University Course

Math 320
Linear algebra and Differential Equations

University of Wisconsin, Madison

Spring 2017

My Class Notes

Nasser M. Abbasi

Fall Spring 2017

Contents

Chapter 1

Introduction

1.1 links

1. Professor Leslie M. Smith web page
2. TA web site
3. piazza class page Needs login

1.2 syllabus

Math 320, Lecture 5: Syllabus
Linear Algebra and Differential Equations
TR 9:30-10:45 in Van Vleck B239

Textbook: *Differential Equations and Linear Algebra*, 3rd Edition, Edwards and Penney, Prentice Hall.

Pre-requisite: Math 222.

Credit toward the math major may not be received for both Math 320 and Math 340.

Professor: Leslie Smith, Departments of Mathematics and Engineering Physics, Office Hours in Van Vleck 825 TR 12:30-2:00, lsmith@math.wisc.edu.

Teaching Assistant:

Jingrui Cheng, Office Hours MWF 7:00-8:00 PM in Van Vleck 516

Exams: There will be two evening exams: **Tuesday February 21** and **Tuesday, March 28**, during the time 7:15-8:30 PM. Please let me know IMMEDIATELY if you have a conflict with these dates. Each exam is 25% of the final grade.

Final Exam: Sunday May 7, 7:25-9:25 PM, 35% of grade.

Piazza: There will be a Piazza course page where all course materials will be posted, and to facilitate peer-group discussions.

Piazza Sign-Up Page: piazza.com/wisc/spring2017/math320005
Piazza Course Page: piazza.com/wisc/spring2017/math320005/home

Weekly Problem Sets: Homework is **due at the beginning of class**, typically on Thursday. Homework will be available on piazza approximately one week prior to the due date. Roughly 15 problems will be assigned each week (most of the time from the book, but not always).

Please write your name and section number clearly on each homework set, stapled please! Unstapled homework will not be accepted.

Grading of Homework: The TA and/or a grader will grade a subset of the homework problems given out each week, with some points also given for completeness. Typically (but not necessarily always), there will be 2 problems graded on a scale of 0-10, with 6 points for completeness. The homework scores will count for 15% of the grade.

Late Policy: Homework turned in after the beginning of class will be considered late and will be graded at 80% credit. Late homework will be accepted until 5 PM on the due date (no credit thereafter, no exceptions). The policy is intended to keep everyone as current as possible.

Please email Jingrui Cheng directly to make arrangements regarding late homework submission: jcheng37@wisc.edu

Calculators: Calculators and/or computer software may be used to help with homework problems but are not permitted during exams.

Grading Scale for Final Grade: 92-100 A, 89-91 AB, 82-88 B, 79-81 BC, 70-78 C, 60-69 D, 59 and below F

Course description: Differential equations are the fundamental tools that scientists and engineers use to model physical reality. The importance of differential equations to science and engineering cannot be over-emphasized. A distinct subject in its own right, linear algebra is a part of mathematics concerned with the structure inherent in mathematical systems. We shall study these subjects together for three reasons: (1) The viewpoint of linear algebra is immensely helpful in uncovering the order underlying the topic of differential equations; it helps us understand the "why" and not just the "how" of our calculations; (2) Linear algebra is essential to the theory of differential equations; (3) Linear algebra is crucial to the computer approximations which are often the only way to solve the most challenging differential equations.

Throughout this course, we will seek to answer the following basic questions:

- When does a differential equation have a solution? When is that solution unique?
- Can one construct the (unique) solution of a differential equation in terms of elementary functions? If not, can one approximate its solution numerically and/or understand it qualitatively?
- How does one choose the differential equation(s) used to model a physical system? What are the strengths and limitations of such models? Specifically, what is the significance of *linearity* in our models and applications?

Course outline: The course covers material in Chapters 1-8 of the text. The topics are listed below with corresponding chapter.

Chapter 1: First-Order ODEs (continuing from 221/222 with some review).

Chapter 2: Mathematical Models and Numerical Methods.

Chapter 3: Linear Systems and Matrices.

Chapter 4: Vectors Spaces.

Chapter 5: Higher-Order Linear ODEs.

Chapter 6: Eigenvalues and Eigenvectors (sections 6.1-6.2).

Chapter 7: Homogeneous Linear Systems of ODEs.

Chapter 8: Nonhomogeneous Linear Systems of ODEs (sections 8.1-8.2).

Learning Outcomes:

- Students will state and apply the Theorem of Existence and Uniqueness for first-order ODEs, and the Theorem of Existence and Uniqueness for second-order linear ODEs.
- Students will find analytical solutions to first-order ODEs, including (but not limited to) separable ODEs and linear ODEs.
- Students will construct approximate solutions to first-order ODEs using numerical methods.

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- Students will state and apply rules for the algebra of matrices.
- Students will demonstrate knowledge of coupled linear algebraic equations, determine when the system has solution(s), and be able to find solution(s) when applicable.
- Students will find analytical solutions to second-order linear ODEs in simple cases.
- Students will demonstrate the relation between higher-order linear ODEs and coupled first-order linear ODEs.
- Students will apply knowledge of linear algebra and differential equations to solve coupled first-order linear ODEs with constant coefficients.

Chapter 2

Exams

Local contents

2.1 First exam

2.1.1 crib sheet for first exam

Math 320 Exam Crib Sheet

1. **Integration by Parts Formula**

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int x\exp(x)dx = x\exp(x) - \int \exp(x)dx + C = x\exp(x) - \exp(x) + C$$

with $u = x$, $dv = \exp(x)\,dx$, $du = dx$, and $v = \exp(x)$.

2. **Example of Partial Fractions**

$$\int \frac{5}{(x^2-5x+6)}dx = \int \frac{5}{(x-2)(x-3)}dx$$

Let

$$\frac{5}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

Therefore

$$(A+B)x = 0 \quad \text{and} \quad -3A-2B = 5.$$

Solving $A+B=0$ and $-3A-2B=5$ gives $A=-5$ and $B=5$. So finally

$$\int \frac{5}{(x^2-5x+6)}dx = \int \frac{-5}{(x-2)}dx + \int \frac{+5}{(x-3)}dx = -5\ln|x-2| + 5\ln|x-3| + C.$$

3. **Exponentials and the Natural Logarithm**: All arguments of ln are assumed greater than zero.

$$\ln(1) = 0$$

$$\ln(a/b) = \ln(a) - \ln(b)$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^r) = r\ln(a)$$

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$$\int \frac{1}{u} du = \ln |u| + C, \quad u \neq 0$$

$$\exp(\ln(x)) = x$$

$$\ln(\exp(x)) = x$$

$$\exp(a+b) = \exp(a)\exp(b)$$

$$\exp(a-b) = \frac{\exp(a)}{\exp(b)}$$

$$\exp(ab) = (\exp(a))^b = (\exp(b))^a$$

4. **Taylor Series for $f(x)$ about the point $x = x_o$:**

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(x)|_{x=x_o} \frac{(x-x_o)^n}{n!}$$

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2.1.2 First Practice exam for first exam

Math 320 (Smith): Practice Exam 1

1. The autonomous ODE given by

$$\frac{dP(t)}{dt} = -(bP^2(t) - aP(t) + h), \quad a > 0, \ b > 0, \ h > 0 \tag{1}$$

models a logistic population with harvesting, for example, the population of fish in a lake from which h fish per year are removed by fishing.

(a) Consider $a = 6$ and $b = 1$. How does the number of critical points depend on the parameter h? What are the values of h that yield real-valued critical point(s)?

(b) Consider $a = 6$, $b = 1$ and $h = 7$. Find and classify the critical points. Make a (rough) sketch of the direction field.

(c) For $a = 6$, $b = 1$, $h = 7$, and starting from the initial condition $P(0) = 3$, find the limiting behavior for large time $t \to \infty$.

2. The following augmented coefficient matrix results from elementary row operations on a 3×3 system of linear algebraic equations $\mathbf{Ax} = \mathbf{b}$.

$$\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & p+3 \end{bmatrix} \tag{2}$$

Consider 2 different values of the parameter p: (a) $p = -3$, and (b) $p = -2$.

Determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.

FOR PART (a) ONLY when $p = -3$: Find all solutions in cases (i) and/or (iii), and write the solution $\mathbf{x}$ in vector form.

3. Given

$$\frac{dy}{dx} = -\frac{y(x)}{(x-1)} + \frac{\exp(-x)}{(x-1)}, \quad y(0) = 2. \tag{3}$$

(a) Find the exact solution. For what values of x is the solution defined?

(b) Use one step of the Forward Euler method with step size h to find an approximation for $y(h)$.

4. (20 points) Consider the initial value problem

$$\frac{dy}{dx} = -\frac{5}{2}x^4y^3, \quad y(0) = -1. \tag{4}$$

(a) Find $y(x)$ explicitly. For what values of x is the solution defined?

(b) Use one step of the Modified Euler (Improved Euler, RK2) method with step size h to find an approximation for $y(h)$.

5. (5 points) TRUE or FALSE: The initial value problem

$$\frac{dy}{dt} = (y-1)^{3/2}, \quad y(1) = 2 \tag{5}$$

is guaranteed to have a unique solution in a subrange of $-\infty < t < \infty$.

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2.1.3 Second Practice exam for first exam

Math 320 (Smith): Practice Exam 1

1. (16 points - 10 min.) For

$$\frac{dy}{dt} = (e^{y+2} - 1)(e^{y} - 1)(y - 2), \quad y(0) = y_o, \quad -\infty < y_o < \infty \tag{1}$$

(a) (14 points) Sketch, roughly, a direction field and classify all critical points.
(b) (2 points) Determine (from your sketch), the asymptotic behavior of the solution for $y_o = -1$, $t \to \infty$.

2. (21 points - 15-20 min.) Solve (15 points)

$$y' - y^3 x \exp(x^2) = 0 \tag{2}$$

for $y(0) = -2$. **Give the range of validity of the solution** (6 points).

3. (24 points - 15-20 min) Write the following systems as $\mathbf{Ax} = \mathbf{b}$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.
(a) (12 points)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= k \\ -2x_1 + x_2 - 3x_3 &= 2 \end{aligned}$$

(b) (12 points)

$$\begin{aligned} x_1 + 3x_3 &= 8 \\ -x_1 + kx_2 - x_3 &= 4 \\ 3x_1 + x_2 + 10x_3 &= 0 \end{aligned}$$

4. (39 points - 20-25 min) Given

$$\frac{dy}{dx} = y + \exp(x), \quad y(0) = 2. \tag{4}$$

(a) (15 points) Find the exact solution and state the region of validity of the exact solution.
(b) (8 points) Use one step of the Forward Euler method with step h to find an approximation for $y(x_o + h)$.
(c) (8 points) Use one step of the Improved Euler method with step h to find an approximation for $y(x_o + h)$.
(d) (8 points) Compare the Taylor series expansions for $y(x_o + h)$ using (i) the exact solution, (ii) the Forward Euler approximation and (iii) the Improved Euler approximation. Explain what these Taylor series expansions tell us about the truncation error of the Forward Euler and Improved Euler methods.

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2.1.4 my solution for first practice exam for first midterm

2.1.4.1 Problem 1

$$\frac{dP(t)}{dt} = -\left(bP^2(t) - aP(t) + h\right)$$

<u>Part(a)</u>

For $a = 6, b = 1$ the ODE becomes

$$\frac{dP(t)}{dt} = -\left(P^2(t) - 6P(t) + h\right)$$

Critical points are given by $\frac{dP(t)}{dt} = 0$. Hence solving for P from

$$P^2 - 6P + h = 0 \tag{1}$$

$$P_c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 4h}}{2} = 3 \pm \sqrt{9 - h}$$

We see now how P_c depends on h. For real valued P_c we want $9 - h > 0$ or

$$h < 9$$

Part(b)

For $a=6, b=1, h=7$ then

$$\frac{dP(t)}{dt} = -\left(P^2(t) - 6P(t) + 7\right)$$

And the critical P_c values are from (1)

$$\begin{aligned} P_c &= 3 \pm \sqrt{9-7} \\ &= 3 \pm \sqrt{2} \\ &= \{4.4142, 1.5858\} \end{aligned}$$

To classify P_c we look at little above and little below each critical value and see what the slope is there. Depending on the sign of the slope around each critical point, we will know if it stable, not stable, or semi-stable. For $P_c = 4.4142$, lets look at $P=5$ and $P=4$

$$\left(-\left(P^2(t) - 6P(t) + 7\right)\right)_{P=5} = -(25 - 6(5) + 7) = -2$$
$$\left(-\left(P^2(t) - 6P(t) + 7\right)\right)_{P=4} = -(16 - 6(4) + 7) = 1$$

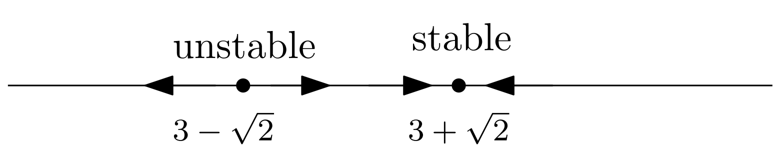
Since the slope is negative to the right of $P_c = 4.4142$ and the slope is positive to the left of $P_c = 4.4142$, this means $P_c = 4.4142$ is stable.

For $P_c = 1.5858$, let look at $P=2$ and $P=1$

$$\left(-\left(P^2(t) - 6P(t) + 7\right)\right)_{P=2} = -(4 - 6(2) + 7) = 1$$
$$\left(-\left(P^2(t) - 6P(t) + 7\right)\right)_{P=1} = -(1 - 6(1) + 7) = -2$$

Since the slope is positive to the right of $P_c = 1.5858$ and the slope is negative to the left of $P_c = 1.5858$, this means $P_c = 1.5858$ is unstable.

Here is the phase plot



Here is sketch of the slope field diagram using the computer showing the two critical values of $P(t)$ found above, confirming that one is stable, and the other is not stable.

```
In[405]:= f[t_, y_] := - (y^2 - 6 y + 7)
p1 = StreamPlot[{1, f[t, y]}, {t, -2, 6}, {y, -1, 7}, Frame → False, Axes → True,
  AspectRatio → 1 / GoldenRatio, AxesLabel → {"t", "P(t)"}, BaseStyle → 14,
  PlotLabel → "Problem 1, part a", TicksStyle → Red, ImageSize → 400]
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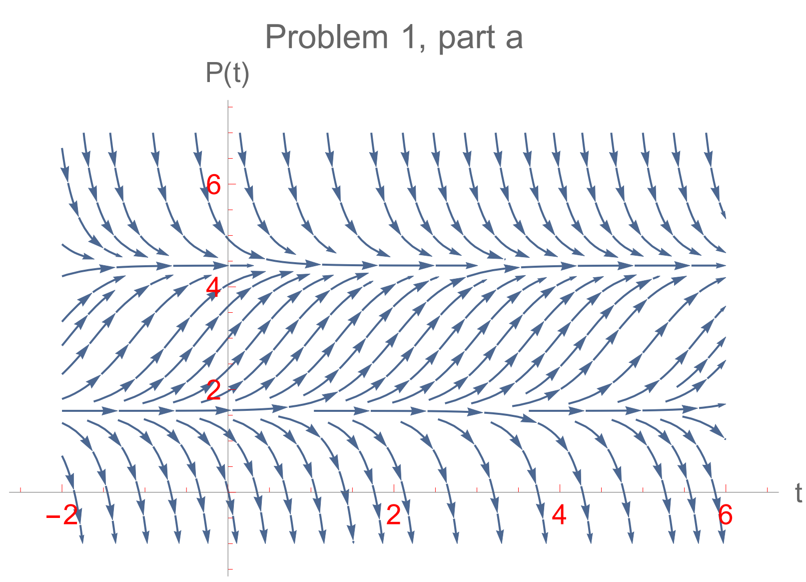

Part(c)

For $a=6, b=1, h=7$ then

$$\frac{dP(t)}{dt} = -\left(P^2(t) - 6P(t) + 7\right)$$

Since $P(0) = 3$, then we see from part(b) sketch of slope field, that the solution curve will move to the critical point $P_c = 3 + \sqrt{2}$. Therefore for $t \to \infty$, $P(t) = 3 + \sqrt{2}$. Here is the slope field diagram, with the solution curve marked as red showing it is moving to the equilibrium solution.

```
In[403]:= f[t_, y_] := -(y^2 - 6 y + 7)
p1 = StreamPlot[{1, f[t, y]}, {t, -2, 6}, {y, -1, 7}, Frame → False, Axes → True,
  AspectRatio → 1 / GoldenRatio, AxesLabel → {"t", "P(t)"}, BaseStyle → 14,
  StreamPoints → {{{{0, 3}, Red}, Automatic}}, PlotLabel → "Problem 1, part a",
  TicksStyle → Red, ImageSize → 400]
```

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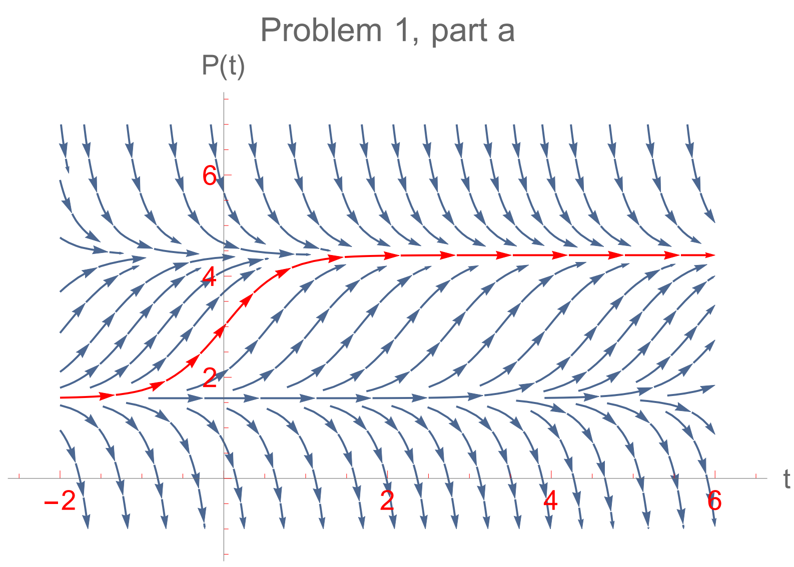


2.1.4.2 Problem 2

$$\begin{pmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & p+3 \end{pmatrix}$$

Part (a)

Using $p = -3$

$$\begin{pmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & 0 \end{pmatrix}$$

case (i) Last equation says that $kx_3 = 0$. If $k \neq 0$, then only $x_3 = 0$ will satisfy the equation. Which gives, from second equation $5x_2 - kx_3 = 4$ or $x_2 = \frac{4}{5}$. And from first equation $-x_1 + x_2 + x_3 = 2$ or $-x_1 = 2 - x_2 = 2 - \frac{4}{5}$. Hence $x_1 = \frac{4}{5} - 2 = -\frac{6}{5}$ Therefore $k \neq 0$ gives unique solution. The solution in vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$$

case (ii) There is no value of k which gives no solution.

case (iii) If $k = 0$ then we have $0(x_3) = 0$. Hence any x_3 value will satisfy this. So there are infinite number of solutions. Let $x_3 = t$, hence from second equation $5x_2 - kt = 4$ or $x_2 = \frac{4+kt}{5}$ and from the first equation $-x_1 + \frac{4+kt}{5} + t = 2$ or $-x_1 = 2 - t - \frac{4+kt}{5}$, hence $x_1 = t + \frac{1}{5}kt - \frac{6}{5}$. The solution in vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t + \frac{1}{5}kt - \frac{6}{5} \\ \frac{4+kt}{5} \\ t \end{pmatrix} = \begin{pmatrix} t - \frac{6}{5} \\ \frac{4}{5} \\ t \end{pmatrix}_{k=0}$$

Part (b)

Using $p = -2$

$$\begin{pmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & 1 \end{pmatrix}$$

case (i) Last equation says that $kx_3 = 1$. If $k \neq 0$, then unique solution exist. But if $k = 0$, we have $(0)\, x_3 = 1$ which is not possible. So for unique solution we need $k \neq 0$ for unique solution.

case (ii) If $k = 0$ we have $(0)\, x_3 = 1$ which is not possible. Hence $k = 0$ gives no solutions.

case (iii) There is no value of k which gives infinite number of solutions.

2.1.4.3 Problem 3

$$\frac{dy}{dx} = -\frac{y}{(x-1)} + \frac{e^{-x}}{x-1}; y(0) = 2$$

part (a)

$$\frac{dy}{dx} = \frac{-y + e^{-x}}{(x-1)}$$

Hence

$$f(x, y) = \frac{-y + e^{-x}}{(x-1)}$$

This is continuous in x except at $x = 1$. And continuous for all y. Hence solution exist in region that does not include $x = 1$. Now $\frac{\partial f}{\partial y} = \frac{-1}{(x-1)}$. We see also here that This is continuous in x except at $x = 1$. No dependency on y. Hence solution exist and unique in some region that do not include $x = 1$. So solve, we use integrating factor

$$\frac{dy}{dx} + \frac{y}{(x-1)} = \frac{e^{-x}}{x-1} \tag{1}$$

$$\mu = e^{\int \frac{1}{x-1} dx} = e^{\ln(x-1)} = (x-1)$$

Therefore, by multiplying both sides of (1) by μ, we obtain

$$\frac{d}{dx}(\mu y) = \mu \frac{e^{-x}}{x-1}$$

$$\frac{d}{dx}((x-1)\, y) = (x-1) \frac{e^{-x}}{x-1}$$

$$= e^{-x}$$

Integrating both sides

$$(x-1)\, y = -e^{-x} + c$$

$$y(x) = \frac{e^{-x}}{1-x} + \frac{c}{x-1}$$

From initial conditions

$$2 = \frac{1}{1} + \frac{c}{-1}$$

$$c = -1$$

Hence the exact solution is

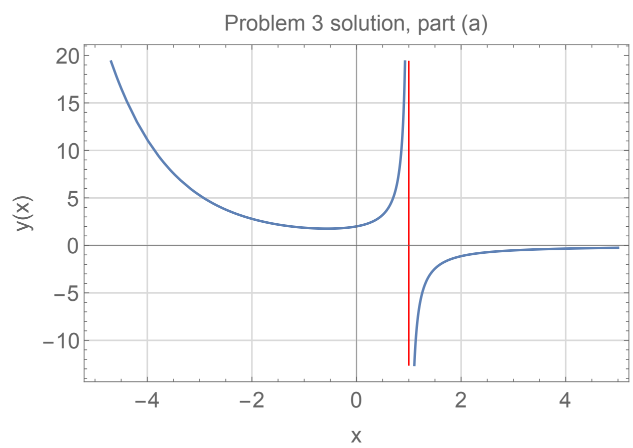
$$\begin{aligned} y(x) &= \frac{e^{-x}}{1-x} + \frac{1}{1-x} \\ &= \frac{e^{-x}+1}{1-x} \end{aligned}$$

Since initial conditions is at $x=0$ and since we found above that solution region can not include point $x=1$, then the solution region is $\boxed{-\infty < x < 1}$

Here is a plot of the solution showing the singularity at $x=1$. For our case, the solution curve is the one to the left of $x=1$ in this diagram

```
In[424]:= s = y[x] /. First@DSolve[{y'[x] + y[x] / (x - 1) == Exp[-x] / (x - 1), y[0] == 2}, y[x], x];
Plot[s, {x, -5, 5}, Frame → True, FrameLabel → {{"y(x)", None}, {"x", "Problem 3 solution, part (a)"}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray, ImageSize → 400,
 ExclusionsStyle → Red]
```

Out[425]=



Part (b)

In Forward Euler, we have

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)$$

In this problem $f\left(x,y\right) = -\frac{y}{(x-1)} + \frac{e^{-x}}{x-1}$, hence

$$y_{n+1} = y_n + h\left(-\frac{y_n}{(x_n-1)} + \frac{e^{-x_n}}{x_n-1}\right)$$

For $n=0$, we have

$$y_1 = y_0 + h\left(-\frac{y_0}{(x_0-1)} + \frac{e^{-x_0}}{x_0-1}\right)$$

But $y_0 = 2$ at $x_0 = 0$, hence the above becomes

$$\begin{aligned} y_1 &= y_0 + h\left(-\frac{2}{-1} + \frac{1}{0-1}\right) \\ &= y_0 + h \end{aligned}$$

Therefore, after one step

$$y(h) = y(0) + h$$

2.1.4.4 Problem 4

$$\frac{dy}{dx} = -\frac{5}{2}x^4y^3; y(0) = -1$$

Part (a)

$f\left(x,y\right) = -\frac{5}{2}x^4y^3$. We see that this is continuous for all x and all y. $\frac{\partial f}{\partial y} = -\frac{5}{2}3x^4y^2$. This is also continuous for all x and all y. Therefore a solution exist and is unique in some region inside $-\infty < x < \infty$.

Now we solve the ODE. This is separable. Hence

$$\frac{dy}{y^3} = -\frac{5}{2}x^4 dx$$

Integrating

$$\frac{-1}{2y^2} = -\frac{1}{2}x^5 + c$$

Applying initial conditions

$$\frac{-1}{2} = c$$

Hence exact solution is

$$\begin{aligned}\frac{-1}{2y^2} &= -\frac{1}{2}x^5 - \frac{1}{2}\\ &= \frac{-x^5 - 1}{2}\end{aligned}$$

Hence $\frac{-1}{y^2} = -x^5 - 1$ or

$$\begin{aligned}y^2 &= \frac{-1}{-x^5 - 1}\\ &= \frac{1}{x^5 + 1}\\ y &= \pm\sqrt{\frac{1}{x^5 + 1}}\end{aligned}$$

But since $y(0) = -1$, then at this point, using the above solution, we see that $-1 = \pm\sqrt{\frac{1}{1}}$. Hence only the negative sign can be used, to satisfy the initial conditions. Therefore, the solution becomes

$$y = -\sqrt{\frac{1}{x^5 + 1}}$$

Since the solution must be real, then $x^5 = -1$ is not allowed (or $x = -1$ is not allowed). And since we started at $x = 0$, then the solution is valid for

$$\boxed{-1 < x < \infty}$$

Here is a plot of the solution curve

```
In[449]:= ClearAll[y, x]
ode = y'[x] == -5/2 x^4 y[x]^3;
s = y[x] /. First@DSolve[{ode, y[0] == -1}, y[x], x]
Plot[s, {x, -2, 5}, Frame → True, FrameLabel → {{"y(x)", None}, {"x", "Problem 4 solution, part (a)"}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray, ImageSize → 400, ExclusionsStyle → Red]
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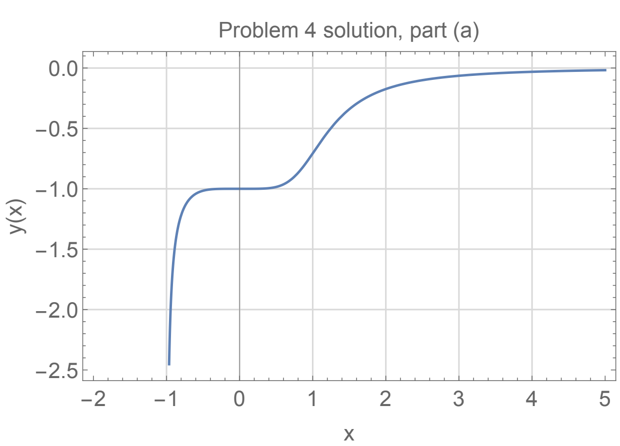


Part (b)

In rk2, we have

$$\begin{aligned}k_1 &= f\left(x_n, y_n\right)\\ u_{n+1} &= y_n + hk_1\\ k_2 &= f\left(x_{n+1}, u_{n+1}\right)\\ y_{n+1} &= y_n + h\frac{1}{2}\left(k_1 + k_2\right)\end{aligned}$$

In this problem $f\left(x,y\right)=-\frac{5}{2}x^4y^3$, hence

$$k_1=-\frac{5}{2}x_n^4y_n^3$$

For $n=0$, we have

$$k_1=-\frac{5}{2}x_0^4y_0^3$$

But $y_0=-1$ at $x_0=0$, hence the above becomes

$$k_1=0$$

Hence

$$\begin{aligned}u_1&=y_0+hk_1\\&=y_0\\&=-1\end{aligned}$$

And

$$\begin{aligned}k_2&=f\left(x_1,u_1\right)\\&=-\frac{5}{2}x_1^4u_1^3\\&=-\frac{5}{2}h^4\left(-1\right)^3\\&=\frac{5}{2}h^4\end{aligned}$$

Hence

$$\begin{aligned}y_1&=y_0+h\frac{1}{2}\left(k_1+k_2\right)\\&=-1+h\frac{1}{2}\left(0+\frac{5}{2}h^4\right)\\&=\frac{5}{4}h^5-1\end{aligned}$$

2.1.4.5 Problem 5

$$\frac{dy}{dt}=\left(y-1\right)^{\frac{3}{2}};y\left(1\right)=2$$

Here

$$f\left(t,y\right)=\left(y-1\right)^{\frac{3}{2}}$$

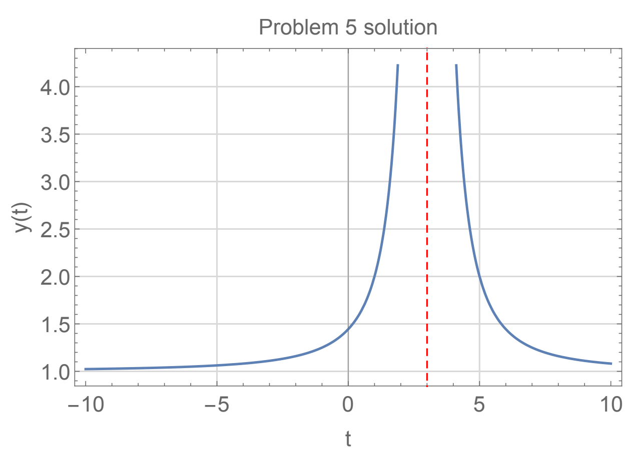
This does not depend on t. If $y<1$, then $\left(y-1\right)^{\frac{3}{2}}$ will be complex valued. Hence for real solution, we want $y\geq 1$. $\frac{\partial f}{\partial y}=\frac{3}{2}\left(y-1\right)^{\frac{1}{2}}$. This does not depend on t. Therefore a solution exist and is unique in some region $-\infty<t<\infty$. As long as $y\geq 1$. Hence TRUE

Note: When solving this, the solution came out to be $y\left(t\right)=\frac{t^2-6t+13}{\left(t-3\right)^2}$, which means the solution below up at $t=3$. i.e the solution is singular at $t=3$. Therefore, the subrange is $-\infty<t<-3$. (we were not asked to find the subrange?) Just to answer that there exist some subrange. Here is a plot of the solution

```
In[504]:= ClearAll[y, x]
ode = y'[t] == (y[t] - 1)^(3/2);
s = y[t] /. First@DSolve[{ode, y[1] == 2}, y[t], t]
Plot[s, {t, -10, 10}, Frame → True, FrameLabel → {{"y(t)", None}, {"t", "Problem 5 solution"}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray, ImageSize → 400, ExclusionsStyle → Red,
 ExclusionsStyle → Red, Epilog → {Dashed, Red, Line[{{3, 0}, {3, 5}}]}]
```

Out[506]= $\frac{13 - 6t + t^2}{(-3 + t)^2}$

Out[507]=



2.2 second exam

2.2.1 practice exam questions

Math 320 (Smith): Practice Problems for Exam 2

1. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 3 \\ -1/3 & -3/2 & -1 \\ -1/2 & -1/4 & -3/2 \end{bmatrix}, \qquad (1)$$

for what vectors $\mathbf{b}$ does $\mathbf{Ax} = \mathbf{b}$ have a solution?

2. (a) For what vectors $\mathbf{b}$ does $\mathbf{Ax} = \mathbf{b}$ have a solution, with $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1/2 \\ 3 & 1 & 2 \\ 0 & 6 & 3 \end{bmatrix}. \qquad (3)$$

(b) Find all possible solutions (or no solution) for $\mathbf{b}^T = [0\ \ 1\ \ 12/5]$ and for $\mathbf{b}^T = [0\ \ 12/5\ \ 1]$.

3. Consider $\mathbf{Ax} = \mathbf{b}$ for

$$\mathbf{A} = \begin{bmatrix} 2/3 & a_{12} & -2 \\ -1/5 & -1/3 & 3/5 \\ 1/2 & 5/6 & -3/2 \end{bmatrix}. \qquad (3)$$

(a) For what values of a_{12} is $\mathbf{A}$ non-singular?
(b) For what values of a_{12} is $\mathbf{A}$ singular?
(c) In all cases of $\mathbf{A}$ singular, analyze the system $\mathbf{Ax} = \mathbf{b}$. What vectors $\mathbf{b}$ lead to solutions $\mathbf{x}$? What are those solutions $\mathbf{x}$?

4. Given that two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, are $3\mathbf{u} - 5\mathbf{v}$ and $\mathbf{v}$ linearly dependent or linearly independent? Prove your answer.

5. Are the following statements TRUE or FALSE? If the statement is false, correct it.

(a) A square matrix with two identical rows is row equivalent to the identity matrix.

(b) The inverse of a square matrix $\mathbf{A}$ exists if $\mathbf{A}$ is row equivalent to the identity matrix $\mathbf{I}$ with the same dimensions.

(c) The determinant of an upper triangular square matrix is the sum of the diagonal elements.

6. Prove Property 4 of the seven properties of determinants.

7. Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \qquad (7)$$

(a) Find a condition on a_{32} and a_{33} such that $\mathbf{A}^{-1}$ exists.

(b) Find the value of the determinant for $a_{32} = 1$ and $a_{33} = -2$. How many columns of $\mathbf{A}$ are independent for $a_{32} = 1$ and $a_{33} = -2$?

(c) For $a_{32} = 5$ and $a_{33} = -4$, can $\mathbf{p}^T = [3\ \ 5\ \ 0]$ be expressed as a linear combination of the columns of $\mathbf{A}$? Support your answer with a calculation (no work, no credit).

(d) Find the value of the determinant for $a_{32} = 5$ and $a_{33} = -4$. How many columns of $\mathbf{A}$ are independent for $a_{32} = 5$ and $a_{33} = -4$?

1

8. (a) Consider a 3×3 matrix $\mathbf{A}$. Show that $\det(\mathbf{A}^T) = \det(\mathbf{A})$.
[In fact, $\det(\mathbf{A}^T) = \det(\mathbf{A})$ for $\mathbf{A}$ $n \times n$.]

(b) The square matrix $\mathbf{A}$ is called orthogonal if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that the determinant of an orthogonal matrix is either $+1$ or -1. You may use the fact that $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.

9. Find the determinant Hint: Use elementary row operations.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ -1 & 2 & 3 & 4 \\ 1 & 3 & 1 & -2 \\ -1 & -3 & 0 & -4 \end{bmatrix} \tag{9}$$

10. Using elementary row operations, find the inverse of

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}. \tag{10}$$

11. (a) Show that any plane through the origin is a subspace of $\mathbf{R}^3$.
(b) Show that the plane $x + 3y - 2z = 5$ is not a subspace of $\mathbf{R}^3$.

2

2.2.2 my solution to second midterm practice exam

2.2.2.1 Problem 1

Question: Given Matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ \frac{-1}{3} & \frac{-3}{2} & -1 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} \end{pmatrix}$$

for what vectors $\bar{b}$ does $A\bar{x} = \bar{b}$ have a solution?

answer Let $\bar{b} = (b_1, b_2, b_3)$. We start by setting up the augmented matrix. The augmented matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ \frac{-1}{3} & \frac{-3}{2} & -1 & b_2 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} & b_3 \end{pmatrix}$$

Applying row operation: $R_2 = R_2 + \frac{1}{3}R_1$ gives

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ 0 & -\frac{4}{3} & 0 & b_2 + \frac{b_1}{3} \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_3 = R_3 + \frac{1}{2}R_1$ gives

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ 0 & -\frac{4}{3} & 0 & b_2 + \frac{b_1}{3} \\ 0 & 0 & 0 & b_3 + \frac{1}{2}b_1 \end{pmatrix}$$

The above is Echelon form. Therefore, from last row, we see that $0x_3 = b_3 + \frac{1}{2}b_1$. For solution to exist, we need $b_3 + \frac{1}{2}b_1 = 0$ or $b_3 = -\frac{1}{2}b_1$. Hence any vector b where the third entry is $-\frac{1}{2}$ the first entry, will result in $A\bar{x} = \bar{b}$ having (infinite) solutions. So $\bar{b}$ needs to have this form

$$\begin{aligned} \bar{b} &= \begin{pmatrix} b_1 \\ b_2 \\ \frac{-1}{2}b_1 \end{pmatrix} \\ &= b_1 \begin{pmatrix} 1 \\ 0 \\ \frac{-1}{2} \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

2.2.2.2 Problem 2

part a For what vector $\bar{b}$ does $A\bar{x} = \bar{b}$ have solution

$$A = \begin{pmatrix} 2 & -1 & \frac{1}{2} \\ 3 & 1 & 2 \\ 0 & 6 & 3 \end{pmatrix}$$

answer

Let $\bar{b} = (b_1, b_2, b_3)$ then the augmented matrix is

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 3 & 1 & 2 & b_2 \\ 0 & 6 & 3 & b_3 \end{pmatrix}$$

Applying row operations: $R_2 = R_2 - \frac{3}{2}R_1$ gives

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2}b_1 \\ 0 & 6 & 3 & b_3 \end{pmatrix}$$

$R_3 = R_3 - \frac{6}{\left(\frac{5}{2}\right)}R_2$ gives

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2}b_1 \\ 0 & 0 & 0 & \frac{18}{5}b_1 - \frac{12}{5}b_2 + b_3 \end{pmatrix}$$

The above is Echelon form. Last row says that $0x_3 = \frac{18}{5}b_1 - \frac{12}{5}b_2 + b_3$. Therefore for solution to exist, we need

$$\frac{18}{5}b_1 - \frac{12}{5}b_2 + b_3 = 0$$

This will generate infinite number of solutions. Any $\bar{b}$ vector of 3 elements where the above constraint is satisfied, will make $A\bar{x} = \bar{b}$ have (infinite) number of solutions. Solving for b_1 in terms of b_2, b_3

$$b_1 = \frac{12}{18}b_2 - \frac{5}{18}b_3$$

Hence $\bar{b}$ can be written as

$$\bar{b} = \begin{pmatrix} \frac{12}{18}b_2 - \frac{5}{18}b_3 \\ b_2 \\ b_3 \end{pmatrix}$$

One such example of $\bar{b}$ can be

$$\bar{b} = \begin{pmatrix} \frac{7}{18} \\ 1 \\ 1 \end{pmatrix}$$

part b Find all possible solutions (or no solution) for

$$\bar{b} = \begin{pmatrix} 0 \\ \frac{12}{5} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{12}{5} \end{pmatrix}$$

We need to first check if these vectors meet the constraint found in part (a), which is $b_1 = \frac{12}{18}b_2 - \frac{5}{18}b_3$. For the first vector given, we get

$$0 \stackrel{?}{=} \frac{12}{18}\left(\frac{12}{5}\right) - \frac{5}{18}(1)$$
$$0 \stackrel{?}{=} \frac{119}{90}$$

Which is not valid. Therefore, $\bar{b} = \begin{pmatrix} 0 \\ \frac{12}{5} \\ 1 \end{pmatrix}$ will produce no solution for when used in $A\bar{x} = \bar{b}$.

Now we check the second vector to see if it meets the constraint or not.

$$0 \stackrel{?}{=} \frac{12}{18}(1) - \frac{5}{18}\left(\frac{12}{5}\right)$$
$$0 \stackrel{?}{=} 0$$

Yes. It satisfies the constraint. Hence this vector will produce solution for $A\bar{x} = \bar{b}$. To find the solution, we plugin this $\bar{b}$ vector and solve for x

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 \\ \frac{-1}{3} & \frac{-3}{2} & -1 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{12}{5} \end{pmatrix}$$

Following the row operation we did above, the output is

$$\left(\begin{array}{ccc|c} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2}b_1 \\ 0 & 0 & 0 & \frac{18}{5}b_1 - \frac{12}{5}b_2 + b_3 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 1 \\ 0 & 0 & 0 & -\frac{12}{5}(1) + \frac{12}{5} \end{array}\right)$$
$$= \begin{pmatrix} 2 & -1 & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence from last row $x_3 = t$, and from second row $\frac{5}{2}x_2 + \frac{5}{4}t = 1$ or $x_2 = \frac{2}{5} - \frac{1}{2}t$ and from first row $2x_1 - x_2 + \frac{1}{2}x_3 = 0$ or $2x_1 = \left(\frac{2}{5} - \frac{1}{2}t\right) - \frac{1}{2}t$ hence solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} - \frac{t}{2} \\ \frac{2}{5} - \frac{1}{2}t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2}t \\ 1 \end{pmatrix}$$

2.2.2.3 Problem 3

Consider $A\bar{x} = \bar{b}$ for

$$A = \begin{pmatrix} \frac{2}{3} & a_{12} & -2 \\ -\frac{1}{5} & \frac{-1}{\frac{3}{5}} & \frac{3}{5} \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} \end{pmatrix}$$

(a) for what values of a_{12} is A non-singular? (b) For what values of a_{12} is A singular? (c) In all cases of A singular, analyze the system $A\bar{x} = \bar{b}$. For what vectors $\bar{b}$ lead to solution $\bar{x}$? What are those solutions?

Answer (a). Expanding along first row gives

$$\begin{aligned} |A| &= \frac{2}{3}A_{11} + a_{12}A_{12} - 2A_{13} \\ &= \frac{2}{3}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} - 2(-1)^{1+3} M_{13} \\ &= \frac{2}{3}M_{11} - a_{12}M_{12} - 2M_{13} \\ &= \frac{2}{3}\begin{vmatrix} \frac{-1}{\frac{3}{5}} & \frac{3}{5} \\ \frac{5}{6} & \frac{-3}{2} \end{vmatrix} - a_{12}\begin{vmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{1}{2} & \frac{-3}{2} \end{vmatrix} - 2\begin{vmatrix} -\frac{1}{5} & \frac{-1}{\frac{3}{5}} \\ \frac{1}{2} & \frac{5}{6} \end{vmatrix} \\ &= \frac{2}{3}(0) - a_{12}(0) - 2(0) \\ &= 0a_{12} \end{aligned}$$

Therefore, there are no values of a_{12} will make A non-singular, since anything times zero is zero.

(b) This follows from part (a). For any value a_{12}, the matrix A remains singular.

(c) Let $\bar{b} = (b_1, b_2, b_3)$, then the augmented matrix is

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ -\frac{1}{5} & \frac{-1}{\frac{3}{5}} & \frac{3}{5} & b_2 \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_2 = R_2 - \left(\frac{-\frac{1}{5}}{\frac{2}{3}}\right) R_1$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_3 = R_3 - \frac{\frac{1}{2}}{\frac{2}{3}} R_1$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ 0 & \frac{5}{6} - \frac{3}{4}a_{12} & 0 & b_3 - \frac{3}{4}b_1 \end{pmatrix}$$

$R_3 = R_3 - \frac{\frac{5}{6}-\frac{3}{4}a_{12}}{\frac{3}{10}a_{12}-\frac{1}{3}} R_2$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ 0 & 0 & 0 & \frac{5}{2}b_2 + b_3 \end{pmatrix}$$

From last row, we see that $0x_3 = \frac{5}{2}b_2 + b_3$. Hence we need (for infinite solutions) to have the constraint

$$\frac{5}{2}b_2 + b_3 = 0$$
$$b_2 = -\frac{2}{5}b_3$$

In which case we assume $x_3 = t$ in this case (parameter). The second row says that

$$\left(\frac{3}{10}a_{12} - \frac{1}{3}\right)x_2 = \frac{3}{10}b_1 + b_2$$

Here we have to consider the case where $a_{12} = \frac{10}{9}$ (which can happen, since a_{12} can be any value for A singular). In this case, we end up with $0x_2 = \frac{3}{10}b_1 + b_2$. Then now, for solution to exist, we need $\frac{3}{10}b_1 + b_2 = 0$ or $b_1 = -\frac{10}{3}b_2$ and now we set $x_2 = s$, second parameter.

On the other hand, if $a_{12} \neq \frac{10}{9}$ then this leads to $\left(\frac{3}{10}a_{12} - \frac{1}{3}\right)x_2 = \frac{3}{10}b_1 + b_2$ and now $x_2 = \frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}$. Therefore in summary

$$x_2 = \begin{cases} s & a_{12} = \frac{10}{9} \text{ and} \frac{3}{10}b_1 + b_2 = 0 \\ \frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}} & a_{12} \neq \frac{10}{9} \end{cases}$$

Finally, first row gives

$$\frac{2}{3}x_1 + a_{12}x_2 - 2x_3 = b_1$$
$$x_1 = b_1 - a_{12}x_2 + 2x_3$$
$$= \frac{3}{2}b_1 - \frac{3}{2}a_{12}x_2 + 3t$$

If $a_{12} = \frac{10}{9}$ and $\frac{3}{10}b_1 + b_2 = 0$ then $x_2 = s$ and above becomes

$$x_1 = \frac{3}{2}b_1 - \frac{3}{2}\left(\frac{10}{9}\right)s + 3t$$
$$= 3t - \frac{5}{3}s + \frac{3}{2}b_1$$

If $a_{12} \neq \frac{10}{9}$ then $x_2 = \frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}$ and x_1 becomes

$$x_1 = \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}}\right) + 3t$$

Therefore in summary

$$x_1 = \begin{cases} 3t - \frac{5}{3}s + \frac{3}{2}b_1 & a_{12} = \frac{10}{9} \text{ and} \frac{3}{10}b_1 + b_2 = 0 \\ \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\right) + 3t & a_{12} \neq \frac{10}{9} \end{cases}$$

Hence solution vector is,

for case $a_{12} = \frac{10}{9}$ and $\frac{3}{10}b_1 + b_2 = 0$ and $\frac{5}{2}b_2 + b_3 = 0$ then solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3t - \frac{5}{3}s + \frac{3}{2}b_1 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 3t - \frac{5}{3}s + 2b_3 \\ s \\ t \end{pmatrix}$$
$$= t\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -\frac{5}{3} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2b_3 \\ 0 \\ 0 \end{pmatrix}$$

And the $\bar{b}$ vector now is

$$\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3}b_2 \\ -\frac{2}{5}b_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}b_3 \\ -\frac{2}{5}b_3 \\ b_3 \end{pmatrix}$$
$$= b_3\begin{pmatrix} \frac{4}{3} \\ -\frac{2}{5} \\ 1 \end{pmatrix}$$

For case $a_{12} \neq \frac{10}{9}$ and $\frac{5}{2}b_2 + b_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\right) + 3t \\ \frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}} \\ t \end{pmatrix}$$
$$= t\begin{pmatrix} 3 \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\right) \\ \frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}} \\ 0 \end{pmatrix}$$

And the $\bar{b}$ vector now is

$$\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ -\frac{2}{5}b_3 \\ b_3 \end{pmatrix}$$
$$= b_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_3\begin{pmatrix} 1 \\ -\frac{2}{5} \\ 1 \end{pmatrix}$$

2.2.2.4 Problem 4

Given that two vectors $\bar{u}, \bar{v}$ are L.I., are $3\bar{u} - 5\bar{v}$ and $\bar{v}$ L.I. or L.D.? prove your answer.

answer

The two vectors are L.I. if the only solution to

$$c_1(3\bar{u} - 5\bar{v}) + c_2(\bar{v}) = \bar{0}$$

is $c_1 = 0, c_2 = 0$. Therefore

$$\begin{aligned} c_1(3\bar{u} - 5\bar{v}) + c_2(\bar{v}) &= 3c_1\bar{u} - 5c_1\bar{v} + c_2\bar{v} \\ &= 3c_1\bar{u} + \bar{v}(c_2 - 5c_1) \end{aligned} \tag{1}$$

Let

$$\begin{aligned} 3c_1 &= k_1 \\ c_2 - 5c_1 &= k_2 \end{aligned} \tag{2}$$

And (1) becomes

$$c_1(3\bar{u}-5\bar{v})+c_2(\bar{v})=k_1\bar{u}+k_2\bar{v}$$

But $\bar{u},\bar{v}$ are L.I., hence $k_1\bar{u}+k_2\bar{v}=\bar{0}$ implies that $k_1=k_2=0$. This means (from (2)) that

$$\begin{aligned}3c_1&=0\\c_2-5c_1&=0\end{aligned}$$

First equation gives $c_1=0$. The second equation now gives $c_2=0$. Hence this shows that $3\bar{u}-5\bar{v}$ and $\bar{v}$ are L.I.

2.2.2.5 Problem 5

Are the following statements true or false? If false, correct it.

1. Square matrix with two identical rows is row equivalent to identity matrix
2. Inverse of square matrix A exists if A is row equivalent to identity matrix I with the same dimension.
3. Determinant of upper triangle square matrix is sum of diagonal elements.

Answer

1. False. Since two rows are identical, the matrix is singular which means there are no row operations which leads to reduced Echelon form.
2. True.
3. False. Determinant of upper triangle square matrix is product (not sum) of diagonal elements.

2.2.2.6 Problem 6

Prove property 4 of the seven properties of determinants.

Answer

Property 4 says that if A,B,C are identical except for one row i, and that row is such that $A(i)+B(i)=C(i)$ then $|A|+|B|=|C|$

Let the three matrices be

$$A=\begin{pmatrix}\times&\times&\times&\times\\a_{i1}&a_{i2}&\cdots&a_{in}\\\times&\times&\times&\times\\\times&\times&\times&\times\end{pmatrix},B=\begin{pmatrix}\times&\times&\times&\times\\b_{i1}&b_{i2}&\cdots&b_{in}\\\times&\times&\times&\times\\\times&\times&\times&\times\end{pmatrix},C=\begin{pmatrix}\times&\times&\times&\times\\c_{i1}&c_{i2}&\cdots&c_{in}\\\times&\times&\times&\times\\\times&\times&\times&\times\end{pmatrix}$$

Where in the above, the i^{th} is shown. We are also told that $A(i)+B(i)=C(i)$ which implies

$$\begin{aligned}a_{i1}+b_{i1}&=c_{i1}\\a_{i2}+b_{i2}&=c_{i2}\\\vdots&=\vdots\\a_{in}+b_{in}&=c_{in}\end{aligned}\tag{1}$$

Taking the determinant of each matrix, and expanding along the i^{th} row gives

$$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$$

Similarly for B and C

$$|B|=b_{i1}B_{i1}+b_{i2}B_{i2}+\cdots+b_{in}B_{in}$$

And

$$|C|=c_{i1}C_{i1}+c_{i2}C_{i2}+\cdots+c_{in}C_{in}$$

Where But since $A_{ij}=B_{ij}=C_{ij}$ is the submatrix for all matrices, we are told the matrices are identical in all other rows (and columns) except for the i^{th} row. Then we can just use any one of them. Lets use C_{ij} for each case. Therefore from above, we can write

$$\begin{aligned} |A| + |B| &= (a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}) + (b_{i1}C_{i1} + b_{i2}C_{i2} + \cdots + b_{in}C_{in}) \\ &= (a_{i1} + b_{i1})\, C_{i1} + (a_{i2} + b_{i2})\, C_{i2} + \cdots + (a_{in} + b_{in})\, C_{in} \end{aligned} \tag{2}$$

Substituting (1) into (2) gives

$$\begin{aligned} |A| + |B| &= c_{i1}C_{i1} + c_{i2}C_{i2} + \cdots + c_{in}C_{in} \\ &= |C| \end{aligned}$$

QED.

2.2.2.7 Problem 7

Consider matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

1. Find condition on a_{32}, a_{33} such that A^{-1} exist.
2. Find value of determinant for $a_{32} = 1$ and $a_{33} = -2$. How many columns of A are independent for $a_{32} = 1, a_{33} = -2$?
3. For $a_{32} = 5, a_{33} = -4$, can $p^T = (3, 5, 0)$ be expressed as linear combination of columns of A?
4. Find value of the determinant for $a_{32} = 5, a_{33} = -4$. How many columns of A are independent?

Answer

(1) Expanding along last row gives

$$\begin{aligned} |A| &= a_{32}A_{32} + a_{33}A_{33} \\ &= a_{32}\,(-1)^{3+2}\,M_{32} + a_{33}\,(-1)^{3+3}\,M_{33} \\ &= -a_{32}M_{32} + a_{33}M_{33} \\ &= -a_{32}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + a_{33}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= -4a_{32} - 5a_{33} \end{aligned}$$

Hence for A^{-1} to exist, we want $|A| \neq 0$, which means we want $-4a_{32} - 5a_{33} \neq 0$ or

$$4a_{32} + 5a_{33} \neq 0$$

(2) When $a_{32} = 1$ and $a_{33} = -2$, the matrix becomes

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

Expanding along last row gives

$$\begin{aligned} |A| &= a_{32}A_{32} + a_{33}A_{33} \\ &= (-1)^{3+2}\,M_{32} - 2\,(-1)^{3+3}\,M_{33} \\ &= -M_{32} - 2M_{33} \\ &= -\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= -4 + 10 \\ &= 6 \end{aligned}$$

Since $|A| \neq 0$, Hence all columns are L.I. (Matrix is full rank).

(3) For $a_{32} = 5, a_{33} = -4$ the matrix becomes

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 5 & -4 \end{pmatrix}$$

To find if $p^T = (3, 5, 0)$ can be expressed are linear combinations of columns of A, implies

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Has solution in c. The above can be written as

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 5 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Setting up the augmented matrix gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 0 & 5 & -4 & 0 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & 5 & -4 & 0 \end{pmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, last row gives $0c_3 = 0$. Hence c_3 can be any value, say t. Second row gives

$$\begin{aligned} -5c_2 + 4c_3 &= 0 \\ c_2 &= \frac{4}{5}t \end{aligned}$$

And from first row

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 3 \\ c_1 &= 3 - 2c_2 + c_3 \\ &= 3 - 2\frac{4}{5}t + t \\ &= 3 - \frac{9}{5}t \end{aligned}$$

Hence there are infinite solutions.

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} &= \begin{pmatrix} 3 - \frac{9}{5}t \\ \frac{4}{5}t \\ t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{9}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} \end{aligned}$$

For any t we can find linear combination of columns of A which gives p^T. For example, using $t=0$ results in solution $c_1=3, c_2=0, c_3=0$. To verify

$$
\begin{aligned}
c_1\begin{pmatrix}1\\2\\0\end{pmatrix}+c_2\begin{pmatrix}2\\-1\\5\end{pmatrix}+c_3\begin{pmatrix}-1\\2\\-4\end{pmatrix} &= 3\begin{pmatrix}1\\2\\0\end{pmatrix}\\
&= \begin{pmatrix}3\\6\\0\end{pmatrix}\\
&= p
\end{aligned}
$$

(4). For $a_{32}=5, a_{33}=-4$ the matrix becomes

$$
A=\begin{pmatrix}1 & 2 & -1\\2 & -1 & 2\\0 & 5 & -4\end{pmatrix}
$$

The determinant is zero, this is because from part (3), we ended up with one zero pivot in Echelon form, which implies $|A|=0$. Since solution has one parameter family, and matrix is 3×3, then there are now 2 L.I. columns in A. This is the same as saying rank of A is 2.

2.2.2.8 Problem 8

Consider 3×3 matrix A. Show that $|A|^T=|A|$

Answer Let A be

$$
A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}
$$

Expanding along first row gives

$$
\begin{aligned}
|A| &= a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{12}+a_{13}(-1)^{1+3}M_{13}\\
&= a_{11}\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{vmatrix}\\
&= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})\\
&= a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31} \qquad (1)
\end{aligned}
$$

Now

$$
A^T=\begin{pmatrix}a_{11} & a_{21} & a_{31}\\a_{12} & a_{22} & a_{32}\\a_{13} & a_{23} & a_{33}\end{pmatrix}
$$

Expanding along first row gives

$$
\begin{aligned}
\left|A^T\right| &= a_{11}(-1)^{1+1}M_{11}+a_{21}(-1)^{1+2}M_{12}+a_{31}(-1)^{1+3}M_{13}\\
&= a_{11}\begin{vmatrix}a_{22} & a_{32}\\a_{23} & a_{33}\end{vmatrix}-a_{21}\begin{vmatrix}a_{12} & a_{32}\\a_{13} & a_{33}\end{vmatrix}+a_{31}\begin{vmatrix}a_{12} & a_{22}\\a_{13} & a_{23}\end{vmatrix}\\
&= a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{21}(a_{12}a_{33}-a_{32}a_{13})+a_{31}(a_{12}a_{23}-a_{22}a_{13})\\
&= a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}-a_{21}a_{12}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}-a_{31}a_{22}a_{13} \qquad (2)
\end{aligned}
$$

Examining (1) and (2), we see they are the same. Hence $|A|=\left|A^T\right|$

2.2.2.9 Problem 9

Find $|A|$

$$
A=\begin{pmatrix}1 & 2 & -2 & 5\\-1 & 2 & 3 & 4\\1 & 3 & 1 & -2\\-1 & -3 & 0 & -4\end{pmatrix}
$$

Answer

$R_2 = R_2 + R_1$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 1 & 3 & 1 & -2 \\ -1 & -3 & 0 & -4 \end{pmatrix}$$

$R_3 = R_3 - R_1$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 1 & 3 & -7 \\ -1 & -3 & 0 & -4 \end{pmatrix}$$

$R_4 = R_4 + R_1$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 1 & 3 & -7 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

$R_3 = R_3 - \frac{1}{4}R_2$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & \frac{11}{4} & -\frac{37}{4} \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

$R_4 = R_4 + \frac{1}{4}R_2$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & \frac{11}{4} & -\frac{37}{4} \\ 0 & 0 & -\frac{7}{4} & \frac{13}{4} \end{pmatrix}$$

$R_4 = R_4 - \frac{-\frac{7}{4}}{\frac{11}{4}}R_3$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & \frac{11}{4} & -\frac{37}{4} \\ 0 & 0 & 0 & -\frac{29}{11} \end{pmatrix}$$

Hence

$$\begin{aligned} |A| &= 1 \times 4 \times \frac{11}{4} \times -\frac{29}{11} \\ &= -29 \end{aligned}$$

2.2.2.10 Problem 10

Using elementary row operations, find the inverse of

$$A = \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}$$

Answer

Set up augmented matrix

$$C=\begin{pmatrix}3&5&6&1&0&0\\2&4&3&0&1&0\\2&3&5&0&0&1\end{pmatrix}$$

$R_2=R_2-\frac{2}{3}R_1$ gives

$$C=\begin{pmatrix}3&5&6&1&0&0\\0&\frac{2}{3}&-1&-\frac{2}{3}&1&0\\2&3&5&0&0&1\end{pmatrix}$$

$R_3=R_3-\frac{2}{3}R_1$ gives

$$C=\begin{pmatrix}3&5&6&1&0&0\\0&\frac{2}{3}&-1&-\frac{2}{3}&1&0\\0&-\frac{1}{3}&1&-\frac{2}{3}&0&1\end{pmatrix}$$

$R_3=R_3-\frac{-\frac{1}{3}}{\frac{2}{3}}R_2$ gives

$$C=\begin{pmatrix}3&5&6&1&0&0\\0&\frac{2}{3}&-1&-\frac{2}{3}&1&0\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

Start backward elimination now. $R_1=R_1-\frac{5}{\frac{2}{3}}R_2$ gives

$$C=\begin{pmatrix}3&0&\frac{27}{2}&6&-\frac{15}{2}&0\\0&\frac{2}{3}&-1&-\frac{2}{3}&1&0\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

$R_1=R_1-\frac{\frac{27}{2}}{\frac{1}{2}}R_3$ gives

$$C=\begin{pmatrix}3&0&0&33&-21&-27\\0&\frac{2}{3}&-1&-\frac{2}{3}&1&0\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

$R_2=R_2-\frac{-1}{\frac{1}{2}}R_3$ gives

$$C=\begin{pmatrix}3&0&0&33&-21&-27\\0&\frac{2}{3}&0&-\frac{8}{3}&2&2\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

Divide each row by diagonal element to make LHS identity matrix. $R_1=\frac{R_1}{3}$ gives

$$C=\begin{pmatrix}1&0&0&11&-7&-9\\0&\frac{2}{3}&0&-\frac{8}{3}&2&2\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

$R_2=\frac{R_2}{\frac{2}{3}}$ gives

$$C=\begin{pmatrix}1&0&0&11&-7&-9\\0&1&0&-4&3&3\\0&0&\frac{1}{2}&-1&\frac{1}{2}&1\end{pmatrix}$$

$R_3 = \frac{R_3}{\frac{1}{2}}$ gives

$$C = \begin{pmatrix} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix}$$

Hence

$$A^{-1} = \begin{pmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{pmatrix}$$

2.2.2.11 Problem 11

(a) Show that any plane through the origin is subspace of $\mathbb{R}^3$

(b) Show that the plane $x + 3y - 2z = 5$ is not subspace of $\mathbb{R}^3$

Answer

part(a) The plane through the origin is the set W of all vectors $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, such that $ax+by+cz = 0$, where x, y, z are the coordinates of the vector v and a, b, c are any arbitrary constants not all zero. To show that W is subspace of $\mathbb{R}^3$, we need to show that additions of *any* two vectors $\boldsymbol{u}, \boldsymbol{v} \in W$ gives vector $\boldsymbol{w} \in W$ (closed under addition) and multiplying any vector $\boldsymbol{u} \in W$ by *any* scalar k gives vector $k\boldsymbol{u} \in W$ (closed under scalar multiplication). We are told the zero vector $0 \in W$ already, so we do not have to show this. (since the plane passes though origin).

To show closure under addition, consider any two vectors $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. Since these two vectors are taken from W, then we know they satisfy the equation of the plane already. i.e.

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned} \tag{1}$$

Now lets add these two vectors

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{u} &= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \end{aligned} \tag{2}$$

We now need to check if the above vector in still in W (i.e. in the plane passing through the origin). To do so, we take the original equation of the plane $ax + by + cz = 0$ and replace x, y, z in this equation by the coordinates in (2) and see if we still get zero in the RHS. This results in

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) &\stackrel{?}{=} 0 \\ ax_1 + ax_2 + by_1 + by_2 + cz_1 + cz_2 &\stackrel{?}{=} 0 \\ (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) &\stackrel{?}{=} 0 \end{aligned}$$

Substituting (1) into the above gives

$$0 + 0 \stackrel{?}{=} 0$$

Yes. Therefore $\boldsymbol{v} + \boldsymbol{u} \in W$. To check closure under scalar multiplication.

$$
\begin{aligned}
k\boldsymbol{v} &= k \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} \qquad (3)
\end{aligned}
$$

We now need to check if the above vector in still in W (i.e. in the plane passing through the origin). To do so, we take the original equation of the plane $ax + by + cz = 0$ and replace x, y, z in this equation by the coordinates in (3) and see if we still get zero in the RHS. This results in

$$
\begin{aligned}
a(kx_1) + b(ky_1) + c(kz_1) &\stackrel{?}{=} 0 \\
k(ax_1 + by_1 + cz_1) &\stackrel{?}{=} 0
\end{aligned}
$$

But since $ax_1 + by_1 + cz_1 = 0$ from (1). Therefore $k(ax_1 + by_1 + cz_1) = 0$. So closed under scalar multiplication.

Part b A subspace must include the zero vector $0 = (0, 0, 0)$. Replacing the coordinates of this vector into LHS of $x + 3y - 2z = 5$ gives

$$
\begin{aligned}
0 + 3(0) - 2(0) &\stackrel{?}{=} 5 \\
0 &\stackrel{?}{=} 5
\end{aligned}
$$

No. Hence not satisfied. Therefore not subspace of $\mathbb{R}^3$.

2.3 final exam

2.3.1 My solution for final exam practice.

2.3.1.1 Problem 1

(a) Find general solution to $x^2y'' - 3xy + 4y = 0$ with $x > 0$. (b) For initial conditions $y(2) = a, y'(2) = b$ give a 2×2 matrix-vector equation to determine the coefficients of the unique solution. Solve the system them write the solution to the initial value problem. (c) Show that general solution contains two L.I. solutions y_1, y_2 with $x > 0$

Solution

2.3.1.1.1 Part(a) Let $y = x^r$ then $y' = Arx^{r-1}, y'' = Ar(r-1)x^{r-2}$. Substituting these into the ODE gives

$$\begin{aligned} x^2 r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r &= 0 \\ r(r-1)x^r - 3rx^r + 4x^r &= 0 \end{aligned}$$

Since $x > 0$, we can cancel x^r and obtain the characteristic equation

$$\begin{aligned} r(r-1) - 3r + 4 &= 0 \\ r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0 \end{aligned}$$

Hence $r = 2$ double root. Therefore

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^2 \ln x \end{aligned}$$

And the homogenous solution is

$$y_h(x) = c_1x^2 + c_2x^2 \ln x \tag{1A}$$

2.3.1.1.2 Part(b) Applying $y(2) = a$ gives

$$a = 4c_1 + 4c_2 \ln 2 \tag{1}$$

Taking derivative of $y_h(x)$

$$y_h'(x) = 2c_1x + 2c_2x \ln x + c_2x$$

Applying $y'(2) = b$ gives

$$b = 4c_1 + c_2(4\ln 2 + 2) \tag{2}$$

Using (1,2), we write them in matrix form to solve for c_1, c_2

$$\begin{pmatrix} 4 & 4\ln 2 \\ 4 & 4\ln 2 + 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{pmatrix} 4 & 4\ln 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b - a \end{pmatrix}$$

From second row,

$$\begin{aligned} 2c_2 &= (b - a) \\ c_2 &= \frac{b-a}{2} \end{aligned}$$

From first row

$$\begin{aligned} 4c_1 + 4\ln 2 c_2 &= a \\ c_1 &= \frac{a - 4c_2 \ln 2}{4} \\ &= \frac{a - 4\left(\frac{b-a}{2}\right)\ln 2}{4} \\ &= \frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2 \end{aligned}$$

Therefore

$$
\begin{aligned}
c_1 &= \frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2\\
c_2 &= \frac{b-a}{2}
\end{aligned}
$$

Pugging these into the $y_h(x) = c_1 x^2 + c_2 x^2 \ln x$ found in part(a) gives

$$
y_h(x) = \left(\frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2\right)x^2 + \left(\frac{b-a}{2}\right)x^2\ln x
$$

2.3.1.1.3 Part (c) We found that

$$
\begin{aligned}
y_1 &= x^2\\
y_2 &= x^2\ln x
\end{aligned}
$$

Hence the Wronskian is

$$
\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x\\ 2x & 2x\ln x + x\end{vmatrix}\\
&= 2x^3\ln x + x^3 - 2x^3\ln x\\
&= x^3
\end{aligned}
$$

Since $x > 0$, hence $W(x)$ never zero. Therefore y_1, y_2 are L.I.

2.3.1.2 Problem 2

Given one solution $y_1(x) = x$, find general solution of $x^2y'' - x(x+2)y' + (x+2)y = 0$ for $x > 0$

Solution

Assume solution is $y(x) = vy_1(x)$. Hence

$$
\begin{aligned}
y' &= v'y_1 + vy_1'\\
y'' &= v''y_1 + v'y_1' + v'y_1' + vy_1''\\
&= v''y_1 + 2v'y_1' + vy_1''
\end{aligned}
$$

Plugging the second solution into the original ODE gives

$$
\begin{aligned}
x^2y'' - x(x+2)y' + (x+2)y &= 0\\
x^2\left(v''y_1 + 2v'y_1' + vy_1''\right) - x(x+2)\left(v'y_1 + vy_1'\right) + (x+2)\left(vy_1\right) &= 0
\end{aligned}
$$

Collecting terms on v, v', v'' gives

$$
v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) + v\overbrace{\left(x^2y_1'' - x(x+2)y_1' + (x+2)y_1\right)}^{0} = 0
$$

Hence

$$
v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) = 0
$$

But $y_1 = x$, hence $y_1' = 1$ and the above becomes

$$
\begin{aligned}
x^3v'' + v'\left(2x^2 - x^2(x+2)\right) &= 0\\
x^3v'' - x^3v' &= 0
\end{aligned}
$$

Since $x > 0$ then above reduces to

$$
v'' - v' = 0
$$

Let $v' = z$ then the above becomes $z' - z = 0$ or $\frac{dz}{dx} = z$ which is separable. Hence the solution is $\ln|z| = x + c_1$ or $z = c_1e^x$. Therefore

$$
v' = c_1e^x
$$

Integrating

$$
v = c_1e^x + c_2
$$

Hence, since

$$\begin{aligned} y &= vy_1 \\ &= (c_1 e^x + c_2)\, x \end{aligned}$$

Therefore the solution is

$$y(x) = c_1 x e^x + c_2 x$$

2.3.1.3 Problem 3

Find general solution to $x^2 y'' - 3xy' + 4y = x^2 \ln(x)$ with $x > 0$

Solution We first solve the homogenous part

$$x^2 y'' - 3xy' + 4y = 0$$

We solved this in problem 1, the solution is

$$y_h(x) = c_1 x^2 + c_2 x^2 \ln x$$

To find particular solution, we will use variation of parameters since $\ln(x)$ is not one of the good functions to guess for. Writing the ODE in standard form

$$y'' - 3\frac{1}{x} y' + \frac{4}{x^2} y = \ln(x)$$

We see from the homogeneous solution that $y_1(x) = x^2, y_2 = x^2 \ln(x)$. Hence we assume the complete solution (including particular solution) is

$$y = y_1 u_1 + y_2 u_2 \tag{1A}$$

Where

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \tag{1}$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx \tag{2}$$

Where in the above, $f(x) = \ln(x)$ and not $x^2 \ln(x)$ since we divide by x^2 in order to make the ODE standard form. W is the Wronskian. We found the Wronskian for this ODE in part(c) problem 1, which is $W = x^3$, Hence (1) becomes

$$u_1 = -\int \frac{x^2 \ln^2(x)}{x^3} dx = -\int \frac{\ln^2(x)}{x} dx$$

Let $z = \ln(x)$ hence $\frac{dz}{dx} = \frac{1}{x}$ or $dx = xdz.$, Hence the integral becomes

$$u_1 = -\int \frac{z^2}{x} x dz = -\int z^2 dz = -\frac{z^3}{3} + c_1$$

Replacing back gives

$$u_1 = -\frac{1}{3} \ln^3(x) + c_1$$

And from (2)

$$u_2 = \int \frac{x^2 \ln(x)}{x^3} dx = \int \frac{\ln(x)}{x} dx$$

Let $z = \ln(x)$ hence $\frac{dz}{dx} = \frac{1}{x}$ or $dx = xdz.$, Hence the integral becomes

$$u_2 = \int \frac{z}{x} x dz = \int z dz = \frac{z^2}{2} + c_2$$

Replacing back gives

$$u_2 = \frac{1}{2} \ln^2(x) + c_2$$

Hence from (1A)

$$\begin{aligned} y &= y_1 u_1 + y_2 u_2 \\ &= x^2 \left(-\frac{1}{3} \ln^3(x) + c_1 \right) + x^2 \ln(x) \left(\frac{1}{2} \ln^2(x) + c_2 \right) \\ &= -\frac{1}{3} x^2 \ln^3(x) + c_1 x^2 + \frac{1}{2} x^2 \ln^3(x) + c_2 x^2 \ln(x) \end{aligned}$$

Hence

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln(x) + \frac{1}{6} x^2 \ln^3(x)}$$

2.3.1.4 Problem 4

Consider the equation $2y'' - 5y' + cy = 0$ with $-\infty < x < \infty$ for c real and constant. (a) For what values of c does characteristic equation have 2 different real roots? (b) for what values of c does the characteristic equation have 1 real repeated root? (c) Find general solution for $c = 2$. (d) for $c = 2$ and initial conditions $y(x_0) = p, y'(x_0) = q$ write a 2×2 matrix equation to determined the coefficients of general solutions.

Solution

2.3.1.4.1 Part (a) Assuming $y = Ae^r$ and substituting into the ODE gives the characteristic equation is

$$2r^2 - 5r + cr = 0$$

The roots are

$$\begin{aligned} r &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{5}{4} \pm \frac{1}{4}\sqrt{25 - 8c} \end{aligned}$$

For two different real roots we want $25 - 8c > 0$. Therefore $25 > 8c$ or

$$\boxed{c < \frac{25}{8}}$$

2.3.1.4.2 Part (b) For repeated real root, we want $r = \frac{5}{4}$. Which means we want $25 - 8c = 0$ or

$$\boxed{c = \frac{25}{8}}$$

2.3.1.4.3 Part (c) When $c = 2$ the ODE becomes

$$2y'' - 5y' + 2y = 0$$

The characteristic equation is

$$\begin{aligned} 2r^2 - 5r + 2 &= 0 \\ (r - 2)\left(r - \frac{1}{2}\right) &= 0 \end{aligned}$$

Hence the solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{\frac{x}{2}}$$

2.3.1.4.4 Part(d) From above,

$$y_h' = c_1 2e^{2x} + \frac{1}{2} c_2 e^{\frac{x}{2}}$$

Applying first initial conditions gives the equation

$$p = c_1 e^{2x_0} + c_2 e^{\frac{x_0}{2}} \tag{1}$$

Applying second initial conditions gives the equation

$$q = c_1 2e^{2x_0} + \frac{1}{2} c_2 e^{\frac{x_0}{2}} \tag{2}$$

Writing (1,2) in matrix form gives

$$\begin{pmatrix} e^{2x_0} & e^{\frac{x_0}{2}} \\ 2e^{2x_0} & \frac{1}{2} e^{\frac{x_0}{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

We are asked not to solve if. Solution of the above gives c_1, c_2 which completes the solution.

2.3.1.5 Problem 5

Consider

$$x' = \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x$$

(a) find general solution. (b) Write the solution in terms of real functions only. (c) using method of undetermined coefficients, write the particular solution for

$$x' = \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix}$$

(d) Find the algebraic equation that given the undetermined coefficients. Do not solve.

Solution

2.3.1.5.1 Part (a) The first step is to determine the eigenvalues from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -3-\lambda & 5 \\ -5 & 3-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

For $\underline{\lambda_1 = 4i}$ we solve $(A - \lambda I)\, v_1 = 0$

$$\begin{pmatrix} -3-\lambda_1 & 5 \\ -5 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3-4i & 5 \\ -5 & 3-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find $(-3-4i)\, v_1 + 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{3+4i}{5}$, hence

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{3+4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 3+4i \end{pmatrix}$$

For $\underline{\lambda_2 = -4i}$ we solve $(A - \lambda I)\, v_2 = 0$

$$\begin{pmatrix} -3-\lambda_2 & 5 \\ -5 & 3-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+4i & 5 \\ -5 & 3+4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find $(-3+4i)\, v_1 + 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{3-4i}{5}$, hence

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ \frac{3-4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 3-4i \end{pmatrix}$$

Therefore, the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1 e^{\lambda_1 t} + c_2 \boldsymbol{v}_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} 5 \\ 3+4i \end{pmatrix} e^{4it} + c_2 \begin{pmatrix} 5 \\ 3-4i \end{pmatrix} e^{-4it} \end{aligned}$$

Convert to new basis.

$$
\begin{aligned}
x_1(t) &= \operatorname{Re}(x_1(t)) \\
&= \operatorname{Re}\begin{pmatrix} 5 \\ 3+4i \end{pmatrix} e^{4it} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ (3+4i)(\cos 4t + i\sin 4t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ 3\cos 4t + 3i\sin 4t + 4i\cos 4t - 4\sin 4t \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ (3\cos 4t - 4\sin 4t) + i(3\sin 4t + 4\cos 4t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos 4t \\ (3\cos 4t - 4\sin 4t) \end{pmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
x_2(t) &= \operatorname{Im}\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ (3\cos 4t - 4\sin 4t) + i(3\sin 4t + 4\cos 4t) \end{pmatrix} \\
&= \begin{pmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{pmatrix}
\end{aligned}
$$

Hence the solution using the new basis is

$$
x_h(t) = C_1\begin{pmatrix} 5\cos 4t \\ (3\cos 4t - 4\sin 4t) \end{pmatrix} + C_2\begin{pmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{pmatrix}
$$

2.3.1.5.2 part (c) Since the RHS is $\begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix}$ then we try to see what we would do in the scalar case and then convert it to vector form. In scalar case, when RHS is te^{4t}, then the guess for t is $(a+bt)$ and the guess for e^{4t} is ce^{4t}. Therefore for the product, it will be $(a+bt)\left(ce^{4t}\right) = ace^{4t} + cbte^{4t}$. Let $ac = A, cb = B$, then the guess will becomes $Ae^{4t} + Bte^{4t}$ or $(A+Bt)e^{4t}$. We convert this to vector form now

$$
\begin{aligned}
x_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{4t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^{4t} \\
&= \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t}
\end{aligned}
$$

Therefore

$$
x_p' = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{4t} + 4\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t}
$$

Plugging this into the ODE

$$
\begin{aligned}
x_p' &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x_p + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix} \\
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{4t} + 4\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix} \\
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + 4\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 - 3tb_1 + 5tb_2 \\ 3a_2 - 5a_1 - 5tb_1 + 3tb_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 - 3tb_1 + 5tb_2 + t \\ 3a_2 - 5a_1 - 5tb_1 + 3tb_2 + 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 + t(5b_2 + 1 - 3b_1) \\ 3a_2 - 5a_1 + 1 + t(3b_2 - 5b_1) \end{pmatrix}
\end{aligned}
$$

From first row in the above, we get two equations. And from the second row in the above, we get two equations. These are

$$
\begin{aligned}
4a_1 + b_1 &= 5a_2 - 3a_1 \\
4b_1 &= 5b_2 + 1 - 3b_1 \\
4a_2 + b_2 &= 3a_2 - 5a_1 + 1 \\
4b_2 &= 3b_2 - 5b_1
\end{aligned}
$$

Or

$$
\begin{aligned}
7a_1 - 5a_2 + b_1 &= 0 \\
7b_1 - 5b_2 &= 1 \\
5a_1 + a_2 + b_2 &= 1 \\
b_2 + 5b_1 &= 0
\end{aligned}
$$

In system form, these equations are

$$
\begin{pmatrix} 7 & -5 & 1 & 0 \\ 0 & 0 & 7 & -5 \\ 5 & 1 & 0 & 1 \\ 0 & 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}
$$

We are asked not to solve this. But to verify the solution with computer solution, here is the complete solution. Solving the above gives

$$
\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{23}{128} \\ \frac{33}{128} \\ \frac{1}{32} \\ \frac{-5}{32} \end{pmatrix}
$$

Using these values, the particular solution becomes

$$
\begin{aligned}
x_p &= \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} \\
&= \begin{pmatrix} \frac{23}{128} + \frac{1}{32} t \\ \frac{33}{128} - \frac{5}{32} t \end{pmatrix} e^{4t}
\end{aligned}
$$

And the full solution is

$$
\begin{aligned}
x(t) &= x_h(t) + x_p(t) \\
x(t) &= C_1 \begin{pmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{pmatrix} + \begin{pmatrix} \frac{23}{128} + \frac{1}{32} t \\ \frac{33}{128} - \frac{5}{32} t \end{pmatrix} e^{4t}
\end{aligned}
$$

Or

$$\begin{aligned}
x_1(t) &= 5C_1\cos 4t + 5C_2\sin 4t + \left(\frac{23}{128} + \frac{1}{32}t\right)e^{4t} \\
x_2(t) &= C_1(3\cos 4t - 4\sin 4t) + C_2(3\sin 4t + 4\cos 4t) + \left(\frac{33}{128} - \frac{5}{32}t\right)e^{4t}
\end{aligned}$$

2.3.1.6 Problem 6

Consider

$$x' = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} x + \begin{pmatrix} 48 \\ 9t \end{pmatrix}$$

(a) find homogeneous solution. (b) using undetermined coefficients, find particular solution. (c) find Wronskian (fundamental matrix). (d) Derive the variation of parameters formula for the solution $x(t)$.

Solution

2.3.1.6.1 Part (a)

The first step is to determine the eigenvalues from

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -4 \\ \frac{1}{4} & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

Hence $\lambda = 3$ repeated. Let us see if complete eigenvalue or defective. We solve $(A - \lambda I) v_1 = 0$

$$\begin{pmatrix} 2-\lambda & -4 \\ \frac{1}{4} & 4-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -4 \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First row gives $-v_1 - 4v_2 = 0$. Assuming $v_1 = -1$ gives $v_2 = \frac{1}{4}$, hence

$$\boxed{\mathbf{v}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}} \tag{1A}$$

We see that we can only get one eigenvector since the second row gives same result. Therefore we need a way to find the second eigenvector v_2. We start by assuming

$$x_2(t) = v_1 t e^{\lambda t} + v_2 e^{\lambda t}$$

We plug this back into the ODE and by comparing terms we find that

$$(A - \lambda I) v_2 = v_1$$

And now we solve for v_2 from the above equation (since we know v_1 already)

$$\begin{pmatrix} -1 & -4 \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

First row gives $-v_1 - 4v_2 = -4$. let $v_1 = 0$, then $v_2 = 1$ Hence

$$\boxed{\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Which is the same as book method. Now that we found $x_1(t)$ and $x_2(t)$ (using either method), then $\boldsymbol{x}_h(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$. Or

$$\begin{aligned}\boldsymbol{x}_h(t) &= c_1\boldsymbol{v}_1 e^{\lambda t} + c_2(\boldsymbol{v}_1 t + \boldsymbol{v}_2)e^{\lambda t} \\ &= c_1\begin{pmatrix}-4\\1\end{pmatrix}e^{3t} + c_2\left(\begin{pmatrix}-4\\1\end{pmatrix}t + \begin{pmatrix}0\\1\end{pmatrix}\right)e^{3t} \\ &= c_1\begin{pmatrix}-4\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}-4t\\t+1\end{pmatrix}e^{3t} \qquad (2)\end{aligned}$$

2.3.1.6.2 part (b) The RHS is $\begin{pmatrix}48\\9t\end{pmatrix} = \begin{pmatrix}48\\0\end{pmatrix} + \begin{pmatrix}0\\9\end{pmatrix}t$. Hence the guess is

$$\begin{aligned}\boldsymbol{x}_p &= \begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix}t \\ &= \boldsymbol{a} + \boldsymbol{b}t\end{aligned}$$

Therefore

$$\boldsymbol{x}_p' = \boldsymbol{b}$$

Substituting into ODE and balancing terms, we solve for $\boldsymbol{a}, \boldsymbol{b}$ as follows

$$\begin{aligned}\boldsymbol{x}_p' &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\boldsymbol{x}_p + \begin{pmatrix}48\\0\end{pmatrix} + \begin{pmatrix}0\\9\end{pmatrix}t \\ \boldsymbol{b} &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}(\boldsymbol{a} + \boldsymbol{b}t) + \begin{pmatrix}48\\0\end{pmatrix} + \begin{pmatrix}0\\9\end{pmatrix}t\end{aligned}$$

<u>Balance constants</u>

$$\begin{aligned}\boldsymbol{b} &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\boldsymbol{a} + \begin{pmatrix}48\\0\end{pmatrix} \\ \begin{pmatrix}b_1\\b_2\end{pmatrix} &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} \\ \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} &= \begin{pmatrix}b_1 - 48\\ b_2\end{pmatrix} \qquad (1)\end{aligned}$$

<u>Balance t</u>

$$\begin{aligned}0 &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\boldsymbol{b} + \begin{pmatrix}0\\9\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} &= \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} + \begin{pmatrix}0\\9\end{pmatrix} \\ \begin{pmatrix}2 & -4\\ \frac{1}{4} & 4\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} &= \begin{pmatrix}0\\-9\end{pmatrix}\end{aligned}$$

Solve for $\begin{pmatrix}b_1\\b_2\end{pmatrix}$ by elimination. $R_2 = R_2 - \frac{1}{8}R_1$

$$\begin{pmatrix}2 & -4\\ 0 & \frac{9}{2}\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}0\\-9\end{pmatrix}$$

Hence $\frac{9}{2}b_2 = -9$, or $b_2 = -2$, from first row $2b_1 - 4b_2 = 0$ or $b_1 = -4$. hence

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

Substituting this in (1) above gives equation to solve for $\boldsymbol{a}$

$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -4-48 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -52 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -4 \\ 0 & \frac{9}{2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -52 \\ \frac{9}{2} \end{pmatrix}$$

From second row $a_2 = 1$ and from first row $2a_1 - 4a_2 = -52$ or $a_1 = \frac{-52+4}{2} = -24$, hence

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -24 \\ 1 \end{pmatrix}$$

Hence the particular solution is

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t \\ &= \begin{pmatrix} -24 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \end{pmatrix} t \\ &= \begin{pmatrix} -24-4t \\ 1-2t \end{pmatrix} \end{aligned}$$

And the complete solution is

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_h + \boldsymbol{x}_p \\ &= c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -4t \\ t+1 \end{pmatrix} e^{3t} + \begin{pmatrix} -24-4t \\ 1-2t \end{pmatrix} \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= -4c_1 e^{3t} - 4c_2 t e^{3t} - 24 - 4t \\ x_2(t) &= c_1 e^{3t} + c_2 (1+t) e^{3t} + 1 - 2t \end{aligned}$$

2.3.1.6.3 part (c) The fundamental matrix $\Phi(t)$ is

$$\begin{aligned} \Phi(t) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \\ &= \begin{pmatrix} -4e^{3t} & -4te^{3t} \\ e^{3t} & (t+1)e^{3t} \end{pmatrix} \end{aligned}$$

2.3.1.6.4 part (d) The derivation is given in textbook at 498.

2.3.1.7 Problem 7

Write down the form of the solution including homogenous and particular parts to the following ODE's

$$
\begin{aligned}
y'' - 2y' + 4y &= e^x \left(x \sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x} \right) \\
y'' - 6y' &= x^2 + x \cosh(6x)
\end{aligned}
$$

solution

For the first ODE

$$
y'' - 2y' + 4y = e^x x \sin\left(\sqrt{3}x\right) + e^{x\left(1+\sqrt{3}\right)}
$$

We start by finding the homogenous solution for $y'' - 2y' + 4y = 0$. The characteristic equation is $r^2 - 2r + 4 = 0$, which has roots

$$
\begin{aligned}
r_1 &= 1 + i\sqrt{3} \\
r_2 &= 1 - i\sqrt{3}
\end{aligned}
$$

Hence

$$
y_h(x) = c_1 \overbrace{e^x \cos\left(\sqrt{3}x\right)}^{y_1} + c_2 \overbrace{e^x \sin\left(\sqrt{3}x\right)}^{y_2}
$$

To find particular solution, we need to find a guess. since the RHS is $e^x x \sin\left(\sqrt{3}x\right) + e^{x\left(1+\sqrt{3}\right)}$, the guess for $e^{x\left(1+\sqrt{3}\right)}$ is $C_0 e^{x\left(1+\sqrt{3}\right)}$ and the guess for x is $c_3 + c_4 x$ and the guess for $\sin\left(\sqrt{3}x\right)$ is $c_5 \sin\left(\sqrt{3}x\right) + c_6 \cos\left(\sqrt{3}x\right)$. Hence the guess for $e^x x \sin\left(\sqrt{3}x\right)$ term only is

$$
\begin{aligned}
e^x x \sin\left(\sqrt{3}x\right) &\to e^x (c_3 + c_4 x) \left(c_5 \sin\left(\sqrt{3}x\right) + c_6 \cos\left(\sqrt{3}x\right)\right) \\
&\to \left(c_3 e^x + c_4 x e^x\right) \left(c_5 \sin\left(\sqrt{3}x\right) + c_6 \cos\left(\sqrt{3}x\right)\right) \\
&= c_3 c_5 e^x \sin\left(\sqrt{3}x\right) + c_3 c_6 e^x \cos\left(\sqrt{3}x\right) + c_4 c_5 x e^x \sin\left(\sqrt{3}x\right) + c_4 c_6 x e^x \cos\left(\sqrt{3}x\right)
\end{aligned}
$$

Rename the constants, and the hence the guess for $e^x x \sin\left(\sqrt{3}x\right)$ term only

$$
e^x x \sin\left(\sqrt{3}x\right) \to C_1 e^x \sin\left(\sqrt{3}x\right) + C_2 e^x \cos\left(\sqrt{3}x\right) + C_3 x e^x \sin\left(\sqrt{3}x\right) + C_4 x e^x \cos\left(\sqrt{3}x\right)
$$

Now that we found the initial guess, we have to look at it again and see if y_1 or y_2 are in the guess just made. If so, we add x. We see that since $e^x x \sin\left(\sqrt{3}x\right)$ is y_1, and $x e^x \cos\left(\sqrt{3}x\right)$ is y_2 so we need to multiply these terms in the guess by x, therefore the above becomes

$$
e^x x \sin\left(\sqrt{3}x\right) \to C_1 x e^x \sin\left(\sqrt{3}x\right) + C_2 x e^x \cos\left(\sqrt{3}x\right) + C_3 x^2 e^x \sin\left(\sqrt{3}x\right) + C_4 x^2 e^x \cos\left(\sqrt{3}x\right)
$$

Therefore the final guess is

$$
\begin{aligned}
y_p &= C_0 e^{x\left(1+\sqrt{3}\right)} + C_1 x e^x \sin\left(\sqrt{3}x\right) + C_2 x e^x \cos\left(\sqrt{3}x\right) + C_3 x^2 e^x \sin\left(\sqrt{3}x\right) + C_4 x^2 e^x \cos\left(\sqrt{3}x\right) \\
&= C_0 e^{x\left(1+\sqrt{3}\right)} + (C_1 + C_3 x)\, x e^x \sin\left(\sqrt{3}x\right) + (C_2 + C_4 x)\, x e^x \cos\left(\sqrt{3}x\right)
\end{aligned}
$$

We are asked to stop here and not solve for the coefficients. (good, since this is hard). Another option to find y_p is to use the Wronskian. But this generates hard to evaluate integral. The Wronskian is

$$
\begin{aligned}
W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} e^x \cos\left(\sqrt{3}x\right) & e^x \sin\left(\sqrt{3}x\right) \\ e^x \cos\left(\sqrt{3}x\right) - \sqrt{3}\sin\left(\sqrt{3}x\right) e^x & e^x \sin\left(\sqrt{3}x\right) + \sqrt{3}\cos\left(\sqrt{3}x\right) e^x \end{vmatrix} \\
&= e^x \cos\left(\sqrt{3}x\right) \left(e^x \sin\left(\sqrt{3}x\right) + \sqrt{3}\cos\left(\sqrt{3}x\right) e^x\right) - e^x \sin\left(\sqrt{3}x\right) \left(e^x \cos\left(\sqrt{3}x\right) - \sqrt{3}\sin\left(\sqrt{3}x\right) e^x\right) \\
&= e^{2x} \sin\left(\sqrt{3}x\right) \cos\left(\sqrt{3}x\right) + \sqrt{3}e^{2x} \cos^2\left(\sqrt{3}x\right) - \left(e^{2x} \cos\left(\sqrt{3}x\right) \sin\left(\sqrt{3}x\right) - \sqrt{3}e^{2x} \sin^2\left(\sqrt{3}x\right)\right) \\
&= \sqrt{3}e^{2x} \cos^2\left(\sqrt{3}x\right) + \sqrt{3}e^{2x} \sin^2\left(\sqrt{3}x\right) \\
&= \sqrt{3}e^x
\end{aligned}
$$

Assume now y_p is

$$
y_p = y_1 u_1 + y_2 u_2
$$

Where

$$u_1 = -\int \frac{y_2 f(x)}{W} dx$$
$$u_2 = \int \frac{y_1 f(x)}{W} dx$$

Where $f(x) = e^x \left(x \sin\left(\sqrt{3}x\right) + \exp\left(\sqrt{3}x\right)\right)$. Hence

$$u_1 = -\int \frac{e^x \sin\left(\sqrt{3}x\right) e^x \left(x \sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x}\right)}{\sqrt{3}e^x} dx$$
$$= -\frac{1}{\sqrt{3}} \int x e^x \sin^2\left(\sqrt{3}x\right) + \sin\left(\sqrt{3}x\right) e^{1+\sqrt{3}x} dx$$

And

$$u_2 = \int \frac{e^x \cos\left(\sqrt{3}x\right) e^x \left(x \sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x}\right)}{\sqrt{3}e^x} dx$$
$$= \frac{1}{\sqrt{3}} \int x e^x \sin\left(\sqrt{3}x\right) \cos\left(\sqrt{3}x\right) + e^{1+\sqrt{3}x} \cos\left(\sqrt{3}x\right) dx$$

For the second ODE

$$y'' - 6y' = x^2 + x \cosh(6x)$$

We start by finding the homogenous solution for $y'' - 6y' = 0$. The characteristic equation is $r^2 - 6r = 0$, or $r(r-6) = 0$ which has roots $r_1 = 0, r_2 = 6$ hence

$$y_h = c_1 + c_2 e^{6x}$$

Therefore $y_1 = 1, y_2 = e^{6x}$. Since RHS is

$$f(x) = x^2 + x \cosh(6x)$$
$$= x^2 + x \frac{e^{6x} + e^{-6x}}{2}$$
$$= x^2 + \frac{1}{2} x e^{6x} + \frac{1}{2} x e^{-6x}$$

Then we see that y_2 which is solution of the homogenous solution is part of the forcing function. Let us find y_p now.

For x^2 we guess $c_1 + c_2 x + c_3 x^2$. But now we see that c_1 which is constant, is just $y_1 = 1$ (scalar multiple of). So we have to multiply the whole guess by x, resulting in $\left(c_1 x + c_2 x^2 + c_3 x^3\right)$.

For $x e^{6x}$ we guess $(c_4 + c_5 x) e^{6x}$ but since $y_2 = e^{6x}$ we have to multiply the guess by x giving $\left(c_4 x + c_5 x^2\right) e^{6x}$.

For $x e^{-6x}$ the guess is $(c_6 + c_7 x) e^{-6x}$. Hence we collect all these and obtain the guess y_p as

$$y_p = \left(c_1 x + c_2 x^2 + c_3 x^3\right) + (c_4 + c_5 x) x e^{6x} + (c_6 + c_7 x) e^{-6x}$$

Another option to find y_p is to use the Wronskian. The Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{6x} \\ 0 & 6e^x \end{vmatrix} = 6e^x$$

Assume now y_p is

$$y_p = y_1 u_1 + y_2 u_2$$

Where

$$u_1 = -\int \frac{y_2 f(x)}{W} dx$$
$$u_2 = \int \frac{y_1 f(x)}{W} dx$$

Where $f(x) = x^2 + x \cosh(6x)$. Hence

$$u_1 = -\frac{1}{6} \int \frac{x^2 + x \cosh(6x)}{e^x} dx$$

And

$$
\begin{aligned}
u_2 &= \int \frac{e^{6x}\left(x^2 + x\cosh(6x)\right)}{6e^x} dx \\
&= \frac{1}{6}\int e^{6x}\left(x^2 + x\cosh(6x)\right) dx
\end{aligned}
$$

2.3.1.8 Problem 8

Find general solution to

$$
y^{(6)} + 4y^{(5)} + 8y^{(4)} + 16y''' + 20y'' + 16y' + 16y = 0
$$

The characteristic equation is

$$
r^6 + 4y^5 + 8r^4 + 16r^3 + 20r^2 + 16r + 16 = 0
$$

Using the hint

$$
\begin{aligned}
(r+2)^2\left(r^2+2\right)^2 &= 0 \\
(r+2)(r+2)\left(r^2+2\right)\left(r^2+2\right) &= 0
\end{aligned}
$$

Hence the roots are $r_1 = -2$ multiplicity 2 and $r_2 = \pm i\sqrt{2}$ multiplicity 2. hence the solution is

$$
y_h = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^{i\sqrt{2}x} + c_4 e^{-i\sqrt{2}x} + x\left(c_5 e^{i\sqrt{2}x} + c_6 e^{-i\sqrt{2}x}\right)
$$

Or as real functions, using Euler relation

$$
y_h = c_1 e^{-2x} + c_2 x e^{-2x} + c_3\cos\left(\sqrt{2}x\right) + c_4\sin\left(\sqrt{2}x\right) + c_5 x\cos\left(\sqrt{2}x\right) + c_6 x\sin\left(\sqrt{2}x\right)
$$

Where constants labels kept the same for simplicity (in practice these are not the same). The solution is analytic everywhere, hence range of solution is $-\infty < x < \infty$

2.3.2 key for final practice exam

#1;

a) First we find a particular solution of the form $y = x^r$. Plug in, we get.

$$x^2\left(r(r-1)x^{r-2}\right) - 3x\left(rx^{r-1}\right) + 4x^r = 0$$

i.e $x^r\left[r^2 - 4r + 4\right] = 0$. $x > 0$

So we get $r^2 - 4r + 4 = 0$, $r_1 = r_2 = 2$.

So we get one particular soln $y_1(x) = x^2$.

To find the other soln, we use reduction of order. Put

$$y_2(x) = x^2 v(x)$$

$y_2' = 2xv + x^2 v'$, $y_2'' = 2v + 4xv' + x^2 v''$.

Plug in: $x^2(2v + 4xv' + x^2 v'') - 3x(2xv + x^2 v') + 4x^2 v = 0$.

Collect terms: $x^3 v' + x^4 v'' = 0$.

i.e $v' + xv'' = 0$.

We solve this using integral factors.

$$v'' + \frac{v'}{x} = 0.$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = \sout{\ln x}\ x.$$

multiply $\mu(x)$ to both sides:

$$(xv')' = xv'' + v' = 0 \Rightarrow xv' = c_1$$

$$v = \int \frac{c_1}{x}dx = c_1 \ln x.$$

So another particular soln $y_2(x) = x^2 \ln x.$

general soln: $y(x) = c_1 x^2 + c_2 x^2 \ln x.$

b). $y'(x) = 2c_1 x + c_2 x + 2c_2 x \ln x.$

then the eqn is:

$$4c_1 + 4c_2 \ln 2 = a.$$

$$2(2c_1 + c_2) + 4c_2 \ln 2 = b.$$

or $$\begin{pmatrix} 4 & 4\ln 2 \\ 4 & 2+4\ln 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Next we solve c_1, c_2.

$$\begin{pmatrix} 4 & 4\ln 2 & a \\ 4 & 2+4\ln 2 & b \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 4 & 4\ln 2 & a \\ 0 & 2 & b-a \end{pmatrix}$$

$$\xrightarrow{r_1 - 2\ln 2\, r_2} \begin{pmatrix} 4 & 0 & (1+2\ln 2)a - 2\ln 2\, b \\ 0 & 2 & b-a \end{pmatrix}$$

i.e $c_1 = \frac{1}{4}\left((1+2\ln 2)a - (2\ln 2)b\right)$

$c_2 = \frac{1}{2}(b-a)$

So the unique soln:

$$y(x) = \frac{x^2}{4}\left((1+2\ln 2)a - (2\ln 2)b\right) + \frac{1}{2}(b-a)x^2\ln x.$$

c). To show linear independence, we show $W(y_1, y_2) \neq 0$.

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix}$$

$$= x^3 + 2x^3\ln x - 2x^3\ln x = x^3 \neq 0 \text{ for } x > 0.$$

So y_1, y_2 linearly indep.

#2. we do reduction of order.

Put $y_2(x) = x\,v(x)$.

then $y_2' = v + xv'$, $y_2'' = 2v' + xv''$.

plug in,

$$x^2(2v' + xv'') - x(x+2)(v + xv') + (x+2)xv = 0$$

Collect terms:

$$x^3 v'' - x^3 v' = 0$$ ~~since x>0~~

$x > 0 \Rightarrow \quad v'' - v' = 0.$

integral factor: $\mu(x) = e^{\int -1\,dx} = e^{-x}$.

$$e^{-x}v'' - e^{-x}v' = (e^{-x}v')' = 0.$$

$\Rightarrow e^{-x}v' = C_1 \quad v' = C_1 e^{x} \Rightarrow v = C_1 e^{x} + C_2.$

So general solution

$$y(x) = x(C_1 e^{x} + C_2).$$

#3. we have found in problem 1 the general soln to homogeneous eqn:

$$y_c(x) = C_1 x^2 + C_2 x^2 \ln x.$$

can write the eqn as:

$$y'' - \frac{3y'}{x} + \frac{4y}{x^2} = \ln x.$$

We use variation of parameters:

$$y_p(x) = \left[-\int \frac{y_2 f}{W}\right] y_1 + \left[\int \frac{y_1 f}{W}\right] y_2.$$

Also we found in problem 1 $W(x) = x^3$.

$$y_p(x) = \left[-\int \frac{x^2 \ln x \cdot \ln x}{x^3} dx\right] \cdot x^2 + \left[\int \frac{x^2 \cdot \ln x}{x^3} dx\right] x^2 \ln x$$

$$= \left[-\int \frac{(\ln x)^2}{x} dx\right] x^2 + \left[\int \frac{\ln x}{x} dx\right] x^2 \ln x.$$

$$\int \frac{(\ln x)^2}{x} dx \overset{u=\ln x}{=\!=} \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3.$$

$$\int \frac{\ln x}{x} dx \overset{u=\ln x}{=} \int u\, du = \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2.$$

So $y_p(x) = -\frac{1}{3}(\ln x)^3 \cdot x^2 + \frac{1}{2} x^2 (\ln x)^3 = \frac{1}{6} x^2 (\ln x)^3$.

So general soln

$$y(x) = y_p(x) + y_c(x)$$

$$= c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3.$$

#4.

Characteristic eqn:

$$2r^2 - 5r + c = 0.$$

a). this means $25 - 4 \times 2 \times c > 0$.

i.e $c < \frac{25}{8}$.

b). this means $25 - 8c = 0$. i.e $c = 25/8$.

c). $2r^2 - 5r + 2 = 0$ $\quad r_1 = 2,\ r_2 = \frac{1}{2}$.

So general sdn $y(x) = C_1 e^{2x} + C_2 e^{\frac{1}{2}x}$.

d). $y'(x) = 2C_1 e^{2x} + \frac{1}{2} C_2 e^{\frac{1}{2}x}$.

So

$$C_1 e^{2x_0} + C_2 e^{\frac{1}{2}x_0} = p.$$

$$2C_1 e^{2x_0} + \frac{1}{2} C_2 e^{\frac{1}{2}x_0} = q.$$

or

$$\begin{pmatrix} e^{2x_0} & e^{\frac{1}{2}x_0} \\ 2e^{2x_0} & \frac{1}{2} e^{\frac{1}{2}x_0} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

(5) 2-1

$$\frac{dx}{dt} = \begin{bmatrix} -3 & 5 \\ -5 & 3 \end{bmatrix} \underline{x}$$

$$(-3-\lambda)(3-\lambda) + 25 = -9 - 3\lambda + 3\lambda + \lambda^2 + 25 = 0$$

$$\lambda^2 + 16 = 0, \quad \lambda^2 = -16, \quad \lambda = \pm 4i$$

$\boxed{\lambda_1 = 4i}$ $\begin{bmatrix} -3-4i & 5 \\ -5 & 3-4i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Top Row $(-3-4i)\xi_1 + 5\xi_2 = 0$

Choose, e.g. $\xi_1 = 5$ $\xi_2 = 3+4i \Rightarrow$

$$(-3-4i)5 + 5(3+4i) = -15 - 20i + 15 + 20i = 0$$

$\underline{\xi}^{(1)} = \begin{bmatrix} 5 \\ 3+4i \end{bmatrix}$ $\underline{x}^{(1)} = \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} e^{4it}$

2-2

$$\underline{x} = C_1 \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} e^{4it} + C_2 \begin{bmatrix} 5 \\ 3-4i \end{bmatrix} e^{-4it}$$

(b) $$\mathrm{Re}\left(\underline{X}^{(1)}\right) = \mathrm{Re}\left\{ \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} \left(\cos 4t + i \sin 4t\right) \right\}$$

$$= \begin{bmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{bmatrix}$$

$$\mathrm{Im}\left(\underline{X}^{(1)}\right) = \begin{bmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{bmatrix}$$

$$\underline{x} = C_3 \begin{bmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{bmatrix} + C_4 \begin{bmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{bmatrix}$$

2-3

(c) $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{g}_1 + \underline{g}_2$

$\underline{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{4t} \qquad \underline{g}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$

Let $\underline{x}_p = \underline{a}te^{4t} + \underline{b}e^{4t}$

$\underline{x}_p' = \underline{a}e^{4t} + 4\underline{a}te^{4t} + 4\underline{b}e^{4t}$

$$\underline{a}e^{4t} + 4\underline{a}te^{4t} + 4\underline{b}e^{4t} = \underline{\underline{A}}\left(\underline{a}te^{4t} + \underline{b}e^{4t}\right) + \underline{g}_1 + \underline{g}_2$$

e^{4t} : $\quad \underline{a} + 4\underline{b} = \underline{\underline{A}}\,\underline{b} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

te^{4t} : $\quad 4\underline{a} = \underline{\underline{A}}\,\underline{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(6) 5-1

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} + \underline{g}(t) \qquad \underline{A} = \begin{bmatrix} 5 & 3 \\ -\frac{1}{3} & 3 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 3 \\ -\frac{1}{3} & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) + 1$$
$$= 15 - 3\lambda - 5\lambda + \lambda^2 + 1$$
$$= \lambda^2 - 8\lambda + 16 = 0$$
$$= (\lambda - 4)^2$$

$\lambda = 4$ repeated

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \xi_1 + 3\xi_2 = 0$$

$$\underline{\xi} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\underline{x}_h = c_1 \underline{\xi} e^{4t} + c_2 \left\{ \underline{\xi} t e^{4t} + \underline{\chi} e^{4t} \right\}$$

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

5-2

$x_1 + 3x_2 = -3$ e.g. $x_1 = 0$ $x_2 = -1$

$$\underline{x}_h = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t} \right\}$$

(b) Let $\underline{\underline{\Psi}}$ be the fundamental matrix. Each column of $\underline{\underline{\Psi}}$ satisfies $\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$; there are n linearly independent columns.

Let $\underline{x} = \underline{\underline{\Psi}}\,\underline{u}$, $\underline{u}$ unknown

$$\underline{x}' = \underline{\underline{\Psi}}'\underline{u} + \underline{\underline{\Psi}}\,\underline{u}' = \underline{\underline{A}}\,\underline{\underline{\Psi}}\,\underline{u} + \underline{g}$$

Then $\underline{\underline{\Psi}}'\underline{u} = \underline{\underline{A}}\,\underline{\underline{\Psi}}\,\underline{u}$ because

$\underline{\underline{\Psi}}' = \underline{\underline{A}}\,\underline{\underline{\Psi}}$ by assumption [column by column]

$$\Rightarrow \quad \underline{\underline{\Psi}}\,\underline{u}' = \underline{g} \quad \Rightarrow \quad \underline{u}' = \underline{\underline{\Psi}}^{-1}\underline{g}$$

$$\Rightarrow \quad \underline{u} = \int \underline{\underline{\Psi}}^{-1}\underline{g}\,dt + \underline{c}$$

5-3

$$\underline{x} = \underline{\underline{\Psi}} \int \underline{\underline{\Psi}}^{-1} \underline{g}\, dt + \underline{\underline{\Psi}}\, \underline{c} \quad \text{or}$$

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int \underline{\underline{\Psi}}^{-1}(t') \underline{g}(t')\, dt' + \underline{\underline{\Psi}}(t)\, \underline{c} \quad \text{or}$$

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int_{t_0}^{t} \underline{\underline{\Psi}}^{-1}(t') \underline{g}(t')\, dt' + \underline{\underline{\Psi}}(t) \underline{\underline{\Psi}}^{-1}(t_0) \underline{x}(t_0)$$

(c) $$\underline{\underline{\Psi}} = \begin{bmatrix} -3e^{4t} & -3te^{4t} \\ e^{4t} & te^{4t} - e^{4t} \end{bmatrix}$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t \\ 1 & t-1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ 0 & e^{-4t} \end{bmatrix} R_2 - R_1$$

$$\begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ \frac{1}{3}e^{-4t} & e^{-4t} \end{bmatrix} R_2(-1)$$

5-4

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} \quad R_1 - tR_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{3}+\frac{t}{3}\right)e^{-4t} & te^{-4t} \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} = \underline{\underline{\Psi}}^{-1}$$

Check $\underline{\Psi}^{-1}\underline{\Psi}$

$$\begin{bmatrix} \left(-\frac{1}{3}+\frac{t}{3}\right)e^{-4t} & te^{-4t} \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -3e^{4t} & -3te^{4t} \\ e^{4t} & (t-1)e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

(7) (a) $y'' - 2y' + 4y = e^x\left[x\sin\sqrt{3}x + e^{\sqrt{3}x}\right]$ (7)

$= xe^x\sin\sqrt{3}x + e^{(1+\sqrt{3})x}$

$y'' - 2y' + 4y = 0 \implies r^2 - 2r + 4 = 0$

$r = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$y_h = C_1 e^x\cos\sqrt{3}x + C_2 e^x\sin\sqrt{3}x$

$y_p = (Ax+B)xe^x\cos\sqrt{3}x + (Cx+D)xe^x\cos\sqrt{3}x$

$+ Ee^{(1+\sqrt{3})x}$

(7-2)

(7)(b) $y'' - 6y' = x^2 + x\cosh 6x$

$$= x^2 + x\left(\frac{e^{6x} + e^{-6x}}{2}\right)$$

$y'' - 6y' = 0 \implies r^2 - 6r = 0 = r(r-6)$

$y_h(x) = c_1 + c_2 e^{6x}$

$y_p = (Ax^3 + Bx^2 + Cx)$

$+ (Dx + E)x e^{6x} + (Fx + G)e^{-6x}$

(8) $(r+2)^2(r^2+2)^2=0$

$r=-2$ repeated

$r^2=-2 \Rightarrow r=\pm i\sqrt{2}$ repeated

$y(x)= c_1 e^{-2x} + c_2 x e^{-2x}$

$+ c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x$

$+ c_5 x \cos\sqrt{2}x + c_6 x \sin\sqrt{2}x$

$-\infty < x < \infty$

2.3.3 final exam questions

Math 320 (Smith): Final Exam, Part I

Sunday May 7, 7:25-9:25 PM, Social Sciences 5206

YOUR NAME:

PLEASE WRITE YOUR NAME ON EVERY PAGE.

YOUR SECTION NUMBER:

Prob 1 /20	20
Prob 2 /20	15
Prob 3 /25	25
Prob 4 /20	20
Prob 5 /15	12
TOTAL /100	

1

1. Find the general solution:

$$y'' + 4y' + 4y = t^{-2} \exp(-2t), \quad t > 0.$$

2

2. Solve the initial value problem:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\,\mathbf{x}(t), \quad \mathbf{A} = \begin{bmatrix} 5 & -2 \\ 1/2 & 3 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

5

4. Given the solution $y_1(x) = \exp(x)$, use the method of Reduction of Order to find the solution to the following initial value problem, and state where the solution is defined.

$$xy'' - (1+x)y' + y = 0, \quad y(1) = 2, \;\; y'(1) = 5$$

Show your work! No work, no credit.

2

5. (a) Find the general solution:

$$2x^2y'' - 3xy' + 2y = 0$$

(b) Find the form of the general solution

$$y^{(v)} + 4y^{(iv)} + 4y''' + y'' + 4y' + 4y = x^2 \exp(-2x) + \exp(-x)\sin(x)$$

given that the characteristic equation is

$$(r+2)^2(r^3+1) = 0.$$

You do not need to solve for the coefficients of the particular solution.

5

1
20
15
25
20
12
92

$-2e^{4}c_2 = \frac{1}{2}$ ~~$c_2 = -e$~~

$-e^{4}c_2 = \frac{1}{4}$ $\quad c_2 = -\frac{e^{-4}}{4}$

2 ✓ **#3** $\dot{x} = Ax + g(t)$ $\quad A = \begin{pmatrix} 2 & 1/2 \\ 2 & 2 \end{pmatrix}$, $\quad g(t) = \begin{pmatrix} e^{-t} + 2 \\ 5 - e^{-t} \end{pmatrix}$

$$\underline{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t} + \begin{pmatrix} -1/2 \\ -2 \end{pmatrix} + \begin{pmatrix} -7/16 \\ 5/8 \end{pmatrix} e^{-t}$$

✓ **#1** $y'' + 4y' + 4y = t^{-2}e^{-2t}$

$$y = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t}(\ln t + 1)$$

$\frac{1}{2}$ **#2** $\dot{x} = Ax$, $\quad A = \begin{pmatrix} 5 & -2 \\ 1/2 & 3 \end{pmatrix}$, $\quad x(1) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$y = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{4t-4} - \frac{1}{4} \begin{pmatrix} 2 + 2t \\ t - 1 \end{pmatrix} e^{4t-4}$$

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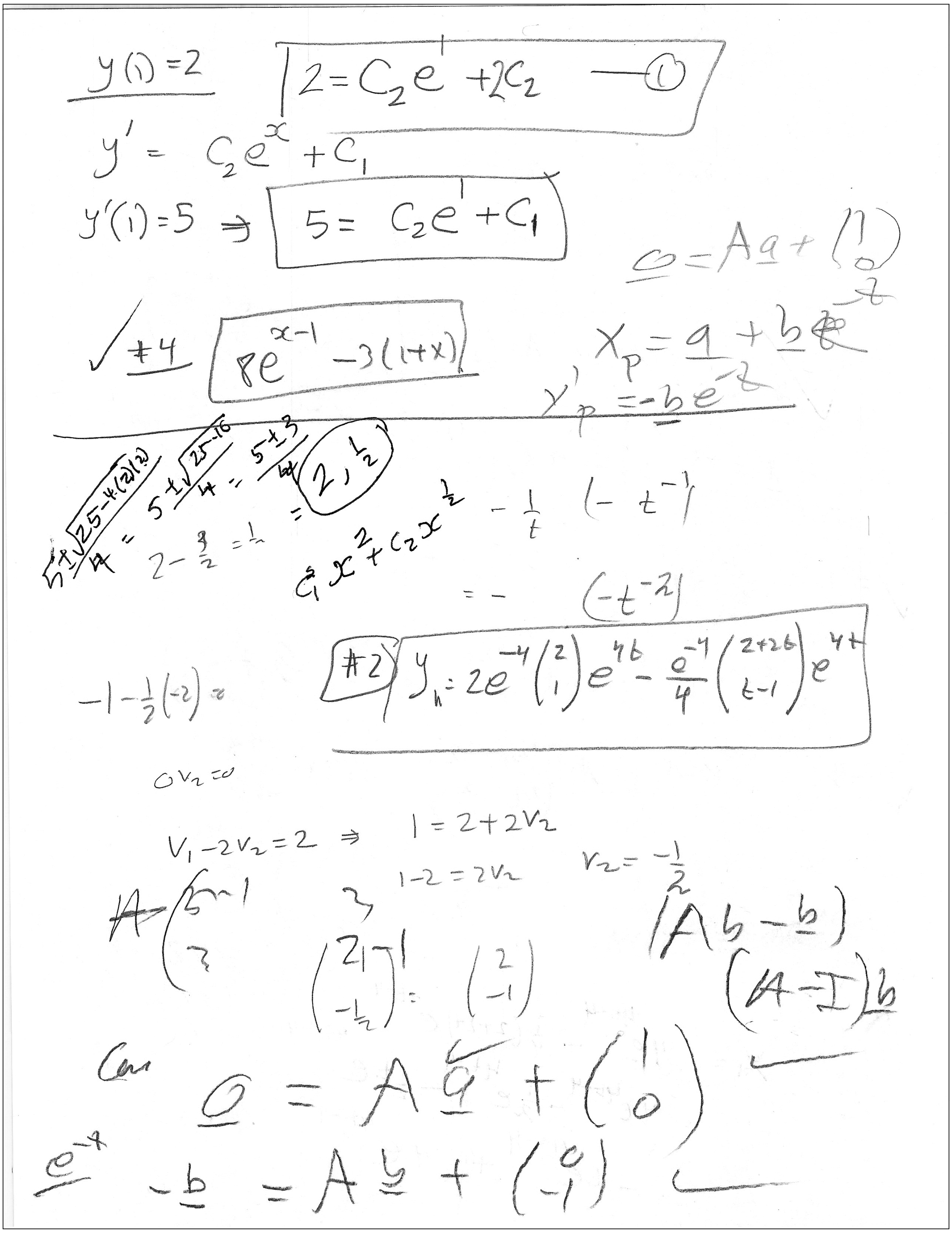
$$x_1 = 4e^{4t-4} - \frac{1}{4}(2+2t)e^{4t-4}$$

$$= 4e^{4t-4} - \frac{1}{2}e^{4t-4} - \frac{1}{2}te^{4t-4}$$

$$= -te^{4t-4} + 4e^{4t-4}$$



$y(1)=2$ $\quad 2 = C_2 e^1 + 2C_2$ — (1)

$y' = C_2 e^x + C_1$

$y'(1) = 5 \Rightarrow 5 = C_2 e^1 + C_1$

✓ #4 $8e^{x-1} - 3(1+x)$

$x_p = \underline{a} + \underline{b} e^{-t}$

$y'_p = -\underline{b} e^{-t}$

$\frac{5 \pm \sqrt{25 - 4(2)(2)}}{4} = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4} = 2, \frac{1}{2}$

$c_1 x^2 + c_2 x^{\frac{1}{2}}$

#2 $y_h = 2e^{-4}\begin{pmatrix}2\\1\end{pmatrix} e^{4t} - \frac{e^{-4}}{4}\begin{pmatrix}2+2t\\t-1\end{pmatrix} e^{4t}$

$v_1 - 2v_2 = 2 \Rightarrow 1 = 2 + 2v_2$

$1 - 2 = 2v_2 \quad v_2 = -\frac{1}{2}$

$(A - I)\underline{b}$

$\underline{0} = A\underline{a} + \begin{pmatrix}1\\0\end{pmatrix}$

$-\underline{b} = A\underline{b} + \begin{pmatrix}0\\-1\end{pmatrix}$

2.3.4 sheat sheet for final exam

Math 320 Exam Crib Sheet

1. **Integration by Parts Formula**

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int x\exp(x)dx = x\exp(x) - \int \exp(x)dx + C = x\exp(x) - \exp(x) + C$$

with $u = x$, $dv = \exp(x)\,dx$, $du = dx$, and $v = \exp(x)$.

2. **Example of Partial Fractions**

$$\int \frac{5}{(x^2-5x+6)}dx = \int \frac{5}{(x-2)(x-3)}dx$$

Let

$$\frac{5}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$
$$= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

Therefore

$$(A+B)x = 0 \quad \text{and} \quad -3A-2B = 5.$$

Solving $A+B=0$ and $-3A-2B=5$ gives $A=-5$ and $B=5$. So finally

$$\int \frac{5}{(x^2-5x+6)}dx = \int \frac{-5}{(x-2)}dx + \int \frac{+5}{(x-3)}dx = -5\ln|x-2| + 5\ln|x-3| + C.$$

3. **Exponentials and the Natural Logarithm**: All arguments of ln are assumed greater than zero.

$$\ln(1) = 0$$

$$\ln(a/b) = \ln(a) - \ln(b)$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^r) = r\ln(a)$$

1

$$\int \frac{1}{u} du = \ln |u| + C, \quad u \neq 0$$

$$\exp(\ln(x)) = x$$

$$\ln(\exp(x)) = x$$

$$\exp(a + b) = \exp(a)\exp(b)$$

$$\exp(a - b) = \frac{\exp(a)}{\exp(b)}$$

$$\exp(ab) = (\exp(a))^b = (\exp(b))^a$$

4. **Taylor Series for $f(x)$ about the point $x = x_o$:**

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(x)|_{x=x_o} \frac{(x - x_o)^n}{n!}$$

6. **Sines and cosines for some angles:**

$$\cos(\pi/6) = \sqrt{3}/2, \quad \sin(\pi/6) = 1/2$$

$$\cos(\pi/3) = 1/2, \quad \sin(\pi/3) = \sqrt{3}/2$$

$$\cos(2\pi/3) = -1/2, \quad \sin(2\pi/3) = \sqrt{3}/2$$

$$\cos(4\pi/3) = -1/2, \quad \sin(4\pi/3) = -\sqrt{3}/2$$

$$\cos(5\pi/3) = 1/2, \quad \sin(5\pi/3) = -\sqrt{3}/2$$

$$\cos(\pi/4) = \sqrt{2}/2, \quad \sin(\pi/4) = \sqrt{2}/2$$

$$\cos(3\pi/4) = -\sqrt{2}/2, \quad \sin(3\pi/4) = \sqrt{2}/2$$

$$\cos(5\pi/4) = -\sqrt{2}/2, \quad \sin(5\pi/4) = -\sqrt{2}/2$$

$$\cos(7\pi/4) = \sqrt{2}/2, \quad \sin(7\pi/4) = -\sqrt{2}/2$$

7. **Definition of sinh(x) and cosh(x)**: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$

2

Chapter 3

study notes

3.1 cheat sheet

3.1.1 Summary of content of what will be in exam 2

section 3.1

1. Possible solutions of $Ax = b$ are: no solution, unique solution, infinite number of solutions.
2. Elementray row operations: multiply one row by non-zero constant, interchange two rows, add multiple of one row to another row

section 3.2

1. Matrices, Gaussian elimination.
2. Setting up augumented matrix $(A|b)$
3. Two matrices are row equivelent if we can do operations on one matrix, and obtain the other matrix
4. Echelon form. Backsubstitution to obtain solution.

section 3.3

1. Reduced Echeoln form: Each leadeing entry in row must be one. All entries in same column as leading entry, above it or below it must be zero. Gauss-Jordan elimination generates Reduced Echeoln form. We basically do Gaussian elimination, followed by backward elimnination, then normalize all diagonal elements to 1.

section 3.4 This section is mainly on matrix operations. Multiplications. How to multiply matrices. How to write system is linear equations as $Ax = b$. All basic stuff.

section 3.5 Inverses of matrices.

1. To find A^{-1}. Set up the $(A|I)$ and generate reduced echelon form.
2. Definition of matrix inverse. B is inverse of A if $AB = I$ and $BA = I$.
3. Matrix inverse is unique. (theorem 1)
4. Theorem 3: $\left(A^{-1}\right)^{-1} = A, (AB)^{-1} = B^{-1}A^{-1}$
5. If A is square and $Ax = b$ has unique solution then $x = A^{-1}b$ (thm 4)
6. square Matrix is invertible, iff it is row equivalent to I_n. Invertible matrix is also called non-singular.

section 3.6 Determinants.

1. To find determinants. Do cofactor expansion along a row or column. Pick one with most zeros in it, to save time.

(a) Property 1: If we multiply one row (or column) of A by k then $|A|$ becomes $k|A|$

(b) property 2: interchanging two rows, introduces a minus sign in $|A|$

(c) property 3: If two rows or columns are the same then $|A| = 0$

(d) property 4: if A_1, A_2, B are identical, except that i^{th} row of B is the sum of the i^{th} of A_1 and A_2, then $|B| = |A_1| + |A_2|$

(e) property 5: Adding constant multiple of one row (or column) to another row (or column) do not change the determinant.

(f) property 6: for upper or lower triangle matrix, $|A|$ is the product of all diagonal elements.

2. Matrix transpose. (but we did not use this much in class).
3. Thm 2. Matrix A is invertible iff $|A| \neq 0$
4. thm 3. $|AB| = |A||B|$., But in general $|A + B| \neq |A| + |B|$
5. $\left|A^{-1}\right| = \frac{1}{|A|}$
6. Cramer rule. But we did not use it. Thm 4. We also did not do thm 5 (adjoint matrices).

Section 3.7 Linear equations, curve fitting. Did not cover.

section 4.1 Vector spaces.

1. Define $\mathbb{R}^3$ as set of all ordered triples (a, b, c) of real numbers. (coordinates)
2. Thm 1. If u, v, w are vectors in $\mathbb{R}^3$ then we have properties of communtativity, associativity, additive inverse and zero element, and distrbutivity. See page 230 for list.
3. Thm 2. Two vectors $\boldsymbol{u}, \boldsymbol{v}$ are Linearly dependent iff there exist scalars a, b not both zero, such that $a\boldsymbol{u} + b\boldsymbol{v} = 0$
4. 3 vectors in $\mathbb{R}^3$ are L.D. if one vector is linear combination of the other two vectors.
5. THM 4. If we put 3 vectors as columns of A and then find $|A| = 0$ then the 3 vectors are L.D.
6. For square matrix, if $Ax = 0$ has only trivial solution, then columns of A are L.I.
7. THM 5. If 3 vectors in $\mathbb{R}^3$ are L.I., then they are basis vectors.
8. subspaces of $\mathbb{R}^3$. None empty subset W of vectors of $\mathbb{R}^3$ is subspace iff it is closed under addition and closed under scalar multiplication. Basic problems here, is to show if vectors make subspace or not. By seeing if the space is closed under additon and scalar multiplication.

section 4.2 Vector space $\mathbb{R}^n$ and subspaces. (page 238).

1. Definition of $\mathbb{R}^n$ vector space. Page 240. 7 points listed.
2. THM 1. Subspace. A subset of $\mathbb{R}^n$ which is also a vector space is called subspace. We only need to verify closed under additions and closed under multiplication for subspace.
3. Solution space: The space in which solution of $A_{m\times n} x_{n\times 1} = 0_{m\times 1}$ live. THis will always be subspace of $\mathbb{R}^n$. To find it, do G.E. and find the free variables. The number of free variables, tell us the dimension of the subspace. If there are 2 free variables, then there will be two basis for the solution space. Each vector will be n length. So the solution space is subspace of $\mathbb{R}^n$
4. Solution space of $A_{m\times n} x = 0$ is always subspace of $\mathbb{R}^n$

section 4.3 Linear combinations and independence of vectors

1. Given a vector w and set of L.I. vectors v_i, find if that vector can be expressed as linear combination of the set of vectors. Set up $w = c_1 v_1 + c_2 v_2 + \cdots$ and solve $Ac = w$ and see if c is all zeros or not.

2. Definition: L.I. of vectors. Solve $Ac = 0$ and see if $c = 0$ or not. If $c = 0$ is solution, then L.I.

3. For square matrix, the columns are L.I. if $|A| \neq 0$.

4. For $A_{m \times n}$, with $m > n$, then if rank A is n, then the columns of A are L.I.

section 4.4 Basis and dimensions of vector spaces. Did not cover for exam.

3.1.2 possible questions and how to answer them

Question Given a set of linear equations in form $Ax = b$ and asks if the system is consistent or not.

Answer System is consistent if it has solution. The solution can be either unique or infinite number of them. To answer this, setup the augmented matrix $(A|b)$ and generate Echelon form (using Gaussian elimination). Then look at the last row. Lets say A had m rows. If last entry in last row is $0 = 0$, then there are infinite solutions, so consistent because this means $0x_m = 0$ and x_m can be anything.

If last entry in last row looks like $0 = r$ where r is a number not zero, then no solution, hence not consistent. If last entry in last row looks line *number* = *anything* then unique solution. So consistent.

So we really need to check if last entry in last row is $0 = r$ to decide. Be careful, do not check to see if $|A|$ not equal to zero and then say it is consistent. Because $|A| = 0$ can still be consistent, since we can have infinite number of solutions. $|A| = 0$ does not necessarily mean no solution.

For example, this system

$$\begin{aligned} 3x_1 + x_2 - 3x_3 &= -4 \\ x_1 + x_2 + x_3 &= 1 \\ 5x_1 + 6x_2 + 8x_3 &= 0 \end{aligned}$$

For the above $|A| = 0$. And it happened that this system has no solution hence not consistent. And the following system

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 13 \\ 2x_1 + 5x_2 + 4x_3 &= 23 \\ 2x_1 + 7x_2 + 8x_3 &= 29 \end{aligned}$$

has also $|A| = 0$. But the above has infinite number of solutions. Hence consistent. So bottom line, do not use $|A|$ to answer questions about consistent or not. (also $|A|$ only works for square matrices any way). So what does $|A|$ give? If $|A|$ is not zero, it says the solution is unique. So if the question gives square matrix, and asks if solution is unique, only then check $|A| = 0$ or not.

Question Problem gives set of linear equations in form $Ax = b$ and asks if system has unique solution, no solution, or infinite solution.

Answer Same as above. Follow same steps.

Question Problem gives square matrix A and asks to find A^{-1}.

Answer Set up the augmented matrix $(A|I)$ where I is the identity matrix. Go through the forward elimination to reach echelon form. Then go though the backward elimination to each reduced Echelon form. Then make all diagonals in A be 1. While doing these row operations, always do them on the whole $(A|I)$ system, not just on A. At then end, A^{-1} will be where I was sitting.

Question Problem gives square matrix A and and square matrix B and asks if B is the inverse of A

Answer Start by multiplying AB and see if you can get I as result. Also need to do BA and see if you can get I as well. If so, then B is the inverse of A. To get to I need to do some matrix manipulation in the middle. But it is all algebra. This is all based on $AA^{-1} = A^{-1}A = I$. (if A is invertible ofcourse). Remember also that A^{-1} is unique. i.e. given a matrix, it has only one matrix which is its inverse.

Question Problem asks to proof that matrix inverse is unique.

Answer Let A be invertible. Let B be its inverse. Assume now that C is also its inverse but $C \neq B$. Then $C = CI = C(AB) = (CA)B = IB = B$, hence $C = B$. Proof by contradiction. So only unique inverse.

Question

Chapter 4

HWs

Local contents

4.1 HW1

4.1.1 Section 1.3 problem 12

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$\frac{dy}{dx} = x \ln y; y(1) = 1$$

Solution

$$f(x,y) = x \ln y$$

Sine $f(x,y)$ is continuous in x for all x and continuous in y for $y > 0$ and since initial condition is at point $(1,1)$, then a solution exist in some interval that contains $(1,1)$.

$$\frac{\partial f(x,y)}{\partial y} = \frac{x}{y}$$

Since $\frac{\partial f(x,y)}{\partial y}$ is continuous in x for all x and continuous in y for $y \neq 0$ and since initial condition is at point $(1,1)$, then the solution is unique in some interval that contains $(1,1)$. The following the the slope field for $f(x,y) = x \ln y$ showing small interval that contains $(1,1)$

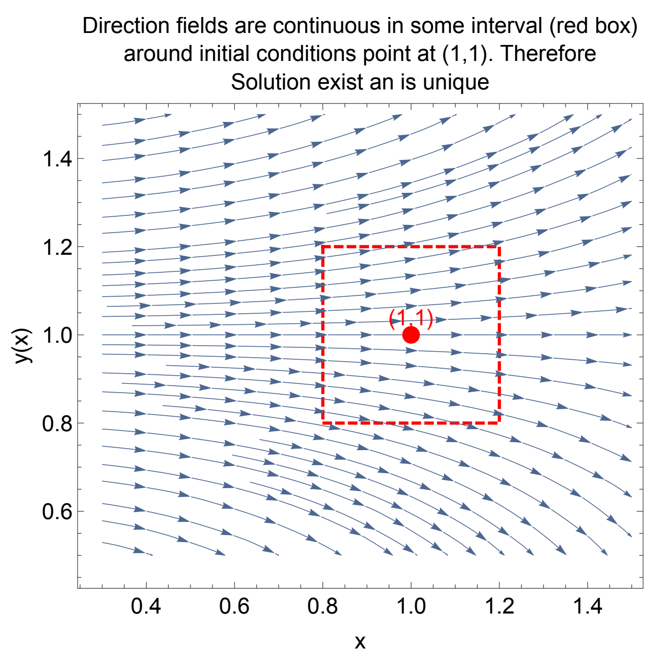


Figure 4.1: Problem 1.3, 11

4.1.2 Section 1.3 problem 17

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$\frac{dy}{dx} = x - 1; y(0) = 1$$

Solution

$$f(x,y) = x - 1$$

$f(x,y)$ is continuous for all x (there is no y dependency to check), then a solution exist in some interval that contains $(0,1)$.

$$\frac{\partial f(x,y)}{\partial y} = 0$$

No dependency on x or y to check. Hence solution is unique in some interval that contains $(0,1)$. The following the the slope field for $f(x,y) = x-1$ showing small interval that contains $(0,1)$

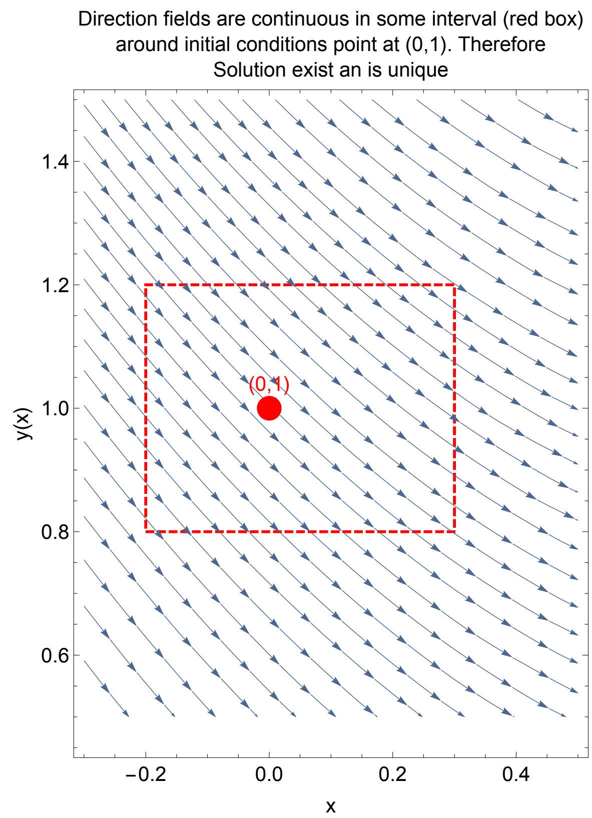


Figure 4.2: Problem 1.3, 17

4.1.3 Section 1.3 problem 18

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$y\frac{dy}{dx} = x - 1; y(1) = 0$$

Solution

$$f(x,y) = \frac{x-1}{y}$$

$f(x,y)$ is continuous for all x, and continuous for all y except at $y = 0$. But since the initial point itself is at $(x = 1, y = 0)$, therefore, the theory can not decide on existence or uniqueness of solution in an intervals containing $(1,0)$.

4.1.4 Section 1.3 problem 22

Use the method of example 2 (page 20) to construct slope field then sketch solution curve corresponding to the given initial condition. Finally use this solution curve to estimate the desired value of the solution $y(x)$.

$$\frac{dy}{dx} = y - x$$
$$y(4) = 1$$
$$y(-4) = ?$$

Solution

$$f(x,y) = y - x$$

By making slope field for $f(x,y) = y - x$, then locating initial point $(4,1)$ and tracing the slope back to $x = -4$, we can then read the y value to be -3. Here is a plot showing trace of the slope field to the point $x = -4$, where $y = -3$. Hence $y(-4) \approx -3$.

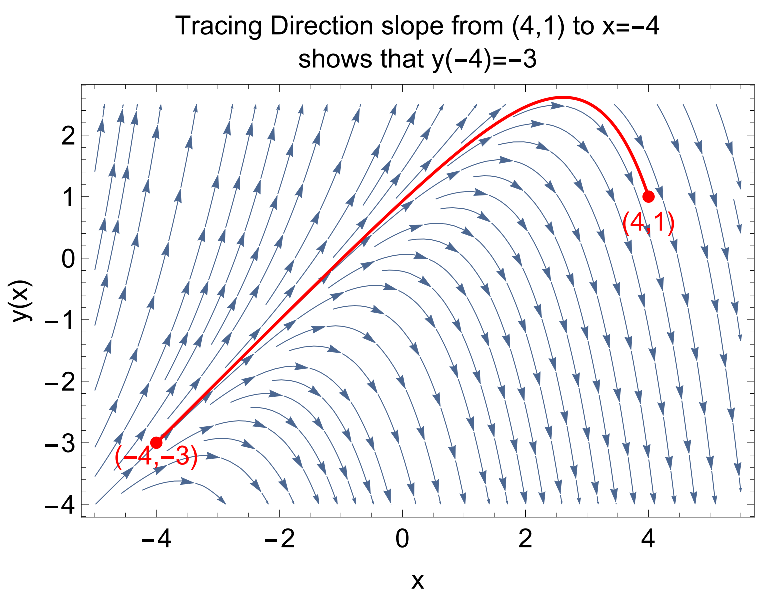


Figure 4.3: Problem 1.3, 22

4.1.5 Section 1.3 problem 26

Suppose the deer population $P(t)$ in small forest satisfies logistic equation $\frac{dp}{dt} = 0.0225p - 0.0003p^2$. Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take for the number of deer to double? What will be the limiting deer population?

Solution

The slope field was first drawn. Then the point $(0,25)$ was located. Then the slope field was traced until $y = 50$, which is double the number of deer from the initial starting time. Now the t component was read from the slope field to answer the first part of the question.

$$f(t,p) = 0.0225p - 0.0003p^2$$

Here is a plot showing trace of the slope field. This shows at about $t = 60$ months, the deer population will be 50.

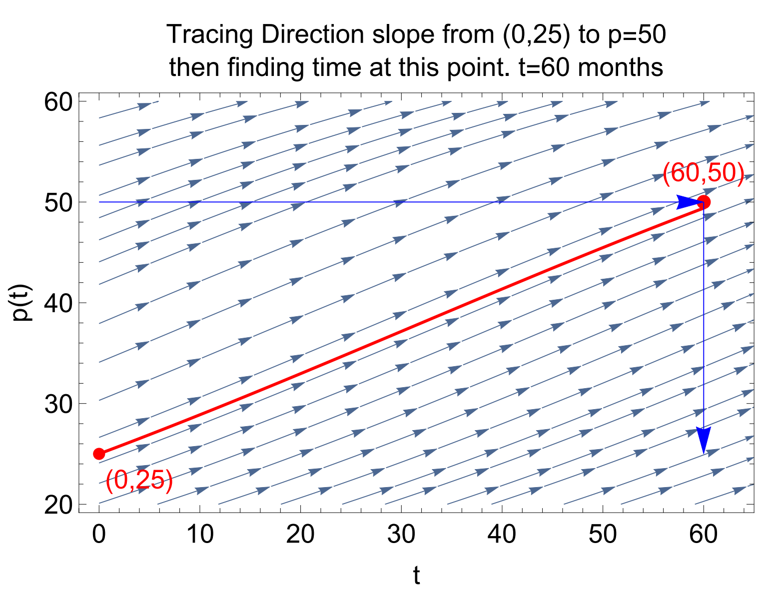


Figure 4.4: Problem 1.3, 26

4.1.6 Section 1.3 problem 28

Verify that if k is constant, then the function $y(x) = kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of the these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y; y(a) = b$ has. One, none or infinitely many.

Solution

To verify that $y(x) = kx$ satisfies the differential equation, we plug-in this solution into the ODE and check that we get the same RHS as given. We see that $y'(x) = k$. Therefore $xy' = y$ becomes $x(k) = y = kx$. Hence satisfied.

$$f(x,y) = \frac{y}{x}$$

This is continuous for all x except at $x = 0$ and continuous for all y. Therefore solution exist in interval which do not contain $x = 0$. In addition $\frac{\partial f(x,y)}{\partial y} = \frac{1}{x}$ which is continuous for all x except at $x = 0$. Hence there is a solution and the solution is unique in an interval that do not contain $x = 0$. Here is a plot of the slope field in region around the origin.

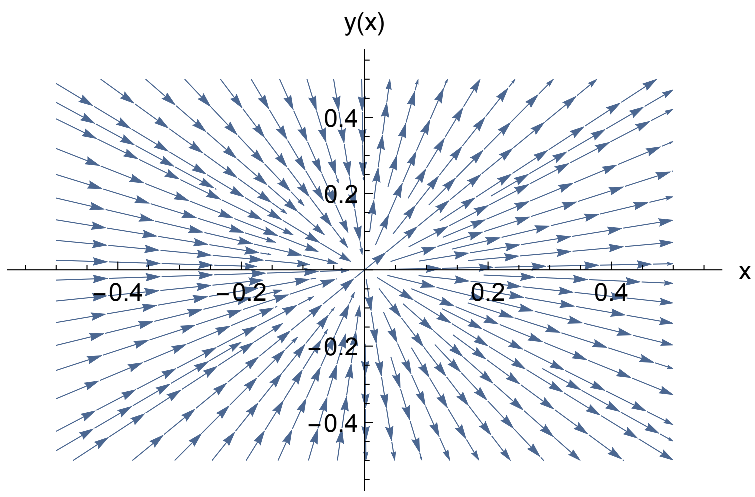


Figure 4.5: Problem 1.3, 28

We see from the above, that if we start from $x = 0, y = 0$, then there are ∞ number of solutions, since there are ∞ number of slope lines starting or ending at $(0,0)$. For any point (a,b) where $a \neq 0$, there is unique solution, since we can find interval around (a,b) in this case with unique slope line. Finally, if $a = 0$ but $b \neq 0$, which means the initial condition is at the y axis, then there is no solution, since the slop is ∞ in this case. Hence

1. Infinite number of solution if $a = 0$ and $b = 0$
2. No solution if $a = 0, b \neq 0$
3. Unique solution if $a \neq 0$ and $b \neq 0$.

4.1.7 Section 1.3 problem 30

Verify that if c is constant, then the function defined piecewise by

$$y(x) = \begin{cases} 1 & x \leq c \\ \cos(x-c) & c < x < c+\pi \\ -1 & x \geq c+\pi \end{cases}$$

Satisfies $y' = -\sqrt{1-y^2}$ for all x. (Perhaps an preliminary sketch with $c = 0$ will be helpful). Sketch a variety of such solution curves. Then determine (in terms of a and b how many different solutions the initial value problem $y' = -\sqrt{1-y^2}; y(a) = b$ has.

Solution

The solution $y(x)$ is plotted for $c = 0, -1, +1$. The following show the result. The effect of c is that it causes a shift to the left or right depending on value of c.

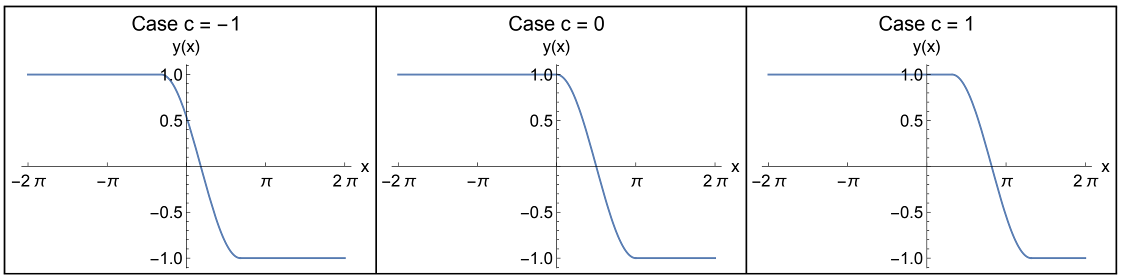


Figure 4.6: Problem 1.3, 30

Since

$$f(x,y) = -\sqrt{1-y^2}$$

Then the above is real, when $|y| < 1$ otherwise the value under the root will be become negative. To show that $y(x)$ satisfies the ODE, we plug-in each branch of the piecewise, one at a time, into the ODE and see if it satisfies it. When $x \leq c$, then $y(x) = 1$. Plugging this into the ODE gives $0 = 0$. Verified. When $c < x < c + \pi$, then $y(x) = \cos(x - c)$. Plugging this into the ODE gives

$$\begin{aligned} -\sin(x-c) &= -\sqrt{1-(\cos(x-c))^2} \\ &= -\sqrt{\sin(x-c)^2} \\ &= -\sin(x-c) \end{aligned}$$

Hence satisfied. When $x \geq c + \pi$ then $y(x) = -1$ and plugging this into the ODE gives

$$\begin{aligned} 0 &= -\sqrt{1-(-1)^2} \\ &= -\sqrt{1-1} \\ &= 0 \end{aligned}$$

Hence solution $y(x)$ satisfies the ODE. The slope field is now plotted

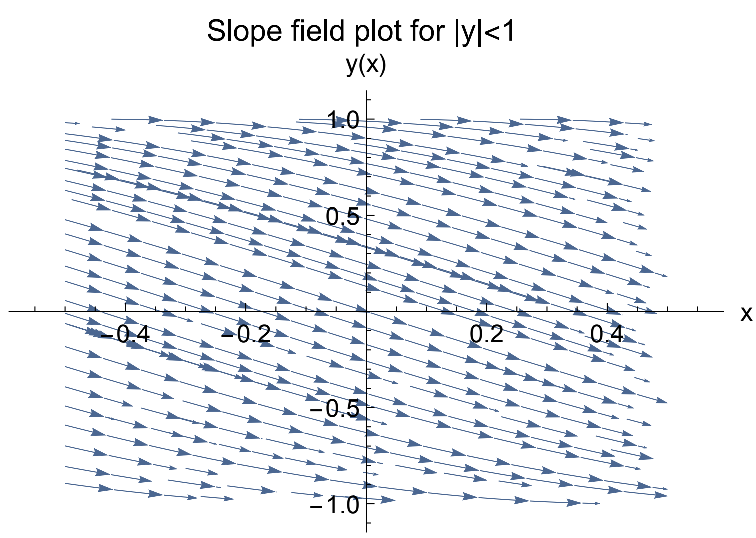


Figure 4.7: Problem 1.3, 30

We see from the slope plot, that starting at any point in a region, as long as $|y| < 0$, then the solution is unique. When $y = 1$ or $y = -1$, then $y' = 0$, and this gives infinite number of solutions since $y = c$ for any constant is a solution. For real solution, y can not be larger than 1. Hence in summary

1. Infinite number of solutions if $b = \pm 1$
2. Unique solution for any (a, b) where $|b| < 1$
3. No real solution for $|b| > 1$

4.1.8 Section 1.4 problem 6

Final general solution of $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

This is separable.

$$\frac{dy}{\sqrt{y}} = 3\sqrt{x}dx$$
$$y^{\frac{-1}{2}}dy = 3x^{\frac{1}{2}}dx$$

Integrating

$$\frac{y^{\frac{1}{2}}}{\frac{1}{2}} = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$2y^{\frac{1}{2}} = 2x^{\frac{3}{2}} + c$$
$$y^{\frac{1}{2}} = x^{\frac{3}{2}} + c_1$$
$$y = \left(x^{\frac{3}{2}} + c_1\right)^2$$

4.1.9 Section 1.4 problem 10

Final general solution of $(1+x)^2 \frac{dy}{dx} = \left(1+y\right)^2$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f\left(x,y\right) = \frac{\left(1+y\right)^2}{(1+x)^2}$$

$f\left(x,y\right)$ is continuous for all y but not continuous for $x = -1$. Therefore solution exist as long as solution interval or initial conditions do not include $x = -1$.

$$\frac{\partial f\left(x,y\right)}{\partial y} = \frac{2\left(1+y\right)}{(1+x)^2}$$

$f\left(x,y\right)$ is continuous for all y but not continuous for $x = -1$. Therefore solution exist and is unique as long as solution interval or initial conditions do not include $x = -1$. The slope field is given below

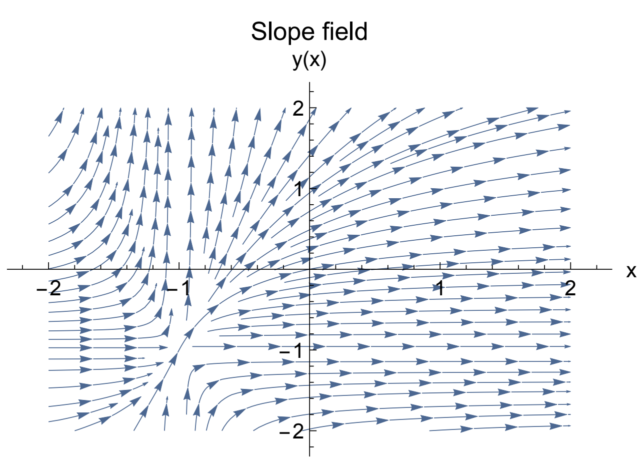


Figure 4.8: Problem 1.4, 10

Now the ODE is solved. This is separable.

$$\frac{dy}{\left(1+y\right)^2} = \frac{dx}{(1+x)^2}$$

Integrating

$$\int \frac{dy}{\left(1+y\right)^2} = \int \frac{dx}{(1+x)^2}$$

Let $u = 1 + y$ then $\frac{du}{dy} = 1$. Hence $\int \frac{dy}{\left(1+y\right)^2} \to \int \frac{du}{u^2} = -\frac{1}{u} \to \frac{-1}{1+y}$. Similarly, $\int \frac{dx}{(1+x)^2} = \frac{-1}{1+x}$.
Therefore the above becomes

$$\frac{-1}{1+y} = \frac{-1}{1+x} + c$$
$$\frac{1}{1+y} = \frac{1}{1+x} + c_1$$
$$\frac{1}{1+y} = \frac{1 + c_1(1+x)}{1+x}$$
$$1 + y = \frac{1+x}{1 + c_1(1+x)}$$

Hence

$$\boxed{y = \frac{1+x}{1+c_1(1+x)} - 1}$$

For $x \neq -1$.

4.1.10 Section 1.4 problem 22

Find explicit particular solution of $\frac{dy}{dx} = 4x^3 y - y; y(1) = -3$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f\left(x,y\right) = 4x^3 y - y$$

$f\left(x,y\right)$ is continuous for all y and continuous for all x.

$$\frac{\partial f\left(x,y\right)}{\partial y} = 4x^3 - 1$$

$\frac{\partial f(x,y)}{\partial y}$ is continuous for all x. It does not depend on y. Hence solution is exist and is unique in some interval that contain initial point $(1,-3)$. Now the ODE is solved.

$$\frac{dy}{dx} = y\left(4x^3 - 1\right)$$

This is now separable

$$\frac{dy}{y} = \left(4x^3 - 1\right) dx$$

Integrating

$$\ln\left|y\right| = 4\frac{x^4}{4} - x + c$$
$$\ln\left|y\right| = x^4 - x + c$$
$$y = e^{x^4 - x + c}$$

Let $e^c = c_1$, then the above can be written as

$$y = c_1 e^{x^4 - x}$$

Now the constant of integration is found from initial conditions. $y(1) = -3$, therefore

$$-3 = c_1 e^{1-1} = c_1$$

Hence the solution becomes

$$y(x) = -3e^{x^4 - x}$$

Here is a plot of the solution in small interval around $x = 1$

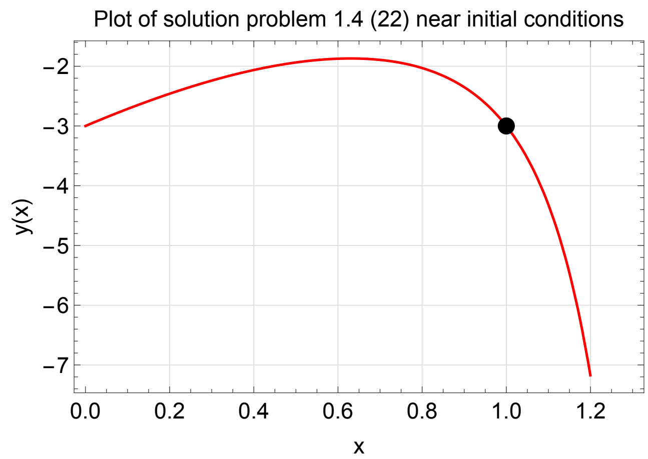


Figure 4.9: Problem 1.4, 22

4.1.11 Section 1.4 problem 26

Find explicit particular solution of $\frac{dy}{dx} = 2xy^2 + 3x^2y^2; y(1) = -1$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f(x,y) = 2xy^2 + 3x^2y^2$$

$f(x,y)$ is continuous for all y and continuous for all x.

$$\frac{\partial f(x,y)}{\partial y} = 4xy + 6x^2y$$

$\frac{\partial f(x,y)}{\partial y}$ is continuous for all x and for all y. Hence a solution is exist and is unique in some interval that contain initial point $(1,-1)$. Now the ODE is solved.

$$\frac{dy}{dx} = y^2\left(2x + 3x^2\right)$$

This is separable.

$$\frac{dy}{y^2} = 2x + 3x^2dx$$

Integrating

$$\begin{aligned} -\frac{1}{y} &= x^2 + x^3 + c \\ \frac{1}{y} &= -\left(x^2 + x^3 + c\right) \\ y &= \frac{-1}{x^2 + x^3 + c} \end{aligned}$$

Applying initial conditions to find c gives

$$\begin{aligned} -1 &= \frac{-1}{1+1+c} \\ -2 - c &= -1 \\ c &= -1 \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= \frac{-1}{x^2 + x^3 - 1} \\ &= \frac{1}{1 - x^2 - x^3} \end{aligned}$$

Here is a plot of the solution in small interval around $x = 1$

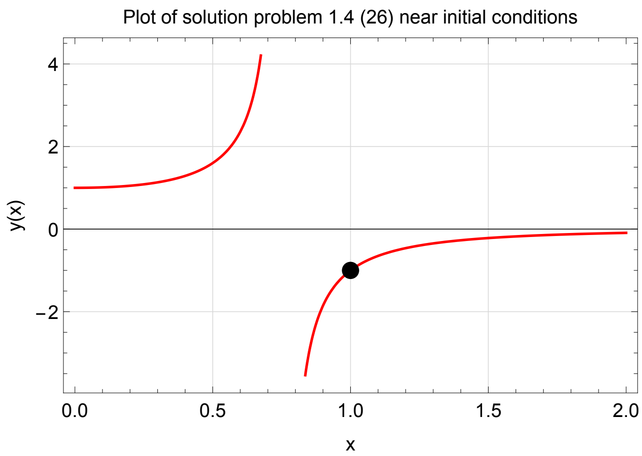


Figure 4.10: Problem 1.4, 26

We notice that at the real root of $1-x^2-x^3$, the solution $y(x)$ goes to $\pm\infty$. This happens at $x \approx 0.75487$.

4.1.12 Section 1.4 problem 30

Solve $\left(\frac{dy}{dx}\right)^2 = 4y$ to verify the general solution curves and singular solution curve that are illustrated in fig 1.4.5. Then determine the points (a,b) in the plane for which the initial value problem $\left(y'\right)^2 = 4y; y(a) = b$ has (a) No solution, (b) infinitely many solutions that are defined for all x, (c) on some neighborhood of the point $x=a$, only finitely many solutions.

Solution

Figure 1.4.5 is below

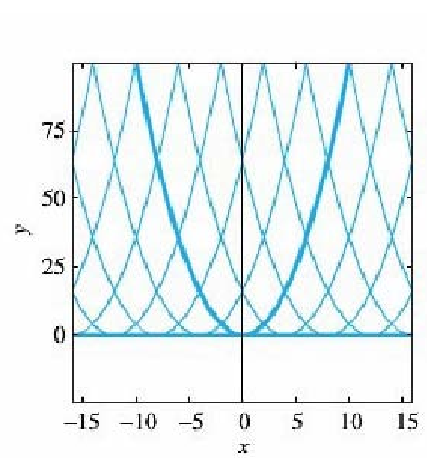


FIGURE 1.4.5. The general solution curves $y=(x-C)^2$ and the singular solution curve $y=0$ of the differential equation $(y')^2 = 4y$.

$$f(x,y) = \pm 2\sqrt{y}$$

Hence $f(x,y)$ is continuous in y for $y>0$. Hence solutions exist for $y>0$. $\frac{\partial f(x,y)}{\partial y} = \pm 2\frac{1}{\sqrt{y}}$ and this is also continuous in y for $y>0$. Therefore, unique solution exist for $y>0$. (Interval can be found around initial conditions (a,b) as long as $b>0$). Here is slope field plot

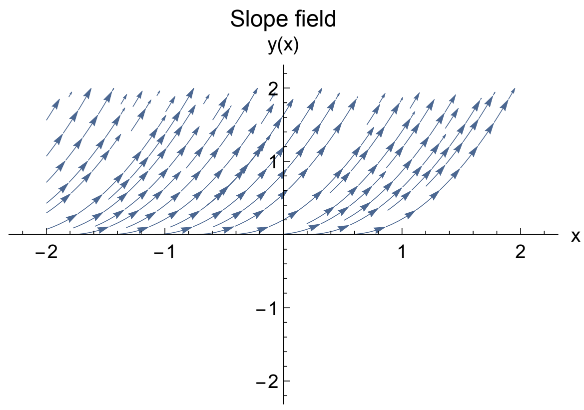


Figure 4.11: Problem 1.4, 30

$$\frac{dy}{dx} = \pm 2\sqrt{y}$$

For the negative case, we obtain

$$\begin{aligned} y^{\frac{-1}{2}} dy &= -2dx \\ 2y^{\frac{1}{2}} &= -2x + c \\ y^{\frac{1}{2}} &= -x + c_1 \\ y &= (c_1 - x)^2 \end{aligned}$$

For the positive case

$$\begin{aligned} y^{\frac{-1}{2}} dy &= 2dx \\ 2y^{\frac{1}{2}} &= 2x + c \\ y^{\frac{1}{2}} &= x + c_1 \\ y &= (c_1 + x)^2 \end{aligned}$$

Hence the solutions are

$$y(x) = \begin{cases} (c_1 - x)^2 & \\ (c_1 + x)^2 & \\ 0 & \text{singular solution} \end{cases}$$

The solution $y(x) = 0$ is singular, since it can not be obtained from the general solution $(c_1 - x)^2$ for arbitrary c. Summary:

1. No solution for $y < 0$
2. singular solution for $y = 0$
3. Two general solutions $(c_1 - x)^2$ and $(c_1 + x)^2$ for all x and $y > 0$.

The following is plot of $y(x) = (c_1 - x)^2$ for few values of c_1 to show the shape of the solution curves. This agrees with the figure given in the book.

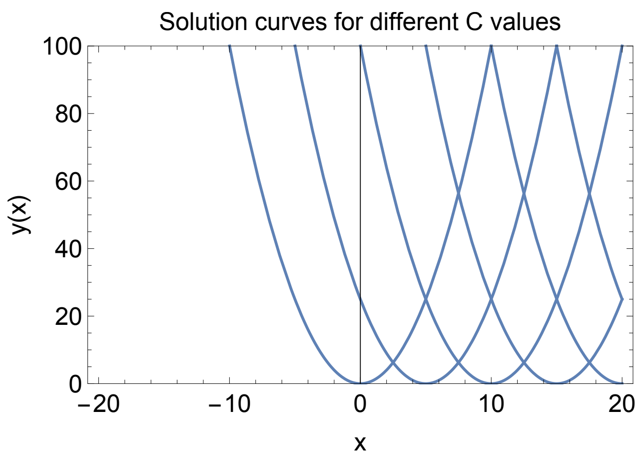


Figure 4.12: Problem 1.4, 30

4.1.13 Section 1.4 problem 42

A certain moon rock was found to contain equal number of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half like is about 1.28×10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

Solution

Half life is the time for a quantity to reduce to half its original number. Let $T = 1.28 \times 10^9$ years in this example. Let $P(0)$ be the number of potassium atoms at time $t = 0$. Hence the formula for half life decay is

$$P(t) = P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}}$$

Where in the above $P(t)$ is number of potassium atoms that remain after time t. Let $g(t)$ be the number of argon atoms at time t. Since $\frac{1}{9}$ of the decayed potassium atoms changed to argon, then

$$\begin{aligned} g(t) &= \frac{1}{9}(P(0) - P(t)) \\ &= \frac{1}{9}\left(P(0) - P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) \\ &= \frac{1}{9}P(0)\left(1 - \left(\frac{1}{2}\right)^{\frac{t}{T}}\right) \end{aligned}$$

Since we want to find t when $g(t) = P(t)$, then we solve from

$$
\begin{aligned}
g(t) &= P(t)\\
\frac{1}{9}P(0)\left(1-\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) &= P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}}\\
\frac{1}{9}\left(1-\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) &= \left(\frac{1}{2}\right)^{\frac{t}{T}}\\
1-\left(\frac{1}{2}\right)^{\frac{t}{T}} &= 9\left(\frac{1}{2}\right)^{\frac{t}{T}}\\
1 &= 9\left(\frac{1}{2}\right)^{\frac{t}{T}} + \left(\frac{1}{2}\right)^{\frac{t}{T}}\\
1 &= 10\left(\frac{1}{2}\right)^{\frac{t}{T}}\\
\frac{1}{10} &= \left(\frac{1}{2}\right)^{\frac{t}{T}}
\end{aligned}
$$

Taking log

$$
\begin{aligned}
\log\left(\frac{1}{10}\right) &= \frac{t}{T}\log\left(\frac{1}{2}\right)\\
t &= T\frac{\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{2}\right)}\\
&= 1.28\times 10^{9}\left(\frac{-2.3}{-0.693}\right)\\
&= 4.2482\times 10^{9}
\end{aligned}
$$

Hence it will take 4.2482 billion years.

4.1.14 Section 1.4 problem 46

The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $\frac{dp}{dx} = (-0.2)p; p(0) = 29.92$. (a) Calculate the barometric pressure at 10,000 ft. and again at 30,000 ft. (b) Without prior conditioning, few people can survive when the pressure drops to less than 15 in. Of mercury. How high is that?

Solution

4.1.14.1 Part (a)

$$
\frac{dp}{dx} = (-0.2)p
$$

This is separable.

$$
\begin{aligned}
\frac{dp}{p} &= -0.2dx\\
\ln|p| &= -0.2x + c\\
p &= ce^{-0.2x}
\end{aligned}
$$

To find c, we apply initial conditions. At $x = 0, p = 29.92$ in, hence

$$
29.92 = c
$$

Therefore the general solution is

$$
p = 29.92e^{-0.2x}
$$

Now, when $x = 10000$ ft or $10000/5280 = 1.894$ miles, then

$$\begin{aligned} p &= 29.92e^{-0.2(1.894)} \\ &= 20.486 \text{ in} \end{aligned}$$

when $x = 30000$ ft or $30000/5280 = 5.6818$ miles, then

$$\begin{aligned} p &= 29.92e^{-0.2(5.6818)} \\ &= 9.6039 \text{ in} \end{aligned}$$

4.1.14.2 Part (b)

We solve for x from

$$\begin{aligned} 15 &= 29.92e^{-0.2x} \\ \frac{15}{29.92} &= e^{-0.2x} \end{aligned}$$

Taking natural log

$$\begin{aligned} \ln \frac{15}{29.92} &= -0.2x \\ -0.69047 &= -0.2x \end{aligned}$$

Hence

$$\begin{aligned} x &= \frac{0.69047}{0.2} = 3.4524 \text{ miles} \\ &= (3.4524)(5280) = 18229 \text{ ft} \end{aligned}$$

4.2 HW2

4.2.1 Section 1.5 problem 18 (page 56)

Problem Find general solution for $xy' = 2y + x^3 \cos x$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned} y' &= f\left(x,y\right) \\ &= \frac{2y + x^3 \cos x}{x} \end{aligned}$$

We see that $f\left(x,y\right)$ is continuous for all $x \neq 0$ and for all y. And

$$\frac{\partial f\left(x,y\right)}{\partial y} = \frac{2}{x}$$

This is continuous for all $x \neq 0$. Therefore solution exist and unique in some interval which do not include $x = 0$. Now we will solve the ODE.

$$xy' = 2y + x^3 \cos x$$

Dividing by $x \neq 0$ and rearranging gives

$$y' - \frac{2}{x} y = x^2 \cos x$$

We see that the integrating factor $\mu = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = \frac{1}{x^2}$. Hence the above ODE can now be written as exact differential by multiplying both side with μ

$$\begin{aligned} \frac{d}{dx}\left(\mu y\right) &= \mu\left(x^2 \cos x\right) \\ \frac{d}{dx}\left(\frac{1}{x^2} y\right) &= \frac{1}{x^2}\left(x^2 \cos x\right) \\ \frac{d}{dx}\left(\frac{1}{x^2} y\right) &= \cos x \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \frac{1}{x^2} y &= \sin x + c \\ y &= x^2 \left(\sin x + c\right); \qquad x \neq 0 \end{aligned}$$

4.2.2 Section 1.5 problem 22

Problem Find solution for $y' = 2xy + 3x^2 e^{x^2}; y(0) = 5$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned} y' &= f\left(x,y\right) \\ &= 2xy + 3x^2 e^{x^2} \end{aligned}$$

We see that $f\left(x,y\right)$ is continuous for all x and for all y. And

$$\frac{\partial f\left(x,y\right)}{\partial y} = 2x$$

This is continuous for all x. Therefore solution exist and unique in some interval. Now we will solve the ODE.

$$y' - 2xy = 3x^2 e^{x^2}$$

We see that the integrating factor $\mu = e^{\int -2x dx} = e^{-x^2}$. Hence the above ODE can now be written as exact differential by multiplying both side with μ

$$\frac{d}{dx}\left(\mu y\right)=\mu\left(3x^2e^{x^2}\right)$$
$$\frac{d}{dx}\left(e^{-x^2}y\right)=e^{-x^2}\left(3x^2e^{x^2}\right)$$
$$\frac{d}{dx}\left(e^{-x^2}y\right)=3x^2$$

Integrating both sides

$$e^{-x^2}y=x^3+c$$

Hence

$$y=e^{x^2}\left(x^3+c\right)$$

Now initial conditions $y(0)=5$ are applied to find c. This gives

$$5=c$$

Hence the complete solution (or the particular solution for this initial conditions) is

$$y=e^{x^2}\left(x^3+5\right)$$

4.2.3 Section 1.5 problem 25

Problem Find solution for $\left(x^2+1\right)y'+3x^3y=6xe^{\frac{-3}{2}x^2};y(0)=1$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned}y'&=f\left(x,y\right)\\&=\frac{6xe^{\frac{-3}{2}x^2}-3x^3y}{x^2+1}\end{aligned}$$

We see that $f\left(x,y\right)$ is continuous for all x except when $x^2=-1$ or $x=\pm i$. But this is not on real line hence it will not affect us. It is also continuous for all y.

$$\frac{\partial f\left(x,y\right)}{\partial y}=\frac{3x^3}{x^2+1}$$

Again, this is continuous for all x except when $x^2=-1$ or $x=\pm i$. But this is not on real line hence it will not affect us.

Therefore solution exists and unique for all x and y. Now we will solve the ODE.

$$y'+\frac{3x^3}{x^2+1}y=\frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}$$

Integration factor is $\mu=e^{\int\frac{3x^3}{x^2+1}dx}$. To evaluate the integral:

$$\begin{aligned}\int\frac{3x^3}{x^2+1}dx&=3\int\frac{x^3}{x^2+1}dx\\&=3\int x-\frac{x}{x^2+1}dx\\&=\frac{3}{2}x^2-3\int\frac{x}{x^2+1}dx\end{aligned}$$

Since $\frac{d}{dx}\ln\left(x^2+1\right)=\frac{2x}{x^2+1}$ then by comparing this to the second integral, we see that $\int\frac{x}{x^2+1}=\frac{1}{2}\ln\left(x^2+1\right)$, hence

$$\int\frac{3x^3}{x^2+1}dx=\frac{3}{2}x^2-\frac{3}{2}\ln\left(x^2+1\right)$$

Therefore

$$\begin{aligned}
\mu &= e^{\int \frac{3x^3}{x^2+1}dx} \\
&= \exp\left(\frac{3}{2}x^2 - \frac{3}{2}\ln\left(x^2+1\right)\right) \\
&= \exp\left(\frac{3}{2}x^2\right)\exp\left(-\frac{3}{2}\ln\left(x^2+1\right)\right) \\
&= \exp\left(\frac{3}{2}x^2\right)\exp\left(\ln\left(x^2+1\right)^{\frac{-3}{2}}\right) \\
&= \frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}
\end{aligned}$$

Multiplying both sides of the ODE with this integration factor gives

$$\begin{aligned}
\frac{d}{dx}\left(\mu y\right) &= \frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}\mu \\
\frac{d}{dx}\left(\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}y\right) &= \frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}} \\
\frac{d}{dx}\left(\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}y\right) &= \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}
\end{aligned}$$

Integrating both sides

$$\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}y = \int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}dx + c \tag{1}$$

To evaluate $\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}dx$, let $u = x^2+1$ hence $du = 2xdx$, therefore the integral becomes

$$\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}dx = \int \frac{6x}{u^{\frac{5}{2}}}\frac{du}{2x} = 3\int \frac{1}{u^{\frac{5}{2}}}du = 3\int u^{-\frac{5}{2}}du = 3\left(\frac{u^{-\frac{3}{2}}}{-\frac{3}{2}}\right) = -2u^{-\frac{3}{2}}$$

Hence

$$\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}dx = -2\left(x^2+1\right)^{-\frac{3}{2}} = \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}}$$

Hence (1) becomes

$$\begin{aligned}
\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}y &= \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}} + c \\
y &= \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}}\frac{\left(x^2+1\right)^{\frac{3}{2}}}{e^{\frac{3}{2}x^2}} + c\frac{\left(x^2+1\right)^{\frac{3}{2}}}{e^{\frac{3}{2}x^2}} \\
y &= -2e^{\frac{-3}{2}x^2} + c\left(x^2+1\right)^{\frac{3}{2}}e^{\frac{-3}{2}x^2}
\end{aligned}$$

Applying $y(0) = 1$ gives

$$\begin{aligned}
1 &= -2 + c \\
c &= 3
\end{aligned}$$

Hence the particular solution is

$$\begin{aligned}
y &= -2e^{\frac{-3}{2}x^2} + 3\left(x^2+1\right)^{\frac{3}{2}}e^{\frac{-3}{2}x^2} \\
&= e^{\frac{-3}{2}x^2}\left(3\left(x^2+1\right)^{\frac{3}{2}} - 2\right)
\end{aligned}$$

4.2.4 Section 1.5 problem 27

Problem Solve the differential equation by regarding y as the independent variable rather than x

$$\left(x\left(y\right)+ye^{y}\right)\frac{dy}{dx\left(y\right)}=1$$

Solution

$$\begin{aligned}\frac{dy}{dx\left(y\right)}&=\frac{1}{x\left(y\right)+ye^{y}}\\ \frac{dx\left(y\right)}{dy}&=x\left(y\right)+ye^{y}\\ \frac{dx\left(y\right)}{dx}-x\left(y\right)&=ye^{y}\end{aligned}$$

For $x\left(y\right)\neq ye^{y}$. Hence

$$\begin{aligned}\frac{dx\left(y\right)}{dy}&=x\left(y\right)+ye^{y}\\ \frac{dx\left(y\right)}{dx}-x\left(y\right)&=ye^{y}\end{aligned}$$

Integrating factor is $\mu=e^{-\int dy}=e^{-y}$. Multiplying both sides with μ gives

$$\begin{aligned}\frac{d}{dy}\left(\mu x\right)&=\mu ye^{y}\\ \frac{d}{dy}\left(e^{-y}x\right)&=y\end{aligned}$$

Integrating both sides

$$e^{-y}x\left(y\right)=\frac{y^{2}}{2}+c$$

Therefore

$$x\left(y\right)=\left(\frac{y^{2}}{2}+c\right)e^{y}$$

4.2.5 Section 1.5 problem 31

Problem (a) show that $y_{c}\left(x\right)=Ce^{-\int P(x)dx}$ is a general solution of $\frac{dy}{dx}+P\left(x\right)y=0$. (b) Show that $y_{p}\left(x\right)=e^{-\int P(x)dx}\int\left(Q\left(x\right)e^{\int P(x)dx}\right)dx$ is a particular solution of $\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$. (c) Suppose that $y_{c}\left(x\right)$ is any general solution of $\frac{dy}{dx}+P\left(x\right)y=0$ and that $y_{p}\left(x\right)$ is any particular solution of $\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$. Show that $y\left(x\right)=y_{c}\left(x\right)+y_{p}\left(x\right)$ is a general solution of $\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$

Solution

4.2.5.1 Part (a)

Given

$$\frac{dy}{dx}+P\left(x\right)y=0$$

Then

$$\frac{dy}{y}=-P\left(x\right)dx$$

Integrating both sides

$$\begin{aligned}\ln\left|y\right|&=-\int P\left(x\right)dx+C\\ y\left(x\right)&=Ce^{-\int P(x)dx}\end{aligned}$$

QED. We can also solve this by substituting $y(x) = Ce^{-\int P(x)dx}$ into $\frac{dy}{dx} + P(x)y = 0$ which gives

$$\Delta = \frac{d}{dx}\left(Ce^{-\int P(x)dx}\right) + P(x)Ce^{-\int P(x)dx} \tag{1}$$

But $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$, hence

$$\begin{aligned}\frac{d}{dx}\left(Ce^{-\int P(x)dx}\right) &= C\frac{d}{dx}\left(-\int P(x)\,dx\right)e^{-\int P(x)dx}\\ &= -CP(x)e^{-\int P(x)dx}\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\Delta &= -CP(x)e^{-\int P(x)dx} + P(x)Ce^{-\int P(x)dx}\\ &= 0\end{aligned}$$

Hence the solution $y(x) = Ce^{-\int P(x)dx}$ satisfies the ODE. Therefore it is solution.

4.2.5.2 Part(b)

Given $\frac{dy}{dx} + P(x)y = Q(x)$, the integrating factor is $\mu = e^{\int P(x)dx}$. Multiplying this by both sides of the ODE gives

$$\begin{aligned}e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y &= e^{\int P(x)dx}Q(x)\\ \frac{d}{dx}\left(e^{\int P(x)dx}y(x)\right) &= e^{\int P(x)dx}Q(x)\end{aligned}$$

Integrating both sides

$$\begin{aligned}e^{\int P(x)dx}y(x) &= \int e^{\int P(x)dx}Q(x)\,dx + C\\ y(x) &= e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right) + Ce^{-\int P(x)dx}\end{aligned}$$

For particular $C = 0$, we obtain

$$y_p(x) = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)$$

Which is what we asked to show.

4.2.5.3 Part(c)

Let

$$\begin{aligned}y(x) &= y_c(x) + y_p(x)\\ &= Ce^{-\int P(x)dx} + e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\end{aligned}$$

We need now to substitute this in $\frac{dy}{dx} + P(x)y = Q(x)$ and see if it satisfies it. First we find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[Ce^{-\int P(x)dx}\right] + \frac{d}{dx}\left[e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right]\\ &= Ce^{-\int P(x)dx}(-P(x)) + \frac{d}{dx}e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right) + e^{-\int P(x)dx}\frac{d}{dx}\int e^{\int P(x)dx}Q(x)\,dx\\ &= -CP(x)e^{-\int P(x)dx} + e^{-\int P(x)dx}(-P(x))\left(\int e^{\int P(x)dx}Q(x)\,dx\right) + e^{-\int P(x)dx}\left(e^{\int P(x)dx}Q(x)\right)\\ &= -CP(x)e^{-\int P(x)dx} - P(x)e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right) + e^{-\int P(x)dx}\left(e^{\int P(x)dx}Q(x)\right)\\ &= -CP(x)e^{-\int P(x)dx} - P(x)e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right) + Q(x)\\ &= -P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx + C\right] + Q(x)\end{aligned}$$

Substituting the above into the left hand side of the given $\frac{dy}{dx} + P(x)y = Q(x)$

$$
\begin{aligned}
LHS &= \frac{dy}{dx} + P(x)y \\
&= -P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx + C\right] + Q(x) + P(x)\left[Ce^{-\int P(x)dx} + e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right] \\
&= \overbrace{-P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx + C\right]} + Q(x) + \overbrace{P(x)e^{-\int P(x)dx}\left[C + \left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right]}
\end{aligned}
$$

We see that the first term in the RHS above and the third term cancel each others. Hence

$$LHS = Q(x)$$

Which is the right side of the ODE. Hence the solution $y(x) = y_c(x) + y_p(x)$ satisfies the ODE.

QED.

4.2.6 Section 1.5 problem 37

Problem A 400 gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at rate 5 gal/s and the well mixed brine in the tank flows out at rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Solution

To reduce confusion, let x be the substance which causes the concentration in the Brine. Let $Q(t)$ be the mass (normally called the amount, but saying mass is more clear than saying amount) of x at time t . Hence $Q(0) = 50$ lb. The goal is to find an ODE that describes how $Q(t)$ changes in time. That is, how the mass of x in the tank changes in time. Using

$$\frac{dQ}{dt} = R_{in} - R_{out}$$

Where R_{in} rate of mass of salt entering the tank per second. And R_{out} is rate of mass of salt leaving the tank per second. But

$$R_{in} = 5 \text{ lb/sec}$$

And

$$R_{out} = \frac{Q(t)}{V(t)}\frac{[\text{lb}]}{[\text{gal}]} \times 3\frac{[\text{gal}]}{[\text{second}]} = \frac{3}{V(t)}Q(t)$$

Where $V(t)$ is *current* volume of brine in tank at time t. Hence the ODE is

$$\frac{dQ}{dt} = 5 - \frac{3}{V(t)}Q(t)$$

$$\frac{dQ}{dt} + \frac{3}{V(t)}Q(t) = 5 \tag{1}$$

But we can find $V(t)$. Since initially $V(0) = 100$ gal, and in one second 5 gal enters, and 3 gal exists, then

$$V(t) = 100 + 2t$$

Hence (1) becomes

$$\boxed{\frac{dQ}{dt} + \frac{3}{100+2t}Q(t) = 5}$$

Integrating factor is

$$\mu = e^{\int \frac{3}{100+2t}dt} = e^{3\int \frac{1}{100+2t}dt} = e^{\frac{3}{2}\ln(100+2t)} = (100+2t)^{\frac{3}{2}}$$

Hence (1) becomes

$$\frac{d}{dt}(\mu Q) = 5\mu$$

Integrating both sides

$$\mu Q = 5\int \mu dt + c$$
$$(100+2t)^{\frac{3}{2}} Q = 5\int (100+2t)^{\frac{3}{2}} dt + c$$
$$(100+2t)^{\frac{3}{2}} Q = (100+2t)^{\frac{5}{2}} + c$$

Hence

$$Q(t) = (100+2t) + c(100+2t)^{\frac{-3}{2}}$$

But at $t=0$, $Q(0)=50$, hence

$$50 = 100 + c(100)^{\frac{-3}{2}}$$
$$c = -50\,000$$

Hence the solution is

$$Q(t) = (100+2t) - 50\,000(100+2t)^{\frac{-3}{2}} \tag{2}$$

This gives us the mass of salt at time t. What we need now to find out is the time it will take to fill the tank say t_{end}, and use that time to find $Q(t_{end})$ from above. Since initially the tank had 300 gallons remains to be filled, and the flow in is at rate of 5 gal/sec and flow out is at 3 gal/sec, then in one second, the tank will fill up with 2 gallons. Hence it will take

$$t = \frac{300}{2} = 150 \text{ sec}$$

To fill the tank. Using this value of t in (2) gives

$$\begin{aligned} Q(150) &= (100+2(150)) - 50\,000(100+2(150))^{\frac{-3}{2}} \\ &= \frac{1575}{4} \\ &= 393.75 \text{ lb} \end{aligned}$$

4.2.7 Section 1.5 problem 44

Problem: Figure 1.5.8 shows a slope field and typical solution curves for $y' = x+y$. (a) show that every curve approaches the straight line $y=-x-1$ as $x\to-\infty$. (b) for each of the five values $y_1 = -10,-5,0,5,10$, determined the initial value y_0 (accurate to 4 decimal points) such that $y(5)=y_1$ for the solution satisfying the initial condition $y(-5)=y_0$

Solution:

4.2.7.1 Part(a)

$$y' = x+y$$
$$y' - y = x$$

Integrating factor is $\mu = e^{-\int dx} = e^{-x}$. Multiplying the above with μ results in

$$\frac{d}{dx}(\mu y) = \mu x$$
$$\frac{d}{dx}(e^{-x}y) = e^{-x}x$$

Integrating both sides

$$e^{-x}y = \int xe^{-x}dx + c$$

But $\int xe^{-x}dx = e^{-x}(-1-x)$ using integration by parts. Hence the above becomes

$$e^{-x}y = e^{-x}(-1-x) + c$$
$$y = (-1-x) + ce^{x} \tag{1}$$

But

$$\lim_{x\to-\infty} e^{x} = 0$$

Hence solution becomes (at large negative x)

$$y = -1 - x$$

Therefore, solution curves approach line $-1 - x$.

4.2.7.2 Part(b)

The solution is $y = (-1 - x) + ce^x$ from part (a). Using $y(-5) = y_0$, then

$$\begin{aligned} y_0 &= (-1+5) + ce^{-5} \\ y_0 &= 4 + ce^{-5} \\ c &= \left(y_0 - 4\right) e^5 \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= (-1 - x) + \left(y_0 - 4\right) e^5 e^x \\ &= (-1 - x) + \left(y_0 - 4\right) e^{x+5} \end{aligned} \tag{2}$$

Now we need to find y_0 such as $\underline{y(5) = -10}$. From (2)

$$\begin{aligned} -10 &= (-1-5) + \left(y_0 - 4\right) e^{10} \\ y_0 &= (-10+6) e^{-10} + 4 \\ &= 3.99982 \end{aligned}$$

For $\underline{y(5) = -5}$, from (2)

$$\begin{aligned} -5 &= (-1-5) + \left(y_0 - 4\right) e^{10} \\ y_0 &= (-5+6) e^{-10} + 4 \\ &= 4.00005 \end{aligned}$$

For $\underline{y(5) = 0}$ from (2)

$$\begin{aligned} 0 &= (-1-5) + \left(y_0 - 4\right) e^{10} \\ y_0 &= 6e^{-10} + 4 \\ &= 4.00027 \end{aligned}$$

For $\underline{y(5) = 5}$ from (2)

$$\begin{aligned} 5 &= (-1-5) + \left(y_0 - 4\right) e^{10} \\ y_0 &= (5+6) e^{-10} + 4 \\ &= 4.00050 \end{aligned}$$

For $\underline{y(5) = 10}$ from (2)

$$\begin{aligned} 10 &= (-1-5) + \left(y_0 - 4\right) e^{10} \\ y_0 &= (10+6) e^{-10} + 4 \\ &= 4.00073 \end{aligned}$$

4.2.8 Section 2.1 problem 3

Problem: Solve $\frac{dx}{dt} = 1 - x^2; x(0) = 3$ and sketch solution

Solution:

$$\begin{aligned} \frac{dx}{dt} &= 1 - x^2 \\ \frac{dx}{1 - x^2} &= dt \end{aligned} \tag{1}$$

For $1 - x^2 \neq 0$ or for $x \neq \pm 1$. But

$$\int \frac{dx}{1 - x^2} = \int \frac{dx}{(1+x)(1-x)}$$

Where $\frac{1}{(1+x)(1-x)} = \frac{A}{(1+x)} + \frac{B}{(1-x)}$. But $A = \left(\frac{1}{(1-x)}\right)_{x=-1} = \frac{1}{2}$ and $B = \left(\frac{1}{(1+x)}\right)_{x=1} = \frac{1}{2}$, hence

$$\begin{aligned}\int \frac{dx}{(1+x)(1-x)} &= \frac{1}{2}\int \frac{dx}{(1+x)} + \frac{1}{2}\int \frac{dx}{(1-x)} \\ &= \frac{1}{2}\ln|(1+x)| - \frac{1}{2}\ln|(1-x)|\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\frac{1}{2}\ln|(1+x)| - \frac{1}{2}\ln|(1-x)| &= \int dt \\ \ln\left|\frac{(1+x)}{(1-x)}\right| &= \int 2dt \\ \ln\left|\frac{(1+x)}{(1-x)}\right| &= 2t + c \\ \frac{(1+x)}{(1-x)} &= ce^{2t} \\ (1+x) &= (1-x)\,ce^{2t} \\ 1+x &= ce^{2t} - xce^{2t} \\ x + xce^{2t} &= ce^{2t} - 1 \\ x &= \frac{ce^{2t}-1}{1+ce^{2t}}\end{aligned}$$

Now we use initial conditions $x(0) = 3$ to find c

$$\begin{aligned}3 &= \frac{c-1}{1+c} \\ c &= -2\end{aligned}$$

Hence solution is

$$\begin{aligned}x &= \frac{-2e^{2t}-1}{1-2e^{2t}} \\ &= \frac{1+2e^{2t}}{2e^{2t}-1}\end{aligned}$$

Here is a plot of the above solution and two other solutions starting from different initial conditions

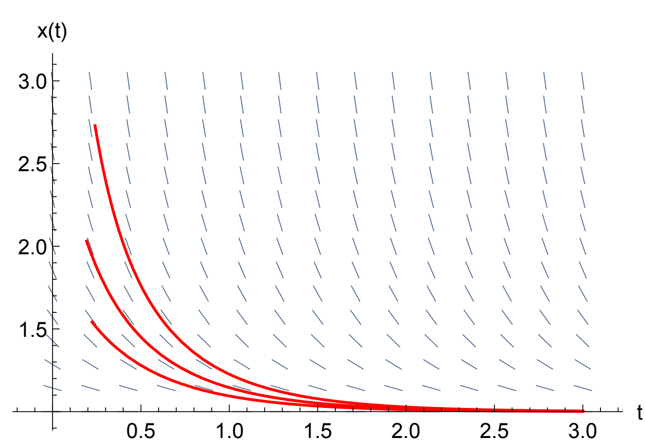


Figure 4.13: Problem 2.1, 3

4.2.9 Section 2.1 problem 13

Problem: Consider a breed of rabbits whose birth and death rates β, δ are each proportional to the rabbit population $P = P(t)$ with $\beta > \delta$. (a) Show that $P(t) = \frac{P(0)}{1-kP(0)t}$, where k constant. Note that $P(t) \to \infty$ as $t \to \frac{1}{(kP(0))}$. This is the doomsday. (b) Suppose that $P(0) = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

4.2.9.1 Part (a)

For doomsday, per book page 86, we use the model that birth rate occur at rate $\beta \propto P^2(t)$ per unit time per population, but in this problem, since death rate is not constant, but also proportional to the rabbit population, then we also make $\delta \propto P^2(t)$ where $\beta > \delta$. Hence we write

$$\frac{dP(t)}{dt} = kP^2(t)$$

Where k is the combined constant of proportionality. This is separable.

$$\begin{aligned}\frac{dP(t)}{P^2(t)} &= kdt \\ \int \frac{dP(t)}{P^2(t)} &= \int kdt \\ -\frac{1}{P} &= kt + c \\ P(t) &= \frac{1}{c - kt} \qquad (1)\end{aligned}$$

Using initial conditions, $t = 0, P(0)$ we find c

$$\begin{aligned}P(0) &= \frac{1}{c} \\ c &= \frac{1}{P(0)}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}P(t) &= \frac{1}{\frac{1}{P(0)} - kt} \\ &= \frac{P(0)}{1 - P(0)kt} \qquad (2)\end{aligned}$$

4.2.9.2 Part (b)

Applying initial conditions to (2) in part (a)

$$\begin{aligned}P(t) &= \frac{P(0)}{1 - kP(0)t} \\ 9 &= \frac{6}{1 - k(6)(10)} \\ k &= \frac{1}{180}\end{aligned}$$

Hence solution becomes

$$P(t) = \frac{6}{1 - \frac{6}{180}t}$$

When $t = \frac{180}{6} = 30$ months, then $P(t) \to \infty$. Hence 30 months is doomsday.

4.2.10 Section 2.1 problem 15

Problem Consider population $P(t)$ satisfying logistic equation $\frac{dP}{dt} = aP - bP^2$ where $B = aP$ is the time rate at which birth occur and $D = bP^2$ is the rate at which death occur. If the initial population is $P(0)$ and $B(0), D(0)$ are the rates per month at $t = 0$, show that the limiting population is $M = \frac{B(0)P(0)}{D(0)}$

Solution

For the limiting model, per book page 82 (limiting population and carrying capacity), we can use

$$\begin{aligned}\frac{dP}{dt} &= aP - bP^2 \\ &= a\left(1 - \frac{b}{a}P\right)P \\ &= a\left(1 - \frac{P}{M}\right)P\end{aligned}$$

note: In class lecture, the above is written as $\frac{dP}{dt} = r\left(1 - \frac{P}{k}\right)p$, where $r = a$ and $k = M$) But book uses different notations. M is the limiting capacity (or also called equilibrium population). Hence from the above, we see that

$$M = \frac{a}{b} \tag{1}$$

But a, which is the growth rate per time per population is

$$a = \frac{B_0}{P_0}$$

And $D(0) = bP^2(0)$, hence

$$b = \frac{D_0}{P_0^2}$$

Therefore (1) becomes

$$\begin{aligned}M &= \frac{\frac{B_0}{P_0}}{\frac{D_0}{P_0^2}} \\ &= \frac{B_0}{D_0}P_0\end{aligned}$$

QED.

4.2.11 Section 2.1 problem 17

Problem Consider rabbit population $P(t)$ satisfying the logistic equation as in problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 death per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

Solution The logistic equation, from problem 15 is

$$\frac{dP}{dt} = aP - bP^2$$

From problem 15: Where

$$B = aP$$

Is the time rate at which birth occur and

$$D = bP^2$$

Is the rate at which death occur and $P(t)$ is current size of population. Per problem 15, we know that the limiting population is

$$M = \frac{B(0)P(0)}{D(0)} = \frac{B(0)P(0)}{D(0)}$$

But we are given here, that $P(0) = 240, B(0) = 9$ per month and $D(0) = 12$ per month. This means

$$M = \frac{9(240)}{12} = 180$$

The above is the limiting population size. We now need to solve (1) in order to answer the question

$$\frac{dP}{dt} = aP - bP^2$$

This is separable

$$\begin{aligned}
\frac{dP}{aP - bP^2} &= dt \\
\int \frac{dP}{aP - bP^2} &= t + c \\
\int \frac{dP}{P(a - bP)} &= t + c \\
\int \frac{1}{aP} - \frac{b}{a(bP - a)} dP &= t + c \\
\frac{1}{a}\ln|P| - \frac{1}{a}\ln|bP - a| &= t + c \\
\frac{1}{a}\ln\left|\left(\frac{bP}{bP - a}\right)\right| &= t + c \\
\frac{1}{a}\ln\left|\left(\frac{bP}{bP - a}\right)\right| &= t + c \\
\ln\left|\left(\frac{bP}{bP - a}\right)\right| &= at + ac \\
\frac{bP}{bP - a} &= c_1 e^{at}
\end{aligned}$$

Where the sign is determined by constant c_1. Hence the above becomes

$$\begin{aligned}
bP &= c_1 e^{at}(bP - a) \\
&= c_1 e^{at} bP - c_1 a e^{at} \\
bP - c_1 e^{at} bP &= -c_1 a e^{at} \\
P\left(b - c_1 e^{at} b\right) &= -c_1 a e^{at} \\
P(t) &= \frac{-c_1 a e^{at}}{b - c_1 e^{at} b} \\
&= \frac{c_1 a e^{at}}{c_1 e^{at} b - b} \\
P(t) &= \frac{a}{b - \frac{b}{c_1} e^{-at}}
\end{aligned}$$

We now need to find c_1 from initial conditions. At $t = 0$, $P(0) = 240$, hence since $B = aP$ then

$$\begin{aligned}
a(0) &= \frac{B(0)}{P(0)} = \frac{9}{240} \\
&= \frac{3}{80}
\end{aligned}$$

And since $D = bP^2$ then

$$\begin{aligned}
b(0) &= \frac{D(0)}{p(0)^2} = \frac{12}{240^2} \\
&= \frac{1}{4800}
\end{aligned}$$

Therefore, at $t=0$, the above solution becomes

$$
\begin{aligned}
P(0) &= \frac{c_1 a(0) e^{at}}{c_1 e^{at} b(0) - b(0)} \\
240 &= \frac{c_1 a(0)}{c_1 b(0) - b(0)} = \frac{c_1 \frac{3}{80}}{\frac{1}{4800}(c_1 - 1)} \\
240\left(\frac{1}{4800}(c_1-1)\right) &= c_1 \frac{3}{80} \\
\frac{1}{20}c_1 - \frac{1}{20} &= c_1 \frac{3}{80} \\
\frac{1}{20}c_1 - c_1\frac{3}{80} &= \frac{1}{20} \\
c_1\left(\frac{1}{20} - \frac{3}{80}\right) &= \frac{1}{20} \\
c_1\left(\frac{1}{80}\right) &= \frac{1}{20} \\
c_1 &= 4
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
P(t) &= \frac{4ae^{at}}{4e^{at}b - b} \\
&= \frac{4\left(\frac{3}{80}\right)e^{\frac{3}{80}t}}{4e^{\left(\frac{3}{80}\right)t}\left(\frac{1}{4800}\right) - \frac{1}{4800}}
\end{aligned}
$$

We now solve for t when $P(t) = 105\%$ of M

$$
\begin{aligned}
\frac{105}{100}(180) &= \frac{4\left(\frac{3}{80}\right)e^{\frac{3}{80}t}(4800)}{4e^{\left(\frac{3}{80}\right)t} - 1} \\
189\left(4e^{\left(\frac{3}{80}\right)t} - 1\right) &= 720e^{\frac{3}{80}t} \\
756e^{\left(\frac{3}{80}\right)t} - 189 &= 720e^{\frac{3}{80}t} \\
756e^{\left(\frac{3}{80}\right)t} - 720e^{\frac{3}{80}t} &= 189 \\
e^{\frac{3}{80}t} &= \frac{189}{36} \\
\frac{3}{80}t &= \ln\frac{189}{36} \\
t &= \frac{80}{3}\ln\frac{189}{36} \\
&= 44.219 \text{ months}
\end{aligned}
$$

4.2.12 Section 2.1 problem 30

Problem A tumor may be regarded as population of multiplying cells. The birth rate of cells in a tumor decreases exponentially with time so that $\beta(t) = \beta_0 e^{-\alpha t}$ where α, β_0 are positive constants. Hence $\frac{dP}{dt} = \beta_0 e^{-\alpha t} P$ with $P(0) = P_0$. Solve the initial value problem for $P(t) = P_0 e^{\left(\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)\right)}$. Observe that $P(t)$ approaches finite limiting population $P_0 e^{\left(\frac{\beta_0}{\alpha}\right)}$ as $t \to \infty$.

Solution

$$
\frac{dP}{dt} = \beta_0 e^{-\alpha t} P
$$

This is separable.

$$
\frac{dP}{P} = \beta_0 e^{-\alpha t} dt
$$

Integrating

$$\begin{aligned}\ln|P| &= \beta_0 \int e^{-\alpha t} dt \\ &= \beta_0 \frac{e^{-\alpha t}}{-\alpha} + C\end{aligned}$$

Hence

$$P(t) = Ce^{-\beta_0 \frac{e^{-\alpha t}}{\alpha}} \tag{1}$$

Applying initial condition on the above gives

$$\begin{aligned}P(0) = P_0 &= Ce^{-\beta_0 \frac{1}{\alpha}} \\ C &= P_0 e^{\beta_0 \frac{1}{\alpha}}\end{aligned}$$

Therefore the solution (1) becomes

$$\begin{aligned}P(t) &= P_0 e^{\beta_0 \frac{1}{\alpha}} e^{-\beta_0 \frac{e^{-\alpha t}}{\alpha}} \\ &= P_0 e^{-\beta_0 \frac{e^{-\alpha t}}{\alpha} + \frac{\beta_0}{\alpha}} \\ &= P_0 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)}\end{aligned}$$

As $t \to \infty$ then $e^{-\alpha t} \to 0$ since $\alpha > 0$, hence the above becomes

$$P(\infty) = M = P_0 e^{\frac{\beta_0}{\alpha}}$$

The above is the limiting population.

4.2.13 Section 2.1 problem 31

Problem For tumor in problem 30, suppose that at $t = 0$, there are $P_0 = 10^6$ cells and that $P(t)$ is then increasing at rate 3×10^5 cells per month. After 6 months the tumor has doubled (in size and number of cells). Solve numerially for α and then find the limiting population of tumor.

Solution From problem (30) we found

$$\begin{aligned}P(t) &= P_0 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)} \\ &= 10^6 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)}\end{aligned}$$

Then, at $t = 0$, we are told $\left(\frac{dP(t)}{dt}\right)_{t=0} = 3\times10^5$ (cells per month). Hence, since $\frac{dP}{dt} = \beta_0 e^{-\alpha t} P$ then at $t = 0$

$$\begin{aligned}3\times10^5 &= \beta_0 P_0 \\ &= \beta_0 10^6\end{aligned}$$

Therefore

$$\beta_0 = \frac{3\times10^5}{10^6} = 0.3$$

We also told that after 6 months, the number of cells has doubled. This means, using $t = 6$ (with units of month) that

$$\begin{aligned}P(6) &= 2P_0 \\ 10^6 e^{\frac{\beta_0}{\alpha}\left(1-e^{-6\alpha}\right)} &= 2\times10^6\end{aligned}$$

But $\beta_0 = 0.3$, hence the above becomes

$$\begin{aligned}e^{\frac{3}{10\alpha}\left(1-e^{-6\alpha}\right)} &= 2 \\ \frac{3}{10\alpha}\left(1-e^{-6\alpha}\right) &= \ln 2 \\ 10\alpha \ln 2 &= 3 - 3e^{-6\alpha} \\ 10\alpha \ln 2 + 3e^{-6\alpha} &= 3\end{aligned}$$

Using a computer, the solutions are $\alpha = 0$ or $\underline{\alpha = 0.3915}$

Now the limiting population is found. From $P(t) = P_0 e^{\frac{\beta_0}{\alpha}(1-e^{-\alpha t})}$, for large t and since $\alpha > 0$ this becomes

$$\begin{aligned}\lim_{t\to\infty} P(t) &= P_0 e^{\frac{\beta_0}{\alpha}}\\ &= 10^6 e^{\frac{0.3}{0.3915}}\\ &= 2.1518 \times 10^6\end{aligned}$$

The above is limit of number of cells for large t.

4.2.14 Section 2.2 problem 7

Problem Solve for $f(x) = 0$ to find critical points. Then analyze the sign of $f(x)$ to determine if each critical point is stable or not and construct the phase diagram for the differential equation. Next solve the ODE. Finally plot the slope field and verify visually the stability of each critital point.

$$\frac{dx}{dt} = f(x) = (x-2)^2$$

Solution The critial points are x values (dependent variable values) where $f(x) = 0$. Hence

$$\begin{aligned}(x-2)^2 &= 0\\ x &= 2\end{aligned}$$

Since $f(x)$ is always positive, this means if x started at something just below $x = 2$, say $x = 1.5$, then eventually x will reach $x = 2$ and stay there. But if x is started at something just about $x = 2$, say $x = 2.5$, then x will keep increasing away from $x = 2$. This means $x = 2$ is semi stable critial since if we start below it, we reach it, but not if we start about it. Hence the phase diagram is

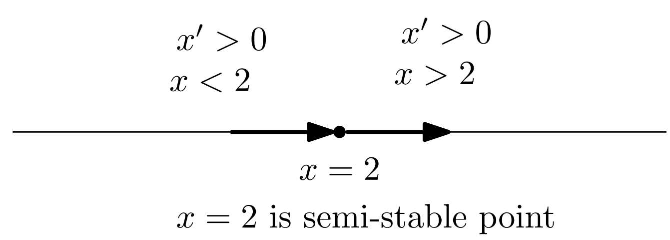


Figure 4.14: Phase diagram, 2.2 problem 7

Now the ODE is solved $\frac{dx}{dt} = (x-2)^2$. This is non-linear seperable

$$\begin{aligned}\frac{dx}{(x-2)^2} &= dt \qquad x \neq 2\\ \int \frac{dx}{(x-2)^2} &= \int dt\end{aligned}$$

Let $x - 2 = u \to \frac{du}{dx} = 1$, therefore $\int \frac{dx}{(x-2)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x-2}$ and the above becomes

$$\begin{aligned}-\frac{1}{x-2} &= t + c\\ x &= 2 - \frac{1}{t+c}\end{aligned}$$

Let $x(0) = x_0$, therefore

$$\begin{aligned}x_0 &= 2 - \frac{1}{c}\\ c &= \frac{1}{2-x_0}\end{aligned}$$

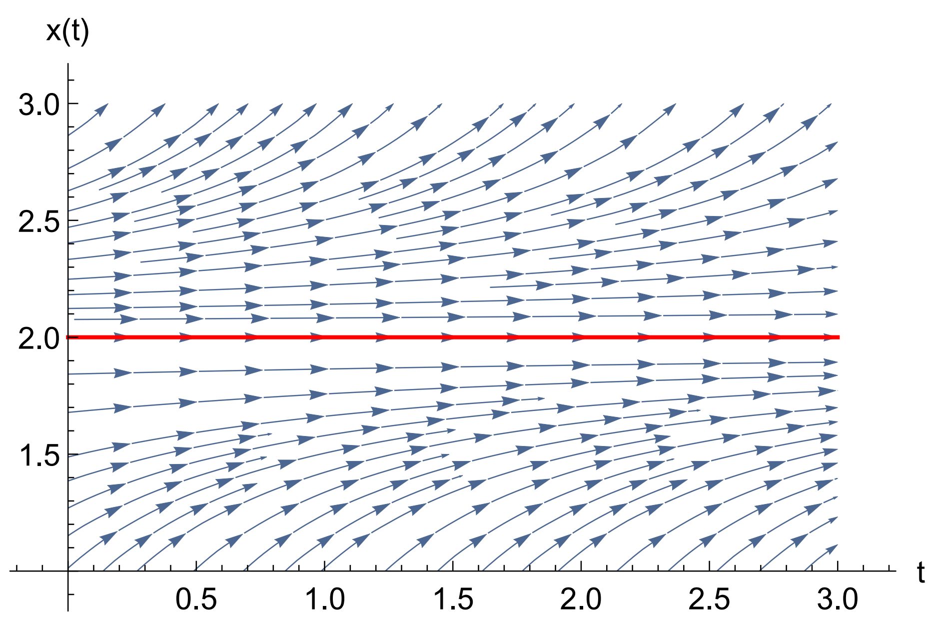
And the solution becomes

$$\begin{aligned} x &= 2 - \frac{1}{t + \frac{1}{2-x_0}} \\ &= 2 - \frac{2 - x_0}{t(2 - x_0) + 1} \\ &= \frac{2(t(2 - x_0) + 1) - 2 + x_0}{t(2 - x_0) + 1} \\ &= \frac{2t(2 - x_0) + x_0}{t(2 - x_0) + 1} \\ &= \frac{4t - 2tx_0 + x_0}{2t - x_0 t + 1} \end{aligned}$$

Hence

$$\boxed{x(t) = \frac{(2t-1)x_0 - 4t}{tx_0 - 2t - 1}}$$

Here is slope field plot



From the above plot, we see the solution lines are moving away from $x = 2$ when they start from $x > 2$ but move towards $x = 2$ when starting from $x < 2$.

4.2.15 Section 2.2 problem 10

Problem Solve for $f(x) = 0$ to find critical points. Then analyze the sign of $f(x)$ to determine if each critical point is stable or not and construct the phase diagram for the differential equation. Next solve the ODE. Finally plot the slope field and verify visually the stability of each critital point.

$$\frac{dx}{dt} = f(x) = 7x - x^2 - 10$$

Solution

The critial points are x values (dependent variable values) where $f(x) = 0$. Hence

$$\begin{aligned} 7x - x^2 - 10 &= 0 \\ x_1 &= 2 \\ x_2 &= 5 \end{aligned}$$

The phase diagram is

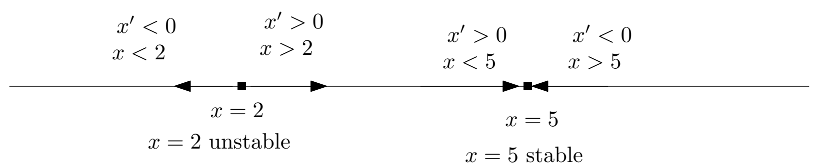


Figure 4.15: Phase diagram, 2.2 problem 7

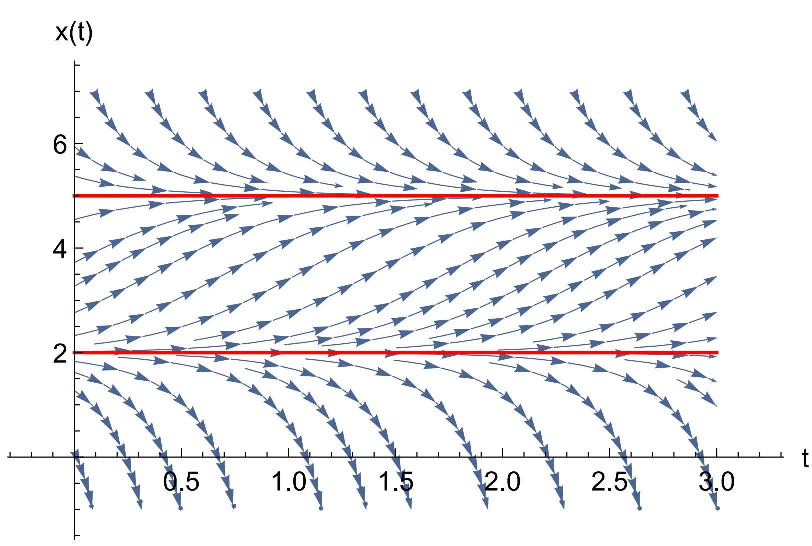
Now the ODE is solved $\frac{dx}{dt} = 7x - x^2 - 10$. This is non-linear seperable

$$\begin{aligned} \frac{dx}{7x - x^2 - 10} &= dt \qquad x \neq 2, x \neq 5 \\ \frac{-dx}{x^2 - 7x + 10} &= dt \\ -\int \frac{dx}{x^2 - 7x + 10} &= \int dt \end{aligned}$$

But $\frac{1}{(x-2)(x-5)} = \frac{A}{(x-2)} + \frac{B}{(x-5)}$, hence $A = \left(\frac{1}{(x-5)}\right)_{x=2} = \left(\frac{1}{-3}\right)$ and $B = \left(\frac{1}{(x-2)}\right)_{x=5} = \frac{1}{3}$ and the above becomes

$$\begin{aligned} -\int \left(\frac{1}{-3(x-2)} + \frac{1}{3(x-5)}\right) &= \int dt \\ \int \frac{1}{3(x-2)} - \int \frac{1}{3(x-5)} &= \int dt \\ \frac{1}{3}\int \frac{dx}{(x-2)} - \frac{1}{3}\int \frac{dx}{(x-5)} &= \int dt \\ \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x-5| &= \int dt \\ \ln|x-2| - \ln|x-5| &= \int 3dt \\ \ln\left|\frac{x-2}{x-5}\right| &= 3t + c \\ \frac{x-2}{x-5} &= ce^{3t} \\ x - 2 &= xce^{3t} - 5ce^{3t} \\ x - xce^{3t} &= 2 - 5ce^{3t} \\ x &= \frac{2 - 5ce^{3t}}{1 - ce^{3t}} \end{aligned}$$

Here is slope field plot



From the above plot, we see the solution lines are moving away from $x = 2$ indicating it is unstable and move towards $x = 5$ indicating it is stable.

4.2.16 Section 2.2 problem 23

Problem Suppose that logistic equation $\frac{dx}{dt} = kx(M-x)$ models a population $x(t)$ of fish in lake that after t months during which no fishing occurs. Now suppose that because of fishing, fish are removed from lake at rate of hx fish per month, with $h > 0$. Thus fish are harvested at a rate propertional to existing fish population, rather than at constant rate of example 4. (a) if $0 < h < kM$, show that population is still logistic. What is the new limiting population. (b) if $h \geq kM$., show that $x(t) \to 0$ at $t \to \infty$ so that lake is eventually fished out.

Solution

Part (a)

Since fish is removed at rate of hx fish per month, then

$$\begin{aligned}
\frac{dx}{dt} &= kx(M-x) - hx \\
&= kx\left((M-x) - \frac{h}{k}\right) \\
&= kx\left(M - \frac{h}{k} - x\right) \\
&= kx\left(\left(M - \frac{h}{k}\right) - x\right)
\end{aligned}$$

But $M - \frac{h}{k} > 0$ since $0 < h < kM$, therefore, if we let $\left(M - \frac{h}{k}\right) = \lambda$, then $\frac{dx}{dt} = kx(\lambda - x)$ is still logistic just as $\frac{dx}{dt} = kx(M-x)$ since $\lambda > 0$. $\lambda = M - \frac{h}{k}$ is the new limiting population.

Part (b)

In this case

$$\begin{aligned}
\frac{dx}{dt} &= kx\left(\left(M - \frac{h}{k}\right) - x\right) \\
&= kx(\lambda - x)
\end{aligned}$$

Now $\lambda < 0$. Solving this ode

$$\begin{aligned}
\frac{dx}{x(\lambda - x)} &= k \\
\frac{1}{\lambda x} - \frac{1}{\lambda}\frac{1}{(\lambda - x)} &= k
\end{aligned}$$

Integrating

$$\begin{aligned}
\frac{1}{\lambda}\ln|x| - \frac{1}{\lambda}\ln|(\lambda - x)| &= \int kdt \\
\ln\left|\frac{x}{\lambda - x}\right| &= \int \lambda kdt \\
\ln\left|\frac{x}{\lambda - x}\right| &= \lambda kt + c \\
\frac{x}{\lambda - x} &= Ce^{\lambda kt} \\
x + xCe^{\lambda kt} &= \lambda Ce^{\lambda kt} \\
x(t) &= \frac{\lambda Ce^{\lambda kt}}{1 + Ce^{\lambda kt}}
\end{aligned}$$

Now, since $\lambda < 0$, then as $t \to \infty$ then $x(t) \to \frac{0}{1} = 0$. Hence the population of fish will die out. (no need to find C first, as the whole term goes to zero). This is what we are asked to show.

4.3 HW3

4.3.1 Section 2.4 problem 8 (page 122)

Problem Apply Euler method twice to approximate solution on interval $\left[0,\frac{1}{2}\right]$ first with step size $h=0.25$ then with step size $h=0.1$. Compare to three decimal places values of the two approximation at $x=\frac{1}{2}$ with the value $y\left(\frac{1}{2}\right)$ of the exact solution. $y'=e^{-y};y(0)=0$. Exact solution is $y(x)=\ln(x+1)$

Solution

Using forward Euler method, we write

$$y_{n+1}=y_n+hf\left(x_n,y_n\right)$$

Here $f\left(x,y\right)=e^{-y}$.

$h=0.25$

$$\begin{aligned}
y(0)&=0\\
y(h)&=y(0.25)=y(0)+he^{-y(0)}=0+0.25e^{0}=0.25\\
y(2h)&=y(0.5)=y(0.25)+he^{-y(0.25)}=0.25+0.25e^{-0.25}=0.445
\end{aligned}$$

$h=0.1$

$$\begin{aligned}
y(0)&=0\\
y(h)&=y(0.1)=y(0)+he^{-y(0)}=0+0.1e^{-0}=0.1\\
y(2h)&=y(0.2)=y(0.1)+he^{-y(0.1)}=0.1+0.1e^{-0.1}=0.190\\
y(3h)&=y(0.3)=y(0.2)+he^{-y(0.2)}=0.190+0.1e^{-0.190}=0.273\\
y(4h)&=y(0.4)=y(0.3)+he^{-y(0.3)}=0.273+0.1e^{-0.273}=0.349\\
y(5h)&=y(0.5)=y(0.4)+he^{-y(0.4)}=0.349+0.1e^{-0.349}=0.420
\end{aligned}$$

Exact solution is $y(0.5)=\ln(0.5+1)=0.405$

h size	$y\left(\frac{1}{2}\right)$
0.25	0.445
0.1	0.420
exact	0.405

4.3.2 Section 2.4 problem 13 (page 122)

Problem Find the exact solution, then apply Euler method twice to approximate to 4 decimal places values the solution on the given interval. First with step $h=0.01$ then with step $h=0.005$. Make table showing the approximate values and the actual values, together with percentage error in the more accurate approximation for x an integral multiple of 0.2. $yy'=2x^3;y(1)=3;1\leq x\leq 2$.

Solution

$$y'=\frac{2x^3}{y}=f\left(x,y\right)$$

Looking at $f(x)$ we see that solution is not defined at $y=0$. Otherwise, $f\left(x,y\right)$ is continuous everywhere. Hence solution exist for $y\neq 0$. Also $\frac{\partial f}{\partial y}=-\frac{2x^3}{y^2}$, hence we see solution is unique, on some interval that does not include $y=0$. Now we will solve the ODE

$$\begin{aligned}
y\frac{dy}{dx}&=2x^3\\
ydy&=2x^3dx
\end{aligned}$$

Integrating

$$\frac{1}{2}y^2 = \frac{1}{2}x^4 + c$$

Applying initial conditions

$$\frac{1}{2}(9) = \frac{1}{2} + c$$
$$\frac{9}{2} - \frac{1}{2} = c$$
$$c = 4$$

Hence exact solution is

$$\frac{1}{2}y^2 = \frac{1}{2}x^4 + 4$$
$$y^2 = x^4 + 8$$
$$y = \pm\sqrt{x^4 + 8}$$

Since $1 \leq x \leq 3$ and $y(1) = 3$, then y can not become negative (else it will have to cross $y = 0$). Therefore solution is just the positive branch

$$y_{exact} = \sqrt{x^4 + 8}$$

Using Euler, we write

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)$$

But $f\left(x_n, y_n\right) = \frac{2x_n^3}{y_n}$ and $x_n = 1 + nh$ where h is the step size. The above becomes

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)$$

Using initial conditions, where $n = 0$, the given values $y_0 = 3$ at $x_0 = 1$. A small function was written to implement Euler method and print table. Source code is given below. Here is the final table generated

x	1.	1.2	1.4	1.6	1.8	2.
h=0.01	3	3.1718	3.4368	3.8084	4.2924	4.889
h=0.005	3	3.1729	3.439	3.8117	4.2967	4.894
exact	3.	3.1739	3.4412	3.8149	4.3009	4.899
% error	0.	0.032303	0.062773	0.085478	0.098183	0.10218

Source code listing:

```
In[69]:= SetOptions[$FrontEndSession, PrintPrecision → 5]
(*HW 3, Math 320. By Nasser M. Abbasi. Problem 2.4 13*)
f[x_, y_] := 2 x^3 / y;
makeTable[h_, from_, to_, y0_] := Module[{nSteps = (to - from) / h, data, y, x, skip},
  Array[y, Rationalize@nSteps, 0];
  Array[x, Rationalize@nSteps, 0];
  y[0] = y0; x[0] = from;

  Do[(*Euler loop*)
   y[n + 1] = y[n] + h f[x[n], y[n]];
   x[n + 1] = x[n] + h,
   {n, 0, nSteps}
  ];

  skip = Round[0.2 / h];
  Table[{x[n], y[n]}, {n, 0, nSteps, skip}]
 ]

In[76]:= data1 = makeTable[0.01, 1, 2, 3];
data2 = makeTable[0.005, 1, 2, 3];
exact = Sqrt[#^4 + 8] & /@ data2[[All, 1]];
p = Grid[{
    {"x", Sequence @@ data1[[All, 1]]},
    {"h=0.01", Sequence @@ data1[[All, 2]]},
    {"h=0.005", Sequence @@ data2[[All, 2]]},
    {"exact", Sequence @@ exact},
    {"% error", Sequence @@ (Abs[data2[[All, 2]] - exact] / Abs[exact] * 100)}
  }, Frame → All]
```

4.3.3 Section 2.4 problem 25

Problem Apply Euler method for $\frac{dv}{dt} = 32 - 1.6v$ with $v(0) = 0$. For $0 \le t \le 2$, using step size $h = 0.01, h = 0.005$, round v to one decimal point. What percentage of limiting velocity 20 ft/sec has been attained after 1 second? After 2 seconds?

Solution

The exact solution is

$$\frac{dv}{dt} + 1.6v = 32$$

$\mu = e^{\int 1.6dt} = e^{1.6t}$, hence

$$\frac{d}{dt}\left(e^{1.6t}v\right) = 32e^{1.6t}$$

Integrating

$$\begin{aligned} e^{1.6t}v &= 32\int e^{1.6t}dt \\ &= \frac{32}{1.6}e^{1.6t} + c \end{aligned}$$

Hence

$$v(t) = \frac{32}{1.6} + ce^{-1.6t}$$

Applying initial conditions

$$\begin{aligned} 0 &= \frac{32}{1.6} + c \\ c &= -\frac{32}{1.6} \end{aligned}$$

Therefore, exact solution is

$$\begin{aligned} v(t) &= \frac{32}{1.6} - \frac{32}{1.6}e^{-1.6t} \\ &= 20\left(1 - e^{-1.6t}\right) \end{aligned}$$

The Euler method is

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)$$

Where here

$$f\left(x_n, y_n\right) = 32 - 1.6y_n$$

Small function was written to find $v(t)$ at $t = 1, 2$ seconds using Euler, with the different step sizes. It prints the value of v when iteration reaches 1 and 2 seconds. Here is the screen output

```
data1 = makeTable[0.01, 0, 2, 0];
At one second, using h=0.01 speed is 16.078 at step n = 100
At 2 seconds, using h=0.01 speed is 19.206 at step n = 200
data2 = makeTable[0.005, 0, 2, 0];
At one second, using h=0.005 speed is 16.02 at step n = 200
At 2 seconds, using h=0.005 speed is 19.195 at step n = 400
```

Therefore (where percentage below, is percentage of limiting speed of 20 ft/sec)

h	speed at 1 second	speed at 2 seconds
0.01	16.078	19.206
0.005	16.02 (80.1%)	19.195 (95.98%)

The source code written for this problem is given below

```
In[80]:= SetOptions[$FrontEndSession, PrintPrecision → 5]
(*HW 3, Math 320. By Nasser M. Abbasi. Problem 2.4 25*)
f[t_, y_] := 32 - 1.6 y;
makeTable[h_, from_, to_, y0_] := Module[{nSteps = Rationalize[(to - from) / h], data, t, skip, y},
  Array[y, nSteps, 0];
  Array[t, nSteps, 0];
  y[0] = y0; t[0] = from;

  Do[(*Euler loop*)
   y[n + 1] = y[n] + h f[t[n], y[n]];
   If[t[n] == 1, Print["At one second, using h=", h, " speed is ", y[n + 1], " at step n = ", n]];
   t[n + 1] = t[n] + h,
   {n, 0, nSteps}
  ];
  Print["At 2 seconds, using h=", h, " speed is ", y[nSteps], " at step n = ", nSteps];
  (*skip=Round[0.2/h];*)
  skip = 1;
  Table[{t[n], y[n]}, {n, 0, nSteps, skip}]
 ]

In[83]:= data1 = makeTable[0.01, 0, 2, 0];
At one second, using h=0.01 speed is 16.078 at step n = 100
At 2 seconds, using h=0.01 speed is 19.206 at step n = 200

In[84]:= data2 = makeTable[0.005, 0, 2, 0];
At one second, using h=0.005 speed is 16.02 at step n = 200
At 2 seconds, using h=0.005 speed is 19.195 at step n = 400
```

4.3.4 Section 2.4 problem 30

Problem Apply Euler method with successively smaller step sizes on the interval $[0, 2]$ to verify empirically that the solution of $y' = x^2 + y^2; y(0) = 0$ has vertical asymptote near $x = 2.003147$. Contrast this with example 2, in which $y(0) = 1$.

Solution

Small function was written to implement Forward Euler for this problem.

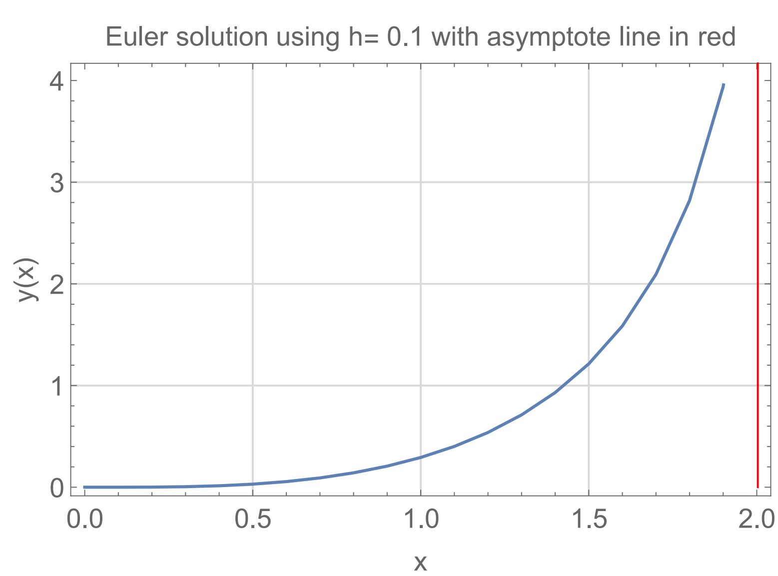
```
In[85]:= SetOptions[$FrontEndSession, PrintPrecision → 5]
(*HW 3, Math 320. By Nasser M. Abbasi. Problem 2.4 30*)
f[x_, y_] := x^2 + y^2;
makeTable[h_, from_, to_, y0_] := Module[{nSteps = Rationalize[(to - from) / h], data, x, y, skip},
  Array[y, nSteps, 0];
  Array[x, nSteps, 0];
  (*Print["number of steps is ",nSteps];*)
  y[0] = y0; x[0] = from;

  Do[(*Euler loop*)
   y[n + 1] = y[n] + h f[x[n], y[n]];
   x[n + 1] = x[n] + h,
   {n, 0, nSteps}
  ];
  skip = 1;
  Table[{x[n], y[n]}, {n, 0, nSteps, skip}]
 ]
```

The above function was called for $h = 0.1, 0.01, 0.001$ which showed that better and better approximation, the numerical solution approached asymptote near $x = 2.003147$. For $h = 0.1$, here is the output

```
In[88]:= h = 0.1;
data1 = makeTable[h, 0, 2, 0];
p1 = ListLinePlot[data1,
  Frame → True,
  FrameLabel →
   {{"y(x)", None},
    {"x", Row[{"Euler solution using h= ", h, " with asymptote line in red"}]}},
  BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray,
  Epilog → {Red, Line[{{2.003147, 0}, {2.003147, 7}}]}, ImageSize → 400]
```

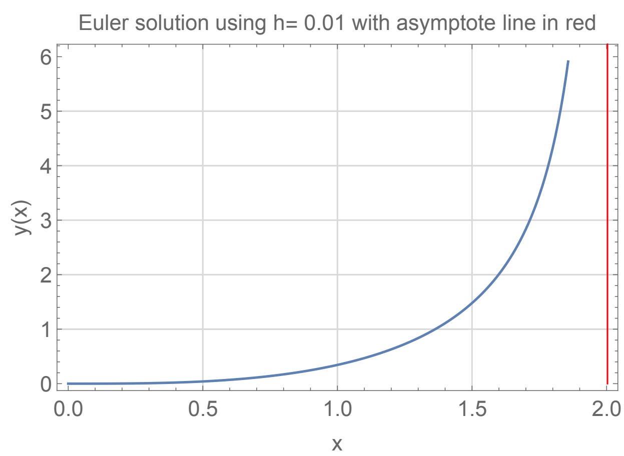
Out[90]=



For $h = 0.01$, here is the output

```
In[48]:= h = 0.01;
data1 = makeTable[h, 0, 2, 0];
p1 = ListLinePlot[data1,
  Frame → True,
  FrameLabel →
   {{"y(x)", None},
    {"x", Row[{"Euler solution using h= ", h, " with asymptote line in red"}]}},
  BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray,
  Epilog → {Red, Line[{{2.003147, 0}, {2.003147, 7}}]}, ImageSize → 400]
```

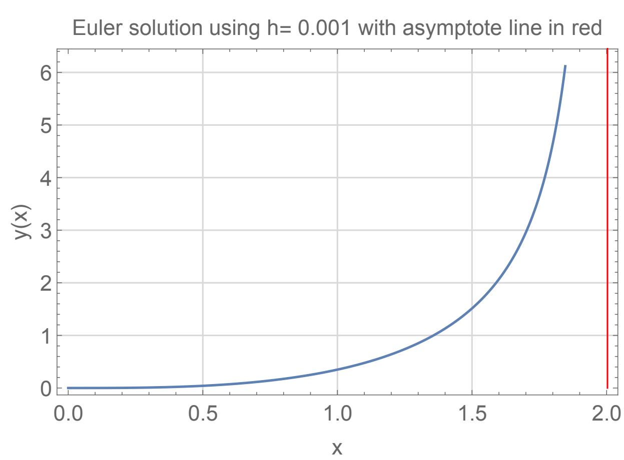
Out[50]=



For $h = 0.001$, here is the output

```
In[51]:= h = 0.001;
data1 = makeTable[h, 0, 2, 0];
p1 = ListLinePlot[data1,
  Frame → True,
  FrameLabel →
   {{"y(x)", None},
    {"x", Row[{"Euler solution using h= ", h, " with asymptote line in red"}]}},
  BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray,
  Epilog → {Red, Line[{{2.003147, 0}, {2.003147, 7}}]}, ImageSize → 400]
```

Out[53]=



4.3.5 Section 2.5 problem 8

Apply improved Euler method to approximate solution on interval $\left[0, \frac{1}{2}\right]$ with step size $h = 0.1$ Construct table showing 4 decimal places values of approximation solution and exact solution at points $0.1, 0.2, 0.3, 0.4, 0.5$.

$$y' = e^{-y}; y(0) = 0$$

Exact solution is $y(x) = \ln(x+1)$

Solution

Improved Euler method uses

$$k_1 = f\left(x_n, y_n\right)$$
$$u_{n+1} = y_n + hk_1$$
$$k_2 = f\left(x_{n+1}, u_{n+1}\right)$$
$$y_{n+1} = y_n + h\frac{k_1 + k_2}{2}$$

A small function was written to implement the above improved Euler method. The following is source code

```
(*HW 3, Math 320. By Nasser M. Abbasi. Problem 2.5 8, improved Euler*)
f[x_, y_] := Exp[-y];
makeTableImproved[h_, from_, to_, y0_] :=
 Module[{nSteps = Rationalize[(to - from) / h], data, x, y, skip, k1, k2, predictor},
  Array[y, nSteps, 0];
  Array[x, nSteps, 0];
  y[0] = y0; x[0] = from;

  Do[(*Euler loop*)
   k1 = f[x[n], y[n]];
   predictor = y[n] + h k1;
   x[n + 1] = x[n] + h;
   k2 = f[x[n + 1], predictor];
   y[n + 1] = y[n] + h (1 / 2 * (k1 + k2)),
   {n, 0, nSteps}
  ];
  skip = Round[0.1 / h];
  Table[{x[n], y[n]}, {n, 0, nSteps, skip}]
 ]
```

This function was called to generate the table and format it. Here is the result

```
In[270]:= h = 0.1;
data1 = makeTable[h, 0, .5, 0];
exact = Log[# + 1] & /@ data1[[All, 1]];
p = Grid[{
   {"x", Sequence @@ data1[[All, 1]]},
   {"h=0.01", Sequence @@ data1[[All, 2]]},
   {"exact", Sequence @@ exact},
   {"% error",
    Sequence @@ ((exact - data1[[All, 2]]) / (If[exact == 0, 1, exact, 1]) * 100)}
  }, Frame -> All]
```

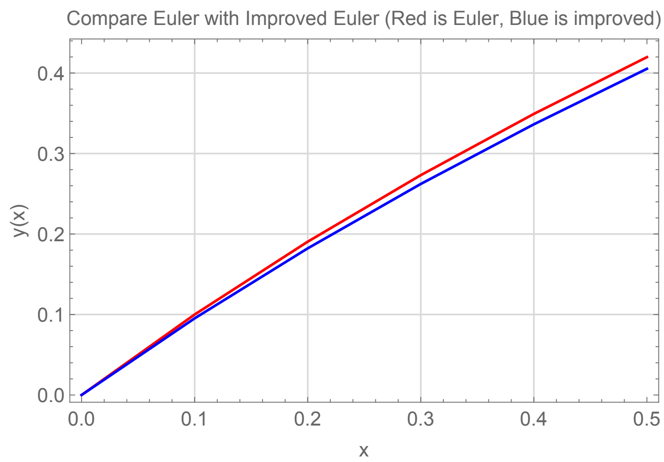
Out[273]=

x	0	0.1	0.2	0.3	0.4	0.5
h=0.01	0	0.09524187	0.1822067	0.2622174	0.3363033	0.405281
exact	0	0.09531018	0.1823216	0.2623643	0.3364722	0.4054651
% error	0	0.00683089	0.01148799	0.01469129	0.01689781	0.01840683

Then Euler method was compared to Improved Euler for the same step size $h = 0.1$, by plotting them on the same figure. Here is the result. The red line is the Euler method, and the blue line is the improved Euler method. We see the difference between them increases as more steps are taken.

```
In[216]:= h = 0.1;
dataEuler = makeTableEuler[h, 0, .5, 0];
dataEulerImproved = makeTableImproved[h, 0, .5, 0];
p1 = ListLinePlot[{dataEuler, dataEulerImproved},
  Frame → True, PlotStyle → {Red, Blue},
  FrameLabel →
   {{"y(x)", None}, {"x", Row[{"Compare Euler with Improved Euler (Red is Euler, Blue is improved)"}]}},
  BaseStyle → 12, GridLines → Automatic, GridLinesStyle → LightGray, ImageSize → 400]
```

Out[219]=



4.3.6 Section 2.5 problem 13

Problem Find the exact solution, then apply improved Euler method twice to approximate to 5 decimal places values the solution on the given interval. First with step $h = 0.01$ then with step $h = 0.005$. Make table showing the approximate values and the actual values, together with percentage error in the more accurate approximation for x an integral multiple of 0.2. $yy' = 2x^3; y(1) = 3; 1 \le x \le 2$.

Solution

The analytical solution is the same as in problem 13, section 2.5 and hence will not be repeated again. The improved Euler function, which was written for problem 8 above, was now used for $h = 0.01$ and $h = 0.005$. Source code is given above in problem 8. Here is the final table generated

```
In[285]:= h = 0.01;
data1 = makeTableImproved[h, 1, 2, 3];
h = 0.005;
data2 = makeTableImproved[h, 1, 2, 3];
exact = Sqrt[#^4 + 8] & /@ data2[[All, 1]];
p = Grid[{
   {"x", Sequence @@ data1[[All, 1]]},
   {"h=0.01", Sequence @@ data1[[All, 2]]},
   {"h=0.005", Sequence @@ data2[[All, 2]]},
   {"exact", Sequence @@ exact},
   {"% error", Sequence @@ ((exact - data2[[All, 2]]) / exact * 100)}
  }, Frame → All]
```

Out[290]=

x	1	1.2	1.4	1.6	1.8	2.
h=0.01	3	3.1739	3.44118	3.81494	4.30091	4.89901
h=0.005	3	3.1739	3.44117	3.81492	4.30089	4.89899
exact	3	3.17389	3.44116	3.81492	4.30088	4.89898
% error	0	-0.0000547372	-0.000101625	-0.000134386	-0.000151819	-0.000156696

To better compare the improved Euler method, with the Euler method, a new table was generated. This gives result only for $h = 0.01$. Here is the result. This used the Euler function which was written for section 2.4 and listed above. The table also includes the difference at each x between the two methods. We see from this table, that as more steps are made (at $x = 2$) that the difference between the improved Euler and Euler method has increased.

```
In[311]:= h = 0.01;
dataEuler = makeTableEuler[h, 1, 2, 3];
dataImproved = makeTableImproved[h, 1, 2, 3];
p = Grid[{
   {"x", Sequence @@ dataEuler[[All, 1]]},
   {"h=0.01, Euler", Sequence @@ dataEuler[[All, 2]]},
   {"h=0.01, Improved Euler", Sequence @@ dataImproved[[All, 2]]},
   {"Absolute Difference", Sequence @@ (dataEuler[[All, 2]] - dataImproved[[All, 2]])}
  }, Frame → All, Alignment → Left]
```

Out[314]=

x	1	1.2	1.4	1.6	1.8	2.
h=0.01, Euler	3	3.171843	3.436841	3.808392	4.292431	4.88896
h=0.01, Improved Euler	3	3.1739	3.441177	3.814939	4.30091	4.89901
Absolute Difference	0	-0.002057166	-0.004335663	-0.006546691	-0.008478646	-0.01005077

4.3.7 Section 2.5 problem 25

Problem Apply improved Euler method for $\frac{dv}{dt} = 32 - 1.6v$ with $v(0) = 0$. For $0 \leq t \leq 2$, using step size $h = 0.01, h = 0.005$, round v to one decimal point. What percentage of limiting velocity 20 ft/sec has been attained after 1 second? After 2 seconds?

Solution The exact solution we derived in section 2.4 above. The improved Euler method, implemented in the function shown above, was used in this problem to generate similar table to section 2.4, problem 25. But now using the improved Euler. Here is the resulting table.

```
data1 = makeTableImproved[0.01, 0, 2, 0];
At one second, using h=0.01 speed is 15.96179 at step n = 100
At 2 seconds, using h=0.01 speed is 19.18464 at step n = 200
data2 = makeTableImproved[0.005, 0, 2, 0];
At one second, using h=0.005 speed is 15.962 at step n = 200
At 2 seconds, using h=0.005 speed is 19.18473 at step n = 400
```

Therefore, improved Euler method result is

h	speed at 1 second	speed at 2 seconds
0.01	15.96179	19.18464
0.005	15.962 (79.81%)	19.18473 (95.923%)

This can be compared with Euler method in problem 2.4.25. We see small difference in speeds at 1 and 2 seconds. The improved Euler result should be taken as the more accurate. Here is the Euler method result, copied from 2.4.25 to make it easier to compare with

h	speed at 1 second	speed at 2 seconds
0.01	16.078	19.206
0.005	16.02 (80.1%)	19.195 (95.98%)

4.3.8 Section 2.5 problem 26

Problem Deer population $P(t)$ in small forest initially numbered 25 and satisfies logistic equation $\frac{dP}{dt} = 0.0225P(t) - 0.0003P^2$. With t in months. Use improved Euler method to approximate solution for 10 years. First with step $h = 1$ and then with $h = 0.5$ rounding off P to 3 decimal points. What percentage of the limiting population of 75 deer has been attained after 5,10 years?

Solution The improved Euler method

$$
\begin{aligned}
k_1 &= f(x_n, y_n) \\
u_{n+1} &= y_n + hk_1 \\
k_2 &= f(x_{n+1}, u_{n+1}) \\
y_{n+1} &= y_n + h\frac{k_1 + k_2}{2}
\end{aligned}
$$

With initial conditions $y_0 = 25$ was used to solve this ODE with $f(x,y) = 0.0225y - 0.0003y^2$. The same improved Euler method function listed earlier was used. The following table summarizes the results

h (moths)	$p(t)$ at 5 years	$p(t)$ at 10 years
1	49.3909 (65.85%)	66.1129 (88.15%)
0.5	49.39135 (65.85%)	66.11343 (88.15%)

In[347]:=

```
(*HW 3, Math 320. By Nasser M. Abbasi. Problem 2.5 26, improved Euler*)
f[t_, y_] := 0.0225 y - 0.0003 y^2;
makeTableImproved[h_, from_, to_, y0_] :=
 Module[{nSteps = Rationalize[(to - from) / h], data, t, y, skip, k1, k2, predictor},
  Array[y, nSteps, 0];
  Array[t, nSteps, 0];
  y[0] = y0; t[0] = from;

  Do[(*Euler loop*)
   k1 = f[t[n], y[n]];
   predictor = y[n] + h k1;
   t[n + 1] = t[n] + h;
   k2 = f[t[n + 1], predictor];
   y[n + 1] = y[n] + h (1 / 2 * (k1 + k2)),
   {n, 0, nSteps}
  ];
  skip = 1; (*Round[0.2/h];*)
  Table[{t[n], y[n]}, {n, 0, nSteps, skip}]
 ]
```

4.4 HW4

4.4.1 Section 3.1 problem 11 (page 155)

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

Solution

We set up the augmented matrix and do forward elimination. The row operations are given on top of each arrow. For example $R_2 = -\frac{1}{2}R_1 + R_2$ mean that row 2 is replaced by $-\frac{1}{2}$ of the first row added to the second row.

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{pmatrix} \underset{R_3=R_3-\frac{3}{2}R_1}{\overset{R_2=R_2-\frac{1}{2}R_1}{\Longrightarrow}} \begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & -0.5 & 0.5 & -3.5 \\ 0 & -3.5 & 4.5 & -28.5 \end{pmatrix} \overset{R_2=-7R_2}{\Longrightarrow}$$

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & 3.5 & -3.5 & 24.5 \\ 0 & -3.5 & 4.5 & -28.5 \end{pmatrix} \overset{R_3=R_3+R_2}{\Longrightarrow} \begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & 3.5 & -3.5 & 24.5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Hence the system of equation now is (from the last matrix above)

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ 3.5y - 3.5z &= 24.5 \\ z &= -4 \end{aligned}$$

Since at the last row, we did not get $0 =$ some number, then the system is consistent. This means the system has either a unique solution, or has infinite number of solutions. But since we did not get $0z = 0$, then the system has a unique solution. Now we will find the unique solution by backward substitution. From last equation, we obtain

$$z = -4$$

From the second equation

$$\begin{aligned} 3.5y - 3.5(-4) &= 24.5 \\ y &= 3 \end{aligned}$$

And from the first equation

$$\begin{aligned} 2x + 7(3) + 3(-4) &= 11 \\ x &= 1 \end{aligned}$$

Hence the solution is

$$\begin{aligned} x &= 1 \\ y &= 3 \\ z &= -4 \end{aligned}$$

4.4.2 Section 3.1 problem 16

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} x - 3y + 2z &= 6 \\ x + 4y - z &= 4 \\ 5x + 6y + z &= 20 \end{aligned}$$

Solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & -3 & 2 & 6 \\ 1 & 4 & -1 & 4 \\ 5 & 6 & 1 & 20 \end{pmatrix} \overset{R_2=R_2-R_1}{\Longrightarrow} \begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 5 & 6 & 1 & 20 \end{pmatrix} \overset{R_3=-5R_1+R_3}{\Longrightarrow}$$

$$\begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 21 & -9 & -10 \end{pmatrix} \overset{R_3=-3R_2+R_3}{\Longrightarrow} \begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Hence the last equation of last matrix above, we see that $0z = -4$. Since this result implies $0 = -4$, which is not possible, then there is no solution. The system is inconsistent. There are no solutions.

4.4.3 Section 3.1 problem 21

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} x + y - z &= 5 \\ 3x + y + 3z &= 11 \\ 4x + y + 5z &= 14 \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & 1 & -1 & 5 \\ 3 & 1 & 3 & 11 \\ 4 & 1 & 5 & 14 \end{pmatrix} \overset{R_2=-3R_1+R_2}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 4 & 1 & 5 & 14 \end{pmatrix} \overset{R_3=-4R_1+R_3}{\Longrightarrow}$$

$$\begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & -3 & 9 & -6 \end{pmatrix} \overset{R_3=-1.5R_2+R_3}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the final equation has $0z = 0$, this means there are infinite number of solutions. Since any z will satisfy this. The system is therefore consistent. Let $z = t$, hence from the second equation we obtain

$$\begin{aligned} -2y + 6z &= -4 \\ -2y + 6t &= -4 \\ y &= \frac{-4 - 6t}{-2} = 2 + 3t \end{aligned}$$

First equation gives

$$\begin{aligned} x + y - z &= 5 \\ x &= 5 - y + z \\ &= 5 - (2 + 3t) + t \\ &= 3 - 2t \end{aligned}$$

Hence solution is

$$\begin{aligned} x &= 3 - 2t \\ y &= 2 + 3t \\ z &= t \end{aligned}$$

4.4.4 Section 3.1 problem 31

Problem A system has the form

$$\begin{aligned} a_1 x + b_1 y &= 0 \\ a_2 x + b_2 y &= 0 \end{aligned}$$

Explain by geometric reasoning why such a system has either a unique solution or infinitely many solutions. In the former case, what is the unique solution?

solution These two equations represent two lines in 2D space. These can be written in standard form as

$$y = -\frac{a_1}{b_1}x$$
$$y = -\frac{a_2}{b_2}x$$

We see now, when we compare each equation above to the equation of a line of the form

$$y = mx + c$$

Where m is the slope, and c is the intercept with the y axis, we see that both lines have zero intercept.

This means both lines pass through the origin, but with possibly different slope. Therefore, since both lines pass though one point, then there is either a unique solution, which is the origin in this case, when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$, or the other case is the infinite number of solutions when the slope is the same, i.e. $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, which means both lines are on top of each others. (same line).

4.4.5 Section 3.1 problem 33

Problem The linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

of three equations in two unknowns, represents three lines L_1, L_2, L_3 in xy plane. Figure 3.1.5 shows six possible configurations of these three lines. In each case describe the solution set of the system.

solution

case a No solution. Since there is not one point where the three lines meet at.

case b Unique solution. Since there is a single point where the three lines intersect at.

case c No solution. Since there is not one point where the three lines meet at.

case d No solutions. All lines are parallel. There is not one point where the three lines meet at

case e Unique solution. There is one single point where the three lines intersect. Even though lines L_1, L_2 are on top of each others.

case f Infinite number of solutions. The three lines are on top of each others.

4.4.6 Section 3.1 problem 34

Problem Consider the linear system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

of three equations in three unknowns to represent three planes P_1, P_2, P_3 in xyz plane. Describe the solution in each of the following cases. (a) Three planes are parallel and distinct. (b) The three planes coincide. $P_1 = P_2 = P_3$. (c) P_1 and P_2 coincide and are parallel to P_3. (d) P_1, P_2 intersect in a line L that is parallel to P_3. (e) P_1, P_2 intersect in line L that lies in P_3. (f) P_1, P_2 intersect in a line L that intersect P_3 in a single point.

solution

case a No solution exist. Since three planes do not intersect.

case b There are infinite number of solutions. Since intersection is line.

case c No solution Since P_1, P_2 are parallel to P_3

case d No solution. This is similar to case c.

case e Infinite number of solution, since the intersection between all three planes is a line.

case f Unique solution. Since a single point is found on the three planes.

4.4.7 Section 3.2 problem 11

Problem Use elementary row operations to transform each augmented coefficient matrix to echelon form then solve the system by back substitution

$$
\begin{aligned}
2x_1 + 8x_2 + 3x_3 &= 2\\
x_1 + 3x_2 + 2x_3 &= 5\\
2x_1 + 7x_2 + 4x_3 &= 8
\end{aligned}
$$

solution

We set up the augmented matrix and do forward elimination

$$
\begin{pmatrix} 2 & 8 & 3 & 2\\ 1 & 3 & 2 & 5\\ 2 & 7 & 4 & 8 \end{pmatrix} \overset{R_2=-\frac{1}{2}R_1+R_2}{\Longrightarrow} \begin{pmatrix} 2 & 8 & 3 & 2\\ 0 & -1 & \frac{1}{2} & 4\\ 2 & 7 & 4 & 8 \end{pmatrix} \overset{R_3=-R_1+R_3}{\Longrightarrow}
$$

$$
\begin{pmatrix} 2 & 8 & 3 & 2\\ 0 & -1 & \frac{1}{2} & 4\\ 0 & -1 & 1 & 6 \end{pmatrix} \overset{R_3=-R_2+R_3}{\Longrightarrow} \begin{pmatrix} 2 & 8 & 3 & 2\\ 0 & -1 & \frac{1}{2} & 4\\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix}
$$

The above final matrix is now in echelon form. Since the final equation says that $\frac{1}{2}x_3 = 2$, therefore the system is consistent. Doing backward substitution gives

$$x_3 = 4$$

From second equation

$$
\begin{aligned}
-x_2 + \frac{1}{2}x_3 &= 4\\
-x_2 + \frac{1}{2}(4) &= 4\\
x_2 &= -2
\end{aligned}
$$

And from first equation

$$
\begin{aligned}
2x_1 + 8x_2 + 3x_3 &= 2\\
2x_1 + 8(-2) + 3(4) &= 2\\
x_1 &= 3
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
x_1 &= 3\\
x_2 &= -2\\
x_3 &= 4
\end{aligned}
$$

4.4.8 Section 3.2 problem 18

Problem Use elementary row operations to transform each augmented coefficient matrix to echelon form then solve the system by back substitution

$$
\begin{aligned}
3x_1 - 6x_2 + x_3 + 13x_4 &= 15\\
3x_1 - 6x_2 + 3x_3 + 21x_4 &= 21\\
2x_1 - 4x_2 + 5x_3 + 26x_4 &= 23
\end{aligned}
$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix}3 & -6 & 1 & 13 & 15\\3 & -6 & 3 & 21 & 21\\2 & -4 & 5 & 26 & 23\end{pmatrix}\overset{R_2=-R_1+R_2}{\Longrightarrow}\begin{pmatrix}3 & -6 & 1 & 13 & 15\\0 & 0 & 2 & 8 & 6\\2 & -4 & 5 & 26 & 23\end{pmatrix}\overset{R_3=-2R_1+3R_3}{\Longrightarrow}$$

$$\begin{pmatrix}3 & -6 & 1 & 13 & 15\\0 & 0 & 2 & 8 & 6\\0 & 0 & 13 & 52 & 39\end{pmatrix}\overset{R_3=-13R_2+2R_3}{\Longrightarrow}\begin{pmatrix}3 & -6 & 1 & 13 & 15\\0 & 0 & 2 & 8 & 6\\0 & 0 & 0 & 0 & 0\end{pmatrix}$$

The above final matrix is now in echelon form. Since last row gives $0x_i = 0$, then there are infinite number of solutions, as any x will satisfy this. System is therefore consistent.

Let

$$x_4 = t$$

Hence from the second row, we obtain

$$\begin{aligned}2x_3 + 8x_4 &= 6\\2x_3 + 8t &= 6\\x_3 &= \frac{6-8t}{2} = 3 - 4t\end{aligned}$$

And from the first equation

$$\begin{aligned}3x_1 - 6x_2 + x_3 + 13x_4 &= 15\\3x_1 - 6x_2 &= 15 - x_3 - 13x_4\\3x_1 - 6x_2 &= 15 - (3 - 4t) - 13t\end{aligned}$$

Let $x_2 = s$ then

$$\begin{aligned}3x_1 - 6s &= 12 - 9t\\x_1 &= \frac{12 - 9t + 6s}{3}\\&= 4 - 3t + 2s\end{aligned}$$

Hence the final solution is

$$\begin{aligned}x_1 &= 4 - 3t + 2s\\x_2 &= s\\x_3 &= 3 - 4t\\x_4 &= t\end{aligned}$$

4.4.9 Section 3.2 problem 24

problem Determine for what value of k each system has (a) unique solution (b) no solution (c) infinite solutions

$$\begin{aligned}3x + 2y &= 0\\6x + ky &= 0\end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix}3 & 2 & 0\\6 & k & 0\end{pmatrix}\overset{R_2=-2R_1+R_2}{\Longrightarrow}\begin{pmatrix}3 & 2 & 0\\0 & -4+k & 0\end{pmatrix}$$

Hence, the last equation says that

$$(-4 + k)\, y = 0$$

case a A unique solution exist if $k \neq 4$, since in this case y must be zero. Giving the unique solution $\{y \to 0, x \to 0\}$

case b There is no value of k which causes no solution to exist. Since the RHS is zero in the last equation.

case c If $k=4$, then we have $0y=0$. Then any value of y will satisfy this. Hence infinite number of solutions.

4.4.10 Section 3.2 problem 27

problem Determine for what value of k each system has (a) unique solution (b) no solution (c) infinite solutions

$$\begin{aligned} x+2y+z&=3\\ 2x-y-3z&=5\\ 4x+3y-z&=k \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix}1&2&1&3\\2&-1&-3&5\\4&3&-1&k\end{pmatrix}\overset{R_2=-2R_1+R_2}{\Longrightarrow}\begin{pmatrix}1&2&1&3\\0&-5&-5&-1\\4&3&-1&k\end{pmatrix}$$

$$\overset{R_3=-4R_1+R_3}{\Longrightarrow}\begin{pmatrix}1&2&1&3\\0&-5&-5&-1\\0&-5&-5&k-12\end{pmatrix}\overset{R_3=-R_2+R_3}{\Longrightarrow}\begin{pmatrix}1&2&1&3\\0&-5&-5&-1\\0&0&0&k-11\end{pmatrix}$$

Hence, the last equation says that

$$(0)\,z=k-11$$

case a No k exist which gives unique solution. For if $k=11$, then we have $(0)\,z=0$ and this gives infinite solutions. And if $k\neq 11$, then we have $(0)\,z=\text{number}$. Which says there are no solution.

case b If $k\neq 11$, then we have $(0)\,z=\text{number}$. Which says there are no solution.

case c if $k=11$, then we have $(0)\,z=0$ and this gives infinite solutions

4.4.11 Section 3.2 problem 28

Problem Under what conditions on the constants a,b,c does the systems

$$\begin{aligned} 2x-y+3z&=a\\ x+2y+z&=b\\ 7x+4y+9z&=c \end{aligned}$$

Have unique solution, no solution, infinite number of solutions?

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix}2&-1&3&a\\1&2&1&b\\7&4&9&c\end{pmatrix}\overset{R_2=2R_2}{\Longrightarrow}\begin{pmatrix}2&-1&3&a\\2&4&2&2b\\7&4&9&c\end{pmatrix}\overset{R_2=R_2-R_1}{\Longrightarrow}\begin{pmatrix}2&-1&3&a\\0&5&-1&2b-a\\7&4&9&c\end{pmatrix}$$

$$\underset{R_3=2R_3}{\overset{R_1=7R_1}{\Longrightarrow}}\begin{pmatrix}14&-7&21&7a\\0&5&-1&2b-a\\14&8&18&2c\end{pmatrix}\overset{R_3=R_3-R_1}{\Longrightarrow}\begin{pmatrix}14&-7&21&7a\\0&5&-1&2b-a\\0&15&-3&2c-7a\end{pmatrix}$$

$$\overset{R_3=R_3-3R_2}{\Longrightarrow}\begin{pmatrix}14&-7&21&7a\\0&5&-1&2b-a\\0&0&0&2c-4a-6b\end{pmatrix}$$

Hence, the last equation says

$$(0)\,z = 2c - 4a - 6b$$
$$(0)\,z = c - 2a - 3b$$
$$0\,(z) = c - (2a + 3b)$$

If the RHS is zero, then we have infinite number of solutions, since then we end up with $(0)\,z = 0$, which means any z will satisfy this equation. But if the RHS is not zero, then we end up with $(0)\,z =$ some number. Which is not possible. Therefore we conclude that

If $c = (2a + 3b)$ then infinite number of solutions.

If $c \neq (2a + 3b)$ then no solution.

It is not possible to obtain a unique solution.

4.4.12 Problem 3

Write the following as $Ax = b$ and determine for what values of the parameter k the system has (i) unique solution (ii) no solution, (iii) infinite solutions. (a)

$$\begin{aligned} x_1 + 3x_2 &= 8 \\ -x_1 + 2x_2 - x_3 &= 4 \\ 3x_1 + x_2 + 10x_3 &= k \end{aligned}$$

(b)

$$\begin{aligned} -x_2 + 0.5x_3 &= 0 \\ 4x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 0.5x_3 &= k \end{aligned}$$

4.4.12.1 Part (a)

We write it first as $Ax = b$

$$\overbrace{\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & -1 \\ 3 & 1 & 10 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} 8 \\ 4 \\ k \end{pmatrix}}^{b}$$

We next set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & 0 & 3 & 8 \\ -1 & 2 & -1 & 4 \\ 3 & 1 & 10 & k \end{pmatrix} \overset{R_2=R_1+R_2}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 3 & 1 & 10 & k \end{pmatrix}$$

$$\overset{R_3=3R_1-R_3}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 0 & -1 & -1 & 24-k \end{pmatrix} \overset{R_3=R_2+2R_3}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 0 & 0 & 0 & 60-2k \end{pmatrix}$$

Therefore, from last equation we see that

$$(0)\,x_3 = 30 - k$$

<u>case (i)</u> It is not possible to have unique solution.

<u>case (iI)</u> If $(30 - k) \neq 0$ then there is no solution, since then we have $0x_3 =$ some number, which is not possible. <u>Hence for $k \neq 30$</u>, there is no solution.

<u>case (iii)</u> If $(30 - k) = 0$ or <u>$k = 30$</u>, then there are infinite number of solutions.

4.4.12.2 Part (b)

We write it first as $Ax = b$

$$\overbrace{\begin{pmatrix} 0 & -1 & 0.5 \\ 4 & 2 & 3 \\ 2 & 3 & 0.5 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} 0 \\ 2 \\ k \end{pmatrix}}^{b}$$

We next set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 0 & -1 & 0.5 & 0 \\ 4 & 2 & 3 & 2 \\ 2 & 3 & 0.5 & k \end{pmatrix} \overset{\text{swap}(R_2,R_1)}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 2 & 3 & 0.5 & k \end{pmatrix} \overset{R_3=R_1-2R_3}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 0 & -4 & 2 & 2-2k \end{pmatrix}$$

$$\overset{R_3=-4R_2+R_3}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 2-2k \end{pmatrix}$$

Therefore, from last equation we see that

$$(0)\, x_3 = 1 - k$$

case (i) It is not possible to have unique solution.

case (iI) If $(1-k) \neq 0$ then there is no solution, since then we have $0x_3 =$ some number, which is not possible. Hence for $k \neq 1$, there is no solution.

case (iii) If $(1-k) = 0$ or $k = 1$, then there are infinite number of solutions.

4.5 HW5

4.5.1 Section 3.3 problem 8 (page 174)

Problem Find Reduced Echelon form for

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

Solution

The first step is to obtain the Echelon form, then convert that to Reduced Echelon form

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix} \underset{R_3=R_3-R_1}{\overset{R_2=R_2-3R_1}{\longrightarrow}} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{pmatrix} \overset{R_3=R_3-\frac{2}{3}R_2}{\longrightarrow} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 0 & -4 \end{pmatrix}$$

Now it is in Echelon form, we make it Reduced Echelon form. First we make each leading element 1

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 0 & -4 \end{pmatrix} \overset{R_2=\frac{1}{3}R_2}{\longrightarrow} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & -4 \end{pmatrix} \overset{R_3=\frac{-1}{4}R_3}{\longrightarrow} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1=R_1+4R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 19 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_2=R_2-6R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1=R_1-19R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4.5.2 Section 3.3 problem 9

Problem Find Reduced Echelon form for

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{pmatrix}$$

Solution

The first step is to obtain the Echelon form, then we convert that to Reduced Echelon form

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{pmatrix} \overset{R_3=R_3-\frac{4}{5}R_1}{\longrightarrow} \begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & -\frac{3}{5} & -\frac{12}{5} \end{pmatrix} \overset{R_3=R_3+\frac{3}{5}R_2}{\longrightarrow} \begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Now it is in Echelon form, we make it Reduced Echelon form. First we make each leading element 1

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1=\frac{1}{5}R_1}{\longrightarrow} \begin{pmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1=R_1-\frac{2}{5}R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

4.5.3 Section 3.3 problem 31

Problem Show that the two matrices in (1) are both row equivalent to the 3×3 identity matrix (and hence by theorem 1, to each others)

Solution The two matrices in (1) are

$$\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix},\begin{pmatrix}1&1&1\\0&2&2\\0&0&3\end{pmatrix} \tag{1}$$

We now reduce each matrix to Reduced Echelon form and see if we obtain the 3×3 identity matrix. Starting with the first matrix above, and since the matrices are already in Echelon form, we just need to do the reduction steps.

First we make each leading element 1

$$\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}\overset{R_2=\frac{1}{4}R_2}{\longrightarrow}\begin{pmatrix}1&2&3\\0&1&\frac{5}{4}\\0&0&6\end{pmatrix}\overset{R_3=\frac{1}{6}R_3}{\longrightarrow}\begin{pmatrix}1&2&3\\0&1&\frac{5}{4}\\0&0&1\end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix}1&2&3\\0&1&\frac{5}{4}\\0&0&1\end{pmatrix}\overset{R_1=R_1-2R_2}{\longrightarrow}\begin{pmatrix}1&0&\frac{1}{2}\\0&1&\frac{5}{4}\\0&0&1\end{pmatrix}\overset{R_2=R_2-\frac{5}{4}R_3}{\longrightarrow}\begin{pmatrix}1&0&\frac{1}{2}\\0&1&0\\0&0&1\end{pmatrix}\overset{R_1=R_1-\frac{1}{2}R_3}{\longrightarrow}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{2}$$

Now we work on the second matrix. First we make each leading element 1

$$\begin{pmatrix}1&1&1\\0&2&2\\0&0&3\end{pmatrix}\overset{R_2=\frac{1}{2}R_2}{\longrightarrow}\begin{pmatrix}1&1&1\\0&1&1\\0&0&3\end{pmatrix}\overset{R_3=\frac{1}{3}R_3}{\longrightarrow}\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}\overset{R_1=R_1-R_2}{\longrightarrow}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\overset{R_2=R_2-R_3}{\longrightarrow}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{3}$$

Comparing (2) and (3) we see that the Reduced Echelon form in both case came out to be

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Hence both matrices in (1) are row equivalent.

4.5.4 Section 3.3 problem 32

Problem Show that the matrix

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

is row equivalent to the 2×2 identity matrix, provided $ad-bc\neq 0$

Solution let us convert the given matrix to Reduced Echelon form. Assuming $a\neq 0$ then

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\overset{R_2=R_2-\frac{c}{a}R_1}{\longrightarrow}\begin{pmatrix}a&b\\0&d-\frac{c}{a}b\end{pmatrix}=\begin{pmatrix}a&b\\0&\frac{ad-cb}{a}\end{pmatrix}$$

Now we need to make each leading element 1.

$$\begin{pmatrix}a&b\\0&\frac{ad-cb}{a}\end{pmatrix}\overset{R_1=\frac{1}{a}R_1}{\longrightarrow}\begin{pmatrix}1&\frac{b}{a}\\0&\frac{ad-cb}{a}\end{pmatrix}\overset{R_2=\frac{a}{ad-cb}R_2}{\longrightarrow}\begin{pmatrix}1&\frac{b}{a}\\0&1\end{pmatrix}$$

Now, assuming that $ad-cb\neq 0$, only then we can do the next step, since we dividing by $ad-cb$

$$\begin{pmatrix}1&\frac{b}{a}\\0&\frac{ad-cb}{a}\end{pmatrix}\overset{R_2=\frac{a}{ad-cb}R_2}{\longrightarrow}\begin{pmatrix}1&\frac{b}{a}\\0&1\end{pmatrix}$$

Now we make all entries in column above each leading element zero.

$$\begin{pmatrix}1 & \frac{b}{a}\\ 0 & 1\end{pmatrix} \xrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

So we see, that unless $ad-cb\neq 0$, we would not have been able to complete the Reduced Echelon form process, since in one the steps above, we would have divided by zero. We conclude that any 2×2 matrix is row equivalent to 2×2 identity matrix provided the determinant is not zero. Since $|A| = ad-cb$.

4.5.5 Section 3.3 problem 36

Problem Suppose that $ad-bc\neq 0$ in the homogeneous system of problem 35. Use problem 32 to show that its only solution is the trivial solution.

Solution Problem 35 gives

$$\begin{aligned} ax+by &= 0\\ cx+dy &= 0 \end{aligned}$$

In matrix form

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Since $ad-cb\neq 0$, then using problem 32, we know A is row equivalent to 2×2 identity matrix. Which means the original system can now be written as

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Which means the solution is $x=0$ and $y=0$. The trivial solution.

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

4.5.6 Section 3.3 problem 37

Problem Show that the system in problem 35 has a non-trivial solution iff $ad-bc=0$

solution Problem 35 gives

$$\begin{aligned} ax+by &= 0\\ cx+dy &= 0 \end{aligned}$$

In matrix form

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

The augment matrix is

$$\begin{pmatrix}a & b & 0\\ c & d & 0\end{pmatrix}$$

We start by reducing it to Echelon form. We assume all along that $a\neq 0$.

$$\begin{pmatrix}a & b & 0\\ c & d & 0\end{pmatrix} \xrightarrow{R_2=R_2-\frac{c}{a}R_1} \begin{pmatrix}a & b & 0\\ 0 & d-\frac{c}{a}b & 0\end{pmatrix} = \begin{pmatrix}a & b & 0\\ 0 & \frac{ad-cb}{a} & 0\end{pmatrix}$$

Now we can solve by backward substitution. There are two cases to consider.

case 1 $ad-cb=0$. In this case, the Echelon form becomes

$$\begin{pmatrix}a & b & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence the second equation says $0\left(y\right)=0$. This implies infinite number of solutions, since any y will do the job.

case 2 $ad - cb \neq 0$. Lets say $ad - cb = N$, some non-zero value. In this case, the Echelon form becomes

$$\begin{pmatrix} a & b & 0 \\ 0 & N & 0 \end{pmatrix}$$

Hence the second equation says $N\left(y\right) = 0$. The solution to this is $\underline{y = 0}$. Therefore, from the first equation we obtain

$$\begin{aligned} ax + by &= 0 \\ ax &= 0 \\ x &= 0 \end{aligned}$$

So we see that the solution vector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is the trivial solution.

Conclusion The system has infinite number of solution iff $ad - cb = 0$ (this is the non-trivial solution case). And the system has unique solution, which is the trivial solution iff $ad - cb \neq 0$.

4.6 practice $Ax = b$

4.6.1 Problem 1

1. Write the following system as $\mathbf{Ax} = \mathbf{b}$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions. In the case of (i) or (iii), find the solution(s).

$$2x_1 + 2x_2 - x_3 = 1$$
$$3x_2 + 3x_3 = 3$$
$$4x_1 + x_2 + kx_3 = -1 \tag{1}$$

solution

$$\begin{pmatrix} 2 & 2 & -1 \\ 0 & 3 & 3 \\ 4 & 1 & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 4 & 1 & k & -1 \end{pmatrix}$$

We start by converting the above to Echelon form

$$\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 4 & 1 & k & -1 \end{pmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -3 & k+2 & -3 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & k+5 & 0 \end{pmatrix}$$

We see that the last equation now has the form

$$(k+5)\, x_3 = 0$$

If $k + 5 = n \neq 0$ then the equation becomes $nx_3 = 0$, which means $x_3 = 0$ is only choice, since $n \neq 0$. This means, from the second equation, $3x_2 + 3x_3 = 3$ or $x_2 = 1$ and from the first equation, $2x_1 + 2x_2 - x_3 = 1$ or $2x_1 + 2 = 1$ or $x_1 = \frac{-1}{2}$. Hence a unique solution. But if $k + 5 = 0$ then last equation gives $0x_3 = 0$, which means any x_3 will do the job. Hence infinite number of solutions.

Therefore, (i) $k \neq -5$ gives unique solution. (ii) Not possible. (iii) $k = -5$ gives infinite solutions.

4.6.2 Problem 2

2. For what values of k does $\mathbf{Ax} = \mathbf{b}$ have (i) no solution, (ii) a unique solution, or (iii) an infinite number of solutions? In the case of (ii) or (iii), find the solution(s).

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 1 & k \\ 3 & 1 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix} \tag{2}$$

solution

$$\begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & k \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 20 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ -1 & 1 & k & 4 \\ 3 & 1 & 4 & 20 \end{pmatrix}$$

We start by converting the above to Echelon form. Swap the second and third row

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ 3 & 1 & 4 & 20 \\ -1 & 1 & k & 4 \end{pmatrix}$$

Now

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ 3 & 1 & 4 & 20 \\ -1 & 1 & k & 4 \end{pmatrix} \underset{R_3=R_3+\frac{1}{2}R_1}{\overset{R_2=R_2-\frac{3}{2}R_1}{\longrightarrow}} \begin{pmatrix} 2 & 0 & -2 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & k-1 & 8 \end{pmatrix} \overset{R_3=R_3-R_2}{\longrightarrow} \begin{pmatrix} 2 & 0 & -2 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & k-8 & 0 \end{pmatrix}$$

From last equation, we obtain $(k-8)\,x_3 = 0$.

(i) No solution case is not possible.

(ii) When $k \neq 8$, then unique solution. Hence $x_3 = 0$. Which means from second equation that $x_2 = 8$ and from first equation, $2x_1 = 8$ or $x_1 = 4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix}$$

(iii) infinite number of solutions when $k = 8$. This gives $0\,(x_3) = 0$, hence any x_3 will do the job. Let $x_3 = t$, the second equation gives $x_2 + 7t = 8$ or $x_2 = 8 - 7t$. and the first equation gives $2x_1 - 2t = 8$ or $x_1 = 4 - t$. Hence solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4-t \\ 8-7t \\ t \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -7 \\ 1 \end{pmatrix}$$

4.6.3 Problem 3

3. In the following exercises, we write the augmented coefficient matrix for $\mathbf{Ax} = \mathbf{b}$. Determine for what values of the parameter p the system has (i) an unique solution, (ii) no solution, (iii) an infinite number of solutions. In case (i), find the unique solution. In case (iii), determine if there is a one-parameter family of solutions, or a two-parameter family of solutions, and find an expression for the solutions $\mathbf{x}$ in terms of the parameter(s).

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ -1 & -2 & 4 & 3 \end{bmatrix} \tag{3a}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & p & 4 & 2 \\ 3 & p+1 & 6 & p+1 \end{bmatrix} \tag{3b}$$

$$\begin{bmatrix} -2 & 3 & p & 1 \\ 4 & 3/2 & 2 & 2 \\ 3 & 3 & 5/2 & 5/2 \end{bmatrix} \tag{3c}$$

solution

4.6.3.1 Part a

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & p & 0 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ -1 & -2 & 4+\frac{3}{2} & 3 \end{pmatrix} \xrightarrow{R_3=R_3+\frac{1}{2}R_1} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ 0 & -\frac{3}{2} & \frac{11}{2} & \frac{7}{2} \end{pmatrix} \xrightarrow{R_3=R_3+\frac{3}{2p}R_2} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ 0 & 0 & \frac{11}{2} & \frac{1}{2p}\left(7p+3\right) \end{pmatrix}$$

We now convert the above to reduced Echelon form. First we make each leading entry 1

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{p} \\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11} \end{pmatrix}$$

Now we zero out all entries in column above leading entries

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{p} \\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11} \end{pmatrix} \xrightarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2p}\left(p-1\right) \\ 0 & 1 & 0 & \frac{1}{p} \\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11} \end{pmatrix} \xrightarrow{R_1=R_1-\frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{11p}\left(p+2\right) \\ 0 & 1 & 0 & \frac{1}{p} \\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11} \end{pmatrix}$$

Hence, the last equation says

$$x_3 = \frac{1}{p}\frac{\left(7p+3\right)}{11}$$

Therefore, if $7p+3 \neq 0$ then x_3 is parameterized by p and we have infinite number of solutions. In this case the solution vector is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{11p}\left(p+2\right) \\ \frac{1}{p} \\ \frac{1}{p}\frac{(7p+3)}{11} \end{pmatrix}$$

But if $7p+3=0$ then $x_3 = 0$, and this means $p = -\frac{3}{7}$. Then from second equation we obtain $x_2 = \frac{-7}{3}$ and from first equation $x_1 = -\frac{5}{11\left(-\frac{3}{7}\right)}\left(-\frac{3}{7}+2\right) = \frac{5}{3}$. Hence in this case the solution is unique

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{-7}{3} \\ 0 \end{pmatrix}$$

In both cases, we assumed $p \neq 0$. It is no possible to obtain the case (ii) which is no solution.

4.6.3.2 Part b

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & p & 4 \\ 3 & p+1 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ p+1 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & p & 4 & 2 \\ 3 & p+1 & 6 & p+1 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & p-2 & 0 & 0 \\ 0 & p-2 & 0 & p-2 \end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & p-2 & 0 & 0 \\ 0 & 0 & 0 & p-2 \end{pmatrix}$$

We see from last equation that $0\left(x_3\right) = p-2$. This means that if $p-2 \neq 0$ then there is no solution. This means if $p \neq 2$ then no solution. On the other hand, if $p = 2$ then last equation becomes $0\left(x_3\right) = 0$, which means any x_3 will do. Let $x_3 = t$. From second equation, we have

$$\left(p-2\right)x_2 = 0$$
$$0\left(x_2\right) = 0$$

So any x_2 will do. Let $x_2 = s$. Then the first equation becomes $x_1 + s + 2t = 1$ or $x_1 = 1 - s - 2t$.

Hence solution vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-s-2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) do not apply. This is two family solution.

4.6.3.3 Part c

$$\begin{pmatrix} -2 & 3 & p \\ 4 & \frac{3}{2} & 2 \\ 3 & 3 & \frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \frac{5}{2} \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} -2 & 3 & p & 1 \\ 4 & \frac{3}{2} & 2 & 2 \\ 3 & 3 & \frac{5}{2} & \frac{5}{2} \end{pmatrix} \xrightarrow[R_3=R_3+\frac{3}{2}R_1]{R_2=R_2+2R_1} \begin{pmatrix} -2 & 3 & p & 1 \\ 0 & \frac{15}{2} & 2+2p & 4 \\ 0 & \frac{15}{2} & \frac{5}{2}+\frac{3}{2}p & 4 \end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix} -2 & 3 & p & 1 \\ 0 & \frac{15}{2} & 2+2p & 4 \\ 0 & 0 & \frac{1}{2}-\frac{1}{2}p & 0 \end{pmatrix}$$

Last equation gives $\left(\frac{1}{2}-\frac{1}{2}p\right)x_3 = 0$. If $\frac{1}{2}-\frac{1}{2}p = 0$ or $p = 1$, then there are infinite number of solutions. Let $x_3 = t$. From second equation, $\frac{15}{2}x_2 + (2+2p)x_3 = 4$ or $\frac{15}{2}x_2 + 4t = 4$, which gives $x_2 = \frac{8}{15} - \frac{8}{15}t$ and from first equation $-2x_1 + 3x_2 + x_3 = 1$ or $-2x_1 + 3\left(\frac{8}{15}-\frac{8}{15}t\right) + t = 1$, hence $x_1 = \frac{3}{10} - \frac{3}{10}t$. The solution vector is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}-\frac{3}{10}t \\ \frac{8}{15}-\frac{8}{15}t \\ t \end{pmatrix}$$

If $\frac{1}{2}-\frac{1}{2}p \neq 0$, then last equation gives $nx_3 = 0$ which is only possible if $x_3 = 0$. This means if $p \neq 1$, then $x_3 = 0$. Second equation gives $\frac{15}{2}x_2 = 4$ or $x_2 = \frac{8}{15}$ and first equation gives $-2x_1 + 3x_2 + x_3 = 1$ or $-2x_1 + 3\left(\frac{8}{15}\right) = 1$, or $x_1 = \frac{3}{10}$, hence solution vector is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{8}{15} \\ 0 \end{pmatrix}$$

case (ii) is not possible.

4.7 HW6

4.7.1 Section 3.4 problem 8 (page 186)

Problem Calculate AB and BA if defined.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix}$$

solution A is 2×3 and B is 3×2, Since inner dimensions agree, then AB is defined and given by 2×2 matrix

$$\begin{aligned} C &= AB \\ &= \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 21 & 15 \\ 35 & 0 \end{pmatrix} \end{aligned}$$

Now B is 3×2 and A is 2×3, hence inner dimensions agree, and BA is 3×3

$$\begin{aligned} C &= BA \\ &= \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{pmatrix} \end{aligned}$$

4.7.2 Section 3.4 problem 15

Problem ABC matrices are given, verify by computation, that $A(BC) = (AB)C$

$$A = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}$$

solution A is $2 \times 1, B$ is 1×3 and C is 3×2.

$$BC = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}$$

Hence

$$\begin{aligned} A(BC) &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 15 \\ 8 & 10 \end{pmatrix} \end{aligned} \tag{1}$$

Now we will do $(AB)C$ and see if we get same result as above

$$AB = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$$

Hence

$$(AB)C = \begin{pmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 12 & 15 \\ 8 & 10 \end{pmatrix} \tag{2}$$

Comparing (1) and (2), we see they are the same. QED.

4.7.3 Section 3.4 problem 20

Problem Write the system as $Ax = 0$ and the find the solution in vector form

$$x_1 - 3x_2 + 7x_5 = 0$$
$$x_3 - 2x_5 = 0$$
$$x_4 - 10x_5 = 0$$

Solution

$$\overbrace{\begin{pmatrix} 1 & -3 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -10 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}^{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To find solution, we need to do Gaussian elimination to obtain Echelon form. But A is already in Echelon form. Hence we start with back substitution phase. From last equation

$$x_4 - 10x_5 = 0$$

Let $x_5 = t$, hence

$$x_4 = 10t$$

From second equation

$$x_3 - 2x_5 = 0$$
$$x_3 = 2t$$

From first equation

$$x_1 - 3x_2 + 7x_5 = 0$$
$$x_1 - 3x_2 = -7t$$

Let $x_2 = s$ then

$$x_1 = 3s - 7t$$

Hence solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3s - 7t \\ s \\ 2t \\ 10t \\ t \end{pmatrix}$$
$$= s\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \end{pmatrix} + t\begin{pmatrix} -7 & 0 & 2 & 10 & 1 \end{pmatrix}$$

4.7.4 Section 3.4 problem 27

Problem A diagonal matrix is square matrix of form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

in which every element off the diagonal is zero. Show that the product AB of two $n \times n$ diagonal matrices is again a diagonal matrix. State concise rule for quickly computing AB. Is it clear that $AB = BA$? Explain.

Solution

We want to perform (using 3×3 for illustration) the following.

$$C = AB = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

Let use the matrix multiplication method, where we multiply A by each column of B at a time, to produce one column of the result C. This means the first column of C is

$$c_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ 0 \\ 0 \end{pmatrix}$$

And the second column of C is

$$c_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ b_{22} \\ 0 \end{pmatrix}$$

And third column of C is

$$c_3 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b_{33} \end{pmatrix}$$

And so on for larger matrices. Using the above view, shows that c_1 will come out to be (using rules of matrix times vector now)

$$c_1 = \begin{pmatrix} a_{11}b_{11} \\ 0 \\ 0 \end{pmatrix}$$

And c_2 and c_3 will come out to be

$$c_2 = \begin{pmatrix} 0 \\ a_{22}b_{22} \\ 0 \end{pmatrix}$$

$$c_3 = \begin{pmatrix} 0 \\ 0 \\ a_{33}b_{33} \end{pmatrix}$$

And so one for larger matrices. Now we uses these columns to make up C and obtain

$$C = \begin{pmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$$

We see that C is diagonal matrix as well. If we reverse the order of multiplications, BA

and follow the same process as above, we will obtain

$$C = \begin{pmatrix} b_{11}a_{11} & 0 & 0 \\ 0 & b_{22}a_{22} & 0 \\ 0 & 0 & b_{33}a_{33} \end{pmatrix}$$

We see if the same Matrix, since number $a_{ii}b_{ii}$ is same as $b_{ii}a_{ii}$. A quick rule to make C is this: Start with C which is all zeros, then multiply each corresponding diagonal elements in A and B and move the result in the diagonal of resulting matrix C. So basically, we just need to multiply diagonal elements.

$$c_{ii} = \begin{cases} a_{ii}b_{ii} & i = 1,2,3\cdots n \\ 0 & \text{otherwise} \end{cases}$$

4.7.5 Section 3.4 problem 29

Problem If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then show that $A^2 = (a+d)A - (ad-bc)I_2$ where I_2 is the 2×2 identity matrix. Thus every 2×2 matrix A satisfies the equation $A^2 - (trace\ A)A + (\det A)I = 0$ where $\det(A) = ad - bc$ and trace is sum of diagonal elements.

solution

First we find A^2 using matrix-matrix multiplication

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix} \end{aligned} \tag{1}$$

Now $trace(A) = a + d$. Hence

$$(trace\ A)A = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This is scalar times matrix. Hence

$$\begin{aligned} (trace\ A)A &= \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} \\ &= \begin{pmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{pmatrix} \end{aligned}$$

And $\det(A)I_2$ is

$$\det(A)I_2 = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is scalar times matrix. Hence

$$\det(A)I_2 = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

From the above, we see that

$$\begin{aligned} (trace\ A)A - \det(A)I_2 &= \begin{pmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{pmatrix} - \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \begin{pmatrix} (a^2+ad)-(ad-bc) & ab+db \\ ac+dc & (ad+d^2)-(ad-bc) \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+db \\ ac+dc & d^2+bc \end{pmatrix} \end{aligned} \tag{2}$$

If we compare (1) and (2), we see they are the same. Hence we showed that

$$A^2 = (trace\ A)A - \det(A)I_2$$

4.7.6 Section 3.4 problem 30

Problem The formula $A^2 = (trace\ A)\,A - \det(A)\,I_2$ can be used to compute A^2 without explicit matrix multiplication. It follows that $A^3 = (trace\ A)\,A^2 - \det(A)\,A$ and $A^4 = (trace\ A)\,A^3 - \det(A)\,A^2$ and so on. Use this method to determine A^2, A^3, A^4, A^5 given that $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

solution

$$\begin{aligned} trace\ A &= 2 + 2 = 4 \\ \det A &= 4 - 1 = 3 \end{aligned}$$

Hence

$$\begin{aligned} A^2 &= (trace\ A)\,A - \det(A)\,I_2 \\ &= 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^3 &= (trace\ A)\,A^2 - \det(A)\,A \\ &= 4\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^4 &= (trace\ A)\,A^3 - \det(A)\,A^2 \\ &= 4\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} - 3\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^5 &= (trace\ A)\,A^4 - \det(A)\,A^3 \\ &= 4\begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix} - 3\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix} \end{aligned}$$

4.7.7 Section 3.4 problem 32

Problem (a) Suppose that $A = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$. Show that $(A + B)^2 \neq A^2 + 2AB + B^2$. (b) Suppose that A, B are square matrices such that $AB = BA$. Show that $(A + B)^2 = A^2 + 2AB + B^2$

solution

4.7.7.1 Part (a)

First we find the LHS

$$\begin{aligned}(A+B)^2 &= \left[\begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix} + \begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\right]^2\\ &= \begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}^2\\ &= \begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}\begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}\\ &= \begin{pmatrix}5 & 52\\ -13 & 96\end{pmatrix} \qquad (1)\end{aligned}$$

Now

$$\begin{aligned}A^2 &= \begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\\ &= \begin{pmatrix}8 & -5\\ -20 & 13\end{pmatrix}\end{aligned}$$

And

$$\begin{aligned}B^2 &= \begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\\ &= \begin{pmatrix}16 & 40\\ 24 & 64\end{pmatrix}\end{aligned}$$

And

$$\begin{aligned}AB &= \begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\\ &= \begin{pmatrix}-1 & 3\\ 5 & 1\end{pmatrix}\end{aligned}$$

Hence

$$2AB = 2\begin{pmatrix}-1 & 3\\ 5 & 1\end{pmatrix} = \begin{pmatrix}-2 & 6\\ 10 & 2\end{pmatrix}$$

Therefore, the RHS $A^2 + 2AB + B^2$ is

$$\begin{aligned}A^2 + 2AB + B^2 &= \begin{pmatrix}8 & -5\\ -20 & 13\end{pmatrix} + \begin{pmatrix}-2 & 6\\ 10 & 2\end{pmatrix} + \begin{pmatrix}16 & 40\\ 24 & 64\end{pmatrix}\\ &= \begin{pmatrix}22 & 41\\ 14 & 79\end{pmatrix} \qquad (2)\end{aligned}$$

Comparing (1) and (2) we see that are not the same. Hence we showed that, in this example, $(A+B)^2 \neq A^2 + 2AB + B^2$

4.7.7.2 Part (b)

Now, we assume that $AB = BA$. But since $(A+B)^2 = A^2 + B^2 + AB + BA$ and we are told that $AB = BA$, then

$$\begin{aligned}(A+B)^2 &= A^2 + B^2 + AB + AB\\ &= A^2 + B^2 + 2AB\end{aligned}$$

So only in the case when $AB = BA$ is $(A+B)^2 = A^2 + B^2 + 2AB$. In Part (a), $AB = \begin{pmatrix}-1 & 3\\ 5 & 1\end{pmatrix}$,

But $BA = \begin{pmatrix}-18 & 14\\ -22 & 18\end{pmatrix}$, so in part (a), $AB \neq BA$ and that is why equality failed.

4.7.8 Section 3.5 problem 13 (page 199)

Problem Find A^{-1} for $\begin{pmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{pmatrix}$

solution

We set up AI_3 and perform row operations on A and I at same time, to convert A to I_3. Then A^{-1} will be the on the right side

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2=R_1]{R_1=R_2} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 3 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{R_3=R_3+2R_2} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-3R_2}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 7 & 0 \\ 1 & -2 & 0 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_2=R_2+R_3} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 7 & 0 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-5R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix}$$

Since the left side is I_3 we stop. Hence

$$A^{-1} = \begin{pmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix}$$

4.7.9 Section 3.5 problem 19

Problem Find A^{-1} for $\begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

solution

We set up AI_3 and perform row operations on A and I at same time, to convert A to I_3. Then A^{-1} will be the on the right side

$$\begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 0 & 2 \\ 0 & -3 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \xrightarrow[R_3=R_2]{R_3=R_2}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3=\frac{R_3}{2}]{R_2=\frac{R_2}{-3}} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{R_1=R_1-4R_2}$$

$$\begin{pmatrix} 1 & 0 & -\frac{11}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{5}{3} & 0 & \frac{4}{3} \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{R_2=R_2-\frac{5}{3}R_3} \begin{pmatrix} 1 & 0 & -\frac{11}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{5}{3} & 0 & \frac{4}{3} \\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{R_1=R_1+\frac{11}{3}R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Since the left side is I_3 we stop. Hence

$$A^{-1}=\begin{pmatrix} -\frac{7}{2} & \frac{11}{6} & \frac{4}{3}\\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3}\\ -\frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$$

4.7.10 Section 3.5 problem 24

Problem Use method of example 8 to find matrix X such that $AX=B$

$$A=\begin{pmatrix}7 & 6\\ 8 & 7\end{pmatrix}, B=\begin{pmatrix}2 & 0 & 4\\ 0 & 5 & -3\end{pmatrix}$$

solution

$$AX=B$$

Pre multiply both sides by A^{-1}

$$\begin{aligned} A^{-1}AX &= A^{-1}B\\ I_3X &= A^{-1}B\\ X &= A^{-1}B \end{aligned} \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)}\begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix}\\ &= \frac{1}{(7\times 7)-(6\times 8)}\begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix}\\ &= \begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix}\begin{pmatrix}2 & 0 & 4\\ 0 & 5 & -3\end{pmatrix}\\ &= \begin{pmatrix}14 & -30 & 46\\ -16 & 35 & -53\end{pmatrix} \end{aligned}$$

4.7.11 Section 3.5 problem 30

Problem Suppose that A,B,C are invertible matrices of same size, show that product ABC is invertible and that $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

solution

$$\begin{aligned} (ABC)\left(C^{-1}B^{-1}A^{-1}\right) &= (AB)\left(CC^{-1}\right)\left(B^{-1}A^{-1}\right)\\ &= (AB)\,I\left(B^{-1}A^{-1}\right)\\ &= (AB)\left(B^{-1}A^{-1}\right)\\ &= A\left(BB^{-1}\right)A^{-1}\\ &= AIA^{-1}\\ &= AA^{-1}\\ & I \end{aligned}$$

And

$$\begin{aligned}\left(C^{-1}B^{-1}A^{-1}\right)(ABC) &= C^{-1}B^{-1}\left(A^{-1}A\right)BC\\ &= C^{-1}B^{-1}IBC\\ &= C^{-1}B^{-1}BC\\ &= C^{-1}\left(B^{-1}B\right)C\\ &= C^{-1}(I)\,C\\ &= C^{-1}C\\ &= I\end{aligned}$$

Thus we get I when we multiply ABC on either side by $C^{-1}B^{-1}A^{-1}$. Because the inverse of ABC is unique, this proves that ABC is invertible and that its inverse is $C^{-1}B^{-1}A^{-1}$. QED

4.7.12 Section 3.5 problem 32

Problem Show that if A is invertible matrix and $AB = AC$ then $B = C$. Thus invertible matrices can be canceled.

solution

Pre multiplying both sides of $AB = AC$ by A^{-1} (which we can do, since we are told A is invertible, then

$$\begin{aligned}A^{-1}AB &= A^{-1}AC\\ \left(A^{-1}A\right)B &= \left(A^{-1}A\right)C\\ IB &= IC\\ B &= C\end{aligned}$$

QED

4.7.13 Section 3.5 problem 34

Problem Show that a diagonal matrix is invertible iff each diagonal element is non-zero. In this case, state concisely how the inverse matrix is obtained.

solution

An $n \times n$ Matrix A is invertible, if there are elementary row operations which converts A to the identity matrix I_n. Since for a diagonal matrix, we just need to divide each row by its diagonal element in order to make the diagonal element 1 (if it was not already so), then we see immediately, that any diagonal matrix can be converted to I_n this way, unless the diagonal element happened to be zero. Since we can not divide by zero. There are no other operations to make the diagonal element, which is zero, become one. Since all entries above and below the diagonal element (i.e. all elements on the same column as the current zero diagonal element) are zero also by definition. So we are stuck with the zero on the diagonal, and unable to make it 1 using row operations.

Another way to proof this is the following. Since the determinant of diagonal matrix is obtained by just multiplying all the diagonal elements with each others, then if one element is zero, then the whole product is zero, and this means $\det(A) = 0$. But a matrix whose determinant is zero is singular and do not have an inverse. QED.

To obtain the inverse matrix for diagonal matrix with non-zero elements, we simply invert each element on the diagonal. For example

$$\begin{aligned}A &= \begin{pmatrix} a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ 0 & 0 & a_{33}\end{pmatrix}\\ A^{-1} &= \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0\\ 0 & \frac{1}{a_{22}} & 0\\ 0 & 0 & \frac{1}{a_{33}}\end{pmatrix}\end{aligned}$$

4.7.14 Section 3.5 problem 35

Problem Let A be $n \times n$ matrix with either row or column consisting of all zeros. Show that A is not invertible.

solution

An $n \times n$ that has at least one row all zeros, or at least one column all zero, is singular. Meaning its determinant is zero. This is from properties of determinants. Therefore, the matrix is not invertible.

Another proof: A matrix with row all zero, can not have a pivot of 1. Hence it is not possible to transform A to I_n using elementary row operations. Since it is square matrix, if the column is all zeros, then by transposing it, we end up with row which is all zero. Which is the same.

4.8 HW7

4.8.1 Section 3.6 problem 4 (page 216)

Problem Use cofactor expansion along row of column which minimize the amount of computation to find determinant of

$$A = \begin{pmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{pmatrix}$$

solution Since the 3$^{\text{rd}}$ row has most zeros(as well as first column), expansion is carried on the last row. Therefore

$$\det(A) = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34}$$

But $a_{31} = a_{32} = a_{33} = 0$. Hence the above simplifies to

$$\begin{aligned} \det(A) &= a_{34}A_{34} \\ &= a_{34}(-1)^{3+4} M_{34} \\ &= -3(-1)^7 M_{34} \\ &= 3M_{34} \end{aligned} \tag{1}$$

Now we need to find M_{34}, which is determinant of the matrix obtained from A by removing the third row and fourth column. Let this new matrix be called B

$$B = \begin{pmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{pmatrix}$$
$$M_{34} = \det(B)$$

We expand this along the 3$^{\text{rd}}$ row of B, since that is the one with most zeros.

$$M_{34} = \det(B) = b_{31}B_{31} + b_{32}B_{32} + b_{33}B_{33}$$

But $b_{31} = b_{33} = 0$. So the above simplifies to

$$\begin{aligned} M_{34} &= b_{32}B_{32} \\ &= b_{32}(-1)^{3+2} M_{32} \\ &= 4(-1)^5 M_{32} \\ &= -4M_{32} \end{aligned} \tag{2}$$

But

$$\begin{aligned} M_{32} &= \begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix} \\ &= 30 - 24 \\ &= 6 \end{aligned}$$

Therefore from (2), $M_{34} = -4(6) = -24$ and from (1)

$$\begin{aligned} \det(A) &= 3M_{34} \\ &= 3(-24) \end{aligned}$$

Hence

$$\boxed{\det(A) = -72}$$

4.8.2 Section 3.6 problem 8

Problem Evaluate determinant of

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{pmatrix}$$

after first simplifying the computation by adding multiple of some row of column to another.

solution The determinant of matrix do not change by adding multiple of one row or multiple of a column to another row or to another column. In the above, we see that adding the second row to the first row gives

$$B = \begin{pmatrix} 0 & 0 & 5 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{pmatrix}$$

Now, expanding on the first row, since that is the one with most zeros, gives

$$\det(B) = b_{11}B_{11} + b_{12}B_{12} + b_{13}B_{13}$$

But $b_{11} = b_{12} = 0$, hence

$$\begin{aligned} \det(B) &= b_{13}B_{13} \\ &= 5\,(-1)^{1+3} M_{13} \\ &= 5M_{13} \end{aligned}$$

But

$$M_{13} = \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} = -4 + 9 = 5$$

Hence $\det(B) = 5\,(5) = 25$. But since $\det(B) = \det(A)$, then

$$\boxed{\det(A) = 25}$$

4.8.3 Section 3.6 problem 19

Problem Use the method of elimination to evaluate the determinant of

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{pmatrix}$$

solution The idea is to use Forward elimination to produce an upper triangle matrix. The determinant of upper triangle matrix is then easily found as the product of elements on the diagonal. Since determinant do not change when adding multiple of a row to another, this method works. So we need first to produce the Echelon form (triangle matrix)

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{pmatrix} \xrightarrow{R_3=R_3+2R_1} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & -3 & 3 & 3 \end{pmatrix} \xrightarrow[R4=R_4+3R_2]{R_3=R_3-3R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & -3 & 3 \end{pmatrix} \xrightarrow{R_4=R_4+\frac{3}{4}R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & \frac{39}{4} \end{pmatrix}$$

Hence

$$\begin{aligned} \det(A) &= 1 \times 1 \times 4 \times \frac{39}{4} \\ &= 39 \end{aligned}$$

4.8.4 Section 3.6 problem 46

<u>Problem</u> Verify the property

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13}\\ a_{21}+ka_{22} & a_{22} & a_{23}\\ a_{31}+ka_{32} & a_{32} & a_{33}\end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

<u>solution</u> This property is saying that adding k times the second columns of A to the first column of A do not change the determinant. This is <u>property 5</u>. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ and let $B = \begin{pmatrix} a_{11}+ka_{12} & a_{12} & a_{13}\\ a_{21}+ka_{22} & a_{22} & a_{23}\\ a_{31}+ka_{32} & a_{32} & a_{33}\end{pmatrix}$. Then

$$\begin{aligned}\det(B) &= \begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13}\\ a_{21}+ka_{22} & a_{22} & a_{23}\\ a_{31}+ka_{32} & a_{32} & a_{33}\end{vmatrix}\\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + k\begin{vmatrix} a_{12} & a_{12} & a_{13}\\ a_{22} & a_{22} & a_{23}\\ a_{32} & a_{32} & a_{33}\end{vmatrix}\end{aligned}$$

But $k\begin{vmatrix} a_{12} & a_{12} & a_{13}\\ a_{22} & a_{22} & a_{23}\\ a_{32} & a_{32} & a_{33}\end{vmatrix} = 0$ since the first column is the same as the second column. Hence $\det(B) = \det(A)$. Now we will show this is true by actual expansion, since this is what the problem is asking. Expanding B along the first column, gives

$$\begin{aligned}\det(B) &= b_{11}B_{11} + b_{21}B_{21} + b_{31}B_{31}\\ &= (a_{11}+ka_{12})(-1)^{1+1}M_{11} + (a_{21}+ka_{22})(-1)^{2+1}M_{21} + (a_{31}+ka_{32})(-1)^{3+1}M_{31}\\ &= (a_{11}+ka_{12})M_{11} - (a_{21}+ka_{22})M_{21} + (a_{31}+ka_{32})M_{31}\\ &= (a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}) + k(a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31})\end{aligned}$$

But $(a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}) = \det(A)$, hence above becomes

$$\det(B) = \det(A) + k(a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31}) \tag{1}$$

But

$$\begin{aligned}a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31} &= a_{12}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{22}\begin{vmatrix} a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix} + a_{32}\begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}\\ &= a_{12}(a_{22}a_{33} - a_{23}a_{32}) - a_{22}(a_{12}a_{33} - a_{13}a_{32}) + a_{32}(a_{12}a_{23} - a_{13}a_{22})\\ &= \overbrace{a_{12}a_{22}a_{33}} - \widehat{a_{12}a_{23}a_{32}} - \overbrace{a_{22}a_{12}a_{33}} + \overleftrightarrow{a_{22}a_{13}a_{32}} + \widehat{a_{32}a_{12}a_{23}} - \overleftrightarrow{a_{32}a_{13}a_{22}}\end{aligned}$$

We see from the above, that all terms cancel out, and we obtain

$$a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31} = 0$$

Hence (1) becomes

$$\begin{aligned}\det(B) &= \det(A) + k(0)\\ &= \det(A)\end{aligned}$$

QED.

4.8.5 Section 3.6 problem 49

<u>Problem</u> Let $A = (a_{ij})$ be 3×3 matrix. Show that $\det(A^T) = \det(A)$ by expanding $\det(A)$ along its first row and $\det(A^T)$ along its first column.

solution Let

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Expanding $\det(A)$ along first row gives

$$\begin{aligned}
\det(A) &= a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}\\
&= a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{12}+a_{13}(-1)^{1+3}M_{13}\\
&= a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\\
&= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31}) \qquad (1)
\end{aligned}$$

But

$$B=A^{T}=\begin{pmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}$$

Finding $\det\left(A^{T}\right)$ by expanding along first column gives

$$\begin{aligned}
\det(B) &= b_{11}B_{11}+b_{21}B_{21}+b_{31}B_{31}\\
&= a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{21}+a_{13}(-1)^{1+3}M_{31}\\
&= a_{11}M_{11}-a_{12}M_{21}+a_{13}M_{31}\\
&= a_{11}\begin{vmatrix} a_{22} & a_{32}\\ a_{23} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31}\\ a_{23} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31}\\ a_{22} & a_{32}\end{vmatrix}\\
&= a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22}) \qquad (2)
\end{aligned}$$

Examining (1) and (2) shows that they are the same expression. Hence

$$\det(A)=\det\left(A^{T}\right)$$

QED.

4.8.6 Section 3.6 problem 52

Problem The square matrix A is called orthogonal provided that $A^{T}=A^{-1}$. Show that the determinant of such matrix must be either $+1$ or -1.

solution

We are given $A^{T}=A^{-1}$. Premultiplying both sides by A gives

$$\begin{aligned} AA^{T} &= AA^{-1}\\ AA^{T} &= I\end{aligned}$$

Taking the determinant of both sides gives

$$\det\left(AA^{T}\right)=\det(I)$$

But $\det(I)=1$ hence

$$\det\left(AA^{T}\right)=1$$

But $\det\left(AA^{T}\right)=\det\left(A^{T}\right)\det(A)$ by property of determinant of products, therefore the above becomes

$$\det\left(A^{T}\right)\det(A)=1$$

But by property of determinant, we know that $\det(A)=\det\left(A^{T}\right)$, therefore the above becomes

$$\begin{aligned}\det(A)\det(A)&=1\\ (\det(A))^{2}&=1\end{aligned}$$

Therefore

$$\det(A)=\pm 1$$

QED

4.8.7 Section 3.6 problem 53

Problem The matrices A, B are said to be similar provided that $A = P^{-1}BP$ for some invertible matrix P. Show that if A and B are similar then $|A| = |B|$

solution

Since

$$A = P^{-1}BP \tag{1}$$

Pre multiplying both sides by P gives

$$\begin{aligned} PA &= PP^{-1}BP \\ &= \left(PP^{-1}\right)BP \\ &= IBP \\ &= BP \end{aligned}$$

Now, taking determinant of both sides gives

$$\det(PA) = \det(BP)$$
$$\det(P)\det(A) = \det(B)\det(P)$$

Since P is invertible, then $\det(P) \neq 0$, therefore, we can divide both sides by $\det(P)$ and this gives

$$\det(A) = \det(B)$$

QED.

4.9 HW8

4.9.1 Section 4.1 problem 7 (page 237)

problem Determine if u, v are linearly dependent or not

$$\bar{u} = (2, 2)$$
$$\bar{v} = (2, -2)$$

solution

Two vectors $\bar{u}, \bar{v}$ are L.D if there exist scalars a, b, not both zero such that

$$a\bar{u} + b\bar{v} = \bar{0}$$
$$a\begin{pmatrix}2\\2\end{pmatrix} + b\begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}2 & 2\\2 & -2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

The above is now is in $Ax = 0$ format. The determinant of A is $|A| = -4 - 4 = -8$. Since $|A| \neq 0$, then a unique exist. Since $\bar{0}$ vector is is always solution to $Ax = 0$, and so it is the only solution here (since solution is unique). This means that only $a = 0, b = 0$ satisfy $a\bar{u} + b\bar{v} = \bar{0}$. Therefore, $\bar{u}, \bar{v}$ are linearly independent.

4.9.2 Section 4.1 problem 12

problem Express w as linear combination of u, v.

$$\bar{u} = (4, 1)$$
$$\bar{v} = (-2, -1)$$
$$\bar{w} = (2, -2)$$

solution

Need to find scalars a, b such that $a\bar{u} + b\bar{v} = \bar{w}$, hence

$$a\begin{pmatrix}4\\1\end{pmatrix} + b\begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$
$$\begin{pmatrix}4 & -2\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$

Applying Gaussian elimination

$$\begin{pmatrix}4 & -2 & 2\\1 & -1 & -2\end{pmatrix} \overset{R_2 = R_2 - \frac{1}{4}R_1}{\longrightarrow} \begin{pmatrix}4 & -2 & 2\\0 & -\frac{1}{2} & -\frac{5}{2}\end{pmatrix}$$

Hence, from last equation

$$-\frac{1}{2}b = -\frac{5}{2}$$
$$b = 5$$

From first equation

$$4a - 2b = 2$$
$$4a = 2(5) + 2$$
$$a = 3$$

Therefore

$$5\bar{u} - 3\bar{v} = \bar{w}$$

4.9.3 Section 4.1 problem 18

problem Apply theorem 4 (that is calculate a determinant) to determine whether the given vectors $\bar{u}, \bar{v}, \bar{w}$ are L.D. or L.I.

$$\begin{aligned}\bar{u} &= (1,1,0)\\ \bar{v} &= (4,3,1)\\ \bar{w} &= (3,-2,-4)\end{aligned}$$

solution

Let a, b, c be scalars, such that $a\bar{u} + b\bar{v} + c\bar{w} = \bar{0}$. The goal now is to determine a, b, c and see they are are all zero or not. Setting up $A\bar{x} = \bar{0}$ system gives

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}4\\3\\1\end{pmatrix} + c\begin{pmatrix}3\\-2\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4 & 3\\1 & 3 & -2\\0 & 1 & -4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now the $|A|$ is found. Subtracting row one from second row first, gives

$$\begin{pmatrix}1 & 4 & 3\\0 & -1 & -5\\0 & 1 & -4\end{pmatrix}$$

Performing cofactor expansion on the first column gives

$$\begin{aligned}|A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}\\ &= a_{11}A_{11}\\ &= a_{11}(-1)^{1+1}M_{11}\\ &= 1 \times M_{11}\\ &= M_{11}\\ &= \begin{vmatrix}-1 & -5\\1 & -4\end{vmatrix}\\ &= 4 + 5\\ &= 9\end{aligned}$$

Since $|A|$ is not zero, then solution of $Ax = 0$ is unique. Hence only solution is

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Which implies that $\bar{u}, \bar{v}, \bar{w}$ are linearly independent.

4.9.4 Section 4.1 problem 24

problem Use the method of example 3 to determine whether the given vectors $\bar{u}, \bar{v}, \bar{w}$ are L.D. or L.I. If they are L.D. then find scalars a, b, c not all zero such that $a\bar{u} + b\bar{v} + c\bar{w} = 0$

$$\begin{aligned}\bar{u} &= (1,4,5)\\ \bar{v} &= (4,2,5)\\ \bar{w} &= (-3,3,-1)\end{aligned}$$

solution

Let a,b,c be scalars, such that $a\bar{u}+b\bar{v}+c\bar{w}=\bar{0}$. Setting up $A\bar{x}=\bar{0}$ system gives

$$a\begin{pmatrix}1\\4\\5\end{pmatrix}+b\begin{pmatrix}4\\2\\5\end{pmatrix}+c\begin{pmatrix}-3\\3\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&4&-3\\4&2&3\\5&5&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Applying Gaussian elimination, $R_2=R_2-4R_1$ gives

$$\begin{pmatrix}1&4&-3\\0&-14&15\\5&5&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$R_3=R_3-5R_1$ gives

$$\begin{pmatrix}1&4&-3\\0&-14&15\\0&-15&14\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$R_3=R_3-\frac{15}{14}R_2$ gives

$$\begin{pmatrix}1&4&-3\\0&-14&15\\0&0&-\frac{29}{14}\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Since the final Echelon form has no zero pivot, therefore $|A|\neq 0$. This means the solution is unique. Hence

$$\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Which implies $\bar{u},\bar{v},\bar{w}$ are linearly independent.

4.9.5 Section 4.1 problem 28

problem Express vector t as linear combination of vectors u,v,w

$$\begin{aligned}\bar{t}&=(7,7,7)\\\bar{u}&=(2,5,3)\\\bar{v}&=(4,1,-1)\\\bar{w}&=(1,1,5)\end{aligned}$$

solution

Let $a\bar{u}+b\bar{v}+c\bar{w}=\bar{t}$, hence

$$a\begin{pmatrix}2\\5\\3\end{pmatrix}+b\begin{pmatrix}4\\1\\-1\end{pmatrix}+c\begin{pmatrix}1\\1\\5\end{pmatrix}=\begin{pmatrix}7\\7\\7\end{pmatrix}$$

$$\begin{pmatrix}2&4&1\\5&1&1\\3&-1&5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}7\\7\\7\end{pmatrix}$$

Applying Gaussian elimination. $R_2=R_2-\frac{5}{2}R_1$ gives

$$\begin{pmatrix}2&4&1\\0&-9&-\frac{3}{2}\\3&-1&5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}7\\-\frac{21}{2}\\7\end{pmatrix}$$

$R_3 = R_3 - \frac{3}{2}R_1$ gives

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -9 & -\frac{3}{2} \\ 0 & -7 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{21}{2} \\ -\frac{7}{2} \end{pmatrix}$$

$R_3 = R_3 - \frac{7}{9}R_2$ gives

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -9 & -\frac{3}{2} \\ 0 & 0 & \frac{14}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{21}{2} \\ \frac{14}{3} \end{pmatrix}$$

Hence

$$\begin{aligned} \frac{14}{3}c &= \frac{14}{3} \\ c &= 1 \end{aligned}$$

And from second row

$$\begin{aligned} -9b - \frac{3}{2}c &= -\frac{21}{2} \\ -9b - \frac{3}{2} &= -\frac{21}{2} \\ b &= 1 \end{aligned}$$

And from first row

$$\begin{aligned} 2a + 4b + c &= 7 \\ 2a + 4 + 1 &= 7 \\ a &= 1 \end{aligned}$$

Hence

$$\bar{t} = \bar{u} + \bar{v} + \bar{w}$$

4.9.6 Section 4.1 problem 31

problem Show that the given set V is closed under addition and under multiplication by scalars and is therefore subspace of $\mathbb{R}^3$. V is the set of all (x, y, z) such that $2x = 3y$

solution

What the above says, that given any vector in this space, such as $\bar{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ then $y_1 = \frac{2}{3}x_1$.

Hence any vector in this space can be written as $\bar{v}_1 = \begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix}$. Given two vectors $\bar{v}_1, \bar{v}_2$ in this space, the sum, should also be in this space. let $\bar{u} = \bar{v}_1 + \bar{v}_2$, therefore

$$\begin{aligned} \bar{u} &= \begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ \frac{2}{3}x_2 \\ z_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 \\ \frac{2}{3}x_1 + \frac{2}{3}x_2 \\ z_1 + z_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 \\ \frac{2}{3}(x_1 + x_2) \\ z_1 + z_2 \end{pmatrix} \end{aligned}$$

Hence $\bar{u}$ is also in this space, since its y coordinate is also $\frac{2}{3}$ of its x coordinate. Now check is made for multiplication by scaler. Let $\bar{u} = c\bar{v}$, hence

$$\bar{u} = c\begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ \frac{2}{3}cx_1 \\ z_1 \end{pmatrix}$$

Hence $\bar{u}$ is also in this space, since its y coordinate is also $\frac{2}{3}$ of its x coordinate. Therefore set V is closed under addition and under multiplication by scalars.

4.9.7 Section 4.1 problem 35

problem Show that the given set V is not a subspace of $\mathbb{R}^3$. V is the set of all (x, y, z) such that $z \geq 0$.

solution

The above set V is the upper half of the 3D space. (all vectors in the positive z part of 3D). But for this to be subspace, it must be closed under scalar multiplication. Let $\bar{u}$ be a vector in the set V. Multiplying this vector by $c = -1$, will result in this vector having negative z component, and it will therefore leave the set V. Therefore the set V is not closed under scalar multiplication. Hence V is not a subspace of $\mathbb{R}^3$.

4.9.8 Section 4.1 problem 40

problem Suppose that $\bar{u}, \bar{v}, \bar{w}$ are vectors in $\mathbb{R}^3$ such that $\bar{u}, \bar{v}$ are L.I. but $\bar{u}, \bar{v}, \bar{w}$ are L.D. Show that there exist scalars a, b such that $\bar{w} = a\bar{u} + b\bar{v}$

solution

Let $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \bar{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Consider the sum $a\bar{u}+b\bar{v}$. Since $\bar{u}, \bar{v}$ are L.I. then $a\bar{u}+b\bar{v}$ will produce non-zero vector unless a, b are both zero. Let this vector be $\bar{w}$. Hence $a\bar{u} + b\bar{v} = \bar{w}$. By definition, $\bar{w}$ is linear combination of $\bar{u}, \bar{v}$, hence the three vectors $\bar{u}, \bar{v}, \bar{w}$ are L.D.

A geometrical proof is as follows. Since $\bar{u}, \bar{v}$ are L.I. then they span a plane in 3D. This means $\bar{u}, \bar{v}$ are basis vector for this 2D plane inside $\mathbb{R}^3$. Now since $\bar{u}, \bar{v}, \bar{w}$ are L.D. then the vector $\bar{w}$ must also be in the same plane that $\bar{u}, \bar{v}$ are its basis. Hence the vector $\bar{w}$ can be expressed in terms of $\bar{u}, \bar{v}$. Therefore there exist a, b such that $a\bar{u} + b\bar{v} = \bar{w}$.

4.9.9 Section 4.2 problem 2 (page 244)

problem A subset W of some n space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^3$ such that $x_1 = 5x_2$

solution

From theorem 1, for the subset W to be subspace, it has to at least satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and since $x_1 = 5x_2$ therefore $\bar{x} = \begin{pmatrix} 5x_2 \\ x_2 \\ x_3 \end{pmatrix}$ Hence adding any two such vectors in this space gives

$$\begin{aligned}\bar{x}+\bar{y} &= \begin{pmatrix}5x_2\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}5y_2\\y_2\\y_3\end{pmatrix}\\ &= \begin{pmatrix}5x_2+5y_2\\x_2+y_2\\x_3+y_3\end{pmatrix}\\ &= \begin{pmatrix}5\left(x_2+y_2\right)\\x_2+y_2\\x_3+y_3\end{pmatrix}\end{aligned}$$

Therefore the sum is also in this set, since its first coordinate is also 5 times its second coordinate. Now scalar multiplication is checked for being closed. Let

$$\begin{aligned}c\bar{x} &= c\begin{pmatrix}5x_2\\x_2\\x_3\end{pmatrix}\\ &= \begin{pmatrix}5\left(cx_2\right)\\\left(cx_2\right)\\cx_3\end{pmatrix}\end{aligned}$$

Therefore multiplication by scalar is also in this set, since the first coordinate is also 5 times its second coordinate. Therefore W is subspace of $\mathbb{R}^3$

4.9.10 Section 4.2 problem 8

problem A subset W of some space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^2$ such that $x_1^2+x_2^2=0$.

solution

From theorem 1, for the subset W to be subspace, it has to satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}, \bar{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}$ be two such vectors such that where $x_1^2+x_2^2=0, y_1^2+y_2^2=0$. Hence adding any two such vectors in this space gives

$$\begin{aligned}\bar{x}+\bar{y} &= \begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}y_1\\y_2\end{pmatrix}\\ &= \begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}\end{aligned}$$

Checking if $\left(x_1+y_1\right)^2+\left(x_2+y_2\right)^2=0$ or not. Expanding

$$\left(x_1+y_1\right)^2+\left(x_2+y_2\right)^2=\left(x_1^2+y_1^2+2x_1y_1\right)+\left(x_2^2+y_2^2+2x_2y_2\right) \tag{1}$$

But $x_1^2=-x_2^2$ and $y_1^2=-y_2^2$ by definition. Substituting this into (1) gives

$$\begin{aligned}\left(x_1+y_1\right)^2+\left(x_2+y_2\right)^2 &= \left(-x_2^2-y_2^2+2x_1y_1\right)+\left(x_2^2+y_2^2+2x_2y_2\right)\\ &= 2\left(x_1y_1+x_2y_2\right)\end{aligned}$$

Now $x_1=ix_2$ and $y_1=\pm iy_2$. Hence the above becomes

$$\begin{aligned}\left(x_1+y_1\right)^2+\left(x_2+y_2\right)^2 &= 2\left(\left(\pm ix_2\right)\left(\pm iy_2\right)+x_2y_2\right)\\ &= 2\left(\left(ix_2\right)\left(iy_2\right)+x_2y_2\right)\\ &= 2\left(-x_2y_2+x_2y_2\right)\\ &= 0\end{aligned}$$

Therefore closed under multiplication. Checking now if closed under scalar multiplication.

$$c\bar{x} = c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}$$

Hence

$$(cx_1)^2 + (cx_2)^2 = c^2x_1^2 + c^2x_2^2$$
$$= c^2\left(x_1^2 + x_2^2\right)$$

But $x_1^2 + x_2^2 = 0$. Therefore $(cx_1)^2 + (cx_2)^2 = 0$ and it is closed under scalar multiplication as well. Therefore W is subspace of $\mathbb{R}^2$.

4.9.11 Section 4.2 problem 11

problem A subset W of some n space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^4$ such that $x_1 + x_2 = x_3 + x_4$.

solution

From theorem 1, for the subset W to be subspace, it has to satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ be two such vectors such that $x_1 + x_2 = x_3 + x_4$ and $y_1 + y_2 = y_3 + y_4$. Adding any two such vectors in this space gives

$$\bar{x} + \bar{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$
$$= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}$$

Checking now if $\left(x_1 + y_1\right) + \left(x_2 + y_2\right) = \left(x_3 + y_3\right) + \left(x_4 + y_4\right)$ or not.

$$\left(x_1 + y_1\right) + \left(x_2 + y_2\right) = (x_1 + x_2) + \left(y_1 + y_2\right)$$

But $x_1 + x_2 = x_3 + x_4$ and $y_1 + y_2 = y_3 + y_4$, therefore the above becomes

$$\left(x_1 + y_1\right) + \left(x_2 + y_2\right) = (x_3 + x_4) + \left(y_3 + y_4\right)$$

Hence closed under addition. Checking now if closed under scalar multiplication.

$$c\bar{x} = c\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
$$= \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \\ cx_4 \end{pmatrix}$$

Hence

$$(cx_1) + (cx_2) = c\,(x_1 + x_2)$$

But $x_1 + x_2 = x_3 + x_4$ hence

$$\begin{aligned}(cx_1) + (cx_2) &= c(x_3 + x_4)\\ &= (cx_3) + (cx_4)\end{aligned}$$

And therefore it is closed under scalar multiplication as well. Hence W is subspace of $\mathbb{R}^4$

4.9.12 Section 4.2 problem 16

problem Apply method of example 5 to find two solution vectors u, v such that the solution space is the set of all linear combinations of the form $su + tv$

$$\begin{aligned}x_1 - 4x_2 - 3x_3 - 7x_4 &= 0\\ 2x_1 - x_2 + x_3 + 7x_4 &= 0\\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0\end{aligned}$$

solution

$$\begin{pmatrix}1 & -4 & -3 & -7\\ 2 & -1 & 1 & 7\\ 1 & 2 & 3 & 11\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Let $R_2 = R_2 - 2R_1$, hence

$$\begin{pmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 1 & 2 & 3 & 11\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Let $R_3 = R_3 - R_1$, hence

$$\begin{pmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 0 & 6 & 6 & 18\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Let $R_3 = R_3 - \frac{6}{7}R_2$, hence

$$\begin{pmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Last row gives $0x_3 + 0x_4 = 0$. Therefore $x_4 = t, x_3 = s$ are the free parameters. From second row

$$\begin{aligned}7x_2 + 7x_3 + 21x_4 &= 0\\ 7x_2 &= -7s - 21t\\ x_2 &= -s - 3t\end{aligned}$$

From first equation

$$\begin{aligned}x_1 - 4x_2 - 3x_3 - 7x_4 &= 0\\ x_1 &= 4x_2 + 3x_3 + 7x_4\\ &= 4(-s - 3t) + 3s + 7t\\ &= -s - 5t\end{aligned}$$

Hence solution vector is

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s-5t \\ -s-3t \\ s \\ t \end{pmatrix} &= \begin{pmatrix} -s \\ -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ -3t \\ 0 \\ t \end{pmatrix} \\
&= s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \\
&= s\bar{u} + t\bar{v}
\end{aligned}
$$

4.9.13 Section 4.2 problem 22

problem Reduce the given system to echelon form to find a single solution vector u such that the solution space is the set of all scalar multiples of u

$$
\begin{aligned}
x_1 + 3x_2 + 3x_3 + 3x_4 &= 0 \\
2x_1 + 7x_2 + 5x_3 - x_4 &= 0 \\
2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0
\end{aligned}
$$

solution

$$
\begin{pmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$R_2 = R_2 - 2R_1$ gives

$$
\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 2 & 7 & 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$R_2 = R_3 - 2R_1$ gives

$$
\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -2 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$R_3 = R_3 - R_2$ gives

$$
\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Last row gives $-x_3 - 3x_4 = 0$. Therefore let $x_4 = t$ be the free parameter. Hence $x_3 = -3t$. From second equation

$$
\begin{aligned}
x_2 - x_3 - 7x_4 &= 0 \\
x_2 &= x_3 + 7x_4 \\
&= -3t + 7t \\
&= 4t
\end{aligned}
$$

And from first equation

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0 \\ x_1 &= -3x_2 - 3x_3 - 3x_4 \\ &= -3(4t) - 3(-3t) - 3t \\ &= -6t \end{aligned}$$

Hence solution vector is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -6t \\ 4t \\ -3t \\ t \end{pmatrix} \\ &= t \begin{pmatrix} -6 \\ 4 \\ -3 \\ 1 \end{pmatrix} \\ &= t\bar{u} \end{aligned}$$

4.9.14 Section 4.2 problem 26

problem Prove: If $\bar{u}$ is a fixed vector in vector space V, then the set of W of all scalar multiples $c\bar{u}$ of $\bar{u}$ is subspace of V

solution

Let

$$W = \{c\bar{u}\}$$

Closure under addition gives

$$\begin{aligned} c_1\bar{u} + c_2\bar{u} &= (c_1 + c_2)\bar{u} \\ &= b\bar{u} \end{aligned}$$

Where $b = c_1 + c_2$. Hence closed under additions. Closure under multiplication by scalar b gives

$$\begin{aligned} bc\bar{u} &= (bc)\bar{u} \\ &= C_1\bar{u} \end{aligned}$$

Where $C_1 = bc$ a new constant. Hence closed under multiplication by scalar b. Therefore W is subspace of V

4.9.15 Section 4.2 problem 27

problem Let $\bar{u}$ and $\bar{v}$ be fixed vectors in vector space V. Show that the set W of all linear combinations $a\bar{u} + b\bar{v}$ is subspace of V

solution

The set W is $W = \{a\bar{u} + b\bar{v}\}$ which is linear combinations of $\bar{u}$ and $\bar{v}$ where a, b are arbitrary scalar. Closure under addition gives

$$(a_1\bar{u} + b_1\bar{v}) + (a_2\bar{u} + b_2\bar{v}) = (a_1 + a_2)\bar{u} + (b_1 + b_2)\bar{v}$$

But $a_1 + a_2$ and $b_1 + b_2$ are arbitrary scalars, say C_1, C_2 respectively. Hence the above becomes $C_1\bar{u} + C_2\bar{v}$ and this is in W. Hence W is closed under addition. Closure under multiplication by scalar c gives

$$c(a\bar{u} + b\bar{v}) = ca\bar{u} + cb\bar{v}$$

But ca and cb are arbitrary scalars, say C_1, C_2 respectively. Hence the above becomes $C_1\bar{u} + C_2\bar{v}$ and this is in W. Therefore W is subspace of V.

4.9.16 Section 4.2 problem 28

problem Suppose A is $n \times n$ matrix and k is constant scalar. Show that the set of all vectors $\bar{x}$ such that $A\bar{x} = k\bar{x}$ is subspace of $\mathbb{R}^n$

solution

Let $W = \{\bar{x}\}$ where $A\bar{x} = k\bar{x}$. To determine if closed under addition, we consider the vector $\bar{x}_1 + \bar{x}_2$. This vector should also satisfy $A(\bar{x}_1 + \bar{x}_2) = k(\bar{x}_1 + \bar{x}_2)$ for it to be closed. Let us check if this is the case or not.

$$\begin{aligned} A(\bar{x}_1 + \bar{x}_2) &= A\bar{x}_1 + A\bar{x}_2 \\ &= k\bar{x}_1 + k\bar{x}_2 \\ &= k(\bar{x}_1 + \bar{x}_2) \end{aligned}$$

Hence it is closed under addition. We will now check closure under scalar multiplication.

$$\begin{aligned} A(c\bar{x}_1) &= cA\bar{x}_1 \\ &= ck\bar{x} \\ &= k(c\bar{x}) \end{aligned}$$

Hence closed under scalar multiplication. Therefore W is subspace of V.

4.9.17 Section 4.3 problem 6 (page 252)

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$\begin{aligned} \bar{v}_1 &= (1,0,0) \\ \bar{v}_2 &= (1,1,0) \\ \bar{v}_3 &= (1,1,1) \end{aligned}$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ gives

$$\begin{aligned} c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) &= (0,0,0) \\ (c_1 + c_2 + c_3, c_2 + c_3, c_3) &= (0,0,0) \end{aligned}$$

Hence $c_3 = 0$ and $c_2 = 0$ and $c_1 = 0$ is the only solution. Therefore definition of linear independence (page 248), the vectors are linearly independent.

4.9.18 Section 4.3 problem 7

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$\begin{aligned} v_1 &= (2,1,0,0) \\ v_2 &= (3,0,1,0) \\ v_3 &= (4,0,0,1) \end{aligned}$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ gives

$$\begin{aligned} c_1(2,1,0,0) + c_2(3,0,1,0) + c_3(4,0,0,1) &= (0,0,0,0) \\ (2c_1 + 3c_2 + 4c_3, c_1, c_2, c_3) &= (0,0,0,0) \end{aligned}$$

Therefore, we see by inspection (comparing terms) that $c_3 = 0, c_2 = 0, c_1 = 0$. Therefore definition of linear independence (page 248), the vectors are linearly independent.

4.9.19 Section 4.3 problem 15

problem Express the indicated vector w as linear combination of the given vectors v_i if this is possible. If not, show it is impossible

$$\begin{aligned}\bar{w} &= (4,5,6)\\ \bar{v}_1 &= (2,-1,4)\\ \bar{v}_2 &= (3,0,1)\\ \bar{v}_3 &= (1,2,-1)\end{aligned}$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{w}$ gives (in matrix form)

$$\begin{pmatrix} 2 & 3 & 1\\ -1 & 0 & 2\\ 4 & 1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix}$$

We now solve for c_1, c_2, c_3. Let $R_2 = R_2 + \frac{1}{2}R_1$ therefore

$$\begin{pmatrix} 2 & 3 & 1\\ 0 & \frac{3}{2} & \frac{5}{2}\\ 4 & 1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}4\\7\\6\end{pmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{pmatrix} 2 & 3 & 1\\ 0 & \frac{3}{2} & \frac{5}{2}\\ 0 & -5 & -3\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}4\\7\\-2\end{pmatrix}$$

$R_3 = R_3 - \frac{10}{3}R_2$ gives

$$\begin{pmatrix} 2 & -1 & 4\\ 0 & \frac{3}{2} & \frac{5}{2}\\ 0 & 0 & \frac{16}{3}\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}4\\7\\\frac{64}{3}\end{pmatrix}$$

Therefore, since there are no zero pivots at end of forward Gaussian elimination, the solution is unique and not zero. (by backward substitution,

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}3\\-2\\4\end{pmatrix}$$

Hence

$$\begin{aligned}\bar{w} &= c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3\\ &= 3\bar{v}_1 - 2\bar{v}_2 + 4\bar{v}_3\end{aligned}$$

4.9.20 Section 4.3 problem 20

problem Three vectors v_1, v_2, v_3 are given. If they are L.I., show this. Otherwise, find a nontrivial linear combination of them that is equal to the zero vector.

$$\begin{aligned}\bar{v}_1 &= (1,1,-1,1)\\ \bar{v}_2 &= (2,1,1,1)\\ \bar{v}_3 &= (3,1,4,1)\end{aligned}$$

solution

Here the space is $\mathbb{R}^4$, but only 3 vectors are given. Therefore theorem 3 at page 252 is used. This theorem says that, if we set the A matrix, with its columns as the given vectors above, then the vectors are L.I. iff there is a 3×3 submatrix inside A which has nonzero determinant. To show this, Gaussian eliminating is used.

$$A=\begin{pmatrix}1&2&3\\1&1&1\\-1&1&4\\1&1&1\end{pmatrix}\xrightarrow{R_2=R_2-R_1}\begin{pmatrix}1&2&3\\0&-1&-2\\-1&1&4\\1&1&1\end{pmatrix}\xrightarrow{R_3=R_3+R_1}\begin{pmatrix}1&2&3\\0&-1&-2\\0&3&7\\1&1&1\end{pmatrix}\xrightarrow{R_4=R_4-R_1}$$

$$\begin{pmatrix}1&2&3\\0&-1&-2\\0&3&7\\0&-1&-2\end{pmatrix}\xrightarrow{R_3=R_3+3R_2}\begin{pmatrix}1&2&3\\0&-1&-2\\0&0&1\\0&-1&-2\end{pmatrix}\xrightarrow{R_4=R_4-R_2}\begin{pmatrix}1&2&3\\0&-1&-2\\0&0&1\\0&0&0\end{pmatrix}$$

The above shows that there is a submatrix of size 3×3 which has nonzero determinant. It is the matrix of the first 3 rows

$$\begin{pmatrix}1&2&3\\0&-1&-2\\0&0&1\end{pmatrix}$$

This has nonzero determinant. Since it is diagonal, its determinant is the product of diagonal elements. Since no diagonal element is zero, the determinant is not zero. This implies vectors are linearly independent.

4.9.21 Section 4.3 problem 24

problem The vectors $\bar{v}_i$ are known to be L.I., apply the definition of L.I. to show that the vectors u_i are also L.I.

$$\begin{aligned}\bar{u}_1 &= \bar{v}_1+\bar{v}_2\\ \bar{u}_2 &= 2\bar{v}_1+3\bar{v}_2\end{aligned}$$

solution

We will examine

$$a\bar{u}_1+b\bar{u}_2=\bar{0}$$

To see if this is satisfied only for $a=0,b=0$.

$$\begin{aligned}a\bar{u}_1+b\bar{u}_2&=\bar{0}\\ a\left(\bar{v}_1+\bar{v}_2\right)+b\left(2\bar{v}_1+3\bar{v}_2\right)&=\bar{0}\\ \bar{v}_1\left(a+2b\right)+\bar{v}_2\left(a+3b\right)&=\bar{0}\end{aligned}$$

But since we are told that $\bar{v}_1,\bar{v}_2$ are L.I., then this implies that $a+2b=0$ and $a+3b=0$. These two equations we solve now for a,b. These two equations show that $2b=3b$, which means $b=0$. Hence $a=0$ as well. Therefore only solution for $a\bar{u}_1+b\bar{u}_2=\bar{0}$ is that $a=b=0$. This is the same as saying $\bar{u}_1,\bar{u}_2$ are linearly independent.

QED

4.10 HW9

4.10.1 Section 4.3 problem 6 (page 252)

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$\bar{v}_1 = (1,0,0)$$
$$\bar{v}_2 = (1,1,0)$$
$$\bar{v}_3 = (1,1,1)$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ gives

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (0,0,0)$$
$$(c_1 + c_2 + c_3, c_2 + c_3, c_3) = (0,0,0)$$

Hence $c_3 = 0$ and $c_2 = 0$ and $c_1 = 0$ is the only solution. Therefore definition of linear independence (page 248), the vectors are linearly independent.

4.10.2 Section 4.3 problem 7

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$v_1 = (2,1,0,0)$$
$$v_2 = (3,0,1,0)$$
$$v_3 = (4,0,0,1)$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ gives

$$c_1(2,1,0,0) + c_2(3,0,1,0) + c_3(4,0,0,1) = (0,0,0,0)$$
$$(2c_1 + 3c_2 + 4c_3, c_1, c_2, c_3) = (0,0,0,0)$$

Therefore, we see by inspection (comparing terms) that $c_3 = 0, c_2 = 0, c_1 = 0$. Therefore definition of linear independence (page 248), the vectors are linearly independent.

4.10.3 Section 4.3 problem 15

problem Express the indicated vector w as linear combination of the given vectors v_i if this is possible. If not, show it is impossible

$$\bar{w} = (4,5,6)$$
$$\bar{v}_1 = (2,-1,4)$$
$$\bar{v}_2 = (3,0,1)$$
$$\bar{v}_3 = (1,2,-1)$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{w}$ gives (in matrix form)

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

We now solve for c_1, c_2, c_3. Let $R_2 = R_2 + \frac{1}{2}R_1$ therefore

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & -5 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$$

$R_3 = R_3 - \frac{10}{3}R_2$ gives

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{16}{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ \frac{64}{3} \end{pmatrix}$$

Therefore, since there are no zero pivots at end of forward Gaussian elimination, the solution is unique and not zero. (by backward substitution,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

Hence

$$\begin{aligned} \bar{w} &= c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 \\ &= 3\bar{v}_1 - 2\bar{v}_2 + 4\bar{v}_3 \end{aligned}$$

4.10.4 Section 4.3 problem 20

problem Three vectors v_1, v_2, v_3 are given. If they are L.I., show this. Otherwise, find a nontrivial linear combination of them that is equal to the zero vector.

$$\begin{aligned} \bar{v}_1 &= (1,1,-1,1) \\ \bar{v}_2 &= (2,1,1,1) \\ \bar{v}_3 &= (3,1,4,1) \end{aligned}$$

solution

Here the space is $\mathbb{R}^4$, but only 3 vectors are given. Therefore theorem 3 at page 252 is used. This theorem says that, if we set the A matrix, with its columns as the given vectors above, then the vectors are L.I. iff there is a 3×3 submatrix inside A which has nonzero determinant. To show this, Gaussian eliminating is used.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 3 & 7 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_4=R_4-R_1}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 3 & 7 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_3=R_3+3R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_4=R_4-R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The above shows that there is a submatrix of size 3×3 which has nonzero determinant. It is the matrix of the first 3 rows

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

This has nonzero determinant. Since it is diagonal, its determinant is the product of diagonal elements. Since no diagonal element is zero, the determinant is not zero. This implies vectors are linearly independent.

4.10.5 Section 4.3 problem 24

problem The vectors $\bar{v}_i$ are known to be L.I., apply the definition of L.I. to show that the vectors u_i are also L.I.

$$
\begin{aligned}
\bar{u}_1 &= \bar{v}_1 + \bar{v}_2 \\
\bar{u}_2 &= 2\bar{v}_1 + 3\bar{v}_2
\end{aligned}
$$

solution

We will examine

$$a\bar{u}_1 + b\bar{u}_2 = \bar{0}$$

To see if this is satisfied only for $a = 0, b = 0$.

$$
\begin{aligned}
a\bar{u}_1 + b\bar{u}_2 &= \bar{0} \\
a(\bar{v}_1 + \bar{v}_2) + b(2\bar{v}_1 + 3\bar{v}_2) &= \bar{0} \\
\bar{v}_1(a + 2b) + \bar{v}_2(a + 3b) &= \bar{0}
\end{aligned}
$$

But since we are told that $\bar{v}_1, \bar{v}_2$ are L.I., then this implies that $a + 2b = 0$ and $a + 3b = 0$. These two equations we solve now for a, b. These two equations show that $2b = 3b$, which means $b = 0$. Hence $a = 0$ as well. Therefore only solution for $a\bar{u}_1 + b\bar{u}_2 = \bar{0}$ is that $a = b = 0$. This is the same as saying $\bar{u}_1, \bar{u}_2$ are linearly independent.

QED

4.11 HW10

4.11.1 Section 5.1 problem 10 (page 299)

problem

Verify that y_1, y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the initial conditions. $y'' - 10y' + 25y = 0$ with $y_1 = e^{5x}, y_2 = xe^{5x}$ and $y(0) = 3, y'(0) = 13$

solution

To verify that y_1 or y_2 is solution to the ODE, we plug it into the ODE and see if it gives zero, which is what the RHS is. Since $y_1' = 5e^{5x}, y_1'' = 25e^{5x}$, then substituting this into the ODE gives

$$\begin{aligned} y_1'' - 10y_1' + 25y_1 &= 0 \\ 25e^{5x} - 10\left(5e^{5x}\right) + 25\left(e^{5x}\right) &= 0 \\ 25e^{5x} - 50e^{5x} + 25e^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Now we do the same for y_2. Since $y_2' = e^{5x} + 5xe^{5x}, y_2'' = 5e^{5x} + 5e^{5x} + 25xe^{5x}$, then substituting this into the ODE gives

$$\begin{aligned} y_2'' - 10y_2' + 25y_2 &= 0 \\ \left(5e^{5x} + 5e^{5x} + 25xe^{5x}\right) - 10\left(e^{5x} + 5xe^{5x}\right) + 25\left(xe^{5x}\right) &= 0 \\ 5e^{5x} + 5e^{5x} + 25xe^{5x} - 10e^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 25xe^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Therefore the general solution is

$$y(x) = c_1 y_1(x) + c_2 y(x)$$

Where the constants are found from initial conditions. Using the first initial condition gives

$$\begin{aligned} y(0) &= 3 \\ c_1 y_1(0) + c_2 y_2(0) &= 3 \\ c_1\left(e^{5x}\right)_{x=0} + c_2\left(xe^{5x}\right)_{x=0} &= 3 \\ c_1 &= 3 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y(x) &= 3y_1(x) + c_2 y_2(x) \\ y' &= 3y_1' + c_2 y_2' \\ &= 3\left(5e^{5x}\right) + c_2\left(e^{5x} + 5xe^{5x}\right) \end{aligned}$$

Applying the second boundary conditions gives

$$
\begin{aligned}
y'(0) &= 13 \\
3\left(5e^{5x}\right)_{x=0} + c_2\left(e^{5x} + 5xe^{5x}\right)_{x=0} &= 13 \\
3(5) + c_2 &= 13 \\
c_2 &= 13 - 15 \\
&= -2
\end{aligned}
$$

Therefore the particular solution is

$$
\begin{aligned}
y(x) &= c_1 y_1(x) + c_2 y(x) \\
&= 3y_1(x) - 2y(x) \\
&= 3e^{5x} - 2xe^{5x} \\
&= e^{5x}(3 - 2x)
\end{aligned}
$$

4.11.2 Section 5.1 problem 19

problem Show that $y_1 = 1, y_2 = \sqrt{x}$ are solutions to $yy'' + \left(y'\right)^2 = 0$ but that their sum $y = y_1 + y_2$ is not a solution

solution To show that y_1 and y_2 are solution to the ODE, we plug them into the ODE and see if the result is the same as the RHS. Since $y_1 = 1$ then $y_1' = 0, y_1'' = 0$. Then ODE becomes

$$
\begin{aligned}
y_1 y_1'' + \left(y_1'\right)^2 &= 0 \\
1(0) + 0 &= 0 \\
0 &= 0
\end{aligned}
$$

Hence verified. For y_2, we have $y_2' = \frac{1}{2x^{\frac{1}{2}}}, y_2'' = \frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}$. Hence the ODE becomes

$$
\begin{aligned}
y_2 y_2'' + \left(y_2'\right)^2 &= 0 \\
x^{\frac{1}{2}}\left(\frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}\right) + \left(\frac{1}{2x^{\frac{1}{2}}}\right)^2 &= 0 \\
\left(\frac{-1}{4}\frac{1}{x}\right) + \left(\frac{1}{4x}\right) &= 0 \\
0 &= 0
\end{aligned}
$$

Hence verified. Now we plugin the sum into the ODE.

$$
\begin{aligned}
\left(y_1 + y_2\right)\left(y_1 + y_2\right)'' + \left(\left(y_1 + y_2\right)'\right)^2 &= 0 \\
\left(y_1 + y_2\right)\left(y_1'' + y_2''\right) + \left(y_1' + y_2'\right)^2 &= 0 \\
\left(y_1 y_1'' + y_1 y_2''\right) + \left(y_2 y_1'' + y_2 y_2''\right) + \left(y_1'\right)^2 + \left(y_2'\right)^2 + 2y_1' y_2' &= 0 \\
y_1 y_1'' + y_1 y_2'' + y_2 y_1'' + y_2 y_2'' + \left(y_1'\right)^2 + \left(y_2'\right)^2 + 2y_1' y_2' &= 0
\end{aligned}
$$

But we found that $y_1 y_1'' + \left(y_1'\right)^2 = 0$ and $y_2 y_2'' + \left(y_2'\right)^2 = 0$ from earlier. Using these into the LHS of the above simplifies it to

$$
y_1 y_2'' + y_2 y_1'' + 2y_1' y_2' = 0
$$

But $y_2'' = \frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}, y_1'' = 0, y_1' = 0, y_1 = 1$, then the above becomes

$$\frac{-1}{4}\frac{1}{x^{\frac{3}{2}}} = 0$$

We see that the LHS is not zero. Hence $y_1 + y_2$ is not a solution to the ODE.

4.11.3 Section 5.1 problem 24

problem Determine whether the pairs of functions are linearly independent or not on the real line. $f(x) = \sin^2 x, g(x) = 1 - \cos 2x$

solution The two functions are L.I. if $c_1 f(x) + c_1 g(x) = 0$ for each x, only when $c_1 = c_2 = 0$. Or stated differently, two functions are L.D. if there exist c_1, c_2 not all zero, such that $c_1 f(x) + c_1 g(x) = 0$ for each x. To show this, we set up the Wronskian W and see if it is zero or not. If $W = 0$ then this mean that the functions are L.D.

$$\begin{aligned} W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\ \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix} \\ = 2\sin^2 x \sin 2x - (1 - \cos 2x)(2\sin x \cos x) \\ = 2\sin^2 x \sin 2x - 2\sin x \cos x + 2\cos 2x \sin x \cos x \end{aligned}$$

The RHS of the above simplifies to 0.

$$W = 0$$

Therefore, the functions are linearly dependent.

4.11.4 Section 5.1 problem 26

problem Determine whether the pairs of functions are linearly independent or not on the real line. $f(x) = 2\cos x + 3\sin x, g(x) = 3\cos x - 2\sin x$

solution To show this, we set up the Wronskian W and see if it is zero or not. If $W = 0$ then this mean that the functions are L.D.

$$\begin{aligned} W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\ \begin{vmatrix} 2\cos x + 3\sin x & 3\cos x - 2\sin x \\ -2\sin x + 3\cos x & -3\sin x - 2\cos x \end{vmatrix} \\ = (2\cos x + 3\sin x)(-3\sin x - 2\cos x) - (3\cos x - 2\sin x)(-2\sin x + 3\cos x) \\ = -13\cos^2 x - 13\sin^2 x \\ = -13\left(\cos^2 x + \sin^2 x\right) \\ = -13 \end{aligned}$$

Since $W \neq 0$ then the functions are Linearly independent.

4.11.5 Section 5.1 problem 27

problem Ley y_p be a particular solution of the nonhomogeneous equation $y'' + py' + qy = f(x)$ and let y_h be the homogenous solution. Show that $y = y_h + y_p$ is a solution of the given ODE.

solution since y_h satisfies the homogenous ODE then we can write

$$y_h'' + py_h' + qy_h = 0 \tag{1}$$

And since y_p satisfies the nonhomogeneous ODE then we can write

$$y_p'' + py_p' + qy_p = f(x) \tag{2}$$

Adding (1)+(2) gives

$$\left(y_p'' + y_h''\right) + p\left(y_p' + y_p'\right) + q\left(y_p + y_h\right) = f(x)$$

But due to linearity of differentiation, then the above can be written as

$$\left(y_p + y_h\right)'' + p\left(y_p + y_p\right)' + q\left(y_p + y_h\right) = f(x)$$

Let $Y = y_p + y_h$ then

$$Y'' + pY' + qY = f(x)$$

Therefore we showed that $Y = y_p + y_h$ satisfies the original ODE, hence it is a solution. QED

4.11.6 Section 5.1 problem 31

problem Show that $y_1 = \sin x^2$ and $y = \cos x^2$ are L.I. functions, but their Wronskian vanishes are $x = 0$. Why does this implies that there is no differential equation of the form $y'' + p(x)y' + q(x)y = 0$ with both p, q continuous everywhere, having both y_1, y_2 are solutions?

solution

$$\begin{aligned} W = \begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix} &= \\ \begin{vmatrix} \sin x^2 & \cos x^2 \\ (2x)\cos x^2 & -(2x)\sin x^2 \end{vmatrix} \\ &= -2x\sin x^2 \sin x^2 - 2x\cos^2 \cos x^2 \\ &= -2x\left(\left(\sin x^2\right)^2 + \left(\cos x^2\right)^2\right) \\ &= -2x \end{aligned}$$

The Wronskian is zero at $x = 0$ but not zero at other points. It is only when $W = 0$ everywhere, we say that y_1, y_2 are L.D. We can have L.I. functions, but also have $W(x_0) = 0$ at some x_0 as in this problem. What this mean, is that $x = 0$ can not be in the domain of the solution for y_1, y_2 to be solutions to the ODE. Hence, since the domain of the solution is everywhere, this means $x = 0$ is part of the domain, then we conclude that y_1, y_2 can not be both solutions, since they are L.I. at $x = 0$.

4.11.7 Section 5.1 problem 32

problem Let y_1, y_2 be two solutions of $A(x)y'' + B(x)y' + C(x)y = 0$ on open interval I where A, B, C are continuous and $A(x)$ is never zero. (a) Let $W = W\left(y_1, y_2\right)$. Show that $A(x)\frac{dW}{dx} = y_1\left(Ay_2''\right) - y_2\left(Ay_1''\right)$ then substitute for Ay_2'' and Ay_1'' from the original ODE to show that $A(x)\frac{dW}{dx}=-B(x)W(x)$ (b) Solve this first order ODE equation to deduce Abel's formula $W(x)=k\exp\left(-\int\frac{B(x)}{A(x)}dx\right)$ where k is constant. (c) Why does Abel's formula imply that the Wronskian $W(y_1,y_2)$ is either zero everywhere or non-zero everywhere (as stated in theorem 3)?

solution

4.11.7.1 Part (a)

By definition

$$W(x)=y_1y_2'-y_2y_1'$$

Hence

$$\begin{aligned}\frac{dW}{dx}&=y_1'y_2'+y_1y_2''-y_2'y_1'-y_2y_1''\\&=y_1y_2''-y_2y_1''\end{aligned}$$

Therefore

$$\begin{aligned}A(x)\frac{dW}{dx}&=A(x)\left(y_1y_2''-y_2y_1''\right)\\&=y_1\left(A(x)y_2''\right)-y_2\left(A(x)y_1''\right)\end{aligned}\tag{1}$$

But from original ODE, $A(x)y_1''+B(x)y_1'+C(x)y_1=0$, therefore

$$A(x)y_1''=-B(x)y_1'-C(x)y_1\tag{2}$$

And also from original ODE, $A(x)y_2''+B(x)y_2'+C(x)y_2=0$, therefore

$$A(x)y_2''=-B(x)y_2'-C(x)y_2\tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}A(x)\frac{dW}{dx}&=y_1\left(-B(x)y_2'-C(x)y_2\right)-y_2\left(-B(x)y_1'-C(x)y_1\right)\\&=-B(x)y_1y_2'-C(x)y_1y_2+B(x)y_2y_1'+C(x)y_2y_1\\&=-B(x)y_1y_2'+B(x)y_2y_1'\\&=-B(x)\left(y_1y_2'-y_2y_1'\right)\\&=-B(x)W(x)\end{aligned}\tag{4}$$

QED.

4.11.7.2 Part (b)

Solving (4).

$$\begin{aligned}A(x)\frac{dW}{dx}+B(x)W(x)&=0\\\frac{dW}{dx}+\frac{B(x)}{A(x)}W(x)&=0\end{aligned}$$

Integrating factor is $\mu=e^{\int\frac{B(x)}{A(x)}dx}$, hence the above becomes

$$\frac{d}{dx}\left(\mu W(x)\right)=0$$

Integrating gives

$$\mu W(x) = k$$
$$W(x) = ke^{-\int \frac{B(x)}{A(x)} dx}$$

4.11.7.3 Part (c)

Since an exponential function is never zero (for bounded $\frac{B(x)}{A(x)}$), then $W(x) = ke^{(\cdot)}$ can only be zero if $k = 0$. This makes $W = 0$ everywhere when $k = 0$. But if $k \neq 0$, then $W \neq 0$ everywhere. So W can only be zero everywhere, or not zero everywhere.

4.11.8 Section 5.1 problem 34

problem Apply theorem 5 and 6 to find general solutions of the differential equation $y'' + 2y' - 15y = 0$

solution The characteristic equation is $r^2 + 2r - 15 = 0$, and the roots are

$$r_1 = 3$$
$$r_2 = -5$$

Therefore the solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
$$= c_1 e^{3x} + c_2 e^{-5x}$$

4.11.9 Section 5.1 problem 42

problem Apply theorem 5 and 6 to find general solutions of the differential equation $35y'' - y' - 12y = 0$

solution The characteristic equation is $35r^2 - r - 12 = 0$, and the roots are

$$r_1 = \frac{3}{5}$$
$$r_2 = -\frac{4}{7}$$

Therefore the solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
$$= c_1 e^{\frac{3}{5}x} + c_2 e^{-\frac{4}{7}x}$$

4.11.10 Section 5.1 problem 48

problem Problem gives a general solution $y(x)$ of a homogeneous second order ODE $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation $y(x) = e^x \left(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}}\right)$

solution We compare the above solution to the general form of the solution given by

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
$$= c_1 e^{x\left(1+\sqrt{2}\right)} + c_2 e^{x\left(1-\sqrt{2}\right)}$$

We see that

$$r_1 = 1 + \sqrt{2}$$
$$r_2 = 1 - \sqrt{2}$$

This implies that the characteristic equation is

$$
\begin{aligned}
(r - r_1)(r - r_2) &= 0 \\
\left(r - \left(1 + \sqrt{2}\right)\right)\left(r - \left(1 - \sqrt{2}\right)\right) &= 0 \\
r^2 - 2r - 1 &= 0
\end{aligned}
$$

Therefore the ODE is

$$y'' - 2y' - y = 0$$

Where $a = 1, b = -2, c = -1$.

4.12 HW11

4.12.1 Section 5.2 problem 12 (page 311)

Problem: Use the Wronskian to prove that the given functions are linearly independent on the given interval. $f(x) = x, g(x) = \cos(\ln x), h(x) = \sin(\ln x)$ for $x > 0$

solution The Wronskian is

$$
\begin{aligned}
W(x) &= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \\
&= \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\sin(\ln x)\frac{1}{x} & \cos(\ln x)\frac{1}{x} \\ 0 & -\cos(\ln x)\frac{1}{x^2} + \sin(\ln)\frac{1}{x^2} & -\sin(\ln x)\frac{1}{x^2} - \cos(\ln x)\frac{1}{x^2} \end{vmatrix}
\end{aligned}
$$

Expanding along the last row

$$
\begin{aligned}
W(x) &= W_{32}(-1)^{3+2} A_{32} + W_{33}(-1)^{3+3} A_{33} \\
&= -\left(-\cos(\ln x)\frac{1}{x^2} + \sin(\ln)\frac{1}{x^2}\right)\begin{vmatrix} x & \sin(\ln x) \\ 1 & \cos(\ln x)\frac{1}{x} \end{vmatrix} + \left(-\sin(\ln x)\frac{1}{x^2} - \cos(\ln x)\frac{1}{x^2}\right)\begin{vmatrix} x & \cos(\ln x) \\ 1 & -\sin(\ln x)\frac{1}{x} \end{vmatrix} \\
&= \left(\cos(\ln x)\frac{1}{x^2} - \sin(\ln)\frac{1}{x^2}\right)(\cos(\ln x) - \sin(\ln x)) + \left(\sin(\ln x)\frac{1}{x^2} + \cos(\ln x)\frac{1}{x^2}\right)(\sin(\ln x) + \cos(\ln x))
\end{aligned}
$$

Let $\sin(\ln)\frac{1}{x^2} = A, \cos(\ln x)\frac{1}{x^2} = B$, $\cos(\ln x) = a, \sin(\ln x) = b$ then the above is

$$
\begin{aligned}
W(x) &= (B - A)(a - b) + (A + B)(b + a) \\
&= 2Ab + 2Ba
\end{aligned}
$$

Transforming back

$$
\begin{aligned}
W(x) &= 2\sin(\ln)\frac{1}{x^2}\sin(\ln x) + 2\cos(\ln x)\frac{1}{x^2}\cos(\ln x) \\
&= 2\sin^2(\ln)\frac{1}{x^2} + 2\cos^2(\ln x)\frac{1}{x^2} \\
&= \frac{2}{x^2}
\end{aligned}
$$

Hence, for $x > 0$ the Wronskian is not zero. Therefore the functions are L.I.

4.12.2 Section 5.2 problem 16

Problem: A third order ODE is given, and three L.I. solutions are given. Find a particular solution satisfying the given initial conditions $y''' - 5y'' + 8y' - 4y = 0$ and $y(0) = 1, y'(0) = 4, y''(0) = 0$ and $y_1 = e^x, y_2 = e^{2x}, y_3 = xe^{2x}$

solution The general solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 + c_3 y_3 \\
&= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}
\end{aligned}
$$

Hence

$$y' = c_1 e^x + 2c_2 e^{2x} + c_3\left(e^{2x} + 2xe^{2x}\right)$$

And

$$y'' = c_1 e^x + 4c_2 e^{2x} + c_3\left(2e^{2x} + 2e^{2x} + 4xe^{2x}\right)$$

From first initial condition we obtain

$$1 = c_1 + c_2 \tag{1}$$

From second initial condition we obtain

$$4 = c_1 + 2c_2 + c_3 \tag{2}$$

And from the third initial condition

$$0 = c_1 + 4c_2 + 4c_3 \tag{3}$$

We have three equations (1,2,3) to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

Augmented matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{pmatrix} \underset{R_3=R_3-R_1}{\overset{R_2=R_2-R_1}{\longmapsto}} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{pmatrix} \overset{R_3=R_3-3R_2}{\longmapsto} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{pmatrix}$$

We see that $|A| = 1$, since reduced matrix is upper diagonal matrix. Hence the solution is unique. From last row we obtain $c_3 = -10$ and from second row $c_2 + c_3 = 3$ or $c_2 = 3 + 10 = 13$ and from first row $c_1 + c_2 = 1$ or $c_1 = 1 - 13 = -12$, hence the particular solution is

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \\ &= -12e^x + 13e^{2x} - 10xe^{2x} \end{aligned}$$

4.12.3 Section 5.2 problem 24

Problem: A nonhomogeneous ODE, homogeneous solution y_h and particular solution y_p are given. Find solution that satisfy the initial conditions. $y'' - 2y' + 2y = 2x$ with $y(0) = 4, y'(0) = 8$ and $y_h = c_1 e^x \cos x + c_2 e^x \sin x$ and $y_p = x + 1$

solution The general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \end{aligned}$$

Therefore

$$y' = c_1 \left(e^x \cos x - e^x \sin x\right) + c_2 \left(e^x \sin x + e^x \cos x\right) + 1$$

First initial conditions gives

$$\begin{aligned} 4 &= c_1 + 1 \\ c_1 &= 3 \end{aligned}$$

Second initial conditions gives

$$8 = c_1 + c_2 + 1$$

Hence $c_2 = 7 - c_1 = 4$. Therefore the general solution becomes

$$\begin{aligned} y &= 3e^x \cos x + 4e^x \sin x + x + 1 \\ &= e^x \left(3\cos x + 4\sin x\right) + x + 1 \end{aligned}$$

4.12.4 Section 5.2 problem 28

Problem: Show that $1, x, x^2, \cdots, x^n$ are L.I.

solution Using the Wronskian

$$W(x) = \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \cdots & n(n-1)x^{n-2} \\ \vdots & \vdots & 0 & 6 & \cdots & n(n-1)(n-2)x^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & n! \end{vmatrix}$$

Therefore, the resulting Wronskian is an upper diagonal. The determinant of an upper diagonal matrix is the product of the diagonal. We see that there can be no zero element on the diagonal. Hence the determinant is never zero. Therefore $1, x, x^2, \cdots, x^n$ are L.I.

4.12.5 Section 5.2 problem 30

Problem: Verify that $y_1 = x$ and $y_2 = x^2$ are L.I. solutions on the entire line of the equation $x^2y'' - 2xy + 2y = 0$ but that $W\left(x, x^2\right)$ vanishes at $x = 0$. Why does these observations not contradicts part (b) of theorem 3?

solution To verify that y_1, y_2 are solution of the ODE, we plugin each into the ODE and see if they satisfy it. For y_1, we obtain

$$x^2y_1'' - 2xy_1 + 2y_1 = 0$$

But $y_1' = 1, y_1'' = 0$, therefore the above becomes

$$\begin{aligned} -2x + 2x &= 0 \\ 0 &= 0 \end{aligned}$$

Verified. For y_2, where $y_2' = 2x, y_2'' = 2$, we obtain

$$\begin{aligned} x^2y_2'' - 2xy_2 + 2y_2 &= 0 \\ 2x^2 - 4x^2 + 2x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Now we need to show that y_1, y_2 are L.I.

$$c_1y_1 + c_2y_2 = 0$$

We now solve for c_1, c_2

$$c_1x + c_2x^2 = 0$$

Comparing coefficients on the LHS and RHS, we see that $c_1 = 0, c_2 = 0$. Hence this shows that y_1, y_2 are L.I. We now find the Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

Hence $W(x) = 0$ at $x = 0$. This does not contradicts part (b) of theorem 3, because when we write the ODE in the standard form

$$y_1'' - \frac{2}{x}y_1 + \frac{2}{x^2}y_1 = 0$$

We see that $p_1(x) = -\frac{2}{x}, p_2(x) = \frac{2}{x^2}$. These functions are not continuous at $x = 0$ (there is singularity at $x = 0$). But theorem 3 applies only to the interval where $p_i(x)$ are continuous. Hence does not apply in this case. If $W(x)$ was zero at location other than $x = 0$, only then this will be a contradiction.

4.12.6 Section 5.3 problem 9 (page 323)

Problem: Find the general solution of the ODE $y'' + 8y' + 25y = 0$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^2 + 8r + 25 = 0$. The roots (using quadratic formula) are

$$\begin{aligned} r_1 &= -4 + 3i \\ r_2 &= -4 - 3i \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= c_1e^{r_1x} + c_2e^{r_2x} \\ &= c_1e^{(-4+3i)x} + c_2e^{(-4-3i)x} \\ &= e^{-4x}\left(c_1\cos 3x + c_2\sin 3x\right) \end{aligned}$$

4.12.7 Section 5.3 problem 16

Problem: Find the general solution of the ODE $y^{(4)} + 18y'' + 81y = 0$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^4 + 18r^2 + 81 = 0$. Let $r^2 = z$, hence $z^2 + 18z + 81 = 0$. This can be factored to $(z + 9)^2 = 0$. Hence the roots are -9 repeated. Therefore $r^2 = -9$ or $r = \pm 3i$. Therefore, the 4 roots are

$$\{3i, -3i, 3i, -3i\}$$

Hence the solution is

$$y = c_1 e^{3ix} + c_2 e^{-3ix} + c_3 x e^{3ix} + c_4 x e^{-3ix}$$

Or

$$\begin{aligned} y &= c_1 \cos 3x + c_2 \sin 3x + x\left(c_3 e^{3ix} + c_4 e^{-3ix}\right) \\ &= c_1 \cos 3x + c_2 \sin 3x + x\left(c_3 \cos 3x + c_4 \sin 3x\right) \end{aligned}$$

Or

$$y = (c_1 + x c_3)\cos 3x + c_2 \sin 3x\,(c_2 + x c_4)$$

4.12.8 Section 5.3 problem 23

Problem: Solve the initial value problem $y'' - 6y' + 25y = 0, y(0) = 3, y'(0) = 1$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^2 - 6r + 25 = 0$. Using quadratic formula, the roots are

$$\begin{aligned} r_1 &= 3 + 4i \\ r_2 &= 3 - 4i \end{aligned}$$

Hence the general solution is

$$y(x) = e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right)$$

Hence

$$y'(x) = 3e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right) + e^{3x}\left(-c_1 4 \sin 4x + c_2 4 \cos 4x\right)$$

Applying first initial conditions gives

$$3 = c_1$$

Applying second initial conditions gives

$$\begin{aligned} 1 &= 3(3) + 4c_2 \\ c_2 &= \frac{1-9}{4} = -2 \end{aligned}$$

Hence the solution is

$$y(x) = e^{3x}\left(3\cos 4x - 2\sin 4x\right)$$

4.12.9 Section 5.3 problem 26

Problem: Solve the initial value problem $y^{(3)} + 10y'' + 25y' = 0, y(0) = 3, y'(0) = 4, y''(0) = 5$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^3 + 10r^2 + 25r = 0$ or $r\left(r^2 + 10r + 25\right) = 0$, or $r(r+5)^2 = 0$. Hence the roots are $\{0, -5, -5\}$. There are repeated root. Hence the solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 x e^{r_2 x} \\ &= c_1 + c_2 e^{-5x} + c_3 x e^{-5x} \end{aligned}$$

Hence

$$y' = -5c_2 e^{-5x} + c_3\left(e^{-5x} - 5xe^{-5x}\right)$$

And

$$y'' = 25c_2 e^{-5x} + c_3\left(-5e^{-5x} - 5\left(e^{-5x} - 5xe^{-5x}\right)\right)$$

Applying first IC gives

$$3 = c_1 + c_2$$

Applying second IC gives

$$4 = -5c_2 + c_3$$

Applying third IC gives

$$\begin{aligned} 5 &= 25c_2 + c_3(-5 - 5) \\ &= 25c_2 - 10c_3 \end{aligned}$$

We have three equations to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 25 & -10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Therefore $R_3 = R_3 + 5R_2$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 25 \end{pmatrix}$$

Therefore, from third row, $-5c_3 = 25$ or

$$c_3 = -5$$

From second row

$$\begin{aligned} -5c_2 + c_3 &= 4 \\ -5c_2 &= 4 + 5 = 9 \\ c_2 &= -\frac{9}{5} \end{aligned}$$

From first row

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 &= 3 + \frac{9}{5} \\ &= \frac{24}{5} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= c_1 + c_2 e^{-5x} + c_3 x e^{-5x} \\ &= \frac{24}{5} - \frac{9}{5} e^{-5x} - 5xe^{-5x} \\ &= \frac{1}{5}\left(24 - 9e^{-5x} - 25xe^{-5x}\right) \end{aligned}$$

4.12.10 Section 5.3 problem 35

Problem: One solution of the differential equation is given, find the second solution $6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0$, and $y_1 = \cos 2x$

solution The characteristic equation is $6r^4 + 5r^3 + 25r^2 + 20r + 4 = 0$. Since $\cos 2x$ is solution, then this implies the roots for this solution must be $r = \pm 2i$, since this is what will give $\cos 2x$ solution. Therefore, there must be factor $\left(r^2 + 4\right)$. Doing long division

$$\frac{6r^4 + 5r^3 + 25r^2 + 20r + 4}{\left(r^2 + 4\right)} = 6r^2 + 5r + 1$$

Hence characteristic equation is

$$\begin{aligned} &\left(r^2 + 4\right)\left(6r^2 + 5r + 1\right) \\ &\left(r^2 + 4\right)(2r + 1)(3r + 1) \end{aligned}$$

Hence the roots are $r_1 = 2i, r_2 = -2i, r_3 = \frac{-1}{2}, r_4 = \frac{-1}{3}$. Therefore the solution is

$$\begin{aligned} y &= c_1 e^{2ix} + c_2 e^{-2ix} + c_3 e^{-\frac{1}{2}x} + c_4 e^{\frac{-1}{3}x} \\ &= c_1 \cos 2x + c_2 \sin 2x + c_3 e^{-\frac{1}{2}x} + c_4 e^{\frac{-1}{3}x} \end{aligned}$$

4.12.11 Section 5.3 problem 38

Problem: Solve $y^{(3)} - 5y'' + 100y' - 500y = 0$ with $y(0) = 0, y'(0) = 10, y''(0) = 250$ given that $y_1(x) = e^{5x}$ is one particular solution of the differential equation.

solution The characteristic equation is $r^3 - 5r^2 + 100r - 500 = 0$. Since $y_1(x) = e^{5x}$ is one solution, then $(r-5)$ is one of the roots. Hence by long division

$$\frac{r^3 - 5r^2 + 100r - 500}{r-5} = r^2 + 100$$

Therefore the factoring of the characteristic equation is

$$(r-5)\left(r^2+100\right) = 0$$

Therefore the roots are $r_1 = 5, r_2 = 10i, r_3 = -10i$ and therefore the solution is

$$y = c_1 e^{5x} + c_2 \cos 10x + c_3 \sin 10x$$

Hence

$$\begin{aligned} y' &= 5c_1 e^{5x} - 10c_2 \sin 10x + 10c_3 \cos 10x \\ y'' &= 25c_1 e^{5x} - 100c_2 \cos 10x - 100c_3 \sin 10x \end{aligned}$$

Applying first IC gives

$$0 = c_1 + c_2$$

Second IC gives

$$10 = 5c_1 + 10c_3$$

Third IC gives

$$250 = 25c_1 - 100c_2$$

We have three equations to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 5 & 0 & 10 \\ 25 & -100 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 250 \end{pmatrix}$$

$R_2 = R_2 - 5R_1$ and $R_3 = R_3 - 25R_1$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 10 \\ 0 & -125 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 250 \end{pmatrix}$$

$R_3 = R_3 - 25R_2$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 10 \\ 0 & 0 & -250 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}$$

Hence from last row $c_3 = 0$. From second row $-5c_2 = 10$ or $c_2 = -2$ and from first row $c_1 + c_2 = 0$ or $c_1 = 2$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{5x} + c_2 \cos 10x + c_3 \sin 10x \\ &= 2e^{5x} - 2\cos 10x \end{aligned}$$

4.12.12 Section 5.3 problem 40

Problem: Find linear homogeneous constant coefficient equation with the given general solution $y(x) = Ae^{2x} + B\cos 2x + C\sin 2x$

solution From the solution, we see that the roots are $r_1 = 2, r_2 = 2i, r_3 = -2i$. Hence the characteristic equation is

$$\begin{aligned} (r-2)\left(r^2+4\right) &= 0 \\ r^3 - 2r^2 + 4r - 8 &= 0 \end{aligned}$$

Therefore the original ODE is

$$y'''(x) - 2y'' + 4y' - 8y = 0$$

4.12.13 Section 5.3 problem 49

Problem: Solve $y^{(4)} - y^{(3)} - y'' - y' - 2y = 0$ with $y(0) = 0, y'(0) = 0, y''(0) = 0, y^{(3)} = 30$

solution The characteristic equation is

$$r^4 - r^3 - r^2 - r - 2 = 0$$

By inspection, we see that $r = -1$ is a root. Hence by long division, we have

$$\frac{r^4 - r^3 - r^2 - r - 2}{(r+1)} = r^3 - 2r^2 + r - 2$$

Therefore characteristic equation is

$$(r+1)\left(r^3 - 2r^2 + r - 2\right) = 0$$

By inspection, one of the roots of $r^3 - 2r^2 + r - 2 = 0$ is 2, hence by long division

$$\frac{r^3 - 2r^2 + r - 2}{r - 2} = r^2 + 1$$

Therefore characteristic equation becomes

$$(r+1)(r-2)\left(r^2+1\right) = 0$$

Hence roots are $r_1 = -1, r_2 = 2, r_3 = -i, r_4 = i$ and therefore the solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x$$

Initial conditions are now applied to find the constants.

$$\begin{aligned} y' &= -c_1 e^{-x} + 2c_2 e^{2x} - c_3 \sin x + c_4 \cos x \\ y'' &= c_1 e^{-x} + 4c_2 e^{2x} - c_3 \cos x - c_4 \sin x \\ y''' &= -c_1 e^{-x} + 8c_2 e^{2x} + c_3 \sin x - c_4 \cos x \end{aligned}$$

From $y(0) = 0$ we obtain

$$0 = c_1 + c_2 + c_3$$

From $y'(0) = 0$

$$0 = -c_1 + 2c_2 + c_4$$

From $y''(0) = 0$

$$0 = c_1 + 4c_2 - c_3$$

And from $y'''(0) = 30$

$$30 = -c_1 + 8c_2 - c_4$$

The 4 equations are solved for c_i

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & 4 & -1 & 0 \\ -1 & 8 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_2 = R_2 + R_1, R_3 = R_3 - R_1, R_4 = R_4 + R_1$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 9 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_3 = R_3 - R_2, R_4 = R_4 - 3R_2$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_4 = R_4 - \frac{2}{3} R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -\frac{10}{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

Hence from last row $-\frac{10}{3}c_4 = 30$,then

$$c_4 = -9$$

From 3rd row

$$\begin{aligned} -3c_3 - c_4 &= 0 \\ c_3 &= 3 \end{aligned}$$

From second row

$$\begin{aligned} 3c_2 + c_3 + c_4 &= 0 \\ 3c_2 + 3 - 9 &= 0 \\ c_2 &= 2 \end{aligned}$$

From first row

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 2 + 3 &= 0 \\ c_1 &= -5 \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x \\ &= -5e^{-x} + 2e^{2x} + 3\cos x - 9\sin x \end{aligned}$$

4.13 HW12

4.13.1 Section 5.1 problem 52 (page 299)

Problem Make the substitution $v = \ln x$ to find general solution for $x > 0$ of the Euler equation $x^2y'' + xy' - y = 0$

solution Let $v = \ln x$. Hence $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$ and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{x}\right)\\ &= \frac{d^2y}{dv^2}\frac{dv}{dx}\frac{1}{x} + \frac{dy}{dv}\frac{d}{dx}\left(\frac{1}{x}\right)\\ &= \frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}\end{aligned}$$

Hence the ODE becomes

$$\begin{aligned}x^2y'' + xy' - y &= 0\\ x^2\left(\frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}\right) + x\left(\frac{dy}{dv}\frac{1}{x}\right) - y(v) &= 0\\ \frac{d^2y}{dv^2} - \frac{dy}{dv} + \frac{dy}{dv} - y(v) &= 0\\ \frac{d^2y}{dv^2} - y(v) &= 0\end{aligned}$$

This can now be solved using characteristic equation. $r^2 - 1 = 0$ or $r^2 = 1$ or $r = \pm 1$. Hence the solution is

$$y(v) = c_1e^{v} + c_2e^{-v}$$

But $v = \ln x$, hence

$$\begin{aligned}y(x) &= c_1e^{\ln x} + c_2e^{-\ln x}\\ &= c_1x + c_1\frac{1}{x}\end{aligned}$$

The above is the solution.

But an easier method is the following. Let $y = x^r$. Hence $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$. Substituting this into the ODE gives

$$\begin{aligned}r(r-1)x^r + rx^r - x^r &= 0\\ x^r(r(r-1) + r - 1) &= 0\end{aligned}$$

Since $x^r \neq 0$, we simplify the above and obtain the characteristic equation

$$\begin{aligned}r(r-1) + r - 1 &= 0\\ r^2 - 1 &= 0\\ r^2 &= 1\\ r &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}y(x) &= c_1x^{r_1} + c_2x^{r_2}\\ &= c_1x + c_2x^{-1}\end{aligned}$$

For $x > 0$.

4.13.2 Section 5.1 problem 54

Problem Make the substitution $v = \ln x$ to find general solution for $x > 0$ of the Euler equation $4x^2y'' + 8xy' - 3y = 0$

solution Let $v=\ln x$. Hence $\frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{dx}=\frac{dy}{dv}\frac{1}{x}$ and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{x}\right)\\ &= \frac{d^2y}{dv^2}\frac{dv}{dx}\frac{1}{x}+\frac{dy}{dv}\frac{d}{dx}\left(\frac{1}{x}\right)\\ &= \frac{d^2y}{dv^2}\frac{1}{x^2}-\frac{dy}{dv}\frac{1}{x^2}\end{aligned}$$

Hence the ODE becomes

$$\begin{aligned}x^2y''+xy'-y &= 0\\ 4x^2\left(\frac{d^2y}{dv^2}\frac{1}{x^2}-\frac{dy}{dv}\frac{1}{x^2}\right)+8x\left(\frac{dy}{dv}\frac{1}{x}\right)-3y(v) &= 0\\ 4\frac{d^2y}{dv^2}-4\frac{dy}{dv}+8\frac{dy}{dv}-3y(v) &= 0\\ 4\frac{d^2y}{dv^2}+4\frac{dy}{dv}-3y(v) &= 0\end{aligned}$$

This can now be solved using characteristic equation. $4r^2+4r-3=0$, whose roots are $r_1=\frac{-3}{2}, r_2=\frac{1}{2}$ Hence the solution is

$$y(v)=c_1e^{\frac{-3}{2}v}+c_2e^{\frac{1}{2}v}$$

But $v=\ln x$, hence

$$\begin{aligned}y(x) &= c_1e^{\frac{-3}{2}\ln x}+c_2e^{\frac{1}{2}\ln x}\\ &= c_1x^{\frac{-3}{2}}+c_1x^{\frac{1}{2}}\end{aligned}$$

4.13.3 Section 5.2 problem 40 (page 311)

Problem Use reduction of order to find second L.I. solution y_2. $x^2y''-x(x+2)y'+(x+2)y=0$ with $y_1=x$ and $x>0$

solution Let $y=vy_1$, hence

$$\begin{aligned}y' &= v'y_1+vy_1'\\ y'' &= v''y_1+v'y_1'+v'y_1'+vy_1''\\ &= v''y_1+2v'y_1'+vy_1''\end{aligned}$$

Therefore the original ODE becomes

$$\begin{aligned}x^2y''-x(x+2)y'+(x+2)y &= 0\\ x^2\left(v''y_1+2v'y_1'+vy_1''\right)-x(x+2)\left(v'y_1+vy_1'\right)+(x+2)\left(vy_1\right) &= 0\\ v''\left(x^2y_1\right)+v'\left(2x^2y_1'-x(x+2)y_1\right)+v\overbrace{\left(x^2y_1''-x(x+2)y_1'+(x+2)y_1\right)}^{0} &= 0\end{aligned}$$

Hence

$$v''\left(x^2y_1\right)+v'\left(2x^2y_1'-x(x+2)y_1\right)=0$$

But $y_1=x$, hence the above becomes

$$\begin{aligned}x^3v''+v'\left(2x^2-x(x+2)x\right) &= 0\\ x^3v''-x^3v' &= 0\end{aligned}$$

Since we are told $x>0$ when we can divide by x^3 and obtain

$$v''-v'=0$$

To solve the above, let

$$z=v'$$

Therefore $z'-z=0$ or $\frac{d}{dx}(ze^x)=0$ or $ze^x=c_1$ or $z=c_1e^{-x}$. Therefore the above becomes

$$v'=c_1e^{-x}$$

Integrating

$$v = c_2 - c_1 e^{-x}$$

Since $y = vy_1$ therefore

$$y = y_1 (c_2 - c_1 e^{-x})$$

But $y_1 = x$, hence the complete solution is

$$y = c_2 x - c_1 x e^{-x}$$

Therefore, we see now that the two basis solutions are

$$\begin{aligned} y_1 &= x \\ y_2 &= xe^x \end{aligned}$$

These can be shown to be L.I. using the Wronskian as follows

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= xe^x + x^2 e^x - xe^x \\ &= x^2 e^x \end{aligned}$$

Which is not zero since we are told $x > 0$. Hence indeed the second basis solution y_2 found is L.I. to y_1.

4.13.4 Section 5.5 problem 9 (page 351)

Problem Find the particular solution for $y'' + 2y' - 3y = 1 + xe^x$

solution First we find the homogenous solution. This will tell us if e^x is one of the basis solutions of not, so we know what to guess. The characteristic equation is

$$\begin{aligned} r^2 + 2r - 3 &= 0 \\ (r-1)(r+3) &= 0 \end{aligned}$$

Hence $y_1 = e^x, y_2 = e^{-3x}$. e^x is a solution to the homogeneous ODE. The guess is therefore

$$\begin{aligned} y_p &= A + (B + Cx) xe^x \\ &= A + \left(Bx + Cx^2\right) e^x \qquad (1) \end{aligned}$$

Hence

$$\begin{aligned} y_p' &= (B + 2Cx) e^x + \left(Bx + Cx^2\right) e^x \\ &= e^x \left(B + 2Cx + Bx + Cx^2\right) \\ y_p'' &= (2C + B + 2Cx) e^x + e^x \left(B + 2Cx + Bx + Cx^2\right) \\ &= e^x \left(2C + B + 2Cx + B + 2Cx + Bx + Cx^2\right) \\ &= e^x \left(2C + 2B + 4Cx + Bx + Cx^2\right) \end{aligned}$$

Plugging into the ODE

$$\begin{aligned} e^x \left(2C + 2B + 4Cx + Bx + Cx^2\right) + 2e^x \left(B + 2Cx + Bx + Cx^2\right) - 3\left(A + \left(Bx + Cx^2\right) e^x\right) &= 1 + xe^x \\ e^x (2C + 2B + 2B) + xe^x (4C + B + 4C + 2B - 3B) + x^2 e^x (C + 2C - 3C) - 3A &= 1 + xe^x \\ e^x (2C + 4B) + xe^x (8C) - 3A &= 1 + xe^x \end{aligned}$$

Hence $-3A = 1$ or $A = -\frac{1}{3}$ and

$$8C = 1$$

Or

$$C = \frac{1}{8}$$

And

$$2C + 4B = 0$$

Or

$$B = -\frac{1}{16}$$

Hence particular solution becomes, from (1)

$$\begin{aligned} y_p &= -\frac{1}{3} + \left(-\frac{1}{16}x + \frac{1}{8}x^2\right)e^x \\ &= -\frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right)e^x \end{aligned}$$

4.13.5 Section 5.5 problem 10

Problem Find the particular solution for $y'' + 9y = 2\cos 3x + 3\sin 3x$

solution First we find the homogenous solution. The characteristic equation is

$$\begin{aligned} r^2 + 9 &= 0 \\ r^2 &= -9 \\ r &= \pm 3i \end{aligned}$$

Hence $y_1 = e^{3ix}, y_2 = e^{-3ix}$ or $y_h = c_1\cos 3x + c_2\sin 3x$. We see that $\cos 3x$ and $\sin 3x$ are already in the homogeneous solution. Therefore the guess is

$$y_p = Ax\cos 3x + Bx\sin 3x$$

Hence

$$\begin{aligned} y_p' &= A\cos 3x - 3Ax\sin 3x + B\sin 3x + 3Bx\cos 3x \\ y_p'' &= -3A\sin 3x - 3A\sin 3x - 9Ax\cos 3x + 3B\cos 3x + 3B\cos 3x - 9Bx\sin 3x \end{aligned}$$

Substitution into the ODE gives

$$\begin{aligned} &(-3A\sin 3x - 3A\sin 3x - 9Ax\cos 3x + 3B\cos 3x + 3B\cos 3x - 9Bx\sin 3x) \\ &\qquad + 9(Ax\cos 3x + Bx\sin 3x) = 2\cos 3x + 3\sin 3x \end{aligned}$$

Or

$$\begin{aligned} -6A\sin 3x - 9Ax\cos 3x + 6B\cos 3x - 9Bx\sin 3x + 9Ax\cos 3x + 9Bx\sin 3x &= 2\cos 3x + 3\sin 3x \\ \sin 3x(-6A) + \cos 3x(6B) + x\sin 3x(-9B + 9B) + x\cos 3x(-9A + 9A) &= 2\cos 3x + 3\sin 3x \\ -6A\sin 3x + 6B\cos 3x &= 2\cos 3x + 3\sin 3x \end{aligned}$$

Hence $-6A = 3$ or $A = \frac{-1}{2}$ and $6B = 2$ or $B = \frac{1}{3}$, therefore the particular solution is

$$\begin{aligned} y_p &= \frac{-1}{2}x\cos 3x + \frac{1}{3}x\sin 3x \\ &= \frac{1}{6}(2x\sin 3x - 3x\cos 3x) \end{aligned}$$

4.13.6 Section 5.5 problem 16

Problem Find the particular solution for $y'' + 9y = 2x^2e^{3x} + 5$

solution From the above problem, we found $y_h = c_1\cos 3x + c_2\sin 3x$. Therefore there are no basis solutions in the RHS which are in the homogenous solution. The guess for the constant term is A. The guess for $2x^2e^{3x}$ is $\left(B_0 + B_1x + B_2x^2\right)e^{3x}$, hence

$$\begin{aligned} y_p &= A + \left(B_0 + B_1x + B_2x^2\right)e^{3x} \\ y_p' &= (B_1 + 2B_2x)e^{3x} + 3\left(B_0 + B_1x + B_2x^2\right)e^{3x} \\ y_p'' &= 2B_2e^{3x} + 3(B_1 + 2B_2x)e^{3x} + 3(B_1 + 2B_2x)e^{3x} + 9\left(B_0 + B_1x + B_2x^2\right)e^{3x} \end{aligned}$$

Simplifying

$$\begin{aligned} y_p'' &= e^{3x}(2B_2 + 3B_1 + 3B_1 + 9B_0) + xe^x(6B_2 + 6B_2 + 9B_1) + x^2e^{3x}(9B_2) \\ &= e^{3x}(2B_2 + 6B_1 + 9B_0) + xe^x(12B_2 + 9B_1) + x^2e^{3x}(9B_2) \end{aligned}$$

Substitution into the ODE gives

$$e^{3x}(2B_2+6B_1+9B_0)+xe^{x}(12B_2+9B_1)+x^2e^{3x}(9B_2)+9\left(A+\left(B_0+B_1x+B_2x^2\right)e^{3x}\right)=2x^2e^{3x}+5$$
$$e^{3x}(2B_2+6B_1+9B_0)+xe^{x}(12B_2+9B_1)+x^2e^{3x}(9B_2)+9A+\left(9B_0+9B_1x+9B_2x^2\right)e^{3x}=2x^2e^{3x}+5$$
$$e^{3x}(2B_2+6B_1+18B_0)+xe^{x}(12B_2+18B_1)+x^2e^{3x}(18B_2)+9A=2x^2e^{3x}+5$$

Comparing coefficients gives

$$\begin{aligned}9A&=5\\2B_2+6B_1+18B_0&=0\\12B_2+18B_1&=0\\19B_2&=2\end{aligned}$$

From last equation $B_2=\frac{1}{9}$. Hence from third equation $18B_1=-\frac{12}{9}$, or $B_1=-\frac{2}{27}$. And from second equation

$$\begin{aligned}2B_2+6B_1+18B_0&=0\\2\left(\frac{1}{9}\right)+6\left(-\frac{2}{27}\right)+18B_0&=0\\B_0&=\frac{1}{81}\end{aligned}$$

And $A=\frac{5}{9}$. Therefore

$$\begin{aligned}y_p&=A+\left(B_0+B_1x+B_2x^2\right)e^{3x}\\&=\frac{5}{9}+\left(\frac{1}{81}-\frac{2}{27}x+\frac{1}{9}x^2\right)e^{3x}\\&=\frac{5}{9}+\left(\frac{1}{81}-\frac{6}{81}x+\frac{9}{81}x^2\right)e^{3x}\\&=\frac{45}{81}+\left(\frac{1}{81}-\frac{6}{81}x+\frac{9}{81}x^2\right)e^{3x}\\&=\frac{1}{81}\left(45+e^{3x}-6xe^{x}+9x^2e^{3x}\right)\end{aligned}$$

4.13.7 Section 5.5 problem 25

Problem Setup the form for the particular solution but do not determine the values of the coefficients. $y''+3y'+2y=xe^{-x}-xe^{-2x}$

solution First we find the homogenous solution. The characteristic equation is

$$\begin{aligned}r^2+3r+2&=0\\(r+1)(r+2)&=0\end{aligned}$$

Hence $y_1=e^{-x}, y_2=e^{-2x}$. We see that the basis solutions are part of the RHS. Therefore the guess solution is

$$y_p=x(A_1+A_2x)e^{-x}+x(A_3+A_4x)e^{-2x}$$

4.13.8 Section 5.5 problem 26

Problem Setup the form for the particular solution but do not determine the values of the coefficients. $y''-6y'+13y=xe^{3x}\sin 2x$

solution First we find the homogenous solution. The characteristic equation is

$$r^2-6r+13=0$$

The roots are $3\pm 2i$. Hence the homogenous solution is $y_h=c_1e^{3x}\cos 2x+c_2e^{3x}\sin 2x$. We see that $e^{3x}\sin 2x$ is already in the homogenous solution. Hence the guess is

$$\begin{aligned}y_p&=\overbrace{(A_1+A_2x)x}^{x\text{ guess}}\overbrace{(A_3\sin 2x+A_4\cos 2x)e^{3x}}^{\sin 2xe^{3x}\text{ guess}}\\&=\left(A_1x+A_2x^2\right)e^{3x}\cos 2x+\left(A_3x+A_4x^2\right)e^{3x}\sin 2x\end{aligned}$$

4.13.9 Section 5.5 problem 37

Problem Solve the initial value problem $y''' - 2y'' + y' = 1 + xe^x$ with $y(0) = 0, y'(0) = 0, y''(0) = 1$

solution First we find the homogenous solution. The characteristic equation is

$$
\begin{aligned}
r^3 - 2r^2 + r &= 0 \\
r\left(r^2 - 2r + 1\right) &= 0
\end{aligned}
$$

For $r^2 - 2r + 1 = 0$, it factors into $(r-1)(r-1)$, hence roots are $r_1 = 0, r_2 = 1, r_3 = 1$. Since double roots, the homogenous solution is

$$y_h = c_1 + c_2 e^x + c_3 x e^x$$

We notice that both e^x and xe^x is in the RHS. Therefore we need to multiply by x^2. The guess is therefore

$$
\begin{aligned}
y_p &= Ax + x^2 (B + Cx) e^x \\
&= Ax + \left(Bx^2 + Cx^3\right) e^x
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_p' &= A + \left(2Bx + 3Cx^2\right) e^x + \left(Bx^2 + Cx^3\right) e^x \\
y_p'' &= (2B + 6Cx) e^x + \left(2Bx + 3Cx^2\right) e^x + \left(2Bx + 3Cx^2\right) e^x + \left(Bx^2 + Cx^3\right) e^x
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
y_p' &= A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) \\
y_p'' &= e^x (2B) + xe^x (6C + 4B) + x^2 e^x (6C + B) + x^3 e^x (C) \\
y_p''' &= e^x (2B) + e^x (6C + 4B) + xe^x (6C + 4B) + 2xe^x (6C + B) + x^2 e^x (6C + B) + 3x^2 e^x (C) + x^3 e^x (C) \\
&= e^x (6B + 6C) + xe^x (6C + 4B + 12C + 2B) + x^2 e^x (6C + B + 3C) + Cx^3 e^x \\
&= e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x
\end{aligned}
$$

Substitution into the ODE gives

$$y_p''' - 2y_p'' + y_p' = 1 + xe^x$$

Hence

$$
\begin{aligned}
& e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x \\
& \quad - 2\left(e^x (2B) + xe^x (6C + 4B) + x^2 e^x (6C + B) + x^3 e^x (C)\right) + \\
& \quad A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) = 1 + xe^x
\end{aligned}
$$

Or

$$
\begin{aligned}
& e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x \\
& \quad - e^x (4B) - xe^x (12C + 8B) - x^2 e^x (12C + 2B) - x^3 e^x (2C) + \\
& \quad A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) = 1 + xe^x
\end{aligned}
$$

Or

$$
\begin{aligned}
& e^x (6B + 6C - 4B) + xe^x (18C + 6B - 12C - 8B + 2B) + \\
& \quad x^2 e^x (9C + B - 12C - 2B + 3C + B) + x^3 e^x (C - 2C + C) + A = 1 + xe^x
\end{aligned}
$$

Or

$$e^x (2B + 6C) + xe^x (6C) + A = 1 + xe^x$$

Hence

$$
\begin{aligned}
6C &= 1 \\
2B + 6C &= 0 \\
A &= 1
\end{aligned}
$$

Therefore, $C = \frac{1}{6}, B = -\frac{1}{2}$, and the particular solution is

$$
\begin{aligned}
y_p &= Ax + x^2 (B + Cx) e^x \\
&= x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x
\end{aligned}
$$

Hence the complete solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 + c_2 e^x + c_3 x e^x + x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x \end{aligned}$$

Applying initial conditions. $y(0) = 0$ gives

$$0 = c_1 + c_2 \tag{1}$$

And

$$y' = c_2 e^x + c_3 e^x + c_3 x e^x + 1 + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x$$

Applying second initial conditions $y'(0) = 0$ gives

$$0 = c_2 + c_3 + 1 \tag{2}$$

And

$$y'' = c_2 e^x + c_3 e^x + c_3 e^x + c_3 x e^x + (-1 + x) e^x + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x$$

Applying initial conditions $y''(0) = 1$ gives

$$\begin{aligned} 1 &= c_2 + 2c_3 - 1 \\ 2 &= c_2 + 2c_3 \end{aligned}$$

The solution is $c_1 = 4, c_2 = -4, c_3 = 3$, hence the general solution is

$$\begin{aligned} y &= c_1 + c_2 e^x + c_3 x e^x + x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x \\ &= 4 - 4e^x + 3xe^x + x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x \end{aligned}$$

4.13.10 Section 5.5 problem 49

Problem Use method of variation of parameters to find particular solution $y'' - 4y' + 4y = 2e^{2x}$

solution We need to first find the homogenous solution. The characteristic equation is

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ (r-2)(r-2) &= 0 \end{aligned}$$

Hence $r_1 = 2$, double root. Therefore

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= xe^{2x} \end{aligned}$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

Where $f(x) = 2e^{2x}$ and

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} \\ &= e^{2x}\left(e^{2x} + 2xe^{2x}\right) - 2xe^{4x} \\ &= e^{4x} + 2xe^{4x} - 2xe^{4x} \\ &= e^{4x} \end{aligned}$$

Hence

$$u_1 = -\int \frac{xe^{2x}\left(2e^{2x}\right)}{e^{4x}} dx = -\int 2x dx = -x^2$$

And

$$u_2 = \int \frac{e^{2x}\left(2e^{2x}\right)}{e^{4x}} dx = 2x$$

Therefore

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= -x^2 e^{2x} + 2x^2 e^{2x} \\ &= x^2 e^{2x} \end{aligned}$$

4.13.11 Section 5.5 problem 50

Problem Use method of variation of parameters to find particular solution $y'' - 4y = \sinh 2x$

solution We need to first find the homogenous solution. The characteristic equation is

$$\begin{aligned} r^2 - 4 &= 0 \\ r &= \pm 2 \end{aligned}$$

Therefore

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^{-2x} \end{aligned}$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

Where $f(x) = \sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$ and

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} \\ &= -2 - 2 = -4 \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= -\int \frac{e^{-2x}\left(\frac{e^{2x} - e^{-2x}}{2}\right)}{-4} dx \\ &= \frac{1}{4}\int e^{-2x}\left(\frac{e^{2x} - e^{-2x}}{2}\right) dx \\ &= \frac{1}{8}\int \left(1 - e^{-4x}\right) dx \\ &= \frac{1}{8}\left(x + \frac{e^{-4x}}{4}\right) \end{aligned}$$

And

$$\begin{aligned} u_2 &= \int \frac{e^{2x}\left(\frac{e^{2x} - e^{-2x}}{2}\right)}{-4} dx \\ &= -\frac{1}{8}\int e^{2x}\left(e^{2x} - e^{-2x}\right) dx \\ &= -\frac{1}{8}\int \left(e^{4x} - 1\right) dx \\ &= -\frac{1}{8}\left(\frac{e^{4x}}{4} - x\right) \end{aligned}$$

Therefore

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \frac{1}{8}\left(x + \frac{e^{-4x}}{4}\right)e^{2x} - \frac{1}{8}\left(\frac{e^{4x}}{4} - x\right)e^{-2x} \\
&= \left(\frac{1}{8}xe^{2x} + \frac{e^{-2x}}{32}\right) - \frac{1}{8}\left(\frac{e^{2x}}{32} - \frac{xe^{-2x}}{8}\right) \\
&= \frac{1}{8}xe^{2x} + \frac{e^{-2x}}{32} - \frac{e^{2x}}{32} + \frac{xe^{-2x}}{8} \\
&= \frac{1}{4}x\left(\frac{e^{2x}+e^{-2x}}{2}\right) + \frac{1}{16}\left(\frac{e^{-2x}-e^{2x}}{2}\right) \\
&= \frac{1}{4}x\left(\frac{e^{2x}+e^{-2x}}{2}\right) - \frac{1}{16}\left(\frac{e^{2x}-e^{-2x}}{2}\right) \\
&= \frac{1}{4}x\cosh 2x - \frac{1}{16}\sinh 2x \\
&= \frac{1}{16}\left(4x\cosh 2x - \sinh 2x\right)
\end{aligned}$$

4.13.12 Section 5.5 problem 53

Problem Use method of variation of parameters to find particular solution $y'' + 9y = 2\sec 3x$

solution We need to first find the homogenous solution. The characteristic equation is

$$\begin{aligned}
r^2 + 9 &= 0 \\
r &= \pm 3i
\end{aligned}$$

Therefore

$$\begin{aligned}
y_1(x) &= \sin 3x \\
y_2(x) &= \cos 3x
\end{aligned}$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned}
u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\
u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx
\end{aligned}$$

Where $f(x) = 2\sec 3x = \frac{2}{\cos 3x}$ and

$$\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} \\
&= -3\sin^2 3x - 3\cos^2 x \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
u_1 &= -\int \frac{\cos 3x\left(\frac{2}{\cos 3x}\right)}{-3} dx \\
&= \frac{1}{3}\int 2dx \\
&= \frac{2}{3}x
\end{aligned}$$

And

$$\begin{aligned} u_2 &= \int \frac{\sin 3x \left(\frac{2}{\cos 3x}\right)}{-3} dx \\ &= \frac{-2}{3} \int \tan 3x dx \\ &= \frac{-2}{3} \left(\frac{1}{6} \ln \frac{1}{\cos^2(3x)}\right) \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{2}{3} x (\sin 3x) + \frac{-2}{3} \left(\frac{1}{6} \ln \frac{1}{\cos^2(3x)}\right) \cos 3x \\ &= \frac{2}{3} x (\sin 3x) - \frac{1}{9} \cos(3x) \ln \left(\frac{1}{\cos^2(3x)}\right) \\ &= \frac{2}{3} x (\sin 3x) + \frac{1}{9} \cos(3x) \ln \left(\cos^2(3x)\right) \\ &= \frac{2}{3} x (\sin 3x) + \frac{2}{9} \cos(3x) \ln |\cos(3x)| \end{aligned}$$

4.13.13 Section 5.5 problem 61

Problem Find a particular solution to the Euler ODE $x^2 y'' + xy' + y = \ln x$ with homogenous solution $y_h = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

solution We see that

$$\begin{aligned} y_1 &= \cos(\ln x) \\ y_2 &= \sin(\ln x) \end{aligned}$$

Using variation of parameters on the ODE

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = \frac{\ln x}{x^2}$$

Where now we use $f(x) = \frac{\ln x}{x^2}$. Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

And

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x} \sin(\ln x) & \frac{1}{x} \cos(\ln x) \end{vmatrix} \\ &= \frac{1}{x} \cos^2(\ln x) + \frac{1}{x} \sin^2(\ln x) \\ &= \frac{1}{x} \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= -\int \frac{\sin(\ln x) \left(\frac{\ln x}{x^2}\right)}{\frac{1}{x}} dx \\ &= -\int \frac{\ln x \sin(\ln x)}{x} dx \\ &= \ln(x) \cos(\ln x) - \sin(\ln x) \end{aligned}$$

And

$$\begin{aligned} u_2 &= \int \frac{\cos(\ln x)\left(\frac{\ln x}{x^2}\right)}{\frac{1}{x}} dx \\ &= \int \frac{\cos(\ln x)(\ln x)}{x} dx \\ &= \ln(x)\sin(\ln x) + \cos(\ln x) \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\ln(x)\cos(\ln x) - \sin(\ln x))\cos(\ln x) + (\ln(x)\sin(\ln x) + \cos(\ln x))\sin(\ln x) \\ &= \ln(x)\cos^2(\ln x) - \sin(\ln x)\cos(\ln x) + \ln(x)\sin^2(\ln x) + \sin(\ln)\cos(\ln x) \\ &= \ln(x)\cos^2(\ln x) + \ln(x)\sin^2(\ln x) \\ &= \ln x \end{aligned}$$

4.13.14 Section 5.5 problem 62

Problem Find a particular solution to the Euler ODE $\left(x^2-1\right)y'' - 2xy' + 2y = x^2 - 1$ with homogenous solution $y_h = c_1 x + c_2\left(1+x^2\right)$

solution We see that

$$\begin{aligned} y_1 &= x \\ y_2 &= 1 + x^2 \end{aligned}$$

Using variation of parameters on the ODE

$$y'' - 2\frac{x}{\left(x^2-1\right)}y' + \frac{2}{\left(x^2-1\right)}y = 1$$

Where now we use $f(x) = 1$. Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

And

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1+x^2 \\ 1 & 2x \end{vmatrix} \\ &= 2x^2 - \left(1+x^2\right) \\ &= x^2 - 1 \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= -\int \frac{\left(1+x^2\right)(1)}{x^2-1} dx \\ &= -x - \ln(x-1) + \ln(x+1) \end{aligned}$$

And

$$\begin{aligned} u_2 &= \int \frac{x}{x^2-1} dx \\ &= \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) \end{aligned}$$

Therefore

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= (-x - \ln(x-1) + \ln(x+1))\, x + \left(\frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1)\right)\left(1+x^2\right) \\
&= -x^2 + x \ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\left(1+x^2\right)\ln|(x-1)(x+1)| \\
&= -x^2 + x \ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\left(1+x^2\right)\ln\left|x^2-1\right|
\end{aligned}
$$

4.14 HW13

4.14.1 Section 7.1 problem 3 (page 404)

problem Transform the following problem or system to set of first order ODE $t^2x'' + tx' + \left(t^2-1\right)x = 0$

solution Since this is second order ODE, we need two state variables, say x_1, x

Let $x_1 = x, x_2 = x'$, hence

$$\left.\begin{array}{l} x_1 = x \\ x_2 = x' \end{array}\right\} \xrightarrow{\text{take derivative}} \left.\begin{array}{l} x_1' = x' \\ x_2' = x'' \end{array}\right\} \xrightarrow{\text{replace RHS}} \begin{array}{c} x_1' = x_2 \\ x_2' = -\frac{x'}{t} - \frac{\left(t^2-1\right)x}{t} = -\frac{x_2}{t} - \frac{\left(t^2-1\right)x_1}{t} \end{array}$$

Hence the two first order ODE's are (now coupled)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{x_2}{t} - \frac{\left(t^2-1\right)x_1}{t} \end{aligned}$$

The matrix form of the above is

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{t^2-1}{t} & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

4.14.2 Section 7.1 problem 8

problem Transform the following problem or system to set of first order ODE $x'' + 3x' + 4x - 2y = 0; y'' + 2y' - 3x + y = \cos t$

solution We have two second order ODE's, hence we need 4 state variables. Let $x_1 = x, x_2 = x', x_3 = y, x_4 = y'$, therefore

$$\left.\begin{array}{l} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{array}\right\} \xrightarrow{\text{take derivative}} \left.\begin{array}{l} x_1' = x' \\ x_2' = x'' \\ x_3' = y \\ x_4' = y'' \end{array}\right\} \xrightarrow{\text{replace RHS}} \begin{array}{c} x_1' = x_2 \\ x_2' = -3x' - 4x + 2y = -3x_2 - 4x_1 + 2x_3 \\ x_3' = x_3 \\ x_4' = -2y' + 3x - y + \cos t = -2x_4 + 3x_1 - x_3 + \cos t \end{array}$$

Hence the 4 first order ODE's are

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_2 - 4x_1 + 2x_3 \\ x_3' &= x_3 \\ x_4' &= -2x_4 + 3x_1 - x_3 + \cos t \end{aligned}$$

The matrix form of the above is

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} + \mathbf{f} \\ \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix} \end{aligned}$$

4.14.3 Section 7.2 problem 9 (page 417)

problem Write the given system in form $x' = P(t)x + f(t)$

$$\begin{aligned} x' &= 3x - 4y + z + t \\ y' &= x - 3z + t^2 \\ z' &= 6y - 7z + t^3 \end{aligned}$$

solution The dependent variables are x, y, z and the independent variable is t. The matrix form is seen by inspection to be

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

4.14.4 Section 7.2 problem 10

problem Write the given system in form $x' = A(t)x + f(t)$

$$\begin{aligned} x' &= tx - y + e^t z \\ y' &= 2x + t^2 y - z \\ z' &= e^{-t} x + 3ty + t^3 z \end{aligned}$$

solution The dependent variables are x, y, z and the independent variable is t. The matrix form is seen by inspection to be

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice that P matrix is time dependent and not constant as the last problem. This is time varying system.

4.14.5 Section 7.2 problem 25

problem Find the complete solution that satisfies the initial conditions. $\boldsymbol{x}(0) = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \\ \boldsymbol{x}_2 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \end{aligned}$$

solution

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \end{aligned} \tag{1}$$

At $t = 0$ the above becomes

$$\begin{aligned} \begin{pmatrix} 11 \\ -7 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

Gaussian elimination. $R_2 = R_2 + R_1$ gives

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}$$

Hence $-c_2 = 4$ or $c_2 = -4$. From first row, $c_1 + c_2 = 11$ or $c_1 = 11 - c_2 = 11 + 4 = 15$, hence the complete solution from (1) is

$$\begin{aligned} \mathbf{x}(t) &= 15\mathbf{x}_1(t) - 4\mathbf{x}_2(t) \\ &= 15\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} - 4\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} 15 \\ -15 \end{pmatrix} e^{3t} + \begin{pmatrix} -4 \\ 8 \end{pmatrix} e^{2t} \end{aligned}$$

4.14.6 Section 7.3 problem 7 (page 429)

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = -3x_1 + 4x_2; x_2' = 6x_1 - 5x_2$

solution

The system in matrix form is

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} -3 & 4 \\ 6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} &= 0 \\ (-3-\lambda)(-5-\lambda) - 24 &= 0 \\ \lambda^2 + 8\lambda - 9 &= 0 \\ (\lambda + 9)(\lambda - 1) &= 0 \end{aligned}$$

Hence $\lambda_1 = 1, \lambda_2 = -9$. For λ_1, we now solve

$$\begin{aligned} (A - \lambda_1 I)\mathbf{v}_1 &= 0 \\ \begin{pmatrix} -3-\lambda_1 & 4 \\ 6 & -5-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3-1 & 4 \\ 6 & -5-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $-4v_1 + 4v_2 = 0$. Picking $v_1 = 1$, then $v_2 = 1$, hence the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For λ_2, we now solve

$$\begin{aligned} (A - \lambda_2 I)\mathbf{v}_2 &= 0 \\ \begin{pmatrix} -3-\lambda_2 & 4 \\ 6 & -5-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3+9 & 4 \\ 6 & -5+9 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $6v_1 + 4v_2 = 0$. Picking $v_1 = 1$, then $v_2 = -\frac{3}{2}$, hence the second eigenvector is $v_2 = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ Therefore the solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t}$$

Or

$$\begin{aligned} x_1(t) &= c_1 e^t + 2c_2 e^{-9t} \\ x_2(t) &= c_1 e^t - 3c_2 e^{-9t} \end{aligned}$$

No initial conditions are given.

4.14.7 Section 7.3 problem 9

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = 2x_1 - 5x_2; x_2' = 4x_1 - 2x_2; x_1(0) = 2, x_2(0) = 3$

solution

The system in matrix form is

$$\begin{aligned} \boldsymbol{x}' &= A\boldsymbol{x} \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(-2-\lambda) + 20 &= 0 \\ \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

Hence $\lambda_1 = 4i, \lambda_2 = -4i$. For λ_1, we now solve

$$\begin{aligned} (A - \lambda_1 I)\boldsymbol{v}_1 &= 0 \\ \begin{pmatrix} 2-\lambda_1 & -5 \\ 4 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $(2-4i)v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{2-4i}{5}$, hence the eigenvector is $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{2-4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$

For λ_2, we now solve

$$\begin{aligned} (A - \lambda_2 I)\boldsymbol{v}_2 &= 0 \\ \begin{pmatrix} 2-\lambda_2 & -5 \\ 4 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2+4i & -5 \\ 4 & -2+4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $(2+4i)v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{(2+4i)}{5}$, hence the eigenvector is $v_2 = \begin{pmatrix} 1 \\ \frac{2+4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2+4i \end{pmatrix}$ Therefore the solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{4it} + c_2 \begin{pmatrix} 5 \\ 2+4i \end{pmatrix} e^{-4it}$$

Or

$$\begin{aligned} x_1(t) &= c_1 5 e^{4it} + c_2 5 e^{-4it} \\ x_2(t) &= c_1(-2+4i) e^{4it} - c_2(2+4i) e^{-4it} \end{aligned}$$

Convert to new basis.

$$\begin{aligned} \Re(\boldsymbol{x}_1) = \Re\begin{pmatrix} 5e^{4it} \\ (2-4i)e^{4it} \end{pmatrix} &= \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ \left(2e^{4it} - 4ie^{4it}\right) \end{pmatrix} \\ &= \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ (2(\cos 4t + i\sin 4t) - 4i(\cos 4t + i\sin 4t)) \end{pmatrix} \\ &= \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ 2\cos 4t + i2\sin 4t - 4i\cos 4t + 4\sin 4t \end{pmatrix} \\ &= \Re\begin{pmatrix} 5\cos 4t + i5\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t) \end{pmatrix} \\ &= \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} \Im(\boldsymbol{x}_1) &= \Im\begin{pmatrix} 5\cos 4t + i5\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t) \end{pmatrix} \\ &= \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{pmatrix} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_3 \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{pmatrix} \tag{1}$$

Or

$$\begin{aligned} x_1(t) &= c_3 5\cos 4t + c_4 5\sin 4t \\ x_2(t) &= c_3(2\cos 4t + 4\sin 4t) + c_4(2\sin 4t - 4\cos 4t) \end{aligned}$$

We now apply the initial conditions. From (1), at $t = 0$ we obtain

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_3 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Or

$$\begin{pmatrix} 5 & 0 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

From first row, $5c_3 = 2$ or $c_3 = \frac{2}{5}$. From second row $2c_3 - 4c_4 = 3$ or $c_4 = -\frac{3-2\left(\frac{2}{5}\right)}{4} = -\frac{11}{20}$. Hence the solution (1) becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{pmatrix} - \frac{11}{20} \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{pmatrix} \tag{1A}$$

Or

$$x_1(t) = \frac{2}{5}5\cos 4t - \frac{11}{20}5\sin 4t$$
$$x_2(t) = \frac{2}{5}(2\cos 4t + 4\sin 4t) - \frac{11}{20}(2\sin 4t - 4\cos 4t)$$

Or

$$x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t$$
$$x_2(t) = \frac{4}{5}\cos 4t + \frac{8}{5}\sin 4t - \frac{22}{20}\sin 4t + \frac{11}{5}\cos 4t$$

Or

$$x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t$$
$$x_2(t) = 3\cos 4t + \frac{1}{2}\sin 4t$$

4.14.8 Section 7.3 problem 12

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = x_1 - 5x_2; x_2' = x_1 + 3x_2;$

solution The system in matrix form is

$$\boldsymbol{x}' = A\boldsymbol{x}$$
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The eigenvalues are found from solving

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(3-\lambda) + 5 = 0$$
$$\lambda^2 - 4\lambda + 8 = 0$$
$$\lambda = 2 \pm 2i$$

Hence $\lambda_1 = 2 + 2i, \lambda_2 = 2 - 2i$. For λ_1, we now solve

$$(A - \lambda_1 I)\boldsymbol{v}_1 = 0$$
$$\begin{pmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using first equation, we see that $(-1-2i)v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{(-1-2i)}{5}$, hence the eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ \frac{-1-2i}{5} \end{pmatrix}$$
$$= \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$$

For λ_2, we now solve

$$(A-\lambda_2 I)\,v_2 = 0$$
$$\begin{pmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1-(2-2i) & -5 \\ 1 & 3-(2-2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1+2i & -5 \\ 1 & 1+2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using first equation, we see that $(-1+2i)\,v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{-1+2i}{5}$, hence the second eigenvector is

$$\begin{aligned} v_2 &= \begin{pmatrix} 1 \\ \frac{-1+2i}{5} \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t)\, e^{\lambda_1 t} + c_2 v_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ -1-2i \end{pmatrix} e^{(2+2i)t} + c_2 \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} e^{(2-2i)t}$$

Convert to new basis.

$$\begin{aligned} \Re(x_1) &= \Re\begin{pmatrix} 5e^{(2+2i)t} \\ (-1-2i)\,e^{(2+2i)t} \end{pmatrix} = \Re\begin{pmatrix} 5e^{2t}(\cos 2t + i\sin 2t) \\ -e^{(2+2i)t} - 2ie^{(2+2i)t} \end{pmatrix} \\ &= \Re\begin{pmatrix} 5e^{2t}(\cos 2t + i\sin 2t) \\ -e^{2t}e^{2it} - 2ie^{2t}e^{2it} \end{pmatrix} \\ &= \Re\begin{pmatrix} 5e^{2t}(\cos 2t + i\sin 2t) \\ -e^{2t}(\cos 2t + i\sin 2t) - 2ie^{2t}(\cos 2t + i\sin 2t) \end{pmatrix} \\ &= \Re\begin{pmatrix} 5e^{2t}(\cos 2t + i\sin 2t) \\ -e^{2t}(\cos 2t + i\sin 2t) - 2e^{2t}(i\cos 2t - \sin 2t) \end{pmatrix} \\ &= \Re\begin{pmatrix} 5e^{2t}\cos 2t + i\left(5e^{2t}\sin 2t\right) \\ -e^{2t}\cos 2t - ie^{2t}\sin 2t - i2e^{2t}\cos 2t + 2e^{2t}\sin 2t \end{pmatrix} \\ &= \Re\begin{pmatrix} 5e^{2t}\cos 2t + i\left(5e^{2t}\sin 2t\right) \\ \left(-e^{2t}\cos 2t + 2e^{2t}\sin 2t\right) + i\left(-e^{2t}\sin 2t - 2e^{2t}\cos 2t\right) \end{pmatrix} \end{aligned}$$

Hence

$$\Re(x_1) = \begin{pmatrix} 5e^{2t}\cos 2t \\ -e^{2t}\cos 2t + 2e^{2t}\sin 2t \end{pmatrix}$$

And

$$\Im(x_1) = \begin{pmatrix} 5e^{2t}\sin 2t \\ -e^{2t}\sin 2t - 2e^{2t}\cos 2t \end{pmatrix}$$

Therefore the solution in the new basis is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 5e^{2t}\cos 2t \\ -e^{2t}\cos 2t + 2e^{2t}\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} 5e^{2t}\sin 2t \\ -e^{2t}\sin 2t - 2e^{2t}\cos 2t \end{pmatrix}$$

Or

$$\begin{aligned} x_1(t) &= C_1 5e^{2t}\cos 2t + C_2 5e^{2t}\sin 2t \\ x_2(t) &= C_1\left(-e^{2t}\cos 2t + 2e^{2t}\sin 2t\right) + C_2\left(-e^{2t}\sin 2t - 2e^{2t}\cos 2t\right) \end{aligned}$$

Or

$$x_1(t) = e^{2t}(5C_1 \cos 2t + 5C_2 \sin 2t)$$
$$x_2(t) = e^{2t}(-C_1 \cos 2t + 2C_1 \sin 2t - C_2 \sin 2t - 2C_2 \cos 2t)$$

Or

$$\begin{aligned} x_1(t) &= e^{2t}(5C_1 \cos 2t + 5C_2 \sin 2t) \\ x_2(t) &= e^{2t}(\cos 2t(-C_1 - 2C_2) + \sin 2t(2C_1 - C_2)) \end{aligned} \tag{1}$$

Note, book must have used the other choice of eigenvalues ordering since it has the signs all flipped the other way from what I have above. flipping all the signs in the solution given above in equation (1), then the book solution results:

$$x_1(t) = e^{2t}(-5C_1 \cos 2t - 5C_2 \sin 2t)$$
$$x_2(t) = e^{2t}(\cos 2t(C_1 + 2C_2) + \sin 2t(-2C_1 + C_2))$$

4.14.9 Section 7.3 problem 14

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = 3x_1 - 4x_2; x_2' = 4x_1 + 3x_2;$

solution The system in matrix form is

$$\boldsymbol{x}' = A\boldsymbol{x}$$
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The eigenvalues are found from solving

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(3-\lambda) + 16 = 0$$
$$\lambda^2 - 6\lambda + 25 = 0$$
$$\lambda = 3 \pm 4i$$

Hence $\lambda_1 = 3 + 4i, \lambda_2 = 3 - 4i$. For λ_1, we now solve

$$(A - \lambda_1 I)\boldsymbol{v}_1 = 0$$
$$\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3-(3+4i) & -4 \\ 4 & 3-(3+4i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using first equation, we see that $(-4i)v_1 - 4v_2 = 0$. Let $v_1 = 1$, then $v_2 = -i$, hence the eigenvector is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For λ_2, we now solve

$$
\begin{aligned}
(A-\lambda_1 I)\,\boldsymbol{v}_1 &= 0\\
\begin{pmatrix}3-\lambda & -4\\ 4 & 3-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\
\begin{pmatrix}3-(3-4i) & -4\\ 4 & 3-(3-4i)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\
\begin{pmatrix}4i & -4\\ 4 & 4i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}
\end{aligned}
$$

Using first equation, we see that $(4i)\,v_1 - 4v_2 = 0$. Let $v_1 = 1$, then $v_2 = i$, hence the eigenvector is

$$\boldsymbol{v}_2 = \begin{pmatrix}1\\ i\end{pmatrix}$$

Therefore the solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)\\
&= c_1\boldsymbol{v}_1(t)\,e^{\lambda_1 t} + c_2\boldsymbol{v}_2(t)\,e^{\lambda_2 t}
\end{aligned}
$$

Therefore

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = c_1\begin{pmatrix}1\\ -i\end{pmatrix}e^{(3+4i)t} + c_2\begin{pmatrix}1\\ i\end{pmatrix}e^{(3-4i)t}$$

Convert to new basis.

$$
\begin{aligned}
\Re(\boldsymbol{x}_1) &= \Re\begin{pmatrix}e^{(3+4i)t}\\ -ie^{(3+4i)t}\end{pmatrix} = \Re\begin{pmatrix}e^{3t}(\cos 4t + i\sin 4t)\\ e^{3t}(-i\cos 4t + \sin 4t)\end{pmatrix}\\
&= \begin{pmatrix}e^{3t}\cos 4t\\ e^{3t}\sin 4t\end{pmatrix}
\end{aligned}
$$

And

$$\Im(\boldsymbol{x}_1) = \begin{pmatrix}e^{3t}\sin 4t\\ -e^{3t}\cos 4t\end{pmatrix}$$

Therefore the solution in the new basis is

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = C_1\begin{pmatrix}e^{3t}\cos 4t\\ e^{3t}\sin 4t\end{pmatrix} + C_2\begin{pmatrix}e^{3t}\sin 4t\\ -e^{3t}\cos 4t\end{pmatrix}$$

Or

$$
\begin{aligned}
x_1(t) &= e^{3t}(C_1\cos 4t + C_2\sin 4t)\\
x_2(t) &= e^{3t}(C_1\sin 4t - C_2\cos 4t)
\end{aligned}
$$

4.14.10 Section 7.3 problem 28

problem TO DO

solution

4.14.11 Section 7.3 problem 30

problem TO DO

solution

4.14.12 Section 7.3 problem 39

problem Find general solution $\boldsymbol{x}' = A\boldsymbol{x}$

$$A = \begin{pmatrix}-2 & 0 & 0 & 9\\ 4 & 2 & 0 & -10\\ 0 & 0 & -1 & 8\\ 0 & 0 & 0 & 1\end{pmatrix}$$

solution The eigenvalues are found from solving

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 0 & 0 & 9 \\ 4 & 2-\lambda & 0 & -10 \\ 0 & 0 & -1-\lambda & 8 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the last row since it has most zeros then

$$\begin{aligned} \det(A-\lambda I) &= (1-\lambda)(-1)^{4+4} \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)(-1)^{3+3} \begin{vmatrix} -2-\lambda & 0 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda) \begin{vmatrix} -2-\lambda & 0 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)(-2-\lambda)(2-\lambda) \end{aligned}$$

Hence the eigenvalues are (distinct case, no repeated)

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2, \lambda_4 = -2$$

For $\lambda_1 = 1$

$$\begin{pmatrix} -2-\lambda_1 & 0 & 0 & 9 \\ 4 & 2-\lambda_1 & 0 & -10 \\ 0 & 0 & -1-\lambda_1 & 8 \\ 0 & 0 & 0 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 0 & 9 \\ 4 & 1 & 0 & -10 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_4 = 1$. Hence from third row

$$\begin{aligned} -2v_3 + 8v_4 &= 0 \\ v_3 &= 4 \end{aligned}$$

From first row

$$\begin{aligned} -3v_1 + 9v_4 &= 0 \\ v_1 &= 3 \end{aligned}$$

From second row

$$\begin{aligned} 4v_1 + v_2 - 10v_4 &= 0 \\ v_2 &= 10 - 12 \\ &= -2 \end{aligned}$$

Hence first eigenvector is

$$v_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1$

$$\begin{pmatrix} -2-\lambda_2 & 0 & 0 & 9 \\ 4 & 2-\lambda_2 & 0 & -10 \\ 0 & 0 & -1-\lambda_2 & 8 \\ 0 & 0 & 0 & 1-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 9 \\ 4 & 3 & 0 & -10 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $2v_4 = 0$, hence $v_4 = 0$. From third row it also says that $v_4 = 0$. from first row we also obtain that $v_1 = 0$. From second row

$$4v_1 + 3v_2 = 0$$

Since $v_1 = 0$ then $v_2 = 0$. We notice that v_3 is left undetermined as there is no equation to determine it. (this happens when there is a column of all zeros, as in this case). Hence we can pick any value for v_3. Lets choose $v_3 = 1$. Therefore the second eigenvector is

$$\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2$

$$\begin{pmatrix} -2-\lambda_3 & 0 & 0 & 9 \\ 4 & 2-\lambda_3 & 0 & -10 \\ 0 & 0 & -1-\lambda_3 & 8 \\ 0 & 0 & 0 & 1-\lambda_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 0 & 0 & 9 \\ 4 & 0 & 0 & -10 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $-v_4 = 0$, hence $v_4 = 0$. From third row it says that $v_3 = 0$ since $v_4 = 0$. from second and first row obtain that $v_1 = 0$.

We notice that v_2 is left undetermined as there is no equation to determine it. Hence we can pick any value for v_2. Lets choose $v_2 = 1$. Therefore the eigenvector is

$$\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_4 = -2$

$$\begin{pmatrix} -2-\lambda_4 & 0 & 0 & 9 \\ 4 & 2-\lambda_4 & 0 & -10 \\ 0 & 0 & -1-\lambda_4 & 8 \\ 0 & 0 & 0 & 1-\lambda_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 9 \\ 4 & 4 & 0 & -10 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $v_4 = 0$. From third row it says that $v_3 = 0$ since $v_4 = 0$. Second row gives

$4v_1 + 4v_2 = 0$. Let $v_1 = 1$ hence $v_2 = -1$. Therefore the eigenvector is

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

We found all the eigenvectors, The solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) + c_3 \boldsymbol{x}_3(t) + c_4 \boldsymbol{x}_4(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} + c_3 \boldsymbol{v}_2(t) e^{\lambda_3 t} + c_2 \boldsymbol{v}_4(t) e^{\lambda_4 t} \end{aligned}$$

Or

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} e^{-2t}$$

Hence

$$\begin{aligned} x_1(t) &= 3c_1 e^{t} + c_4 e^{-2t} \\ x_2(t) &= -2c_1 e^{t} + c_3 e^{2t} - c_4 e^{-2t} \\ x_3(t) &= 4c_1 e^{t} + c_2 e^{-t} \\ x_4(t) &= c_1 e^{t} \end{aligned}$$

4.14.13 Section 7.5 problem 3

problem Find general solution of $\boldsymbol{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \boldsymbol{x}$

solution The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(5-\lambda) + 4 &= 0 \\ \lambda^2 - 6\lambda + 9 &= 0 \\ (\lambda - 3)^2 &= 0 \end{aligned}$$

Hence $\lambda = 3$. repeated root, multiplicity $k = 2$. Let us first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$\begin{aligned} (A - \lambda I) \boldsymbol{v} &= 0 \\ \begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

We see that the first row and the second row give the same eigenvector. $-2v_1 - 2v_2 = 0$. Let $v_1 = 1$, hence $v_2 = -1$. So we can only find one eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solving for v_2 from

$$(A-\lambda I)^2 v_2 = 0$$
$$\begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ can be any value. Let $v_1 = 1, v_2 = 0$ and therefore

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We now find $\boldsymbol{v}_1$ from

$$\begin{aligned} \boldsymbol{v}_1 &= (A-\lambda I)\,\boldsymbol{v}_2 \\ &= \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \end{aligned}$$

Hence the solution is

$$\boldsymbol{x}(t) = c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \tag{1}$$

Where now

$$\begin{aligned} \boldsymbol{x}_1(t) &= \boldsymbol{v}_1 e^{\lambda t} \\ \boldsymbol{x}_2(t) &= (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t} \end{aligned}$$

Plugging these into (1) gives

$$\boldsymbol{x}(t) = c_1 \boldsymbol{v}_1 e^{\lambda t} + c_2 (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t} \tag{2}$$

Replacing the result we found earlier for $\boldsymbol{v}_1, \boldsymbol{v}_2$ into the above, and using $\lambda = 3$ gives

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{3t} + c_2 \left(\begin{pmatrix} -2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{3t}$$

Hence

$$\begin{aligned} x_1(t) &= (-2c_1 + c_2 - 2c_2 t)\, e^{3t} \\ x_2(t) &= (2c_1 + 2c_2 t)\, e^{3t} \end{aligned}$$

4.14.14 Section 7.5 problem 5

problem Find general solution of $\boldsymbol{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \boldsymbol{x}$

solution The eigenvalues are found from solving

$$\begin{aligned} |A-\lambda I| &= 0 \\ \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} &= 0 \\ (7-\lambda)(3-\lambda) + 4 &= 0 \\ (\lambda-5)^2 &= 0 \end{aligned}$$

Hence $\lambda = 5$, repeated root, multiplicity $k = 2$. Let us first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$
\begin{aligned}
(A-\lambda I)\,\boldsymbol{v} &= 0\\
\begin{pmatrix} 7-\lambda & 1\\ -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 7-5 & 1\\ -4 & 3-5 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}
\end{aligned}
$$

From first row we obtain $2v_1 + v_2 = 0$. Let $v_1 = 1$ then $v_2 = -2$. Hence eigenvector is $\boldsymbol{v} = \begin{pmatrix} 1\\ -2 \end{pmatrix}$. We can only find this one eigenvector. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solve for $\boldsymbol{v}_2$ from

$$
\begin{aligned}
(A-\lambda I)^2\,\boldsymbol{v}_2 &= 0\\
\begin{pmatrix} 7-\lambda & 1\\ -4 & 3-\lambda \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 7-5 & 1\\ -4 & 3-5 \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}
\end{aligned}
$$

Hence $\begin{pmatrix} v_1\\ v_2 \end{pmatrix}$ can be any value. Let $v_1 = 1, v_2 = 0$ and therefore

$$
\boldsymbol{v}_2 = \begin{pmatrix} 1\\ 0 \end{pmatrix}
$$

We now find $\boldsymbol{v}_1$ from

$$
\begin{aligned}
\boldsymbol{v}_1 &= (A-\lambda I)\,\boldsymbol{v}_2\\
&= \begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1\\ 0 \end{pmatrix}\\
&= \begin{pmatrix} 2\\ -4 \end{pmatrix}
\end{aligned}
$$

Hence the solution is

$$
\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) \tag{1}
$$

Where now

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \boldsymbol{v}_1 e^{\lambda t}\\
\boldsymbol{x}_2(t) &= (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t}
\end{aligned}
$$

Plugging these into (1) gives

$$
\boldsymbol{x}(t) = c_1\boldsymbol{v}_1 e^{\lambda t} + c_2(\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t} \tag{2}
$$

Replacing the result we found earlier for $\boldsymbol{v}_1, \boldsymbol{v}_2$ into the above, and using $\lambda = 3$ gives

$$
\boldsymbol{x}(t) = c_1\begin{pmatrix} 2\\ -4 \end{pmatrix} e^{5t} + c_2\left(\begin{pmatrix} 2\\ -4 \end{pmatrix} t + \begin{pmatrix} 1\\ 0 \end{pmatrix}\right) e^{5t}
$$

Hence

$$
\begin{aligned}
x_1(t) &= (2c_1 + c_2 + 2c_2 t)\, e^{3t}\\
x_2(t) &= (-4c_1 - 4c_2 t)\, e^{3t}
\end{aligned}
$$

4.14.15 Section 7.5 problem 7

Problem Find general solution of $x' = \begin{pmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{pmatrix} x$

Solution The eigenvalues are found from solving

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding along last row since it has most zeros

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-1)^{3+3} \begin{vmatrix} 2-\lambda & 0 \\ -7 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ -7 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda)(2-\lambda)(9-\lambda) \end{aligned}$$

Hence roots are $\lambda_1 = 2, \lambda_2 = 9$, where now λ_1 has multiplicity $k = 2$, and λ_2 is the good one with no multiplicity. To find associated eigenvector for λ_2 we follow the normal method.

For $\lambda_2 = 9$

$$(A - \lambda_2 I) \mathbf{v}_{\lambda_2} = 0$$

$$\begin{pmatrix} 2-\lambda_2 & 0 & 0 \\ -7 & 9-\lambda_2 & 7 \\ 0 & 0 & 2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-9 & 0 & 0 \\ -7 & 9-9 & 7 \\ 0 & 0 & 2-9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row says $-7v_3 = 0$ or $v_3 = 0$. second row says $-7v_1 = 0$ or $v_1 = 0$. First row adds nothing new. So we see that there is no equation to find v_2 (this is because the second column is all zeros). Hence we pick v_2 anything we want. Let $v_2 = 1$ and therefore

$$\mathbf{v}_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now we go back and look at $\lambda_1 = 2$, this is the one with multiplicity $k = 2$. Let first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$(A-\lambda_1 I)\,\boldsymbol{v}_{\lambda_1} = 0$$
$$\begin{pmatrix} 2-\lambda_1 & 0 & 0 \\ -7 & 9-\lambda_1 & 7 \\ 0 & 0 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2-2 & 0 & 0 \\ -7 & 9-2 & 7 \\ 0 & 0 & 2-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row says v_3 is arbitrary. Let $v_3 = s$. Second row says $-v_1 + v_2 + s = 0$, hence $v_1 = v_2 + s$. No other information can be obtained from first row. So v_2 is arbitrary, say $v_2 = r$, hence the solution is

$$\begin{aligned} \boldsymbol{v}_{\lambda_1} &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} r+s \\ r \\ s \end{pmatrix} \\ &= r\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

So we see that we have linear combination of two eigenvectors for λ_1. Hence this eigenvalue is complete and not defective. No need to use the defective eigenvalue algorithm. These are the two L.I. eigenvector we are looking for. We got lucky here. Hence

$$\boldsymbol{v}_{\lambda_1}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$\boldsymbol{v}_{\lambda_1}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The solution is

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + c_3\boldsymbol{x}_3(t) \tag{1}$$

Where now

$$\begin{aligned} \boldsymbol{x}_1(t) &= \boldsymbol{v}_{\lambda_1}^{(1)} e^{\lambda_1 t} \\ \boldsymbol{x}_2(t) &= \boldsymbol{v}_{\lambda_1}^{(2)} e^{\lambda_1 t} \\ \boldsymbol{x}_3(t) &= \boldsymbol{v}_{\lambda_2} e^{\lambda_2 t} \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned} \boldsymbol{x}(t) &= c_1\boldsymbol{v}_{\lambda_1}^{(1)} e^{\lambda_1 t} + c_2\boldsymbol{v}_{\lambda_1}^{(2)} e^{\lambda_1 t} + c_3\boldsymbol{v}_{\lambda_2} e^{\lambda_2 t} \\ &= c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{9t} \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= (c_1 + c_2)\, e^{2t} \\ x_2(t) &= c_1 e^{2t} + c_3 e^{9t} \\ x_3(t) &= c_2 e^{2t} \end{aligned}$$

4.14.16 Section 8.2 problem 5 (page 502)

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 6x - 7y + 10; y' = x - 2y - 2e^{-t}$

solution

The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0 \end{aligned}$$

Hence $\lambda_1 = 5, \lambda_2 = -1$

For $\lambda_1 = 5$

$$\begin{aligned} \begin{pmatrix} 6-\lambda_1 & -7 \\ 1 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $v_1 - 7v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{1}{7}$, hence the eigenvector is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -1$

$$\begin{aligned} \begin{pmatrix} 6-\lambda_2 & -7 \\ 1 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $7v_1 - 7v_2 = 0$. Let $v_1 = 1$ then $v_2 = 1$, hence the eigenvector is

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} \end{aligned}$$

Or

$$\boldsymbol{x}_h(t) = c_1 \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Or

$$\begin{aligned} x_h(t) &= 7c_1 e^{5t} + c_2 e^{-t} \\ y_h(t) &= c_1 e^{5t} + c_2 e^{-t} \end{aligned} \tag{1}$$

We now see that one of the basis solution in the homogenous part $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$, is also present in the forcing function (RHS of the original ODE). So to use the method of undetermined coefficients, we need to multiply by $t^0 e^{-t}$ and $t^1 e^{-t}$. Therefore, since the RHS is $\begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$, then we guess

$$
\begin{aligned}
\boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} te^{-t} \\
&= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} tc_1 \\ tc_2 \end{pmatrix} \right) e^{-t} \\
&= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t}
\end{aligned}
$$

Note that in systems, for duplication, we multiplied by $t^0 e^{-t}$ and $t^1 e^{-t}$. Hence the need for the $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t}$ term in the above. This is little different than in the scalar case where we just needed one multiplication. See the note in middle of page 497 of textbook on this. Now that we have the guess, we plug it into the system and solve for the coefficients.

$$
\begin{aligned}
\boldsymbol{x}_p' &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{-t} - \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t} \\
&= \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t}
\end{aligned}
$$

Plugging the above into original system, which is

$$
\boldsymbol{x}_p' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}_p + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}
$$

Gives

$$
\begin{aligned}
\begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t} \right) + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\
\begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 e^{-t} + te^{-t} c_1 \\ b_2 e^{-t} + te^{-t} c_2 \end{pmatrix} \right) + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\
\begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 + b_1 e^{-t} + te^{-t} c_1 \\ a_2 + b_2 e^{-t} + te^{-t} c_2 \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\
\begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6a_1 - 7a_2 + 6b_1 e^{-t} - 7b_2 e^{-t} + 6tc_1 e^{-t} - 7tc_2 e^{-t} \\ a_1 - 2a_2 + b_1 e^{-t} - 2b_2 e^{-t} + tc_1 e^{-t} - 2tc_2 e^{-t} \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
(c_1 - b_1 - tc_1) e^{-t} &= e^{-t} (6b_1 - 7b_2 + 6tc_1 - 7tc_2) + 6a_1 - 7a_2 + 10 \\
(c_2 - b_2 - tc_2) e^{-t} &= e^{-t} (b_1 - 2b_2 + tc_1 - 2tc_2 - 2) + a_1 - 2a_2
\end{aligned}
$$

Comparing terms, we obtain

$$
\begin{aligned}
c_1 - b_1 - tc_1 &= 6b_1 - 7b_2 + 6tc_1 - 7tc_2 \\
6a_1 - 7a_2 + 10 &= 0 \\
c_2 - b_2 - tc_2 &= b_1 - 2b_2 + tc_1 - 2tc_2 - 2 \\
a_1 - 2a_2 &= 0
\end{aligned}
$$

Or

$$
\begin{aligned}
c_1 - b_1 - tc_1 &= 6b_1 - 7b_2 + t(6c_1 - 7c_2) \\
6a_1 - 7a_2 + 10 &= 0 \\
c_2 - b_2 - tc_2 &= b_1 - 2b_2 + t(c_1 - 2c_2) - 2 \\
a_1 - 2a_2 &= 0
\end{aligned}
$$

Therefore, from the first and third equation above, we see we get additional two equations when we compare terms in t. Hence

$$\begin{aligned}
c_1 - b_1 &= 6b_1 - 7b_2\\
-c_1 &= 6c_1 - 7c_2\\
6a_1 - 7a_2 + 10 &= 0\\
c_2 - b_2 &= b_1 - 2b_2 - 2\\
-c_2 &= c_1 - 2c_2\\
a_1 - 2a_2 &= 0
\end{aligned}$$

Or

$$\begin{aligned}
c_1 - b_1 &= 6b_1 - 7b_2\\
c_1 &= c_2\\
6a_1 - 7a_2 + 10 &= 0\\
c_2 - b_2 &= b_1 - 2b_2 - 2\\
c_2 &= c_1\\
a_1 - 2a_2 &= 0
\end{aligned}$$

Or

$$\begin{aligned}
c_1 - 7b_1 + 7b_2 &= 0\\
c_1 - c_2 &= 0\\
6a_1 - 7a_2 &= -10\\
c_2 + b_2 - b_1 &= -2\\
a_1 - 2a_2 &= 0
\end{aligned}$$

The systems can be written as

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0\\ 6 & -7 & 0 & 0 & 0 & 0\\ 0 & 0 & -7 & 7 & 1 & 0\\ 0 & 0 & -1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2 \end{pmatrix} = \begin{pmatrix} 0\\ -10\\ 0\\ -2\\ 0 \end{pmatrix}$$

$R_2 = R_2 - 6R_1$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0\\ 0 & 5 & 0 & 0 & 0 & 0\\ 0 & 0 & -7 & 7 & 1 & 0\\ 0 & 0 & -1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2 \end{pmatrix} = \begin{pmatrix} 0\\ -10\\ 0\\ -2\\ 0 \end{pmatrix}$$

$R_4 = R_4 - \frac{1}{7}R_3$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0\\ 0 & 5 & 0 & 0 & 0 & 0\\ 0 & 0 & -7 & 7 & 1 & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{7} & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2 \end{pmatrix} = \begin{pmatrix} 0\\ -10\\ 0\\ -2\\ 0 \end{pmatrix}$$

$R_5 + 7R_4$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \\ -2 \\ -14 \end{pmatrix}$$

From last row we obtain that $6c_2 = -14$ or

$$c_2 = \frac{-7}{3}$$

From 4th row

$$\begin{aligned} -\frac{1}{7}c_1 + c_2 &= -2 \\ -\frac{1}{7}c_1 &= \frac{7}{3} - 2 \\ -c_1 &= \frac{49}{3} - 14 \\ c_1 &= 14 - \frac{49}{3} \\ &= -\frac{7}{3} \end{aligned}$$

From 3rd row

$$\begin{aligned} -7b_1 + 7b_2 + c_1 &= 0 \\ -7b_1 &= -7b_2 - \frac{7}{3} \\ b_1 &= b_2 - \frac{1}{3} \end{aligned}$$

From second row

$$\begin{aligned} 5a_2 &= -10 \\ a_2 &= -2 \end{aligned}$$

From first row

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ a_1 &= 2a_2 \\ &= -4 \end{aligned}$$

Therefore the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ b_2 - \frac{1}{3} \\ b_2 \\ -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = b_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -\frac{1}{3} \\ 0 \\ -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

Where b_1 is arbitrary. If we let $b_2 = 0$ then

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -\frac{1}{3} \\ 0 \\ \frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

Therefore, we go back to the particular solution

$$\boldsymbol{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} te^{-t}$$

And substitute these values found in the solution above and obtain

$$\boldsymbol{x}_p = \begin{pmatrix} -4 \\ -2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix} te^{-t}$$

Or

$$\begin{aligned} x_p(t) &= -4 - \frac{1}{3}e^{-t} - \frac{7}{3}te^{-t} \\ y_p(t) &= -2 - \frac{7}{3}te^{-t} \end{aligned}$$

Or

$$\begin{aligned} x_p(t) &= \frac{1}{3}\left(-12 - e^{-t} - 7te^{-t}\right) \\ y_p(t) &= \frac{1}{3}\left(-6 - 7te^{-t}\right) \end{aligned}$$

Hence the complete solution (using the homogenous solution found in (1)) is

$$\begin{aligned} x(t) &= 7c_1e^{5t} + c_2e^{-t} + \frac{1}{3}\left(-12 - e^{-t} - 7te^{-t}\right) \\ y_h(t) &= c_1e^{5t} + c_2e^{-t} + \frac{1}{3}\left(-6 - 7te^{-t}\right) \end{aligned}$$

4.14.17 Section 8.2 problem 9

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = x - 5y + \cos 2t; y' = x - y$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos 2t \\ 0 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 4 &= 0 \\ \lambda &= \pm 2i \end{aligned}$$

For $\lambda_1 = 2i$ we solve $(A - \lambda_1 I)\, v_1 = 0$

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$(1-2i)\, v_1 - 5v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{(1-2i)}{5}$, therefore

$$v_1 = \begin{pmatrix} 1 \\ \frac{(1-2i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

For $\lambda_1 = -2i$ we solve $(A - \lambda_2 I)\, v_2 = 0$

$$\begin{pmatrix} 1+2i & -5 \\ 1 & -1+2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$(1+2i)\, v_1 - 5v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{(1+2i)}{5}$, therefore

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ \frac{(1+2i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t)\, e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Or

$$\boldsymbol{x}_h(t) = c_1 \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 5 \\ 1+2i \end{pmatrix} e^{-2it}$$

Convert to new basis

$$\begin{aligned} \boldsymbol{x}_1(t) &= \operatorname{Re}(\boldsymbol{x}_1(t)) \\ &= \operatorname{Re}\begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} = \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ (\cos 2t + i\sin 2t) - 2i(\cos 2t + i\sin 2t) \end{pmatrix} \\ &= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ (\cos 2t + i\sin 2t) - 2(i\cos 2t - \sin 2t) \end{pmatrix} \\ &= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t - 2i\cos 2t + 2\sin 2t \end{pmatrix} \\ &= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \end{pmatrix} \\ &= \begin{pmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} \boldsymbol{x}_2(t) &= \operatorname{Im}(\boldsymbol{x}_1(t)) \\ &= \begin{pmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \end{aligned}$$

Hence the homogeneous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= C_1 \boldsymbol{x}_1(t) + C_2 \boldsymbol{x}_2(t) \\ &= C_1 \begin{pmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \end{aligned}$$

Or

$$\begin{aligned} x_h(t) &= 5C_1 \cos 2t + 5C_2 \sin 2t \\ y_h(t) &= (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t \end{aligned}$$

We now see that one of the basis solutions for the homogenous part contains $\cos 2t$ which is also in the forcing function of the original system. Hence we need to pick a guess where we multiply by extra t. Since the forcing function is $\begin{pmatrix}\cos 2t\\0\end{pmatrix}$ then guess

$$\begin{aligned}
\boldsymbol{x}_p &= \begin{pmatrix}a_1\\a_2\end{pmatrix}\sin 2t + \begin{pmatrix}b_1\\b_2\end{pmatrix}\cos 2t + \begin{pmatrix}c_1\\c_2\end{pmatrix}t\sin 2t + \begin{pmatrix}d_1\\d_2\end{pmatrix}t\cos 2t \qquad (1)\\
&= \begin{pmatrix}a_1+tc_1\\a_2+tc_2\end{pmatrix}\sin 2t + \begin{pmatrix}b_1+td_1\\b_2+td_2\end{pmatrix}\cos 2t
\end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{x}_p' &= \begin{pmatrix}c_1\\c_2\end{pmatrix}\sin 2t + 2\begin{pmatrix}a_1+tc_1\\a_2+tc_2\end{pmatrix}\cos 2t + \begin{pmatrix}d_1\\d_2\end{pmatrix}\cos 2t - 2\begin{pmatrix}b_1+td_1\\b_2+td_2\end{pmatrix}\sin 2t\\
&= \begin{pmatrix}c_1-2(b_1+td_1)\\c_2-2(b_2+td_2)\end{pmatrix}\sin 2t + \begin{pmatrix}d_1+2(a_1+tc_1)\\d_2+2(a_2+tc_2)\end{pmatrix}\cos 2t \qquad (2)
\end{aligned}$$

We now substitute (1) and (2) into

$$\boldsymbol{x}_p' = \begin{pmatrix}1&-5\\1&-1\end{pmatrix}\boldsymbol{x}_p + \begin{pmatrix}\cos 2t\\0\end{pmatrix}$$

Hence

$$\begin{pmatrix}c_1-2(b_1+td_1)\\c_2-2(b_2+td_2)\end{pmatrix}\sin 2t + \begin{pmatrix}d_1+2(a_1+tc_1)\\d_2+2(a_2+tc_2)\end{pmatrix}\cos 2t =\\
\begin{pmatrix}1&-5\\1&-1\end{pmatrix}\left(\begin{pmatrix}a_1+tc_1\\a_2+tc_2\end{pmatrix}\sin 2t + \begin{pmatrix}b_1+td_1\\b_2+td_2\end{pmatrix}\cos 2t\right) + \begin{pmatrix}\cos 2t\\0\end{pmatrix}$$

Hence

$$\begin{pmatrix}(c_1-2(b_1+td_1))\sin 2t\\(c_2-2(b_2+td_2))\sin 2t\end{pmatrix} + \begin{pmatrix}(d_1+2(a_1+tc_1))\cos 2t\\(d_2+2(a_2+tc_2))\cos 2t\end{pmatrix} =\\
\begin{pmatrix}1&-5\\1&-1\end{pmatrix}\begin{pmatrix}(a_1+tc_1)\sin 2t+(b_1+td_1)\cos 2t\\(a_2+tc_2)\sin 2t+(b_2+td_2)\cos 2t\end{pmatrix} + \begin{pmatrix}\cos 2t\\0\end{pmatrix}$$

Or

$$\begin{pmatrix}(c_1-2(b_1+td_1))\sin 2t+(d_1+2(a_1+tc_1))\cos 2t\\(c_2-2(b_2+td_2))\sin 2t+(d_2+2(a_2+tc_2))\cos 2t\end{pmatrix} =\\
\begin{pmatrix}(\cos 2t)(b_1+td_1)-5(\cos 2t)(b_2+td_2)+(\sin 2t)(a_1+tc_1)-5(\sin 2t)(a_2+tc_2)\\(\cos 2t)(b_1+td_1)-(\cos 2t)(b_2+td_2)+(\sin 2t)(a_1+tc_1)-(\sin 2t)(a_2+tc_2)\end{pmatrix} + \begin{pmatrix}\cos 2t\\0\end{pmatrix}$$

Therefore

$$(c_1-2(b_1+td_1))\sin 2t+(d_1+2(a_1+tc_1))\cos 2t =\\
(\cos 2t)(b_1+td_1)-5(\cos 2t)(b_2+td_2)+(\sin 2t)(a_1+tc_1)-5(\sin 2t)(a_2+tc_2)+\cos 2t \qquad (3)$$

And

$$(c_2-2(b_2+td_2))\sin 2t+(d_2+2(a_2+tc_2))\cos 2t =\\
(\cos 2t)(b_1+td_1)-(\cos 2t)(b_2+td_2)+(\sin 2t)(a_1+tc_1)-(\sin 2t)(a_2+tc_2) \qquad (4)$$

Equation (3,4) are solved for the unknowns. We need 8 equations in total. Looking at (3) for now. Comparing coefficients of $\sin 2t$ in (3)

$$\begin{aligned}
(c_1-2(b_1+td_1)) &= (a_1+tc_1)-5(a_2+tc_2)\\
c_1-2b_1-2td_1 &= a_1-5a_2+tc_1-5tc_2\\
c_1-2b_1+t(-2d_1) &= a_1-5a_2+t(c_1-5c_2)
\end{aligned}$$

Comparing coefficients we see

$$\begin{aligned}
c_1-2b_1 &= a_1-5a_2\\
a_1-5a_2-c_1+2b_1 &= 0 \qquad (1A)
\end{aligned}$$

And

$$\begin{aligned} -2d_1 &= c_1 - 5c_2 \\ c_1 - 5c_2 + 2d_1 &= 0 \end{aligned} \tag{2A}$$

We do the same for $\cos 2t$ in equation (3) and compare coefficients

$$\begin{aligned} (d_1 + 2(a_1 + tc_1)) &= (b_1 + td_1) - 5(b_2 + td_2) + 1 \\ 2a_1 + d_1 + 2tc_1 &= b_1 - 5b_2 + td_1 - 5td_2 + 1 \\ 2a_1 + d_1 + t(2c_1) &= b_1 - 5b_2 + 1 + t(d_1 - 5d_2) \end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned} 2a_1 + d_1 &= b_1 - 5b_2 + 1 \\ 2a_1 + d_1 - b_1 + 5b_2 &= 1 \end{aligned} \tag{3A}$$

And

$$\begin{aligned} 2c_1 &= d_1 - 5d_2 \\ 2c_1 - d_1 + 5d_2 &= 0 \end{aligned} \tag{4A}$$

We have obtained 4 equations from (3). We do the same on (4) to obtain the other 4 equations. Comparing $\sin 2t$ terms in (4) gives

$$\begin{aligned} (c_2 - 2(b_2 + td_2)) &= (a_1 + tc_1) - (a_2 + tc_2) \\ c_2 - 2b_2 - 2td_2 &= a_1 - a_2 + tc_1 - tc_2 \\ c_2 - 2b_2 + t(-2d_2) &= a_1 - a_2 + t(c_1 - c_2) \end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned} c_2 - 2b_2 &= a_1 - a_2 \\ a_1 - a_2 - c_2 + 2b_2 &= 0 \end{aligned} \tag{5A}$$

And

$$\begin{aligned} -2d_2 &= c_1 - c_2 \\ c_1 - c_2 + 2d_2 &= 0 \end{aligned} \tag{6A}$$

Finally, Comparing $\cos 2t$ terms in (4) gives

$$\begin{aligned} (d_2 + 2(a_2 + tc_2)) &= (b_1 + td_1) - (b_2 + td_2) \\ 2a_2 + d_2 + 2tc_2 &= b_1 - b_2 + td_1 - td_2 \\ 2a_2 + d_2 + t(2c_2) &= b_1 - b_2 + t(d_1 - d_2) \end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned} 2a_2 + d_2 &= b_1 - b_2 \\ 2a_2 + d_2 - b_1 + b_2 &= 0 \end{aligned} \tag{7A}$$

And

$$\begin{aligned} 2c_2 &= d_1 - d_2 \\ d_1 - d_2 - 2c_2 &= 0 \end{aligned} \tag{8A}$$

Equations (1A) to (8A) are now solved for $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$.

$$\begin{aligned} a_1 - 5a_2 - c_1 + 2b_1 &= 0 \\ c_1 - 5c_2 + 2d_1 &= 0 \\ 2a_1 + d_1 - b_1 + 5b_2 &= 1 \\ 2c_1 - d_1 + 5d_2 &= 0 \\ a_1 - a_2 - c_2 + 2b_2 &= 0 \\ c_1 - c_2 + 2d_2 &= 0 \\ 2a_2 + d_2 - b_1 + b_2 &= 0 \\ d_1 - d_2 - 2c_2 &= 0 \end{aligned}$$

Writing the equations in matrix form

$$\begin{pmatrix} 1 & -5 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5 & 2 & 0 \\ 2 & 0 & -1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 5 \\ 1 & -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 2 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the above using the computer gives

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Solve 8.2 problem 9 final Matrix equation

```
In[112]:= mat = {{1, -5, 2, 0, -1, 0, 0, 0},
    {0, 0, 0, 0, 1, -5, 2, 0},
    {2, 0, -1, 5, 0, 0, 1, 0},
    {0, 0, 0, 0, 2, 0, -1, 5},
    {1, -1, 0, 2, 0, -1, 0, 0},
    {0, 0, 0, 0, 1, -1, 0, 2},
    {0, 2, -1, 1, 0, 0, 0, 1},
    {0, 0, 0, 0, 0, -2, 1, -1}};
  b = {0, 0, 1, 0, 0, 0, 0, 0};
  LinearSolve[mat, b]

Out[114]= {1/4, 0, 0, 0, 1/4, 1/4, 1/2, 0}
```

We now go back to (1) and plugging these values into the particular solution

$$\begin{aligned} x_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin 2t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos 2t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t \sin 2t + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} t \cos 2t \\ &= \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} t \sin 2t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} t \cos 2t \end{aligned}$$

Hence

$$\begin{aligned} x_p(t) &= \frac{1}{4} \sin 2t + \frac{1}{4} t \sin 2t + \frac{1}{2} t \cos 2t \\ y_p(t) &= \frac{1}{4} t \sin 2t \end{aligned}$$

Or

$$\begin{aligned} x_p(t) &= \frac{1}{4} (\sin 2t + t \sin 2t + 2t \cos 2t) \\ y_p(t) &= \frac{1}{4} t \sin 2t \end{aligned}$$

Earlier we obtained the homogenous solution as

$$x_h(t) = 5C_1 \cos 2t + 5C_2 \sin 2t$$
$$y_h(t) = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t$$

Therefore the general solution is

$$x(t) = 5C_1 \cos 2t + 5C_2 \sin 2t + \frac{1}{4}(\sin 2t + t\sin 2t + 2t\cos 2t)$$
$$y(t) = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t + \frac{1}{4}t\sin 2t$$

4.14.18 Section 8.2 problem 11

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 2x + 4y + 2; y' = x + 2y + 3; x(0) = 1, y(0) = -1$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 4\lambda = 0$$
$$(\lambda - 4)\lambda = 0$$

Hence $\lambda_1 = 0, \lambda_2 = 4$.

For $\lambda_1 = 0$ we solve $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} 2-\lambda_1 & 4 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$2v_1 + 4v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{-1}{2}$, therefore

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda_1 = 4$ we solve $(A - \lambda_2 I)\mathbf{v}_2 = 0$

$$\begin{pmatrix} 2-\lambda_2 & 4 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$-2v_1 + 4v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{1}{2}$, therefore

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) \\ &= c_1\boldsymbol{v}_1(t)e^{\lambda_1 t} + c_2\boldsymbol{v}_2(t)e^{\lambda_2 t} \end{aligned}$$

Or

$$\boldsymbol{x}_h(t) = c_1\begin{pmatrix}2\\-1\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix}e^{4t} \tag{1}$$

Since constant term exist in both homogenous solution and in forcing function then guess

$$\boldsymbol{x}_p = \begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix}t$$

Therefore

$$\boldsymbol{x}_p' = \begin{pmatrix}b_1\\b_2\end{pmatrix}$$

Substituting this into

$$\boldsymbol{x}_p' = \begin{pmatrix}2 & 4\\1 & 2\end{pmatrix}\boldsymbol{x}_p + \begin{pmatrix}2\\3\end{pmatrix}$$

Gives

$$\begin{aligned} \begin{pmatrix}b_1\\b_2\end{pmatrix} &= \begin{pmatrix}2 & 4\\1 & 2\end{pmatrix}\left(\begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix}t\right) + \begin{pmatrix}2\\3\end{pmatrix} \\ \begin{pmatrix}b_1\\b_2\end{pmatrix} &= \begin{pmatrix}2 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}a_1 + b_1 t\\a_2 + b_2 t\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} \\ \begin{pmatrix}b_1\\b_2\end{pmatrix} &= \begin{pmatrix}2a_1 + 4a_2 + 2tb_1 + 4tb_2\\a_1 + 2a_2 + tb_1 + 2tb_2\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} b_1 &= 2a_1 + 4a_2 + 2tb_1 + 4tb_2 + 2 \\ b_2 &= a_1 + 2a_2 + tb_1 + 2tb_2 + 3 \end{aligned}$$

Or

$$\begin{aligned} b_1 &= 2a_1 + 4a_2 + 2 + t(2b_1 + 4b_2) \\ b_2 &= a_1 + 2a_2 + 3 + t(b_1 + 2b_2) \end{aligned}$$

So by comparing coefficients in each equation we obtain 4 equations as follows

$$\begin{aligned} b_1 &= 2a_1 + 4a_2 + 2 \\ 2b_1 + 4b_2 &= 0 \\ b_2 &= a_1 + 2a_2 + 3 \\ b_1 + 2b_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} 2a_1 + 4a_2 - b_1 &= -2 \\ 2b_1 + 4b_2 &= 0 \\ a_1 + 2a_2 - b_2 &= -3 \\ b_1 + 2b_2 &= 0 \end{aligned}$$

Hence the matrix form is

$$\begin{pmatrix}2 & 4 & -1 & 0\\0 & 0 & 2 & 4\\1 & 2 & 0 & -1\\0 & 0 & 1 & 2\end{pmatrix}\begin{pmatrix}a_1\\a_2\\b_1\\b_2\end{pmatrix} = \begin{pmatrix}-2\\0\\-3\\0\end{pmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_1$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - \frac{1}{4}R_2$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_4 = R_4 - \frac{1}{2}R_2$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

Third row gives $-2b_2 = -2$ or $b_2 = 1$. From second row $2b_1 + 4b_2 = 0$, or $b_1 = -2b_2 = -2$. First row gives

$$\begin{aligned} 2a_1 + 4a_2 - b_1 &= -2 \\ 2a_1 + 4a_2 &= -2 + b_1 \\ 2a_1 + 4a_2 &= -4 \\ \frac{1}{2}a_1 + a_2 &= -1 \end{aligned}$$

Hence a_1 or a_2 are arbitrary. Let $a_2 = 0$ then $a_2 = -2$. Hence the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t \\ &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t \end{aligned}$$

Using (1) the complete solution is

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t \tag{2}$$

At $t = 0$

$$\begin{aligned} \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 + 2c_2 - 2 \\ -c_1 + c_2 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} 2c_1 + 2c_2 - 2 &= 1 \\ -c_1 + c_2 &= -1 \end{aligned}$$

Or

$$\begin{aligned} 2c_1 + 2c_2 &= 3 \\ -c_1 + c_2 &= -1 \end{aligned}$$

Which gives $c_1 = \frac{5}{4}, c_2 = \frac{1}{4}$, therefore (2) becomes

$$\boldsymbol{x}(t) = \frac{5}{4}\begin{pmatrix}2\\-1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}2\\1\end{pmatrix}e^{4t} + \begin{pmatrix}-2\\0\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix}t$$

Or

$$\begin{aligned} x(t) &= \frac{10}{4} + \frac{1}{2}e^{4t} - 2 - 2t \\ y(t) &= -\frac{5}{4} + \frac{1}{4}e^{4t} + t \end{aligned}$$

Or

$$\begin{aligned} x(t) &= \frac{1}{2}\left(1 - 4t + e^{4t}\right) \\ y(t) &= \frac{1}{4}\left(-5 + 4t + e^{4t}\right) \end{aligned}$$

4.14.19 Section 8.2 problem 13

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 2x + y + 2e^t; y' = x + 2y - 3e^t$

solution The matrix form of the system is

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2e^t\\-3e^t\end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix}2-\lambda & 1\\1 & 2-\lambda\end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 3)(\lambda - 1) &= 0 \end{aligned}$$

Hence $\lambda_1 = 1, \lambda_2 = 3$.

For $\lambda_1 = 1$ we solve $(A - \lambda_1 I)\boldsymbol{v}_1 = 0$

$$\begin{aligned} \begin{pmatrix}2-\lambda_1 & 1\\1 & 2-\lambda_1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \\ \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \end{aligned}$$

From first equation $v_1 + v_2 = 0$. Let $v_1 = 1$, hence $v_2 = -1$ and therefore

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$$

For $\lambda_1 = 3$ we solve $(A - \lambda_2 I)\boldsymbol{v}_2 = 0$

$$\begin{aligned} \begin{pmatrix}2-\lambda_2 & 1\\1 & 2-\lambda_2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \\ \begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \end{aligned}$$

From first equation $-v_1 + v_2 = 0$. Let $v_1 = 1$, hence $v_2 = 1$ and therefore

$$\boldsymbol{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) \\ &= c_1\boldsymbol{v}_1(t)e^{\lambda_1 t} + c_2\boldsymbol{v}_2(t)e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \tag{1}$$

Since the forcing function is $\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$ and e^t is a basis solution for the homogenous part, then we guess

$$\begin{aligned} x_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t \\ &= \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \end{pmatrix} e^t \end{aligned}$$

Hence

$$\begin{aligned} x_p' &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \end{pmatrix} e^t \\ &= \begin{pmatrix} b_1 + a_1 + tb_1 \\ b_2 + a_2 + tb_2 \end{pmatrix} e^t \end{aligned}$$

Plugging this back into

$$x_p' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x_p + \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

Gives

$$\begin{aligned} \begin{pmatrix} b_1 + a_1 + tb_1 \\ b_2 + a_2 + tb_2 \end{pmatrix} e^t &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \end{pmatrix} e^t + \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} \\ \begin{pmatrix} b_1 + a_1 + tb_1 \\ b_2 + a_2 + tb_2 \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\ \begin{pmatrix} b_1 + a_1 + tb_1 \\ b_2 + a_2 + tb_2 \end{pmatrix} &= \begin{pmatrix} 2a_1 + a_2 + 2tb_1 + tb_2 \\ a_1 + 2a_2 + tb_1 + 2tb_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} b_1 + a_1 + tb_1 &= 2a_1 + a_2 + 2tb_1 + tb_2 + 2 \\ b_2 + a_2 + tb_2 &= a_1 + 2a_2 + tb_1 + 2tb_2 - 3 \end{aligned}$$

or

$$\begin{aligned} b_1 + a_1 + tb_1 &= 2a_1 + a_2 + 2 + t(2b_1 + b_2) \\ b_2 + a_2 + tb_2 &= a_1 + 2a_2 - 3 + t(b_1 + 2b_2) \end{aligned}$$

Comparing coefficients in the above two equations generates 4 equations to solve for the unknowns

$$\begin{aligned} b_1 + a_1 &= 2a_1 + a_2 + 2 \\ b_1 &= 2b_1 + b_2 \\ b_2 + a_2 &= a_1 + 2a_2 - 3 \\ b_2 &= b_1 + 2b_2 \end{aligned}$$

Or

$$\begin{aligned} a_1 + a_2 - b_1 &= -2 \\ b_1 + b_2 &= 0 \\ a_1 + a_2 - b_2 &= 3 \\ b_1 + b_2 &= 0 \end{aligned}$$

Second and third equation are the same. Using the first 3 equations, the matrix equations are

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$

This is undetermined system. It will either have infinite number of solutions or no solution.

Let $R_2 = R_2 - R_1$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -5 \end{pmatrix}$$

Last row gives $2b_2 = -5$ or $b_2 = \frac{-5}{2}$. Second row gives $b_1 - b_2 = 5$ or $b_1 = 5 + b_2 = 5 - \frac{5}{2} = \frac{5}{2}$. First row gives $a_1 + a_2 - b_1 = -2$ or $a_1 = -a_2 + b_1 - 2$ or

$$\begin{aligned} a_1 &= -a_2 + b_1 - 2 \\ &= -a_2 + \frac{5}{2} - 2 \\ &= -a_2 + \frac{1}{2} \end{aligned}$$

a_2 is arbitrary. Let $a_2 = 0$ and we obtain $a_1 = \frac{1}{2}$. Hence the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix}$$

Therefore since $\boldsymbol{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t$ then

$$\boldsymbol{x}_p = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix} te^t$$

Or

$$\begin{aligned} x_p(t) &= \frac{1}{2}(1 + 5t) e^t \\ y_p(t) &= \frac{-5}{2} te^t \end{aligned}$$

And the general solution is

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_0(t) \\ \boldsymbol{x}_h(t) &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix} te^t \end{aligned}$$

Or

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{3t} + \frac{1}{2}(1 + 5t) e^t \\ y(t) &= -c_1 e^t + c_2 e^{3t} - \frac{5}{2} te^t \end{aligned}$$

4.14.20 Section 8.2 problem 19

problem Use the method of variation of parameters to solve $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}; f(t) = \begin{pmatrix} 180t \\ 90 \end{pmatrix}$$
$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 180t \\ 90 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + \lambda - 6 &= 0 \\ (\lambda - 2)(\lambda + 3) &= 0 \end{aligned}$$

Hence $\lambda_1 = 2, \lambda_2 = -3$.

For $\lambda_1 = 2$ we solve $(A - \lambda_1 I) v_1 = 0$

$$\begin{aligned} \begin{pmatrix} 1-\lambda_1 & 2 \\ 2 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $-v_1 + 2v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{1}{2}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -3$ we solve $(A - \lambda_2 I) v_2 = 0$

$$\begin{aligned} \begin{pmatrix} 1-\lambda_2 & 2 \\ 2 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $4v_1 + 2v_2 = 0$. Let $v_1 = 1$ then $v_2 = -2$ and

$$v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Therefore

$$\begin{aligned} x_h(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t) e^{\lambda_1 t} + c_2 v_2(t) e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} \tag{1}$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$\begin{aligned} W &= \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \end{aligned}$$

Therefore

$$x_p(t) = W \int W^{-1} f(t)\, dt \tag{2}$$

Where

$$
\begin{aligned}
W^{-1} &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{2}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{2}{5}e^{3t} \end{pmatrix}
\end{aligned}
$$

Hence, (2) becomes

$$
\begin{aligned}
\boldsymbol{x}_p(t) &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \int \begin{pmatrix} \frac{2}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{2}{5}e^{3t} \end{pmatrix} \begin{pmatrix} 180t \\ 90 \end{pmatrix} dt \\
&= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \int \begin{pmatrix} 18e^{-2t} + 72te^{-2t} \\ 36te^{3t} - 36e^{3t} \end{pmatrix} dt \\
&= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \begin{pmatrix} -9e^{-2t}(4t+3) \\ 4e^{3t}(3t-4) \end{pmatrix} \\
&= \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \qquad (3) \\
&= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix}
\end{aligned}
$$

At $t = 0$

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -70 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 2c_1 + c_2 - 70 \\ c_1 - 2c_2 + 5 \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 70 \\ -5 \end{pmatrix}
$$

$$
\begin{pmatrix} 2 & 1 \\ 0 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \end{pmatrix}
$$

Last row gives $\frac{-5}{2}c_2 = -40$, or $c_2 = 16$. First row gives $2c_1 + c_2 = 70$, hence $c_1 = \frac{70-c_2}{2} = \frac{70-16}{2} = 27$. Hence the solution from (3) becomes

$$
\boldsymbol{x}(t) = 27 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} + 16 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix}
$$

Or

$$
\begin{aligned}
x(t) &= 54e^{2t} + 16e^{-3t} - 60t - 70 \\
y(t) &= -27e^{2t} - 32e^{-3t} + 5 - 60t
\end{aligned}
$$

4.14.21 Section 8.2 problem 22

problem Use the method of variation of parameters to solve $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$.

$$
A = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}; \boldsymbol{f}(t) = \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix}
$$

$$
\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

solution The matrix form of the system is

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix}
$$

The eigenvalues of the homogenous system are found from

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
\begin{vmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} &= 0 \\
\lambda^2 - 2\lambda - 3 &= 0 \\
(\lambda+1)(\lambda-3) &= 0
\end{aligned}
$$

Hence $\lambda_1 = -1, \lambda_2 = 3$.

For $\lambda_1 = -1$ we solve $(A - \lambda_1 I)\, \boldsymbol{v}_1 = 0$

$$
\begin{aligned}
\begin{pmatrix} 4-\lambda_1 & -1 \\ 5 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first equation $5v_1 - v_2 = 0$. Let $v_1 = 1$ then $v_2 = 5$ and

$$
\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}
$$

For $\lambda_2 = 3$ we solve $(A - \lambda_2 I)\, \boldsymbol{v}_2 = 0$

$$
\begin{aligned}
\begin{pmatrix} 4-\lambda_2 & -1 \\ 5 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first equation $v_1 - v_2 = 0$. Let $v_1 = 1$ then $v_2 = 1$ and

$$
\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

Therefore

$$
\begin{aligned}
\boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\
&= c_1 \boldsymbol{v}_1(t)\, e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t)\, e^{\lambda_2 t}
\end{aligned}
$$

Or

$$
\boldsymbol{x}_h(t) = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \tag{1}
$$

The Wronskian W is, (which is the same as fundamental matrix Φ) is

$$
\begin{aligned}
W &= \begin{pmatrix} \boldsymbol{x}_1(t) & \boldsymbol{x}_2(t) \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix}
\end{aligned}
$$

Therefore

$$
\boldsymbol{x}_p(t) = W \int W^{-1} \boldsymbol{f}(t)\, dt \tag{2}
$$

Where

$$
\begin{aligned}
W^{-1} &= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -\frac{1}{4}e^{t} & \frac{1}{4}e^{t} \\ \frac{5}{4}e^{-3t} & -\frac{1}{4}e^{-3t} \end{pmatrix}
\end{aligned}
$$

Hence, (2) becomes

$$\begin{aligned}
x_p(t) &= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \int \begin{pmatrix} -\frac{1}{4}e^{t} & \frac{1}{4}e^{t} \\ \frac{5}{4}e^{-3t} & -\frac{1}{4}e^{-3t} \end{pmatrix} \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \int \begin{pmatrix} 5e^{4t} - 7 \\ 35e^{-4t} - 5 \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{5}{4}e^{4t} - 7t \\ -5t - \frac{35}{4}e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
x(t) &= x_h(t) + x_p(t) \qquad (3) \\
&= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}
\end{aligned}$$

At $t = 0$

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{4} - \frac{35}{4} \\ \frac{25}{4} - \frac{35}{4} \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{15}{2} \\ -\frac{5}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1 + c_2 - \frac{15}{2} \\ 5c_1 + c_2 - \frac{5}{2} \end{pmatrix}
\end{aligned}$$

Hence

$$\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ -35 \end{pmatrix}$$

Last row gives $-4c_2 = -35$, or $c_2 = \frac{35}{4}$. First row gives $c_1 + c_2 = \frac{15}{2}$, hence $c_1 = \frac{15}{2} - \frac{35}{4} = -\frac{5}{4}$. Hence the solution from (3) becomes

$$x(t) = \frac{-5}{4} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + \frac{35}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}$$

Or

$$\begin{aligned}
x(t) &= \frac{-5}{4}e^{-t} + \frac{35}{4}e^{3t} + \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\
y(t) &= -\frac{25}{4}e^{-t} + \frac{35}{4}e^{3t} + \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t}
\end{aligned}$$

Or

$$\begin{aligned}
x(t) &= -10e^{-t} + 10e^{3t} - 7te^{-t} - 5te^{3t} \\
y(t) &= -15e^{-t} + 15e^{3t} - 35te^{-t} - 5te^{3t}
\end{aligned}$$

4.14.22 Section 8.2 problem 25

problem Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$
$$f(t) = \begin{pmatrix} 4t \\ 1 \end{pmatrix}$$
$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4t \\ 1 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

Hence $\lambda_1 = -i, \lambda_2 = i$.

For $\lambda_1 = -i$ we solve $(A - \lambda_1 I)\, v_1 = 0$

$$\begin{pmatrix} 2-\lambda_1 & -5 \\ 1 & -2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $(2+i)\,v_1 - 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{(2+i)}{5}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{(2+i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$$

For $\lambda_2 = i$ we solve $(A - \lambda_2 I)\, v_2 = 0$

$$\begin{pmatrix} 2-\lambda_2 & -5 \\ 1 & -2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $(2-i)\,v_1 - 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{(2-i)}{5}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{(2-i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

Therefore

$$\begin{aligned} x_h(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t)\, e^{\lambda_1 t} + c_2 v_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 5 \\ 2+i \end{pmatrix} e^{-it} + c_2 \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{it} \tag{1}$$

Change to new basis

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \operatorname{Re}\begin{pmatrix} 5 \\ 2+i \end{pmatrix} e^{-it} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ (2+i)(\cos t - i\sin t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2(\cos t - i\sin t) + (i\cos t + \sin t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2\cos t + \sin t + i(-2\sin t + \cos t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\boldsymbol{x}_2(t) &= \operatorname{Im}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2\cos t + \sin t + i(-2\sin t + \cos t) \end{pmatrix} \\
&= \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix}
\end{aligned}
$$

Hence the homogenous solution in the new basis is

$$
\boxed{\mathbf{x}_h(t) = C_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2\begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix}} \tag{1A}
$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$
\begin{aligned}
W &= \begin{pmatrix} \boldsymbol{x}_1(t) & \boldsymbol{x}_2(t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}
\end{aligned}
$$

Therefore

$$
\boldsymbol{x}_p(t) = W\int W^{-1}\boldsymbol{f}(t)\,dt \tag{2}
$$

Where

$$
\begin{aligned}
W^{-1} &= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{5}(\cos t - 2\sin t) & \sin t \\ \frac{1}{5}(-2\cos t + \sin t) & \cos t \end{pmatrix}
\end{aligned}
$$

Hence, (2) becomes

$$
\begin{aligned}
\boldsymbol{x}_p(t) &= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\int\begin{pmatrix} \frac{1}{5}(\cos t - 2\sin t) & \sin t \\ \frac{1}{5}(-2\cos t + \sin t) & \cos t \end{pmatrix}\begin{pmatrix} 4t \\ 1 \end{pmatrix} dt \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\int\begin{pmatrix} \sin t + 4t\left(\frac{1}{5}\cos t - \frac{2}{5}\sin t\right) \\ \cos t - 4t\left(\frac{2}{5}\cos t - \frac{1}{5}\sin t\right) \end{pmatrix} dt \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\begin{pmatrix} \frac{8}{5}t\cos t - \frac{8}{5}\sin t - \frac{1}{5}\cos t + \frac{4}{5}t\sin t \\ \frac{9}{5}\sin t - \frac{8}{5}\cos t - \frac{4}{5}t\cos t - \frac{8}{5}t\sin t \end{pmatrix}
\end{aligned}
$$

Which, with little help of computer algebra, simplifies to

$$
\boldsymbol{x}_p(t) = \begin{pmatrix} 8t - 1 \\ 4t - 2 \end{pmatrix}
$$

Therefore the general solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \qquad (3)\\
&= C_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2 \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix} + \begin{pmatrix} 8t-1 \\ 4t-2 \end{pmatrix}
\end{aligned}
$$

At $t = 0$

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= C_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}\\
&= \begin{pmatrix} 5C_1 - 1 \\ 2C_1 + C_2 - 2 \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

First row gives $C_1 = \frac{1}{5}$ and last row gives $2C_1 + C_2 = 2$ or $C_2 = 2 - \frac{2}{5} = \frac{8}{5}$. Hence the solution becomes

$$
\boldsymbol{x}(t) = \frac{1}{5} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + \frac{8}{5} \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix} + \begin{pmatrix} 8t-1 \\ 4t-2 \end{pmatrix}
$$

Or

$$
\begin{aligned}
x(t) &= \cos t - 8\sin t + 8t + 8t - 1\\
y(t) &= 2\cos t - 3\sin t + 4t - 2
\end{aligned}
$$

4.14.23 Section 8.2 problem 28

problem Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\\
f(t) &= \begin{pmatrix} 4\ln t \\ \frac{1}{t} \end{pmatrix}\\
\boldsymbol{x}(1) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}
$$

solution The matrix form of the system is

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4\ln t \\ t^{-1} \end{pmatrix}
$$

The eigenvalues of the homogenous system are found from

$$
\begin{aligned}
|A - \lambda I| &= 0\\
\begin{vmatrix} 2-\lambda & -4 \\ 1 & -2-\lambda \end{vmatrix} &= 0\\
\lambda^2 &= 0\\
\lambda &= 0
\end{aligned}
$$

Hence zero eigenvalue. Let see if this is complete eigenvalue or not.

For $\lambda = 0$ we solve $(A - \lambda_1 I)\boldsymbol{v}_1 = 0$

$$
\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

From first equation $2v_1 - 4v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{1}{2}$ and

$$
\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
$$

We can only find this one eigenvector. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solving for v_2 from

$$
\begin{aligned}
(A-\lambda I)^2 v_2 &= 0\\
\begin{pmatrix} 2-\lambda & -4 \\ 1 & -2-\lambda \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Hence $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ can be any value. Let $v_1 = 1, v_2 = 0$ and therefore

$$
\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

We now find $\boldsymbol{v}_1$ from

$$
\begin{aligned}
\boldsymbol{v}_1 &= (A-\lambda I)\,\boldsymbol{v}_2\\
&= \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

Where now

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \boldsymbol{v}_1 e^{\lambda t}\\
\boldsymbol{x}_2(t) &= (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t}
\end{aligned}
$$

Hence the homogenous solution is

$$
\begin{aligned}
\boldsymbol{x}_h(t) &= c_1 \boldsymbol{v}_1 e^{\lambda t} + c_2 (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{\lambda t}\\
&= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)\\
&= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2t+1 \\ t \end{pmatrix}
\end{aligned}
$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$
\begin{aligned}
W &= \begin{pmatrix} \boldsymbol{x}_1(t) & \boldsymbol{x}_2(t) \end{pmatrix}\\
&= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix}
\end{aligned}
$$

Therefore

$$
\boldsymbol{x}_p(t) = W \int W^{-1} \boldsymbol{f}(t)\, dt \tag{2}
$$

Where

$$
\begin{aligned}
W^{-1} &= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix}^{-1}\\
&= \begin{pmatrix} -t & 2t+1 \\ 1 & -2 \end{pmatrix}
\end{aligned}
$$

Hence, (2) becomes

$$
\begin{aligned}
x_p(t) &= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \int \begin{pmatrix} -t & 2t+1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4\ln t \\ \frac{1}{t} \end{pmatrix} dt \\
&= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \int \begin{pmatrix} \frac{1}{t}(2t+1) - 4t\ln t \\ 4\ln t - \frac{2}{t} \end{pmatrix} dt \\
&= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \begin{pmatrix} 2t + \ln t - 2t^2 \ln t + t^2 \\ 4t\ln t - 2\ln t - 4t \end{pmatrix} \\
&= \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
x(t) &= x_h(t) + x_p(t) \qquad (3) \\
&= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} + \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}
\end{aligned}
$$

At $t = 1$

$$
\begin{aligned}
\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 2c_1 + 3c_2 - 6 \\ c_1 + c_2 - 1 \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{7}{2} \end{pmatrix}
$$

second row gives $-\frac{1}{2}c_2 = -\frac{7}{2}$ or $c_2 = 7$ and first row gives $2c_1 + 3c_2 = 7$ or $c_1 = \frac{7-3c_2}{2} = \frac{7-21}{2} = -7$. Hence the solution becomes (from (3))

$$
x(t) = -7 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} + \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}
$$

Or

$$
\begin{aligned}
x(t) &= -7 + 14t + 2t^2(2\ln t - 3) \\
y(t) &= -7 + 9t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2
\end{aligned}
$$

4.14.24 Example on page 500, textbook (Edwards&Penny, 3rd edition)

problem This problem was solved in textbook using matrix exponential. Here is solved using the fundamental matrix only. Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$
\begin{aligned}
A &= \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \\
\bar{f}(t) &= \begin{pmatrix} -15 \\ 4 \end{pmatrix} te^{-2t} \\
\bar{x}(0) &= \begin{pmatrix} 7 \\ 3 \end{pmatrix}
\end{aligned}
$$

Solution

The homogeneous solution was found in the book as

$$\bar{x}_h = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

Following scalar case, the guess would be $\bar{x}_p = \left(\bar{b} + \bar{a}t\right) e^{-2t}$ but since e^{-2t} is in the homogeneous, we have to adjust to be $\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right) e^{-2t} + \bar{c}e^{5t}$. Notice we had to add $\bar{c}e^{5t}$, else it will not work if we just guessed $\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right) e^{-2t}$ based on what we would do in scalar case, we will find we get $\bar{a} = \bar{b} = 0$. This seems to be a trial and error stage and one just have to try to find out. This is why undermined coefficients for systems is not as easy to use as with scalar case. Hence

$$\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right) e^{-2t} + \bar{c}e^{5t}$$

Now we plug-in this back into the ODE and solve for $\bar{a}, \bar{b}, \bar{c}$. But an easier method is to use Variation of parameters. The fundamental matrix is

$$\begin{aligned} \Phi &= \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -2e^{-2t} & e^{5t} \end{pmatrix} \end{aligned}$$

And

$$\Phi^{-1} = \frac{\begin{pmatrix} e^{5t} & 2e^{-2t} \\ -2e^{5t} & e^{-2t} \end{pmatrix}^T}{|\Phi|} = \frac{\begin{pmatrix} e^{5t} & -2e^{5t} \\ 2e^{-2t} & e^{-2t} \end{pmatrix}}{e^{3t} + 4e^{3t}} = \frac{1}{5} \begin{pmatrix} e^{2t} & -2e^{2t} \\ 2e^{-5t} & e^{-5t} \end{pmatrix}$$

Hence using

$$\begin{aligned} \bar{x}_p &= \Phi \int \Phi^{-1} \bar{f}(t)\, dt \\ &= \frac{1}{5} \Phi \int \begin{pmatrix} e^{2t} & -2e^{2t} \\ 2e^{-5t} & e^{-5t} \end{pmatrix} \begin{pmatrix} -15te^{-2t} \\ 4te^{-2t} \end{pmatrix} dt \\ &= \frac{1}{5} \Phi \int \begin{pmatrix} -23t \\ -26te^{-7t} \end{pmatrix} dt \end{aligned}$$

The integral of $\int -23t dt = \frac{-23}{2} t^2$ and $\int -26te^{-7t} dt = \frac{26}{49} e^{-7t} (7t + 1)$ (using integration by parts) hence the above simplifies to

$$\begin{aligned} \bar{x}_p &= \Phi \begin{pmatrix} \frac{-23}{10} t^2 \\ \frac{26}{245} e^{-7t} + \frac{26}{35} te^{-7t} \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -2e^{-2t} & e^{5t} \end{pmatrix} \begin{pmatrix} \frac{-23}{10} t^2 \\ \frac{26}{245} e^{-7t} + \frac{26}{35} te^{-7t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{52}{245} e^{-2t} + \frac{52}{35} te^{-2t} - \frac{23}{10} t^2 e^{-2t} \\ \frac{26}{245} e^{-2t} + \frac{26}{35} te^{-2t} + \frac{23}{5} t^2 e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{490} e^{-2t} \left(-1127t^2 + 728t + 104\right) \\ \frac{1}{245} e^{-2t} \left(1127t^2 + 182t + 26\right) \end{pmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \bar{x} &= \bar{x}_h + \bar{x}_p \\ &= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} \frac{1}{490} e^{-2t} \left(-1127t^2 + 728t + 104\right) \\ \frac{1}{245} e^{-2t} \left(1127t^2 + 182t + 26\right) \end{pmatrix} \end{aligned}$$

To find the constants, we apply initial conditions. At $t = 0$

$$\begin{pmatrix}7\\3\end{pmatrix} = c_1\begin{pmatrix}1\\-2\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}\frac{52}{245}\\\frac{26}{245}\end{pmatrix}$$

$$c_1\begin{pmatrix}1\\-2\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}7\\3\end{pmatrix} - \begin{pmatrix}\frac{52}{245}\\\frac{26}{245}\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\frac{1663}{245}\\\frac{709}{245}\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\0 & 5\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\frac{1663}{245}\\\frac{807}{49}\end{pmatrix}$$

Hence $5c_2 = \frac{807}{49}$ or $c_2 = \frac{807}{245}$ and $c_1 + 2c_2 = \frac{1663}{245}$, hence $c_1 = \frac{1663}{245} - 2\left(\frac{807}{245}\right) = \frac{1}{5}$. Therefore the solution becomes

$$\bar{x} = \frac{1}{5}\begin{pmatrix}1\\-2\end{pmatrix}e^{-2t} + \frac{807}{245}\begin{pmatrix}2\\1\end{pmatrix}e^{5t} + \begin{pmatrix}\frac{1}{490}e^{-2t}\left(-1127t^2 + 728t + 104\right)\\\frac{1}{245}e^{-2t}\left(1127t^2 + 182t + 26\right)\end{pmatrix}$$