Collection of PDE animations

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Contents

These are collection of PDE problems solved analytically and animated. Most of the animations where done in Mathematica, some in Maple and Matlab.

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Laplace and Poisson PDE

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CHAPTER 4

Burger's PDE

Solve

$$u_t + u\ u_x = Du_{xx} \tag{1}$$

BC

$$\begin{aligned} u(0,t) &= 0 \qquad t > 0 \\ u(L,t) &= 0 \qquad t > 0 \end{aligned}$$

Initial conditions

$$u(x,0) = f(x) \qquad 0 < x < L$$

Where D is the diffusion constant.

Solution

Using Cole-Hopf, let

$$u(x,t) = -2D\frac{\phi_x}{\phi} \tag{2}$$

where $\phi \equiv \phi(x,t)$. Rewriting equation (1) as

$$\begin{aligned} u_t &= Du_{xx} - u\ u_x \\ &= \left(Du_x - \frac{u^2}{2}\right)_x \end{aligned} \tag{3}$$

Substituting (2) into (3) gives

$$\begin{aligned}
\left(-2D\frac{\phi_x}{\phi}\right)_t &= \left[D\left(-2D\frac{\phi_x}{\phi}\right)_x - \frac{1}{2}\left(-2D\frac{\phi_x}{\phi}\right)^2\right]_x \\
-2D\left(\frac{\phi_x}{\phi}\right)_t &= \left(-2D^2\left(\frac{\phi_x}{\phi}\right)_x - 2D^2\left(\frac{\phi_x}{\phi}\right)^2\right)_x \\
-2D\left(\frac{\phi_x}{\phi}\right)_t &= -2D^2\left(\left(\frac{\phi_x}{\phi}\right)_x + \left(\frac{\phi_x}{\phi}\right)^2\right)_x
\end{aligned} \tag{4}$$

But

$$\left(\frac{\phi_x}{\phi}\right)_t = -\frac{1}{\phi^2}\phi_t\phi_x + \frac{1}{\phi}\phi_{xt}$$

And

$$\left(\frac{\phi_t}{\phi}\right)_x = -\frac{1}{\phi^2}\phi_x\phi_t + \frac{1}{\phi}\phi_{tx}$$

Therefore $\left(\frac{\phi_x}{\phi}\right)_t = \left(\frac{\phi_t}{\phi}\right)_x$. Using this in LHS of (4) gives

$$-2D\left(\frac{\phi_t}{\phi}\right)_x = -2D^2\left(\left(\frac{\phi_x}{\phi}\right)_x + \left(\frac{\phi_x}{\phi}\right)^2\right)_x \tag{5}$$

And

$$\begin{aligned}
\left(\frac{\phi_x}{\phi}\right)_x + \left(\frac{\phi_x}{\phi}\right)^2 &= -\frac{1}{\phi^2}\phi_x^2 + \frac{\phi_{xx}}{\phi} + \frac{\phi_x^2}{\phi^2} \\
&= \frac{\phi_{xx}}{\phi}
\end{aligned}$$

Using the above in the RHS of (5) gives

$$-2D\left(\frac{\phi_t}{\phi}\right)_x = -2D^2\left(\frac{\phi_{xx}}{\phi}\right)_x \tag{6}$$

Integrating both side w.r.t. x gives

$$-2D\frac{\phi_t}{\phi} = -2D^2\frac{\phi_{xx}}{\phi} + G(t)$$

Where $G(t)$ is the constant of integration since $\phi \equiv \phi(x,t)$. The above simplifies to the heat PDE in $\phi(x,t)$

$$\phi_t = D\phi_{xx} + G(t)\,\phi \tag{6A}$$

Let

$$\psi = \phi e^{-\int G(t)dt} \tag{6B}$$

Then

$$\begin{aligned}\psi_t &= \phi_t e^{-\int G(t)dt} - \phi G(t)\, e^{-\int G(t)dt}\\ &= (\phi_t - \phi G(t))\, e^{-\int G(t)dt}\end{aligned}$$

But from (6A), we see that $\phi_t - \phi G(t) = D\phi_{xx}$. Therefore the above becomes

$$\psi_t = D\phi_{xx} e^{-\int G(t)dt}$$

But from (6B), we see that $\phi_{xx} e^{-\int G(t)dt} = \psi_{xx}$, therefore the above becomes

$$\psi_t = D\psi_{xx}$$

Which is the heat PDE. The original BC and initial conditions are now transformed to ψ to solve the above. Since $u = -2D\frac{\phi_x}{\phi}$, then solving this first for ϕ

$$\begin{aligned}\frac{\phi_x}{\phi} &= -\frac{1}{2D}u\\ \frac{\partial\phi}{\partial x}\frac{1}{\phi} &= -\frac{1}{2D}u\\ \frac{d\phi}{\phi} &= -\frac{1}{2D}udx\end{aligned}$$

Integrating gives

$$\begin{aligned}\ln\phi &= -\frac{1}{2D}\int_0^x u\ ds + C_0\\ \phi(x,t) &= Ce^{-\frac{1}{2D}\int_0^x u\ ds}\end{aligned}$$

Since $u = -2D\frac{\phi_x}{\phi}$, then the constant C cancels out. Then we it can be set to any value as it does not affect the solution. Let $C = 1$ and the above becomes

$$\phi(x,t) = e^{-\frac{1}{2D}\int_0^x u\ ds} \tag{7}$$

(7) is now used to transform the initial conditions. When $u(x,0) = f(x)$ the above becomes

$$\begin{aligned}\phi(x,0) &= e^{-\frac{1}{2D}\int_0^x u(s,0)\ ds} \\ &= e^{-\frac{1}{2D}\int_0^x f(s)\ ds}\end{aligned}$$

Since from (6B), $\psi(x,t) = \phi e^{-\int G(t)dt} = \phi e^{-\int_0^t G(s)ds}$ therefore

$$\begin{aligned}\psi(x,0) &= \phi(x,0)\, e^0 \\ &= \phi(x,0) \\ &= e^{-\frac{1}{2D}\int_0^x f(s)\ ds}\end{aligned}$$

To transform the boundary conditions, $u = -2D\frac{\phi_x}{\phi}$ is used. When $u(0,t) = 0$ then $0 = -2D\frac{\phi_x(0,t)}{\phi(0,t)}$ or

$$\phi_x(0,t) = 0$$

But $\psi = \phi e^{-\int_0^t G(s)ds}$, then $\psi_x = \phi_x e^{-\int_0^t G(s)ds}$ and therefore

$$\begin{aligned}\psi_x(0,t) &= \phi_x(0,t)\, e^{-\int_0^t G(s)ds} \\ &= 0\end{aligned}$$

Similarly, when $u(L,t) = 0$ then $0 = -2D\frac{\phi_x(L,t)}{\phi(L,t)}$ or

$$\phi_x(L,t) = 0$$

Which gives

$$\psi_x(L,t) = 0$$

Hence the heat PDE to solve is

$$\psi_t = D\psi_{xx}$$

BC

$$\begin{aligned} \psi_x(0,t) &= 0 \qquad t > 0 \\ \psi_x(L,t) &= 0 \qquad t > 0 \end{aligned}$$

Initial conditions

$$\psi(x,0) = e^{-\frac{1}{2D}\int_0^x f(s)\ ds} \qquad 0 < x < L$$

The above heat PDE is now solved for $\psi(x,t)$. This solution is transformed back to $u(x,t)$. First using $\psi = \phi e^{-\int G(t)dt}$ to find $\phi(x,t)$, then using $u(x,t) = -2D\frac{\phi_x}{\phi}$, to find $u(x,t)$.

So in summary, there are two transformations needed. Going from $u(x,t) \rightarrow \phi(x,t)$ uses Cole-Hopf. Going from $\phi(x,t) \rightarrow \psi(x,t)$ uses $\psi(x,t) = \phi(x,t)\, e^{-\int G(t)dt}$. It is $\psi(x,t)$ which is solved as the heat PDE $\psi_t = D\psi_{xx}$ and not $\phi(x,t)$, which is just an intermediate transformation.

4.1 Example 1

Let $L = 2\pi, 0 < x < 2\pi, D = \frac{1}{10}$,

$$u(x,0) = f(x) = \sin x$$

And boundary conditions $u(0,t) = u(2\pi,0) = 0$. From above, after carrying the forward transformation, the PDE to solve is found to be

$$\psi_t = D\psi_{xx}$$

With transformed boundary conditions

$$\begin{aligned} \psi_x(0,t) &= 0 \qquad t > 0 \\ \psi_x(2\pi,t) &= 0 \qquad t > 0 \end{aligned}$$

And transformed initial conditions

$$\psi(x,0) = e^{-\frac{1}{2D}\int \sin(x)\ dx} = e^{\frac{1}{2D}\cos(x)}$$

This heat PDE is standard and has known solution by separation of variables which is

$$\psi(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-D\lambda_n t}\cos\left(\sqrt{\lambda_n}x\right)$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$

Or, since $L = 2\pi$,

$$\psi(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-D\frac{n^2}{4}t}\cos\left(\frac{n}{2}x\right)$$
$$\lambda_n = \frac{n^2}{4} \qquad n = 1,2,3,\cdots$$

Where

$$\begin{aligned} c_0 &= \frac{1}{L}\int_0^L \psi(x,0)\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\cos(x)}{2D}}\,dx \\ &= \text{BesselI}\left(0,\frac{1}{2D}\right) \end{aligned}$$

And

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L \psi(x,0)\cos\left(\sqrt{\lambda_n}x\right)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} e^{\frac{\cos(x)}{2D}}\cos\left(\frac{n}{2}x\right)dx \end{aligned}$$

The after integral has no closed form solution. Hence the solution is

$$\psi(x,t) = \text{BesselI}\left(0,\frac{1}{2D}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{2\pi} e^{\frac{\cos(x)}{2D}}\cos\left(\frac{n}{2}x\right)dx\right)e^{-D\frac{n^2}{4}t}\cos\left(\frac{n}{2}x\right)$$

But $\psi = \phi e^{-\int G(t)dt}$, therefore

$$\phi(x,t) = \psi(x,t)\, e^{\int G(t)dt}$$

But I do not know what $G(t)$ is. This was used during the forward transformation only and was eliminated. So how to find $\phi(x,t)$? Need to find $\phi(x,t)$ to be able to find $u(x,t)$ from $u(x,t) = -2D\frac{\phi_x}{\phi}$.

CHAPTER 5

MISC. PDE'S

5.1 FitzHugh-Nagumo in 2D

This was solved numerically in Matlab.

5.1.1 Example 1

The equations to solve are the following on the unit square in 2D.

$$\begin{aligned}\frac{\partial v(x,y,t)}{\partial t} &= D\nabla^2 v + (a-v)(v-1)\,v - w + I\\ \frac{\partial w(x,y,t)}{\partial t} &= \epsilon(v-\gamma w)\end{aligned}$$

Using $a = 0.1, \gamma = 2, \epsilon = 0.005, D = 5\times 10^{-5}, I = 0$, hence the PDE's are

$$\begin{aligned}\frac{\partial v(x,y,t)}{\partial t} &= \left(5\times 10^{-5}\right)\nabla^2 v + (0.1-v)(v-1)\,v - w\\ \frac{\partial w(x,y,t)}{\partial t} &= 0.005(v-2w)\end{aligned}$$

Initial conditions, $t = 0$

$$\begin{aligned}v(x,y,0) &= \exp\left(-100\left(x^2+y^2\right)\right)\\ w(x,y,0) &= 0\end{aligned}$$

Boundary conditions are homogeneous Neumann for v. (I solved this numerically, fractional step method. ADI for the diffusion solve).

5.1.2 Example 2

The equations to solve are the following on the unit square in 2D.

$$\begin{aligned}\frac{\partial v(x,y,t)}{\partial t} &= D\nabla^2 v + (a-v)(v-1)v - w + I\\ \frac{\partial w(x,y,t)}{\partial t} &= \epsilon(v-\gamma w)\end{aligned}$$

Using $a=0.1, \gamma=2, \epsilon=0.005, D=5\times10^{-5}, I=0$, hence the PDE's are

$$\begin{aligned}\frac{\partial v(x,y,t)}{\partial t} &= \left(5\times10^{-5}\right)\nabla^2 v + (0.1-v)(v-1)v - w\\ \frac{\partial w(x,y,t)}{\partial t} &= 0.005(v-2w)\end{aligned}$$

Initial conditions, $t=0$

$$\begin{aligned}v(x,y,0) &= 1-2x\\ w(x,y,0) &= 0.05y\end{aligned}$$

Boundary conditions are homogeneous Neumann for v. (I solved this numerically, fractional step method. ADI for the diffusion solve). See my Matlab web page for source code.

CHAPTER 6

Appendix

6.1 Summary table

Heat PDE $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ in $1D$ (in a rod)

Left side	Right side	$u(x,0)$	$\lambda = 0$	$\lambda > 0$
$u(0) = 0$	$u(L) = 0$	triangle	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$u(L) = 0$	100	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$u(0) = T_0$	$u(L) = 0$	x	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = T_0 - \frac{T_0}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$\frac{\partial u(L)}{\partial x} = 0$	x	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = T_0$	0	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1, 3, 5, \cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = T_0 + \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = 0$	$f(x)$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1, 3, 5, \cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1,3,5\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$\frac{\partial u(L)}{\partial x} = 0$	$f(x)$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1, 3, 5, \cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1,3,5\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$u(L) + \frac{\partial u(L)}{\partial x} = 0$	$f(x)$	No	$\tan\left(\sqrt{\lambda_n}L\right) + \sqrt{\lambda_n} = 0$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}$
$u(0) + \frac{\partial u(0)}{\partial x} = 0$	$u(0) = 0$	0	$\lambda_0 = 0$ $X_0 = A_0$	$\tan\left(\sqrt{\lambda_n}L\right) - \sqrt{\lambda_n} = 0$
$u(-1) = 0$	$u(1) = 0$	$f(x)$	No	$\sqrt{\lambda_n} = \frac{n\pi}{2} \qquad n = 1, 2, 3, \cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right), \sin\left(\sqrt{\lambda_n}x\right)$ $u(x,t) = \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-\lambda_n t} + \sum_{n=2,4,\cdots}^{\infty} B$

Heat PDE $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \beta u$ in $1D$ (in a rod) with $\alpha, \beta > 0$ for $0 < x < \pi$

Left side	Right side	initial condition	$\lambda = 0$	$\lambda > 0$	analytical solution $u(x,$
$\frac{\partial u(0,t)}{\partial x} = 0$	$\frac{\partial u(\pi,t)}{\partial x} = 0$	$u(x,0) = x$	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = n^2, n = 1,2,3,\cdots$ $X(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$	$\frac{\pi}{2} + c_0(e^{-\beta t} - 1) + \frac{2}{\pi} \sum$

(TO DO) Heat PDE for periodic conditions $u(-L) = u(L)$ and $\frac{\partial u(-L)}{\partial x} = \frac{\partial u(L)}{\partial x}$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$$

$$u(x,t) = \overbrace{a_0} + \overbrace{\sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t} + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}}$$

6.2 Using Mathematica to obtain the eigenvalues and eigenfunctions for heat PDE in 1D

6.2.1 $u(0) = 0, u(L) = 0$

For eigenvalue

```
ClearAll[y,x,L];
op={-y''[x]   ,DirichletCondition[y[x]==0,x==0],
DirichletCondition[y[x]==0,x==L]};
(*or simply*)
op={-y''[x],DirichletCondition[y[x]==0,True]};
eig=DEigenvalues[op,y[x],{x,0,L},5]
```

$$\left\{\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \frac{16\pi^2}{L^2}, \frac{25\pi^2}{L^2}\right\}$$

```
FindSequenceFunction[eig,n]
```

$$\frac{\pi^2 n^2}{L^2}$$

For eigenfunctions

```
ClearAll[y,x,L];
op={-y''[x]   ,DirichletCondition[y[x]==0,x==0],
```

```
DirichletCondition[y[x]==0,x==L]};
eigf=Last@DEigensystem[op,y[x],{x,0,L},5]
```

$$\left\{\sin\left(\frac{\pi x}{L}\right),\sin\left(\frac{2\pi x}{L}\right),\sin\left(\frac{3\pi x}{L}\right),\sin\left(\frac{4\pi x}{L}\right),\sin\left(\frac{5\pi x}{L}\right)\right\}$$

```
FullSimplify[FindSequenceFunction[eigf,n]];
```

$$\sin\left(\frac{\pi n x}{L}\right)$$

6.2.2 $u'(0)=0,u'(L)=0$

For eigenvalue

```
ClearAll[y,x,L];
op={-y''[x]+NeumannValue[0,True]};
eig=DEigenvalues[op,y[x],{x,0,L},5]
```

$$\left\{0,\frac{\pi^2}{L^2},\frac{4\pi^2}{L^2},\frac{9\pi^2}{L^2},\frac{16\pi^2}{L^2}\right\}$$

```
FindSequenceFunction[eig,n]
```

$$\frac{\pi^2(n-1)^2}{L^2}$$

```
ClearAll[y,x,L];
op={-y''[x]+NeumannValue[0,True]};
eigf=Last@DEigensystem[op,y[x],{x,0,L},5]
```

$$\left\{1,\cos\left(\frac{\pi x}{L}\right),\cos\left(\frac{2\pi x}{L}\right),\cos\left(\frac{3\pi x}{L}\right),\cos\left(\frac{4\pi x}{L}\right)\right\}$$

```
FullSimplify[FindSequenceFunction[eigf, n]];
```

$$\cos\left(\frac{\pi(n-1)x}{L}\right)$$

6.2.3 $u'(0) = 0, u(L) = 0$

For eigenvalue

```
ClearAll[y,x,L];
op={-y''[x]+NeumannValue[0,x==0],DirichletCondition[y[x]==0,x==L]};
eig=DEigenvalues[op,y[x],{x,0,L},6]
\begin{MMAinline}

\[
\left\{\frac{\pi^2}{4 L^2},\frac{9 \pi^2}{4 L^2},\frac{25 \pi^2}{4 L^2}%
,\frac{49 \pi^2}{4 L^2},\frac{81 \pi^2}{4 L^2},\frac{121 \pi^2}{4 L^2}\right\}
\]

\begin{MMAinline}
Simplify[FindSequenceFunction[eig,n]]
```

$$\frac{\pi^2(1-2n)^2}{4L^2}$$

For eigenfunctions

```
eigf=Last@DEigensystem[op,y[x],{x,0,L},7]
```

$$\left\{\cos\left(\frac{\pi x}{2L}\right), \cos\left(\frac{3\pi x}{2L}\right), \cos\left(\frac{5\pi x}{2L}\right), \cos\left(\frac{7\pi x}{2L}\right), \cos\left(\frac{9\pi x}{2L}\right)\right\}$$

```
FullSimplify[FindSequenceFunction[eigf,n]];
```

$$\cos\left(\frac{\pi(1-2n)x}{2L}\right)$$

6.2.4 $u(0) = 0, u'(L) = 0$

Eigenvalue are the same as above

```
ClearAll[y,x,L];
op={-y''[x]+NeumannValue[0,x==L],DirichletCondition[y[x]==0,x==0]};
eig=DEigenvalues[op,y[x],{x,0,L},6]
```

$$\left\{\frac{\pi^2}{4L^2}, \frac{9\pi^2}{4L^2}, \frac{25\pi^2}{4L^2}, \frac{49\pi^2}{4L^2}, \frac{81\pi^2}{4L^2}, \frac{121\pi^2}{4L^2}\right\}$$

```
FindSequenceFunction[eig,n]
```

$$\frac{\pi^2(1-2n)^2}{4L^2}$$

For eigenfunctions

```
eigf=Last@DEigensystem[op,y[x],{x,0,L},6]
```

$$\left\{\sin\left(\frac{\pi x}{2L}\right), \sin\left(\frac{3\pi x}{2L}\right), \sin\left(\frac{5\pi x}{2L}\right), \sin\left(\frac{7\pi x}{2L}\right), \sin\left(\frac{9\pi x}{2L}\right), \sin\left(\frac{11\pi x}{2L}\right)\right\}$$

```
FullSimplify[FindSequenceFunction[eigf, n]];
```

$$-\sin\left(\frac{\pi(1-2n)x}{2L}\right)$$

6.2.5 $u(0) = 0, u(L) + u'(L) = 0$

For eigenvalue

```
(*can only find them numerially*)
ClearAll[y,x];
op={-y''[x]+NeumannValue[y[x],x==1],DirichletCondition[y[x]==0,x==0]};
eig=DEigenvalues[op,y[x],{x,0,1},6]//N
(* {4.11586,24.1393,63.6591,122.889,201.851,300.55}*)
NSolve[Tan[Sqrt[lam]]+Sqrt[lam]==0&& 0<lam<130,lam]
(*{{lam->4.11586},{lam->24.1393},{lam->63.6591},{lam->122.889}}*)
```

For eigenfunctions

```
ClearAll[y,x];
op={-y''[x]+NeumannValue[y[x],x==1],DirichletCondition[y[x]==0,x==0]};
eig=Last@DEigensystem[op,y[x],{x,0,1},6]//N
```

$$\{\sin(2.02876x),\sin(4.91318x),\sin(7.97867x),\sin(11.0855x),\sin(14.2074x),\sin(17.3364x)\}$$

6.2.6 $u(0)+u'(0)=0, u'(L)=0$

For eigenvalue

```
(*can only find them numerially. Notice the sign difference now. *)
ClearAll[y,x];
op={-y''[x]+NeumannValue[-y[x],x==0],DirichletCondition[y[x]==0,x==1]};
eig=DEigenvalues[op,y[x],{x,0,1},6]//N
(* {0.,20.1908,59.6814,118.914,197.924,296.774}*)
NSolve[Tan[Sqrt[lam]]-Sqrt[lam]==0&& 0<=lam<130,lam]
(*{{lam->0.},{lam->20.1907},{lam->59.6795},{lam->118.9},{lam->197.858}%
,{lam->296.554}}*)
```

For eigenfunctions, mathematica only gives plots, so not shown.

6.2.7 $u(-L)=0, u(L)=0$

For eigenvalue

```
ClearAll[y,x,L];
op={-y''[x],DirichletCondition[y[x]==0,x==-L],DirichletCondition[y[x]==0,x==L]}%
;
eig=DEigenvalues[op,y[x],{x,-L,L},6]
```

$$\left\{\frac{\pi^2}{4L^2},\frac{\pi^2}{L^2},\frac{9\pi^2}{4L^2},\frac{4\pi^2}{L^2},\frac{25\pi^2}{4L^2},\frac{9\pi^2}{L^2}\right\}$$

```
Simplify[FindSequenceFunction[eig,n]]
```

$$\frac{\pi^2 n^2}{4L^2}$$

For eigenfunctions

```
eigf= Last@DEigensystem[op,y[x],{x,-L,L},6]
```

$$\left\{\sin\left(\frac{\pi(L+x)}{2L}\right),\sin\left(\frac{\pi(L+x)}{L}\right),\sin\left(\frac{3\pi(L+x)}{2L}\right),\sin\left(\frac{2\pi(L+x)}{L}\right),\sin\left(\frac{5\pi(L+x)}{2L}\right)\right\}$$

```
FullSimplify[FindSequenceFunction[eigf,n]];
Assuming[Element[n,Integers]&&Element[L,Reals],FullSimplify[
```

$$\frac{1}{2}i\left(\left(-ie^{-\frac{i\pi x}{2L}}\right)^n-\left(ie^{\frac{i\pi x}{2L}}\right)^n\right)$$