My Math 322 page
Introduction to Partial differential equations
Spring 2018
University of Wisconsin
Milwaukee

Nasser M. Abbasi

May 2018 Compiled on October 28, 2018 at 9:49pm [public]

Contents

Chapter 1: Introduction

Took this course in Spring 2018 to learn more about PDE's.

1.1 syllabus

Introduction to Partial Differential Equations, Math 322, Section 1

Prerequisites: jr. st., Math 320; or grad st

Classes: Tuesday and Thursday 2:00–3:15 in EMS E160

Instructor: Hans Volkmer, Office: EMS E451, Phone: 229–5950

Email: volkmer@uwm.edu

Course page: on D2L

Office hours: Tuesday and Thursday 1:00–2:00 and 3:15–4:00 and by appointment

Textbook: Elementary Differential Equations by W. Boyce and R. DiPrima, 10th edition

Chapter 10 Partial Differential Equations and Fourier Series

Chapter 11 Boundary Value Problems and Sturm-Liouville Theory

Grading policy: Midterm exam on Chapter 10, Tuesday, 2:00–3:15, March 13, 2018.

Final exam on Chapters 10 and 11, Thursday, 10:00–12:00. May 17, 2018.

There will 4 take-home quizzes assigned on February 8, February 27, April 10, April 26. You have a week to do these quizzes.

The midterm exam will be worth 120 points, the final exam 180 points and each quiz 40 points with a total of 460 points. 390 points will certainly be enough for an A, 360 will be an A- etc.

The midterm and final exam will be closed book and open notes. You can use calculators on the exams.

Makeup policy: makeups are possible.

If you feel you are a student with a disability please feel free to contact me early in the semester for any help or accommodations which you may need.

Chapter 2: HWs

2.1 my solved problems

2.1.1 Chapter 10.1, Problem 9

Problem Either solve $y'' + 4y = \cos x$ with $y'(0) = 0, y'(\pi) = 0$ or show it has no solution.
Solution The homogeneous solution y_h can be easily found to be

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the basis solutions are

$$\begin{aligned} y_1 &= \cos 2x \\ y_2 &= \sin 2x \end{aligned}$$

And

$$\begin{aligned} y_1' &= -2\sin 2x \\ y_2' &= 2\cos 2x \end{aligned}$$

Hence

$$y_h'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$$

To find particular solution, let

$$y_p = A\cos x$$

The original ODE becomes

$$\begin{aligned} -A\cos x + 4A\cos x &= \cos x \\ 3A\cos x &= \cos x \\ A &= \frac{1}{3} \end{aligned}$$

Hence the full solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= -2c_1 \sin 2x + 2c_2 \cos 2x + \frac{1}{3}\cos x \end{aligned}$$

Therefore

$$y'(x) = -4c_1 \cos 2x - 4c_2 \sin 2x - \frac{1}{3}\sin x$$

First B.C. gives

$$\begin{aligned} y'(0) = 0 &= -4c_1 \\ c_1 &= 0 \end{aligned}$$

Therefore the solution now becomes $y(x) = 2c_2 \cos 2x + \frac{1}{3}\cos x$ and $y'(x) = -4c_2 \sin 2x - \frac{1}{3}\sin x$. The second B.C. gives

$$\begin{aligned} y'(\pi) = 0 &= -4c_2(0) \\ 0 &= -4c_2(0) \end{aligned}$$

Hence c_2 can be any value. Therefore, there is no unique solution. There are infinite number of solutions. Final solution is

$$y(x) = 2c_2 \cos 2x + \frac{1}{3}\cos x$$

Since $2c_2$ is constant, we can rename it to A and write the above as

$$\boxed{y(x) = A\cos 2x + \tfrac{1}{3}\cos x}$$

To verify that there is no unique solution, we set up W where $y_1 = \cos 2x, y_2 = \sin 2x$, and y_1, y_2 as found above. These are the two basis solutions for the homogeneous ODE.

$$W = \begin{vmatrix} y_1'(0) & y_2'(0) \\ y_1'(\pi) & y_2'(\pi) \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0$$

Since $W = 0$, this implies there is no unique solution. Therefore the ODE can has no solution, or it can have an infinite number of solutions. In this case, as shown above, it has infinite number of solutions.

2.1.2 Chapter 10.1, Problem 12

Problem Either solve $x^2y'' + 3xy' + y = x^2$ with $y(1) = 0, y(e) = 0$ or show it has no solution.
Solution The homogeneous solution is first found. This is a Euler ODE. Let $y_h = x^r$, then $y_h' = rx^{r-1}, y_h'' = r(r-1)x^{r-2}$ and the homogeneous ODE becomes

$$\begin{aligned} r(r-1)x^r + 3rx^r + x^r &= 0 \\ r(r-1) + 3r + 1 &= 0 \\ r^2 - r + 3r + 1 &= 0 \\ r^2 + 2r + 1 &= 0 \\ (r+1)(r+1) &= 0 \end{aligned}$$

Hence double roots. Therefore the solution is

$$y_h = c_1\frac{1}{x} + c_2\frac{1}{x}\ln x$$

To find particular solution, let $y_p = c_1 + c_2x + c_3x^2$. Plugging this in original ODE gives

$$\begin{aligned} x^2(2c_3) + 3x(c_2 + 2c_3x) + \left(c_1 + c_2x + c_3x^2\right) &= x^2 \\ x^2(2c_3) + c_1 + x(3c_2 + c_2) + x^2(6c_3 + c_3) &= x^2 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} c_1 &= 0 \\ 4c_2 &= 0 \\ 9c_3 &= 1 \end{aligned}$$

Hence solution is $c_2 = 0, c_1 = 0, c_3 = \frac{1}{9}$. Therefore $y_p = \frac{1}{9}x^2$ and the full solution is

$$\boxed{y(x) = c_1\frac{1}{x} + c_2\frac{1}{x}\ln x + \frac{1}{9}x^2} \tag{1}$$

Boundary conditions are now applied to find c_1, c_2. First BC gives

$$\begin{aligned} 0 &= c_1 + c_2\ln 1 + \frac{1}{9} \\ 0 &= c_1 + \frac{1}{9} \\ c_1 &= -\frac{1}{9} \end{aligned}$$

Second BC $y(e) = 0$ gives

$$\begin{aligned} 0 &= c_1\frac{1}{e} + c_2\frac{1}{e}\ln e + \frac{1}{9}e^2 \\ 0 &= -\frac{1}{9e} + c_2\frac{1}{e} + \frac{1}{9}e^2 \\ c_2 &= \frac{1}{9} - \frac{1}{9}e^3 \\ &= \frac{1-e^3}{9} \end{aligned}$$

Therefore the solution (1) becomes

$$\boxed{y(x) = -\frac{1}{9x} + \frac{x^2}{9} + \left(\frac{1-e^3}{9}\right)\frac{1}{x}\ln x}$$

Therefore solution exist and is unique. This is verified using W where now $y_1 = \frac{1}{x}, y_2 = \frac{1}{x}\ln x$. These are found above as the bases solutions for the homogeneous ODE.

$$W = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1(e) & y_2(e) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{e} & \frac{1}{e} \end{vmatrix} = \frac{1}{e} \neq 0$$

This confirms that a unique solution exists.

2.1.3 Chapter 10.1, Problem 14

Problem Find eigenvalue and eigenfunction of $y'' + \lambda y = 0$ with $y(0) = 0, y'(\pi) = 0$.
Solution
Assuming the solution is $y = Ae^{rx}$, then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$
In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sinh(\mu x)$$

Second BC gives

$$\begin{aligned} y'(x) &= \mu c_2 \cosh(\mu x) \\ 0 &= \mu c_2 \cosh(\mu\pi) \end{aligned}$$

But $\cosh \mu\pi \neq 0$, hence only other choice is $c_2 = 0$, leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.
Case $\lambda = 0$, then the homogenous solution is

$$y(x) = c_1 + c_2 x$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 x$$

Second BC gives

$$\begin{aligned} y'(x) &= c_2 \\ 0 &= c_2 \end{aligned}$$

Leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.
Case $\lambda > 0$, then the homogenous solution is

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sin\left(\sqrt{\lambda} x\right)$$

Second BC gives

$$\begin{aligned} y'(x) &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} x\right) \\ 0 &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} \pi\right) \end{aligned}$$

Non-trivial solution requires $\cos\left(\sqrt{\lambda}\pi\right) = 0$ or $\sqrt{\lambda}\pi = \frac{n\pi}{2}$ for $n = 1, 3, 5, \cdots$. Therefore

$$\begin{aligned} \sqrt{\lambda_n}\pi &= \frac{n\pi}{2} \\ \sqrt{\lambda_n} &= \frac{n}{2} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

Hence the eigenvalues are

$$\lambda_n = \left(\frac{n}{2}\right)^2 \qquad n = 1, 3, 5, \cdots$$

And the corresponding eigenfunction is $\sin\left(\frac{n}{2}x\right)$ for $n = 1, 3, 5, \cdots$. The solution is

$$y(x) = \sum_{n=1,3,5,\cdots}^{\infty} c_n \sin\left(\frac{n}{2}x\right)$$

2.1.4 Chapter 10.1, Problem 20

Problem Find eigenvalue and eigenfunction of $x^2y'' - xy' + \lambda y = 0$ with $y(1) = 0, y(L) = 0, L > 1$
Solution
This is Euler type ODE. Using standard substitution, ley $y = x^r$. The ODE now becomes

$$\begin{aligned} x^2 r(r-1)x^{r-2} - xrx^{r-1} + \lambda x^r &= 0 \\ r(r-1) - r + \lambda &= 0 \\ r^2 - 2r + \lambda &= 0 \end{aligned}$$

The above is called the characteristic equations. Its roots give the solution. The roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}$$

case $1 - \lambda > 0$
Let $1 - \lambda = \mu^2$ for some real μ. Then the roots are $1 \pm \mu$ and hence the solution is

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{1+\mu} + c_2 x^{1-\mu} \\ &= x\left(c_1 x^{\mu} + c_2 \frac{1}{x^{\mu}}\right) \end{aligned}$$

At first BC $y(1) = 0$ the above gives

$$0 = c_1 + c_2$$

At second BC $y(L) = 0$

$$\begin{aligned} 0 &= L\left(c_1 L^{\mu} + c_2 \frac{1}{L^{\mu}}\right) \\ 0 &= c_1 L^{\mu} + c_2 \frac{1}{L^{\mu}} \\ 0 &= \frac{c_1 L^{2\mu} + c_2}{L^{\mu}} \end{aligned}$$

Hence

$$c_1 L^{2\mu} + c_2 = 0$$

But $c_2 = -c_1$, therefore

$$\begin{aligned} c_1 L^{2\mu} - c_1 &= 0 \\ c_1\left(L^{2\mu} - 1\right) &= 0 \end{aligned}$$

For arbitrary $L > 0$ the above can only be satisfied if $c_1 = 0$. This means both c_1, c_2 are zero. Hence $1 - \lambda > 0$ is not possible.
case $1 - \lambda = 0$
Hence the roots now are $r = 1$. Double root. We now in the case of double root the solution can be written as

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_1} \ln x \\ &= c_1 x + c_2 x \ln x \end{aligned}$$

At first BC $y(1) = 0$ the above gives

$$0 = c_1$$

Therefore the solution now becomes $y = c_2 x \ln x$. At second BC $y(L) = 0$

$$\begin{aligned} 0 &= c_2 L \ln L \\ 0 &= c_2 \ln L \end{aligned}$$

Since $L > 0$ then only possibility is that $c_2 = 0$.This means both c_1, c_2 are zero. Hence $1 - \lambda = 0$ is not possible.
case $1 - \lambda < 0$
Let $1 - \lambda = -\mu^2$ for some real μ. Then the roots are $1 \pm i\mu$ and hence the solution is

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{1+i\mu} + c_2 x^{1-i\mu} \\ &= x\left(c_1 x^{i\mu} + c_2 x^{-iu}\right) \end{aligned}$$

The above can be written as

$$\begin{aligned} y &= x\left(c_1 e^{\ln x^{i\mu}} + c_2 e^{\ln x^{-i\mu}}\right) \\ &= x\left(c_1 e^{i\mu \ln x} + c_2 e^{-i\mu \ln x}\right) \end{aligned}$$

Hence $c_1 e^{i\mu \ln x} + c_2 e^{-i\mu \ln x}$ can be written as $C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)$. This is done using Euler relation and the new constants C_1, C_2 are not the same as c_1, c_2. The solution becomes

$$y = x\left(C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right)$$

First BC $y(1) = 0$ the above becomes

$$\begin{aligned} 0 &= C_1 \cos(\mu \ln 1) + C_2 \sin(\mu \ln 1) \\ &= C_1 \end{aligned}$$

Therefore the solution is

$$y = xC_2 \sin(\mu \ln x) \tag{1}$$

For second BC $y(L) = 0$ the above becomes

$$\begin{aligned} 0 &= LC_2 \sin(\mu \ln L) \\ 0 &= C_2 \sin(\mu \ln L) \end{aligned}$$

Non-trivial solution requiressin $(\mu \ln L) = 0$ or $\mu \ln L = n\pi$ for $n = 1, 2, 3, \cdots$. This means

$$\mu = \frac{n\pi}{\ln L} \qquad n = 1, 2, 3, \cdots$$

But $1 - \lambda = -\mu^2$, or $\lambda = 1 + \mu^2$, therefore

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln L}\right)^2 \qquad n = 1, 2, 3, \cdots \tag{2}$$

These are the eigenvalues. The corresponding eigenfunctions are from (1)

$$\begin{aligned} y_n(x) &= c_n x \sin(\mu_n \ln x) \\ &= c_n x \sin\left(\sqrt{\lambda_n - 1} \ln x\right) \\ &= c_n x \sin\left(\sqrt{1 + \left(\frac{n\pi}{\ln L}\right)^2 - 1} \ln x\right) \\ &= c_n x \sin\left(\sqrt{\left(\frac{n\pi}{\ln L}\right)^2} \ln x\right) \\ &= c_n x \sin\left(\frac{n\pi}{\ln L} \ln x\right) \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Hence the solution is

$$y(x) = x \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ln L} \ln x\right)$$

2.1.5 Chapter 10.1, Problem 22

22. Consider a horizontal metal beam of length L subject to a vertical load $f(x)$ per unit length. The resulting vertical displacement in the beam $y(x)$ satisfies the differential equation

$$EI \frac{d^4 y}{dx^4} = f(x),$$

where E is Young's modulus and I is the moment of inertia of the cross section about an axis through the centroid perpendicular to the xy-plane. Suppose that $f(x)/EI$ is a constant k. For each of the boundary conditions given below, solve for the displacement $y(x)$, and plot y versus x in the case that $L = 1$ and $k = -1$.

(a) Simply supported at both ends: $y(0) = y''(0) = y(L) = y''(L) = 0$

(b) Clamped at both ends: $y(0) = y'(0) = y(L) = y'(L) = 0$

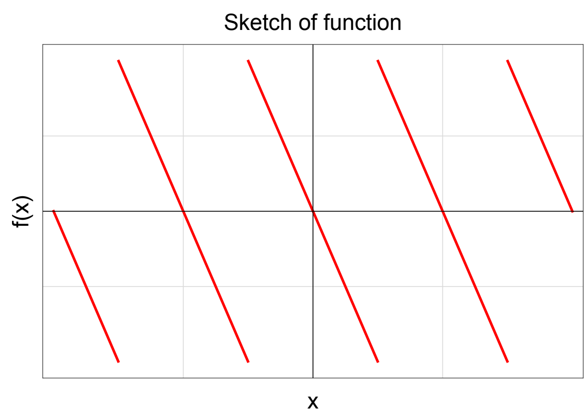
(c) Clamped at $x = 0$, free at $x = L$: $y(0) = y'(0) = y''(L) = y'''(L) = 0$

This is standard ODE with constant coefficients. Just integrating and substitutions.

2.1.6 Chapter 10.2, Problem 13 (With interactive animation)

Problem Sketch the graph of $f(x) = -x, -L \leq x < L$ where $f(x+2L) = f(x)$ and find the Fourier series of the function

Solution



This is an odd function. Only b_n needs to be evaluated.

$$b_n = \frac{1}{T/2} \int_{-T/2}^{T/2} f(x) \sin\left(n\frac{2\pi}{T}x\right)$$

T is the period of $f(x)$ which is $2L$. The above becomes

$$b_n = \frac{1}{L} \int_{-L}^{L} -x \sin\left(n\frac{\pi}{L}x\right)$$

Since x is odd and sin is odd then the product is even and the above simplifies to

$$b_n = \frac{-2}{L} \int_{0}^{L} x \sin\left(n\frac{\pi}{L}x\right) \tag{1}$$

Using integration by parts $\int u dv = uv - \int v du$ where $u = x, dv = \sin\left(n\frac{\pi}{L}x\right)$, therefore $du = 1$ and

$$v = -\frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}} = \frac{-L}{n\pi}\cos\left(n\frac{\pi}{L}x\right)$$

Integral (1) becomes

$$\begin{aligned}
b_n &= \frac{-2}{L}\left(\left[\frac{-L}{n\pi}x\cos\left(n\frac{\pi}{L}x\right)\right]_0^L - \int_0^L \frac{-L}{n\pi}\cos\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{-2}{L}\left(\left[\frac{-L^2}{n\pi}\cos(n\pi) - 0\right] + \frac{-L}{n\pi}\int_0^L \cos\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{-2}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) + \frac{-L}{n\pi}\frac{1}{n\frac{\pi}{L}}\left[\sin\left(n\frac{\pi}{L}x\right)\right]_0^L\right) \\
&= \frac{-2}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) + \frac{-L^2}{n^2\pi^2}\left[\sin(n\pi) - 0\right]\right) \\
&= \frac{-2}{L}\frac{-L^2}{n\pi}\cos(n\pi) \\
&= \frac{2L}{n\pi}\cos(n\pi)
\end{aligned}$$

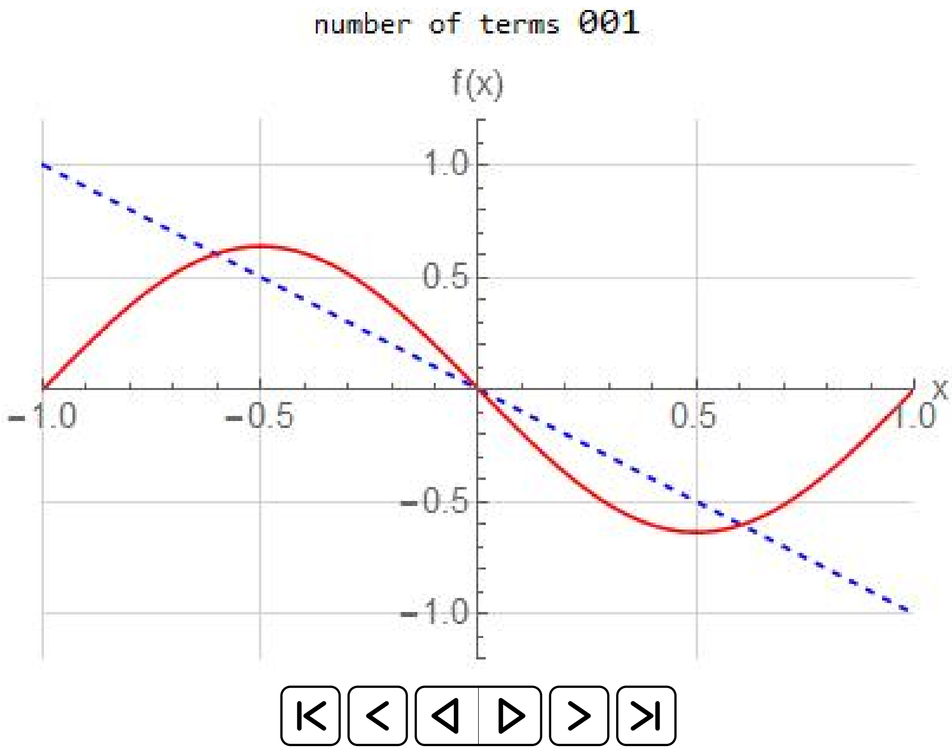
For $n = 1, 2, 3, \cdots$. Looking at few n values gives

$$\begin{aligned}
b_n &= \frac{2L}{\pi}(-1), \frac{2L}{2\pi}, \frac{2L}{3\pi}(-1), \cdots \\
&= \frac{2L}{n\pi}(-1)^n
\end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^n \sin\left(\frac{n\pi}{L}x\right) \\
&= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{L}x\right)
\end{aligned}$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



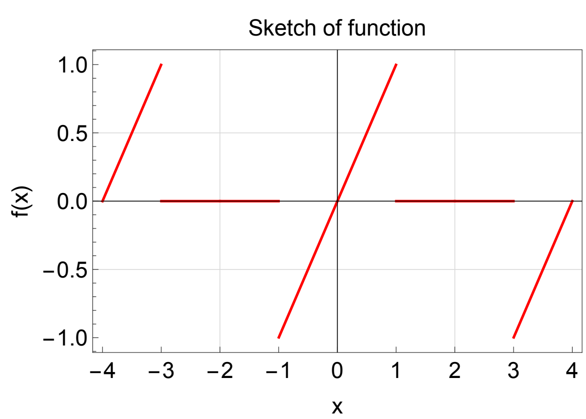
2.1.7 Chapter 10.2, Problem 18 (With interactive animation)

Problem Sketch the graph and find the Fourier series of the function

$$f(x) = \begin{cases} 0 & -2 \leq x \leq -1 \\ x & -1 < x < 1 \\ 0 & 1 \leq x < 2 \end{cases}$$

And $f(x+4) = f(x)$

Solution



$f(x)$ is an odd function. Therefore only b_n needs to be evaluated.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)$$

$2L$ is the period of $f(x)$ which is 4. Hence $L = 2$. The above becomes

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right) \\ &= \frac{1}{2}\left(\int_{-2}^{-1} f(x)\sin\left(\frac{n\pi}{2}x\right) + \int_{-1}^{1} f(x)\sin\left(\frac{n\pi}{2}x\right) + \int_{1}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)\right) \\ &= \frac{1}{2}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi}{2}x\right) \\ &= \frac{1}{2}\int_{-1}^{1} x\sin\left(\frac{n\pi}{2}x\right) \end{aligned}$$

Since x is odd and sin is odd then the product is even and the above simplifies to

$$b_n = \int_{0}^{1} x\sin\left(\frac{n\pi}{2}x\right) \tag{1}$$

Using integration by parts $\int udv = uv - \int vdu$ where $u = x, dv = \sin\left(\frac{n\pi}{2}x\right)$, therefore $du = 1$ and

$$v = -\frac{\cos\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}} = \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$$

Integral (1) becomes

$$\begin{aligned} b_n &= \frac{-2}{n\pi}\left[x\cos\left(n\frac{\pi}{2}x\right)\right]_0^1 - \int_0^1 \frac{-2}{n\pi}\cos\left(n\frac{\pi}{2}x\right)dx \\ &= \frac{-2}{n\pi}\left[\cos\left(\frac{n}{2}\pi\right)\right] + \frac{2}{n\pi}\int_0^1 \cos\left(n\frac{\pi}{2}x\right)dx \\ &= \left(\frac{-2}{n\pi}\cos\left(\frac{n}{2}\pi\right) + \frac{2}{n\pi}\frac{1}{n\frac{\pi}{2}}\left[\sin\left(n\frac{\pi}{2}x\right)\right]_0^1\right) \\ &= \left(\frac{-2}{n\pi}\cos\left(\frac{n}{2}\pi\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n}{2}\pi\right)\right) \end{aligned}$$

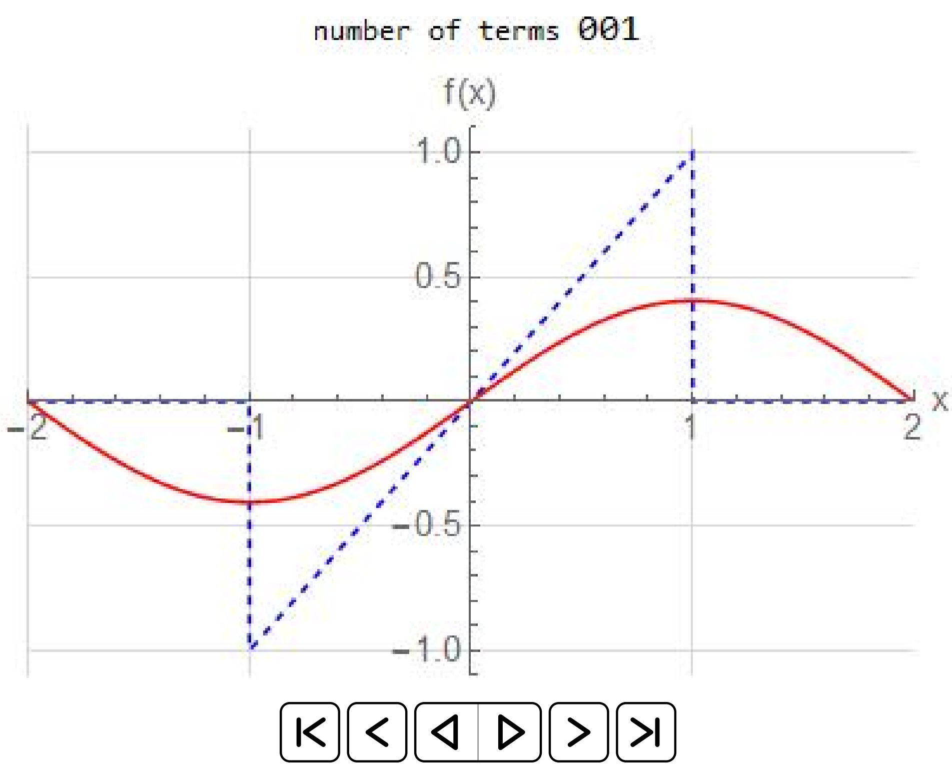
Therefore

$$b_n = \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n}{2}\pi\right) - \frac{2}{n\pi}\cos\left(\frac{n}{2}\pi\right) \qquad n = 1, 2, 3, \cdots$$

The Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n}{2}\pi\right) - \frac{2}{n\pi}\cos\left(\frac{n}{2}\pi\right)\right]\sin\left(\frac{n\pi}{2}x\right)$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).

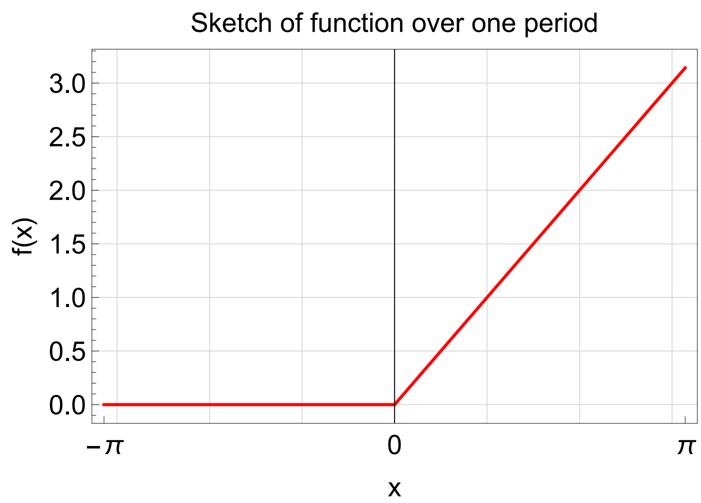


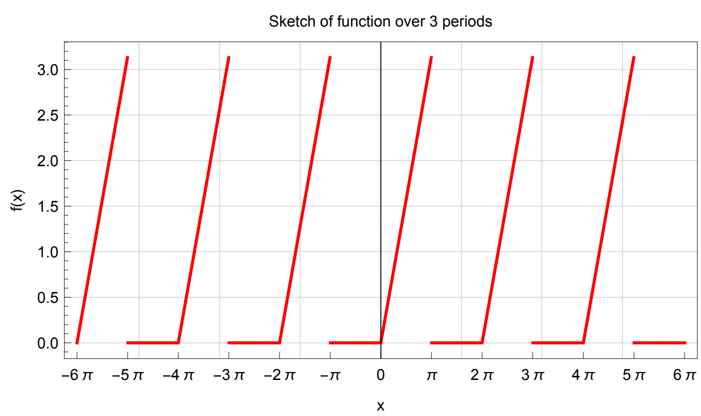
2.1.8 Chapter 10.3, Problem 2

Problem Assume the function is periodically extended outside the original interval. (a) Find the Fourier series of the extended function. (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$$

Solution
This is plot of the above function for one period, and then for 3 periods





part a

The calculation of the Fourier series will have a_n, b_n and will follow same methods as before. The period here is 2π.

part b

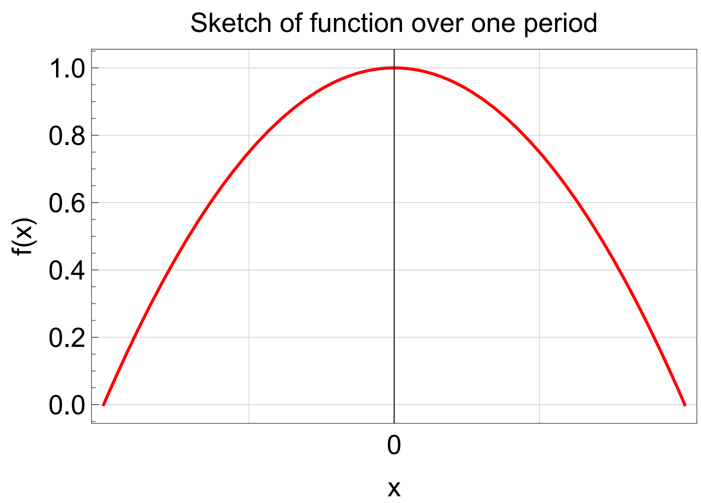
Since both $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series will converge to the function $f(x)$. But at the points where $f(x)$ has jumps (such as at $x = \pm\pi$) the Fourier series will converge to the average value of $f(x)$ at these points.

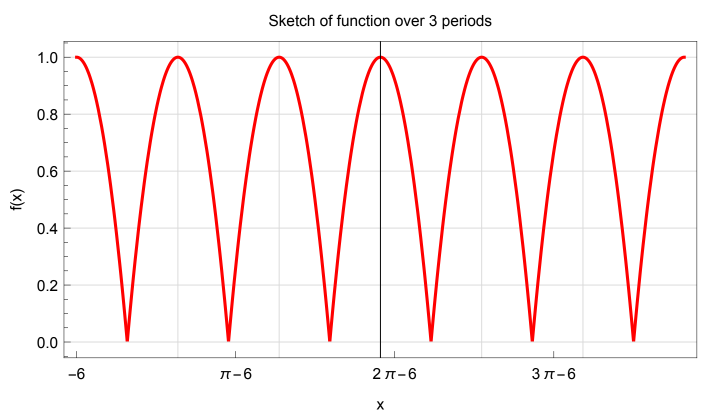
2.1.9 Chapter 10.3, Problem 4

Problem Assume the function is periodically extended outside the original interval. (a) Find the Fourier series of the extended function. (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 - x^2 \qquad -1 \leq x < 1$$

Solution
This is plot of the above function for one period, and then for 3 periods





part a

The calculation of the Fourier series will have only a_n since $f(x)$ is even, and will follow same methods as before. The period here is 2.

part b

Since both $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series will converge to the function $f(x)$ for all x.

2.1.10 Chapter 10.4, Problem 17

Problem (a) Find the Fourier series of the given function (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 \qquad 0 \leq x \leq \pi$$

Use cosine series, with period 2π.
Solution
Extending this as even function gives

$$f_e(x) = 1 \qquad -\pi < x \leq \pi$$

Hence, since period is 2π, then $L = \pi$ now and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f_e(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx = \frac{2}{\pi}\int_0^{\pi} dx = 2$$

And

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{2}{n\pi}\left(-\sin(nx)\right)_0^{\pi} = 0$$

Therefore the cosine extension Fourier series is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \\ &= \frac{a_0}{2} \\ &= 1 \end{aligned}$$

2.1.11 Chapter 10.4, Problem 18 (With interactive animation)

Problem (a) Find the Fourier series of the given function (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 \qquad 0 < x < \pi$$

Use sin series, with period 2π.
Solution
Extending this as odd function gives

$$f_o(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x \leq 0 \end{cases}$$

Hence, since period is 2π, then $L = \pi$ now and, since this is an odd function, only b_n terms will show up

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,dx \end{aligned}$$

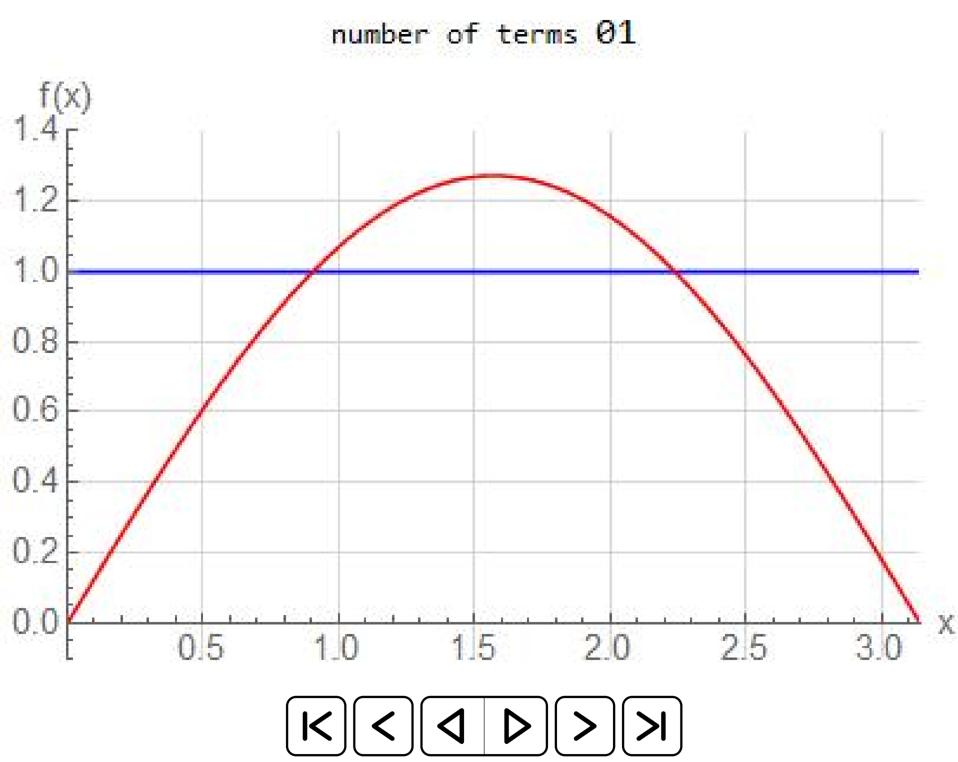
But now $f_o(x)\sin\left(\frac{n\pi}{L}x\right)$ is even, therefore the above simplifies to

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{0}^{\pi} f_o(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx \\ &= \frac{-2}{\pi}\left(\frac{\cos(nx)}{n}\right)_0^{\pi} \\ &= \frac{-2}{n\pi}(\cos(n\pi)-1) \\ &= \frac{-2}{n\pi}\left(-1^n-1\right) \end{aligned}$$

Therefore the sine extension Fourier series is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n\sin(nx) \\ &= \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(-1^n-1\right)\sin(nx) \end{aligned}$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.12 Chapter 10.5, Problem 7

Problem Find solution to $u_t = 100u_{xx}$ with $0 < x < 1, t > 0$ and boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions $u(x,0) = \sin(2\pi x) - \sin(5\pi x)$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 100$ and $\lambda_n = (n\pi)^2, n = 1,2,3,\ldots$. The c_n terms is the Fourier sine coefficients of the initial conditions. But the initial conditions is already expressed as sum of sine terms. Therefore the c_n coefficient can be read directly from $f(x)$, giving $c_2 = 1, c_5 = -1$. Therefore only two terms exist in the sum above, leading to the solution

$$\begin{aligned} u(x,t) &= c_2 e^{-(2\pi)^2(100)t} \sin(2\pi x) + c_5 e^{\,-(5\pi)^2(100)t} \sin(5\pi x) \\ &= e^{-400\pi^2 t} \sin(2\pi x) - e^{-25000\pi t} \sin(5\pi x) \end{aligned}$$

2.1.13 Chapter 10.5, Problem 10 (With interactive animation)

Problem Solve $u_t = u_{xx}$, with $0 < x < L$ and $L = 40cm$ and boundary conditions $u(0,t) = u(L,t) = 0^0$ with initial conditions

$$u(x,0) = \begin{cases} x & 0 \le x < 20 \\ 40 - x & 20 \le x \le 40 \end{cases}$$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\ldots$and $L = 40$ cm. To find c_n, initial conditions are used. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

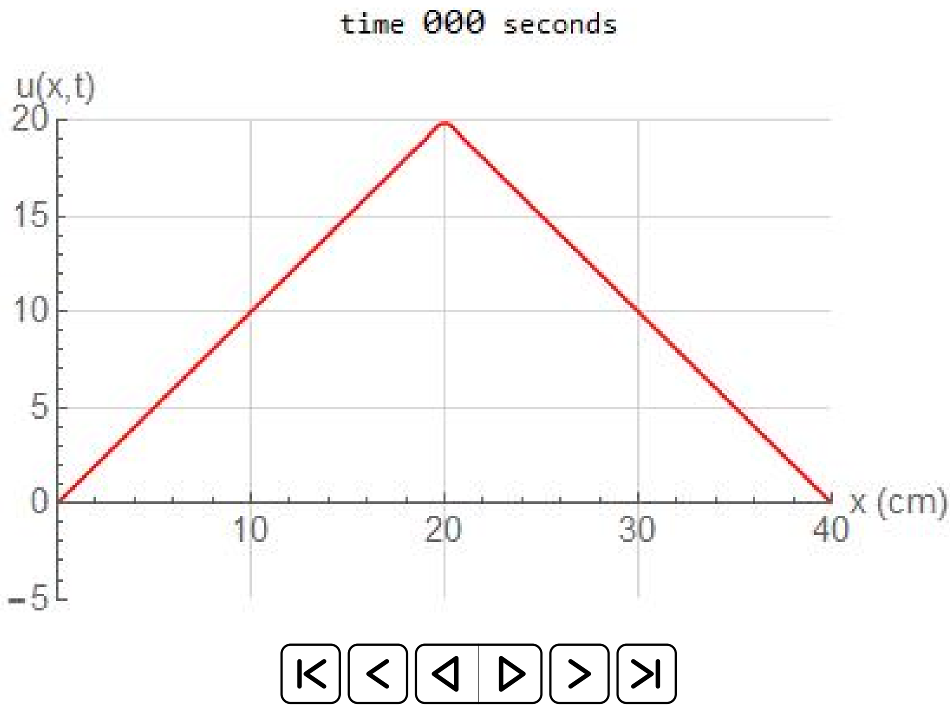
Applying orthogonality result in

$$\begin{aligned} c_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \\ &= \frac{2}{40} \left(\int_0^{20} x \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{20}^{40} (40 - x) \sin\left(\sqrt{\lambda_n} x\right) dx \right) \\ &= \frac{2}{40} \left(\frac{3200}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right) \\ &= \frac{160}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Hence the solution is

$$u(x,t) = \frac{160}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi}{40} x\right)$$

The following is an animation of the above solution for 510 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.14 Chapter 10.5, Problem 11 (With interactive animation)

Problem

Solve $u_t = u_{xx}$, with $0 < x < L$ and $L = 40cm$ and boundary conditions $u(0,t) = u(L,t) = 0^0$ with initial conditions

$$u(x,0) = \begin{cases} 0 & 0 \le x < 10 \\ 50 & 10 \le x \le 30 \\ 0 & 30 \le x \le 40 \end{cases}$$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$ .and $L = 40$ cm. To find c_n, initial conditions are used. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

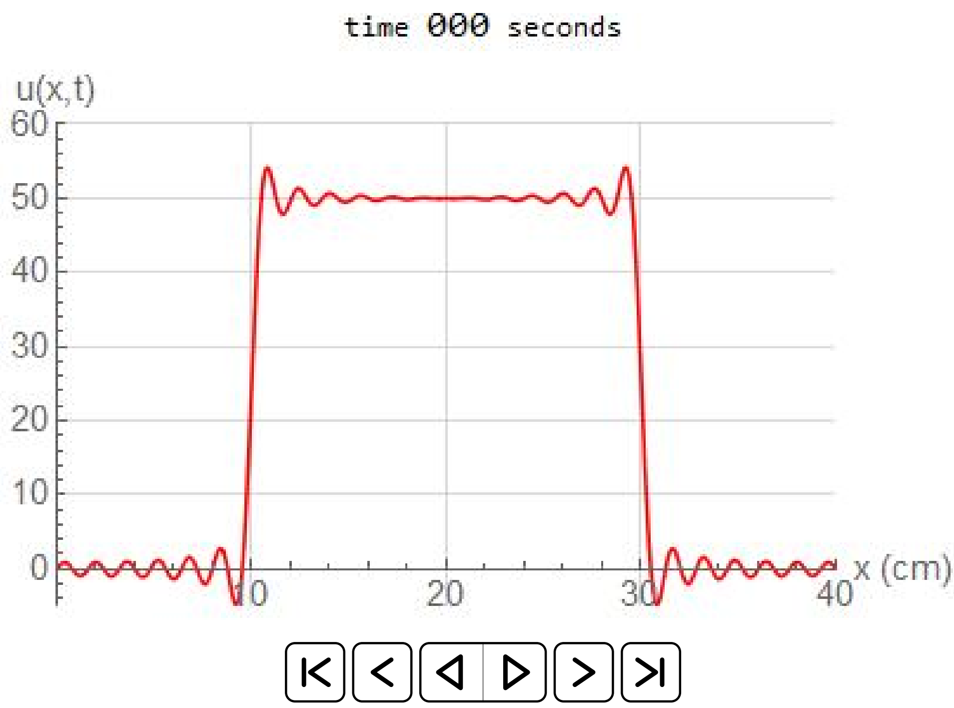
Applying orthogonality result in

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \\ &= \frac{2}{40}\left(\int_0^{10} 0 \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{10}^{30} 50 \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{30}^{40} 0 \sin\left(\sqrt{\lambda_n} x\right) dx\right) \\ &= \frac{2}{40}\int_{10}^{30} 50 \sin\left(\sqrt{\lambda_n} x\right) dx \\ &= \frac{200}{n\pi} \sin\frac{n\pi}{4} \sin\frac{n\pi}{2} \end{aligned}$$

Hence the solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi}{40} x\right)$$

The following is an animation of the above solution for 510 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.15 Chapter 10.6, Problem 5

Problem Find steady state solution that satisfies the given boundary conditions $u_t = \alpha^2 u_{xx}$ with $u(0,t) = 0, u_x(L,t) = 0$
solution at steady state

$$
\begin{aligned}
v''(x) &= 0 \\
v(0) &= 0 \\
v'(L) &= 0
\end{aligned}
$$

Solution to the above ODE is $v(x) = c_1 x + c_2$. At $x = 0$, this leads to $c_2 = 0$. Therefore the solution now becomes $v(x) = c_1 x$ and $v'(x) = c_1$. Second boundary condition implies $c_1 = 0$ as well. Therefore

$$v(x) = 0$$

is the steady state solution.

2.1.16 Chapter 10.6, Problem 7

Problem Find steady state solution that satisfies the given boundary conditions $u_t = \alpha^2 u_{xx}$ with $u_x(0,t) - u(0,t) = 0, u(L,t) = T$
solution at steady state

$$
\begin{aligned}
v''(x) &= 0 \\
v'(0) - v(0) &= 0 \\
v(L) &= T
\end{aligned}
$$

Solution to the above ODE is $v(x) = c_1 x + c_2$. At $x = 0$, this leads to $c_1 - c_2 = 0$. Second boundary condition implies $c_1 L + c_2 = 0$. Two equations in 2 unknowns

$$
\begin{aligned}
c_1 - c_2 &= 0 \\
c_1 L + c_2 &= T
\end{aligned}
$$

From first equation, $c_1 = c_2$. Second equation becomes $c_2(1+L) = T$ or $c_2 = \frac{T}{1+L}$. Therefore the steady state solution

$$
\begin{aligned}
v(x) &= \frac{T}{1+L} x + \frac{T}{1+L} \\
&= \frac{T}{1+L}(1+x)
\end{aligned}
$$

2.1.17 Chapter 10.6, Problem 9 (With interactive animation)

Problem Let $L = 20$ cm, with initial temperature $25^0 C$, an initial conditions $u(0,x) = 0, u(L,0) = 60^0 C$. (a) Find $u(x,t)$. (b) Plot initial temperature distribution, final steady state solution and solution are two intermediate times on same axes. (c) Plot u vs. t for $x = 5, 10, 15$. (d) determine how much time has elapsed before the temperature at $x = 5$ cm comes and remains with 1% of the steady state value. Use $\alpha^2 = 0.86$

solution

$$
\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,x) &= 0 \\
u(L,0) &= 60
\end{aligned}
$$

Let solution be $u(x,t) = w(x,t) + v(x)$ where $v(x)$ is solution to $v''(x) = 0$ with boundary conditions $v(0) = 0, v(L) = 60$. Hence the solution is

$$v(x) = c_1 x + c_2$$

At $x = 0$, this leads to $c_2 = 0$. Therefore solution is $v(x) = c_1 x$. At $x = L$, $60 = c_1 L$ or $c_1 = \frac{60}{L} = \frac{60}{20} = 3$. Therefore

$$v(x) = 3x$$

Hence the complete solution is

$$u(x,t) = \left(\sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \sin\left(\sqrt{\lambda_n} x\right)\right) + 3x$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \cdots$. c_n is now found from initial conditions. At $t = 0$

$$
\begin{aligned}
25 &= \left(\sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)\right) + 3x \\
25 - 3x &= \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)
\end{aligned}
$$

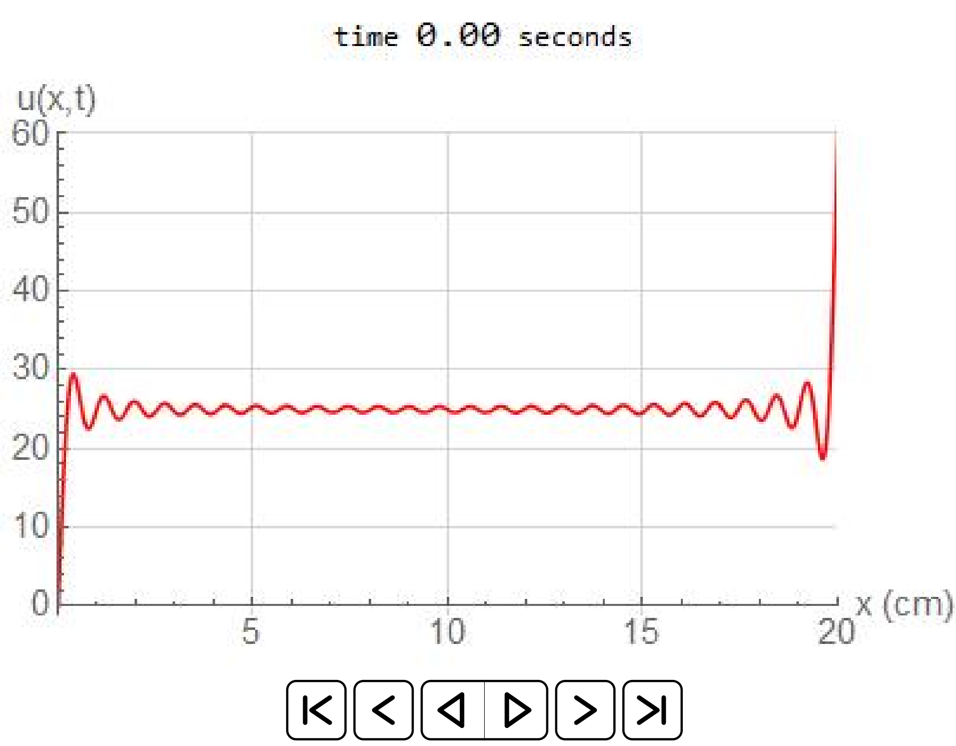
Applying orthogonality gives

$$
\begin{aligned}
\int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx &= c_n \frac{L}{2} \\
c_n &= \frac{2}{L} \int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx \\
&= \frac{2}{20} \int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx
\end{aligned}
$$

Integrating gives $c_n = \frac{50 + 70(-1)^n}{n\pi}$. Therefore the solution is

$$u(x,t) = \left(\sum_{n=1}^{\infty} \frac{50 + 70(-1)^n}{n\pi} e^{-\alpha^2 \lambda_n t} \sin\left(\sqrt{\lambda_n} x\right)\right) + 3x$$

The following is an animation of the above solution for 20 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.18 Chapter 10.6, Problem 10

10. (a) Let the ends of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100°C by an external heat source and that this situation is maintained until a steady state results. Find this steady state temperature distribution.

    (b) At a time $t = 0$ [after the steady state of part (a) has been reached], let the heat source be removed. At the same instant let the end $x = 0$ be placed in thermal contact with a reservoir at 20°C, while the other end remains at 0°C. Find the temperature as a function of position and time.

    (c) Plot u versus x for several values of t. Also plot u versus t for several values of x.

    (d) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1°C of its limiting value?

solution
To do.

2.1.19 Chapter 10.7, Problem 3 (With interactive animation)

Problem Consider elastic string of length L with ends held fixed. Let initial position $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Let $L = 10$, $a = 1$. (a) Find $u(x,t)$. (b) Plot $u(x,t)$ vs x for $0 \leq x \leq 10$ and for several values of time between $t = 0$ and $t = 20$ (c) Plot $u(x,t)$ vs. t for $0 \leq t \leq 20$ and for several values of x (d) Construct an animation of the solution for at least one period. (e) Describe the motion of the string. Let $f(x) = \frac{8x(L-x)^2}{L^3}$

solution Since domain is finite, it is easier to use the series solution for wave equation than D'Alembert solution. This is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ and $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\sqrt{\lambda_n} x\right) dx$. Hence, since $a = 1$ and $L = 10$, the solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{10}t\right) \sin\left(\frac{n\pi}{10}x\right) \\ c_n &= \frac{2}{10}\int_0^{10} \frac{8x(L-x)^2}{L^3} \sin\left(\frac{n\pi}{10}x\right) dx \\ &= \frac{2}{10}\int_0^{10} \frac{8x(10-x)^2}{10^3} \sin\left(\frac{n\pi}{10}x\right) dx \end{aligned}$$

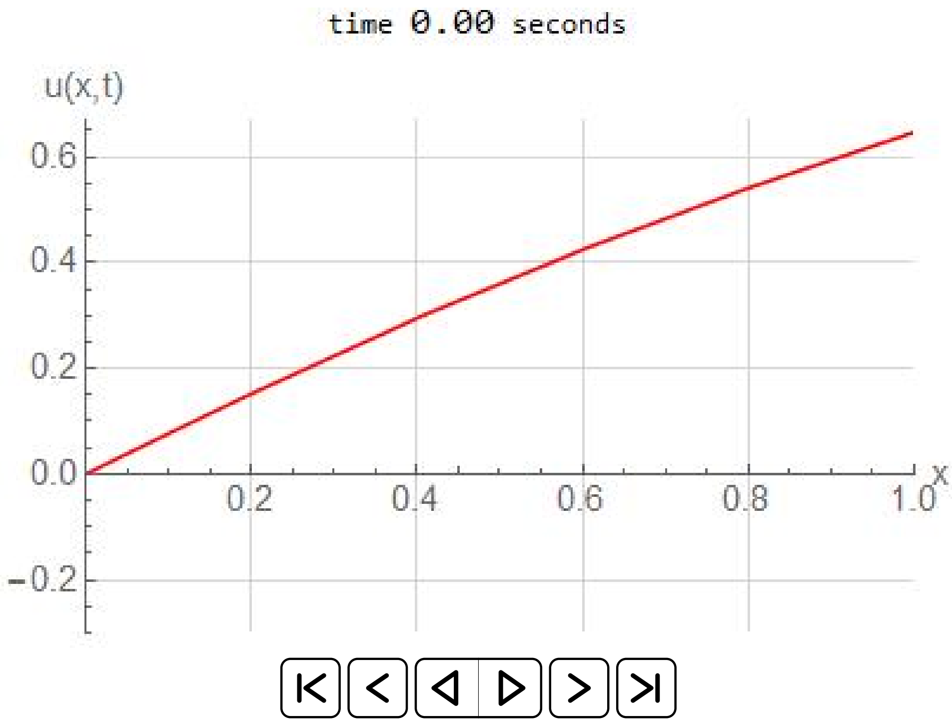
Integrating gives

$$c_n = \frac{32\left(2 + (-1)^n\right)}{n^3 \pi^3}$$

Hence solution is

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{n^3} \cos\left(\frac{n\pi}{10}t\right) \sin\left(\frac{n\pi}{10}x\right)$$

The following is an animation of the above solution for 50 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.20 Chapter 10.7, Problem 7 (With interactive animation)

Problem Consider elastic string of length L with ends held fixed. Let initial position $u(x,0) = 0$ and $u_t(x,0) = g(x)$. Let $L = 10, a = 1$. (a) Find $u(x,t)$. (b) Plot $u(x,t)$ vs x for $0 \leq x \leq 10$ and for several values of time between $t = 0$ and $t = 20$ (c) Plot $u(x,t)$ vs. t for $0 \leq t \leq 20$ and for several values of x (d) Construct an animation of the solution for at least one period. (e) Describe the motion of the string. Let $g(x) = \frac{8x(L-x)^2}{L^3}$

solution Since domain is finite, it is easier to use the series solution for wave equation than D'Alembert solution. The eigenvalue ODE is gives solution for $\lambda > 0$ as

$$X_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ The time solution is $T_n(t) = A_n \cos\left(\sqrt{\lambda_n} a t\right) + B_n \sin\left(\sqrt{\lambda_n} a t\right)$. At $t = 0$, this gives $0 = A_n$. Therefore $T_n(t) = B_n \sin\left(\sqrt{\lambda_n} a t\right)$. Hence the complete solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n T_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

To find c_n, time derivative of the above is taken giving

$$\frac{\partial}{\partial t} u(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right)$$

At $t = 0$ the above becomes

$$g(x) = \sum_{n=1}^{\infty} c_n \sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n} x\right)$$

Applying orthogonality

$$\begin{aligned} \int_0^L g(x) \sin\left(\sqrt{\lambda_n} x\right) dx &= \sqrt{\lambda_n} c_n \frac{L}{2} \\ c_n &= \frac{2}{L\sqrt{\lambda_n}} \int_0^L g(x) \sin\left(\sqrt{\lambda_n} x\right) dx \end{aligned}$$

Hence since $g(x) = \frac{8x(L-x)^2}{L^3}$, $L = 10, a = 1$ the above becomes

$$c_n = \frac{2}{10\left(\frac{n\pi}{10}\right)} \int_0^{10} \frac{8x(10-x)^2}{10^3} \sin\left(\frac{n\pi}{10} x\right) dx$$

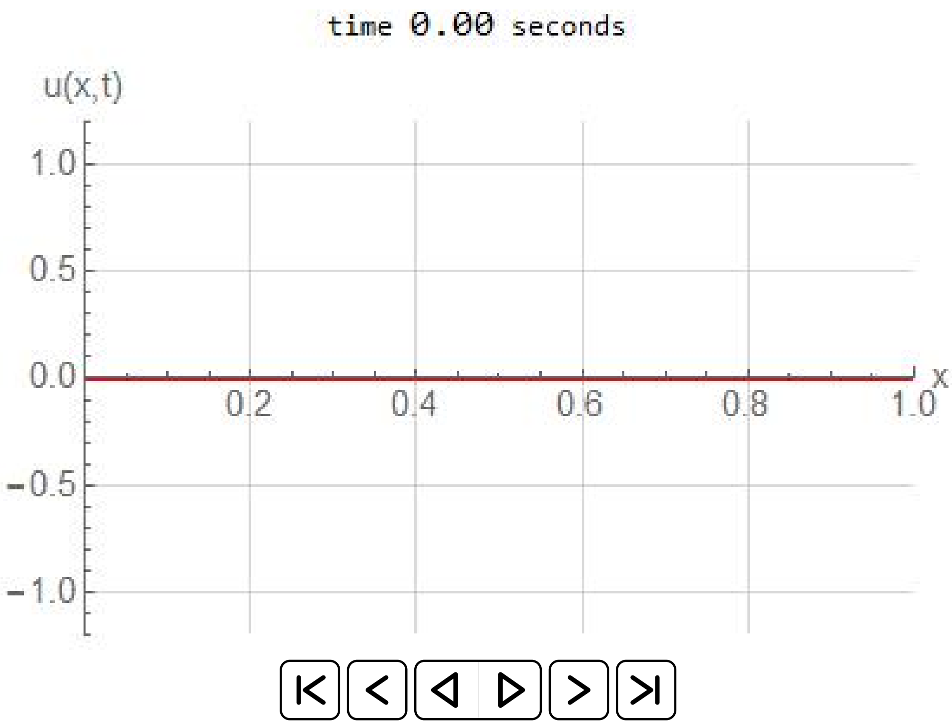
Integrating the above gives

$$c_n = \frac{320\left(2 + (-1)^n\right)}{n^4 \pi^4}$$

Therefore the solution is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{320\left(2+(-1)^n\right)}{n^4\pi^4} T_n(t) X_n(x) \\
&= \frac{320}{\pi^4} \sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^4} \sin\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right)
\end{aligned}
$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$
The following is an animation of the above solution for 40 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.21 Chapter 10.7, Problem 9

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x,t)$ in an elastic string of length L, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x,0) = f(x)$, where f is a given function.
Hint: Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x,t) = \sin \lambda_n x \cos \lambda_n at,$$

where $\lambda_n = (2n-1)\pi/(2L), n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

solution
The eigenvalue ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Boundary condition at $x = 0$ gives

$$0 = A$$

Therefore the solution becomes $X(x) = B\sin\left(\sqrt{\lambda}x\right)$. And $X'(x) = B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$. Applying boundary conditions at $x = L$ gives

$$0 = B\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)$$

Therefore

$$\sqrt{\lambda}L = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}$$

Hence

$$\sqrt{\lambda_n} = \frac{n\pi}{2L} \qquad n = 1, 3, 5, \cdots$$
$$\sqrt{\lambda_n} = \frac{(2n-1)\pi}{2L} \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X_n(x) = c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

And

$$T_n(t) = A_n \cos\left(\sqrt{\lambda_n}at\right) + B_n \sin\left(\sqrt{\lambda_n}at\right)$$
$$T_n'(t) = -A_n a\sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n}at\right) + B_n a\sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n}at\right)$$

Since initial velocity is zero, the above gives

$$0 = B_n a\sqrt{\lambda_n}$$

Which means $B_n = 0$. Hence

$$T_n(t) = A_n \cos\left(\sqrt{\lambda_n}at\right)$$

Therefore the complete solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{(2n-1)\pi}{2L}at\right) \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

c_n is found from initial position by applying orthogonality.

2.1.22 Chapter 10.7, Problem 10

10. Consider an elastic string of length L. The end $x = 0$ is held fixed, while the end $x = L$ is free; thus the boundary conditions are $u(0,t) = 0$ and $u_x(L,t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the displacement $u(x,t)$.

(b) With $L = 10$ and $a = 1$, plot u versus x for $0 \le x \le 10$ and for several values of t. Pay particular attention to values of t between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.

(c) With $L = 10$ and $a = 1$, plot u versus t for several values of x.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

Solution
Straight forward.

2.1.23 Chapter 10.8, Problem 3

3. (a) Find the solution $u(x,y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = f(y), \qquad 0 < y < b,$$
$$u(x,0) = h(x), \qquad u(x,b) = 0, \qquad 0 \le x \le a.$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a,y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x,0) = h(x)$.

(b) Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - (y/b)$.

(c) Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x, u versus y, u versus both x and y, and a contour plot.

Solution
To do.

2.1.24 Chapter 11.1, problem 12

Convert to form $(py')' + q(x)y = 0$

$$y'' - 2xy' + \lambda y = 0$$

Solution
Writing the ODE as $p(x)y'' + Q(x)y' + R(x)y = 0$, hence

$$\begin{aligned} p(x) &= 1 \\ Q(x) &= -2x \\ R(x) &= \lambda \end{aligned}$$

Then the new form is $(\mu(x)p(x)y')' + \mu(x)R(x)y = 0$, where

$$\begin{aligned} \mu(x) &= \frac{1}{p(x)} e^{\int^x \frac{Q(s)}{P(s)} ds} \\ &= e^{\int^x -2s ds} \\ &= e^{-x^2} \end{aligned}$$

Therefore the new form is

$$\left(e^{-x^2} y'\right)' + e^{-x^2} \lambda y = 0$$

2.1.25 Chapter 11.1, problem 13

Convert to form $(py')' + q(x)y = 0$

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0$$

Solution
Writing the ODE as $p(x)y'' + Q(x)y' + R(x)y = 0$, hence

$$\begin{aligned} p(x) &= x^2 \\ Q(x) &= x \\ R(x) &= \left(x^2 - v^2\right) \end{aligned}$$

The new form is $(\mu(x)p(x)y')' + \mu(x)R(x)y = 0$, where

$$\begin{aligned} \mu(x) &= \frac{1}{p(x)} e^{\int^x \frac{Q(s)}{P(s)} ds} \\ &= \frac{1}{x^2} e^{\int^x \frac{1}{s} ds} \\ &= \frac{1}{x^2} e^{|\ln x|} \\ &= \frac{1}{x^2} x \\ &= \frac{1}{x} \end{aligned}$$

Therefore the new form is

$$\left(\frac{1}{x} y'\right)' + \frac{1}{x}\left(x^2 - v^2\right) y = 0$$

2.1.26 Chapter 11.1, problem 18

18. Consider the boundary value problem

$$y'' + 4y' + (4 + 9\lambda)y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$$

(a) Determine, at least approximately, the real eigenvalues and the corresponding eigenfunctions by proceeding as in Problem 17(a, b).
(b) Also solve the given problem directly (without introducing a new variable).
Hint: In part (a) be sure to pay attention to the boundary conditions as well as the differential equation.

Solution

part (a)

Let $y(x) = s(x)u(x)$. Then $y' = s'u + su'$ and $y'' = s''u + s'u' + s'u' + su'' = s''u + 2(s'u') + su''$. Therefore the original ODE becomes

$$s''u + 2(s'u') + su'' + 4(s'u + su') + (4 + 9\lambda)su = 0$$

Collecting terms in u gives

$$su'' + u'(2s' + 4s) + (s'' + 4s' + (4 + 9\lambda)s)u = 0$$

Making u' term vanish requires that $2s' + 4s$ or $s' + 2s = 0$. Hence $\frac{d}{dx}\left(se^{2x}\right) = 0$ or $s = e^{-2x}$. Hence $s' = -2e^{-2x}, s'' = 4e^{-2x}$. Substituting these into the above gives

$$\begin{aligned} e^{-2x}u'' + \left(4e^{-2x} + 4\left(-2e^{-2x}\right) + (4 + 9\lambda)e^{-2x}\right)u &= 0 \\ u'' + (4 + 4(-2) + (4 + 9\lambda))u &= 0 \\ u'' + (4 - 8 + 4 + 9\lambda)u &= 0 \\ u'' + 9\lambda u &= 0 \end{aligned}$$

Let $9\lambda = \hat{\lambda}$ so the above becomes

$$u'' + \hat{\lambda}u = 0$$

With boundary conditions $u(0) = \frac{y(0)}{s(0)} = 0$ and $u'(L) = \frac{y'(L)}{s'(L)} = 0$. This was solved before, the eigenfunctions of the above are

$$\begin{aligned} \Phi_n(x) &= \sin\left(\sqrt{\hat{\lambda}_n}x\right) \\ \hat{\lambda}_n &= \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 3, 5, \cdots \end{aligned}$$

But $\hat{\lambda}_n = 9\lambda_n$, therefore the above becomes

$$\begin{aligned} \Phi_n(x) &= \sin\left(3\sqrt{\lambda_n}x\right) \\ \lambda_n &= \frac{1}{9}\left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Or

$$\Phi_n(x) = \sin\left(\frac{n\pi}{2L}x\right)$$

Now the eigenfunction is normalized

$$\begin{aligned} \int_0^1 (k_n\Phi_n(x))^2\,dx &= 1 \\ k_n^2 \int_0^1 \Phi_n(x)^2\,dx &= 1 \\ k_n^2 \int_0^1 \sin^2\left(\frac{n\pi}{2L}x\right)dx &= 1 \\ k_n^2\frac{L}{2} &= 1 \\ k_n &= \sqrt{\frac{2}{L}} \end{aligned}$$

Hence

$$k_n = \sqrt{\frac{2}{L}}$$

And

$$\hat{\Phi}_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{2L}x\right)$$

Mapping back to $y(x) = s(x)u(s)$, and since $s(x) = e^{-2x}$ then the eigenfunction in y space is

$$\Phi_n(x) = e^{-2x}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1, 3, 5, \cdots$$

Part b

Now the ODE is solved directly. $y'' + 4y' + (4 + 9\lambda)\, y = 0$. The characteristic equation is

$$r^2 + 4r + (4 + 9\lambda) = 0$$

Hence roots are

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(4 + 9\lambda)}}{2} \\ &= \frac{-4 \pm \sqrt{16 - 16 - 36\lambda}}{2} = -2 \pm 3\sqrt{-\lambda} \end{aligned}$$

We know that $\lambda > 0$. So the roots are $r = -2 \pm i\sqrt{\lambda}$ and the solution is

$$y(x) = e^{-2x}\left(A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)\right)$$

Applying boundary conditions $y(0) = 0$ leads to $A = 0$. So the solution becomes

$$y(x) = e^{-2x} B \sin\left(\sqrt{\lambda}x\right)$$

Hence

$$y'(x) = -2e^{-2x} B\sin\left(\sqrt{\lambda}x\right) + e^{-2x} B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

Applying second B.C. $y'(L) = 0$ the above becomes

$$\begin{aligned} 0 &= -2e^{-2L} B\sin\left(\sqrt{\lambda}L\right) + e^{-2L} B\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right) \\ &= B\left(-2\sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)\right) \end{aligned}$$

Non-trivial solution requires that

$$\begin{aligned} -2\sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right) &= 0 \\ -2\tan\sqrt{\lambda}L + \sqrt{\lambda} &= 0 \\ \tan\sqrt{\lambda}L &= \frac{1}{2}\sqrt{\lambda} \end{aligned}$$

Hence the direct method finds that the eigenvalues λ_n are the solutions to the above nonlinear equation and the corresponding eigenfunctions are $e^{-2x}\sin\left(\sqrt{\lambda_n}x\right)$.

2.1.27 Chapter 11.1, problem 19

Determine the real eigenvalues and eigenfunctions.

$$\begin{aligned} y'' + y' + \lambda(y' + y) &= 0 \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

Writing the ODE as

$$y'' + (1 + \lambda)\, y' + \lambda y = 0$$

Case $\lambda = 0$

$$y'' + y' = 0$$

The characteristic equation is

$$\begin{aligned} r^2 + r &= 0 \\ r(r + 1) &= 0 \end{aligned}$$

The roots are $r = 0, -1$. Hence the solution is $y = c_1 + c_2 e^{-x}$. Hence $y' = -c_2 e^{-x}$. First BC gives $y'(0) = 0 \to 0 = -c_2$. Therefore the solution becomes $y = c_1$. Second BC gives $y(1) = 0 \to 0 = c_1$. Therefore trivial solution and $\lambda = 0$ is not eigenvalue.

Case $\lambda < 0$ Let $\lambda = -m^2$ for some real m. The ODE becomes

$$y'' + \left(1 - m^2\right) y' - m^2 y = 0$$

The characteristic equation is

$$r^2 + \left(1 - m^2\right) r - m^2 = 0$$

The roots are

$$\begin{aligned} r &= \frac{-\left(1-m^2\right)}{2} \pm \frac{1}{2}\sqrt{\left(1-m^2\right)^2+4m^2} \\ &= \frac{-\left(1-m^2\right)}{2} \pm \frac{1}{2}\sqrt{1+m^4-2m^2+4m^2} \\ &= \frac{-\left(1-m^2\right)}{2} \pm \frac{1}{2}\sqrt{\left(1+m^2\right)^2} \\ &= \frac{-\left(1-m^2\right)}{2} \pm \frac{1}{2}\left(1+m^2\right) \end{aligned}$$

Hence roots are $r_1 = \frac{-(1-m^2)}{2} + \frac{1}{2}\left(1+m^2\right) = m^2$ and $r_2 = \frac{-(1-m^2)}{2} - \frac{1}{2}\left(1+m^2\right) = -1$. Therefore the solution is

$$y = c_1 e^{m^2 x} + c_2 e^{-x}$$

Hence $y' = m^2 c_1 e^{m^2 x} - c_2 e^{-x}$. First BC gives $y'(0) = 0 \to 0 = m^2 c_1 - c_2$ or $c_2 = m^2 c_1$. Therefore the solution becomes

$$\begin{aligned} y &= c_1 e^{m^2 x} + m^2 c_1 e^{-x} \\ &= c_1\left(e^{m^2 x} + m^2 e^{-x}\right) \end{aligned}$$

Second BC gives $y(1) = 0 \to 0 = c_1\left(e^{m^2} + m^2 e^{-1}\right)$ therefore $c_1 = 0$ and trivial solution. Hence $\lambda < 0$ is not eigenvalue.

<u>Case $\lambda > 0$</u> The characteristic equation is

$$r^2 + (1+\lambda) r + \lambda = 0$$

The roots are

$$\begin{aligned} r &= \frac{-(1+\lambda)}{2} \pm \frac{1}{2}\sqrt{(1+\lambda)^2 - 4\lambda} \\ &= \frac{-(1+\lambda)}{2} \pm \frac{1}{2}\sqrt{1+\lambda^2+2\lambda-4\lambda} \\ &= \frac{-(1+\lambda)}{2} \pm \frac{1}{2}\sqrt{(1-\lambda)^2} \\ &= \frac{-(1+\lambda)}{2} \pm \frac{1}{2}(1-\lambda) \end{aligned}$$

Hence roots are $r_1 = \frac{-1}{2}(1+\lambda) + \frac{1}{2}(1-\lambda) = -\lambda$ and $r_2 = \frac{-1}{2}(1+\lambda) - \frac{1}{2}(1-\lambda) = -1$. Therefore the solution is

$$y = c_1 e^{-\lambda x} + c_2 e^{-x}$$

Hence $y' = -\lambda c_1 e^{\lambda x} - c_2 e^{-x}$. First BC gives $y'(0) = 0 \to 0 = -\lambda c_1 - c_2$ or $c_2 = -\lambda c_1$. Therefore the solution becomes

$$\begin{aligned} y &= c_1 e^{-\lambda x} - \lambda c_1 e^{-x} \\ &= c_1\left(e^{-\lambda x} - \lambda e^{-x}\right) \end{aligned}$$

Second BC gives $y(1) = 0 \to 0 = c_1\left(e^{-\lambda} - \lambda e^{-1}\right)$ For non-trivial solution, we need $e^{-\lambda} - \lambda e^{-1} = 0$. The solution to this is $\lambda = 1$.
When $\lambda = 1$ the eigenfunction is

$$y(x) = c_1\left(e^{-x} - e^{-x}\right) = 0$$

But eigenfunction can not be zero. Therefore there is eigenvalue when $\lambda > 0$. Hence for all cases, there is no eigenvalue with corresponding nonzero eigenfunction.

2.1.28 Chapter 11.1, problem 20

Determine the real eigenvalues and eigenfunctions.

$$\begin{aligned} x^2 y'' - \lambda\left(xy' - y\right) &= 0 \\ y(1) &= 0 \\ y(2) - y'(2) &= 0 \end{aligned}$$

<u>Solution</u>
This is a Euler ODE. $x^2 y'' - \lambda x y' + \lambda y = 0$. Let $y = x^r$, then $y' = r x^{r-1}, y'' = r(r-1) x^{r-2}$. The ODE becomes

$$\begin{aligned} x^2 r(r-1) x^{r-2} - \lambda x r x^{r-1} + \lambda x^r &= 0 \\ r(r-1) x^r - \lambda r x^r + \lambda x^r &= 0 \\ r(r-1) - \lambda r + \lambda &= 0 \end{aligned}$$

Case $\lambda = 0$
The characteristic equation becomes

$$r(r-1) = 0$$

The roots are $r = 0, r = 1$, hence the solution is

$$y = c_1 + c_2 x$$

At BC $y(1) = 0 \to 0 = c_1 + c_2$. Hence $c_1 = -c_2$ and the solution becomes $y = c_1 - c_1 x = c_1(1-x)$. Hence $y' = -c_1$. Second BC $y(2) - y'(2) = 0$ gives

$$\begin{aligned}0 &= c_1(1-2) + c_1 \\ 0 &= -c_1 + c_1 \\ 0 &= 0\end{aligned}$$

Therefore any c_1 will work. Giving a solution

$$y = c_1(1-x)$$

Therefore $\lambda = 0$ is an eigenvalue with eigenfunction $\Phi_0(x) = 1 - x$.
Case $\lambda < 0$ Let $\lambda = -m^2$. The characteristic equation becomes

$$\begin{aligned}r(r-1) + m^2 r - m^2 &= 0 \\ r^2 - r + m^2 r - m^2 &= 0 \\ r^2 + r\left(m^2 - 1\right) - m^2 &= 0\end{aligned}$$

The roots are

$$\begin{aligned}r &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{\left(m^2-1\right)^2 + 4m^2} \\ &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{m^4 - 2m^2 + 1 + 4m^2} \\ &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{\left(1+m^2\right)^2} \\ &= -\frac{1}{2}\left(m^2-1\right) \pm \frac{1}{2}\left(1+m^2\right)\end{aligned}$$

Roots are $r = -\frac{1}{2}\left(m^2-1\right) + \frac{1}{2}\left(1+m^2\right) = 1$ or $r = -\frac{1}{2}\left(m^2-1\right) - \frac{1}{2}\left(1+m^2\right) = -m^2$. Hence solution is

$$y = c_1 x + c_2 x^{-m^2}$$

At BC $y(1) = 0 \to 0 = c_2$. Therefore the solution is $y = c_1 x$ and $y' = c_1$. Second BC gives $y(2) - y'(2) = 0$ or

$$\begin{aligned}0 &= 2c_1 - c_1 \\ 0 &= c_1\end{aligned}$$

Hence trivial solution. So $\lambda < 0$ is not an eigenvalue.
Case $\lambda > 0$
The characteristic equation becomes

$$\begin{aligned}r^2 - r - \lambda r + \lambda &= 0 \\ r^2 - r(1+\lambda) + \lambda &= 0\end{aligned}$$

The roots are

$$\begin{aligned}r &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{(1+\lambda)^2 - 4\lambda} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{1 + \lambda^2 - 2\lambda} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{(1-\lambda)^2} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}(1-\lambda)\end{aligned}$$

Roots are $r = \frac{1}{2}(1+\lambda) + \frac{1}{2}(1-\lambda) = 1$ or $r = \frac{1}{2}(1+\lambda) - \frac{1}{2}(1-\lambda) = \lambda$. Hence solution is

$$y = c_1 x + c_2 x^{\lambda}$$

This is similar to the case above for $\lambda < 0$. Hence there is no eigenvalue for $\lambda > 0$.

2.1.29 Chapter 11.2, problem 1

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y'(1) &= 0 \end{aligned} \tag{1}$$

Solution
The eigenfunction for the above problem can be easily found using chapter 10 methods to be

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

Where

$$\lambda_n = \frac{n\pi}{2L} = \frac{n\pi}{2}$$

The normalized $\hat{\Phi}_n(x) = k_n \Phi_n(x)$. Where

$$\int_0^1 \hat{\Phi}_n^2(x)\,dx = 1$$

Hence solving the above for k_n gives

$$\begin{aligned} \int_0^1 (k_n \Phi_n(x))^2\,dx &= 1 \\ k_n^2 \int_0^1 \Phi_n^2(x)\,dx &= 1 \end{aligned}$$

But $\int_0^1 \Phi_n^2(x)\,dx = \int_0^1 \sin^2\left(\sqrt{\lambda_n} x\right) dx = \int_0^1 \sin^2\left(\frac{n\pi}{2} x\right) dx = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned} k_n^2 \frac{1}{2} &= 1 \\ k_n &= \sqrt{2} \end{aligned}$$

Therefore

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\sin\left(\frac{n\pi}{2}x\right) \qquad n = 1, 3, 5, \cdots \\ &= \left\{\sqrt{2}\sin\left(\frac{\pi}{2}x\right), \sqrt{2}\sin\left(\frac{3\pi}{2}x\right), \sqrt{2}\sin\left(\frac{5\pi}{2}x\right), \cdots\right\} \end{aligned}$$

2.1.30 Chapter 11.2, problem 2

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned} \tag{1}$$

Solution
The eigenfunction for the above problem can be found using chapter 10 methods to be

$$\Phi_n(x) = \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

Where

$$\lambda_n = \frac{n\pi}{2L} = \frac{n\pi}{2}$$

The normalized $\hat{\Phi}_n(x) = k_n \Phi_n(x)$. Where

$$\int_0^1 \hat{\Phi}_n^2(x)\,dx = 1$$

Hence solving the above for k_n gives

$$\begin{aligned} \int_0^1 (k_n \Phi_n(x))^2\,dx &= 1 \\ k_n^2 \int_0^1 \Phi_n^2(x)\,dx &= 1 \end{aligned}$$

But $\int_0^1 \Phi_n^2(x)\,dx = \int_0^1 \cos^2\left(\sqrt{\lambda_n}x\right)dx = \int_0^1 \cos^2\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned} k_n^2 \frac{1}{2} &= 1 \\ k_n &= \sqrt{2} \end{aligned}$$

Therefore

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\cos\left(\frac{n\pi}{2}x\right) \qquad n = 1,3,5,\cdots \\ &= \left\{\sqrt{2}\cos\left(\frac{\pi}{2}x\right), \sqrt{2}\cos\left(\frac{3\pi}{2}x\right), \sqrt{2}\cos\left(\frac{5\pi}{2}x\right), \cdots\right\} \end{aligned}$$

2.1.31 Chapter 11.2, problem 3

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(1) &= 0 \end{aligned} \tag{1}$$

Solution
The eigenfunctions are first found. Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$
In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned} y &= c_1\cosh(\mu x) + c_2\sinh(\mu x) \\ y' &= c_1\mu\sinh(\mu x) + c_2\mu\cosh(\mu x) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\mu \\ c_2 &= 0 \end{aligned}$$

Hence solution becomes

$$y(x) = c_1\cosh(\mu x)$$

Second B.C. $y'(1) = 0$ gives

$$0 = c_1\mu\sinh(\mu)$$

But $\sinh(\mu)$ can not be zero since $\mu \neq 0$, hence $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.
Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y'(1) = 0$ gives

$$0 = 0$$

Therefore c_1 can be any value. Therefore $\lambda = 0$ is an eigenvalue and the corresponding eigenfunction is any constant, say 1.
Case $\lambda > 0$, The solution is

$$\begin{aligned} y(x) &= c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right) \\ y'(x) &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\sqrt{\lambda} \\ c_2 &= 0 \end{aligned}$$

The solution becomes

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right)$$

Second B.C. $y'(1) = 0$ gives

$$0 = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda}\right) = 0$ or $\sqrt{\lambda} = n\pi$ for $n = 1, 2, 3, \cdots$ Therefore

$$\lambda_n = (n\pi)^2 \qquad n = 1, 2, 3, \cdots$$

And the corresponding eigenfunctions are

$$\Phi_n(x) = \cos\left(\sqrt{\lambda} x\right) \qquad n = 1, 2, 3, \cdots$$

Hence

$$\begin{aligned} \Phi_0(x) &= 1 \\ \Phi_n(x) &= \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots \end{aligned}$$

The normalized $\hat{\Phi}_0(x) = k_0\Phi_0(x)$. Where

$$\int_0^1 r(x)\hat{\Phi}_0^2(x)\,dx = 1$$

But $r(x) = 1$. Therefore solving the above for k_0 gives

$$\begin{aligned} \int_0^1 (k_0\Phi_0(x))^2\,dx &= 1 \\ k_0^2\int_0^1 dx &= 1 \\ k_0 &= 1 \end{aligned}$$

And for $n = 1, 2, 3, \cdots$ we obtain

$$\begin{aligned} \int_0^1 \hat{\Phi}_n^2(x)\,dx &= 1 \\ \int_0^1 (k_n\Phi_n(x))^2\,dx &= 1 \\ k_n^2\int_0^1 \Phi_n^2(x)\,dx &= 1 \\ k_n^2\int_0^1 \cos^2\left(\sqrt{n\pi x}\right)\,dx &= 1 \end{aligned}$$

But $\int_0^1 \cos^2\left(\sqrt{n\pi x}\right) = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned} k_n^2\frac{1}{2} &= 1 \\ k_n &= \sqrt{2} \end{aligned}$$

Therefore

$$\hat{\Phi}_0(x) = 1$$

And for $n = 1, 2, 3, \cdots$

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\cos(n\pi x) \\ &= \left\{\sqrt{2}\cos(\pi x), \sqrt{2}\cos(2\pi x), \sqrt{2}\cos(3\pi x), \cdots\right\} \end{aligned}$$

2.1.32 Chapter 11.2, problem 4

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(1) + y(1) &= 0 \end{aligned} \tag{1}$$

Solution
The eigenfunctions for the above problem are first found. Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$
In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned} y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\ y' &= c_1\mu \sinh(\mu x) + c_2\mu \cosh(\mu x) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\mu \\ c_2 &= 0 \end{aligned}$$

Hence solution becomes

$$y(x) = c_1 \cosh(\mu x)$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$0 = c_1 (\cosh(\mu) + \mu \sinh(\mu))$$

But $\sinh(\mu)$ can not be negative since its argument is positive here. And $\cosh\mu$ is always positive. In addition $\cosh(\mu) + \mu\sinh(\mu)$ can not be zero since $\sinh(\mu)$ can not be zero as $\mu \neq 0$ and $\cosh(\mu)$ is not zero. Therefore $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.
Case $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$0 = c_1$$

This gives trivial solution. Therefore $\lambda = 0$ is not eigenvalue.
Case $\lambda > 0$, The solution is

$$\begin{aligned} y(x) &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \\ y'(x) &= -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\sqrt{\lambda} \\ c_2 &= 0 \end{aligned}$$

The solution becomes

$$y(x) = c_1 \cos\left(\sqrt{\lambda}x\right)$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$\begin{aligned} 0 &= c_1 \cos\left(\sqrt{\lambda}\right) - c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}\right) \\ &= c_1 \left(\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda} \sin\left(\sqrt{\lambda}\right)\right) \end{aligned}$$

For non-trivial solution the above implies

$$\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = 0 \tag{1}$$

Therefore the eigenvalues are the solution to the above nonlinear equation. And the corresponding eigenfunctions are

$$\Phi_n = \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Where λ_n are the roots of equation (1).
The normalized $\hat{\Phi}_n = k_n\Phi_n$ eigenfunctions are now found.

$$\int_0^1 r(x)\,\hat{\Phi}_n^2 dx = 1$$

Since the weight function is $r(x) = 1$, then

$$\begin{aligned}
\int_0^1 \hat{\Phi}_n^2 dx &= 1\\
\int_0^1 k_n^2\Phi_n^2 dx &= 1\\
k_n^2\int_0^1 \Phi_n^2 dx &= 1\\
k_n^2\int_0^1 \cos^2\left(\sqrt{\lambda_n}x\right) dx &= 1
\end{aligned}$$

But $\int_0^1 \cos^2(ax)\,dx = \left(\frac{x}{2} + \frac{\sin 2ax}{4a}\right)_0^1 = \left(\frac{x}{2} + \frac{\sin\left(2\sqrt{\lambda_n}x\right)}{4\sqrt{\lambda_n}}\right)_0^1 = \left(\frac{1}{2} + \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right) = \left(\frac{2\sqrt{\lambda_n}+\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)$. Hence the above becomes

$$\begin{aligned}
k_n^2 &= \frac{1}{\frac{2\sqrt{\lambda_n}+\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}}\\
&= \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} + \sin\left(2\sqrt{\lambda_n}\right)}
\end{aligned}$$

But $\sin(2a) = 2\sin a\cos a$ and the above can be written as

$$k_n^2 = \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} + 2\sin\left(\sqrt{\lambda_n}\right)\cos\sqrt{\lambda_n}}$$

But from (1) earlier, we found $\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = 0$ or $\cos\left(\sqrt{\lambda}\right) = \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)$. Substituting this into the above gives

$$k_n^2 = \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} + 2\sqrt{\lambda_n}\sin^2\left(\sqrt{\lambda_n}\right)}$$

And since $\lambda_n \neq 0$ the above simplifies to

$$\begin{aligned}
k_n^2 &= \frac{2}{1 + \sin^2\left(\sqrt{\lambda_n}\right)}\\
&= \frac{4}{4 + \sin^2\left(\sqrt{\lambda_n}\right)}
\end{aligned}$$

Therefore

$$k_n = \sqrt{\frac{2}{1 + \sin^2\left(\sqrt{\lambda_n}\right)}}$$

Since there is no closed form solution to λ_n as it is a root of nonlinear equation $\sqrt{\lambda_n}\tan\left(\sqrt{\lambda_n}L\right) = 1$. Hence the normalized eigenfunctions are

$$\begin{aligned}
\hat{\Phi}_n &= k_n\Phi_n\\
&= \frac{\sqrt{2}}{\sqrt{1 + \sin^2\left(\sqrt{\lambda_n}\right)}}\cos\left(\sqrt{\lambda_n}x\right)
\end{aligned}$$

2.1.33 Chapter 11.2, problem 5

Determine the normalized eigenfunction for

$$\begin{aligned} y'' - 2y' + (1+\lambda)\,y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned} \tag{1}$$

Solution

Let $y(x) = s(x)\,u(x)$. Then $y' = s'u + su'$ and $y'' = s''u + s'u' + s'u' + su'' = s''u + 2(s'u') + su''$. Therefore the original ODE becomes

$$s''u + 2(s'u') + su'' - 2(s'u + su') + (1+\lambda)\,su = 0$$

Collecting terms in u the above becomes

$$su'' + u'(2s' - 2s) + u\left((1+\lambda)\,s + s'' - 2s'\right) = 0$$

To get rid of u' we therefore want $2s' - 2s = 0$ or $s' - s = 0$. Hence the integrating factor is $I = e^{-x}$ and the solution is obtained from $\frac{d}{dx}(se^{-x}) = 0$ or $s = e^{x}$. Therefore, if $s = e^{x}$ then the original ODE becomes

$$\begin{aligned} e^{x}u'' + u\left((1+\lambda)\,e^{x} + e^{x} - 2e^{x}\right) &= 0 \\ u'' + u\left((1+\lambda) + 1 - 2\right) &= 0 \\ u'' + u\left((1+\lambda) - 1\right) &= 0 \\ u'' + \lambda u &= 0 \end{aligned}$$

With the boundary conditions $u(0) = \frac{y(0)}{s(0)} = \frac{y(0)}{e^{0}} = 0$ and $u(1) = \frac{y(1)}{s(1)} = 0$. Hence we need to find the eigenfunctions for

$$\begin{aligned} u'' + \lambda u &= 0 \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

But this we did before. It has $\Phi_n(x) = \sin(n\pi x)$ for $n = 1, 2, \cdots$. And the normalized $\hat{\Phi}_n(x) = \sqrt{2}\sin(n\pi x)$. Mapping this normalized eigenfunction back to $y(x)$ using the transformation $y(x) = s(x)\,u(x)$ gives the normalized eigenfunction in y space as

$$\hat{\Phi}_n(x) = e^{x}\sqrt{2}\sin(n\pi x) \qquad n = 1, 2, 3, \cdots$$

2.1.34 Chapter 11.2, Example 1 redone. page 690

Here, example 1 is solved again, but without using normalization. Showing that one does not need to normalize the eigenfunctions as the book shows and will get same answer. Solve

$$y'' + 2y = -x \tag{1}$$

With boundary conditions $y(0) = 0, y(1) + y'(1) = 0$. Using the method of eigenfunction expansion without normalization.

Solution

The idea behind solving using eigenfunction expansion, is that

$$-(py')' + q(x)\,y(x) = \mu r(x)\,y(x) + f(x) \tag{1A}$$

Is solved using the eigenfunctions of the corresponding homogeneous eigenvalue ODE

$$-(py')' + q(x)\,y(x) = \lambda r(x)\,y(x) \tag{2A}$$

Where in (1A) μ is just a constant. And in (2A), λ is an eigenvalue. Writing (1) in same form as (1A) leads to

$$\begin{aligned} -(y')' - 2y &= x \\ -(y')' &= 2y + x \end{aligned} \tag{3A}$$

Therefore $\mu = 2$ and $r(x) = 1$. The corresponding homogeneous eigenvalue problem is

$$-(y')' = \lambda y(x)$$

Or

$$y'' + \lambda y(x) = 0$$

With boundary conditions $y(0) = 0, y(1) + y'(1) = 0$. The solution of the above is used to solve (3A), which is the original ODE. The solution to the above eigenvalue problem was done before. The result is that λ_n is the solution of nonlinear equation

$$\sin\left(\sqrt{\lambda} x\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda} x\right) = 0$$

Solving this numerically for the first 10 eigenvalues gives

$$\lambda_n = \{4.116, 24.139, 63.659, 122.889, 201.851, 300.55, 418.987, 557.162, 715.077, 892.73\}$$

And the eigenfunctions are

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

Notice that the eigenfunction above is not normalized as in the text book. Now assuming that the solution of the original nonhomogeneous ODE (3A) is given by

$$y(x) = \sum_{n=1}^{\infty} b_n \Phi_n(x)$$

Where b_n is unknown as of now and substituting the above into (3A) gives

$$-\frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n \Phi_n(x) = 2\sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} q_n \Phi_n(x)$$

Where $\sum_{n=1}^{\infty} q_n \Phi_n(x)$ is the eigenfunction expansion of the forcing terms $-x$. In this expression q_n is still not known. Now assuming that differentiation can be moved inside the summation above (this needs conditions which assumed valid here). The above equation now becomes

$$-\sum_{n=1}^{\infty} b_n \Phi_n''(x) - 2\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{n=1}^{\infty} q_n \Phi_n(x) \tag{1A}$$

q_n is now found. This is done by applying orthogonality as follows. Let $x = \sum_{n=1}^{\infty} q_n \Phi_n(x)$. Multiplying both sides by $\Phi_m(x)$ and integrating over the domain gives

$$\int_0^1 x\Phi_m(x)\,dx = \sum_{n=1}^{\infty} q_n \int_0^1 \Phi_n(x)\Phi_m(x)\,dx$$

$$\int_0^1 x\Phi_m(x)\,dx = q_n \int_0^1 \Phi_m^2(x)\,dx \tag{2}$$

Since $\Phi_n(x)$ is not normalized, one can not replace the integral by 1 as in the book. But since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$, the integrals can be evaluated as follows. The right side of (2) is

$$\int_0^1 \sin^2\left(\sqrt{\lambda_n} x\right) dx = \frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}} \tag{3}$$

And the left side of (2) is found by integration by parts

$$\begin{aligned}\int_0^1 x\Phi_m(x)\,dx &= \int_0^1 x\sin\left(\sqrt{\lambda_n} x\right) dx \\ &= \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n}\end{aligned} \tag{4}$$

Using (3) and (4) in (2) q_n is solved for giving

$$\frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n} = q_n\left(\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)$$

$$q_n = \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n\left(\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)} \tag{5}$$

Now that q_n is known, b_n is found from (1A)

$$-\sum_{n=1}^{\infty} b_n \Phi_n''(x) - 2\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{n=1}^{\infty} q_n \Phi_n(x)$$

Since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ then $\Phi_n'(x) = \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right)$, $\Phi_n''(x) = -\lambda_n \sin\left(\sqrt{\lambda_n}x\right) = -\lambda_n\Phi_n(x)$ and the above simplifies to

$$\sum_{n=1}^{\infty} b_n\lambda_n\Phi_n(x) - 2\sum_{n=1}^{\infty} b_n\Phi_n(x) = \sum_{n=1}^{\infty} q_n\Phi_n(x)$$

Canceling summations and also $\Phi_n(x)$ since $\Phi_n(x) \neq 0$ the above simplifies to

$$\begin{aligned} b_n\lambda_n - 2b_n &= q_n \\ b_n &= \frac{q_n}{\lambda_n - 2} \end{aligned}$$

Hence the solution to the original ODE is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} b_n\Phi_n(x) \\ &= \sum_{n=1}^{\infty}\left(\frac{q_n}{\lambda_n - 2}\right)\sin\left(\sqrt{\lambda_n}x\right) \end{aligned}$$

Using the value found for q_n in (5), the above becomes

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n - 2)} \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}} \sin\left(\sqrt{\lambda_n}x\right) \tag{6}$$

The above is the solution, found without normalization. The book solution is

$$y(x) = 4\sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n - 2)} \frac{1}{\left(1 + \cos^2\left(\sqrt{\lambda_n}\right)\right)} \sin\left(\sqrt{\lambda_n}x\right) \tag{7}$$

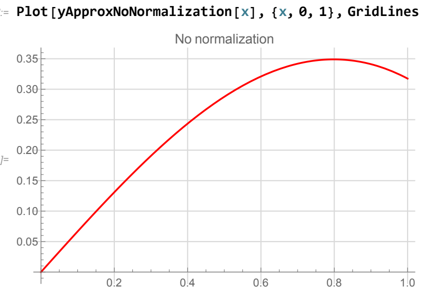
To show that (6) and (7) are actually the same, they are plotted against each others, using 10 terms in the sum, which is more than enough. The result shows identical plots.

Find eigenvalues numerically

```
In[◦]:= ClearAll[y, z, x, λ]
        eigenvalues = x /. NSolve[Sin[x] + x Cos[x] == 0 && 0 < x < 30, x];
        z = eigenvalues^2
Out[◦]= {4.11586, 24.1393, 63.6591, 122.889, 201.851, 300.55, 418.987, 557.162, 715.077, 892.73}
```

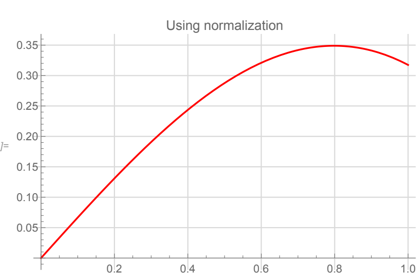
This is the solution without normalization

```
In[◦]:= max = Length@z;

        yApproxNoNormalization[x_] := Sum[λ = z[[n]]; 1/(λ (λ - 2)) ((Sin[√λ] - √λ Cos[√λ])/(1/2 - Sin[2 √λ]/(4 √λ))) Sin[√λ x], {n, 1, max}]

In[◦]:= Plot[yApproxNoNormalization[x], {x, 0, 1}, GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, PlotLabel → "No normalization"]
```

Out[◦]=


This is the solution using normalization (book solution)

```
In[◦]:= yApproxBook[x_] := 4 Sum[λ = z[[n]]; Sin[√λ]/(λ (λ - 2) (1 + Cos[√λ]^2)) Sin[√λ x], {n, 1, max}];

        Plot[yApproxBook[x], {x, 0, 1}, GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, PlotLabel → "Using normalization"]
```

Out[◦]=


They also plotted against the solution found using standard methods, which is

$$y = \frac{\sin\left(\sqrt{2}x\right)}{\sin\left(\sqrt{2}\right) + \sqrt{2}\cos\left(\sqrt{2}\right)} - \frac{x}{2}$$

And both (6,7) matched exactly the above solution.

2.1.35 Chapter 11.2 Problem 14

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned} y'' + y' + 2y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution
The ODE can be written as $(y'+y)' + 2y = 0$. Hence the operator is

$$L[y] = (y'+y)' + 2y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned} \langle L[u], v\rangle &= \int_0^1 L[u]\, v dx \\ &= \int_0^1 \left[(u'+u)' + 2u\right] v dx \\ &= \int_0^1 (u'+u)' v + uv dx \\ &= \int_0^1 \overbrace{(u'+u)'}^{dv} \overbrace{v}^{u} dx + \int_0^1 uv dx \end{aligned} \tag{1}$$

integration by parts of the above gives

$$\begin{aligned} \langle L[u], v\rangle &= [(u'+u)v]_0^1 - \int_0^1 (u'+u) v' dx + \int_0^1 uv dx \\ &= [(u'+u)v]_0^1 - \int_0^1 (u'v' + uv') dx + \int_0^1 uv dx \\ &= [(u'+u)v]_0^1 - \left(\int_0^1 u'v' dx + \int_0^1 uv' dx\right) + \int_0^1 uv dx \end{aligned}$$

Integrating by parts the term $\int_0^1 u'v'dx = [uv']_0^1 - \int_0^1 uv''dx$ the above becomes

$$\begin{aligned} \langle L[u], v\rangle &= [(u'+u)v]_0^1 - \left([uv']_0^1 - \int_0^1 uv''dx + \int_0^1 uv'dx\right) + \int_0^1 uv dx \\ &= [(u'+u)v - uv']_0^1 - \left(-\int_0^1 uv''dx + \int_0^1 uv'dx\right) + \int_0^1 uv dx \\ &= [(u'+u)v - uv']_0^1 + \int_0^1 uv''dx - \int_0^1 uv'dx + \int_0^1 uv dx \\ &= [(u'+u)v - uv']_0^1 + \int_0^1 (v'' - v' + v) u dx \end{aligned}$$

The above can never be $\langle u, L[v]\rangle$ even if the boundary terms vanish, since $\int_0^1 (v''-v'+v)udx \neq \int_0^1 (v''+v'+v)udx$. There is a different sign in the operator obtained. Hence the ode is not self adjoint.

2.1.36 Chapter 11.2, Problem 15

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned} \left(1+x^2\right) y'' + 2xy' + y &= 0 \\ y'(0) &= 0 \\ y(1) + 2y'(1) &= 0 \end{aligned}$$

Solution
The ODE can be written as

$$\left(\left(1+x^2\right) y'\right)' + y = 0$$

The operator

$$L[y] = \left(\left(1+x^2\right) y'\right)' + y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u], v\rangle &= \int_0^1 L[u]\, v dx\\ &= \int_0^1 \left[\left(\left(1+x^2\right)u'\right)' + u\right] v dx\\ &= \int_0^1 \left(\left(1+x^2\right)u'\right)' v + uv dx\\ &= \int_0^1 \left(\left(1+x^2\right)u'\right)' v dx + \int_0^1 uv dx \end{aligned} \tag{1}$$

Starting with the first integral in (1) and using integration by parts

$$\int_0^1 \left(\left(1+x^2\right)u'\right)' v dx = \int_0^1 \overbrace{\left(\left(1+x^2\right)u'\right)'}^{dv} \overbrace{v}^{u} dx$$

By integration by parts, where $\int u dv = |uv| - \int v du$, the above becomes

$$\begin{aligned}\int_0^1 \left(\left(1+x^2\right)u'\right)' v dx &= \left[\left(1+x^2\right)u'v\right]_0^1 - \int_0^1 \left(1+x^2\right)u'v' dx\\ &= \left[\left(1+x^2\right)u'v\right]_0^1 - \int_0^1 \overbrace{\left(1+x^2\right)v'}^{u} \overbrace{u'}^{dv} dx\end{aligned}$$

Doing integration by parts again. But notice the choice of u and dv made above. This is important in order to get to the form needed. The above becomes

$$\begin{aligned}\int_0^1 \left(\left(1+x^2\right)u'\right)' v dx &= \left[\left(1+x^2\right)u'v\right]_0^1 - \left(\left[u\left(1+x^2\right)v'\right]_0^1 - \int_0^1 \left(\left(1+x^2\right)v'\right)' u dx\right)\\ &= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \int_0^1 \left(\left(1+x^2\right)v'\right)' u dx\end{aligned}$$

Going back to (1) and adding the second integral which is left there gives

$$\begin{aligned}\langle L[u], v\rangle &= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \int_0^1 \left(\left(1+x^2\right)v'\right)' u dx + \int_0^1 uv dx\\ &= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \int_0^1 \left[\left(\left(1+x^2\right)v'\right)' + v\right] u dx\end{aligned}$$

But $\int_0^1 \left[\left(\left(1+x^2\right)v'\right)' + v\right] u dx = \langle u, L[v]\rangle$, hence the above becomes

$$\langle L[u], v\rangle = \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \langle u, L[v]\rangle \tag{2}$$

We are almost there. If the boundary terms above all go to zero, then it is self-adjoint. If the boundary terms do not vanish, then the problem is not self adjoint. Evaluating the boundary terms in (2)

$$\begin{aligned}\Delta &= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1\\ &= \left[2u'(1)v(1) - 2u(1)v'(1)\right] - \left[u'(0)v(0) - u(0)v'(0)\right]\end{aligned}$$

Since $u'(0) = 0$ and $v'(0) = 0$, from the given boundary conditions, then above simplifies to

$$\Delta = 2\left(u'(1)v(1) - u(1)v'(1)\right)$$

But $u(1) = -2u'(1)$ and $v(1) = -2v'(1)$, hence the above becomes

$$\begin{aligned}\Delta &= 2\left(u'(1)\left(-2v'(1)\right) - \left(-2u'(1)\right)v'(1)\right)\\ &= 4\left(-u'(1)v'(1) + u'(1)v'(1)\right)\\ &= 0\end{aligned}$$

Since the boundary terms Δ vanish, then from (2)

$$\langle L[u], v\rangle = \langle u, L[v]\rangle \tag{3}$$

Hence the ODE is self-adjoint.

2.1.37 Chapter 11.2, Problem 16

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned} y'' + y &= \lambda y \\ y(0) - y'(1) &= 0 \\ y'(0) - y(1) &= 0 \end{aligned}$$

Solution
The operator is

$$L[y] = y'' + y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned} \langle L[u], v\rangle &= \int_0^1 L[u]\, v dx \\ &= \int_0^1 (u'' + u)\, v dx \\ &= \int_0^1 u'' v dx + \int_0^1 uv dx \\ &= \int_0^1 \overbrace{u''}^{dv} \overbrace{v}^{u} dx + \int_0^1 uv dx \end{aligned} \tag{1}$$

Integrating by parts

$$\langle L[u], v\rangle = [u'v]_0^1 - \int_0^1 \overbrace{u'}^{dv} \overbrace{v'}^{u} dx + \int_0^1 uv dx$$

Integrating by parts again

$$\begin{aligned} \langle L[u], v\rangle &= [u'v]_0^1 - \left([uv']_0^1 - \int_0^1 uv'' dx\right) + \int_0^1 uv dx \\ &= [u'v - uv']_0^1 + \int_0^1 uv'' dx + \int_0^1 uv dx \\ &= [u'v - uv']_0^1 + \int_0^1 (v'' + v)\, u dx \\ &= [u'v - uv']_0^1 + \langle u, L[v]\rangle \end{aligned} \tag{2}$$

Hence if the boundary terms vanish, then it is self adjoint else it is not. Evaluating the boundary terms in (2)

$$\begin{aligned} \Delta &= [u'v - uv']_0^1 \\ &= [u'(1)\, v(1) - u(1)\, v'(1)] - [u'(0)\, v(0) - u(0)\, v'(0)] \end{aligned}$$

But $u'(1) = u(0)$ and $v'(1) = v(0)$ and $u'(0) = u(1)$ and $v'(0) = v(1)$ from the given boundary conditions. Substituting these into the above gives

$$\begin{aligned} \Delta &= [u(0)\, v(1) - u(1)\, v(0)] - [u(1)\, v(0) - u(0)\, v(1)] \\ &= 2u(1)\, v(0) \\ &\neq 0 \end{aligned}$$

Since the boundary terms Δ do not vanish, then from (2)

$$\langle L[u], v\rangle \neq \langle u, L[v]\rangle$$

Hence the ODE is not self-adjoint.

2.1.38 Chapter 11.2, Problem 17

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned} \left(1 + x^2\right) y'' + 2xy' + y &= \lambda \left(1 + x^2\right) y \\ y(0) - y'(1) &= 0 \\ y'(0) + 2y(1) &= 0 \end{aligned}$$

Solution
The ode can be written as

$$\left(\left(1+x^2\right)y'\right)'+y=\lambda\left(1+x^2\right)y$$

Hence the operator is

$$L[y]=\left(\left(1+x^2\right)y'\right)'+y$$

The ODE is self-adjoint if

$$\langle L[u],v\rangle=\langle u,L[v]\rangle$$

For any two functions u,v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u],v\rangle &= \int_0^1 L[u]\,v\,dx\\ &= \int_0^1\left(\left(\left(1+x^2\right)u'\right)'+u\right)v\,dx\\ &= \int_0^1 \overbrace{\left(\left(1+x^2\right)u'\right)'}^{dv}\overbrace{v}^{u}\,dx+\int_0^1 uv\,dx\end{aligned}$$

Integrating by parts

$$\begin{aligned}\langle L[u],v\rangle &= \left[\left(1+x^2\right)u'v\right]_0^1-\int_0^1\left(1+x^2\right)u'v'dx+\int_0^1 uv\,dx\\ &= \left[\left(1+x^2\right)u'v\right]_0^1-\int_0^1 \overbrace{\left(1+x^2\right)v'}^{u}\overbrace{u'}^{dv}\,dx+\int_0^1 uv\,dx\end{aligned}$$

Integrating by parts

$$\begin{aligned}\langle L[u],v\rangle &= \left[\left(1+x^2\right)u'v\right]_0^1-\left(\left[u\left(1+x^2\right)v'\right]_0^1-\int_0^1\left(\left(1+x^2\right)v'\right)'u\,dx\right)+\int_0^1 uv\,dx\\ &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\int_0^1\left(\left(1+x^2\right)v'\right)'u\,dx+\int_0^1 uv\,dx\\ &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\int_0^1\left[\left(\left(1+x^2\right)v'\right)'+v\right]u\,dx\\ &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\langle u,L[v]\rangle\end{aligned}$$

Therefore, if the boundary terms vanish, then the ODE is self adjoint.

$$\Delta=\left[2u'(1)v(1)-2u(1)v'(1)\right]-\left[u'(0)v(0)-u(0)v'(0)\right]$$

But $u'(1)=u(0)$ and $v'(1)=v(0)$ and $u'(0)=2u(1)$ and $v'(0)=2v(1)$, from the given boundary conditions. Substituting these in the above gives

$$\begin{aligned}\Delta &= \left[2u(0)v(1)-2u(1)v(0)\right]-\left[2u(1)v(0)-u(0)2v(1)\right]\\ &= 2u(0)v(1)-2u(1)v(0)-2u(1)v(0)+u(0)2v(1)\\ &= 4u(0)v(1)-4u(1)v(0)\\ &= 0\end{aligned}$$

Hence $\langle L[u],v\rangle=\langle u,L[v]\rangle$, therefore the ODE is self-adjoint.

2.1.39 Chapter 11.2, Problem 18

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned}y''+\lambda y&=0\\ y(0)&=0\\ y(\pi)+y'(\pi)&=0\end{aligned}$$

Solution
The ode can be written as

$$y''=-\lambda y$$

Hence $L[y]=y''$. The ODE is self-adjoint if

$$\langle L[u],v\rangle=\langle u,L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u], v\rangle &= \int_0^\pi L[u] v dx \\ &= \int_0^\pi u'' v dx\end{aligned}$$

Integrating by parts once

$$\langle L[u], v\rangle = [u'v]_0^\pi - \int_0^\pi u'v'dx$$

Integrating by parts again

$$\begin{aligned}\langle L[u], v\rangle &= [u'v]_0^\pi - \left([uv']_0^\pi - \int_0^\pi uv''dx\right) \\ &= [u'v - uv']_0^\pi + \int_0^\pi uv''dx \\ &= [u'v - uv']_0^\pi + \langle u, L[v]\rangle\end{aligned}$$

Now we will check if the boundary terms vanish or not.

$$\begin{aligned}\Delta &= [u'v - uv']_0^\pi \\ &= [u'(\pi) v(\pi) - u(\pi) v'(\pi)] - [u'(0) v(0) - u(0) v'(0)]\end{aligned}$$

Since $u(0) = 0, v(0) = 0$ then the above simplifies to

$$\Delta = u'(\pi) v(\pi) - u(\pi) v'(\pi)$$

But $u'(\pi) = -u(\pi)$ and $v'(\pi) = -v(\pi)$ the above becomes

$$\begin{aligned}\Delta &= -u(\pi) v(\pi) + u(\pi) v(\pi) \\ &= 0\end{aligned}$$

Hence $\langle L[u], v\rangle = \langle u, L[v]\rangle$ and the ODE is self adjoint.

2.1.40 Chapter 11.3, Problem 1

Solve by method of eigenfunction expansion

$$\begin{aligned}y'' + 2y &= -x \\ y(0) &= 0 \\ y(1) &= 0\end{aligned}$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y(0) = 0, y(1) = 0$. This was solved before.

$$\begin{aligned}\hat{\Phi}_n(x) &= \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right) \\ \lambda_n &= (n\pi)^2 \qquad n = 1, 2, 3, \cdots\end{aligned}$$

Hence eigenvalues are $\lambda_n = \{\pi^2, 4\pi^2, 9\pi^2, \cdots\}$. None of the eigenvalues is 2. Therefore the solution to the original ODE can be assumed to be

$$y = \sum_{n=1}^\infty b_n \hat{\Phi}_n(x) \tag{1}$$

Substituting this into the original ODE gives

$$\sum_{n=1}^\infty b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^\infty b_n \hat{\Phi}_n(x) = -x$$

Expanding $-x$ using same basis function as the solution gives

$$\sum_{n=1}^\infty b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^\infty b_n \hat{\Phi}_n(x) = \sum_{n=1}^\infty q_n \hat{\Phi}_n(x) \tag{2}$$

Where q_n is found by applying orthogonality on

$$-x = \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x)$$
$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} q_n \int_0^1 \hat{\Phi}_n(x)\,\hat{\Phi}_m(x)\,dx$$
$$= q_m \int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since normalized, $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = q_m$$

But $\hat{\Phi}_m(x) = \sqrt{2}\sin(n\pi x)$ and the above becomes

$$-\sqrt{2}\int_0^1 x\sin(n\pi x)\,dx = q_n$$

Using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$ the above gives

$$-\sqrt{2}\left(\frac{\sin(n\pi x)}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right)_0^1 = q_n$$
$$-\sqrt{2}\left(\frac{\sin(n\pi)}{(n\pi)^2} - \frac{\cos(n\pi)}{n\pi}\right) = q_n$$
$$\sqrt{2}\left(\frac{\cos(n\pi)}{n\pi}\right) = q_n$$
$$\sqrt{2}\left(\frac{-1^n}{n\pi}\right) = q_n$$

Now that q_n is found, then b_n can be solved for form (2) above giving

$$\sum_{n=1}^{\infty} b_n\hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n\hat{\Phi}_n(x) = \sum_{n=1}^{\infty}\sqrt{2}\left(\frac{-1^n}{4n\pi}\right)\hat{\Phi}_n(x) \tag{2A}$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ since the eigenfunction satisfy the ode $y'' = -\lambda y$ and the above simplifies to

$$-\sum_{n=1}^{\infty} b_n\lambda_n\hat{\Phi}_n(x) + 2\sum_{n=1}^{\infty} b_n\hat{\Phi}_n(x) = \sum_{n=1}^{\infty}\sqrt{2}\left(\frac{-1^n}{4n\pi}\right)\hat{\Phi}_n(x)$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$-b_n\lambda_n + 2b_n = \sqrt{2}\left(\frac{-1^n}{n\pi}\right)$$

Therefore

$$b_n = \frac{\sqrt{2}\left(\frac{-1^n}{n\pi}\right)}{2-\lambda_n}$$
$$= \frac{\sqrt{2}(-1)^n}{\left(2-(n\pi)^2\right)n\pi}$$

Therefore the solution from (1) is

$$y = \sum_{n=1}^{\infty}\frac{\sqrt{2}(-1)^n}{\left(2-(n\pi)^2\right)n\pi}\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n(x) = \sqrt{2}\Phi_n(x) = \sqrt{2}\sin(n\pi x)$ and the above becomes

$$y = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(2-(n\pi)^2\right)n\pi}\sin(n\pi x)$$

Or

$$y = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\left((n\pi)^2-2\right)n\pi}\sin(n\pi x)$$

2.1.41 Chapter 11.3, Problem 2

Solve by method of eigenfunction expansion

$$
\begin{aligned}
y'' + 2y &= -x \\
y(0) &= 0 \\
y'(1) &= 0
\end{aligned}
$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y(0) = 0, y'(1) = 0$. This was solved before.

$$
\begin{aligned}
\Phi_n(x) &= \sin\left(\sqrt{\lambda_n} x\right) \\
\lambda_n &= \left(\frac{n\pi}{2}\right)^2 \qquad n = 1, 3, 5, \cdots
\end{aligned}
$$

Or, to keep the sum continuous, it can be written as

$$
\lambda_n = \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n = 1, 2, 3, \cdots
$$

The normalized eigenfunctions weight k_n is found from solving $\int_0^1 k_n^2 \sin^2\left(\frac{n\pi}{2}x\right) dx = 1$ which results in $k_n = \sqrt{2}$

Hence

$$
\hat{\Phi}_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \qquad n = 1, 2, 3, \cdots
$$

The eigenvalues are $\lambda_n = \left\{\left(\frac{\pi}{2}\right)^2, 9\left(\frac{\pi}{2}\right)^2, 25\left(\frac{\pi}{2}\right)^2, \cdots\right\}$. None of the eigenvalues is 2. Therefore the solution to the original ODE can be assumed to be

$$
y = \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}
$$

Substituting this into the original ODE gives

$$
\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = -x
$$

Expanding $-x$ using same basis function as the solution gives

$$
\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \tag{2}
$$

Where q_n is found by applying orthogonality on

$$
\begin{aligned}
-x &= \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \\
-\int_0^1 x\hat{\Phi}_m(x)\,dx &= \sum_{n=1}^{\infty} q_n \int_0^1 \hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx \\
&= q_m \int_0^1 \hat{\Phi}_m^2(x)\,dx
\end{aligned}
$$

Since normalized, $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$
-\int_0^1 x\hat{\Phi}_m(x)\,dx = q_m
$$

But $\hat{\Phi}_m(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)$ and the above becomes

$$
-\sqrt{2}\int_0^1 x\sin\left((2n-1)\frac{\pi}{2}x\right) dx = q_n
$$

Using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$ the above gives

$$
\begin{aligned}
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{x\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right)_0^1 &= q_n \\
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right)_0^1 &= q_n \\
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) &= q_n
\end{aligned}
$$

Using $\sin\left((2n-1)\frac{\pi}{2}\right) = -\cos(n\pi)$ which for $n = 1, 2, 3, \cdots$ can be written as $-(-1)^n$ or $(-1)^{n+1}$. The above simplifies to

$$-\sqrt{2}\left(\frac{(-1)^{n+1}}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) = q_n$$
$$\sqrt{2}\left(\frac{(-1)^{n}}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) = q_n$$

Now that q_n is found, then b_n can be solved for form (2) above giving

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x) \tag{2A}$$

But $\hat{\Phi}_n''(x) = -\lambda_n \hat{\Phi}_n(x)$ since the eigenfunction satisfy the ode $y'' = -\lambda y$ and the above simplifies to

$$-\sum_{n=1}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x)$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$-b_n \lambda_n + 2b_n = \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}$$

Therefore

$$\begin{aligned} b_n &= \frac{\frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}}{2-\lambda_n} \\ &= \frac{\frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}}{\left(2-\left((2n-1)\frac{\pi}{2}\right)^2\right)} \\ &= \frac{(-1)^n \sqrt{2}}{\left(2-(2n-1)^2\left(\frac{\pi}{2}\right)^2\right)\left((2n-1)\frac{\pi}{2}\right)^2} \end{aligned}$$

Therefore the solution from (1) is

$$y = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left(2-(2n-1)^2\left(\frac{\pi}{2}\right)^2\right)\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x)$$

But $\hat{\Phi}_n(x) = \sqrt{2}\Phi_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)$ and the above becomes

$$y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left((2n-1)^2\left(\frac{\pi}{2}\right)^2-2\right)\left((2n-1)\frac{\pi}{2}\right)^2} \sin\left((2n-1)\frac{\pi}{2}x\right)$$

Since $(2n-1)\frac{\pi}{2} = \left(n-\frac{1}{2}\right)\pi$, the above can also be written as (to match back of book solution)

$$y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left(\left(n-\frac{1}{2}\right)^2\pi^2-2\right)\left(\left(n-\frac{1}{2}\right)\pi\right)^2} \sin\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

2.1.42 Chapter 11.3, Problem 3

Solve by method of eigenfunction expansion

$$\begin{aligned} y'' + 2y &= -x \\ y'(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y'(0) = 0, y'(1) = 0$. This was solved above in Chapter 11.2, problem 3. The eigenvalues are

$$\begin{aligned} \lambda_n &= \left\{0, \pi^2, (2\pi)^2, (3\pi)^2, \cdots\right\} \\ &= (n\pi)^2 \qquad n = 0, 1, 2, \cdots \end{aligned}$$

The normalized eigenfunctions are

$$\hat{\Phi}_0(x) = 1$$

And for $n = 1, 2, 3, \cdots$

$$\begin{aligned}\hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x)\\ &= \sqrt{2}\cos(n\pi x)\\ &= \left\{\sqrt{2}\cos(\pi x), \sqrt{2}\cos(2\pi x), \sqrt{2}\cos(3\pi x), \cdots\right\}\end{aligned}$$

Since none of the eigenvalues is 2, the solution to the original ODE can be assumed to be

$$y = \sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}$$

Substituting this into the original ODE gives

$$\sum_{n=0}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = -x$$

Expanding $-x$ using same basis function as the solution gives

$$\sum_{n=0}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x) \tag{2}$$

Where c_n is found by applying orthogonality on

$$\begin{aligned}-x &= \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x)\\ -\int_0^1 x\hat{\Phi}_m(x)\,dx &= \sum_{n=0}^{\infty} c_n \int_0^1 \hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx\\ &= c_m \int_0^1 \hat{\Phi}_m^2(x)\,dx\end{aligned}$$

Since normalized then $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$-\int_0^1 x\hat{\Phi}_n(x)\,dx = c_n$$

For $n = 0$ the eigenfunction is $\hat{\Phi}_0(x) = 1$ and the above gives $c_0 = -\frac{1}{2}\left[x^2\right]_0^1 = -\frac{1}{2}$ and for $n > 0$ the eigenfunction is $\hat{\Phi}_n(x) = \sqrt{2}\cos(n\pi x)$ and the integrals becomes

$$-\sqrt{2}\int_0^1 x\cos(n\pi x)\,dx = c_n$$

Using $\int x\cos(ax)\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$ the above gives

$$\begin{aligned}c_n &= -\sqrt{2}\left(\frac{\cos(n\pi x)}{(n\pi)^2} + \frac{x\sin(n\pi x)}{n\pi}\right)_0^1\\ &= -\sqrt{2}\left(\frac{\cos(n\pi)}{(n\pi)^2} + \frac{\sin(n\pi)}{n\pi} - \frac{1}{(n\pi)^2}\right)\\ &= -\sqrt{2}\left(\frac{\cos(n\pi)}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right)\\ &= \frac{-\sqrt{2}}{(n\pi)^2}(\cos(n\pi) - 1) \qquad n = 1, 2, \cdots\end{aligned}$$

When n is odd then $c_n = \frac{2\sqrt{2}}{(n\pi)^2}$ and when n is even it is zero. Now that q_n is found, then b_n can be solved for form (2) above giving

$$\sum_{n=0}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x) \tag{2A}$$

But $\hat{\Phi}_n''(x) = -\lambda_n \hat{\Phi}_n(x)$ since the eigenfunction satisfies the ode $y'' = -\lambda y$ and the above simplifies to

$$-\sum_{n=0}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x)$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$\begin{aligned} -b_n\lambda_n + 2b_n &= c_n \\ b_n &= \frac{c_n}{2-\lambda_n} \end{aligned}$$

Therefore the solution from (1) is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{c_n}{2-\lambda_n}\hat{\Phi}_n(x) \\ &= \frac{c_0}{2-\lambda_0}\hat{\Phi}_0(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{c_n}{2-\lambda_n}\hat{\Phi}_n(x) \end{aligned}$$

But $\lambda_0 = 0, c_0 = -\frac{1}{2}$and $\hat{\Phi}_0(x) = 1$, therefore the above becomes

$$\begin{aligned} y(x) &= -\frac{1}{4} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{\frac{2\sqrt{2}}{(n\pi)^2}}{2-(n\pi)^2}\sqrt{2}\cos(n\pi x) \\ &= -\frac{1}{4} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{\left(2-(n\pi)^2\right)(n\pi)^2}\sqrt{2}\cos(n\pi x) \\ &= -\frac{1}{4} - 4\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{\left((n\pi)^2-2\right)(n\pi)^2}\cos(n\pi x) \end{aligned}$$

To make the sum continuous, let $m = (2n-1)$ and now m runs from $1, 2, 3, \cdots$ and above becomes

$$y(x) = -\frac{1}{4} - 4\sum_{n=1,3,5,\cdots}^{\infty} \frac{\cos((2n-1)\pi x)}{\left(((2n-1)\pi)^2-2\right)((2n-1)\pi)^2}$$

2.1.43 Chapter 11.3, Problem 10

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned} y'' + \pi^2 y &= a + x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution
The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore one can see that λ_1 is eigenvalue in the original ODE $y'' + \pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to the specific eigenfunction $\Phi_1(x)$. Therefore the condition is

$$\begin{aligned} \int_0^1 f(x)\Phi_1(x)\,dx &= 0 \\ \int_0^1 (a+x)\sin(\pi x)\,dx &= 0 \\ \int_0^1 a\sin(\pi x)\,dx + \int_0^1 x\sin(\pi x)\,dx &= 0 \\ a\left(-\frac{\cos\pi x}{\pi}\right)_0^1 + \left[\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}\right]_0^1 &= 0 \\ -\frac{a}{\pi}(\cos\pi - 1) + \left[\frac{\sin\pi}{\pi^2} - \frac{\cos\pi}{\pi}\right] &= 0 \\ -\frac{a}{\pi}(-1-1) + \left[-\frac{-1}{\pi}\right] &= 0 \\ \frac{2a}{\pi} + \frac{1}{\pi} &= 0 \end{aligned}$$

Hence

$$a = \frac{-1}{2}$$

Only when a is the above value, is there a solution. The original ODE is now solved using the direct method (meaning, not eigenfunction expansion) when $a = \frac{-1}{2}$ as follows. Solve

$$\begin{aligned} y'' + \pi^2 y &= -\frac{1}{2} + x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

The homogeneous solution is easily found to be $y_h = A\cos(\pi x) + B\sin(\pi x)$. Since the RHS is a polynomial, let the particular solution be $y_p = c_1 + c_2 x$. Then $y_p' = c_2$ and $y_p'' = 0$. Then

$$\begin{aligned}\pi^2(c_1 + c_2 x) &= -\frac{1}{2} + x\\ c_1\pi^2 + c_2\pi^2 x &= -\frac{1}{2} + x\end{aligned}$$

Therefore $c_2\pi^2 = 1$ or $c_2 = \frac{1}{\pi^2}$ and $c_1\pi^2 = -\frac{1}{2}$ or $c_1 = -\frac{1}{2\pi^2}$. Hence $y_p = -\frac{1}{2\pi^2} + \frac{1}{\pi^2}x$. The solution is

$$\begin{aligned}y &= y_h + y_p\\ &= A\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi^2} + \frac{1}{\pi^2}x\end{aligned}$$

Applying boundary conditions, at $y(0) = 0$ the above becomes

$$\begin{aligned}0 &= A - \frac{1}{2\pi^2}\\ A &= \frac{1}{2\pi^2}\end{aligned}$$

Hence the solution becomes

$$y(x) = \frac{1}{2\pi^2}\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi^2} + \frac{1}{\pi^2}x$$

At $y(1) = 0$ the above gives

$$\begin{aligned}0 &= \frac{1}{2\pi^2}\cos(\pi) + B\sin(\pi) - \frac{1}{2\pi^2} + \frac{1}{\pi^2}\\ 0 &= \frac{-1}{2\pi^2} - \frac{1}{2\pi^2} + \frac{1}{\pi^2}\\ 0 &= 0\end{aligned}$$

Therefore B can be any value. Hence the final solution is

$$\boxed{y(x) = \frac{1}{2\pi^2}\cos(\pi x) + B\sin(\pi x) + \frac{1}{\pi^2}\left(x - \frac{1}{2}\right)}$$

The solution is not unique as expected. Any arbitrary value of B gives a solution.

2.1.44 Chapter 11.3, Problem 11

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned}y'' + 4\pi^2 y &= a + x\\ y(0) &= 0\\ y(1) &= 0\end{aligned}$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_2 = 4\pi^2$ is eigenvalue in the original ODE $y'' + 4\pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to the eigenfunction $\Phi_2(x)$. Therefore the condition is

$$\begin{aligned}\int_0^1 f(x)\Phi_2(x)\,dx &= 0\\ \int_0^1 (a + x)\sin(2\pi x)\,dx &= 0\\ \int_0^1 a\sin(2\pi x)\,dx + \int_0^1 x\sin(2\pi x)\,dx &= 0\\ a\left(-\frac{\cos 2\pi x}{2\pi}\right)_0^1 + \left[\frac{\sin(2\pi x)}{4\pi^2} - \frac{x\cos(2\pi x)}{2\pi}\right]_0^1 &= 0\\ -\frac{a}{2\pi}(\cos 2\pi - 1) + \left[\frac{\sin 2\pi}{4\pi^2} - \frac{\cos 2\pi}{2\pi}\right] &= 0\\ -\frac{a}{2\pi}(1 - 1) + \left[-\frac{1}{2\pi}\right] &= 0\\ -\frac{1}{2\pi} &= 0\end{aligned}$$

But this is not possible. Hence there is no a which makes $\int_0^1 (a + x)\sin(2\pi x)\,dx = 0$. This means there is no solution for any a.

2.1.45 Chapter 11.3, Problem 12

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$
\begin{aligned}
y'' + \pi^2 y &= a \\
y'(0) &= 0 \\
y'(1) &= 0
\end{aligned}
$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_0 = 0$ and $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_1 = \pi^2$ is eigenvalue in the original ODE $y'' + \pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to $\Phi_1(x)$. The eigenfunctions in this case are $\Phi_n(x) = \cos(n\pi x)$. Therefore the condition is

$$
\begin{aligned}
\int_0^1 f(x)\,\Phi_1(x)\,dx &= 0 \\
\int_0^1 a\cos(\pi x)\,dx &= 0 \\
a\left(\frac{\sin \pi x}{\pi}\right)_0^1 &= 0 \\
\frac{a}{\pi}(0) &= 0
\end{aligned}
$$

Hence any a will satisfy this. Therefore there is a solution for any a. The solution is

$$
y = A\cos(\pi x) + B\sin(\pi x) + y_p
$$

Since the RHS is a constant, let $y_p = k$. This leads to $\pi^2 k = a$ or $k = \frac{a}{\pi^2}$. Hence the solution is

$$
y = A\cos(\pi x) + B\sin(\pi x) + \frac{a}{\pi^2}
$$

Or

$$
y'(x) = -\pi A\sin(\pi x) + B\pi\cos(\pi x)
$$

At $y'(0) = 0$ the above becomes

$$
0 = B\pi
$$

Hence $B = 0$ and the solution now becomes

$$
\begin{aligned}
y &= A\cos(\pi x) + \frac{a}{\pi^2} \\
y' &= -A\pi\sin(\pi x)
\end{aligned}
$$

At $y(1) = 0$ the above becomes

$$
\begin{aligned}
0 &= -A\pi\sin\pi \\
&= -A(0)
\end{aligned}
$$

Therefore A is arbitrary. Any A will give a solution. Hence the final solution is

$$
\boxed{y = A\cos(\pi x) + \frac{a}{\pi^2}}
$$

For any A and where a is the given a in the original ODE which can take in any value.

2.1.46 Chapter 11.3, Problem 13

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$
\begin{aligned}
y'' + \pi^2 y &= a - \cos\pi x \\
y(0) &= 0 \\
y(1) &= 0
\end{aligned}
$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_0 = 0$ and $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_1 = \pi^2$ is eigenvalue in the original ODE $y'' + \pi^2 y = a - \cos\pi x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to $\Phi_1(x)$. The eigenfunctions in this case are $\Phi_n(x) = \sin(n\pi x)$. Therefore the condition is

$$\begin{aligned}\int_0^1 f(x)\Phi_1(x)\,dx &= 0\\ \int_0^1 (a-\cos\pi x)\sin(\pi x)\,dx &= 0\\ \int_0^1 a\sin(\pi x)\,dx - \int_0^1 \cos(\pi x)\sin(\pi x)\,dx &= 0\end{aligned}$$

Using $\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$ then $\sin(\pi x)\cos(\pi x) = \frac{1}{2}(\sin(0)+\sin(2\pi x)) = \frac{1}{2}\sin(2\pi x)$ and the above becomes

$$\begin{aligned}\int_0^1 a\sin(\pi x)\,dx - \frac{1}{2}\int_0^1 \sin(2\pi x)\,dx &= 0\\ -\frac{a}{\pi}[\cos\pi x]_0^1 + \frac{1}{4\pi}[\cos(2\pi x)]_0^1 &= 0\\ -\frac{a}{\pi}(\cos\pi - 1) + \frac{1}{4\pi}(\cos(2\pi)-1) &= 0\\ \frac{2a}{\pi} &= 0\end{aligned}$$

Hence $a = 0$. Therefore there is a solution only when $a = 0$. The original ODE then becomes

$$y'' + \pi^2 y = -\cos\pi x$$

The homogenous solution is

$$y_h = A\cos(\pi x) + B\sin(\pi x)$$

Since the forcing function matches one of the basis solution, then the particular solution guess is multiplied by extra x. Therefore

$$\begin{aligned}y_p &= x(c_1\cos(\pi x) + c_2\sin(\pi x))\\ y_p' &= c_1\cos(\pi x) + c_2\sin(\pi x) + x(-c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x))\\ y_p'' &= -c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x) + (-c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x)) + x\left(-c_1\pi^2\cos(\pi x) - c_2\pi^2\sin(\pi x)\right)\\ &= \sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2\right)\end{aligned}$$

Substituting back into the ODE gives

$$\begin{aligned}\sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2\right) + \pi^2(x(c_1\cos(\pi x) + c_2\sin(\pi x))) &= -\cos\pi x\\ \sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x + \pi^2 x c_2\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2 + \pi^2 x c_1\right) &= -\cos\pi x\\ -2c_1\pi\sin(\pi x) + 2c_2\pi\cos(\pi x) &= -\cos\pi x\end{aligned}$$

Hence

$$\begin{aligned}-2c_1\pi &= 0\\ 2c_2\pi &= -1\end{aligned}$$

Or

$$\begin{aligned}c_1 &= 0\\ c_2 &= -\frac{1}{2\pi}\end{aligned}$$

Therefore

$$y_p = -\frac{1}{2\pi}x\sin(\pi x)$$

And the general solution is

$$y(x) = A\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi}x\sin(\pi x)$$

At $y(0) = 0$ the above becomes

$$0 = A\cos(\pi x)$$

Hence $A = 0$ and the solution now becomes

$$\boxed{y(x) = B\sin(\pi x) - \frac{1}{2\pi}x\sin(\pi x)}$$

One can stop here, since it is known that the solution is not unique and must contain an arbitrary constant. It is not possible to solve for B using the second boundary conditions.

2.1.47 Chapter 11.3, Problem 16

Show that the problem $y'' + \pi^2 y = \pi^2 x, y(0) = 1, y(1) = 0$ has solution $y = c_1 \sin \pi x + c_2 \cos \pi x + x$ also show that the solution can not be obtained by splitting the problem as suggested in problem 15 since neither of the two subsidiary problems can be solve in this case.

Solution

To attempt to solve the problem by splitting, the solution is first assumed to be $y = u + v$ where u is the solution to $u'' + \pi^2 u = 0, u(0) = 1, u(1) = 0$ and v is the solution to $v'' + \pi^2 v = \pi^2 x, v(0) = 0, v(1) = 0$. Let us now try to solve the u ODE. The solution is

$$u(x) = A \cos \pi x + B \sin \pi x$$

Applying first BC $u(0) = 1$ gives $A = 1$. Hence the solution becomes $u = \cos \pi x + B \sin \pi x$. Applying second BC $u(1) = 0$ gives

$$\begin{aligned}
0 &= \cos \pi + B \sin \pi \\
0 &= 1 + B \tan \pi \\
B &= \frac{-1}{\tan \pi} = \frac{-1}{0}
\end{aligned}$$

Therefore there is no solution for u. Hence no solution is possible by splitting it was suggested in problem 15 for this problem. Now the problem is solved using the direct method. The homogeneous solution is

$$y_h = A \cos \pi x + B \sin \pi x$$

Since the forcing function $\pi^2 x$ is a polynomial, let y_p guess be $y_p = kx$substituting this back into the ODE gives $k = 1$. Hence the solution becomes

$$\begin{aligned}
y &= y_h + y_p \\
&= A \cos \pi x + B \sin \pi x + x
\end{aligned}$$

Applying first BC $y(0) = 1$ gives $1 = A$. Hence the solution now becomes $y = \cos \pi x + B \sin \pi x + x$. Applying second BC $y(1) = 0$ gives

$$\begin{aligned}
0 &= \cos \pi + B \sin \pi + 1 \\
0 &= -1 + B \tan \pi + 1 \\
0 &= B \tan \pi \\
0 &= B(0)
\end{aligned}$$

Therefore, any B will work. Hence the solution is not unique. Let $B = 1$. Therefore the final solution is

$$y = \cos \pi x + \sin \pi x + x$$

This is solution is not unique. This is also a solution $y = \cos \pi x + 3 \sin \pi x + x$ and also this $y = \cos \pi x + 100 \sin \pi x + x$ and also $y = \cos \pi x + x$ and so on.

2.1.48 Chapter 11.3, Problem 19 (With interactive animation)

Use eigenfunction expansion to solve

$$u_t = u_{xx} - x$$

With initial condition $u(x, 0) = \sin\left(\frac{\pi x}{2}\right)$ and boundary conditions $u(0, t) = 0, u_x(1, t) = 0$

Solution

The homogenous PDE is first solved to find the eigenfunctions, and these are used to expand the non-homogenous term $-x$ in the PDE. By separation of variables, the spatial eigenvalue ODE is

$$\begin{aligned}
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X'(1) &= 0
\end{aligned}$$

The eigenfunctions for this ODE are $\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$ with $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1, 3, 5, \cdots$. or $\lambda_n = (2n-1)^2 \left(\frac{\pi}{2}\right)^2$ for $n = 1, 2, 3, \cdots$. with now $\Phi_n(x) = \sin\left((2n-1)\frac{\pi}{2} x\right)$.

The normalized eigenfunctions are $\hat{\Phi}_n(x) = \sqrt{2} \sin\left(\sqrt{\lambda_n} x\right)$. Using these, the original PDE is now solved by assuming the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \hat{\Phi}_n(x)$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$$

Where $\sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$ is the eigenfunction expansion of $-x$. Assuming term by term differentiation is allowed (can be shown to be justified here), the above becomes

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ then the above becomes

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x) \tag{1}$$

Now c_n is found. Since $-x = \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$, then applying orthogonality gives

$$-\int_0^1 r(x)\,x\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx$$

But the weight $r(x) = 1$, hence the above simplifies to

$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = c_n\int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$\begin{aligned}
c_n &= -\int_0^1 x\hat{\Phi}_n(x)\,dx\\
&= -\int_0^1 x\sqrt{2}\sin\left((2n-1)\frac{\pi}{2}\right)dx\\
&= -\sqrt{2}\left[\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{x\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right]_0^1\\
&= -\sqrt{2}\left[\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{\cos\left((2n-1)\frac{\pi}{2}\right)}{(2n-1)\frac{\pi}{2}}\right]
\end{aligned}$$

But $\cos\left((2n-1)\frac{\pi}{2}\right) = 0$ for all n, and the above now simplifies to

$$\begin{aligned}
c_n &= -\sqrt{2}\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2}\\
&= -4\sqrt{2}\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\pi\right)^2}
\end{aligned}$$

But $\sin\left((2n-1)\frac{\pi}{2}\right) = (-1)^{n-1}$ for $n = 1, 2, 3, \cdots$, hence the above becomes

$$\begin{aligned}
c_n &= -4\sqrt{2}\frac{(-1)^{n-1}}{\left((2n-1)\pi\right)^2}\\
&= 4\sqrt{2}\frac{(-1)^n}{\left((2n-1)\pi\right)^2}
\end{aligned}$$

Now that c_n is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = c_n$$

The integrating factor is $e^{\int\lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)\,e^{\lambda_n t}\right) = c_n e^{\lambda_n t}$. Integrating gives

$$\begin{aligned}
b_n(t)\,e^{\lambda_n t} &= b(0) + c_n\int_0^t e^{\lambda_n s}ds\\
b_n(t) &= b(0)\,e^{-\lambda_n t} + c_n e^{-\lambda_n t}\int_0^t e^{\lambda_n s}ds\\
&= b(0)\,e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{\left(e^{\lambda_n t} - 1\right)}{\lambda_n}\\
&= b(0)\,e^{-\lambda_n t} + \frac{c_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right)
\end{aligned}$$

Therefore the solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\,\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty} \left(b(0)\,e^{-\lambda_n t} + \frac{c_n}{\lambda_n}\left(1-e^{-\lambda_n t}\right)\right)\hat{\Phi}_n(x) \end{aligned} \tag{2}$$

At $t=0$, the initial conditions is $u(x,0)=\sin\left(\frac{\pi x}{2}\right)$, therefore the above becomes

$$\begin{aligned} \sin\left(\frac{\pi x}{2}\right) &= \sum_{n=1}^{\infty}\left(b(0)+\frac{c_n}{\lambda_n}(1-1)\right)\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty} b(0)\,\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty} b(0)\sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Hence only $n=1$ gives a solution for $b(0)$, and therefore the above becomes

$$\sin\left(\frac{\pi x}{2}\right) = b(0)\sqrt{2}\sin\left(\frac{\pi}{2}x\right)$$

Or

$$\boxed{b(0)=\frac{1}{\sqrt{2}}}$$

Therefore the solution (2) now becomes

$$u(x,t) = \left(b(0)\,e^{-\lambda_1 t} + c_1 e^{-\lambda_1 t}\frac{\left(e^{\lambda_1 t}-1\right)}{\lambda_1}\right)\hat{\Phi}_1(x) + \sum_{n=2}^{\infty}\frac{c_n}{\lambda_n}\left(1-e^{-\lambda_n t}\right)\hat{\Phi}_n(x) \tag{3}$$

Where

$$\begin{aligned} c_n &= 4\sqrt{2}\frac{(-1)^n}{((2n-1)\pi)^2} \\ b(0) &= \frac{1}{\sqrt{2}} \\ \lambda_n &= \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n=1,2,3,\cdots \\ \hat{\Phi}_n(x) &= \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Hence the solution (3) becomes

$$\begin{aligned} u(x,t) = &\left(\frac{1}{\sqrt{2}}e^{-\frac{\pi^2}{4}t} + c_1 e^{-\frac{\pi^2}{4}t}\frac{\left(e^{\frac{\pi^2}{4}t}-1\right)}{\frac{\pi^2}{4}}\right)\sqrt{2}\sin\left(\frac{\pi}{2}x\right) \\ &+ \sum_{n=2}^{\infty}\frac{c_n}{(2n-1)^2\frac{\pi^2}{4}}\left(1-e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

To make it the same as back of the book solution, some more manipulation is needed.

$$\begin{aligned} u(x,t) = &\, e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2}e^{-\frac{\pi^2}{4}t}\left(e^{\frac{\pi^2}{4}t}-1\right)\sin\left(\frac{\pi}{2}x\right) \\ &+ \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\left(e^{\left((2n-1)\frac{\pi}{2}\right)^2 t}-1\right)\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Or

$$\begin{aligned} u(x,t) = &\, e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2}\left(1-e^{-\frac{\pi^2}{4}t}\right)\sin\left(\frac{\pi}{2}x\right) \\ &+ \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}\left(1-e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Or

$$\begin{aligned} u(x,t) = &\, e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2}\sin\left(\frac{\pi}{2}x\right) - 4\sqrt{2}\frac{c_1}{\pi^2}e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right) \\ &+ \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}\left(1-e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Or

$$u(x,t) = \sqrt{2}\left[4\frac{c_1}{\pi^2} + \left(\frac{1}{\sqrt{2}} - 4\frac{c_1}{\pi^2}\right)e^{-\frac{\pi^2}{4}t}\right]\sin\left(\frac{\pi}{2}x\right) + \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}\left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sin\left((2n-1)\frac{\pi}{2}x\right)$$

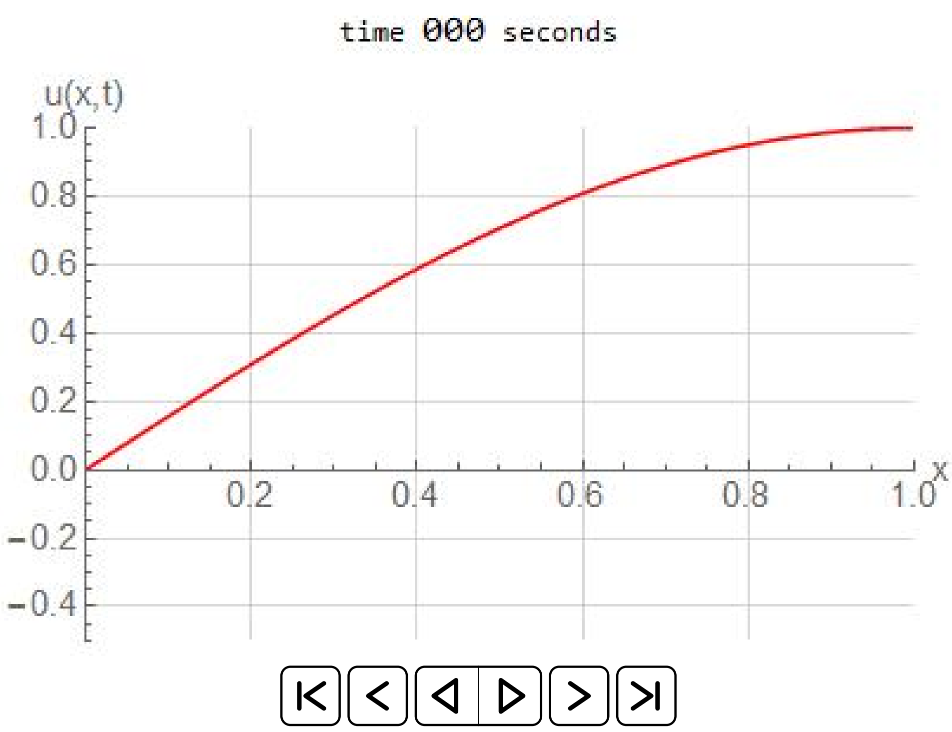
The back of the book uses $c_n = 4\sqrt{2}\frac{(-1)^{n+1}}{((2n-1)\pi)^2}$ instead of $c_n = 4\sqrt{2}\frac{(-1)^n}{((2n-1)\pi)^2}$ as was done in this solution. Therefore, changing c_n to be as the back of the book means flipping the sign of each c_n. (or multiplying by -1). Hence the solution becomes now the same as the back of the book

$$u(x,t) = \sqrt{2}\left[-4\frac{c_1}{\pi^2} + \left(\frac{1}{\sqrt{2}} + 4\frac{c_1}{\pi^2}\right)e^{-\frac{\pi^2}{4}t}\right]\sin\left(\frac{\pi}{2}x\right) - \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}\left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sin\left((2n-1)\frac{\pi}{2}x\right)$$

Where in the above,

$$c_n = 4\sqrt{2}\frac{(-1)^{n+1}}{((2n-1)\pi)^2}$$

Both solutions are the same. The sign is either added to c_n or left outside. This completes the solution. The following is an animation of the above solution for 1.8 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.49 Chapter 11.3, Problem 20 (With interactive animation)

Use eigenfunction expansion to solve

$$u_t = u_{xx} + e^{-t}$$

With initial condition $u(x,0) = 1 - x$ and boundary conditions $u_x(0,t) = 0, u_x(1,t) + u(1,t) = 0$
Solution
The homogenous PDE is solved first to obtain the eigenfunctions. These are then used to expand the non-homogenous term e^{-t} in the PDE. By separation of variables, the spatial eigenvalue ODE is

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(1) + X(1) = 0$$

The eigenfunctions for this ODE were found earlier in problem 4, Chapter 11.2. They are

$$\hat{\Phi}_n = k_n\Phi_n = \frac{\sqrt{2}}{\sqrt{1 + \sin^2\left(\sqrt{\lambda_n}\right)}}\cos\left(\sqrt{\lambda_n}x\right)$$

Where λ_n are the roots of

$$\cos\left(\sqrt{\lambda_n}\right) - \sqrt{\lambda_n}\sin\left(\sqrt{\lambda_n}\right) = 0$$

For $n = 1, 2, 3, \cdots$. Using these, the original PDE is now solved by assuming the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$$

Where $\sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$ is the eigenfunction expansion of e^{-t}. In the above $c_n(t)$ is now a function of time, since the forcing function depends on time in this problem. Assuming term by term differentiation is allowed the above becomes

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ therefore

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x) \tag{1}$$

Now $c_n(t)$ is found. Since $e^{-t} = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$, then applying orthogonality gives

$$\int_0^1 r(x)e^{-t}\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n(t)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx$$

But the weight $r(x) = 1$, hence the above simplifies to

$$e^{-t}\int_0^1 \hat{\Phi}_m(x)\,dx = c_n(t)\int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$e^{-t}\int_0^1 \hat{\Phi}_m(x)\,dx = c_n(t)$$

Hence

$$\begin{aligned}
c_n(t) &= e^{-t}\int_0^1 k_n\cos\left(\sqrt{\lambda_n}x\right)dx \\
&= e^{-t}\frac{k_n}{\sqrt{\lambda_n}}\left[\sin\left(\sqrt{\lambda_n}x\right)\right]_0^1 \\
&= e^{-t}\frac{k_n}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}\right) \qquad (2)
\end{aligned}$$

To make it match the way the back of the book expressed the above, let us write

$$c_n(t) = e^{-t}c_n$$

Where now

$$c_n = \frac{k_n}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}\right)$$

This makes it easier to verify the final solution found here is the same as the back of the book. Now that $c_n(t)$ is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} e^{-t}c_n\hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = e^{-t}c_n$$

The integrating factor is $e^{\int \lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)e^{\lambda_n t}\right) = e^{-t}c_n e^{\lambda_n t}$. Integrating gives

$$\begin{aligned}
b_n(t)e^{\lambda_n t} &= b(0) + c_n\int_0^t e^{-s}e^{\lambda_n s}ds \\
b_n(t) &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\int_0^t e^{(\lambda_n-1)s}ds \\
&= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{\left[e^{(\lambda_n-1)s}\right]_0^t}{\lambda_n - 1} \\
&= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \qquad (3)
\end{aligned}$$

Using the above in $u(x,t)=\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)$ gives the solution as

$$u(x,t)=\sum_{n=1}^{\infty}\left(b(0)e^{-\lambda_n t}+c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t}-1}{\lambda_n-1}\right)\hat{\Phi}_n(x) \tag{4}$$

At $t=0$, the above simplifies to

$$1-x=\sum_{n=1}^{\infty} b(0)\hat{\Phi}_n(x)$$

Applying orthogonality gives

$$\begin{aligned}\int_0^1 r(x)(1-x)\hat{\Phi}_m(x)\,dx &= \sum_{n=1}^{\infty} b(0)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx\\ \int_0^1 r(x)(1-x)\hat{\Phi}_m(x)\,dx &= b(0)\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx\end{aligned}$$

But $r(x)=1$ and $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx=1$ therefore

$$\begin{aligned}b(0) &= \int_0^1 (1-x)\hat{\Phi}_n(x)\,dx\\ &= \int_0^1 \hat{\Phi}_n(x)\,dx-\int_0^1 x\hat{\Phi}_n(x)\,dx\\ &= k_n\left(\int_0^1 \Phi_n(x)\,dx-\int_0^1 x\Phi_n(x)\,dx\right)\end{aligned}$$

But $\Phi_n(x)=\cos\left(\sqrt{\lambda_n}x\right)$, hence the above becomes

$$\begin{aligned}b(0) &= k_n\left(\int_0^1 \cos\left(\sqrt{\lambda_n}x\right)dx-\int_0^1 x\cos\left(\sqrt{\lambda_n}x\right)dx\right)\\ &= k_n\left(\left[\frac{\sin\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1-\left[\frac{\cos\left(\sqrt{\lambda_n}x\right)}{\lambda_n}+\frac{x\sin\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1\right)\\ &= k_n\left(\left[\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}}\right]-\left[\frac{\cos\left(\sqrt{\lambda_n}\right)}{\lambda_n}+\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}}-\frac{1}{\lambda_n}\right]\right)\\ &= k_n\left(\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}}-\frac{\cos\left(\sqrt{\lambda_n}\right)}{\lambda_n}-\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}}+\frac{1}{\lambda_n}\right)\\ &= \frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)\end{aligned}$$

Now that $b(0)$ is found, then the solution (4) becomes

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)e^{-\lambda_n t}+c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t}-1}{\lambda_n-1}\right)\hat{\Phi}_n(x)\\ &= \sum_{n=1}^{\infty}\left(\frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)e^{-\lambda_n t}+\frac{c_n}{\lambda_n-1}\left(e^{-t}-e^{-\lambda_n t}\right)\right)k_n\cos\left(\sqrt{\lambda_n}x\right)\end{aligned}$$

But $k_n=\frac{\sqrt{2}}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}$, hence the above becomes

$$u(x,t)=\sqrt{2}\sum_{n=1}^{\infty}\left(\alpha_n e^{-\lambda_n t}+\frac{c_n}{\lambda_n-1}\left(e^{-t}-e^{-\lambda_n t}\right)\right)\frac{\cos\left(\sqrt{\lambda_n}x\right)}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}$$

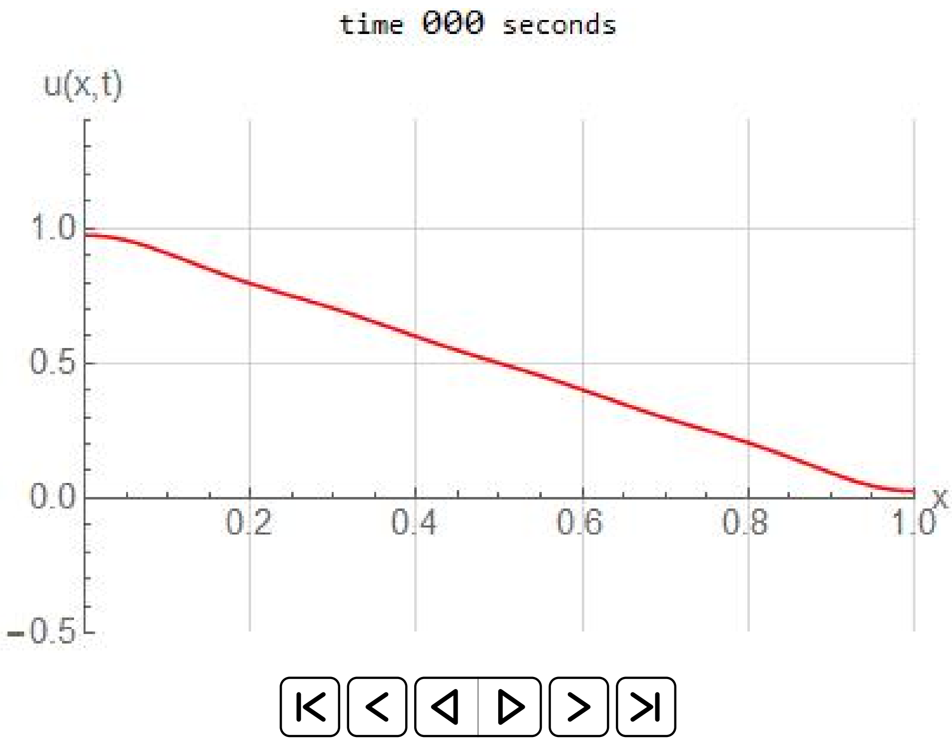
Where

$$\begin{aligned}\alpha_n &= \frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)\\ &= \frac{\sqrt{2}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)}{\lambda_n\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}\end{aligned}$$

And

$$\begin{aligned} c_n &= \frac{k_n}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\right) \\ &= \frac{\sqrt{2}}{\sqrt{\lambda_n}} \frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}} \end{aligned}$$

The following is an animation of the above solution for 6 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.50 Chapter 11.3, Problem 22 (With interactive animation)

Use eigenfunction expansion to solve

$$u_t = u_{xx} + e^{-t}(1-x)$$

With initial condition $u(x,0) = 0$ and boundary conditions $u(0,t) = 0, u_x(1,t) = 0$
Solution
The homogenous PDE is solved first to obtain the eigenfunctions. These are then used to expand the non-homogenous term $e^{-t}(1-x)$ in the PDE. By separation of variables, the spatial eigenvalue ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(1) &= 0 \end{aligned}$$

The eigenfunctions for this ODE were found earlier. They are

$$\begin{aligned} \hat{\Phi}_n &= k_n \Phi_n \\ &= \sqrt{2} \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

Where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1, 3, 5, \cdots$. Or

$$\begin{aligned} \hat{\Phi}_n &= \sqrt{2} \sin\left(\sqrt{\lambda_n} x\right) \\ \lambda_n &= \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n = 1, 2, 3, \cdots \end{aligned}$$

The original PDE is now solved by assuming the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \hat{\Phi}_n(x)$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$\sum_{n=1}^{\infty} b_n'(t) \hat{\Phi}_n(x) = \frac{d^2}{dx^2} \sum_{n=1}^{\infty} b_n(t) \hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n(t) \hat{\Phi}_n(x)$$

Where $\sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$ is the eigenfunction expansion of $e^{-t}(1-x)$. In the above $c_n(t)$ is now a function of time, since the forcing function depends on time in this problem. Assuming term by term differentiation is allowed the above becomes

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ therefore

$$\sum_{n=1}^{\infty} \left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x) \tag{1}$$

Now $c_n(t)$ is found. Since $e^{-t}(1-x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$, then applying orthogonality gives

$$\int_0^1 r(x)e^{-t}(1-x)\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n(t)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx$$

But the weight $r(x) = 1$, hence the above simplifies to

$$e^{-t}\int_0^1 (1-x)\hat{\Phi}_m(x)\,dx = c_n(t)\int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$e^{-t}\int_0^1 (1-x)\hat{\Phi}_m(x)\,dx = c_n(t)$$

Hence

$$\begin{aligned}
c_n(t) &= e^{-t}\int_0^1 (1-x)k_n\sin\left(\sqrt{\lambda_n}x\right)dx \\
&= e^{-t}\sqrt{2}\left(\int_0^1 \sin\left(\sqrt{\lambda_n}x\right)dx - \int_0^1 x\sin\left(\sqrt{\lambda_n}x\right)dx\right) \\
&= e^{-t}\sqrt{2}\left(\left[\frac{-\cos\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1 - \left[\frac{\sin\sqrt{\lambda_n}x}{\lambda_n} - \frac{x\cos\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\right]_0^1\right) \\
&= e^{-t}\sqrt{2}\left(\left[\frac{-\cos\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}}\right] - \left[\frac{\sin\sqrt{\lambda_n}}{\lambda_n} - \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right]\right) \\
&= e^{-t}\sqrt{2}\left(\frac{-\cos\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}} - \frac{\sin\sqrt{\lambda_n}}{\lambda_n} + \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right) \\
&= e^{-t}\sqrt{2}\left(\frac{1}{\sqrt{\lambda_n}} - \frac{\sin\sqrt{\lambda_n}}{\lambda_n}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}\right)
\end{aligned} \tag{2}$$

But $\lambda_n = (2n-1)\frac{\pi}{2}$, therefore $\sin\left((2n-1)\frac{\pi}{2}\right) = \{1,-1,1,-1,\cdots\}$ or $(-1)^{n-1}$ and the above becomes

$$\begin{aligned}
c_n(t) &= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} - (-1)^{n-1}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^{n}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^{n}\right)
\end{aligned}$$

To make it match the way the back of the book expressed the above, let us write

$$c_n(t) = e^{-t}c_n$$

Where

$$c_n = \frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^n\right) \tag{2A}$$

Now that $c_n(t)$ is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty} \left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} e^{-t}c_n\hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = e^{-t} c_n$$

The integrating factor is $e^{\int \lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)\, e^{\lambda_n t}\right) = e^{-t} c_n e^{\lambda_n t}$. Integrating gives

$$\begin{aligned} b_n(t)\, e^{\lambda_n t} &= b(0) + c_n \int_0^t e^{-s} e^{\lambda_n s} ds \\ b_n(t) &= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \int_0^t e^{(\lambda_n - 1)s} ds \\ &= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{\left[e^{(\lambda_n-1)s}\right]_0^t}{\lambda_n - 1} \\ &= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \end{aligned} \tag{3}$$

Using the above in $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \hat{\Phi}_n(x)$ gives the solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left(b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \right) \hat{\Phi}_n(x) \tag{4}$$

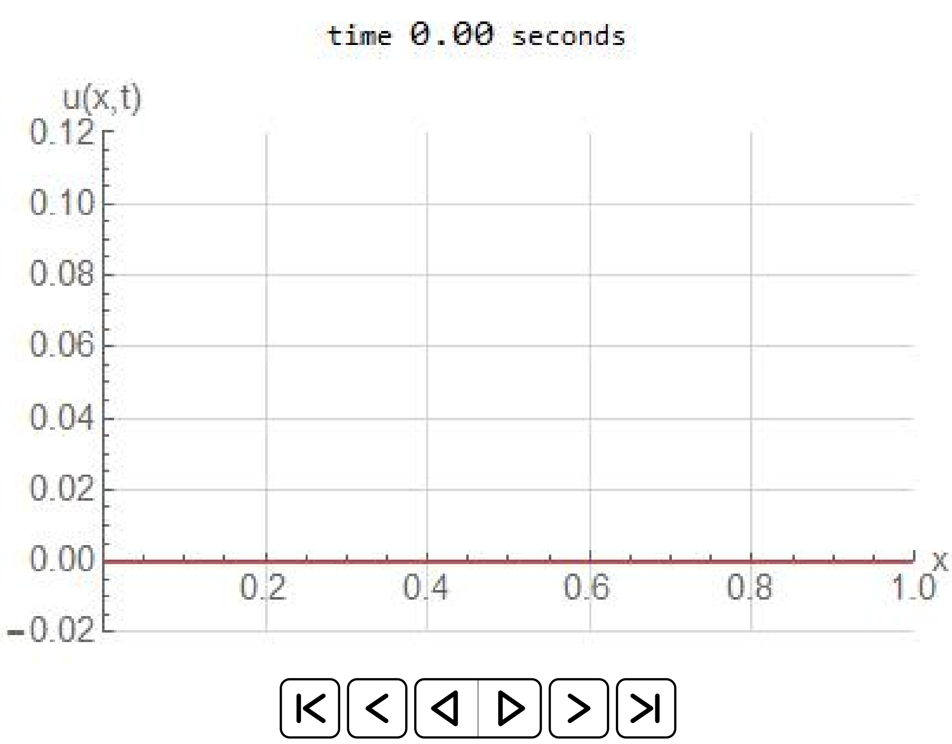
At $t = 0$, the initial conditions are zero, and above simplifies to

$$0 = \sum_{n=1}^{\infty} b(0)\, \hat{\Phi}_n(x)$$

Which implies $b(0) = 0$. Now that $b(0)$ is found, then the solution (4) becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \hat{\Phi}_n(x) \\ &= \sqrt{2} \sum_{n=1}^{\infty} c_n \left(\frac{e^{-t} - e^{-\lambda_n t}}{\lambda_n - 1} \right) \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

Where $c_n = \frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^n\right)$ and $\lambda_n = \left((2n-1)\frac{\pi}{2}\right)^2$. This completes the solution.
The solution was animated and verified it is correct against a numerical solution.
The following is an animation of the above solution for 5 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



2.1.51 Chapter 11.3, Problem 24 (With interactive animation)

Solve

$$u_t = u_{xx} - 2$$

With initial condition $u(x,0) = x^2 - 2x + 2$ and boundary conditions $u(0,t) = 1, u(1,t) = 0$
Solution
Let

$$u(x,t) = w(x,t) + v(x)$$

where $v(x)$ is steady state solution which only needs to satisfy the non-homogenous boundary conditions and $w(x,t)$ is the transient solution which needs to satisfy the homogeneous boundary conditions. At steady state, the PDE becomes an ODE

$$0 = v''(x) - 2$$

This has the solution

$$v(x) = c_1 + c_2 x + x^2$$

Where x^2 is the particular solution. From boundary conditions $v(0) = 1, v(1) = 0$, the solution becomes

$$v(x) = 1 - 2x + x^2$$

Hence $u(x,t) = w(x,t) + 1 - 2x + x^2$. Substituting this into the PDE $u_t = u_{xx} - 2$ results in

$$\begin{aligned} w_t &= w_{xx} + v''(x) - 2 \\ &= w_{xx} + 2 - 2 \\ &= w_{xx} \end{aligned}$$

Hence the PDE to solve is $w_t = w_{xx}$ with $w(0,t) = 0, w(1,t) = 0$. This heat PDE was solved before. Its solution is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin\left(\sqrt{\lambda_n} x\right) \tag{1}$$

Where $\lambda_n = (n\pi)^2$ for $n = 1, 2, 3, \cdots$. At $t = 0$, since $u(x,0) = w(x,0) + v(x)$ then $w(x,0) = u(x,0) - v(x)$ which gives

$$\begin{aligned} w(x,0) &= \left(x^2 - 2x + 2\right) - \left(1 - 2x + x^2\right) \\ &= 1 \end{aligned}$$

Hence at $t = 0$, (1) becomes

$$1 = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right) \tag{1A}$$

Applying orthogonality gives

$$\begin{aligned} \int_0^1 \sin\left(\sqrt{\lambda_n} x\right) dx &= \frac{1}{2} c_n \\ c_n &= 2 \int_0^1 \sin\left(\sqrt{\lambda_n} x\right) dx \\ &= -2 \left[\frac{\cos\left(\sqrt{\lambda_n} x\right)}{\sqrt{\lambda_n}} \right]_0^1 \\ &= \frac{-2}{\sqrt{\lambda_n}} \left[\cos\left(\sqrt{\lambda_n}\right) - 1\right] \\ &= \frac{-2}{n\pi} \left[\cos(n\pi) - 1\right] \end{aligned}$$

For even n the above is zero. And for odd n the above becomes

$$c_n = \frac{4}{n\pi} \qquad n = 1, 3, 5, \cdots$$

Therefore from (1) the solution to $w(x,t)$ is

$$w(x,t) = \frac{4}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} e^{-\lambda_n t} \sin\left(\sqrt{\lambda_n} x\right)$$

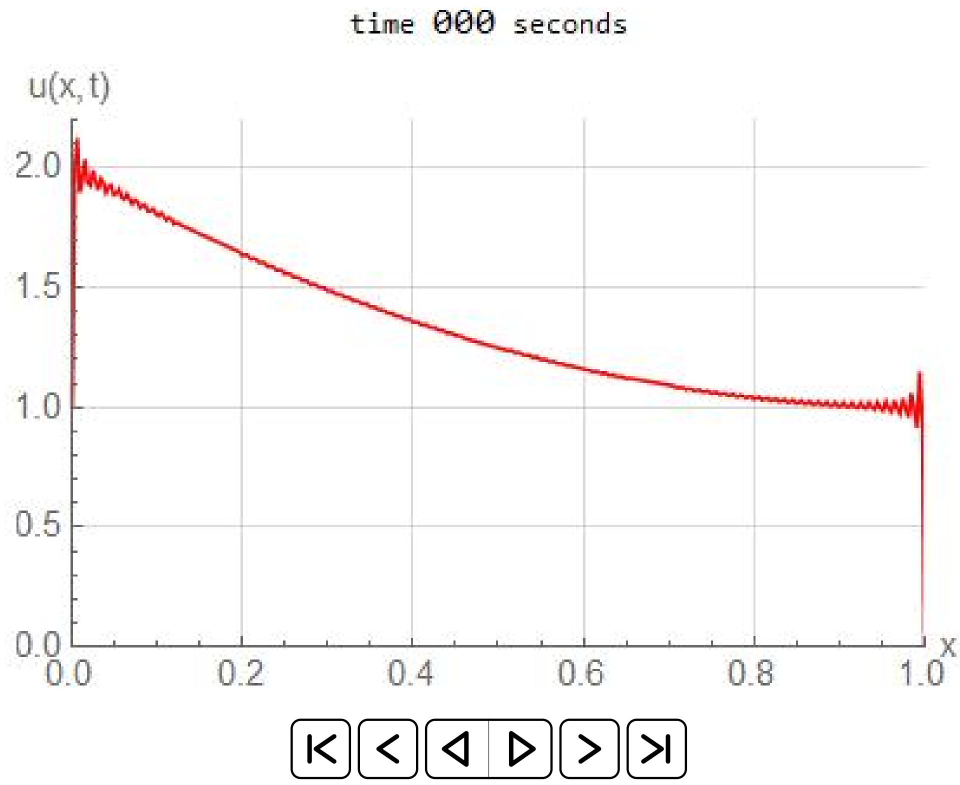
The above can also be written as

$$w(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-(2n-1)^2 \pi^2 t} \sin((2n-1)\pi x)$$

Now, since $u(x,t) = w(x,t) + v(x)$, then the final solution is

$$u(x,t) = x^2 - 2x + 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-(2n-1)^2 \pi^2 t} \sin((2n-1)\pi x)$$

The following is an animation of the above solution for half second. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



2.1.52 Chapter 11.3, Problem 25 (With interactive animation)

Solve

$$u_t = u_{xx} - \pi^2 \cos \pi x$$

With initial condition $u(x,0) = \cos\left(\frac{3\pi}{2}x\right) - \cos(\pi x)$ and boundary conditions $u_x(0,t) = 0, u(1,t) = 1$

Solution

Let

$$u(x,t) = w(x,t) + v(x)$$

where $v(x)$ is steady state solution which only needs to satisfy the non-homogenous boundary conditions and $w(x,t)$ is the transient solution which needs to satisfy the homogeneous version of boundary conditions.

At steady state, the PDE becomes an ODE

$$0 = v''(x) - \pi^2 \cos \pi x$$

This ODE can be easily solved giving

$$v(x) = -\cos(\pi x)$$

Hence $u(x,t) = w(x,t) - \cos(\pi x)$. Substituting this into the PDE $u_t = u_{xx} - \pi^2 \cos \pi x$ results in

$$w_t = w_{xx} + v''(x) - \pi^2 \cos \pi x$$

But $v'(x) = \pi \sin(\pi x)$ and $v''(x) = \pi^2 \cos(\pi x)$. The above becomes

$$w_t = w_{xx}$$

With boundary conditions $w_x(0,t) = 0, w(1,t) = 0$. This was solved before. It has the solution

$$w(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} c_n e^{-\lambda_n t} \cos\left(\sqrt{\lambda_n} x\right) \tag{1}$$

Where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ with $n = 1,3,5,\cdots$. At $t = 0$, from $u(x,0) = w(x,0) + v(x)$, then $w(x,0) = u(x,0) - v(x)$ or

$$\begin{aligned} w(x,0) &= \cos\left(\frac{3\pi}{2}x\right) - \cos(\pi x) + \cos(\pi x) \\ &= \cos\left(\frac{3\pi}{2}x\right) \end{aligned}$$

Therefore, from (1) and at $t = 0$ we obtain

$$\begin{aligned} w(x,0) &= \sum_{n=1,3,5,\cdots}^{\infty} c_n \cos\left(\sqrt{\lambda_n} x\right) \\ \cos\left(\frac{3\pi}{2}x\right) &= \sum_{n=1,3,5,\cdots}^{\infty} c_n \cos\left(\frac{n\pi}{2}x\right) \end{aligned}$$

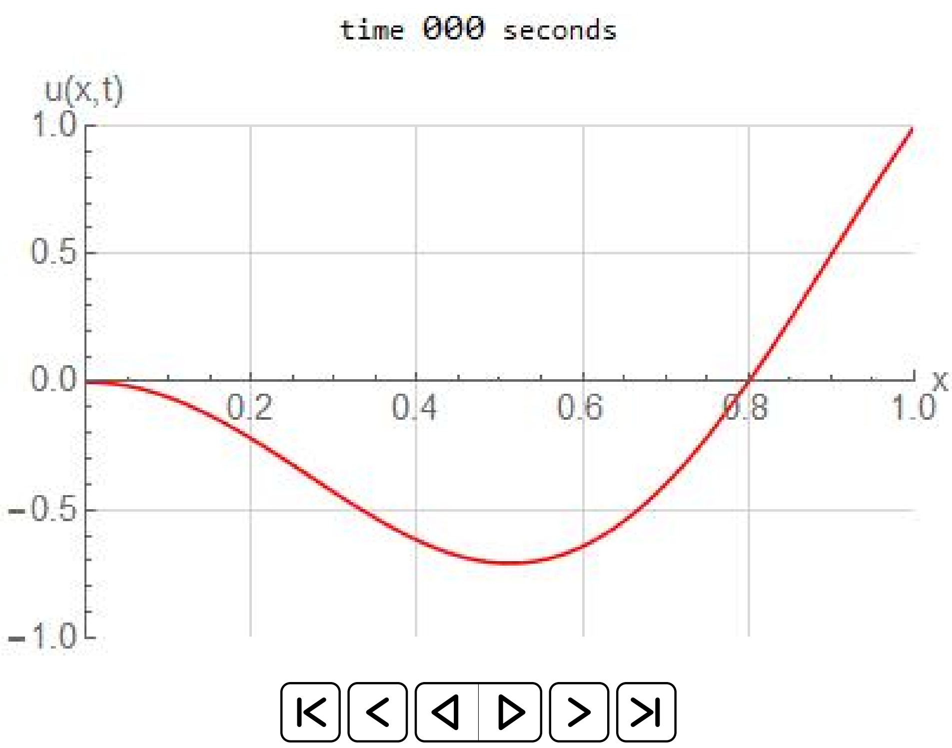
Therefore, only for $n = 3$ is there a solution. Therefore $c_3 = 1$. Hence (1) becomes

$$
\begin{aligned}
w(x,t) &= e^{-\lambda_3 t}\cos\left(\sqrt{\lambda_3}x\right)\\
&= e^{-\left(\frac{3\pi}{2}\right)^2 t}\cos\left(\frac{3\pi}{2}x\right)
\end{aligned}
$$

Therefore the final solution is

$$
\begin{aligned}
u(x,t) &= w(x,t) + v(x)\\
&= -\cos(\pi x) + e^{-\frac{9\pi^2}{4}t}\cos\left(\frac{3\pi}{2}x\right)
\end{aligned}
$$

The following is an animation of the above solution for half second. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



2.1.53 Chapter 11.3, Problem 28

Part (a) Show that by method of variation of parameters that general solution to $y''(x) = -f(x)$ can be written as

$$
y = c_1 + c_2 x - \int_0^x (x-s) f(s)\, ds
$$

part (b). Let the solution required to satisfy boundary conditions $y(0) = 0, y(1) = 0$. Show that $c_1 = 0, c_2 = \int_0^1 (1-x) f(s)\, ds$

part (c). Defining $G(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$ show that the solution can be written as $y(x) = \int_0^1 G(x,s) f(s)\, ds$

Solution

Part (a)

The solution is $y = y_h + y_p$. Where $y_h'' = 0$. This has the solution

$$
y_h = c_1 + c_2 x
$$

In this expression, the basis solutions are

$$
\begin{aligned}
y_1 &= 1\\
y_2 &= x.
\end{aligned}
$$

The particular solution is now found using variation of parameters, where it is assumed that

$$
y_p = y_1 u_1 + y_2 u_2 \tag{1}
$$

And u_1, u_2 are two functions to be determined. Using the standard formulas for finding u_1, u_2 gives

$$
u_1 = \int_0^x \frac{-y_2 F(s)}{W(s)} ds \tag{2}
$$

Where in the above, $F(s)$ is the forcing function in the RHS of the original ODE which is $-f(x)$ here, and W is the Wronskian. The Wronskian is found as follows

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Substituting $y_1 = 1, y_2 = x$ in the above gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$$

Therefore (2) becomes

$$\begin{aligned} u_1 &= \int_0^x -s(-f(s))\,ds \\ &= \int_0^x sf(s)\,ds \end{aligned} \tag{3}$$

Similarly, u_2 is found using

$$\begin{aligned} u_2 &= \int_0^x \frac{y_1 F(s)}{W(s)} ds \\ &= \int_0^x -f(s)\,ds \end{aligned} \tag{4}$$

Using (3,4) in (1) gives the particular solution as

$$\begin{aligned} y_p &= y_1 \int_0^x sf(s)\,ds - y_2 \int_0^x f(s)\,ds \\ &= \int_0^x sf(s)\,ds - x\int_0^x f(s)\,ds \\ &= \int_0^x sf(s)\,ds - \int_0^x xf(s)\,ds \\ &= \int_0^x (s-x)\,f(s)\,ds \\ &= -\int_0^x (x-s)\,f(s)\,ds \end{aligned}$$

Now that particular solution is found, the complete solution is found from $y = y_h + y_p$ giving

$$y = c_1 + c_2 x - \int_0^x (x-s)\,f(s)\,ds \tag{5}$$

Part (b)

Using the BC $y(0) = 0$ on (5) gives

$$\begin{aligned} 0 &= c_1 - \int_0^0 -sf(s)\,ds \\ c_1 &= 0 \end{aligned}$$

Hence $c_1 = 0$ and the solution (5) now becomes

$$y = c_2 x - \int_0^x (x-s)\,f(s)\,ds \tag{6}$$

Using the second BC $y(1) = 0$ the above becomes

$$\begin{aligned} 0 &= c_2 - \int_0^1 (1-s)\,f(s)\,ds \\ c_2 &= \int_0^1 (1-s)\,f(s)\,ds \end{aligned}$$

Hence the solution (6) now becomes

$$\begin{aligned} y &= x\int_0^1 (1-s)\,f(s)\,ds - \int_0^x (x-s)\,f(s)\,ds \\ &= \int_0^1 x(1-s)\,f(s)\,ds - \int_0^x (x-s)\,f(s)\,ds \end{aligned}$$

Writing $\int_0^1 x(1-s) f(s)\,ds = \int_0^x x(1-s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$ then the above becomes

$$y = \int_0^x x(1-s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

Combining the first and third integrals gives

$$\begin{aligned} y &= \int_0^x [x(1-s) - (x-s)] f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x [x - xs - x + s] f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x (-xs + s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \qquad (7) \end{aligned}$$

Which is the result required to show.

Part (c)

From part (b) above, the solution in (7) can be written as

$$y = \int_0^x G_L(x,s) f(s)\,ds + \int_x^1 G_R(x,s) f(s)\,ds \qquad (8)$$

Where

$$G(x,s) = \begin{cases} G_L(x,x) \\ G_L(x,s) \end{cases} = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$$

Hence (8) can be combined into one integral

$$y = \int_0^1 G(x,s) f(s)\,ds$$

2.1.54 Chapter 11.3, Problem 29

By using procedure in problem 28 show that solution to $y'' + y = -f(x), y(0) = 0, y(1) = 0$ is

$$y = \int_0^1 G(x,s) f(s)$$

Where

$$G(x,s) = \begin{cases} \frac{\sin(s)\sin(1-x)}{\sin(1)} & 0 \le s \le x \\ \frac{\sin(x)\sin(1-s)}{\sin(1)} & x \le s \le 1 \end{cases}$$

Solution

Let $y = y_h + y_p$. Where y_h is solution to $y_h'' + y_h = 0$. This has the solution $y_h = c_1 \cos x + c_2 \sin x$. Hence the bases solutions are

$$\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}$$

And therefore the Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Hence

$$u_1 = \int_0^x \frac{-y_2 F(s)}{W(s)} ds$$

Where in the above, $F(s)$ is the forcing function in the RHS of the original ODE which is $-f(x)$ here, and W is the Wronskian. Therefore

$$\begin{aligned} u_1 &= \int_0^x -\sin(s)(-f(s))\,ds \\ &= \int_0^x \sin(s) f(s)\,ds \end{aligned}$$

Similarly, u_2 is found using

$$\begin{aligned}u_2 &= \int_0^x \frac{y_1 F(s)}{W(s)} ds \\ &= \int_0^x \cos(s)(-f(s))\, ds\end{aligned}$$

Hence the particular solution is

$$\begin{aligned}y_p &= y_1 u_1 + y_2 u_2 \\ &= \cos(x) \int_0^x \sin(s) f(s)\, ds - \sin(x) \int_0^x \cos(s) f(s)\, ds \\ &= \int_0^x \cos(x)\sin(s) f(s)\, ds - \int_0^x \sin(x)\cos(s) f(s)\, ds \\ &= \int_0^x (\cos(x)\sin(s) - \sin(x)\cos(s)) f(s)\, ds\end{aligned}$$

Applying $(\sin A \cos B - \cos A \sin B) = \sin(A - B)$ to the integrand above, where $A = x, B = s$ gives

$$y_p = -\int_0^x \sin(x - s) f(s)\, ds$$

Therefore the solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 \cos x + c_2 \sin x) - \int_0^x \sin(x - s) f(s)\, ds \qquad (1)\end{aligned}$$

Applying BC $y(0) = 0$ the above becomes

$$\begin{aligned}0 &= c_1 - \int_0^0 \sin(-s) f(s)\, ds \\ c_1 &= 0\end{aligned}$$

And the solution (1) simplifies to

$$y(x) = c_2 \sin x - \int_0^x \sin(x - s) f(s)\, ds \qquad (2)$$

Applying BC $y(1) = 0$ the above becomes

$$y(x) = c_2 \sin 1 - \int_0^1 \sin(1 - s) f(s)\, ds$$

Hence

$$c_2 = \frac{1}{\sin 1} \int_0^1 \sin(1 - s) f(s)\, ds$$

The solution in (2) now becomes

$$\begin{aligned}y(x) &= \frac{\sin x}{\sin 1} \int_0^1 \sin(1 - s) f(s)\, ds - \int_0^x \sin(x - s) f(s)\, ds \\ &= \frac{1}{\sin 1} \int_0^1 \sin x \sin(1 - s) f(s)\, ds - \int_0^x \sin(x - s) f(s)\, ds\end{aligned}$$

Writing $\int_0^1 \sin x \sin(1 - s) f(s)\, ds = \int_0^x \sin x \sin(1 - s) f(s)\, ds + \int_x^1 \sin x \sin(1 - s) f(s)\, ds$ then the above becomes

$$\begin{aligned}y(x) &= \frac{1}{\sin 1} \left(\int_0^x \sin x \sin(1 - s) f(s)\, ds + \int_x^1 \sin x \sin(1 - s) f(s)\, ds \right) - \int_0^x \sin(x - s) f(s)\, ds \\ &= \int_0^x \frac{\sin x \sin(1 - s)}{\sin(1)} f(s) - \int_0^x \sin(x - s) f(s)\, ds + \int_x^1 \frac{\sin x \sin(1 - s)}{\sin(1)} f(s)\, ds \\ &= \int_0^x \left[\frac{\sin x \sin(1 - s)}{\sin(1)} - \sin(x - s) \right] f(s)\, ds + \int_x^1 \frac{\sin x \sin(1 - s)}{\sin(1)} f(s)\, ds \\ &= \frac{1}{\sin(1)} \int_0^x (\sin x \sin(1 - s) - \sin(1)\sin(x - s)) f(s)\, ds + \int_x^1 \frac{\sin x \sin(1 - s)}{\sin(1)} f(s)\, ds \qquad (3)\end{aligned}$$

Using $\sin(A - B) = \sin A \cos B - \cos A \sin B$, where now $A = 1, B = s$, then

$$\sin(1 - s) = \sin 1 \cos s - \cos 1 \sin s$$

And also

$$\sin(x-s) = \sin x \cos s - \cos x \sin s$$

Using the above two relations in first integral of (3) which is $I = \int_0^x (\sin x \sin(1-s) - \sin(1)\sin(x-s)) f(s)\,ds$ gives

$$\begin{aligned}
I &= \int_0^x (\sin x(\sin 1 \cos s - \cos 1 \sin s) - \sin 1(\sin x \cos s - \cos x \sin s)) f(s)\,ds \\
&= \int_0^x (\sin x \sin 1 \cos s - \sin x \cos 1 \sin s - \sin 1 \sin x \cos s + \sin 1 \cos x \sin s) f(s)\,ds \\
&= \int_0^x (-\sin x \cos 1 \sin s + \sin 1 \cos x \sin s) f(s)\,ds \\
&= \int_0^x (\sin s(\sin 1 \cos x - \sin x \cos 1)) f(s)\,ds \\
&= \int_0^x (\sin s \sin(1-x)) f(s)\,ds
\end{aligned}$$

Substituting the above result in (3) results in

$$y(x) = \int_0^x \frac{\sin s \sin(1-x)}{\sin 1} f(s)\,ds + \int_x^1 \frac{\sin x \sin(1-s)}{\sin(1)} f(s)\,ds \tag{4}$$

Let

$$G(x,s) = \begin{cases} \frac{\sin(s)\sin(1-x)}{\sin(1)} & 0 \le s \le x \\ \frac{\sin(x)\sin(1-s)}{\sin(1)} & x \le s \le 1 \end{cases}$$

Then the solution (4) can be written as

$$y(x) = \int_0^1 G(x,s) f(s)\,ds$$

2.1.55 Chapter 11.3, Problem 31

By using procedure in problem 30 find Green function and express solution as definite integral for

$$\begin{aligned}
-y'' &= f(x) \\
y'(0) &= 0 \\
y(1) &= 0
\end{aligned}$$

<u>Solution</u>
The first step is to determine $y_1(x), y_2(x)$. These are the two fundamental solutions of $y'' = 0$. As the book says, to simplify the derivation, $y_1(x)$ is selected to be the solution that satisfies the boundary conditions at the left end of domain ($x = 0$ in this problem) and $y_2(x)$ satisfies the boundary condition on the right end ($x = 1$).
The homogeneous solution to $y'' = 0$ is

$$y_h(x) = c_1 + c_2 x$$

Therefore $y_1'(0) = 0$. This gives $c_2 = 0$. Hence

$$y_1(x) = 1$$

The second boundary conditions $y_2(1) = 0$ gives $0 = c_1 + c_2$, or $c_1 = -c_2$ and this leads to $y_2(x) = c_2(-1+x)$. Or

$$y_2(x) = x - 1$$

Given y_1, y_2 found above, the next step is to determine the Wronskian as follows

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & x-1 \\ 0 & 1 \end{vmatrix} = 1$$

Therefore, Green function is now computed using equation (iv) on page 701 of text book giving

$$G(x,s) = \frac{-1}{p(x)W(x)} \begin{cases} y_1(s)\,y_2(x) & 0 \le s \le x \\ y_1(x)\,y_2(s) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = 1$, and using values found earlier for y_1, y_2, the above becomes

$$\begin{aligned}
G(x,s) &= -1 \begin{cases} (x-1) & 0 \le s \le x \\ (s-1) & x \le s \le 1 \end{cases} \\
&= \begin{cases} x-1 & 0 \le s \le x \\ s-1 & x \le s \le 1 \end{cases}
\end{aligned}$$

Hence the solution is

$$y(x,s)=\int_0^1 G(x,s)\,f(s)\,ds \tag{1}$$

To verify this solution, it is compared to solution to same ODE using the direct method. Let $f(x)=x$. Hence the ODE is

$$\begin{aligned}-y'' &= x\\ y'(0)&=0\\ y(1)&=0\end{aligned}$$

The solution found above in (1) can now be found as

$$\begin{aligned}y(x) &= \int_0^x G(x,s)\,s ds+\int_x^1 G(x,s)\,s ds\\ &= \int_0^x (1-x)\,s ds+\int_x^1 (1-s)\,s ds\\ &= \left(\frac{s^2}{2}-x\frac{s^2}{2}\right)_0^x+\left(\frac{s^2}{2}-\frac{s^3}{3}\right)_x^1\\ &= \left(\frac{x^2}{2}-\frac{x^3}{2}\right)+\left(\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\right)\\ &= \frac{1}{6}-\frac{1}{6}x^3 \end{aligned}\tag{2}$$

Verification The solution is verified by solving the same problem using the direct method. The homogenous solution is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, let the particular solution be $y_p = kx^3, y_p' = 3kx^2, y'' = 6kx$. Therefore $6kx = -x$ or $k=\frac{-1}{6}$. Therefore the particular solution is $y_p=\frac{-1}{6}x^3$ and the general solution is

$$y(x)=c_1+c_2x-\frac{1}{6}x^3$$

Applying BC $y'(0)=0$ gives

$$c_2=0$$

Hence the solution becomes $y(x)=c_1-\frac{1}{6}x^3$. Applying BC $y(1)=0$ gives $0=c_1-\frac{1}{6}$ or $c_1=\frac{1}{6}$. Therefore the solution is

$$y(x)=\frac{1}{6}-\frac{1}{6}x^3 \tag{3}$$

Which is the same answer found using Green function method. Of course in this case the direct method is much simpler and easier to find. The advantage of Green method, is that once the $G(x,s)$ is found, then for any new $f(x)$ only integration is needed to find the new solution, since $G(x,s)$ does not change when $f(x)$ changes. The direct method requires one to find the particular solution each time, and to determine the constants c_1, c_2 again from boundary conditions each time $f(x)$ changes since the particular solution changes when $f(x)$ changes. With Green function method, all the work in using $G(x,y)$ is done in the integration step only. The solution found using Green function already incorporated the boundary conditions in it.

2.1.56 Chapter 11.3, Problem 32

By using procedure in problem 30 find Green function and express solution as definite integral for

$$\begin{aligned}-y''&=f(x)\\ y(0)&=0\\ y(1)+y'(1)&=0\end{aligned}$$

Solution
The first step is to determine $y_1(x), y_2(x)$, where these are the fundamental solutions of $y''=0$ where $y_1(x)$ satisfies the boundary conditions at the left end of domain ($x=0$) and $y_2(x)$ satisfies the boundary condition on the right end ($x=1$).
Since the homogeneous solution to $y''=0$ is

$$y_h(x)=c_1+c_2x$$

Then $y_1(0)=0$ gives $c_1=0$. Therefore

$$y_1(x)=x$$

And to satisfy $y_2(1)+y_2'(1)=0$ then

$$\begin{aligned}0&=(c_1+c_2)+c_2\\ c_1&=-2c_2\end{aligned}$$

Therefore

$$\begin{aligned} y_2(x) &= -2c_2 + c_2 x \\ &= c_2(x-2) \end{aligned}$$

Hence

$$y_2(x) = x - 2$$

Now that y_1, y_2 are found, the next step is to find the Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x-2 \\ 1 & 1 \end{vmatrix} = x - (x-2) = 2$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G(x,s) = \frac{-1}{p(x)W(x)} \begin{cases} y_1(s) y_2(x) & 0 \le s \le x \\ y_1(x) y_2(s) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = 1$, and using values found earlier for y_1, y_2, then the above becomes

$$\begin{aligned} G(x,s) &= \frac{-1}{2} \begin{cases} s(x-2) & 0 \le s \le x \\ x(s-2) & x \le s \le 1 \end{cases} \\ &= \begin{cases} \frac{s(2-x)}{2} & 0 \le s \le x \\ \frac{x(2-s)}{2} & x \le s \le 1 \end{cases} \end{aligned}$$

And the solution is

$$y(x,s) = \int_0^1 G(x,s) f(s)\, ds \tag{1}$$

To verify this solution, it is compared to solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE is

$$\begin{aligned} -y'' &= x \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

The solution found above in (1) is now found as

$$\begin{aligned} y(x) &= \int_0^x G(x,s)\, s ds + \int_x^1 G(x,s)\, s ds \\ &= \int_0^x \frac{s(2-x)}{2} s ds + \int_x^1 \frac{x(2-s)}{2} s ds \\ &= \frac{1}{2}\int_0^x \left(2s^2 - xs^2\right) ds + \frac{1}{2}\int_x^1 \left(2xs - xs^2\right) ds \\ &= \frac{1}{2}\left(\frac{2s^3}{3} - x\frac{s^3}{3}\right)_0^x + \frac{1}{2}\left(xs^2 - x\frac{s^3}{3}\right)_x^1 \\ &= \frac{1}{6}\left(2x^3 - x^4\right) + \frac{1}{2}\left(\left(x - \frac{x}{3}\right) - \left(x^3 - \frac{x^4}{3}\right)\right) \\ &= \frac{1}{6}\left(2x - x^3\right) \end{aligned} \tag{2}$$

<u>Verification</u> The solution is now verified by solving the same problem using the direct method. The homogenous solution is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, let the particular solution be $y_p = kx^3, y_p' = 3kx^2, y'' = 6kx$. Therefore $6kx = -x$ or $k = \frac{-1}{6}$. Therefore the particular solution is $y_p = \frac{-1}{6}x^3$ and the general solution is

$$y(x) = c_1 + c_2 x - \frac{1}{6}x^3$$

Applying BC $y(0) = 0$ gives

$$c_1 = 0$$

Hence the solution becomes

$$\begin{aligned} y(x) &= c_2 x - \frac{1}{6}x^3 \\ y'(x) &= c_2 - \frac{1}{2}x^2 \end{aligned}$$

Applying BC $y(1) + y'(1) = 0$ gives

$$\begin{aligned} 0 &= \left(c_2 - \frac{1}{6}\right) + \left(c_2 - \frac{1}{2}\right) \\ 0 &= 2c_2 - \frac{2}{3} \\ c_2 &= \frac{1}{3} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= \frac{1}{3}x - \frac{1}{6}x^3 \\ &= \frac{1}{6}\left(2x - x^3\right) \end{aligned} \tag{3}$$

Which is the same as (2) using Green function.

2.1.57 Chapter 11.3, Problem 33

By using procedure in problem 30 find Green function and express solution as definite integral for

$$\begin{aligned} -\left(y'' + y\right) &= f(x) \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

The first step is to determine $y_1(x), y_2(x)$, where these are the fundamental solutions of $y'' + y = 0$ where $y_1(x)$ satisfies the boundary conditions at the left end of domain ($x = 0$) and $y_2(x)$ satisfies the boundary condition on the right end ($x = 1$).

Since the homogeneous solution to $y'' + y = 0$ is

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

Then $y_1' = -c_1 \sin x + c_2 \cos x$ and $y_1'(0) = 0$ leads to $c_2 = 0$, therefore

$$y_1(x) = \cos x$$

And to satisfy $y_2(1) = 0$ then $0 = c_1 \cos 1 + c_2 \sin 1$, hence $c_2 = -c_1 \frac{\cos(1)}{\sin(1)}$. therefore

$$\begin{aligned} y_2(x) &= c_1 \cos x - c_1 \frac{\cos(1)}{\sin(1)} \sin x \\ &= c_1 \left(\cos x - \frac{\cos(1)}{\sin(1)} \sin x\right) \end{aligned}$$

Hence

$$y_2(x) = \cos x - \frac{\cos(1)}{\sin(1)} \sin x$$

Now that y_1, y_2 are found, the next step is to determine the Wronskian.

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \left(\cos x - \frac{\cos(1)}{\sin(1)} \sin x\right) \\ -\sin x & -\left(\sin x + \frac{\cos(1)}{\sin(1)} \cos x\right) \end{vmatrix} \\ &= -\cos x \left(\sin x + \frac{\cos(1)}{\sin(1)} \cos x\right) + \sin x \left(\cos x - \frac{\cos(1)}{\sin(1)} \sin x\right) \\ &= -\cos x \sin x - \frac{\cos(1)}{\sin(1)} \cos^2 x + \sin x \cos x - \frac{\cos(1)}{\sin(1)} \sin^2 x \\ &= -\frac{\cos(1)}{\sin(1)} \left(\cos^2 x + \sin^2 x\right) \\ &= -\frac{\cos(1)}{\sin(1)} \end{aligned}$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G(x,s) = \frac{-1}{p(x)W(x)} \begin{cases} y_1(s)\, y_2(x) & 0 \le s \le x \\ y_1(x)\, y_2(s) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = 1$, and using values found earlier for y_1, y_2, then the above becomes (using $p(x) = 1$)

$$
\begin{aligned}
G(x,s) &= \frac{-1}{-\frac{\cos(1)}{\sin(1)}} \begin{cases} \cos s\left(\cos x - \frac{\cos(1)}{\sin(1)}\sin x\right) & 0 \le s \le x \\ \cos x\left(\cos s - \frac{\cos(1)}{\sin(1)}\sin s\right) & x \le s \le 1 \end{cases} \\
&= \frac{\sin(1)}{\cos(1)} \begin{cases} \cos s\left(\cos x - \frac{\cos(1)}{\sin(1)}\sin x\right) & 0 \le s \le x \\ \cos x\left(\cos s - \frac{\cos(1)}{\sin(1)}\sin s\right) & x \le s \le 1 \end{cases} \\
&= \begin{cases} \frac{\cos s}{\cos(1)}\left(\sin(1)\cos x - \cos(1)\sin x\right) & 0 \le s \le x \\ \frac{\cos x}{\cos(1)}\left(\sin(1)\cos s - \cos(1)\sin s\right) & x \le s \le 1 \end{cases}
\end{aligned}
$$

Using $\sin A\cos B - \cos A\sin B = \sin(A-B)$ then $\sin(1)\cos x - \cos(1)\sin x = \sin(1-x)$ and $\sin(1)\cos s - \cos(1)\sin s = \sin(1-s)$ and the above becomes

$$
G(x,s) = \begin{cases} \frac{\cos s}{\cos(1)}\sin(1-x) & 0 \le s \le x \\ \frac{\cos x}{\cos(1)}\sin(1-s) & x \le s \le 1 \end{cases}
$$

And the solution is

$$
y(x,s) = \int_0^1 G(x,s)\,f(s)\,ds
$$

To verify this solution, it is compared to the solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE is

$$
\begin{aligned}
-(y'' + y) &= x \\
y'(0) &= 0 \\
y(1) &= 0
\end{aligned}
$$

The solution found above in (1) is now computed as

$$
\begin{aligned}
y(x) &= \int_0^x G(x,s)\,s ds + \int_x^1 G(x,s)\,s ds \\
&= \int_0^x \frac{\cos s}{\cos(1)}\sin(1-x)\,s ds + \int_x^1 \frac{\cos x}{\cos(1)}\sin(1-s)\,s ds \\
&= I_1 + I_2 \qquad (1)
\end{aligned}
$$

The first integral is

$$
\begin{aligned}
I_1 &= \frac{\sin(1-x)}{\cos(1)}\int_0^x s\cos s ds \\
&= \frac{\sin(1-x)}{\cos(1)}\left(\cos s + s\sin s\right)_0^x \\
&= \frac{\sin(1-x)}{\cos(1)}\left(\cos x + x\sin x - 1\right)
\end{aligned}
$$

The second integral is

$$
\begin{aligned}
I_2 &= \frac{\cos x}{\cos(1)}\int_x^1 s\sin(1-s)\,ds \\
&= \frac{\cos x}{\cos(1)}\left(s\cos(s-1) - \sin(s-1)\right)_x^1 \\
&= \frac{\cos x}{\cos(1)}\left(\left(\cos(1-1) - \sin(1-1)\right) - \left(x\cos(x-1) - \sin(x-1)\right)\right) \\
&= \frac{\cos x}{\cos(1)}\left(1 - \left(x\cos(x-1) - \sin(x-1)\right)\right)
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
y(x) &= \frac{\sin(1-x)}{\cos(1)}\left(\cos x + x\sin x - 1\right) + \frac{\cos x}{\cos(1)}\left(1 - \left(x\cos(x-1) - \sin(x-1)\right)\right) \\
&= \frac{1}{\cos(1)}\left(\cos x\sin(1-x) + x\sin x\sin(1-x) - \sin(1-x) + \cos x - x\cos x\cos(x-1) - \cos x\sin(x-1)\right) \\
&= \frac{1}{\cos(1)}\left(x\sin x\sin(1-x) - \sin(1-x) + \cos x - x\cos x\cos(x-1)\right) \\
&= \frac{1}{\cos 1}\left(x\left(\sin x\sin(1-x) - \cos x\cos(x-1)\right) - \sin(1-x) + \cos x\right)
\end{aligned}
$$

But $\sin A\sin B-\cos A\cos B=-\cos(A+B)$, using this in the above, where now $x=A, B=(1-x)$ gives

$$\begin{aligned} y(x) &= \frac{1}{\cos 1}\left(x\left(-\cos(x+1-x)\right)-\sin(1-x)+\cos x\right) \\ &= \frac{1}{\cos 1}\left(-x\cos(1)-\sin(1-x)+\cos x\right) \\ &= \frac{\cos x}{\cos(1)}-\frac{\sin(1-x)}{\cos(1)}-x \end{aligned} \tag{2}$$

Verification The solution is now verified by solving the same problem using the direct method. The homogenous solution to $y''+y=0$ is $y_h=c_1\cos x+c_2\sin x$. Since the forcing function is $-x$, let the particular solution be $y_p=k_1x, y_p'=k_1, y''=0$. Therefore $k_1x=-x$ or $k=-1$. Therefore the particular solution is $y_p=-x$ and the general solution is

$$y(x)=c_1\cos x+c_2\sin x-x$$

Now BC $y'(0)=0$ is applied. $y'(x)=-c_1\sin x+c_2\cos x-1$, therefore

$$\begin{aligned} 0 &= c_2-1 \\ c_2 &= 1 \end{aligned}$$

Hence the solution becomes

$$y(x)=c_1\cos x+\sin x-x$$

Applying BC $y(1)=0$ gives

$$\begin{aligned} 0 &= c_1\cos(1)+\sin(1)-1 \\ c_1 &= \frac{1-\sin(1)}{\cos(1)} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= \frac{(1-\sin(1))}{\cos(1)}\cos x+\sin(x)-x \\ &= \frac{\cos x}{\cos(1)}+\frac{-\cos x\sin(1)}{\cos(1)}+\sin(x)-x \\ &= \frac{\cos x}{\cos(1)}+\frac{\sin(x)\cos(1)-\cos x\sin(1)}{\cos(1)}-x \end{aligned}$$

But $\sin(x)\cos(1)-\cos x\sin(1)=\sin(x-1)=-\sin(1-x)$, hence the above becomes

$$\boxed{y(x)=\frac{\cos x}{\cos(1)}-\frac{\sin(1-x)}{\cos(1)}-x} \tag{3}$$

Which is the same solution in (2) found using Green function.

2.1.58 Chapter 11.3, Problem 34

By using procedure in problem 30 find Green function and express solution as definite integral for

$$\begin{aligned} -y'' &= f(x) \\ y(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

Solution
The first step is to determine $y_1(x), y_2(x)$, where these are the fundamental solutions of $y''=0$ where $y_1(x)$ satisfies the boundary conditions at the left end of domain ($x=0$) and $y_2(x)$ satisfies the boundary condition on the right end ($x=1$).
Since the homogeneous solution to $y''=0$ is

$$y_h(x)=c_1+c_2x$$

Then $y_1(0)=0$ gives $c_1=0$. Therefore

$$y_1(x)=x$$

And to satisfy $y_2'(1)=0$ then $0=c_2$. and this leads to

$$y_2(x)=1$$

Now that y_1, y_2 are found, the next step is to find the Wronskian.

$$W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix}=-1$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G(x,s) = \frac{-1}{p(x)W(x)}\begin{cases} y_1(s)\,y_2(s) & 0 \le s \le x \\ y_1(x)\,y_2(x) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = -1$, and using values found earlier for y_1, y_2, then the above becomes

$$G(x,s) = \begin{cases} s & 0 \le s \le x \\ x & x \le s \le 1 \end{cases}$$

And the solution is

$$y(x,s) = \int_0^1 G(x,s)\,f(s)\,ds \tag{1}$$

To verify this solution, it is now compared to the solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE now is

$$\begin{aligned} -y'' &= x \\ y(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

The solution found above in (1) is now computed as

$$\begin{aligned} y(x) &= \int_0^x G(x,s)\,s ds + \int_x^1 G(x,s)\,s ds \\ &= \int_0^x (s)\,s ds + \int_x^1 (x)\,s ds \\ &= \left(\frac{s^3}{3}\right)_0^x + x\left(\frac{s^2}{2}\right)_x^1 \\ &= \frac{1}{3}x^3 + \frac{x}{2}\left(1 - x^2\right) \\ &= \frac{1}{2}x - \frac{1}{6}x^3 \end{aligned} \tag{2}$$

Verification The above solution is now verified by solving the same problem using the direct method. The homogenous solution to $y'' = 0$ is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, the particular solution is $y_p = \frac{-1}{6}x^3$ and the general solution is

$$y(x) = c_1 + c_2 x - \frac{1}{6}x^3$$

BC $y(0) = 0$ gives $c_1 = 0$. The solution becomes $y(x) = c_2 x - \frac{1}{6}x^3$ and $y'(x) = c_2 - \frac{1}{2}x^2$. BC $y'(1) = 0$ gives

$$\begin{aligned} 0 &= c_2 - \frac{1}{2} \\ c_2 &= \frac{1}{2} \end{aligned}$$

Hence the solution becomes

$$y(x) = \frac{1}{2}x - \frac{1}{6}x^3$$

Which is the same solution in (2) found using Green function.

2.1.59 Chapter 11.4, Problem 1

Find formal solution to

$$-(xy')' = \mu xy + f(x)$$

where y, y' bounded as $x \to 0$ and $y(1) = 0$

Solution

The given ODE can be written as

$$-\frac{1}{x}(xy')' = \mu y + \frac{f(x)}{x} \tag{1}$$

The corresponding homogeneous ODE

$$-\frac{1}{x}(xy')' = \lambda y \tag{2}$$

Where $p = x, q = 0, r = x$. This was solved in the textbook at page 707. The fundamental solution is given by $y_n = \Phi_n(x) = J_0\left(\sqrt{\lambda_n}x\right)$ where the eigenvalues λ_n are the roots of $J_o\left(\sqrt{\lambda_n}\right) = 0$. These eigenfunctions are not normalized. Therefore, the solution of the inhomogeneous ODE (1) can be now written as

$$y(x) = \sum_{n=1}^{\infty} b_n \Phi_n(x)$$

Using this in (1) gives

$$-\frac{1}{x}(xy')' = \mu \sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} c_n \Phi_n(x)$$

But from (2), $-\frac{1}{x}(xy')'$ can be replaced by λy, so the above becomes

$$\sum_{n=1}^{\infty} \lambda_n b_n \Phi_n(x) = \mu \sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} c_n \Phi_n(x) \tag{3}$$

Where

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) = \frac{f(x)}{x}$$

c_n is now found by orthogonality. Multiplying both sides of the above by $r(x)\Phi_m(x)$, where the weight $r(x) = x$, and integrating gives

$$\begin{aligned}
\int_0^1 x\frac{f(x)}{x}\Phi_m(x)\,dx &= \sum_{n=1}^{\infty} c_n \int_0^1 x\Phi_n(x)\Phi_m(x)\,dx \\
\int_0^1 f(x)\Phi_m(x)\,dx &= \sum_{n=1}^{\infty} c_n \int_0^1 x\Phi_n(x)\Phi_m(x)\,dx
\end{aligned}$$

Due to orthogonality of the eigenfunctions, the above simplifies to

$$c_n = \frac{\int_0^1 f(x)\Phi_n(x)\,dx}{\int_0^1 x\Phi_n^2(x)\,dx} \tag{4}$$

Since $\Phi_n(x)$ is not normalized, $\int_0^1 x\Phi_n^2(x)\,dx$ can not be replaced by 1. The above is left as is. Substituting (4) in (3) and simplifying gives

$$\begin{aligned}
\lambda_n b_n &= \mu b_n + c_n \\
b_n &= \frac{c_n}{(\lambda_n - \mu)}
\end{aligned}$$

Where $\lambda_n \neq \mu$. Hence the formal solution $y = \sum_{n=1}^{\infty} b_n \Phi_n(x)$ can be written as

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{(\lambda_n - \mu)} J_o\left(\sqrt{\lambda_n} x\right)$$

Using (4) in the above gives

$$y(x) = \sum_{n=1}^{\infty} \left(\frac{\int_0^1 f(x)\Phi_n(x)\,dx}{\int_0^1 x\Phi_n^2(x)\,dx} \right) \frac{J_o\left(\sqrt{\lambda_n} x\right)}{(\lambda_n - \mu)}$$

2.1.60 Chapter 11.4, Problem 2

Consider BVP

$$-(xy')' = \lambda xy$$

where y, y' bounded as $x \to 0$ and $y'(1) = 0$. (a) Show that $\lambda_0 = 0$ is eigenvalue corresponding to $\Phi_0 = 1$. If $\lambda > 0$ show formally that the eigenfunctions are given by $\Phi_n = J_0\left(\sqrt{\lambda_n} x\right)$ where $\sqrt{\lambda_n}$ is the n^{th} positive root in increasing order of $J_0'\left(\sqrt{\lambda_n}\right) = 0$. It is possible to show there are infinite sequence of such roots.
(b) Show that if $m = 0, 1, 2, \cdots$ then $\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx = 0, m \neq n$.
(c) Find formal solution to nonhomogeneous problem $-(xy')' = \mu xy + f(x)$, where y, y' bounded as $x \to 0$ and $y'(1) = 0$, where f is given continuous function on $0 \leq x \leq 1$ and μ is not eigenvalue of the corresponding homogeneous ODE.
Solution

Part (a)

The given ODE can be written as

$$xy'' + y' + \lambda xy = 0 \tag{1}$$

Let $t = \sqrt{\lambda}x$, then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\sqrt{\lambda}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\sqrt{\lambda}\right) = \sqrt{\lambda}\frac{d^2y}{dt^2}\frac{dt}{dx} = \sqrt{\lambda}\frac{d^2y}{dt^2}\sqrt{\lambda} = \lambda\frac{d^2y}{dt^2}$. Hence (1) becomes

$$\frac{t}{\sqrt{\lambda}}\lambda y''(t) + \sqrt{\lambda}y'(t) + \lambda\frac{t}{\sqrt{\lambda}}y(t) = 0$$
$$t\sqrt{\lambda}y''(t) + \sqrt{\lambda}y'(t) + \sqrt{\lambda}ty(t) = 0$$

Since problem says that $\lambda > 0$, then dividing by $\sqrt{\lambda}$ the above simplifies to

$$ty''(t) + y'(t) + ty(t) = 0$$

This is Bessel ODE of zero order. Its solution is $y(t) = c_1 J_0(t) + c_2 Y_0(t)$. Where $J_0(0) = 0$ and $\lim_{t\to 0} Y_0(t) \to \infty$. Hence a bounded solution requires that $c_2 = 0$. Therefore the solution becomes

$$y(t) = c_1 J_0(t)$$

or in terms of x

$$y(x) = c_1 J_0\left(\sqrt{\lambda}x\right)$$

To satisfy the second boundary condition, since $y'(x) = c_1 J_0'\left(\sqrt{\lambda}x\right) = -c_1 J_1\left(\sqrt{\lambda}x\right)$. Therefore the eigenvalues are roots of

$$J_1\left(\sqrt{\lambda}x\right) = 0$$

Plotting $J_1\left(\sqrt{\lambda}x\right)$ shows that the first roots are $\lambda = 0$. Numerically, the first few eigenvalues are

$$\lambda = \{0, 14.682, 49.2185, 103, 499, 177.532, \cdots\} \tag{2}$$

Hence the fundamental solution is $y(x) = J_0\left(\sqrt{\lambda_n}x\right)$ where λ_n is given by above. When $\lambda = 0$, $J_0(0) = 1$. Therefore the eigenfunction associated with $\lambda = 0$ is $\Phi_0(x) = 1$. Since there are infinite eigenvalues (2), there are infinite eigenfunctions $\Phi_n(x) = J_0\left(\sqrt{\lambda_n}x\right)$ where $n = 0, 1, 2, 3, \cdots$

Part (b)

Let $\Phi_n(x), \Phi_m(x)$ be any two eigenfunctions of $(xy')' + \lambda xy = 0$. Therefore each satisfies the ODE. Hence

$$\left(x\Phi_n'\right)' + \lambda_n x\Phi_n(x) = 0 \tag{3A}$$
$$\left(x\Phi_m'\right)' + \lambda_m x\Phi_m(x) = 0 \tag{3B}$$

Multiplying (3A) by Φ_m and (3B) by Φ_n and subtracting gives

$$\Phi_m\left(x\Phi_n'\right)' + \lambda_n x\Phi_m\Phi_n(x) - \Phi_n\left(x\Phi_m'\right)' - \lambda_m x\Phi_n\Phi_m(x) = 0$$
$$\Phi_m\left(x\Phi_n'\right)' - \Phi_n\left(x\Phi_m'\right)' + (\lambda_n - \lambda_m)x\Phi_n\Phi_m(x) = 0$$

Integrating from $0\cdots 1$ gives

$$\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx - \int_0^1 \Phi_n\left(x\Phi_m'\right)' dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{4}$$

Integrating $\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_m}^{u}\,\overbrace{\left(x\Phi_n'\right)' dx}^{dv} = \left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right) dx \tag{5A}$$

And similarly, Integrating $\int_0^1 \Phi_n\left(x\Phi_m'\right)' dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_n}^{u}\,\overbrace{\left(x\Phi_m'\right)' dx}^{dv} = \left[\Phi_n x\Phi_m'\right]_0^1 - \int_0^1 \Phi_n'\left(x\Phi_m'\right) dx \tag{5B}$$

Substituting (5A,5B) back in (4) gives

$$\left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right) dx - \left[\Phi_n x\Phi_m'\right]_0^1 + \int_0^1 \Phi_n'\left(x\Phi_m'\right) dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

The above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{6}$$

The boundary terms above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = \left[\Phi_m(1)\Phi_n'(1) - \Phi_n(1)\Phi_m'(1)\right]$$

But $\Phi_n'(1)$ and $\Phi_m'(1)$ are zero. This is because of the given boundary conditions $y'(1) = 0$. Hence $\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = 0$. Therefore (6) now simplifies to

$$(\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, then one concludes that

$$\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

Which is the result asked to show.

Part (c)

The problem to solve is written as

$$-\frac{1}{x}(xy')' = \mu y + \frac{f(x)}{x} \tag{A}$$

The solution to the corresponding homogeneous ODE $-\frac{1}{x}(xy')' = \lambda y$ was found in part (a). Using eigenfunction expansion, the solution of the nonhomogeneous ODE (A) can then be written as

$$y(x) = \sum_{n=0}^{\infty} b_n\Phi_n(x) \tag{7}$$

Where $\Phi_n(x) = J_0\left(\sqrt{\lambda_n}x\right), n = 0, 1, 2, \cdots$ and λ_n are roots of $-J_1\left(\sqrt{\lambda}\right) = 0$. Using (7) in $-(xy')' = \mu xy + f(x)$ gives

$$-\frac{1}{x}(xy')' = x\sum_{n=0}^{\infty} b_n\Phi_n(x) + \sum_{n=0}^{\infty} c_n\Phi_n(x)$$

But since $-\frac{1}{x}(xy')' = \lambda y$ from part (a), then the above becomes

$$\sum_{n=0}^{\infty}\lambda_n b_n\Phi_n(x) = \mu\sum_{n=0}^{\infty} b_n\Phi_n(x) + \sum_{n=0}^{\infty} c_n\Phi_n(x) \tag{8}$$

Where

$$\sum_{n=0}^{\infty} c_n\Phi_n(x) = \frac{f(x)}{x}$$

c_n is now found by orthogonality. Multiplying both sides of the above by $r(x)\Phi_m(x)$, where the weight $r(x) = x$, and integrating gives

$$\begin{aligned}
\int_0^1 x\frac{f(x)}{x}\Phi_m(x)\,dx &= c_0\int_0^1 x\Phi_0(x)\Phi_m(x)\,dx + \sum_{n=1}^{\infty} c_n\int_0^1 x\Phi_n(x)\Phi_m(x)\,dx\\
\int_0^1 f(x)\Phi_m(x)\,dx &= c_0\int_0^1 x\Phi_0(x)\Phi_m(x)\,dx + \sum_{n=1}^{\infty} c_n\int_0^1 x\Phi_n(x)\Phi_m(x)\,dx
\end{aligned} \tag{9}$$

For $m = 0$, the eigenfunction is $\Phi_0(x) = 1$, and the above becomes

$$\begin{aligned}
\int_0^1 f(x)\,dx &= c_0\int_0^1 x\,dx\\
&= c_0\left[\frac{x^2}{2}\right]_0^1 = \frac{c_0}{2}
\end{aligned}$$

Therefore

$$c_0 = 2\int_0^1 f(x)\,dx \tag{10}$$

For $m > 0$, (9) becomes

$$\int_0^1 f(x)\Phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n\int_0^1 x\Phi_n(x)\Phi_m(x)\,dx$$

Due to orthogonality of the eigenfunctions from part (b) $\int_0^1 x\Phi_n(x)\Phi_m(x)\,dx = 0$ for $m \neq n$, and the above simplifies to

$$c_n = \frac{\int_0^1 f(x)\Phi_n(x)\,dx}{\int_0^1 x\Phi_n^2(x)\,dx} \tag{11}$$

Since $\Phi_n(x)$ is not normalized, $\int_0^1 x\Phi_n^2(x)\,dx$ can not be replaced by 1. The above is left as is. Substituting (10,11) in (8) and simplifying gives

$$\sum_{n=0}^{\infty}\lambda_n b_n\Phi_n(x) = \mu\sum_{n=0}^{\infty} b_n\Phi_n(x) + \sum_{n=0}^{\infty} c_n\Phi_n(x) \tag{12}$$

For $n = 0$ only, and since $\lambda_n = 0$ then (12) gives

$$0 = \mu b_0\Phi_0(x) + c_0\Phi_0(x)$$

But $\Phi_0(x) = 1$, hence

$$\begin{aligned} 0 &= \mu b_0 + c_0 \\ b_0 &= -\frac{c_0}{\mu} \end{aligned}$$

For $n > 0$, then (12) gives

$$\begin{aligned} \sum_{n=1}^{\infty}\lambda_n b_n\Phi_n(x) &= \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x) \\ \lambda_n b_n &= \mu b_n + c_n \\ b_n &= \frac{c_n}{(\lambda_n - \mu)} \end{aligned}$$

Where $\lambda_n \neq \mu$. Hence the formal solution $y = \sum_{n=0}^{\infty} b_n\Phi_n(x)$ can be written as

$$\begin{aligned} y(x) &= b_0\Phi_0(x) + \sum_{n=1}^{\infty} b_n\Phi_0(x) \\ &= -\frac{c_0}{\mu} + \sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n - \mu)} J_o\left(\sqrt{\lambda_n}x\right) \\ &= -\frac{2}{\mu}\int_0^1 f(x)\,dx + \sum_{n=1}^{\infty}\frac{1}{(\lambda_n - \mu)}\frac{\int_0^1 f(x) J_o\left(\sqrt{\lambda_n}x\right)dx}{\int_0^1 xJ_o^2\left(\sqrt{\lambda_n}x\right)dx} J_o\left(\sqrt{\lambda_n}x\right) \end{aligned}$$

But $\int_0^1 xJ_o^2\left(\sqrt{\lambda_n}x\right)dx = \frac{1}{2}\left(J_o^2\left(\sqrt{\lambda_n}\right) + J_1^2\left(\sqrt{\lambda_n}\right)\right)$, hence the above becomes

$$y(x) = -\frac{2}{\mu}\int_0^1 f(x)\,dx + 2\sum_{n=1}^{\infty}\frac{1}{(\lambda_n - \mu)}\frac{\int_0^1 f(x) J_o\left(\sqrt{\lambda_n}x\right)dx}{J_o^2\left(\sqrt{\lambda_n}\right) + J_1^2\left(\sqrt{\lambda_n}\right)} J_o\left(\sqrt{\lambda_n}x\right)$$

2.1.61 Chapter 11.4, Problem 3

Consider $-(xy')' + \frac{k^2}{x}y = \lambda xy$. with y, y' bounded as $x \to 0$ and $y(1) = 0$, where k is positive integer. (a) using $t = \sqrt{\lambda}x$ show the ODE reduces to Bessel of order k. (b) show formally that the eigenvalues $\lambda_1, \lambda_2, \cdots$ of the given differential equation are the squares of positive zeros of $J_k\left(\sqrt{\lambda}\right)$ and that the corresponding eigenfunctions are $\Phi_n(x) = J_k\left(\sqrt{\lambda}x\right)$. It is possible to show there as infinite sequence of such zeros. (c) Show that the eigenfunctions $\Phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx = 0 \qquad m \neq n$$

(d) Determine the coefficients of the formal series expansion $f(x) = \sum_{n=1}^{\infty} a_n\Phi_n(x)$. (e) Final formal solution of the nonhomogeneous problem

$$-(xy')' + \frac{k^2}{x}y = \mu xy + f(x)$$

With y, y' bounded as $x \to 0$ and $y(1) = 0$, where f is given continuous function on $0 \leq x \leq 1$ and μ is eigenvalue of the corresponding homogeneous problem.
Solution

part (a)

The ODE to solve is

$$-(xy')' + \frac{k^2}{x}y - \lambda xy = 0$$

Note: The problem seems to not have mentioned that $\lambda > 0$ here as well, as in the problem above it. This condition is needed to fully solve this problem with y, y' bounded as $x \to 0$ and $y(1) = 0$. The ODE can be written as

$$-xy'' - y' + y\left(\frac{k^2}{x} - \lambda x\right) = 0$$

$$xy'' + y' + y\left(\lambda x - \frac{k^2}{x}\right) = 0 \tag{1}$$

Let $t = \sqrt{\lambda}x$, then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\sqrt{\lambda}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\sqrt{\lambda}\right) = \sqrt{\lambda}\frac{d^2y}{dt^2}\frac{dt}{dx} = \sqrt{\lambda}\frac{d^2y}{dt^2}\sqrt{\lambda} = \lambda\frac{d^2y}{dt^2}$. Hence (1) becomes

$$\frac{t}{\sqrt{\lambda}}\lambda y''(t) + \sqrt{\lambda}y'(t) + y(t)\left(\lambda\frac{t}{\sqrt{\lambda}} - \frac{k^2}{t}\sqrt{\lambda}\right) = 0$$

$$t\sqrt{\lambda}y''(t) + \sqrt{\lambda}y'(t) + \sqrt{\lambda}y(t)\left(t - \frac{k^2}{t}\right) = 0$$

$$t^2y'' + ty' + \left(t^2 - k^2\right)y = 0$$

This is Bessel ODE of k order.

Part (b)

The solution to the above ODE is known to be

$$y(t) = c_1J_k(t) + c_2Y_k(t)$$

Where $J_k(0) = 0$ and $\lim_{t\to 0} Y_k(t) \to \infty$. Hence a bounded solution requires that $c_2 = 0$. Therefore the solution becomes

$$y(t) = c_1J_k(t)$$

Or in terms of x

$$y(x) = c_1J_k\left(\sqrt{\lambda}x\right)$$

To satisfy the second boundary condition $y(1) = 0$ gives

$$c_1J_k\left(\sqrt{\lambda}\right) = 0$$

Non-trivial solution implies $J_k\left(\sqrt{\lambda}\right) = 0$. Therefore the eigenvalues are the square of positive roots of this equation. Even though there are negative and positive roots for $J_k\left(\sqrt{\lambda}\right) = 0$ but for real root, λ must be non-negative. It assumed $\lambda > 0$. There are infinite number of positive roots for $J_k\left(\sqrt{\lambda}\right) = 0$. Hence the eigenfunctions are

$$\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Where λ_n are square of the all positive zeros of $J_k\left(\sqrt{\lambda}\right) = 0$.

Part (c)

Show that the eigenfunctions $\Phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx = 0 \qquad m \neq n$$

Let $\Phi_n(x), \Phi_m(x)$ be any two eigenfunctions of $-(xy')' + \frac{k^2}{x}y = \lambda xy$ where now $\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right)$ and $\Phi_m(x) = J_k\left(\sqrt{\lambda_m}x\right)$. Therefore each satisfies the ODE. Hence

$$-\left(x\Phi_n'\right)' + \frac{k^2}{x}\Phi_n(x) - \lambda_n x\Phi_n(x) = 0 \tag{3A}$$

$$-\left(x\Phi_m'\right)' + \frac{k^2}{x}\Phi_m(x) - \lambda_m x\Phi_m(x) = 0 \tag{3B}$$

Multiplying 3A by Φ_m and 3B by Φ_n and subtracting gives

$$
\begin{aligned}
-\Phi_m\left(x\Phi_n'\right)' + \Phi_m\frac{k^2}{x}\Phi_n(x) - \lambda_n x\Phi_m\Phi_n(x) - \left(-\left(\Phi_n x\Phi_m'\right)' + \frac{k^2}{x}\Phi_n\Phi_m(x) - \lambda_m x\Phi_n\Phi_m(x)\right) &= 0\\
-\Phi_m\left(x\Phi_n'\right)' + \frac{k^2}{x}\Phi_m\Phi_n(x) - \lambda_n x\Phi_m\Phi_n(x) + \Phi_n\left(x\Phi_m'\right)' - \frac{k^2}{x}\Phi_n\Phi_m(x) + \lambda_m x\Phi_n\Phi_m(x) &= 0\\
-\left(x\Phi_n'\right)' + \left(x\Phi_m'\right)' + \left(\lambda_m - \lambda_n\right)x\Phi_n\Phi_m(x) &= 0
\end{aligned}
$$

Integrating from $0\cdots 1$ gives

$$
\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx - \int_0^1 \Phi_n\left(x\Phi_m'\right)' dx + \left(\lambda_n - \lambda_m\right)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{4}
$$

Integrating $\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx$ by parts gives

$$
\int_0^1 \overbrace{\Phi_m}^{u}\overbrace{\left(x\Phi_n'\right)'}^{dv}dx = \left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right)dx \tag{5A}
$$

And similarly, Integrating $\int_0^1 \Phi_n\left(x\Phi_m'\right)' dx$ by parts gives

$$
\int_0^1 \overbrace{\Phi_n}^{u}\overbrace{\left(x\Phi_m'\right)'}^{dv}dx = \left[\Phi_n x\Phi_m'\right]_0^1 - \int_0^1 \Phi_n'\left(x\Phi_m'\right)dx \tag{5B}
$$

Substituting (5A,5B) back in (4) gives

$$
\left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right)dx - \left[\Phi_n x\Phi_m'\right]_0^1 + \int_0^1 \Phi_n'\left(x\Phi_m'\right)dx + \left(\lambda_n - \lambda_m\right)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0
$$

The above simplifies to

$$
\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 + \left(\lambda_n - \lambda_m\right)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{6}
$$

Let $\Delta = \left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1$, then the boundary terms above simplifies to

$$
\Delta = \left[\Phi_m(1)\Phi_n'(1) - \Phi_n(1)\Phi_m'(1)\right] - \lim_{x\to 0}\left[x\Phi_m(x)\Phi_n'(x) - x\Phi_n(x)\Phi_m'(x)\right]
$$

But $\Phi_n(1)$ and $\Phi_m(1)$ are zero. This is because of the given boundary conditions. Hence the above simplifies to

$$
\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = -\lim_{x\to 0}\left(x\left(\Phi_m(x)\Phi_n'(x) - \Phi_n(x)\Phi_m'(x)\right)\right)
$$

But since both $\Phi_m(x), \Phi_n(x), \Phi_n'(x), \Phi_m'(x)$ are bounded as $x \to 0$ then the above vanishes. This means the all the boundary terms are zero and (6) simplifies to

$$
\left(\lambda_n - \lambda_m\right)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0
$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, therefore

$$
\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0
$$

Which is the result asked to show.

Part (d,e)

This is both parts combined. To solve $-(xy')' + \frac{k^2}{x}y = \mu xy + f(x)$, we start with dividing by x to get the ODE to the form

$$
-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \mu y + \frac{f(x)}{x} \tag{1}
$$

The homogeneous ode $-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \lambda y$ was solved in part (a,b). And since the problem says that $\lambda \neq \mu$, then the solution to the above nonhomogeneous ODE is

$$
y(x) = \sum_{n=1}^{\infty} b_n\Phi_n(x) \tag{1}
$$

Where $\Phi_n(x)$ are eigenfunctions of the homogeneous ODE found above to be

$$
\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots
$$

Substituting (2) in RHS of (1) gives

$$-\frac{1}{x}\left(xy'\right)'+\frac{k^2}{x^2}y=\mu\sum_{n=1}^{\infty}b_n\Phi_n\left(x\right)+\sum_{n=1}^{\infty}c_n\Phi_n\left(x\right)$$

Where $\sum_{n=1}^{\infty}c_n\Phi_n\left(x\right)=\frac{f(x)}{x}$. But $-\frac{1}{x}\left(xy'\right)'+\frac{k^2}{x^2}y=\lambda y$ from part (a,b). Therefore the above becomes

$$\sum_{n=1}^{\infty}\lambda_n b_n\Phi_n\left(x\right)=\mu\sum_{n=1}^{\infty}b_n\Phi_n\left(x\right)+\sum_{n=1}^{\infty}c_n\Phi_n\left(x\right)$$

Or

$$\begin{aligned}\lambda_n b_n&=\mu b_n+c_n\\ b_n&=\frac{c_n}{\lambda_n-\mu}\end{aligned}$$

What is left is to to find c_n (called a_n in this problem). Since $\sum_{n=1}^{\infty}c_n\Phi_n\left(x\right)=\frac{f(x)}{x}$, then applying orthogonality gives

$$c_n\int_0^1 r\left(x\right)\Phi_n^2\left(x\right)dx=\int_0^1 r\left(x\right)\frac{f\left(x\right)}{x}\Phi_n\left(x\right)dx$$

But $r\left(x\right)=x$, and the above becomes

$$\begin{aligned}c_n\int_0^1 xJ_k^2\left(\sqrt{\lambda_n}x\right)dx&=\int_0^1 f\left(x\right)J_k\left(\sqrt{\lambda_n}x\right)dx\\ c_n&=\frac{\int_0^1 f\left(x\right)J_k\left(\sqrt{\lambda_n}x\right)dx}{\int_0^1 xJ_k^2\left(\sqrt{\lambda_n}x\right)dx}\end{aligned}$$

This complete the solution.

$$\begin{aligned}y\left(x\right)&=\sum_{n=1}^{\infty}b_nJ_k\left(\sqrt{\lambda_n}x\right)\\ &=\sum_{n=1}^{\infty}\frac{c_n}{\lambda_n-\mu}J_k\left(\sqrt{\lambda_n}x\right)\\ &=\sum_{n=1}^{\infty}\frac{\int_0^1 f\left(x\right)J_k\left(\sqrt{\lambda_n}x\right)dx}{\int_0^1 xJ_k^2\left(\sqrt{\lambda_n}x\right)dx}\frac{J_k\left(\sqrt{\lambda_n}x\right)}{\lambda_n-\mu}\end{aligned}$$

2.1.62 Chapter 11.4, Problem 4

Consider Legendre equation $-\left(\left(1-x^2\right)y'\right)'=\lambda y$ subject to boundary conditions $y\left(0\right)=0$ with y,y' bounded as $x\longrightarrow 1$ and $\Phi_1\left(x\right)=P_1\left(x\right),\Phi_2\left(x\right)=P_3\left(x\right)$, $\Phi_n\left(x\right)=P_{2n-1}\left(x\right)$ corresponding to eigenvalues $\lambda_1=2,\lambda_2=4\cdot 3,\cdots,\lambda_n=2n\left(2n-1\right)$. (a) Show that the eigenfunctions $\Phi_n\left(x\right)$ satisfy the orthogonality relation

$$\int_0^1\Phi_m\left(x\right)\Phi_n\left(x\right)dx=0\qquad m\neq n$$

(b) Final formal solution of the nonhomogeneous problem $-\left(\left(1-x^2\right)y'\right)'=\mu y+f\left(x\right)$ where $y\left(0\right)=0$ with y,y' bounded as $x\longrightarrow 1$ where $f\left(x\right)$ is continuous function on $0\leq x\leq 1$ and μ is not eigenvalue of $-\left(\left(1-x^2\right)y'\right)'=\lambda y$
Solution

Part (a)

Let $\Phi_n\left(x\right),\Phi_m\left(x\right)$ be any two eigenfunctions of $-\left(\left(1-x^2\right)y'\right)'=\lambda y$ associated with eigenvalues λ_n,λ_m, where $\Phi_n\left(x\right)=P_n\left(x\right)$ and $\Phi_m\left(x\right)=P_m\left(x\right)$. Therefore each satisfies the ODE. Hence

$$\left(\left(1-x^2\right)\Phi_n'\left(x\right)\right)'+\lambda_n\Phi_n=0\tag{3A}$$
$$\left(\left(1-x^2\right)\Phi_m'\left(x\right)\right)'+\lambda_m\Phi_m=0\tag{3B}$$

Multiplying 3A by Φ_m and 3B by Φ_n and subtracting gives

$$\begin{aligned}\Phi_m\left(\left(1-x^2\right)\Phi_n'\left(x\right)\right)'+\lambda_n\Phi_m\Phi_n-\left(\Phi_n\left(\left(1-x^2\right)\Phi_m'\left(x\right)\right)'+\lambda_m\Phi_n\Phi_m\right)&=0\\ \Phi_m\left(\left(1-x^2\right)\Phi_n'\left(x\right)\right)'-\Phi_n\left(\left(1-x^2\right)\Phi_m'\left(x\right)\right)'+\left(\lambda_n-\lambda_m\right)\Phi_n\Phi_m&=0\end{aligned}$$

Integrating from $0\cdots 1$ gives (all upper limits below show be $\lim_{\varepsilon\rightarrow 0^-}\int_0^{1-\varepsilon}$ instead of $\int_0^1$ but to simplify notation, the latter is used and at the end, it is switched back to former.

$$\int_0^1\Phi_m\left(\left(1-x^2\right)\Phi_n'\left(x\right)\right)'dx-\int_0^1\Phi_n\left(\left(1-x^2\right)\Phi_m'\left(x\right)\right)'dx+\left(\lambda_n-\lambda_m\right)\int_0^1\Phi_n\Phi_m\left(x\right)dx=0\tag{4}$$

The first integral in (4) $\int_0^1 \overbrace{\Phi_m}^{u} \overbrace{\left(\left(1-x^2\right)\Phi_n'(x)\right)'}^{dv} dx$ is integrated by parts, giving

$$\begin{aligned}\int_0^1 \Phi_m \left(\left(1-x^2\right)\Phi_n'(x)\right)' dx &= \left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \int_0^1 \Phi_m'\left(\left(1-x^2\right)\Phi_n'(x)\right) dx \\ &= \left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \int_0^1 \Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right) dx \qquad \text{(4A)}\end{aligned}$$

Similarly, the second integral in (4) $\int_0^1 \overbrace{\Phi_n}^{u} \overbrace{\left(\left(1-x^2\right)\Phi_m'(x)\right)'}^{dv} dx$ is integrated by parts, giving

$$\begin{aligned}\int_0^1 \Phi_n \left(\left(1-x^2\right)\Phi_m'(x)\right)' dx &= \left[\Phi_n\left(1-x^2\right)\Phi_m'(x)\right]_0^1 - \int_0^1 \Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right) dx \\ &= \left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \int_0^1 \Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right) dx \qquad \text{(4B)}\end{aligned}$$

Substituting (4A) and (4B) back into (4) gives

$$\begin{aligned}&\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \int_0^1 \Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right) dx - \\ &\qquad \left(\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \int_0^1 \Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right) dx\right) \\ &\qquad\qquad + (\lambda_n-\lambda_m)\int_0^1 \Phi_n\Phi_m(x)\,dx = 0\end{aligned}$$

Terms cancel and the above reduces to

$$\begin{aligned}\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 - \left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 + (\lambda_n-\lambda_m)\int_0^1 \Phi_n\Phi_m(x)\,dx &= 0 \\ \left[\Phi_m\left(1-x^2\right)\Phi_n'(x) - \Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1 + (\lambda_n-\lambda_m)\int_0^1 \Phi_n\Phi_m(x)\,dx &= 0 \qquad (5)\end{aligned}$$

Let $\Delta = \left[\Phi_m\left(1-x^2\right)\Phi_n'(x) - \Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1$. The boundary terms above are evaluated as follows

$$\Delta = \lim_{x\to 1}\left[\Phi_m(x)\left(1-x^2\right)\Phi_n'(x) - \Phi_m(x)\left(1-x^2\right)\Phi_n'(x)\right] - \left(\Phi_m(0)\Phi_n'(0) - \Phi_m(0)\Phi_n'(0)\right)$$

Since $\Phi_m(0)=0, \Phi_m(0)=0$, the above simplifies to

$$\begin{aligned}\Delta &= \lim_{x\to 1}\left[\Phi_m(x)\left(1-x^2\right)\Phi_n'(x) - \Phi_m(x)\left(1-x^2\right)\Phi_n'(x)\right] \\ &= \lim_{x\to 1}\left(1-x^2\right)\left[\Phi_m(x)\Phi_n'(x) - \Phi_m(x)\Phi_n'(x)\right]\end{aligned}$$

Since $\Phi_m(x), \Phi_n'(x), \Phi_m(x)\Phi_n'(x)$ are all bounded as $x\to 1$ then the above goes to zero in the limit. Which means all boundary conditions term vanish. Hence (5) reduces to

$$(\lambda_n-\lambda_m)\int_0^1 \Phi_n\Phi_m(x)\,dx = 0$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, therefore

$$\int_0^1 \Phi_n\Phi_m(x)\,dx = 0$$

Which is the result asked to show.

Part (b)

Since $\lambda \neq \mu$, then the the solution to nonhomogeneous ODE is

$$y(x) = \sum_{n=1}^{\infty} b_n\Phi_n(x) \qquad (1)$$

Where $\Phi_n(x)$ are eigenfunctions $\Phi_n(x) = P_{(2n-1)}(x)$. Substituting (1) in $-\left(\left(1-x^2\right)y'\right)' = \mu y + f(x)$ gives

$$-\left(\left(1-x^2\right)y'\right)' = \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x)$$

Where $\sum_{n=1}^{\infty} c_n \Phi_n(x) = f(x)$. But $-\left(\left(1-x^2\right) y'\right)' = \lambda y$, therefore the above becomes

$$\sum_{n=1}^{\infty} \lambda_n b_n \Phi_n(x) = \mu \sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} c_n \Phi_n(x)$$

Or

$$\begin{aligned} \lambda_n b_n &= \mu b_n + c_n \\ b_n &= \frac{c_n}{\lambda_n - \mu} \end{aligned}$$

What is left is to to find c_n. Since $\sum_{n=1}^{\infty} c_n \Phi_n(x) = f(x)$, then applying orthogonality gives

$$c_n \int_0^1 r(x) \Phi_n^2(x)\, dx = \int_0^1 r(x) f(x) \Phi_n(x)\, dx$$

But $r(x) = 1$, and the above becomes

$$\begin{aligned} c_n \int_0^1 P_{(2n-1)}^2(x)\, dx &= \int_0^1 f(x) P_{(2n-1)}(x)\, dx \\ c_n &= \frac{\int_0^1 f(x) P_{(2n-1)}(x)\, dx}{\int_0^1 P_{(2n-1)}^2(x)\, dx} \end{aligned}$$

This complete the solution.

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} b_n P_{(2n-1)}(x) \\ &= \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} P_{(2n-1)}(x) \\ &= \sum_{n=1}^{\infty} \frac{\int_0^1 f(x) P_{(2n-1)}(x)\, dx}{\int_0^1 P_{(2n-1)}^2(x)\, dx} \frac{P_{(2n-1)}(x)}{\lambda_n - \mu} \end{aligned}$$

2.1.63 Chapter 11.4, Problem 5

Equation $\left(1-x^2\right) y'' - xy' + \lambda y = 0$ is Chebyshev's equation. (a) show it can be written as

$$-\left(\sqrt{1-x^2} y'\right)' = \frac{\lambda}{\sqrt{1-x^2}} y \qquad -1 < x < 1$$

(b) consider boundary conditions y, y' bounded as $x \rightarrow -1$ and $x \rightarrow +1$. Show that the problem is self adjoint. (c) Show that

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = 0$$

Where $T_n(x)$ are the eigenfunctions :$T_0(x) = 1, T_1(x) = x, T_2(x) = 1 - 2x^2, \cdots$ and eigenvalues are $\lambda_n = n^2$ for $n = 0, 1, 2, \cdots$
Solution

Part (a)

Writing the ODE $\left(1-x^2\right) y'' - xy' + \lambda y = 0$ as

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

Where $P(x) = \left(1-x^2\right), Q(x) = -x, R(x) = \lambda$, then the integrating factor is

$$\begin{aligned} \mu &= \frac{1}{P} e^{\int \frac{Q(x)}{P(x)} dx} \\ &= \frac{1}{\left(1-x^2\right)} e^{\int \frac{-x}{\left(1-x^2\right)} dx} \end{aligned}$$

But $\int \frac{x}{\left(1-x^2\right)} dx = \frac{1}{2} \ln\left|1-x^2\right|$, therefore $e^{\frac{1}{2}\ln\left|1-x^2\right|} = \sqrt{1-x^2}$ and the above becomes $\mu = \frac{1}{\sqrt{1-x^2}}$. Hence the SL form is

$$\begin{aligned} (\mu P y')' + \mu R(x) y &= 0 \\ \left(\frac{1}{\sqrt{1-x^2}} \left(1-x^2\right) y'\right)' + \frac{1}{\sqrt{1-x^2}} \lambda y &= 0 \\ -\left(\sqrt{1-x^2} y'\right)' &= \frac{1}{\sqrt{1-x^2}} \lambda y \end{aligned}$$

Part (b)

A problem is self adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

Where u, v are any two arbitrary eigenfunctions of the ODE which therefore by definition satisfy the ODE and the boundary conditions as given. Starting with $\langle L[u], v\rangle$ and it is evaluated to see if it leads to $\langle u, L[v]\rangle$. The operator is defined as (from part (a)) as

$$L[y] = -\left(\sqrt{1-x^2}y'\right)' = \frac{1}{\sqrt{1-x^2}}\lambda y$$

Therefore

$$\langle L[u], v\rangle = \int_{-1}^{1} \overbrace{-\left(\sqrt{1-x^2}u'\right)'}^{dv} \overbrace{v}^{u} dx$$

Integrating by parts gives

$$\begin{aligned}\langle L[u], v\rangle &= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \int_{-1}^{1} -\left(\sqrt{1-x^2}u'\right)v' dx\\ &= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \int_{-1}^{1} \overbrace{-\left(\sqrt{1-x^2}v'\right)}^{u} \overbrace{u'}^{dv} dx\end{aligned}$$

Integrating by parts again gives

$$\begin{aligned}\langle L[u], v\rangle &= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \left(\left[-\left(\sqrt{1-x^2}v'\right)u\right]_{-1}^{1} - \int_{-1}^{1} -\left(\sqrt{1-x^2}v'\right)' udx\right)\\ &= \left[-\sqrt{1-x^2}u'v + \sqrt{1-x^2}v'u\right]_{-1}^{1} + \int_{-1}^{1} -\left(\sqrt{1-x^2}v'\right)' udx\\ &= \left[\sqrt{1-x^2}\left(v'u - u'v\right)\right]_{-1}^{1} + \langle u, L[v]\rangle\end{aligned}$$

Therefore the ODE is self adjoint if the boundary terms vanish. Let $\Delta = \left[\sqrt{1-x^2}\left(v'u-u'v\right)\right]_{-1}^{1}$. Evaluating this gives

$$\Delta = \lim_{x\to 1}\sqrt{1-x^2}\left(v'(x)u(x) - u'(x)v(x)\right) - \lim_{x\to -1}\sqrt{1-x^2}\left(v'(x)u(x) - u'(x)v(x)\right)$$

But since u, u' are bounded as $x \to -1$ and $x \to +1$ and also v, v' are bounded as $x \to -1$ and $x \to +1$, then this shows that $\Delta \to 0$. Therefore

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

Hence the ODE is self adjoint.

Part (c)

Since $T_n(x), T_m(x)$ are two eigenfunctions of $-\left(\sqrt{1-x^2}y'\right)' = \frac{1}{\sqrt{1-x^2}}\lambda y$ then each satisfies the ODE. Hence

$$\left(\sqrt{1-x^2}T_n'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_n T_n = 0 \tag{3A}$$

$$\left(\sqrt{1-x^2}T_m'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_m T_m = 0 \tag{3B}$$

Multiplying 3A by T_m and 3B by T_n and subtracting gives

$$\begin{aligned}T_m\left(\sqrt{1-x^2}T_n'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_n T_m T_n - \left(T_n\left(\sqrt{1-x^2}T_m'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_m T_n T_m\right) &= 0\\ T_m\left(\sqrt{1-x^2}T_n'\right)' - T_n\left(\sqrt{1-x^2}T_m'\right)' + \left(\lambda_n - \lambda_m\right)\frac{1}{\sqrt{1-x^2}}T_m T_n &= 0\end{aligned}$$

Integrating from $-1\cdots 1$ gives

$$\int_{-1}^{1} T_m\left(\sqrt{1-x^2}T_n'\right)' dx - \int_{-1}^{1} T_n\left(\sqrt{1-x^2}T_m'\right)' dx + \left(\lambda_n - \lambda_m\right)\int_{-1}^{1}\frac{T_m T_n}{\sqrt{1-x^2}}dx = 0 \tag{1}$$

Integrating by parts the first integral in (1) above gives

$$\int_{-1}^{1} T_m\left(\sqrt{1-x^2}T_n'\right)' dx = \left[T_m\sqrt{1-x^2}T_n'\right]_{-1}^{1} - \int_{-1}^{1} T_m'\left(\sqrt{1-x^2}T_n'\right) dx \tag{1A}$$

Integrating by parts the second integral in (1) gives

$$\int_{-1}^{1} T_n \left(\sqrt{1-x^2}T_m'\right)' dx = \left[T_n\sqrt{1-x^2}T_m'\right]_{-1}^{1} - \int_{-1}^{1} T_n' \left(\sqrt{1-x^2}T_m'\right) dx \tag{1B}$$

Substituting (1A) and (1B) back into (1) and simplifying gives

$$\begin{aligned}
\left[T_m\sqrt{1-x^2}T_n'\right]_{-1}^{1} - \left[T_n\sqrt{1-x^2}T_m'\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1}\frac{T_mT_n}{\sqrt{1-x^2}}dx &= 0\\
\left[T_m\sqrt{1-x^2}T_n' - T_n\sqrt{1-x^2}T_m'\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1}\frac{T_mT_n}{\sqrt{1-x^2}}dx &= 0\\
\left[\sqrt{1-x^2}\left(T_mT_n' - T_nT_m'\right)\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1}\frac{T_mT_n}{\sqrt{1-x^2}}dx &= 0 \qquad (1C)
\end{aligned}$$

Let $\Delta = \left[\sqrt{1-x^2}\left(T_mT_n' - T_nT_m'\right)\right]_{-1}^{1}$, then

$$\Delta = \lim_{x\to 1}\sqrt{1-x^2}\left(T_m(x)T_n'(x) - T_n(x)T_m'(x)\right) - \lim_{x\to -1}\sqrt{1-x^2}\left(T_m(x)T_n'(x) - T_n(x)T_m'(x)\right)$$

But since $T_n(x), T_m(x), T_n'(x), T_m'(x)$ are all bounded as $x \to -1$ and as $x \to +1$, then $\Delta \to 0$. Therefore (1C) becomes

$$(\lambda_n - \lambda_m)\int_{-1}^{1}\frac{T_mT_n}{\sqrt{1-x^2}}dx = 0$$

But since $\lambda_n \neq \lambda_m$, since $m \neq n$, then

$$\int_{-1}^{1}\frac{T_mT_n}{\sqrt{1-x^2}}dx = 0$$

Which is what we are asked to show.

2.1.64 Chapter 11.5, Problem 2 (With interactive animation)

Find displacement $u(r,t)$ in vibrating circular elastic membrane of radius 1 that satisfies the boundary conditions

$$u(1,t) = 0 \qquad t \geq 0$$

And initial conditions

$$\begin{aligned}
u(r,0) &= 0\\
u_t(r,0) &= g(r)
\end{aligned}$$

For $0 \leq r \leq 1$, where $g(1) = 0$.
<u>Solution</u>
The wave equation is $u_{tt} = a^2\left(u_{xx} + u_{yy}\right)$. In polar coordinates this becomes

$$\frac{1}{a^2}u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Due to circular symmetry, the above simplifies to

$$\frac{1}{a^2}u_{tt} = u_{rr} + \frac{1}{r}u_r$$

Applying separation of variables. Let $u = T(t)R(r)$. Substituting this in the above PDE gives

$$\frac{1}{a^2}T''R = R''T + \frac{1}{r}R'T$$

Dividing by RT results in

$$\frac{1}{a^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

Where λ is the sepration constant. For $\lambda > 0$ (it is known $\lambda = 0$ is not eigenvalue, as well as there are no negative eigenvalues.) The above gives two ODE

$$T'' + \lambda^2 a^2 T = 0$$

And

$$rR''(r) + R'(r) + \lambda^2 rR(r) = 0 \tag{1}$$

With the boundary conditions $R(1) = 0$ and to $R(0)$ is bounded. This comes from physics, since one expects the vibration not to blow up in the center of the membrane. The ODE (1) is now transformed to Bessel ODE using

$$\xi = \lambda r$$

Hence $\frac{dR}{dr}=\frac{dR}{d\xi}\frac{d\xi}{dr}=\lambda\frac{dR}{d\xi}$ and $\frac{d^2R}{dr^2}=\lambda^2\frac{d^2R}{d\xi^2}$. Therefore (1) becomes

$$\frac{\xi}{\lambda}\lambda^2R''(\xi)+\lambda R'(\xi)+\lambda^2\frac{\xi}{\lambda}R(\xi)=0$$

The above simplifies to

$$\xi R''(\xi)+R'(\xi)+\xi R(\xi)=0$$

The above is Bessel ODE of order zero. Its solution is

$$R(\xi)=c_1J_0(\xi)+c_2Y_0(\xi)$$

Converting back to r the above becomes

$$R(r)=c_1J_0(r\lambda)+c_2Y_0(r\lambda)$$

Since $R(r)$ is bounded as $r\to 0$, then $c_2=0$ as $Y_0(r\lambda)$ blows up at $r=0$. Therefore the radial solution becomes

$$R(r)=c_1J_0(r\lambda)$$

At boundary conditions $R(1)=0$ the above becomes

$$0=c_1J_0(\lambda)$$

Non trivial solution requires $J_0(\lambda)=0$. Therefore the eigenvalues are the the positive roots of $J_0(\lambda)=0$. The first few eigenvalues are $\lambda_1=5.78319,\lambda_2=30.4713,\lambda_3=74.887,\cdots$. Hence

$$R_n(r)=c_nJ_0(\lambda_nr)\qquad n=1,2,3,\cdots$$

Now the time ODE is

$$T''+\lambda^2a^2T=0$$

Since $\lambda>0$ then the solution is

$$T_n(t)=A_n\cos(\lambda_nat)+B_n\sin(\lambda_nat)$$

Therefore the fundamental solution is

$$u_n(r,t)=T_n(t)R_n(r)$$

And by superposition, the general solution is

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos(\lambda_nat)+B_n\sin(\lambda_nat)\right)J_0(\lambda_nr)\tag{1A}$$

Where the c_n is merged into A_n,B_n due to the product. At $t=0$ and since $u(r,0)=0$, the above becomes

$$0=\sum_{n=1}^{\infty}A_nJ_0(\lambda_nr)$$

Hence $A_n=0$. The solution simplifies to

$$u(r,t)=\sum_{n=1}^{\infty}B_n\sin(\lambda_nat)J_0(\lambda_nr)$$

Taking time derivative gives

$$u_t(r,t)=\sum_{n=1}^{\infty}B_n\lambda_na\cos(\lambda_nat)J_0(\lambda_nr)$$

At $t=0$, and from initial conditions, the above becomes

$$g(r)=\sum_{n=1}^{\infty}B_n\lambda_naJ_0(\lambda_nr)$$

Applying orthogonality, and since the weight is r, therefore

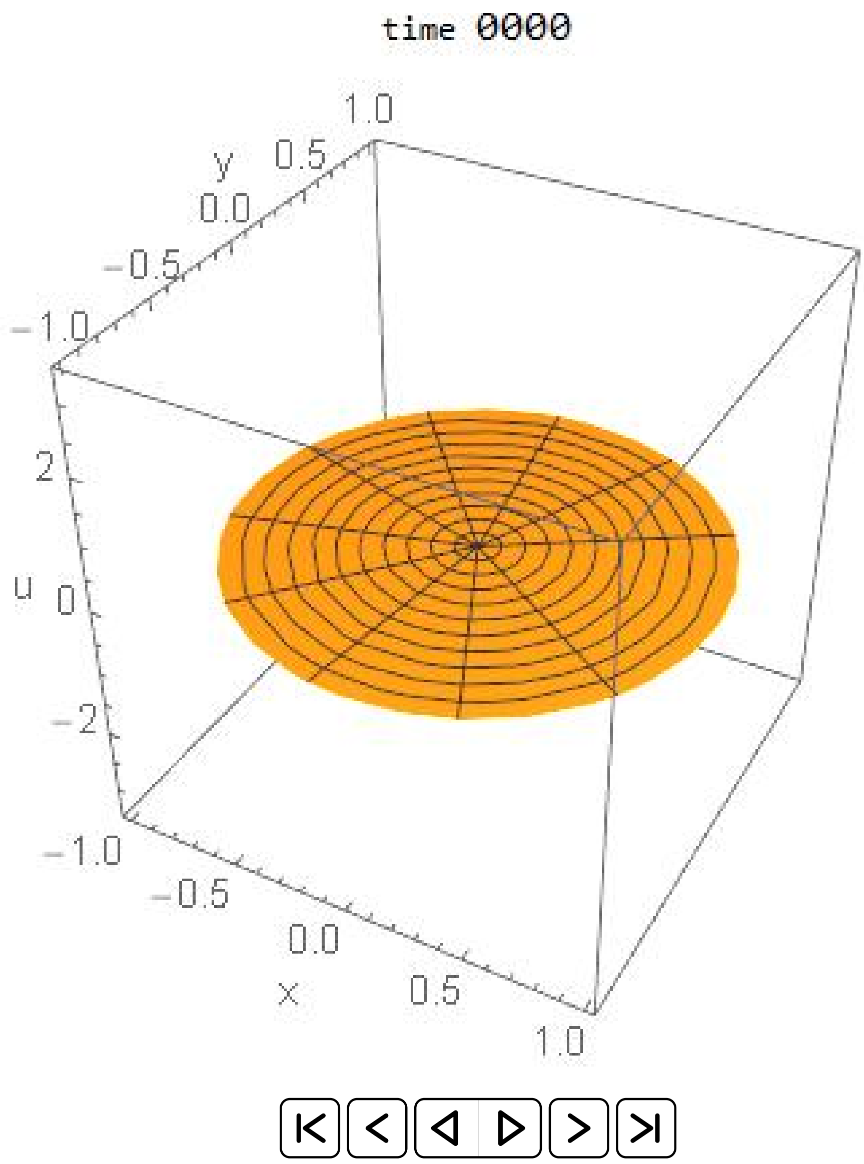
$$\int_0^1 rg(r)J_0(\lambda_nr)\,dr=B_n\lambda_na\int_0^1 rJ_0^2(\lambda_nr)\,dr$$

$$B_n=\frac{1}{\lambda_na}\frac{\int_0^1 rg(r)J_0(\lambda_nr)\,dr}{\int_0^1 rJ_0^2(\lambda_nr)\,dr}\tag{2}$$

Therefore the final solution is

$$u(r,t)=\sum_{n=1}^{\infty}B_n\sin(\lambda_nat)J_0(\lambda_nr)$$

With B_n given by (2).
The following is an animation of the above solution. $a=0.2$ and $g(r)=r$ was used. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



2.1.65 Chapter 11.5, Problem 3 (With interactive animation)

Find displacement $u(r,t)$ in vibrating circular elastic membrane of radius 1 that satisfies the boundary conditions

$$u(1,t) = 0 \qquad t \geq 0$$

And initial conditions

$$u(r,0) = f(r)$$
$$u_t(r,0) = g(r)$$

For $0 \leq r \leq 1$, where $g(1) = 0$.
<u>Solution</u>
The same steps are used to reach the general solution as was done in the above problem. The difference is when initial conditions are used to determine the coefficients.
The general solution from the above problem was found to be

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)) J_0(\lambda_n r) \tag{1A}$$

At $t = 0$

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

Applying orthogonality, and since the weight is r results in

$$\int_0^1 r f(r) J_0(\lambda_n r) \, dr = A_n \int_0^1 r J_0^2(\lambda_n r) \, dr$$
$$A_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r) \, dr}{\int_0^1 r J_0^2(\lambda_n r) \, dr} \tag{2}$$

Taking time derivative of the solution (1A)

$$u_t(r,t) = \sum_{n=1}^{\infty} -A_n \sqrt{\lambda_n} a \sin(\lambda_n a t) + B_n \lambda_n a \cos(\lambda_n a t) J_0(\lambda_n r)$$

At $t = 0$, and from initial conditions, the above becomes

$$g(r) = \sum_{n=1}^{\infty} B_n \lambda_n a J_0(\lambda_n r)$$

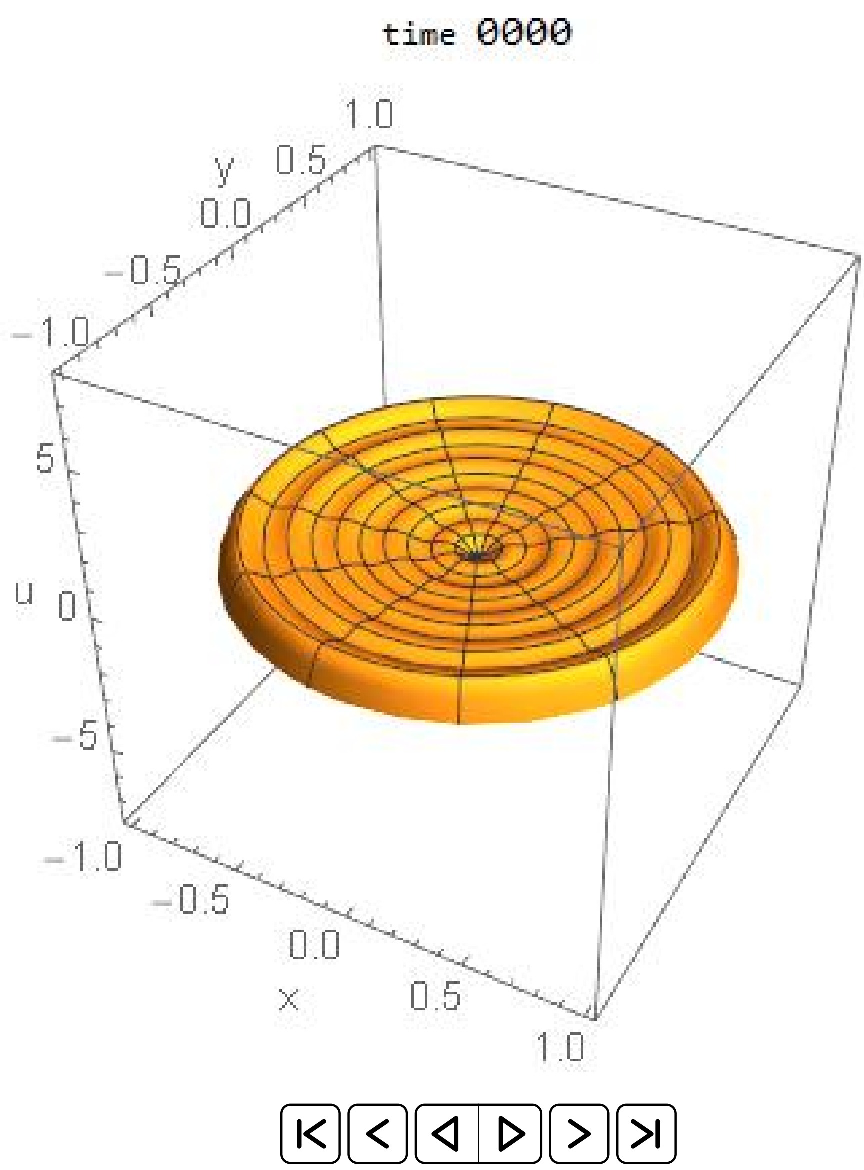
Applying orthogonality, and since the weight is r, therefore

$$\int_0^1 r g(r) J_0(\lambda_n r)\,dr = B_n \lambda_n a \int_0^1 r J_0^2(\lambda_n r)\,dr$$
$$B_n = \frac{1}{\lambda_n a} \frac{\int_0^1 r g(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr} \tag{3}$$

The two coefficients A_n, B_n are now found. Therefore the final solution is

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} a t\right) + B_n \sin\left(\sqrt{\lambda_n} a t\right)\right) J_0\left(\sqrt{\lambda_n} r\right)$$

With A_n given by (2) and B_n given by (3)
The following is an animation of the above solution. $a = 0.2$, $g(r) = r$ and $f(r) = 1 - r$ was used. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



2.1.66 Chapter 11.5, Problem 4

The wave equation in polar coordinates is

$$\frac{1}{a^2} u_{tt} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

Show that if $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ then R, Θ, T satisfy the ODE's

$$r^2 R'' + rR' + \left(\lambda^2 r^2 - n^2\right) R = 0$$
$$\Theta'' + n^2 \Theta = 0$$
$$T'' + \lambda^2 a^2 T = 0$$

Solution

Let $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$. Substituting in the wave PDE gives

$$\frac{1}{a^2}T''R\Theta=R''T\Theta+\frac{1}{r}R'T\Theta+\frac{1}{r^2}\Theta''RT$$

dividing by $R\Theta T$ gives

$$\frac{1}{a^2}\frac{T''}{T}=\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=-\lambda^2$$

Where λ is separation constant. The above now become

$$\begin{aligned}\frac{1}{a^2}\frac{T''}{T}&=-\lambda^2 &(1)\\ \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}&=-\lambda^2\end{aligned}$$

The second ODE above can now be written as

$$\begin{aligned}r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}&=-r^2\lambda^2\\ r^2\frac{R''}{R}+r\frac{R'}{R}+r^2\lambda^2&=-\frac{\Theta''}{\Theta}=n^2\end{aligned}$$

Where n is the new separation constant (I do not like using n for this, but this is what the book did). The above now gives the ODE's

$$\begin{aligned}-\frac{\Theta''}{\Theta}&=n^2 &(2)\\ r^2\frac{R''}{R}+r\frac{R'}{R}+r^2\lambda^2&=n^2 &(3)\end{aligned}$$

Therefore (1,2,3) becomes

$$\begin{aligned}T''+a^2\lambda^2T&=0 &(1A)\\ \Theta''+n^2\Theta&=0 &(2A)\\ r^2R''+rR'+\left(r^2\lambda^2-n^2\right)R&=0 &(3A)\end{aligned}$$

Which is what the problem asked to show.

2.1.67 Chapter 11.5, Problem 5

In the circular cylindrical coordinates r,θ,z defined by

$$\begin{aligned}x&=r\cos\theta\\ y&=r\sin\theta\\ z&=z\end{aligned}$$

Laplace equation is

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0$$

(a) Show that if $u(r,\theta,t)=R(r)\Theta(\theta)Z(z)$ then R,Θ,Z satisfy the ODE's

$$\begin{aligned}r^2R''+rR'+\left(\lambda^2r^2-n^2\right)R&=0\\ \Theta''+n^2\Theta&=0\\ Z''-\lambda^2Z&=0\end{aligned}$$

(b) Show that if $u(r,\theta,z)$ is independent of θ then the first equation in (a) becomes

$$r^2R''+rR'+\lambda^2r^2R=0$$

The second is omitted altogether and the third is unchanged.
Solution

Part (a)

Let $u(r,\theta,z)=R(r)\Theta(\theta)Z(z)$. Substituting in the wave PDE $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0$ gives

$$R''\Theta Z+\frac{1}{r}R'\Theta Z+\frac{1}{r^2}\Theta''RZ+Z''R\Theta=0$$

dividing by $R\Theta Z$ gives

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=-\frac{Z''}{Z}=-\lambda^2$$

Where λ is separation constant. The above now become

$$Z'' - \lambda^2 Z = 0 \tag{1}$$
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

The second ODE above can now be written as

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = -r^2\lambda^2$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = n^2$$

Where n is the new separation constant. The above now gives the ODE's

$$-\frac{\Theta''}{\Theta} = n^2 \tag{2}$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = n^2 \tag{3}$$

Therefore (1,2,3) becomes

$$Z'' - \lambda^2 Z = 0 \tag{1A}$$
$$\Theta'' + n^2\Theta = 0 \tag{2A}$$
$$r^2R'' + rR' + \left(r^2\lambda^2 - n^2\right) R = 0 \tag{3A}$$

Part (b)

When no dependency on θ then the ODE becomes $u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$. Let $u(r,z) = R(r)Z(z)$. Substituting into the wave PDE

$$R''Z + \frac{1}{r}R'Z + Z''R = 0$$

dividing by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -\lambda^2$$

The above gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$
$$-\frac{Z''}{Z} = -\lambda^2$$

Or

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0$$
$$Z'' - \lambda^2 Z = 0$$

2.1.68 Chapter 11.5, Problem 6

Find steady state solution in semi-infinite rod $0 < z < \infty, 0 \leq r \leq 1$ if the temperature is independent of θ and approaches zero as $z \to \infty$. Assume $u(r,z)$ satisfies boundary conditions

$$u(1,z) = 0 \qquad z > 0$$
$$u(r,0) = f(r) \qquad 0 \leq r \leq 1$$

Solution
The PDE is

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$$

By separation of variables, as was done in problem 5 above, this gives

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0 \tag{1}$$
$$R(1) = 0$$
$$\lim_{r\to 0} R(0) \to \text{bounded}$$

And

$$Z'' - \lambda^2 Z = 0 \tag{2}$$
$$Z(0) = f(r)$$
$$\lim_{z\to\infty} Z(r) \to 0$$

The solution to (2) is known to be

$$R(r) = c_n J_0(\lambda_n r)$$

Where λ_n are the positive roots of $J_0(\lambda_n) = 0$. The solution to (2) is

$$Z(z) = A_n e^{\lambda_n z} + B_n e^{-\lambda_n z}$$

Since u goes to zero as $z \longrightarrow \infty$, then this implies $A_n = 0$. Hence

$$Z(z) = B_n e^{-\lambda_n z}$$

Hence the overall solution becomes

$$u(r,z) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n z} J_0(\lambda_n r)$$

Where c_n is combined with B_n. To find B_n, using the final boundary condition $u(r,0) = f(r)$ gives

$$f(r) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r)$$

Applying orthogonality and using the weight of r gives

$$\begin{aligned} \int_0^1 r f(r) J_0(\lambda_n r)\, dr &= B_n \int_0^1 r J_0^2(\lambda_n r)\, dr \\ B_n &= \frac{\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr} \end{aligned}$$

Hence the solution is now complete. It is given by

$$u(r,z) = \sum_{n=1}^{\infty} \frac{\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr} e^{-\lambda_n z} J_0(\lambda_n r)$$

2.1.69 Chapter 11.5, Problem 7

7. The equation

$$v_{xx} + v_{yy} + k^2 v = 0$$

is a generalization of Laplace's equation and is sometimes called the Helmholtz[12] equation.

(a) In polar coordinates the Helmholtz equation is

$$v_{rr} + (1/r)v_r + (1/r^2)v_{\theta\theta} + k^2 v = 0.$$

If $v(r,\theta) = R(r)\Theta(\theta)$, show that R and Θ satisfy the ordinary differential equations

$$r^2 R'' + rR' + (k^2 r^2 - \lambda^2)R = 0, \qquad \Theta'' + \lambda^2 \Theta = 0.$$

(b) Consider the Helmholtz equation in the disk $r < c$. Find the solution that remains bounded at all points in the disk, that is periodic in θ with period 2π, and that satisfies the boundary condition $v(c,\theta) = f(\theta)$, where f is a given function on $0 \le \theta < 2\pi$.
Hint: The equation for R is a Bessel equation. See Problem 3 of Section 11.4.

Solution

Part (a)

Substituting $v(r,\theta) = R(r)\Theta(\theta)$ into the PDE gives

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' R + k^2 R\Theta = 0$$

Dividing by $R\Theta$ gives

$$\begin{aligned} \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2 &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} + r^2 k^2 &= -\frac{\Theta''}{\Theta} = \lambda^2 \end{aligned}$$

Where λ is the separation constant. This gives

$$r^2\frac{R''}{R}+r\frac{R'}{R}+r^2k^2-\lambda^2=0$$

And

$$-\frac{\Theta''}{\Theta}=\lambda^2$$

Hence

$$\begin{aligned} r^2R''+rR'+R\left(r^2k^2-n^2\right)&=0\\ \Theta''+\lambda^2\Theta&=0\end{aligned}$$

Part (b)

Starting with $\Theta''+\lambda^2\Theta=0$. The eigenvalue λ can not be negative. The following two cases are considered.
Case $\lambda=0$
Solution is

$$\Theta(\theta)=c_1\theta+c_2$$

The boundary conditions are periodic with period 2π, meaning

$$\begin{aligned}\Theta(0)&=\Theta(2\pi)\\ \Theta'(0)&=\Theta'(2\pi)\end{aligned}$$

Applying first BC gives

$$c_2=c_12\pi+c_2 \tag{1}$$

Applying second BC gives

$$c_1=c_1 \tag{2}$$

So c_1 can be any value. But to solve (1) c_1 must be zero. Hence first BC now gives

$$c_2=c_2$$

Which means c_2 can be any value, say 1. Therefore $\lambda=0$ is an eigenvalue with eigenfunction $\Phi_0(\theta)=1$
Case $\lambda>0$
The solution now is

$$\Theta(\theta)=A\cos(\lambda\theta)+B\sin(\lambda\theta)$$

The boundary conditions are periodic with period 2π, meaning

$$\begin{aligned}\Theta(0)&=\Theta(2\pi)\\ \Theta'(0)&=\Theta'(2\pi)\end{aligned}$$

Applying the above boundary conditions gives

$$\begin{aligned}A&=A\cos(\lambda2\pi)+B\sin(\lambda2\pi)\\ B\lambda&=A\lambda\sin(\lambda2\pi)+B\lambda\cos(\lambda2\pi)\end{aligned}$$

This means λ must be an integer $n=1,2,\cdots$ for the above relations be satisfied. Since only when n is an integer, the above gives $A=A$ and $B\lambda=B\lambda$. Hence the eigenfunction in this case is

$$\Phi_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)\qquad n=1,2,\cdots$$

Now that the eigenvalues are found, the solution to the R ODE is found. Summary of the above result: The eigenvalues are $n=0$ with eigenfunction $\Phi_0(\theta)=1$ and $n=1,2,3,\cdots$ with eigenfunction $\Phi_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)$.
Case $\lambda=n=0$
In this case, the R ODE above $r^2R''+rR'+R\left(r^2k^2-\lambda^2\right)=0$ reduces to

$$r^2R''+rR'+Rr^2k^2=0$$

let

$$t=rk$$

Therefore $R'(r)=R'(t)k$ and $R''(r)=R''(t)k^2$. Substituting these in the above ODE gives

$$\begin{aligned}\frac{t^2}{k^2}k^2R''(t)+\frac{t}{k}kR'(r)+R\frac{t^2}{k^2}k^2&=0\\ t^2R''(t)+tR'(t)+t^2R(t)&=0\end{aligned}$$

This is now Bessel ODE of order zero. Its solution is

$$R_0(t) = A_0 J_0(t) + B_0 Y_0(t)$$

Converting back to r, the above becomes

$$R_0(r) = A_0 J_0(rk) + B_0 Y_0(rk)$$

Since R is bounded at $r = 0$, this implies $B_0 = 0$, since $Y_0(rk)$ blows up at $r = 0$. Hence

$$R_0(r) = A_0 J_0(rk)$$

This is the solution for eigenvalue $n = 0$.
Case $\lambda = n > 0$
The Bessel PDE now has the form $r^2 R''(r) + rR'(r) + (r^2k^2 - n^2) R(r) = 0$. To convert the ODE to standard Bessel form let

$$t = rk$$

Therefore $R'(r) = R'(t)k$ and $R''(r) = R''(t)k^2$. Substituting these in the above ODE gives

$$\begin{aligned}\frac{t^2}{k^2}k^2 R''(t) + \frac{t}{k}kR'(r) + R\left(\frac{t^2}{k^2}k^2 - n^2\right) &= 0\\ t^2R''(t) + tR'(t) + R(t)(t^2 - n^2) &= 0\end{aligned}$$

This is now Bessel ODE of order n. Its solution is

$$R_n(t) = A_n J_n(t) + B_n Y_n(t)$$

Converting back to r, the above becomes

$$R_n(r) = A_n J_n(rk) + B_n Y_n(rk)$$

Since R is bounded at $r = 0$, this implies $B_n = 0$, since $Y_n(rk)$ blows up at $r = 0$. Hence $R(r) = A_n J_n(rk)$.
This is the solution for eigenvalue $n > 0$.
Hence the fundamental solution is

$$\begin{aligned}v_0(r,\theta) &= \Phi_0(\theta) R_0(r)\\ &= A_0 J_0(rk)\end{aligned}$$

Since $\Phi_0(\theta) = 1$ and

$$\begin{aligned}v_n(r,\theta) &= \Phi_n(\theta) R_n(r)\\ &= (A_n\cos(n\theta) + B_n\sin(n\theta)) J_n(rk)\end{aligned}$$

Where the constants are combined. Therefore the general solution becomes

$$v(r,\theta) = A_0 J_0(rk) + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta)) J_n(rk) \tag{3}$$

Constants A_0, A_n, B_n are found from boundary conditions. At $r = c$, $u(c,\theta) = f(\theta)$ and the above becomes

$$f(\theta) = A_0 J_0(ck) + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta)) J_n(ck)$$

For $n = 0$ only and applying orthogonality

$$\begin{aligned}\int_0^{2\pi} f(\theta)\,d\theta &= \int_0^{2\pi} A_0 J_0(ck)\,d\theta\\ \int_0^{2\pi} f(\theta)\,d\theta &= A_0 J_0(ck)\int_0^{2\pi} d\theta\\ &= 2\pi A_0 J_0(ck)\end{aligned}$$

Hence

$$A_0 = \frac{\int_0^{2\pi} f(\theta)\,d\theta}{2\pi J_0(ck)}$$

And for $n > 0$

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= \sum_{n=1}^{\infty}\int_0^{2\pi}(A_n\cos(n\theta) + B_n\sin(n\theta))\sin(m\theta) J_n(ck)\,d\theta\\ &= \sum_{n=1}^{\infty} J_n(ck) A_n\int_0^{2\pi}\cos(n\theta)\sin(m\theta)\,d\theta + B_n\sum_{n=1}^{\infty} J_n(ck)\int_0^{2\pi}\sin(n\theta)\sin(m\theta)\,d\theta\end{aligned}$$

But $\int_0^{2\pi}\cos(n\theta)\sin(m\theta)\,d\theta = 0$ for all n, m and the above now is solved for B_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= B_n\sum_{n=1}^{\infty} J_n(ck)\int_0^{2\pi}\sin(n\theta)\sin(m\theta)\,d\theta\\ &= B_m J_m(ck)\int_0^{2\pi}\sin^2(m\theta)\,d\theta\\ &= B_m J_m(ck)\,\pi\end{aligned}$$

Hence

$$B_n = \frac{\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}{\pi J_n(ck)}$$

Similarly, to find A_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta &= \sum_{n=1}^{\infty}\int_0^{2\pi}(A_n\cos(n\theta)+B_n\sin(n\theta))\cos(m\theta)J_n(ck)\,d\theta\\ &= \sum_{n=1}^{\infty} J_n(ck)A_n\int_0^{2\pi}\cos(n\theta)\cos(m\theta)\,d\theta + B_n\sum_{n=1}^{\infty}J_n(ck)\int_0^{2\pi}\sin(n\theta)\cos(m\theta)\,d\theta\end{aligned}$$

But $\int_0^{2\pi}\sin(n\theta)\cos(m\theta)\,d\theta = 0$ for all n, m and the above now is solved for A_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= A_n\sum_{n=1}^{\infty} J_n(ck)\int_0^{2\pi}\cos(n\theta)\cos(m\theta)\,d\theta\\ &= A_m J_m(ck)\int_0^{2\pi}\cos^2(m\theta)\,d\theta\\ &= A_m J_m(ck)\,\pi\end{aligned}$$

Hence

$$A_n = \frac{\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}{\pi J_n(ck)}$$

The complete solution from (3) becomes

$$\begin{aligned}v(r,\theta) &= A_0J_0(rk) + \sum_{n=1}^{\infty}(A_n\cos(n\theta)+B_n\sin(n\theta))J_n(rk)\\ A_0 &= \frac{\int_0^{2\pi} f(\theta)\,d\theta}{2\pi J_0(ck)}\\ B_n &= \frac{\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}{\pi J_n(ck)}\\ A_n &= \frac{\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}{\pi J_n(ck)}\end{aligned}$$

2.1.70 Chapter 11.5, Problem 8

8. Consider the flow of heat in a cylinder $0 \le r < 1, 0 \le \theta < 2\pi, -\infty < z < \infty$ of radius 1 and of infinite length. Let the surface of the cylinder be held at temperature zero, and let the initial temperature distribution be a function of the radial variable r only. Then the temperature u is a function of r and t only and satisfies the heat conduction equation

$$\alpha^2[u_{rr} + (1/r)u_r] = u_t, \qquad 0 < r < 1, \quad t > 0,$$

and the following initial and boundary conditions:

$$u(r,0) = f(r), \qquad 0 \le r \le 1,$$
$$u(1,t) = 0, \qquad t > 0.$$

Show that

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)e^{-\alpha^2\lambda_n^2 t},$$

where $J_0(\lambda_n) = 0$. Find a formula for c_n.

Solution
Let $u(r,t) = R(r)T(t)$. Substituting into the PDE gives

$$\frac{1}{\alpha^2}T'R = R''T + \frac{1}{r}R'T$$

Dividing by RT gives

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

Where λ is the separation constant. This gives the ODE

$$\begin{aligned} R'' + \frac{1}{r}R + \lambda^2 R &= 0 \\ rR'' + R + \lambda^2 rR &= 0 \\ (rR')' + \lambda^2 rR &= 0 \end{aligned} \tag{1}$$

With BC

$$\begin{aligned} R(1) &= 0 \\ \lim_{r\to 0} R(r) &\to \text{bounded} \end{aligned}$$

And

$$T' + \alpha^2\lambda^2 R = 0 \tag{2}$$

ODE (1) is Sturm Liouville ODE where $p = r, q = 0$ and the weight is r. The eigenvalue can not be negative. Two cases to consider.
Case $\lambda = 0$
The ODE becomes $(rR')' = 0$ which has solution $rR' = c_1$ or $r\frac{dR}{dr} = c_1$ or $dR = \frac{c_1}{r}dr$. Integrating gives

$$R(r) = c_1 \ln r + c_2$$

Since R is bounded at $r = 0$, then $c_1 = 0$. The solution becomes $R(r) = c_2$. Since $R(1) = 0$ then $c_2 = 0$. Hence trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.
Case $\lambda > 0$
The ODE now becomes $rR''(r) + R(r) + \lambda^2 rR(r) = 0$. Let $t = \lambda r$. Hence $R'(r) = \lambda R'(t)$ and $R''(r) = \lambda^2 R''(t)$ and the ODE becomes

$$\begin{aligned} \frac{t}{\lambda}\lambda^2 R''(t) + \lambda R'(t) + \lambda^2\frac{t}{\lambda}R(t) &= 0 \\ t\lambda R''(t) + \lambda R'(t) + \lambda t R(t) &= 0 \\ tR''(t) + R'(t) + tR(t) &= 0 \end{aligned}$$

This is Bessel ODE of order zero. Its solution is

$$R(t) = c_1 J_o(t) + c_2 Y_o(t)$$

Converting back to r

$$R(r) = c_1 J_o(\lambda r) + c_2 Y_o(\lambda r)$$

Since R is bounded at $r = 0$ then $c_2 = 0$ and the solution becomes

$$R(r) = c_1 J_o(\lambda r)$$

Since $R(1) = 0$ then

$$0 = c_1 J_o(\lambda)$$

For nontrivial solution, $J_o(\lambda) = 0$. This gives the eigenvalues as the positive roots of $J_o(\lambda) = 0$. Hence the solution is

$$R_n(r) = c_n J_o(\lambda_n r)$$

Where λ_n are roots of $J_o(\lambda) = 0$ for $n = 1, 2, 3, \cdots$. The Time ODE (2) has solution

$$T_n(t) = A_n e^{-\lambda_n^2 \alpha^2 t}$$

Hence the final solution is

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2\alpha^2 t} J_0(\lambda_n r)$$

Where constants A_n, c_n are combined into c_n. c_n is now found from initial conditions. At $t = 0$ the above becomes

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

The weight is r, since the R ODE in S.L. form is $(rR')' + \lambda^2 rR = 0$. Therefore, applying orthogonality gives

$$\int_0^1 rf(r) J_0(\lambda_n r)\,dr = c_n \int_0^1 rJ_0^2(\lambda_n r)\,dr$$
$$c_n = \frac{\int_0^1 rf(r) J_0(\lambda_n r)\,dr}{\int_0^1 rJ_0^2(\lambda_n r)\,dr}$$

This completes the solution.

$$u(r,t) = \sum_{n=1}^{\infty} \frac{\int_0^1 rf(r) J_0(\lambda_n r)\,dr}{\int_0^1 rJ_0^2(\lambda_n r)\,dr} e^{-\lambda_n^2 \alpha^2 t} J_0(\lambda_n r)$$

2.1.71 Chapter 11.5, Problem 9

9. In the spherical coordinates ρ, θ, ϕ ($\rho > 0, 0 \le \theta < 2\pi, 0 \le \phi \le \pi$) defined by the equations

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi,$$

Laplace's equation is

$$\rho^2 u_{\rho\rho} + 2\rho u_\rho + (\csc^2\phi) u_{\theta\theta} + u_{\phi\phi} + (\cot\phi) u_\phi = 0.$$

(a) Show that if $u(\rho,\theta,\phi) = \mathrm{P}(\rho)\Theta(\theta)\Phi(\phi)$, then P, Θ, and Φ satisfy ordinary differential equations of the form

$$\rho^2 \mathrm{P}'' + 2\rho\mathrm{P}' - \mu^2\mathrm{P} = 0,$$
$$\Theta'' + \lambda^2\Theta = 0,$$
$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi - \lambda^2)\Phi = 0.$$

The first of these equations is of the Euler type, while the third is related to Legendre's equation.

(b) Show that if $u(\rho,\theta,\phi)$ is independent of θ, then the first equation in part (a) is unchanged, the second is omitted, and the third becomes

$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi)\Phi = 0.$$

(c) Show that if a new independent variable is defined by $s = \cos\phi$, then the equation for Φ in part (b) becomes

$$(1 - s^2)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} + \mu^2\Phi = 0, \qquad -1 \le s \le 1.$$

Note that this is Legendre's equation.

Solution

Part (a)

Let

$$u(\rho,\theta,\phi) = P(\rho)\Theta(\theta)\Phi(\phi)$$

Substituting the above in the Laplace PDE given results in

$$\rho^2 P''\Theta\Phi + 2\rho P'\Theta\Phi + \left(\csc^2\phi\right)\Theta'' P\Phi + \Phi'' P\Theta + \cot(\phi)\Phi' P\Theta = 0$$

Dividing by $P\Theta\Phi$ gives

$$\rho^2\frac{P''}{P} + 2\rho\frac{P'}{P} + \left(\csc^2\phi\right)\frac{\Theta''}{\Theta} + \frac{\Phi''}{\Phi} + \cot(\phi)\frac{\Phi'}{\Phi} = 0$$
$$\rho^2\frac{P''}{P} + 2\rho\frac{P'}{P} = -\left(\csc^2\phi\right)\frac{\Theta''}{\Theta} - \frac{\Phi''}{\Phi} - \cot(\phi)\frac{\Phi'}{\Phi} = \mu^2$$

Where μ is the first separation constant. The above gives

$$\rho^2\frac{P''}{P}+2\rho\frac{P'}{P}=\mu^2$$
$$-\left(\csc^2\phi\right)\frac{\Theta''}{\Theta}-\frac{\Phi''}{\Phi}-\cot(\phi)\frac{\Phi'}{\Phi}-\mu^2=0$$

The first ODE above becomes

$$\rho^2P''+2\rho P'-P\mu^2=0$$

And the second equation is now separated again into two additional ODE's as follows

$$-\frac{\Theta''}{\Theta}-\frac{1}{\csc^2\phi}\frac{\Phi''}{\Phi}-\frac{\cot\phi}{\csc^2\phi}\frac{\Phi'}{\Phi}-\frac{\mu^2}{\csc^2\phi}=0$$
$$\frac{1}{\csc^2\phi}\frac{\Phi''}{\Phi}+\frac{\cot(\phi)}{\csc^2\phi}\frac{\Phi'}{\Phi}+\frac{\mu^2}{\csc^2\phi}=-\frac{\Theta''}{\Theta}=\lambda^2$$

Where λ is the second separation constant. The above gives the following two ODE's

$$\Theta''+\lambda^2\Theta=0$$

And, since $\csc^2\phi=\frac{1}{\sin^2\phi}$ and $\cot(\phi)=\frac{1}{\tan\phi}$, the third ODE is

$$\sin^2\phi\frac{\Phi''}{\Phi}+\frac{\sin^2\phi}{\tan\phi}\frac{\Phi'}{\Phi}+\mu^2\sin^2\phi=\lambda^2$$
$$\sin^2\phi\frac{\Phi''}{\Phi}+\sin\phi\cos\phi\frac{\Phi'}{\Phi}+\mu^2\sin^2\phi=\lambda^2$$
$$\left(\sin^2\phi\right)\Phi''+\left(\sin\phi\cos\phi\right)\Phi'+\left(\mu^2\sin^2\phi-\lambda^2\right)\Phi=0$$

Part (b)

If u is independent of θ then the PDE simplifies to

$$\rho^2u_{\rho\rho}+2\rho u_\rho+u_{\phi\phi}+\cot\phi u_\phi=0 \tag{1}$$

Let

$$u(\rho,\phi)=P(\rho)\Phi(\phi)$$

Substituting the above in the Laplace PDE (1) results in

$$\rho^2P''\Phi+2\rho P'\Phi+\Phi''P+\cot(\phi)\Phi'P=0$$

Dividing by $P\Phi$ gives

$$\rho^2\frac{P''}{P}+2\rho\frac{P'}{P}+\frac{\Phi''}{\Phi}+\cot(\phi)\frac{\Phi'}{\Phi}=0$$
$$\rho^2\frac{P''}{P}+2\rho\frac{P'}{P}=-\frac{\Phi''}{\Phi}-\cot(\phi)\frac{\Phi'}{\Phi}=\mu^2$$

Where μ is the first separation constant. The above gives

$$\rho^2\frac{P''}{P}+2\rho\frac{P'}{P}=\mu^2$$
$$-\frac{\Phi''}{\Phi}-\cot(\phi)\frac{\Phi'}{\Phi}-\mu^2=0$$

The first ODE above becomes

$$\rho^2P''+2\rho P'-\mu^2P=0$$

And the second ODE becomes

$$-\Phi''-\cot(\phi)\Phi'-\mu^2\Phi=0$$
$$\Phi''+\frac{1}{\tan\phi}\Phi'+\mu^2\Phi=0$$
$$\Phi''+\frac{\cos\phi}{\sin\phi}\Phi'+\mu^2\Phi=0$$
$$\left(\sin\phi\right)\Phi''+\left(\cos\phi\right)\Phi'+\left(\mu^2\sin\phi\right)\Phi=0$$

Multiplying again by $\sin\phi$ to get it to the form needed gives

$$\sin^2\phi\Phi''+\left(\sin\phi\cos\phi\right)\Phi'+\left(\mu^2\sin^2\phi\right)\Phi=0 \tag{2}$$

Therefore the first PDE in $P(\rho)$, the second ODE in $\Theta(\theta)$ is now eliminated, and the third ODE changes to the above.

Part (c)

The equation for Φ found in part (b) is

$$\sin^2\phi\frac{d^2\Phi}{d\phi^2}+(\sin\phi\cos\phi)\frac{d\Phi}{d\phi}+\left(\mu^2\sin^2\phi\right)\Phi=0 \tag{1}$$

Let $s=\cos\phi$, then

$$\begin{aligned}\frac{d\Phi}{d\phi}&=\frac{d\Phi}{ds}\frac{ds}{d\phi}\\&=\frac{d\Phi}{ds}(-\sin\phi)\end{aligned} \tag{2}$$

And

$$\begin{aligned}\frac{d^2\Phi}{d\phi^2}&=\frac{d}{d\phi}\left(\frac{d\Phi}{d\phi}\right)\\&=\frac{d}{d\phi}\left(\frac{d\Phi}{ds}(-\sin\phi)\right)\\&=\frac{d^2\Phi}{ds^2}\left(\sin^2\phi\right)-\frac{d\Phi}{ds}(\cos\phi)\end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives

$$\sin^2\phi\left(\frac{d^2\Phi}{ds^2}\left(\sin^2\phi\right)-\frac{d\Phi}{ds}(\cos\phi)\right)+(\sin\phi\cos\phi)\left(\frac{d\Phi}{ds}(-\sin\phi)\right)+\left(\mu^2\sin^2\phi\right)\Phi=0$$

Dividing by $\sin^2\phi$ gives

$$\begin{aligned}\frac{d^2\Phi}{ds^2}\sin^2\phi-\frac{d\Phi}{ds}\cos\phi-\cos\phi\frac{d\Phi}{ds}+\mu^2\Phi&=0\\\frac{d^2\Phi}{ds^2}\sin^2\phi-2\frac{d\Phi}{ds}\cos\phi+\mu^2\Phi&=0\end{aligned}$$

But $\cos\phi=s$ and $\sin^2\phi=1-\cos^2\phi=1-s^2$, therefore the above reduces to

$$\left(1-s^2\right)\frac{d^2\Phi}{ds^2}-2s\frac{d\Phi}{ds}+\mu^2\Phi=0$$

Which is Legendre's equation.

2.1.72 Chapter 11.5, Problem 10

10. Find the steady state temperature $u(\rho,\phi)$ in a sphere of unit radius if the temperature is independent of θ and satisfies the boundary condition

$$u(1,\phi)=f(\phi),\qquad 0\le\phi\le\pi.$$

Hint: Refer to Problem 9 and to Problems 22 through 29 of Section 5.3. Use the fact that the only solutions of Legendre's equation that are finite at both ±1 are the Legendre polynomials.

Solution
TO DO

Chapter 3: Quizzes

3.1 Quizz 1

3.1.1 Problem 1

Problem Solve the boundary value problem

$$y''(x) - y(x) = x \tag{1}$$

with $y(0) = 1, y(1) = 1$
solution
The general solution is the sum of the homogeneous and the particular solution

$$y = y_h + y_p \tag{2}$$

Where $y_h(x)$ is the homogeneous solution of $y_h'' - y_h = 0$. Since this is a constant coefficients ODE, the characteristic equation is found by assuming $y_h = e^{rx}$ and substituting this into $y''(x) - y(x) = 0$ and finding the roots. This results in

$$\begin{aligned} r^2 - 1 &= 0 \\ r &= \pm 1 \end{aligned}$$

Therefore the two linearly independent basis solutions are $y_1 = e^x$ and $y_2 = e^{-x}$. The homogeneous solution is a linear combination of these two basis solutions. In other words

$$y_h(x) = c_1 e^x + c_2 e^{-x}$$

Before proceeding to find the general solution, a check is made now to determine if a unique solution exists or not. The Wronskian $W(x)$ is

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1(1) & y_2(2) \end{vmatrix} = \begin{vmatrix} e^0 & e^{-0} \\ e^1 & e^{-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^1 & e^{-1} \end{vmatrix} = e^{-1} - e \neq 0$$

Since $W(x) \neq 0$, then a unique solution exists.
The particular solution is now found using the method of undetermined coefficients. Since the RHS is polynomial, let the particular solution guess be the following polynomial

$$y_p = A + Bx + Cx^2$$

Therefore $y_p' = B + 2Cx$ and $y_p'' = 2C$. Substituting these into the original ODE (1) gives

$$\begin{aligned} 2C - \left(A + Bx + Cx^2\right) &= x \\ x^2(-C) + x(-B) + (2C - A) &= x \end{aligned}$$

Comparing coefficients of both sides results in

$$\begin{aligned} -C &= 0 \\ -B &= 1 \\ 2C - A &= 0 \end{aligned}$$

Solving for the coefficients gives

$$\begin{aligned} C &= 0 \\ B &= -1 \\ A &= 0 \end{aligned}$$

Therefore the particular solution is now found as

$$\begin{aligned} y_p &= A + Bx + Cx^2 \\ &= -x \end{aligned}$$

The full solution from (2) becomes

$$y = \overbrace{c_1 e^x + c_2 e^{-x}}^{y_h} - x \tag{3}$$

Boundary conditions are now used to determine c_1 and c_2. At $x = 0$ the above becomes

$$1 = c_1 + c_2 \tag{4}$$

And at $x = 1$ (3) gives

$$\begin{aligned} 1 &= c_1 e + c_2 e^{-1} - 1 \\ c_1 e + c_2 e^{-1} &= 2 \end{aligned} \tag{5}$$

Equations (4,5) are now solved for c_1, c_2. From (4), $c_1 = 1 - c_2$. Substituting this into (5) gives

$$\begin{aligned} (1 - c_2)\, e + c_2 e^{-1} &= 2 \\ c_2 \left(-e + e^{-1}\right) + e &= 2 \\ c_2 &= \frac{2 - e}{e^{-1} - e} \end{aligned}$$

Therefore

$$c_1 = 1 - \frac{2 - e}{e^{-1} - e}$$

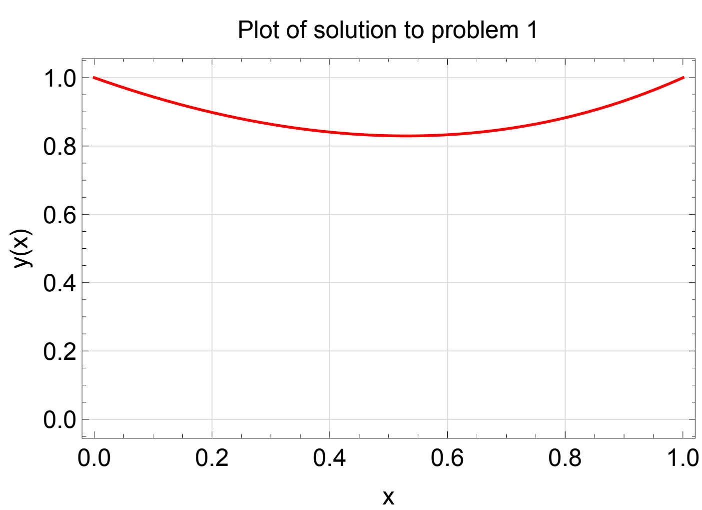
Hence the general solution (3) becomes

$$\begin{aligned} y(x) &= \left(1 - \frac{2 - e}{e^{-1} - e}\right) e^{x} + \left(\frac{2 - e}{e^{-1} - e}\right) e^{-x} - x \\ &= \frac{\left(e^{-1} - e - 2 + e\right)}{e^{-1} - e} e^{x} + \frac{2 - e}{e^{-1} - e} e^{-x} - x \end{aligned}$$

Or

$$y(x) = \frac{\left(e^{-1} - 2\right)}{e^{-1} - e} e^{x} + \frac{(2 - e)\, e^{-x}}{e^{-1} - e} - x$$

This is a plot of the above solution



3.1.2 Problem 2

Problem Find the Fourier sine series for $f(x) = x(1-x), 0 \leq x \leq 1$. Use the result to evaluate the infinite series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \cdots$

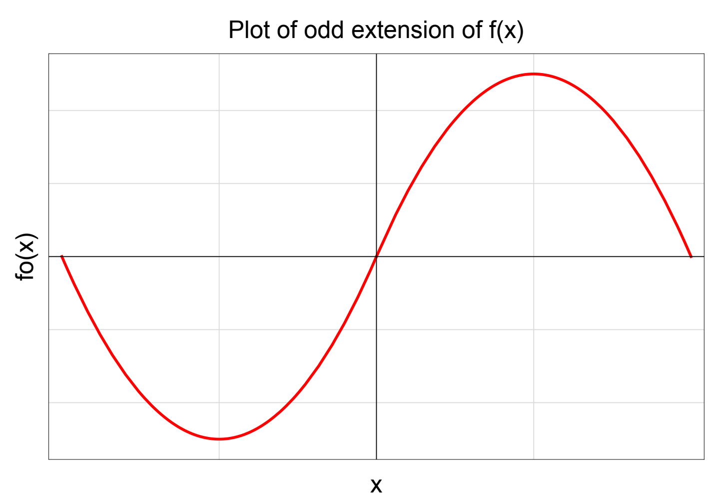
solution

This is a plot of the function $f(x) = x(1-x), 0 \leq x \leq 1$

In the above

$$L = 1$$

To obtain the Fourier sine series, the function is first odd extended to $-1 \leq x < 0$ and after the extension is made, it is repeated using a period $2L$ so that it becomes a periodic function. Here is a plot of the periodic function, called $f_o(x)$ now. One period is shown in this plot for illustration.



Since $f_o(x)$ is an odd function, its Fourier series will contain b_n terms only. The b_n terms are given by the standard formula

$$b_n = \frac{1}{L} \int_{-L}^{L} f_o(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

But $f_o(x)$ is odd function and sine is also odd, therefore the product is an even function, and the above simplifies to

$$b_n = \frac{2}{L} \int_{0}^{L} f_o(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

But over $0 < x < 1$, the function $f_o(x)$ is the same as the original function $f(x)$ which is the non-periodic function given. Therefore the above can be written as

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Since $L = 1$ in this problem, the above simplifies to

$$\boxed{b_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx}$$

And since $f(x) = x(1-x)$, and the above becomes

$$\begin{aligned} b_n &= 2\int_0^1 \left(x - x^2\right) \sin(n\pi x)\, dx \\ &= 2\left(\int_0^1 x\sin(n\pi x)\, dx - \int_0^1 x^2 \sin(n\pi x)\, dx\right) \\ &= 2(I_1 - I_2) \end{aligned} \tag{1}$$

These two integrals are solved using integration by parts. Considering $I_1 = \int_0^1 x\sin(n\pi x)\, dx$ and using $\int u dv = uv - \int v du$. Let $u = x, dv = \sin(n\pi x)$, then $du = 1$ and $v = -\left(\frac{1}{n\pi}\right)\cos(n\pi x)$. Hence

$$\begin{aligned} I_1 &= uv - \int v du \\ &= \left(-x\left(\frac{1}{n\pi}\right)\cos(n\pi x)\right)_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\, dx \\ &= \left(\frac{-1}{n\pi}\cos(n\pi)\right) + \frac{1}{(n\pi)^2}\left(\sin(n\pi x)\right)_0^1 \\ &= \left(\frac{-1}{n\pi}(-1)^n\right) + \frac{1}{(n\pi)^2}\left(\sin(n\pi) - 0\right) \\ &= \left(\frac{-1}{n\pi}(-1)^n\right) \\ &= \frac{(-1)^{n+1}}{n\pi} \end{aligned}$$

For the second integral, let $I_2 = \int_0^1 x^2 \sin(n\pi x)\, dx$ and $u = x^2, dv = \sin(n\pi x)$, therefore $du = 2x, v = -\frac{1}{n\pi}\cos(n\pi x)$. Hence

$$\begin{aligned} I_2 &= uv - \int v du \\ &= \left(-x^2 \frac{1}{n\pi}\cos(n\pi x)\right)_0^1 + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)\, dx \\ &= \left(-\frac{1}{n\pi}(-1)^n\right) + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)\, dx \end{aligned}$$

The above integral in the RHS is also found by integration by parts. Let $u = x, dv = \cos(n\pi x)$ or $du = 1, v = \frac{1}{n\pi}\sin(n\pi x)$. The above becomes

$$\begin{aligned} I_2 &= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left[\left(x\frac{1}{n\pi}\sin(n\pi x)\right)_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\, dx\right] \\ &= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left[0 - \frac{1}{n\pi}\left(-\frac{1}{n\pi}\cos(n\pi x)\right)_0^1\right] \\ &= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left[\frac{1}{(n\pi)^2}\left(\cos(n\pi) - 1\right)\right] \\ &= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}\left((-1)^n - 1\right) \end{aligned}$$

Substituting I_1, I_2 found above back into equation (1) gives the final result

$$\begin{aligned} b_n &= 2\left(\frac{(-1)^{n+1}}{n\pi} - \left(\frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}\left((-1)^n - 1\right)\right)\right) \\ &= 2\left(\frac{(-1)^{n+1}}{n\pi} - \frac{(-1)^{n+1}}{n\pi} - \frac{2}{(n\pi)^3}\left((-1)^n - 1\right)\right) \\ &= 2\left(\frac{(-1)^{n+1}}{n\pi} - \frac{(-1)^{n+1}}{n\pi} - \frac{2(-1)^n}{(n\pi)^3} + \frac{2}{(n\pi)^3}\right) \\ &= 4\left(-\frac{(-1)^n}{(n\pi)^3} + \frac{1}{(n\pi)^3}\right) \\ &= 4\left(\frac{1-(-1)^n}{(n\pi)^3}\right) \end{aligned}$$

For odd n, the above gives

$$\begin{aligned} b_n &= \left\{4\left(\frac{2}{\pi^3}\right), 4\left(\frac{2}{(3\pi)^3}\right), 4\left(\frac{2}{(5\pi)^3}\right), \cdots\right\} \\ &= 8\left\{\left(\frac{1}{\pi^3}\right), \left(\frac{1}{(3\pi)^3}\right), \left(\frac{1}{(5\pi)^3}\right), \cdots\right\} \end{aligned}$$

And for even n all $b_n = 0$. Therefore

$$b_n = \begin{cases} \frac{8}{(n\pi)^3} & n = 1, 3, 5, \cdots \\ 0 & \text{otherwise} \end{cases}$$

The Fourier sine series for $f(x)$ can now be written as

$$\begin{aligned} f(x) &= \sum_{n=1,3,5,\cdots} b_n \sin(n\pi x) \\ &= \frac{8}{\pi^3} \sum_{n=1,3,5,\cdots} \frac{1}{n^3} \sin(n\pi x) \end{aligned}$$

Since $f(x) = x(1-x)$, the above is the same as

$$x(1-x) = 8\left(\frac{1}{\pi^3}\sin(\pi x) + \frac{1}{3^3\pi^3}\sin(3\pi x) + \frac{1}{5^3\pi^3}\sin(5\pi x) + \frac{1}{7^3\pi^3}\sin(7\pi x) + \cdots\right)$$

To obtain the required result, let $x = \frac{1}{2}$ in the above, which gives

$$\begin{aligned} \frac{1}{2}\left(1 - \frac{1}{2}\right) &= 8\left(\frac{1}{1^3\pi^3}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3^3\pi^3}\sin\left(\frac{3}{2}\pi\right) + \frac{1}{5^3\pi^3}\sin\left(\frac{5}{2}\pi\right) + \frac{1}{7^3\pi^3}\sin\left(\frac{7}{2}\pi\right) + \cdots\right) \\ \frac{1}{4} &= \frac{8}{\pi^3}\left(\frac{1}{1^3}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3^3}\sin\left(\frac{3}{2}\pi\right) + \frac{1}{5^3}\sin\left(\frac{5}{2}\pi\right) + \frac{1}{7^3}\sin\left(\frac{7}{2}\pi\right) + \cdots\right) \\ \frac{\pi^3}{32} &= \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \end{aligned}$$

The above can also be written as

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(-1+2n)^3}$$

3.1.3 Problem 3

Problem Find the solution to heat equation $u_t = u_{xx}$ with initial conditions $u(x,0) = f(x)$ with $f(x) = x(1-x), 0 \leq x \leq 1$ and boundary conditions $u(0,t) = u(1,t) = 0$. Approximate $u\left(\frac{1}{2},1\right)$ to 10 decimal places.
solution
Using separation of variables, let $u(x,t) = X(x)T(t)$. Substituting this back into the PDE gives

$$\begin{aligned} T'X &= X''T \\ \frac{T'}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where the separation constant is some real value $-\lambda$. This gives the following two ODE's to solve

$$T' + \lambda T = 0 \tag{1}$$
$$X'' + \lambda X = 0 \tag{2}$$

Starting with the spatial ODE in order to obtain the eigenvalues and eigenfunctions. The boundary conditions on the spatial ODE become

$$\begin{aligned} X(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

Since equation (2) is a constant coefficient ODE, its characteristic equation is $r^2 + \lambda = 0$, which has the solution $r = \pm\sqrt{-\lambda}$, therefore its solution is given by

$$\begin{aligned} X(x) &= c_1 e^{rx} + c_2 e^{-rx} \\ &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \end{aligned} \tag{3}$$

There are three cases to consider, depending on if $\lambda < 0, \lambda = 0, \lambda > 0$. Each one of these cases gives a different solution that needs to be examined to see if the solution satisfies the boundary conditions.
Case 1 Assuming $\lambda < 0$. Therefore $-\lambda$ is positive and $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$, where μ is some positive number. The solution (3) can now be written as

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} \tag{3A}$$

This can be rewritten in terms of the hyperbolic trig functions (to make it easier to manipulate) as

$$X(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \tag{3B}$$

Where the constants c_i in (3A) are different from the constants in (3B), but kept the same for simplicity of notation so as not to introduce new constants. Applying left boundary conditions to (3B) results in

$$0 = c_1$$

The solution (3B) now reduces to

$$X(x) = c_2 \sinh(\mu x)$$

Applying right side boundary conditions to the above results in

$$0 = c_2 \sinh(\mu)$$

But $\sinh(\mu) \neq 0$ since it was assumed μ is not zero and sinh is only zero when its argument is zero. The only possibility then is $c_2 = 0$, which leads to trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.
Case 2. Assuming $\lambda = 0$. The ODE becomes $X'' = 0$, which has the solution

$$X(x) = c_1 x + c_2$$

Applying left side B.C. gives

$$0 = c_2$$

The solution now reduces to

$$X(x) = c_1 x$$

Applying right side B.C. gives

$$0 = c_1$$

Leading to the trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.
Case 3 Assuming $\lambda > 0$. In this case equation $\sqrt{-\lambda}$ is complex and equation (3) can be expressed in terms of trig functions using Euler relation which results in

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \tag{4}$$

Applying left side B.C. gives

$$0 = c_1$$

Solution (4) now reduces to

$$X(x) = c_2 \sin\left(\sqrt{\lambda} x\right) \tag{5}$$

Applying right side B.C. gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\right)$$

Non-trivial solution implies $\sin\left(\sqrt{\lambda}\right) = 0$ or $\sqrt{\lambda} = n\pi$ for $n = 1, 2, 3, \cdots$. Therefore the eigenvalues are

$$\lambda_n = (n\pi)^2 \qquad n = 1, 2, 3, \cdots$$

And the corresponding eigenfunctions from (5) are

$$X_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right) \tag{6}$$

Now that the eigenvalues are known, the solution to the time ODE (1) can be found.

$$T' + \lambda_n T = 0$$

This has the solution (using an integrating factor method)

$$T_n(t) = e^{-\lambda_n t} \tag{7}$$

The constant of integration is not needed for (7) since it will be absorbed with the constant of integration coming from solution of the spatial ODE (6) when these solutions are multiplied with each others below. Therefore the fundamental solution is

$$u_n(x,t) = T_n(t) X_n(x)$$

Linear combination of fundamental solutions is also a solution (since this is a linear PDE). Therefore the general solution is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n \\ &= \sum_{n=1}^{\infty} T_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin\left(\sqrt{\lambda_n} x\right) \end{aligned} \tag{8}$$

Initial conditions is now used to determine c_n, . At $t = 0$, $u(x,0) = f(x)$ and the above becomes

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

Multiplying both sides of the above equation by eigenfunction $\sin\left(\sqrt{\lambda_m} x\right)$ and integrating over the domain of $f(x)$ gives

$$\int_0^1 f(x) \sin\left(\sqrt{\lambda_m} x\right) dx = \int_0^1 \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right) \sin\left(\sqrt{\lambda_m} x\right) dx$$

Interchanging the order of summation and integration gives

$$\int_0^1 f(x) \sin\left(\sqrt{\lambda_m} x\right) dx = \sum_{n=1}^{\infty} c_n \int_0^1 \sin\left(\sqrt{\lambda_n} x\right) \sin\left(\sqrt{\lambda_m} x\right) dx$$

By the orthogonality of the sine functions, all terms in the right side vanish except when $n = m$, leading to

$$\begin{aligned} \int_0^1 f(x) \sin\left(\sqrt{\lambda_m} x\right) dx &= c_m \int_0^1 \sin^2\left(\sqrt{\lambda_m} x\right) dx \\ &= c_m \frac{1}{2} \end{aligned}$$

Therefore (replacing m back to n now, since it is arbitrary)

$$c_n = 2 \int_0^1 f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1, 2, 3, \cdots$$

But $\sqrt{\lambda_n} = n\pi$, hence

$$c_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \qquad n = 1,2,3,\cdots$$

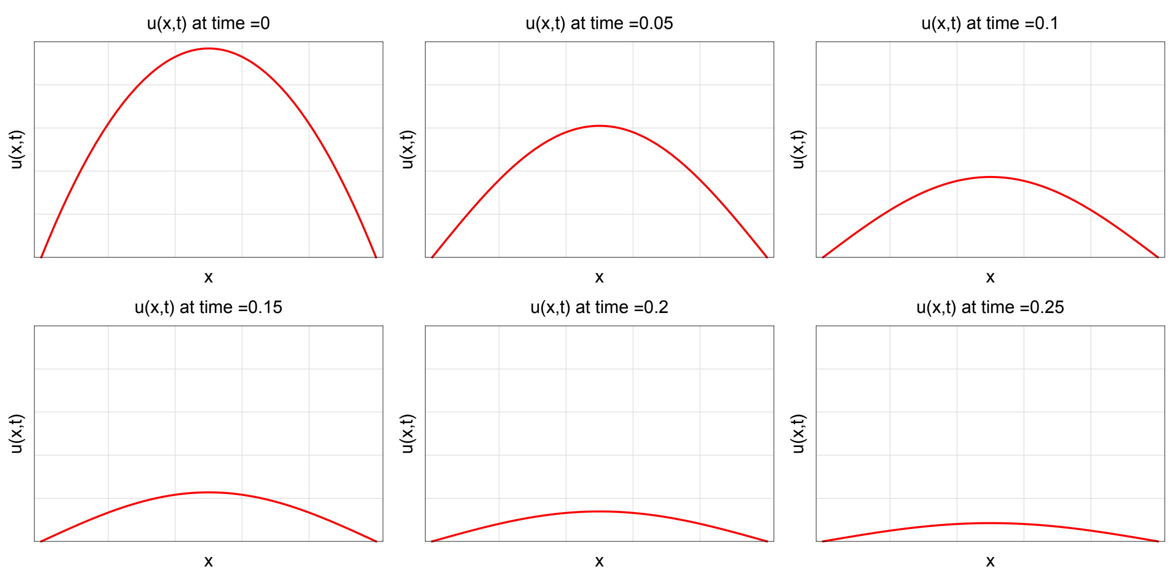
Since $f(x)$ is the same as in problem 2, the above shows that c_n is the same as b_n found in problem 2 above. This means c_n is the sine Fourier series coefficients of $f(x)$ which was found in problem 2. Using that result obtained earlier

$$c_n = b_n = \begin{cases} \frac{8}{(n\pi)^3} & n = 1,3,5,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Using the above in (8), the general solution is therefore

$$\begin{aligned} u(x,t) &= \frac{8}{\pi^3}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3}e^{-\lambda_n t}\sin\left(\sqrt{\lambda_n}x\right) \\ &= \frac{8}{\pi^3}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3}e^{-n^2\pi^2 t}\sin(n\pi x) \end{aligned}$$

The Following is plot of the solution for increasing values of time starting from $t = 0$, using 10 terms in the sum. At about $t = 0.3$ seconds, the temperature reduces to almost zero.



To approximate $u\left(\frac{1}{2},1\right)$ to 10 decimal places, first the solution is written at $x = \frac{1}{2}$ and $t = 1$. From above, the solution is

$$u\left(\frac{1}{2},1\right) = \frac{8}{\pi^3}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3}e^{-n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)$$

Due to the fast convergence, only one term was needed. Result for $n = 1$ and $n = 3$ are

$$\begin{aligned} u_1\left(\frac{1}{2},1\right) &= \frac{8}{\pi^3}\left(e^{-\pi^2}\sin\left(\frac{\pi}{2}\right)\right) = 0.000013345216966776341 \\ u_3\left(\frac{1}{2},1\right) &= \frac{8}{\pi^3}\left(e^{-\pi^2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{27}e^{-9\pi^2}\sin\left(3\frac{\pi}{2}\right)\right) = 0.000013345216966776341 \end{aligned}$$

The above shows that the solution $u_1\left(\frac{1}{2},1\right)$ did not change beyond the first 10 decimal points when adding one more term in the series. Therefore, only one term is needed. Therefore, the final result (rounded to 10 decimal points) is

$$\boxed{u\left(\tfrac{1}{2},1\right) = 0.0000133452}$$

3.1.4 Problem 4

Problem Solve $u_t + u = u_{xx}$ with initial conditions $u(x,0) = f(x)$ and boundary conditions $u(0,t) = u(L,t) = 0$.

solution

Using separation of variables, let $u(x,t) = X(x)T(t)$. Substituting this back into the PDE gives

$$\begin{aligned} T'X + TX &= X''T \\ \frac{T'}{T} + 1 &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where the separation constant is some real value $-\lambda$. This gives the following two ODE's to solve

$$T' + (1+\lambda)T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Starting with the spatial ODE in order to obtain the eigenvalues. The boundary conditions on the spatial ODE become

$$\begin{aligned} X(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

The above boundary value ODE was solved in problem 3. The eigenvalues were found to be

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$

And the corresponding eigenfunctions are

$$X_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right)$$

The solution to the time ODE (1) using integrating factor method is

$$T(t) = e^{-(1+\lambda_n)t}$$

Therefore, as before, the general solution is obtained by linear combination of the fundamental solutions giving

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(1+\lambda_n)t} \sin\left(\sqrt{\lambda_n} x\right) \tag{3}$$

Initial conditions are used to determine c_n. At $t = 0$, $u(x,0) = f(x)$ and the above becomes

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

Multiplying both sides by $\sin\left(\sqrt{\lambda_m} x\right)$ and integrating over the domain of $f(x)$ gives

$$\int_0^L f(x) \sin\left(\sqrt{\lambda_m} x\right) dx = \int_0^L \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right) \sin\left(\sqrt{\lambda_m} x\right) dx$$

Interchanging the order of summation and integrating gives

$$\int_0^L f(x) \sin\left(\sqrt{\lambda_m} x\right) dx = \sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\sqrt{\lambda_n} x\right) \sin\left(\sqrt{\lambda_m} x\right) dx$$

By orthogonality of sine functions, all terms in the right side vanish except when $n = m$, leading to

$$\begin{aligned} \int_0^L f(x) \sin\left(\sqrt{\lambda_m} x\right) dx &= c_m \int_0^L \sin^2\left(\sqrt{\lambda_m} x\right) dx \\ &= c_m \frac{L}{2} \end{aligned}$$

Therefore

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1,2,3,\cdots \tag{4}$$

But $\sqrt{\lambda_n} = \frac{n\pi}{L}$, hence

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \qquad n = 1,2,3,\cdots$$

The above shows that c_n is the Fourier sine series of $f(x)$. Since $f(x)$ is not given, explicit solution for c_n can not be found. Therefore the final solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-(1+\lambda_n)t} \sin\left(\sqrt{\lambda_n} x\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx\right) e^{-(1+\lambda_n)t} \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

With $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

3.1.5 key solution

Homework 1, Math 322

1. Solve the boundary value problem

$$y'' - y = x, \quad y(0) = 0,\, y(1) = 1.$$

Solution: The general solution is

$$y = c_1 \cosh x + c_2 \sinh x - x.$$

The boundary condition give $c_1 = 0$, $c_2 \sinh 1 = 2$. The solution of the BVP is

$$y = \frac{2}{\sinh 1} \sinh x - x.$$

2. Find the Fourier sine series for the function $f(x) = x(1-x)$, $0 \le x \le 1$. Use the result to evaluate the infinite series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \pm \dots.$$

Solution: The Fourier coefficients are

$$\begin{aligned}
c_n &= 2\int_0^1 x(1-x)\sin n\pi x\,dx \\
&= -\frac{2}{n\pi}\, x(1-x)\cos n\pi x\Big|_{x=0}^{x=1} + \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x\,dx \\
&= \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x\,dx \\
&= \frac{2}{n^2\pi^2}\,(1-2x)\sin n\pi x\,dx\Big|_{x=0}^{x=1} + \frac{2}{n^2\pi^2}\int_0^1 (-2)\sin n\pi x\,dx \\
&= -\frac{4}{n^2\pi^2}\int_0^1 \sin n\pi x\,dx \\
&= \begin{cases} \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

By the convergence theorem, we have, for all $0 \le x \le 1$,

$$x(1-x) = \frac{8}{\pi^3}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}\sin(2k-1)\pi x.$$

If we choose $x = \frac{1}{2}$, we get

$$\frac{1}{4} = \frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3},$$

so

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

3. Find the solution to the heat equation $u_t = u_{xx}$ with initial condition $u(x,0) = f(x)$ with $f(x)$ as in problem 2, and boundary conditions $u(0,t) = u(1,t) = 0$. Approximate $u(\frac{1}{2},1)$ to 10 decimal places.
Solution: The solution is

$$u(x,t) = \frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}e^{-(2k-1)^2\pi^2 t}\sin((2k-1)\pi x).$$

Then

$$u(\tfrac{1}{2},1) = \frac{8}{\pi^3}\sum_{k=1}^{\infty}e^{-(2k-1)^3\pi^2}\frac{(-1)^{k+1}}{(2k-1)^3}.$$

This is an alternating series $s = \sum_{k=1}^{\infty}(-1)^{k+1}a_k$ with $a_k \geq 0$, $a_k \geq a_{k+1}$, $a_k \to 0$. Then $\left|s - \sum_{k=1}^{K}(-1)^{k+1}a_k\right| \leq a_{K+1}$. Therefore,

$$\left|u(\tfrac{1}{2},1) - \frac{8}{\pi^3}\sum_{k=1}^{K}e^{-(2k-1)^3\pi^2}\frac{(-1)^{k+1}}{(2k-1)^3}\right| < \frac{8}{\pi^3}e^{-(2K+1)^3\pi^2}\frac{1}{(2K+1)^3}.$$

When we choose $K = 1$, the error is less than 10^{-40}. Therefore, we obtain

$$u(\tfrac{1}{2},1) \approx \frac{8}{\pi^3}e^{-\pi^2} = 0.00001334521692\ldots$$

with an error less than 10^{-40}.

4. Solve the partial differential equation $u_t + u = u_{xx}$ with initial condition $u(x,0) = f(x)$ and boundary conditions $u(0,t) = u(L,t) = 0$ using Fourier series.
Solution: Using the method of separation of variables $u(x,t) = X(x)T(t)$ we find

$$\frac{T'(t)}{T(t)} + 1 = \frac{X''(x)}{X(x)} = -\lambda.$$

Therefore, we obtain

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0,$$

and

$$T' + (\lambda + 1)T(t) = 0.$$

This gives

$$u_n(x,t) = e^{-(n^2\pi^2/L^2+1)t}\sin(n\pi x/L).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}c_n e^{-(n^2\pi^2/L^2+1)t}\sin(n\pi x/L),$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx.$$

3.2 Quizz 2

3.2.1 Problem 1

Problem Solve the wave equation $u_{tt} = u_{xx}$ for infinite domain $-\infty < x < \infty$ with initial position $u(x,0) = f(x) = \frac{1}{1+x^2}$ and zero initial velocity $g(x) = 0$. Plot the solution for $t = 0, 1, 2$ seconds.
solution
The solution for wave PDE $u_{tt} = a^2 u_{xx}$ on infinite domain can be written as a series solution or as general solution using D'Alembert form. Using D'Alembert, the solution is

$$u(x,t) = \frac{1}{2}(f(x-at) + f(x+at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

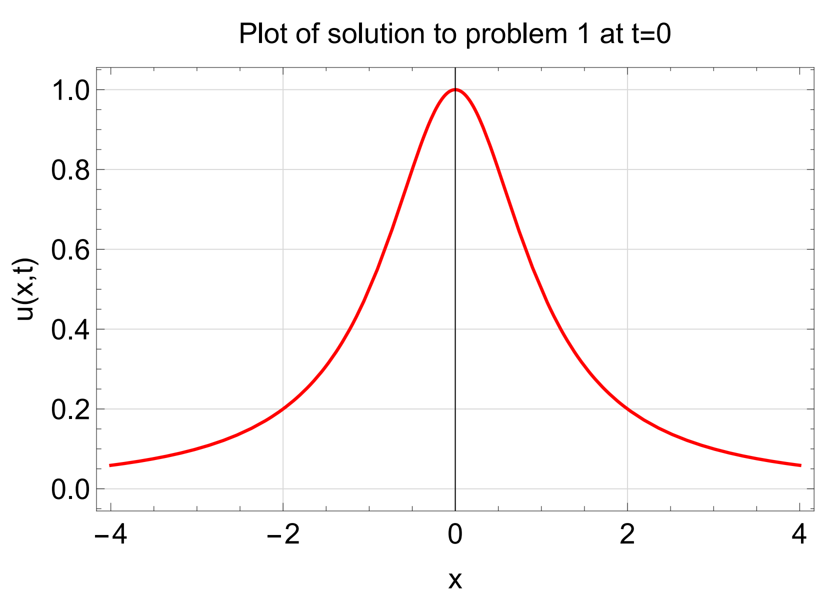
Where in this problem $a = 1$ and $g(x) = 0$. Therefore the above simplifies to

$$u(x,t) = \frac{1}{2}(f(x-t) + f(x+t))$$

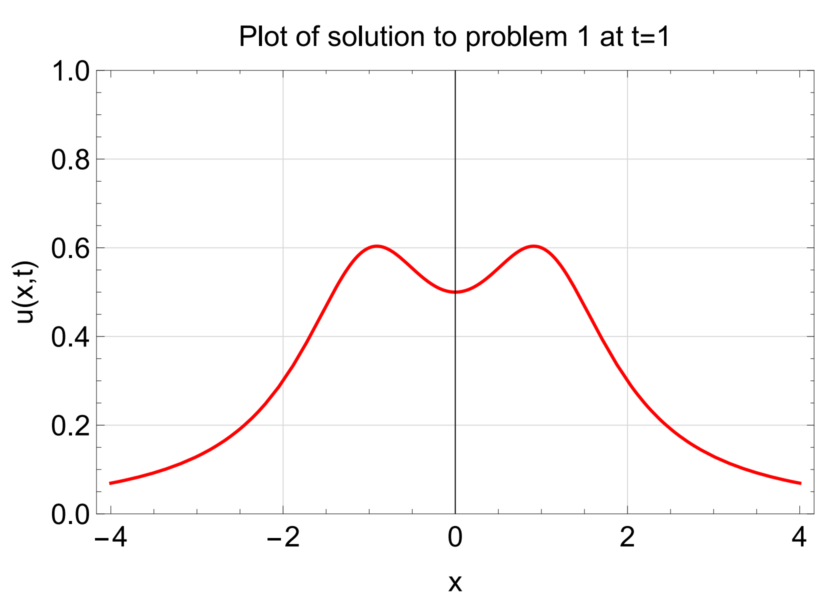
$f(x-t)$ is the initial position shifted to the right by t and $f(x+t)$ is the initial position shifted to the left by t. Since $f(x) = \frac{1}{1+x^2}$, the above solution becomes

$$u(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right)$$

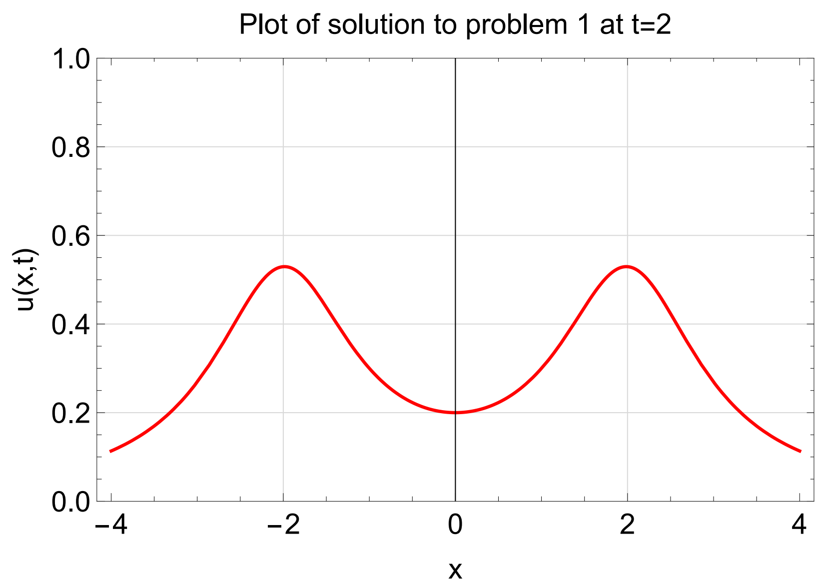
This is a plot of the solution at time $t = 0$ (which is just $\frac{1}{1+x^2}$)



This is a plot of the solution at time $t = 1$



This is a plot of the solution at time $t = 2$



The above shows that, eventually, the initial position splits into two halves, where one half moves to the right and one half moves to the left, but the sum (energy) of the parts remain equal to that at $t = 0$ since there is no damping. An Animation was also made of this solution for better illustration.

3.2.2 Problem 2

Problem Apply the method of separation of variables to the damped wave equation $u_{tt} + 2u_t = u_{xx}$ on finite domain with fixed ends $u(0,t) = 0$ and $u(\pi,t) = 0$. Let initial position be $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = 0$. Determine the first term in the series solution.

solution

Let the solution be $u(x,t) = X(x)T(t)$. Substituting this back into the PDE gives

$$T''X + 2T'X = X''T$$

Dividing throughout by $XT \neq 0$ and simplifying gives

$$\frac{T''}{T} + 2\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence the eigenvalue ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \qquad (1)\\ X(0) &= 0\\ X(\pi) &= 0 \end{aligned}$$

And the corresponding time ODE

$$T'' + 2T' + \lambda T = 0 \qquad (2)$$

The eigenvalue ODE for the homogeneous boundary condition was solved before. The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Since $L = \pi$, the above becomes

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots \qquad (3)$$

The corresponding eigenfunctions are

$$X_n(x) = c_n \sin(nx)$$

Now that the eigenvalues are found, the time ODE (2) is solved.

$$T_n'' + 2T_n' + n^2 T_n = 0$$

This is constant coefficient ODE. The characteristic equation is

$$r^2 + 2r + n^2 = 0$$

The roots are

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-2 \pm \sqrt{4 - 4n^2}}{2}\\ &= -1 \pm \sqrt{1 - n^2} \end{aligned}$$

For $n = 1$ the root becomes $r = -1$ (double root), hence the solution is

$$T_1(t) = A_1 e^{-t} + B_1 t e^{-t} \qquad (4)$$

And for the remaining value $n = 2, 3, \cdots$, the term $\sqrt{1 - n^2}$ becomes complex. Therefore the roots can now be written as $r = -1 \pm i\sqrt{n^2 - 1}$. This implies that the solution can be expressed using trigonometric functions as

$$T_n(t) = e^{-t}\left(A_n \cos\left(t\sqrt{n^2 - 1}\right) + B_n \sin\left(t\sqrt{n^2 - 1}\right)\right) \qquad n = 2, 3, \cdots \qquad (5)$$

Since initial velocity is zero at $t = 0$, then (4) leads to $T_1' = -A_1 e^{-t} + B_1 e^{-t} - tB_1 e^{-t}$. At $t = 0$ this gives $0 = -A_1 + B_1$. Therefore solution (4) becomes

$$T_1(t) = A_1\left(e^{-t} + te^{-t}\right) \qquad (4A)$$

Taking time derivative for (5) gives

$$\begin{aligned} T_n'(t) = -e^{-t}\left(A_n \cos\left(\sqrt{n^2 - 1}t\right) + B_n \sin\left(\sqrt{n^2 - 1}t\right)\right) +\\ e^{-t}\left(-A_n\sqrt{n^2 - 1}\sin\left(\sqrt{n^2 - 1}t\right) + B_n\sqrt{n^2 - 1}\cos\left(\sqrt{n^2 - 1}t\right)\right) \end{aligned}$$

At $t=0$ the above becomes

$$0=-A_n+B_n\sqrt{n^2-1}$$

Hence $A_n=B_n\sqrt{n^2-1}$ and (5) reduces to

$$T_n(t)=B_ne^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\qquad n=2,3,\cdots \tag{5A}$$

Therefore the fundamental solution is

$$\begin{aligned}
u_n(x,t)&=T_n(t)X_n(x)\\
u(x,t)&=\sum_{n=1}^{\infty}T_n(t)X_n(x)\\
&=T_1(t)X_1(x)+\sum_{n=2}^{\infty}T_n(t)X_n(x)\\
&=c_1\left(\left(e^{-t}+te^{-t}\right)\sin x\right)+\sum_{n=2}^{\infty}c_ne^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx)
\end{aligned}\tag{6}$$

Where the constant A_1 was combined into c_1 and B_n combined into c_n. The constants c_1 and c_n are now found from initial position. At $t=0$ (6) becomes

$$f(x)=c_1\sin x+\sum_{n=2}^{\infty}c_n\sqrt{n^2-1}\sin(nx)$$

Multiplying both sides by $\sin(mx)$ and Integrating gives

$$\int_0^{\pi}f(x)\sin(mx)\,dx=\int_0^{\pi}c_1\sin x\sin(mx)\,dx+\sum_{n=2}^{\infty}c_n\sqrt{n^2-1}\left(\int_0^{\pi}\sin(nx)\sin(mx)\,dx\right)\tag{7}$$

For $m=1$ the above reduces to

$$\begin{aligned}
\int_0^{\pi}f(x)\sin xdx&=\int_0^{\pi}c_1\sin^2 xdx\\
\int_0^{\pi}f(x)\sin xdx&=\frac{\pi}{2}c_1\\
c_1&=\frac{2}{\pi}\int_0^{\pi}f(x)\sin xdx
\end{aligned}$$

And for $m=2,3,\cdots$ (7) becomes

$$\begin{aligned}
\int_0^{\pi}f(x)\sin(mx)\,dx&=\overbrace{\int_0^{\pi}c_1\sin x\sin(mx)\,dx}^{0}+c_m\sqrt{m^2-1}\left(\int_0^{\pi}\sin^2(mx)\,dx\right)\\
&=c_m\sqrt{m^2-1}\left(\int_0^{\pi}\sin^2(mx)\,dx\right)
\end{aligned}$$

Hence for $n=2,3,\cdots$ the above gives

$$\int_0^{\pi}f(x)\sin(nx)\,dx=c_n\sqrt{n^2-1}\left(\frac{\pi}{2}\right)$$

Therefore

$$c_n=\frac{2}{\pi\sqrt{n^2-1}}\int_0^{\pi}f(x)\sin nxdx\qquad n=2,3,\cdots$$

This completes the solution. The final solution from (6) becomes

$$\begin{aligned}
u(x,t)=&\left(\frac{2}{\pi}\int_0^{\pi}f(x)\sin xdx\right)\left(e^{-t}+te^{-t}\right)\sin(x)\\
&+\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\int_0^{\pi}f(x)\sin(nx)\,dx}{\sqrt{n^2-1}}e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx)
\end{aligned}$$

To test the solution, it is compared to numerical differential equation solution. Using $f(x)=x(\pi-x)$ as an example. The result showed an exact match. An animation was also made. Therefore the first term is

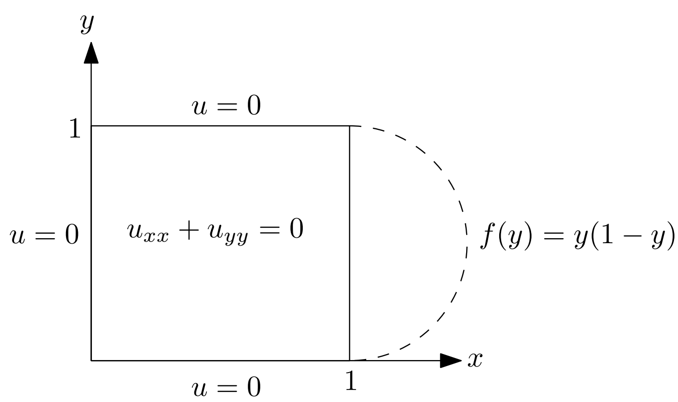
$$\left(\frac{2}{\pi}\int_0^{\pi}f(x)\sin xdx\right)\left(e^{-t}+te^{-t}\right)\sin(x)$$

And for $n=2,3,\cdots$ the n^{th} term is

$$\left(\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx\right)\frac{e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx)}{\sqrt{n^2-1}}$$

3.2.3 Problem 3

Problem Solve $u_{xx} + u_{yy} = 0$ on the square $0 \leq x, y \leq 1$. If $u(0,y) = u(x,0) = u(x,1) = 0$ and $u(1,y) = y - y^2$. Find an approximate value for $u\left(\frac{1}{2},\frac{1}{2}\right)$
solution To make the solution steps more useful and general, a is used for the length of the x dimension and b for the length of the y dimension, then these are replaced by 1 at the very end. This is a plot of boundary conditions



Let $u(x,y) = X(x)\,Y(x)$. Substituting this into the PDE gives

$$X''Y + Y''X = 0$$

Dividing throughout by $XY \neq 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

This gives the eigenvalue ODE

$$\begin{aligned} Y'' + \lambda Y &= 0 \\ Y(0) &= 0 \\ Y(b) &= 0 \end{aligned} \tag{1}$$

The solution to (1) gives the eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1,2,3\cdots$ and since $L = b$, this becomes

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2 \qquad n = 1,2,\cdots$$

And the corresponding eigenfunction

$$\begin{aligned} Y_n(y) &= c_n \sin\left(\sqrt{\lambda_n} y\right) \\ &= c_n \sin\left(\frac{n\pi}{b} y\right) \end{aligned}$$

Therefore the corresponding nonhomogeneous $X(x)$ ODE

$$\begin{aligned} X_n'' - \lambda_n X_n &= 0 \\ X_n(0) &= 0 \\ X_n(a) &= y - y^2 \end{aligned} \tag{2}$$

The solution to (2), since λ_n is positive is

$$\begin{aligned} X_n(x) &= A_n \cosh\left(\sqrt{\lambda_n} x\right) + B_n \sinh\left(\sqrt{\lambda_n} x\right) \\ &= A_n \cosh\left(\frac{n\pi}{b} x\right) + B_n \sinh\left(\frac{n\pi}{b} x\right) \end{aligned}$$

Boundary conditions $X(0) = 0$ gives

$$0 = A_n$$

The solution (3) now simplifies to

$$X_n(x) = B_n \sinh\left(\frac{n\pi}{b} x\right)$$

Hence the fundamental solution is

$$\begin{aligned} u_n(x,y) &= X_n Y_n \\ &= c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right) \end{aligned}$$

Where the constants B_n is merged with c_n. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{3}$$

c_n is now found by applying the boundary condition at $x = a$. The above becomes

$$y - y^2 = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{b}y\right)$ and integrating gives

$$\int_0^b \left(y-y^2\right)\sin\left(\frac{m\pi}{b}y\right)dy = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right)\left(\int_0^b \sin\left(\frac{m\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)dy\right)$$

By orthogonality the above reduces to

$$\begin{aligned} \int_0^b \left(y-y^2\right)\sin\left(\frac{m\pi}{b}y\right)dy &= c_n \sinh\left(\frac{m\pi}{b}a\right)\int_0^b \sin^2\left(\frac{m\pi}{b}y\right)dy \\ &= \frac{b}{2}c_m \sinh\left(\frac{m\pi}{b}a\right) \end{aligned}$$

Therefore

$$c_n = \frac{2}{b}\frac{1}{\sinh\left(\frac{m\pi}{b}a\right)}\int_0^b \left(y-y^2\right)\sin\left(\frac{n\pi}{b}y\right)dy$$

Now replacing $a = 1, b = 1$, the above becomes

$$\begin{aligned} c_n &= \frac{2}{\sinh(n\pi)}\int_0^1 \left(y-y^2\right)\sin(n\pi y)\,dy \\ &= \frac{2}{\sinh(n\pi)}\left(\frac{-2\left(-1+(-1)^n\right)}{n^3\pi^3}\right) \\ &= \frac{-4}{\sinh(n\pi)}\frac{\left(-1+(-1)^n\right)}{n^3\pi^3} \end{aligned}$$

Hence the solution (3) becomes

$$u(x,y) = \frac{-4}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(-1+(-1)^n\right)}{n^3}\frac{\sinh(n\pi x)}{\sinh(n\pi)}\sin(n\pi y)$$

At $x = \frac{1}{2}, y = \frac{1}{2}$ the above becomes

$$u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{-4}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(-1+(-1)^n\right)}{n^3}\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sin\left(\frac{n\pi}{2}\right)$$

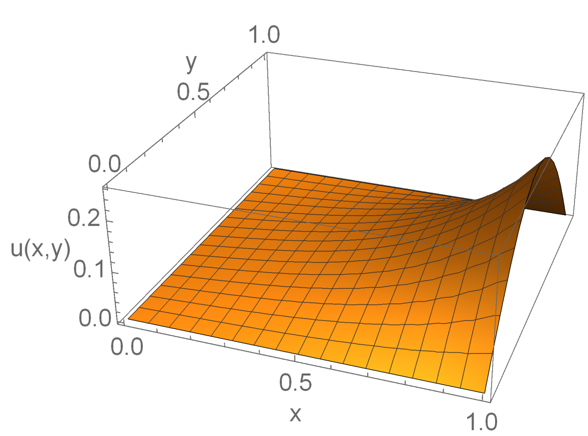
For $n = 1$, the above gives 0.0514136952911346 and for $n = 2$ the value do not change beyond 16 decimal points. So only need to use one term to get very good approximation value as

$$u\left(\frac{1}{2},\frac{1}{2}\right) = 0.0514136952911346$$

This value is between zero and 0.25, where 0.25 is the maximum value at the boundary and zero is the minimum value at the boundary. This agrees with the min-max principle. This is a 3D plot of the solution over the whole square.

```
In[40]:= mySol[x_, y_] := -4 / Pi^3 Sum[((-1 + (-1) ^n)/(n^3)) (Sinh[n Pi x]/Sinh[n Pi]) Sin[n Pi y] , {n, 1, 2}]
         Plot3D[mySol[x, y], {x, 0, 1}, {y, 0, 1}, AxesLabel → {"x", "y", "u(x,y)"}, BaseStyle → 14]
```

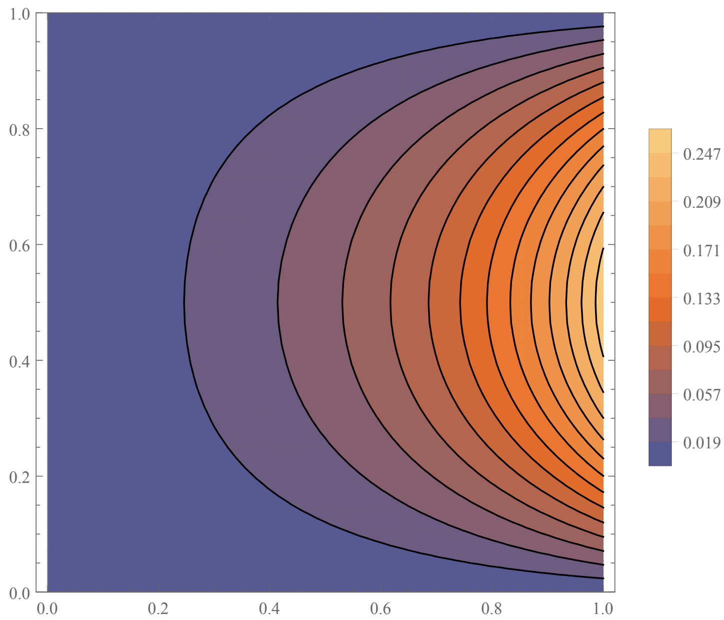
Out[41]=



This is a contour plot

```
ContourPlot[Evaluate[mySol[x, y]], {x, 0, 1}, {y, 0, 1}, AxesLabel → {x, y},
  PlotRange → {-1, 1}, Contours → 100, PlotTheme -> "Scientific", PlotLegends → Automatic]
```

Out[16]=



3.2.4 Problem 4

Problem Solve $u_{xx} + u_{yy} = 0$ on disk $x^2 + y^2 < 1$ with boundary condition xy^2 when $x^2 + y^2 = a$. Where $a = 1$ in this problem. Express solution in x, y

solution The first step is to convert the boundary condition to polar coordinates. Since $x = r\cos\theta, y = r\sin\theta$, then at the boundary $u(r,\theta) = r\cos\theta\,(r\sin\theta)^2$. But $r = 1$ (the radius). Hence at the boundary, $u(1,\theta) = f(\theta)$ where

$$\begin{aligned} f(\theta) &= \cos\theta\sin^2\theta \\ &= \cos\theta\left(1-\cos^2\theta\right) \\ &= \cos\theta - \cos^3\theta \end{aligned}$$

But $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$. Therefore the above becomes

$$\begin{aligned} f(\theta) &= \cos\theta - \left(\frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta\right) \\ &= \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta \qquad (1) \end{aligned}$$

The above is also seen as the Fourier series of $f(\theta)$. The PDE in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

The solution is known to be

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n\left(c_n\cos(n\theta) + k_n\sin(n\theta)\right) \qquad (2)$$

Since the above solution is the same as $f(\theta)$ when $r = 1$, then equating (2) when $r = 1$ to (1) gives

$$\frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(c_n\cos(n\theta) + k_n\sin(n\theta)\right)$$

By comparing terms on both sides, this shows by inspection that

$$\begin{aligned} c_0 &= 0 \\ c_1 &= \frac{1}{4} \\ c_3 &= \frac{-1}{4} \end{aligned}$$

And all other c_n, k_n are zero. Using the above result back in (2) gives the solution as

$$\boxed{u(r,\theta) = \frac{r}{4}\cos\theta - \frac{r^3}{4}\cos 3\theta} \qquad (3)$$

This solution is now converted to xy using the formula

$$\begin{aligned} r^n\cos n\theta &= \sum_{\substack{k=0 \\ even}}^{n}\binom{n}{k} x^{n-k}(-1)^{\frac{k}{2}} y^k \\ &= \sum_{\substack{k=0 \\ even}}^{n}\frac{n!}{k!\,(n-k)!}x^{n-k}(-1)^{\frac{k}{2}} y^k \end{aligned}$$

For $n = 1$ the above gives

$$\begin{aligned} r\cos\theta &= \frac{1!}{0!\,(1-0)!}x^{1-0}(-1)^0 y^0 \\ &= x \qquad (4) \end{aligned}$$

And for $n = 3$

$$\begin{aligned} r^3\cos 3\theta &= \frac{3!}{0!\,(3-0)!}x^{3-0}(-1)^0 y^0 + \frac{3!}{2!\,(3-2)!}x^{3-2}(-1)^1 y^2 \\ &= x^3 - 3xy^2 \qquad (5) \end{aligned}$$

Using (4,5) in (3) gives the solution in x, y

$$\boxed{u(x,y) = \frac{1}{4}x - \frac{1}{4}\left(x^3 - 3xy^2\right)} \qquad (6)$$

This is now verified that is satisfies the PDE $u_{xx}+u_{yy}=0$.

$$\frac{\partial u}{\partial x}=\frac{1}{4}-\frac{1}{4}\left(3x^2-3y^2\right)$$
$$\frac{\partial^2 u}{\partial x^2}=-\frac{6}{4}x$$

And

$$\frac{\partial u}{\partial y}=\frac{6}{4}xy$$
$$\frac{\partial^2 u}{\partial y^2}=\frac{6}{4}x$$

Therefore $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.
Now the boundary conditions $u(x,y)=xy^2$ are also verified. This condition applies when $x^2+y^2=1$ or $y^2=1-x^2$. Substituting this into (6) gives

$$u(x,y)_{@D}=\frac{1}{4}x-\frac{1}{4}\left(x^3-3x\overbrace{\left(1-x^2\right)}^{y2}\right)$$

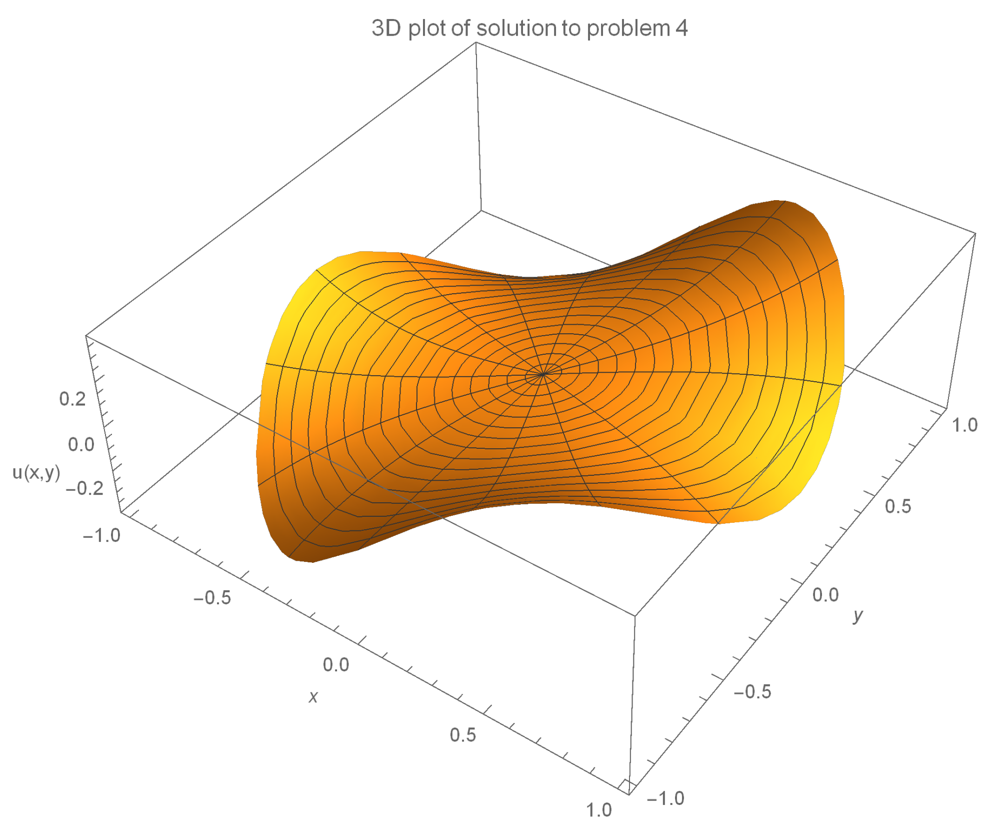
Simplifying gives

$$\begin{aligned}
u(x,y)_{@D}&=\frac{1}{4}x-\frac{1}{4}\left(x^3-\left(3x-3x^3\right)\right)\\
&=\frac{1}{4}x-\frac{1}{4}x^3+\frac{1}{4}\left(3x-3x^3\right)\\
&=\frac{1}{4}x-\frac{1}{4}x^3+\frac{3}{4}x-\frac{3}{4}x^3\\
&=x-x^3\\
&=x\left(1-x^2\right)\\
&=xy^2
\end{aligned}$$

Verified. This is 3D plot of the solution

```
In[76]:= ParametricPlot3D[{r Cos[t], r Sin[t], r / 4 Cos[t] - r^3 / 4 Cos[3 t]},
    {r, 0, 1}, {t, 0, 2 Pi}, AxesLabel → {x, y, "u(x,y)"},
    PlotLabel → "3D plot of solution to problem 4", ImageSize → 500]
```

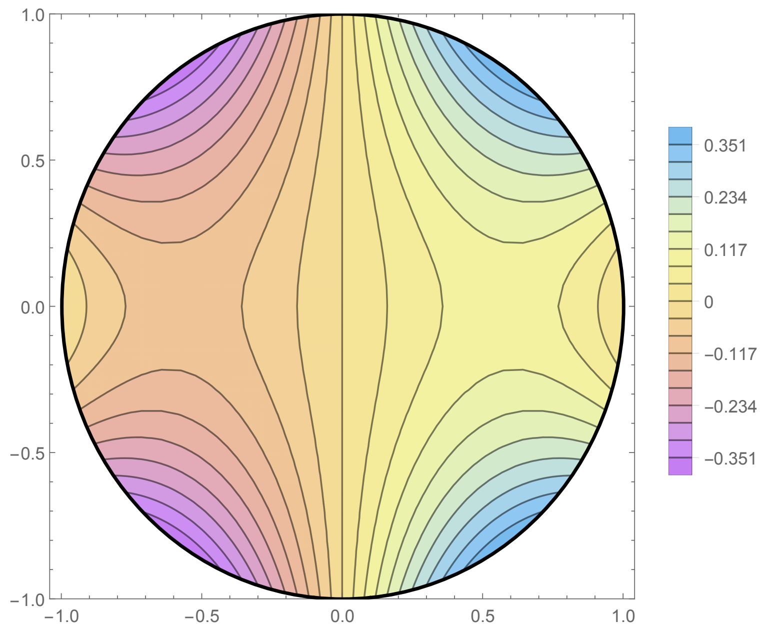
Out[76]=



This is a contour plot

In[96]:=
```
ContourPlot[1 / 4 x - 1 / 4 (x^3 - 3 x y^2), {x, -1, 1}, {y, -1, 1}, AxesLabel → {x, y},
  Contours → 50, PlotLegends → Automatic, ColorFunction → "Pastel",
  Epilog → {Thick, Circle[]},
  PlotRange → {-1, 1},
  RegionFunction → Function[{x, y, z}, Norm[{x, y}] < 1.]]
```

Out[96]=



3.2.5 key solution

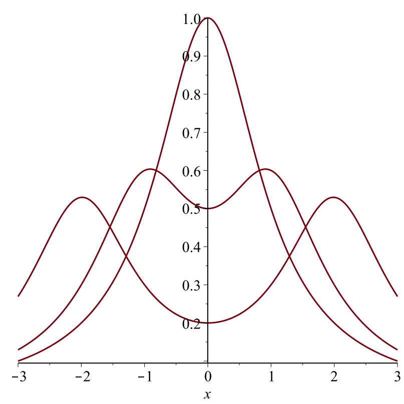
Homework 2, Math 322

1. Solve the wave equation $u_{tt} = u_{xx}$ for $-\infty < x < \infty$, $t \geq 0$ with initial conditions

$$u(x,0) = \frac{1}{1+x^2}, \quad u_t(x,0) = 0.$$

Plot the solutions $u(x,t)$ for $t = 0$, $t = 1$, $t = 2$.
Solution: The d'Alembert solution is

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) = \frac{1}{2}\left(\frac{1}{(1+(x-t)^2} + \frac{1}{1+(x-t)^2}\right).$$



2. Apply the method of separation of variables to the damped wave equation $u_{tt} + 2u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$. Determine the first term in the solution $u(x,t) = \sum_{n=1}^{\infty} \cdots$
Solution: By separation of variables we obtain the solutions

$$y_n(x,t) = \sin(nx)T_n(t),$$

where T_n is the solution of

$$T_n'' + 2T_n' + n^2 T_n = 0, \quad T_n(0) = 1, T_n'(0) = 0.$$

Thus

$$T_1(t) = (1+t)e^{-t}$$

and, for $n \geq 2$,

$$T_n(t) = e^{-t}\frac{\sin(t\sqrt{n^2-1})}{\sqrt{n^2-1}} + e^{-t}\cos(t\sqrt{n^2-1}).$$

By superposition, we obtain the solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n T_n(t)\sin(nx)$$

where b_n are the Fourier sine coefficients of $f(x)$. The first term in the solution formula is

$$\frac{2}{\pi}\left(\int_0^\pi f(s)\sin s\,ds\right)e^{-t}(t+1)\sin x.$$

3. Solve the Dirichlet problem $u_{xx}+u_{yy}=0$ on the square $0\le x,y\le 1$ if $u(0,y)=u(x,0)=u(x,1)=0$ and $u(1,y)=y(1-y)$. Find an approximate value for $u(\frac{1}{2},\frac{1}{2})$.

Solution: The Fourier sine series for $y(1-y)$, $0\le y\le 1$, is

$$y(1-y)=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\sin(n\pi y).$$

According to Section 10.8, the solution of the Dirichley problem is

$$u(x,y)=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\frac{\sinh n\pi x}{\sinh n\pi}\sin(n\pi y).$$

Then

$$u(\tfrac{1}{2},\tfrac{1}{2})=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\frac{\sinh n\pi\frac{1}{2}}{\sinh n\pi}(-1)^{(n-1)/2}.$$

Taking two terms of the series, we find

$$u(\tfrac{1}{2},\tfrac{1}{2})\approx 0.05132\ldots$$

4. Solve the Dirichlet problem

$$\begin{aligned}u_{xx}+u_{yy}&=0\quad\text{if } x^2+y^2<1,\\ u(x,y)&=xy^2\quad\text{if } x^2+y^2=1.\end{aligned}$$

Express the solution $u(x,y)$ in terms of x,y.

Solution: We use the terminating Fourier series

$$\cos\theta\sin^2\theta=\cos\theta-\cos^3\theta=\cos\theta-\left(\frac{3}{4}\cos\theta+\frac{1}{4}\cos(3\theta)\right)=\frac{1}{4}\cos\theta-\frac{1}{4}\cos(3\theta).$$

Then from Section 10.8

$$v(r,\theta)=u(r\cos\theta,r\sin\theta)=\frac{1}{4}r\cos\theta-\frac{1}{4}r^3\cos(3\theta),$$

or

$$v(r,\theta)=\frac{1}{4}r\cos\theta-\frac{1}{4}r^3(-3\cos\theta+4\cos^3\theta).$$

Then

$$u(x,y)=\frac{1}{4}x+\frac{3}{4}x(x^2+y^2)-x^3=\frac{1}{4}x-\frac{1}{4}x^3+\frac{3}{4}xy^2.$$

3.3 Quizz 3

3.3.1 Problem 1

Problem Find the eigenvalues and normalized eigenfunctions of the RSL problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) - y'(0) &= 0 \\ y(\pi) - y'(\pi) &= 0 \end{aligned} \tag{1}$$

solution
The characteristic equation for $y'' + \lambda y = 0$ is given by $r^2 + \lambda = 0$. Hence the roots are

$$r = \pm\sqrt{-\lambda}$$

There are 3 cases to consider.
case $\lambda = 0$ This implies that $r = 0$ is a double root. The solution becomes

$$\begin{aligned} y &= c_1 + c_2 x \\ y' &= c_2 \end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $c_1 - c_2 = 0$ or $c_1 = c_2$. The above solution now becomes

$$\begin{aligned} y &= c_1 (1 + x) \\ y' &= c_1 \end{aligned}$$

The second boundary conditions $y(\pi) - y'(\pi) = 0$ gives $c_1 (1 + \pi) - c_1 = 0$ or $\pi = 0$. Which is not possible. Therefore $\lambda = 0$ is not an eigenvalue.
case $\lambda < 0$ Let $\lambda = -\omega^2$ for some real ω. Hence the roots now are $r = \pm\sqrt{\omega^2} = \pm\omega$. Therefore the solution is

$$y = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Since the exponents are real, the solution can be written in terms of hyperbolic trigonometric functions as

$$\begin{aligned} y &= c_1 \cosh \omega x + c_2 \sinh \omega x \\ y' &= c_1 \omega \sinh \omega x + c_2 \omega \cosh \omega x \end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $0 = c_1 - c_2\omega$ or $c_1 = c_2\omega$. Therefore the above solution becomes

$$\begin{aligned} y &= c_2 \omega \cosh \omega x + c_2 \sinh \omega x \\ &= c_2 (\omega \cosh \omega x + \sinh \omega x) \end{aligned} \tag{2}$$

Hence

$$y' = c_2 \left(\omega^2 \sinh \omega x + \omega \cosh \omega x\right)$$

The second boundary conditions $y(\pi) - y'(\pi) = 0$ gives

$$\begin{aligned} 0 &= c_2 (\omega \cosh \omega\pi + \sinh \omega\pi) - c_2 \left(\omega^2 \sinh \omega\pi + \omega \cosh \omega\pi\right) \\ &= c_2 \left(\omega \cosh \omega\pi + \sinh \omega\pi - \omega^2 \sinh \omega\pi - \omega \cosh \omega\pi\right) \\ &= c_2 \left(\sinh \omega\pi - \omega^2 \sinh \omega\pi\right) \\ &= c_2 \left(1 - \omega^2\right) \sinh \omega\pi \end{aligned}$$

Non-trivial solution implies either $\left(1 - \omega^2\right) = 0$ or $\sinh \omega\pi = 0$. But $\sinh \omega\pi = 0$ only when its argument is zero. But $\omega \neq 0$ in this case. The other option is that $\left(1 - \omega^2\right) = 0$. This implies $\omega^2 = 1$ or, since $\lambda = -\omega^2$, that $\lambda = -1$. Hence $\lambda = -1$ is an eigenvalue. Therefore the solution from (2) above becomes

$$\begin{aligned} y(x) &= c_2 \cosh x + c_2 \sinh x \\ &= c_2 (\cosh x + \sinh x) \end{aligned}$$

But $e^x = \cosh x + \sinh x$, hence the solution can be written as

$$y = c_2 e^x$$

The eigenfunction in this case is therefore

$$\Phi_{-1}(x) = e^x$$

To obtain the normalized eigenfunction, let $\hat{\Phi}_{-1}(x) = k_{-1}\Phi_{-1}(x)$. The normalization factor k_{-1} is found by setting $\int_0^{\pi}\left(r(x)\hat{\Phi}_{-1}(x)\right)^2 dx = 1$. But the weight $r(x) = 1$ in this problem from looking at the Sturm Liouville form given. Therefore solving

$$\begin{aligned}\int_0^{\pi}\hat{\Phi}_{-1}^2(x)\,dx &= 1\\ \int_0^{\pi}(k_{-1}e^x)^2\,dx &= 1\\ k_{-1}^2\int_0^{\pi}e^{2x}dx &= 1\\ k_{-1}^2\left(\frac{e^{2x}}{2}\right)_0^{\pi} &= 1\\ \frac{k_{-1}^2}{2}\left(e^{2\pi}-1\right) &= 1\end{aligned}$$

Therefore

$$k_{-1} = \frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}$$

Hence the normalized eigenfunction is

$$\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right)e^x$$

case $\lambda > 1$ Since λ is positive, then the roots are $r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$. This gives the solution

$$y = c_1e^{i\sqrt{\lambda}x} + c_2e^{-i\sqrt{\lambda}x}$$

Since the exponents are complex, the above solution can be written in terms of the circular trigonometric functions as

$$\begin{aligned}y &= c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\sqrt{\lambda}x\\ y' &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x\end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $0 = c_1 - c_2\sqrt{\lambda}$ or $c_1 = c_2\sqrt{\lambda}$. The above solution becomes

$$\begin{aligned}y &= c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) + c_2\sin\sqrt{\lambda}x \qquad (3)\\ &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) + \sin\sqrt{\lambda}x\right)\end{aligned}$$

Therefore

$$y' = c_2\left(-\lambda\sin\left(\sqrt{\lambda}x\right) + \sqrt{\lambda}\cos\sqrt{\lambda}x\right)$$

Applying second boundary condition $y(\pi) - y'(\pi) = 0$ to the above gives

$$\begin{aligned}0 &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) + \sin\left(\sqrt{\lambda}\pi\right)\right) - c_2\left(-\lambda\sin\left(\sqrt{\lambda}\pi\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\right)\\ &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) + \sin\left(\sqrt{\lambda}\pi\right) + \lambda\sin\left(\sqrt{\lambda}\pi\right) - \sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\right)\\ &= c_2\left(\sin\left(\sqrt{\lambda}\pi\right) + \lambda\sin\left(\sqrt{\lambda}\pi\right)\right)\\ &= c(1+\lambda)\sin\left(\sqrt{\lambda}\pi\right)\end{aligned}$$

For non-trivial solution, either $1+\lambda = 0$ or $\sin\left(\sqrt{\lambda}\pi\right) = 0$. But $1+\lambda = 0$ implies $\lambda = -1$. But it is assumed that λ is positive. The other possibility is that $\sin\left(\sqrt{\lambda}\pi\right) = 0$ which implies

$$\sqrt{\lambda}\pi = n\pi \qquad n = 1, 2, 3, \cdots$$

Or

$$\lambda_n = n^2 \qquad 1, 2, 3, \cdots$$

The corresponding solution from (3) becomes

$$y_n(x) = c_n\left(n\cos(nx) + \sin(nx)\right)$$

Therefore the eigenfunctions are

$$\Phi_n(x) = n\cos(nx) + \sin(nx)$$

To obtain the normalized eigenfunctions, as was done above, $\int_0^\pi \left(r(x)\hat{\Phi}_n(x)\right)^2 dx = 1$ is solved for k_n giving

$$
\begin{aligned}
\int_0^\pi (k_n \Phi_n(x))^2\, dx &= 1 \\
k_n^2 \int_0^\pi (n\cos(nx) + \sin(nx))^2\, dx &= 1 \\
k_n^2 \int_0^\pi \left(n^2\cos^2(nx) + \sin^2(nx) + 2n\cos(nx)\sin(nx)\right) dx &= 1 \\
n^2 \int_0^\pi \cos^2(nx)\, dx + \int_0^\pi \sin^2(nx)\, dx + 2n \int_0^\pi \cos(nx)\sin(nx)\, dx &= \frac{1}{k_n^2} \qquad (4)
\end{aligned}
$$

But $\int_0^\pi \cos^2(nx)\, dx = \frac{\pi}{2}$ and $\int_0^\pi \sin^2(nx)\, dx = \frac{\pi}{2}$ and for the last integral above

$$
\begin{aligned}
\int_0^\pi \cos(nx)\sin(nx)\, dx &= \int_0^\pi \frac{1}{2}\sin(2nx)\, dx \\
&= \frac{1}{2}\left(\frac{-\cos(2nx)}{2n}\right)_0^\pi \\
&= \frac{-1}{4n}(\cos(2nx))_0^\pi \\
&= \frac{-1}{4n}(\cos(2n\pi) - 1)
\end{aligned}
$$

But $\cos(2n\pi) = 1$ because $n = 1, 2, 3, \cdots$. Therefore the above simplifies to $\int_0^\pi \cos(nx)\sin(nx)\, dx = 0$. Using these results in (4) gives

$$
k_n^2\left(n^2\frac{\pi}{2} + \frac{\pi}{2}\right) = 1
$$

Or

$$
k_n = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}
$$

The normalized eigenfunctions are therefore

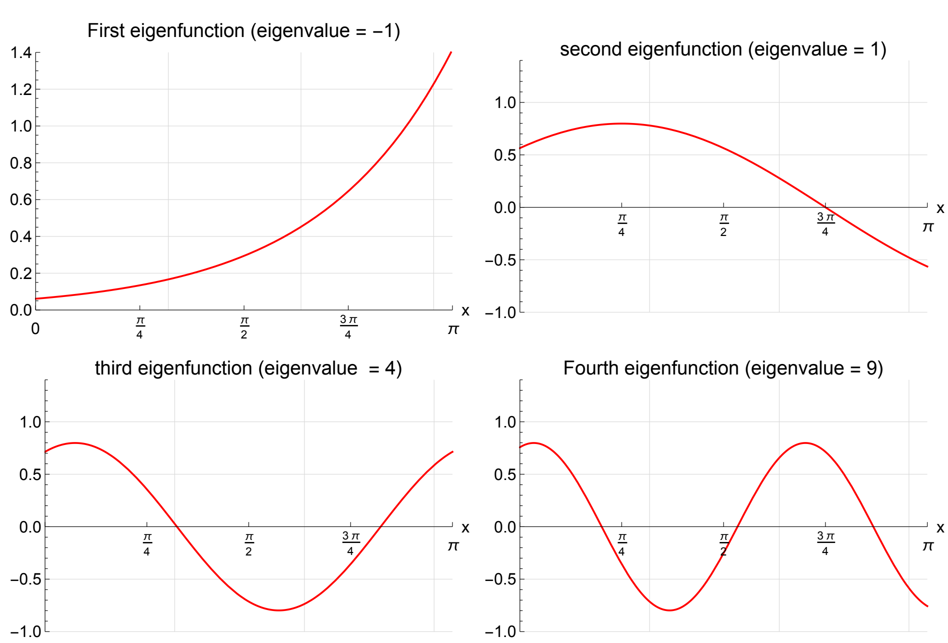
$$
\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx)) \qquad n = 1, 2, 3, \cdots
$$

In <u>summary</u>

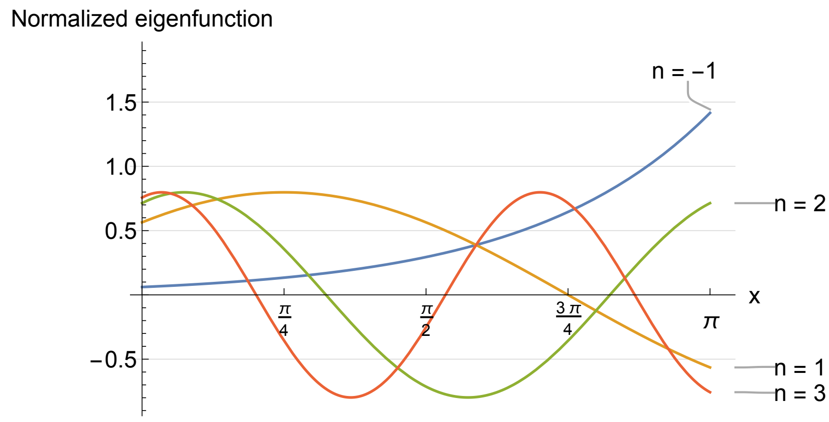
$\lambda = -1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x$

$\lambda_n = n^2$ for $n = 1, 2, \cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))$.

The normalized eigenfunctions $\hat{\Phi}_{-1}, \hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3$ are plotted next to each others below



The normalized eigenfunctions $\hat{\Phi}_{-1}, \hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3$ are plotted on the same plot below as well for illustration.



Some observations: The first eigenfunction $\hat{\Phi}_{-1}(x)$ has no root in $[0,\pi]$, the second eigenfunction $\hat{\Phi}_1$ has one root in $[0,\pi]$ and the third eigefunction has two roots in $[0,\pi]$ and so on. This is what is to be expected. The n^{th} ordered eigenfunction will have $(n-1)$ number of roots (or x axis crossings) inside the domain.

3.3.2 Problem 2

Problem Expand $f(x)=1$ in a series of eigenfunctions of problem 1
solution
Let

$$f(x)=b_{-1}\hat{\Phi}_{-1}(x)+\sum_{n=1}^{\infty}b_n\hat{\Phi}_n(x) \tag{1}$$

The goal is to determine $b_{-1},b_1,b_2,\cdots$. This is done by applying orthogonality. Multiplying both sides of (1) by $r(x)\hat{\Phi}_{-1}(x)$ and integrating over the domain gives

$$\int_0^{\pi}r(x)f(x)\hat{\Phi}_{-1}(x)\,dx=\int_0^{\pi}b_{-1}r(x)\hat{\Phi}_{-1}^2(x)\,dx+\sum_{n=1}^{\infty}b_n\int_0^{\pi}r(x)\hat{\Phi}_{-1}(x)\hat{\Phi}_n(x)\,dx$$

But $r(x)=1$ and due to orthogonality of eigenfunctions, all terms in the sum are zero. The above simplifies to

$$\int_0^{\pi}f(x)\hat{\Phi}_{-1}(x)\,dx=b_{-1}\int_0^{\pi}\hat{\Phi}_{-1}^2(x)\,dx$$

But $f(x)=1$ and $\int_0^{\pi}\hat{\Phi}_{-1}^2(x)\,dx=1$ since normalized eigenfunctions. Hence the above becomes

$$b_{-1}=\int_0^{\pi}\hat{\Phi}_{-1}(x)\,dx$$

From problem one, $\hat{\Phi}_{-1}(x)=\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right)e^x$, therefore the above becomes

$$\begin{aligned}b_{-1}&=\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\int_0^{\pi}e^x dx\\&=\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\left[e^x\right]_0^{\pi}\\&=\frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}\end{aligned}$$

Going back to equation (1), but now the equation is multiplied by $r(x)\hat{\Phi}_m(x)$ for $m>0$ and integrated using $r(x)=1$ and $f(x)=1$ giving

$$\int_0^{\pi}\hat{\Phi}_m(x)\,dx=\int_0^{\pi}b_{-1}\hat{\Phi}_{-1}(x)\hat{\Phi}_m(x)\,dx+\sum_{n=1}^{\infty}b_n\int_0^{\pi}\hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx$$

Due to orthogonality of eigenfunctions, the above simplifies to

$$\int_0^{\pi}\hat{\Phi}_m(x)\,dx=b_m\int_0^{\pi}\hat{\Phi}_m^2(x)\,dx$$

But $\int_0^{\pi}\hat{\Phi}_m^2(x)\,dx=1$, therefore the above becomes

$$b_n=\int_0^{\pi}\hat{\Phi}_n(x)\,dx$$

From problem one, using $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))$ the above becomes

$$
\begin{aligned}
b_n &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \int_0^{\pi} (n\cos(nx) + \sin(nx))\,dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(\int_0^{\pi} n\cos(nx)\,dx + \int_0^{\pi} \sin(nx)\,dx \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(n\left[\frac{\sin(nx)}{n}\right]_0^{\pi} - \left[\frac{\cos(nx)}{n}\right]_0^{\pi} \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(\sin(n\pi) - \frac{1}{n}\left[\cos(n\pi) - 1\right] \right)
\end{aligned}
$$

But $\sin(n\pi) = 0$ since n is integer and $\cos(n\pi) = (-1)^n$. The above becomes

$$
\begin{aligned}
b_n &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(-\frac{1}{n}\left[-1^n - 1\right] \right) \\
&= \frac{\sqrt{2}}{n\sqrt{\pi(1+n^2)}} \left((-1)^{n+1} + 1 \right)
\end{aligned}
$$

For $n = 1, 3, 5, \cdots$ the above simplifies to

$$
b_n = \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}}
$$

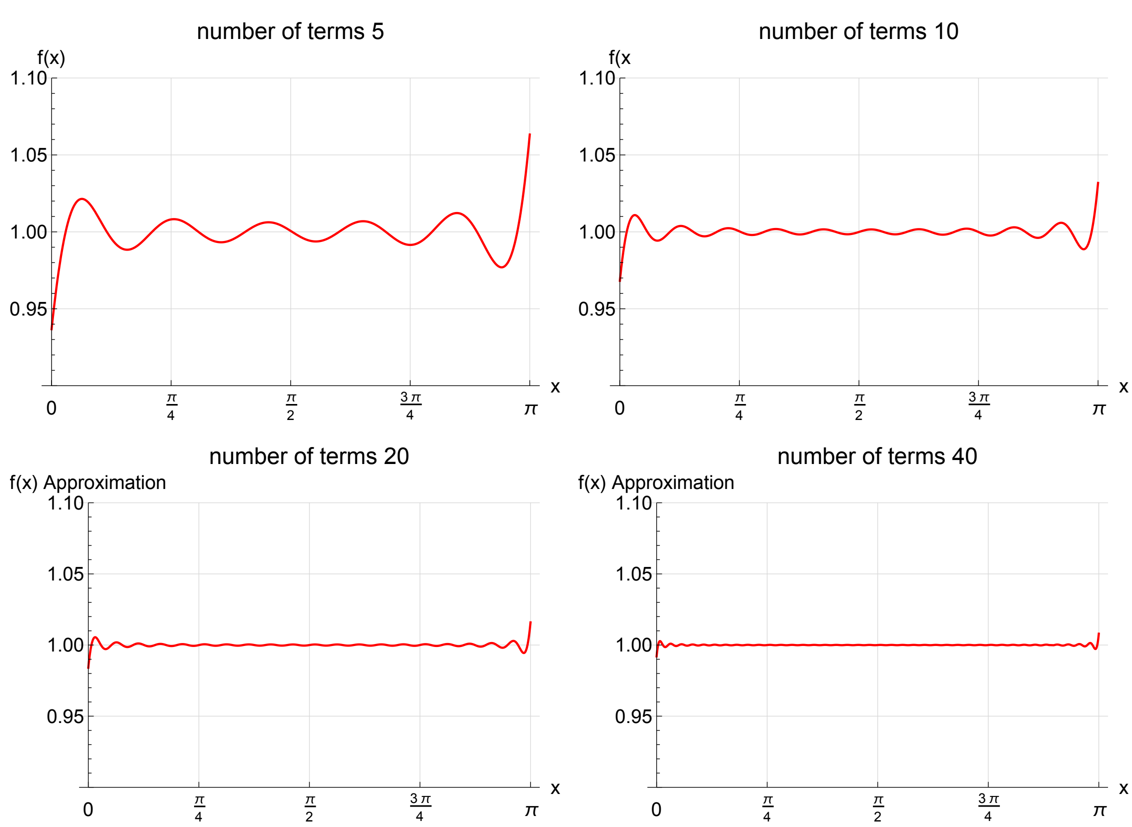
And for $n = 2, 4, 6, \cdots$ gives $b_n = 0$. Therefore the expansion (1) becomes

$$
\begin{aligned}
f(x) &= \frac{\sqrt{2}(e^{\pi} - 1)}{\sqrt{e^{2\pi} - 1}} \hat{\Phi}_{-1}(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}} \hat{\Phi}_n(x) \\
1 &= \frac{\sqrt{2}(e^{\pi} - 1)}{\sqrt{e^{2\pi} - 1}} \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}} \right) e^{x} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}} \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} (n\cos(nx) + \sin(nx)) \\
1 &= \frac{2(e^{\pi} - 1)}{e^{2\pi} - 1} e^{x} + \frac{4}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n(1+n^2)} (n\cos(nx) + \sin(nx))
\end{aligned}
$$

The above can also be written as

$$
1 = \frac{2(e^{\pi} - 1)}{e^{2\pi} - 1} e^{x} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\left(1 + (2n-1)^2\right)} \left((2n-1)\cos((2n-1)x) + \sin((2n-1)x) \right)
$$

To verify the above result, it is plotted for increasing number of n and compared to $f(x) = 1$ to see how well it converges.



Some observations: As more terms are added, the series approximation approaches $f(x) = 1$ more. The convergence is more rapid in the internal of the domain than near the edges. Near the edges at $x = 0$ and $x = 1$, more terms are needed to get better approximation. More oscillation is seen near the edges. This is due to Gibbs phenomenon. Converges is of the order of $O\left(\frac{1}{n^2}\right)$ and the converges is to the mean of $f(x)$.

3.3.3 Problem 3

Problem Consider the regular SL problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ 2y(1) - y'(1) &= 0 \end{aligned} \tag{1}$$

Show that the problem has exactly one negative eigenvalue and compute numerically.
solution
The characteristic equation is $r^2 + \lambda = 0$. Therefore the roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases to consider. This problem is asking only for the negative eigenvalues. Therefore only the case $\lambda < 0$ is considered. Let $\lambda = -\omega^2$ for some real constant. The roots are $r = \pm\sqrt{\omega^2} = \pm\omega$. The solution becomes

$$y = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Since the exponents are real, the solution can be written in terms of hyperbolic trigonometric functions

$$y = c_1 \cosh \omega x + c_2 \sinh \omega x$$

The first boundary conditions $y(0) = 0$ gives $0 = c_1$. The solution becomes

$$\begin{aligned} y &= c_2 \sinh \omega x \\ y' &= c_2 \omega \cosh \omega x \end{aligned} \tag{2}$$

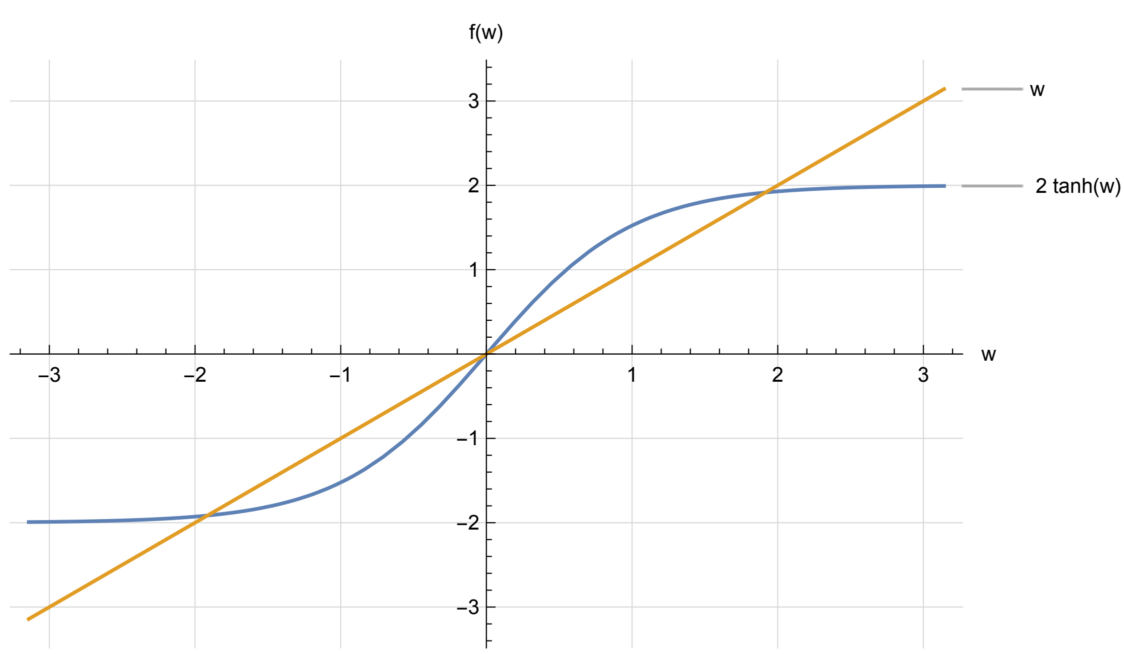
Applying the second boundary conditions $2y(1) - y'(\pi) = 0$ gives

$$\begin{aligned} 0 &= 2c_2 \sinh \omega - c_2 \omega \cosh \omega \\ &= c_2 (2 \sinh \omega - \omega \cosh \omega) \end{aligned}$$

Non trivial solution requires that

$$\begin{aligned} 2 \sinh \omega - \omega \cosh \omega &= 0 \\ 2 \tanh \omega &= \omega \end{aligned}$$

The above equation needs to be solved numerically to find its real roots ω. One root is $\omega = 0$, but this implies $\lambda = 0$. To find if there are other real roots, the function $2 \tanh \omega$ and ω were plotted and where they intersect is located. Root finding was then used to obtain the exact numerical value of the roots. The plot below shows that near $\omega = \pm 2$ there is an intersection. There are no other roots since the line $f(\omega) = \omega$ will keep increasing/decreasing and will not intersect $f(\omega) = 2 \tanh \omega$ any more after these two roots.



Numerical root finding was used to find the roots near points of intersections. It shows that the exact value of $\omega = \pm 1.91501$. Since $\lambda = -\omega^2$, therefore

$$\boxed{\lambda = -3.66726}$$

Is the only negative eigenvalue.

3.3.4 Problem 4

Problem Solve the inhomogeneous B.V.P.

$$\begin{aligned} -y'' &= \mu y + 1 \qquad (1)\\ y(0) - y'(0) &= 0 \\ y(\pi) - y'(\pi) &= 0 \end{aligned}$$

for $\mu = 0$, $\mu = 1$ by methods of section 11.3

Part (a)

$$\begin{aligned} -y'' - \mu y &= 1 \\ y'' + \mu y &= -1 \end{aligned}$$

Using chapter 11.3 method, first the eigenfunctions for the corresponding homogenous ODE $y'' + \mu y = 0$ are found for the same boundary conditions. In problem one, it was found that $\lambda = -1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x$ and $\lambda_n = n^2$ for $n = 1, 2, \cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))$. Since $\lambda = 0$ is not an eigenvalue of the corresponding homogeneous B.V.P., then there is a solution which is by eigenfunction expansion is given by

$$y = b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \qquad (1)$$

Substituting this back into the original ODE gives

$$\left(b_{-1}\hat{\Phi}''_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}''_n(x)\right) + \mu\left(b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x)\right) = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

Where $-1 = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$ is the eigenfunction expansion of -1. Since $\mu = 0$, and $\hat{\Phi}''_n(x) = -\lambda_n \hat{\Phi}_n(x)$, the above simplifies to

$$-\lambda_{-1} b_{-1}\hat{\Phi}_{-1}(x) - \sum_{n=1}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

Therefore, equating coefficients gives

$$\begin{aligned} -\lambda_{-1} b_{-1} &= c_{-1} \\ -b_n \lambda_n &= c_n \end{aligned}$$

Or

$$\begin{aligned} b_{-1} &= -\frac{c_{-1}}{\lambda_{-1}} \qquad (2)\\ b_n &= -\frac{c_n}{\lambda_n} \end{aligned}$$

What is left is to find c_{-1}, c_n. These are found by applying orthogonality since

$$-1 = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

This was done in problem 2. The difference is the minus sign. Therefore the result from problem 2 is used but c_{-1}, c_n from problem 2 are now multiplied by -1 giving

$$\begin{aligned} c_{-1} &= -\frac{\sqrt{2}(e^{\pi} - 1)}{\sqrt{e^{2\pi} - 1}} \\ c_n &= -\frac{2\sqrt{2}}{n\sqrt{\pi(1 + n^2)}} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

Now that c_{-1}, c_n are found, using equation (2) b_{-1}, b_n are can now be found

$$b_{-1} = \frac{\frac{\sqrt{2}(e^{\pi}-1)}{\sqrt{e^{2\pi}-1}}}{(-1)} = -\frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}$$

$$b_n = \frac{\frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}}}{n^2} = \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1+n^2\right)}} \qquad n = 1,3,5,\cdots$$

Hence the solution (1) becomes

$$\begin{aligned} y &= b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \\ &= -\frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}\hat{\Phi}_{-1}(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1+n^2\right)}}\hat{\Phi}_n(x) \\ &= -\frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right)e^x + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1+n^2\right)}}\frac{\sqrt{2}}{\sqrt{\pi\left(1+n^2\right)}}\left(n\cos(nx)+\sin(nx)\right) \\ &= -\frac{2\left(e^{\pi}-1\right)}{e^{2\pi}-1}e^x + \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3\left(1+n^2\right)}\left(n\cos(nx)+\sin(nx)\right) \end{aligned} \tag{2A}$$

The above can also be also be written as

$$y(x) = -\frac{2\left(e^{\pi}-1\right)}{e^{2\pi}-1}e^x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3\left(1+(2n-1)^2\right)}\left((2n-1)\cos((2n-1)x)+\sin((2n-1)x)\right) \tag{2A}$$

To verify the above solution, it was plotted against the solution of $y'' = -1$ found using the direct method to see if they match. The solution using the direct method is found as follows: The homogenous solution is $y_h = c_1 + c_2 x$. Let $y_p = kx^2, y_p' = 2kx, y_p'' = 2k$. Substituting these back into $y'' = -1$ gives $2k = -1$ or $k = -\frac{1}{2}$. Hence $y_p = -\frac{x^2}{2}$ and the solution becomes

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 + c_2 x - \frac{x^2}{2} \end{aligned}$$

Boundary conditions are now applied to determine c_1, c_2. From above, $y'(x) = c_2 - x$. Applying $y(0) - y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_1 - c_2 \\ c_2 &= c_1 \end{aligned}$$

Therefore the solution becomes

$$\begin{aligned} y(x) &= c_1(1+x) - \frac{x^2}{2} \\ y'(x) &= c_1 - x \end{aligned}$$

Applying second BC $y(\pi) - y'(\pi) = 0$ gives

$$\begin{aligned} 0 &= c_1(1+\pi) - \frac{\pi^2}{2} - c_1 + \pi \\ 0 &= c_1(1+\pi-1) - \frac{\pi^2}{2} + \pi \\ c_1 &= \frac{\frac{\pi^2}{2} - \pi}{\pi} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

Therefore, the solution, using direct method is

$$\begin{aligned} y(x) &= \left(\frac{\pi}{2} - 1\right)(1+x) - \frac{x^2}{2} \\ &= \frac{\pi}{2} + \frac{\pi}{2}x - 1 - x - \frac{x^2}{2} \end{aligned}$$

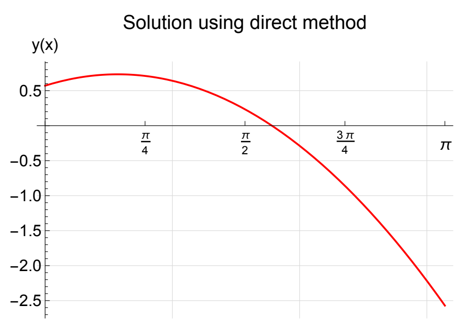
Or

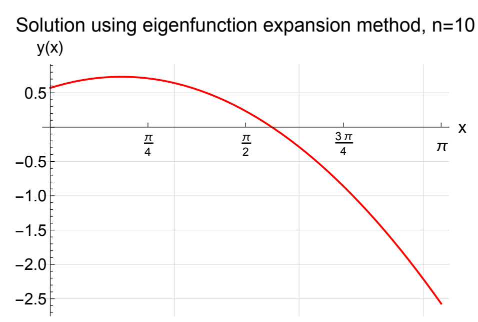
$$\boxed{y(x) = -\frac{x^2}{2} + x\left(\frac{\pi}{2} - 1\right) - 1 + \frac{\pi}{2}} \tag{3}$$

What the above says, is that if (2A) solution is correct, it will converge to solution (3) as more terms are added. In other words

$$-\frac{x^2}{2}+x\left(\frac{\pi}{2}-1\right)-1+\frac{\pi}{2}\approx-\frac{2\left(e^{\pi}-1\right)}{e^{2\pi}-1}e^{x}+\frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3\left(1+n^2\right)}\left(n\cos\left(nx\right)+\sin\left(nx\right)\right)$$

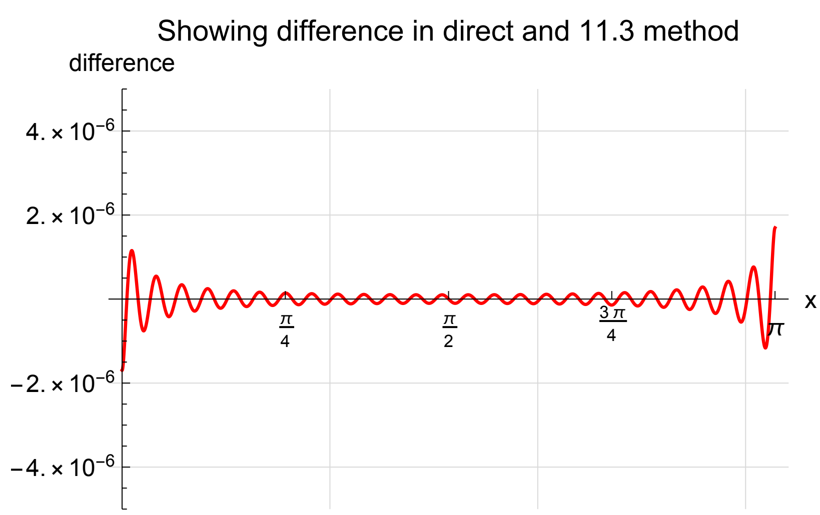
To verify this, the solution from both the direct and the series method were plotted next to each other. Using only $n=10$ in the sum shows that the plots are identical.





Then the difference between these two solution was plotted. A maximum of $n=50$ is used in the sum. The plot shows the difference is almost zero in the internal region and near the edges of the domain the difference of order 10^{-7}. This is expected due to Gibbs phenomenon. Adding more terms made the difference smaller. The converges is of order $O\left(\frac{1}{n^2}\right)$.

```
mySol[max_, x_] := -(2 (Exp[Pi] - 1))/(Exp[2 Pi] - 1) Exp[x] + 4/Pi Sum[1/(n^3 (1 + n^2)) (n Cos[n x] + Sin[n x]), {n, 1, max, 2}]
direct[x_] := -x^2/2 + x (Pi/2 - 1) - 1 + Pi/2;
```



Part (b)

Now the same process as in part (a) is repeated for $\mu=1$

$$\begin{aligned}-y''-\mu y&=1\\ y''+\mu y&=-1\end{aligned}$$

Using 11.3 method, first the eigenfunctions for the corresponding homogenous ODE $y''+\mu y=0$ are found for the same boundary conditions. In problem one, it was found that $\lambda=-1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x)=\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right)e^{x}$ and $\lambda_n=n^2$ for $n=1,2,\cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x)=\frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}\left(n\cos\left(nx\right)+\sin\left(nx\right)\right)$. Therefore $\lambda=1$ is an eigenvalue that corresponds to $\mu=1$. In this case, a solution will exist (and will not be unique) only if the forcing function -1 is orthogonal to $\hat{\Phi}_1(x)$. This is verified as follows. Since $r(x) = 1$, and $n = 1$, then

$$\begin{aligned}
\int_0^{\pi} (-1)\, r(x)\, \hat{\Phi}_1(x)\, dx &= -\int_0^{\pi} \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} (n\cos(nx) + \sin(nx))\, dx \\
&= -\int_0^{\pi} \frac{\sqrt{2}}{\sqrt{\pi(1+1)}} (\cos(x) + \sin(x))\, dx \\
&= \frac{-\sqrt{2}}{\sqrt{2\pi}} \int_0^{\pi} \cos(x) + \sin(x)\, dx \\
&= \frac{-1}{\sqrt{\pi}} \left((\sin x)_0^{\pi} - (\cos x)_0^{\pi} \right) \\
&= \frac{-1}{\sqrt{\pi}} (0 - (-1 - 1)) \\
&= \frac{-2}{\sqrt{\pi}}
\end{aligned}$$

Which is not zero. This means there is no solution.

3.3.5 key solution

Homework 3, Math 322

1. Find the eigenvalues and normalized eigenfunctions of the regular Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0,\ y(\pi) - y'(\pi) = 0.$$

Solution: Let $\lambda = -\omega^2$ with $\omega > 0$, and $y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$. Then the boundary conditions give

$$(c_1 + c_2) - \omega(c_1 - c_2) = 0, \quad (c_1 e^{\omega\pi} + c_2 e^{-\omega\pi}) - (c_1 \omega e^{\omega\pi} - c_2 \omega e^{-\omega\pi}) = 0.$$

In order to get a nontrivial solution c_1, c_2 we need

$$\begin{vmatrix} 1-\omega & 1+\omega \\ (1-\omega)e^{\omega\pi} & (1+\omega)e^{-\omega\pi} \end{vmatrix} = (1-\omega^2)(e^{-\omega\pi} - e^{\omega\pi}) = 0.$$

The only solution $\omega > 0$ is $\omega = 1$. Then $c_2 = 0$. Therefore, $\lambda = -1$ is an eigenvalue and $\phi_0(x) = k_0 e^x$ is a corresponding eigenfunction.
If $\lambda = 0$ then $y(x) = c_1 + c_2 x$. The boundary conditions give

$$c_1 - c_2 = 0, \quad c_1 + c_2\pi - c_2 = 0.$$

It follows that $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.
Let $\lambda = \omega^2$, $\omega > 0$, and $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$. The boundary conditions give

$$c_1 - \omega c_2 = 0, \quad (c_1 \cos \omega\pi + c_2 \sin \omega\pi) - (-c_1\omega \sin \omega\pi + c_2\omega \cos \omega\pi) = 0.$$

In order to get a nontrivial solution we need

$$\begin{vmatrix} 1 & -\omega \\ \cos\omega\pi + \omega \sin\omega\pi & \sin\omega\pi - \omega\cos\omega\pi \end{vmatrix} = (1+\omega^2)\sin\omega\pi.$$

The solutions $\omega > 0$ are $\omega = n = 1, 2, \ldots$. Then $c_1 = nc_2$. Therefore, we found the eigenvalues $\lambda_n = n^2$ with corresponding eigenfunctions $\phi_n(x) = k_n(n \cos nx + \sin nx)$. We calculate

$$1 = k_0^2 \int_0^\pi (e^x)^2\,dx = k_0^2 \frac{1}{2}(e^{2\pi} - 1),$$

$$1 = k_n^2 \int_0^\pi (n\cos nx + \sin nx)^2\,dx = k_n^2(1+n^2)\frac{\pi}{2},$$

and find the normalized eigenfunctions

$$\hat\phi_0(x) = \frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}}\, e^x,$$

$$\hat\phi_n(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1+n^2}}(n\cos nx + \sin nx), \quad n = 1, 2, \ldots$$

2. Expand the function $f(x) = 1$ in a series of eigenfunctions of problem 1.
Solution: For general $f(x)$ the expansion is

$$f(x) = \sum_{n=0}^{\infty} c_n \hat\phi_n(x),$$

where

$$c_n = \int_0^\pi f(t)\hat{\phi}_n t)\,dt.$$

If $f(x) = 1$ then

$$c_0 = k_0 \int_0^\pi e^t\,dt = k_0(e^\pi - 1),$$
$$c_n = k_n \int_0^\pi (n\cos nt + \sin nt)\,dt = k_n \frac{1+(-1)^{n+1}}{n}.$$

Therefore,

$$1 = \frac{2}{e^\pi + 1}e^x + \frac{4}{\pi}\sum_{n\geq 1 \text{ odd}} \frac{1}{n(1+n^2)}(n\cos nx + \sin nx).$$

3. Consider the regular Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad 2y(1) - y'(1) = 0.$$

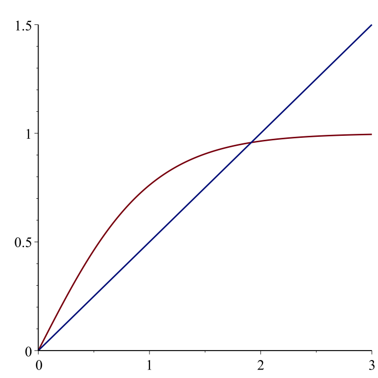
Show that this problem has exactly one negative eigenvalue and compute it numerically.
Solution: We set $\lambda = -\omega^2$ with $\omega > 0$. The condition $y(0) = 0$ gives $y(x) = c\sinh\omega x$. The boundary condition at $x = 1$ shows that λ is an eigenvalue if and only if

$$2\sinh\omega = \omega\cosh\omega$$

or

$$\tanh\omega = \frac{1}{2}\omega.$$

The function $\tanh\omega$ is concave for $\omega > 0$ so it is clear from the picture that there is exactly one positive solution $\omega = 1.9150\ldots$ The negative eigenvalue is $\lambda = -3.66725\ldots$



4. Solve the inhomogeneous boundary value problem

$$-y'' = \mu y + 1, \quad y(0) - y'(0) = 0,\ y(\pi) - y'(\pi) = 0$$

for $\mu = 0$ and $\mu = 1$ by the method of Section 11.3.
Solution: If μ is not an eigenvalue, then the solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n - \mu} \hat{\phi}_n(x).$$

Therefore, if $\mu = 0$ the solution is

$$y(x) = -\frac{2}{e^{\pi}+1} e^x + \frac{4}{\pi} \sum_{n \geq 1 \text{ odd}} \frac{1}{n^3(1+n^2)} (n \cos nx + \sin nx).$$

$\mu = 1$ agrees with the eigenvalue λ_1. There exists a solution only if $c_1 = 0$. But in our example, $c_1 \neq 0$ so there is no solution.

3.4 Quizz 4

3.4.1 Problem 1

Problem Solve the PDE

$$u_t = u_{xx} + xt \qquad 0 \leq x \leq 1, t \geq 0 \tag{1}$$

With boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

And initial condition

$$u(x,0) = \sin(\pi x)$$

Solution
The corresponding homogeneous PDE $u_t = u_{xx}$ with the same homogeneous boundary conditions was solved before. It was found to have eigenfunctions

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$$

With corresponding eigenvalues

$$\lambda_n = n^2\pi^2 \qquad n = 1, 2, 3, \cdots$$

Using eigenfunction expansion, it is now assumed that the solution to the given inhomogeneous PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Substituting the above into the original PDE (1), and since term by term differentiation is justified (eigenfunctions are continuous) results in

$$\sum_{n=1}^{\infty} b_n'(t)\Phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Phi_n''(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x) \tag{1A}$$

Where $\sum_{n=1}^{\infty}\gamma_n(t)\Phi_n(x)$ is the expansion of the forcing function xt using same eigenfunctions

$$xt = \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x) \tag{1B}$$

But $\Phi_n''(x) = -\lambda_n\Phi_n(x)$ since the eigenfunctions satisfy the eigenvalue ODE $X'' = -\lambda_n X$. Therefore (1A) simplifies to

$$\sum_{n=1}^{\infty} b_n'(t)\Phi_n(x) = \sum_{n=1}^{\infty} -\lambda_n b_n(t)\Phi_n(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x)$$
$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t) \tag{2}$$

$\gamma_n(t)$ is now found by applying orthogonality to (1B), and using the weight $r(x) = 1$ gives

$$t\int_0^1 x\Phi_n(x)\,dx = \gamma_n(t)\int_0^1 \Phi_n^2(x)\,dx$$

Using $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right) = \sin(n\pi x)$ and $\int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}$, the above simplifies to

$$t\int_0^1 x\sin(n\pi x)\,dx = \gamma_n(t)\frac{1}{2}$$
$$\gamma_n(t) = 2t\int_0^1 x\sin(n\pi x)\,dx \tag{3}$$

The integral on the right side above is found using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$, therefore

$$\begin{aligned}
\int_0^1 x\sin(n\pi x)\,dx &= \left(\frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi}\right)_0^1 \\
&= \left(\frac{\sin n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n\pi}\right) \\
&= -\frac{\cos n\pi}{n\pi} \\
&= \frac{-(-1)^n}{n\pi} \\
&= \frac{(-1)^{n+1}}{n\pi}
\end{aligned}$$

Hence equation (3) now can be written as

$$\gamma_n(t) = \frac{2(-1)^{n+1}}{n\pi} t$$

Substituting the above in (2) gives the first order ODE to solve for $b_n(t)$

$$b_n'(t) + (n\pi)^2 b_n(t) = \frac{2(-1)^{n+1}}{n\pi} t$$

The integrating factor is $I = e^{n^2\pi^2 t}$. Hence the above becomes, after multiplying both sides by I

$$\frac{d}{dt}\left(e^{n^2\pi^2 t} b_n(t)\right) = \frac{2(-1)^{n+1}}{n\pi} t e^{n^2\pi^2 t}$$

Integrating both sides gives

$$e^{n^2\pi^2 t} b_n(t) = \frac{2(-1)^{n+1}}{n\pi}\int_0^t s e^{n^2\pi^2 s} ds + b_n(0) \tag{4}$$

Where $b_n(0)$ is the constant of integration. Dividing both sides by $e^{n^2\pi^2 t}$ gives

$$b_n(t) = \frac{2(-1)^{n+1}}{n\pi}\int_0^t s e^{n^2\pi^2(s-t)} ds + b_n(0) e^{-n^2\pi^2 t}$$

But $\int_0^t s e^{n^2\pi^2(s-t)} ds = \frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^4\pi^4}$ by integration by parts. The above now becomes

$$b_n(t) = 2(-1)^{n+1}\left(\frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^5\pi^5}\right) + b_n(0) e^{-n^2\pi^2 t}$$

Now that $b_n(t)$ is found, the final solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\Phi_n(x) \\ &= \sum_{n=1}^{\infty}\left(2(-1)^{n+1}\left(\frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^5\pi^5}\right) + b_n(0) e^{-n^2\pi^2 t}\right)\sin(n\pi x) \end{aligned} \tag{5}$$

$b_n(0)$ is determined from the given initial conditions $u(x,0) = \sin \pi x$. The above becomes at $t = 0$

$$\begin{aligned} \sin \pi x &= \sum_{n=1}^{\infty}\left(2(-1)^{n+1}\left(\frac{-1+1}{n^5\pi^5}\right) + b_n(0)\right)\sin(n\pi x) \\ &= \sum_{n=1}^{\infty} b_n(0)\sin(n\pi x) \end{aligned}$$

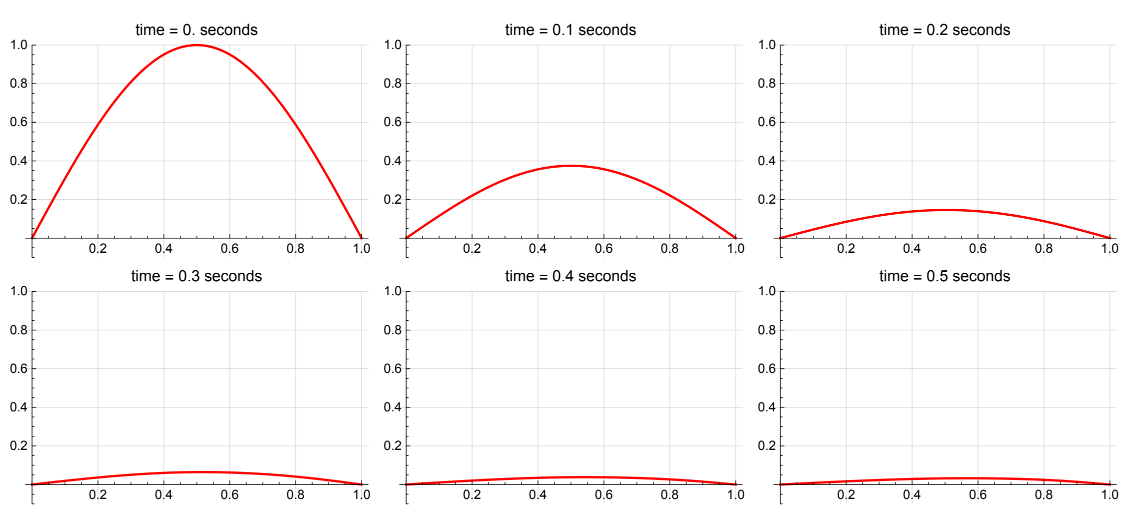
Therefore when $n = 1$ (since LHS is $\sin \pi x$) the above gives

$$b_1(0) = 1$$

And $b_n(0) = 0$ for all other n. Equation (5) now simplifies to

$$u(x,t) = \overbrace{\left(2\left(\frac{\pi^2 t - 1 + e^{-\pi^2 t}}{\pi^5}\right) + e^{-\pi^2 t}\right)\sin(\pi x)}^{n=1 \text{ term}} + \frac{1}{\pi^5}\sum_{n=2}^{\infty}\frac{2}{n^5}(-1)^{n+1}\left(n^2\pi^2 t + e^{-n^2\pi^2 t} - 1\right)\sin(n\pi x)$$

To verify the above solution, it was plotted against numerical solution for different instances of time and also animated. It gave an exact match. A small number of terms was needed in the summation since convergence was fast and is of order $O\left(\frac{1}{n^3}\right)$. The following is a plot of the above solution for different instances of times using 5 terms.



3.4.2 Problem 2

Problem Show that

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx=\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right)-\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

Hint: Use the same method that proves orthogonality of eigenfunctions in 11.4
Solution
In the above, λ and μ are the eigenvalues, with the corresponding eigenfunctions

$$\Phi_\lambda(x)=J_o\left(\sqrt{\lambda}x\right)\tag{1}$$
$$\Phi_\mu(x)=J_o\left(\sqrt{\mu}x\right)\tag{2}$$

These come from the Sturm Liouville equation

$$-(xy')'=\lambda xy\tag{3}$$

Where

$$\begin{aligned}p(x)&=x\\q(x)&=0\\r(x)&=x\end{aligned}$$

In operator form

$$L[\Phi_\lambda]=-\left(\Phi_\lambda'\right)'=\lambda x\Phi_\lambda\tag{4}$$

Similarly for any other eigenvalue such as μ. Multiplying both sides of (4) by $\Phi_\mu(x)$ and integrating gives

$$\int_0^1 L[\Phi_\lambda]\Phi_\mu dx=\int_0^1\overbrace{-\left(\Phi_\lambda'\right)'}^{dv}\overbrace{\Phi_\mu}^{u}\,dx$$

Integrating by part the right side results in

$$\int_0^1 L[\Phi_\lambda]\Phi_\mu dx=\left[-\Phi_\lambda'\Phi_\mu\right]_0^1-\int_0^1-\Phi_\lambda'\Phi_\mu'dx$$

Integrating by parts again the second integral above, where now $dv=-\Phi_\lambda',u=\Phi_\mu'$ gives

$$\begin{aligned}\int_0^1 L[\Phi_\lambda]\Phi_\mu dx&=\left[-\Phi_\lambda'\Phi_\mu\right]_0^1-\left(\left[-\Phi_\lambda\Phi_\mu'\right]_0^1-\int_0^1-\Phi_\lambda\Phi_\mu''dx\right)\\&=\left[-\Phi_\lambda'\Phi_\mu\right]_0^1-\left[-\Phi_\lambda\Phi_\mu'\right]_0^1+\int_0^1-\Phi_\lambda\Phi_\mu''dx\\&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1+\int_0^1\Phi_\lambda\left(-\Phi_\mu'\right)'dx\end{aligned}$$

But $\left(-\Phi_\mu'\right)'=L\left[\Phi_\mu\right]$. Hence the above can be written as

$$\begin{aligned}\int_0^1 L[\Phi_\lambda]\Phi_\mu dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1+\int_0^1 L\left[\Phi_\mu\right]\Phi_\lambda dx\\\int_0^1 L[\Phi_\lambda]\Phi_\mu dx-\int_0^1 L\left[\Phi_\mu\right]\Phi_\lambda dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1\\\int_0^1\left(L[\Phi_\lambda]\Phi_\mu-L\left[\Phi_\mu\right]\Phi_\lambda\right)dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1\end{aligned}$$

But $L[\Phi_\lambda]=\lambda x\Phi_\lambda$ and $L\left[\Phi_\mu\right]=\mu x\Phi_\mu$, therefore the above can be written as

$$\begin{aligned}\int_0^1\left(\lambda x\Phi_\lambda\Phi_\mu-\mu x\Phi_\mu\Phi_\lambda\right)dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1\\\int_0^1(\lambda-\mu)\left(x\Phi_\lambda\Phi_\mu\right)dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1\\(\lambda-\mu)\int_0^1 x\Phi_\lambda\Phi_\mu dx&=\left[-\Phi_\lambda'\Phi_\mu+\Phi_\lambda\Phi_\mu'\right]_0^1\end{aligned}\tag{5}$$

Since $\Phi_\lambda(x)=J_o\left(\sqrt{\lambda}x\right),\Phi_\lambda'(x)=\sqrt{\lambda}J_o'\left(\sqrt{\lambda}x\right)$ and $\Phi_\mu(x)=J_o\left(\sqrt{\mu}x\right),\Phi_\mu'(x)=\sqrt{\mu}J_o'\left(\sqrt{\mu}x\right)$, then the above simplifies to

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx=\left[-\sqrt{\lambda}J_o'\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)+J_o\left(\sqrt{\lambda}x\right)\sqrt{\mu}J_o'\left(\sqrt{\mu}x\right)\right]_0^1$$

What is left is to evaluate the boundary terms $\Delta = \left[-\sqrt{\lambda}J_o'\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right) + J_o\left(\sqrt{\lambda}x\right)\sqrt{\mu}J_o'\left(\sqrt{\mu}x\right)\right]_0^1$. This gives

$$\Delta = \left[-\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right) + J_o\left(\sqrt{\lambda}\right)\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)\right] - \left[-\sqrt{\lambda}J_o'(0)J_o(0) + J_o(0)\sqrt{\mu}J_o'(0)\right]$$

But $J_o'(0) = 0$ (since $J_o'(x) = -J_1(x)$ and $J_1(0) = 0$). Therefore the boundary terms reduces to

$$\Delta = J_o\left(\sqrt{\lambda}\right)\sqrt{\mu}J_o'\left(\sqrt{\mu}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

Substituting this back in (5) gives the desired result

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

3.4.3 Problem 3

Problem By letting $\mu \to \lambda$ in the formula of problem 2, derive a formula for $\int_0^1 xJ_0^2\left(\sqrt{\lambda}x\right)dx$. Then show that the normalized eigenfunctions of the eigenvalue problem in section 11.4 is

$$\hat{\Phi}_n(x) = \frac{\sqrt{2}J_0(j_n x)}{\left|J_0'(j_n)\right|}$$

where $0 < j_1 < j_2 < j_3 < \cdots$ denote the positive zeros of J_0

Solution

Part (a)

From problem 3, the formula obtained is

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

Moving $(\lambda-\mu)$ to the right side gives

$$\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \frac{\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)}{(\lambda-\mu)}$$

Taking the limit $\lim \mu \to \lambda$ then the integral on the left becomes $\int_0^1 x\Phi_\lambda^2 dx$ resulting in

$$\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx = \lim_{\mu\to\lambda}\frac{\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)}{(\lambda-\mu)} \tag{1}$$

When $\mu \to \lambda$ the right side becomes indeterminate form $\frac{0}{0}$. Therefore L'hospital rule is used, which says that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Comparing the above to (1) shows that μ is now like x and λ is like a. Therefore $f'(x)$ is like

$$\begin{aligned}
f'(x) &\equiv \frac{d}{d\mu}\left(\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)\right)\\
&\equiv \frac{d}{d\mu}\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{d}{d\mu}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)\\
&\equiv \frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \sqrt{\mu}\frac{1}{2\sqrt{\mu}}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{2\sqrt{\mu}}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)
\end{aligned}$$

And $g'(x)$ is like $\frac{d}{d\mu}(\lambda-\mu) = -1$. Using the above result back in (1) gives

$$\begin{aligned}
\int_0^1 xJ_o^2\left(\sqrt{\lambda}\right)dx &\equiv \lim_{x\to a}\frac{f'(x)}{g'(x)}\\
&= \lim_{\mu\to\lambda}\left(-\frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\mu}\frac{1}{2\sqrt{\mu}}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \frac{1}{2\sqrt{\mu}}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)\right)\\
&= \lim_{\mu\to\lambda}\left(-\frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{2}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \frac{1}{2}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)\right)
\end{aligned}$$

Now the limit is taken, since there is no indeterminate form. The above becomes

$$\begin{aligned}\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx &= -\frac{1}{2}\frac{1}{\sqrt{\lambda}}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)-\frac{1}{2}J_o''\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)+\frac{1}{2}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\lambda}\right)\\ &= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2-\frac{1}{\sqrt{\lambda}}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)-J_o''\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)\right) \qquad (2)\end{aligned}$$

To simplify the above, the following relations were obtained from dlmf.NIST.gov to simplify the above

$$\begin{aligned}J_n'(x) &= J_{n-1}(x)-\frac{(n+1)}{x}J_n(x)\\ J_n'(x) &= -J_{n+1}(x)+\frac{n}{x}J_n(x)\end{aligned}$$

Using these, then $J_o'\left(\sqrt{\lambda}\right) = -J_1\left(\sqrt{\lambda}\right)$ and $J_0''\left(\sqrt{\lambda}\right) = -J_0\left(\sqrt{\lambda}\right)+\frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)$. Equation (2) now simplifies to

$$\begin{aligned}\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx &= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2-\frac{1}{\sqrt{\lambda}}\left(-J_1\left(\sqrt{\lambda}\right)\right)J_o\left(\sqrt{\lambda}\right)-\left(-J_0\left(\sqrt{\lambda}\right)+\frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)\right)J_o\left(\sqrt{\lambda}\right)\right)\\ &= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2+\frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)+J_0\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)-\frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)\right)\end{aligned}$$

The second term cancels with the last term above giving the final result

$$\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx = \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2+J_0^2\left(\sqrt{\lambda}\right)\right) \qquad (3)$$

Part (b)

$\sqrt{\lambda_n}$ are the positive zeros of $J_0\left(\sqrt{\lambda_n}\right) = 0$. Below, $\sqrt{\lambda_n}$ is replaced by j_n where now j_n are the zeros of $J_0(j_n)$. One way to find the normalized eigenfunction $\hat{J}_0(j_nx)$ is by dividing $J_0(j_nx)$ by its norm. In other words,

$$\hat{J}_0(j_nx) = \frac{J_0(j_nx)}{\|J_0(j_nx)\|} \qquad (1A)$$

But

$$\|J_0(j_nx)\| = \sqrt{\int_0^1 r(x)J_0^2(j_nx)\,dx}$$

Which is by the definition of the norm of a function with the corresponding weight $r(x)$. But from part(a) $\|J_0(j_nx)\| = \int_0^1 r(x)J_0^2(j_nx)\,dx$ was found to be $\frac{1}{2}\left(\left[J_o'(j_n)\right]^2+J_0^2(j_n)\right)$. Therefore (1A) becomes

$$\begin{aligned}\hat{J}_0(j_nx) &= \frac{J_0(j_nx)}{\sqrt{\frac{1}{2}\left([J_o'(j_n)]^2+J_0^2(j_n)\right)}}\\ &= \frac{\sqrt{2}J_0(j_nx)}{\sqrt{[J_o'(j_n)]^2+J_0^2(j_n)}}\end{aligned}$$

But since j_n are the zeros of $J_0(j_n)$, then all the $J_0(j_n)$ terms above vanish giving

$$\begin{aligned}\hat{J}_0(j_nx) &= \frac{\sqrt{2}J_0(j_nx)}{\sqrt{[J_o'(j_n)]^2}}\\ &= \frac{\sqrt{2}J_0(j_nx)}{|J_o'(j_n)|} \qquad (1)\end{aligned}$$

Another way to find the normalized eigenfunctions $\hat{J}_0(j_nx)$ is as was done in the text book, which is to first determine k_n as follows. Let $\hat{J}_0(j_nx) = k_nJ_0(j_nx)$, then the following equation is solved for k_n

$$\int_0^1 r(x)\left[\hat{J}_0(j_nx)\right]^2 dx = 1 \qquad (2)$$

But the weight $r(x) = x$, equation (2) becomes

$$k_n^2\int_0^1 xJ_0^2(j_nx)\,dx = 1$$

But from part(a), $\int_0^1 xJ_0^2(j_nx)\,dx = \frac{1}{2}\left(\left[J_o'(j_n)\right]^2+J_0^2(j_n)\right)$. Hence the above becomes

$$\begin{aligned}k_n^2 &= \frac{1}{\frac{1}{2}\left([J_o'(j_n)]^2+J_0^2(j_n)\right)}\\ k_n &= \frac{\sqrt{2}}{\sqrt{[J_o'(j_n)]^2+J_0^2(j_n)}}\end{aligned}$$

As above, since all $J_0(j_n) = 0$ then

$$k_n = \frac{\sqrt{2}}{\sqrt{[J_o'(j_n)]^2}}$$

And the normalized eigenfunction become

$$\begin{aligned}\hat{J}_0(j_n x) &= k_n J_0(j_n x)\\ &= \frac{\sqrt{2} J_0(j_n x)}{\sqrt{[J_o'(j_n)]^2}}\\ &= \frac{\sqrt{2} J_0(j_n x)}{|J_o'(j_n)|}\end{aligned}$$

Which is the same result as (1).

3.4.4 Problem 4

Problem Solve the inhomogeneous differential equation

$$-\left(\left(1-x^2\right) y'\right)' = y + x^3 \qquad -1 < x < 1$$

With boundary conditions $y(x), y'(x)$ bounded as $x \longrightarrow -1^+$ and $x \longrightarrow 1^-$.
Solution
This problem is solved using 11.3 method (Eigenfunction expansion). The ODE is written as

$$-\left(\left(1-x^2\right) y'\right)' = \mu y + x^3 \tag{1}$$

Where $\mu = 1$ in this case. The corresponding homogeneous eigenvalue ODE to solve is then

$$-\left(\left(1-x^2\right) y'\right)' = \lambda y \tag{2}$$

Comparing to Sturm-Liouville form $-(py')' + qy = r\lambda y$, then $p(x) = \left(1-x^2\right), q = 0, r = 1$. Since $p(x)$ must be positive over all points in the domain, and since in this problem $p(-1) = 0$ and $p(1) = 0$, then both $x = -1, +1$ are singular points. They can be shown to be regular singular points.
Equation (2), where λ is now is an eigenvalue, is the Legendre equation

$$\left(1-x^2\right) y'' - 2xy' + \lambda y = 0$$

Comparing to the standard Legendre equation form in chapter 5

$$\left(1-x^2\right) y'' - 2xy' + n(n+1) y = 0 \tag{3}$$

There are two cases to consider. n is integer and n is not an integer.
Case n is not an integer. It is know that now the solution to (3) is

$$y(x) = c_1 \bar{P}_n(x) + c_2 \bar{Q}_n(x)$$

Where $\bar{P}_n(x)$ is called the Legendre function of order n and $\bar{Q}_n(x)$ is called the Legendre function of the second kind of order n. These solutions are valid for $|x| < 1$ since series expansion was about point $x = 0$. But both of these functions are unbounded at the end points ($\bar{Q}_n(x)$ blows up at $x = \pm 1$ and $\bar{P}_n(x)$ blows up at $x = -1$) leading to trivial solution.
This means n must be an integer. When n is an integer, then $\lambda_n = n(n+1)$. It is known (from chapter 5), that in this case the solution to (3) becomes a terminating power series (a polynomial), which is called the Legendre polynomial $P_n(x)$. These polynomials are there bounded everywhere, including at the end points $x = \pm 1$, and therefore these solutions satisfy the boundary conditions. Hence the Legendre $P_n(x)$ are the eigenfunctions to (3). This table summaries the result found

n	eigenvalue	eigenfunctions
0	$\lambda_0 = 0$	$P_0(x) = 1$
1	$\lambda_1 = 2$	$P_1(x) = x$
2	$\lambda_2 = 6$	$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$
3	$\lambda_3 = 12$	$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$
$\vdots$	$\vdots$	$\vdots$
n	$\lambda_n = n(n+1)$	$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^2}\left(x^2 - 1\right)^n$

What the above says, is that the solution to

$$\left(1-x^2\right) P_n''(x) - 2xP_n'(x) + \lambda_n P_n(x) = 0$$

Is $P_n(x)$ with the corresponding eigenvalue $\lambda_n = n(n+1)$ as given by the above table. Now that the eigenfunctions of the corresponding homogeneous eigenvalue ODE are found, they are used to solve the given inhomogeneous ODE

$$-\left(\left(1-x^2\right) y'\right)' = \mu y + x^3 \tag{4}$$

Using eigenfunction expansion method. Since $\mu = 1$ and since there is no eigenvalue which is also 1, then a solution exists. Let the solution be

$$y(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

Substituting this solution into (4), and noting that $L[y] = -\left(\left(1-x^2\right) y'\right)' = \lambda_n y$ gives

$$\lambda_n \sum_{n=0}^{\infty} c_n P_n(x) = \mu \sum_{n=0}^{\infty} c_n P_n(x) + x^3$$

Expanding x^3 using the same eigenfunctions (this can be done, since x^3 is continuous function and the eigenfunctions are complete), then the above becomes

$$\begin{aligned}
\lambda_n \sum_{n=0}^{\infty} c_n P_n(x) &= \mu \sum_{n=0}^{\infty} c_n P_n(x) + \sum_{n=0}^{\infty} d_n P_n(x) \\
\lambda_n c_n &= \mu c_n + d_n \\
c_n &= \frac{d_n}{\lambda_n - \mu}
\end{aligned}$$

What is left is to determine d_n from

$$x^3 = \sum_{n=0}^{\infty} d_n P_n(x)$$

The above can be solved for d_n using orthogonality, or by direct expansion (otherwise called undetermined coefficients method). Since the force x^3 is already a polynomial in x and of a small order, then direct expansion is simpler. The above then becomes

$$x^3 = d_0 P_0(x) + d_1 P_1(x) + d_2 P_2(x) + d_3 P_3(x)$$

There is no need to expand for more than $n = 3$, since the LHS polynomial is of order 3. Substituting the known $P_n(x)$ expressions into the above equation gives

$$\begin{aligned}
x^3 &= d_0 + d_1 x + d_2 \frac{1}{2}\left(3x^2 - 1\right) + d_3 \frac{1}{2}\left(5x^3 - 3x\right) \\
&= d_0 + d_1 x + d_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + d_3\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)
\end{aligned}$$

Collecting terms of equal powers in x results in

$$x^3 = x^0\left(d_0 - \frac{1}{2}d_2\right) + x\left(d_1 - \frac{3}{2}d_3\right) + x^2\left(\frac{3}{2}d_2\right) + x^3\left(\frac{5}{2}d_3\right)$$

Or

$$\begin{aligned}
d_0 - \frac{1}{2}d_2 &= 0 \\
d_1 - \frac{3}{2}d_3 &= 0 \\
\frac{3}{2}d_2 &= 0 \\
\frac{5}{2}d_3 &= 1
\end{aligned}$$

From third equation, $d_2 = 0$. From first equation $d_0 = 0$, and substituting last equation in the second equation give $d_1 = \frac{3}{2}$. Therefore

$$\begin{aligned}
d_1 &= \frac{3}{5} \\
d_3 &= \frac{2}{5}
\end{aligned}$$

And all other d_n are zero. Now the c_n are found using $c_n = \frac{d_n}{\lambda_n - \mu}$. For $n = 1$

$$c_1 = \frac{d_1}{\lambda_1 - \mu} = \frac{\frac{3}{5}}{2-1} = \frac{3}{5}$$

And for $n=3$

$$c_3 = \frac{d_3}{\lambda_3 - \mu} = \frac{\frac{2}{5}}{12-1} = \frac{2}{55}$$

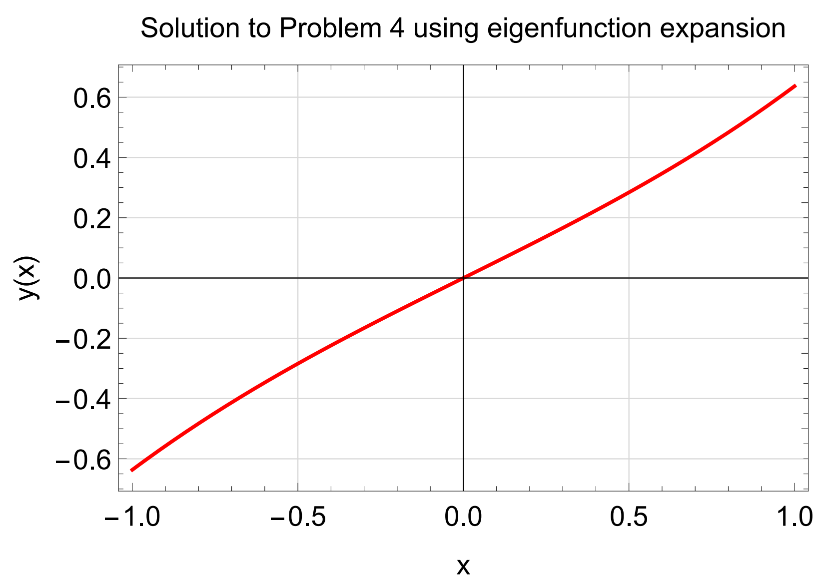
And all other c_n are zero. Hence the final solution from $y(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ reduces to only two terms in the sum

$$\begin{aligned} y(x) &= c_1 P_1(x) + c_3 P_3(x) \\ &= \frac{3}{5}x + \frac{2}{55}\left(\frac{1}{2}\left(5x^3 - 3x\right)\right) \end{aligned}$$

Giving the final solution as

$$\boxed{y(x) = \tfrac{1}{11}x\left(x^2+6\right)}$$

This is a plot of the solution



Appendix for problem 4

Initially I did not know we had to use eigenfunction expansion, so solved it directly as follows. Let the solution to

$$\left(1-x^2\right)y'' - 2xy' + y = x^3$$

Be

$$y(x) = y_h(x) + y_p(x)$$

Where $y_h(x)$ is the homogeneous solution to $\left(1-x^2\right)y'' - 2xy' + y = 0$ and $y_p(x)$ is a particular solution. Now, since $\left(1-x^2\right)y'' - 2xy' + y = 0$ is a Legendre ODE but with a non-integer order, then its solution is not a terminating polynomials, but instead is given by

$$y_h(x) = c_1\bar{P}_n(x) + c_2\bar{Q}_n(x)$$

Where $\bar{P}_n(x)$ is called the Legendre function of order n and $\bar{Q}_n(x)$ is called the Legendre function of the second kind of order n, and $y_p(x)$ is a particular solution. The particular solution can be found, using method of undetermined coefficients to be $y_p(x) = \frac{1}{11}x^3 + \frac{6}{11}x$. Hence the general solution becomes

$$y(x) = c_1\bar{P}_n(x) + c_2\bar{Q}_n(x) + \frac{1}{11}x\left(x^2+6\right)$$

Now since the solution must be bounded as $x \to \pm 1$, then we must set $c_1 = 0$ and $c_2 = 0$, because both $\bar{P}_n(x)$ and $\bar{Q}_n(x)$ are unbounded at the end points ($\bar{Q}_n(x)$ blows up at $x = \pm 1$ and $\bar{P}_n(x)$ blows up at only $x = -1$), therefore the final solution contains only the particular solution

$$y(x) = \frac{1}{11}x\left(x^2+6\right)$$

Which is the same solution found using eigenfunction expansion. At first I thought I made an error somewhere, since I did not think all of the homogenous solution basis could vanish leaving only a particular solution.

3.4.5 key solution

Homework 4, Math 322

1. Solve the the partial differential equation

$$u_t = u_{xx} + xt, \quad 0 \le x \le 1, t \ge 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = \sin \pi x.$$

Solution: The eigenvalue problem $-y'' = \lambda y$, $y(0) = y(1) = 0$ has eigenvalues $\lambda_n = n^2\pi^2$ and normalized eigenfunctions $\hat{\phi}_n(x) = \sqrt{2}\sin(nx)$, $n = 1, 2, \ldots$ Therefore, we are looking for a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\phi}_n(x).$$

We know from the textbook, Section 11.3, that $b_n(t)$ is determined by

$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t), \quad b_n(0) = B_n,$$

where

$$\gamma_n(t) = \int_0^1 xt\hat{\phi}_n(x)\,dx$$

and the sequence B_n is determined by

$$u(x,0) = \sum_{n=1}^{\infty} B_n\hat{\phi}_m(x).$$

Therefore, in our problem, $B_1 = \frac{1}{\sqrt{2}}$ and $B_n = 0$ for $n \ge 2$. Moreover,

$$\gamma_n(t) = t\sqrt{2}\int_0^1 x\sin(n\pi x)\,dx = t\sqrt{2}(-1)^{n+1}\frac{1}{n\pi}.$$

Solving the differential equation for $b_n(t)$ we find

$$b_1(t) = \frac{\sqrt{2}}{\pi^5}\left(\pi^2 t - 1 + e^{-\pi^2 t}\left(\frac{1}{2}\pi^5 + 1\right)\right).$$

and, for $n \ge 2$,

$$b_n(t) = \frac{\sqrt{2}(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right).$$

The solution is

$$\begin{aligned}u(x,t) =& \frac{2}{\pi^5}\left(\pi^2 t - 1 + e^{-\pi^2 t}\left(\frac{1}{2}\pi^5 + 1\right)\right)\sin \pi x \\ &+ \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right)\sin(n\pi x) \\ =& e^{-\pi^2 t}\sin \pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right)\sin(n\pi x).\end{aligned}$$

2. Show that

$$(\lambda - \mu)\int_0^1 xJ_0(\sqrt{\lambda}x)J_0(\sqrt{\mu}x)\,dx = \sqrt{\mu}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})J_0(\sqrt{\mu})$$

for $\lambda, \mu > 0$.

Solution: Let $y = J_0(\sqrt{\lambda}x)$ and $z = J_0(\sqrt{\mu}x)$. Then

$$-(xy')' = \lambda xy, \quad -(xz')' = \mu xz.$$

We multiply the first equation by z, the second by y, subtract and integrate, to find

$$\int_0^1 \left((xz')'y - (xy')'z\right)\,dx = (\lambda - \mu)\int_0^1 xyz\,dx.$$

Integration by parts gives

$$xz'y - xy'z\big|_0^1 = (\lambda - \mu)\int_0^1 xyz\,dx$$

which is the desired identity.

3. By letting $\mu \to \lambda$ in the formula of Problem 2, derive a formula for $\int_0^1 xJ_0(\sqrt{\lambda}x)^2\,dx$. Then show that the normalized eigenfunctions of the eigenvalue problem in Section 11.4 are

$$\hat{\phi}_n(x) = \frac{\sqrt{2}J_0(j_n x)}{|J_0'(j_n)|},$$

where $0 < j_1 < j_2 < j_3 < \ldots$ denote the positive zeros of J_0.

Solution: We divide the identity from problem 2 by $\lambda - \mu$, and let $\mu \to \lambda$. Using L'Hospital's rule we find

$$\begin{aligned}\int_0^1 xJ_0(\sqrt{\lambda}x)^2\,dx =& \lim_{\mu\to\lambda}\frac{\sqrt{\mu}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})J_0(\sqrt{\mu})}{\lambda - \mu} \\ =& -\lim_{\mu\to\lambda}\frac{1}{2}\mu^{-1/2}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) + \frac{1}{2}J_0''(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})\frac{1}{2}\mu^{-1/2}J_0'(\sqrt{\mu}) \\ =& -\frac{1}{2}\lambda^{-1/2}J_0'(\sqrt{\lambda})J_0(\sqrt{\lambda}) - \frac{1}{2}J_0''(\sqrt{\lambda})J_0(\sqrt{\lambda}) + \frac{1}{2}J_0'(\sqrt{\lambda})^2.\end{aligned}$$

Now we use

$$\sqrt{\lambda}J_0''(\sqrt{\lambda}) + J_0'(\sqrt{\lambda}) + \sqrt{\lambda}J_0(\sqrt{\lambda}) = 0.$$

Then we obtain

$$\int_0^1 xJ_0(\sqrt{\lambda}x)^2\,dx = \frac{1}{2}J_0(\sqrt{\lambda})^2 + \frac{1}{2}J_0'(\sqrt{\lambda})^2.$$

If $\lambda = j_n^2$ then this formula simplifies to

$$\int_0^1 xJ_0(j_nx)^2\,dx = \frac{1}{2}J_0'(j_n)^2.$$

The normalized eigenfunctions are

$$\frac{J_0(j_nr)}{\left(\int_0^1 xJ_0(j_nx)^2\,dx\right)^{1/2}} = \frac{\sqrt{2}J_0(j_nr)}{|J_0'(j_n)|}.$$

4. Solve the inhomogeneous differential equation

$$-((1-x^2)y')' = y + x^3, \quad -1 < x < 1$$

with boundary condition

$$y(x), y'(x) \text{ bounded as } x \to -1^+ \text{ and } x \to 1^-.$$

Solution We use the method from Section 11.3. It is stated for regular Sturm-Liouville problems but it works equally well for our singular Sturm-Liouville problem. We look for the solution in the form

$$y = \sum_{n=0}^{\infty} b_nP_n(x).$$

The b_n satisfy

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

The sequence c_n is determined by

$$x^3 = \sum_{n=0}^{\infty} c_nP_n(x).$$

Since $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ and $P_1(x) = x$, we find

$$c_3 = \frac{2}{5}, \quad c_1 = \frac{3}{5},$$

and all other $c_n = 0$. The eigenvalues are $\lambda_n = n(n+1)$ and $\mu = 1$. Therefore,

$$y = \frac{c_1}{2-1}P_1(x) + \frac{c_3}{12-1}P_3(x) = \frac{3}{5}x + \frac{2}{5}\frac{1}{11}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = \frac{6}{11}x + \frac{1}{11}x^3.$$

Chapter 4: Handouts

4.1 Fourier hand out, Jan 31, 2018

THE FOURIER CONVERGENCE THEOREM

Before we can prove the Fourier convergence theorem we need some preparations.

Lemma 1. *Let g be a T-periodic function which is integrable on $[0, T]$. Then, for all a,*

$$\int_0^T g(x)\,dx = \int_a^{a+T} g(x)\,dx.$$

Proof. There is an integer k such that $(k-1)T \le a < kT$. Then

$$\int_a^{a+T} g(x)\,dx = \int_a^{kT} g(x)\,dx + \int_{kT}^{a+T} g(x)\,dx.$$

In the first integral on the right-hand side we substitute $x = t - T$ and use $g(t-T) = g(t)$. Then we obtain

$$\int_a^{a+T} g(x)\,dx = \int_{a+T}^{(k+1)T} g(t)\,dt + \int_{kT}^{a+T} g(x)\,dx.$$

Therefore,

$$\int_a^{a+T} g(x)\,dx = \int_{kT}^{(k+1)T} g(x)\,dx = \int_0^T g(s)\,ds,$$

where we substituted $x = s + kT$. □

The Dirichlet kernel D_n, $n = 0, 1, 2, \dots$, is defined by

$$D_n(t) = \frac{1}{2} + \cos t + \cos(2t) + \cdots + \cos(nt).$$

This is an even function with period 2π. The graph of D_5 is shown in Figure 1.

Lemma 2. *If $t \ne 0, \pm 2\pi, \pm 4\pi, \dots$ then*

$$D_n(t) = \frac{\sin(2n+1)\frac{1}{2}t}{2\sin\frac{1}{2}t}.$$

Otherwise, $D_n(t) = n + \frac{1}{2}$.

Proof. Using $\cos t = \frac{1}{2}(e^{it} + e^{-it})$, we have

$$D_n(t) = \frac{1}{2}\sum_{m=-n}^{n} e^{imt}.$$

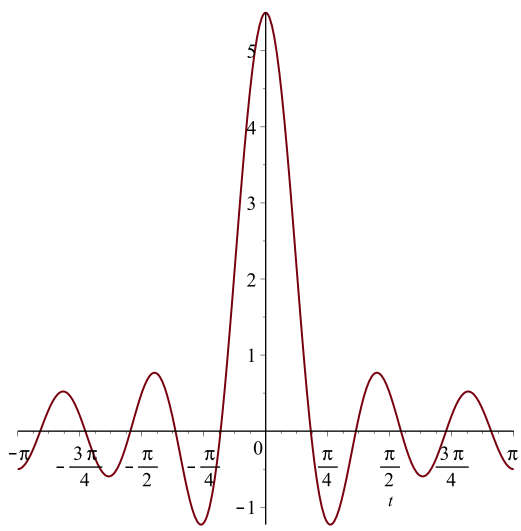


FIGURE 1. Graph of $D_5(t)$

We set $z = e^{it}$. Then

$$\begin{aligned} D_n(t) &= \frac{1}{2} z^{-n}(1 + z + z^2 + \cdots + z^{2n}) \\ &= \frac{1}{2} z^{-n} \frac{z^{2n+1} - 1}{z - 1} \\ &= \frac{1}{2} e^{-int} \frac{e^{(2n+1)it} - 1}{e^{it} - 1} \\ &= \frac{1}{2} \frac{e^{i(2n+1)\frac{1}{2}t} - e^{-i(2n+1)\frac{1}{2}t}}{e^{i\frac{1}{2}t} - e^{-i\frac{1}{2}t}} \\ &= \frac{\sin((2n+1)\frac{1}{2}t)}{2\sin\frac{1}{2}t}, \end{aligned}$$

where we used $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$. □

Lemma 3 (Bessel's inequality)**.** *Let f be a $2L$-periodic function which is integrable on $[-L, L]$ with Fourier coefficients*

(1) $$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{m\pi t}{L}\,dt.$$

Then

(2) $$\frac{1}{2}a_0^2 + \sum_{m=1}^{\infty}(a_m^2 + b_m^2) \le \frac{1}{L}\int_{-L}^{L} f(t)^2\,dt.$$

In particular,

$$\lim_{m\to\infty} a_m = 0, \quad \lim_{m\to\infty} b_m = 0.$$

Proof. Let n be a positive integer, and consider

$$s_n(t) = \frac{1}{2}a_0 + \sum_{m=1}^{n} \left(a_m \cos \frac{m\pi t}{L} + b_m \sin \frac{m\pi t}{L} \right).$$

Then

$$0 \leq \frac{1}{L}\int_{-L}^{L} (f(t)-s_n(t))^2\, dt = \frac{1}{L}\int_{-L}^{L} f(t)^2\, dt - \frac{2}{L}\int_{-L}^{L} f(t)s_n(t)\, dt + \frac{1}{L}\int_{-L}^{L} s_n(t)^2\, dt.$$

Now, using the definition of s_n,

$$\frac{2}{L}\int_{-L}^{L} f(t)s_n(t)\, dt = 2\left(\frac{1}{2}a_0^2 + \sum_{m=1}^{n}(a_m^2 + b_m^2)\right).$$

By orthogonality,

$$\frac{1}{L}\int_{-L}^{L} s_n(t)^2\, dt = \frac{1}{2}a_0^2 + \sum_{m=1}^{n}(a_m^2 + b_m^2).$$

Therefore,

$$0 \leq \frac{1}{L}\int_{-L}^{L} f(t)^2\, dt - \left(\frac{1}{2}a_0^2 + \sum_{m=1}^{n}(a_m^2 + b_m^2)\right).$$

This is true for all n so (2) follows. □

Actually, equality holds in (2) (Parseval's equation) but we do not need this result right now.

A function f is said to be piecewise continuous on the interval $[a, b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ so that
1. f is continuous on the open interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, n$;
2. the one-sided limits $f(x_{i-1}^+) = \lim_{x\to x_{i-1}^+} f(x) =$ and $f(x_i^-) = \lim_{x\to x_i^-} f(x)$ exist and are finite for each $i = 1, 2, \ldots, n$.

Theorem 4 (Fourier convergence theorem)**.** *Let f be a function with period $2L$ such that f and f' are piecewise continuous on $[-L, L]$. Let a_m, b_m be the Fourier coefficients of f as defined in* (1)*. Then, for all real x,*

$$\frac{1}{2}(f(x^+) + f(x^-)) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

In particular, if f is continuous at x,

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

Proof. In order to simplify the writing we assume that $L = \pi$ (consider $f(\frac{L}{\pi}t)$ in place of f.) In the following x denotes a fixed real number. For a positive integer n we define the partial sum of the Fourier series

$$s_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) \, .$$

Then using (1)

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{m=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos mx \cos mt + \sin mx \sin mt)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{m=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos m(t-x)\,dt. \end{aligned}$$

By definition of D_n,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\,dt.$$

We substitute $t - x = u$. Then

$$s_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)D_n(u)\,du.$$

By Lemma 1,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du.$$

We split the integral in two

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(x+u)D_n(u)\,du + \frac{1}{\pi}\int_{0}^{\pi} f(x+u)D_n(u)\,du.$$

It follows easily from the definition of D_n that

$$\frac{1}{\pi}\int_{-\pi}^{0} D_n(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} D_n(t)\,dt = \frac{1}{2}.$$

Therefore,

$$s_n(x) - \frac{1}{2}(f(x^+) + f(x^-)) = I_n + J_n,$$

where

$$I_n = \frac{1}{\pi}\int_{-\pi}^{0} (f(x{+}u){-}f(x^-))D_n(u)\,du, \quad J_n = \frac{1}{\pi}\int_{0}^{\pi} (f(x{+}u){-}f(x^+))D_n(u)\,du.$$

We now show that the two integrals I_n, J_n converge to 0 as $n \to \infty$ which completes the proof. We do this only for J_n, I_n is treated similarly. Now, using Lemma 2,

$$J_n = \frac{1}{\pi}\int_{0}^{\pi} (f(x+u) - f(x^+))\frac{\sin(2n+1)\frac{1}{2}u}{2\sin\frac{1}{2}u}\,du.$$

Substituting $u = 2t$ we can write this as

$$J_n = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} g(t) \sin(2n+1)t \, dt,$$

where

$$g(t) = \frac{f(x+2t) - f(x^+)}{2t} \frac{t}{\sin t} \quad \text{for } 0 < t \leq \frac{1}{2}\pi.$$

Since we assumed that f' is piecewise continuous, the limit $\lim_{t \to 0^+} g(t)$ exists as a finite number (to see this one has to apply the mean-value theorem). Therefore, the function g is piecewise continuous and thus integrable on $[0, \frac{1}{2}\pi]$. It follows from Lemma 3 (with $L = \frac{1}{2}\pi$ and $g(t) = 0$ for $-L < t < 0$) that $\lim_{n \to \infty} J_n = 0$. □

Remark: In the proof we did not directly use that f' is piecewise continuous. It would be simpler to just assume that the limits

$$\lim_{t \to 0^+} \frac{f(x+t) - f(x^+)}{t}, \quad \lim_{t \to 0^-} \frac{f(x+t) - f(x^-)}{t}$$

exist and are finite.

THE DIRICHLET PROBLEM ON AN ELLIPSE

We want to solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \text{ for } (x, y) \text{ in } D,$$
$$u(x, y) = f(x, y) \text{ for } (x, y) \text{ on the boundary of } D$$

when D is the region inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We assume that $a > b > 0$. The focal points of the ellipse are $(\pm c, 0)$. We introduce elliptic coordinates

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta.$$

Usually $\xi > 0$ and $0 \leq \eta < 2\pi$ or $-\pi < \eta \leq \pi$.

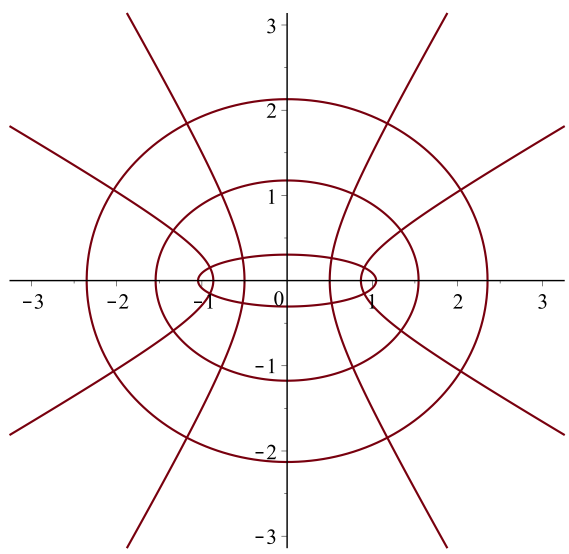


FIGURE 1. Elliptic Coordinates

We set $v(\xi, \eta) = u(c \cosh \xi \cos \eta, c \sinh \xi \sin \eta)$. Then, by the chain rule,

$$\begin{aligned} v_{\xi\xi} &= (c \sinh \xi \cos \eta)^2 u_{xx} + 2c^2 \cosh \xi \sinh \xi \cos \eta \sin \eta \, u_{xy} \\ &\quad + (c \cosh \xi \sin \eta)^2 u_{yy} + c \cosh \xi \cos \eta \, u_x + c \sinh \xi \sin \eta \, u_y, \\ v_{\eta\eta} &= (c \cosh \xi \sin \eta)^2 u_{xx} - 2c^2 \cosh \xi \sinh \xi \cos \eta \sin \eta \, u_{xy} \\ &\quad + (c \sinh \xi \cos \eta)^2 u_{yy} - c \cosh \xi \cos \eta \, u_x - c \sinh \xi \sin \eta \, u_y. \end{aligned}$$

so
$$v_{\xi\xi}+v_{\eta\eta}=c^2(\cosh^2\xi-\cos^2\eta)(u_{xx}+u_{yy}).$$
Therefore, the equation $u_{xx}+u_{yy}=0$ is equivalent to $v_{\xi\xi}+v_{\eta\eta}=0$. We use separation of variables
$$v(\xi,\eta)=\Xi(\xi)E(\eta).$$
Then we find
$$\Xi''-\lambda\Xi=0,\quad E''+\lambda E=0.$$
The equation for E has to have nontrivial 2π periodic solutions. Therefore, $\lambda=n^2$, $n=0,1,2,\dots$ and
$$E_n(\eta)=c_n\cos(n\eta)+d_n\sin(n\eta).$$
The general solution of the differential equation for Ξ with $\lambda=n^2$ is
$$\Xi(\xi)=a_n\cosh(n\xi)+d_n\sinh(n\xi).$$
If we consider the function
$$v(\xi,\eta)=\cosh n\xi\sin n\eta,$$
then we notice that $v(\xi,-\eta)=-v(\xi,\eta)$ so $u(x,-y)=-u(x,y)$. But then $u(x,0)$ should be zero on the focal line which is not true. Therefore, $u(x,y)$ is discontinuous at the focal line $[-c,c]$. Similarly, the function $v(\xi,\eta)=\sinh n\xi\cos n\eta$ has a discontinuous derivative u_y. Therefore, we consider only

(1) $$v_n(\xi,\eta)=c_n\cosh n\xi\cos n\eta+d_n\sinh n\xi\sin n\eta.$$

In fact, we show below that the corresponding function $u_n(x,y)$ is a polynomial in x,y. Therefore, by superposition, we find the solution

(2) $$v(\xi,\eta)=\frac{c_0}{2}+\sum_{n=1}^{\infty}(c_n\cosh n\xi\cos n\eta+d_n\sinh n\xi\sin n\eta).$$

The boundary of D is given by $\xi=\xi_0$, where $\xi_0>0$ is determined from $c\cosh\xi_0=a$. Therefore, in order to satisfy the boundary condition
$$F(\eta):=f(c\cosh\xi_0\cos\eta,c\sinh\xi_0\sin\eta)=v(\xi_0,\eta)$$
we set
$$c_n\cosh n\xi_0=\frac{1}{\pi}\int_0^{2\pi}F(\eta)\cos n\eta\,d\eta,\quad n\ge 0$$
and
$$d_n\sinh n\xi_0=\frac{1}{\pi}\int_0^{2\pi}F(\eta)\sin n\eta\,d\eta,\quad n\ge 1.$$
Substituting these values of c_n,d_n in (2) we find the solution of the Dirichlet problem for the ellipse. We see that the series in (2) converges very well for $\xi<\xi_0$. The quality of convergence on the boundary ellipse $\xi=\xi_0$ is the same as that of the Fourier series for $F(\eta)$.

The function v_n defined in (1) ic called an ellipsoidal harmonic of degree n. These functions are polynomials in x,y as we show below. We use the Chebyshev polynomials T_n defined by $\cos n\theta=T_n(\cos\theta)$. They also satisfy

$\cosh nz = T_n(\cosh z)$. The Chebyshev polynomials can be calculated from the recursion

$$T_0(z) = 1,\ T_1(z) = z,\ T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$$

so

$$T_2(z) = 2z^2 - 1, \quad T_3(z) = 4z^3 - 3z, \quad T_4(z) = 8z^4 - 8z^2 + 1.$$

Then

$$\begin{aligned}\cosh n\xi \cos n\eta + i \sinh n\xi \sin n\eta &= \cosh n(\xi + i\eta)\\ &= T_n(\cosh(\xi + i\eta))\\ &= T_n(\cosh \xi \cos \eta + i \sinh \xi \sin \eta)\\ &= T_n(c^{-1}(x + iy)).\end{aligned}$$

For example,

$$\begin{aligned}\cosh 2\xi \cos 2\eta &= \operatorname{Re}(2c^{-2}(x+iy)^2 - 1) = 2\left(\frac{x}{c}\right)^2 - 2\left(\frac{y}{c}\right)^2 - 1,\\ \sinh 2\xi \sin 2\eta &= \operatorname{Im}(2c^{-2}(x+iy)^2 - 1) = 2\frac{xy}{c^2}.\end{aligned}$$

Example: Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with boundary condition $u(x,y) = \frac{1}{3}x^2$.
Solution: We have $a = 3$, $b = 2$ and $c = \sqrt{5}$. The ellipse is given by $\xi = \xi_0$ where $c \cosh \xi_0 = a$, $c \sinh \xi_0 = b$. The boundary condition is given by the function $F(\eta) = \frac{1}{3}a^2 \cos^2 \eta = 3\cos^2 \eta$. Its Fourier expansion is

$$f(\eta) = 3\cos^2 \eta = \frac{3}{2} + \frac{3}{2}\cos 2\eta.$$

The solution of the Dirichlet problem in elliptic coordinates is

$$v(\xi, \eta) = \frac{3}{2} + \frac{3}{2}\frac{\cosh 2\xi}{\cosh 2\xi_0}\cos 2\eta.$$

Transforming to cartesian coordinates we get

$$u(x,y) = \frac{3}{2} + \frac{3}{2}\frac{5}{13}\left(2\frac{x^2}{5} - 2\frac{y^2}{5} - 1\right) = \frac{12}{13} + \frac{3}{13}(x^2 - y^2).$$

LOWER BOUND FOR EIGENVALUES

We want to derive a lower bound for the eigenvalues of a regular Sturm-Liouville problem. We first need a lemma (related to the Sobolev embedding theorem.)

Lemma 1. *Let $f \in C^1[a,b]$, $x \in [a,b]$, $h > 0$. Then*

$$f(x)^2 \le \left(\frac{1}{b-a} + \frac{1}{h}\right) \int_a^b f(t)^2\,dt + h \int_a^b f'(t)^2\,dt.$$

Proof. For all $x, s \in [a,b]$ we have

$$f(x)^2 - f(s)^2 = \int_s^x 2f(t)f'(t)\,dt \le \int_a^b 2|f(t)||f'(t)|\,dt.$$

We estimate

$$2|f(t)||f'(t| = 2\left(\frac{|f(t)|}{\sqrt{h}}\right)(\sqrt{h}|f'(t)|) \le \frac{1}{h} f(t)^2 + hf'(t)^2.$$

Therefore,

$$f(x)^2 - f(s)^2 \le \frac{1}{h}\int_a^b f(t)^2\,dt + h\int_a^b f'(t)^2\,dt.$$

We integrate this inequality from $s = a$ to $s = b$. Then we obtain

$$(b-a)f(x)^2 - \int_a^b f(s)^2\,ds \le \frac{b-a}{h}\int_a^b f(t)^2\,dt + (b-a)h\int_a^b f'(t)^2\,dt.$$

This is equivalent to the inequality in the statement of the lemma. □

We consider the regular Sturm-Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y = 0,$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

We define

$$c_1 = \begin{cases} 0 & \text{if } \alpha_2 = 0 \text{ or } \frac{\alpha_1}{\alpha_2} \le 0, \\ p(a)\frac{\alpha_1}{\alpha_2} & \text{if } \frac{\alpha_1}{\alpha_2} > 0, \end{cases}$$

and

$$c_2 = \begin{cases} 0 & \text{if } \beta_2 = 0 \text{ or } \frac{\beta_1}{\beta_2} \ge 0, \\ -p(b)\frac{\beta_1}{\beta_2} & \text{if } \frac{\beta_1}{\beta_2} < 0. \end{cases}$$

Then we set $c = c_1 + c_2 \ge 0$.

Theorem 2. *Every eigenvalue λ of a regular Sturm-Liouville problem, satisfies the inequality*

$$\lambda \geq \frac{1}{\min r}\left(-\frac{c}{b-a} - \frac{c^2}{\min p}\right) + \min \frac{q}{r}. \tag{1}$$

Proof. Let $\phi(x)$ be an eigenfunction corresponding to the eigenvalue λ. Then, using integration by parts,

$$\lambda \int r\phi^2 = -\int (p\phi')'\phi + \int q\phi^2 = -\left. p(x)\phi'(x)\phi(x)\right|_a^b + \int p(\phi')^2 + \int q\phi^2,$$

where $\int f$ denotes $\int_a^b f(x)\,dx$. Now

$$-\left. p(x)\phi'(x)\phi(x)\right|_a^b = p(b)\frac{\beta_1}{\beta_2}\phi(b)^2 - p(a)\frac{\alpha_1}{\alpha_2}\phi(a)^2,$$

where $\frac{\beta_1}{\beta_2} = 0$ if $\beta_2 = 0$ and $\frac{\alpha_1}{\alpha_2} = 0$ if $\alpha_2 = 0$. Using the definition of c, we find

$$-\left. p(x)\phi'(x)\phi(x)\right|_a^b \geq -c\max\left\{\phi(a)^2, \phi(b)^2\right\}.$$

Therefore,

$$\lambda \int r\phi^2 \geq -c\max\left\{\phi(a)^2, \phi(b)^2\right\} + \min p \int (\phi')^2 + \min \tfrac{q}{r} \int r\phi^2. \tag{2}$$

If $c = 0$ then

$$\lambda \int r\phi^2 \geq \min \tfrac{q}{r} \int r\phi^2.$$

This gives (1) after division by $\int r\phi^2 > 0$.

If $c > 0$ then we use Lemma 1, and obtain from (2)

$$\lambda \int r\phi^2 \geq -c\left(\frac{1}{b-a} + \frac{1}{h}\right)\int \phi^2 - ch\int (\phi')^2 + \min p \int (\phi')^2 + \min \frac{q}{r}\int r\phi^2, \tag{3}$$

where h can be any positive number. We choose $h = \frac{\min p}{c}$. Then (3) gives

$$\lambda \int r\phi^2 \geq \left(-\frac{c}{b-a} - \frac{c^2}{\min p}\right)\frac{1}{\min r}\int r\phi^2 + \min \frac{q}{r}\int r\phi^2$$

which again gives (1) after division by $\int r\phi^2$. □

THE PRÜFER ANGLE

We consider a regular Sturm-Liouville eigenvalue problem

$$-(p(x)y')' + q(x)y = \lambda\, r(x)y, \quad x \in [a,b] \tag{1}$$

with boundary conditions of the form

$$\cos\alpha\, y(a) = \sin\alpha\, p(a)y'(a), \tag{2}$$

$$\cos\beta\, y(b) = \sin\beta\, p(b)y'(b), \tag{3}$$

where

$$0 \le \alpha < \pi, \quad 0 < \beta \le \pi.$$

Let $y(x)$ be a nontrivial solution of (1). Then we set

$$\xi(x) = p(x)y'(x) = \rho(x)\cos\phi(x), \quad \eta(x) = y(x) = \rho(x)\sin\phi(x).$$

Then

$$\rho(x) = \sqrt{\xi(x)^2 + \eta(x)^2}, \quad \phi(x) = \arctan\frac{\eta(x)}{\xi(x)} = \operatorname{arccot}\frac{\xi(x)}{\eta(x)}.$$

ϕ is called Prüfer angle, and ρ is called Prüfer radius. In order to determine $\phi(x)$ we first choose $\phi(a)$, for example, $-\pi < \phi(a) \le \pi$. Then we use the arctan-formula if $\xi \ne 0$ and the arccot-formula if $\eta \ne 0$. We have to choose the proper branch of the multi-valued arctan, arccot, so that $\phi(x)$ becomes a continuous function (and then also continuously differentiable.)

From the equations

$$\xi' = \rho'\cos\phi - \rho\phi'\sin\phi, \quad \eta' = \rho'\sin\phi + \rho\phi'\cos\phi,$$

we obtain

$$\eta'\cos\phi - \xi'\sin\phi = \rho\phi'.$$

Since $\xi' = (pu')' = (q - \lambda r)\rho\sin\phi$, $\eta' = \frac{\xi}{p} = \frac{\rho}{p}\cos\phi$, it follows that

$$\phi' = \frac{1}{p}\cos^2\phi + (\lambda r - q)\sin^2\phi. \tag{4}$$

A similar calculation shows that

$$\rho' = (\tfrac{1}{p} + q - \lambda r)\rho\cos\phi\sin\phi.$$

It is important to note that (4) is a first order differential equation for the Prüfer angle. In order to satisfy the first boundary condition (2), we choose $\phi(a) = \alpha$. Then $\phi(x,\lambda)$ is uniquely determined by (2). The second boundary condition (3) is satisfied if

$$\phi(b,\lambda) = \beta + n\pi,$$

where n is an integer. One can show that $\lim_{\lambda\to-\infty}\phi(b,\lambda) = 0$, $\lim_{\lambda\to\infty}\phi(b,\lambda) = \infty$ and $\phi(b,\lambda)$ is an increasing function of λ. Therefore, for every $n =$

$0, 1, 2, \ldots,$ there is a unique solution $\lambda = \lambda_n$ of $\phi(b, \lambda) = \beta + n$ and the sequence $\{\lambda_n\}_{n=0}^{\infty}$ represents all the eigenvalues of the regular Sturm-Liouville problem.

Example: Consider

$$-((1+x)y'(x))' + xy = \lambda(1+x^2)y, \quad y(0) = 0,\ y'(1) = 0.$$

Then $p(x) = 1 + x$, $q(x) = x$, $r(x) = 1 + x^2$, $\alpha = 0$, $\beta = \pi/2$.

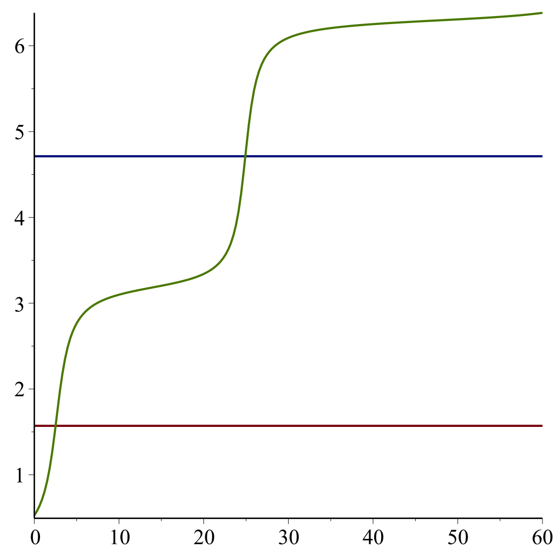


FIGURE 1. Prüfer angle $\phi(1, \lambda)$

The smallest two eigenvalues are

$$\lambda_0 = 2.51173, \quad \lambda_1 = 24.9158$$

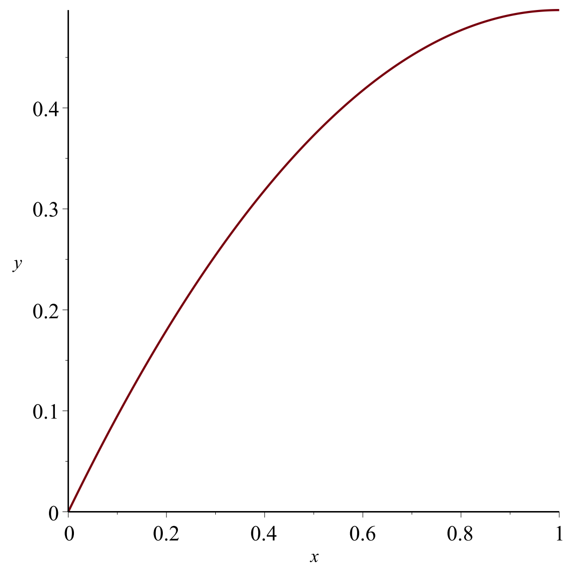


FIGURE 2. Eigenfunction for $\lambda = \lambda_0$

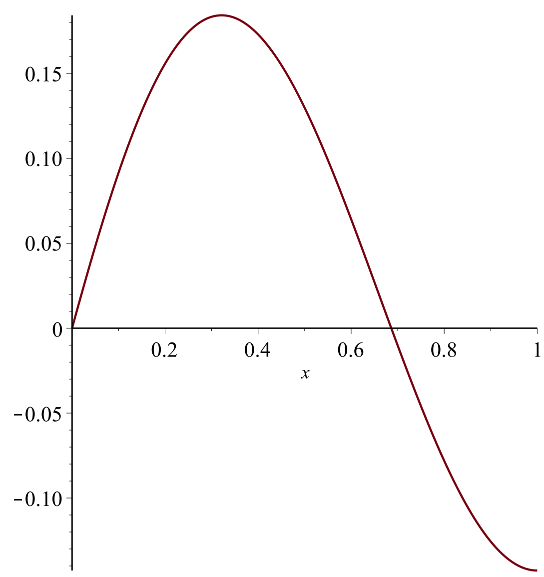


FIGURE 3. Eigenfunction for $\lambda = \lambda_1$

Chapter 5: Exams

5.1 First exam

5.1.1 First exam practice questions

Review for midterm exam Math 322, Tuesday, March 13, 2018

The midterm exam is on sections 10.1–10.8 of our textbook.

1. (Section 10.1) Find the solution (if it exists) of the boundary value problem

$$y'' - y = e^x, \quad y(0) = 1, y(1) = 0.$$

2. (Sections 10.2–10.4) Find the Fourier cosine series for the function

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 < x < 2. \end{cases}$$

Choose $L = 2$. Apply the Fourier convergence theorem. What do we get at $x = 1$?

3. (Sections 10.2–10.4) Find the Fourier sine series for the function $f(x)$ of Problem 2. Choose $L = 2$. Apply the Fourier convergence theorem.

4. (Section 10.5) Solve the heat equation

$$u_t = u_{xx}$$

with boundary conditions

$$u_x(0, t) = 0, \quad u_x(2, t) = 0$$

and initial condition

$$u(x, 0) = f(x)$$

with $f(x)$ from Problem 2. Find the steady-state temperature.

5. (Section 10.6) Solve the heat equation

$$u_t = u_{xx}$$

with boundary conditions

$$u(0, t) = t, \quad u(\pi, t) = 0$$

and initial condition

$$u(x, 0) = 0.$$

6. (Section 10.7) Solve the wave equation

$$u_{tt} = 4u_{xx}, \quad 0 < x < \pi, t > 0$$

with boundary conditions

$$u(0, t) = 0, \quad u(\pi, t) = 0$$

and initial conditions

$$u(x, 0) = \sin^2 x, \quad u_t(x, 0) = 0.$$

Find the d'Alembert solution and the Fourier series solution.

7. (Section 10.7) Find d'Alembert's solution for the wave equation

$$u_{tt}(x, t) = 4u_{xx}(x, t), \quad -\infty < x < \infty, t > 0$$

with initial conditions

$$u(x,0) = \sin x, \quad u_t(x,0) = \cos x.$$

8. (Section 10.8) Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ in the disk $x^2 + y^2 < 1$ and

$$u(x,y) = \begin{cases} 20 & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

on the unit circle $x^2 + y^2 = 1$. Find $u(0,0)$ and $u(0, \frac{1}{2})$.

9. (Section 10.8) Find the solution $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $0 < x < a, y > 0$, that satisfies $u(0,y) = 0$, $u(a,y) = 0$ for $y > 0$ and $u(x,0) = F(x)$, $0 < x < a$ and the additional condition that $u(x,y) \to 0$ as $y \to \infty$.

5.1.2 My solution to first exam practice questions

Problem 1

Find the solution to $y'' - y = e^x, y(0) = 1, y(1) = 0$
solution
The solution to the homogeneous ODE is $y_h = Ae^x + Be^{-x}$. Let the particular be $y_p = Cxe^x$. Hence $y_p' = Ce^x + Cxe^x$ and $y_p'' = Ce^x + Ce^x + Cxe^x$. Substituting into the ODE gives

$$\begin{aligned}2Ce^x + Cxe^x - Cxe^x &= e^x\\ 2C &= 1\\ C &= \frac{1}{2}\end{aligned}$$

Hence $y_p = \frac{1}{2}xe^x$ and the complete solution is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x$$

A, B are now found from boundary conditions. At $x = 0$

$$1 = A + B \tag{1}$$

And at $x = 1$

$$0 = Ae + Be^{-1} + \frac{1}{2}e \tag{2}$$

(1,2) are now solved for A, B. From (1), $A = 1 - B$. (2) becomes

$$\begin{aligned}0 &= (1 - B)e + Be^{-1} + \frac{1}{2}e\\ &= e - Be + Be^{-1} + \frac{1}{2}e\\ &= B\left(e^{-1} - e\right) + \frac{3}{2}e\\ B &= -\frac{3}{2}\frac{e}{e^{-1} - e}\\ &= \frac{3}{2}\frac{e}{e - e^{-1}}\end{aligned}$$

Hence

$$\begin{aligned}A = 1 - \frac{3e}{2(e - e^{-1})} &= \frac{2\left(e - e^{-1}\right) - 3e}{2(e - e^{-1})}\\ &= \frac{2e - 2e^{-1} - 3e}{2(e - e^{-1})}\\ &= \frac{-e - 2e^{-1}}{2(e - e^{-1})}\\ &= \frac{e + 2e^{-1}}{2(e^{-1} - e)}\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= Ae^x + Be^{-x} + \frac{1}{2}xe^x\\ &= \frac{e + 2e^{-1}}{2(e^{-1} - e)}e^x + \frac{3}{2}\frac{e}{e - e^{-1}}e^{-x} + \frac{1}{2}xe^x\end{aligned}$$

Problem 2

Find Fourier cosine series for

$$f(x) = \begin{cases} x & 0 < x < 1\\ 1 & 1 < x < 2\end{cases}$$

Choose $L = 2$. Apply the Fourier convergence theorem. What do we get at $x = 1$?
solution
For cosine series, the function is even extended from $x = -2 \cdots 2$. Therefore only a_n terms exist.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where $L = 2$. But $\frac{a_0}{2}$ is average value. Since the area is $2\left(\frac{1}{2} + 1\right) = 3$, then the average is $\frac{3}{4}$, since the extent is 4. Therefore $a_0 = \frac{3}{2}$. To find a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

But $f(x)$ and cosine are even. Hence the above simplifies to

$$\begin{aligned} a_n &= \int_0^2 f(x) \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \left(\int_0^1 x\cos\left(\frac{n\pi}{2}x\right) dx + \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx\right) \end{aligned}$$

But $\int x \cos ax dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$, therefore

$$\begin{aligned} \int_0^1 x\cos\left(\frac{n\pi}{2}x\right) dx &= \left(\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{x\sin\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2 \left(\cos\left(\frac{n\pi}{2}x\right) + \frac{n\pi}{2}x\sin\left(\frac{n\pi}{2}x\right)\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2 \left(\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right) \end{aligned}$$

And

$$\begin{aligned} \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx &= \left(\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right)_1^2 \\ &= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) \\ &= -\frac{2}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

Hence

$$\begin{aligned} a_n &= \left(\frac{2}{n\pi}\right)^2 \left(\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right) - \frac{2}{n\pi}\sin\frac{n\pi}{2} \\ &= \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\ &= \frac{2}{n^2\pi^2}\left(-2 + 2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right)\right) \end{aligned}$$

Which simplifies to $a_n = -\frac{8\sin\left(\frac{n\pi}{4}\right)^2}{n^2\pi^2}$. Therefore

$$\begin{aligned} f(x) &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2 \cos\left(\frac{n\pi}{2}x\right) \\ &= \frac{3}{4} - \frac{8}{\pi^2}\sin\left(\frac{\pi}{4}\right)^2\cos\left(\frac{\pi}{2}x\right) - \frac{8}{\pi^2}\frac{1}{4}\sin\left(\frac{2\pi}{4}\right)^2\cos(\pi x) - \cdots \\ &= \frac{3}{4} - \frac{4}{\pi^2}\cos\left(\frac{\pi}{2}x\right) - \frac{2}{\pi^2}\cos(\pi x) - \cdots \end{aligned}$$

At $x = 1$

$$\begin{aligned} f(1) &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right) \\ &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right) \end{aligned}$$

In the limit, $\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right) = -\frac{\pi^2}{32}$. Therefore the above becomes

$$\begin{aligned} f(1) &= \frac{3}{4} + \frac{8}{\pi^2}\frac{\pi^2}{32} \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \end{aligned}$$

Which is the value of original $f(x)$ at 1 as expected.
To apply Fourier convergence theorem. The function $f(x)$ is piecewise continuous over $-2 < x < 2$.

$$f'(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$f'(x)$ is also piecewise continuous. Therefore, the Fourier series of $f(x)$ will converge to the average of $f(x)$ at each point.

Problem 3

Find Fourier sine series for

$$f(x)=\begin{cases} x & 0<x<1\\ 1 & 1<x<2\end{cases}$$

Choose $L=2$.
solution
For sine series, the function is odd extended from $x=-2\cdots 2$. Therefore only b_n terms exist.

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

Where $L=2$. To find b_n

$$b_n=\frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

But $f(x)$ is now odd, and sine is odd, hence the product is even and the above simplifies to

$$\begin{aligned} b_n &= \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx\\ &= \left(\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx+\int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx\right)\end{aligned}$$

But $\int x\sin ax dx=\frac{\sin ax}{a^2}-\frac{x\cos ax}{a}$, therefore

$$\begin{aligned}\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx &= \left(\frac{\sin\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2}-\frac{x\cos\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right)_0^1\\ &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}x\right)-\frac{n\pi}{2}x\cos\left(\frac{n\pi}{2}x\right)\right)_0^1\\ &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}\right)-\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right)\end{aligned}$$

And

$$\begin{aligned}\int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx &= -\left(\frac{\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right)_1^2\\ &= -\frac{2}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\end{aligned}$$

Therefore

$$\begin{aligned} b_n &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}\right)-\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right)-\frac{2}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\\ &= -\frac{2\left(n\pi\cos n\pi-2\sin\frac{n\pi}{2}\right)}{n^2\pi^2}\end{aligned}$$

Therefore

$$f(x)=\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{2\sin\frac{n\pi}{2}-n\pi\cos n\pi}{n^2}\sin\left(\frac{n\pi}{L}x\right)$$

As in problem 2, both $f(x)$ and $f'(x)$ are P.W.C. So F.S. converges to average of $f(x)$ at all points.

Problem 4

Solve heat PDE $u_t=u_{xx}$ with boundary conditions $u_x(0,t)=0, u_x(2,t)=0$ and initial conditions $u(x,0)=f(x)$ with $f(x)$ from problem 2. Find steady state solution.

$$f(x)=\begin{cases} x & 0<x<1\\ 1 & 1<x<2\end{cases}$$

solution
When both ends are insulated the solution to the heat PDE is

$$u(x,t)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n e^{-\lambda_n t}\cos\left(\sqrt{\lambda_n}x\right)$$

Where $\lambda_n=\left(\frac{n\pi}{L}\right)^2$ with $n=1,2,3,\cdots$. Since $L=2$, then

$$u(x,t)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n e^{-\left(\frac{n\pi}{2}\right)^2 t}\cos\left(\frac{n\pi}{2}x\right)$$

At $t=0$

$$f(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\cos\left(\frac{n\pi}{2}x\right) \tag{1}$$

But the F.S. of $f(x)$ was found in problem 2, with even extension. It is

$$f(x)=\frac{3}{4}-\sum_{n=1}^{\infty}\frac{8}{\pi^2n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}x\right) \tag{2}$$

Comparing (1) and (2) gives

$$\begin{aligned}\frac{c_0}{2}&=\frac{3}{4}\\ c_n&=\frac{8}{\pi^2n^2}\sin\left(\frac{n\pi}{4}\right)^2\end{aligned}$$

Hence solution is

$$u(x,t)=\frac{3}{4}+\sum_{n=1}^{\infty}\frac{8}{\pi^2n^2}\sin\left(\frac{n\pi}{4}\right)^2e^{-\left(\frac{n\pi}{2}\right)^2t}\cos\left(\frac{n\pi}{2}x\right)$$

At steady state, the solution is

$$u(x,\infty)=\frac{3}{4}$$

Since as $t\to\infty$, the term $e^{-\left(\frac{n\pi}{2}\right)^2t}\to 0$.

Problem 5

Solve heat PDE $u_t=u_{xx}$ with boundary conditions $u(0,t)=t, u(\pi,t)=0$ and initial conditions $u(x,0)=0$

solution

Since boundary conditions are nonhomogeneous, the PDE is converted to one with homogenous BC using a reference function. The reference function needs to only satisfy the nonhomogeneous B.C.
In this case, it is clear that the following function satisfies the nonhomogeneous B.C.

$$r(x,t)=t\left(1-\frac{x}{\pi}\right)$$

Therefore

$$u(x,t)=w(x,t)+r(x,t)$$

Substituting this back into $u_t=u_{xx}$ gives

$$w_t+r_t=w_{xx}+r_{xx}$$

but $r_t=1-\frac{x}{\pi}$ and $r_{xx}=0$, therefore the above simplifies to

$$\begin{aligned}w_t&=w_{xx}+\frac{x}{\pi}-1\\ w_t&=w_{xx}+Q(x)\end{aligned} \tag{1}$$

Where $Q(x)=\frac{x}{\pi}-1$ and where now this PDE now has now homogenous B.C

$$\begin{aligned}w(0,t)&=0\\ w(\pi,t)&=0\end{aligned}$$

Since a source term exist in the PDE (nonhomogeneous in the PDE itself), then equation (1) is solved using the method of eigenfunction expansion. Let

$$w(x,t)=\sum a_n(t)\Phi_n(x)$$

Where $\Phi_n(x)$ is the eigenfunction of the homogeneous PDE $w_t=w_{xx}$, which is known to be have the eigenfunction $\Phi_n(x)=\sin\left(\sqrt{\lambda_n}x\right)=\sin nx$ where the eigenvalues are known to be $\lambda_n=\left(\frac{n\pi}{\pi}\right)^2=n^2$ with $n=1,2,3,\cdots$. Therefore the above becomes

$$w(x,t)=\sum a_n(t)\sin(nx) \tag{1A}$$

Substituting this back into (1) gives

$$\sum a_n'(t)\Phi_n(x)=\sum a_n(t)\Phi_n''(x)+\sum q_n\Phi_n(x)$$

Where $Q(x) = \sum q_n \Phi_n(x)$ is the eigenfunction expansion of the source term. In the above, and after replacing $\Phi_n''(x)$ by $-\lambda_n \Phi_n(x)$ since $\Phi_n(x)$ satisfies the eigenvalue PDE $\Phi_n''(x) + \lambda_n \Phi_n(x) = 0$ the above becomes

$$\begin{aligned}\sum a_n'(t)\Phi_n(x) &= -\sum a_n(t)\lambda_n\Phi_n(x) + \sum q_n\Phi_n(x)\\ a_n'(t) &= -a_n(t)\lambda_n + q_n\\ a_n'(t) + a_n(t)\lambda_n &= q_n \end{aligned} \tag{2}$$

q_n is now found by applying orthogonality on $Q(x) = \sum q_n \Phi_n(x)$ as follows

$$\begin{aligned}Q(x) &= \sum_{n=1}^{\infty} q_n\Phi_n(x)\\ \int_0^{\pi} Q(x)\Phi_n(x)\,dx &= \frac{\pi}{2}q_n\\ q_n &= \frac{2}{\pi}\int_0^{\pi}\left(\frac{x}{\pi}-1\right)\sin(nx)\,dx\\ &= \frac{2}{\pi}\left(\frac{-n\pi+\sin(n\pi)}{n^2\pi}\right)\\ &= \frac{2}{\pi}\left(\frac{-n\pi}{n^2\pi}\right)\\ &= \frac{-2}{n\pi}\end{aligned}$$

Equation (2) becomes

$$a_n'(t) + a_n(t)n^2 = \frac{-2}{n\pi}$$

The solution to this first order ODE can be easily found as

$$a_n(t) = -\frac{2}{n^3\pi} + a_n(0)e^{-n^2t} \tag{3}$$

Therefore (1A) becomes

$$w(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2}{n^3\pi} + a_n(0)e^{-n^2t}\right)\sin(nx) \tag{4}$$

At time $t=0$ the above becomes

$$w(x,0) = \sum_{n=1}^{\infty}\left(-\frac{2}{n^3\pi} + a_n(0)\right)\sin(nx) \tag{5}$$

But

$$\begin{aligned}w(x,0) &= u(x,0) - r(x,0)\\ &= 0-0\\ &= 0\end{aligned}$$

Therefore (5) becomes

$$0 = \sum_{n=1}^{\infty}\left(-\frac{2}{n^3\pi} + a_n(0)\right)\sin(nx)$$

Which implies

$$a_n(0) = \frac{2}{n^3\pi}$$

Hence from (4)

$$w(x,t) = \sum_{n=1}^{\infty}\frac{2}{n^3\pi}\left(e^{-n^2t}-1\right)\sin(nx) \tag{6}$$

The complete solution is therefore

$$\begin{aligned}u(x,t) &= w(x,t) + r(x,t)\\ &= t\left(1-\frac{x}{\pi}\right) + \sum_{n=1}^{\infty}\frac{2}{n^3\pi}\left(e^{-n^2t}-1\right)\sin(nx)\end{aligned}$$

Problem 6

Solve wave PDE $u_{tt} = 4u_{xx}$ on bounded domain $0 < x < \pi, t > 0$ with boundary conditions $u(0,t) = 0, u(\pi,t) = 0$ and initial conditions $u(x,0) = \sin^2 x, u_t(x,0) = 0$. Find d'Alembert solution and Fourier series solution.

solution

Putting the PDE in standard form $u_{tt} = a^2 u_{xx}$ shows that $a = 2$. Let $f(x) = u(x,0) = \sin^2 x$ and $g(x) = u_t(x,0) = 0$, then the d'Alembert solution is (per key solution, one must use the sign function). Let $F(x) = sign(\sin x)\sin^2 x$, then the solution becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2}(F(x+at) + F(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds \\ &= \frac{1}{2}(F(x+at) + F(x-at)) \end{aligned}$$

Now the Fourier solution is found. Applying separation of variables gives

$$\begin{aligned} T''X &= 4X''T \\ \frac{1}{4}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

The eigenvalue ODE is $X'' + \lambda X = 0$ with $X(0) = 0, X(\pi) = 0$. This has eigenfunctions $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ with $\lambda_n = n^2$ where $n = 1,2,3,\cdots$. The time ODE becomes

$$T'' + 4\lambda_n T = 0$$

Since $\lambda_n > 0$, the solution is

$$\begin{aligned} T(t) &= A_n \cos\left(\sqrt{4\lambda_n}t\right) + B_n \sin\left(\sqrt{4\lambda_n}t\right) \\ &= A_n \cos(2nt) + B_n \sin(2nt) \end{aligned}$$

And

$$T' = -2nA_n \sin(2nt) + 2nB_n \cos(2nt)$$

Since $T'(0) = 0$, then the above implies that $B_n = 0$. Therefore the solution simplifies to

$$T_n(t) = A_n \cos(2nt)$$

And the fundamental solution becomes

$$\begin{aligned} u_n &= T_n X_n \\ &= c_n \cos(2nt)\sin(nx) \end{aligned}$$

Hence by superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos(2nt)\sin(nx)$$

At $t = 0$, $u(x,0) = \sin^2 x$, therefore the above becomes

$$\sin^2 x = \sum_{n=1}^{\infty} c_n \sin(nx)$$

Applying orthogonality gives

$$\int_0^{\pi} \sin^2 x \sin(nx)\,dx = c_n \frac{\pi}{2} \tag{1}$$

$$\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\sin(nx)\,dx = c_n \frac{\pi}{2}$$

To evaluate $\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\sin(nx)\,dx$, it is split into $\int_0^{\pi}\left(\frac{1}{2}\sin(nx) - \frac{1}{2}\cos 2x \sin(nx)\right)dx$. But the first part is

$$\begin{aligned} \int_0^{\pi} \frac{1}{2}\sin(nx)\,dx &= -\frac{1}{2n}(\cos(nx))_0^{\pi} \\ &= -\frac{1}{2n}(\cos(n\pi) - 1) \end{aligned}$$

For even $n = 2,4,\cdots$ the above vanishes. For odd $n = 1,3,5,\cdots$ the above becomes

$$\int_0^{\pi} \frac{1}{2}\sin(nx)\,dx = \frac{1}{n}$$

Now the second integral is evaluated

$$\int_0^\pi -\frac{1}{2}\cos 2x \sin(nx)\,dx = -\frac{1}{2}\int_0^\pi \cos 2x \sin(nx)\,dx$$

Using $\int_0^\pi \sin(px)\cos(qx)\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$, then the above becomes, where $p = n, q = 2$

$$\begin{aligned}
-\frac{1}{2}\int_0^\pi \sin(nx)\cos 2x dx &= -\frac{1}{2}\left(-\frac{\cos(n-2)x}{2(n-2)} - \frac{\cos(n+2)x}{2(n+2)}\right)_0^\pi \\
&= \frac{1}{2}\left(\frac{\cos(n-2)x}{2(n-2)} + \frac{\cos(n+2)x}{2(n+2)}\right)_0^\pi \\
&= \frac{1}{2}\left(\frac{\cos(n-2)\pi}{2(n-2)} + \frac{\cos(n+2)\pi}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right)
\end{aligned}$$

For even $n = 2,4,\cdots$ the above vanishes, since it becomes $\frac{1}{2}\left(\frac{1}{2(n-2)} + \frac{1}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right)$, and for odd $n = 1,3,5,\cdots$, the above becomes

$$\begin{aligned}
-\frac{1}{2}\int_0^\pi \sin(nx)\cos 2x dx &= \frac{1}{2}\left(\frac{-1}{2(n-2)} - \frac{1}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right) \\
&= \frac{1}{2}\left(\frac{-2}{2(n-2)} + \frac{-2}{2(n+2)}\right) \\
&= \frac{-1}{2(n-2)} + \frac{-1}{2(n+2)} \\
&= -\frac{n}{n^2-4}
\end{aligned}$$

Therefore, the final result of integration is

$$\begin{aligned}
\int_0^\pi \sin^2 x \sin(nx)\,dx &= \frac{1}{n} - \frac{n}{n^2-4} \qquad n = 1,3,5,\cdots \\
&= -\frac{4}{n(n^2-4)} \qquad n = 1,3,5,\cdots
\end{aligned}$$

Hence from (1), this results in

$$\begin{aligned}
c_n &= -\frac{2}{\pi}\frac{4}{n(n^2-4)} \\
&= -\frac{8}{\pi n(n^2-4)} \qquad n = 1,3,5,\cdots
\end{aligned}$$

Hence the final solution is

$$u(x,t) = \frac{-8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3-4n}\cos(2nt)\sin(nx)$$

The above solution was verified against numerical solution. The result gave an exact match (20 terms was used in the sum).

Problem 7

Find d'Alembert solution for wave PDE $u_{tt} = 4u_{xx}$ on infinite domain with initial position $u(x,0) = \sin x$ and initial velocity $u_t(x,0) = \cos x$

solution

Putting the PDE in standard form $u_{tt} = a^2 u_{xx}$ shows that $a = 2$. Let $f(x) = u(x,0) = \sin x$ and $g(x) = u_t(x,0) = \cos x$, then the d'Alembert solution is

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x+at) + f(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds \\
&= \frac{1}{2}(\sin(x+2t) + \sin(x-2t)) + \frac{1}{4}\int_{x-2t}^{x+2t}\cos(s)\,ds \\
&= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}\sin(s)_{x-2t}^{x+2t} \\
&= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}(\sin(x+2t) - \sin(x-2t)) \\
&= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}\sin(x+2t) - \frac{1}{4}\sin(x-2t) \\
&= \frac{3}{4}\sin(x+2t) + \frac{1}{4}\sin(x-2t)
\end{aligned}$$

Problem 8

Solve the Dirichlet problem $u_{xx}+u_{yy}=0$ inside the disk $x^2+y^2<1$ and $u(x,y)=\begin{cases}20 & y>0\\ 0 & y<0\end{cases}$ on the unit circle $x^2+y^2=1$. Find $u(0,0)$ and $u\left(0,\frac{1}{2}\right)$

solution

The PDE in polar coordinates is

$$u_{rr}+\frac{1}{r}u_r+u_{\theta\theta}=0 \tag{1}$$

Where r is radial distance and θ the polar angle. The boundary conditions in polar coordinates become

$$f(\theta)=\begin{cases}20 & 0<\theta<\pi\\ 0 & \pi<\theta<2\pi\end{cases}$$

The solution to (1) is

$$u(r,\theta)=\frac{c_0}{2}+\sum_{n=1}^{\infty}r^n\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)$$

At $r=1$ (on the boundary) the above solution become

$$f(\theta)=\frac{c_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\theta)+b_n\sin(n\theta)$$

By orthogonality on cosine the above becomes

$$\int_0^{2\pi}f(\theta)\cos(m\theta)\,d\theta=\int_0^{2\pi}\frac{c_0}{2}\cos(m\theta)\,d\theta+\sum_{n=1}^{\infty}a_n\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta+b_n\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta \tag{2}$$

For $n=0$

$$\begin{aligned}\int_0^{2\pi}f(\theta)\,d\theta&=\int_0^{2\pi}\frac{c_0}{2}d\theta\\ \int_0^{\pi}20d\theta&=\frac{c_0}{2}(2\pi)\\ 20\pi&=\frac{c_0}{2}(2\pi)\\ c_0&=20\end{aligned}$$

For $n>0$ (2) becomes

$$\int_0^{2\pi}f(\theta)\cos(m\theta)\,d\theta=\sum_{n=1}^{\infty}a_n\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta+b_n\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta$$

But $\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta=0$ for all n,m and the above reduces to

$$\begin{aligned}\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta&=a_n\pi\\ \int_0^{\pi}20\cos(n\theta)\,d\theta&=a_n\pi\\ \frac{20}{n}\left[\sin(n\theta)\right]_0^{\pi}&=a_n\pi\\ \frac{20}{n}\left(\sin(n\pi)-0\right)&=a_n\pi\end{aligned}$$

Hence $a_n=0$ for all $n>0$. By orthogonality on sine, for $n>0$, (2) becomes

$$\int_0^{2\pi}f(\theta)\sin(m\theta)\,d\theta=\sum_{n=1}^{\infty}a_n\int_0^{2\pi}\sin(m\theta)\cos(n\theta)\,d\theta+b_n\int_0^{2\pi}\sin(m\theta)\sin(n\theta)\,d\theta$$

But $\int_0^{2\pi}\sin(m\theta)\cos(n\theta)\,d\theta=0$ for all m,n and the above reduces to

$$\begin{aligned}\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta&=b_n\pi\\ \int_0^{\pi}20\sin(n\theta)\,d\theta&=b_n\pi\\ -\frac{20}{n}\left(\cos(n\theta)\right)_0^{\pi}&=b_n\pi\\ -\frac{20}{n}\left(\cos(n\pi)-1\right)&=b_n\pi\\ \frac{20}{n}\left(1-\cos(n\pi)\right)&=b_n\pi\end{aligned}$$

When $n = 2, 4, 6, \cdots$ the above gives $b_n = 0$. For $n = 1, 3, 5, \cdots$ the above gives

$$\frac{40}{n} = b_n \pi$$
$$b_n = \frac{40}{n\pi}$$

Therefore the complete solution is

$$u(r,\theta) = 10 + \frac{40}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{r^n}{n} \sin(n\theta)$$

At $u(0,0)$, which corresponds to $r = 0, \theta = 0$, the above gives $u(0,0) = 10$. At $u\left(0, \frac{1}{2}\right)$ which corresponds to $r = \frac{1}{2}, \theta = \frac{\pi}{2}$ the solution gives

$$u(r,\theta) = 10 + \frac{40}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

Evaluated numerically, it converges to 15.90381156
To convert to x, y, the solution is first written as

$$u(r,\theta) = 10 + \frac{40}{\pi} \left(r\sin(\theta) + \frac{1}{3} r^3 \sin(3\theta) + \frac{1}{5} r^5 \sin(5\theta) + \cdots \right)$$

But

$$r\sin(\theta) = y$$

And

$$\begin{aligned} r^3 \sin(3\theta) &= \sum_{\substack{k=1 \\ odd}}^{3} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k-1}{2}} y^k \\ &= \frac{6}{2} x^2 y - y^3 \end{aligned}$$

And

$$\begin{aligned} r^5 \sin(5\theta) &= \sum_{\substack{k=1 \\ odd}}^{5} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k-1}{2}} y^k \\ &= \frac{120}{24} x^4 y - \frac{120}{12} x^2 y^3 + x y^5 \end{aligned}$$

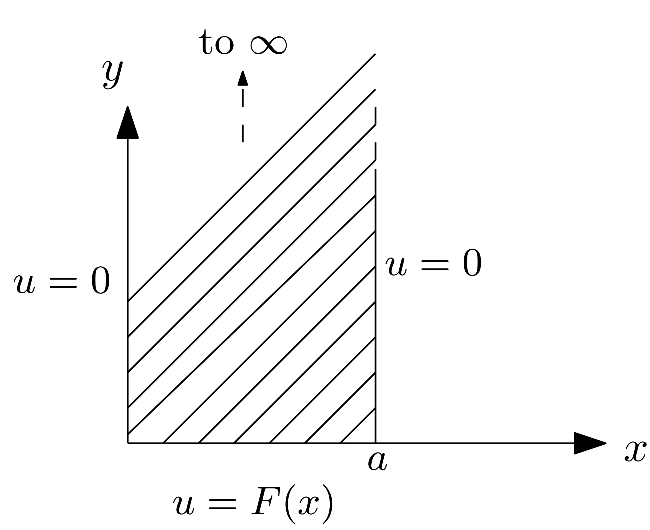
And so on. Hence the solution in xy is

$$u(x,y) = 10 + \frac{40}{\pi} \left(y + \frac{1}{3}\left(3x^2 y - y^3\right) + \frac{1}{5}\left(5x^4 y - 10x^2 y^3 + xy^5\right) + \cdots \right)$$

To verify is the above 3 terms give good approximation, the value at $x = 0, y = \frac{1}{2}$ is now evaluated from the above, which gives 15.8356812467. Which is very close to the above result. One more term can be added to improve this. I am not sure now if there is a way to obtain closed form expression in x, y as the case was with the solution in polar coordinates.

Problem 9

Solve $u_{xx} + u_{yy} = 0$ inside semi-infinite strip $0 < x < a, y > 0$ with $u(0,y) - 0, u(a,y) = 0, u(x,0) = F(x)$ and additional conditions that $u(x,y) \to 0$ as $y \to \infty$
solution
This is a plot of the boundary conditions.



Let $u = X(x)Y(y)$. Substituting this in the PDE gives

$$\begin{aligned} X''Y + Y''X &= 0 \\ \frac{X''}{X} &= -\frac{Y''}{Y} = -\lambda \end{aligned}$$

Which gives the eigenvalue ODE

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(a) &= 0 \end{aligned}$$

which gives the eigenfunction $\Phi_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right)$ where $\lambda_n = \left(\frac{n\pi}{a}\right)^2$ for $n = 1, 2, 3, \cdots$. The corresponding Y ODE is

$$Y'' - \lambda_n Y = 0$$

Since $\lambda_n > 0$, then the solution to this ODE is

$$Y_n = A_n e^{\sqrt{\lambda_n} y} + B_n e^{-\sqrt{\lambda_n} y}$$

Since $\lambda_n > 0$ and the solution goes to zero for large y, then A_n must be zero. Therefore the above simplifies to

$$Y_n(y) = B_n e^{-\sqrt{\lambda_n} y}$$

And the complete solution becomes

$$u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\sqrt{\lambda_n} y} \sin\left(\sqrt{\lambda_n} x\right)$$

Where constants are combined into c_n. Since $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, the above becomes

$$u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi}{a} y} \sin\left(\frac{n\pi}{a} x\right)$$

At $y = 0$, the above becomes

$$F(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} x\right)$$

Applying orthogonality gives

$$\begin{aligned} \int_0^a F(x) \sin\left(\frac{n\pi}{a} x\right) dx &= c_n \frac{a}{2} \\ c_n &= \frac{2}{a} \int_0^a F(x) \sin\left(\frac{n\pi}{a} x\right) dx \end{aligned}$$

Hence the complete solution is

$$u(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} \left(\int_0^a F(x) \sin\left(\frac{n\pi}{a} x\right) dx \right) e^{-\frac{n\pi}{a} y} \sin\left(\frac{n\pi}{a} x\right)$$

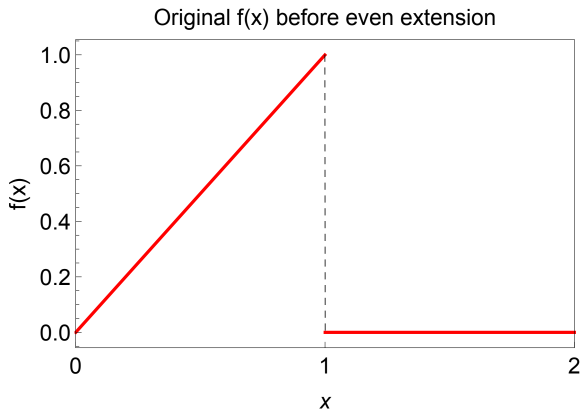
5.1.3 My post-exam solution to first exam

Problem 1

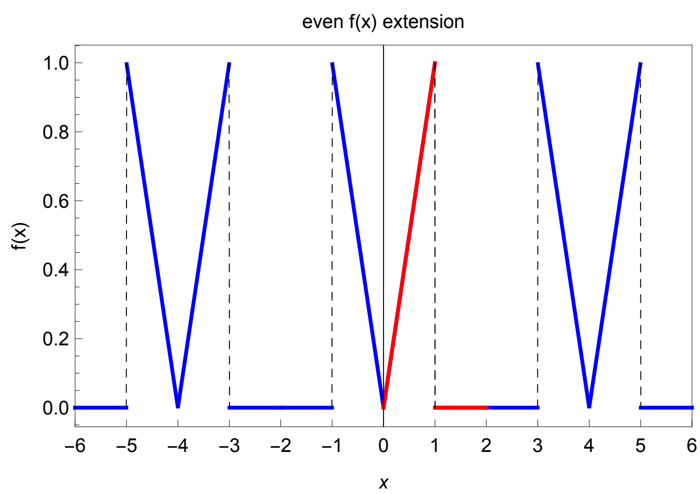
Problem
Find the Fourier cosine series of

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$$

Take $L = 2$.
solution
To obtain the Fourier cosine series, the function $f(x)$ is first even extended to $-2 < x < 2$ with period $2L$ or 4. Then repeated again with period $2L$ over the whole x domain. The following plot shows the original $f(x)$



The following plot shows then even extended $f_e(x)$ over 3 periods for illustrations



The Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

Since extension is even, then the above simplifies to

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

But $L = 2$, therefore

$$\begin{aligned} a_0 &= \int_0^2 f(x)\,dx = \int_0^1 x dx + \int_1^2 0 dx \\ &= \frac{1}{2}\left(x^2\right)_0^1 \\ &= \frac{1}{2} \end{aligned}$$

And

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right) dx$$

Since cosine is even, and $f(x)$ extension is even, then the product is even and the above simplifies to

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}\right) dx$$

Since $L = 2$

$$\begin{aligned} a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi}{2}\right) dx \\ &= \int_0^1 x\cos\left(\frac{n\pi}{2}\right) dx + \int_1^2 0\cos\left(\frac{n\pi}{2}\right) dx \\ &= \int_0^1 x\cos\left(\frac{n\pi}{2}\right) dx \end{aligned}$$

But

$$\int x\cos(ax)\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

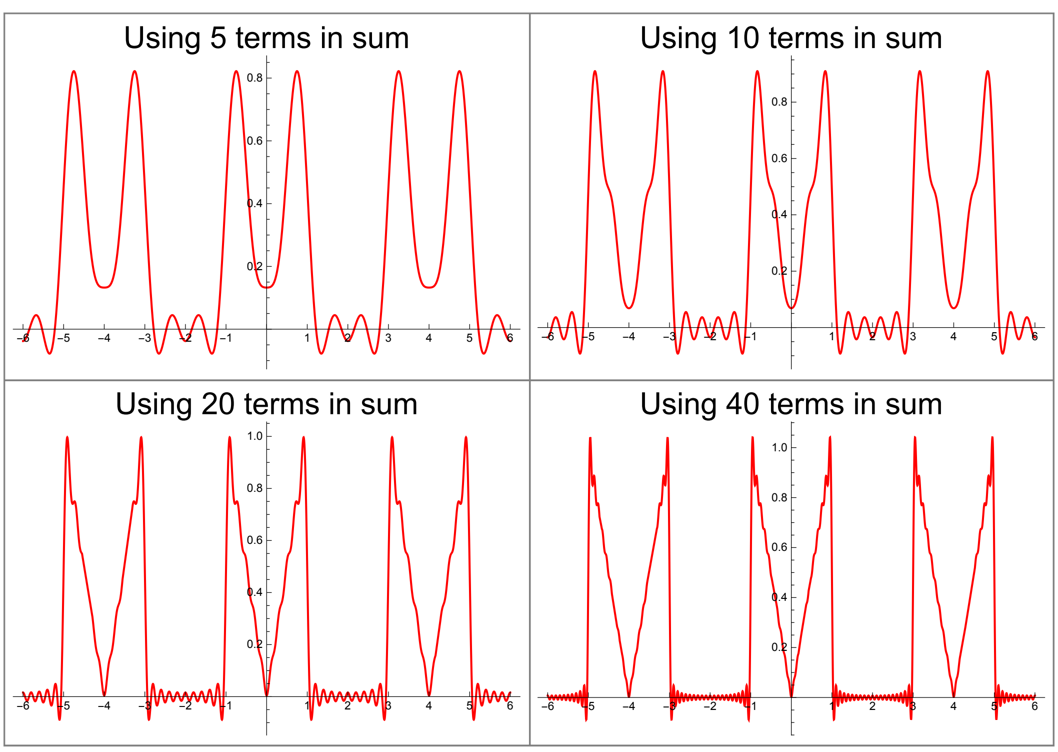
Where $a = \frac{n\pi}{2}$ here. Therefore the integral becomes

$$\begin{aligned}
a_n &= \int_0^1 x\cos\left(\frac{n\pi}{2}\right)dx \\
&= \left(\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{x\sin\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)}\right)_0^1 \\
&= \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{1}{\left(\frac{n\pi}{2}\right)^2} \\
&= \frac{4\cos\left(\frac{n\pi}{2}\right)}{(n\pi)^2} + \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{4}{(n\pi)^2} \\
&= \frac{4\cos\left(\frac{n\pi}{2}\right) + 2n\pi\sin\left(\frac{n\pi}{2}\right) - 4}{n^2\pi^2} \\
&= \frac{2}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right) - 2\right)
\end{aligned}$$

Therefore the Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right) - 2\right)\cos\left(\frac{n\pi}{2}x\right)$$

By Fourier convergence theorem, since $f(x)$ and $f'(x)$ are piecewise contiguous, the Fourier series will converge to each point of $f(x)$ where there is no jump discontinuity, and will converge to the average of $f(x)$ at the point where there is a jump. In this example, it will converge to $\frac{1}{2}$ at the points where is a jump discontinuity There are $x = 1, 3, 5, \cdots$ and at $x = -1, -3, -5, \cdots$. At all other points, Fourier series will converge to $f(x)$. This is a plot of the above Fourier series for increasing number of terms



Problem 2

<u>Problem</u> Solve heat PDE $u_t = 9u_{xx}$ on $0 < x < \pi, t > 0$ with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and initial conditions $u(x,0) = f(x) = 5\sin^2 x$

<u>solution</u>

The solution to the heat PDE with isolated end points is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} c_n e^{-\lambda_n a^2 t}\cos\left(\sqrt{\lambda_n}x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3\cdots$. But $L = \pi$ here. Hence $\lambda_n = n^2$ and $a = 3$. Therefore the above solution becomes

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} c_n e^{-9n^2 t}\cos(nx) \tag{1}$$

At $t = 0$ the above becomes

$$f(x) = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$
$$5\sin^2 x = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

But $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$, therefore the above becomes

$$\frac{5}{2} - \frac{5}{2}\cos(2x) = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

Hence $A_0 = \frac{5}{2}$ and $c_2 = -\frac{5}{2}$ and all other $c_n = 0$. Therefore the solution (1) becomes

$$u(x,t) = \frac{5}{2} - \frac{5}{2}e^{-36t}\cos(2x)$$

At steady state when $t \to \infty$, the solution becomes $u(x) = \frac{5}{2}$. The solution $u\left(\frac{\pi}{2}, t\right)$ becomes

$$\begin{aligned}
u\left(\frac{\pi}{2}, t\right) &= \frac{5}{2} - \frac{5}{2}e^{-36t}\cos\left(2\frac{\pi}{2}\right)\\
&= \frac{5}{2} - \frac{5}{2}e^{-36t}\cos(\pi)\\
&= \frac{5}{2} + \frac{5}{2}e^{-36t}\\
&= \frac{5}{2}\left(1 + e^{-36t}\right)
\end{aligned}$$

Problem 3

Problem
Solve the wave equation $u_{tt} = u_{xx}$ on string, where initial position $f(x) = 0$ and initial velocity is $g(x) = \sin(x) + \sin(2x)$. The string is fixed at both ends.
solution
$a = 1$ in this problem. Using D'Alembert method

$$u(x,t) = \frac{1}{2}\left(f(x+at) + f(x-at)\right) + \frac{1}{2}\int_{x-at}^{x+at} g(s)\,ds$$

Where f, g above are the odd extensions. Since $f(x)$ is zero and $a = 1$, the above simplifies to

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds\\
&= \frac{1}{2}\int_{x-t}^{x+t} \sin(s) + \sin(2s)\,ds\\
&= \frac{1}{2}\left(-\cos(s) - \frac{1}{2}\cos(2s)\right)_{x-t}^{x+t}\\
&= -\frac{1}{2}\left(\cos(s) + \frac{1}{2}\cos(2s)\right)_{x-t}^{x+t}\\
&= -\frac{1}{2}\left(\cos(x+t) + \frac{1}{2}\cos(2(x+t)) - \cos(x-t) - \frac{1}{2}\cos(2(x-t))\right)\\
&= -\frac{1}{2}\cos(x+t) - \frac{1}{4}\cos(2(x+t)) + \frac{1}{2}\cos(x-t) + \frac{1}{4}\cos(2(x-t))\\
&= \frac{1}{2}\left(\cos(x-t) - \cos(x+t)\right) + \frac{1}{4}\left(\cos(2(x-t)) - \cos(2(x+t))\right)
\end{aligned}$$

Using Fourier series method. The solution with initial position zero is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} at\right)\sin\left(\sqrt{\lambda_n} x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $n = 1, 2, 3, \cdots$. Since $L = \pi$ and $a = 1$, the above solution simplifies to

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nt)\sin(nx) \tag{1}$$

To determine c_n, the velocity from the above solution is $\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n n\cos(nt)\sin(nx)$. And at $t = 0$, this becomes

$$f(x) = \sum_{n=1}^{\infty} nc_n \sin(nx)$$

But $f(x)=\sin(x)+\sin(2x)$. Hence the above becomes

$$\sin(x)+\sin(2x)=\sum_{n=1}^{\infty} nc_n\sin(nx)$$

Therefore by inspection $c_1=1$ and $2c_2=1$ or $c_2=\frac{1}{2}$. Therefore the solution (1) becomes

$$u(x,t)=\sin(t)\sin(x)+\frac{1}{2}\sin(2t)\sin(2x)$$

Since the Fourier series and the D'Alembert must be the same, then this implies that

$$\sin(t)\sin(x)+\frac{1}{2}\sin(2t)\sin(2x)=\frac{1}{2}(\cos(x-t)-\cos(x+t))+\frac{1}{4}(\cos(2(x-t))-\cos(2(x+t)))$$

This was confirmed on the computer as well. In this problem, it turned out that it is easier to use the Fourier method, since the initial velocity was given as a Fourier sine series already.

Problem 4

Problem
Solve Laplace PDE $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$ inside annulus $a<r<b$ where $a>0$. The boundary conditions is $u(a\cos\theta,a\sin\theta)=0$ and $u(b\cos\theta,b\sin\theta)=f(\theta)$.
solution
Let $u(r,\theta)=R(r)\Theta(\theta)$. Substituting this back into the PDE gives

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}=0$$

Or

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

The eigenvalue ODE is

$$\begin{aligned}\Theta''+\lambda\Theta&=0\\ \Theta(0)&=\Theta(2\pi)\\ \Theta'(0)&=\Theta'(2\pi)\end{aligned}$$

The solution to the above is known to be

$$\Theta_n(\theta)=c_n\cos\left(\sqrt{\lambda_n}\theta\right)+k_n\sin\left(\sqrt{\lambda_n}\theta\right) \tag{1}$$

Where $\lambda_n=n^2$ and $n=0,1,2,3,\cdots$. Therefore solution (1) becomes

$$\Theta_n(\theta)=c_n\cos(n\theta)+k_n\sin(n\theta)\qquad n=1,2,3,\cdots \tag{1A}$$
$$\Theta_n(\theta)=c_0\qquad n=0 \tag{1B}$$

Therefore the solution to the $\Theta_n(\theta)$ ode is

$$\Theta_n(\theta)=\begin{cases} c_0 & n=0\\ c_n\cos(n\theta)+k_n\sin(n\theta) & n=1,2,3,\cdots\end{cases}$$

The solution to the $R(r)$ ode (this is a Euler ODE) will have two solutions, one when $\lambda_0=0$ when $n=0$ and another solution for $\lambda_n=n^2$ when $n>0$. When eigenvalue is zero, the $R(r)$ ODE becomes

$$\begin{aligned}r^2\frac{R''}{R}+r\frac{R'}{R}&=0\\ r^2R''+rR'&=0\\ rR''+R'&=0\end{aligned}$$

This has the solution

$$R_0(r)=A_0\ln(r)+B_0 \tag{2}$$

Applying the boundary conditions $r=a$ to the above gives

$$\begin{aligned}0&=A_0\ln(a)+B_0\\ B_0&=-A_0\ln(a)\end{aligned}$$

Therefore (2) becomes

$$\begin{aligned}R_0(r)&=A_0\ln(r)-A_0\ln(a)\\ &=A_0(\ln(r)-\ln(a)) \end{aligned}\tag{3}$$

The above is only for the zero eigenvalue. When $n>0$, the $R(r)$ ode becomes the Euler ODE

$$r^2R''+rR'-\lambda_nR=0$$
$$r^2R''+rR'-n^2R=0$$

The solution to this ODE is

$$R_n(r)=A_nr^n+D_nr^{-n} \tag{4}$$

Here the term D_nr^{-n} does not vanish as the case with the solution to the disk. But using the boundary condition that $u=0$ when $r=a$, the above ODE at $r=a$ becomes

$$\begin{aligned}R_n(a)=0&=A_na^n+D_na^{-n}\\ D_n&=-A_n\frac{a^n}{a^{-n}}\\ &=-A_na^{2n}\end{aligned}$$

Substituting the above back in (4) gives

$$\begin{aligned}R_n(r)&=A_nr^n-A_na^{2n}r^{-n}\\ &=A_n\left(r^n-a^{2n}r^{-n}\right)\end{aligned} \tag{4A}$$

Therefore the solution to the $R(r)$ ode is

$$R_n(r)=\begin{cases}A_0(\ln(r)-\ln(a)) & n=0\\ A_n\left(r^n-a^{2n}r^{-n}\right) & n=1,2,3,\cdots\end{cases}$$

The fundamental solution is

$$\begin{aligned}u_n(r,\theta)&=R_n(r)\Theta_n(\theta)\\ &=\overbrace{c_0A_0(\ln(r)-\ln(a))}^{\text{zero eigenvalue}}+\overbrace{\left(r^n-a^{2n}r^{-n}\right)(c_n\cos(n\theta)+k_n\sin(n\theta))}^{n>0\text{ eigenvalues}}\end{aligned}$$

By superposition, the complete solution is

$$u(r,\theta)=c_0A_0(\ln(r)-\ln(a))+\sum_{n=1}^{\infty}A_n\left(r^n-a^{2n}r^{-n}\right)(c_n\cos(n\theta)+k_n\sin(n\theta))$$

Combining c_0A_0 into c_0 and A_nc_n into c_n and A_nk_n into k_n the above simplifies to

$$u(r,\theta)=c_0(\ln(r)-\ln(a))+\sum_{n=1}^{\infty}\left(r^n-a^{2n}r^{-n}\right)(c_n\cos(n\theta)+k_n\sin(n\theta)) \tag{5}$$

Now the boundary condition at $r=b$ is used to determined c_0,c_n and k_n. At $r=b$ and for <u>$n=0$ case</u>, the above becomes, by orthogonality

$$\int_0^{2\pi}f(\theta)\,d\theta=(2\pi)c_0(\ln(b)-\ln(a))$$
$$c_0=\frac{1}{2\pi(\ln(b)-\ln(a))}\int_0^{2\pi}f(\theta)\,d\theta \tag{6}$$

And for $n>0$, solution (5) becomes

$$f(\theta)=\sum_{n=1}^{\infty}\left(b^n-a^{2n}b^{-n}\right)(c_n\cos(n\theta)+k_n\sin(n\theta)) \tag{7}$$

By orthogonality with $\cos(n\theta)$ equation (7) becomes

$$\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta=\left(b^n-a^{2n}b^{-n}\right)c_n\pi$$
$$c_n=\frac{1}{\left(b^n-a^{2n}b^{-n}\right)\pi}\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta$$

And by orthogonality with $\sin(n\theta)$ equation (4) becomes

$$\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta=\left(b^n-a^{2n}b^{-n}\right)k_n\pi$$
$$k_n=\frac{1}{\left(b^n-a^{2n}b^{-n}\right)\pi}\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta$$

This completes the solution. Solution (5) becomes

$$u(r,\theta)=\frac{1}{2\pi}\frac{\ln(r)-\ln(a)}{\ln(b)-\ln(a)}\int_0^{2\pi}f(\theta)\,d\theta+\sum_{n=1}^{\infty}\left(r^n-a^{2n}r^{-n}\right)(c_n\cos(n\theta)+k_n\sin(n\theta))$$
$$c_n=\frac{1}{\left(b^n-a^{2n}b^{-n}\right)\pi}\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta$$
$$k_n=\frac{1}{\left(b^n-a^{2n}b^{-n}\right)\pi}\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta$$

5.2 Final exam

5.2.1 questions

Final exam, Math 322
due Thursday, May 17, 12 noon, 2018

10

1. Solve the heat equation

$$u_t = 9u_{xx}, \quad 0 \le x \le 1,\ t \ge 0$$

with boundary conditions

$$u(0,t) = 0,\ u(1,t) = 1$$

and initial condition

$$u(x,0) = x^2.$$

2. Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ on the disk $x^2 + y^2 \le 1$ with boundary condition $u(\cos\theta, \sin\theta) = \pi^2 - \theta^2,\ -\pi < \theta \le \pi$.

11

3. Solve the inhomogeneous wave equation

$$u_{tt} = u_{xx} + x \sin t, \quad 0 \le x \le 1, t \ge 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = u_t(x,0) = 0.$$

4. Solve the wave equation $u_{tt} = u_{xx} + u_{yy}$ on the unit disk $x^2 + y^2 \le 1$ with boundary condition

$$u(x,y) = 0 \text{ if } x^2 + y^2 = 1,$$

and initial conditions

$$u(x,y,0) = 0, \quad u_t(x,y,0) = \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } x^2 + y^2 \le \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < \epsilon < 1$. Hint: The formula $\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$ may be used. Extra credit: Plot the solution $u(r,t)$ for $\epsilon = \frac{1}{2}$, $t = 1$ and $t = 2$.

5. Find the radial eigenfunctions and corresponding eigenvalues of the Laplace operator Δ on the unit ball subject to Dirichlet boundary conditions. A radial eigenfunction is one which depends only on $r = \sqrt{x^2 + y^2 + z^2}$. That is, solve

$$u_{xx} + u_{yy} + u_{zz} + \lambda^2 u = 0$$

where $u(x,y,z) = R(r)$ with boundary condition

$$u(x,y,z) = 0 \quad \text{if } x^2 + y^2 + z^2 = 1.$$

Hint: The substitution $rR(r) = \tilde{R}(r)$ is useful.

5.2.2 Problem 1

Solve the heat equation

$$u_t = 9u_{xx}$$

For $0 \leq x \leq 1, t \geq 0$ with boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 1$$

And initial conditions

$$u(x,0) = x^2$$

solution

Since the one of the boundary conditions are inhomogeneous, the solution is broken into two parts. Let the solution be

$$u(x,t) = w(x,t) + v(x) \tag{1}$$

Where $w(x,t)$ is the solution to the PDE with homogeneous boundary conditions and $v(x)$ is a reference solution which is only required to satisfy the inhomogeneous boundary condition[1].

Let $v(x) = Ax + B$. At $x = 0$ then $v(0) = 0$ which gives $B = 0$. Hence $v(x) = Ax$. At $x = 1$, $v(1) = 1$ or $A = 1$. Therefore

$$v(x) = x$$

And (1) becomes

$$u(x,t) = w(x,t) + x \tag{1A}$$

Where now $w(x,t)$ satisfies the PDE

$$w_t = 9w_{xx} \tag{1B}$$

For $0 \leq x \leq 1, t \geq 0$ but with the following homogeneous boundary conditions

$$w(0,t) = 0$$
$$w(1,t) = 0$$

And initial conditions given by

$$\begin{aligned} w(x,0) &= u(x,0) - v(x) \\ &= x^2 - x \end{aligned} \tag{2}$$

The PDE (1B) is the heat PDE with homogeneous boundary conditions. This was solved before. It has the solution

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n \alpha^2 t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $\alpha^2 = 9$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3 \cdots$. But $L = 1$, therefore the above solution reduces to

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-9n^2\pi^2 t} \sin(n\pi x) \tag{3}$$

c_n is now found from the initial conditions (2). At $t = 0$ the above becomes

$$x^2 - x = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

Applying orthogonality gives

$$\begin{aligned} \int_0^1 \left(x^2 - x\right) \sin(n\pi x)\, dx &= c_n \int_0^1 \sin^2(n\pi x)\, dx \\ &= c_n \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} c_n &= 2\int_0^1 \left(x^2 - x\right) \sin(n\pi x)\, dx \\ &= 2\left(\int_0^1 x^2 \sin(n\pi x) - \int_0^1 x \sin(n\pi x)\, dx\right) \end{aligned} \tag{3A}$$

Applying the following rule based on integration by parts $\int x^2 \sin(ax) = \frac{2x \sin ax}{a^2} + \left(\frac{2}{a^3} - \frac{x^2}{a}\right) \cos ax$, the first integral above becomes (where $a = n\pi$)

$$\begin{aligned} \int_0^1 x^2 \sin(n\pi x) &= \left[\frac{2x \sin n\pi x}{(n\pi)^2} + \left(\frac{2}{(n\pi)^3} - \frac{x^2}{n\pi}\right) \cos n\pi x\right]_0^1 \\ &= \left[\frac{2 \sin n\pi}{(n\pi)^2} + \left(\frac{2}{(n\pi)^3} - \frac{1}{n\pi}\right) \cos n\pi - \frac{2}{(n\pi)^3}\right] \end{aligned}$$

[1] $w(x,t)$ is called the transient solution with homogeneous boundary conditions, and $v(x)$ the steady state solution with the inhomogeneous boundary conditions.

But $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, therefore the above becomes

$$\begin{aligned}\int_0^1 x^2 \sin(n\pi x) &= \left(\frac{2}{n^3\pi^3} - \frac{1}{n\pi}\right)(-1)^n - \frac{2}{n^3\pi^3} \\ &= \frac{2(-1)^n}{n^3\pi^3} - \frac{(-1)^n}{n\pi} - \frac{2}{n^3\pi^3} \\ &= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left(-1 + (-1)^n\right) \qquad \text{(3A1)}\end{aligned}$$

Applying the following rule based on integration by parts $\int x \sin(ax) = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$, then the second integral in (3A) becomes (where $a = n\pi$)

$$\begin{aligned}\int_0^1 x \sin(n\pi x)\, dx &= \left[\frac{\sin n\pi x}{n^2\pi^2} - \frac{x \cos n\pi x}{n\pi}\right]_0^1 \\ &= \frac{\sin n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n\pi} \\ &= \frac{(-1)^{n+1}}{n\pi} \qquad \text{(3A2)}\end{aligned}$$

Substituting (3A1) and (3A2) into (3A) gives

$$\begin{aligned}c_n &= 2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left(-1 + (-1)^n\right) - \frac{(-1)^{n+1}}{n\pi}\right) \\ &= \frac{4}{n^3\pi^3}\left(-1 + (-1)^n\right)\end{aligned}$$

Therefore, the solution $w(x,t)$ from (3) becomes

$$w(x,t) = \sum_{n=1}^{\infty} 4\frac{\left(-1 + (-1)^n\right)}{n^3\pi^3} e^{-9n^2\pi^2 t} \sin(n\pi x)$$

And the solution $u(x,t)$ from (1A) is

$$u(x,t) = x + \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{\left(-1 + (-1)^n\right)}{n^3} e^{-9n^2\pi^2 t} \sin(n\pi x)$$

Only few terms are needed to obtain a very good approximation, since the convergence is of order $O\left(\frac{1}{n^3}\right)$.

5.2.3 Problem 2

Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ on the unit disk $x^2 + y^2 \leq 1$ with boundary conditions $u(\cos\theta, \sin\theta) = \pi^2 - \theta^2$ and $-\pi < \theta \leq \pi$

solution

This is Laplace PDE on disk. Where $a = 1$ is the radius and r, θ are polar coordinates. The Laplacian in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

With boundary conditions on r being

$$u(a,\theta) = f(\theta) = \pi^2 - \theta^2$$
$$u(0,\theta) < \infty$$

And with standard periodic boundary conditions on θ

$$u(r,-\pi) = u(r,\pi)$$
$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

This PDE was solved before and the solution to the Laplace PDE inside a disk is known to be

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + d_n \sin n\theta) \tag{1}$$

With Fourier coefficients given by

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$c_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta$$
$$d_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta$$

Since the radius $a = 1$ in this problem, then the above become

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta$$
$$d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta$$

The coefficients are now calculated[2].

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right) d\theta$$

But $\int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right) d\theta = \int_{-\pi}^{\pi} \pi^2 d\theta - \int_{-\pi}^{\pi} \theta^2 d\theta = 2\pi^3 - \left[\frac{\theta^3}{3}\right]_{-\pi}^{\pi} = 2\pi^3 - \frac{1}{3}\left[\pi^3 + \pi^3\right] = 2\pi^3 - \frac{2}{3}\pi^3 = \frac{4}{3}\pi^3$. Therefore

$$A_0 = \frac{1}{2\pi}\left(\frac{4}{3}\pi^3\right) = \frac{2\pi^2}{3}$$

And

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right)\cos(n\theta)\, d\theta \\ &= \pi\int_{-\pi}^{\pi} \cos(n\theta)\, d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2\cos(n\theta)\, d\theta \end{aligned}$$

But $\int_{-\pi}^{\pi} \cos(n\theta)\, d\theta = 0$ and by integration by parts as was done earlier $\int_{-\pi}^{\pi} \theta^2 \cos(n\theta)\, d\theta = \frac{4(-1)^n \pi}{n^2}$, hence the above simplifies to

$$c_n = -\frac{4(-1)^n}{n^2}$$

[2] It is important to use integration limit $-\pi \cdots \pi$ and not $0 \cdots 2\pi$.

And

$$
\begin{aligned}
d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\pi^2-\theta^2\right)\sin(n\theta)\,d\theta \\
&= \pi\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\theta^2\sin(n\theta)\,d\theta
\end{aligned}
$$

But $\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta = 0$, and by integration by parts as was done earlier, $\int_{-\pi}^{\pi}\theta^2\sin(n\theta)\,d\theta = 0$, hence

$$
d_n = 0
$$

Using the value of A_0, c_n, d_n found above the solution (1) becomes

$$
\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} r^n\left(c_n\cos n\theta + d_n\sin n\theta\right) \\
&= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} r^n\cos n\theta
\end{aligned}
$$

5.2.4 Problem 3

Solve the inhomogeneous wave equation

$$u_{tt} = u_{xx} + x \sin t$$

For $0 \leq x \leq 1, t \geq 0$ with boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

And initial conditions

$$\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{aligned}$$

solution

Since the inhomogeneity is in the PDE itself (rather than in the boundary conditions), then the method of eigenfunction expansion is used to obtain the solution. Let the solution be

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions of the spatial eigenvalue ODE problem that comes from solving the homogeneous wave equation with the given homogeneous boundary conditions, which is $u_{tt} = u_{xx}$. This wave PDE with the given homogeneous boundary conditions was solved before using separation of variables. The eigenfunctions were found to be

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

With eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

But $L = 1$ here, therefore

$$\lambda_n = n^2\pi^2 \qquad n = 1, 2, 3, \cdots$$

Now that the eigenvalues and eigenfunctions are found, equation (1) is substituted back into the PDE resulting in

$$\sum_{n=1}^{\infty} b_n''(t)\Phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Phi_n''(x) + x\sin t$$

Since $x \sin t$ is a piecewise continuous function in x, it can be represented using the same eigenfunctions[3] and the above equation becomes

$$\sum_{n=1}^{\infty} b_n''(t)\Phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Phi_n''(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x)$$

Since $\Phi_n''(x) = -\lambda_n\Phi_n(x)$, which comes from the eigenvalue ODE, the above simplifies to

$$\begin{aligned} \sum_{n=1}^{\infty} b_n''(t)\Phi_n(x) &= \sum_{n=1}^{\infty} -b_n(t)\lambda_n\Phi_n(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x) \\ b_n''(t) + b_n(t)\lambda_n &= \gamma_n(t) \end{aligned} \tag{2}$$

To solve the above ODE for $b_n(t)$, $\gamma_n(t)$ needs to be found first. Using

$$x\sin t = \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x)$$

Applying orthogonality gives

$$\sin(t)\int_0^1 x\Phi_n(x)\,dx = \gamma_n(t)\int_0^1 \Phi_n^2(x)\,dx$$

Since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$, then $\int_0^1 \Phi_n^2(x)\,dx = \frac{1}{2}$ and the above reduces to

$$\sin(t)\int_0^1 x\sin(n\pi x)\,dx = \frac{1}{2}\gamma_n(t)$$

[3] This is the same as saying the eigenfunctions are complete.

But $\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi}$ by integration by part as was done before, and the above becomes

$$\begin{aligned}\frac{(-1)^{n+1}}{n\pi}\sin(t) &= \frac{1}{2}\gamma_n(t)\\ \gamma_n(t) &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\end{aligned}$$

Using the above back in (2), ODE (2) now becomes

$$b_n''(t) + b_n(t)\,n^2\pi^2 = 2\frac{(-1)^{n+1}}{n\pi}\sin(t) \tag{3}$$

This is a second order, inhomogeneous, linear, with constant coefficients ODE. The solution is the sum of the homogeneous and particular solutions (the subscript n is removed for now from $b_n(t)$, to simplify the notations, then added back after the solution is obtained). Let the solution to (3) be

$$b(t) = b_h(t) + b_p(t)$$

The homogeneous solution is seen to be (since $n^2\pi^2$ is always positive)

$$b_h(t) = A\cos(n\pi t) + B\sin(n\pi t)$$

To find the particular solution, the method of undetermined coefficients is used. let

$$b_p(t) = C\cos(t) + D\sin(t) \tag{4}$$

Hence $b_p' = -C\sin(t) + D\cos(t)\,, b_p'' = -C\cos(t) - D\sin(t)$. Substituting these into (3) gives

$$\begin{aligned}-C\cos(t) - D\sin(t) + (C\cos(t) + D\sin(t))\,n^2\pi^2 &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\\ \cos(t)\left[-C + Cn^2\pi^2\right] + \sin(t)\left[-D + Dn^2\pi^2\right] &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\end{aligned}$$

Therefore, by comparing coefficients

$$\begin{aligned}-C + Cn^2\pi^2 &= 0\\ C\left(n^2\pi^2 - 1\right) &= 0\\ C &= 0\end{aligned}$$

And

$$\begin{aligned}-D + Dn^2\pi^2 &= 2\frac{(-1)^{n+1}}{n\pi}\\ D\left(n^2\pi^2 - 1\right) &= 2\frac{(-1)^{n+1}}{n\pi}\\ D &= 2\frac{(-1)^{n+1}}{n\pi\,(n^2\pi^2 - 1)}\end{aligned}$$

Hence the particular solution (4) is

$$\begin{aligned}b_p(t) &= C\cos(t) + D\sin(t)\\ &= 2\frac{(-1)^{n+1}}{n\pi\,(n^2\pi^2 - 1)}\sin(t)\end{aligned}$$

Now that the particular solution is found, the final solution to (3) becomes

$$b_n(t) = A_n\cos(n\pi t) + B_n\sin(n\pi t) - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2 - 1)}\sin(t) \tag{4}$$

Using the above in the solution (1) gives

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\,\Phi_n(x)\\ &= \sum_{n=1}^{\infty}\left(A_n\cos(n\pi t) + B_n\sin(n\pi t) - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2 - 1)}\sin(t)\right)\sin(n\pi x)\end{aligned}$$

A_n, B_n are found from initial conditions. At $t = 0$ the above simplifies to

$$0 = \sum_{n=1}^{\infty} A_n\sin(n\pi x)$$

Therefore $A_n = 0$ and the solution above reduces to

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \sin(n\pi t) - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \sin(t) \right) \sin(n\pi x) \tag{5}$$

Taking time derivative gives

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(B_n n\pi \cos(n\pi t) - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \cos(t) \right) \sin(n\pi x)$$

At $t = 0$ the above simplifies to

$$0 = \sum_{n=1}^{\infty} \left(B_n n\pi - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \right) \sin(n\pi x)$$

Since this is valid for each n, then $\left(B_n n\pi - 2\frac{(-1)^n}{n\pi(n^2\pi^2-1)} \right) = 0$ or

$$B_n = 2\frac{(-1)^n}{n^2\pi^2(n^2\pi^2 - 1)}$$

Using the above in (5), the final solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} 2\left(\frac{(-1)^n}{n^2\pi^2(n^2\pi^2 - 1)} \sin(n\pi t) - \frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \sin(t) \right) \sin(n\pi x) \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \left(\frac{\sin(n\pi t)}{n\pi} - \sin(t) \right) \sin(n\pi x) \end{aligned}$$

5.2.5 Problem 4

Solve the wave equation

$$u_{tt} = u_{xx} + u_{yy}$$

On the unit disk $x^2 + y^2 \leq 1$ with with boundary conditions

$$u(x,y) = 0 \qquad \text{if } x^2 + y^2 = 1$$

And initial conditions

$$\begin{aligned} u(x,y,0) &= 0 \\ u_t(x,y,0) &= \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } \sqrt{x^2+y^2} \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Where $0 < \epsilon < 1$. Hint: The formula $\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$ may be used. Extra credit: Plot the solution $u(r,t)$ for $\epsilon = \frac{1}{2}, t = 1$ and $t = 2$.

solution

The PDE and initial and boundary conditions are converted to polar coordinates to become

$$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \tag{1}$$

On the unit disk with radius 1. The boundary conditions are

$$\begin{aligned} u(1,\theta,t) &= 0 \\ u(0,\theta,t) &< \infty \end{aligned}$$

Where $u(0,\theta,t) < \infty$ means the solution is bounded at center of disk $r = 0$. The boundary conditions on θ are the standard periodic boundary conditions

$$\begin{aligned} u(r,-\pi,t) &= u(r,\pi,t) \\ u_\theta(r,-\pi,t) &= u_\theta(r,\pi,t) \end{aligned}$$

And initial conditions are[4]

$$\begin{aligned} u(r,\theta,0) &= 0 \\ u_t(r,\theta,0) &= \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } r \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The above PDE is solved by separation of variables. Let $u(r,\theta,t) = T(t)R(r)\Theta(\theta)$. Substituting this in the PDE (1) gives

$$T''R\Theta = R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}\Theta''RT$$

Dividing by $RT\Theta$

$$\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

Where λ is the first separation variable. This results in two equations

$$\frac{T''}{T} = -\lambda^2 \tag{1}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2 \tag{2}$$

The time ODE (1) is

$$T'' + \lambda^2 T = 0 \tag{1A}$$

Multiplying (2) by r^2 and rearranging

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = \mu^2$$

Where μ is the second separation constant. This gives the R ODE as

$$r^2R'' + rR' + \left(r^2\lambda^2 - \mu^2\right)R = 0 \tag{3}$$

And the Θ ODE as

$$\Theta'' + \mu^2\Theta = 0 \tag{4}$$

[4] The original $r^2 \leq \epsilon$ was changed to $r \leq \epsilon$

The eigenvalues for (4) determine the Bessel equation (3) order. Therefore (4) needs to be solved first to determined the order. The ODE boundary conditions for (4) are periodic

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi)\\ \Theta'(-\pi) &= \Theta'(\pi)\end{aligned}$$

case $\mu = 0$. This leads to solution

$$\begin{aligned}\Theta &= c_1\theta + c_2\\ \Theta' &= c_1\end{aligned}$$

First BC gives

$$\begin{aligned}-c_1\pi + c_2 &= c_1\pi + c_2\\ c_1 &= 0\end{aligned}$$

And since second BC $\Theta' = c_1$, this implies $\Theta(\theta)$ is constant. So $\mu = 0$ is an eigenvalue, with $\Theta_0(\theta) = 1$ being the eigenfunction.
Case $\mu > 0$ The solution to (4) becomes

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

To satisfy the periodic boundary conditions, μ must be an integer, and since $\mu > 0$, then $\mu = n$ for $n = 1, 2, 3, \cdots$. Therefore

$$\Theta_0(\theta) = 1 \qquad n = 0 \tag{5A}$$
$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 1, 2, 3, \cdots \tag{5B}$$

The above solution can be combined to one

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 0, 1, 2, \cdots \tag{5}$$

Because when $n = 0$ the above solution gives $\Theta_0(\theta) = A_0$ which is the constant eigenfunction. Now that μ is found, Bessel ODE (3) can be solved.

$$\begin{aligned}r^2R''(r) + rR'(r) + \left(r^2\lambda^2 - n^2\right)R(r) &= 0 \qquad n = 0, 1, 2, 3, \cdots \\ R(1) &= 0\\ R(0) &< \infty\end{aligned} \tag{5C}$$

$\lambda = 0$ is not a possible eigenvalue. This can be shown as follows. When $\lambda = 0$ equation (5C) becomes the Euler ODE

$$r^2R''(r) + rR'(r) + n^2R(r) = 0 \qquad n = 0, 1, 2, 3, \cdots$$

Now, when $n = 0$, then the ODE becomes $r^2R''(r) + rR'(r) = 0$ whose solution is $R(r) = c_1 + c_2\ln(r)$. Since solution is bounded at $r = 0$, then $R(r) = c_1$. And since $R(1) = 0$ then $c_1 = 0$ also, leading to trivial solution. When $n > 0$, the ODE becomes $r^2R''(r) + rR'(r) + n^2R(r) = 0$ whose solution is $R(r) = c_1r^n + c_2\frac{1}{r^n}$. Since solution is bounded at $r = 0$, then $c_2 = 0$ and the solution now becomes $R(r) = c_1r^n$. Using BC $R(1) = 0$ gives $c_1 = 0$ leading again to trivial solution. This shows that $\lambda = 0$ is not eigenvalue. Now that λ is is shown not to be zero, the Bessel ODE (5C) is solved . The first step is to convert the ODE to a Bessel ODE in the classical form in order to use the standard solution. Let $t = \lambda r$, then $R'(r) = R'(t)\lambda$ and $R''(r) = R''(t)\lambda^2$. ODE (5C) becomes

$$\begin{aligned}\frac{t^2}{\lambda^2}\lambda^2R''(t) + \frac{t}{\lambda}\lambda R'(t) + \left(\frac{t^2}{\lambda^2}\lambda^2 - n^2\right)R(t) &= 0\\ t^2R''(t) + tR'(t) + \left(t^2 - n^2\right)R(t) &= 0\end{aligned}$$

This is now in standard Bessel ODE form. This is of order n, where n is $n = 0, 1, 2, 3, \cdots$. Since the order is integer, then the solution is given by

$$R_n(t) = C_nJ_n(t) + D_nY_n(t)$$

Where $J_n(t)$ is the Bessel function of order n and $Y_n(t)$ is the Bessel function of second kind of order n. In terms of r the above solution becomes

$$R_n(r) = C_nJ_n(\lambda r) + D_nY_n(\lambda r)$$

Because the solution is bounded at $r = 0$ and since $Y_n(0)$ blows up, then $D_n = 0$. The above solution simplifies to

$$R_n(r) = C_nJ_n(\lambda r)$$

Applying the second boundary conditions, when $r = 1$ then

$$0 = C_n J_n(\lambda)$$

For non-trivial solution $J_n(\lambda) = 0$. Hence λ are the positive zeros of $J_n(z)$. Let the positive zeros of $J_n(z)$ be j_{nm}. For $m = 1, 2, 3, \cdots$. Therefore

$$\lambda_{nm} = j_{nm} \qquad n = 0, 1, 2, \cdots, m = 1, 2, 3, \cdots$$

This means that j_{nm} is the m^{th} eigenvalue for the n^{th} order Bessel function $J_n(z)$. So there are two indices to handle in this problem. The order of the Bessel function is determined from the $\Theta_n(\theta)$ eigenvalues, and then once this order n is fixed, the second eigenvalue λ_{nm} is determined from the zeros of the Bessel function $J_n(z)$. Hence the $R_{nm}(r)$ solution is

$$R_{nm}(r) = C_{nm} J_n(\lambda_{nm} r) \qquad n = 0, 1, 2, 3, \cdots, m = 1, 2, 3, \cdots$$

Now that λ_{nm} is known, the time ODE (1) can be solved

$$\begin{aligned} T''_{nm} + \lambda_{nm}^2 T_{nm} &= 0 \\ T_{nm} &= A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t) \qquad n = 0, 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{aligned}$$

The fundamental solution is therefore

$$u_{nm}(r, \theta, t) = \Theta_n(\theta) T_{nm}(t) R_{nm}(r)$$

The complete solution is the superposition of the fundamental solutions given by

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \Theta_n(\theta) T_{nm}(t) R_{nm}(r) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\{A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)\} C_{nm} J_n(\lambda_{nm} r) \end{aligned}$$

The above can now be written as

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta ((A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)) C_{nm} J_n(\lambda_{nm} r)) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta ((A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)) C_{nm} J_n(\lambda_{nm} r)) \end{aligned}$$

Or

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta A_{nm}\cos(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta B_{nm}\sin(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta A_{nm}\cos(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta B_{nm}\sin(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \end{aligned} \tag{6}$$

Constants are now merged and renamed as follows in order to simplify the rest of the solution. Let

$$\begin{aligned} A_n A_{nm} C_{nm} &= \bar{A}_{nm} \\ A_n B_{nm} C_{nm} &= \bar{B}_{nm} \\ B_n A_{nm} C_{nm} &= \bar{C}_{nm} \\ B_n B_{nm} C_{nm} &= \bar{D}_{nm} \end{aligned}$$

Equation (6) can now be written as

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{A}_{nm}\cos(n\theta)\cos(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{B}_{nm}\cos(n\theta)\sin(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{nm}\sin(n\theta)\cos(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{D}_{nm}\sin(n\theta)\sin(\lambda_{nm} t) J_n(\lambda_{nm} r) \end{aligned} \tag{7}$$

Initial conditions are used to determine the 4 new constants above. Using initial condition at $t = 0, u(r,\theta,0) = 0$ the above equation becomes

$$0 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{A}_{nm}\cos(n\theta) J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{nm}\sin(n\theta) J_n(\lambda_{nm} r)$$

Applying orthogonality on $\cos(n\theta)$ and $\sin(n\theta)$ in turn shows that $\bar{A}_{nm} = 0$ and $\bar{C}_{nm} = 0$. Therefore the solution (7) reduces to the following two sums only

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{B}_{nm}\cos(n\theta)\sin(\lambda_{nm}t) J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{D}_{nm}\sin(n\theta)\sin(\lambda_{nm}t) J_n(\lambda_{nm} r) \tag{8}$$

Taking time derivative gives

$$u_t(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{B}_{nm}\cos(n\theta)\lambda_{nm}\cos(\lambda_{nm}t) J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{D}_{nm}\sin(n\theta)\lambda_{nm}\cos(\lambda_{nm}t) J_n(\lambda_{nm} r)$$

Applying the second initial condition at $t = 0$ gives

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{B}_{nm}\cos(n\theta)\lambda_{nm} J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{D}_{nm}\sin(n\theta)\lambda_{nm} J_n(\lambda_{nm} r) = \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } r \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Case $n = 0$ (9) becomes

$$\sum_{m=1}^{\infty} \bar{B}_{0m}\lambda_{0m} J_0(\lambda_{0m} r) = \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } r \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Applying orthogonality on $J_0(\lambda_{0m} r)$ results in

$$\bar{B}_{0m}\lambda_{0m}\int_0^1 r J_0^2(\lambda_{0m} r)\,dr = \frac{1}{\pi\epsilon^2}\int_0^{\epsilon} r J_0(\lambda_{0m} r)\,dr$$

$$\bar{B}_{0m} = \frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\int_0^{\epsilon} r J_0(\lambda_{0m} r)\,dr}{\int_0^1 r J_0^2(\lambda_{0m} r)\,dr} \tag{9A}$$

Case $n > 1$ Applying orthogonality on $\cos(n\theta)$, equation (9) becomes

$$\sum_{m=1}^{\infty} \bar{B}_{nm}\left(\int_{-\pi}^{\pi}\cos^2(n\theta)\,d\theta\right)\lambda_{nm} J_n(\lambda_{nm} r) = \begin{cases} \frac{1}{\pi\epsilon^2}\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta & \text{if } r^2 \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{m=1}^{\infty} \pi\bar{B}_{nm}\lambda_{nm} J_n(\lambda_{nm} r) = \begin{cases} 0 & \text{if } r^2 \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Hence $\bar{B}_{nm} = 0$ for all $n > 0$.
The same is now done to find $\bar{D}_{nm}$. Applying orthogonality on $\sin(n\theta)$, equation (9) becomes

$$\sum_{m=1}^{\infty} \bar{D}_{nm}\left(\int_{-\pi}^{\pi}\sin^2(n\theta)\,d\theta\right)\lambda_{nm} J_n(\lambda_{nm} r) = \begin{cases} \frac{1}{\pi\epsilon^2}\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta & \text{if } r^2 \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{m=1}^{\infty} \bar{D}_{nm}\left(\int_{-\pi}^{\pi}\sin^2(n\theta)\,d\theta\right)\lambda_{nm} J_n(\lambda_{nm} r) = \begin{cases} 0 & \text{if } r^2 \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Hence all $\bar{D}_{nm} = 0$ for all $n > 0$.
Therefore the solution (8) reduces to only using $n = 0, m = 1, 2, 3, \cdots$. The solution can now be written as

$$u(r,\theta,t) = \sum_{m=1}^{\infty} \bar{B}_{0m}\sin(\lambda_{0m}t) J_0(\lambda_{0m} r) \tag{10}$$

Where $\bar{B}_{0m} = \frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\int_0^{\epsilon} r J_0(\lambda_{0m} r)dr}{\int_0^1 r J_0^2(\lambda_{0m} r)dr}$ And λ_{0m} are all the positive zeros of $J_0(z)$, $m = 1, 2, 3, \cdots$.
$\bar{B}_{0m}$ is now simplified more. Considering first the numerator of $\bar{B}_{0m}$ which is $\int_0^{\epsilon} r J_0(\lambda_{0m} r)\,dr$. The hint given says that

$$\frac{d}{dr}(r J_1(r)) = r J_0(r)$$

This is the same as saying

$$r J_1(r) = \int r J_0(r)\,dr \tag{10A}$$

However the integral in $\bar{B}_{0m}$ is $\int r J_0(\lambda_{0m} r)\,dr$ and not $\int r J_0(r)\,dr$. To transform it so that the hint can be used, let $\lambda_{0m} r = \bar{r}$, then $\frac{dr}{d\bar{r}} = \frac{1}{\lambda_{0m}}$ or $dr = \frac{d\bar{r}}{\lambda_{0m}}$. Now $\int r J_0(\lambda_{0m} r)\,dr$ becomes $\int \frac{\bar{r}}{\lambda_{0m}} J_0(\bar{r})\frac{d\bar{r}}{\lambda_{0m}}$ or $\frac{1}{\lambda_{0m}^2}\int \bar{r} J_0(\bar{r})\,d\bar{r}$ and now the hint (10A) can be used on this integral giving

$$\frac{1}{\lambda_{0m}^2}\left(\int \bar{r} J_0(\bar{r})\,d\bar{r}\right) = \frac{1}{\lambda_{0m}^2}(\bar{r} J_1(\bar{r}))$$

Replacing $\bar{r}$ back by $\lambda_{0m}r$, gives the result needed

$$\begin{aligned}\frac{1}{\lambda_{0m}^2}\left(\bar{r}J_1\left(\bar{r}\right)\right) &= \frac{1}{\lambda_{0m}^2}\left(\lambda_{0m}rJ_1\left(\lambda_{0m}r\right)\right)\\ &= \frac{1}{\lambda_{0m}}rJ_1\left(\lambda_{0m}r\right)\end{aligned}$$

Now the limits are applied, using the fundamental theory of calculus

$$\begin{aligned}\int_0^{\epsilon} rJ_0\left(\lambda_{0m}r\right)dr &= \frac{1}{\lambda_{0m}}\left[rJ_1\left(\lambda_{0m}r\right)\right]_0^{\epsilon}\\ &= \frac{\epsilon}{\lambda_{0m}}J_1\left(\lambda_{0m}\epsilon\right) \qquad \text{(10B)}\end{aligned}$$

This completes finding the numerator integral in $\bar{B}_{0m}$. The denominator integral in $\bar{B}_{0m}$ is $\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr$. This was found in HW4, from problem 3, which is

$$\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr = \frac{1}{2}\left[J_0'\left(\lambda_{0m}\right)\right]^2$$

But $J_0'\left(\lambda_{0m}\right) = -J_1\left(\lambda_{0m}\right)$, hence the above becomes

$$\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr = \frac{1}{2}J_1^2\left(\lambda_{0m}\right) \qquad \text{(10C)}$$

Applying (10B) and (10C), $\bar{B}_{0m}$ simplifies to the following expression

$$\begin{aligned}\bar{B}_{0m} &= \frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\frac{\epsilon}{\lambda_{0m}}J_1\left(\lambda_{0m}\epsilon\right)}{\frac{1}{2}J_1^2\left(\lambda_{0m}\right)}\\ &= \frac{2}{\pi\epsilon\lambda_{0m}^2}\frac{J_1\left(\lambda_{0m}\epsilon\right)}{J_1^2\left(\lambda_{0m}\right)}\end{aligned}$$

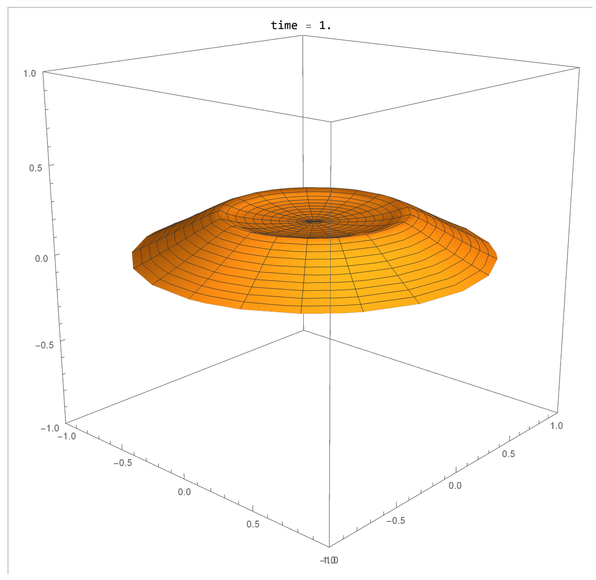
Therefore the final solution becomes

$$\begin{aligned}u\left(r,\theta,t\right) &= \sum_{m=1}^{\infty}\bar{B}_{0m}\sin\left(\lambda_{0m}t\right)J_0\left(\lambda_{0m}r\right)\\ u\left(r,\theta,t\right) &= \frac{2}{\pi\epsilon}\sum_{m=1}^{\infty}\frac{1}{\lambda_{0m}^2}\frac{J_1\left(\lambda_{0m}\epsilon\right)}{J_1^2\left(\lambda_{0m}\right)}J_0\left(\lambda_{0m}r\right)\sin\left(\lambda_{0m}t\right) \qquad \text{(11)}\end{aligned}$$

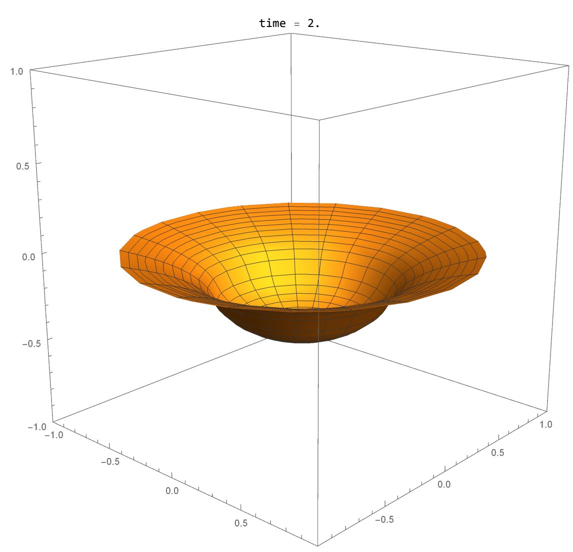
Plotting. When $\epsilon = \frac{1}{2}$, the above solution (11) becomes

$$u(r,\theta,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{\lambda_{0m}^2}\frac{J_1\left(\frac{1}{2}\lambda_{0m}\right)}{J_1^2(\lambda_{0m})}J_0(\lambda_{0m}r)\sin(\lambda_{0m}t) \tag{11A}$$

This is the 3D plot at $t = 1$ second



This is the 3D plot at $t = 2$ seconds

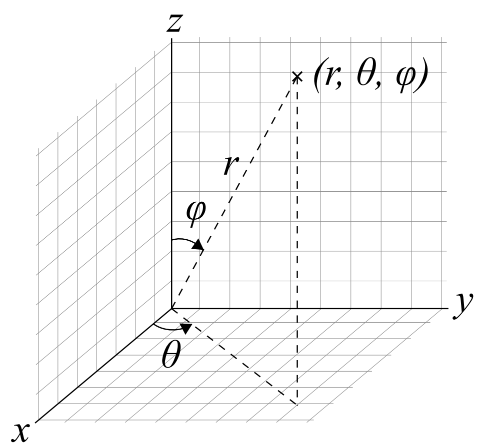


5.2.6 Problem 5

Find the radial eigenfunctions and corresponding eigenvalues of the Laplace operator on the unit ball subject to Dirichlet boundary conditions. A radial eigenfunction is one which depends only on $r = \sqrt{x^2 + y^2 + z^2}$. That is, solve

$$u_{xx} + u_{yy} + u_{zz} + \lambda^2 u = 0$$

Where $u(x, y, z) = R(r)$ with boundary conditions $u(x, y, z) = 0$ when $x^2 + y^2 + z^2 = 1$.
Hint: The substitution $rR(r) = \bar{R}(r)$ is useful.
solution
This is Helmholtz PDE $\nabla^2 u + \lambda^2 u = 0$ in 3D. (Steady state of the wave equation, or standing waves).
The following spherical coordinates system are used [5]



The Laplace operator in 3D using spherical coordinates (r, θ, ϕ) is given by

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Therefore $\nabla^2 u + \lambda^2 u = 0$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \lambda^2 u = 0$$

The problem says that $u(x, y, z) = R(r)$. This implies that solution depends only on r. This means there is no dependency on θ nor on ϕ. In this case, the PDE above simplifies to an ODE in r only.

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + \lambda^2 u &= 0\\ \frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + \lambda^2 r^2 u &= 0\\ r^2 \frac{d^2 u}{dr^2} + 2r\frac{du}{dr} + \lambda^2 r^2 u &= 0\end{aligned}$$

And since $u(r, \theta, \phi) \equiv R(r)$, then the above can be written as

$$r^2 R''(r) + 2rR'(r) + \lambda^2 r^2 R(r) = 0 \tag{1}$$

With the boundary conditions $R(1) = 0$. Now the eigenvalue will be found.
case $\lambda = 0$
The ODE (1) becomes $r^2 R'' + 2rR' = 0$. Let $R'(r) = v(r)$, and the ODE becomes $v' + \frac{2}{r}v = 0$. The integrating factor is $e^{\int \frac{2}{r}dr} = e^{2\ln|r|} = r^2$. $\frac{d}{dr}\left(r^2 v\right) = 0$ or $v = \frac{c_1}{r^2}$. Therefore $R'(r) = \frac{c_1}{r^2}$. Integrating again gives

$$R(r) = c_2 - \frac{c_1}{r}$$

At $R(1) = 0$, the above becomes

$$\begin{aligned}0 &= c_2 - c_1\\ c_2 &= c_1\end{aligned}$$

Hence the solution becomes

$$R(r) = c_1\left(1 - \frac{1}{r}\right)$$

[5] Image obtained from Wikepedia

The solution must be bounded as $r \to 0$, therefore only choice is $c_1 = 0$, leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Case $\lambda \neq 0$ [6]

The ODE is

$$r^2 R''(r) + 2rR'(r) + \lambda^2 r^2 R(r) = 0$$

Using standard transformation $t = \lambda r$, then $R'(r) = \lambda R'(t)$ and $R''(r) = \lambda^2 R''(t)$. The above ODE becomes

$$\begin{aligned}\lambda^2 r^2 R''(t) + 2\lambda r R'(t) + \lambda^2 r^2 R(t) &= 0\\ t^2 R''(t) + 2tR'(t) + t^2 R(t) &= 0 \qquad (2)\end{aligned}$$

This looks like a Bessel ODE of zero order, except Bessel ODE is $t^2 R''(t) + tR'(t) + t^2 R(t) = 0$. The difference is (2) has $2t$ instead of t. To convert it to Bessel ODE, there is another transformation in the dependent variable to achieve this. Let $R(t) = \frac{Z(t)}{\sqrt{t}}$, then

$$\begin{aligned}R'(t) &= \frac{Z'(t)}{\sqrt{t}} - \frac{1}{2} Z(t) \frac{1}{t^{\frac{3}{2}}} \qquad (3)\\ R''(t) &= \frac{Z''(t)}{\sqrt{t}} - \frac{1}{2} Z'(t) \frac{1}{t^{\frac{3}{2}}} - \frac{1}{2} Z'(t) \frac{1}{t^{\frac{3}{2}}} - \frac{1}{2}\left(-\frac{3}{2}\right) Z(t) \frac{1}{t^{\frac{5}{2}}} \qquad (4)\\ &= \frac{Z''(t)}{\sqrt{t}} - Z'(t) \frac{1}{t^{\frac{3}{2}}} + \frac{3}{4} Z(t) \frac{1}{t^{\frac{5}{2}}}\end{aligned}$$

Substituting (3,4) back in (2) gives

$$t^2 \left(\frac{Z''(t)}{\sqrt{t}} - Z'(t) \frac{1}{t^{\frac{3}{2}}} + \frac{3}{4} Z(t) \frac{1}{t^{\frac{5}{2}}}\right) + 2t \left(\frac{Z'(t)}{\sqrt{t}} - \frac{1}{2} Z(t) \frac{1}{t^{\frac{3}{2}}}\right) + t^2 \frac{Z(t)}{\sqrt{t}} = 0$$

Multiplying by $\sqrt{t}$ gives

$$\begin{aligned}t^2 \left(Z''(t) - Z'(t) \frac{1}{t} + \frac{3}{4} Z(t) \frac{1}{t^2}\right) + 2t \left(Z'(t) - \frac{1}{2} Z(t) \frac{1}{t}\right) + t^2 Z(t) &= 0\\ \left(t^2 Z''(t) - tZ'(t) + \frac{3}{4} Z(t)\right) + (2tZ'(t) - Z(t)) + t^2 Z(t) &= 0\\ t^2 Z''(t) + tZ'(t) + \frac{3}{4} Z(t) - Z(t) + t^2 Z(t) &= 0\\ t^2 Z''(t) + tZ'(t) + \left(\frac{3}{4} - 1 + t^2\right) Z(t) &= 0\\ t^2 Z''(t) + tZ'(t) + \left(t^2 - \frac{1}{4}\right) Z(t) &= 0\end{aligned}$$

Or

$$t^2 Z''(t) + tZ'(t) + \left(t^2 - \frac{1}{4}\right) Z(t) = 0$$

This is now in standard Bessel ODE form. To find the order, comparing it to $t^2 Z''(t) + tZ'(t) + \left(t^2 - n^2\right) Z(t) = 0$ shows that $n^2 = \frac{1}{4}$, hence the order is $\frac{1}{2}$. (the negative root, give Bessel function that blow up at zero. Therefore only $\frac{1}{2}$ root is used as the order. The solution of the above Bessel ODE is known to be

$$Z(t) = c_1 J_{\frac{1}{2}}(t) + c_2 Y_{\frac{1}{2}}(t)$$

From above, $R(t) = \frac{Z(t)}{\sqrt{t}}$. Therefore the solution now becomes

$$R(t) = c_1 \frac{J_{\frac{1}{2}}(t)}{\sqrt{t}} + c_2 \frac{Y_{\frac{1}{2}}(t)}{\sqrt{t}}$$

And converting back to $R(r)$ finally gives the radial solution as

$$R(r) = c_1 \frac{J_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}} + c_2 \frac{Y_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}}$$

Since the solution is bounded at $r = 0$, then $c_2 = 0$ and the solution simplifies to

$$R(r) = c_1 \frac{J_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}} \qquad (5)$$

Using $R(1) = 0$ gives

$$0 = c_1 \frac{J_{\frac{1}{2}}(\lambda)}{\sqrt{\lambda}}$$

[6] I am assuming λ is real eigenvalue. Not complex.

For non-trivial solution then

$$J_{\frac{1}{2}}(\lambda) = 0$$

Hence λ are the positive zeros of $J_{\frac{1}{2}}(\lambda)$. These are the eigenvalues. The zeros of $J_{\frac{1}{2}}(\lambda)$ are multiple of π. Hence the first zero is π, the second zero is 2π and so on.

$$\lambda_n = n\pi \qquad n = 1, 2, 3, \cdots$$

Therefore, the eigenfunctions (5) becomes

$$R_n(r) = \sqrt{\frac{1}{n\pi r}} J_{\frac{1}{2}}(n\pi r) \qquad n = 1, 2, 3, \cdots \tag{6}$$

These are also called spherical Bessel functions, since half integer order. There is a known relation between spherical Bessel functions and circular trigonometric functions which says

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

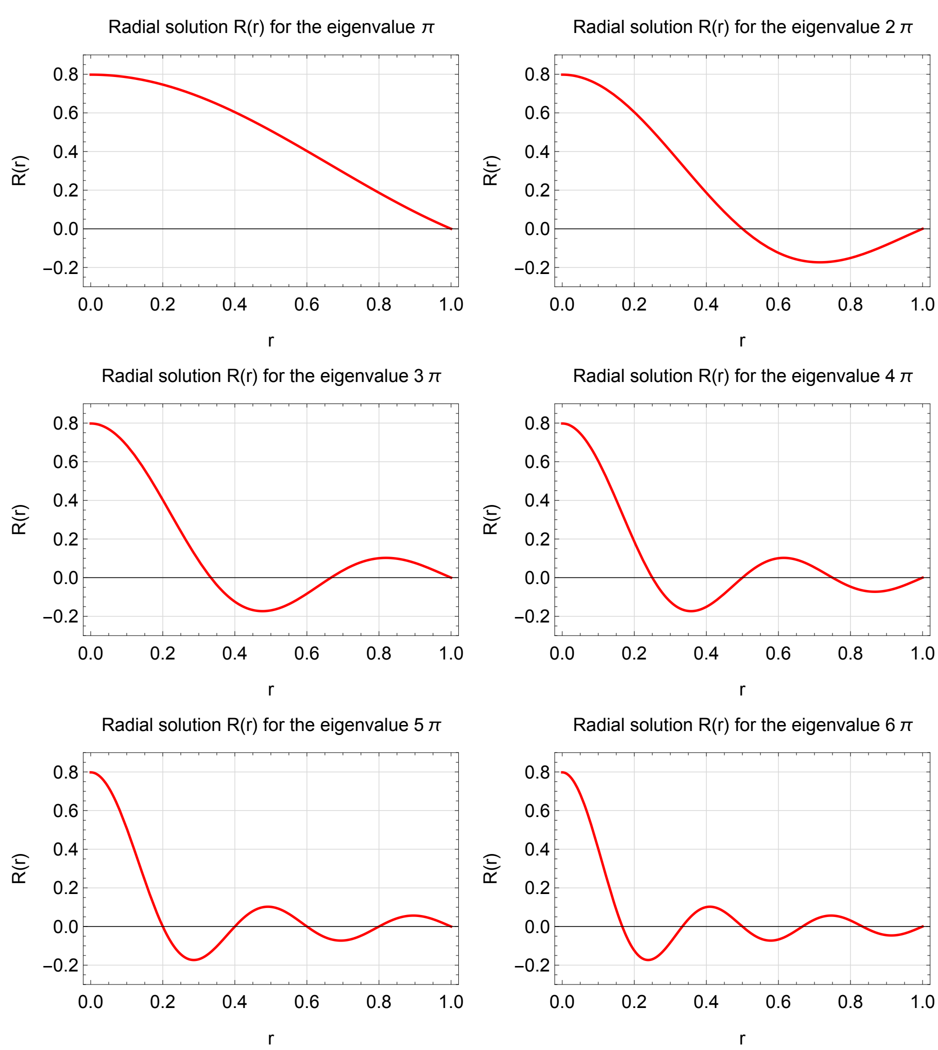
Using the above, the eigenfunctions (6) can also be written as

$$R_n(r) = \sqrt{\frac{2}{\pi^3}} \frac{\sin(n\pi r)}{nr} \qquad n = 1, 2, 3, \cdots$$

Note that

$$\begin{aligned} \lim_{r\to 0} \sqrt{\frac{2}{\pi^3}} \frac{\sin(n\pi r)}{nr} &= \sqrt{\frac{2}{\pi}} \\ &= 0.797885 \end{aligned}$$

For all n. Below is a plot of the first 6 eigenfunctions



References

In working on this exam, I have used a number of references such as Wikepidia, Wolfram Mathworld and the NIST Digital Library of Mathematical Functions.

Chapter 6: Study notes, cheat sheets

6.1 cheat sheets

6.1.1 First exam cheat sheet

Cheatsheet. Math 322. Intro to PDE. Nasser M ABBASI

Fourier Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

. note $f(x)$ assume to have $2L$ Period.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

Fourier Convergence Theorem

IF $f(x)$ is periodic with period $2L$, and integrable over $-L \cdots L$ and if and f' are Piecewise Continuous on $-L \le x < L$, Then its Fourier Series converges to the average value of $f(x)$ at each point.

Solutions to heat PDE on Rod.

$u_t = \alpha^2 u_{xx}$, $0 < x < L$, $t > 0$

$u=0$ —— L —— $u=0$, $f(x) @ u(x,0)$:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n \alpha^2 t} \sin(\sqrt{\lambda_n}\,x); \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n=1,2,\cdots$$

$\frac{\partial u}{\partial x}=0$ —— $\frac{\partial u}{\partial x}=0$:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} C_n e^{-\lambda_n \alpha^2 t} \cos(\sqrt{\lambda_n}\,x); \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n=1,2,\cdots$$

$\frac{\partial u}{\partial x}=0$ —— $u=0$:

$$u(x,t) = \sum_{n=1,3,5\ldots}^{\infty} C_n e^{-\lambda_n \alpha^2 t} \cos(\sqrt{\lambda_n}\,x); \quad \lambda_n = \left(\frac{n\pi}{2L}\right)^2 \; n=1,3,5 ?$$

$u=0$ —— $\frac{\partial u}{\partial x}=0$:

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} C_n e^{-\lambda_n \alpha^2 t} \sin(\sqrt{\lambda_n}\,x); \quad \lambda_n = \left(\frac{n\pi}{2L}\right)^2, n=1,3,5$$

in all above Cases $C_n = \frac{2}{L}\int_0^L f(x)\,\Phi_n(x)\,dx$. Where $\Phi_n(x) = \sin\sqrt{\lambda_n}x$ or $\cos\sqrt{\lambda_n}x$

IF problem has $u=0$ at $-L/2$, $u=0$ at $L/2$, $f(x)$, 0 at center, Then solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n \alpha^2 t} \sin\left(\sqrt{\lambda_n}\left(x+\frac{L}{2}\right)\right); \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with
$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\sin\left(\sqrt{\lambda_n}\left(x+\frac{L}{2}\right)\right)dx$$

B.C. nonhomogeneous: solve $u(x,t) = w(x,t) + r(x)$, where $r(x)$ is steady state solution, that satisfies the nonhomogeneous B.C. and $w(x,t)$ is standard one.

IF given point source $u=0$ ——$\downarrow Q$ at $L/2$—— L, $u=0$

Then Solve. 2 problems:

$u=0$ —— $L/2$, $u=Q$ using steady state $u_1 = w_1 + r_1$

and then solve

$u=Q$ at $L/2$ —— L, $u=0$ also using steady state $u_2 = w_2 + r_2$

Other Sources

IF source do not depend on time, use steady state trick.

$$T_1 \text{ ——}L\text{—— } T_2 \Rightarrow r(x) = T_1 + (T_2 - T_1)\frac{x}{L}$$ ← this function satisfies B.C.

IF there is source. $u=0$ —— $Q(x)=K$ —— $u=0$ Then $r(x) = \frac{KL}{2}x - \frac{Kx^2}{2}$.

This comes from solving $r'' = K$ and finding the particular Solution.

Wave PDE $u_{tt} = a^2 u_{xx}$. bounded domain. on string

Series solution: both ends fixed

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos(\sqrt{\lambda_n}\, at)\, \sin(\sqrt{\lambda_n}\, x)$$ case initial Velocity zero

$$= \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}\, at)\, \sin(\sqrt{\lambda_n}\, x)$$ case initial position zero

$\lambda_n = \left(\frac{n\pi}{L}\right)^2,\; n = 1, 2, \ldots$

in First case: $C_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\, x)\,dx$.

in second case $C_n = \frac{2}{L\sqrt{\lambda_n}}\int_0^L g(x)\sin\sqrt{\lambda_n}\, x\,dx$

Left end Fixed, right end free (i.e. $\frac{\partial u}{\partial x} = 0$ at $x = L$). initial Velocity Zero.

$$u(x,t) = \sum_{n=1,3,5}^{\infty} C_n \cos(\sqrt{\lambda_n}\, at)\, \sin(\sqrt{\lambda_n}\, x);\; \lambda_n = \left(\frac{n\pi}{2L}\right)^2,\; n = 1, 3, 5, \ldots$$

using d'Alembert Solution:

$$u(x,t) = \frac{1}{2}\left(h(x+at) + h(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

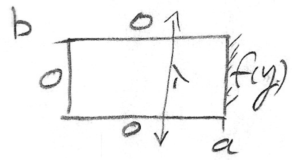
where $h(x)$ is the odd extension of $f(x)$

where $f(x)$ is original position on $0 \le x \le L$. and $h(x)$ is periodic with period $2L$

sheat sheet #2. Math 322. UNM. by Nasser M. Abbasi

Laplace PDE $\nabla^2 u = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

on square

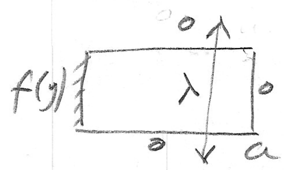


$\Phi_n(y) = \sin(\sqrt{\lambda_n} y)$, $\lambda_n = \left(\frac{n\pi}{b}\right)^2 \Rightarrow \Phi_n(y) = \sin\left(\frac{n\pi}{b} y\right)$

$u(x,y) = \sum_{n=1} C_n \sinh(\sqrt{\lambda_n} x)\, \Phi_n(y)$

$C_n \sinh(\sqrt{\lambda_n} a) = \frac{2}{b}\int_0^b f(y)\, \Phi_n(y)\, dy$.

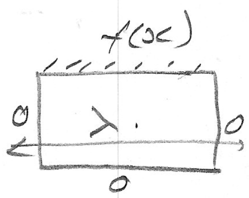
$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{b} \frac{\sinh(\sqrt{\lambda_n} x)}{\sinh(\sqrt{\lambda_n} a)} \left(\int_0^b f(y)\, \Phi_n(y)\, dy\right) \Phi_n(y).$$



$\Phi_n(y) = \sin(\sqrt{\lambda_n} y)$, $\lambda_n = \left(\frac{n\pi}{b}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n}(a-x))\, \Phi_n(y)$.

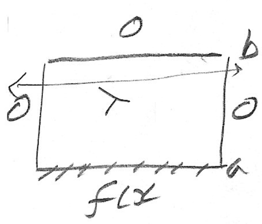
where $C_n \sinh(\sqrt{\lambda_n}(a-x)) = \frac{2}{b}\int_0^b f(y)\, \Phi_n(y)\, dy$.



$\Phi_n(x) = \sin(\sqrt{\lambda_n} x)$, $\lambda_n = \left(\frac{n\pi}{a}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n} y)\, \Phi_n(x)$.

where $C_n \sinh(\sqrt{\lambda_n} b) = \frac{2}{a}\int_0^a f(x)\, \Phi_n(x)\, dx$

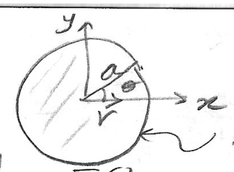


$\Phi_n(x) = \sin(\sqrt{\lambda_n} x)$, $\lambda_n = \left(\frac{n\pi}{a}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n}(b-y))\, \Phi_n(x)$

where $C_n \sinh(\sqrt{\lambda_n}(b-y)) = \frac{2}{a}\int_0^a f(x)\, \Phi_n(x)\, dx$

Laplace inside disk



$f(\theta)$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$\lambda_n = n^2,\ n = 1, 2, \ldots$ also $n = 0$

Solution is $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + K_n \sin n\theta)$

at origin, u = average of $f(\theta)$.

$$\boxed{A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta}$$

and

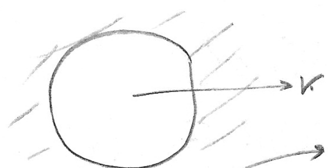
$$C_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$K_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

Same as heat PDE in steady state

Functions $r^n \cos n\theta$, $r^n \sin n\theta$ are called spherical harmonics.

outside disk



only difference

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \boxed{r^{-n}} (C_n \cos n\theta + K_n \sin n\theta)$$

exterior harmonics: $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$.

to convert solution in (r,θ) back to (x,y) use

$$r^n \cos n\theta = \sum_{\substack{k=0 \\ \text{even}}}^{n} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k}{2}} y^k$$

$$r^n \sin \theta = \sum_{\substack{k=1 \\ \text{odd}}}^{n} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k-1}{2}} y^k$$

For exterior use

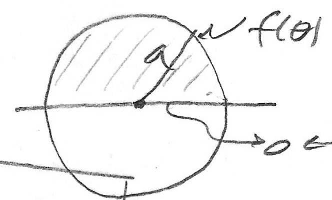
$$r^{-n}\cos n\theta = \frac{r^n \cos n\theta}{r^{2n}} = \frac{r^n \cos n\theta}{(x^2+y^2)^n}$$

$$r^{-n}\sin n\theta = \frac{r^n \sin n\theta}{r^{2n}} = \frac{r^n \sin n\theta}{(x^2+y^2)^n}$$

Cheat sheet #3. Math 322. UWM. Nasser M Abbasi

Laplace on semi-circle

$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$



$\begin{cases} u(r,0)=0 \\ u(r,\pi)=0 \end{cases}$

$u(a,\theta)=f(\theta)$

$0<\theta<\pi$

Solution is $\boxed{u(r,\theta)=\sum_{n=1}^{\infty} r^n C_n \sin n\theta}$

To find C_n. at $r=a$, $f(\theta)=\sum_{n=1}^{\infty} a^n C_n \sin n\theta$.

by orthogonality $\int_0^{\pi} f(\theta)\sin n\theta = a^n C_n\cdot\frac{\pi}{2} \Rightarrow C_n=\frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$

integration table.

$\int x\cos ax\,dx = \frac{\cos ax}{a^2}+\frac{x\sin ax}{a}$

$\int x\sin ax\,dx = \frac{\sin ax}{a^2}-\frac{x\cos ax}{a}$

$\int x^2\sin ax\,dx = \frac{2x\sin ax}{a^2}+\left(\frac{2}{a^3}-\frac{x^2}{a}\right)\cos ax$

$\int x^2\cos ax\,dx = \frac{2x\cos ax}{a^2}+\left(\frac{x^2}{a}-\frac{2}{a^3}\right)\sin ax$

$\int \sin^2 ax\,dx = \frac{x}{2}-\frac{\sin 2ax}{4a}$

$\int \cos^2 ax\,dx = \frac{x}{2}+\frac{\sin 2ax}{4a}$

$\int \sin ax\cos ax\,dx = \frac{\sin^2 ax}{2a}$

$\int \sin ax\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}$

$\int \frac{dx}{ax+b}dx = \frac{1}{a}\ln(ax+b)$

$\int \frac{x}{ax+b}dx = \frac{x}{a}-\frac{b}{a^2}\ln(ax+b)$

$$\int (x-b)\sin ax\,dx = \int x\sin ax - b\int \sin ax$$
$$= \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + \frac{b}{a}\cos ax$$

$$\int (x+b)\sin ax\,dx = \int x\sin ax + b\int \sin ax\,dx$$
$$= \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} - \frac{b}{a}\cos ax.$$

<u>trig identities</u>

$e^{i\theta} = \cos\theta + i\sin\theta$. $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$,

$\cos^2 x + \sin^2 x = 1$; $\cosh^2 x - \sinh^2 x = 1$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$.

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

$\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

$\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

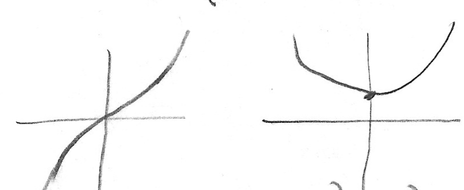
$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$

$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$



sinh(x) cosh(x)

Cheat sheet #4. math 322. Nasser M ABBASI.

$d(uv) = u\,dv + v\,du$

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$z = f(x,y)$. $dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

integration by parts: $\int u\,dv = uv - \int v\,du$

$\int u^n du = \frac{u^{n+1}}{n+1}$, $n \neq -1$. $\int \frac{1}{u} du = \ln u$, if $u > 0$
$= \ln|u|$ otherwise

$\int a^u du = \frac{a^u}{\ln a}$. $a > 0$, $a \neq 1$

$\int \sec^2 u\,du = \tan u$

$\int \tan^2 du = \tan u - u$

Fourier Series related integrals

$\int_0^L \sin^2\left(\frac{\pi}{L}x\right) dx = \frac{L}{2}$ same with $\cos^2\left(\frac{\pi}{L}x\right)$
↳ period is 2L

$\int_{-L}^{L} \sin^2\left(\frac{\pi}{L}x\right) dx = L$
↳ period is 2L

$\int_{-L}^{L} \sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}x\right) dx = 0$

$\int_{-\pi}^{\pi} \sin^2(x)\,dx = \pi$
↳ period 2π

$\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

$$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2}{a}x + \frac{2}{a^2}\right)$$

$$\int e^{ax} \sin bx = \frac{e^{ax}}{(a^2+b^2)} \left(a \sin bx - b \cos bx\right)$$

$$\int e^{ax} \cos bx = \frac{e^{ax}}{(a^2+b^2)} \left(a \cos bx + b \sin ax\right)$$

$$\int e^{ax} \ln x = \frac{e^{ax}}{a} \ln x - \frac{1}{a} \int \frac{e^{ax}}{x} dx$$

$$\int \ln x \, dx = x \ln x - x$$

$$\int x \ln x \, dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$\int \frac{1}{x} \ln x = \frac{1}{2} \ln^2 x$$

Cheat sheet #5

to convert $ay'' + by' + (c + \lambda d)y = 0$ to

$(Py')' - qy + \lambda r y = 0$ do

$\boxed{\mu(x) = e^{\int \frac{b}{a} dx}}$ then

$$P = \mu$$
$$q = -\mu \frac{c}{a}$$
$$r = \frac{\mu}{a} d$$

to convert ODE to exact form:

$a_0 y'' + a_1 y' + a_2 y = 0 \Rightarrow \left(a_0 y' + (a_1 - a_0')y\right)'$

Check if these are the same. if yes, Then exact.

$\to$ $P(x) y'' + Q(x) y' + R(x) y = 0$

This is book formula but for non eigen value ODE.

$\Rightarrow (\mu(x) P y')' + \mu(x) R(x) y = 0$

where $\mu(x) = \frac{1}{P} e^{\int^x \frac{Q(s)}{P(s)} ds}$

Green Function

$$G(x,s) = \frac{\pm 1}{PW} \begin{cases} y_1(s)\, y_2(x) \\ y_1(x)\, y_2(s) \end{cases}$$

impulse (s), measurement at (x)

where $y_1(x)$ satisfies BC. at (a)

$y_2(x)$ satisfies BC. at (b).

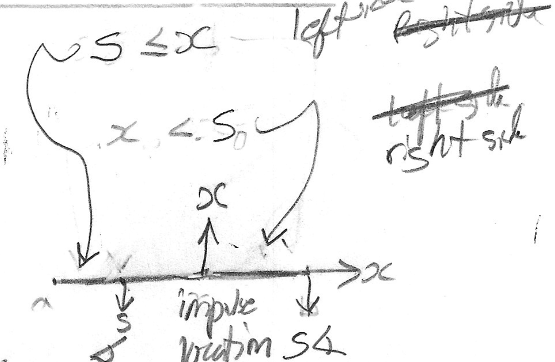
$y_1(x), y_2(x)$ are solutions to $L[y] = 0$

$y_1(a) + y_1'(a) = 0$

$y_2(b) + y_2'(b) = 0$

so first find y_1, y_2.

$G(x, s)$ — s → impulse location; x → response measured at.



$y = \int_0^1 G(x,s) f(s)\, ds$

Bessel ODE $r^2R'' + rR' + (r^2 - n^2)R = 0$

The order is $\boxed{n}$. for order zero. The above becomes

$$R'' + \frac{1}{r}R' + R = 0 \quad \text{or} \quad r^2R'' + rR' + r^2R = 0$$

if we have $R'' + \frac{1}{r}R' + \lambda^2 R = 0$. to convert to Bessel use

$\boxed{t = r\lambda}$ Then $\frac{dR}{dr} = \frac{dR}{dt}\frac{dt}{dr} = \lambda\frac{dR}{dt}$; $\frac{d^2R}{dr^2} = \frac{d^2R}{dt^2}\lambda^2$.

so ODE becomes $R''(t) + \frac{1}{t}\lambda^2 R'(t) + \lambda^2 R = 0$ or

$$\boxed{R''(t) + \frac{1}{t}R'(t) + R = 0}$$

Legendre equation $(1-x^2)y'' - 2xy + n(n+1) = 0$

For integer n, solutions are $P_n(x)$, $P_m(x)$.

even orders odd orders.

Recurrence $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

orthogonality $\int_{-1}^{1} P_n P_m \, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$

$$J_n' = J_{n-1} - \frac{n+1}{x}J_n$$

$$J_n' = -J_{n+1} + \frac{n}{x}J_n$$