Sturm Liouville Equations

$$0 < x < 1$$

$$-\frac{d}{dx}\left\{p(x)\frac{dy(x)}{dx}\right\} + q(x)\,y(x) = \lambda\, r(x)\, y(x)$$

with $a_1 y(0) + a_2 y'(0) = 0$ — separated boundary conditions

$$b_1 y(1) + b_2 y'(1) = 0$$

a_1, a_2, b_1, b_2 real

p, p', q, r real continuous in $0 \le x \le 1$

$p(x) > 0$, $r(x) > 0$ in $0 \le x \le 1$

$L[y] = \lambda\, r(x) y$, $\quad L = -\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

$\boxed{1}$ The operator together with the boundary conditions is symmetric

[also called Hermitian or self-adjoint]

Notation: $(Lu, v) = (u, Lv)$

For real valued functions u, v satisfying the boundary conditions

$$\int_0^1 v\, L[u]\, dx = \int_0^1 u\, L[v]\, dx$$

For complex valued u, v satisfying the boundary conditions

$$\int_0^1 \bar{v}\, L[u]\, dx = \int_0^1 u\, L[\bar{v}]\, dx$$

where bar is complex conjugate.

Lets show this for real valued functions using integration by parts

$$\int_0^1 v\,L[u]\,dx = \int_0^1 \left[-(pu')'v + quv\right] dx$$

$$= -v(pu')\Big|_0^1 + \int_0^1 (pu')v'\,dx + \int_0^1 quv\,dx$$

by parts again $\quad s = pv' \qquad dt = u'dx$

$$ds = (pv')'dx \qquad t = u$$

$$= -v(pu')\Big|_0^1 + puv'\Big|_0^1 - \int_0^1 u(pv')'dx$$

$$+ \int_0^1 quv\,dx$$

$$= -p\left[u'v - uv'\right]\Big|_0^1 + \int_0^1 u\,L[v]\,dx$$

Lagrange's Identity:

$$\int_0^1 \left(L[u]v - uL[v]\right)dx = -p\left[u'v - uv'\right]\Big|_0^1$$

Now u, v both satisfy the boundary conditions

$$u'(0) = -\frac{a_1}{a_2} u(0) \qquad u'(1) = -\frac{b_1}{b_2} u(1)$$

$$v'(0) = \frac{-a_1}{a_2} v(0) \qquad v'(1) = \frac{-b_1}{b_2} v(1)$$

$$-p(1)\left[u'(1)v(1) - u(1)v'(1)\right]$$
$$+ p(0)\left[u'(0)v(0) - u(0)v'(0)\right]$$

$$= -p(1)\left[\frac{-b_1}{b_2} u(1)v(1) + \frac{b_1}{b_2} u(1)v(1)\right]$$
$$+ p(0)\left[\frac{-a_1}{a_2} u(0)v(0) + \frac{a_1}{a_2} u(0)v(0)\right] = 0$$

$$\Rightarrow \int_0^1 v L[u] dx = \int_0^1 u L[v] dx$$

where u, v satisfy the boundary conditions

For complex valued functions, one needs to split into real and imaginary parts and continue...

Please do this for an exercise...

$\boxed{2}$ All eigenvalues λ are real

Suppose λ is complex, with corresponding eigenfunction $\phi(x)$ also possibly complex

$\lambda = \mu + i\nu$, $\phi(x) = U(x) + iV(x)$

We know $(L[\phi], \phi) = (\phi, L[\phi])$

and $L[\phi] = \lambda r \phi \Rightarrow$

$$(\lambda r \phi, \phi) = (\phi, \lambda r \phi)$$

Since ϕ is complex, λ complex, this inner product means

$$\int_0^1 \lambda r \phi \bar{\phi}\, dx = \int_0^1 \phi \bar{\lambda} \bar{r} \bar{\phi}\, dx$$

but $r(x)$ real, $r(x) > 0$ $\Rightarrow$

$$(\lambda - \bar{\lambda}) \int_0^1 r \phi \bar{\phi} \, dx = 0 \quad \text{or}$$

$$(\lambda - \bar{\lambda}) \int_0^1 r (u^2 + v^2) \, dx = 0$$

Since the integrand is non-negative and continuous by assumption, and not identically zero $\Rightarrow$ the integral is positive

$$\Rightarrow (\lambda - \bar{\lambda}) = 2 i \nu = 0$$

$\Rightarrow$ λ real

$\boxed{3}$ For each eigenvalue, there is only one linearly independent eigenfunction

Assume $L[\phi_1] = \lambda_0 r \phi_1$, $L[\phi_2] = \lambda_0 r \phi_2$

with ϕ_1, ϕ_2 linearly independent for $\lambda = \lambda_0$, and ϕ_1, ϕ_2 satisfy the boundary conditions

By definition of linear independence on $0 \le x \le 1$ $\Longrightarrow$

$$\left.\begin{aligned} c_1 \phi_1(x) + c_2 \phi_2(x) = 0 \\ c_1 \phi_1'(x) + c_2 \phi_2'(x) = 0 \end{aligned}\right\} \quad 0 \le x \le 1$$

can only be satisfied with $c_1 = c_2 = 0$

$$\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 0 \le x \le 1$$

has unique soln. $c_1 = c_2 = 0$ only if

$$W(x) = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x) \neq 0$$

However $W(x)$ satisfies the 1st-order ODE

$$\frac{dW(x)}{dx} + \frac{p'(x)}{p(x)} W(x) = 0 \quad \text{in } 0 \le x \le 1$$

as can be checked by direct substitution.

Therefore $$W(x) = C_W \exp\left[-\int \frac{p'(x)}{p(x)}\, dx\right]$$

$$0 \le x \le 1$$

Since the exponential function is never zero, then

$W(x) = 0$ for all $0 \le x \le 1$ $\{$ if $C_W = 0\}$

or $W(x) \neq 0$ for any $0 \le x \le 1$ $\{$ if $C_W \neq 0\}$

Using the boundary conditions, we can compute

$$W(0) = \phi_1(0)\left(\frac{-a_1}{a_2}\right)\phi_2(0) - \left(\frac{-a_1}{a_2}\right)\phi_1(0)\,\phi_2(0) = 0$$

$$W(1) = \phi_1(1)\left(-\frac{b_1}{b_2}\right)\phi_2(1) - \left(-\frac{b_1}{b_2}\right)\phi_1(1)\phi_2(1)$$

$$= 0$$

and therefore $W(x) = 0$ for all $0 \leq x \leq 1$

$\Rightarrow$ $\phi_1(x)$, $\phi_2(x)$ cannot be linearly independent in $0 \leq x \leq 1$

$\boxed{4}$ The eigenfunctions are real

Assume $\phi = u + iV$ is complex

We know $\lambda = \lambda_0$ real by $\boxed{2}$

and $L[\phi] = \lambda_0 r \phi$ $\Rightarrow$

$L[u + iV] = \lambda_0 r [u + iV]$ can only be satisfied if

$$L[u] = \lambda_0 r u \; ; \; L[V] = \lambda_0 r V$$

Also U, V must separately satisfy the boundary conditions

$$a_1\left(U(0)+iV(0)\right)+a_2\left(U'(0)+iV'(0)\right)=0$$

$\Rightarrow$ $a_1U(0)+a_2U'(0)=0$ and

$$a_1V(0)+a_2V'(0)=0$$

$$b_1\left(U(1)+iV(1)\right)+b_2\left(U'(1)+iV'(1)\right)=0$$

$\Rightarrow$ $b_1U(1)+b_2U'(1)=0$ and

$$b_1V(1)+b_2V'(1)=0$$

It follows from $\boxed{3}$ that $U(x)$ $V(x)$ are linearly dependent :

$U+iV=(1+iC)U$ and we may

choose C pure imaginary so that

$U+iV=AU$ with A real

$\boxed{5}$ The eigenfunctions are orthogonal with respect to the weight functions $r(x)$

Let λ_1, λ_2 be different eigenvalues with eigenfunctions $\phi_1(x), \phi_2(x)$ $\Rightarrow$

$$L[\phi_1] = \lambda_1 r \phi_1 \quad , \quad L[\phi_2] = \lambda_2 r \phi_2$$

$$(L[\phi_1], \phi_2) - (\phi_1, L[\phi_2]) = 0 \qquad [1]$$

$$(\lambda_1 r \phi_1, \phi_2) - (\phi_1, \lambda_2 r \phi_2) = 0$$

$$\lambda_1 \int_0^1 r \phi_1 \bar{\phi}_2 \, dx - \bar{\lambda}_2 \int_0^1 \phi_1 \bar{r} \bar{\phi}_2 \, dx = 0$$

but λ, r, ϕ real $\Rightarrow$

$(\lambda_1 - \lambda_2) \int_0^1 r \phi_1 \phi_2 \, dx = 0$ and by

assumption $\lambda_1 \neq \lambda_2$ $\Rightarrow \int_0^1 r \phi_1 \phi_2 \, dx = 0$

(orthogonality)