HW 5 (Second HW on ODE's), Math 601
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Contents

1 problems description

MATH 601 HW #2 ODEs given out oct 10, 2018

[1] Solve the initial value problems

a) $y' + 4y = 20$, $y(0) = 2$

b) $y' + 3y = \sin x$, $y(\pi/2) = 0.3$

c) $y' - y\left(1 + \frac{3}{x}\right) = x + 2$, $y(1) = e - 1$

[2] Solve the Bernoulli equation

$$y' + \frac{y}{3} = \frac{1-2x}{3} y^4$$

[3] Consider the model
$$\begin{cases} m \frac{dv}{dt} = W - B - kv \\ v(0) = 0 \end{cases}$$
for the sinking of a container in the ocean.

/ $v = v(t)$ - velocity, W - weight, B buoyancy force, k - drag coefficient, W, B, k constants /

If container hits bottom at $v_c = 12$ m/sec critical velocity or less it will not break. Determine the critical time t_c when the container reaches critical velocity, assuming

$W = 2254$ N, $B = 2090$ N, $k = 0.637$ kg/sec. $\left(N = \frac{kg\, m}{sec^2}\right)$

What is the critical depth beyond which the container might break up?

[4] Consider the Ricatti equation

$$y' = x^3 (y-x)^2 + y/x$$ and solve it

/ Hint consider a substitution $w(x) = y(x) - x$ /

[5] Under what conditions for the constants A, B, C, D is $(Ax + By) + (Cx + Dy)y' = 0$ exact diff. eq. Solve the equation!

2 problem 1

2.1 part a

The ODE to solve is

$$\frac{d}{dx}y(x) + 4\,y(x) = 20$$

with initial conditions $y(0) = 2$.

Trying separable ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x, y) \\ &= -4\,y + 20 \end{aligned}$$

The ODE $\frac{dy}{dx} = -4\,y + 20$, is separable. It can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

Where $f(x) = 1$ and $g(y) = -4\,y + 20$. Therefore

$$\frac{dy}{dx} = -4\,y + 20$$

Hence

$$\begin{aligned} (-4\,y + 20)^{-1}\,dy &= dx \\ \int (-4\,y + 20)^{-1}\,dy &= \int dx \\ -1/2\,\ln(2) - 1/4\,\ln(|y - 5|) &= x + C_1 \end{aligned}$$

Solving for y gives

$$y = -1/4\,e^{-4\,x - 4\,C_1} + 5$$

The solution above can be written as

$$\boxed{y = -1/4\,C_1 e^{-4\,x} + 5} \tag{1}$$

Initial conditions are now used to solve for C_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\begin{aligned} 2 &= -1/4\,C_1 e^{0} + 5 \\ &= -1/4\,C_1 + 5 \end{aligned}$$

Hence

$$C_1 = 12 \left(e^0\right)^{-1}$$

Which is simplified to

$$C_1 = 12$$

Substituting C_1 found above back in the solution gives

$$y(x) = -3\,e^{-4x} + 5$$

2.2 part b

The ODE to solve is

$$\frac{d}{dx} y(x) + 3\,y(x) = \sin(x)$$

with initial conditions $y(\pi/2) = 3/10$.

Trying Linear ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x,y) \\ &= -3\,y + \sin(x) \end{aligned}$$

The ODE is linear in y and has the form

$$y' = yf(x) + g(x)$$

Where $f(x) = -3$ and $g(x) = \sin(x)$.

Writing the ODE as

$$\begin{aligned} y' - (-3\,y) &= \sin(x) \\ y' + 3\,y &= \sin(x) \end{aligned}$$

Therefore the integrating factor μ is

$$\mu = e^{\int 3\,dx} = e^{3x}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= \mu(\sin(x)) \\ \frac{d}{dx}\left(ye^{3x}\right) &= \sin(x)\,e^{3x} \\ d\left(ye^{3x}\right) &= \left(\sin(x)\,e^{3x}\right)\,dx \end{aligned}$$

Integrating both sides gives

$$ye^{3x} = -1/10\,\cos(x)\,e^{3x} + 3/10\,\sin(x)\,e^{3x} + C_1$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = \frac{-1/10\ \cos(x)\,e^{3x} + 3/10\ \sin(x)\,e^{3x}}{e^{3x}} + \frac{C_1}{e^{3x}}$$

Simplifying the solution gives

$$\boxed{y = 3/10\ \sin(x) - 1/10\ \cos(x) + C_1 e^{-3x}}$$

Initial conditions are now used to solve for C_1. Substituting $x = \pi/2$ and $y = 3/10$ in the above solution gives an equation to solve for the constant of integration.

$$\begin{aligned}3/10 &= 3/10\ \sin(\pi/2) - 1/10\ \cos(\pi/2) + C_1 e^{-3/2\,\pi}\\ &= 3/10 + C_1 e^{-3/2\,\pi}\end{aligned}$$

Hence

$$C_1 = -1/10\,\frac{3\ \sin(\pi/2) - \cos(\pi/2) - 3}{e^{-3/2\,\pi}}$$

Which is simplified to

$$C_1 = 0$$

Substituting C_1 found above back in the solution gives

$$y(x) = 3/10\ \sin(x) - 1/10\ \cos(x)$$

2.3 part c

The ODE to solve is

$$\frac{d}{dx}y(x) - y(x)\left(1 + 3x^{-1}\right) = x + 2$$

with initial conditions $y(1) = e - 1$.

Trying Linear ODE.

In canonical form, the ODE is written as

$$\begin{aligned}y' &= F(x, y)\\ &= \frac{x^2 + xy + 2x + 3y}{x}\end{aligned}$$

The ODE is linear in y and has the form

$$y' = yf(x) + g(x)$$

Where $f(x) = \frac{x+3}{x}$ and $g(x) = \frac{x^2+2x}{x}$.

Writing the ODE as

$$y' - \left(\frac{(x+3)\,y}{x}\right) = \frac{x^2+2\,x}{x}$$
$$y' - \frac{(x+3)\,y}{x} = \frac{x^2+2\,x}{x}$$

Therefore the integrating factor μ is

$$\mu = e^{\int -\frac{x+3}{x}\,\mathrm{d}x} = \mathrm{e}^{-x-3\,\ln(x)}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = \mu\left(\frac{x^2+2\,x}{x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-x-3\,\ln(x)}\right) = \frac{\left(x^2+2\,x\right)\mathrm{e}^{-x-3\,\ln(x)}}{x}$$
$$\mathrm{d}\left(y\mathrm{e}^{-x-3\,\ln(x)}\right) = \left(\frac{\left(x^2+2\,x\right)\mathrm{e}^{-x-3\,\ln(x)}}{x}\right)\mathrm{d}x$$

Integrating both sides gives

$$y\mathrm{e}^{-x-3\,\ln(x)} = -\mathrm{e}^{-x-3\,\ln(x)}x + C_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x-3\,\ln(x)}$ results in

$$y = -x + \frac{C_1}{\mathrm{e}^{-x-3\,\ln(x)}}$$

Simplifying the solution gives

$$\boxed{y = -x + C_1 x^3 \mathrm{e}^x}$$

Initial conditions are now used to solve for C_1. Substituting $x = 1$ and $y = \mathrm{e} - 1$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} - 1 = -1 + C_1\mathrm{e}$$

Hence

$$C_1 = 1$$

Substituting C_1 found above back in the solution gives

$$y(x) = -x + x^3\mathrm{e}^x$$

3 problem 2

The ODE to solve is

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) + 1/3\,y(x) = 1/3\,(1-2\,x)\,(y(x))^4$$

Trying Bernoulli ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x, y) \\ &= -y/3 - 2/3\, y^4 x + 1/3\, y^4 \end{aligned}$$

This is a Bernoulli ODE. Comparing the ODE to solve

$$y' = -y/3 - 2/3\, y^4 x + 1/3\, y^4$$

With Bernoulli ODE standard form

$$y' = f_0(x) y + f_1(x) y^n$$

Shows that $f_0(x) = -1/3$ and $f_1(x) = -2/3\, x + 1/3$ and $n = 4$.

Dividing the ODE by y^4 gives

$$y' y^{-4} = -1/3 y^{-3} + -2/3\, x + 1/3 \tag{1}$$

Let

$$v = y^{-3} \tag{2}$$

Taking derivative of (2) w.r.t x gives

$$\begin{aligned} v' &= -3\, y^{-4} y' \\ y^{-4} &= \frac{v'}{-3\, y'} \end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$\begin{aligned} \frac{v'}{(-3)} &= (-1/3)\, v + -2/3\, x + 1/3 \\ v' &= (-3)\,(-1/3)\, v + (-3)\,(-2/3\, x + 1/3) \\ &= v + 2\, x - 1 \end{aligned}$$

The above now is a linear ODE in $v(x)$ which can be easily solved using an integrating factor.

In canonical form, the ODE is written as

$$\begin{aligned} v' &= F(x, v) \\ &= v + 2\, x - 1 \end{aligned}$$

The ODE is linear in v and has the form

$$v' = v f(x) + g(x)$$

Where $f(x) = 1$ and $g(x) = 2\, x - 1$.

Writing the ODE as

$$v' - (v) = 2x - 1$$
$$v' - v = 2x - 1$$

Therefore the integrating factor μ is

$$\mu = e^{\int -1 \, dx} = e^{-x}$$

The ode becomes

$$\frac{d}{dx}\mu v = \mu (2x - 1)$$
$$\frac{d}{dx}(v e^{-x}) = (2x - 1) e^{-x}$$
$$d(v e^{-x}) = ((2x - 1) e^{-x}) dx$$

Integrating both sides gives

$$v e^{-x} = -(2x + 1) e^{-x} + C_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$v = -2x - 1 + \frac{C_1}{e^{-x}}$$

Simplifying the solution gives

$$\boxed{v = -2x - 1 + C_1 e^x}$$

Replacing v in the above by y^{-3} from equation (2), gives the final solution.

$$y^{-3} = -2x - 1 + C_1 e^x$$

Solving for y gives

$$y = \frac{1}{\sqrt[3]{-2x - 1 + C_1 e^x}}$$
$$y = -1/2 \frac{1}{\sqrt[3]{-2x - 1 + C_1 e^x}} + \frac{i/2\sqrt{3}}{\sqrt[3]{-2x - 1 + C_1 e^x}}$$
$$y = -1/2 \frac{1}{\sqrt[3]{-2x - 1 + C_1 e^x}} - \frac{i/2\sqrt{3}}{\sqrt[3]{-2x - 1 + C_1 e^x}}$$

4 problem 3

The ODE to solve is

$$m \frac{d}{dx} v(x) = w - B - k v(x)$$

with initial conditions $v(0) = 0$.

Trying separable ODE.

In canonical form, the ODE is written as

$$\begin{aligned} v' &= F(x, v) \\ &= -\frac{kv + B - w}{m} \end{aligned}$$

The ODE $\frac{dv}{dx} = -\frac{kv+B-w}{m}$, is separable. It can be written as

$$\frac{dv}{dx} = f(x)g(v)$$

Where $f(x) = 1$ and $g(v) = \frac{-kv-B+w}{m}$. Therefore

$$\frac{dv}{dx} = \frac{-kv - B + w}{m}$$

Hence

$$\begin{aligned} \left(\frac{m}{-kv - B + w}\right) dv &= dx \\ \int \left(\frac{m}{-kv - B + w}\right) dv &= \int dx \\ -\frac{m \ln(|kv + B - w|)}{k} &= x + C_1 \end{aligned}$$

Solving for v gives

$$v = \frac{1}{k}\left(-e^{-\frac{k(x+C_1)}{m}} - B + w\right)$$

Initial conditions are now used to solve for C_1. Substituting $x = 0$ and $v = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{k}\left(-e^{-\frac{kC_1}{m}} - B + w\right)$$

Hence

$$C_1 = -\frac{m \ln(-B + w)}{k}$$

Substituting C_1 found above back in the solution gives

$$v(x) = \frac{1}{k}\left(-e^{-\frac{k}{m}\left(x - \frac{m \ln(-B+w)}{k}\right)} - B + w\right)$$

The solution $\frac{1}{k}\left(-e^{-\frac{k}{m}\left(x - \frac{m \ln(-B+w)}{k}\right)} - B + w\right)$ can be simplified to

$$\boxed{v(x) = \frac{1}{k}\left(-e^{\frac{m \ln(-B+w) - xk}{m}} - B + w\right)} \tag{2}$$

5 problem 4

The ODE to solve is

$$\frac{d}{dx}y(x) = x^3 (y(x) - x)^2 + \frac{y(x)}{x}$$

Trying Riccati ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^6 - 2x^5 y + x^4 y^2 + y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^5 - 2x^4 y + x^3 y^2 + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^5$, $f_1(x) = \frac{-2x^5+1}{x}$ and $f_2(x) = x^3$.

Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u x^3} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 3x^2 \\ f_1 f_2 &= (-2x^5 + 1) x^2 \\ f_2^2 f_0 &= x^{11} \end{aligned}$$

Substituting the above terms back in (2) gives

$$x^3 \frac{d^2}{dx^2}u(x) - \left(3x^2 + (-2x^5 + 1) x^2\right) \frac{d}{dx}u(x) + x^{11} u(x) = 0$$

Solving the above ODE gives

$$u(x) = \mathrm{e}^{-1/5\,x^5}\left(x^5 C_2 + C_1\right)$$

The above shows that

$$u'(x) = -x^4 \mathrm{e}^{-1/5\,x^5}\left(x^5 C_2 + C_1 - 5\,C_2\right)$$

Hence, using the above in (1) gives the solution

$$y(x) = \frac{x\left(x^5 C_2 + C_1 - 5\,C_2\right)}{x^5 C_2 + C_1}$$

Dividing both numerator and denominator by C_2 gives, after renaming the constant $\frac{C_1}{C_2} = C_0$ the following

$$\boxed{y(x) = \frac{x\left(x^5 + C_0 - 5\right)}{x^5 + C_0}}$$

6 Key solution