Note on how to calculate Discrete time Fourier transform for 2D data

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Given data

$$f(n,m) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

To find its DFT, we compute the DFT of each column at a time, which generates a temporary matrix. Then compute the DFT of each row of the temporary matrix. This gives the DFT of the above.

The DFT of 1D is given by

$$F[s] = \frac{1}{n} \sum_{r=1}^{n} f[r] \, e^{\frac{2\pi}{n} i((r-1)(s-1))}$$

Hence for first column in the data, which is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and using $n = 2$ in this example (same number of rows as columns). Then the above becomes

$$\begin{aligned} F[s] &= \frac{1}{2} \sum_{r=1}^{2} f[r] \, e^{\pi i((r-1)(s-1))} \\ &= \frac{1}{2} \left(f[1] + f[2] \, e^{\pi i(s-1)} \right) \\ &= \frac{1}{2} \left(1 + 3e^{\pi i(s-1)} \right) \end{aligned}$$

Therefore

$$\begin{aligned} F[s=1] &= \frac{4}{2} = 2 \\ F[s=2] &= \frac{1}{2} \left(1 + 3e^{\pi i} \right) = \frac{1}{2} (1 - 3) = -1 \end{aligned}$$

Hence the first column of the temporary matrix in F space is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Now we find the DFT of the second column of the input which is $\begin{pmatrix}2\\4\end{pmatrix}$. we have (since $n=2$ in this example)

$$\begin{aligned}
F[s] &= \frac{1}{2}\sum_{r=1}^{2} f[r]\, e^{\frac{2\pi}{2} i((r-1)(s-1))} \\
&= \frac{1}{2}\left(f[1] + f[2]\, e^{\pi i(s-1)}\right) \\
&= \frac{1}{2}\left(2 + 4e^{\pi i(s-1)}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
F[s=1] &= \frac{1}{2}(2+4) = 3 \\
F[s=2] &= \frac{1}{2}\left(2+4e^{\pi i}\right) = \frac{1}{2}(2-4) = -1
\end{aligned}$$

Hence the second column of the temporary matrix in F space is $\begin{pmatrix}3\\-1\end{pmatrix}$ which means after first pass, the temporary matrix in F space is now

$$\begin{pmatrix}2 & 3\\ -1 & -1\end{pmatrix}$$

Now we apply DFT to each row of the above. This is the second pass. For the first row of the above, the DFT is

$$\begin{aligned}
F[s] &= \frac{1}{2}\sum_{r=1}^{2} f[r]\, e^{\frac{2\pi}{2} i((r-1)(s-1))} \\
&= \frac{1}{2}\left(f[1] + f[2]\, e^{\pi i(s-1)}\right) \\
&= \frac{1}{2}\left(2 + 3e^{\pi i(s-1)}\right)
\end{aligned}$$

Therefore the DFT of the first row becomes

$$\begin{aligned}
F[s=1] &= \frac{1}{2}(2+3) = 2.5 \\
F[s=2] &= \frac{1}{2}\left(2+3e^{\pi i}\right) = \frac{1}{2}(2-3) = -0.5
\end{aligned}$$

Which is $\begin{pmatrix}2.5 & -0.5\end{pmatrix}$, and the DFT of the second is

$$\begin{aligned}
F[s] &= \frac{1}{2}\sum_{r=1}^{2} f[r]\, e^{\frac{2\pi}{2} i((r-1)(s-1))} \\
&= \frac{1}{2}\left(f[1] + f[2]\, e^{\pi i(s-1)}\right) \\
&= \frac{1}{2}\left(-1 - e^{\pi i(s-1)}\right)
\end{aligned}$$

which is

$$F[s=1] = \frac{1}{2}(-1-1) = -1$$
$$F[s=2] = \frac{1}{2}\left(-1-e^{\pi i}\right) = \frac{1}{2}(-1+1) = 0$$

Which is $\begin{pmatrix}-2 & 1\end{pmatrix}$. Therefore the final DFT is

$$\begin{pmatrix}2.5 & -0.5\\ -1 & 0\end{pmatrix}$$