there are eight solutions $u_1, \ldots, u_8$ of the Riccati equation which are algebraic of degree 4 over $\mathbf{Q}^{cl}(x)$ and there are two monic irreducible polynomials P, Q of degree 4 over $\mathbf{Q}^{cl}(x)$, such that $(PQ)(u_i) = 0$ for $i = 1, \ldots, 8$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_8\} = \{u_1, \ldots, u_8\}$ and thus $(\tilde{s}\sigma)(PQ) = PQ$ and $(\tilde{s})\sigma\{P, Q\} = \{P, Q\}$. Hence the coefficients of P and Q lie in $K(x)$ where K satisfies $[K : \mathbf{Q}] \leq 2$.

4. Suppose $n = 6$. Now G is the octahedral group and $\#G = 48$. There are three cylic subgroups of order 8. Hence there are six solutions $u_1, \ldots, u_6$ of the Riccati equation which are algebraic of degree 6 over $\mathbf{Q}^{cl}(x)$ and there is a unique monic irreducible polynomial P such that $P(u_i) = 0$ for $i = 1, \ldots, 6$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_6\} = \{u_1, \ldots, u_6\}$. Therefore the coefficients of P are in $\mathbf{Q}(x)$.

5. Suppose $n = 12$. Now G is the icosahedral group and $\#G = 120$. There are six cylic subgroups of order 10. Hence there are twelve solutions $u_1, \ldots, u_{12}$ of the Riccati equation which are algebraic of degree 12 over $\mathbf{Q}^{cl}(x)$ and there is a unique monic irreducible polynomial P such that $P(u_i) = 0$ for $i = 1, \ldots, 12$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_{12}\} = \{u_1, \ldots, u_{12}\}$. It follows that the coefficients of P are in $\mathbf{Q}(x)$.

In section 4 we will see that in the tetrahedral case quadratic field extensions of $\mathbf{Q}$ can occur. However in the tetrahedral case there are exactly 6 solutions of the Riccati equation of degree 6. These solutions have the same minimum polynomial P over $\mathbf{Q}^{cl}$. The coefficients of this polynomial are in $\mathbf{Q}(x)$.

3 Remarks on lemma (2.1)

Let V denote the two-dimensional vector space over $\mathbf{Q}^{cl}$ of the solutions of $y'' = ry$ in F. The two-dimensional space $V_0 := \mathbf{Q}y_0 + \mathbf{Q}y_1$ has the property that $V_0 \otimes_{\mathbf{Q}} \mathbf{Q}^{cl} = V$ and the natural action of $Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ on this tensor product coincides with the action on V by the splitting homomorphism $\tilde{s}$.

For simplicity we have made the choice of Q and Q(x) in Thm 1.2. This choice can be replaced (without any change in the proof) by a field C of constants of characteristic zero and the function field $C(X)$ of an absolutely irreducible curve X over C such that X has a C-rational point x_0 which is a regular point of the differential equation. Indeed, the Picard-Vessiot field of the differential equation can be found inside the field of fractions of the completion of the local ring of $X \otimes C^{cl}$ at x_0 and the Galois group of C^{cl}/C acts in a natural way on this field.

In the general case: K a differential field with a (non algebraically closed) field of constants C one denotes by $K^+ = KC^{cl}$ the compositum of the fields K and C^{cl}. The Picard-Vessiot field F of a differential equation over K is seen as a field extension of K^+. One can show (using an algebraic construction of the Picard-Vessiot field) that the following natural sequence is exact:

$$1 \to DGal(F/K^+) \to DGal(F/K)$$

$$\to Gal(K^+/K) \to 1$$

We note that in general $Gal(K^+/K)$ is a proper subgroup of $Gal(C^{cl}/C)$ since K and C^{cl} need not be linearly disjunct over C. We do not know in this general situation that the sequence above splits. However, Thm 1.2 remains valid in this case and takes the form:

Let $n \in \{1, 2, 4, 6, 12\}$ denote the integer corresponding to the differential Galois group $G = DGal(F/K^+)$. There is a subfield L, $K \subset L \subset K^+$ with $[L : K] \leq 2$ and a solution u of the Riccati equation which is algebraic over K^+ with degree n such that the monic minimal equation of u over K^+ has its coefficients in L.

In stead of using the splitting, one can lift any element of $Gal(K^+/K)$ to a differential automorphism of F over K. That suffices for the proof of the various cases, except for the case $n = 1$ and G is a group of order 1 or 2.
There one needs the following ad hoc arguments: If $G = 1$ then $F = K^+$. Let $y \neq 0$ be a solution of the differential equation. Then $K(y)$ is an extension of K of some degree d and equals $K(z)$ where z is a constant. Write $y = f_0 + f_1 z + \dots + f_{d-1} z^{d-1}$ with all $f_i \in K$. Clearly $f_i'' = r f_i$ and since some $f_i \neq 0$ the Riccati equation has a solution in K.
If G has order 2 then by using an algebraic construction for the Picard-Vessiot field one can show the existence of L as in the statement with $[L : K] \leq 2$.

4 Examples of the quadratic extension

We will show that for $n = 1$ and G the multiplicative group or a finite cyclic group of order > 2 and $n = 4$ and G is the tetrahedral group any quadratic extension K of $\mathbf{Q}$ does occur.

1. G is the multiplicative group $\mathbf{G_m}$ or a finite cyclic group of order > 2. Let the field K be given as $K = \mathbf{Q}(\lambda)$ where $\lambda^2 \in \mathbf{Z}$ is a square free integer ($\neq 0, 1$). Take $u_0, u_1 \in \mathbf{Q}(x), u_1 \neq 0$ and write $u = u_0 + \lambda u_1$. The equation $u' + u^2 = r \in \mathbf{Q}(x)$ is equivalent to $u_0 = -1/2\frac{u_1'}{u_1}$ and $r = u_0^2 + u_0' + \lambda^2 u_1^2$. Any choice of $u_1 \neq 0$ determines some u_0 and r and an equation $y'' = ry$. The corresponding Riccati equation has at least the two solutions $u_0 \pm \lambda u_1$. The equation $y' = (-1/2\frac{u_1'}{u_1} + \lambda u_1)y$ determines the differential Galois group which can be a finite cyclic group or the multiplicative group $\mathbf{G}_m$.
   For a general choice of u_1 the differential Galois group will be $\mathbf{G}_m$.
   If one chooses $u_1 = \frac{a}{b(x^2-\lambda^2)}$ with $a, b \in \mathbf{Z} \setminus \{0\}$, $b \geq 1$ and $g.c.d(a, b) = 1$ then $y = (x - \lambda)^{(\frac{1}{2}+\frac{a}{2b})}(x + \lambda)^{(\frac{1}{2}-\frac{a}{2b})}$ satisfies the equation $y' = (-1/2\frac{u_1'}{u_1} + \lambda u_1)y$ and G is a finite cyclic group of order b if a is odd and b is odd and of order $2b$ if a is even or b is even.

2. Let L_R denote the differential operator $(\frac{d}{dx})^2 - R$, where $R = \frac{3}{16x(x-1)} - \frac{3}{16x^2} - \frac{2}{9(x-1)^2}$. The differential Galois group of L_R is the tetrahedral group. There are exactly eight solutions $U_1, \dots, U_8$ of the Riccati equation $U' + U^2 = R$ which are algebraic of degree 4 over $\mathbf{Q}^{cl}(x)$ and there are two monic irreducible polynomials $P_R, Q_R \in \mathbf{Q}^{cl}(x)[T]$ of degree 4 such that $(P_R Q_R)(U_i) = 0$ for $i = 1, \dots, 8$. One can show that these polynomials satisfy $P_R, Q_R \in \mathbf{Q}(x)[T]$ and $P_R(x, T) = Q_R(\frac{x}{x-1}, T)$. An explicit calculation of P_R, Q_R is done in [Kov86], section 5.2.

   According to F.Klein, the differential operator L_R with the tetrahedral group as differential Galois group is the 'universal' in the following sense:
   Let L_r denote the differential operator $(\frac{d}{dt})^2 - r$ with $r \in \mathbf{Q}^{cl}(t)$ and suppose that the differential Galois group of this equation is also the tetrahedral group. Then there exists exactly two $\mathbf{Q}^{cl}$-linear field endomorphisms $\phi_1, \phi_2 : \mathbf{Q}^{cl}(x) \to \mathbf{Q}^{cl}(t)$ such that $(\phi_i)_* L_R = L_r$. Moreover $\phi_1 = \phi_2 \circ \theta$ where θ is the $\mathbf{Q}^{cl}$-linear field automorphism of $\mathbf{Q}^{cl}(x)$ given by $x \mapsto \frac{x}{x-1}$. The explicit expression for r is

   $$r = R(\phi(x))(\phi'(x))^2 - \frac{1}{2}(\frac{\phi''(x)}{\phi'(x)})' + \frac{1}{4}(\frac{\phi''(x)}{\phi'(x)})^2$$

   with $\phi = \phi_1$ or $\phi = \phi_2$ and $' = \frac{d}{dt}$.
   We refer to [BD79], theorems 3.4 and 3.7, for the statement above and we note that **the 'uniqueness of the pullback' claimed in theorem 3.7 does not hold in the tetrahedral case because $\theta_* L_R = L_R$. In the other two cases (with octahedral or icosahedral group) the uniqueness of the pullback is valid.**

Let ϕ be ϕ_1 or ϕ_2. Let $u_1, \dots, u_8$ be the eight solutions of the Riccati equation $u' + u^2 = r$,which are algebraic of degree 4 over $\mathbf{Q}^{cl}(t)$. If $P_r = P_R(\phi(x), T)$ and $Q_r = Q_R(\phi(x), T)$ then P_r, Q_r are the unique monic irreducible polynomials such that $(P_r Q_r)(u_i) = 0$ for $i = 1, \dots, 8$.

We come now to the construction of the example. Fix a field $K = \mathbf{Q}(\lambda)$, where $\lambda^2 \in \mathbf{Z}$ is a square free integer ($\neq 0,1$). Let $\phi : \mathbf{Q}^{cl}(x) \to \mathbf{Q}^{cl}(t)$ be the $\mathbf{Q}^{cl}$-linear field isomorphism given by $x \mapsto \frac{2}{1+2\lambda t}$. We note that $(\phi \circ \theta)(x) = \frac{2}{1-2\lambda t} = \tau(\phi(x))$, where τ is any automorphism of $\mathbf{Q}^{cl}(t)$ satisfying $\tau(t) = t, \tau(\lambda) = -\lambda$. Then $L := \phi_* L_R$ has the form $(\frac{d}{dx})^2 - r_\lambda$ with

$$r = -\frac{32\lambda^2}{9(1-4\lambda^2 t^2)^2} + \frac{3\lambda^2}{4(1-4\lambda^2 t^2)} \in \mathbf{Q}(t)$$

. The differential Galois group of L is of course the tetrahedral group. Let $P := P_R(\phi(x), T)$ and $Q := Q_R(\phi(x), T)$. Then P and Q are the minimum polynomials of the eight solutions of the Riccati equation $u' + u^2 = r$ which are algebraic of degree 4 over $\mathbf{Q}^{cl}(t)$. Clearly $P, Q \in K(x)[T]$. Take a τ as above and extend τ as an automorphism of $\mathbf{Q}^{cl}(t)[T]$ by $\tau(T) = T$. Then

$$\tau P = P_R(\tau(\phi(x)), T) = P_R(\frac{\phi(x)}{\phi(x) - 1}, T)$$
$$= Q_R(\phi(x), T) = Q$$

Hence P, Q do not belong to $\mathbf{Q}(t)[T]$. This finishes the example.

5 Differential equations of order 3

Consider the third order linear differential equation $y''' + py' + qy = 0$, with $p, q \in \mathbf{Q}(x)$. Let $G \subset Sl(3, \mathbf{Q}^{cl})$ be the differential galois group over $\mathbf{Q}^{cl}(x)$ of this equation. For any solution $y \neq 0$ of the equation the element $u = \frac{y'}{y}$ satisfies the Riccati equation $u'' + 3uu' + u^3 + pu + q = 0$. The analogue of theorem 1.1 can be found in [SU92]. We will use their terminology and description of finite primitive groups.

Definition 5.1 *A group $H \subset Sl(3, \mathbf{Q}^{cl})$ is called 1-reducible if the elements of the group have a common eigenvector.*

Theorem 5.2 *(See [SU92].)*

1. *If there is no algebraic solution over $\mathbf{Q}^{cl}(x)$ of the Riccati equation then $G = Sl(3, \mathbf{Q}^{cl})$ or $G/Z(G) = PSl(2, \mathbf{Q}^{cl})$ or G is a reducible but not a 1-reducible group.*
2. *If there is an algebraic solution of the Riccati equation then the minimal degree n of such an equation can be 1, 3, 6, 9, 21, 36 and*
   - *(a) if $n = 1$ then G is a 1-reducible group.*
   - *(b) if $n = 3$ then G is an imprimitive group.*
   - *(c) if $n = 6$ then $G/Z(G)$ is isomorphic to F_{36} or A_5.*
   - *(d) if $n = 9$ then $G/Z(G)$ is isomorphic to H_{72} or H_{216}.*
   - *(e) if $n = 21$ then $G/Z(G) = G_{168}$*
   - *(f) if $n = 36$ then $G/Z(G) = A_6$*

Theorem 5.3 *Suppose that the Riccati equation $u'' + 3uu' + u^3 + pu + q = 0$ has a solution, which is algebraic over $\mathbf{Q}^{cl}(x)$. Then there exists an algebraic solution u of minimal degree n of the Riccati equation such that the coefficients of the minimum monic polynomial of u over $\mathbf{Q}^{cl}(x)$ lie in a field $K(x)$ with*

1. *$[K : \mathbf{Q}] \leq 6$ if the equation is reducible.*
2. *$[K : \mathbf{Q}] \leq 2$ in case $G/Z(G) = F_{36}$.*
3. *$K = \mathbf{Q}$ in all other cases.*

Proof. The method of the proof in section 2 carries over for order 3 equations. Assume that the Riccati equation has at least one algebraic solution. We distinguish three cases.

(1) G is a reducible group. We will restrict ourselves to the worst case and leave the other cases to the reader. Suppose that the vectorspace of solutions of the third order linear differential equation decomposes into a direct sum of three one dimensional G-stable subspaces and suppose that the corresponding Riccati equation has exactly three solutions $u_1, u_2, u_3 \in \mathbf{Q}^{cl}(x)$. Then $\tilde{s}\sigma$ permutes these three solutions for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$. We conclude that $u_1, u_2, u_3 \in K(x)$, where $K \subset \mathbf{Q}^{cl}$ is a field such that $Gal(K/\mathbf{Q})$ is isomorphic to a subgroup of S_3 and $[K : \mathbf{Q}] \leq 6$. A rather trivial example is the following:
Let $T^3 + pT + q \in \mathbf{Q}[T]$ denote an irreducible polynomial with Galois group S_3. Then the differential equation $y''' + py' + qy = 0$ has as basis for the solutions $y_i := e^{\alpha_i x}$ where $\alpha_1, \alpha_2, \alpha_3$ are the three zeros of the polynomial $T^3 + pT + q$. The 'only' relation satisfied by y_1, y_2, y_3 over the field $\mathbf{Q}^{cl}(x)$ is $y_1 y_2 y_3 = 1$. The differential Galois group is therefore a maximal torus in $SL(3, \mathbf{Q}^{cl})$ and there are precisely three solutions of the Riccati equation, namely $\alpha_1, \alpha_2, \alpha_3$. This shows that $K = \mathbf{Q}(\alpha_1, \alpha_2)$ is the smallest possible field such that the rational solutions of the Riccati equation are in $K(x)$.

(2) G is an imprimitive group. In this case there are three solutions of the Riccati equation which are cubic over $\mathbf{Q}^{cl}(x)$ and there are no algebraic solutions of lower degree. These three solutions have the same minimum polynomial over $\mathbf{Q}^{cl}(x)$. Hence the coefficients of the minimum polynomial are fixed under $\tilde{s}\sigma$ for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ and therefore the coefficients of the minimum polynomial are in $\mathbf{Q}(x)$.

(3) G is a finite primitive group. Let n be the minimal degree of an algebraic solution of the Ricatti equation. Define the two sets

$$\mathcal{U} = \{u \mid u \text{ is an algebraic solution of the Riccati equation of degree } n\}$$

and

$$\mathcal{H} = \{H \subset G \mid H \text{ is reducible and } [G : H] = n\}.$$

The order of the first set is a multiple of n. Consider the map from $\mathcal{U}$ to $\mathcal{H}$ given by $u \mapsto H_u$ where $H_u := \{\sigma \in G \mid \sigma(u) = u\}$. Clearly H_u is a reducible group and $[G : H_u] = [\mathbf{Q}^{cl}(x, u) : \mathbf{Q}^{cl}(x)] = n$. Conversely one can verify for each primitive group separately that any $H \in \mathcal{H}$ is a non-commutative subgroup of G and therefore fixes only one line in the space of solutions of the third order linear differential equation. Hence H fixes precisely one $u \in \mathcal{U}$. Then one has to count the number of elements in $\mathcal{H}$. Using [SU92], one counts that the order of $\mathcal{H}$ is n except for the case F_{36}. In that case $n = 6$ and the order of $\mathcal{H}$ (and hence of $\mathcal{U}$) is 12. It follows that only in this last case one can possibly have a field K which is a quadratic extension of $\mathbf{Q}$.

Remark. In theorem (5.3) we have for simplicity given a formulation with $\mathbf{Q}$ as field of constants. The remarks of section 3 for the order 2 equations apply also to order 3 equations.

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1.14 NonLiouvillian Solutions for Second Order Linear ODEs

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Non-Liouvillian Solutions for Second Order Linear ODEs

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ABSTRACT

There exist sound literature and algorithms for computing Liouvillian solutions for the important problem of linear ODEs with rational coefficients. Taking as sample the 363 second order equations of that type found in Kamke's book, for instance, 51% of them admit Liouvillian solutions and so are solvable using Kovacic's algorithm. On the other hand, special function solutions not admitting Liouvillian form appear frequently in mathematical physics, but there are not so general algorithms for computing them. In this paper we present an algorithm for computing special function solutions which can be expressed using the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ hypergeometric functions. The algorithm is easy to implement in the framework of a computer algebra system and systematically solves 91% of the 363 Kamke's linear ODE examples mentioned.

Categories and Subject Descriptors

I.1 [**Symbolic and algebraic manipulation**]: Algorithms.

General Terms

Algorithms, design, theory.

Keywords

Linear ordinary differential equations, Non-Liouvillian solutions, hypergeometric solutions.

Introduction

Given a second order linear ODE

$$y'' + A(x)\, y' + B(x)\, y = 0 \tag{1}$$

where the quantity[1] $A'/2 + A^2/4 - B$ is a rational function of x, the problem under consideration is that of systematically computing solutions for this ODE even when the solutions admit no Liouvillian form[2].

The first thing to note is that non-Liouvillian solutions which are representable symbolically not as unknown infinite sums can be represented using special functions, e.g. Bessel, Hermite or Legendre functions [1]. In turn, these and most of the special functions frequently appearing in mathematical physics happen to be particular cases of the ${}_p\mathrm{F}_q$ hypergeometric function for p equal to 0, 1 or 2 and q equal to 0 or 1 (see [2]). For example, the Bessel functions can be expressed in terms of ${}_0\mathrm{F}_1$, all cylindrical functions as well as the Hermite, Laguerre, Whittaker and error family of functions can be expressed in terms of ${}_1\mathrm{F}_1$, and all Chebyshev, Gegenbauer, Jacobi, Legendre and some others can be expressed in terms of ${}_2\mathrm{F}_1$.

One natural approach is then to directly attempt the computation of hypergeometric function solutions of these ${}_0\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_2\mathrm{F}_1$ types, since in this way we cover at once solutions involving all the related special functions. Such an approach was developed during the year 2001 (see [3]), it became the main algorithm of the Maple computer algebra system for this type of problem since then and it is the subject of this paper. The algorithm consists of an equivalence approach to the ${}_p\mathrm{F}_q$ differential equations, is formulated in sec. 1, 2 and 3, and computes solutions of the form

$$y = P(x)\ {}_p\mathrm{F}_q\ \left(..;..;\frac{\alpha\, x^k + \beta}{\gamma\, x^k + \delta}\right) \tag{2}$$

where $P(x)$ is an arbitrary function and $\{\alpha, \beta, \gamma, \delta, k\}$ are constants.

It is important to note that the idea of seeking hypergeometric function solutions for (1) or using an equivalence approach for that is not new. In '89 Kamran and Olver [4] showed how to use an equivalence approach to compute Bessel function solutions to eigenvalue problems. Hypergeometric solutions were also discussed by Petkovsek and Salvy [5] in '93. Some of the more recent developments were presented as computer algebra algorithms too. For instance, a classic invariant theory approach was presented during 2000 by von Bülow in [6]; in 2001 Willis [7] presented a semi-heuristic algorithm for computing special functions

[1] This quantity is an invariant under transformations of the dependent variable - see (12).

[2] Functions that can be expressed in terms of exponentials, integrals, and algebraic functions, are called Liouvillian functions. The typical example is $\exp(\int R(x), dx)$ where $R(x)$ is rational or an algebraic function representing the roots of a polynomial.

solutions. In 2002 Bronstein and Lafaille [8] presented an approach for resolving an equivalence under rational transformations, between two linear equations in normal form, whenever one of them has an irregular singularity[3].

There is natural intersection between what these algorithms can solve but none can claim to extensively cover the portions of the problem covered by all the others. If compared with the algorithm presented in this paper - we called it hyper3 - these other algorithms, both those developed before and after hyper3:

- Do not resolve in a systematic manner all of the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ equivalences;
- Do not handle the problem of an invariant involving fractional or abstract powers;
- Do not explore automorphisms to avoid uncomputed integrals in the solution.

Also, hyper3 does not require solving systems of algebraic equations nor computing Groebner basis nor running differential elimination processes nor eliminating parameters by composing resultants (all of them expensive computational processes), thus resulting in a fast and smooth algorithm with little computational cost. These facts, combined with the range of problems it solves, for instance taking Kamke's book [12] as a testing arena, are at the base of the role hyper3 has today in the Maple differential equation libraries.

1. COMPUTING ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ AND ${}_0\mathrm{F}_1$ HYPERGEOMETRIC SOLUTIONS

To compute ${}_p\mathrm{F}_q$ solutions to (1), the idea is to formulate an equivalence approach to the ${}_p\mathrm{F}_q$ underlying hypergeometric differential equations; that is, to determine whether a given linear ODE can be obtained from one of the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ ODEs, respectively given by

$$\begin{array}{rcl} \left(x^2-x\right)y''+\left((a+b+1)\,x-c\right)y'+b\,a\,y &=& 0,\\ xy''+(c-x)\,y'-a\,y &=& 0,\\ xy''+cy'-y &=& 0, \end{array} \tag{3}$$

where $\{a, b, c\}$ are arbitrary constants, by means of a transformation of a certain type. If so, the solution to the given linear ODE is obtained by applying the same transformation to the solution of the corresponding ${}_p\mathrm{F}_q$ ODE above.

This approach of course also requires determining the values of the hypergeometric parameters $\{a, b, c\}$ for which the equivalence exists, and it is clear that its chances of success depend crucially on how general is the class of transformations being considered. For instance, one can verify that for linear transformations[4]

$$x \to F(x), \quad y \to P(x)\,y \tag{4}$$

with arbitrary $F(x), P(x)$, the problem is too general in that to solve it requires solving first the given ODE, so that the approach is of no practical use [6].

[3]That also leads to ${}_1\mathrm{F}_1$ solutions of the form (2), including its particular ${}_0\mathrm{F}_1$ case, whenever the point of application of ${}_1\mathrm{F}_1$ is rational in the independent variable.

[4]The problem of equivalence under transformations $\{x \to F(x),\ \ y \to P(x)\,y + Q(x)\}$ for linear ODEs can always be mapped into one with $Q(x) = 0$, see [9].

The transformations considered in this work are

$$x \to \frac{\alpha\,x^k+\beta}{\gamma\,x^k+\delta}, \quad y \to P(x)\,y \tag{5}$$

with $P(x)$ arbitrary and $\{\alpha, \beta, \gamma, \delta, k\}$ constant with respect to x. These transformations, which do not conform a class in the strict sense[5], can be obtained by sequentially composing three different transformations each of which does constitute a class. The sequence starts with linear fractional - also called Möbius - transformations

$$M := x \to \frac{\alpha\,x+\beta}{\gamma\,x+\delta}, \tag{6}$$

is followed by power transformations

$$x \to x^k, \tag{7}$$

and ends with linear homogeneous transformations of the dependent variable

$$y \to P\,y \tag{8}$$

So, we are talking of an algorithm that systematically computes, when they exist, solutions of the form

$$y = P(x)\ {}_p\mathrm{F}_q\left(..;..;\frac{\alpha\,x^k+\beta}{\gamma\,x^k+\delta}\right) \tag{9}$$

where ${}_p\mathrm{F}_q$ is any of ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$.

1.1 Transformations $y \to P(x)\,y$ of the dependent variable

The first thing to note is that transformations of the form (8) can easily be factored out of the problem: if two equations of the form (1), with coefficients $\{A(x), B(x)\}$ and $\{C(x), D(x)\}$ respectively, can be obtained from each other by means of (8), the transformation relating them is computable from these coefficients. For that purpose, we rewrite both equations in normal form, for instance for (1) use

$$y = u\,e^{-\int A/2\,dx} \tag{10}$$

to obtain

$$u'' = \left(\frac{A'}{2}+\frac{A^2}{4}\ -B\right)u \tag{11}$$

and the transformation relating the two hypothetical ODEs exists when the two normalized equations are equal; the transformation relating them being $y = u\,e^{\int (C-A)/2\,dx}$. In what follows we will refer to

$$I(x) = \frac{A'}{2}+\frac{A^2}{4}\ -B, \tag{12}$$

the coefficient of u in (11), as *the invariant* [10], regardless of the fact that this object is only an absolute invariant under (8) and not under (6) or (7).

[5]By class of transformations we mean a set of transformations closed under composition.

1.2 Transformations $x \to F(x)$ of the independent variable

By changing $x \to F(x)$ in (1), the invariant I_1 of the changed ODE can be expressed in terms of the invariant I_0 of (1) by

$$I_1(x) = F'^2 I_0(F(x)) + S(F') \tag{13}$$

where $S(x)$ is the Schwarzian [11]

$$S(F') = \frac{3F''^2}{4F'^2} - \frac{F'''}{2F'}; \tag{14}$$

The form of $S(F')$ is particularly simple when $F(x)$ is a power transformation (see (23)) and also when $F(x)$ is a Möbius transformation (6), in which case $S(F') = 0$. These are key facts permitting a simple formulation and resolution of the equivalence.

2. MÖBIUS TRANSFORMATIONS AND A CLASSIFICATION OF SINGULARITIES

The first ODE in (3) has 3 regular singularities, at 0, 1 and ∞. The second ODE in (3), also known as the confluent hypergeometric equation, has a regular singularity at 0 and an irregular one at ∞. The third ODE in (3) also has one regular and one irregular singularity at 0 and ∞, but we considered the case separately in order to obtain solutions directly expressed in terms of simpler (Bessel) functions. As we shall see, the structure of the singularities of these equations is a key for resolving related equivalences and Möbius transformations preserve that structure. These transformations only move the location of the poles. For example, the ${}_0\mathrm{F}_1$ hypergeometric equation

$$x\,y'' + c\,y' - y = 0 \tag{15}$$

has one regular singularity at the origin and one irregular at infinity. The transformed ODE, obtained from (15) by means of (6)

$$\begin{aligned} y'' &+ \frac{(\alpha\,(\delta\,c + 2\,\gamma\,x) + \gamma\,(2-c)\,\beta)}{(\alpha\,x+\beta)\,(\gamma\,x+\delta)}\,y' \\ &- \frac{(\alpha\,\delta - \gamma\,\beta)^2}{(\gamma\,x+\delta)^3\,(\alpha\,x+\beta)}\,y = 0 \end{aligned} \tag{16}$$

also has one regular and one irregular singularity, respectively located at $-\beta/\alpha$ and $-\delta/\gamma$. In the case of the ${}_2\mathrm{F}_1$ equation (see (3)), under (6) the three regular singularities move from $\{0, 1, \infty\}$ to $\{-\delta/\gamma, -\beta/\alpha, (\delta-\beta)/(\alpha-\gamma)\}$. So, from the structure of the singularities of an ODE, not only one can tell with respect to which of the three differential equations (3) could the equivalence under (6) be resolved, but also one can extract information regarding the values of the parameters $\{\alpha, \beta, \gamma, \delta\}$ entering the transformation.

Reversing the line of reasoning, through Mobius transformations one can formulate a classification of singularities of the linear ODEs "equivalent" to the ${}_p\mathrm{F}_q$ equations (3), based on how the invariant of each of these equations is transformed. Concretely, after transforming the ${}_2\mathrm{F}_1$ equation, the invariant of the resulting equation has the form

$$I_{{}_2\mathrm{F}_1} = \frac{\omega_2 x^2 + 2\,\omega_1 x + \omega_0}{(\sigma_1 x + \sigma_2)^2\,(\sigma_3 x+\sigma_4)^2\,(\sigma_5 x + \sigma_6)^2} \tag{17}$$

where all $\{\omega_i, \sigma_j\}$ can be expressed in terms of $\{a, b, c\}$ and $\{\alpha, \beta, \delta, \gamma\}$ respectively entering the ${}_2\mathrm{F}_1$ equation (3) and the transformation (6). The invariant of the transformed ${}_1\mathrm{F}_1$ equation has the form

$$I_{{}_1\mathrm{F}_1} = \frac{\omega_2 x^2 + 2\,\omega_1 x + \omega_0}{(\sigma_3 x+\sigma_4)^2\,(\sigma_5 x + \sigma_6)^4} \tag{18}$$

and that of the transformed ${}_0\mathrm{F}_1$ equation has the form

$$I_{{}_0\mathrm{F}_1} = \frac{\omega_1 x + \omega_0}{(\sigma_3 x+\sigma_4)^2\,(\sigma_5 x + \sigma_6)^3} \tag{19}$$

These transformed invariants are all of the form

$$I_{{}_p\mathrm{F}_q} = \frac{\prod_{i=1}^{m}(a_i x + b_i)}{\prod_{i=1}^{n}(c_i x + d_i)^{q_i}} \tag{20}$$

Cancellations between factors in the numerator and denominators of (20) may also happen and, independent of that, some coefficients $\{a_i, c_i\}$ can be zero[6]. So the degrees with respect to x of the numerators and denominators of (17), (18) and (19) can be lower than the maximum implicit by these equations; in this way the problem splits into cases.

Taking these possible cancellations into account, from the structure of the invariants (17), (18) and (19), the different cases for each of the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$, ${}_0\mathrm{F}_1$ classes were determined. With this classification in hands, from the knowledge of the degrees with respect to x of the numerator and denominator of the invariant (20) of a given ODE, one can tell whether or not it can be obtained from the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ equations (3) using (6). These observations can be summarized in a classification table as follows, using the symbol

$$[\le p,\ [q_1*, q_2*, ..., q_n*]]$$

where p is the degree in x of the numerator of (20) and q_i are the powers of the factors entering the denominator of it. The symbol $\le$, when present, refers to the value of p (can be less or equal to). The symbol $*$, when present, means there can be factors canceling between numerator and denominator, so that the actual value of the related q_i can be lower (provided p is also lower by the same amount). For example,

$$[\le 2*, [2*, 2*]] \tag{21}$$

represents the following possible seven different "lists of values" (herein referred as cases) for the degrees of the numerator and denominator of the invariant

$$\begin{aligned} [2*, [2*, 2*]] &= [2,[2,2]], [1,[1,2]], [0,[1,1]], [0,[0,2]] \\ [1*, [2*, 2*]] &= [1,[2,2]], [0,[1,2]] \\ [0,[2,2]] & \end{aligned} \tag{22}$$

With this notation, the classification of all the possible cases equivalent to the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ equations under Möbius transformations is as shown in Table 1.

[6] Provided that, in (6), $\alpha\delta - \gamma\beta \neq 0$ and also that in (1) the invariant remains finite, i.e. its denominator is not zero.

Class	Cases	Number of cases
${}_2F_1$	[<= 2∗, [2∗, 2∗, 2∗]], [<= 2∗, [2∗, 2∗]]	14
${}_1F_1$	[2∗, [2∗, 4]], [<= 2, [6]], [<= 2, [4]], [2∗, [2∗]], [2, [0]]	13
${}_0F_1$	[1∗, [2∗, 3]], [<= 1, [5]], [<= 1, [3]], [1∗, [2∗]], [1, [0]]	9

Table 1. *Classification of linear ODEs equivalent to ${}_pF_q$ ODEs under Möbius*

3. TRANSFORMATIONS $X \to X^K$ OF THE INDEPENDENT VARIABLE

Using the results of the previous sections it is possible to resolve the equivalence of a given linear ODE (1) and the hypergeometric equations (3) under compositions of transformations (8) of the dependent variable $y(x)$ and Möbius transformations (6) of the independent variable x. In this section a worth additional level of generalization is obtained by composing those two transformations with transformations $x \to x^k$ of the independent variable.

The first thing to note regarding power transformations is that, unlike Möbius transformations, they *do not* preserve the structure of singularities. The change in the invariant due to $x \to x^k$, however, has a simple and tractable structure. The Schwarzian (14) is given by:

$$S(F') = \frac{k^2 - 1}{4\,x^2} \tag{23}$$

So, the changed invariant I_1 shown in (13) can be expressed in terms of I_0 by

$$x^2 I_1(x) + \frac{1}{4} = \left(\left(x^k\right)^2 I_0(x^k) + \frac{1}{4}\right) k^2 \tag{24}$$

This naturally suggests the introduction of a “shifted” invariant $J(x)$

$$J_i(x) = x^2 I_i(x) + \frac{1}{4} \tag{25}$$

for which the transformation rule under $x \to x^k$ has the simple form

$$J_1(x) = k^2 J_0(x^k) \tag{26}$$

The equivalence of two linear ODEs A and B under $x \to x^k$ can then be formulated as follows: Given $J_{1A}(x)$ and $J_{1B}(x)$, compute k_A and k_B entering (26) such that the degrees with respect to x of $J_{0A}(x)$ and $J_{0B}(x)$ are minimized. This approach is systematic: equations A and B are related through power transformations only when $J_{0A} = J_{0B}$ and, if so, the mapping relating A and B is just $x \to x^{k_A - k_B}$.

The computation of k minimizing the degrees of J_0 in (26) is formulated as follows. Given the set

$$A := \frac{p_i}{q_i}, \qquad i = 1 \text{ to } m \tag{27}$$

of (possibly rational) numbers entering as exponents in the powers of the independent variable found in J_1, compute the smallest rational number $\tilde{k}$ such that multiplying by it each element of A, all of them become integers. Then the value of k minimizing the degrees of J_0 is $k = 1/\tilde{k}$.

4. SUMMARY OF HYPER3 - EXAMPLES

An itemized description of the algorithm, discussed in the previous subsections to resolve the equivalence proposed in the introduction, is as follows.

1. Rewrite the given equation (1) we want to solve in normal form

$$y'' = I(x)\,y \tag{28}$$

   where $I(x)$ is the invariant (12).

2. Compute $J_1(x)$, the shifted invariant (25), and use transformations $x \to x^k$ to reduce to the integer minimal values the exponents of powers entering $J_0(x)$; i.e., compute k and with it compute $J_0(x)$ in (26).

3. From (25), compute $I_0(x)$ and classify its structure of singularities according to Table 1, to tell whether an equivalence under Möbius transformations is possible and to which of the ${}_2F_1$, ${}_1F_1$ or ${}_0F_1$ equations (3).

4. When the equivalence is possible, from the singularities of $I_0(x)$ and by comparing it with the invariant (20) of the transformed ${}_pF_q$ equation[7], compute the parameters $\{a, b, c\}$ entering the ${}_pF_q$ equation (3) such that the equivalence exists as well as the parameters $\{\alpha, \beta, \gamma, \delta\}$ entering the Möbius transformation (6).

5. Compose the three transformations to obtain one of the form

$$x \to \frac{\alpha x^k + \beta}{\gamma x^k + \delta}, \quad y \to P(x)\,y$$

   mapping the ${}_pF_q$ equation involved into the ODE being solved.

6. Apply this transformation to the known solution of the ${}_pF_q$ equation resulting in the desired ODE solution.

An example of the ${}_2F_1$ class

Consider the second order linear ODE

$$\begin{aligned} y'' \;=\; & \frac{2\,(\nu - \mu)\,x^2 - 3\,x^4 - 2\,(\mu + \nu) - 1}{x^5 - x}\, y' \\ & + \frac{\nu\,\left(\nu + 2\,(\mu + 1)\right)}{x^6 - x^2}\, y \end{aligned} \tag{29}$$

This equation has regular singularities at $\{0, 1, -1, i, -i\}$. Following the steps outlined in the Summary, we rewrite the equation in normal form and then compute the value

[7] At this point, $J_0(x)$ and the shifted invariant of the ${}_pF_q$ equation have the same degrees.

of k leading to an equation with minimal degrees for the powers entering $J_0(x)$ in (26). The value found is $k = 2$. So, using[8]

$$t = x^2, \quad u = \sqrt{x}\,\mathrm{e}^{\left(\int \frac{2(\nu-\mu)x^2 - 3\,x^4 - 2\,(\mu+\nu) - 1}{2\,(x - x^5)}dx\right)}\,y, \tag{30}$$

the given equation (29) can be obtained from

$$u'' = \frac{\left(\mu^2 + 2\left(\nu^2 - \mu - 2\right)\right)t^2 + 2\left(\mu^2 - \nu^2\right)t + \mu\,(\mu + 2)}{4\,t^2(t-1)^2\,(t+1)^2}\,u, \tag{31}$$

which is in normal form and has an invariant with "minimal degrees" with respect to power transformations (7).

In step 3, analyzing the invariant of (31) (coefficient of u in its right-hand-side), the equation has now three regular singular points, at $\{0, 1, -1\}$. Using the notation of sec. 1.2, the degrees with respect to t of the numerator and of each of the linear factors entering the denominator are $[2, [2, 2, 2]]$. The equation matches the classification Table 1 presented in sec. 1.2 and is identified as equivalent to the $_2\mathrm{F}_1$ equation under Möbius transformations (6).

So we proceed with step 4, equating the invariant of (31) with the invariant (17) written in terms of $\{a, b, c, \alpha, \beta, \gamma, \delta\}$, from where we compute the values of the hypergeometric parameters $\{a, b, c\}$ entering the $_2\mathrm{F}_1$ equation (3), such that the equivalence under Möbius exists, as well as the Möbius transformation itself, obtaining

$$\{a = \frac{\nu}{2},\ b = \frac{\nu}{2} - \mu,\ c = -\mu\} \qquad M := x = \frac{2\,t}{t-1}$$

The transformation mapping the $_2\mathrm{F}_1$ equation (3) at these values of the parameters $\{a, b, c\}$ into (31) is then obtained composing the Möbius transformation above with one of the form (8), computed as explained in sec. 1.1, resulting in

$$x = \frac{2\,t}{t-1}, \quad y = \frac{t^{\mu/2}\,(t-1)^{(\nu-\mu-1)/2}}{(t+1)^{(\nu+1)/2}}\,u(t) \tag{32}$$

At this point, we have the transformation (32) mapping (3) into (31), and the transformation (30), mapping (31) into the equation (29) we want to solve. Composing these transformations, in step six we obtain the solution of (29)

$$\begin{aligned} y \ =\ & \frac{x^\nu}{(x^2-1)^{\frac{\nu}{2}}}\,{}_2\mathrm{F}_1\left(\frac{\nu}{2},\, \frac{\nu}{2} - \mu;\, -\mu;\, \frac{2\,x^2}{x^2-1}\right) C_1 \\ +\ & \frac{x^{\nu+2\,\mu+2}}{(x^2-1)^{1+\mu+\nu/2}} \\ & {}_2\mathrm{F}_1\left(\frac{\nu}{2} + 1,\, 1 + \mu + \frac{\nu}{2};\, 2 + \mu;\, \frac{2\,x^2}{x^2-1}\right) C_2 \end{aligned} \tag{33}$$

where C_1 and C_2 are arbitrary constants.

As mentioned in the introduction, an implementation of the algorithm being presented is at the core of the current Maple ability to solve this type of problem. The time consumed by this Maple implementation to compute the solution (33) performing all the steps mentioned is 0.4 seconds in a Pentium IV, 2 GigaHertz computer. The Maple command line to compute this solution directly using hyper3 is: `> dsolve(ode,[hyper3]);`.

An example of the $_1\mathrm{F}_1$ class

As an example which also requires an extension of the algorithm to handle symbolic powers in the invariant (12), consider Kamke's second order linear equation 2.15:

$$y'' + \left(\mu\,x^{2\,\sigma} + \nu\,x^{\sigma-1}\right)y = 0 \tag{34}$$

where μ, ν and σ are constants with respect to x. This equation is already in normal form and the shifted invariant (25) for it is

$$J_1(x) = 1/4 - x^2\left(\mu\,x^{2\,\sigma} + \nu\,x^{\sigma-1}\right) \tag{35}$$

To compute the values of k entering (26) and leading to $J_0(x)$ with minimized integer powers, in (27), instead of restricting $\tilde{k}$ to be a rational number, we allow it to depend on symbolic variables. So we compute $\tilde{k}$ such that the set of exponents entering (35), $A := \{2\sigma + 2,\, \sigma + 1\}$, becomes a set of integers after multiplying each element of it by $\tilde{k}$, resulting in[9] $\tilde{k} = 1/(\sigma + 1)$. In summary, using $\left\{t = x^{\sigma+1},\, u(t) = x^{\sigma/2}\,y(x)\right\}$, Kamke's equation (34) can be obtained from the following equation, which is already in normal form and has an invariant with minimized integer degrees, free of symbolic powers

$$u'' = -\frac{\left(4\,\mu\,t^2 + 4\,\nu\,t + \sigma^2 + 2\,\sigma\right)}{4\,(\sigma+1)^2\,t^2}\,u \tag{36}$$

Proceeding with step 3, the invariant is the coefficient of u in the above and the degrees with respect to t of its numerator and factors in its denominator match the Table 1 of sec. 1.2, identifying (36) as equivalent to the $_1\mathrm{F}_1$ equation under Möbius transformations (6).

As in the previous example, in step 4, comparing the invariant of (36) with the invariant (18) of the transformed $_1\mathrm{F}_1$ equation, we compute the values of the parameters entering the $_1\mathrm{F}_1$ equation (3) such that the equivalence exists, as well as the parameters entering the Möbius transformation. Composing all the transformations, we arrive at the solution for Kamke's example 2.15

$$\begin{aligned} y = \ & \mathrm{e}^{\left(-\frac{i\,\sqrt{\mu}\,x^{\sigma+1}}{\sigma+1}\right)}\Bigg(\\ & {}_1\mathrm{F}_1\left(\frac{\sqrt{\mu}\,\sigma + i\,\nu}{2\,\sqrt{\mu}\,(\sigma+1)};\, \frac{\sigma}{\sigma+1};\, \frac{2\,i\,\sqrt{\mu}}{\sigma+1}\,x^{\sigma+1}\right) C_1 \\ & + {}_1\mathrm{F}_1\left(\frac{\sqrt{\mu}\,(\sigma+2) + i\,\nu}{2\,\sqrt{\mu}\,(\sigma+1)};\, \frac{\sigma+2}{\sigma+1};\, \frac{2\,i\,\sqrt{\mu}}{\sigma+1}\,x^{\sigma+1}\right) x\,C_2\Bigg) \end{aligned} \tag{37}$$

where C_1 and C_2 are arbitrary constants. The time consumed by the implementation in Maple to perform these steps and return the solution above is again 0.4 seconds, as in the previous example. This also illustrates that, for typical problems, the additional handling of symbolic powers does not imply on any important performance cost.

[8] This transformation is the composition of $t \equiv x^k = x^2$ with a transformation of the form (10) so that (31) is normalized.

[9] To perform this computation, it suffices to sequentially take the **gcd** between each of the elements of A.

5. ON THE COMPUTATION OF THE SECOND INDEPENDENT SOLUTION

The algorithm presented is based on computing a transformation mapping a ${}_pF_q$ equation into a given linear ODE, then applying that transformation to the solution of the ${}_pF_q$ equation to obtain the solution for the given problem. This process has a subtlety: depending on the values of the hypergeometric parameters, we may have only one independent solution available for the ${}_pF_q$ equation. In these cases, the second independent solution can be obtained through integration: if $y = S(x)$ is a solution of (1), then

$$y = \int \frac{e^{(\int A(x)dx)}}{S(x)^2}\, dx\, S(x) \tag{38}$$

is a second independent solution directly computable from $S(x)$ and $A(x)$.

This approach, however, frequently introduces uncomputable integrals, thus complicating further manipulations and undermining the usefulness of the result. As an example of this situation, for the ${}_2F_1$ equation,

$$\left(x^2 - x\right) y'' + \left((a+b+1)\, x - c\right) y' + b\, a\, y = 0, \tag{39}$$

the two independent solutions are:

$$\begin{aligned} y \;&=\; {}_2F_1\left(a, b;\, c;\, x\right) C_1 \\ &\quad + x^{1-c}\, {}_2F_1\left(b-c+1, a-c+1;\, 2-c;\, x\right) C_2 \end{aligned} \tag{40}$$

but for $c = 1$ these two solutions are equal. Using the integration recipe (38), a second independent solution is

$$y = \int \frac{e^{\left(\int \frac{(a+b+1)\, x - 1}{x^2 - x}\, dx\right)}}{{}_2F_1\left(a, b;\, 1;\, x\right)^2}\, dx\; {}_2F_1\left(a, b;\, 1;\, x\right) \tag{41}$$

Although the inner integral, with rational integrand, is easy to compute, the outer integral, with ${}_2F_1\left(a, b;\, 1;\, x\right)^2$ in its denominator, is uncomputable in current computer algebra systems.

The approach used in hyper3 to minimize the occurrence of uncomputable integrals consists of exploring the group of automorphisms of the ${}_2F_1$ equation in order to make c not an integer when that is possible. Recalling, the group elements and their action are

Group element	Action on the plane
$g_1 : x \to x$	$(0 \to 0, 1 \to 1, \infty \to \infty)$
$g_2 : x \to 1 - x$	$(0 \to 1, 1 \to 0, \infty \to \infty)$
$g_3 : x \to 1/x$	$(0 \to \infty, 1 \to 1, \infty \to 0)$
$g_4 : x \to 1/(1 - x)$	$(0 \to 1, 1 \to \infty, \infty \to 0)$
$g_5 : x \to (x - 1)/x$	$(0 \to \infty, 1 \to 0, \infty \to 1)$
$g_6 : x \to x/(x - 1)$	$(0 \to 0, 1 \to \infty, \infty \to 1)$

Table 2. *Group of automorphisms of the ${}_2F_1$ equation*

These transformations, known to act as permutations on the set $\{0, 1, \infty\}$, also act as permutations on a set $\{\lambda, \mu, \kappa\}$ related to the hypergeometric parameters $\{a, b, c\}$ by

$$\lambda = 1 - c, \quad \mu = a + b - c, \quad \kappa = a - b \tag{42}$$

These three parameters are the exponent differences of the normal form of the ${}_2F_1$ equation (3), at $\{0, 1, \infty\}$ respectively. The action of each g_i on these parameters is obtained from Table 2 by respectively changing $\{0, 1, \infty\}$ by $\{\lambda, \mu, \kappa\}$. Hence, the solution (40) can be written in different manners, by changing the application point of the ${}_2F_1$ function using the g_i, permuting accordingly the parameters $\{\lambda, \mu, \kappa\}$ entering the ${}_2F_1$ function and multiplying the result by the proper non-constant factor[10].

For example, when c is an integer but $a + b$ is not an integer, applying g_2 and permuting the parameters $\mu \leftrightarrow \lambda$, the power x^{1-c} entering (40) becomes a power with non-integer exponent. Using this mechanism, for (39) at $c = 1$, instead of the solution with integrals (41) we obtain two independent solutions free of uncomputed integrals:

$$\begin{aligned} y \;&=\; {}_2F_1\left(a, b;\, a+b;\, 1-x\right) C_1 \\ &\quad + (x-1)^{1-b-a}\, {}_2F_1\left(1-b, 1-a;\, 2-b-a;\, 1-x\right) C_2 \end{aligned} \tag{43}$$

When c and $a + b$ are both integers, g_2 does not resolve the problem, but if $a - b$ is not an integer then g_3 does, since it permutes the integer $\lambda = 1 - c$ with the non-integer $\kappa = a - b$. For example, for $a = 2/3$, $b = 1/3$, $c = 1$, (39) becomes

$$2\, y/9 + (2\, x - 1)\, y' + \left(x^2 - x\right) y'' = 0 \tag{44}$$

Applying g_3 and permuting the parameters λ and κ, we obtain the following two independent solutions free of integrals

$$\begin{aligned} y \;&=\; x^{-1/3}\, {}_2F_1\left(1/3, 1/3;\, 2/3;\, 1/x\right) C_1 \\ &\quad + x^{2/3}\, {}_2F_1\left(2/3, 2/3;\, 4/3;\, 1/x\right) C_2 \end{aligned} \tag{45}$$

When all of c, $a + b$ and $a - b$ are integers, these permutations are in principle of no use, but still for some cases the solution can be represented free of integrals. This is the case of Legendre's equation. Recalling the relationship between the associated Legendre function of the first kind and the hypergeometric ${}_2F_1$ function[11],

$$\begin{aligned} &\mathrm{LegendreP}\left(a, b, z\right) \;= \\ &\quad \frac{(z+1)^{1/2\, b}\, {}_2F_1\left(a+1, -a;\, 1-b;\, (1-z)/2\right)}{(z-1)^{1/2\, b}\, \Gamma\left(1-b\right)}, \end{aligned} \tag{46}$$

whenever the group elements of Table 2 can map the ${}_2F_1$ function solution into one of the form above, one independent solution can be expressed using LegendreP and the second one is obtained from the first one replacing LegendreP by the associated function of the second kind LegendreQ. For example, for

$$y/4 + (2\, x - 1)\, y' + \left(x^2 - x\right) y'' = 0 \tag{47}$$

we have $\mu = \kappa = \lambda = 0$, so $c = 1$ and both $a + b$ and $a - b$ are integers. A solution free of integrals is

$$\begin{aligned} y \;&=\; \mathrm{LegendreP}\left(-1/2, 2\, x - 1\right) C_1 \\ &\quad + \mathrm{LegendreQ}\left(-1/2, 2\, x - 1\right) C_2 \end{aligned} \tag{48}$$

---

[10] These multiplicative factors are different for each g_1; we omit them here for brevity.

[11] We use here the Maple convention for the branch cuts of LegendreP; the idea being discussed is independent of that.

Conclusions

In this presentation we discussed an algorithm for second order linear ODEs, we called it hyper3, for computing non-Liouvillian solutions by resolving an equivalence to the ${}_2F_1$, ${}_1F_1$ and ${}_0F_1$ equations. Taking Kamke's book as testing arena, this algorithm is the most successful one of the current set of linear ODE algorithms of the Maple system. From the 363 corresponding examples of Kamke's book having rational coefficients, hyper3 alone solves 331 (91 %), followed by Kovacic's algorithm solving 181 (50 %). Moreover, from these 181 examples admitting Liouvillian solutions, hyper3 solves 163 (90 %).

The fact that, for 90% of these equations admitting Liouvillian solutions, the solution can also be computed as a hypergeometric one of the form (9) is a good indication that the restriction used to make the algorithm feasible is appropriate. The fact that around one half of Kamke's examples only admit special function solutions of non-Liouvillian form also illustrates the relevance of this type of solution in the general framework of linear ODE problems popping up in applications.

Despite the simplicity of the approach, till the end of 2001, when the routines for this algorithm were developed, no equivalent or similar algorithms were available in any of the Axiom, Maple, Mathematica, MuPAD or Reduce computer algebra systems (CAS). These CAS failed in computing special function solutions but for occasional success, e.g., by previous to hyper3 Maple routines able to resolve an equivalence under only power transformations of the form (7) [13], or an equivalence under only Möbius transformations and only with respect to the ${}_2F_1$ class [14].

Since at the core of hyper3 there is the concept of singularities, two natural extensions of this work consist of applying the same ideas to compute solutions for linear ODEs of order three and higher [15] and for second order equations of Heun type. The latter have four regular singular points or any combination of singularities derived from that case through confluence processes [16]; one example of these are Mathieu equations. Related work is in progress [17, 18].

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1.15 Galois Groups of Second and Third Order Linear Differential Equations

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Galois Groups of Second and Third Order Linear Differential Equations

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Using the representation theory of groups, we are able to give simple necessary and sufficient conditions regarding the structure of the galois groups of second and third order linear differential equations. These allow us to give simple necessary and sufficient conditions for a second order linear differential equation to have liouvillian solutions and for a third order linear differential equation to have liouvillian solutions or be solvable in terms of second order equations. In many cases these conditions also allow us to determine the group.

1. Introduction

Let k be a differential field[††] with algebraically closed field of constants $\mathcal{C}$ and $L(y) = 0$ a linear differential equation[‡‡] with coefficients in this field. One can form the m^{th} symmetric power $L^{Ⓢm}(y)$ of $L(y)$ which is the smallest order nonzero linear differential equation satisfied by the m^{th} power of any solution of $L(y) = 0$. In this paper we show how factorization properties of these symmetric powers can be used to determine structural properties of the galois groups of second and third order linear differential equation. This in turn will allow us to give necessary and sufficient conditions for these linear differential equations to have liouvillian solutions. For example we show (Corollary 4.4):

> *Let $L(y) = y'' + ry = 0$ be a second order linear differential equation with $r \in k$. $L(y) = 0$ has liouvillian solutions if and only if $L^{Ⓢ6}(y)$ is reducible.*

For third order equations we have similar conditions and are also able to characterize those equations that are solvable in terms of lower order equations (Corollary 4.8):

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[††] of characteristic zero as are all the fields in this paper.

[‡‡] All linear differential equations in this paper are homogeneous

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Let $L(y) = y''' + ry' + sy = 0$ be a third order linear differential equation with $r, s \in k$. $L(y) = 0$ is solvable in terms of lower order linear differential equations if and only if $L^{ⓢ4}(y)$ has order less than 15 or is reducible.

Factorization properties can also be used to determine galois groups in many cases. For example (see Section 2.2 for the definition of the Tetrahedral Group and Theorem 4.3):

Let $L(y) = y'' + ry = 0$ be a second order linear differential equation with $r \in k$. The galois group of $L(y) = 0$ is the Tetrahedral Group $A_4^{SL_2}$ if and only if $L^{ⓢ2}(y)$ is irreducible and $L^{ⓢ3}(y)$ is reducible.

Our results show that one can reduce many questions concerning the galois groups of linear differential equations to factoring associated differential equations. This underscores the importance of finding efficient factorization algorithms, (c.f., [0, 0]).

The main tool of this paper is representation theory and the results spring from the following facts. The first fact (due to Chevalley) is that if one is given an algebraic subgroup H of $GL(n, \mathcal{C})$ then there is a faithful representation $\Phi : GL(n, \mathcal{C}) \rightarrow GL(m, \mathcal{C})$ for some m such that $\Phi(H)$ is uniquely determined by its set of invariant subspaces in $\mathcal{C}^m$ (c.f., Theorem 11.2 of [0]). The second fact is that given a faithful representation of an algebraic group, any other representation can be constructed from this representation using the tools of linear algebra, i.e., tensor product, duals, direct sums and subspaces. Furthermore, if the group is the galois group of a linear differential equation and the representation is the representation on the solution space of the linear differential equation, then one can mimic this construction at the level of the equation to produce an equation whose solution space corresponds to the other representation (c.f., [0, 0, 0, 0, 0]). The final fact that we use is that the solution space of a linear differential equation has a subspace of dimension m invariant under the action of the galois group if and only if the equation has a factor of order m, [0]. Combining these facts one sees that one should be able to determine the galois group of a linear differential equation by considering the factorization properties of certain associated operators. This philosophy has been successfully used in [0, 0, 0, 0, 0]. In this paper, we apply this philosophy to the study of second and third order linear differential equations. Except for the last fact we do not use the full theoretical power of the above facts, but rather calculate directly for the groups involved. In particular we show that in this case it is enough to consider just symmetric powers of small order.

The paper is organized in the following manner. Section 2 contains a description of the groups that can appear as galois groups of second and third order linear differential equations as well as facts about their representation theory. Section 3 reviews facts from the formal theory of linear differential equations and galois theory. In section 4, we present the main results. Section 5 is devoted to examples.

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2. Group Theory

To any homogeneous linear differential equation $L(y) = 0$ of order n with coefficients in a differential field k (with algebraically closed field of constants $\mathcal{C}$), one can associate a group of $n \times n$ matrices $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ called the differential galois group of $L(y) = 0$ (see [0] or [0] for an exposition of this theory; for concreteness, one may let $k = \overline{\mathbf{Q}}(x)$ and $\mathcal{C} = \overline{\mathbf{Q}}$) . Differential and algebraic properties of the equation are mirrored by group theoretic properties of this group. This section is devoted to studying properties of subgroups of $SL(n, \mathcal{C})$. The reader interested only in the form of the algorithms and willing to accept certain group theoretic facts, may proceed to the next section.

Let V denote a finite dimensional vector space of dimension n over an algebraically closed field $\mathcal{C}$.

Definition 2.1. A subgroup G of $GL(V)$ is said to *act irreducibly* if the only G-invariant subspaces of V are $\{0\}$ and V. The group G is *completely reducible* if there are minimal G-invariant subspaces $V_1, \cdots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$.

According to Maschke's theorem, every finite subgroup of $GL(V)$ is completely reducible. We shall see (in sections 3 and 4) that $L(y) = 0$ has a galois group that acts reducibly if and only if $L(y)$ is reducible and that one can test directly if this occurs (A differential equation $L(y)$ with coefficients in k is called *reducible* if $L(y)$ can be written as $L_1(L_2(y))$, where $L_1(y)$ and $L_2(y)$ are differential equations with coefficients in k of order > 0. $L(y) = 0$ is *irreducible* if it is not reducible). We shall need finer group theoretic information when the equation (and therefore the galois group) is irreducible. The following definitions are crucial to studying this situation.

Definition 2.2. Let G be a subgroup of $GL(n, \mathcal{C})$ acting irreducibly, i.e. G is a linear group acting irreducibly on the vector space V of dimension n over $\mathcal{C}$. Then G is called *imprimitive* if, for $k > 1$, there exist subspaces $V_1, \cdots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$ and, for each $g \in G$, the mapping $V_i \to g(V_i)$ is a permutation of the set $\mathcal{S} = \{V_1, \ldots, V_k\}$. The set $\mathcal{S}$ is called a *system of imprimitivity* of G. If all the subspaces V_i are one dimensional, then G is called *monomial*. An irreducible group $G \subseteq GL(n, \mathcal{C})$ which is not imprimitive is called *primitive*.

We note that since an imprimitive group G is assumed to act irreducibly on V, we have that G acts transitively on the V_i. In particular, all the V_i have the same dimension.

Definition 2.3. A group $G \subseteq GL(n, \mathcal{C})$ whose elements have a common eigenvector is called 1-*reducible.*

In [0] it is proven that, if an irreducible differential equation $L(y) = 0$ has a liouvillian solution, then $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ has a 1-reducible subgroup H of finite index and that there is a solution z of $L(y) = 0$ such that the algebraic degree of $u = z'/z$ over k is $\leq [\mathcal{G}(L) : H]$.

In this section we will analyse imprimitive and primitive subgroups of $G \subseteq GL(n, \mathcal{C})$ and also see what consequences we can draw from assuming that such a group has a 1-reducible subgroup of finite index.

2.1. Imprimitive Groups

When n is prime any system of imprimitivity for an imprimitive subgroup $G \subseteq GL(n, \mathcal{C})$ contains only subspaces of dimension one (and therefore G must be a monomial group). The subgroup leaving one of these subspaces fixed will be a 1-reducible subgroup of index n. We therefore have the following:

Proposition 2.1. ([0]) *Let n be a prime number and let $G \subseteq GL(n, \mathcal{C})$ be an imprimitive group. Then G is a monomial group and contains a 1-reducible subgroup of index n.*

2.2. Primitive Groups

The differential galois group $\mathcal{G}(L)$ of a linear differential equation is a linear algebraic group which, after a suitable change of variables (cf. Theorem 3.3), can be assumed to be unimodular, i.e. $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$. We thus restrict ourselves to linear algebraic subgroups of $SL(n, \mathcal{C})$ (see [0], [0] or [0] for the appropriate definitions). We have the following general result:

Lemma 2.2. *Let $G \subseteq GL(n, \mathcal{C})$ be a primitive group. If H is a normal 1-reducible subgroup of G, then H is a subgroup of the group of scalar matrices.*

Proof. We say a subspace $W \subseteq \mathcal{C}^n$ is a maximal eigenspace of H if each element of H acts by scalar multiplication on W and W is maximal with respect to this property. Let $\mathcal{W}$ be the set of maximal eigenspaces of H. By hypothesis, this set is non-empty. If $W_1, \ldots, W_{m+1}$ are elements of $\mathcal{W}$ such that $W_{m+1} \cap (W_1 + \ldots + W_m) \neq \{0\}$, then one can easily show that $m = 1$ and $W_1 = W_2$. This implies that $\mathcal{W}$ is finite and that the sum V' of the elements of $\mathcal{W}$ is a direct sum. Note that H is normal in G so G permutes the elements of $\mathcal{W}$ and so leaves V' invariant. Since G is irreducible, we have $V' = \mathcal{C}^n$ and so $\mathcal{W}$ is a system of imprimitivity of G, unless $\mathcal{W}$ contains just one element. Therefore, we can conclude that the elements of H are all scalar matrices. □

Proposition 2.3. *Let $G \subseteq SL(n, \mathcal{C})$ be a primitive linear algebraic group. Then:*

1 *either G is finite or G^o, the connected component of the identity of G, is a semisimple subgroup of $GL(n, \mathcal{C})$,*
2 *if G also contains a 1-reducible subgroup of finite index, G must be finite,*
3 *if $n = 2$ or 3, and G^o is semisimple, then G^o acts irreducibly on $\mathcal{C}^n$.*

Proof. (c.f., [0] p. 301 for a similar result) Assume that G is primitive and not finite. Let $R(G^o)$ be the radical of G^o. Note that $R(G^o)$ is normal in G. Since $R(G^o)$ is connected and solvable, the Lie-Kolchin Theorem ([0], p. 113) implies that the elements of $R(G^o)$ have a common eigenvector, i.e. $R(G^o)$ is 1-reducible. Therefore, we can conclude from Lemma 2.2 that the elements of $R(G^o)$ are all scalar matrices. Since there are only a finite number of such matrices in $SL(n, \mathcal{C})$, we must have that $R(G^o)$ is trivial and so G^o is semisimple. This proves 1.

If G also contains a 1-reducible subgroup of finite index, then it contains a 1-reducible *normal* subgroup of finite index. Lemma 2.2 implies that this latter group consists only of scalar matrices and, since $G \subseteq SL(n, \mathcal{C})$, must be finite. Therefore, G is finite. This proves 2.

If G^o is semisimple, then any invariant subspace has a complementary invariant subspace. If $n = 2$ or 3, and G^o has a non-trivial invariant subspace, then G^o must have an invariant subspace of dimension 1. This means that G^o is 1-reducible and so by 2., G is finite, a contradiction. Therefore G^o acts irreducibly. □

Proposition 2.3 reduces the question of finding the primitive subgroups of $SL(n,\mathcal{C})$ to the question of finding the finite primitive subgroups and the semisimple subgroups. We begin with the latter. A connected semisimple group is a quotient (by a finite group) of a direct product of simple groups ([0], p. 167). The simple algebraic groups and their representations are well understood. In particular, by comparing dimensions one can see that the only semisimple subgroup of $SL(2,\mathcal{C})$ is $SL(2,\mathcal{C})$. Therefore any primitive proper subgroup of $SL(2,\mathcal{C})$ is finite. For $n = 3$, it is shown in ([0], p. 674) that the only connected proper semisimple subgroup of $SL(3,\mathcal{C})$ that acts irreducibly on $\mathcal{C}^3$ is conjugate to the representation of $SL(2,\mathcal{C})$ given by

$$\rho_3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

This is just the irreducible three dimensional representation of $SL(2,\mathcal{C})$ (see Proposition 2.4 and the discussion after it). $\rho_3(SL(2,\mathcal{C})) \cong SL(2,\mathcal{C})/\{\pm 1\}$ and we shall refer to this group as PSL_2. The normalizer of PSL_2 in $SL(3,\mathcal{C})$ is $PSL_2 \times C_3$ where C_3 is the three element subgroup of scalar matrices ([0], p. 674). Therefore any non-finite proper primitive subgroup of $SL(3,\mathcal{C})$ is conjugate to either PSL_2 or $PSL_2 \times C_3$.

We now turn to the finite primitive subgroups of $SL(2,\mathcal{C})$ and $SL(3,\mathcal{C})$.

One knows the finite primitive subgroups of $PGL(3,\mathcal{C})$ (c.f., [0]). From this list, one can derive the primitive subgroups of $SL(3,\mathcal{C})$ (c.f., [0]). Any finite primitive group of $SL(3,\mathcal{C})$ is isomorphic to one of the following groups:

1 The Valentiner Group $A_6^{SL_3}$ of order 1080 generated as a transitive permutation group of 18 letters by:

$$(1,2,4)(3,8,13)(5,7,9)(6,10,12)(11,15,14),$$
$$(1,3)(2,6)(4,5)(7,12)(8,9)(10,13),$$
$$(1,4)(3,8)(5,9)(6,11)(10,14)(12,15),$$
$$(1,4,8,3,5,9)(2,7,13)(6,12,10)(11,16,14,17,15,18),$$
$$(1,5,8)(2,7,13)(3,4,9)(6,12,10)(11,15,14)(16,18,17).$$

We have $A_6^{SL_3}/Z(A_6^{SL_3}) \cong A_6$.

2 The simple group G_{168} of order 168 defined by:

$$\{X,Y \mid X^7=(X^4Y)^4=(XY)^3=Y^2=id\}.$$

3 $G_{168} \times C_3$, the direct product of G_{168} with the cyclic group C_3 of order 3.

4 A_5, the alternating group of five letters.

5 $A_5 \times C_3$, the direct product of A_5 with a cyclic group C_3 of order 3.

6 The group $H_{216}^{SL_3}$ of order 648 defined by:

$$\{U,V,S,T \mid U^9=V^4=T^3=S^3=(UV)^3=id, VS=TV, VT=S^2V, [U^6,V]=[U^6,T]=[U,S]=id, [U,V^2]=S\}.$$

The group $H_{216}^{SL_3}/Z(H_{216}^{SL_3})$ is the *hessian group* of order 216.

7 The group $H_{72}^{SL_3}$ of order 216 generated by the elements S, T, V and UVU^{-1} of $H_{216}^{SL_3}$.
8 The group $F_{36}^{SL_3}$ of order 108 generated by the elements S, T and V of $H_{216}^{SL_3}$.

If $G/Z(G)$ is a simple group, then G also is a perfect group (i.e. equals its commutator group). In this case any representation of G belongs to $SL(n, \mathcal{C})$. But for the groups $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$ there exist irreducible representations in $GL(3, \mathcal{C})$ which do not belong to $SL(3, \mathcal{C})$. Note that if $g \in SL(3, \mathcal{C})$ has order 2, then the trace of g is -1, and if g has order 4, then the trace cannot be I, $-I$ or -1. Considering the character tables of these groups, one sees that this restriction implies that there are only two irreducible characters of degree 3 left for $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$. For the computations in the rest of the paper we will only have to consider these characters.

The finite primitive subgroups of $SL(2, \mathcal{C})$ are isomorphic to one of the following groups (c.f., [0, 0]):

1 The icosahedral group $A_5^{SL_2}$ of order 120 generated as a transitive permutation group of 24 letters by:

(1,4,2)(3,20,5)(6,13,14)(7,22,8)(9,24,10)(11,21,12)(15,23,16)(17,19,18),

(1,6,5,3,15,2)(4,12,21,20,22,7)(8,19,18,11,10,9)(13,17,16,23,24,14),

(1,3)(2,5)(4,20)(6,15)(7,21)(8,11)(9,18)(10,19)(12,22)(13,23)(14,16)(17,24),

(1,3)(2,5)(4,20)(6,15)(7,21)(8,11)(9,18)(10,19)(12,22)(13,23)(14,16)(17,24).

2 The octahedral $S_4^{SL_2}$ of order 48 given by:

$$\{X, Y \mid X^3 = Y^8 = id, YX^2 = XY^3\}.$$

3 The tetrahedral group $A_4^{SL_2}$ of order 24 generated by X and Y^2.

We note that, the tetrahedral group has two faithfull irreducible representations in $GL(2, \mathcal{C})$ which do not belong to $SL(2, \mathcal{C})$.

2.3. The characters of symmetric products

The main idea of this paper is that for $n = 2$ or 3, one can distinguish between the various primitive groups by decomposing small symmetric powers of the original representation. We do this calculation in this section.

The first step is to calculate the characters χ_m of the m^{th} symmetric powers. We follow the presentation in ([0], p. 181). Let z be a variable and define the functions on $GL(n, \mathcal{C})$, $q_0, \ldots, q_n$, via the formula:

$$det(I - zg) = q_0 - q_1 z + q_2 z^2 - \ldots \pm q_n z^n$$

for $g \in G$. Note that $q_0 = 1$ and $q_n = det(g)$. The characters χ_m of the symmetric power $\mathcal{S}^m(\mathcal{C}^n)$, then satisfy the following recursion:

$$\begin{aligned} \chi_0 &= 1 \\ \chi_l - q_1\chi_{l-1} + q_2\chi_{l-2} - \ldots \pm q_n\chi_{l-n} &= 0 \end{aligned}$$

for $l = 1, 2, \ldots$ and $\chi_{-1}, \chi_{-2}, \ldots$ are set equal to zero. If $G \subseteq SL(2, C)$ we have that

$q_0 = q_2 = 1$ and $q_1 = \chi$, the character of the representation of G on $\mathcal{C}^2$. We get the following:

$$\begin{aligned}\chi_2 &= \chi^2 - 1\\ \chi_3 &= \chi^3 - 2\chi\\ \chi_4 &= \chi^4 - 3\chi^2 + 1\\ \chi_5 &= \chi^5 - 4\chi^3 + 3\chi\\ \chi_6 &= \chi^6 - 5\chi^4 + 6\chi^2 - 1\end{aligned}$$

For $G \subseteq SL(3, C)$, we have that $q_0 = q_3 = 1$, $q_1 = \chi$, the character of the representation of G on $\mathcal{C}^3$ and $q_2 = \bar{\chi}$, where $\bar{\chi}(g) = \chi(g^{-1})$. We get the following:

$$\begin{aligned}\chi_2 &= \chi^2 - \bar{\chi}\\ \chi_3 &= \chi^3 - 2\chi\bar{\chi} + 1\\ \chi_4 &= \chi^4 - 3\bar{\chi}\chi^2 + 2\chi + \bar{\chi}^2\\ \chi_5 &= \chi^5 - 4\bar{\chi}\chi^3 + 3\chi^2 + 3\bar{\chi}^2\chi - 2\bar{\chi}\end{aligned}$$

For higher dimensional representations of G the *power maps* (cf. [0]) are needed in the formula for χ_i (see e.g. [0] for details and further references).

Proposition 2.4. (See [0], pages 150, 180) *$SL(n, \mathcal{C})$ acts irreducibly on $\mathcal{C}^n$ and all symmetric powers $\mathcal{S}^m(\mathcal{C}^n)$ of $\mathcal{C}^n$.*

In fact, any irreducible representation of $SL(2, \mathcal{C})$ is of the form $\mathcal{S}^m(\mathcal{C}^2)$ for $n = 0, 1, 2, \ldots$ ($\mathcal{S}^0(\mathcal{C}^2)$ is the trivial one dimensional representation). Note that PSL_2 is just the image of $SL(2, \mathcal{C})$ in $SL(\mathcal{S}^2(\mathcal{C}^2)) \cong SL(3, \mathcal{C})$. For m odd, these representations are faithful representations of $SL(2, \mathcal{C})$ and for m even, these have kernel $\{\pm 1\}$ (and so factor through PSL_2). Therefore the irreducible representations of PSL_2 are $\mathcal{S}^m(\mathcal{C}^2), m = 0, 2, 4, \ldots$. A character of a representation of $SL(2, \mathcal{C})$ is determined by its behavior on the diagonalizable elements of $SL(2, \mathcal{C})$, since these are Zariski dense in this group. If ϕ_m is the character associated with $\mathcal{S}^m(\mathcal{C}^2)$ and $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is a diagonal element, then $\phi_m(g) = a^m + a^{m-2} + \ldots + a^{-m}$. We use this notation in the following:

Proposition 2.5. *Let χ be the character of the irreducible representation of PSL_2 on $\mathcal{C}^3$ and χ_m the character of the representation on $\mathcal{S}^m(\mathcal{C}^3)$. We then have the following decompositions:*

1. $\chi_2 = \phi_4 + \phi_0$
2. $\chi_3 = \phi_6 + \phi_2$
3. $\chi_4 = \phi_8 + \phi_4 + \phi_0$
4. $\chi_5 = \phi_{10} + \phi_6 + \phi_2$

Proof. To verify these formulas, it is enough to evaluate the χ_m on the diagonal elements of PSL_2. Such a diagonal element is of the form $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}$. Therefore,

$\chi(g) = a^2 + 1 + a^{-2} = \bar{\chi}(g)$. Using the formulas $\chi_2 = \chi^2 - \bar{\chi}, \chi_3 = \chi^3 - 2\chi\bar{\chi} + 1$, etc., we are able to verify the above formulas. □

For each finite primitive subgroup of $SL(2, \mathcal{C})$ and $SL(3, \mathcal{C})$, using its character table (computed using the group theory system Cayley [0]) and the orthogonality relations of characters, we can decompose the characters of the symmetric product (computed using the computer algebra system AXIOM, cf. [0]). The result is summarised in the following two tables, where the numbers $4, 3^2$ in the column A_5 and the row 3 of Table 2 means that the 3-th symmetric product of the character of any faithfull irreducible representation of A_5 in $SL(3, \mathbf{C})$ has an irreducible summand of degree 4 and two irreducible summands of degree 3.

	$A_5^{SL_2}$	$S_4^{SL_2}$	$A_4^{SL_2}$
2	3	3	3
3	4	4	2^2
4	5	$3, 2$	$3, 1^2$
5	6	$4, 2$	2^3
6	$4, 3$	$3^2, 1$	$3^2, 1$

Table 1

	PSL_2 $PSL_2 \times C_3$	A_5 $A_5 \times C_3$	$F_{36}^{SL_3}$	G_{168} $G_{168} \times C_3$	$A_6^{SL_3}$	$H_{72}^{SL_3}$	$H_{216}^{SL_3}$
2	$5, 1$	$5, 1$	$3, 3$	6	6	6	6
3	$7, 3$	$4, 3^2$	$1^2, 4^2$	$7, 3$	10	$8, 2$	$8, 2$
4	$9, 5, 1$	$5^2, 4, 1$	3^5	$8, 6, 1$	$9, 6$	$6^2, 3$	$6^2, 3$
5	$11, 7, 3$	$5, 4, 3^4$	3^7	$8, 7, 3^2$	$15, 3^2$	$6, 3^5$	$9, 6, 3^2$

Table 2

For later use we note that in the decomposition of the third symmetric product of $F_{36}^{SL_3}$, the two one dimensional characters ψ_i are of order 4, i.e. $\psi_i^4 = 1$ but $\psi_i^2 \neq 1$.

3. Differential Equations

3.1. Galois Theory

In this section we first briefly review some facts about differential algebra and the existing algorithms for computing liouvillian solutions of linear differential equations. For a more complete exposition we refer to [0, 0, 0, 0].

A *differential field* (k, δ) is a field k together with a derivation δ on k. A *differential field extension* of (k, δ) is a differential field (K, Δ) such that K is a field extension of k and Δ is an extension of the derivation δ to a derivation on K. In this paper we always assume that k is a field of characteristic 0 and that the field $\mathcal{C} = \ker_k(\delta)$ of constants of δ in k is algebraically closed (e.g. $(\overline{\mathbf{Q}}(x), \frac{d}{dx})$).
We also write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \ldots$ for $\delta(y), \delta^2(y), \ldots$. Unless otherwise stated, a differential equation $L(y) = 0$ over k always means an ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k).$$

Definition 3.1. A differential field extension (K, Δ) of (k, δ) is a *liouvillian extension* if there is a tower of fields

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K,$$

where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

η_i is algebraic over K_i, or
$\delta(\eta_i) \in K_i$ (extension by an integral), or
$\delta(\eta_i)/\eta_i \in K_i$ (extension by the exponential of an integral).

A function contained in a liouvillian extension of k is called *a liouvillian function* over k.

In [0] J. Kovacic gives an algorithm to find a basis of the liouvillian solutions of a second order linear differential equation with coefficients in $k_0(x)$, where k_0 is a finite algebraic extension of $\mathbf{Q}$. In [0] the first author gives a procedure to find a basis of the liouvillian solutions of a linear differential equation $L(y) = 0$ of arbitrary degree n with coefficients belonging to a finite algebraic extension of $\mathbf{Q}(x)$.

Definition 3.2. Let K_1 and K_2 be two differential extensions of k. A *differential k-isomorphism* between K_1 and K_2 is a field isomorphism that leaves k fixed and commutes with δ. The *differential galois group* $\mathcal{G}(K/k)$ of a differential field extension K of k is the set of all differential k-automorphisms of K.

Definition 3.3. Let $L(y) = 0$ be a homogeneous linear differential equation of degree n with coefficients in a differential field k. A differential field extension K of k is called a *Picard-Vessiot extension (PVE)* of k for $L(y) = 0$ if the following holds:

$K = k < y_1, y_2, \ldots, y_n >$, the differential field generated by k and $y_1, y_2, \ldots, y_n$, where $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of $L(y) = 0$.
K and k have the same field of constants.

We denote $\mathcal{G}(L)$ the differential galois group of a PVE associated to $L(y) = 0$.

A PVE of k associated with $L(y) = 0$ is well defined and unique up to differential k-isomorphisms if $\ker_k(\delta)$ is an algebraically closed field of characteristic 0. It may be viewed as the splitting field for the equation $L(y) = 0$. The differential galois group $\mathcal{G}(L)$ of $L(y) = 0$ is a linear algebraic group, and there is a galois correspondence between differential subfields of K/k and linear algebraic subgroups of $\mathcal{G}(L)$ (see, e.g. [0, 0]). If we choose a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of the equation $L(y) = 0$, then for each $\sigma \in \mathcal{G}(L)$ we get $\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j$, where $c_{ij} \in \mathcal{C}$. This gives a faithful representation of $\mathcal{G}(L)$ as a subgroup of $GL(n, \mathcal{C})$. Different choices of basis $\{y_1, y_2, \ldots, y_n\}$ give equivalent representations. This equivalence class of representations is fundamental to our approach. In the sequel we always consider this representation as *the* representation of $\mathcal{G}(L)$.

The following known results show that many properties of the equation $L(y) = 0$ and of its solutions are related to the structure of the group $\mathcal{G}(L)$:

Theorem 3.1. (see e.g. [0], §22, [0], §33) *The differential equation $L(y) = 0$ of degree n over k factors as a differential operator over k if and only if $\mathcal{G}(L) \in GL(n, \mathcal{C})$ is a reducible linear group. Let V be the solution space of $L(y) = 0$. $\mathcal{G}(L)$ leaves an m dimensional subspace of V invariant if and only if $L(y) = L_{n-m}(L_m(y))$ where $L_{n-m}(y)$ and $L_m(y)$ have coefficients in k and are of order $m - n$ and m.*

Theorem 3.2. (see e.g. [0]) *A differential equation $L(y) = 0$ with coefficients in k has*

only solutions which are algebraic over k if and only if $\mathcal{G}(L)$ is a finite group,
only liouvillian solutions over k if and only if the component of the identity $\mathcal{G}(L)^\circ$ of $\mathcal{G}(L)$ in the Zariski topology is solvable. In this case $L(y) = 0$ has a solution whose logarithmic derivative is algebraic over k.

The following theorem will enable us to always assume that the differential galois group $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ of a differential equation $L(y) = 0$ of degree n is unimodular.

Theorem 3.3. ([0], p. 41) *The differential galois group of an differential equation of the form*

$$L(y) = y^{(n)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k) \tag{3.1}$$

is a unimodular group (i.e. $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$).

Using the variable transformation $y = z \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$ it is always possible to transform a given differential equation $L(y)$ into an equation $L_{SL}(y)$ of the form (3.1). For $L(y) = y''' + a_2 y'' + a_1 y' + a_0 y$ we get:

$$L_{SL}(y) = y''' + \left(a_1 - \frac{{a_2}^2}{3} - a_2'\right) y' + \left(a_0 - \frac{a_1 a_2}{3} - \frac{a_2''}{3} + \frac{2{a_2}^3}{27}\right) y.$$

The above form is a sufficient but not necessary condition for $\mathcal{G}(L)$ to be unimodular.

3.2. Linear Operators

We first collect some basic facts on linear differential operators. Linear differential operators can be seen as skew polynomials which can be manipulated almost in the same way as ordinary polynomials.

3.2.1. Factorization of linear differential equations

Let k be a field and δ be a derivation on k. In order to define the notions of irreducibility and factorization for a linear differential equation

$$L(y) = a_n\delta^n(y) + a_{n-1}\delta^{(n-1)}(y) + \cdots + a_0 y = 0$$

of degree n and coefficients in k we look at the associated differential operator:

$$p(\delta) \quad = \quad a_n\delta^n + a_{n-1}\delta^{(n-1)} + \cdots + a_0$$

We now replace D^n by δ^n in $p(\delta)$ and consider

$$p(D) \quad = \quad a_n D^n + a_{n-1}D^{(n-1)} + \cdots + a_0$$

as a *skew polynomial* in D. From $\delta(ay) = \delta(a)y + a\delta(y)$ one gets the rule $Da = aD + a^\delta$. We denote $k[D,\delta]$ the set of all such skew polynomials. This is an example of what is called an Ore ring in the litterature.

In [0] an algebraic theory of $k[D,\delta]$ is given. It is shown there that the usual polynomial addition and a multiplication defined by $Da = aD + a^\delta$ and distributivity makes $k[D,\delta]$ into a (non commutative) ring which has a left and right euclidean algorithm. The degree of $p(D)$ is defined to be the usual polynomial degree of $p(D)$ in D. Since k is a field, the degree of a product is the sum of the degrees.

Definition 3.4. *A linear differential operator $p(D) \in k[D,\delta]$ is* reducible*, if $\exists q_1(D)$, $q_2(D) \in k[D,\delta]$ of degrees > 0 such that $p(D) = q_1(D)\, q_2(D)$. If $p(D)$ is not reducible, then $p(D)$ is called irreducible. A linear differential equation is called reducible (resp. irreducible) if the associated differential operator is reducible (resp. irreducible).*

If $L(y)$ is reducible, then $L(y)$ can be writen as $L_1(L_2(y))$, where $L_1(y)$ and $L_2(y)$ are differential equations of degree > 0. We point out that a factorization of differential equation is usually not unique:

Example: We give two irreducible decompositions of a third order differential operator:

$$\begin{aligned}
&\frac{\mathrm{d}^3}{\mathrm{d}x^3} - \frac{8x^2-2x-3}{2x(4x+1)}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{12x+11}{4x(4x+1)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{4x+5}{4x(4x+1)}\\
&= \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{4x+3}{2x(4x+1)}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{4x+5}{4x(4x+1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{4x^2+x-1}{x(4x+1)}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{4x}\right)
\end{aligned}$$

This shows that different irreducible decompositions, where the degrees are permuted, are possible. ∎

In general, although the irreducible factors are not unique, their degrees are unique (up to permutation) (see [0]). By Theorem 3.1, any irreducible subspace of the solution space of $L(y) = 0$ corresponds to a right factor of $L(y)$. Therefore for a completely reducible galois group (for example, a primitive galois group) the irreducible factors correspond to irreducible invariant subspaces. In this case, any permutation of the degrees gives a (possibly different) factorization. Further properties of factorizations and an algorithm computing **a** factorization of a reducible differential equation $L(y)$ with coefficients in $\mathbf{C}(x)$ can be found in [0, 0, 0].

An Eisenstein criterium for linear differential equation is given in [0]. Another condition for irreducibility is given in [0], p. 293.

3.2.2. Symmetric power of a differential equation

In this section we show how, given some linear differential equations, one can construct the following equations:

THEOREM 3.4. (cf. [0]) *Let $L_1(y) = 0$ and $L_2(y) = 0$ be linear differential equations of degree respectively n_1 and n_2 and fundamental system respectively $S_1 = \{u_1, \cdots, u_{n_1}\}$ und $S_2 = \{v_1, \cdots, v_{n_2}\}$. Then one can construct a differential equation:*

$L(y) = L_1(y) \circledS L_2(y) = 0$ of degree $n_3 \leq n_1 n_2$, whose solution space is spanned by $S = \{u_1 v_1, \cdots, u_{n_1} v_1, \cdots, u_{n_1} v_{n_2}\}$.
$L^{(1)}(y) = 0$ of degree $n \leq n_1$, whose solution space is spanned by the set $S^{(1)} = \{\delta(u_1), \cdots, \delta(u_{n_1})\}$.

PROOF. In order to construct $L_1(y) \circledS L_2(y) = 0$ we take two "arbitrary" solutions u and v of $L_1(y) = 0$ resp. $L_2(y) = 0$ and differentiate their product:

$$\begin{aligned} Y &= uv \\ \delta(Y) &= \delta(u)v + u\delta(v) \\ &\cdots \\ \delta^m(Y) &= \sum_{j=0}^{m} \binom{m}{j} \delta^j(u)\delta^{m-j}(v). \end{aligned}$$

On the right side we can always replace terms $\delta^{n_1}(u)$ and $\delta^{n_2}(v)$ by derivatives of lower order using $L_1(u) = 0$ and $L_2(v) = 0$. On the right side there are then at most $n_1 n_2$ different terms $\delta^i(u)\delta^j(v)$ where $i < n_1$ and $j < n_2$. This shows that for some $m \leq n_1 n_2$ the set $\{Y, \delta(Y), \cdots, \delta^m(Y)\}$ is linear dependent over k, which gives a differential equation for $Y = uv$.

For $L_1 = \sum_i^{n_1} a_i \delta^i(y)$ the differential equation $L^{(1)}(y)$ is given by:

If $a_0 = 0$, then $L^{(1)}(y) = \sum_{i=1}^{n_1} a_i \delta^{i-1}(y)$.
If $a_0 \neq 0$ then

$$L^{(1)}(y) = a_n \delta^n(y) + \sum_{i=0}^{n-1} (\delta(a_{i+1}) + a_i)\delta^i(y) - \frac{\delta(a_0)}{a_0} \left(\sum_{i=1}^{n} a_i \delta^{i-1}(y) \right).$$

In [0] it is shown that S (resp. $S^{(1)}$) spans the solution space of $L_1(y) ⓢ L_2(y) = 0$ (resp. $L^{(1)}(y) = 0$). □

An important special case of the above construction is:

DEFINITION 3.5. *The linear differential equation*

$$L^{ⓢ m}(y) = \overbrace{L(y) ⓢ \cdots ⓢ L(y)}^{m} = 0$$

is called symmetric power of order m of $L(y) = 0$.

In order to compute $L^{ⓢ m}(y)$, one can also differentiate u^m, where u is an "arbitrary" solution of $L(y) = 0$. It can be shown that the differential equation of lowest degree for u^m is just $L^{ⓢ m}(y)$ (note that u^m has to be a solution of $L^{ⓢ m}(y)$). The order of $L^{ⓢ m}$ is at most $\binom{n+m-1}{n-1}$, where n is the degree of $L(y)$ (cf. [0]).

Example: For the the Airy equation $L(y) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy = 0$ we get

$$\begin{aligned} L^{ⓢ 2}(y) &= \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y \\ L^{ⓢ 6}(y) &= \frac{\mathrm{d}^7 y}{\mathrm{d}x^7} - 56x\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} - 140\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 784x^2\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2352x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ &\quad -4(576x^3 - 295)\frac{\mathrm{d}y}{\mathrm{d}x} - 3456x^2 y. \end{aligned}$$

Note that $\left(L^{ⓢ 2}\right)^{ⓢ 3} = L^{ⓢ 6}(y)$. ∎

Let $L(y)$ have order n and let $L(y) = 0$ have solution space V in some Picard-Vessiot K extension of k. There is a natural map Φ_m of the m^{th} symmetric power $\mathcal{S}^m(V)$ (c.f., [0], p. 586) into K given by sending $z_1 ⓢ \ldots ⓢ z_m$ to $z_1 \cdot \ldots \cdot z_m$. The image of this map is the solution space of $L^{ⓢ m}(y) = 0$. The following lemma summarizes the properties of Φ_m needed later:

LEMMA 3.5. *1 Φ_m is a $\mathcal{G}(L)$ morphism of $\mathcal{G}(L)$ modules.*

2 If all representations of $\mathcal{G}(L)$ are completely reducible, then the solution space of $L^{ⓢ m}(y) = 0$ is $\mathcal{G}(L)$-isomorphic to a direct summand of $\mathcal{S}^m(V)$.

3 If $n = 2$, then Φ_m is a bijection for all m. In particular, $L^{ⓢ m}(y) = 0$ has order $m + 1$.

4 If $n = 3$ and Φ_i is a bijection for $i < m$, then the dimension of the kernel of Φ_m is at most 1. In this case, the order of $L^{ⓢ m}(y) = 0$ is either $\frac{1}{2}(m+2)(m+1)$ or $\frac{1}{2}(m+2)(m+1) - 1$.

5 If $n = 3$ and Φ_i is a bijection for $i < m-1$, then the dimension of the kernel of Φ_m is at most 3. In this case, the order of $L^{ⓢ m}(y) = 0$ is at least $\frac{1}{2}(m+2)(m+1) - 3$.

PROOF. 1. is obvious and 2. follows from 1. and complete reducibility. Now assume $n = 2$. If Φ_m is not injective, then there is a homogeneous polynomial F of degree m with coefficients in $\mathcal{C}$ such that $F(y_1, y_2) = 0$ for some linearly independent solutions y_1 and y_2 of $L(y) = 0$. Since $\mathcal{C}$ is algebraically closed, F may be written as a product of linear polynomials, so $F(y_1, y_2) = 0$ would imply that y_1 and y_2 are linearly dependent, a contradiction. This proves 3.

Assume $n = 3$ and assume Φ_i is a bijection for $i < m$. Let $\{y_1, y_2, y_3\}$ be a basis of V. We then have that if $P \neq 0$ is a homogeneous polynomial of degree $i, i < m$, then $P(y_1, y_2, y_3) \neq 0$. Let $W = \{F \mid F$ is a homogeneous polynomial of degree m with coefficients in $\mathcal{C}$ such that $F(y_1, y_2, y_3) = 0\}$. Each non-zero F in W must be irreducible, since otherwise we would have $P(y_1, y_2, y_3) = 0$ for some homogeneous P of degree less than m. If the kernel of Φ_m has dimension at least 2, then there would be two relatively prime irreducible homogeneous polynomials F_1 and F_2 such that $F_1(y_1, y_2, y_3) = F_2(y_1, y_2, y_3) = 0$. The resultant $Res_{y_3}(F_1, F_2)$, of F_1 and F_2 with respect to y_3, is an homogeneous polynomial $F(y_1, y_2)$ of degree m^2 in y_1 and y_2 which must be zero. As in the previous case, a factorization of $F(y_1, y_2)$ yields a contradiction. This proves 4.

Again assume $n = 3$ and assume Φ_i is a bijection for $i < m - 1$. Let $\{y_1, y_2, y_3\}$ be a basis of V. If Φ_{m-1} is a bijection then by what we have just shown, the kernel of Φ_m has dimension at most 1. Assume Φ_{m-1} is not a bijection. This means that the kernel of Φ_{m-1} has dimension 1. Identifying the symmetric powers with spaces of homogeneous polynomials, we let P be a homogeneous polynomial of degree $m - 1$ that spans this kernel. We see, as above, that P must be irreducible. Let $W = \{F \mid F$ is a homogeneous polynomial of degree m with coefficients in $\mathcal{C}$ such that $F(y_1, y_2, y_3) = 0\}$. If $F \in W$ and P does not divide F, then arguing with resultants as in the previous case, we would have a contradiction. Therefore, P divides all the elements of W. This means that W is a subspace of the space of homogeneous polynomials of degree m spanned by Y_1P, Y_2P and Y_3P. Therefore the dimension of W is at most 3. □

The following will give us a criterium to test if the galois group is monomial. As we have noted at the beginning of section 2.1, if n is prime, then a subgroup of $GL(n, \mathbf{C})$ is imprimitive if and only if it is monomial. If n is not prime, there are always non-monomial imprimitive subgroups of $GL(n, \mathbf{C})$.

Proposition 3.6. *If an irreducible linear differential equation $L(y) = 0$ of order n with coefficients in k has a monomial differential galois group $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$, then the n-th symmetric power $L^{\circledS n}(y) = 0$ of $L(y) = 0$ has a solution which is the square root of an element of k.*

Proof. If $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$ is a monomial group, then there is a basis $\{y_1, \cdots, y_n\}$ of the solution space of $L(y) = 0$ such that all matrices $\sigma \in \mathcal{G}(L)$ contain only one non zero element in any row and any column. Such a matrix σ has n non zero entries $a_1, \cdots, a_n$, and since it is an element of a unimodular group, its determinant $\pm a_1 a_2 \cdots a_n$ is 1. For any $\sigma \in \mathcal{G}(L)$ we get

$$\begin{aligned} \sigma(y_1 y_2 \cdots y_n) &= \pm(a_1 a_2 \cdots a_n)(y_1 y_2 \cdots y_n) = \pm det(\sigma) \cdot (y_1 y_2 \cdots y_n) \\ &= \pm y_1 y_2 \cdots y_n. \end{aligned}$$

This shows that $(y_1 y_2 \cdots y_n)^2$ is invariant under $\mathcal{G}(L)$ and thus belongs to k. Since $y_1 y_2 \cdots y_n$ is a solution of $L^{\circledS n}(y) = 0$, we get that $L^{\circledS n}(y) = 0$ has a solution which is the square root of an element of k. □

4. Main results

4.1. Galois groups and symmetric powers

In this section we describe the behavior of the galois group in terms of properties of various symmetric powers of the differential equation. This will give necessary and sufficient conditions for a second or third order linear differential equation to have a liouvillian solution. We start with second order equations. In what follows k will always be a differential field with algebraically closed field of constants $\mathcal{C}$.

Theorem 4.1. *Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficients in k and unimodular differential galois group.*

1 *$L(y)$ is reducible if and only if $L(y) = 0$ has a solution $y \neq 0$ such that $y'/y \in k$. In this case $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$ is reducible.*
2 *Assume $L(y)$ is irreducible. Then $\mathcal{G}(L)$ is imprimitive if and only if $L^{ⓢ2}(y) = 0$ is reducible. In this case $L^{ⓢ2}(y) = 0$ has a solution $y \neq 0$ such that $y^2 \in k$. and $\mathcal{G}(L) \cong \mathcal{C}^* \rtimes \mathbf{Z}/2\mathbf{Z}$ or the dihedral group D_{2n}.*
3 *Assume $\mathcal{G}(L)$ is primitive. Then $L^{ⓢ6}(y) = 0$ is reducible if and only if $\mathcal{G}(L)$ is a finite group.*
4 *$\mathcal{G}(L) \cong SL(2, C)$ if none of the above hold.*

Proof. Theorem 3.1 handles case 1. Therefore assume that $L(y)$ is irreducible. In this case the galois group is either primitive or imprimitive. As we noted in the discussion following Proposition 2.3, the only primitive subgroups of $SL(2, C)$ are either finite or all of $SL(2, C)$.

Assume $L^{ⓢ2}(y)$ is reducible. Lemma 3.5 implies that the solution space of $L^{ⓢ2}(y)$ is $\mathcal{G}(L)$-isomorphic to the second symmetric power of the solution space of $L(y) = 0$. Therefore this symmetric power must be reducible. Table 1 shows that $\mathcal{G}(L)$ cannot be a finite primitive group. Proposition 2.4 shows that $\mathcal{G}(L)$ cannot be $SL(2, C)$. Therefore $\mathcal{G}(L)$ must be imprimitive. Proposition 3.6 implies that $L^{ⓢ2}(y)$ has a solution $y \neq 0$ such that $y^2 \in k$. Furthermore $\mathcal{G}(L)$ must be a monomial group which in this case means that it is a subgroup of $\mathcal{C}^* \rtimes \mathbf{Z}/2\mathbf{Z}$. Either it is the full group or it must be a proper subgroup, in which case it is finite and must be a dihedral group. Conversely, if $\mathcal{G}(L)$ is imprimitive, proposition 3.6 implies that $L^{ⓢ2}(y)$ is reducible.

Now assume that $\mathcal{G}(L)$ is primitive. We then have that $\mathcal{G}(L)$ is one of the finite primitive groups or all of $SL(2, C)$. Table 1 implies that $L^{ⓢ6}(y)$ is reducible. Conversely, Proposition 2.4 implies that $L^{ⓢ6}(y)$ is irreducible if $\mathcal{G}(L) \cong SL(2, C)$.

Finally, any proper subgroup of $SL(2, C)$ is either reducible, imprimitive or a finite primitive group so the final statement above is true. □

In cases 1, 3, and 4 of the above, one can give simple criteria to determine the galois group. We do this in the next three propositions. Case 2 is more problematic and we discuss this following these three results.

Proposition 4.2. *Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficients in k and unimodular differential galois group. Assume $L(y)$ is reducible (and so has a solution $y \neq 0$ such that $y'/y \in k$).*

1 *If $L(y) = 0$ has a unique (up to constant multiple) solution $y \neq 0$ such that $y'/y \in k$, then $\mathcal{G}(L)$ is conjugate to a subgroup of*

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathcal{C}, a \neq 0 \right\}$$

Furthermore, $\mathcal{G}(L)$ is a proper subgroup of T if and only if $y^m \in k$ for some positive integer m. In this case, $\mathcal{G}(L)$ is conjugate to

$$T_m = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathcal{C}, a^m = 1 \right\}$$

where m is the smallest positive integer such that $y^m = 1$.

2 *If $L(y) = 0$ has two linearly independent solutions y_1 and y_2 such that $y_i'/y_i \in k, i = 1, 2$, then $\mathcal{G}(L)$ is conjugate to a subgroup of*

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathcal{C}, a \neq 0 \right\}$$

In this case, $y_1 y_2 \in k$. Furthermore, $\mathcal{G}(L)$ is conjugate to a proper subgroup of D if and only if $y_1^m \in k$ for some positive integer m. In this case $\mathcal{G}(L)$ is a cyclic group of order m where m is the smallest positive integer such that $y_1^m \in k$.

Proof. If $L(y) = 0$ has a solution $y \neq 0$ such that $y'/y \in k$ then y is an eigenvector for all the elements of $\mathcal{G}(L)$ so $\mathcal{G}(L)$ is conjugate to a subgroup of T. If $L(y) = 0$ has two linearly independent solutions $y \neq 0$ such that $y'/y \in k$ then the elements of $\mathcal{G}(L)$ have two independent common eigenvectors so $\mathcal{G}(L)$ is conjugate to a subgroup of D.

Assume case 1 holds and select a basis of the solution space such that $\mathcal{G}(L) \subseteq T$. The map sending $\sigma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ to a is a homomorphism of $\mathcal{G}(L)$ to $\mathcal{C}^*$. If the image of $\mathcal{G}(L)$ is a proper subgroup of $\mathcal{C}^*$, then this image is a finite cyclic group. Therefore, $\mathcal{G}(L) = T_m$. Let y be a common eigenvector of $\mathcal{G}(L)$. We then have $\sigma(y^m) = (ay)^m = y^m$ for any $\sigma \in \mathcal{G}(L)$ since a is an m^{th} root of 1. Clearly, m is the smallest positive integer such that $y^m \in k$. This proves 1.

Assume $L(y) = 0$ has two linearly independent solutions y_1 and y_2 such that $y_i'/y_i \in k, i = 1, 2$. With respect to y_1 and y_2 $\mathcal{G}(L)$ may be identified with a subgroup of D. D is isomorphic to $\mathcal{C}^*$ so any proper subgroup is a finite cyclic group. Furthermore, for any $\sigma \in \mathcal{G}(L), \sigma(y_1 y_2) = a y_1 a^{-1} y_2 = y_1 y_2$ so $y_1 y_2 \in k$. The remaining claim follows as above. □

When $k = \mathbf{C}(x), x' = 1$, one has algorithms to decide this question (see the factorization algorithms mentioned above and also [0, 0]). [0] also mentions the idea of seeing how many solutions one has of a specified form to determine the galois group.

Proposition 4.3. *Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficients in k and unimodular differential galois group. Assume $\mathcal{G}(L)$ is primitive.*

1 *$L^{\circledS 3}(y)$ factors over k if and only if $\mathcal{G}(L) \cong A_4^{SL_2}$. In this case, $L^{\circledS 3}(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ have order 2.*

2 Assuming $L^{ⓢ 3}(y)$ *irreducible, then* $L^{ⓢ 4}(y)$ *factors over* k *if and only if* $\mathcal{G}(L) \cong S_4^{SL_2}$*. In this case,* $L^{ⓢ 4}(y) = L_1(L_2(y))$ *where* $L_1(y)$ *and* $L_2(y)$ *have orders* 3 *and* 2.

3 Assuming $L^{ⓢ 4}(y)$ *irreducible, then* $L^{ⓢ 6}(y)$ *factors over* k *if and only if* $\mathcal{G}(L) \cong A_5^{SL_2}$*. In this case,* $L^{ⓢ 6}(y) = L_1(L_2(y))$ *where* $L_1(y)$ *and* $L_2(y)$ *have orders* 4 *and* 3.

4 $\mathcal{G}(L) \cong SL(2, \mathcal{C})$ *if and only if* $L^{ⓢ 6}(y)$ *is irreducible over* k*.*

Proof. Lemma 3.5 states that the solution space of $L^{ⓢ m}(y) = 0$ is isomorphic to the m^{th} symmetric power of V, the solution space of $L(y) = 0$. Table 1 gives the decompositions of these spaces for small m and Proposition 3.1 gives the first three results. Proposition 2.4 gives the final result. □

Proposition 4.4. *Let* $L(y) = 0$ *be a second order homogeneous linear differential equation with coefficients in* k *and unimodular differential galois group.* $\mathcal{G}(L) \cong SL(2, \mathcal{C})$ *if and only if* $L^{ⓢ 6}(y)$ *is irreducible over* k*. Therefore,* $L(y) = 0$ *has liouvillian solutions if and only if* $L^{ⓢ 6}(y)$ *is reducible.*

Proof. If $\mathcal{G}(L) \cong SL(2, \mathcal{C})$, then Proposition 2.4 implies $L^{ⓢ 6}(y)$ is irreducible over k. Conversely, if $\mathcal{G}(L)$ is a proper subgroup of $SL(2, \mathcal{C})$, then Theorem 4.1 implies that $L(y), L^{ⓢ 2}(y)$, or $L^{ⓢ 6}(y)$ are reducible. If $L(y)$ is reducible then the solution space V of $L(y) = 0$ has a $\mathcal{G}(L)$ invariant subspace W. $\mathcal{S}^6(W)$ will be a $\mathcal{G}(L)$ invariant subspace of $\mathcal{S}^6(V)$, so $L^{ⓢ 6}(y)$ will be reducible. Similarly, if W is a proper $\mathcal{G}(L)$ invariant subspace of $\mathcal{S}^2(V)$, then $\mathcal{S}^3(W)$ will be a proper $\mathcal{G}(L)$ invariant subspace of $\mathcal{S}^6(V)$. The final statement follows from the fact that any proper subgroup of $SL(2, \mathcal{C})$ has a component of the identity that has dimension less than 3. □

Proposition 4.4 can be improved if one knows a priori that $L(y) = 0$ has no algebraic solutions (for example, if $k = \mathbf{C}(x)$ and $L(y) = 0$ is not fuchsian). In this case, the proof shows that $\mathcal{G}(L) \cong SL(2, \mathcal{C})$ if and only if $L^{ⓢ 2}(y)$ is irreducible over k (in fact, if and only if $L^{ⓢ 2}(y) = 0$ has a solution $y \neq 0$ such that $y'/y \in k$). This fact is the basis of the necessary conditions for liouvillian solutions developed by Kaplansky [0]. The above shows that they are also sufficient in this case.

We note that it is not so simple to distinguish between the cases of a finite and an infinite group when $\mathcal{G}(L)$ is imprimitive. This question is discussed in [0] and depends on being able to decide : given an element u algebraic over k, determine if there is a non-zero integer n such that $y'/y = nu$ has a solution y algebraic over k. This question is decidable when $k = \mathbf{C}(x), x' = 1$.

We now discuss the relationship between our ideas and Kovacic's algorithm and we assume the reader is familliar with [0]. Kovacic's algorithm deals with four cases. His first case corresponds to the case when the equation (and therefore also the group) is reducible. The second case corresponds to the galois group being an imprimitive group. Kovacic shows that in this case the linear differential equation $L(y) = 0$ has a solution $y \neq 0$ such that $u = y'/y$ satisfies an irreducible polynomial equations $u^2 + a_1 u + a_0 = 0$. Kovacic's algorithm attempts to find this equation. The coefficient a_1 is of the form $a_1 = \frac{y_1'}{y_1} + \frac{y_2'}{y_2} = \frac{(y_1 y_2)'}{y_1 y_2}$ for some solutions y_1, y_2 of $L(y) = 0$. In particular, this implies that $L^{ⓢ 2}(z) = 0$ has a nonzero solution z such that z'/z is rational. By considering the structure of the galois group, Kovacic can in fact show that z^2 is rational. Kovacic further shows that the coefficient a_0 is completely determined once a_1 is known. Therefore, a careful reading of Kovacic's proof shows that he has proven parts 1. and 2. of Theorem 4.1. Case 3 of Kovacic's algorithm corresponds to the galois group being a primitive proper subgroup of $SL(2, \mathcal{C})$ and therefore being finite. Kovacic shows that in this case, $L(y) = 0$ has a solution $y \neq 0$ such that $u = y'/y$ satisfies an irreducible polynomial equation $u^m + a_{m-1}u^{m-1} + \ldots + a_0 = 0$ for $m = 4, 6$, or 12. The form of a_{m-1} shows that for $m = 4, 6$, or 12 $L^{ⓢ m}(z) = 0$ has a nonzero solution z such that z'/z is rational. In particular, $L^{ⓢ m}(z) = 0$ will have a factor of order 1 in one of these cases. Kovacic again shows that the other coefficients a_i are completely determined by a_{n-1} and so, he shows that (assuming cases 1 and 2 do not hold) a necessary and sufficient condition that $L(y) = 0$ has an algebraic solution is that for $m = 4, 6$, or 12 the equation $L^{ⓢ m}(z) = 0$ has a nonzero solution z such that z'/z is rational. Kovacic shows this using the internal structure (e.g., existence of "large" abelian subgroups) of the finite primitive subgroups of $SL(2, \mathcal{C})$. This could also be shown just from the representation theory in the spirit of Theorem 4.1. When one uses the representation theory as we did, one is naturally led to consider higher order factors of the symmetric powers. One can then find necessary and sufficient conditions in terms of the factorization of symmetric powers of relatively small order. This leads to our distinction of the cases of the algorithm in [0] using $L^{ⓢ m}(z) = 0$ for $m \in \{2, 3, 4, 6\}$ instead of $m \in \{2, 4, 6, 12\}$ (see [0]: p. 5 and p. 32):

Corollary 4.5. *Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficients in k and unimodular differential galois group. Then one of the following holds:*

1. *The equation $L(y) = 0$ has a solution of the form $e^{\int \omega}$ where $\omega \in k$ if and only if $L(y) = 0$ is reducible.*
2. *Assume that the above does not hold. The equation $L(y) = 0$ has a solution of the form $e^{\int \omega}$ where ω is algebraic over k of degree 2 if and only if $L^{ⓢ 2}(y) = 0$ is reducible. In this case $\mathcal{G}(L)$ is an imprimitive subgroup of $SL(2, \mathcal{C})$.*
3. *Assume that the above does not hold, then*
   - (a) *The equation $L(y) = 0$ has an algebraic solution of the form $e^{\int \omega}$ where ω is algebraic over k of degree 4 if and only if $L^{ⓢ 3}(y) = 0$ is reducible. In this case $\mathcal{G}(L) \cong A_4^{SL_2}$.*
   - (b) *The equation $L(y) = 0$ has an algebraic solution of the form $e^{\int \omega}$ where ω is algebraic over k of degree 6 if and only if $L^{ⓢ 4}(y) = 0$ is reducible and $L^{ⓢ 3}(y) = 0$ is irreducible. In this case $\mathcal{G}(L) \cong S_4^{SL_2}$*
   - (c) *The equation $L(y) = 0$ has an algebraic solution of the form $e^{\int \omega}$ where ω is algebraic over k of degree 12 if and only if $L^{ⓢ 6}(y) = 0$ is reducible and $L^{ⓢ 4}(y) = 0$ is irreducible. In this case $\mathcal{G}(L) \cong A_5^{SL_2}$.*
4. *The differential equation has no liouvillian solutions. In this case $\mathcal{G}(L) \cong SL(2, \mathcal{C})$.*

Proof. The first case is trivial, so assume $\mathcal{G}(L)$ acts irreducibly.

If $\mathcal{G}(L) \subseteq SL(2, \mathcal{C})$ is an irreducible algebraic group which is not imprimitive and not finite, then the last case holds.

If $\mathcal{G}(L) \subseteq SL(2, \mathcal{C})$ is a finite primitive group, then $\mathcal{G}(L) \cong A_4^{SL_2}$, $S_4^{SL_2}$ or $A_5^{SL_2}$. Table 1 proves the facts about the decomposition of the symmetric powers. That the algebraic degree 4, 6, 12 for ω is best possible follows from [0] p. 32 (or [0, 0]).

Since $\mathcal{G}(L)$ acts irreducibly (i.e. case 1 does not hold), we get from Theorem 4.1 and Table 1 that $\mathcal{G}(L)$ is imprimitive if and only if $L^{\circledS 2}(y) = 0$ is reducible. The fact that the algebraic degree 2 for ω is best possible follows from [0] p. 32 (or [0, 0]). □

We do not make any claims that the above conditions yield, at present, an algorithm that is better than Kovacic's. Kovacic analyses the situation much further and gives more information than we do above (see [0] for improvements of Kovacic's algorithm and applications and [0] for generalizations to higher order equations of some of Kovacic's other ideas and necessary conditions). We do claim that our results show the importance of factorization algorithms and the need for finding more efficient ways to factor linear operators. They are also readily generalized to higher order equations and can be used over any differential field in which there exists an algorithm to factor differential operators.

We now turn to third order equations. We state our results first for groups that are not primitive and then for primitive groups.

Theorem 4.6. *Let $L(y) = 0$ be a third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants whose differential galois $\mathcal{G}(L)$ group is unimodular.*

1 *$L(y) = 0$ is reducible if and only if $L(y) = 0$ has a solution $y \neq 0$ such that $y'/y \in k$ or $L^*(y) = 0$, the adjoint of $L(y) = 0$, has a solution $y \neq 0$ such that $y'/y \in k$ (if $L(y) = y''' + py' + qy = 0$, then $L^*(y) = y''' + py' - (q - p')y = 0$) .*
2 *Assume $L(y)$ is irreducible. Then $\mathcal{G}(L)$ is imprimitive if and only if $L^{\circledS 3}(y) = 0$ has a solution $y \neq 0$ such that $y^2 \in k$. In this case $\mathcal{G}(L)$ is isomorphic to a subgroup of $C^* \rtimes S_3$, where S_3 is the symmetric group on three letters. If $\mathcal{G}(L)$ is isomorphic to a subgroup of $C^* \rtimes A_3$, where A_3 is the alternating group on three letters, then the above solution y is already in k.*
3 *Assume $L(y)$ is irreducible and 2. does not hold, then $\mathcal{G}(L)$ is a primitive group.*

Proof. $L(y) = 0$ is reducible if and only if $L(y) = L_2(L_1(y))$ or $L(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ are of order 1 and 2 respectively with coefficients in k. If $L(y) = L_1(L_2(y))$ then taking adjoints, we have $L^*(y) = L_2^*(L_1^*(y))$. Therefore (1) holds.

Now assume that $\mathcal{G}(L)$ is irreducible. If $\mathcal{G}(L)$ is imprimitive, then Proposition 2.1 implies that $L^{\circledS 3}(y) = 0$ has a solution $y \neq 0$ such that $y^2 \in k$. Now assume the conditions of 2. hold. Table 2 implies that these conditions cannot hold if $\mathcal{G}(L)$ is primitive, unless $\mathcal{G}(L) \cong F_{36}^{SL_3}$. Since $y^2 \in k$, we must have that $\chi^2 = 1$ where χ is the associated character of the one dimensional invariant subspace generated by y of the third symmetric power of the solution space S of $L(y) = 0$. If $\mathcal{G}(L) \cong F_{36}^{SL_3}$, then all one dimensional characters in the decomposition of the third symmetric power of S have degree 4, i.e. $\chi^2 \neq 1, \chi^4 = 1$ (see remark after Table 2). We cannot be in this case. □

Theorem 4.7. *Let $L(y) = 0$ be a third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants, whose differential galois group $\mathcal{G}(L)$ is unimodular. Assume that $\mathcal{G}(L)$ is primitive.*

1 *If $L^{\circledS 2}(y)$ has order 5 or factors then $\mathcal{G}(L)$ is isomorphic to PSL_2, $PSL_2 \times C_3$, A_5, $A_5 \times C_3$ or $F_{36}^{SL_3}$. In this case one of the following holds*

$\mathcal{G}(L) \cong F_{36}^{SL_3}$ if and only if $L^{\circledS 2}(y)$ has a factor of order 3, or
$\mathcal{G}(L) \cong A_5$ or $A_5 \times C_3$ if and only if $L^{\circledS 3}(y)$ has a factor of order 3 and a factor of order 4, or
$\mathcal{G}(L) \cong PSL_2$ or $\mathcal{G}(L) \cong PSL_2 \times C_3$ if and only if the previous two cases do not hold.

2 If $L^{\circledS 2}(y)$ has order 6 and is irreducible, then one of the following holds

$\mathcal{G}(L) \cong G_{168}$ or $G_{168} \times C_3$ if and only if $L^{\circledS 3}(y)$ has a factor of order 3.
$\mathcal{G}(L) \cong A_6^{SL_3}$ if and only if $L^{\circledS 4}(y)$ is reducible and $L^{\circledS 3}(y)$ is irreducible.
$\mathcal{G}(L) \cong H_{72}^{SL_3}$ if and only if $L^{\circledS 5}(y)$ has more than 2 factors of order 3.
$\mathcal{G}(L) \cong H_{216}^{SL_3}$ if and only if $L^{\circledS 5}(y)$ has exactly 2 factors of order 3 and $L^{\circledS 2}(y)$ has a factor of degree 2.
The galois group is $SL(3,\mathcal{C})$ if and only if none of the above happen.

Proof. The proof proceeds by examining Table 2. Let V be the solution space of $L(y) = 0$. If $L^{\circledS 2}(y)$ has order 5 or factors, then $\mathcal{S}^2(V)$ must have an invariant subspace. This can only happen if $\mathcal{G}(L)$ is one of the groups mentioned. Lemma 3.5.3 implies that the order of $L^{\circledS 2}(y)$ is at least 5, so $\mathcal{G}(L) \cong F_{36}^{SL_3}$ if and only if $L^{\circledS 2}(y)$,has a factor of order 3. Lemma 3.5.4 implies that $L^{\circledS 3}(y)$ has order at least 7. Therefore if $\mathcal{G}(L) \cong A_5$ or $A_5 \times C_3$ then $L^{\circledS 3}(y)$ has a factor of order 3. This does not happen in the other cases considered, therefore 1. holds.

Assume $L^{\circledS 2}(y)$ has order 6 and is irreducible. Table 2 implies that $\mathcal{G}(L) \cong G_{168}$ or $G_{168} \times C_3$ or $A_6^{SL_3}$ or $H_{72}^{SL_3}$ or $H_{216}^{SL_3}$ or $SL(3,C)$. From Table 2 we get that $\mathcal{G}(L) \cong G_{168}$ or $G_{168} \times C_3$ if and only if $L^{\circledS 3}(y)$ has a factor of order 3. If $\mathcal{G}(L) \ncong G_{168}$ or $G_{168} \times C_3$, then none of the m^{th} symmetric powers, $m = 2, 3, 4, 5$ of V have a 1 dimensional invariant subspace for these groups, so the m^{th} symmetric powers of $L(y)$ have order exactly $\frac{1}{2}(m+2)(m+1)$. Table 2 describes how these symmetric powers factor and 2. summarizes the distinguishing cases.

If the galois group is $SL(3,C)$ then all symmetric powers are irreducible, so the theorem follows. □

One can use Table 2 to state other necessary and sufficient conditions for the primitive groups. For example, if $L^{\circledS 2}(y)$ has order 6 and is irreducible then $\mathcal{G}(L) \cong A_6^{SL_3}$ if and only if $L^{\circledS 3}(y)$ is irreducible and $L^{\circledS 4}(y)$ factors. Since it is not clear which criteria will be most usefull, we have just stated one set of criteria to give a taste of what can be done. The above theorems allows us to give criteria for a third order linear differential equation to be solvable in terms of lower order linear differential equations (c.f., [0], for third order equations this concept coincides with the concept of "solving in terms of second order equations" or "eulerian" [0, 0]).

Corollary 4.8. *A third order linear differential equation $L(y) = 0$ with coefficients in a differential field k with algebraically closed field of constants and whose differential galois group $\mathcal{G}(L)$ is unimodular, is solvable in terms of lower order linear differential equations if and only if $L^{\circledS 4}(y)$ has order less than* 15 *or factors.*

Proof. We first note that $L(y) = 0$ is solvable in terms of lower order linear differential equations if and only if $\mathcal{G}(L)$ is a proper subgroup of $SL(3,C)$. To see this we use the criterion of [0], Theorem 1 (c.f., [0] Theorem 5.1, p. 48): $L(y) = 0$ cannot be solved in terms of lower order linear differential equations if and only if $\mathcal{G}(L)$ has a lie algebra $\mathtt{g}$ that is simple and such that if $\rho : \mathtt{g} \to \mathtt{gl}(m, C)$ with $m < n$, then $\rho \equiv 0$. The simple lie subalgebras of `sl(3,C)` are `sl(3,C)` and `sl(2,C)` (c.f., [0]). Since `sl(2,C)` has a representation of smaller degree, the only simple lie algebra having no nontrivial representations of lower order is `sl(3,C)`. Therefore $L(y) = 0$ cannot be solved in terms of lower order linear differential equations if and only if $\mathcal{G}(L) = SL(3, C)$. Therefore if $L^{ⓢ 4}(y)$ has order less than 15 or factors, then $\mathcal{G}(L) \neq SL(3, C)$, so $L(y) = 0$ is solvable in terms of lower order linear differential equations.

Now we note that $\mathcal{G}(L)$ is a proper subgroup of $SL(3, C)$ if and only if $\mathcal{G}(L)$ acts reducibly or is imprimitive or is a primitive proper subgroup of $SL(3, C)$. If $\mathcal{G}(L)$ acts reducibly, then the solution space V of $L(y) = 0$ has an invariant one or two dimensional subspace W. $\mathcal{S}^4(W)$ will be a proper invariant subspace of $\mathcal{S}^4(V)$, so $L^{ⓢ 4}(y)$ has order less than 15 or factors. If $\mathcal{G}(L)$ is imprimitive, then Theorem 4.6 implies that $L^{ⓢ 3}(y)$ has a factor of order 1. Therefore the solution space Z of $L^{ⓢ 3}(y) = 0$ has a one dimensional invariant subspace. Let p span this space and y_1, y_2, y_3 be a basis of V. Then y_1p, y_2p, y_3p spans an invariant subspace of the solution space of $L^{ⓢ 4}(y) = 0$ and so must factor or have order at most 3. Finally, if $\mathcal{G}(L)$ is a primitive proper subgroup of $SL(3, C)$, then Table 2 shows that $L^{ⓢ 4}(y)$ has order less than 15 or factors. □

One also can give necessary and sufficient conditions for the existence of liouvillian solutions:

Corollary 4.9. *Let $L(y) = 0$ be an irreducible third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants whose differential galois $\mathcal{G}(L)$ group is unimodular. $L(y) = 0$ has a liouvillian solution if and only if*

1 $L^{ⓢ 4}(y)$ has order less than 15 or factors, and
2 one of the following holds:

$L^{ⓢ 2}(y)$ has order 6 and is irreducible, or
$L^{ⓢ 3}(y)$ has a factor of order 4.

Proof. $L(y) = 0$ has a liouvillian solution if and only if it is solvable in terms of lower order linear differential equations and its galois group is not PSL_2 or $PSL_2 \times C_3$. The result now follows from Theorem 4.6 and Table 2. □

We now show how our apporach can be used to distinguish the different cases for the algebraic degree of the logarithmic derivative of a liouvillian solution in the algorithm given in [0] using the bounds given in [0] Theorem 5.2 and the improvement of this bounds given in [0].

Corollary 4.10. *Let $L(y) = 0$ be an irreducible third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants whose differential galois $\mathcal{G}(L)$ group is unimodular.*

1 L(y) has a solution whose logarithmic derivative is algebraic of degree 3 if and only if $L^{ⓢ 3}(y)$ has a solution $y \neq 0$ such that $y^2 \in k$. In this case $\mathcal{G}(L)$ is an imprimitive subgroup of $SL(3, \mathcal{C})$.

2 If the above does not hold, then

(a) *L(y) has an algebraic solution whose logarithmic derivative is algebraic of degree 6 if and only if* $L^{\circledS 3}(y)$ *has an irreducible factor of order* 4. *In this case* $\mathcal{G}(L) \cong A_5$, $A_5 \times C_3$ *or* $F_{36}^{SL_3}$.

(b) *L(y) has an algebraic solution whose logarithmic derivative is algebraic of degree 9 if and only if* $L^{\circledS 3}(y)$ *has an irreducible factor of order* 2. *In this case* $\mathcal{G}(L) \cong H_{216}^{SL_3}$ *or* $H_{72}^{SL_3}$.

(c) *L(y) has an algebraic solution whose logarithmic derivative is algebraic of degree 21 if and only if* $L^{\circledS 3}(y)$ *has an irreducible factor of order* 3 *and* $L^{\circledS 2}(y)$ *is irreducible of degree* 6. *In this case* $\mathcal{G}(L) \cong G_{168}$ *or* $G_{168} \times C_3$.

(d) *L(y) has an algebraic solution whose logarithmic derivative is algebraic of degree 36 if and only if* $L^{\circledS 3}(y)$ *is irreducible of degree* 10 *and* $L^{\circledS 4}(y)$ *is reducible. In this case* $\mathcal{G}(L) \cong A_6^{SL_3}$.

3 If none of the above holds, then $L(y) = 0$ *has no liouvillian solutions.*

Proof. If $L^{\circledS 3}(y)$ has a solution $y \neq 0$ such that $y^2 \in k$, then $\mathcal{G}(L)$ is an imprimitive subgroup of $SL(3, \mathcal{C})$ (Theorem 4.6). In this case $L(y) = 0$ has a solution whose logarithmic derivative is algebraic of degree 3 (Theorem 5.2 of [0]). The only if part follows from the fact that the finite primitive group the bounds given in the Theorem are best possible (cf. [0]), Theorem 4.4).
If the first case does not hold, then $\mathcal{G}(L)$ is a primitive subgroup of $SL(3, \mathcal{C})$. If $L^{\circledS 3}(y)$ has a factor of order 4, or $L^{\circledS 2}(y)$ is irreducible of degree 6 or $L^{\circledS 3}(y)$ is irreducible of degree 10, then $\mathcal{G}(L) \ncong PSL_2$ or $PSL_2 \times C_3$. If $L^{\circledS 3}(y)$ or $L^{\circledS 4}(y)$ is reducible, then $\mathcal{G}(L) \ncong SL(3, \mathcal{C})$ (cf. Proposition 2.4). Thus $\mathcal{G}(L)$ is a finite primitive subgroup of $SL(3, \mathcal{C})$. The conditions of the symmetric powers of $L(y) = 0$ now follow from Table 2 and the algebraic degree of the logarithmic derivative from Theorem 4.4 of [0]. This proves 2.

If $\mathcal{G}(L)$ is not an imprimitive group and not a finite primitive subgroup of $SL(3, \mathcal{C})$, then the irreducible equation $L(y) = 0$ has no liouvillian solutions (cf. [0]), Corollary 3.7). □

The above result shows that with the necessary and sufficient conditions given in this paper, one has to look for at most one possible degree of logarithmic derivative of the solution of $L(y) = 0$. This gives a substantial simplification of the algorithm given in [0] for third order differential equations.

5. Examples

Using our results, we want to decide if the differential equation

$$L(y) \quad = \quad \frac{d^3 y}{dx^3} + \frac{32x^2 - 27x + 27}{36x^2(x-1)^2} \frac{d\,y}{dx} - \frac{64x^3 - 81x^2 + 135x - 54}{72x^3(x-1)^3} y = 0$$

has a liouvillian solution.

The equation $L(y) = 0$ is reducible if and only if $L(y) = 0$ or its adjoint $L^*(y) = 0$ has a right factor of order 1. This is equivalent to saying that either $L(y) = 0$ or $L^*(y) = 0$ has a solution whose logarithmic derivative is rational (cf. [0]). Since no such solution exists (this could be computed for example using an algorithm implemented by Manuel

Bronstein in the computer algebra system AXIOM, cf. [0]), we get that $L(y) = 0$ is irreducible.

We now test if the differential galois group is an imprimitive subgroup of $SL(3, \mathcal{C})$. This is the case (cf. Theorem 4.6) if and only if $L^{\circledS 3}(y) =$

$$\begin{aligned}
&\frac{\mathrm{d}^7 y}{\mathrm{d}x^7} + \frac{224x^2 - 189x + 189}{18x^2(x-1)^2}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \frac{-2240x^3 + 2835x^2 - 4725x + 1890}{36x^3(x-1)^3}\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \\
&+ \frac{340480x^4 - 574560x^3 + 1263465x^2 - 969570x + 280665}{1296x^4(x-1)^4}\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \\
&+ \frac{-358400x^5 + 756000x^4 - 2036475x^3 + 2275560x^2 - 1284255x + 289170}{432x^5(x-1)^5}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\
&+ \Bigg(\frac{1003520x^6 - 2540160x^5 + 8042895x^4 - 11711070x^3}{576x^6(x-1)^6} \\
&\qquad + \frac{9723735x^2 - 4309200x + 793800}{576x^6(x-1)^6}\Bigg)\frac{\mathrm{d}\,y}{\mathrm{d}x} \\
&+ \Bigg(\frac{-1576960x^7 + 4656960x^6 - 16875810x^5 + 30150225x^4}{864x^7(x-1)^7} \\
&\qquad + \frac{-32863320x^3 + 21565845x^2 - 7858620x + 1224720}{864x^7(x-1)^7}\Bigg)\, y
\end{aligned}$$

has a solution y such that $y^2 \in \overline{\mathbf{Q}}(x)$.

Since $x^2(x-1)^2$ is a solution of $L^{\circledS 3}(y) = 0$, we get that $\mathcal{G}(L)$ is an imprimitive subgroup of $SL(3, \mathcal{C})$. Thus $L(y) = 0$ has a solution whose logarithmic derivative is algebraic of degree 3 (cf., Corollary 4.10).

We note that our approach does not determine the group $\mathcal{G}(L)$ in the imprimitive case. But since in this example $L(y) = 0$ is the second symmetric power of the equation

$$\frac{d^2\, y}{dx^2} + \left(\frac{3}{16x^2} + \frac{2}{9(x-1)^2} - \frac{3}{16x(x-1)}\right) y \quad = \quad 0$$

whose differential galois group is $A_4^{SL_2}$ ([0], p. 23), we get by construction that $\mathcal{G}(L) \cong A_4$ (from Table 1 it now also follow that $L(y) = 0$ is irreducible).

For third order differential equations very few examples can be found in the literature. We shall show how one can construct such examples for the primitive groups. Assume, we are given a finite group G and a differential equation of arbitrary order with G as its galois group. Let us also assume we know that G has an irreducible representation of degree n. We shall show how to construct a differential equation of order n having the image of G in $GL(n)$ as its galois group. The idea behind this construction is that such a differential equation will occur as a factor of some other equation that we can construct. This will also allow us to construct a differential equation for a group G from the knowledge of an irreducible polynomial $P(Y) \in \overline{\mathbf{Q}}(x)[Y]$ whose Galois group is G.

The validity of our construction depends on the following result of Burnside which shows that if V is a faithful G-module, then any irreducible G-module is a G-summand of $V^{\otimes n} = \underbrace{V \otimes \ldots \otimes V}_{n\ times}$ for some $n \geq 1$:

THEOREM 5.1. ([0],[0] p. 25) *Let V be a finite dimensional vector space and $G \subset GL(V)$ a finite group. If W is a finite dimensional vector space on which G acts irreducibly, then for some $n \geq 1$, W appears as a direct summand of $V^{\otimes n}$.*

We shall also need the following result which shows that one can construct a linear differential equation whose solution space is isomorphic to the tensor product of the solution spaces of two given linear differential equations.

PROPOSITION 5.2. *Let $L_1(y) = 0$ and $L_2(y) = 0$ be linear differential equations with coefficients in $\overline{\mathbf{Q}}(x)$ of orders n and m respectively. One can effectively construct a linear differential equation $L_1 \otimes L_2(y) = 0$ with coefficients in $\overline{\mathbf{Q}}(x)$ having the following property: if K is a Picard-Vessiot extension of $\overline{\mathbf{Q}}(x)$ such that $L_1(y) = 0$ (resp. $L_2(y) = 0$) has n (resp. m) linearly independent solutions in K, then the solution space of $L_1 \otimes L_2(y) = 0$ is $\mathcal{G}(K/\overline{\mathbf{Q}}(x))$ isomorphic to $V_1 \otimes V_2$ where V_1 (resp. V_2) is the solution space of $L_1(y) = 0$ (resp. $L_2(y) = 0$) in K.*

PROOF. Let $L_2^{(i)}(y) = 0$ be the differential equation whose solution space is $\{y^{(i)} \mid y \in V_2\}$ and let α be a nonsingular point of $L_1 \circledS L_2^{(i)}(y)$ for $i = 0, \ldots, m-1$. Such a point exists and can be effectively found since each $L_2^{(i)}(y) = 0$ and so, each $L_1 \circledS L_2^{(i)}(y)$ can be effectively constructed. For notational convenience, we assume $\alpha = 0$. Let t be any integer greater than or equal to the orders of the $L_1 \circledS L_2^{(i)}(y)$. Note that if z is a solution of $L_1 \circledS L_2^{(i)}(y) = 0$ and $z(0) = z'(0) = \ldots = z^{(t)}(0) = 0$, then $z = 0$.
Let K be a Picard-Vessiot extension of $\overline{\mathbf{Q}}(x)$ and let $w_1, \ldots, w_m$ be a basis of the solution space V_2 of $L_2(y) = 0$ in K. Let $u_i = \sum_{j=1}^{m-1} x^{j \cdot t} w_i^{(j)}$. One can show that the u_i are linearly independent (since $det(w_i^{(j)}) \neq 0$) and that they form a basis of a $\mathcal{G}(K/\overline{\mathbf{Q}}(x))$ module isomorphic to V_2. Let $y_1, \ldots, y_n$ be a basis of V_1 in K and consider the elements $\{y_i u_j\}$ in K. We claim that these are linearly independent over the constants. To see this, let $\sum_{i,j} c_{ij} y_i u_j = 0$ for some constants c_{ij}. This implies that $\sum_{k=0}^{m-1} x^{k \cdot t} \sum_{i,j} c_{ij} y_i w_j^{(k)} = 0$. Note that each $z_k = \sum_{i,j} c_{ij} y_i w_j^{(k)}$ is a solution of $L_1 \circledS L_2^{(k)}(y) = 0$ and so has a zero of order at most $t-1$ at $x = 0$ if $z_k \neq 0$. Therefore each term $x^{k \cdot t} \sum_{i,j} c_{ij} y_i w_j^{(k)}$ is either zero or has a zero of order between $k \cdot t$ and $k \cdot t + t - 1$. Since these terms sum to zero, each of them must equal zero. Therefore, for each $k, 0 \leq k \leq m-1, \sum_{i,j} c_{ij} y_i w_j^{(k)} = 0$. Since $det(w_j^{(k)}) \neq 0$, we have for each $j, 1 \leq j \leq m, \sum_{i=1}^{n} c_{ij} y_i = 0$. Since the y_i are linearly independent, we have all $c_{ij} = 0$. Therefore the elements $\{y_i u_j\}$ are linearly independent over the constants. One now sees that they form the basis of a $\mathcal{G}(K/\overline{\mathbf{Q}}(x))$ module isomorphic to $V_1 \otimes V_2$. Proceeding as in Theorem 3.4, one sees that these form the basis of the solution space of a linear differential equation $L_1 \otimes L_2(y) = 0$ with coefficients in $\overline{\mathbf{Q}}(x)$ and that this operator can be constructed just knowing $L_1(y), L_2(y)$ and the integer t. □

COROLLARY 5.3. *Let $L(y) = 0$ be a linear differential equation of order n with coefficients in $\overline{\mathbf{Q}}(x)$. For any m, one can effectively construct a linear differential equation $L_1^{\otimes m}(y) = 0$ with coefficients in $\overline{\mathbf{Q}}(x)$ having the following property: if K is a Picard-Vessiot extension of $\overline{\mathbf{Q}}(x)$ such that $L_1(y) = 0$ has n linearly independent solutions in*

K, then the solution space of $L_1^{\otimes m}(y) = 0$ is $\mathcal{G}(K/\overline{\mathbf{Q}}(x))$ isomorphic to $V^{\otimes m}$ where V is the solution space of $L(y) = 0$.

Combining Theorem 5.1 and Corollary 5.3, we see that the desired differential equation will eventually occur as a factor of some $L^{\otimes m}(y) = 0$. The orders of these equations grow very quickly. Sometimes one can find the desired differential equation by looking at only symmetric powers:
Example: The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{21}{100}\frac{x^2 - x + 1}{x^2(x-1)^2}\, y \quad = \quad 0$$

is irreducible and $\mathcal{G}(L) \cong A_5^{SL_2}$ (see [0], p. 342). According to Table 1, the third symmetric power of this differential equation

$$L(y) \quad = \quad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{21(x^2 - x + 1)}{25x^2(x-1)^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{21(-2x^3 + 3x^2 - 5x + 2)}{50x^3(x-1)^3}\, y$$

is irreducible and has galois group A_5. In order to prove that $\mathcal{G}(L) \cong A_5$ using factorization of differential operators over over $\overline{\mathbf{Q}}(x)$, it is enough (cf., Theorem 4.6 and 4.7) to show that:

1 $L(y)$ is irreducible.
2 $L^{ⓢ 3}$ has no solution y such that $y^2 \in \overline{\mathbf{Q}}(x)$.
3 $L^{ⓢ 2}$ has order 5 or factors.
4 $L^{ⓢ 3}$ has a factor of order 3.

We note that, since $L^{ⓢ 2}$ is the fourth symmetric power of the above second order equation, $L^{ⓢ 2}$ will be of order 5 in this case (cf., Lemma 3.5). In this case, the fact that $L^{ⓢ 3}$ has no solution y such that $y^2 \in \overline{\mathbf{Q}}(x)$ will follow from a factorization of $L^{ⓢ 3}$, which (if $\mathcal{G}(L) \cong A_5$) will have no factor of order 1. ∎

Assume we are given an irreducible polynomial $P(Y) \in \overline{\mathbf{Q}}(x)$ such that the galois group of $P(Y) = 0$ is G and an irreducible representation of $G \in GL(V)$. Assume $P(Y)$ has degree n. Differentiating $P(Y) = 0$, and successively solving for the derivatives of Y and reducing mod $P(Y)$. we get for $i = 0, \ldots, n$ expressions of the form $Y^{(i)} = a_{i,o} + a_{i,1}Y + \ldots + a_{i,n-1}Y^{n-1}$, with the $a_{i,j} \in \overline{\mathbf{Q}}(x)$. These $n+1$ expressions in the n terms Y^j must be linearly dependent over $\overline{\mathbf{Q}}(x)$, so we can find a linear differential equation $L(y) = y^{(n)} + \ldots + a_0 y = 0$ with $n' \leq n$ and coefficients in $\overline{\mathbf{Q}}(x)$ whose solution space is spanned by the roots of $P(Y) = 0$. Note that $\mathcal{G}(L)$ is G. For some value of m, $L^{\otimes m}(y) = 0$ has a solution space having a subspace $\mathcal{G}(L)$ isomorphic to W. Therefore some factor of $L^{\otimes m}(y) = 0$ will have a solution space $\mathcal{G}(L)$ isomorphic to W.

There are two problems in using the above method. The first is that one needs to determine the representation (or at least its character) of $\mathcal{G}(L)$ on the solution space of $L(y) = 0$ in order to be able to predict for which value of m $L^{\otimes m}(y) = 0$ has a solution space having a subspace isomorphic to W. The second problem is that $L^{\otimes m}(y) = 0$ may have many factors of the same order whose solution spaces are different G-modules. One is faced with the problem of determining which factor gives the desired representation. Nonetheless, the above argument shows that such an operator always exists. We now give an example, where some of these problems can be avoided.

Example: We will show how a third order differential equation with differential galois group G_{168} can be constructed using the polynomial

$$Y^7 - 56Y^6 + 609Y^5 + 1190Y^4 + 6356Y^3 + 4536Y^2 - 6804Y - xY^3(Y+1) - 5832,$$

which is irreducible over $\overline{\mathbf{Q}}(x)$ and has galois group $G_{168} \cong PSL_2(7)$ (se e.g. [0], p. 188).

Using the variable transformation $Z = Y - \frac{1}{7}$ we get the irreducible polynomial

$$Y^7 - 735\,Y^5 - (x+10290)\,Y^4 - (33x-4116)\,Y^3 - (408x-979608)\,Y^2 \\ -(2240x-7020524)\,Y - (4608x-15731352),$$

which we denote $P(Y)$, whose galois group over $\overline{\mathbf{Q}}(x)$ is also G_{168}. We denote $y_1, \ldots, y_7$ the solutions of $P(Y) = 0$ and K the splitting field of $P(Y) = 0$ over $\overline{\mathbf{Q}}(x)$. The functions $y_1, \ldots, y_7$, whose derivatives all belong to K, will satisfy a differential equation $L(y) = 0$ of order at most 7 with coefficients in $\overline{\mathbf{Q}}(x)$. If $L(y) = 0$ is of degree 7, then $\mathcal{G}(L)$ is equivalent to a permutation representation of degree 7 of G_{168}. Such a permutation representation has an invariant subspace generated by $y_1 + \ldots + y_7$. Since $y_1 + \ldots + y_7 = 0$ by construction, the functions $y_1, \ldots, y_7$ must satisfy a differential equation $L(y) = 0$ of degree at most 6, and computation shows that $L(y) = 0$ is in fact of degree 6. This differential equation is not of the form given in Theorem 3.3, but since G_{168} is a perfect group, the corresponding differential galois group will be unimodular. The group G_{168} has 6 irreducible characters: the trivial charcater χ_1, two characters $\chi_{3,1}$ and $\chi_{3,2}$ of degree 3 and the characters χ_6, χ_7, χ_8 of degree 6, 7 and 8. According to these characters, if $L(y) = 0$ is reducible, then it either has an irreducible factor of order 3 or only irreducible factors of order 1. The last case clearly can not happen. Thus, if a factorization of $L(y) = 0$ does not produce an irreducible third order equation (one can show that this is not possible), then $L(y) = 0$ is an irreducible equation of order 6 with $\mathcal{G}(L) \cong G_{168}$. Using the above Corrolary 5.3 we can construct an equation $L^{\otimes m}(y) = 0$ whose solution space is isomorphic to $V^{\otimes m}$ where V is the solution space of $L(y) = 0$. From Theorem 5.1 we get that for some m the character of $\mathcal{G}(L^{\otimes m})$ contains a character of degree 3 of G_{168}. Decomposing powers of χ_6 we get:

$$\begin{aligned} (\chi_6)^2 &= \chi_1 + 2\chi_6 + \chi_7 + 2\chi_8 \\ (\chi_6)^3 &= 2\chi_1 + 3\chi_{3,1} + 3\chi_{3,2} + 10\chi_6 + 8\chi_7 + 10\chi_8 \end{aligned}$$

Thus by factoring $L^{\otimes 3}(y) = 0$ one will get an irreducible third order differential equation with galois group G_{168}. ∎

The above example shows that working with tensor products instead of symmetric products leads to differential equations of very large order containing a large amount of redundancy. This is the reason why we have stated our main results using symmetric powers instead of tensor powers.

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1.16 Liouvillian and Algebraic Solutions of Second and Third Order Linear Differential Equations

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Liouvillian and Algebraic Solutions of Second and Third Order Linear Differential Equations

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In this paper we show that the index of a 1-reducible subgroup of the differential Galois group of an ordinary homogeneous linear differential equation $L(y) = 0$ yields the best possible bound for the degree of the minimal polynomial of an algebraic solution of the Riccati equation associated to $L(y) = 0$. For an irreducible third order equation we show that this degree belongs to $\{3, 6, 9, 21, 36\}$. When the Galois group is a finite primitive group, we reformulate and generalize work of L. Fuchs to show how to compute the minimal polynomial of a solution instead of the minimal polynomial of the logarithmic derivative of a solution. These results lead to an effective algorithm to compute Liouvillian solutions of second and third order linear differential equations.

0. Introduction

The computation of the algebraic solutions of a linear differential equation $L(y) = 0$ over the field of rational functions was a problem of great interest of the end of last century. P. Pepin, H. Schwarz, L. Fuchs, F. Klein, C. Jordan and others worked on this problem and gave a solution for second order equations (cf. (Baldassarri and Dwork (1979)), the intoduction of Boulanger (1898), and Gray (1986)). Many of the earliest contributions to the representation theory of finite groups have been made in connection with differential equation (e.g. Jordan's Theorem) and it was the starting point for the classification of the finite primitive groups. In this paper we will focus on the ideas of Fuchs. In Fuchs (1878), Fuchs showed how the (then new) tools of invariant theory could be used to construct, in many cases, the minimal polynomial of an algebraic solution of a second order linear differential equation.

The more general question of finding the liouvillian solutions of a linear differential equation, in which case the differential Galois group can be infinite, leads to the theory of linear algebraic groups. But for a primitive unimodular Galois group, all liouvillian solutions are algebraic (cf. Ulmer (1992)) and in this case the approach of Fuchs can

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be used. This leads to an effective method for computing the minimal polynomial of a solution in this case. This computation is much more *linear* than the computation of the minimal polynomial of the logarithmic derivative of a solution which is performed in the algorithm proposed by Kovacic for second order equations[†] and in the general algorithm proposed by the first author (cf. Kovacic (1986) and Singer (1981)). In the direct computation of a minimal polynomial of a solution, the knowledge of the finitely many possibilities for the differential Galois group can be used not only to bound the degree of the minimal polynomial, but also to compute the coefficients of this polynomial. In this paper we propose the following method for the computation of liouvillian solutions:

i) **Case 1:** If the differential Galois group is a reducible linear group, then a factorisation of the differential equation is used to reduce the problem to a linear differential equation of lower order. In this paper we show how this can be done for third order equations.
ii) **Case 2:** If the differential Galois group is an imprimitive linear group, then the algorithm proposed in Singer (1981) by the first author is used. For second (resp. third) order equations, this leads to the computation of a solution whose logarithmic derivative is algebraic of degree 2 (resp. 3), in which case this general algorithm is still practicable.
iii) **Case 3:** If the differential Galois group is a primitive finite linear group, then we show how the method of Fuchs can be extended to compute the minimal polynomial of an algebraic solution in a very efficient way.

In our approach, we assume that, over the differential field k of coefficients of $L(y) = 0$, algorithms computing a factorisation, a solution whose logarithmic derivative is in k (for case 2) and a solution which is in k (for case 3) of a of linear differential equation exist (see Section 1 for a discussion and references).

In this paper we discuss explicitely second and third order differential equation, but the extension of the method of Fuchs for case 3 to higher order equations is now straightfoward.

The paper is organized as follow: in the first section we derive some results from differential Galois theory. In the second section we show how, using factorisation, case 1 of a reducible third order linear differential equation can be be reduced to the problem of finding liouvillian solutions of a second order equation. In the next section we derive exact possible algebraic degrees of the logarithmic derivative of a second or third order equation. We then briefly discuss the algorithm given by the first author which is used in case 2, where the Galois group is an imprimitive linear group. In the last and main section we focus on differential equations with primitive differential Galois groups. We first compute a bound for the algebraic degree of a solution and then use the semi-invariants of the Galois group to compute the coefficients of the minimal polynomial of an algebraic solution. We also apply the method to a second and a third order linear differential equation with primitive Galois group and compute the minimal polynomial of a solution in both cases.

[†] In fact, an algorithm (with some mistakes) to find the minimal polynomial of the logarithmic derivatives of a solution of a second order linear differential equation was first given by Pepin one hundred years before Kovacic (1986) and Singer (1981) (cf. Pépin (1881)). Furthermore, in Pépin (1881), Pépin is able to use his method to verify the Schwarz list of hypergeometric equations with algebraic solutions (cf. Boulanger (1898))

1. Differential Galois Theory

In this section we first briefly review some facts about differential algebra and the existing algorithms for computing liouvillian solutions of linear differential equations. For a more complete exposition we refer to Kaplansky (1957), Kovacic (1986), Singer (1981) or Singer (1990). In the following we will use the same notation as in Ulmer (1992) or Singer and Ulmer (1992).

A *differential field* (k, δ) is a field k together with a derivation δ on k. A *differential field extension* of (k, δ) is a differential field (K, Δ) such that K is a field extension of k and Δ is an extension of the derivation δ to a derivation on K. In this paper we always assume that k is a field of characteristic 0 and that the field $\mathcal{C} = \ker_k(\delta)$ of constants of δ in k is algebraically closed (e.g. $(\overline{\mathbf{Q}}(x), \frac{d}{dx})$).
We also write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \ldots$ for $\delta(y), \delta^2(y), \ldots$. Unless otherwise stated, a differential equation $L(y) = 0$ over k always means an ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k).$$

In the following we will have to compute rational solutions z of $L(y) = 0$ (i.e. $z \in k$), and solutions of $L(y) = 0$ whose logarithmic derivative is rational (i.e. $z'/z \in k$). Algorithms computing such solutions for various coefficient fields are described in Bronstein. (1992), Liouville (1833), Schlesinger (1895) (volume II, §177) and Singer (1991) (cf. Proposition (2.3)). In the following we always assume that k is a differential field over which such solutions can be computed (e.g. $\mathbf{C}(x), \frac{d}{dx})$). The computation of a solution whose logarithmic derivative is rational is usually much more difficult than the computation of a rational solution.

Definition 1.1. *A differential field extension (K, Δ) of (k, δ) is a liouvillian extension if there is a tower of fields*

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K,$$

where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

i) η_i is algebraic over K_i, or
ii) $\delta(\eta_i) \in K_i$ (extension by an integral), or
iii) $\delta(\eta_i)/\eta_i \in K_i$ (extension by the exponential of an integral).

A function contained in a liouvillian extension of k is called a liouvillian function over k.

In Kovacic (1986) an algorithm is given to find a basis of the liouvillian solutions of a second order linear differential equation with coefficients in $k_0(x)$, where k_0 is a finite algebraic extension of $\mathbf{Q}$. In Singer (1981) the first author gives a procedure to find a basis of the liouvillian solutions of a linear differential equation $L(y) = 0$ of arbitrary degree n with coefficients belonging to a finite algebraic extension of $\mathbf{Q}(x)$.

We refer to Kaplansky (1957), Kovacic (1986), Singer (1981), Singer (1990), Ulmer (1992) or Singer and Ulmer (1992) for the definition of a Picard Vessiot extension (PVE) K associated with $L(y) = 0$, which can be viewed as a splitting field of $L(y) = 0$, and of the differential Galois group $\mathcal{G}(L)$ of $L(y) = 0$, which consists of the automorphisms of a PVE K of k that commute with δ.

If we choose a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of the equation $L(y) = 0$, then for each $\sigma \in \mathcal{G}(L)$ we get $\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j$, where $c_{ij} \in \mathcal{C}$. This gives a faithful representation of $\mathcal{G}(L)$ as a subgroup of $GL(n, \mathcal{C})$. Different choices of bases $\{y_1, y_2, \ldots, y_n\}$ give equivalent representations. This equivalence class of representations is fundamental to our approach. In the sequel we always consider this representation as *the* representation of $\mathcal{G}(L)$.

Many properties of the equation $L(y) = 0$ and of its solutions are related to the structure of the group $\mathcal{G}(L)$:

THEOREM 1.1. (see e.g. Kolchin (1948)) *A differential equation $L(y) = 0$ with coefficients in k has*

i) only solutions which are algebraic over k if and only if $\mathcal{G}(L)$ is a finite group,
ii) only liouvillian solutions over k if and only if the component of the identity $\mathcal{G}(L)^\circ$ of $\mathcal{G}(L)$ in the Zariski topology is solvable. In this case $L(y) = 0$ has a solution whose logarithmic derivative is algebraic over k.

The following theorem will enable us to always assume that the differential Galois group $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ of a differential equation $L(y) = 0$ of degree n is unimodular.

THEOREM 1.2. (Kaplansky (1957), p. 41) *The differential Galois group of a differential equation of the form*

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k) \tag{1.1}$$

is a unimodular group (i.e. $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$) if and only if $\exists W \in k$, such that $W'/W = a_{n-1}$.

Using the variable transformation $y = z \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$ it is always possible to transform a given differential equation $L(y)$ into an equation $L_{SL}(y)$ of the form:

$$L(y) = y^{(n)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in k). \tag{1.2}$$

For $L(y) = y''' + a_2 y'' + a_1 y' + a_0 y$ we get:

$$L_{SL}(y) = y''' + \left(a_1 - \frac{{a_2}^2}{3} - a_2'\right) y' + \left(a_0 - \frac{a_1 a_2}{3} - \frac{a_2''}{3} + \frac{2{a_2}^3}{27}\right) y.$$

LEMMA 1.3. *Let $k \subset K$ be differential fields of characteristic zero with the same field of constants and $y \in K$ with $y'/y \in k$. If y is algebraic over k, then the minimal polynomial of y over k is of the form $y^n - a = 0$ for some $a \in k$,*

PROOF. Let $y'/y = u \in k$ and

$$y^n + a_{n-1} y^{n-1} + \ldots + a_0 = 0$$

be the minimal polynomial of y over k. Differentiating we have:

$$nuy^n + (a_{n-1}' + (n-1) u a_{n-1}) y^{n-1} + \ldots + a_0' = 0.$$

Comparing coefficients we have

$$nua_i \quad = \quad a_i' + iua_i \qquad (i = 1, \ldots, n-1).$$

If for some i, $0 < i < n$, $a_i \neq 0$, we have that

$$(n-i)u \quad = \quad \frac{a_i'}{a_i},$$

we then have

$$\frac{(y^{n-i}a_i^{-1})'}{y^{n-i}a_i^{-1}} \quad = \quad 0.$$

Therefore $y^{n-i}a_i^{-1}$ is a constant (in k). This further implies that y would satisfy a polynomial of degree less than n, a contradiction. Therefore, for each i, $0 < i < n$, we have $a_i = 0$. □

Corollary 1.4. *Let $k \subset K$ be as above, where the field of constants is algebraically closed.*

i) If for $y \in K$ algebraic over k we have y'/y algebraic over k of degree m, then the minimal polynomial $P(\mathbf{Y}) = 0$ of y over k is of the form

$$\mathbf{Y}^{i \cdot m} + a_{m-1}\mathbf{Y}^{i\cdot(m-1)} + \ldots + a_1\mathbf{Y}^i + a_0 \quad (a_j \in k,\, m = [k(y'/y)/k])$$

ii) The extension $k(y)/k(y'/y)$ is a normal extension. If H is the maximal subgroup of the Galois group G of K/k with the property that $\forall h \in H, h(y'/y) = y'/y$, then there is a normal subgroup N of H such that H/N is a cyclic group of order i.

iii) If $\mathcal{T}$ is a set of left coset representatives of H in G, then $P(\mathbf{Y}) = 0$ can be written in the following way:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y}^i - (\sigma(y))^i\right) \tag{1.3}$$

Proof. i) By the previous theorem, the minimal polynomial of y over $k(y'/y)$ is of the form $y^i - a = 0$ for some $a \in k(y'/y)$. Let $m = [k(y'/y) : k]$, then y is a solution of a polynomial of the form:

$$a_m\mathbf{Y}^{i \cdot m} + a_{m-1}\mathbf{Y}^{i\cdot(m-1)} + \ldots + a_1\mathbf{Y}^i + a_0 \quad (a_j \in k)$$

Since $[k(y) : k] = [k(y) : k(y'/y)] \cdot [k(y'/y) : k] = i \cdot m$, y cannot be a solution of a polynomial of lower degree over k. Thus the above polynomial is the minimal polynomial of y over k.

ii) To the tower of fields $k \subseteq k(y'/y) \subseteq k(y) \subseteq K$ corresponds the tower of groups $G \supseteq H \supseteq N \supseteq \{id\}$. Since k contains all the i-th roots of unity, the polynomial $y^i - a = 0$ splits over $k(y'/y)$ and thus $k(y)$ is a normal extension of $k(y'/y)$. Thus N is a normal subgroup of H and the Galois group of $k(y)/k(y'/y)$ is isomorphic to H/N and is a cyclic group.

iii) Since y^i is left fixed by the elements of H, we can use a set of left coset representatives $\mathcal{T}$ of H in G to write the minimal polynomial of y^i is the following way:

$$\prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y} - \sigma(y^i)\right).$$

This gives the following polynomial for y:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y}^i - (\sigma(y))^i\right).$$

Comparing degrees as above, we get that $P(\mathbf{Y})$ is the minimal polynomial of y.
□

In the following we will need some differential equations associated to $L(y) = 0$:

THEOREM 1.5. (cf. Singer (1980)) *Let $L_1(y) = 0$ and $L_2(y) = 0$ be linear differential equations of degrees respectively n_1 and n_2 and fundamental systems respectively $S_1 = \{u_1, \cdots, u_{n_1}\}$ und $S_2 = \{v_1, \cdots, v_{n_2}\}$. Then one can construct a differential equation:*

i) $L(y) = L_1(y) \circledS L_2(y) = 0$ of degree $n_3 \le n_1 n_2$, whose solution space is spanned by $S = \{u_1 v_1, \cdots, u_{n_1} v_1, \cdots, u_{n_1} v_{n_2}\}$.
ii) $L_\delta(y) = 0$ of degree $n \le n_1$, whose solution space is spanned by the set $S_\delta = \{\delta(u_1), \cdots, \delta(u_{n_1})\}$.

In Singer (1980) and Singer and Ulmer (1992) algorithms to construct the above equations are given. The equation

$$L^{\circledS m}(y) = \overbrace{L(y) \circledS \cdots \circledS L(y)}^{m} = 0$$

is called the symmetric power of order m of $L(y) = 0$ and is of order at most $\binom{n+m-1}{n-1}$.

Let $L(y)$ have order n and let $L(y) = 0$ have solution space V in some Picard-Vessiot extension K of k. There is a natural $\mathcal{G}(L)$ morphism Φ_m of $\mathcal{G}(L)$ modules from the m^{th} symmetric power $\mathcal{S}^m(V)$ (c.f., Lang (1984), p. 586) into K given by sending $z_1 \circledS \ldots \circledS z_m$ to $z_1 \cdot \ldots \cdot z_m$. The image of this map is the solution space of $L^{\circledS m}(y) = 0$. If all representations of $\mathcal{G}(L)$ are completely reducible (e.g. if $\mathcal{G}(L)$ is finite), then the solution space of $L^{\circledS m}(y) = 0$ is $\mathcal{G}(L)$-isomorphic to a direct summand of $\mathcal{S}^m(V)$ (cf. Singer and Ulmer (1992), Lemma 3.5). If I is an (semi-) invariant of degree m of *the* representation of $\mathcal{G}(L)$, then by the *computation* of the (semi-) invariant I we will always mean the computation of the image $\Phi_m(I)$ up to a constant multiple. For an invariant I of degree m, $\Phi_m(I)$ is a rational solution of $L^{\circledS m}(y) = 0$. If I is a semi-invariant, then there exists a one dimensional character χ of the group $\mathcal{G}(L)$ such that

$$\forall \sigma \in \mathcal{G}(L),\ \sigma(I) = \chi(\sigma) \cdot I.$$

In particular, if j is the smallest integer j such that χ^j is the trivial character, then $\Phi_m(I^j)$ is a rational solution of $L^{\circledS (m \cdot j)}(y) = 0$. A bound for j follows from the character table of the group $\mathcal{G}(L)$. The one dimensional characters χ corresponding to the semi-invariants and the number of linear independent semi-invariants corresponding to a given character of degree one can be found by decomposing the character of the representation of $\mathcal{G}(L)$ on the m^{th} symmetric power $\mathcal{S}^m(V)$ (cf. Singer and Ulmer (1992), section 2.3). For later reference, we summarize those simple facts:

LEMMA 1.6. *Let $L(y) = 0$ be a linear differential equation with coefficients in k whose differential Galois group $\mathcal{G}(L) \subset GL(n, \mathcal{C})$ is finite. If I is a semi-invariant of degree m of $\mathcal{G}(L)$ and $\Phi_m(I) \neq 0$, then $\Phi_m(I)$ is a non trivial rational solution of $L^{\circledS (m \cdot i)}(y) = 0$,* *where i divides the order of a one dimensional character χ of $\mathcal{G}(L)$. If I is an invariant, then $i = 1$. The possible characters χ corresponding to the semi-invariants can be found by decomposing the character of the representation of $\mathcal{G}(L)$ on the m^{th} symmetric power $\mathcal{S}^m(V)$.*

2. Case 1: a reducible differential Galois group

The equation $L(y)$ factors as a linear differential operator, if and only if $\mathcal{G}(L)$ is a reducible linear group (see e.g. Kolchin (1948)). The factorisation of a differential operator is not unique (see e.g. Singer and Ulmer (1992), section 3.2.1), but an algorithm for computing a factorisation of a differential operator with coefficients in $\overline{\mathbf{Q}}(x)$ is well known (see e.g. Grigor'ev (1990) and Schlesinger (1895)). For third order equations a factorisation can be found by computing the rational solutions of the Riccati equation of both $L(y) = 0$ and of its adjoint. In this section we show how, for a third order differential equation, one can use only one factorisation of $L(y)$ in order to find all liouvillian solutions of $L(y) = 0$.

We will use the well known reduction method of d'Alembert, which allows one to reduce the order of a linear differential equation $L(y) = \sum_{i=0}^{n} a_i y^{(i)} = 0$ using a non trivial solution y_1. The problem of finding further solutions of $L(y) = 0$ reduces to finding the solutions of

$$\tilde{L}(y) \quad = \quad \sum_{i=0}^{n} a_i \left(y_1 \int y \right)^{(i)} = 0,$$

since from a fundamental set of solutions $y_1^*, \cdots, y_{n-1}^*$ of $\tilde{L}(y) = 0$ we get a fundamental system of solutions

$$y_1, y_1 \cdot \int (y_1^*), \cdots, y_1 \cdot \int (y_{n-1}^*)$$

of $L(y) = 0$.

If a second order equation is reducible, then after computing a solution whose logarithmic derivative is rational, one gets a second linearly independent liouvillian solution using the above. Thus, for second order equations, either none or all solutions are liouvillian. This is no longer true for higher order reducible equations:

Lemma 2.1. *Let $L(y) = y''' + Ay' + By = 0$ be a reducible third order differential equation with $A, B \in \overline{\mathbf{Q}}(x)$.*

i) If $L(y) = 0$ has a solution z such that $z'/z = u \in \overline{\mathbf{Q}}(x)$, then the reduction method of d'Alembert gives the equation

$$\tilde{L}(y) \quad = \quad y'' + 3u'y' + \left(3u'' + 3(u')^2 + A\right) y.$$

If $\tilde{L}(y) = 0$ has no non zero liouvillian solutions, then z is, up to a constant multiple, the only liouvillian solution of $L(y) = 0$. If $\tilde{L}(y) = 0$ has a non zero liouvillian solution, then applying again the method of d'Alembert gives 3 linear independent liouvillian solutions.

ii) If $L(y) = 0$ has no solution z such that $z'/z = u \in \overline{\mathbf{Q}}(x)$, then any factorisation algorithm will give a factorisation $L(y) = L_1(L_2(y))$, where $L_2(y)$ is of order 2.

> *Either* $L_2(y) = 0$ *has only liouvillian solutions, in which case the procedure of d'Alembert will produce a third liouvillian solution of* $L(y) = 0$ *which is not a solution of* $L_2(y) = 0$*, or* $L(y) = 0$ *has no liouvillian solution.*

Furthermore, one can determine algorithmically which of these cases hold.

PROOF. If $L(y) = 0$ has a solution z such that $z'/z = u \in \overline{\mathbf{Q}}(x)$, then the reduction method of d'Alembert always gives an equation

$$\tilde{L}(y) \quad = \quad y'' + 3u'y' + \big(3u'' + 3(u')^2 + A\big)\, y.$$

whose coefficients belong to $\overline{\mathbf{Q}}(x)$. The equation $\tilde{L}(y) = 0$ is a second order equation which can be solved using the Kovacic algorithm or the algorithm presented in this paper. Since a second order linear differential equation has either only liouvillian solution or no liouvillian solutions, we get the result.

If a third order differential equation $L(y) = 0$ has no solution z such that $z'/z = u \in \overline{\mathbf{Q}}(x)$, then any factorisation of $L(y)$ will be of the form $L_1(L_2(y))$, where $L_2(y)$ is a second order linear differential equation. We now apply the Kovacic algorithm to $L_2(y) = 0$. If $L_2(y) = 0$ has a liouvillian solution, then $L_2(y) = 0$ and thus $L(y) = 0$ will have two linear independent liouvillian solutions. Using the reduction method of d'Alembert we will get a third solution of $L(y) = 0$ which is not a solution of $L_2(y) = 0$. If V is the subspace of liouvillian solutions of $L(y) = 0$, then $L_2(y)$ maps V into the solution space of $L_1(y) = 0$. If $L_2(y) = 0$ has no liouvillian solutions, then $L_2(y)$ cannot vanish on V. So V has dimension at most 1. Since V is a $\mathcal{G}(L)$ invariant subspace of the solution space of $L(y) = 0$, there is a non zero $z \in V$ so that $z'/z \in \overline{\mathbf{Q}}(x)$. Since we assume that there are no such solutions, V has dimension zero. □

3. Optimal bounds for the logarithmic derivative and case 2: an imprimitive differential Galois group

It is well known, that for a differential equation $L(y) = 0$, one can construct a non linear differential equation $R(u) = 0$, called the *Riccati equation* associated to $L(y)$, such that the logarithmic derivative $u = z'/z$ of any solution of $L(y) = 0$ is a solution of $R(u) = 0$. The Riccati equation associated to $L(y) = y''' + a_2y'' + a_1y' + a_0y$ is $R(u) = u'' + 3uu' + a_2u' + u^3 + a_2u^2 + a_1u + a_0$.

The known algorithms computing liouvillian solutions of a linear differential equation $L(y) = 0$ use the fact that if $L(y) = 0$ has a liouvillian solution, then $L(y) = 0$ has a solution z such that z'/z is algebraic of bounded degree. In Ulmer (1992) a sharp bound for the degree of the minimal polynomial of an algebraic solution of $R(u) = 0$ is derived.

In this section we will first derive the exact degrees of the minimal polynomial $P(u)$ of an algebraic solution of $R(u) = 0$ for a third order differential equation and then present the general method given in Singer (1981) to compute the coefficients of $P(u)$. If $L(y) = 0$ has a liouvillian solution, this, of course, allows us to find a liouvillian solution of the form $y = e^{\int u}$. When $\mathcal{G}(L)$ is an imprimitive linear group, we show that the minimal degree of $P(u)$ is 3 and we offer no alternative to the general method of Singer (1981). On the other hand, when $\mathcal{G}(L)$ is a finite primitive linear group (in which case the minimal degree of $P(u)$ is much larger), we shall show in the next section how to determine directly the minimal polynomial of a solution of $L(y) = 0$. Nonetheless, we shall need the information found in this section.

3.1. The degree of an algebraic logarithmic derivative of a solution

In this section we assume the reader familiar with the notion of a reducible, imprimitive or primitive linear group and with the notion of a projective representation (see e.g. Huppert (1983), (Curtis and Reiner, I. (1962)), Issacs (1976) or Ulmer (1992)).

Since a normal abelian subgroup of a primitive group G is contained in the center $Z(G)$ of G, we get from Jordan's Theorem (see Jordan (1878) and (Curtis and Reiner (1962))) that for a finite primitive group G, there are only finitely many possible groups $G/Z(G)$. If a group $\widetilde{G} \subseteq PGL(n, \mathbf{C})$ is the image (under the canonical map) of a primitive subgroup of $GL(n, \mathbf{C})$, we call $\widetilde{G}$ a primitive subgroup of $PGL(n, \mathbf{C})$.

Definition 3.1. A group $G \subseteq GL(n, \mathcal{C})$ whose elements have a common eigenvector is called 1-*reducible.*

In Ulmer (1992) Theorem 3.4 it is proven that, if an irreducible differential equation $L(y) = 0$ has a liouvillian solution, then $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ has a 1-reducible subgroup H of finite index and that there is a solution z of $L(y) = 0$ such that the algebraic degree of $u = z'/z$ over k is $\leq [\mathcal{G}(L) : H]$. In fact the minimal index of a 1-reducible subgroup of $\mathcal{G}(L)$ is the best possible bound for the degree of an algebraic solution of the Riccati equation of $L(y) = 0$:

Lemma 3.1. *If a differential equation $L(y) = 0$ of degree n has a solution z such that $u = z'/z$ is algebraic of degree m, then $\mathcal{G}(L) \subseteq GL(n, \mathcal{C})$ has a* 1*-reducible subgroup H of index m.*

Proof. Let H be the subgroup of $\mathcal{G}(L)$ which keeps $u = z'/z$ fixed. For any $\sigma \in G$ we have

$$\begin{aligned} \left(\frac{\sigma(z)}{z}\right)' &= \frac{\sigma(z')z - z'\sigma(z)}{z^2} \\ &= \sigma(z)\frac{\frac{\sigma(z')}{\sigma(z)} - \frac{z'}{z}}{z} \\ &= \frac{\sigma(z)}{z}\left(\sigma\left(\frac{z'}{z}\right) - \frac{z'}{z}\right) \\ &= 0. \end{aligned}$$

Thus $\sigma(z)/z = c_\sigma \in \mathcal{C}$ or $\sigma(z) = c_\sigma z$. This shows that z is a common eigenvector of H and that H must be a 1-reducible subgroup of $\mathcal{G}(L)$. Since H is the stabiliser of u, the orbit of u under the action of $\mathcal{G}(L)$ is of length $[\mathcal{G}(L) : H]$. Thus $[k(u) : k] = [\mathcal{G}(L) : H]$. □

A Schur representation group (Γ, π) of a group G (see e.g. Huppert (1983) p. 630) is a central extension of G having the universal property that, if a projective representation $\mathcal{P}$ of G of degree n is given, there exists a representation $\mathcal{D}$ of Γ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mathcal{D}} & GL(n,\mathbf{C}) \\ \downarrow \pi & & \downarrow \mathcal{P}_n \\ G & \xrightarrow{\mathcal{P}} & PGL(n,\mathbf{C}) \end{array}$$

where $\mathcal{P}_n : GL(n,\mathbf{C}) \longmapsto PGL(n,\mathbf{C}) = GL(n,\mathbf{C})/Z(GL(n,\mathbf{C}))$ denotes the canonical homomorphism.

A Schur represention group is usually not uniquely defined, but for our purposes, the knowledge of only one Schur representation group (which by a theorem of I. Schur exists for any finite group G) is necessary (see Ulmer (1992)). There is a routine to construct **a** Schur representation group of a finite group in the group theory system CAYLEY, see Cannon (1984). We make the following definition:

DEFINITION 3.2. We denote by $\mathcal{F}$ a function whose value $\mathcal{F}(n)$ gives the minimal value, such that for each finite primitive subgroup $G \subseteq PGL(n,\mathcal{C})$, any primitive representation of degree n of a Schur representation group of G has a 1-reducible subgroup of index $\leq \mathcal{F}(n)$.

In Ulmer (1992) it is shown that the above function $\mathcal{F}(n)$ is well defined. The following result of Ulmer (1992) shows that the bound in the imprimitive case is always small compared to the bound in the primitive case:

THEOREM 3.2. *If an irreducible differential equation $L(y) = 0$ of degree n with coefficients in a differential field k, whose field of constants is algebraic closed, has a liouvillian solution over k, then $L(y) = 0$ has a solution z such that*

i) if $\mathcal{G}(L) \subseteq GL(n,\mathcal{C})$ is an imprimitive group, then $u = z'/z$ is algebraic over k of degree at most $\max\limits_{d|n,d>1} \{d! \cdot \mathcal{F}(n/d)\}$.
ii) if $\mathcal{G}(L) \subseteq GL(n,\mathcal{C})$ is a primitive group, then $u = z'/z$ is algebraic over k of degree at most $\mathcal{F}(n)$.

We note that if n is prime, one can get a better bound in the imprimitive case (Ulmer (1992), Lemma 4.2). In this case $u = z'/z$ can be chosen to be algebraic of degree n.

In order to compute the bound $\mathcal{F}(n)$ one needs a list of the finite primitive subgroups of $PGL(n,\mathbf{C})$. For $n = 3$ such a list is given for example in Blichfeld (1917):

(i) A_6, the alternating permutation group of 6 letters.
(ii) G_{168}, the simple group of order 168.
(iii) A_5, the alternating permutation group of 5 letters.
(iv) H_{216}, the Hessian group of order 216, which is isomorphic to the permutation group of 9 letters generated by the permutations $(4,5,6)(7,9,8)$ and $(1,2,4)(5,6,8)(3,9,7)$.

(v) H_{72}, the normal subgroup of order 72 of the group H_{216}.
(vi) F_{36}, a normal subgroup of order 36 of the group H_{72} (there are 3 such groups, which are all isomorphic).

From such a (finite) list of the finite primitive subgroups of $PGL(n, \mathbf{C})$ the bound $\mathcal{F}(n)$ can always be computed using the characters of the subgroups of corresponding Schur representation groups. Let G be a finite primitive subgroup of $PGL(n, \mathbf{C})$, Γ_G a Schur representation group of G and $\rho(\Gamma_G)$ an irreducible representation of degree n of Γ_G with character ζ. The restriction of $\rho(\Gamma_G)$ to a subgroup H is 1-reducible if and only if there is a one dimensional character ψ of H such that the scalar product $(\psi, \zeta|H) \neq 0$, where $\zeta|H$ denotes the restriction of ζ to H (cf. Curtis and Reiner (1962), §38). Considering the finitely many primitive groups G and the finitely many subgroups H of Γ_G will give $\mathcal{F}(n)$.

For $n = 3$ the computation is simplified by the fact that a 1-reducible subgroup $\rho(H)$ of a irreducible finite group $\rho(\Gamma_G) \subseteq GL(3, \mathcal{C})$ is either abelian or $\mathcal{C}^3$ is a direct sum of an irreducible one dimensional representation of H and an irreducible two dimensional represention of H. Let ζ be the character of $\rho(\Gamma_G)$. If $\rho(H)$ is 1-reducible, then either $\zeta|H$ is the sum of 3 characters of degree 1, or $\zeta|H$ is the sum of two characters χ_1 and χ_2 of H, where $\chi_1(1) = 1$ and $\chi_2(1) = 2$. We note also that, if a subgroup H of Γ has no irreducible character of degree 3, then $\rho(H)$ must be 1-reducible.

We now look at a Schur representation group Γ of the above groups, constructed using Cayley (for the non simple groups these groups are not all isomorphic, so we will only give the generators and relations of the groups which have been used). Using character tables (also computed in Cayley) we performed the following case-by-case study (In the appendix the character tables of the subgroups of index ≤ 6 of the Schur representation group of A_5 are given):

(i) $\Gamma/Z(\Gamma) \cong A_6$. All subgroups of index 36 have no irreducible character of degree 3 and thus are 1-reducible. In order to see that 36 is the smallest index of a 1-reducible subgroup of $\rho(\Gamma)$ we need to look at all subgroups whose index is less than 36 (We note that the kernel of an irreducible representation of degree 3 of Γ is always of order 2 and thus elements of order 3 have trace $\neq 3$):

(a) Any subgroup H of index 30 contains an element g of order 4. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$. Thus $\zeta|H$ can not be the sum of 3 character of degree 1 and since $\chi_1(g) + \chi_2(g) = 1$, we have $\chi_1 + \chi_2 \neq \zeta|H$. Thus $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(b) Any subgroup H of index 20 contains an element g of order 3 with the property that for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 2$. But for any irreducible character ζ of Γ, we have $\zeta(g) \neq 3$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(c) Any subgroup H of index 18 contains an element g of order 4. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(d) Any subgroup H of index 15 contains an element g of order 4. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$ or $\chi_2(g) = 2$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(e) Any subgroup H of index 10 has no irreducible character of degree 2. For an element of order 3 of H and any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$ and $\zeta|H$ can not be the sum of 3 character of degree 1. Thus $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(f) Any subgroup H of index 6 contains an element g of order 4. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(ii) $\Gamma/Z(\Gamma) \cong G_{168}$. All subgroups of index 21 of G have no irreducible character of degree 3, and thus are 1-reducible. In order to see that 21 is the smallest index of a 1-reducible subgroup of Γ we need to look at all subgroups whose index is less than 21 (We note that the kernel of an irreducible representation of degree 3 of Γ is always of order 2 and thus elements of order 7 have trace $\neq 3$):

(a) The subgroups of index 16 are all conjugate and have no representation of degree 2. For an element g of order 7 of H and any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$ and $\zeta|H$ can not be the sum of 3 character of degree 1. Thus no subgroup of index 16 can be a 1-reducible subgroup of $\rho(\Gamma)$.

(b) Any subgroup H of index 14 contains an element g of order 4. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(c) The subgroups of index 8 are all conjugate and have no irreducible character of degree 2. For an element of order 7 of H and any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$ and $\zeta|H$ can not be the sum of 3 character of degree 1. Thus no subgroup of index 8 can be a 1-reducible subgroup of $\rho(\Gamma)$.

(d) Any subgroup H of index 7 contains an element g of order 4 whose conjugacy class contains 6 elements. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$ or $\chi_2(g) = 2$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(iii) $\Gamma/Z(\Gamma) \cong A_5$. All subgroups of index 6 of Γ have no irreducible character of degree 3, and thus are 1-reducible. In order to see that 6 is the smallest index of a 1-reducible subgroup of Γ we need to look at all subgroups whose index is less than 6. These non abelian groups are all conjugate and of index 5. A subgroup H of index 5 contains an element g of order 4. If ζ is an irreducible character of degree 3 of

Γ, then $\zeta(g) = -1$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 0$. Since $\zeta|H$ can not be the sum of 3 character of degree 1 and $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(iv) From the presentation $\{a, b \mid a^3 = b^3 = (ab)^4 = [(aba)^2, a] = id\}$ of H_{216}, CAYLEY computes the following Schur representation group

$$\{a, b, c, d \quad | \quad a^3c = b^3 = (ab)^4c = a^{-1}b^{-1}a^{-1}(a^{-1}b^{-1})^2(aba)^2bd^{-1}$$
$$= [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = id\}.$$

We will only considere the faithfull representation of degree 3, since the non faithfull represenation of degree 3 of Γ has non central elements in its kernel. Any subgroup H of index 9 has an element g of order 4. If ζ is a faithfull irreducible character of degree 3 of Γ, then $\zeta(g) = 1$. But for any irreducible character χ' of degree 3 of H we must have $\chi'(g) = -1$. Thus $\chi' \neq \zeta|H$ and H must be a 1-reducible subgroup of Γ. In order to see that 9 is the smallest index of a 1-reducible subgroup of Γ we need to look at all subgroups whose index is less than 9. Since those groups are all non abelian and the irreducible representation $\rho(\Gamma)$ of degree 3 is assumed faithfull, $\zeta|H$ can not be the sum of 3 character of degree 1.

(a) The subgroups of index 8 are all conjugate and have no irreducible character of degree 2. Thus no subgroup of index 8 can be a 1-reducible subgroup of $\rho(\Gamma)$.
(b) The subgroups of index 6 are all conjugate and have no irreducible character of degree 2. Thus no subgroup of index 6 can be a 1-reducible subgroup of $\rho(\Gamma)$.
(c) Any subgroup H of index 4 contains an element g of order 3 whose conjugacy class in H contains 1 element. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) \neq 3$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 2$. Since $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.
(d) Any subgroup H of index 3 contains an element g of order 3 whose conjugacy class in H contains 24 elements. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) \neq 3$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 2$. Since $\chi_1 + \chi_2 \neq \zeta|H$, $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(v) From the presentation $\{a, b, c \mid a^2b^{-2} = aba^{-1}b = c^4 = acb^{-1}c^{-2} = id\}$ of H_{72}, CAYLEY computes the following Schur representation group Γ:

$$\{a, b, c, d \quad | \quad a^2b^{-2} = aba^{-1}b = c^4d = acb^{-1}c^{-2}$$
$$= [a, d] = [b, d] = [c, d] = id\}.$$

All subgroups of index 9 of Γ have no irreducible character of degree 3, and thus are 1-reducible. In order to see that 9 is the smallest index of a 1-reducible subgroup of Γ we need to look at all subgroups whose index is less than 9. Since those groups are all non abelian and all irreducible representations $\rho(\Gamma)$ of degree 3 are faithfull, $\zeta|H$ can not be the sum of 3 character of degree 1.

(a) The subgroups of index 8 are all conjugate and have no irreducible character of degree 2. Thus no subgroup of index 8 can be a 1-reducible subgroup of $\rho(\Gamma)$.

(b) Any subgroup H of index 4 contains an element g of order 3 whose conjugacy class contains one element. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) \neq 3$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 2$. Thus $\chi_1 + \chi_2 \neq \chi$ and $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

(c) Any subgroup of index 2 has no irreducible character of degree 2. Thus no subgroup of index 2 can be a 1-reducible subgroup of $\rho(\Gamma)$.

(vi) From the presentation $\{a, b \mid a^4 = (ab^{-1})^2 = b^4 = (ab)^3 = id\}$ of F_{36}, CAYLEY computes the following Schur representation group Γ:

$$\{a, b, c, d \mid a^4c^{-1}d^2 = (ab^{-1})^2d = b^4 = (ab)^3d = [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = id\}.$$

All subgroups of index 6 of Γ have no irreducible character of degree 3, and thus are 1-reducible. In order to see that 6 is the smallest index of a 1-reducible subgroup of Γ we need to look at all subgroups whose index is less than 9. Since those groups are all non abelian and all irreducible representations $\rho(\Gamma)$ of degree 3 are faithfull, $\zeta|H$ can not be the sum of 3 character of degree 1.

(a) The subgroups of index 4 are all conjugate and have no irreducible character of degree 2. Thus no subgroup of index 4 can be a 1-reducible subgroup of $\rho(\Gamma)$.

(b) Any subgroup H of index 2 contains an element g of order 3 whose conjugacy class in H contains 1 element. If ζ is an irreducible character of degree 3 of Γ, then $\zeta(g) \neq 3$. But for any irreducible character χ_1 of degree 1 of H we get $\chi_1(g) = 1$, and for any irreducible character χ_2 of degree 2 of H we get $\chi_2(g) = 2$. Thus $\chi_1 + \chi_2 \neq \zeta|H$ and $\rho(H)$ can not be a 1-reducible subgroup of $\rho(\Gamma)$.

Since, as noted above, any imprimitive subgroup of $GL(3, \mathbf{C})$ has a 1-reducible subgroup of index 3 (Ulmer (1992), Lemma 4.2) we get:

THEOREM 3.3. *If an irreducible third order linear differential equation $L(y) = 0$ with coefficients in k has a liouvillian solution, then $L(y) = 0$ has a solution z, such that for the logarithmic derivative $u = z'/z$ of z one of the following holds:*

i) u is algebraic of degree 36 over k and $\mathcal{G}(L)/Z(\mathcal{G}(L)) \cong A_6$.
ii) u is algebraic of degree 21 over k and $\mathcal{G}(L))/Z(\mathcal{G}(L) \cong G_{168}$.
iii) u algebraic of degree 9 over k and $\mathcal{G}(L)/Z(\mathcal{G}(L))$ is isomorphic to H_{72} or H_{216}.
iv) u is algebraic of degree 6 over k and $\mathcal{G}(L)/Z(\mathcal{G}(L))$ is isomorphic to F_{36} or A_5.
v) u is algebraic of degree 3 over k and $\mathcal{G}(L) \subseteq GL(3, \mathbf{C})$ is an imprimitive group.

For each group $\mathcal{G}(L) \subseteq GL(3, \mathcal{C})$ the numbers given above are best possible.

PROOF. That there exists a solution whose logarithmic derivative is of the given degree follows from Theorem 3.4 of Ulmer (1992) and the previous discussion. From Lemma 3.1 we know that the degree $[k(u) : k]$ is precisely the index of a 1-reducible subgoup of $\mathcal{G}(L)$. Since the above numbers are the minimal index of a 1-reducible subgroup of $\mathcal{G}(L)$, they are best possible. □

A similar calculation can be done to show that for second order equations, the best possible degrees are: 4 if $\mathcal{G}(L)/Z(\mathcal{G}(L)) \cong A_4$, 6 if $\mathcal{G}(L)/Z(\mathcal{G}(L)) \cong S_4$ and 12 if $\mathcal{G}(L)/Z(\mathcal{G}(L)) \cong A_5$ (cf. Ulmer (1992)). This gives an alternative proof of theorem 1 of Kovacic (1986).

We point out that the above result is derived without explicitely determining the possible finite primitive unimodular Galois groups of $L(y) = 0$ but follows just from the knowledge of the list of the finite primitive subgroups of $PGL(3, \mathcal{C})$.

3.2. Computing the coefficients of a minimal polynomial of an algebraic solution of known degree of the Riccati

Since we have just produced the exact minimal degrees of an algebraic solution u of the Riccati equation, we now briefly review the method given in Singer (1981) to computes the coefficients of the minimal polynomial of u. This method is the only known method which can be used in the case of an imprimitive differential Galois group of order $n \geq 3$.

We start by describing an algorithm for finding all solutions y of $L(y) = 0$ such that $\delta(y)/y \in \mathbf{C}(x)$. Let $\mathcal{S}$ be the set of singular points of $L(y)$. At each point $c \in \mathcal{S}$ one can determine a finite set $\mathcal{P}_c$ of elements of $\mathbf{C}(x)$ of the form

$$f_c = \frac{\alpha_{1c}}{x-c} + \frac{\alpha_{2c}}{(x-c)^2} + \ldots + \frac{\alpha_{nc}}{(x-c)^n}$$

or $f_c = \alpha_{1c}x + \ldots + \alpha_{nc}x^n$ if $c = \infty$, such that if y is a solution of $L(y) = 0$ such that $y'/y \in \mathbf{C}(x)$ then

$$\begin{aligned} y &= P(x)e^{\left(\int \sum_{c\in\mathcal{S}} f_c\right)} \\ &= P(x)\prod_{c\in\mathcal{S}}(x-c)^{\alpha_{1c}} e^{\left(\frac{\alpha_{2c}}{x-c} + \frac{\alpha_{3c}}{(x-c)^2} + \ldots + \frac{\alpha_{nc}}{(x-c)^n}\right)} \end{aligned}$$

for some choice of $f_c \in \mathcal{P}$ and $P(x) \in \mathbf{C}[x]$. Futhermore, the degree of any possible $P(x)$ can be bounded in terms of the α_{1c}. A method for determining the sets $\mathcal{P}_c$ is given in (Schlesinger (1895), Vol. II.1 Section 177). A modern presentation using Newton polygons and emphasizing computational aspects and an implementation in DESIRE is given in Grigor'ev (1990) and Tournier (1987). This reduces the problem of finding such solutions to the problem of determining the coefficients of the possible $P(x)$, a problem in linear algebra. A related method is given in Singer (1991).

Let $L(y) = 0$ be an irreducible differential equation and $P(u)$ the minimal polynomial of an algebraic logarithmic derivative $\delta(y_1)/y_1$ of a solution y_1 of $L(y) = 0$. Any solution of $P(u) = 0$ is then a logarithmic derivative of a solution of $L(y) = 0$ and we get:

$$\begin{aligned} P(u) &= u^m + c_{m-1}u^{m-1} + \ldots + c_0 \\ &= (u - \frac{y_1'}{y_1})\,(u - \frac{y_2'}{y_2})\ldots(u - \frac{y_m'}{y_m})\ , \end{aligned}$$

where the y_i are solutions of $L(y) = 0$. The coefficients $c_i \in k$ are homogeneous forms of degree $m - i$ in the m logarithmic derivatives $\{y_1'/y_1, \cdots, y_m'/y_m\}$.

We have

$$
\begin{aligned}
c_{m-1} &= \frac{y_1'}{y_1} + \frac{y_2'}{y_2} + \cdots + \frac{y_m'}{y_m} \\
&= \frac{(y_1 y_2 \cdots y_m)'}{y_1 y_2 \cdots y_m}.
\end{aligned}
$$

The product $y_1 y_2 \cdots y_m$ is a solution of $L^{\circledS m}(y) = 0$. Thus c_{m-1} is a rational logarithmic derivative of a solution of $L^{\circledS^m}(y) = 0$.

Using c_{m-1} we can now compute the other coefficients c_i. The coefficient c_i can be written as:

$$
c_i = \sum_{1 \le k_1 < \cdots < k_i \le m} \left(\frac{y_{k_1}'}{y_{k_1}} \cdots \frac{y_{k_i}'}{y_{k_i}} \right).
$$

Let

$$
\begin{aligned}
v_i &= c_i y_1 y_2 \ldots y_n \\
&= \sum_{1 \le k_1 < \cdots < k_i \le m} y_{k_1}' y_{k_2}' \ldots y_{k_i}' \prod_{j \ne k_1, \cdots, k_i} y_j.
\end{aligned}
$$

Note that v_i is a solution of $(L_\delta)^{\circledS^i}(y) \circledS L^{\circledS (m-i)}(y)$, where $L_\delta(y)$ is a differential equation which is satisfied by the derivatives of solutions $L(y) = 0$. We then have:

$$
\begin{aligned}
\frac{v_i'}{v_i} &= \frac{c_i'}{c_i} + \frac{(y_1 y_2 \ldots y_n)'}{y_1 y_2 \ldots y_n} \\
&= \frac{c_i'}{c_i} + c_{m-1}.
\end{aligned}
$$

Therefore, v_i is the solution of a linear differential equation (that we can construct) and the logarithmic derivative of v_i is rational. We can describe all such solutions. Simillarly, $y_1 y_2 \ldots y_n$ is also a solution of a linear differential equation and its logarithmic derivative is also rational. Since $c_i = v_i/(y_1 y_2 \ldots y_n)$, we can determine the degrees of the numerator and denominator of c_i using the algorithm described at the beginning of this section to determine the possible candidates for v_i and $(y_1 y_2 \ldots y_n)$.

We therefore are able to determine bounds on the degrees of the numerators and denominators of the coefficients of a minimal polynomial. To determine the actual numbers that can appear as coefficients of these numerators and denominators, one must differentiate $P(u) = 0$ repeatedly, solve for the higher derivatives of u, and reduce the Riccati equation $R(u) = 0 \bmod P(u)$. This will give algebraic conditions on the numbers appearing in the coefficients of $P(u)$ (a similar method is used in the last section).

We note that for second order linear differential equations, $R(u)$ has order 1. In this case , it is showed in Kovacic (1986) that for $0 \le i \le n-2$ a simple recursion gives each c_i in terms of the c_j with $j > i$. Therefore in the second order case it suffices to just find the possible c_{m-1}. We do not know a similar statement for higher order equations.

4. Case 3: a primitive unimodular Galois group

In this section we show that in this case, where the bound for the algebraic degree of an algebraic solution of the Riccati is large compared to the imprimitive case (cf.

Theorem 3.2 and 3.3), the difficult computation of a rational solution of some Riccati can be avoided. We will reduce the problem to the computation of a rational solution of some symmetric power and a Gröbner basis computation. In contrast to the previous section where only a projective representation of $\mathcal{G}(L)$ was used, a list of the possible Galois groups will be needed. We start by giving this list (taken from Blichfeld (1917) and (Miller, Blichfeld and Dickson (1938))) for second and third order equations.

4.1. The primitive unimodular groups of degree 2 and 3

4.1.1. The primitive unimodular groups of degree 2

Up to isomorphism, there are 3 primitive unimodular groups of degree 2. According to Miller, Blichfeld and Dickson (1938) we define the following matrices:

$$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad S = \begin{pmatrix} \frac{i-1}{2} & \frac{i-1}{2} \\ \frac{i+1}{2} & \frac{-i-1}{2} \end{pmatrix} \quad U = \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}$$

$$S' = \begin{pmatrix} \xi^3 & 0 \\ 0 & \xi^2 \end{pmatrix} \quad U' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T' = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix},$$

where $\xi^5 = 1$, $\alpha = \dfrac{\xi^4 - \xi}{\sqrt{5}}$ and $\beta = \dfrac{\xi^2 - \xi^3}{\sqrt{5}}$.

Then the groups are the following:

(i) The icosahedral group $A_5^{SL_2}$ of order 120 is generated by S', U' and T'. We have $A_5^{SL_2}/\{\pm 1\}$ is isomorphic to the alternating group A_5 of 5 letters.

(ii) The octahedral group $S_4^{SL_2}$ of order 48 is generated by S and U. We have $S_4^{SL_2}/\{\pm 1\}$ is isomorphic to the symmetric group S_4 of 4 letters.

(iii) The Tetrahedral group of $A_4^{SL_2}$ of order 24 is generated by S and T. We have $A_4^{SL_2}/\{\pm 1\}$ is isomorphic to the alternating group A_4 of 4 letters.

When one looks at a character table of $A_4^{SL_2}$, one sees 3 irreducible representations of degree 3, but since in $SL(2, \mathbf{C})$ the trace of an element of order 3 is -1, only one of these is in $SL(2, \mathbf{C})$. The groups $A_5^{SL_2}$ and $S_4^{SL_2}$ have two non conjugate representation in $SL(2, \mathbf{C})$. But the non equivalent representations can be obtained from each other using the Galois group of $\mathbf{Q}(\sqrt{2}, \xi)$ over $\mathbf{Q}$. This follows from the fact that under the automorphism sending $\sqrt{2}$ to $-\sqrt{2}$ the trace of U will change and that under the automorphism sending $\sqrt{5} \in \mathbf{Q}(\xi)$ to $-\sqrt{5}$ the trace of an element of order 10 will change. This will allow us to work with only one representation and to get the complete result by applying the corresponding automorphism.

4.1.2. The primitive unimodular groups of degree 3

Up to isomorphism, there are 8 primitive unimodular groups of degree 3. According to Miller, Blichfeld and Dickson (1938)[†] we define the following matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^4 & 0 \\ 0 & 0 & \xi \end{pmatrix} \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad E_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & s & t \\ 1 & t & s \end{pmatrix}$$

$$E_4 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2\lambda_2 & 2\lambda_2 \\ \lambda_1 & s & t \\ \lambda_1 & t & s \end{pmatrix} \quad S = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix} \quad R = \frac{1}{\sqrt{-7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}$$

$$V = \rho \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \qquad Z = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

where $\xi^5 = 1$, $s = \xi^2 + \xi^3$, $t = \xi + \xi^4$, $\sqrt{5} = t - s$, $\varepsilon^6 + \varepsilon^3 + 1 = 0$ ($\varepsilon^9 = 1$), $\omega = -\varepsilon^3 - 1$ ($\omega^3 = 1$), $\beta^7 = 1$, $a = \beta^4 - \beta^3$, $b = \beta^2 - \beta^5$, $c = \beta - \beta^6$, $\frac{1}{\sqrt{-7}} = \frac{\beta + \beta^2 + \beta^4 - \beta^6 - \beta^5 - \beta^3}{7}$, $\lambda_1 = \frac{-1 \pm \sqrt{-15}}{4}$, $\lambda_2 = \frac{-1 \mp \sqrt{-15}}{4}$ and $\rho = \frac{1}{\omega - \omega^2}$.

Then the groups are the following:

(i) The Valentiner group $A_6^{SL_3}$ of order 1080 is generated by E_1, E_2, E_3 and E_4. We have $A_6^{SL_3} / < Z >$ is isomorphic to the alternating group A_6 of 6 letters.

(ii) The alternating group A_5 of five letters generated by E_1, E_2 and E_3.

(iii) The direct product $A_5 \times C_3$, of A_5 and the cyclic group C_3 of three elements, generated by E_1, E_2, E_3 and Z.

(iv) The simple group G_{168} of order 168 generated by S, T and R.

(v) The direct product $G_{168} \times C_3$, of G_{168} and the cyclic group C_3 of three elements, generated by S, T, R and Z.

(vi) The group $H_{216}^{SL_3}$ of order 648 generated by S_1, T, V and U, whose projective representation is the Hessian group of order 216.

(vii) The group $H_{72}^{SL_3}$ of order 216 generated by S_1, T, V and UVU^{-1}.

(viii) The group $F_{36}^{SL_3}$ of order 108 generated by S_1, T and V.

As in the previous case, all non equivalent representations of these groups in $SL(3, \mathbf{C})$ can be obtained using the Galois group of the field to which the coefficients of the matrices belong. The group $H_{72}^{SL_3}$ has two faithfull non conjugate representations in $SL(3, \mathcal{C})$ which

[†] The matrix T used here corresponds to the inverse of the matrix T used in Miller, Blichfeld and Dickson (1938) and the definition of $A_6^{SL_3}$ and A_5 correspond to the definitions given in exercise 3 and 4 p. 252 of Miller, Blichfeld and Dickson (1938)

are sent to each other by the automorphism $\sigma : \varepsilon \mapsto \varepsilon^2$ of the Galois group of $\mathbf{Q}(\varepsilon)/\mathbf{Q}$, since the trace of $M = UVU^{-1}VT^{-1}UVU^{-1}$ is ω, while the trace of $\sigma(M)$ is ω^2. The automorphism σ also sends the two faithfull non conjugate representations of $F_{36}^{SL_3}$ in $SL(3,\mathcal{C})$ to each other, since the trace of $M = VS^2TST^{-1}$ is ω, while the trace of $\sigma(M)$ is ω^2. This will allow us to work with only one representation and to get the complete result by applying the corresponding automorphism. We also note that for $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$ only the representations where the elements of order 4 all have trace 1 belong to $SL(3,\mathbf{C})$.

We point out that the above groups do not in general correspond to the Schur representation groups used in the previous section. For example, the group $A_6^{SL_3}$ is of order 1080 while the Schur representation group of A_6 is of order 2160.

4.2. Algebraic degree of a solution

From Theorem 3.3 and Theorem 3.8 of Ulmer (1992) we see that for the computation of an algebraic logarithmic derivative $u = y'/y$ of a solution y, the most difficult cases, where the algebraic degree of u is large, are those of a finite primitive unimodular differential Galois group. In this case all solutions will be algebraic and we shall show how to compute the minimal polynomial of such a solution. From Corollary 1.4 we get that the number of coefficients of the minimal polynomials of y and y'/y are the same. In this section we will derive a bound for the algebraic degree of a solution y of $L(y) = 0$. We note that from $k(y'/y) \subseteq k(y)$ the index of a 1 reducible subgroup of $\mathcal{G}(L)$ is a lower bound for the degree of a solution (cf. Lemma 3.1).

For a second order differential equation $L(y) = 0$ an old result of P. Pepin and L. Fuchs (see Fuchs (1875) and the introduction in Boulanger (1898)) shows that the degree of $P(y)$ is always the largest possible degree, which is the order of $\mathcal{G}(L)$. This can be seen in the following way: If $L(y) = y'' - ry$ $(r \in k)$, then the Wronskian $y_1'y_2 - y_1y_2'$ of two solutions y_1 and y_2 of $L(y) = 0$ is a constant c (see e.g. Kaplansky (1957), p. 40). Thus

$$\frac{y_2}{y_1} = c\int \frac{1}{y_1^2}$$

If y_1 and y_2 are algebraic over k, then the integral on the right hand side must be algebraic and thus a rational function in y_1 over k. Since any solution can be used as y_1, we get that for any solution y_1 of $L(y) = 0$ the field $k(y_1)$ is the full Picard-Vessiot extension K associated to $L(y) = 0$. Thus any solution is a primitive element of K and must be of degree $\mathcal{G}(L)$.

The following Theorem shows that the above result of Pepin and Fuchs no longer holds for third order differential equations:

Theorem 4.1. *Let $L(y)$ be an irreducible third order linear differential equation with Galois group a primitive group $\mathcal{G}(L) \subset SL(3,\mathbf{C})$. If $L(y) = 0$ has a liouvillian solution then all solution are algebraic and there is a solution z whose minimal polynomial $P(\mathbf{Y})$ is of the form*

$$\mathbf{Y}^{d\cdot m} + a_{m-1}\mathbf{Y}^{d\cdot(m-1)} + \ldots + a_1\mathbf{Y}^d + a_0 \quad (a_i \in k)$$

such that one of the following holds:

(i) If $\mathcal{G}(L) \cong A_6^{SL_3}$, $m = 36$ and $d = 6$.

(ii) *If* $\mathcal{G}(L) \cong A_5$, $m = 6$ *and* $d = 2$.
(iii) *If* $\mathcal{G}(L) \cong A_5 \times C_3$, $m = 6$ *and* $d = 6$.
(iv) *If* $\mathcal{G}(L) \cong G_{168}$, , $m = 21$ *and* $d = 2$.
(v) *If* $\mathcal{G}(L) \cong G_{168} \times C_3$, , $m = 21$ *and* $d = 6$.
(vi) *If* $\mathcal{G}(L) \cong H_{216}^{SL_3}$, $m = 9$ *and* $d = 9$.
(vii) *If* $\mathcal{G}(L) \cong H_{72}^{SL_3}$, $m = 9$ *and* $d = 3$.
(viii) *If* $\mathcal{G}(L) \cong F_{36}^{SL_3}$, $m = 6$ *and* $d = 6$.

The above numbers are also the minimal degree of an algebraic solution, except for the group $F_{36}^{SL_3}$*, where there also exists a solution whose minimal polynomial is of degree* 27 *instead of* 36.

In order to prove the above result we need a result linking the permutation representation of $\mathcal{G}(L)$ on the solutions of $P(Y) = 0$ and the linear representation of $\mathcal{G}(L)$ on the solutions of $L(y) = 0$.

Lemma 4.2. *Let* G *be a finite group and* V *a finite irreducible* G*-module over a field of characteristic* 0 *with character* χ*. Let* $\{v_1, \ldots, v_m\}$ *be a* G*-invariant subset of* V *and* V_m *be the associated permutation* G*-module. Then* V *is a direct summand of* V_m *and*

$$\sum_{g \in G} \chi(g) \cdot fix(g) \quad = \quad t \cdot |G| > 0,$$

where $fix(g)$ *is the number of vectors in* $\{v_1, \ldots, v_m\}$ *left fixed by* g *and* t *is the multiplicity of* V *in* V_m.

Proof. By the definition of V_m we can identify a basis $\{z_1, \ldots, z_m\}$ of V_m with the set $\{v_1, \ldots, v_m\}$ in such a way that the action of G on these two sets are the same. We now define a map $\varphi : V_m \longmapsto V$ by $\varphi(z_i) = v_i$ $(i \in \{1, \ldots m\})$. This clearly defines a G-morphism. According to Maschke's Theorem, V_m is completely reducible and thus the direct sum of the image $\varphi(V_m)$ and the kernel of φ. Since V is an irreducible G-module, $\varphi(V_m) = V$. This shows that V is a direct summand of V_m.
If we denote χ_m the character of V_m, then the orthogonality relations give us

$$\sum_{g \in G} \chi(g) \cdot \overline{\chi_m(g)} \quad = \quad t \cdot |G| > 0.$$

With respect to the basis $\{z_1, \ldots, z_m\}$, an element $g \in G$ has a 1 in the (i, i) place if z_i is left fixed and a 0 otherwise. Therefore the trace of this matrix is $fix(g)$ and the formula now follows. □

Proof of Theorem 4.1 For each possible group $\mathcal{G}(L)$ we know the index m of a one reducible subgroup H of smallest index (Theorem 3.3). If $\mathcal{G}(L) \not\cong F_{36}^{SL_3}$ and $\mathcal{G}(L) \not\cong G_{168} \times C_3$, then using Cayley one can show that all groups H of index m are conjugate so that one can choose any of these groups to perform the computations in the following. If $\mathcal{G}(L) \cong F_{36}^{SL_3}$, then two non conjugate groups H have to be considered. If $\mathcal{G}(L) \cong G_{168} \times C_3$, then $m = 21$, but the groups of index 21 which have an irreducible representation of degree 3 cannot be 1-reducible, since from the character tables we get that in such a representation there is an element of order 2 whose trace is -1, but in any 1-dimensional representation of $G_{168} \times C_3$ this element has a trace 1 and in any

2-dimensional representation this element has trace 2. Those subgroups of $G_{168} \times C_3$ of index 21 which are always 1-reducible are all congugate and we can choose any of them.

From Theorem 3.3 we get that $L(y) = 0$ has a solution y such that $[k(y'/y) : k] = m$. From Corollary 1.4 we get that the differential Galois group of $k(y)/k(y'/y)$ is cyclic. If K denotes the PVE of $L(y) = 0$, then the extension $k(y)$ is an intermediate field of $K/k(y'/y)$. The differential Galois group of $K/k(y'/y)$ is isomorphic to H and the Galois group of $k(y)/k(y'/y)$ is a cyclic factor of H (cf. Corollary 1.4). Since we know the possible groups H, using CAYLEY we can compute the order d of all possible cyclic factor groups of H. We get:

i) If $\mathcal{G}(L) \cong A_6^{SL_3}$, $A_5 \times C_3$, $G_{168} \times C_3$, $H_{72}^{SL_3}$ or $F_{36}^{SL_3}$, then d belongs to $\{1, 2, 3, 6\}$.
ii) If $\mathcal{G}(L) \cong A_5$ or G_{168} then d belongs to $\{1, 2\}$.
iii) If $\mathcal{G}(L) \cong H_{216}^{SL_3}$, then d belongs to $\{1, 3, 9\}$.

From Corollary 1.4 we get that the minimal polynomial of y must be of the form:

$$P(\mathbf{Y}) = \mathbf{Y}^{d \cdot m} + a_{m-1}\mathbf{Y}^{d(m-1)} + \ldots + a_1\mathbf{Y}^d + a_0 \quad (a_j \in k)$$

Since $\mathcal{G}(L) \subseteq SL(3, \mathbf{C})$ is irreducible, the splitting field of $P(\mathbf{Y}) = 0$ is a Picard Vessiot extension for $L(y) = 0$ (cf. Ulmer (1992), Corollary 2.4). Thus $\mathcal{G}(L)$ is the (classical) Galois group of $P(\mathbf{Y}) = 0$ (cf. Ulmer (1992), Lemma 1.1) and must have a faithfull representation as a transitive permutation group of degree $d \cdot m$. Using CAYLEY one can see that for a faithfull transitive representation of

(i) $A_6^{SL_3}$ of degree $36 \le j \le (6 \cdot 36)$, j must be 45, 90, 108 135, 180 or 216.
(ii) A_5 of degree $6 \le j \le (2 \cdot 6)$, j must be 6, 10 or 12.
(iii) $A_5 \times C_3$ of degree $6 \le j \le (6 \cdot 6)$, j must be 15, 18, 30 or 36.
(iv) G_{168} of degree $21 \le j \le (2 \cdot 21)$, j must be 21, 24, 28 or 42.
(v) $G_{168} \times C_3$ of degree $21 \le j \le (6 \cdot 21)$, j must be 21, 24, 42, 63, 72, 84 or 126.
(vi) $H_{216}^{SL_3}$ of degree $9 \le j \le (9 \cdot 9)$, j must be 81.
(vii) $H_{72}^{SL_3}$ of degree $9 \le j \le (6 \cdot 9)$, j must be 27, 36 or 54.
(viii) $F_{36}^{SL_3}$ of degree $6 \le j \le (6 \cdot 6)$, j must be 18, 27 or 36.

For each possible j above, we construct all transitive permutation representations of the corresponding differential Galois group $\mathcal{G}(L)$ (note that it is enougth to let $\mathcal{G}(L)$ act on the cosets of one representant of each set of conjugate subgroups) and compute the corresponding permutation representation P with character χ_P of degree j. If for all irreducible characters χ of degree 3 of G, the scalar product χ and χ_P is 0, then from Lemma 4.2 we get that $\mathcal{G}(L)$ cannot permute the j solutions according to P. If this is the case for all transitive permutation respresentations of degree j, then we can exclude the possibility j.

For the groups $A_6^{SL_3}$, A_5, G_{168}, $G_{168} \times C_3$ and $H_{216}^{SL_3}$ only the numbers $j = d \cdot m$ given in the Theorem are still possible and must thus be minimal.

For the groups $H_{72}^{SL_3}$ the values $3 \cdot 9$ and $6 \cdot 9$ and for $F_{36}^{SL_3}$ the values 27 and $6 \cdot 6$ are still possible. We will show by examining the matrices defining the three dimensional representations that in both cases there is always a solution of degree 27 which will then be the minimal degree.

A representation G in $SL(3, \mathcal{C})$ of the group $H_{72}^{SL_3}$ is generated by the matrices S_1, T,

V and UVU^{-1} given in section 4.1.2. A 1-reducible subgroup H of index 9 is generated by S_1TUVU^{-1}, $S_1TUVU^{-1}V$ and $S_1T^{-1}S_1^{-1}T$. The solution $y = (0, -1, 1)$ is left invariant by the first two matrices and sent to ωy by the last matrix. This shows that y^3 is left invariant by H and, since by the above no solution of lower degree is possible, that for this representation of $\mathcal{G}(L) \cong H_{72}^{SL_3}$ the equation $L(y) = 0$ has a solution of the form stated. Since we can map this representation of $H_{72}^{SL_3}$ to a non equivalent one by the automorphism $\sigma : \varepsilon \mapsto \varepsilon^2$ of the Galois group of $\mathbf{Q}(\varepsilon)/\mathbf{Q}$ and that the given eigenvector and eigenvalues of the elements of H belong to $\mathbf{Q}(\varepsilon)$, it follows that for the representation $\sigma(G)$ the group $\sigma(H)$ will have the above properties of H. This shows that a differential equation whose Galois group is isomorphic to $H_{72}^{SL_3}$ will always have a solution of degree $3 \cdot 9$ and of the form stated.

A representation G in $SL(3, \mathcal{C})$ of the group $F_{36}^{SL_3}$ is generated by S_1, T and V of section 4.1.2. The cyclic subgroup F generated by $S_1^{-1}T^{-1}S_1V^{-1}$ is of order 4 and index 27. The solution $y = (-\varepsilon^3, 1, 0)$ is left invariant by F. Since from the above we know that no solution of lower degree is possible, the degree of y must be 27. This shows that for this representation of $F_{36}^{SL_3}$, a solution of degree 27 always exists. Since we can map this representation of $H_{72}^{SL_3}$ to a non equivalent one by the automorphism $\sigma : \varepsilon \mapsto \varepsilon^2$ of the Galois group of $\mathbf{Q}(\varepsilon)/\mathbf{Q}$ and that the given eigenvector and eigenvalues of the elements of H belong to $\mathbf{Q}(\varepsilon)$, it follows that for the representation $\sigma(G)$ the group $\sigma(F)$ will have the above properties of F. This shows that a differential equation whose Galois group is isomorphic to $F_{36}^{SL_3}$ will always have a solution of degree 27. On the other hand, since $6 \cdot 6$ is the only possibility left of the form $6 \cdot i$, there will also always exists a solution of degree $6 \cdot 6$. This completes the proof of Theorem 4.1.

We also note that the result of Fuchs and Pepin, which states that for second order equations the degree of an algebraic solution always corresponds to the order of the primitive unimodular group $\mathcal{G}(L)$ (i.e. any solution is a primitive element of the PVE) can also be proven using the method above.

4.3. Decomposition of the coefficients in terms of (semi-)invariants

In this section as in the previous one we deal with the case of a differential equation $L(y) = 0$ whose Galois group $\mathcal{G}(L)$ is a finite primitive unimodular group. We show how the coefficients of the minimal polynomial of a solution of $L(y) = 0$ can be computed using a basis of the ring of invariants of $\mathcal{G}(L)$ (see e.g. Cox, Little and O'Shea (1992), Chapter 7). This approach is not new and has been successfully used in Fuchs (1875) for the case of second order differential equation. In this section we will describe this procedure and show how it can be generalized to higher order equations.

Let $L(y) = 0$ be a differential equation of degree n with finite primitive differential Galois group $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$. Let $\{y_1, y_2, \ldots, y_n\}$ be a basis of the solution space of $L(y) = 0$ corresponding to the representation $\mathcal{G}(L)$, H a 1-reducible subgroup of $\mathcal{G}(L)$ of minimal index m and $\mathcal{T}$ a set of left coset representatives of H in $\mathcal{G}(L)$. Let y be a common eigenvector of H, then by Corollary 1.4 we get that the minimal polynomial of y is the form

$$\begin{aligned} P(\mathbf{Y}) &= \prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y}^d - (\sigma(y))^d \right) \\ &= \mathbf{Y}^{d \cdot m} + \alpha_{d(m-1)} \mathbf{Y}^{d(m-1)} + \ldots + \alpha_d \mathbf{Y}^d + \alpha_0 \quad (m = |\mathcal{G}(L)/H|), \end{aligned}$$

where any coefficient α_i is a polynomial of degree $d\cdot m-i$ in $\{y_1, y_2, \ldots, y_n\}$. By construction these polynomials are invariant under the action of $\mathcal{G}(L)$ and thus can be expressed in terms of the elements of a basis of the ring of invariants (or semi-invariants) of $\mathcal{G}(L)$. This can be done in the following way:

i) Choose a representation $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$ of the differential Galois group of $L(y) = 0$ (i.e. fix a basis of the solution space) and compute a basis $\{b_1(y_1, \ldots, y_n), \ldots, b_j(y_1, \ldots, y_n)\}$ of the ring of invariants of $\mathcal{G}(L)$.
ii) Compute a 1-reducible subgroup H of minimal index of $\mathcal{G}(L)$ and a common eigenvector y for the matices of H.
iii) Compute a set of left coset representatives $\mathcal{T}$ of H in $\mathcal{G}(L)$.
iv) Compute the polynomial

$$\begin{aligned} P(\mathbf{Y}) &= \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^d - \sigma(z)^d) \\ &= \mathbf{Y}^{d\cdot m} + \gamma_{m-1}(y_1, \ldots, y_n)\mathbf{Y}^{d\cdot(m-1)} + \ldots + \gamma_0(y_1, \ldots, y_n), \end{aligned}$$

where $m = |\mathrm{G(L)}/H|$ and d is the index of a normal subgroup F of H such that H/F is a cyclic group.
v) Using the Gröbner basis algorithm, express $\gamma_i(y_1, \ldots, y_n)$ in terms of polynomials in the invariants $\{b_1(y_1, \ldots, y_n), \ldots b_j(y_1, \ldots, y_n)\}$ (cf. Cox, Little and O'Shea (1992), Chapter 7, §3, Prop. 7. In practice an *Ansatz* turned out to be more effective).

The above computation has to be done once for the finitely many primitive finite subgroups of $SL(n, \mathbf{C})$.

In the following we will use semi-invariants of $\mathcal{G}(L)$ to represent the coefficients of $P(Y)$, since they are usually of lower degree.

4.3.1. Second order equations

For second order equations the decomposition of the coefficients $\alpha_i(y_1, \ldots, y_n)$ in terms of $\{b_1(y_1, \ldots, y_n), \ldots b_j(y_1, \ldots, y_n)\}$ has been computed by Fuchs in Fuchs (1875) for $A_4^{SL_2}$ and $S_4^{SL_2}$. In the following we show that the result for second order equations can be obtained by our approach and restate Fuchs' results.

The second order case is simplified by the following facts:

i) According to the result of Pepin and Fuchs, we must have $d \cdot m = |\mathcal{G}(L)|$ (i.e. any solution is a primitive element of the PVE associated to $L(y) = 0$).
ii) Any one reducible subgroup is abelian and (assuming $\mathcal{G}(L)$ unimodular) is a cyclic group, so that a common eigenvector is just an eigenvector of a generator.

We have to deal with each group separetely.

The tetrahedral group $A_4^{SL_2}$:

We consider the algebraic extension $\mathbf{Q}(\omega)$ of the rational numbers, where ω is a root of $\omega^4 - 2\omega^3 + 5\omega^2 - 4\omega + 1$. We have $i = \sqrt{-1} = -2\omega^3 + 3\omega^2 - 9\omega + 4$ and $\sqrt{-3} = 4\omega^3 - 6\omega^2 + 16\omega - 7$. The group $A_4^{SL_2}$ is generated by the matrices S and T of section 4.1.1 which are defined in $\mathbf{Q}(\omega)$.

We denote $\{y_1, y_2\}$ the basis corresponding to the above representation. In this representation, the ring of semi-invariants of $A_4^{SL_2}$ is generated by (see Miller, Blichfeld and Dickson (1938), p. 224):

$$\begin{aligned} I_1 &= y_1^4 + 2\sqrt{-3}y_1^2y_2^2 + y_2^4, \\ I_2 &= y_1y_2(y_1^4 - y_2^4), \\ I_3 &= y_1^4 - 2\sqrt{-3}y_1^2y_2^2 + y_2^4, \end{aligned}$$

together with the relation $12\sqrt{-3}I_2^2 - I_1^3 + I_3^3 = 0$. We will only need I_1 and I_2 to represent the coefficients of $P(Y)$.

A maximal 1-reducible subgroup of $A_4^{SL_2}$ is the cyclic group of order 6 generated by the matrix TS^{-1} which has an eigenvector $z = (\omega^3 - \omega^2 + 3\omega - 1)y_1 + y_2$. A set of left coset representatives $\mathcal{T}$ of $< TS^{-1} >$ in $A_4^{SL_2}$ is

$$\left\{id, S, S^{-1}, TS\right\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma \in \mathcal{T}} (\mathbf{Y}^6 - \sigma(z)^6),$$

which is:

$$\begin{aligned}
&\mathbf{Y}^{24} + 48\omega^3(y_2y_1^5 - y_2^5y_1)\ \mathbf{Y}^{18} \\
&+ \big((-780\omega^2 + 780\omega - 210)y_1^{12} + (-1824\omega^3 + 2736\omega^2 - 1368\omega + 228)y_2^2y_1^{10} \\
&\quad +(25740\omega^2 - 25740\omega + 6930)y_2^4y_1^8 + (3648\omega^3 - 5472\omega^2 + 2736\omega - 456)y_2^6y_1^6 \\
&\quad +(25740\omega^2 - 25740\omega + 6930)y_2^8y_1^4 \\
&\quad +(-1824\omega^3 + 2736\omega^2 - 1368\omega + 228)y_2^{10}y_1^2 + (-780\omega^2 + 780\omega - 210)y_2^{12}\big)\ \mathbf{Y}^{12} \\
&+ \big((6816\omega^3 + 2496\omega^2 - 7488\omega + 2496)y_2y_1^{17} \\
&\quad +(97088\omega^3 - 326784\omega^2 + 255744\omega - 61568)y_2^3y_1^{15} \\
&\quad +(-231744\omega^3 - 84864\omega^2 + 254592\omega - 84864)y_2^5y_1^{13} \\
&\quad +(-291264\omega^3 + 980352\omega^2 - 767232\omega + 184704)y_2^7y_1^{11} \\
&\quad +(291264\omega^3 - 980352\omega^2 + 767232\omega - 184704)y_2^{11}y_1^7 \\
&\quad +(231744\omega^3 + 84864\omega^2 - 254592\omega + 84864)y_2^{13}y_1^5 \\
&\quad +(-97088\omega^3 + 326784\omega^2 - 255744\omega + 61568)y_2^{15}y_1^3 \\
&\quad +(-6816\omega^3 - 2496\omega^2 + 7488\omega - 2496)y_2^{17}y_1\big)\ \mathbf{Y}^6 \\
&+ \big((780\omega^2 - 780\omega + 209)y_1^{24} + (-8688\omega^3 + 13032\omega^2 - 6672\omega + 1164)y_2^2y_1^{22} \\
&\quad +(-135720\omega^2 + 135720\omega - 36366)y_2^4y_1^{20} \\
&\quad +(304080\omega^3 - 456120\omega^2 + 233520\omega - 40740)y_2^6y_1^{18} \\
&\quad +(1134900\omega^2 - 1134900\omega + 304095)y_2^8y_1^{16} \\
&\quad +(-295392\omega^3 + 443088\omega^2 - 226848\omega + 39576)y_2^{10}y_1^{14} \\
&\quad +(1194960\omega^2 - 1194960\omega + 320188)y_2^{12}y_1^{12} \\
&\quad +(-295392\omega^3 + 443088\omega^2 - 226848\omega + 39576)y_2^{14}y_1^{10} \\
&\quad +(1134900\omega^2 - 1134900\omega + 304095)y_2^{16}y_1^8 \\
&\quad +(304080\omega^3 - 456120\omega^2 + 233520\omega - 40740)y_2^{18}y_1^6 \\
&\quad +(-135720\omega^2 + 135720\omega - 36366)y_2^{20}y_1^4 \\
&\quad +(-8688\omega^3 + 13032\omega^2 - 6672\omega + 1164)y_2^{22}y_1^2 \\
&\quad +(780\omega^2 - 780\omega + 209)y_2^{24}\big)
\end{aligned}$$

Using the Gröbner basis algorithm in Axiom (cf. Jenks and Sutor (1992)) we can represent the coefficients of $P(Y)$ as polynomials in the invariants I_1 and I_2:

$$\mathbf{Y}^{24} + \left(48\omega^3 I_2\right)\ \mathbf{Y}^{18}$$
$$+\left((-780\omega^2 + 780\omega - 210)I_1^3 + (-6144\omega^3 + 9216\omega^2 - 4608\omega + 768)I_2^2\right)\ \mathbf{Y}^{12}$$
$$+\left((6816\omega^3 + 2496\omega^2 - 7488\omega + 2496)I_2 I_1^3\right.$$
$$\left. + (167936\omega^3 - 565248\omega^2 + 442368\omega - 106496)I_2^3\right)\ \mathbf{Y}^6$$
$$+(780\omega^2 - 780\omega + 209)I_1^6$$

The above representation shows that the semi-invariant I_2 is a rational function and thus an invariant, and that I_1 is the cube root of a rational function. This last fact can also be derived using the one dimensional characters of $A_4^{SL_2}$ (cf. Lemma 1.6), since in the decomposition of the character of the sixth symmetric product of a two dimensional character of $A_4^{SL_2}$ there is exactly 1 one dimensional character ϕ which is of order 3 (i.e. $\phi^3 = 1$, $\phi \neq 1$). This also shows that, up to a constant, there is exactly one solution of $L^{\circledS 6}(y) = 0$ which is the cube root of a rational function.

We note that there are other minimal polynomials of solutions that can be derived using either another representation of the group, another eigenvector of a cyclic subgroup of order 4 or another cyclic subgroup of order 4. In Fuchs (1878) p. 21 the following decomposition of a polynomial $P(Y)$ is given:

$$\mathbf{Y}^{24} - 3\varphi\,\mathbf{Y}^{18} + (-3\chi_1^3 - 78\chi^3)\,\mathbf{Y}^{12} + (-\chi_1^3\varphi + 10\chi^3\varphi)\,\mathbf{Y}^6 - 27\chi^6,$$

where χ is an invariant of degree 4 of $A_4^{SL_2}$, χ_1 (the Hessian of χ) is another invariant of degree 4 and φ (the jacobian of χ_1 and χ) is an invariant of degree 6.
We note that our invariants differ from those of Fuchs because we select a different basis for our representation.

The octahedral group $S_4^{SL_2}$:

We consider the algebraic extension $\mathbf{Q}(i, \sqrt{2})$ of the rational numbers. The group $S_4^{SL_2}$ is generated by the matrices S and U of section 4.1.1.

We denote $\{y_1, y_2\}$ the basis corresponding to the above representation. In this representation, the ring of semi-invariants of $S_4^{SL_2}$ is generated by (see Miller, Blichfeld and Dickson (1938), p. 224):

$$\begin{aligned} I_1 &= y_1 y_2 (y_1^4 - y_2^4), \\ I_2 &= y_1^8 + 14 y_1^4 y_2^4 + y_2^8, \\ I_3 &= y_1^{12} - 33 y_1^8 y_2^4 - 33 y_1^4 y_2^8 + y_2^{12}, \end{aligned}$$

Together with the relation $108 I_1^4 - I_2^3 + I_3^2 = 0$. We will only need I_1 and I_2 to represent the coefficients.

A maximal 1-reducible subgroup of $S_4^{SL_2}$ is the cyclic group of order 8 generated by the above matrix U and y_1 is an eigenvector of U. A set of left coset representatives $\mathcal{T}$ of $< U >$ in $S_4^{SL_2}$ is

$$\left\{id, S, US, U^2S, S^{-1}U, SUS\right\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^8 - \sigma(z)^8)$$

Using the Gröbner basis algorithm in AXIOM II we can represent the coefficients of $P(Y)$ as polynomials in I_1 and I_2. We get the following representation for $P(Y)$:

$$\mathbf{Y}^{48} - \frac{5}{4} I_2\,\mathbf{Y}^{40} + \frac{35}{128} I_2^2\,\mathbf{Y}^{32} + \left(\frac{1351}{128} I_1^4 - \frac{25}{1024} I_2^3\right)\mathbf{Y}^{24}$$
$$+\left(\frac{265}{1024} I_2 I_1^4 + \frac{65}{65536} I_2^4\right)\mathbf{Y}^{16} + \left(\frac{39}{32768} I_2^2 I_1^4 - \frac{1}{65536} I_2^5\right)\mathbf{Y}^8 + \frac{1}{65536} I_1^8$$

The above representation shows that I_2 is a rational function and that I_1 is the fourth root of a rational function. Since in the decomposition of the character of the sixth symmetric product of a faithfull two dimensional character of $S_4^{SL_3}$ there is exactly 1 one dimensional character ϕ which is of order 2 (i.e. $\phi^2 = 1$, $\phi \neq 1$), we get that I_1 is the square root of a rational function (cf. Lemma 1.6). This is also derived by L. Fuchs (cf. Fuchs (1878) p. 13). The decomposition of the characters also shows that there will be, up to a constant, exactly one rational solution of $L^{\circledS 6}(y) = 0$ and exactly one solution of $L^{\circledS 6}(y) = 0$ which is the square root of a rational function, and thus, up to a constant, exactly one choice for I_1 and I_2.

The above polynomial was obtained using only one representation of $S_4^{SL_2}$, which has in fact two faithfull non equivalent representations in $SL(2,\mathcal{C})$. Since there is an automorphism σ of the Galois group of $\mathbf{Q}(i,\sqrt{2})/\mathbf{Q}$ sending this representation to a non equivalent representation, $\sigma(P(\mathbf{Y}))$ would be the minimal polynomial of a solution for this representation. Since the above decomposition of $P(\mathbf{Y})$ contains only rational coefficients, any representation of $S_4^{SL_2}$ in $SL(2,\mathcal{C})$ will lead to a solution whose minimal polynomial is of the above form.

We again note that there are other minimal polynomials of solutions that can be derived using either another representation of the group, another eigenvector of a cyclic subgroup of order 6 or another cyclic subgroup of order 6. In Fuchs (1878) p. 21 the following decomposition of a polynomial $P(\mathbf{Y})$ is given:

$$\mathbf{Y}^{48} - 20\chi_1\,\mathbf{Y}^{40} + 70\chi_1^2\,\mathbf{Y}^{32} + (-100\chi_1^3 - 14\cdot 3088\chi^4)\,\mathbf{Y}^{24}$$
$$+(65\chi_1^4 + 40\cdot 424\chi_1\chi^4)\,\mathbf{Y}^{16} + (-16\chi_1^5 - 1248\chi_1^2\chi^4)\,\mathbf{Y}^8 - 16^2\chi^8,$$

where χ is an invariant of degree 6 of $S_4^{SL_2}$ and χ_1 (the Hessian of χ) is another invariant of degree 8.

The icosahedral group $A_5^{SL_2}$:

We consider the algebraic extension $\mathbf{Q}(\xi)$ of the rational numbers, where $\xi^5 = 1$. The group $A_5^{SL_2}$ is generated by the matrices S', U' and T' of section 4.1.1.

We denote $\{y_1, y_2\}$ the basis corresponding to the above representation. In this representation, the ring of invariants of $S_4^{SL_2}$ is generated by (see Miller, Blichfeld and Dickson (1938), p. 224):

$$\begin{aligned}
I_1 &= y_1 y_2 (y_1^{10} + 11 y_1^5 y_2^5 - y_2^{10}),\\
I_2 &= -y_1^{20} - y_2^{20} + 228(y_1^{15} y_2^5 - y_1^5 y_2^{15}) - 494 y_1^{10} y_2^{10},\\
I_3 &= y_1^{30} + y_2^{30} + 522(y_1^{25} y_2^5 - y_1^5 y_2^{25}) - 10005(y_1^{20} y_2^{10} + y_1^{10} y_2^{20}),
\end{aligned}$$

Together with the relation $I_3^2 + I_2^3 - 1728 I_1^5 = 0$.

A maximal 1-reducible subgroup of $A_5^{SL_2}$ is the cyclic group of order 10. Such a group H is generated by the matrix $U^2 S$, which has an eigenvector $z = y_1$. A set of left coset representatives $\mathcal{T}$ of H in $A_5^{SL_2}$ is

$$\left\{id, U, T^{-1}, UT^{-1}, ST^{-1}, TST, S^{-2}T, TS^{-2}T, USTS^{-2}, US^{-1}T, S^2T^{-1}, US^{-2}T^{-1}\right\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^{10} - \sigma(z)^{10})$$

Using an *Ansatz* we can represent the coefficients of $P(Y)$ as polynomials in the invariants I_1, I_2 and I_3 we get the following representation for $P(Y)$:

$$\begin{aligned}
&\mathbf{Y}^{120} + \frac{374}{625} I_2 \mathbf{Y}^{100} - \frac{1001}{3125} I_3 \mathbf{Y}^{90} - \frac{142373}{1953125} I_2^2 \mathbf{Y}^{80} \\
&+ \frac{78254}{9765625} I_2 I_3 \mathbf{Y}^{70} + \left(\frac{832814147}{5273437500000} I_2^3 - \frac{8910209}{42187500000} I_3^2\right) \mathbf{Y}^{60} \\
&- \frac{81631}{30517578125} I_3 I_2^2 \mathbf{Y}^{50} + \left(\frac{-39788034}{152587890625} I_1^5 I_2 - \frac{158499}{19073486328125} I_2^4\right) \mathbf{Y}^{40} \\
&+ \left(\frac{-611864}{762939453125} I_1^5 I_3 + \frac{1254}{95367431640625} I_2^3 I_3\right) \mathbf{Y}^{30} \\
&+ \left(\frac{103862}{476837158203125} I_1^5 I_2^2 + \frac{3124}{298023223876953125} I_2^5\right) \mathbf{Y}^{20} \\
&+ \left(\frac{4}{2384185791015625} I_1^5 I_2 I_3 - \frac{1}{298023223876953125} I_2^4 I_3\right) \mathbf{Y}^{10} \\
&+ \frac{1}{298023223876953125} I_1^{10}
\end{aligned}$$

Since $A_5^{SL_2}$ has only one irreducible character of degree 1, any invariant will be rational and thus not hard to compute. Since there is no polynomial relation between I_1, I_2 and I_3, we get that, up to a constant multiple, there will be one polynomial solution of $L^{\circledS 12}(y) = 0$, $L^{\circledS 20}(y) = 0$ and $L^{\circledS 30}(y) = 0$. This can also be derived from the decomposition of the characters of the 12-th, 20-th and 30-th symmetric product of the irreducible characters of degree 3 of $A_5^{SL_2}$.

As for $S_4^{SL_2}$ there is an element σ of the Galois group of $\mathbf{Q}(\xi)/\mathbf{Q}$ sending the above representation in a non equivalent representation. Thus $\sigma(P(\mathbf{Y}))$ would be the minimal polynomial of a solution for this representation. Since the above decomposition of $P(\mathbf{Y})$ contains only rational coefficients, any representation of $A_5^{SL_2}$ in $SL(2, \mathcal{C})$ will lead to a solution whose minimal polynomial is of the above form.

In Fuchs (1878) (cf. p. 16) no explicit decomposition of a polynomial $P(Y)$ is given.

4.3.2. Third order equations

For third order equations it is not the case that any solution of $L(y) = 0$ is a primitive element of the PVE extension associated to $L(y) = 0$. For $\mathcal{G}(L) \cong H_{216}^{SL_3}$ the order of the minimal polynomial of a primitive element (which always exists) is 648, while from Corollary 4.1 there is a solution whose monic minimal polynomial is of order 81 where at most 9 non zero coefficients have to be computed.

In this section we present results involving groups, so that only the decomposition of the coefficients in terms of the fundamental invariants remains to be done. To illustrate the procedure for third order differential equations, we perform the decomposition of the minimal polynomial in the case $\mathcal{G}(L) \cong A_5$.

We consider each group separetely:

The Valentiner group $A_6^{SL_3}$:

The group $A_6^{SL_3}$ is generated by the matrices E_1, E_2, E_3 and E_4 given in section 4.1.2. The 1-reducible subgroups of index 36 are all conjugate. Such a 1-reducible group H is generated by E_1, $(E_3E_1^2E_4E_1^{-1})^2$ and E_2. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution $z = y_1$ spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in $A_6^{SL_3}$ is

$$\begin{aligned}
\{&id, E_3, E_4, E_3E_4, E_1E_3, E_3E_1E_3, E_1E_4, E_4E_1E_4, E_1^{-2}E_3, E_1^{-2}E_4, E_4E_1^{-1}E_3,\\
&E_1E_4E_1^{-1}E_3, E_1^2E_4E_1^{-1}E_3, E_1^2E_4E_1^{-1}, E_3E_1^2E_4E_1^{-1}, E_1^{-1}E_3E_1^2E_4E_1^{-1},\\
&E_1^{-1}E_3E_4, E_4E_1^{-1}E_3E_4, E_1E_4E_1^{-1}E_3E_4, E_2E_1^2E_4E_1^{-1}E_3, E_1^2E_3, E_1E_4E_3,\\
&E_4E_1E_4E_3, E_1^{-2}E_4E_3, E_4E_1^{-2}E_4E_3, E_2E_1E_4E_1^{-2}E_3, E_4E_1^2E_3E_4E_1^{-1},\\
&E_3E_1^{-2}E_3E_4, E_1E_3E_1^{-2}E_4, E_1^2E_3E_4, E_3E_1^{-1}E_3E_4, E_2E_4E_1^{-1}E_3,\\
&E_1^{-1}E_4E_1E_3, E_3E_1^2E_3E_4E_1^{-1}, E_1E_4E_1^{-1}E_3E_4E_1E_3, E_1E_4E_1^2E_4E_3E_1^{-1}\}
\end{aligned}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) = \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^6 - \sigma(z)^6)$$

The simple group A_5:

The group A_5 is generated by the matrices E_1, E_2 and E_3 given in section 4.1.2. The 1-reducible subgroups of index 6 are all conjugate. Such a 1-reducible group H is generated by E_1 and E_2. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution $z = y_1$ spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in A_5 is

$$\{id, E_3, E_1E_3, E_3E_1E_3, E_1^{-2}E_3, E_1^2E_3\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) = \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^2 - \sigma(z)^2)$$

In the given representation (cf. reference to F. Klein in Miller, Blichfeld and Dickson (1938), p. 254), the ring of invariants of $A_5 \subset SL(3, \mathbf{C})$ is generated by:

$$\begin{aligned}
I_1 &= y_1^2 + y_2y_3,\\
I_2 &= 8y_1^4y_2y_3 - 2y_1^2y_2^2y_3^2 + y_2^3y_3^3 - y_1(y_2^5 + y_3^5)\\
I_3 &= 320y_1^6y_2^2y_3^2 - 160y_1^4y_2^3y_3^3 + 20y_1^2y_2^4y_3^4 + 6y_2^5y_3^5\\
&\quad -4y_1(y_2^5 + y_3^5)(32y_1^4 - 20y_1^2y_2y_3 + 5y_2^2y_3^2) + y_2^{10} + y_3^{10},
\end{aligned}$$

and an invariant I_4 of degree 15. Using an *Ansatz* we can represent the coefficients of $P(Y)$ as polynomials in the invariants I_1, I_2 and I_3. We get the following representation for $P(Y)$:

$$\mathbf{Y}^{12} - 2I_1\,\mathbf{Y}^{10} + \frac{7}{5}I_1^2\,\mathbf{Y}^8 + \left(-\frac{12}{25}I_1^3 + \frac{2}{25}I_2\right)\mathbf{Y}^6 + \left(\frac{11}{125}I_1^4 - \frac{6}{125}I_1I_2\right)\mathbf{Y}^4$$
$$+\left(-\frac{26}{3125}I_1^5 + \frac{6}{625}I_1^2I_2 - \frac{64}{625\cdot 320}I_3\right)\mathbf{Y}^2 + \left(\frac{1}{3125}I_1^6 + \frac{1}{3125}I_2^2 - \frac{2}{3125}I_2I_1^3\right)$$

The group $A_5 \times C_3$:

This group is the direct product of the previous group with the center of $SL(3, \mathbf{C})$ generated by Z. The 1-reducible subgroups of index 6 are all conjugate. From the previous case we get a 1-reducible group $H \times C_3$ generated by E_1, E_2 and Z which has the same set of left coset representatives $\mathcal{T}$ in $A_5 \times C_3$ has H in A_5. Since C_3 consists of scalar multiplications, the common eigenvector $z = y_1$ of H given in the previous case will be a common eigenvector for $H \times C_3$.
The function z^3 of the PVE is left invariant by the normal subgroup C_3, which shows that $k(z^3)/k$ has Galois group A_5. Since z'/z is left fixed by H, the same holds for $(z^3)'/z^3$. From Corollary 1.4 and the bound for d computed in Theorem 4.1 we get that the minimal polynomial $P_3(Y)$ of z^3 over k is given by:

$$\prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^2 - \sigma(z^3)^2)$$

The following polynomial $P(Y)$ has z as a solution:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^6 - \sigma(z)^6)$$

and comparing degrees (cf. Theorem 4.1), we get that $P(Y)$ is the minimal polynomial of z.

The simple group G_{168}:

The group G_{168} is generated by the matrices S, T and R given in section 4.1.2. The 1-reducible subgroups of index 21 are all conjugate. Such a 1-reducible group H is generated by $S^{-2}RS$ and $RS^{-1}RTS$. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution

$$z \quad = \quad \left(\beta^5 + \beta^4 + \beta^2 + 1\right) y_1 + \left(\beta^5 + \beta\right) y_2 + y_3$$

spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in G_{168} is

$$\big\{id, T, T^{-1}, S^{-1}T^{-1}, S^{-2}, S^{-1}, SR, RSR, RS^{-1}RS^{-1}, RS^{-3}, S, TS^{-1}, S^{-1}T,$$
$$SRT, RSRT, RS^{-1}RS^{-1}T, STSR, RSTSR, RTS^2, TS^2, RSRT^{-1}\big\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma\in\mathcal{T}} (\mathbf{Y}^2 - \sigma(z)^2)$$

The group $G_{168} \times C_3$:

This group is the direct product of the previous group with the center of $SL(3,\mathbf{C})$ generated by Z. The 1-reducible subgroups of index 21 are **not** all conjugate. From the previous case we get a 1-reducible group $H \times C_3$ generated by $S^{-2}RS$, $RS^{-1}RTS$ and Z which has the same set of left coset representatives $\mathcal{T}$ has H in G_{168}. Since C_3 consists of scalar multiplications, the common eigenvector z of H given in the previous case will be a common eigenvector for $H \times C_3$.
As in the case $A_5 \times C_3$ we get the following minimal polynomial for z over k.

$$P(\mathbf{Y}) = \prod_{\sigma \in \mathcal{T}} (\mathbf{Y}^6 - \sigma(z)^6).$$

The group $H_{216}^{SL_3}$:

The group $H_{216}^{SL_3}$ is generated by the matrices S_1, T, V and UV given in section 4.1.2. The 1-reducible subgroups of index 9 are all conjugate. Such a 1-reducible group H is generated by $U^2V^{-1}S_1$ and $V^{-1}U^2S_1$. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution $-\varepsilon^3 y_1 + y_2$ spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in $H_{216}^{SL_3}$ is

$$\left\{id, V, V^2, V^{-1}, T^{-1}, S^{-1}, V^{-1}T^{-1}, VS^{-1}, V^2UT^{-1}S\right\}$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) = \prod_{\sigma \in \mathcal{T}} (\mathbf{Y}^9 - \sigma(z)^9)$$

The group $H_{72}^{SL_3}$:

The group $H_{72}^{SL_3}$ is generated by the matrices S_1, T, V and UVU^{-1} given in section 4.1.2. The 1-reducible subgroups of index 9 are all conjugate. Such a 1-reducible group H is generated by S_1TUVU^{-1}, $S_1TUVU^{-1}V$ and $S_1T^{-1}S_1^{-1}T$. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution $z = -y_2 + y_3$ spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in $H_{72}^{SL_3}$ is

$$\left\{id, T, T^{-1}, S, S^{-1}, TS, U^{-2}VU^{-1}, STS, U^{-1}V^{-1}U\right\},$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) = \prod_{\sigma \in \mathcal{T}} (\mathbf{Y}^3 - \sigma(z)^3)$$

The group $F_{36}^{SL_3}$:

The group $F_{36}^{SL_3}$ is generated by S_1, T and V of section 4.1.2. A 1-reducible group H is generated by $S^{-1}T$, STS and V^2T. If we denote $\{y_1, y_2, y_3\}$ the basis of the solution space corresponding to the above representation, then the solution $y_1 + y_2 + \omega y_3$ spanns a one dimensional invariant subspace of H.
A set of left coset representatives $\mathcal{T}$ of H in $F_{36}^{SL_3}$ is

$$\left\{id, V, V^2, V^{-1}, T^{-1}, V^{-1}T^{-1}\right\},$$

and the minimal polynomial of z is given by:

$$P(\mathbf{Y}) \quad = \quad \prod_{\sigma \in \mathcal{T}} (\mathbf{Y}^6 - \sigma(z)^6)$$

5. Computing a solution

Let $L(y) = 0$ be an equation of degree n with coefficients in k and finite primitive Galois group $\mathcal{G}(L) \subset SL(n, \mathbf{C})$. Then the decomposition of the coefficients of the minimal polynomial $P(\mathbf{Y})$ of a solution given in the previous section reduces the computation of the coefficients of $P(\mathbf{Y})$ to the computation of finitely many semi-invariants of $\mathcal{G}(L)$ and of the constants of the semi-invariants, which can be done using a Gröbner basis (for second order equation only gcd computation are needed). In fact, since the coefficients are invariants of $\mathcal{G}(L)$, only powers of the semi-invariants which are rational functions have to be computed. This reduces the computations to the computation of rational solutions of some symmetric power of $L(y) = 0$.

To compute the coefficients of $P(\mathbf{Y})$ one can proceed in the following way:

From the a representation of the minimal polynomial $P(Y)$ of a solution y of $L(y) = 0$ in terms of the semi-invariants of G one can compute the coefficients of $P(Y)$ in the following way:

i) Compute the set $\mathcal{L}$ of i-th symmetric powers $L^{\circledS i}(y)$ of $L(y) = 0$, where i belongs to the set of orders of the semi-invariants appearing in the decomposition of the coefficients of $P(Y)$.
ii) for each equation in $\mathcal{L}$, compute a solution f_i such that $f_i^j \in k$, where j belongs to the set of orders of the one dimensional characters of $\mathcal{G}(L)$ (cf. Lemma 1.6). In general the possible j's can be further restricted by looking at the decomposition of $P(Y)$ in terms of the semi-invariants of $\mathcal{G}(L)$ and noting that the coefficients of $P(Y)$ must be rational.
iii) For each possible value f_i (defined up to a constant c_i) of the invariants I_i obtained, replace I_i by $c_i \cdot f_i$ in $P(Y) = 0$. This gives a new polynomial $Q(Y) = 0$ whose coefficients are polynomials in the variables c_i over k.
iv) For $s \leq n$, using $Q(Y) = 0$, express the derivatives $Y^{(s)}$ of Y as a polynomial in Y and replace $Y^{(s)}$ by this value in $L(y) = 0$. This gives a polynomial $\overline{P}(Y) = 0$ whose coefficients are polynomials in the variables c_i over k.
v) compute the rest $R(Y)$ of the division of $\overline{P}(Y)$ by $Q(Y)$ and determine the constants c_i by setting all coefficients of $R(Y)$ equal to 0. For $i \geq 2$ this can be done using a Gröbner basis.
vi) If a non trivial solution set $\{c_i\}$ is found, then replacing c_i by these values in $Q(Y)$ gives the minimal polynomial of a solution of $L(y) = 0$.

The above method is based on the fact that the polynomial $Q(Y)$ is at least square free. If the cases are considered without knowing the group $\mathcal{G}(L)$, then one has to start with the cases where the polynomial to be constructed is of smallest degree (e.g. first the case $A_4^{SL_2}$, then $S_4^{SL_2}$ and then $A_5^{SL_2}$) and test if the resuling polynomial is square free. Another possibility would be to use the method given in Singer and Ulmer (1992) to determine the differential Galois group of $L(y) = 0$ and make sure that the assumption on $\mathcal{G}(L)$ is correct.

Example. We now apply the above method to compute the solutions of the differential equation

$$L(y) \;=\; \frac{d^2 y}{dx^2} + \left(\frac{3}{16x^2} + \frac{2}{9(x-1)^2} - \frac{3}{16x(x-1)}\right) y = 0$$

which is also studied in Kovacic (1986), p. 23 and Ulmer (1991), p 452. From the result of Kovacic (1986) we know that $\mathcal{G}(L) \cong A_4^{SL_2}$. This could also be computed using the result of Singer and Ulmer (1992) by showing that $L^{\circledS 2}(y)$ is irreducible and $L^{\circledS 3}(y)$ is reducible.

In Section 4.3.1 we decomposed the coefficients of $P(Y)$ and noted that the only semi-invariants present are I_1, a semi-invariant of degree 4, and I_2, a semi-invariant of degree 6. Furthermore, one can see from the form of the coefficients of $P(Y)$ (or from the orders of the associated characters) that ${I_1}^3$ and I_2 are rational. To compute I_1 we compute the 4-th symmetric power of $L(y) = 0$:

$$\begin{aligned} L^{\circledS 4}(y) \;=\; & \frac{d^5 y}{dx^5} \\ & + \frac{5(32x^2 - 27x + 27)}{36x^2(x-1)^2}\,\frac{d^3 y}{dx^3} \\ & - \frac{5(64x^3 - 81x^2 + 135x - 54)}{24x^3(x-1)^3}\,\frac{d^2 y}{dx^2} \\ & + \frac{5(1760x^4 - 2970x^3 + 6615x^2 - 5103x + 1458)}{324x^4(x-1)^4}\,\frac{dy}{dx} \\ & - \frac{5(1792x^5 - 3780x^4 + 10395x^3 - 11718x^2 + 6561x - 1458)}{324x^5(x-1)^5}\,y \\ \;=\; & 0\,. \end{aligned}$$

We must find a solution y of this equation such that $y^3 \in \mathbf{C}(x)$. Since only ${I_1}^3$ is needed, it is enough to compute the rational solution y^3 of $L^{\circledS 12}(y) = 0$ (cf. Lemma 1.6). To compute y one can either use the algorithm described at the begining of Section 3.2 or more simply proceed as follows: Let

$$y \;=\; \left(P(x)\prod_i (x-\alpha_i)^{n_i}\right)^{1/3},$$

where $P(x)$ is a polynomial, $\{\alpha_i\}$ are the singular points of $L^{\circledS 4}(y)$ and n_i are non-negative integers. This implies that for each i, $n_1/3$ is an exponent at α_i and that the exponent at infinity is $\frac{-1}{3}(deg(P) + \sum n_i)$. Checking the possibilities shows that $P(x)$ must be constant and that I_2 must be a constant multiple of $x(x-1)^{4/3}$ or $x(x-1)^{5/3}$.

From a similar computation we get the solution $x^2(x-1)^2$ for $L^{\circledS 6}(y) = 0$ (see e.g. Ulmer (1991)).

We set $I_1 = c_1 x(x-1)^{4/3}$ and $I_2 = c_2 x^2(x-1)^2$ in $P(Y)$ and get a polynomial $Q(Y)$ whose coefficients are polynomials in c_1 and c_2. Using $Q(Y) = 0$ we write Y' and Y'' as a fraction of polynomials in Y and substitute those in $L(y) = 0$. The numerator of the rational function in Y obtained in this way is a polynomial $\overline{P}(Y)$ having a common solution with $Q(Y) = 0$. The pseudo remainder $R(Y)$ of $\overline{P}(Y)$ and $Q(Y) = 0$ is a polynomial in Y of degree at most 23 which must be zero. Thus all coefficients of $R(Y)$ must be zero, which gives a set of polynomials in c_1, c_2 and x. Equating coefficients of powers of x to zero, gives a set of polynomials in c_1 and c_2. We now can compute a Gröbner basis to find the constants c_1 and c_2 and get:

$$[432c_1^2c_2^4 + c_1^4, (-48\omega^3 + 72\omega^2 - 192\omega + 84)c_1^2c_2^4 + c_1^3c_2^2]$$

In order for $Q(Y) = 0$ to be an irreducible polynomial we must have $c_1 \neq 0$ and $c_2 \neq 0$. Setting $c_2 = 1$ we get $c_1 = 48\omega^3 - 72\omega^2 + 192\omega - 84$, which we also write $\sqrt{-432}$. This gives the following minimal polynomial of a solution of $L(y) = 0$:

$$\begin{aligned}
&\mathbf{Y}^{24} + \left(48\omega^3x^2(x-1)^2\right)\ \mathbf{Y}^{18}\\
&+\Big((-780\omega^2 + 780\omega - 210)\left(\sqrt{-432}\right)^3 x^3(x-1)^4\\
&\qquad +(-6144\omega^3 + 9216\omega^2 - 4608\omega + 768)x^4(x-1)^4\Big)\ \mathbf{Y}^{12}\\
&+\Big((6816\omega^3 + 2496\omega^2 - 7488\omega + 2496)\left(\sqrt{-432}\right)^3 x^5(x-1)^6\\
&\qquad + (167936\omega^3 - 565248\omega^2 + 442368\omega - 106496)x^6(x-1)^6\Big)\ \mathbf{Y}^6\\
&+(780\omega^2 - 780\omega + 209)\left(\sqrt{-432}\right)^6 x^6(x-1)^8
\end{aligned}$$

□

The above shows that for the Tetrahedral group (as for any second order equation with primitive unimodular Galois group) the computation of the minimal polynomial of a solution is reduced to the computation of two semi-invariants (i.e. solutions of symmetric powers whose power is rational) and two constants. In fact one constant can be choosen arbitrary so that only one constant remains to be computed. This shows that for second order equations the constant can be computed using only gcd computations. In the paper of Fuchs it is shown that using the Hessian $H(I_1)$ and Jacobian $J(I_1, H(I_1))$ of an invariant I_1 of lowers degree, one gets the other invariants. For the tetrahedral group $A_4^{SL_2}$ one has (Miller, Blichfeld and Dickson (1938), p. 226)

$$I_3 = \frac{H(I_1)}{48\sqrt{-3}} = \frac{1}{48\sqrt{-3}} \begin{vmatrix} \dfrac{\partial^2 I_1}{\partial y_1 \partial y_1} & \dfrac{\partial^2 I_1}{\partial y_1 \partial y_2} \\ \dfrac{\partial^2 I_1}{\partial y_2 \partial y_1} & \dfrac{\partial^2 I_1}{\partial y_2 \partial y_2} \end{vmatrix}$$

$$I_2 = \frac{J(I_1, I_3)}{-32\sqrt{-3}} = \frac{1}{-32\sqrt{-3}} \begin{vmatrix} \dfrac{\partial I_1}{\partial y_1} & \dfrac{\partial I_1}{\partial y_2} \\ \dfrac{\partial I_3}{\partial y_1} & \dfrac{\partial I_3}{\partial y_2} \end{vmatrix}$$

Fuchs then shows how, as a function in x, for a given differential equation $L(y) = y'' - r(x)y = 0$ the Hessian χ_1 and Jacobian φ of an invariant χ of minimal degree can be written as a polynomial in $\chi(x)$ and derivatives of $\chi(x)$. He proves the following relations

(Fuchs (1878), pp. 21-22):

$$\begin{aligned}\chi_1(x) &= c\left[\left(\frac{d\log\chi(x)}{dx}\right)^2 + 4\frac{d^2\log\chi(x)}{dx^2} - 16r\right]\chi(x)^2\\ \varphi(x) &= \sqrt{-\chi_1(x)^3 + 64\chi(x)^3},\end{aligned}$$

where c is a constant that can be computed (Fuchs (1878), p. 22). If we let χ be the invariant I_1 used in Miller, Blichfeld and Dickson (1938), these formulas also yield expressions for I_2 and I_3.

This reduces the above to the computation of one semi-invariant of lowest degree and one constant.

We now compute an example of a third order differential equation:

Example. We now apply the above method to compute the solutions of the differential equation

$$L(y) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{21(x^2-x+1)}{25x^2(x-1)^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{21(-2x^3+3x^2-5x+2)}{50x^3(x-1)^3}\,y\,,$$

which is irreducible and has Galois group A_5 (cf. Singer and Ulmer (1992), section 5).

Since the invariants of $A_5 \subset SL(3,\mathbf{C})$ are of order 2, 6 and 10, we have to compute rational solutions of the second, 6-th and 10-th symmetric powers of $L(y)=0$. The equation $L^{ⓢ 2}(y)=0$ has no non trivial rational solution. The subspaces of rational solutions of $L^{ⓢ 6}(y)=0$ and $L^{ⓢ 10}(y)=0$ each are one dimensional and generated by $x^4(x-1)^4$ and $x^6(x-1)^6(x^2-x+1)$ respectively.

We set $I_1=0$, $I_2=c_2x^4(x-1)^4$ and $I_3=c_3x^6(x-1)^6(x^2-x+1)$ in $P(Y)$ and get a polynomial $Q(Y)$ whose coefficients are polynomials in c_2 and c_3. Using $Q(Y)=0$ we write Y' and Y'' as a fraction of polynomials in Y and substitute those in $L(y)=0$. The numerator of the rational function in Y obtained this way is a polynomial $\overline{P}(Y)$ having a common solution with $Q(Y)=0$. The pseudo remainder $R(Y)$ of $\overline{P}(Y)$ and $Q(Y)=0$ is a polynomial in Y of degree at most 11 which must be zero. Thus all coefficients of $R(Y)$ must be zero, which gives a set of polynomials in c_2 and c_3. We now can compute a Gröbner basis to find the constants c_2 and c_3 and get:

$$[-\frac{1}{256}c_2^3c_3^3 + c_2^8, -\frac{1}{256}c_2c_3^4 + c_2^6c_3, -\frac{1}{256}c_3^5 + c_2^5c_3^2]$$

Since $I_1=0$, we must have $c_2 \neq 0$ and $c_3 \neq 0$ in order for $Q(Y)=0$ to be an irreducible polynomial with Galois group A_5. Setting $c_3=1$ we get $c_2^5 = 1/256$. This gives the following minimal polynomial of a solution of $L(y)=0$:

$$\mathbf{Y}^{12} + \frac{2}{25}\frac{x^4(x-1)^4}{2\sqrt[5]{8}}\mathbf{Y}^6 - \frac{64x^6(x-1)^6(x^2-x+1)}{625\cdot 320}\mathbf{Y}^2 + \frac{1}{3125}\left(\frac{x^4(x-1)^4}{2\sqrt[5]{8}}\right)^2$$

□

The above method turned out to be very efficient. The computation of a minimal polynomial (assuming known the decomposition of the minimal polynomial in term of the invariants) could be done in both cases in much less than 1 CPU hour on an IBM RISC 6000 with 64MB of main memory. An attempt to compute the minimal polynomial of the logarithmic derivative of a solution of the above second order equation (which is used as an example in Kovacic (1986)) using the implementation in MAPLE of the Kovacic algorithm on a SUN 4 with 20MB of main memory did not give any result within 10 CPU hours.

We note, that the above computations can always be done in an algebraic extension $\mathbf{Q}(\alpha)$ of $\mathbf{Q}$ containing the finite singular points of $L(y) = 0$, the entries of the matrices in the transversal $\mathcal{T}$, the coordinates of the common eigenvector of the choosen 1-reducible subgroup H and the coefficients of the invariants of $\mathcal{G}(L)$. Thus no additional algebraic extension is needed at runtime. However, as show in both examples, the coefficients of $P(Y) = 0$ (i.e. the result of the Gröbner basis computation) do not in general belong to $\mathbf{Q}(\alpha)(x)$. Note that the final Gröbner basis computation will yield polynomials whose roots generate an extension F of $\mathbf{Q}(\alpha)$ such that the coefficients belong to $F(x)$.

A. The character tables of the subgroups of index ≤ 6 of the Schur representation group of A_5

We denote G the Schur representation group of A_5 wihch is of order 120. The table of the irreducible characters of degree ≤ 3 of G produced by Cayley is:

class	1	2	3	4	5	6	7	8	9
conj	1	1	20	30	12	12	20	12	12
order	1	2	3	4	5	5	6	10	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	-1	0	z_1	$-1-z_1$	1	$-z_1$	$1+z_1$
χ_3	2	-2	-1	0	$-1-z_1$	z_1	1	$1+z_1$	$-z_1$
χ_4	3	3	0	-1	$-z_1$	$1+z_1$	0	$-z_1$	$1+z_1$
χ_5	3	3	0	-1	$1+z_1$	$-z_1$	0	$1+z_1$	$-z_1$

where $z_1 = -1 - \omega^2 - \omega^3$ and $\omega = e^{2\pi i/5}$.

The subgroups of G of index ≤ 6 are of index 5 or 6.

The subgroups of index 5 are all congugate and the table of the irreducible characters of degree ≤ 3 of of such a group produced by Cayley is:

class	1	2	3	4	5	6	7
conj	1	1	4	4	6	4	4
order	1	2	3	3	4	6	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	$-1-J$	J	1	$-1-J$	J
χ_3	1	1	J	$-1-J$	1	J	$-1-J$
χ_4	2	-2	-1	-1	0	1	1
χ_5	2	-2	$-J$	$1+J$	0	J	$-1-J$
χ_6	2	-2	$1+J$	$-J$	0	$-1-J$	J
χ_7	3	3	0	0	-1	0	0

where $J = e^{2\pi i/3}$.

The subgroups of index 6 are all congugate and the table of the irreducible characters of degree ≤ 3 of of such a group produced by Cayley is:

class	1	2	3	4	5	6	7	8
conj	1	1	5	5	2	2	2	2
order	1	2	4	4	5	5	10	10
χ_1	1	1	1	1	1	1	1	1
χ_2	1	-1	$-I$	I	1	1	-1	-1
χ_3	1	1	-1	-1	1	1	1	1
χ_4	1	-1	I	$-I$	1	1	-1	-1
χ_5	2	-2	0	0	z_1	$-1-z_1$	$-z_1$	$1+z_1$
χ_6	2	-2	0	0	$-1-z_1$	z_1	$1+z_1$	$-z_1$
χ_7	2	2	0	0	z_1	$-1-z_1$	z_1	$-1-z_1$
χ_8	2	2	0	0	$-1-z_1$	z_1	$-1-z_1$	z_1

where $I = e^{2\pi i/4}$, $z_1 = -1 - \omega^2 - \omega^3$ and $\omega = e^{2\pi i/5}$.

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1.17 Symbolic analysis of second-order ordinary differential equations with polynomial coefficients

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Symbolic analysis of second-order ordinary differential equations with polynomial coefficients

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Abstract

The singularity structure of a second-order ordinary differential equation with polynomial coefficients often yields the type of solution. If the solution is a special function that is studied in the literature, then the result is more manageable using the properties of that function. It is straightforward to find the regular and irregular singular points of such an equation by a computer algebra system. However, one needs the corresponding indices for a full analysis of the singularity structure. It is shown that the θ-operator method can be used as a symbolic computational approach to obtain the indicial equation and the recurrence relation. Consequently, the singularity structure which can be visualized through a Riemann P-symbol leads to the transformations that yield a solution in terms of a special function, if the equation is suitable. Hypergeometric and Heun-type equations are mostly employed in physical applications. Thus only these equations and their confluent types are considered with SageMath routines which are assembled in the open-source package symODE2.

Keywords: Ordinary differential equations, symbolic analysis, special functions

1 Introduction

Mathematical analysis of physical problems generally requires the methods of solving ordinary differential equations (ODEs). Numerical solutions of the initial and boundary value problems are often sufficient to give a concrete idea about the behavior of the system, whereas the analytic solution of an ODE in closed form, especially in terms of special functions may have more importance than constituting an exact solution. In some systems, the type of solution has the potential to reveal the symmetries of the system. For example, the emergence of the hypergeometric function in the solutions may indicate the conformal symmetry [1–4].

The applications of the general and confluent hypergeometric equation have dominated the 20th-century [5]. Although the mathematical theory of the Heun equation and its confluent forms are far from complete, these functions have been known and employed by experts in the area for many years [6,7]. Besides, the number of their applications increased substantially after the implementation of the Heun functions in the computer algebra system Maple [8] in 2005 [9–12]. The implementation of the Heun type functions in Mathematica in 2020 is another big leap for the Heun community [13,14].

Most of the free and open-source computer algebra systems and packages involve the solutions of the hypergeometric equation, its confluent form, and related equations at least numerically [15–18]. However, the symbolic solutions of the Heun type equations are defined only in Maple and Mathematica which are commercial systems. The numerical evaluation of the general Heun and singly confluent Heun functions are studied by Motygin using the freely available

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GNU Octave language [19–21]. The recent work by Giscard and Tamar also deals with the numerical calculation of the Heun type functions [22]. The work is in progress for numerical treatment of the Heun type equations under Python, and the initial results of the work are presented in [23].

A historical review on computer algebra systems is given in [24] along with gravitational applications. The general methods for obtaining symbolic solutions to ODEs and a review of comprehensive literature before the year 2000 are given in [25]. Among the papers released prior to 2000, we should cite the seminal paper of Kovacic [26] and Duval and Loday-Richaud's work in which the hypergeometric and Heun type equations are studied in particular [27]. Among the papers on the solutions of ODEs in terms of special functions that are published after 2000, we can cite Bronstein and Lafaille [28], Chan and Cheb-Terrab [29], and van Hoeij with his collaborators [30–32].

SageMath is a free and open-source general-purpose computer algebra system licensed under the GPL [15]. SageMath offers a Python-based language and it is built on many open-source packages such as Maxima, SciPy, NumPy, and matplotlib. A variety of modules are present for many areas such as differential geometry and tensor calculus [33, 34] that enable calculations on quantum field theory and general relativity [35].

We will focus on the singularity analysis and symbolic solutions of the hypergeometric and Heun type equations using the SageMath system and present the open-source package symODE2 which allows users to analyze these equations symbolically without using a commercial program.

This paper is organized in the following way: In the second section, we present the code structure of our package and give a comparison with the existing codes. In the third, fourth, and fifth sections, we briefly explain the analysis of the singularity structure, series solutions, and change of variables for a second order ODE with polynomial coefficients, respectively. In the sixth section, we explain our approach for finding the symbolic solutions of the hypergeometric and Heun type equations. Section seven involves our conclusions and we describe the standard forms of the equations in the appendix.

2 The code structure of the symODE2 package

The symODE2 package is written under SageMath 9.1 using a laptop computer with Intel(R) Core(TM) i7-6500U CPU @ 2.50GHz and 8 GB memory. The operating system is Windows 10 Enterprise ver.1909. It is also tested under SageMath 9.2.

The package consists of two main parts:

- `ode2analyzer.sage` for the general analysis and,
- `hypergeometric_heun.sage` for the symbolic solutions of the equations.

`hypergeometric_heun.sage` calls the routines defined in `ode2analyzer.sage` when needed.

We suggest the user put these two files in the same directory. The parts of the package and a sample worksheet can be downloaded from the address [36]:

`https://github.com/tbirkandan/symODE2`

The User Manual which can be found at the same address contains detailed explanations of the routines and the analysis of the cases in the sample SageMath worksheet.

2.1 General analysis (ode2analyzer):

The first part, `ode2analyzer` contains the routines that

– finds the singularity structure of the input ODE. The output is an array that involves the locations of the singularities, indices of the regular singularities, and the ranks of the irregular singularities.

– finds the indices and/or the recurrence relation with respect to a regular singular point using the θ-operator method which will be defined below.

– performs a change of variables.

– finds the normal form of a second-order ODE. For an input in the form (1), the output is in the form (4).

2.2 Symbolic solutions of special ODEs (hypergeometric_heun):

The second part, `hypergeometric_heun` contains the routines that

– finds the type of the ODE using its singularity structure and solves it using the routines defined below.

– uses a change of variables list in order to bring the input ODE into a special form that is recognized in the package.

– solves a hypergeometric equation.

– solves a confluent hypergeometric equation.

– solves a general Heun equation.

– solves a (singly) confluent Heun equation.

– solves a double confluent Heun equation.

– solves a biconfluent Heun equation.

– solves a triconfluent Heun equation.

2.3 Comparison with the existing codes

symODE2 is the first freely available, open-source package that allows a symbolic treatment of the Heun-type equations and it is written on SageMath which is also a freely available, open-source program. Therefore, symODE2 provides a free alternative to the commercial programs Maple and Mathematica when the problem is expressing the solutions of these equations symbolically.

symODE2 uses the internal functions of SageMath for the numerical analysis of the hypergeometric-type equations. The numerical treatment of the Heun-type equations is not implemented in SageMath. However, this work is in progress under Python in order to reach a larger community, and SageMath will be able to use the Python code directly [23]. The commercial programs Maple and Mathematica provide numerical operations as well as symbolics for both hypergeometric and Heun-type equations. Maple and Mathematica also provide the derivatives of the Heun-type functions, unlike symODE2.

The Heun-type equations are implemented in Mathematica in 2020. Figure (2) that will be given in Section (4) below is created using symODE2 and it can be used as a comparison with the Mathematica results as explained.

The Maple implementation of the Heun-type equations goes back to 2005 and many of the Heun-related applications in the literature published after this year are likely to be done by employing this program. The literature-based cases given in the sample SageMath worksheet [36] and described in the User Manual show that the results obtained by symODE2 agree with the ones in the literature.

3 Singularity analysis

A second order ODE can be written in the form,

$$f_1(x)\frac{d^2y(x)}{dx^2} + f_2(x)\frac{dy(x)}{dx} + f_3(x)y(x) = 0. \tag{1}$$

From this point on, the coefficient functions $f_{1,2,3}(x)$ will be regarded as polynomials in the parameter x, and the x-dependence of the function y will be omitted.

We can denote $p(x) = f_2(x)/f_1(x)$ and $q(x) = f_3(x)/f_1(x)$ to obtain,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{2}$$

If the functions $p(x)$ and $q(x)$ are analytic at a point $x = x_0$, then x_0 is an "ordinary point" for this ODE.

The points that make $p(x)$ or $q(x)$ divergent are called the singular points or singularities of the ODE. If x_* is a singular point and if $(x-x_*)p(x)$ and $(x-x_*)^2q(x)$ are both analytic at $x = x_*$, then x_* is called a "regular singular point". Otherwise, the singular point is "irregular" [37]. The singularity behavior at $x \to \infty$ can be analyzed by performing the transformation $\tilde{x} = 1/x$ and checking the behavior at $\tilde{x} = 0$. If all the singular points of an ODE are regular, then the ODE is said to be a "Fuchsian equation".

If the singularity at $x = x_*$ is irregular but $(x-x_*)^k p(x)$ and $(x-x_*)^{2k} q(x)$ are analytic where k is the least integer satisfying this condition, then the irregular singular point at $x = x_*$ has a rank $(k-1)$ [37]. Consequently, a regular singular point is of rank-0 as $k = 1$.

We can write the equation (2) in normal form in which the coefficient of the first derivative vanishes. We define $y(x) = g(x)w(x)$ and for

$$g(x) = e^{-\frac{1}{2}\int^x p(x')dx'}, \tag{3}$$

we obtain

$$\frac{d^2w}{dx^2} + q'(x)w = 0, \tag{4}$$

where

$$q'(x) = q(x) - \frac{1}{2}\frac{dp(x)}{dx} - \frac{p(x)^2}{4}. \tag{5}$$

Our code transforms the input equation into the normal form in order to deal with the singular points of only one function, namely $q'(x)$. Although the package involves a routine for general change of variables, the results of the transformation $\tilde{x} = 1/x$ is included in the function that finds the singularity behavior as the analysis of $x \to \infty$ case is inevitable.

4 Series solution around a regular singular point

An indicial equation can be defined for a regular singular point [7]. For a finite regular singular point x_* we have,

$$r(r-1) + p_*r + q_* = 0, \tag{6}$$

where p_* and q_* are the residues of $p(x)$ and $(x-x_*)q(x)$ at $x = x_*$, respectively. For the regular singularity at infinity, one can write

$$r(r+1) - p_\infty r + q_\infty = 0, \tag{7}$$

where p_∞ and q_∞ are the residues of $p(x)$ and $xq(x)$ at $x \to \infty$, respectively.

The roots $r = r_1$ and $r = r_2$ of the indicial equation are called the "indices" or "characteristic exponents", or "Frobenius exponents" of the corresponding regular singularity [7]. The sum of the all $(2n)$ indices corresponding to all (n) regular singular points in a Fuchsian equation should be equal to $n-2$ [38].

One can find at least one series solution, the "Frobenius solution" of the form,

$$y = \sum_{n=0}^{\infty} C_n (x - x_*)^{n+r}, \tag{8}$$

near a finite regular singular point x_*, r being a characteristic exponent associated with x_*. The details on the second solution and the solution around infinity can be found in [7]. We substitute the solution (8) into equation (1) to obtain a recurrence relation among the coefficients C_n. For example, the hypergeometric equation admits a two-term recurrence relation which connects C_n with C_{n-1}, while the Heun equation has a three-term recurrence relation connecting C_n, C_{n-1} and C_{n-2}.

The θ-operator method yields the indicial equation and the recurrence relation for a regular singular point with less effort than the formal Frobenius series calculation. Following [39], we define $D = \frac{d}{dx}$ and,

$$\theta = x\frac{d}{dx} = xD, \tag{9}$$

$$\theta(\theta - 1) = x^2 D^2. \tag{10}$$

Similarly, we have

$$x^n D^n = (-1)^n (-\theta)_n, \tag{11}$$

where

$$(-\theta)_n = \prod_{j=0}^{n-1} (-\theta + j), \tag{12}$$

is the generalized factorial notation. A general n^{th} order ODE can be written as

$$[a_0(x)D^n + a_1(x)D^{n-1} + ... + a_n(x)]y = 0. \tag{13}$$

Using eq.(11), we can write the equation in the θ-form, namely

$$[A_0(\theta) + xA_1(\theta) + x^2 A_2(\theta) + ... + x^m A_m(\theta)]y = 0, \tag{14}$$

if the coefficients of the original equation are polynomials in x. Here, $A_{0,1,...,m}(\theta)$ are polynomials in θ.

Let us assume that $x = 0$ is a regular singular point and seek a series solution in the form (8) for $x_* = 0$. It is known that any polynomial expression in θ operating on x^n yields the same polynomial in n times x^n, thus $P(\theta)x^n = P(n)x^n$ [39]. Using this property, we get

$$\sum_{n=0}^{\infty} C_n [A_0(n+r)x^{n+r} + A_1(n+r)x^{n+r+1} + ... + A_m(n+r)x^{n+r+m}] = 0. \tag{15}$$

In order to have an arbitrary C_0, the indicial equation is obtained as $A_0(r) = 0$. Let us shift the indices in the sum,

$$\sum_{n=0}^{\infty} [C_n A_0(n+r) + C_{n-1}A_1(n+r-1) + ... + C_{n-m}A_m(n+r-m)]x^{n+r} = 0, \tag{16}$$

and the recurrence relation reads

$$C_n A_0(n+r) + C_{n-1}A_1(n+r-1) + ... + C_{n-m}A_m(n+r-m) = 0. \tag{17}$$

For a regular singular point x_* other than zero, one should make the transformation $x' = x - x_*$ and do the calculation for x'. For the details of the θ-operator method, proofs and examples, we refer the reader to [39].

In our code, the function that finds the indices and the recurrence relation for a given regular singular point is based on the θ-operator method.

The hypergeometric function is implemented in SageMath as `hypergeometric([a,b],[c],x)`. Therefore we can verify our recurrence relation graphically as seen in Figure (1).

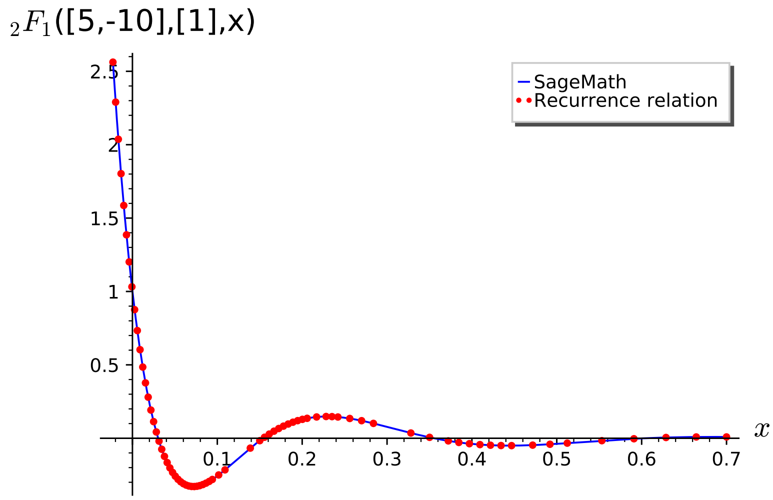

Figure 1: Plot of the hypergeometric function found by our recurrence relation result and the internal command of SageMath.

We should note that finding a solution using the recurrence relation generally requires more effort than presented here. Methods such as analytic continuation should be carefully applied in order to deal with the circle of convergence of the series solution [20, 21].

We can also plot the series solution for the general Heun equation in Figure (2) using a similar code with an array of plots and compare it with the plot given in the Wolfram Blog post [14] to see that they are similar. The details of this analysis can be found in the User Manual and the sample SageMath session [36].

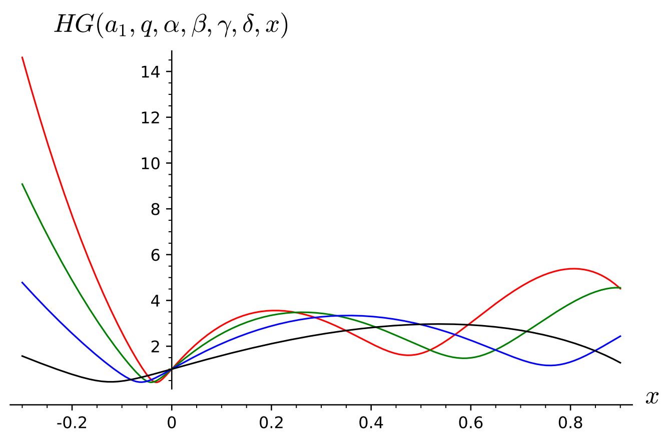

Figure 2: Plot of the general Heun function similar to the one given in [14].

5 Change of variables for a second order ODE

For an ODE of the form (1), we can change the independent variable from x to $t(x)$ using

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt}, \tag{18}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dt}{dx}\right)^2 \frac{d^2y}{dt^2} + \frac{d^2t}{dx^2}\frac{dy}{dt}. \tag{19}$$

As an example, let us make the change $x \to \sqrt{t}$. Then $t = x^2$ and

$$\frac{dy}{dx} = 2\sqrt{t}\frac{dy}{dt}, \tag{20}$$

$$\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}, \tag{21}$$

and the ODE becomes

$$4tf_1(t)\frac{d^2y(t)}{dt^2} + 2\big[f_1(t) + \sqrt{t}f_2(t)\big]\frac{dy(t)}{dt} + f_3(t)y(t) = 0. \tag{22}$$

6 Hypergeometric and Heun-type equations

The code attempts to find symbolic solutions of some special ODES, namely, the hypergeometric equation, the Heun equation, and their confluent forms. The analysis of the equations is based on the singularity structure. Locations of the singularities and corresponding characteristic exponents play a major role in the method. Using particular substitutions and transformations, the input equation is brought into a standard form that can be recognized by the routines.

A basic example of our approach can be given by using the hypergeometric equation. The Riemann P-symbol for the standard form of the hypergeometric equation (29) is

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix}. \tag{23}$$

Here, the locations of the singular points are given in the first row and each column exhibits the characteristic exponents of the corresponding singular points as found by our code in the sample worksheet.

Let us study a more general second order ODE with three regular singular points (x_1, x_2, x_3) and corresponding indices $(c_{i1}, c_{i2},\ i = 1, 2, 3)$, namely,

$$P\begin{Bmatrix} x_1 & x_2 & x_3 & \\ c_{11} & c_{21} & c_{31} & x \\ c_{12} & c_{22} & c_{32} & \end{Bmatrix}. \tag{24}$$

The substitution,

$$u(x) \to \begin{cases} \left(\frac{x-x_1}{x-x_3}\right)^{c_{11}}\left(\frac{x-x_2}{x-x_3}\right)^{c_{21}}u(x), & \text{if } x_3 \neq \infty, \\ (x-x_1)^{c_{11}}(x-x_2)^{c_{21}}u(x), & \text{if } x_3 = \infty, \end{cases} \tag{25}$$

brings the P-symbol in the form

$$P\begin{Bmatrix} x_1 & x_2 & x_3 & \\ 0 & 0 & c_{31}+c_{11}+c_{21} & x \\ c_{12}-c_{11} & c_{22}-c_{21} & c_{32}+c_{11}+c_{21} & \end{Bmatrix}. \tag{26}$$

The transformation,

$$x \to \frac{(x_2-x_3)(x-x_1)}{(x_2-x_1)(x-x_3)}, \tag{27}$$

moves the locations of the singular points from (x_1, x_2, x_3) to $(0, 1, \infty)$ as in the standard form of the hypergeometric equation [42]. Now we have,

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & c_{31}+c_{11}+c_{21} & \frac{(x_2-x_3)(x-x_1)}{(x_2-x_1)(x-x_3)} \\ c_{12}-c_{11} & c_{22}-c_{21} & c_{32}+c_{11}+c_{21} & \end{matrix}\right\}, \tag{28}$$

which corresponds to the standard form of the hypergeometric equation. We note that the sum of the indices is not changed. A similar analysis of the general Heun equation can be found in [42].

Our approach is similar to this example for other equations: we change the indices of the regular singular points and move the locations of the singular points in order to obtain a standard form. After reaching the standard singularity structure of an equation that can be recognized by the code, the parameters are read either from the characteristic exponents or by matching the final form of the input equation with the standard equation in their normal forms.

The polynomial coefficients of the normal forms are matched in the confluent cases. The parameters of these ODEs can be found by solving single equations, i.e. the code finds some parameters by solving algebraic equations that depend only on one parameter. The rest of the parameters are found by substitution. For the Fuschian ODEs, the parameters are read from the characteristic exponents, e.g. the non-zero exponent of the singular point at zero in the hypergeometric equation yields $1 - c$, etc.

We find the parameters with this method and use the Maple or Mathematica forms of the solutions to substitute these parameters. The standard forms of the equations are given in the Appendix.

The hypergeometric equation has three pairs of Frobenius solutions around its three regular singular points and these solutions can be transformed into other solutions via specific transformations [41]. The number of all solutions of the hypergeometric equation is 24. The number of total solutions is 192 for the general Heun equation [43]. The user of our code may need to use some transformations or function identities in order to obtain the desired form of the solution [38, 41, 43].

The results of the hypergeometric and confluent hypergeometric equations are numerically usable as these functions are defined in SageMath. However, the numerical solutions of the Heun-type functions are not defined. The numerical solutions of the general Heun and (singly) confluent Heun functions are defined by Motygin for GNU Octave/MATLAB [20, 21]. GNU Octave/MATLAB commands can be run in a SageMath session. However, this procedure is not straightforward and it is beyond the scope of this work. The method given in [22] can also be employed in order to obtain numerical results. An optimized implementation of Giscard and Tamar's method is in progress, and the initial results of this work are presented in [23].

Several applications with symODE2 can be found in the User Manual and the sample worksheet of the code [36] based on the results obtained in [5, 41, 44–53].

7 Conclusion

We proposed an open-source package under SageMath for symbolic analysis of second-order ordinary differential equations with polynomial coefficients. Our approach was based on the singularity structure, namely, the locations of the singularities, corresponding characteristic exponents, and the ranks of the irregular singular points of the equation.

The singularity structure, indices, and recurrence relations associated with the regular singular points, and symbolic solutions of the hypergeometric equation, Heun equation, and their confluent forms could be found using the package.

Using particular substitutions and transformations, the singularity structure of the input equation was brought in a standard form that could be recognized by the routines. After being reached the standard singularity structure of an equation, the parameters were obtained either using the characteristic exponents or by matching the final form of the input equation with the normal form of the standard equation.

As they were defined in SageMath, the results of the hypergeometric and confluent hypergeometric equations were numerically usable, unlike the Heun-type functions.

We presented that our code worked properly with a number of tests. We also mentioned that some transformations, substitutions, or identities might be needed in order to reach the results of the literature.

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A Appendix: Hypergeometric and Heun type equations in DLMF, Maple, Mathematica, and symODE2

The standard forms of the equations may be defined differently in the literature and in the computer algebra systems. Here, we consider DLMF, a well-known library of mathematical functions [41], and two computer algebra systems, Maple and Mathematica which can work with hypergeometric and Heun type functions. We also note the standard forms of the equations used in the symODE2 code.

In the Appendix of the User Manual, we also give a lists of correspondence of the parameters in these programs [36].

A.1 Hypergeometric equation

In DLMF [41],

$$x(1-x)\frac{d^2y}{dx^2} + \left[c - (a+b+1)y\right]\frac{dy}{dx} - aby = 0. \tag{29}$$

In Maple [54],

$$x(x-1)\frac{d^2y}{dx^2} + [(a+b+1)x - c]\frac{dy}{dx} + aby = 0, \tag{30}$$

with solution `hypergeom([a,b],[c],x)`.

In Mathematica [55],

$$-\left(x(x-1)\frac{d^2y}{dx^2} + [(a+b+1)x - c]\frac{dy}{dx} + aby\right) = 0, \tag{31}$$

with solution `Hypergeometric2F1[a,b,c,z]`.

All equations coincide and they have three regular singularities located at $\{0,\, 1,\, \infty\}$. symODE2 uses this form of the equation as well.

A.2 Confluent hypergeometric equation

In DLMF [41],

$$x\frac{d^2y}{dx^2} + (b-x)\frac{dy}{dx} - ay = 0. \tag{32}$$

In Maple [54],

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0, \tag{33}$$

with solution `hypergeom([a],[c],x)`.

In Mathematica [56],

$$x\frac{d^2y}{dx^2}+(b-x)\frac{dy}{dx}-ay=0, \tag{34}$$

with solution `HypergeometricU[a,b,x]`.

All have one regular singularity located at $\{0\}$ and one irregular singularity of rank-1 at $\{\infty\}$. The equations coincide and symODE2 also uses this form of the equation.

A.3 (General) Heun equation

In DLMF [41],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-a}\right)\frac{dy}{dx}+\frac{\alpha\beta x-q}{x(x-1)(x-a)}y=0, \tag{35}$$

and in Maple [57],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-a}\right)\frac{dy}{dx}+\frac{\alpha\beta x-q}{x(x-1)(x-a)}y=0, \tag{36}$$

where $\epsilon=\alpha+\beta+1-\gamma-\delta$ with solution $\texttt{HeunG}(a,q,\alpha,\beta,\gamma,\delta,x)$.

In Mathematica [58],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\alpha+\beta+1-\gamma-\delta}{x-a}\right)\frac{dy}{dx}+\frac{\alpha\beta x-q}{x(x-1)(x-a)}y=0, \tag{37}$$

with solution $\texttt{HeunG}(a,q,\alpha,\beta,\gamma,\delta,x)$.

All have four regular singularities located at $\{0,\ 1,\ a,\ \infty\}$. symODE2 uses the same structure of the equation.

A.4 (Singly) Confluent Heun equation

In DLMF [41],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\epsilon\right)\frac{dy}{dx}+\frac{\alpha x-q}{x(x-1)}y=0. \tag{38}$$

In Maple [57],

$$\begin{aligned}&\frac{d^2y}{dx^2}-\frac{-x^2\alpha+(-\beta+\alpha-\gamma-2)\,x+\beta+1}{x\,(x-1)}\frac{dy}{dx}\\&-\frac{[(-\beta-\gamma-2)\,\alpha-2\,\delta]\,x+(\beta+1)\,\alpha+(-\gamma-1)\,\beta-2\,\eta-\gamma}{2x\,(x-1)}y=0,\end{aligned} \tag{39}$$

with solution $\texttt{HeunC}(\alpha,\beta,\gamma,\delta,\eta,x)$. This form can be transformed into

$$\frac{d^2y}{dx^2}+\left(\frac{\beta+1}{x}+\frac{\gamma+1}{x-1}+\alpha\right)\frac{dy}{dx}+\left(\frac{\mu}{x}+\frac{\nu}{x-1}\right)y=0, \tag{40}$$

where

$$\delta = \mu+\nu-\alpha\frac{\beta+\gamma+2}{2}, \tag{41}$$

$$\eta = \frac{(\alpha-\gamma)(\beta+1)-\beta}{2}-\mu. \tag{42}$$

In Mathematica [58],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\epsilon\right)\frac{dy}{dx}+\frac{\alpha x-q}{x(x-1)}y=0, \tag{43}$$

with solution HeunC$(q, \alpha, \gamma, \delta, \epsilon, x)$.

All have two regular singularities located at {0,1} and an irregular singularity of rank-1 at $\{\infty\}$. symODE2 uses the Maple form.

A.5 Double confluent Heun equation

In DLMF [41],

$$\frac{d^2y}{dx^2} + \left(\frac{\delta}{x^2} + \frac{\gamma}{x} + 1\right)\frac{dy}{dx} + \frac{\alpha x - q}{x^2}y = 0. \tag{44}$$

This equation has two irregular singular points of rank-1 located at $\{0, \infty\}$.

In Maple [57],

$$\frac{d^2y}{dx^2} - \frac{\alpha\, x^4 - 2\, x^5 + 4\, x^3 - \alpha - 2\, x}{(x-1)^3\,(x+1)^3}\frac{dy}{dx} - \frac{-x^2\beta + (-\gamma - 2\,\alpha)\, x - \delta}{(x-1)^3\,(x+1)^3}y = 0, \tag{45}$$

with solution HeunD$(\alpha, \beta, \gamma, \delta, x)$. This equation has two irregular singular points of rank-1 located at {-1, 1}.

In Mathematica [58],

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x^2} + \frac{\delta}{x} + \epsilon\right)\frac{dy}{dx} + \frac{\alpha x - q}{x^2}y = 0, \tag{46}$$

with solution HeunD$(q, \alpha, \gamma, \delta, \epsilon, x)$. This equation has two irregular singular points of rank-1 located at $\{0, \infty\}$. symODE2 uses the Mathematica form.

A.6 Biconfluent Heun equation

In DLMF [41],

$$\frac{d^2y}{dx^2} - \left(\frac{\gamma}{x} + \delta + x\right)\frac{dy}{dx} + \frac{\alpha x - q}{z}y = 0. \tag{47}$$

In Maple [57],

$$\frac{d^2y}{dx^2} - \frac{\beta\, x + 2\, x^2 - \alpha - 1}{x}\frac{dy}{dx} - \frac{(2\,\alpha - 2\,\gamma + 4)\, x + \beta\,\alpha + \beta + \delta}{2x}y = 0, \tag{48}$$

with solution HeunB$(\alpha, \beta, \gamma, \delta, x)$.

In Mathematica [58],

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \delta + \epsilon x\right)\frac{dy}{dx} + \frac{\alpha x - q}{x}y = 0, \tag{49}$$

with solution HeunB$(q, \alpha, \gamma, \delta, \epsilon, x)$.

All have one regular singularity located at {0} and one irregular singularity of rank-2 at $\{\infty\}$. symODE2 uses the Mathematica form.

A.7 Triconfluent Heun equation

In DLMF [41],

$$\frac{d^2y}{dx^2} + (\gamma + x)\, x\frac{dy}{dx} + (\alpha x - q)\, y = 0. \tag{50}$$

In Maple [57],

$$\frac{d^2y}{dx^2} - (3x^2 + \gamma)\frac{dy}{dx} + [(\beta - 3)\, x + \alpha]y = 0, \tag{51}$$

with solution HeunT$(\alpha, \beta, \gamma, x)$.

In Mathematica [58],

$$\frac{d^2y}{dx^2} + (\gamma + \delta x + \epsilon x^2)\frac{dy}{dx} + (\alpha x - q)y = 0, \tag{52}$$

with solution $\texttt{HeunT}(q, \alpha, \gamma, \delta, \epsilon, x)$.

All have one irregular singularity of rank-3 at $\{\infty\}$. symODE2 uses the Mathematica form.

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1.18 AN INTRODUCTION TO DIFFERENTIAL GALOIS THEORY

By BRUCE SIMON. San Francisco State University. December 17, 2015.

AN INTRODUCTION TO DIFFERENTIAL GALOIS THEORY

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Abstract. Differential Galois theory takes the approach of algebraic Galois theory and applies it to differential field extensions generated by appending solutions to differential equations. In doing so, it uncovers both the same relationship between the solutions to differential equations and the structure of the differential splitting field, and the same solubility conditions for differential equations, as the algebraic Galois theory found for polynomial equations.

This paper provides an informal exposition of the equivalence, through the presentation of simple, concrete examples of each differential analogue. Most of the literature is purely abstract and the algebraic theory employed is heavy. Hopefully, this introduction will be accessible to anyone with a basic knowledge of algebraic Galois theory and differential equations, helping them to more comfortably approach more rigorous treatments of the subject.

Contents

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1. Introduction

In the early 19th century, Evariste Galois discovered a relationship between the structure of the splitting field of an irreducible polynomial and the roots of the polynomial. In particular, he found that the subfields of the splitting field are in bijection with the subgroups of the group of automorphisms of the splitting field that fix the base field. This group is called the **Galois group** of the polynomial and the splitting field is said to be a **Galois extension** of the base field or, simply, **Galois**. This relationship between the Galois group and the Galois extension is given by the **Fundamental Theorem of Galois Theory**.

Theorem 1.1. *If L/K is Galois with Galois group G, $\mathcal{F} = \{F \text{ is a field} \mid K \subseteq F \subseteq L\}$, and $\mathcal{H} = \{H | H \leq G\}$ then*

(1) *There is a bijective map $\mathcal{H} \to \mathcal{F}$ defined by $H \to F \iff \sigma(x) = x \ \forall \sigma \in H, x \in F$.*

(2) *The map is order inverting. If $H_1 \to F_1$, $H_2 \to F_2$ then $F_1 \subset F_2 \iff H_2 < H_1$.*

(3) *$F \in \mathcal{F}$ is Galois over $K \iff F$ is the image of a normal subgroup of G*

Note that if $H \leq G$ then the set $L^H = \{x \in L | \sigma(x) = x \ \forall \sigma \in H\}$ is always a subfield of L and L^H is referred to as the fixed field of H.

Galois' work uncovered a solvability condition for polynomial equations: a polynomial is solvable by radicals if and only if its Galois group is solvable. By demonstrating that the Galois group of a general polynomial of degree 5 or higher was not solvable, Galois confirmed Abel's proof that a general polynomial of degree 5 or more not solvable by radicals.

Example 1.2. Consider $P(x) = (x^2 - 2)(x^2 - 3) \in \mathbb{Q}[x]$.

$P(x)$ has four real roots over $\mathbb{Q}$, $\pm\sqrt{2}$ and $\pm\sqrt{3}$, so $L = \mathbb{Q}\langle\sqrt{2}, \sqrt{3}\rangle$ is the splitting field for $P(x)$ over $\mathbb{Q}$. The extension $L/\mathbb{Q}$ is Galois with the intermediate fields $\mathbb{Q}\langle\sqrt{2}\rangle$, $\mathbb{Q}\langle\sqrt{3}\rangle$, and $\mathbb{Q}\langle\sqrt{6}\rangle$.

To identify the Galois group, $G = Gal(L/\mathbb{Q})$, note that since $\pm\sqrt{2}$ are the roots of $x^2 - 2$ and $\pm\sqrt{3}$ are the roots of $x^2 - 3$, any $\mathbb{Q}$-automorphism of L must map $\pm\sqrt{2} \to \pm\sqrt{2}$ and $\pm\sqrt{3} \to \pm\sqrt{3}$.

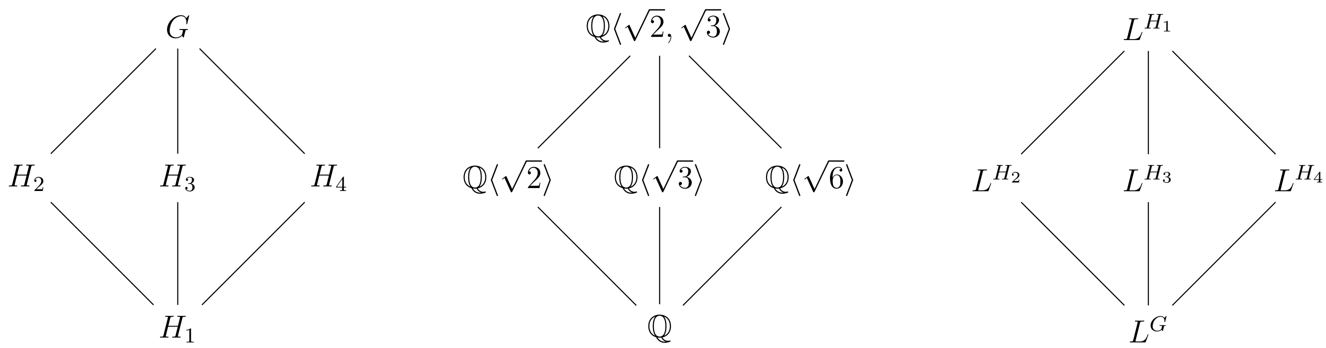
Thus, the Galois group is $G = \{1, \sigma, \tau, \sigma\tau\}$ where

$$\sigma : \begin{cases} \sqrt{2} \to -\sqrt{2} \\ \sqrt{3} \to \sqrt{3} \end{cases} \qquad \tau : \begin{cases} \sqrt{2} \to \sqrt{2} \\ \sqrt{3} \to -\sqrt{3} \end{cases} \qquad \sigma\tau : \begin{cases} \sqrt{2} \to -\sqrt{2} \\ \sqrt{3} \to -\sqrt{3} \end{cases}$$

The subgroups of G are $H_1 = 1, H_2 = \{1, \tau\}, H_3 = \{1, \sigma\}, H_4 = \{1, \sigma\tau\}$, and G.

As σ and τ are automorphisms $\sigma(\sqrt{6}) = \sigma(\sqrt{2})\sigma(\sqrt{3}) = -\sqrt{6}$ and $\tau(\sqrt{6}) = \tau(\sqrt{2})\tau(\sqrt{3}) = -\sqrt{6}$. And, since the group operation in G is composition, $\sigma\tau(\sqrt{6}) = \tau\sigma(\sqrt{6}) = \sqrt{6}$. The bijection promised in (1.1) maps $G \to \mathbb{Q}$, $H_2 \to \mathbb{Q}\langle\sqrt{2}\rangle$, $H_3 \to \mathbb{Q}\langle\sqrt{3}\rangle$,$H_4 \to \mathbb{Q}\langle\sqrt{6}\rangle$, and $H_1 \to \mathbb{Q}\langle\sqrt{2}, \sqrt{3}\rangle$.

The equivalence of the structures of the Galois extension and the Galois group can be seen in the diagrams:



Clearly $\mathbb{Q}\langle\sqrt{2}\rangle$, $\mathbb{Q}\langle\sqrt{3}\rangle$, and $\mathbb{Q}\langle\sqrt{6}\rangle$ are all Galois over $\mathbb{Q}$ and it is easy to verify that H_2, H_3, and H_4 are all normal subgroups of G.

The idea of developing an analogue to algebraic Galois theory for differential equations originated with Sophus Lie in the early 1870's. Lie found that if a group of transformations under composition permutes the integral curves of a differential equation of the form $Xdy - Ydx = 0$, the group may be used to find an integrating factor for the equation. Additionally, he discovered necessary and sufficient conditions for the existence of the transformation group. [Ref5]

In the late 19$^{\text{th}}$ Emile Picard and Ernst Vessiot applied the theory of Lie groups to uncover a solvability condition for first order linear, homogeneous differential equations almost exactly equivalent to Galois' solvability condition for polynomials. Analyzing the relationship between the differential field extensions obtained by appending solutions to differential equations over a base field, and the group of symmetries of those roots they found that an ordinary, linear, homogeneous differential equation is solvable by quadratures if and only if its differential Galois group is solvable.

In a striking parallel to the development of algebraic Galois theory Ellis Kolchin, believing Picard-Vessiot theory was limited by the lack of a formal theory of linear algebraic groups, extended the work of Joseph Ritt in differential algebra to develop this theory. Using his new theory of algebraic matrix groups, Kolchin was able to formalize and extend the work of Picard and Vessiot. The Fundamental Theorem of Picard-Vessiot Theory stated below is due to Kolchin and it is his work that is generally referred to as Picard-Vessiot theory, or differential Galois theory, today. [Ref9]

Theorem 1.3. *If L/K is a Picard-Vessiot extension with differential Galois group G then*

(1) *There is an inclusion reversing bijective map between the set of Zariski closed subgroups H of G and the set of differential fields F with $K \subset F \subset L$ given by $H \to L^H$*
(2) *An intermediate differential field $F = L^H$ is a Picard-Vessiot extension $\iff$ $H \trianglelefteq G$.*

In this paper, we will informally explore the Picard-Vessiot theory. Each object in the Picard-Vessiot theory will be introduced and developed as an analogue to its algebraic counterpart.

Algebraic Galois Theory	**Picard-Vessiot Theory**
Polynomial	Linear Differential Operator
Root of polynomial	Solution to differential equation
Splitting field	Picard-Vessiot extension
Galois group	Differential Galois Group
Solvable Galois Group	Solvable Galois Group

As each object is introduced, it will be illustrated by example. Wherever possible, the same examples will be carried through the introduction of multiple objects.

2. Differential Rings and Fields

Before we can discuss the differential Galois theory, we need a few definitions.

Definition 2.1. Here are the basic definitions of differential rings and fields we will need:

(1) A **derivation** of a ring R is a map $d : R \to R$ such that $\forall r, s \in R$
  (a) $d(r+s) = d(r) + d(s)$
  (b) $d(rs) = d(r)s + rd(s)$

(2) A **differential ring** is a commutative ring with identity and a defined derivation.

(3) A **differential field** is a field with a defined derivation. Of course, every differential field is also a differential ring.

(4) An element in a differential ring or field is **constant** $\iff$ its derivation is 0.

(5) An ideal $I \subseteq R$ is a **differential ideal** if it is closed under the derivation.

(6) A **differential automorphism** is an automorphism, σ, that respects the derivation: $\sigma(r') = [\sigma(r)]' \ \ \forall r \in R$

Proposition 2.2. *Here are some basic facts of differential rings and fields we will need:*

(1) *If R is an integral domain with derivation d, d extends uniquely to the quotient field with the usual quotient rule: $d(\frac{r}{s}) = \frac{d(r)s - rd(s)}{s^2}$.*

(2) *If R is a commutative differential ring and A is a multiplicative system of R the derivation of R extends to the ring $A^{-1}R$ uniquely in the same way.*

(3) *If R is a differential ring then the derivation of R can be extended to the polynomial ring $R[X_1, X_2, \ldots, X_n]$ such that $(\Sigma a_i X^i)' = \Sigma(a_i' X^i + a_i i X^{i-1} X')$*

(4) *If K is a differential field and L/K is a separable algebraic extension, the derivation of K extend uniquely to L and every K-automorphism of L is a differential automorphism.*

(5) *If K is a differential field and $R \supset K$ is a differential ring then any maximal ideal, $I \subset R$, is a prime ideal.*

Examples:

(1) Any commutative ring, R, with identity may be given a differential structure by defining $d(r) = 0 \ \forall r \in R$. Thus, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all differential fields in which every element is constant.

(2) The polynomial rings $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$ with the usual derivation $d(x) = 1$ are all differential rings. Likewise, the polynomial rings in n indeterminates $\mathbb{Q}[x_1, \ldots, x_n]$, $\mathbb{R}[x, \ldots, x_n]$, and $\mathbb{C}[x, \ldots, x_n]$ with $d(x_i) = 1$ $\forall i$ are all differential rings.

(3) The fields of rational functions, $\mathbb{Q}(x)$, $\mathbb{R}(x)$, and $\mathbb{C}(x)$ are all differential fields with the usual derivative.

(4) The ring of infinitely differentiable real valued functions with their usual derivatives is a differential ring and the field of meromorphic functions with their usual derivatives is a differential field.

(5) If R is a differential ring then the ring $R[x_1, \ldots, x_n]$ with the derivation extended by defining $d(x_i) = x_{i+1}$ for $i \leq n-1$ and $d(x_n)$ to be a member of $R[x_1, \ldots, x_n]$ is a differential ring.

In this construction, each x_i is a **differential indeterminate** and the elements of $R[x_1, \ldots, x_n]$ are **differential polynomials** in the indeterminate x_1.
If R was a field, this derivation extends uniquely to the quotient field $R\langle x_1, \ldots, x_n\rangle$

All of the above propositions and examples were taken from chapter 5 of [Ref1]. Some proofs are available there.

The original Picard-Vessiot theory was established in the case where the base field is of characteristic 0 and the field of constants is algebraically closed. Kolchin was able to prove his results for fields of arbitrary characteristic. Recently, the results have been shown to obtain for any closed real field of constants. [Ref10]

3. Linear Differential Operators

The **ring of differential operators** over a differential field K is simply a polynomial ring in one indeterminate, the derivation d, and is analogous to the usual polynomial ring $K[x]$ over a field.

Definition 3.1. The **ring of differential operators** over a differential field K is the noncommutative ring of all polynomials in d with coefficients in K .

$$K[d] = \{\mathcal{L} = a_n d^n + a_{n-1} d^{n-1} + \ldots + a_1 d + a_0 \mid a_i \in K \ \forall i\}$$

The product $d \cdot a$ in the ring of differential operators is defined by $d \cdot a = a' + ad$.

Powers of d act on members of K and its differential extensions as repeated applications of the derivation. Every $\mathcal{L} \in K[d]$ acts on any differential extension of K to create a degree n differential polynomial in one differential indeterminate, giving rise to a **linear homogeneous differential equation** of order n.

$$\mathcal{L}(Y) = a_n Y^{(n)} + a_{n-1} Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0.$$

A differential equation written in the form of a differential polynomial is called a **scalar** equation.

Every order n ordinary linear homogeneous differential equation may also be represented as the 1^{st} order matrix differential equation $Y' = AY$, $A \in GL_n(K)$. The derivation of $M = (m_{ij}) \in gl_n(K)$ is given by $M' = (m_{ij})'$.

Letting $b_i = \frac{a_i}{a_n}$, it is clear that y is a solution of $\mathcal{L}(Y) = a_n Y^{(n)} + a_{n-1} Y^{(n-1)} + \ldots +$

$a_1Y' + a_0Y = 0$ if and only if $\begin{pmatrix} y, & y', & \dots & y^{n-1}, & y^n \end{pmatrix}^T$ satisfies the matrix equation

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{n-1} \\ y^n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{n-1} \\ y^n \end{pmatrix}$$

As all the differential equations we consider in this paper will be linear homogenous ordinary equations, we will not continue to qualify each equation as such.

A solution to a differential equation is the equivalent of a root of a polynomial in the algebraic Galois theory. Unlike the algebraic setting, there are three distinct types of solutions to a differential equation with coefficients in the differential field K.

(1) y is algebraic over K.
(2) y is the integral of a member of K.
(3) y is the exponential of an integral of a member of K

If C_K is the field of constants of K then all C_K-linear combinations of solutions to a differential equation with coefficients in K are also solutions of the differential equation.

Examples:

(1) Let $K = \mathbb{C}(x)$ with the standard derivation and consider the scalar differential equation $\mathcal{L}(Y) = Y'' + \frac{1}{x}Y' = 0$.
By inspection, $y_1 = 1$ and $y_2 = ln\, x$ are two solutions that are linearly independent over $\mathbb{C}$. $y_1 = 1$ is an algebraic element of K and $y_2 = ln\, x$ is the integral of $\frac{1}{x} \in K$. If $c_1, c_2 \in \mathbb{C}$, then the linear combination $c_1 + c_2 \cdot ln\, x$ is also a solution of $\mathcal{L}(Y) = 0$.

(2) The first order matrix equation for the differential equation in example 1 is $\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{x} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$. The solution vectors are $(1, 0)^T$ and $(ln\, x, \frac{1}{x})^T$.

(3) Let $K = \mathbb{C}(x)$ with the standard derivation and consider the scalar equation $\mathcal{L}(Y) = Y'' + Y = 0$.
By inspection, $y_1 = sin\ x$ and $y_2 = cos\ x$ are two solutions of $\mathcal{L}(Y) = 0$ that are linearly independent over $\mathbb{C}$. Additionally, it's easy to see that $y_3 = e^{ix}$ and $y_4 = e^{-ix}$ are another pair of solutions that are linearly independent over $\mathbb{C}$.
Note that the set of solutions $\{y_1, y_2, y_3, y_4\}$ is not linearly independent over $\mathbb{C}$ as, for example, $y_1 = \frac{i}{2}(y_4 - y_3)$

(4) The matrix equation for the differential operator in example 3 is $\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$.
The solution vectors are $(sinx, cosx)^T$, $(cosx, -sinx)^T$ or $(e^{ix}, ie^{ix})^T$, $(e^{-ix}, -ie^{-ix})^T$

As a polynomial of degree n has at most n distinct roots, a differential equation of order n has at most n linearly independent solutions. It is a well known fact that an n^{th} order differential equation will always have a full set of n linearly independent solutions.

Definition 3.2. If K is a field of characteristic 0 and $\mathcal{L}(Y) = 0$ is an n^{th} order differential equation with coefficients in K, then

(1) $\{y_1, \ldots, y_n\}$ is a **fundamental set of solutions** if and only if $\mathcal{L}(y_i) = 0$ for $1 \le i \le n$ and the y_i are linearly independent over K.

(2) If $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions to $\mathcal{L}(Y) = 0$ with the associated matrix equation $Y' = AY$ then the **fundamental solution matrix** for $Y' = AY$ is

$$M_A = \begin{pmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1} \end{pmatrix}$$

(3) The **Wronskian determinant** of any set $\{y_1, \ldots, y_n\}$ is

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1} \end{vmatrix} = det M_A$$

Since $W(y_1, \ldots, y_n)$ is the determinant of an n by n matrix, $W(y_1, \ldots, y_n) \neq 0$ if $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions to the differential operator $\mathcal{L}(Y) = 0$.

Note that any fundamental set of solutions to an n^{th} order differential equation with coefficients in K forms a basis for an n-dimensional vector space over C_K. This is the **solution space** of $\mathcal{L}(Y) = 0$

Examples:

(1) $\{1, ln\, x\}$ is a fundamental solution set for $\mathcal{L}(Y) = Y'' + \frac{1}{x}Y' = 0$.
$M_A = \begin{pmatrix} 1 & ln\, x \\ 0 & \frac{1}{x} \end{pmatrix}$ is a fundamental solution matrix for $\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{x} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$.
$W(1, ln\, x) = det M_A = \frac{1}{x} \neq 0$ for all x in the domain of $ln\, x$.

(2) $\{sin\, x, cos\, x\}$ and $\{e^{ix}, e^{-ix}\}$ are fundamental solution sets for $\mathcal{L}(Y) = Y'' + Y = 0$.
The matrices $\begin{pmatrix} sin\ x & cos\ x \\ cos\ x & -sin\ x \end{pmatrix}$ and $\begin{pmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{pmatrix}$ are fundamental solution matrices for the first order matrix equation $\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$.
$W(sin\ x, cos\ x) = det \begin{pmatrix} sin\ x & cos\ x \\ cos\ x & -sin\ x \end{pmatrix} = -1$
$W(e^{ix}, e^{-ix}) = det \begin{pmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{pmatrix} = -2i.$

4. Picard-Vessiot Extensions

The **Picard-Vessiot field** of $\mathcal{L}(Y) = 0$ with coefficients in a differential field K is analogous to the splitting field of the polynomial $P(x)$ over K. It is the smallest extension of K that contains a fundamental solution set for $\mathcal{L}(Y) = 0$.

Definition 4.1. If $\mathcal{L}(Y) = 0$ has order n with coefficients in the differential field K then a differential extension $L \supseteq K$ is a **Picard-Vessiot extension** if

(1) $L = K\langle y_1, \ldots, y_n\rangle$ where $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions to $\mathcal{L}(Y) = 0$.

(2) L contains no constants that were not in K; $C_L = C_K$.

If L/K is a Picard-Vessiot extension then L is the Picard-Vessiot field of $\mathcal{L}(Y) = 0$.

The condition that $C_L = C_K$ insures that L is the minimal extension of K that contains a fundamental set of solutions to $\mathcal{L}(Y) = 0$.

Examples:

(1) If $K = \mathbb{C}(x)$ with standard derivation, the Picard-Vessiot field of $\mathcal{L}(Y) = Y'' + \frac{1}{x}Y' = 0$ is $L = K\langle ln\, x\rangle$.
(2) The Picard-Vessiot field of $\mathcal{L}(Y) = Y''+Y = 0$ over $K = \mathbb{C}(x)$ is $L = K\langle sinx, cosx\rangle$.

Theorem 4.2. *There exists a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over K if*

(1) *K has characteristic 0 and the field of constants C_K is algebraically closed.*
(2) *If C_K is a closed real field.*
(3) *$\mathcal{L}(Y) = 0$ has an irreducible auxiliary polynomial $P(x) = 0$ with coefficients in C_K. In this case, the Picard-Vessiot extension of $\mathcal{L}(Y) = 0$ is isomorphic to the splitting field of $P(x) = 0$.*

The Picard-Vessiot field of a differential equation $\mathcal{L}(Y) = 0$ is unique up to isomorphism.

If C_K is algebraically closed, the Picard-Vessiot field L/K for $\mathcal{L}(Y) = 0$ may be constructed as follows:

(1) Adjoin a fundamental solution set $\{y_1, \ldots y_n\}$ and their first $n - 1$ derivatives to obtain $K[y_{ij}]$, a differential ring in n^2 indeterminates.
These may be structured as the matrix (y_{ij}) where y_{0j} is the j^{th} solution and $y_{(i+1)j} = y'_{ij}$ for $0 \leq i \leq n-2$, $y_{nj} = -a_{n-1}y_{(n-1)j} - \ldots - a_1 y_{1j} - a_0 y_{0j}$.
(2) Localize by $W(y_1, \ldots, y_n)$ to obtain $R = K[y_{ij}][W^{-1}]$ the **full universal solution algebra** for $\mathcal{L}$.
(3) Any maximal ideal P of a full universal solution algebra is a prime ideal [Crespo] so the quotient R/P, the **Picard-Vessiot ring**, is an integral domain.
(4) The Picard-Vessiot field L is the field of quotients of the Picard-Vessiot ring.

This procedure is described in more detail, with proof of (3), in [Ref1].

Examples:

(1) Let $K = \mathbb{C}(x)$ with the usual derivative, $a \in \mathbb{C}$ and consider the differential equation $\mathcal{L}(Y) = Y' - \frac{a}{x}Y = 0$. If y is a solution to $\mathcal{L}(Y) = 0$ and $W = y$. If $a \notin \mathbb{Z}$ then $y \notin K$. [Ref2]

(a) If $a = \frac{n}{m} \in \mathbb{Q}$ then adjoining y to create the full universal solution algebra $K[y, \frac{1}{y}]$ introduces the relationship $y^m - x^n = 0$. Note that $(y^m - x^n)$ is a maximal differential ideal so is a prime ideal. The Picard-Vessiot field is the field of fractions of the quotient $K[y, \frac{1}{y}]/(y^m - x^n), K\langle x^{\frac{n}{m}}\rangle$.

(b) If $a \notin \mathbb{Q}$ then there is no non-trivial proper differential ideal so the Picard-Vessiot extension is the field of fractions of $K[y, \frac{1}{y}]$.

(2) Let $K = \mathbb{C}(x)$ with the usual derivative and consider $Y^{(3)} - 2Y = 0$ which has the auxiliary polynomial $P(x) = x^3 - 2$. The roots of P are $\sqrt[3]{2}$, $\rho\sqrt[3]{2}$, and $\rho^2\sqrt[3]{2}$ where $\rho = e^{\frac{2\pi i}{3}}$ so $\{e^{\sqrt[3]{2}x}, e^{\rho\sqrt[3]{2}x}, e^{\rho^2\sqrt[3]{2}x}\}$ form a fundamental set of solutions. Thus, the Picard-Vessiot extension is $\mathbb{C}\langle e^{\sqrt[3]{2}x}, e^{\rho\sqrt[3]{2}x}, e^{\rho^2\sqrt[3]{2}x}\rangle$.

5. The Differential Galois Group

Definition 5.1. If L/K is a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$, the **differential Galois group** of $L \supset K$ is $G(L/K) = G_K(\mathcal{L})$, the group of all differential K-automorphisms of L.

As the members of the algebraic Galois group are exactly those transformations under which the polynomial is invariant, the members of $G_K(\mathcal{L})$ are the automorphisms under which the differential operator is invariant. If $\sigma \in G_K(\mathcal{L})$ and y is a solution of $\mathcal{L}(Y) = 0$, $\sigma(y)$ is also a solution. Thus, σ maps each y_i in the fundamental solution set to a C_K-linear combination of the y_i: $\sigma(y_i) = \Sigma c_j y_i$ where $c_j \in C_K$ for $j = 1, \ldots, n$. Like the algebraic Galois group, the members of the differential Galois group are completely determined by their action on the generators of L/K.

Definition 5.2. A **linear algebraic group** is a subgroup $G \subseteq GL_n(C_K)$ that is the set of zeros of of a system of polynomials in n^2 variables with coefficients in C_K.

While algebraic Galois groups are subgroups of S_n, differential Galois groups are linear algebraic groups. In particular, if $\mathcal{L}(Y) = 0$ has order n, $G_K(\mathcal{L})$ is a Lie subgroup of $GL_n(C_K)$.

Examples:

(1) Let $K = \mathbb{C}(x)$, $L = K\langle ln\, x\rangle$ be the Picard-Vessiot field for $\mathcal{L}(Y) = Y'' + \frac{1}{x}Y' = 0$. Any differential K-automorphism in $G_K(\mathcal{L})$ must fix the solution $y = 1 \in C_K$ and map $ln\, x$ to $c_1 + c_2 ln\, x$.

Further, $\sigma \in G_K(\mathcal{L}) \implies \sigma(d(ln\, x)) = d(\sigma(ln\, x)) \implies \frac{c_2}{x} = \frac{1}{x} \implies c_2 = 1$.

Thus $\sigma \in G_K(\mathcal{L})$ maps $ln\, x \to ln\, x + c$ where $c \in \mathbb{C}$ and $G_K(\mathcal{L})$ is isomorphic to $\mathbb{C}$. $G_K(\mathcal{L})$ is the subgroup of $GL_2(\mathbb{C})$ generated by $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ for $c \in \mathbb{C}$.

To verify this is a subgroup of $GL_2(\mathbb{C})$ we compute $\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1 + c_2 \\ 0 & 1 \end{pmatrix}$

(2) Let $K = \mathbb{C}(x)$ and $L = K\langle sin\, x, cos\, x\rangle$ be the Picard-Vessiot field for $\mathcal{L}(Y) = Y'' + Y = 0$.

Any K-automorphism of L, must map $sinx \to asinx + bcosx$, $cosx \to csinx + dcosx$ and satisfy:

$$\sigma(d(sin\,x)) = d(\sigma(sin\,x)) \qquad \sigma(d(cos\,x)) = d(\sigma(cos\,x))$$
$$\sigma(cos\,x) = d(asin\,x + bcos\,x) \qquad \sigma(-sin\,x) = d(csin\,x + dcos\,x)$$
$$csin\,x + dcos\,x = acos\,x - bsin\,x \qquad -asin\,x - bcos\,x = ccos\,x - dsin\,x$$

From which it immediately follows that $a = d$ and $b = -c$.

$G_K(\mathcal{L})$ is the subgroup of $GL_2(\mathbb{C})$ generated by $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $a, b \in \mathbb{C}$ not both zero.

As $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$, $G_K(\mathcal{L})$ is a subgroup of $GL_2(\mathbb{C})$.

(3) Let $K = \mathbb{C}(x)$, $L = K\langle x^a \rangle$ be the Picard-Vessiot field for $\mathcal{L}(Y) = Y' - \frac{a}{x}Y = 0$. If y is a solution of $\mathcal{L}(Y) = 0$ then $\sigma \in G_K(\mathcal{L})$ must be defined by $\sigma(y) = cy$.

(a) If $a = \frac{n}{m}$, then y is algebraic over K with minimal polynomial of degree m so $\sigma \in G_K(\mathcal{L})$ must map $y \to cy$ where $c \in \mathbb{C}$ is an m^{th} root of unity. Thus $G_K(\mathcal{L})$ is cyclic of order m.

(b) If $a \notin \mathbb{Q}$ then y is not algebraic over K then $\sigma(y) = cy$ is a differential K-automorphism $\forall c \in \mathbb{C}$ and $G_K(\mathcal{L}) = GL_1(\mathbb{C})$.

(4) If $K = \mathbb{C}(x)$ and $L = \mathbb{C}\langle e^{\sqrt[3]{2}x}, e^{\rho\sqrt[3]{2}x}, e^{\rho^2\sqrt[3]{2}x} \rangle$ is the Picard-Vessiot field for $\mathcal{L}(Y) = Y^{(3)} - 2Y = 0$.
Suppose $\sigma y_i = c_1y_1 + c_2y_2 + c_3y_3$. Computing $d\sigma(y_i)$ and $\sigma(dy_i)$ as in example (2) above shows that any K-automorphism of L must be of the form $\sigma : \begin{cases} y_1 \to c_1y_1 \\ y_2 \to c_2y_2 \\ y_3 \to c_3y_3 \end{cases}$
Thus $G_K(\mathcal{L}) = (\mathbb{C}^*)^3$, the group of invertible, diagonal matrices in $GL_3(\mathbb{C})$.

6. The Galois Correspondence

The correspondence between the subgroups of the differential Galois group and the intermediate fields in the Picard-Vessiot extension is exactly analogous to the Galois correspondence in algebraic Galois Theory.

Theorem 6.1. *If L/K is a Picard-Vessiot extension with differential Galois group G then*

(1) *There is an inclusion reversing bijective map between the set of Zariski closed subgroups H of G and the set of differential fields F with $K \subset F \subset L$ given by $H \to L^H$*

(2) *An intermediate differential field $F = L^H$ is itself a Picard-Vessiot extension $\iff$ $H \trianglelefteq G$.*

A proof of this theorem is section 6.3 in [Ref1].

Definition 6.2. A subgroup $H \subseteq G$ is **Zariski closed** if H is a linear algebraic group.

Example:

(1) Let $K = \mathbb{C}(x)$ and consider the linear differential operator $\mathcal{L}(Y) = Y' - Y = 0$. $y = e^x$ is a solution to $\mathcal{L}(Y) = 0$ and $L = K\langle e^x \rangle$ is the Picard-Vessiot extension. The differential Galois group is $G_K(\mathcal{L}) = C^*$.

The Zariski closed proper subgroups of C^* are the groups of units of order n: $\mu_n = (e^{\frac{2\pi i}{n}}), n \geq 2$. μ_n is the set of simultaneous solutions to $x^n - 1 = 0$.

The intermediate differential fields of L/K are $L \supset E_n \supset K$ where $E_n = K\langle e^{nx} \rangle$, $n \geq 2$. If m divides n, the Galois correspondence is given by:

Differential Fields		Zariski Closed Subgroups
$K\langle e^x \rangle = L$	$\Longleftrightarrow$	1
$\vert$		$\vert$
$K\langle e^{mx} \rangle$	$\Longleftrightarrow$	$(e^{\frac{2\pi i}{m}})$
$\vert$		$\vert$
$K\langle e^{nx} \rangle$	$\Longleftrightarrow$	$(e^{\frac{2\pi i}{n}})$
$\vert$		$\vert$
$\mathbb{C}(x) = K$	$\Longleftrightarrow$	$G_K(\mathcal{L}) = C^*$

Since C^* is commutative, $H = \mu_n \trianglelefteq C^*$ for all n. $L^H = K\langle e^{nx} \rangle$ is the Picard-Vessiot extension for $\mathcal{L}(Y) = Y' - nY = 0$.

7. Solvability

The algebraic Galois' theory established a solvability condition for polynomials given by the following theorem.

Theorem 7.1. *A polynomial over a field of characteristic 0 is solvable by radicals* $\Longleftrightarrow$ *it has a solvable Galois group.*

Definition 7.2. G is **solvable** if there is a chain of subgroups $1 = G_0 \subset G_1 \ldots \subset G_n = G$ such that $G_{i+1} \trianglelefteq G_i$ and G_{i+1}/G_i is abelian.

Definition 7.3. Let K be a differential field, $\mathcal{L}(Y) = 0$ a differential equation over K. A solution $y \notin K$ is **Liouvillian** if

(1) y is algebraic over K
(2) y is the integral of an element in K
(3) y is the exponential of an element in K

Picard and Vessiot stated an almost identical condition for a linear differential equation to be solvable in terms of Liouvillian functions. Their statement was "given a formal modern proof" by Kolchin. [Ref2]

Theorem 7.4. *Let K be a differential field, L the Picard-Vessiot field for $\mathcal{L}(Y) = 0$ over K. $\mathcal{L}(Y) = 0$ is solvable by Liouvillian functions* $\Longleftrightarrow$ *the identity component of $G_K(L)$ is solvable.*

Singer provides several equivalent and stronger statements, with proofs, as well as examples in section 1.5 of [Ref2].

8. Concluding Thoughts

We have confined ourselves here to the basics of the direct question of differential Galois theory: given an easily solved differential equation, what is the Picard-Vessiot extension, the differential Galois group and the Galois correspondence. There was a great deal of interest in developing algorithms for finding differential Galois groups in the late 1990's and early 2000's. The article by van der Put referenced below provides a summary of those activities and some applications. More recently, work seems to be focused on algorithms for parameterized differential Galois theory.

The inverse question, "given a differential field K with field of constants C and a linear algebraic group G defined over C find a linear differential equation defined over$K$ whose differential Galois group is G" is also being studied. Crespo and Hajto give several suggested references in the last chapter of their book (Pg. 213). The preliminary paper by Harbater, Hartmann, and Maier referenced below claims a positive solution to the problem over Laurent series fields of characteristic 0, namely that every algebraic group over such a field is the Galois group of a differential equation.

Singer discusses several applications of differential Galois theory in mathematics in section 1.3 of his lectures beginning on page 18.

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1.19 Solving Second Order Linear Differential Equations with Klein's Theorem

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Solving Second Order Linear Differential Equations with Klein's Theorem

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ABSTRACT

Given a second order linear differential equations with coefficients in a field $k = C(x)$, the Kovacic algorithm finds all Liouvillian solutions, that is, solutions that one can write in terms of exponentials, logarithms, integration symbols, algebraic extensions, and combinations thereof. A theorem of Klein states that, in the most interesting cases of the Kovacic algorithm (i.e when the projective differential Galois group is finite), the differential equation must be a pullback (a change of variable) of a standard hypergeometric equation. This provides a way to represent solutions of the differential equation in a more compact way than the format provided by the Kovacic algorithm. Formulas to make Klein's theorem effective were given in [4, 2, 3]. In this paper we will give a simple algorithm based on such formulas. To make the algorithm more easy to implement for various differential fields k, we will give a variation on the earlier formulas, namely we will base the formulas on invariants of the differential Galois group instead of semi-invariants.

Categories and Subject Descriptors

G.4 [**Mathematical Software**]: Algorithm design and analysis

General Terms

Algorithms

Keywords

Linear Differential Equations, Liouvillian solutions, Klein's Theorem

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1. INTRODUCTION

The Kovacic algorithm [19] computes closed form (*Liouvillian*) solutions of second order linear differential equations over $k = C(x)$. Since the appearance of [19], many papers have studied and refined the method. The version given in [27] uses invariants instead of the semi-invariants, which is easier to implement especially for differential fields k more complicated than $C(x)$. The paper [15] gives good formulas for computing algebraic solutions (after [25]). The common basis of these algorithms is to derive solutions from *(semi)-invariants* of the *differential Galois group* (see section 3).

Another approach is the Klein pullback method: Klein ([18], also [1, 5, 2]) showed that if the projective differential Galois group is finite, then the equation is a *pullback* of an equation in a finite list of well-known standard hypergeometric equations. This means that the solutions are of the form $e^{\int g} H(f)$ where $f, g \in k$ and H is a standard hypergeometric function $H(x) = {}_2F_1([a, b], [c], x)$ whose parameters a, b, c appear in a finite list. Interest in this method has recently been revived [5, 20, 21] for classifying work, but finding pullback functions still relied on skill.

In [4, 2, 3] Berkenbosch and the authors of this paper give (surprisingly simple) formulas for computing the pullback function f (as well as the function g). In [2, 3] Berkenbosch generalizes Klein's theorem to third order operators.

Our formulas from [4, 2, 3] rely on computing *semi-invariants* of the differential Galois groups, which is well-mastered for differential equations with coefficients in $C(x)$. For more general differential fields, however, it may be easier (as noted in [27]) to use algorithms that compute *invariants* of the differential Galois group instead of semi-invariants. In order to use invariants, we will need to give formulas that are slightly different from those given in [4, 2, 3].

The contribution in this paper is of algorithmic nature: we give an algorithm for solving second order differential by pullbacks for a general differential field k by constructing new formulas which rely on invariants only. A field k is admissible for our algorithm if:

k is an effective (computable) field (this includes extracting square roots), one has an algorithm for computing rational solutions of linear differential equations with coefficients in k and an algorithm for computing exponential solutions of second order differential equations.

Examples of admissible fields are Liouvillian extensions of $C(x)$ ([24]). Implementations of the above assumed algorithms are available for fields such as $C(x)$, $C(x, \exp(f))$ ([8]), quadratic extensions of $C(x)$ ([10]), etc. For those fields k, the algorithm proposed here for computing Liouvillian solutions will be easy to implement.

Although we recall the main ideas in sections 3, we assume in this paper that the reader has an elementary knowledge of differential Galois theory ([23]) and of the Kovacic algorithm [19, 23]. The algorithm in section 2 below follows the lines of the rational version of the Kovacic algorithm given in [27].

Section 2 contains the algorithm. Most of the remainder of the paper is devoted to its correctness and optional improvements. Section 3 contains material and definitions from differential Galois theory and Kovacic's algorithm; section 4 recalls the pullback formulas from [4, 2, 3] for the case $k = C(x)$, Section 5 proves the pullback formulas for a general differential field and the correctness of the algorithm.

Finally, we remark that some recent papers [7, 12] showed how to solve certain classes of second linear differential equations as pullbacks of differential equations corresponding to special functions (Airy, Whittaker, etc). The present work is complementary to those whenever the differential equation has more than 3 singularities and the projective differential Galois group is not PSL_2.

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2. THE ALGORITHM

In this section, we state the algorithm assuming the reader is familiar with notations and concepts from differential Galois theory and Kovacic's algorithm; unfamiliar readers should proceed first to the next sections for explanations and come back to this section afterward.

Let k denote a differential field of characteristic 0. We consider the differential operator

$$L = \partial^2 + A_1\partial + A_0 \in k[\partial] \qquad (2.1)$$

This corresponds to the differential equation $y'' + A_1 y' + A_0 y = 0$. We assume that there exists $w \in k$ such that $A_1 = -\frac{w'}{w}$ (this is not restrictive since after a simple transformation one may assume the stronger condition $A_1 = 0$, see section 3).

We define the following *standard* differential operators

$$St^s_{D_n} = \partial^2 + \frac{x}{x^2-1}\partial - \frac{1}{4n^2(x^2-1)}, n \in \mathbb{N} \qquad (2.2)$$

$$St^s_{\mathcal{G}} = \partial^2 + \frac{(8x+3)}{6(x+1)x}\partial + \frac{(6\nu-1)(6\nu+1)}{144(x+1)^2 x} \qquad (2.3)$$

$$\text{for} \quad (\mathcal{G}, \nu) \in \{(A_4, 1/3), (S_4, 1/4), (A_5, 1/5)\}.$$

$$St^i_{D_2} = \partial^2 + \frac{4}{3}\frac{x}{(x^2-1)}\partial - \frac{5}{144}\frac{x^2+3}{(x^2-1)^2} \qquad (2.4)$$

$$St^i_{D_n} = St^s_{D_n}, \; n > 2 \qquad (2.5)$$

$$St^i_{A_4} = \partial^2 + \frac{2(3x^2-1)}{3x(x^2-1)}\partial + \frac{5}{144x^2(x^2-1)} \qquad (2.6)$$

$$St^i_{S_4} = \partial^2 + \frac{1}{4}\frac{(5x-2)}{(x-1)x}\partial - \frac{7}{576}\frac{1}{(x-1)^2 x} \qquad (2.7)$$

and $St^i_{A_5} = St^s_{A_5}$. These are well studied hypergeometric operators and their solutions are well-known. There are various ways to express the solutions of the above operators, one can use the hypergeometric function ${}_2F_1$, or algebraic functions, or (if $\mathcal{G}$ is not A_5) nested radicals. We propose the ${}_2F_1$ representation as the default choice because it is the most compact representation. Moreover, converting these ${}_2F_1$'s to algebraic functions or nested radicals is easier to implement (table lookup) than the reverse conversion.

The m-th symmetric power $L^{\circledS m}$ of L is the operator whose solutions are spanned by products of m solutions of L. Given differential operators $L \in k[\partial]$ and $\partial - b$, $b \in k$, the notation $L \otimes (\partial - b)$ refers to the operator whose solutions are the solutions of L multiplied by the solution $e^{\int b}$ of $\partial - b$. Given a differential operator $\mathcal{L} = \partial^2 + a_1\partial + a_0$, we define its g-invariant to be $g_{\mathcal{L}} := 2a_1 + \frac{a_0'}{a_0}$.

We can now state the algorithm. The steps have to be performed in the given order, and the algorithm exits when a solution is found.

Pullback Algorithm, general k:

Input: L with $G(L) \subset \mathrm{SL}_2(C)$

Output: Liouvillian solutions, expressed via solutions of the above standard operators

1. Determine if L has a solution y such that $y'/y \in k$ (an *exponential* solution). If so, return a basis of Liouvillian solutions of L [15, 2, 19, 27, 23]
2. Let B_4 be a basis of solutions in k of $L^{\circledS 4}$
   (a) If B_4 contains one element i_4, let $\partial^2 + a_1\partial + a_0 := L \otimes (\partial + \frac{i_4'}{4i_4})$. Return $\sqrt[4]{i_4}\, e^{\pm\int\sqrt{-a_0}}$ or use section 5.3.
   (b) (implementation of this step is optional). If B_4 contains two elements then let $m = 6$ and take solutions as in step 3 below (B_6 will have one element i_6), or use section 5.4.
3. For m in $6, 8, 12$, let B_m be a basis of solutions in k of $L^{\circledS m}$. If B_m contains one element i_m, then let $\mathcal{L} = \partial^2 + a_1\partial + a_0 := L \otimes (\partial + \frac{i_m'}{m i_m})$. Now return the following basis of solutions of L
   $$\sqrt[m]{i_m}\, H_1(f), \quad \sqrt[m]{i_m}\, H_2(f)$$
   where $H_1(x), H_2(x)$ is a basis of solutions of $St^i_{\mathcal{G}}$ and where $\mathcal{G}$ and f are determined as follows:
   (a) If $m = 6$, then $\mathcal{G} := A_4$ and $f := \sqrt{1 + \frac{64}{5}\frac{a_0}{g_{\mathcal{L}}^2}}$. This f will be in k.
   (b) If $m = 8$, $\mathcal{G} := S_4$ and $f = -\frac{7}{144}\frac{g_{\mathcal{L}}^2}{a_0}$.
   (c) If $m = 12$, $\mathcal{G} := A_5$ and $f = \frac{11}{400}\frac{g_{\mathcal{L}}^2}{a_0}$.

   The name of the standard operators refers to the projective differential Galois group $PG(L)$ (see section 3 below) of L.
4. Otherwise the operator has no Liouvillian solutions.

The above algorithm is correct but improvements are possible. In step 2a where B_4 has one element, we have $PG(L) = D_n$ for some $n > 2$. If an integration algorithm for the field $k(\sqrt{-a_0})$ is available, then we could use it to try to simplify the expression $e^{\pm\int\sqrt{-a_0}}$. However, if $n \neq \infty$ then there is an alternative that is likely to be more efficient. To implement this alternative, one starts by running a subroutine of the integration algorithm ([6]) that determines n. When

n is found, if $n \neq \infty$, then instead of running the remainder of the integration algorithm one proceeds by using the formulas in section 5.4.

Implementation of step 2b is optional. In step 2b, the projective Galois group is D_2 (this denotes $C_2 \times C_2$). If step 2b is not implemented, then in the D_2 case the algorithm will proceed to step 3a and compute solutions using formulas meant for A_4. Although these formulas give correct solutions for the D_2 case (note that $D_2 \lhd A_4$ and that these two groups have the same invariants of degree 6) one can find better (more compact) solutions in this case by using equation (2.4) and the formula from section 5.4.

3. DIFFERENTIAL GALOIS THEORY

For completeness and to set notations, we briefly recall the rational Kovacic algorithm from [27]. Let $L = \partial^2 + A_1\partial + A_0$ where $A_0, A_1 \in k$. We consider a second order ordinary linear differential equation

$$Ly = 0, \quad y'' + A_1 y' + A_0 y = 0. \tag{3.8}$$

We assume that $A_1 = \frac{f'}{f}$ for some $f \in k$; this can be achieved after a change of variable $y \mapsto ye^{\int \frac{A_1}{2}}$ which turns the equation (3.8) into the *reduced form* $y'' - ry = 0$ with $r = \frac{A_1^2}{4} + \frac{A_1'}{2} - A_0$.

Given two linearly independent solutions of (3.8), say y_1, y_2 (either "formal" or "actual functions on some open set"), the field $K := k(y_1, y_2, y_1', y_2')$ is a differential field (a field closed under differentiation) and is generated, as a differential field, by y_1 and y_2 over k. This field K is called a *Picard-Vessiot extension* of (3.8). The solution space in K is the C vector space generated by y_1 and y_2, denoted by V in all that follows. The group of differential automorphisms of K over k (i.e., automorphisms of K over k that commute with ∂) is called the *differential Galois group of* (3.8) *over* k. We denote it by $G(L) = \mathrm{Gal}_{K/k}(L)$. The condition $A_1 = \frac{f'}{f}$ ensures that $G(L) \subset \mathrm{SL}_2(C)$.

The *projective Galois group* is defined by

$$PG(L) := G(L)/(G(L) \cap C^*),$$

where $G(L) \cap C^*$ denotes the subgroup of those $g \in G$ that act on V as scalar multiplication.

Multiplying the solutions by $e^{\int b}$ for b in k changes the Galois group $G(L)$ but not the projective Galois group $PG(L)$. The operator whose solutions are $y \cdot e^{\int b}$, with y solution of $L(y) = 0$, is denoted $L \otimes (\partial - b)$. We will say that two operators L_1 and L_2 are *projectively equivalent* when there exists $b \in k$ such that $L_1 = L_2 \otimes (\partial - b)$. It is easy to see that L_1, L_2 are projectively equivalent if and only if they have the same reduced form. If L_1, L_2 are projectively equivalent then $PG(L_1) = PG(L_2)$.

3.1 Invariants and Semi-Invariants

The key to Kovacic's algorithm is that the existence of Liouvillian solutions is (for second order equations) equivalent with the existence of a *semi-invariant* of the differential Galois group.

Definition 3.1. *Fix a basis y_1, y_2 of the solution space V of L.*

1. *A homogeneous polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is called an invariant with respect to the differential operator L if its evaluation $h := I(y_1, y_2)$ is invariant under the action of the differential Galois group $G(L)$ of L. In other words $h \in k$. This function h is then called the value of the invariant polynomial I.*

2. *A homogeneous polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is called a semi-invariant with respect to a differential operator L if $\dfrac{h'}{h} \in k$ where $h := I(y_1, y_2)$.*

We will list a few well known facts, for more details see [25, 23, 19]. For second order operators, there is a one to one correspondence between the (semi)-invariants of degree m and their values (for higher order operators this need not be the case). The values of invariants of degree m are precisely the *rational* solutions of $L^{\circledS m}$, i.e solutions in k. The values of the semi-invariants of degree m are the so-called *exponential* solutions of $L^{\circledS m}$, that is, those solutions h of $L^{\circledS m}$ for which $h'/h \in k$.

The operator $L^{\circledS m}$ can be easily computed from the recursion given in (1.14) in [11] (see also [9]): Let $L_0 = 1$, $L_1 = \partial$ and

$$L_{i+1} = (\partial + iA_1)L_i + i(m - (i-1))A_0 L_{i-1}$$

for $0 < i \leq m$, then $L_{m+1} = L^{\circledS m}$.

3.2 The Subgroups of $\mathrm{SL}_2(C)$

Invariants and semi-invariants are elements of $C[Y_1, Y_2]$. In the algorithm we will not calculate the invariants themselves, but only their values. For each semi-invariant, we will only compute the logarithmic derivative h'/h of the value h of a semi-invariant. So in the following, when we write that there are n semi-invariants of degree m, we are counting the number of distinct $h'/h \in k$ for which h is a solution of $L^{\circledS m}$. And when we write that there are n invariants of degree m, we mean that the set of solutions of $L^{\circledS m}$ in k has a basis with n elements.

We recall the classification of subgroups of $\mathrm{SL}_(C)$ (see e.g [19, 25, 27, 23]) and the invariants and semi-invariants of lowest degree. The group is reducible if there is at least one invariant line in V. A non-zero element of that line is an exponential solution, i.e., a solution whose logarithmic derivative is in k (see [23, 27, 15, 2] for more on this case). The rest of the classification (irreducible cases) is in the above references:

Lemma 3.2 (Imprimitive groups). *Assume that $G(L) \subset SL_2(C)$ and that $G(L)$ is imprimitive, i.e. irreducible and there exist two lines $l_1, l_2 \subset V$ such that $G(L)$ acts on $\{l_1, l_2\}$ by permutation. Then $PG(L) \subset D_\infty$ (infinite dihedral group). Three cases are to be considered.*

1. *$PG(L) = D_2$. Three semi-invariants $S_{2,a}, S_{2,b}, S_{2,c}$ of degree 2 ($S_{2,x}^2$ is invariant), two invariants $I_{4,a}, I_{4,b}$ of degree 4. One invariant I_6 of degree 6, with $I_6 = S_{2,a}S_{2,b}S_{2,c}$. Note that the notation D_2 does not refer to the cyclic group C_2 but to $C_2 \times C_2$.*

2. *$PG(L) = D_n$, $n > 2$. One semi-invariant S_2 of degree 2, one invariant $I_4 = S_2^2$ of degree 4, and another invariant I_{2n} of degree $2n$.*

3. *$PG(L) = D_\infty$ has only one semi-invariant S_2 of degree 2 and one invariant $I_4 = S_2^2$ of degree 4.*

LEMMA 3.3 (PRIMITIVE GROUPS). *Assume G is primitive, i.e neither reducible nor imprimitive, and $G(L) \subset SL_2(C)$. Four cases are to be considered.*

1. *$PG(L) = A_4$; two semi-invariant $S_{4,a}, S_{4,b}$ of degree 4, one invariant I_6 of degree 6, and one invariant I_8 of degree 8, with $I_8 = S_{4,a}S_{4,b}$*

2. *$PG(L) = S_4$; one semi-invariant S_6 of degree 6, one invariant I_8 of degree 8.*

3. *$PG(L) = A_5$; one invariant I_{12} of degree 12.*

4. *$G = SL_2(C)$; no semi-invariants and no Liouvillian solutions.*

The degrees for the (semi)-invariants of these groups allow to give a list of possible symmetric powers $L^{\circledS m}$ to investigate. This is the key to the Kovacic algorithm (semi-invariants) or its Ulmer-Weil rational variant [27] (invariants). Computing invariants (or semi-invariants), one can find the type of the differential Galois group (a little more needs to be done to discriminate D_n from D_∞, see section 4.4). We summarize this in the following immediate corollary

COROLLARY 3.4. *In the Pullback algorithm from section 2, in the case of step 1 the group is reducible, in case of step 2a the projective Galois group is D_∞ or some D_n, $n > 2$. It is D_2 in case of step 2b, A_4 in step 3a, S_4 in step 3b, A_5 in step 3c, and PSL_2 otherwise.*

For each possible finite projective group, pullback formulas can be computed; this is done in the next section.

4. PULLBACK FORMULAS, CASE $K = C(X)$

In this section, we recall our work with Maint Berkenbosch from [4, 2]. The next subsection is standard material [1, 2, 5, 20, 21]

4.1 Standard equations

If y_1, y_2 is a basis of solutions of L, then define $C_L := C(\frac{y_1}{y_2})$, which is a subfield of the Picard-Vessiot extension K. The field C_L does not depend on the choice of basis (replacing y_1, y_2 by another basis corresponds to a Möbius transformation of $\frac{y_1}{y_2}$). Replacing y_1, y_2 by $e^{\int v}y_1, e^{\int v}y_2$ for some function v does not affect C_L either. In fact, given two operators L_1 and L_2, one has $C_{L_1} = C_{L_2}$ if and only if L_1 and L_2 are projectively equivalent.

The projective Galois group $PG(L)$ acts faithfully on C_L. The field $C_L^{PG(L)}$ of invariants under this action can, by Luroth's theorem, be written as $C(f)$ for some $f \in k$. We say that an operator St is *a standard equation* for $PG(St)$ if $C_{St}^{PG(St)}$ equals $C(z)$ for some z with $z' = 1$.

Now assume that L has projective group PG and St is a standard equation with projective Galois group PG. If $C_L^{PG} = C(f)$, then $z \mapsto f$ maps C_{St}^{PG} to C_L^{PG}. This, and the fact that C_L determines L up to projective equivalence, are key ideas in Klein's theorem below. Before stating this, we set a family of standard equations. All other standard equations can then be found using Möbius $x \mapsto (ax+b)/(cx+d)$ and projective equivalence $L \mapsto L \otimes (\partial + v)$ transformations.

A standard equation for each finite projective differential Galois group can be found among the hypergeometric equations

$$St_{PG} = \partial^2 + \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{x(x-1)}$$

where the coefficients a, b, c are related to the differences λ, μ, ν of the exponents at $0, 1$, and ∞ by the relations

$$a = \frac{1-\lambda^2}{4} \quad b = \frac{1-\mu^2}{4} \quad \text{and} \quad c = \frac{1-\nu^2+\lambda^2+\mu^2}{4}.$$

More precisely, one can choose $(\lambda, \mu, \nu) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n}\right)$ for $PG = D_n$, $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{3}\right)$ for $PG = A_4$, $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for $PG = S_4$ and $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{5}\right)$ for $PG = A_5$.

The index PG refers to the projective differential Galois group of St_{PG} corresponding to the chosen values of a, b, c. These equations and their solutions are well known.

4.2 Klein's theorem

DEFINITION 4.1. *Let $L_1 \in C(z)\left[\frac{d}{dz}\right]$ and $L_2 \in k\left[\partial\right]$ be linear differential operators.*

1. *L_2 is a proper pullback of L_1 by $f \in k$ if the change of variable $z \mapsto f$ changes L_1 into L_2.*

2. *L_2 is a (weak) pullback of L_1 by $f \in k$ if there exists $v \in k$ such that $L_2 \otimes (\partial + v)$ is a proper pullback of L_1 by f.*

THEOREM 4.2 (KLEIN, [18, 1, 2]). *Let L be a second order irreducible linear differential operator over k with projective differential Galois group $PG(L)$. Then, $PG(L) \in \{D_n, A_4, S_4, A_5\}$ if and only if L is a (weak) pullback of $St_{PG(L)}$.*

Let L have a projective differential Galois group $PG(L)$ and suppose the standard equation with projective differential Galois group $PG(L)$ has H_1, H_2 as a C-basis of solutions. The theorem of Klein says that L is a pullback of $St_{PG(L)}$. Suppose we know f and v as in definition 4.1, then a C-basis of solutions of $Ly = 0$ is given by $H_1(f)e^{\int v}$ and $H_2(f)e^{\int v}$.

H_1 and H_2 are known for all standard equations. To get the solutions in explicit form one should then determine the projective differential Galois group and, in case it is finite, determine f and v. It was remarked in [1, 5] (and somehow in [18]) that f can be expressed as a quotient of invariants of the differential Galois group, but this idea was not used algorithmically. We will build f (and v) using semi-invariants in section 4, and using invariants in section 5.

The difficulty lies in the fact that L is a *weak* pullback of a standard equation, i.e it is only *projectively equivalent* to a proper pullback of the standard equation. The key to formulas is to compute a normal form such that the normal form of L will be a proper pullback of its standard form.

Suppose that L has a differential Galois group G (and projective group PG) with semi-invariant S of degree m and value σ. And suppose the value of S with respect to the standard operator St_{PG} equals σ_0 (modulo C^*). Then, the value of S w.r.t. both the differential operator $S_G = St_{PG} \otimes (\partial_z + \dfrac{\sigma_0'}{m\sigma_0})$ and the differential operator $\mathcal{L} = L \otimes (\partial_x + \frac{\sigma'}{m\sigma})$ is equal to 1 and the following property holds.

LEMMA 4.3. *$\mathcal{L}$ is a proper pullback of S_G.*

PROOF. The (semi)-invariant of S_G corresponding to σ (in the above notations) has value 1 so it is mapped to 1 under any pullback transformation $z \mapsto f$. L is a weak pullback by Klein's theorem, so $L \otimes (\partial - v)$ will be a proper pullback for some v; but its (semi)-invariant is $e^{\int mv}$, which should be 1, so v must be 0 and hence L must be a proper pullback. □

A direct examination (and relevant choices of standard equations) in each case will provide the pullback function f.

4.3 Formulas: the primitive case

The projective Galois group is in $\{A_4, S_4, A_5\}$ in this section. The standard equation in reference is $St_{PG}\, y = 0$ where the differences of exponents are $\lambda = \dfrac{1}{3}$ at $x = 0$, $\mu = \dfrac{1}{2}$ at $x = 1$, and $\nu = \dfrac{1}{3}$ for A_4, $\dfrac{1}{4}$ for S_4 and $\dfrac{1}{5}$ for A_5 at $x = \infty$.

The differential Galois group of this equation has a semi-invariant S of degree $m = 4$ in the case of A_4, degree $m = 6$ in the case of S_4 and $m = 12$ in the case of A_5 with value $\sigma_0(x) = x^{-m/3}(x-1)^{-m/4}$. The new equation $S_G = St_{PG} \otimes (\partial + \dfrac{1}{3x} + \dfrac{1}{4(x-1)})$ now has an invariant of degree m with value 1. Rearranging it (via a Möbius transform, to obtain nicer formulas), we get the normalized standard equation:

$$St^s_{PG} := \partial^2 + \frac{1}{6}\,\frac{(8\,x+3)}{(x+1)\,x}\partial + \frac{s}{(x+1)^2\,x}$$

with $s = \frac{(6\nu-1)(6\nu+1)}{144}$ (recall that ν is $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ for cases A_4, S_4, A_5 respectively). It has exponents $(\frac{\nu}{2} + \frac{1}{12}, -\frac{\nu}{2} + \frac{1}{12})$ at -1, $(0, \frac{1}{2})$ at 0 and $(0, \frac{1}{3})$ at ∞ where ν has the previous value in each case.

LEMMA 4.4. *Let $\mathcal{L} = \partial^2 + a_1\partial + a_0$ be a normalized operator with $PG(\mathcal{L}) \in \{A_4, S_4, A_5\}$ (i.e it has an invariant of degree m with value 1 for the above values of m). Define $g_{\mathcal{L}} := 2a_1 + \frac{a_0'}{a_0}$. Then $\mathcal{L}$ is a proper pullback of St^s_{PG} and the pullback mapping is*

$$f := 9s\,\frac{g_{\mathcal{L}}^2}{a_0}$$

PROOF. Lemma 4.3 shows that $\mathcal{L}$ is a proper pullback $z \mapsto f$ of St^s_{PG} for some f. Computing this pullback and equating it to $\mathcal{L}$ gives the relations $a_1 = \frac{f'}{2f} + \frac{5f'}{6(f+1)} - \frac{f''}{f'}$ and $a_0 = \frac{sf'^2}{(f+1)^2 f}$ whence $\frac{a_0'}{a_0} = -\frac{2f'}{f+1} - \frac{f'}{f} + \frac{2f''}{f'}$ and the formula follows by simple elimination. □

In fact, the formula was not obtained that way: as we know that $\mathcal{L}$ is a proper pullback and that the solution f is unique (by Klein's theorem and our normalization), we compute the expression of the image of St^s_{PG} under a generic pullback and perform differential elimination [13, 14] (there are other ways to find the formula but this way was the least amount of work). In the same way one can obtain formulas for other choices of standard equations but those turn out to be larger.

So, given $L = \partial^2 + A_1\partial + A_0$ with finite primitive projective group, the pullback function is found the following way:

Pullback for A_4, S_4, A_5, semi-invariant version
Input: $L = \partial^2 + A_1\partial + A_0$ with $PG(L) \in \{A_4, S_4, A_5\}$.
Output: Pullback function f.

1. For $m \in \{4, 6, 12\}$ check for a semi-invariant of degree m and call v its logarithmic derivative.
2. If yes, the projective group $PG(L)$ is known. Let $\mathcal{L} = L \otimes (\partial + \frac{1}{m}v)$; this is a proper pullback of St^s_{PG} with invariant value 1.
3. Write $\mathcal{L} = \partial^2 + a_1\partial + a_0$. Compute $g_{\mathcal{L}} := 2a_1 + \frac{a_0'}{a_0}$, and the pullback mapping is $f := 9s\frac{g_{\mathcal{L}}^2}{a_0}$

REMARK 4.5. *The change of variable $z \mapsto f$ changes g_{St} to $g_{St}(f) \cdot f'$. Now, $g_{St} = -\frac{1}{3(x+1)}$ and the relation $g_{\mathcal{L}} = -\frac{f'}{3(f+1)}$ yields another method to find f. This approach will fail for imprimitive groups because then $g_{\mathcal{L}}$ will be zero.*

4.4 Formulas: the imprimitive case

In this case, the projective Galois group is $PG(L) = D_n$ for $n \in \mathbb{N}$. To simplify formulas, here, we choose the standard equation with exponent differences $\dfrac{1}{2}$ at $+1$ and -1 and $\dfrac{1}{n}$ at infinity. It has a semi-invariant $S_2 = Y_1Y_2$ of degree 2 and two semi-invariants $S_{n,a} = Y_1^n + Y_2^n$ and $S_{n,b} = Y_1^n - Y_2^n$ of degree n. The chosen standard equation

$$St^s_{D_n} = \partial^2 - \frac{z}{z^2-1}\partial - \frac{1}{4n^2}\,\frac{1}{z^2-1}$$

has exponents $\left(0, \dfrac{1}{2}\right)$ at $+1$ and -1 and $\left(\dfrac{-1}{2n}, \dfrac{1}{2n}\right)$ at ∞; it has a semi-invariant of degree 2 and value 1.
An operator $\mathcal{L} = \partial^2 + a_1\partial + a_0$ is a proper pullback of S_{D_n} if $a_0 = -\dfrac{1}{4n^2}\dfrac{f'^2}{f^2-1}$ and $a_1 = -\dfrac{1}{2}\dfrac{a_0'}{a_0}$. The equation $\mathcal{L}y = 0$ admits the solutions $y_1, y_2 = \exp \displaystyle\int \pm\sqrt{-a_0}$ i.e. $y_1 = \sqrt[2n]{f + \sqrt{f^2-1}}$ and $y_2 = 1/y_1$. The number n can thus be determined with (a subroutine of) the algorithm of elementary integration ([6]) applied to $\sqrt{-a_0}$.
For $N \in \mathbb{N}$, the expressions y_1^N and y_2^N are permuted by the Galois group and are found to be a basis of solutions of $\mathcal{L}_N := \partial^2 + a_1\partial + N^2 a_0$. In particular L_{2n} has solutions f (rational) and $\sqrt{f^2-1}$. Once n is known, we would like to compute f from a rational solution F of L_{2n}. However, we would only know it up to a constant so we use its logarithmic derivative:

LEMMA 4.6. *Let $\mathcal{L} = \partial^2 + a_1\partial + a_0$ be an irreducible operator with an invariant of degree 2 with value 1. Assume that $PG(\mathcal{L}) = D_n$. Let F be a rational solution of $\partial^2 + a_1\partial + 4n^2 a_0$ and let $u := \frac{F'}{F}$. Then the solutions of $\mathcal{L}$ are $y_1 = \sqrt[2n]{f + \sqrt{f^2-1}}$ and $y_2 = \sqrt[-2n]{f + \sqrt{f^2-1}}$ with $f = \sqrt{\frac{1}{1 + \frac{u^2}{4n^2 a_0}}}$.*

PROOF. By the above discussion, $\partial^2 + a_1\partial + 4n^2 a_0$ has a rational solution and $F = cf$ for some constant f. Now we have $f'^2 = -4n^2 a_0(f^2 - 1)$. Dividing out by f^2 yields the formula. □

REMARK 4.7. *Despite the square root in the expression of f, the function is rational. However, if the constant field of k is not algebraically closed, a quadratic extension of the constants may be needed in computing this square root (see also [2, 16] and references therein).*

Pullback Formula for D_n, semi-invariant version
Input: $L = \partial^2 + A_1\partial + A_0$ with $PG(L) = D_n$ (n unknown).
Output: Pullback function f and the solutions.

1. Compute a semi-invariant of degree 2 and compute its logarithmic derivative v.
2. If yes, let $\mathcal{L} = L \otimes (\partial + \frac{1}{2}v)$; it is a proper pullback of S_{D_n} with invariant value 1.
3. Denote $\mathcal{L} = \partial^2 + a_1\partial + a_0$. Determine a candidate for (a multiple of) n. (note: if there is more than one semi-invariant of degree 2, then $n = 2$)
4. Compute a rational solution F of $\mathcal{L}_n := \partial^2 + a_1\partial + 4n^2 a_0$ and let $u = \frac{F'}{F}$.
5. Return the solutions $y_1 = e^{\int \frac{v}{2}} \sqrt[2n]{f + \sqrt{f^2 - 1}}$ and $y_2 = e^{\int \frac{v}{2}} \sqrt[-2n]{f + \sqrt{f^2 - 1}}$ with $f = \sqrt{\frac{1}{1 + \frac{u^2}{4n^2 a_0}}}$.

5. PULLBACK FORMULAS, GENERAL K

5.1 Standard Equations

The algorithm for general k uses only invariants (not semi-invariants). Hence, the relevant normal form for the standard and target equations will be the one for which an appropriate invariant (often one with the lowest degree) has value 1. For a projective group PG, a standard equation with semi-invariant of lowest degree with value 1 (resp. with invariants of lowest degree value 1) will be denoted St^s_{PG} (resp. St^i_{PG}).
A second idea that we will use is the fact that $D_2 \subset A_4 \subset S_4$. So, a standard equation for D_2 (resp. A_4) is a pullback of some St_{A_4} (resp. St_{S_4}). Transformations between those equations can be found in [26] (or can be recomputed, as below).

Like in the previous section, we will proceed in reverse order of the classification to give the pullback formulas

5.2 Primitive Cases

5.2.1 Icosaedral case A_5

The group is determined by an invariant of degree 12, as in the $C(x)$ case, so we use the formula from section 4.3.

5.2.2 Octaedral case S_4

Let $St^s_{S_4}$ denote the standard equation from section 4.3 with projective Galois group S_4. It has an invariant of degree 6 with value 1. However our target differential operator L has $G(L) \subset SL_2$. It only has a semi-invariant S_6 of degree 6 and an invariant I_8 of degree 8. Having computed the value of the (semi)-invariant of degree 8 of $St^s_{S_4}$, we tensor $St^s_{S_4}$ with $\partial - \frac{1}{24(x+1)}$ (and, via a Möbius transform, change the singularities to 0, 1 and ∞ to simplify the formula of lemma 5.1) to obtain the standard operator

$$St^i_{S_4} = \partial^2 + \frac{1}{4}\frac{(5\,x - 2)}{(x-1)\,x}\partial - \frac{7}{576}\frac{1}{(x-1)^2\,x}$$

Its exponents are $(0, \frac{1}{2})$ at 0, $(-\frac{1}{24}, \frac{7}{4})$ at 1, and $(0, \frac{1}{4})$ at ∞; it has an invariant of degree 8 with value 1.
We assume that the differential operator L has projective Galois group S_4 and $G(L) \subset \mathrm{SL}_2(C)$. Thus L has an invariant of degree 8 with value σ. We normalize L by tensoring with $\partial + \frac{\sigma'}{8\sigma}$ so its normal form has an invariant of degree 8 with value 1.

LEMMA 5.1. *Let $\mathcal{L} = \partial^2 + a_1\partial + a_0 \in k[\partial]$ be a normalized differential operator with projective Galois group $PG(\mathcal{L}) = S_4$ ($\mathcal{L}$ is normalized to have an invariant of degree 8 with value 1). Define $g_{\mathcal{L}} := 2a_1 + \frac{a_0{}'}{a_0}$. Then $\mathcal{L}$ is a proper pullback of $St^i_{S_4}$ and the pullback mapping is*

$$f = -\frac{7}{144}\frac{g_{\mathcal{L}}^2}{a_0}$$

PROOF. That $\mathcal{L}$ is a proper pullback of $St^i_{S_4}$ follows from lemma 4.3. Pick an unknown function f and form the change of variable $x = f$ in $St^i_{S_4}$. We obtain $a_0 = -\frac{7}{576}\frac{f'^2}{(f-1)^2 f}$ and $a_1 = -\frac{f''}{f'} + \frac{1}{2}\frac{f'}{f} + \frac{3}{4}\frac{f'}{f-1}$ Performing standard differential elimination on the latter, see [13, 14] and references therein, yields the above formula. ☐

With this formula, the algorithm in section 4.3 is straightforward to adapt (compute an invariant of degree 8 of L instead of a semi-invariant of degree 6).

5.2.3 Tetrahedral case A_4

Let $St^s_{A_4}$ denote the standard equation from section 4.3 with projective Galois group A_4. It has an invariant of degree 4 with value 1. As $G(L) \subset SL_2(C)$, our L has only semi-invariants in degree 4, but it has an invariant in degree 6. So, proceeding as in section 5.2.2 (with lemma 3.3.1 in mind) yields a new standard operator $St^i_{A_4}$ for A_4 with an invariant of degree 6 having value 1:

$$St^i_{A_4} = \partial^2 + \frac{2\left(3\,x^2 - 1\right)}{3x\,(x^2-1)}\partial + \frac{5}{144}\frac{1}{x^2\,(x^2-1)}$$

Its exponents are $(0, \frac{1}{3})$ at 1 and -1, and $(-\frac{1}{12}, \frac{5}{12})$ at 0 (the point ∞ is non-singular).

We assume that the differential operator L has projective Galois group A_4 and $G(L) \subset \mathrm{SL}_2(C)$. Thus L has an invariant of degree 6 with value σ. We normalize L by tensoring with $\partial + \frac{\sigma'}{6\sigma}$ so the resulting normal form $\mathcal{L}$ has an invariant of degree 6 with value 1.

LEMMA 5.2. *Let $\mathcal{L} = \partial^2 + a_1\partial + a_0 \in k[\partial]$ be a normalized differential operator with projective Galois group $PG(L) = A_4$, i.e L has an invariant of degree 6 with value 1. Then $\mathcal{L}$ is a proper pullback of $St^i_{A_4}$. Let $g_{\mathcal{L}} := 2a_1 + \frac{a_0{}'}{a_0}$. Then the pullback mapping is*

$$f = \pm\sqrt{1 + \frac{64}{5}\frac{a_0}{g_{\mathcal{L}}^2}}$$

PROOF. One can use the same differential elimination argument as for lemma 5.1. Note that Klein's theorem shows that $1 + \frac{64}{5}\frac{a_0}{g^2}$ must be the square of an element of k. ☐

REMARK 5.3. *The appearance of a square-root is no surprise because the standard equation for A_4 has a symmetry (exchange 1 and -1) so there are two solutions to the pullback problem (see [16, 2] and references therein), each "attached" to one of the two semi-invariants of degree 4. In the algorithm in section 4.3 we need to choose one of the two semi-invariants, hence the (apparent) uniqueness of the pullback formula there.*

An alternative approach to find and prove the formula in the lemma 5.2 is the following. As L is a pullback of St^i_{A4}, it is also a pullback of $St^i_{S_4}$ because $A_4 \subset S_4$. Now apply the S_4 formula to the A_4 standard equation, solve, and one obtains lemma 5.2. The same idea can also be used for D_2.

5.3 Dihedral Groups D_n, $n > 2$

The case $PG(L) \subset D_\infty$ is characterized by the existence of an invariant I_4 of degree 4. We assume that $PG(L) \neq D_2$ so the space of invariants of degree 4 has dimension 1 (and I_4 is the square of a semi-invariant of degree 2). Tensoring L with $\partial + \frac{I_4'}{4I_4}$, we obtain a normalized operator $\mathcal{L}$ which has an invariant of degree 2 with value 1. So we can use the algorithm from section 4.4 (start at step 3) and obtain the pullback function.

REMARK 5.4. *The difficulty in this subsection lies in deciding whether $PG(L)$ is some D_n or D_∞. Computing n is achieved by computing the torsion of some divisor from the integration algorithm, which can be achieved under our assumptions on k, see [6] or [2, 3].*

5.4 Quaternion Group D_2

There is a problem to choose a relevant normalization because the space of invariants of degree 4 is two-dimensional and, in our normalizations, we would need to choose one among those that is a square of a semi-invariant of degree 2 in order to use the formulas from section 4.4. Although this is possible (e.g [27]), we propose a few simpler approaches (the reader is welcome to select whichever one she likes best). As $G(L) \subset \mathrm{SL}_2(C)$, the operator has a unique (up to constants) invariant of degree 6 with value σ (the product of the three semi-invariants of degree 2). Tensoring L with $\partial + \frac{\sigma'}{6\sigma}$, we obtain a normalized operator $\mathcal{L}$ whose invariant of degree 6 has value 1.

Approach 1: We have $D_2 \subset A_4$. Moreover, $\mathcal{L}$ has an invariant of degree 6 with value 1. So $\mathcal{L}$ is a proper pullback of $St^i_{A_4}$ from section 5.2.3 and the pullback is computed directly with the algorithm from section 5.2.3. The good point is that no work is needed; the bad point is that the solutions will be given in terms of the solutions of $St^s_{A_4}$ which is not very good if, for example, we want the minimal polynomial or an expression by radicals.

Approach 2: In approach 1, we have computed a pullback F from $St^i_{A_4}$ so solutions of $\mathcal{L}$ are $\widetilde{H}_i(F)$ with $\widetilde{H}_i$ solutions of $St^i_{A_4}$. Now we precompute the pullback from D^i_2 to A^i_4. First send singularities to $0, 1, \infty$ by a Möbius transform; next, tensor by a first order operator so that the exponents are $(0, 1/3)$ at 0 and ∞. Changing x to x^3, the preimages of 0 and ∞ will have exponents $(0, 1)$ so they will be ordinary, while the preimages of 1 (i.e $1, j, j^2$) will have exponent differences 1/2: the resulting equation is thus a standard D_2 equation. Sending the singularities to $-1, 1, \infty$ and tensoring by a first order operator finally sends us to the standard operator St_{D_2}. We find that $\widetilde{H}_i(\frac{3\sqrt{-3}(x^2-1)}{x^3-9x}) = H_i(x)$ with H_i solutions of $St^i_{D_2}$. So the solutions of $\mathcal{L}$ will be $H_i(f)$ where f is a root of the third degree equation

$$\left(3\sqrt{-3}(f^2-1)\right) - F\left(f^3 - 9f\right) = 0 \qquad (5.9)$$

By Klein's theorem, the latter has three roots f in k which can be computed, e.g by factoring the above. We note that, because the solution is not unique, factoring is inevitable in this process.

6. CONCLUSION

THEOREM 6.1. *The algorithm of section 2 is correct.*

PROOF. The steps compute the projective Galois group by [27] or corollary 3.4. Step 2a is sections 5.3 and 4.4; Step 2b is sections 5.4 and 4.4; Step 3a is section 5.2.3; Step 3b is section 5.2.2; and Step 3c is sections 5.2.1 and 4.3. □

The algorithm presented here is very easy to implement for an admissible differential field. Further improvements and speedups can be provided in the case when $k = C(x)$. The algorithm is implemented in Maple 9.5. A draft implementation (and a maple worksheet to check most formulas of this paper) can be consulted at `http://www.unilim.fr/pages_perso/jacques-arthur.weil/issac05/`

Denote $H(x) = {}_2F_1([-1/60, 11/60], [2/3], 1/(x+1))$ which is one of the solutions of $St^s_{A_5}$. The Kovacic algorithm produces the minimal polynomial m_K of y'/y for some solution y of $St^s_{A_5}$, whereas Fakler's algorithm [15] produces the minimal polynomial m_F of a solution y of $St^s_{A_5}$. Note that m_F is preferable over m_K.

Now consider the following example: $L = 48x(x-1)(75x-139)\partial^2 + (2520x^2 - 47712x/5 + 3336)\partial - 19x + 36001/75$ which has projective Galois group A_5. The pullback function f is rather large (the degree is 31). By default our algorithm uses hypergeometric functions to denote the answer. In essence this means that x in the expression $H(x)$ above is being replaced by f. To get a solution of L in the same format as would have been produced by Kovacic's resp. Fakler's algorithm, one essentially has to substitute f for x in the solution that these algorithms provided for $St^s_{A_5}$. However, this substitution will lead to a large expression because x occurs many times in the expression m_K resp. m_F and all those occurrences are replaced by f. We compared the `kovacicsols` command in Maple 9.5 (which follows the usual Kovacic algorithm) with the algorithm presented here. The size of the output (measured with the command `length`) in Maple 9.5 was 236789 whereas for the new algorithm the size is only 1360. Note that this new algorithm is scheduled to appear in the `kovacicsols` command in the next version of Maple.

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1.20 Formal Solutions of Differential Equations

By MICHAEL F. SINGER. North Carolina State University. ISSAC '89, Portland, Oregon, July 17, 1989.

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Formal Solutions of Differential Equations

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We give a survey of some methods for finding formal solutions of differential equations. These include methods for finding power series solutions, elementary and liouvillian solutions, first integrals, Lie theoretic methods, transform methods, asymptotic methods. A brief discussion of difference equations is also included.

In this paper, I shall discuss the problem of finding formal expressions that represent solutions of differential equations. By using the term "formal", I wish to emphasize the fact that most of the time I will not be concerned with questions of where power series converge or in what domains the expressions represent functions. I shall talk about power series solutions, solutions that can be expressed in terms of special functions such as exponentials, logarithms, or error functions, solutions given implicitly in terms of elementary first integrals and Lie theoretic techniques. I shall briefly mention transform methods, asymptotic expansions and devote a final section to a short discussion of formal solutions of difference equations.

There are many open problems in these areas and I have included my favorite ones. I hope they will stimulate further work.

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I. Power Series Solutions of Differential Equations

My aim here is to contrast what is known about linear differential equations with what is known about non-linear differential equations. Good general references for information about linear differential equations are Poole (1960) and Schlesinger (1895). Consider the linear differential equation

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

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where the $a_i(x) \in \mathbb{C}((x))$, the field of formal Laurent series with finite principal parts. The point $x = 0$ is called an ordinary point of $L(y)$ if 0 is not a pole of any of the $a_i(x)$. When this is the case $L(y) = 0$ will have n linearly independent solutions $y_i = \Sigma a_{ij}x^j$, $0 \leq i \leq n-1$, in $\mathbb{C}[[x]]$, the ring of formal power series (furthermore, each of these solutions will converge in some neighborhood of 0, if the $a_i(x)$ converge in this neighborhood). Such a fundamental set of solutions can be found by setting $a_{ij} = 0$ for $0 \leq i$, $j \leq n-1$, $j \neq i$, and $a_{ii} = 1$, and using the differential equation to find a_{ij}, $j \geq n$, by recursion. If some $a_i(x)$ has a pole at 0, we say 0 is a singular point of $L(y)$. We say the 0 is a regular singular point if in any open angular sector Ω at 0 all solutions y of $L(y) = 0$, analytic in Ω, satisfy $\lim_{z \to 0} z^N y = 0$ for some $N \geq 0$. Fuchs showed that this is equivalent to saying that the order of the pole of each $a_i(x)$ at 0 is $\leq n - i$. If we let $\delta = x\frac{d}{dx}$, we may write

$$x^n L(y) = \delta^n y + b_{n-1}(x)\delta^{n-1}y + \ldots + b_0(x)y$$

for some $b_i(x) \in \mathbb{C}((x))$. In these terms, 0 is a regular point if and only if 0 is not a pole of any of the $b_i(x)$. Let $b_i(x) = \Sigma b_{ij}x^j$ and

$$P(\lambda) = \lambda^n + b_{n-1,0}\lambda^{n-1} + \ldots + b_{0,0}.$$

$P(\lambda)$ is called the indicial polynomial of $L(y)$ at 0. If 0 is a regular singular point of $L(y)$, then there exist n linearly independent solutions of $L(y) = 0$ of the form

$$y_i = x^{\lambda_i}(\varphi_{i0} + \varphi_{i1}\log x + \ldots + \varphi_{is_i}(\log x)^{s_i})$$

with $\varphi_{ij} \in \mathbb{C}((x))$ and λ_i a root of $P(\lambda) = 0$ of multiplicity s_i (Coddington & Levinson (1955), Ch. 4). Once the λ_i are determined, the φ_{ij} can be found using a recursive procedure, due to Frobenius, similar to the ordinary point case. This method has been implemented by several people (e.g. Lafferty (1977), Davenport (1988), Della Dora (1981a), (1981b), Watanabe (1970), Tournier (1987)). When 0 is not a regular singular point, we say that 0 is an irregular singular point. In this case, there exists n linearly independent solutions of the form

$$y_i = e^{Q_i(x)}x^{\gamma_i}(\varphi_{i0} + \varphi_{i1}\log x + \ldots + \varphi_{is_i}(\log x)^{s_i}) \tag{1}$$

where $Q_i(x)$ is a polynomial in x^{-1/q_i}, q_i a positive integer, $\gamma_i \in \mathbb{C}$, s_i a positive integer, and $\varphi_{ij} \in \mathbb{C}[[x^{1/q_i}]]$. Schlesinger (1987) (Vol. I, Sec. 110) describes a method for finding q_i and Q_i and a more modern algorithm based on Newton polygon calculations is given in Della Dora (1981c) (see also Levelt (1975)). Once q_i and Q_i are found, one makes a change of variable $y = e^{Q_i(x)}z$ and proceeds as in the regular singular point case. An implementation, in the DESIR system, is described in Della Dora (1981c) and Tournier (1987). One can make similar definitions with respect to any point $x = \alpha$ or even the point of infinity (this latter case reduces to the point $t = 0$ after we make a change of coordinates $t = \frac{1}{x}$ and

$\frac{d}{dt} = -t^2 \frac{d}{dt}$. All the above algorithms force one to work with algebraic numbers, even if the original equation has coefficients that are polynomials with rational coefficients. A method that minimizes the amount of factorizations needed to do these calculations is presented in (Della Dora *et al.*, (1985).

We now consider a system of linear differential equations

$$Y' = \frac{A(x)}{x^q} Y \tag{2}$$

where $A(x) \in M_n(\mathbb{C}[[x]])$, the ring of $n \times n$ matrices with entries in $\mathbb{C}[[x]]$, and q is a non-negative integer. If $q = 0$, we say that 0 is an ordinary point and if $q \geq 1$, we say that 0 is a singular point. The definitions (in terms of the growth of solutions near 0) of regular and irregular singular point carry over to this case, but there is no analogue of the criteria of Fuchs to distinguish between these two. To do so, we can proceed in several ways. One way (the cyclic vector method) is to convert the system (2) to a single nth order equation and then use Fuchs' criteria (Adjamagbo (1988), Bertrand (1985), Katz (1986), Malgrange (1974), (1981), Ramis (1978), (1984)). A second method is due to Moser (1960). This method considers transforms $Y \to BY$ with $B \in M_n(\mathbb{C}(x))$ and their effect on (2). One gets an equation of a similar form with a possibly different value of q. One tries to find a matrix B so that the resulting q is minimal. When this happens, it is known that $q = 1$ if and only if 0 is a regular singular point. Both methods are discussed, with implementations in mind, in Hillali (1982), (1983), (1986), (1987a), (1987b), (1987c), Hillali & Wazner (1983), (1986a), (1986b). Other criteria and methods for determining if a singular point is regular are discussed in Gerard and Levelt (1973) (c.f. in particular Theorem 4.5). These papers also discuss how one can use either method to calculate other invariants of (2) (e.g. Malgrange index, Katz invariants).

The formal solutions given by (1) do not necessarily involve convergent series. It is known that if the $a_i(x)$ are analytic in a neighborhood of the origin, then in any sufficiently small sector at the origin, there are analytic solutions having (1) as asymptotic expansions. Questions regarding calculating these solutions are addressed in Loday-Richaud (1988), Ramis (1978), (1984), (1985a), (1985b), and Ramis & Thomas (1981).

Before leaving linear differential equations, it should be noted that some work has been done to implement methods of expressing solutions of linear differential equations in terms of series involving Chebyshev polynomials Geddes (1977) or other special functions Cabay & Labahn (1989) and Chaffy (1986).

We now turn to nonlinear differential equations. Although some work regarding algorithms for finding series solutions of nonlinear differential equations has been done in the past (e.g. Fitch, Norman & Moore (1981), (1986) and Geddes (1981)) the first general algorithm was presented in Denef & Lipshitz (1984). They show that given a set S of ordinary polynomial differential equations in $y_1, \ldots, y_m$ with coefficients in $\mathbb{Q}[x]$ and initial conditions, one can decide if S has a solution in $K[[x]]$ satisfying these initial conditions, where $K = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$. Their basic idea in this is to show how one can find an integer N such that the system S is solvable if and only if S has a solution $\mod x^N$. This latter condition reduces to checking the solvability of a system of linear equations. Although their algorithm is very explicit, it does not seen to be efficient.

Deciding if a system of ordinary polynomial differential equations has a power series solution is a delicate question. In Denef & Lipshitz (1984), it is also shown that there is no algorithm to decide if such a system has a convergent solution or if such a system has a non-zero solution. The situation for partial differential equations is worse. Denef and Lipshitz show that there is no algorithm to decide if a linear partial differential equation with coefficients in $\mathbb{Q}[x_1, \ldots, x_9]$ has a solution in $\mathbb{C}[[x_1, \ldots, x_9]]$. Furthermore, there are systems of partial differential equations having infinitely many power series solutions, none of which are computable (i.e., the sequence of coefficients cannot be generated by a Turing machine).

In Grigor'ev & Singer (1988), the authors consider a Newton polygon method to find solutions of differential equations of the form $y = \sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ where the $\alpha_i \in \mathbb{C}$ and the β_i are real with $\beta_0 > \beta_1 > \ldots$. They show that if such an expression satisfies a polynomial differential equation $p(x, y, y', \ldots) = 0$, then $\lim \beta_i = -\infty$. Furthermore, given any such y and $p(x, y, y', \ldots)$, there exists an N such that for any $z = \Sigma\gamma_i x^{\delta_i}$ satisfying $p(x, z, z', \ldots) = 0$ with $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for all i with $\beta_i > N$, then $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for all i (that is, each y is finitely determined). The authors give a method for enumerating solutions of this form of a differential equation and show that it is an undecidable problem to determine if a system of polynomial differential equations has a solution of this form.

We have not yet mentioned power series solutions of algebraic equations. Algorithms for finding the Puiseux expansions (power series in rational powers of x) of algebraic functions are well known Knuth (1981), Ch. 4.7. The fastest to date is due to the Chudnovskys (1985). They have shown how algorithms for finding power series solutions of linear differential equations can be used to find Puiseux expansions of algebraic functions. The key observation is that if y satisfies an irreducible equation $f(x, y) = 0$ of degree n over $\mathbb{C}(x)$, then $[\mathbb{C}(x, y) : \mathbb{C}(x)] = n$ and $y' = -f_x/f_y \in \mathbb{C}(x, y)$, so $\mathbb{C}(x, y)$ is closed under the derivation $'$. This implies that $y, y', \ldots, y^{(n)}$ must be linearly dependent over $\mathbb{C}(x)$, so y satisfies nth order linear differential equation over this field. This equation can be calculated from $f(x, y)$ and then using an efficient version of the Frobenius algorithm one can calculate the Puiseux expansion of y. They are able to show that one can compute the first N terms of this expansion in $O(dN)$ operations and $O(dN)$ space, where d is the total degree of $f(x, y)$.

Other papers concerning power series solutions of differential equations are Bogen (1977), Fateman (1977), Lamnabhi-Lagarrigue & Lamnabhi (1982), (1983), Norman (1975), and Stoutemyer (1977).

II. Closed Form Solutions

We are concerned here with expressing the solutions of differential equations in terms of some given class of functions (e.g., exponentials, integrals and algebraic functions). We begin by considering the simplest differential equation

$$y' = \alpha$$

and ask when a solution (i.e., $y = \int \alpha$) can be expressed in terms of elementary functions, that is, in terms of sin, cos, exp, log, arctan, etc.; the functions of elementary calculus. For example, $y' = (2x)\exp(x^2)$ has an elementary solution $y = \exp(x^2)$ but $y' = \exp(x^2)$ does not (although this is not obvious). We wish to give the informal notion of expressible in elementary terms some mathematical rigor. This is done using the notion of a differential field. A field F is said to be a differential field with derivation $'$, if $' : F \to F$ satisfies $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all $a, b \in F$. The constants of F are $\{c \mid c \in F \text{ and } c' = 0\}$ and are denoted by $C(F)$. For example, $\mathbb{C}(x)$ with the derivation d/dx in a differential field as is the field of meromorphic functions on a connected open set in $\mathbb{C}$ with the usual derivation. To formalize the notion of elementary function, first notice that if one thinks in terms of functions of a complex variable, then sin, cos, tan, arctan, etc. can all be expressed in terms of exp and log. This motivates the following definition. Let $F \subset E$ be differential fields. We say E is an elementary extension of F if there is a tower of fields $F = E_0 \subset \ldots \subset E_n = E$ where $E_i = E_{i-1}(t_i)$ and either (i) t_i is algebraic over E_{i-1}, or (ii) $t_i'/t_i = u_i'$ for some $u_i \in E_{i-1}$ (i.e., $t_i = \exp(u_i)$), or (iii) $t_i' = u_i'/u_i$ for some $u_i \in E_{i-1}$ (i.e., $t_i = \log(u_i)$). We say that y is elementary over F if y belongs to an elementary extension of F. For example, $y = \exp(x\log(x+\sqrt{x}))$ is elementary over $\mathbb{C}(x)$, since y belongs to the last member of the tower $\mathbb{C}(x) \subset \mathbb{C}(x,\sqrt{x}) \subset \mathbb{C}(x,\sqrt{x},\log(x+\sqrt{x})) \subset \mathbb{C}(x,\sqrt{x},\log(x+\sqrt{x}),\exp(x\log(x+\sqrt{x})))$.

Our naive question "When can we express a solution of $y' = \alpha$ in terms of elementary functions?" can now be formalized as "Given a differential field F and an element α of F, when does $y' = \alpha$ have a solution in an elementary extension of F?" The answer is given by Liouville's Theorem: Let F be a differential field of characteristic zero and $\alpha \in F$. If $y' = \alpha$ has a solution in an elementary extension K of F, with $C(F) = C(K)$, then

$$\alpha = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}$$

where v and the u_i are in F and c_i are constants of F. In other words, if α has an elementary antiderivative, then $\int \alpha = v + \Sigma c_i \log(u_i)$, where v and the u_i only involve those functions that already appear in α. The condition on the constants is technical but necessary (if we work over the complex numbers, there is no problem; see Risch (1969) and Davenport, Siret & Tournier (1988) for a further discussion of this issue). Special cases of the above theorem were originally proved by Liouville (1833), (1835). Ostrowski gave a proof of this theorem in the context of differential fields in Ostrowski (1946). The work of Liouville and Ostrowski is discussed in Ritt (1948), along with additional work of Mordukhai-Boltovski and Ritt. A completely algebraic proof was first given by Rosenlicht in Rosenlicht (1968) (see also Rosenlicht (1976)). The best place to read a proof of this theorem is Rosenlicht (1972).

To get a feeling for why Liouville's Theorem is true, one should consider the following pieces of evidence. First, the theorem is true when α is in $\mathbb{C}(x)$. In this case we may expand α in partial fractions $\alpha = p(x) + \sum_i \sum_j a_{ij}(x - b_i)^{-n_{ij}}$. When we integrate α, each term contributes something in $\mathbb{C}(x)$, except if $n_{ij} = 1$, in which case we get $\log(x - b_i)$, which appears linearly. Secondly, we can look at the general case and ask: If we need a new algebraic, log or exponential to integrate an expression, how can this new function appear in the antiderivative. For example, if $\int \alpha$ is an algebraic function of α, then we can sum the conjugates of $\int \alpha$ and divide by their number and get a new antiderivative of α that is a rational function of α. Since the antiderivative is unique up to additive constant, the original algebraic function must be a rational function of α (i.e. no non-rational algebraic functions are needed). Now assume that we needed a new logarithm or an exponential to express our antiderivative. For example, assume that $\int \alpha = (\exp(u))^n + \ldots$. When we differentiate both sides of this equation, we get $\alpha = nu'(\exp(u))^n + \ldots$. Since we are assuming that $\exp(u)$ does not already occur in α, we must have $n = 0$. If $\int \alpha = (\log(u))^n \ldots$, then $\alpha = n(u'/u)(\log(u))^{n-1} + \ldots$. Since we assume that $\log(u)$ does not appear in α, we must have $n = 1$, i.e. the new log appears linearly. This heuristic argument can be formalized and is the basis of the argument in Rosenlicht (1972).

Liouville's Theorem gives a criterion for a function to have an elementary antiderivative and in Rosenlicht (1972) this is used to show that $\int \exp(x^2)$ is not elementary. A general algorithm to decide if a function, elementary over $\mathbb{C}(x)$ has an elementary antiderivative was given by Risch in a series of paper (Risch (1968), (1969), (1970)). The algorithm takes as input an elementary tower $K(x) \subset E_1 = K(x, t_1) \subset \ldots \subset E_m = K(x, t_1, \ldots, t_m)$ (where K is a finitely generated field of characteristic zero) and an element α in E_m and decides if it is of the form prescribed by Liouville's Theorem. If it is, the algorithm produces such an expression. Risch (1969) treated the case of a purely transcendental integrand. Improvements of this algorithm were made by many people (Bronstein (1988), Davenport, Siret & Tournier (1988), Davenport (1983) (this has a large and useful bibliography), Davenport (1986), Epstein (1975), Geddes & Stefanus (1989), Horowitz (1969), (1971), Kaltofen (1984), Norman (1983), Norman & Davenport (1979), Norman & Moore (1977), Rothstein (1976), (1977), Trager (1976), (1984), Yun (1977)). In Risch (1968) and Risch (1970), Risch outlined an algorithm for the mixed case; the case where algebraics are also allowed in the defining tower of α. This algorithm is much more complex than the previous one. When α is algebraic over $\mathbb{C}(x)$, new ideas and improvements were given by Trager and Davenport (Trager (1979), (1984), Davenport (1981)). Bronstein has generalized and applied these ideas to the general case in Bronstein (1990). The Risch algorithm for purely transcendental elementary functions has been implemented in most computer algebra systems. Bronstein's algorithm is being implemented at present in the SCRATCHPAD system.

All algorithms proceed by induction on the length of the defining elementary tower for α (the method of Norman & Moore (1977) does not, but it is known not to be an algorithm, see Norman & Davenport (1977) and Davenport (1986)). A particular α can belong to several different elementary towers. For example $\sqrt{x}\exp(x)$ belongs to both $\mathbb{C}(x, \sqrt{x}, \exp(x))$ and $\mathbb{C}(x, \log(x), \exp(x + (1/2)\log(x)))$. The first of these fields is built up using algebraic elements, while the second is purely transcendental. The efficiency of the algorithms depends heavily on the particular choice of defining tower. Some work has been done with regards to selecting a good defining tower (Davenport (1986) and Bronstein (1988)) but much more can be done. This motivates the following problem:

Problem 1. What is the "best" field of definition for an elementary function? Can one decide if a given elementary function belongs to a purely transcendental elementary extension of $\mathbb{C}(x)$?

Several generalizations of the Liouville Theorem have been made. Risch (1976) gives a Liouville type theorem for integration in terms of real elementary functions and Bronstein gives an algorithm in Bronstein (1989). In Singer, Saunders & Caviness (1985), a Liouville type theorem is presented, along with algorithmic considerations, that deals with integration in terms of a class of functions that includes the elementary functions as well as the error function and the logarithmic integral. This work has been generalized by Cherry (Cherry (1985), (1986)) and Knowles (1986). In these papers the structure of the defining tower plays a crucial role in the algorithmic results and these algorithms only treat certain classes of functions (in particular, they do not handle functions that are built up using algebraic functions). Using ideas developed in algebraic K-theory, Baddoura (1989) gives a Liouville type theorem and algorithm for integration in terms of elementary functions and dilogarithms. Baddoura's work also only deals with a restricted class of functions. There are still many open problems concerning generalizations of Liouville's Theorem and the interested reader is referred to the above papers. Some heuristics are also given in Picquette (1989).

So far we have only considered indefinite integrals. Heuristic techniques for evaluating definite integrals are discussed in Geddes & Scott (1989), Kolbig (1985) and Wang (1971). Recently, Almkvist & Zeilberger (1989) have proposed a method for evaluating expressions of the form $f(x) = \int_a^b F(x,y)dy$, for example

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{y^2} - y^2\right) dy = \sqrt{\pi}\exp(-2x).$$

They consider functions $F(x,y)$ that satisfy a pair of linear partial differential equations of the form

$$P(x,y,\partial/\partial x)F = p_n(x,y)\frac{\partial^n F}{\partial x^n} + \ldots + p_0(x,y)F = 0, \text{ and}$$
$$Q(x,y,\partial/\partial y)F = q_m(x,y)\frac{\partial^m F}{\partial x^m} + \ldots + q_0(x,y)F = 0,$$

with coefficients that are polynomials in x and y (these functions are called D-finite (Lipshitz (1988)). In this case it is known that $f(x)$ will satisfy an ordinary linear differential equation

$$L(x, d/dx)f = a_N(x)\frac{d^N f}{dx^N} + \ldots + a_0(x)f = 0$$

(see Lipshitz (1988)). L can be found using an elimination algorithm. One then solves $L(x, d/dx)f = 0$ in terms of some class of functions (if this is possible, see below) and compares initial conditions to get a closed form expression for $f(x)$.

We now turn to the problem of solving more complicated differential equations in closed form. We start by considering linear differential equations

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0.$$

When the a_i are constants, we teach our undergraduates how to express all solutions as sums of products of polynomials and exponentials. An implementation of an algorithm to do this is described in Tournier (1979). When the a_i are rational functions, some heuristics and special cases are discussed in Malm (1982) and Schmidt (1979) and implementations of variation of parameters and the method of undetermined coefficients are discussed in Schmidt (1976) and Rand (1984).

We now turn to some general algorithms. Assume that the $a_i \in k(x)$, where k is some finitely generated extension of $\mathbb{Q}$. The question of when $L(y) = 0$ has only solutions that are algebraic over $k(x)$ was originally treated by F. Klein in 1877 when $n = 2$. Klein showed that if $L(y) = 0$ has only algebraic solutions then there is a change of variables $x = \varphi(t)$ such that the new equation is of a very special form, that is it appears in a list of all linear differential equations with three singular points and only algebraic solutions discovered by H. A. Schwarz around 1870 (see Gray (1986) for a discussion of the work of Klein, Schwarz and their contemporaries). A modern discussion of Schwarz's list and related material appears in Matsuda (1985). Klein's method was made effective by Baldassarri and Dwork in Baldassarri & Dwork (1979) and Baldassarri (1980). For $n \geq 2$, P. Painlevé and his student A. Boulanger gave a decision procedure in 1898 (a similar procedure was rediscovered by the present author in 1979, see Singer (1980)).

The next natural class of functions are the liouvillian functions. These are the functions that can be built up from $k(x)$ using integration, exponentiation, algebraic functions and composition (a formal definition is given below). These functions are named after J. Liouville, who was the first to give necessary and sufficient conditions for a second order homogeneous linear differential equation to have a solution of this form, Liouville (1839), (1841) and Ritt (1948). When $n = 2$, Kovacic (1986) gave an algorithm to decide if all solutions of $L(y) = 0$ can be expressed in terms of liouvillian functions and showed how to exhibit a basis when this is the case. Kovacic's algorithm is very explicit and parts of it have been implemented in MACSYMA (Saunders (1981)) and MAPLE (Char (1986)) (see also Smith (1984)). Improvements to this algorithm have been given in Duval & Loday-Richaud (1989). For $n \geq 2$, an algorithm is presented in Singer (1981) to decide if $L(y) = 0$ has a non-zero liouvillian solution and, if so, shows how to construct a vector space basis for the space of all such solutions (some of the ideas already occur in Marotte (1898), but I was not aware of this at the time Singer (1981) was written). A natural generalization of this is to find an effective procedure to produce for a given linear differential equation $L(y)$, with coefficients in a liouvillian extension of $\mathbb{Q}(x)$, a basis for the liouvillian solutions of $L(y) = 0$. I have recently shown (Singer (1988c)) that one can do this if the linear differential equation has coefficients in a purely transcendental liouvillian extension of $\mathbb{C}(x)$ or in an elementary extension of $\mathbb{C}(x)$. The algorithm presented there is extremely inefficient and can use improvement and generalization to handle the complete liouvillian case.

Problem 2. Find an efficient algorithm to decide if an nth order linear differential equation with rational function (or liouvillian) coefficients has a liouvillian solution.

Some progress has been made on this problem. A problem that comes up in Singer (1981) is the problem of factoring linear differential equations. Schwarz discusses an algorithm (with implementation) for this in Schwarz (1989) and Grigor'ev discusses another algorithm and gives complexity bounds in Grigor'ev (1988). In Singer (1981), group theoretical methods were used to obtain certain bounds (see below) and Ulmer (1989) shows how stronger techniques from group theory yield better bounds.

Other work on deciding if linear differential equations have liouvillian solutions appears in Watanabe (1981), where techniques are developed to transform a given linear equation to a hypergeometric equation and Watanabe (1984), where change of variable techniques are discussed that will take a linear differential equation with coefficients in a liouvillian extension of $\mathbb{C}(x)$ to one with coefficients in $\mathbb{C}(x)$.

I will now give a sketch of some of the ideas involved in Kovacic (1986) and Singer (1981), and start by defining some notions from differential algebra (Kaplansky (1957) and Kolchin (1973) are good references for this). Let F be a differential field of characteristic 0. If $L(y) = 0$ is an nth order linear differential equation with coefficients in F, we can formally adjoin to F a set of n solutions $y_1, \dots, y_n$ of $L(y) = 0$, linearly independent over $\mathbb{C}$, and their derivatives. When C is algebraically closed, we can choose $y_1, \dots, y_n$ so that the field $K = F(y_1, \dots, y_n, y_1', \dots, y_n', \dots, y_1^{(n-1)}, \dots, y_n^{(n-1)})$ contains no new constants (note that this field is closed under $'$ since $y_i^{(m)}$, $m \geq n$, can be expressed in terms of lower order derivatives of y_i using $L(y_i) = 0$). Such a K is unique up to a differential F-isomorphism and is called the Picard-Vessiot extension of F corresponding to $L(y) = 0$. Let $G = \{\sigma \mid \sigma$ is an automorphism of K, $\sigma(u)' = \sigma(u')$ for all $u \in K$ and $\sigma(v) = v$ for all $v \in F\}$. G is called the galois group of the equation $L(y) = 0$ over F (or of the field K over F). If $y \in K$ is any solution of $L(y) = 0$ and $\sigma \in G$, then $\sigma(y)$ is also a solution of $L(y) = 0$. One can show that this implies that $y = \Sigma c_j y_j$ for some $c_j \in C$. Therefore, for each i, $\sigma(y_i) = \Sigma c_{ij} y_j$ for some $c_{ij} \in K$. In this way we may associate a matrix (c_{ij}) with every $\sigma \in G$. (c_{ij}) is invertible, so this gives us an isomorphism of G into $GL(n, C)$, the group of invertible $n \times n$ matrices over C. Identifying G with its image, it can be shown that $G = GL(n, C) \cap V$, where $V \subset C^{n^2}$ is the zero set of some collection of polynomials (such a set is said to be closed in the Zariski topology). There is a galois theory that identifies differential subfields K_1, $F \subset K_1 \subset K$, with Zariski closed subgroups of G (a closed subgroup corresponds to the field of elements left fixed by all its members; in particular F corresponds to G). We can formalize the notion of solvable in terms of liouvillian functions. K is said to be a liouvillian extension of k if there is a tower of fields $k = K_0 \subset \dots \subset K_n = K$ such that $K_i = K_{i-1}(t_i)$, where either $t_i' \in K_{i-1}$ or $t_i'/t_i \in K_{i-1}$ or t_i is algebraic over K_{i-1} (the first two cases correspond to t_i being an integral or an exponential). A fundamental theorem states that $L(y) = 0$ is solvable in terms of liouvillian functions (i.e. its Picard-Vessiot extension lies in a liouvillian extension of F) if and only if its galois group contains a solvable subgroup of finite index (Kaplansky (1957), Kolchin (1973), Singer (1988b)).

Let us now consider the problem of finding liouvillian solutions of $L(y) = 0$. For simplicity, let us just try to decide if all solutions of $L(y) = 0$ are liouvillian. The galois theory implies that this is the case if and only if the galois group of $L(y)$ has a solvable subgroup of finite index. An effective version of the Lie-Kolchin Theorem asserts that in this case G will have a subgroup H such that the elements of H can simultaneously be put in upper triangular form and such that the index of H is bounded by $I(n)$, a computable function of n. If y is a common eigenvector of H, then $\sigma(y'/y) = cy'/cy = y'/y$ so y'/y is left fixed by H. This implies that y'/y is algebraic over F of degree bounded by $I(n)$. Therefore if $L(y) = 0$ is solvable in terms of liouvillian functions, $L(y) = 0$ will have a solution y such that y'/y is algebraic over F of degree bounded by $I(n)$. We now must decide if $L(y) = 0$ has such a solution. The idea is to look for candidates for the minimal polynomial of $u = y'/y$. If $p(u) = u^N + b_{N-1}(x)u^{N-1} + \dots + b_0(x)$ $(N \leq I(n))$ is the minimal polynomial of such a u, then one can show that there exist solutions $z_1, \dots, z_N$ of $L(y) = 0$ such that each b_i will be the ith symmetric function of the z_j'/z_j. By studying the poles of the coefficients of $L(y) = 0$, we can bound the number and order of the poles and zeroes of the b_i. This allows us to bound the degrees of the numerators and denominators of the b_i. Therefore if $L(y) = 0$ has only liouvillian solutions, it will have a solution y such that $u = y'/y$ satisfies a polynomial over $k(x)$ of degree $\leq I(n)$ whose coefficients have numerators and denominators of effectively bounded degrees. Elimination theory then allows us to decide if such a solution exists and produces u. We then use the change of variables $y = ze^{\int u}$ to get a new equation $L^*(z) = 0$ of lower order and proceed via induction. Actually, to make the induction work we prove a stronger result: given a linear differential equation with coefficients in an algebraic extension of $\mathbb{Q}(x)$, one can find in a finite number of steps a basis for the space of liouvillian solutions of $L(y) = 0$. This is done in Singer (1981).

So far, we have only been considering homogeneous linear differential equations, but one can ask the same questions about non-homogeneous linear differential equations $L(y) = b$. Such questions are considered in Davenport (1984), (1985), Davenport & Singer (1985), (1986), where in addition some open problems are mentioned.

We now turn to the general problem of solving a homogeneous linear differential equation $L(y) = 0$ of order n in terms of algebraic combinations and superpositions of solutions of linear differential equations of lower order (not necessarily homogeneous). In this context, asking for liouvillian solutions of a linear differential equation is the same as asking: When can it be solved in terms of first order linear equations (all solutions of first order linear equations are liouvillian and liouvillian functions are built up using algebraic combinations of solutions of $y' = a$ and $y' - ay = 0$)?

One can next ask: When can the solutions of a homogeneous linear differential equation be expressed in terms of solutions of linear differential equations of order at most two. Special cases of this question have been considered by Clausen, Goursat, Bailey, Ramanujan and others (Erdelyi *et al.* (1953)), who tried to understand when the product of two generalized hypergeometric functions is again a generalized hypergeometric function. They discovered beautiful formulas, such as

$$ {}_1F_1(\alpha,\rho;z)\,{}_1F_1(\alpha,\rho;-z) = {}_2F_3(\alpha,\rho-\alpha;\rho,(1/2)\rho,(1/2)(\rho+1);z^2/4) $$

I formalized the notion of solvability in terms of second order linear differential equations in Singer (1985). Briefly, a homogeneous linear differential equation is said to be solvable in terms of second order linear differential equations if the associated Picard-Vessiot extension lies in a tower of fields, each generated over the previous one by either an algebraic element or a solution of second order linear differential equation (we consider first order linear differential equations to be degenerate second order equations and allow them as well). In Singer (1985), I gave a criterion in terms of the galois group, for a homogeneous linear differential equation $L(y) = 0$, with coefficients in an arbitrary differential field k of characteristic 0, to be solvable in terms of second order linear differential equations. For example, if $L(y)$ has order 3, then it is solvable in terms of second order linear differential equations if and only if one of the following holds: (i) $L(y) = L_1(L_2(y))$ or $L(y) = L_2(L_1(y))$, where $L_1(y)$ and $L_2(y)$ are linear homogeneous differential polynomials of orders 1 and 2 respectively, with coefficients algebraic over k, or (ii) $L(y) = 0$ has a basis of its solution space of the form

$$\begin{aligned} y_1 &= b_0u^2 + b_1(u^2)' + b_2(u^2)'' \\ y_2 &= b_0uv + b_1(uv)' + b_2(uv)'' \\ y_3 &= b_0v^2 + b_1(v^2)' + b_2(v^2)'' \end{aligned}$$

where the b_i are algebraic over k and $\{u, v\}$ is a basis of the solution space of a second order homogeneous linear differential equation of order 2 with coefficients in k (for example, the solution space of $y''' - 4xy' - 6y = 0$ is spanned by $(u^2)', (uv)'$, and $(v^2)'$, where $\{u, v\}$ is a basis for the solutions of $y'' - xy = 0$). In Singer (1985), I show how this can be used to give a decision procedure to determine if an arbitrary third order homogeneous linear differential with coefficients in $\mathbb{Q}(x)$ can be solved in terms of second order linear differential equations.

The general problem of solving homogeneous linear differential equations in terms of lower order linear differential equations is considered in Singer (1988a) (see Singer & Tretkoff (1985) for a discussion of a related problem). Again this notion can be formalized in terms of towers of fields. Necessary and sufficient conditions can be given in terms of the Lie algebra of the galois group. One result is that a homogeneous linear differential equation <u>cannot</u> be solved in terms of lower order linear differential equations if and only if the Lie algebra of its galois group is simple and has no non-zero representations of smaller degree. I do not know of any general algorithms, and pose

<u>Problem 3</u>. Give an effective procedure to decide if a homogeneous linear differential equation can be solved in terms of linear differential equations of lower order. (One can show that a solution of Problem 6 below would yield a solution of this.)

When we consider the question of solving a third order homogeneous linear differential equation in terms of second order linear equations, the algorithm given in Singer (1985) does not allow us to restrict in advance the kind of second order equations we can use. This suggests

<u>Problem 4</u>. Give a procedure to decide if a homogeneous linear differential equation can be solved in terms of solutions of a restricted class of linear differential equations (e.g., Bessel functions).

Recall that a power series $F(x, y)$ in two variables is D-finite if f satisfies a system of non-zero differential equations of the form $P(x, y, \partial/\partial x)F = 0$ and

$Q(x, y, \partial/\partial y)F = 0$. For example, algebraic functions of two variables are D-finite. As mentioned above if F is such a function then $f(x) = \int_b^a F(x, y)dy$ will satisfy a linear differential equation over $\mathbb{C}(x)$. The solutions of the hypergeometric equation can be expressed in this form

$$f(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt$$

Poole (1960). This leads to the following question.

Problem 5. Find a procedure to decide if a linear differential equation has a nonzero solution of the form $\int_b^a f(x, y)dy$, where f is an algebraic function of two variables, and produce one if it does.

Related to the problem of solving linear differential equations in finite terms is the problem of deciding if two linear differential equations are equivalent under a change of coordinates (Berkovich, Gerdt, Kostova&Nechaevsky (1989), Kamran & Olver (1986), Neuman (1984), (1985)) and finding linear differential operators that commute with a given linear differential operator (which then can be used to find solutions of the original operator, see Gerdt & Kostov (1989)).

In most of the above considerations, the galois group of a homogeneous linear differential equation plays a crucial role. Yet unlike the situation with algebraic equations, there is no known algorithm to calculate the galois group of a homogeneous linear differential equation (i.e. produce a set of polynomials defining this group in $GL(n, C) \subset C^{n^2}$) or even its dimension as an algebraic variety (for $n = 2$ or 3 this can be done as a consequence of the algorithms described above, but for $n > 3$, nothing is known). This suggests

Problem 6. Give an algorithm that will find the galois group of any homogeneous linear differential equation with coefficients in $\mathbb{Q}(x)$, or at least calculate its dimension.

There has been some recent activity concerning calculation of the differential galois groups of certain classes of linear differential equations. In Beukers, Bronawell & Heckman (1988), Beukers & Heckman (1987) and Katz (1987), the authors are able to extract representation theoretic information about the galois groups from information at the singular points of the differential equation and combining this with information about root systems of simple Lie groups, can give usable sufficient conditions for an nth order linear differential equation to have a "large" galois group (i.e. the galois group contains $SL(n, \mathbb{C})$ or $SP(n, \mathbb{C})$). Katz is able to refine these techniques in Katz (1989) to calculate the Lie algebra of the galois groups of many differential equations. In Duval & Mitschi (1988) and Mitschi (1989a), (1989b), the authors use the theory of the "savage π_1" (see below) developed by Ramis (1985a), (1985b), (1988) and Martinet & Ramis (1988) to explicitly calculate the galois groups of generalized confluent hypergeometric equations.

Related to the galois group is the notion of the monodromy group. Given a homogeneous linear differential equation $L(y)$ with coefficients in $\mathbb{C}(x)$, let $\{a_1, \ldots, a_m\}$ be the singular points (possibly including ∞) and let $y_1, \ldots, y_n$ be a fundamental set of solutions at a regular point a_0. Given any path γ in the Riemann sphere $S^2 - \{a_1, \ldots, a_m\}$, we can analytically continue $y_1, \ldots, y_n$ around γ and get a new fundamental set of solutions. This new set is a linear combination of the old set, so we can associate to γ an invertible matrix A_γ. A_γ depends only on the homotopy class of γ and we get a homomorphism from $\pi_1(S^2 - \{a_1, \ldots, a_m\})$ to $GL(n, \mathbb{C})$ called the monodromy representation of the differential equation. The image of this homomorphism is called the monodromy group (see Poole (1960) and Katz (1976)). In general, it is very difficult to compute this group. Problem 6 can therefore be restated for monodromy groups.

When all the singular points of $L(y)$ are regular singular points, we know, (e.g., Tretkoff & Tretkoff (1979)) that the Zariski closure of the monodromy group is the galois group. This is not the case when we have irregular singular points (e.g. the monodromy group of $y' - y = 0$ is trivial but the galois group is $\mathbb{C}^*$). Recall that at a singular point (for simplicity, we assume this to be 0), there are n linearly independent solutions of the form

$$y_i = e^{Q_i(x)} x^{\gamma_i} (\varphi_{i0} + \varphi_{i\ell} \log x + \ldots + \varphi_{is_i} (\log x)^{s_i})$$

where $Q_i(x)$ is a polynomial in x^{-1/q_i}, q_i a positive integer, $\gamma_i \in \mathbb{C}$, s_i a positive integer, and $\varphi_{ij} \in \mathbb{C}[[x^{1/q_i}]]$. Let $\nu = LCM\{q_i\}$ and $t = x^{1/\nu}$. Let $K = \mathbb{C}\{t\}[t^{-1}]$, the ring of meromorphic functions in t and $\hat{K} = \mathbb{C}[[t]][t^{-1}] = \mathbb{C}((t))$. In this situation, Ramis defines a group to replace the classical monodromy group. This group is generated by three subsets: the exponential torus, the formal monodromy and the Stokes matrices. Let E be the Picard-Vessiot extension of $\mathbb{C}\{x\}[x^{-1}]$ generated by the y_i. The exponential torus is defined as follows: $K(e^{Q_1(x)}, \ldots, e^{Q_n(x)})$ is a Picard-Vessiot extension of K whose galois group over K is $(\mathbb{C}^*)^r$ for some r. Ramis calls this group the exponential torus $\mathcal{T}$ and shows that it is a subgroup of the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. One can also form the extension $F = \hat{K}(\log t, \{t^{\gamma_i}\}, \{e^{Q_i}\})$ of $\hat{K}$. Note that E is a subfield of this extension. The map $t \to t \cdot \exp(2\pi i/v)$ induces an automorphism of F, which in turn induces an automorphism of E. In this way we can consider $\mathbb{Z}/\nu\mathbb{Z}$ a subgroup of the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$; this is called the formal monodromy. Although the φ_{ij} above are formal series, it is known that in sufficiently small angular sectors, they are the asymptotic expansions of analytic functions. Ramis shows that by demanding a special kind of asymptotic expansion (this is the notion of k-summability) then one can canonically select the sectors and canonically select the analytic functions representing these formal solutions (strictly speaking this statement is only true under an additional assumption on the Newton polygon of the linear differential equation dual to the one under consideration. The technically correct statement can be found in the above references, but the above statement gives the flavor of the result). These sectors overlap and on the overlap the respective solutions are related to each other by a matrix change of basis. The matrices gotten in this way are called the Stokes matrices and Ramis shows that they are also in the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. Ramis is finally able to show that the Zariski closure of the group generated by the exponential torus, the formal monodromy and the Stokes matrices is the local galois group, i.e. the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. Ramis also shows that one can formally construct a group $\prod$, the "savage π_1" such that any group generated by the exponential torus, formal monodromy and Stokes matrices of a singular point is a representation of $\prod$. This gives a generalization of the classical monodromy representation at a point.

The exponential torus and the formal monodromy can be calculated from the formal expressions (above) for the solutions y_i. When one has integral representations of the solutions y_i (for example as G-functions) then one can also calculate the Stokes matrices. Furthermore, if the differential equation has only two singular points, one regular and one irregular, then the local galois group at the irregular singular point is the same as the global galois group (i.e. the galois group over $\mathbb{C}(x)$). This is the ideal used in Duval (1989), Duval & Mitschi (1988) and Mitschi (1989a), (1989b).

We now turn to non-linear differential equations. A liouville type theorem describing the form of elementary solutions of such equations was given by Mordukhai-Boltovski (see Ritt (1948)) for first order non-linear differential equations with coefficients in $\mathbb{C}(x)$, and generalized to higher order equations in Singer (1975), Risch (1979) and Rosenlicht (1977). Mordukhai-Boltovski's theorem states that if $f(x, y, y') = 0$ is a polynomial first order differential equation with coefficients in $\mathbb{C}$ that has an elementary solution, then the equation has a solution of the form

$$y = G(x, \varphi_0 + a_1 \log \varphi_1 + \ldots + a_r \log \varphi_r)$$

or

$$y = G(x, \exp(\varphi_0 + a_1 \log \varphi_1 + \ldots + a_r \log \varphi_r))$$

where the a_i are in $\mathbb{C}$, G is an algebraic function of two variables and the φ_i are algebraic functions of one variable. Except in special cases, I do not know how to make this result effective.

Problem 7. Give a procedure to decide if a polynomial first order differential equation $f(x, y, y') = 0$ has an elementary solution and to find one if it does.

The final issue I wish to bring up in this section is the general question of deciding if a set of polynomial differential equations $\{p_\alpha = 0\}$ in $y_1, \ldots, y_n$ (say with coefficients in $\mathbb{Q}$) is consistent, that is, if the equations have any solutions at all. Closely related to this problem is the problem of determining if every solution of a set of differential equations $\{p_\alpha = 0\}$ is also a solution of another differential equation $q = 0$. Ritt gave an effective procedure for this (Ritt (1966)) and in the process initiated the study of differential ideals and differential algebra in general.

Note that when we say solution, we mean an analytic solution (Rubel (1983) discusses the failure of differential algebra to deal with non-analytic solutions). This procedure was generalized by Seidenberg (1956) and Grigor'ev (1989). Recently Wu has implemented parts of Ritt's procedure (Wu (1987a), (1987b), (1989)) (see also Wang (1987)). In particular, he can show that Newton's laws can be mechanically derived from Kepler's laws. Besides considering the efficiency of Ritt's algorithms, there are still problems in effective differential ideal theory that are open and deserve more attention. We mention one and refer the reader to Ritt (1966) and Kolchin (1973) for relevant definitions

Problem 8. Give an algorithm that finds the minimal prime components of a radical differential ideal.

There are well known algorithms that find the prime components of a radical ideal of (nondifferential) polynomials, but this problem is open in the differential case.

Related to the ideal theory of differential equations is the question of finding Groebner basis for systems of linear partial differential equations (Galligo (1985) and Chen (1989), Kandri-Rody & Weispfenning (1990)), the general problem of simplifying systems of differential equations (Wolf (1985a), (1985b)), and the problem of generating all integrability conditions for systems of partial differential equations (Schwarz (1984)). In Galligo (1985), the author also mentions other problems concerning D-modules, that is modules over the ring $\mathbb{C}[x_1, \ldots, x_n, \partial/\partial x_1, \ldots, \partial/\partial x_n$ These modules have been useful in studying properties of solutions of systems of linear differential equations.

III. First Integrals

In elementary courses in differential equations, I discuss the predator-prey equations

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{aligned}$$

and show that the function

$$F(x,y) = dx + by - c\log x - a\log y$$

is constant on solution curves $(x(t), y(t))$. By studying the critical points of $F(x,y)$ one can then show that all solution curves are closed, that is, all solutions are periodic. A non-constant function that is constant on solution curves is called a first integral. In Singer (1977) and Prelle & Singer (1983), we showed that if differential equations have elementary first integrals, they must be of a very special form. For example, if

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{3}$$

where P and Q are polynomials with complex coefficients, has an elementary first integral, it has one of the form

$$F(x,y) = v_0(x,y) + \Sigma c_i \log(v_i(x,y))$$

where the c_i are constants and the v_i are algebraic functions of two variables. Furthermore, we showed in Prelle & Singer (1983), that if (3) has an elementary integral then there exists an R with $R^n \in \mathbb{C}(x,y)$ for some nonzero integer n, such that $d(RQ\,dx - RP\,dy) = 0$ (i.e. $\frac{\partial(RQ)}{\partial y} + \frac{\partial(RP)}{\partial x} = 0$). Such an R is called an <u>integrating factor</u> and once one is determined, we show in Prelle & Singer (1983) how to determine if (3) has an elementary first integral. Let R be an integrating factor and write $R^n = \prod f_i^{n_i}$ where f_i are irreducible polynomials and n_i are nonzero integers. One can show (Prelle & Singer (1983)) that since R is an integrating factor of (3) we must have $f_i \mid Df_i$ where $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$. Conversely, Darboux showed that if one could find all irreducible f such that $f \mid Df$, then one could decide if there is an integrating factor (see Ince (1944), p. 31). We also know, (Jouanolou (1979), p. 109 and Singer (1988)) that for each system (3) there is an integer N such that if f is irreducible and $f \mid Df$, then the degree of f is less than N, but we do not know any effective procedure for determining N. N does not depend only on the degrees of P and Q in (3) but also on the coefficients as the following example shows. Let $P = (n+1)x$ and $Q = ny$, then $D = (n+1)x\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y}$. One checks that $f = x^n - y^{n+1}$ satisfies $f = n(n+1)Df$. The problem of finding integrating factors and elementary first integrals reduces to

<u>Problem 9</u>. Given $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, with $P, Q \in \mathbb{C}[x,y]$, effectively bound the degrees of all f in $\mathbb{C}[x,y]$ that are irreducible and satisfy $f \mid Df$.

Both Poincaré (1934) and Painlevé (1972) worked on this problem and gave partial results. A modern account of related work appears in Jouanolou (1979).

Even without solving Problem 9, one can use the algorithm outlined in Prelle & Singer (1983) by arbitrarily assigning a bound to the degree of the f's such that $f \mid Df$. The drawback is that the algorithm will sometimes not find a first integral when one exists. This approach has been implemented in Shtokhamer, Glinos & Caviness (1986) with surprising success.

Prelle & Singer (1983) also contains results that imply that if an nth order differential equation $f(x, y, y', \ldots, y^{(n)}) = 0$ has an elementary first integral, it must be of a very special form. These other results have not been made effective. Risch (1976) contains related results.

Singer (1988) contains the foundations of a theory of liouvillian first integrals, that is liouvillian functions of several variables that are constant on solution curves of differential equations. This paper also contains algorithmic considerations. For example, I show that one can decide if (3) has a liouvillian first integral if one can decide the following question:

Problem 10. Given P, Q in $\mathbb{C}[x, y]$ and a, b in $\mathbb{C}(x, y)$, decide if $DU + aU = b$ has a solution a in $\mathbb{C}(x, y)$, where $D = P(\partial/\partial x) + Q(\partial/\partial y)$, and if so find such a solution.

Except in special cases I am unable to give such a procedure, nor am I able to reduce this question to the previous question.

There are several other approaches to finding first integrals. The approach using Lie methods is described below. Schwarz (1985) and Wolf (1987a), (1987b) describe methods that search for polynomial first integrals of an a priori bounded degree. In Goldman (1987a), (1987b) and Sit (1989), the authors describe a method to find polynomial first integrals (or more generally, first integrals that are sums of monomials with real or complex exponents) with an a priori bounded number of terms.

IV. Lie Methods

Both the problem of finding closed form solutions of differential equations and the problem of finding integrating factors can be attacked using Lie group methods. The basic idea is to find a group of symmetries of the differential equations and then use this group to reduce the order or the number of variables appearing in the equation. I will exhibit this idea by discussing Lie's discovery that the knowledge of a one-parameter group of symmetries of an ordinary differential equation of order n allows us to reduce the problem of solving this equation to that of solving a new differential equation of order $n-1$ and integrating. In the case of a first order equation, I will also discuss how the knowledge of a one-parameter group of symmetries allows one to construct an integrating factor. I will be closely following the expositions in Markus (1960), pp. 1–80 and Olver (1979), (1986), Ch. 2, although most of the results mentioned here can be found (in one form or another) in Lie's original works (for example, the comments following Example IV.5 appear as Satz 3 of Lie (1922).

There seem to be no totally general methods for finding the symmetry group of a differential equation, but there are methods that do handle large classes of equations. In Schwarz (1988), Schwarz gives an introduction to Lie methods and differential equations with a special emphasis on the use of computer algebra in computing symmetries. Sample programs and many examples, including symmetries of partial differential equations are also given there. Implementations are also discussed in Char (1980) and the works of Steinberg. Olver (1986), Schwarz (1988) and Steinberg (1983), (1985) are a good source of additional references.

I start with several key definitions. A local one-parameter group acting on $\mathbb{R}^2$ is an open set V, $\{0\} \times \mathbb{R}^2 \subset V \subset \mathbb{R} \times \mathbb{R}^2$ and a C^∞ map $\phi : V \to \mathbb{R}^2$ such that (1) $\phi(0,(x,y)) = (x,y)$ for all $(x,y) \in \mathbb{R}^2$, and (2) $\phi(g,\phi(h,(x,y))) = \phi(g+h,(x,y))$ whenever $g,h \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$ and $(h,(x,y))$, $(g,\phi(h,(x,y)))$ and $(g+h,(x,y))$ are in V (i.e., whenever (2) makes sense). If $V = \mathbb{R} \times \mathbb{R}^2$, we say ϕ is global. We sometimes will write $\phi_t(x,y)$ for $\phi(t,(x,y))$.

EXAMPLE IV.1: (Olver (1979), p. 204) Let $V = \{(t,(x,y)) \mid ty \neq 1\}$ and let

$$\phi(t,(x,y)) = \left(\frac{x}{1-ty}, \frac{y}{1-ty}\right)$$

Note that this cannot be extended to a global group acting on $\mathbb{R}^2$.

An infinitesimal one-parameter group is a system of differential equations $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ (or, more geometrically, the vector field $f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$). Given a local one parameter group $\phi(t,(x,y)) = (F(t,(x,y)), G(t,(x,y)))$, we can define an infinitesimal one parameter group by $\frac{dx}{dt} = \frac{\partial}{\partial t}(F(t,(x,y))\Big|_{t=0}$, $\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(G(t,(x,y))\Big|_{t=0}$. Conversely, given an infinitesimal one-parameter group, if $x(t,(x_0,y_0))$, $y(t,(x_0,y_0))$ are the solutions corresponding to $x(0) = x_0$ and $y(0) = y_0$, then $\phi(t,(x_0,y_0)) = (x(t,(x_0,y_0)), y(t,(x_0,y_0)))$ defines a local one-parameter group acting on $\mathbb{R}^2$. This allows us to move back and forth between these two notions.

EXAMPLE IV.2: In Example IV.1, the infinitesimal one-parameter group is

$$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

If $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ is an infinitesimal one-parameter group, we say that (x_0,y_0) is a critical point if $f(x_0,y_0) = g(x_0,y_0) = 0$. If (x_0,y_0) is not a critical point, it is called a regular point and one can show (Markus (1960), p. 14) that there is a change of coordinates $u(x,y), v(x,y)$ near (x_0,y_0) such that in these new coordinates $X = \frac{\partial}{\partial v}$.

Given a local one-parameter group ϕ_t and a differential equation $F(x,y,y') = 0$, we say that ϕ_t is a symmetry group of $F(x,y,y') = 0$ if the following holds: if Γ is the graph of a solution $y(x)$ of $F(x,y,y') = 0$ through (x_0,y_0) then, if t is close to 0, there is an open neighborhood U_t of (x_0,y_0) such that $\phi_t(\Gamma \cap U_t)$ is the graph of a solution of $f(x,y,y') = 0$ through $\phi_t(x_0,y_0)$ (i.e. ϕ_t takes solutions to solutions). Luckily, one never needs to verify this condition directly. If $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ is the infinitesimal one-parameter group associated with ϕ_t, one can show that ϕ_t is a symmetry group of $F(x,y,y') = 0$ if and only if $X_1(F(x,y,y')) = 0$ whenever $F(x,y,y') = 0$ where

$$X_1 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + \left(\frac{\partial g}{\partial x} + \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right)y' - \left(\frac{\partial f}{\partial y}\right)(y')^2\right)\frac{\partial}{\partial y'}$$

Here we are thinking of x, y, and y' as three independent variables. (To understand what is happening geometrically, it is convenient to think in terms of manifolds. A local one parameter group acting on $\mathbb{R}^2$ is a local action of the Lie group $(\mathbb{R}, +)$ on $\mathbb{R}^2$. One can define the local action ϕ_t of $\mathbb{R}$ on any manifold. As with one-parameter groups, such an action corresponds to a vector field X on the manifold. The action ϕ_t induces an action of $\mathbb{R}$ on the 1st jet space of the manifold and X_1 is the corresponding vector field on this jet space. $F(x, y, y') = 0$ defines a submanifold of the jet space and the condition that the action of $\mathbb{R}$ leave this invariant is precisely that $X_1(F(x, y, y')) = 0$. For details and generalizations of this approach, see Olver (1986).)

<u>EXAMPLE IV.3</u>: (Olver (1986), p. 136) Let ϕ_t be the one-parameter group defined by $\phi_t(x, y) = (e^t x, e^t y)$. The associated infinitesimal one-parameter group is $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Consider a differential equation $y' = F\left(\frac{y}{x}\right) = 0$, that is, a homogeneous equation. One easily checks that $X = X_1$ and $X_1\left(y' - F\left(\frac{y}{x}\right)\right) = 0$. One can also see directly that solutions of a homogeneous equation are mapped to other solutions under the groups of dilations.

We have already mentioned that at a regular point (x_0, y_0), one can choose coordinates $u(x, y), v(x, y)$ so that $X = \frac{\partial}{\partial v}$. In this coordinate system we also have $X_1 = \frac{\partial}{\partial v}$. Assume that $y' = F(x, y)$ is a differential equation such that $X_1(y' - F(x, y)) = 0$ when $y' - F(x, y) = 0$. If we write the differential equation in the new coordinates, say $\frac{dv}{du} = G(u, v)$, then the condition $X_1\left(\frac{dv}{du} - G(u, v)\right) = 0$ when $\frac{dv}{du} = G(u, v)$ implies that $\frac{\partial}{\partial v}G(u, v) = 0$. Therefore, $G(u, v) = H(u)$ is independent of v and $v = \int H\, du + c$. Rewriting this in terms of the original coordinates gives us a solution of the differential equation.

<u>EXAMPLE IV.4</u>: This is a continuation of the previous example. If we let $u = \frac{y}{x}$ and $v = \log x$, then $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ becomes $X = \frac{\partial}{\partial v}$. Assuming that $y = y(x)$ and $v = v(u)$, we have that

$$\frac{dy}{dx} = \frac{1 + u\frac{dv}{du}}{\frac{dv}{du}}$$

so the equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ becomes $\frac{dv}{du} = \frac{1}{F(u) - u}$. This has a solution $v = \int \frac{du}{F(u)-u} + c$.

For example, if

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}$$

then $F(u) = u^2 + 2u$. In the coordinates $u = \frac{y}{x}$, $v = \log x$, we have

$$\frac{dv}{du} = \frac{1}{u^2 + u}$$

The solution is $v = -\log\left(1 + \frac{1}{u}\right) + c$, so $y = \frac{x^2}{d - x}$.

This idea can be generalized to higher order equations. Let $F(x, y, \dots, y^{(n)}) = 0$ be an nth order differential equation. The definition of a one-parameter group being a symmetry group of $F(x, y, \dots, y^{(n)}) = 0$ is the same as before. This again can be stated in terms of the associated infinitesimal one-parameter group $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} : \phi_t$ is a local one-parameter symmetry group of $F(x, y, \dots, y^{(n)}) = 0$ if and only if

$$X_n F = f\frac{\partial}{\partial x} + \sum_{j=0}^{n} g_j \frac{\partial F}{\partial y^{(j)}} = 0 \text{ when } F(x, y, \dots, y^{(n)}) = 0,$$

$$\text{where } g_0 = g$$

$$\text{and } g_j = \frac{\partial g_{j-1}}{\partial x} + \sum_{k=0}^{j-1} \frac{\partial g_{k-1}}{\partial y^{(y)}} y^{(k+1)} - \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'\right) y^{(j)}$$

When this happens, one chooses local coordinates $u(x, y), v(x, y)$ such that $X = \frac{\partial}{\partial v}$ and writes the equation in the new coordinates as $G(u, v, \dots, v^{(n)}) = 0$ (where $v' = \frac{dv}{du}$). The condition $X_n G = 0$ becomes $\frac{\partial G}{\partial v} = 0$, so the equation really is $G(u, v', \dots, v^{(n)}) = 0$. Letting $w = v'$, we see that finding a solution of $G(u, w, \dots, w^{(n-1)}) = 0$, integrating $w = \int v\, du$ and rewriting in the old coordinate system, solves the original equation. Therefore, the existence of a one-parameter group of symmetries of the equation allows us to reduce the order of the equation.

<u>EXAMPLE IV.5</u>: (Olver (1986), p. 142) Consider the equation $y'' + p(x)y' + q(x)y = 0$. The group $\phi_t(x, y) = (x, e^t y)$ is a one-parameter group of symmetries of this equation. The associated one-parameter infinitesimal group is $X = y\frac{\partial}{\partial y}$. If we let $u = x$ and $v = \log y$, then $X = \frac{\partial}{\partial v}$. Since $y = e^v$, $y' = v'e^v$ and $y'' = (v'' + (v')^2)e^v$, the equation becomes $v'' + (v')^2 + pv' + q = 0$. Letting $w = v'$ we get the usual Riccati equation $w' + w^2 + pw + q = 0$. Solving this and letting $y = e^v = e^{\int w}$ solves the original equation.

I now mention a result related to Section III. Consider a differential equation $y' = \frac{Q(x,y)}{P(x,y)}$ which we write as $Q(x, y)dx - P(x, y)dy = 0$. One can show (Olver (1986), p. 139 or Markus (1960), p. 18) that if this differential equation has a local one parameter symmetry group with associated infinitesimal group $f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$, then

$$R(x, y) = \frac{1}{f(x, y)Q(x, y) - g(x, y)P(x, y)}$$

is an integrating factor, that is $d(RQ\, dx - RP\, dy) = 0$.

<u>EXAMPLE IV.6</u>: Consider again a homogeneous equation $y' = F\left(\frac{y}{x}\right)$ but write this as $F\left(\frac{y}{x}\right) dx - dy = 0$. Since the group associated with $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a symmetry group of this equation, $R = \left(xF\left(\frac{y}{x}\right) - y\right)^{-1}$ is an integrating factor.

This result is the basis of many heuristics (Char (1980)). The main problem with applying the above ideas is that it is difficult, in general, to find an infinitesimal one-parameter group satisfying the appropriate conditions and once such a group is found, finding the change of coordinates to make $X = \frac{\partial}{\partial v}$. This problem is discussed in Olver (1979), (1986) and Char (1980), (1981). One can also find non-trivial applications in these references as well as the works of Miller, Schwarz and Steinberg listed in the references. Other works of interests are Belinfante & Kolman (1979), Beyer (1979), Bluman & Cole (1974), Campbell (1966), Cohen (1911), Fushchich & Kornyala (1989), Kersten (1986), Ovsiannikov (1982), Reiman (1981), Roseman & Schwarzmeier (1979), and Winternitz (1983).

I close this section by mentioning the equivalence problem and the method of Cartan. The equivalence problem is the problem of determining when two systems of ordinary or partial differential equations can be mapped to each other by an appropriate change of coordinates and the method of Cartan is a method to solve this problem. This method was turned into an algorithm by Gardner and applied to a diverse collection of problems (Gardner (1983), (1989), Kamran (1988)). Cartan's equivalence method has been used to determine possible symmetry groups of differential equations in Hsu & Kamran (1988) and Kamran & Olver (1988).

V. Transform Methods

The basic idea behind transform methods is to transform a differential equation into an algebraic equation, solve the algebraic equation and then transform back (occasionally, one just transforms the original equation into a simpler differential equation and then tries to solve the simpler equation). An elementary example is the effect of the Laplace Transform on linear differential equations with constant coefficients. The Laplace Transform of a function f, defined on $[0, \infty)$, is $L(f) = F(z) = \int_0^\infty e^{-zt} f(t)dt$. Using the fact that $L(f^{(n)}) = z^n L(f) - \sum_{k=1}^{n} z^{n-k} f^{(k-1)}(0)$, one can easily transform any system of linear differential equations with constant coefficients into a system of linear (algebraic) equations with polynomial coefficients. One solves this and inverts the transform to get solutions of the original equations. This has been implemented in MACSYMA, see Avgoustis (1977), Clarkson (1989) and Rand (1984). More general transform techniques are discussed in Glinos & Saunders (1984), where implementations of techniques from the operational calculus are discussed.

VI. Asymptotics

A problem here is to find algorithms that will generate formulas such as

$$\int_a^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \ldots + (n-1)! \frac{x}{(\log x)^n} + o\left(\frac{x}{(\log x)^n}\right)$$

or other expressions that describe the growth behavior of solutions of linear differential equations. There have been various attempts to give algebraic substance to asymptotic expansions and estimates, that is, make a calculus of asymptotic expressions. Early work includes the considerations of du Bois-Reymond (see the bibliography in Hardy (1910), (1912)). Recently, this area has been given a firm algebraic footing in the works of Boshernitzan, Rosenlicht and van den Dries (see the references). I hope that some of their work can be made effective. Along these lines I propose the following problems

Problem 11. Find an algorithm that solves the following: Given a real elementary function f, find a real elementary function F such that $\int_a^x f \sim F$ (i.e. $\lim_{x\to\infty} \frac{\int_a^x f}{F} = 1$) if such an F exists.

Some work on this problem appears in Bourbaki (1961) and Rosenlicht (1980), and a solution of this would be a first step towards algorithmically generating expressions like (4). For an overview of the many pitfalls associated with attempts to make a calculus of these generalized asymptotic expansions, as well as other useful information on asymptotics, see Olver (1974), especially Ch. 1, Sec. 10 and Olver (1980), especially Sec. 3.

Let P and Q be polynomials in y with coefficients that are real liouvillian functions. All solutions of

$$y' = \frac{P(y)}{Q(y)}$$

that are differentiable in a neighborhood of $+\infty$ are ultimately monotonic (Rosenlicht (1983a)). When P and Q have coefficients in $\mathbb{R}[x]$, Hardy showed that any such solution y satisfies either $y \sim ax^b e^{p(x)}$ or $y \sim ax^b(\log x)^{1/c}$ where b is a real number, $p(x)$ a polynomial and c an integer (Hardy (1910), Bellman (1969)).

Problem 12. Find an analogue of Hardy's result in the general case of P and Q having real liouvillian functions as coefficients.

Formal methods involving asymptotics have been very useful in perturbation theory. Here we are given a differential equation that depends on a parameter ϵ and we wish to find series in ϵ that represent quantities associated with this equation (e.g. solutions, limit cycles, Poincaré maps). This usually is done by substituting the power series in ϵ into an equation, equating powers of ϵ, deriving new equations for the coefficients and solving these new equations. Computer algebra systems such as MACSYMA have been successfully used in this problem. There is an enormous literature on this subject and the reader is referred to Rand (1984) and Rand & Armbruster (1987) for details and a large bibliography.

VII. Difference Equations

The general problem here is: Consider the questions raised in I–VI above in the context of difference equations. Aside from heuristics (Cohen & Katcoff (1977), Hayden & Lamagna (1986), Ivie (1977) and Moenck (1977)), there are a few recent algorithmic results. In 1977, Gosper (Gosper (1977), (1978)) gave an algorithm which gives a closed form expression for $S(n) = \sum_{x=1}^{n} f(x)$ when $S(n)/S(n-1)$ is a rational function. This algorithm has been successfully used to generate and generalize some very interesting formulas. Problems of this kind can be given a formal setting using difference fields. A difference field is a field F with an automorphism σ (Cohn (1966)). If $F = \mathbb{C}(x)$ the automorphism one usually has in mind is $\sigma(f(x)) = f(x+1)$. We can define the usual difference operator by $\Delta f = \sigma(f) - f$. The problem of finding a closed form expression for $S(n) = \sum_{x=1}^{n} f(x)$, then becomes: Given a difference field F and $f \in F$, compute, if it exists, an element g in a suitable extension of F such that $\Delta g = f$. Karr has investigated this problem in Karr (1981), (1985). He rigorously defines what is meant by "summation in finite terms" in terms of towers of difference fields. These towers are called $\prod \Sigma$ fields and are the analogue of elementary extensions in the theory of integration in finite terms. Karr shows how to solve an arbitrary first order linear difference equation in a given $\prod \Sigma$ field and how to make a judicious choice of such an extension. He also gives a liouville type theorem for summation in finite terms. An exposition of some aspects of Gosper's and Karr's work can be found in Lafon (1982).

Problem 13. Generalize Karr's work to nth order linear difference equations.

Recently, Zeilberger (Zeilberger (1989)) uses a setting similar to that in his work on integrals to give an algorithm for evaluating sums of the form $a(n) = \sum_{k=1}^{n} F(n,k)$ where $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k.

Della Dora, Tournier and Wazner have considered the problem of finding power series solutions of linear difference equations $L(y) = \sum_{i=1}^{n} a_i \delta^i y = 0$, where $a_i \in \mathbb{C}(x)$ and $\delta(f(x)) = f(x-1)$. In Della Dora & Tournier (1984) they look for solutions of the form $y(x) = \mu^x \left(\sum_{j=0}^{\infty} a_j (x)_{\nu+j} \right)$, where $(x)_\lambda = \Gamma(x+1)/\Gamma(x-\lambda+1)$. They pursue the method of Boole, a method similar to the Frobenius method for solving linear differential equations. This method only works under certain regularity condition imposed on the coefficients of $L(y)$. In Della Dora & Wazner (1985), they pursue a Newton polygon method that handles a more general case. In Barkatou (1989), Barkatou considers systems of linear difference equations and gives an algorithm (along the lines of Moser's algorithm for differential equations) to reduce such a system and decide if it has a regular singularity.

Another approach to difference equations is discussed in Della Dora & Tournier (1986) and Tournier (1987) based on ideas of Pincherle and recent improvements of J. P. Ramis and A. Duval. The idea is to use the transform $P[\varphi] = \int_\gamma t^{-x-1}\varphi(t)dt$, where γ is a suitably chosen path, to transform the difference equation into a linear differential equation, use the techniques developed to understand the solutions of this new differential equation, and then transform back. The original motivation for Della Dora *et al*'s interest in difference equations was to understand the growth properties of the coefficients appearing in the formal expansions of solutions of a linear differential equation at irregular points. These coefficients satisfy difference equations. The Pincherle-Ramis method converts this problem back to a more tractible problem again involving linear differential equations, gives a remarkable and very pretty circle of ideas.

In Maeda (1987), Maeda discusses Lie method for difference equations.

Final Comments

In the previous sections, I have mentioned how techniques for finding formal solutions have been implemented in computer algebra systems. Besides solving differential equations, computer algebra can be used to generate differential equations and manipulate differential equations (of course, generating, manipulating and solving are not mutually exclusive). In Wang (1986) and Tan (1989), the authors describe the symbolic software FINGER that automatically generates the element equations for the finite element method (see also Roache & Steinberg (1985), (1988)). Another example of using symbol manipulation packages to generate equations is in Hirschberg & Schramm (1989), where the authors describe a package that generates the equations of motion of certain robot systems given the masses, moments of inertia, position of mass centers and connection joint locations. A good example of using a computer algebra systems to manipulate differential equations can be found in Davenport, Siret & Tournier (1988), p. 29, where the authors show how to use MACSYMA to obtain successive derivatives of y with respect to x, starting from $g(x, y) = 0$. They get expressions containing partial derivatives of g and are then able to specialize this to a particular g. Other examples can be found in Rand (1984) and Rand & Armbruster (1987). Another example of manipulation is given in Grossman & Larson (1989), where the authors give an efficient algorithm for evaluating higher order differential operators (such as $E_3E_2E_1 - E_3E_1E_2 - E_2E_1E_3 + E_1E_2E_3$, where $E_i = \Sigma a_i^j \frac{\partial}{\partial x_i}$).

All the problems discussed here have their roots in the 19th century and many of them have effective solutions that were outlined at that time. With the rise of symbolic computation systems, these solutions take on a new relevance. I have included the following textbooks and guides to the old literature in the references:

Bieberbach (1935), Gray (1986), Hilb (1915a), (1915b), Hille (1976), Ince (1944), Kamke (1971), Poole (1960), Schlesinger (1895), (1909), Vessiot (1910), Zwillinger (1989).

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1.21 ON SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC SOLUTIONS

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ON SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC SOLUTIONS

By F. BALDASSARRI *and* B. DWORK

Introduction. We consider second order linear differential operators

$$L = D^2 + B \cdot D + C,$$

$D = d/dx$, $B, C \in \mathbf{C}(x)$. The singular points of L consist of the poles of B and C and possibly the point at infinity.

H. A. Schwarz [15] determined all such operators with three singular points whose kernel consists of algebraic functions. His method was to show that if B and C lie in $\mathbf{R}(x)$ then the monodromy group can be calculated from the group generated by the reflections relative to three circles which meet at angles determined by the exponent differences of L. He used this to show that the solutions of L are all algebraic if and only if these angles coincide with the angles of a spherical triangle whose vertices are fixed points of three rotations any pair of which generates a finite rotation group.

The Schwarz list of 15 reduced curvilinear triangles contains a basic sublist of exponent differences

				group type	order $= M$
I	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{n}$	dihedral	$2n$
II	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	tetrahedral	12
IV	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	octahedral	24
VI	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	icosohedral	60

the first case corresponding to an infinite series of dihedral groups.

Klein asserted [10, pp. 132–135] that he had determined all differential equations of second order with only algebraic solutions. This widely accepted assertion is quite misleading. One may pose two questions

(0.1) Given $n + 1$ points $\{a_1, \ldots, a_n, \infty\}$, determine all second order linear differential equations with only algebraic solutions and with these points as singularities.

(0.2) Given a second order linear differential equation L, determine in a finite number of steps whether the solutions of L are all algebraic.

The method of Klein is better adapted to problem (0.2) than to problem (0.1). His method (cf. Theorem 3.4 below) is based on the fact that (excluding the cyclic case) L has only algebraic solutions if and only if it is a weak pullback by a rational map of an element in the basic Schwarz list. To decide whether L as given by (3.3) is the pullback of L_0 is equivalent to the determination of the existence of a rational solution x of the non-linear third order differential equation (3.7.3). With an a-priori bound for the degree of x this can be reduced to a purely algebraic problem. Klein illustrated this program [12] by exhibiting cases XII, XIV, XV of Schwarz as pullbacks of VI (the other cases on the full Schwarz list had been previously checked off in that way by Brioschi [2]). Klein did not for example show by his method that the Schwarz list is exhaustive, a reasonable step in illustrating a credible program for problem (0.1).

It is our understanding that Problem 0.1 is still open for $n \geq 3$ (i.e. four or more singular points). In the present article we carry out the Klein program for Problem (0.2). The new ingredients of our work are

(0.3) A general formula (Section 1) for the degree of the pullback mapping.

(0.4) A decision procedure (Section 6) for the existence of an algebraic solution of a *linear* first order differential equation over a Riemann Surface. (Correction: This procedure is not new. See R. H. Hisch, Bull. Amer. Math. Soc. 76 (1970), pp. 605–608).

The degree calculation is needed to carry out Klein's method for checking whether L is a pullback of II, IV or VI but the method breaks down if we must test whether the group of L is one of the two infinite series. The cyclic case is treated in Section 4 but the main point in the dihedral case is the reduction (Section 5) to problem (0.4) mentioned above.

In a subsequent article we shall explain how Klein's program may be extended to second order differential equations defined over a Riemann Surface.

In Section 8 we indicate connections with Grothendiecke's conjecture and the work of Katz. We note that the conjecture is still open for four or more singular points (unless the differential equation "comes" from algebraic geometry). Even if verified it would not respond to Problem (0.2).

We note that Forsyth [3, p. 184] discussed Problem (0.2) along the lines of Klein. His treatment cannot be taken seriously.

We regard the present article as expository. Indeed a superior decision procedure (over the sphere) has been given by L. Fuchs (cf. items 19, 20, 21, 22, 23, 25, Vol. II of his collected works). This may be explained quite easily.

For each integer m let L_m be the linear differential equation of order $m + 1$ satisfied by all homogeneous forms of degree m in solutions y_1, y_2 of the second order linear differential equation L given by (4.1). If L has only algebraic solutions then (cf. Theorem 3.8 below) L_{12} must have at least one solution which is the radical of a rational function. If L_{12} has such a solution but L_2 does not then (Fuchs p. 45 loc cit) all solutions of L are algebraic. If L_2 has such a solution, ϕ, but L does not then ϕ^2 is a rational function and $\phi = y_1 y_2$, a product of independent solutions of L. Putting $\tau = y_2/y_1$ we obtain

$$\tau' = \frac{w}{\phi}\tau \tag{0.5}$$

where $w = y_1 y_2' - y_2 y_1'$ is a constant which may be calculated from

$$-w^2 = \phi^2\left[\left(\frac{\phi'}{\phi}\right)^2 + 2\left(\frac{\phi'}{\phi}\right)' - 4Q\right] \tag{0.6}$$

In this case Problem (0.2) is reduced to determining whether τ as given by (0.5) is algebraic, a problem evidently untreated by Fuchs. Our Section 6 serves then to fill in this gap in the treatment of Fuchs. If

L has itself a solution which is the radical of a rational function then the solutions of L are all algebraic only if L satisfies the criteria for the cyclic case (cf. Section 4 below). Finally we note that the question of whether L_{12} has a solution, v, which is the radical of a rational function is elementary since v must be of the form

$$\prod_{i=1}^{n} (x - a_i)^{\alpha_i} g(x)$$

where g is a polynomial whose degree as well as the values of the α_1 are determined up to a finite set of possibilities by the exponents of L at its singularities $\{a_1, \ldots, a_n, \infty\}$.

1. **Pullbacks on Riemann Surfaces.** Let L be a second order differential equation with coefficients in an algebraic function field K of characteristic zero and algebraically closed field of constants k. Hence choosing $x \in K$, $x \notin k$, we may write

$$L = D^2 + AD + B \tag{1.1}$$

where $A, B \in K$ and $D = d/dx$. Letting Γ be the Riemann surface corresponding to K, we view L as being *equivalent* to L_1 also defined over K if a ratio of solutions (at some point of Γ) of L is also a ratio of solutions of L_1. Thus in particular we identify L with

$$L_1 = D^2 + B_1 \tag{1.2}$$

where

$$B_1 = B - \frac{1}{2} A' - \frac{1}{4} A^2 \qquad \left(A' = \frac{d}{dx} A\right),$$

since

$$L_1 = \frac{1}{\theta} \circ L \circ \theta \quad \text{where} \quad \theta'/\theta = -\frac{1}{2} A$$

(notice that θ does not lie in K but is defined locally on Γ).

We shall assume that at each point, P, of Γ, L has two independent solutions of the form

$$y_1 = t^{\alpha_i}(1 + b_{i1}t + b_{i2}t^2 + \cdots) \qquad i = 1, 2 \tag{1.3}$$

where t is a local uniformizing parameter, $\alpha_i \in k$ and the $b_{i,j}$ also lie in k. Because of our definition, the pair (α_1, α_2) is not well defined but their difference is well defined up to sign. For a fixed model we have $(\alpha_1, \alpha_2) = (0, 1)$ for almost all points and so if we choose any archimedean valuation of k and setting $\gamma(P) = |\alpha_1 - \alpha_2|$ for each point P of Γ, we see that $\gamma(P) - 1 = 0$ for almost all P. Let S be any finite subset of Γ and put

$$\Delta(S, L) = \sum_{P \in S} (\gamma(P) - 1). \tag{1.4}$$

For S large enough we obtain a limiting value $\Delta(L)$ independent of S, but depending on the valuation of k chosen for the construction.

Now let K_0 be an algebraic extension of K and let π denote the mapping $\Gamma_0 \to \Gamma$ of the corresponding Riemann surface. Let L_0 be the pullback of L by π (i.e. if τ is ratio of two solutions of L then $\tau \circ \pi$ is ratio of two solutions of L_0). Using the same valuation of k we may define $\Delta(L_0)$.

LEMMA 1.5 *Let g (resp: g_0) be the genus of Γ (resp: Γ_0), let M be the mapping degree, then*

$$M[\Delta(L) - 2(g - 1)] = \Delta(L_0) - 2(g_0 - 1). \tag{1.5.1}$$

Proof. For $P \in \Gamma$, let P_0 be a point of Γ_0 above P. Let $e(P_0)$ be the relative ramification. If α_1, α_2 are exponents at P of a model of L, then $\alpha_i \cdot e(P_0)$ are exponents at P_0 of L_0. Thus

$$\gamma(P_0) = \gamma(P) \cdot e(P_0)$$

which shows that

$$\sum_{P_0 | P} \gamma(P_0) = M \cdot \gamma(P). \tag{1.6}$$

If S is any finite subset of Γ and S_0 is the set of all points of Γ above S, then

$$\Delta(S_0, L_0) + \text{card}\, S_0 = M(\Delta(S, L) + \text{card}\, S). \tag{1.7}$$

The Hurwitz genus formula states that

$$2(g_0 - 1) - 2M(g - 1) = M\, \text{card}\, S - \text{card}\, S_0 \tag{1.8}$$

provided S contains all points of Γ which ramify in Γ_0. The lemma now follows by taking S, so large that in (1.7) we may replace $\Delta(S_0, L_0)$ (resp $\Delta(S, L)$) by $\Delta(L_0)$ (resp: $\Delta(L)$) and using (1.8) to eliminate the terms involving cardinality.

In the application below $g_0 = g_1 = 0$ and the exponent differences are all rational. Besides, coordinate functions t, x will be fixed on Γ_0, Γ (that is $K_0 = k(t)$, $K = k(x)$) and by the *weak pullback* of L to (Γ_0, t) we will denote the model of the pullback of L to Γ_0, which is in the form (1.2) with respect to the coordinate function t.

The point in the lemma is that L, L_0, Γ and Γ_0 may be given without the relation π between Γ and Γ_0 being known. In that situation for given set S of Γ there is no way to determine the lifting, S_0, to Γ_0. We can determine the set T of singularities of L and T_0 of singularities of L_0 but T_0 need not coincide with $\pi^{-1}(T)$, the problem being that ramification points of π may produce singular points of L_0 and again if $\gamma(P) = 1/e$ and P_0 covers P with ramification e then P_0 is not a singularity of L_0 since the exponent difference at P_0 would be 1. The situation would be different if we viewed L as determining the solution space rather than ratios of solutions.

2. **Finite homography groups.** We review (following Halphen [6]) the theory of such groups. Let G be a finite group of homographies

$$t \to \frac{at + b}{ct + d}$$

where a, b, c, d lie in $\mathbf{C}$ and $ad - cb \neq 0$. Thus G may be viewed as a finite set of automorphisms of the function field $H = \mathbf{C}(t)$. By Lüroth's theorem, the fixed field H^G is generated by a single element y and so is of genus zero. Let $P_1, \ldots, P_s$ be the points of H^G which ramify in H, let M be the order of G and let e_i be the relative ramification of H over H^G at one point of H above P_i. It then follows that there are M/e_i places of H above P_i and each place has ramification e_i, which shows that if we use the Hurwitz genus formula we have

$$0 = 2(1 - M) + \sum_{i=1}^{s} (e_i - 1)M/e_i.$$

and so

$$\sum_{i=1}^{s} 1/e_i = s - 2 + \frac{2}{M}. \tag{2.1}$$

Since each $e_i \geq 2$, we obtain

$$s/2 \geq \sum 1/e_i > s - 2$$

which shows that $s = 2, 3$.

If $s = 2$, we use $e_i \leq M$ to conclude from (2.1) that

$$e_1 = e_2 = M$$

which shows that the Galois group G coincides with the ramification group at P_1 and hence G is cyclic.

If $s = 3$ then (2.1) becomes

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} = 1 + \frac{2}{M} \tag{2.2}$$

a diophantine equation with only 4 types of solutions as given by the table in the Introduction. (The point is that we cannot have $e_i \geq 3$ for all i and so can assume $e_1 = 2$. Furthermore e_2, e_3 cannot both exceed 3. Thus $e_2 = 2$ or 3.)

Let G be a finite group of homographies. Then for each $\theta \in G$ we may choose a matrix

$$A_\theta = \begin{pmatrix} a_\theta & b_\theta \\ c_\theta & d_\theta \end{pmatrix} \in SL(2, \mathbf{C}) \tag{2.3}$$

such that

$$\theta(t) = \frac{a_\theta t + b_\theta}{c_\theta t + d_\theta}. \tag{2.4}$$

Of course A_θ is only determined mod ± 1 and hence if $\theta, \varphi \in G$, we have

$$A_\theta A_\varphi = h(\theta, \varphi) A_{\theta\varphi} \tag{2.5}$$

for suitable $h(\theta, \varphi) \in \{\pm 1\}$. It is clear that h is a 2-cocycle for G with coefficients in $\{\pm 1\}$ with trivial action of G on $\{\pm 1\}$. (The example of the 4-group generated by $x \to \pm 1/x$ shows that h need not be cohomologically trivial). For later use we define a mapping k of $G \times G$ into ± 1 and a mapping k_0 of G into ± 1 by the formulas

$$k(\phi, \theta) A_{\phi^{-1}\theta} = A_\phi^{-1} A_0 \tag{2.6.1}$$

$$k_0(\phi) = \prod_{\theta \in G} k(\phi, \theta) \tag{2.6.2}$$

For $u \in \mathbf{C}$, we define

$$F_u(t) = \prod_{\theta \in G} (t - \theta u) \tag{2.7}$$

if θu is finite for all θ. Whenever θu is infinite we replace the factor $t - \theta u$ by $-(a_\theta u + b_\theta)$. Thus F_u is a polynomial of degree $M - \sigma_u$ where σ_u is the number of $\theta \in G$ such that $\theta u = \infty$. We note that

$$\pm F_{\varphi u}(t) = F_u(t)/(c_\varphi u + d_\varphi)^{\sigma_u}$$

for all ϕ in G, the sign being given by $\prod h(\theta\phi^{-1}, \phi)$, the product being over all θ such that $\theta u = \infty$.

If we put $t = t_1/t_2$, we obtain

$$\Delta_u t_2^M F_u(t) = \prod_{\theta \in G} [t_1(c_\theta u + d_\theta) - t_2(a_\theta u + b_\theta)] \tag{2.8}$$

where $M =$ order of G

$$\Delta_u = \prod (c_\theta u + d_\theta)$$

the product being over all $\theta \in G$ such that $\theta u \neq \infty$. If $\varphi \in G$ then putting

$$\varphi_1(t_1, t_2) = a_\varphi t_1 + b_\varphi t_2$$
$$\varphi_2(t_1, t_2) = c_\varphi t_1 + d_\varphi t_2$$

we obtain $\varphi(t) = \varphi_1(t_1, t_2)/\varphi_2(t_1, t_2)$ and so

$$\begin{aligned}\Delta_u \varphi_2(t_1, t_2)^M F_u(\varphi(t)) &= \prod_{\theta \in G} [\varphi_1(t_1, t_2) \cdot (c_\theta u + d_\theta) \\ &\quad - \varphi_2(t_1, t_2)(a_\theta u + b_\theta)] \\ &= \prod_{\theta \in G} [t_1(c_{\varphi^{-1}\theta} u + d_{\varphi^{-1}\theta}) \\ &\quad - t_2(a_{\varphi^{-1}\theta} u + b_{\varphi^{-1}\theta})] k(\phi, \theta) \\ &= \Delta_u t_2^M F_u(t) k_0(\phi).\end{aligned}$$

This shows that

$$F_u(\varphi(t)) = \frac{k_0(\phi)}{(c_\varphi t + d_\varphi)^M} F_u(t). \tag{2.9}$$

The key point is that the ratio between $F_u(\varphi(t))$ and $F_u(t)$ is independent of u. Thus if v lies in another orbit, we see that F_u/F_v is an absolute invariant under G, and so lies in H^G, the subfield of $H = \mathbf{C}(t)$ fixed by G.

Putting $x = F_u(t)/F_v(t)$ we see that $H \supset H^G \supset \mathbf{C}(x)$ and hence

$$\deg H/\mathbf{C}(x) \geq M$$

while the equation

$$0 = F_u(t) - xF_v(t)$$

is a polynomial in t of degree M, which shows that

$$\deg \mathbf{C}(t)/\mathbf{C}(x) \leq M.$$

This shows that

$$H^G = \mathbf{C}(x).$$

LEMMA 2.10. *If u, v, w are points of* $\mathbf{C}$ *then the polynomials F_u, F_v, F_w are linearly dependent over* $\mathbf{C}$.

Proof. We may assume u, v, w to lie in distinct G-orbits and deg $F_w = M$. Then H^G is generated over $\mathbf{C}$ by F_u/F_w and also by F_u/F_v. This shows that F_u/F_w and F_u/F_v are related by a homography. Hence there exist a, b, c, d in $\mathbf{C}$ such that

$$\frac{F_u}{F_w} = \frac{aF_u + bF_v}{cF_u + dF_v}$$

Since F_u and F_w are relatively prime, we conclude that F_w divides $cF_u + dF_v$ but deg $F_w = M \geq$ deg $(cF_u + dF_v)$ which shows that up to a non-zero constant factor,

$$F_w = cF_u + dF_v.$$

This completes the proof of the lemma.

THEOREM 2.11. *If G is a finite non-cyclic group of homographies of order M then there exist polynomials P_1, P_2, P_3 of degree M/e_i $(i = 1, 2, 3)$ (or possibly $M/e_i - 1$ for one i) such that*

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} = 1 + \frac{2}{M} \tag{2.2}$$

$$P_1^{e_1} + P_2^{e_2} + P_3^{e_3} = 0, \qquad (P_i, P_j) = 1 \quad \text{for} \quad i \neq j \tag{2.11.1}$$

$$P_i(\theta t) = P_i(t)/(c_\theta t + d_\theta)^{M/e_i} \qquad \forall \theta \in G. \tag{2.11.2}$$

Proof. As before let $H = \mathbf{C}(t)$, H^G the fixed field under G. The extension H/H^G ramifies at 3 valuations of H^G. If a_i is such a place of H^G, let P_i be the polynomial whose roots are the finite extensions of a_i to H. This means that the degree of P_i is M/e_i unless the infinite place of H lies above a_i. It is clear that $P_i^{e_i}$ is an invariant constructed from an orbit and after adjusting constant factors, equation (2.11.1) follows from Lemma 2.10, equation (2.11.2) follows from (2.9) and (2.2) has been previously established.

To formulate the converse of the above theorem we consider solutions of the equation

$$Q_1^{e_1} + Q_2^{e_2} + Q_3^{e_3} = 0 \tag{2.12}$$

in $\mathbf{C}[t]$ such that Q_1, Q_2, Q_3 are relatively prime in pairs and such that if (say)

$$\deg Q_3^{e_3} < \deg Q_1^{e_1} = \deg Q_2^{e_2} = M \tag{2.13}$$

then

$$\deg Q_3^{e_3} \equiv M \bmod e_3.$$

LEMMA 2.14.

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \geq 1 + \frac{2}{M} \tag{2.14.1}$$

and equality holds if and only if the equation

$$-x = \frac{Q_1^{e_1}}{Q_3^{e_3}}(t) \tag{2.15}$$

defines an extension $\mathbf{C}(t)$ *of* $\mathbf{C}(x)$ *ramified only at 0, 1,* ∞ *with ramification* e_1, e_2, e_3 *at each point of t-sphere above the indicated points of the x-sphere. If equality holds then there exists a constant c such that*

$$Q_2^{e_2-1} = c(e_1 Q_1' Q_3 - e_3 Q_3' Q_1) \tag{2.15.1}$$

Proof. We assume that either $\deg Q_i^{e_i} = M$ for $i = 1, 2, 3$ or equation 2.13 holds. We put $(w_1, w_2, w_3) = (0, 0, 0)$ in the first case and $(0, 0, M\text{-}\deg Q_3^{e_3})/e_3$ in the second case.

We put $\sigma_\infty = 0$, in the first case and $\sigma_\infty = w_3 e_3 - 1$ in the second case. In either case it represents the contribution of the point $t = \infty$ to the Hurwitz genus formula for the genus of $\mathbf{C}(t)$ as an extension of $\mathbf{C}(x)$. This formula may be written

$$0 = 1 - M + \frac{1}{2}\sum_{i=1}^{3} N_i(e_i - 1) + \frac{1}{2}\sigma_\infty + E \tag{2.16}$$

where $N_i = \deg Q_i$ and $E \geq 0$ is a correction term introduced to allow for the possibility that

(a) Ramification may occur at points other than $x = 0, 1, \infty$
(b) the zeros of Q_i may be non-simple.

Since

$$M = (N_i + w_i)e_i,$$

we deduce from (2.16)

$$M\left(\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1\right) = 2E + \begin{cases} 2 & \text{if} \quad w_3 = 0 \\ 1 + w_3 & \quad w_3 \neq 0. \end{cases} \tag{2.17}$$

This proves the inequality which appears in the statement of the lemma. Equality holds if and only if both $E = 0$ and $w_3 = 0, 1$. This completes the proof of the lemma.

Now assume equality holds. Let

$$H = e_1Q_1'Q_3 - e_3Q_3'Q_1$$
$$K = e_2Q_2'Q_3 - e_3Q_3'Q_2$$

By dividing equation (2.12) by $Q_3^{e_3}$ and differentiating

$$Q_1^{e_1-1}H + Q_2^{e_2-1}K = 0.$$

Since Q_1 and Q_2 are relatively prime, $Q_2^{e_2-1}$ divides H. Since Q_1 and Q_3 are relatively prime, H cannot be zero. Equation (2.15.1) now follow by comparison of degrees of two sides.

PROPOSITION 2.18. *Let e_1, e_2, e_3 be given positive integers. Let $\mathbf{C}(t)$ be algebraic over $\mathbf{C}(x)$ ramified at $x = a_1, a_2, a_3$ (and possibly* *elsewhere). Suppose that at each place of* $\mathbf{C}(t)$ *above* a_i *the ramification index is divisible by* e_i.

Let z be a multivalued function on the x-sphere having only a_1, a_2, a_3 as critical points, suppose each branch of z is locally algebraic and that each branch at a_i lies in $\mathbf{C}(((x - a_i)^{1/e_i}))$. Then $z \in \mathbf{C}(t)$.

Proof. The function z is everywhere uniform on the t-sphere and hence is rational function of t.

Definition. A solution of (2.12) is called primitive if equality holds in (2.14.1).

LEMMA 2.19. *If* (Q_1, Q_2, Q_3) *is a primitive solution of* (2.12) *and* (P_1, P_2, P_3) *is an arbitrary one, then there exist polynomials* f, g *such that*

$$P_i = Q_i\left(\frac{f}{g}\right) \cdot g^{M/e_i} \qquad i = 1, 2, 3.$$

Proof. Define z, t algebraic over $\mathbf{Z}(x)$ by setting

$$\begin{cases} -x = \dfrac{Q_1^{e_1}}{Q_3^{e_3}}(z) \\ \\ -x = \dfrac{P_1^{e_1}}{P_3^{e_3}}(t) \end{cases} \tag{2.19.1}$$

We apply proposition 2.18 and lemma 2.14 and conclude that

$$z \in \mathbf{C}(t), \qquad \text{i.e.} \quad z = f/g,$$

where f, $g \in \mathbf{C}[t]$, $(f, g) = 1$. Equation (2.19.1) gives

$$\frac{P_1^{e_1}}{P_3^{e_3}} = \frac{Q_1^{e_1}(f/g)}{Q_3^{e_3}(f/g)} = \frac{[g^{N_1}Q_1(f/g)]^{e_1}}{[g^{N_3}Q_3(f/g)]^{e_3}\ g^{e_3 w_3}}$$

$$\frac{P_2^{e_2}}{P_3^{e_3}} = x - 1 = \frac{Q_2^{e_2}(f/g)}{Q_3^{e_3}(f/g)} = \frac{[g^{N_2}Q_2(f/g)]^{e_2}}{[g^{N_3}Q_3(f/g)]^{e_3}\ g^{e_3 w_3}}$$

(with no loss in generality we take 3 as in the proof of Lemma 2.14).

By hypothesis $1 = (Q_1, Q_2)$. From this we deduce that $g^{N_1}Q_1(f/g)$ and $g^{N_2}Q_2(f/g)$ are relatively prime. Again $(P_1, P_2) = 1$ by hypothesis and the lemma follows from our relations.

COROLLARY 2.20. *If (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are both primitive solutions then the rational function f/g represents a homography.*

Proof. In this situation, equation (2.19.1) shows that z and t are rational functions of each other, hence z is homographic image of t. We can now formulate a converse of Theorem 2.11.

THEOREM 2.21. *If (Q_1, Q_2, Q_3) is a primitive solution of* (2.12) *then the equation*

$$-x = Q_1^{e_1}(t)/Q_3^{e_3}(t) \tag{2.21.1}$$

defines a galois extension $\mathbf{C}(t)$ *of* $\mathbf{C}(x)$ *of degree M.*

Proof. Consider

$$h(Y) = xQ_3^{e_3}(Y) + Q_1^{e_1}(Y)$$

as polynomial in Y with coefficients in $\mathbf{C}(x)$. Clearly t is a root and by the corollary if t_1 is another root then t_1 is homographic image of t. Thus h splits in $\mathbf{C}(t)$. It is easy to check that h is irreducible in $\mathbf{C}(x)[Y]$ and so deg $\mathbf{C}(t)/\mathbf{C}(x) = m$ as asserted.

Note: The extension $\mathbf{C}(t)/\mathbf{C}(x)$ is ramified only at $0, 1, \infty$. This completes the demonstration that the finite subgroups of the group of homographies are uniquely determined up to inner automorphisms by the invariants e_1, e_2, e_3.

The existence of the indicated groups is demonstrated by writing down a primitive solution in each case.

For future use we note that these groups correspond to the case in which $s = 3$ in equation (2.1) and hence to the *non-cyclic* finite groups of homographies.

1. Dihedral group of order $2n$

$$(X_1^n + X_2^n)^2 - (X_1^n - X_2^n)^2 = 4(X_1X_2)^n$$

2. Tetrahedral

$$12\sqrt{-3}f^2 = \phi_1^3 - \phi_2^3$$
$$f = X_1X_2(X_1^4 - X_2^4)$$
$$\phi_1 = X_1^4 + 2\sqrt{-3}\,X_1^2X_2^2 + X_2^4$$
$$\phi_2 = X_1^4 - 2\sqrt{-3}\,X_1^2X_2^2 + X_2^4$$

3. Octahedral

$$W^3 - K^2 = 108f^4$$
$$f = X_1X_2(X_1^4 - X_2^4)$$
$$W = X_1^8 + 14X_1^4X_2^4 + X_2^8$$
$$K = X_1^{12} - 33X_1^8X_2^4 - 33X_1^4X_2^8 + X_2^{12}$$

4. Icosahedral

$$1728f^5 = T^2 + H^3$$
$$f = X_1X_2(X_1^{10} + 11X_1^5X_2^5 - X_2^{10})$$
$$H = -(X_1^{20} + X_2^{20}) + 228(X_1^{15}X_2^5 - X_1^5X_2^{15}) - 494X_1^{10}X_2^{10}$$
$$T = X_1^{30} + X_2^{30} + 522(X_1^{25}X_2^5 - X_1^5X_2^{25}) - 10005(X_1^{20}X_2^{10} - X_1^{10}X_2^{20})$$

3. **Pullbacks on the gauss sphere.** We consider hypergeometric differential equations in normalized form: For λ, μ, ν elements of **C**, let

$$L_{\lambda,\mu,\nu} = D^2 + \frac{A}{x^2} + \frac{B}{(x-1)^2} + \frac{C}{x(x-1)} \tag{3.1}$$

with

$$D = \frac{d}{dx}$$

$$4A = 1 - \lambda^2$$
$$4B = 1 - \mu^2$$
$$4C = \lambda^2 + \mu^2 - \nu^2 - 1.$$

This is the unique second order differential equation with rational coefficients, singular points only at 0, 1, ∞, with constant wronskian and with exponent differences λ, μ, ν at 0, 1, ∞ respectively.

If (Q_1, Q_2, Q_3) is a primitive solution of (2.12) then by setting

$$-x = Q_1^{e_1}(t)/Q_3^{e_2}(t) \tag{3.2}$$

we obtain an extension $K_0 = \mathbf{C}(t)$ of $\mathbf{C}(x) = K$ such that the pullback of $L_{\lambda,\mu,\nu} = L$ in the sense of Section 1 has $e_1\lambda$, $e_2\mu$, $e_3\nu$ as exponent differences. In particular if

$$(\lambda, \mu, \nu) = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}\right)$$

then the pullback has only unity as exponent difference, and no logarithmic singularities and is Fuchsian. Hence the pullback has no singularities and hence is d^2/dt^2.

The function t is a ratio of solutions of this equation and so this algebraic function of x is a ratio of solutions of $L_{1/e_1,1/e_2,1/e_3}$. We summarize:

THEOREM 3.2.1. *If* $\sum_{i=1}^{3}(1/e_i) > 1$ *then all solutions of* $L_{1/e_1,1/e_2,1/e_3}$ *are algebraic and t given by* (3.2) *is a ratio of solutions and the group of* $\mathbf{C}(t, x)/\mathbf{C}(x)$ *is dihedral, tetrahedral, octahedral, icosahedral depending on the values of* (e_1, e_2, e_3).

Let

$$L = \frac{d}{dt^2} + Q(t) \tag{3.3}$$

be a *normalized* second order linear differential operator defined over $\mathbf{C}(t)$ whose solutions are *locally algebraic*. Let τ be a ratio of solutions and let G be the group of homographies corresponding to the action of monodromy upon τ. As is well known the finiteness of G is equivalent to the condition that all solutions of L be algebraic functions. We will refer to G as the *projectivized* monodromy group of L.

THEOREM 3.4. (Klein) *If G is finite but not cyclic then L is the pullback by a rational map of one of the hypergeometric operators $L_{1/e_1,1/e_2,1/e_3}$ with $\sum_{i=1}^3 (1/e_i) > 1$. Conversely such a pullback has a finite projectivized monodromy group.*

Proof. The converse follows directly from Theorem 3.2.1.

If G is finite non-cyclic then by Theorem 2.11 there exists a primitive solution of (2.12) for suitable e_1, e_2, e_3 such that $Q_1^{e_1}/Q_3^{e_3}$ is invariant under G. Thus

$$-\xi(t) = \frac{Q_1^{e_1}}{Q_3^{e_3}}(\tau(t)) \tag{3.5}$$

is invariant under analytic continuation on the t-sphere, and hence is a rational function of t. Comparing this with equation (3.2) we see that $\tau(t)$ is a ratio of solutions of $L_{1/e_1,1/e_2,1/e_3}$ if we replace x by $\xi(t)$. This completes the proof of the theorem.

Suppose now that the projectivized group of L is finite and not cyclic. Hence according to the theorem L is a pullback of some $L_{1/e_1,1/e_2,1/e_3}$ by a rational map. This map need not be unique. To see this let $L_0 = d^2/dt^2$ so that L_0 is pullback of $L_{1/e_1,1/e_2,1/e_3}$ ($\sum (1/e_i) > 1$) by means of equation (3.2). If we put

$$t' = \sigma t$$

where σ is an arbitrary homography and put

$$-X_1 = \frac{Q_1^{e_1}}{Q_3^{e_3}}(t') \tag{3.6}$$

then L_0 is again pullback of $L_{1/e_1,1/e_2,1/e_3}$ by X_1. If X_1 were to equal X for all σ, we could conclude that X is a constant.

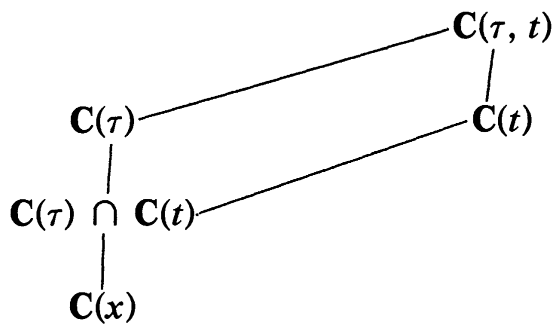
The point in this example is that L_0 has the trivial projectivized group while $L_{1/e_1,1/e_2,1/e_3}$ has a non-trivial group. In general if L is a pullback of $L_{1/e_1,1/e_2,1/e_2}$ then the group of L is a subgroup of that of $L_{1/e_1,1/e_2,1/e_3}$.

THEOREM 3.7. (Klein [12]). *If L and $L_{1/e_1,1/e_2,1/e_3}$ have the same group and if $e_1 \neq e_2$ then the pullback mapping is unique.*

Proof. By hypothesis there exists $x \in \mathbf{C}(t)$ such that τ, a ratio of solutions of L, satisfies

$$-x = \frac{Q_1^{e_1}}{Q_3^{e_3}}(\tau). \tag{3.7.1}$$

Field theoretically we have the lattice shown.



The projectivized group G of L coincides with the galois group of $\mathbf{C}(\tau, t)/\mathbf{C}(t)$ and hence by the usual identification with the group of $\mathbf{C}(\tau)/(\mathbf{C}(\tau) \cap \mathbf{C}(t))$. On the other hand the group of $L_{1/e_1,1/e_2,1/e_3}$ coincides with the group of $\mathbf{C}(\tau)/\mathbf{C}(x)$. The hypothesis of equality of groups is equivalent to the assertion that

$$C(\tau) \cap \mathbf{C}(t) = \mathbf{C}(x). \tag{3.7.2}$$

This condition fixes x up to homography. By an elementary calculation, L as given by (3.3) is the pullback of

$$L_0 = \frac{d^2}{dx^2} + q(x)$$

under the mapping $x = x(t)$ if

$$Q(t) = q(x) \cdot x'^2 + \frac{1}{2}\left(\frac{x''}{x'}\right)' - \frac{1}{4}\left(\frac{x''}{x'}\right)^2, \quad \left(x' = \frac{dx}{dt}\right) \cdot \tag{3.7.3}$$

a formula which may also be written in terms of the Schwarzian derivative. The key point is that

$$\frac{1}{2}\left(\frac{x''}{x'}\right)' - \frac{1}{4}\left(\frac{x''}{x'}\right)^2 = \frac{1}{2}\{x, t\} \tag{3.7.4}$$

remains invariant under homographies

$$x \mapsto \frac{ax + b}{cx + d}, \qquad ad - bc = 1$$

Since x is fixed up to homography, equation 3.7.3 implies that if $y = y(t)$ is also a pullback mapping then

$$q(x)x'^2 = q(y)y'^2 \tag{3.7.5}$$

$$y = \frac{ax + b}{cx + d} \tag{3.7.6}$$

and q is given by (3.1) with $(\lambda, \mu, \nu) = (1/e_1, 1/e_2, 1/e_3)$. Note that A, B, C in (3.1) are each *distinct* from zero. This shows that

$$\frac{A}{x^2} + \frac{B}{(x-1)^2} + \frac{C}{x(x-1)}$$

$$= \left[\frac{A}{y^2} + \frac{B}{(y-1)^2} + \frac{C}{y(y-1)}\right]\frac{1}{(cx+d)^4} \tag{3.7.7}$$

We consider four cases. Since a and c cannot simultaneously be zero we exhaust all possibilities.

Case 1 c, a, a-c all distinct from zero.

The right side of (3.7.7) lies in $x^{-4}\,\mathbf{C}[[x^{-1}]]$. The corresponding Laurent series for the right side is

$$(A + B + C)x^{-2} + (2B + C)x^{-3} \bmod x^{-4}$$

This shows that

$$A + B + C = 0$$
$$2B + C = 0$$

and hence $A = B$ contrary to hypothesis.

Case 2 $a = 0, c \neq 0$.

Here we may let $c = 1, b = -1$. The right side of 3.7.7 is

$$\frac{A}{(x+d)^2} + \frac{B}{(x+1+d)^2(x+d)^2} + \frac{C}{(x+1+d)(x+d)^2}$$

Comparing poles we see that as sets $\{0, 1\} = \{-d, -d-1\}$. Hence $d = -1$ and now multiplying both sides of (3.7.7) by x^2 and setting $x = 0$ we obtain $A = B$ contrary to hypothesis.

Case 3 $c = 0, a \neq 0$.

We may set $a = 1 = d$. Proceeding as in the previous case we obtain $\{0, 1\} = \{-b, 1-b\}$ as sets. Hence $b = 0$ and therefore $y = x$ as asserted.

Case 4 $a = c \neq 0$.

Here we may let $a = c = 1$, $d = b + 1$. Proceeding as in Case 2, $\{0, 1\} = \{-b, -b-1\}$ as sets. Hence $b = -1$ and now multiplying both sides of (3.7.7) by $(x-1)^2$ and setting $x = 1$, we obtain $A = B$ contrary to hypothesis.

THEOREM 3.8 (Klein [12]) *In the notation of equation 3.7.1, let*

$$\tau = y_1/y_2$$

with y_1, y_2 solutions of L, and let

$$w = y_2 y_1' - y_1 y_2',$$

(prime denotes differentiation with respect to t). Let $Q_3(y_1, y_2)$ denote the form of degree M/e_3,

$$Q_3(y_1, y_2) = y_2{}^{M/e_3} Q_3(\tau)$$

Then

$$Q_3(y_1, y_2) = C\left[\frac{w}{x'}\, x^{1-(1/e_1)}(1 - x)^{1-(1/e_2)}\right]^{M/2e_3} \tag{3.8.1}$$

for some constant C.

Proof. Taking the derivative of both sides of equation (3.7.1) and applying (2.15.1) gives

$$x' = C\frac{w}{y_2{}^2}\,\frac{Q_1{}^{e_1-1}Q_2{}^{e_2-1}}{Q_3{}^{e_3+1}}$$

We multiply by $y_2{}^2 Q_3{}^{2e_3/M}/x'$. This gives

$$Q_3(y_1, y_2)^{2e_3/M} = \frac{Cw}{x'}\left(\frac{Q_1{}^{e_1}}{Q_3{}^{e_3}}\right)^{1-(1/e_1)}\left(\frac{Q_2{}^{e_2}}{Q_3{}^{e_3}}\right)^{1-(1/e_2)}$$

Equation (3.8.1) now follows from 2.15, 2.12.

4. **Cyclic Case.** Let

$$L = D^2 - Q, \qquad D = \frac{d}{dt} \tag{4.1}$$

be a given 2nd order differential operator with $Q \in \mathbf{C}(t)$. We again assume that all solutions of L are locally algebraic. Our object is to decide in a finite number of steps whether the projectivized group is cyclic. While there are of course an infinite set of cyclic groups, in the present situation there is an a-priori bound for the order of the group. The group G is generated in any case by the transformations corresponding to local monodromy around singular points. In general this gives no upper bound for the order of G but if G is cyclic then its order is bounded by the least common multiple of the denominators of the exponent differences. This will not be used in the following.

Let τ be a ratio of solutions of L. If G is cyclic then replacing τ by some homographic image if necessary, we may suppose that the elements of G are of the form

$$\tau \mapsto w\tau \tag{4.2}$$

where w runs through the mth roots of unity in $\mathbf{C}$. Hence τ^m is invariant under the monodromy group and thus an element of $\mathbf{C}(t)$, say

$$\tau^m = \xi \in \mathbf{C}(t). \tag{4.3}$$

Then by a classical calculation two independent solutions of L are given by

$$\begin{aligned} u &= \sqrt[2]{1/(\xi^{1/m})}, \\ v &= u\xi^{1/m}. \end{aligned} \tag{4.4}$$

This shows that u'/u and v'/v lie in $\mathbf{C}(t)$.

THEOREM 4.4. *Let L as given by* (4.1) *have solutions which are locally algebraic. Then the projectivized group of L is cyclic if and only if the Riccati equation*

$$y' + y^2 = Q \tag{4.5}$$

has two distinct rational solutions.

Proof. The necessity of this condition has been explained. Conversely if y is a rational solution of (4.5) then y can have only simple poles since by hypothesis the singularities of L are all regular which means that the poles of Q are at most of order two. If we write

$$Q = \sum_{i=1}^{n} \frac{A_i}{(t-\alpha_i)^2} + \frac{B_i}{(t-\alpha_i)} \tag{4.6}$$

then we must have

$$\sum_{i=1}^{n} B_i = 0 \tag{4.7.1}$$

$$-4A_i = 1 - \lambda_i^2 \tag{4.7.2}$$

where λ_i is the exponent difference at α_i ($i = 1, \ldots, n$)

$$1 - \lambda_\infty^2 = -4 \sum_{i=1}^{n} (A_i + \alpha_i B_i). \tag{4.7.3}$$

We may write a rational solution of (4.5) in the form

$$\eta = \sum_{i=1}^{n} \frac{c_i}{t - \alpha_i} + \sum_{j=1}^{N} \frac{1}{t - \theta_j} \tag{4.8}$$

where $\theta_1, \ldots, \theta_N$ are all distinct and are distinct from the singularities of L. Indeed equation (4.5) implies that

$$L = (D + \eta)(D - \eta) \tag{4.9}$$

and so each exponent of $D - \eta$ must be an exponent of L which shows that

$$c_i = \frac{1 \pm \lambda_i}{2} \qquad i = 1, \ldots, n \tag{4.10}$$

and these numbers are rational by hypothesis. Now corresponding to the rational solution of (4.5) given by (4.8), there is an algebraic solution of L given by

$$y = \prod_{j=1}^{N} (t - \theta_j) \cdot \prod_{i=1}^{n} (t - \alpha_i)^{c_i} \tag{4.11}$$

If equation (4.5) has two distinct solutions, then the ratio, τ, of the corresponding solutions of L would be the ratio of two functions of the form given by 4.11 and hence $\tau^m \in \mathbf{C}(t)$ for suitable integer m. This completes the proof of the Theorem.

COROLLARY 4.12. *Each rational solution of* (4.5) *is of the form of equation* (4.8) *where* $(c_1, \ldots, c_n)$ *satisfies* (4.10) *and*

$$N + \sum_{i=1}^{n} c_i + \frac{-1 \pm \lambda_\infty}{2} = 0.$$

Proof. We need only check the estimate for N. As in the proof of the theorem, the exponent of $D - \eta$ at infinity must also be an exponent of L at infinity. The corollary follows directly.

What is the set of all rational solutions of the Riccati equation?

THEOREM 4.13. *If the Riccati equation* (4.5) *has more than two solutions in* $\mathbf{C}(t)$ *then the projectivized group of L is trivial,* (*i.e. all the exponent differences are integers*).

Proof. Let η_1, η_2, η_3 be solutions of equation (4.5) and let u_i be solution of L, such that

$$\eta_i = u_i'/u_i \qquad (i = 1, 2, 3)$$

Then

$$\eta_i - \eta_j = w_{i,j}/u_i u_j$$

where $w_{i,j}$ is the associated wronskian of L. Let $\tau = u_2/u_1$, so

$$\tau = \frac{1}{u_1 u_2}\bigg/\frac{1}{u_2 u_3} = \text{const.} \cdot \eta_1 - \eta_3/\eta_2 - \eta_3.$$

Thus τ lies in $\mathbf{C}(\eta_1, \eta_2, \eta_3)$. This completes the proof.

5. **Dihedral case. Part I.** Let L be as in equation (4.1). We now wish to determine whether the projectivized group is dihedral.

THEOREM 5.1. *If the projectivized group of L is dihedral then the Riccati equation* (4.5) *has a solution in a quadratic extension of* $\mathbf{C}(t)$.

Proof. The conventional hypergeometric differential operator

$$x(1 - x)D^2 + (\gamma - (\alpha + \beta + 1)x)D - \alpha\beta \tag{5.1.1}$$

has exponent differences $1 - \gamma$, $\gamma - \alpha - \beta$, $\alpha - \beta$ at 0, 1, ∞ and no other singularities. Putting

$$\alpha = \frac{1 - \mu}{2}, \qquad \beta = -\frac{\mu}{2}, \qquad \gamma = \frac{1}{2}$$

we obtain a differential equation with exponent differences ½, μ, ½ which by explicit calculations (going back to Gauss) has $(1 + \sqrt{x})^\mu$ and $(1 - \sqrt{x})^\mu$ as solutions. If we replace x by $x(t)$ then the normalized pullback is satisfied by $(1 \pm \sqrt{x})^\mu/\sqrt{w}$ where $w = (1 - x)^{\mu-1}(\sqrt{x})'$. From this the Riccati equation has solution

$$\eta = \frac{\mu}{2}\frac{x'}{x(1-x)}\sqrt{x} - \frac{1}{4}\frac{R'}{R} \tag{5.2}$$

where

$$\sqrt{R} = \frac{\mu}{2}\frac{x'}{x(1-x)}\sqrt{x}.$$

Now by Theorem 3.2, if L has a dihedral group of order $2m$ then L is the pullback of $L_{1/2,1/m,1/2}$ by a rational map $x = x(t)$. This completes the proof of the theorem.

We now study the existence of solutions of equations (4.5) in quadratic extensions.

LEMMA 5.3 (Fuchs). *Let L_2 be the third order linear differential equation satisfied by all binary quadratic forms in y_1, y_2, a pair of independent solutions of L. Suppose that the Ricatti equation* (4.5) *has no solution in* $\mathbf{C}(t)$. *Then equation* (4.5) *has a solution in a quadratic extension field of* $\mathbf{C}(t)$ *if and only if L_2 has a solution whose square lies in* $\mathbf{C}(t)$.

Proof. If equation (4.5) has a solution η_1 in a quadratic extension field, then we may write

$$\eta_1 = \gamma + \sqrt{R} \tag{5.3.1}$$

with γ and R in $\mathbf{C}(t)$. By hypothesis $\sqrt{R} \notin \mathbf{C}(t)$ and hence $\eta_2 = \gamma - \sqrt{R}$ is a distinct solution of (4.5). We may choose y_i solution of L $(i = 1, 2)$ such that

$$\eta_i = y_1'/y_i \tag{5.3.2}$$

and then

$$2\sqrt{R} = \eta_1 - \eta_2 = w/y_1y_2 \tag{5.3.3}$$

where w is the wronskian of L. Since w is a constant, we conclude that $1/\sqrt{R}$ is a constant multiple of y_1y_2 and hence satisfies L_2. This proves the assertion is one direction.

Conversely if $z \in \mathbf{C}(t)$ and $\sqrt{z}$ is a solution of L_2 then $\sqrt{z}$ is a quadratic form in solutions of L. Since such a quadratic form may be factored there are two possibilities:

$$\sqrt{z} = u_1{}^2 \tag{5.3.4}$$

$$\sqrt{z} = u_1u_2 \tag{5.3.5}$$

where u_1, u_2 are independent solutions of L. The first case is ruled out by the hypothesis that equation 4.5 has no solution $\frac{1}{4}\, z'/z \in \mathbf{C}(t)$. We now put $\eta_i = u_i'/u_i$ $(i = 1, 2)$ and calculate from (5.3.5),

$$\eta_1 + \eta_2 = \frac{1}{2}\frac{z'}{z}, \tag{5.3.6}$$

while precisely as 5.3.3

$$\eta_1 - \eta_2 = w/u_1u_2 = w/\sqrt{z} \tag{5.3.3$'$}$$

where again w is a determination of the wronskian of L. Equations (5.3.3′), (5.3.6) show that (4.5) has a solution in a quadratic extension of $\mathbf{C}(t)$. This completes the proof of the lemma.

We now briefly discuss the existence of a solution z of L_2 whose square lies in $\mathbf{C}(t)$. Such a solution must be of the form

$$t = \prod_{i=1}^{n} (t - \alpha_i)^{k_i} \circ g \tag{5.4}$$

where $\alpha_1, \ldots, \alpha_n$ are the finite singularities of L, g is a polynomial different from zero at each α_i. Each k_i must be a half integer with

$$k_i = 1 \pm \lambda_i, 1 \tag{5.5}$$

while

$$\Sigma\, k_i + \deg g = 1, 1 \pm \lambda_\infty. \tag{5.6}$$

This gives a finite set of possibilities for the k_i and the degree of g and the determination of g is reduced to a problem in linear algebra.

6. **Dihedral Case. Part II.** Let L be a 2nd order differential operator as in equation (4.1). In Section 5 we showed that if L has the dihedral group then L decomposes into linear factors in a quadratic extension of $\mathbf{C}(t)$. Such decompositions were studied in that section. To determine whether the group is indeed dihedral we must now be prepared to answer the following question: Given η algebraic over $\mathbf{C}(t)$ does the equation

$$y' = y\eta \tag{6.1}$$

have a solution algebraic over $\mathbf{C}(t)$? (For our application we may assume η is in a quadratic extension of $\mathbf{C}(t)$). This is equivalent to the condition that there exist $m \in \mathbf{N}$ such that $m\eta$ is logarithmic derivative of element of $\mathbf{C}(t, \eta)$. The difficulty is that of finding an a-priori bound for m.

In the classical theory of hypergeometric functions this problem did not arise, as, for $n = 2$, the field $\mathbf{C}(t, \eta)$ is either $\mathbf{C}(\sqrt{t})$, $\mathbf{C}(\sqrt{t-1})$ or $\mathbf{C}(t, \sqrt{t(t-1)})$ and so in all cases of genus zero and hence equation (6.1) may be analyzed without difficulty.

We now consider a generalization of the above problem.

Let ω be a differential with at most simple poles and with rational residues at each pole on a curve C defined over a field K of characteristic zero. To determine in a finite number of steps whether $m\omega$ is a logarithmic differential for some (unknown) $m \in \mathbf{Z}$.

By solving this question we solve the problem of Section 5. As general references for this discussion see [13, 16]. At each point P of C we write

$$\omega = n_p \frac{dt_p}{t_p} + \omega_p. \tag{6.2}$$

where t_p is a local parameter at P, ω_p is regular at t_p and n_p is a rational number which we may assume to be in $\mathbf{Z}$. Let

$$\mathcal{L} = \Sigma\, n_p P \tag{6.3}$$

a divisor of C of degree zero.

There are two steps

I. Find an integer m such that either $m\mathfrak{L}$ is principal or $\mathfrak{L}$ is of infinite order in the Jacobian of C.

II. Decide whether $m\mathfrak{L}$ is principal and if it is find θ in the function field of C such that

$$(\theta) = m\mathfrak{L}$$

Suppose I and II carried out, then put

$$\omega_1 = \frac{d\theta}{\theta} - m\omega. \tag{6.4}$$

We note that ω_1 is a differential of the first kind. If $\omega_1 = 0$ then the problem has an affirmative solution. If $\omega_1 \neq 0$ then $N\omega_1$ is never logarithmic regardless of $N \in \mathbf{N}$ and hence the problem has a negative solution.

I. We choose a non-singular model for C in $\mathbf{P}^3$. Let K be a field of definition of both C and of the divisor $\mathfrak{L}$.

Case 1. K is an algebraic number field.

Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be primes of K extending distinct rational primes p_1, p_2 such that the reductions $\overline{C_1}$, $\overline{C_2}$ of C are non-singular. Let G_j $(j = 1, 2)$ be the group of points on the Jacobian $J(\overline{C_j})$ of the reduced curve $\overline{C_j}$ which are rational over the residue class field $\overline{K_j}$ and which are of order prime to p_j. Let e_j be the exponent of G_j. Thus $(e_j, p_j) = 1$ and

$$e_j G_j = 0. \tag{6.5}$$

LEMMA 6.6. *If the image of $\mathfrak{L}$ in $J(C)$ is of finite order then*

$$e_1 e_2 \mathfrak{L} \sim 0.$$

Proof. If $\mathfrak{L}$ is of finite order then there exists $p_1{}^i m$, $(p_1, m) = 1$ such that

$$p_1{}^i m \mathfrak{L} \sim 0. \tag{6.6.1}$$

Hence $p_1{}^i\mathcal{L}$ is of order prime to p_1. But the mapping

$$J(C)(K) \to J(\overline{C_1})(\overline{K_1})$$

of the group of K-rational points of $J(C)$ into the group of $\overline{K_1}$-rational points of $J(\overline{C_1})$, is injective on the subgroup of elements of order prime to p_1. The image of $p_1{}^i\mathcal{L}$ is rational over $\overline{K_1}$ and of order prime to p_1, hence is annihilated by e_1. Thus

$$e_1 p_1{}^i \mathcal{L} \sim 0 \tag{6.6.2}$$

likewise

$$e_2 p_2{}^j \mathcal{L} \sim 0$$

for some unknown j. We choose $a, b \in \mathbf{Z}$ such that

$$a p_1{}^i + b p_2{}^j = 1$$

and conclude that

$$e_1 e_2 \mathcal{L} = e_1 e_2 (a p_1{}^i \mathcal{L} + b p_2{}^j \mathcal{L}) \sim 0.$$

This completes the proof of the lemma.

Case 2. K is finitely generated over $\mathbf{Q}$.

We choose a specialization $K \to K'$ such that K' is an algebraic number field and $C \to C'$, a non-singular curve. Since

$$J(C)(K) \underset{\text{tors}}{\to} J(C')(K')_{\text{tors}}$$

is an injection, we may repeat case 1. This completes the treatment of problem I.

II. We choose a plane curve model for C with only ordinary singular points $P_1, \ldots, P_n$ the multiplicity of P_j being s_j. Let $D = \sum_{j=1}^n s_j P_j$. Put

$$L = L_0 - L_\infty$$

L_0 and L_∞ being positive divisors.

Construct a curve B having intersection with C containing $D + L_0$. Write

$$C \cap B = D + L_0 + X,$$

where X is a positive divisor. Let $l =$ degree B. Then

$$L_0 \sim L_\infty$$

if and only if there exists a curve B' of degree l such that

$$B' \cap C = L_\infty + D + X.$$

This construction involves only linear algebra and the calculation of X. If $L_0 \sim L_\infty$ then $L_0 - L_\infty = (B/B')$. The solution of part II involves the application of this procedure to $m\mathfrak{L}$.

7. **Decision Procedure.** Let L be a second order differential operator defined over $\mathbf{C}(t)$. We may suppose L is given by (3.3). Our object is to give a decision procedure for determining whether all the solutions are algebraic.

We may assume that all singularities of L are regular, and that all exponents are rational numbers. Our object is to determine whether the projective monodromy group G of L is one of the five finite types. We apply the procedure of Section 4 to determine whether G is cyclic. If it is we are done. Conversely if we exclude the possibility that G is cyclic, we then use the procedure of Section 5, Section 6 to determine whether G is dihedral. Having decided that G is neither cyclic nor dihedral we consider successively the three remaining finite possibilities, i.e. G may be the tetrahedral, octahedral or icosahedral group. The reason for proceeding in this sequence is we wish to use the uniqueness theorem of Klein (3.7).

We now consider whether L is the pullback by $x = x(t)$ of $L_{1/e_1,1/e_2,1/e_3}$ (cf. equation 3.1) where

$$\left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}\right) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$$

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$$

and where we may assume that we have already excluded previous elements in this list of three possibilities.

We are to decide whether there exists a rational function x which satisfies (3.7.3). Here $q(x)$ is given explicitly by (3.1), i.e.

$$4q(x) = \frac{1 - \dfrac{1}{e_1{}^2}}{x^2} + \frac{1 - \dfrac{1}{e_2{}^2}}{(x-1)^2} + \frac{\dfrac{1}{e_1{}^2} + \dfrac{1}{e_2{}^2} - \dfrac{1}{e_3{}^2} - 1}{x(x-1)}$$

Let $a_1, \ldots, a_n, \infty$ be the singularities of L. Let the exponent differences be $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_\infty$. We know that these numbers are rational and we take them to be *positive.* We define

$$\Delta(L) = (\lambda_1 + \cdots + \lambda_n + \lambda_\infty) - (n-1)$$

Thus

$$\Delta(L_{1/e_1,1/e_2,1/e_3}) = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1.$$

It follows from Section 1 that if L is a pullback of $L_{1/e_1,1/e_2,1/e_3}$ by $x = x(t)$ then the mapping degree (i.e. the maximum of the degrees of numerator and denominator) must be

$$M = \Delta(L)/\Delta(L_0).$$

We normalize by insisting that the denominator is monic and we interpret equation 3.7.3 as a set of algebraic relations among the at most $2M + 1$ unknown coefficients of the numerator and denominator of x (we must consider $M + 1$ possibilities as the exact degree of the denominator is not fixed). We thus arrive at $M + 1$ algebraic sets which have in all, at most one rational point. By elimination theory we should not only be able to decide whether a rational solution of 3.7.3 exists but we should indeed be able to determine the solution itself.

8. **Conjecture of Grothendieck.** Let L again be a second order differential operator but for simplicity we suppose that the coefficients lie in $\mathbf{Q}(t)$. For almost all p we may consider L_p, the reduction of L modulo p on $\mathbf{F}_p(t)$. Clearly L_p acts as linear operator on $\mathbf{F}_p(t)$ as vector space over $\mathbf{F}_p(t^p)$. Let K_p be the dimension of the kernel of L_p viewed as vector space over $\mathbf{F}_p(t^p)$.

As a special case of a general conjecture of Grothendieck, it is conjectured:

(8.1) If $K_p = 2$ for almost all p then all the solutions of L are algebraic functions.

The converse of this statement is known.

The general conjecture has been verified by Katz [8] for the case in which L is a suitable direct factor of the Fuchs-Picard equation. In particular it is known to be valid in the case of second order operators with 3 singular points. The conjecture is completely open in the case of Heun's differential equation [17, p. 576] (second order, four singular points), the point being that in the case of 3 singular points, the integral formula of Euler reveals the cohomological nature of the gauss hypergeometric functions but no integral formula is known for the solutions of the Heun differential equation.

For the case of 3 singular points there is a simple algorithm for the calculation of K_p. For 4 or more singular points no such algorithm is known. Thus even if Grothendieck's conjecture were confirmed, it is not clear that it would give a response to problem (0.2).

It may be useful to explain briefly how one could use Katz's result to verify Schwarz's list. We know from Ihara that the differential equation for $F(a, b; c; x)$ has $K_p = 2$ if and only if the minimal representative mod p of c lies between the corresponding representatives of a and of b. Using the criterion of Grothendieck it is easy to check that each equation in Schwarz's list has only algebraic solutions.

To show that Schwarz's list is exhaustive one must start with the fact that the list consists of *classes* of exponent differences. Two sets $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1', \lambda_2', \lambda_3')$ of exponent differences are said to be equivalent [14, p. 119] if

$$\lambda_1' = \epsilon_1 \lambda_i + \mu_i$$

where $\epsilon_i = \pm 1$, each $\mu_i \in \mathbf{Z}$ and $\mu_1 + \mu_2 + \mu_3 \equiv 0 \bmod 2$. The central point is that if the sets of exponent differences are equivalent then the monodromy groups are isomorphic. This can be shown either by Riemann's method of calculating the monodromy group or by Gauss's relations among contiguous hypergeometric functions [5, Section 34].

Consequently we may assume that $(\lambda_1, \lambda_2, \lambda_3)$ form a reduced set of exponent differences, i.e.

$$i \geq \lambda_1 > 0 \tag{8.2}$$

$$1 \geq \lambda_1 + \lambda_j \quad \text{for} \quad i \neq j. \tag{8.3}$$

We follow Schwarz's theory of curvilinear triangles to show that if all solutions are algebraic then

$$\sum_{i=1}^{3} \lambda_i > 1. \tag{8.4}$$

We now use Section 4 to show that the cyclic case can only occur with exponent differences $(1/n, 1, 1/n)$, and the method of Section 5 to show that the dihedral monodromy group can occur only if the exponent differences are $1/2$, $1/2$, $1/m$. This leaves only the tetrahedral, octahedral and icosahedral groups. The elements of these groups have orders 2, 3, 4, 5, and hence the λ_i have these integers as denominators. This condition together with (8.2) shows that there are only a finite number of possibilities; use of (8.3) and (8.4) further reduces the list. Furthermore orders 4 and 5 do not occur simultaneously. By these considerations we arrive at the Schwarz list together with the additional candidates:

Exponent differences	Spherical excess
$\frac{1}{2}, \frac{2}{5}, \frac{2}{5}$	$\frac{3}{10}$
$\frac{2}{3}, \frac{1}{3}, \frac{1}{4}$	$\frac{1}{4}$
$\frac{2}{5}, \frac{2}{5}, \frac{1}{3}$	$\frac{2}{15}$
$\frac{2}{3}, \frac{2}{5}, \frac{1}{5}$	$\frac{4}{15}$
$\frac{3}{5}, \frac{2}{5}, \frac{1}{5}$	$\frac{1}{5}$
$\frac{3}{5}, \frac{2}{5}, \frac{2}{5}$	$\frac{2}{5}$
$\frac{3}{5}, \frac{1}{3}, \frac{1}{3}$	$\frac{4}{15}$

These are excluded by means of Grothendieck's criterion, i.e. by the converse of (8.1).

It is not clear that this mixture of methods of Schwarz and Katz is any improvement over the original method of Schwarz. We note that for the case of 4 singular points there is no theory of reduced set of exponent differences. In the case of 4 singular points, the monodromy group depends not only on the exponential differences and singular points but also upon an additional parameter. One does not know how to infer isomorphism of monodromy groups from relations concerning the parameters [1, p. 311–329].

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1.22 On second order homogeneous linear differential equations with Liouvillian solutions

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On second order homogeneous linear differential equations with Liouvillian solutions[1]

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Abstract

We determine all minimal polynomials for second order homogeneous linear differential equations with algebraic solutions decomposed into invariants and we show how easily one can recover the known conditions on differential Galois groups [10, 17, 24] using invariant theory. Applying these conditions and the differential invariants of a differential equation we deduce an alternative method to the algorithms given in [10, 18, 24] for computing Liouvillian solutions. For irreducible second order equations our method determines solutions by formulas in all but three cases.

Keywords: Differential Galois theory; Differential equations; Liouvillian solutions; Invariants; Differential invariants

1. Introduction

Algorithms computing algebraic solutions of second order differential equations are well-known since last century. Already in 1839, J. Liouville published such a procedure. However, the degree of the minimal polynomial of a solution must be known. Among other renowned mathematicians, Fuchs [4, 5] developed from 1875 to 1877 a method for computing algebraic solutions, which is based only on binary forms. He wanted to clear up the question of when a second order linear differential equation has algebraic solutions and he solved it by determining the possible orders of symmetric powers associated with the given differential equation for which at least one needs to have a root of a rational function as a solution (see e.g. [4, No. 22, Satz]). Thereby, he gave a method – presumably, without taking note of it – that remains valid for determining Liouvillian solutions of irreducible linear differential equations of second order.

Modern algorithms for computing Liouvillian solutions are based on differential Galois theory. These algorithms determine a minimal polynomial of the logarithmic derivative of a Liouvillian solution since one knows that these derivatives are algebraic

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of bounded degree (see [15, Theorem 2.4]). This approach for second order equations stems from Kovacic [10] and has been implemented in Maple and some other computer algebra systems. A more accessible version of this algorithm was given by Ulmer and Weil [24] and is implemented in Maple, too.

Even when the solutions are algebraic, one can determine the minimal polynomial of a solution. In Singer and Ulmer [18] this is used to solve equations with a finite primitive unimodular Galois group by extending the Fuchsian method to arbitrary order. For this, one first has to compute a minimal polynomial decomposed into invariants for every possible Galois group.

In this paper we take up the ideas from Fuchs once again. Applying invariant theory we reformulate these ideas and state them more precisely. From that we obtain an alternative method for computing Liouvillian solutions. Unlike the known algorithms [10, 18, 24] we compute for irreducible second order equations – except for three cases – all Liouvillian solutions directly by formulas and not via their minimal polynomials (Theorem 11).

In the three exceptional cases we get a minimal polynomial of a solution using *exclusively* absolute invariants and their syzygies by computing – depending on the case – one rational solution of the 6th, 8th or 12th symmetric power of the differential equation and determining its corresponding constant (Theorem 16). There is no need for a Gröbner basis computation in these cases. In [4, p. 100] and [18, p. 67] one needs to substitute in these cases a minimal polynomial decomposed into invariants in the differential equation. But this is very expensive.

We note, that it is possible to extend the algorithm presented here at least to all linear differential equations of prime power order.

The paper is organized as follows. In the rest of this section we briefly introduce differential Galois theory and the concept of invariants. In Section 2 we summarize important properties of linear differential equations with algebraic solutions, which we use in Section 3 to compute minimal polynomials decomposed into invariants. In Section 4 we show, how easily one can obtain the known criteria for differential Galois groups [10, 17, 24] using invariant theory. These criteria result in an algorithm for computing Liouvillian solutions of a second order linear differential equation which is presented in Section 5. Finally we give for every (irreducible) case an example.

The rest of this section and the following one contains nothing new, but are included to complete the picture.

1.1. Differential Galois theory

For the exact definitions of the following concepts we refer to [8, 9, 16].

Functions, which one gets from the rational functions by successive adjunctions of nested integrals, exponentials of integrals and algebraic functions, are the *Liouvillian functions*.

A *differential field* $(k, ')$ is a field k together with a derivation $'$ in k. The set of all constants $\mathscr{C} = \{a \in k \mid a' = 0\}$ is a subfield of $(k, ')$.

Let $\mathscr{C}$ be algebraically closed and k be of characteristic 0. Consider the following ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad (a_i \in k) \tag{1}$$

over k with a system $\{y_1, \ldots, y_n\}$ of fundamental solutions.

By extending the derivation $'$ to a system of fundamental solutions and by adjunction of these solutions and their derivatives to k in a way the field of constants does not change, one gets $K = k\langle y_1, \ldots, y_n \rangle$, the so-called *Picard–Vessiot extension* (PVE) of $L(y) = 0$. With the above assumptions, the PVE of $L(y) = 0$ always exists and is unique up to differential isomorphisms. This extension plays the same role for a differential equation as a splitting field for a polynomial equation.

The set of all automorphisms of K, which fix k elementwise and commute with the derivation in K, is a group, the *differential Galois group* $\mathscr{G}(K/k) = \mathscr{G}(L)$ of $L(y) = 0$. Since the automorphisms must commute with the derivation, they map a solution to a solution. Therefore $\mathscr{G}(L)$ operates on the $\mathscr{C}$-vector space of the fundamental solutions and from that one gets a faithful matrix representation of $\mathscr{G}(L)$, hence $\mathscr{G}(L)$ is isomorphic to a linear subgroup of $\mathrm{GL}(n, \mathscr{C})$. Moreover, it is isomorphic to a linear algebraic group. Furthermore, there is a (differential) Galois correspondence between the linear algebraic subgroups of $\mathscr{G}(L)$ and the differential subfields of K/k (see [8, Theorems 5.5 and 5.9]).

The choice of another system of fundamental solutions leads to an equivalent representation. Hence, for every differential equation $L(y) = 0$, there is exactly one representation of $\mathscr{G}(L)$ up to equivalence.

Many properties of $L(y) = 0$ and its solutions can be found in the structure of $\mathscr{G}(L)$. Such an important property is: The component of the identity of $\mathscr{G}(L)^\circ$ of $\mathscr{G}(L)$ in the Zariski topology is solvable, if and only if K is a Liouvillian extension of k (see Kolchin [9, Section 25, Theorem]). By this, we have a criterion to decide whether a linear differential equation $L(y) = 0$ has Liouvillian solutions. An ordinary homogeneous linear differential polynomial $L(y)$ is called *reducible* over k, if there are two homogeneous linear differential polynomials $L_1(y)$ and $L_2(y)$ of positive order over k with $L(y) = L_2(L_1(y))$, otherwise $L(y)$ is called *irreducible*. $L(y) = 0$ is reducible, if and only if the corresponding representation of $\mathscr{G}(L)$ is reducible (see [9, Section 22, Theorem 1]). If an irreducible linear differential equation $L(y) = 0$ has a Liouvillian solution over k, then all solutions of $L(y) = 0$ are Liouvillian (see [15, Theorem 2.4]). However, if $L(y) = 0$ is reducible then Liouvillian solutions only possibly exist. Against this, a second order linear differential equation has either only Liouvillian solutions or no Liouvillian solutions (see e.g. [24, Section 1.2]).

1.2. Invariants

In this section we introduce informally some concepts of invariant theory. For the exact definitions we refer the reader to [20, 19] or [14].

Let V be a finite dimensional $\mathscr{C}$-vector space and G a linear subgroup of $\mathrm{GL}(V)$. An (*absolute*) *invariant* is a polynomial function $f \in \mathscr{C}[V]$ which remains unchanged under the group action, i.e. $f = f \circ g$ for all $g \in G$. If, for some $g \in G$, f and $f \circ g$ differ from each other only by a constant factor then the polynomial function f is called a *relative invariant*. The set of all invariants of G forms the *ring of invariants* $\mathscr{C}[V]^G$. For irreducible groups $G \in \mathrm{GL}(V)$, the rings of invariants $\mathscr{C}[V]^G$ are finitely generated by Hilbert's finiteness theorem (see e.g. [20]).

For finite groups $G \in \mathrm{GL}(V)$ the *Reynolds operator* $R_G(f) = (1/|G|)\sum_{g\in G} f \circ g$ maps a polynomial function $f \in \mathscr{C}[V]$ to the invariant $R_G(f) \in \mathscr{C}[V]^G$. With the *Hessian* $H(I_1) = \det(\partial^2 I_1/\partial v_i\, \partial v_j)$ and the *Jacobian* $J(I_1,\dots,I_n) = \det(\partial I_i/\partial v_j)$ it is possible to generate new invariants from the invariants $I_1(\mathbf{v}),\dots,I_n(\mathbf{v})$ (see e.g. [20, 19, 14]).

Molien and *Hilbert series* (see e.g. [20]) of a ring of invariants allow us to decide whether a set of invariants already generates the whole ring.

Let V be the $\mathscr{C}$-vector space of a system of fundamental solutions of $L(y) = 0$ and let $I(\mathbf{v}) \in \mathscr{C}[V]^{\mathscr{G}(L)}$ be an invariant of $\mathscr{G}(L)$. If one evaluates the invariant $I(\mathbf{v})$ with the fundamental solutions and takes into account that exactly the elements $a \in k$ are invariant under the Galois group $\mathscr{G}(L)$ then $I(y_1,\dots,y_n)$ must be an element of k. An important tool for computing such an element are the symmetric powers of $L(y) = 0$.

The mth *symmetric power* $L^{Ⓢm}(y) = 0$ of $L(y) = 0$ is the differential equation whose solution space consists exactly of all mth power products of solutions of $L(y) = 0$. There is an efficient algorithm to construct symmetric powers described e.g. in [17, p. 20] or [2, p. 14].

2. Algebraic solutions

In this section we briefly give some important properties of linear differential equations with algebraic solutions.

Theorem 1 (Ulmer [22, Theorem 2.2]; Singer [15, Theorem 2.4]). *Let k be a differential field of characteristic 0 with an algebraically closed field of constants. If an irreducible linear differential equation $L(y) = 0$ has an algebraic solution, then*
- *all solutions are algebraic;*
- *$\mathscr{G}(L)$ is finite;*
- *the PVE of $L(y) = 0$ is a normal extension and coincides with the splitting field $k(y_1,\dots,y_n)$.*

For many statements on differential equations it is assumed that the Galois group corresponding to $L(y) = 0$ is unimodular (i.e. $\subseteq \mathrm{SL}(n, \mathscr{C})$).

Theorem 2 (Kaplansky [8, p. 41]; Singer and Ulmer [18, Theorem 1.2]). *Let $L(y)$ be the linear differential equation (1), then $\mathscr{G}(L)$ is unimodular, if and only if there is a $W \in k$ such that $W'/W = a_{n-1}$.*

Using the variable transformation $y = z \cdot \exp(-\int a_{n-1}/n)$, it is always possible to transform the equation $L(y) = 0$ into the equation

$$L_{\mathrm{SL}}(z) = z^{(n)} + b_{n-2}z^{(n-2)} + \cdots + b_1 z' + b_0 z = 0 \quad (b_i \in k).$$

According to Theorem 2, $\mathscr{G}(L_{\mathrm{SL}})$ is unimodular. For second order equations we get $L_{\mathrm{SL}}(z) = z'' + (a_0 - a_1^2/4 - a_1'/2)z = 0$.

Under such a transformation it is clear that $L(y) = 0$ has Liouvillian solutions if and only if $L_{\mathrm{SL}}(z) = 0$ has Liouvillian solutions. Furthermore, if $L(y) = 0$ has only algebraic solutions, then $L_{\mathrm{SL}}(z) = 0$ has only algebraic solutions (cf. [22, p. 184]).

Theorem 3 (Singer and Ulmer [18, Corollary 1.4]). *Let $k \subset K$ be a differential field of characteristic 0 and let the common field of constants of k and K be algebraically closed. If $y \in K$ is algebraic over k and y'/y is algebraic of degree m over k, then the minimal polynomial $P(\mathbf{Y}) = 0$ of y over k can be written in the following way*

$$P(\mathbf{Y}) = \mathbf{Y}^{d \cdot m} + a_{m-1}\mathbf{Y}^{d \cdot (m-1)} + \cdots + a_0 = \prod_{\sigma \in \mathscr{T}} (\mathbf{Y}^d - (\sigma(y))^d), \tag{2}$$

where $[k(y) : k(y'/y)] = d = |H/N|$, H/N is cyclic, $a_j \in k$, $H = \mathscr{G}(K/k(y'/y))$ is a 1-reducible subgroup of $G = \mathscr{G}(K/k)$ and $\mathscr{T}$ is a set of left coset representatives of H in G of minimal index m.

3. Minimal polynomials decomposed into invariants

Theorems 3 and 1 imply that any irreducible linear differential equation $L(y) = 0$ with algebraic solutions has a minimal polynomial $P(\mathbf{Y})$ of the form (2). Therefore, it remains to compute for any finite differential Galois group such a minimal polynomial.

In this section, we compute for any finite unimodular group a minimal polynomial written in terms of invariants. The restriction to unimodular groups is necessary, since only these groups are all known. However, Theorem 2 ensures that we can construct a linear differential equation with unimodular Galois group from any linear differential equation $L(y) = 0$.

3.1. Imprimitive unimodular groups of degree 2

The finite imprimitive algebraic subgroups of $\mathrm{SL}(2, \mathscr{C})$ are the binary dihedral groups $D_n^{\mathrm{SL}_2}$ of order $4n$ [24]. These are central extensions of the dihedral groups D_n. They are generated by [19, p. 89]

$$u_n = \begin{pmatrix} \mathrm{e}^{\pi \mathrm{i}/n} & 0 \\ 0 & \mathrm{e}^{-\pi \mathrm{i}/n} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

A simple calculation shows that these representations are irreducible. The invariants of the binary dihedral groups are generated by

$$I_4 = y_1^2 y_2^2, \qquad I_{2n} = y_1^{2n} + (-1)^n y_2^{2n}, \qquad I_{2n+2} = y_1 y_2 (y_1^{2n} - (-1)^n y_2^{2n})$$

and they satisfy the relation

$$I_{2n+2}^2 - I_4 I_{2n}^2 + (-1)^n 4 I_4^{n+1} = 0, \tag{3}$$

see [19, p. 95]. Let $\{y_1, y_2\}$ be a set of fundamental solutions of an equation $L(y) = 0$ of second order.

Theorem 4. *Let $L(y) = 0$ be an irreducible second order linear differential equation over k with a finite unimodular Galois group $\mathscr{G}(L) \cong D_n^{\mathrm{SL}_2}$. Then*

$$P(\mathbf{Y}) = \mathbf{Y}^{4n} - I_{2n}\mathbf{Y}^{2n} + (-1)^n I_4^n$$

is a minimal polynomial decomposed into invariants for a solution of $L(y) = 0$.

Proof. The degree of a minimal polynomial for a solution of $L(y) = 0$ of order 2 equals the order of the group $\mathscr{G}(L)$, see e.g. [18, p. 55]. Comparing this with $P(\mathbf{Y})$ from Theorem 3 shows that $d \cdot m = |\mathscr{G}(L)|$. $H = \langle u_n \rangle$ with $|H| = 2n$ is a maximal subgroup of $\mathscr{G}(L)$. H is a cyclic group and hence Abelian and 1-reducible and the elements of H have the common eigenvector $z = y_1$ (z is a solution of $L(y) = 0$). $\mathscr{T} = \{u_n^n, v u_n^n\}$ is a set of left coset representatives of H in $\mathscr{G}(L)$.

Together with $m = [\mathscr{G}(L) : H] = 2$ and thus $d = 2n$ one can calculate the minimal polynomial in the following way:

$$\begin{aligned} P(\mathbf{Y}) &= \prod_{\sigma \in \mathscr{T}} (\mathbf{Y}^{2n} - \sigma(z)^{2n}) \\ &= (\mathbf{Y}^{2n} - (-y_1)^{2n})(\mathbf{Y}^{2n} - (-\mathrm{i}y_2)^{2n}) \\ &= \mathbf{Y}^{4n} - (y_1^{2n} + (-1)^n y_2^{2n})\mathbf{Y}^{2n} + (-1)^n y_1^{2n} y_2^{2n}. \end{aligned}$$

Decomposing this expression into the above mentioned invariants completes the proof. □

3.2. Primitive unimodular groups of degree 2

Up to isomorphisms, there are three finite primitive unimodular linear algebraic groups of degree 2. These groups are the tetrahedral group ($A_4^{\mathrm{SL}_2}$), the octahedral group ($S_4^{\mathrm{SL}_2}$) and the icosahedral group ($A_5^{\mathrm{SL}_2}$), see e.g. [24].

In contrast to Fuchs, the minimal polynomials in this section are determined using exclusively absolute invariants. The definitions of the matrix groups stem from Miller et al. [11, p. 221], while the necessary 1-reducible subgroups, left coset representatives and eigenvectors are found in [18]. All the fundamental invariants are computed with the algorithms and implementations given in Fakler [2, 3] (see also the relative invariants given in [11, p. 225]).

3.2.1. The tetrahedral group

$$\mathbf{Y}^{24} + 48I_1\mathbf{Y}^{18} + (90I_3 + 228I_1^2)\mathbf{Y}^{12} + (288I_1I_3 + 2368I_1^3)\mathbf{Y}^6$$
$$- 3I_3^2 + 36I_1^2I_3 - 108I_1^4$$

is a minimal polynomial decomposed into invariants for the tetrahedral group. The invariants of this group are generated by

$$I_1 = \tfrac{1}{2}R_{A_4^{\mathrm{SL}_2}}(y_1^5y_2)(\mathbf{y}) = y_1y_2^5 - y_1^5y_2,$$
$$I_2 = -\tfrac{1}{25}H(I_1) = y_2^8 + 14y_1^4y_2^4 + y_1^8,$$
$$I_3 = \tfrac{1}{8}J(I_1,I_2) = y_2^{12} - 33y_1^4y_2^8 - 33y_1^8y_2^4 + y_1^{12}$$

and they satisfy the relation $I_3^2 - I_2^3 + 108I_1^4 = 0$.

Using Molien and Hilbert series one can show that the ring of invariants can be written as the direct sum of graded $\mathscr{C}$-vector spaces

$$\mathscr{C}[y_1,y_2]^{A_4^{\mathrm{SL}_2}} = \mathscr{C}[I_1,I_2,I_3] = \mathscr{C}[I_1,I_2] \oplus I_3 \cdot \mathscr{C}[I_1,I_2].$$

In this expression for the minimal polynomial, I_1 was multiplied by $-\mu^3$ and I_3 by the factor $-\frac{26}{3}\mu^2 + \frac{26}{3}\mu - \frac{7}{3}$, where $\mu^4 - 2\mu^3 + 5\mu^2 - 4\mu + 1 = 0$,[2] and $i = \sqrt{-1} = 2\mu^3 - 3\mu^2 + 9\mu - 4$.

The above representation needs an algebraic extension. It can be an advantage to choose a representation which is less sparse but does not require an algebraic extension. One obtains such a representation e.g. by computing a lexicographical Gröbner basis from the three equations of the fundamental invariants for $y_2 \succ y_1 \succ I_3 \succ I_2 \succ I_1$:

$$\mathbf{Y}^{24} + 10I_2\mathbf{Y}^{16} + 5I_3\mathbf{Y}^{12} - 15I_2^2\mathbf{Y}^8 - I_2I_3\mathbf{Y}^4 + I_1^4. \tag{4}$$

In this expression for a minimal polynomial decomposed into invariants for the tetrahedral group I_1 was multiplied by $\frac{1}{4}$, I_2 by $-\frac{5}{80}$ and I_3 by the factor $-\frac{1}{16}$.

3.2.2. The octahedral group

$$\mathbf{Y}^{48} + 20I_1\mathbf{Y}^{40} + 70I_1^2\mathbf{Y}^{32} + (2702I_2^2 + 100I_1^3)\mathbf{Y}^{24} + (-1060I_1I_2^2 + 65I_1^4)\mathbf{Y}^{16}$$
$$+ (78I_1^2I_2^2 + 16I_1^5)\mathbf{Y}^8 + I_2^4$$

is a minimal polynomial decomposed into invariants for the octahedral group. The ring of invariants of this group is generated by

$$I_1 = \tfrac{1}{24}R_{S_4^{\mathrm{SL}_2}}(y_1^4y_2^4)(\mathbf{y}) = y_2^8 + 14y_1^4y_2^4 + y_1^8,$$
$$I_2 = \tfrac{1}{9408}H(I_1) = y_1^2y_2^{10} - 2y_1^6y_2^6 + y_1^{10}y_2^2,$$
$$I_3 = -\tfrac{1}{16}J(I_1,I_2) = y_1y_2^{17} - 34y_1^5y_2^{13} + 34y_1^{13}y_2^5 - y_1^{17}y_2.$$

[2] This algebraic extension becomes necessary for computing an eigenvector.

These three invariants satisfy the sysygy $I_3^2 + 108 I_2^3 - I_1^3 I_2 = 0$. That this syzygy is the only relation among the fundamental invariants is confirmed by the Molien and the Hilbert series. They also show, that the ring of invariants decomposes as the direct sum of graded $\mathscr{C}$-vector spaces

$$\mathscr{C}[y_1, y_2]^{S_4^{\mathrm{SL}_2}} = \mathscr{C}[I_1, I_2, I_3] = \mathscr{C}[I_1, I_2] \oplus I_3 \cdot \mathscr{C}[I_1, I_2].$$

In the above-mentioned expression for the minimal polynomial I_1 was multiplied by $-\frac{1}{16}$ and I_2 by the factor $\frac{1}{16}$.

3.2.3. The icosahedral group

$$\begin{aligned}
&\mathbf{Y}^{120} + 20570 I_2 \mathbf{Y}^{100} + 91 I_3 \mathbf{Y}^{90} - 86135665 I_2^2 \mathbf{Y}^{80} - 78254 I_2 I_3 \mathbf{Y}^{70} \\
&\quad + (14993701690 I_2^3 + 11137761250 I_1^5)\mathbf{Y}^{60} + 897941 I_2^2 I_3 \mathbf{Y}^{50} \\
&\quad + (-11602919295 I_2^4 + 273542733750 I_1^5 I_2)\mathbf{Y}^{40} \\
&\quad + (-151734 I_2^3 - 6953000 I_1^5) I_3 \mathbf{Y}^{30} + (503123324 I_2^5 - 7854563750 I_1^5 I_2^2)\mathbf{Y}^{20} \\
&\quad + (1331 I_2^4 + 500 I_1^5 I_2) I_3 \mathbf{Y}^{10} + 3125 I_1^{10}
\end{aligned}$$

is a minimal polynomial decomposed into invariants for the icosahedral group. The three invariants

$$\begin{aligned}
I_1 &= -\tfrac{1}{25} R_{A_5^{\mathrm{SL}_2}}(y_1^6 y_2^6)(\mathbf{y}) = y_1 y_2^{11} - 11 y_1^6 y_2^6 - y_1^{11} y_2, \\
I_2 &= -\tfrac{1}{121} H(I_1) = y_2^{20} + 228 y_1^5 y_2^{15} + 494 y_1^{10} y_2^{10} - 228 y_1^{15} y_2^5 + y_1^{20}, \\
I_3 &= \tfrac{1}{20} J(I_1, I_2) = y_2^{30} - 522 y_1^5 y_2^{25} - 10005 y_1^{10} y_2^{20} \\
&\quad - 10005 y_1^{20} y_2^{10} + 522 y_1^{25} y_2^5 + y_1^{30},
\end{aligned}$$

are the fundamental invariants of the icosahedral group and satisfy the algebraic relation $I_3^2 - I_2^3 + 1728 I_1^5 = 0$.

Molien and Hilbert series verify that this relation is the only syzygy and show that the ring of invariants decomposes as the direct sum of graded $\mathscr{C}$-vector spaces

$$\mathscr{C}[y_1, y_2]^{A_5^{\mathrm{SL}_2}} = \mathscr{C}[I_1, I_2, I_3] = \mathscr{C}[I_1, I_2] \oplus I_3 \cdot \mathscr{C}[I_1, I_2].$$

In the above-mentioned expression for the minimal polynomial I_1 was multiplied by $\frac{1}{125}$, I_2 by $-\frac{1}{275 \cdot 125}$ and I_3 by the factor $-\frac{11}{25 \cdot 125}$.

4. Criteria for differential Galois groups

The numbers and degrees of the invariants of all finite unimodular linear algebraic groups determined in the previous section yield conditions for the Galois group of a second order differential equation. In this section, we show how easily one can recover the known results (see [10, 17, 24]) using invariant theory.

If the Galois group $\mathscr{G}(L)$ is an imprimitive group, it is not easy to distinguish between a finite and an infinite group (see [17, p. 25]). The only infinite imprimitive unimodular Galois group of degree 2 is

$$D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \right\} \quad \text{where } a \in \mathbb{C}^*.$$

This group has only one fundamental invariant $I_4 = y_1^2 y_2^2$ (see [24, Section 3.2]).

The following lemma allows a simple method to distinguish all Galois groups $\mathscr{G}(L)$ for which an irreducible second order linear differential equation $L(y)=0$ has Liouvillian solutions. This is no longer true in higher order.

Lemma 5 (cf. Sturmfels [20, Lemma 3.6.3]; Schur and Grunsky [14, p. 47]). *A binary form of positive degree over k cannot vanish identically. In particular, this holds for homogeneous invariants in two independant variables.*

Rational solutions of the mth symmetric power $L^{Ⓢm}(y)=0$ correspond to homogeneous invariants of degree m of $\mathscr{G}(L)$ (cf. [2, 18]). Hence, as a consequence of Lemma 5, any invariant of degree m corresponds bijectively to a non-trivial rational solution of the mth symmetric power of $L(y)=0$ (see [18, Lemma 3.5 (iii)]).

Corollary 6 (see Ulmer and Weil [24, Lemma 3.2]). *Let $L(y)=0$ be an irreducible second order linear differential equation over k with $\mathscr{G}(L) \cong D_n^{\mathrm{SL}_2}$. Then $L^{Ⓢ4}(y)=0$ has a non-trivial rational solution. In particular,*

(1) *$L^{Ⓢ4}(y)=0$ has two non-trivial rational solutions, if and only if $\mathscr{G}(L) \cong D_2^{\mathrm{SL}_2}$.*

(2) *Otherwise, $L^{Ⓢ4}(y)=0$ has exactly one non-trivial rational solution.*

Proof. $D_2^{\mathrm{SL}_2}$ has two fundamental invariants of degree 4 (see Section 3.1). All the other binary dihedral groups $D_n^{\mathrm{SL}_2}$ have exactly one fundamental invariant of fourth degree. □

The determination of the fundamental invariants of all finite unimodular groups in the last section allows the following result.

Proposition 7. *Let $L(y)=0$ be a second order linear differential equation over k with an unimodular Galois group $\mathscr{G}(L)$. If $L^{Ⓢm}(y)=0$ has a non-trivial rational solution for $m=2$ or odd $m \in \mathbb{N}$, then $L(y)=0$ is reducible.*

Proof. If $L(y)=0$ is irreducible, $L^{Ⓢm}(y)=0$ has at most non-trivial rational solutions for even $m \geqslant 4$. □

It ought to be clear that the practical use of such a statement is restricted. However, the following proposition allows effective computations.

Proposition 8 (see Ulmer and Weil [24, Lemmata 3.2 and 3.3]). *Let $L(y)=0$ be an irreducible second order linear differential equation over k with an unimodular Galois group $\mathscr{G}(L)$. Then the following holds*

(1) *$\mathscr{G}(L)$ is imprimitive, if and only if $L^{\circledS 4}(y)=0$ has a non-trivial rational solution.*

(2) *$\mathscr{G}(L)\cong D_\infty$, if and only if $L^{\circledS 4m}(y)=0$ has exactly one non-trivial rational solution for any $m\in\mathbb{N}$.*

(3) *$\mathscr{G}(L)\cong D_n^{\mathrm{SL}_2}$, if and only if $L^{\circledS 4}(y)=0$ has one and $L^{\circledS 2n}(y)=0$ has two or exactly one non-trivial rational solution depending on whether $4|2n$ or not.*

(4) *$\mathscr{G}(L)$ is primitive and finite, if and only if $L^{\circledS 4}(y)=0$ has none and $L^{\circledS 12}(y)=0$ has at least one non-trivial rational solution.*

(5) *$\mathscr{G}(L)\cong A_4^{\mathrm{SL}_2}$ (tetrahedral group), if and only if $L^{\circledS 4}(y)=0$ has none and $L^{\circledS 6}(y)=0$ has a non-trivial rational solution.*

(6) *$\mathscr{G}(L)\cong S_4^{\mathrm{SL}_2}$ (octahedral group), if and only if $L^{\circledS m}(y)=0$ for $m\in\{4,6\}$ has none and $L^{\circledS 8}(y)=0$ has a non-trivial rational solution.*

(7) *$\mathscr{G}(L)\cong A_5^{\mathrm{SL}_2}$ (icosahedral group), if and only if $L^{\circledS m}(y)=0$ for $m\in\{4,6,8\}$ has none and $L^{\circledS 12}(y)=0$ has a non-trivial rational solution.*

(8) *$\mathscr{G}(L)\cong \mathrm{SL}(2,\mathscr{C})$, if none of the above cases hold.*

Proof. From Corollary 6 and the above remarks on the infinite imprimitive group D_∞ one gets immediately (1)–(3).

The Galois group of an irreducible linear differential equation $L(y)=0$ is irreducible (see [9, Section 22, Theorem 1]). An irreducible group is either imprimitive or primitive. Comparing the degrees of the fundamental invariants of the three finite primitive unimodular linear algebraic groups of degree 2 and the fact that there is no infinite primitive algebraic subgroup of $\mathrm{SL}(2,\mathscr{C})$ (see [17, p. 13]) together with Lemma 5 yields (4). (5)–(7) are simple consequences of Lemma 5 and the invariants computed in the previous section.

If none of the above cases hold, then $\mathscr{G}(L)$ is primitive and infinite and thus, as stated above, equals $\mathrm{SL}(2,\mathscr{C})$. □

As a consequence, we get a nice criterion to decide, whether an irreducible second order linear differential equation has Liouvillian solutions (cf. [17, Proposition 4.4; 10; 4, Satz II, No. 17 and Satz I & II, No. 20]).

Corollary 9. *Let $L(y)=0$ be an irreducible second order linear differential equation over k with an unimodular Galois group $\mathscr{G}(L)$. Then $L(y)=0$ has a Liouvillian solution, if and only if $L^{\circledS 12}(y)=0$ has a non trivial rational solution.*

In particular, $L(y)=0$ has a Liouvillian solution, if and only if $L^{\circledS m}(y)=0$ has a non-trivial rational solution for at least one $m\in\{4,6,8,12\}$.

Proof. $L(y)=0$ has a Liouvillian solution, if and only if the corresponding Galois group is either imprimitive, or primitive and finite. Now, the result follows from Proposition 8. □

5. An alternative algorithm

In this section we derive a direct method to compute Liouvillian solutions of irreducible second order linear differential equations with an imprimitive unimodular Galois group. Computing a minimal polynomial is no longer necessary, but to compute it is still possible. When the differential equation has a primitive unimodular Galois group, we show how one can determine a minimal polynomial of a solution by knowing the group explicitly and using all the fundamental invariants. There is no longer a need to substitute a minimal polynomial decomposed into invariants in the differential equation as it is in [4, p. 100; 18, p. 67].

Let $\{y_1,\ldots,y_n\}$ be a system of fundamental solutions of $L(y)=0$ and

$$\Delta=\begin{vmatrix} y_1 & \cdots & y_n & y \\ y_1' & \cdots & y_n' & y' \\ \vdots & \ddots & \vdots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} & y^{(n)} \end{vmatrix}.$$

Further let $W_i=\partial\Delta/\partial y^{(i)}$ $(i=0,\ldots,n)$, and let $W=W_n$, the Wronskian, and $W'=W_{n-1}$ its first derivative. With this, the differential equation $L(y)=0$ is uniquely determined by

$$L(y)=\frac{\Delta}{W}=y^{(n)}-\frac{W'}{W}y^{(n-1)}+\frac{W_{n-2}}{W}y^{(n-2)}+\cdots+(-1)^n\frac{W_0}{W}y=0$$

or

$$a_i=(-1)^{n-i}\frac{W_i}{W_n}\quad (i=0,\ldots,n-1).$$

Transforming a fundamental system into another system of fundamental solutions of $L(y)=0$ does not change $L(y)=0$, e.g. the coefficients are differentially invariant under the general linear group $\mathrm{GL}(n,\mathscr{C})$. Because these transformations depend on $L(y)=0$, we will denote their group with $\mathrm{G}(L)$.

The coefficients a_k are nth order differential invariants. They form a basis for the differential invariants of $\mathrm{G}(L)$, see [13, p. 16]. Hence, one can represent any differential invariant of $\mathrm{G}(L)$ as a rational function in the $a_0,\ldots,a_{n-1}$ and their derivatives.

Definition 10. Let $L(y)=0$ be a linear differential equation with Galois group $\mathscr{G}(L)$ and I an invariant of degree m of $\mathscr{G}(L)$. The rational solution R of the mth symmetric power $L^{\circledS m}(y)=0$ corresponding to I, is called the *rationalvariant* of I. An algebraic equation, which determines the constant c $(c\in\mathscr{C},\ c\neq 0)$ for $I\mapsto c\cdot R,\ R\neq 0$ is the *determining equation* for the rationalvariant R.

5.1. The imprimitive case

All imprimitive Galois groups possess the common invariant $I_4 = y_1^2 y_2^2$ (see Sections 3.1 and 4), which consists of a single monomial. This common invariant allows to compute Liouvillian solutions with ease.

Theorem 11. *Let $L(y)=0$ be an irreducible second order linear differential equation with an imprimitive unimodular Galois group $\mathscr{G}(L)$. Then $L(y)=0$ has a fundamental system in the following two solutions*

$$y_1 = \sqrt[4]{r}\exp\left[-\frac{C}{2}\int\frac{W}{\sqrt{r}}\right] \quad \textit{and} \quad y_2 = \sqrt[4]{r}\exp\left[\frac{C}{2}\int\frac{W}{\sqrt{r}}\right].$$

Thereby, W is the Wronskian, r is the rationalvariant of the invariant $I_4 = \frac{1}{C^2}\cdot r$ ($C \in \mathscr{C}$, $C \neq 0$) and

$$\frac{4r''r - 3(r')^2}{16r^2} + \frac{W^2}{4r}C^2 + \frac{r'}{4r}a_1 + a_0 = 0 \tag{5}$$

its determining equation.

In particular, (*cf.* [4, *p.* 118]), *if $a_1 = 0$ then*

$$y_1 = \sqrt[4]{r}\exp\left[-\frac{\bar{C}}{2}\int\frac{1}{\sqrt{r}}\right] \quad \textit{and} \quad y_2 = \sqrt[4]{r}\exp\left[\frac{\bar{C}}{2}\int\frac{1}{\sqrt{r}}\right] \quad (\bar{C} = CW)$$

form a system of fundamental solutions, where $\bar{C}$ is determined by Eq. (5).

Proof. Let r be a rational solution of $L^{ⓢ4}(y)=0$ with $I_4 = y_1^2 y_2^2 = c\cdot r$ ($c \in \mathscr{C}, c \neq 0$). Hence, it is $y_2 = \frac{\sqrt{c\cdot r}}{y_1}$. If we substitute this expression for y_2 and for y_2' its derivative in the Wronskian $W = y_1 y_2' - y_1' y_2$, we have

$$\frac{y_1'}{y_1} = \frac{r'}{4r} - \frac{W}{2\sqrt{c\cdot r}} \tag{6}$$

or

$$y_1 = \sqrt[4]{r}\exp\left[-\frac{1}{2\sqrt{c}}\int\frac{W}{\sqrt{r}}\right],$$

respectively. Substituting y_1 in the differential equation $L(y)=0$ we obtain the determining Eq. (5) for the constant $c = 1/C^2$.

If $a_1 = 0$ e.g. W is constant, then y_1 is simplified to $\sqrt[4]{r}\exp[-\frac{W}{2\sqrt{c}}\int\frac{1}{\sqrt{r}}]$ and we get with $\bar{C} = W/\sqrt{c}$ for Eq. (5)

$$\frac{4r''r - 3(r')^2}{16r^2} + \frac{1}{4r}\bar{C}^2 + a_0 = 0. \qquad \square$$

Remark 12. Eq. (6) is already the solved minimal polynomial of the logarithmic derivative of a solution, which is computed in the second case of Kovacic's algorithm [10]. Indeed, Kovacic has used the invariant I_4 to prove the second case of his algorithm [10, p. 10].

In the case of an imprimitive unimodular Galois group, $L^{ⓢ4}(y)=0$ has exactly one non-trivial rational solution except for $D_2^{\mathrm{SL}_2}$ by Proposition 8. Now, suppose $L^{ⓢ4}(y)=0$ has exactly one non-trivial rational solution. Then, using Theorem 11, we can directly compute both Liouvillian solutions of $L(y)=0$. Since the determining equation for the constant C must be valid for all regular points of $L(y)=0$, we only have to evaluate this equation for an arbitrary regular point.

When $L^{ⓢ4}(y)=0$ has two linearly independent non-trivial rational solutions r_1 and r_2 (e.g. $\mathscr{G}(L)\cong D_2^{\mathrm{SL}_2}$) then we have two ways to compute Liouvillian solutions. In the first way we only set $r=c_1r_1+c_2r_2$ and $C=1$ and get the solutions by solving the determining Eq. (5).

The second possibility is to compute a further non-trivial rational solution r_3 of $L^{ⓢ6}(y)=0$. With this rational solutions one makes the ansatz

$$I_{4a}=c_1r_1+c_2r_2, \qquad I_{4b}=c_3r_1+c_4r_2, \qquad I_6=c_5r_3$$

and substitute into the syzygy

$$I_6^2-I_{4a}I_{4b}^2+4I_{4a}^3=0.$$

From the numerator of the thereby obtained rational function we get a system of polynomial equations for the constants $c_1,\ldots,c_5$. Solving this system can be done by computing a lexicographical Gröbner basis (cf. [20]). This gives a necessary condition for the previous invariants. It can be made sufficient by choosing the constants in a way that makes I_{4a}, I_{4b} and I_6 non-trivial and furthermore I_{4a} and I_{4b} linear independent. Since there are infinite many solutions for the invariants this is always possible. Using Theorem 11 we can now compute the Liouvillian solutions from the just constructed invariant I_{4a}.

Another way to compute the solutions is to solve the minimal polynomial of Theorem 4 explicitly.

The condition that a linear differential equation in the imprimitive case has algebraic solutions is based on a Theorem of Abel, see [4, p. 118]. One can state this condition more precisely as follows.

Lemma 13. *Let $L(y)=0$ be a second order linear differential equation with a finite imprimitive unimodular Galois group $\mathscr{G}(L)$. Then the following equation holds*:

$$\int\frac{W}{\sqrt{I_4}}=\frac{1}{2n}\log\frac{I_{2n+2}+I_{2n}\sqrt{I_4}}{I_{2n+2}-I_{2n}\sqrt{I_4}}. \tag{7}$$

Proof. Theorem 4 implies that the solutions of $L(y)=0$ are of the form

$$y_{1,2}=\sqrt[2n]{\frac{1}{2}\left(I_{2n}\pm\sqrt{I_{2n}^2-(-1)^n4I_4^n}\right)}. \tag{8}$$

Substituting I_{2n}^2 by syzygy (3) together with further manipulations gives

$$y_{1,2} = \sqrt[4]{I_4}\ \sqrt[2n]{\frac{\pm I_{2n+2} + I_{2n}\sqrt{I_4}}{2\sqrt{I_4^{n+1}}}}.$$

Once more applying syzygy (3) on I_4^{n+1} and manipulating we get by Theorem 11

$$y_{1,2} = \sqrt[4]{I_4}\ \sqrt[4n]{(-1)^{n+1}\frac{\pm I_{2n+2} + I_{2n}\sqrt{I_4}}{\pm I_{2n+2} - I_{2n}\sqrt{I_4}}} = \sqrt[4]{I_4}\exp\left[\pm\frac{1}{2}\int\frac{W}{\sqrt{I_4}}\right]$$

and therefore

$$\pm\frac{1}{2}\int\frac{W}{\sqrt{I_4}} = \pm\frac{1}{4n}\log\frac{I_{2n+2} + I_{2n}\sqrt{I_4}}{I_{2n+2} - I_{2n}\sqrt{I_4}}. \qquad \square$$

The solutions of $L(y)=0$ are algebraic, if and only if one can write the integral $\int W/\sqrt{I_4}$ in the form (7).

Remark 14. It seems Lemma 13 allows us to determine explicitly the (imprimitive) Galois group of $L(y)=0$. We will study this in a seperate paper.

5.2. The primitive case

This section presents the tools for determining the rationalvariant of an invariant of degree m. The idea stems from [5, p. 22].

Lemma 15. *Let y_1, y_2 be independent functions in x, and let $f(y_1,y_2)$ and $g(y_1,y_2)$ be binary forms of degree m and n, respectively. Then the following identities hold:*
(1) *for the Hessian of $f(y_1,y_2)$*

$$H(f) = \frac{m-1}{W^2}\left[\left(\frac{f'}{f}\right)^2 + m\left(\frac{f'}{f}\right)' + ma_1\left(\frac{f'}{f}\right) + m^2a_0\right]f^2$$

(*for $a_1=0$, cf.* [5, *p.* 22]) *and*
(2) *for the Jacobian of $f(y_1,y_2)$ and $g(y_1,y_2)$*

$$J(f,g) = \frac{mfg' - nf'g}{W}.$$

Thereby, W is the Wronskian of y_1 and y_2 and further $a_0 = W_0/W$ and $a_1 = -W_1/W$ are differential invariants of second order.

Proof. For an arbitrary binary form $f(y_1,y_2) = \sum_{i=0}^m b_i y_1^{m-i} y_2^i$ the following identity holds

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\cdot\begin{pmatrix} f_{y_1} \\ f_{y_2} \end{pmatrix} = \begin{pmatrix} mf \\ f' \end{pmatrix} \quad \text{resp.} \quad \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\cdot\begin{pmatrix} mf \\ f' \end{pmatrix} = \begin{pmatrix} f_{y_1} \\ f_{y_2} \end{pmatrix}.$$

In particular, this is valid for the forms $\partial f/\partial y_1 = f_{y_1}$ and $\partial f/\partial y_2 = f_{y_2}$ of degree $m-1$:

$$\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} (m-1)f_{y_1} \\ f_{y_1}' \end{pmatrix} = \begin{pmatrix} f_{y_1 y_1} \\ f_{y_1 y_2} \end{pmatrix},$$

$$\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} (m-1)f_{y_2} \\ f_{y_2}' \end{pmatrix} = \begin{pmatrix} f_{y_2 y_1} \\ f_{y_2 y_2} \end{pmatrix}.$$

From this one gets the identities by reverse substitution in $H(f) = f_{y_1 y_1} f_{y_2 y_2} - f_{y_1 y_2} f_{y_2 y_1}$ and $J(f,g) = f_{y_1} g_{y_2} - f_{y_2} g_{y_1}$ if one takes the Wronskian and the differential equation $\Delta/W = 0$ for $n=2$ into account. □

Thus, it suffices to compute the non-trivial rational solution of the smallest possible symmetric power of $L(y)=0$. The two remaining fundamental rationalvariants can be determined with Lemma 15. If the rationalvariants are known, one gets the constants from the sygyzies.

Theorem 16. *Let $L(y)=0$ be an irreducible second order linear differential equation over k with finite primitive unimodular Galois group $\mathscr{G}(L)$ and let r be the smallest rationalvariant (e.g. $I_1 = c\cdot r$ ($c \in \mathscr{C}$, $c \neq 0$)). If one sets the Wronskian $W=1$ in the case of $a_1 = 0$, then a determining equation for the rationalvariant r for each case is given by*

$$\mathscr{G}(L) \cong A_4^{\mathrm{SL}_2} : (25J(r,H(r))^2 + 64H(r)^3)c^2 + 10^6 \cdot 108r^4 = 0,$$

$$\mathscr{G}(L) \cong S_4^{\mathrm{SL}_2} : (49J(r,H(r))^2 + 144H(r)^3)c - 118013952r^3H(r) = 0,$$

$$\mathscr{G}(L) \cong A_5^{\mathrm{SL}_2} : (121J(r,H(r))^2 + 400H(r)^3)c + 708624400 \cdot 1728r^5 = 0.$$

Proof. Let us denote $H(f) = (1/W^2)\tilde{H}(f)$, $J(f,g) = (1/W)\tilde{J}(f,g)$ and for constant W let $J(f,H(f)) = (1/W^3)\tilde{J}(f,\tilde{H}(f))$. Then

$$H(c\cdot r) = c^2 H(r) = \frac{c^2}{W^2}\tilde{H}(r),$$

$$J(c\cdot r, H(c\cdot r)) = c^3 J(r,H(r)),$$

and for constant W (e.g. $a_1 = 0$)

$$J(c\cdot r, H(c\cdot r)) = \frac{c^3}{W^3}\tilde{J}(r,\tilde{H}(r)).$$

Furthermore, let $I_1 = c\cdot r$. Substituting the respective expressions for the fundamental invariants in the corresponding syzygies, see Section 3.2, one obtains in the case of

$a_1 = 0$

$$\mathscr{G}(L) \cong A_4^{\mathrm{SL}_2} : \left(\left(\frac{\tilde{J}(r,\tilde{H}(r))}{8 \cdot 25} \right)^2 + \left(\frac{\tilde{H}(r)}{25} \right)^3 \right) c^2 + 108 r^4 W^6 = 0,$$

$$\mathscr{G}(L) \cong S_4^{\mathrm{SL}_2} : \left(\left(\frac{\tilde{J}(r,\tilde{H}(r))}{16 \cdot 9408} \right)^2 + 108 \left(\frac{\tilde{H}(r)}{9408} \right)^3 \right) c - \frac{r^3 \tilde{H}(r)}{9408} W^4 = 0,$$

$$\mathscr{G}(L) \cong A_5^{\mathrm{SL}_2} : \left(\left(\frac{\tilde{J}(r,\tilde{H}(r))}{20 \cdot 121} \right)^2 + \left(\frac{\tilde{H}(r)}{121} \right)^3 \right) c + 1728 r^5 W^6 = 0.$$

For satisfying these equations one can arbitrary choose one of the two non-zero constants c and W, respectively. The assertion follows from the previous relations by setting $W = 1$ in each of them. In a similiar way, one gets for $a_1 \neq 0$ the equations

$$\mathscr{G}(L) \cong A_4^{\mathrm{SL}_2} : \left(\left(\frac{J(r,H(r))}{8 \cdot 25} \right)^2 + \left(\frac{H(r)}{25} \right)^3 \right) c^2 + 108 r^4 = 0,$$

$$\mathscr{G}(L) \cong S_4^{\mathrm{SL}_2} : \left(\left(\frac{J(r,H(r))}{16 \cdot 9408} \right)^2 + 108 \left(\frac{H(r)}{9408} \right)^3 \right) c - \frac{r^3 H(r)}{9408} = 0,$$

$$\mathscr{G}(L) \cong A_5^{\mathrm{SL}_2} : \left(\left(\frac{J(r,H(r))}{20 \cdot 121} \right)^2 + \left(\frac{H(r)}{121} \right)^3 \right) c + 1728 r^5 = 0. \qquad \square$$

It is possible to solve the determining equation for the smallest rationalvariant through evaluation of an arbitrary regular point of $L(y) = 0$, since it must hold for all regular points.

Consequently, Theorem 16 allows to determine for second order linear differential equations with primitive unimodular Galois group a minimal polynomial of a solution without a Gröbner basis computation.

5.3. *The algorithm*

Based on the results of the previous two sections, we propose the following method as an alternative to the already known algorithms of [10, 18, 24]. Thereby, for solving a reducible differential equation we refer to one of these procedures. Computing rational solutions can be done e.g. with the algorithm described in [1]. Moreover, rationalvariants can be determined by the method of van Hoeij and Weil [25] without computing any symmetric power.

Algorithm 1

Input: A linear differential equation $L(y) = 0$ with $\mathscr{G}(L) \subseteq \mathrm{SL}(2, \mathscr{C})$.

Output: Fundamental system of solutions $\{y_1, y_2\}$ of $L(y) = 0$ or minimal polynomial of a solution.

(i) Test, if $L(y)=0$ is reducible. If yes, then compute an exponential and a further Liouvillian solution by applying e.g. one of the previous algorithms.

(ii) Test, if $L^{\circledS 4}(y)=0$ has a non-trivial rational solution.

  (a) If the rational solution space is one-dimensional: Apply Theorem 11.

  (b) If the rational solution space is two-dimensional: *Either* set $r=c_1r_1+c_2r_2$, $C=1$ and apply Theorem 11, *or* compute the rational solution of $L^{\circledS 6}(y)=0$ and determine the three rationalvariants I_{4a}, I_{4b} and I_6 (with a Gröbner basis computation) from syzygy (3) for $n=2$. Subsequently[3] substitute the rationalvariants in Eq. (8).

(iii) Test successively, for $m\in\{6,8,12\}$, if $L^{\circledS m}(y)=0$ has a non-trivial rational solution. If yes, then: compute both remaining rationalvariants with Lemma 15 and determine their constants (Proposition 8) by Theorem 16. Substituting the rationalvariants in the matching minimal polynomial decomposed into invariants from Section 3.2 gives the minimal polynomial of a solution.

(iv) $L(y)=0$ has no Liouvillian solution.

In the following we solve for each of the cases 2(a), 2(b) and 3 of Algorithm 1 an example with the computer algebra system AXIOM 1.2 (see [7]).

Example 17 (*see Ulmer and Weil* [24, *p.* 193], *Weil* [26, *p.* 93]). The differential equation

$$L(y)=y''-\frac{2}{2x-1}y'+\frac{(27x^4-54x^3+5x^2+22x+27)(2x-1)^2}{144x^2(x-1)^2(x^2-x-1)^2}y=0$$

is irreducible and has a unimodular Galois group, since $W'/W=2/(2x-1)$ and $W\in k$. Its fourth symmetric power $L^{\circledS 4}(y)=0$ has a one-dimensional rational solution space generated by $r=x(x-1)(x^2-x-1)^2$.

The constant C is determined by

$$\frac{(36C^2-4)x^2+(-36C^2+4)x+9C^2-1}{36x^6-108x^5+36x^4+108x^3-36x^2-36x}=0$$

or e.g. for the regular point $x_0=2$ by

$$9C^2-1=0.$$

For the integral $\int W/\sqrt{9r}$ one gets

$$\int\frac{2x-1}{\sqrt{9x(x-1)(x^2-x-1)^2}}=\frac{1}{3}\log\frac{(-2x-1)\sqrt{x(x-1)}+2x^2-1}{(-2x+3)\sqrt{x(x-1)}+2x^2-4x+1}.$$

Therefore $L(y)=0$ has a fundamental system in the solutions

$$y_{1,2}=\sqrt[4]{x(x-1)(x^2-x-1)^2}\left(\frac{(-2x+3)\sqrt{x(x-1)}+2x^2-4x+1}{(-2x-1)\sqrt{x(x-1)}+2x^2-1}\right)^{\pm 1/6}.$$

[3] Or apply Theorem 11 to the rationalvariant of I_{4a}.

To this fundamental system corresponds the invariant $I_4 = x(x-1)(x^2-x-1)^2$. Substituting both solutions in I_{2n} for $n=3$ we get

$$I_6 = 4x^2(x-1)^2(x^2-x-1)^2.$$

Hence, $\mathscr{G}(L) \cong D_3^{\mathrm{SL}_2}$. By the relation (3) we obtain the remaining fundamental invariant

$$I_8 = \sqrt{I_4 I_6^2 + 4I_4^4} = 2x^2(x^2-x+1)(x-1)^2(x^2-x-1)^3. \qquad \square$$

Example 18 (*see Ulmer* [23, *p.* 396; 27]). Consider the irreducible differential equation

$$L(y) = y'' + \frac{27x}{8(x^3-2)^2}\, y = 0$$

constructed from Hendriks. Its 4th symmetric power $L^{Ⓢ4}(y)=0$ has a two-dimensional rational solution space, generated by $r_1 = x^3-2$ and $r_2 = x(x^3-2)$. Corollary 6 implies that $\mathscr{G}(L) \cong D_2^{\mathrm{SL}_2}$ is the corresponding Galois group of $L(y)=0$. The rational solution space of $L^{Ⓢ6}(y)=0$ is generated by $r_3 = (x^3-2)^2$.

Substituting the ansatz

$$I_{4a} = c_1(x^3-2) + c_2 x(x^3-2), \qquad I_{4b} = c_3(x^3-2) + c_4 x(x^3-2),$$

$$I_6 = c_5(x^3-2)^2$$

in the relation (3) for $n=2$ gives the necessary condition:

$$\begin{aligned}
&(c_5^2 - c_2c_4^2 + 4c_2^3)x^{12} + (-c_1c_4^2 - 2c_2c_3c_4 + 12\, c_1c_2^2)x^{11} \\
&\quad + (-2c_1c_3c_4 - c_2c_3^2 + 12c_1^2c_2)x^{10} + (-8c_5^2 + 6c_2c_4^2 - c_1c_3^2 - 24c_2^3 + 4c_1^3)x^9 \\
&\quad + (6c_1c_4^2 + 12c_2c_3c_4 - 72c_1c_2^2)x^8 + (12c_1c_3c_4 + 6c_2c_3^2 - 72c_1^2c_2)x^7 \\
&\quad + (24c_5^2 - 12c_2c_4^2 + 6c_1c_3^2 + 48c_2^3 - 24c_1^3)x^6 \\
&\quad + (-12c_1c_4^2 - 24c_2c_3c_4 + 144c_1c_2^2)x^5 + (-24c_1c_3c_4 - 12c_2c_3^2 + 144c_1^2c_2)x^4 \\
&\quad + (-32c_5^2 + 8c_2c_4^2 - 12c_1c_3^2 - 32c_2^3 + 48c_1^3)x^3 \\
&\quad + (8c_1c_4^2 + 16c_2c_3c_4 - 96c_1c_2^2)x^2 \\
&\quad + (16c_1c_3c_4 + 8c_2c_3^2 - 96c_1^2c_2)x + 16c_5^2 + 8c_1c_3^2 - 32c_1^3 = 0.
\end{aligned}$$

In order to satisfy this condition all the coefficients must vanish identically. For instance, we can add $c_4 - \lambda = 0$ to the coefficient equations and compute for this system a lexicographical Gröbner basis for $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$. If one computes an ideal decomposition from this result with the algorithm *groebnerFactorize* and take therein the secondary condition $c_5 \neq 0$ into account, one gets the (parametrized) ideal ($\lambda \neq 0$)

$$\{\lambda^3 c_1 + \tfrac{3}{4}c_3c_5^2,\ \lambda^2 c_2 - \tfrac{3}{4}c_5^2,\ c_3^3 - 2\lambda^3,\ c_4 - \lambda,\ c_5^4 + \tfrac{4}{27}\lambda^6\},$$

or the variety

$$\mathscr{P}=\left\{\begin{array}{c}\left\{c_1=-\frac{\frac{3}{4}c_3c_5^2}{\lambda^3}\right\},\quad \left\{c_2=\frac{\frac{3}{4}c_5^2}{\lambda^2}\right\},\\ \left\{c_3=\sqrt[3]{2\lambda^3}, c_3=(\pm\frac{1}{2}\sqrt{-1}\sqrt{3}-\frac{1}{2})\sqrt[3]{2\lambda^3}\right\},\quad \{c_4=\lambda\},\\ \left\{c_5=\pm\sqrt[4]{-\frac{4}{27}\lambda^6},\ c_5=\pm\sqrt{-1}\sqrt[4]{-\frac{4}{27}\lambda^6}\right\}\end{array}\right\}.$$

$\mathscr{P}$ contains all possible choices for the constants of the fundamental invariants. For instance, the points $(c_1, c_2, c_3, c_4)=(\frac{1}{6}\sqrt{-3}\sqrt[3]{2}\lambda, -\frac{1}{6}\sqrt{-3}\lambda, \sqrt[3]{2}\lambda, \lambda)$ satisfy the sufficient condition for the rationalvariants. Substituting these points in Eq. (8) for $n=2$, we get the two solutions

$$y_{1,2}=\sqrt[4]{\frac{1}{6}\lambda(x^3-2)(3x+3\sqrt[3]{2}\pm 2\sqrt{3(x^2+\sqrt[3]{2}x+\sqrt[3]{2}^2)})}. \qquad \square$$

Example 19 (*see Singer and Ulmer* [18, *p.* 68]; *Kovacic* [10, *p.* 23], [24, 6]). In order to illustrate the given method in the primitive case, we consider the irreducible differential equation [10]

$$L(y)=y''+\left(\frac{3}{16x^2}+\frac{2}{9(x-1)^2}-\frac{3}{16x(x-1)}\right)y=0.$$

Its 4th symmetric power $L^{\circledS 4}(y)=0$ has no non-trivial rational solutions. While $L^{\circledS 6}(y)=0$ has the rationalvariant $r=x^2(x-1)^2$ which generates its one-dimensional rational solution space. Therefore, by Proposition 8 $\mathscr{G}(L)\cong A_4^{\mathrm{SL}_2}$ is the corresponding Galois group of $L(y)=0$ (cf. [10]). For $W=1$, the further two rationalvariants are computed with

$$H(r)=\tfrac{25}{4}x^2(x-1)^3$$

and

$$J(r,H(r))=-\tfrac{25}{2}x^3(x-1)^4(x-2).$$

From these rationalvariants one gets the determining equation of r:

$$\begin{aligned}&(c^2+27648)x^{16}+(-8c^2-221184)x^{15}+(28c^2+774144)x^{14}\\ &\quad+(-56c^2-1548288)x^{13}+(70c^2+1935360)x^{12}\\ &\quad+(-56c^2-1548288)x^{11}+(28c^2+774144)x^{10}\\ &\quad+(-8c^2-221184)x^9+(c^2+27648)x^8=0,\end{aligned}$$

respectively, e.g. for the regular point $x_0=2$ the equation

$$c^2+27648=0.$$

Hence, $c = \pm 96\sqrt{-3}$. Substituting

$$I_1 = \frac{1}{4}\cdot c\cdot r = 24\sqrt{-3}\; x^2(x-1)^2,$$

$$I_2 = -\frac{5}{80}\cdot\frac{-1}{25}c^2 H(r) = 432x^2(x-1)^3,$$

$$I_3 = -\frac{1}{16}\cdot\frac{1}{8}\cdot\frac{-1}{25}c^3 J(r,H(r)) = 10368\sqrt{-3}\; x^3(x-1)^4(x-2),$$

in the minimal decomposed into invariants (4), we obtain the minimal polynomial of a solution:

$$\mathbf{Y}^{24} - 4320x^2(x-1)^3\mathbf{Y}^{16} + 51840\sqrt{-3}\; x^3(x-1)^4(x-2)\mathbf{Y}^{12} - 2799360x^4(x-1)^6\mathbf{Y}^8 + 4478976\sqrt{-3}\; x^5(x-1)^7(x-2)\mathbf{Y}^4 + 2985984x^8(x-1)^8. \quad \square$$

6. Conclusions

The work of Fuchs is difficult to read. The author has first developed the algorithm presented here by himself and noticed afterwards that it is basically a reformulation and improvement of the Fuchsian method. Nevertheless, our method is essentially more efficient. The reason for this lies in using *all* absolute fundamental invariants of the Galois group associated with the differential equation; this enables us to compute the constants from the syzygies.

But in principle our algorithm cannot be more efficient than the algorithm given by Ulmer and Weil [24]. Indeed, both methods have the same time complexity. The algorithm from Ulmer and Weil computes a minimal polynomial of the logarithmic derivative of a solution via a recursion for the coefficients in all cases, while our method tries to determine the solutions explicity as much as possible. If the associated Galois group is the tetrahedral or the octahedral group one can represent both algebraic solutions in radicals. [4]

We feel that this paper shows the connection between determining the Galois group, the rationalvariants and the Liouvillian solutions of a given (irreducible) second order differential equation very clearly. For instance, in the imprimitive case it is easier to compute first the Liouvillian solutions and determine from them the (possibly) missing rationalvariants and the Galois group. Against it, in the primitive case the better way is to compute first the Galois group and to determine from it the remaining rationalvariants and the minimal polynomial of a solution. The behaviour in the case of $D_2^{\mathrm{SL}_2}$ is somehow special (cf. [23]). Also it becomes clear, that a Liouvillian solution or a minimal polynomial of a solution always contains *all* fundamental rationalvariants.

[4] The basic ideas to solve this problem are described in [20, Problem 2.7.5] (cf. also [26, Section III.5]).

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1.23 Solutions of linear ordinary differential equations in terms of special functions

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Solutions of linear ordinary differential equations in terms of special functions

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ABSTRACT

We describe a new algorithm for computing special function solutions of the form $y(x) = m(x)F(\xi(x))$ of second order linear ordinary differential equations, where $m(x)$ is an arbitrary Liouvillian function, $\xi(x)$ is an arbitrary rational function, and F satisfies a given second order linear ordinary differential equation. Our algorithm, which is based on finding an appropriate point transformation between the equation defining F and the one to solve, is able to find all rational transformations for a large class of functions F, in particular (but not only) the ${}_0F_1$ and ${}_1F_1$ special functions of mathematical physics, such as Airy, Bessel, Kummer and Whittaker functions. It is also able to identify the values of the parameters entering those special functions, and can be generalized to equations of higher order.

1. INTRODUCTION

Algorithms and software for computing closed form solutions of linear ordinary differential equations have improved significantly in the past decade, but mostly in the direction of computing their Liouvillian solutions (see e.g. [1, 8, 9]). In particular, computing the Liouvillian solutions of second order linear ordinary differential equations has become a routine task in recent versions of several computer algebra systems. The situation is different with respect to solving such equations in terms of non–Liouvillian special functions. While it is possible to detect whether the solutions of an equation can be expressed in terms of the solutions of equations of second order [7], there is no complete algorithm for deciding whether such solutions can be expressed in terms of the solutions of *specific* equations, usually the ones defining known special functions. This is a restricted instance of the equivalence problem for second-order linear ODEs [3]: given a target equation $y'' = uy$ with $u \in C(x)$ and a known fundamental solution set $\{F_1, F_2\}$ (for example the Airy equation $y'' = xy$), and an arbitrary input equation $y'' = vy$ with $v \in C(x)$, to find functions $m(x)$ and $\xi(x)$ such that $\{m(x)F_1(\xi(x)), m(x)F_2(\xi(x))\}$ is a fundamental solution set of $y'' = vy$. This is the equivalent to looking for a point transformation of the form

$$x \to \xi(x) \quad y \to m(x)y \tag{1}$$

that transforms $y'' = uy$ into $y'' = vy$. It is classically known that all second-order linear ODEs are equivalent under the group of transformations of the form (1), hence that an appropriate transformation always exists [4]. However, the functions $\xi(x)$ and $m(x)$ are given implicitely by differential equations themselves, so this does not provide explicit solutions in terms of F. We are interested in this paper in determining whether an *explicit* transformation of the form (1) exists, with $\xi \in C(x)$ and m a Liouvillian function, and to compute it when it exists. Applying the transformation (1) to $y'' = uy$ and matching the coefficients of the resulting equation with $y'' = vy$ (or equivalently, substituting $y = m(x)F(\xi(x))$ in $y'' = vy$) one obtains the equations $m = \xi'^{-1/2}$ and

$$3\xi''^2 - 2\xi'\xi''' + 4u(\xi)\xi'^4 - 4v\xi'^2 = 0\,, \tag{2}$$

so the remaining problem is to solve the above equation explicitely. Methods using that approach have appeared, in particular [10], who proceeds heuristically by trying various candidates functions ξ with undetermined constants parameters in (2). Each attempt yields systems of algebraic equations for the undetermined constants (and parameters of the special functions), and those equations can then be solved by existing computer algebra systems.

Our main contribution in this paper is an algorithm for computing all the solutions $\xi \in C(x)$ of (2). Our algorithm is applicable whenever the target equation $y'' = uy$ has an irregular singularity at infinity, in addition to any number of affine singularities of arbitrary type. This allows our algorithm to handle the ${}_0F_1$ and ${}_1F_1$ special functions of mathematical physics (*e.g.* the Airy, Bessel, Kummer and Whittaker functions) as well as non-hypergeometric ones. We also show that if the input equation has no Liouvillian solution, then our algorithm decides whether there is any solution of the form $m(x)F(\xi(x))$ for F any solution of the target equation. Our algorithm has been implemented in the computer algebra system Maple and our implementation can be tried interactively on the web[1]. While the abilities

[1] `http://www.inria.fr/cafe/Manuel.Bronstein/cathode/kovacic_demo.html`

of the MAPLE 7 differential equations solver have also been improved regarding solutions in terms of special functions[2] our algorithm is able to solve a larger class of examples, *e.g.*

$$4(x-1)^8\frac{d^2y}{dx^2} = (3-50x+61x^2-60x^3+45x^4-18x^5+3x^6)y(x)\,,$$

whose solutions can be expressed in terms of Airy functions with rational functions as arguments (see examples below).

We would like to thank the referees for their numerous comments, in particular for pointing out the link with the equivalence problem.

2. FORMAL CHANGE OF VARIABLE

The differential equations for ξ and m that result from having (1) map a given operator to another given one can always be obtained by substituting $y = m(x)F(\xi(x))$ in the corresponding differential equation, and this is a classic construction. We describe it in this section using differential polynomials and linear algebra, in a way that is easily performed in a computer algebra system for linear operators of arbitrary order.

Let $(k,')$ be a differential field, $k[D;']$ be the ring of differential operators with coefficients in k, and $L = D^n + \sum_{i=0}^{n-1} a_i D^i \in k[D;']$ be an operator of order $n > 0$. Let M, Z be differential indeterminates over k, $G_0,\ldots,G_{n-1}$ be algebraic indeterminates over $k\langle M,Z\rangle$ and extend the derivation $'$ to $k\{M,Z\}[G_0,\ldots,G_{n-1}]$ via $G_i' = Z'G_{i+1}$ for $0 \le i < n-1$ and $G_{n-1}' = -Z'\sum_{i=0}^{n-1} a_i G_i$. Let $y = MG_0$. Since y is a linear form in $G_0,\ldots,G_{n-1}$ and $'$ preserves the total degree in $G_0,\ldots,G_{n-1}$, the successive derivatives of y are all linear forms in $G_0,\ldots,G_{n-1}$, so $y, y',\ldots,y^{(n)}$ are linearly dependent over $k\langle M,Z\rangle$. Since for $i < n$, G_i appears with the nonzero coefficient MZ'^i in $y^{(i)}$ but does not appear in $y^{(i-1)}$, the elements $y, y',\ldots,y^{(n-1)}$ must be linearly independent over $k\langle M,Z\rangle$, so there is a unique linear dependence of the form $y^{(n)} + \sum_{i=0}^{n-1} b_i y^{(i)} = 0$, which can be computed by linear algebra over $k\langle M,Z\rangle$. Define then

$$L_{M,Z} = D^n + \sum_{i=0}^{n-1} b_i D^i \quad \in k\langle M,Z\rangle[D;']$$

to be the *generic* $M-Z$ *associate of* L.

Given a differential extension K of k and any $m,\xi \in K$ such that $m\xi' \neq 0$, we can specialize $L_{M,Z}$ at $M = m$ and $Z = \xi$, and we denote the resulting operator $L_{m,\xi}$. If k contains an element x such that $x' = 1$ and if the elements of k can be viewed as functions[3] in x, then for any $f \in k$, we write $f(\xi)$ for the result of evaluating f at $x = \xi$. Replacing each a_i by $a_i(\xi)$ in $L_{m,\xi}$, we obtain a new operator, which we denote $L_{x\to\xi,y\to my}$. By construction, it has the following property: if $L(y) = 0$ for some y in a differential extension of k, then $L_{x\to\xi,y\to my}(my(\xi)) = 0$. So if $F_1,\ldots,F_n$ is a fundamental solution set of L, then $mF_1(\xi),\ldots,mF_n(\xi)$ are solutions of $L_{x\to\xi,y\to my}$. Since,

$$\mathrm{Wr}(mF_1(\xi),\ldots,mF_n(\xi)) = m^n \xi'^N \mathrm{Wr}(F_1,\ldots,F_n)(\xi)$$

for some integer $N > 0$, it follows that $mF_1(\xi),\ldots,mF_n(\xi)$ is a fundamental solution set of $L_{x\to\xi,y\to my}$ (in other words, the transformation (1) sends L into $L_{x\to\xi,y\to my}$).

Let now $R = D^n + \sum_{i=0}^{n-1} c_i D^i \in k[D;']$ be another operator and suppose that there exist m,ξ in a differential extension of k such that $m\xi' \neq 0$ and $mF_1(\xi),\ldots,mF_n(\xi)$ are solutions of R. Then, $mF_1(\xi),\ldots,mF_n(\xi)$ is a fundamental solution set of both R and of $L_{x\to\xi,y\to my}$. Since they are both monic and of order n, we must have $R = L_{x\to\xi,y\to my}$. Equating the coefficients of the same powers of D in R and $L_{x\to\xi,y\to my}$ yield a system of n nonlinear ordinary differential equations that m and ξ must satisfy. Finding a fundamental solution set of the form $mF_1(\xi),\ldots,mF_n(\xi)$ of R is thus reduced to solving those equations.

We can also ask a weaker question, namely does R admit some solution of the form $mF(\xi)$ where F is a nonzero solution of L and $m\xi' \neq 0$. In that case, we can only say that R and $L_{x\to\xi,y\to my}$ have a nontrivial right factor in $k\langle m,\xi\rangle[D;']$, so we cannot generate equations for m and ξ. However, if we request in addition that R be irreducible in $k\langle m,\xi\rangle[D;']$, then the existence of such a solution implies that $R = L_{x\to\xi,y\to my}$, hence that m and ξ satisfy the n equations generated. In particular, a second order equation with no Liouvillian solution over k must be irreducible over any Liouvillian extension of k, so if such an equation has a solution of the form $mF(\xi)$ with $m\xi' \neq 0$ and m and ξ Liouvillian over k, then $R = L_{x\to\xi,y\to my}$.

3. SECOND ORDER EQUATIONS

We carry out explicitly in this section the derivation of the above nonlinear differential equations in the case of second-order operators. Computing the generic $M-Z$ associate of $L = D^2 + a_1 D + a_0 \in k[D;']$, we get

$$y = MG_0, \qquad y' = M'G_0 + MG_0' = M'G_0 + MZ'G_1\,,$$

and

$$\begin{aligned} y'' &= M''G_0 + M'G_0' + M'Z'G_1 + MZ''G_1 + MZ'G_1' \\ &= M''G_0 + (2M'Z' + MZ'')G_1 - MZ'^2(a_0G_0 + a_1G_1) \\ &= (M'' - a_0MZ'^2)G_0 + (2M'Z' + MZ'' - a_1MZ'^2)G_1\,. \end{aligned}$$

A calculation of the linear dependence between y, y' and y'' shows that

$$\begin{aligned} L_{M,Z} &= D^2 - \left(2\frac{M'}{M} + \frac{Z''}{Z'} - a_1Z'\right)D \qquad (3) \\ &\quad - \left(\left(\frac{M'}{M}\right)' - \frac{M'^2}{M^2} - \frac{M'}{M}\frac{Z''}{Z'} + a_1Z'\frac{M'}{M} - a_0Z'^2\right). \end{aligned}$$

Let now $v \in k$ be given. As explained in the previous section, if there are m and ξ in a differential extension of k such that $m\xi' \neq 0$ and either

- $mF_1(\xi)$ and $mF_2(\xi)$ are solutions of $y'' = vy$, where and F_1, F_2 is a fundamental solution set of L, or
- $mF(\xi)$ is a solution of $y'' = vy$, where F is some solution of L, m and ξ are Liouvillian over k and $y'' = vy$ has no Liouvillian solution,

[2]See *e.g.* `http://lie.uwaterloo.ca/odetools/hyper3.htm` where the candidate $\xi = (ax^k + b)/(cx^k + d)$ is tried.

[3]This is obviously the case when $k = C(x)$ for some constant field C, and Seidenberg's Embedding Theorem [5, 6] implies that is is also the case when k is a finitely generated differential extension of $\mathbb{Q}(x)$.

then $D^2 - v = L_{x\to\xi, y\to my}$. Using (3) and equating the coefficients of D^1 and D^0 on both sides, we get

$$2\frac{m'}{m} + \frac{\xi''}{\xi'} - a_1(\xi)\xi' = 0 \tag{4}$$

and

$$\left(\frac{m'}{m}\right)' - \frac{m'^2}{m^2} - \frac{m'}{m}\frac{\xi''}{\xi'} + a_1(\xi)\xi'\frac{m'}{m} - a_0(\xi)\xi'^2 = v\,. \tag{5}$$

Equation (4) implies that

$$\frac{m'}{m} = \frac{1}{2}\left(a_1(\xi)\xi' - \frac{\xi''}{\xi'}\right) \tag{6}$$

and using that to eliminate m'/m from (5) we obtain

$$3\xi''^2 - 2\xi'\xi''' + \left(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)\right)\xi'^4 - 4v\xi'^2 = 0\,, \tag{7}$$

which is equation (2) when $a_1 = 0$ and $a_0 = -u$.

4. RATIONAL SOLUTIONS FOR ξ

We now proceed to show that for a large class of target operators L, there is an algorithm for computing all the rational solutions ξ of (7). Suppose from now on that our differential field k is a rational function field $k = C(x)$ where $x' = 1$ and $c' = 0$ for all $c \in C$. Recall that the *order at* ∞ is the function $\nu_\infty(q) = -\deg(q)$ for $q \in C[x]\setminus\{0\}$, and given an irreducible $p \in C[x]$, the *order at* p is the function

$$\nu_p(q) = \max\{n \in \mathbb{Z} \text{ such that } p^n | q\}$$

for $q \in C[x]\setminus\{0\}$. Both functions are extended to fractions via $\nu_\infty(a/b) = \nu_\infty(a) - \nu_\infty(b)$ and $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$. By convention, $\nu_\infty(0) = \nu_p(0) = +\infty$. Furthermore, for $a, b \in C(x)$, they satisfy the following properties (where ν stands for either ν_∞ or ν_p):

- $\nu(ab) = \nu(a) + \nu(b)$,
- $\nu(a+b) \ge \min(\nu(a), \nu(b))$
- $\nu(a) \ne \nu(b) \implies \nu(a+b) = \min(\nu(a), \nu(b))$,
- $\nu(a) < 0 \implies \nu(b(a)) = -\nu_\infty(b)\nu(a)$,
- $\nu_\infty(a) < 0 \implies \nu_\infty(a') = \nu_\infty(a) + 1$,
- $\nu_p(a) < 0 \implies \nu_p(a') = \nu_p(a) - 1$.

Given an hypothesis on the pair (a_0, a_1), the following gives an ansatz with a finite number of undetermined constants for the rational solutions of (7).

THEOREM 1. *Let $\prod_i Q_i^i$ be the squarefree decomposition of the denominator of $v \in C(x)$. If $\nu_\infty(a_1^2 + 2a_1' - 4a_0) < 2$, then any solution $\xi \in C(x)$ of (7) can be written as $\xi = P/Q$ where*

$$Q = \prod_i Q^i_{(2-\nu_\infty(a_1^2+2a_1'-4a_0))i+2} \quad \in C[x]\,, \tag{8}$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ *or*

$$\deg(P) = \deg(Q) + \frac{2 - \nu_\infty(v)}{2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0)} \tag{9}$$

PROOF. Write

$$\Delta = a_1^2 + 2a_1' - 4a_0\,, \quad \delta = \nu_\infty(\Delta)$$

and suppose that $\delta < 2$. The solution $\xi = 0$ can certainly be written in the above form, so let $\xi \in C(x)^*$ be a nonzero solution of (7), and $p \in C[x]$ be an irreducible such that $\nu_p(\xi) < 0$. Then, $\nu_p(\xi''^2) = \nu_p(\xi'\xi''') = 2\nu_p(\xi) - 4$ and $\nu_p(\xi'^4) = 4\nu_p(\xi) - 4$. In addition,

$$\nu_p(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)) = \nu_p(\Delta(\xi)) = -\delta\nu_p(\xi)\,,$$

so

$$\nu_p\left((a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_p(\xi) - 4\,.$$

Since $\delta < 2$, $(4-\delta)\nu_p(\xi) - 4 < 2\nu_p(\xi) - 4$, so

$$\nu_p\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_p(\xi) - 4\,.$$

Thus, we must have $\nu_p(4v\xi'^2) = (4-\delta)\nu_p(\xi) - 4$. Since $\nu_p(4v\xi'^2) = \nu_p(v) + 2\nu_p(\xi) - 2$, we get

$$\nu_p(v) = (2-\delta)\nu_p(\xi) - 2 \le -3\,.$$

This implies that the affine poles of ξ are among the poles of v of multiplicity 3 or more. Furthermore,

$$\nu_p(\xi) = \frac{\nu_p(v) + 2}{2 - \delta} \tag{10}$$

so ξ must be of the form $\xi = P/Q$ where $P \in C[x]$ and

$$Q = \prod_i Q^i_{(2-\delta)i+2}$$

Suppose now that $\deg(P) > \deg(Q) + 1$. Then, $\nu_\infty(\xi) < -1$, so $\nu_\infty(\xi'^4) = 4\nu_\infty(\xi) + 4$ and

$$\nu_\infty(a_1(\xi)^2 - 4a_0(\xi)) = \nu_\infty(\Delta(\xi)) = -\delta\nu_\infty(\xi)\,,$$

which implies that

$$\nu_\infty\left((a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_\infty(\xi) + 4\,.$$

In addition, $\nu_\infty(\xi''^2) = 2\nu_\infty(\xi) + 4$ and either $\nu_\infty(\xi'\xi''') = 2\nu_\infty(\xi) + 4$ when $\nu_\infty(\xi) < -2$, or $\nu_\infty(\xi'\xi''') \ge -1$ when $\nu_\infty(\xi) = -2$. Since $\delta < 2$, $(4-\delta)\nu_\infty(\xi) + 4 < 2\nu_\infty(\xi) + 4$, and $(4-\delta)\nu_\infty(\xi) + 4 = 2\delta - 4 < -1$ when $\nu_\infty(\xi) = -2$, so

$$\nu_\infty\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_\infty(\xi) + 4$$

in any case. We must then have $\nu_\infty(4v\xi'^2) = (4-\delta)\nu_\infty(\xi) + 4$. Since $\nu_\infty(4v\xi'^2) = \nu_\infty(v) + 2\nu_\infty(\xi) + 2$, we get

$$\nu_\infty(v) = (2-\delta)\nu_\infty(\xi) + 2 \tag{11}$$

and the theorem follows. □

We note that the upper bound $\deg(P) \le \deg(Q) + 1$ can be improved when $\delta < 0$. In that case, if $\deg(P) = \deg(Q) + 1$, then $\nu_\infty(\xi) = -1$, so an argument similar to the above shows that

$$\nu_\infty\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = \delta < 0\,.$$

We must then have $\nu_\infty(4v\xi'^2) = \delta$, so $\nu_\infty(v) = \delta$ and (9) holds. Therefore, when $\nu_\infty(a_1^2 + 2a_1' - 4a_0) < 0$, either $\deg(P) \le \deg(Q)$ or $\deg(P)$ is given by (9).

When it is applicable, Theorem 1 yields an immediate algorithm for computing all the solutions $\xi \in C(x)$ of (7) given $v \in C(x)$ as input: we substitute $\sum_{j=0}^{n} c_j x^j / Q$ for ξ in (7), where Q is given by (8), n is the upper bound on $\deg(P)$ given by Theorem 1 and the c_j are undetermined constants. This yields a nonlinear system Σ of algebraic equations for the c_j, whose solutions correspond to all the solutions $\xi \in C(x)$ of (7). Since any constant satisfies (7), Σ always has the line of solutions $(c_0, \ldots, c_n) = \lambda(q_0, \ldots, q_n)$ where $Q = q_0 + q_1 x + \cdots + q_n x^n$ (note that n is always at least $\deg(Q)$). Those solutions do not satisfy the condition $m\xi' \neq 0$, so we adjoin to Σ the additional equation

$$\sum_{j=0}^{n} (q_j c_N - q_N c_j) w_j = 1 \tag{12}$$

where $w_0, \ldots, w_n$ are new indeterminates and N is chosen such that $q_N \neq 0$. Any solution of this augmented system must satisfy $q_j c_N \neq q_N c_j$ for some j, which implies that the corresponding $\xi \in C(x)$ is a nonconstant solution of (7). In addition, when a_0 and a_1 contain parameters (as in the case of families of special functions, *e.g.* Bessel functions), considering them as unknowns in Σ allows the values of those parameters to be found also (this is illustrated in the examples below). Essentially all the computation time of our algorithm is spent finding a solution of Σ, a problem whose complexity is exponential in $\deg(P)$.

Our approach can obviously be used to find all the rational solutions $\xi \in \overline{C}(x)$ of (7), it just means searching for solutions of Σ in $\overline{C}$ rather than C. Of more interest, it can also be used to find some algebraic function solutions of (7). Indeed, equations (10) and (11) provide the ramifications of ξ at the singularities of the equation and at infinity, so it is natural to look for solutions of the form

$$\xi = P\left(x^{1/(2-\nu)}\right) \prod_{i>2} Q_i^{(i-2)/(2-\nu)} \tag{13}$$

where $\nu = \nu_\infty(a_1^2 + 2a_1' - 4a_0)$ and $P \in C[x]$. To bound $\deg(P)$, we note that (11) is valid for $\nu_\infty(\xi) \leq -2$ only, so either

$$\deg(P) < (2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0))(\deg(Q) + 2)$$

or

$$\deg(P) = (2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0)) \deg(Q) + 2 - \nu_\infty(v)\,.$$

As for rational functions, substituting a candidate with undetermined constant coefficients for ξ yields a nonlinear algebraic system for those coefficients. This method does not yield all the algebraic functions solutions of (7) however.

Once a nonconstant solution ξ is found (rational or otherwise), the corresponding m is given by (6), which can be integrated yielding

$$m = \xi'^{-\frac{1}{2}} e^{\frac{1}{2}\int a_1(\xi)\xi'} \tag{14}$$

5. CLASSICAL SPECIAL FUNCTIONS

We now apply the algorithm of the previous section to classical classes of ${}_0F_1$ and ${}_1F_1$ special functions, all satisfying the hypothesis of Theorem 1. Although Kummer and Whittaker functions are rationally equivalent, we explicit the solving algorithm for both of them, allowing users to choose one over the other.

5.1 Airy functions

The operator defining the Airy functions is $L = D^2 - x$, so $a_1 = 0$, $a_0 = -x$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + 4\xi\xi'^4 - 4v\xi'^2 = 0\,. \tag{15}$$

Since $a_1^2 + 2a_1' - 4a_0 = 4x$, $\nu_\infty(a_1^2 + 2a_1' - 4a_0) = -1 < 0$, so by Theorem 1 and the remark following it, any solution of (15) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{3i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \leq \deg(Q)$ or

$$\deg(P) = \deg(Q) + \frac{2 - \nu_\infty(v)}{3}\,.$$

Finally, since $a_1 = 0$, equation (14) becomes

$$m = \sqrt{\frac{1}{\xi'}} \tag{16}$$

5.2 Bessel functions

The operator defining the Bessel and modified Bessel functions is

$$L = D^2 + \frac{1}{x} D + \epsilon - \frac{\nu^2}{x^2}$$

where $\epsilon = 1$ for the Bessel functions and $\epsilon = -1$ for the modified Bessel functions. Therefore, $a_1 = 1/x$ and $a_0 = \epsilon - \nu^2/x^2$, so equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + (4\nu^2 - 1)\frac{\xi'^4}{\xi^2} - 4\epsilon\xi'^4 - 4v\xi'^2 = 0\,. \tag{17}$$

Since

$$a_1^2 + 2a_1' - 4a_0 = \frac{4\nu^2 - 1}{x^2} - 4\epsilon\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (17) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \leq \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = 1/x$, equation (14) becomes

$$m = \sqrt{\frac{\xi}{\xi'}} \tag{18}$$

5.3 Kummer functions

The operator defining the Kummer functions is

$$L = D^2 + \left(\frac{\nu}{x} - 1\right) D - \frac{\mu}{x}\,,$$

so $a_1 = \nu/x - 1$, $a_0 = -\mu/x$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + (\nu^2 - 2\nu)\frac{\xi'^4}{\xi^2} + (4\mu - 2\nu)\frac{\xi'^4}{\xi} + \xi'^4 - 4v\xi'^2 = 0\,. \tag{19}$$

Since

$$a_1^2 + 2a_1' - 4a_0 = 1 + \frac{4\mu - 2\nu}{x} + \frac{\nu^2 - 2\nu}{x^2}\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (19) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = \nu/x - 1$, equation (14) becomes

$$m = e^{-\frac{1}{2}\int \xi} \sqrt{\frac{\xi^\nu}{\xi'}}$$

5.4 Whittaker functions

The operator defining the Whittaker functions is

$$L = D^2 - \left(\frac{1}{4} - \frac{\mu}{x} - \frac{1/4 - \nu^2}{x^2}\right),$$

so $a_1 = 0$, $a_0 = -1/4 + \mu/x + (1/4 - \nu^2)/x^2$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + \left(1 - \frac{4\mu}{\xi} - \frac{1 - 4\nu^2}{\xi^2}\right)\xi'^4 - 4v\xi'^2 = 0\,. \quad (20)$$

Since

$$a_1^2 + 2a_1' - 4a_0 = 1 - \frac{4\mu}{x} - \frac{1 - 4\nu^2}{x^2}\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (17) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = 0$, equation (14) becomes

$$m = \sqrt{\frac{1}{\xi'}} \quad (21)$$

as in the case of Airy functions.

6. EXAMPLES

6.1 Airy functions

We start by solving the equation given at the end of the introduction in terms of Airy functions. The equation is $y'' = vy$ with

$$v = \frac{3 - 50x + 61x^2 - 60x^3 + 45x^4 - 18x^5 + 3x^6}{4(x-1)^8}\,,$$

so $\nu_\infty(v) = 2$ and its denominator is $4(x-1)^8$. Therefore, any solution $\xi \in C(x)$ of (17) must be of the form

$$\xi = \frac{P}{(x-1)^3}$$

where $P \in C[x]$ is of degree $0, 1, 2$ or 3. Substituting $\xi = (c_0 + c_1x + c_2x^2 + c_3x^3)/(x-1)^3$ in (15) yields a system of 14 algebraic equations. The nonconstant condition (12) becomes

$$(3c_0 + c_1)w_1 + (-3c_0 + c_2)w_2 + (c_0 + c_3)w_3 - 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2$ and w_3 yields the 3 solutions

$$\xi = \frac{x(x-2)}{(x-1)^2} \quad \text{and} \quad \xi = -(1 \pm \sqrt{-3})\frac{x(x-2)}{2(x-1)^2}\,.$$

Using (16) we compute

$$m = \sqrt{\frac{1}{\xi'}} = c(x-1)^{3/2}$$

for some constant c. Therefore, a basis of the solutions of $y'' = vy$ is given by

$$(x-1)^{3/2} Ai\left(\frac{x(x-2)}{(x-1)^2}\right) \text{ and } (x-1)^{3/2} Bi\left(\frac{x(x-2)}{(x-1)^2}\right)$$

where Ai and Bi are Airy functions.

6.2 Bessel functions

We now look for solutions in terms of modified Bessel functions of

$$y'' - (v_0 + v_1x)^n y = 0 \quad \text{where } n > 0 \text{ and } v_1 \ne 0\,. \quad (22)$$

Letting $v = (v_0 + v_1x)^n$, $\nu_\infty(v) = -n$ and its denominator is 1, so any solution $\xi \in C(x)$ of (17) must be a polynomial of degree $0, 1$, or $1 + n/2$. Substituting $\xi = c_0 + c_1x$ in (17) yields

$$\frac{c_1^4(1 + 4\epsilon c_0^2 - 4\nu^2) + 8\epsilon c_0 c_1^5 x + 4\epsilon c_1^6 x^2}{(c_0 + c_1x)^2} = -4c_1^2(v_0 + v_1x)^n\,,$$

whose only solution for $n > 0$ and $v_1 \ne 0$ is $c_1 = 0$. Therefore, any nonconstant solution must be a polynomial of degree exactly $1 + n/2$, which implies that there can be such solutions only when n is even. We proceed with $n = 4$, which is the smallest even value for which MAPLE 7 is unable to solve the above equation. Substituting $\xi = c_0 + c_1x + c_2x^2 + c_3x^3$ in (17) yields a system of 15 algebraic equations. The nonconstant condition (12) becomes

$$c_1w_1 + c_2w_2 + c_3w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3$ and ν with parameters v_0 and v_1 and $\epsilon = -1$ yields the 4 solutions

$$\nu = \pm\frac{1}{6}\,, \quad \xi = \pm\frac{1}{3}\frac{(v_0 + v_1x)^3}{v_1}\,.$$

Using (18) we compute

$$m = \sqrt{\frac{\xi}{\xi'}} = c\sqrt{v_0 + v_1x}$$

for some constant c. Therefore, a basis of the solutions of (22) for $n = 4$ is given by

$$\sqrt{v_0 + v_1x}\; I_{1/6}\left(\frac{1}{3}\frac{(v_0 + v_1x)^3}{v_1}\right)$$

and

$$\sqrt{v_0 + v_1x}\; K_{1/6}\left(\frac{1}{3}\frac{(v_0 + v_1x)^3}{v_1}\right)$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds.

6.3 Whittaker functions

For an example with two parameters to identify, we look for solutions in terms of Whittaker functions of

$$y'' + (ax^4 + bx)y = 0 \quad \text{where } a \neq 0\,, \tag{23}$$

which is Kamke's example 2.16 [2] with a specific integer choice for c. Letting $v = -(ax^4 + bx)$, $\nu_\infty(v) = -4$ and its denominator is 1, so any solution $\xi \in C(x)$ of (20) must be a polynomial of degree $0, 1$, or 3. Substituting $\xi = c_0 + c_1 x$ in (20) yields

$$4ac_1^4 x^6 + \text{ lower terms } = 0\,,$$

which implies $c_1 = 0$ whenever $a \neq 0$. Therefore, any nonconstant solution must be a polynomial of degree exactly 3. Substituting $\xi = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ in (20) yields a system of 15 algebraic equations. The nonconstant condition (12) becomes

$$c_1 w_1 + c_2 w_2 + c_3 w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3, \nu$ and μ with parameters a and b yields the 2 solutions

$$\mu = \frac{1}{6}\frac{b}{\sqrt{-a}}\,, \quad \nu = \pm\frac{1}{6}\,, \quad \xi = \frac{2}{3}x^3\sqrt{-a}\,.$$

Using (21) we compute

$$m = \sqrt{\frac{1}{\xi'}} = \frac{c}{x}$$

for some constant c. Therefore, a basis of the solutions of (23) is given by

$$\frac{1}{x} M_{\frac{b}{6\sqrt{-a}},\frac{1}{6}}\left(\frac{2}{3}x^3\sqrt{-a}\right) \text{ and } \frac{1}{x} W_{\frac{b}{6\sqrt{-a}},\frac{1}{6}}\left(\frac{2}{3}x^3\sqrt{-a}\right)$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are Whittaker functions.

6.4 An algebraic transformation ξ

We illustrate the use of the algebraic candidate (13) by solving the Airy equation $y'' = xy$ in terms of modified Bessel functions, thereby recovering classical expressions of Airy functions as Bessel functions. Letting $v = x$, $\nu_\infty(v) = -1$ and its denominator is 1, so any solution $\xi \in C(x)$ of (17) must be a polynomial of degree 0 or 1. Substituting $\xi = c_0 + c_1 x$ in (17) yields

$$-4c_1^4 x^3 + \text{ lower terms } = 0\,,$$

which implies $c_1 = 0$, hence that (17) has no nonconstant rational solution. However, formula (13) yields the algebraic candidate $\xi = P(\sqrt{x})$ where P is a polynomial of degree $0, 1, 2$ or 3. Substituting $\xi = c_0 + c_1 x^{1/2} + c_2 x + c_3 x^{3/2}$ in (17) yields a system of 17 algebraic equations. The nonconstant condition (12) becomes

$$c_1 w_1 + c_2 w_2 + + c_3 w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3$ and ν with $\epsilon = -1$ yields the 4 solutions

$$\nu = \pm\frac{1}{3}\,, \quad \xi = \pm\frac{2}{3}x^{3/2}\,.$$

Using (18) we compute

$$m = \sqrt{\frac{\xi}{\xi'}} = c\sqrt{x}$$

for some constant c. Therefore, a basis of the solutions of the Airy equation $y'' = xy$ is given by

$$\sqrt{x}\; I_{1/3}\left(\frac{2}{3}x^{3/2}\right) \quad \text{and} \quad \sqrt{x}\; K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds. It follows that the Airy functions Ai and Bi can be expressed as linear combinations

$$\sqrt{x}\left(c_1 I_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2 K_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right)$$

and the constants c_1 and c_2 can be found by looking at their values at two points.

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1.24 AN IMPLEMENTATION OF KOVACIC'S ALGORITHM FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS IN FORMAC

By A.Yu.Zherkov. Saratov State University, Saratov, USSR. 1987

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AN IMPLEMENTATION OF KOVACIC'S ALGORITHM FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS IN FORMAC

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1987

1. INTRODUCTION

In the recent paper [1] an algorithm for finding a "closed-form" solution of the following differential equations is given

$$y'' + a(x)y' + b(x)y = 0 . \qquad (1)$$

where a and b are rational functions of the independent variable x . The "closed-form" solution means the Liouvillian solution, i.e. one that can be, expressed in terms of algebraic functions, exponentials and indefinite integrals, see (Kovacic [1]), for precise definition. Kovacic's algorithm provides a Liouvillian solution of (1) or reports that no such solution exists. The main result obtained by Kovacic is the following.

Theorem. Equation (1) has a Liouvillian solution if and only if it has a solution of the form

$$y = \exp\{\int(\omega - a/2)\,dx\},$$

where ω is an algebraic function of x of degree 1, 2, 4, 6 or 12. The last means that ω, satisfying the Ricatti equation

$$\omega' + \omega^2 = R(x), \quad R = a'/2 + a^2/4 - b$$

solves a polynomial equation $G(\omega, x) = 0$, where $G(\omega, x) = \sum_{i=0}^{N} g_i(x)\,\omega^i$, g_i are rational functions of x and $N \in \{1,2,4,6,12\}$.

An algorithm for finding the polynomial G is based on the knowledge of the even order poles of R and consists in constructing and testing a finite number of possible candidates for G. If each candidate is not a desired polynomial then eq.(1) has no Liouvillian solutions.

Kovacic's algorithm has been already implemented in the Computer Algebra Systems MACSYMA [2] and MAPLE [3]. Our implementation is based on the Computer Algebra System FORMAC, see [4]. In the second section of this paper we give the complete algorithm description in such a way that an interested reader can immediately start to implement it using a suitable

Computer Algebra System. In the third (final) section some implementation aspects and the computational experience in the FORMAC are discussed.

2. ALGORITHM DESCRIPTION

Notation:

C denotes the complex numbers; **C**(x), the rational functions over **C**; **C**[x], the polynomials over **C**, **Z** the integers, L a finite set. For $s, t \in \mathbf{C}[x]$ $\gcd(s, t)$ denotes the greatest common divisor of s and t, deg t the leading degree of t and lc(t) the leading coefficient of t.

Problem:

Given $R \in \mathbf{C}(x)$

Find $G(\omega, x) = \sum_{i=0}^{N} g_i(x)\omega^i$, $g_i \in \mathbf{C}(x)$, $N \in \{1, 2, 4, 6, 12\}$, such that $\omega' + \omega^2 = R$ and $G(\omega, x) = 0$.

Algorithm:

1. Partitioning of R
   $L := \emptyset$
   $R := s/t$; $[s, t \in \mathbf{C}[x],\ \gcd(s, t) = 1,\ lc(t) = 1]$
   $m := \deg s - \deg t$;
   Computer the square-free factorization of t:
   $t := t_1 \cdot t_2^2 \cdot \ldots \cdot t_\ell^\ell$; $[t_\ell \neq 1]$.
2. Necessary conditions for N
   if $\forall_{k>0}\ t_{2k+1} = 1$ and ($m/2 \in \mathbf{Z}$ or $m < -2$) then $L := L \cup \{1\}$;
   if $\exists_{k>0}\ t_{2k+1} \neq 1$ or $t_2 \neq 1$ then $L := L \cup \{2\}$;
   if $\forall_{k>2}\ t_k = 1$ and $m \leq -2$ then $L := L \cup \{4, 6, 12\}$;

if $L = \emptyset$ then return 'no solution exists';

3. Constructing of candidates
   $d_0 := \frac{1}{4}(\min(2, -m) - \deg t - 3\deg t_1)$; $\theta_0 := \frac{1}{4}(t'/t + 3t_1'/t_1)$;
   Find the roots c_i of t_2; $[i = 1, 2, \ldots, n_2]$
   for i := 1 to n_2 do begin $d_i := \sqrt{1 + 4\lim_{x \to c_i}(x - c_i)^2 R}$;
   $\theta_i := d_i/(x - c_i)$; end;
   if $m \leq -2$ then begin $n_2 := n_2 + 1$; $\theta_{n_2} := 0$; end
   if $m < -2$ then $d_{n_2} := 1$ else if $m = -2$ then
   $d_{n_2} := \sqrt{1 + 4\,lc(s)/lc(t)}$;

```
if 1 ∈ L then begin
   Find the roots c_i of t_4·t_6·...·t_ℓ; [i = n_2+1, n_2+2, ..., n]
   for i := n_2+1 to n do begin
      ν := m_i/2;  [m_i is the order of the pole c_i in R]
```

$$\text{for } k := 0 \text{ to } \nu - 1 \text{ do } \lambda_{\nu-k} := \frac{d^k}{dx^k}\left((x - c_i)^{\nu}\sqrt{R(x)}\right)\Big|_{x=c_i};$$

$$d_i := 2\lambda_1; \quad \theta_i := 2\sum_{k=1}^{\nu} \lambda_k/(x - c_i)^k;$$

```
   end;
   if m > -2 then begin
      n := n + 1;
      ν := m/2;
```

$$\text{for } k := 0 \text{ to } \nu + 1 \text{ do } \lambda_{\nu-k} := \frac{d^k}{dx^k}\left(x^{\nu}\sqrt{R(1/x)}\right)\Big|_{x=0};$$

$$d_n := 2\lambda_{-1}; \quad \theta_n := 2\sum_{k=0}^{\nu} \lambda_k x^k;$$

```
   end;
end 'if 1 ∈ L';
```

4. Testing of candidates

```
for each N ∈ L [in increasing order] do begin
   if N = 1 then k := n else k := n_2;
   for j := 0 to k do s_j := -N/2;
   for ℓ := 1 to (N+1)^k do begin
      j := 1;
      while s_j = N/2 do begin s := -N/2; j := j + 1; end;
      s_j := s_j + 1;
```

$$d := N \cdot d_0 - \sum_{i=1}^{k} s_i d_i;$$

```
      if d ∈ Z and d ≥ 0 then begin
```

$$\theta := N \cdot \theta_0 + \sum_{i=1}^{k} s_i \theta_i;$$

$$P := \sum_{i=0}^{d} a_i x^i; \text{ [with undetermined } a_i\text{]}$$

$$P_N := P;$$

$$\text{for } i := N \text{ step } -1 \text{ to } 0 \text{ do } P_{i-1} := -P_i' - \theta P_i - (N+i)(i+1)RP_{i+1};$$

```
         Solve equation 'P_{-1} = 0' for P
                        [linear algebraic system for a_i]
         if solution P = P̃ is found
```

$$\text{then return } Q := \sum_{i=0}^{N} \frac{\omega^i}{(N-i)!} P_i \Big|_{P=\tilde{P}};$$

```
      end 'if d...';
```

```
    end 'for ℓ';
end 'for each N;
return 'no solution exists';
```

3. IMPLEMENTATION IN FORMAC

The above algorithm is implemented in the FORMAC Computer Algebra System. The choice of FORMAC is caused by its high execution velocity and comparatively small memory needed, so that one can run our program in IBM and derivative computers with 512K memory. To implement Kovacic's algorithm we developed a number of routines extending the capabilities of FORMAC, such as polynomial division, polynomial gcd's computation, square-free polynomial factorization, determination of the rational roots of polynomials, partial fraction decomposition of rational functions, solving the linear algebraic systems.

The algorithm requires exact numeric computations to be carried out in a quadratic extension of the initial number field **F**, which includes the coefficients and the even order poles of **R**. The current version of our program can be applied to the limited class of eqs.(1) with $\mathbf{F}=\mathbf{Q}$ (rational number field). It means that the program works with the numbers of the form $q_1+q_2\sqrt{q_3}$ which are automatically simplified to the canonical form: $q_1'+q_2'\sqrt{m}$ ($q_1, q_1' \in \mathbf{Q}$, m is a square-free integer). To extend the class of input equations it's sufficient to modify the procedure SIMP implementing the simplification of "numeric expressions".

The program has been tested successfully on examples in [1,5] Moreover we tested 70 equations for which the infinite power series solutions are given in [6] Among them about 30 equations were found to have Liouvillian solutions. For example, a power series solution of the equation

$$y'' + \frac{3-x}{x}y' - \frac{5}{x}y = 0 \quad \text{presented in}^{/6/} \text{ is } y = \sum_{n=0}^{\infty}(n+3)(n+4)x^n/n!$$

Using the program developed we found a closed-form solution: $y=(x^2+8x+12)e^x$. All tested examples take from 3 to 10 sec. of ES-1061 running time and less than 200K memory.

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Жарков А.Ю. E11-87-455

Реализация на языке FORMAC алгоритма Ковачича для решения обыкновенных дифференциальных уравнений

Рассмотрена реализация на языке аналитических вычислений FORMAC алгоритма Ковачича для нахождения лиувиллевских решений дифференциальных уравнений вида $y'' + a(x)y' + b(x)y = 0$, где a, b - рациональные функции x. Приведено формальное описание алгоритма, позволяющее легко реализовать его в любой подходящей системе компьютерной алгебры.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Zharkov A.Yu. E11-87-455

An Implementation of Kovacic's Algorithm for Solving Ordinary Differential Equations in FORMAC

An implementation of Kovacic's algorithm for finding Liouvillian solutions of the differential equations $y'' + a(x)y' + b(x)y = 0$ with rational coefficients a(x) and b(x) in the Computer Algebra System FORMAC is described. The algorithm description is presented in such a way that one can easily implement it in a suitable Computer Algebra System.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1.25 Algorithm implementation in FriCAS

See `https://github.com/fricas/fricas/blob/master/src/algebra/kovacic.spad`

1.26 Maxima implementation of Kovacic algorithm

By Nijso Beishuizen.

See `https://sourceforge.net/p/maxima/mailman/message/32164642/`