Finding roots of unity using Euler and De Moivreś

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To find the roots of

$$f(x) = x^n - 1$$

Solving for x from

$$\begin{aligned} 0 &= x^n - 1 \\ x^n &= 1 \\ x &= 1^{\frac{1}{n}} \end{aligned} \tag{1}$$

Now $1^{\frac{1}{n}}$ is evaluated. Since

$$1 = e^{i(2\pi)}$$

Substituting (2) in the RHS of (1) gives

$$\begin{aligned} x &= \left(e^{i2\pi}\right)^{\frac{1}{n}} \\ &= (\cos 2\pi + i \sin 2\pi)^{\frac{1}{n}} \end{aligned} \tag{3}$$

Using De Moivre's formula

$$(\cos\alpha + i\sin\alpha)^{\frac{1}{n}} = \cos\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right) \qquad k = 0, 1, \cdots n-1$$

Therefore (3) is rewritten as

$$x = \cos\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) \qquad k = 0, 1, \cdots n-1$$

The above gives the roots of $f(x) = x^n - 1$. The following examples illustrate the use of the above.

1. Solve $f(x) = x^2 - 1$. Here $n = 2$, therefore $k = 0, 1$. For $k = 0$

$$\begin{aligned} x &= \cos\left(\frac{2\pi}{2}\right) + i\sin\left(\frac{2\pi}{2}\right) \\ &= -1 \end{aligned}$$

And for $k = 1$

$$
\begin{aligned}
x &= \cos\left(\frac{2\pi}{2} + \frac{2\pi}{2}\right) + i\sin\left(\frac{2\pi}{2} + \frac{2\pi}{2}\right) \\
&= 1
\end{aligned}
$$

Hence the two roots are $\{1, -1\}$

2. Solve $f(x) = x^3 - 1$. Here $n = 3$, hence for $k = 0$

$$
\begin{aligned}
x &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \\
&= -\frac{1}{2} + i\frac{\sqrt{3}}{2}
\end{aligned}
$$

And for $k = 1$

$$
\begin{aligned}
x &= \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) \\
&= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\
&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}
$$

And for $k = 2$

$$
\begin{aligned}
x &= \cos\left(\frac{2\pi}{3} + 2\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + 2\frac{2\pi}{3}\right) \\
&= \cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) \\
&= 1
\end{aligned}
$$

Therefore the roots are $\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$

Here is another example. Let us solve

$$
\begin{aligned}
x - (-8)^{\frac{1}{3}} &= 0 \\
x &= (-8)^{\frac{1}{3}} \\
&= (-8)^{\frac{1}{n}}
\end{aligned}
$$

Where $n = 3$. But $8 = 8(1) = 8e^{2\pi i}$. Hence the above becomes

$$
\begin{aligned}
x &= \left(-8e^{2\pi i}\right)^{\frac{1}{n}} \\
&= -8^{\frac{1}{n}} e^{\frac{2\pi i}{n}} \\
&= -8^{\frac{1}{n}} (\cos 2\pi + i\sin 2\pi)^{\frac{1}{n}} \qquad (1)
\end{aligned}
$$

But by De Moivre's formula

$$(\cos 2\pi + i\sin 2\pi)^{\frac{1}{n}} = \cos\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n} + k\frac{2\pi}{n}\right) \qquad k = 0 \cdots n-1$$

Computing the above gives

$$(\cos 2\pi + i\sin 2\pi)^{\frac{1}{n}} = \left\{-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, 1\right\}$$

Hence from (1)

$$\begin{aligned} x &= -8^{\frac{1}{3}}\left\{-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, 1\right\} \\ &= \left\{-8^{\frac{1}{3}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), -8^{\frac{1}{3}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), -8^{\frac{1}{3}}\right\} \\ &= \{1 - 1.732i, 1 + 1.73i, -2\} \end{aligned}$$