Collection of papers related to finding integrating factors and first integrals for second and higher order differential equations

Nasser M. Abbasi

September 7, 2023 Compiled on September 7, 2023 at 2:32pm

Contents

These are collection of papers related to finding integrating factors for higher order differential equations. Mainly for second order ode's and higher. As I was studying this subject I thought it will be good idea to have any related paper I find on the subject in one place for easy access. The HTML version has only the outlines. Only the PDF version has the actual papers shown.

1 Classification and integration of ordinary differential equations between xy allowing a set of transformations. Sophus Lie (Mathematische Annalen volume 32, pages 21328 1888)

Classification und Integration von gewöhnlichen Differentialgleichungen zwischen xy, die eine Gruppe von Transformationen gestatten.

Von

Sophus Lie in Leipzig.

(Die nachstehende Arbeit erschien zum ersten Male im Frühling 1883 im norwegischen Archiv.)

In einer kurzen Note zur Gesellschaft der Wissenschaften in Göttingen (3. December 1874) gab ich u. A. eine Aufzählung aller continuirlichen Gruppen von Transformationen zwischen zwei Variabeln x und y. Ich lenkte ausdrücklich und stark die Aufmerksamkeit darauf, dass sich hierauf eine Classification und eine rationelle Integrationstheorie aller Differentialgleichungen

$$f(xyy' \cdots y^{(m)}) = 0,$$

die eine continuirliche Transformationsgruppe gestatten, begründen lässt. Später habe ich nun das hiermit scizzirte grosse Programm mehr im Detail ausgeführt. So gab ich in den Abhandlungen der Gesellschaft der Wissenschaften zu Christiania (1874*)) eine rationelle Methode zur Integration von linearen partiellen Differentialgleichungen mit einer Reihe bekannter infinitesimaler Transformationen; hiermit hatte ich dann gleichzeitig eine vollständige Integrationstheorie von gewöhnlichen Differentialgleichungen

$$f(xyy' \cdots y^{(m)}) = 0$$

mit *bekannten* infinitesimalen Transformationen erhalten. Ich gab ferner in mehreren Abhandlungen**) in diesem Archiv (1876, 78) eine Darstellung von denjenigen Methoden, vermöge deren ich in den

---

*) Die betreffende Arbeit ist im Wesentlichen reproducirt in den Math. Ann. Bd. XI.

**) Diese Abhandlungen sind theilweise (aber nicht vollständig) in den Math. Ann. Bd. XVI in neuer Bearbeitung reproducirt worden.

Jahren 1873—1874 alle continuirlichen Gruppen von Transformationen einer zweifach ausgedehnten Mannigfaltigkeit bestimmt hatte.

Hiermit war indess keineswegs mein 1874 scizzirtes Programm, selbst auf gewöhnliche Differentialgleichungen zwischen x und y beschränkt, zur Ausführung gebracht. Nicht allein hatte ich die angekündigte Classification noch nicht durchgeführt, sondern es stand auch noch zurück nachzuweisen, einerseits, wie man entscheidet, ob eine vorgelegte Differentialgleichung eine continuirliche Gruppe gestattet, anderseits wie man diejenigen Differentialgleichungen in rationeller Weise integrirt*), die zur Bestimmung der betreffenden Gruppe dienen. Der Hauptzweck dieser Abhandlung ist, diese beiden wichtigen Capitel meiner Theorie eingehend zu entwickeln. Gleichzeitig halte ich es für zweckmässig, einige Theile meiner Theorie, die ich allerdings früher schon im Wesentlichen gegeben habe, auf's Neue und mehr ausführlich zu behandeln.

Im ersten Abschnitte führe ich die von mir 1874 angekündigte Classification von gewöhnlichen Differentialgleichungen, die eine continuirliche Gruppe von Transformationen zwischen x und y gestatten, vollständig durch. Ich betrachte successiv alle derartigen Gruppen, reducirt auf canonische Formen, und bestimme die zugehörigen invarianten Differentialgleichungen**). Darnach zeige ich, dass eine beliebige vorgelegte Gruppe im Allgemeinen ohne Integration von Differentialgleichungen und jedenfalls durch Integration einer Differentialgleichung 1. O. auf ihre canonische Form gebracht werden kann. Hierdurch gelingt es, alle bei einer beliebig vorgelegten Gruppe invariante Differentialgleichungen anzugeben***).

---

*) Siehe die Abhandlungen der Gesellschaft der Wissenschaften zu Christiania 1881. Siehe auch meine Begründung einer Invariantentheorie der Berührungstransformationen. Math. Ann. Bd. VIII, 1874.

**) Die Gesichtspunkte der citirten Note führen indess noch weiter. Man kann u. A. jede Gruppe von *Berührungs*transformationen zwischen xyy' auf gewisse von mir bestimmte canonische Formen bringen, und darnach die zu jeder canonischen Form entsprechenden invarianten Gleichungen angeben u. s. w.

***) Als ich 1874 in meiner mehrmals besprochenen Note hervorhob, dass auf meine Bestimmung aller Gruppen von Transformationen der Ebene eine Classification aller Gleichungen $f(xyy' \dots y^{(m)}) = 0$ mit einer Gruppe gegründet werden kann, hatte ich diese Classification noch nicht im Detail ausgeführt. Ich hatte die Möglichkeit einer Classification, d. h. die Möglichkeit der Aufstellung der Typen aller Differentialgleichungen $f = 0$, die eine Gruppe gestatten, erkannt. Die hierzu erforderlichen Rechnungen hatte ich aber nicht im Detail ausgeführt, und noch weniger publicirt. Indem ich dies ausdrücklich hervorhebe, bemerke ich, dass der berühmte französische Geometer Halphen in seinen ausgezeichneten Untersuchungen über Differentialinvarianten (Liouvilles Journal Bd. 2 (Serie 3) 1876, Sur les invariants diff., Thèse, Paris 1878 u. s. w.) im Grunde einen wichtigen, wenn auch sehr speciellen Theil meines Programms ausgeführt

Im zweiten Abschnitte dieser Arbeit wende ich meine allgemeine längst publicirte Integrationstheorie von linearen partiellen Differentialgleichungen mit bekannten infinitesimalen Transformationen auf gewöhnliche Differentialgleichungen $f(xyy' \cdots y^{(m)}) = 0$ mit einer *bekannten* Gruppe an. Derartige Gleichungen können in zwei etwas verschiedenen Weisen behandelt werden. Entweder kann man meine allgemeine Theorie direct anwenden und muss dann successiv eine Reihe vollständiger Systeme aufstellen. Oder auch man fängt damit an, die vorgelegte Gruppe auf ihre canonische Form zu bringen; dadurch erhält $f = 0$ ebenfalls eine canonische Form; hierbei hat man nur gewöhnliche Differentialgleichungen zwischen zwei Variabeln zu behandeln. Die Entwickelungen dieses Abschnittes sind grösstentheils nur als Beispiele und Illustrationen zu meiner alten allgemeinen Theorie zu betrachten.

Im dritten Abschnitte denke ich mir eine ganz beliebige Gleichung $f(xyy' \cdots) = 0$ vorgelegt und stelle die Frage, ob dieselbe infinitesimale Transformationen gestattet. Ist dies der Fall, so werden diese Transformationen bestimmt durch gewisse lineare partielle Differentialgleichungen erster und höherer Ordnung, deren Integration in den meisten Fällen durch successive Quadraturen oder durch Integration einer Riccatischen Gleichung 1. O. geleistet werden kann. Es giebt nur zwei Fälle, in denen die Bestimmung der infinitesimalen Transformationen von $f = 0$ nicht in dieser einfachen Weise geleistet werden kann. Wenn $f = 0$ eine Gruppe gestattet, als deren canonische Form die allgemeine projective Gruppe der Ebene gewählt werden kann, so verlangt die Bestimmung dieser Gruppe im Allgemeinen die Integration einer linearen Gleichung dritter Ordnung. Gestattet andererseits $f = 0$ eine Gruppe, als deren canonische Form die Gruppe einer linearen Gleichung*) gewählt werden kann, so verlangt die Bestimmung unserer Gruppe die Integration einer gewöhnlichen linearen Differentialgleichung.

---

hat (siehe § 1, Nummer 3 dieser Arbeit) allerdings mit schönen Anwendungen, die mir theilweise ferner lagen. Halphen macht aufmerksam auf die Beziehungen zwischen seinen Untersuchungen und Klein's und meinen gemeinsamen früheren Untersuchungen über solche Curven, die eine infinitesimale lineare Transformation gestatten. Dagegen kannte er nicht meine anderen viel weiter reichenden Arbeiten, insbesondere nicht meine Note in den Göttinger Nachr., wie auch nicht meine 1874 publicirte Theorie der Integration von linearen partiellen Differentialgleichungen, die eine bekannte Gruppe von Transformationen gestatten.

*) Besonders merkwürdig ist der Fall, dass die lineare Gleichung des Textes in eine mit constanten Coefficienten sich umwandeln lässt. In diesem Falle geschieht wiederum die Bestimmung der gesuchten inf. Transformationen durch Quadratur; kann jedoch die besprochene lineare Gleichung mit constanten Coefficienten die Form $y^{(r)} = 0$ erhalten, so ist die Integration einer Riccatischen Gleichung 1. O. erforderlich.

In weiteren Abschnitten gedenke ich einige verwandte Theorien, die ich schon seit einiger Zeit im Detail ausgeführt habe, zu entwickeln. Insbesondere werde ich meine Theorien auf solche Gleichungen $f(xy\cdots y^{(m)})=0$ anwenden, in denen die Grösse $y^{(m-1)}$ nicht vorkommt. Anderseits werde ich alle Gruppen von Berührungstransformationen der Ebene in canonischer Form betrachten, und ihre invarianten Differentialgleichungen aufstellen; hieran schliesst sich eine rationelle Integrationstheorie solcher Gleichungen $f=0$, die eine beliebige Gruppe von *Berührungs*transformationen gestatten.

Abschnitt I.

Classification aller gewöhnlichen Differentialgleichungen zwischen xy, die eine Gruppe von Transformationen zwischen diesen Variabeln gestatten.

Bestimmen die Gleichungen

$$x_1=f(xya_1a_2\cdots a_r)$$
$$y_1=\varphi(xya_1a_2\cdots a_r)$$

zwischen den alten Variabeln xy, den neuen Variabeln x_1y_1 und den Parametern $a_1a_2\cdots a_r$ eine (continuirliche) Gruppe von Transformationen, so liefert eine Relation der Form

$$\Omega(f\varphi b_1\cdots b_\varrho)=0$$

mit $r+\varrho$ Parametern $a_1\cdots a_r\ b_1\cdots b_\varrho$ eine Schaar und zwar die allgemeinste Schaar von Curven, deren Inbegriff die vorgelegte Gruppe gestattet. Dies folgt unmittelbar aus dem Begriffe Transformationsgruppe. Zu bemerken ist allerdings dabei, dass die $r+\varrho$ Parameter a_i, b_k nicht sämmtlich wesentlich zu sein brauchen*).

Wählt man die Function Ω in bestimmter Weise, so kann man durch wiederholte Differentiation hinsichtlich x so viele Gleichungen zwischen xy, den Differentialquotienten

$$y^{(i)}=\frac{d^iy}{dx^i}$$

---

*) Die $r+\varrho$ Parameter sind wesentlich, wenn eine beliebige Curve der Schaar

$$\Omega(xyb_1\cdots b_\varrho)=0$$

mit ϱ wesentlichen Parametern durch keine infinitesimale Transformation der Gruppe in sich transformirt wird und auch nicht in eine benachbarte Curve dieser Schaar übergeführt wird; giebt es dagegen σ unabhängige infinitesimale Transformationen der Gruppe, welche eine beliebige Curve der Schaar wiederum in eine Curve der Schaar überführt, so sind unter den $r+\varrho$ Parametern $a_1\cdots a_\varrho\ b_1\cdots b_\varrho$ nur $r+\varrho-\sigma$ wesentlich.

und den Parametern $a_i b_k$ bilden, dass es möglich wird, diese Parameter wegzuschaffen. Hierdurch findet man in jedem einzelnen Falle eine Differentialgleichung, die unsere Gruppe gestattet. Und offenbar kann jede derartige Differentialgleichung in dieser Weise gebildet werden. Diese Methode ist indess nicht zweckmässig, indem sie uns keine Uebersicht über die Gestalt und die Eigenschaften der betreffenden Differentialgleichungen liefert. Zweckmässiger ist es, wie ich seit 1874 bei allen meinen Untersuchungen über Transformationsgruppen zu thun pflege, die *infinitesimalen* Transformationen der Gruppe einzuführen, und vermöge derselben die Bestimmung der betreffenden Differentialgleichungen durchzuführen.

Unsere Gruppe mit den r Parametern a_k enthält nach mir r unabhängige infinitesimale Transformationen*), etwa

$$B_i f = \xi_i(xy)\frac{df}{dx} + \eta_i(xy)\frac{df}{dy}.$$
$$(i = 1, 2 \cdots r)$$

Bei einer solchen inf. Transformation erhält x das Increment $\delta x = \xi_i \delta t$, y das Increment $\delta y = \eta_i \delta t$; gleichzeitig erhält y' ein Increment $\delta y'$, y'' ein Increment $\delta y''$ und überhaupt $y^{(i)}$ ein Increment $\delta y^{(i)}$. Wir werden diese Incremente berechnen. Es ist

$$\frac{\delta y'}{\delta t} = \frac{\delta}{\delta t}\,\frac{dy}{dx} = \frac{dx\,\frac{\delta}{\delta t}\,dy - dy\cdot\frac{\delta}{\delta t}\,dx}{dx^2}$$

oder

$$\frac{\delta y'}{\delta t} = \frac{dx\cdot d\,\frac{\delta y}{\delta t} - dy\cdot d\,\frac{\delta x}{\delta t}}{dx^2} = \frac{d\eta - y'\,d\xi}{dx}$$
$$= \frac{\partial\eta}{\partial x} + y'\left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right) - y'^2\,\frac{\partial\xi}{\partial y} = \eta^{(1)}.$$

Dementsprechend ist

$$\frac{\delta y''}{\delta t} = \frac{d\eta^{(1)} - y''\,d\xi}{dx} = \eta^{(2)}$$

und überhaupt

$$\frac{\delta y^{(m)}}{\delta t} = \frac{d\eta^{(m-1)} - y^{(m)}\,d\xi}{dx} = \eta^{(m)}.$$

---

*) Die infinitesimale Transformation, bei der x und y die Incremente

$$\delta x = \xi(xy)\,\delta t,\quad \delta y = \eta(xy)\,\delta t$$

erhalten, bezeichne ich immer mit dem Symbol

$$\xi\,\frac{\partial f}{\partial x} + \eta\,\frac{\partial f}{\partial y}.$$

Setze ich nun

$$B_i^{(m)} f = \xi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)} \frac{\partial f}{\partial y^{(m)}},$$
$$(i = 1, 2 \cdots r)$$

so sind $B_1^{(m)}f$, $B_2^{(m)}f$, $\cdots$ $B_r^{(m)}f$ die r infinitesimalen Transformationen unserer Gruppe, aufgefasst als transformirend nicht allein xy, sondern zugleich die Differentialquotienten $y' \cdots y^{(m)}$. Und also (Götting. Nachr. 1874, p. 537; Math. Ann. XVI, p. 462—463) bestehen $\frac{r(r-1)}{1 \cdot 2}$ Relationen von der Form

$$B_i^{(m)}\left(B_k^{(m)}(f)\right) - B_k^{(m)}\left(B_i^{(m)}(f)\right) = \sum c_{iks} B_s^{(m)} f,$$

in denen die c_{iks} Constanten sind, die überdies von der Zahl m unabhängig sind.

Soll nun eine Differentialgleichung

$$f(xyy' \cdots y^{(m)}) = 0$$

unsere Gruppe gestatten, so ist hierzu erforderlich und auch hinreichend, dass sie die r inf. Transformationen $B_i^{(m)}f$ gestattet; denn dann gestattet $f = 0$ jede infinitesimale Transformation $\Sigma c_k B_k^{(m)} f$ der Gruppe und also zugleich jede endliche Transformation derselben, die ja durch Wiederholung einer inf. Transformation erzeugt werden kann. Und dies kommt darauf hinaus, dass die r Gleichungen

$$(1) \qquad \xi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)} \frac{\partial f}{\partial y^{(m)}} = 0$$

vermöge $f = 0$ identisch bestehen sollen.

Die weitere Discussion stellt sich verschieden jenachdem r gleich, kleiner oder grösser als $m + 2$ ist.

Nehmen wir zunächst an, dass $m + 2 = r$ ist. Dann ist zum Bestehen der r Gleichungen (1) erforderlich, dass die Determinante

$$\Delta = |\xi_i \eta_i \eta_i^{(1)} \cdots \eta_i^{(r-2)}|$$

verschwindet. Dabei können wir vorläufig von dem Falle, dass Δ identisch verschwindet, absehen, indem dies, wie wir später zeigen, nur ganz ausnahmsweise eintritt. Daher muss die Gleichung $\Delta = 0$ vermöge $f = 0$ bestehen. Es ist anderseits nicht schwierig zu beweisen, dass die Differentialgleichung $\Delta = 0$ immer unsere Transformationsgruppe gestattet. Für eine synthetische Auffassung ist dies unmittelbar evident. Ein Werthsystem $xyy' \cdots y^{(r-2)}$ genügt nämlich der Gleichung $\Delta = 0$ dann und nur dann, wenn dasselbe nicht vermöge der Gruppe in jedes benachbarte Werthsystem übergeführt werden kann. Wünscht man einen analytischen Beweis, so bemerke ich, dass ich für den Fall $m = 1$ schon einen solchen Beweis geliefert habe (siehe Math. Ann. Bd. XVI p. 475), dass ferner der Beweis für einen allgemeinen Werth von m in ganz entsprechender Weise geführt wird. Hierauf näher einzugehen halte ich hier nicht für nothwendig. Ich bemerke nur, dass man im Folgenden bei jeder Anwendung des betreffenden Satzes seine Richtigkeit leicht direct verificirt.

Wir wollen sodann annehmen, dass $m + 2 < r$ ist. Dann ist zum Bestehen der Gleichungen (1) erforderlich, dass alle in der Matrix

$$|\xi_i \, \eta_i \, \eta_i^{(1)} \cdots \eta_i^{(m)}|$$

enthaltenen $(m+2)$-reihigen Determinanten gleichzeitig verschwinden; und da dieselben nicht identisch gleich Null sein können, indem Δ nach unserer Voraussetzung nicht identisch verschwinden soll, so müssen die soeben besprochenen Determinanten, die offenbar ganze Functionen der Grössen $y^{(k)}$ sind, einen gemeinsamen Factor (Δ) enthalten; dabei ist klar, dass diese Grösse (Δ) ebenfalls ein Factor von Δ sein muss. Dies giebt uns nun zunächst den Satz:

Satz. *Sucht man alle bei der vorgelegten Gruppe $B_1 f \cdots B_r f$ invarianten Differentialgleichungen*

$$f(xyy' \cdots y^{(m)}) = 0,$$

deren Ordnungszahl m nicht grösser als $r - 2$ ist, so muss man die Determinante

$$\Delta = |\xi_i \, \eta_i \, \eta_i^{(l)} \cdots \eta_i^{(r-2)}|$$

bilden. Verschwindet dieselbe nicht identisch, so liefern ihre Factoren gleich Null gesetzt die gesuchten Differentialgleichungen.

Als Corollar fliesst hieraus der Satz:

Gestattet eine Differentialgleichung m^{ter} Ordnung $f = 0$ $m + 2$ oder noch mehr infinitesimale Punkttransformationen, so kann man ohne Beschränkung annehmen, dass f eine ganze Function der Grössen $y^{(i)}$ ist.

Dieser Satz ist im Vorangehenden nur unter der Voraussetzung erwiesen, dass die Determinante Δ nicht identisch verschwindet. Derselbe ist indess allgemein gültig, wie wir später nachweisen werden.

Es erübrigt noch alle bei der Gruppe invarianten Differentialgleichungen $f(xyy' \cdots y^{(m)}) = 0$, deren Ordnungszahl m grösser als $r - 2$ ist, zu finden. Dabei schliessen wir wie früher vorläufig den Ausnahmefall $\Delta = 0$ aus. Unter dieser Voraussetzung bilden die r Gleichungen (1) nach meinem früher citirten Satze ein vollständiges System mit $m + 2 - r$ gemeinsamen Lösungen. Sei zunächst $m + 2 = r + 1$, dann giebt es eine Lösung φ_1, die durch Integration gefunden wird*). Dabei hängt φ_1 nur von $xyy' \cdots y^{(r-1)}$ ab. Sei

*) Diese Integration kann immer geleistet werden, wenn die *endlichen* Transformationen der Gruppe bekannt sind.

darnach $m+2=r+2$, dann giebt es zwei Lösungen, *unter denen* φ_1 *die eine ist;* die zweite Lösung φ_2 hängt von $xyy'\cdots y^{(r)}$ ab. Ist $m+2=r+3$, so giebt es drei Lösungen φ_1, φ_2 und φ_3, welch' letztere von $xyy'\cdots y^{(r+1)}$ abhängt. Für einen beliebigen Werth von m giebt es $m-r+2$ Lösungen $\varphi_1\varphi_2\cdots\varphi_{m-r+2}$. Es ist nun leicht zu sehen, dass *jedenfalls* nur die beiden ersten φ_k, nämlich φ_1 und φ_2, durch Integration bestimmt zu werden brauchen. Kennt man φ_1 und φ_2, so findet man die übrigen φ_k folgendermassen durch Differentiation.

Es ist die Gleichung

$$\varphi_2 - a\varphi_1 + b = 0$$

mit den beiden arbiträren Constanten a und b eine invariante Differentialgleichung r^{ter} Ordnung. Differentiirt man nun hinsichtlich x, so ist die hervorgehende Gleichung

$$\frac{d\varphi_2}{dx} - a\frac{d\varphi_1}{dx} = 0$$

oder die äquivalente

$$\frac{\frac{d\varphi_2}{dx}}{\frac{d\varphi_1}{dx}} = a = \frac{d\varphi_2}{d\varphi_1}$$

eine invariante Gleichung $(r+1)^{\text{ter}}$ Ordnung mit einer arbiträren Constante.

Also kann die Grösse

$$\frac{d\varphi_2}{dx} : \frac{d\varphi_1}{dx}$$

als Grösse φ_3 gewählt werden. Dementsprechend kann

$$\frac{d\varphi_3}{dx} : \frac{d\varphi_1}{dx}$$

als Grösse φ_4 gewählt werden u. s. w. Dieses Bildungsgesetz zeigt, dass φ_3 hinsichtlich $y^{(r+1)}$, dass φ_4 hinsichtlich $y^{(r+2)}$ linear ist u. s. w.

Satz. *Jede bei der Gruppe $B_1f\cdots B_rf$ invariante Differentialgleichung, deren Ordnung grösser als $r-2$ ist, besitzt die Form*

$$\Omega(\varphi_1\varphi_2\varphi_3\cdots) = 0.$$

Zuletzt nur noch einige weitere Bemerkungen über invariante Differentialgleichungen

$$f(xy\cdots y^{(\varrho)}) = 0,$$

deren Ordnung ϱ nicht $r-1$ übersteigt. Nehmen wir wieder an, dass Δ nicht identisch verschwindet, und sei Δ_i ein Factor von Δ, der von den Grössen $xyy' \ldots y^{r-2-i}$ abhängt. Dann enthält das Integral von $\Delta_i = 0$, $r - i - 2$ arbiträre Constanten:

$$\varphi(xy\alpha_1 \ldots \alpha_{r-i-2}) = 0,$$

d. h. die Gleichung $\Delta_i = 0$ hat ∞^{r-i-2} Integralcurven. Bei den Transformationen der Gruppe werden diese Integralcurven unter sich vertauscht, und zwar wird jede einzelne Integralcurve durch $i + 2$ unabhängige infinitesimale Transformationen der Gruppe in sich selbst transformirt.

Ist insbesondere Δ_0 ein Factor von Δ, der die Grösse $y^{(r-2)}$ enthält*), so ist $\Delta_0 = 0$ eine invariante Differentialgleichung, von deren Integralcurven jede *zwei* unabhängige infinitesimale Transformationen der Gruppe gestattet. Besitzt Δ keinen Factor Δ_0, der $y^{(r-2)}$ wirklich enthält, so heisst dies, dass es keine Curve giebt, die zwei und nur zwei infinitesimale Transformationen unserer Gruppe gestattet.

Betrachten wir endlich die invariante Differentialgleichung $(r-1)^{\text{ter}}$ Ordnung

$$\varphi_1 = a_0,$$

deren arbiträre Constante a_0 einen bestimmten Werth erhalten hat. Diese Differentialgleichung hat ∞^{r-1} Integralcurven, die durch die r unabhängigen infinitesimalen Transformationen der Gruppe unter sich vertauscht werden. Also schliessen wir, dass jede Integralcurve durch eine und nur eine infinitesimale Transformation der Gruppe in sich transformirt wird.

In den drei ersten Paragraphen dieses Abschnittes betrachten wir successiv alle Gruppen von Punkttransformationen der Ebene, indem wir sie auf die von mir bestimmten canonischen Formen gebracht voraussetzen. Für jede solche canonische Gruppe bestimmen wir die zugehörigen invarianten Differentialgleichungen. In dem letzten Paragraphen zeigen wir, wie eine beliebige vorgelegte Gruppe auf ihre canonische Form gebracht wird.

§ 1.

Gruppen, die keine Differentialgleichung 1. O. invariant lassen.

In meiner Aufzählung aller Gruppen von Punkttransformationen einer Ebene (Göttinger Nachr. 1874, Math. Ann. Bd. XVI) theilte ich alle derartigen Gruppen in gewisse Hauptclassen, jenachdem die betreffende Gruppe keine, eine, oder mehrere Differentialgleichungen *erster* Ordnung invariant lässt.

In diesem Paragraphen betrachte ich jede Gruppe, die keine Differentialgleichung erster Ordnung invariant lässt, und bestimme

*) Die Form der Determinante Δ zeigt, dass Δ_0 hinsichtlich $y^{(r-2)}$ linear ist.

alle zugehörigen invarianten Differentialgleichungen höherer Ordnung, unter denen sich immer eine von zweiter Ordnung findet (welche durch passende Coordinatenwahl die lineare Form $y'' = 0$ erhalten kann).

Die betreffende Gruppe enthält entweder acht oder sechs oder fünf Parameter. Sie ist ähnlich mit der allgemeinen projectiven Gruppe der Ebene oder mit einer Untergruppe derselben, die sechs oder fünf Parameter enthält. Wir denken uns im Folgenden unsere Gruppen auf die soeben genannten canonischen Formen gebracht.

1. Jede fünfgliedrige Gruppe, die keine Differentialgleichung erster Ordnung invariant lässt, kann auf die canonische Form*)

$$p, \; q, \; xq, \; xp - yq, \; yp$$

gebracht werden. Die Determinante Δ erhält für diese canonische Form den nicht identisch verschwindenden Werth:

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & x & 1 & 0 & 0 & \\ y & -y & -2y' & -3y'' & -4y''' & \\ y & 0 & -y'^2 & -3y'y'' & -4y'y''' & -3y''^2 \end{vmatrix} = 9y''^3.$$

Daher ist $y'' = 0$ die einzige invariante Differentialgleichung, deren Ordnung kleiner als vier ist.

Zur Bestimmung der Grössen φ_1 und φ_2 bilden wir die folgenden linearen partiellen Gleichungen, in denen wir, wie immer im Folgenden, y_k statt $y^{(k)}$ schreiben:

$$B_1 f = \frac{\partial f}{\partial x} = 0$$

$$B_2 f = \frac{\partial f}{\partial y} = 0$$

$$B_3 f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y_1} = 0$$

$$B_4 f = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - 2y_1\frac{\partial f}{\partial y_1} - 3y_2\frac{\partial f}{\partial y_2} \cdots - 6y_5\frac{\partial f}{\partial y_5} = 0$$

$$B_5 f = y\frac{\partial f}{\partial x} - y_1^2\frac{\partial f}{\partial y_1} - 3y_1y_2\frac{\partial f}{\partial y_2} - (4y_1y_3 + 3y_2^2)\frac{\partial f}{\partial y_3}$$
$$- (5y_1y_4 + 10y_2y_3)\frac{\partial f}{\partial y_4} - (6y_1y_5 + 15y_2y_4 + 10y_3^2)\frac{\partial f}{\partial y_5} = 0$$

---

*) Statt $\frac{\partial f}{\partial z}$ und $\frac{\partial f}{\partial y}$ pflege ich zu schreiben p und q. So z. B. schreibe ich $xp - yq$ statt $x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}$, um die infinitesimale Transformation

$$\delta x = x\delta t, \quad \delta y = -y\delta t$$

zu bezeichnen.

und suchen ihre beiden gemeinsamen Lösungen. Die drei ersten Gleichungen zeigen, dass φ_1 und φ_2 nicht von x, y oder y_1 abhängen. Die beiden letzten Gleichungen erhalten durch Reduction die einfachere Form

$$B_4 f = 3y_2 \frac{\partial f}{\partial y_2} + 4y_3 \frac{\partial f}{\partial y_3} + 5y_4 \frac{\partial f}{\partial y_4} + 6y_5 \frac{\partial f}{\partial y_5} = 0$$

$$B_5 f = 3y_2^2 \frac{\partial f}{\partial y_3} + 10y_2 y_3 \frac{\partial f}{\partial y_4} + (15y_2 y_4 + 10y_3^2) \frac{\partial f}{\partial y_5} = 0.$$

Wir integriren $B_5 f = 0$ in der gewöhnlichen Weise, führen sodann ihre Lösungen

$$y_2, \ \varrho_3 = 3y_2 y_4 - 5y_3^2, \ \varrho_3 = 3y_2^2 y_5 - 5\varrho_2 y_3 - \frac{35}{3} y_3^3 {}^{*})$$

als neue unabhängige Variabeln in

$$B_4 f = B_4 y_2 \cdot \frac{\partial f}{\partial y_2} + B_4 \varrho_2 \frac{\partial f}{\partial \varrho_2} + B_4 \varrho_3 \frac{\partial f}{\partial \varrho_3} = 0$$

ein und erhalten hierdurch die Gleichung

$$3y_2 \frac{\partial f}{\partial y_2} + 8\varrho_2 \frac{\partial f}{\partial \varrho_2} + 12\varrho_3 \frac{\partial f}{\partial \varrho_3} = 0,$$

deren Lösungen

$$\varphi_1 = \frac{\varrho_2}{y_2^{\frac{8}{3}}}, \quad \varphi_2 = \frac{\varrho_3}{y_2^4}$$

eben die gesuchten Grössen φ_1 **) und φ_2 sind. Es bleibt nur übrig, die früher gefundenen Werthe der Grössen ϱ_2 und ϱ_3 einzusetzen. Man bemerkt, dass φ_1 linear hinsichtlich y_4 und anderseits φ_2 linear hinsichtlich y_5 ist.

Da φ_1 hinsichtlich y_2 irrational ist, so kann es zuweilen zweckmässiger sein die Grösse φ_1^3 als φ_1 zu wählen. Eine ähnliche Bemerkung lässt sich mehrmals später machen.

2. Jede sechsgliedrige Gruppe, die keine Differentialgleichung erster Ordnung invariant lässt, kann die canonische Form

$$p, \ q, \ xq, \ yq, \ xp, \ yp$$

erhalten. Die Determinante Δ wird für diese canonische Form gleich

---

*) Die Grösse ϱ_3 erhält durch die Substitution $\varrho_2 = 3y_2 y_4 - 5y_3^2$ den Werth

$$\varrho_3 = 3y_2^2 y_5 - 15y_2 y_3 y_4 + \frac{40}{3} y_3^3.$$

Es ergiebt sich später, dass die Gleichung $\varrho_3 = 0$ eine bemerkenswerthe geometrische Bedeutung besitzt.

**) Man verificirt leicht, dass jede Integralcurve einer Gleichung $\varphi_1 =$ Const. wirklich eine infinitesimale lineare Transformation unserer Gruppe gestattet. Ich erinnere daran, dass *Klein* und ich in einer gemeinsamen Arbeit die Theorie dieser Curven eingehend entwickelt haben.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \\ 0 & y & y_1 & y_2 & y_3 & y_4 \\ x & 0 & -y_1 & -2y_2 & -3y_3 & -4y_4 \\ y & 0 & -y_1^2 & -3y_1y_2 & -4y_1y_3-3y_2^2 & -5y_1y_4-10y_2y_3 \end{vmatrix} = 2y_2^2(5y_3^2 - 3y_2y_4).$$

Es giebt daher zwei invariante Differentialgleichungen, deren Ordnungszahl kleiner als fünf ist, nämlich

$$y_2 = 0, \text{ und } 5y_3^2 - 3y_2y_4 = 0^{*)}.$$

Zur Bestimmung der Grössen φ_1 und φ_2 müssen wir nach unseren gewöhnlichen Regeln sechs lineare partielle Differentialgleichungen zwischen $xyy_1y_2\ldots y_6$ bilden. Drei unter diesen Gleichungen

$$B_1f = \frac{\partial f}{\partial x} = 0, \quad B_2f = \frac{\partial f}{\partial y} = 0, \quad B_3f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y_1} = 0$$

sagen nur aus, dass φ_1 und φ_2 von x, y und y_1 unabhängig sind; die drei übrigen Gleichungen erhalten durch eine einfache Reduction die Form

$$B_4f = y_2\frac{\partial f}{\partial y_2} + y_3\frac{\partial f}{\partial y_3} + \cdots + y_6\frac{\partial f}{\partial y_6} = 0$$

$$B_5f = y_3\frac{\partial f}{\partial y_3} + 2y_4\frac{\partial f}{\partial y_4} + \cdots + 4y_6\frac{\partial f}{\partial y_6} = 0$$

$$B_6f = 3y_2^2\frac{\partial f}{\partial y_3} + 10y_2y_3\frac{\partial f}{\partial y_4} + (15y_2y_4 + 10y_3^2)\frac{\partial f}{\partial y_5} + (21y_2y_5 + 35y_3y_4)\frac{\partial f}{\partial y_6} = 0.$$

Die Lösungen von $B_6f = 0$, nämlich

$$3y_2y_4 - 5y_3^2 = \sigma_2$$

$$3y_2^2y_5 - 5\sigma_2y_3 - \frac{35}{3}y_3^3 = \sigma_3 = 3y_2^2y_5 - 15y_2y_3y_4 + \frac{40}{3}y_3^3$$

$$3y_2^3y_6 - 7\sigma_3y_3 - \frac{70}{3}\sigma_2y_3^2 - 35y_3^4 = \sigma_4 = 3y_2^3y_6 - 21y_2^2y_3y_5 + 35y_2y_3^2y_4 - \frac{35}{3}y_3^4$$

führen wir als neue Variabeln in die Gleichung $B_5f = 0$ ein und bringen sie hierdurch auf die Form

---

*) In der gewählten canonischen Form besteht unsere Gruppe aus allen projectiven Transformationen, bei denen die unendlich entfernte Gerade ihre Lage behält. Die Integralcurven der Gleichung $5y_3^2 = 3y_2y_4$ sind alle Parabeln, d. h. Kegelschnitte, welche jene Gerade berühren. Jede solche Curve gestattet wirklich zwei unabhängige inf. Transformationen unserer Gruppe.

$$2\sigma_2 \frac{\partial f}{\partial \sigma_2} + 3\sigma_3 \frac{\partial f}{\partial \sigma_3} - 4\sigma_4 \frac{\partial f}{\partial \sigma_4} = 0.$$

Die entsprechenden Lösungen, nämlich

$$\frac{\sigma_3}{\sigma_2^{\frac{3}{2}}}, \quad \frac{\sigma_4}{\sigma_2^2},$$

befriedigen als Grössen nullter Ordnung hinsichtlich $y_2 y_3 \ldots y_6$ ebenfalls $B_4 f = 0$ und können daher als die gesuchten Invarianten φ_1 und φ_2

$$\varphi_1 = \frac{\sigma_3}{\sigma_2^{\frac{3}{2}}} \text{*)}, \quad \varphi_2 = \frac{\sigma_4}{\sigma_2^2}$$

gewählt werden. Auch jetzt sind φ_1 und φ_2 ganze Functionen von $y_5 y_6$ und dabei ist φ_1 linear hinsichtlich y_5, φ_2 linear hinsichtlich y_6.

3. Wenn eine achtgliedrige Gruppe keine Differentialgleichung 1. O. invariant lässt, so kann sie auf die canonische Form

$$p,\ q,\ xq,\ yq,\ xp,\ yp,\ x^2p + xyq,\ xyp + y^2q$$

gebracht werden. Die Determinante Δ erhält durch Ausführung und einfache Reduction die Form

$$\Delta = \begin{vmatrix} y_2 & y_3 & y_4 & y_5 & y_6 \\ 0 & y_3 & 2y_4 & 3y_5 & 4y_6 \\ 0 & 3y_2^2 & 10y_2y_3 & 15y_2y_4 + 10y_3^2 & 21y_2y_5 + 35y_3y_4 \\ 0 & 3y_2 & 8y_3 & 15y_4 & 24y_5 \\ 0 & 0 & 6y_2^2 & 30y_2y_3 & 60y_2y_4 + 40y_3^2 \end{vmatrix}.$$

Zur Berechnung derselben subtrahirt man von den Gliedern der dritten Reihe zuerst diejenigen der vierten Reihe, multiplicirt mit y_2, und darnach diejenigen der fünften Reihe, multiplicirt mit $\frac{y_3}{3y_2}$; dann verschwinden alle Glieder der dritten Reihe ausgenommen das letzte, und es wird

$$\Delta = \left(15y_2y_3y_4 - 3y_2^2y_5 - \frac{40}{3}y_3^3\right) \begin{vmatrix} y_3 & 2y_4 & 3y_5 \\ 3y_2 & 8y_3 & 15y_4 \\ 0 & 6y_2^2 & 30y_2y_3 \end{vmatrix}$$

oder

$$\Delta = -2y_2(9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3)^2,$$

sodass Δ nicht identisch gleich Null ist. Es giebt *zwei* invariante

*) Auch jetzt gestattet jede Integralcurve einer Gleichung $\varphi_1 = \text{Const.}$ eine infinitesimale Transformation und gehört somit der von **Klein** und mir untersuchten Categorie an.

Differentialgleichungen, deren Ordnung kleiner als 7 ist*), nämlich $y_2 = 0$ und

(2) $$9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^2 = 0.$$

Um jetzt die Grössen Φ_1 und Φ_2 zu berechnen, müssen wir nach unseren gewöhnlichen Regeln acht lineare partielle Differentialgleichungen in den Variabeln $x\, y\, y_1 \dots y_8$ (den acht infinitesimalen Transformationen $p,\ q,\ xq,\ yq,\ xp,\ x^2p + xyp,\ yp,\ xyp + y^2q$ entsprechend) bilden. Die drei ersten unter diesen Gleichungen

$$B_1 f = \frac{\partial f}{\partial x} = 0, \quad B_2 f = \frac{\partial f}{\partial y} = 0, \quad B_3 f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y_1} = 0$$

sagen nur aus, dass Φ_1 und Φ_2 von $x\, y$ und y_1 unabhängig sind. Diejenige Gleichung, die der inf. Transformation $xyp + y^2 q$ entspricht, brauchen wir nicht zu bilden; sie ist nämlich wegen der Relation

$$(yp,\ x^2p + xyq) = xyp + y^2 q$$

eine Consequenz der übrigen. Die Grössen $\Phi_1\ \Phi_2$ sind daher bestimmt als Functionen von $y_2\, y_3 \dots y_8$ durch die Gleichungen

$$B_4 f = y_2 \frac{\partial f}{\partial y_2} + y_3 \frac{\partial f}{\partial y_3} + \cdots + y_8 \frac{\partial f}{\partial y_8} = 0$$

$$B_5 f = y_3 \frac{\partial f}{\partial y_3} + 2y_4 \frac{\partial f}{\partial y_4} + \cdots + 6y_8 \frac{\partial f}{\partial y_8} = 0$$

$$B_6 f = 3y_2 \frac{\partial f}{\partial y_3} + 8y_3 \frac{\partial f}{\partial y_4} + 15 y_4 \frac{\partial f}{\partial y_5} + 24 y_5 \frac{\partial f}{\partial y_6} + 35 y_6 \frac{\partial f}{\partial y_7} + 48 y_7 \frac{\partial f}{\partial y_8} = 0$$

$$B_7 f = 3y_2^2 \frac{\partial f}{\partial y_3} + 10 y_2 y_3 \frac{\partial f}{\partial y_4} + (15 y_2 y_4 + 10 y_3^2)\frac{\partial f}{\partial y_5} + (21 y_2 y_5 + 35 y_3 y_4)\frac{\partial f}{\partial y_6} + (28 y_2 y_6 + 56 y_3 y_5 + 35 y_4^2)\frac{\partial f}{\partial y_7} + (36 y_2 y_7 + 84 y_3 y_6 + 126\, y_4 y_5)\frac{\partial f}{\partial y_8} = 0.$$

Zur Bestimmung der gemeinsamen Lösungen Φ_1 und Φ_2 derselben bilden wir zuerst die Lösungen von $B_6 = f$, nämlich

$$y_2 = y_2$$

$$\varrho_2 = 3y_2 y_4 - 4y_3^2$$

$$\varrho_3 = 3y_2^2 y_5 - 5y_3 \varrho_2 - \frac{20}{3} y_3^3 = \frac{1}{3}(9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^3)$$

*) Das Resultat des Textes war a priori evident. Denn es giebt ja nur zwei Curven, die gerade Linie und der Kegelschnitt, die mehr als eine infinitesimale und lineare Transformation in sich gestatten. Halphen hat zuerst die obenstehende Differentialgleichung (2) der Kegelschnitte wirklich aufgestellt. Ebenso hat Halphen zuerst die später aufgestellen Grössen Φ_1 und Φ_2 berechnet. [Ich erfahre nachträglich, dass schon Monge die Differentialgleichung der Kegelschnitte berechnet hat. Januar 1888].

$$\varrho_4 = 3y_2{}^3y_6 - 8y_3\varrho_3 - 20y_3{}^2\varrho_2 - \frac{40}{3}y_3{}^4 = 3y_2{}^3y_6 - 24y_2{}^2y_3y_5 + 60y_2y_3{}^2y_4 - 40y_3{}^4$$

$$\varrho_5 = 9y_2{}^4y_7 - 35y_3\varrho_4 - 140y_3{}^2\varrho_3 - \frac{700}{3}y_3{}^3\varrho_2 - \frac{280}{3}y^5$$

$$= 9y_2{}^4y_7 - 105y_2{}^3y_3y_6 + 420y_2{}^2y_3{}^2y_5 - 700y_2y_3{}^3y_4 + \frac{1120}{3}y_3{}^5$$

$$\varrho_6 = 27y_2{}^5y_8 - 48y_3\varrho_5 - 24\cdot 35y_3{}^2\varrho_4 - 16\cdot 140y_3{}^3\varrho_3 - 2800y_3{}^4\varrho_2 - \frac{8\cdot 280}{3}y_3{}^6$$

und führen sie darnach zusammen mit y_3 als neue Variabeln in $B_4f=0$, $B_5f=0$, $B_6f=0$ und $B_7f=0$ ein. Nun ist, wie man leicht sieht

$$B_4y_2 = y_2,\ B_4y_3 = y_3,\ B_4\varrho_k = k\varrho_k;$$
$$B_5y_2 = 0,\ B_5y_3 = y_3,\ B_5\varrho_i = i\varrho_i$$

also erhält $B_4f=0$ in den neuen Variabeln die Form

$$y_2\frac{\partial f}{\partial y_2} + y_3\frac{\partial f}{\partial y_3} + 2\varrho_2\frac{\partial f}{\partial \varrho_2} + 3\varrho_3\frac{\partial f}{\partial \varrho_3} + \cdots + 6\varrho_6\frac{\partial f}{\partial \varrho_6} = 0$$

und $B_5f=0$ die Form

$$y_3\frac{\partial f}{\partial y_3} + 2\varrho_2\frac{\partial f}{\partial \varrho_2} + \cdots + 6\varrho_6\frac{\partial f}{\partial \varrho_6} = 0,$$

sodass $B_4f=0$ sich auf

$$\frac{\partial f}{\partial y_2} = 0$$

reducirt. Ferner ist klar, dass $B_6f=0$ die Form

$$\frac{\partial f}{\partial y_3} = 0$$

annimmt. Zur Einführung der ϱ_k als Variabeln in $B_7f=0$ bilden wir die Ausdrücke

$$B_7\varrho_2 = 6y_2{}^2y_3$$
$$B_7\varrho_3 = 0$$
$$B_7\varrho_4 = -3y_2{}^2\varrho_3 + 20y_2{}^2y_3\varrho_2$$
$$B_7\varrho_5 = y_2{}^2(-21\varrho_4 + 35\varrho_2{}^2) + y_2{}^2y_3\cdot 105\varrho_3$$
$$B_7\varrho_6 = y_2{}^2(-36\varrho_5 + 3\cdot 126\varrho_2\varrho_3) + y_2{}^2y_3(504\varrho_4 + 210\varrho_2{}^2);$$

also erhält $B_7f=0$ durch Division mit $y_2{}^2$ die Form

$$y_3\left\{6\frac{\partial f}{\partial \varrho_2} + 20\varrho_2\frac{\partial f}{\partial \varrho_4} + 105\varrho_3\frac{\partial f}{\partial \varrho_5} + (504\varrho_4 + 210\varrho_2{}^2)\frac{\partial f}{\partial \varrho_6}\right\}$$
$$+\left\{-3\varrho_3\frac{\partial f}{\partial \varrho_4} + (-21\varrho_4 + 35\varrho_2{}^2)\frac{\partial f}{\partial \varrho_5} + (-36\varrho_5 + 3\cdot 126\varrho_2\varrho_3)\frac{\partial f}{\partial \varrho_6}\right\} = 0,$$

welch letztere Gleichung sich wegen $\frac{\partial f}{\partial y_3} = 0$ in zwei spaltet. Die gesuchten Grössen Φ_1 und Φ_2 sind daher bestimmt als Functionen von $\varrho_2\ \varrho_3 \ldots \varrho_6$ durch die drei Gleichungen

$$\begin{cases} 2\varrho_2\frac{\partial f}{\partial \varrho_2}+3\varrho_3\frac{\partial f}{\partial \varrho_3}+\cdots+6\varrho_6\frac{\partial f}{\partial \varrho_6}=0 \\ Cf=6\frac{\partial f}{\partial \varrho_2}+20\varrho_2\frac{\partial f}{\partial \varrho_4}+105\varrho_3\frac{\partial f}{\partial \varrho_5}+(504\varrho_4+210\varrho_2{}^2)\frac{\partial f}{\partial \varrho_6}=0 \\ Df=-3\varrho_3\frac{\partial f}{\partial \varrho_4}+(-21\varrho_4+35\varrho_2{}^2)\frac{\partial f}{\partial \varrho_5}+(-36\varrho_5+3\cdot 126\varrho_2\varrho_3)\frac{\partial f}{\partial \varrho_6}=0 \end{cases}$$

[unter denen die erste aussagt, dass Φ_1 und Φ_2 Functionen von den Verhältnissen der Grössen $\varrho_2^{\frac{1}{2}}\,\varrho_2^{\frac{1}{3}}\cdots\varrho_6^{\frac{1}{6}}$ sind]. Wir bestimmen die Lösungen von $Cf=0$, nämlich

$$\varrho_3,\ u_4=\varrho_4-\frac{5}{3}\varrho_2{}^2$$
$$u_5=\varrho_5-\frac{35}{3}\varrho_2\varrho_3$$
$$u_6=\varrho_6-\frac{175}{3}\varrho_2{}^3-84\varrho_2 u_4$$

und führen sie als Variable in $Df=0$ ein. Nun ist

$$D\varrho_3=0,\ Du_4=-3\varrho_3,\ Du_5=-21u_4,\ Du_6=-36u_5$$

und also erhält $Df=0$ durch Division mit -3 die Form

$$\varrho_3\frac{\partial f}{\partial u_4}+7u_4\frac{\partial f}{\partial u_5}+12u_5\frac{\partial f}{\partial u_6}=0.$$

Die entsprechenden Lösungen sind

$$\varrho_3,\ \sigma=2\varrho_3 u_5-7u_4{}^2=2\varrho_3\varrho_5-35\varrho_2\varrho_3{}^2-7\left(\varrho_4-\frac{5}{3}\varrho_2{}^2\right)^2$$

$$\sigma_1=\varrho_3 u_6-\frac{6\sigma}{\varrho_3}u_2-\frac{14}{\varrho_3}u_4{}^3$$
$$=\varrho_3\left(\varrho_6-84\varrho_3\varrho_4+\frac{245}{3}\varrho_2{}^3\right)-12\left(\varrho_5-\frac{35}{2}\varrho_2\varrho_3\right)\left(\varrho_4-\frac{5}{3}\varrho_2{}^2\right)+\frac{28}{\varrho_3}\left(\varrho_4-\frac{5}{3}\varrho_2{}^2\right)^3.$$

Und da die gesuchten Grössen Φ_1 und Φ_2 Functionen von den Verhältnissen der Grössen $\varrho_2^{\frac{1}{2}}\,\varrho_3^{\frac{1}{3}}\cdots\varrho_6^{\frac{1}{6}}$ sind, so können wir setzen:

$$\Phi_1=\frac{\sigma}{\varrho_3^{\frac{8}{3}}},\qquad \Phi_2=\frac{\sigma_1}{\varrho_3{}^3}.$$

Hiermit ist diese Untersuchung zum Abschluss gebracht*).

---

*) Im Laufe dieser Abhandlung benutze ich häufig den folgenden bekannten Satz: „Bilden $A_1 f=0\ldots A_r f=0$ ein vollständiges System in $x_1\ldots x_n$, so kann die Integration desselben folgendermassen geschehen. Man sucht die Lösungen $\varphi_1\ldots\varphi_{n-1}$ von $A_1 f=0$; bildet sodann

$$A_2 f=0=A_2\varphi_1\frac{\partial f}{\partial\varphi_1}+\cdots+A_2\varphi_{n-1}\frac{\partial f}{\partial\varphi_{n-1}}.$$

Sind die Verhälsnisse der $A_2\varphi_k$ nicht Functionen von $\varphi_1\ldots\varphi_{n-1}$ allein, so zerlegt die Gleichung $A_2 f=0$ sich in mehrere. Wir integriren eine *beliebige* unter ihnen

§ 2.

Gruppen, die zwei und nur zwei Differentialgleichungen erster Ordnung invariant lassen.

Im vorangehenden Paragraphen behandelten wir alle Gruppen, die keine Differentialgleichung erster Ordnung invariant lassen, und bestimmten ihre zugehörigen invarianten Differentialgleichungen höherer Ordnung. Jetzt erledigen wir dasselbe Problem für alle Gruppen mit zwei und nur zwei invarianten Differentialgleichungen 1. O. Dabei können wir nach meinen früheren Untersuchungen (Gött. Nachr. 1874, Math. Ann. Bd. XVI) annehmen, dass diese beiden Gleichungen 1. O. eben sind

$$y_1 = \frac{dy}{dx} = 0, \quad \text{und} \quad \frac{dx}{dy} = \frac{1}{y_1} = 0,$$

und dass dementsprechend die betreffende Gruppe eine der folgenden canonischen Formen besitzt:

q	q	p	q	q	q	q	q	$p+q$
yq	yq	q	yq	yq	yq	yq	yq	$xp+yq$
	p	$xp+cyq$	p	y^2q	y^2q	y^2q	y^2q	x^2p+y^2q
			xp	p	p	p		
				xp	xp			
					x^2p			

Wir werden der Reihe nach diese 9 Gruppen betrachten und ihre zugehörigen invarianten Differentialgleichungen zweiter und höherer Ordnung bestimmen.

4. Zuerst betrachten wir die zweigliedrige Gruppe q, yq. Die zugehörige Determinante

$$\Delta = \begin{vmatrix} 0 & 1 \\ 0 & y \end{vmatrix}$$

verschwindet identisch; dies beruht darauf, dass *jede* Curve (d. h. Gerade) der Schaar $x =$ Const. bei der Gruppe invariant bleibt. Zur Bestimmung der invarianten Differentialgleichungen m^{ter} Ordnung

$$f(xyy_1 \ldots y_m) = 0$$

---

und führen die entsprechenden Lösungen $\psi_1 \ldots \psi_{n-2}$ etwa in $A_3 f = 0$ ein. Die hervorgehende Gleichung $A_3 \psi_1 \frac{\partial f}{\partial \psi_1} + \cdots = 0$ behandeln wir dann in analoger Weise u. s. w.

bilden wir die beiden Gleichungen

$$\frac{\partial f}{\partial y} = 0, \quad y\frac{\partial f}{\partial y} + y_1\frac{\partial f}{\partial y_1} + \cdots + y_m\frac{\partial f}{\partial y_m} = 0.$$

Ist $m > 1$, so ergiebt sich, dass f eine arbiträre Function der Grössen

$$x, \quad \frac{y_2}{y_1}\ \frac{y_3}{y_1} \cdots \frac{y_m}{y_1}$$

ist. Wenn dagegen $m = 1$ ist, so können die beiden Gleichungen

$$\frac{\partial f}{\partial y} = 0, \quad y\frac{\partial f}{\partial y} + y_1\frac{\partial f}{\partial y_1} = 0$$

nur dann gleichzeitig bestehen, wenn $y_1 = 0$ ist; es ist nämlich an sich unmöglich, dass f nur x enthält. Zu den hiermit gefundenen invarianten Differentialgleichungen muss die Gleichung

$$\frac{1}{y_1} = 0$$

gefügt werden. Dieselbe entgeht uns bei unserer Coordinatenwahl. Dieselbe Bemerkung ist bei allen Gruppen dieses Paragraphen zu machen.

5. Die zu der Gruppe p, q, yq gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & y_1 \end{vmatrix}$$

verschwindet nicht identisch. Sie liefert die invariante Differentialgleichung erster Ordnung $y_1 = 0$, wozu wie soeben die Gleichung $\frac{1}{y_1} = 0$ zu fügen ist.

Die invarianten Differentialgleichungen $f = 0$, deren Ordnung grösser als 1 ist, werden bestimmt durch die Relationen

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad y\frac{\partial f}{\partial y} + y_1\frac{\partial f}{\partial y_1} + \cdots + y_m\frac{\partial f}{\partial y_m}$$

und somit ist f eine arbiträre Function von

$$\frac{y_2}{y_1}\ \frac{y_3}{y_1} \cdots \frac{y_m}{y_1}.$$

6. Die zu der Gruppe p, q, yq, xp gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & y_1 & y_2 \\ x & 0 & -y_1 & -2y_2 \end{vmatrix} = -y_1 y_2$$

verschwindet nicht identisch. Es giebt drei invariante Differentialgleichungen, deren Ordnung kleiner als 3 ist, nämlich

$$y_1 = 0, \quad \frac{1}{y_1} = 0, \quad y_2 = 0.$$

Zur Bildung der invarianten Gleichungen höherer Ordnung

$$f(xy\ldots y_m) = 0$$

bilden wir vier lineare partielle Differentialgleichungen, unter denen zwei nur aussagen, dass f von x und y unabhängig ist. Die beiden übrigen Gleichungen

$$y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + \cdots + y_m \frac{\partial f}{\partial y_m} = 0$$

$$y_2 \frac{\partial f}{\partial y_2} + 2 y_3 \frac{\partial f}{\partial y_3} + \cdots + (m-1) y_m \frac{\partial f}{\partial y_m} = 0$$

zeigen, dass f eine arbiträre Function von

$$\frac{y_1 y_3}{y_2^{\,2}}, \quad \frac{y_1^{\,2} y_4}{y_2^{\,3}}, \ldots \frac{y_1^{\,m-2} y_m}{y_2^{\,m-1}}$$

ist.

7. Die zu der Gruppe p, q, $xp + cyq$ gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & cy & (c-1)y_1 \end{vmatrix} = (c-1)y_1$$

verschwindet identisch dann und nur dann, wenn $c = 1$ ist. Diesen Ausnahmefall berücksichtigen wir nicht in diesem Paragraphen, indem die Gruppe p, q, $xp + yq$ einfach unendlich viele Differentialgleichungen erster Ordnung, nämlich jede Gleichung der Form

$$y_1 = \text{Const.}$$

invariant lässt.

Wenn c verschieden von 1 ist, so lässt unsere Gruppe nur zwei Gleichungen erster Ordnung, nämlich

$$y_1 = 0, \quad \text{und} \quad \frac{1}{y_1} = 0$$

invariant. Die invarianten Gleichungen höherer Ordnung $f = 0$ werden bestimmt durch

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ (c-1) y_1 \frac{\partial f}{\partial y_1} + (c-2) y_2 \frac{\partial f}{\partial y_2} + \cdots + (c-m) y_m \frac{\partial f}{\partial y_m} = 0,$$

so dass f eine arbiträre Function der Grössen

$$\frac{y_2}{y_1^{\frac{c-2}{c-1}}} \cdots \frac{y_m}{y_1^{\frac{c-m}{c-1}}}$$

sein muss. Diese Bestimmung bleibt auch dann gültig, wenn c gleich einer unter den Zahlen $2, 3, \ldots m$ ist.

8. Die zu der Gruppe q, yq, y^2q gehörige Determinante

$$\Delta = \begin{vmatrix} 0 & 1 & 0 \\ 0 & y & y_1 \\ 0 & y^2 & 2yy_1 \end{vmatrix}$$

verschwindet identisch, indem jede Curve (d. h. Gerade) der Schaar $x = \text{Const.}$ bei unserer Gruppe invariant bleibt.

Zur Bestimmung der invarianten Differentialgleichungen zweiter und höherer Ordnung bilden wir die Gleichungen

$$\frac{\partial f}{\partial y} = 0,\ \ y_1 \frac{\partial f}{\partial y_1} + \cdots + y_m \frac{\partial f}{\partial y_m} = 0$$

$$y_1^2 \frac{\partial f}{\partial y_2} + 3y_1y_2 \frac{\partial f}{\partial y_3} + (4y_1y_3 + 3y_2^2) \frac{\partial f}{\partial y_4} + \cdots = 0,$$

deren Lösungen sind

$$\varphi_1 = x,\ \ \varphi_2 = \frac{y_1y_3 - \frac{3}{2}y_2^2}{y_1^2},\ \ \varphi_3 = \frac{y_1^2y_4 + 3y_2^3 - 4y_1y_2y_3}{y_1^3} \text{ etc.}$$

Man verificirt leicht, dass φ_3 (Siehe die Einleitung) eine Function von x, φ_2 und $\frac{d\varphi_2}{dx}$ ist, indem

$$\varphi_3 = \frac{d\varphi_2}{dx}$$

ist. Jede bei der Gruppe $q\ yq\ y^2q$ invariante Differentialgleichung dritter oder höherer Ordnung hat somit die Form

$$f\left(x\ \varphi_2\ \frac{d\varphi_2}{dx}\ \frac{d^2\varphi_2}{dx}\ \cdots\right) = 0,$$

wo φ_2 den obenstehenden Werth besitzt.*)

9. Die zu der Gruppe $p\ q\ yq\ y^2q$ gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & y_1 & y_2 \\ 0 & y^2 & 2yy_1 & 2yy_2 + 2y_1^2 \end{vmatrix} = 2y_1^3$$

verschwindet nicht identisch; es giebt, wie man sieht, keine invariante Differentialgleichung zweiter Ordnung. Durch Rechnungen, die mit denen der vorangehenden Nummer fast identisch sind, findet man zur

*) [Die Differentialinvariante φ_2, die schon bei Lagrange auftritt, spielt in Schwarz's schönen Untersuchungen eine wichtige Rolle; Januar 1888.]

Bestimmung der invarianten Differentialgleichungen dritter und höherer Ordnung die Werthe

$$\varphi_1 = \frac{y_1 y_3 - \frac{3}{2} y_2^2}{y_1^2}, \quad \varphi_2 = \frac{d\varphi_1}{dx} = \frac{y_1^2 y_4 - 4 y_1 y_2 y_3 + 3 y_2^3}{y_1^3}$$

und überhaupt

$$\varphi_k = \frac{d^k \varphi_1}{dx_k}.$$

Die betreffenden invarianten Differentialgleichungen haben die Form

$$f(\varphi_1 \varphi_2 \dots \varphi_k) = 0.$$

10. Die zu der Gruppe $p + q$, $xp + yq$, $x^2p + y^2q$ gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ x & y & 0 \\ x^2 & y^2 & 2(y-x)y_1 \end{vmatrix} = 2(y-x)^2 y_1$$

verschwindet nicht identisch. Die Gleichung

$$y - x = 0$$

bestimmt eine invariante Curve. Die invarianten Differentialgleichungen zweiter und dritter Ordnung werden bestimmt durch die Gleichungen

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0,$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y_2\frac{\partial f}{\partial y_2} - 2y_3\frac{\partial f}{\partial y_3} = 0,$$

$$x^2\frac{\partial f}{\partial y} + y^2\frac{\partial f}{\partial y} + 2(y-x)y_1\frac{\partial f}{\partial y_1} + [(2y-4x)y_2 + 2y_1^2 - 2y_1]\frac{\partial f}{\partial y_2}$$
$$+ [(2y-6x)y_3 + 6y_1 y_2 - 6y_2]\frac{\partial f}{\partial y_3} = 0,$$

von denen die erste uns lehrt, dass f die Grössen x und y nur in der Combination $u = x - y$ enthält. Die beiden letzten Gleichungen erhalten durch Einführung von u als Variabeln statt x und y die Form

$$u\frac{\partial f}{\partial u} - y_2\frac{\partial f}{\partial y_2} - 2y_3\frac{\partial f}{\partial y_3} = 0,$$

$$2uy_1\frac{\partial f}{\partial y_1} + [3uy_2 - 2y_1^2 + 2y_1]\frac{\partial f}{\partial y_2} + [4uy_2 - 6y_1y_2 + 6y_2]\frac{\partial f}{\partial y_3} = 0.$$

Wir führen die Lösungen der ersten Gleichung, nämlich

$$y_1, \; uy_2 = v_2, \; u^2 y_3 = v_3$$

als Variabeln in die letzte Gleichung ein; dies giebt

$$2y_1\frac{\partial f}{\partial y_1} + (3v_2 - 2y_1^2 + 2y_1)\frac{\partial f}{\partial v_2} + [4v_3 - 6y_1v_2 + 6v_2]\frac{\partial f}{\partial v_3} = 0.$$

Die entsprechenden Lösungen

$$v_2 y_1^{-\frac{3}{2}} + 2\left(y_1^{\frac{1}{2}} + y_1^{-\frac{1}{2}}\right) = \varphi_1,$$

$$v_3 y_1^{-2} + 6\varphi_1 \left(y_1^{\frac{1}{2}} + y_1^{-\frac{1}{2}}\right) - 6(y_1 + y_1^{-1}) = \varphi_2$$

sind die gesuchten Grössen φ_1 und φ_2. Es bleibt nur übrig die Werthe von v_2 und v_3 einzuführen.

11. Die zu der Gruppe p, q, yq, xp, y^2q gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & y & y_1 & y_2 & y_3 \\ x & 0 & -y_1 & -2y_2 & -3y_3 \\ 0 & y^2 & 2yy_1 & 2yy_2 + 2y_1^2 & 2yy_3 + 6y_1y_2 \end{vmatrix} = 4y_1^2\left(y_1y_3 - \frac{3}{2}y_2^2\right)$$

verschwindet nicht identisch. Es giebt ausser $y_1 = 0$ und $\frac{1}{y_1} = 0$ nur eine invariante Differentialgleichung, deren Ordnung nicht 3 übersteigt, nämlich

$$2y_1y_3 - 3y_2^2 = 0 \text{*)}.$$

Zur Bestimmung der Grössen φ_1, φ_2 bemerken wir, dass sie als Functionen der drei Grössen

$$(3) \quad \begin{cases} w_1 = \frac{y_3}{y_1} - \frac{3}{2}\left(\frac{y_2}{y_1}\right)^2 \\ w_2 = \frac{dw_1}{dx} = \frac{y_4}{y_1} - 4\frac{y_2y_3}{y_1^2} + 3\frac{y_2^3}{y_1^3} \\ w_3 = \frac{d^2w_1}{dx_2} = \frac{y_5}{y_1} - 5\frac{y_2y_4}{y_1^2} - 4\frac{y_3^2}{y_1^2} + 17\frac{y_2^2y_3}{y_1^3} - 9\frac{y_2^4}{y_1^4} \end{cases}$$

durch die (der infinitesimalen Transformation xp entsprechende) Gleichung

$$B_4f = y_2\frac{\partial f}{\partial y_2} + 2y_3\frac{\partial f}{\partial y_3} + 3y_4\frac{\partial f}{\partial y_4} + 4y_5\frac{\partial f}{\partial y_5} = 0$$

bestimmt sind. Nun ist aber

$$B_4w_1 = 2w_1, \quad B_4w_2 = 3w_2, \quad B_4w_3 = 4w_3$$

und also wird

$$B_4f = 0 = 2w_1\frac{\partial f}{\partial w_1} + 3w_2\frac{\partial f}{\partial w_2} + 4w_3\frac{\partial f}{\partial w_3}.$$

Die Lösungen dieser Gleichungen

$$\varphi_1 = \frac{w_2}{w_1^{\frac{3}{2}}}, \quad \varphi_2 = \frac{w_3}{w_1^2}$$

sind die gesuchten Grössen φ_1 und φ_2.

*) Die invariante Differentialgleichung des Textes bestimmt alle Kegelschnitte durch zwei gemeinsame Punkte. Diese Kegelschnitte werden durch passende Coordinatenwahl alle Kreise der Ebene (oder alle Kreise einer Kugel).

12. Die zu der Gruppe

$$p \quad q \quad yq \quad xp \quad y^2q \quad x^2p$$

gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & y & y_1 & y_2 & y_3 & y_4 \\ x & 0 & -y_1 & -2y_2 & -3y_3 & -4y_4 \\ 0 & y^2 & 2yy_1 & 2yy_2+2y_1^2 & 2yy_3+6y_1y_2 & 2yy_4+8y_1y_3+6y_2^2 \\ x^2 & 0 & -2xy_1 & -4xy_2-2y_1 & -6xy_3-6y_2 & -8xy_4-12y_3 \end{vmatrix}$$

hat den nicht identisch verschwindenden Werth

$$\Delta = -4y_1(2y_1y_3 - 3y_2)^2.$$

Es giebt daher ausser $y_1 = 0$ und $\frac{1}{y_1} = 0$ nur eine invariante Differentialgleichung, deren Ordnung kleiner als fünf ist, die folgende nämlich:

$$2y_1y_3 - 3y_2 = 0.$$

Die zu der Gruppe gehörigen Grössen φ_1 und φ_2 sind Functionen von den früher (3) gefundenen Grössen

$$w_1 = \eta_3 - \frac{3}{2}\eta_2^2 \text{ *)}$$
$$w_2 = \eta_4 - 4\eta_2\eta_3 + 3\eta_2^3$$
$$w_3 = \eta_5 - 5\eta_2\eta_4 - 4\eta_3^2 + 17\eta_2^2\eta_3 - 9\eta_2^4$$
$$w_4 = \eta_6 - 6\eta_2\eta_5 - 13\eta_3\eta_4 + 27\eta_2^2\eta_4 + 42\eta_2\eta_3^2 - 87\eta_2^3\eta_3 + 36\eta_2^5$$

und zwar genügen sie (den Transformationen xp und x^2p entsprechend) den beiden Gleichungen

$$Af = \eta_2\frac{\partial f}{\partial \eta_2} + 2\eta_3\frac{\partial f}{\partial \eta_3} + \cdots + 5\eta_6\frac{\partial f}{\partial \eta_6} = 0,$$
$$Bf = \frac{\partial f}{\partial \eta_2} + 3\eta_2\frac{\partial f}{\partial \eta_3} + 6\eta_3\frac{\partial f}{\partial \eta_4} + 10\eta_4\frac{\partial f}{\partial \eta_5} + 15\eta_5\frac{\partial f}{\partial \eta_6} = 0.$$

Nun ist

$$Aw_1 = 2w_1, \quad Aw_2 = 3w_2, \quad Aw_3 = 4w_3, \quad Aw_4 = 5w_4,$$
$$Bw_1 = 0, \quad Bw_2 = 2w_1, \quad Bw_3 = 5w_2, \quad Bw_4 = 9w_3.$$

Daher sind φ_1 und φ_2 bestimmt als Functionen der w_2 durch die Gleichungen

$$2w_1\frac{\partial f}{\partial w_1} + 3w_2\frac{\partial f}{\partial w_2} + 4w_3\frac{\partial f}{\partial w_3} + 5w_4\frac{\partial f}{\partial w_4} = 0,$$
$$2w_1\frac{\partial f}{\partial w_2} + 5w_2\frac{\partial f}{\partial w_3} + 9w_3\frac{\partial f}{\partial w_4} = 0,$$

deren Lösungen sind

---

*) Im Texte setzen wir überall η_i statt $\frac{y_i}{y_1}$.

$$\varphi_1 = \frac{4w_1w_3 - 5w_2^2}{w_1^3},$$

$$\varphi_2 = \frac{4w_1^2w_4 - 18w_1w_2w_3 + 15w_2^3}{w_1^{\frac{9}{2}}}.$$

Führt man hier die Werthe der Grössen w_k ein, so erhält man die Ausdrücke von φ_1 und φ_2 als Functionen von den y_k.

§ 3.

Gruppen, die eine und nur eine Differentialgleichung erster Ordnung invariant lassen.

Gruppen, die eine und nur eine Differentialgleichung 1. O. invariant lassen, können (Göttinger Nachr. 1874; Math. Annalen, Bd. XVI) auf eine der folgenden canonischen Formen gebracht werden, wobei X_i eine Function von x, ε und c Constante bezeichnen.

$X_1 q$	$X_1 q$	$X_1 q$	$X_1 q$	q
$X_2 q$	.	.	.	xq
.	.	.	.	.
.	.	.	.	.
.	$X_r q$	$X_r q$	$X_r q$	$x^r q$
			yq	p
$X_r q$	yq	$p + \varepsilon y q$	p	$xp + cyq$

q	q	q	q
xq	xq	xq	xq
.	.	.	.
.	.	.	.
.	.	.	.
$x^r q$	$x^r q$	$x^r q$	$x^r q$
p	yq	p	p
$xp + [(r+1)y + x^{r+1}]q$	p	$2xp + ryq$	yq
	xp	$x^2p + rxyq$	xp
			$x^2p + rxyq$

p	yq
$xp + yq$	p
$x^2p + 2xyq$	xp
	$x^2p + xyq$

Wir werden successiv diese canonischen Gruppen betrachten und ihre invarianten Differentialgleichungen zweiter und höherer Ordnung bestimmen.

13. Die zu der Gruppe $p, \; xp + yq, \; x^2p + 2xyq$ gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x & y & 0 \\ x^2 & 2xy & 2y \end{vmatrix} = 2y^2$$

verschwindet nicht identisch. Die invariante Gleichung 1. O.:

$$\frac{1}{y_1} = 0$$

entgeht uns bei unserer speciellen Coordinatenwahl.

Die Grössen φ_1 und φ_2 haben als Lösungen der Gleichungen

$$\frac{\partial f}{\partial x} = 0, \quad y\frac{\partial f}{\partial y} - y_2\frac{\partial f}{\partial y_2} - 2y_3\frac{\partial f}{\partial y_3} = 0$$

$$y\frac{\partial f}{\partial y_1} + y_1\frac{\partial f}{\partial y_2} = 0$$

die Werthe

$$\varphi_1 = 2yy_2 - y_1^2, \quad \varphi_2 = y^2y_3.$$

14. Die zu der Gruppe $yq, \; p, \; xp, \; x^2p + xyq$ gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & y & y_1 & y_2 \\ x & 0 & -y_1 & -2y_2 \\ x^2 & xy & y - xy_1 & -3xy_2 \end{vmatrix} = 2y^2y_2$$

verschwindet nicht identisch. Die Gruppe lässt daher eine Gleichung zweiter Ordnung und zwar $y_2 = 0$ invariant, was darauf hinauskommt, dass die Gruppe eine *projective* ist.

Die Grössen φ_1 und φ_2 befriedigen die vier Gleichungen

$$\frac{\partial f}{\partial x} = 0, \quad y\frac{\partial f}{\partial y} + \cdots + y_4\frac{\partial f}{\partial y_4} = 0,$$

$$+ y_1\frac{\partial f}{\partial y_1} + 2y_2\frac{\partial f}{\partial y_2} + \cdots + 4y_4\frac{\partial f}{\partial y_4} = 0,$$

$$- y\frac{\partial f}{\partial y_1} + 3y_2\frac{\partial f}{\partial y_3} + 8y_3\frac{\partial f}{\partial y_4} = 0,$$

und haben somit die Werthe

$$\varphi_1 = \frac{yy_3 + 3y_1y_2}{y^{\frac{1}{2}}y_2^{\frac{3}{2}}}$$

$$\varphi_2 = \frac{3yy_2y_4 - 4yy_3^2}{y_2^3}.$$

15. Die zu der Gruppe $X_1 q \cdots X_r q$ gehörige Determinante

$$\Delta = \begin{vmatrix} 0\, X_1\, X_1' \cdots X_1^{(r-2)} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0\, X_r\, X_r' \cdots X_r^{(r-2)} \end{vmatrix}$$

verschwindet identisch, da jede Curve (d. h. Gerade) der Schaar $x = \text{Const.}$ bei der Gruppe invariant bleibt. Zur Bestimmung der invarianten Differentialgleichungen

$$f(x y y_1 \cdots y_m) = 0$$

bilden wir die r Gleichungen

$$(4) \qquad X_i \frac{\partial f}{\partial y} + X_i' \frac{\partial f}{\partial y_1} + \cdots + X_i^{(m)} \frac{\partial f}{\partial y_m} = 0,$$

die nur wenn $m > r - 1$ ist, andere gemeinsame Lösungen als x besitzen können*). Für $m = r$ ist, wie man leicht verificirt, ausser x zugleich die Determinante

$$D = \begin{vmatrix} X_1 & X_1' & \cdots & X_1^{(r)} \\ \cdot & \cdot & \cdot & \cdot \\ X_r & X_r' & \cdots & X_r^{(r)} \\ y & y_1 & \cdots & y_r \end{vmatrix}$$

eine Lösung. Wir können daher

$$\varphi_1 = x, \quad \varphi_2 = D^{**})$$

setzen. Setzt man überhaupt

$$D_i = \begin{vmatrix} X_1 & X_1' & \cdots & X_1^{(r-1)} & X_1^{(r+i)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_r & X_r' & \cdots & X_r^{(r-1)} & X_r^{(r+i)} \\ y & y_1 & \cdots & y_{r-1} & y_{r+i} \end{vmatrix}$$

so ist D_i immer eine Lösung der Gleichungen (4), dabei vorausgesetzt dass $m \geqq r + i$. Man kann daher

$$\varphi_{i+2} = D_i$$

setzen.

16. Die zu der Gruppe $X_1 q \cdots X_r q\ yq$ gehörige Determinante Δ verschwindet identisch. Die invarianten Differentialgleichungen $f = 0$ sind bestimmt durch die $(r + 1)$ Gleichungen

---

*) Der Schluss im Texte beruht darauf, dass die X_k in dem Sinne unabhängige Functionen von x sind, dass keine Relation der Form $\Sigma c_i X_i = 0$ mit constanten Coefficienten besteht.

**) Ist $r = 1$, so giebt es unbeschränkt viele invariante Differentialgleichungen *erster* Ordnung, indem die invariante Gleichung $D = f(x)$ von erster Ordnung ist.

$$X_i \frac{\partial f}{\partial y} + X_i' \frac{\partial f}{\partial y_1} + \cdots + X_i^{(m)} \frac{\partial f}{\partial y_m} = 0$$

$$y \frac{\partial f}{\partial y} + y_1 \frac{\partial f}{\partial y_1} + \cdots + y_m \frac{\partial f}{\partial y_m} = 0.$$

Dieselben haben (ausser x) keine gemeinsame Lösung, wenn $m < r$. Ist $m = r$, so giebt es eine specielle gemeinsame Lösung, nämlich die lineare und homogene Differentialgleichung

$$D = 0,$$

wobei wir D in derselben Bedeutung wie soeben brauchen. Ist $m = r + i$, so sind die Grössen

$$x \frac{D_1}{D} \frac{D_2}{D} \cdots \frac{D_i}{D}$$

Lösungen unserer linearen partiellen Differentialgleichungen, und wir können daher

$$\varphi_1 = x, \quad \varphi_2 = \frac{D_1}{D}, \cdots \varphi_{i+1} = \frac{D_i}{D}$$

setzen.

17. Die zu der Gruppe

$$X_1 q \cdots X_r q, \quad p + \varepsilon y q \ (\varepsilon = \text{Const.})$$

gehörige Determinante

$$\Delta = \begin{vmatrix} 1 & \varepsilon y & \varepsilon y_1 & \cdots & \varepsilon y_{r-1} \\ 0 & X_1 & X_1' & \cdots & X_1^{(r-1)} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & X_r & X_r' & \cdots & X_r^{(r-1)} \end{vmatrix}$$

verschwindet nicht identisch. Ausser $\frac{1}{y_1} = 0$ giebt es keine invariante Differentialgleichung, deren Ordnung kleiner als r ist*). Die Grössen $\varphi_1 \varphi_2 \cdots$ sind Functionen von $x, D, D_1, D_2 \cdots$ und genügen dabei der Gleichung

$$Bf = \frac{\partial f}{\partial x} + \varepsilon y \frac{\partial f}{\partial y} + \cdots + \varepsilon y_m \frac{\partial f}{\partial y_m} = 0$$

oder der äquivalenten

$$Bx \cdot \frac{\partial f}{\partial x} + BD \cdot \frac{\partial f}{\partial D} + BD_1 \cdot \frac{\partial f}{\partial D_1} + \cdots = 0,$$

---

*) Den Fall $r = 1$ schliessen wir im Texte aus, indem es dann unendlich viele invariante Differentialgleichungen 1. O. giebt (siehe § 4).

wo $Bx = 1$ zu setzen ist. Zur Berechnung der Ausdrücke BD_i erinnern wir (Math. Ann. Bd. XVI p. 499) daran, dass die X_k Relationen der Form

$$(5) \quad \begin{cases} X_1' = \lambda_{11} X_1 \\ X_2' = \lambda_{21} X_1 + \lambda_{22} X_2 \\ X_3' = \lambda_{31} X_1 + \lambda_{32} X_2 + \lambda_{33} X_3 \qquad (\lambda_{ik} = \text{Const.}) \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ X_k' = \lambda_{k1} X_1 + \lambda_{k2} X_2 + \cdots \lambda_k X_k \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \end{cases}$$

erfüllen. Daher ist, wie man durch Ausführung findet,

$$\begin{aligned} BD &= (\lambda_{11} + \lambda_{22} + \cdots + \lambda_{rr} + \varepsilon)\, D = kD \\ BD_1 &= (\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot)\, D_1 = kD_1 \\ &\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ BD_i &= (\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot)\, D_i = kD_i \end{aligned}$$

wo k eine Constante bezeichnet. Die gesuchten Grössen φ_1, φ_2 sind daher bestimmt durch die Gleichung

$$\frac{\partial f}{\partial x} + kD \frac{\partial f}{\partial D} + kD_1 \frac{\partial f}{\partial D_1} + \cdots = 0,$$

sodass

$$\varphi_1 = De^{-kx}, \quad \varphi_2 = D_1 e^{-kx}, \; \cdots \; \varphi_{i+1} = D_i e^{-kx}$$

wird.

Unter Nummer 15 gaben wir die allgemeinen Ausdrücke der Grössen D_i. Diese Ausdrücke können indess vermöge der Formeln (5) wesentlich vereinfacht werden. Denn es ist

$$\frac{\partial D_i}{\partial x} = (\lambda_{11} + \cdots + \lambda_{rr})\, D_i = \Sigma \lambda_{kk} \cdot D_i$$

woraus

$$D_i = e^{x \Sigma \lambda_{kk}} \Omega_1$$

wo Ω_i eine lineare homogene Function mit *constanten* Coefficienten von $y y_1 \cdots y_{r-1} y_{r+i}$ bezeichnet. Daher wird

$$D_i = e^{x \Sigma \lambda_{kk}} (k_{i0}\, y + k_{i1}\, y_1 + \cdots + k_{i,r-1}\, y_{r-1} + k_{i,r+i}\, y_{r+i})$$

und

$$\varphi_{i+1} = e^{-\varepsilon x} (k_{i0}\, y + \cdots + k_{i,r-1}\, y_{r-1} + k_{i,r+i}\, y_{r+i}).$$

Wir gehen hier nicht auf die einfache Berechnung der Constanten k_{ij} ein*). Dagegen heben wir ausdrücklich hervor, dass die Con-

*) $D = 0$ ist, wie wir gesehen haben, im vorliegenden Falle eine lineare homogene Differentialgleichung mit constanten Coefficienten. Es ist dabei klar, dass diese Constante beliebige Werthe haben können.

stante ε ohne wesentliche Beschränkung gleich Null gesetzt werden kann.

Wir bemerken nur noch, dass sich unter den invarianten Differentialgleichungen beliebig viele lineare und homogene mit *constanten* Coefficienten finden. Denn wenn $c_1, c_2 \cdots$ beliebige Constanten bezeichnen, so stellt

$$c_1\varphi_1 + c_2\varphi_2 + \cdots = 0$$

immer eine solche Gleichung dar.

18. Die der Gruppe

$$X_1 q \cdots X_r q\ yq\ p$$

entsprechende Determinante

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & X_1 & X_1' & \cdots & X_1^{(r)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & X_r & X_r' & \cdots & X_r^{(r)'} \\ 0 & y & y_1 & \cdots & y_r \end{vmatrix} = D$$

verschwindet nicht identisch. Es giebt ausser $\frac{1}{y_1} = 0$ eine und nur eine invariante Differentialgleichung, deren Ordnung r nicht übersteigt, nämlich

$$D = 0\text{*)}.$$

Die Grössen $\varphi_1\ \varphi_2 \cdots$ sind Functionen von x, D, $D_1 \cdots$, bestimmt durch die Gleichungen

$$\frac{df}{dx} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial D}\frac{\partial D}{\partial x} + \frac{\partial f}{\partial D_1}\frac{\partial D_1}{\partial x} + \cdots$$

$$0 = y\frac{\partial f}{\partial y} + y_1\frac{\partial f}{\partial y_1} + \cdots + y_m\frac{\partial f}{\partial y_m}.$$

Nun ist, da die X_k durch Relationen der Form

$$X_k' = \lambda_{k1} X_1 + \lambda_{k2} X_2 \cdots + \lambda_{kk} X_k$$

verknüpft sind:

$$\frac{dD_i}{dx} = D_i(\lambda_{11} + \cdots + \lambda_{rr}) = D_i k,$$

woraus

$$D_i = e^{kx}(c_{i0}y + c_{i1}y_1 + \cdots + c_{i,r-1}y_{r-1} + c_{i,r+i}y_{r+i}).$$

Hieraus ergiebt sich, dass wir

$$\varphi_1 = \frac{D_1}{D},\quad \varphi_2 = \frac{D_2}{D}\ \text{etc.}$$

---

*) Ist $r = 1$, so giebt es zwei invariante Gleichungen 1. O.: $\frac{1}{y_1} = 0$ und $D = 0$. Diesen Fall schliessen wir im Texte aus.

setzen können. Die φ_k sind Brüche, deren Zähler und Nenner ganze und lineare Functionen (mit constanten Coefficienten) von $y y_1 y_2 \cdots$ sind.

19. Die Determinante Δ der Gruppe

$$q \; xq \cdots x^{r-1}q \; p \; xp + cyq$$

hat den Werth

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & x & 1 & \cdots & & \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & x^{r-1} & (r-1)x^{r-2} & \cdots & (r-1)\cdots 3\cdot 2\cdot 1 & 0 \\ x & cy & (c-1)y_1 & \cdots & (c-r-1)y_{r-1} & (c-r)y_r \end{vmatrix}$$

d. h. es ist, wenn wir einen nicht verschwindenden Factor wegwerfen,

$$\Delta = (c-r)\,y_r.$$

Δ verschwindet daher nur, wenn $c = r$ ist.

Ist $c \neq r$, so ist $y_r = 0$ $\left(\text{ausser } \frac{1}{y_1} = 0\right)$ die einzige invariante Differentialgleichung, deren Ordnung r nicht übersteigt*). Die Grössen $\varphi_1 \varphi_2 \cdots$ sind (Nummer 15) Functionen von $x D D_1 \cdots$ und erfüllen überdies die Relationen

$$\frac{\partial f}{\partial x} = 0, \quad x\frac{\partial f}{\partial x} + cy\frac{\partial f}{\partial y} + (c-1)\,y_1\frac{\partial f}{\partial y_1} + \cdots = 0.$$

Nun aber ist, wie man durch Ausführung findet, indem man unwesentliche constante Factoren wegwirft,

$$D = y_r, \quad D_1 = y_{r+1}, \quad D_2 = y_{r+2} \quad \cdots$$

Also wird

$$\varphi_1 = \frac{y_{r+1}}{y_r^{\frac{c-r-1}{c-r}}}, \quad \varphi_2 = \frac{y_{r+2}}{y_r^{\frac{c-r-2}{c-r}}}, \quad \cdots.$$

Zurück steht noch der Ausnahmefall $c = r$. In diesem Falle verschwindet Δ identisch. Man findet, dass

$$\frac{1}{y_1} = 0, \quad y_r = \text{Const.}^{**}), \quad y_{r+1} = 0$$

die einzigen invarianten Differentialgleichungen sind, deren Ordnung $r + 1$ nicht übersteigt. Die Grössen $\varphi_1 \varphi_2$ sind Functionen von

$$y_r, \; y_{r+1}, \; y_{r+2} \cdots,$$

---

*) Auch jetzt soll der Fall $r = 1$ ausgeschlossen sein.

**) Ist insbesondere $r = c = 1$, so giebt es ∞^1 invariante Differentialgleichungen 1. O. (siehe § 4).

bestimmt durch die Gleichung

$$y_{r+1}\frac{\partial f}{\partial y_{r+1}} + 2y_{r+2}\frac{\partial f}{\partial y_{r+2}} + 3y_{r+3}\frac{\partial f}{\partial y_{r+3}} + \cdots = 0.$$

Daher wird

$$\varphi_1 = y_r, \quad \varphi_2 = \frac{y_{r+2}}{y_{r+1}^2}, \quad \varphi_3 = \frac{y_{r+3}}{y_{r+1}^3} \text{ etc. } \cdots$$

20. Die Determinante Δ der Gruppe

$$q \;\; xq \cdots x^{r-1}q \;\; p \;\; xp + (ry + x^r)\,q$$

hat den Werth

$$\Delta = r(r-1)\cdots 3\cdot 2\cdot 1$$

und verschwindet somit weder identisch noch für specielle Werthe der Variabeln. Ausser $\frac{1}{y_1} = 0$ giebt es daher keine invariante Differentialgleichung, deren Ordnung r nicht übersteigt. Die Grössen $\varphi_1\,\varphi_2$ sind bestimmt durch die Relation

$$r(r-1)\cdots 3\cdot 2\cdot 1\frac{\partial f}{\partial y_r} - y_{r+1}\frac{\partial f}{\partial y_{r+1}} - 2y_{r+2}\frac{\partial f}{\partial y_{r+2}} - \cdots = 0$$

und haben daher, wenn man zur Abkürzung

$$r(r-1)\cdots 3\cdot 2\cdot 1 = \frac{1}{\omega}$$

setzt, die Werthe

$$\varphi_1 = y_{r+1}e^{\omega y_r}, \quad \varphi_2 = y_{r+2}e^{2\omega y_r}, \quad \varphi_3 = y_{r+3}e^{3\omega y_r} \text{ etc.}$$

21. Die Determinante der Gruppe

$$q \;\; xq \cdots x^{r-1}q \;\; yq \;\; p \;\; xp$$

hat den Werth:

$$\Delta = y_r y_{r+1}.$$

Es giebt daher drei invariante Differentialgleichungen, deren Ordnung kleiner als $r+2$ ist, nämlich

$$\frac{1}{y_1} = 0, \quad y_r = 0, \quad y_{r+1} = 0\,^{*}).$$

Die Grössen $\varphi_1\,\varphi_2\cdots$ hängen nur von $y_r,\, y_{r+1}\, y_{r+2}\cdots$ ab und erfüllen dabei die beiden Gleichungen

$$y_r\frac{\partial f}{\partial y_r} + y_{r+1}\frac{\partial f}{\partial y_{r+1}} + y_{r+2}\frac{\partial f}{\partial y_{r+2}} + \cdots = 0$$

$$y_{r+1}\frac{\partial f}{\partial y_{r+1}} + 2y_{r+2}\frac{\partial f}{\partial y_{r+2}} + \cdots = 0.$$

*) Der Fall $r = 1$ ist im Texte ausgeschlossen.

Sie besitzen somit die Form

$$\varphi_1 = \frac{y_r y_{r+2}}{y_{r+1}^2}, \quad \varphi_2 = \frac{y_r^2 y_{r+3}}{y_{r+1}^3}, \; \ldots .$$

22. Die Determinante Δ der Gruppe

$$q,\ xq, \cdots x^{r-1}q,\ p,\ 2xp + (r-1)yq,\ x^2p + (r-1)\,xyq$$

besitzt (wenn wir von einem nicht verschwindenden constanten Factor absehen) den Werth: y_r^2. Es giebt daher ausser $\frac{1}{y_1} = 0$ nur eine invariante Gleichung, nämlich $y_r = 0$*), deren Ordnung $r+1$ nicht übersteigt. Die Grössen $\varphi_1\,\varphi_2 \cdots$ sind Functionen von $y_r\,y_{r+1} \cdots$ und erfüllen dabei die beiden Gleichungen:

$$Bf = (r+1)y_r \frac{\partial f}{\partial y_r} + (r+3)y_{r+1}\frac{\partial f}{\partial y_{r+1}} + (r+5)y_{r+2}\frac{\partial f}{\partial y_{r+2}} + \cdots = 0$$

$$(r+1)y_r \frac{\partial f}{\partial y_{r+1}} + 2(r+2)y_{r+1}\frac{\partial f}{\partial y_{r+2}} + 3(r+3)y_{r+2}\frac{\partial f}{\partial y_{r+3}} = 0.$$

Die Lösungen der letzten Gleichung sind

$$y_r,\ (r+1)\,y_r y_{r+2} - (r+2)\,y_{r+1}^2 = u$$

$$(r+1)^2 y_r^2 y_{r+3} - 3(r+1)(r+3)y_r y_{r+1} y_{r+2} + 2(r+2)(r+3)y_{r+1}^3 = u_1.$$

Führt man dieselben als Variabeln in $Bf = 0$ ein, so kommt die Gleichung

$$(r+1)\,y_r \frac{\partial f}{\partial y_r} + 2(r+3)\,u\frac{\partial f}{\partial u} + 3(r+3)\,u_1\frac{\partial f}{\partial u_1} = 0$$

mit den Lösungen

$$\varphi_1 = \frac{u}{y_r^{\frac{2(r+3)}{r+1}}}$$

$$\varphi_2 = \frac{u_1}{y_r^{\frac{3(r+3)}{r+1}}}.$$

23. Die Determinante Δ der Gruppe

$$q\ xq \cdots x^{r-1}q\ yq\ p\ xp\ x^2p + (r-1)\ xyq$$

hat den Werth

$$\Delta = y_r[(r+2)\,y_{r+1}^2 - (r+1)\,y_r y_{r+2}];$$

dabei ist vorausgesetzt, dass wir von einem constanten, nicht verschwindenden Factor absehen. Es giebt daher ausser $\frac{1}{y_1} = 0$ nur zwei invariante Differentialgleichungen, deren Ordnung $r+2$ nicht über-

*) Der Fall $r = 1$ soll im Texte ausgeschlossen sein.

steigt. Die eine ist $y_r = 0$*), die zweite kann auf die bemerkenswerthe Form

$$\left(y_r^{-\frac{1}{r+1}}\right)'' = 0$$

gebracht werden.

Die Grössen $\varphi_1\, \varphi_2 \cdots$ unserer Gruppe sind Functionen von den φ_k der vorangehenden Gruppe. Ueberdies erfüllen sie, der Transformation yq entsprechend, die Relation

$$y_r \frac{\partial f}{\partial y_r} + y_{r+1} \frac{\partial f}{\partial y_{r+1}} + y_{r+2} \frac{\partial f}{\partial y_{r+2}} + \cdots = 0;$$

daher können wir, indem wir die Symbole u, u_1, u_2 in derselben Bedeutung wie in der vorangehenden Nummer gebrauchen, den Grössen φ_k unserer Gruppe die folgenden Werthe beilegen:

$$\varphi_1 = \frac{u_1}{u^{\frac{3}{2}}}, \quad \varphi_2 = \frac{u_2}{u^2}.$$

§ 4.

Gruppen, die unendlich viele Differentialgleichungen 1. O. invariant lassen.

Wenn eine Gruppe *unendlich* viele Differentialgleichungen 1. O. invariant lässt, so kann sie auf eine von den drei folgenden canonischen Formen gebracht werden

p q $xp + yq$	q $xp + yq$	p

Wir werden successiv diese drei canonischen Gruppen betrachten und ihre invarianten Differentialgleichungen bestimmen.

24. Die Determinante Δ der Gruppe $p\ q\ xp + yq$:

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 0 \end{vmatrix}$$

verschwindet identisch. Die invarianten Differentialgleichungen sind bestimmt durch die Relationen

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad y_2 \frac{\partial f}{\partial y_2} + 2 y_3 \frac{\partial f}{\partial y_3} + 3 y_4 \frac{\partial f}{\partial y_4} + \cdots = 0.$$

---

*) Der Fall $r = 1$ soll im Texte ausgeschlossen sein.

Daher sind

$$y_1 = \text{Const.}, \quad y_2 = 0$$

die einzigen invarianten Differentialgleichungen, deren Ordnung 2 nicht übersteigt. Die Grössen $\varphi_1 \varphi_2 \cdots$ haben die Werthe

$$\varphi_1 = y_1, \quad \varphi_2 = \frac{y_3}{y_2^2}, \quad \varphi_3 = \frac{y_4}{y_2^3} \cdots .$$

25. Die Determinante Δ der Gruppe $q\ xp + yq$:

$$\Delta = \begin{vmatrix} 0 & 1 \\ x & y \end{vmatrix} = -x$$

verschwindet nicht identisch. Die Gerade $x = 0$ bleibt bei der Gruppe invariant. Die Grössen φ_i sind bestimmt durch die Relationen

$$\frac{\partial f}{\partial y} = 0, \quad x\frac{\partial f}{\partial x} - y_2\frac{\partial f}{\partial y_2} - 2y_3\frac{\partial f}{\partial y_3} - 3y_4\frac{\partial f}{\partial y_4} \cdots = 0$$

und haben daher die Werthe:

$$\varphi_1 = y_2 x, \quad \varphi_2 = y_3 x^2, \quad \varphi_3 = y_4 x^3 \cdots .$$

25. Wenn endlich eine Differentialgleichung die einzige infinitesimale Transformation p gestattet, so besitzt sie die Form

$$f(y y_1 y_2 \cdots y_m) = 0.$$

Hiermit kennen wir canonische Formen aller Differentialgleichungen zwischen x und y, die eine Gruppe von Transformationen zwischen x und y gestatten.

§ 5.

Reduction einer beliebigen Gruppe auf ihre canonische Form.

Wenn eine beliebige Gruppe von Transformationen zwischen x und y vorgelegt ist, so lässt sich immer, wie wir zeigen werden, durch ausführbare Operationen entscheiden, auf welche canonische Form sie gebracht werden kann. Ist diese Bestimmung geleistet, so verlangt die wirkliche Reduction der vorgelegten Gruppe auf ihre canonische Form in den meisten Fällen nur ausführbare Operationen; ausnahmsweise wird jedoch die Integration einer Gleichung 1. O. nothwendig.

Hieraus folgt, dass die Bestimmung der zu einer *beliebigen* Gruppe gehörigen invarianten Differentialgleichungen im ungünstigsten Falle die Integration einer Gleichung 1. O. verlangt.

26. Wir zeigen in dieser Nummer, wie man durch ausführbare Operationen entscheidet, auf welche canonische Form eine vorgelegte Gruppe gebracht werden kann.

Man bestimmt zuerst durch Determinantenbildung, ob es keine, eine, zwei oder unendlich viele invariante Differentialgleichungen 1. O. giebt.

Existirt keine solche Gleichung 1. O., so hat die Gruppe 5, 6 oder acht unabhängige infinitesimale Transformationen. In jedem von diesen drei Fällen giebt es nur eine entsprechende canonische Form, so dass eine weitere Discussion überflüssig wird.

Giebt es zwei und nur zwei invariante Gleichungen 1. O., so fragt es sich zunächst, ob Δ identisch verschwindet oder nicht. Verschwindet Δ nicht identisch, so hat die Gruppe 3, 4, 5 oder 6 unabhängige infinitesimale Transformationen, und dabei ist die canonische Form vollständig bestimmt, wenn die Zahl der inf. Transformationen gleich 5 oder 6 ist. Enthält unsere Gruppe drei infinitesimale Transformationen $B_1 f, B_2 f, B_3 f$, so kann sie entweder die canonische Form $p, q, xp + cyq$ oder die canonische Form $p + q$, $xp + yq$, $x^2 p + y^2 q$ erhalten. Diese beiden Fälle lassen sich dadurch charakterisiren, dass die inf. Transformationen $(B_i B_k)$ im ersten Falle eine zweigliedrige Untergruppe bestimmen, während sie im letzten Falle eine dreigliedrige Gruppe, nämlich die ursprüngliche Gruppe liefern. Enthält unsere Gruppe vier infinitesimale Transformationen, so kann sie entweder die canonische Form $q\ yq\ p\ xp$ oder die Form $q\ yq\ y^2 q\ p$ erhalten; diese Fälle lassen sich dadurch charakterisiren, dass die $(B_i B_k)$ im ersten Falle eine zweigliedrige Untergruppe, im zweiten eine dreigliedrige Untergruppe liefern. Verschwindet Δ identisch, so kann die Gruppe entweder auf die canonische Form q, yq, oder auf die canonische Form q, yq, $y^2 q$ gebracht werden. Die Zahl der unabhängigen infinitesimalen Transformationen entscheidet, welcher Fall vorliegt.

Jetzt setzen wir voraus, dass eine vorgelegte Gruppe $B_1 f \cdots B_s f$ eine und nur eine Gleichung erster Ordnung

$$X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} = Af = 0$$

invariant lässt. Unter den infinitesimalen Transformationen $k_1 B_1 + \cdots + k_s B_s$ giebt es einige, etwa $B_k^{(0)} f$, die eine Relation der Form

$$B_k^{(0)} f = \varphi_k(xy)\, Af$$

erfüllen; denn es giebt ja jedenfalls $r - 3$ inf. Transformationen, die jede Integralcurve von $Af = 0$ invariant lassen. Wir haben also 4 wesentlich verschiedene Möglichkeiten zu berücksichtigen: a) Befriedigt eine jede inf. Transformation $B_k f$ eine Relation der Form

$$B_k f = \varphi_k(xy)\, Af,$$

so kann die Gruppe entweder auf die canonische Form $X_1 q \cdots X_r q$ oder auf die Form $X_1 q \cdots X_r q\ yq$ gebracht werden. Das erste tritt ein, wenn alle $(B_i B_k) = 0$ sind. Die zweite Hypothese findet statt wenn die $(B_i B_k)$ nicht sämmtlich verschwinden. b) Giebt es unter den s Ausdrücken $B_i f\ s-1$, etwa $B_1^0 f \cdots B_{s-1}^0 f$, die eine Relation

$$B_k^{(0)} f = \varphi_k(xy)\ Af$$

erfüllen, so bilden die $s-1$ Transformationen $B_k^{(0)} f$ eine Untergruppe, die der Categorie (a) angehört. Verschwinden alle $(B_i^{(0)} B_k^{(0)})$, so hat die Gruppe $B_k f$ die canonische Form $X_1 q \cdots X_r q\ p + \varepsilon y q$, wo ε ohne Beschränkung gleich Null gesetzt werden kann. Verschwinden die $(B_i^{(0)} B_k^{(0)})$ nicht sämmtlich, so ist $X_1 q \cdots X_r q\ yq\ p$ die gesuchte canonische Form. c) Giebt es $s-2$ Ausdrücke $B_k^{(0)} f$, die eine Relation

$$B_k^{(0)} f = \varphi \cdot Af$$

erfüllen, so bilden die Transformationen $(B_i B_k)$ eine Untergruppe, die der Categorie (b) gehört. Eine zweite Untergruppe bilden alle $B_k^{(0)} f$. Verschwinden die $(B_i^{(0)} B_k^{(0)})$ nicht sämmtlich, so hat die Gruppe die canonische Form $q\ xq \cdots x^{r-1} q\ yq\ p\ xp$. Verschwinden dagegen alle $(B_i^0 B_k^0)$, so kann die Gruppe entweder die Form $q\ xq \cdots x^{r-1}\ p\ xp + Kyq$ oder die Form $q xq \cdots x^{r-1} q\ p\ xp + (ry + x^r) q$ erhalten. Um zwischen diesen beiden Möglichkeiten zu entscheiden, bildet man die Determinante Δ. Ist Δ ein nicht identisch verschwindender Differentialausdruck $(s-2)^{\text{ter}}$ Ordnung, so hat unsere Gruppe die canonische Form

$$q\ \ xq \cdots x^{r-1} q\ \ p\ \ xp + Kyq$$
$$K \neq r.$$

Verschwindet Δ identisch, so hat die Gruppe ebenfalls die eben hingeschriebene Form, nur mit dem Unterschiede, dass $K = r$ ist. Wenn endlich Δ eine nicht identisch verschwindende Function von x, y und y' ist, so kann unsere Gruppe die canonische Form

$$q\ \ xq \cdots x^{r-1} q\ \ p\ \ xp + (ry + x^r)\ q$$

erhalten. d) Giebt es $s-3$ infinitesimale Transformationen $B_k^{(0)}$, so sind vier verschiedene Fälle möglich. Ist $s = 3$, so kann die Gruppe die canonische Form

$$p\ \ xp + yq\ \ x^2 p + 2xyq$$

erhalten. Ist $s = 4$, so ist

$$yq\ \ p\ \ xp\ \ x^2 p + xyq$$

die gesuchte canonische Form. Ist $s > 4$ und ist dabei jedes $(B_i^{(0)} B_k^{(0)}) = 0$, so ist

$$q\ xq \cdots x^{r-1} q\ p,\ \ 2xp + (r-1) yq,\ \ x^2 p + (r-1)\ xyq$$

die gesuchte canonische Form. Sind dagegen die $(B_i^{(0)} B_k^{(0)})$ nicht sämmtlich Null, so kann unsere Gruppe die canonische Form

$$q \quad xq \cdots x^{r-1}q, \quad yq, \quad p, \quad xp, \quad x^2p + (r-1)\,xyq$$

erhalten.

Wenn endlich eine vorgelegte Gruppe unendlich viele Gleichungen erster Ordnung invariant lässt, so kann sie auf eine von den drei Formen

$$q; \quad q, \quad p + cyq; \quad p \quad q \quad x \quad p + yq$$

gebracht werden. Die Anzahl der unabhängigen inf. Transformationen entscheidet, welcher Fall vorliegt.

Also ist es uns wirklich gelungen, durch sehr einfache, immer ausführbare Rechnungen zu entscheiden, welche canonische Form eine vorgelegte Gruppe besitzt.

27. Hat man nach den soeben entwickelten Regeln die zu einer beliebig vorgelegten Gruppe gehörige canonische Form bestimmt, so stellt sich die Frage, wie die Ueberführung auf diese Form wirklich geleistet wird. Ich gebe eine kurzgefasste Erledigung dieser Frage.

Betrachten wir zunächst ein einfaches Beispiel. Sei $B_1 f \cdots B_4 f$ die vorgelegte Gruppe und p_1, q_1, $x_1 p_1$, $y_1 q_1$ ihre canonische Form. Bilde ich dann die $(B_i B_k)$, so erhalte ich eine zweigliedrige Untergruppe, die überdies in der viergliedrigen invariant ist. Sei $B_1\, B_2$ diese Untergruppe. Ich bilde die Gleichungen

$$(c_1 B_1 + c_2 B_2,\ B_3) = k_1(c_1 B_1 + c_2 B_2)$$
$$(c_1 B_1 + c_2 B_2,\ B_4) = k_2(c_1 B_1 + c_2 B_2),$$

in denen $c_1\, c_2\, k_1\, k_2$ Constante bezeichnen sollen. Das Verhältniss $\frac{c_1}{c_2}$ wird bestimmt durch eine quadratische Gleichung, deren Wurzeln ich ohne Beschränkung gleich 0 und ∞ setzen kann. Alsdann sind B_1 und B_2 die beiden einzigen invarianten Transformationen unserer Gruppe; sie entsprechen daher p_1 und q_1. Darnach wähle ich B_3 und B_4 so, dass die folgenden Relationen bestehen:

$$(B_1 B_3) = B_1, \quad (B_1 B_4) = 0, \quad (B_2 B_4) = B_2, \quad (B_2 B_3) = 0, \quad (B_3 B_4) = 0.$$

Setze ich sodann

$$B_1 = \xi_1 p + \eta_1 q = p_1, \qquad B_2 = \xi_2 p + \eta_2 q = q_1,$$
$$B_3 = \xi_3 p + \eta_3 q = x_1 p_1, \quad B_4 = \xi_4 p + \eta_4 q = y_1 q_1,$$

so finde ich

$$x_1 = \frac{\xi_3}{\xi_1} = \frac{\eta_3}{\eta_1}, \quad y_1 = \frac{\xi_4}{\xi_2} = \frac{\eta_4}{\eta_2}.$$

Durch Einführung der hiermit bestimmten Variabeln $x_1\, y_1$ erhält die vorgelegte Gruppe $B_i f$ ihre canonische Form.

Als zweites Beispiel betrachte ich eine dreigliedrige Gruppe $B_1 f\ B_2 f\ B_3 f$, die auf die canonische Form $q_1\ x_1 q_1\ y_1 q_1$ gebracht werden kann. Ich bilde die drei Ausdrücke $(B_i B_k)$, die eine zweigliedrige Untergruppe, etwa $B_1{}^0 B_2{}^0$, bilden. Dabei kann ich ohne Beschränkung annehmen, dass $B_1{}^0$, $B_2{}^0$ und B_3 unabhängige infinitesimale Transformationen unserer Gruppe sind; durch Multiplication von B_3 mit einer zweckmässigen Constante erreicht man, dass Relationen der Form

$$(B_1{}^0 B_2{}^0) = 0, \quad (B_1{}^0 B_3) = B_1{}^0, \quad (B_2{}^0 B_3) = B_2{}^0$$

bestehen. Sodann setze ich

$$\begin{aligned} B_1{}^0 &= \xi_1 p + \eta_1 q = q_1 \\ B_2{}^0 &= \xi_2 p + \eta_2 q = x_1 q_1 \\ B_3{}^0 &= \xi_3 p + \eta_3 q = y_1 q_1, \end{aligned}$$

woraus durch Elimination von q_1

$$\begin{aligned} (\xi_2 - x_1 \xi_1)\, p + (\eta_2 - x_1 \eta_1)\, q = 0 \\ (\xi_3 - y_1 \xi_1)\, p + (\eta_3 - y_1 \eta_1)\, q = 0 \end{aligned}$$

und

$$\begin{aligned} x_1 &= \frac{\xi_2}{\xi_1} = \frac{\eta_2}{\eta_1} \\ y_1 &= \frac{\xi_3}{\xi_1} = \frac{\eta_3}{\eta_1} \end{aligned}$$

folgt. Hiermit kennen wir eine Punkttransformation, vermöge deren unsere Gruppe auf ihre canonische Form gebracht wird.

In den beiden vorangehenden Beispielen hat die Reduction der vorgelegten Gruppe auf ihre canonische Form weder Quadraturen noch Integrationen von Differentialgleichungen, sondern nur Differentiationen und andere ausführbare Operationen verlangt.

Als drittes Beispiel betrachten wir eine dreigliedrige Gruppe $B_1\ B_2\ B_3$ mit der canonischen Form $q\ xq\ p$. Wir bestimmen wie in der vorangehenden Nummer die invariante Gleichung 1. O.: $Af = X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y}$ und suchen darnach alle infinitesimalen Transformationen B^0, die eine Relation der Form

$$B_k{}^0 f = \varphi_k(xy)\, Af$$

erfüllen. Wir wollen annehmen, dass $B_1 f$ und $B_2 f$ solche sind. Dann kann ich ohne Beschränkung voraussetzen, dass die folgenden Relationen bestehen:

$$(B_1 B_3) = 0, \quad (B_2 B_3) = B_1, \quad (B_1 B_2) = 0.$$

Alsdann setze ich

$$\begin{aligned} B_1 &= \xi_1 p + \eta_1 q = q_1 \\ B_2 &= \xi_2 p + \eta_2 q = x_1 q_1 \\ B_3 &= \xi_3 p + \eta_3 q = p_1 \end{aligned}$$

woraus zunächst

$$x_1 = \frac{\xi_2}{\xi_1} = \frac{\eta_2}{\eta_1}$$

hervorgeht. Die Grösse y_1 ist eine Lösung der Gleichung

$$\text{(A)} \qquad \frac{\partial f}{\partial x_1} = 0 = \xi_3 \frac{\partial f}{\partial x} + \eta_3 \frac{\partial f}{\partial y}$$

und diese Gleichung gestattet die bekannte infinitesimale Transformation $q_1 = \xi_1 p + \eta_1 q$; daher findet man ohne weiteres einen Integrabilitätsfactor und für y_1 den Werth

$$y_1 = \int \frac{\xi_3 dy - \eta_3 dx}{\xi_1 \eta_3 - \xi_3 \eta_1}.$$

Als viertes Beispiel betrachten wir eine zweigliedrige Gruppe $B_1 B_2$ mit der canonischen Form q_1, $x_1 p_1 + y_1 q_1$. Wir können ohne Beschränkung annehmen, dass $(B_1 B_2) = B_1$ ist. Wir setzen

$$B_1 = X_1 p + Y_1 q = q_1$$
$$B_2 = X_2 p + Y_2 q = x_1 p_1 + y_1 q_1.$$

Dann ist x_1 eine Lösung der Gleichung

$$\frac{\partial f}{\partial y_1} = 0 = X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y}$$

mit der bekannten infinitesimalen Transformation $B_2 f$. Also ist der Ausdruck

$$w = \int \frac{X_1 dy - Y_1 dx}{X_1 Y_2 - Y_1 X_2}$$

eine Function von x_1 und zwar, wie wir jetzt zeigen, gleich $\log x_1$. Es ist nach meinen bekannten Formeln

$$(q_1 x_1) = 0, \quad (x_1 p_1 + y_1 q_1, x_1) = x_1$$

das heisst

$$X_1 \frac{\partial x_1}{\partial x} + Y_1 \frac{\partial x_1}{\partial y} = 0, \quad X_2 \frac{\partial x_1}{\partial x} + Y_2 \frac{\partial x_1}{\partial y} = x_1,$$

woraus

$$(X_1 Y_2 - X_2 Y_1) \frac{\partial \log x_1}{\partial x} = - Y_1$$
$$(X_1 Y_2 - X_2 Y_1) \frac{\partial \log x_1}{\partial y} = X_1$$

und

$$\log x_1 = \int \frac{X_1 dy - Y_1 dx}{X_1 Y_2 - X_2 Y_1}$$

folgt, wie behauptet wurde. — Zur Bestimmung von y_1 bilden wir die Gleichungen

$$(q_1 y_1) = 1, \quad (x_1 p_1 + y_1 q_1, y_1) = y_1$$

oder die äquivalenten

$$X_1 \frac{\partial y_1}{\partial x} + Y_1 \frac{\partial y_1}{\partial y} = 1, \quad X_2 \frac{\partial y_1}{\partial x} + Y_2 \frac{\partial y_1}{\partial y} = y_1$$

und hieraus die Relationen

$$(X_1 Y_2 - X_2 Y_1) \frac{\partial y_1}{\partial x} + Y_1 y_1 = X_2$$

$$(X_1 Y_2 - X_2 Y_1) \frac{\partial y_1}{\partial y} - X_1 y_1 = - X_2,$$

vermöge deren y_1 durch zwei successive Quadraturen bestimmt wird.

Als fünftes Beispiel betrachten wir eine zweigliedrige Gruppe $B_1 B_2$ mit der canonischen Form q_1, $y_1 q_1$. Dabei können wir annehmen, dass $(B_1 B_2) = B_1$ ist. Wir setzen

$$B_1 = X_1 p + Y_1 q = q_1$$

$$B_2 = X_2 p + Y_2 q = y_1 q_1,$$

woraus

$$y_1 = \frac{X_2}{X_1} = \frac{Y_2}{Y_1}.$$

Die Grösse x_1 ist eine ganz beliebige Lösung der Gleichung

$$X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y} = 0,$$

deren Integration somit erforderlich ist.

Sei jetzt überhaupt $B_1 \dots B_r$,

$$B_k f = X_k \frac{\partial f}{\partial x} + Y_k \frac{\partial f}{\partial y},$$

eine beliebige vorgelegte Gruppe und $C_1 \dots C_r$,

$$C_i f = \xi_i \frac{\partial f}{\partial x_1} + \eta_i \frac{\partial f}{\partial y_1},$$

ihre canonische Form, die nach den Regeln der vorangehenden Nummer bestimmt wird. Wir können in jedem einzelnen Falle die $B_1 f$ derart wählen, dass die r Gleichungen

$$B_1 f = C_1 f,\ B_2 f = C_2 f, \dots B_r f = C_r f$$

bestehen können. Können diese Relationen zwischen $x\, y\, p\, q$ und $x_1\, y_1\, p_1\, q_1$ hinsichtlich $x_1\, y_1$ aufgelöst werden, so ist hiermit die gesuchte Punkttransformation gefunden. Ist eine solche Auflösung unmöglich so bildet man zunächst die Ausdrücke

$$C_i x_1 \quad C_i y_1,$$

die *bekannte* Functionen von x_1 und y_1 sind und setzt sodann

$$B_i x_1 = X_i \frac{\partial x_1}{\partial x} + Y_i \frac{\partial x_1}{\partial y} = C_i x_1,$$

$$B_i y_1 = X_i \frac{\partial y_1}{\partial x} + Y_i \frac{\partial y_1}{\partial y} = C_i y_1.$$

Dies giebt $2r$ Differentialgleichungen 1. O. zwischen $x_1\, y_1\;\, y$ und x, die im Allgemeinen zur Bestimmung von $x_1\, y_1$ durch Quadratur genügen. Nur wenn die canonische Form die eine von den drei folgenden ist

$$q_1;\ q_1\ y_1 q_1;\ q_1\ y_1 q_1\ y_1^2 q_1,$$

ist die Integration einer Differentialgleichung 1. O. nothwendig.

Wenn eine beliebige Gruppe von Transformationen zwischen x und y vorgelegt ist, so entscheidet man zuerst durch Differentiation, auf welche canonische Form sie gebracht werden kann. Ist dies geschehen, so verlangt die Reduction auf diese canonische Form im Allgemeinen nur ausführbare Operationen. Nur wenn die betreffende Form eine von den folgenden ist,

$$q;\quad q, yq;\quad q, yq, y^2q,$$

wird die Integration einer Gleichung 1. *O. nothwendig.*

28. Sucht man alle bei einer beliebig vorgelegten Gruppe zwischen x und y invarianten Differentialgleichungen, so bringt man die Gruppe zuerst auf ihre canonische Form und stellt sodann ohne weiteres die betreffenden Differentialgleichungen auf.

Dies giebt den folgenden Satz, der die wichtigsten Ergebnisse dieser Abhandlung resumirt.

Ist eine ganz beliebige continuirliche Gruppe von Transformationen zwischen x und y vorgelegt, so findet man alle invarianten Differentialgleichungen ohne Integration von Differentialgleichungen, wenn die Gruppe mehr als drei infinitesimale Transformationen enthält. Giebt es drei inf. Transformationen mit der canonischen Form q yq y^2q oder zwei inf. Transformationen mit der canonischen Form q yq oder endlich nur eine inf. Transformation, so wird die Integration einer Gleichung 1. *O. nothwendig. In allen anderen Fällen genügen Differentiationen und Quadraturen.*

In diesem Satze wird vorausgesetzt, dass nur die *infinitesimalen* Transformationen der vorgelegten Gruppe bekannt sind. Kennt man zugleich die *endlichen* Transformationen dieser Gruppe, so kann man immer, auch in den drei Ausnahmefällen, die zugehörigen invarianten Differentialgleichungen ohne Quadratur oder Integration angeben.

Januar 1883.

Abschnitt II.

In dem vorhergehenden Abschnitt bestimmte ich die Form aller Gleichungen

$$f(xy_1 y_2 \ldots y_m) = 0,$$

die eine continuirliche Gruppe von Transformationen gestatten. In diesem zweiten Abschnitt entwickele ich die allgemeine Integrationstheorie aller derartigen Gleichungen, indem ich meine allgemeine Integrationstheorie von linearen partiellen Differentialgleichungen mit bekannten infinitesimalen Transformationen für die betreffenden Beispiele im Detail durchführe.

Dieser Abschnitt zerfällt in mehrere Paragraphen, deren jeder sich an einen bestimmten Paragraphen der ersten Arbeit als Fortsetzung anschliesst.

§ 1.

Integrationstheorie von Differentialgleichungen mit bekannten infinitesimalen Transformationen der Form

$X(x)p + Y(y)q$.

In diesem Paragraphen integrire ich successive alle Differentialgleichungen 2ter und höherer Ordnung mit einer bekannten Gruppe, deren infinitesimale Transformationen sämmtlich die Form

$$X(x)p + Y(y)q$$

besitzen. Es wird dabei vorausgesetzt, dass die betreffende Gruppe keine anderen Differentialgleichungen 1. O. als $y' = 0$ und $\frac{1}{y'} = 0$ invariant lässt.

1. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe q, yq so ist sie, wenn wir $\frac{y_2}{y_1} = u$ setzen, reducibel auf die Form

$$\Omega\left(x u \frac{du}{dx} \cdots \frac{d^{m-2}u}{dx^{m-2}}\right) = 0.$$

Man integrirt diese Gleichung $(m-2)^{ter}$ Ordnung und erhält hierdurch eine Relation mit $m-2$ Constanten

$$y_2 = y_1 f(x a_1 \ldots a_{m-2}),$$

aus der durch wiederholte Integration

$$y_1 = e^{\int f(x)\,dx}$$
$$y = \int dx\, e^{\int f(x)\,dx}$$

hervorgeht.

2. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe p, q, yq, so ist sie, wenn wir

$$\frac{y_2}{y_1} = u, \quad \frac{y_3}{y_1} = v$$

setzen, reducibel auf die Form

$$\Omega\left(u v \frac{dv}{du} \cdots \frac{d^{m-3}v}{du^{m-3}}\right) = 0.$$

Durch Integration dieser Gleichung $(m-3)^{\text{ter}}$ Ordnung erhält man eine Relation mit $m-3$ Constanten

(1) $$f\left(\frac{y_2}{y_1}\ \frac{y_3}{y_1}\ a_1 \cdots a_{m-3}\right) = 0,$$

die wir auch folgendermassen schreiben können

$$\varphi\left(\frac{y_2}{y_1}\ \frac{d}{dx}\left(\frac{y_2}{y_1}\right) a_1 \cdots\right) = 0.$$

Man erhält daher jedenfalls durch eine Quadratur eine Gleichung, $y_2 = y_1 F(x)$, die nach den Regeln der vorangehenden Nummer durch zwei Quadraturen integrirt wird.

Nach der soeben angegebenen Methode verlangt die Integration einer Gleichung der Form

(2) $$\frac{y_3}{y_1} = F\left(\frac{y_2}{y_1}\right)$$

drei und zwar *drei successive* Quadraturen. Ich entwickele jetzt in Uebereinstimmung mit meinen alten Integrationstheorien eine etwas andere Methode, die allerdings ebenfalls drei Quadraturen, nicht aber drei successive Quadraturen verlangt. Die Gleichung (2) ist äquivalent mit der linearen partiellen Differentialgleichung

$$Af = \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y} + y_2 \frac{\partial f}{\partial y_1} + y_1 F \frac{\partial f}{\partial y_2} = 0,$$

welche die drei infinitesimalen Transformationen

$$B_1 f = \frac{\partial f}{\partial x}, \quad B_2 f = \frac{\partial f}{\partial y}, \quad B_3 f = y \frac{\partial f}{\partial y} + y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2}$$

gestattet. Jetzt kann man in *zwei* Weisen ein dreigliedriges vollständiges System mit einer bekannten infinitesimalen Transformation bilden. Einerseits gestattet nämlich das vollständige System

$$Af = 0, \quad B_1 f = 0, \quad B_2 f = 0$$

die infinitesimale Transformation $B_3 f$ und daher (Math. Ann. Bd. XI) hat die äquivalente totale Differentialgleichung

$$y_1 F\, dy_1 - y_2\, dy_2 = 0$$

einen bekannten Integrabilitätsfactor, nämlich $\frac{1}{\Delta}$, wo

$$\Delta = \begin{vmatrix} 1 & y_1 & y_2 & y_1 F \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & y_1 & y_2 \end{vmatrix} = y_2^2 - y_1^2 F.$$

Dies liefert ein Integral von $Af = 0$ nämlich

(3) $$\int \frac{y_1 F dy_1 - y_2 dy_2}{y_2^2 - y_1^2 F} = \text{Const.}$$

Anderseits aber gestattet das vollständige System

$$Af = 0, \quad B_2 f = 0, \quad B_3 f = 0$$

die bekannte infinitesimale Transformation $B_1 f$ und daher liefert meine alte Theorie auch das Integral

$$\int \frac{(y_1^2 F - y_2^2)\, dx + y_2 dy_1 - y_1 dy_2}{y_1^2 F - y_2^2} = \text{Const.}$$

von $Af = 0$. Aus den beiden hiermit gefundenen Integralgleichungen erhält man durch Auflösung y_1 als Function von x, und darnach durch eine neue Quadratur y als Function von x.

3. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $p, q, xp + cyq$, wobei die Constante c von Null und 1 verschieden sein soll, so kann sie, indem wir

$$\varphi_1 = \frac{y_2}{y_1^{\frac{c-2}{c-1}}} \qquad \varphi_2 = \frac{y_3}{y_1^{\frac{c-3}{c-1}}}$$

setzen, auf die Form

$$\Omega\left(\varphi_1\, \varphi_2\, \frac{d\varphi_2}{d\varphi_1} \cdots \frac{d^{m-3}\varphi_2}{d\varphi_1^{m-3}}\right) = 0$$

reducirt werden. Durch Integration dieser Gleichung $(m-3)^{\text{ter}}$ Ordnung erhält man eine Relation mit $m-3$ Constanten

$$f(\varphi_1\, \varphi_2\, a_1 \ldots a_{m-3}) = 0.$$

Kommt in derselben φ_2 nicht vor, so findet man durch Auflösung

$$y_2 = K \cdot y_1^{\frac{c-2}{c-1}}$$

und darnach y als Function von x durch zwei (unabhängige) Quadraturen. Hat man dagegen zur Integration eine Gleichung der Form

$$y_3 = y_1^{\frac{c-3}{c-1}} F\left(\frac{y_2}{y_1^{\frac{c-2}{c-1}}}\right) = \Pi,$$

so setzt man

$$y_3 = \frac{dy_2}{dy_1}\, y_2 = \Pi$$

woraus

$$\frac{dy_2}{dy_1} = y_2^{-1} y_1^{\frac{c-3}{c-1}} F\left(\frac{y_2}{y_1^{\frac{c-2}{c-1}}}\right) = \Phi. \tag{4}$$

Diese Gleichung 1. O. zwischen den Variabeln y_1 und y_2 gestattet die infinitesimale Transformation

$$(c-1) y_1 \frac{\partial f}{\partial y_1} + (c-2) y_2 \frac{\partial f}{\partial y_2},$$

und daher ist

$$\frac{1}{\begin{vmatrix} 1 & \Phi \\ (c-1)\,y_1 & (c-2)\,y_2 \end{vmatrix}} = \frac{1}{(c-2)\,y_2 - (c-1)\,y_1\,\Phi}$$

ein Integrabilitätsfactor und somit

$$\int \frac{dy_2 - \Phi\,dy_1}{(c-1)\,y_2 - (c-1)\,y_1\,\Phi} = \text{Const.}$$

ein Integral von (4). Nachdem hiermit eine Relation zwischen y_1 und y_2 erhalten ist, bestimmt man y als Function von x durch zwei (unabhängige) Quadraturen.

4. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $p\ q\ yq\ xp$, so ist sie, wenn wir von der unmittelbar integrablen Gleichung $y'' = 0$ absehen, reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2\cdots\frac{d^{m-4}\varphi_2}{d\varphi_1^{\,m-4}}\right) = 0,$$

wo φ_1 und φ_2 die Werthe

$$\varphi_1 = \frac{y_1 y_3}{y_2^2}, \quad \varphi_2 = \frac{y_1^2\,y_4}{y_2^3}$$

haben. Durch Integration dieser Gleichung $(m-4)^{\text{ter}}$ Ordnung erhält man eine Relation zwischen $\varphi_1\,\varphi_2$ und $m-4$ Constanten:

$$\varphi_2 = f(\varphi_1),$$

wobei wir von dem einfachen Falle einer Relation $\varphi_1 =$ Const. absehen. Es handelt sich also darum eine Gleichung der Form

$$y_4 = \frac{y_2^3}{y_1^2}\,f\left(\frac{y_1 y_3}{y_2^2}\right)$$

zu integriren. Wir setzen

$$\frac{y_2}{y_1} = v, \quad \frac{y_3}{y_1} = u;$$

dann wird

$$\frac{du}{dv} = \frac{y_1 y_4 - y_2 y_3}{y_1 y_3 - y_2^2} = v\,\frac{\varphi_2 - \varphi_1}{\varphi_1 - 1}$$

$$\varphi_1 = \frac{u}{v^2}, \quad \varphi_2 = f\left(\frac{u}{v^2}\right),$$

sodass wir eine Differentialgleichung 1. O. der Form

$$\frac{du}{dv} = v\,F\left(\frac{u}{v^2}\right)$$

integriren müssen. Dieselbe ist homogen in u und v^2, und also ist

$$\int \frac{du - v\,F\,dv}{2u - v^2\,F} = \text{Const.}$$

eine Integralgleichung. Hiermit ist Alles reducirt auf die Integration einer Differentialgleichung dritter Ordnung der Form $u = \psi(v)$ oder

$$\frac{y_3}{y_1} = \psi\left(\frac{y_2}{y_1}\right).$$

Dieselbe kann nach den Regeln der Nummer 2 erledigt werden. Es ist aber möglich einen anderen und einfacheren Weg zu gehen, wie ich jetzt in Uebereinstimmung mit meiner alten Integrationstheorie zeigen werde.

Die vorgelegte Gleichung

$$y_4 = \frac{y_2^3}{y_1^2} f\left(\frac{y_1 y_3}{y_2^2}\right) = W$$

ist äquivalent mit der linearen partiellen Differentialgleichung

$$Af = \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y} + y_2 \frac{\partial f}{\partial y_1} + y_3 \frac{\partial f}{\partial y_2} + W \frac{\partial f}{\partial y_3} = 0,$$

welche vier bekannte infinitesimale Transformationen, nämlich

$$B_1 f = \frac{\partial f}{\partial x}, \quad B_2 f = \frac{\partial f}{\partial y}$$

$$B_3 f = y \frac{\partial f}{\partial y} + y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + y_3 \frac{\partial f}{\partial y_3}$$

$$B_4 f = x \frac{\partial f}{\partial x} - y_1 \frac{\partial f}{\partial y_1} - 2y_2 \frac{\partial f}{\partial y_2} - 3y_3 \frac{\partial f}{\partial y_3}$$

gestattet. Dabei bilden einerseits die Gleichungen

$$Af = 0, \quad B_1 f = 0, \quad B_2 f = 0, \quad B_3 f = 0$$

ein vollständiges System mit der bekannten infinitesimalen Transformation $B_4 f$ und dem entsprechenden Integrale

$$\int \frac{(y_3^2 - y_2 W)\, dy_1 + (y_1 W - y_2 y_3)\, dy_2 + (y_2^2 - y_1 y_3)\, dy_3}{y_2^2 y_3 - 2 y_1 y_3^2 + y_1 y_2 W};$$

und anderseits bilden die Gleichungen

$$Af = 0, \quad B_1 f = 0, \quad B_2 f = 0, \quad B_4 f = 0$$

ein vollständiges System mit der bekannten infinitesimalen Transformation $B_3 f$ und dem entsprechenden Integrale

$$\int \frac{(3y_3^2 - 2 y_2 W)\, dy_1 + (y_1 W - 3 y_2 y_3)\, dy_2 + (2 y_2^2 - y_1 y_3)\, dy_3}{y_2^2 y_3 - 2 y_1 y_3^2 + y_1 y_2 W}.$$

Eliminirt man y_3 zwischen den beiden gefundenen Integralgleichungen, so erhält man eine Differentialgleichung zweiter Ordnung

$$\varphi(y_2 y_1) = 0,$$

die durch zwei (unabhängige) Quadraturen erledigt wird.

5. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $p+q$, $xp+yq$, x^2p+y^2q, so ist sie, wenn wir

$$\varphi_1 = (x-y)y_2 y_1^{-\frac{3}{2}} + 2\left(y_1^{\frac{1}{2}} + y_1^{-\frac{1}{2}}\right)$$

$$\varphi_2 = (x-y)^2 y_3 y_1^{-2} + 6\varphi_1\left(y_1^{\frac{1}{2}} + y_1^{-\frac{1}{2}}\right) - 6(y_1 + y_1^{-1})$$

setzen, reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2\cdots\frac{d^{m-3}\varphi_2}{d\varphi_1^{m-3}}\right) = 0.$$

Wir integriren diese Differentialgleichung $(m-3)^{\text{ter}}$ Ordnung und erhalten hierdurch eine Relation mit $m-3$ Constanten

$$f(\varphi_1\,\varphi_2\,a_1\cdots) = 0.$$

Enthält dieselbe nicht die Grösse φ_2, so integrirt man die betreffende Gleichung $\varphi_1 = \text{Const.}$, indem man nur die infinitesimalen Transformationen $p+q$ und $xp+yq$ berücksichtigt. Dagegen ist es unmöglich, eine Gleichung der Form

$$\varphi_2 = F(\varphi_1)$$

allgemein zu integriren, während man sie allerdings auf eine *Riccatische* Gleichung erster Ordnung reduciren kann. Dies soll jetzt gezeigt werden.

Als Variabeln wählen wir die Grössen y_1 und φ_1. Es ist, wie eine einfache Rechnung zeigt:

$$\frac{dy_1}{d\varphi_1} = \frac{y_1\left(\varphi_1 - 2y_1^{\frac{1}{2}} - 2y_1^{-\frac{1}{2}}\right)}{\varphi_2 - \frac{3}{2}\varphi_1 - 12}$$

oder, wenn wir $\sqrt{y_1} = z$ setzen,

$$2\,\frac{dz}{d\varphi_1} = \frac{z\varphi_1 - 2z^2 - 2}{F(\varphi_1) - \frac{3}{2}\varphi_1 - 12}.$$

Ist $\Phi(y_1\varphi_1) = \text{Const.}$ eine Integralgleichung der soeben gefundenen Riccatischen Gleichung, so findet man die beiden fehlenden Integralgleichungen von $\varphi_2 = F(\varphi_1)$, durch Differentiation. Setzen wir nämlich

$$Bf = x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + 2(y-x)y_1\frac{\partial f}{\partial y_1} + [(2y-4x)y^2 + 2y_1^2 - 2y_1]\frac{\partial f}{\partial y_2},$$

so sind

$$B(\Phi) = \text{Const.} \quad \text{und} \quad B(B(\Phi)) = \text{Const.}$$

ebenfalls Integralgleichungen von $\varphi_2 = F(\varphi_1)$, und es genügt daher nachzuweisen, dass die drei Grössen Φ, $B\Phi$ und $B(B(\Phi))$ unabhängige Functionen von $x\,y\,y_1$ und y_2 sind. Es ist, da $B\varphi_1$ verschwindet:

$$B(\Phi) = 2(y-x)\cdot y_1\,\frac{\partial\Phi}{\partial y_1}$$

$$B(B(\Phi) = 4(y-x)^2 y_1\,\frac{\partial}{\partial y_1}\Big(y_1\,\frac{\partial\Phi}{\partial y_1}\Big) + 2(y^2-x^2)y_1\,\frac{\partial\Phi}{\partial y_1}$$

und also sind die Grössen Φ, $B\Phi$ und $B(B(\Phi))$ unabhängig hinsichtlich $x\,y\,y_1$ und φ_1, womit der Nachweis geführt ist*).

6. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $q,\ yq,\ y^2q$, so ist sie reducibel auf die Form

$$\Omega\left(x\,w\,\frac{dw}{dx}\cdots\frac{d^{m-3}w}{dx^{m-3}}\right) = 0,$$

wo

$$\frac{y_3}{y_1} - \frac{3}{2}\,\frac{y_2^2}{y_1^2} = w$$

gesetzt ist. Wir integriren die Gleichung $(m-3)^{\text{ter}}$ Ordnung $\Omega = 0$ und erhalten hierdurch eine Differentialgleichung 3. O. der Form

$$\frac{y_3}{y_1} - \frac{3}{2}\,\frac{y_2^2}{y_1^2} = F(x)$$

die wir jetzt auf eine *Riccatische* Gleichung 1. O. reduciren werden. Setzen wir

$$\frac{y_2}{y_1} = z,$$

so wird

$$\frac{dz}{dx} = \frac{y_3}{y_1} - \frac{y_2^2}{y_1^2}$$

oder

(5) $$\frac{dz}{dx} = \frac{1}{2}\,z^2 + F(x).$$

Ist $W(zx) = \text{Const.}$ eine Integralgleichung dieser Riccatischen Gleichung, so findet man die beiden fehlenden Integralgleichungen von $w = F(x)$ folgendermassen durch Differentiation. Setzen wir

$$y^2\,\frac{\partial f}{\partial y} + 2yy_1\,\frac{\partial f}{\partial y_1} + (2y\,y_2 + 2y_1^2)\,\frac{\partial f}{\partial y_2} = Bf,$$

---

*) Die Entwickelungen des Textes liefern das einfachste Beispiel zu einem allgemeinen Theoreme in meiner Theorie der Transformationsgruppen. Gesetzt in der That, dass ein vollständiges System $A_1f = 0 \ldots A_rf = 0$ in den Variabeln $x_1 \ldots x_n$ $n-r$ inf. Transformationen $B_1f \ldots B_{n-r}f$ gestattet, und dass es nicht gelingt, ein Integral durch Differentiation zu bilden. Dann kann man ohne Beschränkung annehmen, dass die B_if eine Gruppe bilden. Sei diese Gruppe *einfach*, und sei $B_1f \ldots B_\varrho f$ eine Untergruppe mit der grösstmöglichen Zahl Parameter. Dann bildet man das vollständige System

$$A_1f = 0 \ldots A_rf = 0,\quad B_1f = 0 \ldots B_\varrho f = 0.$$

Gelingt es dasselbe zu integriren, so findet man immer die fehlenden Lösungen des Systems $A_if = 0$ durch Differentiation. In dieser Arbeit setze ich diesen Satz, den ich im Uebrigen früher in viel allgemeinerer Form aufgestellt habe, nicht als bekannt voraus.

so sind $BW = \text{Const.}$ und $B(B(W)) = \text{Const.}$ bekanntlich Integralgleichungen von $w = F(x)$; es genügt daher nachzuweisen, dass W, BW und $B(B(W))$ unabhängige Functionen von $x\,y\,y_1$ und y_2 sind. Es ist $B(x) = 0$ und

$$B(W) = \frac{\partial W}{\partial z}Bz = 2\frac{\partial W}{\partial z}y_1,$$

$$B(B(W)) = 4\frac{\partial^2 W}{\partial z^2}y_1^2 + 4\frac{\partial W}{\partial z}yy_1,$$

sodass W, BW und $B(B(W))$ wirklich hinsichtlich $y\,x\,y_1$ und z unabhängig sind. Hiermit ist die Integration von $w = F(x)$ auf diejenige der Riccatischen Gleichung (5) zurückgeführt.

7. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $q\;yq\;y^2q\;p$, so ist sie reducibel auf die Form

$$\Omega\left(w\frac{dw}{dx}\cdots\frac{d^{m-3}w}{dx^{m-3}}\right) = 0,$$

wo wiederum

$$\frac{y_3}{y_1} - \frac{3}{2}\frac{y_2^2}{y_1^2} = w$$

gesetzt ist. Man integrirt die Gleichung $(m-3)^{\text{ter}}$ Ordnung, die offenbar immer auf eine Gleichung $(m-4)^{\text{ter}}$ Ordnung reducirbar ist. Die hierdurch gefundene Differentialgleichung 3. O. von der Form

$$w = F(x)$$

wird darnach nach den Regeln der letzten Nummer auf eine Riccatische Gleichung 1. O. zurückgeführt.

8. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q\;yq\;y^2q\;p\;xp,$$

so ist sie*) reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2\,\frac{d\varphi_2}{d\varphi_1}\cdots\frac{d^{m-5}\varphi_2}{d\varphi_1^{m-5}}\right) = 0,$$

wo

$$\varphi_1 = \frac{w'}{w^{\frac{3}{2}}},\quad \varphi_2 = \frac{w''}{w^2},\quad w' = \frac{dw}{dx},\quad w'' = \frac{d^2w}{dx^2}$$

gesetzt ist. Man integrirt die Gleichung $(m-5)^{\text{ter}}$ Ordnung $\Omega = 0$ und findet hierdurch eine Differentialgleichung

$$\varphi_2 = f(\varphi_1),$$

die wir folgendermassen schreiben

---

*) Wenn wir von der unmittelbar integrablen Gleichung: $2y_1y_3 - 3y_2^2 = 0$ absehen.

$$w'' = w^2 f\left(\frac{w'}{w^{\frac{3}{2}}}\right).$$

Diese Differentialgleichung gestattet zwei bekannte infinitesimale Transformationen

$$\frac{\partial f}{\partial x} \text{ und } x\frac{\partial f}{\partial x} - y_1\frac{\partial f}{\partial y_1} - 2y_2\frac{\partial f}{\partial y_2} - 3y_3\frac{\partial f}{\partial y_3} - \cdots,$$

die in den Variabeln x und w die Form

$$\frac{\partial f}{\partial x} \text{ und } x\frac{\partial f}{\partial x} - 2w\frac{\partial f}{\partial w}$$

besitzen. Also ist

$$\int \frac{w'\,dw' - w^2 f.dw}{3w'^2 - 2w^3 f} = \text{Const.}$$

eine erste Integralgleichung. Hiernach findet man durch Auflösung und Quadratur eine Differentialgleichung der Form

$$w = F(x),$$

die nach den Regeln der Nummer 6 auf eine *Riccatische* Gleichung 1. O. reducirt wird.

Man kann im Uebrigen die Integration der Gleichung $\varphi_2 = f(\varphi_1)$ in etwas anderer Weise durchführen, wie hier kurz angedeutet werden soll. In der That, setzt man

$$u = \sqrt{\frac{y_1 y_3}{y_2^2} - \frac{3}{2}},$$

so wird

$$\frac{du}{d\varphi_1} = \frac{u\varphi_1 - 2u^2 - 1}{2\varphi_2 - 3\varphi_1}.$$

Ist diese Riccatische Gleichung integrirt, so findet man durch Differentiation zwei weitere Integralgleichungen der Gleichung $\varphi_2 = f(\varphi_1)$, die man darnach auf eine Differentialgleichung zweiter Ordnung mit den beiden infinitesimalen Transformationen p und xp reducirt. Durch zwei Quadraturen findet man daher endlich y als Function von x.

9. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q \quad yq \quad y^2q \quad p \quad xp \quad x^2p,$$

so ist sie, wenn wir von der integrablen Gleichung $2y_1y_3 - 3y_2^2 = 0$ absehen, reducibel auf die Form

$$\Omega\left(\varphi_1 \varphi_2 \cdots \frac{d^{m-6}\varphi_2}{d\varphi_1^{m-6}}\right) = 0,$$

wo

$$\varphi_1 = \frac{4ww'' - 5w'^2}{w^3}$$

$$\varphi_2 = \frac{4w^2w''' - 18ww'w'' + 15w'^3}{w^{\frac{9}{2}}}$$

und wie früher

$$w = \frac{y_3}{y_1} - \frac{3}{2}\frac{y_2^2}{y_1^2}, \quad w' = \frac{dw}{dx}\ldots$$

gesetzt ist. Durch Integration der Gleichung $(m-6)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Relation der Form

$$\varphi_2 = f(\varphi_1),$$

die eine Differentialgleichung 3. O. in den Variabeln x und w darstellt. Dieselbe gestattet drei bekannte infinitesimale Transformationen p, xp, x^2p, die in den Variabeln $x\,w$ die Form

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x} - 2w\frac{\partial f}{\partial w}, \quad x^2\frac{\partial f}{\partial x} - 4xw\frac{\partial f}{\partial w}$$

besitzen. Zur Integration unserer Differentialgleichung 3. O. führen wir die Grössen

$$u = w^{-\frac{3}{2}}w', \quad \varphi_1 = 4w^{-2}w'' - 5w^{-3}w'^2$$

als neue Variabeln ein. Dann wird

(6) $$\frac{du}{d\varphi_1} = \frac{\varphi_1 - u^2}{\varphi_2}.$$

Ist

$$W(u\varphi_1) = \text{Const.}$$

eine Integralgleichung dieser Riccatischen Gleichung 1. O., so findet man folgendermassen durch Differentiation die beiden fehlenden Integralgleichungen von $\varphi_2 = f(\varphi_1)$, aufgefasst als Differentialgleichung 3. O. in w und x. Setzen wir

$$Bf = x^2\frac{\partial f}{\partial x} - 4xw\frac{\partial f}{\partial w} - (6xw' + 4w)\frac{\partial f}{\partial w'} - (8xw'' + 10w')\frac{\partial f}{\partial w''}$$

so ist $B\varphi_1 = 0$

$$BW = \frac{\partial f}{\partial u}Bu = -4\frac{\partial W}{\partial u}w^{-\frac{1}{2}}$$

$$BBW = 16\frac{\partial^2 W}{\partial u^2}w^{-1} - 8xw^{-\frac{1}{2}}\frac{\partial W}{\partial u},$$

und da die Grössen W, BW und BBW hinsichtlich $u\,\varphi_1\,x$ und w unabhängig sind, so findet man durch Elimination von u und φ_1 zwischen den drei Gleichungen

(7) $$W = \text{Const.}, \quad BW = \text{Const.}, \quad BBW = \text{Const.}$$

die Grösse w bestimmt als Function von x:

$$w = F(x).$$

Diese Gleichung ist nun selbst eine Differentialgleichung 3. O. in y und x, die nach den Regeln der Nummer 6 vermöge einer Riccatischen 1. O. integrirt wird.

Hiermit ist die Gleichung sechster Ordnung $\varphi_2 = f(\varphi_1)$ vermöge *zweier* Riccatischer Gleichungen 1. O. integrirt. Dabei ist indess zu bemerken, dass wir erst nach der Integration der ersten Hülfsgleichung (6) die zweite Hülfsgleichung 1. O. aufstellen konnten. Es ist aber nicht schwierig einzusehen, dass man die Integration von $\varphi_2 = f(\varphi_1)$ auf die Integration zweier *von einander unabhängiger Riccatischer* Gleichungen 1. O. zurückführen kann. Man bemerke in der That nur, dass die beiden Gruppen q, yq, y^2q und $p\ xp\ x^2p$ vollständig gleichberechtigt sind. Vertauscht man daher im Vorangehenden die Grössen x und y, so erhält man eine mit (6) analoge Riccatische Gleichung, deren Integration ebenfalls drei Integralgleichungen

$$W_1 = \text{Const.}, \quad CW_1 = \text{Const.}, \quad CCW_1 = \text{Const.}$$

von $\varphi_2 = f(\varphi_1)$ liefert. Dabei ist es einleuchtend, dass diese drei neuen Integralgleichungen von den drei früheren (7) unabhängig sind. Und also ist wirklich die Gleichung sechster Ordnung $\varphi_2 = f(\varphi_1)$ auf zwei unabhängige Riccatische Gleichungen 1. O. zurückgeführt.

§ 2.

Integration von Differentialgleichungen mit bekannten infinitesimalen Transformationen der Form $X(x)p + Y(xy)q$.

In diesem Paragraphen entwickeln wir die Integrationstheorie aller Differentialgleichungen von zweiter und höherer Ordnung mit einer bekannten Gruppe, deren sämmtliche infinitesimale Transformationen die Form $X(x)p + \eta(xy)q$ besitzen. Dabei wird ausdrücklich vorausgesetzt, dass die betreffende Gruppe keine andere Differentialgleichung erster Ordnung als $\frac{1}{y'} = 0$ invariant lässt.

Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe $p,\ xp + yq,\ x^2p + 2xyq$, so ist sie, wenn wir

$$2yy_2 - y_1^2 = \varphi_1, \quad y^2y_3 = \varphi_2$$

setzen, reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2 \cdots \frac{d^{m-3}\varphi_2}{d\varphi_1^{m-3}}\right) = 0.$$

Man integrirt diese Gleichung $(m-3)$. O. und erhält hierdurch eine Differentialgleichung 3. O.:

$$\varphi_2 = f(\varphi_1),$$

die wir jetzt auf eine Riccatische Gleichung 1. O. reduciren werden. Wir führen neue Variable ein, nämlich y_1 und φ_1; dann wird

$$\frac{dy_1}{d\varphi_1} = \frac{y_1^2 + \varphi_1}{4\varphi_2}.$$

Sei $W(y_1\varphi_1) = $ Const. eine Integralgleichung der soeben erhaltenen Riccatischen Gleichung und sei

$$Bf = x^2\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y} + 2y\frac{\partial f}{\partial y_1} + (2y_1 - 2xy_2)\frac{\partial f}{\partial y_2},$$

so wird

$$BW = 2\frac{\partial W}{\partial y_1}y$$

$$BBW = 4y^2\frac{\partial^2 W}{\partial y_1^2} + 4xy\frac{\partial W}{\partial y_1},$$

und, da die Grössen W, BW und BBW offenbar unabhängig sind, so geben die Integralgleichungen

$$W = \text{Const.},\quad BW = \text{Const.},\quad BBW = \text{Const.}$$

durch Elimination von y_1 und φ_1 die Bestimmung von y als Function von x.

11. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$yq,\quad p,\quad xp,\quad x^2p + xyq$$

so ist sie, wenn wir von der unmittelbar integrablen Gleichung $y_2 = 0$ absehen, reducibel auf die Form

$$\Omega\left(\varphi_1\varphi_2\cdots\frac{d^{m-4}\varphi_2}{d\varphi_1^{m-4}}\right) = 0$$

wo φ_1 und φ_2 die Werthe

$$\varphi_1 = y^{\frac{1}{2}}y_2^{-\frac{3}{2}}y_3 + 3y^{-\frac{1}{2}}y_1y_2^{-\frac{1}{2}}$$

$$\varphi_2 = 3yy_2^{-2}y_4 - 4yy_2^{-3}y_3^2$$

haben. Wir integriren die Gleichung $(m-4)^{\text{ter}}$ Ordnung $\Omega = 0$, und erhalten hierdurch eine Relation

$$\varphi_2 = f(\varphi_1)$$

das heisst eine Differentialgleichung vierter Ordnung, die unsere Gruppe gestattet. Dieselbe soll jetzt auf eine Riccatische Gleichung 1. O. reducirt werden. Wir führen φ_1 und

$$u = (yy_1^{-2}y_2)^{\frac{1}{2}}$$

als neue Variabeln ein. Dann wird

$$\frac{du}{d\varphi_1} = \frac{u\varphi_1 - 2 - 2u^2}{\frac{2}{3}\varphi_2 - \frac{1}{3}\varphi_1^2 + 6}.$$

Ist $W(u\varphi_1)$ eine Integralgleichung der gefundenen Riccatischen Gleichung, so setzen wir

$$Bf = x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + (y - xy_1)\frac{\partial f}{\partial y_1} - 3xy_2\frac{\partial f}{\partial y_2} - (5xy_3 + 3y_2)\frac{\partial f}{\partial y_3}$$

und bilden die Ausdrücke

$$BW = -\frac{\partial W}{\partial u} u y y_1^{-1}$$

$$BBW = u\frac{\partial}{\partial u}\left(u\frac{\partial W}{\partial u}\right) y^2 y_1^{-2} + u\frac{\partial W}{\partial u}(y^2 y_1^{-2} - 2xyy_1^{-1}),$$

die offenbar von einander und von W unabhängig sind. Daher erhält man durch Elimination von u und φ_1 zwischen den Gleichungen

$$W = \text{Const.}, \quad BW = \text{Const.}, \quad BBW = \text{Const.}$$

eine Relation der Form

$$yy_1^{-1} = F(x),$$

woraus als definitive Integralgleichung

$$y = e^{\int\frac{dx}{F}}$$

hervorgeht.

Man kann im Uebrigen die Integration der Gleichung $\varphi_2 = f(\varphi_1)$ in etwas anderer Weise durchführen, wie ich hier kurz angeben werde. Bringt man in der That die vorgelegte Gruppe auf die Form

$$B_1 f = p, \quad B_2 f = 2xp + yq, \quad B_3 f = x^2 p + xyq$$
$$B_4 f = yq,$$

so bilden $B_1 f\, B_2 f\, B_3 f$ eine dreigliedrige Untergruppe und dabei bestehen die Relationen

$$(B_1 B_4) = 0, \quad (B_2 B_4) = 0, \quad (B_3 B_4) = 0$$

(die, wie ich beiläufig bemerke, aussagen, dass $B_1\, B_2\, B_3$ eine *invariante* Untergruppe bilden). Bringe ich daher die Gleichung $\varphi_2 = f(\varphi_1)$ auf die Form

$$y_4 = F(xyy_1 y_2 y_3)$$

und ersetze sie darnach durch die lineare partielle Differentialgleichung

$$Af = \frac{\partial f}{\partial x} + y_1\frac{\partial f}{\partial y} + y_2\frac{\partial f}{\partial y_1} + y_3\frac{\partial f}{\partial y_2} + F\frac{\partial f}{\partial y_3} = 0,$$

so bilden die Gleichungen

$$Bf = 0, \quad B_1 f = 0, \quad B_2 f = 0, \quad B_3 f = 0$$

ein vollständiges System mit der bekannten infinitesimalen Transformation $B_4 f$. Das entsprechende Integral, das man ohne weiteres aufstellen kann, liefert eine Differentialgleichung 3. O. mit der bekannten Gruppe p, $2xp + yq$, $x^2 p + xyq$. Sie wird nach den Regeln der vorangehenden Nummer auf eine Riccatische Gleichung 1. O. reducirt.

13. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$X_1 q, \; X_2 q \ldots X_r q,$$

so ist sie reducibel auf die Form

$$\Omega\left(x\, D\, \frac{dD}{dx} \cdots \frac{d^{m-r} D}{dx^{m-r}}\right) = 0,$$

wo

$$\begin{vmatrix} X_1 & X_1' & \cdots & X_1^{(r)} \\ \cdot & \cdot & \cdot & \cdot \\ X_r & X_r' & \cdots & X_r^{r} \\ y & y_1 & \cdots & y_r \end{vmatrix} = D$$

gesetzt ist. Durch Integration der Gleichung $(m-r)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Relation

$$D = F(x)$$

das heisst eine lineare Differentialgleichung r^{ter} Ordnung, die bekanntlich nach *Lagranges* oder *Cauchys* Regeln integrirt werden kann, indem das allgemeine Integral von $D = 0$ bekannt und gleich $\Sigma c_i X_i$ ist.

Ist die vorgelegte Gleichung m^{ter} Ordnung linear, so ist auch $\Omega = 0$ linear. Der bekannte Satz, dass eine lineare Gleichung m^{ter} Ordnung mit r bekannten Particularintegralen sich auf eine lineare Gleichung $(m-r)^{\text{ter}}$ Ordnung reduciren lässt, ist somit ein sehr specieller Fall unserer soeben entwickelten Theorie.

Auch die oben besprochene Reduction der Gleichung $D = F(x)$ auf die einfachere Gleichung $D = 0$ fliesst als sehr specielles Corollar aus meinen alten Integrationstheorien. Ich werde diesen Zusammenhang in zwei etwas von einander verschiedenen Weisen begründen. Sei die Gleichung $D = F(x)$ auf die Form

$$y_r = V$$

oder die äquivalente Form

$$Af = \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y} + \cdots + V \frac{\partial f}{\partial y_{r-1}} = 0$$

gebracht. Diese lineare partielle Differentialgleichung gestattet r bekannte infinitesimale Transformationen:

$$B_i f = X_i \frac{\partial f}{\partial y} + X_i' \frac{\partial f}{\partial y_1} + \cdots + X_i^{(r-1)} \frac{\partial f}{\partial y_{r-1}}, \quad (i = 1, 2 \cdots r)$$

die paarweise in der Beziehung

$$(B_i B_k) = 0$$

stehen. Also bilden die Gleichungen

$$Af = 0 \quad B_1 f = 0 \quad B_{k-1} f = 0 \quad B_{k+1} f = 0 \cdots B_r f = 0$$

ein vollständiges System mit der bekannten infinitesimalen Transformation $B_k f$; und daher findet man die entsprechende Lösung W_k

durch Quadratur. In dieser Weise findet man r unabhängige Lösungen von $Af = 0$, deren Integration hiermit geleistet ist.

Die hiermit ausgeführte, principiell einfache Integration von $D = F(x)$ ist insofern unvollkommen, als sie nicht die explicite Form der Grösse y als Function von x liefert. Daher füge ich die folgenden Bemerkungen hinzu. Setze ich

$$\begin{vmatrix} X_1 X_1' \cdots X_1^{(r-1)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{r-1} \cdots X_{r-1}^{(r-1)} \\ y \cdots \cdots y_{r-1} \end{vmatrix} = D_r,$$

so kann die Gleichung $D + F(x) = 0$ nach dem Vorangehenden auf die Form

$$\frac{dD_r}{dx} + \varphi(x) D_r + f(x) = 0$$

gebracht werden. Ordnen wir die letzte Gleichung nach den Grössen y_i, so kommt

$$(X_1 X_2' \cdots X_{r-1}^{(r-2)}) \{y_r + \varphi \cdot y_{r-1}\} + \cdots + f(x) = 0$$

und anderseits erhält $D + F(x)$ durch Entwickelung, wenn wir zur Abkürzung

$$(X_1 X_2' \cdots X_r^{r-1}) = \Delta$$

setzen, die Form:

$$\Delta \cdot y_r + \frac{d\Delta}{dx} y_{r-1} + \cdots + F(x).$$

Durch Vergleichung findet man daher die folgenden Werthe von $\varphi(x)$ und $f(x)$:

$$\varphi(x) = \frac{d \log \Delta}{dx}, \quad f(x) = \frac{\Delta_r}{\Delta} F(x),$$

wo

$$(X_1 X_2' \cdots X_{r-1}^{(r-2)}) = \Delta_r$$

gesetzt ist. Also kann $D + F(x) = 0$ die Form

$$\frac{dD_r}{dx} + \frac{d \log \Delta}{dx} D_r + \frac{\Delta_r}{\Delta} F(x) = 0$$

erhalten und durch Integration kommt

$$D_r = -\frac{1}{\Delta} \int \Delta_r F(x) dx.$$

Analoge Ueberlegungen geben uns die r Formeln

$$D_i = -\frac{1}{\Delta} \int \Delta_i F(x)\, dx, \ (i = 1 \cdots r),$$

und da die r Grössen D_i linear und homogen in den Grössen $y y_1 \ldots y_{r-1}$ sind, so findet man durch Auflösung die bekannte Form der Grösse y.

Man sieht leicht, dass diese beiden Integrationstheorien der Gleichung $D + F(x) = 0$ im Wesentlichen identisch sind.

13. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$X_1 q \cdots X_r q \quad y q,$$

so ist sie reducibel auf die Form

$$\Omega\left(x \frac{d \log D}{dx} \cdots \frac{d^{m-r} \log D}{dx^{m-r}}\right) = 0,$$

wo D dieselbe Determinante wie in der vorangehenden Nummer bezeichnet. Durch Integration dieser Gleichung $(m-r-1)^{\text{ter}}$ Ordnung erhält man eine Relation

$$\frac{d \log D}{dx} = F(x)$$

und durch Quadratur die lineare Gleichung

$$D = e^{\int F \cdot dx},$$

die nach den Regeln der vorangehenden Nummer integrirt wird.

14. Gestattet eine Differentialgleichung m^{ter} Ordnung eine Gruppe von der Form

$$X_1 q \cdots X_r q, \ p,$$

so ist sie reducibel auf die Form

$$\Omega\left(\varphi \frac{d\varphi}{dx} \cdots \frac{d^{m-r} \varphi}{dx^{m-r}}\right) = 0$$

wo φ eine lineare und homogene Function mit constanten Coefficienten von $y\, y_1 \ldots y_r$ darstellt:

$$\varphi = cy + c_1 y_1 + \cdots + c_r y_r.$$

Man integrirt $\Omega = 0$, die als eine Gleichung $(m-r-1)^{\text{ter}}$ Ordnung zu betrachten ist. Hierdurch findet man eine Differentialgleichung r^{ter} Ordnung der Form

$$cy + \cdots + c_r y_r = F(x),$$

die in der bekannten Weise integrirt wird.

15. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$X_1 q \cdots X_r q \quad y q \quad p,$$

so ist sie reducibel auf die Form

$$\Omega\left(\varphi_1\, \varphi_2 \cdots \frac{d^{m-r-2} \varphi_2}{d\varphi_1^{\,m-r-2}}\right) = 0;$$

φ_1 und φ_2 haben die Werthe

$$\varphi_1 = \frac{\frac{d\varphi}{dx}}{\varphi} \quad \varphi_2 = \frac{\frac{d^2\varphi}{dx^2}}{\varphi}$$

wo φ wie soeben eine ganze und homogene Function mit constanten Coefficienten von $y\, y_1 \ldots y_r$ bezeichnet.

Durch Integration der Gleichung $(m-r-2)^{\text{ter}}$ Ordnung $\Omega = 0$ kommt eine Relation

$$\varphi_2 = f(\varphi_1)$$

oder

$$\frac{1}{\varphi}\,\frac{d^2\varphi}{dx^2} = f\left(\frac{d\log\varphi}{dx}\right)$$

oder endlich

$$\frac{d^2\log\varphi}{dx^2} = f\left(\frac{d\log\varphi}{dx}\right) - \left(\frac{d\log\varphi}{dx}\right)^2.$$

Diese Differentialgleichung zweiter Ordnung erledigt man durch zwei Quadraturen, und erhält so eine Gleichung

$$\varphi = F(x)$$

die in der bekannten Weise integrirt wird.

16. Gestattet eine Gleichung m^{ter} Ordnung die Gruppe

$$q\ xq \cdots x^{r-1}q,\ p,\ xp + cyq,\ c \neq r,$$

so ist sie reducibel auf die Form

$$\Omega\left(\varphi_1\varphi_2 \cdots \frac{d^{m-r-2}\varphi_2}{d\varphi_1^{m-r-2}}\right) = 0,$$

wo

$$\varphi_1 = \frac{y_{r+1}}{y_r^{\frac{c-r-1}{c-r}}}, \quad \varphi_2 = \frac{y_{r+2}}{y_r^{\frac{c-r-2}{c-r}}}.$$

Durch Integration der Gleichung $(m-r-2)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Relation

$$y_{r+1}\,\frac{dy_{r+1}}{dy_r} = y_r^{\frac{c-r-2}{c-r}}\, f(\varphi_1),$$

die eine Differentialgleichung 1. O. in den Variabeln y_r und y_{r+1} darstellt. Diese Gleichung gestattet die infinitesimale Transformation

$$x\frac{\partial f}{\partial x} + cy\frac{\partial f}{\partial y} + \cdots + (c-r)y_r\frac{\partial f}{\partial y_r} + (c-r-1)y_{r+1}\frac{\partial f}{\partial y_{r+1}},$$

und also giebt eine Quadratur eine Bestimmung von y_{r+1} als Function von y_r. Darnach giebt eine zweite Quadratur y_r als Function von x und endlich findet man vermöge r neuer Quadraturen y als Function von x.

17. Gestattet eine Gleichung m^{ter} Ordnung die Gruppe

$$q\ xq \ldots x^{r-1}q\ p\ xp + ryq,$$

so ist sie, wenn wir von den integrabeln Gleichungen $y_{r+1} = 0$, $y_r =$ Const. absehen, reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2 \cdots \frac{d^{m-r-2}\varphi_2}{d\varphi_1^{m-r-2}}\right) = 0,$$

wo

$$\varphi_1 = y_r, \quad \varphi_2 = \frac{y_{r+2}}{y_{r+1}^2}.$$

Durch Integration der Gleichung $(m-r-2)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Relation

$$y_{r+2} = y_{r+1}^2 f(y_r)$$

oder

$$\frac{dy_{r+1}}{dy_r} = y_{r+1} f(y_r),$$

woraus

$$y_{r+1} = e^{\int f(y_r)\,dy_r}$$

und

$$x = \int dy_r\, e^{-\int f(y_r)\,dy_r}.$$

Hierdurch ist y_r bestimmt als Function von x und daher findet man y durch r weitere Quadraturen.

18. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q\;xq \cdots x^{r-1}q\;p\;xp + (ry + x^r)\,q,$$

so hat sie die Form

$$\Omega\left(\varphi_1\,\varphi_2 \cdots \frac{d^{m-r-2}\varphi_2}{d\varphi_1^{m-r-2}}\right) = 0,$$

wo

$$\varphi_1 = y_{r+1}\,e^{w y_r}, \quad \varphi_2 = y_{r+2}\,e^{2 w y_r}$$

$$\frac{1}{w} = 1 \cdot 2 \cdots (r-1)\,r.$$

Durch Integration der Gleichung $(m-r-2)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Relation $\varphi_2 = f(\varphi_1)$ oder

$$y_{r+1}\,\frac{dy_{r+1}}{dy_r} = e^{-2 w y_r} f\left(y_{r+1}\,e^{w y_r}\right),$$

das heisst eine Differentialgleichung 1. O. zwischen y_r und y_{r+1} mit der bekannten infinitesimalen Transformation

$$x\,\frac{\partial f}{\partial x} + (ry + x^r)\,\frac{\partial f}{\partial y} + \cdots + \frac{1}{w}\,\frac{\partial f}{\partial y_r} - y_{r+1}\,\frac{\partial f}{\partial y_{r+1}}.$$

Daher bestimmt man zuerst y_{r+1} als Function von y_r durch eine

Quadratur, darnach y_r als Function von x durch eine zweite Quadratur und schliesslich y als Function von x vermöge r Quadraturen.

19. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q \; xq \cdots x^{r-1}q \; yq \; p \; xp,$$

so hat sie, wenn wir von den beiden integrabeln Gleichungen $y_r = 0$, $y_{r+1} = 0$ absehen, die Form

$$\Omega\left(\varphi_1 \varphi_2 \cdots \frac{d^{m-r-3}\varphi_2}{d\varphi_1^{\;m-r-2}}\right) = 0,$$

wo

$$\varphi_1 = \frac{y_r y_{r+2}}{y_{r+1}^2}, \quad \varphi_2 = \frac{y_r^2 y_{r+3}}{y_{r+1}^3}.$$

Durch Integration der Gleichung $(m-r-3)^{ter}$ Ordnung $\Omega = 0$ erhält man eine Relation $\varphi_2 = f(\varphi_1)$, das heisst eine Differentialgleichung 2. O. in y_r und y_{r+1}:

$$\frac{y_{r+3}}{y_{r+1}} = \frac{dy_{r+2}}{dy_r} = \frac{y_{r+1}^2}{y_r^2} f\left(\frac{y_r y_{r+2}}{y_{r+1}^2}\right) = \frac{d}{dy_r}\left(y_{r+1} \frac{dy_{r+1}}{dy_r}\right)$$

mit zwei bekannten infinitesimalen Transformationen

$$y_r \frac{\partial f}{\partial y_r} \quad \text{und} \quad y_{r+1} \frac{\partial f}{\partial y_{r+1}}.$$

Hier führen wir

$$\eta = \log y_{r+1}, \quad \xi = \log y_r$$

als neue Variabeln ein, dann wird

$$\varphi_1 = \frac{y_r y_{r+2}}{y_{r+1}^2} = \frac{d\eta}{d\xi}, \quad \varphi_2 = \frac{d^2\eta}{d\xi^2} + 2\frac{d\eta^2}{d\xi} - \frac{d\eta}{d\xi},$$

sodass die Gleichung $\varphi_2 = f(\varphi_1)$ die Form annimmt:

$$\frac{d^2\eta}{d\xi_1} = F\left(\frac{d\eta}{d\xi}\right).$$

Daher geben zwei Quadraturen η als Function von ξ, das heisst y_{r+1} als Function von y_r. Eine neue Quadratur giebt y_r als Function von x, wonach y durch r Quadraturen als Function von x bestimmt wird.

20. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q \; xq \cdots x^{r-1}q \; p \; 2xp + (r-1)\,yq \; x^2p + (r-1)xyq$$

so ist sie, wenn wir von der unmittelbar integrabeln Gleichung $y_r = 0$ absehen, reducibel auf die Form

$$\Omega\left(\varphi_1\,\varphi_2\cdots\frac{d^{m-r-3}\varphi_2}{d\varphi_1^{m-r-3}}\right)=0$$

wo φ_1 und φ_2 die Werthe

$$\varphi_1 = y_r^{\frac{-2(r+3)}{r+1}}\left((r+1)\,y_r y_{r+2}-(r+2)\,y_{r+1}^2\right)=y_r^{\frac{-2(r+3)}{r+1}}\,u$$

$$\varphi_2 = y_r^{\frac{-3(r+3)}{r+1}}\,u_1 =$$

$$y_r^{\frac{-3(r+3)}{r+1}}\left((r+1)^2 y_r^2 y_{r+3}-3(r+1)(r+3)\,y_r y_{r+1} y_{r+2}+2(r+2)(r+3)\,y_{r+1}^3\right)$$

haben. Durch Integration der Gleichung $(m-r-3)^{\text{ter}}$ Ordnung $\Omega=0$ erhält man eine Relation $\varphi_2=f(\varphi_1)$, die nicht allgemein integrabel ist, während sie, wie jetzt gezeigt werden soll, immer auf eine *Riccatische* Gleichung 1. O. reducirt werden kann. Wir wählen φ_1 und

$$v = y_r^{-\frac{r+3}{r+1}}\,y_{r+1}$$

als neue Variabeln. Dann wird

$$\frac{dv}{d\varphi_1}=\frac{v^2+\varphi_1}{\varphi_2}=\frac{v^2+\varphi_1}{f(\varphi_1)}.$$

Ist $W(v\varphi_1)=\text{Const.}$ eine Integralgleichung dieser Riccatischen Gleichung, so findet man zwei weitere Integralgleichungen von $\varphi_2=f(\varphi_1)$ durch Differentiation. In der That setzt man

$$Bf=(r+1)\,x\,y_r\,\frac{\partial f}{\partial y_r}+[(r+3)\,xy_{r+1}+(r+1)\,y_r]\,\frac{\partial f}{\partial y_{r+1}}$$
$$+[(r+5)\,xy_{r+2}+2(r+2)\,y_{r+1}]\,\frac{\partial f}{\partial y_{r+2}},$$

so ist $B\varphi_1=0$, während die Ausdrücke

$$B\,W=\frac{\partial f}{\partial v}\,(r+1)\,y_r$$
$$BB\,W=\frac{\partial^2 f}{\partial v^2}\,(r+1)^2 y_r^2+(r+1)^2\,x\,y_r\,\frac{\partial f}{\partial v}$$

von W unabhängig sind. Daher sind die Relationen $W=\text{Const.}$, $B\,W=\text{Const.}$, $BB\,W=\text{Const.}$ unabhängige Integralgleichungen von $\varphi_2=f(\varphi_1)$. Und daher erhält man durch Elimination von $y_{r+1}y_{r+2}$ und y_{r-3} eine Differentialgleichung der Form

$$y_r=F(x)$$

(mit der bekannten Gruppe $q\;xq\cdots x^{r-1}q$) und schliesslich geben r Quadraturen die Bestimmung von y als Function von x.

21. Gestattet eine Differentialgleichung m^{ter} Ordnung die Gruppe

$$q \;\; xq \cdots x^{r-1}q \;\; p, \;\; xp \;\; yq \;\; x^2p + (r-1)\,xyq$$

so ist sie, wenn wir von den unmittelbar integrabeln Gleichungen

$$y_r = 0, \quad \left(y^{-\frac{1}{r+1}}\right)'' = 0$$

absehen, reducibel auf die Form

$$\Omega\left(\varphi_1 \; \varphi_2 \cdots \frac{d^{m-r-4}\varphi_2}{d\varphi_1^{\,m-r-4}}\right) = 0,$$

wo

$$\varphi_1 = u^{-\frac{3}{2}}u_1, \quad \varphi_2 = u^{-2}u_2$$

während u, u_1 und u_2 dieselben Werthe wie in der vorangehenden Nummer (siehe auch Abschn. I, Nummer 23) haben. Durch Integration der Gleichung $(m-r-4)^{\text{ter}}$ Ordnung $\Omega = 0$ erhält man eine Differentialgleichung $(r-4)^{\text{ter}}$ Ordnung $\varphi_2 = f(\varphi_1)$, die allerdings nicht allgemein integrabel ist, während sie immer, wie jetzt gezeigt werden soll, auf eine Riccatische Differentialgleichung 1. O. reducirt werden kann. Um dies nachzuweisen betrachten wir $\varphi_2 = f(\varphi_1)$ als eine Differentialgleichung *vierter* Ordnung zwischen y_r und x. In diesen Variabeln erhalten die bekannten inf. Transformationen p, xp, yq, $x^2p + (r-1)\,xyq$ die Formen

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, \quad y_r\frac{\partial f}{\partial y_r}, \quad x^2\frac{\partial f}{\partial x} - (r+1)\,xy_r\frac{\partial f}{\partial y_r}.$$

Setzen wir

$$\eta = y_r^{-\frac{1}{r+1}},$$

so erhalten wir eine Differentialgleichung vierter Ordnung zwischen η und x mit den vier bekannten infinitesimalen Transformationen

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, \quad \eta\frac{\partial f}{\partial \eta}, \quad x^2\frac{\partial f}{\partial x} + \eta x\frac{\partial f}{\partial \eta}.$$

Daher findet man nach den Regeln der Nummer 11 vermöge einer Riccatischen Gleichung 1. O. die Grösse η bestimmt als Function von x

$$\eta = y_r^{-\frac{1}{r+1}} = F(x),$$

woraus

$$y_r = F(x)^{-(r+1)};$$

hiernach genügen r Quadraturen zur Bestimmung von y als Function von x.

§ 3.

Integration von Differentialgleichungen mit bekannten infinitesimalen Transformationen der Form $X(xy)p + Y(xy)q$.

In diesem Paragraphen integrire ich alle Differentialgleichungen mit einer bekannten fünfgliedrigen, sechsgliedrigen oder achtgliedrigen Gruppe; dabei wird ausdrücklich vorausgesetzt, dass die betreffende Gruppe keine Curvenschaar $\varphi(xy) = a$ invariant lässt und dass sie daher in Uebereinstimmung mit meinen alten Untersuchungen auf die Form einer projectiven Gruppe gebracht worden ist. Ich sehe ab von den unmittelbar integrabeln Gleichungen

$$y_2 = 0, \quad 5y_3^2 - 3y_2y_4 = 0,$$
$$9y^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0,$$

unter denen die erste alle gerade Linien, die zweite alle Parabeln, die dritte alle Kegelschnitte der Ebene xy bestimmt.

22. Gestattet eine Differentialgleichung, deren Ordnungszahl m grösser als 2 ist, die Gruppe

$$p \quad q \quad xq \quad xp - yq \quad yp,$$

so besitzt sie die Form

$$\Omega\left(\varphi_1\, \varphi_2\, \frac{d\varphi_2}{d\varphi_1} \cdots \frac{d^{m-5}\varphi_2}{d\varphi_1^{m-5}}\right) = 0$$

wo

$$\varphi_1 = y_2^{-\frac{8}{3}}\varrho_2, \quad \varphi_2 = y_2^{-4}\varrho_3$$

und

$$\varrho_2 = 3y_2y_4 - 5y_3^2, \quad \varrho_3 = 3y_2^2y_5 - 15y_2y_3y_4 + \frac{40}{3}y_3^3.$$

Man integrirt die Gleichung $(m-5)^{\text{ter}}$ Ordnung $\Omega = 0$ und erhält hierdurch eine Relation

$$\varphi_2 = F(\varphi_1)$$

das heisst eine Differentialgleichung*) fünfter Ordnung, die wir jetzt auf eine *Riccatische* Gleichung 1. O. reduciren werden.

Wir führen neue Variabeln ein, nämlich φ_1 und

$$u = y_2^{-\frac{4}{3}}y_3;$$

dann wird, wie man leicht findet

$$\frac{du}{d\varphi_1} = \frac{1}{3}\,\frac{\varrho_2 + y_3^2}{y_2^{-\frac{4}{3}}\varrho_3} = \frac{1}{3}\,\frac{\varphi_1 + u^2}{\varphi_2}$$

*) Wir sehen im Texte ab von der unmittelbar integrablen Gleichung $\varphi_1 = \text{Const.}$

oder

$$\frac{du}{d\varphi_1} = \frac{1}{3}\,\frac{u^2+\varphi_1}{F(\varphi_1)}.$$

Ist

$$W(u\varphi_1) = \text{Const.}$$

eine Integralgleichung dieser Riccatischen Gleichung, so findet man folgendermassen zwei neue Integralgleichungen von $\varphi_2 = F(\varphi_1)$ durch Differentiation. Man setzt

$$Bf = y\frac{\partial f}{\partial x} - y_1^2\frac{\partial f}{\partial y_1} - 3y_1y_2\frac{\partial f}{\partial y_2} - (4y_1y_3+3y_2^2)\frac{\partial f}{\partial y_3} - (5y_1y_4+10y_2y_3)\frac{\partial f}{\partial y_4},$$

dann ist

$$B\varphi_1 = 0,\quad Bu = -3y_2^{\frac{2}{3}}$$

und

$$BW = -3\frac{\partial W}{\partial u}y_2^{\frac{2}{3}}$$

$$BBW = 9y_2^{\frac{4}{3}}\frac{\partial^2 W}{\partial u^2} + 6y_1y_2^{\frac{2}{3}}\frac{\partial W}{\partial u},$$

sodass

$$W = \text{Const.},\quad BW = \text{Const.},\quad BBW = \text{Const.}$$

drei unabhängige Integralgleichungen von $\varphi_2 = F(\varphi_1)$ darstellen. Eliminirt man zwischen ihnen die Grössen y_5, y_4 und y_3, so erhält man eine Differentialgleichung zwischen y_1 und y_2, die durch zwei Quadraturen erledigt wird.

23. Gestattet eine Differentialgleichung m^{ter} Ordnung $(m > 4)$ die Gruppe

$$p\quad q\quad xq\quad yq\quad xp\quad yp,$$

so besitzt sie die Form

$$\Omega\left(\varphi_1\,\varphi_2\,\frac{d\varphi_2}{d\varphi_1}\cdots\frac{d^{m-6}\varphi_2}{d\varphi_1^{m-6}}\right) = 0$$

wo

$$\varphi_1 = \varrho_2^{-\frac{3}{2}}\varrho_3,\quad \varphi_2 = \varrho_2^{-2}\varrho_4$$

und

$$\varrho_2 = 3y_2y_4 - 5y_3^2,$$

$$\varrho_3 = 3y_2^2y_5 - 15y_2y_3y_4 + \frac{40}{3}y_3^3,$$

$$\varrho_4 = 3y_2^3y_6 - 21y_2^2y_3y_5 + 35y_2y_3^2y_4 - \frac{35}{3}y_3^4.$$

Wir integriren zuerst die Gleichung $(m-6)^{\text{ter}}$ Ordnung $\Omega = 0$, und erhalten hierdurch eine Relation mit $m-6$ Constanten*)

*) Wir betrachten im Texte nicht die unmittelbar integrable Gleichung $\varphi_1 = \text{Const.}$

$$\varphi_2 = F(\varphi_1) \quad \text{oder} \quad y_6 = W(y_1 \cdots y_5),$$

die selbst eine Differentialgleichung sechster Ordnung darstellt. Wir reduciren dieselbe durch Quadratur auf eine Gleichung fünfter Ordnung, die nach den Regeln der letzten Nummer vermöge einer Riccatischen Gleichung 1. O. erledigt werden kann.

Die bekannte sechsgliedrige Gruppe enthält nämlich die invariante fünfgliedrige Untergruppe

$$p \quad q \quad xq \quad xp - yq \quad yp.$$

Daher bildet die mit der vorgelegten Gleichung $y_6 = W$ äquivalente lineare partielle Differentialgleichung

$$Af = \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y} + \cdots + y_5 \frac{\partial f}{\partial y_4} + W \frac{\partial f}{\partial y_5}$$

zusammen mit den fünf Gleichungen

$$B_1 f = \frac{\partial f}{\partial x} = 0, \quad B_2 f = \frac{\partial f}{\partial y} = 0, \quad B_3 f = x \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y_1} = 0$$

$$B_4 f = x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - 2y_1 \frac{\partial f}{\partial y_1} - \cdots - 5y_5 \frac{\partial f}{\partial y_5} = 0$$

$$B_5 f = y \frac{\partial f}{\partial x} - y_1^2 \frac{\partial f}{\partial y_1} - \cdots = 0$$

ein vollständiges System mit der bekannten infinitesimalen Transformation

$$B_6 f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} - y_2 \frac{\partial f}{\partial y_2} - \cdots - 4y_5 \frac{\partial f}{\partial y_5}.$$

Daher liefern meine alten Theorien durch eine Quadratur ein Integral

$$U(xy \cdots y_5) = \text{Const.}$$

des vollständigen Systems. Nun aber ist $U = \text{Const.}$ eine Differentialgleichung fünfter Ordnung, welche die Gruppe $p \; q \; xq \; xp - yq \; yp$ gestattet, und welche somit nach den Regeln der vorangehenden Nummer vermöge einer Riccatischen Gleichung 1. O. integrirt wird.

Um die hiermit scizzirten Rechnungen in einfachster Weise durchzuführen, ist es zweckmässig, folgendermassen zu verfahren. Wir führen in $\varphi_2 = F(\varphi_1)$ neue Variabeln ein, nämlich

$$\alpha_1 = y_2^{-\frac{8}{3}} \varrho_2 = 3y_2^{-\frac{5}{3}} y_4 - 5y_2^{-\frac{8}{3}} y_3^2$$

$$\alpha_2 = y_2^{-4} \varrho_3 = 3y_2^{-2} y_5 - 15 y_2^{-3} y_3 y_4 + \frac{40}{3} y_2^{-4} y_3^3.$$

Dann wird

$$\frac{d\alpha_2}{d\alpha_1} = \frac{y_2^{-5} \varrho_4 - \frac{5}{3} \alpha_1^2 y_2^{\frac{1}{3}}}{y_2^{\frac{1}{3}} \alpha_2} = \frac{\alpha_1^2 \left(\varphi_2 - \frac{5}{3}\right)}{\alpha_2}$$

und

$$\varphi_1 = \varrho_2^{-\frac{3}{2}} \varrho_3 = \alpha_1^{-\frac{3}{2}} \alpha_2,$$

woraus

$$\varphi_2 = \frac{\alpha_2}{\alpha_1^2} \frac{d\alpha_2}{d\alpha_1} + \frac{5}{3} = F\left(\alpha_1^{-\frac{3}{2}} \alpha_2\right).$$

Die hiermit gefundene Differentialgleichung 1. O. zwischen α_1 und α_2 gestattet die infinitesimale Transformation

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -\frac{4}{3} \alpha_1 \frac{\partial f}{\partial \alpha_1} - 2\alpha_2 \frac{\partial f}{\partial \alpha_2}$$

und wird daher durch eine Quadratur integrirt. Die hervorgehende Relation zwischen α_1 und α_2 ist eine Differentialgleichung fünfter Ordnung, die nach den Regeln der vorangehenden Nummer vermöge einer Riccatischen Gleichung 1. O. erledigt wird.

24. Gestattet eine Differentialgleichung m^{ter} Ordnung $(m > 5)$ die allgemeine projective Gruppe

$$p,\ q,\ xq,\ yq,\ xp,\ yp,\ x^2p + xyq,\ xyp + y^2q$$

so ist sie, wenn wir die Symbole ϱ_k, σ, Φ_1 und Φ_2 in derselben Bedeutung wie in Nummer 3 des ersten Abschnittes brauchen, reducibel auf die Form

$$\Omega\left(\Phi_1 \Phi_2 \frac{d\Phi_2}{d\Phi_1} \cdots \frac{d^{m-8}\Phi_2}{d\Phi_1^{m-8}}\right).$$

Durch Integration dieser Gleichung $(m-8)^{\text{ter}}$ Ordnung erhalten wir eine Differentialgleichung achter Ordnung

$$\Phi_2 = F(\Phi_1),$$

die wir jetzt in Uebereinstimmung mit meinen alten allgemeinen Integrationstheorien auf eine Gleichung zweiter Ordnung*) reduciren werden. Da nämlich die allgemeine achtgliedrige projective Gruppe sechsgliedrige Untergruppen (dagegen keine siebengliedrige Untergruppe) enthält, so ist es nach mir möglich, zwei*) Integralgleichungen von $\Phi_2 = F(\Phi_1)$ durch Integration einer Gleichung zweiter Ordnung herzuleiten. Aus diesen beiden Integralen findet man dann nach meinen allgemeinen Regeln neue durch *Differentiation*, und zwar findet man in dieser Weise alle, da die achtgliedrige Gruppe keine invariante Untergruppe enthält.

Um die Rechnungen in einfachster Weise durchzuführen, ist es zweckmässig, neue Variabeln einzuführen und zwar Φ_1 und die Grössen

$$A = \varrho_2^{-\frac{3}{2}} \varrho_3, \quad B = \varrho_2^{-2} \varrho_4,$$

*) Nach einer neueren Bemerkung von mir, die ich der Gesellschaft der Wissenschaften in Christiania im Septbr. 1882 mittheilte, genügt es sogar, *ein* Integral dieser Gleichung 2. O. aufzufinden.

(wir gebrauchen wie schon gesagt die Bezeichnungen der Nummer 3 des ersten Abschnittes). Wir berechnen die Differentialquotienten von A, B und Φ_1 hinsichtlich x und finden darnach durch Division die Differentialquotienten von A und B hinsichtlich Φ_1. Zur Ausführung dieser Rechnung bestimmen wir zuerst die nachstehenden Werthe der Differentialquotienten der Grössen ϱ_k hinsichtlich x:

$$y_2\varrho_2' = \varrho_3 + \frac{10}{3}\, y_3\varrho_2,$$

$$y_2\varrho_3' = \varrho_4 - \frac{5}{3}\,\varrho_2^2 + 5y_3\varrho_3,$$

$$y_2\varrho_4' = \frac{1}{3}\,\varrho_5 - \frac{8}{3}\,\varrho_2\varrho_3 + \frac{20}{3}\, y_3\varrho_4,$$

$$y_2\varrho_5' = \frac{1}{3}\,\varrho_6 - \frac{35}{3}\,\varrho_2\varrho_4 + \frac{25}{3}\, y_3\varrho_5.$$

Folglich wird

$$\frac{dA}{dx} = \frac{B - \frac{5}{3} - \frac{3}{2}A^2}{y_2\varrho_2^{-\frac{1}{2}}};$$

$$\frac{dB}{dx} = \frac{\Phi_1 A^{\frac{5}{3}} + 19A - 12AB + 7\left(B - \frac{5}{3}\right)^2 A^{-1}}{6y_2\varrho_2^{-\frac{1}{2}}},$$

$$\frac{d\Phi_1}{dx} = \frac{\varrho_3\sigma' - \frac{8}{3}\,\sigma\varrho_3'}{\varrho_3^{\frac{11}{3}}} = \frac{\frac{2}{3}\,\Phi_2 - 35}{y_2\varrho_3^{-\frac{1}{3}}}$$

und

$$\frac{dA}{d\Phi_1} = \frac{\left(B - \frac{5}{3}\right)A^{-\frac{1}{3}} - \frac{3}{2}A^{-\frac{5}{3}}}{\frac{2}{3}\,\Phi_2 - 35},$$

$$\frac{dB}{d\Phi_1} = \frac{\Phi_1 A^{\frac{4}{3}} + 19A^{\frac{2}{3}} - 12A^{\frac{2}{3}}B + 7\left(B - \frac{5}{3}\right)^2 A^{-\frac{4}{3}}}{6\left(\frac{2}{3}\,\Phi_2 - 35\right)}.$$

Da Φ_2 eine gegebene Function von Φ_1 darstellt, so kennen wir hiermit ein gewöhnliches simultanes System zwischen A, B und Φ_1, das offenbar einer Differentialgleichung zweiter Ordnung äquivalent ist.

Unser simultanes System erhält durch die Substitution

$$\alpha = A^{-\frac{4}{3}}\left(B - \frac{5}{3}\right),$$

$$\beta = A^{-\frac{8}{3}}\left(B - \frac{5}{3}\right)^2 - A^{-\frac{2}{3}}$$

die bemerkenswerthe Form

$$\text{(L)} \quad \begin{cases} \dfrac{d\alpha}{d\Phi_1} = \dfrac{\Phi_1 - 2\alpha^2 + \beta}{4\Phi_2 - 210} \\[2ex] \dfrac{d\beta}{d\Phi_1} = \dfrac{2\Phi_1\alpha - 2\alpha\beta - 6}{4\Phi_2 - 210} \text{ *)}. \end{cases}$$

Setzt man endlich

$$\alpha = \frac{x_1}{x_3}, \quad \beta = \frac{x_2}{x_3}$$

$$\frac{dx_3}{d\Phi_1} = \frac{2x_1}{4\Phi_2 - 210},$$

so erhält unser simultanes System die *lineare* Form

$$\frac{dx_1}{\Phi_1 x_3 + x_2} = \frac{dx_2}{2\Phi_1 x_1 - 6x_3} = \frac{dx_3}{2x_1} = \frac{d\Phi_1}{4\Phi_2 - 210}$$

und kann daher, wenn man es vorzieht, durch eine äquivalente lineare Differentialgleichung 3. O. ersetzt werden.

Kennt man die Lösungen W_1, W_2 des Systems (L), so ist nach meinem früher citirten Satze die Integration von $\Phi_2 = F(\Phi_1)$ als geleistet zu betrachten. Dies sieht man auch so ein: Die Gleichungen $W_1 = a$, $W_2 = b$ mit zwei bestimmten Constanten geben ∞^6 Integralcurven, deren Inbegriff alle projective Transformationen gestattet, bei denen die unendlich entfernte Gerade ihre Lage behält. Man führe jetzt durch eine projective Transformation diese Gerade in eine *neue* Lage g_i über. Gleichzeitig erhalten W_1 und W_2 die Werthe $W_1^{(i)}$ $W_2^{(i)}$. Wählt man nun vier Geraden $g_1\ g_2\ g_3\ g_4$, so bestimmen die acht Gleichungen

$$W_1^{(i)} = a_i, \quad W_2^{(i)} = b_i \qquad (i = 1\ 2\ 3\ 4)$$

mit acht bestimmten Constanten eine Schaar von Integralcurven, deren Inbegriff alle projective Transformationen gestattet, bei denen $g_1\ g_2\ g_3\ g_4$ invariant bleiben. Haben daher unsere vier Geraden eine *allgemeine* Lage, so geben die acht Gleichungen eine *einzige* Integralcurve, sodass die Integration geleistet ist.

Aus meinen 1874 gegebenen allgemeinen Integrationstheorien folgt, wie schon gesagt, als Corollar, dass eine Gleichung m^{ter} Ordnung, welche die allgemeine projective Gruppe gestattet, vermöge zweier Hülfsgleichungen von $(m-8)^{\text{ter}}$ und zweiter Ordnung integrirt wird. Dass die Hülfsgleichung zweiter Ordnung mit einer linearen Gleichung 3. O. äquivalent ist, bemerkte *Halphen* in der Sitzung vom 3. Novbr. 1882 der société mathématique.

---

*) Interpretirt man α und β als Cartesische Coordinaten in einer Ebene, Φ_1 als die Zeit, so definiren die Gleichungen (L) eine mit der Zeit variirende projective und infinitesimale Transformation der besprochenen Ebene.

Ist jetzt eine beliebige Gleichung $f(x\,y\,y_1\,\cdots) = 0$ mit einer bekannten Gruppe $B_1 f \cdots B_r f$ vorgelegt, so bestimmt man zuerst nach den Regeln des ersten Abschnittes die canonische Form der Gruppe, und bringt sie darnach auf diese Form. Hiernach verfährt man nach den Regeln dieses Abschnittes. Ich discutire später näher die Fälle der canonischen Formen q; $q\;yq$; $q\;yq\;y^2q$.

Im nächsten Abschnitte zeige ich, wie man die Gruppe einer Gleichung in einfachster Weise bestimmt.

März 1883.

Die vorstehende Abhandlung erschien im Jahre 1883 im norwegischen Archiv. *Sie ist also älter als Sylvesters Untersuchungen über Reciprocanten.* Ebenso ist meine in diesen Annalen Bd. XXIV, 1884 gedruckte Arbeit über Differentialinvarianten älter als die genannten Sylvester'schen Publicationen und die sich daran anschliessenden Untersuchungen.

Juli 1888.

Sophus Lie.

2 Elementry First Integrals of Differential equations by M.J.Prelle, M.F. Singer (1983)

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 279, Number 1, September 1983

ELEMENTARY FIRST INTEGRALS OF DIFFERENTIAL EQUATIONS

BY

M. J. PRELLE AND M. F. SINGER

ABSTRACT. We show that if a system of differential equations has an elementary first integral (i.e. a first integral expressible in terms of exponentials, logarithms and algebraic functions) then it must have a first integral of a very simple form. This unifies and extends results of Mordukhai-Boltovski, Ritt and others and leads to a partial algorithm for finding such integrals.

1. Introduction. It is not always possible and sometimes not even advantageous to write the solutions of a system of differential equations explicitly in terms of elementary functions. Sometimes, though, it is possible to find elementary functions that are constant on solution curves, that is, elementary first integrals. These first integrals allow one to occasionally deduce properties that an explicit solution would not necessarily reveal. Consider the following example:

EXAMPLE 1. The preditor-prey equations

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy, \qquad a, b, c, d \text{ positive real numbers.}$$

Although these cannot be solved explicitly in finite terms, one can show that

$$F(x, y) = dx + by - c\log x - a\log y$$

is constant on solution curves $(x(t), y(t))$. Using the function $F(x, y)$, one can furthermore show that all solution curves in the positive quadrant are closed, that is, all such solutions are periodic.

Note that in this example the first integral is of the form

$$w_0(x, y) + \sum c_i \log w_i(x, y),$$

where the c_i are constants and the w_i are algebraic (in this case, even rational) functions of x and y. Roughly speaking, the main result of this paper is that if a system of differential equations has an elementary first integral, it will then have one of this form. Corollaries of the main result will show that the theory presented here unifies and generalizes a number of results originally due to Mordukhai-Boltovski, Ritt and others. An attempt to do this was made in **[SING: 77]** but the results presented here are more general and the techniques more to the point. Some of these results also appear in **[PRELLE: 82]**. In the following, **Z** stands for the integers, **Q** the rationals and **C** the complex numbers.

Received by the editors September 8, 1982.
1980 *Mathematics Subject Classification*. Primary 12H05.

0002-9947/82/0000-1140/$05.00

2. Main result and corollaries. To fix notation, we let (K, Δ) denote a differential field of characteristic zero with a given set of derivations $\Delta = \{\delta\}_{\delta \in \Delta}$. The constants of (K, Δ), that is, all those elements annihilated by all δ in Δ, will be denoted by $C(K, \Delta)$. We assume that the reader is familiar with the definitions of elementary and liouvillian extensions and related notions. For precise definitions see **[ROS: 76** or **ROSSIN: 77]**.

The following is our main result. Note that, for $\delta_1, \ldots, \delta_n$ in Δ, any K linear combination, $y_1\delta_1 + \cdots + y_n\delta_n$, $y_i \in K$, is a derivation on any Δ-differential extension of K.

THEOREM. *Let (L, Δ) be an elementary extension of the differential field (K, Δ) with $C(L, \Delta) = C(K, \Delta)$. Let $D = y_1\delta_1 + \cdots + y_n\delta_n$ for some $\delta_i \in \Delta$ and $y_i \in K$ and assume that $C(L, \Delta)$ is a proper subset of $C(L, \{D\})$. Then there exist elements of L, $w_0, w_1, \ldots, w_m$, algebraic over K and $c_1, \ldots, c_m$ in $C(K, \Delta)$ such that*

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0 \quad \text{and} \quad \delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0$$

for some $\delta \in \Delta$.

Let us see how Example 1 fits into this scheme.

EXAMPLE 1 REVISITED. Let $K = \mathbf{C}(x, y)$, $\mathbf{C}$ being the complex numbers, x and y indeterminants. Let $\Delta = \{\delta_x, \delta_y\}$ where δ_x (resp. δ_y) is the partial derivative with respect to x (resp. y). Let $D = (ax - bxy)\delta_x + (-cy + dxy)\delta_y$. Let (L, Δ) be an elementary extension of (K, Δ). L then consists of elementary functions of two variables. For g in L, $Dg = 0$ is equivalent to g being constant on solutions of our system of differential equations. For g in L, $\delta_x g \neq 0$ or $\delta_y g \neq 0$ is equivalent to g being not identically constant. Therefore, the hypothesis that $C(L, \Delta)$ is properly contained in $C(L, \{D\})$ is equivalent to the existence of a nonconstant elementary function of two variables that is constant on solutions of our equation. The conclusion states that there must exist $w_0, w_1, \ldots, w_m$ algebraic over $\mathbf{C}(x, y)$ such that $w_0 + \sum_{i=1}^{m} c_i \log w_i$ is constant on solutions of our system (since

$$D\left(w_0 + \sum_{i-1}^{m} c_i \log w_i\right) = Dw_0 + \sum c_i \frac{Dw_i}{w_i} = 0\Bigg)$$

and such that $w_0 + \sum_{i=1}^{m} c_i \log w_i$ is not identically constant. Notice that letting $m = 2$, $w_0 = dx + by$, $c_1 = -c$, $w_1 = x$, $c_2 = -a$, and $w_2 = y$ illustrates the conclusion of the Theorem.

This example may lead one to conjecture that the $w_0, w_1, \ldots, w_m$ guaranteed to exist by the Theorem may be chosen to actually lie in K, rather than being just algebraic over K. This is not necessarily true, even if K is a liouvillian extension of its field of contants.

EXAMPLE 2. Let $k = \mathbf{C}(x)$ with the usual derivation with respect to x which we denote by $'$. Let $E = k\langle \sin^{-1} x\rangle = k((1 - x^2)^{1/2}, \sin^{-1} x)$ and $K = k\langle y \rangle$ where $y = (1 - x^2)^{1/2} \sin^{-1} x$. Notice that y is not algebraic over k and that $y' = 1 - xy/(1 - x^2)$ so $K = k(y)$. K is therefore a purely transcendental extension of $\mathbf{C}$

which can be made into a Δ-differential field by letting $\Delta = \{\delta_x, \delta_y\}$ where δ_x (resp. δ_y) is just the usual partial derivative with respect to x (resp. y). Furthermore, K is a Δ-liouvillian extension of $\mathbf{C}$. Letting $D = \delta_x + (1 - xy/(1 - x^2))\delta_y$, we see that D and $'$ agree on K. E is an algebraic (and therefore elementary) extension of K. In E we have

$$D\frac{y}{\sqrt{1-x^2}} - i\frac{D\left(x + i\sqrt{1-x^2}\right)}{x + i\sqrt{1-x^2}} = 0$$

while

$$\delta_y\frac{y}{\sqrt{1-x^2}} - i\frac{\delta_y\left(x + i\sqrt{1-x^2}\right)}{x + i\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \neq 0.$$

Recall that

$$\int\frac{1}{\sqrt{1-x^2}} = \sin^{-1}x = i\ln\left(x + i\sqrt{1-x^2}\right).$$

Therefore the conclusion of the Theorem is satisfied. Yet it is not true that there exist $w_0, w_1, \ldots, w_m$ in K, such that

$$(1) \qquad Dw_0 + \sum c_i\frac{Dw_i}{w_i} = 0 \quad \text{and} \quad \delta w_0 + \sum c_i\frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta.$$

In fact, it is shown in [**ROSSIN: 77**, p. 335] that if (1) holds for $w_0, w_1, \ldots, w_m$ in K, then $w_0, w_1, \ldots, w_m$ are actually in $\mathbf{C}$ so each $\delta w_i = 0$ for $\delta \in \Delta$.

PROOF OF THE THEOREM. We shall prove a seemingly stronger statement: Let (K, Δ), (L, Δ) and D be as in the Theorem. Assume that there exist $u_0, \ldots, u_m$ in L and $d_1, \ldots, d_m$ in $C(U, \Delta)$ such that

$$(2) \qquad \begin{gathered} Du_0 + \sum_{i=1}^{m} d_i\frac{Du_i}{u_i} = 0 \quad \text{and} \\ \delta u_0 + \sum_{i=1}^{m} d_i\frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \in \Delta. \end{gathered}$$

Then there exist $w_0, \ldots, w_n$ in L, algebraic over K, and $c_1, \ldots, c_n$ in $C(K, \Delta)$ such that

$$Dw_0 + \sum_{i=1}^{m} c_i\frac{Dw_i}{w_i} = 0$$

and

$$\delta w_0 + \sum_{i=1}^{m} c_i\frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta.$$

The hypotheses of statement (2) are certainly satisfied if the hypotheses of the Theorem are satisfied. Conversely, if the hypotheses of statement (2) are satisfied then in some elementary extension of L, $w = u_0 + \sum_{i=1}^{m} c_i \log u_i$ satisfies $Dw = 0$ and

$\delta w \neq 0$ for some $\delta \in \Delta$. Yet it is more convenient to prove statement (2). By induction on the transcendence degree of L over K, we may assume that L is an algebraic extension of $K(t)$ where t is transcendental over K and either $\delta t = (\delta v)t$ for some v in K and all δ in Δ or $\delta t = \delta v/v$ for some $v \neq 0$ in K and all δ in Δ.

First assume that there is a w in L such that $Dw = 0$ and w is not in K. We then must have $\delta w \neq 0$ for some δ in Δ since $C(K, \Delta) = C(L, \Delta)$. Since L is algebraic over $K(t)$, w is a root of an irreducible polynomial $W^n + a_{n-1}W^{n-1} + \cdots + a_0$ with the a_i in $K(t)$. If $\delta a_i = 0$ for each δ in Δ and each i, $0 \leqslant i \leqslant n-1$, we would have $\delta w = 0$ for each δ in Δ. Therefore, for some i, $\delta a_i \neq 0$. Similarly, if $Da_i \neq 0$ for some i, we would have $Dw \neq 0$, so we have $Da_i = 0$ for all i. Therefore there exists an element w in $K(t)$ such that $Dw = 0$ and $\delta w \neq 0$ for some $\delta \in \Delta$. If w is in K, we would satisfy the conclusion of (2), so we can assume that w is not in K. If $\delta t = (\delta v)t$ for some v in K and all $\delta \in \Delta$, we have $Dt = (Dv)t$. Since $C(K, \{D\}) \subsetneq C(K(t), \{D\})$, Proposition 1.26 of **[RISCH: 69]** tells us that there exists an integer n and an element s in K such that $Ds = n(Dv)s$. If $\delta s = n(\delta v)s$ for all δ in Δ, we would have $\delta(st^{-n}) = 0$ for all δ in Δ, which would imply that t is algebraic over K, a contradiction. Letting $w_0 = nv$, $w_1 = s$ and $c_1 = -1$, we have $Dw_1 + c_1 Dw_1/w_1 = 0$ and $w_1 + c_1\delta w_1/w_1 \neq 0$ for some δ in Δ which gives the conclusion of (2). If $\delta t = \delta v/v$ for some $v \neq 0$ in K and all δ in Δ, then $Dt = Dv/v$. Again, since $C(K, \{D\}) \subsetneq C(K(t), \{D\})$, Proposition 1.2a of **[RISCH: 69]** tells us that there exists an s in K such that $Ds = Dv/v$. If $\delta s = \delta v/v$ for all δ in Δ, we would have that $\delta(t - s) = 0$ for all δ in Δ. Since t is not algebraic over K, we must have $\delta s \neq \delta v/v$ for some δ in Δ. Letting $w_0 = s$, $w_1 = v$ and $c_1 = -1$, we have $Dw_0 + c_1 Dw_1/w_1 = 0$ and $\delta w_0 + c_1\delta w_1/w_1 \neq 0$ for some δ in Δ, which gives the conclusion of (2).

Now assume that if w is in L and $Dw = 0$ then w is in K, that is, $C(K, \{D\}) = C(L, \{D\})$. Assume also that the hypotheses of (2) are satisfied. We may furthermore assume that the d_i are linearly independent over $\mathbf{Q}$ (otherwise let $e_1, \ldots, e_k$ be a $\mathbf{Q}$-basis of $\mathbf{Q}d_1 + \cdots + \mathbf{Q}d_m$ such that $d_i = (1/\nu)\sum_{j=1}^{k} \nu_{ij} e_j$ with ν_{ij} and ν in $\mathbf{Z}$. We then have

$$0 = Du_0 + \sum_{i=1}^{m} d_i \frac{Du_i}{u_i} = Du_0 + \frac{1}{\nu}\sum_{i=1}^{k} e_i \frac{D(u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}})}{u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}}},$$

$$0 \neq \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} = \delta u_0 + \frac{1}{\nu}\sum_{i=1}^{k} e_i \frac{\delta(u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}})}{u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}}}$$

and we may use these equations instead of those in the hypotheses of (2)). If we have $\delta t = (\delta v)t$ for all δ in Δ, we then have that $Dt = (Dv)t$. Applying Theorem 2 of **[ROS: 76]**, we have that u_0 is algebraic over K and that there exist integers $\nu_0, \nu_1, \ldots, \nu_n$ with $\nu_0 \neq 0$ such that each $u_i^{\nu_i}/t^{\nu_0}$ is algebraic over K. Let

$$w_0 = u_0 + \frac{1}{\nu_0}\sum_{i=1}^{m} d_i \nu_i w_i, \quad w_i = \frac{u_i^{\nu_0}}{t^{\nu_0}} \quad \text{for } i = 1, \ldots, m, \quad c_i = \frac{1}{\nu_0} d_i.$$

We then have

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = Du_0 + \sum_{i=1}^{m} d_i \frac{Du_i}{u_i} = 0$$

while

$$\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} = \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

This gives the conclusion of (2). If $\delta t = \delta v / v$ for all δ in Δ, then $Dt = Dv/v$. Applying Theorem 2 of [**ROS: 76**] again we have that $u_1, \ldots, u_m$ are algebraic over K, and that there exists a c in K such that $Dc = 0$ and such that $u_0 - ct$ is algebraic over K. If $\delta c \neq 0$ for some δ in Δ we would be done so we can assume c is in $C(K, \Delta)$. Let

$$w_0 = u_0 - ct, \quad w_i = u_i \quad \text{for } i = 1, \ldots, m,$$
$$w_{m+1} = v, \quad c_i = d_i \quad \text{for } i = 1, \ldots, m,$$
$$c_{m+1} = c.$$

We then have

$$Dw_0 + \sum_{i=1}^{m+1} c_i \frac{Dw_i}{w_i} = Du_0 + \sum_{i=1}^{m} c_i \frac{Du_i}{u_i} = 0$$

while

$$\delta w_0 + \sum_{i=1}^{m+1} c_i \frac{\delta w_i}{w_i} = \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

This gives the conclusion of (2) and finishes the proof.

We will now deduce some corollaries. Corollary 1 is a generalization of a theorem of Mordukhai-Boltovski [**M-B: 06**] (also see [**RITT: 48**]), which states: Let $y' = f(x, y)$ be a differential equation with f an algebraic function of x and y. If there exists an elementary function $g(x, y)$ which is constant on solutions of $y' = f(x, y)$, then there exist algebraic functions of two variables $\phi_0, \ldots, \phi_m$ and constants $c_1, \ldots, c_m$ such that $\phi_0(x, y) + \sum_{i=1}^{m} c_i \phi_i(x, y)$ is a first integral of $y' = f(x, y)$, that is, it is not identically constant but is constant on all solutions of $y' = f(x, y)$. By a *differential field of functions in $n + 1$ variables $x_0, x_1, x_2, \ldots, x_n$*, we mean a field of functions, meromorphic in some domain in $\mathbf{C}^{n+1}$, closed under the derivations $\partial/\partial x_i$ and containing the coordinate functions $x_1, \ldots, x_n$.

COROLLARY 1. *Let K be a differential field of functions in $n + 1$ variables and L an elementary extension of K. Let f be in K and assume there exists a nonconstant g in L such that g is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$. Then there exist $w_0, \ldots, w_m$ algebraic over K and constants $c_1, \ldots, c_n$ such that*

$$w_0(x, y, y', \ldots, y^{(n-1)}) + \sum_{i=1}^{m} c_i \log w_i(x, y, y', \ldots, y^{(n-1)})$$

is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

PROOF. Let $D = \partial/\partial x_0 + x_2 \partial/\partial x_1 + x_3 \partial/\partial x_2 + \cdots + f \partial/\partial x_n$ and apply the Theorem, noting that $Dg = 0$ if and only if g is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

Loosely speaking, the next result says that if $u_1, \ldots, u_n$ are elementary functions of a single variable x and $g(x, U_1, \ldots, U_n)$ is an elementary function of $n + 1$ variables

$x, U_1, \ldots, U_n$, such that $g(x, \int u_1\,dx, \ldots, \int u_n\,dx)$ is constant, then some nontrivial linear combination with constant coefficients of the $\int u_i$ is elementary. For $n = 1$, this result is due to [**RITT: 23, RITT: 48**]. The result also appears in [**SING: 77**] and includes the result of [**MOZI: 79**]. To be precise, we let k be an ordinary differential field with derivation δ and let $u_1, \ldots, u_n$ be elements of k. Define a new differential field (K, Δ) as follows: let $K = k(U_1, \ldots, U_n)$ where $U_1, \ldots, U_n$ are indeterminants. Let $\Delta = \{\delta_0, \ldots, \delta_n\}$ where:

(1) δ_0 restricted to k is δ and $\delta_0 U_i = 0$ for $i = 1, \ldots, n$.

(2) For $i = 1, \ldots, n$, let $\delta_i a = 0$ for all a in k and let $\delta_i U_i = 0$ if $i \neq j$ and $\delta_i U_i = 1$.

COROLLARY 2. *Let (K, Δ) be as above and let (L, Δ) be an elementary extension of (K, Δ) so that $C(L, \Delta) = C(K, \Delta)$. Let $D = \delta_0 + \sum_{i=1}^n u_i \delta_i$ and assume $C(L, \Delta)$ is properly contained in $C(L, \{D\})$. Then there exist $v_0, \ldots, v_m$ in k, and constants $c_1, \ldots, c_n, d_1, \ldots, d_m$ in $C(k, \Delta)$, not all the c_i's being zero, so that*

$$\sum_{i=1}^{n} c_i u_i = \delta v_0 + \sum_{i=1}^{m} d_i \frac{\delta v_i}{v_i}.$$

To prove this corollary we need the following lemma. We assume that the reader is familiar with the notation of Theorem 1 of [**ROS: 76**].

LEMMA. *Let (K, Δ) and D be as above and let (E, Δ) be an algebraic extension of K with $C(E, \Delta) = C(K, \Delta)$. If there exists an $\alpha \in E$, such that $\alpha \notin k$ and $D\alpha = 0$ then there exist $c_1, \ldots, c_n \in C(k, \{\delta\})$ and $w \in k$ such that $\sum_{i=1}^n c_i u_i = \delta w$.*

PROOF. Let $\alpha \in E$ such that $\alpha \notin k$ and $D\alpha = 0$. Let $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ be the minimum polynomial of α over $k(U_1, \ldots, U_n)$. Since $D\alpha = 0$, we have $Da_i = 0$ for $i = 0, \ldots, m-1$. If each $a_i \in k$, then α would be algebraic over k and so $0 = D\alpha = \delta\alpha$ and $\delta_i \alpha = 0$ for $i = 1, \ldots, n$. Since $C(E, \Delta) = C(K, \Delta) \subset k$, α would be in k, a contradiction. We can conclude that for some a_i, $a_i \notin k$ and $Da_i = 0$, and therefore we can assume that $\alpha \in k(U_1, \ldots, U_n)$. Proceeding by induction on n, we can assume that $C(k, \{D\}) = C(k(U_1, \ldots, U_{n-1})\{D\})$ while $C(k, \{D\}) \subsetneq C(k(U_1, \ldots, U_n), \{D\})$. Since $DU_n = u_n$ which is in $k(U_1, \ldots, U_{n-1})$, we can conclude that there is an element V in $k(U_1, \ldots, U_{n-1})$ such that $DV = u_n$. Applying Theorem 1 of [**ROS: 76**] to

$$\begin{gathered} DU_1 \in k \\ \vdots \\ DU_{n-1} \in k \\ DV \in k \end{gathered}$$

while noting that $\operatorname{tr\,deg} k(U_1, \ldots, U_{n-1}, V)/k = n - 1$ we can conclude that the n elements $dU_1, \ldots, dU_{n-1}$, dV of $\Omega_{k(U_1, \ldots, U_{n-1}, V)/k}$ are linearly dependent over $C(k, \{D\}) = C(k, \{\delta\})$. Therefore, for some $c_1, \ldots, c_n$ in $C(k, \{\delta\})$ we have $0 = c_1 dU_1 + \cdots + c_{n-1} dU_{n-1} + c_n dV = d(c_1 U_1 + \cdots + c_n V)$. Therefore, $w = c_1 U_1 + \cdots + c_n V$ is algebraic over k and so in k. We also have

$$D(c_1 U_1 + \cdots + c_n V) = c_1 u_1 + \cdots + c_{n-1} u_{n-1} + c_n u_n = Dw = \delta w.$$

PROOF OF COROLLARY 2. We can conclude from the Theorem that there are $w_0, \ldots, w_n$ in L, algebraic over K, such that

$$(3)\qquad \begin{aligned} &Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0, \\ &\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta. \end{aligned}$$

As in the proof of the Theorem we can assume that the c_i are linearly independent over $\mathbf{Q}$.

Letting $E = K(w_0, \ldots, w_n)$, we see that E is a differential extension of K. Furthermore, we can assume that $C(E, \{D\}) = C(k, \{D\})$, since otherwise the Lemma would imply we were done. We now apply Theorem 1 of [**ROS: 76**] to the $n + 1$ equations

$$\begin{gathered} DU_1 \in k \\ \vdots \\ DU_n \in k \\ \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} + Dw_0 \in k \end{gathered}$$

and noting that $\operatorname{tr\,deg} k(U_1, \ldots, U_n, w_0, \ldots, w_n)/k < n + 1$, conclude that there exist constants $f_i \in C(k, \{D\})$ such that

$$f_1 dU_1 + \cdots + f_n dU_n + f_{n+1}\left(\sum_{i=1}^{m} c_i \frac{dw_i}{w_i} + dw_0 \right) = 0.$$

If $f_{n+1} = 0$, we have $d(f_1 U_1 + \cdots + f_n U_n) = 0$, so $v_0 = f_1 U_1 + \cdots + f_n U_n$ is algebraic over k (and therefore in k, since k is relatively algebraically closed in K). Therefore

$$f_1 u_1 + \cdots + f_n u_n = D(f_1 U_1 + \cdots + f_n U_n) = Dv_0 = \delta v_0$$

which gives us the conclusion of the corollary.

If $f_{n+1} \neq 0$, then we can conclude that $f_1 U_1 + \cdots + f_n U_n + f_{n+1} w_0$ and all the w_i, $i = 1, \ldots, m$, are algebraic over k. For some i, $1 \leq i \leq n$, we must have $f_i \neq 0$. If not, we would have $w_0, w_1, \ldots, w_n$ algebraic over k. Since D restricted to k is δ and each δ_j, $1 \leq i \leq n$, restricted to k is 0, we have $Dw_i = \delta w_i$ for $i = 0, \ldots, m$ and $\delta_j w_i = 0$ for $1 \leq j \leq n$, $0 \leq i \leq m$. This would contradict the relations in (3). Let $k_1 = k(f_1 U_1 + \cdots + f_n U_n + f_{n+1} w_0, w_1, \ldots, w_m)$ and let

$$v_0 = \operatorname{Trace}(f_1 U_1 + \cdots + f_n U_n + f_{n+1} w_0) \quad \text{and} \quad v_i = \operatorname{Norm}(w_i) \quad \text{for } i = 1, \ldots, m,$$

where the Trace and Norm are taken in the field k_1 with respect to k. We then have, for some integer p,

$$\frac{Dv_i}{v_i} = p \frac{Dw_i}{w_i}, \qquad i = 1, \ldots, m,$$

$$Dv_0 = pD(f_1 U_1 + \cdots + f_n U_n + f_{n+1} w_0) = pf_1 u_1 + \cdots + pf_n u_n + pf_{n+1} Dw_0.$$

Since $Dw_0 = -\sum c_i Dw_i/w_i$ we have

$$pf_1u_1 + \cdots + pf_nu_n = Dv_0 - pf_{n+1}Dw_0$$

$$= Dv_0 + pf_{n+1}\left(\sum_{i=1}^{m} c_i \frac{Dw_i}{w_i}\right) = Dv_0 + f_{n+1}\sum_{i=1}^{m} c_i \frac{Dv_i}{v_i}.$$

This gives us the conclusion of Corollary 2.

Finally, note that the results of [**MACK: 76**] show how one can decide if there exist constants $c_1, \ldots, c_n$, not all zero such that $\sum c_i u_i$ has an elementary antiderivative, where the u_i lie in a purely transcendental elementary extension of $C(x)$.

In the next corollary, which generalizes a result in [**SING: 75**], we will focus on the differential equation $y' = f(y)$ where $f(y)$ is a nonzero function of one variable. Loosely speaking this corollary says that if $y' = f(y)$ has an elementary first integral then

$$g(y) = \int \frac{1}{f(y)}\, dy$$

is an elementary function of y. The converse is also true, since if $g(y)$ is an elementary function of y and $y(x)$ is a solution of $y' = f(y)$ then $(g(y(x)))' = 1$ so $g(y) - x$ is an elementary function of x and y which is constant on solutions of $y' = f(y)$. If $f(y)$ is an elementary function of y, the Risch integration algorithm allows us to decide if $g(y)$ is elementary and so allows us to determine if the differential equation $y' = f(y)$ has an elementary first integral.

Suppose we have a differential equation $y' = f(y)$, where $f \neq 0$. We model this situation in the following way: let $(K, \{\delta_y\})$ be a differential field such that K is a field of functions in the single variable y which contains the element $f(y)$ and δ_y is a derivation of K such that $\delta_y(y) = 1$. We can extend K to the field $K(x)$ where x is transcendental over K. We extend δ_y to a derivation on this field by letting $\delta_y(x) = 0$ and define a new derivation δ_x on $K(x)$ by letting $\delta_x(a) = 0$ for all a in K and $\delta_x(x) = 1$. Let $\Delta = \{\delta_x, \delta_y\}$ and $D = \delta_x + f\delta_y$.

COROLLARY 3. *Let $(K(x), \Delta)$ and D be as above and let (L, Δ) be an elementary extension of $(K(x), \Delta)$ such that $C(K(x), \Delta) = C(L, \Delta)$. Furthermore, assume $C(K(x), \{D\})$ is a proper subset of $C(L, \{D\})$. Then there exist $u_0, u_1, \ldots, u_m$ in K, with $u_1, \ldots, u_m$ nonzero and $a_1, \ldots, a_n$ in $C(K, \{\delta_y\})$ such that*

$$\frac{1}{f} = \delta_y u_0 + \sum_{i=1}^{m} a_i \frac{\delta_y u_i}{u_i}.$$

PROOF. By the Theorem we know that there exist $w_0, w_1, \ldots, w_m$ in L, algebraic over $K(x)$ and $c_1, \ldots, c_m$ in $C(K(x), \Delta)$ such that

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0$$

while for some δ in Δ

$$\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0.$$

By an argument given previously, we may assume the c_i are linearly independent over **Q**. Since $\delta x \in K$ for $\delta \in \Delta$, we may apply Theorem 2 of [**ROS: 76**]. We can conclude that $w_1, \dots, w_m$ are algebraic over K and that there exists a c in $C(K, \Delta) = C(K, \{\delta_y\})$ such that $w_0 - cx$ is algebraic over K. Let $v_0 = w_0 - cx$ and $v_i = w_i$. We then have

$$cDx + Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} = 0 \tag{4}$$

while for some δ in Δ

$$c\delta x + \delta v_0 + \sum_{i=1}^{m} c_i \frac{\delta v_i}{v_i} \neq 0. \tag{5}$$

Now we shall show that $c \neq 0$. Assume not, then

$$\begin{aligned} 0 = Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} &= \delta_x v_0 \\ &+ \sum_{i=1}^{m} c_i \frac{\delta x v_i}{v_i} + f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right) = f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right) \end{aligned}$$

since $\delta_x v_0 = \delta_x v_1 = \cdots = \delta_x v_m = 0$. We therefore have

$$c\delta x + \delta v_0 + \sum_{i=1} c_i \frac{\delta v_i}{v_i} = \delta v_0 + \sum_{i=1}^{m} \frac{\delta v_i}{v_i} = 0$$

for each δ in Δ, contradicting (5).

Using the fact that $D = \delta_x + f\delta_y$ we can rewrite (4) as

$$0 = cDx + Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} = c + f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right).$$

Let $M = K(v_0, v_1, \dots, v_m)$ and let

$$\begin{aligned} u_0 &= -(1/c) \text{ Trace } v_0, \\ u_i &= \text{Norm } v_i \quad \text{for } i = 1, \dots, m, \\ a_i &= -c_i/c \quad \text{for } i = 1, \dots, m, \end{aligned}$$

where the Norm and Trace are with respect to L over K. We then have

$$\frac{1}{f} = \delta_y u_0 + \sum_{i=1}^{m} a_i \frac{\delta_y u_i}{u_i}$$

with $u_0, \dots, u_m$ in K and a_i in $C(K, \{\delta_y\})$.

3. Algorithmic considerations. The preceding work was motivated by our desire to develop a decision procedure for finding elementary first integrals. These results show that we need only look for elementary integrals of a prescribed form. In this section we shall discuss the problem of finding an elementary first integral for a two-dimensional autonomous system of differential equations and reduce this problem to that of bounding the degrees of algebraic solutions of this system.

Let $K = \mathbf{C}(x, y)$, where x and y are transcendental over $\mathbf{C}$, and let $\Delta = \{\delta_x, \delta_y\}$, where these derivations are just the usual partial derivatives with respect to x and y. Consider the system of differential equations,

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{6}$$

where P and Q are polynomials in $\mathbf{C}[x, y]$ and let $D = P\delta_x + Q\delta_y$. We say that (6) has an elementary first integral if there exists an elementary extension (L, Δ) of (K, Δ) such that $C(L, \Delta) = C(K, \Delta)$ and $C(L, \Delta) \subsetneq C(L, \{D\})$. The existence of an elementary first integral is intimately related to the existence of an algebraic integrating factor for $Qdx - Pdy$. Without explicitly mentioning this 1-form, the following propositions describe this relationship.

PROPOSITION 1. *If the equations of* (6) *have an elementary first integral, then there exists an element $R \neq 0$ algebraic over K such that $DR = -(\delta_x P + \delta_y Q)R$.*

PROOF. Applying the Theorem of §2, there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $C(K, \Delta)$ such that $Dw_0 + \sum_{i=1}^n c_i Dw_i/w_i = 0$ and $\delta w_0 + \sum_{i=1}^n c_i \delta w_i/w_i \neq 0$ for some δ in Δ. Let

$$R_1 = \delta_x w_0 + \sum_{i=1}^n c_i \frac{\delta_x w_i}{w_i}, \qquad R_2 = \delta_y w_0 + \sum_{i=1}^n c_i \frac{\delta_y w_i}{w_i}.$$

We then have $0 = PR_1 + QR_2$. We may assume that at least one of P or Q is nonzero, say Q. Letting $R = R_1/Q$, we have $R_1 = QR$ and $R_2 = -PR$. Since δ_x and δ_y commute in any algebraic extension of K, we see that $\delta_y(QR) = \delta_x(-PR)$. Carrying out this differentiation gives us that $DR = -(\delta_x P + \delta_y Q)R$.

PROPOSITION 2. *Assume that there exists an element $S \neq 0$, algebraic over K such that $DS = -(\delta_x P + \delta_y Q)S$. Then either*

(i) *there exists a w in K such that $Dw = 0$ and $\delta w \neq 0$ for δ in Δ, or*

(ii) *for any $R \neq 0$ algebraic over K such that $DR = -(\delta_x P + \delta_y Q)R$, there exists a c in $\mathbf{C}$ such that $R = cS$ and furthermore R^n is in K for some n in $\mathbf{Z}$.*

If (i) *holds, then the equations in* (6) *obviously have an elementary first integral. If* (i) *does not hold, then the equations in* (6) *have an elementary first integral if and only if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that*

$$\delta_x w_0 + \sum c_i \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y w_0 + \sum c_i \frac{\delta_y w_i}{w_i} = -SP.$$

PROOF. Let $R \neq 0$ be algebraic over K and satisfy $DR = -(\delta_x P + \delta_y Q)R$. Furthermore, assume (i) does not hold, that is, that $Dw = 0$ implies $w \in \mathbf{C}$ for w in K. Since R/S satisfies $D(R/S)/(R/S) = 0$ we have R/S is in $\mathbf{C}$, so $R = cS$ for some c in $\mathbf{C}$. Let E be a normal algebraic extension of K containing R. For any K-automorphism σ of E, we have $D(\sigma R)/\sigma R = -(\delta_x P + \delta_y Q)$. Summing this relation over all σ in the galois group of E over K, we get

$$\frac{D(\text{Norm } R)}{\text{Norm } R} = -n(\delta_x P + \delta_y Q)$$

for some n in $\mathbf{Z}$. Therefore $D(\text{Norm } R/R^n)/(\text{Norm } R/R^n) = 0$ so R^n is a constant multiple of Norm R and therefore in K.

Now, assume that (6) has an elementary first integral. We have shown in the proof of Proposition 1 that there exist $w_0, \ldots, w_n$ and R algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = RQ, \qquad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -RP$$

and

$$DR = -\left(\delta_x P + \delta_y Q\right)R.$$

If (i) does not hold, then by (ii) we have $R = cS$ for some c in $\mathbf{C}$ so

$$\delta_x\left(\frac{w_0}{c}\right) + \sum_{i=1}^{n} \frac{c_i}{c} \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y\left(\frac{w_0}{c}\right) + \sum_{i=1}^{n} \frac{c_i}{c} \frac{\delta_y w_i}{w_i} = -SP.$$

Conversely, if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ \quad \text{and} \quad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP$$

then

$$Dw_0 + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i} = P(SQ) - Q(SP) = 0$$

while

$$\delta w_0 + \sum_{i=1}^{n} c_i \frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

We can then find an elementary extension (L, Δ) of (K, Δ) with $C(L, \Delta) = C(K, \Delta)$ such that L contains an element w with $Dw = 0$ and $\delta w \neq 0$ for some δ in Δ.

Therefore, to decide if (6) has an elementary first integral, we must:

(A) Decide if $Dw = 0$ has a nonconstant solution in $\mathbf{C}(x, y)$ and find one if it does.

(B) If $Dw = 0$ has only constant solutions in $\mathbf{C}(x, y)$, decide if

$$DS = -\left(\delta_x P + \delta_y Q\right)S$$

has a nonconstant algebraic solution S with S^n in $\mathbf{C}(x, y)$ for some n in $\mathbf{Z}$ and find one if it does.

(C) If (B) holds, decide if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ \quad \text{and} \quad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP$$

and find them if they do.

We can solve (C) completely, so we shall deal with it first. Let k be the algebraic closure of $\mathbf{C}(y)$ and let $F = k(x, S)$. Considering F as an ordinary differential field with derivation δ_x, we have $C(F, \{\delta_x\}) = k$. The first step in solving problem (C) is to decide if $SQ = \delta_x u_0 + \sum c_i \delta_x u_i/u_i$ with u_i in F and c_i in k. Since F is algebraic over $k(x)$, a solution of this latter problem was described in [**RISCH: 70**]. If no such u_i and c_i exist, we are done. If u_i and c_i do exist we can assume that the c_i are linearly independent over **Q** and that $\delta_x u_i \neq 0$ for $i = 1,\dots,n$. Next decide if the c_i are in **C**, i.e. $\delta_y c_i = 0$ for all i. If some $\delta_y c_i \neq 0$, then we claim that for any choice of $w_0,\dots,w_n$ in F and $d_1,\dots,d_n$ in k such that $SQ = \delta_x w_0 + \sum d_i \delta_x w_i/w_i$ we have $\delta_y d_i \neq 0$ for some i. To see this assume we have $w_0,\dots,w_n$ in k and d_i in **C** such that $SQ = \delta_x w_0 + \sum d_i \delta_x w_i/w_i$ and assume that $c_1 \notin \mathbf{C}$. We then have

$$\delta_x(u_0 - w_0) + \sum c_i \frac{\delta_x u_i}{u_i} - \sum d_i \frac{\delta_x w_i}{w_i} = 0.$$

If we extend $c_1,\dots,c_n$ to a **Q**-basis of $c_1\mathbf{Q} + \cdots + c_n\mathbf{Q} + d_1\mathbf{Q} + \cdots + d_m\mathbf{Q}$ and rewrite the above equation we get

$$\delta_x(u_0 - w_0) + c_1 \frac{\delta_x u_1}{u_1} + \sum c_i \frac{\delta_x v_i}{v_i} = 0,$$

where the v_i are power products of $u_2,\dots,u_n$, $w_1,\dots,w_m$. Since all the terms appearing here are algebraic over $k(x)$ and $\delta_x x = 1$, we have that u_i is in k so $\delta_x u_1 = 0$, a contradiction. Therefore if some c_i is not in **C**, we can conclude that (6) does not have an elementary integral and we are done. Therefore assume that we have found $u_0,\dots,u_n$ algebraic over F and $c_1,\dots,c_n$ in **C** such that $SQ = \delta_x u_0 + \sum c_i \delta_x u_i/u_i$. Consider the expression

$$I = \delta_y u_0 + \sum c_i \frac{\delta_y u_i}{u_i} + SP.$$

Since δ_x and δ_y commute, we have

$$\delta_x I = \delta_y\left(\delta_x u_0 + \sum c_i \frac{\delta x u_i}{u_i} - SQ\right) = 0.$$

Therefore, I is in k and so in some finite extension of $\mathbf{C}(y)$. Now use [**RISCH: 70**] to decide if there exist $v_0,\dots,v_m$ algebraic over $\mathbf{C}(y)$ and $d_1,\dots,d_m$ in **C** such that

$$I = \delta_y v_0 + \sum_{i=1}^{m} d_i \frac{\delta_y v_i}{v_i}.$$

If such elements exist, we then have

$$\delta_x u_0 - \delta_x v_0 + \sum c_i \frac{\delta_x u_i}{u_i} - \sum d_i \frac{\delta_x v_i}{v_i} = \delta_x u_0 + \sum c_i \frac{\delta_x u_i}{u_i} = SQ$$

and

$$\delta_y u_0 - \delta_y v_0 + \sum c_i \frac{\delta_y u_i}{u_i} - \sum d_i \frac{\delta_y v_i}{v_i} = I - SP - \left(\delta_y v_0 + \sum d_i \frac{\delta_y v_i}{v_i}\right) = -SP$$

and so we are done. If no such elements exist, then we claim that there are no elements $w_0,\dots,w_k$ algebraic over $\mathbf{C}(x, y)$ and $c_1,\dots,c_k$ in **C** such that

$$\delta_y w_0 + \sum e_i \frac{\delta_y w_i}{w_i} = -SP.$$

If such elements existed, then we would have

$$\delta_y u_0 + \sum c_i \frac{\delta_y u_i}{u_i} + \delta_y w_0 + \sum e_i \frac{\delta_y w_i}{w_i} = I.$$

This implies that I has an antiderivative (with respect to δ_y) in an elementary extension of $\mathbf{C}(x, y)$. Note that I is algebraic over $\mathbf{C}(y)$ and $\mathbf{C}(x, y)$ is a δ_y elementary extension of $\mathbf{C}(y)$ with new constants. The strong Liouville Theorem of [**RISCH: 69**] implies that there exist $v_0, \ldots, v_n$ algebraic over $\mathbf{C}(y)$ and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that $I = \delta_y v_0 + \sum c_i \delta_y v_i / v_i$. This completes the procedure for (C). Note that if S is actually in $\mathbf{C}(x, y)$ then problem (C) always has a positive solution.

We now turn to problems (A) and (B). Let w be an element of $\mathbf{C}(x, y)$ and write $w = \prod_{i=1}^m f_i^{n_i}$ with f_i irreducible in $\mathbf{C}[x, y]$ and n_i in $\mathbf{Z}$. If $Dw = 0$, we then see f_i must divide Df_i for each i. Let S be an element algebraic over $\mathbf{C}(x, y)$ with S^n in $\mathbf{C}(x, y)$ such that $DS = -(\delta_x P + \delta_y Q)S$. Write $S = \prod_{i=1}^m f_i^{r_i}$ with f_i irreducible in $\mathbf{C}[x, y]$ and r_i in $\mathbf{Q}$. Since

$$\sum_{i=1}^m r_i \frac{Df_i}{f_i} = \frac{DS}{S} = -\left(\delta_x P + \delta_y Q\right)$$

we again have that each f_i divides Df_i. Results of Darboux [**JOU: 79**, Theorem 3.3, p. 102 and Lemma 3.53, p. 112] imply that the degree of each f_i is bounded. No effective bound is given. This suggests the following problem:

(D) Given P and Q in $\mathbf{C}[x, y]$, let $D = P\delta_x + Q\delta_y$. Given an effective procedure to find an integer N so that if f is irreducible in $\mathbf{C}[x, y]$ and f divides Df, then the degree of f is less than N.

Assuming one has a solution for problem (D), one can solve problems (A) and (B). The results of Darboux quoted above imply that if $f_1, \ldots, f_m$ are irreducible in $\mathbf{C}[x, y]$ and f_i divides Df_i then either $m < ((d + 1)d/2) + 2$ where $d = \max(\deg P, \deg Q)$ or there exist integers n_i, not all zero, such that

$$n_1 \frac{Df_1}{f_1} + \cdots + n_m \frac{Df_m}{f_m} = 0.$$

In the latter case, we would have $Dw = 0$ for $w = \prod_{i=1}^m f_i^{n_i}$. To solve problems (A) and (B), note that once we can find N as in (D), the set of coefficients of polynomials f of degree $\leqslant N$ so that f divides Df forms a projective variety which we can construct. We can therefore decide if this variety has fewer than $d(d + 1)/2 + 2$ points. If it has more points, then we can find them and construct a w in $\mathbf{C}(x, y)$ such that $Dw = 0$. If it has fewer, then we find all of them, thereby finding all polynomials f_i of degree $\leqslant N$ such that f_i divides Df_i. We then decide if we can find r_i in $\mathbf{Q}$, not all zero such that

$$\sum r_i \frac{Df_i}{f_i} = -\left(\delta_x P + \delta_y Q\right).$$

Therefore we have reduced problems (A) and (B) to (D). When $\max(\deg P, \deg Q) = 1$, a solution to this problem appears in [**JOU: 79**, pp. 8–19]. For $\max(\deg P, \deg Q) > 1$ no solution seems to be known. Partial results appear in

[**PAIN: 72**, vol. I, pp. 173–218, vol. II, pp. 433–458 and **POINC: 34**, vol. III, pp. 32–97]. Yet even without a solution to problem (D), the above suggests a heuristic method. Arbitrarily fix an integer N. Find all irreducible polynomials f of degree $\leqslant N$ such that f_i divides Df_i. If the resulting Df_i/f_i are linearly dependent over $\mathbf{Z}$, we then can find a w such that $Dw = 0$. If the Df_i/f_i are not linearly independent over $\mathbf{Z}$, decide if there exist r_i in $\mathbf{Q}$ such that

$$\sum r_i \frac{Df_i}{f_i} = -\left(\delta_x P + \delta_y Q\right).$$

If such r_i exist, let $S = \prod f_i^{r_i}$ and decide if there exist $w_0, \ldots, w_n$ algebraic over $\mathbf{C}(x, y)$ and constants $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP.$$

EXAMPLE 1 REVISITED AGAIN. We again consider the system $\dot{x} = ax - bxy$ and $\dot{y} = -cy + dxy$. Letting $D = (ax - bxy)\delta_x + (-xy + dxy)\delta_y$ we let $N = 1$ and look for all polynomials $f = \alpha x + \beta y + \gamma$ of degree $\leqslant 1$ such that f divides Df. Since $Df = (\beta d - \alpha b)xy + a\alpha x - c\beta y$, if f divides Df we must have that either $\alpha = 0$ or $\beta = 0$. In both cases we get $\gamma = 0$. Therefore, there are just two first degree polynomials f, x and y such that f divides Df. Furthermore,

$$\frac{Dx}{x} = -by + a \quad \text{and} \quad \frac{Dy}{y} = dx - c.$$

One can check that, unless $b = d = 0$, these are linearly independent over $\mathbf{Z}$. We now solve

$$r_1 \frac{Dx}{x} + r_2 \frac{Dy}{y} = r_1(-by + a) + r_2(dx - c) = -\left(a - by + (-c + dx)\right)$$

and find $r_1 = r_2 = -1$. Let $S = x^{-1}y^{-1}$ and find $w_0, \ldots, w_n$ and $c_1, \ldots, c_n$ constants such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ = -\frac{c}{x} + d,$$

$$\delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP = -\frac{a}{y} + b.$$

We get $w_0 = dx + by$, $w_1 = x$, $w_2 = y$, $c_1 = -c$, $c_2 = -a$ and so $dx + by - c \log x - a \log y$ is an elementary first integral.

REFERENCES

[**JOU**] J. P. Jouanolou, *Equations de Pfaff algébriques*, Lecture Notes in Math., vol. 708, Springer-Verlag, Berlin and New York, 1979.

[**MACK**] C. Mack, *Integration of affine forms over elementary functions*, Computer Science Department, University of Utah, Technical Report, VCP–39, 1976.

[**M.-B.**] D. Mordukhai-Boltovski, *Researches on the integration in finite terms of differential equations of the first order*, Communications de la Société Mathématique de Kharkov, X (1906–1909), pp. 34–64, 231–269. (Russian) Translation of pp. 34–64, B. Korenblum and M. J. Prelle, SIGSAM Bulletin, Vol. 15, No. 2, May 1981, pp. 20–32.

[**MOZI**] J. Moses and R. Zippell, *An extension of Liouville's Theorem*, Symbolic and Algebraic Computation (E. W. Ng, ed.), Springer-Verlag, Berlin and New York, 1979, pp. 426–430.

[**PAIN**] *Oeuvres de Paul Painlevé*, Vols. I and II, Editions CNRS, Paris, 1972.

[**POINC**] *Oeuvres de Henri Poincaré*, Vol. III, Gauthier-Villars, Paris, 1934.

[**PRELLE**] M. J. Prelle, Thesis, Rensselaer Polytechnic Institute, 1982.

[**RISCH**] R. H. Risch, *The problem of integration in finite terms*, Trans. Amer. Math. Soc. **139** (1969), 167–189.

[**RISCH**] ______, *The solution of the problem of integration in finite terms*, Bull. Amer. Math. Soc. **76** (1970), 605–608.

[**RITT**] J. F. Ritt, *On the integrals of elementary functions*, Trans. Amer. Math. Soc. **25** (1923), 211–222.

[**RITT**] ______, *Integration in finite terms. Liouville's theory of elementary methods*, Columbia Univ. Press, New York, 1948.

[**ROS**] M. Rosenlicht, *On Liouville's theory of elementary functions*, Pacific J. Math. **65** (1976), 485–492.

[**ROSSIN**] M. Rosenlicht and M. Singer, *On elementary, generalized elementary, and liouvillian extension fields*, Contributions to Algebra, Academic Press, New York, 1977, pp. 329–342.

[**SING**] M. Singer, *Elementary solutions of differential equations*, Pacific J. Math. **59** (1975), 535–547.

[**SING**] ______, *Functions satisfying elementary relations*, Trans. Amer. Math. Soc. **227** (1977), 185–206.

Department of Computer Science, University of Lowell, Lowell, Massachusetts 01854

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650

3 Solving second order ordinary differential equations by extending the Prelle-Singer method. by L.G.S. Duarte, da Mota, J,E,F, Skea (2018)

Solving second order ordinary differential equations by extending the PS method

L.G.S. Duarte[*], L.A. da Mota[†] and J.E.F.Skea[‡]

October 31, 2018

Abstract

An extension of the ideas of the Prelle-Singer procedure to second order differential equations is proposed. As in the original PS procedure, this version of our method deals with differential equations of the form $y'' = M(x, y, y')/N(x, y, y')$, where M and N are polynomials with coefficients in the field of complex numbers C. The key to our approach is to focus not on the final solution but on the first-order invariants of the equation. Our method is an attempt to address algorithmically the solution of SOODEs whose first integrals are elementary functions of x, y and y'.

[*]Universidade do Estado do Rio de Janeiro, Instituto de Física, Departamento de Física Teórica, R. São Francisco Xavier, 524, Maracanã, CEP 20550–013, Rio de Janeiro, RJ, Brazil. E-mail: lduarte@dft.if.uerj.br

[†]idem. E-mail: damota@dft.if.uerj.br

[‡]idem. Email: jimsk@dft.if.uerj.br

1 Introduction

The fundamental position of differential equations (DEs) in scientific progress has, over the last three centuries, led to a vigorous search for methods to solve them. The overwhelming majority of these methods are based on classification of the DE into types for which a method of solution is known, which has resulted in a gamut of methods that deal with specific classes of DEs. This scene changed somewhat at the end of the 19th century when Sophus Lie developed a general method to solve (or at least reduce the order of) ordinary differential equations (ODEs) given their symmetry transformations [1, 2, 3]. Lie's method is very powerful and highly general, but first requires that we find the symmetries of the differential equation, which may not be easy to do. Search methods have been developed [4, 5] to extract the symmetries of a given ODE, however these methods are heuristic and cannot guarantee that, if symmetries exist, they will be found.

On the other hand in 1983 Prelle and Singer (PS) presented a deductive method for solving first order ODEs (FOODE) that presents a solution in terms of elementary functions if such a solution exists [6]. The attractiveness of the PS method lies not only in its basis on a totally different theoretical point of view but, also in the fact that, if the given FOODE has a solution in terms of elementary functions, the method guarantees that this solution will be found (though, in principle it can admittedly take an infinite amount of time to do so). The original PS method was built around a system of two autonomous FOODEs of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with P and Q in $C[x, y]$ or, equivalently, the form $y' = R(x, y)$, with $R(x, y)$ a rational function of its arguments. Here we propose a generalization that allows us to apply the techniques developed by Prelle and Singer to second order differential equations (SOODEs). The key idea is to focus not on the final solution of the equation, but rather its invariants.

This paper is organized as follows: in section 2, the reader is introduced to the PS procedure; section 3 addresses our approach extending the ideas of the PS procedure to the case of SOODEs and discusses how generally applicable the method is to such equations. Section 4 is dedicated to some examples solved via our procedure and, finally, conclusions are presented in section 5.

2 The Prelle-Singer Procedure

Despite its usefulness in solving FOODEs, the Prelle-Singer procedure is not very well known outside mathematical circles, and so we present a brief overview of the main ideas of the procedure.

Consider the class of FOODEs which can be written as

$$y' = \frac{dy}{dx} = \frac{M(x, y)}{N(x, y)} \tag{1}$$

where $M(x, y)$ and $N(x, y)$ are polynomials with coefficients in the complex field C.

In [6], Prelle and Singer proved that, if an elementary first integral of (1) exists, it is possible to find an integrating factor R with $R^n \in C$ for some (possible non-integer) n, such that

$$\frac{\partial(RN)}{\partial x} + \frac{\partial(RM)}{\partial y} = 0. \tag{2}$$

The ODE can then be solved by quadrature. From (2) we see that

$$N\frac{\partial R}{\partial x}+R\frac{\partial N}{\partial x}+M\frac{\partial R}{\partial y}+R\frac{\partial M}{\partial y}=0. \tag{3}$$

Thus

$$\frac{D[R]}{R}=-\left(\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}\right), \tag{4}$$

where

$$D\equiv N\frac{\partial}{\partial x}+M\frac{\partial}{\partial y}. \tag{5}$$

Now let $R=\prod_i f_i^{n_i}$ where f_i are irreducible polynomials and n_i are non-zero integers. From (5), we have

$$\begin{aligned}\frac{D[R]}{R} &= \frac{D[\prod_i f_i^{n_i}]}{\prod_i f_k^{n_k}}=\frac{\sum_i f_i^{n_i-1}n_iD[f_i]\prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}}\\ &= \sum_i\frac{f_i^{n_i-1}n_iD[f_i]}{f_i^{n_i}}=\sum_i\frac{n_iD[f_i]}{f_i}.\end{aligned} \tag{6}$$

From (4), plus the fact that M and N are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (6), we see that $f_i|D[f_i]$.

We now have a criterion for choosing the possible f_i (build all the possible divisors of $D[f_i]$) and, by using (4) and (6), we have

$$\sum_i\frac{n_iD[f_i]}{f_i}=-\left(\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}\right). \tag{7}$$

If we manage to solve (7) and thereby find n_i, we know the integrating factor for the FOODE and the problem is reduced to a quadrature. Risch's algorithm [7] can then be applied to this quadrature to determine whether a solution exists in terms of elementary functions.

3 Extending the Prelle-Singer Procedure

In the previous section, the main ideas and concepts used in the Prelle-Singer procedure were introduced. Here we present an extension of these ideas applicable to SOODEs. The main idea is to focus on the first order invariants of the ODE rather than on the solutions.

3.1 Introduction

Consider the SOODE

$$y''=\frac{d^2y}{dx^2}=\frac{M(x,y,y')}{N(x,y,y')}, \tag{8}$$

where $M(x,y,y')$ and $N(x,y,y')$ are polynomials with coefficients in C. We assume that (8) has a solution in terms of elementary functions, in which case there are two independent elementary functions of x y and y' which are constant on all solutions of (8), namely the first order invariants

$$I_i(x,y,y')=C_i \qquad i=1,2. \tag{9}$$

Without loss of generalization we consider one of these and, dropping the index on I_i we have

$$\mathrm{d}I = \frac{\partial I}{\partial x}\,\mathrm{d}x + \frac{\partial I}{\partial y}\,\mathrm{d}y + \frac{\partial I}{\partial y'}\,\mathrm{d}y' = 0. \tag{10}$$

Now, introducing the notation $\frac{\partial I}{\partial u} \equiv I_u$, we have

$$I_x + I_y y' + I_{y'} y'' = 0, \tag{11}$$

and so

$$y'' = -\frac{I_x + I_y y'}{I_{y'}}, \tag{12}$$

which is (8) in terms of the differential invariant I. Rewriting (8) as

$$\frac{M}{N}\,\mathrm{d}x - \mathrm{d}y' = 0 \tag{13}$$

and observing that

$$y'\,\mathrm{d}x = \mathrm{d}y, \tag{14}$$

we can add the identically null term $S(x,y,y')y'\,\mathrm{d}x - S(x,y,y')\,\mathrm{d}y$ to (13) and obtain the 1-form

$$\left(\frac{M}{N} + Sy'\right)\,\mathrm{d}x - S\,\mathrm{d}y - \mathrm{d}y' = 0. \tag{15}$$

Notice that the 1-form (16) must be proportional to the 1-form (10). So, since the 1-form (10) is exact, we can multiply (16) by the integrating factor $R(x,y,y')$ to obtain

$$\mathrm{d}I = R(\phi + Sy')\,\mathrm{d}x - RS\,\mathrm{d}y - R\,\mathrm{d}y' = 0, \tag{16}$$

where $\phi \equiv M/N$.

Comparing equations (10) and (16),

$$\begin{aligned} I_x &= R(\phi + Sy'), \\ I_y &= -RS, \\ I_{y'} &= -R. \end{aligned} \tag{17}$$

Now equations (17) must satisfy the compatibility conditions $I_{xy} = I_{yx}, I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$. This implies that

$$D[S] = -\phi_y + S\phi_{y'} + S^2, \tag{18}$$

$$D[R] = -R(S + \phi_{y'}), \tag{19}$$

$$R_y = R_{y'}S + S_{y'}R, \tag{20}$$

where the differential operator D is defined as

$$D \equiv \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'}. \tag{21}$$

Combining (18) and (19) we obtain

$$D[RS] = -R\phi_y. \tag{22}$$

So if the product of S and the integrating factor R is a rational function of x, y and y', then $D[RS]$ is too. Since ϕ is rational (and so, therefore, is ϕ_y), equation (22) tells us that R is rational. Using (19) and similar arguments we conclude that S must be a rational function of x, y and y'.

In summary, from (17) it follows that the supposition that RS is rational can be equated to the existence of a first order invariant whose derivatives in relation to x, y and y' are rational functions. With this in mind we restate the original supposition in the form of a conjecture.

3.2 The Conjecture

We first state a result proved in [6]: **Theorem:***Let K be a differential field of functions in $n+1$ variables and L an elementary extension of K. Let f be in K and assume there exists a nonconstant g in L such that g is constant on all solutions of $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$. Then there exist $w_1, \ldots, w_m$ algebraic over K and constants $c_i, \ldots, c_m$ such that*

$$w_0(x, y, y', y'', .., y^{(n-1)}) + \sum_i c_i \log(w_i(x, y, y', y'', \ldots, y^{(n-1)})) \tag{23}$$

is a constant on all solutions of $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$.

This result shows that for the particular case of SOODEs whose solutions are elementary, there are two independent first order invariants of the form

$$w_0(x, y, y') + \sum_i c_i \log w_i(x, y, y'). \tag{24}$$

Our conjecture is that if these two first order invariants exist it is always possible to find a function of them (which will, therefore, itself be a first order invariant) of the form

$$z_0(x, y, y') + \sum_i c_i \log[z_i(x, y, y')], \tag{25}$$

where z_i are *rational* functions of x, y and y'.

Conjecture:*Let K be a differential field of functions in three variables and L an elementary extension of K. Let f be in K and assume there exist two independent nonconstant $\{g_1, g_2\}$ in L such that g_i are constant on all solutions of $y'' = f(x, y, y')$. Then there exists at least one constant of the form*

$$z_0(x, y, y') + \sum_i c_i \log(z_i(x, y, y')) \tag{26}$$

where the z_i are in K.

By the previous reasoning it can be seen that (26) implies that the product RS is a rational function of x, y and y'.

If this conjecture holds, then our extension of the PS method applies to all SOODEs of the form (8). Though we have not been able to prove our conjecture, extensive trials while developing this procedure has not revealed any counter example. Even if the conjecture is false, our experience with real test cases has shown that the method is, at least, applicable to the vast majority of SOODEs of the form (8).

3.3 Finding R and S

Our conjecture implies that, if the SOODE to be solved has an elementary general solution, then S is a rational function which we may write as

$$S = \frac{S_n}{S_d} = \frac{\sum_{i,j,k} a_{ijk} x^i y^j {y'}^k}{\sum_{i,j,k} b_{ijk} x^i y^j {y'}^k}. \tag{27}$$

We can also see that (18) does not involve R. So, given a degree bound on the polynomials S_n and S_d, we may find a set of solutions to this equation which are then candidates to solve the system of equations (18)–(20).

From (19) we have

$$\frac{D[R]}{R} = -(S + \phi_{y'}) = -\frac{S_n}{S_d} - \left(\frac{M}{N}\right)_{y'} = -\frac{S_n N^2 + S_d(N M_{y'} - M N_{y'})}{S_d N^2} \tag{28}$$

which can be rewritten as

$$\frac{\mathcal{D}[R]}{R} = -S_n N^2 + S_d(N M_{y'} - M N_{y'}), \tag{29}$$

where the differential operator $\mathcal{D}$ is defined as

$$\mathcal{D} \equiv (S_d N^2) D. \tag{30}$$

We keep in mind that

- S_n, S_d, N and M are polynomials in x, y and y';
- $\mathcal{D}$ is a linear differential operator whose coefficients of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial y'}$ are polynomials in x, y and $, y'$;
- R is a rational function of x, y and y', which we may write as

$$R = \frac{R_n}{R_d} = \frac{\sum_{i,j,k} c_{ijk} x^i y^j {y'}^k}{\sum_{i,j,k} d_{ijk} x^i y^j {y'}^k}. \tag{31}$$

If we have a theoretical limit on the degrees of R_m and R_d (a *degree bound*), we may use a procedure analogous to that described in section 2 to obtain candidates for the integrating factor R. We simply construct all polynomials in x, y and y' up to the degree bound.

3.4 Reduction of the SOODE

Once R and S have been determined using equations (17) we have all the partial first derivatives of the first order differential invariant, $I(x, y, y')$, which is constant on the solutions. This invariant can then be obtained as

$$\begin{gathered} I(x,y,y') = \int R\,(\phi + S y')\,dx\; - \\ \int \big[RS + \frac{\partial}{\partial y}\int R\,(\phi + S y')\,dx\big]dy\; - \\ \int \Big[R + \frac{\partial}{\partial y'}\Big(\int R\,(\phi + S y')\,dx - \int \big[RS + \frac{\partial}{\partial y}\int R\,(\phi + S y')\,dx\big]dy\Big)\Big]\,dy'. \end{gathered} \tag{32}$$

The equation $I(x, y, y') = C_1$ can then be solved to obtain a FOODE for y': the *reduced* ODE

$$y' = \varphi(x, y, C1). \tag{33}$$

To obtain the general solution of the original ODE, we can apply the Prelle-Singer method in its original form to this reduced ODE. Thus, if our conjecture is correct, the method proposed here (for SOODEs of the form (8)) is as algorithmic as the original PS method for FOODEs. We note that the original PS method fails to be what is strictly an algorithm because no theoretical degree bound is yet known for the candidate polynomials which enter in the prospective solution, and so the procedure has no effective terminating condition for the case when an elementary solution does not exists. In practice, a terminating condition is put in by hand (it is found that polynomials of degree higher than 4 lead to computations which are overly complex for the average desktop computer). However, should such a degree bound be established, and our conjecture shown to be true, then the method proposed here would be an algorithm for deciding whether elementary solutions of SOODEs of the form (8) exist.

4 Examples

In this section we present examples of physically motivated SOODEs that are solved by our procedure[1]. As a simple illustrative example, we begin with the classical harmonic oscillator and then consider some nonlinear SOODEs which arise from astrophysics and general relativity.

Example 1: The Simple Harmonic Oscillator

In its simplest form, the equation for the simple harmonic oscillator is

$$y'' = -y. \tag{34}$$

For this ODE equations (18), (19) and (20) are

$$\begin{aligned} S_x + y'S_y - yS_{y'} &= 1 + S^2, && (35)\\ R_x + y'R_y - yR_{y'} &= -RS, && (36)\\ R_y - R_{y'}S - S_{y'}R &= 0. && (37) \end{aligned}$$

One possible solution to these equations is

$$S = \frac{y}{y'}, \;\; R = y'. \tag{38}$$

From this, and using (32), we get the reduced ODE

$$C1 = y^2 + y'^2, \tag{39}$$

which, of course, represents the energy conservation for the oscillator.

This example is very simple and leads to a form of ϕ which is independent of x and y'. And, as with all linear ODEs, alternative and more straightforward solution methods exist. The other examples illustrate the solution method at work for non-linear SOODEs which can be placed in the form (8).

[1]We present only the reduction of the SOODEs since the integration of the resulting FOODE can be achieved by various methods, including the PS method itself.

Example 2: An Exact Solution in General Relativity

A rich source of non-linear DEs in physics are the highly non-linear equations of General Relativity. In general, Einstein's equations are, of course, partial DEs, but there exist classes of equations where the symmetry imposed reduces these equations to ODEs in one independent variable. One such class is that of static, spherically symmetric solutions for stellar models, which depend only on the radial variable, r. The metric for a general statically spherically spacetime has two free functions, $\lambda(r)$ and $\mu(r)$ say. On imposing the condition that the fluid is a perfect fluid, Einstein's equations reduce to two coupled ODEs for $\lambda(r)$ and $\mu(r)$. Specifying one of these functions reduces the problem to solving an ODE (of first or second order) for the other.

Following this procedure, Buchdahl [8] obtained an exact solution for a relativistic fluid sphere by considering the so-called isotropic metric

$$\dot{s}^2 = (1-f)^2(1+f)^{-2}\dot{t}^2 - (1+f)^4[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2]$$

with $f = f(r)$. The field equations for $f(r)$ reduce to

$$f f'' - 3f'^2 - r^{-1} f f' = 0.$$

Changing notation with $y(x) = f(r)$, equations (18), (19) and (20) assume the form

$$\begin{aligned} S_x + y' S_y + \frac{y'\,(3\,y'\,x + y)}{xy} S_{y'} &= -\frac{y'}{xy} + \frac{y'\,(3\,y'\,x + y)}{xy^2} + \\ &\quad \left(\frac{3\,y'\,x + y}{xy} + 3\,\frac{y'}{y}\right) S + S^2, \qquad (40) \\ R_x + y' R_y + \frac{y'\,(3\,y'\,x + y)}{xy} R_{y'} &= -R\left(S + \frac{3\,y'\,x + y}{xy} + 3\,\frac{y'}{y}\right), \qquad (41) \\ R_y - R_{y'} S - S_{y'} R &= 0. \qquad (42) \end{aligned}$$

One solution of those equations is

$$S = -3\,\frac{y'}{y}, \quad R = \frac{1}{xy^3} \qquad (43)$$

By using (32) we obtain the reduced FOODE:

$$C_1 = y'/(y^3 x) \qquad (44)$$

which is separable and easily integrated to obtain the general solution

$$y(x)^2 = \left(-C_1\,x^2 + C_2\right)^{-1}. \qquad (45)$$

Example 3: A Static Gaseous General-Relativistic Fluid Sphere

In a later paper [9], Buchdahl approaches the problem of the general relativistic fluid sphere using a different coordinate system from the previous example. For ease in comparison of the originals,

Substituting the $\xi(r)$ of the original Writing $y(x)$ instead of the $\xi(r)$ in the original, arrives at the equation

$$y'' = \frac{x^2 y'^2 + y^2 - 1}{x^2 y}, \qquad (46)$$

For this SOODE, eqs (18, 19 and 20) become:

$$
\begin{aligned}
S_x + y' S_y + \frac{x^2 y'^2 + y^2 - 1}{x^2 y} S_{y'} &= -2\, x^{-2} + \frac{x^2 y'^2 + y^2 - 1}{y^2 x^2} \\
&\quad +2\, \frac{y'\, S}{y} + S^2 \qquad (47) \\
R_x + y' R_y + \frac{x^2 y'^2 + y^2 - 1}{x^2 y} R_{y'} &= -R \left(S + 2\, \frac{y'}{y} \right) \qquad (48) \\
R_y - R_{y'} S - S_{y'} R &= 0 \qquad (49)
\end{aligned}
$$

One solution to those equations is:

$$S = \frac{-x^2 y'^2 - x y y' + 1}{x y^2 + x^2 y y'}, \quad R = \frac{y + x y'}{x y^2}. \qquad (50)$$

From this, using eq. (32), we get the reduced FOODE:

$$C_1 = \frac{2\, x y y' + y^2 + x^2 y'^2 - 1}{2 x^2 y^2}. \qquad (51)$$

which can be solved to:

$$y(x)^2 = \frac{\tan(\sqrt{2}\sqrt{C_1}\, (C_2 + x))^2}{\left(2\, C_1 + 2\, \tan(\sqrt{2}\sqrt{C_1}\, (C_2 + x))^2 C_1\right) x^2} \qquad (52)$$

This example has an extra feature: It is not solved by other solvers we have tried (mainly the Maple solver, in the version 5, that we believe to be the best). So, apart from the (already) very interesting fact that our approach is an algorithmic attempt to solve SOODEs, we have also this present fact, i.e., some SOODEs are solved via our method and "escape" from other very powerful solvers.

5 Conclusion

In this paper, we presented an approach that is an extension of the ideas developed by Prelle-Singer [6] to tackle FOODEs. We believe it to be the first technique to address algorithmically the solution of SOODEs with elementary first integrals.

Here, we dealt with a restrict class of SOODEs (namely, the ones of the form (8)). However, we can use our method in solving SOODEs where $\phi(x, y, y')$ depends on elementary functions of x, y, y', following the developments for the Prelle-Singer approach for FOODEs [10, 11]. We are presently working on those ideas.

The generality of our approach is based on a conjecture (see section (3.2)) that we have already proved for many special cases. Even if the conjecture is proven false, our approach is a powerful tool in dealing with SOODEs since we have extensively tested it with many equations, both from mathematics and physical origin. In fact, since all the examples we have encounter have been solved by our approach, we are preparing a computational package implementing the Prelle-Singer procedure (and our present extension) to be submitted to Computer Physics Communications.

References

[1] Stephani, H. *Differential equations: their solution using symmetries*, ed. M.A.H. MacCallum, Cambridge University Press, New York and London (1989).

[2] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Applied Mathematical Sciences **81**, Springer-Verlag, (1989).

[3] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).

[4] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, *Computer Algebra Solving of First Order ODEs Using Symmetry Methods. Comput.Phys.Commun.*, **101**, (1997), 254.

[5] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, *Computer Algebra Solving of Second Order ODEs Using Symmetry Methods. Comput.Phys.Commun.*, **108**, (1998), 90.

[6] M Prelle and M Singer, Elementary first integral of differential equations. *Trans. Amer. Math. Soc.*, **279** 215 (1983).

[7] R H Risch, *The Problem of Integration in Finite Terms Trans. Amer. Math. Soc.*, **139** 167–189 (1969).

[8] H.A.Buchdahl (1964) *A Relativistic Fluid Sphere Resembling the Emden Polytrope of Index 5 Ap. J.* **140** 1512–1516.

[9] H.A.Buchdahl (1967) *General Relativistic Fluid Spheres III. A Static Gaseous Model Ap. J.* **147** 310–316.

[10] Y K Man, Computing Closed Form Solutions of First Order ODEs Using the Prelle-Singer Procedure. *J. Symbolic Computation*, **16** 423–443 (1993)

[11] Y K Man, First Integrals of Autonomous Systems of Differential Equations and the Prelle-Singer Procedure. *J. Phys. A*, **27** L329–332 (1994)

4 Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I. By Nail H. Ibragimov, Sergey V. Meleshko (2005)

ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Communications in Nonlinear Science and Numerical Simulation

ELSEVIER Communications in Nonlinear Science and Numerical Simulation xxx (2006) xxx–xxx

www.elsevier.com/locate/cnsns

Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I

Nail H. Ibragimov [a,*], Sergey V. Meleshko [b]

[a] *Department of Mathematics and Science, Research Centre ALGA: Advances in Lie Group Analysis, Blekinge Institute of Technology, Valhallevegen, SE-371 79 Karlskrona, Sweden*

[b] *School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, 3000, Thailand*

Received 23 November 2005; accepted 23 December 2005

Abstract

Lie's group classification of ODEs shows that the second-order equations can possess one, two, three or eight infinitesimal symmetries. The equations with eight symmetries and only these equations can be linearized by a change of variables. Lie showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. Our aim is to provide a similar description of the equations with three symmetries. There are four different types of such equations. We present here the candidates for all four types. We give an invariant test for existence of three symmetries for one of these candidates.

PACS: 02.30.Jr; 02.20.Tw

Keywords: Lie's group classification; Candidates for equations with three symmetries; Invariants

1. Introduction

According to Lie's classification [1] in the complex domain, any ordinary differential equation of the second order

$$y'' = f(x, y, y') \tag{1}$$

admitting a three-dimensional Lie algebra belongs to one of four distinctly different types. Each of these four types is obtained by a change of variables from the following canonical representatives (see, e.g., [2, Section 8.4]):

* Corresponding author. Tel.: +46 455 385 450; fax: +46 455 385 460.
E-mail address: nib@bth.se (N.H. Ibragimov).

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2005.12.012

$$y'' + Cy^{-3} = 0, \tag{2}$$
$$y'' + Ce^{y'} = 0, \tag{3}$$
$$y'' + Cy'^{(k-2)/(k-1)} = 0, \tag{4}$$
$$y'' + 2\frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0, \tag{5}$$

where $k \neq 0$, 1/2, 1, 2 in (4), and $C = \text{const}$.

Eqs. (2)–(5) admit non-similar three-dimensional Lie algebras L_3 spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{6}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (y - x)\frac{\partial}{\partial y}, \tag{7}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \tag{8}$$

and

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \tag{9}$$

respectively (see, e.g., [2, Section 8.4]).

2. Candidates for equations with three symmetries

Let us subject each of Eqs. (2)–(5) to the arbitrary change of variables

$$t = \varphi(x,y), \qquad u = \psi(x,y), \tag{10}$$

where t is a new independent variable and u is a new dependent variable. Then we obtain from (2)–(5) the equations of the form

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 = 0, \tag{11}$$
$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\exp\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0, \tag{12}$$
$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{(k-2)/(k-1)} = 0, \tag{13}$$

and

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{3/2} = 0, \tag{14}$$

respectively, where $b_i = b_i(t,u)$, $i = 1, \ldots, 12$. Eqs. (11)–(14) are the *candidates for the equations with three symmetries.*

All candidates can be encapsulated in the formula

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)f\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0.$$

Namely, Eqs. (11)–(14) are obtained by letting

$$f(z) = 0, \qquad f(z) = e^z, \qquad f(z) = z^{(k-2)/(k-1)}, \qquad f(z) = z^{3/2}.$$

Using the usual formula for the transformation of derivatives under the change of variables (10), we obtain the following statement.

Theorem 1. *Any equation of the form*

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0$$

is transformed by the change of variables (10) *into an equation of the same form*:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 + (a_5 y'^3 + 3a_6 y'^2 + 3a_7 y' + a_8) f\left(\frac{a_9 y' + a_{10}}{a_{11} y' + a_{12}}\right) = 0.$$

Here a_i and b_i are functions of x, y and t, u, respectively, and are connected by

$$\begin{aligned}
a_1 &= \Delta^{-1}[\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3],\\
a_2 &= \Delta^{-1}[b_4\varphi_x\varphi_y^2 + b_3\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_2\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x)\\
&\quad + b_1\psi_x\psi_y^2 + (\varphi_x\psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y\psi_{xy})/3],\\
a_3 &= \Delta^{-1}[b_4\varphi_x^2\varphi_y + b_3\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_2\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x)\\
&\quad + b_1\psi_x^2\psi_y + (\varphi_y\psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x\psi_{xy})/3],\\
a_4 &= \Delta^{-1}[b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}],\\
a_5 &= \Delta^{-1}[b_8\varphi_y^3 + 3b_7\varphi_y^2\psi_y + 3b_6\varphi_y\psi_y^2 + b_5\psi_y^3],\\
a_6 &= \Delta^{-1}[b_8\varphi_x\varphi_y^2 + b_7\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_6\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_5\psi_x\psi_y^2],\\
a_7 &= \Delta^{-1}[b_8\varphi_x^2\varphi_y + b_7\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_6\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x) + b_5\psi_x^2\psi_y],\\
a_8 &= \Delta^{-1}[b_8\varphi_x^3 + 3b_7\varphi_x^2\psi_x + 3b_6\varphi_x\psi_x^2 + b_5\psi_x^3],\\
a_9 &= b_{10}\varphi_y + b_9\psi_y,\\
a_{10} &= b_{10}\varphi_x + b_9\psi_x,\\
a_{11} &= b_{12}\varphi_y + b_{11}\psi_y,\\
a_{12} &= b_{12}\varphi_x + b_{11}\psi_x,
\end{aligned}$$

where

$$\Delta = (\varphi_x\psi_y - \varphi_y\psi_x) \neq 0$$

is the Jacobian of the change of variables (10).

3. Equations equivalent to Eq. (2)

In this paper, we will dwell on the first candidate, i.e., on equations of the form (11). Other candidates will be considered elsewhere.

We know that all equations obtained from Eq. (2),

$$y'' + Ky^{-3} = 0 \quad (K = \text{const.} \neq 0), \tag{15}$$

by the change of variables (10) are contained in the family of the equations of the form (11):

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0.$$

We also know from Theorem 1 that any Eq. (11) is transformed by the change of variables (10) into an equation of the same form:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 = 0, \tag{16}$$

and that the coefficients of Eqs. (11) and (16) are related by the following equations:

$$\begin{aligned}
a_1 &= \Delta^{-1}[\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3],\\
a_2 &= \Delta^{-1}[b_4\varphi_x\varphi_y^2 + b_3\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_2\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x)\\
&\quad + b_1\psi_x\psi_y^2 + (\varphi_x\psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y\psi_{xy})/3],\\
a_3 &= \Delta^{-1}[b_4\varphi_x^2\varphi_y + b_3\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_2\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x)\\
&\quad + b_1\psi_x^2\psi_y + (\varphi_y\psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x\psi_{xy})/3],\\
a_4 &= \Delta^{-1}[b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}].
\end{aligned} \tag{17}$$

We will use the following information about invariants of Eqs. (11). Lie [1] showed that any second-order equation obtained from the linear equation $y'' = 0$ by the change of variables (10) belongs to the family of Eqs. (11) and obtained the necessary and sufficient conditions for Eqs. (11) to be equivalent to the linear equation. Lie's linearization test can be expressed by means of the equations $L_1 = 0$, $L_2 = 0$ (see, e.g., [3]). These equations are invariant with respect to the change of variables (10). Therefore the quantities L_1 and L_2 are called *relative invariants* for Eq. (11). They involve the coefficients of Eq. (11) and their derivatives of up to second order and can be readily calculated by means of the infinitesimal method [4]. We will write them, using notation from [5], in the following form:

$$\begin{aligned}
L_1 &= -\frac{\partial \Pi_{11}}{\partial u} + \frac{\partial \Pi_{12}}{\partial t} - b_4\Pi_{22} - b_2\Pi_{11} + 2b_3\Pi_{12},\\
L_2 &= -\frac{\partial \Pi_{12}}{\partial u} + \frac{\partial \Pi_{22}}{\partial t} - b_1\Pi_{11} - b_3\Pi_{22} + 2b_2\Pi_{12},
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\Pi_{11} &= 2(b_3^2 - b_2b_4) + b_{3t} - b_{4u},\\
\Pi_{22} &= 2(b_2^2 - 3b_1b_3) + b_{1t} - b_{2u},\\
\Pi_{12} &= b_2b_3 - b_1b_4 + b_{2t} - b_{3u}.
\end{aligned} \tag{19}$$

The change of variables (10) converts the quantities (18) into the following relative invariants for Eq. (11):

$$\widetilde{L}_1 = \Delta(L_1\varphi_x + L_2\psi_x), \qquad \widetilde{L}_2 = \Delta(L_1\varphi_y + L_2\psi_y). \tag{20}$$

For Eq. (15), the relative invariants (20) are written

$$\widetilde{L}_1 = 12Ky^{-5}, \qquad \widetilde{L}_2 = 0. \tag{21}$$

Hence, the following statement is valid.

Lemma 1. *For all Eq.* (11) *obtained from Eq.* (15) *by a change of variables, at least one of the relative invariants* L_1, L_2 *does not vanish, and the corresponding change of the variables* (10) *obeys the equation*

$$L_1\varphi_y + L_2\psi_y = 0. \tag{22}$$

We will use the following relative invariants of higher order given in [5–7]:

$$v_5 = L_2(L_1L_{2t} - L_2L_{1t}) + L_1(L_2L_{1u} - L_1L_{2u}) - b_1L_1^3 + 3b_2L_1^2L_2 - 3b_3L_1L_2^2 + b_4L_2^3 \tag{23}$$

$$w_1 = L_1^{-4}[-L_1^3(\Pi_{12}L_1 - \Pi_{11}L_2) + R_1(L_1^2)_t - L_1^2R_{1t} + L_1R_1(b_3L_1 - b_4L_2)], \tag{24}$$

and

$$I_2 = 3R_1L_1^{-1} + L_{2t} - L_{1u}, \tag{25}$$

where

$$R_1 = L_1L_{2t} - L_2L_{1t} + b_2L_1^2 - 2b_3L_1L_2 + b_4L_2^2.$$

If the relative invariant $I_2 \neq 0$, there is the set of absolute invariants

$$J_{2m} = I_{2m} I_2^{-m} \quad (m \geqslant 1),$$

where

$$I_{2m+2} = L_1 \frac{\partial I_{2m}}{\partial u} - L_2 \frac{\partial I_{2m}}{\partial t} + 2m I_{2m}(L_{2t} - L_{1u}).$$

The similar relative invariants for Eq. (16) are denoted by $\widetilde{v}_5$, $\widetilde{w}_1$, $\widetilde{I}_2$ and $\widetilde{J}_4$. For Eq. (15), invoking Eqs. (21), we obtain:

$$\widetilde{v}_5 = 0, \qquad \widetilde{w}_1 = 0, \qquad \widetilde{I}_2 = 60Ky^{-6}, \qquad \widetilde{J}_4 = 4/5. \tag{26}$$

Hence, we have the following necessary conditions for Eqs. (11) obtained from Eq. (15) by a change of variables:

$$v_5 = 0, \qquad w_1 = 0, \qquad I_2 \neq 0, \qquad J_4 = 4/5. \tag{27}$$

We will obtain now the necessary and sufficient conditions.

One has to find the conditions for the coefficients $b_1(t,u)$, $b_2(t,u)$, $b_3(t,u)$ and $b_4(t,u)$ that guarantee the existence of the functions $\varphi(x,y)$ and $\psi(x,y)$ such that the change of variables (10) transforms the coefficients of Eq. (11) into

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = Ky^{-3},$$

where a_1, a_2, a_3, a_4 are defined by formulae (17). Thus, we have to investigate the consistency of the following over-determined system:

$$\varphi_y \psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3 = 0, \tag{28}$$

$$3b_4\varphi_x\varphi_y^2 + 6b_3\varphi_x\varphi_y\psi_y + 3b_3\varphi_y^2\psi_x + 3b_2\varphi_x\psi_y^2 + 6b_2\varphi_y\psi_x\psi_y + 3b_1\psi_x\psi_y^2 - 2\varphi_{xy}\psi_y + \varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\varphi_y\psi_{xy} = 0, \tag{29}$$

$$3b_4\varphi_x^2\varphi_y + 3b_3\varphi_x^2\psi_y + 6b_3\varphi_x\varphi_y\psi_x + 6b_2\varphi_x\psi_x\psi_y + 3b_2\varphi_y\psi_x^2 + 3b_1\psi_x^2\psi_y - 2\varphi_{xy}\psi_x - \varphi_{xx}\psi_y + 2\varphi_x\psi_{xy} + \varphi_y\psi_{xx} = 0, \tag{30}$$

$$y^3(b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}) - K\Delta = 0. \tag{31}$$

Remark 1. For Eq. (11) equivalent to Eq. (15) one of the values, either L_1 or L_2, is not equal to zero. Notice that if $L_1 = 0$ and $L_2 \neq 0$, then the change

$$t = y, \quad u = -x$$

leads to the change

$$\widetilde{L}_1 = L_2, \qquad \widetilde{L}_2 = L_1, \qquad a_1 = b_4, \qquad a_2 = -b_4, \qquad a_3 = b_2, \qquad a_4 = -b_1.$$

Further without loss of generality it is assumed that $L_1 \neq 0$.

Lemma 2. *Let Eq.* (11) *which is equivalent to Eq.* (15) *has* $L_1 \neq 0$. *Then* $\varphi_y = 0$ *if and only if* $L_2 = 0$.

Proof. The statement follows from Lemma 1. □

Theorem 2. *Eq.* (11) *with* $L_2 = 0$ *is equivalent to Eq.* (15) *if and only if its coefficients satisfy the following equations:*

$$b_1 = 0, \qquad b_{3u} - 2b_{2t} = 0, \tag{32}$$

$$L_{1tt} - \frac{6}{5L_1}(L_{1t})^2 + \frac{3}{5}b_3L_{1t} - 15b_4b_2L_1 + b_4I_2 - 5b_{4u}L_1 + 6b_{3t}L_1 + \frac{54}{5}b_3^2L_1 - \frac{25}{3I_2}L_1^3 = 0, \tag{33}$$

$$I_{2t} - \frac{6}{5L_1} I_2 L_{1t} - \frac{6}{5} b_3 I_2 = 0, \tag{34}$$

$$I_{2u} - 6b_2 I_2 + \frac{6}{5L_1} I_2^2 = 0. \tag{35}$$

Let Eqs. (32)–(35) *be satisfied. Then Eq.* (11) *is mapped to Eq.* (15) *by the change of variables* (10) *of the form*

$$t = \varphi(x), \qquad u = \psi(x,y), \tag{36}$$

where $\varphi(x)$ *is determined by the equation*

$$\varphi_x^2 = \frac{12KI_2}{5L_1^2 y^4} \tag{37}$$

and $\psi(x,y)$ *by the following integrable system:*

$$\psi_y = \frac{5L_1}{I_2 y}, \tag{38}$$

$$\psi_{xx} = \frac{1}{25\varphi_x I_2 L_1^3 y^4} [5\varphi_x \psi_x^2 I_2 L_1^2 y^4 (2I_2 - 15b_2 L_1) + 5\varphi_x K L_1 (25L_1^3 - 12b_4 I_2^2) - 24\psi_x K I_2^2 (L_{1t} + 6L_1 b_3)]. \tag{39}$$

Remark 2. The left-hand sides of Eqs. (32)–(35) are relative invariants with respect to the transformation (36). The equations $v_5 = 0$, $w_1 = 0$ (see (27)) are Eqs. (32), the equation $J_4 = 4/5$ is Eq. (35). In these equations, the variable I_2 is given by $I_2 = 3b_2 L_1 - L_{1u}$.

Theorem 3. *Eq.* (11) *with* $L_2 \neq 0$ *is equivalent to Eq.* (15) *if and only if its coefficients satisfy the following equations:*

$$5L_1 I_{2t} - 6I_2(L_{1t} - b_4 L_2 + b_3 L_1) = 0, \tag{40}$$

$$L_1^2 L_{2u} - b_4 L_2^3 + 3b_3 L_1 L_2^2 - 3b_2 L_1^2 L_2 + b_1 L_1^3 + L_{1t} L_2^2 - L_{1u} L_1 L_2 - L_{2t} L_1 L_2 = 0, \tag{41}$$

$$5L_1^2 I_{2u} - 6I_2(4b_4 L_2^2 - 9b_3 L_1 L_2 + 5b_2 L_1^2 - 4L_{1t} L_2 + 5L_{2t} L_1 - I_2 L_1) = 0, \tag{42}$$

$$\begin{aligned} &15I_2 L_1 L_{1tt} - 63b_4^2 I_2 L_2^2 + 126b_4 b_3 I_2 L_1 L_2 - 225b_4 b_2 I_2 L_1^2 + 81b_4 L_{1t} I_2 L_2 - 90b_4 L_{2t} I_2 L_1 + 15b_4 I_2^2 L_1 \\ &\quad + 162b_3^2 I_2 L_1^2 + 9b_3 L_{1t} I_2 L_1 - 15b_{4t} I_2 L_1 L_2 - 75b_{4u} I_2 L_1^2 + 90b_{3t} I_2 L_1^2 - 18L_{1t}^2 I_2 - 125L_1^4 = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} &L_1^2 L_{2tt} + b_4^2 L_2^3 - 3b_4 b_3 L_1 L_2^2 + 3b_4 b_2 L_1^2 L_2 - b_4 b_1 L_1^3 - 3b_4 L_{1t} L_2^2 + 3b_4 L_{2t} L_1 L_2 + 3b_3 L_{1t} L_1 L_2 - 3b_3 L_{2t} L_1^2 \\ &\quad + b_{4t} L_1 L_2^2 + b_{4u} L_1^2 L_2 - 3b_{3t} L_1^2 L_2 - b_{3u} L_1^3 + 2b_{2t} L_1^3 - L_{1tt} L_1 L_2 + 2L_{1t}^2 L_2 - 2L_{1t} L_{2t} L_1) = 0. \end{aligned} \tag{44}$$

Let Eqs. (40)–(44) *be satisfied. Then Eq.* (11) *is mapped to Eq.* (15) *by the change of variables* (10) *with* $\varphi_y \neq 0$. *The functions* $\varphi(x,y)$ *and* $\psi(x,y)$ *are determined by the following integrable system:*

$$L_1 \varphi_y + L_2 \psi_y = 0, \tag{45}$$

$$5y^4(\varphi_x L_1 + \psi_x L_2)^2 - 12KI_2 = 0, \tag{46}$$

$$\psi_y = \frac{5L_1}{I_2 y}, \tag{47}$$

$$\begin{aligned} 5L_1^2 \psi_{xx} = {} & \psi_x^2(-15b_4 L_2^2 + 30b_3 L_1 L_2 - 15b_2 L_1^2 + 10L_{1t} L_2 - 10L_{2t} L_1 + 2I_2 L_1) \\ & + 24\psi_x \Delta^{-1} y^{-5} K(6b_4 L_2 - 6b_3 L_1 - L_{1t}) + Ky^{-4} I_2^{-1}(-12b_4 I_2^2 + 25L_1^3). \end{aligned} \tag{48}$$

Remark 3. The left-hand sides of Eqs. (40)–(44) are relative invariants with respect to the general change of variables (10). The equations $v_5 = 0$, $w_1 = 0$ (see (27)) are Eqs. (41) and (44), the equation $J_4 = 4/5$ is Eq. (42).

Remark 4. The conditions of Theorem 2 are particular cases of the conditions of Theorem 3 provided that $L_2 = 0$.

4. Proof of Theorem 2

We use the method similar to that employed in [8,9]. Routine calculations were made by means of the system for symbolic calculations *Reduce* [10].

According to Lemma 2, $L_2 = 0$ implies $\varphi_y = 0$. Since $\Delta \neq 0$, one obtains $\varphi_x\psi_y \neq 0$. Eqs. (28)–(31) yield

$$b_1 = 0, \qquad \psi_{yy} = -3\psi_y^2 b_2, \qquad \psi_{xy} = (2\varphi_x)^{-1}(\psi_y\varphi_{xx} - 3\varphi_x^2\psi_y b_3 - 6\varphi_x\psi_x\psi_y b_2),$$
$$\psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} - \varphi_x^2 b_4 - 3\varphi_x\psi_x b_3 - 3\psi_x^2 b_2 + y^{-3}\psi_y K.$$

Equating the mixed derivatives $(\psi_{xy})_x = (\psi_{xx})_y$ and $(\psi_{xy})_y = (\psi_{yy})_x$ one has

$$\varphi_x^4(4b_{4u} - 6b_{3t} + 12b_4b_2 - 9b_3^2) + 6\varphi_x^3\psi_x(b_{3u} - 2b_{2t}) + 12\varphi_x^2 y^{-4}K + 2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2 = 0 \tag{49}$$

and

$$b_{3u} = 2b_{2t}. \tag{50}$$

The derivative φ_{xxx} is found from Eq. (49). The equation $(\varphi_{xxx})_y = 0$ gives

$$\varphi_x^2\psi_y L_1 = 12Ky^{-5}. \tag{51}$$

Differentiating this equation with respect to x and y, one obtains

$$\varphi_x^2(3b_3L_1 - 2L_{1t}) + 2\varphi_x\psi_x(3b_2L_1 - L_{1u}) - 5\varphi_{xx}L_1 = 0, \tag{52}$$
$$\psi_y y(3b_2L_1 - L_{1u}) - 5L_1 = 0. \tag{53}$$

Since $L_1 \neq 0$, one has $(3b_2L_1 - L_{1u}) \neq 0$. Using Eqs. (51) and (53), one finds

$$\psi_y = \frac{5L_1}{y(3b_2L_1 - L_{1u})}, \qquad \varphi_x^2 = \frac{12K(3b_2L_1 - L_{1u})}{5L_1^2y^4}.$$

Substituting them into (52), one obtains

$$10\varphi_x\psi_x L_1^2y^4(3b_2L_1 - L_{1u}) + 12(3b_2L_1 - L_{1u})K(3b_3L_1 - 2L_{1t}) - 25\varphi_{xx}L_1^3y^4 = 0. \tag{54}$$

Since $\varphi_y = 0$, the equation $(\varphi_x^2)_y = 0$ gives

$$5L_1L_{1uu} = 3(-12b_2^2L_1^2 + 3b_2L_{1u}L_1 + 5b_{2u}L_1^2 + 2L_{1u}^2). \tag{55}$$

Using Eq. (54) one can find the derivative φ_{xx}:

$$\varphi_{xx} = 2(3b_2L_1 - L_{1u})(5\varphi_x\psi_xL_1^2y^4 + 6K(3b_3L_1 - 2L_{1t}))/(25L_1^3y^4).$$

By considering the equations

$$(\varphi_x^2)_{xx} - 2(\varphi_{xx}^2 + \varphi_x\varphi_{xxx}) = 0, \tag{56}$$
$$(\varphi_x^2)_x - 2\varphi_x\varphi_{xx} = 0, \tag{57}$$

one can obtain conditions for the coefficients b_1, b_2, b_3, b_4. For example, Eq. (57) gives

$$5L_1L_{1tu} = 3(-6b_3b_2L_1^2 + 2b_3L_{1u}L_1 - b_2L_{1t}L_1 + 5b_{2t}L_1^2 + 2L_{1t}L_{1u}).$$

Invoking that $(\varphi_{xx})_y \equiv 0$ and using the equation $2\varphi_x\varphi_{xx} - (\varphi_x^2)_x \equiv 0$ one obtains

$$L_{1tt} = 3L_{1t}(2L_{1t} - b_3L_1)/(5L_1) + 15b_4b_2L_1 - b_4I_2 + 5b_{4u}L_1 - 6b_{3t}L_1 - 54b_3^2L_1/5 + 25L_1^3/(3I_2),$$

where the relative invariant I_2 (25) becomes

$$I_2 = 3b_2L_1 - L_{1u}.$$

Eqs. (56) and $(\psi_y)_{xx} - (\psi_{xx})_y = 0$ are satisfied. Summing up the above results, we complete the proof of Theorem 2.

5. Proof of Theorem 3

We deal now with the case $L_2 \neq 0$, and hence $\varphi_y \neq 0$. Using Eqs. (28)–(31), one finds the derivatives ψ_{yy}, ψ_{xy}, ψ_{xx}, φ_{xx}:

$$\psi_{yy} = \varphi_y^{-1}(\varphi_{yy}\psi_y - \psi_y^3 b_1) - 3\psi_y^2 b_2 - 3\psi_y b_3\varphi_y - b_4\varphi_y^2,$$

$$2\psi_{xy} = (\varphi_y^2)^{-1}(2\varphi_{xy}\psi_y\varphi_y - \Delta\varphi_{yy} + b_1\psi_y^2(\varphi_x\psi_y - 3\psi_x\varphi_y)) - 3b_3(\varphi_x\psi_y + \psi_x\varphi_y) - 2\varphi_x b_4\varphi_y - 6\psi_x\psi_y b_2,$$

$$\psi_{xx} = \varphi_y^{-3}(-2\varphi_{xy}\varphi_x\psi_y\varphi_y + 2\varphi_{xy}\psi_x\varphi_y^2 + \varphi_{xx}\psi_y\varphi_y^2 + \Delta\varphi_x\varphi_{yy} - b_1\psi_y(\varphi_x^2\psi_y^2 - 3\varphi_x\psi_x\psi_y\varphi_y + 3\psi_x^2\varphi_y^2)) - \varphi_x^2 b_4 - 3\varphi_x\psi_x b_3 - 3\psi_x^2 b_2,$$

$$\varphi_{xx} = \varphi_x\varphi_y^{-2}(2\varphi_{xy}\varphi_y - \varphi_x\varphi_{yy} + \varphi_x\psi_y^2 b_1 - 2\psi_x\psi_y b_1\varphi_y) + \psi_x^2 b_1 + \varphi_y K y^{-3}.$$

Furthermore, the equations $(\psi_{xy})_x = (\psi_{xx})_y$ and $(\psi_{xy})_y = (\psi_{yy})_x$ determine the derivatives φ_{xyy} and φ_{yyy}, respectively,

$$2\varphi_{xyy} = \varphi_y^{-2}(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + \varphi_x\psi_y^4 b_1^2 - 4\psi_x\psi_y^3 b_1^2\varphi_y) + 2\varphi_x\psi_y^2(b_{1t} - 3b_3b_1) + 4\varphi_x\psi_y\varphi_y(2b_{2t} - b_{3u} - 2b_4b_1) + \varphi_x\varphi_y^2(6b_{3t} - 4b_{4u} - 12b_4b_2 + 9b_3^2) + 2\psi_x\psi_y^2(b_{1u} - 6b_2b_1) + 4\psi_x\psi_y\varphi_y(b_{1t} - 3b_3b_1) + 2\psi_x\varphi_y^2(2b_{2t} - b_{3u} - 2b_4b_1),$$

$$2\varphi_{yyy} = \varphi_y^{-1}(3\varphi_{yy}^2 - 3\psi_y^4 b_1^2) + 2\psi_y^3(-6b_2b_1 + b_{1u}) + 6\psi_y^2\varphi_y(-3b_3b_1 + b_{1t}) + 6_y\varphi_y^2(2b_{2t} - 2b_4b_1 - b_{3u}) + \varphi_y^3(6b_{3t} - 12b_4b_2 + 9b_3^2 - 4b_{4u}).$$

The equations $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xyy})_y = (\varphi_{xx})_{yy}$ yield:

$$y^5\Delta(L_2(\varphi_x\psi_y + \psi_x\varphi_y) + 2L_1\varphi_x\varphi_y) - 12\varphi_y K = 0. \tag{58}$$

Eq. (22) and the condition $\Delta \neq 0$ yield that $L_2\psi_y \neq 0$ and $L_1\varphi_x + L_2\psi_x \neq 0$. By virtue of Eq. (22), Eq. (58) becomes

$$y^5\psi_y(\varphi_x L_1 + \psi_x L_2)^2 - 12KL_1 = 0. \tag{59}$$

Differentiating (22) with respect to x and y, and substituting the derivatives ψ_{xx}, ψ_{xy}, ψ_{yy}, and φ_{xx}, one obtains

$$-\varphi_{yy}L_1^2(\varphi_x L_1 + \psi_x L_2) + \psi_y^2\varphi_x(2b_4L_2^3 - 3b_3L_1L_2^2 + b_1L_1^3 - 2L_{1t}L_2^2 + 2L_{2t}L_1L_2) + \psi_y^2\psi_x(3b_3L_2^3 - 6b_2L_1L_2^2 + 3b_1L_1^2L_2 - 2L_{1u}L_2^2 + 2L_{2u}L_1L_2) = 0, \tag{60}$$

$$-b_4L_2^3 + 3b_3L_1L_2^2 - 3b_2L_1^2L_2 + b_1L_1^3 + L_{1t}L_2^2 - L_{1u}L_1L_2 - L_{2t}L_1L_2 + L_{2u}L_1^2 = 0. \tag{61}$$

Eq. (60) yields:

$$\varphi_{yy} = L_1^{-2}(\varphi_x L_1 + \psi_x L_2)^{-1}(\varphi_x\psi_y^2(2b_4L_2^3 - 3b_3L_1L_2^2 + b_1L_1^3 - 2L_{1t}L_2^2 + 2L_{2t}L_1L_2) + \psi_x\psi_y^2L_2(3b_3L_2^2 - 6b_2L_1L_2 + 3b_1L_1^2 - 2L_{1u}L_2 + 2L_{2u}L_1)).$$

Furthermore, Eq. (61) determines L_{2u}:

$$L_{2u} = L_1^{-2}(b_4L_2^3 - 3b_3L_1L_2^2 + 3b_2L_1^2L_2 - b_1L_1^3 - L_{1t}L_2^2 + L_{1u}L_1L_2 + L_{2t}L_1L_2). \tag{62}$$

Using Eq. (59), one obtains the equation $(\varphi_{xy})_x - (\varphi_{xx})_y = 0$. Notice that the equations $(\varphi_y)_{yy} - \varphi_{yyy} = 0$ and $(\varphi_y)_{xy} - \varphi_{xyy} = 0$ are also satisfied. Now we find φ_x^2 from Eq. (59) and substitute it into the equation $\varphi_{xyy} - (\varphi_{yy})_x = 0$. This leads to the following expression for L_{2tt}:

$$L_{2tt} = L_1^{-2}(-b_4^2L_2^3 + 3b_4b_3L_1L_2^2 - 3b_4b_2L_1^2L_2 + b_4b_1L_1^3 + 3b_4L_{1t}L_2^2 - 3b_4L_{2t}L_1L_2 - 3b_3L_{1t}L_1L_2 + 3b_3L_{2t}L_1^2 - b_{4t}L_1L_2^2 - b_{4u}L_1^2L_2 + 3b_{3t}L_1^2L_2 + b_{3u}L_1^3 - 2b_{2t}L_1^3 + L_{1tt}L_1L_2 - 2L_{1t}^2L_2 + 2L_{1t}L_{2t}L_1). \tag{63}$$

Differentiating (59) with respect to x and y, one finds φ_{xy} and ψ_y, respectively,

$$\psi_y = 5L_1(I_2y)^{-1},$$
$$\varphi_{xy} = 12K(5L_1^2y^5(\varphi_xL_1 + \psi_xL_2))^{-1}[6L_{1t}L_2 - 6b_4L_2^2 + 6b_3L_1L_2 - 5L_{2t}L_1] + \psi_x(I_2L_1^2y)^{-1}[10b_4L_2^3 - 15b_3L_1L_2^2 + 5b_1L_1^3 10L_{1t}L_2^2 + 10L_{2t}L_1L_2 - I_2L_1L_2].$$

The equations $(\psi_y)_y = \psi_{yy}$ and $(\varphi_x^2)_x = 2\varphi_x\varphi_{xx}$ yield:

$$5L_1^2I_{2u} = 6I_2(4b_4L_2^2 - 9b_3L_1L_2 + 5b_2L_1^2 - 4L_{1t}L_2 + 5L_{2t}L_1 - I_2L_1), \quad (64)$$
$$5L_1I_{2t} = 6I_2(L_{1t} - b_4L_2 + b_3L_1). \quad (65)$$

Now the equations $(\psi_y)_x = \psi_{xy}$ and $(\varphi_{xy})_y = (\varphi_{yy})_x$ are satisfied.

The equation $(\varphi_{xy})_x = (\varphi_{xx})_y$ yields:

$$L_{1tt} = (63b_4^2I_2L_2^2 - 126b_4b_3I_2L_1L_2 + 225b_4b_2I_2L_1^2 - 81b_4L_{1t}I_2L_2 + 90b_4L_{2t}I_2L_1 - 15b_4I_2^2L_1 - 162b_3^2I_2L_1^2 - 9b_3L_{1t}I_2L_1 + 15b_{4t}I_2L_1L_2 + 75b_{4u}I_2L_1^2 - 90b_{3t}I_2L_1^2 + 18L_{1t}^2I_2 + 125L_1^4)/(15I_2L_1). \quad (66)$$

Notice that the equations $(\psi_y)_x = \psi_{xy}$ and $(\psi_y)_y = \psi_{yy}$ are satisfied.

Summing up the above results, we complete the proof of Theorem 3.

Acknowledgement

One of the authors (SVM) is thankful to R. Conte for fruitful discussions.

References

[1] Lie S. Klassifikation und Integration von gewönlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestaten. III, Archiv for Matematik og Naturvidenskab, 8, Heft 4:371–458, 1883. Reprinted in Lie's Ges Abhandl, vol. 5, 1924, paper XIV, p. 362–427.
[2] Ibragimov NH, editorCRC Handbook of Lie group analysis of differential equations. Symmetries, exact solutions and conservation laws, vol. 1. Boca Roton: CRC Press Inc.; 1994.
[3] Ibragimov NH. Elementary Lie group analysis and ordinary differential equations. Chichester: John Wiley & Sons; 1999.
[4] Ibragimov NH. Invariants of a remarkable family of nonlinear equations. Nonlinear Dyn 2002;30(2):155–66.
[5] Liouville R. Sur les invariants de certaines equations differentielles et sur leurs applications. J Ecole Polytech 1889;59:7–76.
[6] Tresse, AM. Détermination des Invariants Ponctuels de lÉquation Différentielle Ordinaire du Second Ordre $y'' = \omega(x,y,y')$, Preisschriften der fürstlichen Jablonowski'schen Geselschaft XXXII, Leipzig, S. Herzel, 1896.
[7] Babich MV, Bordag LA. Projective differential geometrical structure of the Painleve equations. J Differen Equat 1999;157:452–85.
[8] Ibragimov NH, Meleshko SV. Linearization of third-order ordinary differential equations by point transformations. Arch ALGA 2004;1:71–93.
[9] Ibragimov NH, Meleshko SV. Linearization of third-order ordinary differential equations by point and contact transformations. J Math Anal Appl 2005;308(1):266–89.
[10] Hearn AC. REDUCE Users Manual, ver. 3.3, The Rand Corporation CP 78, Santa Monica, 1987.

5 Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. II. By Nail H. Ibragimov, Sergey V. Meleshko (2005)

Available online at www.sciencedirect.com
ScienceDirect
Communications in Nonlinear Science and Numerical Simulation
Communications in Nonlinear Science and Numerical Simulation xxx (2006) xxx–xxx
www.elsevier.com/locate/cnsns

Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. II

Nail H. Ibragimov [a], Sergey V. Meleshko [b,*]

[a] *Department of Mathematics and Science Research Centre ALGA: Advances in Lie Group Analysis, Blekinge Institute of Technology, SE-371 79 Karlskrona, Sweden*
[b] *School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima 3000, Thailand*

Received 18 February 2006; accepted 18 March 2006

Abstract

The second-order ordinary differential equations can have one, two, three or eight independent symmetries. Sophus Lie showed that the equations with eight symmetries and only these equations can be linearized by a change of variables. Moreover he demonstrated that these equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. We provide a similar description of the equations with three symmetries. There are four different types of such equations. Classes of equations equivalent to one of these equations were studied in [Ibragimov NH, Meleshko SV. Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. Communication in Nonlinear Science and Numerical Simulation, in press], where we presented the candidates for all four types and studied one of these candidates. The present paper is the continuation of the work of Ibragimov and Meleshko and is devoted to other three candidates.

PACS: 02.30.Jr; 02.20.Tw

Keywords: Equations with three symmetries; Candidates; Equivalence test

1. Introduction

According to Lie's classification [2] in the complex domain, any ordinary differential equation of the second order

$$y'' = f(x, y, y'), \tag{1}$$

admitting a three-dimensional Lie algebra belongs to one of four distinctly different types. Each of these four types is obtained by a change of variables from the following canonical representatives (see, e.g. [3], Section 8.4):

* Corresponding author. Tel.: +66 44 224382; fax: +66 44 224185.
E-mail address: sergey@math.sut.ac.th (S.V. Meleshko).

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2006.03.011

Please cite this article in press as: Ibragimov NH, Meleshko SV, Invariants and invariant description of second-order ODEs ..., Commun Nonlinear Sci Numer Simul (2006), doi:10.1016/j.cnsns.2006.03.011

$$y'' + Cy^{-3} = 0, \tag{2}$$
$$y'' + e^{y'} = 0, \tag{3}$$
$$y'' + y'^{(k-2)/(k-1)} = 0, \tag{4}$$
$$y'' + 2\frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0, \tag{5}$$

where k and C are constants such that $k \neq 0, 1/2, 1, 2$ and $C \neq 0$.

Eqs. (2)–(5) admit non-similar three-dimensional Lie algebras L_3 spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{6}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (y - x)\frac{\partial}{\partial y}, \tag{7}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \tag{8}$$

and

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \tag{9}$$

respectively (see, e.g. [3], Section 8.4).

2. Candidates for equations with three symmetries

Let us subject each of Eqs. (2)–(5) to the arbitrary change of variables

$$t = \varphi(x, y), \qquad u = \psi(x, y), \tag{10}$$

where t is a new independent variable and u is a new dependent variable. Then we obtain from (2)–(5) the equations of the form

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0, \tag{11}$$
$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8)\exp\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0, \tag{12}$$
$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8)\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right)^{(k-2)/(k-1)} = 0, \tag{13}$$

and

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8)C\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right)^{3/2} = 0, \tag{14}$$

respectively, where $b_i = b_i(t, u)$, $i = 1, \ldots, 12$. Eqs. (11)–(14) are the candidates for the equations with three symmetries.

All candidates can be encapsulated in the formula

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8)f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0.$$

Namely, Eqs. (11)–(14) are obtained by letting

$$f(z) = 0, \qquad f(z) = e^z, \qquad f(z) = z^{(k-2)/(k-1)}, \qquad f(z) = z^{3/2}. \tag{15}$$

Using the usual formula for the transformation of derivatives under the change of variables (10), we obtain the following statement.

Theorem. *Any equation of the form*

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0 \tag{16}$$

is transformed by the change of variables (10) *into an equation of the same form*:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 + (a_5 y'^3 + 3a_6 y'^2 + 3a_7 y' + a_8) f\left(\frac{a_9 y' + a_{10}}{a_{11} y' + a_{12}}\right) = 0. \tag{17}$$

Here a_i and b_i are functions of x,y and t,u, respectively, and are connected by:

$$\begin{aligned}
a_1 &= \Delta^{-1}[\varphi_y \psi_{yy} - \varphi_{yy}\psi_y + b_4 \varphi_y^3 + 3b_3 \varphi_y^2 \psi_y + 3b_2 \varphi_y \psi_y^2 + b_1 \psi_y^3],\\
a_2 &= \Delta^{-1}[b_4 \varphi_x \varphi_y^2 + b_3 \varphi_y (2\varphi_x \psi_y + \varphi_y \psi_x) + b_2 \psi_y (\varphi_x \psi_y + 2\varphi_y \psi_x)\\
&\quad + b_1 \psi_x \psi_y^2 + (\varphi_x \psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y \psi_{xy})/3],\\
a_3 &= \Delta^{-1}[b_4 \varphi_x^2 \varphi_y + b_3 \varphi_x (\varphi_x \psi_y + 2\varphi_y \psi_x) + b_2 \psi_x (2\varphi_x \psi_y + \varphi_y \psi_x)\\
&\quad + b_1 \psi_x^2 \psi_y + (\varphi_y \psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x \psi_{xy})/3],\\
a_4 &= \Delta^{-1}[b_4 \varphi_x^3 + 3b_3 \varphi_x^2 \psi_x + 3b_2 \varphi_x \psi_x^2 + b_1 \psi_x^3 - \varphi_{xx}\psi_x + \varphi_x \psi_{xx}],
\end{aligned} \tag{18}$$

$$\begin{aligned}
a_5 &= \Delta^{-1}[b_8 \varphi_y^3 + 3b_7 \varphi_y^2 \psi_y + 3b_6 \varphi_y \psi_y^2 + b_5 \psi_y^3],\\
a_6 &= \Delta^{-1}[b_8 \varphi_x \varphi_y^2 + b_7 \varphi_y (2\varphi_x \psi_y + \varphi_y \psi_x) + b_6 \psi_y (\varphi_x \psi_y + 2\varphi_y \psi_x) + b_5 \psi_x \psi_y^2],\\
a_7 &= \Delta^{-1}[b_8 \varphi_x^2 \varphi_y + b_7 \varphi_x (\varphi_x \psi_y + 2\varphi_y \psi_x) + b_6 \psi_x (2\varphi_x \psi_y + \varphi_y \psi_x) + b_5 \psi_x^2 \psi_y],\\
a_8 &= \Delta^{-1}[b_8 \varphi_x^3 + 3b_7 \varphi_x^2 \psi_x + 3b_6 \varphi_x \psi_x^2 + b_5 \psi_x^3],
\end{aligned} \tag{19}$$

$$a_9 = b_{10}\varphi_y + b_9 \psi_y,\ \ a_{10} = b_{10}\varphi_x + b_9 \psi_x,\ \ a_{11} = b_{12}\varphi_y + b_{11}\psi_y,\ \ a_{12} = b_{12}\varphi_x + b_{11}\psi_x, \tag{20}$$

where

$$\Delta = (\varphi_x \psi_y - \varphi_y \psi_x) \neq 0$$

is the Jacobian of the change of variables (10).

It follows from Eqs. (20) that

$$a_9 a_{12} - a_{10} a_{11} = \Delta (b_9 b_{12} - b_{10} b_{11}).$$

Hence the equation

$$a_9 a_{12} - a_{10} a_{11} = 0 \tag{21}$$

is invariant under the change of variables (10). If $a_9 a_{12} - a_{10} a_{11} = 0$, and hence $b_9 b_{12} - b_{10} b_{11} = 0$, the function f disappears in both Eqs. (16) and (17). This leads to the equations equivalent to Eq. (2), i.e. to the case considered in [1]. Therefore, we assume in what follows that

$$b_9 b_{12} - b_{10} b_{11} \neq 0, \qquad a_9 a_{12} - a_{10} a_{11} \neq 0.$$

3. Equations equivalent to Eqs. (3) and (4)

The test for equivalence to both Eqs. (3) and (4) have the same form. The only difference is that Eqs. (3) and (4) have the candidates (12) and (13), respectively, with the different functions f.

Eqs. (3) and (4) have the form (17) with

$$\begin{aligned}
&a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 0,\\
&a_7 = 0, \quad a_8 - 1 = 0, \quad a_9 - a_{12} = 0, \quad a_{10} = 0, \quad a_{11} = 0,
\end{aligned} \tag{22}$$

whereas the function f has the form $f(z) = e^z$ for (3) and $f(z) = z^{(k-2)/(k-1)}$ for (4). Furthermore, the change of variables (10) leaves invariant each candidate. Hence, the equations which are equivalent to (3) and (4) belong to equations of the form (16):

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0.$$

Thus for the functions (10) $\varphi(x,y)$ and $\psi(x,y)$ one obtains the overdetermined system of equations which consists of Eqs. (22), where the coefficients a_i, $(i = 1, 2, \ldots, 12)$ are defined by the relations (18)–(20).

Analysis of compatibility of the overdetermined system depends on the value of b_{12}. If the argument of the function f in (16) is a linear function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 0$ and $b_{12} = 1$. If the argument of the function f in (16) is a rational function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 1$.

Let us consider the first case

$$b_{11} = 1, \qquad b_{12} = 0.$$

In this case the result of compatibility analysis gives that $b_5 b_{10} \neq 0$ and

$$\begin{aligned}
&b_4 = 0, \quad b_6 = 0, \quad b_7 = 0, \quad b_8 = 0,\\
&b_{5t} - 3b_5 b_3 = 0, \quad b_{10} b_{5u} - 3b_5(2b_{10}b_2 + b_{9t}) = 0, \quad b_{9u} + b_{10}b_1 = 0,\\
&b_{10t} + 3b_{10}b_3 = 0, \quad b_{10u} + 3b_{10}b_2 + b_{9t} = 0.
\end{aligned}$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\varphi_x = \frac{b_9}{b_{10}^2 b_5}, \qquad \varphi_y = -\frac{1}{b_{10}^2 b_5}, \qquad \psi_x = -\frac{1}{b_{10}b_5}, \qquad \psi_y = 0.$$

The generators corresponding to (7) are

$$\begin{aligned}
&X_1 = (b_{10}^2 b_5)^{-1}\left[b_9 \frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right], \qquad X_2 = (b_{10}^2 b_5)^{-1}\frac{\partial}{\partial t},\\
&X_3 = (b_{10}^2 b_5)^{-1}\left[((b_9 + 1)x - y)\frac{\partial}{\partial t} - b_{10}x\frac{\partial}{\partial u}\right].
\end{aligned}$$

The generators corresponding to (8) are

$$\begin{aligned}
&X_1 = (b_{10}^2 b_5)^{-1}\left[b_9 \frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right], \qquad X_2 = (b_{10}^2 b_5)^{-1}\frac{\partial}{\partial t},\\
&X_3 = (b_{10}^2 b_5)^{-1}\left[(b_9 x - ky)\frac{\partial}{\partial t} - b_{10}x\frac{\partial}{\partial u}\right].
\end{aligned}$$

In the second case $b_{12} = 1$ one obtains that $b_8 \neq 0$ and

$$\begin{aligned}
&b_{11t}b_{11} - b_{11u} + b_{11}^3 b_4 - 3b_{11}^2 b_3 + 3b_{11}b_2 - b_1 = 0,\\
&3b_2(b_{11}b_{10} - b_9)b_8 + (b_{11}b_{10} - b_9)b_{8u} + 3b_{11}b_8 b_{10u} = 0,\\
&6b_3(b_{11}b_{10} - b_9)b_8 + (b_{11}b_{10} - b_9)b_{8t} + 3b_{11}b_8 b_{10t} + 3b_8 b_{10u} = 0,\\
&b_4(b_{11}b_{10} - b_9) + b_{10t} = 0, \quad b_5 = b_{11}^3 b_8, \quad b_6 = b_{11}^2 b_8, \quad b_7 = b_{11}b_8,\\
&b_8(2b_{11t}b_{10} + b_{10t}b_{11} + b_{10u} - 2b_{9t}) + b_{8t}(b_{11}b_{10} - b_9) = 0,\\
&(b_{11}b_{10} - b_9)(2b_{8u} - 2b_{11t}b_8 - b_{8t}b_{11}) + b_8(2b_{11u}b_{10} - b_{10t}b_{11}^2 + 3b_{10u}b_{11} - 2b_{9u}) = 0.
\end{aligned}$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\begin{aligned}
&\varphi_x = \frac{b_9}{b_8(b_{11}b_{10} - b_9)^2}, \qquad \varphi_y = -\frac{b_{11}}{b_8(b_{11}b_{10} - b_9)^2},\\
&\psi_x = -\frac{b_{10}}{b_8(b_{11}b_{10} - b_9)^2}, \qquad \psi_y = \frac{1}{b_8(b_{11}b_{10} - b_9)^2}.
\end{aligned}$$

Please cite this article in press as: Ibragimov NH, Meleshko SV, Invariants and invariant description of second-order ODEs ..., Commun Nonlinear Sci Numer Simul (2006), doi:10.1016/j.cnsns.2006.03.011

The generators corresponding to (7) are

$$X_1 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_9\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right],$$
$$X_2 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_{11}\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right],$$
$$X_3 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[(b_{11}(x-y) + b_9 x)\frac{\partial}{\partial t} + (y - (1+b_{10})x)\frac{\partial}{\partial u}\right].$$

The generators corresponding to (8) are

$$X_1 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_9\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right],$$
$$X_2 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_{11}\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right],$$
$$X_3 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[(b_9 x - b_{11}ky)\frac{\partial}{\partial t} + (ky - b_{10}x)\frac{\partial}{\partial u}\right].$$

4. Equations equivalent to Eq. (5)

Eq. (5) has the form (17) with

$$\begin{aligned}
&a_1 = 0, \quad 3a_2 - 2/(x-y) = 0, \quad 3a_3 - 2/(x-y) = 0,\\
&a_4 = 0, \quad a_5 = 0, \quad a_6 = 0, \quad a_7 = 0, \quad a_8 - C(x-y)^{-1} = 0,\\
&a_9 - a_{12} = 0, \quad a_{10} = 0, \quad a_{11} = 0,
\end{aligned} \tag{23}$$

and f has the form $f(z) = Cz^{3/2}$. The equations that are equivalent to (5) belong to equations of the form (16):

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + \left(b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8\right) f\left(\frac{b_9 u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0.$$

Thus for the functions (10) $\varphi(x,y)$ and $\psi(x,y)$ one obtains the overdetermined system of equations which consists of Eqs. (23), where the coefficients a_i $(i = 1,2,\ldots,12)$ are defined by the relations (18)–(20).

Analysis of compatibility of the overdetermined system depends on the value of b_{12}. If the argument of the function f in (16) is a linear function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 0$ and $b_{12} = 1$. If the argument of the function f in (16) is a rational function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 1$.

Let us consider the first case

$$b_{11} = 1, \quad b_{12} = 0.$$

In this case the result of compatibility analysis gives that $b_5 b_{10} \neq 0$ and

$$\begin{aligned}
&b_4 = 0, \quad b_6 = 0, \quad b_7 = 0, \quad b_8 = 0,\\
&b_{10t}C - b_{10}(2b_{10}^2 b_5 - 3b_3 C) = 0,\\
&4b_{10}^2 b_9 b_5 + 2b_{10}^2 b_5 - 3b_{10}b_2 C - b_{9t}C - b_{10u}C = 0,\\
&3b_5(-b_{10}^2 b_5 + b_3 C) - b_{5t}C = 0,\\
&3b_5(-3b_{10}^2 b_9 b_5 - b_{10}^2 b_5 + 2b_{10}b_2 C + b_{9t}C) - b_{5u}b_{10}C = 0,\\
&b_{10}(2b_9^2 b_5 + 2b_9 b_5 - b_1 C) - b_{9u}C = 0.
\end{aligned}$$

The functions $\varphi(x, y)$ and $\psi(x, y)$ are found from the compatible system of equations

$$\varphi_x = \frac{b_9 C}{b_{10}^2 b_5 (x-y)}, \qquad \varphi_y = -\frac{C}{b_{10}^2 b_5 (x-y)}, \qquad \psi_x = -\frac{C}{b_{10} b_5 (x-y)}, \qquad \psi_y = 0.$$

The generators corresponding to (9) are

$$X_1 = (b_{10}^2 b_5 (x-y))^{-1}\left[(b_9 - 1)\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right],$$
$$X_2 = (b_{10}^2 b_5 (x-y))^{-1}\left[(b_9 x - y)\frac{\partial}{\partial t} - b_{10} x\frac{\partial}{\partial u}\right],$$
$$X_3 = (b_{10}^2 b_5 (x-y))^{-1}\left[(b_9 x^2 - y^2)\frac{\partial}{\partial t} - b_{10} x^2\frac{\partial}{\partial u}\right].$$

In the second case $b_{12} = 1$ one obtains that $b_8 \neq 0$ and

$$b_5 = b_{11}^3 b_8, \qquad b_6 = b_{11}^2 b_8, \qquad b_7 = b_{11} b_8,$$
$$-b_{10t} C + (b_{11} b_{10} - b_9)(2 b_{10} b_8 (b_{10} + 1) - b_4 C) = 0,$$
$$b_{11t} b_{11} - b_{11u} + b_{11}^3 b_4 - 3 b_{11}^2 b_3 + 3 b_{11} b_2 - b_1 = 0,$$
$$-3 b_{10u} b_{11} b_8 C - b_{8u} C (b_{11} b_{10} - b_9) + 3 b_8^2 (b_{11} + b_9)(b_{11}^2 b_{10}^2 - b_9^2) - 3(b_{11} b_{10} - b_9) b_8 b_2 C = 0,$$
$$-3 b_{10t} b_{11} b_8 C - 3 b_{10u} b_8 C + b_{8t} C(-b_{11} b_{10} + b_9) + 3 b_8^2 (b_{11}^2 b_{10}^3 + 3 b_{11}^2 b_{10}^2 + 2 b_{11} b_{10}^2 b_9 - 2 b_{11} b_{10} b_9$$
$$- 3 b_{10} b_9^2 - b_9^2) - 6(b_{11} b_{10} - b_9) b_3 b_8 C = 0,$$
$$2 b_{11t} b_{10} b_8 C + b_{10t} b_{11} b_8 C + b_{10u} b_8 C - 2 b_{9t} b_8 C + b_{8t} C (b_{11} b_{10} - b_9)$$
$$+ b_8^2 (b_{11}^2 b_{10}^3 - b_{11}^2 b_{10}^2 - 2 b_{11} b_{10}^2 b_9 + 2 b_{11} b_{10} b_9 + b_{10} b_9^2 - b_9^2) = 0,$$
$$2 b_{11t} b_8 C(-b_{11} b_{10} + b_9) + 2 b_{11u} b_{10} b_8 C - b_{10t} b_{11}^2 b_8 C + 3 b_{10u} b_{11} b_8 C + b_{8t} b_{11} C(-b_{11} b_{10} + b_9)$$
$$+ 2 b_{8u} C (b_{11} b_{10} - b_9) + b_8^2 (-b_{11}^3 b_{10}^3 - b_{11}^3 b_{10}^2 + 4 b_{11}^2 b_{10}^2 b_9 + 2 b_{11}^2 b_{10} b_9 - 5 b_{11} b_{10} b_9^2$$
$$- b_{11} b_9^2 + 2 b_9^3) - 2 b_{9u} b_8 C = 0$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\varphi_x = \frac{b_9 C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}, \qquad \varphi_y = -\frac{b_{11} C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8},$$
$$\psi_x = -\frac{b_{10} C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}, \qquad \psi_y = \frac{C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}.$$

The generators corresponding to (9) are

$$X_1 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8\right)^{-1}\left[(b_9 - b_{11})\frac{\partial}{\partial t} + (1 - b_{10})\frac{\partial}{\partial u}\right],$$
$$X_2 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8\right)^{-1}\left[(b_9 x - b_{11} y)\frac{\partial}{\partial t} + (y - b_{10} x)\frac{\partial}{\partial u}\right],$$
$$X_3 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8)\right)^{-1}\left[(b_9 x^2 - b_{11} y^2)\frac{\partial}{\partial t} + (y^2 - b_{10} x^2)\frac{\partial}{\partial u}\right].$$

References

[1] Ibragimov NH, Meleshko SV. Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I. Communication in Nonlinear Science and Numerical Simulation, in press, doi:10.1016/j.cnsns.2005.12.012.

[2] Lie S. Klassifikation und Integration von gewönlichen Differentialgleichungen zwischen x,y, die eine Gruppe von Transformationen gestaten. III. Archiv for Matematik og Naturvidenskab., 8, Heft 1883;4:371–458. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XIV, p. 362–427.

[3] Ibragimov NH, editorCRC Handbook of Lie group analysis of differential equations. Symmetries, exact solutions and conservation laws, vol. 1. Boca Roton: CRC Press Inc.; 1994.

6 On first integrals of second-order ordinary differential equations. By S. V. Meleshko, S. Moyo, C. Muriel, J. L. Romero, P. Guha, A. G. Choudhury (2013)

On first integrals of second-order ordinary differential equations

S. V. Meleshko · S. Moyo · C. Muriel ·
J. L. Romero · P. Guha · A. G. Choudhury

Abstract Here we discuss first integrals of a particular representation associated with second-order ordinary differential equations. The linearization problem is a particular case of the equivalence problem together with a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation. The relationship between the integral form, the associated equations, equivalence transformations, and some examples are considered as part of the discussion illustrating some important aspects and properties.

Keywords Equivalence transformations · Linearization · ODEs · λ-Symmetry

To Professor Peter Leach on his seventieth birthday.

S. V. Meleshko
School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand
e-mail: sergey@math.sut.ac.th

S. Moyo (✉)
Department of Mathematics, Statistics and Physics, Institute for Systems Science, Durban University of Technology,
P O Box 1334, Steve Biko Campus, Durban 4000, South Africa
e-mail: moyos@dut.ac.za

C. Muriel · J. L. Romero
Departamento de Matemáticas, Universitario de Puerto Real, 11510 Puerto Real, Cádiz, Spain
e-mail: concepcion.muriel@uca.es

P. Guha
S.N. Bose National Centre for Basic Sciences, J D Block, Sector III, Salt Lake, Calcutta 700098, West Bengal, India
e-mail: partha@bose.res.in

A. G. Choudhury
Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta 700009, West Bengal, India
e-mail: ghosechoudhury@gmail.com

1 Introduction

Many methods of solving differential equations use a change of variables that transform a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, it is attractive to transform a given differential equation into a linear equation. This problem, which is called a linearization problem, is a particular case of the equivalence problem. The equivalence problem can be formulated as follows. Let a set of invertible transformations be given. One can introduce the equivalence property according to these transformations: two differential equations are equivalent if there is a transformation of the given set that transforms one equation into another. The equivalence problem involves a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation.

1.1 Introduction to the problem

We give a short review of results related to an equivalence problem for a second-order ordinary differential equation (ODE). Furthermore, we distinguish two types of transformations used in the equivalence problem for second-order ODEs, namely point transformations and generalized Sundman transformations. Lie [1] also noted that all second-order ODEs can be transformed into each other by means of contact transformations and that this is not so for third-order equations. Thus this set of transformations cannot be applied to a classification of second-order ODEs.

Among the target equations, two classes of equations can be mentioned. One set of this class was obtained by Lie [2]. Lie's group classification of ODEs shows that the second-order equations can possess one, two, three, or eight infinitesimal symmetries. The equations with eight symmetries can be linearized by a change of variables. Lie showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. A similar description of the equations with three symmetries was provided in [3,4]. Another set of target classes corresponds to the Painlevé equations. Analysis of the classes of equations corresponding to the first and second Painlevé equations was performed in [5,6].

For the linearization problem one studies those classes of equations that are equivalent to linear equations. The first linearization problem for ODEs was solved by Lie [1]. He found the general form of all ODEs of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. Tresse [7] treated the equivalence problem for second-order ODEs in terms of relative invariants of the equivalence group of point transformations. In [8] an infinitesimal technique for obtaining relative invariants was applied to the linearization problem.

A different approach to tackling the equivalence problem of second-order ODEs was developed by Cartan [9]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form. The Cartan approach was further applied by Chern [10] to third-order differential equations. Since none of the conditions given in [10] is an implicit expression that could be used as a test for determining the type of the studied equation, in a series of articles [11–15] the linearization problem was also considered. Linearization with respect to point transformations is studied in [11], with respect to contact transformations in [12–16]. The linearization problem was also investigated with respect to the generalized Sundman transformations [17–19].

The linearization problem via point transformations

$$\tau = \varphi(t,\ x), \quad u = \psi(t,\ x)$$

for a second-order equation $\ddot{x} = F(t,\ x,\ \dot{x})$ is attractive because of the simplicity of the general solution of a linear equation: a linearizable second-order ODE is equivalent to the free particle equation $u'' = 0$. Thus, if one found the linearizing transformation, then the general solution of the original equation could be found easily. Note that for a linearizable equation $\ddot{x} = F(t,\ x,\ \dot{x})$ the expression

$$u' = \frac{\dot{x}\psi_x + \psi_t}{\dot{x}\varphi_x + \varphi_t}$$

is a first integral of the equation. Here subscripts mean derivatives, for example, $\varphi_t = \partial\varphi/\partial t,\ \varphi_x = \partial\varphi/\partial x$ and so on. This motivated the authors of [20–22] to study equations possessing a first integral of the form

$$I = \frac{\dot{x}\tilde{A}(t,\ x) + \tilde{C}(t,\ x)}{\dot{x}\tilde{B}(t,\ x) + \tilde{Q}(t,\ x)}. \tag{1}$$

Notice that a second-order equation equivalent to the free particle equation via the generalized Sundman transformation also possesses a first integral of the form (1).

The authors of [20–22] came to the form of first integral (1) from the study of λ-symmetries for second-order equations that play a fundamental role in the study of λ-symmetries. Although the equation may lack Lie point symmetries, there always exists a λ-symmetry associated to a first integral $I = I(t,\ x,\ \dot{x})$. Such a λ-symmetry can be defined in canonical form by the vector field $\mathbf{v} = \partial_x$ and the function $\lambda = -I_x/I_{\dot{x}}$. When I is of the form

$$I = C(t,\ x) + \frac{1}{A(t,\ x)\dot{x} + B(t,\ x)}, \quad (A \neq 0), \tag{2}$$

such a function λ is given by

$$\lambda(t,\ x,\ \dot{x}) = \gamma(t,\ x)\dot{x}^2 + \alpha(t,\ x)\dot{x} + \beta(t,\ x), \tag{3}$$

where

$$\gamma = AC_x = -a_3, \tag{4a}$$

$$\alpha = 2BC_x - A_x/A = -a_2 - AC_t, \tag{4b}$$

$$\beta = \left(C_x B^2 - B_x\right)/A = -a_1 + A_t/A - 2BC_t. \tag{4c}$$

In this way the study of ODEs that admit first integrals of the form (2) can be seen as a problem of classification of ODEs that admit $\mathbf{v} = \partial_x$ as a λ-symmetry for some function λ of the form (3).

The case where $C_x = 0$,

$$I = C(t) + \frac{1}{\dot{x}A(t,\ x) + C(t,\ x)},$$

was studied in [23]. It must be mentioned here that the case where $\tilde{B} = 0$ in (1) was thoroughly examined in [22].

We denote by $\mathcal{B}$ the class of equations corresponding to the particular case where $\gamma = 0$ in (3). The equations in $\mathcal{B}$ are ODEs of the form

$$\ddot{x} + a_2(t,\ x)\dot{x}^2 + a_1(t,\ x)\dot{x} + a_0(t,\ x) = 0 \tag{5}$$

that admit first integrals of the form (2) with $C_x = 0$.

A significant subclass of ODEs in $\mathcal{B}$, denoted by $\mathcal{A}$, is constituted by the equations that admit first integrals of the form $A(t,\ x)\dot{x} + B(t,\ x)$ [that is, $C = 0$ in (2)]. By (4b), the equations in $\mathcal{A}$ are the equations of the form (5) that admit $\mathbf{v} = \partial_x$ as λ-symmetry for some function $\lambda = -a_2\dot{x} + \beta$. According to the results in [22], the coefficients of the equations in $\mathcal{A}$ must satisfy either $S_1 = S_2 = 0$, where

$$S_1(t,\ x) = a_{1x} - 2a_{2t}, \quad S_2(t,\ x) = (a_0 a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})\, a_1, \tag{6}$$

or, if $S_1 \neq 0$, $S_3 = S_4 = 0$, where

$$S_3(t,\ x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x})\,, \quad S_4(t,\ x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0 a_2 + a_{0x}. \tag{7}$$

The equations in $\mathcal{A}$ such that $S_1 = S_2 = 0$ constitute the subclass $\mathcal{A}_1$, and they admit two functionally independent first integrals of the form $A(t,\ x)\dot{x} + B(t,\ x)$.

Several properties of the linearization through local and nonlocal transformations of the equations in $\mathcal{B}$ are derived in [23,24]. All the equations in $\mathcal{A}_1$ pass the Lie test of linearization (i.e., their coefficients satisfy $L_1 = L_2 = 0$). In contrast, none of the equations in $\mathcal{A}_2$ can be linearized through a local transformation; actually, there exist equations in $\mathcal{A}_2$ that lack Lie point symmetries [see, for example, Eqs. (2.6) and (4.12) in [22]].

Although there exists Eq. (5) whose coefficients satisfy $L_1 = L_2 = 0$, which are not in $\mathcal{A}_1$ (see Example 9 in [24]), they must all belong to $\mathcal{B}$. It is important to remark that there are equations in $\mathcal{B}$ not linearizable through local transformations, apart from the subclass $\mathcal{A}_2$ (as the family appearing in Example 2.1 in [25]). To linearize such types of equations, one must consider nonlocal transformations of the form

$$X = F(t,\ x), \quad \mathrm{d}T = (G_1(t,\ x)\dot{x} + G_2(t,\ x))\,\mathrm{d}t. \tag{8}$$

The equations in $\mathcal{B}$ can be characterized as the unique ODEs (5) that can be linearized through some nonlocal transformation of the form (8). When $G_1(t,\ x) = 0$ in (8), the equation must belong to $\mathcal{A}_2$ and vice versa. In other words, the equations in $\mathcal{A}_2$ are the unique ODEs (5) that can be transformed into the linear equation $X_{TT} = 0$ by means of some nonlocal transformation of the form

$$X = F(t,\ x), \quad \mathrm{d}T = G(t,\ x)\mathrm{d}t. \tag{9}$$

These transformations are known in the literature as *generalized Sundman transformations* (see [17,18,26–30] and references therein). Constructive methods to determine nonlocal linearizing transformations can be derived from the algorithms that calculate the first integrals [23,24]. In particular, local changes of variables that linearize the equations in $\mathcal{A}_1$ can be determined by just dealing with first-order ODEs. We remark that such linearizing point transformations usually appear in the literature as solutions of an involutive system of second-order partial differential equations [31,32].

1.2 Invariants of a class of second-order equations

We recall some known properties of a second-order equation:

$$\ddot{x} + a_3(t,\ x)\dot{x}^3 + 3a_2(t,\ x)\dot{x}^2 + 3a_1(t,\ x)\dot{x} + a_0(t,\ x) = 0. \tag{10}$$

This form of equation is conserved with respect to any change of the independent and dependent variables:

$$\tau = \varphi(t,\ x), \quad u = \psi(t,\ x). \tag{11}$$

In fact, derivatives are changed by the formulae

$$\begin{aligned} u' &= g(t,\ x,\ \dot{x}) = \frac{D_t\psi}{D_t\varphi} = \frac{\psi_t + \dot{x}\psi_x}{\varphi_t + \dot{x}\varphi_x}, \\ u'' &= P(t,\ x,\ \dot{x},\ \ddot{x}) = \frac{D_t g}{D_t\varphi} = \frac{g_t + \dot{x}g_x + \ddot{x}g_{\dot{x}}}{\varphi_t + \dot{x}\varphi_x} \\ &= (\varphi_t + \dot{x}\varphi_x)^{-3}\Big(\ddot{x}\,(\varphi_t\psi_x - \varphi_x\psi_t) + \dot{x}^3\,(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) \\ &\quad + \dot{x}^2\,(\varphi_t\psi_{xx} - \varphi_{xx}\psi_t + 2\,(\varphi_x\psi_{tx} - \varphi_{tx}\psi_x)) \\ &\quad + \dot{x}\,(\varphi_x\psi_{tt} - \varphi_{tt}\psi_x + 2\,(\varphi_t\psi_{tx} - \varphi_{tx}\psi_t)) + \varphi_t\psi_{tt} - \varphi_{tt}\psi_t\Big). \end{aligned} \tag{12}$$

Here D_t is the operator of the total derivative with respect to t, and

$$\Delta = \varphi_t\psi_x - \varphi_x\psi_t \neq 0.$$

Since the Jacobian of the change of variables $\Delta \neq 0$, the equation

$$u'' + b_3(\tau,\ u)u'^3 + 3b_2(\tau,\ u)u'^2 + 3b_1(\tau,\ u)u' + b_0(\tau,\ u) = 0 \tag{13}$$

becomes (10), where

$$\begin{aligned} a_1 &= \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x + \varphi_x^3 b_0 + 3\varphi_x^2\psi_x b_1 + 3\varphi_x\psi_x^2 b_2 + \psi_x^3 b_3\right), \\ a_2 &= \Delta^{-1}\left(3^{-1}\left(\varphi_t\psi_{xx} - \varphi_{xx}\psi_t + 2\left(\varphi_x\psi_{tx} - \varphi_{tx}\psi_x\right)\right) + \varphi_t\varphi_x^2 b_0\right. \\ &\quad \left. + \varphi_x\left(2\varphi_t\psi_x + \varphi_x\psi_t\right)b_1 + \left(\varphi_t\psi_x^2 + 2\varphi_x\psi_t\psi_x\right)b_2 + \psi_t\psi_x^2 b_3\right), \\ a_3 &= \Delta^{-1}\left(3^{-1}\left(\varphi_x\psi_{tt} - \varphi_{tt}\psi_x + 2\left(\varphi_t\psi_{tx} - \varphi_{tx}\psi_t\right)\right) + \varphi_t^2\varphi_x b_0\right. \\ &\quad \left. + \left(\varphi_t^2\psi_x + 2\varphi_t\varphi_x\psi_t\right)b_1 + \left(2\varphi_t\psi_t\psi_x + \varphi_x\psi_t^2\right)b_2 + \psi_t^2\psi_x b_3\right), \\ a_0 &= \Delta^{-1}\left(\varphi_t\psi_{tt} - \varphi_{tt}\psi_t + \varphi_t^3 b_0 + 3\varphi_t^2\psi_t b_1 + 3\varphi_t\psi_t^2 b_2 + \psi_t^3 b_3\right). \end{aligned} \tag{14}$$

Two quantities play a major role in the study of Eqs. (13):

$$L_1 = -\frac{\partial \Pi_{11}}{\partial u} + \frac{\partial \Pi_{12}}{\partial \tau} - b_0\Pi_{22} - b_2\Pi_{11} + 2b_1\Pi_{12},$$
$$L_2 = -\frac{\partial \Pi_{12}}{\partial u} + \frac{\partial \Pi_{22}}{\partial \tau} - b_3\Pi_{11} - b_1\Pi_{22} + 2b_2\Pi_{12},$$

where

$$\Pi_{11} = 2\left(b_1^2 - b_2b_0\right) + b_{1\tau} - b_{0u}, \quad \Pi_{22} = 2\left(b_2^2 - 3b_1b_3\right) + b_{3\tau} - b_{2u},$$
$$\Pi_{12} = b_2b_1 - b_3b_0 + b_{2\tau} - b_{1u}.$$

Under point transformation (11) these components are transformed as follows [33]:

$$\widetilde{L}_1 = \Delta\left(L_1\varphi_t + L_2\psi_t\right), \quad \widetilde{L}_2 = \Delta\left(L_1\varphi_x + L_2\psi_x\right). \tag{15}$$

Here the tilde means that a value corresponds to system (10): the coefficients b_i are exchanged with a_i, the variables τ and u are exchanged with t and x, respectively.

Lie [1] showed that any equation with $L_1 = 0$ and $L_2 = 0$ is equivalent to the equation $u'' = 0$. Liouville [33] also found other relative invariants, for example,

$$v_5 = L_2\left(L_1L_{2\tau} - L_2L_{1\tau}\right) + L_1\left(L_2L_{1u} - L_1L_{2u}\right) - b_3L_1^3 + 3b_2L_1^2L_2 - 3b_1L_1L_2^2 + b_0L_2^3$$

and

$$w_1 = L_1^{-4}\left(-L_1^3\left(\Pi_{12}L_1 - \Pi_{11}L_2\right) + R_1\left(L_1^2\right)_\tau - L_1^2R_{1\tau} + L_1R_1\left(b_1L_1 - b_0L_2\right)\right),$$

where

$$R_1 = L_1L_{2\tau} - L_2L_{1\tau} + b_2L_1^2 - 2b_1L_1L_2 + b_0L_2^2.$$

Notice that for the Painlevé equations $L_1 \neq 0$ and $L_2 = 0$, $v_5 = 0$ and $w_1 = 0$.

Remark 1.1 Without loss of generality one can assume that $L_1 \neq 0$ and $L_2 = 0$; otherwise a change of the dependent and independent variables such that the functions $\varphi(t, x)$ and $\psi(t, x)$ satisfy the equation

$$\varphi_y L_1 + \psi_y L_2 = 0$$

leads to this case. For the sake of simplicity we study equations with $L_1 \neq 0$ and $L_2 = 0$.

1.3 General difficulties of the equivalence problem

Despite the fact that the criteria for linearizability can be simply checked, there are certain difficulties associated with finding the linearizing transformation. Let us consider a second-order ODE

$$y'' + b(x, y)y'^2 + c(x, y)y' + d(x, y) = 0, \tag{16}$$

where the coefficients satisfy the conditions

$$c_y = 2b_x, \quad d_{yy} - b_{xx} - b_xc + b_yd + d_yb = 0. \tag{17}$$

The transformation

$$t = \varphi(x), \quad u = \psi(x, y) \tag{18}$$

mapping Eq. (16) into the equation $u'' = 0$ is found from the compatible conditions

$$\psi_{yy} = \psi_yb, \quad 2\psi_{xy} = \varphi_x^{-1}\psi_y\varphi_{xx} + c\psi_y, \quad \psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} + \psi_yd \tag{19}$$

and

$$\frac{2\varphi'\varphi''' - 3\varphi''^{\,2}}{\varphi'^{\,2}} = H, \tag{20}$$

where $H = 4(d_y + bd) - (2c_x + c^2)$. Notice that by virtue of the second equation of (17), the function $H = H(x)$. To solve systems (19) and (20), one must first solve Eq. (20). The change $\varphi' = g^{-2}$ reduces Eq. (20) to the equation

$$g'' + \frac{1}{4}Hg = 0. \tag{21}$$

It is well known that the Riccati substitution

$$g' = gv$$

reduces Eq. (21) to the Riccati equation

$$v' + v^2 + \frac{1}{4}H = 0.$$

Thus, to solve Eq. (20), one must be able to solve the Riccati equation, which is not solvable in the general case.

The example presented above shows that the solution of the linearization problem is only theoretical: in many applications it becomes impossible to find the linearizing transformation. A similar problem is also encountered in finding the intermediate integral.

2 Existence of first integral

The existence of the first integral of the form

$$I = A(t,\ x) + \frac{1}{B(t,\ x)\dot{x} + Q(t,\ x)}, \quad (B \neq 0), \tag{22}$$

of a second-order equation requires that the necessary form of the equation be (10), where the coefficients are related by the equations

$$\begin{aligned} &a_0 = \left(Q_t - A_t Q^2\right)/B, \quad a_1 = \left(B_t + Q_x - 2A_t BQ - A_x Q^2\right)/(3B), \\ &a_2 = \left(B_x - A_t B^2 - 2A_x BQ\right)/(3B), \quad a_3 = -A_x B. \end{aligned} \tag{23}$$

The sufficient conditions for the existence of an intermediate integral of the form (22) are obtained if one considers (23) as equations for the functions $A(t,\ x)$, $B(t,\ x)$, and $Q(t,\ x)$ with given coefficients $a_i(t,\ x)$, $(i = 0,\ 1,\ 2,\ 3)$.

System (23) gives

$$A_t = B^{-1}G, \quad A_x = -B^{-1}a_3, \quad Q_t = a_0 B + B^{-1}GQ^2, \quad B_t = -Q_x + 2GQ + 3a_1 B - a_3 B^{-1} Q^2, \tag{24}$$

where

$$G = B^{-1}\left(B_x - 3a_2 B + 2a_3 Q\right).$$

The function $G(t,\ x)$ is introduced in order to simplify the calculations.

The equations $(A_x)_t = (A_t)_x$ and $(B_x)_t = (B_t)_x$ give

$$G_x = -B^{-1}Q_x a_3 - a_{3t} + 3a_1 a_3 + 3a_2 G - B^{-2}a_3^2 Q^2 + G^2,$$

$$\begin{aligned} Q_{xx} = B^{-2}\big(&Q_x B\left(3a_2 B - 4a_3 Q + 3BG\right) - G_t B^3 + B^3\left(3a_{1x} - 3a_{2t} + 2a_0 a_3\right) \\ &+ \left(6a_2 a_3 - a_{3x}\right) BQ^2 - 4a_3^2 Q^3 + 4a_3 BGQ^2\big). \end{aligned} \tag{25}$$

The equation $(Q_{xx})_t = (Q_t)_{xx}$ becomes

$$\begin{aligned} G_{tt} = B^{-4}\big(&4G_t Q_x B^3 - 3G_t B^4 a_1 + 4G_t B^2 a_3 Q^2 - 2Q_x^2 B^2 G - 2Ga_3^2 Q^4 \\ &+3Q_x B^3\left(2a_{2t} - a_{1x} - a_0 a_3\right) - 4Q_x BGa_3 Q^2 + B^4 G^2 a_0 + B^4 G\left(a_{0x} + 3a_0 a_2\right) \\ &+B^4\left(a_{0t}a_3 + a_{1tx} + 3a_{1x}a_1 - 2a_{2tt} - 6a_{2t}a_1 + a_{3t}a_0 + 3a_0 a_1 a_3 - \lambda_1\right) \\ &-3B^2 a_3 Q^2\left(a_{1x} - 2a_{2t} + a_0 a_3\right)\big). \end{aligned} \tag{26}$$

The equation $(G_{tt})_x - (G_x)_{tt} = 0$ leads to $S = 0$, where

$$S \equiv 12G_t Q_x B^3 G - 6G_t^2 B^4 - 6Q_x^2 B^2 G^2 + 12G_t B^2 G a_3 Q^2 - 12Q_x B G^2 a_3 Q^2 \\ +12\left(a_{1x} - 2a_{2t} + a_0 a_3\right) B^2 \left(G_t B^2 - Q_x B G - G a_3 Q^2\right) - 6G^2 a_3^2 Q^4 + 3B^4 G \lambda_1 \\ -B^4 \left(\lambda_{1x} - 3a_2\lambda_1 + 6\left(a_{1x} - 2a_{2t} + a_0 a_3\right)^2\right).$$

Furthermore, the equations

$$S_x - 6\left(G + a_2\right) S = 0, \quad B^2 S_t - 6\left(Q_x B - B^2 a_1 + a_3 Q^2\right) S = 0$$

are

$$B a_3 Q_x = 3B^2 G\mu_1 - 5B^2 G^2 + B^2\left(3a_1 a_3 - \mu_2\right) - a_3^2 Q^2, \tag{27a}$$

$$G_t = \left(15B^2\lambda_1\right)^{-1}\left(6Q_x B\lambda_1\left(5G - \mu_1\right) - 3B^2 G\left(\lambda_{1t} + 6a_1\lambda_1\right) + 6a_3\lambda_1 Q^2\left(5G - \mu_1\right) \\ +B^2\left(\lambda_1\mu_{1t} + 12a_{1x}\lambda_1 - 24a_{2t}\lambda_1 + \lambda_{1t}\mu_1 + 12a_0 a_3\lambda_1 + 6a_1\lambda_1\mu_1\right)\right), \tag{27b}$$

where all coefficients μ_i, $(i = 1, 2, \ldots, 7)$ are presented in the Appendix.

For further analysis one needs to consider two cases: (a) $a_3 \neq 0$ and (b)[1] $a_3 = 0$. It is also worth noting that because of the relative invariant υ_5, the property for a_3 which is not equal to zero, is an invariant property of the point transformations conserving $L_2 = 0$.

2.1 Case $a_3 \neq 0$

Let $a_3 \neq 0$; then Eq. (27b) gives

$$Q_x = \left(B a_3\right)^{-1}\left(3B^2 G\mu_1 - 5B^2 G^2 + B^2\left(3a_1 a_3 - \mu_2\right) - a_3^2 Q^2\right).$$

Thus, all first-order derivatives of the unknown functions $A(t, x)$, $B(t, x)$, $Q(t, x)$, and $G(t, x)$ are found:

$$\begin{aligned} &A_t = B^{-1} G, && A_x = -B^{-1} a_3, \\ &Q_t = B a_0 + B^{-1} G Q^2, && Q_x = \left(-5B^2 G^2 + 3B^2 G\mu_1 + B^2\left(3a_1 a_3 - \mu_2\right) - a_3^2 Q^2\right)/(B a_3), \\ &B_t = \left(5BG^2 - 3BG\mu_1 + B\mu_2 + 2G a_3 Q\right)/a_3, && B_x = BG + 3Ba_2 - 2a_3 Q, \\ &G_t = \left(-10G^3 + 8G^2\mu_1 - G\mu_3 + \lambda_1\mu_4\right)/a_3, && G_x = 6G^2 + 3G\left(a_2 - \mu_1\right) - a_{3t} + \mu_2. \end{aligned} \tag{28}$$

The overdetermined system (28) is compatible if the conditions

$$\begin{aligned} &(A_t)_x - (A_x)_t = 0, \quad (B_t)_x - (B_x)_t = 0, \\ &(Q_x)_t - (Q_t)_x = 0, \quad (G_t)_x - (G_x)_t = 0 \end{aligned} \tag{29}$$

are satisfied. Notice also that by virtue of Eqs. (28), (24) are satisfied. Hence, it is not necessary to substitute the first-order derivatives into the intermediate Eqs. (25) and (26).

The conditions in (29) reduce to equations

$$H \equiv 12G^3 a_3 - G^2\mu_5 - G\mu_6 - \mu_7 = 0, \tag{30a}$$

$$75G^4 - 80\mu_1 G^3 + 5q_2 G^2 - q_1 G - q_0 = 0, \tag{30b}$$

where the coefficients q_i $(i = 0, 1, 2)$ are presented in the Appendix. Let us also add to this set of equations the following ones:

$$H_x = 0, \quad H_t = 0. \tag{31}$$

Equation (30a) is a polynomial equation of third degree with respect to G. If we exclude from Eqs. (30b) and (31) the value

$$G^3 = \left(G^2\mu_5 + G\mu_6 + \mu_7\right)/(12a_3),$$

then Eqs. (30b) and (31) become

[1] This case has been studied in [23].

$$5\alpha_1 G^2 + \beta_1 G + \gamma_1 = 0, \quad \alpha_2 G^2 + \beta_2 G + \gamma_2 = 0, \quad 25\alpha_3 G^2 + \beta_3 G + 25\gamma_3 = 0, \tag{32}$$

where all coefficients α_i, β_i, and γ_i, $(i = 1,\ 2,\ 3)$ are presented in the Appendix.

In solving Eq. (30a) with respect to G, one must also satisfy the conditions $G_t = (G)_t$ and $G_x = (G)_x$. Satisfying these conditions is equivalent to satisfying Eq. (31). Thus, further study simply entails an algebraic study of Eqs. (30a) and (32). This study depends on the coefficients α_i, β_i, $(i = 1,\ 2,\ 3)$.

For example, assume that $\alpha_1 \neq 0$. From the first equation of (32) one finds G^2. Substituting G^2 into (30a) and the remaining equations of (32), one obtains linear equations with respect to G. One needs to study resolving these linear equations with respect to G. This depends on the coefficients of these equations.

3 Case $G = 0$

Let us consider the case $G = 0$ without restrictions for λ_2. Then

$$\begin{aligned} &A_t = 0, \quad A_x = -a_3/B, \quad Q_t = a_0 B,\\ &B_t = \left(-Q_x B + 3a_1 B^2 - a_3 Q^2\right)/B, \quad B_x = 3a_2 B - 2a_3 Q. \end{aligned} \tag{33}$$

The equations $(A_x)_t - (A_t)_x = 0$ and $(B_x)_t - (B_t)_x = 0$ give

$$Q_x a_3 B = -a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2, \tag{34a}$$

$$\begin{aligned} &Q_{xx} B^2 + Q_x B\,(2a_3 Q - 3a_2 B) + B^3\,(3a_{2t} - 3a_{1x} - 2a_0 a_3)\\ &+ B Q^2\,(a_{3x} - 6a_2 a_3) + 2B^2 Q\,(3a_1 a_3 - a_{3t}) + 2a_3^2 Q^3 = 0. \end{aligned} \tag{34b}$$

3.1 Case $a_3 \neq 0$

If $a_3 \neq 0$, then Eq. (34a) defines

$$Q_x = \left(-a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2\right)/(a_3 B)\,. \tag{35}$$

This reduces Eq. (34b) and the equation $(Q_x)_t - (Q_t)_x = 0$ to

$$\begin{aligned} &\left(a_{3tx} - 3a_{2t} a_3 + 2a_0 a_3^2\right) a_3 - a_{3t} a_{3x} = 0,\\ &a_{3tt} - 3a_{3t} a_1 + (a_{0x} - 3a_{1t} + 3a_0 a_2)\, a_3 = 0. \end{aligned} \tag{36}$$

Thus, if Eq. (10) satisfies condition (36), then the overdetermined system of equations consisting of Eqs. (33) and (35) is involutive.

For example, for $a_3 = 1$ condition (36) can be solved as follows:

$$a_0 = 3a_{2t}/2, \quad a_1 = a_{2x}/2 + 3a_2^2/4 + \varphi,$$

where $\varphi(x)$ is an arbitrary function. This means that all equations of the form

$$\ddot{x} + \dot{x}^3 + 3a_2 \dot{x}^2 + \left(a_{2x}/2 + 3a_2^2/4 + \varphi\right) \dot{x} + 3a_{2t}/2 = 0 \tag{37}$$

with arbitrary functions $a_2(t,\ x)$ and $\varphi(x)$ have the intermediate integral

$$I = A + \frac{1}{B(\dot{x} + \frac{3}{2}a_2) + H},$$

where the functions $A(x)$, $B(x)$, and $H(x)$ are solutions of the equations

$$A' = -1/B, \quad B' = -H, \quad H' = 3B\varphi - H^2/B.$$

Notice that for $a_{2t} = 0$ Eq. (36) can be reduced to a first-order ODE by the standard change $\dot{x} = y(x)$, whereas for $a_{2t} \neq 0$ this technique is not applicable.

3.2 Case $a_3 = 0$

In this case, Eq. (34a) is satisfied and Eq. (34b) becomes

$$Q_{xx} = 3Q_x a_2 - 3B\left(a_{2t} - a_{1x}\right). \tag{38}$$

The equation $(Q_{xx})_t - (Q_t)_{xx} = 0$ gives

$$3Q_x\eta = B\left(\eta_t + 3a_1\eta - \lambda_1\right), \tag{39}$$

where $\eta = a_{1x} - 2a_{2t}$. Hence, for $\eta = 0$ one has that[2] $\lambda_1 = 0$, and there are no other additional equations for the functions $A(t, x)$, $B(t, x)$, and $Q(t, x)$. This means that the system of equations consisting of Eqs. (33) and (38) is involutive. If $\eta \neq 0$, then one can find Q_x. The equations $(Q_x)_t - (Q_t)_x = 0$ and $(Q_{xx})_x - (Q_x)_x = 0$ give the conditions

$$\begin{aligned} 3\eta\eta_{tt} &= 4\eta_t^2 - 3\eta_t\eta a_1 + 15\eta_t\lambda_1 + 9\eta^2\left(a_{0x} - a_{1t} + 3a_0a_2 - 2a_1^2\right) - 9\eta\left(\lambda_{1t} + a_1\lambda_1\right) + 9\lambda_1^2, \\ \eta\eta_{tx} &= \eta_t\eta_x + 3\eta_x\lambda_1 - 2\eta^3 + 3\eta^2 a_{2t} - 3\eta\lambda_{1x}. \end{aligned} \tag{40}$$

Thus, if Eq. (10) satisfies condition (40), then the overdetermined system of equations consisting of Eqs. (33) and (39) is involutive.

4 Examples

In this section we consider examples of first integrals of the form

$$I = \frac{A(t, x)\dot{x} + B(x, t)}{\dot{x} + Q(x, t)}. \tag{41}$$

Example 4.1 The most general equation associated with the first integral I in (41) is given by

$$\begin{aligned} &\ddot{x} + \frac{A_x}{\Delta}\dot{x}^3 + \frac{1}{\Delta}\left(A_t + B_x + A_xQ - AQ_x\right)\dot{x}^2 \\ &\quad + \frac{1}{\Delta}\left(B_t + A_tQ - AQ_t + B_xQ - BQ_x\right)\dot{x} + \frac{1}{\Delta}\left(QB_t - BQ_t\right) = 0, \end{aligned} \tag{42}$$

where $\Delta = AQ - B$.

Proof Rearranging (41) we obtain

$$I(\dot{x} + Q(x, t)) = A(t, x)\dot{x} + B(x, t),$$

which immediately yields

$$I(\ddot{x} + Q_x\dot{x} + Q_t) = A\ddot{x} + A_x\dot{x}^2 + A_t\dot{x} + B_x\dot{x} + B_t,$$

$$(A\dot{x} + B)(\ddot{x} + Q_x\dot{x} + Q_t) = (\dot{x} + Q(x, t))(A\ddot{x} + A_x\dot{x}^2 + A_t\dot{x} + B_x\dot{x} + B_t).$$ □

This equation is closely related to the (unparameterized) geodesic equations of some connection Γ on $U \in \mathbf{R}^2$

$$\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a\dot{x}^b = v\dot{x}^c$$

for $x^a(t) = (x(t), y(t))$. Eliminating the parameter t yields the second-order ODE for y as a function of x

$$\frac{d^2y}{dx^2} = a(x, y)\left(\frac{dy}{dx}\right)^3 + b(x, y)\left(\frac{dy}{dx}\right)^2 + c(x, y)\left(\frac{dy}{dx}\right) + d(x, y) = 0, \tag{43}$$

where

$$a(x, y) = -\Gamma^2_{11}, \quad b(x, y) = \Gamma^1_{11} - 2\Gamma^2_{12}, \quad c(x, y) = 2\Gamma^1_{12} - \Gamma^2_{22}, \quad d(x, y) = \Gamma^1_{22}.$$

[2] Notice that for Eq. (10), which is not linearizable, one can assume without loss of generality that $\lambda_1 \neq 0$.

In other words, any second-order ODEs with cubic nonlinearity in the first derivatives of the form (42) gives rise to some projective structures.

Example 4.2 A quasimonomial q over $\mathbb{K}$ is defined as

$$q = \mathbf{x}^c = \prod_{i=1}^{n} x_i^{c_i}, \quad c_i \in \mathbb{K}.$$

A quasimonomial function is a finite sum of quasimonomials $f : \mathbb{C} \to \Sigma$, where $\Sigma = \mathbb{C} \cup \{\infty\}$, defined as

$$\mathbf{x} \longrightarrow \sum a_i \prod_{j=1}^{n} x_j^{c_{ij}}.$$

We assume $A = x^\alpha t^\beta$, $B = x^\gamma t^\delta$ and $Q = 1$ in (42) to obtain the second-order equation

$$\ddot{x} + \frac{\alpha}{x(1-x^{\gamma-\alpha})}\dot{x}^3 + \frac{\alpha + \beta\frac{x}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})}\dot{x}^2 + \frac{\beta\frac{x}{t} + \beta\frac{x^{\gamma-\alpha+1}}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})}\dot{x} + \frac{\beta\frac{x^{\gamma-\alpha+1}}{t}}{x(1-x^{\gamma-\alpha})} = 0, \tag{44}$$

which admits the first integral

$$I = \frac{x^\alpha t^\beta(\dot{x} + x^{\gamma-\alpha})}{\dot{x}+1}.$$

4.1 Claim

Setting $\alpha = -1,\ \beta = 1 = \delta,\ \gamma = 0$ we obtain the first integral of

$$\ddot{x} + \frac{1}{x(x-1)}\dot{x}^3 + \left(\frac{1}{x(x-1)} + \frac{1}{t(1-x)}\right)\dot{x}^2 + \frac{1}{t}\left(\frac{1+x}{1-x}\right)\dot{x} + \frac{x}{t(1-x)} = 0 \tag{45}$$

as

$$I = \left(\frac{t}{x}\right)\frac{\dot{x}+x}{\dot{x}+1}.$$

Example 4.3 The first integral of the second-order equation

$$\ddot{x} + \frac{t}{x^2(x-1)}\dot{x}^3 + \left(\frac{2}{1-x} - \frac{(1+x)t}{(1-x)x^2} + \frac{1}{x}\right)\dot{x}^2 + \left(\frac{1}{1-x} + \frac{1-t}{(1-x)x}\right)\dot{x} + \frac{1}{1-x} = 0 \tag{46}$$

is

$$I = \mathrm{e}^{t/x}\frac{\dot{x}+x}{\dot{x}+1}.$$

Example 4.4 Let us set $A = Q^{-1} = \mathrm{e}^{\alpha(x)t}$ and $B = b$ (constant) in (41). Then we obtain the equation

$$(1-b)\ddot{x} + \mathrm{e}^{\alpha(x)t}(\alpha'(x)t\dot{x} + \alpha(x))\dot{x}^2 + 2(\alpha'(x)t\dot{x} + \alpha(x))\dot{x} + \mathrm{e}^{-\alpha(x)t}(\alpha'(x)t\dot{x} + b\alpha(x)) = 0. \tag{47}$$

The corresponding first integral is

$$I = \frac{\mathrm{e}^{\alpha(x)t}\dot{x} + b}{\dot{x} + \mathrm{e}^{-\alpha(x)t}}.$$

4.2 Time-independent case

Consider $A_t = B_t = Q_t = 0$. Thus Eq. (41) becomes

$$\ddot{x} + \frac{A_x}{\Delta}\dot{x}^3 + \frac{1}{\Delta}\left(B_x + A_x Q - A Q_x\right)\dot{x}^2 + \frac{1}{\Delta}\left(A_t Q - A Q_t + B_x Q - B Q_x\right)\dot{x} = 0, \tag{48}$$

which can be expressed as

$$\dot{x} = y, \quad \dot{y} = -\frac{1}{\Delta}\big(A_x y^3 + (B_x + A_x Q - A Q_x)\, y^2 + (B_x Q - B Q_x)\, y\big). \tag{49}$$

This yields the flow equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\Delta}\big(A_x y^2 + (B_x + A_x Q - A Q_x)\, y + (B_x Q - B Q_x)\big).$$

Assume $A = 1$; thus, $\Delta = Q - B$. The flow becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\Delta'(x)}{\Delta(x)}y = \frac{B^2}{\Delta}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q}{B}\right).$$

This immediately yields

$$y = -\frac{1}{Q/B - 1} + C_1.$$

Hence we obtain

$$t = \int \frac{\mathrm{d}x}{C_1 Q - (C_1 + 1)B(x)}.$$

4.3 Reduction

Let $A_x = 0$ and set

$$\frac{1}{\Delta}\left(A_t + B_x - A Q_x\right) = b(x,\, t) = \frac{1}{2}\phi_x, \quad \frac{1}{\Delta}\left(B_t + A_t Q - A Q_t + B_x Q - B Q_x\right) = c(x,\, t) = \phi_t. \tag{50}$$

A large number of second-order ODEs in the Painlevé–Gambier classification system belong to the following class of equations:

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,\, x) = 0.$$

This equation yields the Lagrangian description via Jacobi's last multiplier. If we write this equation in the form

$$\ddot{x} = \mathcal{F}(t,\, x,\, \dot{x}) = -\left[\frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,\, x)\right],$$

then the Jacobi last multiplier M is given by the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln M = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} = \mathrm{e}^{\phi(t,\, x)}.$$

We then obtain the Lagrangian as

$$L(t,\, x,\, \dot{x}) = \mathrm{e}^{\phi(t,\, x)}\frac{\dot{x}^2}{2} + f_1(t,\, x)\dot{x} + f_2(t,\, x).$$

4.4 Conditions for Lagrangians

Let us express ϕ in terms $A,\ B,\ Q$ and find the conditions for the Lagrangian. Defining

$$\phi_x = \frac{2}{\Delta}\left(A_t + B_x - AQ_x\right),$$

$$\phi_t = \frac{1}{\Delta}\left(B_t + A_t Q - AQ_t + B_x Q - BQ_x\right)$$

immediately yields

$$\phi_{xt} = \frac{2}{\Delta}\left(A''(t) + B_{xt} - A'(t)Q_x - AQ_{xt}\right) - \frac{2}{\Delta^2}\left(A'(t)Q + AQ_t - B_t\right)\left(A'(t) + B_x - AQ_x\right),$$

$$\phi_{tx} = \frac{1}{\Delta}\left(B_{tx} + A'Q_x - AQ_{tx} + B_{xx}Q - BQ_{xx}\right) - \frac{1}{\Delta^2}\left(AQ_x - B_x\right)\left(B_t + A'Q - AQ_t + B_x Q - BQ_x\right).$$

4.5 Claim

A second-order nonlinear equation of the form

$$\ddot{x} + b(x,\ t)\dot{x}^2 + c(x,\ t)\dot{x} + d(x,\ t) = 0$$

admits a Lagrangian provided

$$2A'' + B_{xt} - 3A'Q_x - AQ_{xt} - B_{xx}Q + BQ_{xx}(AQ - B)$$
$$= A'Q\left(2A' + 3B_x - 3AQ_x\right) + \left(AQ_t - B_t\right)\left(2A' + B_x - AQ_x\right) - \left(AQ_x - B_x\right)\left(B_x Q - BQ_x\right),$$

where $b(x,\ t)$ and $c(x,\ t)$ are defined as (8) and $d(x,\ t) = \frac{1}{\Delta}(QB_t - BQ_t)$.

Outline of proof. It follows analogously to the argument in Sect. 4.3 and makes use of the compatibility condition $\phi_{xt} = \phi_{tx}$. □

Example 4.5 Set $A = 1$, $\Delta = AQ - B = x^{\alpha}t^{\beta}$, and assume $Q = x^{\gamma}$ in (51); the equation becomes

$$\ddot{x} - \frac{\alpha}{x}\dot{x}^2 + \left((\gamma - \alpha)x^{\gamma-1} - \frac{\beta}{t}\right)\dot{x} - \frac{\beta x^{\gamma}}{t} = 0 \tag{51}$$

whose first integral is

$$I = \frac{\dot{x} + x^{\gamma} - x^{\alpha}t^{\beta}}{\dot{x} + x^{\gamma}}.$$

Let us find the conditions γ and α for which Eq. (51) gives a Lagrangian description.

4.6 Claim

For $\gamma = \alpha$ or $\gamma = 1$ Eq. (51) yields a Lagrangian description.

Proof Equate $\frac{1}{2}\phi_x = -\frac{\alpha}{x}$ and $\phi_t = (\gamma - \alpha)x^{\gamma-1} - \frac{\beta}{t}$. This immediately yields $\phi_{xt} = 0$ and $\phi_{tx} = (\gamma - \alpha)(\gamma - 1)x^{\gamma-2}$. Thus from the compatibility condition we obtain our criteria. □

5 Conclusion

Any second-order ODE that possesses a first integral of the form (1) must be cubic with respect to the first-order derivative (10). This paper gives complete criteria of the existence of a first integral of the form (1) for a second-order ODE (10), which is reduced to an equation with $L_2 = 0$. Despite the fact that any second-order ODE (10) can be reduced to an equation with $L_2 = 0$, the complete solution of the problem requires that sufficient conditions be given using coefficients of the original equation (not reduced). This is still an open problem.

Acknowledgments S. V. Meleshko wishes to thank the Office of the Higher Education Commission under NRU project (SUT) for its partial financial support. S. Moyo thanks the National Research Foundation of South Africa.

Appendix

The following coefficients μ_i appear in Eqs. (27a) and (27b) in Sect. 2:

$$\mu_1 = (\lambda_{1x} - 3a_2\lambda_1)/\lambda_1,$$
$$\mu_2 = \left(\mu_{1x} + 3a_{3t} - 3a_2\mu_1 + \mu_1^2\right)/3,$$
$$\mu_3 = \left(\lambda_{1t}a_3 - 24a_1a_3\lambda_1 + 6\lambda_1\mu_1^2 + 10\lambda_1\mu_2\right)/(5\lambda_1),$$
$$\mu_4 = \left(\mu_{1t}a_3 + 12a_3(a_{1x} - 2a_{2t} + a_0a_3 + a_1\mu_1) - 6\mu_1^3 - 4\mu_1\mu_2 + 5\mu_1\mu_3\right)/(15\lambda_1),$$
$$\mu_5 = a_{3x} - 6a_2a_3 + 10a_3\mu_1,$$
$$\mu_6 = \left(a_{3t}a_3 - 6a_1a_3^2 + 18a_3\mu_1^2 + a_3\mu_2 - a_3\mu_3 - 3\mu_1\mu_5\right)/5,$$
$$\mu_7 = \left(3a_{2t}a_3^2 - \mu_{2x}a_3 - 2a_0a_3^3 - 18a_1a_3^2\mu_1 + 6a_2a_3\mu_2 + a_3\lambda_1\mu_4 + 54a_3\mu_1^3 \right.$$
$$\left. -4a_3\mu_1\mu_2 - 3a_3\mu_1\mu_3 - 9\mu_1^2\mu_5 - 15\mu_1\mu_6 + \mu_2\mu_5\right)/5.$$

In addition, the coefficients α_i, β_i, γ_i appear in Sect. 2.1 and Eq. (32):

$$\alpha_1 = 720a_3^2\mu_1^2 - 432a_1a_3^3 + 144a_3^2\mu_2 - 144a_3^2\mu_3 - 80a_3\mu_1\mu_5 - 300a_3\mu_6 - 5\mu_5^2,$$
$$\beta_1 = 1728a_{1x}a_3^3 - 3456a_{2t}a_3^3 + 1776a_0a_3^4 + 3024a_1a_3^3\mu_1 - 1680a_3^2\lambda_1\mu_4 - 3456a_3^2\mu_1^3$$
$$-1008a_3^2\mu_1\mu_2 + 1008a_3^2\mu_1\mu_3 + 432a_3\mu_1^2\mu_5 + 1040a_3\mu_1\mu_6 - 300a_3\mu_7 - 25\mu_5\mu_6,$$
$$\gamma_1 = 48a_{0x}a_3^4 - 144a_{1t}a_3^4 + 48\mu_{2t}a_3^3 + 144a_0a_2a_3^4 - 432a_1a_3^3\mu_2 - 144a_3^2\lambda_1\mu_1\mu_4 + 96a_3^2\mu_2^2$$
$$+864a_3^2\mu_1^2\mu_2 - 48a_3^2\mu_2\mu_3 - 144a_3\mu_1\mu_2\mu_5 + 320a_3\mu_1\mu_7 - 240a_3\mu_2\mu_6 - 25\mu_5\mu_7,$$
$$\alpha_2 = -72\mu_{5t}a_3^3 + 432a_1a_3^3\mu_5 + 2592a_3^3\lambda_1\mu_4 - 1296a_3^2\mu_1^2\mu_5 + 1152a_3^2\mu_1\mu_6 - 72a_3^2\mu_2\mu_5$$
$$-2160a_3^2\mu_7 + 264a_3\mu_1\mu_5^2 + 180a_3\mu_5\mu_6 - 5\mu_5^3,$$
$$\beta_2 = -72\mu_{6t}a_3^3 + 432a_1a_3^3\mu_6 - 144a_3^2\lambda_1\mu_4\mu_5 - 1296a_3^2\mu_1^2\mu_6 + 1728a_3^2\mu_1\mu_7$$
$$-72a_3^2\mu_2\mu_6 - 72a_3^2\mu_3\mu_6 + 264a_3\mu_1\mu_5\mu_6 - 60a_3\mu_5\mu_7 + 240a_3\mu_6^2 - 5\mu_5^2\mu_6,$$
$$\gamma_2 = -72\mu_{7t}a_3^3 + 432a_1a_3^3\mu_7 - 72a_3^2\lambda_1\mu_4\mu_6 - 1296a_3^2\mu_1^2\mu_7 - 72a_3^2\mu_2\mu_7 - 144a_3^2\mu_3\mu_7$$
$$+264a_3\mu_1\mu_5\mu_7 + 240a_3\mu_6\mu_7 - 5\mu_5^2\mu_7,$$
$$\alpha_3 = -2\mu_{5x}a_3 - 432a_1a_3^3 + 18a_2a_3\mu_5 + 1296a_3^2\mu_1^2 + 144a_3^2\mu_2 - 72a_3^2\mu_3$$
$$-242a_3\mu_1\mu_5 - 336a_3\mu_6 + 3\mu_5^2,$$
$$\beta_3 = -1212a_{1x}a_3^3 + 2424a_{2t}a_3^3 - 10\mu_{5t}a_3^2 + 30\mu_{5x}a_3\mu_1 - 1244a_0a_3^4 + 4644a_1a_3^3\mu_1$$
$$+120a_1a_3^2\mu_5 - 270a_2a_3\mu_1\mu_5 + 1540a_3^2\lambda_1\mu_4 - 18336a_3^2\mu_1^3 - 1548a_3^2\mu_1\mu_2 + 468a_3^2\mu_1\mu_3$$
$$+3312a_3\mu_1^2\mu_5 + 5090a_3\mu_1\mu_6 - 30a_3\mu_2\mu_5 + 20a_3\mu_3\mu_5 + 850a_3\mu_7 + 75\mu_5\mu_6,$$
$$\gamma_3 = -2\mu_{7x}a_3 + 12a_1a_3^2\mu_6 + 30a_2a_3\mu_7 - 36a_3\mu_1^2\mu_6 - 38a_3\mu_1\mu_7 - 4a_3\mu_2\mu_6$$
$$+2a_3\mu_3\mu_6 + 6\mu_1\mu_5\mu_6 + 3\mu_5\mu_7 + 10\mu_6^2.$$

The q_i that follow appear in Eq. (30b):

$$q_0 = a_{0x}a_3^2 - 3a_{1t}a_3^2 - a_{3t}\mu_2 + \mu_{2t}a_3 + 3a_0a_2a_3^2 - 3a_1a_3\mu_2 - 3\lambda_1\mu_1\mu_4 + \mu_2^2,$$
$$q_1 = 36a_{1x}a_3 - 72a_{2t}a_3 + 3a_{3t}\mu_1 + 37a_0a_3^2 + 45a_1a_3\mu_1 - 35\lambda_1\mu_4 - 18\mu_1\left(\mu_1^2 + \mu_2 - \mu_3\right),$$
$$q_2 = a_{3t} + 3a_1a_3 + 3\mu_1^2 - 2\mu_2 + 2\mu_3.$$

References

1. Lie S (1883) Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. III. Arch Matematik og Naturvidenskab 8(4):371–427. Reprinted in Lie's Gessammelte Abhandlungen, 1924, vol 5, paper XIY, pp 362–427

2. Lie S (1891) Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Teubner, Leipzig, Bearbeitet und herausgegeben von Dr. G.Scheffers
3. Ibragimov NH, Meleshko SV (2007) Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. II. Commun Nonlinear Sci Numer Simul 12:1370–1378
4. Ibragimov NH, Meleshko SV (2008) Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I. Commun Nonlinear Sci Numer Simul 13:1015–1020
5. Meleshko SV (2006) To the equivalence problem of second-order ordinary differential equations. Abstract of presentation given at international conference in scientific computing: modelling, simulation and optimization of complex processes, Hanoi, Vietnam, 6th–10th March, 2006. Available online at http://hpsc.iwr.uni-heidelberg.de/HPSCHanoi2006/. Accessed March 2011
6. Dridi R (2009) On the geometry of the first and second Painlevé equations. J Phys A: Math Theor 42(12):125201-1–125201-9
7. Tresse AM (1896) Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre $y'' = \omega(x, y, y')$, Preisschriften der fürstlichen Jablonowski'schen Gesellschaft XXXII, Leipzig, S. Herzel
8. Ibragimov NH (2002) Invariants of a remarkable family of nonlinear equations. Nonlinear Dyn 30(2):155–166
9. Cartan E (1924) Sur les variétés à connexion projective. Bull Soc Math Fr 52:205–241
10. Chern S-S (1940) The geometry of the differential equation $y''' = f(x, y, y', y'')$. Rep Natl Tsing Hua Univ 4:97–111
11. Grebot G (1997) The characterization of third order ordinary differential equations admitting a transitive fiber-preserving point symmetry group. J Math Anal Appl 206:364–388
12. Gusyatnikova VN, Yumaguzhin VA (1999) Contact transformations and local reducibility of ODEs to the form $y''' = 0$. Acta Appl Math 56:155–179
13. Doubrov B, Komrakov B, Morimoto T (1999) Equivalence of holonomic differential equations. Lobachevskii J Math 3:39–71
14. Doubrov B (2001) Contact trivialization of ordinary differential equations. In: Kowalski O, Krupka D, Slovak J (eds) Proceedings of the 8th international conference on differential geometry and its applications. Silesian University in Opava, Opava, pp 73–84
15. Neut S, Petitot M (2002) La géométrie de l'équation $y''' = f(x, y, y', y'')$. C R Acad Sci Paris 335:515–518. Ser I, Équations différentielles/Ordinary Differential Equations
16. Bocharov AV, Sokolov VV, Svinolupov SI (1993) On some equivalence problems for differential equations. Preprint ESI 54, The Erwin Schrödonger International Institute for Mathematical Physics, Wien, Austria, 12 p
17. Euler N, Wolf T, Leach PGL, Euler M (2003) Linearizable third-order ordinary differential equations and generalized Sundman transformations: the case $X''' = 0$. Acta Appl Math 76:89–115
18. Euler N, Euler M (2004) Sundman symmetries of nonlinear second-order and third-order ordinary differential equations. J Nonlinear Math Phys 11(3):399–421
19. Moyo S, Meleshko SV (2011) Application of the generalised Sundman transformation to the linearisation of two second-order ordinary differential equations. J Nonlinear Math Phys 18(1):213–236
20. Muriel C, Romero JL (2008) Integrating factors and λ-symmetries. J Nonlinear Math Phys: Math Theor 15:290–299
21. Muriel c, Romero JL (2009) First integrals, integrating factors and λ-symmetries of second-order differential equations. J Phys A: Math Theor 42(36):365207-1–365207-17
22. Muriel C, Romero JL (2009) Second-order ordinary differential equations and first integrals of the form $a(t, x)\dot{x} + b(t, x)$. J Nonlinear Math Phys 16:209–222
23. Muriel C, Romero JL (2011) Second-order ordinary differential equations with first integrals of the form $c(t) + 1/(a(t, x)\dot{x} + b(t, x))$. J Nonlinear Math Phys 18:237–250
24. Muriel C, Romero JL (2010) Nonlocal transformations and linearization of second-order ordinary differential equations. J Phys A: Math Theor 43:434025. doi:10.1088/1751-8113
25. Muriel C, Romero JL (2009) λ-Symmetries on the derivation of first integrals of ordinary differential equations. In: Greco AM, Rionero S, Ruggeri T (eds) WASCOM 2009: 15th conference on waves and stability in continuous media. World Scientific, Singapore, pp 303–308
26. Chandrasekar VK, Senthilvelan M, Lakshmanan M (2005) On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. Proc R Soc A 461:2451–2476
27. Chandrasekar VK, Senthilvelan M, Lakshmanan M (2006) A unification in the theory of linearization of second-order nonlinear ordinary differential equations. J Phys A 39:L69–L76
28. Nakpim W, Meleshko SV (2010) Linearization of third-order ordinary differential equations by generalized Sundman transformations: the case $X''' + \alpha X = 0$. Commun Nonlinear Sci Numer Simul 15:1717–1723
29. Guha P, Khanra B, Choudhury A (2010) On generalized Sundman transformation method, first integrals, symmetries and solutions of equations of Painlevé–Gambier type. Nonlinear Anal 72:3247–3257
30. Duarte LGS, Moreira IC, Santos FC (1994) Linearization under nonpoint transformations. J Phys A 27:L739–L743
31. Ibragimov NH (2006) A practical course in differential equations and mathematical modelling: classical and new methods, nonlinear mathematical models, symmetry and invariance principles. ALGA Publications, Karlskrona
32. Meleshko SV (2006) On linearization of third-order ordinary differential equations. J Phys A 39:15135–15145
33. Liouville R (1889) Sur les invariants de certaines equations differentielles et sur leurs applications. J Ec Polytech 59:7–76

7 SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS AND FIRST INTEGRALS OF THE FORM $A(t,x)\dot{x}+B(t,x)$ by C. MURIEL, J. L. ROMERO (2009)

Article

Journal of Nonlinear Mathematical Physics, Vol. 16, Suppl. (2009) 209–222

SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS AND FIRST INTEGRALS OF THE FORM $A(t,x)\dot{x}+B(t,x)$

C. MURIEL* and J. L. ROMERO†

Departamento de Matemáticas, Universidad de Cádiz
11510 Puerto Real, Cádiz, Spain
*concepcion.muriel@uca.es
†juanluis.romero@uca.es

Received 24 August 2009
Accepted 14 September 2009

We characterize the equations in the class $\mathcal{A}$ of the second-order ordinary differential equations $\ddot{x}=M(t,x,\dot{x})$ which have first integrals of the form $A(t,x)\dot{x}+B(t,x)$. We give an intrinsic characterization of the equations in $\mathcal{A}$ and an algorithm to calculate explicitly such first integrals. Although $\mathcal{A}$ includes equations that lack Lie point symmetries, the equations in $\mathcal{A}$ do admit λ-symmetries of a certain form and can be characterized by the existence of such λ-symmetries. The equations in a well-defined subclass of $\mathcal{A}$ can completely be integrated by using two independent first integrals of the form $A(t,x)\dot{x}+B(t,x)$. The methods are applied to several relevant families of equations.

Keywords: Ordinary differential equations; first integrals; λ-symmetries; Sundman transformations.

1. Introduction

The search of new methods to solve ordinary differential equations (ODEs) plays a fundamental role in the treatment of physical models. In being faced with the problem of solving a given ODE one may try to transform it into another ODE with known solutions. Usually the considered transformed equations are linear equations and invertible point transformations are the most commonly used. The first linearization problem for ODEs was solved by Lie [11]. He showed that a second-order ODE

$$\ddot{x}=M(t,x,\dot{x}) \tag{1.1}$$

is linearizable by a (local) change of variables if and only if the equation is of the form

$$\ddot{x}+a_3(t,x)\dot{x}^3+a_2(t,x)\dot{x}^2+a_1(t,x)\dot{x}+a_0(t,x)=0 \tag{1.2}$$

and the coefficients a_i, $0\le i\le 3$, satisfy two conditions involving their partial derivatives [12,9,10]. If Eq. (1.1) is linearizable to equation $X_{TT}=0$, then this last equation has two independent first integrals of the form $\widetilde{A}(T,X)\dot{X}+\widetilde{B}(T,X)$, but in terms of variables (t,x) the original equation may lack first integrals of the form

$$A(t,x)\dot{x}+B(t,x). \tag{1.3}$$

In the literature there are plenty of examples of equations with first integrals of the form (1.3) but, as far as we know, no characterizations of these equations have been derived.

Throughout this paper we say that a second-order ODE (1.1) belongs to the class $\mathcal{A}$ if the equation has a first integral of the form (1.3), by using the same variables in (1.1) and (1.3).

In Sec. 2 we prove that equations in $\mathcal{A}$ must have the form (1.2) with $a_3 = 0$, i.e. the equations must be of the form

$$\ddot{x}+a_2(t,x)\dot{x}^2+a_1(t,x)\dot{x}+a_0(t,x)=0. \tag{1.4}$$

The relationships between the coefficients a_i, $1 \le i \le 2$, and the functions A and B are also established. We also consider in this section an equation in $\mathcal{A}$ that lacks Lie point symmetries (see Eq. (2.6)). Another example is given in Sec. 4 (Eq. (4.12)).

In Sec. 3 we give an intrinsic characterization of the equations in the class $\mathcal{A}$. We also provide an algorithm to determine the first integrals of the form 1.3 in terms of the coefficients a_i, $1 \le i \le 2$. The main result in Sec. 3 (Theorem 2) connects $\mathcal{A}$ with the class of equations that are linearizable by a generalized Sundman transformation of the form

$$X = F(t,x), \quad dT = G(t,x)dt. \tag{1.5}$$

This is a consequence of Theorem 2 and the results of Duarte *et al.* in [6]. The three equations appearing in the examples of [6] are also considered here to illustrate our algorithm to determine first integrals.

We have already mentioned that there are equations in $\mathcal{A}$ that lack Lie point symmetries. Recently, several relationships between first integrals and λ-symmetries have been derived ([14,15]). In Sec. 4 we characterize the equations in $\mathcal{A}$ in terms of the λ-symmetries of the equation.

If the vector field ∂_x is a λ-symmetry for two different functions $\lambda_1 = -a_2\dot{x}+\beta_1(t,x)$ and $\lambda_2 = -a_2\dot{x}+\beta_2(t,x)$ then in Sec. 5 we prove that the equation necessarily admits two independent first integrals of the form (1.3) and therefore the equation can completely be integrated.

This complete integrability is illustrated for a large family of equations (Eq. (5.7)) that includes several significant equations of mathematical physics. These equations have been studied by several authors in order to find first integrals by using different approaches. The algorithms presented in this paper allow us to unify the treatment of these equations in a systematic and general procedure. We also characterize the equations of the form (1.4) with $a_2 = 0$ which have two independent first integrals of the form (1.3) as the second-order linear equations.

Several aspects related to the linearization of the equations studied in this paper will be dealt with in a separate work.

2. A First Characterization of the Equations in $\mathcal{A}$

We consider a second-order ordinary differential equation

$$\ddot{x} = M(t,x,\dot{x}). \tag{2.1}$$

If $w = A(t,x)\dot{x} + B(t,x)$ is a first integral of (2.1) and $Z = \partial_t + \dot{x}\partial_x + M(t,x,\dot{x})\partial_{\dot{x}}$ is the linear operator associated to this equation then $Zw = 0$; i.e.

$$Zw = (A_t\dot{x} + B_t) + (A_x\dot{x} + B_x)\dot{x} + M(t,x,\dot{x})A = 0. \tag{2.2}$$

This proves that the equation is of the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{2.3}$$

where the coefficients a_0, a_1, a_2 can be written in terms of A and B as

$$\begin{aligned} a_2(t,x) &= \frac{A_x}{A}, \\ a_1(t,x) &= \frac{B_x + A_t}{A}, \\ a_0(t,x) &= \frac{B_t}{A}. \end{aligned} \tag{2.4}$$

Conversely, we suppose that A and B are two functions verifying (2.4). If we define $w = A\dot{x} + B$ then w is a first integral of (2.3) since

$$D_t w = A \cdot \left(\ddot{x} + \frac{A_x}{A}\dot{x}^2 + \frac{B_x + A_t}{A}\dot{x} + \frac{B_t}{A}\right) = A \cdot (\ddot{x} + a_2\dot{x}^2 + a_1\dot{x} + a_0). \tag{2.5}$$

This proves the following theorem:

Theorem 1. *A function $w = A\dot{x} + B$ is a first integral of* (2.1) *if and only if Eq.* (2.1) *is of the form* (2.3) *and functions A and B satisfy* (2.4). *In this case A is an integrating factor of* (2.3).

Example 1. Ibragimov ([9]) considered the equation

$$\ddot{x} - \frac{\dot{x}^2}{x} - \frac{t^2 + t}{x}\dot{x} + 2t + 1 = 0 \tag{2.6}$$

as an example of an equation that does not admit Lie point symmetries but has an integrating factor $\mu(t,x,\dot{x}) = 1/x$, that could be found by using a method based on variational derivatives. It can be checked that the corresponding equations (2.4) are satisfied by $A(t,x) = 1/x$ and $B(t,x) = (t^2+t)/x$. Therefore $w = \dot{x}/x + (t^2+t)/x$ is a first integral of (2.6).

3. Intrinsic Characterization of the Equations in $\mathcal{A}$ and Construction of First Integrals

Equations (2.4) allows us readily to obtain necessary conditions on a_0, a_1 and a_2 in order that (2.3) have a first integral of the form $A(t,x)\dot{x} + B(t,x)$, with $A \neq 0$. This can be done by eliminating A, B and their derivatives from (2.4) by using the compatibility conditions

$$A_{xt} = A_{tx}, \quad B_{xt} = B_{tx}, \quad C_{xt} = C_{tx}, \tag{3.1}$$

where $C = B_x$ is an auxiliar function. Condition $C_{xt} = C_{tx}$ can be written in the form

$$CS_1 + AS_2 = 0, \tag{3.2}$$

where

$$\begin{aligned} S_1(t,x) &= a_{1x} - 2a_{2t}, \\ S_2(t,x) &= (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1. \end{aligned} \tag{3.3}$$

The analysis of (3.2) leads us to consider two cases:

Case I. If $S_1 = 0$ then necessarily $S_2 = 0$, since $A \neq 0$. In this case a necessary condition for the existence of two functions A and B satisfying (2.4) is $S_1 = S_2 = 0$.

Case II. If $S_1 \neq 0$ then we can write $S_2/S_1 = -C/A$, since $A \neq 0$. By derivating this expression, and by using (2.4), we get:

$$S_3(t,x) \equiv \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}) = 0, \tag{3.4}$$

$$S_4'(t,x) \equiv \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) - (a_{2t} - a_{1x}) + (a_0a_2 + a_{0x}) = 0. \tag{3.5}$$

By using (3.4) in (3.5), this second equation can be written as

$$S_4(t,x) \equiv \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x} = 0. \tag{3.6}$$

Therefore, in this case, a necessary condition for the existence of two functions A and B which satisfy (2.4) is $S_3 = S_4 = 0$.

We now investigate if the former conditions are sufficient for the existence of two functions A and B that verify (2.4). We consider the same two cases as above.

Case I. We suppose that the coefficients of (2.3) are such that $S_1 = S_2 = 0$. Condition $S_1 = 0$ implies that $a_{2t} = \frac{1}{2}a_{1x}$ and therefore the function S_2 can be written as $S_2(t,x) = f_x(t,x)$, where

$$f(t,x) = a_0a_2 + a_{0x} - \frac{1}{2}a_{1t} - \frac{1}{4}a_1^2. \tag{3.7}$$

Condition $S_2 = f_x = 0$ implies that f does not depend on x and, in this case, we can write $f = f(t)$.

Let $P = P(t,x)$ be a function such that

$$P_t = \frac{1}{2}a_1, \quad P_x = a_2. \tag{3.8}$$

The existence of such function P can be ensured, since the compatibility condition $[a_2]_t = [\frac{1}{2}a_1]_x$ is equivalent to condition $S_1 = 0$.

Let $g = g(t)$ be a nonzero solution of the linear equation

$$g''(t) + f(t)\cdot g(t) = 0 \tag{3.9}$$

and let $Q = Q(t,x)$ be a function such that

$$Q_t = a_0 \cdot g \cdot e^P, \quad Q_x = \left(\frac{1}{2}a_1 - \frac{g'}{g}\right) g \cdot e^P. \tag{3.10}$$

There exists a function Q which satisfies (3.10) due to the compatibility condition

$$\left[\left(\frac{1}{2}a_1 - \frac{g'}{g}\right) g \cdot e^P\right]_t = \left[a_0 \cdot g \cdot e^P\right]_x \tag{3.11}$$

is equivalent to (3.9).

If we define

$$A = g \cdot e^P, \quad B = Q \tag{3.12}$$

then it can be verified that A and B satisfy (2.4).

Case II. We suppose that $S_1 \neq 0$ and that the coefficients of (2.3) are such that $S_3 = S_4 = 0$. Since $S_3 = 0$, we have $[a_2]_t = [a_1 + S_2/S_1]_x$ and there exists a function $P = P(t,x)$ such that

$$P_t = a_1 + \frac{S_2}{S_1}, \quad P_x = a_2. \tag{3.13}$$

Let $Q = Q(t,x)$ be such that

$$Q_t = a_0 \cdot e^P, \quad Q_x = -\left(\frac{S_2}{S_1}\right) \cdot e^P. \tag{3.14}$$

There exists a function Q due to the compatibility condition

$$\left[a_0 \cdot e^P\right]_x = \left[-\left(\frac{S_2}{S_1}\right) \cdot e^P\right]_t \tag{3.15}$$

is equivalent to condition $S_4 = 0$.

If we define

$$A = e^P, \quad B = Q \tag{3.16}$$

then the functions A and B satisfy (2.4).

Therefore, we have proved the following theorem:

Theorem 2. *We consider an equation of the form* (2.3) *and let* S_1 *and* S_2 *be the functions defined by* (3.3). *The following alternatives hold*:

(1) *If* $S_1 = 0$ *then the equation has a first integral of the form* (1.3) *if and only if* $S_2 = 0$. *In this case* A *and* B *can be given by* (3.12), *where* P *is a solution of* (3.8), g *is a nonzero solution of* (3.9) *and* Q *is a solution of* (3.10).

(2) *If* $S_1 \neq 0$ *then the equation has a first integral of the form* (1.3) *if and only if* $S_3 = 0$ *and* $S_4 = 0$, *where* S_3 *and* S_4 *are the functions defined by* (3.4) *and* (3.6). *In this case* A *and* B *can be given by* (3.16), *where* P *is a solution of* (3.13) *and* Q *is a solution of* (3.14).

Duarte *et al.* studied in [6] some necessary conditions for which equation (2.3) is linearizable by means of a generalized Sundman transformation of the form

$$X = F(t,x), \quad dT = G(t,x)dt. \tag{3.17}$$

Conditions (10) and (11) in [6] can be written as

$$\begin{aligned} \widetilde{S}_1 &\equiv a_{1x} - 2a_{2t} = 0, \\ \widetilde{S}_2 &\equiv 2a_{0xx} - 2a_{1tx} + 2a_0 a_{2x} - a_{1x}a_1 + 2a_{0x}a_2 + 2a_{2tt} = 0. \end{aligned} \tag{3.18}$$

Some errata appear in expressions (12) and (13) in [6]. The correct expressions are

$$\begin{aligned} \widetilde{S}_3 &\equiv \widetilde{S}_2^2 - 2\widetilde{S}_{1t}\widetilde{S}_2 - 2\widetilde{S}_1^2 a_{1t} + 4\widetilde{S}_1^2 a_{0x} + 4\widetilde{S}_1^2 a_0 a_2 + 2\widetilde{S}_1\widetilde{S}_{2t} - \widetilde{S}_1^2 a_1^2 = 0, \\ \widetilde{S}_4 &\equiv -\widetilde{S}_{1x}\widetilde{S}_2 + \widetilde{S}_1^2 a_{1x} - 2\widetilde{S}_1^2 a_{2t} + \widetilde{S}_1\widetilde{S}_{2x} = 0. \end{aligned} \tag{3.19}$$

It can be verified that conditions (3.18) and (3.19) are equivalent to the conditions given in Theorem 2, since $\widetilde{S}_1 = S_1$, $\widetilde{S}_2 = 2S_2 + a_1 S_1$, $\widetilde{S}_3 = 4S_1^2\, S_4$ and $\widetilde{S}_4 = 2S_1^2\, S_3$.

Other generalizations of Sundman transformations have been considered in the literature (e.g. in [3]). Note also that the generalized Sundman transformation (3.17) was used to define so-called Sundman symmetries [7, 8] of ODEs. In [8] a rich structure of Sundman symmetries was reported for the equations in $\mathcal{A}$, and also for a large class of third-order ODEs. The complete classification of all linearizable third-order ODEs which can be transformed in $X''' = 0$ under the generalized Sundman transformation (3.17) was reported in [7]. For this classification the Sundman symmetries played a fundamental role.

Example 2. The three following equations have been considered in [6] as examples of equations that can be linearized by means of nonlocal transformations of the form (3.17).

$$\ddot{x} - \frac{2\dot{x}^2}{x} + \frac{2x}{t^2} = 0, \tag{3.20}$$

$$\ddot{x} + \left(t - \frac{1}{x}\right)\dot{x}^2 + 2x\dot{x} + \frac{x^2}{t} - \frac{x}{t^2} = 0, \tag{3.21}$$

$$\ddot{x} - \left(\tan(x) + \frac{1}{x}\right)\dot{x}^2 + \left(\frac{1}{t} - \frac{\tan(x)}{xt}\right)\dot{x} - \frac{\tan(x)}{t^2} = 0. \tag{3.22}$$

By the algorithm described in Theorem 2, it is possible to determine first integrals of the form $A(t,x)\dot{x} + B(t,x)$ for each equation listed above. It can be shown that the coefficients of equations (3.20) and (3.21) verify $S_1 = S_2 = 0$ and the coefficients of Eq. (3.22) satisfy $S_1 \neq 0$ but $S_3 = S_4 = 0$.

- For Eq. (3.20) the corresponding system (3.8) is

$$P_t = 0, \quad P_x = -2/x. \tag{3.23}$$

A solution of this system is given by $P(t,x) = -2\ln(x)$. The corresponding equation (3.9) is

$$g''(t) - 2/t^2 g(t) = 0. \tag{3.24}$$

Two linearly independent solutions of this equation are given by $g_1(t) = t^2$, $g_2(t) = 1/t$. By considering $g_1(t)$ the corresponding system (3.14) is

$$Q_t = \frac{2}{x}, \quad Q_x = -\frac{2t}{x^2}. \tag{3.25}$$

A solution of this system is $Q_1(t,x) = 2t/x$. By (3.12) a first integral of (3.20) is given by

$$w^1 = \frac{t^2}{x^2}\dot{x} + \frac{2t}{x}. \tag{3.26}$$

Similarly, by considering $g_2(t) = 1/t$ a second first integral of (3.20) is given by

$$w^2 = \frac{1}{tx^2}\dot{x} - \frac{1}{t^2 x}. \tag{3.27}$$

- For Eq. (3.21), a similar procedure can be followed to obtain two independent first integrals:

$$w^1(t,x,\dot{x}) = \frac{e^{tx}(x+t\dot{x})}{tx}, \quad w^2(t,x,\dot{x}) = \frac{e^{tx}(x+t\dot{x})}{x} - \mathrm{Ei}(tx), \tag{3.28}$$

where $\mathrm{Ei}(z)$ denotes the exponential integral function, i.e. a primitive of e^z/z.

- For Eq. (3.22) we have $S_1 \neq 0$, $S_3 = S_4 = 0$ and $S_2/S_1 = \tan(x) - x/(tx)$. The corresponding system (3.13) is

$$P_t = 0, \quad P_x = -\tan(x) - \frac{1}{x}. \tag{3.29}$$

A solution of this system is given by $P(t,x) = \ln(\cos(x)/x)$. The corresponding system (3.14) is

$$Q_t = -\frac{\sin(x)}{t^2 x}, \quad Q_x = \frac{\cos(x)(x - \tan(x))}{tx^2}, \tag{3.30}$$

a solution of which is $Q(t,x) = \sin(x)/(tx)$. By (3.16),

$$w(t,x,\dot{x}) = \frac{\cos(x)}{x}\dot{x} + \frac{\sin(x)}{tx} \tag{3.31}$$

is a first integral of (3.22).

4. First Integrals of the Form $A(t,x)\dot{x} + B(t,x)$ and λ-Symmetries

We recall [13] that the vector field $v = \partial_x$ is a λ-symmetry of (2.1) if and only if λ is a solution of the equation

$$M_x + \lambda M_{\dot{x}} = \lambda_t + \dot{x}\lambda_x + M\lambda_{\dot{x}} + \lambda^2. \tag{4.1}$$

We suppose that the coefficients a_0, a_1, a_2 of (2.3) are such that either

- $S_1 = S_2 = 0$, or
- $S_1 \neq 0$ and $S_3 = S_4 = 0$.

We now prove that (4.1) has solutions of the form $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$. In this case the following system must be compatible

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{4.2}$$

$$\beta_x + 2(a_2 + \alpha)\beta + a_{1x} + \alpha_t = 0, \tag{4.3}$$

$$\beta_t + \beta^2 + a_1\beta + a_{0x} - a_0\alpha = 0. \tag{4.4}$$

It is obvious that $\alpha(t,x) = -a_2(t,x)$ solves Eq. (4.2). For this α, Eqs. (4.3)–(4.4) are

$$\beta_x + a_{1x} - a_{2t} = 0, \tag{4.5}$$

$$\beta_t + \beta^2 + a_1\beta + a_{0x} + a_0 a_2 = 0. \tag{4.6}$$

Case I. If $S_1 = S_2 = 0$ then we have seen that the function f defined by (3.7) is such that $f = f(t)$. If $h(t)$ is any solution of the Riccati equation

$$h'(t) + h^2(t) + f(t) = 0, \tag{4.7}$$

then $\beta(t,x) = h(t) - \frac{1}{2}a_1(t,x)$ satisfies (4.5) and (4.6), since $S_1 = 0$ and $S_2 = 0$, respectively. Therefore $\lambda = -a_2\dot{x} + \beta$ is, in this case, such that ∂_x is a λ-symmetry of (2.3).

Case II. If $S_1 \neq 0$ and $S_3 = S_4 = 0$ then (3.4) and (3.6) prove that $\beta = S_2/S_1$ is a solution of (4.5)–(4.6). Therefore ∂_x is a λ-symmetry of (2.3) for $\lambda = -a_2\dot{x} + S_2/S_1$.

Conversely, let us suppose that ∂_x is a λ-symmetry for some function λ of the form $\lambda = -a_2\dot{x} + \beta(t,x)$. Then β satisfies Eqs. (4.5)–(4.6).

If we define $\gamma(t,x) = \beta(t,x) + \frac{1}{2}a_1(t,x)$ then γ satisfies the following system

$$\gamma_x + \frac{1}{2}S_1 = 0, \tag{4.8}$$

$$\gamma_t + \gamma^2 + f = 0, \tag{4.9}$$

where S_1 is defined by (3.3) and f is given by (3.7). Since $\gamma_{xt} = \gamma_{tx}$, Eqs. (4.8)–(4.9) imply that

$$S_1\gamma = -\frac{1}{2}S_{1t} + f_x. \tag{4.10}$$

This equation leads us to consider two cases, the same we have considered above.

Case I. If $S_1 = 0$ then (4.8) implies that $\gamma_x = 0$, and therefore $\gamma = \gamma(t)$, and that S_2 can be written as $S_2 = f_x$, where $f(t,x)$ is defined by (3.7). Similarly, (4.9) or (4.10) imply that $f = f(t)$. Therefore $f = f(t)$ and $S_2 = f_x = 0$.

Case II. If $S_1 \neq 0$ then γ is uniquely defined by (4.10):

$$\gamma(t,x) = \frac{1}{S_1}\left(-\frac{1}{2}S_{1t} + f_x\right). \tag{4.11}$$

Therefore, $\beta(t,x) = \gamma(t,x) - \frac{1}{2}a_1(t,x)$ is uniquely defined by the coefficients a_0, a_1, a_2 of (2.3) and it can be checked that $\beta = S_2/S_1$. Since β satisfies Eqs. (4.5)–(4.6), it is necessary that S_2/S_1 satisfies (3.4) and (3.6); i.e. $S_3 = S_4 = 0$.

This proves the following theorem:

Theorem 3. *We consider an equation of the form* (2.3) *and let S_1, S_2, S_3 and S_4 be the functions defined by* (3.3), (3.4) *and* (3.6).

The equation is such that either $S_1 = S_2 = 0$ or $S_3 = S_4 = 0$ if and only if ∂_x is a λ-symmetry of (2.3) *for some $\lambda = -a_2(t,x)\dot{x} + \beta(t,x)$.*

The following theorem sum up our former results:

Theorem 4. *The following conditions on an ODE of the form* (2.1) *are equivalent*:

(1) *Equation* (2.1) *admits a first integral of the form $A(t,x)\dot{x} + B(t,x)$.*
(2) *Equation* (2.1) *is of the form* (2.3) *and there exist two functions $A(t,x)$ and $B(t,x)$ that satisfy* (2.4).
(3) *Equation* (2.1) *is of the form* (2.3) *and its coefficient are such that either $S_1 = S_2 = 0$ or $S_3 = S_4 = 0$.*
(4) *Equation* (2.1) *is of the form* (2.3) *and ∂_x is a λ-symmetry for some function λ of the form $\lambda = -a_2(t,x)\dot{x} + \beta(t,x)$.*

Example 3. We consider the equation

$$\ddot{x} - \frac{2}{x}\dot{x}^2 + \left(\frac{4}{t} - e^{t/x}t\right)\dot{x} - tx^2 - \frac{2x}{t^2} + e^{t/x}\left(\frac{3x^2}{t} + x\right) = 0. \tag{4.12}$$

This equation does not have Lie point symmetries and hence it cannot be integrated by Lie's method ([2]). This equation is of the form (2.3) and its coefficients satisfy $S_1 \neq 0$, and $S_3 = S_4 = 0$. Hence statement 3 of Theorem 4 is satisfied. Consequently:

(1) Equation (4.12) has a first integral of the form $w(t,x,\dot{x}) = A(t,x)\dot{x} + B(t,x)$, where A and B satisfy (2.4) and can be calculated by the algorithm of Sec. 2 (Case II):

$$A(t,x) = \frac{t^3}{x^2}, \qquad B(t,x) = e^{t/x}t^3 - \frac{t^2}{x} - \frac{t^5}{5}. \tag{4.13}$$

(2) Equation (4.12) admits ∂_x as a λ-symmetry for $\lambda = -a_2\dot{x} + S_2/S_1$:

$$\lambda(t,x,\dot{x}) = \frac{2}{x}\dot{x} - \frac{1}{t} + e^{t/x}t. \tag{4.14}$$

5. Complete Integrability in Case I ($S_1 = S_2 = 0$)

We observe that in Case I Eq. (3.9) has two linearly independent solutions $g_1(t)$ and $g_2(t)$. We denote by $W(g_1, g_2)$ the Wronskian of the functions g_1 and g_2. We can construct two functions Q^1 and Q^2 satisfying (3.10):

$$Q^i_t = a_0 \cdot g_i \cdot e^P, \quad Q^i_x = \left(\frac{1}{2}a_1 - \frac{g_i'}{g_i}\right) \cdot g_i \cdot e^P, \quad (i = 1,2). \tag{5.1}$$

Two first integrals of Eq. (2.3) are given by

$$w^1 = g_1\, e^P \dot{x} + Q^1, \quad w^2 = g_2\, e^P \dot{x} + Q^2. \tag{5.2}$$

It can be verified that

$$\begin{vmatrix} w^1_x & w^1_{\dot{x}} \\ w^2_x & w^2_{\dot{x}} \end{vmatrix} = e^P \cdot W(g_1, g_2) \neq 0. \tag{5.3}$$

This proves that in Case I, w^1 and w^2 are two functionally independent first integrals of Eq. (2.3) that are of the form (1.3).

Conversely, let us suppose that Eq. (2.3) has two functionally independent first integrals of the form (1.3):

$$w^1(t,x,\dot{x}) = A^1(t,x)\dot{x} + B^1(t,x), \quad w^2(t,x,\dot{x}) = A^2(t,x)\dot{x} + B^2(t,x). \tag{5.4}$$

The functions A^i and B^i, $i = 1,2$, must satisfy system (2.4):

$$\begin{aligned} a_2 &= \frac{A^1_x}{A^1} = \frac{A^2_x}{A^2}, \\ a_1 &= \frac{A^1_t + B^1_x}{A^1} = \frac{A^2_t + B^2_x}{A^2}, \\ a_0 &= \frac{B^1_t}{A^1} = \frac{B^2_t}{A^2}. \end{aligned} \tag{5.5}$$

The vector field ∂_x is a λ^i-symmetry for

$$\lambda^i = -\frac{w^i_x}{w^i_{\dot{x}}} = -a_2\dot{x} - \frac{B^i_x}{A^i}, \quad (i = 1,2). \tag{5.6}$$

Since w^1 and w^2 are functionally independent, necessarily $\beta^1 = -B^1_x/A^1 \neq -B^2_x/A^2 = \beta^2$ and therefore the system (4.5)–(4.6) has two different solutions. This cannot happen in Case II since in this case the function β such that ∂_x is a λ-symmetry for $\lambda = -a_2\dot{x} + \beta$ is uniquely determined by $\beta = S_2/S_1$. Therefore $S_1 = S_2 = 0$ and we have proved the following result.

Theorem 5. *The following conditions on an equation of the form* (2.3) *are equivalent*:

(1) *The equation admits two functionally independent first integrals of the form* (1.3).
(2) $S_1 = S_2 = 0$.
(3) *The vector field* ∂_x *is a* λ^1*-symmetry and a* λ^2*-symmetry for some functions* $\lambda^1 = -a_2\dot{x} + \beta^1$ *and* $\lambda^2 = -a_2\dot{x} + \beta^2$ *with* $\beta^1 \neq \beta^2$.

5.1. *Some examples*

We now apply Theorem 5 to two families of second-order equations.

1. An equation of the form

$$\ddot{x} + a_2(x)\dot{x}^2 + a_1(t)\dot{x} = 0 \tag{5.7}$$

satisfies any of the conditions given in Theorem 5. By (3.7), $f(t) = -\frac{1}{4}a_1(t)^2 - \frac{1}{2}a_1'(t)$. Two linearly independent solutions $g_1(t)$ and $g_2(t)$ of the corresponding equation (3.9) are given by

$$g_1(t) = \exp\left(\frac{1}{2}\int a_1(t)dt\right), \quad g_2(t) = g_1(t)\int \frac{dt}{g_1^2(t)}. \tag{5.8}$$

We define

$$\begin{aligned} h_1(t) &= \exp\left(-\int a_1(t)dt\right), \quad H_1(t) = \int h_1(t)dt, \\ h_2(x) &= \exp\left(\int a_2(x)dx\right), \quad H_2(x) = \int h_2(x)dx. \end{aligned} \tag{5.9}$$

A solution $P(t,x)$ of the corresponding system (3.8) can be written as $P(t,x) = \ln(g_1(t) \cdot h_2(x))$. Two particular solutions of systems (5.1) are given by $Q^1(t,x) = 0$ and $Q^2(t,x) = -H_2(x)$. Two functionally independent first integrals of Eq. (5.7) are given by (5.10):

$$w^1(t,x,\dot{x}) = \frac{h_2(x)}{h_1(t)}\,\dot{x}, \quad w^2(t,x,\dot{x}) = H_1(t)\cdot\frac{h_2(x)}{h_1(t)}\,\dot{x} - H_2(x). \tag{5.10}$$

By Theorem 1, $\mu_1(t,x) = h_2(x)/h_1(t)$ and $\mu_2(t,x) = H_1(t)\cdot\mu_1(t,x)$ are integrating factors of (5.7).

The general solution of Eq. (5.7) could be found by eliminating $\dot{x}$ from $w^1 = C_1$ and $w^2 = C_2$, $C_1, C_2 \in \mathbb{R}$:

$$C_1\, H_1(t) + H_2(x) = C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{5.11}$$

The vector field ∂_x is a λ^1-symmetry and a λ^2-symmetry for

$$\lambda^1(t,x,\dot{x}) = -a_2(x)\dot{x}, \quad \lambda^2(t,x,\dot{x}) = -a_2(x)\dot{x} + \frac{h_1(t)}{H_1(t)}. \tag{5.12}$$

As a consequence, we have proved the following corollary:

Corollary 1. *Any equation of the form* (5.7) *admits two functionally independent first integrals of the form* (1.3) *that are given by* (5.10), *where* h_1, h_2, H_1 *and* H_2 *are defined by* (5.9). *For Eq.* (5.7), *the vector field* ∂_x *is a* λ^1*-symmetry and a* λ^2*-symmetry for* λ^1, λ^2 *given by* (5.12).

We now consider three equations of the form (5.7) that have previously been used in the literature to illustrate several integration strategies. These equations can be solved by using Corollary 1:

(1) The equation

$$x\ddot{x} = 3\dot{x}^2 + \frac{x}{t}\dot{x} \tag{5.13}$$

was originally derived by Buchdahl [1] in the theory of general relativity. Duarte *et al.* [6] deduced a first integral by applying the extended Prelle–Singer method and Chandrasekar *et al.* derived a second one in [4].

By (5.10)

$$w^1(t,x,\dot{x}) = \frac{\dot{x}}{tx^3}, \quad w^2(t,x,\dot{x}) = \frac{t}{x^3}\dot{x} + \frac{1}{x^2} \tag{5.14}$$

are two independent first integrals.

(2) The equation

$$\ddot{x} + \frac{\dot{x}^2}{x} + 3\frac{\dot{x}}{t} = 0 \tag{5.15}$$

has been considered in [9] to illustrate a method, based on variational derivatives, to find two integrating factors and therefore the general solution of the equation. The method in [9] requires to solve a system of two coupled second-order partial differential equations. By using Corollary 1, two functionally independent first integrals are given by (5.10):

$$w^1(t,x,\dot{x}) = t^3 x\dot{x}, \quad w^2(t,x,\dot{x}) = tx\dot{x} + x^2. \tag{5.16}$$

Two integrating factors of Eq. (5.15) can readily be found by Theorem 1:

$$\mu_1(t,x) = t^3 x, \quad \mu_2(t,x) = tx. \tag{5.17}$$

(3) Equation

$$tx\ddot{x} + (2tx + t)\dot{x}^2 + x\dot{x} = 0 \tag{5.18}$$

was proposed in [5] to show that the extended Prelle–Singer method can be applied to find a non-rational first integral $w = x + \frac{1}{2}\ln(tx\dot{x})$. Since (5.18) has the form (5.7) and a complete system of first integrals is given by the functions w^1, w^2 defined by (5.10), w must be a function of w^1 and w^2 (that are rational first integrals). In fact, it can be checked that $w = \ln(w^1)/2$, where $w^1 = te^{2x}x\dot{x}$. A second independent first integral is given by $w^2 = e^{2x}(t\,\ln(t)\,x\dot{x} + (1-2x)/4)$.

2. Theorem 5 can be used to obtain a characterization of second-order linear ODEs. Suppose that an equation of the form

$$\ddot{x} + a_1(t,x)\dot{x} + a_0(t,x) = 0 \tag{5.19}$$

admits two independent first integrals of the form (1.3). By Theorem 2, the coefficients a_0 and a_1 of (5.19) must satisfy

$$S_1 = a_{1x} = 0 \quad \text{and} \quad S_2 = -a_1 a_{1x} + a_{0xx} - a_{1tx} = 0. \tag{5.20}$$

This implies that $a_1 = a_{11}(t)$ and $a_{0xx} = 0$. Therefore $a_0(t,x) = a_{01}(t)x + a_{02}(t)$ for some functions $a_{01}(t)$ and $a_{02}(t)$ and (5.19) has the form

$$\ddot{x} + a_{11}(t)\dot{x} + a_{01}(t)x + a_{02}(t) = 0. \tag{5.21}$$

Conversely, it is obvious that any equation of the form (5.21) admits two independent first integrals of the form (1.3). We have achieved the following characterization of the second-order linear ODEs in terms of first integrals:

Corollary 2. *An equation of the form* (5.19) *has two independent first integrals of the form* (1.3) *if and only if it is a linear equation, i.e. it has the form* (5.21).

It must be observed that under the conditions of Corollary 2

$$f(t) = \frac{1}{4}(-a_1(t,x)^2 + 4a_{0x}(t,x) - 2a_{1t}(t,x)) = -\frac{a_{11}(t)}{4} - \frac{a_{11}'(t)}{2} + a_{01}(t). \tag{5.22}$$

Therefore, for equations of the form (2.3), the function f defined by (3.7) generalizes the usual invariant that appears in the study of equivalence transformations of second-order linear ODEs ([9]).

6. Conclusions

We have characterized the second-order ODEs that admit first integrals of the form $A(t,x)\dot{x} + B(t,x)$ through an easy-to-check criterion expressed in terms of functions S_1, S_2, S_3 and S_4 given by (3.3) and (3.4)–(3.6).

This criterion and a systematic procedure to calculate such first integrals have been derived in Theorem 2. The considered equations can also be characterized as the equations of the form (1.4) that admit the vector field ∂_x as a λ-symmetry for some $\lambda = -a_2\dot{x}+\beta(t,x)$.

The class of equations such that $S_1 = S_2 = 0$ is composed of the equations that admits two independent first integrals of the form (1.3). The determination of these first integrals requires the solution of a second-order linear ODE (3.9). For these equations there are infinitely many functions λ for which the vector field ∂_x is a λ-symmetry. They are of the form $\lambda = -a_2\dot{x} - a_1/2 + h$, where h denotes a particular solution of the Riccati equation (4.7).

If $S_1 \neq 0$ and $S_3 = S_4 = 0$, the equations have an unique (up to multipliers) first integral of the form (1.3), that can readily be obtained by quadratures. These equations can also be characterized as the equations of the form (1.4) that admit the vector field ∂_x as a λ-symmetry for $\lambda = -a_2\dot{x} + \beta$, where β is uniquely defined by $\beta = S_2/S_1$.

The equations classified in this paper are interestingly related to the equations that can be linearized by generalized Sundman transformations of the form (1.5). This relationship and other aspects of the problem of linearization are studied in detail in a forthcoming paper.

Acknowledgment

The support of DGICYT project MTM2006-05031, Junta de Andalucía group FQM-201 and project P06-FQM-01448 are gratefully acknowledged.

References

[1] H. A. Buchdahl, A relativistic fluid spheres resembling the Emden polytrope of index 5, *Ap. J.* **140** (1964) 1512–1516.

[2] D. Catalano and P. Morando, Local and nonlocal solvable structures in the reduction of ODEs, *J. Phys. A: Math. Theor.* **42** (2009) 035210.

[3] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, A unification in the theory of linearization of second-order nonlinear ordinary differential equations, *J. Phys. A: Math. Gen.* **39** (2006) L69–L76.

[4] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations, *Proc. R. Soc. A* **461** (2005) 2451–2476.

[5] L. G. S. Duarte, S. E. S. Duarte, L. A. C. P. da Mota and J. E. F. Skea, Solving second-order ordinary differential equations by extending the Prelle–Singer method, *J. Phys. A: Math. Gen.* **34** (2001) 3015–3024.

[6] L. G. S. Duarte, I. C. Moreira and F. C. Santos, Linearization under non-point transformations, *J. Phys. A: Math. Gen.* **27** (1994) L739–L743.

[7] N. Euler and M. Euler, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, *J. Nonlinear Math. Phys.* **11** (2004) 399–421.

[8] N. Euler, T. Wolf, P. G. L. Leach and M. Euler, Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case $X''' = 0$, *Acta Appl. Math.* **76** (2003) 89–115.

[9] N. H. Ibragimov, A practical course in differential equations and mathematical modelling: Classical and new methods, nonlinear mathematical models, symmetry and invariance principles, ALGA Publications, Karlskrona (2006).

[10] N. H. Ibragimov and F. Magri, Geometric proof of Lie's linearization theorem, *Nonlinear Dynamics* **36** (2004) 41–46.

[11] S. Lie, Klassifikation und integration von gewhnlichen differentialgleichungen zwischen x, y die eine gruppe von transformationen gestatten. III, *Arch. Mat. Naturvidenskab* **8** (1883) 371–458.

[12] S. V. Meleshko, On linearizaton of third-order ordinary differential equations, *J. Phys. A: Math. Gen.* **39** (2006) 15135–15145.

[13] C. Muriel and J. L. Romero, New methods of reduction for ordinary differential equations, *IMA J. Appl. Math.* **66** (2001) 111–125.

[14] C. Muriel and J. L. Romero, Integrating factors and λ-symmetries, *J. Nonlinear Math. Phys.* **15** (2008) 290–299.

[15] C. Muriel and J. L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42** (2009) 365207.

8 SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS AND FIRST INTEGRALS OF THE FORM $C(t) + \frac{1}{(A(t,x)\dot{x}+B(t,x))}$ by C. MURIEL, J. L. ROMERO (2011)

Second-Order Ordinary Differential Equations with First Integrals of the Form $C(t) + 1/(A(t, x)\dot{x} + B(t, x))$

Article *in* Journal of Nonlinear Mathematical Physics · May 2011

DOI: 10.1142/S1402925111001398

C. Muriel
Universidad de Cádiz

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, Suppl. 1 (2011) 237–250

DOI: 10.1142/S1402925111001398

SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH FIRST INTEGRALS OF THE FORM $C(t) + 1/(A(t, x)\dot{x} + B(t, x))$

C. MURIEL* and J. L. ROMERO†

Mathematics Department, Science Faculty, University of Cádiz
Avda. República Saharaui s/n 11510, Puerto Real, Cádiz, Spain
**concepcion.muriel@uca.es*
†juanluis.romero@uca.es

Received 30 September 2010
Accepted 10 November 2010

We study the class of the ordinary differential equations of the form $\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0$, that admit $v = \partial_x$ as λ-symmetry for some $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$. This class coincides with the class of the second-order equations that have first integrals of the form $C(t)+1/(A(t,x)\dot{x}+B(t,x))$. We provide a method to calculate the functions A, B and C that define the first integral. Some relationships with the class of equations linearizable by local and a specific type of nonlocal transformations are also presented.

Keywords: Ordinary differential equations; symmetries; first integrals; linearization.

Mathematics Subject Classification 2000: 34A05, 34A34, 34A25

1. Introduction

In this paper we consider ordinary differential equations (ODEs) of the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{1.1}$$

where t is the independent variable of the equation, x is the dependent variable and overdot denotes derivation with respect to t.

This class of equations has been studied from several points of view: integrating factors, first integrals, linearizing transformations, λ-symmetries, etc. There are many relationships between the equations that admit some of these kinds of objects. In [10,11] it is shown that, for general second-order equations, the knowledge of a λ-symmetry permits the determination of an integrating factor or a first integral.

In [12] there appears a characterization of second-order equations that admit first integrals of the form $A(t,x)\dot{x} + B(t,x)$. These equations are necessarily of the form (1.1). This class of equations is the same than the class of equations of the form (1.1) that admit $v = \partial_x$ as λ-symmetry for some $\lambda = -a_2(t,x)\dot{x} + \beta(t,x)$.

In this paper we complete the study of equations of the form (1.1) that admit $v = \partial_x$ as λ-symmetry for some $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$. The main result of the paper is a characterization of that class of equations as the class of equations (1.1) that have first integrals of the form

$$I = \frac{1}{A(t,x)\dot{x} + B(t,x)} + C(t). \tag{1.2}$$

This characterization raises the problem of the determination of second-orden equations that admit first integrals of the form

$$I = \frac{1}{A(t,x)\dot{x} + B(t,x)} + C(t,x) = \frac{A_1(t,x)\dot{x} + B_1(t,x)}{A(t,x)\dot{x} + B(t,x)} \tag{1.3}$$

where $A_1 = CA$ and $B_1 = 1 + CB$. However, it can be checked that the class of equations that admit (1.3) as first integral are necessarily of the form

$$\ddot{x} + a_3(t,x)\dot{x}^3 + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0 \tag{1.4}$$

and that $a_3 = 0$ if and only $C_x(t,x) = 0$. The class of Eq. (1.4) is out of the scope of this paper and these equations will be studied in a forthcoming paper.

This paper is organized as follows. In Sec. 2 we establish some notations and recall the known results we need to complete the characterization of equations that admit $v = \partial_x$ as λ-symmetry for some $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$.

In Sec. 3, and in order to simplify our study, we obtain a *canonical* reduction of the equations under consideration. This lets us to obtain a characterization of these equations in terms of first integrals of the form (1.2). In this section we also provide a method to obtain the functions $A(t,x), B(t,x)$ and $C(t)$ that define the first integral (1.2). This method is illustrated with an example.

In Sec. 4 we indicate the steps that could be used to determine whether or not a given Eq. (1.1) is in the class under study. This method could also be used to obtain an intrinsic characterization of the equations. However, a complete study of this intrinsic characterization is rather involved and will be considered in a separate paper.

In Sec. 5 we relate the results in this paper with the problem of the linearization through local and nonlocal transformations. In particular, it is shown that the equations that can be linearized by some local transformations constitute a strict subclass of the equations studied in this paper.

2. Preliminaries

If a second-order ordinary differential equation of the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0 \tag{2.1}$$

admits the vector field $v = \partial_x$ as λ-symmetry for some function λ of the form

$$\lambda(t,x,\dot{x}) = \alpha(t,x)\dot{x} + \beta(t,x) \tag{2.2}$$

then the functions α and β must satisfy the following system of determining equations:

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{2.3}$$

$$\beta_x + 2(a_2 + \alpha)\beta + a_{1x} + \alpha_t = 0, \tag{2.4}$$

$$\beta_t + \beta^2 + a_1\beta - a_0\alpha + a_{0x} = 0. \tag{2.5}$$

The equations of the form (1.1) for which the corresponding system (2.3)–(2.5) admits some solution (α_0, β_0) such that $\alpha_0 = -a_2$ have been studied in [12, 14]. The class of these equations was denoted by $\mathcal{A}$ in [12]. The coefficients of the equations in $\mathcal{A}$ must satisfy one of the two following alternatives: $S_1 = 0$ and $S_2 = 0$ where

$$\begin{aligned} S_1(t,x) &= a_{1x} - 2a_{2t}, \\ S_2(t,x) &= (a_0 a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1, \end{aligned} \tag{2.6}$$

or, if $S_1 \neq 0$, $S_3 = 0$ and $S_4 = 0$, where

$$\begin{aligned} S_3(t,x) &= \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}), \\ S_4(t,x) &= \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0 a_2 + a_{0x}. \end{aligned} \tag{2.7}$$

Let us introduce the following notation:

Definition 2.1. We define $\mathcal{A}_1$ as the class of the equations of the form (1.1) whose coefficients satisfy $S_1 = S_2 = 0$ and $\mathcal{A}_2$ will denote the class of the equations of the form (1.1) whose coefficients satisfy $S_1 \neq 0$ and $S_3 = S_4 = 0$.

We define $\mathcal{B}$ as the class of the equations of the form (1.1) for which system (2.3)–(2.5) is compatible, i.e., the equations of the form (1.1) that admits $v = \partial_x$ as λ-symmetry for some λ of the form (2.2).

It is clear that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A} \subset \mathcal{B}$, but there are equations in $\mathcal{B}$ that are not in $\mathcal{A}$. This is the case of the family of equations ([13])

$$\ddot{x} + \frac{b'(t)}{2x} + \frac{b(t)^2}{4x^3} + a(t)x = 0, \quad b(t) \neq 0. \tag{2.8}$$

It can be checked that the equations in (2.8) with $b'(t) \neq 0$ and

$$\left(\frac{b(t)}{b'(t)}\right)''' + 4\,a(t)\left(\frac{b(t)}{b'(t)}\right)' + 2a'(t)\left(\frac{b(t)}{b'(t)}\right) \neq 0 \tag{2.9}$$

do not have Lie point symmetries. When $b'(t) = 0$, Eq. (2.8) is the well-known Ermakov–Pinney equation ([11]). It can be checked that

$$S_1 = 0, S_2 = \frac{b'(t)x^2 + 3b(t)^2}{x^5} \neq 0 \tag{2.10}$$

and therefore the Eqs. (2.8) do not belong to class $\mathcal{A}$. Since $\alpha = 1/x$ and $\beta = b(t)/x^2$ solve the corresponding system (2.3)–(2.5), the Eqs. (2.8) belong to class $\mathcal{B}$.

Some properties and characterizations of the equations in $\mathcal{A}$ appear in [12]. For the equations in $\mathcal{A}_1$ there are infinitely many solutions of system (2.3)–(2.5) of the form $\alpha_0 = -a_2, \beta_0$ while for the equations in $\mathcal{A}_2$ system (2.3)–(2.5) has a unique solution of the form $\alpha_0 = -a_2, \beta_0$. The equations in $\mathcal{A}$ are the only second-order equations that admit first integrals of the form $A(t,x)\dot{x} + B(t,x)$. Only the equations in the subclass $\mathcal{A}_1$ admit two functionally independent first integrals of this form.

Several aspects on the linerization of the equations in $\mathcal{A}$ have been addressed in [14]. All the equations in subclass $\mathcal{A}_1$ can be linearized by local transformations, i.e., they pass Lie's test of linearization ([6–8]). On the contrary, none equation belonging to $\mathcal{A}_2$ passes Lie's test of linearization. Nevertheless, the equations in $\mathcal{A}_2$ have been characterized as the unique second-order equations that can be linearized through special nonlocal transformations, known in the literature as *generalized Sundman transformations* (see [1–5] and references therein).

In what follows we address the study of properties of the equations in $\mathcal{B}$, dealing with the following topics:

- Characterization of the equations in $\mathcal{B}$.
- Identification of first integrals of the equations in $\mathcal{B}$ and computational methods for them.
- Linearization by nonlocal and local transformations of the equations in $\mathcal{B}$.

3. Order Reduction of Equations in $\mathcal{B}$ Through λ-symmetries

Let us assume that Eq. (1.1) admits the vector field $v = \partial_x$ as λ-symmetry for some function λ of the form (2.2). Let $A^0 = A^0(t,x) \neq 0$ and $B^0 = B^0(t,x)$ be two functions such that

$$A^0_x + \alpha A^0 = 0, \qquad B^0_x + \beta A^0 = 0. \tag{3.1}$$

It is clear that $w_0(t,x,\dot{x}) = A^0(t,x)\dot{x} + B^0(t,x)$ is an invariant of $v^{[\lambda,(1)]} = \partial_x + \lambda\partial_{\dot{x}}$. Since v is a λ-symmetry of (1.1), in terms of $\{t, w_0, \dot{w}_0\}$ Eq. (1.1) takes (locally) the form $\dot{w}_0 = \Delta(t, w_0)$ (see [9] for details). Due to the form of Eq. (1.1), necessarily $\Delta(t,w_0) = H_2(t)w_0^2 + H_1(t)w_0 + H_0(t)$. Let us prove that it is possible to choose suitable solutions A and B of (3.1) for which $\Delta(t, w_0)$ takes simpler forms.

Let $k_2 = k_2(t)$ be such that $k_2' - H_2k_2^2 - H_1k_2 - H_0 = 0$ and let $k_1 = k_1(t)$ be a nonzero function such that $k_1' - (2H_2k_2 + H_1)k_1 = 0$. Since $A = A^0/k_1$ and $B = (B^0 - k_2)/k_1$ are also solutions of system (3.1), $w = A\dot{x} + B$ is an invariant of $v^{[\lambda,(1)]}$. It can be checked that in terms of $\{t, w, \dot{w}\}$ Eq. (1.1) becomes, locally,

$$\dot{w} + J(t)w^2 = 0, \tag{3.2}$$

where $J = -H_2k_1$. When in (3.2) w and $\dot{w}$ are expressed in terms of $\{t, x, \dot{x}, \ddot{x}\}$, the following result is obtained:

Theorem 3.1. *If Eq. (1.1) belongs to the class $\mathcal{B}$ then there exist some functions $A = A(t,x) \neq 0$ and $B = B(t,x)$ and some function $J = J(t)$ such that*

$$\begin{aligned} a_2 &= \frac{A_x}{A} + JA, \\ a_1 &= \frac{A_t}{A} + \frac{B_x}{A} + 2BJ, \\ a_0 &= \frac{B_t}{A} + \frac{B^2}{A}J. \end{aligned} \tag{3.3}$$

In terms of $\{t, w, \dot{w}\}$, where $w = A(t,x)\dot{x} + B(t,x)$, Eq. (1.1) becomes

$$\dot{w} + J(t)w^2 = 0. \tag{3.4}$$

Equation (1.1) belongs to $\mathcal{A}$ if and only if $J(t) = 0$.

In order to check if a given second-order ODE of the form (1.1) belongs to $\mathcal{B}$, the analysis of the compatibility of corresponding system (2.3)–(2.5) can be done in a systematic way. Equation (2.3) is a Riccati-type equation with respect to x with a known particular solution $\alpha = -a_2$. Hence its general solution, depending on an arbitrary function $\rho_1(t)$, can readily be obtained. After substitution, Eq. (2.4) becomes a linear first order ODE, where t is considered as a parameter. Its general solution depends on a function $\rho_2(t)$. Finally, Eq. (2.5) is used to set appropriated functions ρ_1 and ρ_2 in order to get solutions for α and β. Next example illustrates this procedure and shows how to construct the associated reduced Eq. (3.4).

Example 3.1. Let us consider the second-order equation

$$\ddot{x} + \left(x + \frac{1}{x}\right)\dot{x}^2 + \left(t\left(2x + \frac{1}{x}\right) - \frac{1}{t}\right)\dot{x} + x\,t^2 = 0. \tag{3.5}$$

The corresponding Eq. (2.3) becomes

$$\alpha_x + \alpha^2 + \left(x + \frac{1}{x}\right)\alpha - \frac{1}{x^2} + 1 = 0. \tag{3.6}$$

This is a Riccati-type equation and $\alpha = -a_2 = -(x + 1/x)$ is a particular solution; its general solution is given by $\alpha = -x/(e^{\frac{x^2}{2}}\rho_1(t) + 1) - 1/x$ and $\alpha = -1/x$ is a singular solution. For simplicity, we try to find solutions for $\alpha = -1/x$. Then (2.4) becomes

$$\beta_x + 2x\beta + t\left(2 - \frac{1}{x^2}\right) = 0. \tag{3.7}$$

The general solution of this linear equation is given by $\beta(t,x) = e^{-x^2}\rho_2(t) - t/x$. The corresponding Eq. (2.5) becomes

$$tx\rho_2(t)^2 + e^{x^2}\left(2x^2\rho_2(t)t^2 - \rho_2(t)t^2 + x\rho_2'(t)t - x\rho_2(t)\right) = 0. \tag{3.8}$$

Equation (3.8) is satisfied for $\rho_2(t) = 0$. Therefore $\alpha = -1/x$ and $\beta = -t/x$ solve the corresponding system (2.3)–(2.5), i.e., $v = \partial_x$ is a λ-symmetry of (3.5) for $\lambda = -(\dot{x} + t)/x$. This proves that Eq. (3.5) belongs to $\mathcal{B}$.

Now we choose any pair of particular solutions of the corresponding system (3.1):

$$A_x^0 - \frac{A^0}{x} = 0, \qquad B_x^0 - \frac{t}{x}A^0 = 0, \tag{3.9}$$

for example $A^0 = x$ and $B^0 = tx$, and define $w^0 = x\,(\dot{x} + t)$. In terms of $\{t, w_0, \dot{w}_0\}$ Eq. (3.5) becomes $\dot{w}_0 = H_2(t)w_0^2 + H_1(t)w_0 + H_0(t)$, where $H_2(t) = -1, H_1(t) = 1/t, H_0(t) = 0$.

Since $k_2 = 2/t$ is a particular solution of $k_2' - H_2k_2^2 - H_1k_2 - H_0 = 0$ and $k_1 = 1/t^3$ solves $k_1' - (2H_2k_2 + H_1)k_1 = 0$, we finally get that

$$A = \frac{A^0}{k_1} = xt^3, B = \frac{(B^0 - k_2)}{k_1} = t^4x - 2t^2 \text{ and } J = -H_2k_1 = \frac{1}{t^3} \tag{3.10}$$

solve system (3.3) for Eq. (3.5).

The general solution of the corresponding reduced Eq. (3.4) is given by

$$w(t) = \frac{2t^2}{2C_1t^2 - 1}, C_1 \in \mathbb{R}. \tag{3.11}$$

Substituting w by $A\dot{x} + B$ in (3.11), the general solution of Eq. (3.5) arises from the general solution of the Abel equation of the second kind

$$x(\dot{x} + t) = \frac{4C_1t}{2C_1t^2 - 1} \tag{3.12}$$

and can be written in implicit form as

$$\sqrt{2}\varphi\left(\rho(t, x, C_1)\right) - \frac{4C_1}{2C_1t^2 - 1}\exp\left(\rho(t, x, C_1)^2\right) = C_2, C_2 \in \mathbb{R}, \tag{3.13}$$

where $\rho(t, x, C_1) = \frac{\sqrt{2}}{8C_1}(2C_1t^2 - 1 + 4C_1x)$ and $\varphi'(a) = \exp(a^2)$.

3.1. *First integrals of the equations in $\mathcal{B}$*

Let us assume, as above, that Eq. (1.1) is in $\mathcal{B}$ and let us denote by Z the linear operator associated to Eq. (1.1), i.e., $Z = \partial_t + \dot{x}\partial_x - M(t, x, \dot{x})\partial_{\dot{x}}$ where

$$M(t, x, \dot{x}) = a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x). \tag{3.14}$$

By Theorem 3.1, such equation can be written as

$$\dot{w} + J(t)w^2 = 0, \tag{3.15}$$

where $w = A(t, x)\dot{x} + B(t, x)$ and A, B and J satisfy system (3.3). Equation (3.15) can be written as

$$D_t\left(\frac{1}{w} + C(t)\right) = 0, \tag{3.16}$$

where $C(t)$ is any primitive of $-J(t)$. Therefore, by writing $1/w + C(t)$ in terms of the original variables of the equation, we deduce that

$$I(t, x, \dot{x}) = \frac{1}{A(t, x)\dot{x} + B(t, x)} + C(t) \tag{3.17}$$

is a first integral of Z, the linear operator associated to Eq. (1.1). This can also be directly proven by using system (3.3).

Conversely, if (3.17) is a first integral of (1.1) for some $A = A(t, x), B = B(t, x)$ and $C = C(t)$ then

$$0 = Z(I) = \frac{-AM + (A_t + A_x\dot{x})\dot{x} + B_t + B_x\dot{x}}{(A\dot{x} + B)^2} + C' \tag{3.18}$$

and therefore

$$M(t, x, \dot{x}) = \frac{(A^2 + A_x)\dot{x}^2 + (A_t + B_x + 2ABC')\dot{x} + B_t + C'B^2}{A}. \tag{3.19}$$

Equations (3.14) and (3.19) imply that A, B and $J = -C'$ solve system (3.3).

The following result has been proven:

Theorem 3.2. *If system (3.3) is satisfied for some functions $A = A(t,x), B = B(t,x)$ and $J = J(t)$ and $C = C(t)$ is any primitive of $-J(t)$, then $I = 1/(A\dot{x} + B) + C$ is a first integral of (1.1). Conversely, if $I = 1/(A\dot{x} + B) + C$ is a first integral of (1.1) for some $A = A(t,x), B = B(t,x)$ and $C = C(t)$ then A, B and $J(t) = -C'(t)$ solve system (3.3).*

Corollary 3.1. *The equations in $\mathcal{B}$ are characterized as the second-order ordinary differential equations that admit first integrals of the form $I = 1/(A\dot{x} + B) + C$, for some functions $A = A(t,x), B = B(t,x)$ and $C = C(t)$.*

Example 3.2. Theorem 3.2 can be used to calculate a first integral of Eq. (3.5) in Example 3.1: since $A = xt^3, B = t^4x - 2t^2$ and $J = 1/t^3$ solve system (3.3) for Eq. (3.5) and $C = 1/(2t^2)$ is a primitive of $-J$, then $I = 1/(2t^2) + 1/(xt^3\dot{x} + t^4x - 2t^2)$ is a first integral of Eq. (3.5).

The study of the relationships between first integrals and λ-symmetries performed in [11, 13, 10] lets us prove the converse of Theorem 3.1:

Theorem 3.3. *If system (3.3) is satisfied for some A, B and $J(t)$ then $\alpha = -A_x/A$ and $\beta = -B_x/A$ solve system (2.3)–(2.5) and, therefore, the vector field $v = \partial_x$ is a λ-symmetry for $\lambda = \alpha\dot{x} + \beta$.*

Proof. By Theorem 3.2, $I = C + 1/(A\dot{x} + B)$ is a first integral of (1.1), where $C = C(t)$ is any primitive of $-J(t)$. By Theorem 1 in [11], the vector field $v = \partial_x$ is a λ-symmetry of the equation for $\lambda = -I_x/I_{\dot{x}}$. Since $I_x/I_{\dot{x}} = (A_x\dot{x} + B_x)/A$, system (2.3)–(2.5) is satisfied for $\alpha = -A_x/A$ and $\beta = -B_x/A$. □

4. Intrinsic Characterization

Corollary 3.1 gives us a characterization of the equations in $\mathcal{B}$: they are equations of the form (1.1) that admit first integrals of the form (3.17). Therefore, for these equations, the system (3.3) is compatible. Equations in the subclass $\mathcal{A}$ correspond to the case $C(t) = 0$. An intrinsic characterization of these equations, i.e., a characterization of class $\mathcal{A}$ in terms of the coefficients $a_i, 0 \leq i \leq 2$, appear in [12] (Sec. 3). To obtain an intrinsic characterization of equations in $\mathcal{B} \setminus \mathcal{A}$ is a rather involving task: the functions A, B and J and their derivatives must be expressed in terms of $a_i, 0 \leq a_i \leq 2$ and their derivatives. We now show a procedure to obtain such characterization, that could be applied to any given equation of the form (1.1).

For $J \neq 0$, system (3.3) implies that functions A, B and J have to satisfy the following system

$$A_x = a_2A - JA, \tag{4.1}$$

$$B_x = a_1A - A_t - 2ABJ, \tag{4.2}$$

$$B_t = a_0A - JB^2. \tag{4.3}$$

By using Eqs. (4.2) and (4.3), the compatibility condition $(B_x)_t = (B_t)_x$ leads to

$$B^2 + M_2B + M_1 = 0, \tag{4.4}$$

where

$$M_1 = -(a_1 A_t - A(a_0(a_2 + AJ) + a_{0x} - a_{1t}) - A_{tt})/(2AJ^2), \tag{4.5}$$

$$M_2 = -(2A(a_1 J - J') - 4JA_t)/(2AJ^2). \tag{4.6}$$

Equation (4.4) reveals the dependence of B on A, J and the coefficients $a_i, 0 \leq i \leq 2$. To eliminate quadratic dependencies, both members of (4.4) can be derived twice with respect to x and, by using (4.4), we get

$$R_2 B + R_1 = 0, \tag{4.7}$$

where

$$R_2 = a_2 S_1 - S_{1x}, \tag{4.8}$$

for S_1 defined in (2.6) and R_1 is an expression that depends on A, A_t, J and the coefficients of the equation and their derivatives. If $R_2 \neq 0$, Eq. (4.7) determines B in terms of A, J and the coefficients $a_i, 2 \leq i \leq 2$. By using (4.7) and (4.2) we obtain

$$T_2 A_t + T_1 = 0, \tag{4.9}$$

where

$$T_2 = \left(3{a_2}^2 S_1^2 - 2a_2 S_{1x} S_1 + 4(S_{1xx} - a_{2x} S_1) S_1 - 5{S_{1x}}^2\right)/(3(a_2 S_1 - S_{1x})) \tag{4.10}$$

and T_1 is an expression depending on A, A^2, J, the coefficients of the equation and their derivatives. If $T_2 \neq 0$ Eq. (4.9) can actually be written in the form

$$A_t = U_1 A^2 + U_2 A + U_3, \tag{4.11}$$

where U_1, U_2 and U_3 do only depend on J and the coefficients $a_i, 0 \leq i \leq 2$. Equations (4.1) and (4.11) and the compatibility condition $(A_t)_x = (A_x)_t$ lead to an expression of the form

$$Y_3 A^2 + Y_2 A + Y_1 = 0, \tag{4.12}$$

where Y_1, Y_2 and Y_3 are given by

$$\begin{aligned} Y_1 &= a_2 U_1 - U_{1x}, \\ Y_2 &= -2JU_1 - U_{2x} + a_{2t}, \\ Y_3 &= -JU_2 - a_2 U_3 - J' - U_{3x}. \end{aligned} \tag{4.13}$$

If $Y_3 \neq 0$, by derivation of (4.12) with respect to x, we get

$$Z_2 A + Z_1 = 0, \tag{4.14}$$

where Z_1 and Z_2 are defined by

$$\begin{aligned} Z_1 &= -2a_2 Y_1 - Y_{3x} Y_1/Y_3 - JY_2 Y_1/Y_3 + Y_{1x}, \\ Z_2 &= -J{Y_2}^2/Y_3 - a_2 Y_2 - Y_{3x} Y_2/Y_3 + 2JY_1 + Y_{2x}. \end{aligned} \tag{4.15}$$

If $Z_2 \neq 0$, Eq. (4.14) determines A in terms of J and the coefficients of the equation. Through (4.1), J can be calculated in terms of the coefficients $a_i, 0 \leq i \leq 2$. An analogous expression is obtained for B by using (4.7). When these expressions are substituted in (4.1)–(4.3), compatibility conditions on the coefficients of the equation are obtained.

The special cases where R_2, T_2, Y_3 or Z_2 are null must be studied separately. However a complete study of these cases is rather involved and will be considered in a forthcoming paper.

5. On the Linearization of Equations in $\mathcal{B}$

5.1. *Linearization through local transformations*

If a second-order ODE (1.1) is linearizable to equation $X_{TT} = 0$ by means of a local transformation

$$X = R(t,x), T = S(t,x) \tag{5.1}$$

then (1.1) has the form

$$\ddot{x} + a_3(t,x)\dot{x}^3 + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{5.2}$$

where the coefficients $a_i(t,x)$, $0 \le i \le 3$, can be expressed in terms of R, S and their derivatives ([6]) as

$$a_3(t,x) = \frac{S_x R_{xx} - S_{xx} R_x}{S_t R_x - S_x R_t}, \tag{5.3}$$

$$a_2(t,x) = \frac{S_t R_{xx} - R_t S_{xx} + 2(S_x R_{tx} - R_x S_{tx})}{S_t R_x - S_x R_t}, \tag{5.4}$$

$$a_1(t,x) = \frac{S_x R_{tt} - R_x S_{tt} + 2(S_t R_{tx} - R_t S_{tx})}{S_t R_x - S_x R_t}, \tag{5.5}$$

$$a_0(t,x) = \frac{S_t R_{tt} - R_t S_{tt}}{S_t R_x - S_x R_t}. \tag{5.6}$$

Let us introduce the following notation

Definition 5.1. We denote by $\mathcal{L}$ the set of the equations of the form (1.1) that are linearizable to equation $X_{TT} = 0$ by means of a local transformation (5.1).

In this section we prove that $\mathcal{L} \subset \mathcal{B}$, and more precisely, that $\mathcal{L} \subset \mathcal{B} \setminus \mathcal{A}_2$. Since $a_3 = 0$, three possibilities must be considered:

Case (a): $S_x = 0$.
Case (b): $R_x = 0$.
Case (c): $S_x \neq 0$, $R_x \neq 0$, $S_x R_{xx} - S_{xx} R_x = 0$.

In Cases (a) and (b) it has been proven ([14]) that the coefficients of the equation must satisfy $S_1 = S_2 = 0$. Therefore, the equation belongs to subclass $\mathcal{A}_1$ and hence to $\mathcal{B}$. In Case (c), the condition $S_x R_{xx} - S_{xx} R_x = 0$ implies that

$$R(t,x) = g(t)S(t,x) + h(t), \tag{5.7}$$

for some functions $g = g(t)$ and $h = h(t)$. It has been proven ([14]) that in this case the equation belongs to subclass $\mathcal{A}$ if and only if $h = c_1 g + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$ and that the coefficients of the equation must satisfy $S_1 = S_2 = 0$. These results implies that $\mathcal{A}_1 \subset \mathcal{L}$ and $\mathcal{A}_2 \cap \mathcal{L} = \emptyset$. A proof of these statements appears in ([14], Theorem 6).

Let us prove that if $h \neq c_1 g + c_2$, then the equation belongs to $\mathcal{B} \setminus \mathcal{A}_2$. It is clear that $X_T = D_t R(t,x)/D_t S(t,x)$ is a first integral of the equation. By (5.7),

$$X_T = \frac{g'S + h'}{S_t + \dot{x}S_x} + g, \tag{5.8}$$

and this is a first integral of the form (3.17) for

$$A = \frac{S_x}{g'S + h'}, B = \frac{S_t}{g'S + h'}, C(t) = g(t). \tag{5.9}$$

By Theorem 3.3 the equation belongs to $\mathcal{B}$. Thus we have proven the following result:

Theorem 5.1. *If a given equation belongs to $\mathcal{L}$ then the equation belongs to subclass $\mathcal{B} \setminus \mathcal{A}_2$.*

Example 5.1. The second-order equation

$$\ddot{x} + \frac{2}{t - x}\dot{x}^2 + \frac{2}{t - x} = 0 \tag{5.10}$$

was proposed in [14] as an example of an equation in $\mathcal{L}$ that does not belong to $\mathcal{A}$, because $S_1 = 6/(t-x)^2 \neq 0$ and $S_3 = 4/(t-x)^2 \neq 0$. By Theorem 5.1, Eq. (5.10) must belong to $\mathcal{B}$. This can also be directly proven because, for example, $\alpha = \beta = 1/(x-t)$ are particular solutions of the corresponding system (2.3)–(2.5).

Next example shows that $\mathcal{B} \setminus \mathcal{A}$ is strictly wider than $\mathcal{L}$:

Example 5.2. In Sec. 2 it has been proven that the equations

$$\ddot{x} + \frac{b'(t)}{2x} + \frac{b(t)^2}{4x^3} + a(t)x = 0, \quad b'(t) \neq 0. \tag{5.11}$$

belong to the subclass $\mathcal{B} \setminus \mathcal{A}$. If (2.9) is satisfied, these equations do not have Lie point symmetries and hence they do not belong to $\mathcal{L}$. This fact can also be proven by using Lie's test of linearization.

5.2. *Linearization through nonlocal transformations*

Since there are equations in $\mathcal{B} \setminus \mathcal{A}$ that cannot be linearized by local transformations (5.1), it raises the question if such equations could be linearized through transformations involving nonlocal terms. The simplest transformations of this type have been named in [5] *generalized Sundman transformations* (GST) and are of the form

$$X = F(t,x), \quad dT = G(t,x)dt. \tag{5.12}$$

The equations of the form (1.1) that can be linearized through (5.12) have been identified in [14] as the equations in subclass $\mathcal{A}$ and constructive methods to calculate such transformations have been derived (Theorems 2 and 3 in [14]). Hence, in order to linearize the equations of $\mathcal{B} \setminus \mathcal{A}$, we need to consider more general types of nonlocal transformations.

In this section we characterize the equations in $\mathcal{B}$ as the second-order equations of the form (1.1) that can be transformed into $X_{TT} = 0$ through a nonlocal transformation of type

$$X = F(t,x), \quad dT = (G_1(t,x)\dot{x} + G_2(t,x))dt, \tag{5.13}$$

where $G_1 \neq 0$. Second-order equations that can be linearized through (5.13) have been studied by Chandrasekar *et al* in [1]. The authors prove that these equations have to be of the form (1.4) where the coefficients $a_i(t,x)$, $0 \leq i \leq 3$, can be expressed in terms of F, G_1, G_2 and their derivatives (see Eq. (15) in [1]). In particular,

$$a_3(t,x) = \frac{G_1^2}{\Delta}\left(\frac{F_x}{G_1}\right)_x, \tag{5.14}$$

where $\Delta = F_x G_2 - F_t G_1 \neq 0$. The first integral $I_1 = X_T$ of $X_{TT} = 0$ provides, by using (5.13), a first integral of the nonlinear ODE

$$\widetilde{I}_1 = \frac{F_x\dot{x} + F_t}{G_1\dot{x} + G_2} = \frac{F_x}{G_1} - \frac{\Delta/G_1}{G_1\dot{x} + G_2}. \tag{5.15}$$

If the equation is of the form (1.1), i.e. if $a_3 = 0$, then, by (5.14), the funtion F_x/G_1 only depends on t and hence the first integral (5.15) is of the form (1.2). By Corollary 3.1 we deduce that the equations of the form (1.1) that can be linearized by (5.13) are in $\mathcal{B}$.

Conversely, let us prove that any equation in $\mathcal{B}$ can be linearized through a nonlocal transformation of type (5.13). An equation in $\mathcal{B}$ is of the form (1.1) and, by Theorem 3.1, its coefficients satisfy system (3.3) for some functions A, B and J. By Theorem 3.2, $I = C + 1/(A\dot{x} + B)$ is a first integral of the equation where $C' = -J$. By using system (3.3), it can be checked that

$$D_t I = -\frac{A}{(A\dot{x} + B)^2}(\ddot{x} + a_2\dot{x}^2 + a_1\dot{x} + a_0). \tag{5.16}$$

We construct a family of linearizing transformations of the form (5.13) in terms of a nonzero function $M = M(t,x)$ for which the system

$$F_t = (CB + 1)M, F_x = CAM \tag{5.17}$$

is compatible. The compatibility condition $(F_t)_x = (F_x)_t$ implies that $M = M(t,x)$ is a solution of the first order linear partial differential equation

$$(CB + 1)M_x - CAM_t + ((B_x - A_t)C - AC')M = 0. \tag{5.18}$$

Once a nontrivial particular solution M of (5.18) has been chosen, we define $F = F(t,x)$ as any particular solution of system (5.17) and $G_1(t,x) = MA, G_2(t,x) = MB$. It is clear that

$$X_T = \frac{D_t X}{D_t T} = \frac{D_t F}{G_1\dot{x} + G_2} = \frac{M(CA\dot{x} + CB + 1)}{M(A\dot{x} + B)} = I, \tag{5.19}$$

and, by (5.16),

$$X_{TT} = \frac{D_t I}{G_1\dot{x} + G_2} = -\frac{A}{(G_1\dot{x} + G_2)(A\dot{x} + B)^2}(\ddot{x} + a_2\dot{x}^2 + a_1\dot{x} + a_0). \tag{5.20}$$

This proves that F, G_1 and G_2 define a nonlocal transformation of type (5.13) that linearizes the equation in $\mathcal{B}$. We have proven the following result:

Theorem 5.2. *A second-order equation of the form* (1.1) *can be linearized through a nonlocal transformation of the form* (5.13) *if and only if the equation is in class* $\mathcal{B}$.

It should be noted that there can exist other second-order ODEs linearizable by (5.13) that are not in $\mathcal{B}$, but they must be of the form (1.4) with $a_3 \neq 0$ and will be studied in a forthcoming paper.

A disadvantage of the linearization through transformations (5.12) or (5.13) compared to the linearization through local transformations (5.1) is that the general solution of the nonlinear ODE can not be obtained straightforwardly by the two independent integrals of $X_{TT} = 0$

$$I_1 = X_T \quad \text{and} \quad I_2 = X - TX_T, \tag{5.21}$$

due to the nonlocal nature of (5.12) or (5.13) and hence of I_2. It should be pointed out that the linearization of a given ODE through nonlocal transformations (5.12) or (5.13) does not guarantee the integrability of the equation. Euler and Euler presented in [4] an interesting example of a Chazy-type equation which shows that, in general, a generalized Sundman transformation does not preserve the Painlevè property nor does it preserve the Lie symmetry structure of the equations. Another examples of this type with second-order equations of the form (1.1) appear in [14] (example 11). Nevertheless, the first integral $I_1 = X_T$ provides, by using (5.13), the first integral (5.15) that could be used to obtain the general solution of the nonlinear ODE by solving (if possible) the first order ODE corresponding to $\widetilde{I}_1 = C_1, C_1 \in \mathbb{R}$. By (5.19), such first order ODE can be expressed in terms of the functions A, B and C in the form

$$A(t,x)\dot{x} + B(t,x) = \frac{1}{C_1 - C(t)}. \tag{5.22}$$

An alternative method to overcome the problem of the nonlocal nature of the transformation (5.13) to obtain the general solution or a second independent first integral of the nonlinear ODE appears in [1].

Example 5.3. For Eq. (3.5) the corresponding system (3.3) is satisfied for $A = xt^3, B = t^4x - 2t^2$ and $J = 1/t^3$ (see Example 3.1). A primitive of $-J$ is $C = 1/(2t^2)$. It can be checked that $M(t,x) = \varphi\left(\frac{t^2}{2} + x\right)/(tx)$ is the general solution of the corresponding Eq. (5.18), where $\varphi = \varphi(a)$ is an arbitrary function of one variable. The general solution of corresponding system (5.17) becomes

$$F(t,x) = \widetilde{\varphi}\left(\frac{t^2}{2} + x\right) \tag{5.23}$$

where $\widetilde{\varphi}$ is any primitive of $\varphi/2$, i.e., $\widetilde{\varphi}'(a) = \varphi(a)/2, a \in \mathbb{R}$. Therefore

$$X = \widetilde{\varphi}\left(\frac{t^2}{2} + x\right), \quad dT = 2\widetilde{\varphi}'\left(\frac{t^2}{2} + x\right)\left(t^2\dot{x} + t^3 - \frac{2t}{x}\right)dt \tag{5.24}$$

is a family of nonlocal transformations of the form (5.13) that linearize Eq. (3.5). We note that Eq. (3.5) cannot be linearized by local transformations, i.e., it does not pass the Lie test of linearization.

The first order ODE corresponding to (5.22) is Eq. (3.11), which has been used in Example 3.1 to derive the general solution of Eq. (3.5).

Closely related to the concept of a generalized Sundman transformation is the notion of an associated Sundman symmetry. This was introduced by Euler and Euler in [4] who studied the Sundman symmetries of a large class of second-order and third-order nonlinear ODEs. These symmetries can be calculated systematically and can be used to find first integrals of the equations. Hence it would be interesting and tempting to consider symmetries related to the nonlocal transformations (5.12) or (5.13) for the classification of ODEs in terms of their first integrals. It is currently not clear to us what are the connections between Sundman symmetries and λ-symmetries.

6. Conclusions

We have presented some properties and characterizations of the equations in class $\mathcal{B}$ constituted by ODEs (1.1) that admit the vector field $v = \partial_x$ as λ-symmetry for some $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$. This study completes and extends some of the results presented in [12, 14] for the subclass $\mathcal{A}$, that considered the particular case $\alpha = -a_2$.

The equations in class $\mathcal{B}$ can be characterized as the Eqs. (1.1) that admit first integrals of the type (1.2). A method to calculate the functions $A(t,x), B(t,x)$ and $C(t)$ that define such first integrals has been presented. These results complete the study of the second-order equations of the form (1.1) that admit first integrals of the form (1.3). Although there are other second-order equations with first integrals of the type (1.3), they must be of the form (1.4) with $a_3 \neq 0$ and will be studied in a forthcoming paper.

The equations in subclass $\mathcal{A}$ can be characterized in terms of their coefficients $a_i, 0 \leq i \leq 2$, in a useful and compact form through expressions (2.6) and (2.7). To obtain an intrinsic characterization of the equations in $\mathcal{B}\setminus\mathcal{A}$ is a much more complicated task. Some guidelines to deal with this problem have been indicated and a complete study will be considered in a separate paper.

Some aspects on the linearization of the equations in $\mathcal{B}$ have also been considered. Although there are second-order Eqs. (1.1) linearizable by some local transformations that are not in $\mathcal{A}$, it has been proven that all of them are included in $\mathcal{B}$. This is a strict inclusion, because there are equations in $\mathcal{B}$ that do not pass the Lie test of linearization. Nevertheless we have proven, by a constructive method, that such equations can always be linearized through nonlocal transformations of type (5.13) and conversely: the equations in $\mathcal{B}$ are the only second-order equations of the form (1.1) that are linearizable by this type of nonlocal transformations.

Acknowledgment

The support of DGICYT project MTM2009-11875, Junta de Andalucía group FQM–201 and project P06-FQM-01448 are gratefully acknowledged. We wish to thank Prof. Euler for his useful suggestions and comments on the problems studied in this paper.

References

[1] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, A unification in the theory of linearization of second-order nonlinear ordinary differential equations, *J. Phys. A* **39** (2006) L69–L76.

[2] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations, *Proc. R. Soc. A* **461** (2005) 2451–2476.

[3] L. G. S. Duarte, I. C. Moreira and F. C. Santos, Linearization under non-point transformations, *J. Phys. A* **27** (1994) L739–L743.

[4] N. Euler and M. Euler, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, *J. Nonlinear Math. Phys.* **11** (2004) 399–421.

[5] N. Euler, T. Wolf, P. G. L. Leach and M. Euler, Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case $X''' = 0$, *Acta Appl. Math.* **76** (2003) 89–115.

[6] N. H. Ibragimov, *A Practical Course in Differential Equations and Mathematical Modelling: Classical and New Methods, Nonlinear Mathematical Models, Symmetry and Invariance Principles* (ALGA Publications, Karlskrona, 2006).

[7] N. H. Ibragimov and F. Magri, Geometric proof of Lie's linearization theorem, *Nonlinear Dynam.* **36** (2004) 41–46.

[8] S. V. Meleshko, On linearizaton of third-order ordinary differential equations, *J. Phys. A* **39** (2006) 15135–15145.

[9] C. Muriel and J. L. Romero, New methods of reduction for ordinary differential equations, *IMA J. Appl. Math.* **66** (2001) 111–125.

[10] C. Muriel and J. L. Romero, Integrating factors and λ-symmetries, *J. Nonlinear Math. Phys.* **15** (2008) 290–299.

[11] C. Muriel and J. L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42** (2009) 365207 (17 pp).

[12] C. Muriel and J. L. Romero, Second-order ordinary differential equations and first integrals of the form $A(t,x)\dot{x} + B(t,x)$, *J. Nonlinear Math. Phys.* **16** (2009) 209–222.

[13] C. Muriel and J. L. Romero, λ-Symmetries on the derivation of first integrals of ordinary differential equations, in WASCOM 2009: 15th Conference on Waves and Stability in Continuous Media, World Scientific, 2010, pp. 303–308.

[14] C. Muriel and J. L. Romero, Nonlocal transformations and linearization of second-order ordinary differential equations, *J. Phys. A: Math. Theor.* **43** (2010) 13 pp.

9 Connections between symmetries and integrating factors of ODES (M.S. Thesis) by Theodore Kolokolnikov (1999)

CONNECTIONS BETWEEN SYMMETRIES AND INTEGRATING FACTORS OF ODES

by

THEODORE KOLOKOLNIKOV

B.Math. University of Waterloo, 1997

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1999

Department of Mathematics
The University of British Columbia
Vancouver, Canada

Date 30 April 1999

Abstract

In this thesis we examine the connections between conservation laws and symmetries, both for self-adjoint and non self-adjoint ODEs. The goal is to gain a better understanding of how to combine symmetry methods with the method of conservation laws to obtain results not obtainable by either method separately.

We review the concepts of symmetries and integrating factors. We present two known methods of obtaining conservation laws without quadrature, using known conservation laws and symmetries. We show that the two methods yield the same result.

For self-adjoint systems, we examine Noether's theorem in detail and discuss its generalisation for ODEs admitting more than one variational symmetry. We generalise an example from Sheftel [20] and show how to use r-dimensional Lie Algebra of variational symmetries to obtain more then r reductions of order.

We develop an ansatz for finding point variational symmetries. We also develop ansatzes that use a known symmetry to find an integrating factor or another symmetry. These ansatzes are then used to classify ODEs. New solvable cases of Emden-Fowler and Abel ODEs result.

Acknowledgments

I thank George Bluman, my supervisor, for his careful readings and suggestions throughout the preparation of this thesis. Many thanks to Edgardo S. Cheb-Terrab and Austin D. Roche for useful and lively discussions and for suggesting what eventually became Section 4.3. Finally, I would like to thank Greg Reid for his reading of the final draft.

Table of Contents

List of Tables

List of Figures

Preface

One of the most algorithmic methods of finding exact solutions to differential equations is the method of continuous symmetries developed by Sophius Lie in the latter half of the 19th century. Among many applications for ODEs, one can use a continuous symmetry to find a change of variables that leads to a reduction of order.

A more direct approach for reducing the order of ODEs is to use *conservation laws*. Each conservation law leads to a reduction of order without change of variables. This is unlike a symmetry reduction which relies on a change of variables.

In this thesis we explore the connections that exist between symmetry reduction and reduction using conservation laws.

In Chapter 1 we review Lie theory of symmetries, including how to use symmetries to obtain reductions of order of ODEs.

In Chapter 2 we study conservation laws and integrating factors. A conservation law is characterised uniquely by its *integrating factor* and one can obtain a conservation law from an integrating factor using a quadrature. Alternatively, any pair (integrating factor, symmetry) and any pair (conservation law, symmetry) are shown to yield a (possibly trivial) conservation law *without* quadrature.

In Chapter 3 we study self-adjoint ODEs whose integrating factors correspond to a special class of symmetries, *variational symmetries*. In her famous paper [15], Emmy Noether showed how to construct a conservation law from a variational symmetry. Moreover, the resulting conservation law admits the variational symmetry that was used to find it. Thus two reductions of order are possible using a single variational symmetry: one reduction in the original variables and one symmetry reduction. In general, r variational symmetries do not necessarily yield $2r$ reductions of order. In Chapter 3 we will establish a lower bound on how many reductions of order of each type is to be expected. The answer depends on the structure of commutators of admitted variational symmetries.

For a self-adjoint ODE explicit in its highest derivative, it is possible to tell when a point symmetry is variational without looking at the ODE itself. We explore this to give an ansatz for seeking variational symmetries of such ODEs.

In Chapter 4 we consider a *classification problem*: find all ODEs belonging to a given family of ODEs for which a solution can be found. As an example, we classify the Emden-Fowler family of ODEs and find several new solvable cases. We also develop ansatzes that use a known symmetry to help in finding an integrating factor or another symmetry. These ansatzes are then used to find new solvable cases of Abel ODEs.

Chapter 1
Symmetries of ODEs

In this chapter we introduce symmetry methods and show how symmetries can be used to reduce the order of ODEs. Since many physically relevant differential equations admit symmetries, symmetry methods have become increasingly popular since Lie's fundamental work and its rediscovery in the latter half of the 20th century, especially by mathematicians in the former Soviet Union.

Lie gave a common framework and extended different ad-hoc techniques used to find solutions of ODEs. The symmetry methods he developed are highly algorithmic. For instance, using Lie's algorithm one can systematically find point symmetries of differential equations and then find a change of variables which leads to a reduction of order. Many of these methods have been implemented on computer algebra systems (for review, see Hermann [13]).

Some applications of symmetries include: Reduction of order of ODEs, finding special solutions of PDEs, finding conservation laws for self-adjoint systems using Noether's Theorem, and linearizing differential equations. These and other applications are described in [7].

In Section 1.1 we describe how the symmetries arise as well as Lie's algorithm to find them. For ODEs of second or higher order, finding point symmetries leads to solving an overdetermined system of PDEs with only finitely many solutions. Such systems can often be solved completely and computer programs exist to find their solutions.

In Section 1.2 we show how to use symmetries to obtain reduction of order of ODEs using the method of canonical coordinates or the method of differential invariants.

1.1 Symmetries of differential equations

Consider a differential equation

$$y' = f(x), \; x \in \Re \tag{1.1}$$

If we make a change of variables

$$\hat{y} = y + \epsilon \tag{1.2}$$

where ϵ is any real number then $\hat{y}$ will satisfy the same differential equation (1.1). Hence if $y = \phi(x)$ is a solution of (1.1) then so is $\hat{y} = \phi(x) + \epsilon$. This is an example of a continuous symmetry of a differential equation. In general, a symmetry of a differential equation is a transformation that maps the solution of an equation into another solution of the same equation. Symmetries can be discrete (such as reflection or a rotation by 30^o), or continuous (such as in the above example, or a rotation by an arbitrary angle) which depends on a continuous parameter ϵ.

The fact that continuous symmetries are an uncountably infinite family of transformations makes them much more useful for applications than discrete symmetries. It also makes continuous symmetries easier to find. In what follows, we will only discuss continuous symmetries.

The material that we shall present in this section is well-known. See for example [5], [7], [21].

1.1.1 Continuous transformations groups

Since continuous symmetries are continuous transformations, we first study the transformations themselves.

Consider a family of transformations

$$\hat{x}^{\epsilon}(x) : M \times \mathcal{I} \to M$$

indexed by a continuous parameter

$$\epsilon \in \mathcal{I}, \; \mathcal{I} \text{ is an open interval of } \Re \text{ containing zero}$$

which maps $x \in M$ into $\hat{x}^\epsilon \in M$, where M is a smooth manifold[1].

Example 1.1.1 A rotation of the two-dimensional plane is a family of transformations that can be parametrised by an angle $\epsilon \in (-\pi, \pi]$:

$$\hat{x}_1^\epsilon(x_1, x_2) = x_1 \cos\epsilon - x_2 \sin\epsilon$$

$$\hat{x}_2^\epsilon(x_1, x_2) = x_1 \sin\epsilon + x_2 \cos\epsilon$$

Alternatively this can be represented as:

$$\hat{x}^\epsilon(x) = \begin{bmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{bmatrix} x \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.3}$$

Note that in the above example, rotating by angle ϵ and then by angle δ is the same as rotating by angle $\epsilon + \delta$; rotation by zero is the identity transformation. Thus the family of rotations above forms an additive group with respect to ϵ. This motivates the following definition:

Definition 1.1.2 A *flow* is a function

$$\hat{x}^\epsilon(x) : \mathcal{I} \times M \to M$$

with the following properties:

1. $\mathcal{I}$ is an open interval that contains zero with $\epsilon \in \mathcal{I}$ and M is a smooth manifold with $x \in M$
2. $\hat{x}^{\delta+\epsilon}(x) = \hat{x}^\delta(\hat{x}^\epsilon(x))$ whenever $\delta, \epsilon, \delta + \epsilon \in \mathcal{I}$
3. $\hat{x}^0(x) = x$
4. $\hat{x}^\epsilon(x)$ is analytic in ϵ when ϵ is near zero, for every $x \in M$.

A flow forms a *one parameter continuous group of transformations on* M, since $\hat{x}^\epsilon(x)$ is an additive group with respect to ϵ. To show this, we need to show that the inverse of $\hat{x}^\epsilon(x)$ exists and is equal to $\hat{x}^{-\epsilon}(x)$. Indeed,

$$x = \hat{x}^0(x) = \hat{x}^{\epsilon-\epsilon}(x) = \hat{x}^\epsilon(\hat{x}^{-\epsilon}(x)) = \hat{x}^{-\epsilon}(\hat{x}^\epsilon(x)).$$

[1]For our purposes, assume M is an open subset of $\Re^n$

1.1.2 Infinitesimals

A flow is completely characterised by its *infinitesimal.*

Definition 1.1.3 The *infinitesimal* of a flow $\hat{x}^{\epsilon}(x)$ is given by

$$v(x) = \frac{d}{d\epsilon}\hat{x}^{\epsilon}(x)\Big|_{\epsilon=0}$$

Given an infinitesimal, one can recover the corresponding transformation group through the following theorem:

Theorem 1.1.4 (Lie's fundamental Theorem) *A flow is uniquely determined by its infinitesimal and vice-versa.*

If $\hat{x}^{\epsilon}(x)$ is a flow with the infinitesimal $v(x) = \frac{d}{d\epsilon}\hat{x}^{\epsilon}(x)\Big|_{\epsilon=0}$, then $\hat{x}$ satisfies

$$\frac{d}{d\epsilon}\hat{x}^{\epsilon}(x) = v(\hat{x}^{\epsilon}(x)). \tag{1.4}$$

Conversely, if $v(x)$ is analytic and $\hat{x}^{\epsilon}(x)$ satisfies (1.4) with the initial condition $\hat{x}^{0}(x) = x$ then $\hat{x}$ is a flow and v is its infinitesimal.

Proof. First suppose that $\hat{x}$ is a flow. Then $\hat{x}^{\epsilon+\delta}(x) = \hat{x}^{\delta}(\hat{x}^{\epsilon}(x))$ and $\hat{x}^{0}(x) = x$. Differentiating by δ and evaluating at $\delta = 0$ we get (1.4).

Conversely, fix x and suppose $\hat{x}^{\epsilon}(x)$ is a solution of (1.4) with $\hat{x}^{0}(x) = x$. Let $X = \hat{x}^{\epsilon+\delta}(x)$ and $Y = \hat{x}^{\epsilon}(\hat{x}^{\delta}(x))$. Then $X|_{\epsilon=0} = Y|_{\epsilon=0} = \hat{x}^{\delta}(x)$ and both X and Y satisfy

$$\frac{dX}{d\epsilon} = v(X), \frac{dY}{d\epsilon} = v(Y).$$

Since v is analytic, the above equations have an analytic and unique solution near $\epsilon = 0$. Thus $X = Y$ is analytic for ϵ in some neighbourhood of 0 by the existence and uniqueness theorem. This verifies properties (2-4) of definition 1.1.2 □

Example 1.1.5 Let $\hat{x}^{\epsilon}(x)$ be the rotational flow (1.3) as in Example 1.1.1. Then

$$v(x_1, x_2) = \frac{d}{d\epsilon}_{|\epsilon=0}\hat{x}^{\epsilon}(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Conversely, fix x and denote $\hat{x}^\epsilon(x) = y(\epsilon) = \binom{y_1(\epsilon)}{y_2(\epsilon)}$ Then (1.4) with $\hat{x}^0(x) = x$ can be written as a system:

$$y' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y, \; y(0) = x$$

whose solution is precisely the rotational flow.

It is convenient to introduce the *generator* X of $\hat{x}^\epsilon(x)$ to be a first-order differential operator

$$XF = \bigtriangledown F(x) \cdot v(x)$$

acting on any differentiable function $F : M \to \Re$. Then

$$X = v_1 \frac{\partial}{\partial x_1} + \ldots v_n \frac{\partial}{\partial x_n}$$

where v_i is the i-th coordinate of v.

1.1.3 Change of coordinates; canonical coordinates

Given a flow $\hat{x}^\epsilon(x)$, suppose that we make a change of coordinates, $x = x(y)$ and let $y = y(x)$ be the inverse of $x(y)$. In the y-coordinates, the flow $\hat{x}^\epsilon(x)$ becomes

$$\hat{y}^\epsilon(y) = y(\hat{x}^\epsilon(x(y))).$$

Let v be the infinitesimal of $\hat{x}$ and let w be the infinitesimal of $\hat{y}$:

$$v(x) = \frac{d}{d\epsilon}(\hat{x}^\epsilon(x))\Big|_{\epsilon=0}, \; w(y) = \frac{d}{d\epsilon}(\hat{y}^\epsilon(y))\Big|_{\epsilon=0}.$$

Using the chain rule, we obtain:

$$w_i = \frac{d}{d\epsilon} y_i(\hat{x}^\epsilon)\Big|_{\epsilon=0} = v \cdot \bigtriangledown y_i(x)$$

where w_i and y_i denotes the i-th coordinate of w and y respectively.

Using the generator X of $\hat{x}$, $XF = v \cdot \bigtriangledown F$, this can be re-written as

$$w = Xv$$

where Xv is the vector $(Xv_i)^T$.

If we choose a change of coordinates

$$y = (r_1(x), ..., r_{n-1}(x), s(x))^T$$

for which $w = (0, ..., 0, 1)^T$, then by Theorem 1.1.4 the corresponding flow becomes

$$\hat{y}^\epsilon(y) = (y_1, y_2, ..., y_{n-1}, y_n + \epsilon).$$

The resulting coordinates are called *canonical coordinates.* Thus, canonical coordinates "straighten out" the flow.

To find canonical coordinates, proceed as follows.

First, find $n-1$ functionally independent solutions of a linear first-order PDE

$$Xr\,(x) = 0. \tag{1.5}$$

Using the method of characteristics, this is equivalent to solving a system of $n-1$ ODEs of first order.[2] One can then choose $r_1, ..., r_{n-1}$ to be any $n-1$ functionally independent solutions of (1.5)

Second, find a solution to the pde

$$Xs = 1. \tag{1.6}$$

Using the method of characteristics, s can be found by quadrature, once $r_1, \ldots, r_{n-1}$ are found.

1.1.4 Invariance under continuous transformation

Suppose that we are given a function $F : M \to \Re$. A flow $\hat{x}^\epsilon$ that leaves the curves $F(x) = 0$ invariant:

$$F(x) = 0 \Leftrightarrow F(\hat{x}^\epsilon(x)) = 0,$$

is called a *symmetry* of F.

[2]These solutions correspond to $n-1$ constants of motion of the flow $\hat{x}$.

If we differentiate $F(\hat{x}^\epsilon(x)) = 0$ with respect to ϵ and evaluate at $\epsilon = 0$, we obtain:

$$\frac{d}{d\epsilon}F(\hat{x}^\epsilon(x))\bigg|_{\epsilon=0} = \bigtriangledown F(x) \cdot v(x) = 0. \tag{1.7}$$

Thus if $\hat{x}^\epsilon$ is a symmetry of F then

$$\bigtriangledown F(x) \cdot v(x) = 0. \tag{1.8}$$

Conversely,

Theorem 1.1.6 *Suppose that (1.8) holds for some infinitesimal v, for all $x \in M$. Let $\hat{x}^\epsilon(x)$ be the flow that corresponds to the infinitesimal v, given by Lie's Fundamental Theorem 1.1.4. Then $F(\hat{x}^\epsilon(x)) = 0 \Leftrightarrow F(x) = 0$ and thus $\hat{x}^\epsilon(x)$ is a symmetry of $F(x) = 0$.*

Proof. Let $y = \hat{x}^\delta(x)$. Then by assumption,

$$0 = \frac{d}{d\epsilon}F(\hat{x}^\epsilon(y))\bigg|_{\epsilon=0}.$$

Expanding y and using the group property (2) of definition 1.1.2 we obtain:

$$0 = \frac{d}{d\epsilon}F(\hat{x}^\epsilon(y))\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}F(\hat{x}^{\epsilon+\delta}x)\bigg|_{\epsilon=0} = \frac{d}{d\delta}F(\hat{x}^\delta x).$$

Thus $F(\hat{x}^\delta(x))$ is constant for all δ. In particular $F(\hat{x}^0(x)) = F(x) = 0 \Rightarrow F(\hat{x}^\delta(x)) = 0$. □

In terms of the symmetry generator, Theorem 1.1.6 states that a first order differential operator X is a generator of a symmetry of F iff

$$XF = 0. \tag{1.9}$$

Example 1.1.7 Continuing with Example 1.1.1, let

$$F(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

Then $v = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ is a solution to $\bigtriangledown F \cdot v = 0$; the flow $x^\epsilon(x)$ corresponding to the infinitesimal v is just the rotational flow (1.3). In particular if we choose $x_1 = 1, x_2 = 0$ then $\hat{x}^\epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\epsilon \\ \sin\epsilon \end{pmatrix}$. Hence by above theorem, $F\begin{pmatrix} \cos\epsilon \\ \sin\epsilon \end{pmatrix} = 0$, from which follows that $\cos^2\epsilon + \sin^2\epsilon = 1$.

This characterisation of infinitesimals is of fundamental importance: it allows us to find the infinitesimals of a symmetry of F, and thus a symmetry itself.

In the next section we will extend this result to ODEs.

1.1.5 Point symmetries of ODEs

We want to study the invariance of an ODE

$$G(x, y(x), y'(x), ..., y^{(n)}(x)) = 0 \tag{1.10}$$

under a point transformation

$$\begin{cases} \hat{x} = \hat{x}^{\epsilon}(x, y) \\ \hat{y} = \hat{y}^{\epsilon}(x, y) \end{cases} \tag{1.11}$$

which forms a flow and which maps solutions of (1.10) into solutions of (1.10). Thus

$$G(\hat{x}, \hat{y}, \frac{d\hat{x}}{d\hat{y}}, ..., \frac{d^n\hat{y}}{d\hat{x}^n}) = 0 \Leftrightarrow G(x, y, y_1, ..., y_n) = 0$$

where $y = \phi(x)$ denotes any solution of 1.10, and $y_i = \frac{dy_{i-1}}{dx}, i \geq 1$ (using convention $y_0 = y$). The expression $\hat{y}_i = \frac{d^i\hat{y}}{d\hat{x}^n}$ can be written using the *total derivative operator*,

$$D_x = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + ... \tag{1.12}$$

as

$$\hat{y}_i = \frac{D_x\hat{y}_{i-1}}{D_x\hat{x}}, i \geq 1 \text{ with } \hat{y}_0 = \hat{y}. \tag{1.13}$$

The vector $(\hat{x}, \hat{y}, \hat{y}_1, ..., \hat{y}_n)$ is called the n-th extension of the point transformation (1.11).

If (1.11) is a flow then so is its n-th extension, by the following theorem:

Theorem 1.1.8 *Suppose that $(\hat{x}, \hat{y})$ given by (1.11) forms a flow. Let $\hat{y}_i$ be given by (1.13) (using convention $\hat{y}_0 = \hat{y}$). Then $(x, \hat{y}, \hat{y}_1, ..., \hat{y}_n)$ forms a flow on the $n + 2$ dimensional space spanned by $(x, y, y_1, ..., y_n)$.*

Proof. We will prove here the case $n = 1$, the other cases being analogous. To do this we will verify properties 2 and 3 of the definition 1.4.

Property 3: We need to show that $\hat{y}_1^0 = y_1$. Expand $\hat{x}, \hat{y}$ in Taylor series:

$$\hat{x} = x + \epsilon\xi + O(\epsilon^2), \ \hat{y} = y + \epsilon\eta + O(\epsilon^2).$$

Thus

$$\hat{y}_1^0 = \left.\frac{D_x \hat{y}^\epsilon}{D_x \hat{x}^\epsilon}\right|_{\epsilon=0} = \left.\frac{y_1 + \epsilon D_x \xi}{1 + \epsilon D_x \eta}\right|_{\epsilon=0} = y_1.$$

Property 2: Let $\mathbf{x}^\epsilon = (\hat{x}^\epsilon, \hat{y}^\epsilon)$; note that $\mathbf{x}^0 = (x, y)$. We need to show that

$$\hat{y}_1^{\epsilon+\delta}(\mathbf{x}^0, \hat{y}_1^0) = \hat{y}_1^\epsilon(\mathbf{x}^\delta, \hat{y}_1^\delta).$$

Since $\mathbf{x}$ forms a flow, it follows that $\mathbf{x}^{\epsilon+\delta}(\mathbf{x}^0) = \mathbf{x}^\epsilon(\mathbf{x}^\delta)$. Thus

$$\hat{y}_1^{\epsilon+\delta}(\mathbf{x}^0, \hat{y}_1^0) = \frac{D_x \hat{y}^{\epsilon+\delta}(\mathbf{x}^0)}{D_x \hat{x}^{\epsilon+\delta}(\mathbf{x}^0)} = \frac{D_x \hat{y}^\epsilon(\mathbf{x}^\delta)}{D_x \hat{x}^\epsilon(\mathbf{x}^\delta)} = \hat{y}_1^\epsilon(\mathbf{x}^\delta, \hat{y}_1^\delta).$$

□

We can now define:

Definition 1.1.9 A point transformation (1.11) is a *point symmetry* of an n-th order ODE (1.10) if it is a flow and if its n-th extension

$$\begin{cases} \hat{x} = \hat{x}^\epsilon(x, y) \\ \hat{y} = \hat{y}^\epsilon(x, y) \\ \hat{y_1} = \hat{y_1}^\epsilon(x, y, y_1) = \frac{D_x \hat{y}}{D_x \hat{x}} \\ \ldots \\ \hat{y_n} = \hat{y_1}^\epsilon(x, y, y_1, ..., y_n) = \frac{D_x \hat{y}_{n-1}}{D_x \hat{x}} \end{cases} \tag{1.14}$$

satisfies $G(\hat{x}, \hat{y}, ..., \hat{y}_n) = 0$ whenever $G(x, y, ..., y_n) = 0$.

If we are only given infinitesimals

$$\xi = \left.\frac{\partial \hat{x}}{\partial \epsilon}\right|_{\epsilon=0}, \ \eta = \left.\frac{\partial \hat{y}}{\partial \epsilon}\right|_{\epsilon=0}, \ \eta_i = \left.\frac{\partial \hat{y}_i}{\partial \epsilon}\right|_{\epsilon=0}, i = 1..n \tag{1.15}$$

of some flow

$$\begin{cases} \hat{x} = \hat{x}^{\epsilon}(x,y) \\ \hat{y} = \hat{y}^{\epsilon}(x,y) \\ \hat{y_1} = \hat{y_1}^{\epsilon}(x,y,y_1) \\ \dots \\ \hat{y_n} = \hat{y_n}^{\epsilon}(x,y,y_1,...,y_n) \end{cases}, \tag{1.16}$$

then it is possible to verify whether the contact conditions (1.13) hold, without computing the flow itself. To do that we expand (1.16) in Taylor series with respect to ϵ at zero. We obtain:

$$\begin{aligned} \hat{x} &= x + \epsilon\xi + O(\epsilon^2) \\ \hat{y} &= y + \epsilon\eta + O(\epsilon^2) \\ \hat{y_i} &= y_i + \epsilon\eta_i + O(\epsilon^2), i = 1, \dots, n \end{aligned} \tag{1.17}$$

Imposing contact conditions (1.13) we get:

$$\begin{aligned} \hat{y}_1 &= \tfrac{D_x\hat{y}}{D_x\hat{x}} = \tfrac{y_1+\epsilon D_x\eta+O(\epsilon^2)}{1+\epsilon D_x\xi+O(\epsilon^2)} \\ &= (y_1 + \epsilon D_x\eta)(1 - \epsilon D_x\xi + (\epsilon D_x\xi)^2 - ...) + O(\epsilon^2) \\ &= y_1 + \epsilon(D_x\eta - y_1 D_x\xi) + O(\epsilon^2). \end{aligned}$$

Hence

$$\eta_1 = D_x\eta - y_1 D_x\xi.$$

Similarly by induction we obtain the extension formula

$$\eta_i = D_x\eta_{i-1} - y_i D_x\xi, i = 1, 2, ..., n, \text{ with } \eta_0 = \eta. \tag{1.18}$$

Thus if the contact conditions (1.13) hold then the infinitesimals of (1.16) satisfy (1.18).

The converse is also true because of uniqueness of the flow corresponding to a given infinitesimal (see Theorem 1.1.4) and since $(\hat{x}, \hat{y}, ..., \hat{y}_n)$ is a flow (see Theorem 1.1.8).

Hence the infinitesimals of the symmetry, $(\xi, \eta, \eta_1, ..., \eta_n)$, and thus the symmetry itself, is uniquely determined by ξ and η through (1.18).

This leads to a much more useful characterisation of a point symmetry of an ODE:

Theorem 1.1.10 (Lie's algorithm) *To find point symmetries of an n-th order ODE 1.10 it suffices to find* $\xi(x, y), \eta(x, y)$ *such that*

$$\xi G_x + \eta G_y + \eta_1 G_{y_1} + \ldots + \eta_n G_{y_n} = 0 \ (mod\ G{=}0) \tag{1.19}$$

where by $(mod\ G{=}0)$ *we mean that the equality holds whenever* $G(x, y, y_1, \ldots y_n) = 0$ *and* η_i *is given by (1.18). Then* (ξ, η) *is the infinitesimal of a point symmetry of* $G = 0$.

Using the symmetry generator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta_1 \frac{\partial}{\partial y_1} + \ldots,$$

equation (1.19) can be written as

$$XG = 0 \ (\text{mod } G{=}0). \tag{1.20}$$

Often we will omit the extensions $\eta_i, i > 0$ and write

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}.$$

To find a symmetry generator of an ODE in solved form,

$$G = y_n - g(x, y, y_1, \ldots, y_{n-1}) = 0, \tag{1.21}$$

we write down the conditions (1.19) together with (1.18), at the same time replacing any occurrence of y_n by g. When $n > 1$, the resulting linear system of PDEs is *overdetermined* and has only finitely many independent solutions. Consequently, it is often possible to find them. Furthermore, computer programs (for instance "rif" [17], [19] and "diffalg" [8]) are available for simplifying overdetermined systems of PDEs using compartability conditions.

Even though not all second order ODEs have point symmetries, many physically relevant ODEs do.

1.1.6 Lie-Bäcklund symmetries of ODEs

In the future, we will need transformations more general than point symmetries. To generalise the concept of point symmetries we allow the infinitesimals η and ξ of a symmetry of ODE (1.21) to depend on $x, y, y_1, y_2, ... y_{n-1}$. Then $\eta_1 = D_x\eta - y_1 D_x \xi$ may depend on y_n which one can replace by g and similarly for $\eta_2, ..., \eta_n$. The extension law (1.18) remains the same, except that we replace all occurences of y_n by g.

Definition 1.1.11 A *generator of a symmetry* of an ODE (1.21) (or simply a *symmetry*) is first-order differential operator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial y_1} + ...$$

where ξ, η may depend on $y_1, ..., y_{n-1}$:

$$\xi = \xi(x, y, y_1, \ldots, y_{n-1}),\ \eta = \eta(x, y, y_1, \ldots, y_{n-1})$$

and η_i is given by

$$\eta_i = D_x\eta_{i-1} - y_i D_x\xi \text{ (mod } G{=}0), i \geq 1 \text{ with } \eta_0 = \eta. \tag{1.22}$$

and which has the property that

$$XG = 0 \text{ (mod } G{=}0).$$

If ξ or η depend only on x, y then the symmetry is called a *point* symmetry. Otherwise it is called a *Lie-Bäcklund* symmetry.

The generalisation of point symmetries to Lie-Bäcklund symmetries was introduced in [4].

1.1.7 Trivial symmetries

Consider a symmetry with infinitesimals $\xi = 1, \eta = y_1$. The corresponding infinitesimal generator is the *total derivative operator*:

$$D_x = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + ... \tag{1.23}$$

Thus if

$$G(x, y, y_1, ..., y_n) = 0$$

is any ODE then

$$D_x G = \frac{d}{dx} G = 0 \text{ (mod } G{=}0).$$

So D_x is a trivial symmetry of any ODE. Geometrically, a trivial symmetry represents a transformation in the direction of the solution curves of the equation.

More generally,

Proposition 1.1.12 *Let X be a symmetry generator*

$$X = \xi \frac{\partial}{\partial x} + y_1 \xi \frac{\partial}{\partial y} + ...$$

Then X is a (trivial) symmetry of any ODE $G = 0$.

Proof. One can show by induction that the extension formula (1.18) can be rewritten as

$$\eta_i = D_x^{(i)} (\eta - y_1 \xi) + y_{i+1} \xi. \tag{1.24}$$

Thus if $\eta = y_1 \xi$ then $\eta_i = \xi y_{i+1}$ and hence $XG = \xi D_x G = 0$ (mod G=0). □

1.1.8 Evolutionary form for symmetry generators

When a symmetry does not transform the independent variable "x", it is said to be in *evolutionary form.* Geometrically, such a symmetry transforms the curves $y(x)$ in the vertical direction only (see figure 1.1.8).

The generator of a symmetry in evolutionary form is

$$X = v \frac{\partial}{\partial y} + v_1 \frac{\partial}{\partial y_1} + \ldots \tag{1.25}$$

where v may depend on $y_1, y_2, ..., y_{n-1}$. The extension condition (1.18) then becomes

$$v_i = D_x v_{i-1}$$

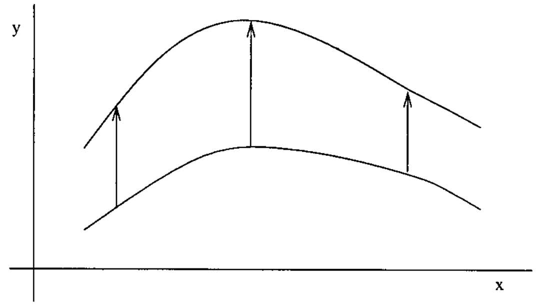


Figure 1.1: An evolutionary symmetry transforms in the vertical direction only.

and hence

$$XG = \frac{\partial}{\partial \epsilon} G(y + \epsilon v)\bigg|_{\epsilon=0}.$$

The expression on the right hand side corresponds to a linearisation of G, in direction v.

Definition 1.1.13 A *Directional (or Lie) Derivative of G in the direction v*, denoted by $\mathcal{D}_v G$, is defined by

$$\mathcal{D}_v G = \frac{\partial}{\partial \epsilon} G(y + \epsilon v)\bigg|_{\epsilon=0} = v\frac{\partial G}{\partial y} + v_1 \frac{\partial G}{\partial y_1} + v_2 \frac{\partial G}{\partial y_2} + ... \tag{1.26}$$

Thus a symmetry is in evolutionary form if and only if its symmetry generator X is a directional derivative $\mathcal{D}_v$ in some direction v.

Let

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

be a symmetry generator of an ODE $G = 0$ and let

$$v = \eta - y_1 \xi.$$

Then

$$X - \xi D_x = (\eta - y_1 \xi)\frac{\partial}{\partial y} + ... = \mathcal{D}_v. \tag{1.27}$$

Since ξD_x is a (trivial) symmetry and a linear combination of two symmetries is also a symmetry, $\mathcal{D}_v$ is also a symmetry generator of $G = 0$. Thus any symmetry can be "rewritten" as a symmetry in evolutionary form.

In general, two symmetry generators are *equivalent* iff they differ by a trivial symmetry iff their evolutionary forms are equal. In particular a symmetry generator in evolutionary form (1.25) is equivalent to a point symmetry iff v depends only on x, y, y_1 and is at most linear in y_1.

Since partial derivatives commute, we have:

Lemma 1.1.14 *A Directional derivative and the total derivative operator commute:*

$$\mathcal{D}_v D_x = D_x \mathcal{D}_v.$$

As a consequence, we have:

Lemma 1.1.15 *Let*

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

be a symmetry generator of G*. Then*

$$X D_x = D_x X + (D_x \xi) D_x.$$

Thus

$$X D_x = D_x X \ (mod\ G{=}0).$$

Proof. Write $X = \mathcal{D}_v + \xi D_x$ where $v = \eta - y_1 \xi$. Then

$$D_x X = D_x \mathcal{D}_v + D_x(\xi D_x) = \mathcal{D}_v D_x + \xi D_x D_x + (D_x \xi) D_x = X D_x + (D_x \xi) D_x$$

□

1.1.9 Lie Algebra of symmetries

It is possible for an ODE $G = 0$ to admit more than one symmetry. However one can show that an ODE of second or higher order admits only finitely many *point* symmetries. In particular, any second order ODE admits at most eight point symmetries, and an ODE of order $n > 2$ admits at most n+4 symmetries (see [7], [21]). Symmetry generators form a vector space since they are solutions of a linear PDE (1.20). More interestingly, a commutator of two symmetry generators is again a symmetry generator:

Theorem 1.1.16 *If X, Y are two symmetry generators of G then their commutator,*

$$Z = [X, Y] = XY - YX,$$

is also a symmetry generator of G.

Proof. Let

$$X = \xi^X \frac{\partial}{\partial x} + \eta^X \frac{\partial}{\partial y} + \eta_1^X \frac{\partial}{\partial y_1} + ..., \; Y = \xi^Y \frac{\partial}{\partial x} + \eta^Y \frac{\partial}{\partial y} + \eta_1^Y \frac{\partial}{\partial y_1} + ...$$

Then

$$Z = \xi^Z \frac{\partial}{\partial x} + \eta^Z \frac{\partial}{\partial y} + \eta_1^Z \frac{\partial}{\partial y_1} + ...$$

with

$$\xi^Z = X\xi^Y - Y\xi^X, \; \eta_i^Z = X\xi_i^Y - Y\xi_i^X, \; i \geq 0, \; \eta_0 = \eta,$$

is a first order differential operator. Furthermore,

$$ZG = XYG - YXG = 0.$$

Thus to show that Z is a symmetry generator, it suffices to show that

$$\eta_i^Z = D_x \eta_{i-1}^Z - y_i D_x \xi^Z, i \geq 1.$$

Since X, Y are symmetry generators, we have

$$\eta_i^Z = X\eta_i^Y - Y\eta_i^X = X(D_x \eta_{i-1}^Y - y_i D_x \xi^Y) - Y(D_x \eta_{i-1}^X - y_i D_x \xi^X).$$

Using Lemma 1.1.15 and some algebra, one can show that this is equal to

$$D_x(X\eta_{i-1}^Y - Y\eta_{i-1}^X) - y_i D_x(X\xi^Y - Y\xi^X) = D_x \eta_{i-1}^Z - y_i D_x \xi^Z$$

□

Let $\hat{x}^\epsilon, \hat{y}^\epsilon$ be two flows with generators X, Y respectively, and let $Z = [X, Y]$. One can show that the generator of the flow

$$\hat{z}^\epsilon = \hat{x}^{-\epsilon} \circ \hat{y}^{-\epsilon} \circ \hat{x}^\epsilon \circ \hat{y}^\epsilon$$

is Z.

Note that if $X = \mathcal{D}_v, Y = \mathcal{D}_w$ are in evolutionary form then their commutator $Z = [\mathcal{D}_v, \mathcal{D}_w]$ is also in evolutionary form, $Z = \mathcal{D}_u$ with u given by

$$u = \mathcal{D}_v w - \mathcal{D}_w v.$$

Note also that a commutator of two point symmetries is a point symmetry.

A *Lie Algebra* of symmetry generators is a vector space of symmetry generators that is closed under the commutation. The *dimension* of a Lie Algebra is its dimension as a vector space. By Theorem 1.1.16, the set of all symmetry generators admitted by an ODE forms a Lie Algebra.

1.2 Reduction of order using symmetries

One of the applications of symmetries is to help in finding explicit solutions of ODEs. If we know a symmetry of an n-th order ODE

$$G = y_n - g(x, y, ..., y_{n-1}) = 0 \tag{1.28}$$

then one can reduce its order by one. For a first order ODE this is is equivalent to finding a general solution of the ODE.

We examine two well-known methods of reduction – canonical coordinates and differential invariants. They are also described in [7] or [21].

1.2.1 Canonical coordinates

Suppose that an n-th order ODE written in solved form,

$$G(r, s, s_1, s_2, ..., s_n) = s_n - g(r, s, s_1, ..., s_{n-1}) = 0 \tag{1.29}$$

admits a point symmetry

$$\hat{r} = r, \hat{s} = s + \epsilon \tag{1.30}$$

whose extensions are $\hat{s}_i = s_i$. Then

$$G(r, s, s_1, s_2, ..., s_n) = 0 \Leftrightarrow G(r, s + \epsilon, s_1, ..., s_n) = 0 \tag{1.31}$$

and hence $g = g(r, s_1, ..., s_{n-1})$ is independent of s. Thus if we let $z(r) = s'(r)$, then the ODE (1.29) becomes an ODE of order $n-1$:

$$z_{n-1} = g(r, z, ..., z_{n-2}). \tag{1.32}$$

If $n = 1$ then (1.32) becomes $z(r) = g(r)$ and hence

$$s(r) = \int^r g(t)dt + C$$

is a general solution to (1.29) for $n = 1$.

In general, any point symmetry can be transformed into (1.30) by using canonical coordinates (cf. Section 1.1.3). Thus if an ODE admits a point symmetry then its order can be reduced by one. We illustrate by example.

Example 1.2.1 Consider an ODE

$$G = y_2 + y_1^3 + \frac{x+1}{x} y_1^2 = 0 \tag{1.33}$$

where $y = y(x), y_1 = y', y_2 = y''$. It admits two point symmetries whose generators are:

$$T = \frac{\partial}{\partial y}$$
$$X = e^y x \frac{\partial}{\partial x} + e^y \frac{\partial}{\partial y} - y_1^2 x e^y \frac{\partial}{\partial y_1} + ...$$

with $[T, X] = X$.

First consider what happens if we use T to reduce the order of (1.33). T is already in canonical form; the reduction of order leads to a first-order Abel ODE after a transformation $z = y_1$:

$$z_1 + z^3 + \frac{x+1}{x} z^2 = 0. \tag{1.34}$$

However the resulting first-order Abel ODE does not have any apparent symmetries. The symmetry X is "lost" because $Xz = z^2 x e^y$ contains y.

However we can use X to reduce (1.33). To do this, we first find canonical coordinates. A solution of

$$Xr(x, y) = e^y x r_x + e^y r_y = 0, \;\; Xs(x, y) = e^y x s_x + e^y s_y = 1$$

is found to be

$$r = xe^{-y}, \; s = -e^{-y} \Leftrightarrow y = -\ln s, \; x = r/s.$$

In new variables $s(r)$, we get

$$G = \frac{s^4}{(rs_1 - s)^3 r}(rs_2 + s_1^2) = 0$$

and the symmetry T becomes

$$T = Tr\frac{\partial}{\partial r} + Ts\frac{\partial}{\partial s} + \ldots = -r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s} + 0\frac{\partial}{\partial s_1} + s_2\frac{\partial}{\partial s_2} + \ldots$$

This time the reduced equation for $z = s_1$,

$$rz_1 + z^2 = 0 \tag{1.35}$$

does inherit a symmetry

$$T^z = (Tr)\frac{\partial}{\partial r} + (Tz)\frac{\partial}{\partial z} = -r\frac{\partial}{\partial r}$$

whose canonical coordinates $(v, w(v))$ with

$$T^z v = 0, T^z w = 1$$

are given by $v = z$, $w = -\ln r$. Transforming (1.35) we get

$$w' = v^{-2}$$

which has a general solution

$$w = -v^{-1} + K_1.$$

Untransforming we get

$$z(r) = \frac{1}{\ln r + K_1}$$

$$s(r) = \int^r \frac{dt}{K_1 + \ln t} + K_2$$

$$e^{-y} = \int^{xe^{-y}} \frac{dt}{K_1 + \ln t} + K_2$$

which is a general solution to (1.33).

The above example illustrates an important point. In general, if an ODE admits two point symmetries X, Y with $[X, Y] = \alpha X$ then one should start reducing its order using X.[3] Then the resulting reduced ODE will "inherit" the symmetry Y. However if Y is used first, then the symmetry X will be "lost"[4]. See [7] or [21] for proof. See also [7] for an algorithm to reduce the order of an ODE admitting an r-dimensional solvable Lie Algebra, by r.

1.2.2 Differential invariants

In this section we shall only consider point symmetries. However the results generalise to Lie-Bäcklund symmetries as well.

Given a point symmetry generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + ...,$$

any solution w of $Xw = 0$ is called an *invariant* of X. An invariant w is an *invariant of order* $i, i \geq 0$ if it depends explicitly y_i (where $y_0 = y$) but does not depend on y_j for $j > i$: $w = w(x, y, \ldots, y_i)$. An invariant is a *differential* invariant if it is of order greater than zero.

An invariant of order zero, $u = u(x, y)$ must satisfy a PDE

$$\xi(x, y)u_x + \eta(x, y)u_y = 0.$$

Hence there is exactly one functionally independent invariant of order zero. Similarly, there are exactly $i + 1$ functionally independent invariants of order at most i.

If two indepedendent invariants are known then one can generate an infinite sequence of independent invariants using the following theorem.

Theorem 1.2.2 *Let u, w be any two invariants of X. Then*

$$\frac{dw}{du} = \frac{D_x w}{D_x u} \tag{1.36}$$

[3]If an ODE admits an n-dimensional Lie Algebra L, then it will be shown in Theorem 4.2.5 that for any $Y \in L$, there exists $X \in L$ with $X \neq Y$ such that either $[X, Y] = \lambda X$ for some possibly complex number λ or else $[X, Y] = Y$. Furthermore, λ is real for $n = 2$. Thus a reduction of two orders is always possible for an ODE of order greater than one that admits a two-dimensional Lie Algebra of symmetries.

[4]In fact, it becomes a Lie-Bäcklund symmetry. See [7], Chapter 7.3 for details.

is also an invariant of X.

Thus if w *is a differential invariant of order* $i \geq 1$ *and* u *is an invariant of order 0, then* $\frac{dw}{du}$ *is a differential invariant of order* $i+1$.

Proof. Since X is a differential operator of first order, the quotient rule holds so that

$$X\left(\frac{D_x w}{D_x u}\right) = \frac{(D_x u)X(D_x w) - (D_x w)X(D_x u)}{(D_x u)^2}$$

and using Lemma 1.1.15 and the fact that $Xv = Xw = 0$ shows that the numerator of the resulting expression is zero. □

Now suppose X is a point symmetry of an n-th order ODE $G = 0$, and let $u = u(x,y), w = w(x,y,y_1)$ be invariants of order zero and one respectively[5] and let

$$w_i = \frac{D_x w_{i-1}}{D_x u} = \frac{dw_{i-1}}{du}, i = 1, ..., n, \text{ with } w_0 = w. \tag{1.37}$$

Then any other invariant of order at most n can be written as a function of $u, w, w_1, ..., w_{n-1}$. Since $XG = 0 \pmod{G{=}0}$ it follows that $G = \hat{G}(u, w, w_1, ..., w_{n-1}) \pmod{G{=}0}$ for some function $\hat{G}$. Thus $G = 0 \Leftrightarrow \hat{G} = 0$. But because of choice (1.37) of w_i,

$$\hat{G}(u, w, w_1, ..., w_{n-1}) = 0 \tag{1.38}$$

is an ODE of order $n-1$ for $w(u)$. Suppose we could solve (1.38) to obtain a general solution

$$w = \phi(u, C_1, C_2, ..., C_{n-1}).$$

Then

$$w(x, y, y_1) = \phi(u(x,y), C_1, ..., C_{n-1})$$

is a first-order ODE for $y(x)$. This first-order ODE still admits X and hence its solution can be found using the method of canonical coordinates.

[5]See [7] for an algorithm of obtaining w through quadrature, once u is known

Example 1.2.3 Consider the scaling symmetry, whose infinitesimals are $\xi = 0, \eta = y$. The corresponding symmetry generator is

$$X = y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}.$$

Note that $u = x$ is an invariant of X (of order 0). To find an invariant $w(x, y, y_1)$ of order 1 we need to find a solution to

$$yw_y + y_1 w_{y_1} = 0.$$

Using the method of characteristics we find:

$$\frac{dy}{y} = \frac{dy_1}{y_1} \Rightarrow \ln(y_1) - \ln(y) = \text{const.} \Rightarrow \frac{y_1}{y} = \text{const.}$$

Thus

$$w = \frac{y_1}{y}$$

is a first-order invariant. An invariant of second order is then given by

$$w_1 = \frac{dw}{du} = \frac{D_x w}{D_x u} = \frac{y_2 y - y_1^2}{y^2} = \frac{y_2}{y} - w^2$$

Note that $w_1 + w^2 = y_2/y$, and in general $y_i/y_j, i, j \geq 0$ are also invariants of X.

Example 1.2.4 Consider the harmonic oscillator

$$G = y_2 + y = 0.$$

G is invariant under $X = y\frac{\partial}{\partial y}$ so we can write G in terms of the invariants

$$u = x,\ w = \frac{y_1}{y},\ w_1 = \frac{D_x w}{D_x u} = \frac{y_2}{y - w^2}$$

of X that were computed in example 1.2.3. We obtain:

$$y_2 = (w_1 + w^2)y$$

and hence

$$G = y_2 + y = (w_1 + w^2 + 1)y = 0.$$

Thus we have to solve the equation

$$\frac{dw}{du} + w^2 + 1 = 0$$

for $w(u)$. This equation is translation-invariant in u and its solution is thus found to be

$$u + \int^w dt/(1+t^2) = C$$

or

$$w = \tan(C - u).$$

Substituting back $u = x, w = y_1/y$ we arrive at the equation

$$y_1 = \tan(C - x)y \tag{1.39}$$

which still admits the scaling symmetry

$$X = y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1}.$$

Using canonical coordinates, $r = x, s = \ln y$ with $Xr = 0, Xs = 1$ the ODE (1.39) becomes

$$s' = \tan(C - r)$$

and thus

$$s = \int \tan(C - r)dr = \ln(\cos(C - r)) + K$$

Untransforming we get the solution:

$$y = C_1 \cos(C_2 - x).$$

In Section 2.1.3 we will present a generalisation of the method of differential invariants that also works for non-point symmetries and will allow us to reduce the order of an ODE by two *without* changing coordinates.

1.3 Discussion

In Sections 1.1.1-1.1.4 we defined what is a symmetry of an ODE. A symmetry can be recovered from its symmetry generator using Lie's Fundamental Theorem. One can find all symmetry generators of an ODE by solving a linear system of PDEs (1.20, 1.18).

For point symmetries of order $n > 1$ the system (1.20, 1.18) is overdetermined and has only finitely many solutions. The problem of finding solutions to overdetermined systems is to a large extent algebraic, and algorithms are available (see [17], [19], [8]) to reduce such systems. The output of such algorithms is an equivalent system, but much simpler, and whose solutions are often trivially found.

In Section 1.1.8 we showed how to represent any symmetry in an equivalent evolutionary form which leaves independent variables unchanged. The advantage of doing this will become clear in Chapter 3.

All symmetries of an ODE form a Lie algebra under commutation. Its structure is important in applications that use many symmetries at once (for instance see [7] where an algorithm for reducing an n-th order ODE admitting a solvable r-dimensional Lie algebra of point symmetries is presented).

In Section 1.2 we have presented two methods of reducing the order of ODEs using symmetries.

The first method, originally developed by Lie (Section 1.2.1), involves "straightening out" the flow using canonical coordinates. In canonical coordinates the order of the ODE can be directly reduced using a quadrature. While this technique works for point symmetries, it does not generalise nicely to Lie-Bäcklund symmetries. However for first order ODEs this technique is general enough, since any symmetry of a first order ODE is a point symmetry. Another feature of this method is that it requires a change of coordinates to work.

Alternatively, for ODEs of second order or higher, one can use the method of differential invariants (Section 1.2.2), also originally developed by Lie. While this method generalises to

Lie-Bäcklund symmetries, it requires one to find the general solution of an auxiliary ODE (1.38) of order $n-1$ before a successful reduction can be obtained!

In Section 2.1.3 a generalisation of the method of differential invariants will be presented, by using the concept of conservation laws.

Chapter 2
Conservation laws and integrating factors

Not all ODEs whose exact solution can be found admit point symmetries. For example, if we start with a Bernoulli equation,

$$\frac{y'}{y} = y^{n-1} + f(x)$$

whose solution is known and differentiate it, we get a second order ODE

$$\frac{y''}{y} - \frac{y'^2}{y^2} = (n-1)y^{n-2}y' + f'(x)$$

which - except for very specific $f(x)$ - does not admit any point symmetries (see [12]). Nonetheless, its solution can be found exactly since it is equivalent to

$$\frac{y'}{y} = y^{n-1} + f(x) + C$$

which is a solvable (Bernoulli) ODE.

The above reduction is an example of a reduction in the *original variables.* All such reductions can be obtained from *conservation laws* of an ODE. In this chapter we study such reductions.

In Section 2.1.1 we define conservation laws and show how one can find them directly from their definition.

The task of finding conservation laws can be simplified when an ODE admits symmetries. In particular, a symmetry generator applied to a conservation law again results in a conservation law (see Section 2.1.2 or [21] or [20]). Based on this fact, two related ansatzes will be developed in Sections 2.1.3 and 2.1.4 that use a symmetry to look for a conservation law.

Instead of seeking the conservation laws, one can seek *integrating factors* which characterise conservation laws. This is done by seeking particular solutions of the *determining equations* that any integrating factor must satisfy. In Section 2.2 we review the developement of these determining equations (see also [16] and [1]). As a consequence of this developement, one can obtain a conservation law corresponding to a given integrating factor, using only integration and arithmetic operations.

In [3] it was shown that one can also use an integrating factor and a symmetry to generate a conservation law. This is discussed in Section 2.3, where we also show that the conservation law thus generated is the same that one gets using the method of the previous paragraph. The relationship between symmetries and integrating factors will be further explored in Chapter 3, which will discuss self-adjoint systems and Noether's Theorem.

2.1 Conservation laws and using symmetries to find them

2.1.1 Conservation laws

A *conservation law* of an ODE

$$G(x, y, y_1, ..., y_n) = 0 \tag{2.1}$$

is an expression $P(x, y, y_1, ..., y_{n-1})$ such that

$$D_x P = 0 \text{ (mod } G{=}0). \tag{2.2}$$

Thus, $P(x, y, y_1, ..., y_{n-1}) = C$ for some constant C, for any solution $y = \phi(x)$ of (2.1).[1] For example $x - \ln y = C$ is a conservation law of a differential equation $y' - y = 0$.

Example 2.1.1 Let $y = \phi(x)$ be any solution of

$$G = y_2 + y = 0 \tag{2.3}$$

and let

$$P = \frac{y^2}{2} + \frac{y_1^2}{2}.$$

[1]According to our definition of a conservation law, a constant is a (trivial) conservation law of any ODE.

We have

$$\frac{dP}{dx} = y_1(y_2 + y) = y_1 G.$$

Thus P is constant for each solution $y = \phi(x)$ of (2.3). Hence P is a conservation law of (2.3).

Note that solving the original equation (2.3) of second order is equivalent to solving $P = C$ which is of first order. Thus by finding a conservation law we have reduced the order of the original ODE.

Example 2.1.2 Let $P = P(x, y, y_1) =$ const. be a conservation law of a second order ODE in solved form,

$$G = y_2 - g(x, y, y_1) = 0.$$

Then (2.2) becomes

$$P_x + y_1 P_y + g P_{y_1} = 0.$$

Any conservation law of an n-th order ODE $G = 0$ is a solution of (2.2) (and vice-versa) which is a linear first order PDE in n variables. Such a PDE has infinitely many solutions (since a function of any solution is again a solution), but only n of them are functionally independent.[2]

Hence any function of a conservation law is once again a conservation law, and an n-th order ODE has infinitely many conservation laws, but only n of them are functionally independent. By using n functionally independent conservation laws to eliminate derivatives of y one can obtain a general solution depending on n arbitrary constants.

Example 2.1.3 Consider

$$G = y_2 + y_1 = 0.$$

Then (2.2) becomes

$$P_x + y_1 P_y - y_1 P_{y_1} = 0.$$

[2]Two expressions $a(x)$ and $b(x)$ are said to be functionally independent if the only solution F to $F(a, b) = 0$ is the zero solution. For example $a = x_1 - x_2, b = x_1^2 - 2x_1x_2 + x_2^2$ are functionally dependent whereas $a = x_1, b = x_2$ are not.

By inspection, $P = P_1 = y_1 + y$ is one of the solutions to the above PDE. By looking for solutions $P = P_2$ independent of y, we find that $P_2 = y_1 e^x$ is another solution of the same PDE independent of P_1. Hence

$$P_1 = y_1 + y = C_1, P_2 = y_1 e^x = C_2$$

are two independent conservation laws of G. Eliminating y_1 we obtain the general solution,

$$y = C_1 - C_2 e^{-x}.$$

2.1.2 Action of symmetry generators on conservation laws

The fundamental relationship between symmetries and conservation laws is provided by the following lemma (see [21], [20])

Lemma 2.1.4 *if X is a symmetry of G and P is a conservation law of G then $X(P)$ (mod G=0) is a conservation law.*

Proof. Let $X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \ldots$, and apply Lemma 1.1.15:

$$D_x X P = X D_x P + (D_x \xi) D_x P = 0 \text{ (mod } G\text{=0)}.$$

□

Below we show that the converse is also true in the following sense:

Lemma 2.1.5 *Let P_0 be a (possibly trivial) conservation law and X be a symmetry of G. Then there exists a conservation law P of G such that $XP = P_0$ (mod G=0).*

Proof. To simplify notation, we shall assume that G is of second order. Any second order ODE has two functionally independent conservation laws (cf. Section 2.1.1). Let Q and R be any two such independent conservation laws.

We first show that at least one of XQ (mod G=0), XR (mod G=0) is non-zero. Note that the system of PDEs

$$\begin{cases} XP = 0 \\ D_x P = 0 \end{cases} \quad \text{(mod } G\text{=0)}. \tag{2.4}$$

has at most one functionally independent solution for P when G is of second order. But R, Q are functionally independent, and both satisfy $Q', R' = 0$ (mod G=0). Thus at most one of R, Q can satisfy (2.4). Thus at least one of XQ (mod G=0), XR (mod G=0) is non-zero.

Now by Lemma (2.1.4), XQ (mod G=0), XR (mod G=0) are again conservation laws. Since there are at most two functionally independent conservation laws of G, any conservation law of G must be a function of Q, R. So let

$$X(Q) = f(Q,R),\ X(R) = g(Q,R),\ P_0 = h(Q,R)$$

for some functions f, g, h, with at least one of f, g non-zero.

Now let $P(Q,R)$ be any non-trivial solution of a PDE

$$f(Q,R)\frac{\partial}{\partial Q}P(Q,R) + g(Q,R)\frac{\partial}{\partial R}P(Q,R) = h(Q,R).$$

Then by chain rule we have

$$\begin{aligned} X(P(Q,R)) =& \ X(Q)\tfrac{\partial}{\partial Q}P(Q,R) + X(R)\tfrac{\partial}{\partial R}P(Q,R) \\ =& \ f(Q,R)\tfrac{\partial}{\partial Q}P(Q,R) + g(Q,R)\tfrac{\partial}{\partial R}P(Q,R) = P_0. \end{aligned}$$

Hence $P(Q,R)$ is the desired conservation law. □

In the following section, this lemma will be used to generalise the method of differential invariants to Lie-Bäcklund symmetries.

2.1.3 Using symmetries to find conservation laws

Lemma 2.1.5 provides an ansatz for looking for a conservation law P of $G = 0$: we seek solutions P of a system

$$\begin{cases} X(P) = 0 \\ D_x P = 0 \end{cases} \quad (\text{mod } G{=}0). \tag{2.5}$$

This ansatz was first used by [12] and [11]. We illustrate with examples.

Example 2.1.6 For a second-order ODE

$$G = y_2 - g(x, y, y_1) = 0$$

the system (2.5), when written out, becomes

$$\begin{cases} P_x\xi + P_y\eta + P_{y_1}\eta_1 = 0 \\ P_x + y_1 P_y + g P_{y_1} = 0 \end{cases}$$

where $P = P(x, y, y_1)$ and $\eta_1 = D_x\eta - y_1 D_x\xi$ (mod G=0).

To solve system (2.5), first find $n-1$ differential invariants $u, w, ...$ of X (which can be obtained from any two independent invariants - see Theorem 1.2.2) and write $P = P(u, w, ...)$. Then substitute into $D_x P = 0$ (mod G=0) and solve the resulting system for $P(u, w, ...)$.

Example 2.1.7 Let's apply this method to the harmonic oscillator

$$G = y_2 + y = 0$$

which admits symmetries

$$T = \frac{\partial}{\partial x}, X = y\frac{\partial}{\partial y}.$$

We first compute a conservation law P with $TP = 0$.

The invariants of T are $u = y, w = y_1$; so $P = P(u, w)$. Plugging this into $P' = 0$ (mod G=0) we obtain $P_u w - P_w u = 0$. Using method of characteristics: $\frac{du}{w} + \frac{dw}{u} = 0, udu + wdw = 0, u^2/2 + w^2/2 = \text{const.}$, we find

$$P = y^2 + y_1^2 = C_1$$

is a conservation law invariant under T. It is now possible to find the solution of $P - C_1 = 0$ since it is invariant under T. Alternately, we will compute a conservation law Q with $XQ = 0$.

As before, $u = x, w = y_1/y$ are differential invariants of X so

$$Q = Q(u, w).$$

Taking total derivative (mod G=0) we obtain

$$u'Q_u + w'Q_w \text{ (mod } G\text{=0)} = Q_u + (-y^2 - y_1^2)/y^2 Q_w = Q_u - (w^2+1)Q_w = 0.$$

Using the method of characteristics we get $du + \frac{dw}{w^2+1} = 0$ or $\frac{dw}{du} + w^2 + 1 = 0$ (which is exactly the same equation as obtained using differential invariants). This can be integrated and we obtain a solution for Q:

$$Q = x + \arctan(y_1/y) = C_2.$$

Solving the system $P = C_1, Q = C_2$ we obtain a full solution of G:

$$x + \arctan(\frac{\sqrt{C_1 - y^2}}{y}) = C_2.$$

Solving for y we obtain

$$y = \pm\sqrt{C_1}\cos(x - C_2).$$

Since this method works for any symmetry (point or Lie-Bäcklund), we can rewrite the symmetry in its evolutionary form, $X = \mathcal{D}_v$. Then the first invariant is just $u = x$ and another invariant is any solution to the PDE

$$\mathcal{D}_v w = 0 \text{ (mod } G\text{=0)}. \tag{2.6}$$

All additional invariants can be obtained through Theorem 1.2.2 which, for a symmetry in evolutionary form, becomes $w_i = D_x w_{i-1}$ (since $u = x$ is an invariant).

Example 2.1.8 For a general second order ODE

$$G = y_2 - g(x, y, y_1) = 0,$$

admitting a symmetry

$$X = \mathcal{D}_v = v(x, y, y_1)\frac{\partial}{\partial y} + ...,$$

(2.6) becomes

$$w_y v + w_{y_1} v' = 0 \text{ (mod } G{=}0) \tag{2.7}$$

whose solution can be obtained using the method of characteristics:

$$\frac{dy}{v} = \frac{dy_1}{v'}$$

where $v = v(x, y, y_1)$, $v' = v_x + y_1 v_y + g v_{y_1}$.

Example 2.1.9 To illustrate that this method works for Lie-Bäcklund symmetries, consider an ODE

$$G = y_2 - g = 0, g = \frac{2y_1}{2x + 2y + 2y_1 - y_1 x}. \tag{2.8}$$

It admits a Lie-Bäcklund symmetry $X = y_1^2 \frac{\partial}{\partial y}$. Equation 2.7 becomes

$$w_y y_1^2 + 2 w_{y_1} g y_1 = 0.$$

Using the method of characteristics we find a solution:

$$w = (y_1 - y_1 x/2 + y + 2) e^{\frac{-y_1}{2}}$$

and thus

$$P = P(x, w)$$

is a conservation law for some P. In particular,

$$D_x w = \frac{1}{2} e^{-y_1/2} (y_1 - y_2 (x + y + y_1 - \frac{1}{2} y_1 x)) = 0 \text{ (mod } G{=}0)$$

and thus w itself is a conservation law. So the original ODE is equivalent to

$$(y_1 - y_1 x/2 + y + 2) e^{\frac{-y_1}{2}} = \text{const.} \tag{2.9}$$

This conservation law is still invariant under X. While in theory the method of canonical coordinates can be used to integrate (2.9) further, in practice it is necessary to isolate y_1 before finding canonical coordinates. We will overcome this difficulty by finding another conservation law of (2.8) directly, using X and the method described in the next section.

2.1.4 Using one symmetry to find two conservation laws

In the previous section we considered an ODE $G = 0$ admitting a symmetry X and showed how it may be possible to find a conservation law P of G by solving the system

$$\begin{cases} XP = 0 \\ D_x P = 0 \end{cases} \quad (\text{mod } G{=}0). \tag{2.10}$$

In this section we will show that if it is possible to find such a P then it may be possible to find another non-trivial conservation law Q for which

$$\begin{cases} XQ = 1 \\ D_x Q = 0 \end{cases} \quad (\text{mod } G{=}0). \tag{2.11}$$

By Lemma 2.1.5, such a Q always exists, since 1 is a (trivial) conservation law. Furthermore, Q must be functionally independent of P.

If G is of second order and P can be found, then Q can always be found using only two quadratures, without having to solve any additional ODEs, as we now illustrate using examples.

Example 2.1.10 Harmonic oscillator, $G = y_2 + y = 0$, admits a symmetry $X = \frac{\partial}{\partial x}$ whose (differential) invariants are $u = y, w = y_1$. In (2.3) we found that

$$P(u, w) = u^2 + w^2$$

satisfies (2.10). We now seek Q of the form

$$Q = F(u, w) + s(x, y, y_1)$$

which satisfies (2.11). Imposing $XQ = 1 \Rightarrow XF + Xs = Xs = 1$ we obtain $s_x = 1$, whose particular solution is

$$s = x.$$

In general, once u, w are known, s can always be obtained through quadrature using the method of characteristics.

Imposing $Q' = 0$ (mod G=0) leads to

$$F_u w - F_w u + 1 = 0. \tag{2.12}$$

We already found a solution $P = u^2+w^2$ to the associated homogeneous problem $P_u w - P_w u = 0$; thus the solution to (2.12) is obtained by solving $F_u w = -1$ with $w = \pm\sqrt{P - u^2}$ from which we find a solution to (2.12):

$$F = \arccos(uP^{-1/2}).$$

Thus

$$Q = \arccos(uP^{-1/2}) + x$$

satisfies (2.11). Solving for u we obtain

$$u = \pm\sqrt{P}\cos(Q - x)$$

which leads to a general solution:

$$y = C_1 \cos(C_2 - x).$$

Example 2.1.11 Consider an ODE from example 1.2.1:

$$G = y_2 + y_1^3 + (1 - x^{-1})y_1 = 0. \tag{2.13}$$

It has two symmetries:

$$X = xe^y \frac{\partial}{\partial x} + e^y \frac{\partial}{\partial y}, \; T = \frac{\partial}{\partial y}.$$

We will use the symmetry X to find a general solution of (2.13).

The invariants of X of zero and first orders are found to be

$$u = xe^{-y}, \; w = \frac{xy_1 - 1}{y_1}.$$

Thus a conservation law P with $XP = 0$ must be a function of u, w: $P = P(u, w)$. In addition it must satisfy

$$D_x P = (D_x u)P_u + (D_x w)P_w = 0 \text{ (mod } G\text{=0)} \tag{2.14}$$

where

$$D_x u = e^{-y}(1 - xy_1), \; D_x w = \frac{1 - xy_1}{x} \text{ (mod } G{=}0)$$

and thus

$$\frac{D_x w}{D_x u} = \frac{1}{u} \text{ (mod } G{=}0).$$

Hence (2.14) is equivalent to:

$$uP_u + P_w = 0.$$

Thus

$$P = ue^{-w} \tag{2.15}$$

is a conservation law of (2.13). We now seek a conservation law Q with $XQ = 1$. Once again, assume Q has form

$$Q = F(u, w) + s$$

where F, s are to be found. Then $XQ = Xs = 1$. By inspection,

$$s = -e^{-y}$$

satisfies $Xs = 1$. It remains to satisfy

$$0 = D_x Q = (D_x u) F_u(u, w) + (D_x w) F_w(u, w) + D_x s \text{ (mod } G{=}0).$$

where

$$D_x s = y_1 e^{-y}$$

and hence

$$\frac{D_x s}{D_x u} = -\frac{1}{w}.$$

It follows that F must satisfy

$$uF_u + F_w = \frac{u}{w}.$$

The solution to the homogeneous part of this equation is given by P found in (2.15). Thus to find F one must solve

$$F_w = \frac{u}{w} \text{ with } ue^{-w} = P$$

or

$$F_w = e^w P w$$

Its solution is given by

$$F = P \int^w \frac{e^t}{t} dt$$

which leads to a conservation law

$$Q = P \int^w \frac{e^t}{t} dt - e^{-y}$$

If we now replace

$$u = xe^{-y}, w = \ln(u/P) = \ln x - y - \ln P$$

we get a general solution

$$Q = P \int^{\ln x - y - \ln P} \frac{e^t}{t} dt - e^{-y}.$$

2.2 Integrating factors and Euler Operator

A conservation law can be characterised by an associated *integrating factor.* There is a close relationship between integrating factors and Euler-Lagrange equations from the variational calculus. This relationship leads to determining equations for integrating factors. In this section we define integrating factors and derive the determining equations for them. This material in standard, see for instance Olver [16]. Following [1], we will also show how to compute the conservation law corresponding to a given integrating factor using only integration and arithmetic operations.

2.2.1 Integrating factors

Definition 2.2.1 The expression $G(x, y, ..., y_n)$ is *exact* or a *divergence* if it is a total derivative of some expresson $P(x, y, y_1, ..., y_{n-1})$:

$$G = D_x(P)$$

For example y_1 and yy_1 are exact since they are total derivatives of y and $y^2/2$ respectively. On the other hand y or yy_2 are not exact.

Definition 2.2.2 w is an *integrating factor* of G if wG is exact.

By definition of exactness, one can then find an expression P such that

$$wG = D_x(P). \tag{2.16}$$

Note that if G is exact then it has an integrating factor 1.

The above condition is equivalent to (2.2): $D_xP = 0$ (mod G=0), when the order of P is less than the order of G. Thus P is a conservation law of G iff there exists an integrating factor w with $D_xP = wG$.

Example 2.2.3 Consider an ODE $G = y_1 - y = 0$, let $P = \ln(y) - x$ and let $w = 1/y$. Then $D_xP = y_1/y - 1 = wG$. Hence $P = C$ is a conservation law of $G = 0$ corresponding to the integrating factor w. Solving $P = C$ for y we obtain the general solution to the ODE, $y = Ce^x$.

2.2.2 Adjoint Directional Derivative and Euler Operator

Recall from Section 1.1.8 that the directional derivative $\mathcal{D}_vG$ is defined by

$$\mathcal{D}_vG = \frac{\partial}{\partial \epsilon}G(y+\epsilon v)\Big|_{\epsilon=0} = v\frac{\partial G}{\partial y} + v'\frac{\partial G}{\partial y_1} + v''\frac{\partial G}{\partial y_2} + \dots$$

Definition 2.2.4 An *adjoint of a directional derivative* is an operator $\mathcal{D}_w^*$ such that $w\mathcal{D}_vG - v\mathcal{D}_w^*G$ is a total derivative, for any w, v, G.

An explicit formula for $\mathcal{D}_w^*$ is obtained using integration by parts:

Theorem 2.2.5 *Let*

$$\mathcal{D}_w^*G = wG_y - D_x(wG_{y_1}) + D_x^2(wG_{y_2}) - \dots \tag{2.17}$$

Then $\mathcal{D}_w^*$ is an adjoint of a directional derivative and satisfies the identity:

$$w\mathcal{D}_vG = v\mathcal{D}_w^*G + D_xS(w, v, G) \tag{2.18}$$

where $\mathcal{S}$ is given by

$$\begin{aligned} \mathcal{S}(w,v,G) \quad &= wG_{y_1}v \\ &+wG_{y_2}v' - (wG_{y_2})'v \\ &+wG_{y_3}v'' - (wG_{y_3})'v' + (wG_{y_3})''v \\ &+wG_{y_4}v''' - (wG_{y_4})'v'' + (wG_{y_4})''v' - (wG_{y_4})'''v \\ &+... \\ &= \Sigma_{i\geq 1}\Sigma_{j=0}^{i-1}(-1)^j D_x^{(j)}(wG_{y_i})D_x^{(i-j-1)}(v) \end{aligned} \tag{2.19}$$

where $(*)' = D_x *$ and $G_{y_i} = \frac{\partial G}{\partial y_i}$.

Proof. The theorem follows from the recursive application of the Leibnitz rule $ba' = (ab)' - ab'$:

$$\begin{aligned} w\mathcal{D}_v G \quad &= \quad wG_y v + wG_{y_1}v' + wG_{y_2}v'' + wG_{y_3}v''' + ... \\ &= \quad wG_y v \\ &\quad\; +(wG_{y_1}v)' - (wG_{y_1})'v \\ &\quad\; +(wG_{y_2}v' - (wG_{y_2})'v)' + (wG_{y_2})''v \\ &\quad\; +(wG_{y_3}v'' - (wG_{y_3})'v' + (wG_{y_3})''v)' - (wG_{y_3})'''v \\ &\quad\; +... \end{aligned}$$

The right border of the above triangle gives $v\mathcal{D}_w^* G$, the rest is $\mathcal{S}'(w,v,G)$. □

Of special interest will be the case $w = 1$:

Definition 2.2.6 The *Euler operator* E is

$$E = \mathcal{D}_1^* = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y_1} + D_x^2 \frac{\partial}{\partial y_2} - ... \tag{2.20}$$

The Euler operator satisfies *Euler identity*:

$$\mathcal{D}_v G = vEG + D_x \mathcal{S}(1,v,G). \tag{2.21}$$

Example 2.2.7 For a general third order ODE we have $\mathcal{D}_w^* G = wG_y - (wG_{y_1})' + (wG_{y_2})'' - (wG_{y_3})'''$ and $\mathcal{S}(w,v,G) = wG_{y_1}v \; + wG_{y_2}v' - (wG_{y_2})'v \; + wG_{y_3}v'' - (wG_{y_3})'v' + (wG_{y_3})''v$.

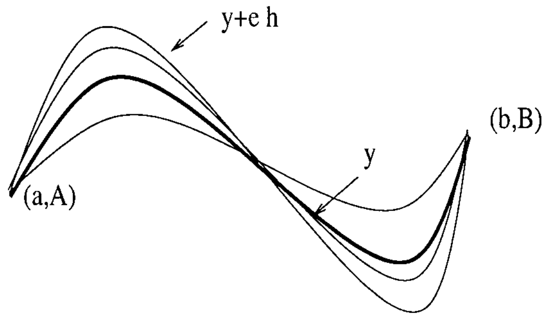


Figure 2.1: A maximiser and curves nearby.

Let $P = y_2 + y,\ G = (y_3 + y_1)/y_2,\ w = y_2,\ v = y$. Then v is a symmetry of $G = 0$, $P' = wG$ and so w is an integrating factor with P being the corresponding conservation law.

To compute $\mathcal{D}_w^* G$ and $\mathcal{S}(w, v, G)$ we first compute $wG_y = 0,\ wG_{y_1} = 1,\ wG_{y_2} = -G,\ wG_{y_3} = 1$, so $\mathcal{D}_w^* G = 0 - 0 + (-G)'' - 0 = 0$ (mod G=0) and $\mathcal{S}(w, v, G) = y - Gy_1 + G'y + y_2 = P$ (mod G=0).

2.2.3 Euler-Lagrange Equations

We have introduced the Euler operator above to treat the following *basic problem of the calculus of variations*: Find a curve $y = \phi(x)$ that minimizes a functional $\Lambda[y]$:

$$\Lambda[y] = \int_a^b L(x, y, y_1, ..., y_n)dx \tag{2.22}$$

with $y_1 = y', ..., y_n = y^{(n)}$. Here L is called a *Lagrangian*, and we minimize over all possible functions $y = \phi(x) \in S$ where S is the set of all smooth[3] functions with prescribed values at fixed endpoints:

$$S = \left\{ \phi(x) \in C^\infty |\ \phi(a) = A, \phi(b) = B, \phi^{(i)}(a) = A_i, \phi^{(i)}(b) = B_i, i = 1..n-1 \right\}. \tag{2.23}$$

Example 2.2.8 If we choose $L(x, y, y_1) = \sqrt{1 + y_1^2}$ then (2.22) is the length of any curve $y = \phi(x)$ from a to b and hence the minimum of Λ is the shortest smooth path between two points (a, A) and (b, B).

[3]To simplify presentation, we define smooth to mean C^∞, although only C^n is required.

The idea to find a minimiser $y = \phi(x)$ is as follows. If $y = \phi(x)$ is a minimiser, $\Lambda[y]$ must be less then Λ[any curve nearby] (see Figure 2.2.3). In particular, if we take any *variation*, i.e. any smooth function h with $y + h \in S$ then we expect that $\Lambda[y] \leq \Lambda[y + \epsilon h]$ for any sufficiently small ϵ. So if we let $f(\epsilon) = \Lambda[y + \epsilon h]$ then f has a minimum at $\epsilon = 0$ and hence $f'(0) = 0$. Thus if $y = \phi(x)$ is a minimizer of (2.22) then

$$f'(0) = \frac{\partial \Lambda[y + \epsilon h]}{\partial \epsilon}|_{\epsilon=0} = \mathcal{D}_h \int_a^b L dx = \int_a^b \mathcal{D}_h L dx = 0$$

where $L = L(x, y, ..., y_n)$. This must hold for all h. Using (2.21) we get:

$$\int_a^b \mathcal{D}_h L dx = \int_a^b h E(L) dx + \int_a^b \mathcal{S}'(1, h, L) dx$$

Note that $\mathcal{S}(1, h, L)$ is linear in h and its total derivatives. Also note that $h_i(a) = 0 = h_i(b), 0 \leq i \leq n - 1$. Thus

$$\int_a^b \mathcal{S}'(1, h, L) dx = \mathcal{S}(1, h, L)_{x=b} - \mathcal{S}(1, h, L)_{x=a} = 0$$

So a necessary condition for $y = \phi(x)$ to be a minimizer of (2.22) is that $\int_a^b h E(L) dx = 0$ for an arbitrary variation h. This forces the integrand to be zero, and hence we obtain

Theorem 2.2.9 *If $y = \phi(x)$ is a minimizer of (2.22) over a set S given by (2.23) then*

$$E(L) = 0. \tag{2.24}$$

Note that the converse does not hold.

Definition 2.2.10 The ODE (2.24) is called *Euler-Lagrange equation.*

Example 2.2.11 For a general $L = L(x, y, y_1)$, $EL = L_y - D_x L_{y_1}$. Taking $L = \sqrt{1 + y_1{}^2}$ as in example (2.2.8), we find $EL = (1 + y_1^2)^{-\frac{3}{2}} y_2 = 0 \Rightarrow y_2 = 0$. Thus the shortest (smooth) path between two points, if it exists, must be a straight line.

2.2.4 Kernel of Euler Operator

We now show that the Euler operator annihilates total derivatives. This will lead to the determining equations for integrating factors. The proof presented here is similar to that in [16] and [1].

Suppose that L is exact, i.e. $L = P'$ for some P. Then the functional (2.22),

$$\Lambda[y] = \int_a^b L dx = P(y)|_{x=b} - P(y)|_{x=a}$$

is independent of the path, depending only on the value of y and its derivatives at the endpoints. So $\Lambda[y]$ is constant for all $y \in S$ (where S is given by (2.23)). Thus any $y \in S$ is a minimiser. Thus by Theorem 2.2.9, $EL = 0$ for $y \in S$. But since the restriction on the endpoints was arbitrary, $EL = 0$ for *all* y.

Conversely, suppose that $EL = 0$. Then by (2.21) we have

$$\mathcal{D}_h L = \frac{d}{dx}\mathcal{S}(1, h, L)$$

and so

$$\mathcal{D}_h L|_{y=y+\lambda h} = \frac{d}{dx}\mathcal{S}(1, h, L)|_{y=y+\lambda h}$$

Thus

$$\frac{d}{d\epsilon}L((y+\lambda h)+\epsilon h)|_{\epsilon=0} = \frac{d}{d\lambda}L(y+\lambda h) = \frac{d}{dx}\mathcal{S}(1, h, L)|_{y=y+\lambda h}.$$

Using the Fundamental Theorem of Calculus we obtain:

$$L(y+h) - L(y) = \int_0^1 \frac{d}{dx}\mathcal{S}(1, h, L)|_{y=y+\lambda h} d\lambda = \frac{d}{dx}\int_0^1 \mathcal{S}(1, h, L)|_{y=y+\lambda h} d\lambda.$$

We now choose $h = -y + \hat{h}(x)$ for some $\hat{h}(x)$ such that $L(\hat{h})$ is finite then we obtain

$$L(y) = -\frac{d}{dx}\int_0^1 \mathcal{S}(1, -y+\hat{h}, L)|_{y=y(1-\lambda)+\hat{h}} d\lambda + \frac{d}{dx}\int L(\hat{h}(x))dx$$

Thus we obtain the following three theorems.

Theorem 2.2.12 (Kernel of Euler Operator) *Let $H(x, y, y_1, ...)$ be any differential expression. Then H is exact if and only if*

$$EH = 0 \textit{ for all } y \in C^\infty.$$

Theorem 2.2.13 (Determining equations for Integrating Factors) *w is an integrating factor of G if and only if*

$$E(wG) = 0 \textit{ for all } y \in C^\infty. \tag{2.25}$$

Theorem 2.2.14 *If H is exact,*

$$H = D_x P,$$

then P is given by

$$P = -\int_0^1 \mathcal{S}(1, -y + \hat{h}, H)|_{y=y(1-\lambda)+\hat{h}} d\lambda + \int H(\hat{h}(x))dx \tag{2.26}$$

where $\hat{h}(x)$ is any function such that the above expression is finite.

Theorem 2.2.14 provides a way of finding conservation laws of an ODE from integrating factors: if w is an integrating factor of G then the corresponding conservation law P with $P' = wG$ can be found by applying formula (2.26) to $H = wG$.

Theorems 2.2.12, 2.2.13 appear in [16] and [3]. Theorem 2.2.14 appears in [3].

The difficult step is to find the integrating factor itself: one needs to seek particular solutions of the PDE $E(wG) = 0$. If one assumes that $G = y_n - g(x, y, y_1, ..., y_{n-1})$ then the integrating factor must be of the form $w = w(x, y, y_1, ..., y_{n-1})$. Consequently, since $E(wG) = 0$ must hold for all $y, y_1, y_2, ...$, it splits into a system of PDEs by equating the coefficients of $y_n, y_{n+1}, ..., y_{2n}$ to zero.

A more natural splitting suggested in [16] and [1] is to first solve $E(wG) = 0$ (mod G=0). Using the product rule for derivatives one obtains:

$$E(wG) = \mathcal{D}_w^* G + \mathcal{D}_G^* w. \tag{2.27}$$

But $\mathcal{D}_G^* w = G w_y - (G w_{y_1})' + (G w_{y_2})'' - ... = 0$ (mod G=0). Thus if w is an integrating factor of G then one must have

$$\mathcal{D}_w^* G = 0 \text{ (mod } G\text{=0)}. \tag{2.28}$$

If w satisfies above then it is called an *adjoint symmetry* of $G = 0$. This leads to the following:

Proposition 2.2.15 *An integrating factor is necessarily an adjoint symmetry. Conversly, an adjoint symmetry w is an integrating factor of an ODE $G = 0$ if it satisfies $E(wG) = 0$ for all functions y.*

Since an ODE admits infinitely many integrating factors, the system $E(wG) = 0$ is not overdetermined. So in general, in order to find a particular integrating factor, one needs to assume some extra condition on w, for example, its polynomial dependence on one of $x, y, y_1, ..., y_{n-1}$.

The situation is similar to that of symmetry methods: to find a Lie-Bäcklund symmetry, one needs to assume some extra condition on its form. For symmetries, a "natural" (geometric) condition is to seek point symmetries. This leads to an overdetermined system for ODEs of order two or higher. In general, no geometric interpretation of integrating factors is known, and hence there is no "natural" extra condition that can be imposed. However an important special case occurs when an ODE is *self-adjoint* as will be discussed in Chapter 3.

2.3 Relationship between integrating factors, conservation laws, and symmetries

Given an integrating factor, a conservation law can be found using a quadrature. A different way of finding conservation laws *without quadrature* is possible when a symmetry is also known. The following theorem was first proved in [3]:

Theorem 2.3.1 *Given an ODE $G = 0$, let w be an adjoint symmetry of G and let $\mathcal{D}_v$ be a symmetry of G. Then $\mathcal{S}(w, v, G)$ (mod G=0) is a conservation law.*

Proof. Since v is a symmetry and w is an adjoint symmetry of G, one has $\mathcal{D}_v G = 0$ (mod G= 0) and $\mathcal{D}_w^* G = 0$ (mod $G = 0$). Hence by (2.18), $D_x \mathcal{S}(w, v, G) = 0$ (mod $G = 0$) and thus $\mathcal{S}(w, v, G)$ (mod G=0) is a conservation law. □

As was shown in Theorem 2.1.4, given a conservation law P and a symmetry X of $G = 0$, $Q = XP$ is also a conservation law of $G = 0$. We now show how this is related to the preceeding theorem:

Theorem 2.3.2 *Given an ODE $G = 0$, let w be an integrating factor of G and let P be the conservation law corresponding to w. Then*

$$\mathcal{D}_v P = \mathcal{S}(w, v, G) \ (mod\ G{=}0). \tag{2.29}$$

where $\mathcal{S}(w,v,G)$ is given by (2.19) for any v. Furthermore, if $\mathcal{D}_v$ is a symmetry of $G=0$, then the above expression is a CL.

To prove this theorem we will need the following lemma.

Lemma 2.3.3 *If G is an expression that depends at most on $x, y, y_1, ..., y_n$ and if $D_x G = 0$ then G is constant.*

Proof. Note that

$$D_x G = G_x + y_1 G_y + y_2 G_{y_1} + ... + y_{n+1} G_{y_n} = 0$$

and since G does not depend on y_{n+1}, one must have $G_{y_n} = 0$. Proceeding by induction we have $G_{y_n} = 0 \Rightarrow G_{y_{n-1}} = 0 \Rightarrow ... \Rightarrow G_{y_1} = 0 \Rightarrow G_y = 0 \Rightarrow G_x = 0$. □

The proof of theorem 2.3.2 now consists of three steps.

Step 1 We first show that

$$D_x \mathcal{D}_v P = D_x \mathcal{S}(w,v,G) \text{ (mod } G\text{=0)}. \quad (2.30)$$

Since D_x and $\mathcal{D}_v$ commute (see Lemma 1.1.14) and since $D_x P = wG$ we have

$$D_x \mathcal{D}_v P = \mathcal{D}_v D_x P = \mathcal{D}_v (wG).$$

We now apply the product rule to the differential operator $\mathcal{D}_v$; we get

$$\mathcal{D}_v(wG) = w\mathcal{D}_v G + G\mathcal{D}_v w.$$

Applying equation (2.18) to both terms on the right hand side, we get:

$$w\mathcal{D}_v G + G\mathcal{D}_v w = D_x \mathcal{S}(w,v,G) + D_x \mathcal{S}(G,v,w) + v\mathcal{D}_w^* G + v\mathcal{D}_G^* w.$$

Combining the last two terms by using (2.27) we obtain

$$D_x \mathcal{S}(w,v,G) + D_x \mathcal{S}(G,v,w) + v\mathcal{D}_w^* G + v\mathcal{D}_G^* w = D_x \mathcal{S}(w,v,G) + D_x \mathcal{S}(G,v,w) + vE(wG).$$

Since w is an integrating factor of G, the term $vE(wG)$ vanishes (see Theorem 2.2.12). Putting it all together, we get

$$D_x \mathcal{D}_v P = D_x \mathcal{S}(w, v, G) + D_x \mathcal{S}(G, v, w), \tag{2.31}$$

this being true for all $x, y, y_1, ...$

Now $\mathcal{S}(G, v, w)$ is linear in G and its total derivatives; it can be written as

$$\mathcal{S}(G, v, w) = Gc_0 + (G)'c_1 + (G)''c_2 + ...$$

where c_i are some expressions involving v, w but independent of G. Thus $\mathcal{S}(G, v, w) = 0$ (mod G=0). This proves (2.30).

Step 2 Let

$$L(v) = \mathcal{D}_v P - \mathcal{S}(w, v, G) - \mathcal{S}(G, v, w).$$

We will show that $L(v) = 0$ for all v. Equation (2.29) then follows since, as was shown in Step 1, $\mathcal{S}(G, v, w) = 0$ (mod G=0).

Since L is linear in v and its partial derivatives, one can write

$$L(v) = va_0 + v_x a_1 + v_y a_2 + v_{y_1} a_3 + ... + v_{xx} a_{11} + v_{xy} a_{12} + v_{xy_1} a_{13} + ... \tag{2.32}$$

where $a_i = a_i(x, y, y_1, ..., y_{2n})$ and the equality holds for arbitrary $v = v(x, y, y_1, ..., y_n)$ and arbitrary $x, y, y_1, ...$

Because of (2.31), $D_x(L(v)) = 0$, for any v. Thus, by Lemma 2.3.3, $L(v)$ is independent of $x, y, y_1, ...$, for *any* $v = v(x, y, y_1, ..., y_m)$. This forces all of the coefficients a_i to be zero, as the following argument demonstrates.

Note that $L(1) = a_0$ and thus a_0 is independent of x. Similarly, $L(x) = a_0 x + a_1$ is independent of x and hence $a_1 = -a_0 x + k$ for some constant k, independent of x. Then $L(x^2) = a_0 x^2 + 2a_1 x + 2a_{11}$ is independent of x and thus a_{11} is at most quadratic in x. Similarly, one can show that all the coefficients a_i are at most polynomial in $x, y, y_1, ...$

If one now takes $v = \ln x$ then $L(\ln x) = a_0 \ln x +$ rational expression in x. But the resulting expression must also be independent of x. Thus $a_0 = 0$. Using similar argument, one can show that all coefficients in (2.32) are zero. Thus $L(v) = 0$.

Step 3 The fact that $\mathcal{D}_v P$ is a conservation law is a direct consequence of Lemma 2.1.4. Alternately, $\mathcal{S}(w, v, G)$ is a conservation law by Theorem 2.3.1. $\square$

An integrating factor by itself always leads to a conservation law by using a quadrature (see Theorem 2.2.14). The advantage of the above theorem is that given an integrating factor and a symmetry, a conservation law can be obtained *without* quadrature. The disadvantage is that often the resulting conservation law may be trivial. For example, it can be shown (see [1]) that a first order ODE $G = y_1 - g(x, y) = 0$ that admits a symmetry $\mathcal{D}_v$ also admits an integrating factor $w = \frac{1}{v}$. In this case $\mathcal{S}(w, v, G) = 1$ is a trivial conservation law.

Example 2.3.4 Consider an ODE

$$G = y_2 - y_1^2 - b(x) = 0. \tag{2.33}$$

A search for integrating factors of the form $w = w(x, y)$ reveals that

$$w = e^{-y} c(x)$$

is an integrating factor of (2.33) iff $c(x)$ satisfies

$$c''(x) + b(x)c(x) = 0. \tag{2.34}$$

As well, the ODE (2.33) is independent of the dependent variable and thus admits a symmetry $\mathcal{D}_v$ with $v = 1$. Thus

$$\mathcal{S}(w, v, G) = (-y_1 c(x) - c'(x))e^{-y} = C$$

is a conservation law of $G = 0$. Let $c_1(x), c_2(x)$ be two independent solutions of (2.34). Then for any solution $y = \phi(x)$ of (2.33), there exist constants C_1, C_2 such that

$$(-y' c_1(x) - c_1'(x))e^{-y} = C_1, \ (-y' c_2(x) - c_2'(x))e^{-y} = C_2$$

Eliminating y' from these equations and using the fact that $c_1 c_2' - c_2 c_1'$ is constant, one obtains

$$y = -\ln(c_1 K_1 + c_2 K_2)$$

is a solution of (2.33) for arbitrary K_1, K_2. Thus

$$y = -\ln(c(x))$$

is a solution of (2.33) iff c satisfies (2.34).

Theorem 2.3.2 can also be used to generate an ansatz: Given an integrating factor w (or a symmetry v), one can seek a symmetry v (or an integrating factor w) for which either $\mathcal{S}(w, v, G) = 0$ or $\mathcal{S}(w, v, G) = 1$. We will illustrate this in Chapter 4, in connection with classification of ODEs.

Theorem 2.3.1 was first discovered by Anco & Bluman in [3]. The connection made in Theorem 2.3.2 is new.

2.4 Discussion

In this chapter we have discussed how to find conservation laws of ODEs. We have discussed a direct way of looking for conservation laws by seeking particular solutions of the PDE (2.2), as well as an indirect approach, by looking for integrating factors. These can be found by seeking particular solutions of the PDE (2.25). Once an integrating factor is found, a corresponding conservation law can be found through a quadrature, using the formula (2.26), due to Anco & Bluman [1].

We have also discussed how to generate a conservation law from a known conservation law and a known symmetry (Lemma 2.1.4). Based on it, we developed a method of using symmetries to generate an ansatz for seeking conservation laws (see Section 2.1.3). Such a method is equivalent to the method of differential invariants for point symmetries, but works for general Lie-Bäcklund symmetries as well. Furthermore, a related method discussed in Section 2.1.4 can be used to find a second conservation law if the method of Section 2.1.3 succeeds. The method of Section 2.1.3 was first presented by Gonzales [12] and by Cheb-Terrab et al. [11].

Similar methods can be developed to use a symmetry to generate an ansatz to look for integrating factors, based on Theorem 2.3.2. This will be discussed Section 4.2 for the purposes of classification of solvable ODEs. Theorem 2.3.2 is also of interest in itself, and will be used in the next chapter, in connection with Noether's Theorem.

Chapter 3
Self-adjoint systems

3.1 Introduction

In this chapter we will study a special class of ODEs that are called *self-adjoint ODEs*, for which an integrating factor is also a symmetry. Such ODEs have a Lagrangian formulation and a celebrated result of Noether characterises precisely those symmetries that lead to conservation laws.

In Section 3.2 we show that Euler-Lagrange equations are self-adjoint. We then state and prove Noether's theorem. Noether's theorem is used to find conservation laws for *variational* symmetries of Euler-Lagrange equations. Furthermore, we show that the resulting conservation law admits the variational symmetry that generated it, and hence every variational symmetry provides a two-fold reduction of order [16], [21]. There are two versions of Noether's theorem: the original version, due to Noether [15] and a more modern presentation due to Bessel-Hagen [6]. In this chapter we will only cover the Bessel-Hagen version.

In its full generality, Noether's theorem relies on a quadrature to find a conservation law (though often the quadrature is trivial to perform). However when a self-adjoint ODE admits two or more variational symmetries, we show in Section 3.2.2 that the conservation law corresponding to their commutator can be obtained without any quadrature [3].

In Section 3.3 we also discuss how a scaling symmetry can be used to obtain conservation laws corresponding to known variational symmetries, without using quadrature.

3.2 Self-adjoint systems and Noether's theorem

3.2.1 Characterisation of a self-adjoint system

In this section we will use the notation from previous chapters. Namely, we will make use of the Euler operator E defined by (2.20), the directional derivative $\mathcal{D}_v$ defined by (1.26) and the adjoint of a directional derivative, $\mathcal{D}_v^*$, defined by (2.17).

Definition 3.2.1 An ODE $G = 0$ is *self-adjoint* iff $\mathcal{D}_v^* G = \mathcal{D}_v G$ for any v.

The motivation for this definition is as follows:

Proposition 3.2.2 *Let $G = 0$ be a self-adjoint ODE and let v be its integrating factor. Then $\mathcal{D}_v$ is a symmetry generator of G.*

Proof. By proposition 2.2.15, If v is an integrating factor of an ODE $G = 0$ then $\mathcal{D}_v^* G = 0$ (mod G=0). Since G is self-adjoint, $\mathcal{D}_v^* G = \mathcal{D}_v G$. Hence $\mathcal{D}_v G = 0$ (mod G=0) and thus $\mathcal{D}_v$ is a symmetry of G. □

There indeed exist non-trivial self-adjoint ODEs as the following theorem shows:

Theorem 3.2.3 *Let $L = L(x, y, y_1, ..., y_n), v = v(x, y, y_1, ...y_m)$ be any expression. Then*

$$\mathcal{D}_v^* EL = \mathcal{D}_v EL$$

for any v.

Olver [16] proves this theorem using Variational Complex. Here we provide a more elementary proof. It consists of a sequence of lemmas.

Lemma 3.2.4 *Let $f = f(x)$ be a function independent of y and its derivatives. Then E and $\mathcal{D}_f$ commute:*

$$E\mathcal{D}_f L = \mathcal{D}_f EL$$

Proof. Because f is independent of y and its derivatives, $\mathcal{D}_f$ and $\frac{\partial}{\partial y_i}$ commute. But total and directional derivatives also commute (see Lemma (1.1.14)). Thus

$$E\mathcal{D}_f L = \sum(-1)^i \frac{d^i}{dx^i}(\frac{\partial}{\partial y_i}\mathcal{D}_f L) = \mathcal{D}_f \sum(-1)^i \frac{d^i}{dx^i}(\frac{\partial}{\partial y_i}L) = \mathcal{D}_f EL$$

for any expression L. □

Lemma 3.2.5 $E\mathcal{D}_v L = \mathcal{D}_v^* EL + \mathcal{D}_{EL}^* v$.

Proof. By (2.21) and Theorem 2.2.12 one has

$$E\mathcal{D}_v L = E(vEL + divergence) = E(vEL).$$

Applying (2.27) to the expression on the right proves the lemma. □

Lemma 3.2.6 *Let $f = f(x)$ be independent of y and its derivatives. Then $\mathcal{D}_f^* EL = \mathcal{D}_f EL$.*

Proof. Using Lemma 3.2.5 we get

$$\mathcal{D}_f^* EL = E\mathcal{D}_f L - \mathcal{D}_{EL}^* f.$$

Since $f(x)$ is independent of $y, y_1, \ldots$ it follows from (2.17) that $\mathcal{D}_{EL}^* f = 0$ and thus the second term on right hand side vanishes. Using Lemma 3.2.4 on the first term completes the proof. □

We now return to the proof of Theorem 3.2.3.

First take $v = f(x)$; then by Lemma 3.2.6 we have:

$$v\mathcal{D}_f EL = v\mathcal{D}_f^* EL.$$

Now apply (2.18) to both sides to obtain

$$f\mathcal{D}_v^* EL = f\mathcal{D}_v EL + R$$

where $R = -\mathcal{S}'(f, v, EL) - \mathcal{S}'(v, f, EL)$. Since $\mathcal{S}$ is linear in the first two arguments and their total derivatives, one can write

$$R = va_0 + v'a_1 + v''a_2 + \ldots + v^{(r)}a_r$$

where a_i are independent of v.

Now if we take any $v = v(x)$ independent of y and its derivatives, then $R = 0$ by Lemma 3.2.6. This, and the fact that the a_i are independent of v implies that $a_i = 0$ for all i (for example choose $v = 1 \Rightarrow R = a_0 = 0$, then choose $v = x$ and so on).

Hence $R = 0$ for all v. $\square$

One can also show that a self-adjoint equation is necessarily an Euler- Lagrange equation; see Olver [16] Theorem 5.68. An explicit formula for the corresponding lagrangian L is also given there.

3.2.2 Variational symmetries and Noether's theorem

Since the Euler-Lagrange equation is self-adjoint, an integrating factor of an Euler-Lagrange equation is also a symmetry of it (see Theorem 3.2.2). However not every symmetry is an integrating factor. Noether's Theorem provides a characterisation of those symmetries of an Euler-Lagrange equation that are integrating factors.

Definition 3.2.7 A symmetry $\mathcal{D}_v$ of an Euler-Lagrange equation $EL = 0$ is *variational* if there exists an expression A such that $\mathcal{D}_v L = D_x A$.

Noether's Theorem provides justification for calling v a symmetry:

Theorem 3.2.8 (Noether's Theorem, Part 1) *Let $G = EL = 0$ be an Euler-Lagrange equation. Then the following are equivalent:*

(a) $\mathcal{D}_v$ is a variational symmetry of G

(b) $E(vG) = 0$

(c) v is an integrating factor of G.

Proof. We first show that $(a) \Rightarrow (b)$. By definition of a variational symmetry, there exists some A for which $\mathcal{D}_v L = D_x A$. Applying Theorem 2.2.12 one obtains:

$$E\mathcal{D}_v L = 0.$$

Using (2.21) this becomes:

$$E(vEL + D_x \mathcal{S}(1, v, L)) = 0$$

Using linearity of E and invoking Theorem 2.2.12 for a second time we get:

$$E(vEL) = 0.$$

To show that $(b) \Rightarrow (a)$ simply run the preceeding implications backwards. The equivalence of (b) and (c) follows by Theorem 2.2.13. □

Once an integrating factor v of a self-adjoint ODE $G = EL = 0$ is known, one can find a corresponding conservation law P using Theorem 2.2.14 with $H = vG$. Alternately one can use the Lagrangian L to significantly reduce the computation of P as follows.

Theorem 3.2.9 (Noether's Theorem, Part 2) *Suppose that v is an integrating factor of a self-adjoint ODE $G = EL = 0$ and, in view of Theorem 3.2.8, let A be such that*

$$\mathcal{D}_v L = D_x A. \tag{3.1}$$

Then

$$P = A - \mathcal{S}(1, v, L) \tag{3.2}$$

is the conservation law corresponding to v so that

$$vEL = D_x P.$$

Proof. The formula for P follows immediately from (2.21):

$$D_x P = vEL = \mathcal{D}_v L - D_x \mathcal{S}(1, v, L) = D_x(A - \mathcal{S}(1, v, L)).$$

□

Note that if v is a variational symmetry of G then the expression A from (3.1) in the preceeding theorem can be obtained through a quadrature by applying Theorem 2.2.14 to $H = \mathcal{D}_v L$. Since the order of EL is in general twice that of L, Theorem 3.2.9 is more effective for computing a conservation law corresponding to a given integrating factor than a direct application of Theorem 2.2.14 to $H = vG$.

In addition to giving a conservation law, an integrating factor of of a self-adjoint ODE is also its symmetry. The following theorem shows how to take advantage of this and get a two-fold reduction of order. This will be illustrated by an example.

Theorem 3.2.10 *Let $G = 0$ be a self-adjoint ODE and let v be its variational symmetry. Let P be a corresponding conservation law: $P' = vG$. Then $\mathcal{D}_v P = 0$ (mod G=0) where C is any constant. Thus v is also a symmetry of $P = C$ where C is any constant.*

Proof. By Theorem 2.29, $\mathcal{D}_v P = \mathcal{S}(v, v, G)$ (mod G=0). Hence the proof follows from the following lemma. □

Lemma 3.2.11 *G is self-adjoint iff $\mathcal{S}(v, v, G) = 0$ for all v*

Proof. By (2.18), $\mathcal{S}'(v, v, G) = v\mathcal{D}_v G - v\mathcal{D}_v^* G = 0$ for all v iff G is self-adjoint. By an argument similar to the argument given in Step II in the proof of Theorem 2.3.2, $RemD(v, v, G) =$ const. for all v iff $S(v, v, G) = 0$ for all v. □

A direct proof of this theorem for ODEs of second order is given in Sheftel [20]. Olver [16] gives another proof of this theorem.

Example 3.2.12 Consider a self-adjoint ODE

$$G = y_2 + x^\alpha y^\beta = 0. \tag{3.3}$$

Its Lagrangian is given by

$$L = \begin{cases} \frac{x^\alpha y^{\beta+1}}{\beta+1} - \frac{y_1^2}{2}, \beta \neq -1 \\ x^\alpha \ln y - \frac{y_1^2}{2}, \beta = -1 \end{cases}.$$

It admits a scaling symmetry

$$X = (1-\beta)x\frac{\partial}{\partial x} + (\alpha+2)y\frac{\partial}{\partial y}. \tag{3.4}$$

Written in evolutionary form, X becomes $\mathcal{D}_v$ with

$$v = (\alpha+2)y - x(1-\beta)y_1.$$

To check if v is a variational symmetry, one can compute

$$E(vG) = -(2\alpha+\beta+3)G$$

and thus v is a variational symmetry iff

$$2\alpha+\beta+3 = 0. \tag{3.5}$$

Alternately, one can check under what conditions $\mathcal{D}_v L$ is exact. By (1.27),

$$\mathcal{D}_v L = XL - \xi D_x L = XL - D_x(\xi L) + (D_x\xi)L \tag{3.6}$$

where $\xi = (1-\beta)x$. By direct computation,

$$XL = 2(\alpha+\beta+1)L$$

and thus

$$\mathcal{D}_v L = (2\alpha+\beta+3)L - D_x(\xi L).$$

Since $EL \neq 0$, L is not exact and thus $\mathcal{D}_v L$ is exact iff (3.5) holds.

Example 3.2.13 Assuming (3.5) holds, G becomes

$$G = y_2 + x^{\alpha}y^{-3-2\alpha} \tag{3.7}$$

and admits a variational symmetry

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.8}$$

The conservation law P with $P' = (y - 2xy_1)G$ can be computed using Theorem 3.2.9:

$$P = -2xL - \mathcal{S}(1, v, L) = -y_1^2 x + y_1 y + \frac{x^{\alpha+1}y^{-2-2\alpha}}{\alpha+1}, \alpha \neq -1.$$

Thus $G = 0$ is reduced to a first order ODE

$$y_1^2 x - y_1 y - \frac{x^{\alpha+1} y^{-2-2\alpha}}{\alpha+1} = C, \alpha \neq -1 \tag{3.9}$$

that, by Theorem 3.2.10, admits X. The canonical coordinates of X are

$$x = e^{2s}, y = re^s$$

under which (3.9) becomes

$$s'(r) = \left((r^2 + 4C) + \frac{4r^{-2-2\alpha}}{\alpha+1} \right)^{-1/2} \quad \alpha \neq -1$$

and hence a general solution to $G = 0$ is given implicitly by

$$\ln x = 2 \int^{yx^{-1/2}} \left((r^2 + C_1) + \frac{4r^{-2-2\alpha}}{\alpha+1} \right)^{-1/2} dr + C_2, \alpha \neq -1$$

where C_1, C_2 are arbitrary constants.

Thus using Noether's Theorem, a single variational symmetry X led to a reduction of order of two. One can show that $G = 0$ has no other point symmetry, except for the trivial cases $\alpha = 0, a = -2, \alpha = -\frac{3}{2}$ (See Section 4.1.1).

Noether's Theorem was first proved in a slightly different version by Noether [15]. She considered general variational symmetries of the Lagrangian, involving both dependent and independent variables and did not note invariance of L to within a divergence. Bessel-Hagen [6] was the first to notice this important generalisation. His version is presented here.

3.3 Obtaining conservation laws without integration

Once a variational symmetry is known, a conservation law can be found through a quadrature using Theorem 3.2. However, when more then one symmetry is known, it is often possible to obtain the corresponding conservation laws without any quadrature.

According to Theorem 2.3.1, if w is an integrating factor of $G = 0$ and v is its symmetry, then $\mathcal{S}(w, v, G)$ given by (2.19) is a conservation law of $G = 0$. However the resulting conservation law may be trivial. In the case when G is self-adjoint, more can be said about such a conservation law. if both v, w are variational, then such a conservation law corresponds to their commutator. This was first observed in [3] where a direct proof was given. In Section 3.3.1 we give an alternative proof which is based on Theorem 5.48 of Olver [16] and on Theorem 2.3.1.

We then consider non-variational symmetries. In Theorem 3.3.5 we give necessary and sufficient conditions for a point symmetry to be variational. As far as we know, this theorem is new. In Section 3.3.3, given any point symmetry v and any variational symmetry w, we define an expression $u(v, w)$ which results in another (possibly new) variational symmetry. When v is also variational, we show that $u(v, w) = [v, w]$. The conservation law corresponding to $u(v, w)$ is given by $P = \mathcal{S}(w, v, G)$.

3.3.1 Commutator of variational symmetries

For convenience, from now on we shall refer to v as a *symmetry* of $G = 0$ when $\mathcal{D}_v G = 0$. Then v is a *point symmetry* iff $\mathcal{D}_v$ is the evolutionary form of a point symmetry iff $v = \eta(x, y) - y_1 \xi(x, y)$ for some $\xi(x, y), \eta(x, y)$. Also v is a variational symmetry iff it is an integrating factor.

Theorem 3.3.1 *Let $G = 0$ be a self-adjoint ODE. Suppose that v, w are variational symmetries of $G = 0$. Then their commutator,*

$$u = [v, w] = \mathcal{D}_v w - \mathcal{D}_w v$$

is also a variational symmetry of $G = 0$. Furthermore, suppose that Q is a conservation law corresponding to w, so that $Q' = wG$. Then the expression

$$R = \mathcal{S}(v, w, G) = \mathcal{D}_v Q \ (mod\ G{=}0),$$

where $\mathcal{S}$ is defined by equation (2.19), is the conservation law corresponding to u: $R' = uG$.

Proof. The proof of this theorem is essentially the same as that given in Theorem 5.48 of Olver [16].

Let Q be the conservation law corresponding to w so that $Q' = wG$. As will be shown in the following lemma, for any v one has:

$$D_x \mathcal{D}_v Q = [v, w]G + D_x \mathcal{S}(G, w, v) + wE(vG).$$

Since a variational symmetry v is an integrating factor, $E(vG) = 0$ (see Theorem 2.2.13) and hence the last term on the right hand side vanishes.

Letting $R = \mathcal{D}_v Q - \mathcal{S}(G, w, v)$ we thus get $D_x R = uG$. But $\mathcal{S}(G, w, v) = 0$ (mod G=0) since $\mathcal{S}$ is linear in its first argument and its total derivatives. Thus $R = \mathcal{D}_v Q$ (mod G=0). An application of Theorem 2.3.2 completes the proof. □

Lemma 3.3.2 *Let G be self-adjoint, and let w, Q be such that $Q' = wG$. Then for any v,*

$$D_x \mathcal{D}_v Q = [v, w]G + D_x \mathcal{S}(G, w, v) + wE(vG).$$

where $[v, w] = \mathcal{D}_v w - \mathcal{D}_w v$.

Proof. Using Lemma 1.1.14, definition of w, Q, product rule for the operator $\mathcal{D}_v$, and self-adjointness of G we obtain:

$$D_x \mathcal{D}_v Q = \mathcal{D}_v D_x Q = \mathcal{D}_v(wG) = G\mathcal{D}_v w + w\mathcal{D}_v G = G\mathcal{D}_v w + w\mathcal{D}_v^* G.$$

Using (2.27) and (2.18) we get

$$w\mathcal{D}_v^* G = -w\mathcal{D}_G^* v + wE(vG) = -G\mathcal{D}_w v + D_x \mathcal{S}(G, w, v) + wE(vG).$$

Thus

$$D_x \mathcal{D}_v Q = G\mathcal{D}_v w - G\mathcal{D}_w v + D_x \mathcal{S}(G, w, v) + wE(vG) = [v, w]G + D_x \mathcal{S}(G, w, v) + wE(vG).$$

□

Theorem 3.3.1 can simplify the application of Noether's Theorem: the conservation law corresponding to the commutator of two variational symmetries can be obtained without any integration. Thus if a Lie Algebra of variational symmetries is simple[1] then all its corresponding conservation laws can be obtained without integration.

Example 3.3.3 (Sheftel) Consider an ODE that was analysed in Sheftel [20], p.116:

$$G = y_4 - y^{-\frac{5}{3}} = 0. \tag{3.10}$$

This equation is self-adjoint since it is in solved form and is independent of odd derivatives of y. A symmetry analysis of this equation (see [20]) reveals that it admits three point symmetries:

$$X_1 = \frac{\partial}{\partial x}, \; X_2 = 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}, \; X_3 = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}$$

with commutators given by

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$2X_1$	X_2
X_2	$-2X_1$	0	$2X_3$
X_3	$-X_2$	$-2X_3$	0

where the entry in the i-th row and the j-th column corresponds to $[X_i, X_j]$. Thus point symmetries of the ODE (3.10) form a simple Lie Algebra.

The evolutionary forms corresponding to X_1, X_2, X_3 are:

$$v_1 = -y_1, \; v_2 = 3y - 2xy_1, \; v_3 = 3xy - y_1x^2.$$

It will be shown by Theorem 3.3.5 that the above symmetries are variational (this is easily checked directly by verifying that $E(v_iG) = 0$). Let P_i be the corresponding conservation laws, so that $P_i' = v_iG$. Then by Theorem (3.3.1) one has:

$$2P_1 = \mathcal{S}(v_1, v_2, G), \; P_2 = \mathcal{S}(v_1, v_3, G), \; 2P_3 = \mathcal{S}(v_2, v_3, G)$$

[1] A Lie Algebra is *simple* if it is equal to its derived algebra. A derived Lie Algebra is the Lie Algebra obtained by taking all possible commutators of the original Lie Algebra

from where one can compute:

$$P_1 = -y_1y_3 + \frac{1}{2}y_2^2 - \frac{3}{2}y^{-2/3}$$

$$P_2 = +2xy_3y_1 + y_2y_1 + 3y^{-2/3}x - y_2^2x - 3yy_3$$

$$P_3 = -3yxy_3 + \frac{3}{2}x^2y^{-2/3} + y_1x^2y_3 + 3yy_2 + xy_1y_2 - \frac{1}{2}x^2y_2^2 - 2y_1^2.$$

The fourth-order ODE $G = 0$ is thus equivalent to the first order ODE

$$H(x, y, y_1, C_1, C_2, C_3) = 0, \tag{3.11}$$

that can obtained by eliminating y_2, y_3 from the system

$$P_1 = C_1, P_2 = C_2, P_3 = C_3. \tag{3.12}$$

The resulting ODE H does not admit X_1, X_2 or X_3. Nonetheless, Sheftel showed how to obtain a symmetry of H, and thus was able to obtain a general solution of G.

The idea is to seek a symmetry

$$X = \lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3$$

such that $XP_i = 0$ (mod G=0), $i = 1, 2, 3$. Since $H = 0$ is equivalent to system (3.12), X is then also a symmetry of H.

By Theorem 3.3.1 and since $\mathcal{D}_{v_i} = X_i$ (mod G=0) one has

$$X_1P_1 = 0,\ X_2P_1 = -2P_1,\ X_3P_1 = -P_2 \text{ (mod } G\text{=0)}$$

since

$$[X_1, X_1] = 0,\ [X_2, X_1] = -2X_1,\ [X_3, X_1] = -X_2.$$

Thus $XP_1 = -2\lambda_2P_1 - \lambda_3P_2 = 0$ Similarly, one obtains a linear system for λ_i:

$$\begin{pmatrix} XP_1 \\ XP_2 \\ XP_3 \end{pmatrix} = \begin{pmatrix} 0 & -2P_1 & -P_2 \\ 2P_1 & 0 & -2P_3 \\ P_2 & 2P_3 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = 0.$$

This system admits a non-zero solution

$$\lambda_1 = -P_3,\ \lambda_2 = P_2, \lambda_3 = -2P_1$$

and hence $X = -C_3X_1 + C_2X_2 - 2C_1X_3$ is a symmetry of (3.11). Using X, one can obtain a general solution to $H = 0$ and thus to $G = 0$.

A generalisation of the preceeding example leads to the following conjecture:

Conjecture 3.3.4 *Let $G = 0$ be a self-adjoint ODE admitting a r-dimensional Lie Algebra of variational symmetries $X_1, ..., X_r$. Let $P_1, ..., P_n$ be the corresponding conservation laws. Let H be the system $P_1 = C_1, ..., P_n = C_n$, equivalent to the ODE G. Let A be the matrix with entries $A_ij = [X_i, X_j] = \sum_k C_{ijk}X_k$ Let R be the rank of A. Then using linear algebra only, one can find $r - R$ symmetries of H.*

In particular, if G admits an r-dimensional abelian Lie Algebra of variational symmetries that correspond to r functionally independent conservation laws then $2r$ reductions of order are possible.

The preceeding conjecture is true for simple variational Lie algebras.

3.3.2 Characterisation of variational point symmetries

Given a self-adjoint ODE $G = 0$ and its symmetry v, one can check if v is a variational symmetry of G by checking if it verifies $E(vG) = 0$. However this check involves G itself. In this section we develop a new, much simpler check which does not reference G. We will prove the following theorem:

Theorem 3.3.5 *Let*

$$G = y_n - g(x, y, y_1, ..., y_{n-1}) = 0 \tag{3.13}$$

be a self-adjoint ODE in solved form[2] and let

$$v = \eta(x, y) - y_1\xi(x, y). \tag{3.14}$$

[2]Note that n must be even for G to be self-adjoint

If v is a point symmetry of G, then

$$E(vG) = (2\eta_y + (1-n)\xi_x - (n+1)\xi_y y_1)G \text{ for all } x, y, y_1, ... \tag{3.15}$$

Consequently, a point symmetry $v = \eta - \xi y_1$ of G is a variational symmetry iff

$$2\eta_y + (1-n)\xi_x = 0 \text{ and } \xi_y = 0. \tag{3.16}$$

The proof of this theorem will be based on the following lemmas.

Lemma 3.3.6 *Let $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ be a point symmetry generator. Then its n-th extension η_n given by (1.18) has the form*

$$\eta_n = y_n(\eta_y - n\xi_x - (n+1)\xi_y y_1) + p(x, y, y_1, ..., y_{n-1})$$

where p is some polynomial in $y_1, ... y_{n-1}$.

Proof. Follows by induction from (1.18). □

Lemma 3.3.7 *Let $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ be a point symmetry of an ODE (3.13). Then*

$$XG = (\eta_y - n\xi_x - (n+1)\xi_y y_1)G.$$

Proof. Since X is a symmetry generator of G, one must have $XG = 0$ (mod G=0). But both XG and G are at most linear in y_n. Thus

$$XG = aG = a(y_n - g) \tag{3.17}$$

for some expression $a = a(x, y, y_1, ..., y_{n-1})$, for all $x, y, y_1, ..., y_n$. Also by Lemma 3.3.6

$$XG = y_n(\eta_y - n\xi_x - (n+1)\xi_y y_1) + f(x, y, y_1, ..., y_{n-1}) \tag{3.18}$$

where f is some expression of order at most $n-1$. Equating (3.17) and (3.18) and then collecting the y_n coefficient, one obtains the desired result. □

We now return to the proof of Theorem 3.3.5. Using (2.27) and self-adjointness of G one has:

$$E(vG) = \mathcal{D}_v^* G + \mathcal{D}_G^* v = \mathcal{D}_v G + \mathcal{D}_G^* v.$$

Using (2.17):

$$\mathcal{D}_G^* v = v_y G - (v_{y_1} G)' = v_y G + (\xi G)'.$$

Using (1.27) and Leibnitz rule:

$$\mathcal{D}_v G = XG - \xi D_x G = XG - (\xi G)' + \xi' G.$$

Thus

$$E(vG) = XG + v_y G + \xi' G.$$

An application of Lemma 3.3.7 proves (3.15). Equation (3.16) follows immediately from (3.15) and Theorem 2.2.13. □

Example 3.3.8 A point symmetry X is a variational symmetry of an n-th order self-adjoint ODE (3.13) iff it is of the form

$$X = \xi(x)\frac{\partial}{\partial x} + \left(\frac{n-1}{2}\xi'(x)y + f(x)\right)\frac{\partial}{\partial y} \tag{3.19}$$

for some functions $\xi(x), f(x)$.

Example 3.3.9 Any translational symmetry of (3.13) is always variational.

A scaling symmetry of (3.13) is variational iff it is a constant multiple of

$$X = 2x\frac{\partial}{\partial x} + (n-1)y\frac{\partial}{\partial y}.$$

This provides another method of checking that a scaling symmetry (3.4) of a self-adjoint ODE (3.3) is variational iff (3.5) holds.

3.3.3 Using non-variational symmetries in conjunction with variational symmetries

As we have seen in Theorem 3.3.1, a commutator of two variational symmetries is a variational symmetry. While a commutator of two non-variational symmetries need not be variational, it sometimes is. The following theorem identifies when this is the case.

Theorem 3.3.10 *Let $G = 0$ be a self-adjoint ODE. Let $v = \eta(x,y) - y_1\xi(x,y)$ be a point symmetry of G and let w be a variational point symmetry of G. Let $[v,w] = \mathcal{D}_v w - \mathcal{D}_w v$ be the commutator of v, w. Then*

$$u = [v,w] + w(2\eta_y + (1-n)\xi_x - (1+n)\xi_y y_1) \tag{3.20}$$

is a variational point symmetry of G. Its conservation law is given by $P = \mathcal{S}(v,w,G)$ (mod G= 0).

Proof. This is a direct consequence of Lemma 3.3.2 and Lemma 3.3.5. □

This theorem is particularly interesting if G admits a scaling symmetry $X = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$ which, expressed in evolutionary form, is $v = by - axy_1$. Using the notation from the preceeding theorem we obtain

$$u = [v,w] + cw, c = 2b + (1-n)a$$

is a variational symmetry corresponding to a conservation law $\mathcal{S}(v,w,G)$. In particular, $[v,w] = u - cw$ is also a variational symmetry since a difference of two variational symmetries is also a variational symmetry. Furthermore, suppose, as is often the case, that v itself is non-variational and that $[v,w] = \alpha v + \beta w$. Then automatically $\alpha = 0$ and $u = (\beta + c)w$ and, provided that $\beta + c \neq 0$, one can obtain a conservation law for u without integration.

Example 3.3.11 A classification of a self-adjoint ODE

$$G = y'' - x^\alpha y^2 = 0$$

(see Section 4.1.1 or [21]) reveals that there are exactly four values of α for which G has point symmetries other than the scaling symmetry

$$v = xy_1 + (\alpha - 2)y.$$

These values are

$$\alpha = 0, -5, -\frac{15}{7}, -\frac{20}{7}.$$

In all four cases, the only other point symmetry admitted is variational. For instance, consider the case $\alpha = -5$. Then G admits a variational symmetry

$$w = xy - x^2 y_1$$

with $[v, w] = -w$. Thus, $c = 7, \beta = -1$, and $u = (\beta + c)w = 6w$. Thus the conservation law for w is

$$P = \frac{1}{6}\mathcal{S}(v, w, G) \text{ (mod } G{=}0) = \frac{1}{6}(vw' - wv') \text{ (mod } G{=}0)$$

from where

$$P = y_1 y x - \frac{1}{2}(y^2 + x^2 y_1^2) + \frac{1}{3} y^3 x^{-3}.$$

Note that no integration was required to obtain P. By comparison, Noether's Theorem relies on Lagrangian formulation as well as being able to find the divergence A from Theorem 3.2.

In [3] the authors showed that a scaling symmetry of a linear self-adjoint PDE can be used to obtain without integration the conservation law corresponding to a given variational symmetry. The preceeding theorem is a generalisation of this.

3.4 Conclusions

In this chapter we have studied symmetries and integrating factors of self-adjoint ODEs. We started by showing that the Euler-Lagrange ODE is self-adjoint and that its integrating factors are variational symmetries and vice-versa.

We presented Noether's Theorem 3.2.9 that can be used to find a conservation law using a variational symmetry and a Lagrangian. The resulting conservation law admits the variational symmetry that was used to find it. Thus two reductions of order are possible using a single variational symmetry: one reduction in original variables and one symmetry reduction. Theorem Theorem 3.3.4 generalises this result to self-adjoint ODEs admitting r variational symmetries. In Example 3.3.3 a simple three-dimensional Lie Algebra of variational symmetries is used to obtain four reductions of order.

For a self-adjoint ODE of the form

$$G = y_n - g(x, y, y_1, ..., y_{n-1}), \tag{3.21}$$

a commutator of a *scaling symmetry* and a variational symmetry is always a variational symmetry. In most cases the conservation law corresponding to such a commutator can be obtained without any integration (see discussion after Theorem 3.3.10). More generally, a commutator of a variational symmetry w and a non-variational point symmetry v need not be a variational symmetry. However, there is an expression $u(v, w)$ given by (3.20) which results in a variational symmetry. When v is also variational, $u(v, w) = [v, w]$.

For a self-adjoint ODE (3.21) it is possible to tell when a given point symmetry is variational, without using G. This provides an ansatz for looking for variational symmetries, given by (3.19).

Chapter 4
Classification of solvable ODEs

An ODE is said to be *solvable* if its general solution can be expressed using quadratures only. Given a family of ODEs, the *classification problem* is to find as many solvable ODEs in that family as possible.

In this chapter we will consider the classification problem for two families of second order ODEs:

$$y'' = Ax^n y^m y'^l \quad \text{and} \quad y'' = f(y)y' + g(y).$$

The first family is known as the Emden-Fowler equation and is chosen because

1. Solvable cases are known which do not admit two point symmetries.[1] (see [22])
2. When $l = 0$ the ODE is self-adjoint.
3. Any ODE in this family admits a scaling symmetry.

The second family is chosen because

1. Any ODE in the family admits a translational symmetry.
2. Any solvable case leads to a solvable Abel equation

$$u'(t) = -g(t)u^3 - f(t)u^2 \tag{4.1}$$

through a change of variables

$$x = s(t), y(x) = t, u(t) = s'(t). \tag{4.2}$$

[1]Note that any ODE admits Lie-Bäcklund symmetries.

To classify these ODEs we will apply the theory of symmetries and integrating factors developed so far. We shall make use of the the symmetries admitted by the above ODEs when seeking other symmetries or integrating factors. To this end, in Section 4.2 we develop ansatzes that use known symmetries or integrating factors.

In Section 4.1 we classify the first family for symmetries and integrating factors. The second family will be classified in Section 4.3.

4.1 Classification of the Emden-Fowler Equation, $y'' = Ax^n y^m y'^l$

The goal of this section is to find solvable cases of the Emden-Fowler Equation,

$$G = y'' - Ax^n y^m y'^l = 0. \tag{4.3}$$

We shall denote such equation by a triple (l, m, n). Before proceeding, we make several useful remarks that hold for any (l, m, n). First, note that a change of variables

$$y(x) = t, x = u(t) \tag{4.4}$$

maps $G = 0$ into another Emden-Fowler Equation,

$$u'' + At^m u^n u'^{3-l} = 0. \tag{4.5}$$

Thus a solvable case (l, m, n) leads to a solvable case $(3 - l, n, m)$. Second, note that G always admits a scaling symmetry

$$(1 - m - l)x\frac{\partial}{\partial x} + (2 + n - l)y\frac{\partial}{\partial y}. \tag{4.6}$$

In Section 4.1.1 we will classify all possible (l, m, n) which admit more then one point symmetry.

In Section 4.1.2 we will find all cases for which G admits an adjoint symmetry of the form

$$w = a(x, y) + b(x, y)y_1. \tag{4.7}$$

Most adjoint symmetries of the form (4.7) will turn out to be integrating factors. [2] This will

[2]Note that by Proposition 2.2.15 an integrating factor is an adjoint symmetry. An adjoint symmetry is not necessarily an integrating factor, but often it is.

lead to more solvable cases, some of which will be different from those found using a symmetry classification.

4.1.1 Point symmetry classification of (4.3)

We begin with the point symmetry classification of (4.3). Using Lie's algorithm, this amounts to solving the overdermined PDE $XG = 0$ (mod $G\!=\!0$) for $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + ...$ Written in full, the resulting PDE is

$$\begin{gathered}\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_1 + (\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3 \\ -Al\eta_x x^n y^m y_1^{l-1} + A(l-3)\xi_y x^n y^m y_1^{l+1} \\ -A\left(m\eta x^n y^{m-1} + n\xi x^{n-1}y^m + (2\xi_x - \eta_y + l\eta_y - l\xi_x)\, x^n y^m\right) y_1^l = 0\end{gathered}.$$

For a fixed l, the resulting system splits by equating the coefficients of the like powers of y_1 to zero. The splitting depends on whether l is arbitrary or one of $l = -1, 0, 1, 2, 3, 4$. Since the change of variables (4.4) maps (4.3) into (4.5), the classification of the cases $l = 2, 3, 4$ can be obtained from the classification of the cases $l = 1, 0, -1$ respectively.

After considering all possible subcases, one eventually obtains the full symmetry classification of (4.3) listed in Table 4.1. Note that the cases $l = -1, 4$ do not yield any additional symmetries.

The cases $(l, m, n) = (0, 2, -\frac{15}{7}), (0, 2, -\frac{20}{7})$ were previously classified in Stephani [21]. Chapter two of the standard reference of solvable ODEs by Kamke [14] lists 246 non-linear ODEs. Of those, seven are are Emden-Fowler ODEs with $n, m \neq 0$. They are ODE number 96 $(0, n, -4)$, 100 (0,3/2, -1/2), 102 $(0, 1-n, n)$, 105 (0,-1,1), 106 (0,-1,2), 205 (0,-2,-1), and 229 (-1,2,-3). Thus the non-trivial cases $(0, m, 3-m)$, (1,1,-1) as well as the cases obtained from them using the transformation (4.4) are not found in [14] or in any other literature cited in the bibliography.

4.1.2 Adjoint symmetry classification of (4.3)

In this section we list all of the cases for which (4.3) admits an adjoint symmetry of the form

$$w(x, y, y_1) = a(x, y) + y_1 b(x, y). \tag{4.8}$$

Condition	Point symmetries of $y'' = Ax^n y^m y'^l$, other than (4.6)
$m = 0$	$\frac{\partial}{\partial y}$
$l = 0, m + n + 3 = 0$	$x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$
$l = 0, m = 2, n = -\frac{15}{7}$	$A\frac{7^3}{12} x^{6/7} \frac{\partial}{\partial x} + \left(1 + A\frac{49}{4} yx^{-1/7}\right) \frac{\partial}{\partial y}$
$l = 0, m = 2, n = -\frac{20}{7}$	$A\frac{7^3}{12} x^{8/7} \frac{\partial}{\partial x} + \left(-x + A\frac{49}{3} yx^{1/7}\right) \frac{\partial}{\partial y}$
$l = 0, m = 1$ or $l = 1, m = 0$	G is linear and thus admits eight symmetries.
$l = 1, m = 1, n = -1$	$-Ax \ln x \frac{\partial}{\partial x} + (1 + Ay) \frac{\partial}{\partial y}$
$n = 0$	$\frac{\partial}{\partial x}$
$l = 3, m + n + 3 = 0$	$xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$
$l = 3, n = 2, m = -\frac{15}{7}$	$\left(1 + A\frac{49}{4} xy^{-1/7}\right) \frac{\partial}{\partial x} + A\frac{7^3}{12} y^{6/7} \frac{\partial}{\partial y}$
$l = 3, n = 2, m = -\frac{20}{7}$	$\left(-y + A\frac{49}{3} xy^{1/7}\right) \frac{\partial}{\partial x} + A\frac{7^3}{12} y^{8/7} \frac{\partial}{\partial y}$
$l = 3, n = 1$ or $l = 2, n = 0$	G is linearisable using (4.4) and admits eight symmetries.
$l = 2, n = 1, m = -1$	$(1 + Ax) \frac{\partial}{\partial x} - Ay \ln y \frac{\partial}{\partial y}$

Table 4.1: Symmetry classification of (4.3)

The process of finding adjoint symmetries is similar to the process of finding symmetries. First write out the determining equation

$$\mathcal{D}_w^* G = 0 \text{ (mod } G\text{=0)}. \tag{4.9}$$

The restriction (4.8) makes (4.9) an overdetermined system. Solving it leads to the classification listed in Table 4.2.

There are two ways in which an adjoint symmetry may lead to a conservation law.

1. Note that (4.3) always admits a scaling symmetry which, written in evolutionary form, is

$$v = (l - n - 2)y - (l + m - 1)xy_1 \tag{4.10}$$

   By Theorem 2.3.1, if w is an adjoint symmetry then the expression $\mathcal{S}(w, v, G)$ (mod G=0) defined by (2.19) is always a (possibly trivial) conservation law. The last column in the Table 4.2 lists such an expression.

2. An integrating factor is always an adjoint symmetry. Conversely, some (but not all) adjoint symmetries are integrating factors. If w is also an integrating factor then the third column in Table 4.2 lists the corresponding conservation law.

This classification identifies two cases which admit an integrating factor without admitting two symmetries.

Case 1: The case $l = 1, n = -1$, corresponding to the ODE

$$G = y'' - \frac{y^m y'}{x} = 0$$

admits an integrating factor $w_3 = x$ leading to a reduction of order $P_3 = C$. From Table 4.2 we see that $\mathcal{S}(w_3, v, G) = 0$ (mod G=0). Thus by Theorem 2.3.2 it follows that $\mathcal{D}_v P_3 = 0$ (mod G= 0) and hence $P_3 = C$ inherits the symmetry v given by (4.10)! Thus in this case, $G = 0$ is completely solvable. Using standard symmetry methods, the solution is found to be

$$x = C_2 \exp\left(\int^y \frac{dt}{t + A\int t^m dt + C_1}\right)$$

Condition	Adjoint symmetry	Conservation law	$\mathcal{S}(w,v,G)$
$l=-1, m=2$	$w_1 = yy_1$	N/A	0
$l=1, n=0$	$w_2 = 1$	$P_2 = y_1 - A\int y^n dy$	$(m+1)P_2$
$l=1, n=-1$	$w_3 = x$	$P_3 = xy_1 - y - A\int y^m dy$	0
$l=1, n=-\frac{1}{2}, m=-2$	$w_4 = 2xy_1 + 2A\frac{x^{\frac{1}{2}}}{y} - y$	$P_4 = y_1^2 x + y_1(2A\frac{x^{1/2}}{y} - y) + A^2\frac{1}{y^2}$	$-P_4$
$l=0$	$v = (n+2)y - (1-m)xy_1$	N/A	0
$l=0, m=-3-2n$	$v = (n+2)y - 2(n+2)xy_1$	$P_5 = -y_1^2 x + yy_1 + A\int x^n dx y^{-2-2n}$	0
$l=0, m=-3-n$	$w_6 = xy - x^2 y_1$	$P_6 = -\frac{1}{2}(y^2 + x^2 y_1^2) + xyy_1$ $-A\int x^{n+1}dx y^{-2-n}$	$(4+2n)P_6$
$l=0, m=2, n=-\frac{15}{7}$	$w_7 = 1 + \frac{49}{4}Ax^{-1/7}y$ $-\frac{7^3}{12}Ax^{6/7}y_1$	$P_7 = y_1 + \frac{49}{4}Ax^{-1/7}yy_1 - \frac{7^3}{24}Ax^{6/7}y_1^2$ $+\frac{7}{8}Ay^2x^{-8/7} + \frac{7^3}{36}A^2x^{-9/7}y^3$	$\frac{6}{7}P_7$
$l=0, m=2, n=-\frac{20}{7}$	$w_8 = -x + \frac{49}{3}Ax^{-1/7}y$ $-\frac{7^3}{12}Ax^{8/7}y_1$	$P_8 = -xy_1 + \frac{49}{3}Ax^{1/7}yy_1 - \frac{7^3}{24}Ax^{8/7}y_1^2$ $-\frac{7}{8}Ay^2x^{-6/7} + \frac{7^3}{36}A^2x^{-12/7}y^3$	$-\frac{6}{7}P_8$

Table 4.2: Adjoint symmetries of (4.3) of the form (4.8). If the adjoint symmetry listed is also an integrating factor, then the third column lists the corresponding conservation law. If an adjoint symmetry is not an integrating factor, N/A is present in the third column. This table does not include cases that are linear or obtained from the linear case by the transformation (4.4). See text for description of the fourth column.

Case 2: The case $l = 1, n = -1/2, m = -2$, corresponding to the ODE

$$G = y'' - x^{-1/2}y^{-2}y' = 0$$

admits an integrating factor w_4 leading to a reduction of order $P_4 - C = 0$. However this equation does not inherit v since $\mathcal{D}_v P_4 = \mathcal{S}(w_4, v, G) = -P_4$ (mod G=0). Nevertheless, one has $\mathcal{D}_v P_4 = 0$ (mod P_4=0) and hence the equation $P_4 = 0$ *does* inherit the symmetry v. Thus one can find a *particular* solution of $G = 0$ by solving $P_4 = 0$ which admits the symmetry v. The resulting particular solution is

$$\frac{1}{4}\ln x + \int^{yx^{-1/4}} \frac{t(1 + 2t(t^2 - 4A)^{-1/2})}{3t^2 + 4A} dt = C$$

Note that the two above cases are not obtainable through a point symmetry classification of (4.3). Neither are they found in Kamke [14] or any other literature cited in the bibliography.

4.2 Using known symmetries or integrating factors as ansatzes

4.2.1 Using a symmetry to generate an ansatz for an integrating factor

In Section 2.1.3 we have discussed how to use a symmetry as an ansatz when looking for conservation laws. Since to every conservation law there corresponds an integrating factor, one can also use a symmetry as an ansatz for an integrating factor directly, without finding a conservation law.

Lemma 4.2.1 *Let* $X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ *be a point symmetry of an ODE*

$$G = y_n - g(x, y, y_1, ..., y_{n-1}). \tag{4.11}$$

Let P *be a conservation law of* G *and* w *the corresponding integrating factor. Let* $Q = XP$ *(mod* G*=0). Then* Q *is a conservation law and*

$$v = w\left(\eta_y - n\xi_y y_1 - (n - 1)\xi_x\right) + Xw \tag{4.12}$$

is the integrating factor corresponding to Q*.*

Proof. Note that $Q = XP$ (mod G=0) is a conservation law by Theorem 2.1.4. Since X is a point symmetry and $P = P(x, y, ..., y_{n-1})$, it follows that XP is independent of y_n. Thus XP (mod G=0) $= XP$. Differentiating, using Lemma 1.1.15 and then using $P' = wG$, we obtain:

$$Q' = (XP)' = X(P') + \xi' P' = X(wG) + \xi' wG.$$

Using $Q' = vG$ and the product rule we get:

$$vG = wXG + GXw + \xi' wG.$$

Application of Lemma 3.3.7 results in (4.12). □

Theorem 4.2.2 *Let X, G be as in Lemma 4.2.1. Then for any given constants α, β, there exists a conservation law P of G which satisfies*

$$XP = \alpha P + \beta \ (mod\ G{=}0). \tag{4.13}$$

Furthermore, let $w = w(x, y, y_1, ..., y_{n-1})$ be an integrating factor of G with $P' = wG$. Then w satisfies the following PDE for all $x, y, y_1, ..., y_{n-1}$:

$$w\left(\eta_y - n\xi_y y_1 - (n-1)\xi_x - \alpha\right) + Xw = 0. \tag{4.14}$$

Conversely, If w satisfies (4.14) then the conservation law P with $P' = wG$ satisfies (4.13) for some constant β. In addition, if $\alpha \neq 0$ then the conservation law P can be obtained without any integration:

$$P = \frac{S(w, v, G)}{\alpha} \ (mod\ G{=}0) \tag{4.15}$$

where $v = \eta - y_1\xi$ is the evolutionary form of X.

Proof. Step 1: We first show that there exists a conservation law P satisfying (4.13). By Lemma 2.1.5 with $P_0 = 1$, there exists a conservation law Q with $XQ = 1$. So $P = f(Q)$ is also a conservation law for any function f. In particular, choose f to be a solution to $f'(Q) = \alpha f(Q) + \beta$. Then

$$X(P) = X(f(Q)) = f'(Q)XQ = f'(Q) = \alpha f(Q) + \beta = \alpha P + \beta.$$

Step 2: Equation (4.14) is obtained by differentiating both sides of (4.13) in exactly the same fashion as in the proof of the preceeding lemma.

Step 3: Equation (4.15) follows from application of Theorem 2.3.2. □

Remark: Suppose that we found a conservation law P of $G = 0$ with $XP = \alpha P + \beta$ and with $\alpha \neq 0$. Letting $Q = P + \frac{\beta}{\alpha}$, we see that $XQ = \alpha Q$ and hence $XQ = 0$ (mod Q=0). Thus X is a symmetry of the ODE

$$P + \frac{\beta}{\alpha} = 0. \tag{4.16}$$

This fact can be used to find a reduction of order of (4.16) which leads to a particular solution depending on an arbitrary constant, if G is of order two. See Section 4.1.2, Case 2 for example.

The following table lists commonly encountered symmetries and the corresponding solution of (4.14), for the case $n = 2$.

$X = \frac{\partial}{\partial x}$:	$w = e^{\alpha x}F(y, y_1)$
$X = \frac{\partial}{\partial y}$:	$w = e^{\alpha y}F(x, y_1)$
$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$:	$w = e^{\lambda x}F(ay - bx, y_1), \lambda = \frac{\alpha}{a}$
$X = y\frac{\partial}{\partial y}$:	$w = y^{\lambda}F(x, \frac{y_1}{y}), \lambda = \alpha - 1$
$X = x\frac{\partial}{\partial x}$:	$w = x^{\lambda}F(y, xy_1), \lambda = \alpha + 1$
$X = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$:	$w = x^{\lambda}F(x^{-b}y^{a}, x^{a-b}y_1^{a}), \lambda = \frac{\alpha+a-b}{a}$

For example, an ODE $G = 0$ admitting $X = \frac{\partial}{\partial x}$ will have an integrating of the form $w = e^{\alpha x}F(y, y_1)$ for any α, for some function F. Note however, that not every integrating factor of G is of that form. Nevertheless, one has the following theorem.

Theorem 4.2.3 *Let* $G = y_n - g(x, y_1, ..., y_{n-1}) = 0$ *be an ODE of order two or higher, and suppose that* G *admits a point symmetry of the form*

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

Suppose also that G admits an integrating factor of the form

$$w = a_0(x,y) + y_1 a_1(x,y) + y_1^2 a_2(x,y) + ... + y_1^N a_N(x,y) \tag{4.17}$$

where N is an arbitrary fixed positive integer if $\xi_y = 0$ and $N = n$ if $\xi_y \neq 0$. Then G admits an integrating factor of the form 4.17 which in addition satisfies (4.14).

Before proving this theorem, we will need the following lemma:

Lemma 4.2.4 *Let $A : \Re^n \to \Re^n$ be a linear transformation and let $x \in \Re^n, x \neq 0$ be such that $Ax = 0$. Then there exists $y \in \Re^n$ independent of x such that either $Ay = \lambda y$ for some constant λ or else $Ay = x$.*

Proof. Any linear transformation must admit at least one eigenvector. If A admits an eigenvector independent of x then choosing y to be such an eigenvector proves the theorem. So assume without loss of generality that A does not admit an eigenvector independent of x. Then x is the only eigenvector of A and zero is its only eigenvalue. By a change of basis, we may assume without loss of generality that A is in its Jordan-Canonical form,

$$A = \begin{bmatrix} 0 & a_1 & & \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1} \\ & & & 0 \end{bmatrix}$$

where a_i is either zero or one, $i = 1..n-1$. By assumption of uniqueness of the eigenvector, it follows that $a_i = 1$ for $i = 1..n-1$. Thus

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \tag{4.18}$$

Also by uniqueness of the eigenvector, $x = (\alpha, 0, ..., 0)^T$ for some $\alpha \neq 0$ is the unique eigenvector of (4.18). Choosing $y = (0, \alpha, 0, ..., 0)^T$ we obtain $Ay = x$ as desired. □

Proof of Theorem 4.2.3. Step 1: Let w be an integrating factor of the form (4.17) with corresponding conservation law P. By Lemma 4.2.1, the integrating factor corresponding to XP is given by

$$w\left(\eta_y - n\xi_y y_1 - (n-1)\xi_x\right) + Xw.$$

Expanding Xw using Lemma 3.3.6 one can show that such an integrating factor is also of the form (4.17).

Step 2: We now show that there exists an integrating factor w satisfying both (4.17) and (4.14). One can show that any ODE of second order or higher admits finitely many linearly independent integrating factors of form (4.17). Let $w_1, ..., w_r$ be such integrating factors and let $P_1, ..., P_r$ be their corresponding conservation laws. Let $P_0 = 1$ be a trivial conservation law. By Step 1, XP_i is a linear combination of $P_0, ..., P_r$. Thus X defines a linear transformation on the vector space spanned by $P_0, ..., P_r$. By Lemma 4.2.4 there exists P independent of P_0 such that either $XP = \alpha P$ for some α or $XP = P_0 = 1$. Since P is independent of P_0, it is non-trivial. Hence there exists a non-trivial conservation P with $XP = \alpha P + \beta$ (where $\beta = 0$ when $\alpha \neq 0$ and $\beta = 1$ when $\alpha = 0$). Applying theorem 4.2.2 concludes the proof. □

See section 4.3.2 for example.

4.2.2 Using a symmetry to generate an ansatz for a symmetry

To find a symmetry X of an ODE $G = 0$, one needs to solve a linear PDE $XG = 0$ (mod G=0) for X. When another symmetry is known, an ansatz can be made that decreases the number of independent variables in the above PDE by one.

Theorem 4.2.5 *Suppose that an ODE $G = 0$ admits a finite Lie algebra $\mathcal{L}$ of symmetries of dimension at least two. Then given any symmetry $X \in \mathcal{L}$, there exists another symmetry $Y \in \mathcal{L}$ independent of X such that either $[X, Y] = \lambda Y$ for some (possibly complex) constant λ or else $[X, Y] = X$.*

Proof. This is a direct consequence of Lemma 4.2.4 applied to the linear transformation $A: Y \to [X, Y]$ and $x = X$. □

Example 4.2.6 Suppose an ODE admits a translational symmetry $X = \frac{\partial}{\partial x}$. If it admits any other point symmetries, then by Theorem 4.2.5 it must also admit either a non-trivial symmetry of the form

$$Y = (a(y) + x)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}.$$

or else a non-trivial symmetry of the form

$$Y = e^{\lambda x}\left(a(y)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}\right)$$

for some functions $a(y), b(y)$ and some (possibly complex) constant λ. See Section 4.3.1 for further application of this example.

In his thesis, Boulton [9] shows how to utilise structure constants of a Lie Algebra to generate ansatzes for symmetries. His algorithm requires an explicit computation of these structure constants.[3] By contrast, no à-priori knowledge of the structure constants is required to use Theorem 4.2.5.

4.3 Classification of $y'' + f(y)y' + g(y) = 0$

Consider the family of ODEs

$$G = y'' + f(y)y' + g(y) = 0. \tag{4.19}$$

Note that (4.19) admits a point symmetry

$$T = \frac{\partial}{\partial x}$$

for any $f(y), g(y)$. We wish to find such $f(y), g(y)$ for which (4.19) admits another symmetry or an integrating factor. We will use ansatzes developed in Section 4.2 to simplify this task.

[3]See Reid [18] for an algorithm that finds structure constants by reducing the overdetermined system of determining equations to standard form and without explicitly solving the determining equations

4.3.1 Symmetry classification of (4.19)

In this section we will classify all possible cases for which (4.19) admits a symmetry other than $\frac{\partial}{\partial x}$. By Example 4.2.6, if (4.19) admits another point symmetry then it must also admit a point symmetry X of the form either

$$X = (a(y) + x)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y} \tag{4.20}$$

or

$$X = e^{\lambda x}\left(a(y)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}\right). \tag{4.21}$$

We analyse the two cases separately.

Case 1: Assuming (4.20), the determining equations $XG = 0$ (mod G=0), simplify to a system of ODEs for $a(y), b(y), f(y), g(y)$:

$$a'' = 0, \quad b'' + 2a'f = 0, \quad g'b - b'g + 2g = 0, \quad f'b + 3a'g + f = 0 \tag{4.22}$$

Solving (4.22) leads to the following.

Theorem 4.3.1 *The ODE $G = y'' + f(y)y' + g(y) = 0$ admits a symmetry $T = \frac{\partial}{\partial x}$ and a symmetry X with $[T, X] = T$ iff*

$$X = (a(y) + x)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}$$

and $a(y), b(y), f(y), g(y)$ satisfy one of:

1. $a(y) = \alpha, \quad b(y) = 0, \quad f(y) = 0, \quad g(y) = 0$
2. $a(y) = \alpha, \quad b(y) = A, \quad f(y) = Be^{-y/A}, \quad g(y) = Ce^{-2y/A}, \quad A \neq 0$
3. $a(y) = \alpha, \quad b(y) = A + By, \quad f(y) = C(A + By)^{-1/B}, \quad g(y) = E(A + By)^{1-2/B}, \quad B \neq 0$
4. $a(y) = Ay + \alpha, \quad f(y) - \frac{1}{2A}b''(y), \quad g(y) = \frac{1}{6A^2}\left(b''(y)b(y) + b''(y)\right), A \neq 0$

   where $b(y)$ satisfies

$$b''''b^2 + 3bb''' - b'b'' + 2b'' = 0 \tag{4.23}$$

where capital letters represent arbitrary constants.

Remark 1: One can show that case 2 can be obtained from case 3 by taking the limit as $B \to 0$. Also, case 1 is just a special case of case 2. Thus there are actually two distict cases: either $a' = 0$ (cases 1,2,3) or $a' \neq 0$ (case 4).

Remark 2: Cases 1,2 and 3 lead to solvable Abel ODEs (4.2) through a transformation (4.1). However the resulting ODE is also solvable by the method of "Abel invariant" described in Kamke [14], page 26.[4] Thus Theorem 4.3.1 does not lead to any solvable Abel ODEs not found in [14].

We now derive a sequence of transformations that transforms the fourth-order ODE (4.23) into an Abel ODE

$$u'(t) + t(t+2)(2t+3)u(t)^3 - (7+3t)u(t)^2 + 3\frac{u(t)}{t} = 0. \tag{4.24}$$

A point symmetry analysis reveals that the ODE (4.23) admits two symmetries:

$$X_1 = y\frac{\partial}{\partial y} + b\frac{\partial}{\partial b},\ X_2 = \frac{\partial}{\partial y}$$

with $[X_2, X_1] = X_2$.

Thus a change of variables

$$r = b(y), s(r) = y$$

leads to a 3rd order ODE for

$$z(r) = s'(r)$$

that inherits the symmetry $X_1 = r\frac{\partial}{\partial r}$. The canonical coordinates

$$v(q) = \ln r, q = z, w(q) = v'(q)$$

then lead to a second order ODE

$$w''wq^2 - 3q^2w'^2 + 3q^2\left(q-1\right)w^2w' - 10qww' + q(9q-10)w^3 - 2q^2\left(q-1\right)^2w^4. \tag{4.25}$$

[4]The method of "Abel invariant" is the only general algorithm in Kamke to solve Abel ODEs. It shows how to find, if exists, a transformation $y = F(t)u(t) + G(t), x = H(t)$ which maps an Abel ODE into a separable ODE.

A search for point symmetries of this ODE reveals a symmetry

$$X_3 = q\,(q-1)\,\frac{\partial}{\partial q} + w(1-3q)\frac{\partial}{\partial w} \tag{4.26}$$

whose canonical coordinates are

$$t = q\,(q-1)^2\,w, p = \ln\left(\frac{q-1}{q}\right)$$

lead to an Abel equation (4.24) with $u(t) = p'(t)$. No solution of this Abel ODE is known.

Using a method described in [7], a symmetry X_3 leads to three particular solutions of (4.25):

$$w(q) = \frac{c}{q\,(q-1)^2} \text{ with } c = -2 \text{ or } -3/2 \text{ or } 0.$$

Corresponding particular solutions for $b(y)$ can then be obtained when $c \neq 0$.

Case 2: Assuming (4.21), the determining equations $XG = 0$ (mod G=0), simplify to a system of ODEs for $a(y), b(y), f(y), g(y)$:

$$a'' = 0, \quad b'' - 2\lambda a' + 2a'f = 0,$$
$$\lambda^2 b + bg' - b'g + \lambda f b + 2\lambda a g = 0,$$
$$bf' + 3a'g + \lambda a f - \lambda^2 a + 2\lambda b' = 0.$$

Several subcases result, summarised below.

Theorem 4.3.2 *The ODE $G = y'' + f(y)y' + g(y) = 0$ admits a symmetry $T = \frac{\partial}{\partial x}$ and a symmetry X with $[T, X] = \lambda X$ iff X is a constant multiple of*

$$e^{\lambda x} a(y)\frac{\partial}{\partial x} + e^{\lambda x} b(y)\frac{\partial}{\partial y}$$

and $a(y), b(y), f(y), g(y)$ satisfy one of:

1. $a(y) = 0, \quad b(y) = 1, \quad f(y) = A, \quad g(y) = -\lambda\,(\lambda + A)\,y + B$
2. $a(y) = 0, \quad b(y) = y + A, \quad f(y) = -2\lambda \ln(y + A) + B,$
   $g(y) = -\lambda\,(y + A)\left((\lambda + B)\ln(y + A) - \lambda \ln^2(y + A) + C\right)$
3. $a(y) = 1, \quad b(y) = 0, \quad f(y) = \lambda, \quad g(y) = 0$
4. $a(y) = 1, \quad b(y) = A, \quad f(y) = \lambda\left(Be^{-\frac{\lambda}{A}y} + 1\right), \quad g(y) = -\lambda\left(A + BAe^{-\frac{\lambda}{A}y} + Ce^{-\frac{2\lambda}{A}y}\right), \quad A \neq 0$
5. $a(y) = 1, \quad b(y) = A + By, \quad f(y) = \lambda - 2B + (A + By)^{-\lambda/B}\,C$
   $g(y) = (A + By)^{1-2\lambda/B}\,E - (A + By)^{1-\lambda/B}\,C - (A + By)\,(\lambda - B)\,, \quad B \neq 0$
6. $a(y) = y + A, \quad f(y) = \lambda - \frac{1}{2}b''(y)$
   $g(y) = \frac{1}{6}b(y)b'''(y) + \frac{\lambda}{6}yb''(y) + \frac{\lambda}{6}Ab''(y) - \frac{2\lambda}{3}b'(y)$
   and $b(y)$ satisfies the fourth order ODE

$$\begin{gathered} b''''b^2 + 12\lambda^2 b + 3\lambda(A + y)b'''b - 6\lambda bb'' + 4\lambda b'^2 - 8\lambda^2(A + y)b' \\ +2\lambda^2(A + y)^2 b'' + -\lambda(A + y)b''b' = 0 \end{gathered} \tag{4.27}$$

where capital letters represent arbitrary constants.

Remark 1: One can show that cases 1-4 can be obtained from case 5 by considering various limiting values of various constants. Thus there are actually two distict cases: either $a' = 0$ (cases 1,2,3,4,5) or $a' \neq 0$ (case 6).

Remark 2: Cases 1-5 lead to solvable Abel ODEs (4.2) through a transformation (4.1). For subcases 2, 4, 5, 6, the resulting ODEs are not solvable by the method of "Abel invariant" described in Kamke [14], page 26. Moreover, none of these ODEs are listed among the 15 solvable ODEs of form (4.2) whose solutions are given in Kamke.[5]

A point symmetry analysis reveals that the ODE (4.27) admits two symmetries:

$$Y_1 = (y + A)\,\frac{\partial}{\partial y} + 2b\frac{\partial}{\partial b}, \quad Y_2 = (y + A)^2\,\frac{\partial}{\partial y} + 3b\,(y + A)\,\frac{\partial}{\partial b}$$

[5]These 15 ODEs are numbers 36, 37, 40, 41, 42, 43, 45, 47, 48, 111, 145, 146, 147, 169, 185 from Chapter 1 of Kamke

with $[Y_1, Y_2] = Y_2$.[6] Using canonical coordinates for Y_2,

$$r = \frac{b}{(y+A)^3}, \quad s(r) = \frac{-1}{y+A}$$

leads to a 3rd order ODE for

$$z(r) = s'(r)$$

that inherits the symmetry $Y_1 = r\frac{\partial}{\partial r}$. Computing the the canonical coordinates of Y_1 we obtain the transformations

$$v(q) = \ln r, q = z, w(q) = v'(q)$$

that lead to a second order ODE

$$w''wq^2 - 3q^2w'^2 + 3q^2(\lambda q - 1)w^2w' - 10qww' + q(9\lambda q - 10)w^3 - 2q^2(\lambda q - 1)^2 w^4. \tag{4.28}$$

A search for point symmetries of this ODE reveals a symmetry

$$Y_3 = (q(\lambda q - 1))\frac{\partial}{\partial q} + w(1 - 3\lambda q)\frac{\partial}{\partial w}$$

whose canonical coordinates

$$t = q(\lambda q - 1)^2 w, p = \ln\left(\frac{\lambda q - 1}{q}\right)$$

lead to the Abel equation (4.24) for $u(t) = p'(t)$. Thus the two ODEs (4.23) and (4.27) are connected through a sequence of (non-local) transformations! It is not immediately obvious whether these two ODEs are also connected by a point transformation.

Setting $\lambda = 0$ in (4.27), we obtain:

Proposition 4.3.3 *The ODE*

$$y'' = (b_2 + 3b_3y)y' - \left(b_0 + b_1y + b_2y^2 + b_3y^3\right)b_3 \tag{4.29}$$

admits symmetries $X = \frac{\partial}{\partial x}$ *and*

$$Y = y\frac{\partial}{\partial x} + \left(b_0 + b_1y + b_2y^3 + b_3y^3\right)\frac{\partial}{\partial y}.$$

with $[X, Y] = 0$.

[6]It is interesting to note that Y_2 satisfies the ansatz for a variational symmetry that was developed in Chapter 3 (see Example 3.3.8) even though neither the ODE (4.27) nor its solved form is self-adjoint.

Remark: One can show that the ODE (4.29) admits an eight-parameter symmetry group and is linearizable.

In summary, the ODE (4.19) admits at least two symmetries iff $f(y), g(y)$ are given either by Theorem 4.3.2 or by Theorem 4.3.1. The subcases 2,4,5,6 of Theorem 4.3.2 lead to families of Abel ODEs whose solution is not given in Kamke [14].

4.3.2 Classification of integrating factors of (4.19)

In this section we will find all cases for which (4.19) admits an integrating factor w of the form

$$w = a(x,y) + y_1 b(x,y). \tag{4.30}$$

Note that (4.19) admits a symmetry $\frac{\partial}{\partial x}$. Thus if (4.19) admits an integrating of the form (4.30) then by Theorem 4.2.3 it must also admit an integrating factor of the form

$$w = e^{\lambda x}\left(a(y) + y_1 b(y)\right)$$

for some $\lambda, a(y), b(y)$. The determining equations $E(wG) = 0$ then simplify to a system of ODEs for $f(y), g(y), a(y), b(y)$:

$$b' = 0, \quad 2a' + \lambda b - 2fb = 0, \quad 2a'' + 2\lambda b' - fb' - bf' = 0, \quad g'a + a'g - \lambda(bg + fa) + \lambda^2 a = 0.$$

This system reduces further, with two cases possible.

Case 1: If $b \neq 0$, one can assume without loss of generality that $b = 1$, leading to the following result.

Proposition 4.3.4 *The ODE*

$$G = y'' + \left(a'(y) + \frac{1}{2}\lambda\right) y' + g(y) = 0 \tag{4.31}$$

admits an integrating factor of the form

$$w_1 = e^{\lambda x}\left(a(y) + y_1\right) \tag{4.32}$$

iff $a(y), g(y)$ satisfy

$$2\lambda(g - aa') + \lambda^2 a + 2(a'g + g'a) = 0. \tag{4.33}$$

If $\lambda \neq 0$ then the conservation law corresponding to w_1 is given by

$$P_1 = S(w, -y_1, G)/\lambda = e^{\lambda x}\left(y_1 a(y) + \frac{1}{2}y_1^2 + \frac{1}{\lambda}a(y)g(y)\right). \tag{4.34}$$

Setting $\lambda = 0$ and then solving (4.33) leads to

Corollary 4.3.5 *The ODE*

$$y'' - A\frac{g'(y)}{g^2(y)}y' + g(y) = 0 \tag{4.35}$$

admits an integrating factor

$$w_2 = \frac{A}{g(y)} + y_1. \tag{4.36}$$

The corresponding conservation law is given by

$$P_2 = \frac{A^2}{2}g^{-2}(y) + \int^y g(t)dt + A\frac{y_1}{g(y)} + \frac{1}{2}y_1^2 + Ax. \tag{4.37}$$

Case 2: If $b = 0$ then $a' = 0$ and so one can assume, without loss of generality, that $a = 1$, which leads to the following two propositions:

Proposition 4.3.6 *The ODE*

$$y'' + \left(\frac{1}{\lambda}g'(y) + \lambda\right)y' + g(y) = 0 \tag{4.38}$$

with $\lambda \neq 0$ admits an integrating factor

$$w_3 = e^{\lambda x}. \tag{4.39}$$

The corresponding conservation law is given by

$$P_3 = S(w, -y_1, G)/\lambda = e^{\lambda x}(y_1 + \frac{1}{\lambda}g(y)). \tag{4.40}$$

Proposition 4.3.7 *The ODE*

$$y'' + f(y)y' + A = 0 \tag{4.41}$$

admits an integrating factor 1. The corresponding conservation law is $y' + \int^y f(t)dt + Ay$.

Note that the ODE $P_3 = 0$ inherits the symmetry $\frac{\partial}{\partial x}$ and thus leads to a particular solution of (4.38). Using the transformation (4.1) one thus obtains:

Corollary 4.3.8 *The Abel ODE*

$$y' = g(x)y^3 + (\frac{1}{\lambda}g'(x) + \lambda)y^2 \tag{4.42}$$

admits a particular solution

$$y = -\frac{\lambda}{g(x)}. \tag{4.43}$$

In summary, we have the following theorem:

Theorem 4.3.9 *The ODE (4.19) admits an integrating factor of the form (4.30) iff the ODE is given by one of (4.31), (4.35), (4.38), (4.41).*

One can also combine the various propositions above to obtain cases for which (4.19) admits two functionally independent conservation laws. For instance if

$$-A\frac{g'(y)}{g^2(y)} = \frac{1}{\lambda}g'(y) + \lambda \tag{4.44}$$

then the ODE (4.35) admits conservation laws (4.37) and (4.40). Furthermore these conservation laws are functionally independent whenever $\lambda \neq 0$. The solution of (4.44) is given by

$$g(y) = \frac{\lambda}{2}\left(-\lambda y - C + \left((\lambda y + C)^2 + \frac{4}{\lambda}A\right)^{1/2}\right) \tag{4.45}$$

where C is an arbitrary constant. As an example, taking $C = 0, \lambda = 1, A = 1$ we get

$$g(y) = \frac{1}{2}\left(-y + (y^2+4)^{1/2}\right).$$

Using

$$\int (y^2+4)^{1/2} dy = \frac{1}{2}y(y^2+4)^{1/2} + 2\ln(y + (y^2+4)^{1/2})$$

the conservation laws (4.37), (4.40) become:

$$P_2 = \frac{2}{\left(-y + (y^2+4)^{1/2}\right)^2} - \frac{1}{4}y^2 + \frac{1}{4}y(y^2+4)^{1/2} + \ln(y + (y^2+4)^{1/2}) + \frac{2y_1}{-y + (y^2+4)^{1/2}} + \frac{1}{2}y_1^2 + x,$$

$$P_3 = e^x \left(y_1 - \frac{1}{2}y + \frac{1}{2}(y^2+4)^{1/2} \right).$$

Eliminating y_1 from the system $P_2 = C_2, P_3 = C_3$ one obtains a general solution to the ODE

$$y'' + \frac{1}{2}\left(1 + \frac{y}{(y^2+4)^{1/2}}\right) y' + \frac{1}{2}\left(-y + (y^2+4)^{1/2}\right) = 0. \qquad (4.46)$$

This ODE does not admit any point symmetries other than $\frac{\partial}{\partial x}$. The corresponding Abel ODE is

$$y' = \frac{1}{2}\left(1 + \frac{x}{(x^2+4)^{1/2}}\right) y^2 + \frac{1}{2}\left(-x + (x^2+4)^{1/2}\right) y^3. \qquad (4.47)$$

4.4 Discussion

In this chapter we have shown how the methods of previous chapters can be applied to find sub-families of solvable ODEs from a given family of ODEs. We have considered Emden-Fowler family of equations as well as the family (4.19). Any solvable case of (4.19) leads to a solvable Abel equation (4.2) using transformations (4.1). For both families, we have obtained several new solvable cases not listed in the standard reference by Kamke [14] or any other reference in the bibliography.

When the family under consideration admits a symmetry, one can use it to generate ansatzes for seeking integrating factors and symmetries. Unlike the symmetry ansatzes studied in [9], our symmetry ansatz does not require a priori knowledge of the Lie algebra structure of the ODE in question.

We have applied such ansatzes to (4.19) which admits a point symmetry $\frac{\partial}{\partial x}$. As a result, we found all cases for which (4.19) admits either another point symmetry or an integrating factor linear in y'.

The ansatz given in Theorem 4.2.2 with $\alpha \neq 0$, if successful, leads to an integrating factor whose corresponding conservation law can be found *without* quadrature using that theorem. The utility of this is demonstrated in Proposition 4.3.4 where the integrating factor w_1 was found depending on an arbitrary function. Nevertheless, the corresponding conservation law

P_1 was found without quadrature. In addition one can always find a particular solution of the resulting conservation law when $\alpha \neq 0$ and the ODE is of order two.

Finding an integrating factor is a task no more difficult than finding a symmetry. However once found, a reduction of order using a symmetry requires finding canonical coordinates. To do this *in general*, one must solve an auxiliary ODE (see Section 1.1.3).[7] By contrast, an integrating factor always leads to a reduction through a quadrature, without ever having to solve any additional ODEs. As well, an integrating factor reduction is a reduction *in the original variables* unlike a symmetry reduction which requires a change of variables involving a differential substitution.

[7] If an n-th order ODE admits a solvable Lie Algebra of n symmetries then there is an algorithm by due to Lie that can find a general solution using quadratures only, never having to solve any additional ODEs (see Stephani [21], Chapter 9.3)

Chapter 5
Conclusions and future work

5.1 Conclusions

In this thesis we have examined the connections between conservation laws and symmetries, both for self-adjoint and non self-adjoint ODEs. The goal was to gain a better understanding of how to combine symmetry methods with the method of conservation laws to obtain results not obtainable by either method separately.

In Chapter 1 we have reviewed symmetry methods and how to use symmetries to obtain reduction of order of ODEs.

In Chapter 2 we have discussed how to find conservation laws. In the absence of symmetries, one can look for conservation laws directly by seeking solutions of a linear PDE (2.2). The situation is more interesting when a symmetry of an ODE is known. Lemma 2.1.4 shows that if X is a symmetry generator and P is a conservation law of an ODE $G = 0$ then XP is also a conservation law of $G = 0$. Conversely, it is shown (Lemma 2.1.5) that if P_0 is any conservation law of $G = 0$ then there exists a conservation law P with $XP = P_0$. Since zero is a (trivial) conservation law, this leads to an ansatz: seek a conservation law P satisfying $XP = 0$. Such an ansatz is equivalent to the method of differential invariants for point symmetries, but works for general Lie-Bäcklund symmetries as well. If such a P is found, one can also seek a conservation law Q with $XQ = 1$. As shown in Section 2.1.4, for second order ODEs, Q can be found using quadratures only.

Instead of looking for conservation laws, one can seek integrating factors that characterise them. An integrating factor w of an ODE $G = 0$ can be found by seeking a particular solution of the determining equation $E(wG) = 0$ (see Theorem 2.2.13). Given an integrating factor, the corresponding conservation law can be computed by a quadrature using Theorem 2.2.14.

When in addition to an integrating factor one also knows a symmetry, it is sometimes possible to compute a non-trivial conservation law *without* quadrature. In particular, given a conservation law P whose integrating factor is w, and given a symmetry X of $G = 0$, the conservation law XP can be obtained from w, X and G without any quadrature, as shown in Theorem 2.3.2.

In Chapter 3 we discussed self-adjoint ODEs. An Euler-Lagrange equation is self-adjoint and an integrating factor of a self-adjoint ODE is also a symmetry of it. Conversely, a *variational* symmetry written in evolutionary form is an integrating factor. Thus there is a one-to-one correspondence between integrating factors and variational symmetries of self-adjoint ODEs.

Given a variational symmetry, one can use Noether's Theorem 3.2.9 to find the corresponding conservation law. Moreover, the resulting conservation law admits the variational symmetry that was used to find it. Thus a two-fold reduction of order is possible using a single variational symmetry: one reduction in original variables and one symmetry reduction. In general, one can ask how many reductions are possible if r variational symmetries are known. The answer is $2r$ if and only if the conservation laws corresponding to variational symmetries are functionally independent, and the variational symmetries form an *abelian* Lie algebra. More generally, this question is answered in Theorem 3.3.4. In Example 3.3.3, three variational symmetries that form a simple Lie Algebra are used to achieve a four-fold reduction of order.

In its full generality, Noether's Theorem requires a quadrature to obtain a conservation law. However when more then one symmetry is known, it is sometimes possible to obtain a conservation law without quadrature. In particular, a commutator of two variational symmetries is shown in Theorem 3.3.1 to be a variational symmetry. Furthermore, a conservation law corresponding to such a commutator can always be obtained without any integration, as shown in Theorem 3.3.1.

A commutator of a variational symmetry and a non-variational symmetry may also be a variational symmetry. In particular we show that for a self-adjoint ODE of the form

$$G = y_n - g(x, y, y_1, ..., y_{n-1}), \tag{5.1}$$

a commutator of a *scaling symmetry* and a variational symmetry is always a variational symmetry. In most cases the conservation law corresponding to such a commutator can be obtained without any integration (see discussion after Theorem 3.3.10). More generally, a commutator of a variational symmetry w and a non-variational point symmetry v need not be a variational symmetry. However, there is an expression $u(v, w)$ given by (3.20) which results in a variational symmetry. When v is also variational, $u(v, w) = [v, w]$.

For a self-adjoint ODE (5.1) it is possible to tell when a given point symmetry is variational, without using G. One merely needs to check if the symmetry satisfies conditions (3.16) of Theorem 3.13. This provides an ansatz for looking for variational symmetries: if G has a variational point symmetry X, then it must be of the form (3.19):

$$X = \xi(x)\frac{\partial}{\partial x} + \left(\frac{n-1}{2}\xi'(x)y + f(x)\right)\frac{\partial}{\partial y}$$

Thus the determining equations for variational point symmetries are immediately reduced from an overdetermined system of PDEs to an overdetermined system of ODEs. Other consequences are given in Example 3.3.9.

In Chapter 4 we have shown how the methods of previous chapters can be applied to find sub-families of solvable ODEs from a given family of ODEs.

We have classified point symmetries of the Emden-Fowler ODE

$$G = y'' - Ax^n y^m y'^l = 0 \tag{5.2}$$

as well as its integrating factors linear in y'. We have identified two cases for which G admits an integrating factor. In the first case, the corresponding conservation law is invariant under the scaling symmetry admitted by G and hence we were able to find a full solution of $G = 0$. In the second case the resulting conservation law does not admit a scaling symmetry in general, but a specific solution was nevertheless found. In both cases symmetry methods alone fail to produce a solution or a reduction of order in the same variables since both cases do not admit point symmetries other than the scaling symmetry. Together, symmetry and integrating factor classification lead to eight new solvable cases of Emden-Fowler equation not found in Kamke [14] or any other literature in bibliography. These cases are:

$$(l, m, n) = (0, m, 3-m), (1, 1, -1), (1, m, -1), (1, -1/2, -2).$$

and the four cases obtained from the above using the transformation (4.4).

We have developed ansatzes that use known symmetries to find new symmetries or integrating factors.

The ansatz for a symmetry using a known symmetry (Theorem 4.2.5) reduces by one the number of independent variables in the determining equations for symmetries, while introducing an extra constant parameter λ (see Example 4.2.6). Thus for point symmetries, the determining equations are reduced from an overdetermined system of PDEs to an overdetermined system of ODEs. This ansatz is general enough in the following sense: if an ODE admits more than one symmetry then it must also admit a symmetry which is different from the known symmetry and which satisfies the ansatz. Conversely, if a symmetry is found using this ansatz and with $\lambda \neq 0$, then it will necessarily be different from the known symmetry.

The ansatz for integrating factors using a known symmetry (Theorems 4.2.2 and 4.2.3) also introduces an additional parameter α, while reducing by one the number of independent variables. If an integrating factor is found using this ansatz with $\alpha \neq 0$, then the corresponding conservation law can be found without any quadrature. Moreover in such a case a particular solution to the ODE depending on an arbitrary constant can then be found if the ODE is of order two. Thus such an ansatz with $\alpha \neq 0$ leads to both a conservation law and a particular solution for ODEs of order two.

We have applied these ansatzes to the equation

$$G = y'' - f(y)y' - g(y) \tag{5.3}$$

which admits a point symmetry $\frac{\partial}{\partial x}$ and which is equivalent to an Abel ODE (4.2) using a transformation (4.1). Any solvable case of G thus leads to a solvable Abel ODE. We found all cases for which G admits either another point symmetry or an integrating factor linear in y'.

A symmetry classification of G resulted in four non-trivial solvable families (cases 2,4,5,6 of Theorem 4.3.2) of Abel ODEs that are distinct from all solvable Abel ODEs reported in Kamke. Each one of these families depends on several arbitrary constants.

An integrating factor classification resulted in several cases for which a *particular* solution to G could be found. As a result, we have singled out a family of Abel equations (4.42) that depends on an arbitrary function. For this family a particular solution (4.43) is given.

We have identified a case for which G admits two functionally independent conservation laws without admitting any point symmetry other then $\frac{\partial}{\partial x}$. This case is also not reported in Kamke.

5.2 Future research

Possible directions for future work include:

1. Extension of ansatzes for the case where more than one symmetry is known.
2. When is a commutator of two non-variational symmetries a variational symmetry?
3. Explain the surprising connection between the two fourth order ODEs (4.23) and (4.27). Section 4.3.
4. Computation of the solutions of the solvable Abel ODEs found in Section 4.3.
5. Extension of the results to PDEs (especially those in Chapter 3).
6. Application of the ansatz (3.19) to find ODEs of the form $y'' = f(x, y)$ that admit a variational symmetry.

Bibliography

[1] Anco, S.C. & Bluman, G., (1998) Integrating factors and first integrals of ordinary differential equations, Eur. J. Appl. Math **9** no. 3, 245–259.

[2] Anco, S.C. & Bluman, G., (1996) Derivation of conservation laws from nonlocal symmetries of differential equations, , J. Math. Phys. **37** no. 5, 2361–2375.

[3] Anco, S.C. & Bluman, G., (1997) Direct Construction algorithm of Conservation Laws from Field Equation, , Phys. Rev. Lett. **78** no. 15, 2869–2873.

[4] Anderson R.L, Kumei S, Wulfman C.E., (1972) Generalization of the concept of invariance of differential equations. Results of applications to some Schrödinger equations, Phys. Rev. Lett. **28** no. 15, 988–991.

[5] Arnold, V.I., (1991) *Ordinary Differential Equations*, Springer-Verlag.

[6] Bessel-Hagen, (1921) Über die Erhaltungssätze det Elektrodynamik., Math. Ann. **84**, 258-276.

[7] Bluman G. & Kumei S., *Symmetries and Differential Equations*, Springer-Verlag, New York. Reprinted with corrections, Springer, New York, 1996.

[8] Boulier F., Lazard D., Ollivier F. and Petitot M., (1995) Representation for the radical of a finitely generated differential ideal, Proc. ISSAC 1995, ACM Press 158-166.

[9] Boulton, A., (1993) New Symmetries from old: Exploiting Lie Algebra Structure, M.Sc. Thesis, University of British Columbia.

[10] Boyer, T.H., (1967) Continuous symmetries and conserved currents, Ann. Phys. 42, 445-466.

[11] Cheb-Terrab S.E, Duarte L.G.S. and da Mota L.A.C.P., (1998) Computer Algebra Solving of Second Order ODEs Using Symmetry Methods, Comput. Phys. Comm. **108** no. 1, 90–114.

[12] Gonzáles-López, A., (1988) Symmetry and integrability by quadratures of ordinary differential equations, Phys.Lett. A **133**, no. 4-5, 190-194.

[13] Hereman, W., (1995) Chapter 13 in vol. 3 of the CRC Handbook of Lie Group Analysis of Differential Equations, Ed.: N.H.Ibragimov, CRC Press, Boca Raton, Florida.

[14] Kamke, E., (1947) *Differentialgleichungen*, N.Y. Chelsea Publ. Co.

[15] Noether, E., (1918) Invariante Variationsprobleme, Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl., 235-257.

[16] Olver P.J., (1993) *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York.

[17] Reid G.J., (1995) Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution, Euro. J. Appl. Math. **2**, 2315-2327.

[18] Reid G.J., Lisle I.G., Boulton A. and Wittkopf A.D., (1992) Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs, Proc. ISSAC 1992, ACM Press, New York.

[19] Reid G.J., Wittkopf A.D. and Boulton A., (1996) Reduction of systems of nonlinear partial differential equations to simplified involutive forms, . Eur. J. Appl. Math. **7** 604-635.

[20] Sheftel, M.B., (1997) *A Course in group analysis of differential equations – in Russian*, University of St.Petersburg of Economics and Finance Press.

[21] Stephani H., (1989) *Differential equations: their solution using symmetries*, Cambridge University Press.

[22] Zaïtsev, V.F., (1988) On discrete-group analysis of ordinary differential equations, Soviet Math. Dokl. **37** no. 2 403-406.

10 On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. By V. K. CHANDRASEKAR, M. SENTHILVELAN AND M. LAKSHMANAN (2005)

PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2005) **461**, 2451–2476
doi:10.1098/rspa.2005.1465
Published online 23 June 2005

On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations

By V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan

Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India
(lakshman@cnld.bdu.ac.in)

A method for finding general solutions of second-order nonlinear ordinary differential equations by extending the Prelle–Singer (PS) method is briefly discussed. We explore integrating factors, integrals of motion and the general solution associated with several dynamical systems discussed in the current literature by employing our modifications and extensions of the PS method. We also introduce a novel way of deriving linearizing transformations from the first integrals to linearize the second-order nonlinear ordinary differential equations to free particle equations. We illustrate the theory with several potentially important examples and show that our procedure is widely applicable.

Keywords: integrability; integrating factor; linearization; equivalence problem

1. Introduction

Solving nonlinear ordinary differential equations (ODEs) is one of the classic but potentially important areas of research in the theory of dynamical systems (Arnold 1978; José & Saletan 2002; Wiggins 2003). Indeed, a considerable amount of research activity in this field was witnessed in the last century. Progress has been made through geometrical analysis and analytical studies. The modern geometrical theory originated with Poincaré and vigorously developed by Arnold, Moser, Birkhoff and others (Percival & Richards 1982; Guckenheimer & Holmes 1983; Litchenberg & Lieberman 1983; Wiggins 2003). Various analytical methods have been concurrently devised to tackle nonlinear ODEs. The ideas developed by Kovalevskaya, Painlevé and his co-workers have been used to integrate a class of nonlinear ODEs and obtain their underlying solutions (Ince 1956). As a consequence of these studies, nonlinear dynamical systems are broadly classified into two categories, namely, (i) integrable and (ii) nonintegrable systems. Indeed, one of the important current problems in nonlinear dynamics is to identify integrable dynamical systems (Ablowitz & Clarkson 1992; Lakshmanan & Rajasekar 2003). Of course, these methods have a close connection with the group theoretical approach introduced by Sophus Lie in the nineteenth century and subsequently extended by Cartan and Tresse to integrate ordinary and partial differential equations (e.g. see Olver (1995) and Bluman & Anco (2002)).

Received 20 July 2004
Accepted 7 February 2005

Different techniques have been proposed for identifying such integrable dynamical systems, including Painlevé analysis (Conte 1999), Lie symmetry analysis (Bluman & Anco 2002) and direct methods of finding involutive integrals of motion (Hietarinta 1987). Each method has its advantages and disadvantages. For a detailed discussion about the underlying theory of each method and its limitations and applications we refer to Lakshmanan & Rajasekar (2003). Also, certain nonlinear ODEs can be solved through transformation to linear ODEs whose solutions are known. In fact, linearization of given nonlinear ODEs is one of the classic problems in ODE theory whose origin dates back to Cartan. For information on recent progress in this direction we refer readers to Olver (1995).

Prelle & Singer (1983) have proposed a procedure for solving first-order ODEs that presents the solution, if such a solution exists, in terms of elementary functions. The attractiveness of the Prelle–Singer (PS) method is that the method guarantees that a solution will be found if the given system of first-order ODEs has a solution in terms of elementary functions. Duarte *et al.* (2001) modified the technique developed by Prelle & Singer and applied it to second-order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second-order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. For a class of systems these authors (Duarte *et al.* 2001) have deduced first integrals through their procedure, in some cases for the first time.

In this paper we show that the theory of Duarte *et al.* (2001) can be extended in different directions to isolate two independent integrals of motion and obtain solutions. In the earlier study it was shown that the theory can be used to derive only one integral. In this work we extend their theory and deduce a general solution from the first integral. Our examples include those considered in Duarte *et al.*'s and certain important equations discussed in the recent literature whose solutions are not known. There are two objectives central to our study. First, it is to show that one can deduce general solutions in a straightforward and simple manner, as well as through finding first integrals. The method we propose is not confined to the PS method alone but can be treated as a general one. If one has a first integral for a given second-order ODE then our method provides the general solution in an algorithmic way for at least a class of equations. The reason for merging our procedure with the PS method, rather than any other method, is owing to the following facts.

(i) It has been conjectured that the PS method is guaranteed to provide first integrals for a given problem if a solution exists.
(ii) The PS method not only gives the first integrals but also the underlying integrating factors, that is, by multiplying the equation with these functions we can rewrite the equation as a perfect differentiable function, which gives the first integrals in a separate way upon integration.
(iii) The PS method can be used to solve nonlinear as well as linear second-order ODEs. As the PS method is based on the equations of motion rather than Lagrangian or Hamiltonian description, the analysis is applicable to both Hamiltonian and non-Hamiltonian systems.

Our second reason is to introduce and demonstrate a novel and straightforward technique for constructing and exploring linearizing transformations. The given second-order nonlinear ODEs can be transformed to linear equations, in particular, to free particle equations by exploring the transformations. As we illustrate below, these transformations can be deduced from the first integral, which is an entirely new technique in the current literature. In a nutshell, once a first integral is known then our procedure provides, at least for a class of problems, the general solutions as well as the linearizing transformations. The ideas proposed here can be applied to a coupled system of second-order ODEs as well as higher order ODEs, which will be presented separately.

The paper is organized as follows. In §2, we briefly describe the PS method applicable for second-order ODEs and indicate new features in finding the integrals of motion. In §3, we have extended the theory in three different directions, which indicates the novelty of the approach. The first significant application is that the second integral can be deduced from the method itself in many cases. The second application is that the general solution can be deduced from the first integral. Finally, we propose a method of identifying linearizing transformations. We emphasize the validity of the theory, with several illustrative examples arising in different areas of physics, in §4. In §5, we demonstrate the method for identifying linearizing transformations with three examples, including one studied in recent literature. We present our conclusions in §6.

2. Prelle–Singer method for second-order ODEs

In this section, we briefly discuss the theory introduced by Duarte *et al.* (2001) for second-order ODEs and extend it so that general solutions can be deduced from the modifications. Let us consider second-order ODEs of the form

$$\ddot{x} = \frac{P}{Q}, \quad P, Q \in \mathbb{C}[t, x, \dot{x}], \tag{2.1}$$

where $\dot{x}$ denotes differentiation with respect to time and P and Q are polynomials in t, x and $\dot{x}$ with coefficients in the field of complex numbers. Let us assume that the ODE (2.1) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential gives

$$\mathrm{d}I = I_t\,\mathrm{d}t + I_x\,\mathrm{d}x + I_{\dot{x}}\,\mathrm{d}\dot{x} = 0, \tag{2.2}$$

where the subscript denotes partial differentiation with respect to that variable. Rewriting equation (2.1) in the form $(P/Q)\mathrm{d}t - \mathrm{d}\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}\,\mathrm{d}t - S(t, x, \dot{x})\,\mathrm{d}x$ to the latter, we obtain the 1-form

$$\left(\frac{P}{Q} + S\dot{x}\right)\mathrm{d}t - S\,\mathrm{d}x - \mathrm{d}\dot{x} = 0. \tag{2.3}$$

Hence, on the solutions, the 1-forms given by (2.2) and (2.3) must be proportional. Multiplying (2.3) by the factor $R(t, x, \dot{x})$, which acts as the integrating factor for equation (2.3), we obtain

$$dI = R(\phi + S\dot{x})dt - RS\,dx - R\,d\dot{x} = 0, \tag{2.4}$$

where $\phi \equiv P/Q$. By comparing equation (2.2) with equation (2.4) we find the relations

$$\left.\begin{aligned} I_t &= R(\phi + \dot{x}S), \\ I_x &= -RS, \\ I_{\dot{x}} &= -R. \end{aligned}\right\} \tag{2.5}$$

Then, the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the equations (2.5) require that

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{2.6}$$

$$D[R] = -R(S + \phi_{\dot{x}}), \tag{2.7}$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \tag{2.8}$$

where

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}.$$

Equations (2.6)–(2.8) can be solved in the following way. One can obtain an expression for S by substituting the given expression of ϕ into equation (2.6) and solving it. Equation (2.7) becomes the determining equation for the function R once S is known. One can get an explicit form for R by solving equation (2.7). Now the functions R and S have to satisfy an extra constraint, that is, equation (2.8). Once a compatible solution satisfying all three equations has been found, then functions R and S fix the integral of motion $I(t, x, \dot{x})$ with the relation

$$\begin{aligned} I(t,x,\dot{x}) = &\int R(\phi + \dot{x}S)dt - \int\left(RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt\right)dx \\ &- \int\left\{R + \frac{d}{d\dot{x}}\left[\int R(\phi + \dot{x}S)dt - \int\left(RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt\right)dx\right]\right\}d\dot{x}. \end{aligned} \tag{2.9}$$

Equation (2.9) can be derived straightforwardly by integrating the equations (2.5). Note that for every independent set (S, R), equation (2.9) defines an integral.

Thus, two independent sets, (S_i, R_i), $i=1, 2$, provide us with two independent integrals of motion through the relation (2.9), which guarantees the integrability of equation (2.1). Since we first solved equations (2.6) and (2.7) and then checked the compatibility of this solution with equation (2.8), we often found that the solutions that satisfied equations (2.6) and (2.7) did not need to satisfy equation (2.8) as equations (2.6)–(2.8) constitute an overdetermined system for the unknowns R and S. In fact, for a class of problems one often gets a set (S_1, R_1) which satisfies equations (2.6)–(2.8) and another set (S_2, R_2) that satisfies only equations (2.6) and (2.7), not equation (2.8). In this situation, we find that one can use the first integral, derived from the set (S_1, R_1), to deduce the second compatible solution $(S_2, \hat{R}_2)$. For example, let the set (S_2, R_2) be a solution of equations (2.6) and (2.7) and not of the constraint equation (2.8). After examining several examples we find that one can make the set (S_2, R_2) compatible by modifying the form of R_2 as

$$\hat{R}_2 = F(t, x, \dot{x})R_2, \tag{2.10}$$

where $\hat{R}_2$ satisfies (2.7), so that we have

$$(F_t + \dot{x}F_x + \phi F_{\dot{x}})R_2 + FD[R_2] = -FR_2(S_2 + \phi_{\dot{x}}). \tag{2.11}$$

Furthermore, if F is a constant of motion (or a function of it), then the first term on the left-hand side vanishes and one gets the same equation (2.7) for R_2, provided F is non-zero. In other words, whenever F is a constant of motion or a function of it, then the solution of equation (2.7) may provide only a factor of the complete solution $\hat{R}_2$ without the factor F in equation (2.10). This general form of $\hat{R}_2$ with S_2 can now form a complete solution to the equations (2.6)–(2.8). In a nutshell, we describe the procedure as follows. First, we determine S and R from equations (2.6) and (2.7). If the set (S,R) satisfies equation (2.8) then we take it as a compatible solution and proceed to construct the associated integral of motion. On the other hand, if it does not satisfy equation (2.8) we then assume the modified form $\hat{R}_2 = F(I_1)R_2$, where I_1 is the first integral which has already been derived through a compatible solution, and find the explicit form of $F(I_1)$ from equation (2.8), which in turn fixes the compatible solution $(S_2, \hat{R}_2)$. This set $(S_2, \hat{R}_2)$ can be utilized to derive the second integral.

3. Generalization

(*a*) *Identifying a second integral of motion*

Duarte *et al.* (2001) have considered certain physically important systems and constructed first integrals. Furthermore, they mentioned that one can deduce the general solution by applying the original PS algorithm to these first integrals (by treating them as first-order ODEs). An interesting observation we make here is that there is no need to invoke the original PS procedure to deduce the general solution. In fact, as we show below, the general solution can be derived in a self-contained way. As the motivation of Duarte *et al.* (2001) was to construct only the first integral, they reported only one set of solutions (S, R) for the equations (2.6)–(2.8). However, we have observed that an additional independent set of solutions, namely, (S_2, R_2), of equations (2.6)–(2.8), may lead to another integral of motion, I_2, and if the latter is an independent function of I_1 then one can write down the general solution for the given problem from these two integrals alone. Now the question is whether one will be able to find a second pair of solutions for the system (2.6)–(2.8) and construct I_2 through the relation (2.9). After investigating several examples we observed the following.

(i) For a class of equations, including harmonic oscillator, equation coming from general relativity and generalized modified Emden equations with constant external forcing, one can easily construct a second pair of solutions (S_2, R_2) and deduce I_2 through the relation (2.9). We call this class Type I.

(ii) For another class of equations we can find (S_2, R_2) explicitly from equations (2.6)–(2.8) but are unable to integrate equation (2.9) exactly and unambiguously obtain the second integral I_2. We call this class Type II. The examples included in this category are Helmhotz oscillator and Duffing oscillator equations. For this class of equations we identify an alternative way to derive the second integration constant.

(iii) There exists another category in which the systems do not even admit a second pair (S_2, R_2) of solutions in simple rational forms for the equations (2.6)–(2.8) and we call this category Type III. An example is the Duffing–van der Pol oscillator, which is one of the prototype examples for the study of nonlinear dynamics in many branches of science. For this class of equations we identify an alternative way to obtain the second integral.

(*b*) *Method of deriving general solution*

To overcome the difficulties in constructing the second constant in Types II and III we propose the following procedure. As our aim is to derive the general solution for the given problem, we split the functional form of the first integral I into two terms so that one involves all the variables $(t, x, \dot{x})$ while the other excludes $\dot{x}$, that is

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \tag{3.1}$$

Now, let us split the function F_1 further in terms of two functions so that F_1 is a function of the product of the two functions, say, a perfect differentiable function $(\mathrm{d}/\mathrm{d}t)G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x)\right) + F_2(G_1(t, x)). \tag{3.2}$$

We note that while rewriting equation (3.1) in the form of equation (3.2), we require that the function $F_2(t, x)$ in equation (3.1) is automatically a function of $G_1(t, x)$. The reason for making such a specific decomposition is that in this case equation (3.2) can be rewritten as a simple first-order ODE for the variable G_1 (see equation (3.4) below). Actually, we originally realized this possibility for the integrable force-free Duffing–van der Pol oscillator equation (Chandrasekar *et al.* 2004), which has been generalized in the present case. Identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is

$$w = G_1(t, x), \quad z = \int_o^t G_2(t', x, \dot{x})\mathrm{d}t', \tag{3.3}$$

one obtains an explicit transformation to remove the time-dependent part in the first integral (2.9). We note here that the integration leading to z on the right-hand side of equation (3.3) can be performed provided the function G_2 is an exact derivative of t, that is, $G_2 = \mathrm{d}z(t, x)/\mathrm{d}t = \dot{x}z_x + z_t$, so that z turns out to be a function of t and x alone. In terms of the new variables, equation (3.2) can be modified to the form

$$I = F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) + F_2(w). \tag{3.4}$$

In other words

$$F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) = I - F_2(w). \tag{3.5}$$

By rewriting equation (3.4) one obtains a separable equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(w), \tag{3.6}$$

which can lead to the solution after integration. By rewriting the solution in terms of the original variables one obtains a general solution for equation (2.1).

(*c*) *Method of deriving linearizing transformations*

Finally, the following interesting point can be noted in the above analysis. Assuming $F_2(w)$ is zero in equation (3.4) obtains the simple equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \hat{I}, \tag{3.7}$$

where $\hat{I}$ is a constant. In other words, we have

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 0, \tag{3.8}$$

which is nothing but the free particle equation. In this case, the new variables z and w helps us to transform the given second-order nonlinear ODE into a second-order linear ODE, which in turn leads to the solution by trivial integration. The new variables z and w turn out to be the linearizing transformations. We discuss this possibility in detail in §5.

4. Applications

In this section, we demonstrate the theory discussed in the previous section with suitable examples. In particular, we consider several interesting examples, including those considered in Duarte *et al.* (2001), derive general solutions and establish complete integrability of these dynamical systems. We split our analysis into three categories. In the first category, we consider examples in which the I_i, $i=1,2$, can be easily derived from the relation (2.9). In the second and third categories, we follow our own procedure detailed in §3*b* and *c*, and deduce the second constant. We note that our procedure can be applied to a wide range of systems with second-order equations similar to equation (2.1) but we consider only a few examples for illustrative purposes.

(*a*) *Type-I systems*

As mentioned earlier, one can obtain the second pair of solutions (S_2, R_2) in an algorithmic way for certain equations from the determining equations (2.6)–(2.8) and construct I_2 through the relation (2.9). We observe that examples 1 and 2 discussed in Duarte *et al.* (2001) can be solved in this way and so we consider these two examples first and then a non-trivial example.

(i) *Example 1: an exact solution in general relativity*

Duarte *et al.* (2001) considered the following equation, which was originally derived by Buchdahl (1964) in the theory of general relativity,

$$x\ddot{x} = 3\dot{x}^2 + \frac{x\dot{x}}{t}, \tag{4.1}$$

and deduced the first integral I through their procedure. In the following, we briefly discuss their results and then illustrate our ideas. Substituting $\phi = (3\dot{x}^2/x) + (\dot{x}/t)$ into equations (2.6)–(2.8) we get

$$S_t + \dot{x}S_x + \frac{\dot{x}(3t\dot{x} + x)}{tx} S_{\dot{x}} = \frac{3\dot{x}^2}{x^2} + \frac{6t\dot{x} + x}{tx} S + S^2, \tag{4.2}$$

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x} + x)}{tx} R_{\dot{x}} = -RS - \frac{6\dot{x}t + x}{tx} R, \tag{4.3}$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \tag{4.4}$$

As mentioned in §2, let us first solve equation (4.2) and obtain an explicit form for the function S. To do so, Duarte *et al.* (2001) considered an ansatz for S of the form

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}, \tag{4.5}$$

where a, b, c and d are arbitrary functions of t and x. A rational form for S can be justified, since from equation (4.5) it may be noted that $S = (I_x/I_{\dot{x}})$. We consider only rational forms for S in $\dot{x}$ for all the examples which we consider in this paper. It may be noted that in certain examples, including the present one and examples 3 and 5 (below), this form degenerates into a polynomial form in $\dot{x}$. However, for other examples such as examples 2 and 4 (below), a rational form like equation (4.5) is required. To be general, we carry out an analysis with the form of equation (4.5).

By substituting equation (4.5) into equation (4.2) and equating the coefficients of different powers of $\dot{x}$ to zero, we get a set of partial differential equations for the variables a, b, c and d. By solving them we find that

$$S_1 = -\frac{3\dot{x}}{x}, \quad S_2 = -\frac{\dot{x}}{x}. \tag{4.6}$$

We note that Duarte *et al.* (2001) have reported the expression S_1 as the only solution for equation (4.2). However, we find S_2 also forms a solution for equation (4.2) and helps to deduce the general solution. Substituting forms S_1 and S_2 into equation (4.3) and solving the latter one can lead to an explicit form for the function R. Let us first consider S_1. By substituting S_1 into equation (4.3) we get the following equation for R:

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x} + x)}{tx} R_{\dot{x}} = \frac{3\dot{x}}{x} R - \frac{6\dot{x}t + x}{tx} R. \tag{4.7}$$

In order to solve equation (4.7) one has to make an ansatz. We assume the following form for R:

$$R = A(t,x) + B(t,x)\dot{x}, \tag{4.8}$$

where A and B are arbitrary functions of (t,x). Since $R = -I_{\dot{x}}$ (*vide* equation (2.5)) the form of R may be a polynomial or rational in $\dot{x}$. Depending upon the problem, one has to choose an appropriate ansatz. To begin with one can consider a simple polynomial (in $\dot{x}$) for R; if that fails one can go for rational forms. Let us start with equation (4.8). By substituting equation (4.8) into equation (4.7) and equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, $R_1 = (1/tx^3)$ can be obtained. The solution $S_1 = -(3\dot{x}/x)$ and $R_1 = (1/tx^3)$ has to satisfy the equation (4.4) in order to be a compatible solution, which it does. Once R and S have been found the first integral I can be fixed easily using the expression (2.9) as

$$I_1 = \frac{\dot{x}}{tx^3}. \tag{4.9}$$

One can easily check that I_1 is constant on the solutions, that is, $(\mathrm{d}I_1/\mathrm{d}t) = 0$. This integral has been deduced in Duarte *et al.* (2001). However, the second expression, S_2 has been ignored by the authors since the corresponding R_2 coming out of equation (4.3) does not form a compatible solution, that is, it does not satisfy equation (4.4). In the following we show how it can be made compatible and use it effectively to deduce the second integration constant.

By substituting the expression $S_2 = -(\dot{x}/x)$ into equation (4.3) and solving it in the same way as outlined in the previous paragraph we obtain the following form for R:

$$R_2 = \frac{1}{x^5 t}. \tag{4.10}$$

However, this set (S_2, R_2) does not satisfy the extra constraint in equation (4.4). In fact, not all forms of R from equation (2.7) satisfy equation (2.8). As we explained in §3, the form of R_2 given in equation (4.10) may not be the 'complete form' but might be a factor of the complete form. To recover the complete form of R it may be assumed that

$$\hat{R} = F(I_1)R, \tag{4.11}$$

where $F(I_1)$ is a function of the first integral I_1, and determine the form of $F(I_1)$ explicitly. For this purpose we proceed as follows. Substituting

$$\hat{R}_2 = F(I_1)R_2 = \frac{1}{tx^5}F(I_1) \tag{4.12}$$

into equation (4.4), we obtain the following equation for F:

$$I_1 F' + 2F = 0, \tag{4.13}$$

where the prime denotes differentiation with respect to I_1. Upon integrating equation (4.13) (after putting the constant of integration to zero) we get

$$F = \frac{1}{I_1^2} = \frac{t^2 x^6}{\dot{x}^2}, \tag{4.14}$$

which fixes the form of $\hat{R}_2$ as

$$\hat{R}_2 = \frac{1}{I_1^2}\frac{1}{x^5 t} = \frac{tx}{\dot{x}^2}. \tag{4.15}$$

It can easily be checked that this set $S_2 = -(\dot{x}/x)$ and $\hat{R}_2 = (tx/\dot{x}^2)$ is a compatible solution for equations (4.2)–(4.4). By substituting S_2 and $\hat{R}_2$ into equation (2.9) we get an explicit form for I_2, namely,

$$I_2 = t\left(t + \frac{x}{\dot{x}}\right). \tag{4.16}$$

From the integrals I_1 and I_2 one can deduce the general solution directly (without performing any further integration) for the problem in the form

$$x = \sqrt{\frac{1}{I_1(I_2 - t^2)}}. \tag{4.17}$$

Of course, the same result can be obtained solving equation (4.9) from the first integral. However, the point we want to emphasize here is that an independent second integral of motion can be deduced to find the solution without any further integration, which can be used profitably when the expression for I_1 cannot be easily solved.

(ii) *Example 2: simple harmonic oscillator*

To illustrate the above procedure also works for linear ODEs, we consider the simple harmonic oscillator and derive the general solution. As the procedure of deriving the first integral has been discussed in detail in Duarte *et al.* (2001), we omit the details and provide only the essential expressions in the following.

The equation of motion for the simple harmonic oscillator is

$$\ddot{x} = -x \tag{4.18}$$

so that equations (2.6)–(2.8) become

$$S_t + \dot{x}S_x - xS_{\dot{x}} = 1 + S^2, \tag{4.19}$$

$$R_t + \dot{x}R_x - xR_{\dot{x}} = -RS, \tag{4.20}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{4.21}$$

As shown in Duarte *et al.* (2001) a simple solution for equations (4.19)–(4.21) can be constructed with the form

$$S_1 = \frac{x}{\dot{x}}, \quad R_1 = \dot{x}, \tag{4.22}$$

which in turn gives the first integral

$$I_1 = \dot{x}^2 + x^2 \tag{4.23}$$

through relation (2.9). However, one can easily check that

$$S_2 = -\frac{\dot{x}}{x}, \quad R_2 = x \tag{4.24}$$

is also a solution for the set equations (4.17) and (4.20) (which has not been reported earlier) but does not satisfy the extra constraint of equation (4.21). Thus as before, let us seek an $\hat{R}_2$ of the form

$$\hat{R}_2 = F(I_1)R_2 = F(I_1)x, \tag{4.25}$$

where $F(I_1)$ is a function of I_1. Substituting equation (4.25) into equation (4.21) and integrating the resultant equation, we get $F=(1/I_1)$. Thus, $\hat{R}_2$ becomes

$$\hat{R}_2 = \frac{x}{I_1} = \frac{x}{x^2+\dot{x}^2}. \tag{4.26}$$

Now, it can be checked that $(S_2, \hat{R}_2)$ satisfies equations (4.19)–(4.21) and furnishes the second integral through relation (2.9) of the form

$$\begin{aligned} I_2 &= -t - \int \frac{\dot{x}}{\dot{x}^2+x^2}\mathrm{d}x - \int\left(\frac{x}{\dot{x}^2+x^2} - \frac{\mathrm{d}}{\mathrm{d}\dot{x}}\int\frac{\dot{x}}{\dot{x}^2+x^2}\mathrm{d}x\right)\mathrm{d}\dot{x}, \\ &= -t - \tan^{-1}\frac{\dot{x}}{x}. \end{aligned} \tag{4.27}$$

Using equations (4.23) and (4.27), we can write down the general solution for the simple harmonic oscillator directly in the form

$$x = \sqrt{I_1}\cos(t + I_2). \tag{4.28}$$

In a similar way, general solutions for a class of physically important systems can be deduced.

It may be noted that I_2 can also be obtained trivially in the above two examples by simply integrating the expressions (4.9) and (4.23) without using the extended procedure. We stress that for certain equations it is not possible to integrate and obtain the general solution in this simple way and the above said procedure has to be followed to obtain the second integral. In the following we discuss one such example for which, to our knowledge, an explicit solution was not previously known.

(iii) *Example 3: modified Emden-type equation with linear term*

It is known that the generalized Emden-type equation with linear and constant external forcing is also linearizable since it admits an eight point Lie symmetry group (Mahomed & Leach 1989*a*; Pandey *et al.* submitted). In the following we explore its general solution through the extended PS algorithm. Let us first consider the equation of the form

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x=0, \tag{4.29}$$

where k and λ_1 are arbitrary parameters. To explore the general solution for the equation (4.29) we again use the PS method. In this case, we have the following determining equations for functions R and S,

$$S_t+\dot{x}S_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)S_{\dot{x}}=k\dot{x}+\frac{k^2}{3}x^2+\lambda_1-Skx+S^2, \tag{4.30}$$

$$R_t+\dot{x}R_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)R_{\dot{x}}=-R(S-kx), \tag{4.31}$$

$$R_x-SR_{\dot{x}}-RS_{\dot{x}}=0. \tag{4.32}$$

As before, let us seek an ansatz for S of the form (4.5) to the first equation in (4.30)–(4.32). By substituting the ansatz (4.5) into equation (4.30) and equating the coefficients of different powers of $\dot{x}$ to zero we get

$$\left.\begin{aligned}db_x-bd_x-kd^2&=0,\\ db_t-bd_t+cb_x-bc_x+a_xd-ad_x-2kcd-\left(\frac{k^2}{3}x^2+\lambda_1\right)d^2+kbdx-b^2&=0,\\ cb_t-bc_t+da_t-ad_t+ca_x-ac_x-kc^2-2\left(\frac{k^2}{3}x^2+\lambda_1\right)cd+2kadx-2ab&=0,\\ ca_t-ac_t-\left(\frac{k^2}{9}x^3+\lambda_1 x\right)(bc-ad)-\left(\frac{k^2}{3}x^2+\lambda_1\right)c^2+kacx-a^2&=0,\end{aligned}\right\} \tag{4.33}$$

where subscripts denote partial derivative with respect to that variable. Solving equation (4.33) we can obtain two specific solutions

$$S_1=\frac{-\dot{x}+\frac{k}{3}x^2}{x},\quad S_2=\frac{kx+3\sqrt{-\lambda_1}}{3}-\frac{k\dot{x}}{kx+3\sqrt{-\lambda_1}}. \tag{4.34}$$

By putting the forms of S_1 and S_2 into equation (4.31) and solving it the respective forms of R can be obtained . To do so let us first consider S_1. By substituting the latter into equation (4.31) we get the following equation for R:

$$R_t+\dot{x}R_x-\left(kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x\right)R_{\dot{x}}=\left(\frac{\dot{x}-\frac{k}{3}x^2}{x}+kx\right)R. \tag{4.35}$$

To solve equation (4.35) we make an ansatz of the form

$$R=\frac{A(t,x)+B(t,x)\dot{x}}{C(t,x)+D(t,x)\dot{x}+E(t,x)\dot{x}^2}. \tag{4.36}$$

By substituting equation (4.36) into equation (4.35), equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, we arrive at

$$R_1 = \mathrm{e}^{-2\sqrt{-\lambda_1}t}\left(\frac{C_0 x}{(3\dot{x}+kx^2-3\sqrt{\lambda_1}x)^2}\right), \tag{4.37}$$

where $C_0 = 18\sqrt{-\lambda_1}$. It can easily be checked that S_1 and R_1 satisfy equation (4.32) and, as a consequence, obtain the first integral

$$I_1 = \mathrm{e}^{-2\sqrt{-\lambda_1}t}\left(\frac{3\dot{x}+kx^2+3\sqrt{-\lambda_1}x}{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}\right). \tag{4.38}$$

We note that, unlike the other two examples, equation (4.38) cannot be easily integrated to provide the second integral (although one can, in fact, explicitly solve the resultant Riccati equation after some effort). We follow the procedure adopted in the previous two examples and construct I_2. By substituting the expression S_2 into equation (4.31) and solving it in the same way as outlined above, we obtain the following form for R:

$$R_2 = C_0\frac{kx+3\sqrt{-\lambda_1}}{k(3\dot{x}+kx^2-3\sqrt{-\lambda_1}x)^2}\mathrm{e}^{-3\sqrt{\lambda_1}t}. \tag{4.39}$$

However, this set (S_2, R_2) does not satisfy the extra constraint (4.32) and so to deduce the correct form of R_2 we assume that

$$\hat{R}_2 = F(I_1)R_2 = C_0\frac{F(I_1)(kx+3\sqrt{-\lambda_1})\mathrm{e}^{-3\sqrt{-\lambda_1}t}}{k(3\dot{x}+kx^2-3\sqrt{-\lambda_1}x)^2}. \tag{4.40}$$

By substituting equation (4.40) into equation (4.32) we obtain $F = (1/I_1^2)$, which fixes the form of $\hat{R}$ as

$$\hat{R}_2 = C_0\frac{kx+3\sqrt{-\lambda_1}}{k(3\dot{x}+kx^2+3\sqrt{-\lambda_1}x)^2}\mathrm{e}^{\sqrt{-\lambda_1}t}. \tag{4.41}$$

Now, one can easily check that this set $(S_2, \hat{R}_2)$ is a compatible solution for the set (4.30)–(4.32), which in turn provides I_2 through the relation (2.9),

$$I_2 = -\frac{2}{k}\mathrm{e}^{\sqrt{-\lambda_1}t}\left(\frac{9\lambda_1+3k\dot{x}+k^2x^2}{3\dot{x}+kx^2+3\sqrt{-\lambda_1}x}\right). \tag{4.42}$$

Using the explicit form of the first integrals I_1 and I_2, the solution can be deduced directly as

$$x = \left(\frac{3\sqrt{-\lambda_1}\left(I_1\mathrm{e}^{2\sqrt{-\lambda_1}t}-1\right)}{kI_1I_2\mathrm{e}^{\sqrt{-\lambda_1}t}+k(1+I_1\mathrm{e}^{2\sqrt{-\lambda_1}t})}\right). \tag{4.43}$$

To our knowledge, this is the first time equation (4.43), which explicitly solves the equation (4.29), has been given. It has several interesting consequences for nonlinear dynamics, which will be discussed separately.

(*b*) *Type-II systems*

In the previous category we considered examples which unambiguously give the integrals I_1 and I_2 through relation (2.9). In the present category we show that there are situations in which an explicit form of I_2 is difficult to obtain through relation (2.9), even though there is a compatible solution (2.6)–(2.8). An alternative method is necessary to obtain the general solution for the given problem. For this purpose, we make use of the method proposed in §3*b*. In the following, we give examples of where such a possibility occurs and how to overcome this situation.

(i) *Example 4: Helmholtz oscillator*

Recently, Almendral & Sanjuán (2003) studied the invariance and integrability properties of the Helmholtz oscillator with friction

$$\ddot{x} + c_1\dot{x} + c_2 x - \beta x^2 = 0, \tag{4.44}$$

where c_1, c_2 and β are arbitrary parameters, which is a simple nonlinear oscillator with quadratic nonlinearity. Using the Lie theory for differential equations, Almendral & Sanjuán (2003) found a parameteric choice, $c_2 = (6c_1^2/25)$, for which the system is integrable and derived the general solution for this parametric value. In the following, we solve this problem through the extended PS method.

Substituting $\phi = -(c_1\dot{x} + c_2 x - \beta x^2)$ into equations (2.6)–(2.8) we obtain

$$S_t + \dot{x}S_x - (c_1\dot{x} + c_2 x - \beta x^2)S_{\dot{x}} = c_2 - 2\beta x - c_1 S + S^2, \tag{4.45}$$

$$R_t + \dot{x}R_x - (c_1\dot{x} + c_2 x - \beta x^2)R_{\dot{x}} = -R(S - c_1), \tag{4.46}$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \tag{4.47}$$

Making the same form of an ansatz, *vide* equations (4.5) and (4.8), we find non-trivial solutions only exist for equations (4.45) and (4.46) for the parametric restrictions $c_2 = \pm(6c_1^2/25)$. However, the case $c_2 = -(6c_1^2/25)$ follows from the case $c_2 = +(6c_1^2/25)$ in equation (4.44) through the simple translation $x = X + (6c_1^2/(25\beta))$. So we consider only the case $c_2 = +(6c_1^2/25)$ in the following

$$S_1 = \frac{\left(\frac{2c_1\dot{x}}{5} + \frac{4c_1^2 x}{25} - \beta x^2\right)}{\dot{x} + \frac{2c_1}{5}x}, \quad R_1 = -\left(\dot{x} + \frac{2c_1}{5}x\right)\mathrm{e}^{(6c_1 t/5)}, \tag{4.48}$$

$$S_2 = \frac{\left(c_1\dot{x} + \frac{6c_1^2 x}{25} - \beta x^2\right)}{\dot{x}}, \quad R_2 = -\dot{x}\mathrm{e}^{c_1 t}. \tag{4.49}$$

Now, it can be easily checked that (S_1, R_1) satisfies the third equation (4.47) and, as a consequence, leads to the first integral of the form

$$I_1 = \mathrm{e}^{(6c_1 t/5)}\left(\frac{\dot{x}^2}{2} + \frac{2c_1 x\dot{x}}{5} + \frac{2c_1^2 x^2}{25} - \frac{\beta x^3}{3}\right). \tag{4.50}$$

However, the second set (S_2, R_2) does not satisfy the extra constraint (4.47) and so we take

$$\hat{R}_2 = F(I)R_2 = -F(I)\dot{x}\,\mathrm{e}^{c_1 t}, \tag{4.51}$$

which in turn gives $F = C_0 I^{-(5/6)}$, where C_0 is an integration constant, so that

$$\hat{R}_2 = -\left(\frac{C_0}{I_1^{(5/6)}}\right)\dot{x}\,\mathrm{e}^{c_1 t} = -\frac{C_0\dot{x}}{\left(\frac{\dot{x}^2}{2} + \frac{2c_1 x\dot{x}}{5} + \frac{2c_1^2 x^2}{25} - \beta\frac{x^3}{3}\right)^{(5/6)}}. \tag{4.52}$$

It can be checked that $(S_2, \hat{R}_2)$ satisfy equations (4.45)–(4.47) and so one can proceed to deduce the second integration constant through relation (2.9). However, upon substituting $(S_2, \hat{R}_2)$ into (2.9) we arrive at

$$I_2 = \int \frac{c_1\dot{x} + \frac{6c_1^2 x}{25} - \beta x^2}{\left(\frac{\dot{x}^2}{2} + \frac{2c_1 x\dot{x}}{5} + \frac{2c_1^2 x^2}{25} - \beta\frac{x^3}{3}\right)^{(5/6)}}\,\mathrm{d}x. \tag{4.53}$$

It is very difficult to evaluate the integral and so an explicit form of I_2 for this problem cannot be obtained. A similar form of I_2 has been also derived by Jones *et al.* (1993) and Bluman & Anco (2002) for the Duffing oscillator problem (that is, the cubic nonlinearity in equation (4.44)).

Unlike the other examples discussed in Type I, the present example presents difficulties in evaluating the second integration constant, in fact, for a class of equations complicated integrals are faced. To overcome this, one has to look for an alternative way that allows the second constant to be deduced in a straightforward and simple manner. We tackled this situation in the following way. As we have seen, in most of the problems, we are able to deduce the first integral, that is, I_1, straightforwardly, and the first integral often admits explicit time-dependent terms. A useful way of overcoming this is to remove the explicit time-dependent terms by transforming the resultant differential equation into an autonomous form and integrate the latter and thus obtain the solution. In order to do this, one needs a transformation, and the latter can often be constructed through ad hoc methods. However, as we have shown in the theory in §3*b*, the required transformation coordinates can be deduced in a simple way from the first integral itself and the problem can be solved in a systematic manner.

Rewriting the first integral I_1 given by equation (4.50) in the form (3.1), we get

$$I_1 = \frac{1}{2}\left(\dot{x} + \frac{2c_1 x}{5}\right)^2 \mathrm{e}^{(6c_1 t/5)} - \frac{\beta x^3}{3}\mathrm{e}^{(6c_1 t/5)}. \tag{4.54}$$

Now, splitting the first term in equation (4.54) further in the form (3.2),

$$I_1 = \mathrm{e}^{(2c_1 t/5)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{2}}x\,\mathrm{e}^{(2c_1 t/5)}\right)\right)^2 - \frac{\beta}{3}(x\,\mathrm{e}^{(2c_1 t/5)})^3, \tag{4.55}$$

and identifying the dependent and independent variables from (4.55) and the relations (3.3), we obtain the transformation

$$w = \frac{1}{\sqrt{2}} x\, \mathrm{e}^{(2c_1 t/5)}, \quad z = -\frac{5}{c_1} \mathrm{e}^{-(c_1 t/5)}. \tag{4.56}$$

It is easy to check that equation (4.44) can be transformed to an autonomous form with the help of the transformation (4.56). We note that the transformation (4.56) exactly coincides with the earlier one constructed via Lie symmetry analysis in Almendral & Sanjuán (2003).

Using transformation (4.56), the first integral (4.54) can be rewritten in the form

$$\hat{I} = w'^2 - \frac{\hat{\beta}}{3} w^3, \tag{4.57}$$

which in turn leads to the solution by an integration. On the other hand, the transformation changes the equation of motion (4.44) to

$$w'' = \hat{\beta} w^2, \tag{4.58}$$

where $\hat{\beta} = 2\sqrt{2}\beta$, which upon integration gives (4.57). From equation (4.57), we obtain

$$w'^2 = 4w^3 - g_3, \tag{4.59}$$

where $z = 2\sqrt{(3/\hat{\beta})}\hat{z}$ and $g_3 = -(12 I_1/\hat{\beta})$. The solution of this differential equation can be represented in terms of Weierstrass function $\wp(\hat{z}; 0, g_3)$ (Gradshteyn & Ryzhik 1980; Almendral & Sanjuán 2003).

(c) *Type-III systems*

In the previous two categories, we met the situation in which we are able to construct a pair of solutions (S_1, S_2) for the equations (2.6), from which R_1 and R_2 have been deduced. However, there are situations where only one set of solutions (R_1, S_1) can be constructed and its corresponding first integral and the second pair of solutions (R_2, S_2) cannot be obtained by a simple rational form of ansatz. In this situation, one can utilize our procedure and deduce the general solution for the given problem. In the following, we illustrate this with a couple of examples.

(i) *Example 5: force-free Duffing–van der Pol oscillator*

One of the well-studied but still challenging equations in nonlinear dynamics is the Duffing–van der Pol oscillator equation. Its autonomous version (force-free) is

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{4.60}$$

where an over-dot denotes differentiation with respect to time and α, β and γ are arbitrary parameters. Equation (4.60) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A vast amount of literature exists on this equation; for details see, for example, Lakshmanan & Rajasekar (2003) and references therein. In this case we have

$$S_t+\dot{x}S_x-((\alpha+\beta x^2)\dot{x}-\gamma x+x^3)S_{\dot{x}}=(2\beta x\dot{x}-\gamma+3x^2)-(\alpha+\beta x^2)S+S^2, \quad (4.61)$$

$$R_t+\dot{x}R_x-((\alpha+\beta x^2)\dot{x}-\gamma x+x^3)R_{\dot{x}}=(\alpha+\beta x^2-S)R, \quad (4.62)$$

$$R_x=SR_{\dot{x}}+RS_{\dot{x}}. \quad (4.63)$$

To solve equations (4.61)–(4.63) we seek an ansatz for S and R of the form

$$S=\frac{a(t,x)+b(t,x)\dot{x}}{c(t,x)+d(t,x)\dot{x}},\quad R=A(t,x)+B(t,x)\dot{x}. \quad (4.64)$$

Upon solving equations (4.61)–(4.63) with the above ansatz, we find that a non-trivial solution exists only for the choice $\alpha=(4/\beta)$, $\gamma=-(3/\beta^2)$, and the corresponding forms of S and R reads

$$S=\frac{1}{\beta}+\beta x^2,\quad R=\mathrm{e}^{(3t/\beta)}. \quad (4.65)$$

For this set, one can construct an invariant through the expression (2.9), which turns out to be (Senthilvelan & Lakshmanan 1995)

$$\dot{x}+\frac{1}{\beta}x+\frac{\beta}{3}x^3=I\mathrm{e}^{-(3t/\beta)}. \quad (4.66)$$

To obtain a second pair of solutions for the equations (4.61)–(4.63), one may seek a more general rational form of S and R by including higher polynomials in $\dot{x}$. However, they all lead to only functionally dependent integrals. As it is not possible to seek the second pair of solutions by a simple ansatz, an alternative way, as indicated in §3*b*, has to be sought. We can deduce the required transformation coordinates from the first integral and transform the latter to an autonomous equation and integrate it.

Using our algorithm given in §3*b*, one can deduce the transformation coordinates from the first integral itself, which turns out to be (Chandrasekar *et al.* 2004)

$$w=-x\,\mathrm{e}^{(1/\beta)t},\qquad z=\mathrm{e}^{-(2/\beta)t}, \quad (4.67)$$

where w and z are new dependent and independent variables, respectively. Substituting (4.67) into (4.60) with the parametric restriction $\alpha=(4/\beta)$, $\gamma=-(3/\beta^2)$, we get

$$w''-\frac{\beta^2}{2}w^2w'=0, \quad (4.68)$$

where prime denotes differentiation with respect to z. Equation (4.68) can be integrated trivially to yield

$$w'-\frac{\beta^2}{6}w^3=I, \quad (4.69)$$

where I is the integration constant. Equivalently, the transformation (4.67) reduces (4.60) to this form. Solving (4.69), we obtain (Gradshteyn & Ryzhik 1980)

$$z - z_0 = \frac{a}{3I}\left[\frac{1}{2}\log\left(\frac{(w+a)^2}{w^2 - aw + a^2}\right) + \sqrt{3}\arctan\left(\frac{w\sqrt{3}}{2a - w}\right)\right], \tag{4.70}$$

where $a = \sqrt[3]{6I/\beta^2}$ and z_0 is the second integration constant. Rewriting w and z in terms of old variables gives the explicit solution for equation (4.60).

We have shown that the systems (4.44) and (4.60) are integrable for certain specific parametric restrictions only. One may also assume that the functions S and R involve higher degree rational functions in $\dot{x}$ and then repeat the analysis. However, such an analysis does not provide any new integrable choice. In fact, the present results coincide exactly with the results obtained through other methods, namely, Painlevé analysis, Lie symmetry analysis and direct methods (Senthilvelan & Lakshmanan 1995; Almendral & Sanjuán 2003; Lakshmanan & Rajasekar 2003).

5. Linearizable equations

In §4 we discussed the complete integrability of nonlinear dynamical systems by constructing a sufficient number of integrals of motion and obtaining the general solutions explicitly. Another way of solving nonlinear ODEs is to transform them to linear ODEs, in particular, to a free particle equation and explore their underlying solutions. Even though this is one of the classic problems in the theory of ODEs, recently, considerable progress has been made (Mahomed & Leach 1989*b*; Steeb 1993; Olver 1995; Harrison 2002). In this direction it has been shown that a necessary condition for a second-order ODE to be linearizable is that it should be of the form (Mahomed & Leach 1989*b*)

$$\ddot{q} = D(t,q) + C(t,q)\dot{q} + B(t,q)\dot{q}^2 + A(t,q)\dot{q}^3, \tag{5.1}$$

where the functions A, B, C and D are analytic. Sufficient condition for the above second-order equation to be linearizable is (Mahomed & Leach 1989*b*)

$$\left.\begin{aligned} &3A_{tt} + 3CA_t - 3DA_q + 3AC_t + C_{qq} - 6AD_q + BC_q - 2BB_t - 2B_{tq} = 0,\\ &B_{tt} + 6DA_t - 3DB_q + 3AD_t - 2C_{tq} - 3BD_q + 3D_{qq} + 2CC_q - CB_t = 0,\end{aligned}\right\} \tag{5.2}$$

where the suffices refer to partial derivatives.

For a given second-order nonlinear ODE, one can easily check whether it can be linearizable or not by using the above necessary and sufficient conditions. However, the non-trivial problem is how to deduce systematically the linearizing transformations if the given equation is linearizable. As far as our knowledge goes, Lie symmetries are often used to extract the linearizing transformations (Mahomed & Leach 1985). As we pointed out in §3, the linearizing transformations can also be deduced from the first integral itself, whenever the system is linearizable, in a simple and straightforward way, and we stress that our procedure is new to the literature. In fact, we use the same procedure discussed in §3*c* and deduce the linearizing transformations. The only difference is, that in the case of linearizing transformations, the function F_2 turns out to be zero in equation (3.2) and as a consequence, the latter becomes $(\mathrm{d}w/\mathrm{d}z)=I$ and the transformation coordinates become the linearizing transformations. We illustrate the theory with certain new examples in the following.

(*a*) *Example 1: general relativity*

To illustrate the underlying ideas let us begin with a simple and physically interesting example, namely, the general relativity equation which we discussed as example 1 in §4. We derived the solution (4.17) using the PS method. In this section we linearize the system and derive its solution. Rewriting the first integral (4.9) in the form (3.1),

$$I = -\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x^2}\right), \tag{5.3}$$

and identifying (5.3) with (3.2), we get

$$G_1 = \frac{1}{x^2}, \quad G_2 = -2t, \quad F_2 = 0. \tag{5.4}$$

With the above choices, equation (3.3) furnishes the transformed variables

$$w = \frac{1}{x^2}, \quad z = -t^2. \tag{5.5}$$

Substituting (5.5) into (4.1), the latter becomes the free particle equation, namely, $(\mathrm{d}^2w/\mathrm{d}z^2)=0$, whose general solution is $w=I_1z+I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of x and t one gets exactly (4.17), which has been derived in a different way.

(*b*) *Example 2: modified Emden-type equations*

Recently, several papers have been devoted to exploring the invariance and integrability properties of the modified Emden-type equations (Mahomed & Leach 1985; Duarte *et al.* 1987),

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 = 0. \tag{5.6}$$

In fact, it is one of the rare second-order nonlinear ODEs which admit eight Lie point symmetries and, as a consequence, is a linearizable one. Recently, Pandey *et al.* (submitted) have obtained the explicit forms of the Lie point symmetries associated with the more general equation

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2 = 0, \tag{5.7}$$

where k, λ_1 and λ_2 are arbitrary parameters. They found that not only the Emden equation (5.6), but also its general form, that is, equation (5.7), admits eight Lie point symmetries. The authors have also reported the explicit forms of the symmetry generators. However, due to the complicated forms of the symmetry generators it is difficult to derive the first integrals and linearizing transformations from the symmetries straightforwardly (although in principle this is always possible). Nevertheless, we discussed the integrability of the case $\lambda_2=0$, $\lambda_1\neq 0$ of equation (5.7) as example 3 in §4 and deduced its general solution. In this section, we transform the equation into a free particle equation and deduce the general solution in an independent manner. We divide our analysis into two cases, namely, (i) $\lambda_1\neq 0$, $\lambda_2=0$ and (ii) $\lambda_1\neq 0$, $\lambda_2\neq 0$, and construct linearizing transformations and general solutions for both cases. As the procedure is the same as given in the previous examples we give only the results.

Case (i) $\lambda_2=0$, $\lambda_1\neq 0$: modified Emden-type equation with linear term

Restricting $\lambda_2=0$ in (5.7), we have

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x=0. \tag{5.8}$$

Since the first integral is already derived, *vide* equation (4.8), we utilize it here to deduce the linearizing transformations. Rewriting the first integral (4.38) in the form

$$I_1=-\frac{\mathrm{e}^{-\sqrt{-\lambda_1}t}kx^2}{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t}\right)\right] \tag{5.9}$$

and identifying (5.9) with (3.2), we get

$$G_1=\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t},\quad G_2=-\frac{3\dot{x}+kx^2-3\sqrt{-\lambda_1}x}{kx^2}\mathrm{e}^{\sqrt{-\lambda_1}t}. \tag{5.10}$$

With the above functions (3.3) furnishes

$$w=\left(\frac{3}{kx}+\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{-\sqrt{-\lambda_1}t},\quad z=\left(\frac{3}{kx}-\frac{1}{\sqrt{-\lambda_1}}\right)\mathrm{e}^{\sqrt{-\lambda_1}t}, \tag{5.11}$$

which is nothing but the linearizing transformation. Note that in this case, while rewriting the first integral I (equation (4.38)) in the form (3.1), the function F_2 disappears, and as a consequence we arrive at (*vide* equation (3.4))

$$\frac{\mathrm{d}w}{\mathrm{d}z}=I, \tag{5.12}$$

which, in turn, gives the free particle equation by differentiation or leads to the solution (4.43) by an integration. On the other hand, vanishing of the function F_2 in this analysis is precisely the condition for the system to be transformed into the free particle equation.

Case (ii) $\lambda_1\neq 0$, $\lambda_2\neq 0$: modified Emden-type equation with linear term and constant external forcing

Finally, we consider the general case, that is

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda_1 x+\lambda_2=0. \tag{5.13}$$

To explore the first integrals associated with the system (5.13), let us again seek the PS algorithm. The determining equations for the functions R and S move to be

$$S_t + \dot{x}S_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)S_{\dot{x}} = k\dot{x} + \frac{k^2}{3}x^2 + \lambda_1 - Skx + S^2, \quad (5.14)$$

$$R_t + \dot{x}R_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)R_{\dot{x}} = (kx - S)R, \quad (5.15)$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (5.16)$$

As before, let us seek an ansatz for S to solve the equation (5.14), namely,

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}. \quad (5.17)$$

Substituting (5.17) into (5.14) and equating the coefficients of different powers of $\dot{x}$ to zero and solving the resultant equations, we arrive at

$$S_1 = \frac{kx + 3\alpha}{3} - \frac{k\dot{x}}{kx + 3\alpha}, \quad S_2 = \frac{kx + 3\beta}{3} - \frac{k\dot{x}}{kx + 3\beta}, \quad (5.18)$$

where $\alpha^3 + \alpha\lambda_1 - (k\lambda_2/3) = 0$ and $\beta = \left(-\alpha \pm \sqrt{-3\alpha^2 - 4\lambda_1}\right)/2$. Putting the forms of S_1 into (5.15) we get

$$R_t + \dot{x}R_x - \left(kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2\right)R_{\dot{x}} = \left(\frac{k\dot{x}}{kx + 3\alpha} - \frac{kx + 3\alpha}{3} + kx\right)R. \quad (5.19)$$

Again, to solve this equation we make an ansatz

$$R = \frac{A(t,x) + B(t,x)\dot{x}}{C(t,x) + D(t,x)\dot{x} + E(t,x)\dot{x}^2}. \quad (5.20)$$

Substituting (5.20) into (5.19) and solving it we obtain the following form of R:

$$R_1 = \frac{C_0(kx + 3\alpha)\mathrm{e}^{\mp\hat{\alpha}t}}{\left(3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2\right)^2}, \quad (5.21)$$

where C_0 is constant and $\hat{\alpha} = \sqrt{-3\alpha^2 - 4\lambda_1}$. We find that the solution (S_1, R_1) satisfies (5.16). Equations (5.18) and (5.21) fix the first integral of the form

$$I_1 = \mathrm{e}^{\mp\hat{\alpha}t}\left(\frac{3k\dot{x} - 3\frac{(3\alpha \mp \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}\right), \quad (5.22)$$

where $C_0 = 9k\hat{\alpha}$. Rewriting the first integral (5.22) in the form (3.1),

$$I_1 = -\frac{\mathrm{e}^{(-3\alpha \mp \hat{\alpha}/2)t}(k_1 x + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2} \times\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \pm \hat{\alpha}}{2(3\alpha^2 + \lambda_1)}\right)\mathrm{e}^{(3\alpha \mp \hat{\alpha}/2)t}\right)\right] \quad (5.23)$$

and identifying (5.23) with (3.2), we get

$$\left.\begin{aligned} G_1 &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\pm\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\mp\hat{\alpha}/2)t},\\ G_2 &= -\frac{3k\dot{x}-3\dfrac{(3\alpha\pm\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2}{(kx+3\alpha)^2}\mathrm{e}^{(3\alpha\pm\hat{\alpha}/2)t}, \end{aligned}\right\} \tag{5.24}$$

so that (3.3) gives

$$\left.\begin{aligned} w &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\pm\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\mp\hat{\alpha}/2)t},\\ z &= \left(\frac{-3}{kx+3\alpha}+\frac{3\alpha\mp\hat{\alpha}}{2(3\alpha^2+\lambda_1)}\right)\mathrm{e}^{(3\alpha\pm\hat{\alpha}/2)t}, \end{aligned}\right\} \tag{5.25}$$

which is nothing but the linearizing transformation. Substituting (5.25) into (5.13) we get the free particle equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=0, \tag{5.26}$$

whose general can be written as $w=I_1z+I_2$. Rewriting w and z in terms of the original variable x and t one obtains

$$x=-\frac{3\alpha}{k}+\frac{6}{k}\left(\frac{(3\alpha^2+\lambda_1)(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})}{3\alpha(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})-2(3\alpha^2+\lambda_1)I_2\mathrm{e}^{(-3\alpha\pm\hat{\alpha}/2)t}\pm\hat{\alpha}(1+I_1\mathrm{e}^{\pm\hat{\alpha}t})}\right). \tag{5.27}$$

On the other hand, the general solution can also be derived by extending the PS method itself. To do so, one has to consider the functionS_2. Thus, substituting the expression S_2 into (5.15) and solving it in the same way as outlined in the previous paragraphs, we obtain the following form for R, that is,

$$R_2=\frac{C_0(kx+3\beta)\mathrm{e}^{3(\alpha\mp\hat{\alpha})t/2}}{\left(3k\dot{x}-3\frac{(3\alpha\pm\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)^2}. \tag{5.28}$$

However, this set, (S_2,R_2), does not satisfy the extra constraint (5.16), and to recover the full form of the integrating factor we assume that

$$\hat{R}_2=F(I_1)R_2. \tag{5.29}$$

Substituting (5.29) into equation (5.16) we obtain an equation for F, that is, $I_1F'+2F=0$, where prime denotes differentiation with respect to I_1. Upon integrating this equation we obtain $F=1/I_1^2$, which fixes the form of $\hat{R}$ as

$$\hat{R}_2=\frac{C_0(kx+3\beta)\mathrm{e}^{3\alpha\pm\hat{\alpha}t/2}}{\left(3k\dot{x}-3\frac{(3\alpha\mp\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)^2}. \tag{5.30}$$

Table 1. *Integral factors, integrals of motion, linearizing transformations and the general solution of equation (5.33)*

null forms and integrating factors

$$S_1=\frac{k_1x+3\alpha}{3}-\frac{k_1\dot{x}}{k_1x+3\alpha},\quad R_1=\frac{C_0(k_1x+3\alpha)\mathrm{e}^{\mp\hat{\alpha}t}}{\left(3k_1\dot{x}-\frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2\right)^2},$$

$$S_2=\frac{k_1x+3\beta}{3}-\frac{k_1\dot{x}}{k_1x+3\beta},\quad R_2=\frac{C_0(k_1x+3\beta)\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2}}{\left(3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2\right)^2},$$

$$\alpha^3-k_2\alpha^2+\alpha\lambda_1-\frac{k_1\lambda_2}{3}=0,\quad \hat{\alpha}=\sqrt{-3\alpha^2+2\alpha^2k_2+k_2^2-4\lambda_1},\quad \hat{\beta}=3\alpha-k_2,$$

$$\beta=\frac{-\alpha+k_2\pm\hat{\alpha}}{2}$$

first integrals

$$I_1=\mathrm{e}^{\mp\hat{\alpha}t}\left(\frac{3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}{3k_1\dot{x}-\frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}\right),\quad C_0=9k_1\hat{\alpha},$$

$$I_2=\frac{-2\hat{\alpha}\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2}}{\hat{\beta}\pm\hat{\alpha}}\left(\frac{3k_1\dot{x}-3k_1x(\alpha-k_2)+k_1^2x^2+9\alpha^2-9\alpha k_2+9\lambda_1}{3k_1\dot{x}-\frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x+9\alpha)+(k_1x+3\alpha)^2}\right)$$

linearizing transformations

$$w=\left(\frac{-3}{k_1x+3\alpha}+\frac{\hat{\beta}\pm\hat{\alpha}}{2(3\alpha^2-2\alpha k_2+\lambda_1)}\right)\mathrm{e}^{\hat{\beta}\pm\hat{\alpha}t/2},\quad z=\left(\frac{-3}{k_1x+3\alpha}+\frac{\hat{\beta}\mp\hat{\alpha}}{2(3\alpha^2-2\alpha k_2+\lambda_1)}\right)\mathrm{e}^{\hat{\beta}\mp\hat{\alpha}t/2}$$

solution

$$x=-\frac{3\alpha}{k_1}+\frac{1}{k_1}\left(\frac{6(3\alpha^2-2\alpha k_2+\lambda_1)(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})}{\hat{\beta}(1-I_1\mathrm{e}^{\pm\hat{\alpha}t})\pm(\hat{\beta}\pm\hat{\alpha})I_1I_2\mathrm{e}^{-\hat{\beta}\pm\hat{\alpha}t/2}\pm\hat{\alpha}(1+I_1\mathrm{e}^{\pm\hat{\alpha}t})}\right)$$

Now, one can easily check that this set S_2 and $\hat{R}_2$ is a compatible solution for the equations (5.14)–(5.16). Substituting S_2 and $\hat{R}_2$ into (2.9), we can obtain an explicit form for the second integral I_2, that is,

$$I_2=-\left(\frac{2\hat{\alpha}(3k\dot{x}-3\alpha kx+k^2x^2+9\alpha^2+9\lambda_1)\mathrm{e}^{3\alpha\pm\hat{\alpha}t/2}}{(3\alpha\pm\hat{\alpha})\left(3k\dot{x}-3\frac{(3\alpha\mp\hat{\alpha})}{2}(kx+3\alpha)+(kx+3\alpha)^2\right)}\right).\tag{5.31}$$

Rewriting equation (5.22) for $\dot{x}$ and substituting it into (5.31) we get the same expression (5.27) as the general solution.

(*c*) *Example 3: generalized modified Emden-type equation*

Recently, Pandey *et al.* (submitted) have considered the following Liénard equation:

$$\ddot{x}+f(x)\dot{x}+g(x)=0,\tag{5.32}$$

where f and g are arbitrary functions of their arguments, and classified systematically, all polynomial forms of f and g which admit eight Lie point symmetry generators with their explicit forms. They found that the most general nonlinear ODE which is linear in $\dot{x}$ whose coefficients are functions of the dependent variable alone should be of the form

$$\ddot{x}+(k_1x+k_2)\dot{x}+\frac{k_1^2}{9}x^3+\frac{k_1k_2}{3}x^2+\lambda_1x+\lambda_2=0, \tag{5.33}$$

where k_i and λ_i, $i=1,2$, are arbitrary parameters, which is consistent with the criteria (5.1) and (5.2) given by Mahomed & Leach (1989*b*). Interestingly, equation (5.33) and all its sub-cases possess $sl(3,R)$ symmetry algebra. For example, we discussed the integrability and linearization of equation (5.33) with $k_2=0$ in the previous example. As the linearizing transformations and the general solution of equation (5.33) are yet to be reported, we include this equation as an example in the present work. As in the previous case we divide our analysis into three cases.

(i) $\lambda_1=0$, $\lambda_2=0$: modified Emden-type equation with quadratic and cubic nonlinearity

$$\ddot{x}+(k_1x+k_2)\dot{x}+\frac{k_1^2}{9}x^3+\frac{k_1k_2}{3}x^2=0. \tag{5.34}$$

(ii) $\lambda_1\neq 0$, $\lambda_2=0$: modified Emden-type equation with quadratic and linear terms

$$\ddot{x}+(k_1x+k_2)\dot{x}+\frac{k_1^2}{9}x^3+\frac{k_1k_2}{3}x^2+\lambda_1x=0. \tag{5.35}$$

(iii) $\lambda_1\neq 0$, $\lambda_2\neq 0$: the full generalized modified Emden-type equation (5.33).

We have derived the integrating factors, integrals of motion, linearizing transformations and the general solutions for all the cases. As the calculations are similar to the ones discussed in the previous case, we present the results in tabular form (table 1), where the results for the most general case (5.33) have been given, from which the results for the limiting cases (5.34) and (5.34) can be deduced.

6. Conclusion

In this paper we have discussed the method of finding general solutions associated with second-order nonlinear ODEs through a modified PS method. The method can be considered as a direct one, complementing the well-known method of Lie symmetries. In particular, we have extended the theory of Duarte *et al.* (2001), such that new integrating factors and their associated integrals of motion can be recovered. These integrals of motion can be utilized to construct the general solution. In the situation where the second integral of motion cannot be recovered, we introduced another approach to derive the second integration constant. Interestingly, we showed that, in this case, it can be derived from the first integral itself, in a simple and elegant way. Apart from the above, we introduced a technique which can be utilized to derive linearizing transformation from the first integral. We illustrated the theory with several new examples and explored their underlying solutions.

In this paper we concentrated our studies only on single second-order ODEs. In principle, the method can also be extended to third-order ODEs and systems of second-order ODEs. The results will be published elsewhere.

The work of V.K.C. is supported by CSIR in the form of a Junior Research Fellowship. The work of M.S. and M.L. forms part of a Department of Science and Technology, Government of India, sponsored research project.

References

Ablowitz, M. J. & Clarkson, P. A. 1992 *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge: Cambridge University Press.

Almendral, J. A. & Sanjuán, M. A. F. 2003 Integrability and symmetries for the Helmholtz oscillator with friction. *J. Phys. A* **36**, 695–710.

Arnold, V. I. 1978 *Mathematical methods of classical mechanics*. New York: Springer.

Bluman, G. W. & Anco, S. C. 2002 *Symmetries and integration methods for differential equations*. New York: Springer.

Buchdahl, H. A. 1964 A relativistic fluid spheres resembling the Emden polytrope of index 5. *Ap. J.* **140**, 1512–1516.

Chandrasekar, V. K., Senthilvelan, M. & Lakshmanan, M. 2004 New aspects of integrability of force-free Duffing–van der Pol oscillator and related nonlinear systems. *J. Phys. A* **37**, 4527–4534.

Conte, R. 1999 *The Painlevé property: one century later*. New York: Springer.

Duarte, L. G. S., Duarte, S. E. S. & Moreira, I. C. 1987 One-dimensional equations with the maximum number of symmetry generators. *J. Phys. A* **20**, L701–L704.

Duarte, L. G. S., Duarte, S. E. S., da Mota, A. C. P. & Skea, J. E. F. 2001 Solving the second-order ordinary differential equations by extending the Prelle–Singer method. *J. Phys. A* **34**, 3015–3024.

Gradshteyn, I. S. & Ryzhik, I. M. 1980 *Table of integrals, series and products*. London: Academic Press.

Guckenheimer, J. & Holmes, P. 1983 *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*. New York: Springer.

Harrison, B. K. 2002, An old problem newly treated with differential forms when and how can the equation $y''=f(x,y,y')$ be linearized? *Proc. Inst. Math. Natl Acad. Sci. Ukraine* **43**, 27–35.

Hietarinta, J. 1987 Direct methods for the search of the second invariant. *Phys. Rep.* **147**, 87–154.

Ince, E. L. 1956 *Ordinary differential equations*. New York: Dover.

Jones, S. E., Butson, G. J. & Lemmers, P. L. 1993 On Abel's identity. Part II. The integration of a non-linear, non-conservative system. *J. Sound Vib.* **160**, 251–257.

José, J. V. & Saletan, E. J. 2002 *Classical dynamics: a contemporary approach*. Cambridge: Cambridge University Press.

Lakshmanan, M. & Rajasekar, S. 2003 *Nonlinear dynamics: integrability, chaos and patterns*. New York: Springer.

Litchenberg, A. J. & Lieberman, M. A. 1983 *Regular and stochastic motion*. New York: Springer.

Mahomed, F. M. & Leach, P. G. L. 1985 The linear symmetries of a nonlinear differential equation. *Quaestiones Math.* **8**, 241–274.

Mahomed, F. M. & Leach, P. G. L. 1989*a* The Lie algebra $sl(3,R)$ and linearization. *Quaestiones Math.* **12**, 121–139.

Mahomed, F. M. & Leach, P. G. L. 1989*b* Lie algebras associated with scalar second-order ordinary differential equation. *J. Math. Phys.* **30**, 2770–2777.

Olver, P. J. 1995 *Equivalence, invariants, and symmetry.* Cambridge: Cambridge University Press.

Pandey, S.N., Senthilvelan, M., Lakshmanan, M. Submitted. Classification of Lie point symmetries for nonlinear dissipative systems.

Percival, I. & Richards, D. 1982 *Introduction to dynamics.* Cambridge: Cambridge University Press.

Prelle, M. & Singer, M. 1983 Elementary first integrals of differential equations. *Trans. Am. Math. Soc.* **279**, 215–229.

Senthilvelan, M. & Lakshmanan, M. 1995 Lie symmetries and infinite dimensional Lie algebras of certain nonlinear dissipative systems. *J. Phys. A* **28**, 1929–1942.

Steeb, W. H. 1993 *Invertible point transformations and nonlinear differential equations.* London: World Scientific.

Wiggins, S. 2003 *Introduction to applied nonlinear dynamical systems and chaos.* New York: Springer.

NOTICE OF CORRECTION

The sentence preceeding equation (4.48) is now present in its correct form.

Equation (4.49) is now present in its correct form.

A detailed erratum will appear at the end of volume 464.

16 September 2008

11 A Simple and Unified Approach to Identify Integrable Nonlinear Oscillators and Systems. By V. K. Chandrasekar, S. N. Pandey, M. Senthilvelan, and M. Lakshmanan (2018)

A Simple and Unified Approach to Identify Integrable Nonlinear Oscillators and Systems

V. K. Chandrasekar[†], S. N. Pandey[††] M. Senthilvelan[†] and M. Lakshmanan[†]

[†] *Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirappalli - 620 024, India and*

[††] *Department of Physics, Motilal Nehru National Institute of Technology, Allahabad-211 004, India*

(Dated: September 7, 2018)

Abstract

In this paper, we consider a generalized second order nonlinear ordinary differential equation of the form $\ddot{x}+(k_1x^q+k_2)\dot{x}+k_3x^{2q+1}+k_4x^{q+1}+\lambda_1x=0$, where k_i's, $i=1,2,3,4$, λ_1 and q are arbitrary parameters, which includes several physically important nonlinear oscillators such as the simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing and Duffing-van der Pol oscillators, modified Emden type equation and its hierarchy, generalized Duffing-van der Pol oscillator equation hierarchy and so on and investigate the integrability properties of this rather general equation. We identify several new integrable cases for arbitrary value of the exponent q, $q \in R$. The $q=1$ and $q=2$ cases are analyzed in detail and the results are generalized to arbitrary q. Our results show that many classical integrable nonlinear oscillators can be derived as sub-cases of our results and significantly enlarge the list of integrable equations that exist in the contemporary literature. To explore the above underlying results we use the recently introduced generalized extended Prelle-Singer procedure applicable to second order ODEs. As an added advantage of the method we not only identify integrable regimes but also construct integrating factors, integrals of motion and general solutions for the integrable cases, wherever possible, and bring out the mathematical structures associated with each of the integrable cases.

I. INTRODUCTION

A. Overview of the problem

In a recent paper[1] we have shown that the force-free Duffing-van der Pol (DVP) oscillator,

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{1}$$

is integrable for the parametric restriction $\alpha = \frac{4}{\beta}$ and $\gamma = -\frac{3}{\beta^2}$. In Eq. (1) over dot denotes differentiation with respect to t and α, β and γ are arbitrary parameters. Under the transformation

$$w = -xe^{\frac{1}{\beta}t}, \qquad z = e^{-\frac{2}{\beta}t}, \tag{2}$$

Eq. (1) with restriction $\alpha = \frac{4}{\beta}$ and $\gamma = -\frac{3}{\beta^2}$ was shown to be transformable to the form

$$w'' - \frac{\beta^2}{2}w^2w' = 0, \tag{3}$$

which can then be integrated[1].

In a parallel direction, while performing the invariance analysis of a similar kind of problem, we find that not only the Eq. (1) but also its generalized version,

$$\ddot{x} + (\frac{4}{\beta} + \beta x^2)\dot{x} + \frac{3}{\beta^2}x + x^3 + \delta x^5 = 0, \quad \delta = \text{arbitrary parameter}, \tag{4}$$

is invariant under the same set of Lie point symmetries[2]. As a consequence one can use the same transformation (2) to integrate the Eq. (4). The transformation (2) modifies Eq. (4) to the form

$$w'' - \frac{\beta^2}{2}w^2w' + \delta w^5 = 0 \tag{5}$$

which is not so simple to integrate straightforwardly. However, we observe that this equation coincides with the second equation in the so called modified Emden equation (MEE) hierarchy, investigated by Feix et al.[3],

$$\ddot{x} + x^l\dot{x} + gx^{2l+1} = 0, \quad l = 1, 2, \ldots, n, \tag{6}$$

where g is an arbitrary parameter.

In fact Feix et al.[3] have shown that through a direct transformation to a third order equation the above Eq. (6) can be integrated to obtain the general solution for the specific choice of the parameter g, namely, for $g = \frac{1}{(l+2)^2}$. For this choice of g, the general solution of (6) can be written as

$$x(t) = \left(\frac{(2+3l+l^2)(t+I_1)^l}{l(t+I_1)^{l+1} + (2+3l+l^2)I_2} \right)^{\frac{1}{l}}, \quad I_1, I_2: \text{ arbitrary constants.} \tag{7}$$

Consequently Eq. (4) can be integrated under the specific parametric choice $\delta = \frac{1}{16}$, and it belongs to the $l = 2$ case of the MEE hierarchy (6) with $g = \frac{1}{16}$. Now the question arises as to whether there exist other new integrable second order nonlinear differential equations which are linear in $\dot{x}$ and containing fifth and other powers of nonlinearity. As far as our knowledge goes only few equations in this class have been shown to be integrable. For example, Smith[4] had investigated a class of nonlinear equations coming under the category

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{8}$$

with $f(x) = (n+2)bx^n - 2a$ and $g(x) = x(c + (bx^n - a)^2)$ where a, b, c and n are arbitrary parameters. He had shown that the Eq. (8) with this specific forms of f and g admits explicit oscillatory solutions. However, one can also expect that there should be a number of integrable equations which also admit solutions which are both oscillatory and non-oscillatory types in the class

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0, \qquad q \in R, \tag{9}$$

where k_i's, $i = 1, 2, 3, 4$ and λ_1 are arbitrary parameters. When $q = 1$, Eq. (9) becomes the generalized MEE

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0, \tag{10}$$

and for $q = 2$ it becomes

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0. \tag{11}$$

We note that Eq. (4) is a special case of (11).

Needless to say Eq. (9) is a unified model for several ground breaking physical systems which includes simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing oscillator, MEE hierarchy, generalized DVP hierarchy and so on.

As noted earlier there exists no rigorous mathematical analysis in the literature for the second order nonlinear differential equations which contain fifth or higher degree nonlinearity in x and linear in $\dot{x}$ and the results are very scarce on integrability or exact solutions. Our motivation to analyze this problem is not only to explore new integrable cases/systems of Eq. (9) but also to synthesize all earlier results under one approach.

Having described the problem and motivation now we can start analyzing the integrability properties of Eq. (9). To identify the integrable regimes we employ the recently introduced extended Prelle-Singer procedure applicable to second order ODEs[5−11]. Through this method we not only identify integrable regimes but also construct integrating factors, integrals of motion and general solution for the integrable cases, wherever possible.

B. Results

We unearth several new integrable equations for any real value of the exponent q in Eq. (9). In the following we summarize the results for the case $q =$ *arbitrary* only and discuss in detail the $q = 1$, $q = 2$ and $q =$ *arbitrary* cases separately in the following sections.

For the choice $q =$ *arbitrary* we find that the following equations are completely integrable (after suitable reparametrizations), all of which appear to be new to the literature:

$$\ddot{x} + (k_1x^q + (q+2)k_2)\dot{x} + k_1k_2x^{q+1} + (q+1)k_2^2x = 0 \tag{12}$$

$$\ddot{x} + ((q+2)k_1x^q + k_2)\dot{x} + k_1^2x^{2q+1} + k_1k_2x^{q+1} + \lambda_1x = 0 \tag{13}$$

$$\ddot{x} + (q+4)k_2\dot{x} + k_4x^{q+1} + 2(q+2)k_2^2x = 0 \tag{14}$$

$$\ddot{x} + ((q+1)k_1x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}[(q+1)k_1^2x^{2q+1} + (q+2)k_1k_2x^{q+1} + k_2^2x] = 0, \quad r \neq 0 \tag{15}$$

$$\ddot{x} + ((q+1)k_1x^q + (q+2)k_2)\dot{x} + (q+1)[\frac{(r-1)}{r^2}k_1^2x^{2q+1} + k_1k_2x^{q+1} + k_2^2x] = 0, \quad r \neq 0, \tag{16}$$

where k_1, k_2, k_4, λ_1 and r are arbitrary parameters. We stress that the above results are true for any arbitrary values of q. We discuss the special cases, namely, $q = 1$ and $q = 2$ separately in detail in sections 3 and 4 in order to put the results of q arbitrary case in proper perspective.

We show that the Eq. (12) is nothing but a generalization of the Duffing-van der Pol oscillator Eq. (1). In a recent work[1,9] three of the present authors have established the integrability of Eq. (12) with $q = 2$. However, in this work we show that the generalized Eq. (12) itself is integrable. Eq. (13) is nothing but the generalized MEE among which the hierarchy of Eq. (6), studied by Feix et al.[3], can be identified as a sub-case. In fact the general solution constructed by Feix et al., Eq. (7), can be derived straightforwardly as a sub-case. Eq. (13) also contains the family of equations studied by Smith[4]. In particular the latter author have derived general solution for the case $k_2^2 < 4\lambda_1$, which turns out to be an oscillatory one. However, in this work, we show that even for arbitrary values of k_2 and λ_1 one can construct the general solution. Interestingly, the system (14) generalizes several physically important nonlinear oscillators. For example, in the case $q = 1$ and 2, Eq. (14) provides us the force-free Helmholtz and Duffing oscillators, respectively, whose nonlinear dynamics is well documented in the literature[12–16]. Here, we present certain integrable generalizations of these nonlinear oscillators. Eq. (15) admits a conservative Hamiltonian for all values of the parameters r, k_1 and k_2 and any integer value of q. We also provide the explicit form of the Hamiltonian for all values of q. As a result we conclude that it is a Liouville integrable system. As far as Eq. (16) is concerned we construct a time dependent integral of motion and transform the latter to time independent Hamiltonian one and thereby ensuring its Liouville integrability.

The plan of the paper is us follows. In the following section we briefly describe the extended Prelle-Singer procedure applicable to second order ODEs. In Sec. III, we consider the case $q = 1$ in (9) and identify the integrable parametric choices of this equation through the extended PS procedure. To do so first we identify the integrable cases where the system admits time independent integrals and construct explicit conservative Hamiltonians for the respective parametric choices. We then identify the cases which admit explicit time dependent integrals of motion. To establish the complete integrability of these cases we use our own procedure and transform the time dependent integrals of motion into time independent integrals of motion and integrate the latter and derive the general solution. In Sec. IV, we repeat the procedure for the case $q = 2$ in Eq. (9) and identify the integrable systems. In Sec. V, we consider the case $q =$ arbitrary in (9) and unearth several new integrable equations and their associated mathematical structures. Finally, we present our conclusions in Sec. VI.

II. GENERALIZED EXTENDED PRELLE-SINGER (PS) PROCEDURE

In this section we briefly recall the generalized extended or modified PS procedure before applying it to the specific problem in hand. Sometime ago, Prelle and Singer[5] have proposed a procedure for solving first order ODEs that admit solutions in terms of elementary functions if such solutions exist. The attractiveness of the PS method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Very recently Duarte et al.[7,8] have modified the technique developed by Prelle and Singer[5,6] and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. For a class of systems these authors have deduced first integrals and in some cases for the first time through their procedure[7]. Recently the present authors have generalized the theory of Duarte et al.[7] in different directions and shown that for the second order ODEs one can isolate even two independent integrals of motion[9–11] and obtain general solutions explicitly without any integration. This theory has also been illustrated for a class of problems[1,9–11]. The authors have also generalized the theory successfully to higher order ODEs[10,17]. For example, in the case of third order ODEs the theory has been appropriately generalized to yield three independent integrals of motion unambiguously so that the general solution follows immediately from these integrals of motion[17].

We stress that the PS procedure has many advantages over other methods. To name a few, we cite: (1) For a given problem if the solution exists it has been conjectured that the PS method guarantees to provide first integrals. (2) The PS method not only gives the first integrals but also the underlying integrating factors, that is, multiplying the equation with these functions we can rewrite the equation as a perfect differentiable function which upon integration gives the first integrals directly. (3) The PS method can be used to solve nonlinear as well as linear second order ODEs. (4) As the PS method is based on the equations of motion rather than on Lagrangian or Hamiltonian description, the analysis is applicable to deal with both Hamiltonian and non-Hamiltonian systems.

A. PS method

Let us rewrite Eq. (9) in the form

$$\ddot{x} = -((k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x) \equiv \phi(x, \dot{x}). \tag{17}$$

Further, we assume that the ODE (17) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential becomes

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \tag{18}$$

where each subscript denotes partial differentiation with respect to that variable. Rewriting Eq. (17) in the form $\phi dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}\, dt - S(t, x, \dot{x})dx$ to the latter, we obtain that on the solutions the 1-form

$$\left(\phi + S\dot{x}\right)dt - Sdx - d\dot{x} = 0. \tag{19}$$

Hence, on the solutions, the 1-forms (18) and (19) must be proportional. Multiplying (19) by the factor $R(t, x, \dot{x})$ which acts as the integrating factors for (19), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{20}$$

Comparing Eq. (18) with (20) we have, on the solutions, the relations

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \tag{21}$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the Eqs. (21), provide us

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}} S + S^2, \tag{22}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -(\phi_{\dot{x}} + S)R, \tag{23}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{24}$$

Solving Eqs. (22)-(24) one can obtain expressions for S and R. It may be noted that any set of special solutions (S, R) is sufficient for our purpose. Once these forms are determined the integral of motion $I(t, x, \dot{x})$ can be deduced from the relation

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2)\right] d\dot{x}, \tag{25}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx.$$

Equation (25) can be derived straightforwardly by integrating the Eq. (21).

The crux of the problem lies in finding the explicit solutions satisfying all the three determining Eqs. (22)-(24), since once a particular solution is known the integral of motion can be readily constructed. The difficulties in constructing admissible set of solutions (S, R) satisfying all the three Eqs. (22)-(24) and possible ways of obtaining the solutions have been discussed in detail in Ref. 9.

III. APPLICATION OF PS PROCEDURE TO EQ. (10)

Let us first consider the case $q = 1$ in Eq. (9) or equivalently (10)

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0. \tag{10}$$

Eq. (10) itself includes several physically important models. For example, choosing $k_i = 0,\ i = 1, ...4$, we get the simple harmonic oscillator equation and the choice $k_1,\ k_2 = 0$ gives us the anharmonic oscillator equation. When $k_1,\ k_4 = 0$ Eq. (10) becomes the force-free Duffing oscillator equation[12]. The choice $k_2,\ k_4,\ \lambda_1 = 0$ provides us the MEE[18]. In the limit $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3}$, Eq. (10) becomes MEE with linear term which is another linearizable equation which we have studied extensively in Refs. 9 and 19. The restriction $k_1,\ k_3 = 0$ leads us to the force-free Helmhotz oscillator[12,13]. In the following we investigate whether the system (10) admits any other integrable case besides the above.

We solve Eq. (10) through the extended PS procedure in the following way. For a given second order equation, (10), the first integral I should be either a time independent or time dependent one. In the former case, it is a conservative system and we have $I_t = 0$ and in the later case we have $I_t \neq 0$. So let us first consider the case $I_t = 0$ and determine the null forms and the corresponding integrating factors and from these we construct the integrals of motion and then we do extend the analysis for the case $I_t \neq 0$.

A. The case $I_t = 0$

1. *Null forms*

In this case one can easily fix the null form S from the first equation in (21) as

$$S = \frac{-\phi}{\dot{x}} = -\frac{((k_1x + k_2)\dot{x} + k_3x^3 + k_4x^2 + \lambda_1 x)}{\dot{x}}. \tag{26}$$

2. *Integrating Factors*

Substituting this form of S, given in (26), into (23) we get

$$\begin{aligned} R_t + \dot{x}R_x - ((k_1x + k_2)\dot{x} + k_3x^3 + k_4x^2 + \lambda_1 x)R_{\dot{x}} \\ = \left((k_1x + k_2) + \frac{((k_1x+k_2)\dot{x}+k_3x^3+k_4x^2+\lambda_1 x)}{\dot{x}}\right)R. \end{aligned} \tag{27}$$

Equation (27) is a first order linear partial differential equation with variable coefficients. As we noted earlier any particular solution is sufficient to construct an integral of motion (along with the function S). To seek a particular solution for R one can make a suitable ansatz instead of looking for the general solution. We assume R to be of the form,

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x})^r}, \tag{28}$$

where A and B are functions of their arguments, and r is a constant which are all to be determined. We demand the above form of ansatz, (28), due to the following reason. To deduce the first integral I we assume a rational form for I, that is, $I = \frac{f(x,\dot{x})}{g(x,\dot{x})}$, where f and g are arbitrary functions of x and $\dot{x}$ and are independent of t. Since we already assumed that I is independent of t, we have, $I_x = \frac{f_x g - f g_x}{g^2}$ and $I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$. From (21) one can see that $R = I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$, $S = \frac{I_x}{I_{\dot{x}}} = \frac{f_x g - f g_x}{f_{\dot{x}} g - f g_{\dot{x}}}$ and $RS = I_x$, so that the denominator of the function S should be the numerator of the function R. Since the denominater of S is $\dot{x}$ (vide Eq. (26)) we fixed the numerator of R as $\dot{x}$. To seek a suitable function in the denominator initially one can consider an arbitrary form $R = \frac{\dot{x}}{h(x,\dot{x})}$. However, it is difficult to proceed with this choice of h. So let us assume that $h(x, \dot{x})$ is a function which is polynomial in $\dot{x}$. To begin with let us consider the case where h is linear in $\dot{x}$, that is, $h = A(x) + B(x)\dot{x}$. Since R is in rational form while taking differentiation or integration the form of the denominator remains same but the power of the denominator decreases or increases by a unit order from that of the initial one. So instead of considering h to be of the form $h = A(x) + B(x)\dot{x}$, one may consider a more general form $h = (A(x) + B(x)\dot{x})^r$, where r is a constant to be determined. Such a generalized form of h and so R leads to several new integrable cases as we see below.

Substituting (28) into (27) and solving the resultant equations, we arrive at the relation

$$r(\dot{x}(A_x + B_x\dot{x}) + \phi B) = (A + B\dot{x})\phi_{\dot{x}}. \tag{29}$$

Solving Eq. (29) with $\phi = -((k_1x+k_2)\dot{x}+k_3x^3+k_4x^2+\lambda_1x)$, we find nontrivial forms for the functions A and B for two choices, namely, (i) k_1, k_2 arbitrary and (ii) $k_1 =$ arbitrary, $k_2 = 0$ with restrictions on other parameters as given below. The respective forms of the functions and the restriction on the parameters are

(i) $\underline{k_1,\ k_2 : \text{arbitrary}}$

$$A(x) = \frac{(r-1)b_0}{r}(\frac{k_1}{2}x^2 + k_2x), \quad B(x) = b_0 = \text{constant}, \quad r = \text{constant},$$
$$k_3 = \frac{b_0(r-1)}{2r^2}k_1^2, \quad k_4 = \frac{3b_0(r-1)}{2r^2}k_1k_2, \quad \lambda_1 = \frac{b_0(r-1)k_2^2}{r^2}, \tag{30a}$$

(ii) $\underline{k_1 = \text{arbitrary},\ k_2 = 0}$

$$A(x) = \frac{(r-1)b_0}{2r}k_1x^2 + \frac{r\lambda_1}{k_1}, \quad B(x) = b_0,$$
$$k_3 = \frac{b_0(r-1)}{2r^2}k_1^2, \quad k_4 = 0, \quad \lambda_1 = \text{arbitrary parameter (here)}. \tag{30b}$$

We note that the case (ii) cannot be derived from case (i) by taking $k_2 = 0$. For example, choosing $k_2 = 0$ in (30a) we get not only $k_4 = 0$ but also $\lambda_1 = 0$ whereas in the case (ii) we have the freedom $\lambda_1 =$ arbitrary, so the cases (30a) and (30b) are to be treated as separate. Making use of the forms of A and B from Eqs. (30a) and (30b) into (28), the integrating factor, 'R', for the two cases can be obtained as

(i) $\underline{k_1,\ k_2 : \text{arbitrary}}$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{r}(\frac{k_1}{2}x^2 + k_2x) + \dot{x}\right]^r}, \quad r \neq 0 \tag{31a}$$

(ii) $\underline{k_1 = \text{arbitrary},\ k_2 = 0}$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{2r}k_1x^2 + \frac{r\lambda_1}{k_1} + \dot{x}\right]^r}, \quad r \neq 0. \tag{31b}$$

We note that b_0 is a common parameter in the above and it is absorbed in the definition of 'R', see Eqs. (23) and (24). While deriving the above forms of R (Eqs. (31a) and (31b)) we assumed that $r \neq 0$ and for the choice $r = 0$ we obtain consistent solution only if both the parameters k_1 and k_2 become zero. Of course, this sub-case can be treated as a trivial one since when $k_1, k_2 = 0$ the damping term in Eq. (10) vanishes and the system becomes an integrable anharmonic oscillator. In this trivial case we have the integrating factor of the form:

(iii) $\underline{k_1,\ k_2 = 0}$

$$R = \dot{x}, \quad r = 0. \tag{31c}$$

Finally one has to check the compatibility of forms S and R with the third Eq. (24). We indeed verified that the sets

$$(i) \quad S = -\frac{((k_1 x + k_2)\dot{x} + \frac{(r-1)}{r^2}(\frac{k_1^3}{2}x^2 + \frac{3k_1k_2}{2}x^2 + k_2^2 x))}{\dot{x}},$$

$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{2}x^2 + k_2 x) + \dot{x})^r}, \quad k_1, k_2 = \text{arbitrary}, \quad r \neq 0 \tag{32a}$$

$$(ii) \quad S = -\frac{(k_1 x\dot{x} + \frac{(r-1)}{2r^2}k_1^2 x^3 + \lambda_1 x)}{\dot{x}},$$

$$R = \frac{\dot{x}}{(\frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1} + \dot{x})^r}, \quad k_1 = \text{arbitrary},\ k_2 = 0, \quad r \neq 0 \tag{32b}$$

and

$$(iii) \quad S = -\frac{(k_3 x^3 + k_4 x^2 + \lambda_1 x)}{\dot{x}}, \quad R = \dot{x}, \quad k_1,\ k_2 = 0, \tag{32c}$$

satisfy the Eq. (24) individually. As a consequence all the three pairs form compatible sets of solution for the Eqs. (22)-(24).

3. Integrals of motion

Having determined the explicit forms of S and R one can proceed to construct integrals of motion using the expressions (25). The parametric restrictions (30a) and (30b) fix the equation of motion (10) to the following specific forms,

$$(i) \quad \ddot{x} + (k_1 x + k_2)\dot{x} + \frac{(r-1)}{2r^2}\left(k_1^2 x^3 + 3k_1k_2x^2 + 2k_2^2 x\right) = 0, \quad r \neq 0, \tag{33a}$$

$$(ii) \quad \ddot{x} + k_1 x\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \lambda_1 x = 0, \quad r \neq 0, \tag{33b}$$

$$(iii) \quad \ddot{x} + k_3x^3 + k_4x^2 + \lambda_1 x = 0, \quad r = 0. \tag{33c}$$

In the above $k_1,\ k_2,\ k_3,\ k_4,\ \lambda_1$ and r are arbitrary parameters.

We note that the transformation $x = y - \frac{k_2}{k_1}$ transforms equation (33a) to the form

$$\ddot{y} + k_1 y\dot{y} + \frac{(r-1)k_1^2}{2r^2}y^3 - \frac{(r-1)k_2^2}{2r^2}y = 0, \quad r \neq 0. \tag{34}$$

Eq. (34) is obtained from Eq. (33b) by fixing $\lambda_1 = -\frac{(r-1)k_2^2}{2r^2}$. So, hereafter, we consider Eq. (33a) as a special case of Eq. (33b) and so discuss only Eq. (33b) as the general one. It may be noted that Eq. (33b) includes several known integrable cases. For example, the choice $r = 3$ and $\lambda_1 = 0$ in Eq. (33b) yields the MEE[18]. On the other hand the choice $r = -1$ leads us to the equation $\ddot{x} + k_1x\dot{x} - k_1^2x^3 + \lambda_1 x = 0$ which can be solved in terms of Weierstrass elliptic function[20]. *The other choices of r lead to new integrable cases* as we see below.

Substituting the forms of S and R (vide Eqs. (32b) and (32c)) into the general form of the integral of motion (25) and evaluating the resultant integrals, we obtain the following time independent first integrals for the cases (33b) and (33c):

$$(iia) \quad I_1 = \left(\dot{x} + \frac{(r-1)}{2r}k_1x^2 + \frac{r\lambda_1}{k_1}\right)^{-r} \tag{35a}$$

$$\times\left[\dot{x}(\dot{x} + \frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1}) + \frac{(r-1)}{r^2}(\frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1})^2\right], \quad r \neq 0, 1, 2,$$

$$(iib) \quad I_1 = \frac{4k_1\dot{x}}{k_1^2x^2 + 4k_1\dot{x} + 8\lambda_1} - \log(k_1^2x^2 + 4k_1\dot{x} + 8\lambda_1), \quad r = 2, \tag{35b}$$

$$(iic) \quad I_1 = \dot{x} + \frac{k_1}{2}x^2 - \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1), \quad r = 1, \tag{35c}$$

$$(iii) \quad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{4}x^4 + \frac{k_4}{3}x^3 + \frac{\lambda_1}{2}x^2, \quad r = 0. \tag{35d}$$

Note that in Eq. (35a), r can take any real value, except $0, 1, 2$. In the above integrals I_1 given by Eqs. (35a) - (35c) correspond to the ODE (33b), while (35d) corresponds to the Eq. (33c).

Due to the fact that the integrals of motion (35) are time independent, one can look for a Hamiltonian description for the respective equations of motion. In fact, we obtain the explicit Hamiltonian forms for all the above cases.

4. Hamiltonian Description of (35)

Assuming the existence of a Hamiltonian

$$I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x}), \tag{36}$$

where $L(x,\dot{x})$ is the Lagrangian and p is the canonically conjugate momentum, we have

$$\begin{aligned} \frac{\partial I}{\partial \dot{x}} &= \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}, \\ \frac{\partial I}{\partial x} &= \frac{\partial H}{\partial x} = \frac{\partial p}{\partial x}\dot{x} - \frac{\partial L}{\partial x}. \end{aligned} \tag{37}$$

From (37) we identify

$$\begin{aligned} p &= \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x}, \\ L &= \int (p_x\dot{x} - I_x)dx + \int [p - \frac{d}{d\dot{x}}\int (p_x\dot{x} - I_x)dx]d\dot{x}. \end{aligned} \tag{38}$$

Plugging the expressions (36) into (38) one can evaluate the canonically conjugate momentum and the associated Lagrangian as well as the Hamiltonian. They read as follows:

(a) The canonical momenta :

$$(iia,b) \quad p = \frac{1}{r-1}\left(\dot{x} + \frac{(r-1)}{r}\frac{k_1}{2}x^2 + \frac{r\lambda_1}{k_1}\right)^{1-r}, \quad r \neq 0,1 \tag{39a}$$

$$(iic) \quad p = \log(k_1\dot{x} + \lambda_1), \quad r = 1 \tag{39b}$$

$$(iii) \quad p = \dot{x}, \quad r = 0. \tag{39c}$$

(Note in the above $r = 2$ is included in Eq. (39b) itself).

(b) The Lagrangian :

$$(iia) \quad L = \frac{1}{(2-r)(r-1)}\left(\dot{x} + \frac{(r-1)}{r}\frac{k_1x^2}{2} + \frac{r\lambda_1}{k_1}\right)^{2-r}, \quad r \neq 0,1,2 \tag{40a}$$

$$(iib) \quad L = \log(4k_1\dot{x} + 8\lambda_1 + k_1^2x^2), \quad r = 2 \tag{40b}$$

$$(iic) \quad L = \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1) + \dot{x}(\log(k_1\dot{x} + \lambda_1) - 1) - \frac{1}{2}k_1x^2, \quad r = 1 \tag{40c}$$

$$(iii) \quad L = \frac{\dot{x}^2}{2} - \frac{k_3}{4}x^4 - \frac{k_4}{3}x^3 - \frac{\lambda_1}{2}x^2, \quad r = 0. \tag{40d}$$

(c) <u>The Hamiltonian</u> :

$$(iia)\quad H=\Big[\frac{\Big((r-1)p\Big)^{\frac{r-2}{r-1}}}{(r-2)}-p(\frac{(r-1)}{2r}k_1x^2+\frac{r\lambda_1}{k_1})\Big],\quad r\neq 0,1,2 \quad (41a)$$

$$(iib)\quad H=\frac{2\lambda_1}{k_1}p+\frac{k_1}{4}x^2p+\log(\frac{4k_1}{p}),\quad r=2 \quad (41b)$$

$$(iic)\quad H=\frac{1}{k_1}(e^p-\lambda_1p+\frac{k_1^2}{2}x^2-\lambda_1),\quad r=1 \quad (41c)$$

$$(iii)\quad H=\frac{p^2}{2}+\frac{k_3}{4}x^4+\frac{k_4}{3}x^3+\frac{\lambda_1}{2}x^2,\quad r=0. \quad (41d)$$

One can check that the Hamilton's equations of motion are indeed equivalent to the appropriate equation (10).

Since Eqs. (33b) and (33c) admit time independent Hamiltonians they can be classified as Liouville integrable systems. *The important fact we want to stress here is that for arbitrary values of* r*, including fractional values, the equation (33b) is integrable.*

5. *Canonical transformation for the Hamiltonian Eqs. (41)*

Interestingly, we also identifed suitable canonical transformation to standard particle in a potential description for the Hamiltonians (41). Now introducing the canonical transformations

$$x=\frac{2rP}{k_1U},\qquad p=-\frac{k_1U^2}{4r},\quad r\neq 0,1, \quad (42)$$

$$x=\frac{P}{k_1},\qquad p=-k_1U,\quad r=1 \quad (43)$$

the Hamiltonian H in Eq. (41) can be recast in the standard form (after rescaling)

$$H=\begin{cases}\frac{1}{2}P^2+\frac{(1-r)}{(r-2)}\left(\frac{(r-1)k_1U^2}{4r}\right)^{\frac{(r-2)}{r-1}}+\frac{(r-1)\lambda_1}{4}U^2, & r\neq 0,1,2\\ \frac{1}{2}P^2+\frac{\lambda_1}{4}U^2+\log(\frac{32}{U^2}), & r=2\\ \frac{1}{2}P^2+e^{-k_1U}+\lambda_1k_1U, & r=1\\ \frac{1}{2}P^2+\frac{k_3}{4}U^4+\frac{\lambda_1}{2}U^2, & r=0.\end{cases} \quad (44)$$

It is straightforward to check that when U and P are canonical so do x and p (and vice versa) and the corresponding equations of motion turn out to be

$$\ddot{U} - 2\bigg(\frac{(r-1)k_1}{4r}\bigg)^{\frac{(2-r)}{(1-r)}} U^{\frac{(3-r)}{(1-r)}} + \frac{(r-1)\lambda_1}{2}U = 0, \quad r \neq 0, 1 \tag{45a}$$

$$\ddot{U} - k_1 e^{-U} + k_1\lambda_1 = 0, \qquad r = 1 \tag{45b}$$

$$\ddot{U} + k_3 U^3 + \lambda_1 U = 0, \qquad r = 0. \tag{45c}$$

One may note that the equations of motion now become standard type anharmonic oscillator equations.

B. The case $I_t \neq 0$

In the previous sub-section we considered the case $I_t = 0$. As a consequence S turns out to be $\frac{-\phi}{\dot{x}}$. However in the case $I_t = 0$, the function S has to be determined from Eq. (22), that is,

$$S_t + \dot{x}S_x - ((k_1x + k_2)\dot{x} + k_3x^3 + k_4x^2 + \lambda_1 x)S_{\dot{x}}$$
$$= (k_1\dot{x} + 3k_3x^2 + 2k_4x + \lambda_1) - (k_1x + k_2)S + S^2. \tag{46}$$

Since it is too difficult to solve Eq. (46) for its general solution, we seek a particular solution for S, which is sufficient for our purpose. In particular, we seek a simple rational expression for S in the form

$$S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}, \tag{47}$$

where a, b, c and d are arbitrary functions of t and x which are to be determined. Of course, the analysis of this form alone does not exhaust all possible cases of interest. We hope to make a more exhaustive study of Eq. (46) separately. Substituting (47) into (46) and equating the coefficients of different powers of $\dot{x}$ to zero, we get

$$\begin{aligned}
& db_x - bd_x - k_1 d^2 = 0, \\
& db_t - bd_t + cb_x - bc_x + a_x d - ad_x - 2k_1 cd - (3k_3x^2 + 2k_4 x + \lambda_1)d^2 \\
& \qquad\qquad\qquad\qquad + (k_1 x + k_2)bd - b^2 = 0, \\
& cb_t - bc_t + da_t - ad_t + ca_x - ac_x - k_1 c^2 - 2(3k_3x^2 + 2k_4 x + \lambda_1)cd \\
& \qquad\qquad\qquad\qquad + 2(k_1 x + k_2)ad - 2ab = 0, \\
& ca_t - ac_t - (k_3x^3 + k_4x^2 + \lambda_1 x)(bc - ad) - (3k_3x^2 + 2k_4 x + \lambda_1)c^2 \\
& \qquad\qquad\qquad\qquad + (k_1 x + k_2)ac - a^2 = 0.
\end{aligned} \tag{48}$$

The determining equation for the functions a, b, c and d have now turned out to be nonlinear. To solve these equations we further assume that the functions a, b, c and d are polynomials in x with coefficients which are arbitrary functions in t. Substituting these forms into Eqs. (48) we obtain another enlarged set of determining equations for the unknowns and solving the latter consistently we obtain nontrivial solutions for the functions a, b, c and d for four sets of parametric choices. We present the explicit forms of the associated null function S given by (47) and the parametric restrictions in Table I.

Now substituting the forms of S into Eq. (23) and solving the resultant equation we obtain the corresponding forms of R. To solve the determining equation for R we again seek the same form of ansatz (28) but with explicit t dependence on the coefficient functions, that is, $R = \frac{S_d}{(A(t,x)+B(t,x)\dot{x})^r}$, where S_d is the denominator of S. We report the resultant forms of R in Table I. Once S and R are determined then one has to verify the compatibility of this set (S, R) with the extra constraint Eq. (24). We find that the forms S and R given in Table I do satisfy the extra constraint equation and form a compatible solution. Now substituting S_i's and R_i's into Eq. (25) one can construct the associated integrals of motion. We report the integrals of motion (I) in Table I along with the forms S and R.

At this stage, we note that the first integral for the *case* (i) with $k_2, \lambda_1 = 0$ has been derived in Ref. 18 through Lie symmetry analysis. However, recently, we have derived[9] the first integral for arbitrary values of k_2 and λ_1. *The case* (ii) *is new to the literature*. The first integral for the case (iii) was reported recently in Refs. 9,12 and 13. *The first integral for the case* (iva) *is new to the literature.* The *case* $r = 0$ discussed as (ivb) is nothing but the force-free Duffing oscillator whose integrability has been discussed in Refs. 12 and 14.

TABLE I: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of $\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3}$ $(k_1,\ k_2,\ \lambda_1 :$ arbitrary$)$	$\frac{(\frac{k_1}{3}x^2 - \dot{x})}{x}$	$\frac{x e^{\mp\omega t}}{(\dot{x} - \frac{(k_2 \pm \omega)}{2}x + \frac{k_1}{3}x^2)^2}$	$(a)\ I = e^{\mp\omega t}\left(\frac{3\dot{x} - \frac{3(-k_2 \mp \omega)}{2}x + k_1 x^2}{3\dot{x} - \frac{3(-k_2 \pm \omega)}{2}x + k_1 x^2}\right),$ $k_2, \lambda_1 \neq 0,\ \omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$ $(b)\ I = -t + \frac{x}{(\frac{k_2}{2}x + \frac{k_1}{3}x^2 + \dot{x})}, \qquad k_2^2 = 4\lambda_1$
(ii)	$k_3 = 0, k_4 = \frac{k_1}{4}(k_2 \pm \omega),$ $(k_1,\ k_2,\ \lambda_1 :$ arbitrary$)$	$\frac{1}{2}(k_2 \mp \omega) + k_1 x,$	$e^{\frac{(k_2 \pm \omega)}{2}t}$	$I = \left(\dot{x} + \frac{k_2 \mp \omega}{2}x + \frac{k_1}{2}x^2\right)e^{(\frac{k_2 \pm \omega}{2})t},$ $\omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1,\ k_3 = 0,\ \lambda_1 = \frac{6k_2^2}{25}$ $(k_2,\ k_4 :$ arbitrary$)$	$\frac{(\frac{2k_2\dot{x}}{5} + \frac{4k_2^2 x}{25} + k_4 x^2)}{(\dot{x} + \frac{2k_2}{5}x)}$	$(\dot{x} + \frac{2k_2}{5}x)e^{\frac{6}{5}k_2 t}$	$I = e^{\frac{6}{5}k_2 t}\left(\frac{\dot{x}^2}{2} + \frac{2k_2}{5}x\dot{x} + \frac{2k_2^2}{25}x^2 + \frac{k_4}{3}x^3\right)$
(iva)	$k_3 = \frac{(r-1)k_1^2}{2r^2},\ k_4 = \frac{k_1 k_2}{3},$ $\lambda_1 = \frac{2k_2^2}{9},\ r \neq 0$ $(k_1,\ k_2,\ r :$ arbitrary$)$	$\frac{k_2}{3} + k_1 x + \frac{3k_3 x^3}{(3\dot{x} + k_2 x)}$	$\frac{(k_2 x + 3\dot{x})e^{(\frac{2(2-r)k_2}{3})t}}{(\frac{k_2}{3}x + r k_3 x^2 + \dot{x})^r}$	$I = \left(\frac{k_3}{2}x^4 + (\dot{x} + \frac{k_2}{3}x)(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2)\right)$ $\times\left(\dot{x} + \frac{k_2}{3}x + r k_3 x^2\right)^{-r} e^{\frac{2(2-r)}{3}k_2 t}, \quad r \neq 2$ $I = \frac{2}{3}k_2 t + \log(4k_2 x + 3k_1 x^2 + 12\dot{x})$ $-\frac{4(k_2 x + 3\dot{x})}{(4k_2 x + 3k_1 x^2 + 12\dot{x})}, \qquad r = 2$
(ivb)	$k_1 = 0,\ k_4 = 0,$ $\lambda_1 = \frac{2k_2^2}{9}, r = 0$ $(k_2,\ k_3 :$ arbitrary$)$	$\frac{(\frac{k_2}{3}\dot{x} + \frac{k_2^2}{9}x + k_3 x^3)}{(\dot{x} + \frac{k_2 x}{3})}$	$e^{\frac{4}{3}k_2 t}(\dot{x} + \frac{k_2 x}{3})$	$I = e^{\frac{4}{3}k_2 t}\left[\frac{\dot{x}^2}{2} + \frac{k_2}{3}x\dot{x} + \frac{k_2^2}{18}x^2 + \frac{k_3}{4}x^4\right]$

Since we obtained only one integral in each case, (except case (i) where we have found second explicit time dependent integral, see Ref. 9), which are also time dependent ones, we need to integrate them further to obtain the second integration constant and prove the complete integrability of the respective systems, which is indeed a difficult task.

In this connection we have introduced a new method[1,9] which can be effectively used to transform the time dependent integral into a time independent one, for a *class of problems*, so that the latter can be integrated easily. We invoke this procedure here in order to integrate the time dependent first integrals and obtain the general solution for all the cases in Table I (except case (iv), see below). For the *case* (iv), we prove the Liouville integrability of it.

C. Method of transforming time dependent first integral to time independent one

Let us assume that there exists a first integral for the equation (10) of the form,

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \tag{49}$$

Now let us split the function F_1 further in terms of two functions such that F_1 itself is a function of the product of the two functions, say, a perfect differentiable function $\frac{d}{dt}G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is,

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{d}{dt}G_1(t, x)\right) + F_2\left(G_1(t, x)\right), \tag{50}$$

where F_1 is a function which involves the variables t, x and $\dot{x}$ whereas F_2 should involve only the variable t and x. We note that while rewriting Eq. (49) in the form (50), we demand that the function $F_2(t, x)$ in (49) automatically to be a function of $G_1(t, x)$. Now identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t, x), \quad z = \int_o^t G_2(t', x, \dot{x})dt', \tag{51}$$

one indeed obtains an explicit transformation to remove the time dependent part in the first integral. We note here that the integration on the right hand side of (51) leading to z can be performed provided the function G_2 is an exact derivative of t, that is, $G_2 = \frac{d}{dt}z(t, x) = \dot{x}z_x + z_t$, so that z turns out to be a function t and x alone. In terms of the new variables, Eq. (50) can be modified to the form

$$I = F_1\left(\frac{dw}{dz}\right) + F_2(w). \tag{52}$$

In other words,

$$F_1\left(\frac{dw}{dz}\right) = I - F_2(w). \tag{53}$$

Now rewriting Eq. (52) one obtains a separable equation

$$\frac{dw}{dz} = f(w), \tag{54}$$

which can lead to the solution after an integration. Now rewriting the solution in terms of the original variables one obtains a general solution for the given equation.

In the following using the above idea we integrate the first integrals given in Table I and deduce the second integration constant and general solution.

D. Application

Case (ia): $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3},\ k_1,\ k_2$ and λ_1 : arbitrary:

The parametric restrictions given above fix the equation of motion (10) in the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0, \tag{55}$$

Let us rewrite the first integral associated for this case (vide case (i) in Table I) in the form

$$I_1 = -\frac{k_1 e^{\frac{k_2 \mp \omega}{2}t} x^2}{(3\dot{x} - \frac{(-k_2 \pm \omega)}{2}3x + k_1 x^2)}\left[\frac{d}{dt}\left((\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1})e^{\frac{-k_2 \mp \omega}{2}t}\right)\right], \tag{56}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. Comparing this with the equation (50), and using (51), we obtain

$$w = (\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1})e^{\frac{-k_2 \mp \alpha}{2}t}, \quad z = (\frac{-3}{k_1 x} + \frac{-k_2 \mp \omega}{2\lambda_1})e^{\frac{-k_2 \pm \omega}{2}t}. \tag{57}$$

Substituting (57) into Eq. (55), the latter becomes the free particle equation, namely, $\frac{d^2 w}{dz^2} = 0$, whose general solution is $w = I_1 z + I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of x and t one gets

$$x(t) = \left(\frac{6\lambda_1(1 - I_1 e^{\omega t})}{k_1\omega(1 + I_1 e^{\omega t}) - (k_2 \pm \omega)I_2 e^{\frac{k_2 \pm \omega}{2}t} - k_1 k_2(1 - I_1 e^{\omega t})}\right), \tag{58}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$.

Interestingly one can consider several sub-cases. In the following we discuss some important ones which are being widely discussed in the current literature. In particular, the difference in dynamics arises mainly depending on the sign of the parameter α $(=\sqrt{k_2^2-4\lambda_1})$. We consider the cases (i) $k_2^2<4\lambda_1$ (ii) $k_2^2>4\lambda_1$ and (iii) $k_2^2=4\lambda_1$ separately. The restriction $k_2^2<4\lambda_1$ reduces the solution (58) to the form[4],

$$x(t)=\frac{A\cos(\omega_0 t+\delta)}{\left(e^{\frac{k_2}{2}t}+\frac{2k_1A}{3(k_2^2+4\omega_0^2)}(2\omega_0\sin(\omega_0 t+\delta)-k_2\cos(\omega_0 t+\delta))\right)}, \tag{59}$$

where $\omega_0=\frac{\sqrt{4\lambda_1-k_2^2}}{2}$ and $\delta,\ A$ are arbitrary constants. A further restriction $k_2=0$ gives us the purely sinusoidally oscillating solution[19]

$$x(t)=\frac{A\sin(\omega_0 t+\delta)}{1-(\frac{k}{3\omega_0})A\cos(\omega_0 t+\delta)},\quad 0\leq A<\frac{3\omega_0}{k},\quad \omega_0=\sqrt{\lambda_1}, \tag{60}$$

where A and δ are arbitrary constants. The associated equation of motion, namely $\ddot{x}+k_1x\dot{x}+\frac{k_1^2}{9}x^3+\lambda_1x=0$, admits very interesting nonlinear dynamics, see for example in Ref. 19.

On the other hand, in the limit $k_2^2>4\lambda_1$ the solution looks like a dissipative/front-like one[19]. A further restriction $\lambda_1=0$ takes us to the solution of the form[11]

$$x(t)=\left(\frac{3k_2(I_1e^{k_2t}-1)}{k_1+k_2(3I_2+k_1I_1t)e^{k_2t}}\right). \tag{61}$$

Case (ib): $k_3=\frac{k_1^2}{9},\ k_4=\frac{k_1k_2}{3},\ k_2^2=4\lambda_1,\ k_1$ and k_2 : arbitrary:

The third choice $k_2^2=4\lambda_1$ in (58) leads us to the solution

$$x(t)=\left(\frac{3(I_1+t)}{3I_2e^{\frac{k_2}{2}t}-\frac{2k_1}{k_2^2}(2+I_1k_2+k_2t)}\right). \tag{62}$$

Further parametric restriction $k_2,\ \lambda_1=0$ provides us the general solution of the form

$$x(t)=\left(\frac{6(I_1+t)}{k_1(I_1+t)^2+6I_2}\right). \tag{63}$$

The underlying equation, that is, $\ddot{x}+k_1x\dot{x}+\frac{k_1^2}{9}x^3=0$, is the $l=1$ integrable case of Eq. (6) with the solution (7) (see for example in Refs. 18 and 19).

Case (ii): $k_3 = 0,\ k_4 = \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2$ and λ_1 : arbitrary:

In this case we have the equation of the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^2 + \lambda_1 x = 0. \tag{64}$$

The associated first integral reads (vide case (*ii*) in Table 1)

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \frac{k_1}{2}x^2\right)e^{\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}t}. \tag{65}$$

Note that Eq. (65) can be rewritten as a Riccati equation of the form[21]

$$\dot{x} = Ie^{(\frac{-k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2})t} - \left(\frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}\right)x - \frac{k_1}{2}x^2. \tag{66}$$

The general solution of the Riccati equation is known to be free from movable critical points and satisfies the Painlevé property. In this sense Eq. (64) can be considered as integrable in the Painlevé criteria sense. However, in the general case, (66), it is not clear whether it can be explicitly integrated further. However, for the special case $\lambda_1 = \frac{2k_2^2}{9}$ it can be integrated as follows.

The restriction $\lambda_1 = \frac{2k_2^2}{9}$ fixes the equation of motion (64) and the first integral (65) in the forms

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \tag{67}$$

and

$$I = \left(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2\right)e^{\frac{2k_2}{3}t}, \tag{68}$$

respectively. Now rewriting (68) in the form (50), we get

$$I = e^{\frac{k_2}{3}t}\left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})\right) + \frac{k_1}{2}(xe^{\frac{k_2}{3}t})^2. \tag{69}$$

Identifying the dependent and independent variables from (69) and using the identities (51), we obtain the transformation

$$w = xe^{\frac{k_2}{3}t}, \quad z = -\frac{3}{k_2}e^{-\frac{k_2}{3}t}. \tag{70}$$

Using the transformation (70) the first integral (68) can be rewritten in the form

$$\hat{I} = w' + \frac{k_1}{2}w^2 \tag{71}$$

which in turn leads to the solution by an integration, that is,

$$w(z) = \sqrt{\frac{2I}{k_1}}\tanh\left[\sqrt{\frac{k_1 I}{2}}(z-z_0)\right], \tag{72}$$

where z_0 is arbitrary constant. Rewriting (72) in terms of old variables we get

$$x(t) = \sqrt{\frac{2I}{k_1}}e^{-(\frac{k_2}{3})t}\tanh\left[\frac{3}{k_2}(\sqrt{\frac{k_1 I}{2}})(e^{-\frac{k_2}{3}t_0} - e^{-\frac{k_2}{3}t})\right], \tag{73}$$

where t_0 is the second integration constant.

Case (iii): $k_1,\ k_3 = 0,\ \lambda_1 = \frac{6k_2^2}{25},\ k_2$ and $k_4:$ arbitrary:

The corresponding equation of motion is

$$\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0. \tag{74}$$

Rewriting the associated first integral I_1, given in Case (iii) in Table I, in the form (49), we get

$$I = \frac{1}{2}\left(\dot{x} + \frac{2k_2}{5}x\right)^2 e^{\frac{6}{5}k_2 t} + \frac{k_4}{3}x^3 e^{\frac{6}{5}k_2 t}. \tag{75}$$

Now splitting the first term in Eq. (75) further in the form (50), we obtain

$$I = e^{\frac{2k_2 t}{5}}\left(\frac{d}{dt}(\frac{1}{\sqrt{2}}xe^{\frac{2k_2 t}{5}})\right)^2 + \frac{k_4}{3}(xe^{\frac{2}{5}k_2 t})^3. \tag{76}$$

Identifying the dependent and independent variables from (76) and using the relations (51), we obtain the transformation

$$w = \frac{1}{\sqrt{2}}xe^{\frac{2k_2 t}{5}}, \quad z = -\frac{5}{k_2}e^{-\frac{k_2 t}{5}}. \tag{77}$$

Using this transformation, (77), the first integral (75) can be rewritten in the form

$$\hat{I} = w'^2 + \frac{\hat{k_4}}{3}w^3, \tag{78}$$

where $\hat{k_4} = 2\sqrt{2}k_4$, which inturn leads to

$$w'^2 = 4w^3 - g_3, \tag{79}$$

where $z = 2\sqrt{\frac{3}{\hat{k_4}}}\hat{z}$ and $g_3 = -\frac{12I_1}{\hat{k_4}}$. The solution of this differential equation can be represented in terms of Weierstrass function[12,13] $\varrho(\hat{z};0,g_3)$.

Case (iv): $k_3 = \frac{(r-1)}{2r^2}k_1^2,\ k_4 = \frac{k_1 k_2}{3},\ \lambda_1 = \frac{2k_2^2}{9},\ k_1,\ k_2$ and r : arbitrary (but not zero):

The above parameters fix the equation of motion (10) in the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \quad r \neq 0. \tag{80}$$

The associated first integral reads (vide case (iva) in Table I)

$$I = \begin{cases} \left(\frac{(r-1)}{4r^2}k_1^2 x^4 + (\dot{x} + \frac{k_2}{3}x)(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2) \right) \\ \qquad \times \left(\dot{x} + \frac{k_2}{3}x + \frac{(r-1)}{2r}k_1 x^2 \right)^{-r} e^{\frac{2(2-r)}{3}k_2 t}, & r \neq 0, 2 \\ \frac{2}{3}k_2 t + \log(4k_2 x + 3k_1 x^2 + 12\dot{x}) - \frac{4(k_2 x + 3\dot{x})}{(4k_2 x + 3k_1 x^2 + 12\dot{x})}, & r = 2. \end{cases} \tag{81}$$

Rewriting Eq. (81) in the form (50), we get

$$I = \begin{cases} \left(\frac{(r-1)k_1^2}{4r^2}(xe^{\frac{k_2}{3}t})^4 + \frac{d}{dt}(xe^{\frac{k_2}{3}t}) \left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t} + \frac{k_1}{2}(xe^{\frac{k_2}{3}t})^2 \right) e^{\frac{k_2}{3}t} \right) \\ \qquad \times \left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t} + \frac{k_1(r-1)}{2r}(xe^{\frac{k_2}{3}t})^2 \right)^{-r}, & r \neq 0, 2 \\ \frac{4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t}}{k_1(xe^{\frac{k_2}{3}t})^2 + 4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t}} - \log\left(k_1(xe^{\frac{k_2}{3}t})^2 + 4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t} \right), & r = 2. \end{cases} \tag{82}$$

Identifying the dependent and independent variables from (82) and the relations (51), we obtain the transformation

$$w = xe^{\frac{k_2}{3}t}, \quad z = -\frac{3}{k_2}e^{-\frac{k_2}{3}t}. \tag{83}$$

In terms of the new variables, (83), the first integral I given above, (82), can be written as

$$I = \begin{cases} \left(w' + \frac{(r-1)}{2r}k_1 w^2 \right)^{-r} \left[\frac{(r-1)}{4r^2}k_1^2 w^4 + w'(w' + \frac{k_1}{2}w^2) \right], & r \neq 0, 2 \\ \frac{4w'}{k_1 w^2 + 4w'} - \log(k_1 w^2 + 4w'), & r = 2. \end{cases} \tag{84}$$

On the other hand the transformation (83) modifies the equation (80) to the form

$$w'' + k_1 w w' + \frac{(r-1)k_1^2}{2r^2}w^3 = 0, \quad r \neq 0 \text{ and } ' = \frac{d}{dz}. \tag{85}$$

Finally, for the case $r = 0$, we have an equation of the form (vide case (ivb) in Table I), $\ddot{x} + k_2\dot{x} + k_3x^3 + \frac{2}{9}k_2^2 x = 0$, which is nothing but the force-free Duffing oscillator equation. Again using the transformation (83), the associated time dependent integral given in Table I can be rewritten as

$$I = \frac{w'^2}{2} + \frac{k_3}{4}w^4, \qquad r = 0. \tag{86}$$

Though it is difficult to integrate the above time independent first integrals, (84), as they are in complicated forms, one can easily check that Eq. (86) ($r = 0$) can be integrated in terms of Jacobian elliptic function[14] and the case $r = 1$ is already discussed as case (ii) in this section. For the other cases one can give a Hamiltonian formulation as in Sec. III A 4 and write the corresponding Hamiltonian as

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - p\left(\frac{(r-1)}{2r}k_1 w^2\right)\right], & r \neq 0, 1, 2, \\ \frac{k_1}{4}w^2 p + \log\left(\frac{4k_1}{p}\right), & r = 2 \\ e^p + \frac{k_1}{2}w^2, & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{4}w^4, & r = 0 \end{cases} \tag{87}$$

where

$$p = \begin{cases} \frac{1}{r-1}\left(\frac{(r-1)}{2r}k_1 w^2 + w'\right)^{1-r}, & r \neq 0, 1 \\ \log(w'), & r = 1 \\ w', & r = 0. \end{cases} \tag{88}$$

Thus one is ensured of Liouville integrability of system (85) and so (80) for all values of r. Further, following the analysis in the above subsection III A 5, one can make a canonical transformation (vide Eqs. (42)-(44)) to standard nonlinear oscillator equations.

E. Summary of results for the $q = 1$ case:

To summarize the results obtained in this section, we have identified six integrable cases in Eq. (10) among which four of them were already known in the literature and the remaining two are new. In the following, we tabulate all of them for convenience.

1. *Integrable equations already known in the literature*

(1) $$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0, \qquad (55)$$

(2) $$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \qquad (67)$$

(3) $$\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0, \qquad (74)$$

(4) $$\ddot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0. \qquad (33c)$$

We note that the dynamics and certain transformation properties of Eq. (55) have been studied in detail by three of the present authors in Refs. 9 and 11 recently. In particular, we have shown that this equation admits certain unusual nonlinear dynamics[19]. The dynamics of Eqs. (67),(74) and (33c) can be found in Ref. 12.

2. *New integrable equations*

(1) $$\ddot{x} + k_1 x\dot{x} + k_3 x^3 + \lambda_1 x = 0, \qquad (33b)$$

(2) $$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \qquad (80)$$

where $r^2 k_3 = \frac{(r-1)k_1^2}{2}$ and k_1, k_2, λ_1 and r are arbitrary parameters. We note that (33b) includes the first equation of MEE hierarchy (6) as a sub-case. Importantly, we showed that (33b) is a Hamiltonian system (see Eq. (41)) and so it is Liouville integrable. Equation (80) can be transformed to the integrable Eq. (85). Explicit general solution of certain special cases, namely, $r = 3$ or $\frac{3}{2}$ and $r = -1$ or $\frac{1}{2}$ are reported in Ref. 20.

IV. GENERALIZED FORCE FREE DVP FORM OF EQUATIONS

Let us now consider the case $q = 2$ in Eq. (9) or equivalently (11), that is,

$$\ddot{x} = -((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x) \equiv \phi(x, \dot{x}). \tag{11}$$

Interestingly Eq. (11) includes another class of physically important nonlinear oscillators. For example, choosing $k_3 = 0$ one can get force-free Duffing-van der Pol oscillator equation. With the choice $k_2,\ k_4,\ \lambda_1 = 0$, it coincides with the second equation in the MEE hierarchy equation. Equation (11) with the restriction $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4}$ and $\lambda_1 = (\omega_0^2 + \frac{k_2^2}{4})$, has been investigated in a different perspective in Ref. 4. However, the general equation of the form (11) has never been considered for integrability test and so we perform the same here.

To identify integrals of motion and the general solution of Eq. (11) we again seek the PS procedure. As the calculations are similar to the $q = 1$ case of Eq. (9) which was carried out in the previous section, in the following, we give only the important steps.

A. The case $I_t = 0$

By considering the same arguments given in Sec. III A 1, the null form S can be fixed easily in the form

$$S = -\frac{((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x)}{\dot{x}}. \tag{89}$$

The respective R equation becomes

$$\begin{aligned} R_t + \dot{x} R_x - ((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x) R_{\dot{x}} \\ = ((k_1 x^2 + k_2) + \frac{((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x)}{\dot{x}}) R. \end{aligned} \tag{90}$$

To seek a particular form for R one may seek a suitable ansatz. We assume R to be of the form (28) and investigate the system (90) as before. Following a similar procedure we find that a nontrivial particular solution for (90) exists in the form

$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2 x) + \dot{x})^r}, \tag{91}$$

where r, k_1 and k_2 are arbitrary parameters and the remaining parameters, k_3, k_4 and λ_1, are fixed by the relations

$$k_3 = \frac{(r-1)}{3r^2} k_1^2, \quad k_4 = \frac{4(r-1)}{3r^2} k_1 k_2, \quad \lambda_1 = \frac{(r-1)}{r^2} k_2^2. \tag{92}$$

Further, we confirmed the compatibility of the functions S and R with the extra constraint (24) also. We note that unlike the earlier case, $q = 1$, we do not get a nontrivial solution for the parametric restriction $k_2,\ k_4 = 0$. The above restrictions fix the Eq. (11) to the following specific forms:

$$(ia) \quad \ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{(r-1)}{3r^2}k_1^2x^5 + \frac{4(r-1)k_1k_2}{3r^2}x^3 + \frac{(r-1)k_2^2}{r^2}x = 0,\ r \neq 0 \quad (93a)$$

$$(ib) \quad \ddot{x} + k_3x^5 + k_4x^3 + \lambda_1 x = 0,\ r = 0 \quad (93b)$$

Now substituting (89) and (91) into (25) and evaluating the integrals we obtain the first integrals in the form

$$(ia) \quad I_1 = \left(\dot{x} + \frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2x)\right)^{-r} \times \left[\dot{x}(\dot{x} + \frac{k_1}{3}x^3 + k_2x) + \frac{(r-1)}{r^2}(\frac{k_1}{3}x^3 + k_2x)^2\right],\quad r \neq 0, 2, \quad (94a)$$

$$(ib) \quad I_1 = \frac{6\dot{x}}{(6\dot{x} + 3k_2x + k_1x^3)} - \log(6\dot{x} + 3k_2x + k_1x^3),\quad r = 2, \quad (94b)$$

$$(ii) \quad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{6}x^6 + \frac{k_4}{4}x^4 + \frac{\lambda_1}{2}x^2,\quad r = 0. \quad (94c)$$

Further, as in the $q = 1$ case in Sec. III A 4, the integrals (94) can be recast into the Hamiltonian form

$$(ia) \quad H = \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}p(\frac{k_1}{3}x^3 + k_2x)\right],\quad r \neq 0, 1, 2, \quad (95a)$$

$$(ib) \quad H = \frac{k_2}{2}xp + \frac{k_1}{6}x^3p + \log(\frac{6}{p}),\quad r = 2, \quad (95b)$$

$$(ic) \quad H = e^p + \frac{k_1}{3}x^3 + k_2x,\quad r = 1, \quad (95c)$$

$$(ii) \quad H = \frac{p^2}{2} + \frac{k_3}{6}x^6 + \frac{k_4}{4}x^4 + \frac{\lambda_1}{2}x^2,\quad r = 0. \quad (95d)$$

where the corresponding canonical momenta respectively are

$$(ia, b) \quad p = \frac{1}{(r-1)}\left(\dot{x} + \frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2x)\right)^{(1-r)},\quad r \neq 0, 1, \quad (96a)$$

$$(ic) \quad p = \log\dot{x},\quad r = 1, \quad (96b)$$

$$(ii) \quad p = \dot{x},\quad r = 0. \quad (96c)$$

Note that in the above the parameters $r,\ k_1,\ k_2,\ k_3$ and λ_1 are all arbitrary. We also note here that unlike the $q = 1$ case discussed in Sec. III, so far we have been unable to find suitable canonical transformations for the above Hamiltonian systems so that the standard 'potential' equation results. The problem is being further investigated.

B. The case $I_t \neq 0$

Now let us study the case $I_t \neq 0$. In this case S has to be determined from Eq. (22), that is,

$$S_t + \dot{x}S_x - ((k_1x^2 + k_2)\dot{x} + k_3x^5 + k_4x^3 + \lambda_1 x)S_{\dot{x}} \\ = (2k_1x\dot{x} + 5k_3x^4 + 3k_4x^2 + \lambda_1) - (k_1x^2 + k_2)S + S^2. \qquad (97)$$

As we did in the $q = 1$ case of Eq. (9) we proceed to solve Eq. (97) with the same form of ansatz (47). Doing so we find that Eq. (97) admits non-trivial forms of solutions for certain specific parametric restrictions. We report both the parametric values and their respective forms of S in Table II.

Now substituting the forms of S into Eq. (23) and solving the resultant equation we obtain the corresponding forms of R. Once S and R are determined then one has to verify the compatibility of this solution with the extra constraint (24). Then one can substitute the null forms and integrating factors into (25) and construct the associated integrals of motion. We report the integrating factors (R) and time-dependent integrals of motion (I) in Table II.

The remaining task is to derive the general solution and establish the complete integrability of Eq. (11) for each parametric restriction. We again adopt the procedure given in Sec. III C and transform the time dependent integrals into time independent ones and integrate the latter and deduce the general solution. As the procedure is exactly the same we provide only the results in the following.

Case (ia): $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1k_2}{4},\ k_1,\ k_2$ and λ_1 : arbitrary:

Substituting the parametric restrictions given above in Eq. (11), we get

$$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1^2}{16}x^5 + \frac{k_1k_2}{4}x^3 + \lambda_1 x = 0. \qquad (98)$$

TABLE II: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of $\ddot{x}+(k_1x^2+k_2)\dot{x}+k_3x^5+k_4x^3+\lambda_1 x=0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3=\frac{k_1^2}{16},\ k_4=\frac{k_1k_2}{4}$ $(k_1,\ k_2,\ \lambda_1: \text{arbitrary})$	$\dfrac{\frac{k_1}{2}x^3-\dot{x}}{x}$	$\dfrac{xe^{\mp\omega t}}{(\dot{x}-\frac{(k_2\pm\omega)}{2}x+\frac{k_1}{4}x^3)^2}$	$(a)\ I=e^{\mp\omega t}\left(\frac{4\dot{x}+2(k_2\pm\omega)x+k_1x^3}{4\dot{x}+2(k_2\mp\omega)x+k_1x^3}\right),$ $k_2,\lambda_1\neq 0,\ \omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$ $(b)\ I=-t+\frac{x}{(\frac{k_2}{2}x+\frac{k_1x^3}{4}+\dot{x})},\quad k_2^2=4\lambda_1$
(ii)	$k_3=0,k_4=\frac{k_1}{6}(k_2\pm\omega),$ $(k_1,\ k_2,\ \lambda_1: \text{arbitrary})$	$\dfrac{1}{2}(k_2\mp\omega)+k_1x^2$	$e^{\frac{(k_2\pm\omega)}{2}t}$	$I=\left(\dot{x}+\frac{k_2\mp\omega}{2}x+\frac{k_1}{3}x^3\right)e^{(\frac{k_2\pm\omega}{2})t},$ $\omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1,\ k_3=0,\ \lambda_1=\frac{2k_2^2}{9}$ $(k_2,\ k_4: \text{arbitrary})$	$\left(\dfrac{\frac{k_2}{3}\dot{x}+\frac{k_2^2}{9}x+k_4x^3}{\dot{x}+\frac{k_2}{3}x}\right)$	$(\dot{x}+\dfrac{k_2}{3}x)e^{\frac{4}{3}k_2t}$	$I=e^{\frac{4}{3}k_2t}\left[\frac{\dot{x}^2}{2}+\frac{k_2}{3}x\dot{x}+\frac{k_2^2}{18}x^2+\frac{k_4}{4}x^4\right]$
(iva)	$k_3=\frac{(r-1)k_1^2}{3r^2},\ k_4=\frac{k_1k_2}{4},$ $\lambda_1=\frac{3k_2^2}{16},\ r\neq 0$ $(k_1,\ k_2,\ r: \text{arbitrary})$	$\dfrac{k_2}{4}+k_1x^2+\dfrac{4k_3x^5}{(4\dot{x}+k_2x)}$	$\dfrac{(k_2x+4\dot{x})e^{\frac{3(2-r)}{4}k_2t}}{(\frac{k_2}{4}x+rk_3x^3+\dot{x})^r}$	$I=\left(\frac{k_3}{3}x^6+(\dot{x}+\frac{k_2}{4}x)(\dot{x}+\frac{k_2}{4}x+\frac{k_1}{3}x^3)\right)$ $\times\left(\dot{x}+\frac{k_2}{4}x+rk_3x^3\right)^{-r}e^{\frac{3(2-r)}{4}k_2t},\ \ r\neq 2$ $I=\frac{3}{4}k_2t+\log(6k_2x+4k_1x^3+24\dot{x})$ $-\dfrac{6(k_2x+4\dot{x})}{(6k_2x+4k_1x^3+24\dot{x})},\qquad r=2$
(ivb)	$k_1=0,\ k_4=0,$ $\lambda_1=\frac{3k_2^2}{16},r=0$ $(k_2,\ k_3: \text{arbitrary})$	$\left(\dfrac{\frac{k_2}{4}\dot{x}+\frac{k_2^2}{16}x+k_3x^5}{\dot{x}+\frac{k_2}{4}x}\right)$	$e^{\frac{3k_2}{2}t}(\dot{x}+\dfrac{k_2}{4}x)$	$I=e^{\frac{3k_2}{2}t}\left(\frac{\dot{x}^2}{2}+\frac{k_2}{4}x\dot{x}+\frac{k_2^2}{32}x^2+\frac{k_3}{6}x^6\right)$

We observed that the first integral of this case (i) (see Table II), when rewritten, is nothing but the Bernoulli equation which can be integrated strightforwardly[21] and it leads to the general solution of the form

$$x(t) = \left(\frac{8k_2\lambda_1(e^{\omega t} - I_1)^2}{I_1^2 k_1 k_2(-k_2+\omega) - e^{2\omega t}k_1k_2(k_2+\omega) + 8I_2k_2\lambda_1 e^{(k_2+\omega)t} + 8I_1k_1\lambda_1 e^{\omega t}} \right)^{\frac{1}{2}}, \quad (99)$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. A sub-case of the Eq. (98), namely, $k_2^2 < 4\lambda_1$ has been studied by Smith[4,22] who showed that the corresponding equation of motion admits a damped oscillatory form of solution, namely,

$$x(t) = \frac{Acos(\omega_0 t + \delta)}{\left(e^{k_2 t} - \frac{k_1 A}{4k_2} + \frac{k_1 A}{4(k_2^2+4\omega_0^2)}\left(2\omega_0 sin2(\omega_0 t+\delta) - k_2 cos2(\omega_0 t + \delta)\right)\right)^{\frac{1}{2}}}, \quad (100)$$

where $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 + k_2^2}$ and $\delta,\ A$ are arbitrary constants.

On the other hand for $k_2^2 > 4\lambda_1$, the solution (99) becomes dissipative type having a front-like structure. In particular, for $\lambda_1 = 0$ we get a solution of the form

$$x(t) = \left(\frac{2\sqrt{k_2}(I_1 e^{k_2 t} - 1)}{(-k_1 + 2k_1 I_1 e^{k_2 t}(2 + k_2 I_1 t e^{k_2 t}) + 4k_2 I_2 e^{2k_2 t})^{\frac{1}{2}}} \right). \quad (101)$$

Case (ib): $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4},\ k_2^2 = 4\lambda_1,\ k_1$ and k_2 : arbitrary:

In this case we get the general solution of the form from (101) as

$$x(t) = \left(\frac{2(I_1+t)^2}{2e^{k_2 t}I_2 - \frac{k_1}{k_2^3}(2 + I_1^2k_2^2 + 2k_2 t + k_2^2 t^2 + 2I_1 k_2(1+k_2 t))} \right)^{\frac{1}{2}}. \quad (102)$$

One may note that a sub-case of this equation, namely, $k_2 = \lambda_1 = 0$ leads us to the second equation in the MEE hierarchy (6) and the corresponding solution follows from Eq. (102) as

$$x(t) = \sqrt{6}\left(\frac{(I_1+t)^2}{6I_2 + k_1 t(3I_1^2 + 3I_1 t + t^2)} \right)^{\frac{1}{2}}. \quad (103)$$

This form exactly coincides with the solution (7) for $l = 2$.

Case (ii): $k_3 = 0,\ k_4 = \frac{k_1}{6}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2$ and λ_1 : arbitrary:

The repective equation of motion and the first integral are (see Table II)

$$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1}{6}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^3 + \lambda_1 x = 0, \tag{104}$$

and

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \frac{k_1}{3}x^3\right)e^{\frac{k_2 \pm \sqrt{k_2^2-4\lambda_1}}{2}t}. \tag{105}$$

Eq. (105) can be rewritten as an Abel's equation in the form

$$\dot{x} = Ie^{(\frac{-k_2 \mp \sqrt{k_2^2-4\lambda_1}}{2})t} - \left(\frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}\right)x - \frac{k_1}{3}x^3. \tag{106}$$

It is not clear whether Eq. (106) can be explicitly integrated in general. However, for the special case $\lambda_1 = \frac{3}{16}k_2^2$ it can be integrated as follows.

The restriction $\lambda_1 = \frac{2k_2^2}{9}$ fixes the equation of motion (104) and the first integral (105) in the forms

$$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \tag{107}$$

and

$$I = \left(\dot{x} + \frac{k_2}{4}x + \frac{k_1}{3}x^3\right)e^{\frac{3k_2}{4}t}, \tag{108}$$

respectively.

Now following our procedure given in Sec. 3.3 one arrives at the general solution[1] as

$$z + z_0 = -\frac{a}{3I}\left[\frac{1}{2}\log\left(\frac{(w-a)^2}{w^2+aw+a^2}\right) + \sqrt{3}\arctan\left(\frac{-w\sqrt{3}}{2a+w}\right)\right], \tag{109}$$

with $w = xe^{\frac{k_2}{4}t}$, $z = -\frac{2}{k_2}e^{-\frac{k_2}{2}t}$ and $a = \sqrt[3]{\frac{3I}{k_1}}$ and z_0 is the second integration constant. Rewriting w and z in terms of old variables one can get the explicit solution.

Case (iii): $k_1,\ k_3 = 0,\ \lambda_1 = \frac{2k_2^2}{9},\ k_2$ and k_4 : arbitrary:

The parametric restrictions given above fix the equation of motion (11) to the force-free Duffing oscillator, namely, $\ddot{x} + k_2\dot{x} + k_4x^3 + \frac{2k_2^2}{9}x = 0$. We have already discussed the general solution of this equation in Sec. III (vide case (*iv*)).

Case (iv): $k_3 = \frac{(r-1)k_1^2}{3r^3},\ k_4 = \frac{k_1k_2}{4},\ \lambda_1 = \frac{3k_2^2}{16},\ k_1,\ k_2$ and r : arbitrary:

The equation of motion turns out to be

$$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{(r-1)k_1^2}{3r^2}x^5 + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \quad r \neq 0. \tag{110}$$

Rewriting the associated first integral I, given in Case (iv) in Table II, in the form (50), we get

$$I = \begin{cases} \left(\frac{(r-1)k_1^2}{9r^2}(xe^{\frac{k_2}{4}t})^6 + \frac{d}{dt}(xe^{\frac{k_2}{4}t})\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t} + \frac{k_1}{3}(xe^{\frac{k_2}{4}t})^3\right)e^{\frac{k_2}{2}t}\right) \\ \qquad \times\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t} + \frac{k_1(r-1)}{3r}(xe^{\frac{k_2}{4}t})^3\right)^{-r}, & r \neq 0, 2, \\ \frac{6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}}{k_1(xe^{\frac{k_2}{4}t})^3 + 6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}} - \log(k_1(xe^{\frac{k_2}{4}t})^3 + 6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}), & r = 2 \\ \frac{1}{2}\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})\right)^2 e^{k_2t} + \frac{k_3}{6}(xe^{\frac{k_2}{4}t})^6, & r = 0 \end{cases} \tag{111}$$

and identifying the dependent and independent variables from (111) and the relations (51), we obtain the transformation

$$w = xe^{\frac{k_2}{4}t}, \quad z = -\frac{2}{k_2}e^{-\frac{k_2}{2}t}. \tag{112}$$

In terms of the new variables (112) the first integral I given above, (111) can be written as

$$I = \begin{cases} \left(w' + \frac{(r-1)}{3r}k_1w^3\right)^{-r}\left[w'(w' + \frac{k_1}{3}w^3) + \frac{(r-1)}{9r^2}k_1^2w^6\right], & r \neq 0, 2 \\ \frac{6w'}{k_1w^3 + 6w'} - \log(k_1w^3 + 6w'), & r = 2, \\ \frac{w'^2}{2} + \frac{k_3}{6}w^6, & r = 0. \end{cases} \tag{113}$$

On the other hand substituting the transformation (112) into the equation of motion (110) we get

$$w'' + k_1w^2w' + \frac{(r-1)k_1^2}{3r^2}w^5 = 0, \quad r \neq 0, \qquad ' = \frac{d}{dz}. \tag{114}$$

In the case $r=0$, we have an equation of the form (vide case (ivb) in Table II)

$$\ddot{x}+k_2\dot{x}+k_3x^5+\frac{3k_2^2}{16}x=0. \tag{115}$$

We note that the Eq. (114) is the $l=2$ case of Eq. (6). As we mentioned in the introduction the general solution of this equation can be obtained only for certain specific choices, namely, $\frac{(r-1)k_1^2}{3r^2}=\frac{1}{16}$. This in turn gives $r=4k_1$ or $\frac{4}{3}k_1$. The respective solutions for these values of r of Eq. (114) can be fixed from Eq. (7) with $l=2$. The other cases do not seem to be amenable to explicit integration. However, all of them can be recast in the Hamiltonian form as we see below.

As the first integrals (113) are now 'time' independent ones, one can give a Hamiltonian formalism for all the integrals (113) by following the ideas given in Sec. III A 4. Doing so we obtain

$$H=\begin{cases}\left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)}-\frac{(r-1)}{3r}k_1w^3p\right], & r\neq 0,1,2,\\ \frac{k_1}{6}w^3p+\log(\frac{6}{p}), & r=2\\ e^p+\frac{k_1}{3}w^3, & r=1\\ \frac{p^2}{2}+\frac{k_3}{6}w^6, & r=0\end{cases} \tag{116}$$

where

$$p=\begin{cases}\frac{1}{(r-1)}\left(w'+\frac{(r-1)}{3r}k_1w^3\right)^{(1-r)}, & r\neq 0,1\\ \log w', & r=1\\ \frac{p^2}{2}+\frac{k_3}{6}w^6, & r=0.\end{cases} \tag{117}$$

In this sense these cases may be considered as Liouville integrable systems. Finally, for $r=0$ case in Eq. (113) can be integrated in terms of Jacobian elliptic function (see for example in Ref. 23). Again, here, we have not been able to identify canonical transformations which can lead to the identification of suitable 'potential' equations.

C. Summary of results in $q = 2$ case:

To summarize the results obtained for the $q = 2$ case, we have identified six integrable cases in Eq. (11) among which three of them were already known in the literature and the remaining three are new. In the following, we tabulate both of them.

1. *Integrable equations already known in the literature*

$$(1) \qquad \ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \qquad (107)$$

$$(2) \qquad \ddot{x} + k_2\dot{x} + k_3x^3 + \frac{2k_2^2}{9}x = 0, \qquad (118)$$

$$(3) \qquad \ddot{x} + k_3x^5 + k_4x^3 + \lambda_1x = 0. \qquad (93b)$$

We mention that Eq. (107) is nothing but the force-free DVP whose integrability is established in Ref. 1 and Eq. (118) is nothing but the force-free Duffing oscillator[12,14].

2. *New integrable equations*

$$(1) \qquad \ddot{x} + (k_1x^2 + k_2)\dot{x} + k_3x^5 + \frac{4(r-1)k_1k_2}{3r^2}x^3 + \frac{(r-1)k_2^2}{r^2}x = 0,\ r \neq 0 \qquad (93a)$$

$$(2) \qquad \ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1^2}{16}x^5 + \frac{k_1k_2}{4}x^3 + \lambda_1x = 0, \qquad (98)$$

$$(3) \qquad \ddot{x} + (k_1x^2 + k_2)\dot{x} + k_3x^5 + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \qquad (110)$$

where $r^2k_3 = \frac{(r-1)k_1^2}{3}$ and k_1, k_2, λ_1 and r are arbitrary parameters. We proved that Eq. (93a) is Liouville integrable one. As far as equation (98) is concerned we derived the general solution for arbitrary values of $k_1,\ k_2$ and λ_1. Finally, for Eq. (110) though we identified only one time dependent integral, we have demonstrated that it can be transformed into time independent Hamiltonian and thereby ensuring its Liouville integrability.

V. EXTENDED PRELLE-SINGER METHOD TO GENERALIZED EQ. (9)

One can investigate the integrability properties of Eq. (9) by considering the cases $q = 3, 4, 5, \ldots,$ one by one and classify the integrable equations. Since the procedure and the mathematical techniques in exploring the integrating factors (R), null forms (S), first integrals (I) and general solution are the same in each case we do not consider each case in detail. We straightaway move to the case $q = arbitrary$, that is, $q \in R$ and not necessarily an integer, and present the outcome of our investigations.

As we did earlier, we consider the cases $I_t = 0$ and $I_t \neq 0$ separately for the choice $q = arbitrary$ also. First let us consider the case $I_t = 0$.

A. The case $I_t = 0$

By considering the same arguments given in Sec. 3.1.1 the null form S and the integrating factor R can be fixed easily in the form

$$S = -\frac{((k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{1+q} + \lambda_1 x)}{\dot{x}},$$
$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{(q+1)}x^{q+1} + k_2 x) + \dot{x})^r}, \tag{119}$$

where k_1 and k_2 are arbitrary and the remaining parameters, k_3, k_4 and λ_1, are related to the parameters k_1 and k_2 through the relations

$$k_3 = \frac{(r-1)}{r^2}(q+1)\hat{k}_1^2, \quad k_4 = \frac{(r-1)}{r^2}(q+2)\hat{k}_1 k_2, \quad \lambda_1 = \frac{(r-1)}{r^2}k_2^2, \tag{120}$$

where $\hat{k}_1 = \frac{k_1}{(q+1)}$. The above restrictions fix Eq. (9) to the following specific forms:

$$(ia) \qquad \ddot{x} + ((q+1)\hat{k}_1 x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}[(q+1)\hat{k}_1^2 x^{2q+1} + (q+2)\hat{k}_1 k_2 x^{q+1} + k_2^2 x] = 0, \ r \neq 0 \tag{15}$$

$$(ib) \qquad \ddot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0, \ r = 0. \tag{121}$$

Now substituting (119) into (25) and evaluating the integrals we obtain the first integrals of the form

$$(ia) \quad I_1 = \left(\dot{x} + \frac{(r-1)}{r}(\hat{k}_1 x^{q+1} + k_2 x)\right)^{-r} \times \left[\dot{x}(\dot{x} + \hat{k}_1 x^{q+1} + k_2 x) + \frac{(r-1)}{r^2}(\hat{k}_1 x^{q+1} + k_2 x)^2\right], \quad r \neq 0, 2, \qquad (122a)$$

$$(ib) \quad I_1 = \frac{\dot{x}}{(\dot{x} + \frac{k_2}{2}x + \frac{\hat{k}_1}{2}x^{q+1})} - \log(\dot{x} + \frac{k_2}{2}x + \frac{\hat{k}_1}{2}x^{q+1}), \quad r = 2, \qquad (122b)$$

$$(ii) \quad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{2(q+1)}x^{2(q+1)} + \frac{k_4}{(q+2)}x^{q+2} + \frac{\lambda_1}{2}x^2, \quad r = 0. \qquad (122c)$$

Further, using the above forms of the first integrals, one can show that the equation of motion (9), with the parametric restrictions (120), can also be derived from the Hamiltonians

$$(ia) \quad H = \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}p(\hat{k}_1 x^{q+1} + k_2 x)\right], \quad r \neq 0, 1, 2, \qquad (123a)$$

$$(ib) \quad H = \frac{k_2}{2}xp + \frac{\hat{k}_1}{2}x^{q+1}p + \log(\frac{2(q+1)}{p}), \quad r = 2 \qquad (123b)$$

$$(ic) \quad H = e^p + \hat{k}_1 x^{q+1} + k_2 x, \quad r = 1, \qquad (123c)$$

$$(ii) \quad H = \frac{p^2}{2} + \frac{k_3}{2(q+1)}x^{2(q+1)} + \frac{k_4}{(q+1)}x^{q+1} + \frac{\lambda_1}{2}x^2, \quad r = 0, \qquad (123d)$$

where the corresponding canonical momenta respectively are

$$(ia, b) \quad p = \frac{1}{(r-1)}\left(\dot{x} + \frac{(r-1)}{r}(\hat{k}_1 x^{q+1} + k_2 x)\right)^{(1-r)}, \quad r \neq 0, 1, \qquad (124a)$$

$$(ic) \quad p = \log \dot{x}, \quad r = 1, \qquad (124b)$$

$$(ii) \quad p = \dot{x}, \quad r = 0. \qquad (124c)$$

With the above Hamiltonian formulation, for the parametric set (120), the integrability of the associated equation of motion is assured for these parametric cases through Liouville theorem.

B. The case $I_t \neq 0$

We use the same ansatz and ideas which we followed for the $q = 1$ and $q = 2$ cases to determine the forms of S and R. As the procedure is exactly the same as in the earlier cases we present the parametric restrictions and the respective form of expressions of the integrating factors, null forms and integrals of motions in Table III without further discussion.

Since we derived only one integral, which is also a time dependent one for each parametric restriction, we need to integrate each one of them further and obtain the second integration constant in order to prove the complete integrability of each of the cases reported in Table III. In the following we deduce the second integral and general solution by utilizing the proceduce given in Sec. III C.

Case (ia): $k_3 = \frac{k_1^2}{(q+2)^2},\ k_4 = \frac{k_1 k_2}{(q+2)},\ k_1,\ k_2$ and $\lambda_1:$ arbitrary:

We have an equation of the form

$$\ddot{x} + ((q+2)\hat{k}_1 x^q + k_2)\dot{x} + \hat{k}_1^2 x^{2q+1} + \hat{k}_1 k_2 x^{q+1} + \lambda_1 x = 0, \tag{13}$$

where $k_1 = (q+2)\hat{k}_1$. The corresponding first integral given in Table 3 is nothing but the Bernoulli equation which can be solved using the standard method[21]. The general solution turns out to be

$$x(t) = \left(e^{\omega t} - I_1\right)\left(e^{\frac{q}{2}(k_2+\omega)t}\left(I_2 + \hat{k}_1 q \int \left(\frac{e^{\omega t} - I_1}{e^{\frac{1}{2}(k_2+\omega)t}}\right)^q dt\right)\right)^{\frac{-1}{q}}, \tag{125}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. We note here that a sub-case of the above, namely, $k_2^2 < 4\lambda_1$, has been studied by Smith[4] who had shown that the corresponding system admits the general solution of the form

$$x(t) = Acos(\omega_0 t + \delta)e^{-\frac{k_2}{2}t}\left(1 + q\hat{k}_1 A \int e^{\frac{-qk_2}{2}t} cos^q(\omega_0 t + \delta) dt\right)^{-\frac{1}{q}}, \tag{126}$$

where $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 + k_2^2}$ and $\delta,\ A$ are arbitrary constants. For $k_2^2 > 4\lambda_1$, the solution become a dissipative type/front-like structure. In particular, for $\lambda_1 = 0$ the general solution takes the form

$$x(t) = \left(e^{k_2 t} I_1 - 1\right)\left[e^{qk_2 t}\left(I_2 + \hat{k}_1 q \int \left(I_1 - e^{-k_2 t}\right)^q dt\right)\right]^{-\frac{1}{q}}. \tag{127}$$

TABLE III: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of $\ddot{x}+(k_1x^q+k_2)\dot{x}+k_3x^{2q+1}+k_4x^{q+1}+\lambda_1 x=0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3=\frac{k_1^2}{(q+2)^2},\ k_4=\frac{k_1k_2}{(q+2)}$ ($k_1,\ k_2,\ \lambda_1$: arbitrary)	$\dfrac{(\frac{qk_1}{(q+2)}x^{q+1}-\dot{x})}{x}$	$\dfrac{xe^{\mp\omega t}}{(\dot{x}-\frac{(k_2\pm\omega)}{2}x+\frac{k_1}{(q+2)}x^{q+1})^2}$	$(a)\ I=e^{\mp\omega t}\left(\frac{\dot{x}-\frac{(-k_2\mp\omega)}{2}x+\frac{k_1}{q+2}x^{q+1}}{\dot{x}-\frac{(-k_2\pm\omega)}{2}x+\frac{k_1}{q+2}x^{q+1}}\right),$ $k_2,\lambda_1\neq 0,\ \omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$ $(b)\ I=-t+\frac{x}{(\frac{k_2}{2}x+\frac{k_1x^{q+1}}{q+2}+\dot{x})},\qquad k_2^2=4\lambda_1$
(ii)	$k_4=\frac{k_1(k_2\pm\omega)}{2(q+1)},\ k_3=0,$ ($k_1,\ k_2,\ \lambda_1$: arbitrary)	$\dfrac{1}{2}(k_2\mp\omega)+k_1x^q,$	$e^{\frac{(k_2\pm\omega)}{2}t}$	$I=\left(\dot{x}+\frac{k_2\mp\omega}{2}x+\frac{k_1}{(q+1)}x^{q+1}\right)e^{(\frac{k_2\pm\omega}{2})t},$ $\omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1,k_3=0,\lambda_1=\frac{2(q+2)k_2^2}{(q+4)^2}$ ($k_2,\ k_4$: arbitrary)	$\dfrac{\frac{2k_2\dot{x}}{(q+4)}+\frac{4k_2^2x}{(q+4)^2}+k_4x^{q+1}}{(\dot{x}+\frac{2k_2x}{(q+4)})}$	$(\dot{x}+\dfrac{2k_2x}{(q+4)})e^{\frac{2(q+2)}{(q+4)}k_2t}$	$I=e^{\frac{2(q+2)}{(q+4)}k_2t}\left[\frac{\dot{x}^2}{2}+\frac{2k_2x\dot{x}}{(q+4)}+\frac{2k_2^2x^2}{(q+4)^2}+\frac{k_4x^{q+2}}{(q+2)}\right]$
(iv)a	$k_3=\frac{(r-1)k_1^2}{(q+1)r^2},$ $k_4=\frac{k_1k_2}{(q+2)},$ $\lambda_1=\frac{(q+1)k_2^2}{(q+2)^2},\ r\neq 0$ ($k_1,\ k_2,\ r$: arbitrary)	$\frac{k_2}{(q+2)}+k_1x^q+\frac{k_3x^{2q+1}}{(\dot{x}+\frac{k_2}{(q+2)}x)}$	$\dfrac{(k_2x+(q+2)\dot{x})e^{\frac{(q+1)(2-r)}{(q+2)}k_2t}}{(\frac{k_2}{(q+2)}x+rk_3x^{q+1}+\dot{x})^r}$	$I=\left(\frac{k_3x^{2(q+1)}}{(q+1)}+(\dot{x}+\frac{k_2x}{q+2})(\dot{x}+\frac{k_2x}{q+2}+\frac{k_1x^{q+1}}{q+1})\right)$ $\times\left(\frac{k_2}{(q+2)}x+rk_3x^{q+1}+\dot{x}\right)^{-r}e^{\frac{(q+1)(2-r)}{(q+2)}k_2t},r\neq 2$ $I=\frac{q+1}{q+2}k_2t+\log(k_1x^{q+1}+2(q+1)(\dot{x}+\frac{k_2}{q+2}x))$ $-(\frac{2(q+1)(\dot{x}+\frac{k_2}{q+2}x)}{k_1x^{q+1}+2(q+1)(\dot{x}+\frac{k_2}{q+2}x)}),\qquad r=2$
(iv)b	$k_1=0,\ k_4=0,$ $\lambda_1=\frac{(q+1)k_2^2}{(q+2)^2},r=0$ ($k_2,\ k_3$: arbitrary)	$\dfrac{k_2}{(q+2)}+\dfrac{k_3x^{2q+1}}{(\dot{x}+\frac{k_2}{(q+2)}x)}$	$e^{\frac{(2q+2)k_2}{(q+2)}t}(\dot{x}+\dfrac{k_2}{(q+2)}x)$	$I=\left(\frac{\dot{x}^2}{2}+\frac{k_2x\dot{x}}{(q+2)}+\frac{k_2^2x^2}{2(q+2)^2}+\frac{k_3x^{2q+2}}{(2q+2)}\right)e^{\frac{(2q+2)k_2}{(q+2)}t}$

Case (ib): $\underline{k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4},\ k_2 = 4\lambda_1,\ k_1 \text{ and } k_2:\ \text{arbitrary:}}$

A general solution for this case can be fixed from (127) as

$$x(t) = (I_1 + t)e^{-\frac{k_2}{2}t}\left(I_2 + q\hat{k}_1 \int e^{-\frac{qk_2}{2}t}(I_1 + t)^q dt\right)^{-\frac{1}{q}}. \tag{128}$$

On the other hand the general solution for the parametric choice $k_2,\ \lambda_1 = 0$ turns out to be

$$x(t) = \left(\frac{(q+1)(I_1+t)^q}{\hat{k}_1 q(I_1+t)^{q+1} + (q+1)I_2}\right)^{\frac{1}{q}}, \tag{129}$$

which exactly coincides with the result (7) obtained by Feix et al.[3] for integer $q(=l)$ values.

Case (ii): $\underline{k_3 = 0,\ k_4 = \frac{k_1}{2(q+1)}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2 \text{ and } \lambda_1:\ \text{arbitrary:}}$

The associated equation of motion and the first integral are (see Table III)

$$\ddot{x} + ((q+1)\hat{k}_1 x^q + k_2)\dot{x} + \frac{\hat{k}_1}{2}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^{q+1} + \lambda_1 x = 0, \tag{130}$$

and

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \hat{k}_1 x^{q+1}\right)e^{(\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2})t}, \tag{131}$$

where $k_1 = (q+1)\hat{k}_1$. Like in the earlier cases, that is, $q = 1$ and $q = 2$, we are able to integrate the first integral (131) explicitly only for a specific parametric restriction, namely, $\lambda_1 = (q+1)\hat{k}_2^2$, where $k_2 = (q+2)\hat{k}_2$. In this case the equation of motion (130) and the first integral, Eq. (131), can be recast in the form

$$\ddot{x} + (k_1 x^q + (q+2)\hat{k}_2)\dot{x} + k_1 \hat{k}_2 x^{q+1} + (q+1)\hat{k}_2^2 x = 0, \tag{12}$$

and

$$I = \left(\dot{x} + \hat{k}_2 x + \hat{k}_1 x^{q+1}\right)e^{(q+1)\hat{k}_2 t}, \tag{132}$$

respectively. Now comparing (132) with (50), we get

$$I = e^{q\hat{k}_2 t}\left(\frac{d}{dt}(xe^{\hat{k}_2 t})\right) + \hat{k}_1 (xe^{\hat{k}_2 t})^{(q+1)}. \tag{133}$$

Next identifying the dependent and independent variables from (133) using the relations (51), we obtain the transformation

$$w = xe^{\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}. \tag{134}$$

Using the transformation (134) the first integral (133) can be rewritten in the form

$$I = w' + \hat{k}_1 w^{(q+1)} \tag{135}$$

which in turn leads to the solution by an integration, that is,

$$z - z_0 = \int \frac{dw}{I - \hat{k}_1 w^{(q+1)}}, \tag{136}$$

where z_0 is an arbitrary constant. Solving Eq. (136) we get[24]

$$z - z_0 = \frac{1}{Ig^{\frac{1}{(q+1)}}} \begin{cases} -\frac{2}{q+1}\sum_{i=0}^{\frac{q-1}{2}} P_i \cos\frac{2i}{q+1}\pi + \frac{2}{q+1}\sum_{i=0}^{\frac{q-1}{2}} Q_i \sin\frac{2i}{q+1}\pi \\ +\frac{1}{q+1} ln\frac{(1+w)}{(1-w)}, \quad \text{q-a positive odd number,} \\ -\frac{2}{q+1}\sum_{i=0}^{\frac{q-2}{2}} R_i \cos\frac{2i+1}{q+1}\pi + \frac{2}{q+1}\sum_{i=0}^{\frac{q-2}{2}} T_i \sin\frac{2i+1}{q+1}\pi \\ +\frac{1}{q+1} ln(1+w), \quad \text{q-a positive even number,} \end{cases} \tag{137}$$

where $g = \frac{\hat{k}_1}{I}$ and

$$P_i = \frac{1}{2}ln\left(w^2 - 2w\cos\frac{2i}{q+1}\pi + 1\right), \quad Q_i = arctan\left[\frac{w - \cos\frac{2i}{q+1}\pi}{\sin\frac{2i}{q+1}\pi}\right],$$

$$R_i = \frac{1}{2}ln\left(w^2 + 2w\cos\frac{2i+1}{q+1}\pi + 1\right), \quad T_i = arctan\left[\frac{w + \cos\frac{2i+1}{q+1}\pi}{\sin\frac{2i+1}{q+1}\pi}\right].$$

Rewriting w and z in terms of old variables one can get the explicit solution.

Case (*iii*): $k_1, k_3 = 0, \ \lambda_1 = \frac{2(q+2)k_2^2}{(q+4)^2}, \ k_2$ and k_4 : arbitrary

The parametric choice given above fixes the equation of motion of the form

$$\ddot{x} + (q+4)\hat{k}_2\dot{x} + k_4 x^{(q+1)} + 2(q+2)\hat{k}_2^2 x = 0, \tag{14}$$

where $k_2 = (q+4)\hat{k}_2$. Rewriting the first integral I given in Case (*iii*) in Table III, in the form (49), we get

$$I = \frac{1}{2}\left(\dot{x} + 2\hat{k}_2 x\right)^2 e^{2(q+2)\hat{k}_2 t} + \frac{k_4 x^{(q+2)}}{(q+2)} e^{2(q+2)\hat{k}_2 t}. \tag{138}$$

Now splitting the first term in Eq. (138) further in the form of (50),

$$I = \left[e^{q\hat{k}_2 t}\frac{d}{dt}\left(\frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}\right)\right]^2 + \frac{2^{(\frac{q+2}{2})}k_4}{(q+2)}\left(\frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}\right)^{(q+2)} \tag{139}$$

and identifying the dependent and independent variables from (139) using the relations (51), we obtain the transformation

$$w = \frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}. \tag{140}$$

Using the transformation (140) the first integral (138) can be brought to the form

$$I = w'^2 + \frac{2^{(\frac{q+2}{2})}k_4}{(q+2)}w^{(q+2)}. \tag{141}$$

Separating the dependent and independent variables and integrating the resultant equation we get

$$z - z_0 = \int \frac{dw}{\sqrt{I - \hat{k}_4 w^{(q+2)}}}, \tag{142}$$

where $\hat{k}_4 = \frac{2^{(\frac{q+2}{2})}}{(q+2)}k_4$ and z_0 is an arbitrary constant.

Case (iv): $k_3 = \frac{(r-1)k_1^2}{(q+1)r^2},\ k_4 = \frac{k_1k_2}{(q+2)},\ \lambda_1 = \frac{(q+1)k_2^2}{(q+2)^2},\ k_1,\ k_2$ and r : arbitrary:

The equation of motion in this case turns out to be

$$\ddot{x} + ((q+1)\hat{k}_1x^q + (q+2)\hat{k}_2)\dot{x} + (q+1)(\frac{(r-1)}{r^2}\hat{k}_1^2x^{2q} + \hat{k}_1\hat{k}_2x^q + \hat{k}_2^2)x = 0, \quad r \neq 0 \tag{16}$$

where $k_1 = (q+1)\hat{k}_1,\ k_2 = (q+2)\hat{k}_2$. Rewriting the associated first integral I, given in Case (iv) in Table III, in the form (50), we get

$$I = \begin{cases} \left(\frac{(r-1)\hat{k}_1^2}{r^2}(xe^{\hat{k}_2t})^{2(q+1)} + \frac{d}{dt}(xe^{\hat{k}_2t})\left(\frac{d}{dt}(xe^{\hat{k}_2t})e^{q\hat{k}_2t} + \hat{k}_1(xe^{\hat{k}_2t})^{q+1}\right)e^{q\hat{k}_2t}\right), \\ \quad \times\left(\frac{d}{dt}(xe^{\hat{k}_2t})e^{q\hat{k}_2t} + \frac{\hat{k}_1(r-1)}{r}(xe^{\hat{k}_2t})^{q+1}\right)^{-r}, & r \neq 0,2 \\ \frac{\frac{d}{dt}(xe^{\hat{k}_2t})e^{q\hat{k}_2t}}{\frac{\hat{k}_1}{2}(xe^{\hat{k}_2t})^{q+1} + \frac{d}{dt}(xe^{\hat{k}_2t})e^{q\hat{k}_2t}} - \log(\frac{\hat{k}_1}{2}(xe^{\hat{k}_2t})^{q+1} + \frac{d}{dt}(xe^{\hat{k}_2t})e^{q\hat{k}_2t}), & r = 2 \\ \frac{1}{2}\left(\frac{d}{dt}(xe^{\hat{k}_2t})\right)^2 e^{2qk_2t} + \frac{k_3}{2(q+1)}(xe^{\hat{k}_2t})^{2(q+1)}, & r = 0. \end{cases} \tag{143}$$

Identifying the dependent and independent variables from (143) and the relations (51), we obtain the transformation

$$w = xe^{\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}. \tag{144}$$

Subsituting the transformation (144) into (16), one obtains

$$w'' + (q+1)\hat{k}_1 w^q w' + (q+1)\frac{(r-1)}{r^2}\hat{k}_1^2 w^{2q+1} = 0, \;\; r \neq 0, \quad ' = \frac{d}{dz}. \tag{145}$$

In terms of the new variables (144) change the time dependent first integral into time independent ones of the form

$$I = \begin{cases} \left(w' + \frac{(r-1)}{r}\hat{k}_1 w^{q+1}\right)^{-r}\left[w'(w' + \hat{k}_1 w^{q+1}) + \frac{(r-1)}{r^2}\hat{k}_1^2 w^{2(q+1)}\right], & r \neq 0, 2, \\ \frac{w'}{w' + \frac{\hat{k}_1}{2}x^{q+1}} - \log(w' + \frac{\hat{k}_1}{2}w^{q+1}), & r = 2, \\ \frac{w'^2}{2} + \frac{k_3}{2(q+1)}w^{2(q+1)}, & r = 0. \end{cases} \tag{146}$$

Once again one can deduce the Hamiltonians in the form

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}\hat{k}_1 w^{q+1} p\right], & r \neq 0, 1, 2, \\ \frac{1}{2}\hat{k}_1 w^{q+1} p + \log(\frac{2(q+1)}{p}), & r = 2, \\ e^p + \hat{k}_1 w^{q+1}, & r = 1, \\ \frac{p^2}{2} + \frac{k_3}{2(q+1)}w^{2(q+1)}, r = 0, \end{cases} \tag{147}$$

with

$$p = \begin{cases} \frac{1}{(r-1)}\left(w' + \frac{(r-1)}{r}\hat{k}_1 w^{q+1}\right)^{(1-r)}, & r \neq 0, 1 \\ \log(w'), & r = 1 \\ w', & r = 0, \end{cases} \tag{148}$$

and thereby ensuring liouville integrability of Eq. (16).

C. Summary of results in $q=$ *arbitrary* case:

To conclude the integrability of Eq. (9), we have established the fact that the following equations, are integrable

$$(1) \qquad \ddot{x}+(k_1x^q+(q+2)k_2)\dot{x}+k_1k_2x^{q+1}+(q+1)k_2^2x=0, \tag{12}$$

$$(2) \qquad \ddot{x}+((q+2)k_1x^q+k_2)\dot{x}+k_1^2x^{2q+1}+k_1k_2x^{q+1}+\lambda_1x=0, \tag{13}$$

$$(3) \qquad \ddot{x}+(q+4)k_2\dot{x}+k_4x^{q+1}+2(q+2)k_2^2x=0, \tag{14}$$

$$(4) \qquad \ddot{x}+((q+1)k_1x^q+k_2)\dot{x}+\frac{(r-1)}{r^2}((q+1)k_1^2x^{2q} +(q+2)k_1k_2x^q+k_2^2)x=0,\ r\neq 0 \tag{15}$$

$$(5) \qquad \ddot{x}+((q+1)k_1x^q+(q+2)k_2)\dot{x}+(q+1)(k_3x^{2q}+k_1k_2x^q+k_2^2)x=0, \tag{16}$$

where $r^2k_3=(r-1)k_1^2$ and $k_1,\ k_2,\ k_4,\ \lambda_1$ and r are arbitrary parameters (for simplicity we have removed hats in k_i's, $i=1,2$, in Eqs. (12)-(16)) . The significance and newness of the equations (12)-(16) are already pointed out in Sec. I B.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have investigated the integrability properties of Eq. (9) and shown that it admits a large class of integrable nonlinear systems. In fact, many classical integrable nonlinear oscillators can be derived as sub-cases of our results. One of the important outcomes of our investigation is that the entire class of Eq. (6) can be derived from a conservative Hamiltonian (vide Eq. (123)) eventhough the system deceptively looks like a dissipative equation.

From our detailed analysis we have shown that Eq. (9) admits both conservative Hamiltonian systems and dissipative systems, depending on the choice of parameters. As far as the former is concerned we have deduced the explicit forms of the Hamiltonians for the respective equations. In fact, for the case, $q=1$, we have constructed suitable canonical transformations and transformed the equations into conservative nonlinear oscillator equations. However, the canonical transformations for the conservative Hamiltonian systems for the cases $q=2,\ldots,$ arbitrary, if at all they exist, still remain to be obtained. Exploring the classical dynamics underlying these conservative Hamiltonian systems is also of considerable interest for further study. As far as dissipative systems are concerned we have not only shown that Eq. (9) contains the well known force-free Helmholtz, Duffing and Duffing-van der Pol oscillators but also have several integrable generalizations which is another important outcome of our investigations. The study of chaotic dynamics of these nonlinear oscillators under further perturbations is one of the current topics[22] in the contemporary literature in nonlinear dynamics. In principle one can extend such analysis to the above generalized equations as well.

In this paper, we have also not touched the question of linearizability of the integrable cases of Eq. (9). In our earlier work, we have shown that the Eq. (55) is linearizable to the free particle equation, $\frac{d^2w}{dz^2} = 0$. Of course one can show that this is the only linearizable equation in (9) through invertible point transformation[9,11,18]. However, linearizablity of other integrable cases through more general transformations still remains to be explored.

In addition to the above, we have also carried out the Painlevé singularity structure analysis of Eq. (9) and compared the results obtained through both the methods. The details of this will be published elsewhere.

As we mentioned at the end of Sec. II, the crux of the PS procedure lies in finding the explicit solutions satisfying all the three determining Eqs. (22)-(24). In this paper we have considered only certain specific ansatz forms to determine the null forms S, and integrating factors R. As a consequence only a specific class of integrable equations have been derived. It is not clear, whether these ansatz forms used in this paper exhaust all possible integrable cases of Eq. (9). One needs to consider more generalized ansatz forms, and if possible to solve Eqs. (22)-(24) for the most general forms of R and S, and try to identify all possible integrable cases underlying Eq. (9). This is being explored further.

VII. ACKNOWLEDGMENTS

The work of VKC is supported by CSIR in the form of a Senior Research Fellowship. The work of MS and ML forms part of a Department of Science and Technology, Government of India, sponsored research project.

[1] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Phys. A **37**, 4527 (2004).

[2] S. N. Pandey, M. Senthilvelan and M. Lakshmanan, *Classification of Lie point symmetries of nonlinear dissipative system of the type* $\ddot{x} + f(x)\dot{x} + g(x) = 0$ preprint, to be submitted for publication.

[3] M. R. Feix, C. Geronimi, L. Cairo, P. G. L. Leach, R. L. Lemmer and S. Bouquet, J. Phys. A **30**, 7437 (1997).

[4] R. A. Smith, J. London Math. Soc. **36**, 33 (1961).

[5] M. Prelle and M. Singer, Trans. Am. Math. Soc. **279**, 215 (1983).

[6] Y. K. Man and M. A. H. MacCallum, J. Symbolic Computation **11**, 1 (1996).

[7] L. G. S. Duarte, S. E. S. Duarte, A. C. P. da Mota and J. E. F. Skea, J. Phys. A **34**, 3015 (2001).

[8] L. G. S. Duarte, S. E. S. Duarte and A. C. P. da Mota, J. Phys. A **35**, 1001 (2002); **35**, 3899 (2002).

[9] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Proc. R. Soc. London **A461**, 2451 (2005).

[10] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Nonlinear Math. Phys. **12**, 184 (2005).

[11] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Chaos, Solitons and Fractals **26**, 1399 (2005).

[12] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics: Integrability Chaos and Patterns* (Springer-Verlag, New York, 2003).

[13] J. A. Almendral and M. A. F. Sanjuán, J. Phys. A **36**, 695 (2003).

[14] S. Parthasarathy and M. Lakshmanan, J. Sound and Vib. **137**, 523 (1990).

[15] G. W. Bluman and S. C. Anco, *Symmetries and Integration Methods for Differential Equations* (Springer-Verlag, New York, 2002).

[16] T. C. Bountis, L. B. Drossos, M. Lakshmanan and S. Parthasarathy J. Phys. A **26**, 6927 (1993).

[17] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *On the complete integrability and linearization of nonlinear ordinary differential equations- part II: third order equations* submitted to Proc. R. Soc. London A.

[18]F. M. Mahomed and P. G. L. Leach, Quaestiones Math. **8**, 241 (1985); **12** 121 (1985); P. G. L. Leach, M. R. Feix and S. Bouquet, J. Phys. A **29**, 2563 (1988); P. G. L. Leach, J. Math. Phys. **26**, 2510 (1985); R. L. Lemmer and P. G. L. Leach, J. Phys. A **26**, 5017 (1993).

[19]V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, arXiv:nlin. SI/0408054 submitted to Phys. Rev. E.

[20]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

[21]G. M. Murphy, *Ordinary Differential Equations and Their Solutions* (Affiliated East-west press, New Delhi, 1969).

[22]D. L. Gonzalez and O. Piro, Phys. Rev. Lett. **50**, 870 (1983); Phys. Rev. **A30**, 2788 (1983).

[23]M. Lakshmanan and J. Prabhakaran, 1973 Lettere al Nuovo Cimento **7**, 689 (1973); M. Lakshmanan, Lettere al Nuovo Cimento **8**, 743 (1973).

[24]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic press, London, 1980).

12 ALGEBRAIC PROPERTIES OF FIRST INTEGRALS FOR SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS OF MAXIMAL SYMMETRY. By A. Aslam, K.S. Mahomed and E. Momoniat. (2016)

Quaestiones Mathematicae 2016: 1–12.
© 2016 NISC (Pty) Ltd
http://dx.doi.org/10.2989/16073606.2015.1134698

This is the final version of the article that is published ahead of the print and online issue

ALGEBRAIC PROPERTIES OF FIRST INTEGRALS FOR SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS OF MAXIMAL SYMMETRY

A. Aslam

DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa, and
Department of Basic Sciences, School of Electrical Engineering and Computer Science, National University of Sciences and Technology, Campus H-12, 44000 Islamabad, Pakistan.
E-Mail adnangrewal@yahoo.com

K.S. Mahomed and E. Momoniat

DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa.
E-Mail komal.mahomed@wits.ac.za ebrahim.momoniat@wits.ac.za

Abstract. Symmetries of the first integrals for scalar linear or linearizable second-order ordinary differential equations (ODEs) have already been derived and shown to exhibit interesting properties. One of these is that the symmetry algebra $sl(3,\mathbb{R})$ is generated by the three triplets of symmetries of the functionally independent first integrals and its quotient. In this paper, we first investigate the Lie-like operators of the basic first integrals for the linearizable maximally symmetric system of two second-order ODEs represented by the free particle system, obtainable from a complex scalar free particle equation, by splitting the corresponding complex basic first integrals and its quotient as well as their associated symmetries. It is proved that the 14 Lie-like operators corresponding to the complex split of the symmetries of the functionally independent first integrals I_1, I_2 and their quotient I_2/I_1 are precisely the Lie-like operators corresponding to the complex split of the symmetries of the scalar free particle equation in the complex domain. Then, it is shown that there are distinguished four symmetries of each of the four basic integrals and their quotients of the two-dimensional free particle system which constitute four-dimensional Lie algebras which are isomorphic to each other and generate the full symmetry algebra $sl(4,\mathbb{R})$ of the free particle system. It is further shown that the $(n+2)$-dimensional algebras of the $n+2$ first integrals of the system of n free particle equations are isomorphic to each other and generate the full symmetry algebra $sl(n+2,\mathbb{R})$ of the free particle system.

Mathematics Subject Classification (2010): 34C14, 34C20, 22E70.

Key words: Systems, maximal symmetry algebra, first integral, complex symmetry, linearization.

1. Introduction. First integrals and integrating factors play a central role in the study of ODEs. In fact, finding a first integral of a given ODE is equivalent to obtaining an integrating factor of the equation. For first-order ODEs, Lie [12] showed how to construct an integrating factor from each admitted point symmetry. Conversely, Lie showed that each integrating factor yields an admitted point symmetry of the equation. In general, for a scalar or systems of ODEs, an integrating factor is a set of functions, multiplying each of the ODEs, which yields a first integral. For a first-order scalar ODE, a first integral is a quadrature. For an nth-order ($n > 1$) scalar ODE, a first integral is an expression relating the independent variable, the dependent variable and derivatives up to order $n-1$, which is constant for all solutions of the ODE. It is well-known that two independent first integrals of a second-order ODE provide the general solution of the equation. First integrals provide means for reducing the order of an ODE, see e.g. [6, 7, 10, 21]. The symmetry of the Emden equation and its first integrals have been discussed in [22] amongst other works. The possible reductions in order for nonlinear ODEs with first integrals and Lie group of symmetries are widely discussed [8]. For a scalar linear second-order ODE represented by the free particle equation, symmetries of three first integrals have been studied in [11] and it was shown there that the three triplets of first integrals have isomorphic algebras and generate the complete symmetry group of the equation itself. Recently, a complete symmetry classification of the first integrals of a scalar linearizable second-order ODE was derived in [13]. First integrals for second-order ODEs are also studied in [17, 18, 19, 20]. In the work [15] a complete symmetry classification of the first integrals for scalar second-order ODEs was studied.

A method, known as complex symmetry analysis, was established in [1, 2]. This method provides a connection between a complex ODE or partial differential equation (PDE) and a system of ODEs/PDEs by splitting the base complex equation into real and imaginary parts.

This paper is organized as follows. A brief review of the mathematical formalism is given in the next section. In the subsequent section we present the relationship between the Lie-like operators and first integrals of the system of two ODEs of maximal symmetry by the complex method. In the fourth section we present the symmetries of the basic first integrals of the free particle system and show how these are utilized to generate the full symmetry algebra of the system. A summary and brief discussion are given in the last section.

2. Preliminaries. It is well-known that the system of second order ODEs

$$E_i(t,x,x',x'',y,y',y'')=0, \qquad i=1,2 \tag{1}$$

is invariant under the infinitesimal generator

$$\mathbf{X}=\xi(t,x,y)\frac{\partial}{\partial t}+\mu(t,x,y)\frac{\partial}{\partial x}+\nu(t,x,y)\frac{\partial}{\partial y} \tag{2}$$

if and only if

$$\mathbf{X}^{[2]}\left(E_i\right)|_{E_i=0} = 0, \qquad i = 1, 2, \tag{3}$$

where

$$\mathbf{X}^{[2]} = \mathbf{X} + \mu_t^{(1)}\frac{\partial}{\partial x'} + \mu_{tt}^{(2)}\frac{\partial}{\partial x''} + \nu_t^{(1)}\frac{\partial}{\partial y'} + \nu_{tt}^{(2)}\frac{\partial}{\partial y''}, \tag{4}$$

with

$$\mu_t^{(1)} = D_t\mu - x' D_t\xi \tag{5}$$

$$\mu_{tt}^{(2)} = D_t\mu_t^{(1)} - x'' D_t\xi \tag{6}$$

$$\nu_t^{(1)} = D_t\nu - y' D_t\xi \tag{7}$$

$$\nu_{tt}^{(2)} = D_t\nu_t^{(1)} - y'' D_t\xi \tag{8}$$

in which D_t is the total differentiation operator, is called the second prolongation of the generator $\mathbf{X}$.

It is the case that (2) is the point symmetry of the system (1), whereas for first integrals, the first integral given by

$$I = f(t, x, y, x', y'), \tag{9}$$

of the system of ODEs (1), is an invariant of $\mathbf{X}$, that is, (2) is the symmetry generator of (9) if and only if

$$\mathbf{X}^{[1]} I = 0. \tag{10}$$

Here $\mathbf{X}$ is said to annihilate I and does not leave it invariant as for symmetries of the equations. Therefore, the procedure for determining symmetries of ODEs is different to that of finding symmetries of their first integrals. As a matter of fact the symmetries of a first integral constitute a subalgebra of the symmetries of the equation(s) itself/themselves that give(s) rise to it as is known from the work [9]. It is important to state that the symmetries of the first integral are not in general the symmetries of the system. We clarify this by means of the example of the Ermakov-Lewis invariant given in [5]. The operator [5]

$$X_a = a(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{a}r\frac{\partial}{\partial r} \tag{11}$$

is a symmetry of the Ermakov-Lewis invariant

$$I = \frac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta)d\theta. \tag{12}$$

This means that I as in (12) has an infinite-dimensional Lie algebra as there is no restriction on a (see [5]). The Ermakov system in plane polar coordinates is

$$\ddot{r} - r\dot{\theta}^2 = F(\theta)/r^3, \tag{13}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = G(\theta)/r^3. \tag{14}$$

The second of these equations give rise to the integral (12) and the infinite symmetries generated by X_a in (11) are (by [9]) symmetries of the equation (14). This can easily be verified. However, it is important to stress that the infinite symmetry algebra of (12) is not the symmetry algebra of the Ermakov system above.

We study the symmetries of the first integrals of second-order linearizable systems which generate the full algebra of the system.

We consider only those classes of the ODE system (1) which are linearizable via point transformations to the simplest system, viz. the free particle system. These are characterized by the class which is cubic in the first derivatives as given in [16].

Thus here we only focus on those systems of second-order ODEs which are reducible to the free particle system by point transformation. In the works [4, 11, 14] the generation and characterization of algebraic properties of scalar second- and higher-order ODEs of maximal symmetries were investigated. In [4], the number of symmetries associated with the ratio of any two linear first integrals of an nth-order scalar ODE is shown to be $n-1$, where $n \geq 4$ (see Proposition 2). The work [14] gives the generation result (see Theorem 4) that the full Lie algebra of the nth-order ODE $y^{(n)} = 0, \quad n \geq 3$ is generated by two subalgebras: an $(n+1)$-dimensional algebra and a three-dimensional subalgebra. For systems of second-order ODEs, we see that the results are quite different to that of the scalar higher-order ODEs (see Proposition 3 in Section 4). However, there is a pattern which is similar to scalar second-order ODEs as in [11] and here we have a natural extension for systems. It is also the case that in this work we also pursue the complex method and Lie-like operators which have interesting consequences as encapsulated in Proposition 1 in Section 3.

We now consider the scalar free particle equation

$$u'' = 0, \quad ' = d/dt. \tag{15}$$

It is well-known that (15) has eight point symmetries which are

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{\partial}{\partial u}, \quad \mathbf{X}_3 = t\frac{\partial}{\partial t}, \\
&\mathbf{X}_4 = u\frac{\partial}{\partial u}, \quad \mathbf{X}_5 = t\frac{\partial}{\partial u}, \quad \mathbf{X}_6 = u\frac{\partial}{\partial t}, \\
&\mathbf{X}_7 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}, \quad \mathbf{X}_8 = tu\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}.
\end{aligned} \tag{16}$$

The free particle equation (15) has two functionally independent first integrals

$$\mathbf{I}_1 = u', \tag{17}$$
$$\mathbf{I}_2 = tu' - u. \tag{18}$$

The first integral (17) has the following symmetries [11]

$$\mathbf{X}_1 = \tfrac{\partial}{\partial t}, \quad \mathbf{X}_2 = \tfrac{\partial}{\partial u}, \quad \mathbf{X}_3 = t\tfrac{\partial}{\partial t} + u\tfrac{\partial}{\partial u}, \tag{19}$$

and the first integral (18) has the symmetries [11]

$$\mathbf{Y}_1 = t\tfrac{\partial}{\partial t}, \quad \mathbf{Y}_2 = t\tfrac{\partial}{\partial u}, \quad \mathbf{Y}_3 = t^2\tfrac{\partial}{\partial t} + tu\tfrac{\partial}{\partial u}. \tag{20}$$

The quotient of both of the first integrals (17) and (18), that is

$$\mathbf{I}_3 = \frac{\mathbf{I}_2}{\mathbf{I}_1} = t - \frac{u}{u'}, \tag{21}$$

is also a first integral and has the following three symmetries [11]

$$\mathbf{Z}_1 = u\tfrac{\partial}{\partial t}, \quad \mathbf{Z}_2 = u\tfrac{\partial}{\partial u}, \quad \mathbf{Z}_3 = tu\tfrac{\partial}{\partial t} + u^2\tfrac{\partial}{\partial u}. \tag{22}$$

An important result was proved in [11] that the three triplets of symmetries (as given above) have isomorphic algebras, denoted as $L^I_{3;5}$ (in the classification of first integrals presented in [13]), which generate the complete algebra $sl(3,\mathbb{R})$ of the free particle equation (15) or linearizable scalar ODEs.

In this paper, inter alia, we focus our attention on systems of ODEs of maximal symmetry which have the property in that the algebras of their integrals generate their full algebra. This is investigated in Section 4.

The system of two free particle equations, as is well-known as well,

$$\begin{aligned} x'' &= 0, \\ y'' &= 0, \end{aligned} \tag{23}$$

has the fifteen point symmetries, which are

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = \frac{\partial}{\partial y}, \\
&\mathbf{X}_4 = t\frac{\partial}{\partial t}, \quad \mathbf{X}_5 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_6 = y\frac{\partial}{\partial y}, \\
&\mathbf{X}_7 = t\frac{\partial}{\partial x}, \quad \mathbf{X}_8 = t\frac{\partial}{\partial y}, \quad \mathbf{X}_9 = x\frac{\partial}{\partial t}, \\
&\mathbf{X}_{10} = y\frac{\partial}{\partial t}, \quad \mathbf{X}_{11} = y\frac{\partial}{\partial x}, \quad \mathbf{X}_{12} = x\frac{\partial}{\partial y}, \\
&\mathbf{X}_{13} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}, \\
&\mathbf{X}_{14} = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \\
&\mathbf{X}_{15} = ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.
\end{aligned} \tag{24}$$

These symmetries (24) constitute the Lie algebra $sl(4,\mathbb{R})$. We show how these or a subset of these symmetries arise in the way the symmetries of the first integrals of the free particle system or the complex split of the complex symmetries of the complex first integrals of the scalar complex free particle equation are used.

3. First integral of the system of two ODEs by the complex method. In this section, we apply the complex symmetry analysis on the scalar complex free particle ODE (15) and its basic first integrals to obtain a system of two ODEs and the first integrals for the free particle system. The scalar second-order free particle ODE (15) is split into a system of two second-order free particle ODEs (23) (as is known [2]) by taking

$$u = x + iy, \tag{25}$$

and the corresponding eight symmetries (16) are transformed into 14 Lie-like operators, which are written as

$$\begin{aligned}
\mathbf{X}_1 &= \tfrac{\partial}{\partial t},\\
\mathbf{X}_2 &= \tfrac{\partial}{\partial x},\\
\mathbf{X}_3 &= \tfrac{\partial}{\partial y},\\
\mathbf{X}_4 &= t\tfrac{\partial}{\partial t},\\
\mathbf{X}_5 &= x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y},\\
\mathbf{X}_6 &= y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y},\\
\mathbf{X}_7 &= t\tfrac{\partial}{\partial x},\\
\mathbf{X}_8 &= t\tfrac{\partial}{\partial y},\\
\mathbf{X}_9 &= x\tfrac{\partial}{\partial t},\\
\mathbf{X}_{10} &= y\tfrac{\partial}{\partial t},\\
\mathbf{X}_{11} &= t^2\tfrac{\partial}{\partial t} + \tfrac{1}{2}tx\tfrac{\partial}{\partial x} + \tfrac{1}{2}ty\tfrac{\partial}{\partial y},\\
\mathbf{X}_{12} &= ty\tfrac{\partial}{\partial x} - tx\tfrac{\partial}{\partial y},\\
\mathbf{X}_{13} &= tx\tfrac{\partial}{\partial t} + \tfrac{1}{2}(x^2 - y^2)\tfrac{\partial}{\partial x} + xy\tfrac{\partial}{\partial y},\\
\mathbf{X}_{14} &= ty\tfrac{\partial}{\partial t} + xy\tfrac{\partial}{\partial x} - \tfrac{1}{2}(x^2 - y^2)\tfrac{\partial}{\partial y}.
\end{aligned} \tag{26}$$

All these 14 Lie-like operators are not the Lie point symmetries of the free particle system (see also discussion on similar systems in [1, 2]). Only $\mathbf{X}_1$ to $\mathbf{X}_{10}$ are the Lie point symmetries. Under the transformation (25), the first integral (17) splits into the following two first integrals

$$\mathbf{I}_{11} = x', \tag{27}$$

$$\mathbf{I}_{12} = y', \tag{28}$$

and the corresponding three symmetries (19) split into five symmetries

$$\begin{aligned}
\mathbf{X}_1 &= \tfrac{\partial}{\partial t},\\
\mathbf{X}_2 &= \tfrac{\partial}{\partial x},\\
\mathbf{X}_3 &= \tfrac{\partial}{\partial y},\\
\mathbf{X}_4 &= t\tfrac{\partial}{\partial t} + \tfrac{1}{2}x\tfrac{\partial}{\partial x} + \tfrac{1}{2}y\tfrac{\partial}{\partial y},\\
\mathbf{X}_5 &= y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}.
\end{aligned} \tag{29}$$

The first integral (18) splits into the two first integrals, under the complex transformation (25), written as

$$\mathbf{I}_{21} = tx' - x, \tag{30}$$
$$\mathbf{I}_{22} = ty' - y, \tag{31}$$

and we have the five operators after splitting the corresponding three symmetries (20), as

$$\begin{aligned}
\mathbf{X}_1 &= t\tfrac{\partial}{\partial t},\\
\mathbf{X}_2 &= t\tfrac{\partial}{\partial x},\\
\mathbf{X}_3 &= t\tfrac{\partial}{\partial y},\\
\mathbf{X}_4 &= t^2\tfrac{\partial}{\partial t} + \tfrac{1}{2}tx\tfrac{\partial}{\partial x} + \tfrac{1}{2}ty\tfrac{\partial}{\partial y},\\
\mathbf{X}_5 &= ty\tfrac{\partial}{\partial x} - tx\tfrac{\partial}{\partial y}.
\end{aligned} \tag{32}$$

Finally, by applying the transformation (25) and splitting the real and imaginary parts, the first integral (21) gives the following two first integrals

$$\mathbf{I}_{31} = t - \frac{xx' + yy'}{x'^2 + y'^2}, \tag{33}$$
$$\mathbf{I}_{32} = \frac{x'y - xy'}{x'^2 + y'^2}, \tag{34}$$

and in this case the three symmetries (22) split into the following six operators

$$\begin{aligned}
\mathbf{X}_1 &= x\tfrac{\partial}{\partial t},\\
\mathbf{X}_2 &= y\tfrac{\partial}{\partial t},\\
\mathbf{X}_3 &= x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y},\\
\mathbf{X}_4 &= y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y},\\
\mathbf{X}_5 &= tx\tfrac{\partial}{\partial t} + \tfrac{1}{2}(x^2 - y^2)\tfrac{\partial}{\partial x} + xy\tfrac{\partial}{\partial y},\\
\mathbf{X}_6 &= ty\tfrac{\partial}{\partial t} + xy\tfrac{\partial}{\partial x} - \tfrac{1}{2}(x^2 - y^2)\tfrac{\partial}{\partial y}.
\end{aligned} \tag{35}$$

We make the following deduction here. The complex split of the scalar free particle equation results in 14 Lie-like operators (only 10 are symmetries). Exactly the same occurs for the complex split of the complex symmetries of the basic first integrals of the complex free particle equation: 14 Lie-like operators arise as the rotation symmetry is repeated and the one scaling generator is a a linear combination of the two scaling operators.

We have therefore the following proposition.

Proposition 1. *The 14 Lie-like operators corresponding to the complex split of the symmetries of the functionally independent first integrals I_1, I_2 and their quotient I_2/I_1 are precisely the Lie-like operators corresponding to the complex split of the symmetries of the scalar free particle equation in the complex domain.*

4. Symmetries of the first integrals of a system of two ODEs of maximal symmetry. In this section, we discuss the symmetries of the basic first integrals and their quotients of a system of two ODEs of maximal symmetry represented by the free particle system and its relation to the complete symmetries of the system of free particle ODEs.

The first integral (27) has the following seven point symmetries

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = \frac{\partial}{\partial y},\\
&\mathbf{X}_4 = y\frac{\partial}{\partial y}, \quad \mathbf{X}_5 = t\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = x\frac{\partial}{\partial y},\\
&\mathbf{X}_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.
\end{aligned} \tag{36}$$

The first integral (28) possesses the seven symmetries

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = \frac{\partial}{\partial y},\\
&\mathbf{X}_4 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = t\frac{\partial}{\partial x}, \quad \mathbf{X}_6 = y\frac{\partial}{\partial x},\\
&\mathbf{X}_7 = t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y}.
\end{aligned} \tag{37}$$

The first integral (30) admits the following seven symmetries

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial y}, \quad \mathbf{X}_2 = t\frac{\partial}{\partial t}, \quad \mathbf{X}_3 = y\frac{\partial}{\partial y},\\
&\mathbf{X}_4 = t\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = t\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = x\frac{\partial}{\partial y},\\
&\mathbf{X}_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}.
\end{aligned} \tag{38}$$

The first integral (31) has the seven symmetries

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = t\frac{\partial}{\partial t}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x},\\
&\mathbf{X}_4 = t\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = t\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = y\frac{\partial}{\partial x},\\
&\mathbf{X}_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}.
\end{aligned} \tag{39}$$

The symmetries for the first integral $\frac{I_{21}}{I_{11}} = t - x/\acute{x}$ are

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial y}, \quad \mathbf{X}_2 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_3 = y\frac{\partial}{\partial y},\\
&\mathbf{X}_4 = t\frac{\partial}{\partial y}, \quad \mathbf{X}_5 = x\frac{\partial}{\partial t}, \quad \mathbf{X}_6 = x\frac{\partial}{\partial y},\\
&\mathbf{X}_7 = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.
\end{aligned} \tag{40}$$

The symmetries for the first integral $\frac{I_{22}}{I_{12}} = t - y/\acute{y}$ are given by

$$\begin{aligned}&\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_3 = y\frac{\partial}{\partial y},\\ &\mathbf{X}_4 = t\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = y\frac{\partial}{\partial t}, \quad \mathbf{X}_6 = y\frac{\partial}{\partial x},\\ &\mathbf{X}_7 = ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.\end{aligned} \tag{41}$$

Guided by the symmetries of the basic first integrals and its quotient of the scalar free particle equation as reviewed in Section 2, we have the following four-dimensional distinguished subalgebras of the seven-dimensional algebras generated by (36) to (41). These have basis vectors

$$\mathbf{Y}_{11} = \frac{\partial}{\partial t},\ \mathbf{Y}_{12} = \frac{\partial}{\partial x},\ \mathbf{Y}_{13} = \frac{\partial}{\partial y},\ \mathbf{Y}_{14} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{42}$$

$$\mathbf{Y}_{21} = t\frac{\partial}{\partial t},\ \mathbf{Y}_{22} = t\frac{\partial}{\partial x},\ \mathbf{Y}_{23} = t\frac{\partial}{\partial y},\ \mathbf{Y}_{24} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}, \tag{43}$$

$$\mathbf{Y}_{31} = x\frac{\partial}{\partial t},\ \mathbf{Y}_{32} = x\frac{\partial}{\partial x},\ \mathbf{Y}_{33} = x\frac{\partial}{\partial y},\ \mathbf{Y}_{34} = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{44}$$

and

$$\mathbf{Y}_{41} = y\frac{\partial}{\partial t},\ \mathbf{Y}_{42} = y\frac{\partial}{\partial x},\ \mathbf{Y}_{43} = y\frac{\partial}{\partial y},\ \mathbf{Y}_{44} = ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{45}$$

The operators (42) come from (36) and (37). Those of (43) arise from (38) and (39). Finally the generators (44) and (45) result from (40) and (41), respectively. We also observe that these symmetries are the symmetries of the free particle system as given in (24) with the proviso that the scaling symmetries are linearly dependent, e.g. $\mathbf{Y}_{14} = \mathbf{Y}_{21} + \mathbf{Y}_{32} + \mathbf{Y}_{43}$. The Lie algebras of the operators (42) to (45) are isomorphic to each other as can be seen by simple changes of bases for each case. The algebra is a dilation algebra (similar to that which occurs for the symmetry algebra of the basic integrals and its quotient of the scalar free particle equation [11] except here we have a four-dimensional algebra) with commutation relations

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0,\ [X_i, X_4] = X_i,\ i = 1, 2, 3. \tag{46}$$

in appropriate basis. We denote this algebra by D_4. It is interesting to point out that in [23] precisely this algebra implies with the representation (42) implies linearization for a nonlinear system that admits it and the construction of the transformation reduces the system to the free particle system.

We thus have the following proposition.

Proposition 2. *The four quadruplets of symmetries (42) to (45) of the basic integrals $I_1 = x'$, $I_2 = tx' - x$ and the quotients $J_1 = t - x/x'$ and $J_2 = t - y/y'$ which have Lie algebra isomorphic to the algebra D_4, generate the full symmetry* *algebra $sl(4,\mathbb{R})$ of the two-dimensional free particle system.*

The above discussion is easily extendible to n-dimensional free particle systems in that now one has that the $n+2$ basic integrals $I_i = x_i'$ and $I_{i+n} = tx_i' - x_i$, for any fixed $i = 1, \ldots, n$ and their quotients $J_i = I_{i+n}/I_i$, $i = 1, \ldots, n$ of the n-dimensional free particle system each has $n+2$ symmetries given by

$$\mathbf{Y}_{11} = \frac{\partial}{\partial t},\ \mathbf{Y}_{1k+1} = \frac{\partial}{\partial x^k},\ k = 1, \ldots, n, \mathbf{Y}_{1n+2} = t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad (47)$$

$$\mathbf{Y}_{21} = t\frac{\partial}{\partial t},\ \mathbf{Y}_{2k+1} = t\frac{\partial}{\partial x^k},\ k = 1, \ldots, n, \mathbf{Y}_{2n+2} = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i}, \quad (48)$$

and

$$\mathbf{Y}_{j+21} = x^j\frac{\partial}{\partial t},\ \mathbf{Y}_{j+2k+1} = x^j\frac{\partial}{\partial x^k},\ k = 1, \ldots, n,$$
$$\mathbf{Y}_{j+2n+2} = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i},\ j = 1, \ldots, n. \quad (49)$$

The symmetries of the n-dimensional $(n \geq 2)$ free particle system comprise of the $n+1$ translations, the $n+1$ dilations, the n solution symmetries, the n symmetries of type $x^i\partial/\partial t$, the $n(n+1)$ symmetries of mixed type in the space variables $x^i\partial/\partial x^j$ and the $n+1$ true projective symmetries $t^2\partial/\partial t + tx^i\partial/\partial x^i$ and $tx^i\partial/\partial t + x^jx^i\partial/\partial x^i$ making a total of n^2+4n+3 symmetries which constitute the symmetry algebra $sl(n+2,\mathbb{R})$. We see that the symmetries (47) to (49) are symmetries of the n-dimensional free particle system except for the linear dependency as previously noted of the dilations. The Lie algebra of the symmetries (47) to (49) are isomorphic to each other. They form the dilation algebra D_{n+2}, $n \geq 2$.

We hence have the following proposition.

Proposition 3. *The $(n+2)^2$ symmetries (47) to (49) of the basic integrals $I_i = x_i'$ and $I_{i+n} = tx_i' - x_i$, for any fixed $i = 1, \ldots, n$ and their quotients $J_i = I_{i+n}/I_i$, $i = 1, \ldots, n$ which have Lie algebra isomorphic to the algebra D_{n+2}, generate the full symmetry algebra $sl(n+2,\mathbb{R})$ of the n-dimensional $(n \geq 2)$ free particle system.*

We remark that the algebra D_{n+2} which has representation given by (47) is precisely the algebra that yields linearization for an n-dimensional $(n \geq 2)$ system which admits it as a symmetry algebra. This is investigated in the work [3]. Also one gets linearization to the free particle system in this case.

5. Conclusion. Symmetries of the fundamental first integrals for a scalar linear or linearizable second-order ordinary differential equation have already been derived and display interesting properties. One of these is that the three isomorphic triplets of symmetries of the integrals generate the full symmetry algebra $sl(3,\mathbb{R})$ of the equation itself. In this paper, we firstly apply complex symmetry analysis on the scalar complex second-order linearizable ODE represented by the free particle equation. We have shown that the 14 Lie-like operators corresponding to the complex split of the symmetries of the basic functionally independent first integrals and its quotient of the free particle equation are precisely the Lie-like operators corresponding to the complex split of the symmetries of the scalar free particle equation in the complex domain. We also have proved that certain $(n+2)$-dimensional $(n \geq 2)$ subalgebras of the symmetry algebra of the basic first integrals and their quotients of the n-dimensional free particle system together generate the full algebra of symmetries of the free particle system which is $sl(n+2, \mathbb{R})$.

Acknowledgments. A. Aslam is grateful to Wits for the hospitality and financial support for his stay during which time this work was commenced, and to NUST for travel support. K.S. Mahomed is grateful to the University of the Witwatersrand and the NRF of South Africa for funding through a postdoctoral innovation fellowship. E. Momoniat thanks the NRF for a research grant support.

REFERENCES

1. S. ALI, F.M. MAHOMED AND A. QADIR, Linearizability criteria for systems of two second-order differential equations by complex methods, *Nonlinear Dynamics* **66**(1–2) (2011), 77–88.
2. ———— Complex Lie symmetries for scalar second-order ordinary differential equations, *Nonlinear Analysis Real World Applications* **10**(6) (2009), 3335–3344.
3. MUHAMMAD AYUB, MASOOD KHAN AND F.M. MAHOMED, Algebraic Linearization Criteria for Systems of Second-Order Ordinary Differential Equations, *Nonlinear Dynamics* **67** (2012), 2053–2062.
4. G.P. FLESSAS, K.S. GOVINDER AND P.G.L. LEACH, Characterisation of the Algebraic Properties of First Integrals of Scalar Ordinary Differential Equations of Maximal Symmetry, *J. Math. Anal. and Appl.* **212** (1997), 349–374.
5. K.S. GOVINDER AND P.G.L. LEACH, Ermakov systems: a group theoretic approach, *Phys. Lett. A* **186** (1994), 391–395.
6. ———— The Algebraic Structures of the First Integrals of Third-Order Linear Equations, *J. Math. Anal. and Appl.* **193** (1995), 114–133.
7. ———— Noetherian Integrals via Nonlocal Transformation, *Phys. Lett. A* **201** (1995), 91–94.
8. A.H. KARA AND F.M. MAHOMED, Action of Lie-B̈acklund Symmetries on Conservation Laws, Modern Group Analysis, *Proceedings of the International Conference at the Sophus Lie Conference Center,* pp. 175–180, Nordfjordeid, Norway, 1997.
9. ———— A basis of conservation laws for partial differential equations, *Journal of Nonlinear Mathematical Physics* **9** supplement 2, (2002), 60–72.
10. P.G.L. LEACH, K.S. GOVINDER AND B. ABRAHAM-SHRAUNER, Symmetries of First Integrals and Their Associated Differential Equations, *J. Math. Anal. and Appl.* **253** (1999), 58–83.
11. P.G.L. LEACH AND F.M. MAHOMED, Maximal Subalgebra Associated with First Integral of a System Possessing $sl(3, R)$ algebra, *J. Math. Phys.* **29** (1988), 1807–1813.
12. S. LIE, *Theorie der Transformationsgruppen,* Vol. III, Teubner, Leipzig, 1893.

13. K.S. Mahomed and E. Momoniat, Symmetry Classification of First Integrals for Scalar Linearizable Second-Order ODEs, *Journal of Applied Mathematics* **2012**, (2012), Article ID 847086, doi:10.1155/2012/847086, 14 pages.

14. ———— Algebraic Properties of First Integrals for Scalar Linear Third-Order ODEs of Maximal Symmetry, *Abstract and Applied Analysis* **2013** (2013), Article ID 530365, 8 pages.

15. ———— Classification of First Integrals for Scalar Dynamical Equations, *Int. J. Non-Linear Mech.* **59** (2014), 52-59.

16. F.M. Mahomed and A. Qadir, Invariant linearization criteria for systems of cubically nonlinear second-order ordinary differential equations, *J. Nonlinear Math. Phys.* **16**(3) (2009), 283–298.

17. S.V. Meleshko, S. Moyo, C. Muriel, J. L. Romero, P. Guha and A.G. Choudhury, On first integrals of second-order ordinary differential equations, *J. Eng. Math.* **82** (2013), 17–30.

18. C. Muriel and J.L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42**(36) (2009), 365207.

19. ———— Second-order ordinary differential equations and first integrals of the form $a(t,x)x + b(t,x)$, *J. Nonlinear Math. Phys.* **16** (2009), 209–222.

20. ———— Second-order ordinary differential equations with first integrals of the form $c(t) + 1/(a(t,x)x + b(t,x))$, *J. Nonlinear Math. Phys.* **18** (2011), 237–250.

21. P.J. Olver, *Applications of Lie Groups to Differential Equations,* Springer-Verlag, New York, 1986.

22. C. Wafo Soh and F.M. Mahomed, Noether Symmetries of $y'' = f(x)y^n$ with Applications to Non-Static Spherically Symmetric Perfect Fluid Solutions, *Classical Quantum Gravity* **16** (1999), 3553–3566.

23. ———— Linearization criteria for a system of second-order ordinary differential equations, *Int. J. Non-Linear Mech.* **36** (2001), 671–677.

Received 8 April, 2015 and in revised form 23 November, 2015.

13 Integrating Factors and First Integrals for Ordinary Differential Equations. By S.C. Anco and G.W. Bluman. (1997)

Euro. Jnl of Applied Mathematics (1998), *vol.* 9, *pp.* 245–259. Printed in the United Kingdom

Integrating factors and first integrals for ordinary differential equations

STEPHEN C. ANCO and GEORGE BLUMAN

Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

(*Received 25 July 1997; revised 13 January 1998*)

We show how to find all the integrating factors and corresponding first integrals for any system of Ordinary Differential Equations (ODEs). Integrating factors are shown to be all solutions of both the adjoint system of the linearised system of ordinary differential equations and a system that represents an extra adjoint-invariance condition. We present an explicit construction formula to find the resulting first integrals in terms of integrating factors, and discuss techniques for finding integrating factors. In particular, we show how to utilize known first integrals and symmetries to find new integrating factors. Illustrative examples are given.

1 Introduction

For first-order scalar Ordinary Differential Equations (ODEs), Sophus Lie (cf. Lie, 1874) showed how to construct an integrating factor from each admitted point symmetry. Conversely, Lie showed that each integrating factor yields an admitted point symmetry.

In general, for systems of one or more ODEs, an integrating factor is a set of functions, multiplying each of the ODEs, which yields a first integral. If the system is self-adjoint, then its integrating factors are necessarily solutions of its linearized system. Such solutions are the symmetries of the given system of ODEs. If a given system of ODEs is not self-adjoint, then its integrating factors are necessarily solutions of the adjoint system of its linearized system. Such solutions are known as *adjoint symmetries* (Gordon, 1986; see also Sarlet *et al.*, 1987, 1990) of the given system of ODEs.

In this paper, we introduce an *adjoint-invariance condition* which is a necessary and sufficient condition for an admitted adjoint symmetry to be an integrating factor. We present an explicit formula for the first integral corresponding to each integrating factor. These results are the counterparts of our work on Partial Differential Equations (PDEs) (Anco & Bluman, 1997, 1998).

For a first-order scalar ODE, a first integral is a quadrature. For an nth-order scalar ODE, a first integral is an expression relating the independent variable, the dependent variable and derivatives to order $n-1$, which is constant for *all* solutions of the ODE. First integrals are defined analogously for systems of ODEs.

If r independent first integrals are known, then an nth-order scalar ODE can be reduced to one or more $(n-r)$th-order ODEs *in terms of r essential constants*[1] *and the given*

[1] Constants are essential if none of them can be reduced in terms of function combinations of the others.

dependent and independent variables. In particular, n independent first integrals yield the general solution involving n essential constants.

Sophus Lie (cf. Lie, 1888; Bluman, 1990) showed that if an nth-order scalar ODE admits an r-parameter solvable group of point symmetries, then it can be reduced to an $(n-r)$th-order ODE plus r quadratures.[2] Lie's reduction uses derived independent and dependent variables, given by invariants and differential invariants to order $n-r$, arising from the admitted point symmetries. Consequently, the 'reduced' ODE *is not* an $(n-r)$th-order ODE in terms of the given dependent and independent variables. Thus, Lie's reduction is not as useful as a reduction in terms of first integrals.

In §2 we establish our framework. We define integrating factors and first integrals for systems of ODEs. We show that each integrating factor must be an adjoint symmetry, and derive the adjoint-invariance condition for an adjoint symmetry to be an integrating factor. We give the explicit formula for the first integral arising from an integrating factor. Finally, we show how our framework treats the well-known situation for first-order scalar ODEs.

In §3 we treat the case of second-order scalar ODEs, and make some remarks about the situation for higher-order scalar ODEs. In §4 we discuss techniques for finding and utilizing adjoint symmetries in conjunction with the adjoint-invariance condition. We show how to use an adjoint symmetry and functions of known first integrals to obtain *new* first integrals. Finally, in §5 we consider various examples.

2 The basic framework

Consider any nth-order system of one or more ODEs

$$G_\sigma(x, y, y', \dots, y^{(n)}) = 0, \quad \sigma = 1, \dots, N \tag{2.1}$$

with any number of dependent variables $y = \{y^1, \dots, y^M\}$ and one independent variable x; y' represents the first-order derivative of y; $y^{(j)}$ represents the jth-order derivative of y. For arbitrary functions $Y = \{Y^1, \dots, Y^M\}$, let $G_\sigma[Y] = G_\sigma(x, Y, Y', \dots, Y^{(n)})$. The aim is to find all factors $\Lambda^\sigma[Y] = \Lambda^\sigma(x, Y, Y', \dots, Y^{(n-1)})$ and functions $\Phi[Y] = \Phi(x, Y, Y', \dots, Y^{(n-1)})$ so that

$$\Lambda^\sigma[Y]\, G_\sigma[Y] = \frac{d}{dx}\Phi[Y] \tag{2.2}$$

holds for all $Y(x)$ for which $\Lambda^\sigma[Y]\, G_\sigma[Y]$ is finite. (Throughout this paper, we use the index notation $\sigma = 1, \dots, N$; $\rho = 1, \dots, M$; and the convention that summation is assumed over any repeated index in all expressions.)

From equation (2.2), it follows that

$$\Phi[y] = \text{const} \tag{2.3}$$

on the solutions $y(x)$ of system (2.1) for which each $\Lambda^\sigma[y]$ is finite. In particular, if $\Lambda^\sigma[Y]$ is finite for arbitrary $Y(x)$, then $\Phi[y] = \text{const}$ holds for all solutions of system (2.1).

We allow $\Lambda^\sigma[Y]$ and $\Phi[Y]$ to depend at most upon $Y^{(n-1)}$, since we assume that the system (2.1) determines $y^{(n)}$ in terms of lower-order derivatives of y.

[2] Lie's method can be extended to invariance under r-parameter solvable groups of higher order symmetries.

Definition 2.1 *A set of factors* $\{\Lambda^\sigma[Y]\}$ *satisfying* (2.2) *is an* integrating factor *of system* (2.1) *and, correspondingly,* $\Phi[y] = \text{const}$ *is a* first integral *of system* (2.1).

Before defining adjoint symmetries and introducing our adjoint-invariance condition, we first consider the linearized system, and its adjoint, obtained from equation (2.1).

The linearized system is given by

$$L_{\sigma\rho}[y]\,v^\rho = 0 \tag{2.4}$$

where

$$L_{\sigma\rho}[Y]\,V^\rho = G_{\sigma\rho}[Y] + G^1_{\sigma\rho}[Y]\frac{dV^\rho}{dx} + \cdots + G^n_{\sigma\rho}[Y]\frac{d^n V^\rho}{dx^n} \tag{2.5}$$

with

$$G_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y^\rho}, \quad G^1_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y'^\rho}, \ldots, G^n_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y^{(n)\rho}}.$$

In equation (2.4), $v = \{v^1, \ldots, v^M\}$ is a solution of the linearized system holding for *all* solutions $y(x)$ of system (2.1); in equation (2.5), $V = \{V^1, \ldots, V^M\}$ and $Y = \{Y^1, \ldots, Y^M\}$ are *arbitrary* functions of x.

The linearized system (2.4) is the set of determining equations for the symmetries of system (2.1). In particular, a solution v of system (2.4) is a symmetry of the system (2.1) with infinitesimal generator $v^\rho\,\partial/\partial y^\rho$.

The adjoint of the linearized system (2.4) is given by

$$L^*_{\rho\sigma}[y]\,w^\sigma = 0, \tag{2.6}$$

where

$$L^*_{\rho\sigma}[Y]\,W^\sigma = G_{\sigma\rho}[Y]\,W^\sigma - \frac{d}{dx}(G^1_{\sigma\rho}[Y]\,W^\sigma) + \cdots + (-1)^n\frac{d^n}{dx^n}(G^n_{\sigma\rho}[Y]\,W^\sigma). \tag{2.7}$$

In system (2.6), $w = \{w^1, \ldots, w^N\}$ is a solution of the adjoint system holding for all solutions $y(x)$ of the given system of ODEs (2.1); in system (2.7), $W = \{W^1, \ldots, W^N\}$ and $Y = \{Y^1, \ldots, Y^M\}$ are arbitrary functions of x.

Definition 2.2 *The adjoint system* (2.6) *is the set of determining equations for the* adjoint symmetries *of system* (2.1). *In particular, a solution w of the adjoint system* (2.6) *is an adjoint symmetry of the system* (2.1).

Definition 2.3 *System* (2.1) *is* self-adjoint *if and only if* $L^*_{\sigma\rho}[Y] \equiv L_{\sigma\rho}[Y]$.

Theorem 2.4 *Every integrating factor of system* (2.1) *satisfies the* adjoint-invariance condition

$$L^*_{\rho\sigma}[Y]\,\Lambda^\sigma[Y] = -\Lambda^\sigma_\rho[Y]\,G_\sigma[Y] + \frac{d}{dx}(\Lambda^{1\sigma}_\rho[Y]\,G_\sigma[Y]) + \cdots + (-1)^{n-2}\frac{d^{n-1}}{dx^{n-1}}(\Lambda^{(n-1)\sigma}_\rho[Y]\,G_\sigma[Y]) \tag{2.8}$$

for arbitrary $Y(x)$ *where*

$$\Lambda^\sigma_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y^\rho}, \quad \Lambda^{1\sigma}_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y'^\rho}, \ldots, \Lambda^{(n-1)\sigma}_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y^{(n-1)\rho}}.$$

Proof Since system (2.2) holds for arbitrary $Y(x)$, it also holds with $Y^\rho(x)$ replaced by the one-parameter (λ) family of functions $Y^\rho(x;\lambda) = Y^\rho(x)+\lambda V^\rho(x)$, where $Y^\rho(x)$, $V^\rho(x)$ are arbitrary functions of x. Thus, we have

$$\Lambda^\sigma[Y(x;\lambda)]\,G_\sigma[Y(x;\lambda)] = \frac{d}{dx}\Phi[Y(x;\lambda)]. \tag{2.9}$$

Now differentiate system (2.9) with respect to λ and set $\lambda = 0$. Then use

$$\left.\frac{\partial G_\sigma[Y(x;\lambda)]}{\partial\lambda}\right|_{\lambda=0} = L_{\sigma\rho}[Y]\,V^\rho,$$

given by the linearizing expression (2.5). This leads to

$$\frac{d}{dx}\left(\left.\frac{\partial}{\partial\lambda}\Phi[Y(x;\lambda)]\right|_{\lambda=0}\right)$$
$$= \Lambda^\sigma[Y]\,(L_{\sigma\rho}[Y]\,V^\rho)+G_\sigma[Y]\left(\Lambda^\sigma_\rho[Y]\,V^\rho+\Lambda^{1\sigma}_\rho[Y]\frac{dV^\rho}{dx}+\cdots+\Lambda^{(n-1)\sigma}_\rho[Y]\frac{d^{n-1}V^\rho}{dx^{n-1}}\right). \tag{2.10}$$

Now apply the Euler operators

$$E_{V^\rho} = \frac{\partial}{\partial V^\rho}-\frac{d}{dx}\frac{\partial}{\partial V'^\rho}+\cdots+(-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\frac{\partial}{\partial V^{(n-1)\rho}} \tag{2.11}$$

to each side of equation (2.10), which is an expression in terms of the arbitrary functions $\{Y^\rho(x)\},\{V^\rho(x)\}$. Since Euler operators annihilate total derivatives, the left-hand side of equation (2.10) vanishes upon action by the Euler operators (2.11). On the right-hand side of equation (2.10), the Euler operators (2.11) applied to $\Lambda^\sigma[Y](L_{\sigma\rho}[Y]\,V^\rho)$ yield $L^*_{\rho\sigma}[Y]\,\Lambda^\sigma[Y]$, given by system (2.7) with $W^\sigma = \Lambda^\sigma[Y]$. The Euler operators (2.11) applied to the rest of the right-hand side of equation (2.10) yield

$$\Lambda^\sigma_\rho[Y]\,G_\sigma[Y]-\frac{d}{dx}(\Lambda^{1\sigma}_\rho[Y]\,G_\sigma[Y])+\cdots+(-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(\Lambda^{(n-1)\sigma}_\rho[Y]\,G_\sigma[Y]).$$

Thus the adjoint-invariance condition (2.8) is obtained. □

Corollary 2.5 *If* $\Phi[y] = \text{const}$ *is a first integral of the system of ODEs* (2.1), *then its integrating factor* $\{\Lambda^\sigma[Y]\}$ *satisfies the adjoint system*

$$L^*_{\rho\sigma}[y]\,\Lambda^\sigma[y] = 0, \tag{2.12}$$

holding for all solutions $y(x)$ *of system* (2.1).

The proof of Corollary 2.5 follows immediately from the adjoint-invariance condition (2.8) with $Y(x) = y(x)$ given by any solution of system (2.1).

An important consequence of Corollary 2.5 is that *all* first integrals arise from solutions of the adjoint system (2.12). If system (2.1) is self-adjoint, then solutions of the adjoint system (2.12) are symmetries of system (2.1). If system (2.1) is not self-adjoint, the solutions of the adjoint system (2.12) are not symmetries of (2.1) but *adjoint symmetries* (Gordon, 1986; Sarlet *et al.*, 1987, 1990) of system (2.1). However, as will be shown in the examples in §5, an adjoint symmetry does not always satisfy the adjoint-invariance condition (2.8), i.e. an adjoint symmetry does not always give rise to a first integral.

For any adjoint symmetry that satisfies the adjoint-invariance condition (2.8), we now derive a formula which yields the corresponding first integral. To proceed, we first need to establish the following identity.

Lemma 2.6 *The operators* $L_{\sigma\rho}[Y]$ *and* $L^*_{\rho\sigma}[Y]$ *satisfy the identity*

$$W^\sigma L_{\sigma\rho}[Y]\, V^\rho - V^\rho L^*_{\rho\sigma}[Y]\, W^\sigma \equiv \frac{d}{dx} S[W, V; G[Y]] \tag{2.13}$$

for arbitrary functions $Y^\rho(x)$, $V^\rho(x)$, $W^\sigma(x)$, *where*

$$S[W, V; G[Y]] = V^\rho W^\sigma G^1_{\sigma\rho} + \left(\frac{dV^\rho}{dx} - V^\rho \frac{d}{dx}\right)(W^\sigma G^2_{\sigma\rho}) + \cdots$$

$$+\left(\frac{d^{n-1} V^\rho}{dx^{n-1}} + \sum_{l=1}^{n-2} (-1)^l \frac{d^{n-l-1} V^\rho}{dx^{n-l-1}} \frac{d^l}{dx^l} + (-1)^{n-1} V^\rho \frac{d^{n-1}}{dx^{n-1}}\right)(W^\sigma G^n_{\sigma\rho}). \tag{2.14}$$

Proof The identity (2.13) follows from a direct expansion of both sides of (2.13), using the definitions of $L_{\sigma\rho}[Y]$ and $L^*_{\rho\sigma}[Y]$ given by equations (2.5) and (2.7), respectively. □

We are now ready to establish the converse of Theorem 2.4.

Theorem 2.7 *Suppose* $\{\Lambda^\sigma[Y]\}$ *satisfies the adjoint-invariance condition* (2.8). *Then* $\{\Lambda^\sigma[Y]\}$ *is an integrating factor for the system of ODEs* (2.1). *In particular,*

$$\Lambda^\sigma[Y]\, G_\sigma[Y] = \frac{d}{dx}\Phi[Y] \tag{2.15}$$

with $\Phi[Y] = \Phi_1(x, Y, Y', \dots, Y^{(2n)}) + \Phi_2(x)$ *given by the formulae:*

$$\Phi_1 = \int_0^1 d\lambda \left(S\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial \lambda}; G[Y(x;\lambda)]\right] + N\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial \lambda}; G[Y(x;\lambda)]\right]\right), \tag{2.16}$$

$$\Phi_2 = \int k(x)\, dx, \tag{2.17}$$

where

$$S\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial\lambda}; G[Y(x;\lambda)]\right] = \frac{\partial Y^\rho(x;\lambda)}{\partial\lambda} \Lambda^\sigma[Y(x;\lambda)]\, G^1_{\sigma\rho}[Y(x;\lambda)]$$

$$+\left(\frac{d}{dx}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right) - \frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\frac{d}{dx}\right)(\Lambda^\sigma[Y(x;\lambda)]\, G^2_{\sigma\rho}[Y(x;\lambda)]) + \cdots$$

$$+\left(\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right) + \sum_{l=1}^{n-2}(-1)^l \frac{d^{n-l-1}}{dx^{n-l-1}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right)\frac{d^l}{dx^l} + (-1)^{n-1}\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\frac{d^{n-1}}{dx^{n-1}}\right)$$

$$\times (\Lambda^\sigma[Y(x;\lambda)]\, G^n_{\sigma\rho}[Y(x;\lambda)]); \tag{2.18}$$

$$N\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial\lambda}; G[Y(x;\lambda)]\right] = \frac{\partial Y^\rho(x;\lambda)}{\partial\lambda} G_\sigma[Y(x;\lambda)]\,\Lambda_\rho^{1\sigma}[Y(x;\lambda)]$$
$$+\left(\frac{d}{dx}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right) - \frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\frac{d}{dx}\right)(G_\sigma[Y(x;\lambda)]\,\Lambda_\rho^{2\sigma}[Y(x;\lambda)]) + \cdots$$
$$+\left(\frac{d^{n-2}}{dx^{n-2}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right) + \sum_{l=1}^{n-3}(-1)^l \frac{d^{n-l-2}}{dx^{n-l-2}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\right)\frac{d^l}{dx^l} + (-1)^{n-2}\frac{\partial Y^\rho(x;\lambda)}{\partial\lambda}\frac{d^{n-2}}{dx^{n-2}}\right)$$
$$\times (G_\sigma[Y(x;\lambda)]\,\Lambda_\rho^{(n-1)\sigma}[Y(x;\lambda)]); \tag{2.19}$$
$$k(x) = \Lambda^\sigma[\tilde{Y}(x)]\,G_\sigma[\tilde{Y}(x)]. \tag{2.20}$$

Here $\tilde{Y}(x) = \{\tilde{Y}^1(x), \ldots, \tilde{Y}^M(x)\}$ *are* any *fixed functions such that the function* $k(x)$ *is finite, and* $Y(x;\lambda)$ *is the one-parameter* (λ) *family of functions* $Y^\sigma(x;\lambda) = \lambda Y^\sigma(x) + (1-\lambda)\,\tilde{Y}^\sigma(x)$, *for arbitrary* $Y^\sigma(x)$, $\sigma = 1, \ldots, M$.

Proof Let $V(x) = (\partial Y(x;\lambda))/\partial\lambda = Y(x) - \tilde{Y}(x)$. From the adjoint-invariance condition (2.8), we obtain

$$V^\rho(x)\Big(L^*_{\rho\sigma}[Y(x;\lambda)]\,\Lambda^\sigma[Y(x;\lambda)] + \Lambda_\rho^\sigma[Y(x;\lambda)]\,G_\sigma[Y(x;\lambda)] - \frac{d}{dx}(\Lambda_\rho^{1\sigma}[Y(x;\lambda)]\,G_\sigma[Y(x;\lambda)]) + \cdots$$
$$+(-1)^n \frac{d^{n-1}}{dx^{n-1}}(\Lambda_\rho^{(n-1)\sigma}[Y(x;\lambda)]\,G_\sigma[Y(x;\lambda)])\Big) = 0. \tag{2.21}$$

Now we manipulate the terms in equation (2.21) as follows. From identity (2.13), the first term in (2.21) becomes $V^\rho L^*_{\rho\sigma}\Lambda^\sigma = \Lambda^\sigma L_{\sigma\rho} V^\rho - dS/dx$, where S is given by expression (2.18). Using the Leibniz rule for d/dx, the third term of equation (2.21) becomes

$$-V^\rho \frac{d}{dx}(\Lambda_\rho^{1\sigma} G_\sigma) = -\frac{d}{dx}(V^\rho \Lambda_\rho^{1\sigma} G_\sigma) + \left(\frac{dV^\rho}{dx}\right)\Lambda_\rho^{1\sigma} G_\sigma,$$

and the other terms of (2.21) become

$$(-1)^q V^\rho \frac{d^q}{dx^q}(\Lambda_\rho^{q\sigma} G_\sigma)$$
$$= (-1)^q \frac{d}{dx}\left\{\left[V^\rho \frac{d^{q-1}}{dx^{q-1}} + \sum_{l=1}^{q-2}(-1)^l\left(\frac{d^l V^\rho}{dx^l}\right)\frac{d^{q-l-1}}{dx^{q-l-1}} + (-1)^{q-1}\left(\frac{d^{q-1} V^\rho}{dx^{q-1}}\right)\right]\Lambda_\rho^{q\sigma} G_\sigma\right\}$$
$$+\left(\frac{d^q V^\rho}{dx^q}\right)\Lambda_\rho^{q\sigma} G_\sigma$$

for $q = 1, \ldots, n-1$.

Hence equation (2.21) becomes

$$\Lambda^\sigma L_{\sigma\rho} V^\rho + \left(\Lambda_\rho^\sigma V^\rho + \Lambda_\rho^{1\sigma}\frac{dV^\rho}{dx} + \cdots + \Lambda_\rho^{(n-1)\sigma}\frac{d^{(n-1)} V^\rho}{dx^{(n-1)}}\right) G_\sigma - \frac{d}{dx}(S+N) = 0, \tag{2.22}$$

where N is given by expression (2.19).

Now observe that $L_{\sigma\rho}[Y(x;\lambda)]\,V^\rho(x) = (\partial G_\sigma[Y(x;\lambda)])/\partial\lambda$, and that

$$\Lambda_\rho^\sigma[Y(x;\lambda)]\,V^\rho(x) + \Lambda_\rho^{1\sigma}[Y(x;\lambda)]\frac{dV^\rho(x)}{dx} + \cdots + \Lambda_\rho^{(n-1)\sigma}[Y(x;\lambda)]\frac{d^{n-1} V^\rho(x)}{dx^{n-1}} = \frac{\partial}{\partial\lambda}\Lambda^\sigma[Y(x;\lambda)].$$

Then equation (2.22) becomes

$$\Lambda^{\sigma}[Y(x;\lambda)]\left(\frac{\partial}{\partial\lambda}G_{\sigma}[Y(x;\lambda)]\right)+\left(\frac{\partial}{\partial\lambda}\Lambda^{\sigma}[Y(x;\lambda)]\right)G_{\sigma}[Y(x;\lambda)] = \frac{\partial}{\partial\lambda}(\Lambda^{\sigma}[Y(x;\lambda)]\,G_{\sigma}[Y(x;\lambda)])$$

$$= \frac{d}{dx}(S[\Lambda[Y(x;\lambda)], V(x); G[Y(x;\lambda)]]+N[\Lambda[Y(x;\lambda)], V(x); G[Y(x;\lambda)]]), \quad (2.23)$$

where S and N are given by expressions (2.18) and (2.19), respectively.

Now integrate equation (2.23) with respect to λ from $\lambda = 0$ to $\lambda = 1$. Then we obtain

$$\Lambda^{\sigma}[Y]\,G_{\sigma}[Y]-\Lambda^{\sigma}[\tilde{Y}]\,G_{\sigma}[\tilde{Y}] = d\Phi_1/dx,$$

where Φ_1 is given by expression (2.16). To complete the proof, we observe that $\Lambda^{\sigma}[\tilde{Y}]\,G_{\sigma}[\tilde{Y}] = k(x) = d\Phi_2/dx$. □

Note that, if $\Lambda^{\sigma}[Y]$, $G_{\sigma}[Y]$, $\sigma = 1,\dots,n$ are finite for $Y^{\rho} = 0$, $\rho = 1,\dots, M$, then we can choose $\tilde{Y}^{\rho} = 0$, $\rho = 1,\dots, M$, and thus simplify the integral for Φ_1. Moreover, if $y^{\rho} = 0$, $\rho = 1,\dots, M$, is a solution of system (2.1), then Φ_2 vanishes.

As a consequence of Theorems 2.4 and 2.7, we see that for any system of ODEs, all first integrals arise from adjoint symmetries that satisfy the adjoint-invariance condition.

2.1 First-order ODEs

We now consider the classical problem of finding the integrating factor for any first-order scalar ODE written in solved form

$$G(x,y,y') = y'-g(x,y) = 0. \quad (2.24)$$

Here the linearized ODE is

$$L[y]\,v = \frac{dv}{dx}-g_y v = 0, \quad (2.25)$$

and the corresponding adjoint ODE is given by

$$L^*[y]\,w = -\frac{dw}{dx}-g_y w = 0. \quad (2.26)$$

The symmetries of ODE (2.24) are the solutions of (2.25), while the adjoint symmetries of ODE (2.24) are the solutions of (2.26), which hold for all solutions $y(x)$ of ODE (2.24).

For arbitrary $Y = Y(x)$, each integrating factor $\Lambda(x, Y)$ of ODE (2.24) satisfies the adjoint-invariance condition

$$-\frac{d\Lambda(x, Y)}{dx}-g_Y(x, Y)\,\Lambda(x, Y) = -(Y'-g(x, Y))\,\Lambda_Y(x, Y), \quad (2.27)$$

which reduces to

$$g_Y\Lambda+\Lambda_x+g\Lambda_Y = 0. \quad (2.28)$$

Theorem 2.8 *If $\Lambda(x,y)$ is an adjoint symmetry of ODE* (2.24), *then $\Lambda(x, Y)$ is an integrating factor of ODE* (2.24).

Proof From (2.26)–(2.27) it follows that $\Lambda(x, Y)$ is a solution of (2.28) if and only if $w = \Lambda(x,y)$ is a solution of (2.26). □

Theorem 2.9 *Each symmetry $v(x,y)$ of ODE* (2.24) *yields an adjoint symmetry $\Lambda(x,y) = 1/(v(x,y))$ of ODE* (2.24). *Conversely, each adjoint symmetry $\Lambda(x,y)$ of ODE* (2.24) *yields a symmetry $v(x,y) = 1/(\Lambda(x,y))$ of ODE* (2.24).

Proof From (2.24)–(2.25), it follows that any symmetry $v(x,y)$ of ODE (2.24) satisfies

$$v_x(x,Y)+g(x,Y)\,v_Y(x,Y)-g_Y(x,Y)\,v(x,Y)=0, \tag{2.29}$$

for arbitrary $Y(x)$. In turn, by direct substitution, one can show that $v(x,Y)$ satisfies (2.29) if and only if $\Lambda(x,Y)=1/(v(x,Y))$ satisfies

$$-g_Y(x,Y)\,\Lambda(x,Y)-\Lambda_x(x,Y)+g(x,Y)\,\Lambda_Y(x,Y)=0. \tag{2.30}$$

Hence $\Lambda(x,Y)$ satisfies the adjoint-invariance condition (2.28). □

For any integrating factor $\Lambda(x,Y)$, the first integral formula (2.16)–(2.20) yields

$$\Phi_1(x,y)+\Phi_2(x)=\text{const},$$

which gives the general solution of ODE (2.24). In terms of any fixed function $\tilde{y}(x)$, one has

$$S=(y-\tilde{y})\,\Lambda(x,\lambda(y-\tilde{y})+\tilde{y}),$$
$$N=0,$$
$$k(x)=\Lambda(x,\tilde{y})\,(\tilde{y}'-g(x,\tilde{y})),$$

which leads to

$$\Phi_1(x,y)=\int_0^1 S\,d\lambda=\int_{\tilde{y}}^{y}\Lambda(x,z)\,dz,\quad \Phi_2(x)=\int k(x)\,dx. \tag{2.31}$$

From the above, we see that for any first-order ODE each adjoint symmetry is an integrating factor and, conversely, each integrating factor is an adjoint symmetry. In the next section, we will show that this is not the case for higher-order ODEs.

3 Second-order and higher-order scalar ODEs

We now show how the framework presented in §2 applies to any second-order scalar ODE

$$y''-g(x,y,y')=0 \tag{3.1}$$

and higher-order scalar ODEs

$$y^{(n)}-g(x,y,y',y'',\ldots,y^{(n-1)})=0. \tag{3.2}$$

3.1 Second-order ODEs

The linearized ODE for equation (3.1) is given by

$$L[y]\,v=\frac{d^2v}{dx^2}-g_{y'}\frac{dv}{dx}-g_y\,v=0, \tag{3.3}$$

and the corresponding adjoint ODE is

$$L^*[y]\,w=\frac{d^2w}{dx^2}+\frac{d}{dx}(g_{y'}\,w)-g_y\,w=\frac{d^2w}{dx^2}+g_{y'}\frac{dw}{dx}+(g_{xy'}+y'\,g_{yy'}+gg_{y'y'}-g_y)\,w=0. \tag{3.4}$$

The solutions $w=\Lambda(x,y,y')$ of ODE (3.4), holding for any $y(x)$ satisfying the second-order

ODE (3.1), are the adjoint symmetries of (3.1). Explicitly, the determining equation for an adjoint symmetry $\Lambda(x, y, y')$ is given by

$$L^*[y]\Lambda(x,y,y') = \Lambda_{xx}+2y'\Lambda_{xy}+2g\Lambda_{xy'}+(y')^2\Lambda_{yy}+2y'g\Lambda_{yy'}+g^2\Lambda_{y'y'}$$
$$+(g_x+y'g_y+2gg_{y'})\Lambda_{y'}+(g+y'g_{y'})\Lambda_y+g_{y'}\Lambda_x+(g_{xy'}+y'g_{yy'}+gg_{y'y'}-g_y)\Lambda = 0, \quad (3.5)$$

which must hold for arbitrary x, y, y'. In turn, an adjoint symmetry $\Lambda(x, y, y')$ of ODE (3.1) yields an integrating factor $\Lambda(x, Y, Y')$ of (3.1) if and only if $\Lambda(x, Y, Y')$ satisfies the adjoint-invariance condition

$$L^*[Y]\Lambda(x,Y,Y') = -(Y''-g)(\Lambda_{xY'}+Y'\Lambda_{YY'}+g\Lambda_{Y'Y'}+2g_{Y'}\Lambda_{Y'}+2\Lambda_Y+g_{Y'Y'}\Lambda), \quad (3.6)$$

which must hold for arbitrary x, Y, Y', Y'' with $g = g(x, Y, Y')$. Thus, the adjoint-invariance condition for $\Lambda(x, Y, Y')$ to be an integrating factor of (3.1) reduces to $\Lambda(x, Y, Y')$, solving the linear system of PDEs

$$\Lambda_{xx}+2Y'\Lambda_{xy}+2g\Lambda_{xy'}+(Y')^2\Lambda_{yy}+2Y'g\Lambda_{yy'}+g^2\Lambda_{y'y'}+(g_x+Y'g_y+2gg_{y'})\Lambda_{y'}$$
$$+(g+Y'g_{y'})\Lambda_y+g_{y'}\Lambda_x+(g_{xy'}+Y'g_{yy'}+gg_{y'y'}-g_y)\Lambda = 0 \quad (3.7)$$

given by equation (3.5) with y replaced by Y, and

$$\Lambda_{xY'}+Y'\Lambda_{YY'}+g\Lambda_{Y'Y'}+2g_{Y'}\Lambda_{Y'}+2\Lambda_Y+g_{Y'Y'}\Lambda = 0. \quad (3.8)$$

given by (3.6). Equations (3.7)–(3.8) must hold for arbitrary values of x, Y, Y'.

Since every second-order ODE (3.1) has an infinite number of integrating factors, it follows that there must exist an infinite number of solutions of the system (3.7)–(3.8). Unlike the situation for a first-order ODE, where each adjoint symmetry yields an integrating factor, solutions of (3.7) are not always integrating factors, since they must also satisfy condition (3.8).

Correspondingly, for each integrating factor the construction formula (2.16)–(2.20) yields the first integral

$$\Phi[y] = \Phi_1(x,y,y')+\Phi_2(x) = \text{const}$$

of equation (3.1). In terms of any fixed function $\tilde{y}(x)$, with $r = \lambda y+(1-\lambda)\tilde{y}$, $\Lambda = \Lambda(x, r, r')$, one has

$$S = ((y'-\tilde{y}')-(y-\tilde{y})g_r(x,r,r'))\Lambda$$
$$-(y-\tilde{y})(\Lambda_x+(\lambda y'+(1-\lambda)\tilde{y}')\Lambda_r+(\lambda g(x,y,y')+(1-\lambda)\tilde{y}'')\Lambda_{r'}),$$
$$N = (y-\tilde{y})(\lambda g(x,y,y')+(1-\lambda)\tilde{y}''-g(x,r,r'))\Lambda_{r'},$$

so that

$$S+N = (y'-\tilde{y}')\Lambda-(y-\tilde{y})(g(x,r,r')\Lambda_{r'}+(\lambda y'+(1-\lambda)\tilde{y}')\Lambda_r+\Lambda_x+g_{r'}(x,r,r')\Lambda), \quad (3.9)$$
$$k(x) = [\tilde{y}''-g(x,\tilde{y},\tilde{y}')]\Lambda(x,\tilde{y},\tilde{y}'). \quad (3.10)$$

Consequently,

$$\Phi_1(x,y,y') = \int_0^1 (S+N)\,d\lambda, \quad (3.11)$$

$$\Phi_2(x) = \int k(x)\,dx. \quad (3.12)$$

One chooses $\tilde{y}$ so that $k(x)$ is finite. If both $g(x,0,0)$ and $\Lambda(x,0,0)$ are finite, one can set $\tilde{y}=0$ provided the corresponding integral $\int_0^1 (S+N)\,d\lambda$ converges. In this case,

$$S+N = y'\Lambda(x,\lambda y,\lambda y')-y[g(x,\lambda y,\lambda y')\,(\Lambda_{r'}(x,\lambda y,r')|_{r'=\lambda y'})+\lambda y'(\Lambda_r(x,r,\lambda y')|_{r=\lambda y}) + \Lambda_x(x,\lambda y,\lambda y')+(g_{r'}(x,\lambda y,r')|_{r'=\lambda y'})\,\Lambda(x,\lambda y,\lambda y')].$$

3.2 Higher-order ODEs

For higher-order scalar ODEs (3.2), the adjoint-invariance condition for an integrating factor $\Lambda(x, Y, Y', \dots, Y^{(n-1)})$ yields a linear determining equation which is a relation involving $x, Y, Y', \dots, Y^{(2n-2)}$, where each of the $2n$ quantities $x, Y, Y', \dots, Y^{(2n-2)}$ are to be treated as independent variables. This relation is a polynomial expression in terms of $Y^{(n)}$, $Y^{(n+1)}, \dots, Y^{(2n-2)}$, whose coefficients depend on $x, Y, Y', \dots, Y^{(n-1)}$. The coefficient of the term independent of $Y^{(n)}, Y^{(n+1)}, \dots, Y^{(2n-2)}$, yields the determining equation for the adjoint symmetries. The coefficients of the other terms in the polynomial expression yield further linear PDEs satisfied by $\Lambda(x, Y, Y', \dots, Y^{(n-1)})$. For $n=2$, as shown in (3.6), this splitting yields one such linear PDE (from the coefficient of the Y'' term). For $n=3$, one can show that this splitting yields three such linear PDEs from the coefficients of the terms involving $Y^{(4)}, (Y''')^2$ and Y'''. For $n=4$, the splitting yields five such linear PDEs from the coefficients of the terms involving $Y^{(6)}, Y^{(4)}\,Y^{(5)}, Y^{(5)}, (Y^{(4)})^2$ and $Y^{(4)}$.

4 Techniques for obtaining adjoint symmetries yielding first integrals

For *any* system of ODEs (2.1), there is an infinite number of linearly independent solutions of its corresponding adjoint system (2.6). Hence, a system of ODEs (2.1) always has an infinite number of adjoint symmetries. Consequently, in practice one must resort to specific ansatze in order to find adjoint symmetries.

We now focus on nth-order scalar ODEs. Here, one such ansatz is to seek solutions of the form $w=\Lambda(x,y,y',\dots,y^{(n-2)})$, which depend upon derivatives of order at most $n-2$ rather than $n-1$, for the corresponding adjoint symmetry determining equation (2.6).

More importantly, if one knows an adjoint symmetry and one or more first integrals arising from other adjoint symmetries, then one can use a second ansatz to seek further first integrals as follows. For a given nth-order scalar ODE, suppose that

$$\Phi_1[y]=C_1,\dots,\Phi_m[y]=C_m$$

are m functionally independent first integrals corresponding to the m integrating factors $\Lambda_1[Y],\dots,\Lambda_m[Y]$, respectively. Note that

$$\Lambda[y]=\frac{\partial\Gamma}{\partial C_1}\Lambda_1[y]+\dots+\frac{\partial\Gamma}{\partial C_m}\Lambda_m[y]$$

for any function $\Gamma(C_1,\dots,C_m)$, generates an inessential first integral $\Gamma(C_1,\dots,C_m)=\text{const}$.

Now suppose $w=\Lambda[y]$ is an adjoint symmetry such that

$$\Lambda[y]\neq\frac{\partial\Gamma}{\partial C_1}\Lambda_1[y]+\dots+\frac{\partial\Gamma}{\partial C_m}\Lambda_m[y], \tag{4.1}$$

for all functions $\Gamma(C_1, \ldots, C_m)$. We observe that for an arbitrary function $F(C_1, \ldots, C_m)$,

$$w = \Lambda_F[y] = F(C_1, \ldots, C_m)\Lambda[y] \tag{4.2}$$

is also an adjoint symmetry.

If we substitute $w = \Lambda_F[Y]$, given by equation (4.2) with y replaced by Y, into the adjoint-invariance condition (2.8), then we obtain a linear determining equation for F. Each solution, if any, of this determining equation yields a *new* integrating factor for the nth-order ODE. This will be illustrated through examples in §5.

A third ansatz which can lead to finding further adjoint symmetries is suggested by the following observation. If a given nth-order ODE admits a point symmetry, then each integrating factor of the ODE can always be expressed as a product of a multiplier expression, and some function of the invariants/differential invariants of the point symmetry. Consequently, the ODE admits adjoint symmetries of such a product form. One can then try using a known adjoint symmetry or integrating factor as the multiplier expression in a trial form in order to seek new adjoint symmetries. In particular, suppose a given nth-order ODE admits an integrating factor $\Lambda[Y]$ and a point symmetry with corresponding invariants/differential invariants $u(x, y), v_1(x, y, y'), \ldots, v_{n-1}(x, y, y', \ldots, y^{(n-1)})$. Let

$$\Lambda_f[y] = f(u, v_1, \ldots, v_{n-1})\Lambda[y], \tag{4.3}$$

for an arbitrary function $f(u, v_1, \ldots, v_{n-1})$. If we substitute $w = \Lambda_f[y]$ into the adjoint symmetry determining equation (2.6), then we obtain a linear determining equation for f. Each solution $f \neq$ const of this determining equation yields a new adjoint symmetry of the nth order ODE. In turn, we feed such a new adjoint symmetry into the second ansatz to seek further first integrals.

The above discussion extends naturally to systems of ODEs.

5 Examples

We now use three examples to illustrate our procedure for obtaining first integrals.

5.1 Harmonic oscillator

Consider the harmonic oscillator equation

$$y'' + y = 0. \tag{5.1}$$

The ODE (5.1) is self-adjoint, so that its adjoint symmetries are symmetries. The corresponding determining equation (3.5) for an adjoint symmetry $w = \Lambda(x, y, y')$ becomes

$$\Lambda_{xx} + 2y'\Lambda_{xy} - 2y\Lambda_{xy'} + (y')^2\Lambda_{yy} - 2yy'\Lambda_{yy'} + y^2\Lambda_{y'y'} - y'\Lambda_{y'} - y\Lambda_y + \Lambda = 0. \tag{5.2}$$

Here the extra adjoint-invariance determining equation (3.8) for $w = \Lambda(x, y, y')$ to yield an integrating factor becomes

$$\Lambda_{xy'} + y'\Lambda_{yy'} - y\Lambda_{y'y'} + 2\Lambda_y = 0. \tag{5.3}$$

Obviously ODE (5.1) admits translations in x and scalings in y which respectively yield adjoint symmetries $\Lambda_1 = y'$ and $\Lambda_2 = y$ satisfying (5.2).

Clearly, $\Lambda = y'$ satisfies the adjoint-invariance condition (5.3). Since $y''+y$ and y' are non-singular for $y=0$, we can set $\tilde{y}=0$ in our construction formula (3.9)–(3.12). Then we have

$$S+N = \lambda[(y')^2+y^2],$$

and hence the corresponding first integral is the energy

$$\Phi = \int_0^1 \lambda((y')^2+y^2)\,d\lambda = \tfrac{1}{2}((y')^2+y^2) = C_1. \tag{5.4}$$

It is easy to check that the adjoint symmetry $\Lambda = y$ does not satisfy the adjoint-invariance condition (5.3). Now we try the second ansatz presented in §4, using the previously-obtained first integral (5.4). Let

$$\Lambda = \Lambda_F = F(C_1)\,Y, \tag{5.5}$$

where $C_1 = \tfrac{1}{2}((Y')^2+Y^2)$. Substituting (5.5) into the adjoint-invariance condition (5.3), we find that $F(C_1)$ satisfies the ODE $C_1F'+F=0$. This yields the integrating factor $\Lambda = Y/((Y')^2+Y^2)$. Since Λ is singular for $Y=0$, we choose $\tilde{y}=1$ in our construction formula (3.9)–(3.12). Correspondingly, $r=\lambda(y-1)+1, r'=\lambda y'$, so that

$$S+N = \frac{y'r-(y-1)r'}{r^2+(r')^2}, \quad k(x)=1.$$

This leads to the first integral

$$\Phi = x+\int_0^1 \frac{y'}{\lambda^2(y')^2+[\lambda(y-1)+1]^2}\,d\lambda = x+\frac{\pi}{2}-\tan^{-1}\left(\frac{y}{y'}\right) = C_2, \tag{5.6}$$

which is the phase.

The first integrals (5.4) and (5.6) lead to the complete reduction $y=\sqrt{2C_1}\sin(x-C_2+\pi/2)$.

5.2 Frequency-damped oscillator

As a second example, we use the frequency-damped oscillator equation

$$y''+y(y')^2=0, \tag{5.7}$$

considered by Gordon (1986), Sarlet *et al.* (1987) and Mimura & Nôno (1994). The ODE (5.7) is not self-adjoint, so that its adjoint symmetries are not symmetries. Here the adjoint symmetry determining equation (3.5) for $w=\Lambda(x,y,y')$ is

$$\begin{aligned}\Lambda_{xx}+2y'\Lambda_{xy}-2y(y')^2\Lambda_{xy'}+(y')^2\Lambda_{yy}-2y(y')^3\Lambda_{yy'}+y^2(y')^4\Lambda_{y'y'}\\ +(4y^2-1)(y')^3\Lambda_{y'}-3y(y')^2\Lambda_y-2yy'\Lambda_x+(2y^2-1)(y')^2\Lambda=0.\end{aligned} \tag{5.8}$$

The extra adjoint-invariance determining equation (3.8) becomes

$$\Lambda_{xy'}+y'\Lambda_{yy'}-y(y')^2\Lambda_{y'y'}-4yy'\Lambda_{y'}+2\Lambda_y-2y\Lambda=0. \tag{5.9}$$

We try the first ansatz $\Lambda=\Lambda(x,y)$. Then equation (5.8) leads to the adjoint symmetry

$$\Lambda = axe^{y^2/2}+l(y), \tag{5.10}$$

with $a = \text{const}$, and $l(y)$ satisfying the ODE $l'' - 3yl' + (2y^2 - 1)l = 0$. Substituting (5.10) into the adjoint-invariance condition (5.9), we obtain $l' - yl = 0$, and thus $l(y) = be^{y^2/2}$, $b = \text{const}$. Hence we get two integrating factors, $\Lambda_1 = e^{y^2/2}$, $\Lambda_2 = xe^{y^2/2}$.

Next, we construct the first integral arising from $\Lambda = \Lambda_1 = e^{y^2/2}$. Clearly, we can set $\tilde{y} = 0$ in the construction formulae (3.9)–(3.12). This leads to the first integral

$$\Phi = y' \int_0^1 [1 + \lambda^2 y^2] e^{\lambda^2 y^2/2} \, d\lambda = y' e^{y^2/2} = C_1, \tag{5.11}$$

after integration by parts on the second term.

Now the first integral arising from $\Lambda = \Lambda_2 = xe^{y^2/2}$ is easy to construct since, again with $\tilde{y} = 0$, the construction formulae (3.9)–(3.12) reduces to

$$\Phi = C_1 x - \int_0^1 ye^{\lambda^2 y^2/2} \, dy = C_1 x - \int_0^y e^{u^2/2} \, du = C_2. \tag{5.12}$$

This yields the general solution $\int_0^y e^{u^2/2} \, du = C_1 x - C_2$ of the ODE (5.7).

5.3 Wave-speed equation

For a third example, we consider the fourth-order wave-speed equation

$$G(y, y', y'', y''' y^{(4)}) = (yy'(y/y')'')' = 0, \tag{5.13}$$

which arises when one seeks potential symmetries for a wave equation with a variable wave speed $y(x)$ (see Bluman & Kumei, 1987). The ODE (5.13) is not self-adjoint. Its adjoint symmetry determining equation for $w = \Lambda(x, y, y', y'', y''')$ is given by

$$((\Lambda' yy')'' y/(y')^2)' + (\Lambda' yy')''/y' - (\Lambda' y(y/y'')')' + \Lambda' y'(y/y')'' = 0. \tag{5.14}$$

The adjoint-invariance condition is

$$((\Lambda' YY')'' Y/(Y')^2)' + (\Lambda' YY')''/Y' - (\Lambda' Y(Y/Y'')')' + \Lambda' Y'(Y/Y')''$$
$$= -((G\Lambda_{Y'''})''' - (G\Lambda_{Y''})'' + (G\Lambda_{Y'})' - G\Lambda_Y), \tag{5.15}$$

with $\Lambda = \Lambda(x, Y, Y', Y'', Y''')$ and $G = (YY'(Y/Y')'')'$.

By inspection, $\Lambda = 1$ satisfies (5.14)–(5.15), which leads to the first integral

$$\Phi = yy'(y/y')'' = C_1. \tag{5.16}$$

Since the ODE (5.13) admits translations in x with corresponding invariants y and y', we employ the third ansatz of §4, in conjunction with the integrating factor $\Lambda = 1$ and these invariants, and seek adjoint symmetries of the form $\Lambda = \Lambda_f = f(y, y')$. Then (5.14) becomes a polynomial in y''', y''. The coefficient of $(y''')^2$ gives $y' f_{y'y'} + 3f_{y'} = 0$. This yields $f_{y'} = h(y)/(y')^3$ for some function $h(y)$. Then the coefficient of y''' gives the equation $12y(y')^2 f_{y'y'y} + 3y^2 f_{y'y'} + 41yy' f_{y'y} + 9y' f_{y'} + 12yf_y = 0$. This leads to $h = \text{const}$, and hence $f = 1/(y')^2$. Once can check that $\Lambda = 1/(y')^2$ satisfies both the adjoint symmetry determining equation (5.14) and the adjoint-invariance condition equation (5.15). The singularity of Λ at $y' = 0$ leads us to choose $\tilde{y} = x$ in our construction formula (3.9)–(3.12). The resulting first integral is

$$\Phi = -y''' y^2/(y')^3 - yy''/(y')^2 + (yy'')^2/(y')^4 = C_2. \tag{5.17}$$

The first integrals (5.16)–(5.17) reduce the ODE (5.13) to the second-order ODE

$$(y'')^2 = ((C_1 - C_2(y')^2)(y')^2)/y^2. \tag{5.18}$$

Now we again use the third ansatz in conjunction with the integrating factors $\Lambda = 1$ and $\Lambda = 1/(y')^2$ together with the differential invariant $\alpha = yy''/(y')^2$ arising from the invariance of ODE (5.13) under scalings in both x and y.

Using the integrating factor 1, we try $\Lambda = \Lambda_f = f(\alpha)$. The adjoint symmetry determining equation (5.14) yields $f = \alpha^2$. Feeding this into the second ansatz, we first substitute $\Lambda = \Lambda_F = F(C_1)\alpha^2$ into the adjoint-invariance restriction (5.15). Unfortunately, this yields $F = 0$. Next we substitute $\Lambda = \Lambda_F = F(C_2)\alpha^2$ into (5.15), which then becomes a polynomial with terms $y^{(6)}, y^{(4)}y^{(5)}, y^{(5)}, (y^{(4)})^3, (y^{(4)})^2, y^{(4)}$. The coefficient of $y^{(6)}$ yields $F = 1/(C_2)^2$. One can then check that

$$\Lambda = \left(\frac{yy''}{(y')^2 C_2}\right)^2$$

satisfies (5.15). From our construction formula we obtain the corresponding first integral

$$\Phi = (yy''/y')^2/C_2 + (y')^2 = C_3.$$

However, one can show that the first integral (5.19) is inessential, since $C_3 = C_1/C_2$.

Finally, using the integrating factor $1/(y')^2$, we try $\Lambda = \Lambda_f = f(\alpha)/(y')^2$. In this case, the adjoint symmetry determining equation (5.14) leads to

$$f = \tan^{-1}(c/\alpha) + \frac{c/\alpha}{1+(c/\alpha)^2} \tag{5.20}$$

with $c = \text{const}$. Here c arises from the scaling symmetry $\alpha \to \alpha/c$ admitted by the determining ODE satisfied by $f(\alpha)$.

One can check that $\Lambda = f(\alpha)/(y')^2$ does not satisfy the adjoint-invariance determining equation (5.15). Now we try a variant of the second ansatz as follows. We substitute $\Lambda = \Lambda_F = F(C_2)f/(y')^2$, where f is given by equation (5.20) with $c = H(C_2)$, into the adjoint-invariance determining equation (5.15). This leads to $F = (C_2)^{-3/2}$ and $H = (C_2)^{1/2}$. Consequently, we obtain the integrating factor

$$\Lambda = (C_2)^{-3/2}\left(\tan^{-1}(\sqrt{C_2}/\alpha) + \frac{\sqrt{C_2}/\alpha}{1+(\sqrt{C_2}/\alpha)^2}\right),$$

and our construction formula yields the first integral

$$\Phi = (C_2)^{-1/2}\tan^{-1}(\sqrt{C_2}/\alpha) - \ln y = C_4. \tag{5.21}$$

The first integrals (5.16), (5.17) and (5.21) reduce the ODE (5.13) to a first order ODE. In particular, we have

$$y'\sqrt{C_1/C_2 - (y')^2} = y\cot(\sqrt{C_2}(C_4 + \ln y)).$$

Isolating y', we obtain

$$y' = \sqrt{C_3}\sin(\sqrt{C_2}(C_4 + \ln y)).$$

6 Conclusion

For any system of ODEs, we have derived determining equations which are necessary and sufficient conditions satisfied by its integrating factors. In particular, the solutions of these determining equations yield all integrating factors. We have also derived a simple explicit formula which yields a first integral for each solution. For an nth-order scalar ODE the determining equations are a linear system of $2n-2$ PDEs consisting of the adjoint of the determining equation for symmetries of the nth-order ODE and an additional $2n-3$ equations when $n \geqslant 2$. No additional equations arise in the case of a first-order scalar ODE.

We have introduced special techniques to seek solutions of the determining equations. These techniques involve the use of known first integrals, eliminations of variables and symmetry considerations. We have exhibited several examples illustrating combinations of these techniques.

References

ANCO, S. C. & BLUMAN, G. (1997) Direct construction of conservation laws from field equations. *Phys. Rev. Lett.* **78**, 2869–2873.

ANCO, S. C. & BLUMAN, G. (1998) Direct construction algorithm for conservation laws of differential equations. Submitted.

BLUMAN, G. (1990) A reduction algorithm for an ordinary differential equation admitting a solvable Lie group. *SIAM J. Appl. Math.* **50**, 1689–1705.

BLUMAN, G. & KUMEL, S. (1987) On invariance properties of the wave equation. *J. Math. Phys.* **28**, 307–318.

GORDON, T. J. (1986) On the symmetries and invariants of the harmonic oscillator. *J. Phys. A* **19**, 183–189.

LIE, S. (1874) *Verhandlungen der Gesselschaft der Wissenschaften zu Christiania.*

LIE, S. (1888) Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. *Math. Ann.* **32**, 213–281.

MIMURA, F. & NÔNO, T. (1994) A new conservation law for a system of second-order differential equations. *Bull. Kyushu Inst. Tech.* **41**, 1–10.

SARLET, W., CANTRIJN, F. & CRAMPIN, M. (1987) Pseudo-symmetries, Noether's theorem and the adjoint equations. *J. Phys. A* **20**, 1365–1376.

SARLET, W., PRINCE, G. E. & CRAMPIN, M. (1990) Adjoint symmetries for time-independent second-order equations. *J. Phys. A* **23**, 1335–1347.

14 SYMMETRY AND INTEGRABILITY BY QUADRATURES OF ORDINARY DIFFERENTIAL EQUATIONS. By Artemio GONZALEZ-LOPEZ (1988)

SYMMETRY AND INTEGRABILITY BY QUADRATURES OF ORDINARY DIFFERENTIAL EQUATIONS

Artemio GONZÁLEZ-LÓPEZ
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Received 27 June 1988; accepted for publication 1 September 1988
Communicated by D.D. Holm

In this paper, the connection between point symmetries and the integrability by quadratures of second-order ordinary differential equations is discussed. An example is given of a family of second-order ordinary differential equations integrable by quadratures whose point symmetry group is, nevertheless, trivial. This refutes the widespread belief that the existence of nontrivial point symmetries is a necessary condition for the integrability by quadratures of ordinary differential equations. The significance of dynamical (versus point) symmetries in this field is illustrated with a few recent results.

1. Introduction

The attempt at understanding the deep reasons underlying the integrability by quadratures of ordinary differential equations was one of the mean reasons that led Sophus Lie to introduce what are now called Lie groups. Basically, Lie's idea was that the symmetry group of a differential equation should play a central role in its integrability by quadratures, much the same as the role played by the Galois group of an algebraic equation in its solubility by radicals. This idea, although never carried out in all its scope, is the leitmotif behind many successful applications of Lie groups to the integration of differential equations. In what follows, we shall only list a couple of key results in this direction, referring the reader to refs. [1–4] for a more comprehensive survey.

(i) *Integrating factor for first-order ordinary differential equations.* An *infinitesimal (point) symmetry* (also called *symmetry vector*) of a differential equation is a vector field generating a one-parameter group of (point) symmetry transformations of the equation [1]. The qualifier "point" in the previous definition means that we are dealing with transformations involving only the dependent and independent variables of the differential equation, but not its derivatives; for a more detailed explanation of this, see e.g. ref. [1]. Perhaps the most general result discovered by Lie connecting symmetry and integrability by quadratures is the fact that the knowledge of a (transverse) infinitesimal symmetry of a first-order ordinary differential equation

$$a(x,y)y'+b(x,y)=0 \qquad (1.1)$$

automatically provides an integrating factor of the equation and, therefore, enables us to integrate it by quadratures. This integrating factor has a very simple expression; indeed, if

$$S=\xi(x,y)\partial_x+\eta(x,y)\partial_y \qquad (1.2)$$

is an infinitesimal symmetry of (1.1), then

$$M=(a\eta+b\xi)^{-1} \qquad (1.3)$$

is its associated integrating factor. As a matter of fact, this result can be used to relate most elementary methods of integrating by quadratures first-order ordinary differential equations to symmetry properties [1, p. 139].

(ii) *Integration by quadratures of second-order ordinary differential equations.* A less well-known result of Sophus Lie is that the knowledge of a two-dimensional group of symmetries of a second-order ordinary differential equation

$$y''=f(x,y,y') \qquad (1.4)$$

makes it possible to integrate it by quadratures

0375-9601/88/$ 03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

[2,4,5]. To give just the flavor of Lie's method, suppose for instance that the generators S_1 and S_2 of the known two-parameter symmetry group of (1.4) commute:

$$[S_1, S_2]=0. \qquad (1.5)$$

Let

$$S_i=\xi_i(x,y)\partial_x+\eta_i(x,y)\partial_y, \quad i=1,2, \qquad (1.6)$$

and

$$\Delta=\begin{vmatrix} 1 & y' & f \\ \xi_1 & \eta_1 & \eta_1'-y'\xi_1' \\ \xi_2 & \eta_2 & \eta_2'-y'\xi_2' \end{vmatrix}, \qquad (1.7)$$

where

$$g'\equiv g_x+y'g_y+fg_{y'}.$$

Then the one-forms

$$\omega_i=\Delta^{-1}\begin{vmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}y' \\ 1 & y' & f \\ \xi_i & \eta_i & \eta_i'-y'\xi_i' \end{vmatrix}, \quad i=1,2, \qquad (1.8)$$

are closed, i.e. locally

$$\omega_i=\mathrm{d}I_i,$$

where the functions I_i can be explicitly computed by integrating (1.8) along suitable paths. Lie showed that I_1 and I_2 are two functionally independent first integrals of (1.4), and therefore this equation can be integrated by quadratures alone.

It is important to emphasize that Lie's method is completely constructive and computational. In fact, it gives an algorithm, requiring only a finite number of differentiations and quadratures, for obtaining two functionally independent first integrals of the differential equation (1.4). This algorithm should not be confused with the general method described in ref. [1] to lower by two the order of an arbitrary ordinary differential equation invariant under a two-parameter Lie group of transformations. Indeed, the latter method, unlike Lie's, requires in general the integration of auxiliary first-order differential equations to find the differential invariants of the symmetry group. Its practical value, therefore, is limited to the cases in which the generators are simple enough that their differential invariants can be found by inspection.

The results described above, and many others, corroborate Lie's belief in the power, generality and elegance of group-theoretical methods for the integration of ordinary differential equations. It has gradually become clear, however, that the connection between symmetry and integrability by quadratures is far less direct and much subtler than these results suggest. To illustrate this point, consider the relatively simple case of linear second-order ordinary differential equations:

$$y''+a_1(x)y'+a_0(x)y+b(x)=0. \qquad (1.9)$$

As is well known [6], the symmetry group of all these equations is isomorphic to SL(3, $\mathbb{R}$). However, it is obvious that not all of them are integrable by quadratures. Hence in this case the structure of the symmetry group is of no help in detecting the integrability of (1.9) by quadratures. It is also remarkable that the symmetry group of any equation of the form (1.9), even if it is not integrable by quadratures, has the *maximum* dimension allowed to any second-order differential equation, namely eight [5,7]. Thus, for example, the equation

$$y''=y^3, \qquad (1.10)$$

which is integrable by quadratures, has a two-dimensional symmetry group (generated by the vector fields ∂_x and $x\partial_x-y\partial_y$), whereas

$$y''+xy=0, \qquad (1.11)$$

which is not integrable by quadratures [8], admits an eight-dimensional group of point symmetries!

This and other examples should make clear that the presence of symmetries is not always sufficient to ensure the integrability by quadratures of a differential equation. It is very tempting, however, to assume the converse, i.e. *that every differential equation which is integrable by quadratures has a nontrivial symmetry group*. In fact, the belief in the latter assertion is so widespread that it has virtually attained the status of a "folk theorem". This is certainly not surprising, since we have not been able to find in the copious literature on the integration of ordinary differential equations a single explicit example of a differential equation with a trivial symmetry group which is integrable by quadratures.

Recently, however, this situation has changed, since there are now available explicit examples of systems of two second-order ordinary differential equations integrable by quadratures but with a trivial symmetry group [9]. It is the goal of this paper to extend the results in ref. [9] to *scalar* second-order ordinary differential equations, presenting examples of equations of this type integrable by quadratures but possessing a trivial symmetry group. The interest of this stems from the fact that the case of second-order scalar differential equations is the simplest from both the mathematical and the physical points of view. Indeed, physically we are dealing with particle motion on a straight line, whereas mathematically scalar second-order equations are the lowest-order and lowest-dimensional differential equations for which the above "folk theorem" is not trivially true. (This is so because, as is well known [5,10], every system of first-order ordinary differential equations possesses an infinite-dimensional symmetry group.) Thus, the existence of scalar second-order differential equations integrable by quadratures and with a trivial symmetry group shows that the lack of a direct and universal connection between symmetry and integrability by quadratures is not due to any subtlety arising from higher-order or higher-dimension peculiarities, but lies at the heart of the matter instead.

We shall end this section with a few words on the organization of this paper. In the next section we shall deal with the details of the construction of second-order differential equations with trivial symmetry group but integrable by quadratures. The concluding section will be devoted to a brief discussion of the possibility of circumventing the limitations on the application of symmetries to the integration by quadratures of differential equations outlined above with the help of dynamical symmetries.

2. Integrable differential equations with a trivial symmetry group

In this section we are going to present a second-order differential equation (or, more precisely, a family of such equations) with no nontrivial point symmetries which is, nevertheless, integrable by quadratures.

Consider, indeed, the differential equation

$$y''=y^{-1}y'^2+pg(x)y^py'+g'(x)y^{p+1} \quad (y>0), \tag{2.1}$$

where $p\neq 0$ is a real constant and $g\neq 0$ is an arbitrary function. Multiplying this equation by the obvious integrating factor y^{-1} and integrating once we obtain the first-order equation

$$y^{-1}y'-g(x)y^p=C,$$

where C is a real constant, or equivalently

$$y'-Cy=g(x)y^{p+1}. \tag{2.2}$$

This is a Bernouilli equation, and therefore [11, p.22] it is integrable by quadratures.

Let us now prove that (2.1) has no nontrivial point symmetries, unless g is of one of the special forms that will be described in a moment. Indeed, let

$$S=\xi(x,y)\partial_x+\eta(x,y)\partial_y \tag{2.3}$$

denote a symmetry vector of (2.1). The necessary and sufficient condition for this is [1,2,5,7]

$$\begin{aligned}\eta^{(2)}&=\eta^{(1)}(2y^{-1}y'+pgy^p)\\&\quad+\eta[-y^{-2}y'^2+p^2gy^{p-1}+(p+1)g'y^p]\\&\quad+\xi(pg'y^py'+g''y^{p+1}).\end{aligned} \tag{2.4}$$

Here, as usual,

$$\eta^{(1)}=\eta'-y'\xi', \quad \eta^{(2)}=(\eta^{(1)})'-f\xi', \tag{2.5}$$

$$'\equiv\partial_x+y'\partial_y+f\partial_{y'}. \tag{2.6}$$

When (2.5) and (2.6) are substituted in (2.4), it is clear that both sides of this equation reduce to polynomials in y'. Equating the coefficients of these polynomials, we obtain the following system of partial differential equations for ξ and η:

$$y\xi_{yy}+\xi_y=0, \tag{2.7}$$

$$y^2\eta_{yy}-y\eta_y+\eta=2y^2(\xi_{xy}+pgy^p\xi_y, \tag{2.8}$$

$$\begin{aligned}2\eta_{xy}-\xi_{xx}&=pgy^p\xi_x+3g'y^{p+1}\xi_y\\&\quad+2y^{-1}\eta_x+p^2gy^{p-1}\eta+pg'y^p\xi,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\eta_{xx}&=(2\xi_x-\eta_y)g'y^{p+1}+pgy^p\eta_x\\&\quad+(p+1)g'y^p\eta+g''y^{p+1}\xi.\end{aligned} \tag{2.10}$$

The first of these equations is easily solved to yield

$$\xi=d(x)+c(x)\ln y. \tag{2.11}$$

Eq. (2.8) is an Euler equation [11,p.141]. Its general solution, taken (2.11) into account, is found to be

$$\eta=y[a(x)+b(x)\ln y+c'(x)(\ln y)^2] + \frac{2}{p}c(x)g(x)y^{p+1}\,. \tag{2.12}$$

Substituting now (2.11) and (2.12) into (2.9) and simplifying we obtain the following equation:

$$-2pcg^2y^{2p}+[3c''-p(gc)'y^p]\ln y - p^2cgy^p(\ln y)^2+(4c'g+cg')y^p = (d''-2b')+py^p(dg)'+2p^2gy^p(a+b\ln y)\,. \tag{2.13}$$

From the first term in the left-hand side of (2.13) we readily obtain

$$c=0. \tag{2.14}$$

Substituting this back in (2.14) yields

$$b=0 \tag{2.15}$$

and

$$d''=0, \tag{2.16}$$

$$(gd)'+pag=0. \tag{2.17}$$

Finally, it is immediate to check that when eqs. (2.14)–(2.16) are taken into account (2.11) reduces to

$$a''=0. \tag{2.18}$$

Now when $ad\neq 0$, i.e. when eq. (2.1) admits nontrivial symmetries, the form of g is obviously restricted by (2.17). It is actually quite easy to find explicitly all the solutions of (2.17), in view of (2.16) and (2.18). The result is

$$g(x)=k_1\,e^{k_2x}(k_3+k_4x)^{k_5}, \tag{2.19}$$

or

$$g(x)=k_6\,e^{k_7x^2}, \tag{2.20}$$

where k_1, ..., k_7 are real constants related to a and d. (When g is of the form (2.20), (2.17) forces d to be constant, whereas when g is of the form (2.19) the most general d is an affine function of x.) Therefore, unless g is of one of the special forms (2.19) or (2.20) (which include, in particular, the case $g=0$ discarded at the outset), (2.1) has no nonzero infinitesimal symmetries. To conclude this section, let us summarize our findings in a theorem:

Theorem. If g is not of the form (2.19) or (2.20), the second-order differential equation (2.1) admits no nontrivial symmetries, but it is nevertheless integrable by quadratures.

3. The role of dynamical symmetries

As the example presented in the previous section shows, *the absence of point symmetries is no obstruction to the integrability by quadratures of a differential equation.* In other words, Lie's original program of explaining integrability by quadratures via symmetry properties, while certainly fruitful and inspiring, cannot be carried out in its entirety, at least within the restricted framework of *point* symmetries.

In this section we shall briefly discuss the possibility of overcoming this difficulty by using dynamical symmetries. A *dynamical symmetry* (also called Lie–Bäcklund or generalized symmetry by other authors, cf. ref. [1]) of a second-order ordinary differential equation

$$y''=f(x,y,y') \tag{3.1}$$

is a vector field

$$X=\xi(x,y,y')\partial_x+\eta(x,y,y')\partial_y + \zeta(x,y,y')\partial_{y'}, \tag{3.2}$$

such that

$$[A,X]=\rho(x,y,y')A, \tag{3.3}$$

where

$$A=\partial_x+y'\partial_y+f(x,y,y')\partial_{y'} \tag{3.4}$$

is the vector field associated to the differential equation (3.1). If

$$S\equiv\xi(x,y)\partial_x+\eta(x,y)\partial_y$$

is an infinitesimal symmetry of (3.1), then its first prolongation

$$S^{(1)}=S+\eta^{(1)}(x,y,y')\partial_{y'} \tag{3.5}$$

(cf (2.5)) is a dynamical symmetry of this equation. Thus the concept of dynamical symmetry generalizes that of infinitesimal (point) symmetry introduced earlier. Out of a given dynamical symmetry (3.2) it is always possible to construct a new dynamical symmetry $\tilde{X}$ whose x-component vanishes, namely

$$\tilde{X}=X-\xi A. \tag{3.6}$$

Indeed, from (3.3) and (3.6) actually follows the stronger equality

$$[A, \tilde{X}]=0. \tag{3.7}$$

In what follows, we shall always implicitly assume that our dynamical symmetries satisfy the above normalization, i.e.

$$\xi=0. \tag{3.8}$$

A dynamical symmetry satisfying (3.8) is often said to be in *evolutionary form*, cf. ref. [1].

The rationale of using dynamical symmetries is that, while differential equations seldom possess nontrivial point symmetries, it can be shown that every ordinary differential equation admits an infinity of linearly independent dynamical symmetries [10]. On the other hand, it can also be shown that Lie's result on the integrability by quadratures of second-order ordinary differential equations invariant under a two-dimensional group of point symmetries quoted in the introduction can be suitably generalized to dynamical symmetries. More precisely [12], we have the following theorem:

Theorem 3.1. If two nonproportional dynamical symmetries – not necessarily generating a Lie algebra – of the second-order differential equation (3.1) are known, then the latter equation can be integrated by two quadratures at most.

It is therefore not unreasonable to conjecture that, whenever a second-order differential equation is integrable by quadratures, it is always possible to effect its integration by the above mechanism. Remarkably enough, this is indeed the case [13]:

Theorem 3.2. If the second-order differential equation (3.1) is integrable by quadratures, it is always possible to find two (commuting) nonproportional dynamical symmetries of the latter equation by quadratures alone.

The preceding theorems strongly suggest that there is a very close link between integrability by quadratures of ordinary differential equations and *dynamical*, instead of *point*, symmetries. This certainly agrees with the spirit – if not entirely with the letter – of Lie's original views on this matter discussed in the introduction. Additional work on this subject, including its generalization to ordinary differential equations of arbitrary order, is currently in progress.

References

[1] P. Olver, Applications of Lie groups to differential equations (Springer, Berlin, 1986).
[2] A. Cohen, An introduction to the Lie theory of one-parameter groups (Heath, New York, 1911).
[3] L.E. Dickson, Ann. Math. 25 (1924) 287.
[4] L. Bianchi, Lezioni sulla teoría dei gruppi continui finiti di trasformazioni (Spoerri, Pisa, 1918).
[5] S. Lie, Differentialgleichungen (Teubner, Leipzig, 1891).
[6] A. González-López, J. Mat. Phys. 29 (1988) 1097.
[7] F. González-Gascón and A. González-López, J. Math. Phys. 24 (1983) 2006.
[8] I. Kaplansky, An introduction to differential algebra (Hermann, Paris, 1976) §26.
[9] F. González-Gascón and A. González-López, Phys. Lett. A 129 (1988) 153.
[10] F. González-Gascón and A. González-López, Lett. Nuovo Cimento 32 (1981) 353.
[11] E.L. Ince, Ordinary differential equations (Dover, New York, 1957) [reprint].
[12] F. González-Gascón and A. González-López, Dynamical symmetries and integration of differential equations, Proc. International Days on Synergetics, order and chaos, Madrid (Spain), October 1987.
[13] A. González-López, Integration by quadratures of ordinary differential equations, in preparation.

15 Integrating Factors and ODE Patterns. By E.S. Cheb-Terraba, A.D. Rochea (1997)

Universidade do Estado do Rio de Janeiro
Instituto de Física

IF-UERJ-2/97
Preprint
November 1997

Integrating Factors and ODE Patterns

E.S. Cheb-Terrab[1,2], A.D. Roche[1]

Abstract

A systematic algorithm for building integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$ for non-linear second order ODEs is presented. When such an integrating factor exists, the algorithm determines it *without solving any differential equations*. Examples of ODEs not having point symmetries are shown to be solvable using this algorithm. The scheme was implemented in Maple, in the framework of the *ODEtools* package and its ODE-solver. A comparison between this implementation and other computer algebra ODE-solvers in tackling non-linear examples from Kamke's book is shown.

(Submitted for publication in Journal of Symbolic Computation)

[1]Department of Computer Science, Faculty of Mathematics, University of Waterloo, Ontario, Canada
[2]Symbolic Computation Group, Departamento de Física Teórica, IF-UERJ.
Available as http://dft.if.uerj.br/preprint/e7-2.tex; also as http://lie.uwaterloo.ca/odetools/e7-2.tex

1 Introduction

¿From a practical point of view, when developing solving methods for ODEs, what one actually does is attempt to determine families of ODEs which can be transformed into algebraic problems or into simple ODEs such as[3] $y' = F(x)$ or $y' = F(y)$ by changes of variables or equivalent processes. For high order ODEs, one hopes that such a simplification of the problem will be possible after successive reductions of order. Some more powerful schemes are also able to exploit other information, as for instance integrating factors or the ODE's symmetries, and so to try a multiple reduction of order at once (see for instance [2] and [3]).

In the specific case of integrating factors, although in principle it can always be determined whether a given ODE is exact (a total derivative), there is no known universal scheme for making ODEs exact. Actually, for n^{th}order ODEs - as in the case of symmetries - integrating factors are determined as solutions of an n^{th}order linear PDE in n+1 variables, and to solve this *determining* PDE is a major problem in itself.

Bearing this in mind, this paper presents a method for obtaining integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ for non-linear second order explicit ODEs[4], using a different approach, based only on a computerized analysis of the pattern of the given ODE. That is, for a given ODE, if an integrating factor with such a functional dependence exists, the scheme returns the integrating factor itself without solving any differential equations.

The exposition is organized as follows. In sec. 2, the use of integrating factors for solving ODEs is briefly reviewed. In sec. 3, the scheme for obtaining the aforementioned integrating factors $\mu(x, y')$ or $\mu(y, y')$ is presented and some examples are given. In sec. 4, some aspects of the integrating factor and symmetry approaches are reviewed, and their complementariness is illustrated with two ODE families not having point symmetries. Sec. 5 contains some statistics concerning the new solving method and the second order non-linear ODEs found in Kamke's book, as well as a comparison of performances of computer algebra packages in solving a related subset of these ODEs. In sec. 6 the computer algebra implementation of the scheme in the framework of the ODEtools package [4] is outlined, and a description of the package's new command, **redode**, is presented. Finally, the conclusions contain some general remarks about this work and its possible extensions.

Aside from this, in the Appendix, a table containing extra information concerning integrating factors for some of Kamke's ODEs is presented.

2 Integrating factors and reductions of order

2.1 First order ODEs

The idea of looking for an integrating factor (μ) is usually presented in the framework of solving a given first order ODE, say,

$$y' = \Phi(x, y) \tag{1}$$

If by multiplying Eq.(1) by a factor $\mu(x, y)$, the ODE becomes a total derivative[5],

$$\mu(x, y)\,(y' - \Phi(x, y)) = \frac{d}{dx}\; R(x, y) \tag{2}$$

for some function R, then one can look for μ as a solution to the first order PDE:

$$\frac{\partial \mu}{\partial x} + \frac{\partial}{\partial y}\left(\mu\,\Phi\right) = 0 \tag{3}$$

which arises as the exactness condition for the problem (see Eq.(7)). Once μ has been obtained, $R(x, y)$ - an implicit form solution - can be calculated as a line integral.

[3]Throughout this article, we use the notation $y = y(x), y' = \frac{dy}{dx}, y^{(n)} = \frac{d^n y}{dx^n}$.

[4]We say that a second order ODE is in explicit form when it appears as $y'' - \Phi(x, y, y') = 0$.

[5]In this paper we use the term "integrating factor" in connection with the explicit form of the ODE, i.e., the ODE, turned exact by taking the product $\mu\, ODE$, is assumed to be of the form $y'' = \Phi(x, y, y')$ or $y'' - \Phi(x, y, y') = 0$.

Although to solve Eq.(3) for μ is as difficult as the original problem, it turns out that for a given $N(x,y)$, when a solution of the form $\mu(x,y) = \widetilde{\mu}\,(q)\,N(x,y)$ exists - q is either x or y only - μ can be determined by solving an auxiliary linear first order ODE. For example, introducing $\mu(x,y) = \widetilde{\mu}\,(x)\,N(x,y)$ and $M(x,y) = N(x,y)\,\Phi(x,y)$, one obtains:

$$\widetilde{\mu}\,(x) = C_1\, e^{-\int \frac{1}{N}\left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dx} \tag{4}$$

and a solution $\widetilde{\mu}\,(x)$ exists only when the integrand in above does not depend on y. This gives both an existence condition and an explicit solution to the problem; however, the advantages of the scheme are only apparent since there is no way to determine in advance what would be the appropriate $N(x,y)$.

2.2 High order ODEs

Integrating factors for high order ODEs are defined as in the first order case. Here, we consider $\mu(x,y,y',...,y^{(n-1)})$ to be an integrating factor for an n^{th}order ODE, say

$$y^{(n)} = \Phi(x,y,y',...,y^{(n-1)}) \tag{5}$$

if after multiplying the explicit ODE by μ we obtain a total derivative:

$$\mu\left(y^{(n)} - \Phi\right) = \frac{dR}{dx} \tag{6}$$

for some function $R(x,y,y',...,y^{(n-1)})$. To determine μ, one can try to solve for it in the exactness condition, obtained applying Euler's operator to the total derivative $H \equiv \mu\left(y^{(n)} - \Phi\right)$:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial y''}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial H}{\partial y^{(n)}}\right) = 0 \tag{7}$$

Now, it can be shown by induction that Eq.(7) is always of the form

$$A(x,y,y',...,y^{(2n-3)}) + y^{(2n-2)} B(x,y,y',...,y^{(n-1)}) = 0 \tag{8}$$

where A is of degree n-1 in $y^{(n)}$ and linear in $y^{(k)}$ for $n < k \leq (2n-3)$, so that Eq.(8) can be split into an overdetermined system of PDEs for μ. For example, for second order ODEs Eq.(7) is of the form

$$A(x,y,y') + y''\ B(x,y,y') = 0 \tag{9}$$

Hence, by taking $A(x,y,y') = 0$ and $B(x,y,y') = 0$ we have a system of two PDEs for μ:

$$\begin{aligned} A(x,y,y') \equiv & \\ & \left(\frac{\partial^2 \mu}{\partial y' \partial x} + \left(\frac{\partial^2 \mu}{\partial y' \partial y}\right) y' - \frac{\partial \mu}{\partial y}\right) \Phi + \left(\frac{\partial^2 \Phi}{\partial y' \partial x} - \frac{\partial \Phi}{\partial y} + \left(\frac{\partial^2 \Phi}{\partial y' \partial y}\right) y'\right) \mu \\ & + \left(\frac{\partial^2 \mu}{\partial y^2}\right) y'^2 + \left(\left(\frac{\partial \mu}{\partial y'}\right) \frac{\partial \Phi}{\partial y} + \left(\frac{\partial \mu}{\partial y}\right) \frac{\partial \Phi}{\partial y'} + 2\left(\frac{\partial^2 \mu}{\partial x \partial y}\right)\right) y' \\ & + \left(\frac{\partial \mu}{\partial y'}\right) \frac{\partial \Phi}{\partial x} + \left(\frac{\partial \mu}{\partial x}\right) \frac{\partial \Phi}{\partial y'} + \frac{\partial^2 \mu}{\partial x^2} = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} B(x,y,y') \equiv & \\ & 2\,\frac{\partial \mu}{\partial y} + \left(\frac{\partial^2 \mu}{\partial y'^2}\right) \Phi + 2\left(\frac{\partial \mu}{\partial y'}\right) \frac{\partial \Phi}{\partial y'} + \mu \frac{\partial^2 \Phi}{\partial y'^2} + \frac{\partial^2 \mu}{\partial y' \partial x} + \left(\frac{\partial^2 \mu}{\partial y' \partial y}\right) y' = 0 \end{aligned} \tag{11}$$

Nonetheless, there are no general rules which might help in solving these PDEs[6].

Alternatively, a possible strategy for directly obtaining R instead of looking for μ can be formulated as follows. Consider the first order linear operator associated to Eq.(5)

$$\mathrm{A} : f \rightarrow \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + ... + \Phi \frac{\partial f}{\partial y^{(n-1)}} \tag{12}$$

where x, y and its derivatives are all treated as independent variables on the same footing. Now

$$A(R) = 0 \tag{13}$$

and there are n functionally independent solutions (first integrals) to the problem. In some cases, a first integral R such that $dR/dy^{(n-1)} \neq 0$ can be obtained as the solution to a subset of the characteristic strip of $A(R) = 0$, or by other means.

3 Integrating Factors and ODE patterns

Since the classical way for determining integrating factors leads to problems similar in difficulty to solving the ODEs themselves, we consider here a different approach, based on a careful matching of an ODE pattern.

The starting point is the observation that it is trivial to solve the inverse problem; i.e., to find the most general ODE having a given μ. In fact, from Eq.(6), we have

$$\mu(x, y, y', ..., y^{(n-1)}) = \frac{\partial R}{\partial y^{(n-1)}} \tag{14}$$

and hence the reduced ODE R is of the form

$$R = G(x, y, ..., y^{(n-2)}) + \int \mu \; dy^{(n-1)} \tag{15}$$

for some function G. Inserting Eq.(15) into Eq.(13) and solving for $y^{(n)}$ leads to the general form of an ODE having μ as integrating factor:

$$y^{(n)} = \frac{-1}{\mu} \left[\frac{\partial}{\partial x} \left(\int \mu \, dy^{(n-1)} + G \right) + ... + y^{(n-1)} \frac{\partial}{\partial y^{(n-2)}} \left(\int \mu \, dy^{(n-1)} + G \right) \right] \tag{16}$$

The expression above then becomes an *ODE pattern* which one can try to match against an input ODE. The generation of such *pattern matching routines* is difficult, even for restricted subfamilies of integrating factor, but once built, they are a powerful and computationally efficient way to reduce the order of the corresponding ODEs (see sec. 5).

3.1 Second order ODEs and the integrating factor family $\mu(x, y')$

In the case of second order ODEs, if instead of considering the general case $\mu(x, y, y')$ we restrict the family of integrating factors under consideration to $\mu(x, y')$, Eq.(15) - the reduced ODE - becomes

$$R(x, y, y') = F(x, y') + G(x, y) \tag{17}$$

for some functions G and F, where

$$\mu(x, y') = F_{y'}(x, y') \tag{18}$$

(we denote $F_{y'} = \frac{\partial F}{\partial y'}$). Eq.(16) can then be written in terms of F and G as

[6] In a recent work by [5] (1997), the authors arrive at Eq.(9) and Eq.(11) - numbered there as (3.5) and (3.8) - departing from the adjoint linearized system corresponding to a given ODE; the possible splitting of Eq.(8) into an overdetermined system for μ is also mentioned. However, in formula (3.5) of that work, y'' of Eq.(9) above appears replaced by $\Phi(x, y, y')$, and the authors discuss possible alternatives to tackle Eqs.(9) and (11) instead of Eqs.(10) and (11).

$$y'' = \Phi(x,y,y') \equiv -\frac{F_x(x,y') + G_x(x,y) + G_y(x,y)\, y'}{F_{y'}(x,y')} \tag{19}$$

The idea is now to build a routine to determine if a given ODE can be written in the form Eq.(19), in which case it will have an integrating factor of the form $\mu(x,y')$, and if so, determine it, leading to the reduced ODE Eq.(17) by means of standard methods (see for instance [6] p.221). The feasibility of such a computational routine is based on the following theorem.

Theorem 1 *Given a nonlinear second order ODE*

$$y'' = \Phi(x,y,y') \tag{20}$$

($\frac{\partial \Phi}{\partial y} \neq 0$) [7] *for which an integrating factor of the form $\mu(x,y')$ exists, such an integrating factor can be systematically determined without solving any differential equations.*

PROOF. We divide the proof in two steps. In the first step we *assume* that, given Eq.(20), it is always possible to determine $\mu(x,y')$ up to a factor depending on x; that is, to find some $\mathcal{F}(x,y')$ satisfying

$$\mathcal{F}(x,y') = \frac{\mu(x,y')}{\widetilde{\mu}\,(x)} \tag{21}$$

for some unknown function $\widetilde{\mu}\,(x)$. We then prove that the knowledge of $\mathcal{F}(x,y')$ is enough to determine $\widetilde{\mu}\,(x)$ by means of a simple integral, hence leading to the desired $\mu(x,y')$.

In a second step, we prove our assumption, that is, we show how to find $\mathcal{F}(x,y')$ satisfying Eq.(21), concluding the proof of the theorem.

3.1.1 Determination of $\widetilde{\mu}\,(x)$ when $\mathcal{F}(x,y')$ is known

Starting with the first aforementioned step, we assume that we can determine $\mathcal{F}(x,y')$. It follows from Eqs.(18), (19) and (21) that

$$\frac{\partial}{\partial y}\Big(\Phi(x,y,y')\ \mathcal{F}(x,y')\Big) = \frac{G_{y\,x}(x,y) + G_{y\,y}(x,y)\, y'}{\widetilde{\mu}\,(x)} \tag{22}$$

so that by taking coefficients of y' in $\frac{\partial \Phi}{\partial y}\mathcal{F}$ we obtain

$$\begin{aligned}
\varphi_1 &\equiv \Phi_y(x,y,y')\,\mathcal{F}(x,y') - y'\,\frac{\partial}{\partial y'}\Big(\Phi_y(x,y,y')\,\mathcal{F}(x,y')\Big) = \frac{G_{y\,x}(x,y)}{\widetilde{\mu}\,(x)} \\
\varphi_2 &\equiv \frac{\partial}{\partial y'}\Big(\Phi_y(x,y,y')\,\mathcal{F}(x,y')\Big) = \frac{G_{y\,y}(x,y)}{\widetilde{\mu}\,(x)}
\end{aligned} \tag{23}$$

Similarly, we obtain

$$\begin{aligned}
\varphi_3 &\equiv -\frac{\partial}{\partial y'}\Big(\Phi(x,y,y')\ \mathcal{F}(x,y')\Big) = \frac{F_{y'\,x}(x,y') + G_y(x,y)}{\widetilde{\mu}\,(x)} \\
\varphi_4 &\equiv \frac{\partial}{\partial y'}\mathcal{F}(x,y') = \frac{F_{y'\,y'}(x,y')}{\widetilde{\mu}\,(x)}
\end{aligned} \tag{24}$$

Now, since Eq.(20) is nonlinear by hypothesis, either φ_2 or φ_4 is different from zero, so that at least one of the pairs of ratios $\{\varphi_1,\ \varphi_2\}$ or $\{\varphi_3,\ \varphi_4\}$ can be used to determine $\widetilde{\mu}\,(x)$ as the solution of an auxiliary first order linear ODE. For example, if $\varphi_2 \neq 0$,

[7] ODEs *missing* y may also have integrating factors of the form $\mu(x,y')$, which cannot be determined using the scheme here presented. However, such integrating factors are not really relevant since these ODEs can always be reduced to first order by a simple change of variables.

$$\frac{\partial}{\partial y}\left(\varphi_1(x,y)\,\widetilde{\mu}(x)\right) = \frac{\partial}{\partial x}\left(\varphi_2(x,y)\,\widetilde{\mu}(x)\right) \tag{25}$$

and we obtain

$$\widetilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_2}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right) dx} \tag{26}$$

If $\varphi_2 = 0$ then $\varphi_4 \neq 0$ and we obtain

$$\widetilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_4}\left(\frac{\partial \varphi_3}{\partial y'} - \frac{\partial \varphi_4}{\partial x}\right) dx} \tag{27}$$

Eqs.(26) and (27) alternatively give both an explicit solution to the problem and an existence condition, since a solution $\widetilde{\mu}(x)$ - and hence an integrating factor of the form $\mu(x,y')$ - exists if the integrand in Eq.(26) or Eq.(27) only depends on x. $\triangle$

Example: Kamke's ODE 37.

$$y'' = -2\,y\,y' - f(x)\left(y' + y^2\right) + g(x) \tag{28}$$

This example is interesting[8] because it has no point symmetries for arbitrary $f(x)$ and $g(x)$ (see sec. 4). For this ODE, $\mathcal{F}(x,y')$ was determined (see sec. 3.1.2) as:

$$\mathcal{F}(x,y') = 1 \tag{29}$$

from which (Eq.(22))

$$\frac{G_{y\,x}(x,y) + G_{y\,y}(x,y)\,y'}{\widetilde{\mu}(x)} = -2y\,f(x) - 2y' \tag{30}$$

and then as in Eq.(23) we obtain

$$\begin{aligned} \varphi_1 &= -2y\,f(x) \\ \varphi_2 &= -2 \end{aligned} \tag{31}$$

Using this in Eq.(26), we get

$$\widetilde{\mu}(x) = e^{\displaystyle\int f(x)\,dx} \tag{32}$$

and so, from Eq.(21), since $\mathcal{F}(x,y') = 1$, $\mu(x,y') = \widetilde{\mu}(x)$.

3.1.2 Determination of $\mathcal{F}(x,y')$

We now prove our assumption, that is, we show how to obtain a function $\mathcal{F}(x,y')$ satisfying Eq.(21) from the knowledge of $\Phi(x,y,y')$, and without solving any differential equations.

Since we have already assumed that the given ODE has an integrating factor of the form $\mu(x,y')$, then there exist some functions $F(x,y')$ and $G(x,y)$ such that it is possible to rewrite $\Phi(x,y,y')$ - the right-hand-side (RHS) of the given ODE - as in Eq.(19). We then start by considering the expression

$$\Upsilon \equiv \frac{\partial \Phi}{\partial y} = -\frac{G_{x\,y}(x,y) + G_{y\,y}(x,y)\;y'}{F_{y'}(x,y')} \tag{33}$$

[8]For ODE 6.37, Kamke shows a reduction of order to a general Riccati ODE, based on the theory for ODEs having solutions with no movable critical points - see [7], and [8] p 331.

and the possible cases.

Case A

The first case happens when the ratio $G_{x\,y}(x,y)/G_{y\,y}(x,y)$ *depends* on y; i.e., $G_{x\,y}(x,y)$ and $G_{y\,y}(x,y)$ are linearly independent w.r.t y. To determine whether this is the case, note that we cannot just analyze the mentioned ratio itself since it is unknown. However, we can always select the factors of Υ containing y, and check if this expression *also* contains y'. If so, we just determine $F_{y'}(x,y')$ up to a factor depending on x, that is, the required $\mathcal{F}(x,y')$, as the reciprocal of the factors of Υ which depend on y' but not y. $\triangle$

Example: Kamke's ODE 226

This ODE is presented in Kamke's book already in exact form, so we start by rewriting it in explicit form as

$$y'' = \frac{x^2 y y' + x y^2}{y'} \tag{34}$$

We determine Υ (Eq.(33)) as

$$\Upsilon = \frac{x(xy' + 2y)}{y'} \tag{35}$$

The only factor of Υ containing y is:

$$xy' + 2y \tag{36}$$

and since this also depends on y', $\mathcal{F}(x,y')$ is immediately given by

$$\mathcal{F}(x,y') = y' \tag{37}$$

Case B

When the expression formed by all the factors of Υ containing y does not contain y', it is impossible to determine *a priori* whether one of the functions $\{G_{x\,y}(x,y),\ G_{y\,y}(x,y)\}$ is zero, or alternatively their ratio does not depend on y. We then proceed by assuming the former, build an expression for $\mathcal{F}(x,y')$ as in Case A, and determine $\widetilde{\mu}\,(x)$ as explained in the previous subsection. If this doesn't lead to the required integrating factor, we then proceed as follows.

Case C

In this case, we assume that neither $G_{x\,y}(x,y)$ nor $G_{y\,y}(x,y)$ are zero and their ratio is a function of just x, so that we have

$$\begin{aligned} G_{x\,y}(x,y) &= v_1(x)\ w(x,y) \\ G_{y\,y}(x,y) &= v_2(x)\ w(x,y) \end{aligned} \tag{38}$$

for some unknown functions $v_1(x)$ and $v_2(x)$, such that Eq.(33) can be factored as

$$\Upsilon = w(x,y)\ \frac{(v_1(x) + v_2(x)\ y')}{F_{y'}(x,y')} \tag{39}$$

for some function $w(x,y)$ which *can always be determined* as the factors of Υ depending on y. To determine $F_{y'}(x,y')$ up to a factor depending on x, we then need to determine the ratio $v_1(x)/v_2(x)$. For this purpose, from Eq.(38) we build an auxiliary PDE for $G_y(x,y)$,

$$G_{x\,y}(x,y) = \frac{v_1(x)}{v_2(x)}\, G_{y,y}(x,y) \tag{40}$$

The general solution of Eq.(40) is given by

$$G_y(x,y) = \mathcal{G}\,(y + p(x)) \tag{41}$$

where $\mathcal{G}$ is an arbitrary function of its argument and for convenience we introduced

$$p'(x) \equiv v_1(x)/v_2(x) \tag{42}$$

We can now determine $p'(x)$, that is, the ratio v_1/v_2 we were looking for, as follows. Taking into account Eq.(38), we arrive at

$$v_2(x)\, w(x,y) = \mathcal{G}'(y + p(x)) \tag{43}$$

By taking the ratio between this expression and its derivative w.r.t y we obtain

$$\mathcal{H}(y + p(x)) \equiv \frac{\partial w}{\partial y}/w = \frac{\mathcal{G}''(y + p(x))}{\mathcal{G}'(y + p(x))} \tag{44}$$

that is, a function of $y + p(x)$ only, which we can determine since we know $w(x,y)$. If $\mathcal{H}' \neq 0$, we obtain $p'(x)$ as

$$p'(x) = \frac{\partial \mathcal{H}}{\partial x}/\frac{\partial \mathcal{H}}{\partial y} = \frac{\left(\frac{\partial^2 w}{\partial y^2 \partial x}\right)\ w - \left(\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}}{\left(\frac{\partial^2 w}{\partial y^2}\right)\ w - \left(\frac{\partial w}{\partial y}\right)^2} \tag{45}$$

Once we determined $p'(x)$, from Eq.(39) we determine $\mathcal{F}(x, y')$ as

$$\mathcal{F}(x, y') = \frac{(p' + y')\ w}{\Upsilon} \tag{46}$$

where Υ, $w(x,y)$ and $p'(x)$ are now all known. △

Example: Kamke's ODE 136.

We begin by writing the ODE in explicit form as

$$y'' = \frac{h(y')}{x - y} \tag{47}$$

This example is interesting since the standard search for point symmetries is frustrated from the very beginning: the determining PDE for the problem will not split due to the presence of an arbitrary function of y'. Here Υ (Eq.(33)) is determined as

$$\Upsilon = -\frac{h(y')}{(x - y)^2} \tag{48}$$

and $w(x,y)$ as

$$w(x,y) = \frac{1}{(x - y)^2} \tag{49}$$

Then $\mathcal{H}(y + p(x))$ (Eq.(44)) becomes

$$\mathcal{H} = \frac{2}{x - y} \tag{50}$$

and hence, from Eq.(45), $p'(x)$ is

$$p'(x) = -1 \tag{51}$$

so from Eq.(46):

$$\mathcal{F}(x, y') = \frac{1 - y'}{h(y')} \tag{52}$$

<u>Case D</u>

We now discuss how to obtain $p'(x)$ when $\mathcal{H}'(y + p(x)) = 0$. We consider at first the case in which $\mathcal{H} = 0$, hence $\mathcal{G}'' = 0$, and so, recalling Eq.(41), we see that

$$G(x,y) = B_1\ (y+p(x))^2 + B_2\ (y+p(x)) + g(x) \tag{53}$$

for some function $g(x)$ and some constants B_1, B_2. Recalling Eq.(19), $\Phi(x,y,y')$ takes the form

$$\Phi(x,y,y') = -\frac{F_x(x,y') + g'(x) + (2B_1\ (y+p(x)) + B_2)(y'+p'(x))}{F_{y'}(x,y')} \tag{54}$$

We can now obtain explicit equations where the only unknown is $p(x)$ as follows. First, from the knowledge of Υ and Φ we build the two explicit expressions:

$$\Lambda \equiv \frac{1}{\Upsilon} = -\frac{F_{y'}}{2B_1\ (y'+p'(x))} \tag{55}$$

and

$$\Psi \equiv \frac{\Phi(x,y,y')}{\Upsilon} - y = \frac{F_x + g'(x)}{2B_1\ (y'+p'(x))} + p(x) + \frac{B_2}{2B_1} \tag{56}$$

It is now clear from Eq.(55) and Eq.(56) that Λ and Ψ are related by the following equation:

$$\frac{\partial}{\partial x}\left((y'+p'(x))\ \Lambda\right) + \frac{\partial}{\partial y'}\left((y'+p'(x))\ \Psi\right) = p(x) + \frac{B_2}{2B_1} \tag{57}$$

where the only unknowns are $p(x)$, B_1, and B_2. By differentiating the equation above w.r.t y' and x we obtain two equations where the only unknown is $p'(x)$:

$$\Lambda_{y'}p''(x) + (\Lambda_{xy'} + \Psi_{y'y'})(y'+p'(x)) + \Lambda_x + 2\Psi_{y'} = 0 \tag{58}$$

$$\Lambda\ p'''(x) + (\Lambda_{xx} + \Psi_{y'x})(y'+p'(x)) + (\Lambda_x + \Psi_{y'})p''(x) + \Psi_x = p'(x) \tag{59}$$

As a shortcut, if $(\Lambda_{xy'} + \Psi_{y'y'})/\Lambda_{y'}$ depends on y', then we can build a linear algebraic equation for $p'(x)$ by solving for $p''(x)$ in Eq.(58) and differentiating w.r.t. y'. Otherwise, in general we obtain $p'(x)$ by solving a linear algebraic equation built by eliminating $p''(x)$ between Eq.(58) and Eq.(59)[9].

If Eq.(58) depends neither on $p'(x)$ nor on $p''(x)$ this scheme will not succeed. However, it is possible to prove that in that case the original ODE is already linear, and easy to solve. To see this, we set to zero the coefficients of $p'(x)$ and $p''(x)$ in Eq.(58), obtaining:

$$\Lambda_{y'} = \Lambda_{xy'} + \Psi_{y'y'} = \Lambda_x + 2\Psi_{y'} = 0 \tag{60}$$

from which Λ is a function of x only, and then

$$\Psi_{y'y'} = 0 \tag{61}$$

If we now rewrite $F(x,y')$ as in

$$F(x,y') = Z(x,y') - g(x) - \Lambda(y'+p')^2\, B_1 \tag{62}$$

and introduce this expression in Eq.(55), we obtain $Z_{y'} = 0$; similarly, using this result, Eq.(56), Eq.(61) and Eq.(62) we obtain $Z_x = 0$. Hence, Z *is a constant.* Finally, taking into account that Z is constant, Eq.(62) and Eq.(54), we see that the ODE Eq.(20) which led us to this case is just a non-homogeneous linear ODE of the form

$$(y+p)'' + (\Lambda'(y+p)' - 2(y+p) - B_2/B_1)/2\Lambda = 0 \tag{63}$$

which homogeneous part does not depend on $p(x)$:

[9] From Eq.(55), $\Lambda \neq 0$, so that Eq.(59) always depends on $p'''(x)$, and solving Eq.(58) for $p''(x)$ and substituting twice into Eq.(59) will lead to the desired equation for $p'(x)$. If Eq.(58) depends on $p'(x)$ but not on $p''(x)$, then Eq.(58) itself is already a linear algebraic equation for $p'(x)$.

$$y'' + \frac{\Lambda'(x)}{2\Lambda(x)}\, y' - \frac{y}{\Lambda(x)} = 0 \tag{64}$$

and which solution is in any case straightforward. $\triangle$

Example: Kamke's ODE 66.

This ODE is given by

$$y'' = a\,(c + bx + y) \left(y'^2 + 1\right)^{3/2} \tag{65}$$

Proceeding as in Case A, we determine Υ, $w(x,y)$, and $\mathcal{H}(y+p(x))$ as

$$\Upsilon = a \left(y'^2 + 1\right)^{3/2}; \qquad w(x,y) = 1; \qquad \mathcal{H} = 0 \tag{66}$$

From the last equation we realize that we are in Case D. We determine Λ and Ψ (Eqs. (55), (56)) as:

$$\begin{aligned} \Lambda &= \frac{1}{\left(y'^2 + 1\right)^{3/2} a} \\ \Psi &= c + b\,x \end{aligned} \tag{67}$$

We then build Eq.(57) for this ODE:

$$\frac{p''(x)}{\left(y'^2 + 1\right)^{3/2} a} + c + b\,x = p(x) + \frac{B_2}{2B_1} \tag{68}$$

Differentiating w.r.t. y' leads to Eq.(58):

$$-\,3 \frac{p''(x)\,y'}{\left(y'^2 + 1\right)^{5/2} a} = 0 \tag{69}$$

from which it follows that $p''(x) = 0$. Using this in Eq.(59) we obtain:

$$p'(x) = b \tag{70}$$

after which Eq.(46) becomes

$$\mathcal{F}(x, y') = \frac{y' + b}{a\left(y'^2 + 1\right)^{3/2}} \tag{71}$$

<u>Case E</u>

We now show how to obtain $p'(x)$ when $\mathcal{H}'(y+p(x)) = 0$ and $\mathcal{H} = \mathcal{G}''/\mathcal{G}'$ is a constant, B_1, which is different from zero; so $\mathcal{G}'$ is an exponential function of its argument $(y+p(x))$ and hence from Eq.(41)

$$G(x,y) = B_2 e^{(y+p(x))B_1} + (y+p(x))B_3 + g(x) \tag{72}$$

for some constants B_2, B_3 and some function $g(x)$. In this case, it is always possible to arrive at an algebraic equation for $p'(x)$, though the case entails some subtleties. First of all, $\Phi(x,y,y')$ will be of the form

$$\Phi(x,y,y') = -\frac{F_x(x,y') + g'(x) + \left(B_2 B_1 e^{(y+p(x))B_1} + B_3\right)(y' + p'(x))}{F_{y'}(x,y')} \tag{73}$$

Now, taking advantage of the fact that we explicitly know B_1, we build our first explicit expression by dividing $B_1 e^{y B_1}$ by Υ:

$$\Lambda \equiv -\frac{F_{y'}}{B_2 e^{p(x) B_1}\ (y' + p'(x))} \tag{74}$$

We now multiply Φ by Λ and subtract $B_1 e^{B_1 y}$ to obtain our second explicit expression:

$$\Psi \equiv \frac{1}{B_2 e^{p(x) B_1}} \left(\frac{F_x + g'(x)}{y' + p'(x)} + B_3 \right) \tag{75}$$

Now, as in Case D, Λ and Ψ are related by

$$\frac{\partial}{\partial x} \Big((y' + p'(x))\, \Lambda \Big) + (y' + p'(x))\ p'(x) \Lambda B_1 + \frac{\partial}{\partial y'} \Big((y' + p'(x))\, \Psi \Big) = \frac{B_3}{B_2 e^{p(x) B_1}} \tag{76}$$

where the only unknowns are B_2, B_3 and $p(x)$. We build a first equation for $p'(x)$ by differentiating Eq.(76) with respect to y'

$$\left(p''(x) + p'(x)^2 B_1 \right) \Lambda_{y'} + p'(x) \left(y' \Lambda_{y'} B_1 + \Lambda B_1 + \Lambda_{x y'} + \Psi_{y' y'} \right) + 2 \Psi_{y'} + \Lambda_x + y' \Lambda_{x y'} + y' \Psi_{y' y'} = 0 \tag{77}$$

The problem now is that, due to the exponential on the RHS of Eq.(76), differently from Case D, we are not able to obtain a second expression for $p'(x)$ by differentiating w.r.t x. The alternative we have found to determine $p'(x)$ can be summarized as follows.
We first note that if $\Lambda_{y'} = 0$, Eq.(77) is already a linear algebraic equation for p'[10], so that we are only worried with the case $\Lambda_{y'} \neq 0$. With this in mind, we divide Eq.(77) by $\Lambda_{y'}$ and, *if* the resulting expression depends on y', we directly obtain a linear algebraic equation in $p'(x)$ just differentiating w.r.t y'.△

Example:

$$y'' = \frac{y'\ (x y' + 1)\,(-2 + e^y)}{y' x^2 + y' - 1} \tag{78}$$

This example is interesting because it involves a non-rational dependency on $y(x)$ - the dependent variable - thus being out of the scope of most of the symmetry analysis software presently available. It is also curious that there are no examples of this type in all of Kamke's set of non-linear second order ODEs. On the other hand, using the algorithm here presented, proceeding as in Case A, we determine Υ, $w(x,y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = \frac{y'(x y' + 1) e^y}{y' x^2 + y' - 1}; \qquad w(x,y) = e^y; \qquad \mathcal{H} = 1 \tag{79}$$

From the last equation we know that we are in Case E. We then determine Λ and Ψ as in Eqs. (74) and (75):

[10] We can see this by assuming $\Lambda_{y'} = 0$ and that Eq.(77) does not contain p', and then arriving at a contradiction as follows. We first set the coefficients of p' in Eq.(77) to zero, arriving at

$$0 = B_1\ \Lambda + \Psi_{y' y'} = 2 \Psi_{y'} + \Lambda_x + \Psi_{y' y'} y' \tag{A}$$

Eliminating $\Psi_{y' y'}$ gives

$$2 \Psi_{y'} = B_1\ \Lambda\ y' - \Lambda_x$$

Differentiating the expression above w.r.t y' and since $\Lambda_{y'} = 0$ we have,

$$2 \Psi_{y' y'} = B_1\ \Lambda$$

Finally using Eq.(A), $0 = \Lambda$, contradicting $F_{y'} \neq 0$.

$$\begin{aligned}\Lambda &= \frac{y'x^2 + y' - 1}{y'(xy'+1)} \\ \Psi &= -2\end{aligned} \tag{80}$$

Now, we build Eq.(76):

$$\frac{1}{xy'+1}\left(\left(p'' + p'^2 + y'^2\frac{(xp'-1)}{xy'+1}\right)\left(x^2+1-\frac{1}{y'}\right) + 2\,xp' - 2\right) = \frac{B_3}{B_2\,e^p} \tag{81}$$

and, differentiating w.r.t. y', we obtain (Eq.(77)),

$$\frac{2\,xy' + 1 - (x^3+x)y'^2}{y'^2(xy'+1)^2}\left(p'' + p'^2\right) + \frac{2\,y' - 1 - 2\,x + xy'}{(xy'+1)^3}(xp'-1) = 0 \tag{82}$$

Proceeding as explained, dividing by $\Lambda_{y'}$ and differentiating w.r.t. y' gives

$$\frac{\partial}{\partial y'}\left(y'^2\frac{2\,y' - 1 - 2\,x + xy'}{(xy'+1)\left(2\,xy' + 1 - (x^3+x)y'^2\right)}\right)(xp'-1) = 0 \tag{83}$$

Solving for $p'(x)$ gives $p'(x) = 1/x$, from which (Eq.(46)):

$$\mathcal{F}(x, y') = \left(y' - \frac{1}{x}\right)\frac{y'x^2 + y' - 1}{y'(xy'+1)} \tag{84}$$

Case F

The final branch occurs when Eq.(77) divided by $\Lambda_{y'}$ does not depend on y' (so that we will not be able to differentiate w.r.t y'). In this case we can build a linear algebraic equation for $p'(x)$ as follows. Let us introduce the label $\beta(x, p', p'')$ for Eq.(77) divided by $\Lambda_{y'}$, so that Eq.(77) becomes:

$$\Lambda_{y'}(x, y')\ \beta(x, p', p'') = 0 \tag{85}$$

Again, since we obtained Eq.(77) by differentiating Eq.(76) with respect to y', we see that Eq.(76) can be written in terms of β by integrating Eq.(85) with respect to y':

$$\Lambda(x, y')\beta(x, p', p'') + \gamma(x, p', p'') = \frac{B_3}{B_2 e^{p(x)B_1}} \tag{86}$$

where $\gamma(x, p', p'')$ is the constant of integration, and can be determined explicitly in terms of x, p' and p'' by comparing Eq.(86) with Eq.(76). Taking into account that $\beta(x, p', p'') = 0$, we see that Eq.(86) reduces to:

$$\gamma(x, p', p'') = \frac{B_3}{B_2 e^{p(x)B_1}} \tag{87}$$

We can remove the unknowns B_2 and B_3 after multiplying Eq.(87) by $e^{p(x)B_1}$, differentiating with respect to x, and then dividing once again by $e^{p(x)B_1}$. We now have our second equation for p', which we can build explicitly in terms of p', since we know $\gamma(x, p', p'')$ and B_1:

$$\frac{d\gamma}{dx} + B_1\ p'\gamma = 0 \tag{88}$$

Eliminating the derivatives of p' between Eq.(85) and Eq.(88) leads to a linear algebraic equation in p'. Once we have p', the determination of $\mathcal{F}(x, y')$ follows directly from Eq.(46). □

3.2 Integrating factors of the form $\mu(y, y')$

Just as in the previous section, from Eq.(16), the ODE family admitting an integrating factor of the form $\mu(y, y')$ is given by

$$y'' = -\frac{y'}{\mu(y, y')}\left(\int \mu_y dy' + G_y\right) - \frac{G_x}{\mu(y, y')} \tag{89}$$

For this ODE family, it would be possible to build a pattern matching routine as done in the previous section for the case $\mu(x, y')$. However, it is straightforward to notice that under the transformation $y(x) \to x,\ x \to y(x)$, Eq.(89) transforms into an ODE of the form Eq.(19) with integrating factor $\mu(x, y'^{-1})/y'^2$. This means that the above pattern can be matched by merely changing variables in the given ODE and matching Eq.(19). It follows that any explicit 2nd order ODE having an integrating factor of the form $\mu(y, y')$ can be reduced to a first order ODE by first changing variables, and then using the scheme outlined in the previous section (unless the resulting ODE is linear).

Example:

$$y'' - \frac{y'^2}{y} + \sin(x)\, y'\, y + \cos(x)\, y^2 = 0 \tag{90}$$

Changing variables as in $y(x) \to x,\ x \to y(x)$ we obtain

$$y'' + \frac{y'}{x} - \sin(y)\, y'^2 x - \cos(y)\, x^2 y'^3 = 0 \tag{91}$$

Using the algorithm outlined in the previous section, an integrating factor of the form $\mu(x, y')$ for Eq.(91) is given by

$$\frac{1}{y'^2 x} \tag{92}$$

from where an integrating factor of the form $\mu(y, y')$ for Eq.(90) is $1/y$, leading to the first integral

$$\sin(x) y + \frac{y'}{y} + C_1 = 0, \tag{93}$$

which is a first order ODE of Bernoulli type. The solution to Eq.(90) then follows directly. This example is particularly interesting since from [9] we know ODE Eq.(90) has no point symmetries.

3.3 Integrating factors of the form $\mu(x, y)$

For completeness, we review here the determination of integrating factors of the form $\mu(x, y)$ for second order ODEs, already found in the literature (see for instance Lemma 3.8 in [10]). Contrary to the cases $\mu(x, y')$ or $\mu(y, y')$, an integrating factor depending only on x and y can easily be found - when it exists - by directly solving the determining equations (10) and (11).

¿From Eq.(16), the general second order ODE having an integrating factor $\mu(x, y)$ takes the form

$$y'' = a(x, y)\, y'^2 + b(x, y)\, y' + c(x, y), \tag{94}$$

where

$$a(x, y) = -\frac{\mu_y}{\mu}, \quad b(x, y) = -\frac{G_y + \mu_x}{\mu}, \quad c(x, y) = -\frac{G_x}{\mu} \tag{95}$$

for some unknown function $G(x, y)$. As a shortcut to solving Eqs. (10) and (11), one can directly tackle Eqs.(95); the calculations are straightforward. There are two cases to be considered.

<u>Case A</u>: $2a_x - b_y \neq 0$

Defining the two auxiliary quantities

$$\varphi \equiv c_y - a\, c - b_x, \qquad \Upsilon \equiv a_{x,x} + a_x\, b + \varphi_y \tag{96}$$

an integrating factor of the form $\mu(x,y)$ exists only when

$$\Upsilon_y - a_x = 0, \qquad \Upsilon_x + \varphi + b\,\Upsilon - \Upsilon^2 = 0 \tag{97}$$

and is then given by

$$\mu(x,y) = \exp\left(\int\left(-\Upsilon + \frac{\partial}{\partial x}\int a\,dy\right)dx - \int a\,dy\right) \tag{98}$$

Case B: $2a_x - b_y = 0$

Redefining $\varphi \equiv c_y - a\,c$, an integrating factor of the form $\mu(x,y)$ exists only when

$$a_{x,x} - a_x\,b - \varphi_y = 0, \tag{99}$$

Then, $\mu(x,y)$ is given by

$$\mu(x,y) = \nu(x)\ \mathrm{e}^{-\int a\,dy} \tag{100}$$

where $\nu(x)$ is either one of the independent solutions of the second order linear ODE

$$\nu'' = A(x)\,\nu' + B(x)\,\nu, \tag{101}$$

and

$$\mathcal{I} \equiv \frac{\partial}{\partial x}\int a\,dy, \qquad A(x) \equiv 2\,\mathcal{I} - b, \qquad B(x) \equiv \varphi + \left(\mathcal{I} - \frac{\partial}{\partial x}\right)(b - \mathcal{I}) \tag{102}$$

It should be noted that when the attempt to solve the linear ODE Eq.(101) is successful, using each of its two independent solutions for integrating factors leads to the general solution of Eq.(94), instead of just a reduction of order. Also, when the original ODE was linear, Eq.(101) is just the corresponding adjoint equation, as was to be expected (see for instance Murphy's book).

3.4 The Connection to PDEs

Let $R(x,y,y')$ be a first integral of Eq.(20). We rewrite Eq.(13) by renaming $y' \equiv z$

$$\frac{\partial R}{\partial x} + z\frac{\partial R}{\partial y} + \Phi(x,y,z)\frac{\partial R}{\partial z} = 0 \tag{103}$$

From Theorem 3.1, if a given PDE of the form Eq.(103) has a particular solution of the form $R(x,y,z) = F(x,z) + G(x,y)$, such that $R(x,y,z)$ is nonlinear in y or z; or $R(x,y,z) = F(y,z) + G(x,y)$, such that $R(x,y,z)$ is nonlinear in y or z^{-1}, then F and G can be determined in a systematic manner.

Although this is a natural consequence of the previous sections, it is worth mentioning that the determination of R using the scheme here presented *does not require* solving the characteristic strip of Eq.(103), thus being a genuine alternative.

4 Integrating factors and symmetries

The main result being presented in this paper is a systematic algorithm for the determination of integrating factors of the form $\mu(x,y')$ and $\mu(y,y')$ *without solving any auxiliary differential equations*, and this last fact is the most relevant point. Nonetheless, it is interesting to briefly review the similarities and differences between the standard integrating factor (μ) and symmetry approaches, so as to have an insight of how complementary these methods can be in practice.

To start with, both methods tackle an n^{th} order ODE by looking for solutions to a linear n^{th} *order determining PDE* in $n+1$ variables (see sec. 2). Any given ODE has infinitely many integrating factors and symmetries. When many solutions to these *determining* PDEs are found, both approaches can, in principle, give a multiple reduction of order.

In the case of integrating factors there is one unknown function, while for symmetries there is a pair of infinitesimals to be found. But symmetries are defined up to an arbitrary function, so that we can always take one of these infinitesimals equal to zero; hence we are facing approaches of equivalent levels of difficulty and actually of equivalent solving power too.

Also valid for both approaches is the fact that, unless some *restrictions* are introduced on the functional dependence of μ or the infinitesimals, there is no hope that the corresponding determining PDEs will be easier to solve than the original ODE. In the case of symmetries, it is usual to restrict the problem to ODEs having *point symmetries*, that is, to consider infinitesimals depending only on x and y. The restriction to the integrating factors here discussed is similar: we considered μ's depending on only two variables.

At this point it can be seen that the two approaches are complementary: the determining PDEs for μ and for the symmetries are different[11], so that even using identical restrictions on the functional dependence of μ and the infinitesimals, problems which may be untractable using one approach may be easy or even trivial using the other one.

As an example of this, consider Kamke's ODE 6.37, appearing in this paper as Eq.(28):

$$y'' + 2\,y\,y' + f(x)\left(y' + y^2\right) - g(x) = 0$$

As mentioned in the exposition, for arbitrary $f(x)$ and $g(x)$, this ODE has an integrating factor depending only on x, easily determined using the algorithm presented. Now, for non-constant $f(x)$ and $g(x)$, this ODE has no point symmetries, that is, no solutions of the form $[\xi(x,y),\ \eta(x,y)]$, except for the particular case in which $g(x)$ can be expressed in terms of $f(x)$ as in[12]

$$g(x) = \frac{f''}{4} + \frac{3\,f\,f'}{8} + \frac{f^3}{16} - \frac{C_2 \exp\left(-3/2 \int f(x)dx\right)}{4\left(2\,C_1 + \int \exp\left(-1/2 \int f(x)dx\right) dx\right)^3} \tag{104}$$

Furthermore, this ODE has no non-trivial symmetries of the form $[\xi(x,y'),\ \eta(x,y')]$ either, and for symmetries of the form $[\xi(y,y'),\ \eta(y,y')]$ the determining PDE does not even split into a system.

Another ODE example of this type is found in a paper by [9] (1988):

$$y'' - \frac{{y'}^2}{y} - g(x)\,p\,y^p y' - g'\,y^{p+1} = 0 \tag{105}$$

In that work it is shown that for constant p, the ODE above only has point symmetries for very restricted forms of $g(x)$. For instance, Eq.(90) is a particular case of the ODE above and has no point symmetries. Nonetheless, *for arbitrary* $g(x)$, Eq.(105) has an obvious integrating factor depending on only one variable: $1/y$, leading to a first integral of Bernoulli type:

$$\frac{y'}{y} - g(x)y^p + C_1 = 0 \tag{106}$$

so that the whole family Eq.(105) is integrable by quadratures.

We note that Eq.(28) and Eq.(105) are respectively particular cases of the general reducible ODEs having integrating factors of the forms[13] $\mu(x)$:

$$y'' = -\frac{(\mu_x + G_y)}{\mu(x)} y' - \frac{G_x}{\mu(x)} \tag{107}$$

where $\mu(x)$ and $G(x,y)$ are arbitrary; and $\mu(y)$:

$$y'' = -\frac{(\mu_y\,y' + G_y)}{\mu(y)}\,y' - \frac{G_x}{\mu(y)} \tag{108}$$

[11] We are considering here ODEs of order greater than one.

[12] To determine $g(x)$ in terms of $f(x)$ we used the *standard form* Maple package by Reid and Wittkopf complemented with some basic calculations.

[13] To obtain the general ODE family reducible by a given integrating factor we used the routine **redode** also presented in this paper in sec. 6

In turn, these are very simple cases if compared with the general ODE families Eq.(19) and Eq.(89), respectively having integrating factors of the forms $\mu(x, y')$ and $\mu(y, y')$, and which can be systematically reduced in order using the algorithm here presented.

It is then natural to conclude that the integrating factor and the symmetry approaches can be useful for solving different types of ODEs, and can be viewed as equivalently powerful and general, and in practice complementary. Moreover, if for a given ODE, an integrating factor and a symmetry are known, in principle one can combine this information to build two first integrals and reduce the order by two at once ([3], chap. 3).

5 Tests

After plugging the reducible-ODE scheme here presented into ODEtools, we tested the scheme and routines using Kamke's non-linear 246 second order ODE examples[14]. The purpose was to confirm the correctness of the returned results and to determine which of these ODEs have integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$. The test consisted of determining μ and testing the exactness condition Eq.(7) of the product μ times ODE.

We then ran a comparison of performances in solving a related subset of Kamke's examples using different computer algebra ODE-solvers (Maple, Mathematica, MuPAD and the Reduce package Convode). The idea was to situate the new scheme in the framework of a sample of relevant packages presently available. As a secondary goal, we were also interested in comparing the solving performance of the new scheme with the one of the symmetry scheme implemented in ODEtools.

Finally we considered the table of integrating factors for second order non-linear ODEs found in Murphy's book and the answers for them returned by all these ODE-solvers.

5.1 The *reducible-ODE* solving scheme and Kamke's ODEs

To run the test with Kamke's ODEs, the first step was to classify these ODEs into: *missing x*, *missing y*, *exact* and *reducible*, where the latter refers to the new scheme. The reason for such a classification is that ODEs missing variables are straightforwardly reducible, so they are not the relevant target of the new scheme. Also, ODEs already in exact form can be easily reduced after performing a simple check for exactness; before running the tests all these ODEs were rewritten in explicit form by isolating y''. For classifying the ODEs we used the **odeadvisor** command from ODEtools. All the integrating factors found satisfied the exactness condition Eq.(7). The classification we obtained for these 246 ODEs is as follows

Classification	ODE numbers as in Kamke's book
99 ODEs are missing x or missing y	1, 2, 4, 7, 10, 12, 14, 17, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32, 40, 42, 43, 45, 46, 47, 48, 49, 50, 54, 56, 60, 61, 62, 63, 64, 65, 67, 71, 72, 81, 89, 104, 107, 109, 110, 111, 113, 117, 118, 119, 120, 124, 125, 126, 127, 128, 130, 132, 137, 138, 140, 141, 143, 146, 150, 151, 153, 154, 155, 157, 158, 159, 160, 162, 163, 164, 165, 168, 188, 191, 192, 197, 200, 201, 202, 209, 210, 213, 214, 218, 220, 222, 223, 224, 232, 234, 236, 237, 243, 246
13 are in exact form	36, 42, 78, 107, 108, 109, 133, 169, 170, 178, 226, 231, 235
40 ODEs are *reducible* with integrating factor $\mu(x, y')$ or $\mu(y, y')$ *and* missing x or y	1, 2, 4, 7, 10, 12, 14, 17, 40, 42, 50, 56, 64, 65, 81, 89, 104, 107, 109, 110, 111, 125, 126, 137, 138, 150, 154, 155, 157, 164, 168, 188, 191, 192, 209, 210, 214, 218, 220, 222, 236
28 ODEs are *reducible* and not missing x or y	36, 37, 51, 66, 78, 97, 108, 123, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215, 226, 235

Table 1. Missing variables, exact and *reducible* Kamke's 246 second order non-linear ODEs.

¿From the table above, $\approx 30\%$ of these 246 ODEs from Kamke's book are reducible to first order using the scheme here presented. Also, although the symmetry scheme implemented in ODEtools - which works with dynamical symmetries and includes heuristic procedures - finds symmetries for 191 of these 246 ODEs, it is unsuccessful in reducing the order of five ODEs which the new scheme does reduce. These are the ODEs numbered 36, 37, 123, 215, and 235. It is interesting to note that ODEs 36, 37 and 123 have no point symmetries; ODE 215 lead to a third order PDE system whose solution - in terms of

[14] Kamke's ODEs 6.247 to 6.249 cannot be made explicit and are then excluded from the tests.

elliptic integrals - can be obtained by *computer plus hand* if one uses trial and error; and for ODE 235, the determining PDE for the symmetries does not split into a system due to the presence of an arbitrary function of y'. Also, none of the other computer algebra ODE-solvers tested during this work succeeded in solving or reducing the order of any of these five ODEs[15], even though the corresponding integrating factors depend on only one variable (see sec. 5.2).

For ODE 215, which we write in explicit form as

$$\text{ode} := \quad y'' = \frac{\left(6\,y^2 - \frac{a}{2}\right) y'^2}{4\,y^3 - a\,y - b} - f(x)\,y' \tag{109}$$

the integrating factor found by the computer algebra implementation of the new scheme (see sec. 6) is[16]:

```
> mu = intfactor(ode,y(x));
```

$$\mu = \frac{1}{y'} \tag{110}$$

This integrating factor leads to a reduced ODE which can be solved as well, resulting in the following implicit solution in terms of an elliptic integral[17]:

```
> odsolve(ode);
```

$$\int e^{-\int f(x)dx}\,dx - C_1 \int^y \frac{1}{\sqrt{-4\,z^3 + z\,a + b}}dz + C_2 = 0 \tag{111}$$

The integrating factor found for ODE 37 (see Eqs.(28) and (32)) leads to a reduction of order resulting in the most general first order Riccati type ODE; in this example **odsolve** just returns the reduction of order obtained by the new integrating factor scheme. For ODE 123, the integrating factor found is $1/y$, and the reduced ODE is also a generic Riccati ODE. Finally, ODE 235 appears in Kamke's book written in exact form, but the ODE is interesting because it contains three arbitrary functions, of the first derivative, the dependent and the independent variables, respectively. Such an arbitrary dependence makes this ODE almost intractable for most computer algebra ODE-solvers and related packages. We then first isolated the highest derivative as to make the ODE *non-exact*

$$\text{ode} := \quad y'' = -\frac{(G(y)\,y' + F(x))}{H(y')} \tag{112}$$

The integrating factor here found is

```
> mu = intfactor(ode,y(x));
```

$$\mu = H(y') \tag{113}$$

Concerning timings, it is worth mentioning that in the specific subset of 28 Kamke's examples which are not missing variables, the average *time consumed* by **odsolve** in solving each ODE using the new scheme was 2.5 sec, while using symmetries this time jumps to 21 sec. These tests were performed using a Pentium 200, 64 Mb RAM, running Windows 95. In summary: for these 28 ODEs having an integrating factor of the form $\mu(x, y')$ or $\mu(y, y')$, the new scheme seems to be, on the average, $\approx$ 10 times faster than the symmetry scheme.

[15] A table of results obtained using the Reduce package *CRACK*, kindly sent to us by Dr. T. Wolf, shows that out of these five, *CRACK* is determining symmetries only for ODE 215, and concluding that there exist no point symmetries for ODEs 6.36, 6.37, 6.123 and 6.235, but perhaps for some undetermined special values of the function parameters entering the ODEs (see details for ODE 6.37 in sec. 4).

[16] In what follows, the *input* can be recognized by the Maple prompt >.

[17] For ODE 6.215, there is a typographical mistake in Kamke's book concerning the reduced ODE: instead of $...\sqrt{4y^3 - g_2 y - g_3}...$, one should read $...\sqrt{4\,y^3 - g_2 y - g_3}....$

5.2 Comparison of performances

With the classification presented in Table 1. in hands, we used different computer algebra systems to run a comparison of performances in solving these ODEs having integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$. For our purposes, the interesting subset is the one comprised of the 28 ODEs not already missing variables (see Table 1.). The results we obtained are summarized in the following table[18]:

	Kamke's ODE numbers			
	Convode	Mathematica 3.0	MuPAD 1.3	ODEtools
Solved:	51, 166, 173, 174, 175, 176, 179.	78, 97, 108, 166, 169, 173, 174, 175, 176, 178, 179, 206.	78, 97, 108, 133, 166, 169, 173, 174, 175, 176, 179,	51, 78, 97, 108, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215.
Totals:	7	12	11	22
Reduced:				36, 37, 66, 123, 226, 235.
Totals:	0	0	0	6

Table 2. *Performances in solving 28 Kamke's ODEs having an integrating factor $\mu(x, y')$ or $\mu(y, y')$*

As shown above, while the scheme here presented is finding first integrals in all the 28 ODE examples, opening the way to solve 22 of them to the end, the next scores are only 12 and 11 ODEs, respectively solved by Mathematica 3.0 and MuPAD 1.3.

Concerning the six reductions of order returned by **odsolve**, it must be said that neither MuPAD nor Mathematica provides a way to convey them, so that perhaps their ODE-solvers are obtaining first integrals for these cases but the routines are giving up when they cannot solve the problem to the end.

Maple R4 is not present in the table since it is not solving any of these 28 ODEs. This is understandable since in R4 the only methods implemented for high order non-linear ODEs are those for ODEs which are missing variables. This situation is being resolved in the upcoming Maple R5, where the ODEtools routines are included in the Maple library, and the previous ODE-solver has been replaced by **odsolve**[19].

Although the primary goal of this work is just to obtain first integrals for second order ODEs, it is also interesting to comment on the six ODEs shown in Table 2. for which the new scheme succeeds in determining integrating factors but the reduced ODEs remain unsolved. First of all, for ODEs 36, 37 and 123 the reduction of order lead to general Riccati type ODEs, so that in these cases no more than a reduction of order should be expected. Concerning ODE 235 (Eq.(112)), the reduced ODE is:

$$\int^{y'} H(z)dz + \int^{y} G(z)dz + \int F(x)dx + C_1 = 0 \tag{114}$$

Methods for solving such a first order ODE are known only for very special explicit combinations of H, G and F. Concerning ODEs 66 and 226, the obtained reduced ODEs are the same as those shown in Kamke's book, and are out of the scope of **odsolve**.

5.3 The *reducible-ODE* scheme and Murphy's table of integrating factors

There is an explicit paragraph in Murphy's book concerning integrating factors of the form $\mu(y')$, where it is shown a table with four second order non-linear ODE families for which $\mu(y')$ is already known. The first two families are trivial in the sense that they are already missing variables. The third of these ODE families is:

$$\text{ode} := \quad y'' = P(x)\, y' + Q(y)\, {y'}^2 \tag{115}$$

where P and Q are arbitrary functions of its arguments; this is actually Liouville's ODE. The integrating factor mentioned in the book is the same found by the scheme here presented: y'; and the corresponding reduced ODE can be solved in implicit form:

[18] When building the statistics for ODEtools, we passed to **odsolve** the optional argument `[reducible]`, meaning: try the reducible scheme, and if it does not solve the problem just give up. To solve the reduced ODE all of **odsolve**'s methods, including symmetries, were used. The input and output in the respective format for all the packages tested are available in http://dft.if.uerj.br/odetools/mu_odes.zip.

[19] However, the scheme here presented was not ready when the development library was closed; the *reducible* scheme implemented in Maple R5 is able to determine, when they exist, integrating factors only of the form $\mu(y')$.

```
> odsolve(ode);
```

$$\int e^{\int P(x)dx}dx - \int^{y} e^{-\int Q(z)dz + C_1}dz + C_2 = 0 \tag{116}$$

The fourth ODE family is the most general second order ODE having $1/y'$ as integrating factor (see sec. 6):

$$\text{ode} := \quad y'' = \frac{\partial R(x,y)}{\partial x} y' + \frac{\partial R(x,y)}{\partial y}\, y'^2 \tag{117}$$

for some function $R(x,y)$. Here the new scheme finds the integrating factor $1/y'$ and returns the reduced ODE

$$\ln(y') - R(x,y) + C_1 = 0 \tag{118}$$

actually a generic first order ODE[20].

6 Computer algebra implementation

We implemented the scheme for finding integrating factors described in sec. 3 in the framework of the ODEtools package [4], taking advantage of its set of programming tool routines specifically designed to work with ODEs. The implementation consists of:

- The plugging of the *reducible-ODE* solving scheme here presented in the block of methods for nonlinear second order ODEs of the ODEtools command **odsolve**;
- The extension of the capabilities of the ODEtools **intfactor** command to determine integrating factors for non-linear second order ODEs using the scheme here presented;
- A new user-level routine, **redode**, returning the most general explicit ODE having a given integrating factor (Eq.(16));

The computational implementation follows straightforwardly the explanations of sec. 3 and includes three main routines, for determining $\mathcal{F}(x,y')$, $\widetilde{\mu}(x)$ and the reduced ODE $R(x,y,y')$, respectively. Callings to these routines were in turn added to both the **intfactor** and **odsolve** commands, so that the scheme becomes available at user-level.

A test of this implementation in **odsolve** and some related examples are found in sec. 5. Since detailed descriptions of the ODEtools commands are found in the On-Line help, we have restricted this section to a description of the new command **redode** followed by two examples.

Description of **redode**

Command name: **redode**
Feature: returns the n^{th}order ODE having a given integrating factor
Calling sequence:

```
> redode(mu, n, y(x));
> redode(mu, n, y(x), R);
```

Parameters:

`n`	- indicates the order of the requested ODE.
`mu`	- an integrating factor depending on x, $y, ..., y^{(n-1)}$.
`y(x)`	- the dependent variable.
`R`	- optional, the expected reduced ODE depending on x, $y, ..., y^{(n-1)}$.

Synopsis:

• Given an integrating factor $\mu(x,y,...y^{(n)})$, **redode**'s main goal is to return the ODE of order n having μ as integrating factor

[20] For the third ODE family, Mathematica 3.0 returns a wrong answer and MuPAD 1.3 gives up, while for the fourth family, Mathematica gives up and MuPAD returns an ERROR message.

$$y^{(n)} = \frac{-1}{\mu}\left[\frac{\partial}{\partial x}\left(\int \mu\, dy^{(n-1)} + G\right) + ... + y^{(n-1)}\frac{\partial}{\partial y^{(n-2)}}\left(\int \mu\, dy^{(n-1)} + G\right)\right]$$

where $G \equiv G(x, y, ...y^{(n-2)})$ is an arbitrary function of its arguments (see sec. 3). This command is useful to identify the general ODE problem related to a given μ, as well as to understand the possible links between the integrating factor scheme for reducing the order and other reduction schemes (e.g., symmetries).

• When the expected *reduced ODE* (differential order $n-1$), here called R, is also given as argument, the routine proceeds as follows. First, a test to see if the requested ODE exists is performed:

$$\mu(x, y, ...y^{(n-1)}) = \nu(x, y, ...y^{(n-2)})\ \frac{\partial}{\partial y^{(n-1)}} R(x, y, ..., y^{(n-1)}) \tag{119}$$

for some function $\nu(x, y, ...y^{(n-1)})$. If the problem is solvable, **redode** will then return an n^{th}order $ODE^{(n)} = y^{(n)} - \Phi(x, y, ..., y^{(n-1)})$ satisfying

$$\mu(x, y, ..., y^{(n-1)})\ ODE^{(n)} = \frac{d}{dx}\left(\nu(x, y, ...y^{(n-2)})\ R(x, y, ..., y^{(n-1)})\right) \tag{120}$$

that is, an ODE having as first integral $\nu\, R + constant$.

• When the given μ does not depend on $y^{(n-1)}$ and R is non-linear in $y^{(n-1)}$, the requested n^{th}order ODE nevertheless exists if R can be solved for $y^{(n-1)}$.

Examples:

The **redode** command is interesting mainly as a tool for generating solving schemes for given ODE families; we illustrate with two examples.

1. Consider the family of second order ODEs having as integrating factor $\mu = F(x)$ - an arbitrary function - such that the reduced ODE has the same integrating factor. We want to set up an algorithm such that, given a second order linear ODE,

$$y'' = \psi_1(x)\, y' + \psi_2(x)\, y + \psi_3(x) \tag{121}$$

where there are no restrictions on $\psi_1(x)$, $\psi_2(x)$ or $\psi_3(x)$, the scheme determines if the ODE belongs to the family just described, and if so it also determines $F(x)$. The knowledge of $F(x)$ will be enough to build a closed form solution for the ODE.

To start with we obtain the first order ODE having $F(x)$ as integrating factor via

```
> ode_1 := redode(F(x), y(x));
```

$$\text{ode}_1 := y' = -\frac{1}{F(x)}\left(y\ \frac{dF(x)}{dx} + \_F1(x)\right) \tag{122}$$

where $\_F1(x)$ is an arbitrary function. To obtain the second order ODE aforementioned we pass `ode_1` as argument (playing the role of the *reduced* ODE) together with the integrating factor $F(x)$ to obtain

```
> ode_2 := redode(F(x), y(x), ode_1);
```

$$\text{ode}_2 := y'' = -\frac{1}{F(x)}\left(2\, y'\ \frac{dF(x)}{dx} + y\frac{d^2F(x)}{dx^2} + \frac{d\_F1(x)}{dx}\right) \tag{123}$$

Taking this general ODE pattern as departure point, we setup the required solving scheme by comparing coefficients in Eq.(121) and Eq.(123), obtaining

$$\frac{-2}{F(x)}\frac{dF(x)}{dx} = \psi_1(x), \qquad \frac{-1}{F(x)}\frac{d^2F(x)}{dx^2} = \psi_2(x) \tag{124}$$

By solving the first equation, we get $F(x)$ as

$$F(x) = C_1\, e^{-\int \frac{\psi_1(x)}{2} dx} \tag{125}$$

and by substituting this result into the second one we get the pattern identifying the ODE family

$$\frac{1}{2}\frac{d\psi_1(x)}{dx} - \frac{1}{4}\left(\psi_1(x)\right)^2 - \psi_2(x) = 0 \tag{126}$$

Among the ODE-solvers of Maple R4, Mathematica 3.0, MuPAD 1.3 or Convode (Reduce), only those of MuPAD and Maple succeed in solving this ODE family.

2. Consider the second order ODE family having as integrating factor $\mu = F(x)$ - an arbitrary function - *also having* the symmetry[21] $[\xi = 0, \eta = F(x)]$, and such that the reduced ODE is the most general first order linear ODE

$$\text{ode}_1 := y' = A(x)\, y + B(x) \tag{127}$$

where $A(x)$ and $B(x)$ are arbitrary functions. To start with, we obtain the aforementioned second order ODE having the integrating factor $F(x)$ as in Example 1.

```
> ode_2 := redode(F(x), y(x), ode_1);
```

$$y'' = \left(\frac{dA(x)}{dx} + \frac{\left(\frac{dF(x)}{dx}\right) A(x)}{F(x)}\right) y + \left(A(x) - \frac{\frac{dF(x)}{dx}}{F(x)}\right) y' + \frac{dB(x)}{dx} + \frac{\left(\frac{dF(x)}{dx}\right) B(x)}{F(x)} \tag{128}$$

In this step, `ode_2` is in fact the most general second order linear ODE. If we now impose the symmetry condition[22] X(`ode_2`)=0, where $X = [0,\, F(x)]$ we arrive at the following restriction on $A(x)$

$$-F(x)\frac{dA(x)}{dx} - 2\left(\frac{dF(x)}{dx}\right) A(x) + \frac{\left(\frac{dF(x)}{dx}\right)^2}{F(x)} + \frac{d^2F(x)}{dx^2} = 0 \tag{129}$$

Solving this ODE for $A(x)$, introducing the result into Eq.(128) and disregarding the non-homogeneous term (irrelevant in the solving scheme) we obtain the homogeneous ODE family pattern:

$$y'' = \left(\frac{1}{2}\frac{dH(x)}{dx} + \frac{3}{4}\frac{\left(\frac{dH(x)}{dx}\right)^2}{\left(H(x)\right)^2} - \frac{1}{2}\frac{\frac{d^2H(x)}{dx^2}}{H(x)}\right) y + H(x)\; y' \tag{130}$$

where we introduced $H(x) = (F(x))^{-2}$. Although this ODE family appears more general than the one treated in Example 1., the setting up of a solving scheme here is easier: one just needs to check if the coefficient of y in a given ODE is related to the coefficient of y' as in equation Eq.(130), in which case the integrating factor is just $\frac{1}{\sqrt{H(x)}}$.

7 Conclusions

This paper presented a systematic method for obtaining integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ - when they exist - for second order non-linear ODEs, as well as its computer algebra implementation in the framework of the ODEtools package. The scheme is new, as far as we know, and the implementation has proven to be a valuable tool since it leads to reductions of order for varied ODE examples, as shown in sec. 5. Actually, the implementation of the scheme solves ODEs not solved by using standard or symmetry methods (see sec. 4) or other computer algebra ODE-solvers (see sec. 5.2); furthermore, it involves only algebraic operations, so that - in principle - it gives answers remarkably faster than the symmetry scheme.

[21] Here we denote the infinitesimal symmetry generator by $[\xi, \eta] \equiv \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$

[22] For linear ODEs, symmetries of the form $[0, F(x)]$ are also symmetries of the homogeneous part.

It is also worth mentioning that restricting the dependence of μ in Eq.(7) to $\mu(x, y')$, *does not lead* to a straightforwardly solvable problem except for few or simple cases. Moreover, when the determination of μ from Eq.(7) is frustrated, there is no way to determine whether such a solution $\mu(x, y')$ exists. It is then a pleasant surprise to see such integrating factors - provided they exist - being systematically determined in all cases and without solving any differential equations, convincing us of the value of the new scheme.

On the other hand, we are restricting the problem to the universe of second order ODEs having integrating factors depending only on two variables - the general case is $\mu(x, y, y')$ - and even so, for integrating factors of the form $\mu(x, y)$ the method may fail in solving the auxiliary linear ODE Eq.(101) which appears in one of the subcases.

Some natural extensions of this work then would be to develop a scheme for building integrating factors of the forms considered in this work, now for higher order ODEs, at least for restricted ODE families yet to be determined. Concerning these extensions, the **redode** routine presented, designed to find the most general n^{th}order ODE having a given integrating factor, optionally reducing to a given k^{th} order ODE ($k < n$), can be of use in investigating further problems. We are presently working on these possible extensions[23], and expect to succeed in obtaining reportable results in the near future.

Acknowledgments

This work was supported by the State University of Rio de Janeiro (UERJ), Brazil, and by the Symbolic Computation Group, Faculty of Mathematics, University of Waterloo, Ontario, Canada. The authors would like to thank K. von Bülow[24] for a careful reading of this paper.

References

[1] Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen.* Chelsea Publishing Co, New York (1959).

[2] MacCallum, M.A.H., *Using computer algebra to solve ordinary differential equations*, Computer algebra in industry 2: Problem solving in practice, ed. A.M. Cohen, L.J. van Gastel and S.M. Verduyn Lunel, pp. 19-41, John Wiley and Sons, Chichester (1995).

[3] Stephani, H. *Differential equations: their solution using symmetries*, ed. M.A.H. MacCallum, Cambridge University Press, New York and London (1989).

[4] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, "Computer Algebra Solving of Second Order ODEs Using Symmetry Methods". Computer Physics Communications 108 (1998) 90

[5] S.C. Anco and G.W. Bluman, *Integrating Factors and First Integrals for Ordinary Differential Equations*, Preprint, University of British Columbia (1997).

[6] Murphy, G.M., *Ordinary Differential Equations and their solutions.* Van Nostrand, Princeton, 1960.

[7] Painleve, P., Acta Math. 25 (1902) 31, Gl (2).

[8] Ince, E.L., *Ordinary Differential Equations.* Dover Publications, 1956.

[9] A. González-López, *Symmetry and integrability by quadratures of ordinary differential equations*, Phys. Lett. A, (1988) 190.

[10] Sheftel, M.B., *A Course in Group Analysis of Differential Equations, Part II: Ordinary Differential Equations.* St. Petersburg Institute of Economics and Finance, 1997. In Russian.

[11] G.J. Reid and A.D. Wittkopf, *Long Guide to the Standard Form Package*, Pre-print, Department of Mathematics, University of British Columbia, Vancouver, Canada, 1993.

[23] See http://dft.if.uerj.br/odetools.html
[24] Symbolic Computation Group, Theoretical Physics Department, IF-UERJ-Brazil.

Appendix A

This appendix contains some additional information which may be useful as a reference for developing computer algebra implementations of this work, or for improving the one here presented.

As explained in sec. 3.1.2, the scheme presented can be subdivided into six different cases: A, B, C, D, E and F. Actually, there are just five cases since case B is always either A or C. From the point of view of a computer implementation of the scheme it is interesting to know what one would expect from such an implementation concerning Kamke's ODEs and the cases aforementioned. We then display here both the integrating factors obtained for the 28 Kamke's ODEs used in the tests (see sec. 5) and the "case" corresponding to each ODE.

Integrating factor	Kamke's book ODE-number	Case
1	36	D
$e^{\int f(x)dx}$	37	A
${y'}^{-1}$	51, 166, 169, 173, 175, 176, 179, 196, 203, 204, 206, 215	C
$\frac{b+y'}{\left(1+{y'}^2\right)^{3/2}}$	66	D
x	78	D
x^{-1}	97	A
y	108	D
y^{-1}	123	A
$\frac{1+y'}{\left(y'-1\right)y'}$	133	C
$\frac{y'-1}{\left(1+y'\right)y'}$	134	C
$\frac{y'-1}{\left(1+y'\right)\left(1+{y'}^2\right)}$	135	C
$\frac{y'-1}{h(y')}$	136	C
$\frac{x}{2\,xy'-1}$	174	C
$(1+y')^{-1}$	178	C
$\frac{1}{y'\left(1+2\,yy'\right)}$	193	C
y'	226	A
$h(y')$	235	C

Table A.1 Integrating factors for Kamke's *reducible* and ODEs not missing variables.

16 Integrating factors for second order ODEs. By E.S. Cheb-Terraba, A.D. Rochea (1999)

Integrating factors for second order ODEs

E.S. Cheb-Terrab[a,b] and A.D. Roche[a]

[a] *Symbolic Computation Group*
Computer Science Department, Faculty of Mathematics
University of Waterloo, Ontario, Canada.

[b] *Department of Theoretical Physics*
State University of Rio de Janeiro, Brazil.

Journal of Symbolic Computation, V. 27, No. 5, pp. 501-519 (1999)
(Original version received 25 November 1997)

Abstract

A systematic algorithm for building integrating factors of the form $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$ for second order ODEs is presented. The algorithm can determine the existence and explicit form of the integrating factors themselves without solving any differential equations, except for a linear ODE in one subcase of the $\mu(x, y)$ problem. Examples of ODEs not having point symmetries are shown to be solvable using this algorithm. The scheme was implemented in Maple, in the framework of the *ODEtools* package and its ODE-solver. A comparison between this implementation and other computer algebra ODE-solvers in tackling non-linear examples from Kamke's book is shown.

1 Introduction

Although in principle it is always possible to determine whether a given ODE is exact (a total derivative), there is no known method which is always successful in making arbitrary ODEs exact. For n^{th} order ODEs - as in the case of symmetries - integrating factors (μ) are determined as solutions of an n^{th} order linear PDE in $n+1$ variables, and to solve this determining PDE is a major problem in itself.

Despite the fact that the determining PDE for μ naturally splits into a PDE system, the problem is - as a whole - too general, and to solve it a restriction of the problem in the form of a more concrete ansatz for μ is required. For example, in a recent work by [2] the authors explore possible ansatzes depending on the given ODE, which are useful when this ODE has known symmetries of certain type. In another work, [3] explores the use of computer algebra to try various ansatzes for μ, no matter the ODE input, but successively increasing the order of the derivatives (up to the $n^{\text{th}} - 1$ order) on which μ depends; the idea is to try to maximize the splitting so as to increase the chances of solving the resulting PDE system by first simplifying it using differential Groebner basis techniques.

Bearing this in mind, this paper presents a method, for second order explicit ODEs[1], which systematically determines the existence and the explicit form of integrating factors when they depend on only two variables, that is: when they are of the form $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$. The approach works without solving any auxiliary differential equations - except for a linear ODE in one subcase of the $\mu(x, y)$ problem - and is based on the use of the forms of the ODE families admitting such integrating factors. It turns out that with this restriction - μ depends on only two variables - the use of differential Groebner basis techniques is not necessary; these integrating factors, when they exist, can be given directly by identifying the input ODE as a member of one of various related ODE families.

The exposition is organized as follows. In sec. 2, the standard formulation of the determination of integrating factors is briefly reviewed and the method we used for obtaining the aforementioned integrating factors $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$ is presented. In sec. 3, some aspects of the integrating factor and symmetry approaches are discussed, and their complementariness is illustrated with two ODE families not having point symmetries. Sec. 4 contains some statistics concerning the new solving method and the second order non-linear ODEs found in Kamke's book, as well as a comparison of performances of some popular computer algebra packages in solving a related subset of these ODEs. Finally, the conclusions contain some general remarks about the work.

Aside from this, in the Appendix, a table containing extra information concerning integrating factors for some of Kamke's ODEs is presented.

2 Integrating Factors and ODE patterns

In this paper we use the term "integrating factor" in connection with the explicit form of an n^{th} order ODE

$$y^{(n)} - \Phi(x, y, y', ..., y^{(n-1)}) = 0 \tag{1}$$

so that $\mu(x, y, y', ..., y^{(n-1)})$ is an integrating factor if

$$\mu \left(y^{(n)} - \Phi \right) = \frac{d}{dx} R(x, y, y', ..., y^{(n-1)}) \tag{2}$$

for some function R. The knowledge of μ is - in principle - enough to determine R by using standard formulas (see for instance Murphy's book). To determine μ, one can try to solve for it in the exactness condition, obtained applying Euler's operator to the total derivative $H \equiv \mu \left(y^{(n)} - \Phi \right)$:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial y''}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial H}{\partial y^{(n)}}\right) = 0 \tag{3}$$

Eq.(3) is of the form

$$A(x, y, y', ..., y^{(2n-3)}) + y^{(2n-2)} B(x, y, y', ..., y^{(n-1)}) = 0 \tag{4}$$

[1]We say that a second order ODE is in explicit form when it appears as $y'' - \Phi(x, y, y') = 0$. Also, we exclude from the discussion the case of a linear ODE and an integrating factor of the form $\mu(x)$, already known to be the solution to the adjoint ODE.

where A is of degree $n-1$ in $y^{(n)}$ and linear in $y^{(k)}$ for $n < k \leq (2n-3)$, so that Eq.(4) can be split into a PDE system for μ. In the case of second order ODEs - the subject of this work - Eq.(3) is of the form

$$A(x,y,y') + y''\ B(x,y,y') = 0 \tag{5}$$

and the PDE system is obtained by taking A and B equal to zero[2]:

$$\begin{aligned} A \;&\equiv\; (y'\mu_{y'y} - \mu_y + \mu_{y'x})\,\Phi + (\Phi_{y'x} + y'\Phi_{y'y} - \Phi_y)\,\mu + {y'}^2\mu_{yy} \\ &\quad + (\mu_y\Phi_{y'} + \mu_{y'}\Phi_y + 2\,\mu_{xy})\,y' + \mu_{y'}\Phi_x + \mu_x\Phi_{y'} + \mu_{xx} = 0 \end{aligned} \tag{6}$$

$$B \;\equiv\; y'\mu_{y'y} + \Phi\,\mu_{y'y'} + \mu\,\Phi_{y'y'} + 2\,\mu_y + 2\,\mu_{y'}\Phi_{y'} + \mu_{y'x} = 0 \tag{7}$$

Regarding the solvability of these equations, unless a more concrete ansatz for $\mu(x,y,y')$ is given, the problem is in principle as difficult as solving the original ODE. We then studied the solution for μ of Eqs.(6) and (7) when μ depends only on two variables, that is: for $\mu(x,y)$, $\mu(x,y')$ and $\mu(y,y')$. Concretely, we searched for the existence conditions for such integrating factors, expressed as a set of equations in Φ, plus an algebraic expression for μ as a function of Φ, valid when the existence conditions hold. Formulating the problem in that manner and taking into account the integrability conditions of the system, Eqs.(6) and (7) turned out to be solvable for $\mu(x,y)$, but appeared to us untractable when μ depends on two variables one of which is y'.

We then considered a different approach, taking into account from the beginning the form of the ODE family admitting a given integrating factor. As shown in the following sections, it turns out that, using that piece of information (Eq.(11) below), when μ depends only on two variables the existence conditions and the integrating factors themselves can be systematically determined; and in the cases $\mu(x,y')$ and $\mu(y,y')$, this can be done without solving any differential equations.

Concerning the ODE families admitting given integrating factors, we note that, from Eq.(2)

$$\mu(x,y,y',...,y^{(n-1)}) = \frac{\partial R}{\partial y^{(n-1)}} \tag{8}$$

and hence the first integral R is of the form

$$R = G(x,y,...,y^{(n-2)}) + \int \mu\ dy^{(n-1)} \tag{9}$$

for some function G. In turn, since R is a first integral, it satisfies

$$R_x + y'R_y + ... + \Phi R_{y^{(n-1)}} = 0 \tag{10}$$

Inserting Eq.(9) into the above and solving for $y^{(n)}$ leads to the general form of an ODE admitting a given integrating factor:

[2]In a recent work by [2], the authors arrive at Eq.(5) and Eq.(7) departing from the adjoint linearized system corresponding to a given ODE; the possible splitting of Eq.(4) into an overdetermined system for μ is also mentioned. However, in that work, y'' of Eq.(5) above appears replaced by $\Phi(x,y,y')$, and the authors discuss possible alternatives to tackle Eqs.(5) and (7) instead of Eqs.(6) and (7).

$$y^{(n)} = \frac{-1}{\mu}\left[\frac{\partial}{\partial x}\left(\int \mu\, dy^{(n-1)} + G\right) + ... + y^{(n-1)}\frac{\partial}{\partial y^{(n-2)}}\left(\int \mu\, dy^{(n-1)} + G\right)\right] \tag{11}$$

2.1 Second order ODEs and integrating factors of the form $\mu(x, y)$

We consider first the determination of integrating factors of the form $\mu(x, y)$, which turns out to be straightforward[3]. The determining equations (6) and (7) for this case are given by:

$$\begin{gathered} {y'}^2\mu_{yy} + 2\,\mu_{xy}y' + \mu\,\Phi_{y'x} + \mu\,\Phi_{y'y}y' - \mu\,\Phi_y - \mu_y\Phi + \mu_y y'\,\Phi_{y'} + \mu_{xx} + \mu_x\Phi_{y'} = 0 \\ \mu\,\Phi_{y'y'} + 2\,\mu_y = 0 \end{gathered} \tag{12}$$

Although the use of integrability conditions is enough to tackle this problem, the solving of Eqs.(12) can be directly simplified if we take into account the ODE family admitting an integrating factor $\mu(x, y)$. From Eq.(11), that ODE family takes the form

$$y'' = a(x, y)\,{y'}^2 + b(x, y)\,y' + c(x, y), \tag{13}$$

where

$$a(x, y) = -\frac{\mu_y}{\mu}, \quad b(x, y) = -\frac{G_y + \mu_x}{\mu}, \quad c(x, y) = -\frac{G_x}{\mu} \tag{14}$$

and $G(x, y)$ is an arbitrary function of its arguments. Hence, as a shortcut to solving Eqs.(12), one can take Eq.(13) as an existence condition - Φ must be a polynomial of degree two in y' - and directly solve Eqs.(14) for μ. The calculations are straightforward; there are two different cases.

Case A: $2a_x - b_y \neq 0$

Defining the two auxiliary quantities

$$\varphi \equiv c_y - a\,c - b_x, \qquad \Upsilon \equiv a_{xx} + a_x\,b + \varphi_y \tag{15}$$

an integrating factor of the form $\mu(x, y)$ exists only when

$$\Upsilon_y - a_x = 0, \qquad \Upsilon_x + \varphi + b\,\Upsilon - \Upsilon^2 = 0 \tag{16}$$

and is then given in solved form, in terms of a, b and c by

$$\mu(x, y) = \exp\left(\int\left(-\Upsilon + \frac{\partial}{\partial x}\int a\,dy\right)dx - \int a\,dy\right) \tag{17}$$

So, in this case, when an integrating factor of this type exists there is only one[4] and it can be determined without solving any differential equations.

Case B: $2a_x - b_y = 0$

[3] The result for Case A presented in this subsection is also presented as lemma 3.8 in [5].

[4] We recall that if μ is an integrating factor leading to a first integral ψ, then the product $\mu F(\psi)$ - where F is an arbitrary function - is also an integrating factor, which however does not lead to a first integral independent of ψ.

Redefining $\varphi \equiv c_y - a\,c$, an integrating factor of the form $\mu(x,y)$ exists only when

$$a_{xx} - a_x\,b - \varphi_y = 0, \tag{18}$$

and then $\mu(x,y)$ is given by

$$\mu(x,y) = \nu(x)\ \mathrm{e}^{-\int a\,dy} \tag{19}$$

where $\nu(x)$ is either one of the independent solutions of the second order linear ODE[5]

$$\nu'' = A(x)\,\nu' + B(x)\,\nu, \tag{20}$$

and

$$A(x) \equiv 2\mathcal{I} - b, \qquad B(x) \equiv \varphi + \left(\mathcal{I} - \frac{\partial}{\partial x}\right)(b - \mathcal{I})\,, \qquad \mathcal{I} \equiv \frac{\partial}{\partial x}\int a\,dy \tag{21}$$

So in this case, to transform Eq.(19) into an explicit expression for μ we first need to solve a second order linear ODE. When the attempt to solve Eq.(20) is successful, using each of its two independent solutions as integrating factors leads to the general solution of Eq.(13), instead of just a reduction of order.

2.2 Second order ODEs and integrating factors of the form $\mu(x, y')$

When the integrating factor is of the form $\mu(x,y')$, the determining equations (6) and (7) become

$$\begin{gathered}(\Phi_y y' + \Phi_x)\,\mu_{y'} + (-\Phi_y + \Phi_{y'x} + \Phi_{y'y}y')\,\mu + \mu_{xx} + \mu_x\Phi_{y'} + \mu_{y'x}\Phi = 0\\ \Phi\,\mu_{y'y'} + \mu\,\Phi_{y'y'} + 2\,\mu_{y'}\Phi_{y'} + \mu_{y'x} = 0\end{gathered} \tag{22}$$

As in the case $\mu(x,y)$, the solution we are interested in is an expression for $\mu(x,y')$ in terms of Φ, as well as existence conditions for such an integrating factor expressed as equations in Φ. However, differently than the case $\mu(x,y)$, we didn't find a way to solve the $\mu(x,y')$ problem just using integrability conditions, neither working by hand nor using the specialized computer algebra packages *diffalg* [6] and *standard_form* [7]. We then considered approaching the problem as explained in the previous subsection, departing from the form of Eq.(9) for $\mu = \mu(x,y')$:

$$y'' = \Phi(x,y,y') \equiv -\frac{F_x + G_x + G_y\,y'}{F_{y'}} \tag{23}$$

where $G(x,y)$ and $F(x,y')$ are arbitrary functions of their arguments and

$$\mu(x,y') = F_{y'} \tag{24}$$

Now, Eq.(23) is not polynomial in either x, y or y', and hence its use to simplify and solve the problem is less straightforward than in the case $\mu(x,y)$. However, in Eq.(23), all the dependence on y comes from $G(x,y)$ in the numerator, and as it is shown below, this fact is a key to solving

[5] When the given ODE is linear, Eq.(20) is just the corresponding adjoint equation.

the problem. Considering ODEs for which $\Phi_y \neq 0$[6], the approach we used can be summarized in the following three lemmas whose proofs are developed separately for convenience.

Lemma 1. *For all linear ODEs of the family Eq.(23), an integrating factor of the form* $\mu(x, y')$ *such that* $\mu_{y'} \neq 0$*, when it exists, can be determined directly from the coefficient of* y *in the input ODE.*

Lemma 2. *For all non-linear ODEs of Eq.(23), the knowledge of* $\mu(x, y')$ *up to a factor depending on* x*, that is, of* $\mathcal{F}(x, y')$ *satisfying*

$$\mathcal{F}(x, y') = \frac{\mu(x, y')}{\widetilde{\mu}(x)} \tag{25}$$

is enough to determine $\widetilde{\mu}(x)$ *by means of an integral.*

Lemma 3. *For all non-linear ODEs members of Eq.(23), it is always possible to determine a function* $\mathcal{F}(x, y')$ *satisfying Eq.(25).*

Corollary. *For all second order ODEs such that* $\Phi_y \neq 0$*, the determination of* $\mu(x, y')$ *(if the ODE is linear we assume* $\mu_{y'} \neq 0$*), when it exists, can be performed systematically and without solving any differential equations.*

2.2.1 Proof of Lemma 1

For Eq.(23) to be linear and not missing y, either G_x or G_y must be linear in y. Both G_x and G_y cannot simultaneously be linear in y since, in such a case, $G_x/F_{y'}$ or $y' G_y/F_{y'}$ would be non-linear in $\{y, y'\}$[7]; therefore, either $G_{yy} = 0$ or $G_{xy} = 0$.

Case A: G_y is linear in y and $G_{xy} = 0$

Hence, G is given by

$$G = C_2 y^2 + C_1 y + g(x) \tag{26}$$

where $g(x)$ is arbitrary. From Eq.(23), in order to have $y' G_y/F_{y'}$ linear in $\{y, y'\}$, $F_{y'}$ must of the form $\nu(x) y'$ for some function $\nu(x)$. Also, $F_x/F_{y'}$ can have a term linear in y', and a term proportional to $1/y'$ to cancel with the one coming from $G_x/F_{y'} = g'/F_{y'}$, so that

$$F_{y'} = \nu y', \qquad F_x = \frac{\nu' y'^2}{2} - g' \tag{27}$$

where the coefficient $\nu'/2$ in the second equation above arises from the integrability conditions between both equations. Eq.(23) is then of the form

$$y'' = -\frac{\nu'}{2\,\nu}\, y' - \frac{2\, C_2}{\nu}\, y - \frac{C_1}{\nu} \tag{28}$$

[6]ODEs *missing* y may also have integrating factors of the form $\mu(x, y')$. Such an ODE however can always be reduced to first order by a change of variables, so that the determination of a $\mu(x, y')$ for it is equivalent to solving a first order ODE problem - not the focus of this work.

[7]We are only interested in the case $\mu_{y'} = F_{y'y'} \neq 0$.

and hence, a linear ODE $y'' = a(x)y' + b(x)y$ has an integrating factor $\mu(x, y') = y'/b$ when $b'/b - 2a = 0$. △

Case B: G_x is linear in y and $G_{yy} = 0$

In this case, in order to have Eq.(23) linear, $F_{y'}$ cannot depend on y', so that the integrating factor is of the form $\mu(x)$ and hence the case is of no interest: we end up with the standard search for $\mu(x)$ as the solution to the adjoint of the original linear ODE. △

2.2.2 Proof of Lemma 2

It follows from Eqs.(23) and (24) that, given $\mathcal{F}$ satisfying (25),

$$\frac{\partial}{\partial y}\left(\Phi(x, y, y')\ \mathcal{F}(x, y')\right) = -\frac{G_{y\,x}(x, y) + G_{y\,y}(x, y)\,y'}{\widetilde{\mu}\,(x)} \tag{29}$$

Hence, by taking coefficients of y' in the above,

$$\begin{aligned}
\varphi_1 \quad &\equiv \quad \Phi_y(x, y, y')\,\mathcal{F}(x, y') - y'\,\frac{\partial}{\partial y'}\left(\Phi_y(x, y, y')\,\mathcal{F}(x, y')\right) = -\frac{G_{y\,x}(x, y)}{\widetilde{\mu}\,(x)} \\
\varphi_2 \quad &\equiv \quad \frac{\partial}{\partial y'}\left(\Phi_y(x, y, y')\,\mathcal{F}(x, y')\right) = -\frac{G_{y\,y}(x, y)}{\widetilde{\mu}\,(x)}
\end{aligned} \tag{30}$$

where the left-hand-sides can be calculated explicitly since they depend only on Φ and the given $\mathcal{F}$. Similarly,

$$\begin{aligned}
\varphi_3 \quad &\equiv \quad -\frac{\partial}{\partial y'}\left(\Phi(x, y, y')\ \mathcal{F}(x, y')\right) = \frac{F_{y'x}(x, y') + G_y(x, y)}{\widetilde{\mu}\,(x)} \\
\varphi_4 \quad &\equiv \quad \frac{\partial}{\partial y'}\mathcal{F}(x, y') = \frac{F_{y'y'}(x, y')}{\widetilde{\mu}\,(x)}
\end{aligned} \tag{31}$$

Now, since in this case the ODE family Eq.(23) is nonlinear by hypothesis, either φ_2 or φ_4 is different from zero, so that at least one of the pairs $\{\varphi_1,\ \varphi_2\}$ or $\{\varphi_3,\ \varphi_4\}$ can be used to determine $\widetilde{\mu}\,(x)$ as the solution of a first order linear ODE. For example, if $\varphi_2 \neq 0$,

$$\frac{\partial}{\partial y}\left(\varphi_1(x, y)\ \widetilde{\mu}\,(x)\right) = \frac{\partial}{\partial x}\left(\varphi_2(x, y)\ \widetilde{\mu}\,(x)\right) \tag{32}$$

from where

$$\widetilde{\mu}\,(x) = e^{\displaystyle\int \frac{1}{\varphi_2}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right) dx} \tag{33}$$

If $\varphi_2 = 0$ then $\varphi_4 \neq 0$ and we obtain

$$\widetilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_4}\left(\frac{\partial \varphi_3}{\partial y'} - \frac{\partial \varphi_4}{\partial x}\right) dx} \tag{34}$$

When combined with Eq.(25), Eqs.(33) and (34) alternatively give both an explicit solution to the problem and an existence condition, since a solution $\widetilde{\mu}(x)$ - and hence an integrating factor of the form $\mu(x, y')$ - exists if the integrand in Eq.(33) or Eq.(34) only depends on x. △

2.2.3 Proof of Lemma 3

We start from Eq.(23) by considering the expression

$$\Upsilon \equiv \Phi_y = -\frac{G_{x\,y}(x, y) + G_{y\,y}(x, y)\ y'}{F_{y'}(x, y')} \tag{35}$$

and develop the proof below splitting the problem into different cases. For each case we show how to find $\mathcal{F}(x, y')$ satisfying Eq.(25). $\mathcal{F}$ will then lead to the required integrating factor when, in addition to the conditions explained below, the existence conditions for $\widetilde{\mu}(x)$ mentioned in the previous subsection are satisfied.

Case A: $G_{x\,y}/G_{y\,y}$ depends on y

To determine whether this is the case, we cannot just analyze the ratio $G_{x\,y}/G_{y\,y}$ itself since it is unknown. However, from Eq.(35), in this case the factors of Υ depending on y will also depend on y', and this condition can be formulated as

$$\frac{\partial}{\partial y'}\left(\frac{\Upsilon_y}{\Upsilon}\right) \neq 0 \tag{36}$$

When this inequation holds, we determine $F_{y'}(x, y')$ up to a factor depending on x, that is, the required $\mathcal{F}(x, y')$, as the reciprocal of the factors of Υ which depend on y' but not y. △

Example: Kamke's ODE 226

This ODE is presented in Kamke's book already in exact form, so we start by rewriting it in explicit form as

$$y'' = \frac{x^2 y y' + x y^2}{y'} \tag{37}$$

We determine Υ (Eq.(35)) as

$$\Upsilon = \frac{x(xy' + 2y)}{y'} \tag{38}$$

The only factor of Υ containing y is:

$$xy' + 2y \tag{39}$$

and since this also depends on y', $\mathcal{F}(x, y')$ is given by

$$\mathcal{F}(x, y') = y' \tag{40}$$

Case B: either $G_{xy} = 0$ or $G_{yy} = 0$

When the expression formed by all the factors of Υ containing y does not contain y', in Eq.(36) we will have $\frac{\partial}{\partial y'}(\frac{\Upsilon_y}{\Upsilon}) = 0$, and it is impossible to determine *a priori* whether one of the functions $\{G_{x\,y},\ G_{y\,y}\}$ is zero, or alternatively their ratio does not depend on y. We then proceed by assuming the former, build an expression for $\mathcal{F}(x, y')$ as in Case A, and check for the existence of $\widetilde{\mu}\,(x)$ as explained in the previous subsection. If $\widetilde{\mu}\,(x)$ exists, the problem is solved; otherwise we proceed as follows.

Case C: $G_{x\,y}/G_{y\,y} \neq 0$ and does not depend on y

In this case, neither $G_{x\,y}$ nor $G_{y\,y}$ is zero and their ratio is a function of just x, so that

$$\begin{aligned} G_{x\,y} &= v_1(x)\ w(x, y) \\ G_{y\,y} &= v_2(x)\ w(x, y) \end{aligned} \tag{41}$$

for some unknown functions $v_1(x)$ and $v_2(x)$. Eq.(35) is then given by

$$\Upsilon = w(x, y)\ \frac{(v_1(x) + v_2(x)\ y')}{F_{y'}(x, y')} \tag{42}$$

for some function $w(x, y)$, which is made up of the factors of Υ depending on y and not on y'. To determine $F_{y'}(x, y')$ up to a factor depending on x, we need to determine the ratio $v_1(x)/v_2(x)$. For this purpose, from Eq.(41) we build a PDE for $G_y(x, y)$,

$$G_{x\,y} = \frac{v_1(x)}{v_2(x)}\, G_{y,y} \tag{43}$$

The general solution of Eq.(43) is

$$G_y = \mathcal{G}\,(y + p(x)) \tag{44}$$

where $\mathcal{G}$ is an arbitrary function of its argument and for convenience we introduced

$$p'(x) \equiv v_1(x)/v_2(x) \tag{45}$$

We now determine $p'(x)$ as follows. Taking into account Eq.(41),

$$v_2(x)\, w(x, y) = \mathcal{G}'(y + p(x)) \tag{46}$$

By taking the ratio between this expression and its derivative w.r.t y we obtain

$$\mathcal{H}(y + p(x)) \equiv \frac{\partial \ln(w)}{\partial y} = \frac{\mathcal{G}''(y + p(x))}{\mathcal{G}'(y + p(x))} \tag{47}$$

that is, a function of $y + p(x)$ only, which we can determine since we know $w(x, y)$. If $\mathcal{H}' \neq 0$, $p'(x)$ is given by

$$p'(x) = \frac{\mathcal{H}_x}{\mathcal{H}_y} = \frac{w_{xy} w - w_x w_y}{w_{yy} w - {w_y}^2} \tag{48}$$

In summary, the conditions for this case are

$$\Upsilon_y \neq 0, \quad \frac{\partial}{\partial y'} \left(\frac{\Upsilon_y}{\Upsilon} \right) = 0, \quad \frac{\partial^2}{\partial y \partial x} \ln(w) \neq 0, \quad \frac{\partial^2}{\partial y \partial y} \ln(w) \neq 0 \tag{49}$$

and then, from Eq.(42), $\mathcal{F}(x, y')$ is given by

$$\mathcal{F}(x, y') = \frac{(p' + y')\ w}{\Upsilon} \tag{50}$$

where at this point Υ, $w(x, y)$ and $p'(x)$ are all known. $\triangle$

Example: Kamke's ODE 136.

We begin by writing the ODE in explicit form as

$$y'' = \frac{h(y')}{x - y} \tag{51}$$

This example is interesting since the standard search for point symmetries is made difficult by the presence of an arbitrary function of y'. Υ (Eq.(35)) is determined as

$$\Upsilon = -\frac{h(y')}{(x - y)^2} \tag{52}$$

and $w(x, y)$ as

$$w(x, y) = \frac{1}{(x - y)^2} \tag{53}$$

Then $\mathcal{H}(y + p(x))$ (Eq.(47)) becomes

$$\mathcal{H} = \frac{2}{x - y} \tag{54}$$

and hence, from Eq.(48), $p'(x)$ is

$$p'(x) = -1 \tag{55}$$

so from Eq.(50):

$$\mathcal{F}(x, y') = \frac{1 - y'}{h(y')} \tag{56}$$

<u>Case D</u>: $\mathcal{H} = 0$

We now discuss how to obtain $p'(x)$ when $\mathcal{H}'(y + p(x)) = 0$. We consider first the case in which $\mathcal{H} = 0$. Then, $\mathcal{G}'' = 0$ and the condition for this case is

$$\Upsilon_y = 0 \tag{57}$$

Recalling Eq.(44), G is given by

$$G(x,y) = C_1\ (y + p(x))^2 + C_2\ (y + p(x)) + g(x) \tag{58}$$

for some function $g(x)$ and some constants C_1, C_2. From Eq.(23), $\Phi(x, y, y')$ takes the form

$$\Phi(x,y,y') = -\frac{F_x(x,y') + g'(x) + (2C_1\ (y + p(x)) + C_2)(y' + p'(x))}{F_{y'}(x,y')} \tag{59}$$

We now determine $p'(x)$ as follows. First, from the knowledge of Υ and Φ we build the two explicit expressions:

$$\Lambda \equiv \frac{1}{\Upsilon} = -\frac{F_{y'}}{2C_1\ (y' + p'(x))} \tag{60}$$

and

$$\Psi \equiv \frac{\Phi(x,y,y')}{\Upsilon} - y = \frac{F_x + g'(x)}{2C_1\ (y' + p'(x))} + p(x) + \frac{C_2}{2C_1} \tag{61}$$

From Eq.(60) and Eq.(61) Λ and Ψ are related by:

$$\frac{\partial}{\partial x}\left((y' + p'(x))\ \Lambda\right) + \frac{\partial}{\partial y'}\left((y' + p'(x))\ \Psi\right) = p(x) + \frac{C_2}{2C_1} \tag{62}$$

where the only unknowns are $p(x)$, C_1, and C_2. By differentiating the equation above w.r.t y' and x we obtain two equations where the only unknown is $p'(x)$:

$$\Lambda_{y'}p''(x) + (\Lambda_{xy'} + \Psi_{y'y'})(y' + p'(x)) + \Lambda_x + 2\Psi_{y'} = 0 \tag{63}$$

$$\Lambda\ p'''(x) + (\Lambda_{xx} + \Psi_{y'x})(y' + p'(x)) + (\Lambda_x + \Psi_{y'})p''(x) + \Psi_x = p'(x) \tag{64}$$

from where we obtain $p'(x)$ by solving a linear algebraic equation built by eliminating $p''(x)$ between Eq.(63) and Eq.(64)[8]. Also, as a shortcut, if $(\Lambda_{xy'} + \Psi_{y'y'})/\Lambda_{y'}$ depends on y', then we can build a linear algebraic equation for $p'(x)$ by solving for $p''(x)$ in Eq.(63) and differentiating w.r.t. y'. $\triangle$

Remark

If Eq.(63) depends neither on $p'(x)$ nor on $p''(x)$ this scheme will not succeed. However, in that case the original ODE is actually linear and given by Eq.(28). To see this, we set to zero the coefficients of $p'(x)$ and $p''(x)$ in Eq.(63), obtaining:

$$\Lambda_{y'} = \Lambda_{xy'} + \Psi_{y'y'} = \Lambda_x + 2\Psi_{y'} = 0 \tag{65}$$

[8]From Eq.(60), $\Lambda \neq 0$, so that Eq.(64) always depends on $p'''(x)$, and solving Eq.(63) for $p''(x)$ and substituting twice into Eq.(64) will lead to the desired equation for $p'(x)$. If Eq.(63) depends on $p'(x)$ but not on $p''(x)$, then Eq.(63) itself is already a linear algebraic equation for $p'(x)$.

which implies that Λ is a function of x only, and then

$$\Psi_{y'y'} = 0 \tag{66}$$

If we now rewrite $F(x, y')$ as

$$F(x, y') = Z(x, y') - g(x) - \Lambda(y' + p')^2\, C_1 \tag{67}$$

and introduce this expression in Eq.(60), we obtain $Z_{y'} = 0$; similarly, using this result, Eq.(61), Eq.(66) and Eq.(67) we obtain $Z_x = 0$. Hence, Z *is a constant*, and taking into account Eq.(67) and Eq.(59), the ODE which led us to this case is just a non-homogeneous linear ODE of the form

$$(y + p)'' + (\Lambda'(y + p)' - 2(y + p) - C_2/C_1)/2\Lambda = 0 \tag{68}$$

whose homogeneous part does not depend on $p(x)$:

$$y'' + \frac{\Lambda'(x)}{2\Lambda(x)}\, y' - \frac{y}{\Lambda(x)} = 0 \tag{69}$$

and as mentioned, it is the same as Eq.(28).

Example: Kamke's ODE 66.

This ODE is given by

$$y'' = a\,(c + bx + y)\left(y'^2 + 1\right)^{3/2} \tag{70}$$

Proceeding as in Case A, we determine Υ, $w(x, y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = a\left(y'^2 + 1\right)^{3/2}; \qquad w(x, y) = 1; \qquad \mathcal{H} = 0 \tag{71}$$

From the last equation we realize that we are in Case D. We determine Λ and Ψ (Eqs. (60), (61)) as:

$$\begin{aligned} \Lambda &= \frac{1}{\left(y'^2 + 1\right)^{3/2} a} \\ \Psi &= c + b\,x \end{aligned} \tag{72}$$

We then build Eq.(62) for this ODE:

$$\frac{p''(x)}{\left(y'^2 + 1\right)^{3/2} a} + c + b\,x = p(x) + \frac{C_2}{2C_1} \tag{73}$$

Differentiating w.r.t. y' leads to Eq.(63):

$$-3\frac{p''(x)\,y'}{\left(y'^2 + 1\right)^{5/2} a} = 0 \tag{74}$$

from which it follows that $p''(x) = 0$. Using this in Eq.(64) we obtain:

$$p'(x) = b \tag{75}$$

after which Eq.(50) becomes

$$\mathcal{F}(x, y') = \frac{y' + b}{a\left(y'^2 + 1\right)^{3/2}} \tag{76}$$

<u>Case E</u>: $\mathcal{H}' = 0$ and $\mathcal{H} \neq 0$

In this case $\mathcal{H}(y+p(x)) = \mathcal{G}''/\mathcal{G}' = C_1$, so $\mathcal{G}'$ is an exponential function of its argument $(y+p(x))$ and hence from Eq.(44)

$$G(x, y) = C_2 e^{(y+p(x))C_1} + (y + p(x))C_3 + g(x) \tag{77}$$

for some constants C_2, C_3 and some function $g(x)$. In this case one of the conditions to be satisfied is

$$\Upsilon_y = \text{constant} \neq 0 \tag{78}$$

and $\Phi(x, y, y')$ will be of the form

$$\Phi(x, y, y') = -\frac{F_x(x, y') + g'(x) + \left(C_2 C_1 e^{(y+p(x))C_1} + C_3\right)(y' + p'(x))}{F_{y'}(x, y')} \tag{79}$$

Taking advantage of the fact that we explicitly know C_1, we build a first expression for p' by dividing $C_1 e^{yC_1}$ by Υ:

$$\Lambda \equiv -\frac{F_{y'}}{C_2 e^{p(x)C_1}\,(y' + p'(x))} \tag{80}$$

We obtain a second expression for p' by multiplying Φ by Λ and subtracting $C_1 e^{C_1 y}$

$$\Psi \equiv \frac{1}{C_2 e^{p(x)C_1}}\left(\frac{F_x + g'(x)}{y' + p'(x)} + C_3\right) \tag{81}$$

As in Case D, Λ and Ψ are related by

$$\frac{\partial}{\partial x}\left((y' + p'(x))\,\Lambda\right) + (y' + p'(x))\; p'(x)\Lambda C_1 + \frac{\partial}{\partial y'}\left((y' + p'(x))\,\Psi\right) = \frac{C_3}{C_2 e^{p(x)C_1}} \tag{82}$$

where the only unknowns are C_2, C_3 and $p(x)$. Differentiating Eq.(82) with respect to y' we have

$$\begin{aligned}\left(p''(x) + p'(x)^2 C_1\right)\Lambda_{y'} + p'(x)\left(y'\Lambda_{y'}C_1 + \Lambda C_1 + \Lambda_{xy'} + \Psi_{y'y'}\right)&\\ +2\Psi_{y'} + \Lambda_x + y'\Lambda_{xy'} + y'\Psi_{y'y'} = 0&\end{aligned} \tag{83}$$

The problem now is that, due to the exponential on the RHS of Eq.(82), differently from Case D, we are not able to obtain a second expression for $p'(x)$ by differentiating w.r.t x. The alternative we have found can be summarized as follows. We first note that if $\Lambda_{y'} = 0$, Eq.(83) is already a linear algebraic equation[9] for p', so that we are only worried with the case $\Lambda_{y'} \neq 0$. With this in mind, we divide Eq.(83) by $\Lambda_{y'}$ and, *if* the resulting expression depends on y', we directly obtain a linear algebraic equation in $p'(x)$ by just differentiating w.r.t y'. $\triangle$

Example[10]*:*

$$y'' = \frac{y' \,(xy'+1)\,(-2+e^{y})}{y'x^2+y'-1} \tag{84}$$

We determine Υ, $w(x,y)$, and $\mathcal{H}(y+p(x))$ as

$$\Upsilon = \frac{y'(xy'+1)e^{y}}{y'x^2+y'-1}; \qquad w(x,y) = e^{y}; \qquad \mathcal{H} = 1 \tag{85}$$

From the last equation we know that we are in Case E. We then determine Λ and Ψ as in Eqs. (80) and (81):

$$\begin{aligned} \Lambda &= \frac{y'x^2+y'-1}{y'(xy'+1)} \\ \Psi &= -2 \end{aligned} \tag{86}$$

Now, we build Eq.(82):

$$\frac{1}{xy'+1}\left(\left(p''+p'^2+y'^2\frac{(xp'-1)}{xy'+1}\right)\left(x^2+1-\frac{1}{y'}\right)+2\,xp'-2\right) = \frac{C_3}{C_2\,e^{p}} \tag{87}$$

and, differentiating w.r.t. y', we obtain Eq.(83):

$$\frac{2\,xy'+1-(x^3+x)y'^2}{y'^2(xy'+1)^2}\left(p''+p'^2\right)+\frac{2\,y'-1-2\,x+xy'}{(xy'+1)^3}(xp'-1) = 0 \tag{88}$$

Proceeding as explained, dividing by $\Lambda_{y'}$ and differentiating w.r.t. y', we have

$$\frac{\partial}{\partial y'}\left(y'^2\frac{2\,y'-1-2\,x+xy'}{(xy'+1)\,(2\,xy'+1-(x^3+x)y'^2)}\right)(xp'-1) = 0 \tag{89}$$

[9] We can see this by assuming that $\Lambda_{y'} = 0$ and that Eq.(83) does not contain p', and then arriving at a contradiction as follows. We first set the coefficients of p' in Eq.(83) to zero, arriving at

$$0 = C_1\,\Lambda + \Psi_{y'y'} = 2\Psi_{y'} + \Lambda_x + \Psi_{y'y'}y' \tag{A}$$

Eliminating $\Psi_{y'y'}$ gives

$$2\Psi_{y'} = C_1\,\Lambda\,y' - \Lambda_x$$

Differentiating the expression above w.r.t y' and since $\Lambda_{y'} = 0$, we have

$$2\Psi_{y'y'} = C_1\,\Lambda$$

Finally, using Eq.(A), $0 = \Lambda$, contradicting $F_{y'} \neq 0$.

[10] There are no examples of this type in all of Kamke's set of non-linear second order ODEs.

Solving for $p'(x)$ gives $p'(x) = 1/x$, from which Eq.(50) becomes:

$$\mathcal{F}(x, y') = \left(y' - \frac{1}{x}\right) \frac{y'x^2 + y' - 1}{y'(xy' + 1)} \tag{90}$$

Case F

The final branch occurs when Eq.(83) divided by $\Lambda_{y'}$ does not depend on y' (so that we will not be able to differentiate w.r.t y'). In this case we can build a linear algebraic equation for $p'(x)$ as follows. Let us introduce the label $\beta(x, p', p'')$ for Eq.(83) divided by $\Lambda_{y'}$, so that Eq.(83) becomes:

$$\Lambda_{y'}(x, y')\ \beta(x, p', p'') = 0 \tag{91}$$

Since we obtained Eq.(83) by differentiating Eq.(82) with respect to y', Eq.(82) can be written in terms of β by integrating Eq.(91) with respect to y':

$$\Lambda(x, y')\beta(x, p', p'') + \gamma(x, p', p'') = \frac{C_3}{C_2 e^{p(x)C_1}} \tag{92}$$

where $\gamma(x, p', p'')$ is the constant of integration, and can be determined explicitly in terms of x, p' and p'' by comparing Eq.(92) with Eq.(82). Taking into account that $\beta(x, p', p'') = 0$, Eq.(92) reduces to:

$$\gamma(x, p', p'') = \frac{C_3}{C_2 e^{p(x)C_1}} \tag{93}$$

We can remove the unknowns C_2 and C_3 after multiplying Eq.(93) by $e^{p(x)C_1}$, differentiating with respect to x, and then dividing once again by $e^{p(x)C_1}$. We now have our second equation for p', which we can build explicitly in terms of p', since we know $\gamma(x, p', p'')$ and C_1:

$$\frac{d\gamma}{dx} + C_1\ p'\gamma = 0 \tag{94}$$

Eliminating the derivatives of p' between Eq.(91) and Eq.(94) leads to a linear algebraic equation in p'. Once we have p', the determination of $\mathcal{F}(x, y')$ follows directly from Eq.(50). △

2.3 Integrating factors of the form $\mu(y, y')$

From Eq.(11), the ODE family admitting an integrating factor of the form $\mu(y, y')$ is given by

$$y'' = -\frac{y'}{\mu}\left(G_y + \frac{\partial}{\partial y}\int \mu\ dy'\right) - \frac{G_x}{\mu} \tag{95}$$

where $\mu(y, y')$ and $G(x, y)$ are arbitrary functions of their arguments. For this ODE family, it would be possible to develop an analysis and split the problem into cases as done in the previous section for the case $\mu(x, y')$. However, it is straightforward to notice that under the transformation $y(x) \to x,\ x \to y(x)$, Eq.(95) transforms into an ODE of the form Eq.(23) with integrating factor $\mu(x, {y'}^{-1})/{y'}^2$. It follows that an integrating factor for any member of the ODE family above can be found by merely changing variables in the given ODE and calculating the corresponding integrating factor of the form $\mu(x, y')$.

Example:

$$y'' - \frac{y'^2}{y} + \sin(x)\, y'\, y + \cos(x)\, y^2 = 0 \tag{96}$$

Changing variables $y(x) \to x,\ x \to y(x)$ we obtain

$$y'' + \frac{y'}{x} - \sin(y)\, y'^2 x - \cos(y)\, x^2 y'^3 = 0 \tag{97}$$

Using the algorithm outlined in the previous section, an integrating factor of the form $\mu(x, y')$ for Eq.(97) is given by

$$\frac{1}{y'^2 x} \tag{98}$$

from where an integrating factor of the form $\mu(y, y')$ for Eq.(96) is $1/y$, leading to the first integral

$$\sin(x)\ y + \frac{y'}{y} + C_1 = 0, \tag{99}$$

which is a first order ODE of Bernoulli type. The solution to Eq.(96) then follows directly. This example is interesting since from [8] Eq.(96) has no point symmetries.

3 Integrating factors and symmetries

Besides the formulas for integrating factors of the form $\mu(x, y)$, the main result presented in this paper is a systematic algorithm for the determination of integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ *without solving any auxiliary differential equations or performing differential Groebner basis calculations*, and these last two facts constitute the relevant point. Nonetheless, it is interesting to briefly compare the standard integrating factor (μ) and symmetry approaches, so as to have an insight of how complementary these methods can be in practice.

To start with, both methods tackle an n^{th} order ODE by looking for solutions to a linear n^{th} *order determining PDE* in $n + 1$ variables. Any given ODE has infinitely many integrating factors and symmetries. When many solutions to these *determining* PDEs are found, both approaches can, in principle, give a multiple reduction of order.

In the case of integrating factors there is one unknown function, while for symmetries there is a pair of infinitesimals to be found. But symmetries are defined up to an arbitrary function, so that we can always take one of these infinitesimals equal to zero[11]; hence we are facing approaches of equivalent levels of difficulty and actually of equivalent solving power.

Also valid for both approaches is the fact that, unless some *restrictions* are introduced on the functional dependence of μ or the infinitesimals, there is no hope that the corresponding determining PDEs will be easier to solve than the original ODE. In the case of symmetries, it is usual to restrict the problem to ODEs having *point symmetries*, that is, to consider infinitesimals depending only

[11] Symmetries $[\xi(x, y, ..y^{(n-1)}), \eta(x, y, ..y^{(n-1)})]$ of an n^{th} order ODE can always be rewritten as $[G, (G - \xi)y' + \eta]$, where $G(x, y, ..y^{(n-1)})$ is an arbitrary function (for first order ODEs, y' must replaced by the right-hand-side of the ODE). Choosing $G = 0$ the symmetry acquires the form $[0, \bar{\eta}]$

on x and y. The restriction to the integrating factors here discussed is similar: we considered μ's depending on only two variables.

At this point it can be seen that the two approaches are complementary: the determining PDEs for μ and for the symmetries are different[12], so that even using identical restrictions on the functional dependence of μ and the infinitesimals, problems which may be untractable using one approach may be easy or even trivial using the other one.

As an example of this, consider Kamke's ODE 6.37

$$y'' + 2\,y\,y' + f(x)\left(y' + y^2\right) - g(x) = 0 \tag{100}$$

For *arbitrary* $f(x)$ and $g(x)$, this ODE has an integrating factor depending only on x, easily determined using the algorithms presented. Now, for *non-constant* $f(x)$ and $g(x)$, this ODE has no point symmetries, that is, no infinitesimals of the form $[\xi(x,y),\ \eta(x,y)]$, except for the particular case in which $g(x)$ can be expressed in terms of $f(x)$ as in[13]

$$g(x) = \frac{f''}{4} + \frac{3\,f\,f'}{8} + \frac{f^3}{16} - \frac{C_2 \exp\left(-3/2\,\int f(x)dx\right)}{4\left(2\,C_1 + \int \exp\left(-1/2\,\int f(x)dx\right)dx\right)^3} \tag{101}$$

Furthermore, this ODE does not have non-trivial symmetries of the form $[\xi(x,y'),\ \eta(x,y')]$ either, and for symmetries of the form $[\xi(y,y'),\ \eta(y,y')]$ the determining PDE does not split into a system.

Another ODE example of this type is found in a paper by [8] (1988):

$$y'' - \frac{y'^2}{y} - g(x)\,p\,y^p y' - g'\,y^{p+1} = 0 \tag{102}$$

In that work it is shown that for constant p, the ODE above only has point symmetries for very restricted forms of $g(x)$. For instance, Eq.(96) is a particular case of the ODE above and has no point symmetries. On the other hand, *for arbitrary* $g(x)$, Eq.(102) has an obvious integrating factor depending on only one variable: $1/y$, leading to a first integral of Bernoulli type:

$$\frac{y'}{y} - g(x)y^p + C_1 = 0 \tag{103}$$

so that the whole family Eq.(102) is integrable by quadratures.

We note that Eq.(100) and Eq.(102) are respectively particular cases of the general reducible ODEs having integrating factors of the form $\mu(x)$:

$$y'' = -\frac{(\mu_x + G_y)}{\mu(x)}y' - \frac{G_x}{\mu(x)} \tag{104}$$

where $\mu(x)$ and $G(x,y)$ are arbitrary; and $\mu(y)$:

[12] We are considering here ODEs of order greater than one.

[13] To determine $g(x)$ in terms of $f(x)$ we used the *standard form* Maple package by Reid and Wittkopf complemented with some basic calculations.

$$y'' = -\frac{(\mu_y\, y' + G_y)}{\mu(y)}\, y' - \frac{G_x}{\mu(y)} \tag{105}$$

In turn, these are very simple cases if compared with the general ODE families Eq.(23) and Eq.(95), respectively having integrating factors of the forms $\mu(x, y')$ and $\mu(y, y')$, and which can be systematically reduced in order using the algorithms here presented.

It is then natural to conclude that the integrating factor and the symmetry approaches are useful for solving different types of ODEs, and can be viewed as equivalently powerful and general, and in practice complementary. Moreover, if for a given ODE, an integrating factor and a symmetry are known, in principle one can combine this information to build two first integrals and reduce the order by two at once (see for instance [9]).

4 Tests

After plugging the reducible-ODE scheme here presented into the ODEtools package [10], we tested the scheme and routines using Kamke's non-linear 246 second order ODE examples[14]. The purpose was to confirm the correctness of the returned results and to determine which of these ODEs have integrating factors of the form $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$. The test consisted of determining μ and testing the exactness condition Eq.(3).

In addition, we ran a comparison of performances in solving a subset of Kamke's examples having integrating factors of the forms $\mu(x, y')$ or $\mu(y, y')$, using different computer algebra ODE-solvers (Maple, Mathematica, MuPAD and the Reduce package Convode). The idea was to situate the new scheme in the framework of a sample of relevant packages presently available.

To run the comparison of performances, the first step was to classify Kamke's ODEs into: *missing* x, *missing* y, *exact* and *reducible*, where the latter refers to ODEs having integrating factors of the forms $\mu(x, y')$ or $\mu(y, y')$. ODEs missing variables were not included in the test since they can be seen as first order ODEs in disguised form, and as such they are not the main target of the algorithm being presented. The classification we obtained for these 246 ODEs is as follows

Classification	ODE numbers as in Kamke's book
99 ODEs are missing x or missing y	1, 2, 4, 7, 10, 12, 14, 17, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32, 40, 42, 43, 45, 46, 47, 48, 49, 50, 54, 56, 60, 61, 62, 63, 64, 65, 67, 71, 72, 81, 89, 104, 107, 109, 110, 111, 113, 117, 118, 119, 120, 124, 125, 126, 127, 128, 130, 132, 137, 138, 140, 141, 143, 146, 150, 151, 153, 154, 155, 157, 158, 159, 160, 162, 163, 164, 165, 168, 188, 191, 192, 197, 200, 201, 202, 209, 210, 213, 214, 218, 220, 222, 223, 224, 232, 234, 236, 237, 243, 246
13 are in exact form	36, 42, 78, 107, 108, 109, 133, 169, 170, 178, 226, 231, 235
40 ODEs are *reducible* with integrating factor $\mu(x, y')$ or $\mu(y, y')$ *and* missing x or y	1, 2, 4, 7, 10, 12, 14, 17, 40, 42, 50, 56, 64, 65, 81, 89, 104, 107, 109, 110, 111, 125, 126, 137, 138, 150, 154, 155, 157, 164, 168, 188, 191, 192, 209, 210, 214, 218, 220, 222, 236
28 ODEs are *reducible* and not missing x or y	36, 37, 51, 66, 78, 97, 108, 123, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215, 226, 235

Table 1. Missing variables, exact and *reducible* Kamke's 246 second order non-linear ODEs.

[14]Kamke's ODEs 6.247 to 6.249 cannot be made explicit and are then excluded from the tests.

For our purposes, the interesting subset is the one comprised of the 28 ODEs not already missing variables. The results we obtained using the aforementioned computer algebra ODE-solvers[15] are summarized as follows[16]:

Kamke's ODE numbers				
	Convode	Mathematica 3.0	MuPAD 1.3	ODEtools
Solved:	51, 166, 173, 174, 175, 176, 179.	78, 97, 108, 166, 169, 173, 174, 175, 176, 178, 179, 206.	78, 97, 108, 133, 166, 169, 173, 174, 175, 176, 179,	51, 78, 97, 108, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215.
Totals:	7	12	11	22
Reduced:				36, 37, 66, 123, 226, 235.
Totals:	0	0	0	6

Table 2. *Performances in solving 28 Kamke's ODEs having an integrating factor* $\mu(x, y')$ *or* $\mu(y, y')$

As shown above, while the scheme here presented is finding first integrals in all the 28 ODE examples, opening the way to solve 22 of them to the end, the next scores are only 12 and 11 ODEs, respectively solved by Mathematica 3.0 and MuPAD 1.3.

Concerning the six reductions of order returned by **odsolve**, it must be said that neither MuPAD nor Mathematica provide a way to convey them, so that perhaps their ODE-solvers are obtaining first integrals for these cases but the routines are giving up when they cannot solve the problem to the end.

5 Conclusions

In connection with second order ODEs, this paper presented a systematic method for determining the existence of integrating factors and their explicit form, when they have the forms $\mu(x, y)$, $\mu(x, y')$ and $\mu(y, y')$. The scheme is new, as far as we know, and its implementation in the framework of the computer algebra package ODEtools has proven to be a valuable tool. Actually, the implementation of the scheme solves ODEs not solved by using standard or symmetry methods (see sec. 3) or some other relevant and popular computer algebra ODE-solvers (see sec. 4).

Furthermore, the algorithms presented involve only very simple operations and do not require solving auxiliary differential equations, except in one branch of the $\mu(x, y)$ problem. So, even for examples where other methods also work, for instance by solving the related PDE system Eqs.(6) and (7) using ansatzes and differential Groebner basis techniques, the method here presented can return answers faster and avoiding potential explosions of memory[17].

On the other hand, we have restricted the problem to the universe of second order ODEs having integrating factors depending only on two variables while packages as CONLAW (in REDUCE) can try and in some cases solve the PDE system Eqs.(6) and (7) by using more varied ansatzes for μ.

[15] Maple R4 is not present in the table since it is not solving any of these 28 ODEs. This situation is being resolved in the upcoming Maple R5, where the ODEtools routines are included in the Maple library, and the previous ODE-solver was replaced by **odsolve**. However, the scheme here presented was not ready when the development library was closed; the *reducible* scheme implemented in Maple R5 is able to determine, when they exist, integrating factors only of the form $\mu(y')$.

[16] Some of these 28 ODEs are given in Kamke in exact form and hence they can be easily reduced after performing a check for exactness; before running the tests all these ODEs were rewritten in explicit form by isolating y''.

[17] Explosions of memory may happen when calculating all the integrability conditions involved at each step in the differential Groebner basis approach.

A natural extension of this work would be to develop a scheme for building integrating factors of restricted but more general forms, now for higher order ODEs. We are presently working on these possible extensions[18], and expect to succeed in obtaining reportable results in the near future.

Acknowledgments

This work was supported by the State University of Rio de Janeiro (UERJ), Brazil, and by the Symbolic Computation Group, Faculty of Mathematics, University of Waterloo, Ontario, Canada. The authors would like to thank K. von Bülow[19] for a careful reading of this paper.

References

[1] Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen.* Chelsea Publishing Co, New York (1959).

[2] S.C. Anco and G.W. Bluman, *Integrating Factors and First Integrals for Ordinary Differential Equations*, Preprint, University of British Columbia (1997).

[3] T. Wolf, *Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions*, J. Symb. Comp. 27 (1999), pp 221-238.

[4] Murphy, G.M., *Ordinary Differential Equations and their solutions.* Van Nostrand, Princeton, 1960.

[5] Sheftel, M.B., *A Course in Group Analysis of Differential Equations, Part II: Ordinary Differential Equations.* St. Petersburg Institute of Economics and Finance, 1997. In Russian.

[6] F. Boulier, Lazard, Ollivier and Petitot, *Representation for the radical of a finitely generated differential ideal*, Proceedings of ISSAC95, pp. 158-166

[7] G.J. Reid and A.D. Wittkopf, *Long Guide to the Standard Form Package*, Pre-print, Department of Mathematics, University of British Columbia, Vancouver, Canada, 1993.

[8] A. González-López, *Symmetry and integrability by quadratures of ordinary differential equations*, Phys. Lett. **A**, V. 133, N 4.5 (1998) 190.

[9] Stephani, H. *Differential equations: their solution using symmetries*, ed. M.A.H. MacCallum, Cambridge University Press, New York and London (1989).

[10] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, "Computer Algebra Solving of Second Order ODEs Using Symmetry Methods". Computer Physics Communications 108 (1998) 90

[18] See http://lie.uwaterloo.ca/odetools.html

[19] Symbolic Computation Group of the Theoretical Physics Department at UERJ - Brazil.

Appendix A

We display here both the integrating factors obtained for the 28 Kamke's ODEs used in the tests (see sec. 4) and the "case" corresponding to each ODE when using just the algorithms for $\mu(x, y')$ or $\mu(y, y')$[20]. As explained in sec. 2.2.3, the algorithm presented is subdivided into different cases: A, B, C, D, E and F, and case B is always either A or C.

Integrating factor	Kamke's book ODE-number	Case
1	36	D
$e^{\int f(x)dx}$	37	A
${y'}^{-1}$	51, 166, 169, 173, 175, 176, 179, 196, 203, 204, 206, 215	C
$\frac{b+y'}{\left(1+{y'}^2\right)^{3/2}}$	66	D
x	78	D
x^{-1}	97	A
y	108	D
y^{-1}	123	A
$\frac{1+y'}{(y'-1)y'}$	133	C
$\frac{y'-1}{(1+y')y'}$	134	C
$\frac{y'-1}{(1+y')\left(1+{y'}^2\right)}$	135	C
$\frac{y'-1}{h(y')}$	136	C
$\frac{x}{2\,xy'-1}$	174	C
$(1+y')^{-1}$	178	C
$\frac{1}{y'\,(1+2\,yy')}$	193	C
y'	226	A
$h(y')$	235	C

Integrating factors for Kamke's ODEs which are *reducible* and not missing x or y.

[20]We note that for non-linear ODEs these two algorithms work as well when $\mu_{y'} = 0$, but in practice these very simple examples are covered by the algorithm for $\mu(x, y)$ presented in sec. 2.1.

17 Integrating factors, adjoint equations and Lagrangians. By Nail H. Ibragimov (2005)

Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 318 (2006) 742–757
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Integrating factors, adjoint equations and Lagrangians

Nail H. Ibragimov

Department of Mathematics and Science, Research Centre ALGA: Advances in Lie Group Analysis, Blekinge Institute of Technology, SE-371 79 Karlskrona, Sweden

Received 30 October 2005
Available online 7 December 2005
Submitted by Steven G. Krantz

Abstract

Integrating factors and adjoint equations are determined for linear and non-linear differential equations of an arbitrary order. The new concept of an adjoint equation is used for construction of a Lagrangian for an arbitrary differential equation and for any system of differential equations where the number of equations is equal to the number of dependent variables. The method is illustrated by considering several equations traditionally regarded as equations without Lagrangians. Noether's theorem is applied to the Maxwell equations.

Keywords: Integrating factor for higher-order equations; Adjoint equation to non-linear equations; Lagrangian; Noether's theorem

1. Introduction

It is a traditional custom to associate adjoint equations exclusively with linear equations. It is also customary to discuss integrating factors for non-linear ordinary differential equations only in the case of first-order equations. Recall that Noether's theorem provides a connection between conservation laws for variational problems with symmetries of the Euler–Lagrange equations. In this introduction, we outline the corresponding definitions and results.

1.1. Integrating factor

The usual approach to integrating factors is as follows. A first-order ordinary differential equation

$$a(x, y)y' + b(x, y) = 0, \tag{1.1}$$

E-mail address: nib@bth.se.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.11.012

where $y' = dy/dx$, is written in the differential form:

$$a(x, y)\,dy + b(x, y)\,dx = 0. \tag{1.2}$$

Equation (1.2) is said to be *exact* if its left-hand side is the differential, i.e.,

$$a(x, y)\,dy + b(x, y)\,dx = d\Phi(x, y) \tag{1.3}$$

with some function $\Phi(x, y)$. If Eq. (1.2) is exact, its solution is defined implicitly by $\Phi(x, y) = C = \text{const}$.

In general, Eq. (1.2) is not exact but it becomes exact upon multiplying by a certain function $\mu(x, y)$:

$$\mu(a\,dy + b\,dx) = d\Phi \equiv \Phi_y\,dy + \Phi_x\,dx, \tag{1.4}$$

where

$$\Phi_y = \frac{\partial \Phi}{\partial y}, \qquad \Phi_x = \frac{\partial \Phi}{\partial x}\cdot$$

The function $\mu(x, y)$ is called an *integrating factor* for Eq. (1.2). It follows from (1.4) that

$$\Phi_y = \mu a, \qquad \Phi_x = \mu b. \tag{1.5}$$

The integrability condition for the system (1.5) is written $\Phi_{xy} = \Phi_{yx}$ and yields the following equation for determining the integrating factors:

$$\frac{\partial(\mu a)}{\partial x} = \frac{\partial(\mu b)}{\partial y}\cdot \tag{1.6}$$

Theoretically, Eq. (1.6) provides an infinite number of integrating factors for Eq. (1.2). Practically, however, the integration of Eq. (1.6) is not usually simpler than the integration of the differential equation (1.2) in question. Nevertheless, the concept of an integrating factor gives us a useful tool since integrating factors for certain particular equations can be found by *ad hoc* methods. If one knows two linearly independent integrating factors, $\mu_1(x, y)$ and $\mu_2(x, y)$, for (1.2) then the general solution of (1.2) is obtained without additional quadratures from the equation

$$\frac{\mu_1(x, y)}{\mu_2(x, y)} = C. \tag{1.7}$$

1.2. Adjoint linear differential operators

Let $x = (x^1, \ldots, x^n)$ be n independent variables and $u = (u^1, \ldots, u^m)$ be m dependent variables with the partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots$ of the first, second, etc. orders, where $u_i^\alpha = \partial u^\alpha/\partial x^i$, $u_{ij}^\alpha = \partial^2 u^\alpha/\partial x^i \partial x^j$. Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{1.8}$$

the total differentiation with respect to x^i, we have

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \quad \ldots.$$

Recall the definition of the adjoint linear operator. Let us consider, e.g., the scalar (i.e., $m = 1$) second-order linear partial differential equations

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = f(x), \tag{1.9}$$

where L is the following linear differential operator:

$$L = a^{ij}(x)D_i D_j + b^i(x)D_i + c(x). \tag{1.10}$$

The summation convention is used throughout the paper. Here, for example, the summation is assumed over $i, j = 1, \dots, n$. The coefficients $a^{ij}(x)$ are symmetric, i.e., $a^{ij} = a^{ji}$.

The *adjoint operator* to L is a second-order linear differential operator L^* such that

$$vL[u] - uL^*[v] = D_i\big(p^i\big) \equiv \operatorname{div} P(x) \tag{1.11}$$

for all functions u and v, where $P(x) = (p^1(x), \dots, p^n(x))$ is any vector. The adjoint operator L^* is uniquely determined and has the form

$$L^*[v] = D_i D_j\big(a^{ij}v\big) - D_i\big(b^i v\big) + cv. \tag{1.12}$$

The operator L is said to be *self-adjoint* if $L[u] = L^*[u]$ for any function $u(x)$. Recall that the operator (1.10) is self-adjoint if and only if

$$b^i(x) = D_j\big(a^{ij}\big), \quad i = 1, \dots, n. \tag{1.13}$$

The linear homogeneous equation

$$L^*[v] \equiv D_i D_j\big(a^{ij}v\big) - D_i\big(b^i v\big) + cv = 0 \tag{1.14}$$

is called the *adjoint equation* to the linear differential equation (1.9), $L[u] = f(x)$.

The definitions of the adjoint operator and the adjoint equation are the same for systems of second-order equations. They are obtained by assuming in Eq. (1.9) that u is an m-dimensional vector-function and that the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ of the operator (1.10) are $m \times m$-matrices.

If $n = m = 1$ we have the definition of the adjoint operator to linear ordinary differential equations. Let us set $u = y$ and consider the first-order equation

$$L[y] \equiv a_0(x)y' + a_1(x)y = f(x). \tag{1.15}$$

The *adjoint operator* $L^*[z]$ to $L[y]$ has the form

$$L^*[z] = -(a_0 z)' + a_1 z. \tag{1.16}$$

The definition of the adjoint operator to higher-order equations is similar. For example, in the case of the second-order equation

$$L[y] \equiv a_0 y'' + a_1 y' + a_2 y = f(x) \tag{1.17}$$

with variable coefficients $a_0(x), a_1(x), a_2(x)$, the adjoint operator $L^*[z]$ to $L[y]$ is

$$L^*[z] = (a_0 z)'' - (a_1 z)' + a_2 z. \tag{1.18}$$

Likewise, in the case of the third-order equation

$$L[y] \equiv a_0 y''' + a_1 y'' + a_2 y' + a_3 y = f(x), \tag{1.19}$$

the adjoint operator $L^*[z]$ to $L[y]$ is given by

$$L^*[z] = -(a_0 z)''' + (a_1 z)'' - (a_2 z)' + a_3 z. \tag{1.20}$$

The homogeneous equation $L^*[z] = 0$ is called the adjoint equation to $L[y] = f(x)$.

1.3. Noether's theorem

Noether's theorem [9] manifests a connection between symmetries and conservation laws for variational problems and provides a simple procedure for construction of conservation laws for Euler–Lagrange equations with known symmetries. The main steps of this procedure are as follows.

For the sake of brevity, consider Lagrangians $\mathcal{L}(x, u, u_{(1)})$ involving, along with the independent variables $x = (x, \ldots, x^n)$ and the dependent variables $u = (u, \ldots, u^m)$, the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ only. Then the Euler–Lagrange equations have the form

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{1.21}$$

They are obtained by variation of the integral $\int \mathcal{L}(x, u, u_{(1)})\,\mathrm{d}x$ taken over an arbitrary n-dimensional domain in the space of the independent variables.

Noether's theorem states that if the variational integral is invariant under a continuous transformation group G with a generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.22}$$

then the vector field $C = (C^1, \ldots, C^n)$ defined by

$$C^i = \xi^i \mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad i = 1, \ldots, n, \tag{1.23}$$

provides a conservation law for the Euler–Lagrange equations (1.21), i.e., obeys the equation $\operatorname{div} C \equiv D_i(C^i) = 0$ for all solutions of (1.21).

The invariance of the variational integral implies that the Euler–Lagrange equations (1.21) admit the group G. Therefore, in order to apply Noether's theorem, one has first of all to find the symmetries of Eqs. (1.21). Then one should single out the symmetries leaving invariant the variational integral (1.21). This can be done by means of the following infinitesimal test for the invariance of the variational integral (proved in [5], see also [6]):

$$X(\mathcal{L}) + \mathcal{L} D_i\left(\xi^i\right) = 0, \tag{1.24}$$

where the generator X is prolonged to the first derivatives $u_{(1)}$ by the formula

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \left[D_i\left(\eta^\alpha\right) - u_j^\alpha D_i\left(\xi^j\right)\right]\frac{\partial}{\partial u_i^\alpha}. \tag{1.25}$$

If Eq. (1.24) is satisfied, then the vector (1.23) provides a conservation law.

The invariance condition (1.24) can be replaced by the divergence condition

$$X(\mathcal{L}) + \mathcal{L} D_i\left(\xi^i\right) = D_i\left(B^i\right). \tag{1.26}$$

Then Eq. (1.21) has a conservation law $D_i(C^i) = 0$, where (1.23) is replaced by

$$C^i = \xi^i \mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{1.27}$$

It is a common belief that the applicability of Noether's theorem is severely restricted because Lagrangians exists only for very special types of differential equations. The aim of the present paper is to dispel this myth.

2. Main constructions

Here, the notion of an integrating factor is extended to higher-order ordinary differential equations. Furthermore, an adjoint equation is defined for non-linear ordinary and partial differential equations of an arbitrary order. Then, using the new concept of an adjoint equation, I obtain a Lagrangian for any ordinary and partial differential equation. It follows that Noether type conservation theorems can be applied to any differential equation as well as to any system where the number of differential equations is equal to the number of the dependent variables.

2.1. Preliminaries

We will use the calculus in the space $\mathcal{A}$ of differential functions introduced in [4] (see also [5, Section 19.1] and [6, Section 8.2]). Let us denote by z the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \ldots) \tag{2.1}$$

with elements z^ν $(\nu \geqslant 1)$, where $z^i = x^i$ $(1 \leqslant i \leqslant n)$, $z^{n+\alpha} = u^\alpha$ $(1 \leqslant \alpha \leqslant m)$ and the remaining elements represent the derivatives of u. Finite subsequences of z are denoted by $[z]$.

A *differential function* f is a locally analytic function $f([z])$ (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables (2.1). The highest order of derivatives appearing in a differential function f is called the order of f and is denoted by $\operatorname{ord}(f)$. Thus, $\operatorname{ord}(f) = s$ means that $f = f(x, u, u_{(1)}, \ldots, u_{(s)})$. The set of all differential functions of finite order is denoted by $\mathcal{A}$. The set $\mathcal{A}$ is a vector space endowed with the usual multiplication of functions. In other words, if $f([z]) \in \mathcal{A}$ and $g([z]) \in \mathcal{A}$ and if a and b any constants, then

$$af + bg \in \mathcal{A}, \quad \operatorname{ord}(af + bg) \leqslant \max\bigl\{\operatorname{ord}(f), \operatorname{ord}(g)\bigr\},$$
$$fg \in \mathcal{A}, \quad \operatorname{ord}(fg) = \max\bigl\{\operatorname{ord}(f), \operatorname{ord}(g)\bigr\}.$$

Furthermore, the space $\mathcal{A}$ is closed under the total derivation: if $f \in \mathcal{A}$, then

$$D_i(f) \in \mathcal{A}, \quad \operatorname{ord}\bigl(D_i(f)\bigr) = \operatorname{ord}(f) + 1.$$

The *Euler–Lagrange operator* in $\mathcal{A}$ is defined by the formal sum

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} + D_i D_j \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \quad \alpha = 1, \ldots, m, \tag{2.2}$$

where, for every s, the summation is presupposed over the repeated indices $i, j, \ldots$ running from 1 to n. The operator $\delta/\delta u^\alpha$ is termed also the *variational derivative*.

The operator (2.2) with one independent variable x is written

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_x \frac{\partial}{\partial u_x^\alpha} + D_x^2 \frac{\partial}{\partial u_{xx}^\alpha} - D_x^3 \frac{\partial}{\partial u_{xxx}^\alpha} + \cdots. \tag{2.3}$$

In the case of one independent variable x and one dependent variable y, we will use the common notation and write $z = (x, y, y', y'', \ldots, y^{(s)}, \ldots)$. Then the total differentiation (1.8) is written as follows:

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots \tag{2.4}$$

and the Euler–Lagrange operator (2.3) becomes

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''} - D_x^3 \frac{\partial}{\partial y'''} + \cdots. \tag{2.5}$$

The main constructions of this section are based on the concept of multipliers and the following lemmas (for the proofs, see [6, Section 8.4]).

Lemma 2.1. *Let $f(x, y, y', \ldots, y^{(s)}) \in \mathcal{A}$. If $D_x(f) = 0$ identically in all variables $x, y, y', \ldots, y^{(s)}$, and $y^{(s+1)}$, then $f = C = const$. Likewise, if $f(x, u, u_{(1)}, \ldots, u_{(s)})$ is a differential function with one independent variable x and several dependent variables $u = (u^1, \ldots, u^m)$, the equation $D_x(f) = 0$ implies that $f = C$.*

Lemma 2.2. *A differential function $f(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ with one independent variable x is a total derivative*:

$$f = D_x(g), \quad g(x, u, \ldots, u_{(s-1)}) \in \mathcal{A}, \tag{2.6}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \ldots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{2.7}$$

Lemma 2.3. *A function $f(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ with several independent variables $x = (x^1, \ldots, x^n)$ and several dependent variables $u = (u^1, \ldots, u^m)$ is a divergence of a vector field $H = (h^1, \ldots, h^n)$, $h^i \in \mathcal{A}$:*

$$f = \operatorname{div} H \equiv D_i\big(h^i\big), \tag{2.8}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \ldots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{2.9}$$

2.2. *Integrating factor for higher-order equations*

Definition 2.1. Consider sth-order ordinary differential equations of the form

$$a\big(x, y, y', \ldots, y^{(s-1)}\big) y^{(s)} + b\big(x, y, y', \ldots, y^{(s-1)}\big) = 0. \tag{2.10}$$

A differential function $\mu(x, y, y', \ldots, y^{(s-1)})$ is called an integrating factor for Eq. (2.10) if the multiplication by μ converts the left-hand side of Eq. (2.10) into a total derivative of some function $\Phi(x, y, y', \ldots, y^{(s-1)}) \in \mathcal{A}$:

$$\mu a y^{(s)} + \mu b = D_x(\Phi). \tag{2.11}$$

Knowledge of an integrating factor allows one to reduce the order of Eq. (2.10). Indeed, Eqs. (2.10)–(2.11) are written $D_x(\Phi) = 0$, and Lemma 2.1 yields the $(s-1)$-order equation

$$\Phi\big(x, y, y', \ldots, y^{(s-1)}\big) = C. \tag{2.12}$$

Definition 2.1 can be readily extended to systems of ordinary differential equations of any order.

Theorem 2.1. *The integrating factors for Eq.* (2.10) *are determined by the following equation*:

$$\frac{\delta}{\delta y}\big(\mu a y^{(s)} + \mu b\big) = 0, \tag{2.13}$$

where $\delta/\delta y$ *is the variational derivative* (2.5). *Equation* (2.13) *involves the variables* $x, y, y', \dots, y^{(2s-2)}$ *and should be satisfied identically in all these variables.*

Proof. Equation (2.13) is obtained from Lemma 2.2. The highest derivative that may appear after the variational differentiation (2.5) has the order $2s-1$. It occurs in the terms

$$(-1)^s D_x^s(\mu a) \quad \text{and} \quad (-1)^{s-1} D_x^{s-1}\left[y^{(s)} \frac{\partial(\mu a)}{\partial y^{(s-1)}}\right].$$

We have, dropping the terms that certainly do not involve $y^{(2s-1)}$:

$$(-1)^s D_x^s(\mu a) = -(-1)^{s-1} D_x^{s-1}\left[y^{(s)} \frac{\partial(\mu a)}{\partial y^{(s-1)}}\right] + \cdots.$$

Thus, the terms containing $y^{(2s-1)}$ annihilate each other, and hence Eq. (2.13) involves only the variables $x, y, y', \dots, y^{(2s-2)}$. This completes the proof. □

For the first-order equation (1.1), $a(x,y)y' + b(x,y) = 0$, Eq. (2.13) is written:

$$\frac{\delta}{\delta y}(\mu a y' + \mu b) = y'(\mu a)_y + (\mu b)_y - D_x(\mu a) = 0.$$

Since $D_x(\mu a) = (\mu a)_x + y'(\mu a)_y$, we arrive at Eq. (1.6), $(\mu b)_y - (\mu a)_x = 0$.

Consider the second-order equation

$$a(x, y, y')y'' + b(x, y, y') = 0. \tag{2.14}$$

The integrating factors μ depend on x, y, y', and Eq. (2.13) for determining $\mu(x, y, y')$ is written:

$$\frac{\delta}{\delta y}(\mu a y'' + \mu b) = y''(\mu a)_y + (\mu b)_y - D_x\big[y''(\mu a)_{y'} + (\mu b)_{y'}\big] + D_x^2(\mu a) = 0.$$

We have

$$\begin{aligned}
D_x(\mu a) &= y''(\mu a)_{y'} + y'(\mu a)_y + (\mu a)_x,\\
D_x^2(\mu a) &= y'''(\mu a)_{y'} + y''^2(\mu a)_{y'y'} + 2y'y''(\mu a)_{yy'} + 2y''(\mu a)_{xy'}\\
&\quad + y''(\mu a)_y + y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx},\\
D_x\big(y''(\mu a)_{y'}\big) &= y'''(\mu a)_{y'} + y''^2(\mu a)_{y'y'} + y'y''(\mu a)_{yy'} + y''(\mu a)_{xy'},\\
D_x\big((\mu b)_{y'}\big) &= y''(\mu b)_{y'y'} + y'(\mu b)_{yy'} + (\mu b)_{xy'},
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\delta}{\delta y}(\mu a y'' + \mu b) &= y''\big[y'(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'}\big]\\
&\quad + y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y.
\end{aligned}$$

Since this expression should vanish identically in x, y, y' and y'', we arrive at the following statement.

Theorem 2.2. *The integrating factors $\mu(x, y, y')$ for the second-order equation* (2.14) *are determined by the following system of two equations*:

$$y'(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'} = 0, \tag{2.15}$$

$$y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y = 0. \tag{2.16}$$

Theorem 2.2 shows that the second-order equations, unlike the first-order ones, may have no integrating factors. Indeed, the integrating factor $\mu(x, y)$ for any first-order equation is determined by the single first-order linear partial differential equation (1.6) which always has infinite number of solutions. In the case of second-order equations (2.14), one unknown function $\mu(x, y, y')$ should satisfy two second-order linear partial differential equations (2.15)–(2.16). An integrating factor exists only if the over-determined system (2.15)–(2.16) is compatible.

Remark 2.1. If a second-order equation (2.14) has two integrating factors, its general solution can be found without additional integration.

Example 2.1. Let us calculate integrating factors for the following equation:

$$y'' + \frac{y'^2}{y} + 3\frac{y'}{x} = 0. \tag{2.17}$$

Equation (2.17) has the form (2.14) with

$$a = 1, \qquad b = \frac{y'^2}{y} + 3\frac{y'}{x}\cdot$$

For the sake of simplicity, we will look for the integrating factors of the particular form $\mu = \mu(x, y)$. Then Eq. (2.15) reduces to $2\mu_y - (\mu b)_{y'y'} = 0$. Since $(\mu b)_{y'y'} = 2\mu/y$, we obtain the equation

$$\frac{\partial\mu}{\partial y} - \frac{\mu}{y} = 0,$$

whence $\mu = \phi(x)y$. Thus, we have

$$\mu = \phi(x)y, \qquad \mu_{yy} = 0, \qquad \mu_{xy} = \phi', \qquad \mu_{xx} = \phi'' y, \qquad \mu b = \phi y'^2 + 3\frac{\phi}{x} yy',$$

$$(\mu b)_y = 3\frac{\phi}{x} y', \qquad (\mu b)_{yy'} = 3\frac{\phi}{x}, \qquad (\mu b)_{xy'} = 2\phi' y' + 3\left(\frac{\phi'}{x} - \frac{\phi}{x^2}\right)y.$$

Substitution in Eq. (2.16) leads to the following Euler's equation:

$$x^2\phi'' - 3x\phi' + 3\phi = 0.$$

Integrating it by the standard change of the independent variable, $t = \ln x$, we obtain two independent solutions, $\phi = x$ and $\phi = x^3$. Thus, Eq. (2.17) has two integrating factors:

$$\mu_1 = xy, \qquad \mu_2 = x^3 y, \tag{2.18}$$

and can be solved without an additional integration (see Remark 2.1).

Indeed, multiplying Eq. (2.17) by the first integrating factor, we have

$$xy\left(y'' + \frac{y'^2}{y} + 3\frac{y'}{x}\right) = xyy'' + xy'^2 + 3yy' = 0.$$

Substituting $xyy'' = D_x(xyy') - yy' - xy'^2$, we reduce it to

$$D_x(xyy') + 2yy' = D_x\big(xyy' + y^2\big) = 0,$$

whence

$$xyy' + y^2 = C_1. \tag{2.19}$$

The similar calculations by using the second integrating factor (2.18) yields

$$x^3 yy' = C_2. \tag{2.20}$$

Eliminating y' from Eqs. (2.19)–(2.20), we obtain the following general solution to Eq. (2.17):

$$y = \pm\sqrt{C_1 - \frac{C_2}{x^2}}. \tag{2.21}$$

2.3. Adjoint equations

Definition 2.2. Consider a system of sth-order partial differential equations

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2.22}$$

where $F_\alpha(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ are differential functions with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$, $u = u(x)$. The system of *adjoint equations* to Eqs. (2.22) is defined by

$$F^*_\alpha(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.23}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, $v = v(x)$.

Remark 2.2. In the case of linear equations, adjoint equations given by Definition 2.2 are identical with the classical adjoint equations discussed in Section 1.2. Therefore, the adjoint equation to a linear equation (or a system) $F(x, u, \ldots, u_{(s)}) = 0$ for $u(x)$ is a linear equation (a system) $F^*(x, v, \ldots, v_{(s)}) = 0$ for $v(x)$, and the relation to be adjoint is symmetric, i.e., $F^{**} = F$. More specifically, if the adjoint equation to $F^*(x, v, \ldots, v_{(s)}) = 0$ is $F^{**}(x, w, \ldots, w_{(s)}) = 0$, then setting $w = u$ in the latter equation we obtain the original equation.

Definition 2.3. A system of equations (2.22) is said to be *self-adjoint* if the system obtained from the adjoint equations (2.23) by the substitution $v = u$:

$$F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m,$$

is identical with the original system (2.22).[1]

Example 2.2. Let us take $n = 1, m = 1$, set $u = y, v = z$, and consider the first-order linear ordinary differential equation (1.15):

$$F(x, y, y') \equiv a_0 y' + a_1 y - f(x) = 0.$$

[1] In general, it does not mean that $F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = F_\alpha(x, u, \ldots, u_{(s)})$, see, e.g., Example 2.6.

Equation (2.23) defining the adjoint equation is written:

$$\frac{\delta(zF)}{\delta y} = \left(\frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'}\right)\bigl(z\bigl[a_0 y' + a_1 y - f(x)\bigr]\bigr) = 0.$$

Since

$$\frac{\partial}{\partial y}\bigl(z\bigl[a_0 y' + a_1 y - f(x)\bigr]\bigr) = a_1 z, \qquad \frac{\partial}{\partial y'}\bigl(z\bigl[a_0 y' + a_1 y - f(x)\bigr]\bigr) = a_0 z,$$

Eq. (2.23) yields the adjoint equation $a_1 z - D_x(a_0 z) = 0$, or

$$a_1 z - (a_0 z)' = 0$$

the left-hand side of which is identical with the adjoint operator (1.16).

Example 2.3. For the second-order equation (1.17),

$$a_0 y'' + a_1 y' + a_2 y = f(x),$$

Definition 2.2 yields the adjoint equation

$$\left(\frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''}\right)\bigl(z\bigl[a_0 y'' + a_1 y' + a_2 y - f(x)\bigr]\bigr) = 0.$$

Proceeding as in the previous example, one obtains the adjoint equation (1.18):

$$(a_0 z)'' - (a_1 z)' + a_2 z = 0.$$

Example 2.4. Consider the second-order linear partial differential equation (1.9):

$$L[u] \equiv a^{ij}(x) u_{ij} + b^i(x) u_i + cu = f(x).$$

The definition (2.23) of the adjoint equation is written

$$\left(\frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}}\right)\bigl(v\bigl[a^{ij}(x) u_{ij} + b^i(x) u_i + cu - f(x)\bigr]\bigr) = 0$$

and yields the adjoint equation (1.14):

$$L^*[u] \equiv D_i D_j\bigl(a^{ij} v\bigr) - D_i\bigl(b^i v\bigr) + cv = 0.$$

Example 2.5. Consider the heat equation

$$u_t - c(x) u_{xx} = 0,$$

where $c(x)$ is a variable or constant coefficient. Equation (2.23) is written (see (2.2)):

$$\frac{\delta}{\delta u}\bigl(v\bigl[c(x) u_{xx} - u_t\bigr]\bigr) = \left(-D_t \frac{\partial}{\partial u_t} + D_x^2 \frac{\partial}{\partial u_{xx}}\right)\bigl(v\bigl[c(x) u_{xx} - u_t\bigr]\bigr) = 0$$

and yields the adjoint equation $D_x^2(c(x)v) + D_t(v) = 0$, or

$$v_t + (cv)_{xx} = 0.$$

Let us calculate by Definition 2.2 the adjoint equations to several well-known non-linear equations from mathematical physics.

Example 2.6. Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x. \tag{2.24}$$

We take $F(t, x, u, \ldots, u_{(3)}) = u_t - u_{xxx} - uu_x$ and write the left-hand side of Eq. (2.23) in the form

$$\frac{\delta}{\delta u}\big(v[u_t - u_{xxx} - uu_x]\big) = -v_t + v_{xxx} - vu_x + D_x(uv) = -v_t + v_{xxx} + uv_x.$$

Hence, $F^*(t, x, u, v, \ldots, u_{(3)}, v_{(3)}) = -(v_t - v_{xxx} - uv_x)$, and the adjoint equation to the Korteweg–de Vries equation (2.24) is

$$v_t = v_{xxx} + uv_x. \tag{2.25}$$

We have

$$F^*(t, x, u, u, \ldots, u_{(3)}, u_{(3)}) = -(u_t - u_{xxx} - uu_x) \equiv -F(t, x, u, \ldots, u_{(3)}).$$

Thus, Eq. (2.24) is self-adjoint (see Definition 2.3).

Let us find the adjoint equation to Eq. (2.25). We have

$$\frac{\delta}{\delta v}\big(w[v_t - v_{xxx} - uu_x]\big) = -w_t + w_{xxx} + D_x(uw) = -w_t + w_{xxx} + uw_x + wu_x.$$

Hence, the adjoint to Eq. (2.25) is $w_t = w_{xxx} + uw_x + wu_x$. Setting here $w = u$, we obtain the equation

$$u_t = u_{xxx} + 2uu_x$$

different from the original Korteweg–de Vries equation (2.24) (cf. Remark 2.2).

Example 2.7. Consider the Burgers equation

$$u_t = uu_x + u_{xx}. \tag{2.26}$$

The left-hand side of Eq. (2.23) is written:

$$\frac{\delta}{\delta u}\big(v[u_t - uu_x - u_{xx}]\big) = -v_t - vu_x + D_x(uv) - v_{xx} = -v_t + uv_x - v_{xx}.$$

Hence, adjoint equation to the Burgers equation (2.26) is (see also [7])

$$v_t = uv_x - v_{xx}. \tag{2.27}$$

Example 2.8. Consider the non-linear heat equation:

$$u_t = \big[k(u)u_x\big]_x. \tag{2.28}$$

The left-hand side of Eq. (2.23) is written:

$$\frac{\delta}{\delta u}\big(v\big[u_t - k(u)u_{xx} - k'(u)u_x^2\big]\big)$$
$$= -v_t - k'(u)vu_{xx} - k''(u)vu_x^2 - D_x^2\big(k(u)v\big) + 2D_x\big(k'(u)vu_x\big). \tag{2.29}$$

We have $D_x(k(u)v) = kv_x + k'vu_x$ and therefore

$$-D_x^2\big(k(u)v\big) + 2D_x\big(k'(u)vu_x\big) = -D_x(kv_x) + D_x\big(k'vu_x\big).$$

Inserting this in Eq. (2.29) and making simple calculations we arrive at the following adjoint equation to the non-linear heat equation (2.28) (see also [7]):

$$v_t + k(u)v_{xx} = 0. \tag{2.30}$$

Let us find the adjoint equation to (2.30). We have

$$\frac{\delta}{\delta v}\big(w\big[v_t + k(u)v_{xx}\big]\big) = -w_t + D_x^2\big[k(u)w\big].$$

Hence, the adjoint equation to (2.30) is $w_t = [k(u)w]_{xx}$ and does not coincide with Eq. (2.28) upon setting $w = u$.

2.4. Lagrangians

Theorem 2.3. *Any system of sth-order differential equations* (2.22)*,*

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2.22}$$

considered together with its adjoint equation (2.23)*,*

$$F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.23}$$

has a Lagrangian. Namely, the simultaneous system (2.22)–(2.23) *with* $2m$ *dependent variables* $u = (u^1, \ldots, u^m)$ *and* $v = (v^1, \ldots, v^m)$ *is the system of Euler–Lagrange equations* (1.21) *with the Lagrangian* $\mathcal{L}$ *defined by*[2]

$$\mathcal{L} = v^\beta F_\beta. \tag{2.31}$$

Proof. Indeed, we have

$$\frac{\delta \mathcal{L}}{\delta v^\alpha} = F_\alpha(x, u, \ldots, u_{(s)}) \tag{2.32}$$

and

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}). \qquad \square \tag{2.33}$$

Let us turn to examples. Consider linear equations, e.g. the homogeneous linear second-order partial differential equation (1.9):

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0. \tag{2.34}$$

The Lagrangian (2.31) is written:

$$\mathcal{L} = vL[u] = v\big(a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u\big). \tag{2.35}$$

We have

$$\frac{\delta \mathcal{L}}{\delta v} = \frac{\partial \mathcal{L}}{\partial v} = L[u] \tag{2.36}$$

[2] See also the concept of a weak Lagrangian introduced in [3].

and

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = D_i D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right) - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i}\right) + \frac{\partial \mathcal{L}}{\partial u}$$
$$= D_i D_j\big(a^{ij} v\big) - D_i\big(b^i v\big) + cv = L^*[v]. \tag{2.37}$$

Theorem 2.4. *Let the linear operator $L[u]$ be self-adjoint, $L^*[u] = L[u]$. Then Eq.* (2.34) *is obtained from the Lagrangian*

$$\mathcal{L} = \frac{1}{2}\big[c(x)u^2 - a^{ij}(x)u_i u_j\big]. \tag{2.38}$$

Proof. We rewrite the Lagrangian (2.35) in the form

$$\mathcal{L} = v\big(a^{ij}u_{ij} + b^i u_i + cu\big) = D_j\big(va^{ij}u_i\big) - vu_i D_j\big(a^{ij}\big) + vb^i u_i - a^{ij}u_i v_j + cuv.$$

The first term at the right-hand side can be dropped by Lemma 2.3, while the second and the third terms annihilate each other by the condition (1.13). Finally, we set $v = u$, divide by two and arrive at the Lagrangian (2.38). □

Example 2.9. For the Helmholtz equation $\Delta u + k^2 u = 0$, (2.38) gives the well-known Lagrangian $\mathcal{L} = (k^2u^2 - |\nabla u|^2)/2$.

If one deals with linear equations that are not self-adjoint or with non-linear equations, one obtains a Lagrangian formulation by considering the equation in question together with its adjoint equation.

Example 2.10. The linear heat equation is not self-adjoint. Therefore, we consider it together with its adjoint equation and obtain the system of two equations:

$$u_t - c(x)u_{xx} = 0, \qquad v_t + (cv)_{xx} = 0 \tag{2.39}$$

which is derived from the Lagrangian

$$\mathcal{L} = vu_t - c(x)vu_{xx}. \tag{2.40}$$

Example 2.11. According to Example 2.6, the Lagrangian

$$\mathcal{L} = v[u_t - u_{xxx} - uu_x] \tag{2.41}$$

leads to the Korteweg–de Vries equation (2.24) and its conjugate (2.25) combined in the following system:

$$u_t = u_{xxx} + uu_x, \qquad v_t = v_{xxx} + uv_x. \tag{2.42}$$

Example 2.12. Likewise, we obtain from Example 2.8 the Lagrangian

$$\mathcal{L} = v\big[u_t - k(u)u_{xx} - k'(u)u_x^2\big] \tag{2.43}$$

that leads to the non-linear heat equation (2.28) and its conjugate (2.30) combined in the following system:

$$u_t = \big[k(u)u_x\big]_x, \qquad v_t + k(u)v_{xx} = 0. \tag{2.44}$$

Example 2.13. One of fundamental equations in quantum mechanics is the Dirac equation

$$\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi = 0, \quad m = \text{const.} \tag{2.45}$$

The dependent variable ψ is a 4-dimensional column vector with complex valued components $\psi^1, \psi^2, \psi^3, \psi^4$. The independent variables compose the four-dimensional vector $x = (x^1, x^2, x^3, x^4)$, where x^1, x^2, x^3 are the real valued spatial variables and x^4 is the complex variable defined by $x^4 = ict$ with t being time and c the light velocity. Furthermore, γ^k are the following 4×4 complex matrices called the Dirac matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Equation (2.45) does not have a Lagrangian. Therefore, it is considered together with the conjugate equation

$$\frac{\partial \tilde{\psi}}{\partial x^k} \gamma^k - m\tilde{\psi} = 0. \tag{2.46}$$

Here $\tilde{\psi} = \overline{\psi}^T \gamma^4$ is the row vector, where $\overline{\psi}$ denotes the complex-conjugate to ψ and T the transposition. The system (2.45)–(2.46) has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[\tilde{\psi}\left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi\right) - \left(\frac{\partial \tilde{\psi}}{\partial x^k}\gamma^k - m\tilde{\psi}\right)\psi\right].$$

Indeed, we have

$$\frac{\delta \mathcal{L}}{\delta \psi} = -\left(\frac{\partial \tilde{\psi}}{\partial x^k}\gamma^k - m\tilde{\psi}\right), \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\psi}} = \gamma^k \frac{\partial \psi}{\partial x^k} + m\psi.$$

3. Application to the Maxwell equations

This section is dedicated to illustration of the method by applying Noether's theorem to the Maxwell equations. Consider the Maxwell equations in vacuum:

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{curl} \boldsymbol{H}, \quad \operatorname{div} \boldsymbol{E} = 0,$$
$$\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{curl} \boldsymbol{E}, \quad \operatorname{div} \boldsymbol{H} = 0. \tag{3.1}$$

The system (3.1) contains six dependent variables, namely, the components of the electric field $\boldsymbol{E} = (E^1, E^2, E^3)$ and the magnetic field $\boldsymbol{H} = (H^1, H^2, H^3)$, and eight equations, i.e. it is *overdetermined*. On the other hand, the number of equations in the Euler–Lagrange equations (1.21) is equal to the number of dependent variables. Consequently, the system (3.1) cannot have a

Lagrangian. What is considered in the literature as a variational problem in electrodynamics (see, e.g. [1,8]) provides a Lagrangian for the wave equation

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

for the vector potential $\boldsymbol{A}$ of the electromagnetic field, but not for the Maxwell equations (3.1).

Let us find a Lagrangian for the electromagnetic field by using Theorem 2.3. First we note that the equations $\operatorname{div} \boldsymbol{E} = 0$, $\operatorname{div} \boldsymbol{H} = 0$ hold at any time provided that they are satisfied at the initial time $t = 0$. Hence, they are merely initial conditions (see, e.g. [2] or [6]). Therefore, we will consider the following *determined system* of differential equations (we set $t' = ct$ and take t' as new t):

$$\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{3.2}$$

We introduce six new dependent variables, namely the components of the vectors $\boldsymbol{V} = (V^1, V^2, V^3)$ and $\boldsymbol{W} = (W^1, W^2, W^3)$, and introduce the Lagrangian

$$\mathcal{L} = \boldsymbol{V} \cdot \left(\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{W} \cdot \left(\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{3.3}$$

in accordance with the definition (2.31).

One can readily verify that the Lagrangian (3.3) yields the system (3.2) together with its adjoint, namely:

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{V}} \equiv \operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \frac{\delta \mathcal{L}}{\delta \boldsymbol{W}} \equiv \operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{3.4}$$

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{E}} \equiv \operatorname{curl} \boldsymbol{V} + \frac{\partial \boldsymbol{W}}{\partial t} = 0, \qquad \frac{\delta \mathcal{L}}{\delta \boldsymbol{H}} \equiv \operatorname{curl} \boldsymbol{W} - \frac{\partial \boldsymbol{V}}{\partial t} = 0. \tag{3.5}$$

If we set $\boldsymbol{V} = \boldsymbol{E}$, $\boldsymbol{W} = \boldsymbol{H}$, Eqs. (3.5) coincide with (3.4). Hence, the operator in (3.2) is self-adjoint. Therefore we set $\boldsymbol{V} = \boldsymbol{E}$, $\boldsymbol{W} = \boldsymbol{H}$ in (3.3), divide by two and obtain the Lagrangian for the Maxwell equations (3.2) (cf. Theorem 2.4):

$$\mathcal{L} = \frac{1}{2}\left[\boldsymbol{E} \cdot \left(\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H} \cdot \left(\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right)\right]. \tag{3.6}$$

In coordinates, the Lagrangian (3.6) is written:

$$\begin{aligned} \mathcal{L} = {} & E^1\left(E_y^3 - E_z^2 + H_t^1\right) + E^2\left(E_z^1 - E_x^3 + H_t^2\right) + E^3\left(E_x^2 - E_y^1 + H_t^3\right) \\ & + H^1\left(H_y^3 - H_z^2 - E_t^1\right) + H^2\left(H_z^1 - H_x^3 - E_t^2\right) + H^3\left(H_x^2 - H_y^1 - E_t^3\right). \end{aligned} \tag{3.7}$$

The symmetries of the Maxwell equations are well known, and one can apply Noether's theorem by using the Lagrangian (3.6). We will employ, as an example, the invariance of Eqs. (3.2) with respect to the group of transformations

$$\boldsymbol{H}' = \boldsymbol{H} \cos\theta + \boldsymbol{E} \sin\theta, \qquad \boldsymbol{E}' = \boldsymbol{E} \cos\theta - \boldsymbol{H} \sin\theta \tag{3.8}$$

with the generator

$$X = \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\frac{\partial}{\partial \boldsymbol{E}} \equiv \sum_{i=1}^{3}\left(E^i\frac{\partial}{\partial H^i} - H^i\frac{\partial}{\partial E^i}\right). \tag{3.9}$$

The prolongation (1.25) of this generator is written

$$X = \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{E}_t\frac{\partial}{\partial \boldsymbol{H}_t} - \boldsymbol{H}_t\frac{\partial}{\partial \boldsymbol{E}_t} + \boldsymbol{E}_x\frac{\partial}{\partial \boldsymbol{H}_x} - \boldsymbol{H}_x\frac{\partial}{\partial \boldsymbol{E}_x} + \cdots. \tag{3.10}$$

Acting by the operator (3.10) on the Lagrangian (3.6), we have

$$X(\mathcal{L}) = \frac{1}{2}\big[-\boldsymbol{H}\cdot(\operatorname{curl}\boldsymbol{E} + \boldsymbol{H}_t) + \boldsymbol{E}\cdot(\operatorname{curl}\boldsymbol{H} - \boldsymbol{E}_t) + \boldsymbol{E}\cdot(-\operatorname{curl}\boldsymbol{H} + \boldsymbol{E}_t) + \boldsymbol{H}\cdot(\operatorname{curl}\boldsymbol{E} + \boldsymbol{H}_t)\big] = 0.$$

Hence, the condition (1.24) is satisfied and one can obtain a conservation law by the formula (1.23). We will write the conservation law in the form

$$D_t(\tau) + \operatorname{div}\boldsymbol{\chi} = 0, \tag{3.11}$$

where $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$, $\operatorname{div}\boldsymbol{\chi} = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3)$. Equation (1.23) yields

$$\tau = \boldsymbol{E}\cdot\frac{\partial\mathcal{L}}{\partial \boldsymbol{H}_t} - \boldsymbol{H}\cdot\frac{\partial\mathcal{L}}{\partial \boldsymbol{E}_t} = \frac{1}{2}\big[\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{H}\cdot(-\boldsymbol{H})\big] = \frac{1}{2}\big[E^2 + H^2\big].$$

Hence, τ is the energy density. Likewise, calculating the spatial coordinates of the conserved vector (1.23), one can verify that $\boldsymbol{\chi}$ is the Poynting vector, $\boldsymbol{\chi} = \boldsymbol{E}\times\boldsymbol{H}$. Thus, we have obtained the conservation of energy (see, e.g. [8]):

$$D_t\left(\frac{E^2 + H^2}{2}\right) + \operatorname{div}(\boldsymbol{E}\times\boldsymbol{H}) = 0. \tag{3.12}$$

References

[1] E. Bessel-Hagen, Über die Erhaltungssätze der Elektrodynamik, Math. Ann. 84 (1921) 258–276.

[2] R. Courant, D. Hilbert, Methods of Mathematical Physics. Vol. II: Partial Differential Equations, by R. Courant, Interscience, Wiley, New York, 1989.

[3] N.H. Ibragimov, Lie–Bäcklund groups and conservation laws, Dokl. Akad. Nauk SSSR 230 (1) (1976) 26–29, English transl.: Soviet Math. Dokl. 17 (5) (1976) 1242–1246.

[4] N.H. Ibragimov, Sur l'équivalence des équations d'évolution, qui admettent une algébre de Lie–Bäcklund infinie, C. R. Acad. Sci. Paris Sér. I 293 (1981) 657–660.

[5] N.H. Ibragimov, Transformation Groups Applied to Mathematical Physics, Nauka, Moscow, 1983, English transl.: Riedel, Dordrecht, 1985.

[6] N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiley, Chichester, 1999.

[7] N.H. Ibragimov, T. Kolsrud, Lagrangian approach to evolution equations: Symmetries and conservation laws, Nonlinear Dynamics 36 (1) (2004) 29–40.

[8] L.D. Landau, E.M. Lifshitz, Field Theory. Course of Theoretical Physics, vol. 2, fourth revised ed., Fizmatgiz, 1962, English transl.: The Classical Theory of Fields, fifth ed., Pergamon, New York, 1971.

[9] E. Noether, Invariante Variationsprobleme, Königliche Gesellschaft der Wissenschaften, Göttingen Math. Phys. Kl. (1918) 235–257, English transl.: Transport Theory and Statistical Physics 1 (3) (1971) 186–207.

18 Exactness of Second Order Ordinary Differential Equations and Integrating Factors. By R. AlAhmad, M. Al-Jararha, H. Almefleh (2014)

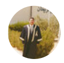
Exactness of Second Order Ordinary Differential Equations and Integrating Factors

Article · June 2015

Source: arXiv

3 authors:

Rami Alahmad
Yarmouk University



Mohammadkheer Al Jararha
Yarmouk University

Hasan Almefleh
Yarmouk University

Exactness of Second Order Ordinary Differential Equations and Integrating Factors

R. AlHahmad,* M. Al-Jararha† and H. Almefleh‡

Department of Mathematics, Yarmouk University, Irbid, Jordan, 21163.

Abstract

A new problem is studied, the concept of exactness of a second order nonlinear ordinary differential equations is established. A method is constructed to reduce this class into a first order equations. If the second order equation is not exact we introduce, under certain conditions, an integrating factor that transform it to an exact one.

Keywords and Phrases: Second order differential equation, Exact equations, Non-exact equations, Integrating factor.
AMS (2000) Subject Classification: 65L05, 49K15, 37C10

1 Introduction

The concept of exactness for a class of a first order nonlinear differential equations was presented [5] with a well-defined method of solution. The notion of integrating factor were introduced to convert differential equation that is not exact into an exact one.

Second order nonlinear differential equations play an important role in Applied Mathematics, Physics, and Engineering [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. To find the general solution of a nonlinear second order differential equation is not an easy problem in the general case. In fact, a very specific class of nonlinear second order differential equations can be solved by using special transformations. Another approach to study the solution of nonlinear second order differential equations is the dynamical systems approach. Using this approach a qualitative solution is given instead of the particular solution of the equation. A class of these equations will be solved in this paper.

The outline of the paper: we give mathematical formulation for the exactness of a class of second order nonlinear equations based on transforming them into a first order equations. Also, we will introduce the idea of integrating factor to convert some differential equations into exact equations, and we will prove some related results.

*rami_thenat@yu.edu.jo
†mohammad.ja@yu.edu.jo
‡almefleh@yu.edu.jo

2 Exact Second Order Differential Equations

Consider the following nonlinear second order differential equation

$$a_2(x,y,y')y'' + a_1(x,y,y')y' + a_0(x,y,y') = 0. \tag{2.1}$$

If a function $\Psi(x,y,y')$ exists with the properties that

$$\frac{\partial\Psi(x,y,y')}{\partial x} = a_0(x,y,y'),\ \frac{\partial\Psi(x,y,y')}{\partial y} = a_1(x,y,y'),\ \text{and}\ \frac{\partial\Psi(x,y,y')}{\partial y'} = a_2(x,y,y'), \tag{2.2}$$

then we have

$$\frac{\partial\Psi(x,y,y')}{\partial y'}y'' + \frac{\partial\Psi(x,y,y')}{\partial y}y' + \frac{\partial\Psi(x,y,y')}{\partial x} = 0.$$

So, by the chain rule, we get

$$\frac{d\Psi(x,y,y')}{dx} = 0.$$

Hence,

$$\Psi(x,y,y') = c$$

reduces Eq. (2.1) into a first order differential equation.

Definition 2.1. *The nonlinear second order differential equation* (2.1) *is called exact equation if there exists a function* $\Psi(x,y,y')$ *such that* (2.2) *holds.*

Theorem 2.1. *Let the functions* $a_2(x,y,y')$, $a_1(x,y,y')$, $a_0(x,y,y')$, $\frac{\partial a_2}{\partial y}$, $\frac{\partial a_2}{\partial x}$, $\frac{\partial a_1}{\partial x}$, $\frac{\partial a_1}{\partial y'}$, $\frac{\partial a_0}{\partial y'}$, *and* $\frac{\partial a_0}{\partial y}$ *be continuous functions in a simply connected region* $R \subseteq \mathbb{R}^3$. *Then Eq.* (2.1) *is exact if and only if*

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'},\ \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'},\ \textit{and}\ \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}. \tag{2.3}$$

Proof. Assume that (2.3) hold. We are going to construct a function $\Psi(x,y,y')$ such that

$$\frac{\partial\Psi(x,y,y')}{\partial x} = a_0(x,y,y').$$

Then by integrating this equation with respect to x, we get

$$\Psi(x,y,y') = \int_{x_0}^{x} a_0(\alpha,y,y')d\alpha + \Phi(y,y'). \tag{2.4}$$

Therefore, by differentiating the above equation with respect to y and using the assumption, we get

$$\frac{\partial\Phi(y,y')}{\partial y} = a_1(x_0,y,y').$$

Hence,

$$\Phi(y,y') = \int_{y_0}^{y} a_1(x_0,\beta,y')d\beta + \xi(y').$$

To find $\xi(y')$, we substitute $\Phi(y,y')$ in Eq. (2.4) to get

$$\Psi(x,y,y') = \int_{x_0}^{x} a_0(\alpha,y,y')d\alpha + \int_{y_0}^{y} a_1(x_0,\beta,y')d\beta + \xi(y').$$

Differentiate this equation with respect to y' and again use the assumptions, to get

$$\xi'(y') = a_2(x_0, y_0, y').$$

Therefore,

$$\xi(y') = \int_{y'_0}^{y'} a_2(x_0, y_0, \gamma)d\gamma.$$

Hence,

$$\Psi(x, y, y') = \int_{x_0}^{x} a_0(\alpha, y, y')d\alpha + \int_{y_0}^{y} a_1(x_0, \beta, y')d\beta + \int_{y'_0}^{y'} a_2(x_0, y_0, \gamma)d\gamma.$$

The proof of the other direction is obvious. In fact, it comes from the assumption that $a_2(x, y, y')$, $a_1(x, y, y')$, and $a_0(x, y, y')$ are continuous with their first partial derivatives.

Remark 2.1. *From the above theorem, we conclude that the nonlinear second order differential equation* (2.1) *is exact equation if the conditions*

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}, \quad \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}, \quad \textit{and} \quad \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y} \tag{2.5}$$

hold.

Example 2.1. *(The Plane Hydrodynamic Jet) Consider the second order nonlinear differential equation*

$$3\epsilon y'' + yy' = 0.$$

This is exact. By using the result in the above theorem, we have

$$\begin{aligned} \Psi(x, y, y') &= \int_0^y \beta d\beta + 3\epsilon \int_0^{y'} d\gamma && (2.6) \\ &= \frac{y^2}{2} + 3\epsilon y'. && (2.7) \end{aligned}$$

Hence, the equation is reduced to $\Psi(x, y, y') = c^2$, *which is equivalent to*

$$3\epsilon y' + \frac{y^2}{2} = c^2.$$

Remark 2.2. *Consider the following nonlinear second order differential equation*

$$y'' + a_1(x, y)y' + a_0(x, y) = 0, \tag{2.8}$$

where $a_1(x, y)$ *and* $a_0(x, y)$ *satisfy the condition* (2.2). *Note that* $a_2(x, y, y') = 1$, $a_1(x, y, y') = a_1(x, y)$, *and* $a_0(x, y, y') = a_0(x, y)$, *and so, it is obvious to see that the conditions* (2.5) *hold. Therefore,* (2.8) *is exact.*

Example 2.2. *The second order nonlinear initial value problem*

$$\begin{cases} y'' + 12xy^3y' + \left(3y^4 - 1\right) = 0 \\ y(0) = 2, \quad y'(0) = 0, \end{cases} \tag{2.9}$$

is exact. Therefore, there exists a function $\Psi(x, y, y')$ *which reduces the above equation into a first order differential equation. By applying the above theorem, we have*

$$\Psi(x, y, y') = \int_{x_0}^{x} a_0(\alpha, y, y')d\alpha + \int_{y_0}^{y} a_1(x_0, \beta, y')d\beta + \int_{y'_0}^{y'} a_2(x_0, y_0, \gamma)d\gamma,$$

and since $x_0 = 0$, $y_0 = 2$, *and* $y_0' = 0$, *we have*

$$\begin{aligned}\Psi(x,y,y') &= \int_0^x a_0(\alpha,y,y')d\alpha + \int_2^y a_1(0,\beta,y')d\beta + \int_0^{y'} a_2(0,2,\gamma)d\gamma, \\ &= \int_0^x (3y^4-1)d\alpha + \int_0^{y'} d\gamma, \\ &= y' + (3y^4-1)x.\end{aligned}$$

Hence, $\Psi(x,y,y') = c$ *reduces Eq.* (2.9) *to*

$$y' + (3y^4-1)x = c.$$

By applying the initial data, we get $c = 0$. *Hence, Eq.* (2.9) *is reduced to the following first order differential equation*

$$y' + 3xy^4 - x = 0.$$

For which an implicit solution can be obtained by separating the variable.

3 Non-exact Second Order Differential Equations and Integrating Factors

In this section, we introduce the notion of the integrating factor for the second order differential equation (2.1). Also, we deduce some conditions for the existence of such integrating factor. First, we start by the following definition for the integrating factor:

Definition 3.1. *An integrating factor of Eq.* (2.1) *is a non zero function* $\mu(x,y,y')$, *such that the equation*

$$\mu(x,y,y')a_2(x,y,y')y'' + \mu(x,y,y')a_1(x,y,y')y' + \mu(x,y,y')a_0(x,y,y') = 0 \tag{3.1}$$

is exact. i.e.,

$$\frac{\partial A_2}{\partial y} = \frac{\partial A_1}{\partial y'}, \quad \frac{\partial A_2}{\partial x} = \frac{\partial A_0}{\partial y'}, \quad \text{and} \quad \frac{\partial A_1}{\partial x} = \frac{\partial A_0}{\partial y}, \tag{3.2}$$

where

$$A_2(x,y,y') = \mu(x,y,y')a_2(x,y,y'),$$

$$A_1(x,y,y') = \mu(x,y,y')a_1(x,y,y'),$$

and

$$A_0(x,y,y') = \mu(x,y,y')a_0(x,y,y').$$

Theorem 3.1. *Assume that Eq.* (2.1) *is not an exact equation. Then, it has no integrating factor of one of the forms* $\mu(x,y,y')$, $\mu(x,y)$, $\mu(x,y')$, *or* $\mu(y,y')$ *if and only if*

$$\left(\frac{\partial a_0}{\partial y} - \frac{\partial a_1}{\partial x}\right)a_2 + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_0}{\partial y'}\right)a_1 + \left(\frac{\partial a_1}{\partial y'} - \frac{\partial a_2}{\partial y}\right)a_0 \neq 0. \tag{3.3}$$

Proof. If such an integrating factor exists, then the conditions in Eq. (3.2) should be hold. A simple calculations shows that the following equations:

$$a_2\frac{\partial \mu}{\partial y}+\mu\frac{\partial a_2}{\partial y}=a_1\frac{\partial \mu}{\partial y'}+\mu\frac{\partial a_1}{\partial y'},$$

$$a_2\frac{\partial \mu}{\partial x}+\mu\frac{\partial a_2}{\partial x}=a_0\frac{\partial \mu}{\partial y'}+\mu\frac{\partial a_0}{\partial y'},$$

and

$$a_1\frac{\partial \mu}{\partial x}+\mu\frac{\partial a_1}{\partial x}=a_0\frac{\partial \mu}{\partial y}+\mu\frac{\partial a_0}{\partial y},$$

must be hold. By solving the above three algebraic equations, simultaneously, we get

$$\left[\left(\frac{\partial a_0}{\partial y}-\frac{\partial a_1}{\partial x}\right)a_2+\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_0}{\partial y'}\right)a_1+\left(\frac{\partial a_1}{\partial y'}-\frac{\partial a_2}{\partial y}\right)a_0\right]\mu(x,y,z)=0.$$

Clearly, if

$$\left[\left(\frac{\partial a_0}{\partial y}-\frac{\partial a_1}{\partial x}\right)a_2+\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_0}{\partial y'}\right)a_1+\left(\frac{\partial a_1}{\partial y'}-\frac{\partial a_2}{\partial y}\right)a_0\right]\neq 0,$$

then

$$\mu(x,y,y')=0.$$

Similarly, for Eq. (2.1), we can show that there is no integrating factor of one of the forms $\mu(x,y)$, $\mu(x,y')$, or $\mu(y,y')$ if (3.3) holds. □

Example 3.1. *Consider the second order nonlinear equation*

$$xy(2x+y)y''+(x^2+xy)y'+(3xy+y^2)=0. \tag{3.4}$$

Theorem 3.1 shows that the above equation has an integrating factor. In fact, the integrating factor is given by $\mu(x,y)=\frac{1}{xy(2x+y)}$. *This integrating factor transforms Eq. (3.4) into an exact equation, which can be reduced into a first order differential equation. In fact, it is reduced into the following equation:*

$$\frac{dy}{dx}+\ln\left(xy\sqrt{y+2x}\right)=c.$$

The following result gives necessary conditions for the integrating factor to be a function of x only.

Remark 3.1. *Through out this paper, we use the notation* $\partial_\eta f:=\dfrac{\partial f}{\partial \eta}$.

Lemma 3.1. *Assume that Eq. (2.1) is not an exact equation. Then, it has an integrating factor*

$$\mu(x)=\exp\left\{\int^x\frac{\partial_y a_0-\partial_x a_1}{a_1}dx\right\}=\exp\left\{\int^x\frac{\partial_{y'} a_0-\partial_x a_2}{a_2}dx\right\}$$

if and only if

$$\frac{\partial_y a_0-\partial_x a_1}{a_1}\ \textit{and}\ \frac{\partial_{y'} a_0-\partial_x a_2}{a_2}\ \textit{depend only on}\ x,$$

$$\frac{\partial_y a_0-\partial_x a_1}{a_1}=\frac{\partial_{y'} a_0-\partial_x a_2}{a_2},$$

and

$$\partial_y a_2=\partial_{y'} a_1.$$

Proof. Assume that Eq. (2.1) has an integrating factor $\mu(x)$. Therefore, conditions (3.2) hold. Hence, we get the following algebraic equations:

$$\mu\frac{\partial a_2}{\partial y} = \mu\frac{\partial a_1}{\partial y'},$$

$$a_2\mu' + \mu\frac{\partial a_2}{\partial x} = \mu\frac{\partial a_0}{\partial y'},$$

and

$$a_1\mu' + \mu\frac{\partial a_1}{\partial x} = \mu\frac{\partial a_0}{\partial y}.$$

Using the first equation, we have a non zero integrating factor, if $\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$. The last two equations implies that

$$\frac{\mu'}{\mu} = \frac{\frac{\partial a_0}{\partial y'} - \frac{\partial a_2}{\partial x}}{a_2} = \frac{\frac{\partial a_0}{\partial y} - \frac{\partial a_1}{\partial x}}{a_1}.$$

By integrating the above equation with respect to x, we get

$$\mu(x) = \exp\left\{\int^x \frac{\partial_y a_0 - \partial_x a_1}{a_1} dx\right\} = \exp\left\{\int^x \frac{\partial_{y'} a_0 - \partial_x a_2}{a_2} dx\right\}. \square$$

Lemma 3.2. *The integrating factor of Eq.* (2.1) *in terms of* y *is given by*

$$\mu(y) = \exp\left\{\int^y \frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} dy\right\} = \exp\left\{\int^y \frac{\partial_x a_1 - \partial_y a_0}{a_0} dy\right\},$$

provided that

$$\frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} \text{ and } \frac{\partial_x a_1 - \partial_y a_0}{a_0} \text{ depend only on } y,$$

$$\frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} = \frac{\partial_x a_1 - \partial_y a_0}{a_0},$$

and

$$\partial_x a_2 = \partial_{y'} a_0.$$

Lemma 3.3. *The integrating factor of Eq.* (2.1) *in terms of* y' *is given by*

$$\mu(y') = \exp\left\{\int^{y'} \frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} dy'\right\} = \exp\left\{\int^{y'} \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0} dy'\right\},$$

provided that

$$\frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} \text{ and } \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0} \text{ depend only on } y',$$

$$\frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} = \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0},$$

and

$$\partial_x a_1 = \partial_y a_0.$$

Example 3.2. *Consider the nonlinear second order differential equation*

$$(1+y^2)yy'' + g(y)y' + (1+y^2)y = 0,$$

where $g(y)$ is an arbitrary function in y. This equation is not exact. In fact, it has an integrating factor $\mu(y) = \frac{1}{y(1+y^2)}$ which transforms this equation into the exact second order differential equation

$$y'' + \frac{g(y)}{y(1+y^2)}y' + 1 = 0.$$

Since the condition (3.3) can not be held easily. i.e., to have an integrating factor of the form $\mu(x,y,y')$, we are looking for an integrating factor of the form $\mu(\alpha(x)\beta(y)\gamma(y'))$, where $\alpha(x), \beta(y)$ and $\gamma(y')$ are arbitrary functions in x, y, and y', respectively. For such an integrating factor to exist, we have the following theorem:

Theorem 3.2. *Assume that Eq. (2.1) is not an exact equation. Then, an integrating factor $\mu(\alpha(x)\beta(y)\gamma(y'))$ of Eq. (2.1) exists and is given by*

$$\begin{aligned}\mu(\xi) = \mu(\alpha(x)\beta(y)\gamma(y')) &= \exp\left\{\int^{\xi} \frac{\partial_{y'}a_1 - \partial_y a_2}{\alpha(x)\left[\beta'(y)\gamma(y')a_2 - \beta(y)\gamma'(y')a_1\right]}d\xi\right\} \\ &= \exp\left\{\int^{\xi} \frac{\partial_y a_0 - \partial_x a_1}{\gamma(y')\left[\alpha(x)\beta'(y)a_1 - \alpha'(x)\beta(y)a_0\right]}d\xi\right\} \\ &= \exp\left\{\int^{\xi} \frac{\partial_x a_2 - \partial_{y'} a_0}{\beta(y)\left[\alpha(x)\gamma'(y')a_0 - \alpha'(x)\gamma(y')a_2\right]}d\xi\right\},\end{aligned}$$

if and only if

$$\begin{aligned}\frac{\partial_{y'}a_1 - \partial_y a_2}{\alpha(x)\left[\beta'(y)\gamma(y')a_2 - \beta(y)\gamma'(y')a_1\right]} &= \frac{\partial_y a_0 - \partial_x a_1}{\gamma(y')\left[\alpha(x)\beta'(y)a_1 - \alpha'(x)\beta(y)a_0\right]} \\ &= \frac{\partial_x a_2 - \partial_{y'} a_0}{\beta(y)\left[\alpha(x)\gamma'(y')a_0 - \alpha'(x)\gamma(y')a_2\right]},\end{aligned}$$

and they depend on $\xi(x,y,y') := \alpha(x)\beta(y)\gamma(y')$.

Proof. The proof is a direct consequence of conditions (3.2).

Using the above theorem, and by either assuming $\gamma(y') = 1$, $\beta(y) = 1$, or $\alpha(x) = 1$, we can deduce that the integrating factors are $\mu(\alpha(x)\beta(y))$, $\mu(\alpha(x)\gamma(y'))$ and $\mu(\beta(y)\gamma(y'))$, respectively. The results are listed in the following corollaries:

Corollary 3.1. *An integrating factor, $\mu(\alpha(x)\beta(y))$, of Eq. (2.1) exists and is given by*

$$\begin{aligned}\mu(\alpha(x)\beta(y)) &= \exp\left\{\int^{\xi} \frac{\partial_{y'}a_1 - \partial_y a_2}{\alpha(x)\beta'(y)a_2}d\xi\right\} \\ &= \exp\left\{\int^{\xi} \frac{\partial_{y'}a_0 - \partial_x a_2}{\alpha'(x)\beta(y)a_2}d\xi\right\} \\ &= \exp\left\{\int^{\xi} \frac{\partial_y a_0 - \partial_x a_1}{\alpha(x)\beta'(y)a_1 - \alpha'(x)\beta(y)a_0}d\xi\right\},\end{aligned}$$

if and only if

$$\frac{\partial_{y'}a_1 - \partial_y a_2}{\alpha(x)\beta'(y)a_2} = \frac{\partial_{y'}a_0 - \partial_x a_2}{\alpha'(x)\beta(y)a_2} = \frac{\partial_y a_0 - \partial_x a_1}{\alpha(x)\beta'(y)a_1 - \alpha'(x)\beta(y)a_0},$$

and they depend on $\xi(x,y) := \alpha(x)\beta(y)$.

Corollary 3.2. *An integrating factor, $\mu(\alpha(x)\gamma(y'))$, of Eq.* (2.1) *exists and is given by*

$$\begin{aligned}
\mu(\xi) &= \mu(\alpha(x)\gamma(y')) \\
&= \exp\left\{\int^{\xi} \frac{\partial_y a_2 - \partial_{y'} a_1}{\alpha(x)\gamma'(y')a_0} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_x a_1 - \partial_y a_0}{\alpha'(x)\gamma(y')a_1} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_{y'} a_0 - \partial_x a_2}{\alpha'(x)\gamma(y')a_2 - \alpha(x)\gamma'(y')a_0} d\xi\right\},
\end{aligned}$$

provided that

$$\frac{\partial_y a_2 - \partial_{y'} a_1}{\alpha(x)\gamma'(y')a_0} = \frac{\partial_x a_1 - \partial_y a_0}{\alpha'(x)\gamma(y')a_1} = \frac{\partial_{y'} a_0 - \partial_x a_2}{\alpha'(x)\gamma(y')a_2 - \alpha(x)\gamma'(y')a_0},$$

and they depend on $\xi(x, y') := \alpha(x)\gamma(y')$.

Corollary 3.3. *An integrating factor, $\mu(\beta(y)\gamma(y'))$, of Eq.* (2.1) *exists and is given by*

$$\begin{aligned}
\mu(\xi) &= \mu(\beta(y)\gamma(y')) \\
&= \exp\left\{\int^{\xi} \frac{\partial_y a_0 - \partial_x a_1}{\beta'(y)\gamma(y')a_1} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_x a_2 - \partial_{y'} a_0}{\beta(y)\gamma'(y')a_0} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_{y'} a_1 - \partial_y a_2}{\beta'(y)\gamma(y')a_2 - \beta(y)\gamma'(y')a_1} d\xi\right\},
\end{aligned}$$

provided that

$$\frac{\partial_y a_0 - \partial_x a_1}{\alpha'(y)\beta(y')a_1} = \frac{\partial_x a_2 - \partial_{y'} a_0}{\alpha(y)\beta'(y')a_0} = \frac{\partial_{y'} a_1 - \partial_y a_2}{\alpha'(y)\beta(y')a_2 - \alpha(y)\beta'(y')a_1},$$

and they depend on $\xi(y, y') := \beta(y)\gamma(y')$.

4 Conclusions and Remarks

In this paper, we imposed conditions on the equation

$$a_2(x, y, y')y'' + a_1(x, y, y')y' + a_0(x, y, y') = 0,$$

so that it is exact. In addition, we introduced an integrating factor in case where the equation is not an exact differential equation. Moreover, we presented some examples showing that this method is powerful in solving a class of second order nonlinear differential equations. For further studies, it is reasonable to improve this definition and this technique to a more complicated class of differential equations. For example, if we consider the general form of the second order nonlinear differential equation $f(x, y, y', y'') = 0$. Also, it is reasonable to improve this method to work for higher order nonlinear differential equations.

References

[1] W. F. Ames, *Nonlinear ordinary differential equations in transport process*, Vol. **42**, Academic Press, New York, 1968.

[2] H. T. Davis, *Introduction to nonlinear differential and integral equations*, Drover, New York, 1965.

[3] S. Ghandrasekhar, *Introduction to a study of stellar structures*, Univ. Chicago Press, Illinois, 1939.

[4] J. K. Hale, *Nonlinear oscillations*, McGraw-Hill, New York, 1963.

[5] D. W. Jordan and P. Smith, *Nonlinear ordinary differential equations: An introduction for scientist and engineers*, 4th edition, Oxford University Press, 2007.

[6] J. P. Laselle and S. Lefschetz, *Nonlinear differential equations and nonlinear mechanics*, Wiley, New York, 1957.

[7] S. Lefschetz, *Differential Equations: Geometric Theory*, Academic Press, New York, 1963.

[8] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, 1955.

[9] R. A. Struble, *Nonlinear differential equations*, McGraw-Hill, New York, 1962.

[10] T. von Karman, *The engineer grapples with nonlinear problems*, Bull. Amer. Math. Soc., **46** (1940), pp. 615–683.

19 A New Approach for Solving Second Order Ordinary Differential Equations. By Laith K. AL-Hwawcha and Namh A. Abid (2008)

Journal of Mathematics and Statistics 4 (1): 58-59, 2008
ISSN 1549-3644

A New Approach for Solving Second Order Ordinary Differential Equations

Laith K. AL-Hwawcha and Namh A. Abid
Al-Zaytoonah University, Amman, Jordan

Abstract: A new approach is presented to solve second order linear differential equations with variable coefficients and some illustrative examples are given.

Key words: Second order equations, general solution, homogeneous and nonhomogeneous equations

INTRODUCTION

Consider the second order linear ordinary differential equation

$$y'' + P(x)\, y' + Q(x)\, y = G(x) \qquad (1)$$

where, P, Q and G are continuous functions. It is known that the power series method is a powerful method for solving Eq.(1). However, this method needs a lot of time, space and high concentration during calculations. In this research, we present a new approach which can be used to a wide class of equations either to find a general solution to the associated homogeneous equation or to find a particular solution to Eq.(1) without requiring the general solution or any solution of the associated homogeneous equation as most methods require. For more details, see[1].

MAIN RESULTS

In this section we introduce our main results.

Theorem 1: Consider the equation

$$y'' + P(x)\, y' + Q(x)\, y = 0 \qquad (2)$$

If $v(x) = y'(x) + \beta(x)y(x)$, where $\beta(x)$ is a solution of the Riccati equation $\beta'(x) = Q(x) - P(x)\,\beta(x) + \beta^2(x)$, then,

$$y(x) = e^{-\int \beta(x)dx} \int e^{\int (2\beta(x) - P(x))dx} dx \qquad (3)$$

is a solution of Eq.(2).

Proof: It is easy to show that $v' = (\beta(x) - P(x))\, v$, where Riccati equation has been used and $v(x) = e^{\int (\beta(x) - P(x))dx}$, then the result is achieved.

Note: It is known that the substitution $v(x) = \dfrac{-y'}{y}$ transfers Eq. (2) to a Riccati equation and $y = e^{-\int v(x)dx}$ is a solution of the equation. This result is included in the theorem (1) and the formula (3) really gives a second linearly independent solution to Eq. (2) and therefore the general solution is constructed. These facts are illustrated in the following example.

Example 1: Find a general solution of the equation

$$x\, y'' - (1+x)\, y' + y = 0 \qquad (4)$$

Solution: Here, $P(x) = \dfrac{-(1+x)}{x}$, $Q(x) = \dfrac{1}{x}$, so the Riccati equation is

$$\beta'(x) = \frac{1}{x} + \left(\frac{1+x}{x}\right)\beta(x) + \beta^2(x)$$

and $\beta(x) = -1$ is a solution of the equation, and then $y_1(x) = e^{\int dx} = e^x$ is a solution of the equation. Thus

$$y_2(x) = e^{\int dx} \int e^{\int \left(-2 + \frac{1+x}{x}\right)dx} dx$$
$$= -x - 1.$$

Hence the general solution is

$$y(x) = c_1 e^x + c_2(x+1)\,.$$

By using the same technique, naturally one can get the following result, which can be used to find a particular solution of Eq. (1). In particular, this procedure can be used easily to find a particular solution of second order ordinary differential equations

Corresponding Author: Laith K. AL-Hwawcha, Al-Zaytoonah University, Amman, Jordan

with constants coefficients and for Cauchy- Euler equation because the associated Riccati equation is solvable.

Theorem 2: Consider the equation

$$y''+P(x)\,y'+Q(x)\,y = G(x) \qquad (5)$$

If $v(x) = y'(x)+\beta(x)y(x)$, where $\beta(x)$ is a solution of the Riccati equation

$$\beta'(x) = Q(x) - P(x)\,\beta(x) + \beta^2(x),$$

then

$$y(x) = e^{-\int \beta(x)dx}\int (e^{\int (2\beta(x)-P(x))dx}\int G(x)e^{-\int (\beta(x)-P(x))dx}dx)dx$$

is a solution of Eq. (5).

Example 2: Find a particular solution of the equation

$$x^2y'' + 3xy' + y = x^2 \ln x, x > 0 \qquad (6)$$

Solution: Here, $P(x) = \frac{3}{x}$ and $Q(x) = \frac{1}{x^2}$, so the Riccati equation is given by:

$$\beta'(x) = \frac{1}{x^2} - \frac{3}{x}\beta(x)+\beta^2(x),$$

and $\beta(x) = \frac{1}{x}$ is a solution of the equation. Thus

$$y_p(x) = e^{-\int \frac{1}{x}dx}\int (e^{-\int \frac{1}{x}dx}\int \ln(x)e^{2\int \frac{1}{x}dx}dx)dx$$
$$= \frac{1}{9}x^2(\ln(x) - \frac{2}{3})$$

is a particular solution of the given equation.

CONCLUSION

In this research we introduce a new approach for solving second order ordinary differential equations, and it seems an easier way to teach these equations than the usual ones.

REFERENCES

1. Boyce, W.E. and R.C. DiPrima, 2000. Elementary Differential Equations and Boundary Value Problems. John Wiley and Sons, Inc.

20 On Exact Solutions of Second Order Nonlinear Ordinary Differential Equations. By Amjed Zraiqat, Laith K. Al-Hwawcha (2015)

Applied Mathematics, 2015, 6, 953-957
Published Online June 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.66087

On Exact Solutions of Second Order Nonlinear Ordinary Differential Equations

Amjed Zraiqat[1], Laith K. Al-Hwawcha[2]

[1]Al-Zaytoonah University of Jordan, Amman, Jordan
[2]German Jordanian University, Amman, Jordan
Email: amjad@zuj.edu.jo, Laith.hawawsheh@gju.edu.jo

Received 26 April 2015; accepted 30 May 2015; published 2 June 2015

Abstract

In this paper, a new approach for solving the second order nonlinear ordinary differential equation $y'' + p(x; y)y' = G(x; y)$ is considered. The results obtained by this approach are illustrated by examples and show that this method is powerful for this type of equations.

Keywords

Nonlinear Ordinary Differential Equation, Partial Differential Equation, Riccati Differential Equation

1. Introduction

Exact solutions have always played and still play an important role in properly understanding the qualitative features of many phenomena and processes in various fields of natural science. Exact solutions of nonlinear equations, including those without a clear physical sense which do not correspond to real phenomena and processes, play an important role of test problems for verifying the correctness and assessment of accuracy of various numerical, asymptotic, and approximate methods. Moreover, the model equations admitting exact solutions serve as the basis for the development of new numerical, asymptotic, and approximate methods, which, in turn, enable us to study more complicated problems having no analytical solutions [1]. In the paper [2], Laith and Nama introduced a new approach for solving second order linear differential equation with variable coefficients

$$y'' + p(x)y' + q(x)y = g(x). \quad (1)$$

To look for exact solution of (1) the authors introduced the substitution

$$v(x) = y' + \beta(x)y \quad (2)$$

How to cite this paper: Zraiqat, A. and Al-Hwawcha, L.K. (2015) On Exact Solutions of Second Order Nonlinear Ordinary Differential Equations. *Applied Mathematics*, **6**, 953-957. http://dx.doi.org/10.4236/am.2015.66087

and have looked for a solution of the Riccati equation

$$\beta' = q(x) - p(x)\beta + \beta^2. \tag{3}$$

In this paper, we generalize the idea of [2] and propose a general approach for solving the nonlinear second order equation

$$y'' + p(x,y)y' + q(x,y)y = g(x,y) \tag{4}$$

which can be written as

$$y'' + p(x,y)y' = G(x,y) \tag{5}$$

where $G(x,y) - g(x,y) - q(x,y)y$.

2. The Main Results

In this section, we propose an algorithm that enables us to reduce the Equations (4) and (5) by looking for solutions of the partial differential equations

$$v_x(x,y) + (v(x,y) - \beta(x,y)y)v_y(x,y) = g(x,y) + (\beta_y(x,y)y - p(x,y) + \beta(x,y))v(x,y) \tag{6}$$

$$\beta_x(x,y) - y\beta(x,y)\beta_y(x,y) = q(x,y) - p(x,y)\beta(x,y) + \beta^2(x,y). \tag{7}$$

Theorem 1. *If* $v(x;y)$ *is any solution of* (6) *where* $(x;y)$ *is a solution of* (7), *then Equation* (4) *can be reduced to a first order equation.*

Proof. In order to prove this theorem, consider the transformation

$$v(x) = y' + \beta(x)y \tag{8}$$

if we differentiate both sides of (8) with respect to x we obtain

$$v_x(x,y) + y'v_y(x,y) = y'' + \beta(x,y)y' + \beta_x(x,y)y + \beta_y(x,y)y'y \tag{9}$$

substituting (4) and (8) in (9), we have

$$\begin{aligned} &v_x(x,y) + (v(x,y) - \beta(x,y)y)v_y(x,y) \\ &= g(x,y) + (\beta_x(x,y) - q(x,y))y + (\beta_y(x,y)y - p(x,y) + \beta(x,y))y' \end{aligned} \tag{10}$$

assuming that $\beta(x,y)$ is a solution of (7), Equation (10) can be reduced to (6), solving (6) for $v(x,y)$ we have the result. ■

Theorem 2. *If* $v(x,y)$ *is any solution of the equation*

$$v_x(x,y) + v(x,y)v_y(x,y) = G(x,y) - p(x,y)v(x,y). \tag{11}$$

Then (5) can be reduced to a first equation.

Proof. From theorem (1) the associated equation with $\beta(x,y)$ is

$$\beta_x(x,y) - y\beta(x,y)\beta_y(x,y) = p(x,y)\beta(x,y) + \beta^2(x,y) \tag{12}$$

which has a solution $\beta(x,y) = 0$, thus the equation associated with $v(x,y)$ is (11), solving for $v(x,y)$ Equation (5) reduced to a first order equation. ■

Theorem 3. *If* $\beta(x,y)$ *is any solution of the equation*

$$\beta_x(x,y) - y\beta(x,y)\beta_y(x,y) = \frac{-G(x,y)}{y} - p(x,y)\beta(x,y) + \beta^2(x,y),\ y \neq 0. \tag{13}$$

Then Equation (5) *can be reduced to first order equation.*

Proof. Equation (5) can be written as

$$y'' + p(x,y)y' - \frac{G(x,y)}{y}y = 0 \tag{14}$$

applying theorem (1), we have that $v(x,y)=0$ is a solution of

$$v_x(x,y)+\left(v(x,y)-\beta(x,y)y\right)v_y(x,y)=\left(\beta(x,y)y-p(x,y)+\beta(x,y)\right)v(x,y) \tag{15}$$

solving (13) for $\beta(x,y)$, the result follows. ■

Theorem 4. *If* $\frac{\partial p(x,y)}{\partial x}=-\frac{\partial G(x,y)}{\partial y}$, *then Equation* (5) *can be reduced to a first order equation.*

Proof. Applying theorem (2) the result follows. ■

3. Examples

In this section, we give some examples on our approach for reduction and finding solutions of nonlinear second order ordinary differential equations, these equations and more equations that can be easily solved by this method can be found in [1] [3]-[7].

Example 1. *Consider the equation*

$$y''=f(y) \tag{16}$$

comparing with Equation (4) *we note that* $p(x,y)=0$, $q(x,y)=0$, $g(x,y)=f(y)$.

First, we solve

$$\beta_x(x,y)-y\beta(x,y)\beta_y(x,y)=\beta^2(x,y) \tag{17}$$

the associated ratios with Equation (17) *are*

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{-\beta_y}=\frac{\mathrm{d}\beta}{\beta^2}$$

from which, we find that $\beta(x,y)=\frac{c_1}{y}$.

Second, we solve

$$v_x(x,y)+\left(v(x,y)-c_1\right)v_y(x,y)=f(y) \tag{19}$$

the associated ratios with Equation (19) *are*

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{v-c_1}=\frac{\mathrm{d}v}{f(y)} \tag{20}$$

from which, we find that $(xy)=c_1\pm\left(c_2+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}}$.

Finally, we substitute $\beta(x,y)$, $v(x,y)$ *in Equation* (8) *to get*

$$x\pm c_3=\int\left(c_2+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}}. \tag{21}$$

Example 2. *Consider the equation*

$$xy''=ny'+x^{2n+1}f(y) \tag{22}$$

this equation can be written as

$$y''-\frac{n}{x}y'=x^{2n}f(y) \tag{23}$$

comparing with Equation (5) *we have that* $p(x,y)=\frac{-n}{x}$, $q(x,y)=0$, $G(x,y)=x^{2n}f(y)$.

The equation associated with $\beta(x,y)$ *is*

$$\beta_x(x,y)-y\beta_y(x,y)=\frac{n}{x}\beta+\beta^2 \tag{24}$$

from which we find that $\beta(x,y)=0$. *The equation associated with* $v(x,y)$ *is*

$$v_x(x,y)+v(x,y)v_y(x,y)=x^{2n}f(y)+\frac{n}{x}v \tag{25}$$

we look for a solution of the form

$$v(x,y)=m(x)n(y) \tag{26}$$

substituting $v(x,y)$ *in Equation* (25), *we have*

$$m'(x)n(y)+m^2(x)n(y)n'(y)=x^{2n}f(y)+\frac{n}{x}m(x)n(y). \tag{27}$$

Thus, $m(x)$ *and* $n(y)$ *must satisfy the following equations*

$$m'(x)=\frac{n}{x}m(x) \tag{28}$$

$$m^2(x)=x^{2n} \tag{29}$$

$$n(y)n'(y)=f(y) \tag{30}$$

from which we find that

$$m(x)=x^n \tag{31}$$

$$n(y)=\pm\left(c_1+2\int f(y)\,\mathrm{d}y\right) \tag{32}$$

so, $v(x,y)=\pm x^n\left(c_1+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}}$. *Finally we solve*

$$y'=\pm x^n\left(c_1+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}} \tag{33}$$

and two cases are considered,

$$n=-1,\ \textit{the solution is}\ \int\left(c_1+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}}=\pm\ln|x|+c_2 \tag{34}$$

$$n\neq-1,\ \textit{the solution is}\ \int\left(c_1+2\int f(y)\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}y=\pm\frac{x^{2n+1}}{n+1}+c_2. \tag{35}$$

Example 3. Consider the equation

$$(1-y)^2y''-\left(x+\frac{3}{2}\right)y'-=1-y \tag{36}$$

Equation (36) can be written as

$$y''-\frac{\left(x+\frac{3}{2}\right)}{(1-y)^2}y'-=\frac{1}{1-y}. \tag{37}$$

Comparing with Equation (5) we have $p(x,y)=\frac{\left(x+\frac{3}{2}\right)}{(1-y)^2}$, $G(x,y)=\frac{1}{1-y}$, furthermore $\frac{\partial_p(x,y)}{\partial x}=-\frac{\partial G(x,y)}{\partial y}$. So, theorem (4) can be applied as follows:

$$v_x(x,y)=\frac{1}{1-y} \tag{38}$$

which implies that

$$v(x,y)=\frac{x}{1-y}+\psi(y). \quad (39)$$

Differentiating both sides of (39), we have

$$v_y(x,y)=\frac{x}{(1-y)^2}+\psi'(y). \quad (40)$$

Assuming that $v_y(x,y)=-p(x,y)$, yields

$$\frac{\left(x+\frac{3}{2}\right)}{(1-y)^2}=\frac{x}{(1-y)^2}+\psi'(y) \quad (41)$$

thus, Equation (36) reduced to the first order exact ordinary differential equation

$$y'=\frac{2x+3}{2-2y} \quad (42)$$

which has the solution

$$x^2+3x+y^2-2y=C. \quad (43)$$

4. Conclusion

In this article, a new method is considered for solving second order nonlinear ordinary differential equations. The small size of computation in comparison with the computational size required by other analytical methods [1], and the dependence on first order partial differential equations show that this method can be improved and introduces a significant improvement in solving this type of differential equations over existing methods. This method is proposed to be considered as an alternative approach being employed to a wide variety of equations.

References

[1] Polyanin, A.D. and Zaitsev, V.F. (2003) Handbook of Exact Solutions for Ordinary Differential Equations. 2nd Edition, Chapman & Hall/CRC Press, Boca Raton.

[2] Al-Hwawcha, L.K. and Abid, N.A. (2008) A New Approach for Solving Second Order Ordinary Differential Equations. *Journal of Mathematics and Statistics*, **4**, 58-59. http://dx.doi.org/10.3844/jmssp.2008.58.59

[3] Polyanin, A.D. and Manzhirov, A.V. (2006) Handbook of Mathematics for Engineers and Scientists. Chapman & Hall/CRC Press, Boca Raton. http://dx.doi.org/10.1201/9781420010510

[4] Zwillinger, D. (1997) Handbook of Differential Equations. 3rd Edition, Academic Press, Boston.

[5] Polyanin, A.D., Zaitsev, V.F. and Moussiaux, A. (2002) Handbook of First Order Partial Differential Equations. Taylor & Francis, London.

[6] Kamke, E. (1977) Differentialgleichungen: Losungsmethoden und Losungen. I. Gewohnliche Differentialgleichungen. B. G. Teubner, Leipzig. http://dx.doi.org/10.1007/978-3-663-05925-7

[7] Boyce, W.E. and Di Prima, R.C. (2000) Elementary Differential Equations and Boundary Value Problems. John Wiley and Sons, Inc., Hoboken.

21 On Solving Some Classes of Second Order ODEs. By R. AlAhmad, M. Al-Jararha (2017)

arXiv:1706.01929v1 [math.CA] 23 May 2017

On Solving Some Classes of Second Order ODEs

R. AlAhmad*and M. Al-Jararha†

Department of Mathematics, Yarmouk University, Irbid, Jordan, 21163.

Abstract

In this paper, we introduce some analytical techniques to solve some classes of second order differential equations. Such classes of differential equations arise in describing some mathematical problems in Physics and Engineering.

AMS Subject Classification: 34A25, 34A30.
Key Words and Phrases: Chebyshev's Differential Equation, Hypergeometric Differential Equation, Cauchy-Euler's Differential Equation, Exact Second Order Differential Equations, Nonlinear Second Order Differential Equations.

1 Introduction

One of the most important applications in the calculus of variation is to maximize (minimize) the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx, \tag{1.1}$$

where $p(x)$ is a positive and differentiable function on some open interval $(a, b) \subset \mathbb{R}$, and $h(x)$ is a differentiable function. In fact, the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx \tag{1.2}$$

attains its extreme values at a function $y(x) \in C^2(a, b)$ that $y(x)$ satisfies the Euler's-Lagrange differential equation [5, 7, 10],

$$\frac{\partial F}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x, y, y')\right) = 0, \tag{1.3}$$

*rami_thenat@yu.edu.jo
†mohammad.ja@yu.edu.jo

where

$$F(x,y,y') := \sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}}. \tag{1.4}$$

Therefore, the problem of maximizing (minimizing) $Q[y]$ is reduced to solve the differential equation (1.3). i.e., to solve

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) = \frac{1}{2}h'(y(x)). \tag{1.5}$$

Hence, it is a matter to solve such differential equations. In the first part of this paper, we solve the following class of second order differential equation:

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\; y'(x), y(x)) = 0 \tag{1.6}$$

which generalizes the differential equation (1.5). Here, we assume that $p(x)$ is a positive and differentiable function on some open interval $(a,b) \subset \mathbb{R}$, and $f(\sqrt{p(x)}\; y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. In fact, equation (1.6) not only generalizes (1.5), but also it generalizes many of well known differential equation. For example,

1. the Chebyshev's Differential Equation [5, 12],

$$(1-x^2)y''(x) - x\, y'(x) + n^2 y(x) = 0, \quad |x| < 1,$$

2. the Cauchy-Euler's Differential Equation [6, 8, 11],

$$ax^2 y''(x) + a\, x\, y'(x) + b\, y(x) = 0, \quad x > 0,$$

3. the Nonlinear Chebyshev's Equation,

$$(1-x^2)y''(x) + \left(\alpha\sqrt{1-x^2} - x\right) y'(x) + f(y(x)) = 0, \qquad |x| < 1,$$

4. the Hypergeometric Differential Equation [4, 12],

$$x(1-x)y''(x) + [c - (a+b+1)x]\, y'(x) - a\, b\, y(x) = 0,\ 0 < x < 1, \quad \text{with } c = 1/2, a = -b,$$

and

5. the Nonlinear Hypergeometric Differential Equation,

$$x(1-x)y''(x) + \left(\frac{1}{2} - x + \alpha\sqrt{x(1-x)}\right) y'(x) + f(y(x)) = 0,\ 0 < x < 1.$$

Throughout this paper, we call the class of differential equation in (1.6) by Chebyshev's-type of differential equations.

In the second part of this paper, we introduce an approach to solve the differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{1.7}$$

where a_0, a_1 and a_2 are constants, and $f \in C^2(a,b)$, for some open interval $(a,b) \subset \mathbb{R}$. We also give an approach to solve the differential equation

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{1.8}$$

where $p(x)$ is a positive and differentiable function on some open interval $(a,b) \subset \mathbb{R}$, and $f \in C^2(c,d)$, for some open interval $(c,d) \subset \mathbb{R}$. Throughout this paper, we call these classes of second order differential equations by $f-$type second order differential equations.

In the third part of this paper, we introduce an approach to solve the second order nonlinear differential equation

$$a_2(x,y,y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x,y,y')(f'(y)y') + a_0(x,y,y') = 0, \tag{1.9}$$

where $f(y)$ is an invertible function ($y = f^{-1}(z)$), and $f \in C^2(a,b)$ where (a,b) is the open interval in $\mathbb{R}$. To solve this class of differential equations, we assume that

$$a_2\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right)z'' + a_1\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right)z' + a_0\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) = 0 \tag{1.10}$$

is exact differential equation. The differential equation (1.10) is called exact if the conditions

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'}, \quad \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'}, \text{ and } \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z}. \tag{1.11}$$

hold [1, 2]. In this case, the first integral of (1.10) exists and it is given by

$$\int_{x_0}^{x} a_0(\alpha, z, z')d\alpha + \int_{z_0}^{z} a_1(x_0, \beta, z')d\beta + \int_{z_0'}^{z'} a_2(x_0, z_0, \gamma)d\gamma = c.$$

Throughout this paper, we call this class of differential equations by $f-$type second order differential equations that can be transformed into exact second order differential equations.

The layout of the paper: In the first section, we solve Chebyshev's-type of Second Order Differential Equations. In the second section, we solve the $f-$type of second order differential equations. In the third section, we solve $f-$type second order differential equation that can be transformed into exact second order differential equations. The fourth section is devoted for the concluding remarks.

2 Solving Chebyshev's-type of Second Order Differential Equations

In this section, we present an approach to solve Chebyshev's-type of second order differential equations

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\,y'(x), y(x)) = 0, \tag{2.1}$$

where $p(x)$ is a positive and differentiable function on some open interval $(a,b) \in \mathbb{R}$, and $f(\sqrt{p(x)}$ $y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. The approach is described in the following theorem:

Theorem 2.1. *Assume that $p(x)$ be a positive and differentiable function on the open interval $(a,b)\subset\mathbb{R}$. Let x_0 be any point in the interval (a,b). Then*

$$t=\int_{x_0}^{x}\frac{d\xi}{\sqrt{p(\xi)}}$$

transforms the differential equation (2.1) into the second order differential equation

$$y''(t)+f(y(t),y'(t))=0. \tag{2.2}$$

Proof. Let

$$t=\int_{x_0}^{x}\frac{d\xi}{\sqrt{p(\xi)}}.$$

Since,

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}.$$

Hence,

$$\frac{dy}{dt}=\sqrt{p(x)}\,\frac{dy}{dx}. \tag{2.3}$$

Therefore,

$$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\sqrt{p(x)}\,\frac{dy}{dx}\right)=\frac{d}{dx}\left(\sqrt{p(x)}\,\frac{dy}{dx}\right)\frac{dx}{dt}=p(x)y''+\frac{1}{2}p'(x)y'. \tag{2.4}$$

By substituting (2.3) and (2.4) in Equation (1.6), we get the result. ∎

Remark 2.1. *The differential equation (2.2) is independent of the variable t, and so, it is easy to solve by setting $\eta(t)=y'(t)$. Hence, it reduces into the following first order differential equation:*

$$\eta\frac{d\eta}{dy}+f(y,\eta)=0. \tag{2.5}$$

In case that $f(\sqrt{p(x)}\,y',y)=f(y)$, we get

$$\eta^2(t)=-2\int^{y}f(\xi)d\xi+c.$$

Hence,

$$y'(t)=\left(c-2\int^{y}f(\xi)d\xi\right)^{\frac{1}{2}}$$

where c is the integration constant.

Next, we present some examples to explain this approach.

Example 2.1. *Consider the nonlinear Chebyshev's differential equation*

$$\begin{cases}(1-x^2)y''(x)-xy'(x)+4\sqrt{1-x^2}\,y'y(x)=0,\\ y(0)=\frac{1}{2},\ y'(0)=-\frac{1}{2}.\end{cases} \tag{2.6}$$

Then

$$t = \int^x \frac{d\xi}{\sqrt{1-\xi^2}} = \arcsin(x)$$

transforms (2.6) *into*

$$y''(t) + 4y'(t)y(t) = 0.$$

Set $\eta(t) = y'(t)$. *The above equation becomes*

$$\eta \frac{d\eta}{dy} + 4\eta y = 0.$$

The solution of this equation is $y(t) = \dfrac{1}{2(t+1)}$. *Therefore,* $y(x) = \dfrac{1}{2(\arcsin(x)+1)}$.

Example 2.2. *Consider the initial value problem*

$$\begin{cases} x^2y'' + xy' - 3y^2 = 0, \ x > 0, \\ y(1) = 2, \ y'(1) = 4. \end{cases} \tag{2.7}$$

Then

$$t = \int_1^x \frac{d\xi}{\xi} d\xi = \ln(x)$$

transforms the (2.7) *into*

$$\begin{cases} y'' - 3y^2 = 0, \\ y(0) = 2, \ y'(0) = 4. \end{cases} \tag{2.8}$$

The solution of the above differential equation is

$$y(t) = \frac{2}{(1-t)^2}.$$

Hence,

$$y(x) = \frac{2}{(1-\ln(x))^2}.$$

Example 2.3. *Consider the linear form of* (1.6)

$$\phi(x)y'' + \frac{1}{2}\phi'(x)y' + \lambda^2 y = 0, \tag{2.9}$$

where $\lambda \in \mathbb{R}$, *and* $\phi(x)$ *is a positive and differentiable function on some open interval* $(a,b) \subset \mathbb{R}$. *By applying the transformation*

$$t = \int_{x_0}^x \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi,$$

equation (2.9) *can be transformed into the following second order differential equation:*

$$\frac{d^2y}{dt^2} + \lambda^2 y = 0.$$

The solution of this different5ial equation is

$$y(t) = C_1 \sin(\lambda t) + C_2 \cos(\lambda t).$$

Hence, the general solution of equation (2.9) *is*

$$y(x) = C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right).$$

Example 2.4. *Consider the following second order linear differential equation (see Eq. 239, p. 335 in [11]):*

$$4xy'' + 2y' + y = 0. \tag{2.10}$$

From the previous example, the general solution of this equation is given by

$$y(x) = C_1 \sin\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right) + C_2 \cos\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right)$$

and so,

$$y(x) = C_1 \sin\left(\sqrt{x}\right) + C_2 \cos\left(\sqrt{x}\right).$$

Remark 2.2. *Consider the second order linear differential equation*

$$(\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x) + \lambda y(x) = 0, \quad x \in (a, b), \tag{2.11}$$

and assume that $\phi(x)$ is a positive and differentiable function on an open interval $(a, b) \subset \mathbb{R}$. Moreover, assume that $\phi(a) = \phi(b) = 0$. Define the linear differential operator

$$L[y] := -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x))\right) = \lambda y(x).$$

Then the boundary value problem

$$\begin{cases} L[y] = -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x)\right) = \lambda y(x), & a < x < b, \\ \phi(a) = \phi(b) = 0, \end{cases} \tag{2.12}$$

satisfies the Lagrange Identity $\int_a^b \phi L[\psi]dx = \int_a^b \psi L[\phi]dx$, where ϕ and ψ satisfy the above boundary value problem. Therefore, the operator $L[y]$ is self-adjoint. Hence, the boundary value problem (2.12) *has an orthogonal set of eigenfunctions $\{\phi_n(x)\}_{n=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Since the above boundary value problem is a special case of* (1.6)*. Then, by using the approach described in Theorem 2.1, it is easy to find its orthogonal set of eigenfunctions.*

By using the same approach described in Theorem 2.1. We can solve the following class of second order linear differential equations:

$$[P(x)]^2 y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y(x) = 0, \tag{2.13}$$

where $P(x) > 0$, $P(x) \in C^1(a, b)$, and α and β are constants. In fact, the transformation

$$t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}, \tag{2.14}$$

where $x_0,\ x \in (a, b)$, transforms Eq. (2.13) into the following second order differential equation:

$$y''(t) + \alpha y'(t) + \beta y(t) = 0.$$

This differential equation is with constant coefficients which can be solved by using the elementary techniques of solving second order differential equations. For illustration, we present the following examples:

Example 2.5. *Consider the well-known Cauchy-Euler's Equation*

$$x^2y''(x) + (\alpha+1)xy'(x) + \beta y = 0, \; x > 0.$$

Then $P(x) = x$*, and the* $t-$*transformation is* $t = \ln(x)$*, which transforms the equation into*

$$y''(x) + \alpha y'(x) + \beta y = 0$$

Example 2.6. *Consider the Chebyshev's Equation*

$$\left[1-x^2\right] y''(x) - 2xy'(x) + n^2 y = 0, \; |x| < 1.$$

Then $P(x) = \sqrt{1-x^2}$*, and the* $t-$*transformation is* $t = \sin^{-1}(x)$*, which transforms the equation into*

$$y''(x) + n^2 y = 0.$$

Using this transformation, the solution of Chebyshev's Equation is given by

$$y(x) = A\cos(n\sin^{-1}(x)) + B\sin(n\sin^{-1}(x)).$$

Example 2.7. *Consider the Hypergeometric Equation*

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 0, \; x \in (0,1). \tag{2.15}$$

Then $P(x) = \sqrt{x(1-x)}$*, and the* $t-$*transformation is* $t = \sin^{-1}(2x-1)$*. This transforms the equation into*

$$y''(t) + a^2 y(t) = 0.$$

Hence, the solution of (2.15) *is given by*

$$y(x) = A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)).$$

For certain functions, $h(x) \in C(a,b)$, for some open interval $(a,b) \in \mathbb{R}$, we can solve the nonhomogeneous second order differential equation

$$\left[P(x)\right]^2 y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y = h(x). \tag{2.16}$$

Particularly, when $h(x)$ can be written in the form $H(t)$, where $t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}$. The following example shows this idea:

Example 2.8. *Consider the nonhomogeneous differential equation*

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 2x, \; x \in (0,1). \tag{2.17}$$

Then $P(x) = \sqrt{x(1-x)}$*. The* $t-$*transformation is* $t = \sin^{-1}(2x-1)$*. This transforms the equation into*

$$y''(t) + a^2 y(t) = 1 + \sin(t).$$

Hence, the solution of equation (2.17) *is given by*

$$y(x) = \begin{cases} A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)) + \dfrac{2x-1}{a^2-1} + \dfrac{1}{a^2}, & \textit{if } a \neq \pm 1, \\ A\cos(\sin^{-1}(2x-1)) + B(2x-1) + \dfrac{1}{2}(1-2x)\sin^{-1}(2x-1) + 1, & \textit{if } a = \pm 1. \end{cases}$$

3 Solving $f-$type Second Order Differential Equations

In this section, we solve the following class of second order nonlinear differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{3.1}$$

where a_2, a_1 and a_0 are constants, and $f \in C^2(a,b)$, for some open interval $(a,b) \subset \mathbb{R}$. In this section, we also solve the following class of second order nonlinear differential equation:

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{3.2}$$

where $p(x)$ is a positive and differentiable function on an open interval $(a,b) \subset \mathbb{R}$, and $f \in C^2(c,d)$, for some open interval $(c,d) \subset \mathbb{R}$. To solve (3.1), let $z = f(y)$. Hence, $z' = f'(y)y'$, and $z'' = f'(y)y'' + (y')^2 f''(y)$. Substitute z, z' and z'' in equation (3.1), we get

$$a_2 z'' + a_1 z' + a_0 z = g(x), \tag{3.3}$$

Similarly, equation (3.2) becomes

$$p(x)z'' + \frac{1}{2}p'(x)z' + a_0 z = 0, \tag{3.4}$$

which is the linear form of (1.6). Therefore, it can be solved by using the technique described in Example 2.3. To illustrate the procedure of solving (3.1) and (3.2), we present the following examples:

Example 3.1. *Consider Langumir Equation, with a slightly modification,*

$$3yy'' + 3(y')^2 + 4yy' + y^2 = 1. \tag{3.5}$$

The original Langumir Equation is given by

$$3yy'' + (y')^2 + 4yy' + y^2 = 1$$

which originally appears in connection with the theory of current flow from hot cathode to an anode in a hight vacuum [3, 9]. To solve (3.5)*, we let* $z = \frac{y^2}{2}$*. Then* $z' = yy'$ *and* $z'' = yy'' + (y')^2$*. Hence, equation* (3.5) *becomes*

$$3z'' + 4z' + 2z = 1.$$

The solution of this equation is

$$z(x) = e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + \frac{1}{2}.$$

Hence, the solution of (3.5) *is given by*

$$y^2 = 2e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + 1.$$

Example 3.2. *Consider the initial value problem*

$$\begin{cases} y'' + (y')^2 + 1 = (\cos \omega x)e^{-y}, \quad \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

This problem is equivalent to

$$\begin{cases} \left(y'' + (y')^2\right) e^y + e^y = (\cos \omega x), \quad \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

Let $z = e^y$. Then $z' = y'e^y$ and $z'' = y''e^y + (y')^2 e^y$. By substituting z, z' and z'' in the above initial value problem, we get

$$\begin{cases} z'' + z = \cos \omega x, \quad \omega \neq \pm 1, \\ z(0) = 1, \; z'(0) = 0. \end{cases}$$

The solution of this problem is $z(x) = \frac{1}{1-\omega^2}\left(\cos \omega x - \omega^2 \cos x\right)$, $\omega \neq \pm 1$. Therefore, $y(x) = \ln\left(\frac{1}{1-\omega^2}\left(\cos \omega x - \omega^2 \cos x\right)\right)$, $\omega \neq \pm 1$.

Example 3.3. *Let $\phi(x)$ be a positive and differentiable function on an open interval $(a, b) \subset \mathbb{R}$, and consider the differential equation*

$$\phi(x)\left(y'' + (y')^2\right) + \frac{1}{2}\phi'(x)y' + \lambda = 0.$$

By multiplying this equation by e^y, we get

$$\phi(x)\left(y'' + (y')^2\right) e^y + \frac{1}{2}\phi'(x)y'e^y + \lambda e^y = 0.$$

Let $z = e^y$. Then $z' = y'e^y$ and $z'' = y''e^y + (y')^2 e^y$. By substituting z, z' and z'' in the above differential equation, we get

$$\phi(x)z'' + \frac{1}{2}\phi'(x)z' + \lambda z = 0.$$

The solution of this equation is (see Example 2.3)

$$z(x) = C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right).$$

Therefore,

$$y(x) = \ln\left[C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right)\right].$$

4 Solving f-type Second Order Differential Equations that can be Transformed into Exact Second Order Differential Equations

In this section, we solve the following class of second order nonlinear differential equations:

$$a_2(x, y, y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x, y, y')(f'(y)y') + a_0(x, y, y') = 0, \tag{4.1}$$

where $f(y)$ is an invertible function and $f \in C^2(a,b)$. To solve this class of differential equations, we let $z = f(y)$. Then $z' = f'(y)y'$ and $z'' = f''(y)(y')^2 + f'(y)y''$. Moreover, we let $y = f^{-1}(z)$. Then $y' = \frac{z'}{f'(f^{-1}(z))}$. Hence, equation (4.1) can be transformed into the following differential equation:

$$a_2\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z'' + a_1\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z' + a_0\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) = 0 \tag{4.2}$$

Assume that (4.2) is exact, then it can be solved. To explain the procedure of solving such differential equations, we consider the following example:

Example 4.1. *Consider the second order nonlinear differential equation*

$$\begin{cases} e^y\left[y'' + (y')^2\right] + 12xe^{4y}y' + \left(3e^{4y} - 1\right) = 0, \\ y(0) = \ln 2,\ y'(0) = 0. \end{cases} \tag{4.3}$$

Let $z = e^y$. Then $z' = e^y y'$ and $z'' = e^y y'' + e^y (y')^2$. Hence, Eq. (4.3) becomes

$$\begin{cases} z'' + 12xz^3 z' + \left(3z^4 - 1\right) = 0, \\ z(0) = 2,\ z'(0) = 0. \end{cases} \tag{4.4}$$

Therefore, $a_2(x,z,z') = 1$, $a_1(x,z,z') = 12xz^3$, and $a_0(x,z,z') = \left(3z^4 - 1\right)$. In addition, we have

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'} = 0, \quad \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'} = 0, \text{ and } \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z} = 12z^3, \tag{4.5}$$

Therefore, equation (4.4) is exact differential equation. Hence, its first integral exists and it is given by

$$z' + 3xz^4 - x = 0.$$

For which an implicit solution of this equation can be obtained by separating the variables, and so, $y(x) = \ln(z(x))$.

Remark 4.1. *Assume that (4.2) is not exact. Then an integrating factor of (4.2) could be exist. Hence, it can be transformed into an exact differential equation (see [1]). To explain the procedure of solving (4.2) in case it is not exact, we present the following example:*

Example 4.2. *Consider the second order nonlinear differential equation*

$$xe^y\left(2x + e^y\right)\left(y'' + (y')^2\right) + x\left(x + e^y\right)y' + \left(3x + e^y\right) = 0. \tag{4.6}$$

By multiplying this equation by e^y, we get

$$xe^{2y}\left(2x + e^y\right)\left(y'' + (y')^2\right) + x\left(x + e^y\right)e^y y' + e^y\left(3x + e^y\right) = 0. \tag{4.7}$$

Let $z = e^y$. Then $z' = e^y y'$ and $z'' = e^y y'' + e^y (y')^2$. Hence, by substituting z, z' and z'' in (4.7), we get

$$xz(2x + z)z'' + x(x + z)z' + z(3x + z) = 0. \tag{4.8}$$

This equation is not exact since $\frac{\partial a_2}{\partial z} = 2(x+z) \neq 0 = \frac{\partial a_1}{\partial z'}$. *An integrating factor of this second order nonlinear differential equation exists, and it is given by* $\mu(x,z) = \dfrac{1}{xz(2x+z)}$. *Multiplying* (4.8) *by* $\mu(x,z)$, *we get*

$$z'' + \frac{(x+z)}{z(2x+z)}z' + \frac{(3x+z)}{x(2x+z)} = 0. \tag{4.9}$$

Clearly,

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'} = 0, \;\; \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'} = 0, \;\; \text{and} \;\; \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z} = \frac{-1}{(2x+z)^2}\,. \tag{4.10}$$

Therefore, the differential equation (4.9) *is exact, and its first integral is given by*

$$c = z' + \ln\left(xz\sqrt{2x+z}\right). \tag{4.11}$$

This first order differential equation can be solved by using the elementary techniques of solving first order differential equations. Hence, $y(x) = \ln(z(x))$.

Finally, we consider the nonhomogeneous second order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x),$$

where $a_2(x) \neq 0$, $a_1(x)$, and $a_0(x)$ are differentiable functions on an open interval $(a,b) \subset \mathbb{R}$. This equation admits an integrating factor $\mu(x) = \dfrac{1}{a_2(x)}$ provided that $W(a_2,a_1)(x) = a_0(x)a_2(x)$, where $W(a_2,a_1)(x) = a_2(x)a_1'(x) - a_1(x)a_2'(x)$. For this case, we present the following example:

Example 4.3. *consider the second order linear differential equation*

$$e^x y'' + \cos x y' - (\cos x + \sin x)y = h(x).$$

By multiplying this equation by the integrating factor e^{-x}, *we get*

$$y'' + e^{-x}\cos x y' - e^{-x}(\cos x + \sin x)y = h(x)e^{-x}.$$

This equation can be written as

$$\frac{d}{dx}\left[y' + (e^{-x}\cos x)y\right] = h(x)e^{-x}$$

Hence, its first integral is given by

$$y' + (e^{-x}\cos x)y = \int^x h(\xi)e^{-\xi}d\xi + c_1$$

which can be solved by using the elementary techniques of solving first order differential equations.

5 Concluding Remarks

In this paper, we solved some classes of second order differential equation. In fact, we solved the following classes of second order differential equations:

1. The Chebyshev's type of second order differential equation

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0, \quad x \in (a,b), \tag{5.1}$$

   where $p(x)$ is a positive and differentiable function on an open interval $(a,b) \subset \mathbb{R}$, and $f(\sqrt{p(x)}\, y'(x), y(x))$ is a continuous function on some domain $D \subset \mathbb{R}^2$.

2. The $f-$type of second order differential equations

   a)
$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{5.2}$$

   where a_2, a_1 and a_0 are constants, and the function $f(y)$ is of C^2-class on some open interval $(a,b) \subset \mathbb{R}$, and

   b)
$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{5.3}$$

   where $p(x)$ is a positive and differentiable function on some open interval $(a,b) \subset \mathbb{R}$, and $f \in C^2(c,d)$, for some open interval $(c,d) \subset \mathbb{R}$.

3. f-type second order differential equations that can be transformed into exact second order Differential Equations

$$a_2(x,y,y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x,y,y')(f'(y)y') + a_0(x,y,y') = 0, \tag{5.4}$$

   where the function $f(y)$ is an invertible function and $f \in C^2(a,b)$, for some open interval $(a,b) \subset \mathbb{R}$.

Moreover, we presented some examples to explain our approach of solving the above classes of second order differential equation.

References

[1] AlAhmad R., Al-Jhararha M., and Almefleh H. (2015) Exactness of Second Order Ordinary Differential Equations and Integrating Factors. JJMS **11**(3), pp. 155-167.

[2] Al-Jararha M. (2015) Exactness of Higher Nonlinear Ordinary Differential Equations. Global Journal of Pure and Applie4d Mathematics, **11**(3), pp. 1467-1473.

[3] Ames, W. F. (1968) *Nonlinear ordinary differential equations in transport process.* Vol. **42**, Academic Press, NY.

[4] Anderws, L. C. (1998) *Special Functions of Mathematics for Engineers*. 2nd edition, Oxford University Press, Oxford.

[5] Arfken, G. (1985) *Mathematical Methods for Physicists*. 3rd edition, Academic Pres, Orlando, FL.

[6] Boyce, W. E. and DiPrima, R. C. (1986) *Elementary Differential Equations and Boundary Value Problems*. 4th edition, Wiley, NY.

[7] Hand, L.N. and Finch, J.D. (1998) *Analytical Mechanics*. Cambridge University Press, Cambridge.

[8] Kreyszig, E. Kreyszig, H. and Norminton, E. J. (2011) *Advanced Engineering Mathematics*. 10th edition, Wiley, NY.

[9] Langmuir, I and Bolgett, K. B. (1923) Currents Limited by Space Charge between Coaxial Cylinders, Phys. Rev., **22**(347).

[10] M. V. Makarets, M. V. and Reshetnyak, V. Y. (1995)*Ordinary differential equations and calculus of variations*. World Scientific Publishing Co. Inc., NJ.

[11] Murphy, G. M. (1960) *Ordinary Differential Equations and their Solutions*. Van Nostrand Reinhold Company, New York.

[12] Zwillinger, D. (1977) *Handbook of Differential Equations*. 3rd edition, Academic Press Boston, MA.

22 λ symmetry and integrating factors for $x''(f(t,x)+g(t,x)x')e^{x}$. By Khodayar Goodarzi and Mehdi Nadjafikhah (2022)

HyperScience International Journal
Original Research Papers
Open Access Journals
ISSN: 2821-3300
HIJ, Vol 2, No 1, pp 7-13, March 2022
https://doi.org/10.55672/hij2022pp7-13

λ-Symmetry and Integrating Factor For $\ddot{x}(f(t,x)+g(t,x)\dot{x})e^{x}$

Khodayar Goodarzi [1]
Mehdi Nadjafikhah [2] *

[1] *Department of Mathematics, Borujerd Branch, Islamic Azad Uinversity, Borujerd, Iran.*
[2] *School of Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, I.R. Iran.*

ABSTRACT

in this paper, we will calculate an integrating factor, first integral and reduce the order the non-Linear second-order ODEs $\ddot{x}(f(t,x)+g(t,x)\dot{x})e^{x}$, through λ-symmetry method. Moreover, we compute an integrating factor, first integral and reduce the order for particular cases of this equation.

Keyword: Symmetry, Integrating Factor, First Integral, Order reduction

INTRODUCTION

Symmetries method have been widely used to reduce the order of an ordinary differential equation (ODE) and to reduce the number of independent variables in a partial differential equation (PDE)[1].
There are many examples of ODEs that have trivial Lie symmetries. In 2001, Muriel and Romero introduced λ-symmetry method to reduce the order of an ODEs and to find general solutions for such examples.
Recently, they [2] presented techniques to obtain first integral, integrating factor, λ-symmetry of second-order ODEs $\ddot{x}=F(t,x,\dot{x})$ and the relationship between them.
In addition, the study of a λ-symmetry method of the ODEs permits us the de termination of an integrating factor and reduce the order of the ODEs and explain the reduction process of many ODEs that lack Lie symmetries.
In this paper, first we will recall some of the foundational results about symmetry and λ-symmetry rather briefly. we present some theorems about an integrating factor, first integral and reduce the order of the ODEs. second, we will calculate an integrating factor, first integral and reduce the or der the non-Linear second-order ODEs $\ddot{x}=(f(t,x)+g(t,x)\dot{x})e^{x}$, through λ symmetry method, which are non-Lie symmetry equation and functions $f(t,x)$ and $g(t,x)$ are arbitrary.
Moreover, we will reduce the order particular cases of the equation $\ddot{x}=(f(t,x)+g(t,x)\dot{x})e^{x}$, which are $\ddot{x}=(f(t,x)+g(t,x)\dot{x})e^{x}$, and $\ddot{x}=(f(t,x)+g(t,x)\dot{x})e^{x}$, through λ-symmetry method. we will present many examples for these equations.

λ-SYMMETRIES ON ODES

In this section we recall some of the foundational results about symmetry and λ-symmetry rather briefly [2-9].
Let $\mathbf{v}$ be a vector field defined on an open subset $M\subset T\times X$.

We denote by $M^{(n)}$ the corresponding jet space $M^{(n)} \subset T \times X^{(n)}$, for $n \in N$. Their elements are $(t, x^{(n)}) = (t, x, x_1, \cdots, x_n)$, where, for $i = 1, 2, \cdots, n, x_i$ denotes the derivative of order i of x with respect to t. Suppose

$$\Delta(t,\ x^{(n)}) = 0 \qquad (1)$$

be an ODE defined over the total space M. The latter characterizes a Lie symmetry of an ODE as a vector filed $\mathbf{v}= \xi(t,x)\partial/\partial t + \eta(t,x)\partial/\partial x$, that satisfies $\mathbf{v}^{(n)}[\Delta(t,x^{(n)})] = 0$, if $\Delta(t,x^{(n)}) = 0$, where $\mathbf{v}^{(n)}$that called $n - th$ prolongation of $\mathbf{v}$ is

$$\mathbf{v}^{(n)} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \eta^{(i)}\left(t,x^{(i)}\right)\frac{\partial}{\partial x_i}$$

Where

$$\eta^{(i)}(t,x^{(i)}) = D_t\left(\eta^{(i-1)}(t,x^{(i-1)})\right) - D_t(\xi(t,x))x_i$$

and $\eta^{(0)}(t,x) = \eta(t,x)$ for $i = 1,\cdots,n$, where D_t denote the total derivative operator with respect to t [9].

If an ODE does not have Lie point symmetry, then we using λ-symmetry method for reduce of order the ODE. λ-symmetry method is as follows [3].

For every function $\lambda \in C^\infty(M^{(1)})$, we will define a new prolongation and Lie symmetry of v in the following way.

Let $\mathbf{v}= \xi(t,x)\partial/\partial t + \eta(t,x)\partial/\partial x$, be a vector field defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. The λ-prolongation of order n of $\mathbf{v}$, denoted by $\mathbf{v}^{[\lambda,(n)]}$, is the vector field defined on M by

$$\mathbf{v}^{[\lambda,(n)]} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \eta^{(i)}\left(t,x^{(i)}\right)\frac{\partial}{\partial x_i}$$

where

$$\eta^{[\lambda,(i)]}(t,x^{(i)}) = (D_t + \lambda)\left(\eta^{[\lambda,(i-1)]}(t,x^{(i-1)})\right) - \left((D_t + \lambda)\xi(t,x)\right)x_i$$

and $\eta^{[\lambda,(0)]}(t,x) = \eta(t,x)$ for $i = 1,\cdots,n$.

A vector field $\mathbf{v}$ is a λ-symmetry of the Eq. (1), if there exists function $\lambda \in C^\infty(M^{(1)})$, such that $\mathbf{v}^{[\lambda,(n)]}[\Delta(t,x^{(n)})] = 0$, if $\Delta(t,x^{(n)}) = 0$.

Note. Suppose vector field $v = \partial/\partial x$ be a λ-symmetry of the Eq.(1), then

$$\eta^{[\lambda,(n-1)]} = \frac{\partial}{\partial x} + (D_t + \lambda)(1)\frac{\partial}{\partial x_1} + (D_t + \lambda)(D_t + \lambda)(1)\frac{\partial}{\partial x_2} + \cdots + (D_t + \lambda)(D_t + \lambda)(1)\frac{\partial}{\partial x_{n-1}}$$

or equivalent

$$v^{[\lambda,(n-1)]} = \sum_{i=1}^{n}(D_t + \lambda)^{(i)}(1)\frac{\partial}{\partial x_i} \qquad (2)$$

An integrating factor of the Eq. (1), is a function $\mu(t,x^{(n-1)})$ such that the equation $\mu.\Delta = 0$ is an exact equation,

$$\mu(t,x^{(n-1)}).\Delta(t,x^{(n)}) = D_t\left(G(t,x^{(n-1)})\right).$$

Function $G(t,x^{(n-1)})$, will be called a first integral of the Eq. (1), and $D_t\left(G(t,x^{(n-1)})\right) = 0$, is a conserved form of the Eq.(1) [6, 10]. Let

$$x_n = F(t,x^{(n-1)}) \qquad (3)$$

be a nth-order ordinary differential equation, where F is an analytic function of its arguments. We denote by $A = \partial_t + x_1\partial_x + x_2\partial_{x^{(1)}} + \cdots + F(t,x^{(n-1)})\partial_{x^{(n-1)}}$ the vector field associated with (3) [3].

Function $I(t,x^{(n-1)})$ is a first integral [7]of (3), such that $A(I) = 0$ and an integrating factor of (3), is any function $\mu(t,x^{(n-1)})$ such that

$$\mu\left((t,x^{(n-1)})x^{(n)} - F(t,x^{(n-1)})\right) = D_t I(t,x^{(n-1)}).$$

By using (2), It can be checked that the vector field $v = \partial_x$ is a λ-symmetry of (3), if the function $\lambda(t, x^{(k)})$ is any particular solution of the equation

$$(D_t + \lambda)^{(n)}(1) = \sum_{i=0}^{n-1}(D_t + \lambda)^{(i)}(1)\frac{\partial F}{\partial x_i} \qquad (4)$$

Theorem 2.1. *If $I(t, x^{(n-1)})$ is a first integral of (3), then $\mu(t,x^{(n-1)}) = I_{x^{(n-1)}}(t,x^{(n-1)})$ is an integrating factor of (3).*

Proof. Let $I(t, x^{(n-1)})$ be a first integral of (3), then

$$A(I) = I_t + x^{(1)}I_x + x^{(2)}I_{x^{(1)}} + \cdots + F(t,x^{(n-1)})I_{x^{(n-1)}} = 0.$$

Therefore

$$I_t + x^{(1)}I_x + x^{(2)}I_{x^{(1)}} + \cdots + x^{(n-1)}I_{x^{(n-2)}} = -F\left(t, x^{(n-1)}\right)I_{x^{(n-1)}}$$

and

$$D_tI = I_t + x^{(1)}I_x + x^{(2)}I_{x^{(1)}} + \cdots + x^{(n-1)}I_{x^{(n-2)}} + x^{(n)}I_{x^{(n-1)}} = -F\left(t, x^{(n-1)}\right)I_{x^{(n-1)}} + x(n)I_{x^{(n-1)}} = I_{x^{(n-1)}}(x^{(n)} - F(t, x^{(n-1)})).$$

Hence $\mu(t, x^{(n-1)}) = I_{x^{(n-1)}}(t, x^{(n-1)})$. The vector field $v = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is a λ-symmetry of equation (3) if and only if $[v^{[\lambda,(n-1)]}, A] = \lambda v^{[\lambda,(n-1)]} + \tau A$ where $\tau = -(A + \lambda)(\xi(t,x))$ [3]. When $v = \partial_x$ is a λ-symmetry of equation 3) if and only if $\left[v^{[\lambda,(n-1)]}, A\right] = \lambda v^{[\lambda,(n-1)]}$.

Theorem 2.2. *If $v = \partial_x$ is a λ-symmetry of (3) for some function $\lambda(t, x^{(n-1)})$, then there is a first integral $I(t, x^{(n-1)})$ of (3) such that* $v^{[\lambda,(n-1)]}(I) = 0$

Proof. If $v = \partial_x$ is a λ-symmetry of (3) for some function $\lambda(t, x^{(n-1)})$, then $[v^{[\lambda,(n-1)]}, A] = \lambda v^{[\lambda,(n-1)]}$.

Therefore $\{v^{[\lambda,(n-1)]}, A\}$ is an involutive set of vector fields in $M^{(n-1)}$ and there is function $I(t, x^{(n-1)})$ such that $v^{[\lambda,(n-1)]}(I) = 0$ and $A(I) = 0$.

Let $\omega(t, x^{(n-1)})$ be a first integral of $v^{[\lambda,(n-1)]}$, i.e, $v^{[\lambda,(n-1)](\omega)} = 0$, then by using of (2), $\omega(t, x^{(n-1)})$ is a solution of PDE:

$$\omega_x + (D_t + \lambda)(1)\omega_{x^{(1)}} + \cdots + (D_t + \lambda)^{n-1}(1)\omega_{x^{(n-1)}} = 0 \tag{5}$$

Let $I\left(t, x^{(n-1)}\right) = G(t, \omega(t, x^{(n-1)}))$ be a first integral of (3), then

$$\begin{aligned} 0 = A(I) &= I_t + x^{(1)}I_x + x^{(2)}I_{x^{(1)}} + \cdots + F\left(t, x^{(n-1)}\right)I_{x^{(n-1)}} \\ &= (G_t + G_\omega\omega_t) + x^{(1)}(G_\omega\omega_x) + x^{(2)}\left(G_\omega\omega_{x^{(1)}}\right) + \cdots + F(t, x^{(n-1)})(G_\omega\omega_{x^{(n-1)}}) \\ &= G_t + \omega t + x(1)\omega x + x(2)\omega x(1) + \cdots + F(t, x^{(n-1)})\omega x(n-1))G\omega \\ &= Gt + A(\omega)G\omega = G_t + H(t,\omega)G_\omega \end{aligned}$$

where $A(\omega) = H(t,\omega)$. Hence, if $G(t,\omega)$ is a particular solution of $Gt + H(t,\omega)G_\omega = 0$ then $I\left(t, x^{(n-1)}\right) = G(t, \omega(t, x^{(n-1)}))$ is a first integral of (3). In summary, a procedure to find a first integral $I(t, x^{(n-1)})$ and consequently an integrating factor $\mu(t, x^{(n-1)})$ of (3), by using λ-symmetry method is as follows.

- The vector field $v = \partial_x$ is a λ-symmetry of (3), if function $\lambda(t, x^{(n-1)})$ is any particular solution of the equation (4).
- Find a first integral $\omega(t, x^{(n-1)})$, i.e. a particular solution of the equation (5).
- Evaluate $A(\omega) = H(t, \omega)$.
- Find a first integral $G(t,\omega)$ from the solution of the equation $G_t + H(t, \omega)G_\omega = 0$.
- The function $I\left(t, x^{(n-1)}\right) = G(t, \omega(t, x^{(n-1)}))$ is a first integral of (3).
- The function $\mu\left(t, x^{(n-1)}\right) = I_{x^{(n-1)}}\ (t, x^{(n-1)})$ is an integrating factor of (3).

We focus our attention on second order ODEs, $n = 2$ in equation (3), i.e.

$$\ddot{u} = F(x, u, \dot{u}) \tag{6}$$

where F is an analytic function of its arguments. A procedure to find a first integral $I(t, x, \dot{x})$ and consequently an integrating factor $\mu(t, x, \dot{x})$ of (3), by using λ-symmetry method is as follows.

- The vector field $v = \partial_x$ is a λ- symmetry of (6), if function $\lambda(t, x, \dot{x})$ is any particular solution of the equation

$$D_t(\lambda) + \lambda^2 = \frac{\partial F}{\partial x} + \lambda\frac{\partial F}{\partial x} \tag{7}$$

- Let v be a λ-symmetry of (6), then $\omega(t, x, \dot{x})$ is a first-order invariant of $v^{[\lambda,1]}$, that is, any particular solution of the equation

$$\omega_x + \lambda(t, x, \dot{x}).\omega\dot{x} = 0 \tag{8}$$

- Evaluate $A(\omega) = H(t, \omega)$.
- Find a first integral $G(t, \omega)$ from the solution of the equation $G_t + H(t, \omega)G_\omega = 0$.
- The function $I(t, x, \dot{x}) = G(t, \omega(t, x, \dot{x}))$ is a first integral of (6).
- The function $\mu(t, x, \dot{x}) = Ix^{\cdot}(t, x, \dot{x})$ is an integrating factor of (6).

REDUCTION OF $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^{x}$, BY λ-SYMMETRY METHOD

Let

$$\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^{x} \tag{9}$$

be a second-order ordinary differential equation, where $F(t,x,\dot{x}) = (f(t,x) + g(t,x)\dot{x})e^{x}$ is an analytic function on its arguments and $f(t,x)$ and $g(t,x)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry. There exists a function $\lambda(t,x,\dot{x})$ such that the vector field $v = \partial_x$ is a λ-symmetry of the equation (9). To determine such functions $\lambda(t,x,\dot{x})$, by (7), λ is any particular solution for the equation.

$$0 = D_t(\lambda) + \lambda^2 - \frac{\partial F}{\partial x} - \lambda\frac{\partial F}{\partial \dot{x}}$$
$$= \lambda_t + \dot{x}\lambda_x + \ddot{x}\lambda_{\dot{x}} + \lambda^2 - (f_x + g_x\dot{x})e^{x} - (f + g\dot{x})e^{x} - \lambda g e^{x}$$

or corresponding to

$$\lambda_t + \dot{x}\lambda_x + (f + g\dot{x})e^{x}\lambda_{\dot{x}} + \lambda^2 - (f_x + g_x\dot{x})e^{x} - (f + g\dot{x})e^{x} - \lambda g e^{x} = 0 \tag{10}$$

For the sake of simplicity, we try to find a solution λ (10) of the form $\lambda(t,x,\dot{x}) = \lambda_1(t,x)\dot{x} + \lambda_2(t,x)$, we obtain the following system:

$$\lambda_1^2 + (\lambda_1)_x = 0$$
$$(\lambda_1)_t + (\lambda_2)_x + 2\lambda_1\lambda_2 - g_x e^{x} - g e^{x} = 0,$$
$$(\lambda_2)_t + f e^{x}\lambda_1 + \lambda_2^2 - f_x e^{x} - f e^{x} - \lambda_2 g e^{x} = 0,$$

A particular solution of the first equation is given by $\lambda_1 = 0$. The second and third equations become

$$(\lambda_2)_x - g_x e^{x} - g e^{x} = 0$$

$$(\lambda_2)_t + \lambda_2^2 - f_x e^{x} - f e^{x} - \lambda_2 g e^{x} = 0$$

For the first equation and the second equation, we have

$$g = (\lambda_2 + 1)e^{-x}, \text{ and } f = \left(\int ((\lambda_2)_t - \lambda_2)dx\right)e^{-x}$$

A particular solution of this system is $\lambda_2 = ge^{x} - 1$, where $g_t - g + e^{-x} = f_x + f$. Hence,

$$\lambda(t,x,\dot{x}) = \lambda_1(t,x)\dot{x} + \lambda_2(t,x) = \lambda_2(t,x) = g(t,x)e^{x} - 1.$$

Therefore, the vector field $v = \partial_x$ is a λ-symmetry of (9) for

$$\lambda(t,x,\dot{x}) = g(t,x)e^{x} - 1 \tag{11}$$

To find an integrating factor associated to λ, first, we find a first integral invariant $\omega(t,x,\dot{x})$ of $v^{[\lambda,1]}$ by the equation that corresponds to (8), which means,

$$\omega_x + (ge^{x} - 1)\omega_{\dot{x}} = 0 \tag{12}$$

For the sake of simplicity, we try to find a solution ω of the form $\omega(t,x,\dot{x}) = \omega_1(t,x)\dot{x} + \omega_2(t,x)$, we have

$$(\omega_1)_x\dot{x} + (\omega_2)_x + (ge^{x} - 1)\omega_1 = 0$$

or corresponding

$$(\omega_1)_x = 0, \qquad (\omega_2)_x + (ge^{x} - 1)\omega_1 = 0$$

A particular solution of the first equation is given by $\omega_1 = 1$. the second equation become $(\omega_2)_x + (ge^{x} - 1) = 0$, the solution of this equation is $\omega_2 = -\int ge^{x}dx + x$. Hence,

$$\omega(t,x,\dot{x}) = \omega_1(t,x)\dot{x} + \omega_2(t,x)\dot{x} - \int g(t,x)e^{x}dx + x \tag{13}$$

is a particular solution for (12). The vector field associated $A = \partial_t + \dot{x}\partial_x + F(t,x,\dot{x})\partial\dot{x}$ acts on ω, then, we have

$$A(\omega) = -\int e^{x}g_t dx + \dot{x} + fe^{x}$$
$$= -\int e^{x}g_t dx + \dot{x} + \left(\int ((\lambda_2)_t - \lambda_2)\,dx\right)$$
$$= -\int e^{x}g_t dx + \dot{x} + \int (g_t e^{x} - ge^{x} + 1)\,dx$$
$$= \dot{x} - \int ge^{x}dx + x = \omega = H(t,\omega)$$

Therefore, $A(\omega) = \omega = H(t,\omega)$. The function

$$G(t,\omega) = \omega e^{-t} \tag{14}$$

is a particular solution for the equation $G_t + \omega G_\omega = 0$. Therefore,

$$I(t,x,\dot{x}) = G(t,\omega(t,x,\dot{x})) = (\dot{x} + x - \int g(t,x)e^{x}dx)e^{-t}$$

is a first integral of (9) also the function

$$\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = e^{-t} \qquad (15)$$

is an integrating factor of (9). Also,

$$D_t\big(G(t,\omega(t,x,\dot{x})\big) = D_t\big(\dot{x}+x-\int g(t,x)e^x dx\big)e^{-t}\big) = 0$$

is a conserved form of (9).

Summation. λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and con sequently an integrating factor $\mu(t,x,\dot{x})$ of (9) is as follows.

- The vector field $v=\partial_x$ is a λ- symmetry of (9), and function $\lambda(t,x,\dot{x}) = g(t,x)e^x - 1$ is a particular solution of the equation $D_t(\lambda)+\lambda^2 = \partial F/\partial x + \lambda\,\partial F/\partial\dot{x}$.
- Let v be a λ-symmetry of (9), then $\omega(t,x,\dot{x}) = \dot{x} + x - \int g(t,x)e^x dx$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (g(t,x)e^x - 1)\omega x^{\cdot} = 0$.
- We have $A(\omega) = H(t,\omega) = \omega$.
- The function $G(t,\omega) = \omega e^{-t}$ is a particular solution for the equation $G_t + \omega G_\omega = 0$.
- The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = (\dot{x}+x-\int g(t,x)e^x dx)e^{-t}$ is a first integral of (9). Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t\left((\dot{x}+x-\int g(t,x)e^x dx)e^{-t}\right) = 0$, is a conserved form of (9).
- The function $\mu(t,x,\dot{x}) = I\dot{x}(t,x,\dot{x}) = e^{-t}$, is an integrating factor of (9).

Corollary 3.1. *Equality* $D_t\left((\dot{x}+x-\int g(t,x)e^x dx)e^{-t}\right) = 0$ *is a conserved form of (9), therefore reduce the order of equation* $\ddot{x} = (f(t,x)+g(t,x)\dot{x})e^x$ *is the equation* $\dot{x}+x-\int g(t,x)e^x dx = 0$.

SPECIAL CASES OF THE EQUATION $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$

Special cases of the equation $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$ are $\ddot{x} = (f(x)+g(x)\dot{x})e^x$. We consider the second-order ODE

$$\ddot{x} = (f(t)+g(t)\dot{x})e^x \qquad (16)$$

where $F(t,x,\dot{x}) = (f(t)+g(t)\dot{x})e^x$ is an analytic function on its arguments and $f(t)$ and $g(t)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry.
Similar of the equation (9), λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and consequently an integrating factor $\mu(t,x,\dot{x})$ of (16) is as follows: The vector field $v=\partial_x$ is a λ-symmetry of (16), and function $\lambda(t,x,x^{\cdot}) = 1/t + g(t)e^x$ is a particular solution of the equation $D_t(\lambda)+\lambda^2 = \partial F/\partial x + \lambda\,\partial F/\partial x^{\cdot}$.
Let v be a λ-symmetry of (16), then $\omega(t,x,\dot{x}) = \dot{x} - g(t)e^x - x/t$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (1/t + g(t)e^x)\omega\dot{x} = 0$. We have $A(\omega) = H(t,\omega) = -(1/t)\omega$.
The function $G(t,\omega) = t\omega$ is a particular solution for the equation $G_t - (1/t)\,\omega\, G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = t\dot{x} - tg(t)e^x - x$, is a first integral of (16).
Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t(t\dot{x} - tg(t)e^x - x) = 0$, is a conserved form of (16). The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = t$ is an integrating factor of (16).

Corollary 4.1. *Equality* $D_t(t\dot{x} - tg(t)e^x - x) = 0$, *is a conserved form of (4.1), therefore reduce the order of the equation* $\ddot{x} = (f(t)+g(t)\dot{x})e^x$, *is the equation* $t\dot{x} - tg(t)ex - x = 0$.

We consider the second-order ODE

$$\ddot{x} = (f(t)+g(t)\dot{x})e^x \qquad (17)$$

where $F(t,x,\dot{x}) = (f(x)+g(x)\,\dot{x})e^x$ is an analytic function on its arguments and $f(x)$ and $g(x)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry.
Similar of the equation (9), λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and consequently an integrating factor $\mu(t,x,\dot{x})$ of (17) is as follows: The vector field $v=\partial_x$ is a λ-symmetry of (17), and function $\lambda(t,x,\dot{x}) = g(x)e^x - 1$ is a particular solution of the equation $D_t(\lambda)+\lambda^2 = \partial F/\partial x + \lambda\partial F/\partial\dot{x}$,
Let v be a λ-symmetry of (17), then $\omega(t,x,\dot{x}) = \dot{x} + x - \int g(x)e^x dx$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (g(x)e^x - 1)\omega_{\dot{x}} = 0$. We have $A(\omega) = H(t,\omega) = \omega$.
The function $G(t,\omega) = \omega e^{-t}$, is a particular solution for the equation $G_t + \omega G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = (\dot{x}+x-\int g(x)e^x dx)e^{-t}$, is a first integral of (4.2). Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t\left((\dot{x}+x-\int g(x)e^x dx)e^{-t}\right) = 0$, is a conserved form of (17). The function $\mu(t,x,\dot{x}) = I\,\dot{x}(t,x,\dot{x}) = e^{-t}$, is an integrating factor of (17).

Corollary 4.2. *Equality* $D_t\big(\dot{x}+x-\int g(x)e^x dx\big) = 0$, *is a conserved form of (4.2), therefore reduce the order of the equation* $\ddot{x} = (f(x)+g(x)\dot{x})e^x$, *is the equation* $\dot{x}+x-\int g(x)e^x dx = 0$.

SOME ILLUSTRATIONS

Example 1. We consider the second-order ordinary differential equation

$$\ddot{x} = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x} \quad (18)$$

where in the Eq. (9), $f(t,x) = (t^3 - 1)\, cos\, xe^{-x}$ and $g(t,x) = (t\, sin\, x + 1)e^{-x}$ and also the function $F(t,x,\dot{x}) = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x}$, is an analytic function on its arguments. It can be checked that this equation does not have Lie point symmetry. Therefore, we have for the equation (18).

The vector field $v = \partial_x$ is a λ-symmetry of (18), and function $\lambda(t,x,\dot{x}) = g(t,x)e^x - 1 = tsin\, x$, is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x + \lambda \partial F/\partial\, \dot{x}$.

Let v be a λ-symmetry of (5.1), then $\omega(t,x,\ \dot{x}) = \dot{x} + x - \int g(t,x)e^x dx = \dot{x} + t\, cos\, x$, is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation ω_x+($t$sin x)$\omega_{\dot{x}}$ = 0. We have $A(\omega) = H(t,\omega) = \omega$. The function $G(t,\ \omega) = \omega e^{-t}$, is a particular solution for the equation $G_t + \omega G_\omega = 0$.

The function

$$I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) =$$

$$(\dot{x} + x - \int g(t,x)e^x dx)e - t\ = (\dot{x} + t\, cos\, x)e^{-t},$$

is a first integral of (5.1). $D_t(G\big(t,\omega(t,x,\dot{x})\big) = Dt((\dot{x} + t\, cos\, x)e^{-t}) = 0$, is a conserved form of (18). The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = e^{-t}$, is an integrating factor of (18). Therefore, we reduce the order of the equation $\ddot{x} = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x}$, to the equation $(\dot{x} + t\, cos\, x)e^{-t} = 0$. This equation does not have Lie point symmetries.

Example 2. Let

$$\ddot{x} = (sinh\, t + cosh\, t/t + cosh\, t\dot{x})e^x \quad (19)$$

where in the Eq. (19), $f(t) = sin\, t + cosh\, t/t$ and $g(t) = cosh\, t$ and also the function $F(t,x,\dot{x}) = (sin\, t + cosht/t + cosh\, t\dot{x})e^x$, is an analytic function on its arguments.

This equation does not have Lie point symmetry. We have for the equation (19).

The vector field $v = \partial_x$ is a λ-symmetry of (19), and function $\lambda(t,x,\dot{x}) = 1/t + g(t)e^x = 1/t + cosh(t)e^x$, is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x\ +\ \lambda \partial F/\partial \dot{x}$.

Let v be a λ-symmetry of (19), then $\omega(t,x,\dot{x}) = \dot{x} - g(t)e^x - x/t = \dot{x} - cosh\, te^x - x/t$, is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (1/t + cosh\, te^x)\omega_{\dot{x}} = 0$. We have $A(\omega) = H(t,\omega) = -(1/t)\omega$.

The function $G(t,\omega) = t\omega$, is a particular solution for the equation $G_t - (1/t)\omega G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = t\dot{x} - tg(t)e^x - x = t\dot{x} - t\, cosh\, te^x - x$, is a first integral of (19). Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t(t\dot{x} - t\, cosh\, te^x - x) = 0$, is a conserved form of (19).

The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = t$, is an integrating factor of (19).

Therefore, we reduce the order of the equation $\ddot{x} = (sinh\, t + cosht/t + cosh\, t\dot{x})e^x$, to the equation $t\dot{x} - t\, cosh\, te^x - x = 0$. This equation dose not have Lie point symmetries.

ACKNOWLEDGMENT

In this paper, we calculated an integrating factor, first integral and reduce the order the non-Linear second-order ODEs $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, through λ-symmetry method. Moreover, we computed an integrating factor, first integral and reduce the order for particular cases of this equation that are $\ddot{x} = (f(t) + g(t)\dot{x})e^x$ and $\ddot{x} = (f(x) + g(x)\, \dot{x})e^x$.

REFERENCES

[1] B. J. I. j. o. a. m. Abraham-Shrauner, "Hidden symmetries and nonlocal group generators for ordinary differential equations," vol. 56, no. 3, pp. 235-252, 1996.

[2] C. Muriel, J. J. J. o. P. A. M. Romero, and Theoretical, "First integrals, integrating factors and λ-symmetries of second-order differential equations," vol. 42, no. 36, p. 365207, 2009.

[3] C. Muriel and J. J. I. J. o. A. m. Romero, "New methods of reduction for ordinary differential equations," vol. 66, no. 2, pp. 111-125, 2001.

[4] C. Muriel and J. J. J. L. t. Romero, "C∞-symmetries and reduction of equations without Lie point symmetries," vol. 13, no. 1, pp. 167-188, 2003.

[5] G. Gaeta, P. J. J. o. P. A. M. Morando, and General, "On the geometry of lambda-symmetries and PDE reduction," vol. 37, no. 27, p. 6955, 2004.

[6] C. Muriel and J. J. J. N.-l. M. P. Romero, "λ-symmetries and integrating factors," vol. 15, pp. 290-299, 2008.

[7] C. Muriel and J. J. J. o. N. M. P. Romero, "Second-order ordinary differential equations and

first integrals of the form A (t, x) ẋ+ B (t, x)," vol. 16, no. supp01, pp. 209-222, 2009.

[8] C. Muriel, J. J. J. o. P. A. M. Romero, and Theoretical, "A λ-symmetry-based method for the linearization and determination of first integrals of a family of second-order ordinary differential equations," vol. 44, no. 24, p. 245201, 2011.

[9] P. J. Olver, "Symmetry groups of differential equations," in *Applications of Lie Groups to Differential Equations*: Springer, 1986, pp. 77-185.

[10] E. J. M. P. i. E. Yaşar, "Integrating factors and first integrals for Liénard type and frequency-damped oscillators," vol. 2011, 2011.

23 On invariants of second-order ordinary differential equations $y'' = f(x, y, y')$ via point transformations. By Ahmad Y. Al-Dweik (2014)

On invariants of second-order ordinary differential equations $y'' = f(x, y, y')$ via point transformations

Ahmad Y. Al-Dweik

Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

aydweik@kfupm.edu.sa

Abstract

Bagderina [1] solved the equivalence problem for a family of scalar second-order ordinary differential equations (ODEs), with cubic nonlinearity in the first-order derivative, via point transformations. However, the question is open for the general class $y'' = f(x, y, y')$ which is not cubic in the first-order derivative. We utilize Lie's infinitesimal method to study the differential invariants of this general class under an arbitrary point equivalence transformations. All fifth order differential invariants and the invariant differentiation operators are determined. As an application, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants are non-zero is provided.

Keywords: Lie's infinitesimal method, differential invariants, second order ODEs, equivalence problem, point transformations, canonical forms, Lie symmetries.

1 Introduction

Lie's group classification of ODEs shows that the second-order equations can possess one, two, three or eight infinitesimal symmetries. According to Lie's classification [2] in the complex domain, any second order ODE

$$y'' = f(x, y, y'), \tag{1.1}$$

is obtained by a change of variables from one of eight canonical forms. The equations with eight symmetries and only these equations can be linearized by a change of variables. The initial seminal studies of scalar second-order ODEs which are linearizable by means of point transformations are due to Lie [2] and Tressé [3]. They showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. Mahomed and Leach [4] proved that Lie linearization conditions are equivalent to the vanishing of the Tressé relative invariants (1.2) as stated in the next theroem

Theorem 1.1. *[5] The equation $y'' = f(x, y, y')$ is equivalent to the normal form $y'' = 0$ with eight symmetries under **point transformations** if and only if the Tressé relative invariants*

$$\begin{aligned} I_1 &= f_{y',y',y',y'} \\ I_2 &= \dot{D}_x^2 f_{y',y'} - 4\dot{D}_x f_{y,y'} - 3 f_y f_{y',y'} + 6 f_{y,y} + f_{y'} \left(4 f_{y,y'} - \dot{D}_x f_{y',y'}\right) \end{aligned} \tag{1.2}$$

both vanish identically. Where $\dot{D}_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + f\frac{\partial}{\partial y'}$.

Regarding the equivalence of the second-order differential equations to the normal form $y'' = 0$ via contact transformations, it is well known that all second-order differential equations have $y'' = 0$ as the sole equivalence class.

The linearization problem is a particular case of the equivalence problem. For the general theory of the equivalence problem including algorithms of construction of differential invariants, the interested reader is referred to [6, 9]. Ibragimov [10, 12] developed a simple method for constructing invariants of families of linear and nonlinear differential equations admitting infinite equivalence transformation groups. Lie's infinitesimal method was applied to solve the equivalence problem for several linear and nonlinear differential equations [13, 14, 15, 16, 17, 18, 19, 20, 21]. Cartan's equivalence method [6, 22] is another systematic approach to solve the equivalence problem for differential equations.

By using Lie's infinitesimal method, Bagderina [1] solved the equivalence problem of second-order ODEs which are at most cubic in the first-order derivative ($I_1 = 0$)

$$y'' = a(x,y)y'^3 + b(x,y)y'^2 + c(x,y)y' + d(x,y) \tag{1.3}$$

with respect to the group of point equivalence transformations

$$\bar{x} = \phi(x,y), \bar{y} = \psi(x,y). \tag{1.4}$$

As an extension, in this paper, we use Lie's infinitesimal method to study the differential invariants of the second-order ODEs (1.1) which are not cubic in the first-order derivative ($I_1 \neq 0$) with respect to the group of point equivalence transformations. The motivation of this study is finding invariant description of the canonical forms for second-order ODEs in the complex plane [6] which are not cubic in the first-order derivative.

Invariant description of the canonical forms for second-order ODEs in the complex plane with three infinitesimal symmetries was given in [8, 15] where they presented the candidates for all four types and then they studied these candidates. In this paper, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x,y,y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided.

The structure of the paper is as follows. In the next section, we give a short description of Lie's infinitesimal method to find the differential invariants and invariant differentiation operators of the class of ODEs (1.1) with respect to the group of point equivalence transformations (1.4). In Section 3, using the methods described in Section 2, the infinitesimal point equivalence transformations are recovered. In Section 4, we find the fifth-order differential invariants and invariant differentiation operators of the class of ODEs (1.1), which are not cubic in the first-order derivative, under point equivalence transformations. In Section 5, invariant description of all the canonical forms in the complex plane [6] for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided. Finally, the conclusion is presented.

Throughout this paper, we use the notation $A = [a_1, a_2, ..., a_n]$ to express any differential operator $A = \sum_{j=1}^{n} a_j \frac{\partial}{\partial b_j}$. Also, we denote y' by p.

2 Lie's infinitesimal method

In this section, we briefly describe the Lie method used to derive differential invariants using point equivalence transformations.

Consider the kth-order system of PDEs of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E_\alpha(x, u, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m \ , \tag{2.5}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$,..., respectively, in which the total differentiation operator with respect to x^i is

$$D_i = \tfrac{\partial}{\partial x^i} + u_i^\alpha \tfrac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \tfrac{\partial}{\partial u_j^\alpha} + ..., \quad i = 1, ..., n \ , \tag{2.6}$$

with the summation convention adopted for repeated indices.

Definition 2.1. *The Lie-Bäcklund operator* is

$$X = \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A \ , \tag{2.7}$$

where A is the space of *differential functions.*

The operator (2.7) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \tfrac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \\ &= \xi^i D_i + W^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \tfrac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \end{aligned} \tag{2.8}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} &\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j) = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ &\zeta^\alpha_{i_1 \ldots i_s} = D_{i_s}(\zeta^\alpha_{i_1 \ldots i_{s-1}}) - u^\alpha_{j i_1 \ldots i_{s-1}} D_{i_s}(\xi^j) = D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{2.9}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.10}$$

Definition 2.2. *The point equivalence transformation of a class of PDEs (2.5)* is an invertible transformation of the independent and dependent variables of the form

$$\bar{x} = \phi(x, u), \bar{u} = \psi(x, u), \tag{2.11}$$

that maps every equation of the class into an equation of the same family, viz.

$$E_\alpha(\bar{x}, \bar{u}, \ldots, \bar{u}_{(k)}) = 0, \quad \alpha = 1, \ldots, m. \tag{2.12}$$

In order to describe Lie's infinitesimal method for deriving differential invariants using point equivalence transformations, we use as example the class of equations (1.1). It is well-known that the point equivalence transformation

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y), \tag{2.13}$$

maps (1.1) into the same family, viz.

$$\bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}'), \tag{2.14}$$

for arbitrary functions $\phi(x, y)$ and $\psi(x, y)$, where $\bar{f}$, in general, can be different from the original function f. The set of all equivalence transformations forms a group denoted by $\mathcal{E}$.

The standard procedure for Lie's infinitesimal invariance criterion [9] is implemented in the next section to recover the continuous group of point equivalence transformations (2.13) for the class of second-order ODEs (1.1) with the corresponding infinitesimal point equivalence transformation operator

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + \mu(x, y, p, f)\partial_f, \tag{2.15}$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary functions obtained from

$$\bar{x} = x + \epsilon\ \xi(x, y) + O(\epsilon^2) = \phi(x, y), \tag{2.16}$$

$$\bar{y} = y + \epsilon\ \eta(x, y) + O(\epsilon^2) = \psi(x, y), \tag{2.17}$$

and

$$\mu = \dot{D}_x^2(W) + \xi(x, y)\dot{D}_x f, \tag{2.18}$$

with $W = \eta - \xi p$ and $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + f\frac{\partial}{\partial p}$.

Definition 2.3. *An invariant of a class of second-order ODEs (1.1)* is a function of the form

$$J = J(x, y, p, f), \tag{2.19}$$

which is invariant under the equivalence transformation (2.13).

Definition 2.4. *A differential invariant of order s of a class of second-order ODEs (1.1)* is a function of the form

$$J = J(x, y, p, f, f_{(1)}, f_{(2)}, ..., f_{(s)}), \tag{2.20}$$

which is invariant under the equivalence transformation (2.13) where $f_{(1)}, f_{(2)}, ..., f_{(s)}$ denote the collections of all first, second,..., sth-order partial derivatives.

Definition 2.5. *An invariant system of order s of a class of second-order ODEs (1.1)* is the system of the form $E_\alpha(x, y, p, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0,\ \alpha = 1, ..., m$ which satisfies the condition

$$Y^{(s)} E_\alpha(x, y, p, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0\ (mod\ E_\alpha = 0,\ \alpha = 1, ..., m),\ \alpha = 1, ..., m. \tag{2.21}$$

An invariant system with $\alpha = 1$ is called an invariant equation.

Now, according to the theory of invariants of infinite transformation groups [9], the invariant criterion

$$YJ(x, y, p, f) = 0, \tag{2.22}$$

should be split by means of the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of PDEs whose solution gives the required invariants.

It should be noted that since the generator Y contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.22) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in Y. We point out that these m PDEs are not necessarily linearly independent.

In order to determine the differential invariants of order s, we need to calculate the prolongations of the operator Y using (2.8) by considering f as a dependent variable and the variables x, y, p as independent variables:

$$\begin{aligned} Y^{(s)} &= Y(x)\tilde{D}_x + Y(y)\tilde{D}_y + Y(p)\tilde{D}_p + \tilde{W}\tfrac{\partial}{\partial f} + \sum_{s\geq 1} \tilde{D}_{i_1}...\tilde{D}_{i_s}(\tilde{W})\tfrac{\partial}{\partial f_{i_1 i_2 \dots i_s}}, \\ & i_1, i_2, ..., i_s \in \{x, y, p\}, \end{aligned} \tag{2.23}$$

where

$$\tilde{D}_k = \partial_k + f_k\partial_f + f_{ki}\partial_{f_i} + f_{kij}\partial_{f_{ij}} + ..., \quad i, j, k \in \{x, y, p\}. \tag{2.24}$$

in which $\tilde{W}$ is the *Lie characteristic function*

$$\tilde{W} = \mu - Y(x)f_x - Y(y)f_y - Y(p)f_p. \tag{2.25}$$

The differential invariants are determined by the equations

$$Y^{(s)}J(x, y, p, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0. \tag{2.26}$$

It should be noted that since the generator $Y^{(s)}$ contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.26) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in $Y^{(s)}$.

For simplicity, from here on, we denote the derivative of $f(x, y, p)$ with respect to the independent variables x, y, p as f_1, f_2, f_3. The same notation will be used for higher-order derivatives.

Now, in order to find all the fifth order differential invariants of the third-order ODE (1.1), one can solve the invariant criterion (2.26) with $s = 5$. However, for compactness of the derived differential invariants, one can replace any partial derivative with respect to x by the total derivative with respect to x. So, we need to solve the following invariant criterion

$$\begin{aligned} &Y^{(5)}J(x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3},\\ &f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10},\\ &d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4},\\ &d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}) = 0 \end{aligned} \tag{2.27}$$

by prolonging the infinitesimal operator $Y^{(5)}$ to the variables $d_{i,j}$ through the infinitesimals

$Y^{(5)}d_{i,j}$, where

$$d_{1,1}=\dot{D}_xf, d_{1,2}=\dot{D}_xf_2, d_{1,3}=\dot{D}_xf_3, d_{1,4}=\dot{D}_xf_{2,2}, d_{1,5}=\dot{D}_xf_{2,3}, d_{1,6}=\dot{D}_xf_{3,3}, d_{1,7}=\dot{D}_xf_{2,2,2},$$
$$d_{1,8}=\dot{D}_xf_{2,2,3}, d_{1,9}=\dot{D}_xf_{2,3,3}, d_{1,10}=\dot{D}_xf_{3,3,3}, d_{1,11}=\dot{D}_xf_{2,2,2,2}, d_{1,12}=\dot{D}_xf_{2,2,2,3}, d_{1,13}=\dot{D}_xf_{2,2,3,3},$$
$$d_{1,14}=\dot{D}_xf_{2,3,3,3}, d_{1,15}=\dot{D}_xf_{3,3,3,3}, d_{2,1}=\dot{D}_x^2f, d_{2,2}=\dot{D}_x^2f_2, d_{2,3}=\dot{D}_x^2f_3, d_{2,4}=\dot{D}_x^2f_{2,2},$$
$$d_{2,5}=\dot{D}_x^2f_{2,3}, d_{2,6}=\dot{D}_x^2f_{3,3}, d_{2,7}=\dot{D}_x^2f_{2,2,2}, d_{2,8}=\dot{D}_x^2f_{2,2,3}, d_{2,9}=\dot{D}_x^2f_{2,3,3}, d_{2,10}=\dot{D}_x^2f_{3,3,3},$$
$$d_{3,1}=\dot{D}_x^3f, d_{3,2}=\dot{D}_x^3f_2, d_{3,3}=\dot{D}_x^3f_3, d_{3,4}=\dot{D}_x^3f_{2,2}, d_{3,5}=\dot{D}_x^3f_{2,3}, d_{3,6}=\dot{D}_x^3f_{3,3},$$
$$d_{4,1}=\dot{D}_x^4f, d_{4,2}=\dot{D}_x^4f_2, d_{4,3}=\dot{D}_x^4f_3, d_{5,1}=\dot{D}_x^5f \tag{2.28}$$

Definition 2.6. *An invariant differentiation operator of a class of second-order ODEs (1.1)* is a differential operator $\mathcal{D}$ which satisfies that if I is a differential invariant of ODE (1.1), then $\mathcal{D}I$ is its differential invariant too.

As it is shown in [9], the number of independent invariant differentiation operators $\mathcal{D}$ equals the number of independent variables x, y and p. The invariant differentiation operators $\mathcal{D}$ should take the form

$$\mathcal{D}=K\tilde{D}_x+L\tilde{D}_y+M\tilde{D}_p, \tag{2.29}$$

with the coordinates K, L and M satisfying the non-homogeneous linear system

$$Y^{(5)}K=\mathcal{D}(Y(x)), \quad Y^{(5)}L=\mathcal{D}(Y(y)), \quad Y^{(5)}M=\mathcal{D}(Y(p)), \tag{2.30}$$

where K, L and M are functions of the following variables

$$x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3},$$
$$f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10},$$
$$d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4},$$
$$d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1} \tag{2.31}$$

In reality, the general solution of the system (2.30) gives both the differential invariants and the differential operators. This general solution can be found by prolonging the infinitesimal operator $Y^{(5)}$ to the variables K, L and M through the infinitesimals $Y^{(5)}K, Y^{(5)}L$ and $Y^{(5)}M$ respectively. Then solving the invariant criterion

$$Y^{(5)}J(x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3}, f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}, K, L, M) = 0 \tag{2.32}$$

gives the implicit solution of the variables K, L and M with the differential invariants.

In this paper, we are interested in finding the fifth order differential invariants and differential operators of the general class $y'' = f(x, y, y')$ under point transformations (2.13). So, according to the theory of invariants of infinite transformation groups [9], the invariant criterion (2.32) should be split by the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of partial differential equations (PDEs):

$$X_iJ = 0,\ T_iJ = 0,\ i = 1 \dots 36, \tag{2.33}$$

where $X_i, i = 1 \dots 36$, are the differential operators corresponding to the coefficients of the following derivatives of $\eta(x, y)$ up to the seven order in the invariant criterion

$$\eta, \eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,1,1}, \eta_{1,1,2}, \eta_{1,2,2}, \eta_{2,2,2}, \eta_{1,1,1,1}, \eta_{1,1,1,2}, \eta_{1,1,2,2}, \eta_{1,2,2,2}, \eta_{2,2,2,2}, \eta_{1,1,1,1,1}, \eta_{1,1,1,1,2}, \eta_{1,1,1,2,2}, \eta_{1,1,2,2,2}, \eta_{1,2,2,2,2}, \eta_{2,2,2,2,2}, \eta_{1,1,1,1,1,1}, \eta_{1,1,1,1,1,2}, \eta_{1,1,1,1,2,2}, \eta_{1,1,1,2,2,2}, \eta_{1,1,2,2,2,2}, \eta_{1,2,2,2,2,2}, \eta_{2,2,2,2,2,2}, \eta_{1,1,1,1,1,1,1}, \eta_{1,1,1,1,1,1,2}, \eta_{1,1,1,1,1,2,2}, \eta_{1,1,1,1,2,2,2}, \eta_{1,1,1,2,2,2,2}, \eta_{1,1,2,2,2,2,2}, \eta_{1,2,2,2,2,2,2}, \eta_{2,2,2,2,2,2,2} \tag{2.34}$$

and $T_i, i = 1 \dots 36$, are the differential operators corresponding to the coefficients of the following derivatives of $\xi(x, y)$ up to the seven order in the invariant criterion

$$\xi, \xi_1, \xi_2, \xi_{1,1}, \xi_{1,2}, \xi_{2,2}, \xi_{1,1,1}, \xi_{1,1,2}, \xi_{1,2,2}, \xi_{2,2,2}, \xi_{1,1,1,1}, \xi_{1,1,1,2}, \xi_{1,1,2,2}, \xi_{1,2,2,2}, \xi_{2,2,2,2}, \xi_{1,1,1,1,1}, \xi_{1,1,1,1,2}, \xi_{1,1,1,2,2}, \xi_{1,1,2,2,2}, \xi_{1,2,2,2,2}, \xi_{2,2,2,2,2}, \xi_{1,1,1,1,1,1}, \xi_{1,1,1,1,1,2}, \xi_{1,1,1,1,2,2}, \xi_{1,1,1,2,2,2}, \xi_{1,1,2,2,2,2}, \xi_{1,2,2,2,2,2}, \xi_{2,2,2,2,2,2}, \xi_{1,1,1,1,1,1,1}, \xi_{1,1,1,1,1,1,2}, \xi_{1,1,1,1,1,2,2}, \xi_{1,1,1,1,2,2,2}, \xi_{1,1,1,2,2,2,2}, \xi_{1,1,2,2,2,2,2}, \xi_{1,2,2,2,2,2,2}, \xi_{2,2,2,2,2,2,2} \tag{2.35}$$

Functionally independent solutions of the system (2.33) provide all independent differential invariants of $y'' = f(x, y, y')$ up to the fifth order under point transformations, as well as an implicit solution of the variables K, L and M which provide the differential operators via (2.29). The solution of system (2.33) is found in many steps using Maple as follows:

First, let us consider the subsystem induced by the sixth and seventh derivatives of ξ and η

$$X_iJ = 0,\ T_iJ = 0,\ i = 22\ldots36. \tag{2.36}$$

where the operators X_i and $T_i,\ i = 22\ldots36$ are given in Appendix A in term of the variables $z_i, i = 1\ldots62$ after relabeling the variables

$$x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3},$$
$$f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10},$$
$$d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4},$$
$$d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}, K, L, M \tag{2.37}$$

by the variables $z_i, i = 1\ldots62$, respectively.

In 62-dimensional space of variables $z_i, i = 1\ldots62$, the rank of the system (2.36) is 16, so it has 46 functionally independent solutions which are given as:

$$l_1 = z_1, l_2 = z_2, l_3 = z_3, l_4 = z_4, l_5 = z_5, l_6 = z_6, l_7 = z_7, l_8 = z_8, l_9 = z_9, l_{10} = z_{10}, l_{11} = z_{11}, l_{12} = z_{12}, l_{13} = z_{13},$$
$$l_{14} = z_{15}, l_{15} = z_{16}, l_{16} = z_{17}, l_{17} = z_{18}, l_{18} = z_{21}, l_{19} = z_{22}, l_{20} = z_{23}, l_{21} = z_{24}, l_{22} = z_{25}, l_{23} = z_{26}, l_{24} = z_{27},$$
$$l_{25} = z_{28}, l_{26} = z_{29}, l_{27} = z_{30}, l_{28} = z_{32}, l_{29} = z_{33}, l_{30} = z_{34}, l_{31} = z_{37}, l_{32} = z_{38}, l_{33} = z_{39}, l_{34} = z_{40}, l_{35} = z_{41},$$
$$l_{36} = z_{42}, l_{37} = z_{44}, l_{38} = z_{45}, l_{39} = z_{48}, l_{40} = z_{49}, l_{41} = z_{50}, l_{42} = z_{52}, l_{43} = z_{55}, l_{44} = z_{60}, l_{45} = z_{61}, l_{46} = z_{62} \tag{2.38}$$

Second, let us consider the subsystem induced by the fifth derivatives of ξ and η

$$X_iJ = 0,\ T_iJ = 0,\ i = 16\ldots21. \tag{2.39}$$

where the inherited operators X_i and $T_i,\ i = 16\ldots21$ are given in Appendix B in term of the new variables $l_i, i = 1\ldots46$.

In 46-dimensional space of variables $l_i, i = 1\ldots46$, the rank of the system (2.39) is 10, so it has 36 functionally independent solutions which are given as:

$$m_1 = l_1, m_2 = l_2, m_3 = l_3, m_4 = l_4, m_5 = l_5, m_6 = l_6, m_7 = l_7, m_8 = l_8, m_9 = l_9,$$
$$m_{10} = l_{11}, m_{11} = l_{12}, m_{12} = l_{13}, m_{13} = l_{15}, m_{14} = l_{16}, m_{15} = l_{17}, m_{16} = l_{19}, m_{17} = l_{20},$$
$$m_{18} = l_{21}, m_{19} = l_{22}, m_{20} = l_{23}, m_{21} = l_{24}, m_{22} = l_{26}, m_{23} = l_{27}, m_{24} = l_{29}, m_{25} = l_{30},$$
$$m_{26} = l_{32}, m_{27} = l_{33}, m_{28} = l_{34}, m_{29} = l_{36}, m_{30} = l_{38}, m_{31} = -4\, l_{28} + 6\, l_{10} + l_{39}, m_{32} = l_{40},$$
$$m_{33} = -4\, l_{37} + 6\, l_{25} + l_{43}, m_{34} = l_{44}, m_{35} = l_{45}, m_{36} = l_{46} \tag{2.40}$$

Third, let us consider the subsystem induced by the fourth derivatives of ξ and η

$$X_iJ = 0,\ T_iJ = 0,\ i = 11 \ldots 15. \tag{2.41}$$

where the inherited operators X_i and T_i, $i = 11 \ldots 15$ are given in Appendix C in term of the new variables $m_i, i = 1 \ldots 36$.

In 36-dimensional space of variables $m_i, i = 1 \ldots 36$, the rank of the system (2.41) is 10, so it has 26 functionally independent solutions which are given as:

$$n_1 = m_1, n_2 = m_2, n_3 = m_3, n_4 = m_4, n_5 = m_5, n_6 = m_6, n_7 = m_8, n_8 = m_9, n_9 = m_{11},$$
$$n_{10} = m_{12}, n_{11} = m_{14}, n_{12} = m_{15}, n_{13} = m_{17}, n_{14} = m_{18}, n_{15} = m_{19}, n_{16} = m_{21}, n_{17} = m_{23},$$
$$n_{18} = m_{25}, n_{19} = m_{27}, n_{20} = 6\, m_7 - 4\, m_{22} + m_{30}, n_{21} = -3\, m_9 m_7 + m_{12} m_{20} - m_6 m_{24} +$$
$$4\, m_6 m_{10} + m_{31}, n_{22} = -2\, m_{24} + 2\, m_{10} + m_{32}, n_{23} = 6\, m_6 m_7 - 3\, m_9 m_{20} + m_{33},$$
$$n_{24} = m_{34}, n_{25} = m_{35}, n_{26} = m_{36} \tag{2.42}$$

Fourth, let us consider the subsystem induced by the third derivatives of ξ and η

$$X_iJ = 0,\ T_iJ = 0,\ i = 7 \ldots 10. \tag{2.43}$$

where the inherited operators X_i and T_i, $i = 7 \ldots 10$ are given in Appendix D in term of the new variables $n_i, i = 1 \ldots 26$.

In 26-dimensional space of variables $n_i, i = 1 \ldots 26$, the rank of the system (2.43) is 8, so it has 18 functionally independent solutions which are given as:

$$t_1 = n_1, t_2 = n_2, t_3 = n_3, t_4 = n_4, t_5 = n_6, t_6 = n_8, t_7 = n_{10}, t_8 = n_{12}, t_9 = n_{13}, t_{10} = n_{14},$$
$$t_{11} = n_{19}, t_{12} = -3\, n_8 n_5 - n_6 n_{17} + 4\, n_7 n_6 + n_{20}, t_{13} = 4\, {n_7}^2 - n_{17} n_7 + 2\, n_5 n_{18} - 6\, n_5 n_9 + n_{21},$$
$$t_{14} = -2\, n_{10} n_5 - n_8 n_{17} + n_7 n_8 + n_{16} n_{10} + n_6 n_{18} + n_{22}, t_{15} = -3\, n_6 n_8 n_5 - n_{17} {n_6}^2 - 3\, n_{17} n_5$$
$$+4\, {n_6}^2 n_7 + 4\, n_{16} n_7 - n_{16} n_{17} + n_{23}, t_{16} = n_{24}, t_{17} = n_{25}, t_{18} = n_{26} \tag{2.44}$$

Finally, let us consider the subsystem induced by the zero, first and second derivatives of ξ and η

$$X_i J = 0,\ T_i J = 0,\ i = 1 \dots 6. \tag{2.45}$$

where the inherited operators X_i and $T_i,\ i = 1 \dots 6$ are given in Appendix E in term of the new variables $t_i, i = 1 \dots 18$ which can be rewritten, by backing substitution, as

$$\begin{aligned} &t_1 = x, t_2 = y, t_3 = y_1, t_4 = f, t_5 = f_3, t_6 = f_{3,3}, t_7 = f_{3,3,3}, t_8 = f_{3,3,3,3}, \\ &t_9 = f_{2,3,3,3,3}, t_{10} = f_{3,3,3,3,3}, t_{11} = \dot{D}_x f_{3,3,3,3}, t_{16} = K, t_{17} = L, t_{18} = M \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} t_{12} =&\ \ 4\, f_3 f_{2,3} - f_3 \dot{D}_x f_{3,3} - 3\, f_{3,3} f_2 + 6\, f_{2,2} + \dot{D}_x^2 f_{3,3} - 4\, \dot{D}_x f_{2,3}, \\ t_{13} =&\ \ \tilde{D}_y t_{12} \\ t_{14} =&\ \ \tilde{D}_p t_{12}, \\ t_{15} =&\ \ f_3\ t_{12} + \dot{D}_x t_{12} \end{aligned} \tag{2.47}$$

It should be noted here that t_8 and t_{12} are the fourth order Tresse relative invariants. It is well known that a scalar second-order ODE is linearizable via a point transformation if and only if they both vanish identically as shown by Theorem 1.1. Moreover, it is noted that t_{13}, t_{14} and t_{15} vanish identically when $t_{12} = 0$.

One can see that the operators X_i and $T_i,\ i = 1 \dots 6$ form a Lie algebra $\mathcal{L}_{12}$ with the nonzero commutators

$$
\begin{array}{llll}
[X_2, X_3] = X_2, & [X_2, X_5] = 2\ X_4, & [X_2, X_6] = X_5, & [X_2, T_2] = -X_2, \\
[X_2, T_3] = T_2 - X_3, & [X_2, T_4] = -X_4, & [X_2, T_5] = -X_5 + 2\ T_4, & [X_2, T_6] = -X_6 + T_5, \\
[X_3, X_4] = -X_4, & [X_3, X_6] = X_6, & [X_3, T_3] = T_3, & [X_3, T_5] = T_5, \\
[X_3, T_6] = 2\ T_6, & [X_4, T_2] = -2\ X_4, & [X_4, T_3] = T_4 - X_5, & [X_5, T_2] = -X_5, \\
[X_5, T_3] = -2\ X_6 + T_5, & [X_6, T_3] = T_6, & [T_2, T_3] = -T_3, & [T_2, T_4] = T_4, \\
[T_2, T_6] = -T_6, & [T_3, T_4] = T_5, & [T_3, T_5] = 2\ T_6 &
\end{array}
\tag{2.48}
$$

Moreover, the projection of the operators X_i and $T_i,\ i = 1 \dots 6$ on the 4-dimensional space of variables $t_i, i = 1 \dots 4$ are the generators of the original infinite Lie algebra spanned by the infinitesimal operators (2.15) before the prolongation to the fifth order.

In section 4, the joint invariants of the operators (2.45) provide all differential invariants of $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, up to the fifth order under point transformations.

3 The infinitesimal point equivalence transformations

In order to find continuous group of equivalence transformations of the class (1.1) we consider the arbitrary function f that appears in our equation as a dependent variable and the variables $x, y, y' = p$ as independent variables and apply the Lie infinitesimal invariance criterion [9], that is we look for the infinitesimal ξ, η and μ of the equivalence operator Y:

$$Y = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \mu(x, y, p, f)\partial_f, \tag{3.49}$$

such that its prolongation leaves the equation (1.1) invariant.

The prolongation of operator Y can be given using (2.8) as

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_{y''} + \mu(x, y, p, f)\partial_f, \tag{3.50}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + y''\frac{\partial}{\partial p} + y'''\frac{\partial}{\partial y''} + ...$$

is the operator of total derivative and $W = \eta(x, y) - \xi(x, y)p$ is the characteristic of infinitesimal operator $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$.

So, the Lie infinitesimal invariance criterion gives $\mu = \dot{D}_x^2(W) + \xi(x, y)\dot{D}_x f$ for arbitrary functions $\xi(x, y)$ and $\eta(x, y)$ where $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + f\frac{\partial}{\partial p}$.

Thus, equation (1.1) admits an infinite continuous group of equivalence transformations generated by the Lie algebra $\mathcal{L}_\mathcal{E}$ spanned by the following infinitesimal operators

$$U = \xi(x, y)\frac{\partial}{\partial x} - pD_x(\xi)\partial_p - (2fD_x(\xi) + p\dot{D}_x^2(\xi))\partial_f, \tag{3.51}$$

$$V = \eta(x, y)\partial_y + D_x(\eta)\partial_p + \dot{D}_x^2(\eta)\partial_f, \tag{3.52}$$

The infinitesimal point equivalence transformations (3.51)-(3.52) can be written in the finite form as in (2.16)-(2.17), respectively, where ϕ and ψ are arbitrary functions of the indicated variables.

4 Fifth-order differential invariants and invariant equations under point transformations

In this section, we derive all the fifth-order differential invariants of the general class $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, under point transformations (2.13). Moreover, the invariant differentiation operators [9] are constructed in order to get some higher-order differential invariants from the lower-order ones. Precisely, we obtain the following theorem.

Theorem 4.1. *Every second-order ODE $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, belongs to one of two classes of equations. For the first class of equation ($\nu_1 \neq 0$), there are three fifth order differential invariants, under point transformations,*

$$\beta_1 = {\nu_2}^4 {\nu_1}^{-\frac{7}{2}}, \quad \beta_2 = {\nu_3}^4 {\nu_1}^{-\frac{11}{2}}, \quad \beta_3 = {\nu_4}^4 {\nu_1}^{-5}, \tag{4.53}$$

and three invariant differential operators

$$\begin{aligned}
\mathcal{D}_1 &= (f_{3,3,3,3})^{-\frac{2}{5}} \nu_1^{\frac{1}{8}} \tilde{D}_p, \\
\mathcal{D}_2 &= (f_{3,3,3,3})^{\frac{1}{5}} \nu_1^{-\frac{3}{8}} \left(\tilde{D}_x + p\ \tilde{D}_y + f \tilde{D}_p \right), \\
\mathcal{D}_3 &= (f_{3,3,3,3})^{-\frac{6}{5}} \nu_1^{-\frac{1}{4}} \left(f_{3,3,3,3,3} \tilde{D}_x + (5 f_{3,3,3,3} + p\ f_{3,3,3,3,3}) \tilde{D}_y + (10 f_3 f_{3,3,3,3} + f f_{3,3,3,3,3} + 5\ \dot{D}_x f_{3,3,3,3}) \tilde{D}_p \right),
\end{aligned} \tag{4.54}$$

which satisfy the higher order relations

$$\begin{aligned}
&\mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H - \rho_1 \mathcal{D}_1 H - \rho_2 \mathcal{D}_2 H - \rho_3 \mathcal{D}_3 H = 0, \\
&\mathcal{D}_1 \mathcal{D}_3 H - \mathcal{D}_3 \mathcal{D}_1 H - \sigma_1 \mathcal{D}_1 H - \sigma_2 \mathcal{D}_2 H - \sigma_3 \mathcal{D}_3 H = 0, \\
&\mathcal{D}_2 \mathcal{D}_3 H - \mathcal{D}_3 \mathcal{D}_2 H - \omega_1 \mathcal{D}_1 H - \omega_2 \mathcal{D}_2 H - \omega_3 \mathcal{D}_3 H = 0,
\end{aligned} \tag{4.55}$$

for any differential invariant H.

However, there is no fifth-order differential invariants for the second class ($\nu_1 = 0$), where ν_1, ν_2, ν_3 and ν_4 are the relative invariants given by (4.56) and the commutator invariants ρ_1, ρ_2, ρ_3, $\sigma_1, \sigma_2, \sigma_3$ and $\omega_1, \omega_2, \omega_3$ can be given by (4.65).

Proof. The joint invariants of the operators (2.45) provide all differential invariants of $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, up to the fifth order under point transformations, as well as an implicit solution of the variables K, L and M which provide the differential operators via (2.29).

The joint invariants of the first derived subgroup of $\mathcal{L}_{12}$ can be given for the case $f_{3,3,3,3} \neq 0$

after backing substitution as an arbitrary function $J(x, y, \nu_1, \nu_2, \nu_3, \nu_4.\nu_5, \nu_6, \nu_7)$, where

$$
\begin{aligned}
\nu_1 = &\quad {t_8}^{\frac{1}{5}} t_{12}, \\
\nu_2 = &\quad {t_8}^{-\frac{6}{5}} \left(t_{12} t_{10} + 5\, t_8 t_{14} \right), \\
\nu_3 = &\quad {t_8}^{-\frac{3}{5}} \left(5\, t_{11} t_{12} + \left(7\, t_5 t_{12} + t_{15} \right) t_8 \right), \\
\nu_4 = &\quad {t_8}^{-2} \Big(\left(10\, t_5 t_{14} - 5\, t_{12} t_6 + 5\, t_{13} \right) {t_8}^2 + \left(\left(-3\, t_5 t_{10} - 5\, t_9 \right) t_{12} + t_{15} t_{10} + 5\, t_{14} t_{11} \right) t_8 \Big)
\end{aligned}
\tag{4.56}
$$

and

$$
\begin{aligned}
\nu_5 &= {f_{3,3,3,3}}^{\frac{1}{5}} \left(L - K y_1 \right), \\
\nu_6 &= \tfrac{1}{5} {f_{3,3,3,3}}^{-\frac{6}{5}} \left(5\, K f_{3,3,3,3} + K y_1 f_{3,3,3,3,3} - f_{3,3,3,3,3} L \right), \\
\nu_7 &= {f_{3,3,3,3}}^{-\frac{3}{5}} \Big(2\, f_{3,3,3,3} K f_3 y_1 - f_{3,3,3,3} K f - 2\, f_{3,3,3,3} f_3 L + f_{3,3,3,3} M + \dot{D}_x f_{3,3,3,3} (K y_1 - L) \Big).
\end{aligned}
\tag{4.57}
$$

The non-zero inheritance of the operators X_i and $T_i,\ i = 1 \ldots 12$ in term of the new variables $x, y, \nu_i, i = 1 \ldots 7$ is

$$
\begin{aligned}
X_1 = &\quad [0, 1, 0, 0, 0, 0, 0, 0, 0], \\
T_1 = &\quad [1, 0, 0, 0, 0, 0, 0, 0, 0], \\
X_3 = &\quad [0, 0, -\tfrac{8}{5}\, \nu_1, -\tfrac{7}{5}\, \nu_2, -\tfrac{11}{5}\, \nu_3, -2\, \nu_4, \tfrac{2}{5}\, \nu_5, \tfrac{3}{5}\, \nu_6, -\tfrac{1}{5} \nu_7], \\
T_2 = &\quad X_3.
\end{aligned}
\tag{4.58}
$$

The joint invariants of the operators (4.58) are the invariants of the operator

$$
Z = \quad 8\, \nu_1 \tfrac{\partial}{\partial \nu_1} + 7\, \nu_2 \tfrac{\partial}{\partial \nu_2} + 11\, \nu_3 \tfrac{\partial}{\partial \nu_3} + 10\, \nu_4 \tfrac{\partial}{\partial \nu_4} - 2\, \nu_5 \tfrac{\partial}{\partial \nu_5} - 3\, \nu_6 \tfrac{\partial}{\partial \nu_6} + \nu_7 \tfrac{\partial}{\partial \nu_7}. \tag{4.59}
$$

The invariants of the operators (4.59) can be given using characteristic method for two classes as follows:

(1) First class of equation ($\nu_1 \neq 0$)

$$
\beta_1 = {\nu_2}^4 {\nu_1}^{-\frac{7}{2}}, \quad \beta_2 = {\nu_3}^4 {\nu_1}^{-\frac{11}{2}}, \quad \beta_3 = {\nu_4}^4 {\nu_1}^{-5}, \tag{4.60}
$$

and

$$
\gamma_1 = {\nu_5}^8 {\nu_1}^2, \ \gamma_2 = {\nu_6}^8 {\nu_1}^3, \ \gamma_3 = \tfrac{{\nu_7}^8}{\nu_1} \tag{4.61}
$$

(2) Second class of equation ($\nu_1 = 0$) does not have fifth-order differential invariants independent from the variables K, L and M. This because of vanishing the variables t_{13}, t_{14} and t_{15} identically when $t_{12} = 0$, and so $\nu_2 = \nu_3 = \nu_4 = 0$ whenever $\nu_1 = 0$.

Regarding the invariant differentiation operators, γ_1, γ_2 and γ_3 are the only invariants depending on the variables K, L and M. Then the general solution of (2.30) can be given implicitly as

$$\gamma_1 = F_1,\ \gamma_2 = F_2,\ \gamma_3 = F_3, \tag{4.62}$$

where F_1, F_2 and F_3 are the arbitrary functions of differential invariants $\beta_i,\ i = 1 \dots 3$.

Solving system (4.62) gives the variables K, L and M in terms of three arbitrary functions F_1, F_2 and F_3 which provide three independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ via (2.29).

Finally, since the matrix

$$A = \begin{pmatrix} \mathcal{D}_1 x & \mathcal{D}_2 x & \mathcal{D}_3 x \\ \mathcal{D}_1 y & \mathcal{D}_2 y & \mathcal{D}_3 y \\ \mathcal{D}_1 p & \mathcal{D}_2 p & \mathcal{D}_3 p \end{pmatrix} \tag{4.63}$$

is an invertible matrix with the non-zero determinant $J = 5\ {f_{3,3,3,3}}^{-\frac{2}{5}} {\nu_1}^{-\frac{1}{2}}$, then the invariant differential operators should satisfy the commutation relations

$$\begin{aligned} [\mathcal{D}_1, \mathcal{D}_2] &= \rho_1 \mathcal{D}_1 + \rho_2 \mathcal{D}_2 + \rho_3 \mathcal{D}_3, \\ [\mathcal{D}_1, \mathcal{D}_3] &= \sigma_1 \mathcal{D}_1 + \sigma_2 \mathcal{D}_2 + \sigma_3 \mathcal{D}_3, \\ [\mathcal{D}_2, \mathcal{D}_3] &= \omega_1 \mathcal{D}_1 + \omega_2 \mathcal{D}_2 + \omega_3 \mathcal{D}_3, \end{aligned} \tag{4.64}$$

where

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_1\mathcal{D}_2x - \mathcal{D}_2\mathcal{D}_1x \\ \mathcal{D}_1\mathcal{D}_2y - \mathcal{D}_2\mathcal{D}_1y \\ \mathcal{D}_1\mathcal{D}_2p - \mathcal{D}_2\mathcal{D}_1p \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_1\mathcal{D}_3x - \mathcal{D}_3\mathcal{D}_1x \\ \mathcal{D}_1\mathcal{D}_3y - \mathcal{D}_3\mathcal{D}_1y \\ \mathcal{D}_1\mathcal{D}_3p - \mathcal{D}_3\mathcal{D}_1p \end{pmatrix},$$
$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_2\mathcal{D}_3x - \mathcal{D}_3\mathcal{D}_2x \\ \mathcal{D}_2\mathcal{D}_3y - \mathcal{D}_3\mathcal{D}_2y \\ \mathcal{D}_2\mathcal{D}_3p - \mathcal{D}_3\mathcal{D}_2p \end{pmatrix}. \tag{4.65}$$

Hence the commutator identities (4.64) can be applied to any differential invariants H to give the higher order relations (4.55). □

5 Application

In this section, invariant description of all the canonical forms in the complex plane [6] for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided. Moreover, one example of the second class ($\nu_1 = 0$) is given from the canonical forms of second order ODE in the real plane [5].

Example 5.1. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = c \ \exp(-y'). \tag{5.66}$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three constant fifth-order differential invariants

$$\beta_1 = -65536, \ \beta_2 = -65536, \ \beta_3 = 2825761. \tag{5.67}$$

Example 5.2. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = c\ y'^{\left(\frac{\alpha-2}{\alpha-1}\right)},\ \alpha \neq 0, \frac{1}{2}, 1, 2. \tag{5.68}$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three fifth-order differential invariants

$$\beta_1 = -4096\,\frac{(\alpha+1)^4}{2\,\alpha^3-5\,\alpha^2+2\,\alpha},\ \beta_2 = -4096\,\frac{(\alpha+1)^4}{2\,\alpha^3-5\,\alpha^2+2\,\alpha},\ \beta_3 = \frac{\left(14\,\alpha^2+13\,\alpha+14\right)^4}{\alpha^2(2\,\alpha-1)^2(\alpha-2)^2}. \tag{5.69}$$

As a special case, when $\alpha = -1$, one have the second order ODE

$$y'' = c\ y'^{\frac{3}{2}}. \tag{5.70}$$

with the three fifth-order differential invariants

$$\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 625. \tag{5.71}$$

Example 5.3. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = 6\ yy' - 4\ y^3 + c\ (y' - y^2)^{\frac{3}{2}},\ c \neq \pm 4i. \tag{5.72}$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three fifth-order differential invariants

$$\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 625\,\frac{c^2}{16+c^2}. \tag{5.73}$$

Example 5.4. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = 6\ yy' - 4\ y^3 + c\ (y' - y^2)^{\frac{3}{2}},\ c = \pm 4i. \tag{5.74}$$

It is an equation of the second class ($\nu_1 = 0$), so it does not have fifth-order differential invariants.

Example 5.5. Consider the canonical form of second order ODE in the real plane [5]

$$x\ y'' = y' + y'^3 + (1 + y'^2)^{\frac{3}{2}}. \tag{5.75}$$

It is an equation of the second class ($\nu_1 = 0$), so it does not have fifth-order differential invariants.

Example 5.6. Consider the canonical form of second order ODE in the complex plane with two infinitesimal symmetries [6]

$$y'' = f(y'). \tag{5.76}$$

It is an equation of the first class ($\nu_1 \neq 0$). It has three non-constant fifth-order differential invariants. However, this class can be characterized by the relation $\beta_1 + \beta_2 = 0$ and the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ where it has rank one.

Example 5.7. Consider the canonical form of second order ODE in the complex plane with two infinitesimal symmetries [6]

$$y'' = y' + f(y' - y). \tag{5.77}$$

For the case ($\nu_1 \neq 0$), it has three non-constant fifth-order differential invariants. However, this class can be characterized by the relation $\beta_1 + \beta_2 \neq 0$ and the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ where it has rank one.

Example 5.8. Consider the canonical form of second order ODE in the complex plane with one infinitesimal symmetries [6]

$$y'' = f(x, y'). \tag{5.78}$$

For the case ($\nu_1 \neq 0$), it has three non-constant fifth-order differential invariants. However, this class can be characterized by the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ where it has rank two.

6 Conclusion

The paper provides an extension of the work of Bagderina [1] who solved the equivalence problem for scalar second-order ordinary differential equations (ODEs), cubic in the first-order derivative, via point transformations. However, the question is open for the general class $y'' = f(x, y, y')$ which is not cubic in the first-order derivative. Lie's infinitesimal method was utilized to study the differential invariants of this general class under an arbitrary point equivalence transformations. All fifth order differential invariants and the invariant differentiation operators were determined. These are stated as Theorems 4.1 in Section 4.

As an application, the symmetry algebra of the second order ODE $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is characterized as follows:

1) The symmetry algebra is 3-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ is zero (the differential invariants β_1, β_2 and β_3 are constant).

2) The symmetry algebra is 2-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ is one.

3) The symmetry algebra is 1-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3)}{\partial(x,y,p)}$ is two.

Moreover, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided.

Acknowledgments

The author would like to thank the King Fahd University of Petroleum and Minerals for its support and excellent research facilities and I also want to thank Drs. Fazal Mahomed, Hassan Azad and Tahir Mustafa for several discussions.

Appendix A: The differential operators of the homogeneous linear system of PDEs (2.36)

$$\begin{aligned}
X_{22} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] \\
X_{23} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 6\,z_3, -z_6, 2, 21\,z_4, 0, 0, 0] \\
X_{24} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 5\,z_3, 0, -z_6, 2, 0, 15\,{z_3}^2, 15\,z_4 - 5\,z_6 z_3, 10\,z_3, 105\,z_4 z_3, 0, 0, 0] \\
X_{25} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,z_3, \\
&\quad 0, 0, -z_6, 2, 0, 0, 0, 10\,{z_3}^2, 0, 10\,z_4 - 4\,z_6 z_3, 8\,z_3, 0, 20\,{z_3}^3, 60\,z_4 z_3 - 10\,z_6 {z_3}^2, 20\,{z_3}^2, 210\,z_4 {z_3}^2, 0, 0, 0] \\
X_{26} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,z_3, 0, 0, 0, -z_6, 2, 0, 0, 0, 0, 0, 0, 6\,{z_3}^2, \\
&\quad 0, 0, -3\,z_6 z_3 + 6\,z_4, 6\,z_3, 0, 0, 0, 10\,{z_3}^3, 0, 30\,z_4 z_3 - 6\,z_6 {z_3}^2, 12\,{z_3}^2, 0, 15\,{z_3}^4, -10\,z_6 {z_3}^3 + 90\,z_4 {z_3}^2, 20\,{z_3}^3, \\
&\quad 210\,z_4 {z_3}^3, 0, 0, 0] \\
X_{27} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\,z_3, 0, 0, 0, 0, -z_6, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,{z_3}^2, 0, 0, 0, 3\,z_4 - 2\,z_6 z_3, 4\,z_3, \\
&\quad 0, 0, 0, 0, 0, 0, 4\,{z_3}^3, 0, 0, 12\,z_4 z_3 - 3\,z_6 {z_3}^2, 6\,{z_3}^2, 0, 0, 0, 5\,{z_3}^4, 0, -4\,z_6 {z_3}^3 + 30\,z_4 {z_3}^2, 8\,{z_3}^3, 0, 6\,{z_3}^5, \\
&\quad 60\,z_4 {z_3}^3 - 5\,z_6 {z_3}^4, 10\,{z_3}^4, 105\,z_4 {z_3}^4, 0, 0, 0] \\
X_{28} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^2, 0, 0, 0, 0, -z_6 z_3 + z_4, 2\,z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^3, 0, 0, 0, \\
&\quad -z_6 {z_3}^2 + 3\,z_4 z_3, 2\,{z_3}^2, 0, 0, 0, 0, 0, 0, {z_3}^4, 0, 0, -z_6 {z_3}^3 + 6\,z_4 {z_3}^2, 2\,{z_3}^3, 0, 0, 0, {z_3}^5, 0, -z_6 {z_3}^4 + 10\,z_4 {z_3}^3, \\
&\quad 2\,{z_3}^4, 0, {z_3}^6, -z_6 {z_3}^5 + 15\,z_4 {z_3}^4, 2\,{z_3}^5, 21\,z_4 {z_3}^5, 0, 0, 0] \\
X_{29} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] \\
X_{30} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 7\,z_3, 0, 0, 0] \\
X_{31} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 6\,z_3, 0, 21\,{z_3}^2, 0, 0, 0] \\
X_{32} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 1, 0, 0, 0, 0, 0, 0, 5\,z_3, 0, 0, 0, 15\,{z_3}^2, 0, 35\,{z_3}^3, 0, 0, 0] \\
X_{33} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 4\,z_3, 0, 0, 0, 0, 0, 0, 10\,{z_3}^2, 0, 0, 0, 20\,{z_3}^3, 0, 35\,{z_3}^4, 0, 0, 0] \\
X_{34} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,z_3, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0, 6\,{z_3}^2, 0, 0, 0, 0, 0, 0, 10\,{z_3}^3, 0, 0, 0, 15\,{z_3}^4, 0, 21\,{z_3}^5, 0, 0, 0] \\
X_{35} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\,z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,{z_3}^2, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 4\,{z_3}^3, 0, 0, 0, 0, 0, 0, 5\,{z_3}^4, 0, 0, 0, 6\,{z_3}^5, 0, 7\,{z_3}^6, 0, 0, 0] \\
X_{36} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^3, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, {z_3}^4, 0, 0, 0, 0, 0, 0, {z_3}^5, 0, 0, 0, {z_3}^6, 0, {z_3}^7, 0, 0, 0]
\end{aligned}$$

$$T_{22} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, -1, -7\,z_4, 0, 0, 0]$$

$$T_{23} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, -z_3, 0, 0, -1, 0, -6\,{z_3}^2, -6\,z_4 + z_6 z_3, -8\,z_3, -63\,z_4 z_3, 0, 0, 0]$$

$$T_{24} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3,$$
$$0, 0, 0, -1, 0, 0, 0, -5\,{z_3}^2, 0, -5\,z_4 + z_6 z_3, -7\,z_3, 0, -15\,{z_3}^3, -45\,z_4 z_3 + 5\,z_6 {z_3}^2, -25\,{z_3}^2, -210\,z_4 {z_3}^2, 0, 0, 0]$$

$$T_{25} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -4\,{z_3}^2,$$
$$0, 0, z_6 z_3 - 4\,z_4, -6\,z_3, 0, 0, 0, -10\,{z_3}^3, 0, 4\,z_6 {z_3}^2 - 30\,z_4 z_3, -18\,{z_3}^2, 0, -20\,{z_3}^4, -120\,z_4 {z_3}^2 + 10\,z_6 {z_3}^3,$$
$$-40\,{z_3}^3, -350\,z_4 {z_3}^3, 0, 0, 0]$$

$$T_{26} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^2, 0, 0, 0, -3\,z_4 + z_6 z_3, -5\,z_3,$$
$$0, 0, 0, 0, 0, 0, -6\,{z_3}^3, 0, 0, -18\,z_4 z_3 + 3\,z_6 {z_3}^2, -12\,{z_3}^2, 0, 0, 0, -10\,{z_3}^4, 0, 6\,z_6 {z_3}^3 - 60\,z_4 {z_3}^2, -22\,{z_3}^3, 0,$$
$$-15\,{z_3}^5, 10\,z_6 {z_3}^4 - 150\,z_4 {z_3}^3, -35\,{z_3}^4, -315\,z_4 {z_3}^4, 0, 0, 0]$$

$$T_{27} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2\,{z_3}^2, 0, 0, 0, 0, -2\,z_4 + z_6 z_3, -4\,z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^3, 0, 0, 0,$$
$$2\,z_6 {z_3}^2 - 9\,z_4 z_3, -7\,{z_3}^2, 0, 0, 0, 0, 0, 0, -4\,{z_3}^4, 0, 0, -24\,z_4 {z_3}^2 + 3\,z_6 {z_3}^3, -10\,{z_3}^3, 0, 0, 0, -5\,{z_3}^5, 0,$$
$$4\,z_6 {z_3}^4 - 50\,z_4 {z_3}^3, -13\,{z_3}^4, 0, -6\,{z_3}^6, 5\,z_6 {z_3}^5 - 90\,z_4 {z_3}^4, -16\,{z_3}^5, -147\,z_4 {z_3}^5, 0, 0, 0]$$

$$T_{28} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^3, 0, 0, 0, 0, z_6 {z_3}^2 - 3\,z_4 z_3, -3\,{z_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^4, 0, 0, 0,$$
$$z_6 {z_3}^3 - 6\,z_4 {z_3}^2, -3\,{z_3}^3, 0, 0, 0, 0, 0, 0, -{z_3}^5, 0, 0, -10\,z_4 {z_3}^3 + z_6 {z_3}^4, -3\,{z_3}^4, 0, 0, 0, -{z_3}^6, 0,$$
$$-15\,z_4 {z_3}^4 + z_6 {z_3}^5, -3\,{z_3}^5, 0, -{z_3}^7, -21\,z_4 {z_3}^5 + z_6 {z_3}^6, -3\,{z_3}^6, -28\,z_4 {z_3}^6, 0, 0, 0]$$

$$T_{29} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0]$$

$$T_{30} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, -7\,{z_3}^2, 0, 0, 0]$$

$$T_{31} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, -6\,{z_3}^2, 0, -21\,{z_3}^3, 0, 0, 0]$$

$$T_{32} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, -z_3, 0, 0, 0, 0, 0, 0, -5\,{z_3}^2, 0, 0, 0, -15\,{z_3}^3, 0, -35\,{z_3}^4, 0, 0, 0]$$

$$T_{33} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, 0, 0, 0$$
$$, 0, 0, -4\,{z_3}^2, 0, 0, 0, 0, 0, 0, -10\,{z_3}^3, 0, 0, 0, -20\,{z_3}^4, 0, -35\,{z_3}^5, 0, 0, 0]$$

$$T_{34} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^2, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, -6\,{z_3}^3, 0, 0, 0, 0, 0, 0, -10\,{z_3}^4, 0, 0, 0, -15\,{z_3}^5, 0, -21\,{z_3}^6, 0, 0, 0]$$

$$T_{35} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2\,{z_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^3, 0, 0, 0, 0$$
$$, 0, 0, 0, 0, 0, 0, -4\,{z_3}^4, 0, 0, 0, 0, 0, 0, -5\,{z_3}^5, 0, 0, 0, -6\,{z_3}^6, 0, -7\,{z_3}^7, 0, 0, 0]$$

$$T_{36} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^4, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, -{z_3}^5, 0, 0, 0, 0, 0, 0, -{z_3}^6, 0, 0, 0, -{z_3}^7, 0, -{z_3}^8, 0, 0, 0]$$

Appendix B: The differential operators of the homogeneous linear system of PDEs (2.39)

$$X_{16} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0]$$

$$X_{17} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,5\,l_3,2,0,0,0,0]$$

$$X_{18} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,4\,l_3,0,2,0,0,0,10\,{l_3}^2,8\,l_3,2,0,0,0]$$

$$X_{19} = [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3\,l_3,0,0,2,0,0,0,0,0,0,6\,{l_3}^2,0,6\,l_3,0,2,0,10\,{l_3}^3,12\,{l_3}^2,6\,l_3,0,0,0]$$

$$X_{20} = [0,0,0,0,0,0,0,0,0,2\,l_3,0,0,0,2,0,0,0,0,0,0,0,0,0,0,3\,{l_3}^2,0,0,4\,l_3,0,0,2,0,0,0,4\,{l_3}^3,0,6\,{l_3}^2,0,4\,l_3,0,5\,{l_3}^4,8\,{l_3}^3,6\,{l_3}^2,0,0,0]$$

$$X_{21} = [0,0,0,0,0,0,0,0,0,{l_3}^2,0,0,0,2\,l_3,0,0,0,2,0,0,0,0,0,0,{l_3}^3,0,0,2\,{l_3}^2,0,0,2\,l_3,0,0,0,{l_3}^4,0,2\,{l_3}^3,0,2\,{l_3}^2,0,{l_3}^5,2\,{l_3}^4,2\,{l_3}^3,0,0,0]$$

$$T_{16} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-l_3,-1,0,0,0,0]$$

$$T_{17} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-l_3,0,-1,0,0,0,-5\,{l_3}^2,-7\,l_3,-4,0,0,0]$$

$$T_{18} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-l_3,0,0,-1,0,0,0,0,0,0,-4\,{l_3}^2,0,-6\,l_3,0,-4,0,-10\,{l_3}^3,-18\,{l_3}^2,-18\,l_3,0,0,0]$$

$$T_{19} = [0,0,0,0,0,0,0,0,0,-l_3,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,-3\,{l_3}^2,0,0,-5\,l_3,0,0,-4,0,0,0,-6\,{l_3}^3,0,-12\,{l_3}^2,0,-14\,l_3,0,-10\,{l_3}^4,-22\,{l_3}^3,-30\,{l_3}^2,0,0,0]$$

$$T_{20} = [0,0,0,0,0,0,0,0,0,-2\,{l_3}^2,0,0,0,-4\,l_3,0,0,0,-4,0,0,0,0,0,0,-3\,{l_3}^3,0,0,-7\,{l_3}^2,0,0,-10\,l_3,0,0,0,-4\,{l_3}^4,0,-10\,{l_3}^3,0,-16\,{l_3}^2,0,-5\,{l_3}^5,-13\,{l_3}^4,-22\,{l_3}^3,0,0,0]$$

$$T_{21} = [0,0,0,0,0,0,0,0,0,-{l_3}^3,0,0,0,-3\,{l_3}^2,0,0,0,-6\,l_3,0,0,0,0,0,0,-{l_3}^4,0,0,-3\,{l_3}^3,0,0,-6\,{l_3}^2,0,0,0,-{l_3}^5,0,-3\,{l_3}^4,0,-6\,{l_3}^3,0,-{l_3}^6,-3\,{l_3}^5,-6\,{l_3}^4,0,0,0]$$

Appendix C: The differential operators of the homogeneous linear system of PDEs (2.41)

$$X_{11} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0]$$

$$X_{12} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,4\,m_3,2,0,-m_{12},0,3\,m_9,0,0,0]$$

$$X_{13} = [0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,3\,m_3,0,2,0,0,0,0,0,6\,{m_3}^2,6\,m_3,2,3\,m_9-3\,m_3m_{12},0,-6\,m_6+9\,m_3m_9,0,0,0]$$

$$X_{14} = [0,0,0,0,0,0,2\,m_3,0,0,2,0,0,0,0,0,0,0,0,0,3\,{m_3}^2,0,4\,m_3,0,2,0,0,0,4\,{m_3}^3,6\,{m_3}^2,4\,m_3,-6\,m_6+6\,m_3m_9-3\,{m_3}^2m_{12},0,-12\,m_6m_3+9\,{m_3}^2m_9,0,0,0]$$

$$X_{15} = [0,0,0,0,0,0,{m_3}^2,0,0,2\,m_3,0,0,2,0,0,0,0,0,0,{m_3}^3,0,2\,{m_3}^2,0,2\,m_3,0,0,0,{m_3}^4,2\,{m_3}^3,2\,{m_3}^2,-6\,m_6m_3+3\,{m_3}^2m_9-{m_3}^3m_{12},0,-6\,m_6{m_3}^2+3\,{m_3}^3m_9,0,0,0]$$

$$T_{11} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-m_3,-1,0,0,0,0,0,0,0]$$

$$T_{12} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-m_3,0,-1,0,0,0,0,0,-4\,{m_3}^2,-6\,m_3,-4,m_3m_{12},0,-3\,m_3m_9,0,0,0]$$

$$T_{13} = [0,0,0,0,0,0,-m_3,0,0,-1,0,0,0,0,0,0,0,0,0,-3\,{m_3}^2,0,-5\,m_3,0,-4,0,0,0,-6\,{m_3}^3,-12\,{m_3}^2,-14\,m_3,-3\,m_3m_9+3\,{m_3}^2m_{12},-6,6\,m_6m_3-9\,{m_3}^2m_9,0,0,0]$$

$$T_{14} = [0,0,0,0,0,0,-2\,{m_3}^2,0,0,-4\,m_3,0,0,-4,0,0,0,0,0,0,-3\,{m_3}^3,0,-7\,{m_3}^2,0,-10\,m_3,0,-6,0,-4\,{m_3}^4,-10\,{m_3}^3,-16\,{m_3}^2,6\,m_6m_3-6\,{m_3}^2m_9+3\,{m_3}^3m_{12},-12\,m_3,12\,m_6{m_3}^2-9\,{m_3}^3m_9,0,0,0]$$

$$T_{15} = [0,0,0,0,0,0,-{m_3}^3,0,0,-3\,{m_3}^2,0,0,-6\,m_3,0,0,-6,0,0,0,-{m_3}^4,0,-3\,{m_3}^3,0,-6\,{m_3}^2,0,-6\,m_3,0,-{m_3}^5,-3\,{m_3}^4,-6\,{m_3}^3,6\,m_6{m_3}^2-3\,{m_3}^3m_9+{m_3}^4m_{12},-6\,{m_3}^2,6\,m_6{m_3}^3-3\,{m_3}^4m_9,0,0,0]$$

Appendix D: The differential operators of the homogeneous linear system of PDEs (2.43)

$$X_7 = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0]$$

$$X_8 = [0,0,0,0,1,0,0,0,0,0,0,0,0,0,3\,n_3,2,0,0,0,3\,n_8,-2\,n_{18}+6\,n_9,0,5\,n_{17}-8\,n_7+3\,n_8n_6,$$
$$0,0,0]$$

$$X_9 = [0,0,0,0,2\,n_3,0,2,0,0,0,0,0,0,0,3\,{n_3}^2,4\,n_3,2,0,0,-6\,n_6+6\,n_3n_8,12\,n_3n_9-4\,n_3n_{18}$$
$$-14\,n_7+2\,n_{17},0,-6\,{n_6}^2+6\,n_8n_6n_3-16\,n_3n_7+10\,n_3n_{17}+6\,n_5-6\,n_{16},0,0,0]$$

$$X_{10} = [0,0,0,0,{n_3}^2,0,2\,n_3,0,2,0,0,0,0,0,{n_3}^3,2\,{n_3}^2,2\,n_3,0,0,-6\,n_6n_3+3\,{n_3}^2n_8,-2\,{n_3}^2n_{18}$$
$$+6\,{n_3}^2n_9+2\,n_3n_{17}-14\,n_3n_7+12\,n_5,0,-6\,{n_6}^2n_3+3\,n_8n_6{n_3}^2+5\,{n_3}^2n_{17}-8\,{n_3}^2n_7$$
$$+6\,n_3n_5-6\,n_3n_{16},0,0,0]$$

$$T_7 = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,-n_3,-1,0,0,0,0,0,n_{10},4\,n_7-n_{17},0,0,0]$$

$$T_8 = [0,0,0,0,-n_3,0,-1,0,0,0,0,0,0,0,-3\,{n_3}^2,-5\,n_3,-4,0,0,-3\,n_3n_8,-6\,n_3n_9+2\,n_3n_{18}$$
$$+4\,n_7-n_{17},-3\,n_8+3\,n_3n_{10},-3\,n_8n_6n_3+20\,n_3n_7-8\,n_3n_{17}-12\,n_5,0,0,0]$$

$$T_9 = [0,0,0,0,-2\,{n_3}^2,0,-4\,n_3,0,-4,0,0,0,0,0,-3\,{n_3}^3,-7\,{n_3}^2,-10\,n_3,-6,0,6\,n_6n_3$$
$$-6\,{n_3}^2n_8,-12\,{n_3}^2n_9+4\,{n_3}^2n_{18}-4\,n_3n_{17}+22\,n_3n_7-12\,n_5,6\,n_6-6\,n_3n_8+3\,{n_3}^2n_{10},$$
$$6\,{n_6}^2n_3-6\,n_8n_6{n_3}^2-13\,{n_3}^2n_{17}+28\,{n_3}^2n_7-30\,n_3n_5+6\,n_3n_{16},0,0,0]$$

$$T_{10} = [0,0,0,0,-{n_3}^3,0,-3\,{n_3}^2,0,-6\,n_3,0,-6,0,0,0,-{n_3}^4,-3\,{n_3}^3,-6\,{n_3}^2,-6\,n_3,0,6\,n_6{n_3}^2$$
$$-3\,{n_3}^3n_8,-6\,{n_3}^3n_9+2\,{n_3}^3n_{18}+18\,{n_3}^2n_7-3\,{n_3}^2n_{17}-24\,n_3n_5,-3\,{n_3}^2n_8+6\,n_6n_3$$
$$+{n_3}^3n_{10},6\,{n_6}^2{n_3}^2-3\,{n_3}^3n_8n_6+12\,{n_3}^3n_7+6\,{n_3}^2n_{16}-18\,{n_3}^2n_5-6\,{n_3}^3n_{17},0,0,0]$$

Appendix E: The differential operators of the homogeneous linear system of PDEs (2.45)

$$
\begin{aligned}
X_1 =\ & [0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]\\
X_2 =\ & [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,t_{16},0]\\
X_3 =\ & [0,0,t_3,t_4,0,-t_6,-2\,t_7,-3\,t_8,-4\,t_9,-4\,t_{10},-3\,t_{11},-t_{12},-2\,t_{13},-2\,t_{14},-t_{15},0,t_{17},t_{18}]\\
X_4 =\ & [0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,t_{16}]\\
X_5 =\ & [0,0,0,2\,t_3,2,0,0,0,-t_{10},0,-3\,t_8,0,-t_{14},0,t_{12},0,0,t_{17}+t_{16}t_3]\\
X_6 =\ & [0,0,0,{t_3}^2,2\,t_3,2,0,0,-3\,t_8-t_3t_{10},0,-3\,t_3t_8,0,-t_3t_{14}-t_{12},0,t_3t_{12},0,0,t_{17}t_3]\\
T_1 =\ & [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]\\
T_2 =\ & [0,0,-t_3,-2\,t_4,-t_5,0,t_7,2\,t_8,2\,t_9,3\,t_{10},t_{11},-2\,t_{12},-2\,t_{13},-t_{14},-3\,t_{15},t_{16},0,-t_{18}]\\
T_3 =\ & [0,0,-{t_3}^2,-3\,t_4t_3,-t_5t_3-3\,t_4,-4\,t_5+t_3t_6,-3\,t_6+3\,t_3t_7,5\,t_3t_8,-t_{11}+t_4t_{10}+6\,t_3t_9,\\
& 5\,t_8+7\,t_3t_{10},5\,t_4t_8+4\,t_3t_{11},-t_3t_{12},t_5t_{12}+t_4t_{14}-t_{15},t_3t_{14}-t_{12},-4\,t_4t_{12}-2\,t_3t_{15},t_{17},0,-2\,t_{18}t_3]\\
T_4 =\ & [0,0,0,-t_3,-1,0,0,0,0,0,2\,t_8,0,0,0,-3\,t_{12},0,0,-t_{16}t_3]\\
T_5 =\ & [0,0,0,-2\,{t_3}^2,-4\,t_3,-4,0,0,t_3t_{10}+2\,t_8,0,7\,t_3t_8,0,t_3t_{14}-2\,t_{12},0,-7\,t_3t_{12},0,0,-t_{16}{t_3}^2-t_{17}t_3]\\
T_6 =\ & [0,0,0,-{t_3}^3,-3\,{t_3}^2,-6\,t_3,-6,0,{t_3}^2t_{10}+5\,t_3t_8,0,5\,{t_3}^2t_8,0,-t_3t_{12}+{t_3}^2t_{14},0,-4\,{t_3}^2t_{12},0,0,-t_{17}{t_3}^2]
\end{aligned}
$$

References

[1] Y.Y. Bagderina, Invariants of a family of scalar second-order ordinary differential equations, J. Phys. A: Math. Theor. 46 (2013) 295201 (36pp).

[2] S. Lie, Klassifikation und Integration von gewohnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten, III. Archiv for Matematik og Naturvidenskab., 8, Heft 1883;4:371458. Reprinted in Lies Ges. Abhandl., vol. 5, 1924, paper XIV, p. 362427.

[3] A. Tressé, Sur les invariant différentiels des groupes continus de transformations, Acta Math. 18 (1894) 1-88.

[4] F.M. Mahomed, P.G.L. Leach. The Lie algebra $sl(3, R)$ and linearization, Quaestiones Mathematicae, (1989), 12–121.

[5] F.M. Mahomed, Symmetry group classification of ordinary differential equations: Survey of some results, Mathematical Mathods in the Applied Sciences, 30 (2007),1995-2012.

[6] P.J. Olver, Equivalence, Invariants and Symmetry, Cambridge University Press, Cambridge, 1995.

[7] N.H. Ibragimov, S.V. Meleshko, Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I, Commun Nonlinear Sci Numer Simul. 12 (2007), 1370-1378.

[8] N.H. Ibragimov, S.V. Meleshko, Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. II, Commun Nonlinear Sci Numer Simul. 13 (2008), 1015-1020.

[9] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, (1982).

[10] N.H. Ibragimov, Infinitesimal method in the theory of invariants of algebraic and differential equations, Not. South Afr. Math. Soc. 29 (1997) 61-70.

[11] N.H. Ibragimov and F.M. Mahomed, Ordinary differential equations, CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3. N.H. Ibragimov ed., CRC Press, Boca Raton (1996) 191.

[12] N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiley, New York, 1999.

[13] N.H. Ibragimov, Laplace type invariants for parabolic equations, Nonlinear Dynam. 28 (2002) 125-133.

[14] N.H. Ibragimov, Invariants of a remarkable family of nonlinear equations, Nonlinear Dynam. 30 (2002) 155-166.

[15] N.H. Ibragimov, C. Sophocleous, Differential invariants of the one-dimensional quasi-linear second-order evolution equation, Commun. Nonlinear Sci. Numer. Simul. 12 (2007) 1133-1145.

[16] N.H. Ibragimov, C. Sophocleous, Invariants for evolution equations, Proc. Inst. Math. NAS Ukraine 50 (2004) 142-148.

[17] N.H. Ibragimov, S.V. Meleshko, Linearization of third-order ordinary differential equations by point and contact transformations, J. Math. Anal. Appl. 308 (2005) 266-289.

[18] I.K. Johnpillai, F.M. Mahomed, Singular invariant equation for the (1 + 1) Fokker-Plank equation, J. Phys. A: Math. Gen. 28 (2001) 11033-11051.

[19] M. Torrisi, R. Tracinà, A. Valenti, On the linearization of semilinear wave equations, Nonlinear Dynam. 36 (2004) 97-106.

[20] M. Torrisi, R. Tracinà, Second-order differential invariants of a family of diffusion equations, J. Phys. A: Math. Gen. 38 (2005) 7519-7526.

[21] R. Tracinà, Invariants of a family of nonlinear wave equations, Commun. Nonlinear Sci. Numer. Simul. 9 (2004) 127-133.

[22] Gardner R B 1989 The Method of Equivalence and Its Applications (Philadelphia: SIAM).

24 Invariance of second order ordinary differential equations under two-dimensional affine subalgebras of ErmakovPinney Lie algebra. By J. F. Carinena, F. Gungor, P. J. Torres (2020)

Invariance of second order ordinary differential equations under two-dimensional affine subalgebras of Ermakov–Pinney Lie algebra

J. F. Cariñena*[1], F. Güngör†[2], and P. J. Torres‡[3]

[1]Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2]Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 34469 Istanbul, Turkey
[3]Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain

February 11, 2020

Abstract

Using the only admissible rank-two realisations of the Lie algebra of the affine group in one dimension in terms of the Lie algebra of Lie symmetries of the Ermakov-Pinney (EP) equation, some classes of second order nonlinear ordinary differential equations solvable by reduction method are constructed. One class includes the standard EP equation as a special case. A new EP equation with a perturbed potential but admitting the same solution formula as EP itself arises. The solution of the dissipative EP equation is also discussed.

1 Introduction

Among the most general second order linear differential equations in normal form

$$\psi'' + q(x)\psi' + p(x)\psi = 0, \tag{1.1}$$

those with $q \equiv 0$ are especially important in both classical and quantum physics and will be said to be of Schrödinger type, because the usual Schrödinger equation for

*jfc@unizar.es
†gungorf@itu.edu.tr
‡ptorres@ugr.es

the determination of stationary states is of this type with a coefficient p given by $p(x) = E - V(x)$, where V is the potential and E is the energy eigenvalue. Then, the equation can be written as

$$\psi'' + p(x)\psi = 0. \tag{1.2}$$

Changing variables to t for the independent variable and x for the dependent variable, and using the dot notation for time derivative, the corresponding equation

$$\ddot{x} + p(t)x = 0 \tag{1.3}$$

is known in a more mathematical context as *Hill's equation* [1–3]. It has been shown in [4] (see also [5]) that there is an infinitesimal point transformation of symmetry of such a Schrödinger type equation of the form

$$X_a(t,x) = a(t)\frac{\partial}{\partial t} + \frac{\dot{a}(t)}{2}x\frac{\partial}{\partial x}, \tag{1.4}$$

where the function a satisfies the following third order linear ODE

$$\mathbb{M}(a) = \dddot{a} + 4p(t)\dot{a} + 2\dot{p}(t)a = 0, \tag{1.5}$$

which was called in [6] projective vector field equation. Moreover, as the differential equation (1.3) is linear, all vector fields of the form $X = b(t)\partial/\partial x$ with b being a solution of (1.3) are also infinitesimal symmetries of the equation.

Similarly, we can consider the nonlinear Ermakov-Pinney (EP) differential equation

$$\ddot{x} + p(t)\,x = \frac{k}{x^3}, \quad x \neq 0, \quad k \in \mathbb{R}. \tag{1.6}$$

One can show (see Section 2) that such differential equation is invariant under a 3-dimensional Lie algebra of Lie symmetries generated by vector fields of the form (1.4) where a satisfies Eq. (1.5). Let us mention that the differential equation (1.5) is very related to the theory of higher order Adler-Gelfand-Dikii differential operators [7, 8] and it plays a key role in the study of projective connections and $\mathfrak{gl}(n,\mathbb{R})$ current algebras [6, 9].

The main objective of this paper is to identify families of second order ordinary differential equations that are invariant under a two-dimensional affine Lie subalgebra of the Lie algebra associated to the EP equation, i.e., the Lie algebra generated by vector fields (1.4). One of the identified families of equations will be seen to include the EP equation (1.6) as a special case. This analysis is performed on Section 3. The presence of the non-Abelian two-dimensional symmetry Lie algebra, which is solvable, is sufficient for a second order ODE to be fully integrable by quadratures. In a final section devoted to conclusions and remarks, we point out that the identified invariant equations are not only of theoretical interest, but they are related to some recent models arising in population dynamics. This connection has been explored in more detail in a separate paper [10].

2 Preliminaries

In order to show our main motivation we will start with the derivation of the Lie algebra of infinitesimal point transformations of symmetry of (1.6) using the prolongation algorithm for differential equations (see for example [11,12]). Given the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ with coordinate expression

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x}, \tag{2.1}$$

its second order prolongation $X^{(2)}$ is given by

$$X^{(2)} = X + \eta^{(1)}(t,x,\dot{x})\frac{\partial}{\partial \dot{x}} + \eta^{(2)}(t,x,\dot{x},\ddot{x})\frac{\partial}{\partial \ddot{x}}, \tag{2.2}$$

where

$$\eta^{(1)} = D_t\eta - \dot{x}D_t\xi, \quad \eta^{(2)} = D_t\eta^{(1)} - \ddot{x}D_t\xi.$$

Here $D_t = d/dt$ is a symbol for the total derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \cdots . \tag{2.3}$$

More explicitly, the coefficients of two first prolongations of (2.1) are:

$$\eta^{(1)} = \eta_t + (\eta_x - \xi_t)\dot{x} - \xi_x\,\dot{x}^2,$$

and

$$D_t\eta^{(1)} = \eta_{tt} + (2\eta_{tx} - \xi_{tt})\dot{x} + (\eta_{xx} - 2\xi_{xt})\dot{x}^2 - \xi_{xx}\,\dot{x}^3 - 2\xi_x\,\dot{x}\ddot{x} + (\eta_x - \xi_t)\ddot{x},$$

and consequently,

$$\begin{aligned} \eta^{(2)} &= D_t\eta^{(1)} - \ddot{x}D_t\xi = \eta_{tt} + (2\eta_{tx} - \xi_{tt})\dot{x} + (\eta_{xx} - 2\xi_{xt})\dot{x}^2 \\ &- \xi_{xx}\,\dot{x}^3 + (\eta_x - 2\xi_t)\ddot{x} - 3\xi_x\,\dot{x}\ddot{x}. \end{aligned}$$

In the particular case of Ermakov-Pinney equation (1.6) with $k \neq 0$, the property characterizing the functions ξ and η such that a vector field, (2.1) is a Lie symmetry of such equation is given by

$$\left(X^{(2)}(\ddot{x} + p(t)x - k\,x^{-3})\right)\Big|_{\ddot{x}+p(t)x-k\,x^{-3}=0} = 0, \qquad x > 0, \tag{2.4}$$

or more explicitly,

$$\left(\eta^{(2)}\right)\Big|_{\ddot{x}+p(t)x-k\,x^{-3}=0} + \dot{p}(t)x\xi + \eta\left(p(t) + 3kx^{-4}\right) = 0. \tag{2.5}$$

The particularly interesting case is when the vector field is a projectable vector field, i.e. like in (2.1) but with $\xi_x = 0$, because its flow is made of bundle map diffeomophisms $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$.

The coefficients of the different powers of $\dot{x}$ in (2.5) must be zero, i.e. taking into account the corresponding form of $\eta^{(2)}$:

$$(k\,x^{-3}-p(t)x)\,(\eta_x-2\xi_t-3\xi_x\dot{x})+\eta_{tt}+(2\eta_{xt}-\xi_{tt})\dot{x}+(\eta_{xx}-2\xi_{xt})\dot{x}^2-\xi_{xx}\dot{x}^3 \\ +\dot{p}(t)x\xi+\eta(p(t)+3kx^{-4})=0,$$

and consequently we find the following set of conditions:

$$\begin{aligned}&\xi_{xx}=0,\\ &\eta_{xx}-2\xi_{tx}=0,\\ &2\eta_{tx}-\xi_{tt}+3(p(t)x-k\,x^{-3})\xi_x=0,\\ &(k\,x^{-3}-p(t)x)(\eta_x-2\xi_t)+\eta_{tt}+\dot{p}(t)x\xi+(p(t)+3kx^{-4})\eta=0.\end{aligned} \tag{2.6}$$

The two first equations lead to the following form for ξ and η

$$\xi(t,x)=d(t)x+b(t),\qquad \eta(t,x)=\dot{d}(t)x^2+c(t)\;x+e(t),$$

and using these expressions in the third equation of the preceding system we find

$$2(2\ddot{d}(t)x+\dot{c}(t))-(\ddot{d}(t)\,x+\ddot{b}(t))+3(p(t)x-k\,x^{-3})d(t)=0. \tag{2.7}$$

This condition implies, first, that the function d must be zero, because the coefficient of x^{-3} is $k\,d(t)$, and furthermore $2\dot{c}(t)-\ddot{b}(t)=0$, and then the expressions of the functions ξ and η are

$$\xi(t,x)=b(t),\qquad \eta=c(t)x+e(t),$$

which shows that X is a projectable vector field.

Finally, the fourth equation reduces to

$$(k\,x^{-3}-p(t)x)(c(t)-2\dot{b}(t))+\ddot{c}(t)x+\ddot{e}(t)+\dot{p}(t)xb(t)+(p(t)+3kx^{-4})(c(t)x+e(t))=0,$$

and for the coefficients of different powers of x to be zero we obtain:

$$\begin{aligned}&e(t)=0,\\ &2k\,(2c(t)-\,\dot{b}(t))=0,\\ &\ddot{c}(t)+2p(t)\,\dot{b}(t)+\dot{p}(t)\,b(t)=0.\end{aligned} \tag{2.8}$$

The second equation shows that

$$c(t)=\frac{1}{2}\dot{b}(t), \tag{2.9}$$

and a substitution in the third equation gives rise to

$$\frac{1}{2}\dddot{b}(t)+2p(t)\,\dot{b}(t)+\dot{p}(t)\,\dot{b}(t)=0, \tag{2.10}$$

i.e. b is a solution of (1.5).

This means that the symmetry vector fields we are looking for are of the form

$$X_b(t,x) = b(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{b}(t)\, x\frac{\partial}{\partial x}, \tag{2.11}$$

where $b(t)$ is a solution of (1.5).

The correspondence $a \mapsto X_a$ mapping each solution of (1.5) into an infinitesimal symmetry of the nonlinear Ermakov-Pinney differential equation (1.6) is $\mathbb{R}$-linear, because $X_{a_1+\lambda a_2} = X_{a_1} + \lambda X_{a_2}$, for each real number $\lambda \in \mathbb{R}$. Consequently, as (1.5) is a linear third order differential equation, the set of vector fields determined by solutions a of the differential equation (1.5) is a three-dimensional real linear space.

Note that if we consider Hill's equation (1.3), it is possible to show that if u_1 and u_2 are two linearly independent solutions of (1.3), then the three functions $f_{ij} = u_i\, u_j$, $i \le j = 1, 2$, are solutions of (1.5).

In fact, remark first that taking derivatives we obtain that

$$\dddot{u}_i + p(t)\,\dot{u}_i + \dot{p}(t)\, u_i = 0, \quad i = 1, 2,$$

and if we make use of these two equations, then the following third-order derivative

$$D_t^3(u_iu_j) = \dddot{u}_i u_j + 3\ddot{u}_i\,\dot{u}_j + 3\dot{u}_i\ddot{u}_j + u_i\dddot{u}_j\,,$$

can be rewritten as follows

$$D_t^3(u_iu_j) = -(p(t)\,\dot{u}_i + \dot{p}(t)\,u_i)u_j - 3p(t)\,u_i\dot{u}_j + 3\dot{u}_i(-p(t)\,u_j) - u_i(p(t)\,\dot{u}_j + \dot{p}(t)\,u_j),$$

that after simplification becomes

$$D_t^3(u_iu_j) = -[2\dot{p}(t)\,u_iu_j + 4p(t)(\dot{u}_iu_j + u_i\dot{u}_j)]\,.$$

We have therefore obtained

$$D_t^3(u_iu_j) + 4p(t)D(u_iu_j) + 2\,\dot{p}(t)\,u_iu_j = 0\,,$$

what proves that the three functions $f_{ij} = u_i\,u_j$, $i \le j = 1, 2$, are solutions of (1.5). Moreover, as the Wronskian of the three functions f_{ij} is

$$W(u_1^2, u_1u_2, u_2^2) = 2(u_1\,\dot{u}_2 - u_2\,\dot{u}_1)^3\,,$$

we see that if $\{u_1, u_2\}$ is a fundamental set of solutions of the second-order equation (1.3), then the functions u_1^2, u_1u_2 and u_2^2 are linearly independent and they span the three-dimensional linear space of solutions of (1.5) whose general solution can be written as a linear combination

$$a(t) = Au_1^2 + 2Bu_1u_2 + Cu_2^2, \qquad A, B, C \in \mathbb{R}. \tag{2.12}$$

We can prove now that the set of vector fields of the form (1.4) that are Lie symmetries of the Ermakov-Pinney (EP) equation (1.6) is a Lie algebra: such Lie symmetries of (1.6) close on the three-dimensional real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ spanned by the vector fields

$$X_{ij} = f_{ij}\frac{\partial}{\partial t} + \frac{1}{2}\dot{f}_{ij}x\frac{\partial}{\partial x}, \quad i \le j = 1, 2. \tag{2.13}$$

In fact, the set of vector fields as in (1.4) is closed under commutator because

$$[X_{a_1}, X_{a_2}] = X_{W(a_1,a_2)}, \tag{2.14}$$

where $W(a_1, a_2)$ denotes the Wronskian $W(a_1, a_2) = a_1\,\dot{a}_2 - a_2\,\dot{a}_1$, and, moreover, if a_1 and a_2 are solutions of (1.5), then the function $w_{12}(t) = W(a_1(t), a_2(t))$ is a solution of (1.5) too, because

$$(W(a_1, a_2))^{\dot{}} = \dot{a}_1\,\dot{a}_2 + a_1\,\ddot{a}_2 - \ddot{a}_1\,a_2 - \dot{a}_2\,\dot{a}_1 = a_1\,\ddot{a}_2 - \ddot{a}_1\,a_2,$$

and then,

$$\dot{w}_{12} = (W(a_1, a_2))^{\dot{}} = a_1\,\ddot{a}_2 - a_2\,\ddot{a}_1,$$

and when taking derivatives in this expression we get

$$\ddot{w}_{12} = a_1\,\dddot{a}_2 - a_2\,\dddot{a}_1 + \dot{a}_1\,\ddot{a}_2 - \dot{a}_2\,\ddot{a}_1,$$

and therefore, if a_1 and a_2 are solutions of (1.5), a simple calculation shows that the preceding relation reduces to

$$\ddot{w}_{12} = -4p(t)\,w_{12} + \dot{a}_1\,\ddot{a}_2 - \dot{a}_2\,\ddot{a}_1,$$

from where we see that

$$\dddot{w}_{12} = -4p(t)\dot{w}_{12} - 4\dot{p}(t)\,w_{12} + \dot{a}_1(-4p(t)\,\dot{a}_2 - 2\dot{p}(t)\,a_2) - \dot{a}_2(-4p(t)\,\dot{a}_1 - 2\dot{p}(t)\,a_1),$$

and simplifying terms we arrive at

$$\dddot{w}_{12} = -4p(t)\dot{w}_{12} - 2\dot{p}(t)\,w_{12}.$$

We note that the same argument with more computational efforts can be used to show that the Wronskian w_{12} of any two independent solutions of the general third order linear PDE

$$\dddot{a} + c_2(t)\ddot{a} + c_1(t)\dot{a} + c_0(t)a = 0, \tag{2.15}$$

is also a solution if and only if the coefficients satisfy $c_2 = 0$, $\dot{c}_1 = 2c_0$, (a formally self-adjoint equation).

Having in mind the mentioned property that for any pair of functionally independent solutions of (1.3), u_1 and u_2, the functions $u_1^2, u_1\,u_2$, and u_2^2 form a basis of the three-dimensional real linear space of solutions of (1.5), we can consider as a basis of the three-dimensional real Lie algebra of infinitesimal symmetries of (1.5) the vector fields $X_{u_1^2}, X_{u_1\,u_2}$, and $X_{u_2^2}$, and as

$$\begin{aligned} W(u_1^2, u_1\,u_2) &= u_1^2\,W(u_1, u_2), \\ W(u_1^2, u_2^2) &= 2u_1\,u_2\,W(u_1\,u_2), \\ W(u_1\,u_2, u_2^2) &= u_2^2\,W(u_1, u_2), \end{aligned}$$

where $W(u_1(t), u_2(t))$ is constant, and we obtain from (2.14) that

$$\begin{array}{l} [X_{u_1^2}, X_{u_1 u_2}] = X_{W(u_1^2, u_1 u_2)} = X_{u_1^2 W(u_1,u_2)}, \\ [X_{u_1^2}, X_{u_2^2}] = X_{W(u_1^2, u_1 u_2)} = 2\, X_{u_1 u_2 W(u_1,u_2)}, \\ [X_{u_1 u_2}, X_{u_2^2}] = X_{W(u_1 u_2, u_2^2)} = X_{u_2^2 W(u_1,u_2)}. \end{array} \tag{2.16}$$

We can conclude from here that: *If u_1 and u_2 are two functionally independent solutions of (1.3) such that $W(u_1, u_2) = 1$, then the vector fields $Y_1 = X_{u_1^2}$, $Y_2 = X_{u_1 u_2}$ and $Y_3 = X_{u_2^2}$ generate a Lie algebra of vector fields of infinitesimal Lie symmetries of (1.5) isomorphic to* $\mathfrak{sl}(2, \mathbb{R})$, because they satisfy the commutation relations

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = 2Y_2, \quad [Y_2, Y_3] = Y_3. \tag{2.17}$$

This leads to the following result: The set of infinitesimal symmetries of (1.6) is a three-dimensional real Lie algebra of vector fields like (1.4) where a is solution of (1.5).

It is also to be remarked that it has been proved in [13] that the Ermakov-Pinney equation

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3},$$

when written as a first-order system

$$\dot{x} = v, \quad \dot{v} = -\omega^2(t)x + \frac{k}{x^3}$$

is a Lie system with associated Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, generated by the vector fields

$$X_1 = x\,\frac{\partial}{\partial v}, \quad X_2 = v\frac{\partial}{\partial x} + \frac{k}{x^3}\frac{\partial}{\partial v}, \quad X_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right),$$

which satisfy the following commutation relations

$$[X_1, X_2] = 2X_3, \qquad [X_1, X_3] = -X_1, \qquad [X_2, X_3] = X_2.$$

As it happens for each Lie system, the flow of generators of its Vessiot Lie algebra, the vector fields X_1, X_2 and X_3, transforms each Lie system defined by them into another one of the same type.

Eq. (1.5) admits the first integral

$$K = \frac{1}{4}(2a\,\ddot{a} - \dot{a}^2) + p(t)a^2, \tag{2.18}$$

because multiplying the left hand side of (1.5) by $\frac{1}{2}a$ we obtain

$$\frac{1}{2}a\,(\dddot{a} + 4p(t)\dot{a} + 2\dot{p}(t)a) = \frac{d}{dt}\left(\frac{1}{4}(2a\,\ddot{a} - \dot{a}^2) + p(t)a^2\right) = 0.$$

The value of K for the general solution of (1.5) written in terms of two linearly independent solutions of (1.3) as in (2.12) is specified as $K = (AC - B^2)w_{12}^2$, because introducing the notation for the bilinear form $\langle\cdot,\cdot\rangle$

$$a = Au_1^2 + 2Bu_1u_2 + Cu_2^2 = (u_1, u_2)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \langle \mathbf{u}, \mathbf{u} \rangle,$$

where $\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2$, with $A = \langle \mathbf{e}_1, \mathbf{e}_1\rangle$, $B = \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ and $C = \langle \mathbf{e}_2, \mathbf{e}_2\rangle$, then

$$\dot{a} = 2\langle \dot{\mathbf{u}}, \mathbf{u}\rangle, \qquad \ddot{a} = -2p(t)\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle,$$

and using the expression (2.18) we find

$$K = \langle \mathbf{u}, \mathbf{u}\rangle(\langle \dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle - p(t)\langle \mathbf{u}, \mathbf{u}\rangle - \langle \dot{\mathbf{u}}, \mathbf{u}\rangle^2) + p\langle \mathbf{u}, \mathbf{u}\rangle^2 = \langle \dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle\langle \mathbf{u}, \mathbf{u}\rangle - \langle \dot{\mathbf{u}}, \mathbf{u}\rangle^2.$$

The right hand side of the preceding expression reminds that of the square of exterior product when $\langle\cdot,\cdot\rangle$ is the Euclidean product. We can then define a skew-symmetric bilinear form F either by this expression for the module when the two vectors have positive orientation and the opposite if the pair of vectors have the inverse orientation. This expression $K = |F(\mathbf{u}, \dot{\mathbf{u}})|^2$ shows that as for the exterior product

$$\|\mathbf{u} \times \dot{\mathbf{u}}\| = |W(u_1, u_2)|\ \|\mathbf{e}_1 \times \mathbf{e}_2\|,$$

and any two skew-symmetric forms are proportional

$$K = |W(u_1, u_2)|^2\ |F(\mathbf{e}_1, \mathbf{e}_2)|^2,$$

and as

$$|F(\mathbf{e}_1, \mathbf{e}_2)|^2 = \langle \mathbf{e}_1, \mathbf{e}_1\rangle\langle \mathbf{e}_2, \mathbf{e}_2\rangle - \langle \mathbf{e}_1, \mathbf{e}_2\rangle^2 = AC - B^2,$$

we find from here the announced result.

We refer the interested readers to [14,15] for solutions and Lie symmetry properties of EP equation (1.6) and projective vector field equation (1.5). Let us comment that Eq. (1.3) still has a $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra of Lie symmetry. In general, the third-order auxiliary equation (1.5) crops up in symmetry analysis of second and higher order linear ODEs with the property of being anti-self adjoint or of maximal Lie symmetry and, as we have seen above, in second order nonlinear ODEs whose solutions are expressed in terms of (1.3) like EP and also its generalisations [14,16], and it is used in the derivation of first integrals for time-dependent Hamiltonian systems [17].

While it is possible to remove the coefficient p from (1.6) by the change of variables $(t, x) \to (\tau, \xi)$ defined by

$$x = \xi(\tau)u_1, \qquad \tau = (W(u_1, u_2))^{-1}\frac{u_2}{u_1}, \tag{2.19}$$

with u_1 and u_2 particular solutions of (1.3), we prefer to keep the potential p to serve our purposes in the current context. Moreover, we can use the orientation-preserving transformation

$$\bar{t} = \tau(t), \quad \bar{a}(\bar{t}) = \dot{\tau}(t)a(t), \quad \dot{\tau} > 0, \tag{2.20}$$

where τ satisfies the third-order Kummer–Schwarz equation

$$\{\tau; t\} = 2\,p(t), \tag{2.21}$$

with $\{\tau; t\}$ being the Schwarz derivative (see [18] for a short introduction), i.e.

$$\{\tau; t\} = \frac{\dddot{\tau}}{\dot{\tau}} - \frac{3}{2}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)^2. \tag{2.22}$$

See e.g. [19] and references therein, and [20–22] for related concepts and their physical applications. Such transformation maps Eq.(1.5) into its Laguerre-Forsyth canonical form $\bar{a}'''(\bar{t}) = 0$ [23], where the prime denotes derivative with respect to the new independent variable $\bar{t}$ (See for example [24, 25]). As remarked by Kummer [26] the solutions of (2.21) can be expressed as the quotient of two linearly independent solutions of (1.3). This implies that transformation (2.20) can be written in the form

$$\bar{t} = \tau(t) = \frac{\alpha u_1 + \beta u_2}{\gamma u_1 + \delta u_2}, \quad \bar{a}(\bar{t}) = -\Delta\, W(u_1, u_2)(\gamma u_1 + \delta u_2)^{-2} a(t), \quad \Delta = \alpha\delta - \beta\gamma \neq 0. \tag{2.23}$$

With the special choice $\alpha = 0$, $\beta = 1$, $\gamma = W(u_1, u_2)$, $\delta = 0$ ($\Delta = -W(u_1, u_2) \neq 0$) and the relationship $x = \sqrt{a}$ between (2.18) and the equation $\ddot{x} + px = Kx^{-3}$, transformation (2.19) is recovered.

We can reobtain the general solution (2.12) from (2.23)

$$a(t) = -\frac{1}{\Delta\, W(u_1, u_2)}(\gamma u_1 + \delta u_2)^2(c_1 + c_2\tau + c_3\tau^2) = Au_1^2 + Bu_1u_2 + Cu_2^2$$

after a redefinition of the arbitrary constants.

In particular, if we choose $p = 0$ ($u_1(t) = t$, $u_2(t) = 1$, $W = -1$) then we obtain the $\mathrm{SL}(2, \mathbb{R})$ subgroup of the symmetry group of the canonical equation $\dddot{a} = 0$

$$\bar{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \Delta = 1, \tag{2.24}$$

together with $\bar{a} = (\gamma t + \delta)^{-2} a$.

The $\mathrm{SL}(2, \mathbb{R})$ symmetry group of the canonical EP equation $\ddot{x} = kx^{-3}$ is thus given by (2.24) with $\bar{x} = (\gamma t + \delta)^{-1} x$.

In passing, we comment that Eq. (2.15) can be reduced to the canonical form $\dddot{a} = 0$ by a point transformation if and only if the following singular invariant equation relative to the general form-preserving transformation $\tau = \tau(t)$, $\bar{a} = \phi(t)a$ of (2.15) is satisfied [27, 28]

$$9\ddot{c}_2 + 18\dot{c}_2 c_2 - 27\dot{c}_1 + 4c_2^3 - 18c_1c_2 + 54c_0 = 0. \tag{2.25}$$

The special case $c_2 = 0$ is equivalent to the formal self-adjointness of the equation.

3 Second order ODEs invariant under the two-dimensional affine algebra

We start this section by looking for the second order differential equations which admit as a Lie algebra of symmetry a Lie subalgebra of the Lie algebra of symmetries of the Ermakov-Pinney equation. The only two-dimensional Lie subalgebra is isomorphic to that of the affine group of transformations of the real line. It is spanned by two vector fields X_1 and X_2 such that $[X_1, X_2] = X_1$. Then, if X_1 and X_2 are vector fields of the form

$$X_{a_1}(t, x) = a_1(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{a}_1(t)\, x\frac{\partial}{\partial x}, \qquad X_{a_2}(t, x) = a_2(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{a}_2(t)\, x\frac{\partial}{\partial x},$$

where a_1 and a_2 are positive solutions of (1.5), then using the relation (2.14) we see that the functions a_1 and a_2 must be related by

$$W(a_1, a_2) = a_1,$$

and therefore,

$$a_1\, \dot{a}_2 - a_2\, \dot{a}_1 = a_1,$$

then, starting from a solution a_1 of (1.5) we obtain that a_2 must be a solution of the inhomogeneous linear differential equation

$$\dot{a}_2 = \frac{\dot{a}_1}{a_1} a_2 + 1.$$

As $a_2 = a_1$ is a solution of the associated linear homogeneous equation we should introduce the change of variable $a_2 = a_1\, s$, and the given equation becomes

$$a_1\, \dot{s} = 1,$$

which gives

$$s(t) = \int^t \frac{1}{a_1(\zeta)}\, d\zeta.$$

Since $a_2 = a_1\, s$ and

$$\frac{d}{dt}(a_1\, s) = \dot{a}_1\, s + a_1\, \dot{s} = \dot{a}_1\, s + 1,$$

this proves that

$$X_{a_2} = s(t)\, X_{a_1} + \frac{1}{2} x\, \frac{\partial}{\partial x}.$$

We are now interested in the most general class of second order ODEs involving functions expressed in terms of arbitrary solutions of (1.5) for a given $p(t)$ and solvable by a pair of quadratures. We start by realising the two-dimensional non-Abelian Lie algebra, generated by two vector fields X_1 and X_2 such that X_1 is of the form (1.4), i.e. is an infinitesimal point transformation of symmetry of both (1.3) and (1.6), and X_2 is a vector field satisfying the commutation relations $[X_1, X_2] = X_1$. Such Lie algebra is generated by

$$X_1 = a(t)\frac{\partial}{\partial t} + \frac{\dot{a}(t)}{2} x \frac{\partial}{\partial x}, \quad X_2 = s(t) X_1 + \beta X_0, \quad X_0 = x\frac{\partial}{\partial x}, \quad s(t) = \int^t \frac{d\zeta}{a(\zeta)}, \tag{3.1}$$

with $\beta \neq 0$ and where a was assumed to be solution of (1.5) for the given $p(t)$ and β a real number. This is so because given X_1 of the above mentioned form, then we can write X_2 as a linear combination of the form

$$X_2 = c(t)\, X_1 + b(t)\, x\frac{\partial}{\partial x} = c(t)\, X_1 + b(t)\, X_0,$$

and then, as $[X_1, X_0] = 0$,

$$[X_1, X_2] = [X_1, c(t)\, X_1] + \left[X_1, b(t)\, x\frac{\partial}{\partial x}\right] = X_1(c)\, X_1 + a(t)\, \dot{b}(t)\, x\frac{\partial}{\partial x},$$

and therefore, in order to have $[X_1, X_2] = X_1$, the functions b and c must satisfy

$$a\dot{c} = 1, \qquad \dot{b} = 0,$$

from where we obtain that $b(t)$ must be constant, $b(t) = \beta$, and $c(t)$ must be given by $s(t)$ as indicated by (3.1). The constant β must be different from zero, otherwise X_2 and X_1 would be proportional in each point

The 2-dimensional Lie algebra spanned by X_1 and X_2 (recall that we assumed $\beta \neq 0$) is isomorphic to the Lie algebra of the affine group in the real line. They define a transitive action of this Lie algebra on the plane (t, x) (no nontrivial ordinary invariants exist). Recall that if a is not constant, only in the particular case $\beta = 1/2$ the vector field X_2 is of the family of vector fields (1.4), in other words we only have a rank-two realisation of the algebra within the class of vector fields (1.4).

Our aim is to construct the general second order ODE invariant under the realisation (3.1) of the two-dimensional affine algebra $\mathfrak{aff}(1, \mathbb{R})$. This is a standard procedure and requires finding invariants for the second prolongation $\mathrm{pr}^{(2)}\,\mathfrak{aff}(1, \mathbb{R})$ by solving a pair of first order linear PDEs by the method of characteristics.

3.1 Invariant equation in the case $\beta = 1/2$.

We should look for the most general second order ODE invariant under the realisation (3.1) with $\beta = 1/2$. We start by looking for the invariant functions for the second prolongation of X_1, $X_1^{(2)}$, given by

$$X_1^{(2)} = a(t)\frac{\partial}{\partial t} + \frac{\dot{a}(t)}{2}x\frac{\partial}{\partial x} + \frac{1}{2}\left(\ddot{a}x - \dot{a}\dot{x}\right)\frac{\partial}{\partial \dot{x}} + \frac{1}{2}\left(\dddot{a}\,x - 3\dot{a}\ddot{x}\right)\frac{\partial}{\partial \ddot{x}}.$$

The can be are computed as characteristic solutions of the partial differential equation $X_1^{(2)}H = 0$ and we find as solution a function $H(J_1, J_2, J_3)$ where

$$J_1 = \frac{x}{\sqrt{a}}, \quad J_2 = \sqrt{a}\left(\dot{x} - \frac{\dot{a}}{2a}x\right) = a\dot{J}_1, \quad J_3 = a^{3/2}(\ddot{x} + px). \tag{3.2}$$

In order to impose X_2 invariance we first remark that as

$$X_2^{(2)}(J_1) = \frac{1}{2}J_1, \quad X_2^{(2)}(J_2) = -\frac{1}{2}J_2$$

and

$$X_2^{(2)}(J_3) = \frac{1}{2}a^{3/2}(-3\ddot{x} + px) + \frac{a^{-1/2}}{2}(2a\ddot{a} - \dot{a}^2)x,$$

that using the first integral (2.18) we can rewrite as

$$X_2^{(2)}(J_3) = -\frac{3}{2}J_3 + 2KJ_1,$$

the differential invariants of order ≤ 2 of the algebra $\mathfrak{aff}(1, \mathbb{R})$ are found by solving the PDE

$$\frac{1}{2}J_1\frac{\partial H}{\partial J_1} - \frac{1}{2}J_2\frac{\partial H}{\partial J_2} + \left(-\frac{3}{2}J_3 + 2KJ_1\right)\frac{\partial H}{\partial J_3} = 0. \tag{3.3}$$

Then we consider the associated system

$$\frac{dJ_1}{J_1} = -\frac{dJ_2}{J_2} = \frac{dJ_3}{-3J_3 + 4KJ_1}$$

and from the first fraction with the second or with the third one we find the invariant functions

$$I = J_1 J_2 = x\dot{x} - \frac{\dot{a}}{2a}x^2, \quad J = J_1^3 J_3 - KJ_1^4 = x^3(\ddot{x} + px) - \frac{K}{a^2}x^4, \tag{3.4}$$

such that the general solution of $X_1^{(2)}H = X_2^{(2)}H = 0$ is an arbitrary function of I and J.

The invariant second order ODE are therefore of the form

$$x^3(\ddot{x} + px) = \frac{K}{a^2}x^4 + G(I), \tag{3.5}$$

with G an arbitrary smooth function, or written in a different way,

$$\ddot{x} + \left(p(t) - Ka(t)^{-2}\right)x = x^{-3}G(I). \tag{3.6}$$

So for a given function p we can produce a class of ODEs integrable by quadratures. The first integral condition (2.18) gives us

$$\ddot{x} - \frac{1}{4a^2}(2a\ddot{a} - \dot{a}^2)x = x^{-3}G\left(x\dot{x} - \frac{\dot{a}}{2a}x^2\right), \tag{3.7}$$

which actually depends on p in a disguise form. The above equation can be written explicitly with $\nu = \dot{a}/a$ as

$$\ddot{x} - \frac{1}{4}(2\dot{\nu} + \nu^2)x = x^{-3}G\left(x\dot{x} - \frac{1}{2}\nu x^2\right). \tag{3.8}$$

It is straightforward to see that Eq. (3.8) allows the following invariant (particular solutions)

$$x(t) = C_0\sqrt{a(t)}, \qquad G(0) = 0, \tag{3.9}$$

$$x(t) = C_0\sqrt{s(t)a(t)}, \qquad C_0 + 4G\left(\frac{C_0^2}{2}\right) = 0. \tag{3.10}$$

It is useful to find an equivalent form of (3.8) under the transformation $x = z^{1/k}$, which takes (3.5) to

$$\ddot{z} - \frac{k}{4}\left(2\dot{\nu} + \nu^2\right)z = \frac{k-1}{k}\frac{\dot{z}^2}{z} + kz^{(k-4)/k}G(I), \tag{3.11}$$

where

$$I = \frac{1}{k}z^{(2-k)/k}\left(\dot{z} - \frac{k}{2}\nu z\right), \quad \nu = \frac{\dot{a}}{a}.$$

Of course, the constant k multiplying the arbitrary function G can be absorbed into G. The symmetry algebra is

$$X_1 = a(t)\frac{\partial}{\partial t}+\frac{k\dot{a}(t)}{2}z\frac{\partial}{\partial z}, \quad X_2 = s(t)X_1+\frac{k}{2}z\frac{\partial}{\partial z} = a(t)s(t)\frac{\partial}{\partial t}+\frac{k}{2}(1+\dot{a}s)z\frac{\partial}{\partial z}. \tag{3.12}$$

It is more convenient to put $k = 4/(1-n)$ for some real $n \neq 1$ for which (3.11) takes the form

$$\ddot{z} + \frac{1}{n-1}(2\dot{\nu}+\nu^2)z = \frac{n+3}{4}\frac{\dot{z}^2}{z} + z^n G(I), \tag{3.13}$$

where

$$I = \frac{1-n}{4}z^{-(n+1)/2}\left(\dot{z} - \frac{2\nu}{1-n}z\right), \quad \nu = \frac{\dot{a}}{a}.$$

Now we will examine some particular cases. For $p = -\lambda^2/4$ we have the possibilities $a = 1$ ($\nu = 0$, $K = -\lambda^2/4$), and $a = e^{\pm\lambda t}$, $\lambda \neq 0$ ($\nu = \pm\lambda$, $K = 0$) and the corresponding invariant equations have the form

$$\ddot{x} = x^{-3}G(x\dot{x}), \tag{3.14}$$

$$\ddot{x} - \frac{\lambda^2}{4}x = x^{-3}G\left(x\dot{x} \pm \frac{\lambda}{2}x^2\right). \tag{3.15}$$

The corresponding symmetry vector fields are

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, \\ X_1 = \exp[\pm\lambda t]\left(\frac{\partial}{\partial t} \pm \frac{\lambda}{2}x\frac{\partial}{\partial x}\right), \quad X_2 = \pm\frac{1}{\lambda}\frac{\partial}{\partial t}. \end{gathered} \tag{3.16}$$

For $p = \lambda^2/4$, we have either $a = \cos(\lambda t)$ (and $\nu = -\lambda\tan(\lambda t)$, $K = -\lambda^2/4$) or $a = \sin(\lambda t)$ (and then $\nu = \lambda\cot(\lambda t)$, $K = -\lambda^2/4$). The value of K is determined either by direct computation from (2.18) or by making use of the relation $K = (AC-B^2)W^2$. For example, comparing the relation

$$a = \cos(\lambda t) = \cos^2\frac{\lambda t}{2} - \cos^2\frac{\lambda t}{2} = u_1^2 - u_2^2$$

with (2.12) implies $A = -C = 1$, $B = 0$ and with $W(u_1, u_2) = \lambda/2$ we find $K = -\lambda^2/4$. For $a = \sin(\lambda t)$, we have $A = C = 0$, $B = 1$. The corresponding equation and symmetries for $a = \cos(\lambda t)$ are

$$\ddot{x} + \frac{\lambda^2}{4}(1+\sec^2(\lambda t))x = x^{-3}G\left(x\dot{x} + \frac{\lambda}{2}\tan(\lambda t)\,x^2\right), \tag{3.17}$$

$$\begin{gathered} X_1 = \cos(\lambda t)\frac{\partial}{\partial t} - \frac{\lambda}{2}\sin(\lambda t)x\frac{\partial}{\partial x}, \\ X_2 = \frac{2}{\lambda}\tanh^{-1}\left(\tan\frac{(\lambda t)}{2}\right)\cos(\lambda t)\frac{\partial}{\partial t} + \frac{1}{2}\left(1 - 2\tanh^{-1}\left(\tan\frac{\lambda t}{2}\right)\sin(\lambda t)\right)x\frac{\partial}{\partial x}. \end{gathered} \tag{3.18}$$

3.2 Reduction to quadrature and solutions

We can introduce the new coordinates (r, s) adapted to the vector field X_1, i.e. such that $X_1 r = 0, X_1 s = 1$, which are therefore given by

$$r = \frac{x}{\sqrt{a}}, \quad s = \int^t \frac{d\zeta}{a(\zeta)}, \tag{3.19}$$

so that $X_1 = \partial/\partial s$. Then as $X_2 r = r/2$ and $X_2 s = s$, the rank-two affine algebra is transformed to the one generated by the vector fields

$$X_1 = \frac{\partial}{\partial s}, \quad X_2 = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r}.$$

If we note the relations

$$r\frac{dr}{ds} = x\left(\dot{x} - \frac{\dot{a}}{2a}x\right), \qquad \frac{d^2r}{ds^2} = a^{3/2}\left(\ddot{x} - \frac{1}{4a^2}(2a\ddot{a} - \dot{a}^2)x\right),$$

the canonical form of invariant equation (3.6) or (3.7) has the form of the generalised Ermakov-Pinney equation

$$\frac{d^2r}{ds^2} = r^{-3}G\left(r\frac{dr}{ds}\right). \tag{3.20}$$

The equivalent form (3.13) is reduced to the canonical form

$$r''(s) = \frac{n+3}{4}\frac{r'^2}{r} + r^n G(\omega), \quad \omega = \frac{(1-n)}{4}r^{-(n+1)/2}r' \tag{3.21}$$

by means of the coordinate transformation

$$r = a^{2/(n-1)}z, \quad s = \int^t \frac{d\zeta}{a(\zeta)}. \tag{3.22}$$

Eq. (3.21) is invariant under the Lie algebra spanned by the vector fields

$$X_1 = \frac{\partial}{\partial s}, \quad X_2 = s\frac{\partial}{\partial s} + \frac{2}{1-n}r\frac{\partial}{\partial r}.$$

When G is restricted to a constant, say $G = 4G_0/(1-n)$, with G_0 a constant, it is known as a special case of second order Kummer-Schwarz equation (see Eq. (3.40)), which has a general solution formula so that solution z of (3.13) is given by

$$z(t) = a^{2/(1-n)}r(s) = (Aa + 2Bas + Cas^2)^{2/(1-n)}, \quad AC - B^2 = G_0. \tag{3.23}$$

The structure of the canonical equation (3.20) or (3.21) for $n = -3$ suggests the special choice $G =$ const. which reduces to the canonical form of the standard Ermakov-Pinney equation, namely

$$r''(s) = G_0 r^{-3}. \tag{3.24}$$

In this case the affine symmetry algebra (3.1) with $\beta \neq 0$ extends to an $\mathfrak{sl}(2,\mathbb{R})$ algebra isomorphic to the second type in Lie's classification list. The additional symmetry vector field is given by

$$X_3 = s^2 \frac{\partial}{\partial s} + sr \frac{\partial}{\partial r}.$$

We already know that Eq. (3.24) admits a general solution formula given by

$$r(s) = (A + 2Bs + Cs^2)^{1/2}, \quad AC - B^2 = G_0. \tag{3.25}$$

From this fact we immediately see that the following equation

$$\ddot{x} + [p(t) - Ka(t)^{-2}]x = G_0 x^{-3} \tag{3.26}$$

admits a $\mathfrak{sl}(2,\mathbb{R})$ symmetry algebra spanned by the vector fields (3.1) and an additional one

$$X_3 = a(t)s(t)^2 \frac{\partial}{\partial t} + \frac{1}{2}(\dot{a}(t)s(t)^2 + 2s(t))x \frac{\partial}{\partial x}. \tag{3.27}$$

We note that the realisation of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra is generated by

$$X_1 = a \frac{\partial}{\partial t} + \frac{\dot{a}}{2} x \frac{\partial}{\partial x}, \quad X_2 = sX_1 + \frac{1}{2} X_0, \quad X_3 = s^2 X_1 + sX_0, \tag{3.28}$$

where the vector field X_0 is $X_0 = x\partial/\partial x$, with commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3, \tag{3.29}$$

which can be derived from the following commutation relations

$$[X_1, X_0] = 0, \quad [X_1, sX_0] = X_0, \quad [X_1, sX_1] = X_1, \tag{3.30}$$

$$[X_1, s^2 X_1] = 2sX_1, \quad [sX_1, sX_0] = sX_0, \quad [sX_1, s^2 X_1] = s^2 X_1, \tag{3.31}$$

from where we see that the symmetry vector fields X_1, X_2, X_3 of (3.28) satisfy the commutation relations (3.29) characteristics of $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra. The general solution of (3.26) is now given by the formula

$$x(t) = \sqrt{a(t)}\, r(s(t)) = \sqrt{Aa + 2Bas + Cas^2}, \quad AC - B^2 = G_0. \tag{3.32}$$

This solution is somewhat surprising because as long as K is a non-vanishing constant we obtain the general solution of the Ermakov-Pinney equation with a considerably modified potential $\tilde{p}(t) = p(t) - Ka(t)^{-2}$, and only when $K = 0$ it coincides with the usual Ermakov-Pinney solution.

As an example we consider a case where $K = -\lambda^2/4 \neq 0$, $p = \lambda^2/4$, $a = \cos(\lambda t)$:

$$\ddot{x} + \frac{\lambda^2}{4}(1 + \sec^2(\lambda t))x = G_0 x^{-3}. \tag{3.33}$$

The general solution of (3.33), despite being too complicated, is given exactly by the formula (3.32) with $s(t)$ being

$$s(t) = \frac{1}{\lambda} \log \left[\frac{1 + \tan \frac{\lambda t}{2}}{1 - \tan \frac{\lambda t}{2}} \right].$$

The choice $p = 1$, $a = 1 + \alpha\cos(2t)$, $|\alpha| < 1$ ($K = 1 - \alpha^2$) leads to the EP equation

$$\ddot{x} + \left(1 + \frac{\alpha^2 - 1}{(1 + \alpha\cos(2t))^2}\right)x = G_0 x^{-3}, \qquad |\alpha| < 1. \tag{3.34}$$

The π-periodic general solution of (3.34) is given by (3.32) with $s(t)$ being

$$s(t) = \frac{1}{\sqrt{1 - \alpha^2}} \arctan\left(\frac{1 - \alpha}{1 + \alpha}\tan t\right).$$

The linear version of (3.34) with $G_0 = 0$ belongs to a one-parameter family of Hill's equations with coefficients periodic of period π (also a subclass of the so-called four-parameter Ince equations [29]).

On the other hand, the special choice $G(I) = 4G_0/(1 - n)$ in (3.13) produces the following important form of a $\mathfrak{sl}(2, \mathbb{R})$-invariant equation that frequently arises in many applications

$$\ddot{z} + \frac{4}{1 - n}\left(p - Ka^{-2}\right)z = \frac{n + 3}{4}\frac{\dot{z}^2}{z} + \frac{4G_0}{1 - n}z^n. \tag{3.35}$$

A basis of the symmetry algebra is given by

$$X_1 = a\frac{\partial}{\partial t} + \frac{k\dot{a}}{2}x\frac{\partial}{\partial x}, \quad X_2 = sX_1 + \frac{k}{2}X_0, \quad X_3 = s^2X_1 + ksX_0, \quad X_0 = x\frac{\partial}{\partial x}, \tag{3.36}$$

where $k = 4/(1 - n)$. The general solution of (3.35) is given by (see solution (3.23))

$$z(t) = (Aa + 2Bas + Cas^2)^{2/(1-n)}, \quad AC - B^2 = G_0. \tag{3.37}$$

This equation can be regarded as a generalisation of the second order Kummer-Schwarz (2KS) equation provided that $K \neq 0$.

The following dissipative form of (3.35) for $K = 0$ can also be of some interest

$$\ddot{w} + r(t)\dot{w} + \frac{4p(t)}{1 - n}w = \sigma\frac{\dot{w}^2}{w} + \frac{4q}{1 - n}\exp\left[-2\int^t r(\zeta)d\zeta\right]w^n, \; n \neq 1, \; q \in \mathbb{R}, \; \sigma = \frac{n + 3}{4}. \tag{3.38}$$

We call (3.38) dissipative second order Kummer-Schwarz (d2KS) equation. The linear transformation

$$w(t) = \phi(t)z(t), \quad \phi(t) = \exp\left[\frac{1}{2(\sigma - 1)}\int^t r(\zeta)d\zeta\right], \quad 2(\sigma - 1) = \frac{n - 1}{2}, \tag{3.39}$$

transforms (3.38) into

$$\ddot{z} + \frac{4}{1 - n}I(t)z = \sigma\frac{\dot{z}^2}{z} + \frac{4q}{1 - n}z^n, \tag{3.40}$$

where

$$I(t) = p - \frac{1}{4}(r^2 + 2\dot{r}).$$

We already know that Eq. (3.40) has the general solution

$$z = (Au_1^2 + 2Bu_1u_2 + Cu_2^2)^{2/(1-n)}, \quad (AC - B^2)W^2(u_1, u_2) = q, \tag{3.41}$$

where u_1, u_2 are two linearly independent solutions of the equation

$$\ddot{z} + I(t)z = \ddot{z} + \left(p - \frac{1}{4}(r^2 + 2\dot{r})\right) z = 0. \tag{3.42}$$

The general solution of (3.38) is given by

$$w(t) = \exp\left[\frac{2}{n-1}\int^t r(\zeta)d\zeta\right](Au_1^2+2Bu_1u_2+Cu_2^2)^{2/(1-n)}, \quad (AC-B^2)W^2(u_1, u_2) = q. \tag{3.43}$$

The d2KS equation (3.38) is invariant under the real Lie algebra of vector fields

$$X_a = a\frac{\partial}{\partial t} + \frac{2}{1-n}(\dot{a} - ar)w\frac{\partial}{\partial w},$$

where the function a is in the real linear space spanned by the functions u_1^2, u_1u_2, u_2^2, where u_1, u_2 are solutions of

$$\ddot{w} + [p - \frac{1}{4}(r^2 + 2\dot{r})]w = 0.$$

The commutation relations between the three components of the algebra satisfy those of the $\mathfrak{sl}(2, \mathbb{R})$ algebra in (2.17).

We note that a Lagrangian L of the 2KS equation (3.40) is provided by

$$L(t, z, \dot{z}) = \left(\frac{1-n}{4}\right)^2 z^{-(n+3)/2}\dot{z}^2 - I(t)z^{(1-n)/2} - qz^{(n-1)/2}. \tag{3.44}$$

3.3 Reduction to quadratures of Eq. (3.21)

We now turn to perform reduction to quadratures of the differential equation (3.21). To this end, we let $R = dr/ds$ and exchange the roles of (r, s). This gives the first order equation

$$\frac{dR}{dr} = \frac{n+3}{4}\frac{R}{r} + \frac{r^n}{R}G(\omega), \quad \omega = \frac{1-n}{4}r^{-(n+1)/2}R.$$

Invariance of this equation under the dilational symmetry generated by the vector field $r\partial/\partial r + \frac{(n+1)}{2}R\,\partial/\partial R$ implies reduction to the separable form

$$\frac{d\omega}{d\xi} = \frac{1-n}{4}\omega + \frac{(1-n)^2}{16\omega}G(\omega), \tag{3.45}$$

which is achieved by changing coordinates to $(\omega, \xi = \ln r)$ and r, s defined by (3.22). Once a solution $\omega = \Phi(\xi, C_1)$ to (3.45) has been found, the general solution is obtained by integrating another separable first order ODE

$$\frac{dr}{ds} = R = \frac{4}{(1-n)}r^{(n+1)/2}\Phi(\ln r, C_1).$$

More conveniently, one can use the change of coordinates $\bar{s} = r^{(1-n)/2}$, $\bar{r} = s + r^{(1-n)/2}$ to transform (3.21) into

$$\bar{s}\frac{d^2\bar{r}}{d\bar{s}^2} = \widehat{G}\left(\frac{d\bar{r}}{d\bar{s}}\right) \tag{3.46}$$

with symmetry Lie algebra generated by $\langle \partial/\partial\bar{r}, \bar{s}\partial/\partial\bar{s} + \bar{r}\partial/\partial\bar{r}\rangle$ and a new arbitrary function $\widehat{G}$. Integration of (3.46) is straightforward.

3.4 Linearizable subclasses by Lie's test

In this subsection, we reconsider the canonical equation (3.20) for $r(s)$

$$r'' = f(r, r') = r^{-3}G(rr') = r^{-3}G(I) \tag{3.47}$$

and apply the Lie's test for a second order ODE in normal form $r'' = f(s, r, p)$, $p = r'$, which determines the necessary and sufficient conditions for transformability to a linear equation by a point transformation. Such conditions are expressed by the vanishing of the following fourth order relative invariants [31]

$$\mathbb{I}_1 = f_{pppp} = 0, \quad \mathbb{I}_2 = \widehat{D}_s^2 f_{pp} - 4\widehat{D}_s f_{rp} - f_p \widehat{D}_s f_{pp} + 6f_{rr} - 3f_r f_{pp} + 4f_p f_{rp} = 0, \tag{3.48}$$

where $\widehat{D}_s = \partial/\partial s + p\partial/\partial r + f\partial/\partial p$. The first condition requires that G must be a cubic polynomial of I, $G(I) = G_0 I^3 + G_1 I^2 + G_2 I + G_3$. The second condition restricts the coefficients in two possible forms

$$G_2 = G_3 = 0, \tag{3.49}$$

$$G_0 = \frac{G_2}{27G_3^3}(G_2^2 - 18G_3), \quad G_1 = \frac{G_2^2 - 5G_3}{3G_3}, \quad G_3 \neq 0. \tag{3.50}$$

The first choice gives the equation $r'' = G_0 r'^3 + G_1 s^{-1} r'^2$, which is equivalent to the linear equation $s''(r) + G_1 r^{-1} s'(r) + G_0 = 0$ by an exchange of the coordinates $s \leftrightarrow r$.

The other possibility gives the linearizable equation

$$r''(s) = \frac{G_2}{6G_3}\left(\frac{G_2^2}{18G_3} - 1\right)\frac{r'^3}{r^3} + 3\left(1 - \frac{G_2^2}{18G_3}\right)\frac{r'^2}{r} + G_2 r r' - 2G_3 r^3. \tag{3.51}$$

Reverting (s, r) back to (t, x) gives us a more general form of a linearizable second order ODE.

The special choice $G_3 = G_2^2/18$, $G_2 = -3\ell$ of the coefficients singles out a well-known second member of the Riccati chain (the modified Emden equation) [32, 33]

$$r'' + 3\ell r r' + \ell^2 r^3 = 0, \tag{3.52}$$

which is generated by the second iteration of the Riccati operator $\mathbb{D} = D_s + \ell r$:

$$\mathbb{D}^2 r = (D_s + \ell r)(D_s + \ell r) r = 0. \tag{3.53}$$

Eq. (3.53) is also recognised as a spacial case of the second order Riccati equation in the sense of Vessiot and Wallenberg [34]. This $\mathfrak{sl}(3,\mathbb{R})$ invariant equation can also be obtained from (3.11) by choosing $k=-2$, $a=1$ ($s(t)=t$) and $G(I)=3I^2-3\ell I+\ell^2/2$. By scaling $r\to\ell r$ we can put $\ell=1$.

Just like the ordinary first order Riccati equation, the Hopf–Cole transformation $r=\rho'/\rho$ linearizes (3.52) to the third order linear equation $\rho'''=0$. Moreover, a point transformation linearizing (3.52) to $R''(S)=0$ is provided by (see Example 5.5 of [12])

$$S=s-\frac{1}{r},\quad R=\frac{s^2}{2}-\frac{s}{r}. \tag{3.54}$$

We comment that though the more general form

$$r''+arr'+br^3=0 \tag{3.55}$$

does not pass Lie test unless $b=a^2/9$, it was shown to be linearizable to

$$\frac{d^2\bar{r}}{d\bar{s}^2}+a\frac{d\bar{r}}{d\bar{s}}+2b\bar{r}=0$$

by the nonlocal transformation $\bar{s}=\displaystyle\int^s r(\zeta)\,d\zeta$, $\bar{r}=r^2$ [35].

Finally, we mention that it was shown in [36] using an ansatz that a special case of second-order Riccati equation, in particular (3.52) with $\ell=1$, admits the (non-natural) Lagrangian

$$L=\frac{1}{r'+r^2}. \tag{3.56}$$

We can recover L by transforming the Lagrangian $L_0=R'^2$ of the free particle equation $R''=0$ by (3.54). The transformed Lagrangian $\bar{L}$ is obtained as

$$\bar{L}=\frac{[s(r'+r^2)-r]^2}{(r'+r^2)^2}D_sS=\frac{1}{r'+r^2}+s^2+\frac{s^2r'}{r^2}-\frac{2s}{r}=L+D_s\left[\frac{s^3}{3}-\frac{s^2}{r}\right].$$

Remark that as the Lagrangians $\bar{L}$ and L differ by a total derivative they give rise to the same Euler-Lagrange equation (3.52). In other words, L and $\bar{L}$ are gauge equivalent Lagrangians [37].

4 Conclusions and outlook

In this paper we have analysed the invariance of second order ODEs under a 2-dimensional affine Lie algebras realised by vector fields (3.1) as extensions of the EP-symmetry vector field (1.4). By construction, these type of equations can be integrated by Lie's standard reduction procedure. It is also possible to give some particular (invariant) solutions. In the rank two case, for a constant choice of the arbitrary function G appearing in the ODE, we have produced an equation of EP type (see (1.6)) but with potential $p(t)$ replaced by $p(t)-Ka^{-2}(t)$, K being some constant fixed by choice of a. The general solution formula for (1.6) remains unchanged. We have introduced a dissipative (damped) version of EP equation and presented its general solution (nonlinear superposition). Linearisable subclasses of the canonical ODEs are obtained by Lie's test.

As a final remark, let us mention that the presented study is not merely academic, for some equations treated here arise in different applications. For example, in the recent paper [38], the authors investigated solutions and first integrals of a second order ODE falling within the class (3.38), based on symmetry approach. This ODE is obtained from elimination of a dynamical system modeling the total population of Easter island [39]. Solutions can be readily recovered from our general results. A separate article [10] has recently been devoted to study integrability properties of a variable coefficient variant of the above-mentioned model by using results of the present work.

Acknowledgments

This work was initiated while one of the authors (F. G.) was visiting the Department of Theoretical Physics, Zaragoza University. This author is much indebted to J. F. Cariñena for invaluable discussions on Lie and Quasi-Lie systems and also to members of the Department for the warm hospitality.

References

[1] G. W. Hill. On the part of the motion of lunar perigee which is a function of the mean motions of the sun and moon. *Acta Math.* 8 (1): 1–36, 1886.

[2] W. Magnus and S. Winkler. Hill's equation, Dover 2004.

[3] G. Teschl. Ordinary differential equations and Dynamical Systems, Graduate Studies in Mathematics, Volume 140, Amer. Math. Soc., Providence, 2012.

[4] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, and P. J. Torres. Lie symmetries and solitons in nonlinear systems with spatially inhomogeneous nonlinearities. *Phys. Rev. Lett.* 98: 064102, 2007.

[5] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, and P. J. Torres. Lie symmetries, qualitative analysis and exact solutions of nonlinear Schrödinger equations with inhomogeneous nonlinearities. *Discrete and Continuous Dynamical Systems - Series B* 9(2): 221–233, 2008.

[6] P. Guha. Diffeomorphisms on S^1, projective structures and integrable systems. *ANZIAM J.* 44: 169–180, 2002.

[7] M. Adler. On a trace functional for formal pseudo-differential operators and the symplectic structure of the KdV. *Invent. Math.* 50: 219–248, 1979.

[8] I. M. Gelfand and L. A. Dikii. A family of Hamiltonian structures connected with integrable nonlinear differential equations. In: I. M. Gelfand, Collected papers, vol. 1, (Springer, 1987).

[9] P. Mathieu. Extended classical conformal algebras and the second Hamiltonian structure of Lax equations. *Phys. Lett. B* 208: 101–106, 1988.

[10] F. Güngör and P. J. Torres. Integrability of the Basener-Ross model with time-dependent coefficients. *SeMA Journal* 76: 485–493, 2019.

[11] P. J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics*. Springer, Springer-Verlag New York, 2nd edition, 1993.

[12] F. Güngör. Notes on Lie symmetry group methods for differential equations, arXiv:math.CA/1901.01543.

[13] J. F. Cariñena, J. de Lucas, and M. F. Rañada. Recent applications of the theory of Lie systems in Ermakov systems. *SIGMA 4*: 031, 2008.

[14] F. Güngör and P. J. Torres. Lie point symmetry analysis of a second order differential equation with singularity. *Journal of Mathematical Analysis and Applications* 451(2): 976–989, 2017.

[15] J. F. Cariñena and J. de Lucas. A nonlinear superposition rule for solutions of Milne–Pinney equation. *Phys. Lett. A* 372: 5385–5389, 2008.

[16] R. Campoamor-Stursberg. Perturbations of Lagrangian systems based on the preservation of subalgebras of Noether symmetries. *Acta Mechanica* 227(7): 1941–1956, 2016.

[17] J. Struckmeier and C. Riedel. Canonical transformations and exact invariants for time-dependent Hamiltonian systems. *Annalen der Physik* 11(1): 15–38, 2002.

[18] V. Ovsienko and S. Tabachnikov. What is the Schwarzian Derivative?. *Notices A.M.S.* 56: 34–36, 2009.

[19] J. F. Cariñena, P. Guha and J. de Lucas. Quasi-Lie Schemes approach to second-order Gambier equations. *SIGMA* 9: 026, 2013.

[20] J. de Lucas and C. Sardón. On Lie systems and Kummer-Schwarz equations. *Journal of Mathematical Physics* 54: 033505, 2013.

[21] J. F. Cariñena, J. Grabowski, J. de Lucas, and C. Sardón. Dirac–Lie systems and Schwarzian equations. *Journal of Differential Equations* 257: 2303–2340, 2014.

[22] M. S. Plyushchay. Schwarzian derivative treatment of the quantum second-order supersymmetry, anomaly, and coupling-constant metamorphosis. *Annals of Physics* 377: 164–179, 2017.

[23] F. Neuman. On Halphen and Laguerre–Forsyth canonical forms of linear differential equations. *Archivum Mathematicum* 26(2): 147–154, 1990.

[24] F. A. Homann. On transformations preserving Laguerre–Forsyth canonical form. *Procs. Amer. Math. Soc. 9(3): 408–411, 1958.*

[25] P. J. Olver. Equivalence, Invariants and Symmetry. Cambridge University Press, Cambridge, 1995.

[26] E. Kummer. Über die hypergeometrische reihe. *Crelle* 15: 3–83: ibid. 127–172, 1836.

[27] J. Krause and L. Michel. Équations différentielles linéaires d'ordre $n > 2$ ayant une algèbre de Lie de Symétrie de dimension $n + 4$. *C. R. Acad. Sci. Paris*, I 307: 905–910, 1988.

[28] J. C. Ndogmo. Coefficient characterization of linear differential equations with maximal symmetries. *Commun. Nonlinear Sci. Numer. Simulat.* 19: 1270–1278, 2014.

[29] C. Athorne. On a subclass of Ince equations. *Journal of Physics A: Mathematical and General*, 23(4): L137–L140, 1990.

[30] P. A. Clarkson and P. J. Olver. Symmetry and the Chazy equation. *Journal of Differential Equations* 124(1): 225–246, 1996.

[31] R. Milson and F. Valiquette. Point equivalence of second-order ODEs: Maximal invariant classification order. *Journal of Symbolic Computation* 67: 16–41, 2015.

[32] A. M. Grundland and D. Levi. On higher-order Riccati equations as Bäcklund transformations. *Journal of Physics A: Mathematical and General* 32(21): 3931–3937, 1999.

[33] J. F. Cariñena, P. Guha, and M. F. Rañada. Higher-order Abel equations: Lagrangian formalism, first integrals and Darboux polynomials. *Nonlinearity* 22(12): 2953–2969, 2009.

[34] H. T. Davis. Introduction to Nonlinear Differential Equations and Integral Equations Dover, New York, 1962.

[35] B. Abraham-Shrauner. Hidden symmetries and linearization of the modified Painlevé–Ince equation. *Journal of Mathematical Physics* 34(10): 4809–4816, 1993.

[36] J. F. Cariñena, M. F. Rañada, and M. Santander. Lagrangian formalism for nonlinear second-order Riccati systems: One-dimensional integrability and two-dimensional superintegrability. *Journal of Mathematical Physics* 46(6): 062703, 2005.

[37] J. F. Cariñena and L. A. Ibort. Non–Noether constants of motion. *J. Phys. A: Math. Gen.* 16: 1–7, 1983.

[38] M. C. Nucci and G. Sanchini. Symmetries, Lagrangians and conservation laws of an Easter island population model. *Symmetry* 7(3): 1613–1632, 2015.

[39] B. Basener and D. S. Ross. Booming and crashing populations and Easter island. *SIAM Journal on Applied Mathematics* 65(2): 684–701, 2004.

25 Solutions of Systems of Ordinary Differential Equations Using Invariants of Symmetry Groups. By Fatma Al-Kindia, Muhammad Ziadb (2019)

Solutions of Systems of Ordinary Differential Equations Using Invariants of Symmetry Groups

Fatma Al-Kindi[a)] and Muhammad Ziad[b)]

Department of Mathematics, College of Science, Sultan Qaboos University, Muscat, Sultanate of Oman

[a)]s18085@student.squ.edu.om
[b)]mziad@squ.edu.om

Abstract. We investigate the use of invariants of the admitted Lie groups of transformation in finding solutions of the systems of ordinary differential equations (ODEs). Bluman's theorem (1990) of invariant solutions of ODEs is extended for systems of ODEs. Differential invariants of a Lie group are used in reducing order of the given system. Examples are given to illustrate the methods.

INTRODUCTION

Invariant curves of Lie groups of transformations admitted by differential equations (DEs) provide a practical way of constructing their solutions and are known as invariant solutions of DEs. Invariants of the one-parameter Lie group admitted by an ordinary differential equation (ODE), if is not an invariant solution, may be used to reduce the order of the ODE. It was proved that if the r-parameter Lie group admitted by an nth order ODE is solvable, then the invariants of the group can reduce that ODE to $(n-r)$th order [1–6]. Bluman built a theorem which establishes a method of using one-parameter Lie groups admitted by a scalar ODE to find its invariant solutions [2, 7]. In the case an invariant solution does not exist, differential invariants play an important role in reduction of order of ODEs. Stephani [3] discussed the advantage of the method using differential invariants of r-parameter group admitted by a scalar ODE to find solution and gave a suggestion to use the method to find solutions of systems of ODEs. Here we extend Bluman's theorem for systems of ODEs, illuminate the use of differential invariants in reduction of order of systems of ODEs, and demonstrate the advantages of these methods by some examples.

Consider a system of k n^{th} order ordinary differential equations (ODEs)

$$F_i(t, \mathbf{x}, \mathbf{x}', ..., \mathbf{x}^{(n)}) = 0, \quad i = 1, ..., k, \tag{1}$$

where $\mathbf{x}^T = [x_1(t), x_2(t), ..., x_k(t)]$, and $\mathbf{x}' = \frac{d\mathbf{x}}{dt}$. Assume the system (1) admits the one parameter Lie group of transformations with infinitesimal generator

$$X = \xi(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_i(t, \mathbf{x})\frac{\partial}{\partial x_i}, \quad i = 1, ..., k. \tag{2}$$

We concentrate on the case where the number of equations in the system (1) is same as the number of dependent variables of the system, which is the case in most applications [4]. In what follows, we give some preliminaries required to be used in the next sections. Theorems mentioned in this section are well known and are given in many references. The notation ∂_x means the partial differential operator $\partial/\partial x$, while d/dt defines the total differential operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + x_i'\frac{\partial}{\partial x_i} + x_i''\frac{\partial}{\partial x_i'} + ... + x_i^{(n)}\frac{\partial}{\partial x_i^{(n-1)}}. \tag{3}$$

A curve $\Phi(t, \mathbf{x}) = [\phi_1(t, \mathbf{x}), \phi_2(t, \mathbf{x}), ..., \phi_k(t, \mathbf{x})]^T$ is an invariant of X if

$$X\Phi(t, \mathbf{x}) = \mathbf{0} \;\text{ when }\; \Phi(t, \mathbf{x}) = \mathbf{0}, \tag{4}$$

Modern Treatment of Symmetries, Differential Equations and Applications (Symmetry 2019)
AIP Conf. Proc. 2153, 020002-1–020002-8; https://doi.org/10.1063/1.5125067
Published by AIP Publishing. 978-0-7354-1898-1/$30.00

whereas $\Phi(t, \mathbf{x}) = \mathbf{0}$ is called an *invariant solution* of the system (1) related to its invariance under X, if it is an invariant curve of X and it satisfies the system (1). The prolonged n^{th} order form of X in (2) is given by

$$X^{(n)} = \xi(t,\mathbf{x})\frac{\partial}{\partial t} + \eta_i(t,\mathbf{x})\frac{\partial}{\partial x_i} + \eta_i'(t,\mathbf{x},\mathbf{x}')\frac{\partial}{\partial x_i'} + ... + \eta_i^{(n)}(t,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(n)})\frac{\partial}{\partial x_i^{(n)}}, \tag{5}$$

where

$$\eta_i^{(m)} = \frac{d\eta_i^{(m-1)}}{dt} - x_i^{(m)}\frac{d\xi}{dt}, \quad m = 1,...,n.$$

According to (4), the symmetry condition for the operator (2) to be admitted by the system of ODEs (1) is

$$X^{(n)}F_i(t,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(n)}) = 0 \;\text{ when }\; F_i(t,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(n)}) = 0, \quad \forall i = 1,...,k. \tag{6}$$

Assume that $\Phi(t, \mathbf{x}, \mathbf{x}', ..., \mathbf{x}^{(n)})$ is an invariant of (5), then

$$X^{(n)}\Phi = \mathbf{0},$$

and the characteristic equations

$$\frac{dt}{\xi} = \frac{dx_i}{\eta_i} = \frac{dx_i'}{\eta_i'} = ... = \frac{dx_i^{(n)}}{\eta_i^{(n)}} \tag{7}$$

provide an invariant Φ of $X^{(n)}$. Equations (7) can be written in the form

$$x_i^{(m)} = \frac{\eta_i^{(m-1)}}{\xi}, \quad m = 1,...,n \tag{8}$$

with the same solution Φ. An invariant of an infinitesimal (5) depending on the derivatives of the dependent variables is called a *differential invariant*. The order of the differential invariant is the highest derivative on which it depends. We conclude this section with the following theorem [3, 8].

Theorem 1 *If $\phi(t, \mathbf{x}, \mathbf{x}', ..., \mathbf{x}^{(p)})$ and $\psi(t, \mathbf{x}, \mathbf{x}', ..., \mathbf{x}^{(q)})$ are two functionally independent differential invariants of the symmetry generator $X^{(n)}$ of orders p and q respectively, where $p < q < n$, then*

$$\rho = \frac{d\psi}{d\phi} = \frac{d\psi/dt}{d\phi/dt}$$

is a differential invariant of $X^{(n)}$ of order $q + 1$.

EXTENSION OF BLUMAN'S THEOREM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Theorem 2 *Suppose that the system of ODEs (1) admits a one parameter Lie group of transformations with infinitesimal generator (2) in Domain $\mathbf{D} \subset \mathbf{R}^{k+1}$. We have two cases.*

case 1: *$\xi(t, \mathbf{x}) \neq 0$ in $\mathbf{D}$. Let*

$$\psi_i(t,\mathbf{x}) = \frac{\eta_i(t,\mathbf{x})}{\xi(t,\mathbf{x})}, \quad Y = \frac{\partial}{\partial t} + \psi_i(t,\mathbf{x})\frac{\partial}{\partial x_i} = \frac{1}{\xi(t,\mathbf{x})}X, \quad i = 1,...,k, \tag{9}$$

and

$$Q_i(t,\mathbf{x}) = F_i(t,\mathbf{x},\psi_1,...,\psi_k,Y\psi_1...,Y\psi_k,...,Y^{n-1}\psi_1,...,Y^{n-1}\psi_k).$$

1. *If any of $Q_i(t, \mathbf{x}) = 0$ is not defined in $\mathbf{D}$, then the system (1) has no invariant solution related to its invariance under (2).*
2. *If $Q_i(t, \mathbf{x}) \equiv 0$ in $\mathbf{D}$, then each invariant curve of (2) is an invariant solution of the system (1).*
3. *If $Q_i(t, \mathbf{x}) \not\equiv 0$ but $Q_i(t, \mathbf{x}) = 0$ defines curves in $\mathbf{D}$, then these curves define invariant solutions for the system (1) in $\mathbf{D}$.*

case 2: *$\xi(t,\mathbf{x}) \equiv 0$ in **D**.*

1. *If $\eta_i(t,\mathbf{x}) = 0$ defines curves $x_i = \phi_i(t)$ in $\mathbf{D}$, then these curves define invariant solutions for the system (1).*
2. *If $\eta_i(t,\mathbf{x}) \neq 0$ in $\mathbf{D}$, then the system (1) has no invariant solution relating to its invariance under (2).*

Proof. **case 1:** $\xi \neq 0$:
The corresponding prolonged vector field of Y in equation (9) is given by

$$Y^{(n)} = \frac{1}{\xi(t,\mathbf{x})}X^{(n)} = \frac{\partial}{\partial t} + \frac{\eta_i}{\xi}\frac{\partial}{\partial x_i} + \frac{\eta_i'}{\xi}\frac{\partial}{\partial x_i'} + ... + \frac{\eta_i^{(n-1)}}{\xi}\frac{\partial}{\partial x_i^{(n-1)}}.$$

Using the invariance condition (8)

$$Y^{(n)} = \frac{\partial}{\partial t} + x_i'\frac{\partial}{\partial x_i} + x_i''\frac{\partial}{\partial x_i'} + ... + x_i^n\frac{\partial}{\partial x_i^{(n-1)}} = \frac{d}{dt}. \tag{10}$$

which is the total differential defined in equation (3). Furthermore, if

$$\psi_i(t,\mathbf{x}) = \frac{\eta_i(t,\mathbf{x})}{\xi(t,\mathbf{x})},$$

then from the invariance condition (8)

$$x_i' = \psi_i, \tag{11}$$

and from (10)

$$x_i'' = \frac{dx_i'}{dt} = \frac{d\psi_i}{dt} = Y\psi_i.$$

Then by induction,

$$x_i^{(m)} = Y^{(m-1)}\psi_i, \quad i = 1,...,k, \;\; m = 1,...,n,$$

and the solution of the above equations are invariants of (5). Using (8), the latter equations can be written as

$$\eta_i^{(m)} = \xi Y^{(m)}\psi_i. \tag{12}$$

Now assume that system (1) admits the symmetry generator given by (2), then the invariance condition (6) holds, i.e.,

$$\xi\left(\frac{\partial F_i}{\partial t} + \frac{\eta_j}{\xi}\frac{\partial F_i}{\partial x_j} + \frac{\eta_j'}{\xi}\frac{\partial F_i}{\partial x_j'} + ... + \frac{\eta_j^n}{\xi}\frac{\partial F_i}{\partial x_j^n}\right) = 0,$$

and from (12),

$$\xi\left(\frac{\partial F_i}{\partial t} + \psi_j\frac{\partial F_i}{\partial x_j} + Y\psi_j\frac{\partial F_i}{\partial x_j'} + ... + Y^{(n)}\psi_j\frac{\partial F_i}{\partial x_j^n}\right) = 0. \tag{13}$$

Writing

$$Q_i(t,\mathbf{x}) = F_i(t,\mathbf{x},\psi_1,...,\psi_k,Y\psi_1...,Y\psi_k,...,Y^{n-1}\psi_1,...,Y^{n-1}\psi_k),$$

then

1. if any of $Q_i(t,\mathbf{x}) = 0$ is not defined in $\mathbf{D}$, then the invariants $\phi_i(t,\mathbf{x})$ are not solutions of (1), and hence there is no invariant solution of (1) related to (2).
2. if $Q_i(t,\mathbf{x}) \equiv 0$, $\forall i$, i.e, Q_i is identically zero, then all solutions of (11) are invariant solutions of system (1).
3. if $Q_i(t,\mathbf{x}) \not\equiv 0$, i.e, Q_i is not identically vanishing, then when $Q_i(t,\mathbf{x}) = 0$ we have

$$X^{(n)}Q_i = \xi\left(\frac{\partial Q_i}{\partial t} + \frac{\eta_j}{\xi}\frac{\partial Q_i}{\partial x_j} + \frac{\eta_j'}{\xi}\frac{\partial Q_i}{\partial x_j'} + ... + \frac{\eta_j^n}{\xi}\frac{\partial Q_i}{\partial x_j^n}\right).$$

   Using equation (12)

$$X^{(n)}Q_i = \xi\left(\frac{\partial F_i}{\partial t} + \psi_j\frac{\partial F_i}{\partial x_j} + Y\psi_j\frac{\partial F_i}{\partial x_j'} + ... Y^{(n)}\psi_j\frac{\partial F_i}{\partial x_j^n}\right),$$

   which, from equation (13) reduces to

$$X^{(n)}Q_i = 0.$$

   Therefore $Q_i(t,\mathbf{x}) = 0$ defines an invariant solution of system (1).

case 2: $\xi \equiv 0$:

1. If $\eta_i(t, \mathbf{x}) = 0$, then solving for x_i gives

$$E_i \equiv x_i - \phi_i(t) = \mathbf{x} - \Phi(t) = 0, \quad i = 1, ..., k, \tag{14}$$

where $\Phi^T = [\phi_1(t), \phi_2(t), ..., \phi_k(t)]$. But since $\xi \equiv 0$, using $\eta_i(t, \mathbf{x}) = \eta_i(t, \Phi(t)) = 0$, we have

$$XE_i = \eta_j(t, \Phi(t)).\delta_i^j = 0, \quad \text{where } \delta_i^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{15}$$

showing that (14) is an invariant of X. Now, we show that (14) satisfies the system (1), i.e., if

$$\mathbf{x} = \Phi(t), \ \mathbf{x}' = \Phi'(t), \ \mathbf{x}'' = \Phi''(t), \ \mathbf{x}^{(n)} = \Phi^{(n)}(t),$$

then

$$F_i = \left(t, \Phi(t), \Phi'(t), ..., \Phi^{(n)}(t)\right) = 0, \quad i = 1, ..., k.$$

From the special form of the generator X, ($\xi = 0$), then any function $\Psi(t)$ of t only will be invariant of X. Let $\Psi = \Phi(t)$ of equation (15). Then the functions of the vector $\Phi(t)$ and all their ordinary derivatives, as well as the simple function $\psi(t) = t$, will be invariants of X. Then the general form of the nth order invariant of X is

$$G\left(t, \Phi(t), \Phi'(t), ..., \Phi^{(n)}(t)\right) = 0,$$

where G is an arbitrary function of $t, \Phi(t), \Phi'(t), ..., \Phi^{(n)}(t)$. Let G be F_i. Then,

$$F_i\left(t, \Phi(t), \Phi'(t), ..., \Phi^{(n)}(t)\right) = 0. \tag{16}$$

Hence from (16) and (15), $\mathbf{x} = \Phi(t)$ are invariant solution system of (1).

2. If $\eta_i(t, \mathbf{x}) \neq 0$ in $\mathbf{D}$, then, it is obvious that there is no invariant solution for system (1) related to its invariance under (2).

□

Example 1 *The system*

$$\ddot{x} = \dot{x} + \frac{\dot{y}}{y}e^{-t}, \tag{17}$$

$$\ddot{y} = \frac{\dot{y}^2}{y} + \dot{y} + y, \tag{18}$$

admits, among others, the symmetry generators

$$X_1 = \frac{\partial}{\partial t} + \frac{1}{2}e^{-t}\frac{\partial}{\partial x} + y(\ln y + t)\frac{\partial}{\partial y}, \quad X_2 = (\ln y + t)\frac{\partial}{\partial x}.$$

We will choose the symmetry generator X_1 since $\xi \neq 0$. Here $\xi = 1$, $\eta_1 = \frac{1}{2}e^{-t}$ and $\eta_2 = y(\ln y + t)$. Then

$$Y = \frac{1}{\xi}X_1 = X_1, \quad \psi_1 = \frac{1}{2}e^{-t}, \quad \psi_2 = y(\ln y + t),$$

and

$$Y\psi_1 = -\frac{1}{2}e^{-t}, \quad Y\psi_2 = y(1 + (\ln y)^2 + \ln y + 2t\ln y + t^2 + t).$$

Therefore

$$Q_1(t, x, y, \psi_1, \psi_2, Y\psi_1, Y\psi_2) = e^{-t}(\ln y + t - 1),$$
$$Q_2(t, x, y, \psi_1, \psi_2, Y\psi_1, Y\psi_2) \equiv 0.$$

$Q_1 = 0$ gives

$$y(t) = e^{1-t},$$

while the second equation means that the solution of $\dot{x} = \psi_1$ is an invariant curve of the system, namely,

$$x(t) = -\frac{1}{2}e^{-t} + c,$$

where c is a constant.
Regarding to the symmetry generator X_2, where $\xi \equiv 0$, $\eta_i = 0$ implies

$$\eta_1(t, x, y) = \ln y + t = 0,$$

giving

$$y(t) = e^{-t}$$

which when substituted into the equations (17) and (18) gives

$$x(t) = -\frac{1}{2}e^{-t} + c_1 e^t + c_2.$$

THE USE OF DIFFERENTIAL INVARIANTS

Scalar ODEs

We consider the case when $\xi = 0$, and use differential invariants either to find invariant first integrals of the ODE, or use them to reduce the order of an ODE. Following theorem illustrates this idea.

Theorem 3 *Let*

$$F(t, x, x', ..., x^{(n)}) = 0 \tag{19}$$

be a single n^{th} order ODE admitting the symmetry generator

$$X = \eta(t, x)\frac{\partial}{\partial x}, \tag{20}$$

and

$$x' - f(t, x, \phi) = 0 \tag{21}$$

is a solution of the ODE

$$\frac{dx'}{dx} = \frac{\eta'(t, x, x')}{\eta(t, x)}.$$

Then (21) is an invariant solution for (19) if and only if

$$F\left(t, x, f, f^{(1)}, ..., f^{(n-1)}\right) \equiv 0,$$

where $f^{(m)} = df^{(m-1)}/dt$, $m = 1, ..., n-1$, $\eta' = d\eta/dt$ and ϕ is the integration constant. If $F\left(t, x, f, f^{(1)}, ..., f^{(n-1)}\right) \not\equiv 0$, then

$$F\left(t, x, f(t, x, \phi), f^{(1)}(t, x, x', \phi'), ..., f^{(n-1)}(t, x, x', ..., x^{(n-1)}, \phi^{(n-1)})\right) = 0$$

is an $(n-1)$th order ordinary differential equation of the variables t, $\phi(t, x, x')$, $\phi'(t, x, x')$,...,$\phi^{(n-1)}(t, x, x', ..., x^{(n-1)})$.

Proof. The first prolongation of the generator (20) is

$$X^{(1)} = \eta_(t, x)\frac{\partial}{\partial x} + \eta'(t, x, x')\frac{\partial}{\partial x'}.$$

whose first order invariant, $\phi = \phi(t, x, x')$ or $x' = f(t, x, \phi)$, can be found by solving the characteristic equation

$$\frac{dx}{\eta(t, x)} = \frac{dx'}{\eta'(t, x, x')}.$$

Since $\xi \equiv 0$, then any function $\phi_\circ = \phi_\circ(t)$ is a zero order invariant of (20). We choose $\phi_\circ(t) = t$. Then by theorem (1), the second order invariant is

$$\phi'(t, x, x', x'') = \frac{d\phi/dt}{d\phi_\circ/dt} = \frac{d\phi/dt}{1} = x'' - \frac{d}{dt}f = 0.$$

Consequently

$$x' = f, x'' = \frac{d}{dt}f = f^{(1)}, ..., x^{(n)} = \frac{d}{dt}f^{(n-2)} = f^{(n-1)}. \tag{22}$$

Now, suppose that (21) is an invariant solution, then it must satisfy the ODE (19). Equations (22) then gives

$$F\left(t, x, f, f^{(1)}, ..., f^{(n-1)}\right) = F\left(t, x, x', ..., x^{(n)}\right) \equiv 0. \tag{23}$$

Conversely, if (23) is satisfied, then

$$x' = f,\ x'' = f^{(1)},\ ..., x^{(n)} = f^{(n-1)},$$

are first integrals of (19). Thus from equations (22), $x' - f(t, x, \phi) = 0$ is an invariant solution of (19). Consequently, from equation (22),

$$\phi^{(n-1)}(t, x, x', ..., x^{(n)}) = \phi^{(n-1)}(t, x, f, f', ..., f^{(n-1)}) \equiv 0.$$

If $F\left(t, x, f, f^{(1)}, ..., f^{(n-1)}\right) \not\equiv 0$, the nth order invariant of the generator X is of the form

$$G(t, \phi, \phi', ..., \phi^{(n-1)}) = 0,$$

where G is arbitrary. Hence

$$G(t, \phi(t, x, x'), \phi'(t, x, x', x''), ..., \phi^{(n-1)}(t, x, x', ..., x^{(n)})) = 0.$$

Since G is arbitrary, we can choose it such that the above equation can be written as

$$F(t, x, x', ..., x^{(n)}) = 0,$$

or

$$F\left(t, x, f, f^{(1)}, ..., f^{(n-1)}\right) = 0.$$

□

Example 2 *The ODE*

$$x'' - \frac{2x'^2}{x} - x' - tx^2 = 0 \tag{24}$$

admits the symmetry generator

$$X^{(1)} = x^2\frac{\partial}{\partial x} + 2xx'\frac{\partial}{\partial x'}.$$

The first order invariant of the above generator is

$$\phi(t, x, x') = \frac{x'}{x^2},$$

which by solving for x' we get

$$x' = f(t, x, \phi) = \phi x^2.$$

Thus $f^{(1)} = 2\phi xx' + \phi' x^2$, while $F(t, x, f, f^{(1)}) = \phi' x^2 + 2\phi^2 x^3 - 2\phi^2 x^3 - \phi x^2 - tx^2 \not\equiv 0$. Hence $F(\phi, \phi_\circ) = 0$ gives the reduced ODE

$$\phi' - \phi - \phi_\circ = 0$$

whose solution is

$$\phi(\phi_\circ) = -(\phi_\circ + 1) + c\ exp(\phi_\circ).$$

Now substituting back the values of ϕ and $\phi_\circ$ into the latter equation gives the reduced first order ODE

$$\frac{x'}{x^2} = -(t + 1) + c_1 e^t,$$

with solution

$$x(t) = \frac{2}{t^2 + 2t - 2c_1 t + c_1},$$

where c_1 and c_2 are constants of integration.

System of second order ODEs

If the differential invariant satisfies the given system of ODEs then it is an invariant solution, namely, first integral for the system of ODEs. If not, we can use functionally independent differential invariants to reduce the order of the system of ODEs and we can obtain from zero to n^{th} order differential invariants by solving the characteristic equations (7). Here we will concentrate on a system of second order ODEs

$$F_i(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'') = 0, \quad i = 1, 2, \tag{25}$$

admitting the symmetry generator

$$X^{(1)} = \eta_i(t, \mathbf{x})\frac{\partial}{\partial x_i} + \eta_i'(t, \mathbf{x}, \mathbf{x}')\frac{\partial}{\partial x_i'}, \quad i = 1, 2, \tag{26}$$

with $\xi \equiv 0$. Any function of the independent variable t will be an invariant of (26) of order zero, and so the function $u_\circ(t) = t$ will be the simplest invariant with $du_\circ/dt = 1$. The first order invariants $u_i = u_i(t, \mathbf{x}, \mathbf{x}')$ are solutions of the characteristic system

$$\frac{dx_i'}{dx_i} = \frac{\eta_i'(t, \mathbf{x}, \mathbf{x}')}{\eta_i(t, \mathbf{x})}, \quad i = 1, 2.$$

Note that in solving the latter equations, we treat the variable t as constant since it is invariant of X in (26) and so constant along it. From theorem (1),

$$u_i'(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'') = \frac{du_i/dt}{du_\circ/dt} = \frac{du_i/dt}{1} = \frac{du_i}{dt}, \quad i = 1, 2,$$

is an invariant of (26) of second order. If both components of x_1 and x_2 are not vanishing in (26), then we can find the zero order differential invariant $u_\circ = u_\circ(t, x_1, x_2)$ as the solution of the characteristic

$$\frac{dx_1}{\eta_1(t, \mathbf{x})} = \frac{dx_2}{\eta_2(t, \mathbf{x})},$$

and in this case

$$u_i'(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'') = \frac{du_i/dt}{du_\circ/dt}, \quad i = 1, 2.$$

Once we find the above differential invariants, we substitute the original variables t, x_1, x_2 and their derivatives using system (25) and the invariants $u_\circ$, u_1 and u_2. This will give a system of first order ODEs

$$u_i' = G_i\left(u_\circ, u_1, u_2\right), \ i = 1, 2.$$

The solution of this latter system gives a reduced form of the system (25)

$$u_i = f_i\left(u_\circ\right), \ i = 1, 2,$$

of order 1.

Example 3 *The system in example (1) admits, in addition to others, the Lie point symmetry generator*

$$X_3^{(1)} := y\frac{\partial}{\partial y} + \dot{y}\frac{\partial}{\partial \dot{y}},$$

whose zero and first order invariants are, respectively,

$$u_\circ = t, \quad u_1 = \frac{\dot{y}}{y}. \tag{27}$$

We have

$$\frac{du_1}{du_\circ} = \frac{\ddot{y}}{y} - \frac{\dot{y}^2}{y^2}, \quad \text{or} \quad u_1' = u_1 + 1.$$

Solving the above first order ODE and substituting back the original variables from (27) we get

$$\dot{y} = y(c_1 e^t - 1). \tag{28}$$

Substituting this result into (17), we get $\ddot{x} - \dot{x} = c_1 - e^t$*, whose first integral*

$$\dot{x} = c_1 + \frac{1}{2}e^{-t} + c_2 e^t. \tag{29}$$

Hence from (28) and (29) we have a first order reduced system which solves to

$$x(t) = -\tfrac{1}{2}e^{-t} + c_1 t + c_2 e^t + c_3,$$
$$y(t) = c_4 e^{\alpha}, \quad \text{where} \quad \alpha = c_1 e^t - 1.$$

CONCLUSION

An ODE admitting a symmetry group may be either completely solved by using the generators of the Lie algebra of the symmetry group or otherwise the generators may be used to reduce the order of the ODE. The invariants of the generators of the Lie algebra admitted by a single ODE is widely discussed in literature [1–9]. The invariant curves when satisfy the ODE appear as invariant solutions. Bluman developed a method of finding invariant solutions for a single ODE and proved a theorem which provides a criterion for a given ODE to admit an invariant solution [2, 7]. In this paper we have extended these methods for finding solutions of a system of ODEs. We have concentrated on the case where the number of dependent variables are identical to the number of ODEs and have been able to prove Theorem 2 which provides conditions for a system of ODEs to admit invariant solutions.

In the case of non-existence of invariant solutions, we have made use of the idea of differential invariants, elaborated in Theorem 1. This explains method of finding all functionally independent differential invariants of a symmetry generator in the case of a single ODE. Using this approach we have been able to prove Theorem 3 which provides a criterion of reducing the order of an n^{th} order scalar ODE either to order one or $n - 1$.

Stephani [3] discussed the advantage of the method of using differential invariants of Lie groups admitted by a scalar ODE to find their solutions and gave a suggestion to use the method to find solutions of systems of ODEs. We are presently working on the methods of finding solutions of systems of ODEs for which invariant solutions cannot be obtained in the light of Theorem 2. For a system of ODEs, Theorem 1 does not guarantee to provide full set of differential invariants [3]. Due to this limitation Theorem 3 cannot be generalised for a system of ODEs. However, in the case where $\xi \equiv 0$ (refer to equation (5)) this approach may be useful to reduce the order of a system of second order ODEs. Examples have been provided to illustrate the results proved in the paper.

ACKNOWLEDGMENTS

This work is undertaken under the internal grant project: IG/SCI/MATH/19/02 of Sultan Qaboos University.

REFERENCES

[1] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, 1982).
[2] G. W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations* (Springer, New York, 2002).
[3] H. Stephani, *Differential Equations: Their Solution Using Symmetries* (Cambridge University Press, Cambridge, 1989).
[4] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations* (John Wiley and Sons, Chichester, 1999).
[5] L. P. Eisenhart, *Continuous Groups Of Transformations* (Princeton University Press, Princeton, 1933).
[6] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986).
[7] G. W. Bluman, SIAM J. Appl. Math. **50**, 1706 (1990).
[8] G. Emanuel, *Solution of Ordinary Differential Equations by Continuous Groups* (Chapman and Hall/CRC, Boca Raton, 2001).
[9] F. Schwarz, *Algorithmic Lie Theory for Solving Ordinary Differential Equations* (Chapman and Hall/CRC, Boca Raton, 2008).

26 On the Method of Differential Invariants for Solving Higher Order Ordinary Differential Equations. By Winter Sinkala, Molahlehi Charles Kakuli (2020)

axioms

MDPI

Review

On the Method of Differential Invariants for Solving Higher Order Ordinary Differential Equations

Winter Sinkala * and Molahlehi Charles Kakuli

Department of Mathematical Sciences and Computing, Faculty of Natural Sciences, Walter Sisulu University, Private Bag X1, Mthatha 5117, South Africa
* Correspondence: wsinkala@wsu.ac.za

Abstract: There are many routines developed for solving ordinary differential Equations (ODEs) of different types. In the case of an nth-order ODE that admits an r-parameter Lie group $(3 \leq r \leq n)$, there is a powerful method of Lie symmetry analysis by which the ODE is reduced to an $(n-r)$th-order ODE plus r quadratures provided that the Lie algebra formed by the infinitesimal generators of the group is solvable. It would seem this method is not widely appreciated and/or used as it is not mentioned in many related articles centred around integration of higher order ODEs. In the interest of mainstreaming the method, we describe the method in detail and provide four illustrative examples. We use the case of a third-order ODE that admits a three-dimensional solvable Lie algebra to present the gist of the integration algorithm.

Keywords: ordinary differential equation; lie symmetry analysis; solvable lie algebra; differential invariant; reduction of order

MSC: 34A05; 34C14; 34C20

Citation: Sinkala, W.; Kakuli, M.C. On the Method of Differential Invariants for Solving Higher Order Ordinary Differential Equations. *Axioms* **2022**, *11*, 555. https://doi.org/10.3390/axioms11100555

Academic Editor: Carlos Escudero

Received: 24 September 2022
Accepted: 12 October 2022
Published: 14 October 2022

1. Introduction

The study of ODEs poses significant challenges, especially in cases involving equations of higher order that are nonlinear. As a result, various methods have been proposed for investigating different types of ODEs. Chandrasekar et al. [1], for example, propose a method that unifies and generalises known linearising transformations for finding general solutions of third-order nonlinear ODEs. Related work is done by Mohanasubha et al. [2] who propose a method of solution that involves deriving linearising transformations for a class of second-order nonlinear ordinary differential equations. In [3], conditions are provided for the linearisation of third-order ODEs by tangent transformations (see also the references in [3] for related work on the problem of transforming a given differential equation into a linear equation). It turns out that "symmetry properties", which are central in Lie symmetry analysis of differential equations, by and large, provide a basis for systematically solving the majority of ordinary differential equations for which exact solutions can be found [3–13].

There are several ways in which the symmetry group associated with a differential equation can be used to analyse the equation. For a given differential equation, the symmetry group may be used to derive new solutions of the equation from old ones [5,7], to reduce the order of the equation [5,7,8] or to establish whether or not the equation can be linearised, and to construct explicit linearisations when such exist [14–16]. Other uses include the derivation of conserved quantities [7].

Many symmetry-based approaches for solving ODEs involve reduction of order, whereby for a given ODE of order n, the problem is reduced to that of solving one or more ODEs of order at most $n-1$. Lie symmetry analysis has well-established algorithms for solution methods based on reduction of order. It is well known, in particular, that if an

nth-order ODE admits a one-parameter Lie symmetry group, then the order of the equation can be reduced by one. The method of differential invariants extends this in that an ODE of order n is reduced to an ODE of order $n-1$ plus r quadratures (where $3 \leq r \leq n$) provided that the ODE is invariant under an r-parameter Lie group whose infinitesimal generators form an r-dimensional solvable Lie algebra [5,12,17]. The method is essentially a general integration procedure for solving (or, at least, reduction of order of) any higher order ODE that admits a solvable lie algebra of the right dimension. It consists of a number of successive iterations that reduce the problem to integration of a number of first-order ODEs each of which has an admitted Lie point symmetry. Therefore, each of the first-order ODEs may be integrated routinely using the admitted Lie point symmetry [4–9]. It seems that the method of differential invariants has not been used widely to study higher order ODEs as we could not find many applications in the literature.

In this paper, we describe the method of differential invariants and provide four instructive examples involving nonlinear third-order ODEs that arise in different contexts.

The rest of the article is organised as follows: In Section 2, we present the algorithm of the method of differential invariants in the case where a third-order ODE admits a three-dimensional solvable Lie algebra. In Section 3, we provide four illustrative examples. We give concluding remarks in Section 4.

2. Reduction Algorithm for an nth-Order ODE ($n \geq 3$) with a Solvable Lie Algebra

Let us assume that an nth-order ODE admits an r-parameter Lie group of transformations. There is a reduction algorithm [5] by means of which the ODE can be reduced to an $(n-1)$th-order ODE plus r quadratures provided that the infinitesimal generators of the admitted Lie group form an r-dimensional solvable Lie algebra. We present the reduction algorithm in the simplified case involving a third-order ODE that admits a 3-parameter solvable Lie algebra. In this case, the reduction algorithm results in the general solution of the ODE.

Consider a third-order

$$f(x, y, y', y'', y''') = 0 \tag{1}$$

that admits a 3-parameter Lie group of point transformations, and for which the associated infinitesimal generators Y_1, Y_2, Y_3 form a solvable Lie algebra. Without loss of generality, we can assume that the infinitesimal generators have the following commutation relations:

$$[Y_i, Y_j] = \sum_{k=1}^{j-1} C_{ij}^k Y_k, \quad 1 \leq i < j, \quad j = 2, 3. \tag{2}$$

for some real structure constants C_{ij}^k [5].

Let $r_1(x, y)$, $v_1(x, y, y')$ be such that

$$Y_1 r_1 = 0, \quad Y_1^{(1)} v_1 = 0,$$

so that

$$w_1 = \frac{dv_1}{dr_1} \tag{3}$$

is a differential invariant, i.e., $Y_1^{(2)} w_1 = 0$. In terms of the invariants r_1 and v_1, and the differential invariant w_1, (1) is reduced to a second-order ODE

$$w_1 = \psi^1(r_1, v_1), \tag{4}$$

for some function ψ^1. Writing $Y_2^{(1)}$ in terms of r_1 and v_1, we obtain

$$Y_2^{(1)} = \alpha_1(r_1) \frac{\partial}{\partial r_1} + \beta_1(r_1, v_1) \frac{\partial}{\partial v_1}, \tag{5}$$

with the first extension given by

$$Y_2^{(2)} = Y_2^{(1)} + \gamma_1(r_1, v_1, w_1)\frac{\partial}{\partial w_1}, \tag{6}$$

where

$$\alpha_1(r_1) = Y_2 r_1, \quad \beta_1(r_1, v_1) = Y_2^{(1)} v_1, \quad \gamma_1(r_1, v_1, w_1) = Y_2^{(2)} w_1,$$

for some functions α_1, β_1 and γ_1. It is noteworthy that (5) is admitted by Equation (4).

Let $r_2(r_1, v_1)$, $v_2(r_1, v_1, w_1)$ be such that

$$Y_2^{(1)} r_2 = 0, \quad Y_2^{(2)} v_2 = 0,$$

so that

$$w_2 = \frac{dv_2}{dr_2} \tag{7}$$

is a differential invariant, i.e., $Y_2^{(3)} w_2 = 0$. In terms of the invariants r_2, v_2 and w_2, the ODE (1) reduces to a first-order ODE

$$w_2 = \psi^2(r_2, v_2), \tag{8}$$

for some function ψ^2. Writing $Y_3^{(2)}$ in terms of r_2 and v_2, we obtain

$$Y_3^{(2)} = \alpha_2(r_2)\frac{\partial}{\partial r_2} + \beta_2(r_2, v_2)\frac{\partial}{\partial v_2}, \tag{9}$$

with the first extension given by

$$Y_3^{(3)} = Y_3^{(2)} + \gamma_2(r_2, v_2, w_2)\frac{\partial}{\partial w_2}, \tag{10}$$

where

$$\alpha_2(r_2) = Y_3^{(1)} r_2, \quad \beta_2(r_2, v_2) = Y_3^{(2)} v_2, \quad \gamma_2(r_2, v_2, w_2) = Y_3^{(3)} w_2,$$

for some functions α_2, β_2 and γ_2. Here also (9) is admitted by Equation (8).

In light of the admitted symmetry (10), the first-order Equation (8) can be integrated routinely to give a solution of the form

$$v_2 = \omega^2(r_2) \tag{11}$$

for some function ω^2. Expressing (11) in terms of v_1 and r_1, we obtain a first-order ODE

$$\frac{dv_1}{dr_1} = \psi^1(v_1, r_1), \tag{12}$$

i.e., we determine the hitherto unknown function ψ^1 in (4). Solving Equation (12), we obtain a solution of the form

$$v_1 = \omega^1(r_1) \tag{13}$$

for some function ω^1. Again, the solution (13) can be expressed in terms of x and y to obtain the last first-order ODE in the form

$$\frac{dy}{dx} = \psi^0(x, y), \tag{14}$$

for some function ψ^0. Equation (14) admits Y_1 and, when solved, provides the general solution of Equation (1).

3. Illustrative Examples

In this section, we use the method of differential invariants to find general solutions of four third-order ODEs, each of which admits a symmetry Lie algebra of order greater than three. In each case, we identify a three-dimensional solvable subalgebra and use it to perform complete integration of the ODE.

Example 1. *Consider the ODE*

$$(y')^2 y'' - 2y(y'')^2 + yy'y''' = 0, \tag{15}$$

which arises in the context of group classification of the $1+1$ *Fokker–Planck diffusion-convection equation [18]*

$$\theta_t = [D(\theta)\theta_z]_z - K'(\theta)\theta_z, \tag{16}$$

where t *is time,* z *is the depth,* $\theta(t,z)$ *is the volumetric soil water content,* $D(\theta)$ *is the soil water diffusivity and* $K(\theta)$ *is the hydraulic conductivity, with* $K'(\theta) = \frac{dK}{d\theta} \neq 0$.

Besides the translation symmetries

$$X_1 = \frac{\partial}{\partial z} \quad \text{and} \quad X_2 = \frac{\partial}{\partial t}, \tag{17}$$

which are clearly admitted by (16), additional symmetries are possible only if D *solves this third-order nonlinear ODE [19]*

$$D'(\theta)^2 D''(\theta) - 2D(\theta)D''(\theta)^2 + D(\theta)D'(\theta)D'''(\theta) = 0, \tag{18}$$

which is Equation (15) with θ *and* D *replaced with* x *and* y*, respectively.*

Equation (15) admits a four-dimensional symmetry Lie algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = y\ln y\frac{\partial}{\partial y}. \tag{19}$$

We use the solvable algebra $\langle X_1, X_3, X_4\rangle$*, for which*

$$[X_3, X_4] = X_3 \tag{20}$$

is the only nonzero Lie bracket. We relabel the symmetries as follows:

$$X_3 \to Y_1, \quad X_4 \to Y_2, \quad X_1 \to Y_3,$$

to ensure that the commutation relations of the operators Y_1, Y_2 *and* Y_3 *satisfy (2).*

To carry out the reduction algorithm, we first need the following extended infinitesimal generators:

$$\left.\begin{aligned} Y_1^{(1)} &= y\tfrac{\partial}{\partial y} + y'\tfrac{\partial}{\partial y'} \\ Y_2^{(2)} &= y\ln y\tfrac{\partial}{\partial y} + y'(1+\ln y)\tfrac{\partial}{\partial y'} + \left(\tfrac{y'^2}{y} + y'' + y''\ln y\right)\tfrac{\partial}{\partial y''} \\ Y_3^{(3)} &= \tfrac{\partial}{\partial x}. \end{aligned}\right\} \tag{21}$$

Starting with $Y_1^{(1)}$*, we solve the corresponding characteristic equations*

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dy'}{y'} \tag{22}$$

to obtain invariants

$$r_1 = x, \quad v_1 = \frac{y'}{y}, \tag{23}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = \frac{yy'' - (y')^2}{y^2}. \tag{24}$$

Writing $Y_2^{(2)}$ in terms of r_1, v_1 and w_1, we obtain

$$Y_2^{(2)} = v_1 \frac{\partial}{\partial v_1} + w_1 \frac{\partial}{\partial w_1}. \tag{25}$$

From the corresponding characteristic equation

$$\frac{dr_1}{0} = \frac{dv_1}{v_1} = \frac{dw_1}{w_1}, \tag{26}$$

we obtain invariants

$$r_2 = r_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1}, \tag{27}$$

which, in view of (23), can be written in terms of x, y, y' and y'' as follows:

$$r_2 = x \quad \text{and} \quad v_2 = \frac{yy'' - (y')^2}{yy'}. \tag{28}$$

From (28) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{(y')^2}{y^2} - \frac{y''}{y} + \frac{y'y''' - (y'')^2}{(y')^2}. \tag{29}$$

Equation (15) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. To find ψ^2, we express Equation (15) as

$$y''' = \frac{2y(y'')^2 - (y')^2 y''}{yy'}. \tag{30}$$

and replace y''' in (29) by the right hand-side of (30). We obtain

$$\frac{dv_2}{dr_2} = \left[\frac{yy'' - (y')^2}{yy'}\right]^2 = v_2^2, \tag{31}$$

which is a first-order ODE that admits $Y_3^{(2)}$ written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = \frac{\partial}{\partial r_2}. \tag{32}$$

Solving (31) we obtain

$$v_2 = -\frac{1}{r_2 + \kappa_1}, \tag{33}$$

where κ_1 is an arbitrary constant. In terms of r_1 and v_1, Equation (33) is transformed, via (27), into another first-order ODE,

$$\frac{dv_1}{dr_1} = -\frac{v_1}{r_1 + \kappa_1}, \tag{34}$$

which admits symmetry (25). Equation (34) is another simple ODE, the solution of which is

$$v_1 = \frac{\kappa_2}{r_1 + \kappa_1}, \tag{35}$$

where κ_2 is another arbitrary constant. Using (23), we write (35) as a first-order ODE in the variables x and y, namely

$$y' = \frac{\kappa_2 y}{x + \kappa_1}, \tag{36}$$

which admits symmetry Y_1 from (21). Equation (36) is the last first-order ODE in the series of iterations and is also a simple variables-separable equation. The solution of (36) is

$$y = \kappa_3 (x + \kappa_1)^{\kappa_2}, \tag{37}$$

where κ_3 is a further arbitrary constant. This is in fact the general solution of Equation (15).

Example 2. *Consider the nonlinear ODE*

$$y''' = \frac{3}{2}\frac{y''2}{y'}, \tag{38}$$

which is the canonical form of every third ODE that admits a transitive fiber-preserving six-dimensional point symmetry group [20].

Equation (38) admits a six-dimensional symmetry Lie algebra L_6 spanned by the operators

$$\left.\begin{array}{lll} X_1 = \frac{\partial}{\partial x} & X_2 = x\frac{\partial}{\partial x} & X_3 = x^2\frac{\partial}{\partial x} \\ X_4 = \frac{\partial}{\partial y} & X_5 = y\frac{\partial}{\partial y} & X_6 = y^2\frac{\partial}{\partial y}. \end{array}\right\} \tag{39}$$

The symmetries X_2, X_3 and X_4 span a solvable Lie algebra which has

$$[X_2, X_3] = X_3 \tag{40}$$

as the only nonzero Lie bracket. With relabelling

$$X_3 \to Y_1, \quad X_2 \to Y_2, \quad X_4 \to Y_3,$$

the commutation relations of the operators Y_1, Y_2 and Y_3 satisfy (2).

We extend the identified infinitesimal generators:

$$\left.\begin{array}{lll} Y_1^{(1)} & = & x^2\frac{\partial}{\partial x} - 2xy'\frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial y}. \end{array}\right\} \tag{41}$$

Solving the characteristic equations

$$\frac{dx}{x^2} = \frac{dy}{0} = \frac{dy'}{-2xy'} \tag{42}$$

arising from $Y_1^{(1)}$, we obtain invariants

$$r_1 = y, \quad v_1 = x^2 y', \tag{43}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = x\left(\frac{xy''}{y'} + 2\right). \tag{44}$$

In terms of r_1, v_1 and w_1, $Y_2^{(2)}$ becomes

$$Y_2^{(2)} = v_1 \frac{\partial}{\partial v_1} + w_1 \frac{\partial}{\partial w_1}. \tag{45}$$

From the corresponding characteristic equation

$$\frac{dr_1}{0} = \frac{dv_1}{v_1} = \frac{dw_1}{w_1}, \tag{46}$$

we obtain the next set of invariants

$$r_2 = r_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1}, \tag{47}$$

which, in view of (43), can be written in terms of x, y, y' and y'' as follows:

$$r_2 = y \quad \text{and} \quad v_2 = \frac{2y' + xy''}{x(y')^2}. \tag{48}$$

From (48) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{y'''}{(y')^3} - \frac{2(y'')^2}{(y')^4} - \frac{2y''}{x(y')^3} - \frac{2}{x^2(y')^2}. \tag{49}$$

Equation (38) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. To find ψ^2, substitute out y''' from (49) using (38) and then use (48) to write the resulting equation in terms of r_2 and v_2. We obtain the first-order ODE

$$\frac{dv_2}{dr_2} = -\frac{v_2^2}{2}, \tag{50}$$

which admits $Y_3^{(2)}$, written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = \frac{\partial}{\partial r_2}. \tag{51}$$

The solution of (50) is

$$v_2 = \frac{2}{r_2 - \kappa_1}, \tag{52}$$

where κ_1 is an arbitrary constant. In terms of r_1 and v_1, Equation (52) is transformed, using (47), into the next first-order ODE

$$\frac{dv_1}{dr_1} = \frac{2v_1}{r_1 - \kappa_1}, \tag{53}$$

which admits symmetry (45). Equation (53) is solved easily. We obtain

$$v_1 = \kappa_2(\kappa_1 - r_1)^2, \tag{54}$$

where κ_2 is another arbitrary constant. Using (43) we write (54) as a first-order ODE in the variables x and y, namely

$$y' = \frac{\kappa_2(y - \kappa_1)^2}{x^2}. \tag{55}$$

Equation (55) admits Y_1, i.e., the symmetry X_4 from (39) and is the last ODE in the series of iterations. Furthermore, it is a variables-separable ODE, the solution of which is

$$y = \frac{x}{\kappa_2 - \kappa_3 x} + \kappa_1, \tag{56}$$

where κ_3 is another arbitrary constant. This is the general solution of Equation (38).

Example 3. *Consider the nonlinear ODE*

$$y''' + x\big(y''\big)^2 + \tfrac{1}{x}\,y'' = 0, \tag{57}$$

an example of third-order ODEs that are equivalent to linear second-order ODEs via tangent transformations [3]. Equation (57) admits a four-dimensional symmetry Lie algebra spanned by the operators

$$X_1 = x^2\tfrac{\partial}{\partial x} + x(y + \ln x - 1)\tfrac{\partial}{\partial y},\ X_2 = x\tfrac{\partial}{\partial x},\ X_3 = \tfrac{\partial}{\partial y},\ X_4 = x\tfrac{\partial}{\partial y}. \tag{58}$$

The commutator relations of X_2, X_3 and X_4 are such that

$$[X_2, X_4] = X_4 \tag{59}$$

is the only nonzero Lie bracket. This means that X_1, X_2 and X_4 span a solvable Lie algebra and satisfy (2), with the following labelling:

$$X_4 \to Y_1, \quad X_2 \to Y_2, \quad X_3 \to Y_3.$$

The extensions of the identified infinitesimal generators are:

$$\left.\begin{array}{rcl} Y_1^{(1)} & = & x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial y}. \end{array}\right\} \tag{60}$$

We solve characteristic equations

$$\frac{dx}{0} = \frac{dy}{x} = \frac{dy'}{1} \tag{61}$$

associated with $Y_1^{(1)}$, we obtain invariants

$$r_1 = x, \quad v_1 = y' - \frac{y}{x}, \tag{62}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = \frac{y}{x^2} - \frac{y'}{x} + y''. \tag{63}$$

Writing $Y_2^{(2)}$ in terms of r_1, v_1 and w_1, we obtain

$$Y_2^{(2)} = r_1\frac{\partial}{\partial r_1} - v_1\frac{\partial}{\partial v_1} - 2w_1\frac{\partial}{\partial w_1}, \tag{64}$$

for which the corresponding characteristic equations are

$$\frac{dr_1}{r_1} = \frac{dv_1}{-v_1} = \frac{dw_1}{-2w_1}. \tag{65}$$

We obtain from the solution of (65) invariants

$$r_2 = r_1 v_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1^2}, \tag{66}$$

which, in view of (62), can be written in terms of x, y, y' and y'' as follows:

$$r_2 = xy' - y \quad \text{and} \quad v_2 = \frac{y + x(xy'' - y')}{(y - xy')^2}. \tag{67}$$

From (67) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{x(yy''' - 3y'y'') + 3yy'' + x^2\Big(2(y'')^2 - y'y'''\Big)}{y''(y - xy')^3}. \tag{68}$$

Equation (57) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. To find ψ^2, we use (57) to substitute out y''' from (68) and then use (67) to write the resulting equation in terms of r_2 and v_2. We obtain the first-order ODE

$$\frac{dv_2}{dr_2} = -\frac{(r_2 + 2)v_2 + 1}{r_2}, \tag{69}$$

that admits $Y_3^{(2)}$ written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = -\frac{\partial}{\partial r_2} + \frac{2r_2 v_2 + 1}{r_2^2}\frac{\partial}{\partial v_2}. \tag{70}$$

The solution of (69) is

$$v_2 = \frac{\kappa_1 e^{-r_2} - r_2 + 1}{r_2^2}, \tag{71}$$

where κ_1 is an arbitrary constant. In terms of r_1 and v_1, Equation (71) is transformed, using (66), into another first-order ODE

$$\frac{dv_1}{dr_1} = \frac{\kappa_1 e^{-r_1 v_1} - r_1 v_1 + 1}{r_1^2}, \tag{72}$$

which admits symmetry (64). The solution of (72) is

$$e^{r_1 v_1} = \kappa_2 r_1 - \kappa_1, \tag{73}$$

where κ_2 is another arbitrary constant. Finally, we use (62) to write (73) as an ODE in the variables x and y. We obtain

$$e^{xy' - y} = x\kappa_2 - \kappa_1, \tag{74}$$

which admits Y_1, i.e., the symmetry X_4 from (58). The solution of (74), namely

$$y = x \ln\left[\left(\frac{\kappa_1}{x} - \kappa_2\right)^{\kappa_2/\kappa_1}(\kappa_2 x - \kappa_1)^{-1/x}\right] + \kappa_3 x, \quad \kappa_1 \neq 0, \tag{75}$$

where κ_3 is another arbitrary constant is the general solution of Equation (57).

Example 4. *The equation we consider here*

$$y''' + \frac{3y'y''}{y} - 3y'' - \frac{3(y')^2}{y} + 2y' = 0, \tag{76}$$

drawn from [1] admits a seven-dimensional symmetry Lie algebra spanned by the operators

$$\left.\begin{array}{l} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{1}{y}\frac{\partial}{\partial y}, \quad X_3 = 2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ X_4 = e^x\frac{\partial}{\partial x} + e^x\left(y + \dfrac{1}{y}\right)\frac{\partial}{\partial y}, \quad X_5 = e^{-x}\frac{\partial}{\partial x}, \quad X_6 = \frac{e^x}{y}\frac{\partial}{\partial y} \\ X_7 = \frac{e^{2x}}{y}\frac{\partial}{\partial y}. \end{array}\right\} \tag{77}$$

Using the solvable algebra $\langle X_1, X_3, X_7\rangle$, *for which nonzero Lie brackets are*

$$[X_1, X_7] = 2X_7 \quad \textit{and} \quad [X_3, X_7] = 2X_7, \tag{78}$$

we relabel the symmetries as follows:

$$X_7 \to Y_1, \quad X_3 \to Y_2, \quad X_1 \to Y_3,$$

to ensure that the commutation relations of Y_1, Y_2 *and* Y_3 *satisfy (2).*

As in the previous examples, the following extensions of Y_1, Y_2 *and* Y_3 *are needed in the calculations that follow:*

$$\left.\begin{array}{lcl} Y_1^{(1)} & = & \frac{e^{2x}}{y}\frac{\partial}{\partial y} + e^{2x}\left(\frac{2}{y} - \frac{y'}{y^2}\right)\frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & 2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial x}. \end{array}\right\} \tag{79}$$

We compute two invariants of $Y_1^{(1)}$,

$$r_1 = x, \quad v_1 = yy' - y^2, \tag{80}$$

from which we derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = y(y'' - 2y') + (y')^2. \tag{81}$$

In terms of r_1, v_1 *and* w_1, $Y_2^{(2)}$ *becomes*

$$Y_2^{(2)} = \frac{\partial}{\partial r_1} + v_1\frac{\partial}{\partial v_1} + w_1\frac{\partial}{\partial w_1}. \tag{82}$$

Invariants of (82) are

$$r_2 = e^{-r_1}v_1 \quad \textit{and} \quad v_2 = \frac{w_1}{v_1}, \tag{83}$$

or, in terms of x, y *and the derivatives,*

$$r_2 = ye^{-x}(y' - y) \quad \textit{and} \quad v_2 = \frac{y(y'' - 2y') + (y')^2}{y(y' - y)}. \tag{84}$$

The differential invariant derived from (84) is

$$\begin{aligned} w_2 \quad = \quad & \frac{dv_2}{dr_2} = e^x\Big[y^3(2y'' - y''') - y^2\Big(2(y')^2 + y'(y'' - y''') + (y'')^2\Big) - (y')^4 \\ & + y(y')^2(2y' + y'')\Big]\Big[y^2(y - y')^2\Big(y^2 + y(y'' - 3y') + (y')^2\Big)\Big]^{-1}. \end{aligned} \tag{85}$$

We now use Equation (76) to substitute out y''' from (85) and then express the resulting equation in terms of r_2 and v_2 using (84). We obtain

$$\frac{dv_2}{dr_2} = -\frac{v_2}{r_2}, \tag{86}$$

a first-order ODE that admits $Y_3^{(2)}$ written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = r_2 \frac{\partial}{\partial r_2}. \tag{87}$$

The solution of (86) is

$$v_2 = \frac{\kappa_1}{r_2}, \tag{88}$$

where κ_1 is an arbitrary constant. We now use (83) to express (88) in terms of r_1 and v_1. We obtain

$$\frac{dv_1}{dr_1} = \kappa_1 e^{r_1}, \tag{89}$$

which admits symmetry (82). Upon solving (89), we obtain

$$v_1 = \kappa_1 e^{r_1} + \kappa_2, \tag{90}$$

where κ_2 is another arbitrary constant. Using (80) we write (90) an order ODE in the variables x and y,

$$y' = \frac{\kappa_1 e^x + \kappa_2 + y^2}{y}, \tag{91}$$

which admits Y_1, i.e., the symmetry X_7 from (77). Equation (91) is easily solved and we obtain

$$y = \left(\kappa_3 e^{2x} - 2\kappa_1 e^x - \kappa_2\right)^{1/2}, \tag{92}$$

where κ_3 is another arbitrary constant. This is in fact the general solution of Equation (76).

4. Concluding Remarks

In this paper, we have provided a clear exposition of the method of differential invariants for integrating (or, at least, reduction of order of) any higher order ODE that admits a solvable Lie algebra. We have included in the paper four illustrative examples that involve nonlinear ODEs of different classes and drawn from different contexts, each of which admits a three-dimensional solvable lie algebra. The presentation of the reduction algorithm in this paper is instructive in that the exposition is based on a third-order ODE, which makes the method easy to appreciate. In this connection, it is our hope that this paper will serve as an invitation to others to consider using the method of differential invariants on ODEs that they encounter.

Author Contributions: Conceptualization, W.S.; methodology, W.S. and M.C.K.; software, W.S. and M.C.K.; formal analysis, W.S. and M.C.K.; writing—original draft preparation, M.C.K. writing—review and editing, W.S. and M.C.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank the Directorate of Research Development and Innovation of Walter Sisulu University for continued support. We also thank the anonymous reviewers for their careful reading of our manuscript and their insightful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Chandrasekar, V.K.; Senthilvelan, M.; Lakshmanan, M. On the complete integrability and linearization of nonlinear ordinary differential equations. II. Third-order equations. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2006**, *462*, 1831–1852. [CrossRef]
2. Mohanasubha, R.; Chandrasekar, V.K.; Senthilvelan, M. A note on deriving linearizing transformations for a class of second-order nonlinear ordinary differential equations. *Nonlinear Anal. Real World Appl.* **2018**, *39*, 202–212. [CrossRef]
3. Nakpim, W. Third-order ordinary differential equations equivalent to linear second-order ordinary differential equations via tangent transformations. *J. Symb. Comput.* **2016**, *77*, 63–77. [CrossRef]
4. Schwarz, F. *Algorithmic Lie Theory for Solving Ordinary Differential Equations*; Chapman & Hall/CRC: New York, NY, USA, 2008.
5. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
6. Bluman, G.W.; Cheviakov, A.F.; Anco, S.C. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA, 2010.
7. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
8. Hydon, P.E. *Symmetry Methods for Differential Equations: A Beginner's Guide*; Cambridge University Press: New York, NY, USA, 2000.
9. Cantwell, B.J. *Introduction to Symmetry Analysis*; Cambridge University Press: Cambridge, UK, 2002.
10. Senthilvelan, M.; Chandrasekar, V.K.; Mohanasubha, R. Symmetries of Nonlinear Ordinary Differential Equations: The Modified Emden Equation as a Case Study. *Pramana—J. Phys.* **2015**, *85*, 755–787. [CrossRef]
11. Ibragimov, N.H.; Meleshko, S.V. Linearization of third-order ordinary differential equations by point and contact transformations. *J. Math. Anal.* **2005**, *308*, 266–289. [CrossRef]
12. Naz, R.; Mahomed, F.M.; Mason, D.P. Symmetry Solutions of a Third-Order Ordinary Differential Equation which Arises from Prandtl Boundary Layer Equations. *J. Nonlinear Math. Phys.* **2008**, *15*, 179–191. [CrossRef]
13. Mahomed, F.M. Symmetry group classification of ordinary differential equations: Survey of some results. *Math. Meth. Appl. Sci.* **2007**, *30*, 1995–2012. [CrossRef]
14. Bluman, G.W.; Kumei, S. Symmetry-based algorithms to relate partial differential equations. I. Local symmetries. *Eur. J. Appl. Math.* **1990**, *1*, 189–216. [CrossRef]
15. Kumei, S.; Bluman, G.W. When nonlinear differential equations are equivalent to linear differential equations. *SIAM J. Appl. Math.* **1982**, *42*, 1157–1173. [CrossRef]
16. Mahomed, F.M.; Leach, P.G.L. The Lie Algebra $SL(3,R)$ and Linearization. *Quaest. Math.* **1989**, *12*, 121–139. [CrossRef]
17. Ibragimov, N.H.; Nucci, M.C. Integration of third-order ordinary differential equations by Lie's method: Equations admitting three-dimensional Lie algebras. *Lie Groups Appl.* **1994**, *1*, 49–64.
18. Broadbridge, P.; White, L. Constant rate rainfall infiltration-A versatile nonlinear model. 1, Analytic solutions. *Water Resour. Res.* **1988**, *24*, 145–154. [CrossRef]
19. Kakuli, M.C. Lie Symmetry Analysis of a Nonlinear Fokker–Planck Diffusion-Convection model. Master's Dissertation, Walter Sisulu University, Mthatha, South Africa, 2017.
20. Grebot, G. The Characterization of third-order Ordinary Differential Equations Admitting a Transitive Fiber-Preserving Point Symmetry Group. *J. Math. Anal. Appl.* **1997**, *206*, 364–388. [CrossRef]

27 On the Eight Dimensional Point Symmetries of Second Order Linear O.D.E's. By Uchechukwu Opara (2020)

Review Article
Volume 14:1, 2020
DOI: 10.37421/glta.2020.14.297

Journal of
Generalized Lie Theory and Applications

ISSN: 1736-4337
Open Access

On the Eight Dimensional Point Symmetries of Second Order Linear O.D.E's

Uchechukwu Opara*

Department of Mathematics and Statistics, Veritas University, Abuja, Nigeria

Abstract

The purpose of this paper is to review Sophus Lie's methods for differential equations, shedding light on the Lie point symmetries of generic linear second order ordinary differential equations (O.D.E's). As a matter of necessity in the overall review, there are important contrasts highlighted in derivations of the infinitesimal generators, with and without the prior application of point transformations.

Keywords: Lie groups • Lie algebras • Linearizable second order ordinary differential equations • Kummer-liouville point transform • Semi-invariants

Introduction

The technique of Lie group theory in resolution of differential equations is relatively modern. Sophus Lie, the originator of this technique, developed its foundations very late into the nineteenth century. The theorems discovered by Lie on second order ordinary differential equations are actually classical; with consequences harnessed by Kummer and Liouville [1,2], creating prospective gateways into functional analytic research (consider the necessity to find the kernel of the Kummer-Liouville transform addressed in the relevant section). As recently as the late twentieth century, there has been a resurgence of attempts to prove a reduction theorem by Lie for linearizable second order O.D.E's (see Results section below for the theorem). Take as an example, the publication by Govinder and Leach [3]. The popular attempts encountered in academic archives fall short of rigorous descriptive detail. This paper offers a well-detailed proof of the above mentioned reduction theorem, with a rare, tactfully systematic and didactic approach.

Given an n^{th} order O.D.E $f(x, y, y'(x),\ldots, y^{(n)}(x))=0$, there could possibly exist a non-trivial, non-degenerate map on its domain of definition $(x, y) \mapsto (X, Y)$ such that we have also $f(X, Y, Y'(X),\ldots, Y^{(n)}(X))=0$. If there exists a one-parameter family of such maps $(P_\lambda)_{\lambda\in R}:(x, y) \mapsto (X(\lambda), Y(\lambda))$ such that the following properties hold-

1. $P_{\lambda 2} \circ P_{\lambda 1}=P_{\lambda 2+\lambda 1}$
2. P_0=Identity
3. P_λ is infinitely many times differentiable with respect to x, y and λ,

then we say that the family $\{ P_\lambda\}$ is a one-parameter symmetry Lie group of transformations that is accommodated by the O.D.E. $(X(\lambda), Y(\lambda))$ is referred to as the global form of the group, and the corresponding infinitesimal form:

$$v=\xi(x,y)\frac{\partial}{\partial x}+\phi(x,y)\frac{\partial}{\partial y}$$

called an infinitesimal generator for the group is obtained by setting

$$\xi := dX/d\lambda|_{\lambda=0} \; and \; \phi:=\frac{dY}{d\lambda}|_{\lambda=0}$$

Given the infinitesimal form, we can as well deduce the global form by integrating the autonomous system of differential equations

$$\frac{dX}{d\lambda}=\xi(\mathrm{X},\mathrm{Y}) \; and \; \frac{dY}{d\lambda}=\phi(X,Y)$$

subject to the initial conditions $\mathrm{X}|_{\lambda=0}=x$ and $\mathrm{Y}|_{\lambda=0}=y$.

Symmetry considerations of differential equations usually simplify these problems, by illuminating their reducibility properties. The method of symmetry groups for differential equations gives rise to certain solutions called group invariant solutions, which may or may not be the entire solution set [4]. The most general technique for discovering Lie symmetries of an equation is by prolonging the infinitesimal vector field action into a *jet space*. For the case of the O.D.E $f(x, y, y(x),\ldots, y^{(n)}(x))=0$,

the jet space will be an open subset of $\mathbf{R}^{2+n}$, in which we will have the prolonged vector field action

$$\mathrm{pr}^{(n)}\, v = v+\Sigma_{i=1}^{n}\phi^{x^i}\,\frac{\partial}{\partial y^{(i)}}$$

The prolonged vector field action on the jet space gives an induced invariance on that space from the underlying O.D.E. For details on how the algorithm to compute the coefficients ϕ^{x^i} is derived, we refer the reader to [5]. Generally, the implementation of this prolongation technique in resolution of O.D.E's involves intuitively equating coefficients of monomials from the process, in line with the theorem given below.

Prolongation theorem Olver [5]

Let $f(x, y, y'(x),\ldots, y^{(n)}(x))=0$, be an O.D.E that is defined over an open subset $M\subseteq R^2$. If G is a local group of transformations with infinitesimal generator $\mathbf{v}$ acting on M and $\mathrm{pr}^{(n)}\mathbf{v}[f(x, y, y'(x),\ldots, y^{(n)}(x))]=0$ whenever $f(x, y, y'(x),\ldots, y^{(n)}(x))=0$, then G is a symmetry group of the O.D.E.

It is substitution of the initial variables with the *canonical co-ordinates* of the accommodated one-parameter symmetries that simplifies a given differential equation. The pair of canonical co-ordinates (μ, ψ) must satisfy $\mu(X, Y)=\mu(x, y)$ and $\psi(X, Y)=\psi(x, y) + \lambda$. The functions μ and $\varphi(\mu)$ for any real-valued analytic function φ are called invariants of the group.

Much work has been done on symmetry considerations in the specific case of linear second order O.D.E's. For instance, we have explicit derivations of eight independent accommodated infinitesimal symmetries [6]. Detailed account of the crucial Kummer-Liouville transform which pertains to this class of equations [1]. In the ensuing content of this paper,

***Address for Correspondence:** Uchechukwu Opara, Department of Mathematics and Statistics, Veritas University, Abuja, Nigeria, E-mail: ucmiop@yahoo.com.*

Received 26 February, 2020; **Accepted** 3 March, 2020; **Published** 17 March, 2020

the importance of semi-invariants of the generic equation is frequently brought up. This is an aspect required in obtaining a quality overview of the symmetries, which often tends to be overlooked. Hence, a rare and accurate proof of Sophus Lie's theorem on linear second order O.D.E's is systematically constructed.

The relevance of Lie symmetries in resolution of higher order O.D.E's is then briefly discussed in conclusion.

Point Symmetries of Generic Second Order Linear O.D.E's

Point symmetries without point transformations

The equation under examination throughout this section will be the homogenous equation

$$y''+a_1(x)y'+a_0(x)y=0 \qquad (1).$$

We need not include the case where a_1 and a_0 are both constants, which is immediately resolved by means of the characteristic quadratic equation. Although the generic case in non-homogenous, the superposition principle for linear O.D.E's emphasizes the need for solving (1). Solutions to (1) exist locally whenever

$$a_1(x)\in C^1(I),\ \ a_0(x)\in C(I)$$

for an open, non-empty subinterval (I) of the real number line, so we take this as given a-priori.

Let **'v'** be a vector field $\xi\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}$ defined on some open subset U $\subseteq$ I × **R**. The second prolongation of **'v'** is given as $pr^{(2)}v:=v+\phi^x\frac{\partial}{\partial y_x}+\phi^{xx}\frac{\partial}{\partial y_{xx}}$.

By the given prolongation theorem, equation (1) accommodates **'v'** if

$pr^{(2)}v[y''+a_1(x)y'+a_0(x)y]=0$ whenever $y''+a_1(x)y'+a_0(x)y=0$,

in which case **'v'** is referred to as an infinitesimal Lie symmetry of (1). The collection of all infinitesimal symmetries accommodated by a differential equation forms a linear space referred to as a Lie algebra. Symmetry considerations of (1) arise from the need to simplify or give more elaborate procedures for computing its solutions, and this leads us to implement the prolongation technique for differential equations. To this end, we compute the coefficients ϕ^x and ϕ^{xx} respectively to be

$$\phi^x=D_x(\phi-\xi y_x)+\xi y_{xx}=\phi_x+\phi_y y_x-[\xi_x y_x+\xi_x(y_x)^2$$
$$+\xi y_{xx}]+\xi y_{xx}=\phi_x+\phi_y y_x-\xi_x y_x-\xi_y(y_x)^2,$$

$$\phi^{xx}=D^2_{xx}(\phi-\xi y_x)+\xi y_{xxx}=\phi_{xx}+2\phi_{xy}y_x+\phi_{yy}(Y_x)^2$$
$$+\phi_y Y_{xx}-\xi_{xx}y_x-2\xi_{xy}(y_x)^2-2\xi_x y_{xx}-3\xi_y y_x y_x-\xi_{yy}(y_x)^3$$

The symbol 'D' stands for the total derivative, subscripts of ξ and ϕ symbolize partial derivation, and otherwise, subscripts denote total derivation with respect to the given variables. We will now go into considerable detail on how to compute the infinitesimal symmetries (or generators) of **(1)** directly. First of all, we determine that

$$pr^{(2)}v[y''+a_1(x)y'+a_0(x)y]$$
$$=a_1'(x).\xi.y'(x)+a_0'(x)\xi.y+a_0(x)\phi+a_1(x)\phi^x+\phi^{xx}$$
$$=\xi a_1'y'+\xi a_0'y+\phi a_0+a_1(\phi^x+\phi_y y'-\xi_x y'$$
$$-\xi_y(y')^2)+\phi_{xx}+2\phi_{xy}y'+\phi_{yy}(y')^2+\phi_y y''$$
$$-\xi_{xx}y'-2\xi_{xy}(y')^2-2\xi_x y''-3\xi_y y'y''-\xi_{yy}(y')^3.$$

Imposing that (1) accommodates **'v'**, we use the symmetry condition

$pr^{(2)}v[y''+a_1(x)y'+a_0(x)y]=0$ whenever, $y''+a_1(x)y'+a_0(x)y=0$ to determine further that

$$-\xi_{yy}(y')^3+[\phi_{yy}-2\xi_{xy}+2\xi_y a_1](y')^2+[\xi a_1'-\xi_x a_1+2\phi_{xy}$$
$$-\xi_{xx}+2\xi_x a_1+3\xi_y a_0 y]y'+\xi a_0'y+\phi a_0+a_1\phi_x+\phi_{xx}+2\xi_x a_0 y=0$$

Because the coefficients from the infinitesimal generators do not depend on derivatives of y, we set the coefficients of $(y')^3$, $(y')^2$ and y' above each equal to zero to realize three equations. A fourth equation also arises by way of the other terms.

From the coefficient of $(y')^3$, we have $\xi_{yy}=0 \leftrightarrow \xi(x,y)=A(x)+B(x).y$.

From the coefficient of $(y')^2$, we have

$\phi(x,y)=[C(x)+2A'(x)-2a_1A(x)]y+B'(x)y^2-a_1B(x)y^2+D(x)$.

From the coefficient of y',

$-3a_1'A(x)-3a_1'B(x)y-3a_1A'(x)-3a_1B'(x)y+2C'(x)+3A''(x)+3B''(x)y+3a_0B(x)y=0$

giving two consequential equations;

$-3a_1A(x)+2C(x)+3A'(x)=k$ (I)

$-a_1'B(x)-a_1B'(x)+B''(x)+a_0B(x)=0$ (II),

from the free terms and coefficients of y respectively, where k is a constant in (I). Afterwards, the terms from the symmetry condition which are not multiplied by y' yield three more consequential equations;

$a_0'B-a_1a_1'B-a_1^2B'+B'''-2a_1'B'-a_1''B+a_0a_1B=0$ (III),

$a_0'A+2a_0A'+a_1C'-2a_1a_1'A-2a_1^2A'+C''+2A''-2a_1'A-4a_1'A'=0$ (IV),

$D'+a_1D'+a_0D=0$ (V).

Equations (II) and (III) are both derived directly from the adjoint of the original equation (1), that is,

$B''-(a_1B)'+a_0B=0$ (II').

By straightforward computations involving (I), equation (IV) can be reduced to the conditions

$a_0'A+2a_0A'=(C''+a_1C')/3$ (IV'),

$A''+(4a_0-a_1^2-2a_1')A'+(2a_0'-a_1a_1'-a_1')A=0$ (IV'').

At this juncture, we have obtained sufficient information to tell the most general appearances of the coefficient functions from the infinitesimal symmetries. Let the constant k in (I) be denoted c_1. In (V), we determine that $D(x)=c_2y_1+c_3y_2$, where y_1 and y_2 are specific linearly independent solutions of (1). From (II'), we have

$$B(x)=\exp(\textstyle\int a_1dx)[c_4y_1+c_5y_2].$$

To solve for A(x), it is helpful to examine the normal form of **(1)**, that is,

$(y_*)''+p(x)y_*=0$ **(1')**

obtained via the point transform

$$y=\exp(-1/2\int^x a_1(\tau)d\tau)y_*$$

Hence, the coefficient of y_* in the normal form, which is identified as the semi-invariant of (1) is given as,

$p(x)=a_0-\frac{a_1'}{2}-\frac{a_1^2}{4}$. Therefore, we can rewrite (IV'') as A''' + 4pA' + 2p'A = 0. It is then easy to check that $y_{*1}.y_{*2}$ is a solution to (IV'') where y_{*1}, y_{*2} are both solutions to the normal form of (1). Therefore, we determine the most general solution to the third order linear O.D.E (IV'') to be:

$$A=\exp(\textstyle\int a_1dx)[c_6y_1^2+2c_7y_1y_2+c_8y_2^2].$$

This is substituted in (I) to give

$$C=\frac{1}{2}(c_1+3a_1A-3A')=\frac{c_1}{2}-3.\exp(\int a_1dx)[c_6y_1y_1'+c_7(y_1'y_2+y_1y_2')+c_8y_2y_2'].$$

Hence, we obtain the most general point symmetry of **(1)**, without any prior point transformation, to be

$$\xi\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}=(A+By)\frac{\partial}{\partial x}+([c+2A'-2a_1A]y+[B'-a_1B]y^2+D)\frac{\partial}{\partial y}.$$

After substituting the values obtained above, we can separate the most general symmetry by the eight constants $\{c_i\}_{i=1}^{8}$ as follows:

$$\sum_{i=1}^{8}c_iv_i$$

Suggestive of an eight-parameter symmetry group of (1). The process of computing the symmetries as done above for this case turns up an obvious problem: almost all the single-parameter symmetries depend on having a specific solution y_1 or y_2 of (1) in hand a-priori. The only single-parameter symmetry which does not depend on any specific solution is $y\frac{\partial}{\partial y}$, which corresponds to the so called *scaling group.* It is accommodated by all linear differential equations. In this study, the scaling group reduces (1) to a Riccati equation of the first order, when applied alone. We observe this development by computing the global form of this one-parameter group to be

$$(X,Y)=(x,e^{\lambda}y),$$

and then a canonical co-ordinate for the group is $\psi=\ln y$ because $\psi(X,Y)=\psi(x,y)+\lambda$ Substituting the dependent variable in (1) with the canonical co-ordinate, we obtain the first order non-linear Riccati equation

$$w'+w^2+a_1(x)w+a_0(x)=0,\quad \text{where}\quad W=\Psi'.$$

In fact, there is also a reverse correspondence in this regard, being that every Riccati equation can be transformed into a linear O.D.E of the second order.

We will hereby make a few further remarks on Riccati equations, as they are relevant to this study. Special Riccati equations have the form

$$y'(x)=ay^2+bx^a,$$

where a, b, α are real constants. When α = 0, the special Riccati equation is integrated by separation of variables:

$$\frac{dy}{ay^2+b}=dx.$$

Another easily solved case is α=-2, in which substitution of the dependent variable z=1/2 maps the above Riccati equation to the form

$$\frac{dz}{dx}=-a-b\left(\frac{z}{x}\right)^2$$

which can then be integrated by quadrature. Riccati and Bernoulli both discovered that the special Riccati equation can be mapped to the form wherein α=0, and can hence be integrated by quadrature in terms of elementary functions, if α takes values in one of the two rational sequences

$$\left(-\frac{4n}{2n-1}\right),\left(-\frac{4n}{2n+1}\right)n=1,2,3,...$$

The limit of both of these sequences is -2. Later, Liouville showed that these Riccati equations can be mapped to a form that can be integrated by quadrature in terms of elementary functions only if α takes a value in one of these two rational sequences.

In the case of a general Riccati equation

$$y'=P(x)+Q(x)y+R(x)y^2$$

it is linearizable by a point transformation of the dependent variable y to a linear O.D.E of the first order, if and only if it has a constant solution [7]. More details on the simplification and integration of such O.D.E's can be readily accessed, but we now return our focus to point symmetries of (1).

To modify the result on symmetries of (1) obtained prior to remarks on Riccati equations, there are point transformations which we may implement before employing the prolongation technique. The most general point transformation which preserves the order and linearity of (1) is called the Kummer-Liouville (KL) transformation, and we will unravel more subtle properties of the infinitesimal symmetries by engaging it.

Point symmetries with KL point transforms

The Kummer-Liouville transform is given by

$$y=v(x)z,\ \ dt=u(x)dx\quad (KL);\qquad u,v\in C^2(I),\quad uv\neq 0\,\forall x\in I$$

which rearranges **(1)** to be of the form

$$\ddot{z}+b_1(t)\dot{z}+b_0(t)z=0\quad (2);\quad b_1(t)\in C^1(J),b_0(t)\in C(J)$$

Where J is an open, non-empty sub-interval of the real number line.

Theorem (*Stäckel - Lie*) [1]

The Kummer-Liouville transform is the most general point transform which preserves the order and linearity of (1).

For clarity, we will use the prime sign (′) to denote differentiation with respect to x and an overset dot to denote differentiation with respect to t. Observe that we need the following three to occur in order to obtain (2) from (1) by way of transform (KL).

(i) We must have the non-commutative factorization

$$Ly=\left(D-\frac{v'}{v}-\frac{u'}{u}-r_2(t)u\right)\left(D-\frac{v'}{v}-r_1(t)u\right)y=0;\ D=\frac{d}{dx}$$

where r_1(t) and r_2(t)satisfy the Riccati equations:

$$\dot{r}_1+r_1^2+b_1(t)r_1+b_0(t)=0;\ \dot{r}_2-r_2^2-b_1(t)r_2+\dot{b}_1-b_0(t)=0$$

(ii) $-2v'v^{-1}-u'u^{-1}+b(t)u=a_1(x)$

(iii) $v''+a_1v'+a_0v-b_0(t)u^2v=0.$

The reduction of (1) to (2) was posed as Kummer's problem, which was to find the set of all KL transformations that could do this. It is known that Kummer's problem is always solvable. As a combination of the above three requirements for the KL transform, we get that (1) can be reduced to (2) if and only if the following two conditions are satisfied

$$v(x)=|u|^{-\frac{1}{2}}\exp(-\frac{1}{2}\int a_1(x)dx)\exp(\frac{1}{2}\int b_1(t)dt)\qquad (E)$$

$$\frac{1}{2}\frac{t'''}{t'}-\frac{3}{4}\left(\frac{t''}{t'}\right)^2+B_0(t)t'^2=A_0(x)\qquad (E_2)$$

where

$$A_0(x)=a_0-\frac{a_1'}{2}-\frac{a_1^2}{4};\ B_0(t)=b_0-\frac{\dot{b}_1}{2}-\frac{b_1^2}{4}$$

are respectively called the semi-invariants of (1) and (2). We solve (ii) over v in order to get (E) and then we substitute (iii) by (E) using the relation u=t′ to get (E_2).

At the crux, we wish to reduce (1) to a linear O.D.E of autonomous form, that is, one with constant coefficients;

$$\ddot{z}+b_1\dot{z}+b_0z=0\qquad (2'),$$

where the coefficient b_0 is a real number, while b_1 may either be real or purely imaginary.

(2′) can be factorized either through the noncommutative operators of the first order-

$$L_y=\left(D-\frac{v'}{v}-\frac{u'}{u}-r_2u\right)\left(D-\frac{v'}{v}-r_1u\right)y=0;$$

or through the commutative operators of the first order-

$$\frac{1}{u^2}L_y = \left(\frac{1}{u}D - \frac{v'}{uv} - r_2\right)\left(\frac{1}{u}D - \frac{v'}{uv} - r_1\right)y = 0$$

Where r_1, r_2 are roots of the characteristic equation; $r^2+b_1r+b_0=0$.

We remark that (1) can be reduced to (2') by transform KL if and only if the following occur.

(i) (1) admits a certain one-parameter Lie symmetry

(ii) u(x) satisfies $\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \left(\frac{-b_1^2}{4} + b_0\right)u^2 = A_0(x)$

(iii) $u''' - 6\frac{u'u''}{u} + 6\frac{(u')^3}{u^2} + 4A_0u' - 2A_0u' = 0$

(iv)The multiplier v and the kernel u of the KL transform are related through the formulas

$$v(x) = |u|^{\frac{-1}{2}} \exp(-\frac{1}{2}\int a_1(x)dx)\exp(\frac{1}{2}b_1\int udx);$$

$$v'' + a_1 v' + a_0 v - b_0(t)u^2 v = 0.$$

(v) The resolvent of (2') is given by the function

$$R(x) = |u|^{-1}\exp\left(-\int a_1 dx\right)$$

and it satisfies

(vi) $R''' + 3a_1 R'' + (4a_0 + a_1' + 2a_1^2)R' + (2a_0' + 4a_0 a_1)R = 0.$

The one-parameter existence follows from the reducibility of (1) to autonomous form (2'), as will soon be discussed. Condition (iii) can be obtained from (ii) by calculation, and then (vi) can be obtained from (iii) by way of the resolvent function

$$R(x) := y_1(x)y_2(x).$$

To be substituted in the resolvent function, we have linearly independent solutions of (1) given by

$$y_{1,2}(x) = |u|^{-1/2}\exp\left(-1/2\int a_1(x)dx\right)\exp\left(\pm\sqrt{\frac{b_1^2}{4} - b_0}\int udx\right);$$

for the case of KL transform to autonomous form (2') .

Focusing now on the general symmetry of (2), we must recall the KL substitutions

y=vz, t=∫udx, so as to observe that

$$y'' + y'\overbrace{\left(\frac{-2v'}{v} - \frac{u'}{u} + b_1u\right)}^{a_1x} + y\overbrace{\left(\frac{2v'^2}{v^2} - \frac{v''}{v} + \frac{v'u'}{vu} - \frac{b_1uv'}{v} + b_0u^2\right)}^{a_0x} = 0 \qquad (3).$$

By applying the second prolongation under the condition $b_0' = b_1' = 0$, we realize that (3) admits the infinitesimal generator

$$\chi_1 = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{vu}y\frac{\partial}{\partial y}$$

which precisely corresponds to the case of reduction to autonomous form (2'). Thus, the canonical coordinates for χ_1 are made of the pair (t, z), where z:=y/v is called an invariant and t=∫ udx. The invariant is obtained by integrating the differentials

$$\frac{dx}{\xi} = \frac{dy}{\phi} \Leftrightarrow udx = \frac{uv}{yv'}dy,$$

Resulting in y/v=constant, which is why y/v is an invariant. The other canonical coordinate is simply

$$\int_{x_0}^{x}\frac{d\tau}{1/u} = \int udx.$$

Since the pair of canonical coordinates results in the reduction of (1) to autonomous form, the Lie symmetry χ_1 is the requirement addressed in condition (i) above. Involving the autonomous case, equation (3) yet again accommodates an eight-parameter Lie symmetry, and so seven other independent single-parameter symmetries besides χ_1 are mentioned as follows:

$$\chi_2 = v\frac{\partial}{\partial y} \qquad \chi_3 = \chi_1\int udx$$

$$\chi_4 = \chi_2\int udx \qquad \chi_5 = \frac{y}{v}\chi_1$$

$$\chi_6 = \frac{y}{v}\chi_2 \qquad \chi_7 = (\int udx)^2\chi_1 + (\frac{y}{v}\int udx)\chi_2$$

$$\chi_8 = (\frac{y}{v}\int udx)\chi_1 + (\frac{y}{v})^2\chi_2.$$

The linear space spanned by χ_1 to χ_8 is a Lie algebra that is stable under the Lie bracket structure [.,.] as shown in the table included below Table 1. Instability under the Lie bracket or commutator would have implied the necessity to include more vector fields, other than $\{\chi_i\}_{i=1}^{8}$, to the infinitesimal generators spanning the Lie algebra accommodated by (3). This is at large, due to the fact that O.D.E's only accommodate finite dimensional Lie algebras. The element in the i'th row and j'th column of the table is the vector field $[\chi_i, \chi_j]$.

Note that the characterization of Lie brackets [X, Y] f = X(Y(f)) - Y(X(f)) for $X = X^i\frac{\partial}{\partial x_i}, Y = Y^j\frac{\partial}{\partial x_j}$ and any C^∞ function *f* gives us the formula:

$$[X, Y] = \sum_i\sum_j\left\{X^j\frac{\partial Y^i}{\partial x_j} - Y^j\frac{\partial X^i}{\partial x_j}\right\}\frac{\partial}{\partial x_i}.$$

Table 1. $[\chi_i, \chi_j]$ is the i[th] row, j[th] column (1 ≤ i, j ≤ 8).

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
χ_1	0	0	χ_1	χ_2	0	0	$2\chi_3+\chi_6$	χ_5
χ_2	0	0	0	0	χ_1	χ_2	χ_4	$\chi_3+2\chi_6$
χ_3	$-\chi_1$	0	0	χ_2	$-\chi_5$	0	χ_7	0
χ_4	$-\chi_2$	0	$-\chi_2$	0	$\chi_3-\chi_6$	χ_4	0	χ_7
χ_5	0	$-\chi_1$	χ_5	$\chi_6-\chi_3$	0	$-\chi_5$	χ_8	0
χ_6	0	$-\chi_2$	0	$-\chi_4$	χ_5	0	0	χ_8
χ_7	$-2\chi_3-\chi_6$	$-\chi_4$	$-\chi_7$	0	$-\chi_8$	0	0	0
χ_8	$-\chi_5$	$-\chi_3-2\chi_6$	0	$-\chi_7$	0	$-\chi_8$	0	0

The Lie brackets of the infinitesimal symmetries in their initial computed forms $\{v_i\}_{i=1}^{8}$ are not so readily determined. Nevertheless, the symmetries $\{\chi_i\}_{i=1}^{8}$ can be obtained as linear combinations of $\{v_i\}_{i=1}^{8}$ from functional specifications for the kernel (u) and multiplier (v) stated above. For instance, we have the chiefly required symmetry for conversion to autonomous form obtained as: $\chi_1 = \frac{1}{2}v_7 + b_1v_1.$

Although the KL transform gives more insight into the symmetry concept being addressed, all the infinitesimal generators except χ_6 (which corresponds to the scaling group) depend on the special kernel function u, and this is still indirectly tantamount to solving (1) beforehand. For this reason, construction of algorithms for computing the kernel has a substantial heuristic value in itself.

Results

It is useful to engage a second point transformation to (1) in discussions of its Lie symmetries, which is the reduction to normal form. After the generic KL transform, we reduce (2) into normal form by changing the dependent variable to $\bar{z}$ where

$$z = (\overline{z})\exp\left[\frac{-1}{2}\int^{t} b_1(\tau)d\tau\right]$$

and the result of this transform is

$$\frac{d^2\overline{z}}{dt^2} + \left(b_0 - \frac{\dot{b}_1}{2} - \frac{b_1^2}{4}\right)\overline{z} = 0 \qquad (2")$$

where $B_0 = b_0 - \frac{\dot{b}_1}{2} - \frac{b_1^2}{4}$ has previously been identified as a semi-invariant of **(2)**. Having presented sufficient pertinent information, this particular transform is geared towards validating Sophus Lie's theorem on linear O.D.E's of the second order, as stated below.

Theorem (Lie's theorem on linear second order O.D.E's) [8]

The O.D.E (1) can be mapped via a point transform into the form $\ddot{Y} = 0$, which implies accommodation of the eight-dimensional Lie algebra sl(3,R).

Now, we can map (1) into the above mentioned form from (2") by finding which value of u solves B_0=0. What we obtain concretely is

$$\ddot{\overline{z}} = 0$$

In (2") after a double point transformation of (1). The details for justification of this transformation are given below.

By setting the semi-invariant of (2) equal to zero, we get the semi-invariant of (1) to be

$$A_0 = \frac{1}{2}\frac{u"}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2.$$

It is clear that the associated homogenous equation A_0=0 accommodates the scaling group, of which the global form is $P_\lambda(x,u)=(x,e^\lambda u)$, so we obtain a canonical coordinate for this one-parameter group to be $\psi = \ln(u)$ .Recall the condition that u ≠ 0 on the interval I of interest. If u is negative then $\psi = \ln(|u|)$,and this sign change will not tamper significantly with the result of the ensuing computation. Consider the first case,

$$u = \exp(\psi) \rightarrow u' = \psi'\exp(\psi) \rightarrow u" = \psi"\exp(\psi) + (\psi')^2\exp(\psi),$$

and we have the following simplification for the semi-invariant of (1);

$$A_0 = \frac{1}{2}(\psi" + (\psi')^2) - \frac{3}{4}(\psi')^2 = \frac{\psi"}{2} - \frac{(\psi')^2}{4} = \frac{w'}{2} - \frac{w^2}{4}$$

where $\psi' = w$. The equation given just above is a Riccati equation, so we hereby employ the correspondence between Riccati equations and second order linear O.D.E's. By substituting w with $-\frac{2\zeta'}{\zeta}$, the Riccati equation becomes $\zeta" + A_0\zeta = 0$.

This linear equation is always solvable for ζ in $C^2(I)$, from which we recover u by reversing the prior substitutions as shown below.

$$w = -\frac{2\zeta'}{\zeta}$$

$$\Rightarrow \int w dx = -2\ln(|\zeta|) + k$$

$$\Rightarrow \psi = -2\ln(|\zeta|) + k$$

$$\Rightarrow u = \exp[-2\ln(|\zeta|) + k] = \frac{e^k}{\zeta^2}$$

where k is a constant of integration and ζ is a non-trivial solution to (1'). This given value of u solves B_0=0.

We should remark that the transform from (1) into its own normal form (1') is only a particular case of the (KL) transform with the functions

$$u(x) \equiv 1; v(x) = \exp\left(-\frac{1}{2}\int a_1(x)dx\right); z = y_*.$$

Therefore, the kernel u of transform KL enables us to restructure the Lie symmetries of (1), so as to reduce this O.D.E into various simpler forms.

We can take u as the auxiliary variable to examine the two most important cases; namely $b_0' = b_1' = 0$ for reducibility of (1) to autonomous form, and B_0=0 for reducibility of (1) to the form $\ddot{Y} = 0$, which corroborates Sophus Lie's theorem.

As a further remark, it is noteworthy that an arbitrary O.D.E of the second order is linearizable if and only if it accommodates an 8-parameter symmetry group, which is the symmetry group of maximal dimension for this class of equations. If it does accommodate such a group, then it can be mapped by point transformation(s) to the equation $\ddot{Y} = 0$. If it does not accommodate a symmetry group of dimension 8, then it accommodates a 0-, 1-, 2-, or 3-parameter symmetry group. This is another aspect of Sophus Lie's categorization of second order ordinary differential equations. For example, as we have seen above, the non-linear differential equation involved in the semi-invariant of (1), which is given as $\frac{1}{2}\frac{u"}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = 0$, is linearizable. The Lie algebra of infinitesimal symmetries accommodated by this equation is spanned by the eight vector fields listed as follows.

$$\chi_1 = xu^{\frac{3}{2}}\frac{\partial}{\partial u} \quad \chi_2 = xu^{\frac{-1}{2}}\frac{\partial}{\partial x} - 2u^{\frac{1}{2}}\frac{\partial}{\partial u} \quad \chi_3 = u^{\frac{-1}{2}}\frac{\partial}{\partial x} \quad \chi_4 = x\frac{\partial}{\partial x}$$

$$\chi_5 = \frac{\partial}{\partial x} \quad \chi_6 = u\frac{\partial}{\partial u} \quad \chi_7 = u^{\frac{3}{2}}\frac{\partial}{\partial u} \quad \chi_8 = -x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}$$

Conclusion: Point Symmetries of Higher Order O.D.E's

It is befitting to pass a few further comments on point symmetries of O.D.E's of order three and higher, following the details elucidated on those of the second order. For each given order, there is a maximal dimension for admissible symmetry groups [9], such as is eight for equations of the second order. Whenever an O.D.E admits a Lie group of one-parameter symmetries of the maximal dimension, then it is linearizable by a point transformation. Moreover, whenever the canonical coordinates from an accommodated one-parameter symmetry are employed by change of variable(s), the original O.D.E is transformed into another form with order one less. For instance, we have already seen as an explicit application of the scaling group in transforming the second order equation (1) into a first-order Riccati equation.

The main challenge that lingers in the midst of an abundance of one-parameter symmetries is that, whenever a given equation is reduced to another form by any one of them, the resulting form usually fails to inherit any of the symmetries which were present at first [10]. To simplify further using the Lie symmetry technique, one would then have to perform the infinitesimal symmetry prolongations again, which may or may not yield any vector fields. Not every differential equation admits a Lie symmetry to begin with, and computer algebra is encouraged for equations with order three or higher due to the rapid growth of the number of computations involved with each increment in order (and degree) of the differential equations. These signal a number of pronounced limitations involved with the approach of Lie groups. Nevertheless, whenever present, the wieldiness of Lie symmetries provides several opportunities for greater in-depth study of differential equations at large, as exemplified above.

References

1. Berkovich, Lev M, and Nikolai H. Rozov. “Transformation of Linear Differential Equations of Second Order and Adjoined Nonlinear Equations.” *Archivum Mathematicum* 33 (1997): 75-98.

2. Berkovich, Lev M, and Fedor L. Berkovich. “Transformation and Factorization of Second Order Linear Ordinary Differential equations and its Implementation in REDUCE.” *Univ Beograd Publ Elektrotehn Fak* 6 (1995): 12-25.

3. Govinder, KS, and P.G.L. Learch. “An Elementary Demonstration of the Existesnce of sl(3,R) Symmetry for all Second-Order Linear Ordinary Differential Equations.” *SIAM Review* 40 (1998): 945-946.

4. Bluman, George. “Invariant Solutions for Ordinary Differential Equations.” *SIAM J Appl Math* 50 (1990): 1706-1715.

5. Olver, Peter J. “Applications of Lie Groups to Differential Equations (Second Edition).” *Springer-Verlag New York, Inc* (1993).

6. Hydon, Peter E. “Symmetry Methods for Differential Equations - A Beginner's Guide.” *Cambridge University Press: C. W. Kilmister* (2000).

7. Ibragimov, Nail H. “Memoirs on Integration of Ordinary Differential Equations by Quadrature.” *Archives of ALGA* 5 (2008): 27-62.

8. Olver, Peter J. “Equivalence, Invariants, and Symmetry.” Cambridge University Press; J. F. Toland, (1995).

9. Stephani, Hans, and M.A.H MacCallum. “Differential Equations: Their Solution Using Symmetries.” Cambridge University Press, *Cambridge [England]: New York* (1989).

10. Olver, Peter J, and Philip Rosenau. “Group-Invariant Solutions to Differential Equations.” *SIAM J Appl Math* 47 (1987): 263-278.

How to cite this article: Opara, Uchechukwu. “On the Eight Dimensional Point Symmetries of Second Order Linear O.D.E's.” J Generalized Lie Theory Appl 14 (2020): 297. doi: 10.37421/GLTA.2020.14.297

28 Review of Symbolic Software for Lie Symmetry Analysis. By W. HEREMAN (1997)

Pergamon

Mathl. Comput. Modelling Vol. 25, No. 8/9, pp. 115–132, 1997

0895-7177/97 $17.00 + 0.00

PII: S0895-7177(97)00063-0

Review of Symbolic Software for Lie Symmetry Analysis

W. HEREMAN
Department of Mathematical and Computer Sciences
Colorado School of Mines, Golden, CO 80401, U.S.A.
whereman@mines.edu

Abstract—Computer algebra packages and tools that aid in the computation of Lie symmetries of differential equations are reviewed. The methods and algorithms of Lie symmetry analysis are briefly outlined. Examples illustrate the use of the symbolic software.

Keywords—Lie symmetry software, Lie symmetry analysis, Symmetry reduction, Group-invariant solutions, Symbolic computations.

1. INTRODUCTION

Sophus Lie (1842–1899) pioneered the study of continuous transformation groups that leave systems of differential equations invariant. Lie's work [1–3] brought diverse and *ad hoc* integration methods for solving special classes of differential equations under a common conceptual umbrella. Indeed, Lie's infinitesimal transformation method provides a widely applicable technique to find closed form solutions of ordinary differential equations (ODEs). Standard solution methods for first-order or linear ODEs can be characterized in terms of symmetries. Through the group classification of ODEs, Lie succeeded in identifying all ODEs that can either be reduced to lower-order ones or be completely integrated via group theoretic techniques.

Applied to partial differential equations (PDEs), Lie's method [2] leads to group-invariant solutions and conservation laws. Exploiting the symmetries of PDEs, new solutions can be derived from known ones, and PDEs can be classified into equivalence classes. Furthermore, group-invariant solutions obtained via Lie's approach may provide insight into the physical models themselves, and explicit solutions can serve as benchmarks in the design, accuracy testing, and comparison of numerical algorithms.

Nowadays, the concept of symmetry plays a key role in the study and development of mathematics and physics. Indeed, the theory of Lie groups and Lie algebras is applied to diverse fields of mathematics including differential geometry, algebraic topology, bifurcation theory, to name a few. Lie's original ideas greatly influenced the study of physically important systems of differential equations in classical and quantum mechanics, fluid dynamics, elasticity, and many other applied areas [4–8].

The application of Lie group methods to concrete physical systems involves tedious computations. Even the calculation of the continuous symmetry group of a modest system of differential

Special thanks go to the editors for inviting me to write this survey paper. F. Verheest is thanked for editorial comments which greatly improved the manuscript. This work was supported in part by Grant # CCR-9300978 of the National Science Foundation of the United States of America.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

equations is prone to errors, if done with pencil and paper. Computer algebra systems (CAS) such as Mathematica, MACSYMA, Maple, REDUCE, AXIOM and MuPAD are extremely useful for such computations. Symbolic packages [9–11], written in the language of these CAS, can find the determining equations of the Lie symmetry group. The most sophisticated packages then reduce these into an equivalent but more suitable system, subsequently solve that system in closed form, and go on to calculate the infinitesimal generators that span the Lie algebra of symmetries.

In Section 2, we discuss methods and algorithms used in the computation of Lie symmetries. We address the computation of determining systems, their reduction to standard form, solution techniques, and the computation of the size of the symmetry group. In Section 3, we look beyond Lie-point symmetries, addressing contact and generalized symmetries, as well as nonclassical or conditional symmetries.

Section 4 is devoted to a review of modern Lie symmetry programs, classified according to the underlying CAS. The review focuses on Lie symmetry software for classical Lie-point symmetries, contact (or dynamical), generalized (or Lie-Bäcklund) symmetries, nonclassical (or conditional) symmetries. Most of these packages were written in the last decade. Researchers interested in details about pioneering work should consult [9,10,12]. In Section 5, two examples illustrate results that can be obtained with Lie symmetry software. In Section 6 we draw some conclusions.

Lack of space forces us to give only a few key references for the Lie symmetry packages. A comprehensive survey of the literature devoted to theoretical as well as computational aspects of Lie symmetries, with over 300 references, can be found elsewhere [11].

2. METHODS AND ALGORITHMS

2.1. Computing the Determining Equations

The classical "Lie symmetry group of a system of differential equations" is a local group of point transformations, meaning diffeomorphisms on the space of independent and dependent variables, which map solutions of the system into other solutions.

There are three major methods to compute Lie symmetries. The first one uses prolonged vector fields, the second utilizes differential forms (wedge products) due to Cartan. The third one uses the notion of "formal symmetry." A detailed account of the technical steps of the first method, which is used in most of the Lie symmetry packages, together with a brief discussion of the two other methods is given in [11].

For a system of m differential equations,

$$\Delta^i\left(x, u^{(k)}\right) = 0, \qquad i = 1, 2, \dots, m, \tag{1}$$

of arbitrary order k, with p independent and q dependent (real) variables, denoted by $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$, $u = (u^1, u^2, \dots, u^q) \in \mathbb{R}^q$, the group transformations have the form $\tilde{x} = \Lambda_G(x, u)$, $\tilde{u} = \Omega_G(x, u)$, where the functions Λ_G and Ω_G are to be determined.

In the method of prolonged vector fields [2], instead of looking for the Lie group G, one looks for its Lie algebra $\mathcal{L}$, realized by vector fields of the form

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u) \frac{\partial}{\partial u^l}. \tag{2}$$

To determine the coefficients $\eta^i(x, u)$ and $\varphi_l(x, u)$ one constructs the k^{th} prolongation $\text{pr}^{(k)}\alpha$ of the vector field α, applies it to the system (1), and requests that the resulting expression vanishes on the solution set of (1).

The result is a system of linear homogeneous PDEs for η^i and φ_l, in which x and u are treated as independent variables, called the determining or defining equations for the symmetries of the system. Solving these by hand, interactively or automatically with a symbolic package, will give the explicit forms of $\eta^i(x,u)$ and $\varphi_l(x,u)$.

This sounds straightforward, but the method involves tedious calculations because the length and complexity of the expressions increase rapidly with p, q, m, and especially k. The procedure, for which the details are in [11,13], consists of two major steps: *deriving* the determining equations, and *solving* them explicitly. From the Lie algebra of symmetry generators, one can obtain the Lie group of point transformations upon integration of a system of first-order characteristic equations.

2.2. Reducing the Determining Equations

A detailed review of innovative ways of classifying, subsequently reducing, and finally solving overdetermined systems of linear homogeneous PDEs is given in [11].

To design a reliable and powerful integration algorithm for a system of determining equations the system needs to be brought into a standard form.

Standard form procedures can be viewed as generalizations to systems of linear PDEs of the Gaussian reduction method for matrices or linear systems, except that integrability conditions are also added to the system. As defined by Reid [14], a standard form is obtained by repeating the following steps:

(i) write each equation in solved form with respect to its highest order derivative,
(ii) replace these highest order derivatives throughout the rest of the system,
(iii) add any new equations arising from integrability conditions.

Reid and collaborators [14–19], Schwarz [20–22], and Wolf and Brand [23–27], partially implemented algorithms to reduce systems of PDEs. Their work led to sophisticated symbolic codes in MACSYMA, Maple, and REDUCE for that purpose.

In [21,22], Schwarz describes the algorithm *InvolutionSystem*, based on the theory of differential equations of Riquier and Janet, to transform a linear system of PDEs into involutive form. If all consistent orderings for the terms in the system of PDEs are known, the algorithm *InvolutionSystem* may be applied repeatedly to determine a universal Gröbner basis [22]. Schwarz designed *InvolutionSystem* to determine the size of the Lie symmetry group of a given system of PDEs without having to integrate the determining equations. We devote Section 2.4 to this important topic.

Reid's algorithm *standard_form* [14,15] also has its roots in the classical Riquier-Janet Theory. The algorithm was first implemented in MACSYMA, and a Maple version became available later [19]. It takes as input the system of PDEs and a matrix which specifies a complete ordering on the derivatives appearing in the system. It then reduces the system of PDEs to an equivalent simplified ordered triangular system with all integrability conditions included and all redundancies (differential and algebraic) eliminated.

In contrast to using the monomials of the Riquier-Janet Theory, Reid's algorithm implements an equivalence class approach to the problem of bringing a system of PDEs into a standard form. Once that is achieved, one can continue with the automatic determination of a Taylor series solution of the system to a specified finite degree.

Reid and Wittkopf's package [19] facilitates automated interfacing with major symmetry packages such as DIMSYM [28,29], LIESYMM [30], and SYMMGRP.MAX [12], and also with the differential Gröbner basis package DIFFGROB2 [31]. A TeX interface between *standard_form* and Hickman's program [32] that uses physical variable notation has been provided by Lisle. Full details and examples of the package, which includes other powerful algorithms for symmetry analysis of PDEs, are given in [17,19].

Reid and McKinnon developed a recursive algorithm called *Rsolve_Pdesys* [33] that builds on Reid's *standard_form* [14], and finds particular solutions of linear systems of PDEs using only ODE solution techniques. Applied to symmetry problems, their algorithm will find all polynomial/rational solutions of the determining equations provided the symmetry group is finite dimensional.

Recently, Reid *et al.* [34] designed a new algorithm which uses a finite number of differentiations and algebraic operations to simplify analytic nonlinear systems of PDEs to what they call 'a reduced involutive form' (*rif*), which includes the integrability conditions of the system and satisfies a constant rank condition. *Rif* combines features of geometric involutive form algorithms and the Reid-Wittkopf standard form algorithm. In [34] an algebraic realization of the *rif* algorithm is given for polynomially nonlinear PDE systems. Called *gröbner_rif*, it uses Buchberger's algorithm [35,36] for its nonlinear simplifier in addition to algorithms for constructing the radical of a polynomial ideal, and, for example, it drastically simplifies nonlinear systems of determining equations arising from nonclassical symmetries. A worked example, involving a coupled cubic nonlinear Schrödinger system in $3 + 1$ dimensions, is given in [34].

In the full computer implementation of "triangulation" algorithms, one can take advantage of a "differential" generalization of Buchberger's algorithm for Gröbner bases [35,36]. That technique reduces systems of nonlinear (and, consequently, also linear) PDEs to standard form.

The Maple program DIFFGROB [37] and its second version DIFFGROB2 by Mansfield [31], are designed to compute the differential Gröbner basis (DGB) for polynomially nonlinear PDE systems. DIFFGROB2 calculates in a systematic way: elimination ideals, integrability conditions, and compatibility conditions of a system of nonlinear PDEs of polynomial type, up to certain technical constraints fully explained in [31].

Fundamental tools in Mansfield's package are the Kolchin-Ritt algorithm, a differential analogue of Buchberger's algorithm with pseudo-reduction instead of reduction (to ensure termination), and the *diffgbasis* algorithm, which takes into account algebraic as well as differential consequences of nonlinear systems. These two algorithms compute the DGB for a wide range of systems of PDEs.

The package DIFFGROB2 has proven to be an effective tool in solving overdetermined systems of linear and nonlinear PDEs in the study of classical and nonclassical symmetries [11]. For more information about DIFFGROB2 and illustrations of its use, the reader should consult [31,38–43], in particular the manual [31], which contain extra references.

Wittkopf's algorithm *diff_reduce* [44] is similar to Mansfield's algorithm in that it attempts to reduce polynomially nonlinear systems of PDEs to the form of a DGB. Wittkopf's algorithm uses reduction rather than pseudo-reduction, and incorporates strategies for efficient memory management.

The program CRACK by Wolf and Brand [25–27,45] also carries out a Gröbner basis analysis but in slightly modified form. First, the algorithm is enriched by the integration of PDEs whenever possible, but in such a way, that the new integrated PDEs are still polynomial in the Gröbner basis. Selective integration can reduce the complexity and aid in solving the determining equations, in particular for systems for which pure Gröbner basis methods would be unfeasible. Second, for efficiency reasons, only a restricted completion algorithm is used.

Several other differential-algebraic and geometric approaches and corresponding implementations are possible to reduce linear and nonlinear systems of PDEs [11,34].

2.3. Solving the Determining Equations

The most challenging part of Lie symmetry analysis by computer, involves the design of an "integrator" for the overdetermined systems of linear homogeneous PDEs. Two other important topics tie in with the integration of the determining equations:

(i) the transformation of the determining equations into standard and passive forms; and
(ii) the computation of the size of the symmetry group,

discussed in Sections 2.2 and 2.4, respectively.

The design of algorithms and programs to bring the determining equations into standard form were a major step forward. Once systems are decoupled or reduced to standard involutive form, subsequent integration is more tractable and reliable. One could use separation of variables, standard techniques for linear differential equations, and specific heuristic rules as given in [11]. The only determining equations left for manual handling should be the "constraint" equations or any other equations whose general solutions cannot be written explicitly in closed form.

In order to be able to make the determination of certain types of Lie generators into a decision procedure, one needs an algorithm for solving linear homogeneous ODEs. An important step towards this goal is the factorization as applied in SPDE [46,47]. An in-depth review of issues related to the implementation of this is given in [48].

Despite the innovative efforts of Reid, Schwarz, Wolf, and Brand, and many others, there is no general algorithm available to integrate an arbitrary (overdetermined) system of determining equations that consists of linear homogeneous PDEs for the η's and the ϕ's. Most of the computer programs use a set of heuristic rules [28,29,46,49–51] for the integration of the determining system. We will not repeat these rules here, they can be found in [11].

2.4. Computing the Size of the Symmetry Group

Schwarz [21,22] and Reid [14–16] independently developed algorithms for determining the dimension of the symmetry algebra from the infinitesimal determining system without having to integrate the system explicitly.

Schwarz's algorithm SYMSIZE [21,22] is available with the computer algebra system REDUCE, as part of the package SPDE (see Section 4.1). Schwarz also translated SYMSIZE into the language of SCRATCHPAD II, the predecessor of AXIOM.

The size of the symmetry group can always be determined with SYMSIZE in a finite number of steps. SYMSIZE accepts a system of PDEs as input, and allows to compute *a priori* the number of free parameters, if the group is finite, and the number of unspecified functions, if the group is infinite. In turn, SYMSIZE allows to test *a posteriori* if the solution of the determining equations is complete. At the heart of SYMSIZE is the procedure *InvolutionSystem*, which transforms the determining system into an involutive system by means of a critical pair/completion algorithm. Similar algorithms are applied in computing Gröbner bases in polynomial ideal theory (see Section 2.2).

Concurrently, yet independent of Schwarz, Reid [14–16] realized that triangularization algorithms may be used to bypass the explicit solution of the determining equations and compute the size of the symmetry group and the commutators immediately. Reid developed the program SYMCAL [14], written originally in MACSYMA, but now converted by Reid and Wittkopf into Maple [19].

In [16], a nonheuristic algorithm *structure_constant* is presented, based on the routines *Taylor* and *standard_form*, which always determines (in a finite number of steps) the dimension and the structure constants of the finite part of the Lie symmetry algebra. The newest Maple algorithm [18] computes the dimension and the commutation relations without Taylor expansions; hence, it is applicable to infinite-dimensional Lie algebras. An extension of the algorithm also classifies differential equations (with variable coefficients) according to the structure of their symmetry groups. Furthermore, the approach advocated by Reid applies to the determination of symmetries of Lie-contact and Lie-Bäcklund types, as well as potential symmetries.

Readers interested in the problem of determining the "size" of the solution space for arbitrary involutive systems should also consult a recent paper by Seiler [52].

Finally, the tools for reducing systems of linear homogeneous PDEs, available within the package CRACK [25–27,45], can also greatly assist in the investigation of the size of the symmetry group.

3. BEYOND LIE-POINT SYMMETRIES

3.1. Contact and Generalized Symmetries

For the computation of generalized symmetries or Lie-Bäcklund symmetries [2] the use of symbolic programs is even more appropriate, for the calculations are lengthier and more time consuming. In a generalized vector field, which still takes the form of (2), the functions η^i and ϕ_l may now depend on a finite number of derivatives of u, i.e.,

$$\alpha = \sum_{i=1}^{p} \eta^i \left(x, u^{(k)}\right) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l \left(x, u^{(k)}\right) \frac{\partial}{\partial u^l}. \tag{3}$$

If $k = 1$ the generalized symmetry determines a classical contact symmetry and vice versa, at least in the case of one dependent variable. The even simpler case $k = 0$, with $u^{(0)} = u$, leads to point symmetries.

3.2. Nonclassical or Conditional Symmetries

Recently it was shown that the "nonclassical method of group-invariant solutions," originally introduced in [53], can determine new solutions of various physically significant nonlinear PDEs.

In contrast to Lie-point symmetries, the transformations corresponding to nonclassical (or conditional) symmetries neither leave the differential equation invariant, nor transform all the solutions into other solutions. They merely transform a subset of solutions into other solutions. For a well-documented perspective on the computation of nonclassical symmetries we recommend [40,54,55].

Accounting for "nonclassical symmetries," the program should automatically add the q invariant surface conditions [53]

$$Q^l \left(x, u^{(1)}\right) = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial u^l}{\partial x_i} - \varphi_l(x, u) = 0, \qquad l = 1, \dots, q, \tag{4}$$

and their differential consequences, to the system (1). However, the inclusion of nonclassical symmetries, and perhaps other types of symmetries [2], requires solving systems of determining equations which are *no longer linear*, for which new integration algorithms must be designed.

4. REVIEW OF SYMBOLIC SOFTWARE

In Table 1, we indicate the scope of the most modern Lie symmetry software packages with key references. In [11], the reader can find detailed information about the developers and distributors of the various packages.

4.1. REDUCE Programs

In the early '80s, Schwarz developed the program SPDE (see references in [46]). The program automatically derives and often successfully solves the determining equations for Lie-point symmetries with minimal intervention by the user. Since 1986 SPDE is distributed together with REDUCE for various types of computers, ranging from PCs to CRAYs. The newest and drastically reworked Version 1.0 of SPDE [56] ensures that all infinitesimal symmetry generators with algebraic coefficients will be obtained if the equations are nonlinear and of order higher than one. Concerning the input, the equations must be algebraic in their arguments. There is no restriction on the number of independent and dependent variables, and the equations can have any number of constant parameters (no arbitrary functions). The program computes the determining equations, then generates a Gröbner basis for the determining system in a term ordering specified

Table 1. Scope of Lie symmetry programs.

Name	System	Developer(s) and Refs.	Point	Gen.	Non-class.	Solves Det. Eqs.
CRACK	REDUCE	Wolf and Brand [24]	–	–	–	Yes
DELiA	Pascal	Bocharov *et al.* [57,58]	Yes	Yes	No	Yes
DIFFGROB2	Maple	Mansfield [31]	–	–	–	Reduction
DIMSYM	REDUCE	Sherring [28,29]	Yes	Yes	No	Yes
LIE	REDUCE	Eliseev *et al.* [59]	Yes	Yes	No	No
LIE	muMATH	Head [49]	Yes	Yes	Yes	Yes
Lie	Mathematica	Baumann [60]	Yes	No	Yes	Yes
LieBaecklund	Mathematica	Baumann [61]	No	Yes	No	Interactive
LIEDF/INFSYM	REDUCE	Gragert and Kersten [62,63]	Yes	Yes	No	Interactive
LIEPDE	REDUCE	Wolf and Brand [25]	Yes	Yes	No	Yes
Liesymm	Maple	Carminati *et al.* [30]	Yes	No	No	Interactive
MathSym	Mathematica	Herod [64]	Yes	No	Yes	Reduction
NUSY	REDUCE	Nucci [65,66]	Yes	Yes	Yes	Interactive
PDELIE	MACSYMA	Vafeades [67,68]	Yes	Yes	No	Yes
SPDE	REDUCE	Schwarz [46]	Yes	No	No	Yes
SYMCAL	Maple/MACSYMA	Reid and Wittkopf [19]	–	–	–	Reduction
SYM_DE	MACSYMA	Steinberg [51]	Yes	No	No	Partially
symgroup.c	Mathematica	Bérubé and de Montigny [69]	Yes	No	No	No
SYMMGRP.MAX	MACSYMA	Champagne *et al.* [12]	Yes	No	Yes	Interactive
SYMSIZE	REDUCE	Schwarz	–	–	–	Reduction

by the user. The integration of the reduced system is carried out automatically, the symmetry generators and their commutator table can be displayed in LaTeX.

Based on Cartan's exterior calculus, Gragert [62], and Gragert, Kersten and Martini [70] used computer algebra systems to calculate the classical Lie symmetries of differential equations. More recently, Gragert [71,72] added a package for more general Lie algebra computations, including code for higher-order and super symmetries and super prolongations. Kersten [50,73] further perfected the software package for the calculation of the Lie algebra of infinitesimal symmetries (including Lie-Bäcklund symmetries) of exterior differential systems. Readers interested in the differential geometrical foundation of Lie symmetry analysis, and related REDUCE algorithms should consult [63,74,75].

Eliseev, Fedorova and Kornyak [59], wrote code in REDUCE-2 to generate (but not solve) the system of determining equations for point and contact symmetries. Their paper discusses the algorithm and shows three worked examples. Fedorova and Kornyak [76,77] generalized the algorithm to include the case of Lie-Bäcklund symmetries.

The interactive REDUCE program NUSY by Nucci [65,66], generates determining equations for Lie-point, nonclassical, Lie-Bäcklund and approximate symmetries and provides interactive tools to solve them.

The package CRACK by Wolf and Brand [25–27,45] solves overdetermined systems of differential equations with polynomial terms. The general purpose package features code for decoupling, separating, and simplifying PDEs. Integration of exact PDEs and differential factorization are also possible. CRACK has many applications that are facilitated via special tools, some of which can aid in the investigation of Lie symmetries of ODEs. However, functions and tools available within CRACK allow simplification and integration of linear homogeneous PDEs, beyond those derivable via symmetry analysis (for examples see [45]).

Upon completion of CRACK, Wolf went on to develop several new REDUCE programs [78], called LIEPDE, APPLYSYM, QUASILINPDE and DETRAFO, which all make use of the tools of CRACK. First, LIEPDE [24,78] finds Lie-point and contact symmetries of PDEs by deriving and solving a few simple determining equations, before continuing with the computation of the more complicated determining equations. This idea, which makes the program highly efficient, was used in Wolf's FORMAC program [23,79,80], and is also implemented in the design of the feedback mechanism of SYMMGRP.MAX [12]. This strategy becomes crucial when symmetry programs are applied to large systems of PDEs, or in the computation of higher-order symmetries, where space and memory limitations come into play.

The aim of QUASILINPDE [78] is to find the general solutions of quasi-linear PDEs. These solutions are then used by APPLYSYM [78], which applies the symmetry to lower the order of ODEs, to calculate similarity variables for PDEs, to reduce the number of independent variables of a system of PDEs, and to generalize special known solutions of ODEs and PDEs. The program APPLYSYM is automatic, but can also be used interactively.

The program DETRAFO, used by APPLYSYM, performs arbitrary point- and contact transformations of ODEs/PDEs, and is applied when the similarity and symmetry variables have been found. To our knowledge, APPLYSYM is one of the first symbolic programs that truly applies point symmetries that can be calculated with the program LIEPDE.

In [81,82], Gerdt introduced the program HSYM for the explicit computation of higher-order symmetries for PDEs. If the given system of equations has arbitrary parameters, the necessary conditions for the existence of higher order symmetries will lead to a system of algebraic equations in the parameters. Via the program ASYS, that algebraic system is reduced to a standard form via a Gröbner basis algorithm. The focus in Gerdt's work is on the investigation of the integrability of polynomial type nonlinear evolution equations, by verifying the existence of higher order symmetries and their associated conservation laws.

Sarlet and Vanden Bonne [83] offer specific procedures to assist in the computation of adjoint symmetries of second-order ODEs. Their software constructs determining equations for certain classes of adjoint symmetries, which are of the same type as for (generalized) symmetries, and relies on other packages such as DIMSYM to solve these.

The program DIMSYM by Sherring [28,29], in collaboration with Prince, finds various types of symmetries, currently, point symmetries, Lie-Bäcklund, and conditional symmetries. DIMSYM can isolate special cases, bring the determining equations in standard form and aid in the solution of group classification problems. An attractive feature of DIMSYM is that the integrator for the determining equations also works for systems of linear homogeneous differential equations not necessarily obtained from symmetry analysis. The overall strategy of the solver is to put the system of determining equations into standard form based on the Reid-Wittkopf algorithm (see Section 2.2), while solving explicitly all equations in the system that the algorithm is capable of solving.

DIMSYM attempts to determine the generators and checks whether or not the generators are correct. It allows to specify the dependence of the symmetry vector field coefficients, which is useful for computing Lie-Bäcklund symmetries. DIMSYM provides a lot of flexibility: ansätze can be made, simplification routines can be called separately, manual intervention is possible, etc.

Finally, the program RELIE by Oliveri [84] interactively computes Lie-point symmetries, via a collection of (algebraic) procedures which automate the steps that would be taken if the calculation were done by hand.

4.2. MACSYMA Programs

The MACSYMA version of the program SYMCON [85,86], originally written in muMATH, tries to compute Lie-point and Bessel-Haagen generalized symmetries (of any order) and their conservation laws. Vafeades later produced PDELIE [67,68,87,88], a drastically improved version of SYMCON [85,86]. The package PDELIE attempts to produce similarity solutions of ODEs, analyze PDEs with a multiplicative or additive scalar parameter, and compute the commutator table and the structure constants of the Lie algebra. Also, the densities of the Noether conservation laws of systems of variational and divergence type can be computed.

PDELIE consists of several subroutines to set up the determining equations, and to compute the generators and structure constants of the Lie group. It uses the Reid-Wittkopf standard form algorithm and a set of heuristic rules to facilitate the integration, and finds the invariants of the symmetry group. Using these invariants, it then reduces the dimension of the given differential equation. In cases where the reduced equation is an ODE, it tries to integrate it explicitly, thus arriving at special similarity solutions of the original equation.

Just as PDELIE, the program SYM_DE by Steinberg [51,89,90] was recently added to the out-of-core library of MACSYMA. Steinberg's program computes infinitesimal symmetry operators and the explicit form of the infinitesimal transformations for simple systems. In cases where the program cannot finish the computation automatically, the user can intervene and, for instance, ask for infinitesimal symmetries of polynomial form. The program solves some (or all) of the determining equations automatically and, if needed, the user can add extra information.

The flexibility within SYMMGRP.MAX, the program by Champagne, Hereman and Winternitz [9,12], and the possibility of using it interactively, allows to find the symmetry group of arbitrarily large and complicated systems of equations on relatively small computers.

SYMMGRP.MAX allows to follow a path that would be taken in manual calculations. That is, obtain in as simple a manner as possible the simplest determining equations, solve them and feed the information back to the computer. Partial information can be extracted rapidly. For instance, one can derive a subset of the determining equations, such as those that occur as coefficients in the highest derivatives in the independent variables. These are usually single-term equations, which imply that the coefficients of the vectorfield are independent of some variables or depend linearly on some of the other variables.

When the prolongation can be applied successfully to the complete system, or a subset thereof, SYMMGRP.MAX produces a list of determining equations, free of trivial factors, duplication, and differential redundancies. A feedback mechanism then facilitates the solution of the determining system step by step on the computer. Typically, users will provide information about the η's and φ's, as it becomes available. In [12], a worked example shows the use of the feedback mechanism within SYMMGRP.MAX to solve the determining equations completely.

Although not designed for that purpose, SYMMGRP.MAX can be easily adapted to nonclassical symmetries [38–40,55]. In [40], Clarkson and Mansfield give a detailed explanation of such an adaptation. Currently, Hereman is adapting SYMMGRP.MAX for the calculation of Lie-point symmetries of difference-differential equations.

4.3. Maple Programs

In [30], Carminati, Devitt, and Fee present LIESYMM for creating the determining equations via the Harrison-Estabrook procedure. Within LIESYMM various interactive tools are available for integrating the determining equations, and for working with Cartan's differential forms.

Vu [91] has translated Head's muMATH program LIE [49] into Maple syntax. Vu's program *Desolv* first automatically generates the determining equations for Lie-point symmetries, subsequently solves them to determine the explicit forms of the coefficients of the vector field, and finally computes the generators. The heuristic procedures implemented in Vu's program perform polynomial decomposition of PDEs, decoupling of PDEs, integration of simple PDEs and ODEs. Vu adopted some of the integration methods from CRACK, the decoupling method from the Reid-Wittkopf standard form algorithm, and ideas from Mansfield's DIFFGROB package.

Hickman [32] wrote a collection of Maple routines that aid in the computation of Lie-point symmetries, nonlocal symmetries, and Wahlquist-Estabrook-type prolongations. His tools for symmetry analysis include user-friendly procedures to enter names of variables, to create total derivatives, to generate and prolong vector fields, and to derive and partially solve determining equations.

Mansfield has developed the package DIRMETH for the computation of symmetries via the direct method proposed by Clarkson and Kruskal [92], as part of DIFFGROB2.

4.4. Mathematica Programs

Herod [70] developed MathSym for deriving the determining equations corresponding to Lie-point symmetries, including nonclassical (or conditional) symmetries. Upon derivation of the determining equations, the program reduces these equations via an algorithm based on the Riquier-Janet method. Herod's doctoral thesis [64] contains the well-documented code of MathSym and applications to various equations from fluid dynamics.

Recently, the packages *Lie.m* and *Baecklund.m* have been added by Baumann [60,61] to MathSource, the Mathematica Program Library. Baumann's program *Lie.m* [60] follows the structure of our MACSYMA program SYMMGRP.MAX [12] very closely. However, the program *Lie.m* can handle transcendental functions in the input equations. The newest version of *Lie.m* can be used to compute point symmetries, contact symmetries and nonclassical symmetries. *Lie.m* brings the determining equations in canonical form via the Riquier-Janet procedure, and goes on to solve the determining equations automatically. A set of integration rules, similar to those mentioned in [11], is implemented.

Once the determining equations are solved, the program continues with the computation of the vector basis, ideals, and commutator table of the Lie algebra, its structure constants, Casimir operators, and its metric tensor.

Baumann's package *Baecklund.m* [61] contains functions that attempt to compute generalized symmetries for PDEs and ODEs, and invariants of ODEs. When applied to second-order ODEs, the program attempts to verify if the computed symmetries are of variational type. If so, the program calculates the corresponding invariants (integrals of motion).

Bérubé and de Montigny [69] produced Lie symmetry code in Mathematica. Their program *symmgroup.c* computes the determining equations for Lie-point symmetries, closely following the structure of SYMMGRP.MAX. The data for the program may consist of PDEs with arbitrary functions. Transcendental functions in both dependent and independent variables are also permitted. In [69], three well-chosen examples are given to illustrate the capabilities of the program.

Finally, Coult (Program in Applied Mathematics, University of Colorado, Boulder, CO 80309, U.S.A.) developed the Mathematica program *symmgroup.m*, for the computation of the determining equations corresponding to Lie-point symmetries of a large class of differential equations with polynomial terms.

4.5. SCRATCHPAD and AXIOM Programs

Schwarz [93] rewrote SPDE [46] for use with Version 1 of SCRATCHPAD II (now superseded by AXIOM).

Seiler and coworkers [94,95] are designing a package that will compute determining equations for classical and nonclassical symmetries. Consult [11] for a description of their program JET for geometry computations based on the jet bundle formalism.

4.6. muMATH Programs

The program LIE by Head [49] is based on Version 4.12 of muMATH. Since LIE comes bundled with a limited version of muMATH, it is a self-contained stand-alone package that runs on IBM compatible PCs. Version 4.4 of Head's program calculates and solves the determining equations (for Lie-point symmetries, contact and generalized (Lie Bäcklund) symmetries, and also nonclassical symmetries) automatically for systems of differential equations. LIE also computes the Lie vectors and their commutators. Interventions by the user are possible but rarely needed. Due to the limitations of muMATH, the program LIE is bounded by the 256 KB of memory for program and workspace. For a program of limited size, LIE is remarkable in its achievements.

The SYMCON package by Vafeades [85,86] also uses muMATH to compute the determining equations for Lie-point symmetries, without solving them. Furthermore, the program verifies whether the symmetry group is of variational or divergence type and computes the conservation laws associated with the symmetries. Unfortunately, the package cannot handle large systems of equations. This limitation motivated Vafeades to rewrite his SYMCON program in MACSYMA syntax [67,68,87,88]. The MACSYMA version can handle generalized symmetries and their conservation laws.

Mikhailov and collaborators [96] developed muMATH software to verify the integrability of systems of PDEs by testing for the existence of higher symmetries. The program computes special symmetries, canonical conservation laws, and carries out conformal transformations to bring PDEs into canonical form.

4.7. Programs for Other Systems

Kornyak and Fushchich [97,98] developed programs in Turbo C and AMP for the computation of Lie-Bäcklund symmetries. Their programs also classify equations with arbitrary parameters and functions with respect to such symmetries, and reduce the determining equations into passive form: all integrability conditions are then explicit and, therefore, the resulting system is in involution.

There are also two FORMAC programs for symmetry analysis. The first program, called *LB*, was written in the PL/1 language by Fedorova and Kornyak [76,99]. The successor, called *LBF*, was developed by Fushchich and Kornyak [97], and written in PL/1-FORMAC. Both programs create the system of determining equations for Lie-Bäcklund symmetries and attempt to reduce and solve these equations. The program *LBF* is completely automatic, and both were designed for low-memory requirements so that they could run on PCs.

The PL/1-based FORMAC package CRACKSTAR developed by Wolf [23,79,80] allows to investigate Lie symmetries of systems of PDEs, besides dealing with dynamical symmetries of ODEs [100], and the like. A good overview of the capabilities of CRACKSTAR is given in [80]; a description of the routines and worked examples can be found in [100]. For efficiency, CRACKSTAR generates and solves first-order determining equations early on, and then continues with the higher-order determining equations.

Gerdt [81,82], Gerdt and Zharkov [101] and Gerdt, Shvachka and Zharkov [102,103] used REDUCE and PL/1-FORMAC to investigate the integrability of nonlinear evolution equations. Their program FORMINT contains algorithms to calculate Lie-Bäcklund symmetries and conserved densities, but does not use the jet bundle formalism.

The calculation of the Lie group by computer was also proposed by Popov [104], who used the program SOPHUS for the calculation of conservation laws of evolution equations.

In [105], Bocharov and Bronstein present SCoLAr, a package written in standard PASCAL that finds infinitesimal symmetries and conservation laws of arbitrary systems of differential equations.

The PC package DELiA, standing for "Differential Equations with Lie Approach," is an outgrowth of the SCoLAr project. DELiA, written in Turbo PASCAL by Bocharov and his collaborators [58], is a stand-alone computer algebra system for investigating differential equations. It performs various tasks based on Lie's approach, such as the computation of Lie-point and Lie-Bäcklund symmetries, canonical conserved densities and generalized conservation laws, simplification and partial integration of overdetermined systems of differential equations, etc.

In order to be able to handle large problems, DELiA first generates and solves first-order determining equations, and then continues to generate and solve the higher-order determining equations. The analyzer/integrator, which is available as a separate tool at the user level, includes a general algorithm for passivization [105], together with a set of integration rules for linear and quasi-linear systems of PDEs. The methods are well described in the user guides [57,58].

Using the algorithmic language REFAL, Topunov [106] developed a software package for symmetry analysis that contains subroutines to reduce determining systems in passive form.

5. EXAMPLES

5.1. The Dym-Kruskal Equation

Consider the Dym-Kruskal equation [107],

$$u_t - u^3 u_{xxx} = 0. \tag{5}$$

Clearly, this is one equation with two independent variables and one dependent variable. The assignments of the variables are as follows:

$$x \longmapsto x[1], \qquad t \longmapsto x[2], \qquad u \longmapsto u[1]. \tag{6}$$

Then, symmetry software such as SPDE, LIE, PDELIE, DIMSYM, SYM_DE, or SYMMGRP.MAX, automatically compute the determining equations for the coefficients eta[1] $= \eta^x$, eta[2] $= \eta^t$, and phi[1] $= \varphi^u$ of the vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}. \tag{7}$$

There are only eight determining equations,

$$\frac{\partial \text{ eta2}}{\partial u[1]} = 0, \qquad \frac{\partial \text{ eta2}}{\partial x[1]} = 0, \qquad \frac{\partial \text{ eta1}}{\partial u[1]} = 0, \qquad \frac{\partial^2 \text{ phi1}}{\partial u[1]^2} = 0, \tag{8}$$

$$\frac{\partial^2 \text{ phi1}}{\partial u[1] \partial x[1]} - \frac{\partial^2 \text{ eta1}}{\partial x[1]^2} = 0, \qquad \frac{\partial \text{ phi1}}{\partial x[2]} - u[1]^3 \frac{\partial^3 \text{ phi1}}{\partial x[1]^3} = 0, \tag{9}$$

$$\begin{gathered} 3u[1]^3 \frac{\partial^3 \text{ phi1}}{\partial u[1] \partial x[1]^2} + \frac{\partial \text{ eta1}}{\partial x[2]} - u[1]^3 \frac{\partial^3 \text{ eta1}}{\partial x[1]^3} = 0, \\ u[1] \frac{\partial \text{ eta2}}{\partial x[2]} - 3u[1] \frac{\partial \text{ eta1}}{\partial x[1]} + 3 \text{ phi1} = 0. \end{gathered} \tag{10}$$

These determining equations are easily solved explicitly, either automatically with SPDE, LIE, DIMSYM, and PDELIE, or with the feedback mechanism within SYM_DE and SYMMGRP.MAX. The general solution, rewritten in the original variables, is

$$\eta^x = k_1 + k_3\, x + k_5\, x^2, \qquad \eta^t = k_2 - 3k_4\, t, \qquad \varphi^u = (k_3 + k_4 + 2k_5\, x)\, u, \tag{11}$$

where $k_1, \dots, k_5$ are arbitrary constants. The five infinitesimal generators are

$$\begin{gathered} G_1 = \partial_x, \qquad G_2 = \partial_t, \qquad G_3 = x\partial_x + u\partial_u, \\ G_4 = -3t\partial_t + u\partial_u, \qquad G_5 = x^2\partial_x + 2xu\partial_u. \end{gathered} \tag{12}$$

Clearly, (5) is invariant under translations (G_1 and G_2) and scaling (G_3 and G_4). The flow corresponding to each of the infinitesimal generators can be obtained via simple integration. As an example, let us compute the flow corresponding to G_5. This requires integration of the first-order system

$$\frac{d\tilde{x}}{d\epsilon} = \tilde{x}^2, \qquad \tilde{x}(0) = x, \qquad \frac{d\tilde{t}}{d\epsilon} = 0, \qquad \tilde{t}(0) = t, \qquad \frac{d\tilde{u}}{d\epsilon} = 2\tilde{x}\tilde{u}, \qquad \tilde{u}(0) = u, \tag{13}$$

where ϵ is the parameter of the transformation group. One readily obtains

$$\tilde{x}(\epsilon) = \frac{x}{(1 - \epsilon x)}, \qquad \tilde{t}(\epsilon) = t, \qquad \tilde{u}(\epsilon) = \frac{u}{(1 - \epsilon x)^2}. \tag{14}$$

Therefore, one concludes that for any solution $u = f(x,t)$ of equation (5), the transformed solution

$$\tilde{u}\left(\tilde{x}, \tilde{t}\right) = (1 + \epsilon\tilde{x})^2 f\left(\frac{\tilde{x}}{1 + \epsilon\tilde{x}}, \tilde{t}\right) \tag{15}$$

will solve $\tilde{u}_{\tilde{t}} - \tilde{u}^3 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$.

5.2. The Nonlinear Schrödinger Equation

As a second example, we take the Nonlinear Schrödinger (NLS) equation [107]

$$iu_t + u_{xx} + u|u|^2 = 0. \tag{16}$$

The complex equation can be replaced by a coupled system

$$v_t + w_{xx} + w\left(v^2 + w^2\right) = 0, \qquad w_t - v_{xx} - v\left(v^2 + w^2\right) = 0 \tag{17}$$

for the real and imaginary parts v, w of the complex variable u.

Now, SYMMGRP.MAX (or, for that matter, any other symmetry program) quickly generates the 20 determining equations for the coefficients of the vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^v \frac{\partial}{\partial v} + \varphi^w \frac{\partial}{\partial w}. \tag{18}$$

The first 11 single-term determining equations are similar to (8), and provide information about the dependencies of the η's and the ϕ's on x, t, v, and w, and their linearity in the latter two dependent variables. The remaining nine determining equations are a bit more complicated, but the entire system is readily solved.

In the original variables, the solution reads

$$\begin{aligned} \eta^x &= k_1 + 2k_4\, t + k_5\, x, & \eta^t &= k_2 + 2k_5\, t, \\ \varphi^v &= k_3\, w - k_4\, xw - k_5\, v, & \varphi^w &= -k_3\, v + k_4\, xv - k_5\, w, \end{aligned} \tag{19}$$

where $k_1, \ldots, k_5$ are arbitrary constants. As in the previous examples, the complete symmetry algebra is spanned by five symmetry generators:

$$\begin{aligned} G_1 = \partial_x, \qquad G_2 = \partial_t, \qquad & G_3 = w\partial_v - v\partial_w, \\ G_4 = 2t\partial_x - x(w\partial_v - v\partial_w), \qquad & G_5 = x\partial_x + 2t\partial_t - v\partial_v - w\partial_w. \end{aligned} \tag{20}$$

Clearly, (16) is invariant under translations in space and time (G_1 and G_2). Generator G_3 corresponds to adding an arbitrary constant to the phase of u. The Galilean boost is generated by G_4. Finally, G_5 indicates invariance of the equation under scaling (or dilation).

Similarity reductions can then be obtained by solving the characteristic equations

$$\frac{dx}{\eta^x} = \frac{dt}{\eta^t} = \frac{dv}{\varphi^v} = \frac{dw}{\varphi^w}, \tag{21}$$

or equivalently, the invariant surface conditions

$$\eta^x v_x + \eta^t v_t - \varphi^v = 0, \qquad \eta^x w_x + \eta^t w_t - \varphi^w = 0. \tag{22}$$

The actual reductions can be found in [108], where a quite general class of nonlinear Schrödinger equations is treated. All the reductions of the NLS can be obtained from G_1 through G_5; in other words, nonclassical symmetries would not lead to new symmetry reductions. To compute nonclassical symmetries of (17), it suffices to replace v_t and w_t from (22). If $\eta^t \neq 0$, we set $\eta^t = 1$ for simplicity. Thus,

$$v_t = -\eta^x v_x + \varphi^v, \qquad w_t = -\eta^x w_x + \varphi^w. \tag{23}$$

The case $\eta^t = 0$ has to be considered separately. Since SYMMGRP.MAX allows the user to give information about the coefficients in the vector field, the computation can now proceed as in the classical case. For worked examples, we refer to [40,55].

6. CONCLUSIONS

The programs reviewed in this paper are easy to use provided the user has access to and knows the basics of the underlying CAS, such as MACSYMA, Maple, Mathematica, and REDUCE.

Apart from the theoretical study of the underlying mathematics, there is a need for further development and implementation of effective algorithms for generating, reducing, simplifying and fully solving the determining equations for (classical and nonclassical) Lie-point symmetries and generalized or Lie-Bäcklund symmetries.

The availability of sophisticated symbolic programs certainly will accelerate the study of symmetries of physically important systems of differential equations in classical mechanics, fluid dynamics, elasticity, and other applied areas.

REFERENCES

1. G.W. Bluman and S. Kumei, Symmetries and differential equations, *Applied Mathematical Science* **81**, Springer-Verlag, New York, (1989).
2. P.J. Olver, Applications of Lie groups to differential equations, *Graduate Texts in Mathematics*, 2nd Edition, Springer-Verlag, New York, (1993).
3. H. Stephani, *Differential Equations: Their Solution Using Symmetries*, Cambridge University Press, Cambridge, UK, (1989).
4. G. Gaeta, Nonlinear symmetries and nonlinear equations, *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1994).
5. N.H. Ibragimov, Group analysis of ordinary differential equations and the invariance principle in mathematical physics, (for the 150th anniversary of Sophus Lie), *Russian Math. Surveys* **47**, 89–156, (1992).
6. N.H. Ibragimov, Editor, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 1: Symmetries, Exact Solutions, and Conservation Laws, CRC Press, Boca Raton, FL, (1994).
7. N.H. Ibragimov, Editor, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 2: Applications in Engineering and Physical Sciences, CRC Press, Boca Raton, FL, (1995).
8. N.H. Ibragimov, Editor, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 3: New Trends in Theoretical Developments and Computational Methods, CRC Press, Boca Raton, FL, (1996).
9. W. Hereman, SYMMGRP.MAX: A symbolic program for symmetry analysis of partial differential equations, In *Exploiting Symmetry in Applied and Numerical Analysis, Proc. AMS-SIAM Summer Seminar*, Fort Collins, CO, 1992; Lectures in Applied Mathematics **29**, (Edited by E. Allgower *et al.*), pp. 241–257, American Mathematical Society, Providence, RI, (1993).
10. W. Hereman, Review of symbolic software for the computation of Lie symmetries of differential equations, *Euromath Bulletin* **1** (2), 45–82, (1994).
11. W. Hereman, Symbolic software for Lie symmetry analysis, In *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 3: New Trends in Theoretical Developments and Computational Methods, (Edited by N.H. Ibragimov), Chapter 13, CRC Press, Boca Raton, FL, (1996).
12. B. Champagne, W. Hereman and P. Winternitz, The computer calculation of Lie point symmetries of large systems of differential equations, *Comp. Phys. Comm.* **66**, 319–340, (1991).
13. W. Hereman and W. Zhuang, Symbolic software for soliton theory, *Acta Appl. Math.* **39**, 361–378, (1995).
14. G.J. Reid, Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution, *Euro. J. Appl. Math.* **2**, 293–318, (1991).
15. G.J. Reid, Algorithmic determination of Lie symmetry algebras of differential equations, In *Lie Theory, Differential Equations and Representation Theory, Proc. Annual Seminar of the Canadian Math. Soc.*, Montréal, Canada, 1989; (Edited by V. Hussin), pp. 363–372, Les Publications de Centre de Recherches Mathématiques, Montréal, Canada, (1990).
16. G.J. Reid, Finding abstract Lie symmetry algebras of differential equations without integrating determining equations, *Euro. J. Appl. Math.* **2**, 319–340, (1991).
17. G.J. Reid and A. Boulton, Reduction of systems of differential equations to standard form and their integration using directed graphs, In *Proc. ISSAC '91*, Bonn, Germany, (Edited by S.M. Watt), pp. 308–312, ACM Press, New York, (1991).
18. G.J. Reid, I.G. Lisle, A. Boulton and A.D. Wittkopf, Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs, In *Proc. ISSAC '92*, Berkeley, CA, (Edited by B. Trager and D. Lazard), pp. 63–68, ACM Press, New York, (1992).
19. G.J. Reid and A.D. Wittkopf, *The Long Guide to the Standard Form Package*, Department of Mathematics, University of British Columbia, Vancouver, Canada, (1993).
20. F. Schwarz, The Riquier-Janet theory and its application to non-linear evolution equations, *Physica D* **11**, 243–251, (1984).
21. F. Schwarz, An algorithm for determining the size of symmetry groups, *Computing* **49**, 95–115, (1992).

22. F. Schwarz, Reduction and completion algorithms for partial differential equations, In *Proc. ACM-SIGSAM 1992 Int. Symp. on Symbolic and Algebraic Computation*, ISSAC '92, Berkeley, CA, (Edited by B. Trager and D. Lazard), pp. 49–56, ACM Press, New York, (1992).
23. T. Wolf, The symbolic integration of exact PDEs, School of Mathematical Sciences, Queen Mary and Westfield College, University of London, London, (Preprint), (1992).
24. T. Wolf, An efficiency improved program LIEPDE for determining Lie-symmetries of PDEs, In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Proc. Int. Workshop Acireale*, Catania, Italy, 1992, (Edited by N.H. Ibragimov *et al.*), pp. 377–385, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1993).
25. T. Wolf and A. Brand, The computer algebra package CRACK for investigating PDEs, In *Proc. ERCIM Advanced Course on Partial Differential Equations and Group Theory*, Bonn, Germany, (Edited by J.F. Pommaret), pp. 1–19, Gesellschaft für Mathematik und Datenverarbeitung, Sankt Augustin, Germany, (1992); *Manual for CRACK added to the REDUCE Network Library*, School of Mathematical Sciences, Queen Mary and Westfield College, University of London, London, (1992).
26. T. Wolf and A. Brand, Heuristics for overdetermined systems of PDEs, In *Proc. Meeting on Artificial Intelligence in Mathematics*, Glasgow, United Kingdom, (Edited by J.H. Johnson *et al.*), Clarendon Press, Oxford, (1994).
27. T. Wolf and A. Brand, Investigating symmetries and other analytical properties of ODEs with the computer algebra package CRACK, School of Mathematical Sciences, Queen Mary and Westfield College, University of London, London, (Preprint), (1994).
28. J. Sherring, DIMSYM—Symmetry determination and linear differential equations package, Department of Mathematics, LaTrobe University, Bundoora, Australia, (Preprint), (1993).
29. J. Sherring, Symmetry and Computer Algebra Techniques for Differential Equations, Ph.D. Thesis, Department of Mathematics, LaTrobe University, Bundoora, Australia, (1993).
30. J. Carminati, J.S. Devitt and G.J. Fee, Isogroups of differential equations using algebraic computing, *J. Sym. Comp.* **14**, 103–120, (1992).
31. E.L. Mansfield, DIFFGROB2: A symbolic algebra package for analyzing systems of PDE using Maple, In *User's Manual for Release 2*, (Preprint), Department of Mathematics, University of Exeter, Exeter, UK, (1993).
32. M. Hickman, The use of Maple in the search for symmetries, In *Proc. 14th IMACS World Congress on Computational and Applied Mathematics*, Vol. 1, Atlanta, GA, (Edited by W.F. Ames), pp. 226–229, IMACS, New Brunswick, NJ, (1994).
33. G.J. Reid and D.K. McKinnon, Solving systems of linear PDEs in their coefficient field by recursively decoupling and solving ODEs, Department of Mathematics, University of British Columbia, Vancouver, Canada, (Preprint), (1992).
34. G.J. Reid, A.D. Wittkopf and A. Boulton, Reduction of systems of nonlinear partial differential equations to simplified involutive forms, *Euro. J. Appl. Math.* **6**, (1995) (to appear).
35. B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory, In *Multidimensional Systems Theory*, (Edited by N.K. Bose), pp. 184–232, Reidel, Dordrecht, The Netherlands, (1985).
36. B. Buchberger, *An Implementation of Gröbner Bases in Mathematica*, MathSource 0205-300, Wolfram Research Inc., Champaign, IL, (1992).
37. E.L. Mansfield, Differential Gröbner Bases, Ph.D. Thesis, Department of Computer Science, University of Sydney, Sydney, Australia, (1991).
38. P.A. Clarkson and E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Physica D* **70**, 250–288, (1993).
39. P.A. Clarkson and E.L. Mansfield, On a shallow water wave equation, *Nonlinearity* **7**, 975–1000, (1994).
40. P.A. Clarkson and E.L. Mansfield, Algorithms for the nonclassical method of symmetry reductions, *SIAM J. Appl. Math.* **54**, 1693–1719, (1994).
41. P.A. Clarkson and E.L. Mansfield, Symmetry reductions and exact solutions of a shallow water wave equation, *Acta Appl. Math.* **39**, 245–276, (1995).
42. E.L. Mansfield and P.A. Clarkson, Applications of the differential algebra package DIFFGROB2 to reductions of PDE, In *Proc. 14th IMACS World Congress on Computational and Applied Mathematics*, Vol. 1, Atlanta, GA, (Edited by W.F. Ames), pp. 336–339, IMACS, New Brunswick, NJ, (1994).
43. E.L. Mansfield and P.A. Clarkson, Applications of the differential algebra package DIFFGROB2 to classical symmetries of differential equations, (Preprint) M94/18, Department of Mathematics, University of Exeter, Exeter, UK, (1994).
44. G.J. Reid, D.T. Weih and A.D. Wittkopf, A point symmetry group of a differential equation which cannot be found using infinitesimal methods, In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Proc. Int. Workshop Acireale*, Catania, Italy, 1992; (Edited by N.H. Ibragimov *et al.*), pp. 311–316, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1993).
45. T. Wolf and A. Brand, Investigating DEs with CRACK and related programs, In *Special AMS Bulletin for ISSAC '95*, (Edited by R. Grossmann), pp. 1–8, School of Mathematical Sciences, Queen Mary and Westfield College, University of London, London, (1995).
46. F. Schwarz, Symmetries of differential equations from Sophus Lie to computer algebra, *SIAM Review* **30**, 450–481, (1988).

47. F. Schwarz, A factorization algorithm for linear ordinary differential equations, In *Proc. ACM-SIGSAM 1989 Int. Symp. on Symbolic and Algebraic Computation, ISSAC '89*, Portland, OR, (Edited by G. Gonnet), pp. 17–25, ACM Press, New York, (1989).
48. F. Schwarz, Computer Algebra Software for Scientific Applications, In *Proc. 1993 CISM Advanced School on Computerized Symbolic Manipulation in Mechanics*, Udine, Italy, (Edited by E. Kreuzer), Springer-Verlag, Berlin, (1994) (to appear).
49. A.K. Head, LIE: A PC program for Lie analysis of differential equations, *Comp. Phys. Comm.* **77**, 241–248, (1993).
50. P.H.M. Kersten, Software to compute infinitesimal symmetries of exterior differential systems, with applications, *Acta Appl. Math.* **16**, 207–229, (1989).
51. S. Steinberg, Symmetries of differential equations, *MACSYMA Newsletter* **7**, 3–7, (1990).
52. W.M. Seiler, On the arbitrariness of the general solution of an involutive partial differential equation, *J. Math. Phys.* **35**, 486–498, (1994).
53. G.W. Bluman and J.D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.* **18**, 1025–1042, (1969).
54. P.A. Clarkson, Nonclassical symmetry reductions of nonlinear partial differential equations, *Mathl. Comput. Modelling* **18** (10), 45–68, (1993).
55. P.A. Clarkson, Nonclassical symmetry reductions of the Boussinesq equation, *Chaos, Solitons and Fractals* **5**, 2261–2301, (1995) (to appear).
56. F. Schwarz, *SPDE: Symmetries of Partial Differential Equations*, Guidelines for the use of SPDE 1.0, Institut SCAI, GMD, Postfach 1316, D-53731 Sankt Augustin, Germany, (1994).
57. A.V. Bocharov, DEliA: A system of exact analysis of differential equations using S. Lie approach, Report by Joint Venture OWIMEX Program Systems Institute of the U.S.S.R., Academy of Sciences, Pereslavl-Zalessky, U.S.S.R., (1989).
58. A.V. Bocharov, *DEliA: A System of Exact Analysis of Differential Equations Using S. Lie Approach*, DELiA 1.5.1 User Guide, Beaver Soft Programming Team, Brooklyn, NY, (1991).
59. V.P. Eliseev, R.N. Fedorova and V.V. Kornyak, A REDUCE program for determining point and contact Lie symmetries of differential equations, *Comp. Phys. Comm.* **36**, 383–389, (1985).
60. G. Baumann, *Lie Symmetries of Differential Equations: A Mathematica Program to Determine Lie Symmetries*, MathSource 0202-622, Wolfram Research Inc., Champaign, IL, (1992).
61. G. Baumann, *Generalized Symmetries: A Mathematica Program to Determine Lie-Bäcklund Symmetries*, MathSource 0204-680, Wolfram Research Inc., Champaign, IL, (1993).
62. P.K.H. Gragert, Symbolic computations in prolongation theory, Ph.D. Thesis, Department of Mathematics, Twente University of Technology, Enschede, The Netherlands, (1981).
63. P.K.H. Gragert and P.H.M. Kersten, Implementation of differential geometry objects and functions with an application to extended Maxwell equations, In *Proc. EUROCAM '82*, Marseille, France, 1982; Lecture Notes in Computer Science, (Edited by J. Calmet), pp. 181–187, Springer-Verlag, New York, (1982).
64. S. Herod, Computer assisted determination of Lie point symmetries with application to fluid dynamics, Ph.D. Thesis, Program in Applied Mathematics, The University of Colorado, Boulder, CO, (1994).
65. M.C. Nucci, *Interactive REDUCE Programs for Calculating, Classical, Nonclassical, and Lie-Bäcklund Symmetries of Differential Equations*, Manual of the Program, (Preprint), GT Math:062090-051, School of Mathematics, Georgia Institute of Technology, Atlanta, GA, (1990).
66. M.C. Nucci, Interactive REDUCE programs for calculating Lie-point, nonclassical, Lie-Bäcklund, and approximate symmetries of differential equations: Manual and floppy disk, In *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 3: New Trends in Theoretical Developments and Computational Methods, Chapter 14, CRC Press, Boca Raton, FL, (1996).
67. P. Vafeades, PDELIE: A partial differential equation solver, *MACSYMA Newsletter* **9** (1), 1–13, (1992).
68. P. Vafeades, PDELIE: A partial differential equation solver II, *MACSYMA Newsletter* **9** (2–4), 5–20, (1992).
69. D. Bérubé and M. de Montigny, *A Mathematica Program for the Calculation of Lie Point Symmetries of Systems of Differential Equations*, (Preprint), CRM-1822, Centre de Recherches Mathématiques, Montréal, Canada, (1996).
70. P.K.H. Gragert, P.H.M. Kersten and A. Martini, Symbolic computations in applied differential geometry, *Acta Appl. Math.* **1**, 43–77, (1983).
71. P.K.H. Gragert, Lie algebra computations, *Acta Math. Appl.* **16**, 231–242, (1989).
72. P.K.H. Gragert, Prolongation algebras of nonlinear PDEs, In *Proc. ERCIM Advanced Course on Partial Differential Equations and Group Theory, Part II*, (Edited by J.F. Pommaret), pp. 1–20, Gesellschaft für Mathematik und Datenverarbeitung, Sankt Augustin, Germany, (1992).
73. P.H.M. Kersten, Infinitesimal symmetries: A computational approach, Ph.D. Thesis, Department of Mathematics, Twente University of Technology, Enschede, The Netherlands, (1985); CWI Tract 34, Center for Mathematics and Computer Science, Amsterdam, (1987).
74. P.K.H. Gragert and P.H.M. Kersten, Graded differential geometry in REDUCE: Supersymmetric structures of the modified KdV equation, *Acta Appl. Math.* **24**, 211–231, (1991).
75. J. Sherring and G. Prince, Geometric aspects of reduction of order, *Trans. Am. Math. Soc.* **314**, 433–453, (1992).
76. R.N. Fedorova and V.V. Kornyak, Applying computer algebra systems for determination of the Lie and Lie-Bäcklund symmetries of differential equations, In *Proc. Int. Conf. on Computer Algebra and Its Applications*

in Theoretical Physics, (in Russian), pp. 248–261, Joint Institute for Nuclear Research, Dubna, U.S.S.R., (1985).
77. R.N. Fedorova and V.V. Kornyak, A REDUCE program for computing determining equations of Lie-Bäcklund symmetries of differential equations, Report R11-87-19, (in Russian), Joint Institute for Nuclear Research, Dubna, U.S.S.R., (1987).
78. T. Wolf, Programs for Applying Symmetries of PDEs, In *Proc. ACM-SIGSAM 1995 Int. Symp. on Symbolic and Algebraic Computation*, ISSAC '95, Montreal, Canada, pp. 7–15, ACM Press, New York, (1995).
79. T. Wolf, Zur analytischen Untersuchung und exakten Lösung von Differentialgleichungen mit Computeralgebrasystemen, Dissertation, Friedrich Schiller Universität, Jena, Germany, (1989).
80. T. Wolf, A package for the analytic investigation and exact solutions of differential equations, In *Proc. EUROCAL '87*, Leipzig, GDR, 1987; Lecture Notes in Computer Science **378**, (Edited by J.H. Davenport), pp. 479–491, Springer-Verlag, Berlin, (1989).
81. V.P. Gerdt, Computer algebra tools for higher symmetry analysis of nonlinear evolution equations, In *Programming Environments for High-Level Problem Solving*, (Edited by P.W. Gaffney and E.N. Houstis), pp. 107–115, Elsevier Science Publishers, Amsterdam, (1993).
82. V.P. Gerdt, Computer algebra, symmetry analysis and integrability of nonlinear evolution equations, *Int. J. Mod. Phys. C* **4**, 279–286, (1993).
83. W. Sarlet and J. Vanden Bonne, REDUCE-procedures for the study of adjoint symmetries of second-order differential equations, *J. Sym. Comp.* **13**, 683–693, (1992).
84. F. Oliveri, RELIE: A REDUCE program for calculating lie symmetries, (Preprint), Department of Mathematics, University of Messina, Messina, Italy, (1994).
85. P. Vafeades, Computer algebraic determination of symmetries and conservation laws of PDEs, In *Proc. ISMM Int. Symposium on Computer Applications in Design, Simulation and Analysis*, New Orleans, LA, (Edited by E.K. Park), pp. 310–312, Acta Press, Anaheim, CA, (1990).
86. P. Vafeades, SYMCON: A MACSYMA package for the determination of symmetries and conservation laws of PDEs, (Preprint), Department of Engineering Sciences, Trinity University, San Antonio, TX, (1990).
87. P. Vafeades, PDELIE: A partial differential equation solver III, *MACSYMA Newsletter*, (1995) (to appear).
88. P. Vafeades, *PDELIE: Symbolic Software for the Analysis of Partial Differential Equations by Lie Group Methods*, User's Manual, (Preprint), San Antonio, TX, (1994).
89. S. Steinberg, Symmetry operators, In *Proc. 1979 MACSYMA User's Conference*, Washington, DC, (Edited by V.E. Lewis), pp. 408–444, MIT Press, Boston, MA, (1979).
90. S. Steinberg, Lie series, Lie transformations, and their applications, In *Lie Methods in Optics, Proc. CIFMO-CIO Workshop*, Leon, Mexico, 1985; Lecture Notes in Physics **250**, (Edited by J. Sánchez Mondragón and K.B. Wolf), pp. 45–103, Springer-Verlag, New York, (1986).
91. K. Vu, *Source code for Desolv Release 1.0*, Department of Mathematics, Monash University, Clayton, Victoria, Australia, (1995).
92. P.A. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.* **30**, 2201–2213, (1989).
93. F. Schwarz, Programming with abstract data types: The symmetry package SPDE in SCRATCHPAD, In *Trends in Computer Algebra, Proc. Int. Symposium*, Bad Neuenahr, Germany, 1987; Lecture Notes in Computer Science **296**, (Edited by R. Janßen), pp. 167–176, Springer-Verlag, New York, (1988).
94. J. Schü, W.M. Seiler and J. Calmet, Algorithmic methods for Lie pseudogroups, In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Proc. Int. Workshop Acireale*, Catania, Italy, 1992; (Edited by N.H. Ibragimov *et al.*), pp. 337–344, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1993).
95. W.M. Seiler, Analysis and application of the formal theory of partial differential equations, Ph.D. Thesis, Department of Physics, University of Lancaster, Lancaster, UK, (1994).
96. A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations, In *What is Integrability?*, Springer Lecture Notes in Nonlinear Dynamics, (Edited by V.I. Zakharov), pp. 115–184, Springer-Verlag, New York, (1990).
97. W.I. Fushchich and V.V. Kornyak, Computer algebra application for determining Lie and Lie-Bäcklund symmetries of differential equations, *J. Symb. Comp.* **7**, 611–619, (1989).
98. V.V. Kornyak and W.I. Fushchich, A program for symmetry analysis of differential equations, In *Proc. ISSAC '91*, Bonn, Germany, (Edited by S.M. Watt), pp. 315–316, ACM Press, New York, (1991).
99. R.N. Fedorova and V.V. Kornyak, Determination of Lie-Bäcklund symmetries of differential equations using FORMAC, *Comp. Phys. Comm.* **39**, 93–103, (1986).
100. H.J. Pohle and T. Wolf, Automatic determination of dynamical symmetries of ordinary differential equations, *Computing* **41**, 297–307, (1989).
101. V.P. Gerdt and A.Yu. Zharkov, Computer generation of necessary integrability conditions for polynomial-nonlinear evolution systems, In *Proc. ISSAC '90*, Tokyo, Japan, (Edited by S. Watanabe and M. Nagata), pp. 250–254, ACM Press, New York, (1990).
102. V.P. Gerdt, A.B. Shvachka and A.Yu. Zharkov, FORMINT—A program for classification of integrable nonlinear evolution equations, *Comp. Phys. Comm.* **34**, 303–311, (1985).
103. V.P. Gerdt, A.B. Shvachka and A.Yu. Zharkov, Computer algebra application for classification of integrable non-linear evolution equations, *J. Symb. Comp.* **1**, 101–107, (1985).

104. M.D. Popov, Automized computing of defining equations for a Lie group, *Izvestiya AN Belor. SSR, Ser. Phys.-Math.*, (in Russian) **2**, 33–37, (1985).
105. A.V. Bocharov and M.L. Bronstein, Efficiently implementing two methods of the geometrical theory of differential equations: An experience in algorithm and software design, *Acta Appl. Math.* **16**, 143–166, (1989).
106. V.L. Topunov, Reducing systems of linear differential equations to a passive form, *Acta Appl. Math.* **16**, 191–206, (1989).
107. M.J. Ablowitz and P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, In *London Mathematical Society Lecture Notes on Mathematics*, Vol. 149, Cambridge University Press, Cambridge, UK, (1991).
108. P.A. Clarkson, Dimensional reductions and exact solutions of a generalized nonlinear Schrödinger equation, *Nonlinearity* **5**, 453–472, (1992).