My Physics 501 page
Special Topics: Mathematical Models of Physical Problems I
Fall 2018
University of Wisconsin, Milwaukee

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Contents

Chapter 1

Introduction

1.1 syllabus

Physics 501 **Mathematical Models of Physical Problems I** Fall, 2018

MW 12:30-1:45 Ken 1130

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Text: *Mathematical Methods for Physics and Engineering*, (third edition) by Riley, Hobson, and Bence (required).

The text is available at the UWM bookstore. *Mathematical Models of Physical Problems* by Anchordoqui and Paul is also a useful reference.

Brief Description

Infinite Series; Complex Analysis; Integral Transforms; Evaluation of Integrals; Ordinary Differential Equations; Special Functions; Partial Differential Equations.

Prerequisite: Physics 210 and Mathematics 234 – linear algebra Differential equations

Homework: Sets of homework will be distributed once every one to two weeks
You are free to talk to the other students and me about the problems but you must write up the solutions yourself.

Grading:	Homework:	40 %
	Test :	25 %
	Final:	35 %

Makeup exams will be given only if there is a documented need to miss the exam (e.g., injury, illness, or on a family death).

Exam is scheduled for Dec 19 from 3:00-5:00 in room Ken 1130. The midterm will be held in class.

See http://uwm.edu/secu/wp-content/uploads/sites/122/2016/12/Syllabus-Links.pdf for University Policies.

1.2 Topics covered

This is list of lectures and topics covered in each

Table 1.1: Topics covered

#	Date	Topics
1	Wed. Sept 4, 2018	1. Infinite series, geometric series. 2. conditions for convergence, 3. harmonic series, alternating series. 4. tests for convergence such as ratio test, integral test.
2	Monday Sept 10, 2018	1. Talked about series solution to $(1-x^2)y''-2xy'+n(n+1)y=0$. 2. Binomial series $(1+x)^r = 1 + rx + \frac{r(r-1)x^2}{2!} + \dots$ 3. series of $e, \sin(x), \cos(x)$. 4. Introducing Bernoulli numbers. 5. Show that alternating series is convergent but not absolutely. 6. Leibniz condition for convergence and its proof. 7. Showed that sum of $1-1/2+1/3-1/4+\dots$ is $\ln 2$. 8. Working with absolutely convergent series.
3	Wed Sept 12, 2018	1. Familiar series, $e, \sin x, \cos x, \ln(1+x), \arctan(x)$ 2. How to get Bernulli numbers. More on Bernulli numbers but I really did not understand these well and how to use them. hopefully they will not be on the exam. 3. Started Complex analysis. Basic introduction. Properties of complex numbers and mapping.
4	Monday Sept 17, 2018	1. complex functions $u(x,y)+iv(x,y)$ 2. continuity in complex domain. 3. Derivative in complex domain and how direction is important. 4. Cauchy-Riemman equation to test for analytical function. 5. Harmonic functions. Exponential function in complex domain. 6. Multivalued functions, such as $\log z$. 7. How to obtain inverse trig function and solve $w = \arcsin(z)$
		Continued on next page

Table 1.1 – continued from previous page

Lecture #	Date	Topics
5	Wed Sept 19, 2018	1. derivative in complex plane. Definition of analytic function. 2. $\log(z)$ and $\sqrt(z)$ in complex plane and multivalued. Branch points and branch cuts. 3. Integration over contour. Parameterization $\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$ 4. Cauchy-Goursat theorem: $\oint f(z) = 0$ for analytical functions. Proof using Cauchy-Riemman equations and Green theorem. 5. Cauchy integral formula $2\pi i f(z_0) = \oint \frac{f(z)}{z-z_0}\,dz$ 6. Like in real, in complex domain, Continuity Does Not Imply Differentiability. 7. More on analytic functions and multivalued functions. Principal value. 8. Power functions $z^p = e^{p \ln z}$ 9. Complex integration.
6	Monday Sept 24, 2018	1. Proof of Cauchy integral formula. 2. Maximum moduli of analytic functions. If $f(z)$ is analytic in D and not constant, then it has no maximum value inside D. The maximum of $f(z)$ is on the boundary. 3. Taylor series for complex functions and Laurent series.
7	Wed Sept 26, 2018	1. If number of terms in principal part of Laurent series is infinite, then it is essential singularity. 2. Proof of Laurent theorem. 3. properties of power series. Uniqueness. 4. Residues, types of singularities. How to find residues and examples
8	Monday Oct 1, 2018	1. Residue theorem $\oint_C f(z)\,dz = 2\pi i \sum$ residues inside C 2. examples using Residue theorem. 3. Analytic continuation. Examples. 4. $\Gamma(z)$ function. Defined for $\Re(z) > 0$. Using analytical continuation to extend it to negative complex plane. Euler representation and Weistrass represenation.

Continued on next page

Table 1.1 – continued from previous page

Lecture #	Date	Topics
9	Wed Oct 3, 2018	1. More on Euler represenation of $\Gamma(z)$ and how to use it for extending definition $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt$ for negative z using $\Gamma(z)=\frac{\Gamma(z+1)}{z}$ for $-1<z$. 2. Euler reflection formula $$\Gamma(x)\Gamma(1-x)=\int_0^\infty \frac{t^{x-1}}{1+t}\,dt=\frac{\pi}{\sin(\pi x)}$$ 3. proof of Euler reflection formula using contour integration. 4. Some useful formulas for $\Gamma(z)$ 5. Method for integrations, some tricks to obtain definite integrations.
10	Monday Oct 8, 2018	No class.
11	Wed Oct 10, 2018	No class.
12	Monday Oct 15, 2018	1. More on method of integration. Starting Contour integration. 2. How to decide that $\int_{C_R} f(z)=0$ on the upper half plane. Using Jordan inquality. 3. More examples of integrals on real line using contour integration.
13	Wed Oct 17, 2018	1. More contour integrations. 2. Starting approximation expansion of integrals. Example using error function $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ by applying Taylor series. 3. Large x expansion by repeated integration by parts. 4. Starting Asymprtotic series. Definition. Example on finding $S(x)$ for $\operatorname{erf}(x)$ for large x. When to truncate. 5. Saddle point methods of integration to approximate integral for large x.
14	Monday Oct 22, 2018	1. More saddle point integration. 2. Saddle point methods of integration to approximate integral for large x. Method of steepest decsent. Example to find $\Gamma(x+1)=\int_0^\infty t^x e^{-t}\,dt=\sqrt{2\pi x}x^x e^{-x}$ 3. extend saddle point method to complex plane. Finding correct angle. Long example.

Continued on next page

Table 1.1 – continued from previous page

Lecture #	Date	Topics
15	Wed Oct 24, 2018	1. More on saddle point in complex plane. Angles. Example applied on $\int \Gamma(1+z) = \int_0^{\infty} \exp -t + z \ln t \, dt$ 2. how to determine coefficients of asymptotic series expansion. 3. Starting new topic. Fourier series. Definitions. 4. proprties of Fourier series. Examples how to find A_n, B_n.
16	Friday Oct 26, 2018	Make up lecture. 1. More on Fourier series. Examples. Fourier series using the complex formula. 2. Parseval identity. 3. Fourier Transform derivation.
17	Monday Oct 29, 2018	1. Fourier transform pairs. 2. How to find inverse fourier transform. Generalization to higher dimensions. 3. Properties of Fourier transform. 4. convolution. 5. Example on driven harmonic oscillator. 6. Statring ODE's. Order and degree of ODE.
18	Wed Oct 31, 2018	1. More on first order ODE's. Separable, exact. How to find integrating factor. 2. Bernulli ODE $y' + f(x)y = g(x)y^n$ 3. Homogeneous functions. defintion. order of. 4. isobaric ODE's.
19	Monday Nov 5, 2018	1. How to find integating factor for exact ODE. 2. Finished example on isobaric first order ODE. $$xy^2(3y\,dx + x\,dy) - (2y\,dx - x\,dy) = 0$$ 3. Higher order ODE's. How to solve. How to find particular solution. Undetermined coefficients. What to do if forcing function has same form as one of the solutions to homogeneous solutions. 4. How to use power series to solve nonlinear ode $y'' = x - y^2$
20	Wed Nov 7, 2018	First exam

Continued on next page

Table 1.1 – continued from previous page

Lecture #	Date	Topics
21	Monday Nov 12, 2018	1. More on higher order ODE's. Series solutions. 2. ordinary point. Regular singular point. Example Legendre ODE $(1-x^2)y''-2xy'+n(n+1)y=0$. 3. Example for regular singular point, Bessel ODE $x^2y''+xy'+(x^2-m^2)y=0$ Use $y=x^2\sum_{n=0}^{\infty}c_nx^n$
22	Wed Nov 14, 2018	1. Continue Bessel ODE $x^2y''+xy'+(x^2-m^2)y=0$ solving using $y=x^2\sum_{n=0}^{\infty}c_nx^n$. How to find second independent solution.
23	Monday Nov 19, 2018	1. Started on Sturm Lioville, Hermetian operators 2. setting Bessel ODE in Sturm Lioville form 3. more on Hermitian operator. 4. Wronskian to check for linear independece of solutions.
24	Wed Nov 21, 2018	Thanks Giving.
25	Monday Nov 26, 2018	1. finding second solution to Bessel ODE for m integer using the Wronskian. $W(x)=\frac{C}{p(x)}=\frac{-2\sin\pi m}{\pi}$ 2. Generating functions to find way to generate Besself functions.
26	Wed Nov 28, 2018	1. Using Generating functions 2. Bessel functions of half integer order, spherical Bessel functions 3. Legendre polynomials, recusrive relations. 4. orthonomalization. 5. physical applications
27	Monday December 3, 2018	1. Second solution to Legendre using Wronskian 2. Spherical harmonics 3. Normalization of eigenfunctions 4. Degenerncy, using Gram-Schmidt to find other L.I. solutions. 5. Expanding function using complete set of basis functions, example using Fourier series 6. Inhomogeneous problems, starting Green function

Continued on next page

Table 1.1 – continued from previous page

Lecture #	Date	Topics
28	Wed December 5, 2018	1. Green function. Solution to the ODE with point source. 2. Example using vibrating string $y'' + k^2 y = 0$. Find Green function. Two Methods. Use second method. 3. Started on PDE.
29	Friday December 7, 2018	(Make up lecture) 1. more PDE's. Solve wave PDE in 1D 2. separation of variables. Solve Wave PDE in 3D in spherical coordiates.
30	monday December 10, 2018	1. Solving wave PDE in 3D in spherical coordinates. Normal modes. 2. Solving wave PDE in 3D in cylindrical coordinates. 3. Inhomogeneous B.C. on heat PDE. Break it into 2 parts.
31	Wed December 12, 2018	Last lecture. 1. Finish Inhomogeneous B.C. on heat PDE. Break it into 2 parts. Final solution, using Fourier series. 2. Last problem. INtegral transform method. Solving heat pde on infinite line using Fourier transform.

Chapter 2

exams

2.1 Practice exam from 2017

2.1.1 Problem 1

i) Find Laurent series for $f(z) = \frac{1}{\left(z^2+1\right)^3}$ around isolated singular pole $z = i$. What is the order of the pole? ii) Use residues to evaluate the integral $\int_0^\infty \frac{dx}{\left(x^2+1\right)^3}$

<u>solution</u>

$z^2 + 1 = 0$ gives $z = \pm i$. Hence there is a pole at $z = i$ of order 3 and also a pole at $z = -i$ of order 3. Hence $g(z) = (z-i)^3 f(z)$ is analytic at $z = i$ and therefore it has a Taylor series expansion around $z = i$ given by

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} a_n (z-i)^n \\ (z-i)^3 f(z) &= \sum_{n=0}^{\infty} a_n (z-i)^n \end{aligned} \tag{1}$$

Where $a_n = \left. \frac{\frac{d^n}{dz^n} g(z)}{n!} \right|_{z=i}$. But

$$\begin{aligned} g(z) &= (z-i)^3 f(z) \\ &= (z-i)^3 \frac{1}{\left(z^2+1\right)^3} \\ &= (z-i)^3 \frac{1}{((z-i)(z+i))^3} \\ &= (z-i)^3 \frac{1}{(z-i)^3 (z+i)^3} \\ &= \frac{1}{(z+i)^3} \end{aligned}$$

To find a_n then $a_n = \frac{1}{n!} \frac{d^n}{dz^n} \frac{1}{(z+i)^3}$ is evaluated for few n terms. Since order is 3, at least 5 terms are needed to see the residue and the first term in the analytical part of the series $(n > 0)$. Starting with $n = 0$

$$a_0 = \left. \frac{1}{(z+i)^3} \right|_{z=i} = \frac{1}{(2i)^3} = \frac{1}{-8i} = \frac{1}{8} i$$

For $n = 1$

$$a_1 = \left. \frac{d}{dz} \frac{1}{(z+i)^3} \right|_{z=i} = \left. \frac{-3}{(z+i)^4} \right|_{z=i} = \frac{-3}{(2i)^4} = \frac{-3}{16}$$

For $n = 2$

$$a_2 = \left. \frac{1}{2} \frac{d}{dz} \frac{-3}{(z+i)^4} \right|_{z=i} = \left. \frac{1}{2} \frac{-3(-4)}{(z+i)^5} \right|_{z=i} = \frac{1}{2} \frac{-3(-4)}{(2i)^5} = \frac{1}{2} \frac{-3(-4)}{2^5 i} = \frac{6}{32i} = \frac{3}{16i} = -\frac{3i}{16}$$

For $n=3$

$$a_3=\frac{1}{3!}\frac{d}{dz}\frac{-3(-4)}{(z+i)^5}\bigg|_{z=i}=\frac{1}{6}\frac{-3(-4)(-5)}{(z+i)^6}\bigg|_{z=i}=\frac{1}{6}\frac{-3(-4)(-5)}{(2i)^6}=\frac{1}{6}\frac{-3(-4)(-5)}{-2^6}=\frac{5}{32}$$

For $n=4$

$$a_4=\frac{1}{4!}\frac{d}{dz}\frac{-3(-4)(-5)}{(z+i)^6}\bigg|_{z=i}=\frac{1}{24}\frac{-3(-4)(-5)(-6)}{(z+i)^7}\bigg|_{z=i}=\frac{1}{24}\frac{-3(-4)(-5)(-6)}{(2i)^7}=\frac{1}{24}\frac{3(4)(5)(6)}{-i2^7}=\frac{15}{128}i$$

Substituting all these back into (1) gives

$$\begin{aligned}(z-i)^3 f(z)&=\sum_{n=0}^{\infty}a_n(z-i)^n\\&=a_0+a_1(z-i)+a_2(z-i)^2+a_3(z-i)^3+a_4(z-i)^4+\cdots\end{aligned}$$

Therefore

$$\begin{aligned}f(z)&=\frac{1}{(z-i)^3}\left(a_0+a_1(z-i)+a_2(z-i)^2+a_3(z-i)^3+a_4(z-i)^4+\cdots\right)\\&=\frac{1}{(z-i)^3}\left(\frac{1}{8}i+\frac{-3}{16}(z-i)-\frac{3i}{16}(z-i)^2+\frac{5}{32}(z-i)^3+\frac{15}{128}i(z-i)^4+\cdots\right)\\&=\frac{1}{8}\frac{i}{(z-i)^3}-\frac{3}{16}\frac{1}{(z-i)^2}-\frac{3}{16}\frac{i}{(z-i)}+\frac{5}{32}+\frac{15}{128}i(z-i)-\cdots\end{aligned}\tag{1A}$$

The residue is the coefficient of the term with $\frac{1}{z-i}$ factor. Hence residue is $-\frac{3i}{16}$. The order is 3 since that is the highest power in $\frac{1}{z-i}$.

The above method always works, but it means having to evaluate derivatives a number of times. For a pole of high order, it means evaluating the derivative for as many times as the pole order and more to reach the analytical part. Another method is to expand the function using binomial expansion

$$(1+x)^p=1+pz+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\cdots\tag{2}$$

The above is valid for real p, which can be negative or positive, but only for $|x|<1$. This is now applied to expand

$$\begin{aligned}f(z)&=\frac{1}{\left(z^2+1\right)^3}\\&=\frac{1}{(z-i)^3(z+i)^3}\end{aligned}$$

Let $z-i=\xi$, or $z=\xi+i$ and the above becomes

$$\begin{aligned}f(z)&=\frac{1}{\xi^3(\xi+2i)^3}\\&=\frac{1}{\xi^3}\frac{1}{(2i)^3\left(1+\frac{\xi}{2i}\right)^3}\\&=\left(\frac{i}{\xi^3}\frac{1}{8}\right)\frac{1}{\left(1+\frac{\xi}{2i}\right)^3}\end{aligned}\tag{3}$$

Now the binomial expansion can be used on $\frac{1}{\left(1+\frac{\xi}{2i}\right)^3}$ term above, which is valid for $\left|\frac{\xi}{2i}\right|<1$, which gives

$$\begin{aligned}\frac{1}{\left(1+\frac{\xi}{2i}\right)^3}&=1-(3)\frac{\xi}{2i}+\frac{(-3)(-4)}{2!}\left(\frac{\xi}{2i}\right)^2-\frac{(-3)(-4)(-5)}{3!}\left(\frac{\xi}{2i}\right)^3+\frac{(-3)(-4)(-5)(-6)}{4!}\left(\frac{\xi}{2i}\right)^4+\cdots\\&=1+\frac{3}{2}i\xi+6\frac{\xi^2}{4i^2}+10\frac{\xi^3}{2^3i^3}+15\frac{\xi^4}{2^4i^4}+\cdots\\&=1+\frac{3}{2}i\xi-\frac{3}{2}\xi^2-\frac{10}{8}i\xi^3+\frac{15}{16}\xi^4+\cdots\end{aligned}$$

Therefore (3) becomes

$$\begin{aligned} f(z) &= \frac{i}{8\xi^3}\left(1+\frac{3}{2}i\xi-\frac{3}{2}\xi^2-\frac{10}{8}i\xi^3+\frac{15}{16}\xi^4+\cdots\right) \\ &= \left(\frac{i}{8\xi^3}-\frac{3}{16}\frac{1}{\xi^2}-\frac{3}{16}\frac{i}{\xi}+\frac{10}{64}+\frac{15}{(16)(8)}i\xi+\cdots\right) \end{aligned}$$

But $z=\xi+i$ or $\xi=z-i$, and the above becomes

$$f(z)=\left(\frac{i}{8(z-i)^3}-\frac{3}{16}\frac{1}{(z-i)^2}-\frac{3}{16}\frac{i}{(z-i)}+\frac{5}{32}+\frac{15}{128}i(z-i)+\cdots\right) \tag{4}$$

Which is valid for $|z-i|<1$. In other words, inside a disk of radius 2, centered around $z=i$.

Comparing (4) with (1A), shows they are the same as expected. Which is the better method? After working both, I think the second method is faster, but requires careful transformation, the first method is more direct but requires more computations.

ii) Let $\int_0^\infty \frac{dx}{(x^2+1)^3}=I$, hence, because $\frac{1}{(x^2+1)^3}$ is even, then

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3} \\ &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^2+1)^3} \\ &= \frac{1}{2}\lim_{R\to\infty}\left(\int_{-R}^{R}\frac{dx}{(x^2+1)^3}+\oint_C\frac{dz}{(z^2+1)^3}\right) \end{aligned}$$

The above is valid as long as one can show $\oint_C \frac{dz}{(z^2+1)^3}\to 0$ as $R\to\infty$. The contour C is from R to $-R$ over semicircle, going anticlock wise. The radius of the circle is R. Since the above integration now includes $z=i$, then by residual theorem, the above is just $-\frac{3i}{16}$. The residue was found in the first part. In other words

$$\frac{1}{2}\lim_{R\to\infty}\overbrace{\left(\int_{-R}^{R}\frac{dx}{(x^2+1)^3}+\oint_C\frac{dz}{(z^2+1)^3}\right)}^{2\pi i\left(-\frac{3i}{16}\right)}=\frac{1}{2}\left[2\pi i\left(-\frac{3i}{16}\right)\right]$$

Letting $z=R\,e^{i\theta}$ and taking $R\to\infty$, then $\oint_C\frac{dz}{(z^2+1)^3}\to 0$ and the above simplifies to

$$\begin{aligned} \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^2+1)^3} &= \frac{1}{2}\left[2\pi\frac{3}{16}\right] \\ \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^2+1)^3} &= \pi\frac{3}{16} \\ \lim_{R\to\infty}\int_{0}^{R}\frac{dx}{(x^2+1)^3} &= \frac{3\pi}{16} \end{aligned}$$

Therefore

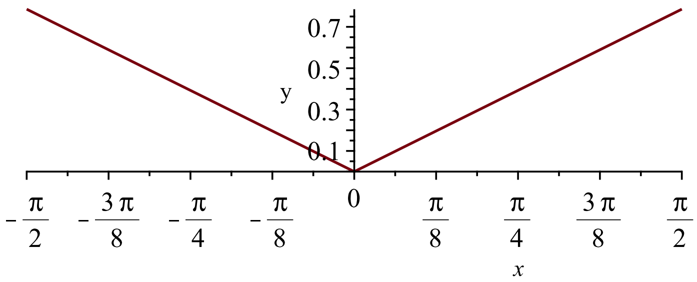
$$\int_0^\infty\frac{dx}{(x^2+1)^3}=\frac{3\pi}{16}$$

2.1.2 Problem 2

Expand $f(x)=\frac{x}{L}$ as Fourier series for $0<x<\frac{\pi}{L}$ and $f(x)=-\frac{x}{L}$ for $-\frac{\pi}{L}<x<0$.

<u>solution:</u>

This function is even. For example, for $L=2$, it looks like this



Hence the Fourier series will not have sin terms.

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{T}x\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}x\right) \end{aligned}$$

Where in the above T is the period of the function. In this problem $T = \frac{2\pi}{L}$, hence the above becomes

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{\frac{2\pi}{L}}x\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nLx) \end{aligned} \tag{1}$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{2}{\frac{2\pi}{L}} \int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} f(x)\,dx = \frac{L}{\pi} \int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} f(x)\,dx \\ &= \frac{L}{\pi}(2) \int_0^{\frac{\pi}{L}} \frac{x}{L} dx \\ &= \frac{2L}{\pi}\frac{1}{L}\left(\frac{x^2}{2}\right)_0^{\frac{\pi}{L}} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{L^2}\right) \\ &= \frac{\pi}{L^2} \end{aligned}$$

And

$$\begin{aligned} a_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} f(x)\cos(nLx)\,dx \\ &= \frac{2}{\frac{2\pi}{L}}(2) \int_0^{\frac{\pi}{L}} f(x)\cos(nLx)\,dx \\ &= \frac{2L}{\pi} \int_0^{\frac{\pi}{L}} \frac{x}{L}\cos(nLx)\,dx \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{L}} x\cos(nLx)\,dx \end{aligned}$$

Using integration by parts $\int udv = uv - \int vdu$. Let $u = x, du = 1$ and let $dv = \cos(nLx), v =$

$\frac{\sin(nLx)}{nL}$, therefore the above becomes

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\left(\left[x\frac{\sin(nLx)}{nL}\right]_0^{\frac{\pi}{L}} - \int_0^{\frac{\pi}{L}} \frac{\sin(nLx)}{nL}dx\right)\\
&= \frac{2}{\pi}\left(\left[\frac{\pi}{L}\frac{\sin\left(nL\frac{\pi}{L}\right)}{nL} - 0\right] - \frac{1}{nL}\int_0^{\frac{\pi}{L}} \sin(nLx)\,dx\right)\\
&= \frac{2}{\pi}\left(-\frac{1}{nL}\int_0^{\frac{\pi}{L}} \sin(nLx)\,dx\right)\\
&= \frac{-2}{\pi nL}\left(\int_0^{\frac{\pi}{L}} \sin(nLx)\,dx\right)\\
&= \frac{-2}{\pi nL}\left(-\frac{\cos(nLx)}{nL}\right)_0^{\frac{\pi}{L}}\\
&= \frac{2}{\pi n^2L^2}\left(\cos(nLx)\right)_0^{\frac{\pi}{L}}\\
&= \frac{2}{\pi n^2L^2}\left(\cos\left(nL\frac{\pi}{L}\right) - 1\right)\\
&= \frac{2}{\pi n^2L^2}\left(\cos(n\pi) - 1\right)\\
&= \frac{2}{\pi n^2L^2}\left((-1)^n - 1\right)\\
&= \frac{-2 + 2(-1)^n}{\pi n^2L^2}
\end{aligned}
$$

Therefore from (1) the Fourier series is

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nLx)\\
&= \frac{\pi}{2L^2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2L^2}\left((-1)^n - 1\right)\cos(nLx)
\end{aligned}
$$

The convergence is of order n^2, so it is fast. Only few terms are needed to obtain very good approximation.

2.1.3 Problem 3

(i) Solve $xy' + 3x + y = 0$. (ii) Solve $y'' - 2y' + y = e^x$

Solution

(i). This is linear first order ODE.

$$
\begin{aligned}
y' + 3 + \frac{y}{x} &= 0 \qquad x \neq 0\\
y' + \frac{y}{x} &= -3
\end{aligned}
$$

Integrating factor is $\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. Multiplying both sides of the above by μ, the left side becomes complete differential and it simplifies to

$$
\begin{aligned}
\frac{d}{dx}\left(\mu y\right) &= -3\mu\\
\frac{d}{dx}\left(xy\right) &= -3x\\
d\left(xy\right) &= -3xdx
\end{aligned}
$$

Integrating gives

$$
xy = -\frac{3}{2}x^2 + C
$$

Hence the solution is

$$
\boxed{y = -\frac{3}{2}x + \frac{C}{x} \qquad x \neq 0}
$$

(ii) $y'' - 2y' + y = e^x$ is linear second order with constant coefficients. The solution to the homogeneous part $y''-2y'+y=0$ can be found by first finding the roots of the characteristic equation $s^2-2s+1=0$, hence $s=\frac{-b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}$ or, $s=\frac{2}{2}\pm\frac{1}{2}\sqrt{4-4}=1$. One double root. Therefore

$$y_h(x)=C_1e^x+C_2xe^x$$

To find the particular solution, the method of undetermined coefficients is used. Since the forcing function is e^x, then a guess $y_p=ke^x$. But e^x is a basis solution. Hence $y_p=kxe^x$ is now selected. But also xe^x is basis solution. Then $y_p=kx^2e^x$ is finally selected. Substituting this into the original ODE in order to solve for k, gives

$$y_p''-2y_p'+y_p=e^x$$

But $y_p'=2kxe^x+kx^2e^x$ and $y_p''=2ke^x+2kxe^x+2kxe^x+kx^2e^x$. Hence the above becomes

$$\begin{aligned}\left(2ke^x+2kxe^x+2kxe^x+kx^2e^x\right)-2\left(2kxe^x+kx^2e^x\right)+kx^2e^x&=e^x\\ \left(2k+2kx+2kx+kx^2\right)-2\left(2kx+kx^2\right)+kx^2&=1\\ 2k+4kx+kx^2-4kx-2kx^2+kx^2&=1\\ 2k&=1\\ k&=\frac{1}{2}\end{aligned}$$

Therefore $y_p=\frac{1}{2}x^2e^x$, and the complete general solution is

$$y(x)=y_h(x)+y_p(x)$$

Therefore

$$\boxed{y(x)=C_1e^x+C_2xe^x+\frac{1}{2}x^2e^x}$$

2.1.4 Problem 4

Problem 4:

Consider the following differential equation (where $0\le x<\infty$)

$$x^2\frac{d^2}{dx^2}y(x)+2x\frac{d}{dx}y(x)+x^2y(x)=0 \tag{5}$$

(i) Identify a regular singular point for this equation.

(ii) Consider the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ (note that a solution of this form exists). Set $c_0=1$. Find the condition for c_1 and then find a recurrence relation for c_m/c_{m-2}.

(iii) Write a closed form expression for the power series solution (the power series should look familiar).

(iv) Using the Wronskian (note that the differential equation is a Sturm Liouville equation) and the above closed form solution, find a second solution.

Figure 2.1: Problem 4 Statement

solution

Part (1)

$$\begin{aligned}x^2y''+2xy'+x^2y&=0\\ y''+\frac{2}{x}y'+y&=0\end{aligned}$$

x is a regular singular point. Becuase $\lim_{x\to0}(x-0)\frac{2}{x}=2$. Since limit exist, then regular singular point.

Part (2)

Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

The ODE becomes

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} &= 0 \\
\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n &= 0
\end{aligned}
$$

Therefore

$$
\sum_{n=0}^{\infty} (n(n-1) + 2n) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0
$$

We start from $n = 1$ since $n = 0$ is used to find the indicial equation. For $n = 1$

$$
\begin{aligned}
2a_1 &= 0 \\
a_1 &= 0
\end{aligned}
$$

For $n \geq 2$

$$
\begin{aligned}
(n(n-1) + 2n) a_n + a_{n-2} &= 0 \\
a_n (n(n-1) + 2n) &= -a_{n-2} \\
a_n &= \frac{-a_{n-2}}{n(n-1) + 2n}
\end{aligned}
$$

For $n = 2$

$$
\begin{aligned}
a_2 &= \frac{-a_0}{2(2-1) + 4} \\
&= -\frac{1}{6} a_0
\end{aligned}
$$

For $n = 3$

$$
a_3 = \frac{-a_1}{3(2) + 6}
$$

Since $a_1 = 0$ then $a_3 = 0$. All odd terms are therefore zero.

For $n = 4$

$$
a_n = \frac{-a_2}{(4)(3) + 8} = -\frac{1}{20} a_2 = -\frac{1}{20}\left(-\frac{1}{6} a_0\right) = \frac{1}{120} a_0
$$

Therefore

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_2 x^2 + a_4 x^4 + \cdots \\
&= a_0 - \frac{1}{6} a_0 x^2 + \frac{1}{60} a_0 x^4 - \cdots \\
&= a_0 \left(1 - \frac{1}{6} x^2 + \frac{1}{1200} x^4 - \cdots\right)
\end{aligned}
$$

Part (3)

setting $a_0 = 1$ as problem says, and since $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$ then the above is

$$
y_1(x) = \frac{\sin x}{x}
$$

Part (4)

$$W(x)=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix} \frac{\sin x}{x} & y_2\\ \frac{x\cos x-\sin x}{x^2} & y_2'\end{vmatrix}=y_2'\frac{\sin x}{x}-y_2\frac{x\cos x-\sin x}{x^2}$$

But from Abel's theorem, $W(x)=Ce^{\int -p(x)dx}$ where $p(x)$ is the coefficient in the ODE $y''+p(x)y'+q(x)y=0$. Since the ODE is $y''-\frac{2}{x}+y=0$ then $p=\frac{-2}{x}$ and $W(x)=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=x^{-2}$. Hence

$$\begin{aligned} y_2'\frac{\sin x}{x}-y_2\frac{x\cos x-\sin x}{x^2}&=\frac{C}{x^2}\\ x\sin(x)y_2'-y_2(x\cos x-\sin x)&=C\\ y_2'-y_2\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)&=\frac{C}{x\sin x}\end{aligned}$$

Integrating factor is $\mu=e^{\int-\frac{\cos x}{\sin x}+\frac{1}{x}dx}=e^{\int\frac{-\frac{d}{dx}(\sin x)}{\sin x}dx}e^{\int\frac{1}{x}dx}=e^{\int\frac{-1}{\sin x}d(\sin x)}+e^{\ln x}=e^{-\ln(\sin x)}x=\frac{x}{\sin x}$. Multiplying both sides by this integrating factor gives

$$\frac{d}{dx}\left(y_2\frac{x}{\sin x}\right)=\frac{C}{\sin^2 x}$$

Integrating gives

$$\begin{aligned} y_2\frac{x}{\sin x}&=\frac{C\cos(x)}{-\sin(x)}+C_2\\ y_2(x)&=-C\frac{\cos x}{x}+C_2\frac{\sin x}{x}\\ &=C_1\frac{\cos x}{x}+C_2\frac{\sin x}{x}\end{aligned}$$

But $C_2\sin x$ is the second solution, so we only keep $y_2(x)=C_1\frac{\cos x}{x}$. Hence the final solution is

$$y(x)=C_1\frac{\cos x}{x}+C_2\frac{\sin x}{x}$$

2.1.5 Problem 5

Find solution to $\frac{d^2}{dx^2}G(x,x_0)+k^2G(x,x_0)=\delta(x-x_0)$ subject to boundary conditions $G(0,x_0)=G(L,x_0)=0$

<u>solution</u>

First we obtain the solution to the homogeneous ODE $y''+k^2y=0$ with $y(0)=0,y(b)=0$ which has the two solutions $y_1(x)=\cos kx, y_2(x)=\sin(kx)$.

Therefore the solution to the Green function is

$$\begin{aligned} G(x,x_0)&=\begin{cases} A_1y_1(x)+A_2y_2(x) & 0<x<x_0\\ B_1y_1(x)+B_2y_2(x) & x<x_0<b\end{cases}\\ &=\begin{cases} A_1\cos(kx)+A_2\sin(kx) & 0<x<x_0\\ B_1\cos(kx)+B_2\sin(kx) & x<x_0<b\end{cases}\end{aligned}$$

Where A_i,B_i are constants to be found. From the condition $G(0,x_0)=0$ then the first solution above gives $A_1\cos(0)=0\rightarrow A_1=0$. And from $G(L,x_0)=0$ the second solution above gives $B_1\cos(kL)+B_2\sin(kLb)=0$ or $B_1=-\frac{B_2\sin(kL)}{\cos(kLb)}$, hence the solution now becomes

$$
\begin{aligned}
G(x,x_0) &= \begin{cases} A_2 \sin(kx) & 0<x<x_0 \\ B_1\cos(kx)+B_2\sin(kx) & x<x_0<L \end{cases}\\
&= \begin{cases} A_2 \sin(kx) & 0<x<x_0 \\ -\frac{B_2\sin(kL)}{\cos(kL)}\cos(kx)+B_2\sin(kx) & x<x_0<L \end{cases}\\
&= \begin{cases} A_2 \sin(kx) & 0<x<x_0 \\ \frac{B_2}{\cos(kL)}\left(\sin(kx)\cos(kL)-\sin(kL)\cos(kx)\right) & x<x_0<L \end{cases}
\end{aligned}
$$

Using $\sin(a-b)=\sin a\cos b - \cos a\sin b$, then $\sin(kx)\cos(kL) - \sin(kL)\cos(kx) = \sin(kx-kL) = \sin(k(x-L))$ and the above becomes

$$
G(x,x_0) = \begin{cases} A_2 \sin(kx) & 0<x<x_0 \\ \frac{B_2}{\cos(kL)}\sin(k(x-L)) & x<x_0<L \end{cases} \tag{1A}
$$

Now from continuity condition $G(x,x_0)_{x=x_0-\varepsilon} = G(x,x_0)_{x=x_0+\varepsilon}$ i.e. at $x=x_0$, then (from now on, we switch to x_0).

$$
A_2\sin(kx_0) = \frac{B_2}{\cos(kL)}\sin(k(x_0-L)) \tag{1}
$$

Now we find the derivative of G at $x=x_0$ gives

$$
\frac{d}{dx}G(x,x_0)\Big|_{x=x_0} = \begin{cases} kA_2\cos(kx_0) & 0<x<x_0 \\ \frac{kB_2}{\cos(kL)}\cos(k(x_0-L)) & x<x_0<L \end{cases}
$$

And from jump discontinuty in derivative of G at $x=x_0$ would obtain, since $G'_{x>x_0+\varepsilon} - G'_{x<x_0-\varepsilon} = \frac{-1}{p(x)}$, then

$$
\frac{kB_2}{\cos(kb)}\cos(k(x_0-L)) - kA_2\cos(kx_0) = \frac{-1}{p(x)}
$$

But since the ODE is $y''+k^2y=0$ then in SL form this is $-\left(y'\right)'+k^2y=0$, and comparing to $-\left(py'\right)'+k^2y=0$ we see that $p=-1$. Then above becomes

$$
\frac{kB_2}{\cos(kL)}\cos(k(x_0-L)) - kA_2\cos(kx_0) = 1 \tag{2}
$$

From (1,2) we solve for A_1, B_2. From (1)

$$
A_2 = \frac{B_2}{\cos(kb)\sin(kx_0)}\sin(k(x_0-L)) \tag{3}
$$

Substituting into (2) gives

$$
\begin{aligned}
\frac{kB_2}{\cos(kL)}\cos(k(x_0-L)) - k\left(\frac{B_2}{\cos(kL)\sin(kx_0)}\sin(k(x_0-L))\right)\cos(kx_0) &= 1\\
kB_2\cos(k(x_0-L)) - kB_2\sin(k(x_0-L))\frac{\cos(kx_0)}{\sin(kx_0)} &= \cos(kL)\\
kB_2\left(\sin(kx_0)\cos(k(x_0-L)) - \sin(k(x_0-L))\cos(kx_0)\right) &= \cos(kL)\sin(kx_0) \quad (4)
\end{aligned}
$$

Using $\sin(a-b)=\sin a\cos b-\cos a\sin b$, then

$$
\begin{aligned}
\sin(kx_0)\cos(k(x_0-L)) - \sin(k(x_0-L))\cos(kx_0) &= \sin(kx_0 - k(x_0-L))\\
&= \sin(kL)
\end{aligned}
$$

Then (3) becomes

$$
\begin{aligned}
kB_2 \sin(kL) &= \cos(kL)\sin(kx_0)\\
B_2 &= \frac{\cos(kL)\sin(kx_0)}{k\sin(kL)}
\end{aligned}
$$

Substituing the above in (3) gives

$$
\begin{aligned}
A_2 &= \frac{\cos(kL)\sin(kx_0)}{k\sin(kL)\cos(kL)\sin(kx_0)}\sin(k(x_0-L))\\
&= \frac{\sin(k(x_0-L))}{k\sin(kL)}
\end{aligned}
$$

Using A_2, B_2 found above in (1A) gives

$$
\begin{aligned}
G(x,x_0) &= \begin{cases} \frac{\sin(k(x_0-L))}{k\sin(kL)}\sin(kx) & 0<x<x_0\\ \frac{\cos(kL)\sin(kx_0)}{k\sin(kL)\cos(kb)}\sin(k(x-L)) & x<x_0<L \end{cases}\\
&= \frac{1}{k\sin(kL)}\begin{cases} \sin(k(x_0-L))\sin(kx) & 0<x<x_0\\ \sin(kx_0)\sin(k(x-L)) & x<x_0<L \end{cases}
\end{aligned}
$$

The following approach seems faster.

second solution

Instead of starting from

$$
G(x,x_0) = \begin{cases} A_1y_1(x) + A_2y_2(x) & 0<x<x_0\\ B_1y_1(x)+B_2y_2(x) & x<x_0<b \end{cases}
$$

We first find the eigenfunction $\Phi_n(x)$ that solves $y'' + k^2y = 0$ which satisfies the boundary conditions $y(0) = 0, y(L) = 0$. Then write

$$
G(x,x_0) = \begin{cases} A\Phi_n(x) & 0<x<x_0\\ B\Phi_n(x-L) & x<x_0<b \end{cases}
$$

So now we have only 2 unknowns to find, A, B using the continutity and jump conditions on G. Let see how this works on this same problem. The solution to $y'' + k^2y = 0$ is $y(x) = A\cos kx + B\sin kx$. At $y(0) = 0$ implies $A = 0$, so the solution becomes $y(x) = B\sin kx$ and at $x(L) = 0$ this gives $0 = B\sin(kL)$, which implies $kL = n\pi$ or $k = \frac{n\pi}{L}$. Hence the solution is $\Phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$. Therefore we set up the Green function as

$$
G(x,x_0) = \begin{cases} A\sin\left(\frac{n\pi}{L}x\right) & 0<x<x_0\\ B\sin\left(\frac{n\pi}{L}(x-L)\right) & x<x_0<b \end{cases}
$$

Or by letting $k_n = \frac{n\pi}{L}$

$$
G(x,x_0) = \begin{cases} A\sin(k_nx) & 0<x<x_0\\ B\sin(k_n(x-L)) & x<x_0<L \end{cases} \tag{1}
$$

Now continutity says

$$
A\sin(k_nx_0) = B\sin(k_n(x_0-L)) \tag{2}
$$

Taking derivative of (1) at $x = x_0$ gives

$$G'(x, x_0) = \begin{cases} Ak_n \cos(k_n x_0) & 0 < x < x_0 \\ Bk_n \cos(k_n (x_0 - L)) & x < x_0 < L \end{cases}$$

Then jump discontinutiy gives

$$Bk_n \cos(k_n (x_0 - L)) - Ak_n \cos(k_n x_0) = 1 \tag{3}$$

Solving (2,3) for A, B gives, as we did earlier

$$A = \frac{\sin(k(x_0 - L))}{k \sin(kL)}$$
$$B = \frac{\sin(kx_0)}{k \sin(kL)}$$

Using these in (1) gives

$$G(x, x_0) = \frac{1}{k \sin(kL)} \begin{cases} \sin(k(x_0 - L)) \sin(k_n x) & 0 < x < x_0 \\ \sin(kx_0) \sin(k_n (x - L)) & x < x_0 < Lb \end{cases}$$

Which is the same result obtained earlier.

2.2 exam 1

2.2.1 exam 1 prep sheet

Preparation sheet for the test: November 7 in class

The test will cover notes and examples on infinite series, complex analysis, and evaluation of integrals (not including the saddle point integration) and problem sets 1-4.

1. Things to know without thought:

(a) Sum of a geometric series.
(b) Definitions of convergence and absolute convergences of infinite series.
(c) Ratio test for convergence.
(d) Series expansions for e^x, $\sin x$, $\cos x$, and $\ln(1 + x)$ (for $|x| < 1$).
(e) Cauchy-Riemann conditions and definition of analytic functions.
(f) The meaning of $\log z$, $Logz$, and z^c
(g) Cauchy-Goursat Theorem and Cauchy Integral formula.
(h) The form of Taylor and Laurent series.
(i) The meaning of analytic continuation.
(j) The residue theorem.
(k) The definition of Cauchy principle value.
(l) The definition of an asymptotic series.

2.2.2 exam 1 writeup

Problem 1

Using a well known sum, find a closed for expression for the following series

$$f(z)=1+2z+3z^2+4z^3+5z^4+\cdots$$

Using the ratio test, find for what values of z this series converges.

Solution

Method 1

Assume that the closed form is

$$(1-z)^a=1+2z+3z^2+4z^3+5z^4+\cdots$$

For some unknown a. Now a will be solved for. Using Binomial series definition $(1-z)^a=1-az+\frac{(a)(a-1)}{2!}z^2-\frac{a(a-1)(a-2)}{3!}z^3+\cdots$. in the LHS above gives

$$1-az+\frac{a(a-1)}{2!}z^2-\frac{a(a-1)(a-2)}{3!}z^3+\cdots=1+2z+3z^2+4z^3+5z^4+\cdots$$

By comparing coefficients of z in the left side and on the right side shows that $a=-2$ from the coefficient of z term. Verifying this on the coefficient of z^2 shows it is correct since it gives $\frac{(-2)(-3)}{2}=3$. Therefore

$$a=-2$$

The closed form is therefore

$$\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+5z^4+\cdots$$

Method 2

Starting with Binomial series expansion given by

$$\frac{1}{1-z}=1+z+z^2+z^3+z^4+\cdots$$

Taking derivative w.r.t. z on both sides of the above results in

$$\begin{aligned}\frac{d}{dz}\left(\frac{1}{1-z}\right)&=\frac{d}{dz}\left(1+z+z^2+z^3+z^4+\cdots\right)\\-(1-z)^{-2}(-1)&=0+1+2z+3z^2+4z^3+\cdots\\\frac{1}{(1-z)^2}&=1+2z+3z^2+4z^3+\cdots\end{aligned}$$

Therefore the closed form expression is

$$\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\cdots$$

Which is the same as method 1.

The series general term of the series is

$$1+2z+3z^2+4z^3+\cdots=\sum_{n=0}^{\infty}(n+1)z^n$$

Applying the ratio test

$$\begin{aligned}L&=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\&=\lim_{n\to\infty}\left|\frac{(n+2)z^{n+1}}{(n+1)z^n}\right|\\&=\lim_{n\to\infty}\left|\frac{(n+2)z}{n+1}\right|\\&=z\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|\\&=z\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right|\end{aligned}$$

But $\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right| = 1$ and the above limit becomes

$$L = z$$

By the ratio test, the series converges when $|L| < 1$. Therefore $1+2z+3z^2+4z^3+\cdots$ converges absolutely when $|z| < 1$. An absolutely convergent series is also a convergent series. Hence the series converges for $|z| < 1$.

Problem 2

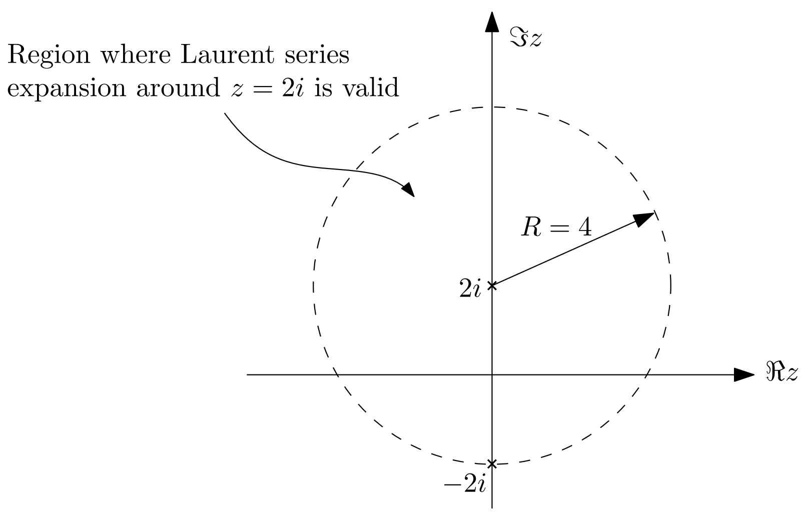
Find the Laurent series for the function

$$f(z) = \frac{1}{\left(z^2+4\right)^3}$$

About the isolated singular pole $z = 2i$. What is the order of this pole? What is the residue at this pole?

Solution

The poles are at $z^2 = 4$ or $z = \pm 2i$. The expansion of $f(z)$ is around the isolated pole at $z = 2i$. This pole has order 3. The region where this expansion is valid is inside a disk centered at $2i$ (but not including the point $z = 2i$ itself) and up to the nearest pole which is located at $-2i$. Therefore the disk will have radius 4.



Let

$$u = z - 2i$$
$$z = u + 2i$$

Substituting this expression for z back in $f(z)$ gives

$$
\begin{aligned}
f(z) &= \frac{1}{\left((u+2i)^2+4\right)^3}\\
&= \frac{1}{\left(u^2-4+4ui+4\right)^3}\\
&= \frac{1}{\left(u^2+4ui\right)^3}\\
&= \frac{1}{\left(u\left(u+4i\right)\right)^3}\\
&= \frac{1}{u^3}\frac{1}{\left(u+4i\right)^3}\\
&= \frac{1}{u^3}\frac{1}{\left[4i\left(\frac{u}{4i}+1\right)\right]^3}\\
&= \frac{1}{u^3}\frac{1}{(4i)^3\left(\frac{u}{4i}+1\right)^3}\\
&= \frac{1}{-i64u^3}\frac{1}{\left(\frac{u}{4i}+1\right)^3}\\
&= \left(\frac{i}{64u^3}\right)\frac{1}{\left(\frac{u}{4i}+1\right)^3} \qquad (1)
\end{aligned}
$$

Expanding the term $\frac{1}{\left(1+\frac{u}{4i}\right)^3}$ using Binomial series, which is valid for $\left|\frac{u}{4i}\right|<1$ or $|u|<4$ gives

$$
\begin{aligned}
\frac{1}{\left(1+\frac{u}{4i}\right)^3} &= 1+(-3)\frac{u}{4i}+\frac{(-3)(-4)}{2!}\left(\frac{u}{4i}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{u}{4i}\right)^3+\frac{(-3)(-4)(-5)(-6)}{4!}\left(\frac{u}{4i}\right)^4+\cdots\\
&= 1-3\frac{u}{4i}+\frac{3\cdot 4}{2!}\frac{u^2}{16i^2}-\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64i^3}+\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256i^4}+\cdots\\
&= 1+3i\frac{u}{4}-\frac{3\cdot 4}{2!}\frac{u^2}{16}-\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64(-i)}+\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256}+\cdots\\
&= 1+3i\frac{u}{4}-\frac{3\cdot 4}{2!}\frac{u^2}{16}-i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64}+\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256}+\cdots \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) and simplifying gives

$$
\begin{aligned}
f(z) &= \left(\frac{i}{64u^3}\right)\left(1+3i\frac{u}{4}-\frac{3\cdot 4}{2!}\frac{u^2}{16}-i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64}+\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256}+\cdots\right)\\
&= \frac{i}{64u^3}+\frac{i}{64u^3}\left(3i\frac{u}{4}\right)-\frac{i}{64u^3}\left(\frac{3\cdot 4}{2!}\frac{u^2}{16}\right)-\frac{i}{64u^3}\left(i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64}\right)+\frac{i}{64u^3}\left(\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256}\right)+\cdots\\
&= \frac{i}{64u^3}-\frac{1}{64u^2}\frac{3}{4}-\frac{i}{64u}\left(\frac{3\cdot 4}{2!}\frac{1}{16}\right)+\frac{1}{64}\left(\frac{3\cdot 4\cdot 5}{3!}\frac{1}{64}\right)+\frac{i}{64}\left(\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u}{256}\right)+\cdots\\
&= \frac{i}{64u^3}-\frac{3}{256u^2}-i\frac{3}{512}\frac{1}{u}+\frac{5}{2048}+i\frac{15}{16384}u+\cdots
\end{aligned}
$$

Replacing u back by $z-2i$ in the above results in

$$
f(z) = \frac{i}{64}\frac{1}{(z-2i)^3}-\frac{3}{256}\frac{1}{(z-2i)^2}-\frac{3i}{512}\frac{1}{(z-2i)}+\frac{5}{2048}+\frac{15i}{16384}(z-2i)+\cdots \qquad (3)
$$

This expansion is valid for $|z-2i|<4$. The above shows that the residue is

$$
\boxed{-\frac{3i}{512}}
$$

Which is the coefficient of the $\frac{1}{(z-2i)}$ term in (3).

Problem 3

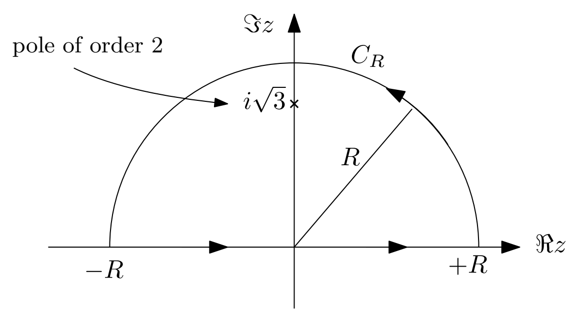
Use residues to evaluate the following integral

$$I=\int_0^{\infty}\frac{dx}{x^4+6x^2+9}$$

Solution

The integrand is an even function. Therefore the integral $\int_{-\infty}^{\infty}\frac{dx}{x^4+6x^2+9}$ is evaluated instead and then the required integral I will be half the value obtained. The poles of $\frac{1}{x^4+6x^2+9}$ are the zeros of the denominator. Factoring the denominator as $\left(x^2+3\right)\left(x^2+3\right)=0$, shows the roots are $x=\pm i\sqrt{3}$ from the first factor and $x=\pm i\sqrt{3}$ from the second factor.

Since the upper half plane will be used, the pole located there is $+i\sqrt{3}$ and it is of order two. Now that pole locations are known, the following contour is used to evaluate $\int_{-\infty}^{\infty}\frac{dx}{x^4+6x^2+9}$ as shown in the plot below



$$\begin{aligned}\oint_C f(z)\,dz &= \lim_{R\to\infty}\int_C f(z)\,dz+\lim_{R\to\infty}\int_{-R}^{+R}f(x)\,dx\\ &=\lim_{R\to\infty}\int_C\frac{dz}{z^4+6z^2+9}+\lim_{R\to\infty}\int_{-R}^{+R}\frac{dx}{x^4+6x^2+9}dx \qquad (2)\end{aligned}$$

Where the integral $\int_{-R}^{+R}$ is Cauchy principal integral. Since the contour C is closed and because $f(z)$ is analytic on and inside C except for the isolated singularity inside at $z=i\sqrt{3}$, then by Cauchy integral formula $\oint_C f(z)\,dz=2\pi i\sum\text{Residue}$. Where the sum of residues is over all poles inside C. Therefore (2) can becomes

$$\int_{-\infty}^{+\infty}\frac{dx}{x^4+6x^2+9}dx=2\pi i\sum\text{Residue}-\lim_{R\to\infty}\int_C f(z)\,dz \qquad (3)$$

But

$$\begin{aligned}\left|\int_C f(z)\,dz\right|_{\max} &\le ML\\ &=\left|f(z)\right|_{\max}\pi R \qquad (4)\end{aligned}$$

Using

$$\left|f(z)\right|_{\max}\le\frac{1}{\left|z^2+3\right|_{\min}\left|z^2+3\right|_{\min}}$$

By inverse triangle inequality $\left|z^2+3\right|\ge|z|^2-3$. But $|z|=R$ on C, therefore $\left|z^2+3\right|\ge R^2-3$ and the above can now be written as

$$\left|f(z)\right|_{\max}\le\frac{1}{\left(R^2-3\right)\left(R^2-3\right)}$$

Using the above in (4) gives

$$\begin{aligned}\left|\int_C f(z)\,dz\right|_{\max} &\leq \frac{\pi R}{\left(R^2-3\right)\left(R^2-3\right)}\\ &= \frac{\pi R}{R^4-6R^2+9}\\ &= \frac{\frac{\pi}{R}}{R^2-6+\frac{9}{R^2}}\end{aligned}$$

In the limit as $R\to\infty$ then $\left|\int_C f(z)\,dz\right|_{\max}\to 0$. Using this result in (3) it simplifies to

$$\int_{-\infty}^{+\infty}\frac{dx}{x^4+6x^2+9}dx = 2\pi i\sum \text{Residue} \tag{5}$$

What is left now is to determine the residue at pole $z_0 = i\sqrt{3}$ which is of order 2. This is done using

$$\text{Residue}\left(z_0\right) = \lim_{z\to z_0}\frac{d}{dz}\left(\left(z-z_0\right)^2 f(z)\right)$$

But $z_0 = i\sqrt{3}$ and the above becomes

$$\begin{aligned}\text{Residue}\left(i\sqrt{3}\right) &= \lim_{z\to i\sqrt{3}}\frac{d}{dz}\left(\left(z-i\sqrt{3}\right)^2\frac{1}{\left(z-i\sqrt{3}\right)^2\left(z+i\sqrt{3}\right)^2}\right)\\ &= \lim_{z\to i\sqrt{3}}\frac{d}{dz}\frac{1}{\left(z+i\sqrt{3}\right)^2}\\ &= \lim_{z\to i\sqrt{3}}\frac{-2}{\left(z+i\sqrt{3}\right)^3}\\ &= \frac{-2}{\left(i\sqrt{3}+i\sqrt{3}\right)^3}\\ &= \frac{-2}{\left(2i\sqrt{3}\right)^3}\\ &= \frac{-2}{-(8)(3)i\sqrt{3}}\\ &= \frac{1}{12i\sqrt{3}}\end{aligned}$$

Using the above value of the residue in (5) gives

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dx}{x^4+6x^2+9}dx &= 2\pi i\left(\frac{1}{12i\sqrt{3}}\right)\\ &= \frac{\pi}{6\sqrt{3}}\end{aligned}$$

Therefore the integral $\int_0^\infty \frac{dx}{x^4+6x^3+9}$ is half of the above result which is

$$\int_0^\infty \frac{dx}{x^4+6x^2+9} = \frac{\pi}{12\sqrt{3}}$$

Problem 4

Find two approximations for the integral $x>0$

$$I(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\cos^2\theta}d\theta$$

One for small x (keeping up to linear order in x) and one for large values of x (keeping only the leading order term).

<u>Solution</u>

The integrand has the form e^z. This has a known Taylor series expansion around zero given by

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

Replacing z by $x\cos^2\theta$ in the above gives

$$e^{x\cos^2\theta} = 1 + x\cos^2\theta + \frac{\left(x\cos^2\theta\right)^2}{2} + \cdots$$

The problem is asking to keep linear terms in x. Therefore

$$e^{x\cos^2\theta} \approx 1 + x\cos^2\theta$$

Replacing the integrand in the original integral by the above approximation gives

$$\begin{aligned}
I(x) &\approx \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + x\cos^2\theta\right) d\theta \\
&\approx \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + x\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta\right) \\
&\approx \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + x\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta d\theta\right) \\
&\approx \frac{1}{2\pi}\left(\pi + x\left(\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{4}\left(2\theta + \sin 2\theta\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{4}\left(2\pi + 0\right)\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{2}\pi\right) \\
&\approx \frac{1}{2}\left(1 + \frac{x}{2}\right)
\end{aligned}$$

For large value of x, The integrand is written as $e^{f(\theta)}$ where $f(\theta) = x\cos^2\theta$. The value of θ where $f(\theta)$ is maximum is first found. Then solving for θ in

$$\begin{aligned}
\frac{d}{d\theta}x\cos^2\theta &= 0 \\
-2x\cos\theta\sin\theta &= 0
\end{aligned}$$

Hence solving for θ in

$$\cos\theta\sin\theta = 0$$

There are two solutions to this. Either $\theta = \frac{\pi}{2}$ or $\theta = 0$. To find which is the correct choice, the sign of $\frac{d^2}{d\theta^2}f(\theta)$ is checked for each choice.

$$\begin{aligned}
\frac{d^2}{d\theta^2}x\cos^2\theta &= \frac{d}{d\theta}\left(-2x\cos\theta\sin\theta\right) \\
&= -2x\frac{d}{d\theta}\left(\cos\theta\sin\theta\right) \\
&= -2x\left(-\sin\theta\sin\theta + \cos\theta\cos\theta\right) \\
&= -2x\left(-\sin^2\theta + \cos^2\theta\right) \qquad (1)
\end{aligned}$$

Substituting $\theta = \frac{\pi}{2}$ in (1) and using $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ gives

$$\begin{aligned}
\left.\frac{d^2}{d\theta^2}x\cos^2\theta\right|_{\theta=\frac{\pi}{2}} &= -2x(-1) \\
&= 2x
\end{aligned}$$

Since the problem says that $x > 0$ then $\left.\frac{d^2}{d\theta^2}x\cos^2\theta\right|_{\theta=\frac{\pi}{2}} > 0$. Therefore this is a minimum.

Using the second choice $\theta = 0$, then (1) becomes (after using $\cos(0) = 1$ and $\sin(0) = 0$)

$$\left.\frac{d^2}{d\theta^2} x \cos^2\theta\right|_{\theta=0} = -2x$$

And because $x > 0$ then $\left.\frac{d^2}{d\theta^2} x \cos^2\theta\right|_{\theta=0} < 0$. Therefore the integrand is maximum at

$$\theta_{peak} = 0$$

Now that peak θ is found, then $f(\theta)$ is expanded in Taylor series around $\theta_{peak} = 0$. Since $f(\theta) = x\cos^2\theta$, then

$$f\left(\theta_{peak}\right) = x$$

And $f'(\theta) = -2x\cos\theta\sin\theta$. At θ_{peak} this becomes $f'\left(\theta_{peak}\right) = 0$. The next term is the quadratic one, given by

$$\begin{aligned} f''(\theta) &= -2x\frac{d}{d\theta}(\cos\theta\sin\theta) \\ &= -2x\left(-\sin^2\theta + \cos^2\theta\right) \end{aligned}$$

Evaluating the above at $\theta_{peak} = 0$ gives

$$f''\left(\theta_{peak}\right) = -2x$$

The problem says to keep leading term, so no need for more terms. Therefore the Taylor series expansion of $f(\theta) = x\cos^2\theta$ around $\theta = \theta_{peak}$ is

$$\begin{aligned} x\cos^2\theta &\approx f\left(\theta_{peak}\right) + f'\left(\theta_{peak}\right)\theta + \frac{1}{2!}f''\left(\theta_{peak}\right)\theta^2 \\ &= x + 0 - \frac{2x}{2!}\theta^2 \\ &= x - x\theta^2 \\ &= x\left(1 - \theta^2\right) \end{aligned}$$

The integral now becomes

$$\begin{aligned} I(x) &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\left(1-\theta^2\right)} d\theta \\ &\approx \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x} e^{-x\theta^2} d\theta \\ &= \frac{1}{2\pi}\left(e^x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x\theta^2} d\theta\right) \end{aligned}$$

Comparing $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x\theta^2} d\theta$ to the Gaussian integral $\int_{-\infty}^{\infty} e^{-a\theta^2} d\theta = \sqrt{\frac{\pi}{x}}$, then the above can be approximated as

$$I(x) = \frac{e^x}{2\pi}\sqrt{\frac{\pi}{x}}$$

Summary of result

Small x approximation	$\frac{1}{2}\left(1 + \frac{x}{2}\right)$
Large x approximation	$\frac{e^x}{2\pi}\sqrt{\frac{\pi}{x}}$

Note that using the computer, the exact solution is

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\cos^2\theta} d\theta = \frac{1}{2}e^{\frac{x}{2}}\,\text{BesselI}\left(0, \frac{x}{2}\right)$$

Problem 5

Use the Cauchy-Riemann equations to determine where the function

$$f(z) = z + \overline{z^2}$$

Is analytic. Evaluate $\oint_C f(z)\,dz$ where contour C is on the unit circle $|z| = 1$ in a counter-clockwise sense.

Solution

Using $z = x + iy$, the function $f(z)$ becomes

$$\begin{aligned} f(z) &= x + iy + \overline{\left(x+iy\right)^2} \\ &= x + iy + \overline{\left(x^2 - y^2 + 2ixy\right)} \\ &= x + iy + \left(x^2 - y^2 - 2ixy\right) \\ &= \left(x + x^2 - y^2\right) + i\left(y - 2xy\right) \end{aligned}$$

Writing $f(z) = u + iv$, and comparing this to the above result shows that

$$\begin{aligned} u &= x + x^2 - y^2 \\ v &= y - 2xy \end{aligned} \tag{1}$$

Cauchy-Riemann are given by

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{-\partial u}{\partial y} &= \frac{\partial v}{\partial x} \end{aligned}$$

Using result in (1), Cauchy-Riemann are checked to see if they are satisfied or not. The first equation above results in

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 + 2x \\ \frac{\partial v}{\partial y} &= 1 - 2x \end{aligned}$$

Therefore $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$. This shows that $f(z)$ is not analytic for all x, y.

Since $f(z)$ is not analytic, Cauchy integral formula can not be used. Instead this can be integrated using parameterization. Let $z = e^{i\theta}$ (No need to use $re^{i\theta}$ since $r = 1$ in this case because it is the unit circle). The function $f(z)$ becomes

$$\begin{aligned} f(z) &= e^{i\theta} + \overline{\left(e^{i\theta}\right)^2} \\ &= e^{i\theta} + \overline{e^{2i\theta}} \\ &= e^{i\theta} + e^{-2i\theta} \end{aligned}$$

And because $z = e^{i\theta}$ then $dz = d\theta e^{i\theta}$. The integral now becomes

$$\begin{aligned} \oint_C f(z)\,dz &= \int_0^{2\pi} \left(e^{i\theta} + e^{-2i\theta}\right) e^{i\theta} d\theta \\ &= \int_0^{2\pi} \left(e^{2i\theta} + e^{-i\theta}\right) d\theta \\ &= \left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi} + \left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi} \\ &= \frac{1}{2i}\left[\cos 2\theta + i\sin 2\theta\right]_0^{2\pi} + i\left[\cos\theta - i\sin\theta\right]_0^{2\pi} \\ &= \frac{1}{2i}\left[(\cos 4\pi + i\sin 4\pi) - (\cos 0 + i\sin 0)\right] + i\left[(\cos 2\pi - i\sin 2\pi) - (\cos 0 - i\sin 0)\right] \\ &= \frac{1}{2i}\left[1 - 1\right] + i\left[1 - 1\right] \end{aligned}$$

Hence

$$\oint_C f(z)\,dz = 0$$

2.3 final exam

2.3.1 exam 1 prep sheet

Preparation sheet for the exam: Dec 19 from 2:45 to 5:15 in class

The exam will cover notes and examples on infinite series, complex analysis, evaluation of integrals, integral transforms, ordinary differential equations, eigenvalue problems, and partial differential equations. The exam will also cover problem sets 1-7. You are allowed to have a 8.5 by 11 sheet of sheet of paper with equations on it.

1. Things to know without thought:

(a) Sum of a geometric series.
(b) Definitions of convergence and absolute convergences of infinite series.
(c) Ratio test for convergence.
(d) Series expansions for e^x, $\sin x$, $\cos x$, and $\ln(1+x)$ (for $|x| < 1$).
(e) Cauchy-Riemann conditions and definition of analytic functions.
(f) The meaning of $\log z$, $Logz$, and z^c
(g) Cauchy-Goursat Theorem and Cauchy Integral formula.
(h) The form of Taylor and Laurent series.
(i) The meaning of analytic continuation.
(j) The residue theorem.
(k) The definition of Cauchy principle value.
(l) The definition of an asymptotic series.
(m) The definition of Fourier series and transform The definition of a separable first order ODE.
(n) The criterium to check if an ODE is exact.
(o) Solution of linear first order differential equation using an integrating factor.
(q) Definition and solution of a homogeneous ODE.
(q) Use of the method of undetermined coefficients to solve inhomogeneous linear ODEs.
(r) Ordinary and regular singular points of a linear differential equation (and the form of the series solutions for these).
(s) Definition of a Hermitian differential operator.
(t) Sturm Lioville differential equation and related orthogonality of eigenfunctions.
(u) Use of the Wronskian to check if two solutions are independent.
(v) The definition of generating functions and how to use these to find recurrence relations.
(w) The completeness relation.
(x) The definition of the Green function in solving inhomogeneous problems.
(y) The eigenfunction expansion of the Green function.
(z) The form of the wave, Laplace, and diffusion equations in 1,2, and 3 dimensions.
(aa) The solution of the 1D wave equation.
(bb) The use of separation of variables. Specifically in 3D spherical coordinates and in 2D polar coordinates.

2.3.2 my final exam cheat sheet

Solve for $\psi(r,\theta,\phi,t)$ where $\frac{\partial\psi^2}{\partial t^2} = c^2\nabla\psi$ (Wave PDE 3D spherical coordinates)

Let $\psi = T(t)X(r,\theta,\phi)$. First seperation gives the following two equations

k^2 is first separation constant

$\omega = ck$

$$T''(t) + c^2k^2T(t) = 0 \xrightarrow{\text{solution}} T(t) = \cos(\omega t) + \sin(\omega t)$$

$$\nabla X + k^2X = 0 \longleftarrow \text{Helmholtz equtiion}$$

Let $X = R(r)\Theta(\theta)\Phi(\phi)$ and apply separation again.

angular equation. Second sepration. Use $l(l+1)$ as separation constant.

solve first, m^2 is separation constant

$$\Phi'' + m^2\Phi = 0 \longrightarrow \Phi(\phi) = \left\{ \begin{array}{c} e^{im\phi} \\ e^{-im\phi} \end{array} \right\}$$

associated Legendre polynomial of first kind

$$\left(1-x^2\right)\Theta'' - 2x\Theta' + \left(l\left(l+1\right) - \frac{m^2}{1+x^2}\right)\Theta = 0 \longrightarrow \Theta(x) = \left\{ \begin{array}{c} P_l^m(x) \\ Q_l^m(x) \end{array} \right\}$$

$x = \cos\theta$

Associated legendre equation
Note: l is integer and $-l \le m \le l$

associated Legendre polynomial of second kind. not used. blows up $x = \pm 1$

Radial equation (Bessel like)

shperical bessel function first kind

$$r^2R'' + 2rR' + (r^2k^2 - l(l+1))R = 0 \longrightarrow R(r) = \left\{ \begin{array}{c} j_l(kr) \\ y_l(kr) \end{array} \right\}$$

Can be converted to standard Bessel ODE by some transformation (not shown)

$\Downarrow$

shperical bessel function second kind. Not used as it blows up at $r = 0$

Final solution is summation of fundamental solution
$\sum j_l(kr)\,P_l^m(\cos(m\phi) + \sin(m\phi))(\cos\omega t + \sin\omega t)$

Solve for $\psi(r,\phi,t)$ where $\frac{\partial\psi^2}{\partial t^2} = c^2\nabla\psi$ (Wave PDE in 2D disk, polar coordinates). Membrane is fixed on edge of disk. Radius a.

Laplacian in 2D polar
$\nabla u = \frac{\partial u^2}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u^2}{\partial\phi^2}$

Let $\psi = T(t)X(r,\phi)$. First seperation gives the following two equations

k^2 is first separation constant

$\omega = ck_{mn}$

$$T''(t) + c^2k^2T(t) = 0 \xrightarrow{\text{solution}} T(t) = \cos(\omega t) + \sin(\omega t)$$

$$\nabla X + k^2X = 0 \longleftarrow \text{Helmholtz equtiion}$$

Let $X = R(r)\Phi(\phi)$ and apply separation again.

solve first, m^2 is separation constant

$$\Phi'' + m^2\Phi = 0 \xrightarrow[\text{solution}]{} \Phi(\phi) = \left\{ \begin{array}{c} e^{im\phi} \\ e^{-im\phi} \end{array} \right\}$$

Due to periodicity, m must be integer.

Radial equation

Bessel function order m

$$r^2R'' + rR' + (r^2k^2 - m^2)R = 0 \xrightarrow[\text{solution}]{} R(r) = \left\{ \begin{array}{c} J_m(kr) \\ Y_m(kr) \end{array} \right\}$$

Bessel ODE

Bessel function second kind. Not used as blow up at $r = 0$

$J_m(ka) = 0$ from boundary conditions. This fixes k. Let Z_{mn} be the n^{nt} zero of the Bessel J_m function. Therefore $k_{mn} = \frac{Z_{mn}}{a}$ are allowed values of k.
$\psi \propto J_m(k_{mn}r)(\cos(m\phi) + \sin(m\phi))$ This gives rise to modal shapes
$\psi(r,\phi,t) = \sum J_m(k_{mn}r)(\cos(m\phi) + \sin(m\phi))(\cos(ck_{mn}t) + \sin(ck_{mn}t))$

wave 1D: $\frac{\partial\psi^2}{\partial t^2} = c^2\frac{\partial\psi^2}{\partial x^2}$
$\psi(x,t) = \frac{1}{2}(f_0(x-ct) + f_0(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(s)\,ds$
Where $f_0(x)$ is initial position of string and $g_0(x)$ is initial velocity

Bessel: $x^2y'' + xy' + (x^2 - m^2)y = 0$, $W = \frac{C}{x}$. When m not intger, solutions $J_m(x), J_{-m}(x)$. When m integer, solutions $J_m(x), Y_m(x)$

Legendre: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, $W = \frac{C}{1-x^2}$. Solutions $P_n(x), Q_n(x)$. Where Q blows at ± 1

Sturm-Lioville: $(py')' - qy + \lambda ry = 0$, operator $L[y] = -(py')' + q$. When $q = 0$, equation becomes $L[y] = \lambda ry$. Inner product $< u_i, u_j >= \int \bar{u_i}u_j\,dx$

Bernuli $y' + f(x)y = g(x)y^n$, where $n \neq 0, 1$. Divide by $y^n \rightarrow \frac{1}{y^n}y' + f(x)y^{1-n} = g(x)$ and let $v = y^{1-n} \rightarrow v' = (1-n)y^{-n}y'$. converts ODE to separable in $v(x)$

isobaric given y weight m and x weight 1. each term must have the same weight if we can find m. Then let $y = vx^m$. Find $\frac{dy}{dx}$. sub into ODE to get rid of y. Separable in v. solve.

Exact Write ode as $Mdx + Ndx = 0$ then check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If so then set up 2 equations $\frac{\partial\Phi}{\partial x} = M$, $\frac{\partial\Phi}{\partial y} = N$. Integrate the first to get $\Phi = \int Mdx + f(y)$, differentiate this w.r.t y, and compare to $\frac{\partial\Phi}{\partial y} = N$ and solve for $f(y)$. Solution is $\Phi(x,y) = c$

Sturm-Liouville To convert $ay'' + by' + (c + \lambda d)y = 0$ to $\left(p(x)\frac{dy}{dx}\right)' - q(x)y(x) + \lambda r(x)y(x) = 0$ use:
$p(x) = e^{\int\frac{b(x)}{a(x)}dx}$ and $q(x) = -p(x)\frac{c(x)}{a(x)}$ and $r(x) = \frac{p(x)}{a(x)}d$

generating functions. Bessel $g(x.t) = e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$, then write $g(x,t) = \sum_{n=-\infty}^{\infty} J_n(x)t^n$. Find
$A_n(x) = \frac{1}{2\pi i}\oint\frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{n+1}}dt = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta$. To find recusive relations, do $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$ add/subtract we get
$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$ and
$J_n'(x) = \frac{n}{x}J_n(x) - J_{n-1}(x)$ For Legendre use
$g(x,t) = \frac{1}{\sqrt{1-2xt-t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n$. Do $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$
add/subtract we get $P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x)$
and $P_{n-1}'(x) = -nP_n(x) + xP_n'(x)$
$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$. Special values,
$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

Laplace PDE on disk:
$y(x) = A_0 + \sum r^n(C_n\cos n\theta + k_n\sin n\theta)$ where
$A_0 = \frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta$ and
$C_na^n = \frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta d\theta$ and
$k_na^n = \frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta d\theta$ where a is radius and $f(\theta)$ is boundary condition.

Euler ode $r^2y'' + ry' + y = 0$, let $y = r^{\alpha}$

Polynomial ODE. $y'(x) = f(ax + by + c)$. Let $V = ax + by + c$, then $dV = adx + bdy$, converts it to separable $\frac{dy}{dv} = \frac{f(v)}{a + bf(v)}$

Green function. For boundary value problem $y'' + y = f(x)$ write it as

$\boxed{\log(z) = \ln|z| + i(\theta_0 + 2n\pi)}$

$$G(x,x_0) = \left\{ \begin{array}{ll} Ay_1(x) & 0 < x < x_0 \\ By_2(x) & x_0 < x < L \end{array} \right.$$

Wronskian $W(x) = det[[y_1,y_2],[y_1',y_2']] = Ce^{-\int p(x)\,dx}$ Where p is from $y'' + py' + qy = 0$

Where $y_1(x)$ is one of the solutions to $y'' + y = 0$ that satisfies left BC and $y_2(x)$ is one which satifies right BC. Then solve for A, B from contuinity condition $Ay_1(x_0) = By_2(x_0)$ and jump discontinuty $By_2'(x_0) - Ay_1'(x_0) = \frac{-1}{p(x)}$ where $p(x)$ is from SL form. (-1) for the above.
Then flipe the x, x_0 roles and then do $y(x) = \int_0^x G(x_0,x)f(x_0)\,dx_0 + \int_x^L G(x_0,x)f(x_0)\,dx_0$
Variation of parameters $y_p(x) = -y_1(x)\int\frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int\frac{y_1(x)f(x)}{W(x)}dx$

Fourier series $f(x) = \frac{A_0}{2} + \sum_{n=0}^{\infty}A_n\cos\left(\frac{2\pi}{T}nx\right) + B_2\sin\left(\frac{2\pi}{T}nx\right)$ where $A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos\left(\frac{2\pi}{T}nx\right)$ and $B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin\left(\frac{2\pi}{T}nx\right)$. Complex form is $f(x) = \sum_{n=-\infty}^{\infty}C_ne^{i\frac{2\pi}{T}nx}, C_n = \int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)e^{-i\frac{2\pi}{T}nx}dx$.
Fourier transform $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}d\omega$ and $F(\omega) = \int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$. Parsvales: $\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$

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Eigenfunction expansion. Startting from $L[y]-ky=f(x)$, assuming $y(x)=\sum_n C_n\Phi_n(x)$ where $\Phi_n(x)$ eigenfunctions found by solving $L[y]-\lambda y=0$ with homogenous B.C. Then $f(x)=\sum_n d_n\Phi_n(x)$. Subtituting in the ode and use that $L[y]=\lambda y$ results in $\sum_n(\lambda_n-k)C_n\Phi_n(x)=\sum_n d_n\Phi_n(x)$. But $d_n=\langle\Phi_n(x),f(x)\rangle=\frac{2}{L}\int_0^L\Phi_n(x)f(x)\,dx$. The $\frac{2}{L}$ due to normalization. This gives $C_n=\frac{d_n}{\lambda_n-k}$. Now that C_n is found the solution is $y(x)=\sum_n\frac{d_n}{\lambda_n-k}\Phi_n(x)=\sum\frac{\langle\Phi_n(x),f(x)\rangle}{\lambda_n-k}\Phi_n(x)$ or $y(x)=\sum\frac{\Phi_n(x)}{\lambda_n-k}\int\overline{\Phi}_n(x')f(x')\,dx'$ or $y(x)=\int\sum\frac{\Phi_n(x')\overline{\Phi}_n(x')}{\lambda_n-k}f(x')\,dx'$. Compare to $y(x)=\int^x G(x,x')f(x')\,dx'$ shows that $G(x,x')=\sum\frac{\Phi_n(x')\overline{\Phi}_n(x')}{\lambda_n-k}$. This is called Green function eigenfunction expansion. This assumes Φ_n are normalized so weight is 1.

completeness relation $\int\bar{u}_i u_j r(x)\,dx=\delta_{ij}$. $f=\sum C_n\Phi_n(x)$ and $r(x')\sum_n\Phi_n(x)\Phi_n(x')=\delta(x-x')$. If $f(x)=\delta(x-x')$, then $L[G(x,x')]-kG(x,x')=\delta(x-x')$

Operator is Hermite if $\int\bar{u}L[v]\,dx=\int\bar{v}L[u]\,dx$

Asymptotic series $S(z)=c_0+\frac{c_1}{z}+\frac{c_2}{z^2}+\cdots$ is series expansion of $f(z)$ gives good approximation for large z. we truncate $S(z)$ before it becomes divergent. n is the number of terms in $S_n(z)$. It is optimally truncated when $n\approx|z|^2$. $S(x)$ has the following two important properties

1. $\lim_{|z|\to\infty}z^n(f(z)-S_n(z))=0$ for fixed n.
2. $\lim_{n\to\infty}z^n(f(z)-S_n(z))=\infty$ for fixed z.

$$\Gamma(x)\Gamma(1-x)=\int_0^\infty\frac{t^{x-1}}{1+t}dt\quad 0<x<1$$
$$=\frac{\pi}{\sin(\pi x)}$$

If pole of order n. to find residue, use
$$\text{Residue}(z_0)=\lim_{z\to z_0}\frac{d^{n-1}}{dz^n}\frac{(z-z_0)^n}{(n-1)!}f(z)$$

$S(z)\sim f(z)$ when $S(z)$ is the asymptotic series expansion of $f(z)$ for large z. common method to find $S(z)$ is by integration by parts

Cauchy theoreom. Cauchy-Goursat: If $f(z)$ is analytic on and inside closed contour C then $\oint_C f(z)\,dz=0$. But remember that if $\oint_C f(z)\,dz=0$ then this does not necessarily imply $f(z)$. If $f(z)$ is analytic on and inside C then and z_0 is a point in C then

$$2\pi i f(z_0)=\oint_C\frac{f(z)}{z-z_0}dz \text{ and } 2\pi i f'(z_0)=\oint_C\frac{f(z)}{(z-z_0)^2}dz \text{ and } \frac{2\pi i}{n!}f^{(n)}(z_0)=\oint_C\frac{f(z)}{(z-z_0)^{n+1}}dz$$

Gaussian $\int_{-\infty}^{\infty}e^{-ax^2}dx=\sqrt{\frac{\pi}{x}}$

residue, use $R(z_0)=\lim_{z\to z_0}(z-z_0)f(z)$

Green's Theorem says $\int_C P\,dx+Q\,dy=\int_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

Laurent series $f(z)=\sum_{n=0}^\infty a_n(z-z_0)^n+\sum_{n=1}^\infty\frac{b_n}{(z-z_0)^n}$ where $a_n=\frac{1}{2\pi i}\oint\frac{f(z)}{(z-z_0)^{n+1}}dz$ and $b_n=\frac{1}{2\pi i}\oint\frac{f(z)}{(z-z_0)^{-n+1}}dz$. Power series of $f(z)$ around z_0 is $\sum_0^\infty a_n(z-z_0)^n$ where $a_n=\frac{1}{n!}f^{(n)}(z)\big|_{z=z_0}$ For Laurent series, lets say singularity is at $z=0$ and $z=1$. To expand about $z=0$, get $f(z)$ to look like $\frac{1}{1-z}$ and use geometric series for $|z|<1$. To expand about $z=1$, there are two choices, to the inside and to the outside. For the outside, i.e. $|z|>1$, get $f(z)$ to have $\frac{1}{1-\frac{1}{z}}$ form, this now valid for $|z|>1$.

if $e^{-y}\leq 1$ does not work, use Jordan inquality and use $e^{-y}\leq\frac{\pi}{R}$ for contour integration

Euler Gamma function
$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt\quad\text{Re}(z)>0$ To extend $\Gamma(z)$ to the left half plane, i.e. for negative values. Let us define, using the above recursive formula
$\bar{\Gamma}(z)=\frac{\Gamma(z+1)}{z}\quad\text{Re}(z)>-1$

$$\Gamma(z)=(z-1)\Gamma(z-1)\quad\text{Re}(z)>1$$
$$\Gamma(1)=1$$
$$\Gamma(n)=(n-1)!$$
$$\Gamma(n+1)=n!$$
$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$
$$\Gamma(z+1)=z\Gamma(z)\quad\text{recursive formula}$$
$$\Gamma(\bar{z})=\overline{\Gamma(z)}$$
$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}$$

$$\int_0^\infty e^{-ax}\cos\lambda x=\frac{a}{a^2+\lambda^2}$$
$$\int_0^\infty e^{-ax}\sin\lambda x=\frac{\lambda}{a^2+\lambda^2}$$

If $f(z)$ satisfies CR everywhere in that region then it is analytic. Let $f(z)=u(x,y)+iv(x,y)$, then these two equations in Cartesian coordinates are $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$ Sometimes it is easier to use the polar form of these. Let $f(z)=r\cos\theta+i\sin\theta$, then the equations become $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}$ and $-\frac{1}{r}\frac{\partial u}{\partial\theta}=\frac{\partial v}{\partial r}$

F.S. (period 2L).
$\int_0^L\sin^2\left(\frac{\pi}{L}x\right)dx=\frac{L}{2}$ and
$\int_{-L}^L\sin^2\left(\frac{\pi}{L}x\right)dx=L$

To $\int_C f(z)\,dz$ where C is some open path, i.e. not closed, such as a straight line or a half circle arc. Use parameterization. This converts the integral to line integration. If C is straight line, use standard t parameterization,
$x(t)=(1-t)x_0+tx_1$ and
$y(t)=(1-t)y_0+ty_1$ where (x_0,y_0) in the line initial point and (x_1,y_1) is the line end point. This works for straight lines. Use the above and rewrite $z=x+iy$ as $z(t)=x(t)+iy(t)$ and plug-in in this $z(t)$ in $f(z)$ to obtain $f(t)$, then the integral becomes
$$\int_C f(z)\,dz=\int_{t=0}^{t=1}f(z)\,z'(t)\,dt$$
Now evaluate this integral.

$$\int x\cos ax=\frac{\cos ax}{a^2}+\frac{x\sin ax}{a}$$
$$\int x\sin ax=\frac{\sin ax}{a^2}\frac{x\cos ax}{a}$$
$$\int\sin^2 ax=\frac{x}{2}-\frac{\sin 2ax}{4a}$$
$$\int\cos^2 ax=\frac{x}{2}+\frac{\sin 2ax}{4a}$$
$$\int\sin ax\cos ax=\frac{\sin^2 ax}{2a}$$
$$\int\frac{1}{ax+b}=\frac{1}{a}\ln(ax+b)$$
$$\int\frac{x}{ax+b}=\frac{x}{a}-\frac{b}{a^2}\ln(ax+b)$$
$$\int\frac{1}{\sqrt{1-x^2}}=\arcsin(x)$$
$$\int\frac{-1}{\sqrt{1-x^2}}=\arccos(x)$$
$$\int\frac{1}{1+x^2}=\arctan(x)$$

Geometric series:
$\sum_{n=0}^N r^n=1+r+r^2+r^3+\cdots+r^N=\frac{1-r^{N-1}}{1-r}$ and
$\sum_{n=0}^\infty r^n=1+r+r^2+r^3+\cdots=\frac{1}{1-r}\quad|r|<1$ and
$\sum_{n=0}^\infty(-1)^n r^n=1-r+r^2-r^3+\cdots=\frac{1}{1+r}\quad|r|<1$
Binomial series: $(x+y)^n=$
$x^n+nx^{n-1}y+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3+\cdots$
From this can generate all other special cases. For

$$\frac{1}{(1+x)}=1-x+x^2-x^3+\cdots\quad|x|<1$$
$$\frac{1}{(1-x)}=1+x+x^2+x^3+\cdots\quad|x|<1$$

$$\sin x=\frac{1}{2}\frac{e^{ix}-e^{-ix}}{2i}\qquad\sin\theta=\frac{z-z^{-1}}{2i}$$
$$\cos x=\frac{1}{2}\frac{e^{ix}+e^{-ix}}{2}\qquad\cos\theta=\frac{z+z^{-1}}{2}$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A$$
$$\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$$
$$\cos A\cos B=\frac{1}{2}(\cos(A+B)+\cos(A+B))$$
$$\sin A\cos B=\frac{1}{2}(\sin(A-B)+\sin(A+B))$$

$$\int xe^{ax}=\frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)$$
$$\int x^2e^{ax}=\frac{e^{ax}}{a}\left(x^2-\frac{2}{a}x+\frac{2}{a^2}\right)$$
$$\int\sin bxe^{ax}=\frac{e^{ax}}{(a^2+b^2)}(a\sin bx-b\cos bx)$$
$$\int\ln x=x\ln x-x$$

$z=re^{i(\theta+2n\pi)}$, hence $z^{\frac{1}{2}}=r^{\frac{1}{2}}e^{\frac{i\theta}{2}}e^{2n\pi}$ for $n=0,1$. Hence 2 roots. $n=0$ gives principle part

$z^c=e^{c\ln z}$

$\log(z)=\ln|z|+i(\theta_0+2n\pi)$
$n=0$ principle part

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots|x|<1$$
$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}-\cdots|x|<1$$
$$\sin x=x-\frac{x^3}{3!}+\cdots$$
$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots$$

Chapter 3

HWs

3.1 HW 1

3.1.1 Problem 1

Part a

For what values of x does the following series converge. $f(x) = 1 + \frac{9}{x^2} + \frac{81}{x^2} + \frac{729}{x^3} + \cdots$

answer

The general term of the series is

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{x}\right)^{2n}$$

The ratio test can be used to determine convergence. Since all the terms are positive (powers are even), then the absolute value is not needed.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{\left(\frac{3}{x}\right)^{2(n+1)}}{\left(\frac{3}{x}\right)^{2n}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{3}{x}\right)^{2n}}{\left(\frac{3}{x}\right)^{2n}} \left(\frac{3}{x}\right)^2 \\
&= \lim_{n\to\infty} \frac{3^2}{x^2} \\
&= \frac{9}{x^2}
\end{aligned}$$

The series converges when $L < 1$, which means $\frac{9}{x^2} < 1$ or $x^2 > 9$. Therefore it convergence for

$$|x| > 3$$

Part b

Does the following series converges or diverges? $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

answer

The terms are $\{\ln(2), \ln\left(2 + \frac{1}{2}\right), \ln\left(2 + \frac{1}{3}\right), \ln\left(2 + \frac{1}{4}\right), \cdots\}$.

Since the terms become monotonically decreasing after some n (after the second term in this case), the integral test could be used. Let

$$I=\int\ln\left(1+\frac{1}{x}\right)dx$$

The above indefinite integral is first evaluated. Since $\ln\left(1+\frac{1}{x}\right)=\ln\left(\frac{1+x}{x}\right)=\ln(1+x)-\ln(x)$, the above can be written as

$$I=\int\ln(1+x)\,dx-\int\ln(x)\,dx \tag{1}$$

To evaluate the first integral in (2) $\int\ln(1+x)\,dx$, let $u=1+x$, then $du=dx$, therefore

$$\begin{aligned}\int\ln(1+x)\,dx&=\int\ln(u)\,du\\&=u\ln(u)-u\end{aligned}$$

Hence

$$\int\ln(1+x)\,dx=(1+x)\ln(1+x)-(1+x) \tag{2}$$

The second integral in (1) is

$$\int\ln(x)\,dx=x\ln(x)-x \tag{3}$$

Using (2,3) back into (1) gives

$$\begin{aligned}I&=((1+x)\ln(1+x)-(1+x))-(x\ln(x)-x)\\&=(1+x)\ln(1+x)-1-x-x\ln(x)+x\\&=(1+x)\ln(1+x)-x\ln(x)-1 \end{aligned}\tag{4}$$

Now that the indefinite integral is evaluated, the limit is taken using

$$R=\lim_{N\to\infty}\int^{N}\ln\left(1+\frac{1}{x}\right)dx$$

Only upper limit is used following the book method[1]. Using the result found in (4), the above becomes

$$R=\lim_{N\to\infty}\left[(1+x)\ln(1+x)-x\ln(x)-1\right]^{N}$$

The above becomes

$$\begin{aligned}R&=\lim_{N\to\infty}\left[(1+x)\ln(1+x)-x\ln(x)-1\right]^{N}\\&=\lim_{N\to\infty}\left[(1+N)\ln(1+N)-N\ln(N)-1\right]\\&=\lim_{N\to\infty}\left[\ln(1+N)+N\ln(1+N)-N\ln(N)-1\right]\\&=\lim_{N\to\infty}\left[\ln(1+N)+N\ln\left(\frac{1+N}{N}\right)-1\right]\\&=\lim_{N\to\infty}\ln(1+N)+\lim_{N\to\infty}N\ln\left(\frac{1+N}{N}\right)-1\end{aligned}\tag{5}$$

But

$$\lim_{N\to\infty}N\ln\left(\frac{1+N}{N}\right)=\lim_{N\to\infty}\frac{\ln\left(\frac{1+N}{N}\right)}{\frac{1}{N}}$$

This gives indeterminate form 1/0. So using L'Hospital's rule, by taking derivatives of numerator and denominator gives

$$\lim_{N\to\infty}\frac{\ln\left(\frac{1+N}{N}\right)}{\frac{1}{N}}=\lim_{N\to\infty}\frac{\frac{\left(1-\frac{1+N}{N}\right)}{1+N}}{-\frac{1}{N^2}}=\lim_{N\to\infty}-\frac{\left(1-\frac{1+N}{N}\right)N^2}{1+N}=\lim_{N\to\infty}-\frac{(N-1-N)N}{1+N}=\lim_{N\to\infty}\frac{N}{1+N}=1$$

[1]See page 131, second edition. Mathematical methods for physics and engineering. Riley, Hobson and Bence.

Therefore $\lim_{N\to\infty} N \ln\left(\frac{1+N}{N}\right) = 1$. Using this result in (5) results in

$$\begin{aligned} R &= \lim_{N\to\infty} \ln(1+N) + 1 - 1 \\ &= \lim_{N\to\infty} \ln(1+N) \\ &= \infty \end{aligned}$$

Therefore, by the integral test the series diverges.

3.1.2 Problem 2

Find closed form for the series $f(x) = \sum_{n=0}^{\infty} n^2 x^{2n}$ by taking derivatives of variant of $\frac{1}{1-x}$. For what values of x does the series converge?

answer

$$f(x) = x^2 + 4x^4 + 9x^6 + 16x^8 + 25x^{10} + \cdots$$

Observing that

$$n^2 x^{2n} = \frac{x}{2}\frac{d}{dx}\left(nx^{2n}\right)$$

Therefore the sum can be written as

$$\begin{aligned} f(x) &= \frac{x}{2}\sum_{n=0}^{\infty}\frac{d}{dx}\left(nx^{2n}\right) \\ &= \frac{x}{2}\frac{d}{dx}\sum_{n=0}^{\infty} nx^{2n} \\ &= \frac{x}{2}\frac{d}{dx}\left(x^2 + 2x^4 + 3x^6 + 4x^8 + 5x^{10} + \cdots\right) \\ &= \frac{x}{2}\frac{d}{dx}\left(x^2\left(1 + 2x^2 + 3x^4 + 4x^6 + 5x^8 + \cdots\right)\right) \end{aligned} \tag{1}$$

To find what $1 + 2x^2 + 3x^4 + 4x^6 + \cdots$ sums to, we compare it to the binomial series

$$(1+z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha-1)z^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)z^3}{3!} + \cdots = 1 + 2x^2 + 3x^4 + 4x^6 + 5x^8 + \cdots$$

Hence, by setting

$$\begin{aligned} z &= -x^2 \\ \alpha &= -2 \end{aligned}$$

shows they are the same. Therefore

$$\begin{aligned} \left(1-x^2\right)^{-2} &= 1 + (-2)\left(-x^2\right) + \frac{(-2)(-3)\left(-x^2\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(-x^2\right)^3}{3!} + \cdots \\ &= 1 + 2x^2 + 3x^2 + 3x^4 + \cdots \end{aligned}$$

The above is valid for $|z| < 1$ which implies $x^2 < 1$ or $|x| < 1$. Hence

$$1 + 2x^2 + 3x^4 + 4x^6 + \cdots = \left(1-x^2\right)^{-2}$$

Using the above result in (1) gives

$$\begin{aligned}
f(x) &= \frac{x}{2}\frac{d}{dx}\left(\frac{x^2}{\left(1-x^2\right)^2}\right)\\
&= \frac{x}{2}\left(\frac{4x^3}{\left(1-x^2\right)^3}+\frac{2x}{\left(1-x^2\right)^2}\right)\\
&= \frac{2x^4}{\left(1-x^2\right)^3}+\frac{x^2}{\left(1-x^2\right)^2}\\
&= \frac{2x^4+x^2\left(1-x^2\right)}{\left(1-x^2\right)^3}\\
&= \frac{2x^4+x^2-x^4}{\left(1-x^2\right)^3}\\
&= \frac{x^4+x^2}{\left(1-x^2\right)^3}
\end{aligned}$$

Therefore

$$\boxed{f(x)=\frac{x^2\left(x^2+1\right)}{\left(1-x^2\right)^3}}$$

Where the above converges for $|x|<1$, from above, where we used Binomial expansion which is valid for $|x|<1$. This result could also be obtained by using the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{2(n+1)}}{n^2x^{2n}}\right|$$

Since all powers are even, the absolute value is not needed. The above becomes

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)^2x^2}{n^2}\\
&= x^2\lim_{n\to\infty}\frac{(n+1)^2}{n^2}\\
&= x^2
\end{aligned}$$

Therefore for the series to converge, we know that $\frac{a_{n+1}}{a_n}$ must be less than 1. Hence $x^2<1$ or $|x|<1$, which is the same result as above.

3.1.3 Problem 3

Part a

Find the sum of $1+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}-\cdots$

solution

We would like to combine each two consecutive negative terms and combine each two consecutive positive terms in the series in order to obtain an alternating series which is easier to work with. but to do that, we first need to check that the series is absolutely convergent. The $|a_n|$ term is $\frac{1}{4^n}$, therefore

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{4^{n+1}}}{\frac{1}{4^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{4^n}{4^{n+1}}\right| \\
&= \left|\frac{1}{4}\right|
\end{aligned}
$$

Since $|L|<1$ then the series is absolutely convergent so we are allowed now to group (or rearrange) terms as follows

$$
\begin{aligned}
S &= \left(1+\frac{1}{4}\right)-\left(\frac{1}{16}+\frac{1}{64}\right)+\left(\frac{1}{256}+\frac{1}{1024}\right)-\left(\frac{1}{4096}+\frac{1}{16384}\right)+\cdots \\
&= \frac{5}{4}-\frac{5}{64}+\frac{5}{1024}-\frac{5}{16\,384}+\cdots \\
&= \frac{5}{4}\left(1-\frac{1}{16}+\frac{1}{256}-\frac{1}{4096}+\cdots\right) \\
&= \frac{5}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n}} \\
&= \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n \qquad (1)
\end{aligned}
$$

But $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n$ has the form $\sum_{n=0}^{\infty}(-1)^n r^n$ where $r=\frac{1}{16}$ and since $|r|<1$ then by the binomial series

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n r^n &= 1-r+r^2-r^3+\cdots \\
&= \frac{1}{1+r}
\end{aligned}
$$

Therefore the sum in (1) becomes, using $r=\frac{1}{16}$

$$
\begin{aligned}
S &= \frac{5}{4}\left(\frac{1}{1+\frac{1}{16}}\right) \\
&= \frac{5}{4}\left(\frac{16}{17}\right)
\end{aligned}
$$

Hence

$$\boxed{S=\frac{20}{17}}$$

Or

$$S\approx 1.176$$

Part b

Find the sum of $\frac{1}{1!}+\frac{8}{2!}+\frac{16}{3!}+\frac{64}{4!}+\cdots$

solution

The sum can be written as

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty}\frac{n^3}{n!} \\
&= \sum_{n=1}^{\infty}\frac{nn^2}{n!}
\end{aligned}
$$

But $\frac{n}{n!} = \frac{n}{(n-1)!n} = \frac{1}{(n-1)!}$ and the above reduces to

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!}$$

Let $n-1 = m$ or $n = m+1$. The above becomes

$$\begin{aligned}
S &= \sum_{m=0}^{\infty} \frac{(m+1)^2}{m!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n^2+1+2n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n^2}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} + 2\sum_{n=0}^{\infty} \frac{n}{n!} \qquad (1)
\end{aligned}$$

Considering the first term $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ which can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n^2}{n!} &= \sum_{n=1}^{\infty} \frac{n^2}{n!} \\
&= \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \qquad (2)
\end{aligned}$$

Again, letting Let $n-1 = m$ then $\sum_{n=1}^{\infty} \frac{n}{(n-1)!}$ becomes $\sum_{m=0}^{\infty} \frac{m+1}{m!}$. Hence (2) becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n^2}{n!} &= \sum_{m=0}^{\infty} \frac{m+1}{m!} \\
&= \sum_{n=0}^{\infty} \frac{n+1}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\
&= \sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}
\end{aligned}$$

But $\sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$. Letting $n-1 = m$, this becomes $\sum_{m=0}^{\infty} \frac{1}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!}$and the above reduces to

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n^2}{n!} &= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\
&= e + e \\
&= 2e \qquad (3)
\end{aligned}$$

The above takes care of the first term in (1). Therefore (1) can now be written as

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} \frac{n^2}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} + 2\sum_{n=0}^{\infty} \frac{n}{n!} \\
&= 2e + e + 2\sum_{n=0}^{\infty} \frac{n}{n!} \\
&= 3e + 2\sum_{n=0}^{\infty} \frac{n}{n!}
\end{aligned}$$

But $\sum_{n=0}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!}$ and $\sum_{n=1}^{\infty} \frac{n}{n!}$ was calculated above. It can be written as $\sum_{n=0}^{\infty} \frac{1}{n!}$. The above now becomes

$$\begin{aligned}
S &= 3e + 2\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right) \\
&= 3e + 2e
\end{aligned}$$

Therefore

$$\boxed{S = 5e}$$

or

$$S \approx 13.5914$$

3.1.4 key solution to HW 1

Problem Set 1 Solutions

1 a) $f(x) = 1 + \frac{9}{x^2} + \frac{81}{x^4} + \frac{729}{x^6} + \cdots$

$$= \sum_{n=0}^{\infty} \left(\frac{9}{x^2}\right)^n = \frac{1}{1 - \frac{9}{x^2}} \quad \text{For } \frac{9}{x^2} < 1$$

$$\Rightarrow x^2 > 9 \quad \text{or} \quad \boxed{|x| > 3}$$

b) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} a_n$

For large n limit look at

$$\frac{a_{n+1}}{a_n} = \frac{\ln\left(1 + \frac{1}{n+1}\right)}{\ln\left(1 + \frac{1}{n}\right)}, \quad \text{expand this in powers } \frac{1}{n} \text{ to } O\left(\frac{1}{n}\right)^2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\Rightarrow \ln\left(1 + \frac{1}{n}\right) \simeq \frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{3}\left(\frac{1}{n}\right)^3$$

$$\ln\left(1 + \frac{1}{n+1}\right) = \ln\left[1 + \frac{1}{n}\left(\frac{1}{1+1/n}\right)\right] \simeq \ln\left[1 + \frac{1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right)\right]$$

$$\simeq \frac{1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right) - \frac{1}{2}\frac{1}{n^2}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right)^2 + \frac{1}{3}\frac{1}{(n)^3}$$

can set $\left(1 - \frac{1}{n} + \frac{1}{n^2}\right) = 1$ here

$$\simeq \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \left(1 + 1 + \frac{1}{3}\right)\frac{1}{(n)^3}$$

$$= \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \frac{7}{3}\frac{1}{n^3}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \simeq \frac{\frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \frac{7}{3}\frac{1}{n^3}}{\frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + \frac{1}{3}\left(\frac{1}{n^3}\right)}$$

$$\simeq \left(1 - \frac{3}{2}\frac{1}{n} + \frac{7}{3}\frac{1}{n^2}\right)\left(1 + \frac{1}{2}\frac{1}{n} - \frac{1}{3}\frac{1}{n^2} + \frac{1}{4}\frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{n} + \left(\frac{7}{3} - \frac{1}{3} + \frac{1}{4} - \frac{3}{4}\right)\frac{1}{n^2}$$

$$= 1 - \frac{1}{n} + \frac{3}{2}\frac{1}{n^2}$$

this diverges using the integral test developped in class

$$\Rightarrow \sum_n \ln\left(1+\frac{1}{n}\right) \longrightarrow \infty$$

2. $f(x) = \sum_{n=0}^{\infty} n^2 x^{2n}$

know $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ and $\frac{d}{dx}\frac{1}{1-x^2} = \sum_{n=0}^{\infty} 2n\, x^{2n-1}$

$$\Rightarrow \frac{1}{2}x\frac{d}{dx}\frac{1}{1-x^2} = \sum_{n=0}^{\infty} n\, x^{2n}$$

$$\Rightarrow \frac{1}{2}x\frac{d}{dx}\left(\frac{1}{2}x\frac{d}{dx}\frac{1}{1-x^2}\right) = \sum_{n=0}^{\infty} n^2 x^{2n}$$

↳ compute this

$$\frac{1}{2}x\frac{d}{dx}\left(\frac{1}{2}x\frac{d}{dx}\frac{1}{1-x^2}\right) = \frac{1}{2}x\frac{d}{dx}\left(x\,\frac{x}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}x\frac{d}{dx}\left(\frac{x^2}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}x\left[\frac{2x}{(1-x^2)^2} - \frac{2x^2(-2x)}{(1-x^2)^3}\right]$$

$$= \frac{x^2}{(1-x^2)^2} + \frac{2x^4}{(1-x^2)^3}$$

$$= \frac{1}{(1-x^2)^3}\left(x^2 - x^4 + 2x^4\right)$$

$$\sum_{n=0}^{\infty} n^2 x^{2n} = \frac{x^2(1+x^2)}{(1-x^2)^3}$$

↳ this converges for $|x|<1$

Exercise ?

a) $1+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}--$

$$= 1-\frac{1}{16}+\frac{1}{256}-\cdots+\frac{1}{4}-\frac{1}{64}+\frac{1}{1024}-\cdots$$

$$= 1-\frac{1}{16}+\frac{1}{256}-\cdots+\frac{1}{4}\left(1-\frac{1}{16}+\frac{1}{256}-\cdots\right)$$

$$= \left(1+\frac{1}{4}\right)\left(1-\frac{1}{16}+\frac{1}{256}-+\cdots\right)$$

$$= \left(1+\frac{1}{4}\right)\frac{1}{1+\frac{1}{16}} = \frac{5}{4}\cdot\frac{16}{17} = \frac{20}{17}$$

b) $\frac{1}{1!}+\frac{8}{2!}+\frac{27}{3!}+\frac{64}{4!}+\cdots = \sum_{n=0}^{\infty}\frac{n^3}{n!}$

now consider $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

then $x\frac{d}{dx}e^x = \sum_{n=0}^{\infty}\frac{nx^n}{n!}$ ← this converges for all x, including $x=1$

$$\Rightarrow x\frac{d}{dx}\left[x\frac{d}{dx}\left(x\frac{d}{dx}e^x\right)\right] = \sum_{n=0}^{\infty}\frac{n^3x^n}{n!}$$

Evaluate this and then set $x=1$

$x\frac{d}{dx}e^x = xe^x$; $x\frac{d}{dx}xe^x = x^2e^x + xe^x$

$$x\frac{d}{dx}\left(x^2e^x+xe^x\right) = 2x^2e^x + x^3e^x + x^2e^x + xe^x = x^3e^x+3x^2e^x+xe^x$$

$$\Rightarrow x^3e^x+3x^2e^x+xe^x = \sum_{n=0}^{\infty}\frac{n^3x^n}{n!} \Rightarrow \boxed{\sum_{n=0}^{\infty}\frac{n^3}{n!} = 5e}$$

3.2 HW 2

3.2.1 Problem 1

Find all possible values for (put into $x+iy$ form)

1. $\log\left(1+\sqrt{3}i\right)$
2. $\left(1+\sqrt{3}i\right)^{2i}$

Answer

Part 1

Let $z = x + iy$, where here $x = 1, y = 3$, then $|z| = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$ and $\arg(z) = \theta_0 = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{6} = 60^0$. The function $\log(z)$ is infinitely multi-valued, given by

$$\log(z) = \ln|z| + i(\theta_0 + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

Where θ_0 is the principal argument, which is 60^0 in this example, which is when $n = 0$. This is done to make $\log(z)$ single valued. This makes the argument of z restricted to $-\pi < \theta_0 < \pi$. This makes the negative real axis the branch cut, including the origin. To find all values, we simply use (1) for all possible n values other than $n = 0$. Each different n values gives different branch cut. This gives, where $\ln|z| = \ln(2)$ in all cases, the following

$$\begin{aligned}
\log(z) &= \ln(2) + i\left(\frac{\pi}{3}\right) && n = 0\\
&= \ln(2) + i\left(\frac{\pi}{3} + 2\pi\right) && n = 1\\
&= \ln(2) + i\left(\frac{\pi}{3} - 2\pi\right) && n = -1\\
&= \ln(2) + i\left(\frac{\pi}{3} + 4\pi\right) && n = 2\\
&= \ln(2) + i\left(\frac{\pi}{3} - 4\pi\right) && n = -2\\
&\vdots
\end{aligned}$$

Or

$$\begin{aligned}
\log(z) &= 0.693 + 1.047i\\
&= 0.693 + 7.330i\\
&= 0.693 - 5.236i\\
&= 0.693 + 13.614i\\
&= 0.693 - 11.519i\\
&\vdots
\end{aligned}$$

These are in $(x + iy)$ form. There are infinite number of values. Picking a specific branch cuts (i.e. specific n value), picks one of these values. The principal value is one associated with $n = 0$.

Part 2

Let $z = 1 + i\sqrt{3}$, hence

$$\begin{aligned}
f(z) &= z^{2i}\\
&= \exp(2i\log(z))\\
&= \exp(2i(\ln|z| + i(\theta_0 + 2n\pi))) \qquad n = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Where in this example, as in first part, $\ln|z| = \ln(2) = 0.693$ and principal argument is $\theta_0 = \frac{\pi}{3} = 60^0$. Hence the above becomes

$$\begin{aligned}
f(z) &= \exp\left(2i\left(\ln(2) + i\left(\frac{\pi}{3} + 2n\pi\right)\right)\right)\\
&= \exp\left(2i\ln(2) - \left(\frac{2\pi}{3} + 4n\pi\right)\right)\\
&= \exp\left(i\ln 4 - \left(\frac{2\pi}{3} + 4n\pi\right)\right)\\
&= \exp(i\ln 4)\exp\left(-\left(\frac{2\pi}{3} + 4n\pi\right)\right)\\
&= e^{-\left(\frac{2\pi}{3} + 4n\pi\right)}(\cos(\ln 4) + i\sin(\ln 4))\\
&= e^{-\left(\frac{2\pi}{3} + 4n\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3} + 4n\pi\right)}\sin(\ln 4)
\end{aligned}$$

Which is now in the form of $x+iy$. First few values are

$$\begin{aligned} f(z) &= e^{-\left(\frac{2\pi}{3}\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}\right)}\sin(\ln 4) && n=0 \\ &= e^{-\left(\frac{2\pi}{3}+4\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}+4\pi\right)}\sin(\ln 4) && n=1 \\ &= e^{-\left(\frac{2\pi}{3}-4\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}-4\pi\right)}\sin(\ln 4) && n=-1 \\ &= e^{-\left(\frac{2\pi}{3}+8\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}+8\pi\right)}\sin(\ln 4) && n=2 \\ &= e^{-\left(\frac{2\pi}{3}-8\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}-8\pi\right)}\sin(\ln 4) && n=-2 \\ &\vdots \end{aligned}$$

Or

$$\begin{aligned} f(z) &= 0.0226 + i0.121 \\ &= 7.878 \times 10^{-8} + i4.222 \times 10^{-7} \\ &= 6478 + i34713 \\ &= 2.748 \times 10^{-13} + i1.472 \times 10^{-12} \\ &= 1.858 \times 10^{9} + i9.954 \times 10^{9} \\ &\vdots \end{aligned}$$

3.2.2 Problem 2

Given that $u\left(x,y\right) = 3x^2y - y^3$ find $v\left(x,y\right)$ such that $f(z)$ is analytic. Do the same for $u\left(x,y\right) = \frac{y}{x^2+y^2}$

Solution

Part (1)

$u\left(x,y\right) = 3x^2y - y^3$. The function $f(z)$ is analytic if it satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Applying the first equation gives

$$6xy = \frac{\partial v}{\partial y}$$

Hence, solving for v by integrating, gives

$$v\left(x,y\right) = 3xy^2 + f(x) \tag{3}$$

Is the solution to (3) where $f(x)$ is the constant of integration since it is a partial differential equation. We now use equation (2) to find $f(x)$. From (2)

$$\begin{aligned} -\left(3x^2 - 3y^2\right) &= \frac{\partial v}{\partial x} \\ -3x^2 + 3y^2 &= \frac{\partial v}{\partial x} \end{aligned}$$

But (3) gives $\frac{\partial v}{\partial x} = 3y^2 + f'(x)$, hence the above becomes

$$\begin{aligned} -3x^2 + 3y^2 &= 3y^2 + f'(x) \\ f'(x) &= -3x^2 + 3y^2 - 3y^2 \\ &= -3x^2 \end{aligned}$$

Integrating gives

$$\begin{aligned} f(x) &= \int -3x^2 dx \\ &= -x^3 + C \end{aligned}$$

Therefore, (3) becomes

$$v\left(x,y\right)=3xy^2+f\left(x\right)$$

Or

$$\boxed{v\left(x,y\right)=3xy^2-x^3+C}$$

Where C is arbitrary constant. To verify, we apply CR again. Equation (1) now gives

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y}\\ 6xy&=6yx\end{aligned}$$

Verified. Equation (2) gives

$$\begin{aligned}-\frac{\partial u}{\partial y}&=\frac{\partial v}{\partial x}\\ -3x^2+3y^2&=-3x^2+3y^2\end{aligned}$$

Verified.

Part (2)

$u\left(x,y\right)=\frac{y}{x^2+y^2}$. The function $f\left(z\right)$ is analytic if it satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{2}$$

Applying the first equation gives

$$-\frac{2xy}{\left(x^2+y^2\right)^2}=\frac{\partial v}{\partial y}$$

Hence, solving for v by integrating, gives

$$\begin{aligned}v&=-2x\int\frac{y}{\left(x^2+y^2\right)^2}dy\\ &=\frac{x}{x^2+y^2}+f\left(x\right)\end{aligned} \tag{3}$$

Is the solution to (3) where $f\left(x\right)$ is the constant of integration since it is a partial differential equation. equation (2) gives

$$-\frac{1}{x^2+y^2}+\frac{2y^2}{\left(x^2+y^2\right)^2}=\frac{\partial v}{\partial x}$$

But (3) gives $\frac{\partial v}{\partial x}=\frac{1}{x^2+y^2}-\frac{2x^2}{\left(x^2+y^2\right)^2}+f'\left(x\right)$, hence the above becomes

$$\begin{aligned}-\frac{1}{x^2+y^2}+\frac{2y^2}{\left(x^2+y^2\right)^2}&=\frac{1}{x^2+y^2}-\frac{2x^2}{\left(x^2+y^2\right)^2}+f'\left(x\right)\\ f'\left(x\right)&=-\frac{2}{x^2+y^2}+\frac{2\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2}\\ &=-\frac{2}{x^2+y^2}+\frac{2}{\left(x^2+y^2\right)}\\ &=0\end{aligned}$$

Hence

$$f\left(x\right)=C$$

where C is arbitrary constant. Therefore, (3) becomes

$$\boxed{v\left(x,y\right)=\frac{x}{x^2+y^2}+C}$$

To verify, CR is applied again. Equation (1) now gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{-2xy}{\left(x^2+y^2\right)^2} = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

Hence verified. Equation (2) gives

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$
$$-\frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2} = \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2}$$
$$\frac{-\left(x^2+y^2\right)+2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2}$$
$$\frac{-x^2+y^2}{\left(x^2+y^2\right)^2} = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$$

Verified.

3.2.3 Problem 3

Evaluate the integral (i) $\oint\limits_C |z|^2\,dz$ and (ii) $\oint\limits_C \frac{1}{z^2}dz$ along two contours. These contours are

1. Line segment with initial point 1 and fixed point i
2. Arc of unit circle with $\operatorname{Im}(z) \geq 0$ with initial point 1 and final point i

Solution

Part (1)

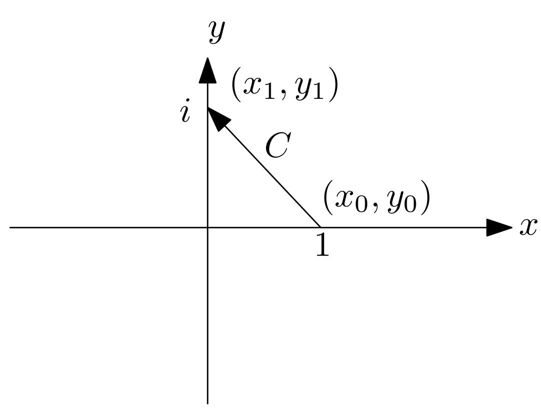


Figure 3.1: Integration path

First integral We start by finding the parameterization. For line segments that starts at (x_0, y_0) and ends at (x_1, y_1), the parametrization is given by

$$x(t) = (1-t)\,x_0 + tx_1$$
$$y(t) = (1-t)\,y_0 + ty_1$$

For $0 \leq t \leq 1$. Hence for $z = x + iy$, it becomes $z(t) = x(t) + iy(t)$. In this case, $x_0 = 1, y_0 = 0, x_1 = 0, y_1 = 1$, therefore

$$x(t) = (1-t)$$
$$y(t) = t$$

Using these, $z(t)$ is found from

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ &= (1-t) + it \end{aligned}$$

And

$$z'(t) = -1 + i$$

Since $|z|^2 = x^2 + y^2$, then in terms of t it becomes

$$|z(t)|^2 = (1-t)^2 + t^2$$

Hence the line integral now becomes

$$\begin{aligned} \int_C |z|^2 \, dz &= \int_0^1 |z(t)|^2 z'(t) \, dt \\ &= \int_0^1 \left((1-t)^2 + t^2\right)(-1+i) \, dt \\ &= (-1+i) \int_0^1 (1-t)^2 + t^2 dt \\ &= (-1+i) \int_0^1 1 + t^2 - 2t + t^2 dt \\ &= (-1+i) \int_0^1 1 + 2t^2 - 2t \, dt \\ &= (-1+i) \left(\int_0^1 dt + \int_0^1 2t^2 dt - \int_0^1 2t \, dt \right) \\ &= (-1+i) \left((t)_0^1 + 2\left(\frac{t^3}{3}\right)_0^1 - 2\left(\frac{t^2}{2}\right)_0^1 \right) \\ &= (-1+i) \left(1 + \frac{2}{3} - 2\left(\frac{1}{2}\right) \right) \end{aligned}$$

Hence

$$\boxed{\int_C |z|^2 \, dz = \tfrac{2}{3}(i-1)}$$

<u>second integral</u>

Using the same parameterization above. But here the integrand is

$$\frac{1}{z^2} = \frac{1}{((1-t)+it)^2}$$

Hence the integral becomes

$$\begin{aligned} \int_C \frac{1}{z^2} dz &= \int_0^1 \frac{1}{((1-t)+it)^2} z'(t) \, dt \\ &= (i-1) \int_0^1 \frac{1}{((1-t)+it)^2} dt \\ &= (i-1)(-i) \end{aligned}$$

Hence

$$\boxed{\int_C \tfrac{1}{z^2} dz = 1 + i}$$

Part (2)

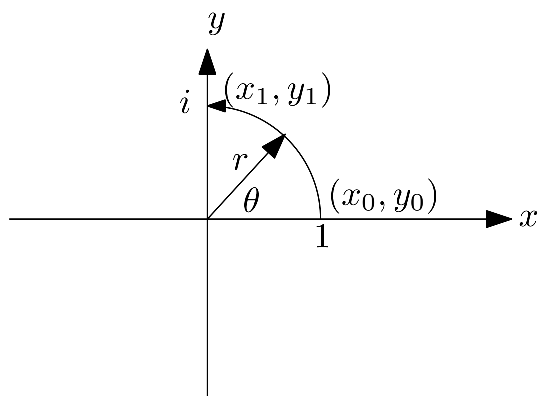


Figure 3.2: Integration path

First integral Let $z = re^{i\theta}$ then $\frac{dz}{d\theta} = rie^{i\theta}$. When $z = 1$ then $\theta = 0$. When $z = i$ then $\theta = \frac{\pi}{2}$, hence we can parameterize the contour integral using θ and it becomes

$$\begin{aligned}
\int_C |z|^2 \, dz &= \int_0^{\frac{\pi}{2}} r^2 \left(rie^{i\theta}\right) d\theta \\
&= ir^3 \int_0^{\frac{\pi}{2}} e^{i\theta} d\theta \\
&= ir^3 \left[\frac{e^{i\theta}}{i}\right]_0^{\frac{\pi}{2}} \\
&= r^3 \left[e^{i\theta}\right]_0^{\frac{\pi}{2}} \\
&= r^3 \left[e^{i\frac{\pi}{2}} - e^0\right] \\
&= r^3 \left[i - 1\right]
\end{aligned}$$

But $r = 1$, therefore the above becomes

$$\boxed{\int_C |z|^2 \, dz = i - 1}$$

second integral

Using the same parameterization above. But here the integrand now

$$\frac{1}{z^2} = \frac{1}{r^2 e^{i2\theta}}$$

Therefore

$$\begin{aligned}
\int_C \frac{1}{z^2} dz &= \int_0^{\frac{\pi}{2}} \frac{1}{r^2 e^{i2\theta}} \left(rie^{i\theta}\right) d\theta \\
&= \frac{i}{r} \int_0^{\frac{\pi}{2}} e^{-i\theta} d\theta \\
&= \frac{i}{r} \left(\frac{e^{-i\theta}}{-i}\right)_0^{\frac{\pi}{2}} \\
&= \frac{-1}{r} \left(e^{-i\theta}\right)_0^{\frac{\pi}{2}} \\
&= \frac{-1}{r} \left(e^{-i\frac{\pi}{2}} - 1\right) \\
&= \frac{-1}{r} \left(-i - 1\right)
\end{aligned}$$

But $r = 1$, hence

$$\boxed{\int_C \frac{1}{z^2} dz = 1 + i}$$

3.2.4 Problem 4

Use the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

To evaluate

$$\oint_C \frac{1}{(z+1)(z+2)} dz$$

Where C is the circular contour $|z+1| = R$ with $R < 1$. Note that if $R > 1$ then a different result is found. Why can't the Cauchy integral formula above be used for $R > 1$?

Solution

The disk $|z+1| = R$ is centered at $z = -1$ with $R < 1$. The function

$$g(z) = \frac{1}{(z+1)(z+2)}$$

has pole at $z = -1$ and at $z = -2$.

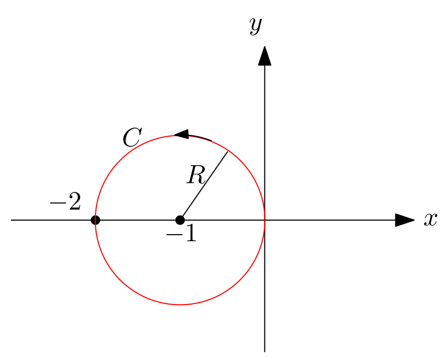


Figure 3.3: Showing location of pole

In the Cauchy integral formula, the function $f(z)$ is analytic on C and inside C. Hence, to use Cauchy integral formula, we need to convert $g(z) = \frac{1}{(z+1)(z+2)}$ to look like $\frac{f(z)}{z-z_0}$ where $f(z)$ is analytic inside C. This is done as follows

$$\begin{aligned}\frac{1}{(z+1)(z+2)} &= \frac{\frac{1}{(z+2)}}{z-(-1)} \\ &= \frac{f(z)}{z-(-1)}\end{aligned}$$

Where now $f(z) = \frac{1}{(z+2)}$. This has pole at $z = -2$. Since this pole is outside C then $f(z)$ is analytic on and inside C and can be used for the purpose of using Cauchy integral formula, which now can be written as

$$\begin{aligned}\oint_C \frac{1}{(z+1)(z+2)} dz &= \oint_C \frac{\frac{1}{(z+2)}}{z-(-1)} dz \\ &= \oint_C \frac{f(z)}{z-(-1)} dz \\ &= (2\pi i) f(-1)\end{aligned}$$

Therefore, we just need to evaluate $f(-1)$ which is seen as 1. Hence

$$\boxed{\oint_C \frac{1}{(z+1)(z+2)} dz = 2\pi i} \tag{1}$$

To verify, we can solve this again using the residue theorem

$$\oint_C g(z)\, dz = 2\pi i \left(\text{sum of residues of } g(z) \text{ inside } C\right)$$

But $g(z) = \frac{1}{(z+1)(z+2)}$ has only one pole inside C, which is at $z = -1$. Therefore the above becomes

$$\oint_C \frac{1}{(z+1)(z+2)} = 2\pi i \left(\text{residue of } g(z) \text{ at } -1\right) \tag{2}$$

To find residue at -1, we can use one of the short cuts to do that. Where we write $\frac{1}{(z+1)(z+2)} = \frac{\Phi(z)}{z+1}$ where $\Phi(z)$ is analytic at $z = -1$ and $\Phi(-1) \neq 0$. Therefore we see that $\Phi(z) = \frac{1}{z+2}$. Hence residue of $\frac{1}{(z+1)(z+2)} = \Phi(z_0) = \frac{1}{(-1)+2} = 1$. Equation (2) becomes

$$\oint_C \frac{1}{(z+1)(z+2)} = 2\pi i$$

Which is same result obtained in (1) by using Cauchy integral formula directly.

To answer last part, when $R > 1$, then now both poles $z = -1$ and $= -2$, are inside C. Therefore, we can't split $\frac{1}{(z+1)(z+2)}$ into one part that is analytic (the $f(z)$ in the above), in order to obtain expression $\frac{f(z)}{z-z_0}$ in order to apply Cauchy integral formula directly. Therefore when $R > 1$ we should use

$$\oint_C g(z)\,dz = 2\pi i \left(\text{sum of residues of } g(z) \text{ inside } C\right)$$

3.2.5 Problem 5

Evaluate the integral

$$\oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) dz$$

Where he contour is the unit circle around origin (counter clockwise direction).

Solution

$$\begin{aligned}\oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) dz &= \oint_C e^{z^2}\left(\frac{z-1}{z^3}\right) dz \\ &= \oint_C \frac{f(z)}{(z-z_0)^3} dz\end{aligned}$$

Where $z_0 = 0$ and where

$$f(z) = e^{z^2}(z-1)$$

But $f(z)$ is analytic on C and inside, since e^{z^2} is analytic everywhere and $z-1$ has no poles. Hence we can use Cauchy integral formula for pole of higher order given by

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Where $n = 2$ in this case. Therefore, since $z_0 = 0$ the above reduces to

$$\oint_C \frac{f(z)}{z^3} dz = \frac{2\pi i}{2} f''(0) \tag{1}$$

Now we just need to find $f''(z)$ and evaluate the result at $z_0 = 0$

$$\begin{aligned} f'(z) &= 2ze^{z^2}(z-1) + e^{z^2} \\ f''(z) &= 2e^{z^2}(z-1) + 2z\left(2ze^{z^2}(z-1) + e^{z^2}\right) + 2ze^{z^2}\end{aligned}$$

Hence

$$f''(0) = -2$$

Therefore (1) becomes

$$\oint_C \frac{e^{z^2}(z-1)}{z^3} dz = -2\pi i \tag{2}$$

To verify, we will do the same integration by converting it to line integration using parameterization on θ. Let $z(\theta) = re^{i\theta}$, but $r = 1$, therefore $z(\theta) = e^{i\theta}, dz = ie^{i\theta}d\theta$. Therefore the integral becomes

$$\oint_C e^{z^2}\left(\frac{z-1}{z^3}\right)dz = \int_0^{2\pi} e^{e^{2i\theta}}\left(\frac{e^{i\theta}-1}{e^{3i\theta}}\right)ie^{i\theta}d\theta$$
$$= i\int_0^{2\pi} e^{e^{2i\theta}}\left(\frac{e^{i\theta}-1}{e^{2i\theta}}\right)d\theta$$

This is a hard integral to solve by hand. Using computer algebra software, it also gave $-2\pi i$. This verified the result. Clearly using the Cauchy integral formula to solve this problem was much simpler that using parameterization.

3.2.6 key solution to HW 2

Problem Set 2 Solutions

Exercise 1 :

i) $\log(1+\sqrt{3}i)$; now $1+\sqrt{3}i = re^{i\theta}$

$\Rightarrow r = 2$ and $\theta = \frac{\pi}{3} + 2n\pi$

or $1+\sqrt{3}i = 2e^{i(\frac{\pi}{3}+2n\pi)}$

$\Rightarrow \boxed{\log(1+\sqrt{3}i) = \ln 2 + i(\frac{\pi}{3} + 2n\pi)}$ $n = 0, \pm 1, \pm 2$, etc

ii) $(1+i\sqrt{3})^{2i}$; again $1+\sqrt{3}i = 2e^{i(\frac{\pi}{3}+2n\pi)}$

$\Rightarrow (1+i\sqrt{3})^{2i} = e^{2i[\ln 2 + i\frac{\pi}{3} + in2\pi]}$

$= e^{i\ln 4}e^{-\frac{2\pi}{3}}e^{-4n\pi}$

$\Rightarrow (1+i\sqrt{3})^{2i} = [\cos(\ln 4) + i\sin(\ln 4)]e^{-\frac{2\pi}{3}}e^{-4n\pi}$

Exercise 2 :

i) $u(x,y) = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6xy$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 3x^2 - 3y^2$

$\Rightarrow \frac{\partial v}{\partial y} = 6xy \Rightarrow v(x,y) = 3y^2x + g(x)$

$\frac{\partial v}{\partial x} = 3y^2 + g'(x) = -(3x^2 - 3y^2)$

$\Rightarrow g'(x) = -3x^2 \Rightarrow g = -x^3 + c$ [can set $c=0$]

$\Rightarrow v(x,y) = 3y^2x - x^3 \Rightarrow \boxed{f(z) = -i(x+iy)^3}$

ii) $u(x,y) = \dfrac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow v(x,y) = \frac{+x}{x^2+y^2} + g(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + g'(x)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + g'(x) = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

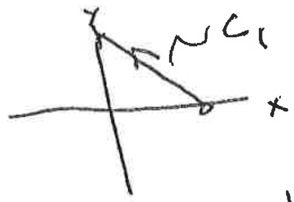
$\Rightarrow g'(x) = 0$, can choose $g = c$

$$\Rightarrow v = \frac{x}{x^2+y^2} \Rightarrow f = \frac{y+ix}{x^2+y^2} = \frac{i(x-iy)}{x^2+y^2}$$

or $$\boxed{f = \frac{i(x-iy)}{(x+iy)(x-iy)} = \frac{i}{x+iy} = \frac{i}{z}}$$

Exercise 3

i) $\int_{C_1} |z|^2 dz$ where



$= \int_{C_1}(x^2+y^2)(dx+i\,dy)$ now $x = 1-y$ along C_1

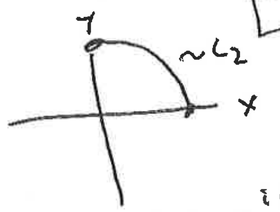
$\Rightarrow y = 1-x \Rightarrow dy = -dx$

$= \int_1^0 (1-i)\,dx\,(x^2+(1-x)^2)$

$= (i-1)\int_0^1 dx\left[2x^2-2x+1\right] = (i-1)\left[\frac{2}{3}x^3 - x^2 + x\right]\Big|_0^1$

$= \boxed{(i-1)\frac{2}{3}}$

$\int_{C_2} |z|^2 dz$ where



$= \int_{C_2}(x^2+y^2)\,dz$ now $z = e^{i\theta}$ $dz = ie^{i\theta}d\theta$

$= \int_0^{\pi/2} i\,d\theta\, e^{i\theta} = \frac{i}{i}e^{i\theta}\Big|_0^{\pi/2} = e^{i\pi/2} - 1 = \boxed{i-1}$

ii) $\int_{C_1}\frac{1}{z^2}dz = (1-i)\int_1^0 \frac{dx}{[x+i(1-x)]^2} = (i-1)\int_0^1 \frac{dx}{[(1-i)x+i]^2}$

$= \frac{i-1}{(i-1)^2}\int_0^1 \frac{dx}{\left[x+\frac{i}{1-i}\right]^2} = \frac{1}{i-1}\int_0^1 \frac{dx}{\left[x+\frac{i-1}{2}\right]^2}$

$= \frac{-1}{i-1}\,\frac{1}{x+\frac{i-1}{2}}\Big|_0^1 = \frac{1}{i-1}\left[\frac{1}{\frac{i-1}{2}} - \frac{1}{1+\frac{i-1}{2}}\right]$

$= \frac{1}{i-1}\left(\frac{2}{i-1} - \frac{2}{1+i}\right) = \frac{2}{i-1}\,\frac{(1+i-(i-1))}{-2} = \frac{-2}{i-1} = \frac{2}{1-i} = \frac{2(1+i)}{2} = 1+i$

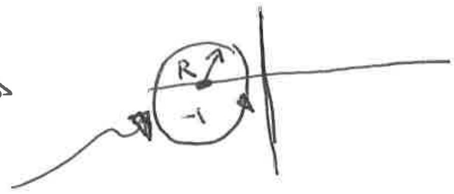
$\int_{C_2}\frac{1}{z^2}dz = \int_0^{\pi/2} i e^{-i\theta}d\theta = -e^{-i\theta}\Big|_0^{\pi/2} = 1 - e^{-i\pi/2} = 1+i$ same result

• notice $|z|^2$ was not analytic while $\frac{1}{z^2}$ was along C_1 and C_2 and $|z|^2$ gave different answers while $\frac{1}{z^2}$ gave the same answer.

Exercise 4

$$\oint_c \frac{dz}{(z+1)(z+2)}$$

this is C



use

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0} dz$$

let $f(z) = \dfrac{1}{z+2}$, then $f(z)$ is analytic on and inside C

$$\Rightarrow \oint \frac{dz\, f(z)}{z-z_0} = \oint_c \frac{dz\, f(z)}{z-(-1)} = 2\pi i f(-1) = 2\pi i$$

$$\therefore \oint_c \frac{dz}{(z+1)(z+2)} = 2\pi i$$

- For $R>1$, and $|z|=R$, we no longer have $\dfrac{1}{z+2}$ analytic everywhere inside the contour C, we cannot use Cauchy integral formula.

Exercise 5

$$\oint_C e^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz = I$$

↳ unit circle about origin

now $e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$

$$\Rightarrow I = \oint_C \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz = \sum_{n=0}^{\infty} \frac{1}{n!} \oint_C \left(z^{2n-2} - z^{2n-3}\right)dz$$

now we know that $\oint_C z^n = 2\pi i\, \delta_{n,-1}$

$$\Rightarrow \quad I = \sum_{n=0}^{\infty} \frac{1}{n!} 2\pi i \left[\underbrace{\delta_{2n-2,-1}}_{\hookrightarrow n=\frac{1}{2}\ (\text{not possible})} - \underbrace{\delta_{2n-3,-1}}_{\hookrightarrow n=1}\right]$$

$$\boxed{I = 2\pi i}$$

3.3 HW 3

3.3.1 Problem 1

Part (a)

Use Cauchy-Riemann equations to determine if $|z|$ analytic function of the complex variable z.

Solution

$$f(z) = |z|$$

Let $z = x + iy$, then

$$\begin{aligned} f(z) &= \left(x^2 + y^2\right)^{\frac{1}{2}} \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= \sqrt{x^2 + y^2} \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}$, which shows that $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$. Therefore $|z|$ is not analytic.

Part (b)

Use Cauchy-Riemann equations to determine if $\operatorname{Re}(z)$ analytic function of the complex variable z.

Solution

$$f(z) = \operatorname{Re}(z)$$

Let $z = x + iy$, then

$$\begin{aligned} f(z) &= x \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= x \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = 1$, which shows that $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$. Therefore $\operatorname{Re}(z)$ is not analytic.

Part (c)

Use Cauchy-Riemann equations to determine if $e^{\sin z}$ analytic function of the complex variable z.

Solution

$f(z) = e^{\sin z}$ is analytic since we can show that $\exp(z)$ is analytic by applying Cauchy-Riemann (C-R), and also show that $\sin(z)$ is analytic using C-R. Theory of analytic functions it says that the composition of analytic functions is also an analytic function, which means $e^{\sin z}$ is analytic.

But this problems seems to ask to use C-R equations directly to show this. Therefore we need to first determine the real and complex parts (u, v) of the function $e^{\sin z}$. Since

$$\sin z = \frac{z - z^{-1}}{2i}$$

Then

$$\begin{aligned} f(z) &= e^{\sin z} \\ &= \exp\left(\frac{z - z^{-1}}{2i}\right) \\ &= \exp\left(\frac{z}{2i}\right)\exp\left(\frac{-1}{2iz}\right) \end{aligned}$$

But $z = x + iy$ and the above expands to

$$\begin{aligned} \exp(\sin z) &= \exp\left(\frac{1}{2i}\left(x+iy\right)\right)\exp\left(\frac{-1}{2i\left(x+iy\right)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{1}{\left(x+iy\right)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x-iy}{\left(x+iy\right)\left(x-iy\right)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x-iy}{x^2+y^2}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right)\exp\left(\frac{i}{2}\left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right)\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2} + \frac{1}{2}\frac{y}{x^2+y^2}\right) \\ &= \exp\left(\frac{-i}{2}x\right)\exp\left(\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2}\right)\exp\left(\frac{1}{2}\frac{y}{x^2+y^2}\right) \end{aligned}$$

Collecting terms gives

$$\begin{aligned} \exp(\sin z) &= \exp\left(\frac{1}{2}y + \frac{1}{2}\frac{y}{x^2+y^2}\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2} - \frac{i}{2}x\right) \\ &= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2} - \frac{i}{2\left(x^2+y^2\right)}x\left(x^2+y^2\right)\right) \\ &= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\exp\left(i\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right) \\ &= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\left[\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right) + i\sin\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\right] \\ &= \exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right) + i\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right) \end{aligned}$$

Therefore, since $\exp(\sin z) = u + iv$, then we see from above that

$$\begin{aligned} u &= \exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right) \\ v &= \exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right) \end{aligned}$$

Now we need to check the Cauchy-Riemann equations on the above u, v functions we found.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Evaluating each partial derivative gives

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{d}{dx}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&+\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&=\frac{1}{2}\frac{2yx\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2x}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(\left(x^2+y^2\right)+1\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\\
&-\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&=\frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(\left(x^2+y^2\right)+1\right)}{x^2+y^2}\right)\cos\left(\frac{x\left(1-\left(x^2+y^2\right)\right)}{2\left(x^2+y^2\right)}\right)\\
&-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{\left(1-3x^2-y^2\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2x}{2\left(x^2+y^2\right)^2}\right)\\
&=\frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(x^2+y^2+1\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\\
&-\exp\left(\frac{1}{2}\frac{y\left(x^2+y^2+1\right)}{x^2+y^2}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)}{2\left(x^2+y^2\right)^2}\right)\end{aligned}$$

The above can be simplified more to become

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{-1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y\left(x^2+y^2+1\right)}{2\left(x^2+y^2\right)}\right)\\
&\qquad\left[2xy\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right]\end{aligned}\tag{3}$$

Now we evaluate $\frac{\partial v}{\partial y}$ to see if it the same as above. Since $v=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$ then

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{d}{dy}\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\frac{d}{dy}\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&=\left(\frac{1}{2}\frac{\left(1+x^2+3y^2\right)\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2y}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{1}{2}\frac{\left(-2xy\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)\left(2y\right)}{\left(x^2+y^2\right)^2}\right)\\
&=\left(\frac{1}{2}\frac{x^4+2x^2y^2+x^2+y^4-y^2}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{1}{2}\frac{-2xy}{\left(x^2+y^2\right)^2}\right)\\
&=\left(\frac{1}{2}\frac{x^4+2x^2y^2+x^2+y^4-y^2}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{xy}{\left(x^2+y^2\right)^2}\right)\end{aligned}$$

Simplifying the above more gives

$$\frac{\partial v}{\partial y}=\frac{-1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)$$
$$\left[2xy\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \tag{4}$$

Comparing (3) and (4) shows they are the same expressions. Therefore the first equation is verified.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

Now we verify the second equation $-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$. Since $u=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$ then

$$\begin{aligned}
\frac{\partial u}{\partial y}&=\frac{d}{dy}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dy}\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&=\frac{\left(1+x^2+3y^2\right)\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2y}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{\left(-2y\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2y}{2\left(x^2+y^2\right)^2}\\
&=\frac{\left(x^4+2x^2y^2+x^2+y^4-y^2\right)}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{\left(-2y\right)\left(x^2+y^2\right)-2yx+2yx\left(x^2+y^2\right)}{2\left(x^2+y^2\right)^2}\\
&=\frac{\left(x^4+2x^2y^2+x^2+y^4-y^2\right)}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{yx}{\left(x^2+y^2\right)^2}
\end{aligned}$$

The above can simplified more to give

$$\frac{\partial u}{\partial y}=\frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)$$
$$\left[\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right]$$

Hence

$$\frac{-\partial u}{\partial y}=\frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)$$
$$\left[-\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}-2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \tag{5}$$

And since $v=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$ then

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{d}{dx}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&= \frac{1}{2}\left(\frac{2xy\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2x}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\left(\frac{\left(1-3x^2-y^2\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2x}{2\left(x^2+y^2\right)^2}\right)\\
&= \frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{-x^4-2x^2y^2-x^2-y^4+y^2}{2\left(x^2+y^2\right)^2}\right)
\end{aligned}
$$

The above can simplified more to give

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad\left[-\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}-2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \qquad (6)
\end{aligned}
$$

Comparing (5,6) shows they are the same, i.e.

$$\frac{-\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

C-R equations are satisfied, and because it is clear that all partial derivatives $\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ are continuous functions in x,y as they are made up of exponential and trigonometric functions which are continuous, then we conclude that $f(z)=e^{\sin z}$ is analytic function everywhere.

3.3.2 Problem 2

Part (a)

Represent $\frac{z+3}{z-3}$ by its Maclaurin series and give the region of validity for the representation. Next expand this in powers of $\frac{1}{z}$ to find a Laurent series. What is the range of validity of the Laurent series?

Solution

Maclaurin series is expansion of $f(z)$ around $z=0$. Since $f(z)$ has simple pole at $z=3$, then the region of validity will be a disk centered at $z=0$ up to the nearest pole, which is at $z=3$. Hence $|z|<3$ is the region.

$$
\begin{aligned}
f(z) &= \frac{z+3}{z-3}\\
&= \frac{z+3}{-3\left(1-\frac{z}{3}\right)}\\
&= \frac{z+3}{-3}\left(\frac{1}{1-\frac{z}{3}}\right)
\end{aligned}
$$

Now we can expand using Binomial to obtain

$$\begin{aligned}
f(z) &= \frac{3+z}{-3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right)\\
&= \left(-1-\frac{z}{3}\right)\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right)\\
&= \left(-1-\frac{z}{3}\right)+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)^2+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)^3+\cdots\\
&= -1-\frac{z}{3}-\frac{z}{3}-\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3-\left(\frac{z}{3}\right)^3-\left(\frac{z}{3}\right)^4+\cdots\\
&= -1-\frac{2}{3}z-\frac{2}{9}z^2-\frac{2}{27}z^3-\frac{2}{81}z^4-\cdots
\end{aligned}$$

Or

$$f(z) = -1-\sum_{n=1}^{\infty}\frac{2}{3^n}z^n$$

To expand in negative powers of z, or in $\frac{1}{z}$, then

$$\begin{aligned}
f(z) &= \frac{z+3}{z\left(1-\frac{3}{z}\right)}\\
&= \frac{z+3}{z}\left(\frac{1}{1-\frac{3}{z}}\right)
\end{aligned}$$

For $\left|\frac{3}{z}\right|<1$ or $|z|<3$ the above becomes

$$\begin{aligned}
f(z) &= \frac{z+3}{z}\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right)\\
&= \left(1+\frac{3}{z}\right)\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right)\\
&= \left(1+\frac{3}{z}\right)+\left(1+\frac{3}{z}\right)\frac{3}{z}+\left(1+\frac{3}{z}\right)\left(\frac{3}{z}\right)^2+\left(1+\frac{3}{z}\right)\left(\frac{3}{z}\right)^3+\cdots\\
&= 1+\frac{3}{z}+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\left(\frac{3}{z}\right)^3+\left(\frac{3}{z}\right)^4+\cdots\\
&= 1+\frac{6}{z}+\frac{18}{z^2}+\frac{54}{z^3}+\cdots
\end{aligned}$$

This is valid for $|z|>3$. The residue is 6, which can be confirmed using

$$\begin{aligned}
\text{Residue}(3) &= \lim_{z\to 3}(z-3)f(z)\\
&= \lim_{z\to 3}(z-3)\frac{z+3}{z-3}\\
&= \lim_{z\to 3}(z+3)\\
&= 6
\end{aligned}$$

Summary

$$\boxed{f(z)=\frac{z+3}{z-3}=\begin{cases}-1-\frac{2}{3}z-\frac{2}{9}z^2-\frac{2}{27}z^3-\frac{2}{81}z^4-\cdots & |z|<3\\ 1+\frac{6}{z}+\frac{18}{z^2}+\frac{54}{z^3}+\cdots & |z|>3\end{cases}}$$

Part (b)

Find Laurent series for $\frac{z}{(z+1)(z-3)}$ in each of the following domains (i) $|z|<1$ (ii) $1<|z|<3$ (iii) $|z|>3$

Solution

The possible region are shown below. Since there is a pole at $z=-1$ and pole at $z=3$, then there are three different regions. They are named A,B,C in the following diagram

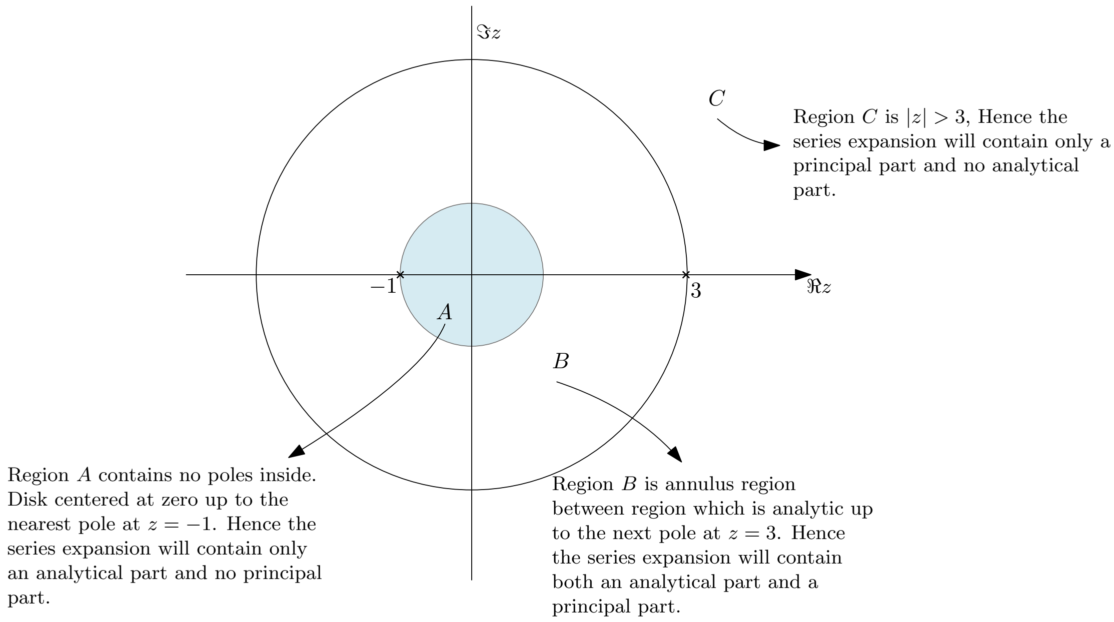


Figure 3.4: Laurent series regions

First the expression $\frac{z}{(z+1)(z-3)}$ is expanded using partial fractions

$$\frac{z}{(z+1)(z-3)} = \frac{A}{(z+1)} + \frac{B}{(z-3)} \tag{1}$$

Hence

$$\begin{aligned} z &= A(z-3) + B(z+1) \\ &= z(A+B) - 3A + B \end{aligned}$$

The above gives two equations

$$\begin{aligned} A + B &= 1 \\ 0 &= -3A + B \end{aligned}$$

First equation gives $A = 1 - B$. Substituting in the second equation gives $0 = -3(1-B) + B$ or $0 = -3 + 4B$, hence $B = \frac{3}{4}$, which implies $A = 1 - \frac{3}{4} = \frac{1}{4}$, therefore (1) becomes

$$\frac{z}{(z+1)(z-3)} = \frac{1}{4}\frac{1}{(z+1)} + \frac{3}{4}\frac{1}{(z-3)}$$

Considering each term in turn. For $\frac{1}{4}\frac{1}{(z+1)}$, we can expand this as

$$\frac{1}{4}\frac{1}{(z+1)} = \frac{1}{4}\left(1 - z + z^2 - z^3 + z^4 + \cdots\right) \qquad |z| < 1 \tag{2a}$$

$$\frac{1}{4}\frac{1}{(z+1)} = \frac{1}{4z}\frac{1}{\left(1+\frac{1}{z}\right)} = \frac{1}{4z}\left(1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \cdots\right) \qquad |z| > 1 \tag{2b}$$

And for the term $\frac{3}{4}\frac{1}{(z-3)}$, we can expand this as

$$\frac{3}{4}\frac{1}{(z-3)} = -\frac{1}{4}\frac{1}{\left(1-\frac{z}{3}\right)} = -\frac{1}{4}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \cdots\right) \qquad |z| < 3 \tag{3a}$$

$$\frac{3}{4}\frac{1}{(z-3)} = \frac{3}{4z}\frac{1}{\left(1-\frac{3}{z}\right)} = \frac{3}{4z}\left(1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right) \qquad |z| > 3 \tag{3b}$$

Now that we expanded all the terms in the two possible ways for each each, we now consider each region of interest, and look at the above 4 expansions, and simply pick for each region the expansion which is valid in for that region of interest.

For (i), region A: In this region, we want $|z| < 1$. From (2,3) we see that (2a) and (3a) are valid expansions in $|z| < 1$. Hence

$$
\begin{aligned}
\frac{z}{(z+1)(z-3)} &= \frac{1}{4}\left(1 - z + z^2 - z^3 + z^4 + \cdots\right) - \frac{1}{4}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \cdots\right) \\
&= \frac{1}{4}\left(1 - z + z^2 - z^3 + z^4 - \cdots\right) - \frac{1}{4}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \frac{z^4}{81} + \cdots\right) \\
&= \left(\frac{1}{4} - \frac{1}{4}z + \frac{1}{4}z^2 - \frac{1}{4}z^3 + \frac{1}{4}z^4 - \cdots\right) - \left(\frac{1}{4} + \frac{z}{12} + \frac{z^2}{36} + \frac{z^3}{108} + \frac{z^4}{324} + \cdots\right) \\
&= -\frac{1}{4}z - \frac{z}{12} + \frac{1}{4}z^2 - \frac{z^2}{36} - \frac{1}{4}z^3 - \frac{z^3}{108} + \frac{1}{4}z^4 - \frac{z^4}{324} - \cdots \\
& -\frac{1}{3}z + \frac{2}{9}z^2 - \frac{7}{27}z^3 + \frac{20}{81}z^4 - \cdots
\end{aligned}
$$

For (ii), region B: This is for $1 < |z| < 3$. From equations (2,3) we see that (2b) and (3a) are valid in this region. Hence

$$
\begin{aligned}
\frac{z}{(z+1)(z-3)} &= \frac{1}{4z}\left(1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \cdots\right) - \frac{1}{4}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \cdots\right) \\
&= \frac{1}{4z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right) - \frac{1}{4}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \frac{z^4}{81} + \cdots\right) \\
&= \left(\frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \cdots\right) - \left(\frac{1}{4} + \frac{z}{12} + \frac{z^2}{36} + \frac{z^3}{108} + \frac{z^4}{324} + \cdots\right) \\
&= \overbrace{\cdots + \frac{1}{4z^5} - \frac{1}{4z^4} + \frac{1}{4z^3} - \frac{1}{4z^2} + \frac{1}{4z}}^{\text{principal part}} \overbrace{- \frac{1}{4} - \frac{z}{12} - \frac{z^2}{36} - \frac{z^3}{108} - \frac{z^4}{324}}^{\text{analytical part}} - \cdots
\end{aligned}
$$

The residue is $\frac{1}{4}$ by looking at the above. The value for the residue can be verified as follows. Using

$$
b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz
$$

Where in the above z_0 is the location of the pole and n is the coefficient of the $\frac{1}{z^n}$ is the principal part. Since we want the residue, then $n = 1$ and the above becomes

$$
b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz
$$

In the above, the contour C is circle somewhere inside the annulus $1 < |z| < 3$. It does not matter that the radius is, as long as it is located in this range. For example, choosing radius 2 will work. The above then becomes

$$
b_1 = \frac{1}{2\pi i}\oint_C \frac{z}{(z+1)(z-3)}dz \tag{5}
$$

However, since $f(z)$ is analytic in this region, then $\oint_C f(z)\,dz = 2\pi i \sum$ (residues inside). There is only one pole now inside C, which is at $z = -1$. So all what we have to do is find the residue at $z = -1$.

$$
\begin{aligned}
\operatorname{Residue}(-1) &= \lim_{z\to -1}(z+1)f(z) \\
&= \lim_{z\to -1}(z+1)\frac{z}{(z+1)(z-3)} \\
&= \lim_{z\to -1}\frac{z}{(z-3)} \\
&= \frac{-1}{(-1-3)} \\
&= \frac{1}{4}
\end{aligned}
$$

Using this in (5) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\left(2\pi i\frac{1}{4}\right) \\ &= \frac{1}{4} \end{aligned}$$

Which agrees with what we found in (4) above.

For (iii), region C: This is for $|z| > 3$. From (2,3) we see that (2b) and (3b) are valid expansions in $z > 3$, Hence

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4z}\left(1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4-\cdots\right)+\frac{3}{4z}\left(1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right) \\ &= \left(\frac{1}{4z}-\frac{1}{4z^2}+\frac{1}{4z^3}-\frac{1}{4z^4}+\frac{1}{4z^5}-\cdots\right)+\frac{3}{4z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\right) \\ &= \left(\frac{1}{4z}-\frac{1}{4z^2}+\frac{1}{4z^3}-\frac{1}{4z^4}+\frac{1}{4z^5}-\cdots\right)+\left(\frac{3}{4z}+\frac{9}{4z^2}+\frac{27}{4z^3}+\frac{81}{4z^4}+\cdots\right) \\ &= \cdots+\frac{20}{z^4}+\frac{7}{z^3}+\frac{2}{z^2}+\frac{1}{z} \end{aligned}$$

This is as expected contains only a principal part and no analytical part. The residue is 1. This above value for the residue can be verified as follows. Using

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$$

Where in the above z_0 is the location of the pole and n is the coefficient of the $\frac{1}{z^n}$ is the principal part. Since we want the residue, then $n = 1$ and the above becomes

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

In the above, the contour C is circle somewhere in $|z| > 3$. It does not matter that the radius is. The above integral then becomes

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{z}{(z+1)(z-3)}dz \tag{7}$$

However, since $f(z)$ is analytic in $|z| > 3$, then $\oint_C f(z)\,dz = 2\pi i \sum$ (residues inside). There are now two poles inside C, one at $z = -1$ and one at $z = 3$. So all what we have to do is find the residues at each. We found earlier that $\operatorname{Residue}(-1) = \frac{1}{4}$. Now

$$\begin{aligned} \operatorname{Residue}(3) &= \lim_{z\to 3}(z-3)f(z) \\ &= \lim_{z\to 3}(z-3)\frac{z}{(z+1)(z-3)} \\ &= \lim_{z\to 3}\frac{z}{(z+1)} \\ &= \frac{3}{4} \end{aligned}$$

Therefore the sum of residues is 1. Using this result in (7) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\left(2\pi i\left(\frac{1}{4}+\frac{3}{4}\right)\right) \\ &= 1 \end{aligned}$$

Which agrees with what result from (6) above.

Summary of results

$$\boxed{f(z) = \frac{z}{(z+1)(z-3)} = \begin{cases} -\frac{1}{3}z+\frac{2}{9}z^2-\frac{7}{27}z^3+\frac{20}{81}z^4-\cdots & |z|<1 \\ \cdots+\frac{1}{4z^5}-\frac{1}{4z^4}+\frac{1}{4z^3}-\frac{1}{4z^2}+\frac{1}{4z}-\frac{1}{4}-\frac{z}{12}-\frac{z^2}{36}-\frac{z^3}{108}-\frac{z^4}{324}-\cdots & 1<|z|<3 \\ \cdots+\frac{20}{z^4}+\frac{7}{z^3}+\frac{2}{z^2}+\frac{1}{z} & |z|>3 \end{cases}}$$

3.3.3 Problem 3

Part (a)

Use residue theorem to evaluate $\oint_C \frac{e^{-2z}}{z^2} dz$ on contour C which is circle $|z| = 1$ in positive sense.

Solution

For $f(z)$ which is analytic on and inside C, the Cauchy integral formula says

$$\oint_C f(z)\, dz = 2\pi i \sum_j \text{Residue}\left(z = z_j\right) \tag{1}$$

Where the sum is over all residues located inside C. for $f(z) = \frac{e^{-2z}}{z^2}$ there is a simple pole at $z = 0$ of order 2. To find the residue, we use the formula for pole or order m given by

$$\text{Residue}(z_0) = \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \frac{(z - z_0)^m}{(m-1)!} f(z)$$

Hence for $m = 2$ and $z_0 = 0$ the above becomes

$$\begin{aligned}
\text{Residue}(0) &= \lim_{z \to 0} \frac{d}{dz} z^2 f(z) \\
&= \lim_{z \to 0} \frac{d}{dz} z^2 \frac{e^{-2z}}{z^2} \\
&= \lim_{z \to 0} \frac{d}{dz} e^{-2z} \\
&= \lim_{z \to 0} \left(-2e^{-2z}\right) \\
&= -2
\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
\oint_C \frac{e^{-2z}}{z^2} dz &= 2\pi i(-2) \\
&= -4\pi i
\end{aligned}$$

Part (b)

Use residue theorem to evaluate $\oint_C z e^{\frac{1}{z}} dz$ on contour C which is circle $|z| = 1$ in positive sense.

Solution

The singularity is at $z = 0$, but we can not use the simple pole residue finding method here, since this is an essential singularity now due to the $e^{\frac{1}{z}}$ term. To find the residue, we expand $f(z)$ around $z = 0$ in Laurent series and look for the coefficient of $\frac{1}{z}$ term.

$$\begin{aligned}
f(z) &= z e^{\frac{1}{z}} \\
&= z\left(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right) \\
&= z + 1 + \frac{1}{2}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots
\end{aligned}$$

Hence residue is $\frac{1}{2}$. Therefore

$$\begin{aligned}
\oint_C z e^{\frac{1}{z}} dz &= 2\pi i\left(\frac{1}{2}\right) \\
&= \pi i
\end{aligned}$$

Part (c)

Use residue theorem to evaluate $\oint_C \frac{z+2}{z^2-\frac{z}{2}}dz$ on contour C which is circle $|z| = 1$ in positive sense.

Solution

$$\begin{aligned} f(z) &= \frac{z+2}{z^2 - \frac{z}{2}} \\ &= \frac{z+2}{z\left(z - \frac{1}{2}\right)} \end{aligned}$$

Hence there is a simple pole at $z = 0$ and simple pole at $z = \frac{1}{2}$

$$\begin{aligned} \text{Residue}(0) &= \lim_{z\to 0}(z)f(z) \\ &= \lim_{z\to 0} z\frac{z+2}{z\left(z-\frac{1}{2}\right)} \\ &= \lim_{z\to 0}\frac{z+2}{\left(z-\frac{1}{2}\right)} \\ &= \frac{2}{-\frac{1}{2}} \\ &= -4 \end{aligned}$$

And

$$\begin{aligned} \text{Residue}\left(\frac{1}{2}\right) &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) \\ &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z+2}{z\left(z-\frac{1}{2}\right)} \\ &= \lim_{z\to\frac{1}{2}}\frac{z+2}{z} \\ &= \frac{\frac{1}{2}+2}{\frac{1}{2}} \\ &= 5 \end{aligned}$$

Therefore

$$\begin{aligned} \oint_C \frac{z+2}{z^2-\frac{z}{2}}dz &= 2\pi i(5-4) \\ &= 2\pi i \end{aligned}$$

3.3.4 key solution to HW 3

Problem Set 3 Solutions

Exercise 1

a) $|z| = \sqrt{x^2+y^2}$ $\Rightarrow$ $u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$

now $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \neq \frac{\partial v}{\partial y} = 0$ (unless $x=0$)

$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \neq -\frac{\partial v}{\partial x} = 0$ (unless $y=0$)

$\Rightarrow$ $|z|$ is only possibly analytic at $(x,y) = (0,0)$

everywhere else it is not analytic. However at $x=y=0$ the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ do not exist at this point

$\Rightarrow$ $|z|$ is nowhere analytic

b) $\operatorname{Re} z = x$ $\Rightarrow$ $u = x$ $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$ for all x, y $\Rightarrow$ $\operatorname{Re} z$ is nowhere analytic

c) $e^{\sin z}$, need to find u and v

$e^{\sin z} = e^{\sin(x+iy)} = e^{[\sin x \cos(iy) + \sin(iy)\cos x]}$

now $\cos(iy) = \cosh(y)$

$\sin(iy) = i\sinh(y)$

$e^{\sin z} = e^{\sin x \cosh(y) + i \sinh(y) \cos x}$

$$\Rightarrow \quad u(x,y) = \cos(\sinh y \cos x)\, e^{\sin x \cosh y}$$

$$v(x,y) = \sin(\sinh y \cos x)\, e^{\sin x \cosh y}$$

now
$$\frac{\partial u}{\partial x} = + \sin(\sinh y \cos x) \sinh y \sin x\, e^{\sin x \cosh y} + \cos(\sinh y \cos x) \cos x \cosh y\, e^{\sin x \cosh y}$$

$$\frac{\partial u}{\partial y} = -\sin(\sinh y \cos x) \cos x \cosh y\, e^{\sin x \cosh y} + \cos(\sinh y \cos x) \sin x \sinh y\, e^{\sin x \cosh y}$$

$$\frac{\partial v}{\partial x} = -\cos(\sinh y \cos x) \sinh y \sin x\, e^{\sin x \cosh y} + \sin(\sinh y \cos x) \cos x \cosh y\, e^{\sin x \cosh y}$$

$$\frac{\partial v}{\partial y} = \cos(\sinh y \cos x) \cosh y \cos x\, e^{\sin x \cosh y} + \sin(\sinh y \cos x) \sin x \sinh y\, e^{\sin x \cosh y}$$

then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\Rightarrow$ $e^{\sin z}$ is everywhere analytic.

Exercise 2

a) $$\frac{z+3}{z-2} = \frac{-(z+3)}{2(1-\frac{z}{2})} = -\left(\frac{z}{2}+\frac{3}{2}\right)\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right)$$

$$= -\frac{3}{2} - \frac{5}{2}\frac{z}{2} - \frac{5}{2}\left(\frac{z}{2}\right)^2 - \frac{5}{2}\left(\frac{z}{2}\right)^3 - \cdots$$

which is valid for $|z|<2$

now consider series for $\frac{1}{z}$

$$\frac{z+3}{z-2} = \frac{1+\frac{3}{z}}{1-\frac{2}{z}} = \left(1+\frac{3}{z}\right)\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right)$$

$$= 1 + 5\frac{1}{z} + 5\cdot\frac{2}{z^2} + 5\cdot\frac{4}{z^3} + 5\cdot\frac{8}{z^4} + \cdots$$

this is valid for $|z|>2$

b) $$\frac{z}{(z+1)(z-3)} = \frac{a}{z+1} + \frac{b}{z-3} = \frac{1/4}{z+1} + \frac{3/4}{z-3}$$

For $|z|<1$

$$\frac{1/4}{z+1} = \frac{1}{4}\sum_{n=0}^{\infty}(-z)^n$$ which converges for $|z|<1$

similarly we have

$$\frac{3/4}{z-3} = -\frac{1}{4}\frac{1}{1-\frac{z}{3}} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$ which converges for $|z|<3$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=0}^{\infty}(-z)^n - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n - \frac{1}{3^n}\right)z^n \qquad \text{for } |z|<1$$

ii) for $1<|z|<3$

- we can still use $\dfrac{3/4}{z-3} = -\dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{3^n} \quad (|z|<3)$

- look at $\dfrac{1/4}{z+1} = \dfrac{1}{4z}\dfrac{1}{1+\frac{1}{z}} = \dfrac{1}{4z}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$

valid for $|z|>1$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

which converges for $1<|z|<3$

iii) For $|z|>3$

we have $\dfrac{1/4}{z+1} = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$ which converges for $|z|>1$

consider $\dfrac{3/4}{z-3} = \dfrac{3}{4z}\dfrac{1}{1-\frac{3}{z}} = \dfrac{3}{4z}\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{z^n} = \frac{3}{4}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$

which converges for $|z|>3$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \frac{3}{4}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} = \frac{1}{4}\sum_{n=1}^{\infty}\left[(-1)^{n+1} + 3^n\right]\frac{1}{z^n}$$

which converges for $|z|>3$

Exercise 3

a) $$\int_C \frac{e^{-2z}}{z^2}\,dz = \int_C \frac{\sum_{n=0}^{\infty}\frac{(-2z)^n}{n!}}{z^2}\,dz = \int_C\left[\frac{1}{z^2}-\frac{2z}{z^2}+\frac{4z^2}{2!\,z^2}-\frac{8z^3}{3!\,z^2}+\cdots\right]dz$$

$$= \int_C\left[\frac{1}{z^2}-\frac{2}{z}+2-\frac{4z}{3}+\cdots\right]dz$$

only term that gives non-zero integral

$$= 2\pi i(-2) = -4\pi i$$

b) $$\int_C z e^{\frac{1}{z}}\,dz = \int_C z\left[1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\cdots\right]dz$$

$$= \int_C\left[z+1+\frac{1}{2z}+\frac{1}{6z^2}+\cdots\right]dz$$

only term that gives non-zero integral

$$= 2\pi i\cdot\frac{1}{2} = \pi i$$

c) $$\int_C \frac{z+2}{z^2-z/2} = \int_C \frac{z+2}{z(z-\frac{1}{2})}\,dz$$

this has poles at $z=1/2$ and $z=0$

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{\frac{1}{2}+2}{\frac{1}{2}} = 5 \qquad \operatorname*{Res}_{z=0} f(z) = \frac{2}{-1/2} = -4$$

$$\Rightarrow \int_C \frac{z+2}{z(z-1/2)} = 2\pi i(5-4) = 2\pi i$$

3.4 HW 4

3.4.1 Problem 1

Using series expansion evaluate the integral $I = \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$

Solution

We first need to find the Taylor series for $\ln\left(\frac{1+x}{1-x}\right)$ expanded around $x = 0$. Since

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln\left((1+x)\left(\frac{1}{1-x}\right)\right)\\ &= \ln(1+x) + \ln\left(\frac{1}{1-x}\right)\\ &= \ln(1+x) - \ln(1-x) \qquad (1)\end{aligned}$$

Looking at $\ln(1+x)$, where now $f(x) = \ln(1+x)$, then we see that $f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2}, f'''(x) = \frac{2}{(1+x)^3}, f^{(4)}(x) = -\frac{2\cdot 3}{(1+x)^4}, \cdots$, therefore

$$\begin{aligned}\ln(1+x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots\\ &= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (2)\end{aligned}$$

Similarly for $\ln(1-x)$, where now $f'(x) = \frac{-1}{1-x}, f''(x) = \frac{-1}{(1-x)^2}, f'''(x) = \frac{-2}{(1-x)^3}, f^{(4)}(x) = -\frac{2\cdot 3}{(1-x)^4}, \cdots$, therefore

$$\begin{aligned}\ln(1-x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots\\ &= 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (3)\end{aligned}$$

Using (2,3) in (1) gives the series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ as

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots \qquad (4)\end{aligned}$$

Using (4) in the integral given results in

$$\begin{aligned}I &= \int_0^1 \left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots\right)\frac{dx}{x}\\ &= \int_0^1 \left(2 + \frac{2}{3}x^2 + \frac{2}{5}x^4 + \frac{2}{7}x^6 + \cdots\right)dx\\ &= \left[2x + \frac{2}{3}\frac{x^3}{3} + \frac{2}{5}\frac{x^5}{5} + \frac{2}{7}\frac{x^7}{7} + \cdots\right]_0^1\end{aligned}$$

Which simplifies to

$$\begin{aligned}I &= 2 + \frac{2}{3}\frac{1}{3} + \frac{2}{5}\frac{1}{5} + \frac{2}{7}\frac{1}{7} + \cdots\\ &= 2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \frac{2}{9^2} + \cdots\\ &= 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots\right)\\ &= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \qquad (5)\end{aligned}$$

The following are two methods to obtain closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. The first method is based on writing

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \qquad (6)$$

Where the sum on the left is broken into odd and even terms on the right, as in

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right) + \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

But, from lecture Sept. 12, 2018, we showed in class that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \qquad (7)$$

(This is called the Basel problem, and the above closed form sum was first given by Euler in 1734). Now using (7) into (6) results in

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\\
&= \frac{3}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\\
&= \frac{3}{4}\left(\frac{\pi^2}{6}\right)\\
&= \frac{\pi^2}{8}
\end{aligned}$$

Another way to obtained closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ is to use Fourier series. Considering the Fourier series for the following periodic function

$$f(x)=\begin{cases}-x & -\pi<x<0\\ 0 & 0\leq x\leq \pi\end{cases}$$

Using

$$f(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos(nx)+\sum_{n=1}^{\infty}B_n\sin(nx)$$

Therefore

$$A_0=\frac{1}{\pi}\int_{-\pi}^{0}-xdx=\frac{-1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi}^{0}=\frac{-1}{2\pi}\left(x^2\right)_{-\pi}^{0}=\frac{-1}{2\pi}\left(-\pi^2\right)=\frac{1}{2}\pi$$

And

$$\begin{aligned}
A_n &= \frac{-1}{\pi}\int_{-\pi}^{0}x\cos(nx)\,dx=\frac{1+(-1)^{n+1}}{n^2}\\
B_n &= \frac{-1}{\pi}\int_{-\pi}^{0}x\sin(nx)\,dx=\frac{(-1)^{n+1}}{n}\pi
\end{aligned}$$

Hence the Fourier series for $f(x)$ is

$$\begin{aligned}
f(x) &= \frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\pi(\sin nx)\\
&= \frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx)-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)
\end{aligned}$$

Evaluating the above at $x=0$ then all the $\sin$ terms vanish and we obtain

$$\begin{aligned}
0 &= \frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\\
&= \frac{\pi}{4}-\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)\\
&= \frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi}{4}\\
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi^2}{8}
\end{aligned}$$

Now that we found closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, we can find the value of the integral.

Since $I = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, then

$$\begin{aligned}\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} &= 2\left(\frac{\pi^2}{8}\right)\\ &= \frac{\pi^2}{4}\end{aligned}$$

3.4.2 Problem 2

Let $I(x) = \int_0^{\infty} e^{xf(t)}dt$ with $f(t) = t - \frac{e^t}{x}$, find a large x approximation for this integral.

Solution

$$\begin{aligned}I &= \int_0^{\infty} \exp\left(xf(t)\right)dt\\ &= \int_0^{\infty} \exp\left(x\left(t - \frac{e^t}{x}\right)\right)dt\\ &= \int_0^{\infty} \exp\left(xt - e^t\right)dt\\ &= \int_0^{\infty} \exp\left(F(t)\right)dt \qquad (1)\end{aligned}$$

Where $F(t) = xt - e^t$. We need to find saddle point where $F(t)$ is maximum. Hence

$$\begin{aligned}\frac{d}{dt}F(t) &= 0\\ x - e^t &= 0\\ e^t &= x\\ t_0 &= \ln(x)\end{aligned}$$

Where t_0 is location of t where $F(t)$ is maximum. We called this in class t_{peak}. We now expand $F(t)$ around t_0 using Taylor series

$$F(t) = F(t_0) + F'(t_0)(t - t_0) + \frac{1}{2}F''(t_0)(t - t_0)^2 + \cdots \qquad (2)$$

But

$$\begin{aligned}F(t_0) &= x\ln(x) - e^{\ln x}\\ &= x\ln x - x\end{aligned}$$

And $F'(t) = x - e^t$, hence as expected $F'(t_0) = 0$. And $F''(t) = -e^t$, therefore $F''(t_0) = -e^{\ln x} = -x$. We see also that $F''(t_0) < 0$, which means the saddle point was a maximum and not a minimum (since x is positive). Using these in (2) gives

$$\begin{aligned}F(t) &\approx (x\ln x - x) + \frac{1}{2}(-x)(t - \ln x)^2\\ &= x\ln x - x - \frac{1}{2}x(t - \ln x)^2\end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned}I &= \int_0^{\infty} \exp\left(x\ln x - x - \frac{1}{2}x(t - \ln x)^2\right)dt\\ &= \int_0^{\infty} \exp(x\ln x)\exp(-x)\exp\left(-\frac{1}{2}x(t - \ln x)^2\right)dt\\ &= \exp(x\ln x)\exp(-x)\int_0^{\infty} \exp\left(-\frac{1}{2}x(t - \ln x)^2\right)dt\\ &= x^x e^{-x}\int_0^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2}dt \qquad (3)\end{aligned}$$

Now, since the peak value where $F(t)$ occurs is on the positive real axis, because $t_0 = \ln(x)$, therefore $x > 1$ to have a maximum, and assuming a narrow peak, then all the contribution to the integral comes from x close to the peak location, so we can change $\int_0^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2}dt$

to $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt$ without affecting the final result. Therefore (3) becomes

$$I = x^x e^{-x} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt \tag{4}$$

Now comparing $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt$ to the Gaussian integral $\int_{-\infty}^{\infty} e^{-a(t-b)^2} dt = \sqrt{\frac{\pi}{a}}$, shows that $a = \frac{x}{2}$ for our case. Hence

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt = \sqrt{\frac{2\pi}{x}}$$

Therefore (4) becomes

$$I \approx x^x e^{-x} \sqrt{\frac{2\pi}{x}}$$

For large x.

3.4.3 Problem 3

Evaluate the following integrals with aid of residue theorem $a \geq 0$. (a) $\int_0^{\infty} \frac{1}{x^4+1} dx$ (b) $\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx$

Part (a)

Since the integrand is even, then

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

Now we consider the following contour

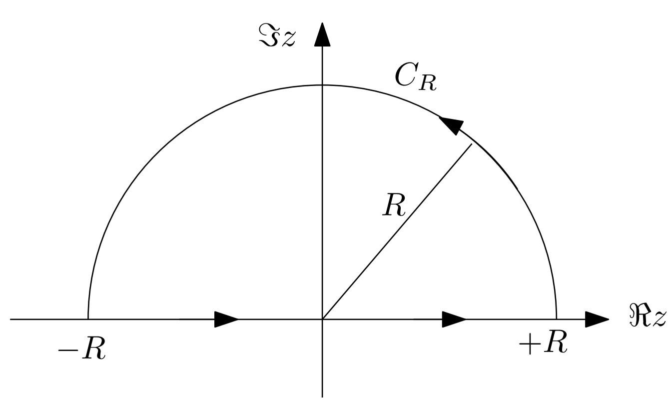


Figure 3.5: contour used for problem 3

Therefore

$$\oint_C f(z)\, dz = \left(\lim_{R\to\infty} \int_{-R}^{0} f(x)\, dx + \lim_{\tilde{R}\to\infty} \int_0^{\tilde{R}} f(x)\, dx \right) + \lim_{R\to\infty} \int_{C_R} f(z)\, dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned} \oint_C f(z)\, dz &= \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx + \lim_{R\to\infty} \int_{C_R} f(z)\, d \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Where $\sum \text{Residue}$ is sum of residues of $\frac{1}{z^4+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} f(z)\, dz$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz \tag{1}$$

Now we will show that $\lim_{R\to\infty}\int_{C_R}\frac{1}{z^4+1}dz=0$. Since

$$\begin{aligned}\left|\int_{C_R}\frac{1}{z^4+1}dz\right| &\leq ML\\ &= |f(z)|_{\max}(\pi R)\end{aligned}\tag{2}$$

But

$$f(z)=\frac{1}{\left(z^2-i\right)\left(z^2+i\right)}$$

Hence, and since $z=R\,e^{i\theta}$ then

$$|f(z)|_{\max}\leq\frac{1}{\left|z^2-i\right|_{\min}\left|z^2+i\right|_{\min}}$$

But but inverse triangle inequality $\left|z^2-i\right|\geq|z|^2+1$ and $\left|z^2+i\right|\geq|z|^2-1$, and since $|z|=R$ then the above becomes

$$\begin{aligned}|f(z)|_{\max} &\leq\frac{1}{\left(R^2+1\right)\left(R^2-1\right)}\\ &=\frac{1}{R^4-1}\end{aligned}$$

Therefore (2) becomes

$$\left|\int_{C_R}\frac{1}{z^4+1}dz\right|\leq\frac{\pi R}{R^4-1}$$

Then it is clear that as $R\to\infty$ the above goes to zero since $\lim_{R\to\infty}\frac{\pi R}{R^4-1}=\lim_{R\to\infty}\frac{\frac{\pi}{R^3}}{1-\frac{1}{R^4}}=\frac{0}{1}=0$. Then (1) now simplifies to

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx=2\pi i\sum\text{Residue}\tag{2A}$$

We just now need to find the residues of $\frac{1}{z^4+1}$ located in upper half plane. The zeros of the denominator $z^4+1=0$ are at $z=-1^{\frac{1}{4}}=\left(e^{i\pi}\right)^{\frac{1}{4}}$, then the first zero is at $e^{i\frac{\pi}{4}}$, and the second zero at $e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)}=e^{i\left(\frac{3}{4}\pi\right)}$ and the third zero at $e^{i\left(\frac{3}{4}\pi+\frac{\pi}{2}\right)}=e^{i\left(\frac{5}{4}\pi\right)}$ and the fourth zero at $e^{i\left(\frac{5}{4}\pi+\frac{\pi}{2}\right)}=e^{i\frac{7}{4}\pi}$. Hence poles are at

$$\begin{aligned}z_1 &= e^{i\frac{\pi}{4}}\\ z_2 &= e^{i\frac{3}{4}\pi}\\ z_3 &= e^{i\frac{5}{4}\pi}\\ z_4 &= e^{i\frac{7}{4}\pi}\end{aligned}$$

Out of these only the first two are in upper half plane z_1 and z_1. Hence

$$\begin{aligned}\text{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)f(z)\\ &= \lim_{z\to z_1}(z-z_1)\frac{1}{z^4-1}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}(z_1) &= \lim_{z\to z_1}\frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}\\ &= \frac{1}{4e^{i\frac{3\pi}{4}}}\end{aligned}$$

Similarly for the other residue

$$\begin{aligned}\text{Residue}\left(z_2\right) &= \lim_{z\to z_2}\left(z-z_2\right)f\left(z\right)\\ &= \lim_{z\to z_1}\left(z-z_2\right)\frac{1}{z^4-1}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\left(z_2\right) &= \lim_{z\to z_2}\frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3}\\ &= \frac{1}{4e^{i\frac{9\pi}{4}}}\\ &= \frac{1}{4e^{i\frac{\pi}{4}}}\end{aligned}$$

Hence (2A) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx &= 2\pi i\left(\frac{1}{4e^{i\frac{3\pi}{4}}}+\frac{1}{4e^{i\frac{\pi}{4}}}\right)\\ &= 2\pi i\left(\frac{\sqrt{2}}{4i}\right)\\ &= \frac{1}{2}\sqrt{2}\pi\end{aligned}$$

But $\int_0^{\infty}\frac{1}{x^4+1}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$, therefore

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^4+1}dx &= \frac{\sqrt{2}}{4}\pi\\ &= \frac{2}{4\sqrt{2}}\pi\\ &= \frac{\pi}{2\sqrt{2}}\end{aligned}$$

Part (b)

Since the integrand is even, then

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\left(ax\right)}{x^2+1}dx$$

We will evaluate $\int_{-\infty}^{\infty}\frac{e^{iaz}}{x^2+1}dx$ and at the end take the real part of the answer. Considering the following contour

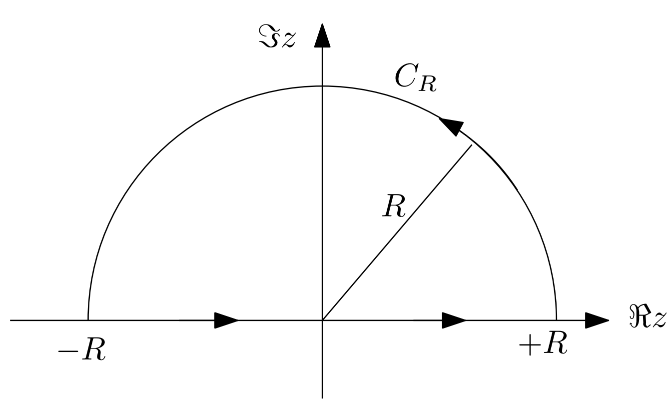


Figure 3.6: contour used for part b

Then

$$\oint\limits_C f(z)\,dz = \left(\lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty}\int_{0}^{\tilde{R}} f(x)\,dx\right) + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned}\oint\limits_C f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,d\\ &= 2\pi i\sum \text{Residue}\end{aligned}$$

Where $\sum \text{Residue}$ is sum of residues of $\frac{e^{iaz}}{x^2+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = 2\pi i\sum \text{Residue} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}dx = 2\pi i\sum \text{Residue} - \lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}}{z^2+1}dz \tag{1}$$

Now we will show that $\lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}}{z^2+1}dz = 0$. Since

$$\begin{aligned}\left|\int_{C_R}\frac{e^{iaz}}{z^2+1}dz\right| &\le ML\\ &= |f(z)|_{\max}(\pi R)\end{aligned} \tag{2}$$

But

$$\begin{aligned}f(z) &= \frac{e^{iaz}}{(z-i)(z+i)}\\ &= \frac{e^{ia(x+iy)}}{(z-i)(z+i)}\\ &= \frac{e^{iax-ay}}{(z-i)(z+i)}\\ &= \frac{e^{iax}e^{-ay}}{(z-i)(z+i)}\end{aligned}$$

Hence

$$\begin{aligned}|f(z)|_{\max} &= \frac{|e^{iaz}|_{\max}|e^{-ay}|_{\max}}{|z-i|_{\min}|z+i|_{\min}}\\ &= \frac{|e^{-ay}|_{\max}}{(R+1)(R-1)}\\ &= \frac{|e^{-ay}|_{\max}}{R^2-1}\end{aligned}$$

Since $a>0$ and since in upper half $y>0$ then $|e^{-ay}|_{\max} = |e^{-aR}|_{\max} = 1$. Jordan inequality was not needed here, since there is no extra x in the numerator of the integrand in this problem. The above now reduces to

$$|f(z)|_{\max} = \frac{1}{R^2-1}$$

Equation (2) becomes

$$\left|\int_{C_R}\frac{e^{iaz}}{z^2+1}dz\right| \le \frac{\pi R}{R^2-1}$$

$R\to\infty$ the above goes to zero since $\lim_{R\to\infty}\frac{\pi R}{R^2-1} = \lim_{R\to\infty}\frac{\frac{\pi}{R^2}}{1-\frac{1}{R^2}} = \frac{0}{1} = 0$. Equation (1) now simplifies to

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+1}dx = 2\pi i\sum \text{Residue}$$

We just now need to find the residues of $\frac{1}{z^2+1}$ that are located in upper half plane. The zeros of the denominator $z^2+1=0$ are at $z=\pm i$, hence poles are at

$$z_1 = i$$
$$z_2 = -i$$

Only z_1 is in upper half plane. Therefore

$$\begin{aligned}
\operatorname{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)\,f(z)\\
&= \lim_{z\to z_1}(z-z_1)\frac{e^{iaz}}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_1}\frac{e^{iaz}}{(z-z_2)}\\
&= \frac{e^{ia(i)}}{(i+i)}\\
&= \frac{e^{-a}}{2i}
\end{aligned}$$

Since $\int_{-\infty}^{\infty}\frac{e^{ax}}{x^4+1}dx = 2\pi i\sum\operatorname{Residue}$ then

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+1}dx &= 2\pi i\left(\frac{e^{-a}}{2i}\right)\\
&= \pi e^{-a}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{\infty}\frac{e^{iax}}{x^4+1}dx &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ax}}{x^4+1}dx\\
&= \frac{\pi}{2}e^{-a}
\end{aligned}$$

But real part of the above is

$$\int_0^{\infty}\frac{\cos(ax)}{x^4+1}dx = \frac{\pi}{2}e^{-a}$$

3.4.4 Problem 4

Using residues evaluate(a) $\int_0^{2\pi}\frac{1}{1+a\cos\theta}d\theta$ for $|a|<1$ (b) $\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta$ for n integer.

Part (a)

Using contour which is anti-clockwise over the unit circle

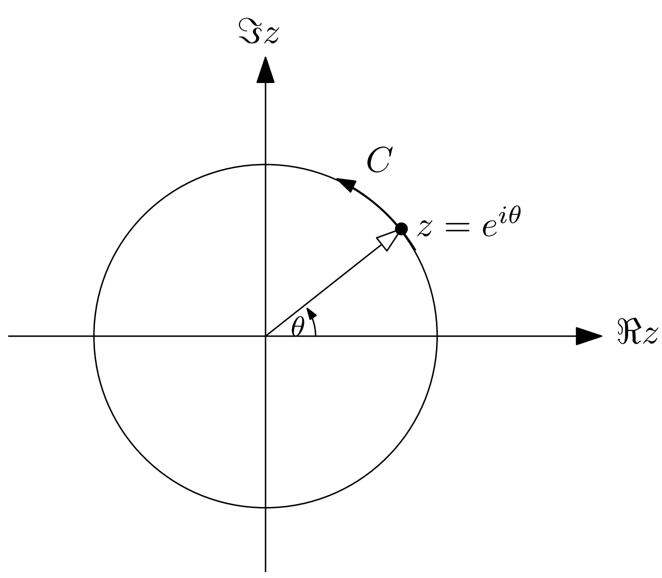


Figure 3.7: contour used for problem 4

Let $z=e^{i\theta}$, hence $dz = d\theta ie^{i\theta} = d\theta iz$. Using $\cos\theta = \frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$
\begin{aligned}
\oint_C \frac{\frac{1}{iz}dz}{1+a\frac{z+z^{-1}}{2}} &= \frac{2}{i}\oint_C \frac{\frac{1}{z}dz}{2+a\left(z+\frac{1}{z}\right)}\\
&= \frac{2}{i}\oint_C \frac{dz}{2z+az^2+a}\\
&= \frac{2}{ai}\oint_C \frac{dz}{z^2+\frac{2}{a}z+1}\\
&= \frac{2}{ai}\oint_C \frac{dz}{(z-z_1)(z-z_2)}
\end{aligned}
$$

Where z_1, z_2 are roots of $z^2+\frac{2}{a}z+1=0$ which are found to be (using the quadratic formula) as

$$
\begin{aligned}
z_1 &= \frac{-1-\sqrt{1-a^2}}{a}\\
z_2 &= \frac{-1+\sqrt{1-a^2}}{a}
\end{aligned}
$$

Since $|a|<1$ then only z_2 will be inside the unit disk for all a values. Therefore

$$
\begin{aligned}
\frac{2}{ai}\oint_C \frac{dz}{(z-z_1)(z-z_2)} &= \left(\frac{2}{ai}\right)2\pi i \operatorname{Residue}(z_2)\\
&= \frac{4}{a}\pi \operatorname{Residue}(z_2) \qquad (1)
\end{aligned}
$$

Now we will find the $\operatorname{Residue}(z_2)$ where in this case $f(z)=\frac{1}{(z-z_1)(z-z_2)}$. Hence

$$
\begin{aligned}
\operatorname{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2)f(z)\\
&= \lim_{z\to z_2}(z-z_2)\frac{1}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_2}\frac{1}{(z-z_1)}\\
&= \frac{1}{\left(\frac{-1+\sqrt{1-a^2}}{a}\right)-\left(\frac{-1-\sqrt{1-a^2}}{a}\right)}\\
&= \frac{a}{2\sqrt{1-a^2}}
\end{aligned}
$$

Using the above result in (1) gives

$$
\begin{aligned}
\int_0^{2\pi}\frac{1}{1+a\cos\theta}d\theta &= \left(\frac{4}{a}\pi\right)\frac{a}{2\sqrt{1-a^2}}\\
&= \frac{2\pi}{\sqrt{1-a^2}} \qquad a\neq 1
\end{aligned}
$$

Using Maple, verified that the above result is correct.

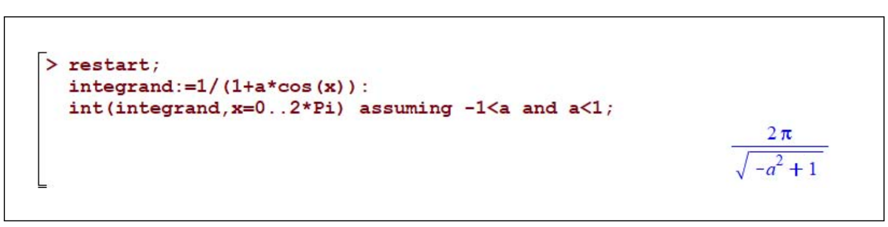


Figure 3.8: Verification using Maple

Part (b)

Since integrand is even, then $\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta=\frac{1}{2}\int_0^{2\pi}(\cos(\theta))^{2n}\,d\theta$. Using same contour as in part (a), and letting $z=e^{i\theta}$, hence $dz=d\theta ie^{i\theta}=d\theta iz$ and using $\cos\theta=\frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$
\begin{aligned}
\int_0^{2\pi} (\cos(\theta))^{2n}\, d\theta &= \oint_C \left(\frac{z+\frac{1}{z}}{2}\right)^{2n} \frac{dz}{iz} \\
&= \frac{1}{i}\oint_C \frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \left(\frac{z^2+1}{z}\right)^{2n} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n}} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} dz
\end{aligned}
$$

Considering $f(z) = \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}$, this has a pole at $z=0$ of order $m = 2n+1$. Therefore

$$
\frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} dz = \left(\frac{1}{4^n i}\right) 2\pi i \operatorname{Residue}(z=0) \tag{1}
$$

So we now need to find residue of $f(z)$ at $z=0$ but for pole of order $m = 2n+1$. Using the formula for finding residue for pole of order m gives

$$
\operatorname{Residue}(z_0 = 0) = \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \frac{(z-z_0)^m f(z)}{(m-1)!}
$$

But $m = 2n+1$, and $z_0 = 0$, hence the above becomes

$$
\begin{aligned}
\operatorname{Residue}(0) &= \lim_{z\to 0} \frac{d^{2n}}{dz^{2n}} \frac{z^{2n+1}}{(2n)!} \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} \\
&= \frac{1}{(2n)!} \lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)
\end{aligned}
$$

Equation (1) becomes

$$
\int_0^{2\pi} (\cos(\theta))^{2n}\, d\theta = \left(\frac{1}{4^n}\right) 2\pi \left(\frac{1}{(2n)!} \lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\right)
$$

Therefore

$$
\begin{aligned}
\int_0^{\pi} (\cos(\theta))^{2n}\, d\theta &= \frac{1}{2}\left(\frac{1}{4^n}\right) 2\pi \left(\frac{1}{(2n)!} \lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\right) \\
&= \frac{1}{4^n}\frac{\pi}{(2n)!} \lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)
\end{aligned}
$$

Will now try to obtained closed form solution. Trying for different n values in order to see the pattern. From few lectures ago, we learned also that

$$
\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n}\sqrt{\pi}
$$

Now will generate a table to see the pattern

n	$\frac{1}{4^n}\frac{\pi}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)$	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$
1	$\frac{1}{4}\frac{\pi}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^2+1\right)^2$	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$
2	$\frac{1}{4^2}\frac{\pi}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^2+1\right)^4$	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$
3	$\frac{1}{4^3}\frac{\pi}{6!}\lim_{z\to 0}\frac{d^6}{dz^6}\left(z^2+1\right)^6$	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$
4	$\frac{1}{4^4}\frac{\pi}{8!}\lim_{z\to 0}\frac{d^8}{dz^8}\left(z^2+1\right)^8$	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$
5	$\frac{1}{4^5}\frac{\pi}{10!}\lim_{z\to 0}\frac{d^{10}}{dz^{10}}\left(z^2+1\right)^{10}$	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Based on the above, we see that $I=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$, which is verified as follows

n	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$	$\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$
1	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$	$\frac{\sqrt{\pi}\left(\frac{\sqrt{\pi}}{2}\right)}{1}=\frac{1}{2}\pi$
2	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$	$\frac{\sqrt{\pi}\left(\frac{3\sqrt{\pi}}{4}\right)}{2!}=\frac{3}{8}\pi$
3	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$	$\frac{\sqrt{\pi}\left(\frac{15\sqrt{\pi}}{8}\right)}{3!}=\frac{15\pi}{(6)(8)}=\frac{15\pi}{48}=\frac{3}{16}\pi$
4	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}\left(\frac{105\sqrt{\pi}}{16}\right)}{4!}=\frac{\sqrt{\pi}\left(105\sqrt{\pi}\right)}{(24)(16)}=\frac{105\pi}{384}=\frac{35}{128}\pi$
5	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$	$\frac{\sqrt{\pi}\left(\frac{945\sqrt{\pi}}{32}\right)}{5!}=\frac{945\pi}{(120)(32)}=\frac{945\pi}{3840}=\frac{63}{256}\pi$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore

$$\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Tried to do pole/zero cancellation on the integrand of $\oint_C\frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}dz$ in order to find a simpler method than the above but was not able to. The above result was verified using the computer

```
In[]:= Assuming[Element[n, Integers] && n > 0, Integrate[Cos[x]^(2 n), {x, 0, π}]];
       TraditionalForm[%]
Out[]//TraditionalForm=
```

$$\frac{\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Figure 3.9: Verification using Mathematica

3.4.5 key solution to HW 4

Problem Set 4

Exercise 1

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right\} - \left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right\}$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5}$$

$$= 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\}$$

ratio test $\quad \frac{a_{n+1}}{a_n} = \frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}} \overset{n\to\infty}{\sim} x^2 \quad$ converges for $|x| < 1$

$$I = 2\int_0^1 \left\{1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots\right\} dx$$

$$= 2\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right\}$$

Recall $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$\Rightarrow = \frac{1}{2} I + \frac{1}{4}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\} = \frac{1}{2} I + \frac{1}{4}\zeta(2)$$

$$\left(\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} + \left\{\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right\}\right.$$

$$\Rightarrow \quad I = \frac{3}{2}\zeta(2) = \frac{3}{2}\frac{\pi^2}{6} = \frac{\pi^2}{4}.$$

Exercise 2 : $I(x) = \int_0^{\infty} e^{x f(t)}\, dt$ with $f(t) = t - \frac{e^t}{x}$

$$f'(t) = 1 - \frac{e^t}{x} \longrightarrow 0 = f'(t_0) = 1 - \frac{e^{t_0}}{x} \longrightarrow t_0 = \ln x$$

$\hat{\text{L}}$ maximum

$$\therefore \quad f(t_0) = \ln x - 1$$
$$f'(t_0) = 0$$
$$f''(t_0) = -\frac{e^{t_0}}{x} = -1$$

$$\therefore \quad f(t) \approx \ln x - 1 - \frac{1}{2}(t - \ln x)^2$$ near maximum

$$\therefore \quad I(x) \approx \int_0^{\infty} e^{x \ln x - x}\, e^{-\frac{1}{2}x(t - \ln x)^2}\, dt$$

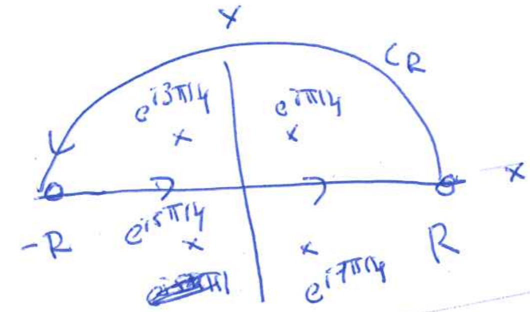
$$\approx x^x e^{-x} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t - \ln x)^2}\, dt$$

$\leftarrow$ sharply peaked about $t = \ln x$

$$= x^x e^{-x} \sqrt{\frac{2\pi}{x}}$$

$$\longrightarrow \quad I(x) \approx \sqrt{2\pi}\, x^{x - \frac{1}{2}} e^{-x}$$ for large x

Exercise 7 a) $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



consider $z_0^4 = -1$ has poles

for $z_0 = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

$\Rightarrow$ $e^{i\pi/4}$ and $e^{i3\pi/4}$ are inside the contour

- find the residues use:

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \frac{1}{4} z_0^{-3} \Rightarrow \operatorname{Res}_{z=e^{i\pi/4}} f(z) = \frac{1}{4} e^{-i3\pi/4}$$

$$\text{and } \operatorname{Res}_{z=e^{i3\pi/4}} f(z) = \frac{1}{4} e^{-i9\pi/4}$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} f(z)\,dz = 2\pi i \sum \operatorname{Res}$$

$$= 2\pi i \, \frac{1}{4}\left(e^{-i3\pi/4} + e^{-i7\pi/4}\right)$$

$$= -\frac{\pi}{2i}\left(-e^{i\pi/4} + e^{-i\pi/4}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

Also $\displaystyle\left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{R^4-1} \to 0$ as $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

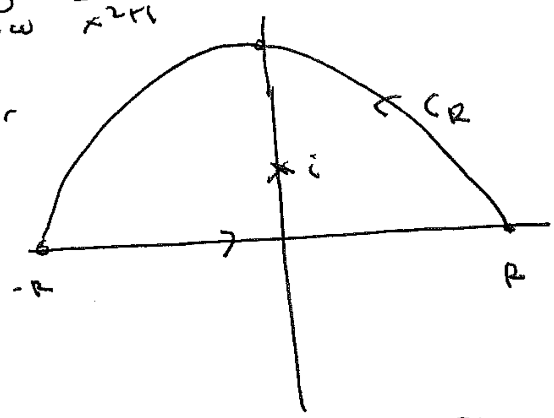
b) $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx \quad a > 0$

$= \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\cos ax}{x^2+1} + i\,\frac{\sin ax}{x^2+1}\right)dx$

odd function, so Cauchy P.V. vanishes

$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx$

now choose contour



$f(z) = \frac{e^{iaz}}{z^2+1}$

$f(z) = \frac{e^{iaz}}{(z+i)(z-i)}$

choose UHP to close contour since $e^{iaz} = e^{-ay}e^{iax}$ ($e^{-ay} < 1$ for $a > 0$)

. the contour contains one pole

$\operatorname{Res}_{z=i} f(z) = \frac{e^{-a}}{2i}$ also on C_R $|f(z)| \le \frac{|e^{iaz}|}{R^2-1} = \frac{e^{-ay}}{R^2-1} \le \frac{1}{R^2-1}$

$\Rightarrow \left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{R^2-1} \to 0$ as $R \to \infty$

$\therefore \int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \frac{1}{2}\, 2\pi i \left(\frac{e^{-a}}{2i}\right) = \frac{\pi}{2} e^{-a}$ for $a > 0$

Exercise 4

a) $I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_C \frac{1}{1+a\frac{z+z^{-1}}{2}} \frac{dz}{iz} = \int_C \frac{2/ai}{z^2+\frac{2}{a}z+1} dz$

roots of denominator are $z_\pm = \frac{1}{a}\left(-1 \pm \sqrt{1-a^2}\right)$

note $z_+ z_- = 1 \Rightarrow z_+ = \frac{1}{z_-}$ also $|z_-| = \frac{1}{|a|}\left(1+\sqrt{1-a^2}\right) > 1$ $\quad |a|<1$

$\Rightarrow |z_+| < 1 \rightarrow z_+$ is the pole within the unit circle

residue at z_+ is $\frac{2/ai}{z_+ - z_-} = \frac{2/ai}{\frac{2}{a}\sqrt{1-a^2}} = \frac{1}{i}\frac{1}{\sqrt{1-a^2}}$

$\therefore \quad I = 2\pi i \left(\frac{1}{i}\frac{1}{\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}$

b) $I = \int_0^{\pi} \cos^{2n}(\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\,d\theta$

$= \frac{1}{2}\int_C \left(\frac{z+z^{-1}}{2}\right)^{2n} \frac{dz}{iz} = \frac{1}{2i}\frac{1}{2^{2n}}\int_C \frac{1}{z}\underbrace{\left(z+z^{-1}\right)^{2n}}_{\text{use binomial expansion}} dz$

$= \frac{1}{2i}\frac{1}{2^{2n}}\int_C \frac{1}{z}\left\{z^{2n} + z^{2n-2} + \cdots + \binom{2n}{n}z^0 + \cdots + z^{-2n}\right\} dz$

this is the only term that gives a pole

$= \frac{1}{2i}\frac{1}{2^{2n}} 2\pi i \binom{2n}{n}$

$= \frac{\pi}{2^{2n}}\binom{2n}{n} = \frac{\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}$

3.5 HW 5

3.5.1 Problem 1

Expand the following functions, which are periodic in $\frac{2\pi}{L}$, in Fourier series (i) $f(x) = 1 - \frac{|x|}{L}$ for $\frac{-L}{2} \leq x \leq \frac{L}{2}$. (ii) $f(x) = e^x$ for $\frac{-L}{2} \leq x \leq \frac{L}{2}$

Solution

Part 1

The following is a plot of the function $f(x) = 1 - \frac{|x|}{L}$. In the plot below $L = 1$ was used for illustration.

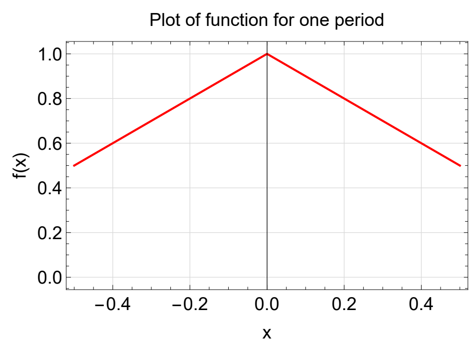


Figure 3.10: Function plot

```
L = 1;
f[x_] := 1 - Abs[x] / L;

p = Plot[f[x], {x, -L / 2, L / 2},
  AxesOrigin → {0, 0}, Frame → True,
  FrameLabel → {{"f(x)", None}, {"x", "Plot of function for one period"}},
  BaseStyle → 14,
  GridLines → Automatic, GridLinesStyle → LightGray,
  PlotStyle → Red]

Export["../images/p1_plot_1.pdf", p]
```

Figure 3.11: Code used

The Fourier series of $f(x) =$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L} n x\right) + b_n \sin\left(\frac{2\pi}{L} n x\right) \tag{1}$$

Where L is the period.

$$a_0 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, dx$$

$\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, dx$ is the area under the curve. Looking at the plot above shows the area is made up of the lower rectangle of area $\frac{1}{2}L$ and a triangle whose area is $\left(\frac{1}{2}L\right)\left(\frac{1}{2}\right)$. Therefore the total area is $\frac{1}{2}L + \frac{1}{4}L = \frac{3}{4}L$. Hence

$$\begin{aligned} a_0 &= \frac{2}{L}\left(\frac{3}{4}L\right) \\ &= \frac{3}{2} \end{aligned}$$

And

$$a_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos\left(\frac{2\pi}{L} n x\right) dx$$

Since $f(x)$ is an even function, the above simplifies to

$$
\begin{aligned}
a_n &= \frac{4}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi}{L}nx\right)dx\\
&= \frac{4}{L}\int_0^{\frac{L}{2}} \left(1-\frac{x}{L}\right)\cos\left(\frac{2\pi}{L}nx\right)dx\\
&= \frac{4}{L}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi}{L}nx\right)dx - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx\right)\\
&= \frac{4}{L}\left(\left[\sin\left(\frac{2\pi}{L}nx\right)\right]_0^{\frac{L}{2}} - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx\right)\\
&= \frac{4}{L}\left(\left[\sin\left(\frac{2\pi}{L}n\left(\frac{L}{2}\right)-0\right)\right] - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx\right)\\
&= \frac{4}{L}\left(\overbrace{[\sin \pi n]}^{0} - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx\right)\\
&= -\frac{4}{L^2}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx
\end{aligned}
$$

Using integration by parts: Let $u = x, dv = \cos\left(\frac{2\pi}{L}nx\right)$ then $du = 1, v = \frac{\sin\left(\frac{2\pi}{L}nx\right)}{\frac{2\pi}{L}n} = \frac{L}{2\pi n}\sin\left(\frac{2\pi}{L}nx\right)$. The above integral becomes

$$
\begin{aligned}
\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx &= \left(\frac{L}{2\pi n}x\sin\left(\frac{2\pi}{L}nx\right)\right)_0^{\frac{L}{2}} - \frac{L}{2\pi n}\int_0^{\frac{L}{2}}\sin\left(\frac{2\pi}{L}nx\right)dx\\
&= \left(\frac{L}{2\pi n}\left(\frac{L}{2}\right)\sin\left(\frac{2\pi}{L}n\frac{L}{2}\right)-0\right) - \frac{L}{2\pi n}\left(-\frac{\cos\left(\frac{2\pi}{L}nx\right)}{\frac{2\pi}{L}n}\right)_0^{\frac{L}{2}}\\
&= \frac{L^2}{4\pi n}\sin(n\pi) + \frac{L^2}{4\pi^2 n^2}\left(\cos\left(\frac{2\pi}{L}n\left(\frac{L}{2}\right)\right)-1\right)\\
&= \frac{L^2}{4\pi^2 n^2}\left(\cos(\pi n)-1\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
a_n &= -\frac{4}{L^2}\left(\frac{L^2}{4\pi^2 n^2}\left(\cos(\pi n)-1\right)\right)\\
&= \frac{1}{\pi^2 n^2}\left(1-\cos(\pi n)\right)
\end{aligned}
$$

The above is zero for even n and $\frac{2L^2}{4\pi^2 n^2}$ for odd n. Therefore the above simplifies to

$$
a_n = \frac{2}{\pi^2 n^2} \qquad n = 1,3,5,\cdots
$$

Because $f(x)$ is an even function, then $b_n = 0$ for all n. The Fourier series from (1) now becomes

$$
f(x) = \frac{3}{4} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2}{\pi^2 n^2}\cos\left(\frac{2\pi}{L}nx\right)
$$

To verify the above result, the Fourier series approximation given above was plotted for increasing n against the original $f(x)$ function in order to see how the approximation improves as n increases. Using $L = 2$, the result is given below.

The original function is in the red color. The plot shows that the convergence is fast (due to the $\frac{1}{n^2}$ term). The convergence is uniform. After only 4 terms, the error between $f(x)$ and its Fourier series approximation becomes very small. As expected, the error is largest at the top and at the lower corners where the original function changes more rapidly and therefore more terms would be needed in those regions compared to the straight edges regions of the function $f(x)$ to get a better approximation.

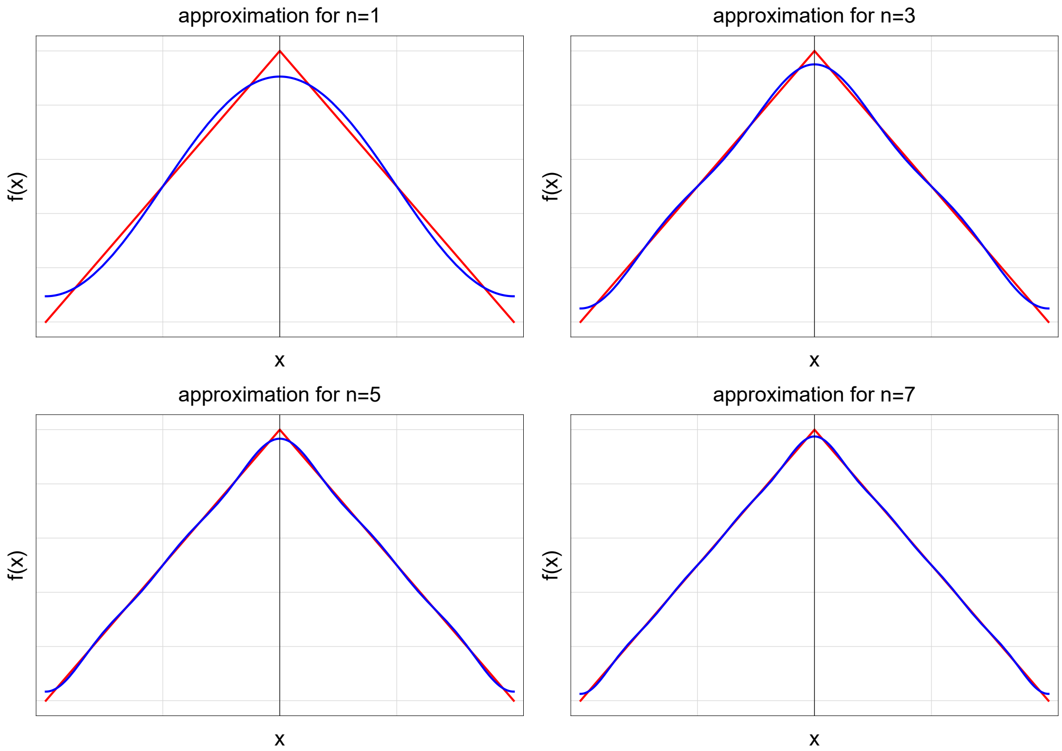


Figure 3.12: Fourier series approximation, part 1

```
ClearAll[L, x, n, a]
L = 2;
a[n_] := 2 / (Pi^2 n^2);

fApprox[x_, nTerms_] := 3 / 4 + Sum[a[n] Cos[2 Pi / L n x], {n, 1, nTerms, 2}]

p = Table[
   Plot[{f[x], fApprox[x, i]}, {x, -L / 2, L / 2},
    Frame → True,
    FrameLabel → {{"f(x)", None}, {"x", Row[{"approximation for n=", i}]}},
    GridLines → Automatic, GridLinesStyle → LightGray,
    PlotStyle → {Red, Blue},
    ImageSize → 400,
    BaseStyle → 16],
   {i, 1, 7, 2}
  ];

p = Grid[Partition[p, 2]]
Export["../images/p1_plot_2.pdf", p]
```

Figure 3.13: Code used

Part 2

The following is a plot of the function $f(x) = e^x$. In this plot, $L = 1$ was used.

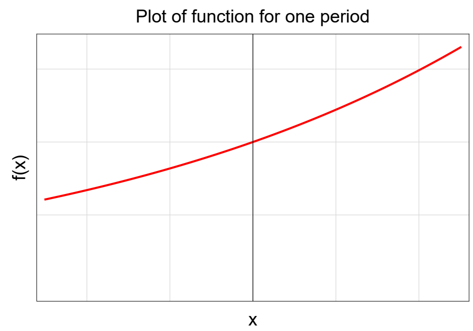


Figure 3.14: Function plot part 2

```
L = 1;
f[x_] := Exp[x];
p = Plot[f[x], {x, -L / 2, L / 2}, AxesOrigin → {0, 0},
  Frame → True,
  FrameLabel → {{"f(x)", None}, {"x", "Plot of function for one period"}},
  BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red]
Export["../images/p1_plot_3.pdf", p]
```

Figure 3.15: Code used

The Fourier series of $f(x) =$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L}nx\right) + b_n \sin\left(\frac{2\pi}{L}nx\right) \tag{1A}$$

Where L is the period and

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,dx \\ &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^x dx \\ &= \frac{2}{L}\left[e^x\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\ &= \frac{2}{L}\left[e^{\frac{L}{2}} - e^{-\frac{L}{2}}\right] \\ &= \frac{4}{L}\left[\frac{e^{\frac{L}{2}} - e^{-\frac{L}{2}}}{2}\right] \\ &= \frac{4}{L}\sinh\left(\frac{L}{2}\right) \end{aligned}$$

And

$$\begin{aligned} a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi}{L}nx\right)dx \\ &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^x \cos\left(\frac{2\pi}{L}nx\right)dx \end{aligned} \tag{1}$$

Integration by parts: Let $u = \cos\left(\frac{2\pi}{L}nx\right), du = -\frac{2\pi n}{L}\sin\left(\frac{2\pi}{L}nx\right)$ and let $dv = e^x, v = e^x$, therefore

$$
\begin{aligned}
I &= \int_{-\frac{L}{2}}^{\frac{L}{2}} e^x \cos\left(\frac{2\pi}{L}nx\right) dx \\
&= \left[e^x \cos\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} -\frac{2\pi n}{L} \sin\left(\frac{2\pi}{L}nx\right) e^x dx \\
&= \left[e^{\frac{L}{2}} \cos\left(\frac{2\pi}{L}n\frac{L}{2}\right) - e^{-\frac{L}{2}} \cos\left(\frac{2\pi}{L}n\left(-\frac{L}{2}\right)\right)\right] + \frac{2\pi n}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{2\pi}{L}nx\right) e^x dx \\
&= \left[e^{\frac{L}{2}} \cos(\pi n) - e^{-\frac{L}{2}} \cos(\pi n)\right] + \frac{2\pi n}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{2\pi}{L}nx\right) e^x dx \\
&= \cos(\pi n)\left(e^{\frac{L}{2}} - e^{-\frac{L}{2}}\right) + \frac{2\pi n}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{2\pi}{L}nx\right) e^x dx \\
&= 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) + \frac{2\pi n}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{2\pi}{L}nx\right) e^x dx
\end{aligned}
$$

Integration by parts again, let $u = \sin\left(\frac{2\pi}{L}nx\right), du = \frac{2\pi n}{L}\cos\left(\frac{2\pi}{L}nx\right)$ and $dv = e^x, v = e^x$. The above becomes

$$
I = 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) + \frac{2\pi n}{L}\left(\left[e^x \sin\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{2\pi n}{L} \cos\left(\frac{2\pi}{L}nx\right) e^x dx\right)
$$

The term $\left[e^x \sin\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}}$ goes to zero since it gives $\sin(n\pi)$ and n is integer. The above simplifies to

$$
\begin{aligned}
I &= 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) + \frac{2\pi n}{L}\left(-\frac{2\pi n}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{2\pi}{L}nx\right) e^x dx\right) \\
&= 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) - \frac{4\pi^2 n^2}{L^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{2\pi}{L}nx\right) e^x dx
\end{aligned}
$$

Since $\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{2\pi}{L}nx\right) e^x dx = I$ the above reduces to

$$
\begin{aligned}
I &= 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) - \frac{4\pi^2 n^2}{L^2} I \\
I\left(1 + \frac{4\pi^2 n^2}{L^2}\right) &= 2\cos(\pi n)\sinh\left(\frac{L}{2}\right) \\
I &= \frac{2\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{1 + \frac{4\pi^2 n^2}{L^2}}
\end{aligned}
$$

Using the above in (1) gives

$$
\begin{aligned}
a_n &= \frac{2}{L} \frac{2\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{1 + \frac{4\pi^2 n^2}{L^2}} \\
&= \frac{2L^2}{L} \frac{2\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{L^2 + 4\pi^2 n^2} \\
&= \frac{4L}{L^2 + 4\pi^2 n^2} \cos(\pi n)\sinh\left(\frac{L}{2}\right)
\end{aligned}
$$

Next, b_n is found:

$$\begin{aligned} b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi}{L}nx\right)dx \\ &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^x \sin\left(\frac{2\pi}{L}nx\right)dx \end{aligned} \tag{2}$$

Integration by parts: Let $u=\sin\left(\frac{2\pi}{L}nx\right), du=\frac{2\pi n}{L}\sin\left(\frac{2\pi}{L}nx\right)$ and let $dv=e^x, v=e^x$, therefore

$$\begin{aligned} I &= \int_{-\frac{L}{2}}^{\frac{L}{2}} e^x\sin\left(\frac{2\pi}{L}nx\right)dx \\ &= \left[e^x\sin\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{2\pi n}{L}\cos\left(\frac{2\pi}{L}nx\right)e^x dx \end{aligned}$$

But $\left[e^x\sin\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}}$ goes to zero as $\sin(\pi n)=0$ for integer n and the above simplifies to

$$I = -\frac{2\pi n}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\cos\left(\frac{2\pi}{L}nx\right)e^x dx$$

Integration by parts again: let $u=\cos\left(\frac{2\pi}{L}nx\right), du=-\frac{2\pi n}{L}\sin\left(\frac{2\pi}{L}nx\right)$ and $dv=e^x, v=e^x$. The above becomes

$$\begin{aligned} I &= -\frac{2\pi n}{L}\left(\left[e^x\cos\left(\frac{2\pi}{L}nx\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} -\frac{2\pi n}{L}\sin\left(\frac{2\pi}{L}nx\right)e^x dx\right) \\ &= -\frac{2\pi n}{L}\left(2\cos(\pi n)\sinh\left(\frac{L}{2}\right) + \frac{2\pi n}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\sin\left(\frac{2\pi}{L}nx\right)e^x dx\right) \end{aligned}$$

But $\int_{-\frac{L}{2}}^{\frac{L}{2}}\sin\left(\frac{2\pi}{L}nx\right)e^x dx = I$ and the above reduces to

$$\begin{aligned} I &= -\frac{2\pi n}{L}\left(2\cos(\pi n)\sinh\left(\frac{L}{2}\right)+\frac{2\pi n}{L}I\right) \\ I &= -\frac{4\pi n}{L}\cos(\pi n)\sinh\left(\frac{L}{2}\right) - \frac{4\pi^2 n^2}{L^2}I \\ I\left(1+\frac{4\pi^2n^2}{L^2}\right) &= -\frac{4\pi n}{L}\cos(\pi n)\sinh\left(\frac{L}{2}\right) \\ I &= \frac{-\frac{4\pi n}{L}\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{1+\frac{4\pi^2n^2}{L^2}} \\ &= \frac{-4\pi nL\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{L^2+4\pi^2n^2} \end{aligned}$$

Using the above in (2) gives

$$\begin{aligned} b_n &= \frac{2}{L}\frac{-4\pi nL\cos(\pi n)\sinh\left(\frac{L}{2}\right)}{L^2+4\pi^2n^2} \\ &= \frac{-8\pi n}{L^2+4\pi^2n^2}\cos(\pi n)\sinh\left(\frac{L}{2}\right) \end{aligned}$$

Therefore, from (1A) the Fourier series is

$$f(x) = \frac{2}{L}\sinh\left(\frac{L}{2}\right) + \sum_{n=1}^{\infty}\frac{4L}{L^2+4\pi^2n^2}\cos(\pi n)\sinh\left(\frac{L}{2}\right)\cos\left(\frac{2\pi}{L}nx\right) - \frac{8\pi n}{L^2+4\pi^2n^2}\cos(\pi n)\sinh\left(\frac{L}{2}\right)\sin\left(\frac{2\pi}{L}nx\right) \tag{3}$$

To verify the result, the above was plotted for increasing n against the original $f(x)$ function to see how the approximation improves as n increases. Using $L=2$, the result is displayed below. The original function is in the red color.

Compared to part (1), more terms are needed here to get good approximation. Since the original function is piecewise continuous when extending over multiple periods, the convergence is no longer a uniform convergence. At the point of discontinuity, the approximation converges to the average value of the original function at that point. At about 20 terms the approximation started to give good results. Due to Gibbs phenomena, at the points of discontinuities, the error is largest. Here is a plot showing one period

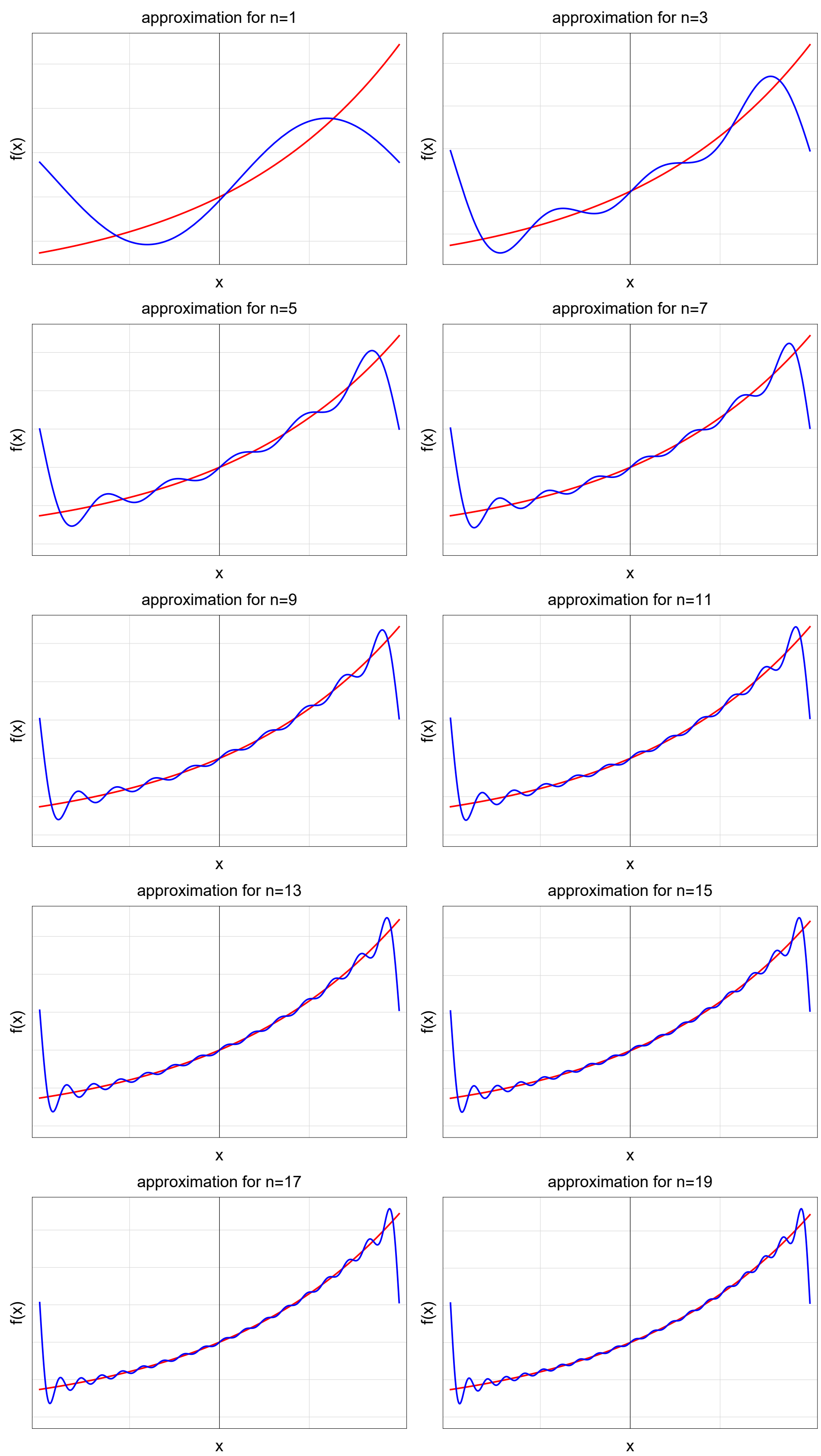


Figure 3.16: Fourier series approximation, showing one period

```
ClearAll[L, x, n, a]
L = 2;
f[x_] := Exp[x];
a[n_] := 4 L / (L^2 + 4 Pi^2 n^2) Cos[Pi n] Sinh[L / 2] ;
b[n_] := -8 Pi n / (L^2 + 4 Pi^2 n^2) Cos[Pi n] Sinh[L / 2] ;
fApprox[x_, nTerms_] := 2 / L Sinh[L / 2] + Sum[a[n] Cos[2 Pi / L n x] + b[n] Sin[2 Pi / L n x], {n, 1, nTerms, 1}]
p = Table[
   Plot[{f[x], fApprox[x, i]}, {x, -L / 2, L / 2},
    Frame → True,
    FrameLabel → {{"f(x)", None}, {"x", Row[{"approximation for n=", i}]}},
    GridLines → Automatic, GridLinesStyle → LightGray,
    PlotStyle → {Red, Blue}, ImageSize → 400, BaseStyle → 16],
   {i, 1, 20, 2}
  ];
p = Grid[Partition[p, 2]]
Export["../images/p1_plot_4.pdf", p]
```

Figure 3.17: Code used

In the following plot, 3 periods are shown to make it easier to see the effect of discontinuities and the Gibbs phenomena

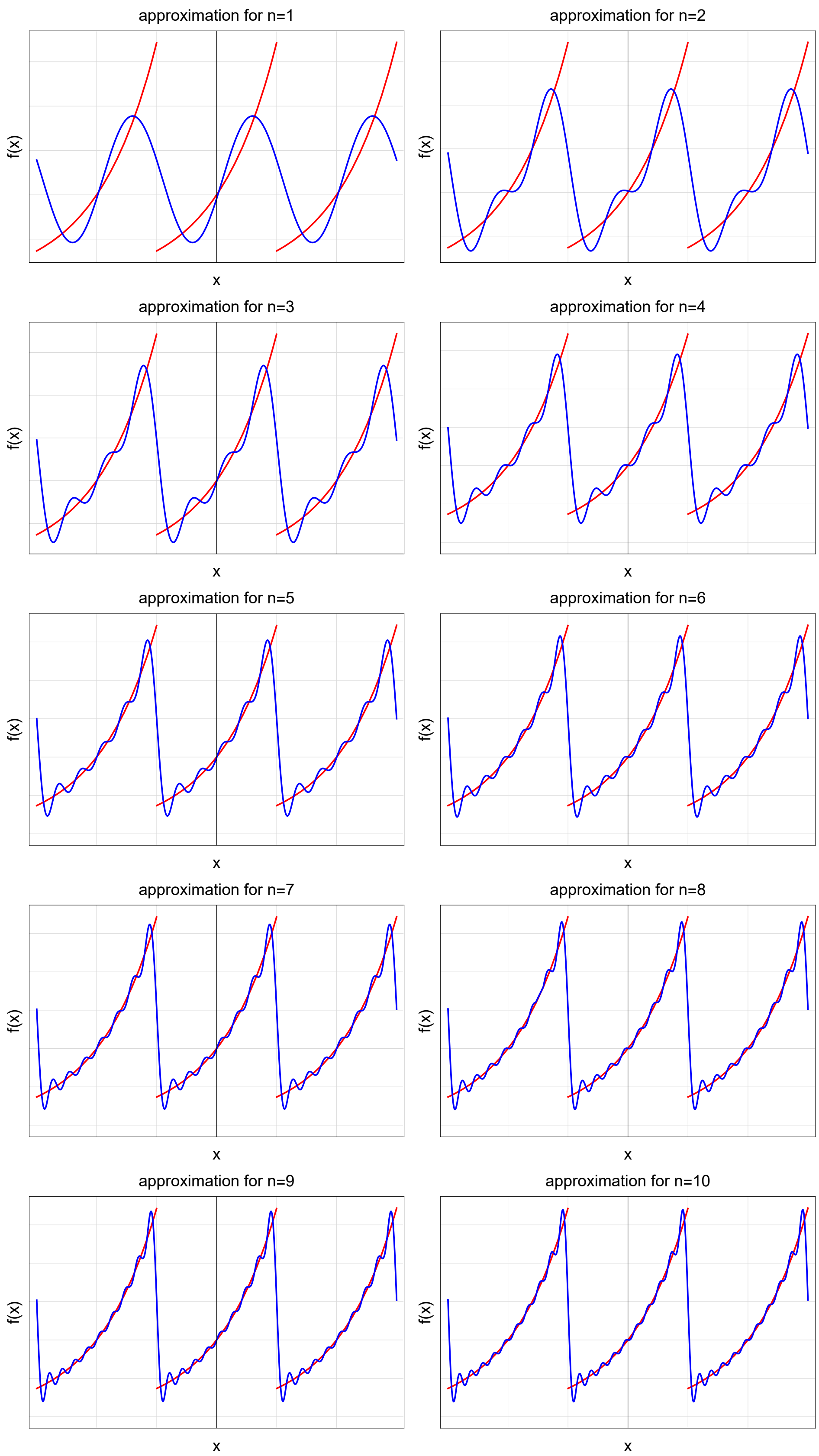


Figure 3.18: Fourier series approximation, showing 3 periods

```
ClearAll[L, x, n, a]
L = 2;
f[x_] := Piecewise[{
    {Exp[x + L], -3 / 2 L < x < -L / 2},
    {Exp[x], -L / 2 < x < L / 2},
    {Exp[x - L], L / 2 < x < 3 / 2 L}}
  ];
a[n_] := 4 L / (L^2 + 4 Pi^2 n^2) Cos[Pi n] Sinh[L / 2] ;
b[n_] := -8 Pi n / (L^2 + 4 Pi^2 n^2) Cos[Pi n] Sinh[L / 2] ;

fApprox[x_, nTerms_] := 2 / L Sinh[L / 2] + Sum[a[n] Cos[2 Pi / L n x] + b[n] Sin[2 Pi / L n x], {n, 1, nTerms, 1}];

p = Table[
   Plot[{f[x], fApprox[x, i]}, {x, -3 / 2 L, 3 / 2 L},
    Frame → True, FrameLabel → {{"f(x)", None}, {"x", Row[{"approximation for n=", i}]}},
    GridLines → Automatic, GridLinesStyle → LightGray,
    PlotStyle → {Red, Blue},
    ImageSize → 400, BaseStyle → 16],
   {i, 1, 10, 1}
  ];
p = Grid[Partition[p, 2]]
Export["../images/p1_plot_5.pdf", p]
```

Figure 3.19: Code used

3.5.2 Problem 2

Find the general solution of

1. $2x^3y' = 1 + \sqrt{1+4x^2y}$
2. $e^x \sin y - 2y \sin x + \left(y^2 + e^x \cos y + 2\cos y\right) y' = 0$
3. $y' + y\cos x = \frac{1}{2}\sin x$

Solution

part 1

This ODE is not separable and it is also not exact (It was checked for exactness and failed the test). The ODE is next checked to see if it is isobaric. An ODE $y' = f\left(x,y\right)$ is isobaric (which is a generalization of a homogeneous ODE) if the substitution

$$y\left(x\right) = v\left(x\right)x^m$$

Changes the ODE to be a separable one in $v\left(x\right)$. To determine if it isobaric, a weight m is assigned to y and to dy, and a weight of 1 is assigned to x and to dx, then if an m could be found such that each term in the ODE will have the same weight, then the ODE is isobaric and it can be made separable using the above substitution. Writing the above ODE as

$$2x^3dy = \left(1+\sqrt{1+4x^2y}\right)dx$$

$$\overbrace{2x^3dy} - \overbrace{dx} - \overbrace{\sqrt{1+4x^2y}dx} = 0$$

Adding the weights of the first term above gives $2x^3dy \to 3+m$. The next term weight is $dx \to 1$. The next term weight is $\sqrt{1+4x^2y}dx \to \frac{1}{2}\left(2+m\right)+1 = 2+\frac{m}{2}$. Therefore the weights of each term are

$$\{3+m, 1, 2+\frac{m}{2}\}$$

Each term weight can be made the same by selecting $\underline{m=-2}$. This value makes each term have weight 1 and the above becomes

$$\{1,1,1\}$$

Therefore the ODE is isobaric. Using this value of m the substitution $y = \frac{v}{x^2}$ is now used to make the original ODE separable

$$\frac{dy}{dx} = \frac{1}{x^2}\frac{dv}{dx} - 2\frac{v}{x^3}$$

The original ODE now becomes (where each y is replaced by $\frac{v}{x^2}$) separable as follows

$$
\begin{aligned}
2x^3\left(\frac{1}{x^2}\frac{dv}{dx}-2\frac{v}{x^3}\right) &= 1+\sqrt{1+4x^2\frac{v}{x^2}}\\
2x\frac{dv}{dx}-4v &= 1+\sqrt{1+4v}\\
2x\frac{dv}{dx} &= 1+\sqrt{1+4v}+4v
\end{aligned}
$$

Solving this ODE for $v(x)$

$$
\frac{dv}{1+\sqrt{1+4v}+4v}=\frac{1}{2x}dx
$$

Integrating both sides gives

$$
\int\frac{dv}{1+\sqrt{1+4v}+4v}=\frac{1}{2}\ln|x|+c \tag{2}
$$

The integral above is solved by substitution. Let $\sqrt{1+4v}=u$, hence $\frac{du}{dv}=\frac{1}{2}\frac{4}{\sqrt{1+4v}}=\frac{2}{u}$ or $dv=\frac{1}{2}udu$. Squaring both sides of $\sqrt{1+4v}=u$ (and assuming $1+4v>0$) gives $1+4v=u^2$ or $v=\frac{u^2-1}{4}$. Therefore the LHS integral in (2) becomes

$$
\begin{aligned}
\int\frac{1}{1+\sqrt{1+4v}+4v}dv &= \frac{1}{2}\int\frac{u}{1+u+4\left(\frac{u^2-1}{4}\right)}du\\
&= \frac{1}{2}\int\frac{u}{u+u^2}du\\
&= \frac{1}{2}\int\frac{1}{1+u}du\\
&= \frac{1}{2}\ln|1+u|
\end{aligned}
$$

Using this result in (2) gives the following (the absolute values are removed because the constant of integration absorbs the sign).

$$
\begin{aligned}
\frac{1}{2}\ln(1+u) &= \frac{1}{2}\ln x+c\\
\ln(1+u) &= \ln x+2c
\end{aligned}
$$

Let $2c=C_0$ be a new constant. The above becomes

$$
\begin{aligned}
\ln(1+u) &= \ln x+C_0\\
e^{\ln(1+u)} &= e^{\ln x+C_0}\\
1+u &= e^{C_0}x\\
1+u &= Cx
\end{aligned}
$$

Where $C=e^{C_0}$ is a new constant. Therefore the solution is

$$
u(x)=Cx-1
$$

Since $u(x)=\sqrt{1+4v}$ then the above becomes

$$
\begin{aligned}
\sqrt{1+4v} &= Cx-1\\
1+4v &= (Cx-1)^2\\
v(x) &= \frac{(Cx-1)^2-1}{4}
\end{aligned}
$$

But $y=\frac{v}{x^2}$ therefore the above gives the final solution as

$$
y(x)=\frac{(Cx-1)^2-1}{4x^2}
$$

Where C is the constant of integration.

Part 2

$$
e^x\sin y-2y\sin x+\left(y^2+e^x\cos y+2\cos x\right)y'=0
$$

The first step is to write the ODE in standard form to check if it is an exact ODE

$$M(x,y)dx + N(x,y)dy = 0$$

Hence

$$\begin{aligned}M(x,y) &= e^x \sin y - 2y \sin x\\ N(x,y) &= y^2 + e^x \cos y + 2 \cos x\end{aligned}$$

Next, the ODE is determined if it is exact or not. The ODE is exact if the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Applying the above on the given ODE results in

$$\begin{aligned}\frac{\partial M}{\partial y} &= e^x \cos y - 2 \sin x\\ \frac{\partial N}{\partial x} &= e^x \cos y - 2 \sin x\end{aligned}$$

Because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are used to solve for the function $\phi\left(x,y\right)$

$$\frac{\partial \phi}{\partial x} = M = e^x \sin y - 2y \sin x \tag{3}$$

$$\frac{\partial \phi}{\partial y} = N = y^2 + e^x \cos y + 2 \cos x \tag{4}$$

Integrating (3) w.r.t x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int e^x \sin y - 2y \sin x dx$$

$$\phi\left(x,y\right) = e^x \sin y + 2y \cos x + f(y) \tag{5}$$

Where $f(y)$ is used as the constant of integration because $\phi\left(x,y\right)$ is a function of both x and y. Taking derivative of (5) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^x \cos y + 2 \cos x + f'(y) \tag{6}$$

But (4) says that $\frac{\partial \phi}{\partial y} = y^2 + e^x \cos y + 2 \cos x$. Therefore by equating (4) and (6) then $f'\left(y\right)$ can be solved for:

$$y^2 + e^x \cos y + 2 \cos x = e^x \cos y + 2 \cos x + f'(y) \tag{7}$$

Solving the above for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating w.r.t y gives $f\left(y\right)$

$$\begin{aligned}\int f' dy &= \int y^2 dy\\ f(y) &= \frac{1}{3}y^3 + C_1\end{aligned}$$

Where C_1 is constant of integration. Substituting the value of $f(y)$ back into (5) gives $\phi\left(x,y\right)$

$$\phi = e^x \sin y + 2y \cos x + \frac{1}{3}y^3 + C_1$$

But since ϕ itself is a constant function, say $\phi = C_0$ where C_0 is new constant, then by combining C_1 and C_0 constants into a new constant C_1, the above gives the solution

$$C_1 = e^x \sin y(x) + 2y(x) \cos x + \frac{1}{3}y^3(x)$$

The above is left in implicit form for simplicity.

Part 3

$$y' + y \cos x = \frac{1}{2}\sin(2x)$$

This ODE is linear in y. It is solved using an integrating factor $\mu = e^{\int \cos x dx} = e^{\sin x}$. Multiplying both sides of the ODE by μ makes the left side an exact differential

$$d\left(y\mu\right) = \frac{1}{2}\mu \sin\left(2\right) x dx$$

Integrating both sides gives

$$\begin{aligned} y\mu &= \frac{1}{2}\int \mu \sin\left(2x\right) dx + C \\ ye^{\sin x} &= \frac{1}{2}\int e^{\sin x} \sin\left(2x\right) dx + C \end{aligned} \tag{1}$$

The above integral can be solved as follows. Since $\sin\left(2x\right) = 2\sin x \cos x$ therefore then

$$I = \frac{1}{2}\int e^{\sin x} \sin\left(2x\right) dx = \int e^{\sin x} \sin x \cos x dx$$

Using the substitution $z = \sin x$, then $dz = dx \cos x$ and the above becomes

$$I = \int e^z z dz$$

Integrating the above by parts: $\int u dv = uv - \int v du$. Let $u = z, dv = e^z \to du = 1, v = e^z$, and the above becomes

$$\begin{aligned} I &= ze^z - \int e^z dz \\ &= ze^z - e^z \\ &= e^z\left(z - 1\right) \end{aligned}$$

Since $z = \sin x$ the above reduces to

$$I = e^{\sin x}\left(\sin\left(x\right) - 1\right)$$

Substituting this back in (1) results in

$$ye^{\sin x} = e^{\sin x}\left(\sin\left(x\right) - 1\right) + C$$

Therefore the final solution is

$$y\left(x\right) = \sin\left(x\right) - 1 + Ce^{-\sin x}$$

Where C is the constant of integration.

3.5.3 Problem 3

Find general solution of

1. $y''' - 4y'' - 4y' + 16 = 8\sin x$
2. $a^2 y'^2 = \left(1 + y'^2\right)^3$

Solution

Part 1

$$y''' - 4y'' - 4y' = 8\sin x - 16$$

This is linear nonhomogeneous ODE with constant coefficients. Solving first the homogeneous ODE $y''' - 4y'' - 4y' = 0$. Since the term y is missing from the ODE then the substitution $y' = u$ reduces the ODE to a second order ODE

$$u'' - 4u' - 4u = 0 \tag{1}$$

Let $u = e^{\lambda x}$. Substituting this into the above and simplifying gives the characteristic equation

$$\lambda^2 - 4\lambda - 4 = 0$$

The Roots are $\lambda = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ or

$$\begin{aligned}
\lambda &= \frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 4(-4)} \\
&= 2 \pm \frac{1}{2}\sqrt{32} \\
&= 2 \pm 2\sqrt{2} \\
&= 2\left(1 \pm \sqrt{2}\right)
\end{aligned}$$

Hence the solution to (1) is given by linear combinations of $e^{\lambda_1 x}, e^{\lambda_2 x}$ as

$$u_h(x) = c_1 e^{2\left(1+\sqrt{2}\right)x} + c_2 e^{2\left(1-\sqrt{2}\right)x}$$

But since $y' = u$, then y is found by integrating the above

$$\begin{aligned}
y_h &= \int c_1 e^{2\left(1+\sqrt{2}\right)x} + c_2 e^{2\left(1-\sqrt{2}\right)x} dx \\
&= c_1 \frac{e^{2\left(1+\sqrt{2}\right)x}}{2\left(1+\sqrt{2}\right)} + c_2 \frac{e^{\left(2-2\sqrt{2}\right)x}}{2\left(1-\sqrt{2}\right)} + C_3
\end{aligned}$$

To simplify the above, let $\frac{c_1}{2\left(1+\sqrt{2}\right)} = C_1, \frac{c_2}{2\left(1-\sqrt{2}\right)} = C_2$, where C_1, C_2 are new constants. The above simplifies to

$$y_h = C_1 e^{2\left(1+\sqrt{2}\right)x} + C_2 e^{\left(2-2\sqrt{2}\right)x} + C_3$$

The above solution is homogeneous solution to the original ODE. Next, the particular solution is found. Since the RHS of the original ODE is $\sin x - 16$ then choosing y_p to have the form

$$y_p = A\sin x + B\cos x + kx$$

Therefore

$$\begin{aligned}
y_p' &= k + A\cos x - B\sin x \\
y_p'' &= -A\sin x - B\cos x \\
y_p''' &= -A\cos x + B\sin x
\end{aligned}$$

Substituting these back into the original ODE $y''' - 4y'' - 4y' = 8\sin x - 16$ gives

$$\begin{aligned}
(-A\cos x + B\sin x) - 4(-A\sin x - B\cos x) - 4(k + A\cos x - B\sin x) &= 8\sin x - 16 \\
-A\cos x + B\sin x + 4A\sin x + 4B\cos x - 4A\cos x + 4B\sin x - 4k &= 8\sin x - 16 \\
\cos x(-A + 4B - 4A) + \sin x(B + 4A + 4B) - 4k &= 8\sin x - 16 \\
\cos x(-5A + 4B) + \sin x(5B + 4A) - 4k &= 8\sin x - 16
\end{aligned}$$

Comparing coefficients gives the following equations to solve for the unknowns A, B, k

$$\begin{aligned}
-4k &= -16 \\
-5A + 4B &= 0 \\
5B + 4A &= 8
\end{aligned}$$

The second equation gives $B = \frac{5}{4}A$. Using this in the third equation gives $5\left(\frac{5}{4}A\right) + 4A = 8$, solving gives $A = \frac{32}{41}$. Hence $B = \frac{5}{4}\left(\frac{32}{41}\right) = \frac{40}{41}$. The first equation gives $k = 4$. Therefore the particular solution is

$$\begin{aligned}
y_p &= A\sin x + B\cos x + kx \\
&= \frac{32}{41}\sin x + \frac{40}{41}\cos x + 4x
\end{aligned}$$

Now that y_h and y_p are found, the general solution is found as

$$\begin{aligned}
y &= y_h + y_p \\
&= C_1 e^{2\left(1+\sqrt{2}\right)x} + C_2 e^{\left(2-2\sqrt{2}\right)x} + C_3 + \frac{32}{41}\sin x + \frac{40}{41}\cos x + 4x
\end{aligned}$$

Where C_1, C_2 are the two constants of integration.

Part 2

$$a^2y'^2 = \left(1+y'^2\right)^3$$

Let $y' = A$, the above becomes

$$\begin{aligned} a^2A^2 &= \left(1+A^2\right)^3 \\ &= 1 + 3A^2 + \frac{(3)(2)}{2!}A^4 + \frac{(3)(2)(1)}{3!}A^6 \\ &= 1 + 3A^2 + 3A^4 + A^6 \end{aligned}$$

Hence the polynomial is

$$A^6 + 3A^4 + A^2\left(3-a^2\right) + 1 = 0$$

Let $A^2 = B$ and the above becomes

$$B^3 + 3B^2 + B\left(3-a^2\right) + 1 = 0$$

With the help of the computer, the cubic roots of the above are

$$B_1 = \sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2} + \frac{1}{3}\frac{a^2}{\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2}} - 1$$

$$B_2 = \frac{1}{2}i\sqrt{3}\left(\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2} - \frac{1}{3}\frac{a^2}{\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2}}\right) - \frac{1}{2}\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2} - \frac{1}{6}\frac{a^2}{\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2}} - 1$$

$$B_3 = -\frac{1}{2}\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2} - \frac{1}{2}i\sqrt{3}\left(\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2} - \frac{1}{3}\frac{a^2}{\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2}}\right) - \frac{1}{6}\frac{a^2}{\sqrt[3]{\sqrt{\frac{1}{4}a^4 - \frac{1}{27}a^6} - \frac{1}{2}a^2}} - 1$$

Therefore $A_1 = \pm\sqrt{B_1}, A_2 = \pm\sqrt{B_2}, A_3 = \pm\sqrt{B_3}$ or, since $y'(x) = A$, then there are 6 solutions, each is a solution for one root.

$$\begin{aligned} \frac{dy_1}{dx} &= +\sqrt{B_1} \\ \frac{dy_2}{dx} &= -\sqrt{B_1} \\ \frac{dy_3}{dx} &= +\sqrt{B_2} \\ \frac{dy_4}{dx} &= -\sqrt{B_2} \\ \frac{dy_5}{dx} &= +\sqrt{B_3} \\ \frac{dy_6}{dx} &= -\sqrt{B_3} \end{aligned}$$

But the roots $\pm B_i$ are constants. Therefore each of the above can be solved by direct integration. The final solution which gives the solutions

$$\begin{aligned} y_1 &= \sqrt{B_1}x + C_1 \\ y_2 &= -\sqrt{B_1}x + C_2 \\ y_3 &= \sqrt{B_2}x + C_3 \\ y_4 &= -\sqrt{B_2}x + C_4 \\ y_5 &= \sqrt{B_3}x + C_5 \\ y_6 &= -\sqrt{B_3}x + C_6 \end{aligned}$$

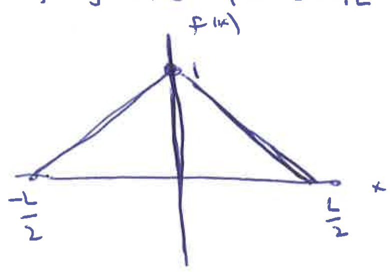
Where the constants B_i are given above.

3.5.4 key solution to HW 5

Problem Set 5 Solutions

Exercise 1:

i) $f(x) = 1 - 2|x|/L$ for $-L/2 \le x \le L/2$



For a Fourier series

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\,dx$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n x}{L}\right)dx$$

• By symmetry $B_n = 0$

$$A_0 = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)dx = \frac{4}{L}\left[x - \frac{x^2}{L}\right]\Big|_0^{L/2} = 1$$

$$A_n = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)\cos\left(\frac{2n\pi x}{L}\right)dx$$

$$= \frac{4}{L}\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Big|_0^{L/2} - \frac{8}{L^2}\left(\frac{L}{2n\pi}\right)^2\left[\cos\left(\frac{2\pi n x}{L}\right) + \left(\frac{2\pi n x}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\right]\Big|_0^{L/2}$$

$$= -\frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \Rightarrow A_n = \frac{4}{\pi^2 n^2} \text{ n odd}, \quad A_n = 0 \text{ , n even}$$

$$\therefore\ f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2\pi n x}{L}\right)$$

b) $f(x) = e^x$ for $-L/2 \le x \le L/2$

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} e^x\,dx = \frac{2}{L} e^x \Big|_{-L/2}^{L/2} = \frac{2}{L}\left(e^{L/2} - e^{-L/2}\right) = 2\,\frac{\sinh(L/2)}{L/2}$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} e^x \cos\left(\frac{2n\pi x}{L}\right)dx = \frac{2}{L}\left[e^x\,\frac{\cos\left(\frac{2n\pi x}{L}\right) + \left(\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi x}{L}\right)}{1+\left(\frac{2n\pi}{L}\right)^2}\right]_{-L/2}^{L/2}$$

$$= 2\,\frac{\sinh(L/2)}{L/2}\,\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} e^x \sin\left(\frac{2n\pi x}{L}\right)dx = \frac{2}{L}\left[e^x\,\frac{\sin\left(\frac{2n\pi x}{L}\right) - \left(\frac{2n\pi}{L}\right)\cos\left(\frac{2n\pi x}{L}\right)}{1+\left(\frac{2n\pi}{L}\right)^2}\right]_{-L/2}^{L/2}$$

$$= -2\,\frac{\sinh(L/2)}{L/2}\,\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}\left(\frac{2n\pi}{L}\right)$$

$$\therefore\ f(x) = \frac{\sinh(L/2)}{L/2}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}\left[\cos\left(\frac{2n\pi x}{L}\right) - \left(\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi x}{L}\right)\right]\right\}$$

Exercise 2 :

i) $2x^3y' = 1 + \sqrt{1+4x^2y}$, the form of this equation suggests x^2y is dimensionless

$\Rightarrow$ let $v = x^2y$ or $y = \frac{v}{x^2}$

$\Rightarrow$ $y' = \frac{v'}{x^2} - \frac{2v}{x^3}$

$\Rightarrow$ $2x\,v' - 4v = 1 + \sqrt{1+4v}$

$\Rightarrow$ $2x\,dv = (1 + 4v + \sqrt{1+4v})\,dx$ ~ seperable

$\frac{dx}{x} = \frac{2dv}{1+4v+\sqrt{1+4v}}$ let $u^2 = 1+4v$

$2u\,du = 4dv$

$\Rightarrow 2dv = u\,du$

$\therefore$ $\frac{dx}{x} = \frac{u\,du}{u^2+u} = \frac{du}{u+1}$

$\Rightarrow$ $\ln x + c = \ln(u+1)$

$\Rightarrow$ $u = 2Ax - 1$ where $2A = e^c$

$\therefore$ $1+4v = (2Ax-1)^2 = 4A^2x^2 - 4Ax + 1$

$\Rightarrow$ $v = A^2x^2 - Ax$

$\Rightarrow$ $\boxed{y = \frac{v}{x^2} = A^2 - \frac{A}{x}}$

ii) $e^x \sin y - 2y\sin x + (y^2 + e^x\cos y + 2\cos x)\,y' = 0$

or $dx\,(e^x \sin y - 2y\sin x) + dy\,(y^2 + e^x\cos y + 2\cos x) = 0$

check if this is exact : $df = A\,dx + B\,dy = 0$

then $\dfrac{\partial A}{\partial y} \overset{?}{=} \dfrac{\partial B}{\partial x}$ or $\dfrac{\partial}{\partial y}\left(e^x\sin y - 2y\sin x\right) \overset{?}{=} \dfrac{\partial}{\partial x}\left(y^2 + e^x\cos y + 2\cos x\right)$

$\Rightarrow\ e^x\cos y - 2\sin x \overset{?}{=} e^x\cos y - 2\sin x$ ✓

$\Rightarrow\ \dfrac{\partial f}{\partial x} = e^x\sin y - 2y\sin x \qquad \dfrac{\partial f}{\partial y} = y^2 + e^x\cos y + 2\cos x$

$\Rightarrow\ f = e^x\sin y + 2y\cos x + g(y)$

and $\dfrac{\partial f}{\partial y} = e^x\cos y + 2\cos x + \dfrac{dg}{dy} = y^2 + e^x\cos y + 2\cos x$

$\Rightarrow\ \dfrac{dg}{dy} = y^2 \Rightarrow\ g = \dfrac{y^3}{3} + \text{cnst}$

$$\Rightarrow\ \boxed{f = e^x\sin y + 2y\cos x + \frac{y^3}{3} + \text{cnst} = 0}$$

iii) $y' + y\cos x = \frac{1}{2}\sin 2x$

For this problem, let us introduce an integrating factor $\lambda(x) = \exp\int^x \cos x'\,dx' = e^{\sin x}$

then $e^{\sin x}\left[y' + y\cos x\right] = e^{\sin x}\sin x\cos x$

$\dfrac{d}{dx}\left(y\,e^{\sin x}\right) = e^{\sin x}\sin x\cos x$

$$\Rightarrow \quad y e^{\sin x} = \int^{x} e^{\sin x'} \sin x' \cos x' \, dx'$$

let $\quad u = \sin x' \quad , \quad du = \cos x' \, dx'$

$$\therefore \quad y e^{\sin x} = \int^{\sin x} u e^{u} \, du = \left[u e^{u} - e^{u} + c \right]^{\sin x}$$

or $\quad y e^{\sin x} = \sin x \, e^{\sin x} - e^{\sin x} + c$

$$\therefore \quad \boxed{y = \sin x - 1 + c e^{-\sin x}}$$

Exercise 3

i) $y''' - 4y'' - 4y' + 16 = \varepsilon \sin x$

- let $p = y'$

$\Rightarrow \quad p'' - 4p' - 4(p-4) = \varepsilon \sin x$

- let $\ell = p - 4$ $\quad$ (so $y' = \ell + 4$)

$\Rightarrow \quad \ell'' - 4\ell' - 4\ell = \varepsilon \sin x$

solve the homogeneous equation $\ell'' - 4\ell' - 4\ell = 0$

- try $\ell = e^{mx}$ $\Rightarrow$ $m^2 - 4m - 4 = 0$

$\Rightarrow \quad m_\pm = \dfrac{4 \pm \sqrt{16+16}}{2} = 2(1 \pm \sqrt{2})$

$\Rightarrow \quad \ell = a e^{m_+ x} + b e^{m_- x}$

- Find a particular solution

$\ell_p'' - 4\ell_p' - 4\ell_p = \varepsilon \sin x$

try $\ell_p = \alpha \cos x + \beta \sin x$

$\Rightarrow \; -(\alpha \cos x + \beta \sin x) - 4(-\alpha \sin x + \beta \cos x) - 4(\alpha \cos x + \beta \sin x) = \varepsilon \sin x$

$\Rightarrow \quad -\alpha - 4\beta - 4\alpha = 0 \qquad (\cos x)$

$\quad -\beta + 4\alpha - 4\beta = \varepsilon$

$\Rightarrow \quad \alpha = \dfrac{32}{41}$ and $\beta = -\dfrac{40}{41}$

$\Rightarrow \quad \ell_p = \dfrac{32}{41} \cos x - \dfrac{40}{41} \sin x$

$\Rightarrow \ell = \cancel{a e^{m_+ x}}\, a e^{m_+ x} + b e^{m_- x} + \dfrac{32}{41} \cos x - \dfrac{40}{41} \sin x \quad ; \quad y' = \ell + 4$

$\Rightarrow \quad \boxed{y = \text{const} + 4x + \dfrac{a}{m_+} e^{m_+ x} + \dfrac{b}{m_-} e^{m_- x} + \dfrac{32}{41} \sin x + \dfrac{40}{41} \cos x}$

iii) $a^2 y'^2 = (1+y'^2)^3$

$\Rightarrow \quad y'^6 + 3y'^4 + 3y'^2 + 1 = a^2 y'^2$

or $\quad y'^6 + 3y'^4 + (3-a^2)y'^2 + 1 = 0$

this is a polynomial in y'^2, for which we can find three roots u_i:

$$u_1 = \frac{P}{6} + \frac{2a^2}{P} - 1$$

$$u_2 = -\frac{P}{12} - \frac{a^2}{P} - 1 + \frac{\sqrt{3}I}{2}\left(\frac{P}{6} - \frac{2a^2}{P}\right)$$

$$u_3 = -\frac{P}{12} - \frac{a^2}{P} - 1 - \frac{\sqrt{3}I}{2}\left(\frac{P}{6} - \frac{2a^2}{P}\right)$$

where $P = \left[a^2\left(-108 + 12\sqrt{81 - 12a^2}\right)\right]^{1/3}$

$\Rightarrow \quad y'^2 = u_i \quad$ or $\quad y' = \pm\sqrt{u_i}$

$\Rightarrow \quad \boxed{y = \pm\sqrt{u_i}\, x + c}$ is the solution

3.6 HW 6

3.6.1 Problem 1

Consider the equation $xy'' + (c-x)y' - ay = 0$. Identify a regular singular point and find two series solutions around this point. Test the solutions for convergence.

Solution

Writing the ODE as

$$y'' + A(x)y' + B(x)y = 0$$

Where

$$A(x) = \frac{(c-x)}{x}$$
$$B(x) = \frac{-a}{x}$$

The above shows that $x_0 = 0$ is a singularity point for both $A(x)$ and $B(x)$. Examining $A(x)$ and $B(x)$ to determine what type of singular point it is

$$\lim_{x\to x_0}(x-x_0)A(x) = \lim_{x\to 0} x\frac{(c-x)}{x} = \lim_{x\to 0}(c-x) = c$$

Because the limit exists, then $x_0 = 0$ is regular singular point for $A(x)$.

$$\lim_{x\to x_0}(x-x_0)^2 B(x) = \lim_{x\to 0} x^2\left(\frac{-a}{x}\right) = \lim_{x\to 0}(-ax) = 0$$

Because the limit exists, then $x_0 = 0$ is also regular singular point for $B(x)$.

Therefore $\underline{x_0 = 0 \text{ is a regular singular point}}$ for the ODE.

Assuming the solution is Frobenius series gives

$$\begin{aligned} y(x) &= x^r \sum_{n=0}^{\infty} C_n (x-x_0)^n \qquad C_0 \neq 0 \\ &= x^r \sum_{n=0}^{\infty} C_n x^n \\ &= \sum_{n=0}^{\infty} C_n x^{n+r} \end{aligned}$$

Therefore

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} \end{aligned}$$

Substituting the above in the original ODE $xy'' + (c-x)y' - ay = 0$ gives

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} + (c-x)\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - a\sum_{n=0}^{\infty}C_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + c\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - x\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty}aC_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty}c(n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)C_n x^{n+r} - \sum_{n=0}^{\infty}aC_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty}((n+r)(n+r-1) + c(n+r))C_n x^{n+r-1} - \sum_{n=0}^{\infty}((n+r)+a)C_n x^{n+r} = 0$$

Since all powers of x have to be the same, adjusting indices and exponents gives (where in the second sum above, the outside index n is increased by 1 and n inside the sum is decreased by 1)

$$\sum_{n=0}^{\infty}((n+r)(n+r-1) + c(n+r))C_n x^{n+r-1} - \sum_{n=1}^{\infty}((n-1+r)+a)C_{n-1}x^{n+r-1} = 0 \tag{1}$$

Setting $n = 0$ gives the indicial equation, which only comes from the first sum above as the second sum starts from $n = 1$.

$$((r)(r-1) + cr)C_0 = 0$$

Since $C_0 \neq 0$ then

$$\begin{aligned} (r)(r-1) + cr &= 0 \\ r^2 - r + cr &= 0 \\ r(r+c-1) &= 0 \end{aligned}$$

The roots are

$$
\begin{aligned}
r_1 &= 1 - c \\
r_2 &= 0
\end{aligned}
$$

Assuming that $r_2 - r_1$ is not an integer, in other words, assuming $1 - c$ is not an integer (problem did not say), then In this case, two linearly independent solutions can be constructed directly. The first is associated with $r_1 = 1 - c$ and the second is associated with $r_2 = 0$. These solutions are

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} C_n x^{n+1-c} \qquad C_0 \neq 0 \\
y_2(x) &= \sum_{n=0}^{\infty} D_n x^{n} \qquad D_0 \neq 0
\end{aligned}
$$

The coefficients are not the same in each solution. For the first one C_n is used and for the second D_n is used.

The solution $y_1(x)$ associated with $r_1 = 1 - c$ is now found. From (1), and replacing r by $1 - c$ gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} ((n+1-c)(n+1-c-1) + c(n+1-c)) C_n x^{n+1-c-1} - \sum_{n=1}^{\infty} ((n-1+1-c) + a) C_{n-1} x^{n+1-c-1} &= 0 \\
\sum_{n=0}^{\infty} ((n+1-c)(n-c) + c(n+1-c)) C_n x^{n-c} - \sum_{n=1}^{\infty} ((n-c) + a) C_{n-1} x^{n-c} &= 0 \\
\sum_{n=0}^{\infty} n(n-c+1) C_n x^{n-c} - \sum_{n=1}^{\infty} ((n-c) + a) C_{n-1} x^{n-c} &= 0
\end{aligned}
$$

For $n > 0$ the above gives the recursive relation ($n = 0$ is not used, since it was used to find r). For $n > 0$ the last equation above gives

$$
\begin{aligned}
n(n-c+1) C_n - ((n-c) + a) C_{n-1} &= 0 \\
C_n &= \frac{((n-c)+a)}{n(n-c+1)} C_{n-1}
\end{aligned}
$$

Few terms are generated to see the pattern. For $n = 1$

$$
C_1 = \frac{(1-c+a)}{1(1-c+1)} C_0 = \frac{(1-c+a)}{(2-c)} C_0
$$

For $n = 2$

$$
\begin{aligned}
C_2 &= \frac{(2-c+a)}{2(2-c+1)} C_1 \\
&= \frac{(2-c+a)}{2(3-c)} \frac{(1-c+a)}{(2-c)} C_0
\end{aligned}
$$

For $n = 3$

$$
\begin{aligned}
C_3 &= \frac{(3-c+a)}{3(3-c+1)} C_2 \\
&= \frac{(3-c+a)}{3(4-c)} \frac{(2-c+a)}{2(3-c)} \frac{(1-c+a)}{(2-c)} C_0
\end{aligned}
$$

And so on. The pattern for general term is

$$
\begin{aligned}
C_n &= \frac{((n-c)+a)}{n(n-c+1)} \cdots \frac{(3-c+a)}{3(3-c+1)} \frac{(2-c+a)}{2(2-c+1)} \frac{(1-c+a)}{1(1-c+1)} C_0 \\
&= \prod_{m=1}^{n} \frac{((m-c)+a)}{m(n-c+1)}
\end{aligned}
$$

Therefore the solution associated with $r_1 = 1 - c$ is

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} C_n x^{n+r} \\
&= \sum_{n=0}^{\infty} C_n x^{n+1-c} \\
&= C_0 x^{1-c} + C_1 x^{2-c} + C_2 x^{3-c} + \cdots
\end{aligned}
$$

Using results found above, and looking at few terms gives the <u>first solution</u> as

$$y_1(x) = C_0 x^{1-c}\left(1+\frac{(1-c+a)}{(2-c)}x+\frac{1}{2}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^2+\frac{1}{6}\frac{(3-c+a)}{(4-c)}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^3+\cdots\right)$$

The second solution associated with $r_2 = 0$ is now found. As above, using (1) but with D_n instead of C_n for coefficients and replacing r by zero gives

$$\sum_{n=0}^{\infty}(n(n-1)+cn)D_n x^{n-1} - \sum_{n=1}^{\infty}((n-1)+a)D_{n-1}x^{n-1} = 0$$

For $n > 0$ the above gives the recursive relation for the second solution

$$\begin{aligned}(n(n-1)+cn)D_n - ((n-1)+a)D_{n-1} &= 0\\ D_n &= \frac{n-1+a}{n(n-1)+cn}D_{n-1}\\ &= \frac{n-1+a}{cn-n+n^2}D_{n-1}\end{aligned}$$

Few terms are now generated to see the pattern. For $n = 1$

$$D_1 = \frac{a}{c}D_0$$

For $n = 2$

$$\begin{aligned}D_2 &= \frac{1+a}{2c-2+4}D_1\\ &= \frac{1+a}{2(c+1)}\frac{a}{c}D_0\end{aligned}$$

For $n = 3$

$$\begin{aligned}D_3 &= \frac{3-1+a}{3c-3+9}D_2\\ &= \frac{2+a}{3(c+2)}\frac{1+a}{2(c+1)}\frac{a}{c}D_0\end{aligned}$$

And so on. Hence the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty}D_n x^n\\ &= D_0 + D_1 x + D_2 x^2 + \cdots\end{aligned}$$

Using result found above gives the <u>second solution</u> as

$$y_2(x) = D_0\left(1+\frac{a}{c}x+\frac{1}{2}\frac{(1+a)a}{c(c+1)}x^2+\frac{1}{6}\frac{a(1+a)(2+a)}{c(c+2)(c+1)}x^3+\cdots\right)$$

The final solution is therefore the sum of the two solutions

$$\begin{aligned}y(x) = C_0 x^{1-c}&\left(1+\frac{(1-c+a)}{(2-c)}x+\frac{1}{2}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^2+\frac{1}{6}\frac{(3-c+a)}{(4-c)}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^3+\cdots\right)\\ &+ D_0\left(1+\frac{a}{c}x+\frac{1}{2}\frac{(1+a)a}{c(c+1)}x^2+\frac{1}{6}\frac{a(1+a)(2+a)}{c(c+2)(c+1)}x^3+\cdots\right)\end{aligned}\tag{2}$$

Where C_0, D_0 are the two constant of integration.

<u>Testing for convergence</u>. For $y_1(x)$ solution, the general term from above was

$$C_n x^n = \frac{((n-c)+a)}{n(n-c+1)}C_{n-1}x^n$$

Hence by ratio test

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{C_n x^n}{C_{n-1}x^{n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{((n-c)+a)}{n(n-c+1)}C_{n-1}x^n}{C_{n-1}x^{n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{((n-c)+a)\,x}{(n\,(n-c+1))}\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{n-c+a}{n^2-nc+n}\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{\frac{1}{n}-\frac{c}{n^2}+\frac{a}{n^2}}{1-\frac{c}{n}+\frac{1}{n}}\right| \\
&= |x|\left|\frac{0}{1}\right| \\
&= 0
\end{aligned}
$$

Therefore the series $y_1(x)$ converges for all x.

Testing for convergence. For $y_2(x)$ solution, the general term is

$$
D_n x^n = \frac{n-1+a}{cn-n+n^2}D_{n-1}x^n
$$

Hence by ratio test

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{D_n x^n}{D_{n-1}x^{n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{n-1+a}{cn-n+n^2}D_{n-1}x^n}{D_{n-1}x^{n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n-1+a}{cn-n+n^2}x\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{n-1+a}{cn-n+n^2}\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{\frac{1}{n}-\frac{1}{n^2}+\frac{a}{n^2}}{\frac{c}{n}-\frac{1}{n}+1}\right| \\
&= |x|\left|\frac{0}{1}\right| \\
&= 0
\end{aligned}
$$

Therefore the series $y_2(x)$ also converges for all x. This means the solution $y(x) = y_1(x) + y_2(x)$ found in (2) above also converges for all x.

3.6.2 Problem 2

The Sturm Liouville equation can be expressed as

$$
L[u(x)] = \lambda\rho(x)\,u(x)
$$

Where L is given as in class. Show L is Hermitian on the domain $a \le x \le b$ with boundary conditions $u(a) = u(b) = 0$. Find the orthogonality condition.

Solution

$$
L = -\left(p\frac{d^2}{dx^2} + p'\frac{d}{dx} - q\right)
$$

The operator L is Hermitian if

$$
\int_a^b \bar{v}L[u]\,dx = \overline{\int_a^b \bar{u}L[v]\,dx}
$$

Where in the above u, v are any two functions defined over the domain that satisfy the boundary conditions given. Starting from the left integral to show it will result in the right integral. Replacing $L[u]$ by $-\left(p\frac{d^2}{dx^2}+p'\frac{d}{dx}-q\right)u$ in the LHS of the above gives

$$
\begin{aligned}
-\int_a^b \bar{v}\left(p\frac{d^2}{dx^2}+p'\frac{d}{dx}-q\right)u\,dx &= -\int_a^b \bar{v}\left(p\frac{d^2u}{dx^2}+p'\frac{du}{dx}-qu\right)dx\\
&= -\int_a^b \bar{v}p\frac{d^2u}{dx^2}+\bar{v}p'\frac{du}{dx}-q\bar{v}u\,dx\\
&= -\overbrace{\int_a^b p\bar{v}\frac{d^2u}{dx^2}dx}^{I_1}-\int_a^b \bar{v}p'\frac{du}{dx}dx+\int_a^b q\bar{v}u\,dx \qquad (1)
\end{aligned}
$$

Looking at the first integral above, which is $I_1=\int_a^b (p\bar{v})\left(\frac{d^2u}{dx^2}\right)dx$. The idea is to integrate this twice to move the second derivative from u to $\bar{v}$. Applying $\int AdB = AB-\int BdA$, where

$$
\begin{aligned}
A &\equiv p\bar{v}\\
dB &\equiv \frac{d^2u}{dx^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
dA &= p\frac{d\bar{v}}{dx}+p'\bar{v}\\
B &= \frac{du}{dx}
\end{aligned}
$$

Therefore the integral I_1 in (1) becomes

$$
\begin{aligned}
I_1 &= \int_a^b p\bar{v}\frac{d^2}{dx^2}u\\
&= \left[p\bar{v}\frac{du}{dx}\right]_a^b-\int_a^b \frac{du}{dx}\left(p\frac{d\bar{v}}{dx}+p'\bar{v}\right)dx
\end{aligned}
$$

But $\bar{v}(a)=0$ and $\bar{v}(b)=0$, hence the boundary terms above vanish and simplifies to

$$
\begin{aligned}
I_1 &= -\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}+p'\bar{v}\,\frac{du}{dx}dx\\
&= -\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx-\int_a^b p'\bar{v}\,\frac{du}{dx}dx \qquad (2)
\end{aligned}
$$

Before integrating by parts a second time, putting the result of I_1 back into (1) first simplifies the result. Substituting (2) into (1) gives

$$
\begin{aligned}
\int_a^b \bar{v}L[u]\,dx &= -I_1-\int_a^b \bar{v}p'\frac{du}{dx}dx+\int_a^b q\bar{v}u\,dx\\
&= -\overbrace{\left(-\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx-\int_a^b p'\bar{v}\,\frac{du}{dx}dx\right)}^{I_1}-\int_a^b \bar{v}p'\frac{du}{dx}dx+\int_a^b q\bar{v}u\,dx\\
&= \int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx+\int_a^b p'\bar{v}\,\frac{du}{dx}dx-\int_a^b \bar{v}p'\frac{du}{dx}dx+\int_a^b q\bar{v}u\,dx
\end{aligned}
$$

The second and third terms above cancel and the result becomes

$$
\int_a^b \bar{v}L[u]\,dx = \overbrace{\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx}^{I_2}+\int_a^b q\bar{v}u\,dx \qquad (3)
$$

Now integration by parts is applied on the first integral above. Let $I_2=\int_a^b \frac{du}{dx}\left(p\frac{d\bar{v}}{dx}\right)dx$. Applying $\int AdB = AB-\int BdA$, where

$$
\begin{aligned}
A &\equiv p\frac{d\bar{v}}{dx}\\
dB &\equiv \frac{du}{dx}
\end{aligned}
$$

Hence

$$\begin{aligned} dA &= p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} \\ B &= u \end{aligned}$$

Therefore the integral I_2 becomes

$$I_2 = \left[p\frac{d\bar{v}}{dx}u\right]_a^b - \int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx$$

But $u(a) = 0, u(b) = 0$, hence the boundary term vanishes and the above simplifies to

$$I_2 = -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx$$

Substituting the above back into (3) gives

$$\begin{aligned} \int_a^b \bar{v}L[u]\,dx &= -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx + \int_a^b q\bar{v}u\,dx \\ &= -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} - q\bar{v}\right)dx \end{aligned}$$

But $-\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} - q\bar{v}\right) = L[\bar{v}]$ by definition, and the above becomes

$$\int_a^b \bar{v}L[u]\,dx = \int_a^b uL[\bar{v}]\,dx$$

But $\int_a^b uL[\bar{v}]\,dx = \overline{\int_a^b \bar{u}L[v]\,dx}$, and the above becomes

$$\int_a^b \bar{v}L[u]\,dx = \overline{\int_a^b \bar{u}\left(L[v]\right)dx}$$

Therefore L is Hermitian.

3.6.3 Problem 3

1. For the equation $y'' + \frac{1-\alpha^2}{4x^2}y = 0$ show that two solutions are $y_1(x) = a_0x^{\frac{1+\alpha}{2}}$ and $y_2(x) = a_0x^{\frac{1-\alpha}{2}}$
2. For $\alpha = 0$, the two solutions are not independent. Find a second solution y_{20} by solving $W' = 0$ (W is the Wronskian).
3. Show that the second solution found in (2) is a limiting case of the two solutions from part (1). That is

$$y_{20} = \lim_{\alpha\to 0}\frac{y_1 - y_2}{\alpha}$$

Solution

Part 1

The point $x_0 = 0$ is a regular singular point. This is shown as follows.

$$\begin{aligned} \lim_{x\to x_0}(x-x_0)^2\frac{1-\alpha^2}{4x^2} &= \lim_{x\to 0}x^2\frac{1-\alpha^2}{4x^2} \\ &= \lim_{x\to 0}\frac{1-\alpha^2}{4} \\ &= \frac{1-\alpha^2}{4} \end{aligned}$$

Since the limit exist, then $x_0 = 0$ is a regular singular point. Assuming the solution is a Frobenius series given by

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad c_0 \neq 0$$

Therefore

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r-1} \\ y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r-2} \end{aligned}$$

Substituting the above 2 expressions back into the original ODE gives

$$\begin{aligned} 4x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r-2}\right)+\left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty}c_n x^{n+r}\right) &= 0 \\ \sum_{n=0}^{\infty}4(n+r)(n+r-1)\,c_n x^{n+r}+\left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty}c_n x^{n+r}\right) &= 0 \end{aligned} \tag{1}$$

Looking at $n=0$ first, in order to obtain the indicial equation gives

$$\begin{aligned} 4(r)(r-1)\,c_0+\left(1-\alpha^2\right)c_0 &= 0 \\ c_0\left(4r^2-4r+\left(1-\alpha^2\right)\right) &= 0 \end{aligned}$$

But $c_0\neq 0$, therefore

$$r^2-r+\frac{\left(1-\alpha^2\right)}{4}=0$$

The roots are $r=\frac{-b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}$, but $a=1,b=-1,c=\frac{\left(1-\alpha^2\right)}{4}$, hence the roots are

$$\begin{aligned} r &= \frac{1}{2}\pm\frac{1}{2}\sqrt{1-\left(1-\alpha^2\right)} \\ &= \frac{1}{2}\pm\frac{1}{2}\sqrt{\alpha^2} \\ &= \frac{1}{2}\pm\frac{1}{2}\alpha \end{aligned}$$

Hence $r_1=\frac{1}{2}(1+\alpha)$ and $r_2=\frac{1}{2}(1-\alpha)$. Each one of these roots gives a solution. The difference is

$$\begin{aligned} r_2-r_1 &= \frac{1}{2}(1+\alpha)-\frac{1}{2}(1-\alpha) \\ &= \alpha \end{aligned}$$

Therefore, to use the same solution form $y_1(x)=\sum_{n=0}^{\infty}c_n x^{n+r_1}$ and $y_2(x)=\sum_{n=0}^{\infty}d_n x^{n+r_2}$ for each, it is assumed that α is not an integer. In this case, the recursive relation for $y_1(x)$ is found from (1) by using $r=\frac{1}{2}(1+\alpha)$ which results in

$$\sum_{n=0}^{\infty}4\left(n+\frac{1}{2}(1+\alpha)\right)\left(n+\frac{1}{2}(1+\alpha)-1\right)c_n x^{n+\frac{1}{2}(1+\alpha)}+\left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty}c_n x^{n+\frac{1}{2}(1+\alpha)}\right)=0$$

For $n>0$ the above becomes

$$\begin{aligned} 4\left(n+\frac{1}{2}(1+\alpha)\right)\left(n+\frac{1}{2}(1+\alpha)-1\right)c_n+\left(1-\alpha^2\right)c_n &= 0 \\ \left(4\left(n+\frac{1}{2}(1+\alpha)\right)\left(n+\frac{1}{2}(1+\alpha)-1\right)+\left(1-\alpha^2\right)\right)c_n &= 0 \\ 4n(n+\alpha)\,c_n &= 0 \end{aligned}$$

The above can be true for all $n>0$ only when $c_n=0$ for $n>0$. Therefore the solution is only the term with c_0

$$y_1(x)=\sum_{n=0}^{\infty}c_n x^{n+r_1}=c_0x^{r_1}=c_0x^{\frac{1}{2}(1+\alpha)}$$

To find the second solution $y_2(x)$, the above is repeated but with

$$y_2(x)=\sum_{n=0}^{\infty}d_n x^{n+r_2}$$

Where the constants are not the same and by replacing r in (1) by $r_2=\frac{1}{2}(1-\alpha)$. This results in

$$\sum_{n=0}^{\infty}4\left(n+\frac{1}{2}(1-\alpha)\right)\left(n+\frac{1}{2}(1-\alpha)-1\right)d_n x^{n+\frac{1}{2}(1-\alpha)}+\left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty}d_n x^{n+\frac{1}{2}(1-\alpha)}\right)=0$$

For $n>0$

$$\left(4\left(n+\frac{1}{2}(1-\alpha)\right)\left(n+\frac{1}{2}(1-\alpha)-1\right)+\left(1-\alpha^2\right)\right)d_n=0$$
$$4n(n-\alpha)d_n=0$$

The above is true for all $n>0$ only when $c_n=0$ for $n>0$. Therefore the solution is just the term with d_0

$$y_2(x)=\sum_{n=0}^{\infty}d_nx^{n+r_2}=d_0x^{r_2}=d_0x^{\frac{1}{2}(1-\alpha)}$$

Therefore the two solutions are

$$y_1(x)=c_0x^{\frac{1}{2}(1+\alpha)}$$
$$y_2(x)=d_0x^{\frac{1}{2}(1-\alpha)}$$

Part 2

When $\alpha=0$ then the ODE becomes

$$4x^2y''+y=0$$

And the two solutions found in part (1) simplify to

$$y_1(x)=c_0\sqrt{x}$$
$$y_2(x)=d_0\sqrt{x}$$

Therefore the two solutions are not linearly independent. Let $y_{20}(x)$ be the second solution. The Wronskian is

$$W(x)=\begin{vmatrix}y_1 & y_{20}\\ y_1' & y_{20}'\end{vmatrix}=y_1y_{20}'-y_{20}y_1' \tag{1}$$

Using Abel's theorem which says that for ODE of form $y''+p(x)y'+q(x)y=0$, the Wronskian is $W(x)=Ce^{-\int p(x)dx}$. Applying this to the given ODE above and since $p(x)=0$ then the above becomes

$$W(x)=C$$

Where C is constant. For y_{20} to be linearly independent from y_1 $W(x)\neq 0$. Using $W(x)=C$ in (1) results in the following equation (here it is also assumed that $y_1\neq 0$, or $x\neq 0$, because the equation is divided by y_1)

$$y_1y_{20}'-y_{20}y_1'=C$$
$$y_{20}'-y_{20}\frac{y_1'}{y_1}=\frac{C}{y_1}$$

Since $y_1=\sqrt{x}$ and $y_1'=\frac{1}{2}\frac{1}{\sqrt{x}}$ the above simplifies to

$$y_{20}'-y_{20}\frac{\frac{1}{2}\frac{1}{\sqrt{x}}}{\sqrt{x}}=\frac{C}{\sqrt{x}}$$
$$y_{20}'-y_{20}\frac{1}{2x}=\frac{C}{\sqrt{x}} \tag{2}$$

But the above is linear first order ODE of the form $Y'+pY=q$, therefore the standard integrating factor to use is $I=e^{\int p(x)dx}$ which results in

$$\begin{aligned}I&=e^{\int\frac{-1}{2x}dx}\\&=e^{-\frac{1}{2}\int\frac{1}{x}dx}\\&=e^{-\frac{1}{2}\ln x}\\&=\frac{1}{\sqrt{x}}\end{aligned}$$

Multiplying both sides of (2) by this integrating factor, makes the left side of (2) an exact differential

$$\frac{d}{dx}\left(y_{20}\frac{1}{\sqrt{x}}\right) = \frac{C}{x}$$

Integrating both sides gives

$$\begin{aligned}
y_{20}\frac{1}{\sqrt{x}} &= C\int \frac{1}{x}dx + C_1\\
y_{20}\frac{1}{\sqrt{x}} &= 2C\ln x + C_1\\
y_{20} &= 2C\ln x\sqrt{x} + C_1\sqrt{x}
\end{aligned}$$

Or

$$y_{20} = C_1 \ln x\sqrt{x} + C_2\sqrt{x} \tag{3}$$

The above is the second solution. Therefore the final solution is

$$y(x) = C_0 y_1(x) + C_3 y_{20}(x)$$

Substituting $y_1 = \sqrt{x}$ and y_{20} found above and combining the common term $\sqrt{x}$ and renaming constants gives

$$y(x) = C_1\sqrt{x} + C_2\ln x\sqrt{x}$$

Another method to find the second solution

This method is called the reduction of order method. It does not require finding $W(x)$ first. Let the second solution be

$$y_{20} = Y = v(x)\, y_1(x) \tag{4}$$

Where $v(x)$ is unknown function to be determined, and $y_1(x) = \sqrt{x}$ which is the first solution that is already known. Therefore

$$\begin{aligned}
Y' &= v'y_1 + vy_1'\\
Y'' &= v''y_1 + v'y_1' + v'y_1' + vy_1''\\
&= v''y_1 + 2v'y_1' + vy_1''
\end{aligned}$$

Since Y is a solution to the ODE $4x^2y'' + y = 0$, then substituting the above equations back into the ODE $4x^2y'' + y = 0$ gives

$$\begin{aligned}
4x^2\left(v''y_1 + 2v'y_1' + vy_1''\right) + vy_1 &= 0\\
v''\left(4x^2y_1\right) + v'\left(8x^2y_1'\right) + v\left(\overbrace{4x^2y_1'' + y_1}^{0}\right) &= 0
\end{aligned}$$

But $4x^2y_1'' + y_1 = 0$ because y_1 is a solution. The above simplifies to

$$v''\left(4x^2y_1\right) + v'\left(8x^2y_1'\right) = 0$$

But $y_1 = x^{\frac{1}{2}}$, hence $y_1' = \frac{1}{2}x^{\frac{-1}{2}}$ and the above simplifies to

$$\begin{aligned}
v''\left(4x^2x^{\frac{1}{2}}\right) + v'\left(4x^2x^{\frac{-1}{2}}\right) &= 0\\
x^{\frac{5}{2}}v'' + v'x^{\frac{3}{2}} &= 0\\
xv'' + v' &= 0\\
v'' + \frac{1}{x}v' &= 0
\end{aligned}$$

This ODE is now easy to solve because the $v(x)$ term is missing. Let $w = v'$ and the above first order ODE $w' + \frac{1}{x}w = 0$. This is linear in w. Hence using integrating factor $I = e^{\int \frac{1}{x}dz} = x$, this ODE becomes

$$\begin{aligned}
\frac{d}{x}(wx) &= 0\\
wx &= C\\
w &= \frac{C}{x}
\end{aligned}$$

Where C is constant of integration. Since $v' = w$, then $v' = \frac{C_1}{x}$. Now $v(x)$ is found by integrating both sides

$$v = C_1 \ln x + C_2$$

Therefore the second solution from (4) becomes

$$\begin{aligned} y_{20} &= C_1 \ln x y_1 + C_2 y_1 \\ &= C_1 \sqrt{x} \ln x + C_2 \sqrt{x} \end{aligned} \tag{5}$$

Comparing the above to (3), shows it is the same solution. Both methods can be used, but reduction of order method is a more common method and it does not require finding the Wronskian first, although it is not hard to find by using Abel's theorem.

Part 3

The solutions we found in part (1) are

$$\begin{aligned} y_1(x) &= C_1 x^{\frac{1}{2}(1+\alpha)} \\ y_2(x) &= C_2 x^{\frac{1}{2}(1-\alpha)} \end{aligned}$$

Therefore

$$\lim_{\alpha\to 0} \frac{y_1 - y_2}{\alpha} = \lim_{\alpha\to 0} \frac{C_1 x^{\frac{1}{2}(1+\alpha)} - C_2 x^{\frac{1}{2}(1-\alpha)}}{\alpha}$$

Applying L'Hopital's

$$\lim_{\alpha\to 0} \frac{y_1 - y_2}{\alpha} = \lim_{\alpha\to 0} \frac{C_1 \frac{d}{d\alpha}\left(x^{\frac{1}{2}(1+\alpha)}\right) - C_2 \frac{d}{d\alpha}\left(x^{\frac{1}{2}(1-\alpha)}\right)}{1} \tag{1}$$

But

$$\begin{aligned} \frac{d}{d\alpha}\left(x^{\frac{1}{2}(1+\alpha)}\right) &= \frac{d}{d\alpha} e^{\frac{1}{2}(1+\alpha)\ln x} \\ &= \frac{d}{d\alpha} e^{\left(\frac{1}{2}\ln x + \alpha \ln x\right)} \\ &= \ln x e^{\left(\frac{1}{2}\ln x + \alpha \ln x\right)} \end{aligned}$$

And

$$\begin{aligned} \frac{d}{d\alpha}\left(x^{\frac{1}{2}(1-\alpha)}\right) &= \frac{d}{d\alpha} e^{\frac{1}{2}(1-\alpha)\ln x} \\ &= \frac{d}{d\alpha} e^{\left(\frac{1}{2}\ln x - \alpha \ln x\right)} \\ &= -\ln x e^{\left(\frac{1}{2}\ln x - \alpha \ln x\right)} \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned} \lim_{\alpha\to 0} \frac{y_1 - y_2}{\alpha} &= \lim_{\alpha\to 0} C_1 \ln x e^{\left(\frac{1}{2}\ln x + \alpha \ln x\right)} + C_2 \ln x e^{\left(\frac{1}{2}\ln x - \alpha \ln x\right)} \\ &= \ln x \left(\lim_{\alpha\to 0} C_1 e^{\left(\frac{1}{2}\ln x + \alpha \ln x\right)} + C_2 e^{\left(\frac{1}{2}\ln x - \alpha \ln x\right)}\right) \\ &= \ln x \left(C_1 e^{\frac{1}{2}\ln x} + C_2 e^{\frac{1}{2}\ln x}\right) \\ &= \ln x \left(C_1 \sqrt{x} + C_2 \sqrt{x}\right) \\ &= C \sqrt{x} \ln x \end{aligned}$$

The above is the same as (3) found in part (2). Hence

$$y_{20}(x) = \lim_{\alpha\to 0} \frac{y_1 - y_2}{\alpha}$$

Which is what the problem asked to show.

3.6.4 key solution to HW 6

Problem Set 6 Solutions

Exercise 1 : $xy'' - (c-x)y' - ay = 0$

- this has a regular singularity at $x=0$
- let $y = x^k \sum_{n=0}^{\infty} d_n x^n$, then

$$\sum_{n=0}^{\infty} d_n (n+k)(n+k-1) x^{n+k-1} + \sum_{n=0}^{\infty} d_n (n+k)(c-x) x^{n+k-1}$$
$$- a \sum_{n=0}^{\infty} d_n x^{n+k} = 0$$

or $$\sum_{n=0}^{\infty} d_n \left\{ \left[(n+k)(n+k-1) + c(n+k)\right] x^{n+k-1} - \left[n+k+a\right] x^{n+k} \right\} = 0$$

the x^{k-1} term gives the indicial equation

$k(k-1) + ck = 0$

$\Rightarrow$ $\boxed{k=0}$ or $\boxed{k=-c+1}$

- next let us find the recurrence relation

$$d_{n+1}\left[(n+k+1)(n+k) + c(n+k+1)\right] - d_n (n+k+a) = 0$$

or $$d_{n+1} = \frac{n+k+a}{(n+k+1)(n+k+c)} d_n$$

$$\Rightarrow \quad d_n = \frac{(n+k+a-1)!\; k!\; (k-1+c)!}{(k+a-1)!\; (n+k)!\; (n+k-1+c)!}$$

For $k=0$, we get

$$d_n = \frac{(n+a-1)!\,(c-1)!}{(a-1)!\,n!\,(n+c-1)!}$$

or $$y = d_0\left[1+\frac{a}{c}x+\frac{(a+1)a}{(c+1)c}\frac{1}{2}x^2+\dots\right]$$

check this for convergence

$$\frac{d_{n+1}\,x^{n+1}}{d_n\,x^n} = \frac{n+a+2}{(n+1)(n+c)}x \underset{\text{as } n\to\infty}{\approx} \frac{x}{n} \Rightarrow \text{converges for all } x$$

second solution $k=-c+1$

$$d_n = \frac{(n+a-c)!\,(1-c)!\,c!}{(a-c)!\,(n-c+1)!\,n!}d_0 \quad \text{or}$$

$$y = a_0\left[1+\frac{a-c+1}{-c}x+\frac{(a-c+1)(a-c+2)}{-c(c+1)2}x^2\dots\right.$$

$$\frac{d_{n+1}\,x^{n+1}}{d_n\,x^n} \approx \frac{x}{n}, \text{ also converges for all } x$$

Exercise 2 :

- $L\,u(x) + \lambda\, p(x)\, u(x) = 0$
- show L is Hermitian :

$$\int_a^b u^+ L v\, dx = \int_a^b u^+ \left[p \frac{d^2}{dx^2} + p' \frac{d}{dx} - q(x) \right] v\, dx$$

$$= \left[u^+ p \cancel{\frac{d}{dx} v} \right]\Big|_a^b - \int_a^b \frac{dv}{dx} \frac{d}{dx}(u^+ p)\, dx + \int_a^b u^+ p' \frac{dv}{dx}\, dx - \int_a^b u^+ q v\, dx$$

$$= -\int_a^b \frac{dv}{dx} \frac{du^+}{dx} p\, dx - \int_a^b \frac{dv}{dx} \cancel{u^+ p'}\, dx + \int_a^b u^+ p' \cancel{\frac{dv}{dx}}\, dx - \int_a^b u^+ q v\, dx$$

$$= \left[-v \cancel{\frac{d}{dx}} \left(p \frac{du^+}{dx} \right) \right]\Big|_a^b + \int_a^b v \frac{d}{dx} \left(p \frac{du^+}{dx} \right) dx - \int_a^b u^+ q v\, dx$$

$$= \left[\int_a^b v^+ L u\, dx \right]^+ \Rightarrow \text{Hermitian}$$

- now let u_i and u_j be eigenfunctions with eigenvalues λ_i and λ_j :

$L u_i = -\lambda_i p u_i$ and $L u_j = -\lambda_j p u_j$

$$\Rightarrow \int_a^b u_j^+ L u_i\, dx = -\lambda_i \int_a^b u_j^+ u_i\, p\, dx$$

and $\int_a^b u_i^+ L u_j\, dx = -\lambda_j \int_a^b u_i^+ u_j\, p\, dx$ now since

L is Hermitian $\int_a^b u_j^+ L u_i\, dx = \left[\int_a^b u_i^+ L u_j\, dx \right]^+$

$\therefore (\lambda_i - \lambda_j^*) \int_a^b u_i^+(x) u_j(x)\, p(x)\, dx = 0$

$\Rightarrow$ when $\lambda_i \neq \lambda_j$ then u_i and u_j are orthogonal $0 = \int_a^b u_i^+(x)\, u_j(x)\, p(x) dx$

Exercise 3

i) $y'' + \frac{1-\alpha^2}{4x^2} y = 0$, let $y = x^p$

$\Rightarrow \quad p(p-1)\, x^{p-2} + \frac{(1-\alpha^2)}{4} x^{p-2} = 0$

$\Rightarrow \quad p^2 - p + \frac{1-\alpha}{2}\,\frac{1+\alpha}{2} = 0$

check that $\quad p_\pm = \frac{1\pm\alpha}{2}$ is a solution

$\left(\frac{1\pm\alpha}{2}\right)^2 - \left(\frac{1\pm\alpha}{2}\right) + \frac{1-\alpha}{2}\,\frac{1+\alpha}{2} \overset{?}{=} 0$

$\left(\frac{1\pm\alpha}{2}\right)\left[\frac{1\pm\alpha}{2} - 1 + \frac{1\mp\alpha}{2}\right] \overset{?}{=} 0$

$\hookrightarrow$ this is indeed $\not{0}$

$\Rightarrow \quad y = x^{\frac{1\pm\alpha}{2}}$ are solutions

ii) For $\alpha = 0 \qquad y_0 = \sqrt{x}$

now $\quad y'' + \frac{1-\alpha^2}{4x^2} y = 0 \quad$ has Sturm-Liouville form with $p(x) = 1 \Rightarrow W = c$, c is a constant

$\Rightarrow \quad y_{20}'\, y_0 - y_{20}\, y_0' = c$

or $\quad y_0^2 \left(\frac{y_{20}}{y_0}\right)' = c \quad \Rightarrow \quad y_{20} = y_0 \int \frac{c}{y_0^2}\, dx'$

or $\quad y_{20} = \sqrt{x}\, c \int \frac{dx'}{x'} \quad$ or $\quad \boxed{y_{20} = c\sqrt{x}\ln x}$

iii) want $\lim_{\alpha \to 0} \frac{x^{\frac{1+\alpha}{2}} - x^{\frac{1-\alpha}{2}}}{\alpha}$, use L'Hopital's rule

$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} x^{\frac{1+\alpha}{2}} \Big|_{\alpha=0} - \frac{d}{d\alpha} x^{\frac{1-\alpha}{2}} \Big|_{\alpha=0}$$

$$= \lim_{\alpha \to 0} \sqrt{x} \left[\frac{d}{d\alpha} x^{\frac{\alpha}{2}} - \frac{d}{d\alpha} x^{-\frac{\alpha}{2}} \right] \Big|_{\alpha=0}$$

now $\frac{d}{d\alpha} x^{\frac{\alpha}{2}} \Big|_{\alpha=0} = \frac{d}{d\alpha} e^{\frac{\alpha}{2}\ln x} \Big|_{\alpha=0} = \frac{1}{2} \ln x$

$$\Rightarrow \lim_{\alpha \to 0} \frac{x^{\frac{1+\alpha}{2}} - x^{\frac{1-\alpha}{2}}}{\alpha} = \frac{\sqrt{x}}{2}(\ln x + \ln x) = \sqrt{x} \ln x = \tilde{x}_0$$

as required.

3.7 HW 7

3.7.1 Problem 1

Exercise 1: Consider Hermite's differential equation valid for $(-\infty < x < \infty)$:

$$y'' - 2xy' + 2ny = 0 \tag{1}$$

i) Assume the existence of a generating function $g(x,t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$. Differentiate $g(x,t)$ with respect to x and use the recurrence relation $H_n'(x) = 2nH_{n-1}(x)$to develop a first order differential equation for $g(x,t)$.
ii) Integrate this equation with respect to x holding t fixed.
iii) Use the relationships $H_{2n}(0) = (-1)^n(2n)!/n!$ and $H_{2n+1}(0) = 0$ to evaluate $g(0,t)$ and show $g(x,t) = \exp(-t^2 + 2tx)$.
iv) Use the generating function to find the recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.
v) By integrating the product $e^{-x^2}g(x,s)g(x,t)$ over all x, show

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\sqrt{\pi}\delta_{nm}. \tag{2}$$

Figure 3.20: Problem statement

Solution

$$y'' - 2xy' + 2ny = 0 \qquad -\infty < x < \infty$$

Part 1

$$g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

Differentiating w.r.t x, and assuming term by term differentiation is allowed, gives

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Using $H_n'(x) = 2nH_{n-1}(x)$ in the above results in

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}$$

But for $n = 0$, the first term is zero, so the sum can start from 1 and give the same result

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=1}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}$$

Now, decreasing the summation index by 1 and increasing the n inside the sum by 1 gives

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial x} &= \sum_{n=0}^{\infty} 2(n+1)H_n(x) \frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} 2(n+1)H_n(x) \frac{t^{n+1}}{(n+1)n!} \\
&= \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} 2t\left(H_n(x) \frac{t^n}{n!}\right) \\
&= 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}
\end{aligned}$$

But $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = g(x,t)$ and the above reduces to

$$\frac{\partial g(x,t)}{\partial x} = 2tg(x,t)$$

The problem says it is supposed to be a first order differential equation and not a first order partial differential equation. Therefore, by assuming x to be a fixed parameter instead of another independent variable, the above can now be written as

$$\frac{d}{dx}g(x,t) - 2tg(x,t) = 0$$

Part 2

From the solution found in part (1)

$$\begin{aligned}\frac{\frac{d}{dx}g(x,t)}{g(x,t)} &= 2t\\ \frac{dg(x,t)}{g(x,t)} &= 2tdx\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\int\frac{dg(x,t)}{g(x,t)} &= \int 2tdx\\ \ln|g(x,t)| &= 2tx + C\\ g(x,t) &= e^{2tx+C}\\ g(x,t) &= C_1e^{2tx}\end{aligned}$$

Where $C_1 = e^C$ a new constant. Let $g(0,t) = g_0$ then the above shows that $C_1 = g_0$ and the above can now be written as

$$g(x,t) = g(0,t)e^{2tx}$$

Part 3

Using the given definition of $g(x,t) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$ and when $x = 0$ then

$$\begin{aligned}g(0,t) &= \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!}\\ &= H_0(0) + H_1(0) + \sum_{n=2}^{\infty} H_n(0)\frac{t^n}{n!}\end{aligned}$$

But $H_0(x) = 1$, hence $H_0(0) = 1$ and $H_1(x) = 2x$, hence $H_1(0) = 0$ and the above becomes

$$g(0,t) = 1 + \sum_{n=2}^{\infty} H_n(0)\frac{t^n}{n!}$$

For the remaining series, it can be written as sum of even and odd terms

$$g(0,t) = 1 + \sum_{n=2,4,6,\cdots}^{\infty} H_n(0)\frac{t^n}{n!} + \sum_{n=3,5,7,\cdots}^{\infty} H_n(0)\frac{t^n}{n!}$$

Or, equivalently

$$g(0,t) = 1 + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n}(0)\frac{t^{2n}}{(2n)!} + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n+1}(0)\frac{t^{2n+1}}{(2n+1)!}$$

But using the hint given that $H_{2n+1}(0) = 0$ and $H_{2n}(0) = \frac{(-1)^n(2n)!}{n!}$ the above simplifies to

$$\begin{aligned}g(0,t) &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} \frac{(-1)^n(2n)!}{n!}\frac{t^{2n}}{(2n)!}\\ &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} (-1)^n\frac{t^{2n}}{n!}\end{aligned}$$

But since $(-1)^n\frac{t^{2n}}{n!} = 1$ when $n = 0$, then the above sum can be made to start as zero and it simplifies to

$$g(0,t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!}$$

Therefore the solution $g(x,t) = g(0,t)e^{tx}$ found in part (2) becomes

$$g(x,t) = \left(\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!}\right)e^{2tx} \tag{1}$$

Now the sum $\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$ and comparing this sum to standard series of $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, then this shows that when $z = -t^2$ and series for e^{-t^2} becomes

$$\begin{aligned} e^{-t^2} &= 1 + \left(-t^2\right) + \frac{\left(-t^2\right)^2}{2!} + \frac{\left(-t^2\right)^3}{3!} + \frac{\left(-t^2\right)^4}{4!}\cdots \\ &= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!}\cdots \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!} = e^{-t^2}$$

Substituting this into (1) gives

$$\begin{aligned} g(x,t) &= e^{-t^2}e^{2tx} \\ &= e^{2tx-t^2} \end{aligned}$$

Part 4

Since $g(x,t) = e^{2tx-t^2}$ from part (3), then

$$\begin{aligned} \frac{\partial}{\partial t}g(x,t) &= (2x-2t)e^{2tx-t^2} \\ &= (2x-2t)g(x,t) \end{aligned}$$

But $g(x,t) = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}$, therefore the above can be written as

$$\begin{aligned} \frac{\partial}{\partial t}g(x,t) &= (2x-2t)\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} \\ &= 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2t\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} \\ &= 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty}H_n(x)\frac{t^{n+1}}{n!} \\ &= 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty}H_{n-1}(x)\frac{t^n}{(n-1)!} \\ &= 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty}nH_{n-1}(x)\frac{t^n}{n(n-1)!} \\ &= 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty}nH_{n-1}(x)\frac{t^n}{n!} \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t}g(x,t) &= \frac{\partial}{\partial t}\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}nH_n(x)\frac{t^{n-1}}{n!} \end{aligned}$$

Since at $n = 0$ the sum is zero, then it can be started from $n = 1$ without changing the result

$$\begin{aligned} \frac{\partial}{\partial t}g(x,t) &= \sum_{n=1}^{\infty}nH_n(x)\frac{t^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)!} \\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)n!} \\ &= \sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} \end{aligned} \tag{2}$$

Equating (1) and (2) gives

$$\sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty} nH_{n-1}(x)\frac{t^n}{n!}$$

But $\sum_{n=1}^{\infty} nH_{n-1}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} nH_{n-1}(x)\frac{t^n}{n!}$ because at $n=0$ it is zero, so it does not affect the result to start the sum from zero, and now the above can be written as

$$\sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} nH_{n-1}(x)\frac{t^n}{n!}$$

Now since all the sums start from $n=0$ then the above means the same as

$$H_{n+1}(x)\frac{t^n}{n!} = 2xH_n(x)\frac{t^n}{n!} - 2nH_{n-1}(x)\frac{t^n}{n!}$$

Canceling $\frac{t^n}{n!}$ from each term gives

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Which is the result required to show.

Part 5

The problem is asking to show that

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,dx = \begin{cases} 0 & n \neq m \\ 2^n n!\sqrt{\pi} & n = m \end{cases}$$

The first part below will show the case for $n \neq m$ and the second part part will show the case for $n = m$

<u>case $n \neq m$</u> This is shown by using the differential equation directly. I found this method easier and more direct. Before starting, the ODE $y'' - 2xy' + 2ny = 0$ is rewritten as

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}y'\right) + 2ny = 0 \tag{1}$$

The above form is exactly the same as the original ODE as can be seen by expanding it. Now, Let $H_n(x)$ be one solution to (1) and let $H_m(x)$ be another solution to (1) which results in the following two ODE's

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_n = 0 \tag{1A}$$

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_m = 0 \tag{2A}$$

Multiplying (1A) by H_m and (2A) by H_n and subtracting gives

$$\begin{aligned} H_m\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_n\right) - H_n\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_m\right) &= 0 \\ \left(H_m e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_nH_m\right) - \left(H_n e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_nH_m\right) &= 0 \\ H_m e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) - H_n e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2(n-m)H_nH_m &= 0 \\ H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right) - H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2(n-m)H_nH_m e^{-x^2} &= 0 \end{aligned} \tag{3}$$

But

$$H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right) = \frac{d}{dx}\left(e^{-x^2}H_n'H_m\right) - e^{-x^2}H_n'H_m'$$

And

$$H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right) = \frac{d}{dx}\left(e^{-x^2}H_m'H_n\right) - e^{-x^2}H_m'H_n'$$

Therefore

$$\begin{aligned} H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right) - H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right) &= \left(\frac{d}{dx}\left(e^{-x^2}H_n'H_m\right) - e^{-x^2}H_n'H_m'\right) - \left(\frac{d}{dx}\left(e^{-x^2}H_m'H_n\right) - e^{-x^2}H_m'H_n'\right) \\ &= \frac{d}{dx}\left(e^{-x^2}H_n'H_m\right) - \frac{d}{dx}\left(e^{-x^2}H_m'H_n\right) \\ &= \frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m - H_m'H_n\right)\right) \end{aligned}$$

Substituting the above relation back into (3) gives

$$\frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m - H_m'H_n\right)\right) + 2\left(n-m\right)H_nH_me^{-x^2} = 0$$

Integrating gives

$$\int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m - H_m'H_n\right)\right)dx + \int_{-\infty}^{\infty}2\left(n-m\right)H_nH_me^{-x^2}dx = 0$$

$$\int_{-\infty}^{\infty}d\left(e^{-x^2}\left(H_n'H_m - H_m'H_n\right)\right) + 2\left(n-m\right)\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx = 0$$

$$\left[e^{-x^2}\left(H_n'H_m - H_m'H_n\right)\right]_{-\infty}^{\infty} + 2\left(n-m\right)\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx = 0$$

But $\lim_{x\to\pm\infty}e^{-x^2}\to 0$ so the first term above vanishes and the above becomes

$$2\left(n-m\right)\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx = 0$$

Since this is the case where $n\neq m$ then the above shows that

$$\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx = 0 \qquad n\neq m$$

Now the case $n=m$ is proofed. When $H_n = H_m$ then the integral becomes $\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx$. Using the known Rodrigues formula for Hermite polynomials, given by

$$H_n\left(x\right) = \left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

Then applying the above the above to one of the $H_n\left(x\right)$ in the integral $\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx$, gives

$$\begin{aligned}\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx &= \int_{-\infty}^{\infty}\left(\left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\right)H_ne^{-x^2}dx\\ &= \left(-1\right)^n\int_{-\infty}^{\infty}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_ndx\end{aligned}$$

Now integration by parts is carried out. $\int udv = uv - \int vdu$. Let $u = H_n$ and let $dv = \frac{d^n}{dx^n}e^{-x^2}$, therefore $du = H_n'\left(x\right) = 2nH_{n-1}\left(x\right)$ and $v = \frac{d^{n-1}}{dx^{n-1}}e^{-x^2}$, therefore

$$\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx = \left(-1\right)^n\left(\left[H_n\left(x\right)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)2nH_{n-1}\left(x\right)dx\right)$$

But $\left[H_n\left(x\right)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right]_{-\infty}^{\infty}\to 0$ as $x\to\pm\infty$ because each derivative of $\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}$ produces a term with e^{-x^2} which vanishes at both ends of the real line. Hence the above integral now becomes

$$\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx = \left(-1\right)^n\left(-2n\int_{-\infty}^{\infty}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)H_{n-1}\left(x\right)dx\right)$$

Now the process is repeated, doing one more integration by parts. This results in

$$\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx = \left(-1\right)^n\left(-2n\left(-2\left(n-1\right)\int_{-\infty}^{\infty}\left(\frac{d^{n-2}}{dx^{n-2}}e^{-x^2}\right)H_{n-2}\left(x\right)dx\right)\right)$$

And again

$$\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx = \left(-1\right)^n\left(-2n\left(-2\left(n-1\right)\left(-2\left(n-2\right)\int_{-\infty}^{\infty}\left(\frac{d^{n-3}}{dx^{n-3}}e^{-x^2}\right)H_{n-3}\left(x\right)dx\right)\right)\right)$$

This process continues n times. After n integrations by parts, the above becomes

$$\begin{aligned}\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx &= \left(-1\right)^n\left(-2n\left(-2\left(n-1\right)\left(-2\left(n-2\right)\left(\cdots\left(\int_{-\infty}^{\infty}e^{-x^2}H_0\left(x\right)dx\right)\right)\right)\right)\right)\\ &= \left(-1\right)^n\left(-2\right)^n n!\int_{-\infty}^{\infty}e^{-x^2}H_0\left(x\right)dx\\ &= 2^n n!\int_{-\infty}^{\infty}e^{-x^2}H_0\left(x\right)dx\end{aligned}$$

But $H_0\left(x\right) = 1$, therefore the above becomes

$$\int_{-\infty}^{\infty}H_nH_ne^{-x^2}dx = 2^n n!\int_{-\infty}^{\infty}e^{-x^2}dx$$

But

$$\int_{-\infty}^{\infty} e^{-x^2} = 2\int_{0}^{\infty} e^{-x^2}$$
$$= 2\frac{\sqrt{\pi}}{2}$$
$$= \sqrt{\pi}$$

Therefore

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = 2^n n! \sqrt{\pi}$$

This completes the case for $n = m$. Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0 & n \neq m \\ 2^n n! \sqrt{\pi} & n = m \end{cases}$$

Which is what the problem asked to show.

3.7.2 Problem 2

Exercise 2: a) Consider the differential equation for $0 < r < \infty$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = 0 \tag{3}$$

where $n = 0, 1, 2, 3, \ldots$ Find two independent solutions, one which vanishes as $r \to 0$ and the other that vanishes as $r \to \infty$. Hint let $x = \ln r$.
b) Given the result of part a), find the solution to the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = \frac{1}{r}\delta(r - r') \tag{4}$$

with the boundary condition that the solution vanishes as $r \to 0$ and $r \to \infty$.

Figure 3.21: Problem statement

Solution

Part (a)

$$y''(r) + \frac{1}{r} y'(r) - \frac{n^2}{r^2} y(r) = 0 \qquad 0 < r < \infty$$

Or

$$r^2 y''(r) + r y'(r) - n^2 y(r) = 0$$

case $n = 0$

The ode becomes $r^2 y''(r) + r y'(r) = 0$. Let $z = y'$ and it becomes $r^2 z'(r) + r z(r) = 0$ or $z'(r) + \frac{1}{r} z(r) = 0$. This is linear in $z(r)$. Integrating factor is $I = e^{\int \frac{1}{r} dr} = r$. Multiplying the ode by I it becomes exact differential $\frac{d}{dr}(zr) = 0$ or $d(zr) = 0$, hence $z = \frac{c_1}{r}$ where c_1 is constant of integration. Therefore

$$y'(r) = \frac{c_1}{r}$$

Integrating again gives

$$y(r) = \frac{c_1}{\ln r} + c_2$$

Since $\lim_{r \to 0}$ the solution is bounded, then c_1 must be zero. Therefore $0 = c_2$ and this implies $c_2 = 0$ also. Therefore when $n = 0$ the solution is

$$y(r) = 0$$

Case $n \neq 0$

Since powers of r is the same as order of derivative in each term, this is an Euler ODE. It is solved by assuming $y = r^{\alpha}$. Hence $y' = \alpha r^{\alpha-1}, y'' = \alpha(\alpha-1) r^{\alpha-2}$. Substituting these into the above ODE gives

$$\begin{aligned} r^2\alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^{\alpha} &= 0 \\ \alpha(\alpha-1) r^{\alpha} + \alpha r^{\alpha} - n^2 r^{\alpha} &= 0 \\ r^{\alpha}\left(\alpha(\alpha-1) + \alpha - n^2\right) &= 0 \end{aligned}$$

Assuming non-trivial solution $r^{\alpha} \neq 0$, then the indicial equation is

$$\begin{aligned} \alpha(\alpha-1) + \alpha - n^2 &= 0 \\ \alpha^2 &= n^2 \\ \alpha &= \pm n \end{aligned}$$

Hence one solution is

$$y_1(r) = r^n$$

And second solution is

$$y_2(r) = r^{-n}$$

And the general solution is linear combination of these solutions

$$y(r) = c_1 r^n + c_2 r^{-n}$$

The above shows that $\lim_{r\to 0} y_1(r) = 0$ and $\lim_{r\to\infty} y_2(r) = 0$.

Part (b)

Short version of the solution

To simplify the notations, r_0 is used instead of r' in all the following.

$$y''(r) + \frac{1}{r}y'(r) - \frac{n^2}{r^2}y(r) = \frac{1}{r}\delta(r-r_0) \qquad 0 < r < \infty$$

Multiplying both sides by r the above becomes

$$ry''(r) + y'(r) - \frac{n^2}{r}y(r) = \delta(r-r_0) \tag{1}$$

But the two solutions[2] to the homogeneous ODE $ry''(r) + y'(r) - \frac{n^2}{r}y(r) = 0$ were found in part (a). These are

$$\begin{aligned} y_1(r) &= r^n \qquad (1A) \\ y_2(r) &= r^{-n} \end{aligned}$$

The Green function is the solution to

$$\begin{aligned} rG(r,r_0) + G(r,r_0) - \frac{n^2}{r}G(r,r_0) &= \delta(r-r_0) \qquad (1B) \\ \lim_{r\to 0} G(r,r_0) &= 0 \\ \lim_{r\to\infty} G(r,r_0) &= 0 \end{aligned}$$

Which is given by (Using class notes, Lecture December 5, 2018) as

$$G(r,r_0) = \frac{1}{C}\begin{cases} y_1(r)\,y_2(r_0) & 0 < r < r_0 \\ y_1(r_0)\,y_2(r) & r_0 < r < \infty \end{cases} \tag{2}$$

Note, I used $\frac{+1}{C}$ and not $\frac{-1}{C}$ as in class notes, since I am using $L = -\left(\left(py'\right)' - qy\right)$ as the operator and not $L = +\left(\left(py'\right)' + qy\right)$. Now C is given by

$$C = p(r_0)\left(y_1(r_0)\,y_2'(r_0) - y_1'(r_0)\,y_2(r_0)\right)$$

[2]All the following is for $n \neq 0$, since for $n = 0$, only trivial solution exist

Where from (1A) we see that

$$
\begin{aligned}
y_1(r_0) &= r_0^n \\
y_2'(r_0) &= -nr_0^{-n-1} \\
y_1'(r_0) &= nr_0^{n-1} \\
y_2(r_0) &= r_0^{-n}
\end{aligned}
$$

Therefore C becomes

$$
\begin{aligned}
C &= p(r_0)\left(-nr_0^{-n-1}r_0^n - nr_0^{n-1}r_0^{-n}\right) \\
&= 2nr_0^{-1}p(r_0)
\end{aligned}
$$

We just need now to find $p(r_0)$. This comes from Sturm Liouville form. We need to convert the ODE $r^2y''(r) + ry'(r) - n^2y(r) = 0$ to Sturm Liouville. Writing this ODE as $ay'' + by' + (c+\lambda)y = 0$ where $a = r^2, b = r, c = 0, \lambda = -n^2$, therefore

$$
\begin{aligned}
p &= e^{\int \frac{b}{a}dr} = e^{\int \frac{r}{r^2}dr} = r \\
q &= -p\frac{c}{a} = 0 \\
\rho &= \frac{p}{a} = \frac{r}{r^2} = \frac{1}{r}
\end{aligned}
$$

Hence the SL form is $\left(py'\right)' - qy + \lambda\rho y = 0$. Hence the SL form is $\left(py'\right)' - qy + \lambda\rho y = 0$ or

$$
\left(ry'\right)' - \frac{1}{r}n^2y = 0 \tag{2A}
$$

Hence the operator is $L\left[y\right] = -\left(\frac{d}{dr}\left(r\frac{d}{dr}\right)\right)\left[y\right]$ and in standard form it becomes $L\left[y\right] + \frac{1}{r}n^2y = 0$.

The above shows that $p(r_0) = r_0$. Therefore

$$
C = 2n
$$

Hence Green function is now found from (2) as, for $n \neq 0$

$$
G(r,r_0) = \frac{1}{2n}\begin{cases} r^n r_0^{-n} & 0 < r < r_0 \\ r_0^n r^{-n} & r_0 < r < \infty \end{cases}
$$

Since $f(r)$ in the original ODE is zero, there is nothing to convolve with. i.e. $y(r) = \int_0^\infty G(r,r_0) f(r_0)\,dr_0$ here is not needed since there is no $f(r)$. Therefore the above is the final solution.

Extended solution

This solution shows derivation of (2) above. It can be considered as an appendix. The Green function is the solution to

$$
rG(r,r_0) + G(r,r_0) - \frac{n^2}{r}G(r,r_0) = \delta(r - r_0) \tag{1B}
$$

$$
\begin{aligned}
\lim_{r\to 0} G(r,r_0) &= 0 \\
\lim_{r\to \infty} G(r,r_0) &= 0
\end{aligned}
$$

In (1B), r_0 is the location of the impulse and r is the location of the observed response due to this impulse. The solution to the above ODE is now broken to two regions

$$
G(r,r_0) = \begin{cases} A_1y_1(r) + A_2y_2(r) & 0 < r < r_0 \\ B_1y_1(r) + B_1y_2(r) & r_0 < r < \infty \end{cases} \tag{2}
$$

Where $y_1(r), y_2(r)$ are the solution to $ry''(r) + y'(r) - \frac{n^2}{r}y(r) = 0$ and these were found in part (a) to be $y_1(r) = r^n, y_2(r) = r^{-n}$ and A_1, A_2, B_1, B_2 needs to be determined. Hence (2) becomes

$$
G(r,r_0) = \begin{cases} A_1r^n + A_2r^{-n} & 0 < r < r_0 \\ B_1r^n + B_2r^{-n} & r_0 < r < \infty \end{cases} \tag{3}
$$

The left boundary condition $\lim_{r\to 0} G(r,r_0) = 0$ implies $A_2 = 0$ and the right boundary condition $\lim_{r\to\infty} G(r,r_0) = 0$ implies $B_1 = 0$. This is needed to keep the solution bounded.

Hence (3) simplifies to

$$G(r,r_0)=\begin{cases} A_1 r^n & 0<r<r_0 \\ B_2 r^{-n} & r_0<r<\infty \end{cases} \tag{4}$$

To determine the remaining two constants A_1, B_2, two additional conditions are needed. The first is that $G(r,r_0)$ is continuous at $r=r_0$ which implies

$$A_1 r_0^n = B_2 r_0^{-n} \tag{5}$$

The second condition is the jump in the derivative of $G(r,r_0)$ given by

$$\left.\frac{d}{dr}G(r,r_0)\right|_{r>r_0} - \left.\frac{d}{dr}G(r,r_0)\right|_{r<r_0} = \frac{-1}{p(r_0)}$$

Where $p(r_0)$ comes from the Sturm Liouville form of the homogeneous ODE. This was found above as $p(r_0)=r_0$. Hence the above condition becomes

$$\left.\frac{d}{dr}G(r,r_0)\right|_{r>r_0} - \left.\frac{d}{dr}G(r,r_0)\right|_{r<r_0} = \frac{-1}{r_0}$$

Equation (4) shows that $\left.\frac{d}{dr}G(r,r_0)\right|_{r>r_0} = -nB_2 r_0^{-n-1}$ and that $\left.\frac{d}{dr}G(r,r_0)\right|_{r<r_0} = nA_1 r_0^{n-1}$. Using these in the above gives the second equation needed

$$-nB_2 r_0^{-n-1} - nA_1 r_0^{n-1} = \frac{-1}{r_0} \tag{6}$$

Solving (5,6) for A_1, B_1: From (5) $A_1 = B_2 r_0^{-2n}$. Substituting this in (6) gives

$$\begin{aligned} -nB_2 r_0^{-n-1} - n\left(B_2 r_0^{-2n}\right) r^{n-1} &= \frac{-1}{r_0} \\ -nB_2 r^{-n-1} - nB_2 r^{-n-1} &= \frac{-1}{r_0} \\ -2nB_2 r_0^{-n-1} &= -r_0^{-1} \\ B_2 &= \frac{-r_0^{-1}}{-2nr_0^{-n-1}} \\ &= \frac{1}{2n} r_0^n \end{aligned}$$

But since $A_1 = B_2 r_0^{-2n}$, then

$$\begin{aligned} A_1 &= \frac{1}{2n} r_0^n r_0^{-2n} \\ &= \frac{1}{2n} r_0^{-n} \end{aligned}$$

Therefore the solution (4), which is the Green function, becomes, for $n \neq 0$

$$G(r,r_0)=\begin{cases} \frac{1}{2n} r_0^{-n} r^n & 0<r<r_0 \\ \frac{1}{2n} r_0^n r^{-n} & r_0<r<\infty \end{cases} \tag{7}$$

3.7.3 key solution to HW 7

Problem Set 7 Solutions

i) $$g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2n\, H_{n-1}(x) \frac{t^n}{n!}$$

$$= 2t \underbrace{\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^{n-1}}_{\text{let } \tilde{n}=n-1} = 2t \sum_{\tilde{n}=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

or $$\boxed{\frac{\partial g}{\partial x} = 2t g}$$

ii) $$\frac{dg}{g} = 2t\,dx \Rightarrow \ln g = 2tx + f(t)$$

$$\Rightarrow \boxed{g = e^{2tx} e^{f(t)}}$$

iii) let $n = 2k$

$$g(0,t) = \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \frac{t^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = e^{-t^2}$$

$$\Rightarrow e^{f(t)} = e^{-t^2}$$

$$\Rightarrow g(x,t) = e^{-t^2 + 2tx}$$

iv) $e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

take $\frac{d}{dx}$ of the above equation

$\Rightarrow (-2t+2x)\, e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{n H_n(x)}{n!} t^{n-1}$

$\Rightarrow (-2t+2x) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{n H_n(x)\, t^{n-1}}{n!} \sim \tilde{n} = n-1$

$\tilde{n} = n+1$

$\Rightarrow -2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} + 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{\tilde{n}=0}^{\infty} \frac{H_{\tilde{n}+1}(x)\, t^{\tilde{n}}}{\tilde{n}!}$

or $-2 \sum_{\tilde{n}=1}^{\infty} \frac{\tilde{n} H_{\tilde{n}-1} t^{\tilde{n}}}{\tilde{n}!} + 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{\tilde{n}=0}^{\infty} \frac{H_{\tilde{n}+1}(x) t^{\tilde{n}}}{\tilde{n}!}$

$\Rightarrow \boxed{-2n H_{n-1} + 2x H_n = H_{n+1}}$ as required

v) $e^{-x^2} g(x,s)\, g(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_m(x) H_n(x) \frac{s^m}{m!} \frac{t^n}{n!} e^{-x^2}$

consider $\int_{-\infty}^{\infty} e^{-x^2} g(x,s)\, g(x,t) = \int_{-\infty}^{\infty} e^{-x^2} e^{-s^2+2xs} e^{-t^2+2tx} dx$

$= \int_{-\infty}^{\infty} e^{-x^2} e^{2x(s+t)} e^{-s^2} e^{-t^2} dx$

$= e^{-s^2} e^{-t^2} e^{(s+t)^2} \int_{-\infty}^{\infty} e^{-(x-s-t)^2} dx$

or $\int_{-\infty}^{\infty} e^{-x^2} g(x,s)\, g(x,t)\, dx = \sqrt{\pi}\, e^{2st}$

$$\Rightarrow \sqrt{\pi} \sum_{p=0}^{\infty} \frac{(st)^p 2^p}{p!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n s^m}{m!\, n!} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx$$

L equating power by power in s and t

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = f_m \delta_{m,n}$$

$$\Rightarrow \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(st)^m 2^m}{m!} = \sum_{m=0}^{\infty} \frac{(st)^m f_m}{(m!)^2}$$

$$\Rightarrow f_m = \sqrt{\pi}\, (m!)\, 2^m \text{ as required.}$$

Exercise 2

a) $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = 0 \qquad 0 < r < \infty$

let $x = \ln r \qquad \frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr} = r\frac{d}{dr}$

note $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = 0$ can be rewritten

as $\underbrace{r\frac{d}{dr}}_{\frac{d}{dx}}\left[\underbrace{r\frac{d}{dr}}_{\frac{d}{dx}} y(r)\right] - n^2 y(r) = 0$

$\rightarrow \quad \frac{d^2}{dx^2} y(x) - n^2 y(x) = 0$

$\Rightarrow \quad y = e^{\pm nx} = \begin{cases} r^n & \text{vanishes as } r \to 0 \\ r^{-n} & \text{vanishes as } r \to \infty \end{cases}$

b) Let $G(r, r') = \begin{cases} A r^n & 0 < r < r' \\ B r^{-n} & r' < r < \infty \end{cases}$

continuity at $r = r' \longrightarrow \quad A(r')^n = B(r')^{-n} = C$

so $A = C(r')^{-n}$ and $B = C(r')^n$

$\therefore \quad G(r, r') = C \begin{cases} (r/r')^n & 0 < r < r' \\ (r'/r)^n & r' < r < \infty \end{cases}$

Also $\frac{d}{dr}\left[r \frac{d}{dr} G(r, r')\right] - \frac{n^2}{r} G(r, r') = \delta(r - r')$

integrate from $r = r' - \varepsilon$ to $r = r' + \varepsilon$ and let $\varepsilon \to 0$

$\lim_{\varepsilon \to 0}\left[r \frac{d}{dr} G(r, r')\right]_{r'-\varepsilon}^{r'+\varepsilon} - \lim_{\varepsilon \to 0}\left[\frac{n^2}{r} G(r, r')\right]_{r'-\varepsilon}^{r'+\varepsilon} = 1$

$$\Rightarrow \quad c(-n)\left(\frac{r'}{r}\right)^n\Big|_{r'} - cn\left(\frac{r}{r'}\right)^n\Big|_{r'} = 1$$

$$\Rightarrow \quad c = \frac{-1}{2n}$$

$$\therefore \quad \boxed{G(r,r') = \frac{-1}{2n}\begin{cases}(r/r')^n & 0 < r < r' \\ (r'/r)^n & r' < r < \infty\end{cases}}$$