Deriving trig identities

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To derive trig identities (something useful in the exam), we will use Euler relation as starting point, which is $e^{ix}=\cos x+i\sin x$.

1 $\cos(A+B)$ and $\sin(A+B)$^^E^^L

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B) \tag{1}$$

But $e^{i(A+B)} = e^{iA}e^{iB}$ therefore

$$\begin{aligned} e^{iA}e^{iB} &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin A\sin B \\ &= (\cos A\cos B - \sin A\sin B) + i(\cos A\sin B + \sin A\cos B) \end{aligned} \tag{2}$$

Now (1) is the same as (2). Hence the real part and the imaginary parts must be the same. Therefore

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{3}$$
$$\sin(A+B) = \cos A\sin B + \sin A\cos B \tag{4}$$

2 $\cos(A-B)$ and $\sin(A-B)$^^E^^L

This can be derived in similar way to the above using $e^{i(A-B)} = \cos(A-B) + i\sin(A-B)$ and so on. But more easily, it can be derived from (3,4) directly by just changing replacing B by $-B$ everywhere and then changing $\sin(-B)$ to $-\sin B$ and leaving $\cos B$ the same since $\cos(-B) = \cos B$. This is because cos is even and sin is odd, then (3) becomes

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{3A}$$
$$\sin(A-B) = -\cos A\sin B + \sin A\cos B \tag{4A}$$

So we really just need to find (3) to find the 4 formulas for addition and subtractions of angles.

3 $\cos(2A)$ and $\sin(2A)$^^E^^L

These also can be found from (3,4). By replacing B with A resulting in

$$\begin{aligned} \cos(A+A) &= \cos A\cos A - \sin A\sin A \\ \sin(A+A) &= \cos A\sin A + \sin A\cos A \end{aligned}$$

Therefore

$$\cos(2A) = \cos^2 A - \sin^2 A \tag{3C}$$
$$\sin(2A) = 2\cos A\sin A \tag{4C}$$

Or we could use Euler formula, but the above is simpler. To use Euler formula, we write

$$e^{i(2A)} = \cos(2A) + i\sin(2A) \tag{5}$$

But $e^{i(2A)} = e^{iA}e^{iA}$ therefore

$$\begin{aligned} e^{iA}e^{iA} &= (\cos A + i\sin A)(\cos A + i\sin A) \\ &= \cos^2 A + 2i\cos A\sin A - \sin^2 A \\ &= \left(\cos^2 A - \sin^2 A\right) + i(2\cos A\sin A) \end{aligned} \tag{6}$$

Comparing (5,6) shows that

$$\begin{aligned} \cos(2A) &= \cos^2 A - \sin^2 A \\ \sin(2A) &= 2\cos A\sin A \end{aligned}$$

Which is the same as (3C,4C) above.

4 $\cos\left(\frac{x}{2}\right)$^^E^^L

From the double angle formula (3C)

$$\cos(2A) = \cos^2 A - \sin^2 A$$

But $\cos^2 A + \sin^2 A = 1$ then $\sin^2 A = 1 - \cos^2 A$ and the above becomes

$$\begin{aligned} \cos(2A) &= \cos^2 A - \left(1 - \cos^2 A\right) \\ &= 2\cos^2 A - 1 \end{aligned}$$

Hence

$$\cos^2 A = \frac{\cos(2A) + 1}{2}$$

Let $A = \frac{x}{2}$ then the above becomes

$$\begin{aligned} \cos^2\left(\frac{x}{2}\right) &= \frac{\cos(x) + 1}{2} \\ \cos\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{\cos(x) + 1}{2}} \end{aligned}$$

The sign depends on the quadrant of $\frac{x}{2}$.

5 $\sin\left(\frac{x}{2}\right)$^^E^^L

From the double angle formula (3C)

$$\cos(2A) = \cos^2 A - \sin^2 A$$

But $\cos^2 A + \sin^2 A = 1$ then $\cos^2 A = 1 - \sin^2 A$ and the above becomes

$$\begin{aligned} \cos(2A) &= 1 - \sin^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}$$

Hence

$$\sin^2 A = \frac{1-\cos(2A)}{2}$$

Let $A = \frac{x}{2}$ then the above becomes

$$\begin{aligned}\sin^2\left(\frac{x}{2}\right) &= \frac{1-\cos(x)}{2}\\ \sin\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos(x)}{2}}\end{aligned}$$

The sign depends on the quadrant of $\frac{x}{2}$.

6 $\sin(\alpha)+\sin(\beta)$^^E^^L

This can be found by adding (4) and (4A). Let

$$\begin{aligned}A+B &= \alpha \qquad (7)\\ A-B &= \beta\end{aligned}$$

Then (4)+(4A) now becomes

$$\begin{aligned}\sin(\alpha)+\sin(\beta) &= (\cos A\sin B+\sin A\cos B)-\cos A\sin B+\sin A\cos B\\ &= 2\sin A\cos B \qquad (8)\end{aligned}$$

Now we solve for A, B from (7). Which gives

$$\begin{aligned}A &= \frac{\alpha+\beta}{2}\\ B &= \frac{\alpha-\beta}{2}\end{aligned}$$

Substituting the above in (8) gives

$$\sin(\alpha)+\sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

7 $\cos(\alpha)+\cos(\beta)$^^E^^L

This can be found by adding (3) and (3A). Let

$$\begin{aligned}A+B &= \alpha \qquad (7)\\ A-B &= \beta\end{aligned}$$

Then (3)+(3A) now becomes

$$\begin{aligned}\cos(\alpha)+\cos(\beta) &= (\cos A\cos B-\sin A\sin B)+(\cos A\cos B+\sin A\sin B)\\ &= 2\cos A\cos B \qquad (9)\end{aligned}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha+\beta}{2}$$
$$B = \frac{\alpha-\beta}{2}$$

Substituting the above in (9) gives

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

8 $\sin(\alpha) - \sin(\beta)$^^^E^^^L

This can be found from (4)-(4A). Let

$$\begin{aligned} A+B &= \alpha \\ A-B &= \beta \end{aligned} \tag{7}$$

Then (4)-(4A) now becomes

$$\begin{aligned} \sin(\alpha) - \sin(\beta) &= (\cos A \sin B + \sin A \cos B) + \cos A \sin B - \sin A \cos B \\ &= 2\cos A \sin B \end{aligned} \tag{10}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha+\beta}{2}$$
$$B = \frac{\alpha-\beta}{2}$$

Substituting the above in (10) gives

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

9 $\cos(\alpha) - \cos(\beta)$^^^E^^^L

This can be found from (3)-(3A). Let

$$\begin{aligned} A+B &= \alpha \\ A-B &= \beta \end{aligned} \tag{7}$$

Then (3)-(3A) now becomes

$$\begin{aligned} \cos(\alpha) - \cos(\beta) &= (\cos A \cos B - \sin A \sin B) - (\cos A \cos B + \sin A \sin B) \\ &= -2\sin A \sin B \end{aligned} \tag{11}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha+\beta}{2}$$
$$B = \frac{\alpha-\beta}{2}$$

Substituting the above in (11) gives

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

10 $\cos(A)\cos(B)$^^^E^^^L

Adding (3)+(3A) gives

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ \cos(A+B) + \cos(A-B) &= 2\cos A\cos B \end{aligned}$$

Hence

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

11 $\sin(A)\cos(B)$^^^E^^^L

Adding (4)+(4A) gives

$$\begin{aligned} \sin(A+B) &= \cos A\sin B + \sin A\cos B \\ \sin(A-B) &= -\cos A\sin B + \sin A\cos B \\ \sin(A+B) + \sin(A-B) &= 2\sin A\cos B \end{aligned}$$

Hence

$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

12 $\sin(A)\sin(B)$^^^E^^^L

(3)-(3A) gives

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ \cos(A+B) - \cos(A-B) &= -2\sin A\sin B \end{aligned}$$

Hence

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$