IIR digital filter design for low pass filter based on impulse invariance and bilinear transformation methods using butterworth analog filter

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A low pass digital filter (IIR) is designed based on given specifications. Butterworth analog filter $H(s)$ is designed first, then it is converted to digital filter $H(z)$

Analytical procedure is illustrated below and simplified to allow one to more easily program the algorithm. Four different numerical examples are used to illustrate the procedure.

1 Introduction

This report derives a symbolic procedure to design a low pass IIR digital filter from an analog Butterworth filter using 2 methods: impulse invariance and bilinear transformation. Two numerical examples are used to illustrate using the symbolic procedure.

There are a total of 13 steps. A Mathematica program with GUI is written to enable one to use this design for different parameters. A Matlab script is written as well.

1.1 Filter specifications

Filter specifications are 5 parameters. The frequency specifications use analog frequencies, while the attenuation for the passband and the stopband are given in db units.

F_s	The sampling frequency in Hz
f_c	The passband cutoff frequency in Hz
f_s	The stopband corner frequency in Hz
δ_p	The attenuation in db at f_c
δ_s	The attenuation in db at f_s

This diagram below illustrates these specifications

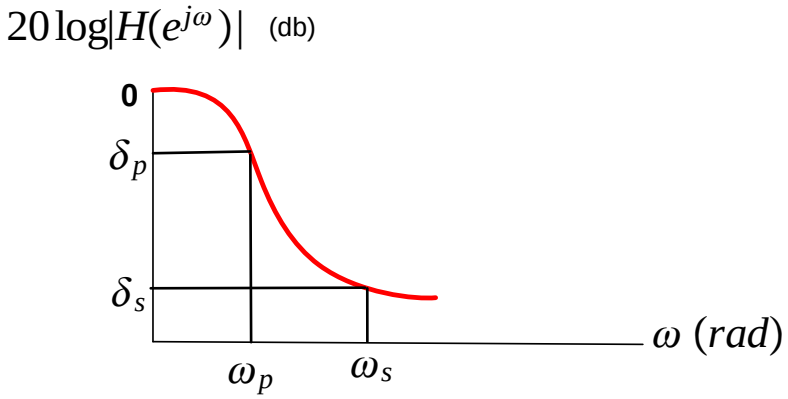

Figure 1: Minimal specifications

The frequency specifications are in hz, and they are first converted to digital frequencies ω where $0 \leq |\omega| \leq \pi$ before using the attenuation specifications. The sampling frequency F_s is used to do this conversion where F_s corresponds to 2π on the digital frequency scale.

2 The design steps

1. Specifications are converted from analog to digital frequencies.
2. Based on design method (impulse invariance of bilinear) the attenuation criteria is applied to determine Ω_c and N (the filter order).
3. Using Ω_c and N the locations of the poles of the butterworth analog filter $H(s)$ are found.
4. $H(z)$ is determined from $H(s)$. The method of doing this depends if impulse invariance or bilinear is used. This step is much simpler for the bilinear method as it does not require performing partial fractions decomposition on $H(s)$.

The analytical design procedure is given below.

2.1 Impulse invariance method

Analog specifications is converted to digital specifications: $\frac{F_s}{2\pi} = \frac{f_p}{\omega_p}$, hence $\omega_p = 2\pi \frac{f_p}{F_s}$ and $\omega_s = 2\pi \frac{f_s}{F_s}$. Next, the criteria relative to the digital normalized scale is found

$$\begin{aligned} 20 \log \left| H\left(e^{j\omega_p}\right) \right| &\geq \delta_p \\ 20 \log \left| H\left(e^{j\omega_s}\right) \right| &\leq \delta_s \end{aligned}$$

Therefore

$$\left| H\left(e^{j\omega_p}\right) \right| \geq 10^{\frac{\delta_p}{20}} T \tag{A}$$

$$\left| H\left(e^{j\omega_s}\right) \right| \leq 10^{\frac{\delta_s}{20}} T \tag{B}$$

Notice that T scaling factor was added above. The Butterworth analog filter squared magnitude Fourier transform is given by

$$|H_a(j\Omega)|^2 = \frac{T^2}{1 + \left(\frac{j\Omega}{j\Omega_c} \right)^{2N}}$$

Equations (A) and (B) above are now written in terms of the analog Butterworth amplitude frequency response giving

$$\frac{T^2}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq \left(10^{\frac{\delta_p}{20}}\right)^2 T^2$$
$$\frac{T^2}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq \left(10^{\frac{\delta_s}{20}}\right)^2 T^2$$

Hence

$$\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq \ 10^{\frac{\delta_p}{10}}$$
$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq 10^{\frac{\delta_s}{10}}$$

For impulse invariance, $\Omega_p = \frac{\omega_p}{T}$ and $\Omega_s = \frac{\omega_s}{T}$. Therefore

$$1+\left(\frac{\omega_p/T}{\Omega_c}\right)^{2N} \leq 10^{-\frac{\delta_p}{10}} \tag{1}$$
$$1+\left(\frac{\omega_s/T}{\Omega_c}\right)^{2N} \geq 10^{-\frac{\delta_s}{10}} \tag{2}$$

Changing inequalities to equalities and simplifying gives

$$\left(\frac{\omega_p/T}{\Omega_c}\right)^{2N} = 10^{-\frac{\delta_p}{10}} - 1$$
$$\left(\frac{\omega_s/T}{\Omega_c}\right)^{2N} = 10^{-\frac{\delta_s}{10}} - 1$$

Dividing the above two equations results in

$$\left(\frac{\omega_p}{\omega_s}\right)^{2N} = \frac{10^{-\frac{\delta_p}{10}} - 1}{10^{-\frac{\delta_s}{10}} - 1}$$
$$2N[\log(\omega_p) - \log(\omega_s)] = \log\left(10^{-\frac{\delta_p}{10}} - 1\right) - \log\left(10^{-\frac{\delta_s}{10}} - 1\right)$$
$$N = \frac{1}{2}\frac{\log\left[10^{-\frac{\delta_p}{10}} - 1\right] - \log\left[10^{-\frac{\delta_s}{10}} - 1\right]}{\log(\omega_p) - \log(\omega_s)}$$

The above is later rounded to the nearest integer using the Ceiling function $N = \lceil N \rceil$.

For impulse invariance method, using equation (1) above to solve for Ω_c gives

$$\begin{aligned}
\left(\frac{\omega_p/T}{\Omega_c}\right)^{2N} &= 10^{-\frac{\delta_p}{10}} - 1 \\
2N\left(\log_{10}\frac{\omega_p/T}{\Omega_c}\right) &= \log_{10}\left(10^{-\frac{\delta_p}{10}} - 1\right) \\
\log_{10}\frac{\omega_p/T}{\Omega_c} &= \frac{1}{2N}\log_{10}\left(10^{-\frac{\delta_p}{10}} - 1\right) \\
\frac{\omega_p/T}{\Omega_c} &= 10^{\left(\frac{1}{2N}\log_{10}\left(10^{-\frac{\delta_p}{10}}-1\right)\right)} \\
\Omega_c &= \frac{\omega_p/T}{10^{\left(\frac{1}{2N}\log_{10}\left(10^{-\frac{\delta_p}{10}}-1\right)\right)}}
\end{aligned}$$

Now that N and Ω_c are found, next the poles of $H(s)$ are determined. Since the butterworth magnitude square of the transfer function is

$$|H_a(s)|^2 = \frac{T^2}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Then $H(s)$ poles are found by setting the denominator of the above to zero which gives

$$\begin{aligned}
1+\left(\frac{s}{j\Omega_c}\right)^{2N} &= 0 \\
\left(\frac{s}{j\Omega_c}\right)^{2N} &= -1 \\
&= e^{j(\pi+2\pi k)} \qquad k = 0,1,2,\cdots 2N-1 \\
\frac{s}{j\Omega_c} &= e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
s &= j\Omega_c\ e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
&= \Omega_c\ e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
&= \Omega_c\ e^{j\left(\frac{\pi(1+2k+N)}{2N}\right)}
\end{aligned}$$

Only the LHS poles needs to be found, and these are located at $i = 0\cdots N-1$. This is because these are the stable poles. Hence the i^{th} pole is

$$s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$$

For an, using $i = 0$, $N = 6$ gives

$$\begin{aligned}
s_0 &= \Omega_c\ e^{j\left(\frac{\pi(1+N)}{2N}\right)} \\
&= \Omega_c\ e^{j\left(\frac{\pi 7}{12}\right)}
\end{aligned}$$

Now the analog filter is generated based on the above selected poles, which for impulse invariance gives

$$H_a(s) = \frac{T\ K}{\prod_{i=0}^{N-1}(s - s_i)} \tag{3}$$

K is found by solving $H_a(0) = T$ which gives

$$k = \prod_{i=0}^{N-1}(-s_i)$$

The poles are written in non-polar form and substituted into (3) which gives

$$s_i = \Omega_c\, e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = \Omega_c\left(\cos\frac{\pi(1+2i+N)}{2N} + j\sin\frac{\pi(1+2i+N)}{2N}\right) \qquad i = 0\cdots N-1$$

Therefore

$$H_a(s) = \frac{T\ K}{\prod_{i=0}^{N-1}\left(s - \Omega_c\left(\cos\frac{\pi(1+2i+N)}{2N} + j\sin\frac{\pi(1+2i+N)}{2N}\right)\right)} \tag{4}$$

Where

$$\Omega_c = \frac{\omega_p/T}{10^{\left(\frac{1}{2N}\log_{10}\left(10^{-\frac{\delta_p}{10}}-1\right)\right)}}$$

And

$$N = \left\lceil \frac{1}{2}\frac{\log\left[10^{-\frac{\delta_p}{10}} - 1\right] - \log\left[10^{-\frac{\delta_s}{10}} - 1\right]}{\log(\omega_p) - \log(\omega_s)} \right\rceil$$

Now that $H(s)$ is found, it is converted to $H(z)$. We need to make sure that we multiply poles of complex conjugates with each others to make the result simple to see.

Now that $H_a(s)$ is found, the $A \to D$ conversion is done. This means to obtain $H(z)$ from the above $H(s)$. When using impulse invariance, partial fraction decomposition is performed on (4) above in order to write $H(s)$ as

$$H(s) = \sum_{i=0}^{N-1}\frac{A_i}{s - s_i}$$

For example, to obtain A_j the result is

$$A_j = \lim_{s \to s_j} H_a(s) = \frac{T\ k}{\prod_{\substack{i=0 \\ i \neq j}}^{N-1}(s - s_i)}$$

Once the $A's$ are found, then $H(z)$ becomes

$$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \exp\left(s_i\ T\right) z^{-1}}$$

This completes the design. The above form of $H(z)$ could also be converted to rational expression as $H(z) = \frac{N(z)}{D(z)}$.

2.2 Bilinear transformation method

First step is to convert analog specifications to digital specifications: $\frac{F_s}{2\pi} = \frac{f_p}{\omega_p}$, hence $\omega_p = 2\pi \frac{f_p}{F_s}$ and $\omega_s = 2\pi \frac{f_s}{F_s}$

Converting the criteria relative to the digital normalized scale gives

$$20 \log \left| H\left(e^{j\omega_p}\right) \right| \geq \delta_p$$
$$20 \log \left| H\left(e^{j\omega_s}\right) \right| \leq \delta_s$$

Hence

$$\left| H\left(e^{j\omega_p}\right) \right| \geq 10^{\frac{\delta_p}{20}} \tag{A}$$
$$\left| H\left(e^{j\omega_s}\right) \right| \leq 10^{\frac{\delta_s}{20}} \tag{B}$$

Butterworth analog filter squared magnitude Fourier transform is given by

$$\left| H_a(j\Omega) \right|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c} \right)^{2N}}$$

Equations (A) and (B) above are now written in terms of the analog Butterworth amplitude frequency response and become

$$\frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c} \right)^{2N}} \geq \left(10^{\frac{\delta_p}{20}} \right)^2 = \ 10^{\frac{\delta_p}{10}}$$
$$\frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c} \right)^{2N}} \leq \left(10^{\frac{\delta_s}{20}} \right)^2 = 10^{\frac{\delta_s}{10}}$$

Values for Ω_p and Ω_s are assigned as follows: $\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$, $\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$. Hence the above becomes

$$1+\left(\frac{\frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)}{\Omega_c}\right)^{2N} \leq 10^{\frac{\delta_p}{10}} \tag{1}$$

$$1+\left(\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c}\right)^{2N} \geq 10^{\frac{\delta_s}{10}} \tag{2}$$

Changing inequalities to equalities and simplifying gives

$$\left(\frac{\frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)}{\Omega_c}\right)^{2N} = 10^{\frac{\delta_p}{10}}-1$$

$$\left(\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c}\right)^{2N} = 10^{\frac{\delta_s}{10}}-1$$

Dividing the above 2 equations results in

$$\left(\frac{\tan\left(\frac{\omega_p}{2}\right)}{\tan\left(\frac{\omega_s}{2}\right)}\right)^{2N} = \frac{10^{\frac{\delta_p}{10}}-1}{10^{\frac{\delta_s}{10}}-1}$$

$$2N\left[\log\left(\tan\left(\frac{\omega_p}{2}\right)\right)-\log\left(\tan\left(\frac{\omega_s}{2}\right)\right)\right] = \log\left(10^{\frac{\delta_p}{10}}-1\right)-\log\left(10^{\frac{\delta_s}{10}}-1\right)$$

$$N = \frac{1}{2}\frac{\log\left(10^{\frac{\delta_p}{10}}-1\right)-\log\left(10^{\frac{\delta_s}{10}}-1\right)}{\log\left(\tan\left(\frac{\omega_p}{2}\right)\right)-\log\left(\tan\left(\frac{\omega_s}{2}\right)\right)}$$

The above is rounded to the nearest integer using the Ceiling function. i.e. $N = \lceil N \rceil$.

For bilinear transformation equation (2) is used to find Ω_c. Solving for Ω_c gives

$$1+\left(\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c}\right)^{2N} = 10^{\frac{\delta_s}{10}}$$

$$2N\left(\log_{10}\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c}\right) = \log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)$$

$$\log_{10}\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c} = \frac{1}{2N}\log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)$$

$$\frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\Omega_c} = 10^{\left(\frac{1}{2N}\log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)\right)}$$

$$\Omega_c = \frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{10^{\left(\frac{1}{2N}\log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)\right)}}$$

Now that N and Ω_c are found, the poles of $H(s)$ can be determined. Since for bilinear the magnitude square of the transfer function is

$$|H_a(s)|^2 = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Hence $H(s)$ poles are found by setting the denominator of the above to zero

$$\begin{aligned}
1+\left(\frac{s}{j\Omega_c}\right)^{2N} &= 0 \\
\left(\frac{s}{j\Omega_c}\right)^{2N} &= -1 \\
&= e^{j(\pi+2\pi k)} \qquad k=0,1,2,\cdots 2N-1 \\
\frac{s}{j\Omega_c} &= e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
s &= j\Omega_c\ e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
&= \Omega_c\ e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
&= \Omega_c\ e^{j\left(\frac{\pi(1+2k+N)}{2N}\right)}
\end{aligned}$$

Only need the LHS poles are needed, which are located at $i=0\cdots N-1$, because these are the stable poles. Hence the i^{th} pole is

$$s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$$

For example for $i=0,\ N=6$ the result is

$$s_0 = \Omega_c\ e^{j\left(\frac{\pi(1+N)}{2N}\right)} = \Omega_c\ e^{j\left(\frac{\pi 7}{12}\right)}$$

For bilinear $H(s)$ is given by

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1}(s-s_i)} \tag{3}$$

K is found by solving $H_a(0)=1$ giving

$$k = \prod_{i=0}^{N-1}(-s_i)$$

The same expression results for k in both cases. Writing poles in non-polar form and substituting them into (3) gives

$$s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = \Omega_c\left(\cos\frac{\pi(1+2i+N)}{2N} + j\sin\frac{\pi(1+2i+N)}{2N}\right) \qquad i=0\cdots N-1$$

Then

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1}\left(s - \Omega_c\left(\cos\frac{\pi(1+2i+N)}{2N} + j\sin\frac{\pi(1+2i+N)}{2N}\right)\right)} \tag{4}$$

Where

$$\Omega_c = \frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{10^{\left(\frac{1}{2N}\log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)\right)}}$$

And

$$N = \left\lceil \frac{1}{2}\frac{\log_{10}\left(10^{\frac{\delta_p}{10}}-1\right) - \log_{10}\left(10^{\frac{\delta_s}{10}}-1\right)}{\log_{10}\left(\tan\left(\frac{\omega_p}{2}\right)\right) - \log_{10}\left(\tan\left(\frac{\omega_s}{2}\right)\right)} \right\rceil$$

Now that $H(s)$ is found, it is converted to $H(z)$. After finding $H(s)$ as shown above then s is replaced by $\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$. This is much simpler than the impulse invariance method. Before doing this substitution, we make sure to multiply poles which are complex conjugate of each others in the denominator of $H(s)$. After this, then the above substitution is made.

2.3 Summary of analytical derivation method

Table with the derivation equations is made to follow to design in either bilinear or impulse invariance. Note that the same steps are used in both designs except for step $5, 6, 8, 13$. This table make it easier to develop a program for the implementation

step	Impulse invariance	common equation	bilinear
1		$\omega_p = 2\pi \frac{f_p}{F_s}$	
2		$\omega_s = 2\pi \frac{f_s}{F_s}$	
3		$\alpha_p = \frac{1}{10^{\frac{\delta_p}{10}}}$	
4		$\alpha_s = \frac{1}{10^{\frac{\delta_s}{10}}}$	
5	$\Omega_p = \frac{\omega_p}{T}$		$\frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$
6	$\Omega_s = \frac{\omega_s}{T}$		$\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$
7		$N = \left\lceil \frac{1}{2}\frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil$	
8	$\Omega_c = \frac{\Omega_p}{10^{\left(\frac{1}{2N}\log_{10}[\alpha_p - 1]\right)}}$		$\Omega_c = \frac{\Omega_s}{10^{\left(\frac{1}{2N}\log_{1}\right.}}$
9		poles of H(S) $s_i = \Omega_c \, e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \qquad i = 0 \cdots N-1$	
10		$k = \prod_{i=0}^{N-1}(-s_i)$	
11	$H_a(s) = \frac{T \; K}{\prod_{i=0}^{N-1}(s - s_i)}$		$H_a(s) = \frac{K}{\prod_{i=0}^{N-1}(s}$
	do partial fractions:		
12	$H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{s - s_i}$		
13	$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \exp(T s_i) z^{-1}}$		$H(z) = H_a(s)$

3 Numerical design examples

3.1 Example 1

Sampling frequency $F_s = 20khz$, passband frequency $f_p = 2khz$, stopband frequency $f_s = 3khz$, with $\delta_p \geq -1db$ and $\delta_{stop} \leq -15db$

3.1.1 using impulse invariance method (using T=1)

step	Impulse invariance
1	$\omega_p = 2\pi \frac{f_p}{F_s} \rightarrow \frac{2\pi(2000)}{20000} \rightarrow 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} \rightarrow \frac{2\pi(3000)}{20000} \rightarrow 0.3\pi$
3	$\alpha_p = \frac{1}{10^{\frac{\delta_p}{10}}} \rightarrow \frac{1}{10^{\frac{-1}{10}}} \rightarrow 1.258\,9$
4	$\alpha_s = \frac{1}{10^{\frac{\delta s}{10}}} \rightarrow \frac{1}{10^{\frac{-15}{10}}} \rightarrow 31.623$
5	$\Omega_p = \frac{\omega_p}{T} \rightarrow \frac{\omega_p}{1} \rightarrow 0.2\pi$
6	$\Omega_s = \frac{\omega_s}{T} \rightarrow \frac{0.3\pi}{1} \rightarrow 0.3\pi$
7	$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil \rightarrow \frac{1}{2} \frac{\log_{10}(1.258\,9 - 1) - \log_{10}(31.623 - 1)}{\log_{10}(0.2\pi) - \log_{10}(0.3\pi)} \rightarrow \lceil 5.885\,9 \rceil \rightarrow 6$
8	$\Omega_c = \frac{\Omega_p}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_p - 1]\right)}} \rightarrow \frac{0.2\pi}{10^{\left(\frac{1}{2\times 6} \log_{10}(1.258\,9 - 1)\right)}} \rightarrow 0.703\,21$
9	poles of H(S) $s_i = \Omega_c \; e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \rightarrow s_i = 0.703\,21 \; e^{j\left(\frac{\pi(7+2i)}{12}\right)} i = 0 \cdots 5$
	$s_0 = -0.182 + j0.679\,25, s_1 = -0.497\,24 + j0.497\,2, s_2 = -0.679\,25 + j0.182$
	$s_3 = -0.679\,25 - j0.182\,, s_4 = -0.497\,24 - j0.497\,24, s_5 = -0.182 - j0.679\,25$
10	$k = \prod_{i=0}^{N-1} (-s_i) \rightarrow (0.182 - j0.679\,25)\,(0.497\,24 - j0.497\,24)\,(0.679\,25 - j0.182\,)$
	$(0.679\,25 + j0.182)\,(0.497\,24 + j0.497\,24)\,(0.182 + j0.679\,25) \rightarrow 0.120\,92$
11	$H_a(s) = \frac{T \; K}{\prod_{i=0}^{N-1} (s - s_i)} \rightarrow$
	$\frac{1 \times 0.120\,92}{(s+0.182-j0.679\,25)(s+0.497\,24-j0.497\,24)(s+0.679\,25-j0.182)(s+0.679\,25+j0.182)(s+0.497\,24+j0.497\,24)(s+0.182+j0.679\,25)}$
	$\rightarrow$multiply complex conjugates$\rightarrow \frac{0.120\,92}{(s^2+0.364\,s+0.494\,5)(s^2+0.994\,48s+0.494\,50)(s^2+1.358\,5s+0.494\,5)}$
	$\rightarrow \frac{0.120\,92}{s^6+2.717\,0s^5+3.691\,0s^4+3.178\,9s^3+1.825\,2s^2+0.664\,38s+0.120\,92}$
12	partial fraction $H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{s - s_i} \rightarrow \frac{0.143\,54+j0.248\,61}{s+0.182-j0.679\,25} - \frac{1.071\,4-j1.166\,8\times 10^{-5}}{s+0.497\,24+j0.497\,24} + \frac{0.927\,85-j1.607\,1}{s+0.679\,25-j0.182}$
	$+\frac{0.927\,85+j1.607\,1}{s+0.679\,25+j0.182} - \frac{1.071\,4+j1.166\,8\times 10^{-5}}{s+0.497\,24-j0.497\,24} + \frac{0.143\,54-j0.248\,61}{s+0.182+j0.679\,25}$
13	$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1-\exp(Ts_i)z^{-1}} \rightarrow \frac{0.143\,54+j0.248\,61}{1-\exp(-0.182+j0.679\,25)z^{-1}} -$
	$\frac{1.071\,4-j1.166\,8\times 10^{-5}}{1-\exp(-0.497\,24-j0.497\,24)z^{-1}} + \frac{0.927\,85-j1.607\,1}{1-\exp(-0.679\,25+j0.182)}$
	$+\frac{0.927\,85+j1.607\,1}{1-\exp(-0.679\,25-j0.182)\,z^{-1}} - \frac{1.071\,4+j1.166\,8\times 10^{-5}}{1-\exp(-0.497\,24+j0.497\,24)z^{-1}} + \frac{0.143\,54-j0.248\,61}{1-\exp(-0.182-j0.679\,25)z^{-1}}$
	$H(z) = \frac{0.143\,54+j0.248\,61}{1-(0.648\,58+j0.523\,68)z^{-1}} - \frac{1.071\,4-j1.166\,8\times 10^{-5}}{1-(0.534\,55-j0.290\,12)z^{-1}} + \frac{0.927\,85-j1.607\,1}{1-(0.498\,62+j9.176\,5\times 10^{-2})z^{-1}}$
	$+\frac{0.927\,85+j1.607\,1}{1-(0.498\,62-j9.176\,5\times 10^{-2})\,z^{-1}} - \frac{1.071\,4+j1.166\,8\times 10^{-5}}{1-(0.534\,55+j0.29012)z^{-1}} + \frac{0.143\,54-j0.248\,61}{1-(0.648\,58-j0.52368)z^{-1}}$

3.1.2 Bilinear method

Using the above design table, these are the numerical values: $T = \frac{1}{F_s} = \frac{1}{20000}$

step	Bilinear
1	$\omega_p = 2\pi \frac{f_p}{F_s} \rightarrow \frac{2\pi(2000)}{20000} \rightarrow 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} \rightarrow \frac{2\pi(3000)}{20000} \rightarrow 0.3\pi$
3	$\alpha_p = \frac{1}{10^{\frac{\delta_p}{10}}} \rightarrow \frac{1}{10^{\frac{-1}{10}}} \rightarrow 1.258\,9$
4	$\alpha_s = \frac{1}{10^{\frac{\delta s}{10}}} \rightarrow \frac{1}{10^{\frac{-15}{10}}} \rightarrow 31.623$
5	$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) \rightarrow 2 \times 20000 \tan\left(\frac{0.2\pi}{2}\right) \rightarrow 12997$
6	$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) \rightarrow 2 \times 20000 \tan\left(\frac{0.3\pi}{2}\right) \rightarrow 20381$
7	$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil \rightarrow \frac{1}{2} \frac{\log_{10}(1.258\,9 - 1) - \log_{10}(31.623 - 1)}{\log_{10}(12997) - \log_{10}(20381)} \rightarrow \lceil 5.304\,8 \rceil \rightarrow 6$
8	$\Omega_c = \frac{\Omega_s}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_s - 1]\right)}} \rightarrow \frac{20381}{10^{\left(\frac{1}{2 \times 6} \log_{10}(31.623 - 1)\right)}} \rightarrow 15325.6$
9	poles of H(S) $s_i = \Omega_c \; e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \rightarrow s_i = 15325 \; e^{j\left(\frac{\pi(7+2i)}{12}\right)} \qquad i = 0 \cdots 5$
	$s_0 = -3966.4 + j14803, s_1 = -10836 + j10836., s_2 = -14803 + j3966.4$
	$s_3 = -14803 - j3966.4, s_4 = -10836 - j10836., s_5 = -3966.4 - j14803$
10	$k = \prod_{i=0}^{N-1} (-s_i) \rightarrow (3966.4 - j14803)\,(10836 - j10836)\,(14803 - j3966.4)$
	$(14803 + j3966.4)\,(10836 + j10836)\,(3966.4 + j14803)$
	$\rightarrow$ multiply complex conjugate terms
	$\rightarrow (2.348\,6 \times 10^8)\,(2.348\,4 \times 10^8)\;(2.348\,6 \times 10^8) \rightarrow 1.295\,4 \times 10^{25}$
11	$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s - s_i)} \rightarrow$
	$\frac{1.295\,4 \times 10^{25}}{(s+3966.4-j14803)(s+10836-j10836)(s+14803-j3966.4)(s+14803+j3966.4)(s+10836+j10836)(s+3966.4+j14803)}$
	$\rightarrow$ multiply complex conjugate
	$\frac{1.295\,4 \times 10^{25}}{(s^2+7932.8s+2.348\,6 \times 10^8)(s^2+21\,672s+234\,837\,792)(s^2+29606.s+2.348\,6 \times 10^8)}$
	$= \frac{6257.4s - 1.355\,8 \times 10^8}{s^2+7932.8s+2.348\,6 \times 10^8} - \frac{46702.s + 5.060\,4 \times 10^8}{s^2+21672.s+2.348\,4 \times 10^8} + \frac{40445.s + 8.765\,4 \times 10^8}{s^2+29606.s+2.348\,6 \times 10^8} \rightarrow$
	$\rightarrow \frac{1.295\,4 \times 10^{25}}{s^6+59211.s^5+1.753\,0 \times 10^9 s^4+3.290\,2 \times 10^{13} s^3+4.116\,9 \times 10^{17} s^2+3.265\,8 \times 10^{21} s+1.295\,3 \times 10^{25}}$
12	
13	$H(z) = H_a(s)\big\vert_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \rightarrow TODO$
	$\cdots$

3.2 Example 2

Sampling frequency $F_s = 10khz$, passband corner frequency $f_p = 1khz$, stopband corner frequency $f_s = 2khz$, with criteria $\delta_p \geq -3db$ and $\delta_{stop} \leq -10db$

3.2.1 using impulse invariance method

$T = 1$

step	Impulse invariance
1	$\omega_p = 2\pi \frac{f_p}{F_s} \rightarrow \frac{2\pi(1000)}{10000} \rightarrow 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} \rightarrow \frac{2\pi(2000)}{10000} \rightarrow 0.4\pi$
3	$\alpha_p = \frac{1}{10^{\frac{\delta_p}{10}}} \rightarrow \frac{1}{10^{\frac{-3}{10}}} \rightarrow 1.995\,3$
4	$\alpha_s = \frac{1}{10^{\frac{\delta s}{10}}} \rightarrow \frac{1}{10^{\frac{-10}{10}}} \rightarrow 10.0$
5	$\Omega_p = \frac{\omega_p}{T} \rightarrow \frac{\omega_p}{1} \rightarrow 0.2\pi$
6	$\Omega_s = \frac{\omega_s}{T} \rightarrow \frac{0.4\pi}{1} \rightarrow 0.4\pi$
7	$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil \rightarrow \frac{1}{2} \frac{\log_{10}(1.995\,3-1) - \log_{10}(10.0-1)}{\log_{10}(0.2\pi) - \log_{10}(0.4\pi)} \rightarrow 1.588\,4 \rightarrow 2$
8	$\Omega_c = \frac{\Omega_p}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_p - 1]\right)}} \rightarrow \frac{0.2\pi}{10^{\left(\frac{1}{2\times 2} \log_{10}(1.995\,3-1)\right)}} \rightarrow 0.629\,06$
9	poles of H(S) $s_i = \Omega_c \; e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \rightarrow s_i = 0.629\,06 \; e^{j\left(\frac{\pi(3+2i)}{4}\right)} \quad i = 0 \cdots 1$
	$s_0 = -0.444\,81 + j0.444\,81, s_1 = -0.444\,81 - j0.444\,81$
10	$k = \prod_{i=0}^{N-1} (-s_i) \rightarrow (0.444\,81 - j0.444\,81)\,(0.444\,81 + j0.444\,81) \rightarrow 0.395\,71$
11	$H_a(s) = \frac{T \;\; K}{\prod_{i=0}^{N-1} (s - s_i)} \rightarrow \frac{0.395\,71}{(s+0.444\,81-j0.444\,81)(s+0.444\,81+j0.444\,81)} \rightarrow \frac{0.395\,71}{s^2+0.889\,62s+0.395\,71}$
12	partial fraction $H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{s - s_i} \rightarrow \frac{0.444\,81j}{s+0.444\,81+0.444\,81j} - \frac{0.444\,81j}{s+0.444\,81-0.444\,81j}$
	$\rightarrow \frac{0.245\,35z}{z^2-1.157\,2z+0.410\,81}$
13	$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1-\exp(Ts_i)z^{-1}} \rightarrow \frac{0.444\,81i}{1-\exp(-0.444\,81-j0.444\,81)z^{-1}} - \frac{0.444\,81i}{1-\exp(-0.444\,81+j0.444\,81)z^{-1}}$

3.2.2 Using bilinear

$T = \frac{1}{10000}$

step	Impulse invariance
1	$\omega_p = 2\pi \frac{f_p}{F_s} \rightarrow \frac{2\pi(1000)}{10000} \rightarrow 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} \rightarrow \frac{2\pi(2000)}{10000} \rightarrow 0.4\pi$
3	$\alpha_p = \frac{1}{10^{\frac{\delta p}{10}}} \rightarrow \frac{1}{10^{\frac{-3}{10}}} \rightarrow 1.995\,3$
4	$\alpha_s = \frac{1}{10^{\frac{\delta s}{10}}} \rightarrow \frac{1}{10^{\frac{-10}{10}}} \rightarrow 10.0$
5	$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) \rightarrow 2 \times 10000 \tan\left(\frac{0.2\pi}{2}\right) \rightarrow 6498.4$
6	$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) \rightarrow 2 \times 10000 \tan\left(\frac{0.4\pi}{2}\right) \rightarrow 14531.$
7	$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil \rightarrow \frac{1}{2} \frac{\log_{10}(1.995\,3 - 1) - \log_{10}(10.0 - 1)}{\log_{10}(6498.4) - \log_{10}(14531.)} \rightarrow 1.368\,1 \rightarrow 2$
8	$\Omega_c = \frac{\Omega_s}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_s - 1]\right)}} \rightarrow \frac{14531}{10^{\left(\frac{1}{2 \times 2} \log_{10}(10.0 - 1)\right)}} \rightarrow 8389.5$
9	poles of H(S) $s_i = \Omega_c \, e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \rightarrow s_i = 8389.5 \, e^{j\left(\frac{\pi(3+2i)}{4}\right)} \quad i = 0 \cdots 1$
	$s_0 = -5932.3 + j5932.3, s_1 = -5932.3 - j5932.3$
10	$k = \prod_{i=0}^{N-1} (-s_i) \rightarrow (5932.3 - j5932.3)\,(5932.3 + j5932.3) \rightarrow 7.038\,4 \times 10^7$
11	$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s - s_i)} \rightarrow \frac{7.038\,4 \times 10^7}{(s + 5932.3 - i5932.3)(s + 5932.3 + i5932.3)}$
12	$\rightarrow \frac{7.038\,4 \times 10^7}{s^2 + 11865.s + 7.038\,4 \times 10^7}$
13	$H(z) = H_a(s)\big\vert_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}} \rightarrow \frac{0.099\,459\,z^2 + 0.198\,92z + 0.099\,459}{z^2 - 0.931\,56z + 0.329\,38}$

Mathematica GUI program is written to implement the above. It can be downloaded from here

Also a Matlab script `nma_filter.m.txt` was written (no GUI) to implement this design.

This script (written on Matlab 2007a). This script does not handle the conversion from H(s) to H(z) well yet, need to work more on this... of course, one can just use Matlab butter() function for this.

Example output of the above Matlab script is `matlab_output.txt`

4 IIR design for minimum order filter

This is another small note on IIR design for minimum order filter.

This document describes how to design an IIR digital filter given a specification in which the filter order is specified.

Given the following diagram, the specifications for the design will be

1. Digital filter order (N)
2. f_{pass}, the passband corner frequency (or the cutoff frequency) at $-3db$. This means $A_{pass} = -3db$

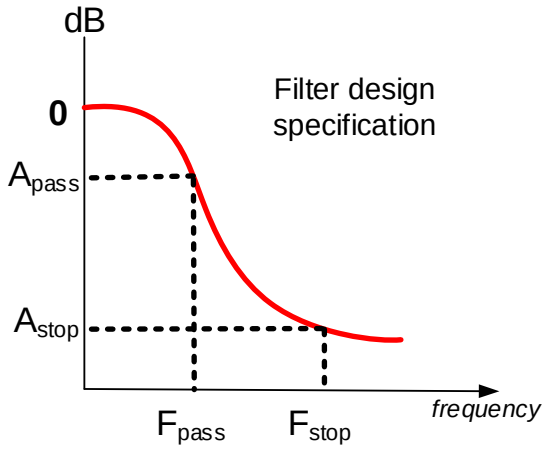

Figure 2: specifications

4.1 Impulse invariance method

We first convert analog specifications to digital specifications: $\frac{F_s}{2\pi} = \frac{f_{pass}}{\omega_p}$, hence $\omega_p = 2\pi \frac{f_{pass}}{F_s}$

Then the cutoff frequency $\Omega_c = \frac{\omega_p}{T}$ for impulse invariance.

Next find the poles of $H(s)$. Since the butterworth magnitude square of the transfer function is

$$|H_a(s)|^2 = \frac{T^2}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Hence $H(s)$ poles are found by setting the denominator of the above to zero

$$1 + \left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

And as I did in the earlier document, the poles of $H(s)$ are found at

$$s = \Omega_c \; e^{j\left(\frac{\pi(1+2k+N)}{2N}\right)}$$

We only need to find the LHS poles, which are located at $i = 0 \cdots N-1$, because these are the stable poles. Hence the i^{th} pole is

$$s_i = \Omega_c \; e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$$

Now we can write the analog filter generated based on the above selected poles, which is, for impulse invariance

$$H_a(s) = \frac{T \; K}{\prod_{i=0}^{N-1} (s - s_i)} \tag{3}$$

K is found by solving $H_a(0) = T$ hence

$$k = \prod_{i=0}^{N-1} (-s_i)$$

Now we need to write poles in non-polar form and plug them into (3)

$$s_i = \Omega_c \; e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = \Omega_c \left(\cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \qquad i = 0 \cdots N-1$$

Hence,

$$H_a(s) = \frac{T \; K}{\prod_{i=0}^{N-1} \left(s - \Omega_c \left(\cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \right)} \tag{4}$$

Now that we have found $H(s)$ we need to convert it to $H(z)$

We need to make sure that we multiply poles of complex conjugates with each others to make the result simple to see.

Now that we have $H_a(s)$, we do the $A \rightarrow D$ conversion. I.e. obtain $H(z)$ from the above $H(s)$. When using impulse invariance, we need to perform partial fraction decomposition on (4) above in order to write $H(s)$ in this form

$$H(s) = \sum_{i=0}^{N-1} \frac{A_i}{s - s_i}$$

For example, to obtain A_j, we write

$$A_j = \lim_{s \rightarrow s_j} H_a(s) = \frac{T \; k}{\prod_{\substack{i=0 \\ i \neq j}}^{N-1} (s - s_i)}$$

Once we find all the $A's$, we now write $H(z)$ as follows

$$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \exp\left(s_i\ T\right) z^{-1}}$$

This completes the design. We can try to convert the above form of $H(z)$ to a rational expression as $H(z) = \frac{N(z)}{D(z)}$

4.2 bilinear transformation method

We first convert analog specifications to digital specifications: $\frac{F_s}{2\pi} = \frac{f_{pass}}{\omega_p}$, hence $\omega_p = 2\pi \frac{f_{pass}}{F_s}$

Now $\Omega_c = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$

Now that we have N and Ω_c we find the poles of $H(s)$. Since for bilinear the magnitude square of the transfer function is

$$\left|H_a(s)\right|^2 = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Hence $H(s)$ poles are found by setting the denominator of the above to zero

$$1 + \left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

Which leads to poles at $s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$. We only need to find the LHS poles, which are located at $i = 0 \cdots N-1$, because these are the stable poles. For bilinear, $H(s)$ is given by

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s - s_i)} \tag{3}$$

K is found by solving $H_a(0) = 1$, hence we obtain

$$k = \prod_{i=0}^{N-1} (-s_i)$$

We see that the same expression results for k for both cases.

Now we need to write poles in non-polar form and plug them into (3)

$$s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = \Omega_c \left(\cos\frac{\pi(1+2i+N)}{2N} + j \sin\frac{\pi(1+2i+N)}{2N}\right) \qquad i = 0 \cdots N-1$$

then

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} \left(s - \Omega_c \left(\cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \right)} \tag{4}$$

Now that we have found $H(s)$ we need to convert it to $H(z)$. After finding $H(s)$ as shown above, we simply replace s by $\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$. This is much simpler than the impulse invariance method. Before doing this substitution, make sure to multiply poles which are complex conjugate of each others in the denominator of $H(s)$. After this, then do the above substitution

5 Appendix

The following is listing of the Matlab script, and a sample run output

5.1 Code listing

source code file

```
%Simple matlab script to design an IIR low pass filter using
%butterworth in either impulse inv. or bilinear method.

%
% EE 420, CSU Fullerton
%
% by Nasser M. Abbasi
% 5/5/201

% Filter SPECIFICATIONS
clear all;
close all;
Fs=10000;
fp=1000;
fs=2000;
dbp=-3;
dbs=-10;
BILINEAR=1;  %set this to 0 to do impulse inv.

%%%%%%%%%%%%%%%%%%

fprintf('***** STARTING DESIGN *******\n');
```

```
fprintf('Sampling frequency=%f Hz\n',Fs);
fprintf('freq at passband=%f Hz\n',fp);
fprintf('freq at stopband=%f Hz\n',fs);
fprintf('db at passband=%f \n',dbp);
fprintf('db at stopband=%f \n',dbs);
if BILINEAR
   fprintf('Doing Bilinear method\n');
else
   fprintf('Doing impulse invariance method\n');
end



if BILINEAR
    T=1/Fs;
else
    T=1;
end

fprintf('T=%f\n',T);

wp=2*pi*fp/Fs;
ws=2*pi*fs/Fs;
alphap=1/(10^(dbp/10));
alphas=1/(10^(dbs/10));

if BILINEAR
    gammap=2/T * tan(wp/2);
else
    gammap=wp/T;
end

if BILINEAR
    gammas=2/T * tan(ws/2);
else
    gammas=ws/T;
end

oldn=.5*(log10(alphap-1)-log10(alphas-1))/(log10(gammap)-log10(gammas));
n=ceil(oldn);
fprintf('n=%f, rounded to %d\n',oldn,n);
```

```
if BILINEAR
    gammaC=gammas/(10^(1/(2*n)*log10(alphas-1)));
else
    gammaC=gammap/(10^(1/(2*n)*log10(alphap-1)));
end

fprintf('Gamma C =%f\n',gammaC);

poles_of_hs=zeros(n,1);
for i=0:n-1
    poles_of_hs(i+1)=gammaC*exp(sqrt(-1)*(pi*(1+2*i+n)/(2*n)));
end

fprintf('POLES Of H(s)\n');
poles_of_hs

k=prod(-poles_of_hs);
den=poly(poles_of_hs);

fprintf('k=%d\n',k);

fprintf('H(s)=\n',k);


if BILINEAR
    hs=tf(k,den)
else
    hs=tf(T*k,den)
end


[r,p,k]=residue(k,den);
hzp=zeros(n,1);
for i=1:n
    hzp(i)=exp(p(i)*T);
end

fprintf('POLES Of H(z)\n');
hzp
```

```
[B,A]=residue(r,hzp);
fprintf('H(z)=\n',k);
hz=tf(T*B',A')
```

5.2 Sample run output

text output from run

```
***** STARTING DESIGN *******
Sampling frequency=20000.000000 Hz
freq at passband=2000.000000 Hz
freq at stopband=3000.000000 Hz
db at passband=-1.000000
db at stopband=-15.000000
Doing impulse invariance method
T=1.000000
n=5.885783, rounded to 6
Gamma C =0.703205
POLES Of H(s)

poles_of_hs =

         -0.182002858631059 +     0.679243915533887i
         -0.497241056902828 +     0.497241056902828i
         -0.679243915533887 +     0.182002858631059i
         -0.679243915533887 -     0.182002858631059i
         -0.497241056902828 -     0.497241056902828i
         -0.182002858631059 -     0.679243915533887i

k=1.209183e-001
H(s)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 92

Transfer function:

                            (0.1209-1.041e-016i)
-------------------------------------------------------------------------
s^6 + (2.717-4.441e-016i) s^5 + (3.691-1.998e-015i) s^4 + (3.179
```

```
       -2.22e-015i) s^3 + (1.825-1.554e-015i) s^2 + (0.6644-4.996e-016i) s

                                                  + (0.1209-1.041e-016i)


POLES Of H(z)

hzp =

          0.534553736986506 +     0.290115961427623i
          0.534553736986506 -     0.290115961427623i
          0.648579932539211 +     0.523670977796743i
          0.648579932539211 -     0.523670977796743i
          0.498626135868541 -    0.0917668874413562i
          0.498626135868543 +    0.0917668874413561i

H(z)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 107

Transfer function:

-(0.9938-0.8577i) s^4 - (0.22-0.703i) s^3 + (1.377+1.095i) s^2 - (0.6574

                                           +2.854i) s - (0.9146-1.114i)

-------------------------------------------------------------------------
-(0.6979+1.411i) s^4 + (0.6009-0.8998i) s^3 - (0.03618-0.9428i) s^2

                                    + (0.1901+0.7751i) s - (0.6018+0.246i)



============================================================

***** STARTING DESIGN *******
Sampling frequency=20000.000000 Hz
freq at passband=2000.000000 Hz
freq at stopband=3000.000000 Hz
```

```
db at passband=-1.000000
db at stopband=-15.000000
Doing Bilinear method
T=0.000050
n=5.304446, rounded to 6
Gamma C =15324.588619
POLES Of H(s)

poles_of_hs =

          -3966.29539304108 +      14802.4159246557i
          -10836.1205316146 +      10836.1205316146i
          -14802.4159246557 +      3966.29539304109i
          -14802.4159246557 -      3966.29539304109i
          -10836.1205316146 -      10836.1205316146i
          -3966.29539304108 -      14802.4159246557i

k=1.295188e+025
H(s)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 90

Transfer function:

                          (1.295e025-1.288e010i)
-------------------------------------------------------------------------
s^6 + (5.921e004-7.276e-012i) s^5 + (1.753e009-3.576e-007i) s^4

        + (3.29e013-0.01172i) s^3 + (4.117e017-224i) s^2 + (3.265e021

                                    -2.49e006i) s + (1.295e025-1.288e010i)


POLES Of H(z)

hzp =

          0.498385402712059 +     0.299971817994199i
           0.49838540271206 -     0.299971817994199i
          0.467705977269569 -    0.0939883945094121i
```

```
          0.605559876472429 +     0.553064536118524i
          0.467705977269569 +    0.0939883945094127i
           0.60555987647243 -     0.553064536118524i

H(z)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 107

Transfer function:

-(0.5053-1.947i) s^4 + (0.02086-1.348i) s^3 - (0.9771+0.04836i) s^2

                                + (0.01411+0.02526i) s - (0.3816-0.3931i)

-------------------------------------------------------------------------
(0.1988-1.53i) s^4 + (0.6374+0.9373i) s^3 - (1.064+0.2778i) s^2

                                  - (0.7076-0.6148i) s + (0.4667-0.6282i)




=================================================================

Sampling frequency=10000.000000 Hz
freq at passband=1000.000000 Hz
freq at stopband=2000.000000 Hz
db at passband=-3.000000
db at stopband=-10.000000
Doing impulse invariance method
T=1.000000
n=1.588388, rounded to 2
Gamma C =0.629065
POLES Of H(s)

poles_of_hs =

        -0.444816082052343 +     0.444816082052343i
        -0.444816082052343 -     0.444816082052343i
```

```
k=3.957227e-001
H(s)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 92

Transfer function:
               (0.3957-8.327e-017i)
-------------------------------------------------
s^2 + (0.8896-5.551e-017i) s + (0.3957-8.327e-017i)

POLES Of H(z)

hzp =

          0.578571949761589 +     0.275792192443573i
          0.578571949761589 -     0.275792192443573i

H(z)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 107

Transfer function:
(-6.9e-016+1.253i)

=====================================

Sampling frequency=10000.000000 Hz
freq at passband=1000.000000 Hz
freq at stopband=2000.000000 Hz
db at passband=-3.000000
db at stopband=-10.000000
Doing Bilinear method
T=0.000100
n=1.368163, rounded to 2
Gamma C =8389.390482
POLES Of H(s)

poles_of_hs =
```

```
          -5932.19490014964 +      5932.19490014964i
          -5932.19490014964 -      5932.19490014964i

k=7.038187e+007
H(s)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 90

Transfer function:
                 (7.038e007-2.235e-008i)
-----------------------------------------------------------
s^2 + (1.186e004-9.095e-013i) s + (7.038e007-2.235e-008i)

POLES Of H(z)

hzp =

          0.458140439181504 -     0.308891358305335i
          0.458140439181504 +     0.308891358305335i

H(z)=
Warning: Transfer function has complex coefficients.
> In tf.tf at 246
  In nma_filter at 107

Transfer function:
(2.058e-016-1.78i)

EDU>>
```

6 References

1. Alan Oppenheim, Ronald Schafer, Digital Signal Processing, Prentice-Hall, inc. 1975, Chapter 5.
2. Mostafa Shiva, Electrical engineering department, California state university, Fullerton, Lecture notes, handout H.
3. John Proakis, Dimitris Manolakis, digital signal processing, 3rd edition, section

8.3.