HW 13, Math 320, Spring 2017

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Contents

0.1 Section 7.1 problem 3 (page 404)

problem Transform the following problem or system to set of first order ODE $t^2x''+tx'+\left(t^2-1\right)x=0$

solution Since this is second order ODE, we need two state variables, say x_1, x

Let $x_1 = x, x_2 = x'$, hence

$$\left.\begin{array}{l} x_1 = x \\ x_2 = x' \end{array}\right\} \xrightarrow{\text{take derivative}} \left.\begin{array}{l} x_1' = x' \\ x_2' = x'' \end{array}\right\} \xrightarrow{\text{replace RHS}} \begin{array}{c} x_1' = x_2 \\ x_2' = -\frac{x'}{t} - \frac{\left(t^2-1\right)x}{t} = -\frac{x_2}{t} - \frac{\left(t^2-1\right)x_1}{t} \end{array}$$

Hence the two first order ODE's are (now coupled)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{x_2}{t} - \frac{\left(t^2-1\right)x_1}{t} \end{aligned}$$

The matrix form of the above is

$$x' = Ax$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{t^2-1}{t} & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

0.2 Section 7.1 problem 8

problem Transform the following problem or system to set of first order ODE $x''+3x'+4x-2y=0; y''+2y'-3x+y=\cos t$

solution We have two second order ODE's, hence we need 4 state variables. Let $x_1 = x, x_2 = x', x_3 = y, x_4 = y'$, therefore

$$\left.\begin{array}{l} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{array}\right\} \xrightarrow{\text{take derivative}} \left.\begin{array}{l} x_1' = x' \\ x_2' = x'' \\ x_3' = y \\ x_4' = y'' \end{array}\right\} \xrightarrow{\text{replace RHS}} \begin{array}{c} x_1' = x_2 \\ x_2' = -3x' - 4x + 2y = -3x_2 - 4x_1 + 2x_3 \\ x_3' = x_3 \\ x_4' = -2y' + 3x - y + \cos t = -2x_4 + 3x_1 - x_3 + \cos t \end{array}$$

Hence the 4 first order ODE's are

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_2 - 4x_1 + 2x_3 \\ x_3' &= x_3 \\ x_4' &= -2x_4 + 3x_1 - x_3 + \cos t \end{aligned}$$

The matrix form of the above is

$$x' = Ax + f$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}$$

0.3 Section 7.2 problem 9 (page 417)

problem Write the given system in form $x' = P(t)x + f(t)$

$$\begin{aligned} x' &= 3x - 4y + z + t \\ y' &= x - 3z + t^2 \\ z' &= 6y - 7z + t^3 \end{aligned}$$

solution The dependent variables are x, y, z and the independent variable is t. The matrix form is seen by inspection to be

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

0.4 Section 7.2 problem 10

problem Write the given system in form $x' = A(t)x + f(t)$

$$\begin{aligned} x' &= tx - y + e^t z \\ y' &= 2x + t^2 y - z \\ z' &= e^{-t}x + 3ty + t^3 z \end{aligned}$$

solution The dependent variables are x, y, z and the independent variable is t. The matrix form is seen by inspection to be

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice that P matrix is time dependent and not constant as the last problem. This is time varying system.

0.5 Section 7.2 problem 25

problem Find the complete solution that satisfies the initial conditions. $\boldsymbol{x}(0) = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \\ \boldsymbol{x}_2 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \end{aligned}$$

solution

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \end{aligned} \tag{1}$$

At $t = 0$ the above becomes

$$\begin{aligned} \begin{pmatrix} 11 \\ -7 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

Gaussian elimination. $R_2 = R_2 + R_1$ gives

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}$$

Hence $-c_2 = 4$ or $c_2 = -4$. From first row, $c_1 + c_2 = 11$ or $c_1 = 11 - c_2 = 11 + 4 = 15$, hence the complete solution from (1) is

$$\begin{aligned} \boldsymbol{x}(t) &= 15\boldsymbol{x}_1(t) - 4\boldsymbol{x}_2(t) \\ &= 15 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} - 4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} 15 \\ -15 \end{pmatrix} e^{3t} + \begin{pmatrix} -4 \\ 8 \end{pmatrix} e^{2t} \end{aligned}$$

0.6 Section 7.3 problem 7 (page 429)

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = -3x_1 + 4x_2; x_2' = 6x_1 - 5x_2$

solution

The system in matrix form is

$$\begin{aligned} \boldsymbol{x}' &= A\boldsymbol{x} \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} -3 & 4 \\ 6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} &= 0 \\ (-3-\lambda)(-5-\lambda) - 24 &= 0 \\ \lambda^2 + 8\lambda - 9 &= 0 \\ (\lambda+9)(\lambda-1) &= 0 \end{aligned}$$

Hence $\lambda_1 = 1, \lambda_2 = -9$. For λ_1, we now solve

$$\begin{aligned} (A - \lambda_1 I)\, v_1 &= 0 \\ \begin{pmatrix} -3-\lambda_1 & 4 \\ 6 & -5-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3-1 & 4 \\ 6 & -5-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $-4v_1 + 4v_2 = 0$. Picking $v_1 = 1$, then $v_2 = 1$, hence the eigenvector is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For λ_2, we now solve

$$\begin{aligned} (A - \lambda_2 I)\, v_2 &= 0 \\ \begin{pmatrix} -3-\lambda_2 & 4 \\ 6 & -5-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3+9 & 4 \\ 6 & -5+9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $6v_1 + 4v_2 = 0$. Picking $v_1 = 1$, then $v_2 = -\frac{3}{2}$, hence the second eigenvector is $v_2 = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ Therefore the solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 v_1(t)\, e^{\lambda_1 t} + c_2 v_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t}$$

Or

$$\begin{aligned} x_1(t) &= c_1 e^{t} + 2c_2 e^{-9t} \\ x_2(t) &= c_1 e^{t} - 3c_2 e^{-9t} \end{aligned}$$

No initial conditions are given.

0.7 Section 7.3 problem 9

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = 2x_1 - 5x_2; x_2' = 4x_1 - 2x_2; x_1(0) = 2, x_2(0) = 3$

solution

The system in matrix form is

$$x' = Ax$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(-2-\lambda) + 20 &= 0 \\ \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

Hence $\lambda_1 = 4i, \lambda_2 = -4i$. For λ_1, we now solve

$$\begin{aligned} (A - \lambda_1 I) v_1 &= 0 \\ \begin{pmatrix} 2-\lambda_1 & -5 \\ 4 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $(2-4i)v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{2-4i}{5}$, hence the eigenvector is $v_1 = \begin{pmatrix} 1 \\ \frac{2-4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$

For λ_2, we now solve

$$\begin{aligned} (A - \lambda_2 I) v_2 &= 0 \\ \begin{pmatrix} 2-\lambda_2 & -5 \\ 4 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2+4i & -5 \\ 4 & -2+4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Using first equation, we see that $(2+4i)v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{(2+4i)}{5}$, hence the eigenvector is $v_2 = \begin{pmatrix} 1 \\ \frac{2+4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2+4i \end{pmatrix}$ Therefore the solution is

$$\begin{aligned} x(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t) e^{\lambda_1 t} + c_2 v_2(t) e^{\lambda_2 t} \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{4it} + c_2 \begin{pmatrix} 5 \\ 2+4i \end{pmatrix} e^{-4it}$$

Or

$$\begin{aligned} x_1(t) &= c_1 5 e^{4it} + c_2 5 e^{-4it} \\ x_2(t) &= c_1(-2+4i) e^{4it} - c_2(2+4i) e^{-4it} \end{aligned}$$

Convert to new basis.

$$\begin{aligned}
\Re(x_1) &= \Re\begin{pmatrix} 5e^{4it} \\ (2-4i)\,e^{4it}\end{pmatrix} = \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ \left(2e^{4it} - 4ie^{4it}\right)\end{pmatrix} \\
&= \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ (2(\cos 4t + i\sin 4t) - 4i(\cos 4t + i\sin 4t))\end{pmatrix} \\
&= \Re\begin{pmatrix} 5(\cos 4t + i\sin 4t) \\ 2\cos 4t + i2\sin 4t - 4i\cos 4t + 4\sin 4t\end{pmatrix} \\
&= \Re\begin{pmatrix} 5\cos 4t + i5\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t)\end{pmatrix} \\
&= \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t\end{pmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\Im(x_1) &= \Im\begin{pmatrix} 5\cos 4t + i5\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t)\end{pmatrix} \\
&= \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t\end{pmatrix}
\end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = c_3\begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t\end{pmatrix} + c_2\begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t\end{pmatrix} \tag{1}$$

Or

$$\begin{aligned}
x_1(t) &= c_3 5\cos 4t + c_4 5\sin 4t \\
x_2(t) &= c_3(2\cos 4t + 4\sin 4t) + c_4(2\sin 4t - 4\cos 4t)
\end{aligned}$$

We now apply the initial conditions. From (1), at $t = 0$ we obtain

$$\begin{pmatrix} 2 \\ 3\end{pmatrix} = c_3\begin{pmatrix} 5 \\ 2\end{pmatrix} + c_2\begin{pmatrix} 0 \\ -4\end{pmatrix}$$

Or

$$\begin{pmatrix} 5 & 0 \\ 2 & -4\end{pmatrix}\begin{pmatrix} c_3 \\ c_4\end{pmatrix} = \begin{pmatrix} 2 \\ 3\end{pmatrix}$$

From first row, $5c_3 = 2$ or $c_3 = \frac{2}{5}$. From second row $2c_3 - 4c_4 = 3$ or $c_4 = -\frac{3-2\left(\frac{2}{5}\right)}{4} = -\frac{11}{20}$. Hence the solution (1) becomes

$$\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \frac{2}{5}\begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t\end{pmatrix} - \frac{11}{20}\begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t\end{pmatrix} \tag{1A}$$

Or

$$\begin{aligned}
x_1(t) &= \frac{2}{5}5\cos 4t - \frac{11}{20}5\sin 4t \\
x_2(t) &= \frac{2}{5}(2\cos 4t + 4\sin 4t) - \frac{11}{20}(2\sin 4t - 4\cos 4t)
\end{aligned}$$

Or

$$\begin{aligned}
x_1(t) &= 2\cos 4t - \frac{11}{4}\sin 4t \\
x_2(t) &= \frac{4}{5}\cos 4t + \frac{8}{5}\sin 4t - \frac{22}{20}\sin 4t + \frac{11}{5}\cos 4t
\end{aligned}$$

Or

$$\begin{aligned}
x_1(t) &= 2\cos 4t - \frac{11}{4}\sin 4t \\
x_2(t) &= 3\cos 4t + \frac{1}{2}\sin 4t
\end{aligned}$$

0.8 Section 7.3 problem 12

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = x_1 - 5x_2; x_2' = x_1 + 3x_2;$

solution The system in matrix form is

$$x' = Ax$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The eigenvalues are found from solving

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) + 5 = 0$$

$$\lambda^2 - 4\lambda + 8 = 0$$

$$\lambda = 2 \pm 2i$$

Hence $\lambda_1 = 2 + 2i, \lambda_2 = 2 - 2i$. For λ_1, we now solve

$$(A - \lambda_1 I)\, v_1 = 0$$

$$\begin{pmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using first equation, we see that $(-1-2i)\,v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{(-1-2i)}{5}$, hence the eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ \frac{-1-2i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$$

For λ_2, we now solve

$$(A - \lambda_2 I)\, v_2 = 0$$

$$\begin{pmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(2-2i) & -5 \\ 1 & 3-(2-2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+2i & -5 \\ 1 & 1+2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using first equation, we see that $(-1+2i)\,v_1 - 5v_2 = 0$. Picking $v_1 = 1$, then $v_2 = \frac{-1+2i}{5}$, hence the second eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ \frac{-1+2i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$$

Therefore the solution is

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 v_1(t)\, e^{\lambda_1 t} + c_2 v_2(t)\, e^{\lambda_2 t}$$

Therefore

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ -1-2i \end{pmatrix} e^{(2+2i)t} + c_2 \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} e^{(2-2i)t}$$

Convert to new basis.

$$\begin{aligned}\Re(x_1) &= \Re\begin{pmatrix}5e^{(2+2i)t}\\(-1-2i)\,e^{(2+2i)t}\end{pmatrix} = \Re\begin{pmatrix}5e^{2t}(\cos 2t + i\sin 2t)\\-e^{(2+2i)t} - 2ie^{(2+2i)t}\end{pmatrix}\\
&= \Re\begin{pmatrix}5e^{2t}(\cos 2t + i\sin 2t)\\-e^{2t}e^{2it} - 2ie^{2t}e^{2it}\end{pmatrix}\\
&= \Re\begin{pmatrix}5e^{2t}(\cos 2t + i\sin 2t)\\-e^{2t}(\cos 2t + i\sin 2t) - 2ie^{2t}(\cos 2t + i\sin 2t)\end{pmatrix}\\
&= \Re\begin{pmatrix}5e^{2t}(\cos 2t + i\sin 2t)\\-e^{2t}(\cos 2t + i\sin 2t) - 2e^{2t}(i\cos 2t - \sin 2t)\end{pmatrix}\\
&= \Re\begin{pmatrix}5e^{2t}\cos 2t + i\left(5e^{2t}\sin 2t\right)\\-e^{2t}\cos 2t - ie^{2t}\sin 2t - i2e^{2t}\cos 2t + 2e^{2t}\sin 2t\end{pmatrix}\\
&= \Re\begin{pmatrix}5e^{2t}\cos 2t + i\left(5e^{2t}\sin 2t\right)\\\left(-e^{2t}\cos 2t + 2e^{2t}\sin 2t\right) + i\left(-e^{2t}\sin 2t - 2e^{2t}\cos 2t\right)\end{pmatrix}\end{aligned}$$

Hence

$$\Re(x_1) = \begin{pmatrix}5e^{2t}\cos 2t\\-e^{2t}\cos 2t + 2e^{2t}\sin 2t\end{pmatrix}$$

And

$$\Im(x_1) = \begin{pmatrix}5e^{2t}\sin 2t\\-e^{2t}\sin 2t - 2e^{2t}\cos 2t\end{pmatrix}$$

Therefore the solution in the new basis is

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = C_1\begin{pmatrix}5e^{2t}\cos 2t\\-e^{2t}\cos 2t + 2e^{2t}\sin 2t\end{pmatrix} + C_2\begin{pmatrix}5e^{2t}\sin 2t\\-e^{2t}\sin 2t - 2e^{2t}\cos 2t\end{pmatrix}$$

Or

$$\begin{aligned}x_1(t) &= C_1 5e^{2t}\cos 2t + C_2 5e^{2t}\sin 2t\\
x_2(t) &= C_1\left(-e^{2t}\cos 2t + 2e^{2t}\sin 2t\right) + C_2\left(-e^{2t}\sin 2t - 2e^{2t}\cos 2t\right)\end{aligned}$$

Or

$$\begin{aligned}x_1(t) &= e^{2t}(5C_1\cos 2t + 5C_2\sin 2t)\\
x_2(t) &= e^{2t}(-C_1\cos 2t + 2C_1\sin 2t - C_2\sin 2t - 2C_2\cos 2t)\end{aligned}$$

Or

$$\begin{aligned}x_1(t) &= e^{2t}(5C_1\cos 2t + 5C_2\sin 2t)\\
x_2(t) &= e^{2t}(\cos 2t(-C_1 - 2C_2) + \sin 2t(2C_1 - C_2))\end{aligned} \tag{1}$$

Note, book must have used the other choice of eigenvalues ordering since it has the signs all flipped the other way from what I have above. flipping all the signs in the solution given above in equation (1), then the book solution results:

$$\begin{aligned}x_1(t) &= e^{2t}(-5C_1\cos 2t - 5C_2\sin 2t)\\
x_2(t) &= e^{2t}(\cos 2t(C_1 + 2C_2) + \sin 2t(-2C_1 + C_2))\end{aligned}$$

0.9 Section 7.3 problem 14

problem Apply the eigenvalue method to find general solution of the given system. For each problem, use a computer to construct direction field and typical solution curve. $x_1' = 3x_1 - 4x_2; x_2' = 4x_1 + 3x_2;$

solution The system in matrix form is

$$x' = Ax$$

$$\begin{pmatrix}x_1'\\x_2'\end{pmatrix} = \begin{pmatrix}3 & -4\\4 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

The eigenvalues are found from solving

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
\begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} &= 0 \\
(3-\lambda)(3-\lambda) + 16 &= 0 \\
\lambda^2 - 6\lambda + 25 &= 0 \\
\lambda &= 3 \pm 4i
\end{aligned}
$$

Hence $\lambda_1 = 3 + 4i, \lambda_2 = 3 - 4i$. For λ_1, we now solve

$$
\begin{aligned}
(A - \lambda_1 I)\, \boldsymbol{v}_1 &= 0 \\
\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 3-(3+4i) & -4 \\ 4 & 3-(3+4i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Using first equation, we see that $(-4i)\, v_1 - 4v_2 = 0$. Let $v_1 = 1$, then $v_2 = -i$, hence the eigenvector is

$$
\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}
$$

For λ_2, we now solve

$$
\begin{aligned}
(A - \lambda_1 I)\, \boldsymbol{v}_1 &= 0 \\
\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 3-(3-4i) & -4 \\ 4 & 3-(3-4i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 4i & -4 \\ 4 & 4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Using first equation, we see that $(4i)\, v_1 - 4v_2 = 0$. Let $v_1 = 1$, then $v_2 = i$, hence the eigenvector is

$$
\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}
$$

Therefore the solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\
&= c_1 \boldsymbol{v}_1(t)\, e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t)\, e^{\lambda_2 t}
\end{aligned}
$$

Therefore

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(3+4i)t} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(3-4i)t}
$$

Convert to new basis.

$$
\begin{aligned}
\Re(\boldsymbol{x}_1) = \Re \begin{pmatrix} e^{(3+4i)t} \\ -ie^{(3+4i)t} \end{pmatrix} &= \Re \begin{pmatrix} e^{3t}(\cos 4t + i \sin 4t) \\ e^{3t}(-i \cos 4t + \sin 4t) \end{pmatrix} \\
&= \begin{pmatrix} e^{3t} \cos 4t \\ e^{3t} \sin 4t \end{pmatrix}
\end{aligned}
$$

And

$$
\Im(\boldsymbol{x}_1) = \begin{pmatrix} e^{3t} \sin 4t \\ -e^{3t} \cos 4t \end{pmatrix}
$$

Therefore the solution in the new basis is

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} e^{3t} \cos 4t \\ e^{3t} \sin 4t \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \sin 4t \\ -e^{3t} \cos 4t \end{pmatrix}
$$

Or

$$x_1(t) = e^{3t}(C_1 \cos 4t + C_2 \sin 4t)$$
$$x_2(t) = e^{3t}(C_1 \sin 4t - C_2 \cos 4t)$$

0.10 Section 7.3 problem 28

problem TO DO

solution

0.11 Section 7.3 problem 30

problem TO DO

solution

0.12 Section 7.3 problem 39

problem Find general solution $x' = Ax$

$$A = \begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

solution The eigenvalues are found from solving

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 0 & 0 & 9 \\ 4 & 2-\lambda & 0 & -10 \\ 0 & 0 & -1-\lambda & 8 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the last row since it has most zeros then

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-1)^{4+4} \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)(-1)^{3+3} \begin{vmatrix} -2-\lambda & 0 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda) \begin{vmatrix} -2-\lambda & 0 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)(-2-\lambda)(2-\lambda) \end{aligned}$$

Hence the eigenvalues are (distinct case, no repeated)

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2, \lambda_4 = -2$$

For $\lambda_1 = 1$

$$\begin{pmatrix} -2-\lambda_1 & 0 & 0 & 9 \\ 4 & 2-\lambda_1 & 0 & -10 \\ 0 & 0 & -1-\lambda_1 & 8 \\ 0 & 0 & 0 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 0 & 9 \\ 4 & 1 & 0 & -10 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_4 = 1$. Hence from third row

$$\begin{aligned} -2v_3 + 8v_4 &= 0 \\ v_3 &= 4 \end{aligned}$$

From first row

$$\begin{aligned} -3v_1 + 9v_4 &= 0 \\ v_1 &= 3 \end{aligned}$$

From second row

$$\begin{aligned} 4v_1 + v_2 - 10v_4 &= 0 \\ v_2 &= 10 - 12 \\ &= -2 \end{aligned}$$

Hence first eigenvector is

$$v_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1$

$$\begin{pmatrix} -2-\lambda_2 & 0 & 0 & 9 \\ 4 & 2-\lambda_2 & 0 & -10 \\ 0 & 0 & -1-\lambda_2 & 8 \\ 0 & 0 & 0 & 1-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 9 \\ 4 & 3 & 0 & -10 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $2v_4 = 0$, hence $v_4 = 0$. From third row it also says that $v_4 = 0$. from first row we also obtain that $v_1 = 0$. From second row

$$4v_1 + 3v_2 = 0$$

Since $v_1 = 0$ then $v_2 = 0$. We notice that v_3 is left undetermined as there is no equation to determine it. (this happens when there is a column of all zeros, as in this case). Hence we can pick any value for v_3. Lets choose $v_3 = 1$. Therefore the second eigenvector is

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2$

$$\begin{pmatrix} -2-\lambda_3 & 0 & 0 & 9 \\ 4 & 2-\lambda_3 & 0 & -10 \\ 0 & 0 & -1-\lambda_3 & 8 \\ 0 & 0 & 0 & 1-\lambda_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 0 & 0 & 9 \\ 4 & 0 & 0 & -10 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $-v_4 = 0$, hence $v_4 = 0$. From third row it says that $v_3 = 0$ since $v_4 = 0$. from second and first row obtain that $v_1 = 0$.

We notice that v_2 is left undetermined as there is no equation to determine it. Hence we can pick any value for v_2. Lets choose $v_2 = 1$. Therefore the eigenvector is

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_4 = -2$

$$\begin{pmatrix} -2-\lambda_4 & 0 & 0 & 9 \\ 4 & 2-\lambda_4 & 0 & -10 \\ 0 & 0 & -1-\lambda_4 & 8 \\ 0 & 0 & 0 & 1-\lambda_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 9 \\ 4 & 4 & 0 & -10 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From last row $v_4 = 0$. From third row it says that $v_3 = 0$ since $v_4 = 0$. Second row gives $4v_1 + 4v_2 = 0$. Let $v_1 = 1$ hence $v_2 = -1$. Therefore the eigenvector is

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

We found all the eigenvectors, The solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) + c_3 \boldsymbol{x}_3(t) + c_4 \boldsymbol{x}_4(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} + c_3 \boldsymbol{v}_2(t) e^{\lambda_3 t} + c_2 \boldsymbol{v}_4(t) e^{\lambda_4 t} \end{aligned}$$

Or

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} e^{-2t}$$

Hence

$$\begin{aligned} x_1(t) &= 3c_1 e^t + c_4 e^{-2t} \\ x_2(t) &= -2c_1 e^t + c_3 e^{2t} - c_4 e^{-2t} \\ x_3(t) &= 4c_1 e^t + c_2 e^{-t} \\ x_4(t) &= c_1 e^t \end{aligned}$$

0.13 Section 7.5 problem 3

problem Find general solution of $\boldsymbol{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \boldsymbol{x}$

solution The eigenvalues are found from solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(5-\lambda) + 4 &= 0 \\ \lambda^2 - 6\lambda + 9 &= 0 \\ (\lambda - 3)^2 &= 0 \end{aligned}$$

Hence $\lambda = 3$. repeated root, multiplicity $k = 2$. Let us first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$(A-\lambda I)\,\boldsymbol{v}=0$$
$$\begin{pmatrix}1-\lambda & -2\\ 2 & 5-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

We see that the first row and the second row give the same eigenvector. $-2v_1-2v_2=0$. Let $v_1=1$, hence $v_2=-1$. So we can only find one eigenvector $\begin{pmatrix}1\\ -1\end{pmatrix}$. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solving for $\boldsymbol{v}_2$ from

$$(A-\lambda I)^2\,\boldsymbol{v}_2=0$$
$$\begin{pmatrix}1-\lambda & -2\\ 2 & 5-\lambda\end{pmatrix}^2\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}^2\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Hence $\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ can be any value. Let $v_1=1, v_2=0$ and therefore

$$\boldsymbol{v}_2=\begin{pmatrix}1\\ 0\end{pmatrix}$$

We now find v_1 from

$$\begin{aligned}v_1&=(A-\lambda I)\,\boldsymbol{v}_2\\&=\begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\&=\begin{pmatrix}-2\\ 2\end{pmatrix}\end{aligned}$$

Hence the solution is

$$\boldsymbol{x}(t)=c_1\boldsymbol{x}_1(t)+c_2\boldsymbol{x}_2(t) \tag{1}$$

Where now

$$\begin{aligned}\boldsymbol{x}_1(t)&=\boldsymbol{v}_1e^{\lambda t}\\ \boldsymbol{x}_2(t)&=(\boldsymbol{v}_1t+\boldsymbol{v}_2)\,e^{\lambda t}\end{aligned}$$

Plugging these into (1) gives

$$\boldsymbol{x}(t)=c_1\boldsymbol{v}_1e^{\lambda t}+c_2(\boldsymbol{v}_1t+\boldsymbol{v}_2)\,e^{\lambda t} \tag{2}$$

Replacing the result we found earlier for v_1, v_2 into the above, and using $\lambda=3$ gives

$$\boldsymbol{x}(t)=c_1\begin{pmatrix}-2\\ 2\end{pmatrix}e^{3t}+c_2\left(\begin{pmatrix}-2\\ 2\end{pmatrix}t+\begin{pmatrix}1\\ 0\end{pmatrix}\right)e^{3t}$$

Hence

$$\begin{aligned}x_1(t)&=(-2c_1+c_2-2c_2t)\,e^{3t}\\ x_2(t)&=(2c_1+2c_2t)\,e^{3t}\end{aligned}$$

0.14 Section 7.5 problem 5

problem Find general solution of $\boldsymbol{x}'=\begin{pmatrix}7 & 1\\ -4 & 3\end{pmatrix}\boldsymbol{x}$

<u>solution</u> The eigenvalues are found from solving

$$
\begin{aligned}
|A-\lambda I| &= 0\\
\begin{vmatrix} 7-\lambda & 1\\ -4 & 3-\lambda \end{vmatrix} &= 0\\
(7-\lambda)(3-\lambda)+4 &= 0\\
(\lambda-5)^2 &= 0
\end{aligned}
$$

Hence $\lambda=5$, repeated root, multiplicity $k=2$. Let us first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$
\begin{aligned}
(A-\lambda I)\,\boldsymbol{v} &= 0\\
\begin{pmatrix} 7-\lambda & 1\\ -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 7-5 & 1\\ -4 & 3-5 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}
\end{aligned}
$$

From first row we obtain $2v_1+v_2=0$. Let $v_1=1$ then $v_2=-2$. Hence eigenvector is $\boldsymbol{v}=\begin{pmatrix} 1\\ -2 \end{pmatrix}$. We can only find this one eigenvector. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solve for $\boldsymbol{v}_2$ from

$$
\begin{aligned}
(A-\lambda I)^2\,\boldsymbol{v}_2 &= 0\\
\begin{pmatrix} 7-\lambda & 1\\ -4 & 3-\lambda \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 7-5 & 1\\ -4 & 3-5 \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}^2\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1\\ v_2 \end{pmatrix} &= \begin{pmatrix} 0\\ 0 \end{pmatrix}
\end{aligned}
$$

Hence $\begin{pmatrix} v_1\\ v_2 \end{pmatrix}$ can be any value. Let $v_1=1, v_2=0$ and therefore

$$
\boldsymbol{v}_2=\begin{pmatrix} 1\\ 0 \end{pmatrix}
$$

We now find $\boldsymbol{v}_1$ from

$$
\begin{aligned}
\boldsymbol{v}_1 &= (A-\lambda I)\,\boldsymbol{v}_2\\
&= \begin{pmatrix} 2 & 1\\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1\\ 0 \end{pmatrix}\\
&= \begin{pmatrix} 2\\ -4 \end{pmatrix}
\end{aligned}
$$

Hence the solution is

$$
\boldsymbol{x}(t)=c_1\boldsymbol{x}_1(t)+c_2\boldsymbol{x}_2(t) \tag{1}
$$

Where now

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \boldsymbol{v}_1 e^{\lambda t}\\
\boldsymbol{x}_2(t) &= (\boldsymbol{v}_1 t+\boldsymbol{v}_2)\,e^{\lambda t}
\end{aligned}
$$

Plugging these into (1) gives

$$x(t) = c_1 v_1 e^{\lambda t} + c_2 (v_1 t + v_2) e^{\lambda t} \tag{2}$$

Replacing the result we found earlier for v_1, v_2 into the above, and using $\lambda = 3$ gives

$$x(t) = c_1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t} + c_2 \left(\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{5t}$$

Hence

$$x_1(t) = (2c_1 + c_2 + 2c_2 t) e^{3t}$$
$$x_2(t) = (-4c_1 - 4c_2 t) e^{3t}$$

0.15 Section 7.5 problem 7

Problem Find general solution of $x' = \begin{pmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{pmatrix} x$

Solution The eigenvalues are found from solving

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding along last row since it has most zeros

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-1)^{3+3} \begin{vmatrix} 2-\lambda & 0 \\ -7 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ -7 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda)(2-\lambda)(9-\lambda) \end{aligned}$$

Hence roots are $\lambda_1 = 2, \lambda_2 = 9$, where now λ_1 has multiplicity $k = 2$, and λ_2 is the good one with no multiplicity. To find associated eigenvector for λ_2 we follow the normal method.

For $\lambda_2 = 9$

$$(A - \lambda_2 I) v_{\lambda_2} = 0$$

$$\begin{pmatrix} 2-\lambda_2 & 0 & 0 \\ -7 & 9-\lambda_2 & 7 \\ 0 & 0 & 2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-9 & 0 & 0 \\ -7 & 9-9 & 7 \\ 0 & 0 & 2-9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row says $-7v_3 = 0$ or $v_3 = 0$. second row says $-7v_1 = 0$ or $v_1 = 0$. First row adds nothing new. So we see that there is no equation to find v_2 (this is because the second column is all zeros). Hence we pick v_2 anything we want. Let $v_2 = 1$ and therefore

$$v_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now we go back and look at $\lambda_1 = 2$, this is the one with multiplicity $k = 2$. Let first check if this is a complete eigenvalue or not. (i.e. if we can find two L.I. eigenvectors from this eigenvalue). If not, we need to use defective algorithm to find the eigenvectors). But we always check if it complete or not.

$$(A-\lambda_1 I)\,\boldsymbol{v}_{\lambda_1}=0$$

$$\begin{pmatrix}2-\lambda_1 & 0 & 0\\ -7 & 9-\lambda_1 & 7\\ 0 & 0 & 2-\lambda_1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2-2 & 0 & 0\\ -7 & 9-2 & 7\\ 0 & 0 & 2-2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0\\ -7 & 7 & 7\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Last row says v_3 is arbitrary. Let $v_3=s$. Second row says $-v_1+v_2+s=0$, hence $v_1=v_2+s$. No other information can be obtained from first row. So v_2 is arbitrary, say $v_2=r$, hence the solution is

$$\begin{aligned}\boldsymbol{v}_{\lambda_1}&=\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\\ &=\begin{pmatrix}r+s\\ r\\ s\end{pmatrix}\\ &=r\begin{pmatrix}1\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}\end{aligned}$$

So we see that we have linear combination of two eigenvectors for λ_1. Hence this eigenvalue is complete and not defective. No need to use the defective eigenvalue algorithm. These are the two L.I. eigenvector we are looking for. We got lucky here. Hence

$$\boldsymbol{v}_{\lambda_1}^{(1)}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\boldsymbol{v}_{\lambda_1}^{(2)}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

The solution is

$$\boldsymbol{x}(t)=c_1\boldsymbol{x}_1(t)+c_2\boldsymbol{x}_2(t)+c_3\boldsymbol{x}_3(t) \tag{1}$$

Where now

$$\begin{aligned}\boldsymbol{x}_1(t)&=\boldsymbol{v}_{\lambda_1}^{(1)}e^{\lambda_1 t}\\ \boldsymbol{x}_2(t)&=\boldsymbol{v}_{\lambda_1}^{(2)}e^{\lambda_1 t}\\ \boldsymbol{x}_3(t)&=\boldsymbol{v}_{\lambda_2}e^{\lambda_2 t}\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\boldsymbol{x}(t)&=c_1\boldsymbol{v}_{\lambda_1}^{(1)}e^{\lambda_1 t}+c_2\boldsymbol{v}_{\lambda_1}^{(2)}e^{\lambda_1 t}+c_3\boldsymbol{v}_{\lambda_2}e^{\lambda_2 t}\\ &=c_1\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t}+c_2\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t}+c_3\begin{pmatrix}0\\1\\0\end{pmatrix}e^{9t}\end{aligned}$$

Or

$$\begin{aligned}x_1(t)&=(c_1+c_2)\,e^{2t}\\ x_2(t)&=c_1e^{2t}+c_3e^{9t}\\ x_3(t)&=c_2e^{2t}\end{aligned}$$

0.16 Section 8.2 problem 5 (page 502)

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 6x - 7y + 10; y' = x - 2y - 2e^{-t}$

solution

The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0 \end{aligned}$$

Hence $\lambda_1 = 5, \lambda_2 = -1$

For $\lambda_1 = 5$

$$\begin{aligned} \begin{pmatrix} 6-\lambda_1 & -7 \\ 1 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $v_1 - 7v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{1}{7}$, hence the eigenvector is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -1$

$$\begin{aligned} \begin{pmatrix} 6-\lambda_2 & -7 \\ 1 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first equation $7v_1 - 7v_2 = 0$. Let $v_1 = 1$ then $v_2 = 1$, hence the eigenvector is

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t) e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t) e^{\lambda_2 t} \end{aligned}$$

Or

$$\boldsymbol{x}_h(t) = c_1 \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Or

$$\begin{aligned} x_h(t) &= 7c_1 e^{5t} + c_2 e^{-t} \\ y_h(t) &= c_1 e^{5t} + c_2 e^{-t} \end{aligned} \tag{1}$$

We now see that one of the basis solution in the homogenous part $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$, is also present in the forcing function (RHS of the original ODE). So to use the method of undetermined coefficients, we need to multiply by $t^0 e^{-t}$ and $t^1 e^{-t}$. Therefore, since the RHS is $\begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$, then we guess

$$\begin{aligned} x_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t e^{-t} \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} tc_1 \\ tc_2 \end{pmatrix} \right) e^{-t} \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t} \end{aligned}$$

Note that in systems, for duplication, we multiplied by $t^0 e^{-t}$ and $t^1 e^{-t}$. Hence the need for the $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t}$ term in the above. This is little different than in the scalar case where we just needed one multiplication. See the note in middle of page 497 of textbook on this. Now that we have the guess, we plug it into the system and solve for the coefficients.

$$\begin{aligned} x_p' &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{-t} - \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} \end{aligned}$$

Plugging the above into original system, which is

$$x_p' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} x_p + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$$

Gives

$$\begin{aligned} \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 + tc_1 \\ b_2 + tc_2 \end{pmatrix} e^{-t} \right) + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\ \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 e^{-t} + te^{-t} c_1 \\ b_2 e^{-t} + te^{-t} c_2 \end{pmatrix} \right) + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\ \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 + b_1 e^{-t} + te^{-t} c_1 \\ a_2 + b_2 e^{-t} + te^{-t} c_2 \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \\ \begin{pmatrix} c_1 - b_1 - tc_1 \\ c_2 - b_2 - tc_2 \end{pmatrix} e^{-t} &= \begin{pmatrix} 6a_1 - 7a_2 + 6b_1 e^{-t} - 7b_2 e^{-t} + 6tc_1 e^{-t} - 7tc_2 e^{-t} \\ a_1 - 2a_2 + b_1 e^{-t} - 2b_2 e^{-t} + tc_1 e^{-t} - 2tc_2 e^{-t} \end{pmatrix} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix} \end{aligned}$$

We obtain

$$\begin{aligned} (c_1 - b_1 - tc_1)\, e^{-t} &= e^{-t} (6b_1 - 7b_2 + 6tc_1 - 7tc_2) + 6a_1 - 7a_2 + 10 \\ (c_2 - b_2 - tc_2)\, e^{-t} &= e^{-t} (b_1 - 2b_2 + tc_1 - 2tc_2 - 2) + a_1 - 2a_2 \end{aligned}$$

Comparing terms, we obtain

$$\begin{aligned} c_1 - b_1 - tc_1 &= 6b_1 - 7b_2 + 6tc_1 - 7tc_2 \\ 6a_1 - 7a_2 + 10 &= 0 \\ c_2 - b_2 - tc_2 &= b_1 - 2b_2 + tc_1 - 2tc_2 - 2 \\ a_1 - 2a_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} c_1 - b_1 - tc_1 &= 6b_1 - 7b_2 + t\,(6c_1 - 7c_2) \\ 6a_1 - 7a_2 + 10 &= 0 \\ c_2 - b_2 - tc_2 &= b_1 - 2b_2 + t\,(c_1 - 2c_2) - 2 \\ a_1 - 2a_2 &= 0 \end{aligned}$$

Therefore, from the first and third equation above, we see we get additional two equations when we compare terms in t. Hence

$$\begin{aligned} c_1 - b_1 &= 6b_1 - 7b_2 \\ -c_1 &= 6c_1 - 7c_2 \\ 6a_1 - 7a_2 + 10 &= 0 \\ c_2 - b_2 &= b_1 - 2b_2 - 2 \\ -c_2 &= c_1 - 2c_2 \\ a_1 - 2a_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} c_1 - b_1 &= 6b_1 - 7b_2 \\ c_1 &= c_2 \\ 6a_1 - 7a_2 + 10 &= 0 \\ c_2 - b_2 &= b_1 - 2b_2 - 2 \\ c_2 &= c_1 \\ a_1 - 2a_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} c_1 - 7b_1 + 7b_2 &= 0 \\ c_1 - c_2 &= 0 \\ 6a_1 - 7a_2 &= -10 \\ c_2 + b_2 - b_1 &= -2 \\ a_1 - 2a_2 &= 0 \end{aligned}$$

The systems can be written as

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 6 & -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 7 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_2 = R_2 - 6R_1$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 7 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_4 = R_4 - \frac{1}{7}R_3$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_5 + 7R_4$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \\ -2 \\ -14 \end{pmatrix}$$

From last row we obtain that $6c_2 = -14$ or

$$c_2 = \frac{-7}{3}$$

From 4th row

$$\begin{aligned}
-\frac{1}{7}c_1 + c_2 &= -2 \\
-\frac{1}{7}c_1 &= \frac{7}{3} - 2 \\
-c_1 &= \frac{49}{3} - 14 \\
c_1 &= 14 - \frac{49}{3} \\
&= -\frac{7}{3}
\end{aligned}$$

From 3rd row

$$\begin{aligned}
-7b_1 + 7b_2 + c_1 &= 0 \\
-7b_1 &= -7b_2 - \frac{7}{3} \\
b_1 &= b_2 - \frac{1}{3}
\end{aligned}$$

From second row

$$\begin{aligned}
5a_2 &= -10 \\
a_2 &= -2
\end{aligned}$$

From first row

$$\begin{aligned}
a_1 - 2a_2 &= 0 \\
a_1 &= 2a_2 \\
&= -4
\end{aligned}$$

Therefore the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ b_2 - \frac{1}{3} \\ b_2 \\ -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = b_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -\frac{1}{3} \\ 0 \\ -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

Where b_1 is arbitrary. If we let $b_2 = 0$ then

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -\frac{1}{3} \\ 0 \\ \frac{7}{3} \\ \frac{-7}{3} \end{pmatrix}$$

Therefore, we go back to the particular solution

$$\boldsymbol{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} te^{-t}$$

And substitute these values found in the solution above and obtain

$$\boldsymbol{x}_p = \begin{pmatrix} -4 \\ -2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} -\frac{7}{3} \\ \frac{-7}{3} \end{pmatrix} te^{-t}$$

Or

$$x_p(t) = -4 - \frac{1}{3}e^{-t} - \frac{7}{3}te^{-t}$$
$$y_p(t) = -2 - \frac{7}{3}te^{-t}$$

Or

$$x_p(t) = \frac{1}{3}\left(-12 - e^{-t} - 7te^{-t}\right)$$
$$y_p(t) = \frac{1}{3}\left(-6 - 7te^{-t}\right)$$

Hence the complete solution (using the homogenous solution found in (1)) is

$$x(t) = 7c_1e^{5t} + c_2e^{-t} + \frac{1}{3}\left(-12 - e^{-t} - 7te^{-t}\right)$$
$$y_h(t) = c_1e^{5t} + c_2e^{-t} + \frac{1}{3}\left(-6 - 7te^{-t}\right)$$

0.17 Section 8.2 problem 9

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = x - 5y + \cos 2t; y' = x - y$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos 2t \\ 0 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 4 = 0$$
$$\lambda = \pm 2i$$

For $\lambda_1 = 2i$ we solve $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$(1-2i)v_1 - 5v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{(1-2i)}{5}$, therefore

$$v_1 = \begin{pmatrix} 1 \\ \frac{(1-2i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

For $\lambda_1 = -2i$ we solve $(A - \lambda_2 I)v_2 = 0$

$$\begin{pmatrix} 1+2i & -5 \\ 1 & -1+2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$(1+2i)v_1 - 5v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{(1+2i)}{5}$, therefore

$$v_2 = \begin{pmatrix} 1 \\ \frac{(1+2i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix}$$

Therefore the homogenous solution is

$$\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$
$$= c_1v_1(t)e^{\lambda_1 t} + c_2v_2(t)e^{\lambda_2 t}$$

Or

$$\mathbf{x}_h(t) = c_1\begin{pmatrix} 5 \\ 1-2i \end{pmatrix}e^{2it} + c_2\begin{pmatrix} 5 \\ 1+2i \end{pmatrix}e^{-2it}$$

Convert to new basis

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \operatorname{Re}\left(\boldsymbol{x}_1(t)\right) \\
&= \operatorname{Re}\begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} = \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ (\cos 2t + i\sin 2t) - 2i(\cos 2t + i\sin 2t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ (\cos 2t + i\sin 2t) - 2(i\cos 2t - \sin 2t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t - 2i\cos 2t + 2\sin 2t \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos 2t + i\sin 2t) \\ \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\boldsymbol{x}_2(t) &= \operatorname{Im}\left(\boldsymbol{x}_1(t)\right) \\
&= \begin{pmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}
\end{aligned}
$$

Hence the homogeneous solution is

$$
\begin{aligned}
\boldsymbol{x}_h(t) &= C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) \\
&= C_1\begin{pmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + C_2\begin{pmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}
\end{aligned}
$$

Or

$$
\begin{aligned}
x_h(t) &= 5C_1\cos 2t + 5C_2\sin 2t \\
y_h(t) &= (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t
\end{aligned}
$$

We now see that one of the basis solutions for the homogenous part contains $\cos 2t$ which is also in the forcing function of the original system. Hence we need to pick a guess where we multiply by extra t. Since the forcing function is $\begin{pmatrix} \cos 2t \\ 0 \end{pmatrix}$ then guess

$$
\begin{aligned}
\boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\sin 2t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\cos 2t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t\sin 2t + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} t\cos 2t \qquad (1) \\
&= \begin{pmatrix} a_1 + tc_1 \\ a_2 + tc_2 \end{pmatrix}\sin 2t + \begin{pmatrix} b_1 + td_1 \\ b_2 + td_2 \end{pmatrix}\cos 2t
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\boldsymbol{x}_p' &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\sin 2t + 2\begin{pmatrix} a_1 + tc_1 \\ a_2 + tc_2 \end{pmatrix}\cos 2t + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\cos 2t - 2\begin{pmatrix} b_1 + td_1 \\ b_2 + td_2 \end{pmatrix}\sin 2t \\
&= \begin{pmatrix} c_1 - 2(b_1 + td_1) \\ c_2 - 2(b_2 + td_2) \end{pmatrix}\sin 2t + \begin{pmatrix} d_1 + 2(a_1 + tc_1) \\ d_2 + 2(a_2 + tc_2) \end{pmatrix}\cos 2t \qquad (2)
\end{aligned}
$$

We now substitute (1) and (2) into

$$
\boldsymbol{x}_p' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\boldsymbol{x}_p + \begin{pmatrix} \cos 2t \\ 0 \end{pmatrix}
$$

Hence

$$
\begin{aligned}
&\begin{pmatrix} c_1 - 2(b_1 + td_1) \\ c_2 - 2(b_2 + td_2) \end{pmatrix}\sin 2t + \begin{pmatrix} d_1 + 2(a_1 + tc_1) \\ d_2 + 2(a_2 + tc_2) \end{pmatrix}\cos 2t = \\
&\qquad \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\left(\begin{pmatrix} a_1 + tc_1 \\ a_2 + tc_2 \end{pmatrix}\sin 2t + \begin{pmatrix} b_1 + td_1 \\ b_2 + td_2 \end{pmatrix}\cos 2t\right) + \begin{pmatrix} \cos 2t \\ 0 \end{pmatrix}
\end{aligned}
$$

Hence

$$\begin{pmatrix}(c_1 - 2(b_1 + td_1))\sin 2t \\ (c_2 - 2(b_2 + td_2))\sin 2t\end{pmatrix} + \begin{pmatrix}(d_1 + 2(a_1 + tc_1))\cos 2t \\ (d_2 + 2(a_2 + tc_2))\cos 2t\end{pmatrix} =$$
$$\begin{pmatrix}1 & -5 \\ 1 & -1\end{pmatrix}\begin{pmatrix}(a_1 + tc_1)\sin 2t + (b_1 + td_1)\cos 2t \\ (a_2 + tc_2)\sin 2t + (b_2 + td_2)\cos 2t\end{pmatrix} + \begin{pmatrix}\cos 2t \\ 0\end{pmatrix}$$

Or

$$\begin{pmatrix}(c_1 - 2(b_1 + td_1))\sin 2t + (d_1 + 2(a_1 + tc_1))\cos 2t \\ (c_2 - 2(b_2 + td_2))\sin 2t + (d_2 + 2(a_2 + tc_2))\cos 2t\end{pmatrix} =$$
$$\begin{pmatrix}(\cos 2t)(b_1 + td_1) - 5(\cos 2t)(b_2 + td_2) + (\sin 2t)(a_1 + tc_1) - 5(\sin 2t)(a_2 + tc_2) \\ (\cos 2t)(b_1 + td_1) - (\cos 2t)(b_2 + td_2) + (\sin 2t)(a_1 + tc_1) - (\sin 2t)(a_2 + tc_2)\end{pmatrix} + \begin{pmatrix}\cos 2t \\ 0\end{pmatrix}$$

Therefore

$$(c_1 - 2(b_1 + td_1))\sin 2t + (d_1 + 2(a_1 + tc_1))\cos 2t =$$
$$(\cos 2t)(b_1 + td_1) - 5(\cos 2t)(b_2 + td_2) + (\sin 2t)(a_1 + tc_1) - 5(\sin 2t)(a_2 + tc_2) + \cos 2t \tag{3}$$

And

$$(c_2 - 2(b_2 + td_2))\sin 2t + (d_2 + 2(a_2 + tc_2))\cos 2t =$$
$$(\cos 2t)(b_1 + td_1) - (\cos 2t)(b_2 + td_2) + (\sin 2t)(a_1 + tc_1) - (\sin 2t)(a_2 + tc_2) \tag{4}$$

Equation (3,4) are solved for the unknowns. We need 8 equations in total. Looking at (3) for now. Comparing coefficients of $\sin 2t$ in (3)

$$\begin{aligned}(c_1 - 2(b_1 + td_1)) &= (a_1 + tc_1) - 5(a_2 + tc_2) \\ c_1 - 2b_1 - 2td_1 &= a_1 - 5a_2 + tc_1 - 5tc_2 \\ c_1 - 2b_1 + t(-2d_1) &= a_1 - 5a_2 + t(c_1 - 5c_2)\end{aligned}$$

Comparing coefficients we see

$$\begin{aligned}c_1 - 2b_1 &= a_1 - 5a_2 \\ a_1 - 5a_2 - c_1 + 2b_1 &= 0 \quad \text{(1A)}\end{aligned}$$

And

$$\begin{aligned}-2d_1 &= c_1 - 5c_2 \\ c_1 - 5c_2 + 2d_1 &= 0 \quad \text{(2A)}\end{aligned}$$

We do the same for $\cos 2t$ in equation (3) and compare coefficients

$$\begin{aligned}(d_1 + 2(a_1 + tc_1)) &= (b_1 + td_1) - 5(b_2 + td_2) + 1 \\ 2a_1 + d_1 + 2tc_1 &= b_1 - 5b_2 + td_1 - 5td_2 + 1 \\ 2a_1 + d_1 + t(2c_1) &= b_1 - 5b_2 + 1 + t(d_1 - 5d_2)\end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned}2a_1 + d_1 &= b_1 - 5b_2 + 1 \\ 2a_1 + d_1 - b_1 + 5b_2 &= 1 \quad \text{(3A)}\end{aligned}$$

And

$$\begin{aligned}2c_1 &= d_1 - 5d_2 \\ 2c_1 - d_1 + 5d_2 &= 0 \quad \text{(4A)}\end{aligned}$$

We have obtained 4 equations from (3). We do the same on (4) to obtain the other 4 equations. Comparing $\sin 2t$ terms in (4) gives

$$\begin{aligned}(c_2 - 2(b_2 + td_2)) &= (a_1 + tc_1) - (a_2 + tc_2) \\ c_2 - 2b_2 - 2td_2 &= a_1 - a_2 + tc_1 - tc_2 \\ c_2 - 2b_2 + t(-2d_2) &= a_1 - a_2 + t(c_1 - c_2)\end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned}c_2 - 2b_2 &= a_1 - a_2 \\ a_1 - a_2 - c_2 + 2b_2 &= 0 \quad \text{(5A)}\end{aligned}$$

And

$$\begin{aligned}-2d_2 &= c_1 - c_2 \\ c_1 - c_2 + 2d_2 &= 0 \quad \text{(6A)}\end{aligned}$$

Finally, Comparing $\cos 2t$ terms in (4) gives

$$\begin{aligned}(d_2 + 2(a_2 + tc_2)) &= (b_1 + td_1) - (b_2 + td_2)\\ 2a_2 + d_2 + 2tc_2 &= b_1 - b_2 + td_1 - td_2\\ 2a_2 + d_2 + t(2c_2) &= b_1 - b_2 + t(d_1 - d_2)\end{aligned}$$

Comparing coefficients on the above gives two new equations

$$\begin{aligned}2a_2 + d_2 &= b_1 - b_2\\ 2a_2 + d_2 - b_1 + b_2 &= 0 \qquad (7A)\end{aligned}$$

And

$$\begin{aligned}2c_2 &= d_1 - d_2\\ d_1 - d_2 - 2c_2 &= 0 \qquad (8A)\end{aligned}$$

Equations (1A) to (8A) are now solved for $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$.

$$\begin{aligned}a_1 - 5a_2 - c_1 + 2b_1 &= 0\\ c_1 - 5c_2 + 2d_1 &= 0\\ 2a_1 + d_1 - b_1 + 5b_2 &= 1\\ 2c_1 - d_1 + 5d_2 &= 0\\ a_1 - a_2 - c_2 + 2b_2 &= 0\\ c_1 - c_2 + 2d_2 &= 0\\ 2a_2 + d_2 - b_1 + b_2 &= 0\\ d_1 - d_2 - 2c_2 &= 0\end{aligned}$$

Writing the equations in matrix form

$$\begin{pmatrix} 1 & -5 & 2 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -5 & 2 & 0\\ 2 & 0 & -1 & 5 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 5\\ 1 & -1 & 0 & 2 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2\\ 0 & 2 & -1 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2\\ d_1\\ d_2 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}$$

Solving the above using the computer gives

$$\begin{pmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2\\ d_1\\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\\ 0\\ 0\\ 0\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{2}\\ 0 \end{pmatrix}$$

Solve 8.2 problem 9 final Matrix equation

```
In[112]:= mat = {{1, -5, 2, 0, -1, 0, 0, 0},
    {0, 0, 0, 0, 1, -5, 2, 0},
    {2, 0, -1, 5, 0, 0, 1, 0},
    {0, 0, 0, 0, 2, 0, -1, 5},
    {1, -1, 0, 2, 0, -1, 0, 0},
    {0, 0, 0, 0, 1, -1, 0, 2},
    {0, 2, -1, 1, 0, 0, 0, 1},
    {0, 0, 0, 0, 0, -2, 1, -1}};
  b = {0, 0, 1, 0, 0, 0, 0, 0};
  LinearSolve[mat, b]
```

Out[114]= $\left\{\frac{1}{4}, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right\}$

We now go back to (1) and plugging these values into the particular solution

$$\begin{aligned} x_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin 2t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos 2t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t \sin 2t + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} t \cos 2t \\ &= \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} t \sin 2t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} t \cos 2t \end{aligned}$$

Hence

$$\begin{aligned} x_p(t) &= \frac{1}{4} \sin 2t + \frac{1}{4} t \sin 2t + \frac{1}{2} t \cos 2t \\ y_p(t) &= \frac{1}{4} t \sin 2t \end{aligned}$$

Or

$$\begin{aligned} x_p(t) &= \frac{1}{4} (\sin 2t + t \sin 2t + 2t \cos 2t) \\ y_p(t) &= \frac{1}{4} t \sin 2t \end{aligned}$$

Earlier we obtained the homogenous solution as

$$\begin{aligned} x_h(t) &= 5C_1 \cos 2t + 5C_2 \sin 2t \\ y_h(t) &= (C_1 - 2C_2) \cos 2t + (2C_1 + C_2) \sin 2t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} x(t) &= 5C_1 \cos 2t + 5C_2 \sin 2t + \frac{1}{4} (\sin 2t + t \sin 2t + 2t \cos 2t) \\ y(t) &= (C_1 - 2C_2) \cos 2t + (2C_1 + C_2) \sin 2t + \frac{1}{4} t \sin 2t \end{aligned}$$

0.18 Section 8.2 problem 11

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 2x + 4y + 2; y' = x + 2y + 3; x(0) = 1, y(0) = -1$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda &= 0 \\ (\lambda - 4)\lambda &= 0 \end{aligned}$$

Hence $\lambda_1 = 0, \lambda_2 = 4$.

For $\lambda_1 = 0$ we solve $(A - \lambda_1 I)\,\mathbf{v}_1 = 0$

$$\begin{pmatrix} 2-\lambda_1 & 4 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$2v_1 + 4v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{-1}{2}$, therefore

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda_1 = 4$ we solve $(A - \lambda_2 I)\,\mathbf{v}_2 = 0$

$$\begin{pmatrix} 2-\lambda_2 & 4 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation

$$-2v_1 + 4v_2 = 0$$

Let $v_1 = 1$, hence $v_2 = \frac{1}{2}$, therefore

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \mathbf{x}_h(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= c_1\mathbf{v}_1(t)\,e^{\lambda_1 t} + c_2\mathbf{v}_2(t)\,e^{\lambda_2 t} \end{aligned}$$

Or

$$\mathbf{x}_h(t) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} \tag{1}$$

Since constant term exist in both homogenous solution and in forcing function then guess

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}t$$

Therefore

$$\mathbf{x}_p' = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Substituting this into

$$\mathbf{x}_p' = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\mathbf{x}_p + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Gives

$$\begin{aligned} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}t\right) + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 2a_1 + 4a_2 + 2tb_1 + 4tb_2 \\ a_1 + 2a_2 + tb_1 + 2tb_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} b_1 &= 2a_1 + 4a_2 + 2tb_1 + 4tb_2 + 2 \\ b_2 &= a_1 + 2a_2 + tb_1 + 2tb_2 + 3 \end{aligned}$$

Or

$$b_1 = 2a_1 + 4a_2 + 2 + t(2b_1 + 4b_2)$$
$$b_2 = a_1 + 2a_2 + 3 + t(b_1 + 2b_2)$$

So by comparing coefficients in each equation we obtain 4 equations as follows

$$\begin{aligned} b_1 &= 2a_1 + 4a_2 + 2 \\ 2b_1 + 4b_2 &= 0 \\ b_2 &= a_1 + 2a_2 + 3 \\ b_1 + 2b_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} 2a_1 + 4a_2 - b_1 &= -2 \\ 2b_1 + 4b_2 &= 0 \\ a_1 + 2a_2 - b_2 &= -3 \\ b_1 + 2b_2 &= 0 \end{aligned}$$

Hence the matrix form is

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_1$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - \frac{1}{4}R_2$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$R_4 = R_4 - \frac{1}{2}R_2$

$$\begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

Third row gives $-2b_2 = -2$ or $b_2 = 1$. From second row $2b_1 + 4b_2 = 0$, or $b_1 = -2b_2 = -2$. First row gives

$$\begin{aligned} 2a_1 + 4a_2 - b_1 &= -2 \\ 2a_1 + 4a_2 &= -2 + b_1 \\ 2a_1 + 4a_2 &= -4 \\ \frac{1}{2}a_1 + a_2 &= -1 \end{aligned}$$

Hence a_1 or a_2 are arbitrary. Let $a_2 = 0$ then $a_2 = -2$. Hence the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Therefore

$$\boldsymbol{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t$$
$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t$$

Using (1) the complete solution is

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t \tag{2}$$

At $t = 0$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2c_1 + 2c_2 - 2 \\ -c_1 + c_2 \end{pmatrix}$$

Hence

$$2c_1 + 2c_2 - 2 = 1$$
$$-c_1 + c_2 = -1$$

Or

$$2c_1 + 2c_2 = 3$$
$$-c_1 + c_2 = -1$$

Which gives $c_1 = \frac{5}{4}, c_2 = \frac{1}{4}$, therefore (2) becomes

$$\boldsymbol{x}(t) = \frac{5}{4} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t$$

Or

$$x(t) = \frac{10}{4} + \frac{1}{2} e^{4t} - 2 - 2t$$
$$y(t) = -\frac{5}{4} + \frac{1}{4} e^{4t} + t$$

Or

$$x(t) = \frac{1}{2} \left(1 - 4t + e^{4t}\right)$$
$$y(t) = \frac{1}{4} \left(-5 + 4t + e^{4t}\right)$$

0.19 Section 8.2 problem 13

problem Apply method of undetermined coefficients to find particular solution system. If initial conditions are given, apply initial conditions to find the complete solution. $x' = 2x + y + 2e^t; y' = x + 2y - 3e^t$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$
$$\lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 3)(\lambda - 1) = 0$$

Hence $\lambda_1 = 1, \lambda_2 = 3$.

<u>For $\lambda_1 = 1$</u> we solve $(A - \lambda_1 I)\,\boldsymbol{v}_1 = 0$

$$\begin{pmatrix} 2-\lambda_1 & 1 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $v_1 + v_2 = 0$. Let $v_1 = 1$, hence $v_2 = -1$ and therefore

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<u>For $\lambda_1 = 3$</u> we solve $(A - \lambda_2 I)\,\boldsymbol{v}_2 = 0$

$$\begin{pmatrix} 2-\lambda_2 & 1 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $-v_1 + v_2 = 0$. Let $v_1 = 1$, hence $v_2 = 1$ and therefore

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore the homogenous solution is

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) \\ &= c_1 \boldsymbol{v}_1(t)\, e^{\lambda_1 t} + c_2 \boldsymbol{v}_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Or

$$\boldsymbol{x}_h(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \tag{1}$$

Since the forcing function is $\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$ and e^t is a basis solution for the homogenous part, then we guess

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t e^t \\ &= \begin{pmatrix} a_1 + t b_1 \\ a_2 + t b_2 \end{pmatrix} e^t \end{aligned}$$

Hence

$$\begin{aligned} \boldsymbol{x}_p' &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} a_1 + t b_1 \\ a_2 + t b_2 \end{pmatrix} e^t \\ &= \begin{pmatrix} b_1 + a_1 + t b_1 \\ b_2 + a_2 + t b_2 \end{pmatrix} e^t \end{aligned}$$

Plugging this back into

$$\boldsymbol{x}_p' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{x}_p + \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

Gives

$$\begin{aligned} \begin{pmatrix} b_1 + a_1 + t b_1 \\ b_2 + a_2 + t b_2 \end{pmatrix} e^t &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 + t b_1 \\ a_2 + t b_2 \end{pmatrix} e^t + \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} \\ \begin{pmatrix} b_1 + a_1 + t b_1 \\ b_2 + a_2 + t b_2 \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 + t b_1 \\ a_2 + t b_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\ \begin{pmatrix} b_1 + a_1 + t b_1 \\ b_2 + a_2 + t b_2 \end{pmatrix} &= \begin{pmatrix} 2a_1 + a_2 + 2t b_1 + t b_2 \\ a_1 + 2a_2 + t b_1 + 2t b_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} b_1 + a_1 + t b_1 &= 2a_1 + a_2 + 2t b_1 + t b_2 + 2 \\ b_2 + a_2 + t b_2 &= a_1 + 2a_2 + t b_1 + 2t b_2 - 3 \end{aligned}$$

or

$$b_1 + a_1 + tb_1 = 2a_1 + a_2 + 2 + t(2b_1 + b_2)$$
$$b_2 + a_2 + tb_2 = a_1 + 2a_2 - 3 + t(b_1 + 2b_2)$$

Comparing coefficients in the above two equations generates 4 equations to solve for the unknowns

$$\begin{aligned} b_1 + a_1 &= 2a_1 + a_2 + 2 \\ b_1 &= 2b_1 + b_2 \\ b_2 + a_2 &= a_1 + 2a_2 - 3 \\ b_2 &= b_1 + 2b_2 \end{aligned}$$

Or

$$\begin{aligned} a_1 + a_2 - b_1 &= -2 \\ b_1 + b_2 &= 0 \\ a_1 + a_2 - b_2 &= 3 \\ b_1 + b_2 &= 0 \end{aligned}$$

Second and third equation are the same. Using the first 3 equations, the matrix equations are

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$

This is undetermined system. It will either have infinite number of solutions or no solution.

Let $R_2 = R_2 - R_1$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -5 \end{pmatrix}$$

Last row gives $2b_2 = -5$ or $b_2 = \frac{-5}{2}$. Second row gives $b_1 - b_2 = 5$ or $b_1 = 5 + b_2 = 5 - \frac{5}{2} = \frac{5}{2}$. First row gives $a_1 + a_2 - b_1 = -2$ or $a_1 = -a_2 + b_1 - 2$ or

$$\begin{aligned} a_1 &= -a_2 + b_1 - 2 \\ &= -a_2 + \frac{5}{2} - 2 \\ &= -a_2 + \frac{1}{2} \end{aligned}$$

a_2 is arbitrary. Let $a_2 = 0$ and we obtain $a_1 = \frac{1}{2}$. Hence the solution is

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix}$$

Therefore since $\boldsymbol{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t$ then

$$\boldsymbol{x}_p = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix} te^t$$

Or

$$\begin{aligned} x_p(t) &= \frac{1}{2}(1 + 5t) e^t \\ y_p(t) &= \frac{-5}{2} te^t \end{aligned}$$

And the general solution is

$$x(t) = x_h(t) + x_0(t)$$

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} \frac{5}{2} \\ \frac{-5}{2} \end{pmatrix} te^t$$

Or

$$x(t) = c_1 e^t + c_2 e^{3t} + \frac{1}{2}(1+5t)e^t$$

$$y(t) = -c_1 e^t + c_2 e^{3t} - \frac{5}{2} te^t$$

0.20 Section 8.2 problem 19

problem Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}; f(t) = \begin{pmatrix} 180t \\ 90 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 180t \\ 90 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda - 2)(\lambda + 3) = 0$$

Hence $\lambda_1 = 2, \lambda_2 = -3$.

For $\lambda_1 = 2$ we solve $(A - \lambda_1 I) v_1 = 0$

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ 2 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $-v_1 + 2v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{1}{2}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -3$ we solve $(A - \lambda_2 I) v_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 2 \\ 2 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $4v_1 + 2v_2 = 0$. Let $v_1 = 1$ then $v_2 = -2$ and

$$v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Therefore

$$\begin{aligned} x_h(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t) e^{\lambda_1 t} + c_2 v_2(t) e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} \tag{1}$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$\begin{aligned} W &= \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \end{aligned}$$

Therefore

$$x_p(t) = W \int W^{-1} f(t)\, dt \tag{2}$$

Where

$$\begin{aligned} W^{-1} &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{2}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{2}{5}e^{3t} \end{pmatrix} \end{aligned}$$

Hence, (2) becomes

$$\begin{aligned} x_p(t) &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \int \begin{pmatrix} \frac{2}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{2}{5}e^{3t} \end{pmatrix} \begin{pmatrix} 180t \\ 90 \end{pmatrix} dt \\ &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \int \begin{pmatrix} 18e^{-2t} + 72te^{-2t} \\ 36te^{3t} - 36e^{3t} \end{pmatrix} dt \\ &= \begin{pmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{pmatrix} \begin{pmatrix} -9e^{-2t}(4t+3) \\ 4e^{3t}(3t-4) \end{pmatrix} \\ &= \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix} \end{aligned} \tag{3}$$

At $t = 0$

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -70 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 + c_2 - 70 \\ c_1 - 2c_2 + 5 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 70 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \end{pmatrix}$$

Last row gives $\frac{-5}{2} c_2 = -40$, or $c_2 = 16$. First row gives $2c_1 + c_2 = 70$, hence $c_1 = \frac{70 - c_2}{2} = \frac{70-16}{2} = 27$. Hence the solution from (3) becomes

$$x(t) = 27 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} + 16 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + \begin{pmatrix} -60t - 70 \\ 5 - 60t \end{pmatrix}$$

Or

$$\begin{aligned} x(t) &= 54e^{2t} + 16e^{-3t} - 60t - 70 \\ y(t) &= -27e^{2t} - 32e^{-3t} + 5 - 60t \end{aligned}$$

0.21 Section 8.2 problem 22

problem Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$A = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}; f(t) = \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda + 1)(\lambda - 3) = 0$$

Hence $\lambda_1 = -1, \lambda_2 = 3$.

For $\lambda_1 = -1$ we solve $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} 4-\lambda_1 & -1 \\ 5 & -2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $5v_1 - v_2 = 0$. Let $v_1 = 1$ then $v_2 = 5$ and

$$v_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

For $\lambda_2 = 3$ we solve $(A - \lambda_2 I)v_2 = 0$

$$\begin{pmatrix} 4-\lambda_2 & -1 \\ 5 & -2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $v_1 - v_2 = 0$. Let $v_1 = 1$ then $v_2 = 1$ and

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{aligned} x_h(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t) e^{\lambda_1 t} + c_2 v_2(t) e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \tag{1}$$

The Wronskian W is, (which is the same as fundamental matrix Φ) is

$$\begin{aligned} W &= \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \end{aligned}$$

Therefore

$$x_p(t) = W \int W^{-1} f(t)\, dt \tag{2}$$

Where

$$
\begin{aligned}
W^{-1} &= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -\frac{1}{4}e^{t} & \frac{1}{4}e^{t} \\ \frac{5}{4}e^{-3t} & -\frac{1}{4}e^{-3t} \end{pmatrix}
\end{aligned}
$$

Hence, (2) becomes

$$
\begin{aligned}
\boldsymbol{x}_p(t) &= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \int \begin{pmatrix} -\frac{1}{4}e^{t} & \frac{1}{4}e^{t} \\ \frac{5}{4}e^{-3t} & -\frac{1}{4}e^{-3t} \end{pmatrix} \begin{pmatrix} 28e^{-t} \\ 20e^{3t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \int \begin{pmatrix} 5e^{4t} - 7 \\ 35e^{-4t} - 5 \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-t} & e^{3t} \\ 5e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{5}{4}e^{4t} - 7t \\ -5t - \frac{35}{4}e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \\
&= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}
\end{aligned} \tag{3}
$$

At $t = 0$

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{4} - \frac{35}{4} \\ \frac{25}{4} - \frac{35}{4} \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{15}{2} \\ -\frac{5}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1 + c_2 - \frac{15}{2} \\ 5c_1 + c_2 - \frac{5}{2} \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{5}{2} \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ -35 \end{pmatrix}
$$

Last row gives $-4c_2 = -35$, or $c_2 = \frac{35}{4}$. First row gives $c_1 + c_2 = \frac{15}{2}$, hence $c_1 = \frac{15}{2} - \frac{35}{4} = -\frac{5}{4}$. Hence the solution from (3) becomes

$$
\boldsymbol{x}(t) = \frac{-5}{4} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-t} + \frac{35}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\ \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \end{pmatrix}
$$

Or

$$
\begin{aligned}
x(t) &= \frac{-5}{4}e^{-t} + \frac{35}{4}e^{3t} + \frac{5}{4}e^{3t} - 7te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t} \\
y(t) &= -\frac{25}{4}e^{-t} + \frac{35}{4}e^{3t} + \frac{25}{4}e^{3t} - 35te^{-t} - 5te^{3t} - \frac{35}{4}e^{-t}
\end{aligned}
$$

Or

$$
\begin{aligned}
x(t) &= -10e^{-t} + 10e^{3t} - 7te^{-t} - 5te^{3t} \\
y(t) &= -15e^{-t} + 15e^{3t} - 35te^{-t} - 5te^{3t}
\end{aligned}
$$

0.22 Section 8.2 problem 25

problem Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

$$f(t) = \begin{pmatrix} 4t \\ 1 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4t \\ 1 \end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

Hence $\lambda_1 = -i, \lambda_2 = i$.

For $\lambda_1 = -i$ we solve $(A - \lambda_1 I)\, v_1 = 0$

$$\begin{pmatrix} 2-\lambda_1 & -5 \\ 1 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $(2+i)\, v_1 - 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{(2+i)}{5}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{(2+i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$$

For $\lambda_2 = i$ we solve $(A - \lambda_2 I)\, v_2 = 0$

$$\begin{pmatrix} 2-\lambda_2 & -5 \\ 1 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation $(2-i)\, v_1 - 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{(2-i)}{5}$ and

$$v_1 = \begin{pmatrix} 1 \\ \frac{(2-i)}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

Therefore

$$\begin{aligned} x_h(t) &= c_1 x_1(t) + c_2 x_2(t) \\ &= c_1 v_1(t)\, e^{\lambda_1 t} + c_2 v_2(t)\, e^{\lambda_2 t} \end{aligned}$$

Or

$$x_h(t) = c_1 \begin{pmatrix} 5 \\ 2+i \end{pmatrix} e^{-it} + c_2 \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{it} \tag{1}$$

Change to new basis

$$\begin{aligned}
\boldsymbol{x}_1(t) &= \operatorname{Re}\begin{pmatrix} 5 \\ 2+i \end{pmatrix} e^{-it} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ (2+i)(\cos t - i\sin t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2(\cos t - i\sin t) + (i\cos t + \sin t) \end{pmatrix} \\
&= \operatorname{Re}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2\cos t + \sin t + i(-2\sin t + \cos t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\boldsymbol{x}_2(t) &= \operatorname{Im}\begin{pmatrix} 5(\cos t - i\sin t) \\ 2\cos t + \sin t + i(-2\sin t + \cos t) \end{pmatrix} \\
&= \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix}
\end{aligned}$$

Hence the homogenous solution in the new basis is

$$\boxed{\mathbf{x}_h(t) = C_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2\begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix}} \tag{1A}$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$\begin{aligned}
W &= \begin{pmatrix} \boldsymbol{x}_1(t) & \boldsymbol{x}_2(t) \end{pmatrix} \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}
\end{aligned}$$

Therefore

$$\boldsymbol{x}_p(t) = W\int W^{-1}\boldsymbol{f}(t)\,dt \tag{2}$$

Where

$$\begin{aligned}
W^{-1} &= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{5}(\cos t - 2\sin t) & \sin t \\ \frac{1}{5}(-2\cos t + \sin t) & \cos t \end{pmatrix}
\end{aligned}$$

Hence, (2) becomes

$$\begin{aligned}
\boldsymbol{x}_p(t) &= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\int\begin{pmatrix} \frac{1}{5}(\cos t - 2\sin t) & \sin t \\ \frac{1}{5}(-2\cos t + \sin t) & \cos t \end{pmatrix}\begin{pmatrix} 4t \\ 1 \end{pmatrix} dt \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\int\begin{pmatrix} \sin t + 4t\left(\frac{1}{5}\cos t - \frac{2}{5}\sin t\right) \\ \cos t - 4t\left(\frac{2}{5}\cos t - \frac{1}{5}\sin t\right) \end{pmatrix} dt \\
&= \begin{pmatrix} 5\cos t & -5\sin t \\ 2\cos t + \sin t & -2\sin t + \cos t \end{pmatrix}\begin{pmatrix} \frac{8}{5}t\cos t - \frac{8}{5}\sin t - \frac{1}{5}\cos t + \frac{4}{5}t\sin t \\ \frac{9}{5}\sin t - \frac{8}{5}\cos t - \frac{4}{5}t\cos t - \frac{8}{5}t\sin t \end{pmatrix}
\end{aligned}$$

Which, with little help of computer algebra, simplifies to

$$\boldsymbol{x}_p(t) = \begin{pmatrix} 8t - 1 \\ 4t - 2 \end{pmatrix}$$

Therefore the general solution is

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \qquad (3) \\
&= C_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2\begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix} + \begin{pmatrix} 8t - 1 \\ 4t - 2 \end{pmatrix}
\end{aligned}$$

At $t=0$

$$\begin{pmatrix}0\\0\end{pmatrix}=C_1\begin{pmatrix}5\\2\end{pmatrix}+C_2\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-1\\-2\end{pmatrix}$$
$$=\begin{pmatrix}5C_1-1\\2C_1+C_2-2\end{pmatrix}$$

Hence

$$\begin{pmatrix}5&0\\2&1\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

First row gives $C_1=\frac{1}{5}$ and last row gives $2C_1+C_2=2$ or $C_2=2-\frac{2}{5}=\frac{8}{5}$. Hence the solution becomes

$$\boldsymbol{x}(t)=\frac{1}{5}\begin{pmatrix}5\cos t\\2\cos t+\sin t\end{pmatrix}+\frac{8}{5}\begin{pmatrix}-5\sin t\\-2\sin t+\cos t\end{pmatrix}+\begin{pmatrix}8t-1\\4t-2\end{pmatrix}$$

Or

$$x(t)=\cos t-8\sin t+8t+8t-1$$
$$y(t)=2\cos t-3\sin t+4t-2$$

0.23 Section 8.2 problem 28

problem Use the method of variation of parameters to solve $\boldsymbol{x}'=A\boldsymbol{x}+\boldsymbol{f}(t)$.

$$A=\begin{pmatrix}2&-4\\1&-2\end{pmatrix}$$
$$\boldsymbol{f}(t)=\begin{pmatrix}4\ln t\\\frac{1}{t}\end{pmatrix}$$
$$\boldsymbol{x}(1)=\begin{pmatrix}1\\-1\end{pmatrix}$$

solution The matrix form of the system is

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}4\ln t\\t^{-1}\end{pmatrix}$$

The eigenvalues of the homogenous system are found from

$$|A-\lambda I|=0$$
$$\begin{vmatrix}2-\lambda&-4\\1&-2-\lambda\end{vmatrix}=0$$
$$\lambda^2=0$$
$$\lambda=0$$

Hence zero eigenvalue. Let see if this is complete eigenvalue or not.

For $\lambda=0$ we solve $(A-\lambda_1 I)\boldsymbol{v}_1=0$

$$\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

From first equation $2v_1-4v_2=0$. Let $v_1=1$ then $v_2=\frac{1}{2}$ and

$$\boldsymbol{v}_1=\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

We can only find this one eigenvector. Second row gives same eigenvector. This means this is defective eigenvalue. We can't use this method. We are stuck. So we switch to the defective eigenvalue method (page 450). We start by solving for v_2 from

$$(A-\lambda I)^2 v_2 = 0$$

$$\begin{pmatrix} 2-\lambda & -4 \\ 1 & -2-\lambda \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ can be any value. Let $v_1 = 1, v_2 = 0$ and therefore

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We now find v_1 from

$$\begin{aligned} v_1 &= (A-\lambda I)\, v_2 \\ &= \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Where now

$$\begin{aligned} x_1(t) &= v_1 e^{\lambda t} \\ x_2(t) &= (v_1 t + v_2)\, e^{\lambda t} \end{aligned}$$

Hence the homogenous solution is

$$\begin{aligned} x_h(t) &= c_1 v_1 e^{\lambda t} + c_2 (v_1 t + v_2)\, e^{\lambda t} \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} \end{aligned}$$

The Wronskian W (which is the same as fundamental matrix Φ) is

$$\begin{aligned} W &= \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \end{aligned}$$

Therefore

$$x_p(t) = W \int W^{-1} f(t)\, dt \tag{2}$$

Where

$$\begin{aligned} W^{-1} &= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -t & 2t+1 \\ 1 & -2 \end{pmatrix} \end{aligned}$$

Hence, (2) becomes

$$x_p(t) = \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \int \begin{pmatrix} -t & 2t+1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4\ln t \\ \frac{1}{t} \end{pmatrix} dt$$
$$= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \int \begin{pmatrix} \frac{1}{t}(2t+1) - 4t\ln t \\ 4\ln t - \frac{2}{t} \end{pmatrix} dt$$
$$= \begin{pmatrix} 2 & 2t+1 \\ 1 & t \end{pmatrix} \begin{pmatrix} 2t + \ln t - 2t^2 \ln t + t^2 \\ 4t\ln t - 2\ln t - 4t \end{pmatrix}$$
$$= \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}$$

Therefore the general solution is

$$x(t) = x_h(t) + x_p(t) \tag{3}$$
$$= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} + \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}$$

At $t = 1$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 2c_1 + 3c_2 - 6 \\ c_1 + c_2 - 1 \end{pmatrix}$$

Hence

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{7}{2} \end{pmatrix}$$

second row gives $-\frac{1}{2}c_2 = -\frac{7}{2}$ or $c_2 = 7$ and first row gives $2c_1 + 3c_2 = 7$ or $c_1 = \frac{7-3c_2}{2} = \frac{7-21}{2} = -7$. Hence the solution becomes (from (3))

$$x(t) = -7 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} + \begin{pmatrix} 2t^2(2\ln t - 3) \\ 2t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2 \end{pmatrix}$$

Or

$$x(t) = -7 + 14t + 2t^2(2\ln t - 3)$$
$$y(t) = -7 + 9t + \ln t + 2t^2 \ln t - 2t\ln t - 3t^2$$

0.24 Example on page 500, textbook (Edwards&Penny, 3rd edition)

problem This problem was solved in textbook using matrix exponential. Here is solved using the fundamental matrix only. Use the method of variation of parameters to solve $x' = Ax + f(t)$.

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$
$$\bar{f}(t) = \begin{pmatrix} -15 \\ 4 \end{pmatrix} te^{-2t}$$
$$\bar{x}(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Solution

The homogeneous solution was found in the book as

$$\bar{x}_h = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

Following scalar case, the guess would be $\bar{x}_p = \left(\bar{b} + \bar{a}t\right)e^{-2t}$ but since e^{-2t} is in the homogeneous, we have to adjust to be $\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right)e^{-2t} + \bar{c}e^{5t}$. Notice we had to add $\bar{c}e^{5t}$, else it will not work if we just guessed $\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right)e^{-2t}$ based on what we would do in scalar case, we will find we get $\bar{a} = \bar{b} = 0$. This seems to be a trial and error stage and one just have to try to find out. This is why undermined coefficients for systems is not as easy to use as with scalar case. Hence

$$\bar{x}_p = \left(\bar{b}t + \bar{a}t^2\right)e^{-2t} + \bar{c}e^{5t}$$

Now we plug-in this back into the ODE and solve for $\bar{a}, \bar{b}, \bar{c}$. But an easier method is to use Variation of parameters. The fundamental matrix is

$$\begin{aligned}\Phi &= \begin{pmatrix}\bar{x}_1 & \bar{x}_2\end{pmatrix}\\ &= \begin{pmatrix}e^{-2t} & 2e^{5t}\\ -2e^{-2t} & e^{5t}\end{pmatrix}\end{aligned}$$

And

$$\Phi^{-1} = \frac{\begin{pmatrix}e^{5t} & 2e^{-2t}\\ -2e^{5t} & e^{-2t}\end{pmatrix}^T}{|\Phi|} = \frac{\begin{pmatrix}e^{5t} & -2e^{5t}\\ 2e^{-2t} & e^{-2t}\end{pmatrix}}{e^{3t} + 4e^{3t}} = \frac{1}{5}\begin{pmatrix}e^{2t} & -2e^{2t}\\ 2e^{-5t} & e^{-5t}\end{pmatrix}$$

Hence using

$$\begin{aligned}\bar{x}_p &= \Phi\int\Phi^{-1}\bar{f}(t)\,dt\\ &= \frac{1}{5}\Phi\int\begin{pmatrix}e^{2t} & -2e^{2t}\\ 2e^{-5t} & e^{-5t}\end{pmatrix}\begin{pmatrix}-15te^{-2t}\\ 4te^{-2t}\end{pmatrix}dt\\ &= \frac{1}{5}\Phi\int\begin{pmatrix}-23t\\ -26te^{-7t}\end{pmatrix}dt\end{aligned}$$

The integral of $\int -23t dt = \frac{-23}{2}t^2$ and $\int -26te^{-7t}dt = \frac{26}{49}e^{-7t}(7t+1)$ (using integration by parts) hence the above simplifies to

$$\begin{aligned}\bar{x}_p &= \Phi\begin{pmatrix}\frac{-23}{10}t^2\\ \frac{26}{245}e^{-7t} + \frac{26}{35}te^{-7t}\end{pmatrix}\\ &= \begin{pmatrix}e^{-2t} & 2e^{5t}\\ -2e^{-2t} & e^{5t}\end{pmatrix}\begin{pmatrix}\frac{-23}{10}t^2\\ \frac{26}{245}e^{-7t} + \frac{26}{35}te^{-7t}\end{pmatrix}\\ &= \begin{pmatrix}\frac{52}{245}e^{-2t} + \frac{52}{35}te^{-2t} - \frac{23}{10}t^2e^{-2t}\\ \frac{26}{245}e^{-2t} + \frac{26}{35}te^{-2t} + \frac{23}{5}t^2e^{-2t}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{490}e^{-2t}\left(-1127t^2 + 728t + 104\right)\\ \frac{1}{245}e^{-2t}\left(1127t^2 + 182t + 26\right)\end{pmatrix}\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\bar{x} &= \bar{x}_h + \bar{x}_p\\ &= c_1\begin{pmatrix}1\\ -2\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}2\\ 1\end{pmatrix}e^{5t} + \begin{pmatrix}\frac{1}{490}e^{-2t}\left(-1127t^2 + 728t + 104\right)\\ \frac{1}{245}e^{-2t}\left(1127t^2 + 182t + 26\right)\end{pmatrix}\end{aligned}$$

To find the constants, we apply initial conditions. At $t = 0$

$$\begin{aligned}\begin{pmatrix}7\\ 3\end{pmatrix} &= c_1\begin{pmatrix}1\\ -2\end{pmatrix} + c_2\begin{pmatrix}2\\ 1\end{pmatrix} + \begin{pmatrix}\frac{52}{245}\\ \frac{26}{245}\end{pmatrix}\\ c_1\begin{pmatrix}1\\ -2\end{pmatrix} + c_2\begin{pmatrix}2\\ 1\end{pmatrix} &= \begin{pmatrix}7\\ 3\end{pmatrix} - \begin{pmatrix}\frac{52}{245}\\ \frac{26}{245}\end{pmatrix}\\ \begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} &= \begin{pmatrix}\frac{1663}{245}\\ \frac{709}{245}\end{pmatrix}\\ \begin{pmatrix}1 & 2\\ 0 & 5\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} &= \begin{pmatrix}\frac{1663}{245}\\ \frac{807}{49}\end{pmatrix}\end{aligned}$$

Hence $5c_2 = \frac{807}{49}$ or $c_2 = \frac{807}{245}$ and $c_1 + 2c_2 = \frac{1663}{245}$, hence $c_1 = \frac{1663}{245} - 2\left(\frac{807}{245}\right) = \frac{1}{5}$. Therefore the solution becomes

$$\bar{x} = \frac{1}{5}\begin{pmatrix}1\\ -2\end{pmatrix}e^{-2t} + \frac{807}{245}\begin{pmatrix}2\\ 1\end{pmatrix}e^{5t} + \begin{pmatrix}\frac{1}{490}e^{-2t}\left(-1127t^2 + 728t + 104\right)\\ \frac{1}{245}e^{-2t}\left(1127t^2 + 182t + 26\right)\end{pmatrix}$$