UNIVERSITY OF LONDON

General Certificate of Education

A level Mathematics (Pure and Applied)

with answers

Twenty First Edition

January 1972 to Summer 1973

Published by the University of London

Price 30p

SBN 7187 0288 3

UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

Advanced Level

MATHEMATICS (PURE AND APPLIED)

INSTRUCTIONS FOR EACH PAPER

THREE hours are allowed.

EIGHT questions are to be answered.

JANUARY 1972

PAPER 1

PURE MATHEMATICS

1. (i) If a is a positive constant, find the set of values of x for which $a(x^2 + 2x - 8)$ is negative. Find the value of a if this function has a minimum value of -27.

(ii) Find two quadratic functions of x which are zero at $x = 1$, which take the value 10 when $x = 0$ and which have a maximum value of 18. Sketch the graphs of these two functions.

2. (i) Solve the simultaneous equations

$$\log(x - 2) + \log 2 = 2\log y,$$
$$\log(x - 3y + 3) = 0.$$

(ii) Show that there are 126 ways in which 10 children can be divided into two groups of 5. Find the number of ways in which this can be done

(*a*) if the two youngest children must be in the same group,

(*b*) if they must not be in the same group.

PAPER II

APPLIED MATHEMATICS

1. The points A,B,C,D,E,F are the vertices of a regular hexagon. Forces each of 2 newtons act along AB and DC, and forces each of 1 newton act along BC and ED, in the directions indicated by the order of the letters. Forces P newtons and Q newtons act along EF and AF respectively. Find P and Q
   (a) if the system reduces to a couple,
   (b) if the resultant of the system is a force acting along EB.

2. A uniform rod AB of length $2a$ and weight W rests with the end A on rough horizontal ground, and the end B against a rough vertical wall. The rod is in a vertical plane perpendicular to the wall, and is inclined at an angle θ to the horizontal, where $\tan\theta = 2$. When a weight $2W$ is hung from a point P on the rod, where $AP = 3a/2$, the rod is in limiting equilibrium. If μ is the coefficient of friction at A and at B, show that $\mu = 0{\cdot}32$ approximately.

3. Show that the distance of the centroid of a uniform circular sector AOB from the centre O is $(2a\sin\theta)/3\theta$, where 2θ is the angle AOB and a is the radius.
   Find the distance from O of the centroid of the segment of which AB is the chord, given that $\theta = \pi/6$. If a uniform lamina in the shape of this segment hangs at rest freely suspended from A, show that the tangent of the angle which AB makes with the downward vertical equals
$$(11 - 2\pi\sqrt{3})/(2\pi - 3\sqrt{3}).$$

4. A car of mass 1 000 kg has a maximum speed of 15 m/s up a slope inclined at an angle θ to the horizontal where $\sin\theta = 0{\cdot}2$. There is a constant frictional resistance equal to one tenth of the weight of the car. Find the maximum speed of the car on a level road.
   If the car descends the same slope with its engine working at half its maximum power, find the acceleration of the car at the moment when its speed is 30 m/s.

5. A motorboat moving at 8 km/h relative to the water travels from a point A to a point B 10 km distant whose bearing from A is 150° (i.e. N150°E). It then travels to a point C 10 km from B and due west of B. If there is a current of constant speed 4 km/h from north to south, find the two courses to be set, and prove that the total time taken to reach C is approximately 2 hours 20 minutes.

6. A smooth sphere A of mass $2m$, moving on a horizontal plane with speed u, collides directly with another smooth sphere B of equal radius and of mass m which is at rest. If the coefficient of restitution between the spheres is e, find their speeds after impact.
   The sphere B later rebounds from a perfectly elastic vertical wall, and then collides directly with A. Prove that after this collision the speed of B is $(2/9)(1 + e)^2u$, and find the speed of A.

7. A particle is projected from the origin O with velocity V at an elevation θ to the horizontal. Show that its height y above O when it has travelled a distance x horizontally is given by
$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}.$$
   A ball thrown from O with speed 1 400 cm/s is caught at a point P, which is 1 000 cm horizontally from O and 187·5 cm above the level of O. Find the two possible angles of projection. If the ball is thrown from O with the same initial speed to pass through a point 562·5 cm vertically above P, show that there is only one possible angle of projection.

8. Two particles P and Q, connected by light inextensible strings AP and AQ to a fixed point A, are moving in the same sense in two horizontal circles. The strings AP and AQ are each of length 80 cm and they make angles of 45° and 60° respectively with the downward vertical. At a certain instant the strings are in the same vertical plane; show that they will next both be in the same vertical plane after an interval of nearly 4 seconds.

9. A particle moves along a horizontal straight line with acceleration proportional to $\cos \pi t$, where t is the time. When $t = 0$ the velocity of the particle is u, and when $t = \frac{1}{2}$ its velocity is $2u$. Find the distance that the particle has travelled when $t = 2$, and draw the velocity-time graph for the interval $0 \leqslant t \leqslant 2$.

10. One end of a light elastic string of natural length 50 cm and modulus mg is attached to a point A on the ceiling of a room of height 300 cm. An identical string is attached at one end to the floor at a point B vertically beneath A. The free ends of the strings are attached to a particle, which can rest in equilibrium at a point 200 cm below A. Find the mass of the particle, and prove that if it is given a small vertical displacement from the position of equilibrium the period of its oscillations will be 1·4 seconds approximately.

SUMMER 1972

PAPER I

PURE MATHEMATICS

1. (i) If $f(x) \equiv 9 + 2(k + 4)x + 2kx^2$ $(k \neq 0)$,

(*a*) find the set of values of k for which $f(x)$ is positive for all real values of x,

(*b*) form the equation whose roots are the reciprocals of the roots of $f(x) = 0$.

(ii) Solve the simultaneous equations

$$\log_y x = 2, \quad 5y = x + 12 \log_x y.$$

2. (i) In how many different ways can the letters of the word MATHEMATICS be arranged? In how many of these arrangements will two A's be adjacent? Find the number of arrangements in which all the vowels come together.
(Answers should be left in factorial form.)

(ii) Show by induction, or otherwise, that 3 is a factor of $n(n^2 + 2)$, where n is a positive integer.

3. (i) Express $(1 + x)^{\frac{1}{2}}$ and $(1 - x)^{-2}$ as series of ascending powers of x, in each case up to and including the term in x^2. Hence show that, if x is small,

$$(1 + x)^{\frac{1}{2}}/(1 - x)^2 = 1 + \frac{5}{2}x + \frac{31}{8}x^2 + \ldots .$$

Use this latter expansion to calculate, to 2 decimal places, the percentage change which occurs in the value of $a^{\frac{1}{2}}/b^2$ if a is increased by 1% and b is decreased by 1%.

(ii) The expression $y = px^2 + qx$ is an approximation to the relation connecting two variables x and y. Using the values given in the following table draw a suitable graph and from it estimate each of p and q to the nearest whole number.

x	1	2	3	4	5	6
y	74	126	162	172	175	144

4. (i) If $z = 3 - 4i$, express z^2 and $1/z$ in the form $a + bi$ where a and b are real and represent them in an Argand diagram. If $w^2 = z$, express the two values of w in the form $a + bi$.

(ii) State de Moivre's theorem for a positive integral power and use it to express $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin \theta$ and $\cos \theta$ respectively. Hence show that, if $\sin 3\theta + \cos 3\theta = 0$, either $\tan \theta = 1$ or $\sin 2\theta = -\frac{1}{2}$.

5. A triangle has its vertices at A (4, 4), B (−4, 0), C (6, 0).

(*a*) Find the equation of the circle through the points A, B, C.

(*b*) Find the coordinates of the point where the internal bisector of the angle BAC meets the x-axis.

(*c*) Find the equation of the circle which passes through B and touches AC at C.

6. Find the equations of the tangent and normal at the point $P\,(t^2, 2t)$ to the parabola $y^2 = 4x$.
The tangent at P meets the x-axis at A and the normal at P meets the x-axis at B. The perpendicular from P to the y-axis meets this axis at L. If O is the origin, show that $OALP$ is a parallelogram.
If the areas of $OALP$ and the triangle OPB are equal, find the coordinates of the possible positions of P.

7. (i) Show that $\cos^6 x + \sin^6 x = 1 - \frac{3}{4}\sin^2 2x$.
(ii) Solve, for $0° \leqslant x \leqslant 180°$, the equation
$$\sin x + \sin 5x = \sin 3x.$$
(iii) Find the general solution of the equation
$$3\cos x + 4\sin x = 2,$$
giving the answer in degrees and minutes.

8. (i) Differentiate (*a*) $(1 + x)^2/(1 + x^2)$, (*b*) $\ln\sqrt{(1 + \sin^2 x)}$, simplifying the answers where possible.
(ii) If $(1 + x)(2 + y) = x^2 + y^2$, find dy/dx in terms of x and y. Find the gradient of the curve $(1 + x)(2 + y) = x^2 + y^2$ at each of the two points where the curve cuts the y-axis. Show that there are two points at which the tangents to this curve are parallel to the y-axis.

9. (i) Find the minimum value of $e^{-x}\sin x$ for $0 \leqslant x \leqslant 2\pi$.
(ii) An isosceles triangle is inscribed in a circle of fixed radius r. Show that the area, Δ, of this triangle is given by $\Delta = r^2(1 + \cos\theta)\sin\theta$, where θ is the angle between the equal sides. Find the greatest possible value of Δ.

10. (i) Evaluate
$$(a)\quad \int_0^{\frac{1}{4}\pi} \frac{1 + \tan^2 x}{1 + \tan x}\,dx,$$
$$(b)\quad \int_1^2 \frac{x^3 + x^2 + x + 1}{x^2 + 1}\,dx.$$
(ii) Sketch the curve $2y = x(x - 4)(2x - 5)$.
The line $y = x$ cuts this curve at the origin O and at A and B, where A is between O and B. Find the area bounded by the arc OA of the curve and the line $y = x$.

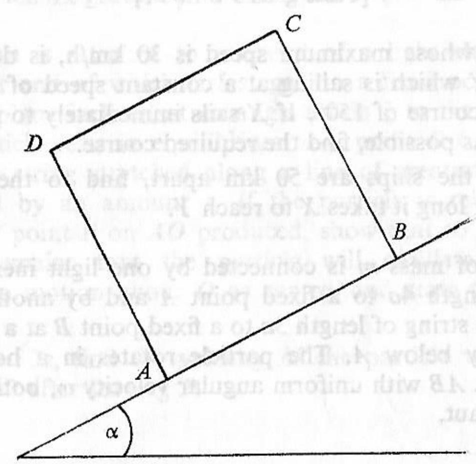
Paper II

Applied Mathematics

1. In a triangle ABC the mid-points of BC, CA, AB are O, P, Q respectively. Express $\overrightarrow{OA} + \overrightarrow{OC}$ in terms of $\overrightarrow{OP}$ and show that $\overrightarrow{OA} - \overrightarrow{OC} = 2\,\overrightarrow{OQ}$.
In a plane quadrilateral $EFGH$ the mid-points of EG, FH are X, Y respectively. Prove that the resultant of forces represented completely by $\overrightarrow{EF}$, $\overrightarrow{GF}$, $\overrightarrow{EH}$ and $\overrightarrow{GH}$ is represented completely by $4\,\overrightarrow{XY}$.

2. A uniform rod AB of length $2a$ and mass M is freely hinged to a fixed point A and is held inclined at an angle of 30° to the horizontal (with B above A) by a light inextensible string attached to the mid-point of the rod and to a point C vertically above A, where $AC = a$. Find the magnitude and direction of the reaction at A. Find also the tension in the string.
If a particle of mass M is now attached at B, show that the tension in the string is trebled.

3.



A uniform cube of weight W is placed as shown in the figure on a rough plane of inclination $\alpha\left(<\frac{\pi}{4}\right)$, the centre of mass of the cube lying in the plane $ABCD$ and the edges perpendicular to this plane being horizontal. If the coefficient of friction between the cube and the plane is μ, show that the cube cannot remain in equilibrium unless $\mu \geqslant \tan \alpha$.

If $\tan \alpha = 1/2$, $\mu = 2/3$ and a horizontal force P, steadily increasing in magnitude from zero, is applied at D (acting from left to right and with line of action lying in the plane $ABCD$) show that equilibrium will be broken by the cube turning about the edge through B before it slides up the plane.

4. $ABCD$ is a uniform square metal plate of side 3 m. Points E, F are taken on AB, BC respectively such that $BE = BF = x$ m and the portion BEF is removed. Find the distance of the centroid of the remainder from AD. Show that the remainder cannot stand in equilibrium on AE as base with AD vertical unless

$$2x^3 - 54x + 81 \geqslant 0 .$$

If the mass of the remainder is 14 kg, find in newtons the least horizontal force applied at C required to maintain it in equilibrium in this position when $x = 2$.

[Take g as $9{\cdot}8$ m/s^2.]

5. A ship X, whose maximum speed is 30 km/h, is due south of a ship Y which is sailing at a constant speed of 36 km/h on a fixed course of 150°. If X sails immediately to intercept Y as soon as possible, find the required course.

If initially the ships are 50 km apart, find to the nearest minute how long it takes X to reach Y.

6. A particle of mass m is connected by one light inextensible string of length $4a$ to a fixed point A and by another light inextensible string of length $3a$ to a fixed point B at a distance $5a$ vertically below A. The particle rotates in a horizontal plane about AB with uniform angular velocity ω, both strings remaining taut.

Show that

$$\omega^2 \geqslant \frac{5g}{16a} .$$

If either string will break when subjected to a tension exceeding $6\,mg$, show further that

$$\omega^2 \leqslant \frac{55g}{16a} .$$

7. A lorry of mass 10 000 kg has a maximum speed of 24 km/h up a slope of 1 in 10 against a resistance of 1200 newtons. Find the effective power of the engine in kilowatts.

If the resistance varies as the square of the speed, find the maximum speed on the level to the nearest km/h.

[Take g as $9{\cdot}8$ m/s^2.]

8. A particle is projected with speed V at an angle α to the horizontal. Show that its greatest height above the point of projection during its flight is $(V^2 \sin^2\alpha)/(2g)$.

A ball is projected from a point at a height a above horizontal ground, with speed V at an angle α to the horizontal. At the highest point of its flight it impinges normally on a vertical wall and rebounds.

Show that the horizontal distance from the point of projection to the wall is $(V^2 \sin \alpha \cos \alpha)/g$ and that the time taken by the ball to reach the ground after the impact is $\sqrt{(V^2 \sin^2 \alpha + 2ga)}/g$.

9. A particle is attached to one end of a light elastic string, the other end of which is fastened to a fixed point A on a smooth plane inclined at an angle $\arcsin \frac{1}{4}$ to the horizontal. The particle rests in equilibrium at a point O on the plane with the string stretched along a line of greatest slope and extended by an amount c. If the particle is released from rest at a point P on AO produced, show that so long as the string remains taut the particle will oscillate in simple harmonic motion about O as centre, and state the periodic time.

If $OP = 2c$, find the velocity of the particle when it first reaches O after leaving P.

10. A particle starts from rest at time $t = 0$ and moves in a straight line with variable acceleration f m/s², where

$$f = \frac{t}{5} \quad (0 \leqslant t \leqslant 5),$$

$$f = \frac{t}{5} + \frac{10}{t^2} \quad (t \geqslant 5),$$

t being measured in seconds. Show that the velocity is $2\frac{1}{2}$ m/s when $t = 5$ and 11 m/s when $t = 10$.

Show also that the distance travelled by the particle in the first 10 seconds is $\left(43\frac{1}{3} - 10 \ln 2\right)$ m.

JANUARY 1973

Paper I

Pure Mathematics

1. (i) Prove that, if k is a real constant and $k \neq -1$, then if $b = 0$ the roots of the equation

$$x^2 + bx - 2 + k(x^2 + 3x + 2) = 0$$

are real and distinct for all values of k. If $b = 3$, find for what values of k the roots are real.

(ii) If α and β are the roots of $x^2 + px + q = 0$, obtain the equation with roots $2\alpha - 1, 2\beta - 1$.

2. (i) Solve for real x the equation

$$4(3^{2x+1}) + 17(3^x) - 7 = 0.$$

(ii) If s and t are positive numbers other than 1, prove that

(a) $\log_s t + \log_{1/s} t = 0$,

(b) $\log_s t = 1/\log_t s$.

3. (i) Find the modulus and one value for the argument of $(i + 1)^2/(i - 1)^4$.

(ii) Find the two square roots of $5 - 12i$ in the form $a + bi$ where a and b are real. Show the points P and Q representing the square roots in an Argand diagram. Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1, PQR_2 are equilateral.

4. (i) A tennis club is to select a team of three pairs, each pair consisting of a man and a woman, for a match. The team is to be chosen from 7 men and 5 women. In how many different ways can the three pairs be selected?

(ii) Prove by induction or otherwise that

$$\sum_{r=2}^{n} (r-1)r(r+2) = \frac{1}{12}(n-1)n(n+1)(3n+10).$$

5. Assuming that $|x| < 1$, expand

$$\frac{1-x}{(1+x)^2} + \frac{1+x}{(1-x)^2}$$

in ascending powers of x as far as and including the term in x^6. Find the coefficient of x^{2n} in the expansion.

By giving a suitable value to x, find

$$\sum_{n=0}^{\infty} \frac{4n+1}{2^{2n}}.$$

6. Find in radians the general solutions of the equations

(a) $\cos x = \sin 3x$,

(b) $\cos x + \cos 7x = \cos 4x$,

(c) $\sin x - \cos x = 1$.

7. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 1 = 0$. Find the area of the square.

Also find the equations of the tangents to the circle from the origin.

8. The parabolas $x^2 = 4ay$ and $y^2 = 4ax$ meet at the origin and at the point P. The tangent to $x^2 = 4ay$ at P meets $y^2 = 4ax$ again at A, and the tangent to $y^2 = 4ax$ at P meets $x^2 = 4ay$ again at B. Prove that the angle APB is $\arctan(3/4)$, and that AB is a common tangent to the two parabolas.

9. (i) Differentiate with respect to x

$$(a)\ \frac{2-x}{1-2x},\quad (b)\ \sec^2 2x,\quad (c)\ \ln(1+e^{3x}).$$

(ii) Prove that the curve $y = 4x^5 + kx^3$ has two turning points when $k < 0$ and none when $k \geqslant 0$. Find the turning points when $k = -5/3$, and distinguish between them. Sketch the curve for this value of k.

10. Sketch the curve with equation $y = x - \dfrac{1}{x}$.

The area bounded by the curve, the x-axis and the lines $x = 2$, $x = 3$ is rotated through 2π radians about the x-axis. Calculate the volume of the solid of revolution so formed.

PAPER II

APPLIED MATHEMATICS

1. Show that a given force P is equivalent to an equal like parallel force, acting at a point not on the line of action of P, together with a couple.

$ABCD$ is a square of side 2 metres and forces of magnitude 1,3,3,5 newtons act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ respectively. Prove that these forces are equivalent to a single force. Find the magnitude of this force and the point where its line of action cuts BA produced.

If now an additional force of magnitude $2\sqrt{2}$ newtons acts along $\overrightarrow{AC}$, prove that the system reduces to a couple. Find the magnitude of this couple.

2. Two uniform ladders AB and BC, each of length $2a$ and weight W, rest in a vertical plane with their upper ends smoothly hinged at B and their lower ends A and C resting on a rough horizontal plane. If the angle $BAC = 60°$ and the coefficient of friction between the lower ends of the ladders and the ground is μ, show that equilibrium is possible when $\mu \geqslant 1/(2\sqrt{3})$.

A man of weight $2W$ now climbs up the ladder AB. Assuming that equilibrium is maintained, show that when he is distant xa from A, where $0 \leqslant x < 2$, the vertical component of the reaction at C is $(2 + x)\ W/2$. Determine also the horizontal component of the reaction at C. Hence find the greatest possible value which x can take without disturbing equilibrium, when $\mu = 5/(8\sqrt{3})$.

3. A heavy uniform circular disc centre X and radius R has a circular hole centre Y and radius r cut from it, where $r < R$. If $XY = R - r$, and the centre of gravity of the crescent-shaped lamina is at a distance $\dfrac{4r}{9}$ from X, show that $R = \dfrac{5r}{4}$.

The lamina is now suspended from a point on its outer rim lying on the perpendicular to XY through X. Find the angle which XY makes with the vertical. If the weight of the crescent is W, find the smallest weight which must be attached to the lamina to maintain XY in a horizontal position.

4. A speed boat which can travel at 20 knots in still water starts from the corner X of an equilateral triangle XYZ of side 10 nautical miles and describes the complete course $XYZX$ in the least possible time. A tide of 5 knots is running in the direction $\overrightarrow{ZX}$.

Find:

(a) the speed along XY,

(b) to the nearest minute, the time taken by the speedboat to traverse the complete course $XYZX$.

5. A particle P of mass m rests on a smooth horizontal table and is attached to fixed points A, B on the table by two light elastic strings PA, PB. The distance $AB = 6l$ and the natural lengths of the strings PA, PB are $2l$ and l respectively. The moduli of the strings are such that a force mg will extend the first string by $2l$ and the second string by l. Show that in the equilibrium position the string PA is of length $4l$.

The particle is now held at C, the mid-point of AB. Prove that, when it is released from rest, its motion is simple harmonic. Determine the period and amplitude of its oscillation.

6. Define the kinetic energy and potential energy of a particle, giving S.I. units in each case.
A particle X of mass m, moving with constant speed on the inner surface of a fixed smooth hemispherical bowl of radius r, describes a horizontal circle at a depth of $r/2$ below the centre of the bowl. Prove that its velocity is $\sqrt{(3gr/2)}$.
A second particle Y of mass $2m$ is moving in a similar manner in a horizontal circle at a depth of $r/3$ below the centre of the bowl. Calculate the difference in potential energy and the difference in kinetic energy between X and Y.

7. A golf ball, initially at rest, is dropped on to a horizontal surface and bounces directly up again with velocity v. If the coefficient of restitution between the ball and the surface is e, show that the ball will go on bouncing for a time $\frac{2v}{g(1-e)}$ after the first impact.
[You may assume $1 + e + e^2 + e^3 + \ldots = (1-e)^{-1}$.]
If the golf ball is dropped from a height of 19·62 m and comes to rest 12 seconds later, find the value of e.
[Take g as 9·81 m/s².]

8. A pump raises water from a depth of 10 m and discharges it horizontally through a pipe of 0·1 m diameter at a velocity of 8 m/s. Calculate the work done by the pump in one second.
If the water impinges directly with the same velocity on a vertical wall, find the force exerted by the water on the wall if it is assumed that none of the water bounces back.
[Take g as 9·81 m/s², π as 3·142 and the mass of 1 m³ of water as 1 000 kg.]

9. Three perfectly elastic uniform spheres of equal size and masses $2m$, m and $3m$ lie at rest at points A, B and C on a straight line ABC on a horizontal table. The sphere at A is projected directly towards B with velocity v and after collision the sphere at B moves directly towards C and collides with the sphere at C. Prove that after the second collision the kinetic energy of the sphere of mass m is twice that of the sphere of mass $2m$ and find their relative velocity. Prove that the sphere of mass $3m$ will experience only one collision.

10. A particle is to be projected under gravity from O, the centre of a horizontal circle of radius 70 m. Calculate the minimum velocity of projection of the particle if its trajectory is to pass through the circumference of the circle.
Another particle is projected from O with a velocity of 108 km/h at an angle of 40° to the horizontal. This particle just clears the top of a vertical post on the circumference of the circle. Find, to the nearest tenth of a metre, the height of the post.
[Take g as 9·81 m/s².]

SUMMER 1973

Paper I

Pure Mathematics

1. (i) Given that x is real and that
$$y = \frac{x^2+2}{2x+1},$$
show that y cannot lie between -2 and 1.
(ii) Find the real values of x for which
$$\log_3 x - 2\log_x 3 = 1.$$

2. (i) A committee of three people is to be chosen from four married couples. Find in how many ways this committee can be chosen
(*a*) if all are equally eligible,
(*b*) if the committee must consist of one woman and two men,
(*c*) if all are equally eligible except that a husband and wife cannot *both* serve on the committee.

(ii) Prove, by induction or otherwise, that
$$\sum_{r=1}^{n} r\,2^{r-1} = 1 + (n-1)2^n.$$

3. (i) Given that the first four terms in the expansion of $(1 + ax)^n$ in ascending powers of x are

$$1 + 6x + 24x^2 + cx^3,$$

calculate the values of the constants a, n and c.

(ii) Find constants p and q such that

$$x^2 + 6x + 10 = (x + p)^2 + q$$

for all values of x. Deduce that, if x is real,

$$0 < \frac{1}{x^2 + 6x + 10} \leqslant 1.$$

4. (i) If $z_1 = 1 - i$ and $z_2 = 7 + i$, find the modulus of

(*a*) $z_1 - z_2$,

(*b*) $z_1 z_2$,

(*c*) $\dfrac{z_1 - z_2}{z_1 z_2}$.

(ii) Sketch on an Argand diagram the locus of a point P representing the complex number z, where

$$|z - 1| = |z - 3i|,$$

and find z when $|z|$ has its least value on this locus.

5. Show that the line

$$(2x - y - 3) + k(x - y - 1) = 0$$

passes through the point P of intersection of the lines

$$2x - y - 3 = 0, \quad x - y - 1 = 0$$

for all values of k.

Show that P lies outside the circle

$$x^2 + y^2 + 4x - 6y + 11 = 0$$

and find the values of k for which the line is a tangent to the circle. Obtain the equations of the two tangents from P to the circle.

6. Find the equation of the tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$.

The tangent and the normal to the parabola at P meet the x-axis at T and G respectively and M is the foot of the perpendicular from P to the line $x = -a$; S is the point $(a, 0)$. Show that $ST = PM = SP$ and deduce that PT bisects the angle SPM. Show also that PG bisects the angle between PS and MP produced.

7. (i) Find the values of θ between 0 and 2π for which

$$\sin 2\theta = \sin \frac{\pi}{6}.$$

(ii) Show that

$$(2 \cos \phi + 3 \sin \phi)^2 \leqslant 13$$

for all values of ϕ.

(iii) The sides BC, CA, AB of a triangle ABC are of length $x + y$, x, $x - y$ respectively. Show that

$$\cos A = \frac{x - 4y}{2(x - y)}.$$

Show also that

$$\sin A - 2 \sin B + \sin C = 0.$$

8. (i) Differentiate with respect to x each of the following functions

(*a*) $e^{2x} \cos(\pi x)$,

(*b*) $\ln\left(\dfrac{x^2 + 1}{2x + 1}\right)$,

(*c*) $\sqrt{(1 + 4x^2)}$.

(ii) Show that, for $x > 0$, the function $(\ln x)/x$ has a maximum at $x = e$ and no other turning value.

9. (i) Evaluate the following integrals:

(*a*) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$,

(*b*) $\displaystyle\int_0^1 \frac{x^2 \, dx}{x + 1}$.

(ii) The function $f(x)$ is defined for $0 \leqslant x \leqslant 2$ by

$$f(x) = x \quad \text{for } 0 \leqslant x \leqslant 1,$$
$$f(x) = (2 - x)^2 \text{ for } 1 < x \leqslant 2.$$

Sketch the graph of this function for $0 \leqslant x \leqslant 2$ and find

$$\int_0^2 f(x) \, dx.$$

10. (i) Calculate the area of the region in the first quadrant enclosed by the curve $y = \tan x$, the line $x = \pi/4$ and the x-axis. Calculate also the volume generated when this region is rotated through 2π radians about the x-axis.

(ii) The curve whose equation is

$$y = e^x(ax^2 + bx + c)$$

where a, b, c are constants, is such that its tangents are parallel to the x-axis at $x = 1$ and $x = 3$ and the curve cuts the y-axis where $y = 9$. Calculate a, b and c.

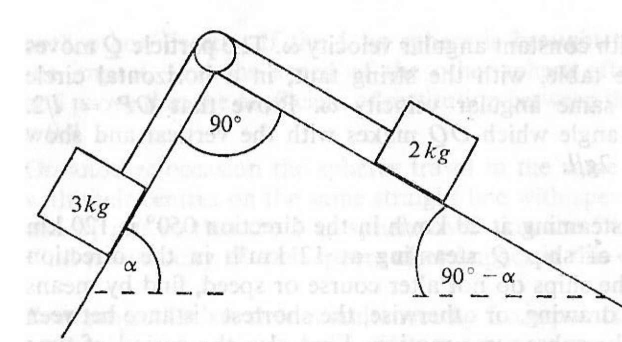
PAPER II

APPLIED MATHEMATICS

1. Forces of magnitude $2P$, P, $2P$, $3P$, $2P$ and P act along the sides AB, BC, CD, ED, EF and AF respectively of a regular hexagon of side $2a$ in the directions indicated by the letters. Prove that this system of forces can be reduced to a single force of magnitude $2P\sqrt{3}$ acting along AC together with a couple. Find the magnitude of the couple.

Show that the system can be reduced to a single force without a couple. If the line of action of this force cuts FA produced at X, calculate the length of AX.

2. The figure shows a block of mass 3 kg resting on a rough plane inclined at an angle of $\alpha\left(\text{where } \tan\alpha = \frac{4}{3}\right)$ to the horizontal. This block is connected by a light inextensible string which passes over a smooth pulley to a block of mass 2 kg resting on an equally rough plane inclined at an angle of $(90° - \alpha)$ to the horizontal. Both parts of the string lie in a vertical plane which meets each of the inclined planes in a line of greatest slope. If the 3 kg block just begins to slide down the plane on which it rests and μ is the coefficient of friction between the blocks and the planes, show that $\mu = 6/17$. Calculate also the least additional mass that must be attached to the 2 kg block in order that the 3 kg block should just begin to slide up the plane.



3. Two fixed ponts A and B on the same horizontal level are 20 cm apart. A light elastic string, which obeys Hooke's law, is just taut when its ends are fixed at A and B. A block of mass 5 kg is attached to the string at a point P where $AP = 15$ cm. The system is then allowed to take up its position of equilibrium with P below AB and it is found that in this position the angle APB is a right angle. If $\angle BAP = \theta$, show that the ratio of the extensions of AP and BP is

$$\frac{4\cos\theta - 3}{4\sin\theta - 1}.$$

Hence show that θ satisfies the equation

$$\cos\theta\,(4\cos\theta - 3) = 3\sin\theta\,(4\sin\theta - 1)\,.$$

4. A uniform right circular solid cone is of height h and the radius of its base is r. Prove that its centre of mass is at a distance $3h/4$ from the vertex.

The cone is made of the same material as a uniform solid cylinder of radius r and height $2h/3$. The circular bases of the cone and cylinder are joined together so that their centres coincide and when the resulting solid is freely suspended from a point on the circular edge of the join, the uppermost slant edge of the cone is horizontal. Show that $5h^2 = 36r^2$.

5. Particles P and Q of equal mass are connected by a light inelastic string of length l threaded through a small hole O in a smooth horizontal table. The particle P is free to move on the table and describes a horizontal circle so that OP

rotates with constant angular velocity ω. The particle Q moves below the table, with the string taut, in a horizontal circle with the same angular velocity ω. Prove that $OP = l/2$. Find the angle which OQ makes with the vertical and show that $\omega^2 > 2g/l$.

6. A ship P steaming at 20 km/h in the direction 050° is 120 km due west of ship Q steaming at 12 km/h in the direction 330°. If the ships do not alter course or speed, find by means of a scale drawing, or otherwise, the shortest distance between them in the subsequent motion. Find also the period of time during which the ships are within a range of 50 km of each other.

7. A particle of mass 4 kg is attached to one end X of a light inextensible string which passes over a smooth light pulley and supports particles of masses 2 kg and 3 kg at the other end Y. The end X is held in contact with a horizontal table at a depth 6 m below the pulley, both portions of the string being vertical and the particles at Y hanging freely. The system is released from rest. When Y has descended a distance of 2·5 m, the particle of mass 2 kg is disconnected and begins to fall freely. Calculate the greatest height reached by X above the table and the momentum of the 4 kg particle when it strikes the table.

[Take g to be 9·81 m/s².]

8. A car of mass 1000 kg whose maximum power is constant at all speeds experiences a constant resistance R newtons. If the maximum speed of the car on the horizontal is 120 km/h and the maximum speed up a slope of angle θ where $\sin\theta = 1/100$ is 60 km/h, calculate the power of the car.

Calculate also the maximum speed of the car (*a*) on the horizontal and (*b*) up the slope when it is pulling a caravan of mass 1000 kg if the total resistance to the motion of the car and the caravan is $3R$ newtons.

[Take g to be 9·81 m/s².]

9. Two spheres of masses 2 kg and 5 kg travel towards each other, with their centres on the same straight line, with speeds 10 m/s and $6\frac{2}{3}$ m/s respectively. The spheres strike each other directly. If the 5 kg sphere is brought to rest by the impact, find the speed of the other sphere after impact and prove that the coefficient of restitution between the spheres is 0·4.

On another occasion the spheres travel in the same direction with their centres on the same straight line with speeds 10 m/s and $6\frac{2}{3}$ m/s so that the 2 kg sphere overtakes the 5 kg sphere. Find the velocity of each sphere after impact in this case.

10. Two projectiles are fired simultaneously from points P and Q on horizontal ground and collide head on when travelling horizontally. The first projectile is fired with speed v m/s at an angle of elevation α and the second is fired with speed $v/2$ m/s at an angle of elevation β. If $PQ = \dfrac{v^2 \sin\beta}{2g}$, show that

(*a*) $2\sin\alpha = \sin\beta$,

(*b*) $2 = 2\cos\alpha + \cos\beta$.

Hence show that $\cos\alpha = \dfrac{7}{8}$.

ANSWERS

JANUARY 1972

Paper I

1. (i) $-4 < x < 2; a = 3$;
(ii) $-2x^2 - 8x + 10, -50x^2 + 40x + 10$.

2. (i) $x = 4, y = 2$ or $x = 10, y = 4$;
(ii) (*a*) 56, (*b*) 70.

3. (i) $2\sqrt{2}$, 135°; $2\sqrt{2}$, $-135°$; $\frac{1}{2}$i.

4. (i) -2048;
(ii) $r = \sqrt{10}$, $\alpha = 18° 26'$; $n(360°) + 53° 8'$, $n(360°) - 90°$.

5. (i) (*a*) 0·994027, (*b*) 2·004994;
(ii) $a = \frac{1}{2}, b = 12$.

6. (i) 1/3.

7. $x \cos\theta + y \sin\theta = 2 + 2\cos\theta$; $(-2, 0)$; $x^2 + y^2 = 2x + 2$.

9. (i) $-2/(1 + \sin 2x)$;
(ii) (*a*) Minimum at (1, 0), maximum at (−1, 4);
(*b*) Minimum at (1, 0), maximum at (−3, 256).

10. (i) (*a*) $\frac{1}{2}x^2 - 2x + 4\ln(x + 2)$, (*b*) $-(\cos 5x + 5\cos x)/10$;
(ii) $x = \ln 3$, $y = 3$; $2 \ln 3$.

Paper II

1. (*a*) 4, 5.
(*b*) 4, 4.

3. $a/(2\pi - 3\sqrt{3})$.

4. 45 m/s, 1·715 m/s².

5. 135° 31′, 300°.

6. $\dfrac{u(2 - e)}{3}$, $\dfrac{2u(1 + e)}{3}$, $-\dfrac{u}{9}\left[e^2 + 8e - 2\right]$.

7. 26° 34′, 74° 03′.

9. $2u$.

10. $2m$.

SUMMER 1972

Paper I

1. (i) (*a*) $2 < k < 8$, (*b*) $9x^2 + 2(k + 4)x + 2k = 0$;
(ii) $x = 4, y = 2$; $x = 9, y = 3$.

2. (i) (11!)/8, (10!)/4, (4!)(7!).

3. (i) $1 + \frac{1}{2}x - \frac{1}{8}x^2 \ldots$, $1 + 2x + 3x^2$, 2·54% increase;
(ii) $p \approx -10$, $q \approx 83$.

4. (i) $-7 - 24\text{i}$, $(3 + 4\text{i})/25$, $\pm(2 - \text{i})$.

5. (*a*) $x^2 + y^2 - 2x - 24 = 0$;
(*b*) (8/3, 0);
(*c*) $x^2 + y^2 - 2x + 5y - 24 = 0$.

6. $(2, 2\sqrt{2})$, $(2, -2\sqrt{2})$.

7. (ii) 0°, 30°, 60°, 120°, 150°, 180°.
(iii) $360n° + 119° 33'$, $360n° - 13° 17'$.

8. (i) (*a*) $\dfrac{2(1 - x^2)}{(1 + x^2)^2}$, (*b*) $\dfrac{\sin x \cos x}{1 + \sin^2 x}$;
(ii) $\dfrac{2x - y - 2}{x - 2y + 1}$; 4/3, −1/3.

9. (i) $-e^{-(5\pi/4)}/\sqrt{2}$; (ii) $\frac{3}{4}r^2\sqrt{3}$.

10. (i) (*a*) $\ln 2$, (*b*) $2\frac{1}{2}$;
(ii) $4\frac{2}{3}$.

PAPER II

2. Mg along rod; Mg.

4. 12·0 N.

5. 36° 52′; 54 min.

7. $73\frac{1}{3}$ kW; 50 km/h.

9. $4\pi\sqrt{c/g}$; $\sqrt{cg}$.

JANUARY 1973

PAPER I

1. (i) $k \leqslant -17$ or $k > -1$.
(ii) $x^2 + 2(p + 1)x + (1 + 2p + 4q) = 0$.

2. (i) -1.

3. (i) $\frac{1}{2}$, $\dfrac{3\pi}{2}$.
(ii) $3 - 2i$, $-3 + 2i$. $\pm(2 + 3i)\sqrt{3}$.

4. (i) 2, 100.

5. $2 + 10x^2 + 18x^4 + 26x^6$, $2(4n + 1)$, $6\frac{2}{9}$.

6. (*a*) $m\pi + \dfrac{\pi}{4}$, $\dfrac{\pi}{2}\left(m + \dfrac{1}{4}\right)$
(*b*) $\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{4}$, $\left(2n\pi \pm \dfrac{\pi}{3}\right)\Big/3$.
(*c*) $(2r + 1)\pi$, $(2s + \frac{1}{2})\pi$.

7. $8, y = 0 , y = \dfrac{4}{3}x$.

9. (i) (*a*) $3/(1 - 2x)^2$.
(*b*) $4 \sec^2 2x \tan 2x$.
(*c*) $3e^{3x}/(1 + e^{3x})$.
(ii) Maximum point at $\left(-\dfrac{1}{2}, \dfrac{1}{12}\right)$.
Minimum point at $\left(\dfrac{1}{2}, -\dfrac{1}{12}\right)$.

10. $9\pi/2$.

PAPER II

1. $2\sqrt{2}$, $AP = 6$, 12.

2. $\left(1 + x\right)\dfrac{W}{2\sqrt{3}}$, $\dfrac{2}{3}$.

3. 70° 26′ , $\dfrac{16}{45}W$.

4. 17·03 knots, 1 h 34 min.

5. $2\pi\sqrt{\dfrac{2l}{3g}}$, l .

6. $\dfrac{mgr}{6}$, $\dfrac{23\ mgr}{12}$.

7. 5/7.

8. 8175 J, 502·7 N.

9. v

10. 26·20 m s^{-1}, 13·2 m.

SUMMER 1973

Paper 1

1. (ii) 9, $\frac{1}{3}$.

2. (i) (*a*) 56, (*b*) 24, (*c*) 32.

3. (i) − 2, − 3, 80.
(ii) 3, 1.

4. (i) (*a*) $2\sqrt{10}$ (*b*) 10 (*c*) $\frac{2}{\sqrt{10}}$
(ii) (− 2 + 6i)/5.

5. $-\frac{3}{2}, -\frac{15}{8}$; $x + y - 3 = 0, x + 7y - 9 = 0.$

6. $ty - x - at^2 = 0.$

7. (i) $\pi/12, 5\pi/12, 13\pi/12, 17\pi/12.$

8. (i) (*a*) $2e^{2x}\cos(\pi x) - \pi e^{2x}\sin(\pi x)$,
(*b*) $\frac{2x^2 + 2x - 2}{(x^2 + 1)(2x + 1)}$, (*c*) $\frac{4x}{\sqrt{(1 + 4x^2)}}$

9. (i) (*a*) $\pi/16$, (*b*) $\ln 2 - \frac{1}{2}$; (ii) $\frac{5}{6}$.

10. (i) $\ln\sqrt{2}, \pi\left(1 - \frac{\pi}{4}\right)$
(ii) 1, − 6, 9.

Paper 2

1. $aP\sqrt{3}, \frac{a}{2}$.

2. $7\frac{5}{9}$ *kg*.

5. $\cos^{-1}(2g/lw^2)$.

6. 14 km, 4 h 28 min.

7. $4\frac{4}{9}$ m, 14·12 kg m/s.

8. 3270 W; (*a*) 40 km/h, (*b*) 24 km/h.

9. $\frac{20}{3}$ m/s; $6\frac{2}{3}$ m/s, 8 m/s.

Printed by Eyre & Spottiswoode Ltd, Thanet Press, Margate
58348/30984/7-73/12000