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# Symbolic solution to expansion of natural frequencies method for determining $e^{At}$

Written by Nasser M. Abbasi, based on lecture given by Professor Barmish, ECE 717, University of Wisconsin, Madison
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The method of finding $e^{At}$ using expansion of natural frequencies is first applied to a $2\times2$ matrix to illustrate the symbolic method. Then a general purpose function is written at the end using the symbolic method and applied to much larger problems for illustrations. This is a standalone function which can be called to determine $e^{At}$ for matrices A which can have repeated eigenvalues. The result is verified by comparing the output to *Mathematica* buildin function MatrixExp[]

## Example 1 from lecture done using symbolics

### Define *A*

```
Clear["Global`*"]
A0 = {{2, 0}, {-3, -3}};
MatrixForm[A0]
```

$$\begin{pmatrix} 2 & 0 \\ -3 & -3 \end{pmatrix}$$

### Find eigenvalues of A

```
λ = {2, -3};
```

### Set the eigenvalue multiplers

```
multipliers = {1, 1}; (*multiplers of eigenvalues*)
```

Apply $e^{At}$ algorithm using expansion of natural frequencies

```
eAt = Total@Flatten@Last@Reap@Do[Do[ Sow[Y[k, i] t^k Exp[t λ⟦i⟧]],
        {k, 0, multipliers⟦i⟧ - 1}], {i, 1, Length[λ]}];
Row[
 {"e^At=",
  eAt}]
```

```
e^At=ⅇ^(2 t) Y[0, 1] + ⅇ^(-3 t) Y[0, 2]
```

Automatic generation of lists of unknowns Y(k,i) and list of symbols to b[i] to use for solving Ax=b

```
varsb = Array[b, Total[multipliers], {0, Total[multipliers] - 1}]
varsX = Flatten@Last@Reap@
    Do[Do[Sow@Y[k, i], {k, 0, multipliers⟦i⟧ - 1}], {i, 1, Length[multipliers]}]
```

```
{b[0], b[1]}
```

```
{Y[0, 1], Y[0, 2]}
```

Take repeated derivatives of $e^{At}$ to generate the set of equations to solve for Y(k,i)

```
eqs = Table[varsb⟦k + 1⟧ == (D[eAt, {t, k}]) /. t → 0, {k, 0, Total[multipliers] - 1}]
```

```
{b[0] == Y[0, 1] + Y[0, 2], b[1] == 2 Y[0, 1] - 3 Y[0, 2]}
```

The problem can be viewed as Ax=b where x=Y(k,i) is the unknowns and b are the symbolic names for the matrix powers.

```
{rightSide, A1} = Normal@CoefficientArrays[eqs, varsX];
Row[
 {MatrixForm[A1], MatrixForm[Transpose@{varsX}], "=", MatrixForm[rightSide]}]
```

$$\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} Y[0, 1] \\ Y[0, 2] \end{pmatrix} = \begin{pmatrix} b[0] \\ b[1] \end{pmatrix}$$

Now solve Ax=b

```
sol = Solve[eqs, varsX]
```

$$\left\{\left\{Y[0,1]\to\frac{1}{5}(3\,b[0]+b[1]),\ Y[0,2]\to\frac{1}{5}(2\,b[0]-b[1])\right\}\right\}$$

Replace symbols b(i) on the right of the above, by the actual numerical values of the matrix powers

```
bN = (b[#] → MatrixPower[A0, #]) & /@ Range[0, Length[varsb] - 1];
Row[{bN[[#, 1]], "=", MatrixForm[bN[[#, -1]]]}] & /@ Range[Length[bN]]
```

$$\left\{b[0]=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ b[1]=\begin{pmatrix}2 & 0\\ -3 & -3\end{pmatrix}\right\}$$

Now apply these numerical values to the solution found earlier in order to obtain numerical values for Y(k,i)

```
sol2 = First[sol /. bN];
Row[{sol2[[#, 1]], "=", MatrixForm@sol2[[#, -1]]}] & /@ Range[Length[sol2]]
```

$$\left\{Y[0,1]=\begin{pmatrix}1 & 0\\ -\frac{3}{5} & 0\end{pmatrix},\ Y[0,2]=\begin{pmatrix}0 & 0\\ \frac{3}{5} & 1\end{pmatrix}\right\}$$

Now since Y(k,i) are now found, evaluate $e^{\mathrm{At}}$ found above using the above solution

```
Row[{"e^At=", MatrixForm[(eAt /. sol2)]}]
```

$$e^{\mathrm{At}}=\begin{pmatrix}e^{2t} & 0\\ \frac{3e^{-3t}}{5}-\frac{3e^{2t}}{5} & e^{-3t}\end{pmatrix}$$

Compare to *Mathematica* buildin function

```
MatrixExp[A0 t] // MatrixForm
```

$$\begin{pmatrix}e^{2t} & 0\\ -\frac{3}{5}e^{-3t}\left(-1+e^{5t}\right) & e^{-3t}\end{pmatrix}$$

# Function to automatically calculate $e^{\mathrm{At}}$ using symbolic

# natural frequencies method

In[459]:=

```
matrixExp[A0_, debug_, Y_, t_] :=
 Module[{eigenvalues, multipliers, tally, eAt, k, i,
   varsb, varsX, rightSide, A1, eqs, sol, sol2, bN},
  eigenvalues = Eigenvalues[A0];
  tally = Tally[eigenvalues];
  eigenvalues = tally[[All, 1]];
  multipliers = tally[[All, 2]];
  If[debug, Print[Grid[Join[{{"λ", "multiplier"}}, tally], Frame → All]]];

  eAt = Total@Flatten@
     Last@Reap@Do[Do[ Sow[Y[ k, eigenvalues[[i]] ] t^k Exp[t eigenvalues[[i]] ]],
         {k, 0, multipliers[[i]] - 1}], {i, 1, Length[eigenvalues]}];

  If[debug, Print@Row[{"e^At=", eAt}]];

  varsb = Array[b, Total[multipliers], {0, Total[multipliers] - 1}];
  varsX = Flatten@
    Last@Reap@Do[Do[Sow@Y[k, eigenvalues[[i]]], {k, 0, multipliers[[i]] - 1}],
       {i, 1, Length[multipliers]}];
  eqs = Table[varsb[[k + 1]] == (D[eAt, {t, k}]) /. t → 0, {k, 0, Length[varsb] - 1}];
  If[debug, Print["eqs=\n", TableForm@eqs]];

  {rightSide, A1} = Normal@CoefficientArrays[eqs, varsX];

  If[debug, Print["A x = b  form"]];
  If[debug, Print@Row[{MatrixForm[-A1],
      MatrixForm[Transpose@{varsX}], "=", MatrixForm[rightSide]}]];

  sol = First@Solve[eqs, varsX];
  bN = (varsb[[# + 1]] → MatrixPower[A0, #]) & /@ Range[0, Length[varsb] - 1];
  sol2 = sol /. bN;
  If[debug, Print[Row[{sol2[[#, 1]], "=", MatrixForm@sol2[[#, -1]]}]] & /@
     Range[Length[sol2]]];
  eAt = eAt /. sol2;
  If[debug, Print@Row[{"e^At=", MatrixForm[eAt, Dividers → All]}]];
  eAt
 ]
```

# Example 2 from lecture solved using the above function

In[460]:=

```
Clear[Y, t];
A0 = {{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {-1, 2, 2, 0, 0, 0},
   {1, 0, -1, 2, 0, 0}, {0, 0, 1, 0, 2, 0}, {0, -1, 1, 0, 0, 3}};
(z1 = matrixExp[A0, True, Y, t]) // MatrixForm
```

| λ | multiplier |
|---|---|
| 3 | 1 |
| 2 | 3 |
| 1 | 2 |

$e^{At} = e^{t}\, Y[0,1] + e^{2t}\, Y[0,2] + e^{3t}\, Y[0,3] + e^{t}\, t\, Y[1,1] + e^{2t}\, t\, Y[1,2] + e^{2t}\, t^{2}\, Y[2,2]$

eqs=

$$
\begin{aligned}
b[0] &== Y[0,1] + Y[0,2] + Y[0,3] \\
b[1] &== Y[0,1] + 2\,Y[0,2] + 3\,Y[0,3] + Y[1,1] + Y[1,2] \\
b[2] &== Y[0,1] + 4\,Y[0,2] + 9\,Y[0,3] + 2\,Y[1,1] + 4\,Y[1,2] + 2\,Y[2,2] \\
b[3] &== Y[0,1] + 8\,Y[0,2] + 27\,Y[0,3] + 3\,Y[1,1] + 12\,Y[1,2] + 12\,Y[2,2] \\
b[4] &== Y[0,1] + 16\,Y[0,2] + 81\,Y[0,3] + 4\,Y[1,1] + 32\,Y[1,2] + 48\,Y[2,2] \\
b[5] &== Y[0,1] + 32\,Y[0,2] + 243\,Y[0,3] + 5\,Y[1,1] + 80\,Y[1,2] + 160\,Y[2,2]
\end{aligned}
$$

A x = b  form

$$
\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 1 \\ 9 & 4 & 4 & 2 & 1 & 2 \\ 27 & 8 & 12 & 12 & 1 & 3 \\ 81 & 16 & 32 & 48 & 1 & 4 \\ 243 & 32 & 80 & 160 & 1 & 5 \end{pmatrix}
\begin{pmatrix} Y[0,3] \\ Y[0,2] \\ Y[1,2] \\ Y[2,2] \\ Y[0,1] \\ Y[1,1] \end{pmatrix}
=
\begin{pmatrix} b[0] \\ b[1] \\ b[2] \\ b[3] \\ b[4] \\ b[5] \end{pmatrix}
$$

$$
\left\{ Y[0,3] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ -\frac{1}{2}&\frac{1}{2}&1&0&0&1 \end{pmatrix},\ 
Y[0,2] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ -1&2&1&0&0&0 \\ 0&2&0&1&0&0 \\ 1&-2&0&0&1&0 \\ 1&-2&-1&0&0&0 \end{pmatrix},\ 
Y[1,2] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&-2&-1&0&0&0 \\ -1&2&1&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \right.
$$

$$
\left. Y[2,2] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix},\ 
Y[0,1] = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 1&-2&0&0&0&0 \\ 0&-2&0&0&0&0 \\ -1&2&0&0&0&0 \\ -\frac{1}{2}&\frac{3}{2}&0&0&0&0 \end{pmatrix},\ 
Y[1,1] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \right\}
$$

$$
e^{At} = \begin{pmatrix}
e^{t} & 0 & 0 & 0 & 0 & 0 \\
0 & e^{t} & 0 & 0 & 0 & 0 \\
e^{t} - e^{2t} & -2e^{t} + 2e^{2t} & e^{2t} & 0 & 0 & 0 \\
e^{2t}\, t & -2e^{t} + 2e^{2t} - 2e^{2t}\, t & -e^{2t}\, t & e^{2t} & 0 & 0 \\
-e^{t} + e^{2t} - e^{2t}\, t & 2e^{t} - 2e^{2t} + 2e^{2t}\, t & e^{2t}\, t & 0 & e^{2t} & 0 \\
-\frac{e^{t}}{2} + e^{2t} - \frac{e^{3t}}{2} & \frac{3e^{t}}{2} - 2e^{2t} + \frac{e^{3t}}{2} & -e^{2t} + e^{3t} & 0 & 0 & e^{3t}
\end{pmatrix}
$$

Out[462]//MatrixForm=

$$
\begin{pmatrix}
e^{t} & 0 & 0 & 0 & 0 & 0 \\
0 & e^{t} & 0 & 0 & 0 & 0 \\
e^{t} - e^{2t} & -2e^{t} + 2e^{2t} & e^{2t} & 0 & 0 & 0 \\
e^{2t}\, t & -2e^{t} + 2e^{2t} - 2e^{2t}\, t & -e^{2t}\, t & e^{2t} & 0 & 0 \\
-e^{t} + e^{2t} - e^{2t}\, t & 2e^{t} - 2e^{2t} + 2e^{2t}\, t & e^{2t}\, t & 0 & e^{2t} & 0 \\
-\frac{e^{t}}{2} + e^{2t} - \frac{e^{3t}}{2} & \frac{3e^{t}}{2} - 2e^{2t} + \frac{e^{3t}}{2} & -e^{2t} + e^{3t} & 0 & 0 & e^{3t}
\end{pmatrix}
$$

# Compare to buildin function to verify

```
(z2 = MatrixExp[A0 t]) // MatrixForm
```

$$\begin{pmatrix} e^{t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 & 0 & 0 \\ -e^{t}\left(-1+e^{t}\right) & 2\,e^{t}\left(-1+e^{t}\right) & e^{2t} & 0 & 0 & 0 \\ e^{2t}\,t & -2\,e^{t}\left(1-e^{t}+e^{t}\,t\right) & -e^{2t}\,t & e^{2t} & 0 & 0 \\ -e^{t}\left(1-e^{t}+e^{t}\,t\right) & 2\,e^{t}\left(1-e^{t}+e^{t}\,t\right) & e^{2t}\,t & 0 & e^{2t} & 0 \\ -\frac{1}{2}\,e^{t}\left(-1+e^{t}\right)^{2} & \frac{1}{2}\,e^{t}\left(3-4\,e^{t}+e^{2t}\right) & e^{2t}\left(-1+e^{t}\right) & 0 & 0 & e^{3t} \end{pmatrix}$$

```
MatrixForm[Simplify[z1 - z2]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Verification Method, suggested by Professor Barmish. We Differentiate Matrix $e^{\mathrm{At}}$ found above and see it is the same as $A^k$ for $t = 0$

In[463]:=

```
tbl = Table[{k, MatrixForm[D[z1, {t, k}] /. t → 0],
    MatrixForm[MatrixPower[A0, k]]}, {k, 0, 5}];
h = {{"k", "d^k/dt e^At at t=0", "A^k"}};
Grid[Join[h, tbl], Frame → All]
```

Out[465]=

| k | $\frac{d^k}{dt}e^{At}$ at t=0 | $A^k$ |
|---|---|---|
| 0 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 3 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 3 \end{pmatrix}$ |
| 2 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 6 & 4 & 0 & 0 & 0 \\ 4 & -2 & -4 & 4 & 0 & 0 \\ -1 & 2 & 4 & 0 & 4 & 0 \\ -1 & -2 & 5 & 0 & 0 & 9 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 6 & 4 & 0 & 0 & 0 \\ 4 & -2 & -4 & 4 & 0 & 0 \\ -1 & 2 & 4 & 0 & 4 & 0 \\ -1 & -2 & 5 & 0 & 0 & 9 \end{pmatrix}$ |
| 3 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -7 & 14 & 8 & 0 & 0 & 0 \\ 12 & -10 & -12 & 8 & 0 & 0 \\ -5 & 10 & 12 & 0 & 8 & 0 \\ -6 & -1 & 19 & 0 & 0 & 27 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -7 & 14 & 8 & 0 & 0 & 0 \\ 12 & -10 & -12 & 8 & 0 & 0 \\ -5 & 10 & 12 & 0 & 8 & 0 \\ -6 & -1 & 19 & 0 & 0 & 27 \end{pmatrix}$ |
| 4 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -15 & 30 & 16 & 0 & 0 & 0 \\ 32 & -34 & -32 & 16 & 0 & 0 \\ -17 & 34 & 32 & 0 & 16 & 0 \\ -25 & 10 & 65 & 0 & 0 & 81 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -15 & 30 & 16 & 0 & 0 & 0 \\ 32 & -34 & -32 & 16 & 0 & 0 \\ -17 & 34 & 32 & 0 & 16 & 0 \\ -25 & 10 & 65 & 0 & 0 & 81 \end{pmatrix}$ |
| 5 | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -31 & 62 & 32 & 0 & 0 & 0 \\ 80 & -98 & -80 & 32 & 0 & 0 \\ -49 & 98 & 80 & 0 & 32 & 0 \\ -90 & 59 & 211 & 0 & 0 & 243 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -31 & 62 & 32 & 0 & 0 & 0 \\ 80 & -98 & -80 & 32 & 0 & 0 \\ -49 & 98 & 80 & 0 & 32 & 0 \\ -90 & 59 & 211 & 0 & 0 & 243 \end{pmatrix}$ |