LOWER BOUND FOR EIGENVALUES

We want to derive a lower bound for the eigenvalues of a regular Sturm-Liouville problem. We first need a lemma (related to the Sobolev embedding theorem.)

Lemma 1. *Let $f \in C^1[a,b]$, $x \in [a,b]$, $h > 0$. Then*

$$f(x)^2 \le \left(\frac{1}{b-a} + \frac{1}{h}\right) \int_a^b f(t)^2\,dt + h \int_a^b f'(t)^2\,dt.$$

Proof. For all $x, s \in [a,b]$ we have

$$f(x)^2 - f(s)^2 = \int_s^x 2f(t)f'(t)\,dt \le \int_a^b 2|f(t)||f'(t)|\,dt.$$

We estimate

$$2|f(t)||f'(t| = 2\left(\frac{|f(t)|}{\sqrt{h}}\right)(\sqrt{h}|f'(t)|) \le \frac{1}{h} f(t)^2 + hf'(t)^2.$$

Therefore,

$$f(x)^2 - f(s)^2 \le \frac{1}{h}\int_a^b f(t)^2\,dt + h\int_a^b f'(t)^2\,dt.$$

We integrate this inequality from $s = a$ to $s = b$. Then we obtain

$$(b-a)f(x)^2 - \int_a^b f(s)^2\,ds \le \frac{b-a}{h}\int_a^b f(t)^2\,dt + (b-a)h\int_a^b f'(t)^2\,dt.$$

This is equivalent to the inequality in the statement of the lemma. $\square$

We consider the regular Sturm-Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y = 0,$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

We define

$$c_1 = \begin{cases} 0 & \text{if } \alpha_2 = 0 \text{ or } \frac{\alpha_1}{\alpha_2} \le 0, \\ p(a)\frac{\alpha_1}{\alpha_2} & \text{if } \frac{\alpha_1}{\alpha_2} > 0, \end{cases}$$

and

$$c_2 = \begin{cases} 0 & \text{if } \beta_2 = 0 \text{ or } \frac{\beta_1}{\beta_2} \ge 0, \\ -p(b)\frac{\beta_1}{\beta_2} & \text{if } \frac{\beta_1}{\beta_2} < 0. \end{cases}$$

Then we set $c = c_1 + c_2 \ge 0$.

Theorem 2. *Every eigenvalue λ of a regular Sturm-Liouville problem, satisfies the inequality*

$$\lambda \geq \frac{1}{\min r}\left(-\frac{c}{b-a}-\frac{c^2}{\min p}\right)+\min\frac{q}{r}. \tag{1}$$

Proof. Let $\phi(x)$ be an eigenfunction corresponding to the eigenvalue λ. Then, using integration by parts,

$$\lambda\int r\phi^2 = -\int (p\phi')'\phi + \int q\phi^2 = -\,p(x)\phi'(x)\phi(x)\big|_a^b + \int p(\phi')^2 + \int q\phi^2,$$

where $\int f$ denotes $\int_a^b f(x)\,dx$. Now

$$-\,p(x)\phi'(x)\phi(x)\big|_a^b = p(b)\frac{\beta_1}{\beta_2}\phi(b)^2 - p(a)\frac{\alpha_1}{\alpha_2}\phi(a)^2,$$

where $\frac{\beta_1}{\beta_2}=0$ if $\beta_2=0$ and $\frac{\alpha_1}{\alpha_2}=0$ if $\alpha_2=0$. Using the definition of c, we find

$$-\,p(x)\phi'(x)\phi(x)\big|_a^b \geq -c\max\left\{\phi(a)^2,\phi(b)^2\right\}.$$

Therefore,

$$\lambda\int r\phi^2 \geq -c\max\left\{\phi(a)^2,\phi(b)^2\right\} + \min p\int(\phi')^2 + \min\tfrac{q}{r}\int r\phi^2. \tag{2}$$

If $c=0$ then

$$\lambda\int r\phi^2 \geq \min\tfrac{q}{r}\int r\phi^2.$$

This gives (1) after division by $\int r\phi^2>0$.

If $c>0$ then we use Lemma 1, and obtain from (2)

$$\lambda\int r\phi^2 \geq -c\left(\frac{1}{b-a}+\frac{1}{h}\right)\int\phi^2 - ch\int(\phi')^2 + \min p\int(\phi')^2 + \min\frac{q}{r}\int r\phi^2, \tag{3}$$

where h can be any positive number. We choose $h=\frac{\min p}{c}$. Then (3) gives

$$\lambda\int r\phi^2 \geq \left(-\frac{c}{b-a}-\frac{c^2}{\min p}\right)\frac{1}{\min r}\int r\phi^2 + \min\frac{q}{r}\int r\phi^2$$

which again gives (1) after division by $\int r\phi^2$. □