Simple examples illustrating the use of the deformation gradient tensor

Nasser M. Abbasi

February 3, 2006 Compiled on September 7, 2023 at 9:22pm

Contents

1 Introduction

This note illustrates using simple examples, how to evaluate the deformation gradient tensor $\tilde{\mathbf{F}}$ and derive its polar decomposition into a stretch and rotation tensors.

Diagrams are used to help illustrate geometrically the effect of applying the stretch and the rotation tensors on a differential vector with the purpose of giving better insight into these operations. For simplicity, only 2D shapes are used.

Starting by selecting some arbitrary differential vector $\mathbf{dR}$ in the undeformed shape. The shape is then assumed to undergo a fixed form of deformation such that $\tilde{\mathbf{F}}$ is constant over the whole body (as opposed to being a field tensor where $\tilde{\mathbf{F}}$ would be a function of the position). Then the tensor $\tilde{\mathbf{F}}$ is computed and shown using diagrams how the differential vector $\mathbf{dR}$ in the undeformed shape is mapped to the vector $\mathbf{dr}$ in the deformed shape by successive application of the stretch tensor $\tilde{\mathbf{U}}$ followed by a parallel translation operation, and followed by the application of the rotation tensor $\tilde{\mathbf{R}}$.

The point that $\mathbf{dR}$ is located at is labeled P in the undeformed shape, and its image will be labeled P' in the deformed shape. The coordinates in the undeformed shape will be upper case X_1, X_2 and in the deformed shape will be lower case x_1, x_2.

One observation found is that if the deformation is such that perpendicular lines in the undeformed shape remain perpendicular to each others in the deformed shape, then this implies that the rotation tensor $\tilde{\mathbf{R}}$ will come out to be the identity tensor. The first 2 examples below illustrate this case. In the third example the rotation tensor $\tilde{\mathbf{R}}$ is not the identity tensor because lines do not remain perpendicular to each others after deformation.

2 Examples

2.1 Square shape becomes longer with width fixed

The following diagram is the undeformed configuration.

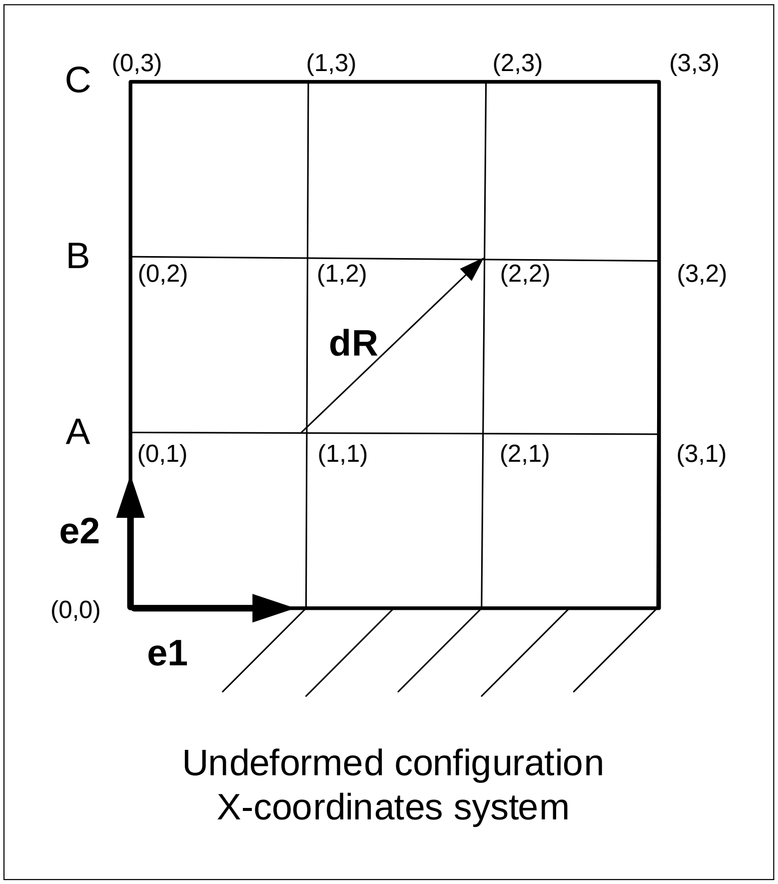

Figure 1: undeformed configuration

In this shape, the vector $\mathbf{dR}$ extends from the point $(1,1)$ to the point $(2,2)$. In this example, we assume a deformation whereby the shape is pulled upwards by some distance, causing the shape to become longer in the vertical direction and we assume the shape remain the same width.

This is the simplest form of deformation. Let us assume for simplicity that the shape becomes 3 times as long as before.

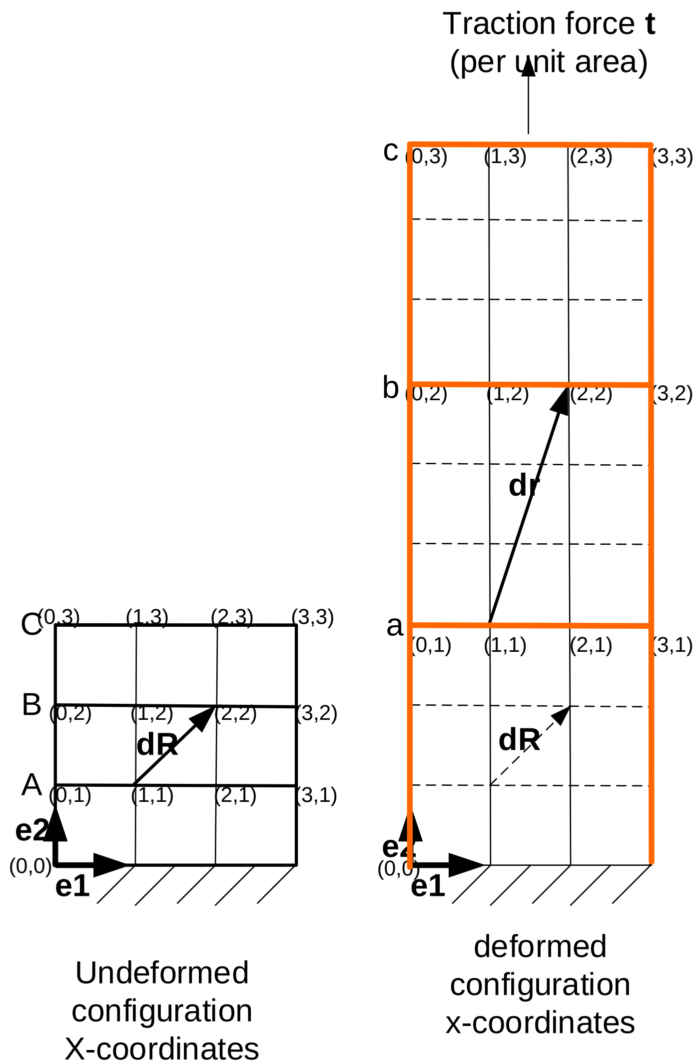

Figure 2: shape becomes 3 times as long

We observe the following. The lines A,B,C have moved to new locations in the deformed configuration. For instance, the line A started at $(0,1)$ and ended at $(3,1)$ in the undeformed shape coordinates. While the same line now labeled lower case a, starts from $(0,3)$ and ends at $(3,3)$ in the deformed shape using the undeformed coordinates system.

The first step in finding $\tilde{\mathbf{F}}$ is to determine the mapping between the X coordinates in the undeformed shape, and the x coordinates in the deformed shape. In this example this mapping is constant over any region of the shape. We see immediately that since the width of the shape did not change, then

$$x_1 = X_1$$

and since the new shape is 3 times as long as before then

$$x_2 = 3X_2$$

And now we can calculate $\tilde{\mathbf{F}}$. Since

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}$$

then given that $\frac{\partial x_1}{\partial X_1} = 1$, $\frac{\partial x_1}{\partial X_2} = 0$, $\frac{\partial x_2}{\partial X_1} = 0$, $\frac{\partial x_2}{\partial X_2} = 3$ we obtain the numerical value for $\tilde{\mathbf{F}}$

$$\tilde{\mathbf{F}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

We note here that $\tilde{\mathbf{F}}$ is the same for any region of the deformed shape. This is because the deformation is uniform.

Now we can find $\mathbf{dr}$.

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR}$$

Since from the undeformed shape we see that

$$\mathbf{dR} = \mathbf{e}_1 + \mathbf{e}_2$$

Then

$$\begin{aligned} \mathbf{dr} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

Hence

$$\mathbf{dr} = \mathbf{e}_1 + 3\mathbf{e}_2$$

Looking at the deformed shape we see that this agrees with the expected shape of the deformed $\mathbf{dr}$ vector.

Now once $\tilde{\mathbf{F}}$ is found, we can determine the stretch tensor $\tilde{\mathbf{U}}$ and the rotation tensor $\tilde{\mathbf{R}}$. We will do this algebraically first, then verify the result geometrically. Since by definition

$$\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$$

Once $\tilde{\mathbf{F}}$ is known, we can find $\tilde{\mathbf{U}}$ using the relation

$$\begin{aligned} \tilde{\mathbf{U}}^2 &= \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \end{aligned}$$

Now we take the square root of the matrix $\tilde{\mathbf{U}}^2$ to find $\tilde{\mathbf{U}}$[1]

$$\tilde{\mathbf{U}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

[1] To obtain the square root of a matrix, say matrix C, follow these steps.

and now that $\tilde{\mathbf{U}}$ is known, we can find $\tilde{\mathbf{R}}$

$$\begin{aligned}\tilde{\mathbf{R}} &= \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

To verify this result algebraically, we write

$$\begin{aligned}\mathbf{dr} &= \tilde{\mathbf{F}} \cdot \mathbf{dR} \\ &= \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}} \cdot \mathbf{dR} \\ &= \tilde{\mathbf{R}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \tilde{\mathbf{R}} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \mathbf{dr} &= \mathbf{e}_1 + 3\mathbf{e}_2\end{aligned}$$

Which agrees with earlier result.

To verify the result geometrically, we first apply the stretch tensor $\tilde{\mathbf{U}}$ to $\mathbf{dR}$, this results in a new differential vector which we call $\mathbf{dr}^*$, then we slide $\mathbf{dr}^*$ without changing its slope (i.e. parallel translation) such that the vector $\mathbf{dr}^*$ starts at the point P' in the deformed configuration, where the point P' is the image of the point P in the undeformed shape, and then we apply the rotation tensor $\tilde{\mathbf{R}}$ to $\mathbf{dr}^*$ to obtain $\mathbf{dr}$.

Hence

$$\begin{aligned}\mathbf{dr}^* &= \tilde{\mathbf{U}} \cdot \mathbf{dR} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \mathbf{e}_1 + \mathbf{3e}_2\end{aligned}$$

Now we apply the rotation of $\tilde{\mathbf{R}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to $\mathbf{dr}^*$, and since the rotation is a unit tensor, then this operation will produce no effect.

---

1. Determine the eigenvalues λ of the matrix.
2. For each eigenvalue λ_n determine the correspending eigenvector V_n
3. Construct the Matrix N whose columns are the eigenvectors V_n. i.e. the first column will be the vector V_1 etc...
4. Construct matrix M with diagonal elements that contains the $\sqrt{\lambda_n}$. i.e. $M(1,1) = \sqrt{\lambda_1}$, $M(2,2) = \sqrt{\lambda_2}$, and so forth. (This is the Jordan form for real distinct eigenvalues)
5. Now $\sqrt{C} = N^T M N$

In Matlab, the command expm() can be used to calculate sqrt of a matrix.

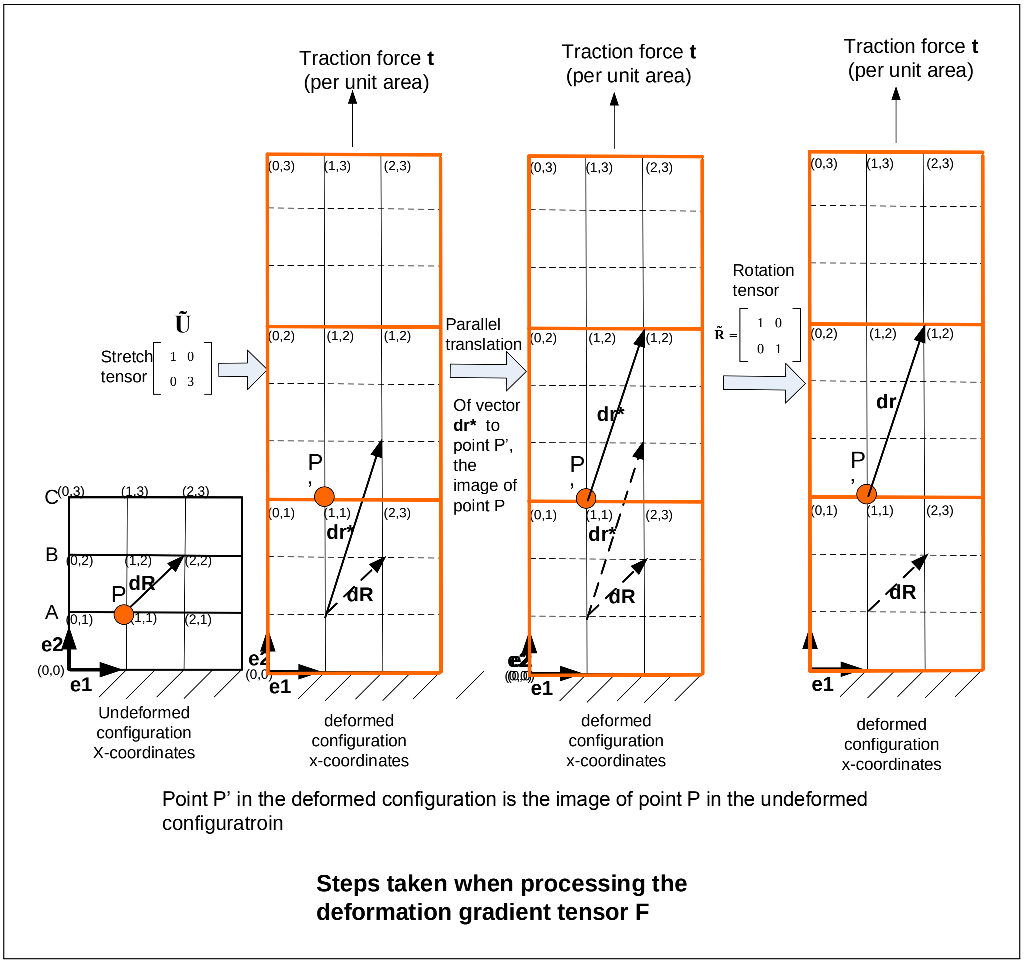

Figure 3: rotation is a unit tensor

2.2 Square shape becomes both longer and wider

In this example we start with the same original shape as above, but we increase both the length and the width of the shape and not just its length. Let the length be 3 times as long as the original length, and the width be 1.5 times as wide as the original width.

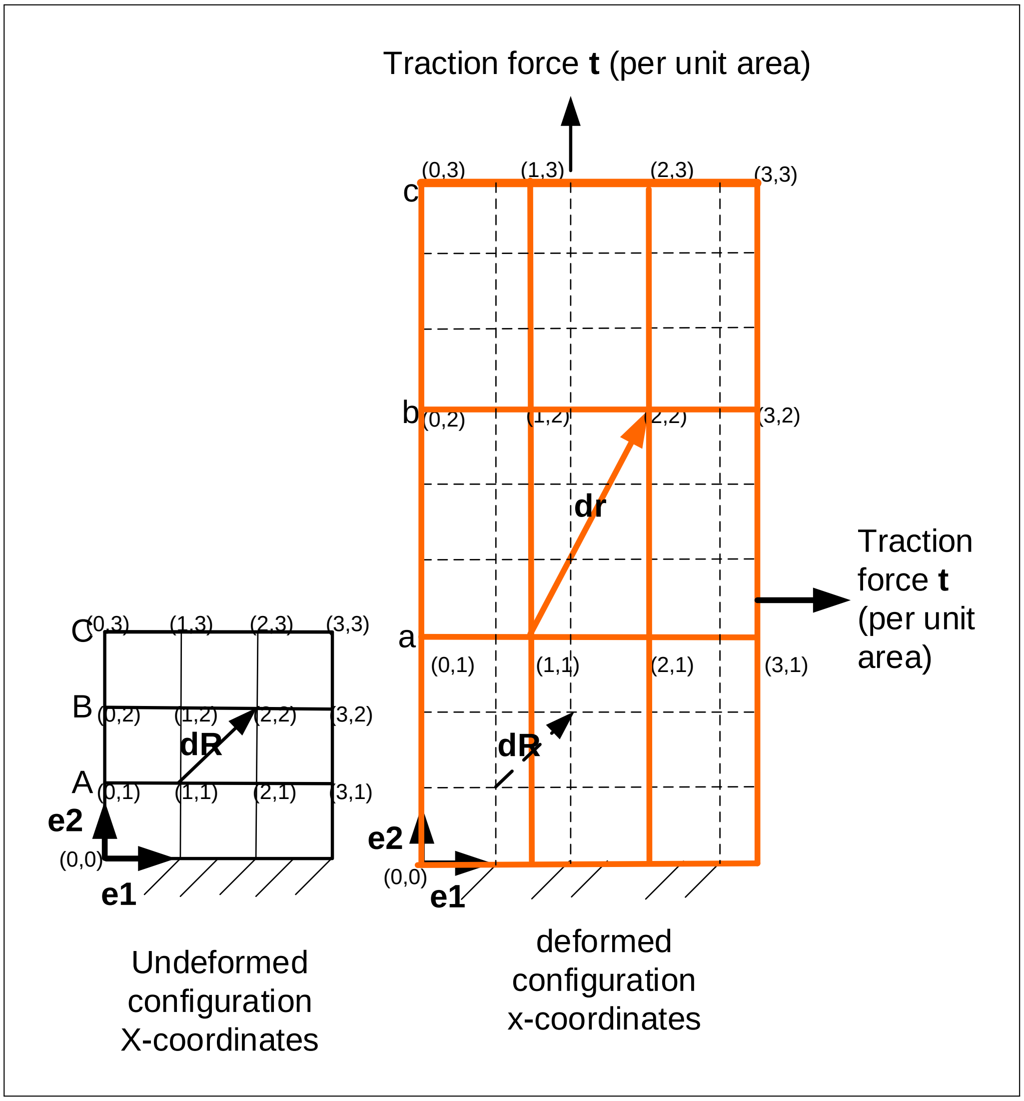

Figure 4: Square shape becomes both longer and wider

As before, the first step in finding $\tilde{\mathbf{F}}$ is to determine the mapping between the X coordinates in the undeformed shape, and the x coordinates in the deformed shape. In this example, this mapping is constant over any region of the shape. We see that

$$x_1 = 1.5X_1$$

and since the new shape is 3 times as long as before then

$$x_2 = 3X_2$$

And now we can calculate $\tilde{\mathbf{F}}$. Since

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}$$

then given that $\frac{\partial x_1}{\partial X_1} = 1.5$, $\frac{\partial x_1}{\partial X_2} = 0$, $\frac{\partial x_2}{\partial X_1} = 0$, $\frac{\partial x_2}{\partial X_2} = 3$ we obtain numerical value for $\tilde{\mathbf{F}}$

$$\tilde{\mathbf{F}} = \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix}$$

Now let us find $\mathbf{dr}$.

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR}$$

From the undeformed shape we see that

$$\mathbf{dR} = \mathbf{e}_1 + \mathbf{e}_2$$

Hence

$$\begin{aligned} \mathbf{dr} &= \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \end{aligned}$$

hence,

$$\mathbf{dr} = 1.5\mathbf{e}_1 + 3\mathbf{e}_2$$

Looking at the deformed shape we see that this is indeed the case.

Now once $\tilde{\mathbf{F}}$ is found, we can determine the stretch tensor $\tilde{\mathbf{U}}$ and the rotation tensor $\tilde{\mathbf{R}}$. We will do this algebraically first, then verify the result geometrically.

$$\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$$

Once $\tilde{\mathbf{F}}$ is known, we can find $\tilde{\mathbf{U}}$

$$\begin{aligned} \tilde{\mathbf{U}}^2 &= \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}} \\ &= \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2.25 & 0 \\ 0 & 9 \end{bmatrix} \end{aligned}$$

Hence

$$\tilde{\mathbf{U}} = \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix}$$

and now that $\tilde{\mathbf{U}}$ is known, we can find $\tilde{\mathbf{R}}$

$$\begin{aligned} \tilde{\mathbf{R}} &= \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}^{-1} \\ &= \begin{bmatrix} 1.5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

To verify the result geometrically, we first apply the stretch $\tilde{\mathbf{U}}$ to $\mathbf{dR}$, this results in a new differential vector which we call $\mathbf{dr}^*$, then we slide $\mathbf{dr}^*$ without changing its slope (i.e. parallel translation) such that the vector $\mathbf{dr}^*$ starts at the point P' in the deformed configuration, where the point P' is the image of the point P, and then we apply the rotation $\tilde{\mathbf{R}}$ to $\mathbf{dr}^*$ to obtain $\mathbf{dr}$. Hence

$$\begin{aligned}\mathbf{dr}^* &= \tilde{\mathbf{U}} \cdot \mathbf{dR} = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}\\ &= \mathbf{e}_1 + \mathbf{3e}_2\end{aligned}$$

Now we apply the rotation of $\tilde{\mathbf{R}} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ to $\mathbf{dr}^*$, and since the rotation is a unit tensor, then no rotation will occur.

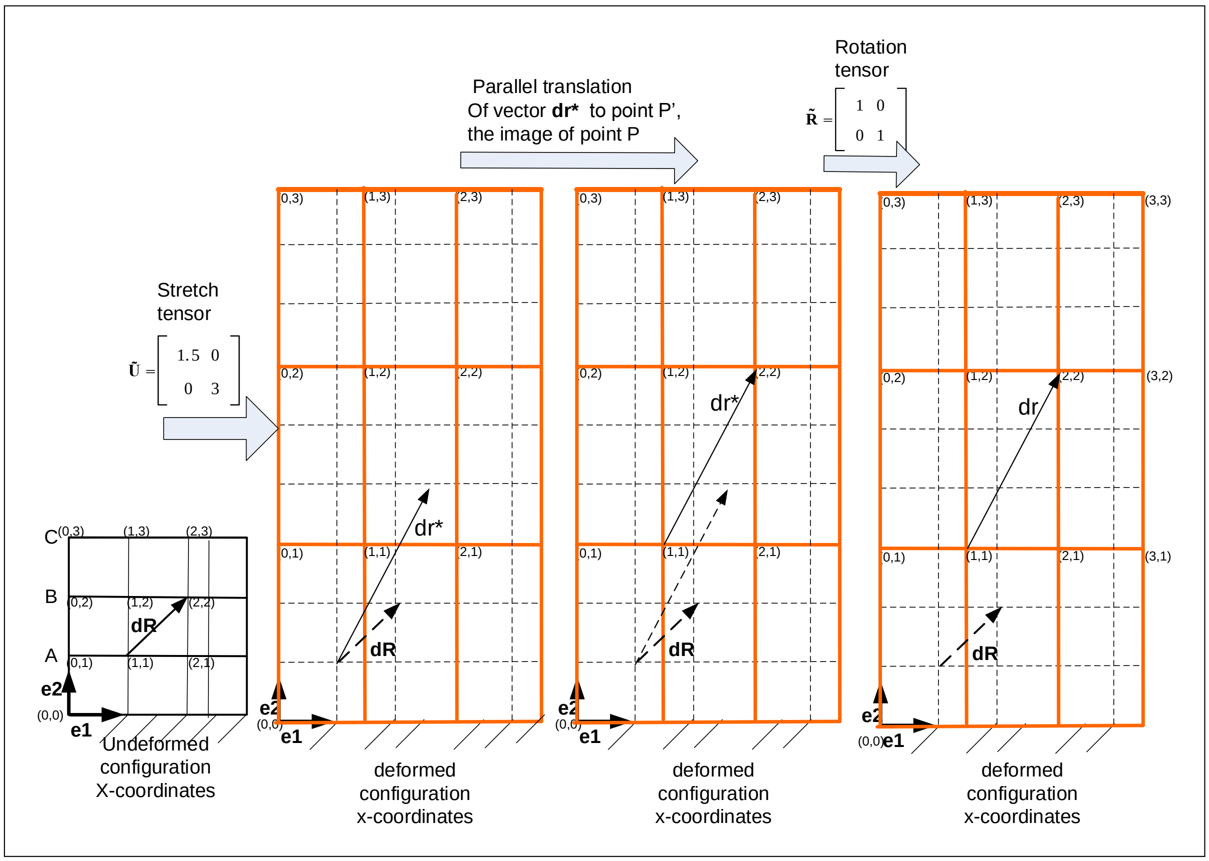

Figure 5: after applying the rotation

2.3 square shape becomes wider and pulled at an angle.

In this example, the same undeformed shape shown in earlier examples will be deformed to cause the rotation tensor to be something other than the identity tensor. We assume the following deformation

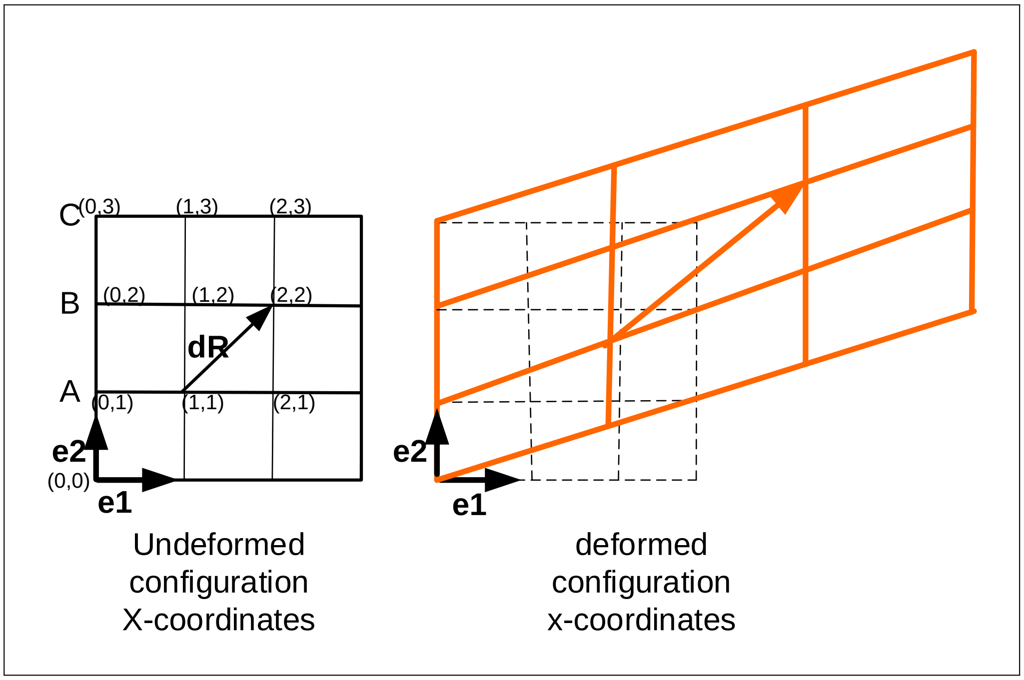

Figure 6: deformation assumed

The above deformation is constructed such that

$$x_1 = 2X_1$$
$$x_2 = X_1 + X_2$$

Now we can calculate $\tilde{\mathbf{F}}$. Since

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}$$

then given that $\frac{\partial x_1}{\partial X_1} = 2$, $\frac{\partial x_1}{\partial X_2} = 0$, $\frac{\partial x_2}{\partial X_1} = 1$, $\frac{\partial x_2}{\partial X_2} = 1$ we obtain numerical value for $\tilde{\mathbf{F}}$

$$\tilde{\mathbf{F}} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Now we can find $\mathbf{dr}$.

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR}$$

From the undeformed shape we see that

$$\mathbf{dR} = \mathbf{e}_1 + \mathbf{e}_2$$

Hence

$$\begin{aligned} \mathbf{dr} &= \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{aligned}$$

Therefore

$$\mathbf{dr} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

Looking at the deformed shape we see that this is indeed the case. Now once $\tilde{\mathbf{F}}$ is found, we can determine the stretch tensor $\tilde{\mathbf{U}}$ and the rotation tensor $\tilde{\mathbf{R}}$.

We will do this algebraically first, then verify the result geometrically.

$$\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$$

Once $\tilde{\mathbf{F}}$ is known, we can find $\tilde{\mathbf{U}}$

$$\begin{aligned}\tilde{\mathbf{U}}^2 =& \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}} \\ =& \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

Hence

$$\tilde{\mathbf{U}} = \begin{bmatrix} 2.2136 & 0.3162 \\ 0.3162 & 0.9487 \end{bmatrix}$$

and now that $\tilde{\mathbf{U}}$ is known, we can find $\tilde{\mathbf{R}}$

$$\begin{aligned}\tilde{\mathbf{R}} =& \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}^{-1} \\ \tilde{\mathbf{R}} =& \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4743 & -0.1581 \\ -0.1581 & 1.1068 \end{bmatrix} \\ \tilde{\mathbf{R}} =& \begin{bmatrix} 0.9487 & -0.3162 \\ 0.3162 & 0.9487 \end{bmatrix}\end{aligned}$$

To verify the result geometrically, we first apply the stretch tensor $\tilde{\mathbf{U}}$ to $\mathbf{dR}$, this results in a new differential vector which we call $\mathbf{dr}^*$, then we slide $\mathbf{dr}^*$ without changing its slope (i.e. parallel translation) such that the vector $\mathbf{dr}^*$ starts at the point P' in the deformed configuration, where the point P' is the image of the point P, and then we apply the rotation tensor $\tilde{\mathbf{R}}$ to $\mathbf{dr}^*$ to obtain $\mathbf{dr}$.

Hence

$$\begin{aligned}\mathbf{dr}^* = \tilde{\mathbf{U}} \cdot \mathbf{dR} =& \begin{bmatrix} 2.2136 & 0.3162 \\ 0.3162 & 0.9487 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5298 \\ 1.2649 \end{bmatrix} \\ =& 2.5298\ \mathbf{e}_1 + 1.2649\ \mathbf{e}_2\end{aligned}$$

Now we apply the rotation to $\tilde{\mathbf{R}}$ to $\mathbf{dr}^*$ to obtain $\mathbf{dr}$

$$\begin{aligned}\mathbf{dr} =& \tilde{\mathbf{R}} \cdot \mathbf{dr}^* \\ =& \begin{bmatrix} 0.9487 & -0.3162 \\ 0.3162 & 0.9487 \end{bmatrix} \begin{bmatrix} 2.5298 \\ 1.2649 \end{bmatrix} \\ =& \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ =& 2\mathbf{e}_1 + 2\mathbf{e}_2\end{aligned}$$

which agrees with the result obtained above.

The following diagram illustrates geometrically the action of $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{U}}$.

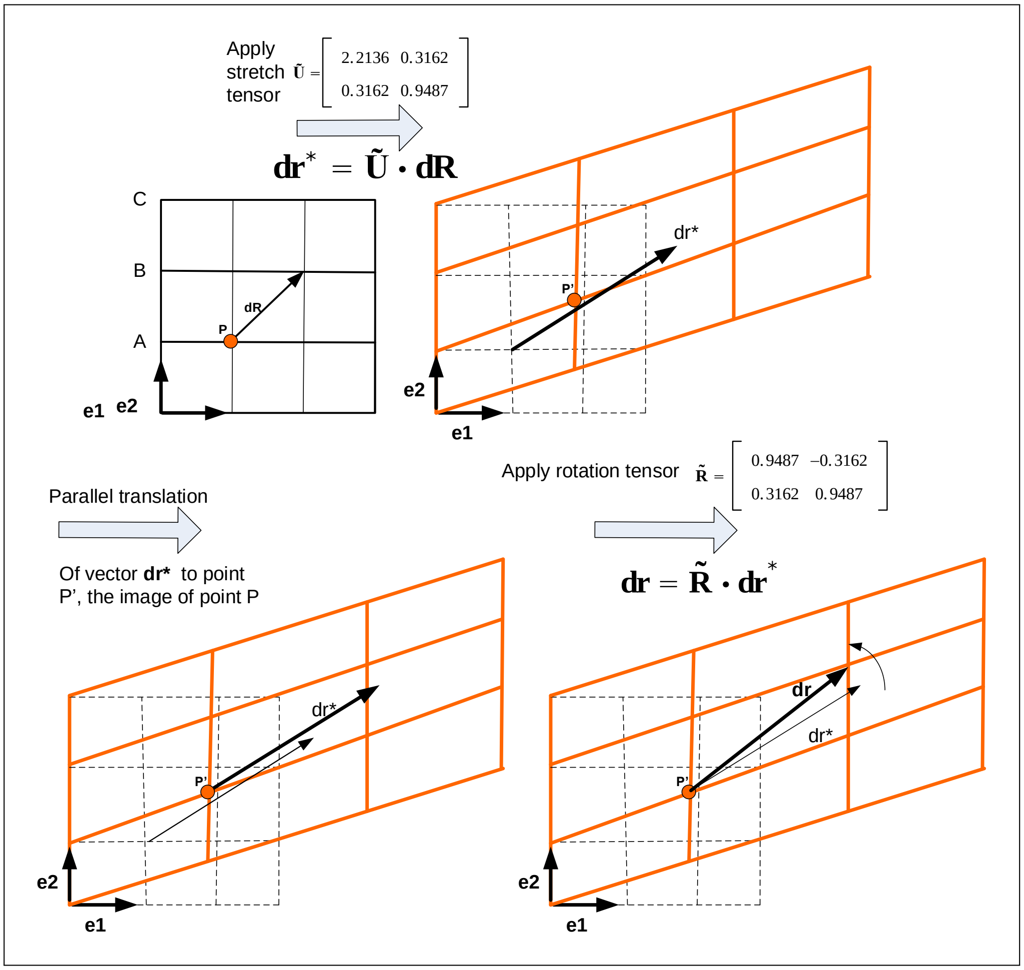

Figure 7: final result