Note on finding the Laplacian in Polar, Cylinderical and Spherical coordinates using Tensor calculus

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April 10, 2019 Compiled on January 30, 2024 at 5:57pm

Contents

1 Introduction

I wrote this note to help me learn tensors. The goal is to derive the Laplacian ∇^2 using tensor calculus for 2D Polar, 3D Cylindrical and in 3D Spherical coordinates.

The Laplacian in Cartesian coordinates is given by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The following diagram shows $\nabla^2 u$ in Polar, Cylindrical and Spherical coordinates

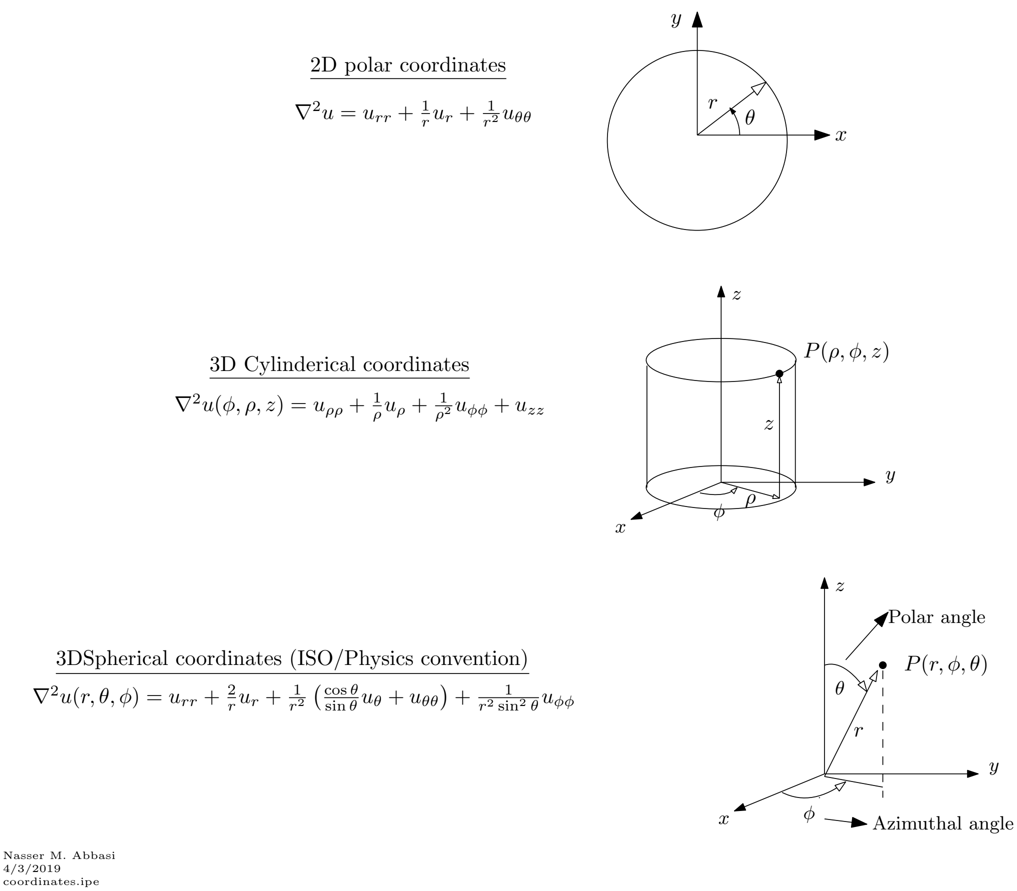

Figure 1: Laplacian in different coordinate systems

The Laplacian operator in any orthogonal coordinate system is given by

$$\nabla^2 = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \tag{1}$$

Where g is the metric tensor and $|g|$ is the determinant of g. The derivation of the above is not shown here. References contains the derivation of the above. Only the use of (1) is shown here.

2 2D Polar

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y$ and the coordinates in the other system (Polar) are $x^1 = r, x^2 = \theta$. The relation between these must be known and invertible also, meaning $\zeta \equiv \zeta(x)$ and $x \equiv x(\zeta)$. This relation can be found from geometry as

$$\begin{aligned}\zeta^1 &= r\cos\theta \\ \zeta^2 &= r\sin\theta\end{aligned}$$

The first step is to determine the metric tensor g for the Polar coordinates. This is given by

$$g_{kl} = \delta_{ij}\frac{\partial \zeta^i}{\partial x^k}\frac{\partial \zeta^j}{\partial x^l}$$

The above using Einstein summation notation. Since the coordinate system is orthogonal, g_{kl} will be diagonal, hence only g_{11}, g_{22} are non zero. This is not the case for all coordinates systems. For general curvilinear coordinates system, g can contain all components. But for the coordinates systems used here, g will always be diagonal.

$$\begin{aligned}g_{11} &= \frac{\partial \zeta^1}{\partial x^1}\frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1}\frac{\partial \zeta^2}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial r}\frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r}\frac{\partial \zeta^2}{\partial r} \\ &= \left(\frac{\partial \zeta^1}{\partial r}\right)^2 + \left(\frac{\partial \zeta^2}{\partial r}\right)^2 \\ &= \cos^2\theta + \sin^2\theta \\ &= 1\end{aligned}$$

And

$$\begin{aligned}g_{22} &= \frac{\partial \zeta^1}{\partial x^2}\frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2}\frac{\partial \zeta^2}{\partial x^2} \\ &= \frac{\partial \zeta^1}{\partial \theta}\frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta}\frac{\partial \zeta^2}{\partial \theta} \\ &= \left(\frac{\partial \zeta^1}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta}\right)^2 \\ &= (-r\sin\theta)^2 + (r\cos\theta)^2 \\ &= r^2\end{aligned}$$

Hence ds^2 in polar coordinates is

$$\begin{aligned} ds^2 &= g_{kl} dx^k dx^l \\ &= g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 \\ &= g_{11}(dr)^2 + g_{22}(d\theta)^2 \\ &= (dr)^2 + r^2(d\theta)^2 \end{aligned}$$

From the above we see that

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Hence $\det(g) = r^2$. We are now ready to apply (1)

$$\begin{aligned} \nabla^2 &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \\ &= \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_1} \left(\frac{\sqrt{r^2}}{g_{11}} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_2} \left(\frac{\sqrt{r^2}}{g_{22}} \frac{\partial}{\partial x^2} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{r}{r^2} \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r} \left(\frac{\partial}{\partial r} + r \frac{\partial^2}{\partial r^2} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

Therefore

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\ &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \end{aligned}$$

3 3D Spherical

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Spherical system are $x^1 = \phi, x^2 = r, x^3 = \theta$. The relation between these is known as (Note that the following depends on convention used for which is θ and which is ϕ.

Physics convention as shown in the diagram above is used here).

$$\begin{aligned}\zeta^1 &= r\sin\theta\cos\phi\\ \zeta^2 &= r\sin\theta\sin\phi\\ \zeta^3 &= r\cos\theta\end{aligned}$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij}\frac{\partial\zeta^i}{\partial x^k}\frac{\partial\zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$\begin{aligned}g_{11} &= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^1} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^1} + \frac{\partial\zeta^3}{\partial x^1}\frac{\partial\zeta^3}{\partial x^1}\\ &= \frac{\partial\zeta^1}{\partial\phi}\frac{\partial\zeta^1}{\partial\phi} + \frac{\partial\zeta^2}{\partial\phi}\frac{\partial\zeta^2}{\partial\phi} + \frac{\partial\zeta^3}{\partial\phi}\frac{\partial\zeta^3}{\partial\phi}\\ &= \left(\frac{\partial\zeta^1}{\partial\phi}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\phi}\right)^2 + \left(\frac{\partial\zeta^3}{\partial\phi}\right)^2\\ &= (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + (0)^2\\ &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\\ &= r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)\\ &= r^2\sin^2\theta\end{aligned}$$

And

$$\begin{aligned}g_{22} &= \frac{\partial\zeta^1}{\partial x^2}\frac{\partial\zeta^1}{\partial x^2} + \frac{\partial\zeta^2}{\partial x^2}\frac{\partial\zeta^2}{\partial x^2} + \frac{\partial\zeta^3}{\partial x^2}\frac{\partial\zeta^3}{\partial x^2}\\ &= \frac{\partial\zeta^1}{\partial r}\frac{\partial\zeta^1}{\partial r} + \frac{\partial\zeta^2}{\partial r}\frac{\partial\zeta^2}{\partial r} + \frac{\partial\zeta^3}{\partial r}\frac{\partial\zeta^3}{\partial r}\\ &= \left(\frac{\partial\zeta^1}{\partial r}\right)^2 + \left(\frac{\partial\zeta^2}{\partial r}\right)^2 + \left(\frac{\partial\zeta^3}{\partial r}\right)^2\\ &= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2\\ &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\\ &= \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\\ &= \sin^2\theta + \cos^2\theta\\ &= 1\end{aligned}$$

And

$$
\begin{aligned}
g_{33} &= \frac{\partial \zeta^1}{\partial x^3}\frac{\partial \zeta^1}{\partial x^3} + \frac{\partial \zeta^2}{\partial x^3}\frac{\partial \zeta^2}{\partial x^3} + \frac{\partial \zeta^3}{\partial x^3}\frac{\partial \zeta^3}{\partial x^3} \\
&= \frac{\partial \zeta^1}{\partial \theta}\frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta}\frac{\partial \zeta^2}{\partial \theta} + \frac{\partial \zeta^3}{\partial \theta}\frac{\partial \zeta^3}{\partial \theta} \\
&= \left(\frac{\partial \zeta^1}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^3}{\partial \theta}\right)^2 \\
&= (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 \\
&= r^2\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^2\theta \\
&= r^2\cos^2\theta + r^2\sin^2\theta \\
&= r^2
\end{aligned}
$$

Hence ds^2 in Spherical coordinates is

$$
\begin{aligned}
ds^2 &= g_{kl}dx^k dx^l \\
&= g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + g_{33}\left(dx^3\right)^2 \\
&= g_{11}(d\phi)^2 + g_{22}(dr)^2 + g_{33}(d\theta)^2 \\
&= r^2\sin^2\theta(d\phi)^2 + (dr)^2 + r^2(d\theta)^2
\end{aligned}
$$

From the above we see that

$$
g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} r^2\sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}
$$

Hence $\det(g) = r^4\sin^2\theta$. We are now ready to apply (1)

$$
\begin{aligned}
\nabla^2 &= \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{\det(g)}}{g_{ii}}\frac{\partial}{\partial x^i}\right) \\
&= \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{11}}\frac{\partial}{\partial x^1}\right) + \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_2}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{22}}\frac{\partial}{\partial x^2}\right) + \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_3}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{33}}\partial\right. \\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{r^2\sin\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{r^2\sin\theta}{r^2}\frac{\partial}{\partial\theta}\right) \\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\
&= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right) + \frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial^2}{\partial\theta^2}\right) \\
&= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}
\end{aligned}
$$

Therefore

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \\ &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta + u_{\theta\theta}\right) + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}\end{aligned}$$

4 3D Cylindrical

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Cylindrical system are $x^1 = \phi, x^2 = r, x^3 = z$. The relation between these is known as

$$\begin{aligned}\zeta^1 &= r\cos\phi \\ \zeta^2 &= r\sin\phi \\ \zeta^3 &= z\end{aligned}$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij}\frac{\partial \zeta^i}{\partial x^k}\frac{\partial \zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$\begin{aligned}g_{11} &= \frac{\partial \zeta^1}{\partial x^1}\frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1}\frac{\partial \zeta^2}{\partial x^1} + \frac{\partial \zeta^3}{\partial x^1}\frac{\partial \zeta^3}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial \phi}\frac{\partial \zeta^1}{\partial \phi} + \frac{\partial \zeta^2}{\partial \phi}\frac{\partial \zeta^2}{\partial \phi} + \frac{\partial \zeta^3}{\partial \phi}\frac{\partial \zeta^3}{\partial \phi} \\ &= \left(\frac{\partial \zeta^1}{\partial \phi}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \phi}\right)^2 + \left(\frac{\partial \zeta^3}{\partial \phi}\right)^2 \\ &= (-r\sin\phi)^2 + (r\cos\phi)^2 + (0)^2 \\ &= r^2\end{aligned}$$

And

$$\begin{aligned}g_{22} &= \frac{\partial \zeta^1}{\partial x^2}\frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2}\frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^3}{\partial x^2}\frac{\partial \zeta^3}{\partial x^2} \\ &= \frac{\partial \zeta^1}{\partial r}\frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r}\frac{\partial \zeta^2}{\partial r} + \frac{\partial \zeta^3}{\partial r}\frac{\partial \zeta^3}{\partial r} \\ &= \left(\frac{\partial \zeta^1}{\partial r}\right)^2 + \left(\frac{\partial \zeta^2}{\partial r}\right)^2 + \left(\frac{\partial \zeta^3}{\partial r}\right)^2 \\ &= (\cos\phi)^2 + (\sin\phi)^2 + (1)^2 \\ &= 1\end{aligned}$$

And

$$
\begin{aligned}
g_{33} &= \frac{\partial\zeta^1}{\partial x^3}\frac{\partial\zeta^1}{\partial x^3} + \frac{\partial\zeta^2}{\partial x^3}\frac{\partial\zeta^2}{\partial x^3} + \frac{\partial\zeta^3}{\partial x^3}\frac{\partial\zeta^3}{\partial x^3} \\
&= \frac{\partial\zeta^1}{\partial z}\frac{\partial\zeta^1}{\partial z} + \frac{\partial\zeta^2}{\partial z}\frac{\partial\zeta^2}{\partial z} + \frac{\partial\zeta^3}{\partial z}\frac{\partial\zeta^3}{\partial z} \\
&= \left(\frac{\partial\zeta^1}{\partial z}\right)^2 + \left(\frac{\partial\zeta^2}{\partial z}\right)^2 + \left(\frac{\partial\zeta^3}{\partial z}\right)^2 \\
&= (0)^2 + (0)^2 + (1)^2 \\
&= 1
\end{aligned}
$$

Hence ds^2 in Cylindrical coordinates is

$$
\begin{aligned}
ds^2 &= g_{kl}dx^k dx^l \\
&= g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + g_{33}\left(dx^3\right)^2 \\
&= g_{11}(d\phi)^2 + g_{22}(dr)^2 + g_{33}(dz)^2 \\
&= r^2(d\phi)^2 + (dr)^2 + (dz)^2
\end{aligned}
$$

From the above we see that

$$
g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Hence $\det(g) = r^2$. We are now ready to apply (1)

$$
\begin{aligned}
\nabla^2 &= \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{\det(g)}}{g_{ii}}\frac{\partial}{\partial x^i}\right) \\
&= \frac{1}{\sqrt{r^2}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{r^2}}{g_{11}}\frac{\partial}{\partial x^1}\right) + \frac{1}{\sqrt{r^2}}\frac{\partial}{\partial x_2}\left(\frac{\sqrt{r^2}}{g_{22}}\frac{\partial}{\partial x^2}\right) + \frac{1}{\sqrt{r^2}}\frac{\partial}{\partial x_3}\left(\frac{\sqrt{r^2}}{g_{33}}\frac{\partial}{\partial x^3}\right) \\
&= \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial}{\partial\phi}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right) \\
&= \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \\
&= \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} \\
&= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + u_{zz}
\end{aligned}
$$

4.1 References

1. Lecture notes, Physics 5041. UMN Spring 2019 by Professor Kapusta
2. Appendix A, Einstein's Theory, A rigorous introduction. By Gron and Naess. Springer publisher.