University Course

Engineering Physics (EP) 548
Engineering analysis II

University of Wisconsin, Madison
Spring 2017

My Class Notes

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Spring 2017

Contents

Chapter 1

Introduction

Local contents

1.1 links

1. Professor Leslie M. Smith web page
2. piazza class page Needs login

1.2 syllabus

NE 548
Engineering Analysis II
TR 11:00-12:15 in Van Vleck B341

Leslie Smith: *lsmith@math.wisc.edu*, Office Hours in Van Vleck Tuesday/Thursday 12:30-2:00, http://www.math.wisc.edu/~lsmith.

Textbook 1 Required: *Advanced Mathematical Methods for Scientists and Engineers*, Bender and Orszag, Springer.

Textbook 2 Recommended: *Applied Partial Differential Equations*, Haberman, Pearson/Prentice Hall. This text is recommended because some of you may already own it (from Math 322). Almost any intermediate-advanced PDEs text would be suitable alternative as reference.

Pre-requisite: NE 547. If you have not taken NE 547, you should have previously taken courses equivalent to Math 319, 320 or 340, 321, 322.

Assessment: Your grade for the course will be based on two take-home midterm exams (35% each) and selected homework solutions (30%). The homework and midterms exams for undergraduate and graduate students will be different in scope; please see the separate learning outcomes below. Graduate students will be expected to synthesize/analyze material at a deeper level.

Undergraduate Learning Outcomes:

- Students will demonstrate knowledge of asymptotic methods to analyze ordinary differential equations, including (but not limited to) boundary layer theory, WKB analysis and multiple-scale analysis.
- Students will demonstrate knowledge of commonly used methods to analyze partial differential equations, including (but not limited to) Fourier analysis, Green's function solutions, similarity solutions, and method of characteristics.
- Students will apply methods to idealized problems motivated by applications. Such applications include heat conduction (the heat equation), quantum mechanics (Schrodinger's equation) and plasma turbulence (dispersive wave equations).

Graduate Learning Outcomes:

- Students will demonstrate knowledge of asymptotic methods to analyze ordinary differential equations, including (but not limited to) boundary layer theory, WKB analysis and multiple-scale analysis.
- Students will demonstrate knowledge of commonly used methods to analyze partial differential equations, including (but not limited to) Fourier analysis, Green's function solutions, similarity solutions, and method of characteristics.
- Students will apply methods to idealized problems motivated by applications. Such applications include heat conduction (the heat equation), quantum mechanics (Schrodinger's equation) and plasma turbulence (dispersive wave equations).

- Students will apply methods to solve problems in realistic physical settings.
- Students will synthesize multiple techniques to solve equations arising from applications.

Grading Scale for Final Grade: 92-100 A, 89-91 AB, 82-88 B, 79-81 BC, 70-78 C, 60-69 D, 59 and below F

Midterm 1: Given out Thursday March 2, 2015 and due Thursday March 9, 2017.

Midterm 2: Given out Thursday April 27, 2017 and due Thursday May 4, 2015.

Homework: Homework problems will be assigned regularly, either each week or every other week, paced for 6 hours out-of-class work every week. Homework groups are encouraged, but each student should separately submit solutions reflecting individual understanding of the material.

Piazza: There will be a Piazza course page where all course materials will be posted. Piazza is also a forum to facilitate peer-group discussions. Please take advantage of this resource to keep up to date on class notes, homework and discussions.

Piazza Sign-Up Page: piazza.com/wisc/spring2017/neema548
Piazza Course Page: piazza.com/wisc/spring2017/neema548/home

Course Outline

Part I: Intermediate-Advanced Topics in ODEs from Bender and Orszag.

1. Review of local analysis of ODEs near ordinary points, regular singular points and irregular singular points (BO Chapter 3, 1.5 weeks)
2. Global analysis using boundary layer theory (BO Chapter 9, 1.5 weeks).
3. Global analysis using WKB theory (BO Chapter 10, 1.5 weeks).
4. Green's function solutions (1 week)
5. Multiple-scale analysis (BO Chapter 11, 1.5 weeks).

Part II: Intermediate-Advanced Topics in PDEs

1. Review of Sturm-Liouville theory and eigenfunction expansions (1.5 weeks)
2. Non-homogeneous problems and Green's function solutions (1.5 weeks)
3. Infinite domain problems and Fourier transforms (1.5 weeks)
4. Quasilinear PDEs (1.5 weeks)
5. Dispersive wave systems (time remaining)

2

Chapter 2

Resources

Local contents

2.1 Review of ODE's

(1)

Review of Basic ODEs

Consider all possible 2^{nd}-order, linear, homogeneous ODEs

$$a(x)\,y''(x) + b(x)\,y'(x) + c(x)\,y(x) = 0$$

in standard form: $y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$

with $p(x), q(x)$ real continuous in $x \in (\alpha, \beta)$

Theory The solution space is a 2-dimensional linear function space $\Rightarrow$ we need 2 linearly independent solutions to describe all possible solutions

The initial or boundary conditions select a solution to the physical problem from among the infinite set.

Defn of Linear Independence

$y_1(x)$ and $y_2(x)$ are linearly independent if

$$a_1 y_1(x) + a_2 y_2(x) = 0 \Rightarrow a_1 = a_2 = 0$$

{then $y_2(x)$ is not proportional to $y_1(x)$}

$$\left.\begin{aligned} a_1 y_1(x) + a_2 y_2(x) &= 0 \\ a_1 y_1'(x) + a_2 y_2'(x) &= 0 \end{aligned}\right\} \quad \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(2)

has solution $a_1 = a_2 = 0$ if $W(y_1, y_2) \neq 0$ $x \in (\alpha, \beta)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

So $y_1(x), y_2(x)$ are linearly independent if $W(y_1, y_2) \neq 0$.

Then $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is the general solution {includes all possible solutions}

Find C_1, C_2 using initial or boundary conditions

Reduction of Order If you only know one solution $y_1(x)$, a 2^{nd} linearly independent solution can be found as $y_2(x) = y_1(x)\, v(x)$. Find $v(x)$ by plugging in:

Example $x^2 y'' + 3xy' + y = 0$ Let $y(x) = Ax^r$

$\Rightarrow r(r-1) + 3r + 1 = 0 \Rightarrow r = -1$ repeated

$y_1(x) = Ax^{-1}$ **check**

To find a 2^{nd} linearly independent solution, let

$y(x) = x^{-1} v(x)$ and plug in

(3)

$y(x) = x^{-1} v(x)$; $y' = -x^{-2} v + x^{-1} v'$;

$y'' = 2x^{-3} v - 2x^{-2} v' + x^{-1} v''$

Plug into $x^2 y'' + 3xy' + y = 0$ $\Rightarrow$

$$y(x) = x^{-1} \{ C_1 + C_2 \ln x \}$$

$\Rightarrow$ $y_2(x) = x^{-1} \ln x$ [check]

Why does it work?

$$W(y_1, y_1 v) = \begin{vmatrix} y_1 & y_1 v \\ y_1' & v y_1' + y_1 v' \end{vmatrix}$$

$$= y_1 \{ y_1' v + y_1 v' \} - y_1 y_1' v = y_1^2 v'$$

y_1, $y_1 v$ will be linearly independent unless

$y_1^2 v' = 0 \Rightarrow y_1^2 = 0$ NO

or $v' = 0 \Rightarrow v = C$ NO

(4)

Special Equations and Solution Techniques

1. Constant Coefficients $ay'' + by' + cy = 0 \quad -\infty < x < \infty$

$y = Ae^{rx} \Rightarrow ar^2 + br + c = 0$

quadratic eqn. for $r \Rightarrow$ 2 different real roots,
1 real repeated root (then use RoO),
2 complex conjugate roots
Everything is known!

2. Euler / Equidimensional Eqns.

$ax^2y'' + bxy' + cy = 0 \quad ; \quad y = Ax^r$

$\Rightarrow \quad ar(r-1) + br + c = 0$

quadratic eqn. for $r \Rightarrow$ Everything is known!
The equation in standard form has a singularity at $x = 0$:

$y'' + \frac{b}{ax} y' + \frac{c}{ax} y = 0$ and $y = Ax^r$ can be singular at $x = 0$.

(5)

3. Regular Sturm Liouville Problems

$$-\left[P_{SL}(x)\, y'(x)\right]' + Q_{SL}(x)\,y(x) = \lambda R_{SL}(x)\, y(x) \qquad 0<x<1$$

Note: this is not standard form, hence caps and subscript "SL" for coefficients

Note: the eigenvalue λ appears; typically comes from separation of variables as in our heat conduction problem

Separated boundary conditions $a_1 y(0) + a_2 y'(0) = 0$

$b_1 y(1) + b_2 y'(1) = 0$

a_1, a_2, b_1, b_2 real

P_{SL}, P'_{SL}, Q_{SL}, R_{SL} real continuous $0 \le x \le 1$

P_{SL}, $R_{SL} > 0$ in $0 \le x \le 1$

Need $P_{SL}(x) > 0$ in $0 \le x \le 1$ for existence [divide by $P_{SL}(x)$ to get standard form]

Need $R_{SL}(x) > 0$ in $0 \le x \le 1$ for reality of eigenvalues λ_n and eigenfunctions $y_n(x)$

(6)

SL often written $L[y] = \lambda R_{SL}(x) y \quad 0<x<1$

with $L = -\frac{d}{dx}\left[P_{SL}(x) \frac{d}{dx} \right] + Q_{SL}(x)$

One can prove the following:

** all eigenvalues λ_n are real

** all eigenfunctions $y_n(x)$ are real

** all eigenfunctions satisfy an orthogonality relation with respect to the weight function $R_{SL}(x)$:

$$\int_0^1 y_n(x)\, y_m(x)\, R_{SL}(x)\, dx = \Omega\, \delta_{nm}$$

Look for proofs in intro ODEs books ...

(7)

Recall Bessel of Order Zero:

$r^2 \varphi''(r) + r\varphi'(r) + \lambda r \varphi(r) = 0$, or standard form

$\varphi''(r) + \frac{1}{r}\varphi'(r) + \lambda\varphi(r) = 0$

In SL form $P_{SL}(r) = r$, $Q_{SL}(r) = 0$, $R_{SL}(r) = r$

$$-\frac{d}{dr}\left[r\frac{d\varphi(r)}{dr}\right] = \lambda r \varphi(r)$$

$-\{\varphi' + r\varphi''\} = \lambda r \varphi$ or

$\varphi'' + \frac{1}{r}\varphi' + \lambda\varphi = 0$ ✓

Bessel is a singular SL problem because

$P_{SL}(0) = R_{SL}(0) = 0$ [not > 0]

but proofs go through with $\varphi(0)$ bounded

(8)

Orthogonality in the familiar setting of sine functions

Consider $y'' = -\lambda y \quad y(0) = 0 \quad y(\pi) = 0$

a regular SL problem ...

$$y = A\sin(\sqrt{\lambda}\, x) + B\cos(\sqrt{\lambda}\, x)$$

$$y(0) = 0 \implies B = 0$$

$$y(\pi) = 0 \implies A\sin(\sqrt{\lambda}\, \pi) = 0 \implies \sqrt{\lambda}\, \pi = n\pi$$
$$\implies \sqrt{\lambda} = n$$

$$y_n(x) = A_n \sin nx$$

Orthogonality : $\displaystyle\int_0^{\pi} \sin nx \, \sin mx \, dx = \frac{\pi}{2}\delta_{nm}$

To represent a general $f(x)$ odd in $[-\pi, \pi]$
we need all of them !

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \, \sin mx \, dx$$

$$= \pi \delta_{nm} b_n = b_m \implies b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

(9)

For eqns. other than constant coefficient or Euler / Equi-dimensional eqns., we typically perform "local analysis" about some point $x = x_0$ [just like we did for Bessel]

We try to approximate the solution near $x = x_0$ with some kind of series solution

3 types of series solutions

(i) $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ if x_0 is an "ordinary point."

where n is integer and we need to find the a_n's.

This Taylor series solution is not singular at $x = x_0$

(ii) $y(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n (x - x_0)^n$ if x_0 is a "regular singular point."

n integer, α is real ; we need to find the a_n's and α.

This is called a Frobenius series

(10)

The solution may be singular at $x = x_0$ depending on the value of α, but the singularity is a power of $(x - x_0)$

(iii) $y(x) = \exp\left[\beta(x - x_0)\right] \sum_{n=0}^{\infty} a_n (x - x_0)^{\gamma n}$

If x_0 is an "irregular singular point"

n integer, $\gamma > 0$ real, $\beta = \beta(x)$;

we need to find $\beta(x)$, γ and the a_n's.

Now the solution may be singular at $x = x_0$ and the singularity may cause $y(x) \to \infty$ faster than a power of $(x - x_0)$

Note the difference between

$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{\alpha + n}$ Frobenius; integer powers apart from an overall factor of $(x - x_0)^{\alpha}$

$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{\gamma n}$ which may be non-integer powers

(11)

In all cases we need to ask: For what values of x does the expansion give a good approximation to the solution?

How many terms do we need to keep in order to obtain a good approximation?

In particular, does the series converge?

Pointwise Convergence =>

$$\lim_{N\to\infty} \left| y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \right| = 0$$

For fixed x as $N \to \infty$

If the series does not converge, can it still be useful?

2.2 Sturm Liouville

(1)

Sturm Liouville Equations

$0 < x < 1$

$$-\frac{d}{dx}\left\{p(x)\frac{dy(x)}{dx}\right\} + q(x)\,y(x) = \lambda\, r(x)\, y(x)$$

with $a_1 y(0) + a_2 y'(0) = 0$ separated boundary conditions

$b_1 y(1) + b_2 y'(1) = 0$

a_1, a_2, b_1, b_2 real

p, p', q, r real continuous in $0 \le x \le 1$

$p(x) > 0$, $r(x) > 0$ in $0 \le x \le 1$

$L[y] = \lambda\, r(x)\, y$, $\quad L = -\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

[I] The operator together with the boundary conditions is symmetric

[also called Hermitian or self-adjoint]

Notation: $(Lu, v) = (u, Lv)$

(2)

For real valued functions u, v satisfying the boundary conditions

$$\int_0^1 v\, L[u]\, dx = \int_0^1 u\, L[v]\, dx$$

For complex valued u, v satisfying the boundary conditions

$$\int_0^1 \bar{v}\, L[u]\, dx = \int_0^1 u\, L[\bar{v}]\, dx$$

Where bar is complex conjugate.

Lets show this for real valued functions using integration by parts

(3)

$$\int_0^1 v\,L[u]\,dx = \int_0^1 \left[-(pu')'v + quv\right] dx$$

$$= -v(pu')\Big|_0^1 + \int_0^1 (pu')v'\,dx + \int_0^1 quv\,dx$$

by parts again $\quad s = pv' \qquad dt = u'dx$

$\qquad ds = (pv')'dx \qquad t = u$

$$= -v(pu')\Big|_0^1 + puv'\Big|_0^1 - \int_0^1 u(pv')'dx$$

$$+ \int_0^1 quv\,dx$$

$$= -p\left[u'v - uv'\right]\Big|_0^1 + \int_0^1 u\,L[v]\,dx$$

Lagrange's Identity:

$$\int_0^1 \left(L[u]v - uL[v]\right)dx = -p\left[u'v - uv'\right]\Big|_0^1$$

④

Now u, v both satisfy the boundary conditions

$$u'(0) = -\frac{a_1}{a_2} u(0) \qquad u'(1) = -\frac{b_1}{b_2} u(1)$$

$$v'(0) = \frac{-a_1}{a_2} v(0) \qquad v'(1) = \frac{-b_1}{b_2} v(1)$$

$$-p(1)\left[u'(1)v(1) - u(1)v'(1)\right] + p(0)\left[u'(0)v(0) - u(0)v'(0)\right]$$

$$= -p(1)\left[\frac{-b_1}{b_2} u(1)v(1) + \frac{b_1}{b_2} u(1)v(1)\right] + p(0)\left[\frac{-a_1}{a_2} u(0)v(0) + \frac{a_1}{a_2} u(0)v(0)\right] = 0$$

$$\Rightarrow \int_0^1 v L[u]\,dx = \int_0^1 u L[v]\,dx$$

where u, v satisfy the boundary conditions

(5)

For complex valued functions, one needs to split into real and imaginary parts and continue...

Please do this for an exercise...

[2] All eigenvalues λ are real

Suppose λ is complex, with corresponding eigenfunction $\phi(x)$ also possibly complex

$\lambda = \mu + i\nu$, $\phi(x) = U(x) + iV(x)$

We know $(L[\phi], \phi) = (\phi, L[\phi])$

and $L[\phi] = \lambda r \phi \implies$

$$(\lambda r \phi, \phi) = (\phi, \lambda r \phi)$$

Since ϕ is complex, λ complex, this inner product means

$$\int_0^1 \lambda r \phi \bar{\phi}\, dx = \int_0^1 \phi \bar{\lambda} \bar{r} \bar{\phi}\, dx$$

⑥

but $r(x)$ real, $r(x) > 0 \Rightarrow$

$$(\lambda - \bar{\lambda}) \int_0^1 r \phi \bar{\phi} \, dx = 0$$ or

$$(\lambda - \bar{\lambda}) \int_0^1 r (u^2 + v^2) \, dx = 0$$

Since the integrand is non-negative and continuous by assumption, and not identically zero $\Rightarrow$ the integral is positive

$$\Rightarrow (\lambda - \bar{\lambda}) = 2iv = 0$$

$\Rightarrow \lambda$ real

(7)

[3] For each eigenvalue, there is only one linearly independent eigenfunction

Assume $L[\phi_1] = \lambda_0 r \phi_1$, $L[\phi_2] = \lambda_0 r \phi_2$ with ϕ_1, ϕ_2 linearly independent for $\lambda = \lambda_0$, and ϕ_1, ϕ_2 satisfy the boundary conditions

By definition of linear independence on $0 \le x \le 1$ $\Longrightarrow$

$$\left.\begin{aligned} c_1 \phi_1(x) + c_2 \phi_2(x) &= 0 \\ c_1 \phi_1'(x) + c_2 \phi_2'(x) &= 0 \end{aligned}\right\} \quad 0 \le x \le 1$$

can only be satisfied with $c_1 = c_2 = 0$

$$\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 0 \le x \le 1$$

has unique soln. $c_1 = c_2 = 0$ only if

$$w(x) = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x) \neq 0$$

(8)

However $W(x)$ satisfies the 1[st]-order ODE

$$\frac{dW(x)}{dx} + \frac{p'(x)}{p(x)} W(x) = 0 \quad \text{in} \quad 0 \leq x \leq 1$$

as can be checked by direct substitution.

Therefore $$W(x) = C_W \exp\left[-\int \frac{p'(x)}{p(x)}\, dx\right]$$

$$0 \leq x \leq 1$$

Since the exponential function is never zero, then

$W(x) = 0$ for all $0 \leq x \leq 1$ $\{$if $C_W = 0\}$

or $W(x) \neq 0$ for any $0 \leq x \leq 1$ $\{$if $C_W \neq 0\}$

Using the boundary conditions, we can compute

$$W(0) = \phi_1(0)\left(\frac{-a_1}{a_2}\right)\phi_2(0) - \left(\frac{-a_1}{a_2}\right)\phi_1(0)\,\phi_2(0) = 0$$

(9)

$$W(1) = \phi_1(1)\left(-\frac{b_1}{b_2}\right)\phi_2(1) - \left(-\frac{b_1}{b_2}\right)\phi_1(1)\phi_2(1) = 0$$

and therefore $W(x) = 0$ for all $0 \le x \le 1$

$\Rightarrow$ $\phi_1(x)$, $\phi_2(x)$ cannot be linearly independent in $0 \le x \le 1$

[4] The eigenfunctions are real

Assume $\phi = u + iV$ is complex

We know $\lambda = \lambda_0$ real by [2]

and $L[\phi] = \lambda_0 r \phi \Rightarrow$

$L[u + iV] = \lambda_0 r [u + iV]$ can only be satisfied if

$$L[u] = \lambda_0 r u \; ; \; L[v] = \lambda_0 r V$$

(10)

Also U, V must separately satisfy the boundary conditions

$$a_1\left(U(0)+iV(0)\right)+a_2\left(U'(0)+iV'(0)\right)=0$$

$$\Rightarrow \quad a_1 U(0)+a_2 U'(0)=0 \quad \text{and}$$

$$a_1 V(0)+a_2 V'(0)=0$$

$$b_1\left(U(1)+iV(1)\right)+b_2\left(U'(1)+iV'(1)\right)=0$$

$$\Rightarrow \quad b_1 U(1)+b_2 U'(1)=0 \quad \text{and}$$

$$b_1 V(1)+b_2 V'(1)=0$$

It follows from [3] that $U(x)$ $V(x)$ are linearly dependent :

$U+iV = (1+iC)U$ and we may choose C pure imaginary so that

$U+iV = AU$ with A real

(11)

[5] The eigenfunctions are orthogonal with respect to the weight function $r(x)$

Let λ_1, λ_2 be different eigenvalues with eigenfunctions $\phi_1(x), \phi_2(x) \Rightarrow$

$$L[\phi_1] = \lambda_1 r \phi_1 \quad , \quad L[\phi_2] = \lambda_2 r \phi_2$$

$$(L[\phi_1], \phi_2) - (\phi_1, L[\phi_2]) = 0 \qquad [1]$$

$$(\lambda_1 r\phi_1, \phi_2) - (\phi_1, \lambda_2 r \phi_2) = 0$$

$$\lambda_1 \int_0^1 r\phi_1 \bar{\phi}_2 \, dx - \bar{\lambda}_2 \int_0^1 \phi_1 \bar{r} \bar{\phi}_2 \, dx = 0$$

but λ, r, ϕ real $\Rightarrow$

$$(\lambda_1 - \lambda_2) \int_0^1 r \phi_1 \phi_2 \, dx = 0$$ and by

assumption $\lambda_1 \neq \lambda_2 \Rightarrow \int_0^1 r \phi_1 \phi_2 \, dx = 0$

(orthogonality)

2.3 Wave equation

①

NE 548 The Wave Equation

Governs the (small) displacement of an elastic string at position x, time t; derived using Newton's Law for small amplitude displacements

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

lets first think about the infinite domain $-\infty < x < \infty$

$u = u(x,t)$

same as $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$

check: $\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x \partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ ✓

also same as $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$

Let $w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$; $v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$

$\Rightarrow$ 2 1st-order wave equations

$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$; $\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$

$w = w(x,t)$ $v = v(x,t)$

(2)

So first we need to understand these 1st-order wave equations, and then return to the 2nd-order case.

Let $x(t)$ be a moving observer

[In fluids, let $x = x(t)$ "follow a fluid particle"]

Then $w = w(x(t), t)$ satisfies

$$\frac{d}{dt} w = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial t} \quad \text{by chain rule}$$

If the observer moves at constant speed $\frac{dx(t)}{dt} = c$

then the 1D wave equation

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad \text{is equivalent to}$$

$$\left.\begin{aligned} &\frac{d}{dt} w = 0 \quad \text{along} \quad \frac{dx}{dt} = c \\ &\text{where } w = w(x(t), t), \quad x = x(t) \end{aligned}\right\}$$

Thus we have converted the PDE into 2 ODEs by introducing the moving observer [by following particles]

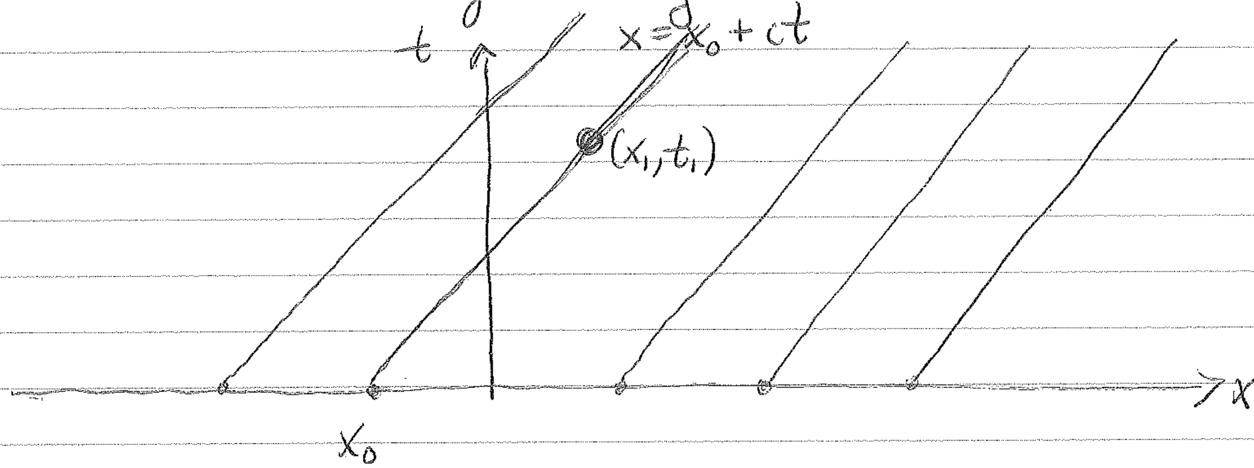
(3)

We are using the "method of characteristics" and the curves given by $\frac{dx(t)}{dt} = c$ are called the "characteristic curves".

In this case $x = x_0 + ct$ are lines with slope c originating at x_0.

$\frac{dw}{dt} = 0$ along $x = x_0 + ct$ means that

w does not change along curves $x = x_0 + ct$:



(4)

In the original problem we will be given an initial condition, say $w(x,0) = P(x)$

so we need to solve

$$\frac{dw(x(t),t)}{dt} = 0 \quad \text{along} \quad \frac{dx(t)}{dt} = 0 \quad \text{with}$$

$$w(x(0),0) = P(x(0))$$

Since w does not change in time along characteristics $\Longrightarrow$

$w(x,t) = P(x_0)$ on $x = x_0 + ct$, same as

$w(x,t) = P(x_0)$ on $x_0 = x - ct$ $\Longrightarrow$

$$\boxed{w(x,t) = P(x-ct)}$$

This is a long-winded way of saying

$w(x,t) = P(x-ct)$ satisfies

$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ which we already knew!

(5)

Check: Let $\xi = x - ct$, $w(x,t) = P(\xi)$

$\frac{\partial w}{\partial t} = -cP'(\xi)$; $c\frac{\partial w}{\partial x} = cP'(\xi)$ ✓

Example 1 $\quad \frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0$

$$w(x,0) = P(x) = \begin{cases} 0 & x < 0 \\ 4x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

w is not changing in time along $x = x_0 + 2t$

$w(x,t) = P(x_0)$ on $x_0 = x - 2t$

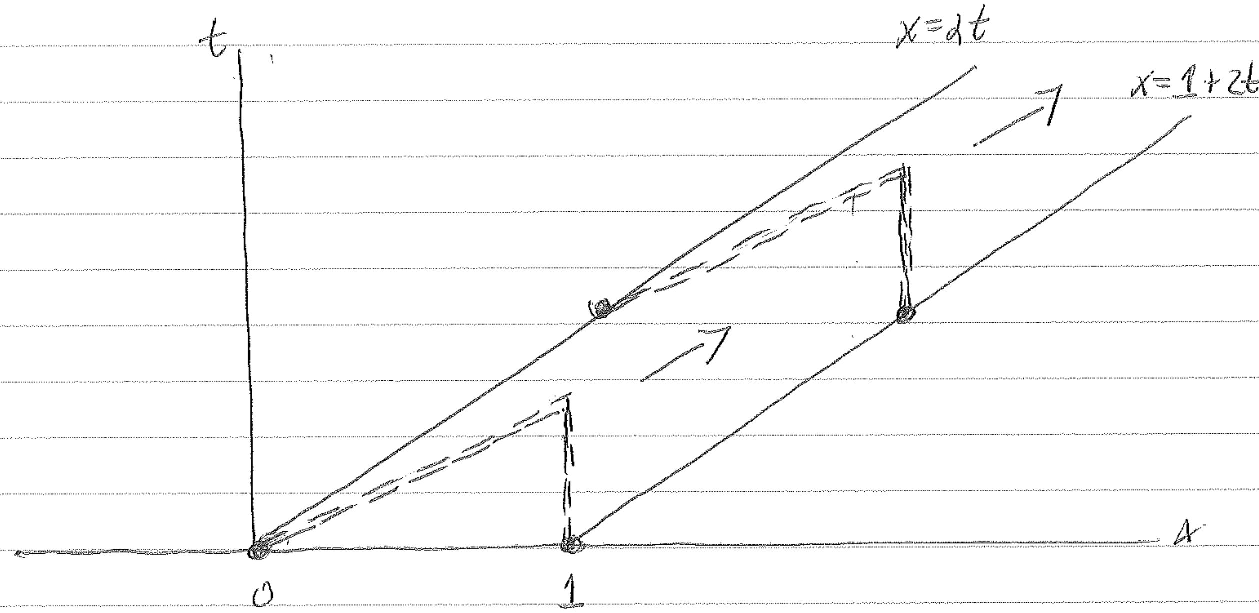
$$= \begin{cases} 0 & x_0 < 0 \\ 4x_0 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$

$$= \begin{cases} 0 & x - 2t < 0 \\ 4(x-2t) & 0 \le x - 2t \le 0 \\ 0 & x - 2t > 1 \end{cases}$$

or also written

6

$$w(x,t) = \begin{cases} 0 & x < 2t \\ 4(x-2t) & 0 \le x-2t \le 1 \\ 0 & x > 1+2t \end{cases}$$



imagine anothe axis out of the page

a wave of fixed shape moving to the right.

max amplitude of the wave is 4

(7)

[Simple Application: Transport with Decay]

Transport of a radioactively decaying solute in a uniform flow with wave speed c

$$\frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x} = -a\,u(x,t) \qquad u(x,0) = F(x)$$

$-\infty < x < \infty$ [$F(x)$ given for all values of the argument]

$a > 0$ is the decay rate

$u(x,t)$ is concentration [amount per unit volume] of the radioactive solute]

Translate to ODEs by letting

$x = x(t)$; $u = u(x(t), t)$

$x_0 = x(0)$; $u(0) = u(x(0), 0) = F(x_0)$

Then $\frac{du(t)}{dt} = -a\,u(t)$ along $\frac{dx(t)}{dt} = c$

with solution

$u(t) = A \exp[-at]$ along $x(t) = ct + x_0$

(8)

The initial condition $u(0) = F(x_0) \Rightarrow$

$A = F(x_0)$ a different constant for each value of x_0. on $-\infty < x_0 < \infty$

Finally $u(t) = F(x_0)\exp(-at)$ along $x(t) = ct + x_0$

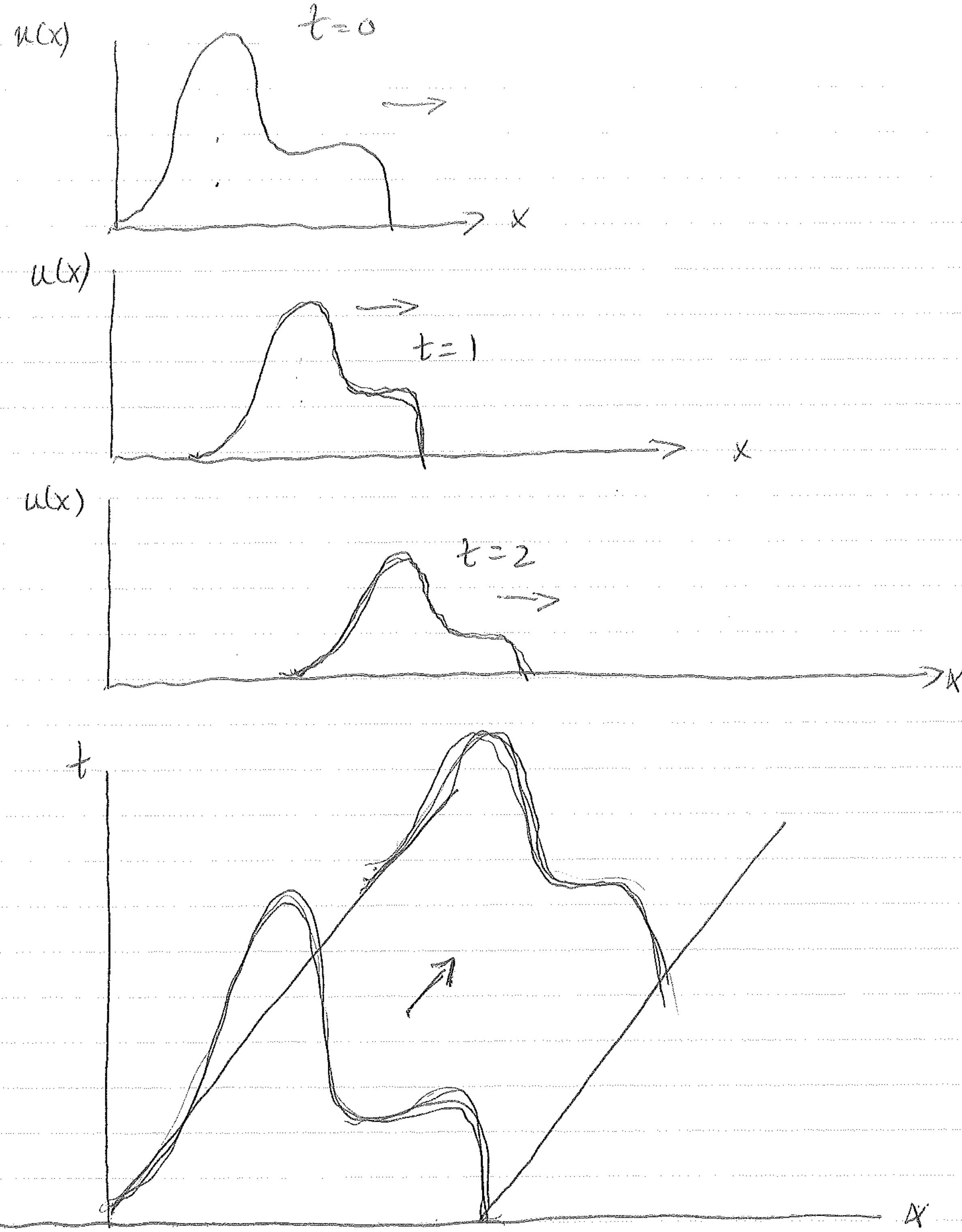
Translate back to the PDE:

$$u(x,t) = F(x-ct)\exp(-at)$$

↑ translation ↖ decay

(9)

Pictures



(10)

Example 3 Linear, non-constant coefficient, 1st-order wave eqn.

$$\frac{\partial w}{\partial t} + 3t^2 \frac{\partial w}{\partial x} = 2tw \qquad w(x,0) = P(x)$$

$$\frac{d}{dt} w(x(t),t) = 2tw \quad \text{along} \quad \frac{dx(t)}{dt} = 3t^2$$

$$\text{or} \quad x(t) = t^3 + x_0$$

$$\Rightarrow \quad \frac{dw}{w} = 2t\,dt \quad \text{along} \quad x(t) = t^3 + x_0$$

$$\Rightarrow \quad \ln|w| = t^2 + C$$

$$w = C^* \exp(t^2) \quad \text{along} \quad x(t) = t^3 + x_0$$

The initial conditions: $w(x(0),0) = P(x(0))$

$$w(x_0, 0) = P(x_0)$$

So $w(x_0,0) = C^* \exp(0) = P(x_0) \Rightarrow C^* = P(x_0)$

$$\Rightarrow \quad w(x,t) = P(x_0)\exp(t^2) \quad \text{or}$$

$$\boxed{w(x,t) = P(x - t^3)\exp(t^2)}$$

Chapter 3

HWs

Local contents

3.1 HW1

3.1.1 problem 3.3 (page 138)

3.1.1.1 Part c

problem Classify all the singular points (finite and infinite) of the following

$$x(1-x)y''+(c-(a+b+1)x)y'-aby=0$$

Answer

Writing the DE in standard form

$$\begin{aligned}
y''+p(x)y'+q(x)y&=0\\
y''+\frac{c-(a+b+1)x}{x(1-x)}y'-\frac{ab}{x(1-x)}y&=0\\
y''+\overbrace{\left(\frac{c}{x(1-x)}-\frac{(a+b+1)}{(1-x)}\right)}^{p(x)}y'-\overbrace{\frac{ab}{x(1-x)}}^{q(x)}y&=0
\end{aligned}$$

$x=0,1$ are singular points for $p(x)$ as well as for $q(x)$. Now we classify what type of singularity each point is. For $p(x)$

$$\begin{aligned}
\lim_{x\to 0}xp(x)&=\lim_{x\to 0}x\left(\frac{c}{x(1-x)}-\frac{(a+b+1)}{(1-x)}\right)\\
&=\lim_{x\to 0}\left(\frac{c}{(1-x)}-\frac{x(a+b+1)}{(1-x)}\right)\\
&=c
\end{aligned}$$

Hence $xp(x)$ is analytic at $x=0$. Therefore $x=0$ is regular singularity point. Now we check for $q(x)$

$$\begin{aligned}
\lim_{x\to 0}x^2q(x)&=\lim_{x\to 0}x^2\frac{-ab}{x(1-x)}\\
&=\lim_{x\to 0}\frac{-xab}{(1-x)}\\
&=0
\end{aligned}$$

Hence $x^2q(x)$ is also analytic at $x=0$. Therefore $x=0$ is regular singularity point. Now we look at $x=1$ and classify it. For $p(x)$

$$\begin{aligned}
\lim_{x\to 1}(x-1)p(x) &= \lim_{x\to 1}(x-1)\left(\frac{c}{x(1-x)}-\frac{(a+b+1)}{(1-x)}\right)\\
&= \lim_{x\to 1}(x-1)\left(\frac{-c}{x(x-1)}+\frac{(a+b+1)}{(x-1)}\right)\\
&= \lim_{x\to 1}\left(\frac{-c}{x}+(a+b+1)\right)\\
&= -c+(a+b+1)
\end{aligned}$$

Hence $(x-1)p(x)$ is analytic at $x=1$. Therefore $x=1$ is regular singularity point. Now we check for $q(x)$

$$\begin{aligned}
\lim_{x\to 1}(x-1)^2 q(x) &= \lim_{x\to 1}(x-1)^2\left(\frac{-ab}{x(1-x)}\right)\\
&= \lim_{x\to 1}(x-1)^2\left(\frac{ab}{x(x-1)}\right)\\
&= \lim_{x\to 1}(x-1)\left(\frac{ab}{x}\right)\\
&= 0
\end{aligned}$$

Hence $(x-1)^2 q(x)$ is analytic also at $x=1$. Therefore $x=1$ is regular singularity point. Therefore $x=0,1$ are regular singular points for the ODE. Now we check for x at ∞. To check the type of singularity, if any, at $x=\infty$, the DE is first transformed using

$$x=\frac{1}{t} \tag{1}$$

This transformation will always results in [1] new ODE in t of this form

$$\frac{d^2y}{dt^2} + \frac{\left(-p\left(t\right) + 2t\right)}{t^2} + \frac{q\left(t\right)}{t^4} y = 0 \tag{2}$$

But

$$\begin{aligned} p\left(t\right) = p\left(x\right)\Big|_{x=\frac{1}{t}} &= \left(\frac{c - \left(a + b + 1\right)x}{x\left(1 - x\right)}\right)_{x=\frac{1}{t}} \\ &= \left(\frac{c - \left(a + b + 1\right)\frac{1}{t}}{\frac{1}{t}\left(1 - \frac{1}{t}\right)}\right) \\ &= \frac{ct^2 - t\left(a + b + 1\right)}{\left(t - 1\right)} \end{aligned} \tag{3}$$

And

$$\begin{aligned} q\left(t\right) = q\left(x\right)\Big|_{x=\frac{1}{t}} &= \left(-\frac{ab}{x\left(1 - x\right)}\right)_{x=\frac{1}{t}} \\ &= -\frac{ab}{\frac{1}{t}\left(1 - \frac{1}{t}\right)} \\ &= -\frac{abt^2}{\left(t - 1\right)} \end{aligned} \tag{4}$$

[1]Let $x = \frac{1}{t}$, then

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -t^2\frac{d}{dt}$$

And

$$\begin{aligned} \frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right) &= -t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) \\ &= -t^2\left(-2t\frac{d}{dt} - t^2\frac{d^2}{dt^2}\right) \\ &= 2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2} \end{aligned}$$

The original ODE becomes

$$\begin{aligned} \left(2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2}\right)y + p\left(x\right)\Big|_{x=\frac{1}{2}}\left(-t^2\frac{d}{dt}\right)y + q\left(x\right)\Big|_{x=\frac{1}{t}} y &= 0 \\ \left(2t^3y' + t^4y''\right) - t^2p\left(t\right)y' + q\left(t\right)y &= 0 \\ t^4y'' + \left(-t^2p\left(t\right) + 2t^3\right)y' + q\left(t\right)y &= 0 \\ y'' + \frac{\left(-p\left(t\right) + 2t\right)}{t^2}y' + \frac{q\left(t\right)}{t^4}y &= 0 \end{aligned}$$

Substituting equations (3,4) into (2) gives

$$y'' + \frac{\left(-\left(\frac{ct^2-t(a+b+1)}{(t-1)}\right)+2t\right)}{t^2} + \frac{\left(-\frac{abt^2}{(t-1)}\right)}{t^4}y = 0$$

$$y'' + \frac{\left(2t(t-1) - t^2c + (a+b+1)t\right)}{t^2(t-1)}y' - \frac{ab}{t^2(t-1)}y = 0$$

Expanding

$$y'' + \overbrace{\left(\frac{2t-1-tc+a+b}{t(t-1)}\right)}^{p(t)}y' - \overbrace{\frac{ab}{t^2(t-1)}}^{q(t)}y = 0$$

We see that $t=0$ (this means $x=\infty$) is singular point for both $p(x), q(x)$. Now we check what type it is. For $p(t)$

$$\begin{aligned}\lim_{t\to 0} tp(t) &= \lim_{t\to 0} t\left(\frac{2}{t} - \frac{c}{(t-1)} + \frac{(a+b+1)}{t(t-1)}\right) \\ &= \lim_{t\to 0}\left(2 - \frac{tc}{(t-1)} + \frac{(a+b+1)}{(t-1)}\right) \\ &= 1-a-b\end{aligned}$$

$tp(t)$ is therefore analytic at $t=0$. Hence $t=0$ is regular singular point. Now we check for $q(t)$

$$\begin{aligned}\lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2\left(-\frac{ab}{t^2(t-1)}\right) \\ &= \lim_{t\to 0}\left(\frac{-ab}{(t-1)}\right) \\ &= ab\end{aligned}$$

$t^2q(t)$ is therefore analytic at $t=0$. Hence $t=0$ is regular singular point for $q(t)$. Therefore $t=0$ is regular singular point which mean that $x\to\infty$ is a regular singular point for the ODE.

Summary

Singular points are $x=0,1$. Both are regular singular points. Also $x=\infty$ is regular singular point.

3.1.1.2 Part (d)

Problem Classify all the singular points (finite and infinite) of the following

$$xy'' + (b-x)y' - ay = 0$$

solution

Writing the ODE in standard for

$$y'' + \frac{(b-x)}{x}y' - \frac{a}{x}y = 0$$

We see that $x = 0$ is singularity point for both $p(x)$ and $q(x)$. Now we check its type. <u>For $p(x)$</u>

$$\begin{aligned}\lim_{x\to 0} xp(x) &= \lim_{x\to 0} x\frac{(b-x)}{x} \\ &= b\end{aligned}$$

Hence $xp(x)$ is analytic at $x = 0$. Therefore $x = 0$ is regular singularity point for $p(x)$. <u>For $q(x)$</u>

$$\begin{aligned}\lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} x^2\left(\frac{-a}{x}\right) \\ &= \lim_{x\to 0}(-ax) \\ &= 0\end{aligned}$$

Hence $x^2 q(x)$ is analytic at $x = 0$. Therefore $x = 0$ is regular singularity point for $q(x)$.

Now we check for x at ∞. To check the type of singularity, if any, at $x = \infty$, the DE is first transformed using

$$x = \frac{1}{t} \tag{1}$$

This results in (as was done in above part)

$$\frac{d^2y}{dt^2} + \frac{\left(-p(t) + 2t\right)}{t^2} + \frac{q(t)}{t^4}y = 0$$

Where

$$\begin{aligned}p(t) &= \left.\frac{(b-x)}{x}\right|_{x=\frac{1}{t}} \\ &= \frac{\left(b - \frac{1}{t}\right)}{\frac{1}{t}} \\ &= (bt-1)\end{aligned}$$

And $q(t) = -\frac{a}{x} = -at$ Hence the new ODE is

$$\begin{aligned}\frac{d^2y}{dt^2} + \frac{(-(bt-1) + 2t)}{t^2} - \frac{at}{t^4}y &= 0 \\ \frac{d^2y}{dt^2} + \frac{-bt + 1 + 2t}{t^2} - \frac{a}{t^3}y &= 0\end{aligned}$$

Therefore $t = 0$ (or $x = \infty$) is singular point. Now we will find the singularity type

$$\begin{aligned}\lim_{t\to 0} tp(t) &= \lim_{t\to 0} t\left(\frac{-bt + 1 + 2t}{t^2}\right) \\ &= \lim_{t\to 0}\left(\frac{-bt + 1 + 2t}{t}\right) \\ &= \infty\end{aligned}$$

Hence $tp(t)$ is not analytic, since the limit do not exist, which means $t = 0$ is an irregular singular point for $p(t)$. We can stop here, but will also check for $q(t)$

$$\begin{aligned} \lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2\left(-\frac{a}{t^3}\right) \\ &= \lim_{t\to 0}\left(-\frac{a}{t}\right) \\ &= \infty \end{aligned}$$

Therefore, $t=0$ is irregular singular point, which means $x=\infty$ is an irregular singular point.

Summary

$x=0$ is regular singular point. $x=\infty$ is an irregular singular point.

3.1.2 problem 3.4

3.1.2.1 part d

problem Classify $x=0$ and $x=\infty$ of the following

$$x^2 y'' = y e^{\frac{1}{x}}$$

Answer

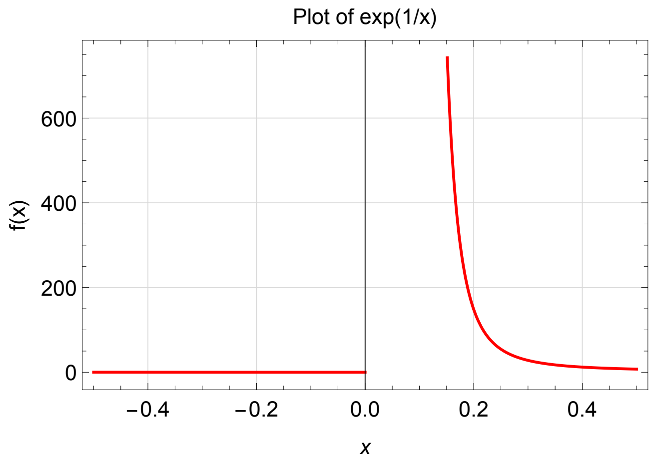
In standard form

$$y'' - \frac{e^{\frac{1}{x}}}{x^2} y = 0$$

Hence $p(x)=0, q(x) = -\frac{e^{\frac{1}{x}}}{x^2}$. The singularity is $x=0$. We need to check on $q(x)$ only.

$$\begin{aligned} \lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} x^2\left(-\frac{e^{\frac{1}{x}}}{x^2}\right) \\ &= \lim_{x\to 0}\left(-e^{\frac{1}{x}}\right) \end{aligned}$$

The above is not analytic, since $\lim_{x\to 0^+}\left(-e^{\frac{1}{x}}\right)=\infty$ while $\lim_{x\to 0^-}\left(-e^{\frac{1}{x}}\right)=0$. This means $e^{\frac{1}{x}}$ is not differentiable at $x=0$. Here is plot $e^{\frac{1}{x}}$ near $x=0$



Therefore $x = 0$ is an irregular singular point. We now convert the ODE using $x = \frac{1}{t}$ in order to check what happens at $x = \infty$. This results in (as was done in above part)

$$\frac{d^2y}{dt^2} + \frac{q(t)}{t^4} y = 0$$

But

$$\begin{aligned} q(t) &= q(x)\Big|_{x=\frac{1}{t}} \\ &= \left(-\frac{e^{\frac{1}{x}}}{x^2}\right)_{x=\frac{1}{t}} \\ &= -t^2 e^t \end{aligned}$$

Hence the ODE becomes

$$\begin{aligned} y'' - \frac{-t^2 e^t}{t^4} y &= 0 \\ y'' + \frac{e^t}{t^2} y &= 0 \end{aligned}$$

We now check $q(t)$.

$$\begin{aligned} \lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2 \frac{e^t}{t^2} \\ &= \lim_{t\to 0} e^t \\ &= 1 \end{aligned}$$

This is analytic. Hence $t = 0$ is regular singular point, which means $x = \infty$ is regular singular point.

Summary $x = 0$ is irregular singular point, $x = \infty$ is regular singular point.

3.1.2.2 Part e

problem

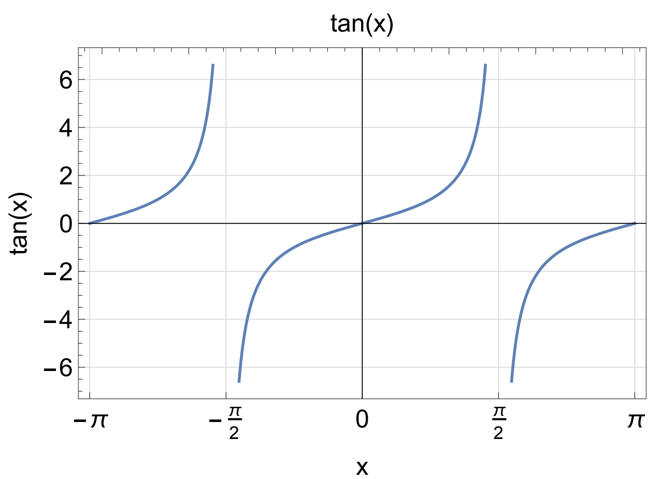
Classify $x = 0$ and $x = \infty$ of the following

$$(\tan x)\, y' = y$$

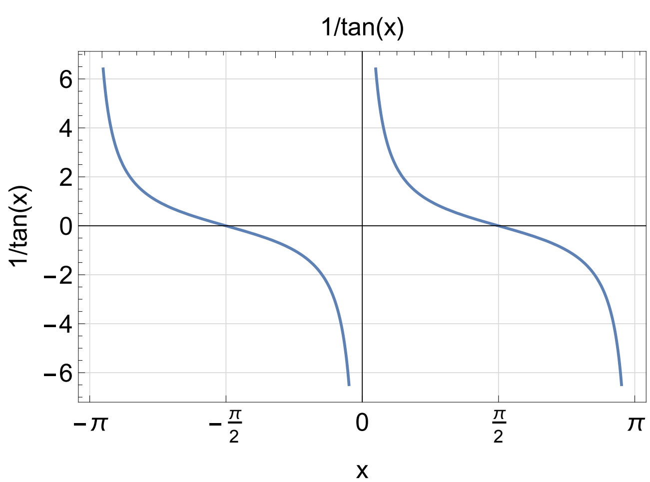
Answer

$$y' - \frac{1}{\tan x} y = 0$$

The function $\tan x$ looks like



Therefore, $\tan(x)$ is not analytic at $x = \left(n - \frac{1}{2}\right)\pi$ for $n \in \mathbb{Z}$. Hence the function $\frac{1}{\tan(x)}$ is not analytic at $x = n\pi$ as seen in this plot



Hence singular points are $x = \{\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots\}$. Looking at $x = 0$

$$\begin{aligned}
\lim_{x\to 0} xp(x) &= \lim_{x\to 0}\left(x\frac{1}{\tan(x)}\right)\\
&= \lim_{x\to 0}\left(\frac{\frac{dx}{dx}}{\frac{d\tan(x)}{dx}}\right)\\
&= \lim_{x\to 0}\frac{1}{\sec^2 x}\\
&= \lim_{x\to 0}\cos^2 x\\
&= 1
\end{aligned}$$

Therefore the point $x = 0$ is <u>regular singular point</u>. To classify $x = \infty$, we use $x = \frac{1}{t}$ substitution. $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -t^2\frac{d}{dt}$ and the ODE becomes

$$\begin{aligned}
\left(-t^2\frac{d}{dt}\right)y - \frac{1}{\tan\left(\frac{1}{t}\right)}y &= 0\\
-t^2 y' - \frac{1}{\tan\left(\frac{1}{t}\right)}y &= 0\\
y' + \frac{1}{t^2\tan\left(\frac{1}{t}\right)}y &= 0
\end{aligned}$$

Hence

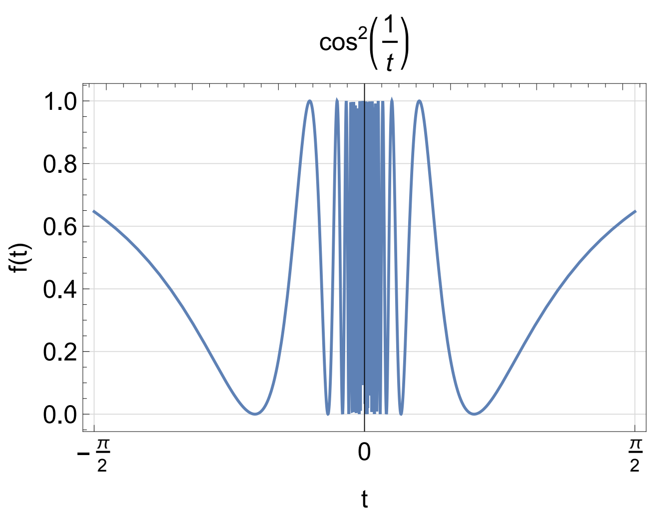
$$p(t) = \frac{1}{t^2\tan\left(\frac{1}{t}\right)}$$

This function has singularity at $t = 0$ and at $t = \frac{1}{n\pi}$ for $n \in \mathbb{Z}$. We just need to consider $t = 0$

since this maps to $x=\infty$. Hence

$$\begin{aligned}\lim_{t\to 0} tp(t) &= \lim_{t\to 0}\left(t\frac{1}{t^2\tan\left(\frac{1}{t}\right)}\right)\\ &= \lim_{t\to 0}\left(\frac{1}{t\tan\left(\frac{1}{t}\right)}\right)\\ &= \lim_{t\to 0}\left(\frac{\frac{1}{t}}{\tan\left(\frac{1}{t}\right)}\right) \underset{\text{L'hopitals}}{\longrightarrow} \lim_{t\to 0}\left(\frac{-\frac{1}{t^2}}{-\frac{\sec^2\left(\frac{1}{t}\right)}{t^2}}\right)\\ &= \lim_{t\to 0}\left(\frac{1}{\sec^2\left(\frac{1}{t}\right)}\right)\\ &= \lim_{t\to 0}\left(\cos^2\left(\frac{1}{t}\right)\right)\end{aligned}$$

The following is a plot of $\cos^2\left(\frac{1}{t}\right)$ as t goes to zero.



This is the same as asking for $\lim_{x\to\infty}\cos^2(x)$ which does not exist, since $\cos(x)$ keeps oscillating, hence it has no limit. Therefore, we conclude that $tp(t)$ is not analytic at $t=0$, hence t is irregular singular point, which means $x=\infty$ is an irregular singular point.

Summary

$x=0$ is regular singular point and $x=\infty$ is an irregular singular point

3.1.3 Problem 3.6

3.1.3.1 part b

Problem Find the Taylor series expansion about $x = 0$ of the solution to the initial value problem

$$\begin{aligned} y'' - 2xy' + 8y &= 0 \\ y(0) &= 0 \\ y'(0) &= 4 \end{aligned}$$

solution

Since $x = 0$ is ordinary point, then we can use power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Hence

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=0}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \end{aligned}$$

Therefore the ODE becomes

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 8 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 8 a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 8 a_n x^n &= 0 \end{aligned}$$

Hence, for $n = 0$ we obtain

$$\begin{aligned} (n+1)(n+2) a_{n+2} x^n + 8 a_n x^n &= 0 \\ 2a_2 + 8a_0 &= 0 \\ a_2 &= -4a_0 \end{aligned}$$

For $n \geq 1$

$$\begin{aligned} (n+1)(n+2) a_{n+2} - 2n a_n + 8 a_n &= 0 \\ a_{n+2} &= \frac{2n a_n - 8 a_n}{(n+1)(n+2)} \\ &= \frac{a_n (2n-8)}{(n+1)(n+2)} \end{aligned}$$

Hence for $n = 1$

$$a_3 = \frac{a_1 (2-8)}{(1+1)(1+2)} = -a_1$$

For $n=2$

$$a_4 = \frac{a_2\left(2\left(2\right)-8\right)}{\left(2+1\right)\left(2+2\right)} = -\frac{1}{3}a_2 = -\frac{1}{3}\left(-4a_0\right) = \frac{4}{3}a_0$$

For $n=3$

$$a_5 = \frac{a_3\left(2\left(3\right)-8\right)}{\left(3+1\right)\left(3+2\right)} = -\frac{1}{10}a_3 = -\frac{1}{10}\left(-a_1\right) = \frac{1}{10}a_1$$

For $n=4$

$$a_6 = \frac{a_4\left(2\left(4\right)-8\right)}{\left(4+1\right)\left(4+2\right)} = 0$$

For $n=5$

$$a_7 = \frac{a_5\left(2\left(5\right)-8\right)}{\left(5+1\right)\left(5+2\right)} = \frac{1}{21}a_5 = \frac{1}{21}\left(\frac{1}{10}a_1\right) = \frac{1}{210}a_1$$

For $n=6$

$$a_8 = \frac{a_6\left(2\left(6\right)-8\right)}{\left(6+1\right)\left(6+2\right)} = \frac{1}{14}a_6 = 0$$

For $n=7$

$$a_9 = \frac{a_7\left(2\left(7\right)-8\right)}{\left(7+1\right)\left(7+2\right)} = \frac{1}{12}a_7 = \frac{1}{12}\left(\frac{1}{210}a_1\right) = \frac{1}{2520}a_1$$

Writing now few terms

$$\begin{aligned}
y\left(x\right) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots \\
&= a_0 + a_1x + \left(-4a_0\right)x^2 + \left(-a_1\right)x^3 + \frac{4}{3}a_0x^4 + \frac{1}{10}a_1x^5 + 0 + \frac{1}{210}a_1x^7 + 0 + \frac{1}{2520}a_1x^9 + \cdots \\
&= a_0 + a_1x - 4a_0x^2 - a_1x^3 + \frac{4}{3}a_0x^4 + \frac{1}{10}a_1x^5 + \frac{1}{210}a_1x^7 + \frac{1}{2520}a_1x^9 + \cdots \\
&= a_0\left(1 - 4x^2 + \frac{4}{3}x^4\right) + a_1\left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \frac{1}{2520}x^9 + \cdots\right) \qquad (1)
\end{aligned}$$

We notice that a_0 terms terminates at $\frac{4}{3}x^4$ but the a_1 terms do not terminate. Now we need to find a_0, a_1 from initial conditions. At $x=0, y\left(0\right)=0$. Hence from (1)

$$0 = a_0$$

Hence the solution becomes

$$y\left(x\right) = a_1\left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \frac{1}{2520}x^9 + \cdots\right) \qquad (2)$$

Taking derivative of (2), term by term

$$y'\left(x\right) = a_1\left(1 - 3x^2 + \frac{5}{10}x^4 + \frac{7}{210}x^6 + \frac{9}{2520}x^8 + \cdots\right)$$

Using $y'\left(0\right)=4$ the above becomes

$$4 = a_1$$

Hence the solution is

$$y(x) = 4\left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \frac{1}{2520}x^9 + \cdots\right)$$

Or

$$\boxed{y(x) = 4x - 4x^3 + \frac{2}{5}x^5 + \frac{2}{105}x^7 + \frac{1}{630}x^9 + \cdots}$$

The above is the Taylor series of the solution expanded around $x = 0$.

3.1.4 Problem 3.7

Problem: Estimate the number of terms in the Taylor series (3.2.1) and (3.2.2) at page 68 of the text, that are necessary to compute

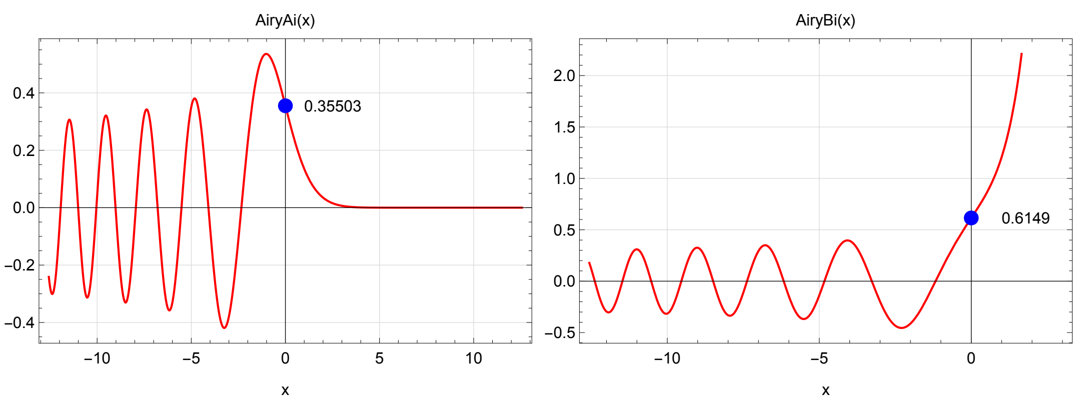
$\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ correct to three decimal places at $x = \pm 1, \pm 100, \pm 10000$

Answer:

$$\mathrm{Ai}(x) = 3^{\frac{-2}{3}} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma\left(n + \frac{2}{3}\right)} - 3^{\frac{-4}{3}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma\left(n + \frac{4}{3}\right)} \tag{3.2.1}$$

$$\mathrm{Bi}(x) = 3^{\frac{-1}{6}} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma\left(n + \frac{2}{3}\right)} - 3^{\frac{-5}{6}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma\left(n + \frac{4}{3}\right)} \tag{3.2.2}$$

The radius of convergence of these series extends from $x = 0$ to $\pm\infty$ so we know this will converge to the correct value of $\mathrm{Ai}(x), \mathrm{Bi}(x)$ for all x, even though we might need large number of terms to achieve this, as will be shown below. The following is a plot of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$



3.1.4.1 AiryAI series

For $\operatorname{Ai}(x)$ looking at the $\frac{(N+1)^{th}}{N^{th}}$ term

$$\Delta = \frac{3^{\frac{-2}{3}}\frac{x^{3(N+1)}}{9^{N+1}(N+1)!\Gamma\left(N+1+\frac{2}{3}\right)} - 3^{\frac{-4}{3}}\frac{x^{3N+2}}{9^{N+1}(N+1)!\Gamma\left(N+1+\frac{4}{3}\right)}}{3^{\frac{-2}{3}}\frac{x^{3N}}{9^{N}N!\Gamma\left(N+\frac{2}{3}\right)} - 3^{\frac{-4}{3}}\frac{x^{3N+1}}{9^{N}N!\Gamma\left(N+\frac{4}{3}\right)}}$$

This can be simplified using $\Gamma(N+1) = N\Gamma(N)$ giving

$$\Gamma\left(\left(N+\frac{2}{3}\right)+1\right) = \left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)$$
$$\Gamma\left(\left(N+\frac{4}{3}\right)+1\right) = \left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)$$

Hence Δ becomes

$$\Delta = \frac{\frac{3^{\frac{-2}{3}}x^{3(N+1)}}{9^{N+1}(N+1)!\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}}x^{3N+2}}{9^{N+1}(N+1)!\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}}x^{3N}}{9^{N}N!\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}}x^{3N+1}}{9^{N}N!\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}}x^{3(N+1)}}{9(N+1)\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}}x^{3N+2}}{9(N+1)\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}}x^{3N}}{\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}}x^{3N+1}}{\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}}x^{3(N+1)}\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}}x^{3N+2}\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}}x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}}x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}}x^{3(N+1)}\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}}x^{3N+2}\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)\left(N+\frac{2}{3}\right)\left(N+\frac{4}{3}\right)}}{3^{\frac{-2}{3}}x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}}x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}$$

For large N, we can approximate $\left(N+\frac{2}{3}\right),\left(N+\frac{4}{3}\right),(N+1)$ to just $N+1$ and the above becomes

$$\Delta=\frac{\frac{3^{\frac{-2}{3}}x^{3(N+1)}\Gamma\left(N+\frac{4}{3}\right)-3^{\frac{-4}{3}}x^{3N+2}\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)^2}}{3^{\frac{-2}{3}}x^{3N}\Gamma\left(N+\frac{4}{3}\right)-3^{\frac{-4}{3}}x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}$$

Or

$$\Delta=\frac{3^{\frac{-2}{3}}x^{3N}x^3\Gamma\left(N+\frac{4}{3}\right)-3^{\frac{-4}{3}}x^{3N}x^2\Gamma\left(N+\frac{2}{3}\right)}{9\left(N+1\right)^2\left(3^{\frac{-2}{3}}x^{3N}\Gamma\left(N+\frac{4}{3}\right)-3^{\frac{-4}{3}}x^{3N}x\Gamma\left(N+\frac{2}{3}\right)\right)}$$

Or

$$\begin{aligned}\Delta&=\frac{0.488x^{3N}x^2\left(x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)\right)}{(0.488)\,9\left(N+1\right)^2x^{3N}\left(\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)\right)}\\&=\frac{x^2}{9\left(N+1\right)^2}\left(\frac{x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)}\right)\end{aligned}$$

We want to solve for N s.t. $\left\|\frac{x^2}{9(N+1)^2}\left(\frac{x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)}\right)\right\|<0.001$.

<u>For $x=1$</u>

$$\begin{aligned}\frac{1}{9\left(N+1\right)^2}\left(\frac{\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}\right)&<0.001\\\frac{1}{9\left(N+1\right)^2}&<0.001\\9\left(N+1\right)^2&>1000\\\left(N+1\right)^2&>\frac{1000}{9}\\N+1&>\sqrt{\frac{1000}{9}}\\N&>9.541\\N&=10\end{aligned}$$

<u>For $x=100$</u>

$$\left\|\frac{100^2}{9\left(N+1\right)^2}\left(\frac{100\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488\left(100\right)\Gamma\left(N+\frac{2}{3}\right)}\right)\right\|<0.001$$

I could not simplify away the Gamma terms above any more. Is there a way? So wrote small function (in Mathematica, which can compute this) which increments N and evaluate the above, until the value became smaller than 0.001. At $N = 11,500$ this was achieved.

For $x = 10000$

$$\left| \frac{10000^2}{9(N+1)^2} \left(\frac{100\Gamma\left(N+\frac{4}{3}\right) - 0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right) - 0.488(10000)\Gamma\left(N+\frac{2}{3}\right)} \right) \right| < 0.001$$

Using the same program, found that $N = 11,500,000$ was needed to obtain the result below 0.001.

For $= -1$ also $N = 10$. For $x = -100$, $N = 10,030$, which is little less than $x = +100$. For $x = -10000$, $N = 10,030,000$

Summary table for AiryAI(x)

x	N
1	10
−1	10
100	11,500
−100	10,030
10000	11,500,000
−10000	10,030,000

The Mathematica function which did the estimate is the following

```
estimateAiryAi[x_, n_] := (x^2/(9*(n + 1)^2))*((x*Gamma[n + 4/3] -
0.488*Gamma[n + 2/3])/(Gamma[n + 4/3] - 0.488*x*Gamma[n + 2/3]))
estimateAiryAi[-10000, 10030000] // N
-0.0009995904
estimateAiryAi[100, 11500] // N
0.000928983646707407
estimateAiryAi[10000, 11500000] // N
0.000929156800155198
```

3.1.4.2 AiryBI series

For $\mathrm{Bi}(x)$ the difference is the coefficients. Hence using the result from above, and just replace the coefficients

$$\Delta = \frac{3^{\frac{-1}{6}} x^{3N} x^3 \Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-5}{8}} x^{3N} x^2 \Gamma\left(N+\frac{2}{3}\right)}{9(N+1)^2\left(3^{\frac{-1}{6}} x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-5}{8}} x^{3N} x \Gamma\left(N+\frac{2}{3}\right)\right)}$$
$$= \frac{x^2\left(x\Gamma\left(N+\frac{4}{3}\right) - 0.604\,39\Gamma\left(N+\frac{2}{3}\right)\right)}{9(N+1)^2\left(\Gamma\left(N+\frac{4}{3}\right) - 0.604\,39x\Gamma\left(N+\frac{2}{3}\right)\right)}$$

Hence for $x=1$, using the above reduces to

$$\frac{1}{9(N+1)^2} < 0.001$$

Which is the same as AiryAI, therefore $N=10$. For $x=100$, using the same small function in Mathematica to calculate the above, here are the result.

Summary table for AiryBI(x)

x	N
1	10
−1	10
100	11,290
−100	9,900
10000	10,900,000
−10000	9,950,000

The result between AiryAi and AiryBi are similar. AiryBi needs a slightly less number of terms in the series to obtain same accuracy.

The Mathematica function which did the estimate for the larger N value for the above table is the following

```
estimateAiryBI[x_, n_] :=  Abs[(x^2/(9*(n + 1)^2))*((x*Gamma[n + 4/3] -
0.60439*Gamma[n + 2/3])/(Gamma[n + 4/3] - 0.6439*x*Gamma[n + 2/3]))]
```

3.1.5 Problem 3.8

Problem How many terms in the Taylor series solution to $y''' = x^3 y$ with $y(0)=1, y'(0) = y''(0) = 0$ are needed to evaluate $\int_0^1 y(x)\,dx$ correct to three decimal places?

Answer

$$y''' - x^3 y = 0$$

Since x is an ordinary point, we use

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
y'' &= \sum_{n=0}^{\infty} n (n+1) a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} n (n+1) a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \\
y''' &= \sum_{n=0}^{\infty} n (n+1)(n+2) a_{n+2} x^{n-1} = \sum_{n=1}^{\infty} n (n+1)(n+2) a_{n+2} x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} x^n
\end{aligned}
$$

Hence the ODE becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} x^n - x^3 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} x^n - \sum_{n=0}^{\infty} a_n x^{n+3} &= 0 \\
\sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} x^n - \sum_{n=3}^{\infty} a_{n-3} x^n &= 0
\end{aligned}
$$

For $n = 0$

$$
\begin{aligned}
(1)(2)(3) a_3 &= 0 \\
a_3 &= 0
\end{aligned}
$$

For $n = 1$

$$
\begin{aligned}
(2)(3)(4) a_4 &= 0 \\
a_4 &= 0
\end{aligned}
$$

For $n = 2$

$$
a_5 = 0
$$

For $n \geq 3$, recursive equation is used

$$
\begin{aligned}
(n+1)(n+2)(n+3) a_{n+3} - a_{n-3} &= 0 \\
a_{n+3} &= \frac{a_{n-3}}{(n+1)(n+2)(n+3)}
\end{aligned}
$$

Or, for $n = 3$

$$
a_6 = \frac{a_0}{(4)(5)(6)}
$$

For $n = 4$

$$
a_7 = \frac{a_1}{(4+1)(4+2)(4+3)} = \frac{a_1}{(5)(6)(7)}
$$

For $n = 5$

$$
a_8 = \frac{a_2}{(5+1)(5+2)(5+3)} = \frac{a_2}{(6)(7)(8)}
$$

For $n=6$

$$a_9 = \frac{a_3}{(6+1)(6+2)(6+3)} = 0$$

For $n=7$

$$a_{10} = \frac{a_4}{(n+1)(n+2)(n+3)} = 0$$

For $n=8$

$$a_{11} = \frac{a_5}{(8+1)(8+2)(8+3)} = 0$$

For $n=9$

$$a_{12} = \frac{a_6}{(9+1)(9+2)(9+3)} = \frac{a_6}{(10)(11)(12)} = \frac{a_0}{(4)(5)(6)(10)(11)(12)}$$

For $n=10$

$$a_{13} = \frac{a_7}{(10+1)(10+2)(10+3)} = \frac{a_7}{(11)(12)(13)} = \frac{a_1}{(5)(6)(7)(11)(12)(13)}$$

For $n=11$

$$a_{14} = \frac{a_8}{(11+1)(11+2)(11+3)} = \frac{a_8}{(12)(13)(14)} = \frac{a_2}{(6)(7)(8)(12)(13)(14)}$$

And so on. Hence the series is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + 0x^3 + 0x^4 + 0x^5 + \frac{a_0}{(4)(5)(6)}x^6 + \frac{a_1}{(5)(6)(7)}x^7 + \frac{a_2}{(6)(7)(8)}x^8 \\
&+ 0 + 0 + 0 + \frac{a_0}{(4)(5)(6)(10)(11)(12)}x^{12} + \frac{a_1}{(5)(6)(7)(11)(12)(13)}x^{13} \\
&+ \frac{a_2}{(6)(7)(8)(12)(13)(14)}x^{14} + 0 + 0 + 0 + \cdots
\end{aligned}$$

Or

$$\begin{aligned}
y(x) = a_0 + a_1 x + a_2 x^2 + \frac{a_0}{(4)(5)(6)}x^6 + \frac{a_1}{(5)(6)(7)}x^7 + \frac{a_2}{(6)(7)(8)}x^8 + \\
\frac{a_0}{(4)(5)(6)(10)(11)(12)}x^{12} + \frac{a_1}{(5)(6)(7)(11)(12)(13)}x^{13} + \frac{a_2}{(6)(7)(8)(12)(13)(14)}x^{14} + \cdots
\end{aligned}$$

Or

$$\begin{aligned}
y(x) &= a_0\left(1 + \frac{x^6}{(4)(5)(6)} + \frac{x^{12}}{(4)(5)(6)(10)(11)(12)} + \cdots\right) \\
&+ a_1\left(x + \frac{x^7}{(5)(6)(7)} + \frac{x^{13}}{(5)(6)(7)(11)(12)(13)} + \cdots\right) \\
&+ a_2\left(x^2 + \frac{x^8}{(6)(7)(8)} + \frac{x^{14}}{(6)(7)(8)(12)(13)(14)} + \cdots\right)
\end{aligned}$$

$$
\begin{aligned}
y(x) &= a_0\left(1+\frac{1}{120}x^6+\frac{1}{158400}x^{12}+\cdots\right)\\
&\quad + a_1\left(x+\frac{1}{210}x^7+\frac{1}{360360}x^{13}+\cdots\right)\\
&\quad + a_2\left(x^2+\frac{1}{336}x^8+\frac{1}{733\,824}x^{14}+\cdots\right)
\end{aligned}
$$

We now apply initial conditions $y(0)=1, y'(0)=y''(0)=0$. When $y(0)=1$

$$1 = a_0$$

Hence solution becomes

$$
\begin{aligned}
y(x) &= \left(1+\frac{1}{120}x^6+\frac{1}{158400}x^{12}+\cdots\right)\\
&\quad + a_1\left(x+\frac{1}{210}x^7+\frac{1}{360360}x^{13}+\cdots\right)\\
&\quad + a_2\left(x^2+\frac{1}{336}x^8+\frac{1}{733\,824}x^{14}+\cdots\right)
\end{aligned}
$$

Taking derivative

$$
\begin{aligned}
y'(x) &= \left(\frac{6}{120}x^5+\frac{12}{158400}x^{11}+\cdots\right)\\
&\quad + a_1\left(1+\frac{7}{210}x^6+\frac{13}{360360}x^{12}+\cdots\right)\\
&\quad + a_2\left(2x+\frac{8}{336}x^7+\frac{12}{733\,824}x^{13}+\cdots\right)
\end{aligned}
$$

Applying $y'(0)=0$ gives

$$0 = a_1$$

And similarly, Applying $y''(0)=0$ gives $a_2=0$. Hence the solution is

$$
\begin{aligned}
y(x) &= a_0\left(1+\frac{x^6}{(4)(5)(6)}+\frac{x^{12}}{(4)(5)(6)(10)(11)(12)}+\cdots\right)\\
&= 1+\frac{x^6}{(4)(5)(6)}+\frac{x^{12}}{(4)(5)(6)(10)(11)(12)}+\frac{x^{18}}{(4)(5)(6)(10)(11)(12)(16)(17)(18)}+\cdots\\
&= 1+\frac{x^6}{120}+\frac{x^{12}}{158\,400}+\frac{x^{18}}{775\,526\,400}+\cdots
\end{aligned}
$$

We are now ready to answer the question. We will do the integration by increasing the number of terms by one each time. When the absolute difference between each increment becomes less than 0.001 we stop. When using <u>one term</u> For

$$
\begin{aligned}
\int_0^1 y(x)\,dx &= \int_0^1 dx\\
&= 1
\end{aligned}
$$

When using two terms

$$\begin{aligned}\int_0^1 1+\frac{x^6}{120}dx &= \left(x+\frac{x^7}{120\,(7)}\right)_0^1\\ &= 1+\frac{1}{840}\\ &= \frac{841}{840}\\ &= 1.001190476\end{aligned}$$

Difference between one term and two terms is 0.001190476. When using three terms

$$\begin{aligned}\int_0^1 1+\frac{x^6}{120}+\frac{x^{12}}{158\,400}dx &= \left(x+\frac{x^7}{840}+\frac{x^{13}}{2059\,200}\right)_0^1\\ &= 1+\frac{1}{840}+\frac{1}{2059\,200}\\ &= \frac{14\,431\,567}{14\,414\,400}\\ &= 1.001190962\end{aligned}$$

Comparing the above result, with the result using two terms, we see that only two terms are needed since the change in accuracy did not affect the first three decimal points. Hence we need only this solution with two terms only

$$y(x) = 1+\frac{1}{120}x^6$$

3.1.6 Problem 3.24

3.1.6.1 part e

Problem Find series expansion of all the solutions to the following differential equation about $x=0$. Try to sum in closed form any infinite series that appear.

$$2xy''-y'+x^2y=0$$

Solution

$$y''-\frac{1}{2x}y'+\frac{x}{2}y=0$$

The only singularity is in $p(x)$ is $x=0$. We will now check if it is removable. (i.e. regular)

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}$$

Therefore $x=0$ is regular singular point. Hence we try Frobenius series

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in $2x^2y'' - xy' + x^3y = 0$ results in

$$
\begin{aligned}
2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^3 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+3} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=3}^{\infty} a_{n-3} x^{n+r} &= 0 \qquad (1)
\end{aligned}
$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from setting $n=0$ in (1) with the assumption that $a_0 \neq 0$. Setting $n=0$ in (1) gives

$$
\begin{aligned}
2(n+r)(n+r-1) a_n - (n+r) a_n &= 0 \\
2(r)(r-1) a_0 - r a_0 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then we obtain the indicial equation (quadratic in r)

$$
\begin{aligned}
2(r)(r-1) - r &= 0 \\
r(2(r-1) - 1) &= 0 \\
r(2r-3) &= 0
\end{aligned}
$$

Hence roots are

$$
\begin{aligned}
r_1 &= 0 \\
r_2 &= \frac{3}{2}
\end{aligned}
$$

Since $r_1 - r_2$ is not an integer, then we know we can now construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \sum_{n=0} a_n x^n \\
y_2(x) &= x^{r_2} \sum_{n=0} b_n x^n
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0} a_n x^n \\
y_2(x) &= \sum_{n=0} b_n x^{n+\frac{3}{2}}
\end{aligned}
$$

Notice that the coefficients are not the same. Since we now know r_1, r_2, we will use the above series solution to obtain $y_1(x)$ and $y_2(x)$.

For $y_1(x)$ where $r_1 = 0$

$$\begin{aligned} y_1(x) &= \sum_{n=0} a_n x^n \\ y'(x) &= \sum_{n=0} n a_n x^{n-1} = \sum_{n=1} n a_n x^{n-1} = \sum_{n=0} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=0} n(n+1) a_{n+1} x^{n-1} = \sum_{n=1} n(n+1) a_{n+1} x^{n-1} = \sum_{n=0} (n+1)(n+2) a_{n+2} x^n \end{aligned}$$

Substituting the above in $2x^2 y'' - xy' + x^3 y = 0$ results in

$$\begin{aligned} 2x^2 \sum_{n=0} (n+1)(n+2) a_{n+2} x^n - x \sum_{n=0} (n+1) a_{n+1} x^n + x^3 \sum_{n=0} a_n x^n &= 0 \\ \sum_{n=0} 2(n+1)(n+2) a_{n+2} x^{n+2} - \sum_{n=0} (n+1) a_{n+1} x^{n+1} + \sum_{n=0} a_n x^{n+3} &= 0 \\ \sum_{n=2} 2(n-1)(n) a_n x^n - \sum_{n=1} n a_n x^n + \sum_{n=3} a_{n-3} x^n &= 0 \end{aligned}$$

For $n = 1$ (index starts at $n = 1$).

$$\begin{aligned} -n a_n x^n &= 0 \\ -a_1 &= 0 \\ a_1 &= 0 \end{aligned}$$

For $n = 2$

$$\begin{aligned} 2(n-1)(n) a_n x^n - n a_n x^n &= 0 \\ 2(2-1)(2) a_2 - 2a_2 &= 0 \\ 2a_2 &= 0 \\ a_2 &= 0 \end{aligned}$$

For $n \geq 3$ we have recursive formula

$$\begin{aligned} 2(n-1)(n) a_n - n a_n + a_{n-3} &= 0 \\ a_n &= \frac{-a_{n-3}}{n(2n-3)} \end{aligned}$$

Hence, for $n = 3$

$$a_3 = \frac{-a_0}{(3)(6-3)} = \frac{-a_0}{9}$$

For $n = 4$

$$a_4 = \frac{-a_1}{n(2n-3)} = 0$$

For $n = 5$

$$a_5 = \frac{-a_2}{n(2n-3)} = 0$$

For $n = 6$

$$a_6 = \frac{-a_3}{n(2n-3)} = \frac{-a_3}{(6)(12-3)} = \frac{-a_3}{54} = \frac{a_0}{(54)(9)} = \frac{1}{486} a_0$$

And for $n=7,8$ we also obtain $a_7=0, a_8=0$, but for a_9

$$a_9=\frac{-a_6}{9\left(2\left(9\right)-3\right)}=\frac{-a_6}{135}=\frac{-a_0}{\left(135\right)\left(486\right)}=-\frac{1}{65\,610}a_0$$

And so on. Hence from $\sum_{n=0} a_n x^n$ we obtain

$$\begin{aligned}y_1\left(x\right)&=a_0-\frac{-a_0}{9}x^3+\frac{1}{486}a_0x^6-\frac{1}{65\,610}a_0x^9+\cdots\\&=a_0\left(1-\frac{1}{9}x^3+\frac{1}{486}x^6-\frac{1}{65\,610}x^9+\cdots\right)\end{aligned}$$

Now that we found $y_1\left(x\right)$.

For $y_2\left(x\right)$ with $r_2=\frac{3}{2}$.

$$\begin{aligned}y\left(x\right)&=\sum_{n=0} b_n x^{n+\frac{3}{2}}\\y'\left(x\right)&=\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}}\\y''\left(x\right)&=\sum_{n=0}\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}}\end{aligned}$$

Substituting this into $2x^2y''-xy'+x^3y=0$ gives

$$\begin{aligned}2x^2\sum_{n=0}\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}}-x\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}}+x^3\sum_{n=0}b_nx^{n+\frac{3}{2}}&=0\\\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}+2}-\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}+1}+\sum_{n=0}b_nx^{n+\frac{3}{2}+3}&=0\\\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}-\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}+\sum_{n=0}b_nx^{n+\frac{9}{2}}&=0\\\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}-\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}+\sum_{n=3}b_{n-3}x^{n+\frac{9}{2}-3}&=0\\\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}-\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}}+\sum_{n=3}b_{n-3}x^{n+\frac{3}{2}}&=0\end{aligned}$$

Now that all the x terms have the same exponents, we can continue.

For $n=0$

$$\begin{aligned}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_0x^{\frac{3}{2}}-\left(n+\frac{3}{2}\right)b_0x^{\frac{3}{2}}&=0\\2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)b_0x^{\frac{3}{2}}-\left(\frac{3}{2}\right)b_0x^{\frac{3}{2}}&=0\\\frac{3}{2}b_0x^{\frac{3}{2}}-\left(\frac{3}{2}\right)b_0x^{\frac{3}{2}}&=0\\0b_0&=0\end{aligned}$$

Hence b_0 is arbitrary.

For $n = 1$

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_n x^{n+\frac{3}{2}} - \left(n+\frac{3}{2}\right)b_n x^{n+\frac{3}{2}} = 0$$
$$2\left(1+\frac{1}{2}\right)\left(1+\frac{3}{2}\right)b_1 - \left(1+\frac{3}{2}\right)b_1 = 0$$
$$5b_1 = 0$$
$$b_1 = 0$$

For $n = 2$

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_n x^{n+\frac{3}{2}} - \left(n+\frac{3}{2}\right)b_n x^{n+\frac{3}{2}} = 0$$
$$2\left(2+\frac{1}{2}\right)\left(2+\frac{3}{2}\right)b_2 - \left(2+\frac{3}{2}\right)b_2 = 0$$
$$14b_2 = 0$$
$$b_2 = 0$$

For $n \geq 3$ we use the recursive formula

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_n - \left(n+\frac{3}{2}\right)b_n + b_{n-3} = 0$$
$$b_n = \frac{-b_{n-3}}{n\,(2n+3)}$$

Hence for $n = 3$

$$b_3 = \frac{-b_0}{3\,(9)} = \frac{-b_0}{27}$$

For $n = 4, n = 5$ we will get $b_4 = 0$ and $b_5 = 0$ since $b_1 = 0$ and $b_2 = 0$.

For $n = 6$

$$b_6 = \frac{-b_3}{6\,(12+3)} = \frac{-b_3}{90} = \frac{b_0}{27\,(90)} = \frac{b_0}{2430}$$

For $n = 7, n = 8$ we will get $b_7 = 0$ and $b_8 = 0$ since $b_4 = 0$ and $b_5 = 0$

For $n = 9$

$$a_9 = \frac{-b_6}{n\,(2n+3)} = \frac{-b_6}{9\,(18+3)} = \frac{-b_6}{189} = \frac{-b_0}{2430\,(189)} = \frac{-b_0}{459270}$$

And so on. Hence, from $y_2(x) = \sum_{n=0} b_n x^{n+\frac{3}{2}}$ the series is

$$\begin{aligned} y_2(x) &= b_0 x^{\frac{3}{2}} - \frac{b_0}{27}x^{3+\frac{3}{2}} + \frac{b_0}{2430}x^{6+\frac{3}{2}} - \frac{b_0}{459\,270}x^{9+\frac{3}{2}} + \cdots \\ &= b_0 x^{\frac{3}{2}}\left(1 - \frac{x^3}{27} + \frac{x^6}{2430} - \frac{x^9}{459270} + \cdots\right) \end{aligned}$$

The final solution is

$$y(x) = y_1(x) + y_2(x)$$

Or

$$\boxed{y(x) = a_0\left(1 - \frac{1}{9}x^3 + \frac{1}{486}x^6 - \frac{1}{65\,610}x^9 + \cdots\right) + b_0 x^{\frac{3}{2}}\left(1 - \frac{x^3}{27} + \frac{x^6}{2430} - \frac{x^9}{459270} + \cdots\right)} \tag{2}$$

Now comes the hard part. Finding closed form solution.

The Taylor series of $\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$ is (using CAS)

$$\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right) \approx 1 - \frac{1}{9}x^3 + \frac{1}{486}x^6 - \frac{1}{65610}x^9 + \cdots \tag{3}$$

And the Taylor series for $\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$ is (Using CAS)

$$\begin{aligned}
\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right) &\approx \frac{1}{3}x^{\frac{3}{2}}\sqrt{2} - \frac{1}{81}x^{\frac{9}{2}}\sqrt{2} + \frac{1}{7290}x^{\frac{15}{2}}\sqrt{2} - \cdots \\
&= x^{\frac{3}{2}}\left(\frac{1}{3}\sqrt{2} - \frac{1}{81}x^3\sqrt{2} + \frac{1}{7290}x^6\sqrt{2} - \cdots\right) \\
&= \frac{1}{3}\sqrt{2}x^{\frac{3}{2}}\left(1 - \frac{1}{27}x^3 + \frac{1}{2430}x^6 - \cdots\right)
\end{aligned} \tag{4}$$

Comparing (3,4) with (2) we see that (2) can now be written as

$$y(x) = a_0\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right) + \frac{b_0}{\frac{1}{3}\sqrt{2}}\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$$

Or letting $c = \frac{b_0}{\frac{1}{3}\sqrt{2}}$

$$\boxed{y(x) = a_0\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right) + c_0\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)}$$

This is the closed form solution. The constants a_0, c_0 can be found from initial conditions.

3.1.6.2 part f

Problem Find series expansion of all the solutions to the following differential equation about $x = 0$. Try to sum in closed form any infinite series that appear.

$$(\sin x)\,y'' - 2(\cos x)\,y' - (\sin x)\,y = 0$$

Solution

In standard form

$$y'' - 2\left(\frac{\cos x}{\sin x}\right)y' - y = 0$$

Hence the singularities are in $p(x)$ only and they occur when $\sin x = 0$ or $x = 0, \pm\pi, \pm 2\pi, \cdots$ but we just need to consider $x = 0$. Let us check if the singularity is removable.

$$\lim_{x\to 0} xp(x) = 2\lim_{x\to 0} x\frac{\cos x}{\sin x} = 2\lim_{x\to 0} x\frac{1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots} = 2\lim_{x\to 0}\frac{1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots} = 2$$

Hence the singularity is regular. So we can use Frobenius series

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

Substituting the above in $\sin(x)y'' - 2\cos(x)y' - \sin(x)y = 0$ results in

$$\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - 2\cos(x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sin(x)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

Now using Taylor series for $\sin x, \cos x$ expanded around 0, the above becomes

$$\begin{aligned} &\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \\ &\quad -2\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \\ &\quad -\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}a_n x^{n+r} = 0 \end{aligned} \tag{1}$$

We need now to evaluate products of power series. Using what is called Cauchy product rule, where

$$f(x)g(x) = \left(\sum_{m=0}^{\infty} b_m x^m\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_m a_n x^{m+n} \tag{2}$$

Applying (2) to first term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1} \tag{3}$$

Applying (2) to second term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)}{(2m)!}a_n x^{2m+n+r-1} \tag{4}$$

Applying (2) to the last term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}a_n x^{n+r} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m a_n}{(2m+1)!}x^{2m+n+r+1} \tag{5}$$

Substituting (3,4,5) back into (1) gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1} -2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+r)}{(2m)!}a_n x^{2m+n+r-1} -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m a_n}{(2m+1)!}x^{2m+n+r+1} = 0$$

We now need to make all x exponents the same. This gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1} -2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+r)}{(2m)!}a_n x^{2m+n+r-1} -\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^m a_{n-2}}{(2m+1)!}x^{2m+n+r-1} = 0 \tag{6}$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from setting $n = m = 0$ in (6) with the assumption that $a_0 \neq 0$. This results in

$$\begin{aligned}\frac{(-1)^m (n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1} - 2\frac{(-1)^m (n+r)}{(2m)!}a_n x^{2m+n+r-1} &= 0\\ (r)(r-1)a_0 - 2ra_0 &= 0\\ a_0\left(r^2 - r - 2r\right) &= 0\end{aligned}$$

Since $a_0 \neq 0$ then we obtain the indicial equation (quadratic in r)

$$\begin{aligned}r^2 - 3r &= 0\\ r(r-3) &= 0\end{aligned}$$

Hence roots are

$$\begin{aligned}r_1 &= 3\\ r_2 &= 0\end{aligned}$$

(it is always easier to make $r_1 > r_2$). Since $r_1 - r_2 = 3$ is now an integer, then this is case II part (b) in textbook, page 72. In this case, the two linearly independent solutions are

$$\begin{aligned}y_1(x) &= x^3\sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+3}\\ y_2(x) &= ky_1(x)\ln(x) + x^{r_2}\sum_{n=0} b_n x^n = ky_1(x)\ln(x) + \sum_{n=0} b_n x^n\end{aligned}$$

Where k is some constant. Now we will find y_1. From (6), where now we set $r = 3$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+3)(n+2)}{(2m+1)!}a_n x^{2m+n++2}$$
$$-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+3)}{(2m)!}a_n x^{2m+n+2}$$
$$-\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^m a_{n-2}}{(2m+1)!}x^{2m+n+2}=0$$

For $m = 0, n = 1$

$$\frac{(-1)^m(n+3)(n+2)}{(2m+1)!}a_n - 2\left(\frac{(-1)^m(n+3)}{(2m)!}a_n\right)=0$$
$$(4)(3)a_1 - 2(4a_1) = 0$$
$$a_1 = 0$$

For $m = 0, n \geq 2$ we obtain recursive equation

$$(n+3)(n+2)a_n - 2(n+3)a_n - a_{n-2} = 0$$
$$a_n = \frac{-a_{n-2}}{(n+3)(n+2)-2(n+3)} = \frac{-a_{n-2}}{n(n+3)}$$

Hence, for $m = 0, n = 2$

$$a_2 = \frac{-a_0}{10}$$

For $m = 0, n = 3$

$$a_3 = \frac{-a_1}{n(n+3)} = 0$$

For $m = 0, n = 4$

$$a_4 = \frac{-a_2}{4(7)} = \frac{a_0}{280}$$

For $m = 0, n = 5$

$$a_5 = 0$$

For $m = 0, n = 6$

$$a_6 = \frac{-a_4}{6(6+3)} = \frac{-a_0}{6(6+3)(280)} = \frac{-a_0}{15120}$$

For $m = 0, n = 7$, $a_7 = 0$ and for $m = 0, n = 8$

$$a_8 = \frac{-a_6}{8(8+3)} = \frac{1}{1330560}a_0$$

And so on. Since $y_1(x) = \sum_{n=0} a_n x^{n+3}$, then the first solution is now found. It is

$$\begin{aligned}
y_1(x) &= a_0x^3 + a_1x^4 + a_2x^5 + a_3x^6 + a_4x^7 + a_5x^8 + a_6x^9 + a_7x^{10} + a_8x^{11} + \cdots \\
&= a_0x^3 + 0 - \frac{a_0}{10}x^5 + 0 + \frac{a_0}{280}x^7 + 0 - \frac{a_0}{15120}x^9 + 0 + \frac{1}{1330560}a_0x^{11} + \cdots \\
&= a_0x^3\left(1 - \frac{x^2}{10} + \frac{x^5}{280} - \frac{x^6}{15120} + \frac{x^8}{1330560} - \cdots\right) \qquad (7)
\end{aligned}$$

The second solution can now be found from

$$y_2 = k y_1(x)\ln(x) + \sum_{n=0} b_n x^n$$

I could not find a way to convert the complete solution to closed form solution, or even find closed form for $y_1(x)$. The computer claims that the closed form final solution is

$$\begin{aligned}
y(x) &= y_1(x) + y_2(x) \\
&= a_0\cos(x) + b_0\left(-\sqrt{\cos^2 x - 1} + \cos x \ln\left(\cos(x) + \sqrt{\cos^2 x - 1}\right)\right)
\end{aligned}$$

Which appears to imply that (7) is $\cos(x)$ series. But it is not. Converting series solution to closed form solution is hard. Is this something we are supposed to know how to do? Other by inspection, is there a formal process to do it?

3.1.7 key solution of selected problems

3.1.7.1 section 3 problem 8

Problem Statement: How many terms are needed in the Taylor series solution to $y''' = x^3 y$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ are needed to evaluate $\int_0^\infty y(x)dx$ correct to 3 decimal places?
Solution: Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n a_n (n-1) x^{n-1}$$

$$y''' = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3} = x^3 y = x^3 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+3}.$$

Shift indices by letting $n \to (n-6)$ in the RHS:

$$\sum_{n=0}^{\infty} a_n x^{n+3} \to \sum_{n=6}^{\infty} a_{n-6} x^{n-3}$$

$$\sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 0 \cdot x^{-3} + 0 \cdot x^{-2} + 0 \cdot x^{-1} + 3 \cdot 2 \cdot 1 \cdot a_3 x^0 + 4 \cdot 3 \cdot 2 \cdot a_4 x^1$$

$$+ \ 5 \cdot 4 \cdot 3 \cdot a_5 x^2 + \sum_{n=6}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

From here,

$$a_3 = 0, \quad a_4 = 0, \quad a_5 = 0$$

$$x^{n-3}\{n(n-1)(n-2)a_n - a_{n-6}\} = 0 \text{ for } n = 6, 7, 8, ...$$

The recurrence relation becomes:

$$a_n = \frac{1}{n(n-1)(n-2)} a_{n-6} \quad n \geq 6$$

Solution to D.E. becomes:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \sum_{n=6}^{\infty} [\frac{1}{n(n-1)(n-2)} a_{n-6} x^n]$$

$$y(0) = a_0 = 1 ==> a_0 = 1$$

1

$$y'(0) = a_1 = 0 ==> a_1 = 0$$

$$y''(0) = 2 \cdot a_2 = 0 ==> a_2 = 0$$

$$y(x) = 1 + \sum_{n=6}^{\infty} \frac{1}{n(n-1)(n-2)} a_{n-6} x^n$$

$$\int_0^1 y(x)dx = [x + \sum_{n=6}^{\infty} \frac{1}{(n+1)n(n-1)(n-2)} a_{n-6} x^{n+1}]_0^1$$

$$= 1 + \sum_{n=6}^{\infty} \frac{1}{(n+1)(n)(n-1)(n-2)} a_{n-6}$$

$$= 1 + \frac{1}{7 \cdot 6 \cdot 5 \cdot 4}(1) + \frac{1}{13 \cdot 12 \cdot 11 \cdot 10}(\frac{1}{6 \cdot \cdot 4}(1)) + ...$$

$$= 1 + \frac{1}{840} + \frac{1}{2,059,200} + ...$$

$$\int_0^1 y(x)dx \simeq 1 + 0.00119 + 4.856 \times 10^{-7}$$

So the 2nd non-zero term is required. All other terms are below the specified tolerance of 0.001.

2

3.1.7.2 section 3 problem 6

①

3.6b $y'' - 2xy' + 8y = 0$, $y(0) = 0$, $y'(0) = 4$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ where the initial conditions give $a_0 = 0$, $a_1 = 4$

Plug in the expansion to obtain

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 8\sum_{n=0}^{\infty} a_n x^n = 0$$

↑ $m = n-2$

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=0}^{\infty} (-2n+8)a_n x^n = 0$$

or equivalently

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (-2n+8)a_n x^n = 0$$

$$\Rightarrow \quad (n+2)(n+1)a_{n+2} - 2(n-4)a_n = 0 \qquad n \geq 0 \text{ integer}$$

Since $a_0 = 0 \Rightarrow$ all even a_n's are zero

②

For odd $n = 2m+1$ $m \geq 0$ integer

$$a_{n+2} = \frac{2(n-4)a_n}{(n+2)(n+1)} \Rightarrow$$

$$a_{2m+3} = \frac{2(2m+1-4)a_{2m+1}}{(2m+3)(2m+1)}$$

$$= \frac{2(2m-3)a_{2m+1}}{(2m+3)(2m+1)}$$

The series does not truncate because the numerator is not zero for any $m \geq 0$ integer

$$y(x) = 4x + \sum_{m=0}^{\infty} a_{2m+3}\, x^{2m+3}$$

a_{2m+3} given above.

3.1.7.3 section 3 problem 4

(1)

BO 3.4 d

$$x^2 y'' = \exp[1/x]\, y$$

consider $x_0 = 0$: $y'' - \frac{\exp[1/x]}{x^2} y = 0$

$x^2 q(x) = -\exp[1/x]$ not analytic at $x_0 = 0$

$\Rightarrow$ $x_0 = 0$ is an irregular singular point

consider $x \to \infty$ with $t = \frac{1}{x}$:

$$t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt} - \exp(t)\, t^2 y = 0$$

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} - \frac{\exp(t)}{t^2} y = 0$$

$t p(t) = 2$, $t^2 q(t) = -\exp(t)$

analytic at $t_0 = 0$ $\Rightarrow$

$x = \infty$ is a regular singular point of the original equation

(2)

BO 3.4 e

$(\tan x)\, y' = y \;;\quad y' - \dfrac{\cos x}{\sin x}\, y = 0$

$x_0 = 0$: does $x p(x)$ have a Taylor series expansion about $x_0 = 0$?

$$x\,p(x) = \frac{-x\cos x}{\sin x}$$

$$= \frac{-x\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}}{\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right\}}$$

$$= \frac{-x\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}}{x\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right\}}$$

$$= -\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}\left\{1 + \frac{x^2}{3!} + \cdots\right\}$$

$$= -\left\{1 - \frac{x^2}{2!} + O(x^4)\right\}\left\{1 + \frac{x^2}{3!} + O(x^4)\right\}$$

converges for $|x| < 1 \Rightarrow x_0 = 0$ is a regular singular point

Consider $x \to \infty$ with $t = \frac{1}{x}$:

$$-t^2 \frac{dy}{dt} - \frac{\cos(1/t)}{\sin(1/t)} y = 0$$

$$\frac{dy}{dt} + \frac{1}{t^2} \frac{\cos(1/t)}{\sin(1/t)} y = 0$$

$t=0$ is irregular singular $\Rightarrow$ $x \to \infty$ is an irregular singular point of the original equation.

3.1.7.4 section 3 problem 24

1 Closed form of Problem 3.24e

In problem 3.24e, we find Frobenius series solutions

$$y_\alpha(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \tag{1}$$

for $\alpha = 0, \frac{3}{2}$, and recurrence relation on the coefficients:

$$a_{n+3} = \frac{-a_n}{(n+\alpha+3)(2n+2\alpha+3)}.$$

For each α, $a_1 = a_2 = 0$, so only the a_{3k} survive. Rewriting the recurrence relation:

$$a_{3k+3} = \frac{-a_{3k}}{(3k+3+\alpha)(6k+3+2\alpha)} = \frac{-a_{3k}}{9(k+1+\alpha/3)(2k+1+2\alpha/3)}.$$

To identify a closed-form sum, we need to find the general form of the coefficients a_{3k}. Towards this end, consider the case $\alpha = 0$. Now

$$a_{3k} = \frac{-1/9}{k(2k-1)} a_{3k-3} = \frac{(-1/9)^2}{k(k-1)\cdot(2k-1)(2k-3)} a_{3k-6} = \cdots = \frac{(-1/9)^k}{k!\cdot(2k-1)!!} a_0.$$

Now note that

$$(2k-1)!! = \frac{(2k)!}{(2k)!!} = \frac{(2k)!}{2^k\cdot k!} \implies k!\cdot(2k-1)!! = 2^{-k}(2k)!,$$

so

$$a_{3k} = \frac{(-1/9)^k}{2^{-k}(2k)!} a_0 = \frac{(-2/9)^k}{(2k)!} a_0.$$

Therefore,

$$y_0(x) = a_0 \sum_{k=0}^{\infty} \frac{(-2/9)^k}{(2k)!} x^{3k} = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^{3/2}\sqrt{2}/3)^{2k} =: a_0 \sum_{k=0}^{\infty} \frac{-z^{2k}}{(2k)!},$$

where $z := x^{3/2}\sqrt{2}/3$. We recognize the remaining series as the Taylor series for $\cos z$, so

$$y_0(x) = a_0 \cos z(x) = a_0 \cos\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

For $\alpha = \frac{3}{2}$,

$$a_{3k+3} = \frac{-a_{3k}}{9(k+3/2)(2k+2)} = \frac{-a_{3k}}{9(k+1)(2k+3)}.$$

Following the same argument as before,

$$a_{3k} = \frac{(-1/9)^k}{k!\cdot(2k+1)!!} a_0.$$

This is the same as before, but divided by $2k+1$, so

$$y_{3/2}(x) = a_0 \sum_{k=0}^{\infty} \frac{(-2/9)^k}{(2k+1)!} x^{3k} = a_0 \sin\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

Therefore, the general solution is

$$y(x) = y_0(x) + y_{3/2}(x) = c_1 \cos\left(\frac{\sqrt{2}}{3} x^{3/2}\right) + c_2 \sin\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

1

2 Closed form of Problem 3.24f

Problem 3.24f asks us to solve $(\sin x)y'' - 2(\cos x)y' - (\sin x)y = 0$. Dividing by $\sin x$,

$$y'' - 2(\cot x)y' - y = 0,$$

so $x = 0$ is a regular singular point ($\cot x$ is not analytic at $x = 0$, but $x \cot x$ is). Thus, we can look for a Frobenius series solution.

Expanding $\sin x$ and $\cos x$:

$$0 = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} - 2\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}\right) \sum_{n=0}^{\infty} (n+\alpha)a_n x^{n+\alpha-1} - \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

The lowest order is $x^{\alpha-1}$:

$$0 = \alpha(\alpha-1)a_0 - 2\alpha a_0 = \alpha(\alpha-3)a_0,$$

so $\alpha = 0, 3$. We are in case II(b) on page 72.

Let $\alpha = 0$. Then

$$0 = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}\right) \sum_{n=0}^{\infty} n a_n x^{n-1} - \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} a_n x^n.$$

We solve order-by-order:

$$\begin{aligned}
x^0 &: 0 = -2a_1 \implies a_1 = 0,\\
x^1 &: 0 = 2a_2 - 2\cdot 2a_2 - a_0 \implies a_2 = -\frac{a_0}{2},\\
x^{2k} &: 0 = \sum_{\ell=0}^{k} (-1)^{k-\ell} \frac{2\ell(2\ell+1)}{(2k-2\ell+1)!} a_{2\ell+1} - 2\sum_{\ell=0}^{k} (-1)^{k-\ell} \frac{2\ell+1}{(2k-2\ell)!} a_{2\ell+1} - \sum_{\ell=0}^{k-1} \frac{(-1)^{k-\ell-1}}{(2k-2\ell-1)!} a_{2\ell+1},\\
x^{2k+1} &: 0 = \sum_{\ell=0}^{k+1} (-1)^{k-\ell+1} \frac{2\ell(2\ell-1)}{(2k-2\ell+3)!} a_{2\ell} - 2\sum_{\ell=0}^{k+1} (-1)^{k-\ell+1} \frac{2\ell}{(2k-2\ell+2)!} a_{2\ell} - \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{(2k-2\ell+1)!} a_{2\ell},
\end{aligned}$$

where $k \geq 1$. From the x^{2k} terms, we see that a_{2n+1} depends only on $a_1, a_3, ..., a_{2n-1}$. As $a_1 = 0$, it is a simple (strong) induction to show that each $a_{2n+1} = 0$.

We use the x^{2k+1} terms for $k = 1$ to see that

$$0 = \left(0a_0 - \frac{2}{6}a_2 + \frac{12}{1}a_4\right) - 2\left(0a_0 - \frac{2}{2}a_2 + \frac{4}{1}a_4\right) - \left(\frac{-1}{6}a_0 + \frac{1}{1}a_2\right) \implies a_4 = -\frac{1}{4}\left(\frac{1}{6}a_0 - \frac{2}{3}a_2\right).$$

As $a_2 = -\frac{1}{2}a_0$, we see that $a_4 = \frac{1}{24}a_0$. We notice a pattern that these coefficients are starting to look like those in the Taylor series expansion of $y_0(x) = a_0 \cos x$, which we now verify by plugging into the ODE:

$$(\sin x)y_0''(x) - 2(\cos x)y_0'(x) - (\sin x)y_0(x) = -a_0 \sin x \cos x + 2a_0 \cos x \sin x - a_0 \sin x \cos x = 0.$$

To find the other solution, we use reduction of order: look for a solution of the form $y_3(x) = v(x)\cos x$ (we keep the $\alpha = 3$ subscript to remind us that we expect the Taylor series of our solution to start at the x^3-term). Plugging into the ODE:

$$(\sin x)(v''(x)\cos x - 2v'(x)\sin x - v(x)\cos x) - (2\cos x)(v'(x)\cos x - v(x)\sin x) - (\sin x)(v(x)\cos x) = 0.$$

Simplifying,

$$(\sin x\cos x)v''(x) - 2v'(x) = 0 \implies v'(x) = c_1\tan^2 x \implies v(x) = c_1(\tan x - x) + c_0.$$

We set $c_0 = 0$, since this corresponds to $y_0(x)$. Then $y_3(x) = c_1(\tan x - x)\cos x = c_1(\sin x - x\cos x)$.

The full solution is therefore

$$y(x) = c_0\cos x + c_1(\sin x - x\cos x).$$

3.2 HW2

3.2.1 problem 3.27 (page 138)

Problem derive 3.4.28. Below is a screen shot from the book giving 3.4.28 at page 88, and the context it is used in before solving the problem

Example 5 *Local behavior of solutions near an irregular singular point of a general nth-order Schrödinger equation.* In this example we derive an extremely simple and important formula for the leading behavior of solutions to the nth-order Schrödinger equation

$$\frac{d^n y}{dx^n} = Q(x)y \tag{3.4.27}$$

near an irregular singular point at x_0.

The exponential substitution $y = e^S$ and the asymptotic approximations $d^kS/dx^k \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \ldots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x [Q(t)]^{1/n}\, dt$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$.

The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$y(x) \sim c[Q(x)]^{(1-n)/2n} \exp\left\{\omega \int^x [Q(t)]^{1/n}\, dt\right\}, \qquad x \to x_0. \tag{3.4.28}$$

If $x_0 \neq \infty$, (3.4.28) is valid if $|(x - x_0)^n Q(x)| \to \infty$ as $x \to x_0$. If $x_0 = \infty$, then (3.4.28) is valid if $|x^n Q(x)| \to \infty$ as $x \to \infty$. This important formula forms the basis of WKB theory and will be rederived perturbatively and in much greater detail in Sec. 10.2. If $Q(x) < 0$, solutions to (3.4.27) oscillate as $x \to \infty$; the nature of asymptotic relations between oscillatory functions is discussed in Sec. 3.7.

Here are some examples of the application of (3.4.28):

(*a*) For $y'' = y/x^5$, $y(x) \sim cx^{5/4} e^{\pm 2x^{-3/2}/3}$ $(x \to 0+)$.
(*b*) For $y''' = xy$, $y(x) \sim cx^{-1/3} e^{3\omega x^{4/3}/4}$ $(x \to +\infty)$, where $\omega^3 = 1$.
(*c*) For $d^4y/dy^4 = (x^4 + \sin x)y$, $y(x) \sim cx^{-3/2} e^{\omega x^2/2}$ $(x \to +\infty)$, where $\omega = \pm 1, \pm i$.

Solution

For n^{th} order ODE, $S_0(x)$ is given by

$$S_0(x) \sim \omega \int^x Q(t)^{\frac{1}{n}}\, dt$$

And (page 497, textbook)

$$S_1(x) \sim \frac{1-n}{2n} \ln(Q(x)) + c \tag{10.2.11}$$

Therefore

$$\begin{aligned} y(x) &\sim \exp(S_0 + S_1) \\ &\sim \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt + \frac{1-n}{2n} \ln(Q(x)) + c\right) \\ &\sim c\,[Q(x)]^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt\right) \end{aligned}$$

Note: I have tried other methods to proof this, such as a proof by induction. But was not able to after many hours trying. The above method uses a given formula which the book did not indicate how it was obtained. (see key solution)

3.2.2 Problem 3.33(b) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $x^4 y''' - 3x^2 y' + 2y = 0$

Solution Let

$$\begin{aligned} y(x) &= e^{S(x)} \\ y' &= S' e^S \\ y'' &= S'' e^S + (S')^2 e^S \\ y''' &= S''' e^S + S'' S' e^S + 2S' S'' e^S + (S')^3 e^S \end{aligned}$$

Hence the ODE becomes

$$x^4 \left[S''' + 3S'S'' + (S')^3 \right] - 3x^2 S' = -2 \tag{1}$$

Now, we define $S(x)$ as sum of a number of leading terms, which we try to find

$$S(x) = S_0(x) + S_1(x) + S_2(x) + \cdots$$

Therefore (1) becomes (using only two terms for now $S = S_0 + S_1$)

$$\begin{aligned} \{S_0 + S_1\}''' + 3\left\{(S_0 + S_1)(S_0 + S_1)''\right\} + \left\{(S_0 + S_1)'\right\}^3 - \frac{3}{x^2}\{S_0 + S_1\}' &= -\frac{2}{x^4} \\ \left\{S_0''' + S_1'''\right\} + 3\left\{(S_0 + S_1)\left(S_0'' + S_1''\right)\right\} + \left\{S_0' + S_1'\right\}^3 - \frac{3}{x^2}\left(S_0' + S_1'\right) &= -\frac{2}{x^4} \\ \left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0 + S_0''S_1' + S_1''S_0'\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} &= -\frac{2}{x^4} \end{aligned} \tag{2}$$

Assuming that $S_0' \ggg S_1', S_0''' \ggg S_1''', \left(S_0'\right)^3 \ggg 3\left(S_0'\right)^2 S_1'$ then equation (2) simplifies to

$$S_0''' + 3S_0''S_0' + \left(S_0'\right)^3 - \frac{3}{x^2} S_0' \sim -\frac{2}{x^4}$$

Assuming $\left(S_0'\right)^3 \ggg S_0''', \left(S_0'\right)^3 \ggg 3S_0''S_0', \left(S_0'\right)^3 \ggg \frac{3}{x^2} S_0'$ (which we need to verify later), then the above becomes

$$\left(S_0'\right)^3 \sim -\frac{2}{x^4}$$

Verification[2]

[2]When carrying out verification, all constant multipliers and signs are automticaly simplified and removed

Since $S_0' \sim \left(\frac{-2}{x^4}\right)^{\frac{1}{3}} = \frac{1}{x^{\frac{4}{3}}}$ then $S_0'' \sim \frac{1}{x^{\frac{7}{3}}}$ and $S_0''' \sim \frac{1}{x^{\frac{10}{3}}}$. Now we need to verify the three assumptions made above, which we used to obtain S_0'.

$$\left(S_0'\right)^3 \ggg S_0'''$$
$$\frac{1}{x^4} \ggg \frac{1}{x^{\frac{10}{3}}}$$

Yes.

$$\left(S_0'\right)^3 \ggg 3S_0''S_0'$$
$$\frac{1}{x^4} \ggg \left(\frac{1}{x^{\frac{7}{3}}}\right)\left(\frac{1}{x^{\frac{4}{3}}}\right)$$
$$\frac{1}{x^4} \ggg \left(\frac{1}{x^{\frac{11}{3}}}\right)$$

Yes.

$$\left(S_0'\right)^3 \ggg \frac{3}{x^2}S_0'$$
$$\frac{1}{x^4} \ggg \left(\frac{1}{x^2}\right)\frac{1}{x^{\frac{4}{3}}}$$
$$\frac{1}{x^4} \ggg \frac{1}{x^{\frac{10}{3}}}$$

Yes. Assumed balance ise verified. Therefore

$$\left(S_0'\right)^3 \sim -\frac{2}{x^4}$$
$$S_0' \sim \omega x^{\frac{-4}{3}}$$

Where $\omega^3 = -2$. Integrating

$$\begin{aligned} S_0 &\sim \omega \int x^{\frac{-4}{3}}\,dx \\ &\sim \omega \int x^{-\frac{4}{3}}dx \\ &\sim -3\omega x^{\frac{-1}{3}} \end{aligned}$$

Where we ignored the constant of integration since subdominant. To find leading behavior, we go back to equation (2) and now solve for S_1.

$$\left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0' + S_0''S_1' + S_1''S_0'\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} = -\frac{2}{x^4}$$

Moving all known quantities (those which are made of S_0 and its derivatives) to the RHS

going from one step to the next, as they do not affect the final result.

and simplifying, gives

$$\left\{S_1'''\right\}+3\left\{S_0''S_1'+S_1''S_0'\right\}+\left\{3\left(S_0'\right)^2S_1'+3S_0'\left(S_1'\right)^2\right\}-\frac{3S_1'}{x^2}\sim -S_0'''-3S_0''S_0'+\frac{3}{x^2}S_0'$$

Now we assume the following (then will verify later)

$$\begin{aligned}3\left(S_0'\right)^2S_1'&\ggg 3S_0'\left(S_1'\right)^2\\3\left(S_0'\right)^2S_1'&\ggg S_1'''\\3\left(S_0'\right)^2S_1'&\ggg S_1''S_0'\\3\left(S_0'\right)^2S_1'&\ggg S_0''S_1'\\3\left(S_0'\right)^2S_1'&\ggg S_1''S_0'\end{aligned}$$

Hence

$$3\left(S_0'\right)^2S_1'-\frac{3S_1'}{x^2}\sim -S_0'''-3S_0''S_0'+\frac{3S_0'}{x^2} \tag{3}$$

But

$$\begin{aligned}S_0'&\sim \omega x^{\frac{-4}{3}}\\\left(S_0'\right)^2&\sim \omega^2x^{\frac{-8}{3}}\\S_0''&\sim -\frac{4}{3}\omega x^{\frac{-7}{3}}\\S_0'''&\sim \frac{28}{9}\omega x^{\frac{-10}{3}}\end{aligned}$$

Hence (3) becomes

$$\begin{aligned}3\left(\omega^2x^{\frac{-8}{3}}\right)S_1'-\frac{3S_1'}{x^2}&\sim\frac{28}{9}\omega x^{\frac{-10}{3}}+3\left(\frac{4}{3}\omega^2x^{\frac{-7}{3}}x^{\frac{-4}{3}}\right)+\frac{3\omega x^{\frac{-4}{3}}}{x^2}\\3\omega^2x^{\frac{-8}{3}}S_1'-3x^{-2}S_1'&\sim\frac{28}{9}\omega x^{\frac{-10}{3}}+4\omega^2x^{\frac{-11}{3}}+3\omega x^{\frac{-10}{3}}\end{aligned}$$

For small x, $x^{\frac{-8}{3}}S_1'\ggg x^{-2}S_1'$ and $x^{\frac{-11}{3}}\ggg x^{\frac{-10}{3}}$, then the above simplifies to

$$\begin{aligned}3\omega^2x^{\frac{-8}{3}}S_1'&\sim 4\omega^2x^{\frac{-11}{3}}\\S_1'&\sim\frac{4}{3}x^{-1}\end{aligned}$$

$$S_1\sim\frac{4}{3}\ln x$$

Where constant of integration was dropped, since subdominant.

<u>Verification</u> Using $S_0' \backsim x^{\frac{-4}{3}}, \left(S_0'\right)^2 \backsim x^{\frac{-8}{3}}, S_0'' \backsim x^{\frac{-7}{3}}, S_1' \sim \frac{1}{x}, S_1'' \sim \frac{1}{x^2}, S_1''' \sim \frac{1}{x^3}$

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg 3S_0'\left(S_1'\right)^2\\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-4}{3}}\frac{1}{x^2}\\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1'''\\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^3}\\
\frac{1}{x^{\frac{8}{3}}} &\ggg \frac{1}{x^2}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1''S_0'\\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2}x^{\frac{-4}{3}}\\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_0''S_1'\\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-7}{3}}\frac{1}{x}\\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes

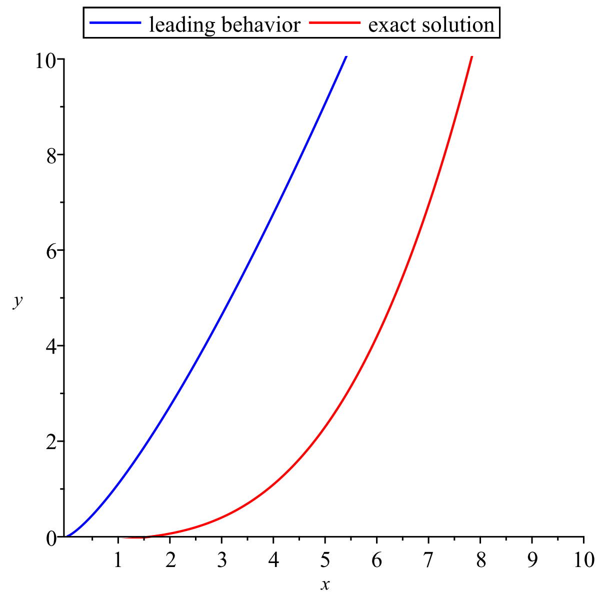
$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1''S_0'\\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2}x^{\frac{-4}{3}}\\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes. All verified. Leading behavior is

$$
\begin{aligned}
y(x) &\backsim e^{S_0(x)+S_1(x)}\\
&= \exp\left(c\omega x^{\frac{-1}{3}} + \frac{4}{3}\ln x\right)\\
&= x^{\frac{4}{3}} e^{c\omega x^{\frac{-1}{3}}}
\end{aligned}
$$

I now wanted to see how Maple solution to this problem compare with the leading behavior near $x = 0$. To obtain a solution from Maple, one have to give initial conditions a little bit removed from $x = 0$ else no solution could be generated. So using arbitrary initial conditions at $x = \frac{1}{100}$ a solution was obtained and compared to the above leading behavior. Another problem is how to select c in the above leading solution. By trial and error a constant was selected. Here is screen shot of the result. The exact solution generated by Maple is very complicated, in terms of hypergeom special functions.

```
ode:=x^4*diff(y(x),x$3)-3*x^2*diff(y(x),x)+2*y(x);
pt:=1/100:
ic:=y(pt)=500,D(y)(pt)=0,(D@@2)(y)(pt)=0:
sol:=dsolve({ode,ic},y(x)):
leading:=(x,c)->x^(4/3)*exp(c*x^(-1/3));
plot([leading(x,.1),rhs(sol)],x=pt..10,y=0..10,color=[blue,red],
legend=["leading behavior","exact solution"],legendstyle=[location=top]);
```



3.2.3 problem 3.33(c) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $y'' = \sqrt{x}y$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$
\begin{aligned}
y(x) &= e^{S_0(x)} \\
y'(x) &= S_0' e^{S_0} \\
y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\
&= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0}
\end{aligned}
$$

Substituting in the ODE gives

$$
S_0'' + \left(S_0'\right)^2 = \sqrt{x} \tag{1}
$$

Assuming $S_0'' \sim \left(S_0'\right)^2$ then (1) becomes

$$S_0'' \sim -\left(S_0'\right)^2$$

Let $S_0' = z$ then the above becomes $z' = -z^2$. Hence $\frac{dz}{dx}\frac{1}{z^2} = -1$ or $\frac{dz}{z^2} = -dx$. Integrating $-\frac{1}{z} = -x + c$ or $z = \frac{1}{x+c_1}$. Hence $S_0' = \frac{1}{x+c_1}$. Integrating again gives

$$S_0(x) \sim \ln|x + c_1| + c_2$$

Verification

$S_0' = \frac{1}{x+c_1}, \left(S_0'\right)^2 = \frac{1}{(x+c_1)^2}, S_0'' = \frac{-1}{(x+c_1)^2}$

$$\begin{aligned} S_0'' &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$.

$$\begin{aligned} \left(S_0'\right)^2 &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$. Verified. Controlling factor is

$$\begin{aligned} y(x) &\sim e^{S_0(x)} \\ &\sim e^{\ln|x+c_1|+c_2} \\ &\sim Ax + B \end{aligned}$$

3.2.4 problem 3.35

Problem: Obtain the full asymptotic behavior for small x of solutions to the equation

$$x^2 y'' + (2x+1)\,y' - x^2\left(e^{\frac{2}{x}} + 1\right)y = 0$$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$\begin{aligned} x^2\left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} + (2x+1) S_0' e^{S_0} - x^2\left(e^{\frac{2}{x}} + 1\right) e^{S_0(x)} &= 0 \\ x^2\left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' - x^2\left(e^{\frac{2}{x}} + 1\right) &= 0 \\ x^2\left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' &= x^2\left(e^{\frac{2}{x}} + 1\right) \\ S_0'' + \left(S_0'\right)^2 + \frac{(2x+1)}{x^2} S_0' &= e^{\frac{2}{x}} + 1 \end{aligned}$$

Assuming balance

$$\begin{aligned} \left(S_0'\right)^2 &\sim \left(e^{\frac{2}{x}} + 1\right) \\ \left(S_0'\right)^2 &\sim e^{\frac{2}{x}} \\ S_0' &\sim \pm e^{\frac{1}{x}} \end{aligned}$$

Where 1 was dropped since subdominant to $e^{\frac{1}{x}}$ for small x.

Verification Since $\left(S_0'\right)^2 \sim e^{\frac{2}{x}}$ then $S_0' \sim e^{\frac{1}{x}}$ and $S_0'' \sim -\frac{1}{x^2} e^{\frac{1}{x}}$, hence

$$\begin{aligned} \left(S_0'\right)^2 &\ggg S_0'' \\ e^{\frac{2}{x}} &\ggg \frac{1}{x^2} e^{\frac{1}{x}} \\ e^{\frac{1}{x}} &\ggg \frac{1}{x^2} \end{aligned}$$

Yes, As $x \to 0^+$

$$\begin{aligned} \left(S_0'\right)^2 &\ggg \frac{(2x+1) S_0'}{x^2} \\ e^{\frac{2}{x}} &\ggg \frac{(2x+1)}{x^2} e^{\frac{1}{x}} \\ e^{\frac{1}{x}} &\ggg \frac{(2x+1)}{x^2} \\ e^{\frac{1}{x}} &\ggg \frac{2}{x} + \frac{1}{x^2} \end{aligned}$$

Yes as $x \to 0^+$. Verified. Hence both assumptions used were verified OK. Hence

$$S_0' \sim \pm e^{\frac{2}{x}}$$
$$S_0 \sim \pm \int e^{\frac{1}{x}} dx$$

Since the integral do not have closed form, we will do asymptotic expansion on the integral. Rewriting $\int e^{\frac{1}{x}} dx$ as $\int \frac{e^{\frac{1}{x}}}{(-x^2)} \left(-x^2\right) dx$. Using $\int u dv = uv - \int v du$, where $u = -x^2, dv = \frac{-e^{\frac{1}{x}}}{x^2}$, gives $du = -2x$ and $v = e^{\frac{1}{x}}$, hence

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= uv - \int v du \\ &= -x^2 e^{\frac{1}{x}} - \int -2x e^{\frac{1}{x}} dx \\ &= -x^2 e^{\frac{1}{x}} + 2 \int x e^{\frac{1}{x}} dx \qquad (1)\end{aligned}$$

Now we apply integration by parts on $\int x e^{\frac{1}{x}} dx = \int x \frac{e^{\frac{1}{x}}}{-x^3} \left(-x^3\right) du = \int \frac{e^{\frac{1}{x}}}{-x^2} \left(-x^3\right) du$, where $u = -x^3, dv = \frac{e^{\frac{1}{x}}}{-x^2}$, hence $du = -3x^2, v = e^{\frac{1}{x}}$, hence we have

$$\begin{aligned}\int x e^{\frac{1}{x}} dx &= uv - \int v du \\ &= -x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx\end{aligned}$$

Substituting this into (1) gives

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} + 2\left(-x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx\right) \\ &= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6 \int 3x^2 e^{\frac{1}{x}} dx\end{aligned}$$

And so on. The series will become

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6x^4 e^{\frac{1}{x}} + 24x^5 e^{\frac{1}{x}} + \cdots + n! x^{n+1} e^{\frac{1}{x}} + \cdots \\ &= -e^{\frac{1}{x}} \left(x^2 + 2x^3 + 6x^4 + \cdots + n! x^{n+1} + \cdots\right)\end{aligned}$$

Now as $x \to 0^+$, we can decide how many terms to keep in the RHS, If we keep one term, then we can say

$$\begin{aligned}S_0 &\sim \pm \int e^{\frac{1}{x}} dx \\ &\sim \pm x^2 e^{\frac{1}{x}}\end{aligned}$$

For two terms

$$S_0 \sim \pm \int e^{\frac{1}{x}} dx \sim \pm e^{\frac{1}{x}} \left(x^2 + 2x^3\right)$$

And so on. Let us use one term for now for the rest of the solution.

$$S_0 \sim \pm x^2 e^{\frac{1}{x}}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$
$$y''(x) = \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Substituting into the given ODE gives

$$x^2\left[\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)''\right] + (2x+1)(S_0(x) + S_1(x))' - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left[\left(S_0' + S_1'\right)^2 + (S_0 + S_1)''\right] + (2x+1)\left(S_0'(x) + S_1'(x)\right) - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left[\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + \left(S_0'' + S_1''\right)\right] + (2x+1)S_0'(x) + (2x+1)S_1'(x) - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left(S_0'\right)^2 + x^2\left(S_1'\right)^2 + 2x^2S_0'S_1' + x^2S_0'' + x^2S_1'' + (2x+1)S_0'(x) + (2x+1)S_1'(x) = x^2\left(e^{\frac{2}{x}} + 1\right)$$

But $x^2\left(S_0'\right)^2 \sim x^2\left(e^{\frac{2}{x}} + 1\right)$ since we found that $S_0' \sim e^{\frac{1}{x}}$. Hence the above simplifies to

$$x^2\left(S_1'\right)^2 + 2x^2S_0'S_1' + x^2S_0'' + x^2S_1'' + (2x+1)S_0'(x) + (2x+1)S_1'(x) = 0$$
$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' + \frac{(2x+1)}{x^2}S_0'(x) + \frac{(2x+1)}{x^2}S_1'(x) = 0 \qquad (2)$$

Now looking at $S_0'' + \frac{(2x+1)}{x^2}S_0'(x)$ terms in the above. We can simplify this since we know $S_0' = e^{\frac{1}{x}}, S_0'' = -\frac{1}{x^2}e^{\frac{1}{x}}$ This terms becomes

$$-\frac{1}{x^2}e^{\frac{1}{x}} + \frac{(2x+1)}{x^2}e^{\frac{1}{x}} = \frac{-e^{\frac{1}{x}} + 2xe^{\frac{1}{x}} + e^{\frac{1}{x}}}{x^2} = \frac{2xe^{\frac{1}{x}}}{x^2} = \frac{2e^{\frac{1}{x}}}{x}$$

Therefore (2) becomes

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_1'' + \frac{(2x+1)}{x^2}S_1'(x) \sim \frac{-2e^{\frac{1}{x}}}{x}$$
$$\frac{\left(S_1'\right)^2}{S_0'} + 2S_1' + \frac{S_1''}{S_0'} + \frac{(2x+1)}{x^2S_0'}S_1'(x) \sim \frac{-2e^{\frac{1}{x}}}{xS_0'}$$
$$\frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}} + 2S_1' + \frac{S_1''}{e^{\frac{1}{x}}} + \frac{(2x+1)}{x^2e^{\frac{1}{x}}}S_1'(x) \sim \frac{-2}{x}$$

Assuming the balance is

$$S_1' \sim \frac{-1}{x}$$

Hence

$$S_1(x) \sim -\ln(x) + c$$

Since c subdominant as $x \to 0^+$ then

$$\boxed{S_1(x) \sim -\ln(x)}$$

Verification

$$\begin{aligned}
S_1' &\ggg \frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}}\\
\frac{1}{x} &\ggg \frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}}\\
e^{\frac{1}{x}} &\ggg \frac{1}{x}
\end{aligned}$$

Yes, for $x \to 0^+$

$$\begin{aligned}
S_1' &\ggg \frac{S_1''}{e^{\frac{1}{x}}}\\
\frac{1}{x} &\ggg \frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}}
\end{aligned}$$

Yes.

$$\begin{aligned}
S_1' &\ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}} S_1'(x)\\
\frac{1}{x} &\ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}}\frac{1}{x}\\
\frac{1}{x} &\ggg \frac{1}{x^3 e^{\frac{1}{x}}}
\end{aligned}$$

Yes. All assumptions verified. Hence leading behavior is

$$\begin{aligned}
y(x) &\sim \exp\left(S_0(x) + S_1(x)\right)\\
&\sim \exp\left(\pm x^2 e^{\frac{1}{x}} - \ln(x)\right)\\
&\sim \frac{1}{x}\left(\exp\left(x^2 e^{\frac{1}{x}}\right) + \exp\left(-x^2 e^{\frac{1}{x}}\right)\right)
\end{aligned}$$

For small x, then we ignore $\exp\left(-x^2 e^{\frac{1}{x}}\right)$ since much smaller than $\exp\left(x^2 e^{\frac{1}{x}}\right)$. Therefore

$$y(x) \sim \frac{1}{x}\exp\left(x^2 e^{\frac{1}{x}}\right)$$

3.2.5 problem 3.39(h)

problem Find leading asymptotic behavior as $x \to \infty$ for $y'' = e^{-\frac{3}{x}} y$

<u>solution</u> Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + (S_0')^2 e^{S_0} \\ &= \left(S_0'' + (S_0')^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$\begin{aligned} \left(S_0'' + (S_0')^2\right) e^{S_0} &= e^{-\frac{3}{x}} e^{S_0} \\ S_0'' + (S_0')^2 &= e^{-\frac{3}{x}} \end{aligned}$$

<u>Assuming</u> $(S_0')^2 \ggg S_0''$ the above becomes

$$\begin{aligned} (S_0')^2 &\sim e^{-\frac{3}{x}} \\ S_0' &\sim \pm e^{-\frac{3}{2x}} \end{aligned}$$

Hence

$$S_0 \sim \pm \int e^{-\frac{3}{2x}} dx$$

Integration by parts. Since $\frac{d}{dx} e^{\frac{-3}{2x}} = \frac{3}{2x^2} e^{-\frac{3}{2x}}$, then we rewrite the integral above as

$$\int e^{-\frac{3}{2x}} dx = \int \frac{3}{2x^2} e^{-\frac{3}{2x}} \left(\frac{2x^2}{3}\right) dx$$

And now apply integration by parts. Let $dv = \frac{3}{2x^2} e^{-\frac{3}{2x}} \to v = e^{-\frac{3}{2x}}, u = \frac{2x^2}{3} \to du = \frac{4}{3}x$, hence

$$\begin{aligned} \int e^{-\frac{3}{2x}} dx &= [uv] - \int v du \\ &= \frac{2x^2}{3} e^{-\frac{3}{2x}} - \int \frac{4}{3} x e^{-\frac{3}{2x}} dx \end{aligned}$$

Ignoring higher terms, then we use

$$S_0 \sim \pm \frac{2x^2}{3} e^{-\frac{3}{2x}}$$

<u>Verification</u>

$$\begin{aligned} (S_0')^2 &\ggg S_0'' \\ \left(e^{-\frac{3}{2x}}\right)^2 &\ggg \frac{3}{2x^2} e^{-\frac{3}{2x}} \\ e^{-\frac{3}{x}} &\ggg \frac{3}{2x^2} e^{-\frac{3}{2x}} \end{aligned}$$

<u>Yes</u>, as $x \to \infty$. To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$
$$y''(x) = \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Using the above, the ODE $y'' = e^{-\frac{3}{x}} y$ now becomes

$$\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)'' = e^{-\frac{3}{x}}$$
$$\left(S_0' + S_1'\right)^2 + S_0'' + S_1'' = e^{-\frac{3}{x}}$$
$$\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = e^{-\frac{3}{x}}$$

But $\left(S_0'\right)^2 \sim e^{-\frac{3}{x}}$ hence the above simplifies to

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

<u>Assuming</u> $\left(2S_0'S_1'\right) \ggg S_1''$ the above becomes

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' = 0$$

<u>Assuming</u> $2S_0'S_1' \ggg \left(S_1'\right)^2$

$$2S_0'S_1' + S_0'' = 0$$
$$S_1' \sim -\frac{S_0''}{2S_0'}$$
$$S_1 \sim -\frac{1}{2}\ln\left(S_0'\right)$$

But $S_0' \sim e^{-\frac{3}{2x}}$, hence the above becomes

$$S_1 \sim -\frac{1}{2}\ln\left(e^{-\frac{3}{2x}}\right) + c$$
$$\sim \frac{3}{4x} + c$$

<u>Verification</u>

$$\left(2S_0'S_1'\right) \ggg S_1''$$
$$\left(2e^{-\frac{3}{2x}}\frac{-3}{4x^2}\right) \ggg \frac{3}{2x^3}$$
$$\frac{3}{2}\frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{3}{2x^3}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^3}$$

Yes.

$$\left(2S_0'S_1'\right) \ggg \left(S_1'\right)^2$$
$$\left(2e^{-\frac{3}{2x}}\frac{-3}{4x^2}\right) \ggg \left(\frac{-3}{4x^2}\right)^2$$
$$\frac{3}{2}\frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{9}{16x^4}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^4}$$

Yes. All verified. Therefore, the leading behavior is

$$\begin{aligned} y(x) &\sim \exp\left(S_0(x)+S_1(x)\right) \\ &\sim \exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}} - \frac{1}{2}\ln\left(e^{-\frac{3}{2x}}\right) + c\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}}\right) \end{aligned} \tag{1}$$

Check if we can use 3.4.28 to verify:

$$\lim_{x\to\infty}|x^nQ(x)| = \lim_{x\to\infty}\left|x^2e^{-\frac{3}{x}}\right| \to \infty$$

We can use it. Lets verify using 3.4.28

$$y(x) \sim c\left[Q(x)\right]^{\frac{1-n}{2n}}\exp\left(\omega\int^x Q[t]^{\frac{1}{n}}\,dt\right)$$

Where $\omega^2=1$. For $n=2, Q(x)=e^{-\frac{3}{x}}$, the above gives

$$\begin{aligned} y(x) &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{1-2}{4}}\exp\left(\omega\int^x\left[e^{-\frac{3}{t}}\right]^{\frac{1}{2}}dt\right) \\ &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{-1}{4}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \end{aligned} \tag{2}$$

We see that (1,2) are the same. Verified OK. Notice that in (1), we use the approximation for the $e^{-\frac{3}{2x}}dx \approx \frac{2x^2}{3}e^{-\frac{3}{2x}}$ we found earlier. This was done, since there is no closed form solution for the integral.

QED.

3.2.6 problem 3.42(a)

Problem: Extend investigation of example 1 of section 3.5 (a) Obtain the next few corrections to the leading behavior (3.5.5) then see how including these terms improves the numerical approximation of $y(x)$ in 3.5.1.

Solution Example 1 at page 90 is $xy'' + y' = y$. The leading behavior is given by 3.5.5 as $(x \to \infty)$

$$y(x) \sim cx^{\frac{-1}{4}} e^{2x^{\frac{1}{2}}} \tag{3.5.5}$$

Where the book gives $c = \frac{1}{2}\pi^{\frac{-1}{2}}$ on page 91. And 3.5.1 is

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \tag{3.5.1}$$

To see the improvement, the book method is followed. This is described at end of page 91. This is done by plotting the leading behavior as ratio to $y(x)$ as given in 3.5.1. Hence for the above leading behavior, we need to plot

$$\frac{\frac{1}{2}\pi^{\frac{-1}{2}} x^{\frac{-1}{4}} e^{2x^{\frac{1}{2}}}}{y(x)}$$

We are given $S_0(x), S_1(x)$ in the problem. They are

$$S_0(x) = 2x^{\frac{1}{2}}$$
$$S_1(x) = -\frac{1}{4}\ln x + c$$

Hence

$$\begin{aligned} S_0'(x) &= x^{\frac{-1}{2}} \\ S_0'' &= \frac{-1}{2}x^{-\frac{3}{2}} \\ S_1'(x) &= -\frac{1}{4}\frac{1}{x} \\ S_1''(x) &= \frac{1}{4x^2} \end{aligned} \tag{1}$$

We need to find $S_2(x), S_3(x), \cdots$ to see that this will improve the solution $y(x) \sim \exp(S_0 + S_1 + S_2 + \cdots)$ as $x \to x_0$ compared to just using leading behavior $y(x) \sim \exp(S_0 + S_1)$. So now we need to find $S_2(x)$

Let $y(x) = e^S$, then the ODE becomes

$$x\left(S'' + (S')^2\right) + S' = 1$$

Replacing S by $S_0(x) + S_1(x) + S_2(x)$ in the above gives

$$\begin{aligned} (S_0 + S_1 + S_2)'' + \left[(S_0 + S_1 + S_2)'\right]^2 + \frac{1}{x}(S_0 + S_1 + S_2)' &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left[\left(S_0' + S_1' + S_2'\right)\right]^2 + \frac{1}{x}\left(S_0' + S_1' + S_2'\right) &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left\{\left[S_0'\right]^2 + 2S_0'S_1' + 2S_0'S_2' + \left[S_1'\right]^2 + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_0' + S_1' + S_2'\right\} &\sim \frac{1}{x} \end{aligned}$$

Moving all known quantities to the RHS, these are $S_0'', S_1'', \left[S_0'\right]^2, 2S_0'S_1', S_0', S_1', \left[S_1'\right]^2$ then the above reduces to

$$\left\{S_2''\right\}+\left\{+2S_0'S_2'+2S_1'S_2'+\left[S_2'\right]^2\right\}+\frac{1}{x}\left\{S_2'\right\}\sim\frac{1}{x}-S_0''-S_1''-\left[S_0'\right]^2-2S_0'S_1'-\frac{1}{x}S_0'-\frac{1}{x}S_1'-\left[S_1'\right]^2$$

Replacing known terms, by using (1) into the above gives

$$\left\{S_2''\right\}+\left\{2S_0'S_2'+2S_1'S_2'+\left[S_2'\right]^2\right\}+\frac{1}{x}\left\{S_2'\right\}\sim$$
$$\frac{1}{x}+\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{4x^2}-\left[x^{\frac{-1}{2}}\right]^2-2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right)-\frac{1}{x}\left(x^{\frac{-1}{2}}\right)-\frac{1}{x}\left(-\frac{1}{4}\frac{1}{x}\right)-\left(-\frac{1}{4}\frac{1}{x}\right)^2$$

Simplifying gives

$$\left\{S_2''\right\}+\left\{2S_0'S_2'+2S_1'S_2'+\left[S_2'\right]^2\right\}+\frac{1}{x}\left\{S_2'\right\}\sim\frac{1}{x}+\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{4x^2}-x^{-1}+\frac{1}{2}x^{\frac{-3}{2}}-x^{\frac{-3}{2}}+\frac{1}{4}\frac{1}{x^2}-\frac{1}{16}\frac{1}{x^2}$$

Hence

$$\left\{S_2''\right\}+\left\{2S_0'S_2'+2S_1'S_2'+\left[S_2'\right]^2\right\}+\frac{1}{x}\left\{S_2'\right\}\sim-\frac{1}{16}\frac{1}{x^2}$$

Lets assume now that

$$2S_0'S_2'\sim-\frac{1}{16}\frac{1}{x^2} \tag{2}$$

Therefore

$$\begin{aligned}S_2'&\sim-\frac{1}{32}\frac{1}{S_0'x^2}\\&\sim-\frac{1}{32}\frac{1}{\left(x^{\frac{-1}{2}}\right)x^2}\\&\sim-\frac{1}{32}x^{\frac{-3}{2}}\end{aligned}$$

We can now verify this before solving the ODE. We need to check that (as $x\to\infty$)

$$\begin{aligned}2S_0'S_2'&\ggg S_2''\\2S_0'S_2'&\ggg 2S_1'S_2'\\2S_0'S_2'&\ggg\left[S_2'\right]^2\\2S_0'S_2'&\ggg\frac{1}{x}S_2'\end{aligned}$$

Where $S_2''\sim x^{\frac{-5}{2}}$, Hence

$$\begin{aligned}2S_0'S_2'&\ggg S_2''\\x^{\frac{-1}{2}}\left(x^{\frac{-3}{2}}\right)&\ggg x^{\frac{-5}{2}}\\x^{-2}&\ggg x^{\frac{-5}{2}}\end{aligned}$$

Yes.

$$
\begin{aligned}
2S_0'S_2' &\ggg 2S_1'S_2' \\
x^{-2} &\ggg \left(\frac{1}{x}\right)\left(x^{\frac{-3}{2}}\right) \\
x^{-2} &\ggg x^{\frac{-5}{2}}
\end{aligned}
$$

Yes

$$
\begin{aligned}
2S_0'S_2' &\ggg \left[S_2'\right]^2 \\
x^{-2} &\ggg \left(x^{\frac{-3}{2}}\right)^2 \\
x^{-2} &\ggg x^{-3}
\end{aligned}
$$

Yes

$$
\begin{aligned}
2S_0'S_2' &\ggg \frac{1}{x}S_2' \\
x^{-2} &\ggg \frac{1}{x}x^{\frac{-3}{2}} \\
x^{-2} &\ggg x^{\frac{-5}{2}}
\end{aligned}
$$

Yes. All assumptions are verified. Therefore we can g ahead and solve for S_2 using (2)

$$
\begin{aligned}
2S_0'S_2' &\sim -\frac{1}{16}\frac{1}{x^2} \\
S_2' &\sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{S_0'} \\
&\sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{x^{\frac{-1}{2}}} \\
&\sim -\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence

$$
\boxed{S_2 \sim \frac{1}{16}\frac{1}{\sqrt{x}}}
$$

The leading behavior now is

$$
\begin{aligned}
y(x) &\sim \exp\left(S_0 + S_1 + S_2\right) \\
&\sim \exp\left(2x^{\frac{1}{2}} - \frac{1}{4}\ln x + c + \frac{1}{16}\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Now we will find S_3. From

$$
x\left(S'' + (S')^2\right) + S' = 1
$$

Replacing S by $S_0 + S_1 + S_2 + S_3$ in the above gives

$$(S_0 + S_1 + S_2 + S_3)'' + \left[(S_0 + S_1 + S_2 + S_3)'\right]^2 + \frac{1}{x}(S_0 + S_1 + S_2 + S_3)' \sim \frac{1}{x}$$

$$\left\{S_0'' + S_1'' + S_2'' + S_3''\right\} + \left[\left(S_0' + S_1' + S_2' + S_3'\right)\right]^2 + \frac{1}{x}\left(S_0' + S_1' + S_2' + S_3'\right) \sim \frac{1}{x}$$

Hence

$$\begin{aligned}&\left\{S_0'' + S_1'' + S_2'' + S_3''\right\} + \\ &\quad \left\{\left[S_0'\right]^2 + 2S_0'S_1' + 2S_0'S_2' + 2S_1'S_2' + \left[S_1'\right]^2 + \left[S_2'\right]^2 + 2S_0'S_3' + 2S_1'S_3' + 2S_2'S_3' + \left[S_3'\right]^2\right\} \\ &\qquad + \frac{1}{x}\left\{S_0' + S_1' + S_2' + S_3'\right\} \sim \frac{1}{x}\end{aligned}$$

Moving all known quantities to the RHS gives

$$\begin{aligned}&\left\{S_3''\right\} + \left\{2S_0'S_3' + 2S_1'S_3' + 2S_2'S_3' + \left[S_3'\right]^2\right\} + \frac{1}{x}\left\{S_3'\right\} \\ &\sim \frac{1}{x} - S_0'' - S_1'' - S_2'' - \left[S_0'\right]^2 - 2S_0'S_1' - 2S_0'S_2' - 2S_1'S_2' - \left[S_1'\right]^2 - \left[S_2'\right]^2 - \frac{1}{x}S_0' - \frac{1}{x}S_1' - \frac{1}{x}S_2' \end{aligned} \tag{3}$$

Now we will simplify the RHS, since it is all known. Using

$$\begin{aligned}S_0'(x) &= x^{\frac{-1}{2}} \\ \left[S_0'\right]^2 &= x^{-1} \\ S_0'' &= \frac{-1}{2}x^{-\frac{3}{2}} \\ S_1'(x) &= -\frac{1}{4}\frac{1}{x} \\ \left[S_1'(x)\right]^2 &= \frac{1}{16}\frac{1}{x^2} \\ S_1''(x) &= \frac{1}{4x^2} \\ S_2'(x) &= -\frac{1}{32}\frac{1}{x^{\frac{3}{2}}} \\ \left[S_2'(x)\right]^2 &= \frac{1}{1024x^3} \\ S_2''(x) &= \frac{3}{64}\frac{1}{x^{\frac{5}{2}}} \\ 2S_0'S_1' &= 2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right) = -\frac{1}{2}\frac{1}{x^{\frac{3}{2}}} \\ 2S_0'S_2' &= 2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right) = -\frac{1}{16x^2} \\ 2S_1'S_2' &= 2\left(-\frac{1}{4}\frac{1}{x}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right) = \frac{1}{64x^{\frac{5}{2}}}\end{aligned}$$

Hence (3) becomes

$$\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}\sim$$
$$\frac{1}{x}+\frac{1}{2x^{\frac{3}{2}}}-\frac{1}{4x^2}-\frac{3}{64}\frac{1}{x^{\frac{5}{2}}}-\frac{1}{x}+\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}+\frac{1}{16x^2}-\frac{1}{64x^{\frac{5}{2}}}-\frac{1}{16}\frac{1}{x^2}-\frac{1}{1024x^3}-\frac{1}{x^{\frac{3}{2}}}+\frac{1}{4}\frac{1}{x^2}+\frac{1}{32}\frac{1}{x^{\frac{5}{2}}}$$

Simplifying gives

$$\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}\sim-\left(\frac{32}{1024x^{\frac{5}{2}}}+\frac{1}{1024x^3}\right)$$

Let us now assume that

$$\begin{aligned}S_0'S_3'&\ggg S_1'S_3'\\ S_0'S_3'&\ggg S_2'S_3'\\ S_0'S_3'&\ggg\left[S_3'\right]^2\\ S_0'S_3'&\ggg\frac{1}{x}\left\{S_3'\right\}\\ S_0'S_3'&\ggg S_3''\end{aligned}$$

Therefore, we end up with the balance

$$\begin{aligned}2S_0'S_3'&\sim-\left(\frac{32}{1024x^{\frac{5}{2}}}+\frac{1}{1024x^3}\right)\\ S_3'&\sim-\left(\frac{32}{1024x^{\frac{5}{2}}S_0'}+\frac{1}{1024x^3S_0'}\right)\\ &\sim-\left(\frac{32}{1024x^{\frac{5}{2}}\left(x^{\frac{-1}{2}}\right)}+\frac{1}{1024x^3\left(x^{\frac{-1}{2}}\right)}\right)\\ &\sim-\left(\frac{1}{32x^2}+\frac{1}{1024x^{\frac{5}{2}}}\right)\end{aligned}$$

Hence

$$\boxed{S_3\sim\frac{1}{1024}\left(\frac{2}{32x^2}+\frac{32}{x}\right)}$$

Where constant of integration was ignored. Let us now verify the assumptions made

$$\begin{aligned}S_0'S_3'&\ggg S_1'S_3'\\ S_0'&\ggg S_1'\end{aligned}$$

Yes.

$$\begin{aligned}S_0'S_3'&\ggg S_2'S_3'\\ S_0'&\ggg S_2'\end{aligned}$$

Yes

$$S_0'S_3' \ggg \left[S_3'\right]^2$$
$$S_0' \ggg S_3'$$

Yes.

$$S_0'S_3' \ggg \frac{1}{x}\left\{S_3'\right\}$$
$$S_0' \ggg \frac{1}{x}$$
$$x^{\frac{-1}{2}} \ggg \frac{1}{x}$$

Yes, as $x \to \infty$, and finally

$$S_0'S_3' \ggg S_3''$$
$$x^{\frac{-1}{2}}\left(\frac{1}{32x^2} + \frac{1}{1024x^{\frac{5}{2}}}\right) \ggg \left(\frac{5}{2048x^{\frac{7}{2}}} + \frac{1}{16x^3}\right)$$
$$\frac{\left(32\sqrt{x}+1\right)}{1024x^3} \ggg \frac{128\sqrt{x}+5}{2048x^{\frac{7}{2}}}$$

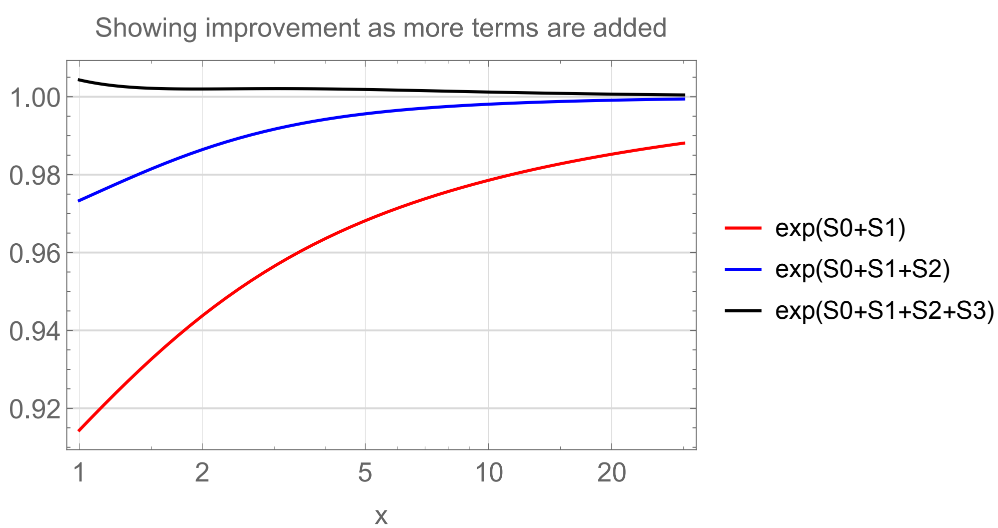
Yes, as $x \to \infty$. All assumptions verified. The leading behavior now is

$$\begin{aligned} y(x) &\sim \exp\left(S_0 + S_1 + S_2 + S_3\right) \\ &\sim \exp\left(2x^{\frac{1}{2}} - \frac{1}{4}\ln x + c + \frac{1}{16}\frac{1}{\sqrt{x}} + \frac{1}{1024}\left(\frac{2}{32x^2} + \frac{32}{x}\right)\right) \\ &\sim cx^{\frac{-1}{4}}\exp\left(2x^{\frac{1}{2}} + \frac{1}{16}\frac{1}{\sqrt{x}} + \frac{1}{1024}\left(\frac{2}{32x^2} + \frac{32}{x}\right)\right) \end{aligned}$$

Now we will show how adding more terms to leading behavior improved the $y(x)$ solution for large x. When plotting the solutions, we see that $\frac{\exp(S_0+S_1+S_2+S_3)}{y(x)}$ approached the ratio 1 sooner than $\frac{\exp(S_0+S_1+S_2)}{y(x)}$ and this in turn approached the ratio 1 sooner than just using $\frac{\exp(S_0+S_1)}{y(x)}$. So the effect of adding more terms, is that the solution becomes more accurate for larger range of x values. Below is the code used and the plot generated.

```
ClearAll[y, x];
s0[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2)])/y[x, 300];
s0s1[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x]])/y[x, 300];
s0s1s2[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x]])/y[x, 300];
s0s1s2s3[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x] + 1/1024 (2/(32 x^2) + 32/x)])/y[x, 300];
y[x_, max_] := Sum[ x^n / (Factorial[n]^2), {n, 0, max}];
LogLinearPlot[Evaluate[{s0s1[x], s0s1s2[x], s0s1s2s3[x]}], {x, 1, 30},
 PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 PlotLegends → {"exp(S0+S1)", "exp(S0+S1+S2)", "exp(S0+S1+S2+S3)"},
 FrameLabel → {{None, None}, {"x", "Showing improvement as more terms are added"}},
 PlotStyle → {Red, Blue, Black}, BaseStyle → 14]
```

Out[59]=



3.2.7 problem 3.49(c)

Problem Find the leading behavior as $x \to \infty$ of the general solution of $y'' + xy = x^5$

Solution This is non-homogenous ODE. We solve this by first finding the homogenous solution (asymptotic solution) and then finding particular solution. Hence we start with

$$y_h'' + xy_h = 0$$

$x = \infty$ is ISP point. Therefore, we assume $y_h(x) = e^{S(x)}$ and obtain

$$S'' + (S')^2 + x = 0 \tag{1}$$

Let

$$S(x) = S_0 + S_1 + \cdots$$

Therefore (1) becomes

$$\left(S_0'' + S_1'' + \cdots\right) + \left(S_0' + S_1' + \cdots\right)^2 = -x$$
$$\left(S_0'' + S_1'' + \cdots\right) + \left(\left[S_0'\right]^2 + 2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) = -x \tag{2}$$

Assuming $\left[S_0'\right]^2 \ggg S_0''$ we obtain

$$\left[S_0'\right]^2 \sim -x$$
$$S_0' \sim \omega\sqrt{x}$$

Where $\omega = \pm i$

Verification

$$\left[S_0'\right]^2 \ggg S_0''$$
$$x \ggg \frac{1}{2}\frac{1}{\sqrt{x}}$$

Yes, as $x \to \infty$. Hence

$$S_0 \sim \frac{3}{2}\omega x^{\frac{3}{2}}$$

Now we will find S_1. From (2), and moving all known terms to RHS

$$\left(S_1'' + \cdots\right) + \left(2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) \sim -x - S_0'' - \left[S_0'\right]^2 \tag{3}$$

Assuming

$$2S_0'S_1' \ggg S_1''$$
$$2S_0'S_1' \ggg \left[S_1'\right]^2$$

Then (3) becomes (where $S_0' \sim \omega\sqrt{x}, \left[S_0'\right]^2 \sim -x, S_0'' \sim \frac{1}{2}\omega\frac{1}{\sqrt{x}}$)

$$\begin{aligned}
2S_0'S_1' &\sim -x - S_0'' - \left[S_0'\right]^2 \\
S_1' &\sim \frac{-x - S_0'' - \left[S_0'\right]^2}{2S_0'} \\
&\sim \frac{-x - \frac{1}{2}\omega\frac{1}{\sqrt{x}} - (-x)}{2\omega\sqrt{x}} \\
&\sim -\frac{1}{4x}
\end{aligned}$$

Verification (where $S_1'' \sim \frac{1}{4}\frac{1}{x^2}$)

$$\begin{aligned}
2S_0'S_1' &\ggg S_1'' \\
\sqrt{x}\left(\frac{1}{4x}\right) &\ggg \frac{1}{4}\frac{1}{x^2} \\
\frac{1}{x^{\frac{1}{2}}} &\ggg \frac{1}{x^2}
\end{aligned}$$

Yes, as $x \to \infty$

$$\begin{aligned}
2S_0'S_1' &\ggg \left[S_1'\right]^2 \\
\frac{1}{x^{\frac{1}{2}}} &\ggg \left(\frac{1}{4x}\right)^2 \\
\frac{1}{x^{\frac{1}{2}}} &\ggg \frac{1}{16x^2}
\end{aligned}$$

Yes, as $x \to \infty$. All validated. We solve for S_1

$$\begin{aligned}
S_1' &\sim -\frac{1}{4x} \\
S_1 &\sim -\frac{1}{4}\ln x + c
\end{aligned}$$

y_h is found. It is given by

$$\begin{aligned}
y_h(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\
&\sim \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}} - \frac{1}{4}\ln x + c\right) \\
&\sim c x^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right)
\end{aligned}$$

Now that we have found y_h, we go back and look at

$$y'' + xy = x^5$$

And consider two cases (a) $y'' \sim x^5$ (b) $xy \sim x^5$. The case of $y'' \sim xy$ was covered above. This is what we did to find $y_h(x)$.

case (a)

$$
\begin{aligned}
y_p'' &\sim x^5 \\
y_p' &\sim \frac{1}{5}x^4 \\
y_p &\sim \frac{1}{20}x^3
\end{aligned}
$$

Where constants of integration are ignored since subdominant for $x \to \infty$. Now we check if this case is valid.

$$
\begin{aligned}
xy_p &\lll x^5 \\
x\frac{1}{20}x^3 &\lll x^5 \\
x^4 &\lll x^5
\end{aligned}
$$

No. Therefore case (a) did not work out. We try case (b) now

$$
\begin{aligned}
xy_p &\sim x^5 \\
y_p &\sim x^4
\end{aligned}
$$

Now we check is this case is valid.

$$
\begin{aligned}
y_p'' &\lll x^5 \\
12x^2 &\lll x^5
\end{aligned}
$$

Yes. Therefore, we found

$$
y_p \sim x^4
$$

Hence the complete asymptotic solution is

$$
\begin{aligned}
y(x) &\sim y_h(x) + y_p(x) \\
&\sim cx^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right) + x^4
\end{aligned}
$$

3.2.8 key solution of selected problems

3.2.8.1 section 3 problem 27

(1)

3.27 $\frac{d^{(n)}}{dx^{(n)}} y = Q(x) y \quad x \to x_0$ ISP

Assume $Q(x) > 0$ for simplicity

Lets do 3rd-order case 1st:

$y''' = Q(x) y$, $y = \exp[S(x)]$, $S \sim S_0 + S_1 + S_2 + \cdots$

$y' = S' e^S$, $y'' = S'' e^S + (S')^2 e^S$

$y''' = \left[S''' + (S')^3 + 3 S'' S' \right] e^S$

Try the dominant balance $(S_0')^3 \sim Q(x) \quad x \to x_0$

$\Rightarrow S_0' \sim \omega [Q(x)]^{1/3} \quad \omega = (1)^{1/3}$

$\Rightarrow S_0 \sim \int^x \omega [Q(t)]^{1/3} dt + C_0$

Lets assume that all dropped terms are smaller than the terms we kept.

Continuing on ...

(2)

$$\left\{ S_0''' + S_1''' + \cdots \right\} + \left\{ (S_0')^3 + 3(S_0')^2 S_1' + \cdots \right\}$$

$$+ 3\left\{ S_0'' S_0' + S_0'' S_1' + S_0' S_1'' + \cdots \right\} \sim \phi(x)$$

Next balance $3(S_0')^2 S_1' \sim -3 S_0'' S_0'$

$$\Rightarrow \quad S_1' \sim -\frac{S_0''}{S_0'} \quad \Rightarrow \quad S_1 \sim -\ln|S_0'| + C_1$$

$$\Rightarrow \quad S_1 \sim -\ln\left[\omega\, \phi(x)^{1/3}\right] + C_1$$

$$y(x) \sim C^* \phi(x)^{-1/3} \exp\left[\int^x \omega\, \phi(t)^{1/3}\, dt\right]$$

where constants have been absorbed into C^*.

Check BO formula for $n = 3$

$$\frac{(1-n)}{2n} = \frac{(1-3)}{6} = -\frac{1}{3} \quad \checkmark$$

$$\frac{1}{n} = \frac{1}{3} \quad \checkmark$$

(3)

Now try 4th-order equation

$$y^{(IV)} = \{S^{(IV)} + (S')^4 + 4S'''S' + 6(S')^2S'' + 3S''S''\}e^S$$

The dominant balance will be $(S_0')^4 \sim \varphi(x)$

$\Rightarrow S_0' \sim \omega\, \varphi(x)^{1/4}, \quad \omega = (1)^{1/4}$

$\Rightarrow S_0 \sim \int^x \omega\, \varphi(t)^{1/4} dt + C_0$

Assuming consistency, the next order will give

$$4(S_0')^3 S_1' \sim -6(S_0')^2 S_0''$$

$$S_1' \sim -\frac{6}{4} \frac{S_0''}{S_0'}$$

$$S_1 \sim -\frac{6}{4} \ln|S_0'| + C_1$$

$$\sim -\frac{6}{4} \ln\left[\omega\, \varphi(x)^{1/4}\right] + C_1$$

(4)

Note that we supress higher-order derivatives in favor of lower-order derivatives raised to some power.

$$y \sim C^* \left[Q(x)^{1/4} \right]^{-6/4} \exp\left[\int^x \omega\, Q(t)^{1/4}\, dt \right]$$

$x \to x_0$

$$y \sim C^* \left[Q(x) \right]^{-3/8} \exp\left[\int^x \omega Q(t)^{1/4}\, dt \right]$$

Check BO formula for $n=4$

$\frac{1-n}{2n} = \frac{1-4}{8} = \frac{-3}{8}$ ✓ $\frac{1}{n} = 1/4$ ✓

Generalizing to larger $n \Rightarrow$

$$y(x) \sim C^* \left[Q(x) \right]^{\frac{(1-n)}{2n}} \exp\left[\int^x \omega\, Q(t)^{1/n}\, dt \right]$$

$x \to x_0$

3.2.8.2 section 3 problem 33 b

NEEP 548: Engineering Analysis II 2/6/2011

Problem 3.33: Bender & Orszag

Instructor: Leslie Smith

1 Problem Statement

Find the leading behavoir of the following equation as $x \to 0^+$:

$$x^4 y''' - 3x^2 y' + 2y = 0 \tag{1}$$

2 Solution

Need to make the substitution: $y = e^{S(x)}$

$$\begin{aligned}
y' &= S' e^{S(x)} \\
y'' &= S'' e^{S(x)} + (S')^2 e^{S(x)} \\
y''' &= S''' e^{S(x)} + 3S'' S' e^{S(x)} + (S')^3 e^{S(x)}
\end{aligned}$$

After substituting and diving through by y, one obtains,

$$S''' + 3S''S' + (S')^3 - \frac{3}{x^2} S' + \frac{2}{x^4} \sim 0 \tag{2}$$

Typically, $S'' << (S')^2 \implies S''' << (S')^3$. Similarly, assume $S''' << (S')^3$
With these assumptions, 2 becomes:

$$(S')^3 - \frac{3}{x^2} S' \sim -\frac{2}{x^4}$$

which is still a difficult problem to solve. Therefore, also want to assume $(S')^3 >> \frac{3}{x^2} S'$ as $x \to 0^+$.
Then,

$$\begin{aligned}
(S')^3 &\sim \frac{-2}{x^4} && x \to 0^+ \\
S' &\sim w x^{-4/3} && x \to 0^+ \\
w &= (-2)^{1/3} \\
S_o(x) &\sim -3wx^{-1/3} && x \to 0^+ \\
S'' &\sim -\frac{4w}{3} x^{-7/3} \\
S''' &\sim \frac{28w}{9} x^{-10/3}
\end{aligned}$$

1

Now, check our assumptions,

$$(S')^3 \sim -2x^{-12/3} >> -3S'x^{-2} \sim -3wx^{-10/3} \qquad x \to 0^+ \qquad \checkmark$$
$$(S')^3 \sim -2x^{-12/3} >> S''' \sim \frac{28w}{9}x^{-10/3} \qquad x \to 0^+ \qquad \checkmark$$
$$(S')^3 \sim -2x^{-12/3} >> S''S' \sim \frac{-4w^2}{3}x^{-11/3} \qquad x \to 0^+ \qquad \checkmark$$

Now estimate the inegrating function $C(x)$ by letting,

$$S(x) = S_o(x) + C(x) \tag{3}$$

and substitute this into 2.

$$S_o''' + C''' + 3\underbrace{(S_o'' + C'')(S_o' + C')}_{\text{term 1}} + \underbrace{(S_o' + C')^3}_{\text{term 2}} - \frac{3}{x^2}(S_o' + C') + \frac{2}{x^4} = 0 \tag{4}$$

term 1: $(S_o''S_o' + C''S_o' + S_o''C' + C''C')$

term 2: $(S_o')^3 + (C')^3 + 3(S')^2C' + 3(S')(C')^2$

From here, notice that we have already balanced the terms $(S_o')^3$ and $\frac{-2}{x^4}$, so they are removed at this point. Now we make the following assumptions:

$$S_o' >> C', \qquad S_o'' >> C'', \qquad S_o''' >> C''' \qquad \text{as} \qquad x \to 0^+$$

These assumptions result in,

$$S_o''' + 3S_o''S_o' + 3(S_o')^2C' \sim \frac{3S_o'}{x^2} + \frac{3C'}{x^2} \qquad x \to 0^+$$

Insert the value of S_o found in the previous step,

$$\frac{28w}{9}x^{-10/3} - 4w^2x^{-11/3} + 3w^2x^{-8/3}C' \sim 3wx^{-10/3} + 3C'x^{-6/3}$$

In keeping with the dominant balance idea, it is clear that the middle two terms dominate, thus leading to the following simplified relation,

$$3w^2x^{-8/3}C' \sim 4w^2x^{-11/3} \qquad x \to 0^+$$
$$C' \sim \frac{4}{3}x^{-1} \qquad x \to 0^+$$
$$C \sim \frac{4}{3}ln(x) \qquad x \to 0^+$$

Then $C'' \sim \frac{-4}{3}x^{-2}$, $C''' \sim \frac{8}{3}x^{-3}$ and once again, check assumptions made,

2

$$S_o' \sim wx^{-4/3} >> C' \sim \frac{4}{3}x^{-1} \qquad x \to 0^+ \qquad \checkmark$$
$$S_o'' \sim \frac{-4}{3}x^{-7/3} >> C'' \sim \frac{-4}{3}x^{-2} \qquad x \to 0^+ \qquad \checkmark$$
$$S_o''' \sim \frac{28w}{9}x^{-10/3} >> C''' \sim \frac{8}{3}x^{-3} \qquad x \to 0^+ \qquad \checkmark$$

With the appearance of the ln(x) term, we have likely found the leading behavior already. However, lets find D, the third term, just to check.

Substitute $y = e^{(S_o+C_o+D)}$, then divide by y,

$$y' = [S_o' + C_o' + D']e^{(S_o+C_o+D)}$$
$$y'' = [S_o'' + C_o'' + D'']e^{(S_o+C_o+D)} + [S_o' + C_o' + D']^2 e^{(S_o+C_o+D)}$$
$$y''' = [S_o''' + C_o''' + D''']e^{(S_o+C_o+D)} + 3[S_o'' + C_o'' + D''][S_o' + C_o' + D']e^{(S_o+C_o+D)} + [S_o' + C_o' + D']^3 e^{(S_o+C_o+D)}$$

And...here we go...

$$S_o''' + \cancel{C_o'''} + \cancel{D'''} + 3[\cancel{S_o''S_o'} + S_o''C_o' + \cancel{S_o''D'} + C_o''S_o' + \cancel{C_o''C_o'} + \cancel{C_o''D'} + \cancel{D''S_o'} + \cancel{D''C_o'} + \cancel{D''D'}]$$

$$+ \cancel{6S_o'C_o'D'} + 3[\cancel{(S_o')^2C_o'} + \underbrace{(S_o')^2D' + (C_o')^2S_o'}_{\text{Unknown balance}} + \cancel{(C_o')^2D'} + \cancel{(D')^2S_o'} + \cancel{(D')2C_o'}] + \cancel{(S_o')^3}$$

$$+ \cancel{(C_o')^3} + \cancel{(D')^3} - \frac{3}{x^2}[S_o' + C_o' + D'] + \cancel{\frac{2}{x^4}} = 0$$

Assumptions: $S_o' >> C_o' >> D'$, $S_o'' >> C_o'' >> D''$, $S_o''' >> C_o''' >> D'''$ all as $x \to 0^+$ and remove previously balanced terms,

$$S_o''' + 3S_o''C_o' + 3C_o''S_o' + 3(S_o')^2D' \sim -3(C_o')S_o' \qquad x \to 0^+$$

Subbing in the previously found values for S_o, C_o,

$$-3wx^{-10/3} + \frac{28w}{9}x^{-10/3} - \cancel{\frac{16w}{3}x^{-10/3}} - 4wx^{-10/3} + 3w^2x^{-8/3}D' \sim \cancel{\frac{-16w}{3}x^{-10/3}}$$

3

$$\begin{aligned}
-3w^2x^{-8/3}D' &\sim \frac{35w}{9}x^{-10/3}\\
D' &\sim \frac{35}{27w}x^{-2/3}\\
D &\sim \frac{35}{27w}(\frac{1}{3})x^{1/3} + d
\end{aligned}$$

So since $x^{1/3} \to 0$ as $x \to 0^+$, the leading order behavior is

$$\boxed{y \sim \exp[-\frac{w}{3}x^{-1/3} + \frac{4}{3}\ln(x) + d] \qquad x \to 0^+}$$

4

3.2.8.3 section 3 problem 33 c

(1)

3.33 c $\quad y'' = x^{\frac{1}{2}} y \quad x \to 0^+$

$$\{S_0'' + S_1'' + \cdots\} + \{(S_0')^2 + 2S_0'S_1' + \cdots\} \sim x^{\frac{1}{2}}$$

Try 2-term balances

[1] $(S_0')^2 \sim x^{\frac{1}{2}} \quad x \to 0^+ \ ; \quad S_0' \sim \pm x^{1/4} \ ;$

$S_0 \sim \pm \frac{4}{5} x^{5/4} + C_0$

Check $(S_0')^2 \gg S_0''$?

$x^{\frac{1}{2}} \gg \pm \frac{1}{4} x^{-3/4}$ NO inconsistent

[2] $S_0'' \sim x^{\frac{1}{2}} \quad x \to 0^+ \ ; \quad S_0' \sim \frac{2}{3} x^{\frac{3}{2}} + C_0 \ ;$

$S_0 \sim \frac{2}{3} \cdot \frac{2}{5} x^{5/2} + C_0 x + C_1$

Check $S_0'' \gg (S_0')^2$?

$x^{\frac{1}{2}} \gg \frac{4}{9} x^3 + \frac{4}{3} C_0 x^{\frac{3}{2}} + C_0^2$

NO inconsistent

(2)

3 $S_0'' \sim -(S_0')^2 \quad x \to 0^+$

Let $S_0' = z \Rightarrow z' = -z^2$; $\frac{dz}{z^2} = -dx$;

$-z^{-1} = -x + C \Rightarrow z = \frac{1}{x+D} \Rightarrow$

$S_0 \sim \ln|x+D| + E$

Check $(S_0')^2 \gg x^{\frac{1}{2}} \quad x \to 0^+$?

$\frac{1}{(x+D)^2} \gg x^{\frac{1}{2}} \quad x \to 0^+$

True for both $D=0$ and $D \neq 0$

So the controlling factor is

$y \sim \exp[S_0] \sim \exp[\ln|x+D| + E]$

$y \sim Ax + B$

(the 1st term in 2 linearly independent solutions)

$y_1 \sim Ax \quad y_2 \sim B$

(3)

Since the lowest order approximation has 2 coefficients A, B to be determined by boundary/initial conditions, from now on we set integration coefficients to zero.

CASE 1) $y_1 \sim Ax \sim A\exp[\ln x]$, $S_0 \sim \ln x$

$$\{ S_0'' + S_1'' + \cdots \} + \{ (S_0')^2 + 2S_0'S_1' + \cdots \} \sim x^{1/2} \quad x \to 0^+$$

Try $S_1'' + 2S_0'S_1' \sim x^{1/2}$, $S_0' \sim \frac{1}{x}$

$$\Rightarrow S_1'' + \frac{2}{x}S_1' \sim x^{1/2}$$

integrating factor $\mu(x) = \exp\left[\int \frac{2}{x}dx\right] = x^2$

$$\frac{d}{dx}\left[x^2 S_1'\right] \sim x^2 x^{1/2} \sim x^{5/2}$$

$$x^2 S_1' \sim \frac{2}{7}x^{7/2} \; ; \; S_1' \sim \frac{2}{7}x^{3/2}$$

$$\Rightarrow S_1 \sim \frac{2}{7}\frac{2}{5}x^{5/2} \sim \frac{4}{35}x^{5/2}$$

(4)

$$y_1 \sim A \exp\left[\ln x + \frac{4}{35} x^{5/2} + \cdots\right]$$

Check: $(S_1')^2 \ll x^{1/2} \quad x \to 0^+$?

$$x^3 \ll x^{1/2} \quad x \to 0^+ \quad \text{TRUE}$$

CASE 2 $y_2 \sim B \sim \exp[C]$, $S_0 \sim C$ constant

$$\{S_0'' + S_1'' + \cdots\} + \{(S_0')^2 + 2S_0'S_1' + \cdots\} \sim x^{1/2} \quad x \to 0^+$$

Try $S_1'' + 2S_0'S_1' \sim x^{1/2} \Rightarrow S_1'' \sim x^{1/2}$

$$\Rightarrow S_1' \sim \frac{2}{3} x^{3/2} \Rightarrow S_1 \sim \frac{4}{15} x^{5/2}$$

with check $(S_1')^2 \ll x^{1/2}$ TRUE

$$y_2 \sim \exp\left[C + \frac{4}{15} x^{5/2} + \cdots\right]$$

$$y = y_1 + y_2 \sim Ax \exp\left[\frac{4}{35} x^{5/2} + \cdots\right] + B \exp\left[\frac{4}{15} x^{5/2} + \cdots\right]$$

$$\sim Ax\left\{1 + \frac{4}{35} x^{5/2} + \cdots\right\}$$

$$+ B\left\{1 + \frac{4}{15} x^{5/2} + \cdots\right\}$$

(5)

$y = y_1 + y_2$

$$Ax \sum_{n=0}^{\infty} a_n x^{5n/2} + B \sum_{n=0}^{\infty} b_n x^{5n/2}$$

$a_0 = 1$, $a_1 = \frac{4}{35}$ $b_0 = 1$, $b_1 = \frac{4}{15}$

continue on to find a_n, $n \geq 2$ b_n, $n \geq 2$

Note: Balance $\boxed{2}$ would be consistent if we drop integration constants, but then we generate only solution y_2; use reduction of order to find y_1 ...

3.2.8.4 section 3 problem 35

Some notes on BO 3.35 (to get the leading behavior):

- a consistent balance at lowest order seems to be

$$(S_o')^2 \sim \exp(2/x), \quad x \to 0$$

- taking the square root and integrating leads to

$$S_o \sim \pm \int_1^x \exp(1/s) ds$$

- then change variables $s = 1/t$ to find

$$S_o \sim \pm \int_1^{(1/x)} \exp(t) \left(-\frac{1}{t^2} \right) dt$$

- integration by parts gives

$$S_o \sim \mp x^2 \exp(1/x) + \pm a_o + \mp \int_1^{(1/x)} \frac{2}{t^3} \exp(t) dt$$

$$\sim \mp x^2 \exp(1/x)$$

  and subdominant terms have been dropped. [Why are these terms subdominant and why is this the only consistent integration by parts?]

3.2.8.5 section 3 problem 42a

[illegible] 3.42 a SOLUTION

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

This satisfies the 2nd order D.E.

[1] $xy'' + y' = y$ w/ ISP @ $x = \infty$

Let $y = e^{S(x)}$

$y' = S' e^S$

$y'' = S'' e^S + (S')^2 e^S$

Plug into [1].

[2] $xS'' + x(S')^2 + S' - 1 = 0$

Assume,

[a1] $xS'' \ll x(S')^2$ as $x \to \infty$

$\Rightarrow$ $x(S')^2 + S' \sim 1$ as $x \to \infty$

or $S'(x) \sim [-1 \pm \sqrt{1+4x}]/2x$

For large x,

$S'(x) \sim \pm x^{-1/2}$ $x \to \infty$

Notice $y(x) = \sum_0^\infty \frac{x^n}{(n!)^2}$ is always (+) for x-large

So, only keep positive sign solution.

$S(x) \sim 2x^{1/2}$ $x \to \infty$

[3] or $S(x) = 2x^{1/2} + C(x)$ where $C(x) \ll 2x^{1/2}$ for $x \to \infty$

[a2] $C(x) \ll 2x^{1/2}$

1

Plug [3] into [2],

$$S' = x^{-1/2} + c'$$
$$S'' = -\tfrac{1}{2} x^{-3/2} + c''$$

$$\Rightarrow \quad -\tfrac{1}{2} x^{-1/2} + xc'' + 1 + 2x^{1/2}c' + x(c')^2 + x^{-1/2} + c' = 1$$

$$\underbrace{xc''}_{a} + \underbrace{x(c')^2}_{b} + (\underbrace{2x^{1/2}}_{c} + \underbrace{1}_{d})c' + \underbrace{\tfrac{1}{2}x^{-1/2}}_{e} = 0$$

From [a2]: $c \ll 2x^{1/2}$

$c' \ll x^{-1/2} \Rightarrow \underbrace{x(c')^2}_{b} \ll \underbrace{x^{1/2}c'}_{c}$ & $\underbrace{c'}_{d} \ll \underbrace{\tfrac{1}{2}x^{-1/2}}_{e}$

$c'' \ll -\tfrac{1}{2} x^{-3/2} \Rightarrow \underbrace{xc''}_{a} \ll \underbrace{-\tfrac{1}{2} x^{-1/2}}_{e}$

So balance becomes,

[3] $2x^{1/2}c' \sim -\tfrac{1}{2} x^{-1/2}$ or $c' \sim -\tfrac{1}{4} x^{-1}$

$$\Rightarrow \quad c(x) \sim -\tfrac{1}{4} \ln x \qquad x \to \infty$$

or $\quad c(x) = -\tfrac{1}{4} \ln x + D(x)$

where

[a3] $\quad D(x) \ll -\tfrac{1}{4} \ln x$

$$\therefore \quad y(x) = e^{S+C+D} = c\, x^{-1/4} e^{[2x^{1/2} + D(x)]}$$

Now look for $y(x) = e^{S+C+D}$ to find D.

$$y' = [S_0' + C_0' + D']e^{(S_0+C_0+D)}$$

$$y'' = [S_0'' + C_0'' + D'']e^{(S_0+C_0+D)} + (S_0' + C_0' + D')^2 e^{(S_0+C_0+D)}$$

$$= [S_0'' + C_0'' + D'' + (S_0')^2 + (C_0')^2 + (D')^2 + 2S_0'C_0' + 2S_0'D' + 2C_0'D']e^{(S_0+C_0+D)}$$

Plug into [1];

$$xS_0''^{[3]} + xC_0'' + xD'' + x(S_0')^{2\,[2]} + x(C_0')^2 + x(D')^2 + 2xS_0'C_0'^{[3]} + 2xS_0'D' + 2xC_0'D'$$

$$+ S_0'^{[2]} + C_0' + D' - 1^{[2]} = 0$$

[2] Gives $x(S_0')^2 + S_0' - 1 \sim 0$

[3] Gives $C' \sim -\frac{1}{4}x^{-1}$ or $x(S_0'' + 2S_0'C_0') = 0$

$$xC_0'' + xD'' + x(C_0')^2 + x(D')^2 + 2xS_0'D' + 2xC_0'D' + C_0' + D' = 0$$

$$S_0 = 2x^{1/2} \qquad C_0 = -\frac{1}{4}\ln x$$

$$S_0' = x^{-1/2} \qquad C_0' = -\frac{1}{4}x^{-1}$$

$$C_0'' = \frac{1}{4}x^{-2}$$

$$\Rightarrow \frac{1}{4}x^{-1} + xD'' + \frac{1}{16}x^{-1} + x(D')^2 + 2x\,x^{-1/2}D' + 2x\left(-\frac{1}{4}x^{-1}\right)D' + \left(-\frac{1}{4}x^{-1}\right) + D' = 0$$

$$xD'' + x(D')^2 + 2x^{1/2}D' + D' - \frac{1}{2}D' + \frac{1}{16}x^{-1} + \frac{1}{4}x^{-1} - \frac{1}{4}x^{-1} = 0$$

$$\underset{(a)}{xD''} + \underset{(b)}{x(D')^2} + \underset{(c)}{(2x^{1/2}} + \underset{(d)}{\tfrac{1}{2})D'} + \underset{(e)}{\tfrac{1}{16}x^{-1}} = 0$$

From [a8]

$D \ll -\frac{1}{4}\ln x$

$D' \ll -\frac{1}{4}x^{-1}$ → $xD'^2 \ll -\frac{1}{4}D'$ → (b) ≪ (d)

$D'' \ll \frac{1}{4}x^{-2}$ → $xD'' \ll x^{-1}$ or (a) ≪ (e)

$\frac{1}{2} \ll 2x^{1/2}$ for $x \to \infty$ or (d) ≪ (c)

3

Therefore,

{5} $2x^{1/2}D' \sim -\frac{1}{16}x^{-1}$

$D' \sim -\frac{1}{32}x^{-3/2}$ ∴ $\boxed{D \sim \frac{1}{16}x^{-1/2}}$

or $D = \frac{1}{16}x^{-1/2} + E(x)$ where $E(x) << \frac{1}{16}x^{-1/2}$

{a4} $E(x) << \frac{1}{16}x^{-1/2}$

Now let $S(x) = S_0 + C_0 + D_0 + E(x)$ $D_0 = \frac{1}{16}x^{-1/2}$

Thus, plugging into [1] gives,

$x S_0'' + x C_0'' + x D_0'' + x E'' + x(S_0')^2 + x(C_0')^2 + x(D_0')^2 + x(E')^2$
$+ 2xS_0'C_0' + 2xS_0'D_0' + 2xC_0'D_0' + 2xS_0'E' + 2xC_0'E' + 2xD_0'E'$
$+ S_0' + C_0' + D_0' + E' = 1$

(crossed-out terms annotated {3}, {5}, {2}, {5}, {3}, {5}, {2}, {5}, {2})

Removing previous balances, including most recent balance from

{5} → $2x^{1/2}D' \sim -\frac{1}{16}x^{-1}$ or $x(C_0'' + (C_0')^2 + 2S_0'D_0') + C_0' \sim 0$

{2} → $x(S_0')^2 + S_0' - 1 \sim 0$

{3} → $c' + \frac{1}{4}x^{-1} \sim 0$

Results in,

$$xD_0'' + xE'' + x(D_0')^2 + x(E')^2 + 2xC_0'D_0' + 2xS_0'E' + 2xC_0'E' + 2xD_0'E' + D_0' + E' = 0$$

$D_0' = -\frac{1}{32}x^{-3/2}$ $C_0' = -\frac{1}{4}x^{-1}$ $S_0' = x^{-1/2}$

$D_0'' = \frac{3}{64}x^{-5/2}$ $C_0'' = \frac{1}{4}x^{-2}$

$$\Rightarrow \frac{3}{64}x^{-3/2} + xE'' + \frac{1}{1024}x^{-2} + x(E')^2 + 2x\left(-\frac{1}{4}x^{-1}\right)\left(-\frac{1}{32}x^{-3/2}\right) + 2x\cdot x^{-1/2}E' + 2x\left(-\frac{1}{4}x^{-1}\right)E'$$
$$+ 2x\left(-\frac{1}{32}x^{-3/2}\right)E'$$
$$+ \left(-\frac{1}{32}x^{-3/2}\right) + E' = 0$$

4

$$\Rightarrow \quad xE'' + x(E')^2 + 2x^{1/2}E' + \tfrac{1}{2}E' - \tfrac{1}{16}x^{-1/2}E' + \tfrac{1}{1024}x^{-2} + \tfrac{1}{32}x^{-3/2} = 0$$

(a) (b) (c) (d) (e) (f) (g)

From [a_4] :

$E \ll \frac{1}{16}x^{-1/2}$

$E' \ll -\frac{1}{32}x^{-3/2} \rightarrow x(E')^2 \ll -\frac{1}{32}x^{-1/2}E'$ or (b) ≪ (e)

$\rightarrow \frac{1}{2}E' \ll \frac{1}{32}x^{-3/2}$ or (d) ≪ g

$E'' \ll \frac{3}{64}x^{-5/2} \rightarrow xE'' \ll \frac{3}{64}x^{-3/2}$ or (a) ≪ (g)

$x^{-2} \ll x^{-3/2}$ or (f) ≪ (g)

$2x^{1/2}E' \gg \frac{1}{16}x^{-1/2}E'$ or (c) ≫ (e)

$$\Rightarrow \quad 2x^{1/2}E' + \tfrac{1}{32}x^{-3/2} \sim 0$$

or $E' \sim -\frac{1}{64}x^{-2}$

Then $\boxed{E \sim \frac{1}{64}x^{-1}}$ or $E = \frac{1}{64}x^{-1} + F(x)$

STOP HERE.

Finally,

$$y(x) = e^{S+C+D+E} = e^{2x^{1/2} - \frac{1}{4}\ln x + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1} + F(x)}$$

or $y(x) \sim c\,x^{-1/4}\exp\left[2x^{1/2} + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1}\right]$

5

42 3/3

$$\frac{1}{16}x^{-\frac{3}{2}} + 2x^{\frac{1}{2}}E_0' - \frac{1}{32}x^{-\frac{3}{2}} + E' = 0$$

$$\frac{1}{32}x^{-\frac{3}{2}} + (2x^{\frac{1}{2}}+1)E_0' = 0$$, And $1 \ll 2x^{\frac{1}{2}}, x\to\infty$, so

$$E_0' = \frac{-1}{64}x^{-2} \Rightarrow E_0 = \int^x E_0'(t)dt = \frac{1}{64}x^{-1}$$, which gives the approximation

$$\boxed{y(x) \sim c x^{-\frac{1}{4}} e^{2x^{1/2} + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1}}, \quad x\to\infty}$$ +8

the improvement in the numerical approximation to $y(x)$ can be seen in computing $y(x)$ for various x values, with $c = \frac{1}{2}(\pi)^{-1/2}$ for equivalence.

For $x=10$;

$$y(10) = \sum_{n=0}^{\infty}\frac{10^n}{(n!)^2} \simeq 90.47595$$

$$y_1(10) = ce^{S_0(10)+C_0(10)} \simeq 88.53491 \quad (2.15\% \text{ error})$$

$$y_2(10) = ce^{S_0(10)+C_0(10)+D_0(10)} \simeq 90.30214 \quad (0.19\% \text{ error})$$

$$y_3(10) = ce^{S_0(10)+C_0(10)+D_0(10)+E_0(10)} \simeq 90.44335 \quad (0.04\% \text{ error})$$

For $x=10{,}000$,

$$y(10{,}000) = \sum_{n=0}^{\infty}\frac{(10{,}000)^n}{(n!)^2} \simeq 2.03968717\times10^{85}$$

$$y_1(10{,}000) \simeq 2.03840157\times10^{85} \quad (6.26\times10^{-2}\% \text{ error})$$

$$y_2(10{,}000) \simeq 2.03968397\times10^{85} \quad (1.57\times10^{-4}\% \text{ error})$$

$$y_3(10{,}000) \simeq 2.03968716\times10^{85} \quad (8.20\times10^{-7}\% \text{ error})$$ +2

3.3 HW3

3.3.1 problem 9.3 (page 479)

problem (a) show that if $a(x) < 0$ for $0 \le x \le 1$ then the solution to 9.1.7 has boundary layer at $x = 1$. (b) Find a uniform approximation with error $O(\varepsilon)$ to the solution 9.1.7 when $a(x) < 0$ for $0 \le x \le 1$ (c) Show that if $a(x) > 0$ it is impossible to match to a boundary layer at $x = 1$

solution

3.3.1.1 Part a

Equation 9.1.7 at page 422 is

$$\begin{aligned} \varepsilon y'' + a(x)y' + b(x)y(x) &= 0 \qquad (9.1.7)\\ y(0) &= A\\ y(1) &= B \end{aligned}$$

For $0 \le x \le 1$. Now we solve for $y_{in}(x)$, but first we introduce inner variable ξ. We assume boundary layer is at $x = 0$, then show that this leads to inconsistency. Let $\xi = \frac{x}{\varepsilon^p}$ be the inner variable. We express the original ODE using this new variable. We also need to determine p. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx}\\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2} \end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and (9.1.7) becomes

$$\begin{aligned} \varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + a(x)\varepsilon^{-p}\frac{dy}{d\xi} + y &= 0\\ \varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + a(x)\varepsilon^{-p}\frac{dy}{d\xi} + y &= 0 \end{aligned}$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore balance gives $1 - 2p = -p$ or $p = 1$. The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2} + a(x)\varepsilon^{-1}\frac{dy}{d\xi} + y = 0 \qquad (1)$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (1) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + a(x)\,\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{1A}$$

Collecting powers of $O\left(\varepsilon^{-1}\right)$ terms, gives the ODE to solve for y_0^{in} as

$$y_0'' \sim -a(x)\,y_0'$$

In the rapidly changing region, because the boundary layer is very thin, we can approximate $a(x)$ by $a(0)$. The above becomes

$$y_0'' \sim -a(0)\,y_0'$$

But we are told that $a(x) < 0$, so $a(0) < 0$, hence $-a(0)$ is positive. Let $-a(0) = n^2$, to make it more clear this is positive, then the ODE to solve is

$$y_0'' \sim n^2 y_0'$$

The solution to this ODE is

$$y_0(\xi) \sim \frac{C_1}{n^2} e^{n^2\xi} + C_2$$

Using $y(0) = A$, then the above gives $A = \frac{C_1}{n^2} + C_2$ or $C_2 = A - \frac{C_1}{n^2}$ and the ODE becomes

$$\begin{aligned} y_0(\xi) &\sim \frac{C_1}{n^2} e^{n^2\xi} + \left(A - \frac{C_1}{n^2}\right) \\ &\sim \frac{C_1}{n^2}\left(e^{n^2\xi} - 1\right) + A \end{aligned}$$

We see from the above solution for the inner layer, that as ξ increases (meaning we are moving away from $x = 0$), then the solution $y_0(\xi)$ and its derivative is increasing and not decreasing since $y_0'(\xi) = C_1 e^{n^2\xi}$ and $y_0''(\xi) = C_1 n^2 e^{n^2\xi}$.

But this contradicts what we assumed that the boundary layer is at $x = 0$ since we expect the solution to change less rapidly as we move away from $x = 0$. Hence we conclude that if $a(x) < 0$, then the boundary layer can not be at $x = 0$.

Let us now see what happens by taking the boundary layer to be at $x = 1$. We repeat the same process as above, but now the inner variable as defined as

$$\xi = \frac{1 - x}{\varepsilon^p}$$

We express the original ODE using this new variable and determine p. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\left(-\varepsilon^{-p}\right)$. Hence $\frac{d}{dx} \equiv \left(-\varepsilon^{-p}\right)\frac{d}{d\xi}$

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\ &= \left(\left(-\varepsilon^{-p}\right)\frac{d}{d\xi}\right)\left(\left(-\varepsilon^{-p}\right)\frac{d}{d\xi}\right) \\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2} \end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and equation (9.1.7) becomes

$$\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} - a(x)\varepsilon^{-p}\frac{dy}{d\xi} + y = 0$$
$$\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} - a(x)\varepsilon^{-p}\frac{dy}{d\xi} + y = 0$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $1-2p = -p$ or

$$\boxed{p = 1}$$

The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2} - a(x)\varepsilon^{-1}\frac{dy}{d\xi} + y = 0 \tag{2}$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (2) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) - a(x)\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{2A}$$

Collecting $O\left(\varepsilon^{-1}\right)$ terms, gives the ODE to solve for y_0^{in} as

$$y_0'' \sim a(x)y_0'$$

In the rapidly changing region, $\alpha = a(1)$, because the boundary layer is very thin, we approximated $a(x)$ by $a(1)$. The above becomes

$$y_0'' \sim \alpha y_0'$$

But we are told that $a(x) < 0$, so $\alpha < 0$, and the above becomes

$$y_0'' \sim \alpha y_0'$$

The solution to this ODE is

$$y_0(\xi) \sim \frac{C_1}{\alpha}e^{\alpha\xi} + C_2 \tag{3}$$

Using

$$\begin{aligned} y(x=1) &= y(\xi=0) \\ &= B \end{aligned}$$

Then (3) gives $B = \frac{C_1}{\alpha} + C_2$ or $C_2 = B - \frac{C_1}{\alpha}$ and (3) becomes

$$\begin{aligned} y_0(\xi) &\sim \frac{C_1}{\alpha}e^{\alpha\xi} + \left(B - \frac{C_1}{\alpha}\right) \\ &\sim \frac{C_1}{\alpha}\left(e^{\alpha\xi} - 1\right) + B \end{aligned} \tag{4}$$

From the above, $y_0'(\xi) = -C_1e^{\alpha\xi}$ and $y_0''(\xi) = C_1\alpha e^{\alpha\xi}$. We now see that as that as ξ increases (meaning we are moving away from $x = 1$ towards the left), then the solution $y_0(\xi)$ is actually changing less rapidly. This is because $\alpha < 0$. The solution is changing less rapidly as we move away from the boundary layer as what we expect. Therefore, we conclude that if $a(x) < 0$ then the boundary layer can not be at $x = 0$ and has to be at $x = 1$.

3.3.1.2 Part b

To find uniform approximation, we need now to find $y^{out}(x)$ and then do the matching. Since from part(a) we concluded that y_{in} is near $x = 1$, then we assume now that $y^{out}(x)$ is near $x = 0$. Let

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (9.1.7) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a(x)\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b(x)\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms of $O(1)$ gives the ODE

$$a(x)\, y_0' + b(x)\, y_0 = 0$$

The solution to this ODE is

$$y_0(x) = C_2 e^{-\int_0^x \frac{b(s)}{a(s)} ds}$$

Applying $y(0) = A$ gives

$$\begin{aligned} A &= C_2 e^{-\int_0^1 \frac{b(s)}{a(s)} ds} \\ &= C_2 E \end{aligned}$$

Where E is constant, which is the value of the definite integral $E = e^{-\int_0^1 \frac{b(s)}{a(s)} ds}$. Hence the solution $y^{out}(x)$ can now be written as

$$y_0^{out}(x) = \frac{A}{E} e^{-\int_0^x \frac{b(s)}{a(s)} ds}$$

We are now ready to do the matching.

$$\lim_{\xi \to \infty} y^{in}(\xi) \sim \lim_{x \to 0} y^{out}(x)$$

$$\lim_{\xi \to \infty} \frac{C_1}{\alpha}\left(e^{\alpha \xi} - 1\right) + B \sim \lim_{x \to 0} \frac{A}{E} e^{-\int_0^x \frac{b(s)}{a(s)} ds}$$

But since $\alpha = a(1) < 0$ then the above simplifies to

$$\begin{aligned} -\frac{C_1}{a(1)} + B &= \frac{A}{E} \\ C_1 &= -a(1)\left(\frac{A}{E} - B\right) \end{aligned}$$

Hence inner solution becomes

$$\begin{aligned}
y_0^{in}(\xi) &\sim \frac{-a(1)\left(\frac{A}{E}-B\right)}{a(1)}\left(e^{a(1)\xi}-1\right)+B\\
&\sim \left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B\\
&\sim B\left(e^{a(1)\xi}-1\right)-\frac{A}{E}\left(e^{a(1)\xi}-1\right)+B\\
&\sim \left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B
\end{aligned}$$

The uniform solution is

$$\begin{aligned}
y_{\text{uniform}}(x) &\sim y^{in}(\xi)+y^{out}(x)-y^{match}\\
&\sim \overbrace{\left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B}^{y^{in}}+\overbrace{\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}}^{y^{out}}-\frac{A}{E}
\end{aligned}$$

Or in terms of x only

$$y_{\text{uniform}}(x)\sim\left(B-\frac{A}{E}\right)\left(e^{a(1)\frac{1-x}{\varepsilon}}-1\right)+B+\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}-\frac{A}{E}$$

3.3.1.3 Part c

We now assume the boundary layer is at $x=1$ but $a(x)>0$. From part (a), we found that the solution for $y_0^{in}(\xi)$ where boundary layer at $x=1$ is

$$y_0(\xi)\sim\frac{C_1}{\alpha}\left(e^{\alpha\xi}-1\right)+B$$

But now $\alpha=a(1)>0$ and not negative as before. We also found that $y_0^{out}(x)$ solution was

$$y_0^{out}(x)=\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}$$

Lets now try to do the matching and see what happens

$$\begin{aligned}
\lim_{\xi\to\infty}y^{in}(\xi) &\sim \lim_{x\to 0}y^{out}(x)\\
\lim_{\xi\to\infty}\frac{C_1}{\alpha}\left(e^{\alpha\xi}-1\right)+B+O(\varepsilon) &\sim \lim_{x\to 0}\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}+O(\varepsilon)\\
\lim_{\xi\to\infty}C_1\left(\frac{e^{\alpha\xi}}{\alpha}-\frac{1}{\alpha}\right) &\sim \frac{A}{E}-B
\end{aligned}$$

Since now $\alpha>0$, then the term on the left blows up, while the term on the right is finite. Not possible to match, unless $C_1=0$. But this means the boundary layer solution is just a constant B and that $\frac{A}{E}=B$. So the matching does not work in general for arbitrary conditions. This means if $a(x)>0$, it is not possible to match boundary layer at $x=1$.

3.3.2 problem 9.4(b)

Problem Find the leading order uniform asymptotic approximation to the solution of

$$\varepsilon y'' + \left(1+x^2\right) y' - x^3 y(x) = 0 \tag{1}$$
$$y(0) = 1$$
$$y(1) = 1$$

For $0 \leq x \leq 1$ in the limit as $\varepsilon \to 0$.

solution

Since $a(x) = \left(1+x^2\right)$ is positive, we expect the boundary layer to be near $x = 0$. First we find $y^{out}(x)$, which is near $x = 1$. Assuming

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + \left(1+x^2\right)\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - x^3\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms in $O(1)$ gives the ODE

$$\left(1+x^2\right) y_0' \sim x^3 y_0$$

The ODE becomes $y_0' \sim \frac{x^3}{\left(1+x^2\right)} y_0$ with integrating factor $\mu = e^{\int \frac{-x^3}{\left(1+x^2\right)} dx}$. To evaluate $\int \frac{-x^3}{\left(1+x^2\right)} dx$, let $u = x^2$, hence $\frac{du}{dx} = 2x$ and the integral becomes

$$\int \frac{-x^3}{\left(1+x^2\right)} dx = -\int \frac{ux}{(1+u)} \frac{du}{2x} = -\frac{1}{2}\int \frac{u}{(1+u)} du$$

But

$$\int \frac{u}{(1+u)} du = \int 1 - \frac{1}{(1+u)} du$$
$$= u - \ln(1+u)$$

But $u = x^2$, hence

$$\int \frac{-x^3}{\left(1+x^2\right)} dx = \frac{-1}{2}\left(x^2 - \ln\left(1+x^2\right)\right)$$

Therefore the integrating factor is $\mu = \exp\left(\frac{-1}{2}x^2 + \frac{1}{2}\ln\left(1+x^2\right)\right)$. The ODE becomes

$$
\begin{aligned}
\frac{d}{dx}\left(\mu y_0\right) &= 0\\
\mu y_0 &\sim c\\
y_{out}(x) &\sim c\exp\left(\frac{1}{2}x^2 - \frac{1}{2}\ln\left(1+x^2\right)\right)\\
&\sim c e^{\frac{1}{2}x^2} e^{\ln\left(1+x^2\right)^{\frac{-1}{2}}}\\
&\sim c\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}
\end{aligned}
$$

To find c, using boundary conditions $y(1) = 1$ gives

$$
\begin{aligned}
1 &= c\frac{e^{\frac{1}{2}}}{\sqrt{2}}\\
c &= \sqrt{2}e^{-\frac{1}{2}}
\end{aligned}
$$

Hence

$$
y_0^{out}(x) \sim \sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}
$$

Now we find $y^{in}(x)$ near $x = 0$. Let $\xi = \frac{x}{\varepsilon^p}$ be the inner variable. We express the original ODE using this new variable and determine p. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$
\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx}\\
&= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\\
&= \varepsilon^{-2p}\frac{d^2}{d\xi^2}
\end{aligned}
$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and $\varepsilon y'' + \left(1+x^2\right)y' - x^3 y(x) = 0$ becomes

$$
\begin{aligned}
\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + \left(1 + (\xi\varepsilon^p)^2\right)\varepsilon^{-p}\frac{dy}{d\xi} - (\xi\varepsilon^p)^3 y &= 0\\
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(1 + \xi^2\varepsilon^{2p}\right)\varepsilon^{-p}\frac{dy}{d\xi} - \xi^3\varepsilon^{3p}y &= 0
\end{aligned}
$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $1 - 2p = -p$ or $p = 1$. The ODE now becomes

$$
\varepsilon^{-1}\frac{d^2y}{d\xi^2} + \left(1 + \xi^2\varepsilon^2\right)\varepsilon^{-1}\frac{dy}{d\xi} - \xi^3\varepsilon^3 y = 0 \tag{2}
$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (2) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(1 + \xi^2\varepsilon^2\right)\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) - \xi^3\varepsilon^3\left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{2A}$$

Collecting terms in $O\left(\varepsilon^{-1}\right)$ gives the ODE

$$y_0''(\xi) \sim -y_0'(\xi)$$

The solution to this ODE is

$$y_0^{in}(\xi) \sim c_1 + c_2 e^{-\xi} \tag{3}$$

Applying $y_0^{in}(0) = 1$ gives

$$\begin{aligned} 1 &= c_1 + c_2 \\ c_1 &= 1 - c_2 \end{aligned}$$

Hence (3) becomes

$$\begin{aligned} y_0^{in}(\xi) &\sim (1 - c_2) + c_2 e^{-\xi} \\ &\sim 1 + c_2\left(e^{-\xi} - 1\right) \end{aligned} \tag{4}$$

Now that we found y_{out} and y_{in}, we apply matching to find c_2 in the y_{in} solution.

$$\begin{aligned} \lim_{\xi\to\infty} y_0^{in}(\xi) &\sim \lim_{x\to 0^+} y_0^{out}(x) \\ \lim_{\xi\to\infty} 1 + c_2\left(e^{-\xi} - 1\right) &\sim \lim_{x\to 0^+} \sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}} \\ 1 - c_2 &\sim \sqrt{\frac{2}{e}} \lim_{x\to 0^+} \frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \\ &\sim \sqrt{\frac{2}{e}} \lim_{x\to 0^+} \frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \\ &= \sqrt{\frac{2}{e}} \end{aligned}$$

Hence

$$c_2 = 1 - \sqrt{\frac{2}{e}}$$

Therefore the $y_0^{in}(\xi)$ becomes

$$\begin{aligned}
y_0^{in}(\xi) &\sim 1+\left(1-\sqrt{\frac{2}{e}}\right)\left(e^{-\xi}-1\right)\\
&\sim 1+\left(e^{-\xi}-1\right)-\sqrt{\frac{2}{e}}\left(e^{-\xi}-1\right)\\
&\sim e^{-\xi}-\sqrt{\frac{2}{e}}\left(e^{-\xi}-1\right)\\
&\sim e^{-\xi}-\sqrt{\frac{2}{e}}e^{-\xi}+\sqrt{\frac{2}{e}}\\
&\sim e^{-\xi}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{\frac{2}{e}}\\
&\sim 0.858+0.142e^{-\xi}
\end{aligned}$$

Therefore, the uniform solution is

$$y_{uniform} \sim y_{in}(x)+y_{out}(x)-y_{match}+O(\varepsilon) \tag{4}$$

Where y_{match} is $y_{in}(x)$ at the boundary layer matching location. (or y_{out} at same matching location). Hence

$$\begin{aligned}
y_{match} &\sim 1-c_2\\
&\sim 1-\left(1-\sqrt{\frac{2}{e}}\right)\\
&\sim \sqrt{\frac{2}{e}}
\end{aligned}$$

Hence (4) becomes

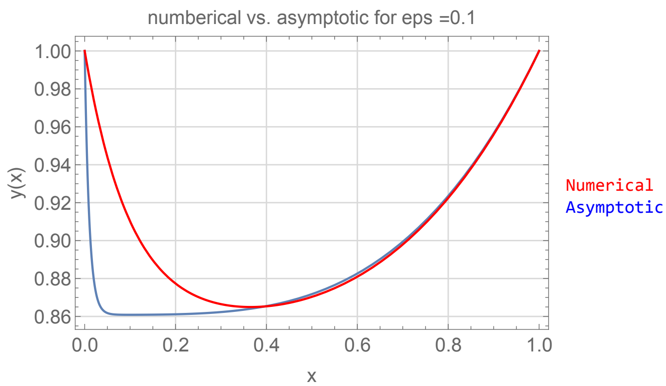
$$\begin{aligned}
y_{uniform} &\sim \overbrace{e^{-\xi}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{\frac{2}{e}}}^{y_{in}}+\overbrace{\sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}}^{y_{out}}-\overbrace{\sqrt{\frac{2}{e}}}^{y_{match}}\\
&\sim e^{-\frac{x}{\varepsilon}}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}+O(\varepsilon)
\end{aligned}$$

This is the leading order uniform asymptotic approximation solution. To verify the result, the numerical solution was plotted against the above solution for $\varepsilon=\{0.1,0.05,0.01\}$. We see from these plots that as ε becomes smaller, the asymptotic solution becomes more accurate when compared to the numerical solution. This is because the error, which is $O(\varepsilon)$, becomes smaller. The code used to generate these plots is

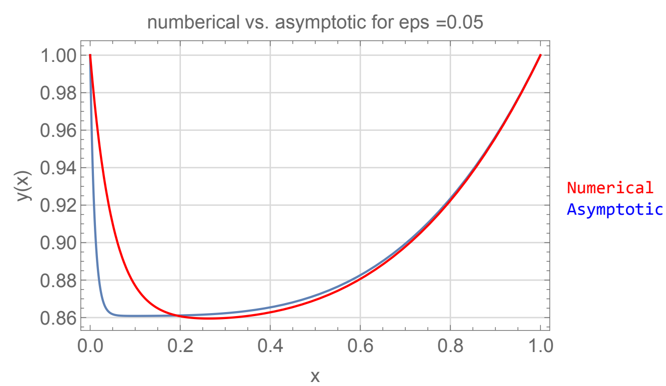
```
In[180]:= eps = 0.1;
sol = NDSolve[{1 / 100 y''[x] + (1 + x^2) y'[x] - x^3 y[x] == 0, y[0] == 1, y[1] == 1}, y, {x, 0, 1}];
p1 = Plot[Evaluate[y[x] /. sol], {x, 0, 1}, Frame → True,
   FrameLabel → {{"y(x)", None}, {"x", Row[{"numberical vs. asymptotic for eps =", eps}]}},
   GridLines → Automatic, GridLinesStyle → LightGray];
mysol[x_, eps_] := Exp[-x/eps] (1 - Sqrt[2/Exp[1]]) + (Sqrt[2] Exp[(x^2-1)/2])/(Sqrt[1 + x^2]);
p2 = Plot[mysol[x, eps], {x, 0, 1}, PlotRange → All, PlotStyle → Red];
Show[Legended[p1, Style["Numerical", Red]], Legended[p2, Style["Asymtotic", Blue]]]
```

The following are the three plots for each value of ε

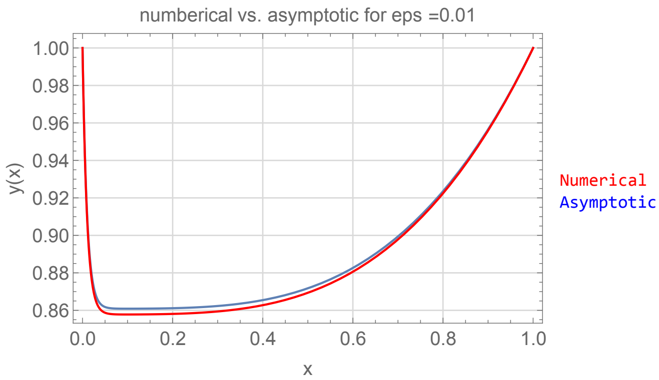
Out[6]=



Out[12]=



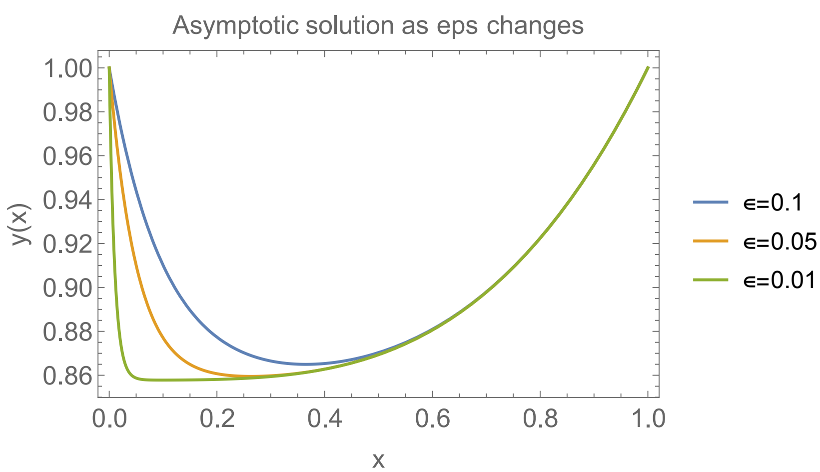
Out[17]=



To see the effect on changing ε on only the asymptotic approximation, the following plot gives the approximation solution only as ε changes. We see how the approximation converges to the numerical solution as ε becomes smaller.

```
In[236]:= Plot[{mysol[x, .1], mysol[x, .05], mysol[x, .01]}, {x, 0, 1},
  PlotLegends → {"ϵ=0.1", "ϵ=0.05", "ϵ=0.01"}, Frame → True,
  FrameLabel → {{"y(x)", None}, {"x", "Asymptotic solution as eps changes"}},
  BaseStyle → 14]
```

Out[236]=



3.3.3 problem 9.6

Problem Consider initial value problem

$$y' = \left(1 + \frac{x^{-2}}{100}\right) y^2 - 2y + 1$$

With $y(1) = 1$ on the interval $0 \le x \le 1$. (a) Formulate this problem as perturbation problem by introducing a small parameter ε. (b) Find outer approximation correct to order ε with errors of order ε^2. Where does this approximation break down? (c) Introduce inner variable and find the inner solution valid to order 1 (with errors of order ε). By matching to the outer solution find a uniform valid solution to $y(x)$ on interval $0 \le x \le 1$. Estimate the accuracy of this approximation. (d) Find inner solution correct to order ε (with errors of order ε^2) and show that it matches to the outer solution correct to order ε.

solution

3.3.3.1 Part a

Since $\frac{1}{100}$ is relatively small compared to all other coefficients, we replace it with ε and the ODE becomes

$$y' - \left(1 + \frac{\varepsilon}{x^2}\right) y^2 + 2y = 1 \tag{1}$$

3.3.3.2 Part b

Assuming boundary layer is on the left side at $x = 0$. We now solve for $y_{out}(x)$, which is the solution near $x = 1$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - \left(1 + \frac{\varepsilon}{x^2}\right)\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)^2 + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1$$

Expanding the above to see more clearly the terms gives

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - \left(1 + \frac{\varepsilon}{x^2}\right)\left(y_0^2 + \varepsilon\left(2y_0 y_1\right) + \varepsilon^2\left(2y_0 y_2 + y_1^2\right) + \cdots\right) + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1 \tag{2}$$

The leading order are those terms of coefficient $O(1)$. This gives

$$y_0' - y_0^2 + 2y_0 \sim 1$$

With boundary conditions $y(1) = 1$.

$$\frac{dy_0}{dx} \sim y_0^2 - 2y_0 + 1$$

This is separable

$$\frac{dy_0}{y_0^2 - 2y_0 + 1} \sim dx$$
$$\frac{dy_0}{\left(y_0 - 1\right)^2} \sim dx$$

For $y_0 \neq 1$. Integrating

$$\int \frac{dy_0}{\left(y_0-1\right)^2} \sim \int dx$$
$$\frac{-1}{y_0-1} \sim x+C$$
$$\left(y_0-1\right)(x+C) \sim -1$$
$$y_0 \sim \frac{-1}{x+C}+1 \tag{3}$$

To find C, from $y(1)=1$, we find

$$1 \sim \frac{-1}{1+C}+1$$
$$1 \sim \frac{C}{1+C}$$

This is only possible if $\underline{C=\infty}$. Therefore from (2), we conclude that

$$\boxed{y_0(x) \sim 1}$$

The above is leading order for the outer solution. Now we repeat everything to find $y_1^{out}(x)$. From (2) above, we now keep all terms with $O(\varepsilon)$ which gives

$$y_1' - 2y_0y_1 + 2y_1 \sim \frac{1}{x^2}y_0^2$$

But we found $y_0(x) \sim 1$ from above, so the above ODE becomes

$$y_1' - 2y_1 + 2y_1 \sim \frac{1}{x^2}$$
$$y_1' \sim \frac{1}{x^2}$$

Integrating gives

$$y_1(x) \sim -\frac{1}{x}+C$$

The boundary condition now becomes $y_1(1)=0$ (since we used $y(1)=1$ earlier with y_0). This gives

$$0 = -\frac{1}{1}+C$$
$$C = 1$$

Therefore the solution becomes

$$\boxed{y_1(x) \sim 1-\tfrac{1}{x}}$$

Therefore, the outer solution is

$$y_{out}(x) \sim y_0 + \varepsilon y_1$$

Or

$$\boxed{y(x) \sim 1 + \varepsilon\left(1 - \frac{1}{x}\right) + O\left(\varepsilon^2\right)}$$

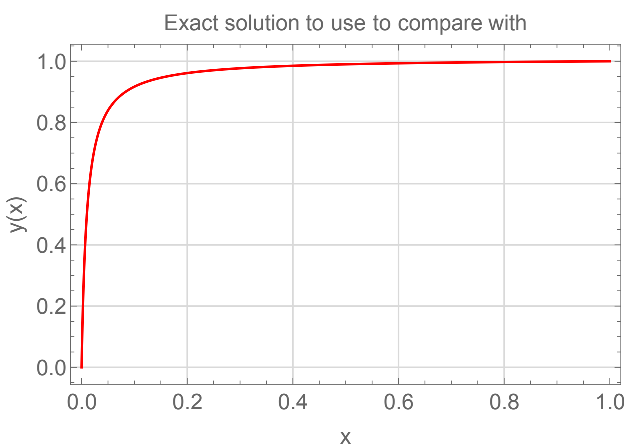
Since the ODE is $y' - \left(1 + \frac{\varepsilon}{x^2}\right)y^2 + 2y = 1$, the approximation breaks down when $x < \sqrt{\varepsilon}$ or $x < \frac{1}{10}$. Because when $x < \sqrt{\varepsilon}$, the $\frac{\varepsilon}{x^2}$ will start to become large. The term $\frac{\varepsilon}{x^2}$ should remain small for the approximation to be accurate. The following are plots of the y_0 and $y_0 + \varepsilon y_1$ solutions (using $\varepsilon = \frac{1}{100}$) showing that with two terms the approximation has improved for the outer layer, compared to the full solution of the original ODE obtained using CAS. But the outer solution breaks down near $x = 0.1$ and smaller as can be seen in these plots. Here is the solution of the original ODE obtained using CAS

```
In[46]:= eps = 1/100;
ode = y'[x] == (1 + eps/x^2) y[x]^2 - 2 y[x] + 1;
sol = y[x] /. First@DSolve[{ode, y[1] == 1}, y[x], x]
```

$$\text{Out[48]=}\ \frac{10\,x\left(-12 - 5\sqrt{6} - 12\,x^{\frac{2\sqrt{6}}{5}} + 5\sqrt{6}\,x^{\frac{2\sqrt{6}}{5}}\right)}{-\sqrt{6} - 120\,x - 50\sqrt{6}\,x + \sqrt{6}\,x^{\frac{2\sqrt{6}}{5}} - 120\,x^{1+\frac{2\sqrt{6}}{5}} + 50\sqrt{6}\,x^{1+\frac{2\sqrt{6}}{5}}}$$

```
In[53]:= Plot[sol, {x, 0, 1}, PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel → {{"y(x)", None}, {"x", "Exact solution to use to compare with"}}, BaseStyle → 14,
  PlotStyle → Red, ImageSize → 400]
```

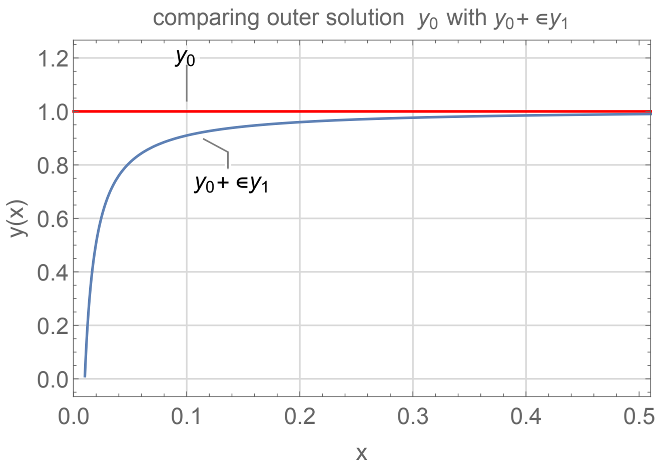
Out[53]=



In the following plot, the y_0 and the $y_0 + \varepsilon y_1$ solutions are superimposed on same figure, to show how the outer solution has improved when adding another term. But we also notice that the outer solution $y_0 + \varepsilon y_1$ only gives good approximation to the exact solution for about $x > 0.1$ and it breaks down quickly as x becomes smaller.

```
In[164]:= outer1 = 1;
outer2 = 1 + eps * (1 - 1 / x);
p2 = Plot[Callout[outer1, "y₀", Scaled[0.1]], {x, 0, 1}, PlotRange → All, Frame → True,
   GridLines → Automatic, GridLinesStyle → LightGray,
   FrameLabel → {{"y(x)", None}, {"x", "comparing outer solution  y₀ with y₀+ ∊y₁"}},
   BaseStyle → 14, PlotStyle → Red, ImageSize → 400];
p3 = Plot[Callout[outer2, "y₀+ ∊y₁", {Scaled[0.5], Below}], {x, 0.01, 1}, AxesOrigin → {0, 0}];
Show[p2, p3, PlotRange → {{0.01, .5}, {0, 1.2}}]
```

Out[168]=



3.3.3.3 Part c

Now we will obtain solution inside the boundary layer $y_{in}(\xi) = y_0^{in}(\xi) + O(\varepsilon)$. The first step is to always introduce new inner variable. Since the boundary layer is on the right side, then

$$\xi = \frac{x}{\varepsilon^p}$$

And then to express the original ODE using this new variable. We also need to determine p in the above expression. Since the original ODE is $y' - \left(1 + \varepsilon x^{-2}\right) y^2 + 2y = 1$, then $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{dy}{d\xi}(\varepsilon^{-p})$, then the ODE now becomes

$$\frac{dy}{d\xi}\varepsilon^{-p} - \left(1 + \frac{\varepsilon}{\xi \varepsilon^{p2}}\right) y^2 + 2y = 1$$
$$\frac{dy}{d\xi}\varepsilon^{-p} - \left(1 + \frac{\varepsilon^{1-2p}}{\xi^2}\right) y^2 + 2y = 1$$

Where in the above $y \equiv y(\xi)$. We see that we have $\left\{\varepsilon^{-p}, \varepsilon^{(1-2p)}\right\}$ as the two biggest terms to match. This means $-p = 1 - 2p$ or

$$p = 1$$

Hence the above ODE becomes

$$\frac{dy}{d\xi}\varepsilon^{-1} - \left(1 + \frac{\varepsilon^{-1}}{\xi^2}\right) y^2 + 2y = 1$$

We are now ready to replace $y(\xi)$ with $\sum_{n=0}^{\infty}\varepsilon^n y_n$ which gives

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right)\varepsilon^{-1} - \left(1+\frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)^2 + 2\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right) = 1$$

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right)\varepsilon^{-1} - \left(1+\frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0^2+\varepsilon\left(2y_0y_1\right)+\cdots\right) + 2\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right) = 1 \quad (3)$$

Collecting terms with $O\left(\varepsilon^{-1}\right)$ gives

$$y_0' \sim \frac{1}{\xi^2}y_0^2$$

This is separable

$$\begin{aligned}\int \frac{dy_0}{y_0^2} &\sim \int \frac{1}{\xi^2}d\xi\\ -y_0^{-1} &\sim -\xi^{-1} + C\\ \frac{1}{y_0} &\sim \frac{1}{\xi} - C\\ \frac{1}{y_0} &\sim \frac{1-C\xi}{\xi}\\ y_0^{in} &\sim \frac{\xi}{1-\xi C}\end{aligned}$$

Now we use matching with y_{out} to find C. We have found before that $y_0^{out}(x) \sim 1$ therefore

$$\begin{aligned}\lim_{\xi\to\infty} y_0^{in}(\xi) + O(\varepsilon) &= \lim_{x\to 0} y_0^{out}(x) + O(\varepsilon)\\ \lim_{\xi\to\infty}\frac{\xi}{1-\xi C} &= 1 + O(\varepsilon)\\ \lim_{\xi\to\infty}(-C) + O\left(\xi^{-1}\right) + O(\varepsilon) &= 1 + O(\varepsilon)\\ -C &= 1\end{aligned}$$

Therefore

$$y_0^{in}(\xi) \sim \frac{\xi}{1+\xi} \quad (4)$$

Therefore,

$$\begin{aligned}y_{uniform} &= y_0^{in} + y_0^{out} - y_{match}\\ &= \overbrace{\frac{\xi}{1+\xi}}^{y_{in}} + \overbrace{1}^{y_{out}} - 1\end{aligned}$$

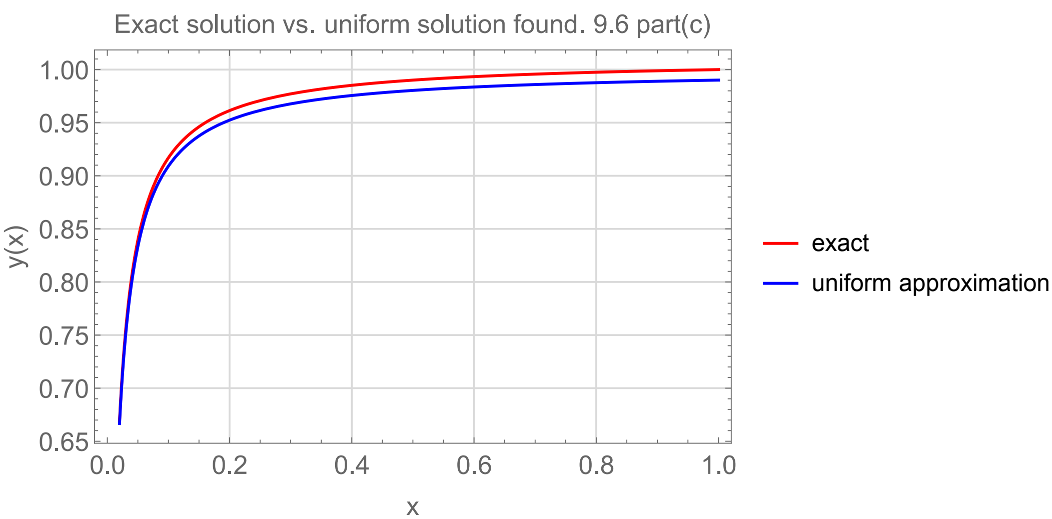
Since $y_{match} = 1$ (this is what $\lim_{\xi\to\infty} y_0^{in}$ is). Writing everything in x, using $\xi = \frac{x}{\varepsilon}$ the above becomes

$$\begin{aligned}y_{\text{uniform}} &= \frac{\frac{x}{\varepsilon}}{1+\frac{x}{\varepsilon}}\\ &= \frac{x}{\varepsilon + x}\end{aligned}$$

The following is a plot of the above, using $\varepsilon = \frac{1}{100}$ to compare with the exact solution.,

```
In[189]:= y[x_, eps_] := x/(x + eps)

p1 = Plot[{exactSol, y[x, 1/100]}, {x, 0.02, 1}, PlotRange → All,
  Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel →
   {{"y(x)", None},
    {"x", "Exact solution vs. uniform solution found. 9.6 part(c)"}},
  BaseStyle → 14, PlotStyle → {Red, Blue}, ImageSize → 400,
  PlotLegends → {"exact", "uniform approximation"}]

Plot[y[x, 1/100], {x, 0, 1}, PlotRange → {{.05, 1}, Automatic}]
```

Out[190]=



3.3.3.4 Part (d)

Now we will obtain y_1^{in} solution inside the boundary layer. Using (3) we found in part (c), reproduced here

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right)\varepsilon^{-1} - \left(1 + \frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0^2 + \varepsilon\left(2y_0y_1\right) + \cdots\right) + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1 \qquad (3)$$

But now collecting all terms of order of $O(1)$, this results in

$$y_1' - y_0^2 - \frac{2}{\xi^2} y_0 y_1 + 2y_0 \sim 1$$

Using y_0^{in} found in part (c) into the above gives

$$y_1' - \frac{2}{\xi}\left(\frac{1}{1+\xi}\right)y_1 \sim 1 - 2\left(\frac{\xi}{1+\xi}\right) + \left(\frac{\xi}{1+\xi}\right)^2$$
$$y_1' - \left(\frac{2}{\xi(1+\xi)}\right)y_1 \sim \frac{1}{(\xi+1)^2}$$

This can be solved using integrating factor $\mu = e^{\int \frac{-2}{\xi+\xi^2} d\xi}$ using partial fractions gives $\mu = \exp\left(-2\ln\xi + 2\ln\left(1+\xi\right)\right)$ or $\mu = \frac{1}{\xi^2}\left(1+\xi\right)^2$. Hence we obtain

$$\begin{aligned}
\frac{d}{dx}\left(\mu y_1\right) &\sim \mu \frac{1}{\left(\xi+1\right)^2} \\
\frac{d}{dx}\left(\frac{1}{\xi^2}\left(1+\xi\right)^2 y_1\right) &\sim \frac{1}{\xi^2}
\end{aligned}$$

Integrating

$$\begin{aligned}
\frac{1}{\xi^2}\left(1+\xi\right)^2 y_1 &\sim \int \frac{1}{\xi^2} d\xi \\
\frac{1}{\xi^2}\left(1+\xi\right)^2 y_1 &\sim \frac{-1}{\xi} + C_2 \\
\left(1+\xi\right)^2 y_1 &\sim -\xi + \xi^2 C_2 \\
y_1 &\sim \frac{-\xi + \xi^2 C_2}{\left(1+\xi\right)^2}
\end{aligned}$$

Therefore, the inner solution becomes

$$\begin{aligned}
y^{in}\left(\xi\right) &= y_0 + \varepsilon y_1 \\
&= \frac{\xi}{1+\xi C_1} + \varepsilon \frac{\xi^2 C_2 - \xi}{\left(1+\xi\right)^2}
\end{aligned}$$

To find C_1, C_2 we do matching with with y^{out} that we found in part (a) which is $y_{out}\left(x\right) \sim 1 + \varepsilon\left(1 - \frac{1}{x}\right)$

$$\lim_{\xi\to\infty}\left(\frac{\xi}{1+\xi C_1} + \varepsilon \frac{\xi^2 C_2 - \xi}{\left(1+\xi\right)^2}\right) \sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right)$$

Doing long division $\frac{\xi}{1+\xi C_1} = \frac{1}{C_1} - \frac{1}{\xi C_1^2} + \frac{1}{\xi^2 C_1^3} + \cdots$ and $\frac{\xi^2 C_2 - \xi}{\left(1+\xi\right)^2} = C_2 - \frac{2C_2+1}{\xi} - \cdots$, hence the above becomes

$$\begin{aligned}
\lim_{\xi\to\infty}\left(\left(\frac{1}{C_1} - \frac{1}{\xi C_1^2} + \frac{1}{\xi^2 C_1^3} + \cdots\right) + \left(\varepsilon C_2 - \varepsilon\frac{2C_2+1}{\xi} + \cdots\right)\right) &\sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right) \\
\frac{1}{C_1} + \varepsilon C_2 &\sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right)
\end{aligned}$$

Using $x = \xi\varepsilon$ on the RHS, the above simplifies to

$$\begin{aligned}
\frac{1}{C_1} + \varepsilon C_2 &\sim \lim_{\xi\to\infty} 1 + \varepsilon\left(1 - \frac{1}{\xi\varepsilon}\right) \\
&\sim \lim_{\xi\to\infty} 1 + \left(\varepsilon - \frac{1}{\xi}\right) \\
&\sim 1 + \varepsilon
\end{aligned}$$

Therefore, $C_1 = 1$ and $C_2 = 1$. Hence the inner solution is

$$\begin{aligned} y^{in}(\xi) &= y_0 + \varepsilon y_1 \\ &= \frac{\xi}{1+\xi} + \varepsilon \frac{\xi^2 - \xi}{(1+\xi)^2} \end{aligned}$$

Therefore

$$\begin{aligned} y_{\text{uniform}} &= y_{in} + y_{out} - y_{match} \\ &= \overbrace{\frac{\xi}{1+\xi} + \varepsilon \frac{\xi^2 - \xi}{(1+\xi)^2}}^{y_{in}} + \overbrace{1 + \varepsilon\left(1 - \frac{1}{x}\right)}^{y_{out}} - (1+\varepsilon) \end{aligned}$$

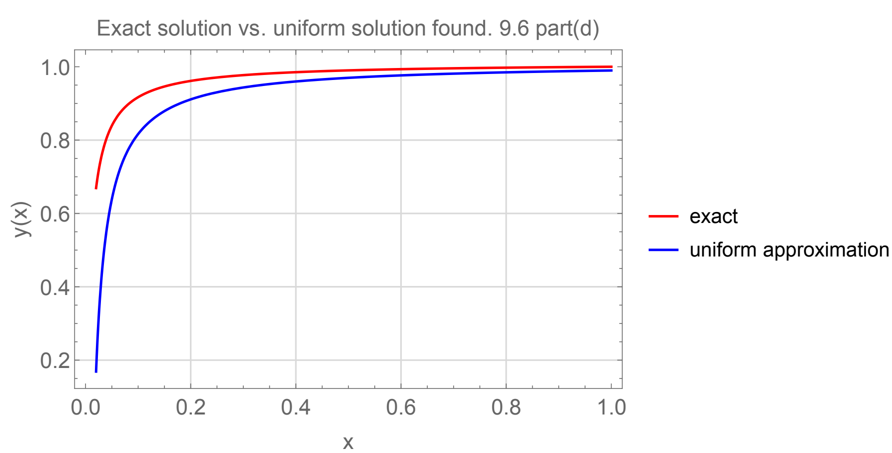
Writing everything in x, using $\xi = \frac{x}{\varepsilon}$ the above becomes

$$\begin{aligned} y_{\text{uniform}} &= \frac{\frac{x}{\varepsilon}}{1 + \frac{x}{\varepsilon}} + \varepsilon \frac{\frac{x^2}{\varepsilon^2} - \frac{x}{\varepsilon}}{\left(1 + \frac{x}{\varepsilon}\right)^2} + 1 + \varepsilon\left(1 - \frac{1}{x}\right) - (1+\varepsilon) \\ &= \frac{x}{\varepsilon + x} + \frac{x^2 - x\varepsilon}{\varepsilon\left(1 + \frac{x}{\varepsilon}\right)^2} + 1 + \varepsilon - \frac{\varepsilon}{x} - 1 - \varepsilon \\ &= \frac{x}{\varepsilon + x} + \frac{x^2 - x\varepsilon}{\varepsilon\left(1 + \frac{x}{\varepsilon}\right)^2} - \frac{\varepsilon}{x} + O\left(\varepsilon^2\right) \end{aligned}$$

The following is a plot of the above, using $\varepsilon = \frac{1}{100}$ to compare with the exact solution.

```
In[192]:= y[x_, eps_] := x/(x + eps) + (x^2 - x eps)/(eps (1 + x/eps)^2) - eps/x

p1 = Plot[{exactSol, y[x, 1/100]}, {x, 0.02, 1}, PlotRange → All, Frame → True,
  GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel → {{"y(x)", None}, {"x", "Exact solution vs. uniform solution found. 9.6 part(d)"}},
  BaseStyle → 14, PlotStyle → {Red, Blue}, ImageSize → 400,
  PlotLegends → {"exact", "uniform approximation"}]

Plot[y[x, 1/100], {x, 0, 1}, PlotRange → {{.05, 1}, Automatic}]
```

Out[193]=



Let us check if $y_{\text{uniform}}(x)$ satisfies $y(1)=1$ or not.

$$
\begin{aligned}
y_{\text{uniform}}(1) &= \frac{1}{\varepsilon+1} + \frac{1-\varepsilon}{\varepsilon\left(1+\frac{1}{\varepsilon}\right)^2} - \varepsilon + O\left(\varepsilon^2\right) \\
&= \frac{1-\varepsilon^3-3\varepsilon^2+\varepsilon}{(\varepsilon+1)^2}
\end{aligned}
$$

Taking the limit $\varepsilon \to 0$ gives 1. Therefore $y_{\text{uniform}}(x)$ satisfies $y(1)=1$.

3.3.4 problem 9.9

problem Use boundary layer methods to find an approximate solution to initial value problem

$$
\begin{aligned}
\varepsilon y'' + a(x)y' + b(x)y &= 0 \qquad (1)\\
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}
$$

And $a>0$. Show that leading order uniform approximation satisfies $y(0)=1$ but not $y'(0)=1$ for arbitrary b. Compare leading order uniform approximation with the exact solution to the problem when $a(x), b(x)$ are constants.

Solution

Since $a(x)>0$ then we expect the boundary layer to be at $x=0$. We start by finding $y_{out}(x)$.

$$
y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots
$$

Substituting this into (1) gives

$$
\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0
$$

Collecting terms with $O(1)$ results in

$$
\begin{aligned}
a y_0' &\sim -b y_0 \\
\frac{dy_0}{dx} &\sim -\frac{b}{a} y_0
\end{aligned}
$$

This is separable

$$
\begin{aligned}
\int \frac{dy_0}{y_0} &\sim -\int \frac{b(x)}{a(x)} dx \\
\ln\left|y_0\right| &\sim -\int \frac{b(x)}{a(x)} dx + C \\
y_0 &\sim C e^{-\int_1^x \frac{b(s)}{a(s)} ds}
\end{aligned}
$$

Now we find y_{in}. First we introduce interval variable

$$
\xi = \frac{x}{\varepsilon^p}
$$

And transform the ODE. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned}\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx}\\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2}\end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE becomes

$$\begin{aligned}\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + a(\xi)\frac{dy}{d\xi}\varepsilon^{-p} + b(\xi)y &= 0\\ \varepsilon^{1-2p}y'' + a\varepsilon^{-p}y' + by &= 0\end{aligned}$$

Balancing $1-2p$ with $-p$ shows that

$$\boxed{p = 1}$$

Hence

$$\varepsilon^{-1}y'' + a\varepsilon^{-1}y' + by = 0$$

Substituting $y_{in} = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ in the above gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with order $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'' \sim -ay_0'$$

Assuming $z = y_0'$ then the above becomes $z' \sim -az$ or $\frac{dz}{d\xi} \sim -az$. This is separable. The solution is $\frac{dz}{z} \sim -ad\xi$ or

$$\begin{aligned}\ln|z| &\sim -\int_0^{\xi} a(s)\,ds + C_1\\ z &\sim C_1 e^{-\int_0^{\xi} a(s)ds}\end{aligned}$$

Hence

$$\begin{aligned}\frac{dy_0}{d\xi} &\sim C_1 e^{-\int_0^{\xi} a(s)ds}\\ dy_0 &\sim \left(C_1 e^{-\int_0^{\xi} a(s)ds}\right)d\xi\end{aligned}$$

Integrating again

$$y_0^{in} \sim \int_0^{\xi}\left(C_1 e^{-\int_0^{\eta} a(s)ds}\right)d\eta + C_2$$

Applying initial conditions at $y(0)$ since this is where the y_{in} exist. Using $y_{in}(0) = 1$ then the above becomes

$$1 = C_2$$

Hence the solution becomes

$$y_0^{in} \sim \int_0^{\xi} \left(C_1 e^{-\int_0^{\eta} a(s)ds} \right) d\eta + 1$$

To apply the second initial condition, which is $y'(0) = 1$, we first take derivative of the above w.r.t. ξ

$$y_0' \sim C_1 e^{-\int_0^{\xi} a(s)ds}$$

Applying $y_0'(0) = 1$ gives

$$1 = C_1$$

Hence

$$y_0^{in} \sim 1 + \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta$$

Now to find constant of integration for y^{out} from earlier, we need to do matching.

$$\lim_{\xi \to \infty} y_0^{in} \sim \lim_{x \to 0} y_0^{out}$$

$$\lim_{\xi \to \infty} 1 + \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta \sim \lim_{x \to 0} Ce^{-\int_1^{x} \frac{b(s)}{a(s)} ds}$$

On the LHS the integral $\int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta$ since $a > 0$ and negative power on the exponential. So as $\xi \to \infty$ the integral value is zero. So we have now

$$1 \sim \lim_{x \to 0} Ce^{-\int_1^{x} \frac{b(s)}{a(s)} ds}$$

Let $\lim_{x \to 0} e^{-\int_1^{x} \frac{b(s)}{a(s)} ds} \to E$, where E is the value of the definite integral $Ce^{-\int_1^{0} \frac{b(s)}{a(s)} ds}$. Another constant, which if we know $a(x), b(x)$ we can evaluate. Hence the above gives the value of C as

$$C = \frac{1}{E}$$

The uniform solution can now be written as

$$\begin{aligned} y_{\text{uniform}} &= y_{in} + y_{out} - y_{\text{match}} \\ &= 1 + \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta + \frac{1}{E} e^{-\int_1^{x} \frac{b(s)}{a(s)} ds} - 1 \\ &= \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta + \frac{1}{E} e^{-\int_1^{x} \frac{b(s)}{a(s)} ds} \end{aligned} \tag{2}$$

Finally, we need to show that $y_{\text{uniform}}(0) = 1$ but not $y'_{\text{uniform}}(0) = 1$. From (2), at $x = 0$ which also means $\xi = 0$, since boundary layer at left side, equation (2) becomes

$$y_{\text{uniform}}(0) = 0 + \frac{1}{E} \lim_{x \to 0} e^{-\int_1^{x} \frac{b(s)}{a(s)} ds}$$

But we said that $\lim_{x \to 0} e^{-\int_1^{x} \frac{b(s)}{a(s)} ds} = E$, therefore

$$y_{\text{uniform}}(0) = 1$$

Now we take derivative of (2) w.r.t. x and obtain

$$y'_{\text{uniform}}(x) = \frac{d}{dx}\left(\int_0^{\frac{x}{\varepsilon}} e^{-\int_0^{\eta} a(s)ds} d\eta\right) + \frac{1}{E}\frac{d}{dx}\left(e^{-\int_1^x \frac{b(s)}{a(s)}ds}\right)$$
$$= e^{-\int_0^{\frac{x}{\varepsilon}} a(s)ds} - \frac{1}{E}\frac{b(x)}{a(x)} e^{-\int_1^x \frac{b(s)}{a(s)}ds}$$

And at $x = 0$ the above becomes

$$y'_{\text{uniform}}(0) = 1 - \frac{1}{E}\frac{b(0)}{a(0)}$$

The above is zero only if $b(0) = 0$ (since we know $a(0) > 0$). Therefore, we see that $y'_{\text{uniform}}(0) \neq 1$ for any arbitrary $b(x)$. Which is what we are asked to show.

Will now solve the whole problem again, when a, b are constants.

$$\varepsilon y'' + ay' + by = 0 \tag{1A}$$
$$y(0) = 1$$
$$y'(0) = 1$$

And $a > 0$. And compare leading order uniform approximation with the exact solution to the problem when $a(x), b(x)$ are constants. Since $a > 0$ then boundary layer will occur at $x = 0$. We start by finding $y_{out}(x)$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with $O(1)$ results in

$$ay_0' \sim -by_0$$
$$\frac{dy_0}{dx} \sim -\frac{b}{a}y_0$$

This is separable

$$\int \frac{dy_0}{y_0} \sim -\frac{b}{a}dx$$
$$\ln|y_0| \sim -\frac{b}{a}x + C$$
$$y_0^{out} \sim C_1 e^{-\frac{b}{a}x}$$

Now we find y_{in}. First we introduce internal variable $\xi = \frac{x}{\varepsilon^p}$ and transform the ODE as we did above. This results in

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with order $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'' \sim -ay_0'$$

Assuming $z = y_0'$ then the above becomes $z' \sim -az$ or $\frac{dz}{d\xi} \sim -az$. This is separable. The solution is $\frac{dz}{z} \sim -ad\xi$ or

$$\ln|z| \sim -a\xi + E_1$$
$$z \sim E_1 e^{-a\xi}$$

Hence

$$\frac{dy_0}{d\xi} \sim E_1 e^{-a\xi}$$
$$dy_0 \sim E_1 e^{-a\xi} d\xi$$

Integrating again

$$y_0^{in} \sim E_1 \left(\frac{-1}{a}\right) e^{-a\xi} + E_2$$

Applying initial conditions at $y(0)$ since this is where the y_{in} exist. Using $y_{in}(0) = 1$ then the above becomes

$$1 = E_1 \left(\frac{-1}{a}\right) + E_2$$
$$a(E_2 - 1) = E_1$$

Hence the solution becomes

$$y_0^{in} \sim (1 - E_2) e^{-a\xi} + E_2 \tag{1B}$$

To apply the second initial condition, which is $y'(0) = 1$, we first take derivative of the above w.r.t. ξ

$$y_0' \sim -a(1 - E_2) e^{-a\xi}$$

Hence $y'(0) = 1$ gives

$$1 = -a(1 - E_2)$$
$$1 = -a + aE_2$$
$$E_2 = \frac{1 + a}{a}$$

And the solution y_{in} in (1B) becomes

$$y_0^{in} \sim \left(1 - \frac{1 + a}{a}\right) e^{-a\xi} + \frac{1 + a}{a}$$
$$\sim \left(\frac{-1}{a}\right) e^{-a\xi} + \frac{1 + a}{a}$$
$$\sim \frac{(1 + a) - e^{-a\xi}}{a}$$

Now to find constant of integration for $y^{out}(x)$ from earlier, we need to do matching.

$$\lim_{\xi\to\infty} y_0^{in} \sim \lim_{x\to 0} y_0^{out}$$
$$\lim_{\xi\to\infty} \frac{(1+a)-e^{-a\xi}}{a} \sim \lim_{x\to 0} C_1 e^{-\frac{b}{a}x}$$
$$\frac{1+a}{a} \sim C_1$$

Hence now the uniform solution can be written as

$$\begin{aligned}
y_{\text{uniform}}(x) &\sim y_{in} + y_{out} - y_{\text{match}} \\
&\sim \overbrace{\frac{(1+a)-e^{-a\frac{x}{\varepsilon}}}{a}}^{y_{in}} + \overbrace{\frac{1+a}{a}e^{-\frac{b}{a}x}}^{y_{out}} - \frac{1+a}{a} \\
&\sim \frac{(1+a)}{a} - \frac{e^{-a\frac{x}{\varepsilon}}}{a} + \frac{1+a}{a}e^{-\frac{b}{a}x} - \frac{1+a}{a} \\
&\sim -\frac{e^{-a\frac{x}{\varepsilon}}}{a} + \frac{1+a}{a}e^{-\frac{b}{a}x} \\
&\sim \frac{1}{a}\left((1+a)e^{-\frac{b}{a}x} - e^{-a\frac{x}{\varepsilon}}\right)
\end{aligned} \tag{2A}$$

Now we compare the above, which is the leading order uniform approximation, to the exact solution. Since now a, b are constants, then the exact solution is

$$y_{exact}(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} \tag{3}$$

Where $\lambda_{1,2}$ are roots of characteristic equation of $\varepsilon y'' + ay' + by = 0$. These are $\lambda = \frac{-a}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{a^2 - 4\varepsilon b}$. Hence

$$\lambda_1 = \frac{-a}{2} + \frac{1}{2}\sqrt{a^2 - 4\varepsilon b}$$
$$\lambda_2 = \frac{-a}{2} - \frac{1}{2}\sqrt{a^2 - 4\varepsilon b}$$

Applying initial conditions to (3). $y(0) = 1$ gives

$$1 = A + B$$
$$B = 1 - A$$

And solution becomes $y_{exact}(x) = Ae^{\lambda_1 x} + (1-A)e^{\lambda_2 x}$. Taking derivatives gives

$$y'_{exact}(x) = A\lambda_1 e^{\lambda_1 x} + (1-A)\lambda_2 e^{\lambda_2 x}$$

Using $y'(0) = 1$ gives

$$1 = A\lambda_1 + (1-A)\lambda_2$$
$$1 = A(\lambda_1 - \lambda_2) + \lambda_2$$
$$A = \frac{1-\lambda_2}{\lambda_1 - \lambda_2}$$

Therefore, $B = 1 - \frac{1-\lambda_2}{\lambda_1-\lambda_2}$ and the exact solution becomes

$$
\begin{aligned}
y_{exact}(x) &= \frac{1-\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 x} + \left(1 - \frac{1-\lambda_2}{\lambda_1-\lambda_2}\right)e^{\lambda_2 x} \\
&= \frac{1-\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 x} + \left(\frac{(\lambda_1-\lambda_2)-(1-\lambda_2)}{\lambda_1-\lambda_2}\right)e^{\lambda_2 x} \\
&= \frac{1-\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 x} + \left(\frac{\lambda_1-1}{\lambda_1-\lambda_2}\right)e^{\lambda_2 x} \qquad (4)
\end{aligned}
$$

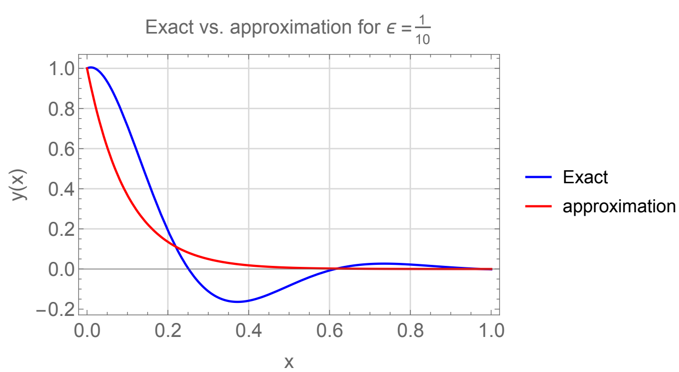
While the uniform solution above was found to be $\frac{1}{a}\left((1+a)\,e^{-\frac{b}{a}x} - e^{-a\frac{x}{\varepsilon}}\right)$. Here is a plot of the exact solution above, for $\varepsilon = \{1/10, 1/50, 1/100\}$, and for some values for a, b such as $a = 1, b = 10$ in order to compare with the uniform solution. Note that the uniform solution is $O(\varepsilon)$. As ε becomes smaller, the leading order uniform solution will better approximate the exact solution. At $\varepsilon = 0.01$ the uniform approximation gives very good approximation. This is using only leading term approximation.

```
In[134]:= ClearAll[x, y]
eps = 1 / 10; a = 1; b = 10;
mySol = 1 / a ((1 + a) * Exp[-b / a x] - Exp[-a x / eps]);
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
```

Out[137]= $\frac{1}{5} e^{-5x} \left(5 \operatorname{Cos}\left[5\sqrt{3}\,x\right] + 2\sqrt{3}\operatorname{Sin}\left[5\sqrt{3}\,x\right]\right)$

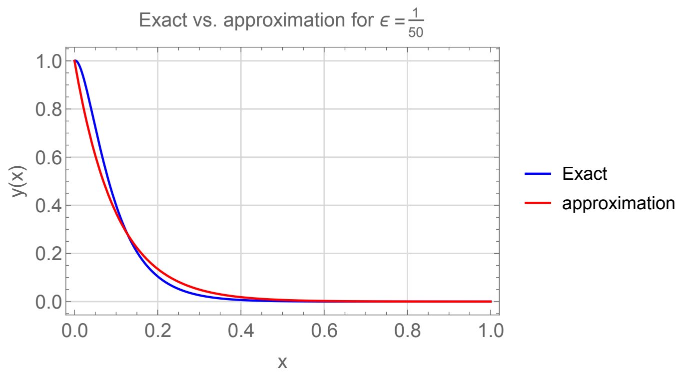
```
In[125]:= Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
 Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[125]=



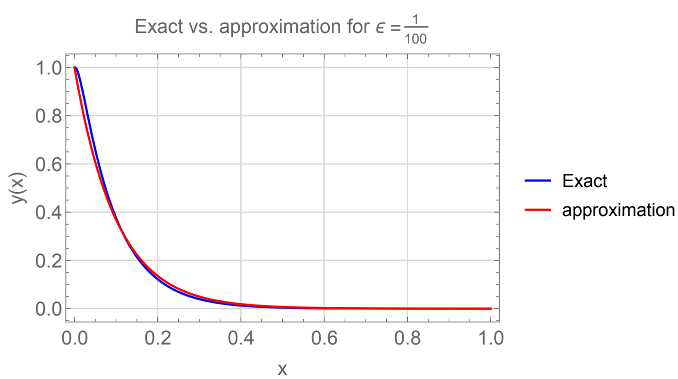
In[126]:=
```
ClearAll[x, y]
eps = 1 / 50; a = 1; b = 10;
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
 Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[128]= $\frac{1}{50}\left(25\, e^{\left(-25-5\sqrt{5}\right)x} - 26\sqrt{5}\, e^{\left(-25-5\sqrt{5}\right)x} + 25\, e^{\left(-25+5\sqrt{5}\right)x} + 26\sqrt{5}\, e^{\left(-25+5\sqrt{5}\right)x}\right)$

Out[129]=


In[130]:=
```
ClearAll[x, y]
eps = 1 / 100; a = 1; b = 10;
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
 Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[132]= $\frac{1}{100}\left(50\, e^{\left(-50-10\sqrt{15}\right)x} - 17\sqrt{15}\, e^{\left(-50-10\sqrt{15}\right)x} + 50\, e^{\left(-50+10\sqrt{15}\right)x} + 17\sqrt{15}\, e^{\left(-50+10\sqrt{15}\right)x}\right)$

Out[133]=


3.3.5 problem 9.15(b)

Problem Find first order uniform approximation valid as $\varepsilon \to 0^+$ for $0 \le x \le 1$

$$\varepsilon y'' + \left(x^2+1\right) y' - x^3 y = 0 \tag{1}$$
$$y(0) = 1$$
$$y(1) = 1$$

Solution

Since $a(x) = \left(x^2+1\right)$ is positive for $0 \le x \le 1$, therefore we expect the boundary layer to be on the left side at $x=0$. Assuming this is the case for now (if it is not, then we expect not to be able to do the matching). We start by finding $y_{out}(x)$.

$$y_{out}(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n = y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

Substituting this into (1) gives

$$\varepsilon\left(y_0''+\varepsilon y_1''+\varepsilon^2 y_2''+\cdots\right)+\left(x^2+1\right)\left(y_0'+\varepsilon y_1'+\varepsilon^2 y_2'+\cdots\right)-x^3\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)=0 \qquad (2)$$

Collecting terms with $O(1)$ results in

$$\left(x^2+1\right)y_0' \sim x^3 y_0$$
$$\frac{dy_0}{dx}\sim\frac{x^3}{\left(x^2+1\right)}y_0$$

This is separable.

$$\int\frac{dy_0}{y_0}\sim\int\frac{x^3}{\left(x^2+1\right)}dx$$
$$\ln|y_0|\sim\int x-\frac{x}{1+x^2}dx$$
$$\sim\frac{x^2}{2}-\frac{1}{2}\ln\left(1+x^2\right)+C$$

Hence

$$y_0\sim e^{\frac{x^2}{2}-\frac{1}{2}\ln\left(1+x^2\right)+C}$$
$$\sim\frac{Ce^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$

Applying $y_0^{out}(1)=1$ to the above (since this is where the outer solution is), we solve for C

$$1\sim\frac{Ce^{\frac{1}{2}}}{\sqrt{2}}$$
$$C\sim\sqrt{2}e^{\frac{-1}{2}}$$

Therefore

$$y_0^{out}\sim\frac{\sqrt{2}e^{\frac{-1}{2}}e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$
$$\sim\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$

Now we need to find y_1^{out}. From (2), but now collecting terms in $O(\varepsilon)$ gives

$$y_0''+\left(x^2+1\right)y_1' \backsim x^3 y_1 \qquad (3)$$

In the above y_0'' is known.

$$
\begin{aligned}
y_0'(x) &= \sqrt{\frac{2}{e}}\frac{d}{dx}\left(\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\right)\\
&= \sqrt{\frac{2}{e}}\frac{x^3 e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

And

$$
y_0''(x) = \sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{5}{2}}}
$$

Hence (3) becomes

$$
\begin{aligned}
\left(x^2+1\right)y_1' &\backsim x^3 y_1 - y_0''\\
\left(x^2+1\right)y_1' &\backsim x^3 y_1 - \sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{5}{2}}}\\
y_1' - \frac{x^3}{\left(x^2+1\right)}y_1 &\backsim -\sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{7}{2}}}
\end{aligned}
$$

Integrating factor is $\mu = e^{-\int \frac{x^3}{\left(x^2+1\right)}dx} = e^{-\frac{x^2}{2}+\frac{1}{2}\ln\left(1+x^2\right)} = \left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}}$, hence the above becomes

$$
\begin{aligned}
\frac{d}{dx}\left(\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}} y_1\right) &\backsim -\sqrt{\frac{2}{e}}\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{7}{2}}}\\
&\backsim -\sqrt{\frac{2}{e}}\frac{x^2\left(x^4+x^2+3\right)}{\left(1+x^2\right)^3}
\end{aligned}
$$

Integrating gives (with help from CAS)

$$
\begin{aligned}
\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}} y_1(x) &\backsim -\sqrt{\frac{2}{e}}\int \frac{x^2\left(x^4+x^2+3\right)}{\left(1+x^2\right)^3}dx\\
&\backsim -\sqrt{\frac{2}{e}}\int 1 - \frac{3}{\left(1+x^2\right)^3} + \frac{4}{\left(1+x^2\right)^2} - \frac{2}{1+x^2}dx\\
&\backsim -\sqrt{\frac{2}{e}}\left(x - \frac{3x}{4\left(1+x^2\right)^2} + \frac{7x}{8\left(1+x^2\right)} - 9\frac{\arctan(x)}{8}\right) + C_1
\end{aligned}
$$

Hence

$$y_1^{out}(x) \backsim -\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{1}{2}}}\left(x-\frac{3x}{4\left(1+x^2\right)^2}+\frac{7x}{8\left(1+x^2\right)}-9\frac{\arctan(x)}{8}\right)+C_1\frac{e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{1}{2}}}$$

Now we find C_1 from boundary conditions $y_1(1)=0$. (notice the BC now is $y_1(1)=0$ and not $y_1(1)=1$, since we used $y_1(1)=1$ already).

$$\sqrt{\frac{2}{e}}\frac{e^{\frac{1}{2}}}{(1+1)^{\frac{1}{2}}}\left(1-\frac{3}{4(1+1)^2}+\frac{7}{8(1+1)}-9\frac{\arctan(1)}{8}\right)=C_1\frac{e^{\frac{1}{2}}}{(1+1)^{\frac{1}{2}}}$$

$$\sqrt{\frac{2}{e}}\frac{e^{\frac{1}{2}}}{\sqrt{2}}\left(1-\frac{3}{16}+\frac{7}{16}-\frac{9}{8}\arctan(1)\right)=C_1\frac{e^{\frac{1}{2}}}{\sqrt{2}}$$

Simplifying

$$\begin{aligned}1-\frac{3}{16}+\frac{7}{16}-\frac{9}{8}\arctan(1) &= C_1\frac{e^{\frac{1}{2}}}{\sqrt{2}}\\ C_1 &= \sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{8}\arctan(1)\right)\\ C_1 &= \sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)\\ &= 0.31431\end{aligned}$$

Hence

$$\begin{aligned}y_1^{out} &\sim -\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{\left(1+x^2\right)}}\left(x-\frac{3x}{4\left(1+x^2\right)^2}+\frac{7x}{8\left(1+x^2\right)}-\frac{9}{8}\arctan(x)\right)+\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)\frac{e^{\frac{x^2}{2}}}{\sqrt{\left(1+x^2\right)}}\\ &\sim \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{\left(1+x^2\right)}}\left(\left(\frac{5}{4}-\frac{9}{32}\pi\right)-\left(x-\frac{3x}{4\left(1+x^2\right)^2}+\frac{7x}{8\left(1+x^2\right)}-\frac{9}{8}\arctan(x)\right)\right)\\ &\sim \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{\left(1+x^2\right)}}\left(\frac{5}{4}-\frac{9}{32}\pi-x+\frac{3x}{4\left(1+x^2\right)^2}-\frac{7x}{8\left(1+x^2\right)}+\frac{9}{8}\arctan(x)\right)\end{aligned}$$

Hence

$$
\begin{aligned}
y^{out}(x) &\sim y_0^{out} + \varepsilon y_1^{out} \\
&\sim \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} + \varepsilon\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{\left(1+x^2\right)}}\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right) + O\left(\varepsilon^2\right) \\
&\sim \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right) + O\left(\varepsilon^2\right)
\end{aligned} \tag{3A}
$$

Now that we found $y^{out}(x)$, we need to find $y^{in}(x)$ and then do the matching and the find uniform approximation. Since the boundary layer at $x=0$, we introduce inner variable $\xi = \frac{x}{\varepsilon^p}$ and then express the original ODE using this new variable. We also need to determine p in the above expression. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$
\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\
&= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\
&= \varepsilon^{-2p}\frac{d^2}{d\xi^2}
\end{aligned}
$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE $\varepsilon y'' + \left(x^2+1\right)y' - x^3 y = 0$ now becomes

$$
\begin{aligned}
\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + \left((\xi\varepsilon^p)^2 + 1\right)\varepsilon^{-p}\frac{dy}{d\xi} - (\xi\varepsilon^p)^3 y &= 0 \\
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi^2\varepsilon^p + \varepsilon^{-p}\right)\frac{dy}{d\xi} - \xi^3\varepsilon^{3p}y &= 0
\end{aligned}
$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $p=1$. The ODE now becomes

$$
\varepsilon^{-1}\frac{d^2y}{d\xi^2} + \left(\xi^2\varepsilon + \varepsilon^{-1}\right)\frac{dy}{d\xi} - \xi^3\varepsilon^3 y = 0 \tag{4}
$$

Assuming that

$$
y_{in}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots
$$

And substituting the above into (4) gives

$$
\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\xi^2\varepsilon + \varepsilon^{-1}\right)\left(y_0' + \varepsilon y_1' + \cdots\right) - \xi^3\varepsilon^3\left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{4A}
$$

Collecting terms in $O\left(\varepsilon^{-1}\right)$ gives

$$
y_0'' \backsim -y_0'
$$

Letting $z = y_0'$, the above becomes

$$\begin{aligned}\frac{dz}{d\xi} &\backsim -z\\ \frac{dz}{z} &\backsim -d\xi\\ \ln|z| &\backsim -\xi + C_1\\ z &\backsim C_1 e^{-\xi}\end{aligned}$$

Hence

$$\begin{aligned}\frac{dy_0}{d\xi} &\backsim C_1 e^{-\xi}\\ y_0 &\backsim C_1 \int e^{-\xi} d\xi + C_2\\ &\backsim -C_1 e^{-\xi} + C_2 \qquad (5)\end{aligned}$$

Applying boundary conditions $y_0^{in}(0) = 1$ gives

$$\begin{aligned}1 &= -C_1 + C_2\\ C_2 &= 1 + C_1\end{aligned}$$

And (5) becomes

$$\begin{aligned}y_0^{in}(\xi) &\backsim -C_1 e^{-\xi} + (1 + C_1)\\ &\backsim 1 + C_1\left(1 - e^{-\xi}\right) \qquad (6)\end{aligned}$$

We now find y_1^{in}. Going back to (4) and collecting terms in $O(1)$ gives the ODE

$$y_1'' \backsim y_1'$$

This is the same ODE we solved above. But it will have different B.C. Hence

$$y_1^{in} \backsim -C_3 e^{-\xi} + C_4$$

Applying boundary conditions $y_1^{in}(0) = 0$ gives

$$\begin{aligned}0 &= -C_3 + C_4\\ C_3 &= C_4\end{aligned}$$

Therefore

$$\begin{aligned}y_1^{in} &\backsim -C_3 e^{-\xi} + C_3\\ &\backsim C_3\left(1 - e^{-\xi}\right)\end{aligned}$$

Now we have the leading order y^{in}

$$\begin{aligned}y^{in}(\xi) &= y_0^{in} + \varepsilon y_1^{in}\\ &= 1 + C_1\left(1 - e^{-\xi}\right) + \varepsilon C_3\left(1 - e^{-\xi}\right) + O\left(\varepsilon^2\right) \qquad (7)\end{aligned}$$

Now we are ready to do the matching between (7) and (3A)

$$\lim_{\xi\to\infty} 1 + C_1\left(1-e^{-\xi}\right) + \varepsilon C_3\left(1-e^{-\xi}\right) \sim$$
$$\lim_{x\to 0}\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1+\varepsilon\left(\frac{5}{4}-\frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right)$$

Or

$$1+C_1+\varepsilon C_3 \sim \sqrt{\frac{2}{e}}\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} + \sqrt{\frac{2}{e}}\varepsilon\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(\frac{5}{4}-\frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)$$

But $\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \to 1$, $\lim_{x\to 0}\frac{3x}{4\left(1+x^2\right)^2} \to 0$ $\lim_{x\to 0}\frac{7x}{8\left(1+x^2\right)} \to 0$ therefore the above becomes

$$1 + C_1 + \varepsilon C_3 \sim \sqrt{\frac{2}{e}} + \sqrt{\frac{2}{e}}\varepsilon\left(\frac{5}{4}-\frac{9}{32}\pi\right)$$

Hence

$$\begin{aligned} 1 + C_1 &= \sqrt{\frac{2}{e}} \\ C_1 &= \sqrt{\frac{2}{e}} - 1 \\ C_3 &= \sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right) \end{aligned}$$

This means that

$$\begin{aligned} y^{in}(\xi) &\sim 1 + C_1\left(1-e^{-\xi}\right) + \varepsilon C_3\left(1-e^{-\xi}\right) \\ &\sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1-e^{-\xi}\right) + \varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)\left(1-e^{-\xi}\right) \end{aligned}$$

Therefore

$$\begin{aligned} y_{\text{uniform}(x)} &\sim y^{in}(\xi) + y^{out}(x) - y_{match} \\ &\sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1-e^{-\xi}\right) + \varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)\left(1-e^{-\xi}\right) \\ &+ \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1+\varepsilon\left(\frac{5}{4}-\frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right) \\ &- \left(\sqrt{\frac{2}{e}} + \sqrt{\frac{2}{e}}\varepsilon\left(\frac{5}{4}-\frac{9}{32}\pi\right)\right) \end{aligned}$$

Or (replacing ξ by $\frac{x}{\varepsilon}$ and simplifying)

$$y_{\text{uniform}(x)} \sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1 - e^{-\xi}\right) - e^{-\xi}\varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right)$$
$$+ \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right)$$
$$- \sqrt{\frac{2}{e}}$$

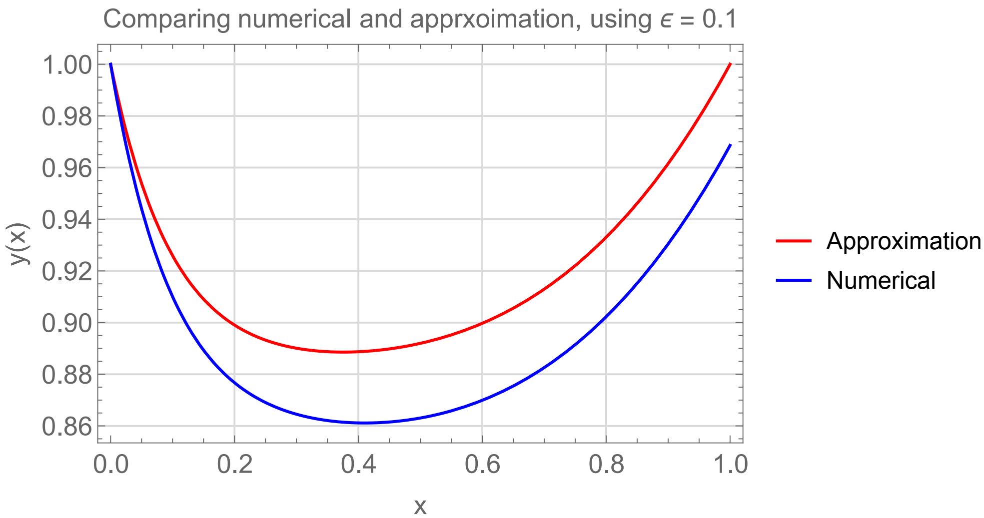
Or

$$y_{\text{uniform}(x)} \sim -\sqrt{\frac{2}{e}}e^{-\xi} + e^{-\xi} - e^{-\xi}\varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right)$$
$$+ \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right)$$

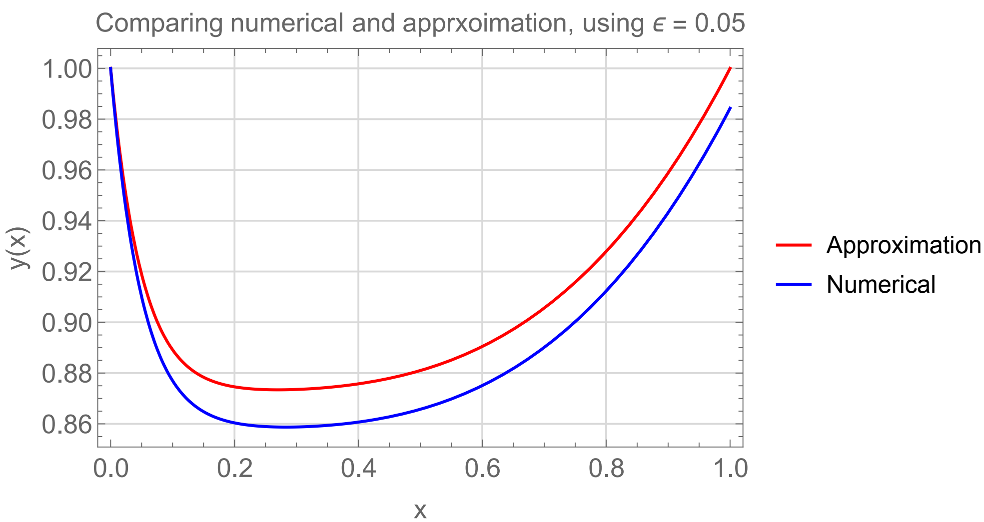
To check validity of the above solution, the approximate solution is plotted against the numerical solution for different values of $\varepsilon = \{0.1, 0.05, 0.01\}$. This shows very good agreement with the numerical solution. At $\varepsilon = 0.01$ the solutions are almost the same.

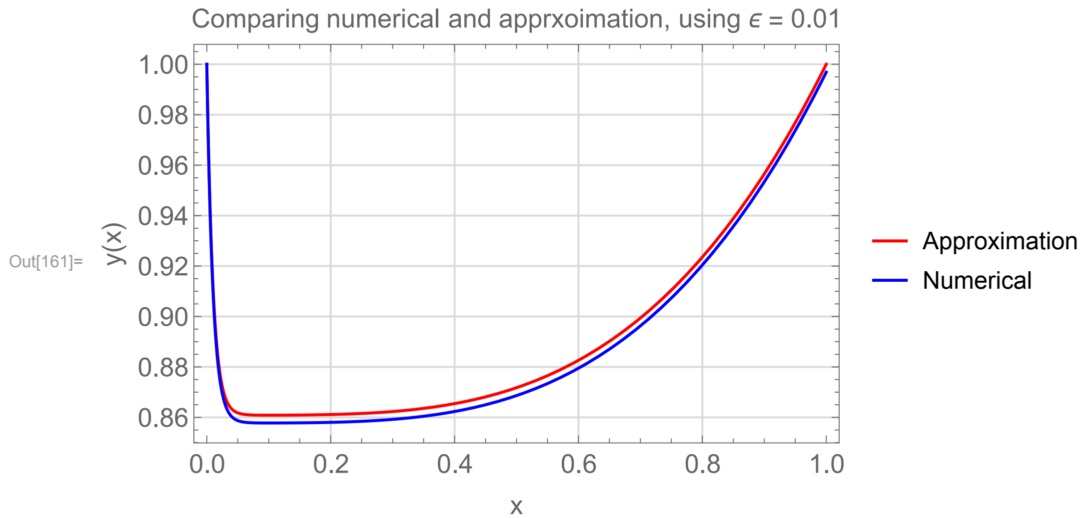
```
ClearAll[x, y, ϵ];
ϵ = 0.01;
r = Sqrt[2/Exp[1]];
ode = ϵ y''[x] + (x^2 + 1) y'[x] - x^3 y[x] == 0;
sol = First@NDSolve[{ode, y[0] == 1, y[1] == 1}, y, {x, 0, 1}];
p1 = Plot[Evaluate[y[x] /. sol], {x, 0, 1}];
mysol[x_, ϵ_] := -r Exp[-x/ϵ] + Exp[-x/ϵ] - ϵ r (5/4 - 9/32 π) + r Exp[x^2/2]/Sqrt[1 + x^2] (1 + ϵ (5/4 - 9/32 π - x + (3 x)/(4 (1 + x^2)^2) - (7 x)/(8 (1 + x^2)) + 9/8 ArcTan[x]));

p2 = Plot[{Evaluate[y[x] /. sol], mysol[x, ϵ]}, {x, 0, 1}, Frame → True, PlotStyle → {Red, Blue},
  FrameLabel → {{"y(x)", None}, {"x", Row[{"Comparing numerical and apprxoimation, using ϵ = ", N@ϵ}]}}, GridLines → Automatic,
  GridLinesStyle → LightGray, PlotLegends → {"Approximation", "Numerical"}, BaseStyle → 14, ImageSize → 400]
```

Out[145]=



Out[153]=





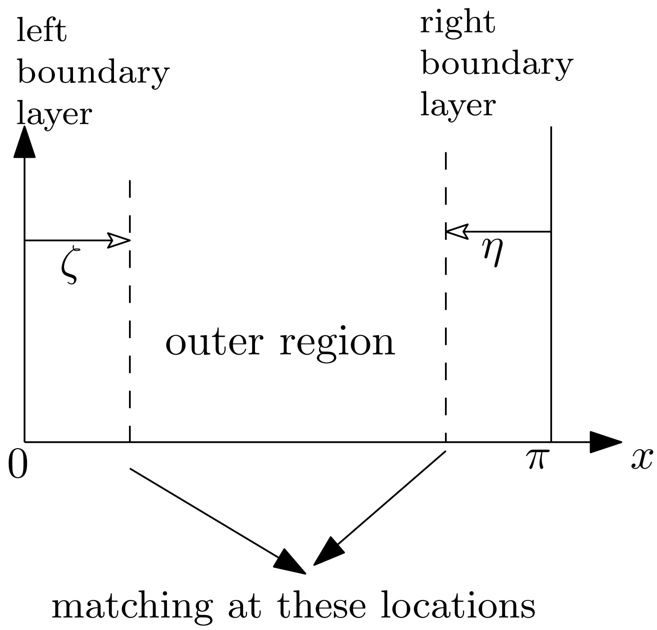
3.3.6 problem 9.19

<u>Problem</u> Find lowest order uniform approximation to boundary value problem

$$\varepsilon y'' + (\sin x)\, y' + y \sin(2x) = 0$$
$$y(0) = \pi$$
$$y(\pi) = 0$$

<u>Solution</u>

We expect a boundary layer at left end at $x = 0$. Therefore, we need to find $y^{in}(\xi), y^{out}(x)$, where ξ is an inner variable defined by $\xi = \frac{x}{\varepsilon^p}$.



Finding $y^{in}(\xi)$

At $x = 0$, we introduce inner variable $\xi = \frac{x}{\varepsilon^p}$ and then express the original ODE using this new variable. We also need to determine p in the above expression. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\
&= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\
&= \varepsilon^{-2p}\frac{d^2}{d\xi^2}
\end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE $\varepsilon y'' + (\sin x)\, y' + y\sin(2x) = 0$ now becomes

$$\begin{aligned}
\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + (\sin(\xi\varepsilon^p))\,\varepsilon^{-p}\frac{dy}{d\xi} + \sin(2\xi\varepsilon^p)\,y = 0 \\
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + (\sin(\xi\varepsilon^p))\,\varepsilon^{-p}\frac{dy}{d\xi} + \sin(2\xi\varepsilon^p)\,y = 0
\end{aligned}$$

Expanding the $\sin$ terms in the above, in Taylor series around zero, $\sin(x) = x - \frac{x^3}{3!} + \cdots$ gives

$$\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi\varepsilon^p - \frac{(\xi\varepsilon^p)^3}{3!} + \cdots\right)\varepsilon^{-p}\frac{dy}{d\xi} + \left(2\xi\varepsilon^p - \frac{(2\xi\varepsilon^p)^3}{3!} + \cdots\right)y = 0$$
$$\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi - \frac{\xi^3\varepsilon^{2p}}{3!} + \cdots\right)\frac{dy}{d\xi} + \left(2\xi\varepsilon^p - \frac{(2\xi\varepsilon^p)^3}{3!} + \cdots\right)y = 0$$

Then the largest terms are $\left\{\varepsilon^{1-2p}, 1\right\}$, therefore $1 - 2p = 0$ or

$$\boxed{p = \frac{1}{2}}$$

The ODE now becomes

$$y'' + \left(\xi - \frac{\xi^3\varepsilon}{3!} + \cdots\right)y' + \left(2\xi\sqrt{\varepsilon} - \frac{\left(2\xi\sqrt{\varepsilon}\right)^3}{3!} + \cdots\right)y = 0 \tag{1}$$

Assuming that

$$y^{left}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Then (1) becomes

$$\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\xi - \frac{\xi^3\varepsilon}{3!} + \cdots\right)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(2\xi\sqrt{\varepsilon} - \frac{\left(2\xi\sqrt{\varepsilon}\right)^3}{3!} + \cdots\right)\left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

Collecting terms in $O(1)$ gives the balance

$$y_0''(\xi) \sim -\xi y_0'(\xi)$$
$$y_0(0) = \pi$$

Assuming $z = y_0'$, then

$$z' \sim -\xi z$$
$$\frac{dz}{z} \sim -\xi$$
$$\ln|z| \sim -\frac{\xi^2}{2} + C_1$$
$$z \sim C_1 e^{\frac{-\xi^2}{2}}$$

Therefore $y_0' \sim C_1 e^{\frac{-\xi^2}{2}}$. Hence

$$y_0(\xi) \sim C_1\int_0^{\xi} e^{\frac{-s^2}{2}}\,ds + C_2$$

With boundary conditions $y(0) = \pi$. Hence

$$\pi = C_2$$

And the solution becomes

$$y_0^{in}(\xi) \sim C_1 \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi \tag{2}$$

Now we need to find $y^{out}(x)$. Assuming that

$$y^{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Then $\varepsilon y'' + (\sin x) y' + y \sin(2x) = 0$ becomes

$$\varepsilon \left(y_0'' + \varepsilon y_1'' + \cdots\right) + \sin(x)\left(y_0' + \varepsilon y_1' + \cdots\right) + \sin(2x)\left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

Collecting terms in $O(1)$ gives the balance

$$\begin{aligned}
\sin(x) y_0'(x) &\sim -\sin(2x) y_0(x) \\
\frac{dy_0}{y_0} &\sim -\frac{\sin(2x)}{\sin(x)} dx \\
\ln|y_0| &\sim -\int \frac{\sin(2x)}{\sin(x)} dx \\
&\sim -\int \frac{2\sin x \cos x}{\sin(x)} dx \\
&\sim -\int 2\cos x dx \\
&\sim -2\sin x + C_5
\end{aligned}$$

Hence

$$\begin{aligned}
y_0^{out}(x) &\sim A e^{-2\sin x} \\
y_0(\pi) &= 0
\end{aligned}$$

Therefore $A = 0$ and $y_0^{out}(x) = 0$.Now that we found all solutions, we can do the matching. The matching on the left side gives

$$\begin{aligned}
\lim_{\xi\to\infty} y^{in}(\xi) &= \lim_{x\to 0} y^{out}(x) \\
\lim_{\xi\to\infty} C_1 \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi &= \lim_{x\to 0} C_5 e^{-2\sin x} \\
\lim_{\xi\to\infty} C_1 \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi &= 0
\end{aligned} \tag{3}$$

But

$$\int_0^{\xi} e^{\frac{-s^2}{2}} ds = \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)$$

And $\lim_{\xi\to\infty}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)=1$, hence (3) becomes

$$\begin{aligned} C_1\sqrt{\frac{\pi}{2}}+\pi &= 0 \\ C_1 &= -\pi\sqrt{\frac{2}{\pi}} \\ &= -\sqrt{2\pi} \end{aligned} \tag{4}$$

Therefore from (2)

$$y_0^{in}(\xi) \sim -\sqrt{2\pi}\int_0^{\xi} e^{\frac{-s^2}{2}}\,ds+\pi \tag{5}$$

Near $x=\pi$, using $\eta=\frac{\pi-x}{\varepsilon^p}$. Expansion $y^{in}(\eta)\sim y_0(\eta)+\varepsilon y_1(\eta)+O(\varepsilon^2)$ gives $p=\frac{1}{2}$. Hence $O(1)$ terms gives

$$\begin{aligned} y_0''(\eta) &\sim \eta y_0'(\eta) \\ y_0^{in}(0) &= 0 \\ y_0^{in}(\eta) &\sim D\int_0^{\eta} e^{\frac{-s^2}{2}}\,ds \end{aligned}$$

And matching on the right side gives

$$\begin{aligned} \lim_{\eta\to\infty} y^{in}(\eta) &= \lim_{x\to\pi} y^{out}(x) \\ \lim_{\eta\to\infty} D\int_0^{\eta} e^{\frac{-s^2}{2}}\,ds &= 0 \\ D &= 0 \end{aligned} \tag{6}$$

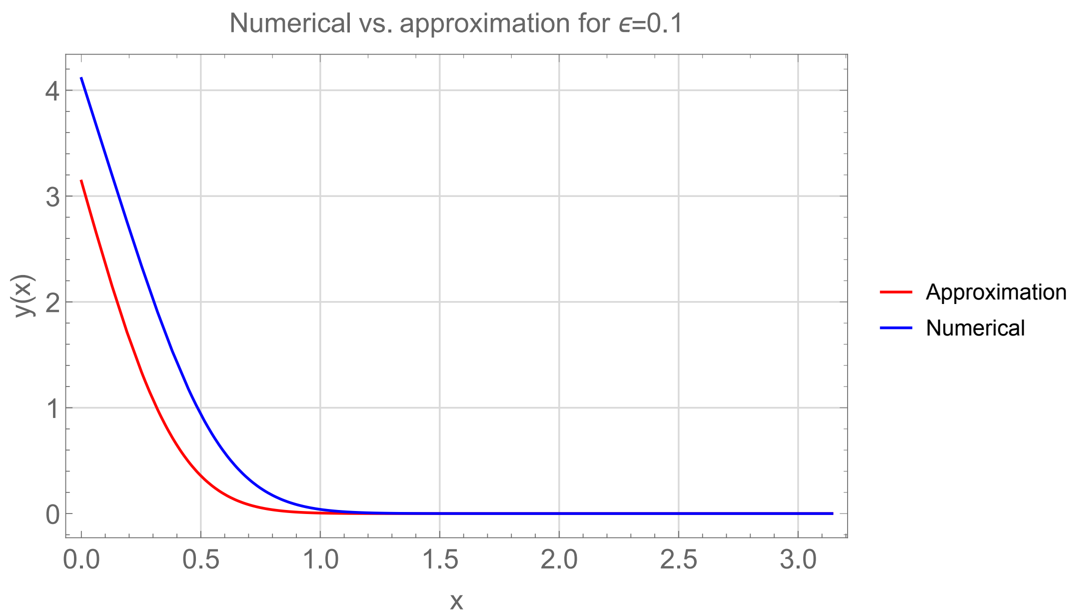
Therefore the solution is

$$\begin{aligned} y(x) &\sim y^{in}(\xi)+y^{in}(\eta)+y^{out}(x)-y^{match} \\ &\sim -\sqrt{2\pi}\int_0^{\xi} e^{\frac{-s^2}{2}}\,ds+\pi+0 \\ &\sim -\sqrt{2\pi}\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)+\pi \\ &\sim -\sqrt{2\pi}\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)+\pi \\ &\sim \pi-\pi\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) \end{aligned} \tag{7}$$

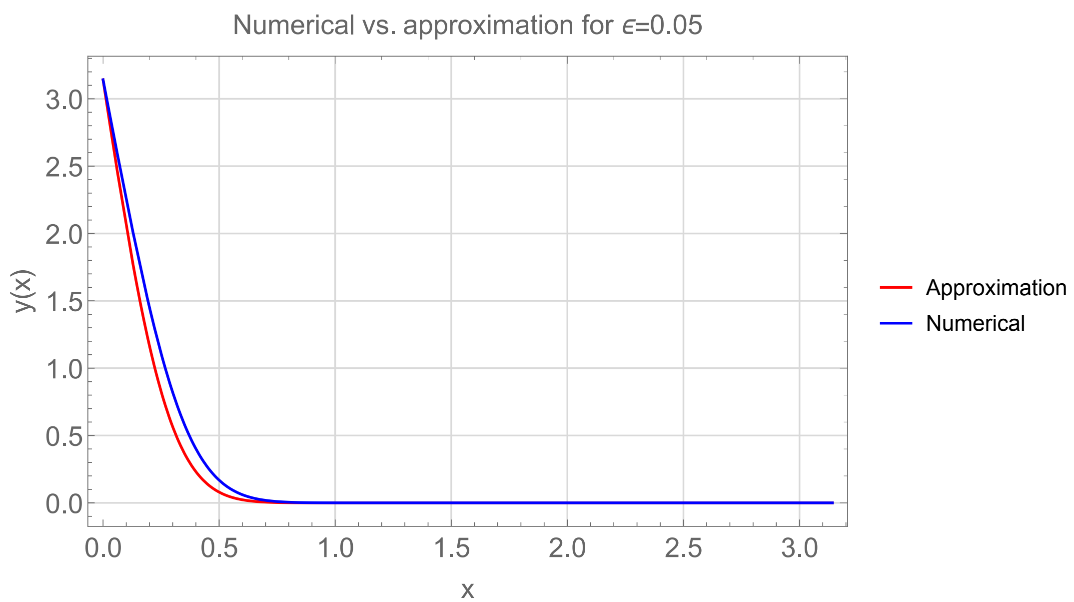
The following plot compares exact solution with (7) for $\varepsilon=0.1, 0.05$. We see from these results, that as ε decreased, the approximation solution improved.

```
In[37]:= ClearAll[x, ϵ, y]
mySol[x_, ϵ_] := Pi - Pi Erf[x/Sqrt[2 ϵ]];
ϵ = 0.1;
ode = ϵ y''[x] + Sin[x] y'[x] + y[x] Sin[2 x] == 0;
sol = NDSolve[{ode, y[0] == π, y'[π] == 0}, y, {x, 0, π}];
Plot[{mySol[x, ϵ], Evaluate[y[x] /. sol]}, {x, 0, Pi}, Frame → True,
 FrameLabel → {{"y(x)", None}, {"x", Row[{"Numerical vs. approximation for ϵ=", ϵ}]}}, GridLines → Automatic,
 GridLinesStyle → LightGray, BaseStyle → 16, ImageSize → 500, PlotLegends → {"Approximation", "Numerical"},
 PlotStyle → {Red, Blue}, PlotRange → All]
```

Out[42]=



Out[54]=



3.3.7 key solution of selected problems

3.3.7.1 section 9 problem 9

Problem 9.9 asks us to use boundary layer theory to find the leading order solution to the initial value problem $\varepsilon y''(x) + ay'(x) + by(x) = 0$ with $y(0) = y'(0) = 1$ and $a > 0$. Then we are to compare to the exact solution. The problem is ambiguous as to whether a and b are functions or constants. For clarity's sake, we assume a and b are constants, but all the following work can be generalized for nonconstant a and b as well.

Since $a > 0$, the boundary layer occurs at $x = 0$, where the initial conditions are specified. We set $x = \varepsilon\xi$,[1] and then in the inner region,

$$\frac{1}{\varepsilon}y''_{\text{in}}(\xi) + \frac{a}{\varepsilon}y'_{\text{in}}(\xi) + by_{\text{in}}(\xi) = 0. \tag{1}$$

Thus, to leading order, $y''_{\text{in}}(\xi) \sim -ay'_{\text{in}}(\xi)$, which has solution $y_{\text{in}}(\xi) = C_0 + C_1e^{-a\xi}$. Solving for C_0 and C_1, we see that $y(0) = 1 \implies C_0 + C_1 = 1$, and

$$y'(x)\bigg|_{x=0} = 1 \implies y'(\xi)\bigg|_{\xi=0} = \varepsilon \implies -aC_1 = \varepsilon \implies C_1 = -\varepsilon/a = O(\varepsilon).$$

This means that $C_1e^{-a\xi} = O(\varepsilon)$ shouldn't appear at this order in the expansion, and $C_1 = 0$. We should throw this information out because we have already thrown out information at $O(\varepsilon)$ in solving the equation, and we have no guarantee that the $O(\varepsilon)$ value for C_1 is actually correct to $O(\varepsilon)$.

So there is no boundary layer at leading order! The inner solution $y_{\text{in}}(x) = C_0 = 1$ does not change rapidly, and it will cancel when we match, just leaving the outer solution. This is okay and happens occasionally when you get lucky.

In the outer region, we have $y_{\text{out}} \sim -\frac{b}{a}y_{\text{out}}$, so $y_{out}(x) = Ce^{-bx/a} + O(\varepsilon)$. Matching to the inner solution, $C = 1$, so $y_{\text{uniform}}(x) = e^{-bx/a} + O(\varepsilon)$. We note that $y'_{\text{uniform}}(0) = -\frac{b}{a} \neq 1$ in general.

Although the problem does not ask for it, we can also go to the next order in our asymptotic expansion. And even though no boundary layer appeared at leading order, one will appear at $O(\varepsilon)$.

Going back to the inner region, let $y_{\text{in}}(\xi) = Y_0(\xi) + \varepsilon Y_1(\xi) + O(\varepsilon^2)$. We already computed that $Y_0(\xi) = 1$. Now looking at (1) at $O(1)$, $Y_1''(\xi) + aY_1'(\xi) + bY_0(\xi) = 0 \implies Y_1''(\xi) + aY_1'(\xi) = -b$. Solving, $Y_1(\xi) = C_2 + C_3e^{-a\xi} - \frac{b}{a}\xi$. We have initial conditions $Y_1(0) = 0$, but since $y'(0) = 1$ was not satisfied, we note that $Y_1'(0) = 1$. Thus, $C_2 + C_3 = 0$ and $-aC_3 - \frac{b}{a} = 1$, so

$$C_3 = -\frac{1}{a} - \frac{b}{a^2} \qquad \text{and} \qquad C_2 = -C_3 = \frac{1}{a} + \frac{b}{a^2}.$$

Therefore,

$$y_{\text{in}}(\xi) = 1 + \varepsilon\left(\left(\frac{1}{a} + \frac{b}{a^2}\right)\left(1 - e^{-a\xi}\right) - \frac{b}{a}\xi\right) + O(\varepsilon^2),$$

so

$$y_{\text{in}}(x) = 1 + \varepsilon\left(\left(\frac{1}{a} + \frac{b}{a^2}\right)\left(1 - e^{-ax/\varepsilon}\right) - \frac{b}{a}\frac{x}{\varepsilon}\right) + O(\varepsilon^2).$$

Turning to the outer solution, let $y_{\text{out}}(x) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$. We already found that $y_0(x) = e^{-bx/a}$. At order ε, the outer equation reads:

$$y_0''(x) + ay_1'(x) + by_1(x) = 0 \implies y_1'(x) + \frac{b}{a}y_1(x) = -\frac{1}{a}e^{-bx/a}.$$

[1] We know that we can use ε instead of ε^p for some unknown constant p because a is positive near zero, and $p \neq 1$ only occurs when $a(x) \to 0$ at the boundary layer.

1

Therefore,

$$y_1(x) = C_4 e^{-bx/a} - \frac{x}{a} e^{-bx/a},$$

and

$$y_{\text{out}}(x) = e^{-bx/a} + \varepsilon \left(\left(C_4 - \frac{x}{a} \right) e^{-bx/a} \right) + O(\varepsilon^2).$$

Now we match. As $x \to 0+$, the outer solution goes to

$$y_{\text{out}}(x) \sim 1 + C_4 \varepsilon + O(\varepsilon^2),$$

while as $\xi \to \infty$, the inner solution approaches

$$y_{\text{in}}(x) \sim 1 + \left(\frac{1}{a} + \frac{b}{a^2} \right) \varepsilon + O(\varepsilon^2),$$

where I have cheated slightly.[2] Matching,

$$C_4 = \frac{1}{a} + \frac{b}{a^2},$$

and thus,

$$y_{\text{uniform}}(x) = e^{-bx/a} + \varepsilon \left(\left(\left(\frac{1}{a} + \frac{b}{a^2} \right) - \frac{x}{a} \right) e^{-bx/a} - \left(\frac{1}{a} + \frac{b}{a^2} \right) e^{-ax/\varepsilon} \right) + O(\varepsilon^2).$$

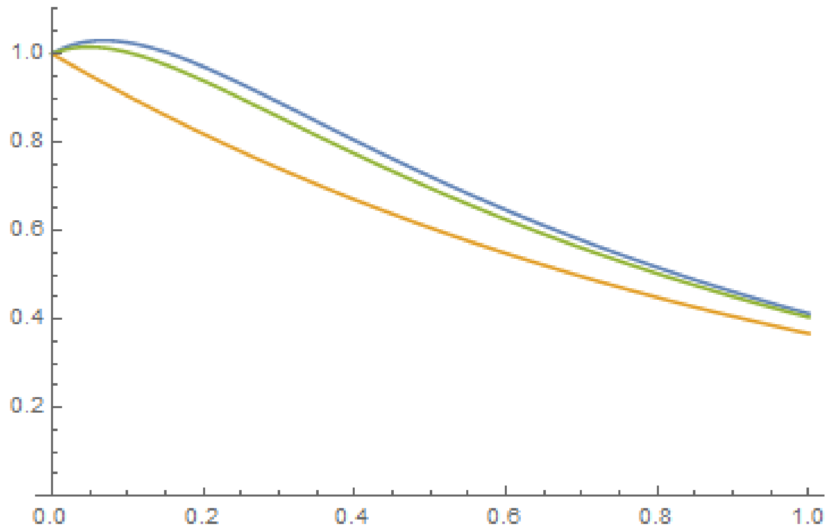
Let us now graphically compare the asymptotic solutions accurate to $O(1)$ and $O(\varepsilon)$ with the exact solution:

$$y_{\text{exact}}(x) = \frac{1}{2\sqrt{a^2 - 4b\varepsilon}} \left(-a e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} - 2\varepsilon e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + \sqrt{a^2 - 4b\varepsilon}\, e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} \right.$$
$$\left. + a e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + 2\varepsilon e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + \sqrt{a^2 - 4b\varepsilon}\, e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} \right).$$

For simplicity, take $a = b = 1$.

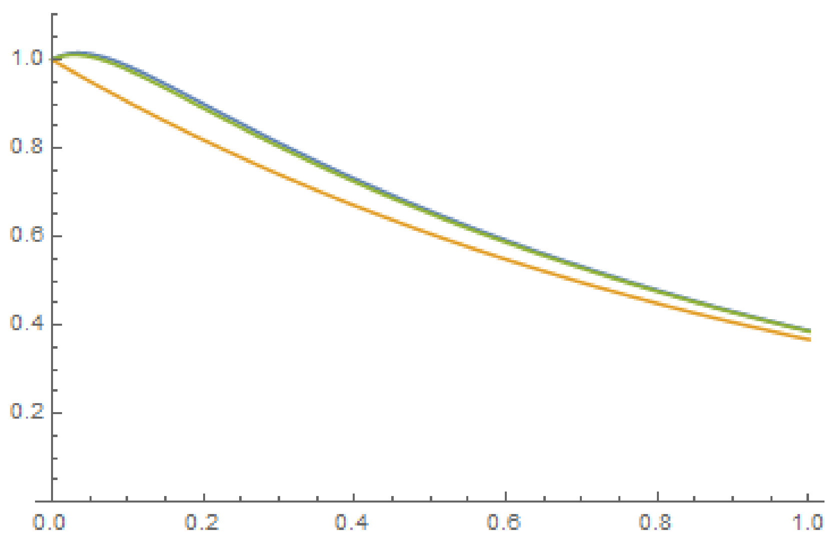
[2] I ignored the term which was linear in ξ, which blows up as $\xi \to \infty$. This term came from ignoring the outer expansion, which changes on the same order, and is called a secular term. This occurs because we ignored the outer expansion, as is conventional when working with the boundary layer, but the outer solution, when Taylor expanded about zero, matches this term exactly at order x, and so we do not have any problems. The outer solution changes at a slow rate compared to ξ, and so the appearance of two time scales in the boundary layer causes problems with the match (which I swept under the rug by cheating). This problem is therefore far better suited for multiscale methods, which form the subject matter of Chapter 11.

For the first graph, $\varepsilon = 0.1$:



The blue curve represents the exact solution, the orange curve the uniform solution to leading order, and the green curve the uniform solution accurate to $O(\varepsilon)$. We see that the differences between the curves remains small always, and that the higher order approximation is much closer to the exact solution. The leading order solution never differs by more than about $0.1 \approx \varepsilon$, while the next order solution differs from the exact solution by a much smaller amount (approximately $O(\varepsilon^2)$).

Now let $\varepsilon = 0.05$:



The color scheme is the same as before. We notice that the same qualitative observations from before hold for this graph as well. The difference between the orange and blue curves is even half as much, in good agreement with our $O(\varepsilon)$ error estimation. This also validates our conclusion that the boundary layer only appears at $O(\varepsilon)$. It is somewhat more difficult to verify pictorially that the error for the green curve is $O(\varepsilon^2)$, but it is also clear that the error in this plot is less than in the first plot, and by a greater factor than two.

3.3.7.2 section 9 problem 15

①

Example 2

Lets set up the problems for

$$\varepsilon y'' + (x^2+1)y' - x^3 y = 0 \qquad y(0) = y(1) = 1$$

Expect a bundary layer near $x=0$. Why?

$$-(x^2+1)y' \approx \varepsilon y''$$

$$v y' \approx \varepsilon y'' \qquad \underset{\text{pushing } y \text{ to left}}{v<0}, \quad \varepsilon > 0$$

Consider the "outer" problem $x > 0$, $\varepsilon \to 0$

Let $y_{out} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$

$$\varepsilon\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)'' + (x^2+1)\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)'$$
$$-x^3\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

$$y(1) = 0$$

order by order $\Longrightarrow$

(2)

ε^0: $(x^2+1)y_0' - x^3 y_0 = 0 \quad y(1) = 1$

ε^1: $(x^2+1)y_1' - x^3 y_1 = -y_0'' \quad y_1(1) = 0$

ε^2: $(x^2+1)y_2' - x^3 y_2 = -y_1'' \quad y_2(1) = 0$ etc.

$$\Rightarrow \quad y_0(x) = \frac{\sqrt{2}}{e^{1/2}} \frac{e^{x^2/2}}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad y_1 = y_0\left[-x + \frac{9}{8}\tan^{-1}x - \frac{7}{8}\frac{x}{(x^2+1)} + \frac{3}{4}\frac{x}{(1+x^2)^2} + \frac{5}{4} - \frac{9}{32}\pi\right]$$

$$\boxed{y_{out} = y_0 + \varepsilon y_1 + O(\varepsilon^2)}$$

Now find inner solution let $\xi = \frac{x}{\varepsilon}$ $\quad x = O(\varepsilon)$, $\varepsilon \to 0^+$

$$\frac{1}{\varepsilon^2}\varepsilon \frac{d^2y}{d\xi^2} + (\varepsilon^2\xi^2+1)\frac{1}{\varepsilon}\frac{dy}{d\xi} + \varepsilon^3\xi^3 y = 0$$

$$\frac{d^2y}{d\xi^2} + (\varepsilon^2\xi^2+1)\frac{dy}{d\xi} + \varepsilon^4\xi^3 y = 0$$

(3) (b)

Let $y_{in}(\xi) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$

ε^0: $y_0'' + y_0' = 0 \quad y_0(0) = 1$

ε^1: $y_1'' + y_1' = 0 \quad y_1(0) = 0$

ε^2: $y_2'' + \xi^2 y_2' = 0 \quad y_2(0) = 0$ etc.

$$y_0 = A(1 - e^{-\xi}) + 1$$
$$y_1 = B(1 - e^{-\xi})$$

$y_{in} = y_0 + \varepsilon y_1 + O(\varepsilon^2)$

Now match order by order

$$\lim_{\frac{x}{\varepsilon} \to \infty} y_{in} = \lim_{x \to 0} y_{out}$$

ε^0: $\frac{\sqrt{2}}{e^{1/2}} = 1 + A \implies A = \frac{\sqrt{2}}{e^{1/2}} - 1$

3.3.7.3 section 9 problem 19

Here is an outline for the solution to BO 9.19.

The boundary value problem is

$$\epsilon y'' + \sin(x)y' + 2\sin(x)\cos(x)y = 0, \quad y(0) = \pi, \quad y(\pi) = 0$$

using $\sin(2x) = 2\sin(x)\cos(x)$. Our heuristic argument based on the 1D advection-diffusion equation suggests a boundary layer on the left at $x = 0$. An outer expansion $y_{\text{out}}(x) \sim y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$ gives

$$O(1): \quad y_o' \sim -2\cos(x)y_o$$

with solution

$$y_o(x) \sim A\exp[-2\sin(x)]$$

which cannot satisfy $y(\pi) = 0$ unless $A = 0$ (see below).

For the inner solution near $x = 0$ let $\xi = x/\epsilon^p$. The expansion $y_{\text{in}}(\xi) \sim y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2)$ leads to $p = 1/2$ and

$$O(1): \quad y_o''(\xi) \sim -\xi y_o'(\xi), \quad y(\xi = 0) = \pi$$

with solution

$$y_o(\xi) \sim \pi + C\int_0^{\xi} \exp(-t^2/2)dt$$

$$y_o(\xi) \sim \pi + \sqrt{2}C\int_0^{\xi/\sqrt{2}} \exp(-s^2)ds = \pi + \sqrt{2}C\frac{\sqrt{\pi}}{2}\text{erf}(\xi/\sqrt{2}).$$

On the other side near $x = \pi$, let $\eta = (\pi - x)/\epsilon^p$. The expansion $y_{\text{in}}(\eta) \sim y_o(\eta) + \epsilon y_1(\eta) + O(\epsilon^2)$ leads to $p = 1/2$ and

$$O(1): \quad y_o''(\eta) \sim \eta y_o'(\eta), \quad y(\eta = 0) = 0$$

using $\sin(\pi - \epsilon^{1/2}\eta) = \sin(\epsilon^{1/2}\eta) \sim \epsilon^{1/2}\eta$ for $\epsilon^{1/2}\eta \to 0$. Then

$$y_o(\eta) \sim D\int_0^{\eta} \exp(t^2/2)dt$$

Now matching

$$\lim_{\xi\to\infty} y_{\text{in}}(\xi) = \lim_{x\to 0} y_{\text{out}}(x)$$

$$\lim_{\eta\to\infty} y_{\text{in}}(\eta) = \lim_{x\to\pi} y_{\text{out}}(x)$$

gives $A = D = 0$ and $C = -\sqrt{2\pi}$. Thus we obtain

$$y(x) \sim \pi - \pi\ \text{erf}(x/\sqrt{2\epsilon}).$$

1

3.3.7.4 section 9 problem 4b

NEEP 548: Engineering Analysis II 2/27/2011

Problem 9.4-b: Bender & Orszag

Instructor: Leslie Smith

1 Problem Statement

Find the boundary layer solution for

$$\epsilon y'' + (x^2+1)y' - x^3 y = 0 \tag{1}$$

2 Solution

We can expect a boundary layer at $x = 0$.

2.1 Outer Solution

$$(x^2+1)y_o' - x^3 y_o = 0$$
$$y_o' - \left(\frac{x^3}{x^2+1}\right) y_o = 0$$

This is in a form that we can find an integrating factor for,

$$\mu = \exp\left(\int \frac{\mathrm{x}^3}{\mathrm{x}^2+1}\mathrm{dx}\right) = \exp\left[-\frac{1}{2}(\ln[\mathrm{x}^2+1] - \mathrm{x}^2)\right] = (x^2+1)^{-1/2}e^{-x^2/2}$$
$$\implies [y_o(x^2+1)^{-1/2}e^{-x^2/2}]' = 0$$

$$y_o = C_1(x^2+1)^{-1/2}e^{x^2/2}$$

B.C.: $y(1) = 1 \implies 1 = C_1(2)^{-1/2}e^{1/2} \implies C_1 = \sqrt{2}e^{-1/2}$. Therefore, the outer solution is:

$$\boxed{y_o = \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2}} \tag{2}$$

1

2.2 Inner Solution

Define $\xi = x/\epsilon$ such that when $x \sim \epsilon$, $\xi \sim 1$

$$\frac{\partial y}{\partial x} = \frac{dy}{d\xi}\frac{d\xi}{dx}$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\epsilon^2}\frac{d^2 y}{d\xi^2}$$

Plug this into the D. E. to get,

$$\frac{1}{\epsilon}\frac{d^2y}{d\xi^2} + \epsilon\xi^2\frac{dy}{d\xi} + \frac{1}{\epsilon}\frac{dy}{d\xi} - \xi^2\epsilon^2 y = 0 \tag{3}$$

The let the inner solution take the form,

$$y_{in} = y_o(\xi) + \epsilon y_1(\xi) + \epsilon^2 y_2(\xi) + ...$$

Balance terms, order by order,

$$O\left(\frac{1}{\epsilon}\right): \quad y_o'' + y_o' = 0 \qquad y(0) = 1$$
$$O(1): \quad y_1'' + y_1' = 0 \qquad y(0) = 0$$

To the lowest order,

$$y_{in}'' + y_{in}' = 0 \qquad \text{let} \qquad z = y_{in}'$$

$$\begin{aligned} \implies \ln z &= -\xi + C_2 \\ \ln y_{in}' &= -\xi + C_2 \\ y_{in}' &= C_3 e^{-\xi} \\ y_{in} &= C_3 e^{\xi} + C_4 \end{aligned}$$

We can solve for one of the constants by imposing the boundary conditions, $y_{in}(0) = 1 \implies 1 = -C_3 + C_4 \implies C_4 = 1 + C_3$

$$\boxed{y_{in} = 1 + C\left(1 - e^{-\xi}\right)} \tag{4}$$

The other constant we get by matching the 'outer' and 'inner' solutions:

2

$$\lim_{x\to 0} y_o = \lim_{\xi\to\infty} y_{in}$$
$$\lim_{x\to 0} \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2} = \lim_{\xi\to\infty}[1 + C(1-e^{-\xi})]$$
$$\sqrt{2}e^{-1/2} = 1 + C$$
$$\implies C = \sqrt{2}e^{-1/2} - 1$$

Note: $y_{match} = \sqrt{2}e^{-1/2}$

$$y_{tot} = y_{in} + y_{out} - y_{match} \tag{5}$$

$$\boxed{y_{tot} = \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2} + e^{-x/\epsilon}(1-\sqrt{2}e^{-1/2})} \tag{6}$$

3.3.7.5 section 9 problem 6

Here is an outline for the solution to BO 9.6.

Write the equation as

$$y' = (1 + \epsilon x^{-2})y^2 - 2y + 1, \quad y(1) = 1$$

and we want to solve this on $0 \leq x \leq 1$. For the outer solution, the expansion $y_{\text{out}}(x) \sim y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$ gives

$$O(1): \quad y_o' \sim (y_o - 1)^2, \quad y_o(1) = 1$$

$$O(\epsilon): \quad y_1' \sim 2y_o y_1 + y_o^2 x^{-2} - 2y_1, \quad y_1(1) = 0$$

with solutions

$$y_o(x) \sim 1, \quad y_1(x) \sim 1 - x^{-1}$$

valid away from zero. So $y_{\text{out}}(x) \sim 1 + \epsilon(1 - x^{-1}) + O(\epsilon^2)$.

For the inner solution let $\xi = x/\epsilon^p$. The expansion $y_{\text{out}}(\xi) \sim y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2)$ leads to $\xi \propto \epsilon$ (so we take $\xi = \epsilon$ for convenience). The biggest terms are:

$$O(\epsilon^{-1}): \quad y_o' \sim y_o^2 \xi^{-2}$$

with solution

$$y_o(\xi) \sim \xi(1 - A\xi)^{-1}.$$

Now matching

$$\lim_{\xi \to \infty} y_o(\xi) + O(\epsilon) = \lim_{x \to 0} y_o(x) + O(\epsilon)$$

$$\lim_{\xi \to \infty} \xi(1 - A\xi)^{-1} + O(\epsilon) = \lim_{x \to 0} 1 + O(\epsilon)$$

$$\lim_{\xi \to \infty} -A^{-1} + O(\xi^{-1}) + O(\epsilon) = \lim_{x \to 0} 1 + O(\epsilon)$$

gives $A = -1$ with matching region $\epsilon \ll x \ll 1$.

The next-order problem is

$$O(1): \quad y_1' \sim y_o^2 + 2y_o y_1 \xi^{-2} - 2y_o + 1.$$

Use $y_o(\xi)$ from above and the integrating factor method to find

$$y_1(\xi) \sim -\xi(1 + \xi)^{-2} + C\xi^2(1 + \xi)^{-2}.$$

Now matching

1

$$\lim_{\xi\to\infty} y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2) = \lim_{x\to 0} y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} \xi(1+\xi)^{-1} + \epsilon\bigg(C\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\bigg) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon\bigg(1 - x^{-1}\bigg) + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} \frac{1}{1+\xi^{-1}} + \epsilon\bigg(C\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\bigg) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon - \frac{\epsilon}{x} + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} 1 - \xi^{-1} + O(\xi^{-2}) + \epsilon C + O(\epsilon\xi^{-1}) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon - \frac{\epsilon}{x} + O(\epsilon^2)$$

and so we choose $C = 1$.

Question for you: what is the matching region? (I think $\epsilon^{1/2} \ll x \ll 1$). Is this correct?

Now we form the uniform approximation $y \sim y_{\text{in}} + y_{\text{out}} - y_{\text{match}} + O(\epsilon^2)$:

$$y \sim 1 + \epsilon(1 - x^{-1}) + \xi(1+\xi)^{-1} + \epsilon\bigg(\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\bigg) - \bigg(1 + \epsilon - \frac{\epsilon}{x}\bigg) + O(\epsilon^2)$$

with $\xi = x/\epsilon$ and simplify. Notice that there is no singularity at $x = 0$ the solution is uniformly valid on $0 \leq x \leq 1$.

More questions for you: Does the solution above satisfy the initial condition $y(1) = 1$? If not, can you add an $O(\epsilon^2)$ correction so that $y(1) = 1$? Will you still have a solution uniformly valid to $O(\epsilon^2)$?

I encourage you to make plots so that you can visualize the solution!

2

3.4 HW4

3.4.1 problem 10.5 (page 540)

problem Use WKB to obtain solution to

$$\varepsilon y'' + a(x) y' + b(x) y = 0 \tag{1}$$

with $a(x) > 0, y(0) = A, y(1) = B$ correct to order ε.

solution

Assuming

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0$$

Therefore

$$y'(x) \sim \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)$$

$$y''(x) \sim \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) + \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2 \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)$$

Substituting the above into (1) and simplifying gives (writing = instead of $\sim$ for simplicity for now)

$$\varepsilon\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x) + \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2\right] + \frac{a}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x) + b = 0$$

$$\frac{\varepsilon}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x) + \frac{\varepsilon}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n'(x)\sum_{n=0}^{\infty}\delta^n S_n'(x)\right) + \frac{a}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x) + b = 0$$

Expanding gives

$$\frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) + \frac{\varepsilon}{\delta^2}\left(\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\right) + \frac{a}{\delta}\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right) + b = 0$$

Simplifying

$$\left(\frac{\varepsilon}{\delta}S_0'' + \varepsilon S_1'' + \varepsilon\delta S_2'' + \cdots\right) + \left(\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2 + \frac{2\varepsilon}{\delta}\left(S_0'S_1'\right) + \varepsilon\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \left(\frac{a}{\delta}S_0' + aS_1' + a\delta S_2' + \cdots\right) + b = 0 \tag{1A}$$

The largest terms in the left are $\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2$ and $\frac{a}{\delta}S_0'$. By dominant balance these must be equal in magnitude. Hence $\frac{\varepsilon}{\delta^2} = O\left(\frac{1}{\delta}\right)$ or $\frac{\varepsilon}{\delta} = O(1)$. Therefore δ is proportional to ε and for simplicity

ε is taken as equal to δ, hence (1A) becomes

$$\left(S_0'' + \varepsilon S_1'' + \varepsilon^2 S_2'' + \cdots\right) + \left(\varepsilon^{-1}\left(S_0'\right)^2 + 2S_0'S_1' + \varepsilon\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \left(a\varepsilon^{-1}S_0' + aS_1' + a\varepsilon S_2' + \cdots\right) + b = 0$$

Terms of $O\left(\varepsilon^{-1}\right)$ give

$$\left(S_0'\right)^2 + aS_0' = 0 \tag{2}$$

And terms of $O(1)$ give

$$S_0'' + 2S_0'S_1' + aS_1' + b = 0 \tag{3}$$

And terms of $O(\varepsilon)$ give

$$2S_0'S_2' + aS_2' + \left(S_1'\right)^2 + S_1'' = 0$$

$$S_2' = -\frac{\left(S_1'\right)^2 + S_1''}{\left(a + 2S_0'\right)} \tag{4}$$

Starting with (2)

$$S_0'\left(S_0' + a\right) = 0$$

There are two cases to consider.

case 1 $S_0' = 0$. This means that $S_0(x) = c_0$. A constant. Using this result in (3) gives an ODE to solve for $S_1(x)$

$$\begin{aligned} aS_1' + b &= 0 \\ S_1' &= -\frac{b(x)}{a(x)} \\ S_1 &\sim -\int_0^x \frac{b(t)}{a(t)} dt + c_1 \end{aligned}$$

Using this result in (4) gives an ODE to solve for $S_2(x)$

$$\begin{aligned} S_2' &= -\frac{\left(-\frac{b(x)}{a(x)}\right)^2 + \left(-\frac{b(x)}{a(x)}\right)'}{a(x)} \\ &= -\frac{\frac{b^2(x)}{a^2(x)} - \left(\frac{b'(x)}{a(x)} - \frac{b(x)a'(x)}{a^2(x)}\right)}{a(x)} \\ &= -\frac{\frac{b^2(x)}{a^2(x)} - \frac{a(x)b'(x)}{a^2(x)} + \frac{a'(x)b(x)}{a^2(x)}}{a(x)} \\ &= \frac{a(x)b'(x) - b^2(x) - a'(x)b(x)}{a^3(x)} \end{aligned}$$

Therefore

$$S_2 = \int_0^x \frac{a(t)\,b'(t) - b^2(t) - a'(t)\,b(t)}{a^3(t)}dt + c_2$$

For case one, the solution becomes

$$\begin{aligned}
y_1(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0 \\
&\sim \exp\left(\frac{1}{\varepsilon}\left(S_0(x) + \varepsilon S_1(x) + \varepsilon^2 S_2(x)\right)\right) \qquad \varepsilon \to 0^+ \\
&\sim \exp\left(\frac{1}{\varepsilon}S_0(x) + S_1(x) + \varepsilon S_2(x)\right) \\
&\sim \exp\left(\frac{1}{\varepsilon}c_0 - \int_0^x \frac{b(t)}{a(t)}dt + c_1 + \varepsilon\int_0^x \frac{a(t)\,b'(t) - b^2(t) - b(t)}{a^3(t)}dt + c_2\right) \\
&\sim C_1 \exp\left(-\int_0^x \frac{b(t)}{a(t)}dt + \varepsilon\int_0^x \frac{a(t)\,b'(t) - b^2(t) - b(t)}{a^3(t)}dt\right) \qquad (5)
\end{aligned}$$

Where C_1 is a constant that combines all $e^{\frac{1}{\varepsilon}c_0 + c_1 + c_2}$ constants into one. Equation (5) gives the first WKB solution of order $O(\varepsilon)$ for case one. Case 2 now is considered.

<u>case 2</u> In this case $S_0' = -a$, therefore

$$S_0 = -\int_0^x a(t)\,dt + c_0$$

Equation (3) now gives

$$\begin{aligned}
S_0'' + 2S_0'S_1' + aS_1' + b &= 0 \\
-a' - 2aS_1' + aS_1' + b &= 0 \\
-aS_1' &= a' - b \\
S_1' &= \frac{b - a'}{a} \\
S_1' &= \frac{b}{a} - \frac{a'}{a}
\end{aligned}$$

Integrating the above results in

$$S_1 = \int_0^x \frac{b(t)}{a(t)}dt - \ln(a) + c_1$$

$S_2(x)$ is now found from (4)

$$\begin{aligned}
S_2' &= -\frac{\left(S_1'\right)^2 + S_1''}{\left(a + 2S_0'\right)} \\
&= -\frac{\left(\frac{b-a'}{a}\right)^2 + \left(\frac{b-a'}{a}\right)'}{\left(a + 2\left(-a\right)\right)} \\
&= -\frac{\frac{b^2+(a')^2-2ba'}{a^2} + \frac{b'-a''}{a} - \frac{a'b-(a')^2}{a^2}}{-a} \\
&= \frac{b^2 + (a')^2 - 2ba' + ab' - aa'' - a'b - (a')^2}{a^3} \\
&= \frac{b^2 - 2ba' + ab' - aa'' - a'b}{a^3}
\end{aligned}$$

Hence

$$S_2 = \int_0^x \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)} dt + c_2$$

Therefore for this case the solution becomes

$$\begin{aligned}
y_2(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0 \\
&\sim \exp\left(\frac{1}{\varepsilon}\left(S_0(x) + \varepsilon S_1(x) + \varepsilon^2 S_2(x)\right)\right) \qquad \varepsilon \to 0^+ \\
&\sim \exp\left(\frac{1}{\varepsilon}S_0(x) + S_1(x) + \varepsilon S_2(x)\right)
\end{aligned}$$

Or

$$y_2(x) \sim \exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt + c_0\right)\exp\left(\int_0^x \frac{b(t)}{a(t)}dt - \ln(a) + c_1\right)$$
$$\exp\left(\varepsilon\int_0^x \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt + c_2\right)$$

Which simplifies to

$$y_2(x) \sim \frac{C_2}{a}\exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt + \int_0^x \frac{b(t)}{a(t)}dt + \varepsilon\int_0^x \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt\right) \tag{6}$$

Where C_2 is new constant that combines c_0, c_1, c_2 constants. The general solution is linear combinations of y_1, y_2

$$y(x) \sim Ay_1(x) + By_2(x)$$

Or

$$y(x) \sim C_1 \exp\left(-\int_0^x \frac{b(t)}{a(t)}dt + \varepsilon \int_0^x \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)}dt\right)$$

$$+\frac{C_2}{a(x)} \exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt + \int_0^x \frac{b(t)}{a(t)}dt + \varepsilon \int_0^x \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt\right)$$

Now boundary conditions are applied to find C_1, C_2. Using $y(0) = A$ in the above gives

$$A = C_1 + \frac{C_2}{a(0)} \tag{7}$$

And using $y(1) = B$ gives

$$B = C_1 \exp\left(-\int_0^1 \frac{b(t)}{a(t)}dt + \varepsilon \int_0^1 \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)}dt\right)$$

$$+\frac{C_2}{a(1)} \exp\left(\frac{-1}{\varepsilon}\int_0^1 a(t)\,dt + \int_0^1 \frac{b(t)}{a(t)}dt + \varepsilon \int_0^1 \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt\right)$$

Neglecting exponentially small terms involving $e^{\frac{-1}{\varepsilon}}$ the above becomes

$$B = C_1 \exp\left(-\int_0^1 \frac{b(t)}{a(t)}dt\right) \exp\left(\varepsilon \int_0^1 \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)}dt\right)$$

$$+ \frac{C_2}{a(1)} \exp\left(\int_0^1 \frac{b(t)}{a(t)}dt\right) \exp\left(\varepsilon \int_0^1 \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt\right) \tag{8}$$

To simplify the rest of the solution which finds C_1, C_2, let

$$z_1 = \int_0^1 \frac{b(t)}{a(t)}dt$$

$$z_2 = \int_0^1 \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)}dt$$

$$z_3 = \int_0^1 \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)}dt$$

Hence (8) becomes

$$B = C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{C_2}{a(1)} e^{z_1} e^{\varepsilon z_3} \tag{8A}$$

From (7) $C_2 = a(0)(A - C_1)$. Substituting this in (8A) gives

$$
\begin{aligned}
B &= C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{a(0)(A - C_1)}{a(1)} e^{z_1} e^{\varepsilon z_3} \\
&= C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{a(0)}{a(1)} A e^{z_1} e^{\varepsilon z_3} - \frac{a(0)}{a(1)} C_1 e^{z_1} e^{\varepsilon z_3} \\
B &= C_1 \left(e^{-z_1} e^{\varepsilon z_2} - \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3} \right) + A \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3} \\
C_1 &= \frac{B - A\frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3}}{e^{-z_1} e^{\varepsilon z_2} - \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3}} \\
&= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \qquad (9)
\end{aligned}
$$

Using (7), now C_2 is found

$$
\begin{aligned}
A &= C_1 + \frac{C_2}{a(0)} \\
A &= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} + \frac{C_2}{a(0)} \\
C_2 &= a(0) \left(A - \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \right) \qquad (10)
\end{aligned}
$$

The constants C_1, C_2, are now found, hence the solution is now complete.

Summary of solution

$$
y(x) \sim C_1 \exp\left(-\int_0^x \frac{b(t)}{a(t)} dt + \varepsilon \int_0^x \frac{a(t) b'(t) - b^2(t) - a'(t) b(t)}{a^3(t)} dt \right)
$$

$$
+ \frac{C_2}{a(x)} \exp\left(\frac{-1}{\varepsilon} \int_0^x a(t) dt + \int_0^x \frac{b(t)}{a(t)} dt + \varepsilon \int_0^x \frac{b^2(t) - 2b(t) a'(t) + a(t) b'(t) - a(t) a''(t) - a'(t) b(t)}{a^3(t)} dt \right)
$$

Where

$$
\begin{aligned}
C_1 &= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \\
C_2 &= a(0) \left(A - \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \right)
\end{aligned}
$$

And

$$
\begin{aligned}
z_1 &= \int_0^1 \frac{b(t)}{a(t)} dt \\
z_2 &= \int_0^1 \frac{a(t) b'(t) - b^2(t) - a'(t) b(t)}{a^3(t)} dt \\
z_3 &= \int_0^1 \frac{b^2(t) - 2b(t) a'(t) + a(t) b'(t) - a(t) a''(t) - a'(t) b(t)}{a^3(t)} dt
\end{aligned}
$$

3.4.2 problem 10.6

problem Use second order WKB to derive formula which is more accurate than (10.1.31) for the n^{th} eigenvalue of the Sturm-Liouville problem in 10.1.27. Let $Q(x) = (x+\pi)^4$ and compare your formula with value of E_n in table 10.1

solution

Problem 10.1.27 is

$$y'' + EQ(x)y = 0$$

With $Q(x) = (x+\pi)^4$ and boundary conditions $y(0) = 0, y(\pi) = 0$. Letting

$$E = \frac{1}{\varepsilon}$$

Then the ODE becomes

$$\varepsilon y''(x) + (x+\pi)^4 y(x) = 0 \tag{1}$$

Physical optics approximation is obtained when $\lambda \to \infty$ or $\varepsilon \to 0^+$. Since the ODE is linear, and the highest derivative is now multiplied by a very small parameter ε, WKB can be used to solve it. Assuming the solution is

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0$$

Then

$$y'(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)$$

$$y''(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2 + \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\right)$$

Substituting these into (1) and canceling the exponential terms gives

$$\varepsilon\left(\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\right) \sim -(x+\pi)^4$$

$$\frac{\varepsilon}{\delta^2}\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right) + \frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) \sim -(x+\pi)^4$$

$$\frac{\varepsilon}{\delta^2}\left(\left(S_0'\right)^2 + \delta\left(2S_1'S_0'\right) + \delta^2\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) \sim -(x+\pi)^4$$

$$\left(\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2 + \frac{\varepsilon}{\delta}\left(2S_1'S_0'\right) + \varepsilon\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \left(\frac{\varepsilon}{\delta}S_0'' + \varepsilon^2 S_1'' + \cdots\right) \sim -(x+\pi)^4 \tag{2}$$

The largest term in the left side is $\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2$. By dominant balance, this term has the same order of magnitude as right side $-(x+\pi)^4$. Hence δ^2 is proportional to ε and for simplicity, δ can be taken equal to $\sqrt{\varepsilon}$ or

$$\delta = \sqrt{\varepsilon}$$

Equation (2) becomes

$$\left(\left(S_0'\right)^2 + \sqrt{\varepsilon}\left(2S_1'S_0'\right) + \varepsilon\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \left(\sqrt{\varepsilon}S_0'' + \varepsilon S_1'' + \cdots\right) \sim -(x+\pi)^4$$

Balance of $O(1)$ gives

$$\left(S_0'\right)^2 \sim -(x+\pi)^4 \tag{3}$$

And Balance of $O\left(\sqrt{\varepsilon}\right)$ gives

$$2S_1'S_0' \sim -S_0'' \tag{4}$$

Balance of $O(\varepsilon)$ gives

$$2S_0'S_2' + \left(S_1'\right)^2 \sim -S_1'' \tag{5}$$

Equation (3) is solved first in order to find $S_0(x)$. Therefore

$$S_0' \sim \pm i(x+\pi)^2$$

Hence

$$\begin{aligned} S_0(x) &\sim \pm i\int_0^x (t+\pi)^2\,dt + C^{\pm} \\ &\sim \pm i\left(\frac{t^3}{3} + \pi t^2 + \pi^2 t\right)_0^x + C^{\pm} \\ &\sim \pm i\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + C^{\pm} \end{aligned} \tag{6}$$

$S_1(x)$ is now found from (4), and since $S_0'' = \pm 2i(x+\pi)$ therefore

$$\begin{aligned} S_1' &\sim -\frac{1}{2}\frac{S_0''}{S_0'} \\ &\sim -\frac{1}{2}\frac{\pm 2i(x+\pi)}{\pm i(x+\pi)^2} \\ &\sim -\frac{1}{x+\pi} \end{aligned}$$

Hence

$$\begin{aligned} S_1(x) &\sim -\int_0^x \frac{1}{t+\pi}dt \\ &\sim -\ln\left(\frac{\pi+x}{\pi}\right) \end{aligned} \tag{7}$$

$S_2(x)$ is now solved from from (5)

$$\begin{aligned} 2S_0'S_2' + \left(S_1'\right)^2 &\sim -S_1'' \\ S_2' &\sim \frac{-S_1'' - \left(S_1'\right)^2}{2S_0'} \end{aligned}$$

Since $S_1' \sim -\frac{1}{x+\pi}$, then $S_1'' \sim \frac{1}{(x+\pi)^2}$ and the above becomes

$$\begin{aligned} S_2' &\sim \frac{-\frac{1}{(x+\pi)^2} - \left(-\frac{1}{x+\pi}\right)^2}{2\left(\pm i\,(x+\pi)^2\right)} \\ &\sim \frac{-\frac{1}{(x+\pi)^2} - \frac{1}{(x+\pi)^2}}{\pm 2i\,(x+\pi)^2} \\ &\sim \pm i\frac{1}{(x+\pi)^4} \end{aligned}$$

Hence

$$\begin{aligned} S_2 &\sim \pm i\left(\int_0^x \frac{1}{(t+\pi)^4}dt\right) + k^{\pm} \\ &\sim \pm\frac{i}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right) + k^{\pm} \end{aligned}$$

Therefore the solution becomes

$$\begin{aligned} y(x) &\sim \exp\left(\frac{1}{\sqrt{\varepsilon}}\left(S_0(x) + \sqrt{\varepsilon}S_1(x) + \varepsilon S_2(x)\right)\right) \\ &\sim \exp\left(\frac{1}{\sqrt{\varepsilon}}S_0(x) + S_1(x) + \sqrt{\varepsilon}S_2(x)\right) \end{aligned}$$

But $E = \frac{1}{\varepsilon}$, hence $\sqrt{\varepsilon} = \frac{1}{\sqrt{E}}$, and the above becomes

$$y(x) \sim \exp\left(\sqrt{E}S_0(x) + S_1(x) + \frac{1}{\sqrt{E}}S_2(x)\right)$$

Therefore

$$y(x) \sim \exp\left(\pm i\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + C^{\pm} - \ln\left(\frac{\pi+x}{\pi}\right) \pm i\frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right) + k^{\pm}\right)$$

Which can be written as

$$\begin{aligned} y(x) \sim \left(\frac{\pi+x}{\pi}\right)^{-1} C\exp\left(i\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right)\right)\right) \\ - C\left(\frac{\pi+x}{\pi}\right)^{-1}\exp\left(-i\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right)\right)\right) \end{aligned}$$

Where all constants combined into $\pm C$. In terms of $\sin/\cos$ the above becomes

$$\begin{aligned} y(x) \sim \frac{\pi A}{\pi+x}\cos\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right)\right) \\ + \frac{\pi B}{\pi+x}\sin\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi+x)^3}\right)\right) \end{aligned} \tag{8}$$

Boundary conditions $y(0) = 0$ gives

$$\begin{aligned} 0 &\sim A\cos\left(0 + \frac{1}{\sqrt{E}}\left(\frac{1}{\pi^3} - \frac{1}{\pi^3}\right)\right) + B\sin\left(0 + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\pi^3}\right)\right) \\ &\sim A \end{aligned}$$

Hence solution in (8) reduces to

$$y(x) \sim \frac{\pi B}{\pi + x}\sin\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi + x)^3}\right)\right)$$

Applying B.C. $y(\pi) = 0$ gives

$$\begin{aligned} 0 &\sim \frac{\pi B}{\pi + \pi}\sin\left(\sqrt{E}\left(\frac{\pi^3}{3} + \pi\pi^2 + \pi^2\pi\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi + \pi)^3}\right)\right) \\ &\sim \frac{B}{2}\sin\left(\sqrt{E}\left(\frac{7}{3}\pi^3\right) + \frac{1}{\sqrt{E}}\left(\frac{7}{24\pi^3}\right)\right) \end{aligned}$$

For non trivial solution, therefore

$$\sqrt{E_n}\left(\frac{7}{3}\pi^3\right) + \frac{1}{\sqrt{E_n}}\left(\frac{7}{24\pi^3}\right) = n\pi \qquad n = 1, 2, 3, \cdots$$

Solving for $\sqrt{E_n}$. Let $\sqrt{E_n} = x$, then the above becomes

$$\begin{aligned} x^2\left(\frac{7}{3}\pi^3\right) + \left(\frac{7}{24\pi^6}\right) &= xn\pi \\ x^2 - \frac{3}{7\pi^2}xn + \frac{1}{8\pi^6} &= 0 \end{aligned}$$

Solving using quadratic formula and taking the positive root, since $E_n > 0$ gives

$$\begin{aligned} x &= \frac{1}{28\pi^3}\left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right) \qquad n = 1, 2, 3, \cdots \\ \sqrt{E_n} &= \frac{1}{28\pi^3}\left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right) \\ E_n &= \left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right)^2 \end{aligned}$$

Table 10.1 is now reproduced to compare the above more accurate E_n. The following table shows the actual E_n values obtained this more accurate method. Values computed from above formula are in column 3.

```
In[207]:= sol = x /. Last@Solve[x^2 - 3/(7 Pi^2) x n + 1/(8 Pi^6) == 0, x]
```

$$\text{Out[207]=}\quad \frac{3 n \pi + \frac{\sqrt{-49+18 n^2 \pi^2}}{\sqrt{2}}}{14 \pi^3}$$

```
In[244]:= lam[n_] := (Evaluate@sol)^2
nPoints = {1, 2, 3, 4, 5, 10, 20, 40};
book = {0.00188559, 0.00754235, 0.0169703, 0.0301694, 0.0471397, 0.188559, 0.754235, 3.01694};
exact = {0.00174401, 0.00734865, 0.0167524, 0.0299383, 0.0469006, 0.188395, 0.753977, 3.01668};
hw = Table[N@(lam[n]), {n, nPoints}];
data = Table[{nPoints[[i]], book[[i]], hw[[i]], exact[[i]]}, {i, 1, Length[nPoints]}];
data = Join[{{"n", "En using S0+S1 (book)", "En using S0+S1+S2 (HW)", "Exact"}}, data];
Style[Grid[data, Frame → All], 18]
```

Out[251]=

n	E_n using S0+S1 (book)	E_n using S0+S1+S2 (HW)	Exact
1	0.0018856	0.0016151	0.001744
2	0.0075424	0.00728	0.0073487
3	0.01697	0.016709	0.016752
4	0.030169	0.029909	0.029938
5	0.04714	0.046879	0.046901
10	0.18856	0.1883	0.1884
20	0.75424	0.75398	0.75398
40	3.0169	3.0167	3.0167

The following table shows the relative error in place of the actual values of E_n to better compare how more accurate the result obtained in this solution is compared to the book result

```
In[252]:= data = Table[{nPoints[[i]],
     100 * Abs@(exact[[i]] - book[[i]]) / exact[[i]],
     100 * Abs@(exact[[i]] - hw[[i]]) / exact[[i]]}, {i, 1, Length[nPoints]}];
data = Join[{{"n", "En using S0+S1 (book) Rel error", "En using S0+S1+S2 (HW) Rel error"}}, data];
Style[Grid[data, Frame → All], 18]
```

Out[254]=

n	E_n using S0+S1 (book) Rel error	E_n using S0+S1+S2 (HW) Rel error
1	8.1181	7.3927
2	2.6359	0.9343
3	1.3007	0.2576
4	0.77192	0.098497
5	0.5098	0.045387
10	0.087051	0.051101
20	0.034219	0.00021643
40	0.0086187	0.000055619

The above shows clearly that adding one more term in the WKB series resulted in more accurate eigenvalue estimate.

3.5 HW5

NE548 Problem: Similarity solution for the 1D homogeneous heat equation

Due Thursday April 13, 2017

1. (a) Non-dimensionalize the 1D homogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

with $-\infty < x < \infty$, and $u(x,t)$ bounded as $x \to \pm\infty$.

(b) Show that the non-dimensional equations motivate a similarity variable $\xi = x/t^{1/2}$.

(c) Find a similarity solution $u(x,t) = H(\xi)$ by solving the appropriate ODE for $H(\xi)$.

(d) Show that the similarity solution is related to the solution we found in class on 4/6/17 for initial condition

$$u(x,0) = 0, \quad x < 0 \qquad u(x,0) = C, \quad x > 0.$$

solution

3.5.1 Part a

Let $\bar{x}$ be the non-dimensional space coordinate and $\bar{t}$ the non-dimensional time coordinate. Therefore we need

$$\bar{x} = \frac{x}{l_0}$$
$$\bar{t} = \frac{t}{t_0}$$
$$\bar{u} = \frac{u}{u_0}$$

Where l_0 is the physical characteristic length scale (even if this infinitely long domain, l_0 is given) whose dimensions is $[L]$ and t_0 of dimensions $[T]$ is the characteristic time scale and $\bar{u}(\bar{x},\bar{t})$ is the new dependent variable, and u_0 characteristic value of u to scale against (typically this is the initial conditions) but this will cancel out. We now rewrite the PDE $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$ in terms of the new dimensionless variables.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t} \\ &= u_0 \frac{\partial \bar{u}}{\partial \bar{t}} \frac{1}{t_0} \end{aligned} \tag{1}$$

And

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} \\ &= u_0 \frac{\partial \bar{u}}{\partial \bar{x}}\frac{1}{l_0}\end{aligned}$$

And

$$\frac{\partial^2 u}{\partial x^2} = u_0 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\frac{1}{l_0^2} \tag{2}$$

Substituting (1) and (2) into $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}$ gives

$$\begin{aligned}u_0 \frac{\partial \bar{u}}{\partial \bar{t}}\frac{1}{t_0} &= v u_0 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\frac{1}{l_0^2} \\ \frac{\partial \bar{u}}{\partial \bar{t}} &= \left(v\frac{t_0}{l_0^2}\right)\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\end{aligned}$$

The above is now non-dimensional. Since v has units $\left[\frac{L^2}{T}\right]$ and $\frac{t_0}{l_0^2}$ also has units $\left[\frac{T}{L^2}\right]$, therefore the product $v\frac{t_0}{l_0^2}$ is non-dimensional quantity.

If we choose t_0 to have same magnitude (not units) as l_0^2, i.e. $t_0 = l_0^2$, then $\frac{t_0}{l_0^2} = 1$ (with units $\left[\frac{T}{L^2}\right]$) and now we obtain the same PDE as the original, but it is non-dimensional. Where now $\bar{u} \equiv \bar{u}(\bar{t}, \bar{x})$.

3.5.2 Part (b)

I Will use the Buckingham π theorem for finding expression for the solution in the form $u(x,t) = f(\xi)$ where ξ is the similarity variable. Starting with $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}$, in this PDE, the diffusion substance is heat with units of Joule J. Hence the concentration of heat, which is what u represents, will have units of $[u] = \frac{J}{L^3}$. (heat per unit volume). From physics, we expect the solution $u(x,t)$ to depend on x, t, v and initial conditions u_0 as these are the only relevant quantities involved that can affect the diffusion. Therefore, by Buckingham theorem we say

$$u \equiv f(x, t, v, u_0) \tag{1}$$

We have one dependent quantity u and 4 independent quantities.The units of each of the above quantities is

$$
\begin{aligned}
[u] &= \frac{J}{L^3}\\
[x] &= L\\
[t] &= T\\
[v] &= \frac{L^2}{T}\\
[u_0] &= \frac{J}{L^3}
\end{aligned}
$$

Hence using Buckingham theorem, we write

$$[u] = \left[x^a t^b v^c u_0^d\right] \tag{2}$$

We now determine a,b,c,d, by dimensional analysis. The above is

$$
\begin{aligned}
\frac{J}{L^3} &= L^a T^b \left(\frac{L^2}{T}\right)^c \left(\frac{J}{L^3}\right)^d\\
(J)\left(L^{-3}\right) &= \left(L^{a+2c-3d}\right)\left(T^{b-c}\right)\left(J^d\right)
\end{aligned}
$$

Comparing powers of same units on both sides, we see that

$$
\begin{aligned}
d &= 1\\
b - c &= 0\\
a + 2c - 3d &= -3
\end{aligned}
$$

From second equation above, $b = c$, hence third equation becomes

$$
\begin{aligned}
a + 2c - 3d &= -3\\
a + 2c &= 0
\end{aligned}
$$

Since $d = 1$. Hence

$$
\begin{aligned}
c &= -\frac{a}{2}\\
b &= -\frac{a}{2}
\end{aligned}
$$

Therefore, now that we found all the powers, (we have one free power a which we can set to any value), then from equation (1)

$$
\begin{aligned}
[u] &= \left[x^a t^b v^c u_0^d\right]\\
\frac{u}{u_0} &= \bar{u} = x^a t^b v^c
\end{aligned}
$$

Therefore $\bar{u}$ is function of all the variables in the RHS. Let this function be f (This is the same as H in problem statement). Hence the above becomes

$$
\begin{aligned}
\bar{u} &= f\left(x^a t^{-\frac{a}{2}} v^{-\frac{a}{2}}\right)\\
&= f\left(\frac{x^a}{v^{\frac{a}{2}} t^{\frac{a}{2}}}\right)
\end{aligned}
$$

Since a is free variable, we can choose

$$a = 1 \tag{2}$$

And obtain

$$\bar{u} \equiv f\left(\frac{x}{\sqrt{\upsilon t}}\right) \tag{3}$$

In the above $\frac{x}{\sqrt{\upsilon t}}$ is now non-dimensional quantity, which we call, the similarity variable

$$\xi = \frac{x}{\sqrt{\upsilon t}} \tag{4}$$

Notice that another choice of a in (2), for example $a = 2$ would lead to $\xi = \frac{x^2}{\upsilon t}$ instead of $\xi = \frac{x}{\sqrt{\upsilon t}}$ but we will use the latter for the rest of the problem.

3.5.3 Part (c)

Using $u \equiv f(\xi)$ where $\xi = \frac{x}{\sqrt{\upsilon t}}$ then

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{df}{d\xi}\frac{\partial \xi}{\partial t} \\
&= \frac{df}{d\xi}\frac{\partial}{\partial t}\left(\frac{x}{\sqrt{\upsilon t}}\right) \\
&= -\frac{1}{2}\frac{df}{d\xi}\left(\frac{x}{\sqrt{\upsilon}t^{\frac{3}{2}}}\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{df}{d\xi}\frac{\partial \xi}{\partial x} \\
&= \frac{df}{d\xi}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{\upsilon t}}\right) \\
&= \frac{df}{d\xi}\frac{1}{\sqrt{\upsilon t}}
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{df}{d\xi}\frac{1}{\sqrt{\upsilon t}}\right)\\ &= \frac{1}{\sqrt{\upsilon t}}\frac{\partial}{\partial x}\left(\frac{df}{d\xi}\right)\\ &= \frac{1}{\sqrt{\upsilon t}}\left(\frac{d^2 f}{d\xi^2}\frac{\partial \xi}{\partial x}\right)\\ &= \frac{1}{\sqrt{\upsilon t}}\left(\frac{d^2 f}{d\xi^2}\frac{1}{\sqrt{\upsilon t}}\right)\\ &= \frac{1}{\upsilon t}\frac{d^2 f}{d\xi^2}\end{aligned}$$

Hence the PDE $\frac{\partial u}{\partial t} = \upsilon\frac{\partial^2 u}{\partial x^2}$ becomes

$$\begin{aligned}-\frac{1}{2}\frac{df}{d\xi}\left(\frac{x}{\sqrt{\upsilon}t^{\frac{3}{2}}}\right) &= \upsilon\frac{1}{\upsilon t}\frac{d^2 f}{d\xi^2}\\ \frac{1}{t}\frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{x}{\sqrt{\upsilon}t^{\frac{3}{2}}}\frac{df}{d\xi} &= 0\\ \frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{x}{\sqrt{\upsilon t}}\frac{df}{d\xi} &= 0\end{aligned}$$

But $\frac{x}{\sqrt{\upsilon t}} = \xi$, hence we obtain the required ODE as

$$\begin{aligned}\frac{d^2 f(\xi)}{d\xi^2} + \frac{1}{2}\xi\frac{df(\xi)}{d\xi} &= 0\\ f'' + \frac{\xi}{2}f' &= 0\end{aligned}$$

We now solve the above ODE for $f(\xi)$. Let $f' = z$, then the ODE becomes

$$z' + \frac{\xi}{2}z = 0$$

Integrating factor is $\mu = e^{\int \frac{\xi}{2}d\xi} = e^{\frac{\xi^2}{4}}$, hence

$$\begin{aligned}\frac{d}{d\xi}(z\mu) &= 0\\ z\mu &= c_1\\ z &= c_1 e^{\frac{-\xi^2}{4}}\end{aligned}$$

Therefore, since $f' = z$, then

$$f' = c_1 e^{\frac{-\xi^2}{4}}$$

Integrating gives

$$f(\xi) = c_2 + c_1 \int_0^{\xi} e^{\frac{-s^2}{4}} ds$$

3.5.4 Part (d)

For initial conditions of step function

$$u(x,0) = \begin{cases} 0 & x < 0 \\ C & x > 0 \end{cases}$$

The solution found in class was

$$u(x,t) = \frac{C}{2} + \frac{C}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4\upsilon t}}\right) \tag{1}$$

Where $\operatorname{erf}\left(\frac{x}{\sqrt{4\upsilon t}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4\upsilon t}}} e^{-z^2} dz$. The solution found in part (c) earlier is

$$f(\xi) = c_1 \int_0^{\xi} e^{\frac{-s^2}{4}} ds + c_2$$

Let $s = \sqrt{4}z$, then $\frac{ds}{dz} = \sqrt{4}$, when $s = 0, z = 0$ and when $s = \xi, z = \frac{\xi}{\sqrt{4}}$, therefore the integral becomes

$$f(\xi) = c_1 \sqrt{4} \int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz + c_2$$

But $\frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz = \operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right)$, hence $\int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz = \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right)$ and the above becomes

$$\begin{aligned} f(\xi) &= c_1 \sqrt{\pi} \operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right) + c_2 \\ &= c_3 \operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right) + c_2 \end{aligned}$$

Since $\xi = \frac{x}{\sqrt{\upsilon t}}$, then above becomes, when converting back to $u(x,t)$

$$u(x,t) = c_3 \operatorname{erf}\left(\frac{x}{\sqrt{4\upsilon t}}\right) + c_2 \tag{2}$$

Comparing (1) and (2), we see they are the same. Constants of integration are arbitrary and can be found from initial conditions.

3.6 HW6

3.6.1 Problem 1

1. Use the Method of Images to solve

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad x > 0$$

(a)

$$u(0,t) = 0, \qquad u(x,0) = 0.$$

(b)

$$u(0,t) = 1, \qquad u(x,0) = 0.$$

(c)

$$u_x(0,t) = A(t), \qquad u(x,0) = f(x).$$

Note, I will use k in place of ν since easier to type.

3.6.1.1 Part (a)

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ x &> 0 \\ u(0,t) &= 0 \\ u(x,0) &= 0 \end{aligned}$$

Multiplying both sides by $G(x,t;x_0,t_0)$ and integrating over the domain gives (where in the following G is used instead of $G(x,t;x_0,t_0)$ for simplicity).

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} Gu_t \, dtdx = \int_{x=0}^{\infty}\int_{t=0}^{\infty} ku_{xx}G \, dtdx + \int_{x=0}^{\infty}\int_{t=0}^{\infty} QG \, dtdx \tag{1}$$

For the integral on the LHS, we apply integration by parts once to move the time derivative from u to G

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} Gu_t \, dtdx = \int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{x=0}^{\infty}\int_{t=0}^{\infty} G_t u \, dtdx$$

And the first integral in the RHS of (1) gives, after doing integration by parts two times

$$\begin{aligned}
\int_{x=0}^{\infty}\int_{t=0}^{\infty} k u_{xx} G \, dtdx &= \int_{t=0}^{\infty} [u_x G]_{x=0}^{\infty} dt - \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u_x G_x \, dtdx \\
&= \int_{t=0}^{\infty} [u_x G]_{x=0}^{\infty} dt - \left(\int_{t=0}^{\infty} [u G_x]_{x=0}^{\infty} dt - \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u G_{xx} \, dtdx\right) \\
&= \int_{t=0}^{\infty} \left([u_x G]_{x=0}^{\infty} - [u G_x]_{x=0}^{\infty}\right) dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u G_{xx} \, dtdx \\
&= \int_{t=0}^{\infty} [u_x G - u G_x]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u G_{xx} \, dtdx \\
&= -\int_{t=0}^{\infty} [u G_x - u_x G]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u G_{xx} \, dtdx
\end{aligned}$$

Hence (1) becomes

$$\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{x=0}^{\infty}\int_{t=0}^{\infty} G_t u \, dtdx = \int_{t=0}^{\infty} [u_x G - u G_x]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} k u G_{xx} \, dtdx + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ \, dtdx$$

Or

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} -G_t u - k u G_{xx} \, dtdx = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{t=0}^{\infty} [u G_x - u_x G]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ \, dtdx \tag{2}$$

We now want to choose $G(x,t;x_0,t_0)$ such that

$$\begin{aligned}
-G_t u - k u G_{xx} &= \delta(x-x_0)\,\delta(t-t_0) \\
-G_t u &= k u G_{xx} + \delta(x-x_0)\,\delta(t-t_0)
\end{aligned} \tag{3}$$

This way, the LHS of (2) becomes $u(x_0,t_0)$. Hence (2) now becomes

$$u(x_0,t_0) = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{t=0}^{t_0} [u G_x - u_x G]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ \, dtdx \tag{4}$$

We now need to find the Green function which satisfies (3). But (3) is equivalent to solution of problem of

$$\begin{aligned}
-G_t u &= k u G_{xx} \\
G(x,0) &= \delta(x-x_0)\,\delta(t-t_0) \\
-\infty &< x < \infty \\
G(x,t;x_0,t_0) &= 0 \qquad t > t_0 \\
G(\pm\infty,t;x_0,t_0) &= 0 \\
G(x,t_0;x_0,t_0) &= \delta(x-x_0)
\end{aligned}$$

But the above problem has a known fundamental solution which we found, but for the forward heat PDE instead of the reverse heat PDE. The fundamental solution to the forward heat PDE is

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t-t_0)}} \exp\left(\frac{-(x-x_0)^2}{4k(t-t_0)}\right) \qquad 0 \le t_0 \le t$$

Hence for the reverse heat PDE the above becomes

$$G(x,t)=\frac{1}{\sqrt{4\pi k(t_0-t)}}\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)\qquad 0\leq t\leq t_0 \tag{5}$$

The above is the infinite space Green function and what we will use in (4). Now we go back to (4) and simplify the boundary conditions term. Starting with the term $\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}dx$. Since $G(x,\infty;x_0,t_0)=0$ then upper limit is zero. At $t=0$ we are given that $u(x,0)=0$, hence this whole term is zero. So now (4) simplifies to

$$u(x_0,t_0)=-\int_{t=0}^{\infty}[uG_x-u_xG]_{x=0}^{\infty}dt+\int_{x=0}^{\infty}\int_{t=0}^{\infty}GQ\,dtdx \tag{6}$$

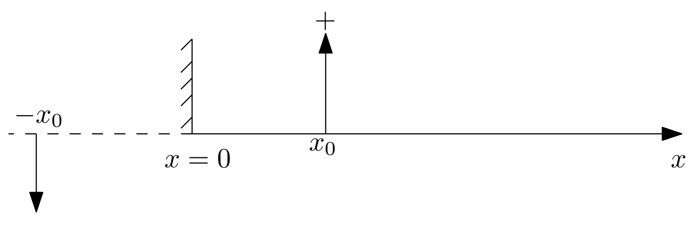
We are told that $u(0,t)=0$. Hence

$$\begin{aligned}[uG_x-u_xG]_{x=0}^{\infty}&=(u(\infty,t)G_x(\infty,t)-u_x(\infty,t)G(\infty,t))-(u(0,t)G_x(0,t)-u_x(0,t)G(0,t))\\&=(u(\infty,t)G_x(\infty,t)-u_x(\infty,t)G(\infty,t))+u_x(0,t)G(0,t)\end{aligned}$$

We also know that $G(\pm\infty,t;x_0,t_0)=0$, Hence $G(\infty)=0$ and also $G_x(\infty)=0$, hence the above simplifies

$$[uG_x-u_xG]_{x=0}^{\infty}=u_x(0,t)G(0,t)$$

To make $G(0,t)=0$ we place an image impulse at $-x_0$ with negative value to the impulse at x_0. This will make G at $x=0$ zero.



Therefore the Green function to use is, from (5) becomes

$$G(x,t)=\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)-\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)\qquad 0\leq t\leq t_0$$

Therefore the solution, from (4) becomes

$$u(x_0,t_0)=\int_{x=0}^{\infty}\int_{t=0}^{t_0}\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)-\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)Q(x,t)\;dtdx \tag{7}$$

Switching the order of x_0,t_0 with x,t gives

$$u(x,t)=\int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right)-\exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\;dt_0dx_0 \tag{8}$$

Notice, for the terms $(x_0-x)^2,(x_0+x)^2$, since they are squared, the order does not matter, so we might as well write the above as

$$u(x,t)=\int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t-t_0)}\right)-\exp\left(\frac{-(x+x_0)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\;dt_0dx_0 \tag{8}$$

3.6.1.2 Part (b)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$x > 0$$
$$u(0,t) = 1$$
$$u(x,0) = 0$$

Everything follows the same as in part (a) up to the point where boundary condition terms need to be evaluated.

$$u(x_0,t_0) = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty}\, dx - \int_{t=0}^{t_0} [uG_x - u_x G]_{x=0}^{\infty}\, dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\ dtdx \tag{4}$$

Where

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0 - t)}} \exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) \qquad 0 \leq t \leq t_0 \tag{5}$$

Starting with the term $\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx$. Since $G(x,\infty;x_0,t_0) = 0$ then upper limit is zero. At $t=0$ we are given that $u(x,0) = 0$, hence this whole term is zero. So now (4) simplifies to

$$u(x_0,t_0) = -\int_{t=0}^{\infty} [uG_x - u_x G]_{x=0}^{\infty}\, dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ\ dtdx \tag{6}$$

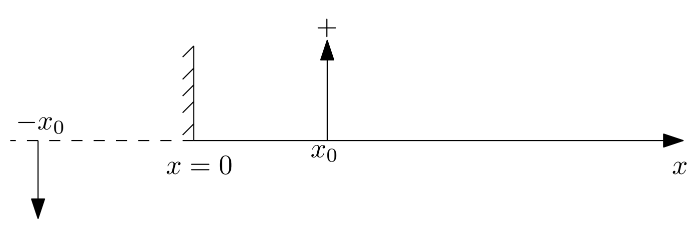
Now

$$[uG_x - u_x G]_{x=0}^{\infty} = (u(\infty,t)G_x(\infty,t) - u_x(\infty,t)G(\infty,t)) - (u(0,t)G_x(0,t) - u_x(0,t)G(0,t))$$

We are told that $u(0,t) = 1$, we also know that $G(\pm\infty,t;x_0,t_0) = 0$, Hence $G(\infty,t) = 0$ and also $G_x(\infty,t) = 0$, hence the above simplifies

$$[uG_x - u_x G]_{x=0}^{\infty} = -G_x(0,t) + u_x(0,t)G(0,t)$$

To make $G(0) = 0$ we place an image impulse at $-x_0$ with negative value to the impulse at x_0. This is the same as part(a)



$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0 - t)}} \left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) - \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right) \qquad 0 \leq t \leq t_0$$

Now the boundary terms reduces to just

$$[uG_x - u_x G]_{x=0}^{\infty} = -G_x(0,t)$$

We need now to evaluate $G_x(0,t)$, the only remaining term. Since we know what $G(x,t)$ from the above, then

$$\frac{\partial}{\partial x}G(x,t)=\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{-(x-x_0)}{2k(t_0-t)}\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)+\frac{2(x+x_0)}{4k(t_0-t)}\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)$$

And at $x=0$ the above simplifies to

$$\begin{aligned}G_x(0,t)&=\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{x_0}{2k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)+\frac{x_0}{2k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\\&=\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{x_0}{k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\end{aligned}$$

Hence the complete solution becomes from (4)

$$\begin{aligned}u(x_0,t_0)&=-\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx-\int_{t=0}^{t_0}[uG_x-u_xG]_{x=0}^{\infty}\,dt+\int_{x=0}^{\infty}\int_{t=0}^{t_0}GQ\;dtdx\\&=-\int_{t=0}^{t_0}-G_x(0,t)\,dt+\int_{x=0}^{\infty}\int_{t=0}^{t_0}GQ\;dtdx\end{aligned}$$

Substituting G and G_x in the above gives

$$\begin{aligned}u(x_0,t_0)=&\int_{t=0}^{t_0}\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{x_0}{k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)dt\\&+\int_{x=0}^{\infty}\int_{t=0}^{t_0}\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)-\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)Q(x,t)\;dtdx\end{aligned}$$

Switching the order of x_0,t_0 with x,t

$$\begin{aligned}u(x,t)=&\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\frac{x}{k(t-t_0)}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)\right)dt_0\\&+\int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right)-\exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\;dt_0dx_0\end{aligned}\tag{7}$$

But

$$\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\frac{x}{k(t-t_0)}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)\right)dt_0=\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

Hence (7) becomes

$$u(x,t)=\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)+\int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right)-\exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\;dt_0dx_0$$

The only difference between this solution and part(a) solution is the extra term $\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$ due to having non-zero boundary conditions in this case.

3.6.1.3 part(c)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$x > 0$$
$$\frac{\partial u(0,t)}{\partial x} = A(t)$$
$$u(x,0) = f(x)$$

Everything follows the same as in part (a) up to the point where boundary condition terms need to be evaluated.

$$u(x_0,t_0) = -\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx - \int_{t=0}^{t_0}[uG_x - u_xG]_{x=0}^{\infty}\,dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\;dtdx \tag{4}$$

Starting with the term $\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx$. Since $G(x,\infty;x_0,t_0) = 0$ then upper limit is zero. At $t = 0$ we are given that $u(x,0) = f(x)$, hence

$$\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx = \int_{x=0}^{\infty} -u(x,0)\,G(x,0)\,dx$$
$$= \int_{x=0}^{\infty} -f(x)\,G(x,0)\,dx$$

Looking at the second term in RHS of (4)

$$[uG_x - u_xG]_{x=0}^{\infty} = (u(\infty,t)\,G_x(\infty,t) - u_x(\infty,t)\,G(\infty,t)) - (u(0,t)\,G_x(0,t) - u_x(0,t)\,G(0,t))$$

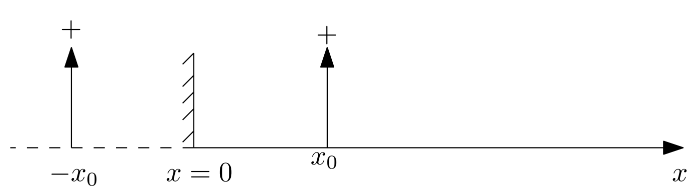
We are told that $u_x(0,t) = A(t)$, we also know that $G(\pm\infty,t;x_0,t_0) = 0$, Hence $G(\infty,t) = 0$ and also $G_x(\infty,t) = 0$. The above simplifies

$$[uG_x - u_xG]_{x=0}^{\infty} = -(u(0,t)\,G_x(0,t) - A(t)\,G(0,t)) \tag{5}$$

We see now from the above, that to get rid of the $u(0,t)\,G_x(0,t)$ term (since we do not know what $u(0,t)$ is), then we now need

$$G_x(0,t) = 0$$

This means we need an image at $-x_0$ which is of same sign as at $+x_0$ as shown in this diagram



Which means the Green function we need to use is the sum of the Green function solutions for the infinite domain problem

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) + \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right) \qquad 0 \le t \le t_0 \tag{6}$$

The above makes $G_x(0,t) = 0$ and now equation (5) reduces to

$$\begin{aligned}
[uG_x - u_xG]_{x=0}^{\infty} &= A(t)\,G(0,t)\\
&= A(t)\left(\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(-x_0)^2}{4k(t_0-t)}\right)+\exp\left(\frac{-(x_0)^2}{4k(t_0-t)}\right)\right)\right)\\
&= \frac{A(t)}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)+\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\\
&= \frac{A(t)}{\sqrt{\pi k(t_0-t)}}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)
\end{aligned}$$

We now know all the terms needed to evaluate the solution. From (4)

$$u(x_0,t_0) = -\int_{x=0}^{\infty} -f(x)\,G(x,0)\,dx - \int_{t=0}^{t_0} A(t)\,G(0,t)\,dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\ dtdx \tag{7}$$

Using the Green function we found in (6), then (7) becomes

$$\begin{aligned}
u(x_0,t_0) = \int_{x=0}^{\infty} f(x)\frac{1}{\sqrt{4\pi k t_0}}&\left(\exp\left(\frac{-(x-x_0)^2}{4kt_0}\right)+\exp\left(\frac{-(x+x_0)^2}{4kt_0}\right)\right)dx\\
&- \int_{t=0}^{t_0}\frac{A(t)}{\sqrt{\pi k(t_0-t)}}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)dt\\
&+ \int_{x=0}^{\infty}\int_{t=0}^{t_0}\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right)+\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)Q\ dtdx
\end{aligned}$$

Changing the roles of x,t and x_0,t_0 the above becomes

$$\begin{aligned}
u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{x_0=0}^{\infty} f(x_0)&\left(\exp\left(\frac{-(x_0-x)^2}{4kt}\right)+\exp\left(\frac{-(x_0+x)^2}{4kt}\right)\right)dx_0\\
&- \int_{t_0=0}^{t}\frac{A(t_0)}{\sqrt{\pi k(t-t_0)}}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)dt_0\\
&+ \int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right)+\exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\ dt_0dx_0
\end{aligned}$$

<u>Summary</u>

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\Delta_1 - \Delta_2 + \Delta_3$$

Where

$$\begin{aligned}
\Delta_1 &= \int_{x_0=0}^{\infty} f(x_0)\left(\exp\left(\frac{-(x_0-x)^2}{4kt}\right)+\exp\left(\frac{-(x_0+x)^2}{4kt}\right)\right)dx_0\\
\Delta_2 &= \int_{t_0=0}^{t}\frac{A(t_0)}{\sqrt{\pi k(t-t_0)}}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)dt_0\\
\Delta_3 &= \int_{x_0=0}^{\infty}\int_{t_0=0}^{t}\frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right)+\exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\ dt_0dx_0
\end{aligned}$$

Where Δ_1 comes from the initial conditions and Δ_2 comes from the boundary conditions and Δ_3 comes from for forcing function. It is also important to note that Δ_1 is valid for only $t > 0$ and not for $t = 0$.

3.6.2 Problem 2

2. (a) Solve by the Method of Characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \quad t \geq 0$$

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad \frac{\partial u(0,t)}{\partial x} = h(t)$$

(b) For the special case $h(t) = 0$, explain how you could use a symmetry argument to help construct the solution.

(c) Sketch the solution if $g(x) = 0$, $h(t) = 0$ and $f(x) = 1$ for $4 \leq x \leq 5$ and $f(x) = 0$ otherwise.

3.6.2.1 Part (a)

The general solution we will use as starting point is

$$u(x,t) = F(x-ct) + G(x+ct)$$

Where $F(x-ct)$ is the right moving wave and $G(x+ct)$ is the left moving wave. Applying $u(x,0) = f(x)$ gives

$$f(x) = F(x) + G(x) \tag{1}$$

And

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{dF}{d(x-ct)} \frac{\partial (x-ct)}{\partial t} + \frac{dG}{d(x+ct)} \frac{\partial (x+ct)}{\partial t} \\ &= -cF' + cG' \end{aligned}$$

Hence from second initial conditions we obtain

$$g(x) = -cF' + cG' \tag{2}$$

Equation (1) and (2) are for valid only for positive argument, which means for $x \geq ct$. $G(x+ct)$ has positive argument always since $x \geq 0$ and $t \geq 0$, but $F(x-ct)$ can have negative argument when $x < ct$. For $x < ct$, we will use the boundary conditions to define $F(x-ct)$. Therefore for $x \geq ct$ we solve (1,2) for G, F and find

$$F(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(s)\, ds \tag{2A}$$

$$G(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(s)\, ds \tag{2B}$$

This results in

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \qquad x \geq ct \tag{3}$$

The solution (3) is only valid for arguments that are positive. This is not a problem for $G(x+ct)$ since its argument is always positive. But for $F(x-ct)$ its argument can become negative when $0 < x < ct$. So we need to obtain a new solution for $F(x-ct)$ for the case when $x < ct$. First we find $\frac{\partial u(x,t)}{\partial x}$

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial x} &= \frac{dF}{d(x-ct)}\frac{\partial(x-ct)}{\partial x} + \frac{dG}{d(x+ct)}\frac{\partial(x+ct)}{\partial x} \\
&= \frac{dF(x-ct)}{d(x-ct)} + \frac{dG(x+ct)}{d(x+ct)}
\end{aligned}$$

Hence at $x=0$

$$\begin{aligned}
h(t) &= \frac{dF(-ct)}{d(-ct)} + \frac{dG(ct)}{d(ct)} \\
\frac{dF(-ct)}{d(-ct)} &= h(t) - \frac{dG(ct)}{d(ct)} \qquad (4)
\end{aligned}$$

Let $z=-ct$, therefore (4) becomes, where $t=\frac{-z}{c}$ also

$$\begin{aligned}
\frac{dF(z)}{dz} &= h\left(-\frac{z}{c}\right) - \frac{dG(-z)}{d(-z)} \qquad z<0 \\
\frac{dF(z)}{dz} &= h\left(-\frac{z}{c}\right) + \frac{dG(-z)}{dz} \qquad z<0
\end{aligned}$$

To find $F(z)$, we integrate the above which gives

$$\begin{aligned}
\int_0^z \frac{dF(s)}{ds}ds &= \int_0^z h\left(-\frac{s}{c}\right)ds + \int_0^z \frac{dG(-s)}{ds}ds \\
F(z) - F(0) &= \int_0^z h\left(-\frac{s}{c}\right)ds + G(-z) - G(0)
\end{aligned}$$

Ignoring the constants of integration $F(0), G(0)$ gives

$$F(z) = \int_0^z h\left(-\frac{s}{c}\right)ds + G(-z) \tag{4A}$$

Replacing $z = x-ct$ in the above gives

$$F(x-ct) = \int_0^{x-ct} h\left(-\frac{s}{c}\right)ds + G(ct-x)$$

Let $r=-\frac{s}{c}$ then when $s=0$, $r=0$ and when $s=x-ct$ then $r=-\frac{x-ct}{c}=\frac{ct-x}{c}$. And $\frac{dr}{ds}=-\frac{1}{c}$. Using these the integral in the above becomes

$$\begin{aligned}
F(x-ct) &= \int_0^{\frac{ct-x}{c}} h(r)(-c\,dr) + G(ct-x) \\
&= -c\int_0^{t-\frac{x}{c}} h(r)\,dr + G(ct-x) \qquad (5)
\end{aligned}$$

The above is $F(\cdot)$ when its argument are negative. But in the above $G(ct-x)$ is the same as we found above in (2b), which just replace it argument in 2B which was $x+ct$ with $ct-x$ and obtain

$$G(ct-x)=\frac{1}{2}f(ct-x)+\frac{1}{2c}\int_0^{ct-x}g(s)\,ds \qquad x<ct$$

Therefore (5) becomes

$$F(x-ct)=-c\int_0^{t-\frac{x}{c}}h(r)\,dr+\frac{1}{2}f(ct-x)+\frac{1}{2c}\int_0^{ct-x}g(s)\,ds \qquad x<ct \tag{7}$$

Hence for $x<ct$

$$u(x,t)=F(x-ct)+G(x+ct) \tag{8}$$

But in the above $G(x+ct)$ do not change, and we use the same solution for G for $x\geq ct$ which is in (2B), given again below

$$G(x+ct)=\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_0^{x+ct}g(s)\,ds \tag{9}$$

Hence, plugging (7,9) into (8) gives

$$\begin{aligned}u(x,t)&=-c\int_0^{\frac{ct-x}{c}}h(s)\,ds+\frac{1}{2}f(ct-x)+\frac{1}{2c}\int_0^{ct-x}g(s)\,ds+\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_0^{x+ct}g(s)\,ds \qquad x<ct\\&=-c\int_0^{\frac{ct-x}{c}}h(s)\,ds+\frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left(\int_0^{ct-x}g(s)\,ds+\int_0^{x+ct}g(s)\,ds\right)\end{aligned}$$

The above for $x<ct$. Therefore the full solution is

$$u(x,t)=\begin{cases}\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds & x\geq ct\\ -c\int_0^{t-\frac{x}{c}}h(s)\,ds+\frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left(\int_0^{ct-x}g(s)\,ds+\int_0^{x+ct}g(s)\,ds\right) & x<ct\end{cases} \tag{10}$$

3.6.2.2 Part (b)

From (10) above, for $h(t)=0$ the solution becomes

$$u(x,t)=\begin{cases}\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds & x\geq ct\\ \frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left(\int_0^{ct-x}g(s)\,ds+\int_0^{x+ct}g(s)\,ds\right) & x<ct\end{cases} \tag{1}$$

The idea of symmetry is to obtain the same solution (1) above but by starting from d'Almbert solution (which is valid only for positive arguments)

$$u(x,t)=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds \qquad x>ct \tag{1A}$$

But by using f_{ext},g_{ext} in the above instead of f,g, where the d'Almbert solution becomes valid for $x<ct$ when using f_{ext},g_{ext}

$$u(x,t)=\frac{f_{ext}(x+ct)+f_{ext}(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g_{ext}(s)\,ds \qquad x<ct \tag{2}$$

Then using (1A) and (2) we show it is the same as (1). We really need to show that (2) leads to the second part of (1), since (1A) is the same as first part of (1).

The main issue is how to determine f_{ext}, g_{ext} and determine if they should be even or odd extension of f, g. From boundary conditions, in part (a) equation (4A) we found that

$$F(z) = \int_0^z h\left(-\frac{s}{c}\right) ds + G(-z)$$

But now $h(t) = 0$, hence

$$F(z) = G(-z) \tag{3}$$

Now, looking at the first part of the solution in (1). we see that for positive argument the solution has $f(x-ct)$ for $x > ct$ and it has $f(ct-x)$ when $x < ct$. So this leads us to pick f_{ext} being even as follows. Let

$$z = x - ct$$

Then we see that

$$\begin{aligned} f(x-ct) &= f(-(x-ct)) \\ f(z) &= f(-z) \end{aligned}$$

Therefore we need f_{ext} to be an <u>even function</u>.

$$f_{ext}(z) = \begin{cases} f(z) & z > 0 \\ f(-z) & z < 0 \end{cases}$$

But since $F(z) = G(-z)$ from (3), then g_{ext} is also <u>even function</u>, which means

$$g_{ext}(z) = \begin{cases} g(z) & z > 0 \\ g(-z) & z < 0 \end{cases}$$

Now that we found f_{ext}, g_{ext} extensions, we go back to (2). For negative argument

$$u(x,t) = \frac{f_{ext}(x+ct) + f_{ext}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\,ds \qquad x < ct$$

Since f_{ext}, g_{ext} are even, then using $g_{ext}(z) = g(-z)$ since now $z < 0$ and using $f_{ext}(z) = f(-z)$ since now $z < 0$ the above becomes

$$u(x,t) = \frac{f(-(x+ct)) + f(-(x-ct))}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\,ds \qquad x < ct$$

But $f(-(x+ct)) = f(x+ct)$ since even and $f(-(x-ct)) = f(ct-x)$, hence the above becomes

$$u(x,t) == \frac{f(x+ct) + f(ct-x)}{2} + \frac{1}{2c}\left(\int_{x-ct}^{0} g(-s)\,ds + \int_0^{x+ct} g(s)\,ds\right) \qquad x < ct$$

Let $r = -s$, then $\frac{dr}{ds} = -1$. When $s = x - ct$, $r = ct - x$ and when $s = 0, r = 0$. Then the first integral on the RHS above becomes

$$\begin{aligned} u(x,t) &= \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{2c}\left(\int_{ct-x}^{0} g(r)(-dr) + \int_{0}^{x+ct} g(s)\,ds\right) \\ &= \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{2c}\left(\int_{0}^{ct-x} g(r)\,dr + \int_{0}^{x+ct} g(s)\,ds\right) \end{aligned}$$

Relabel r back to s, then

$$u(x,t) = \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{2c}\left(\int_{0}^{ct-x} g(s)\,ds + \int_{0}^{x+ct} g(s)\,ds\right) \qquad x < ct \tag{5}$$

Looking at (5) we see that this is the same solution in (1) for the case of $x < ct$. Hence (1A) and (5) put together give

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ \frac{f(ct-x)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_{0}^{ct-x} g(s)\,ds + \int_{0}^{x+ct} g(s)\,ds\right) & x < ct \end{cases} \tag{1}$$

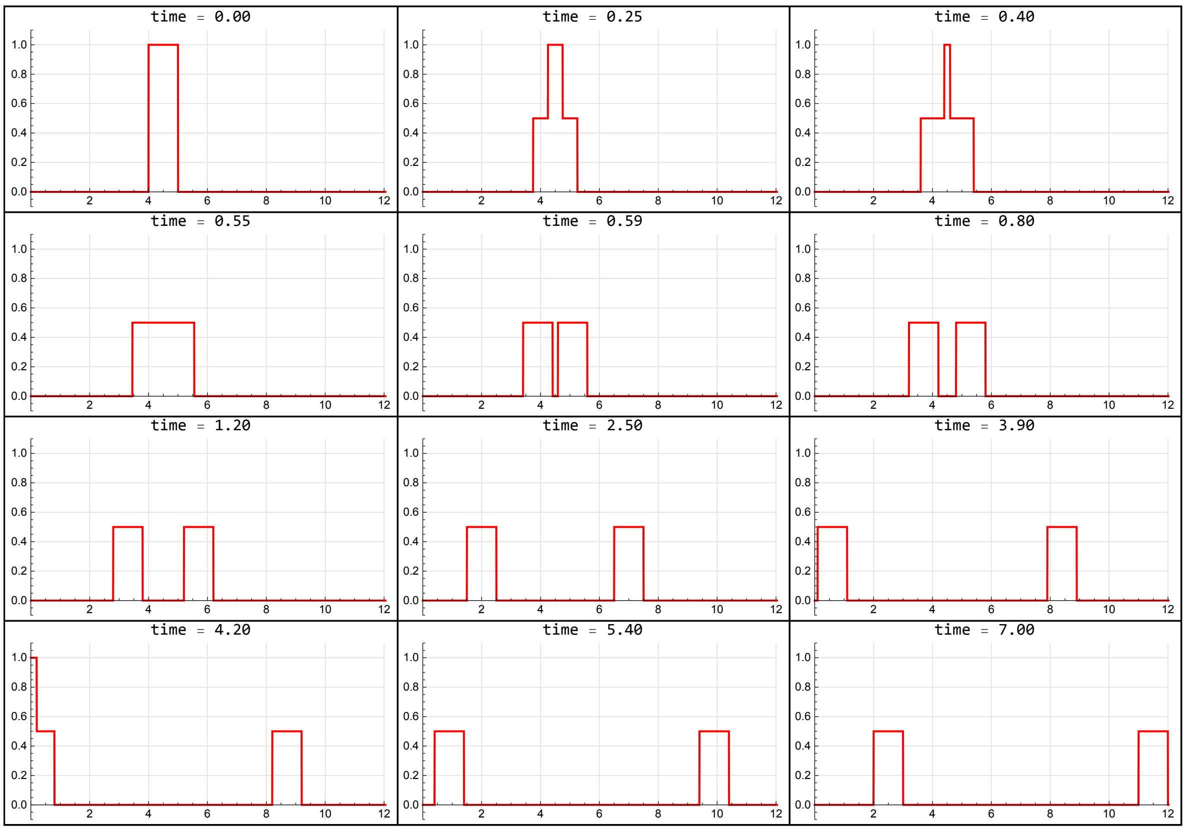
Which is same solution obtain in part(a).

3.6.2.3 Part(c)

For $g(x) = 0, h(t) = 0$ and $f(x)$ as given, the solution in equation (10) in part(a) becomes

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} & x \geq ct \\ \frac{f(ct-x)+f(x+ct)}{2} & x < ct \end{cases} \tag{10}$$

A small program we written to make few sketches important time instantces. The left moving wave $G(x+ct)$ hits the boundary at $x=0$ but do not reflect now as the case with Dirichlet boundary conditions, but instead it remains upright and turns around as shown.


time = 0.00
time = 0.25
time = 0.40
time = 0.55
time = 0.59
time = 0.80
time = 1.20
time = 2.50
time = 3.90
time = 4.20
time = 5.40
time = 7.00

Chapter 4

Study notes

Local contents

4.1 question asked 2/7/2017

Book at page 80 says "it is usually true that $S'' \lll (S')^2$" as $x \to x_0$. What are the exceptions to this? Since it is easy to find counter examples.

For $y'' = \sqrt{x}y$, as $x \to 0^+$. Using $y(x) = e^{S_0(x)}$ and substituting, gives the ODE

$$S_0'' + (S_0')^2 = \sqrt{x} \tag{1}$$

If we follow the book, and drop S_0'' relative to $(S')^2$ then

$$\begin{aligned}(S_0')^2 &= x^{\frac{1}{2}} \\ S_0' &= \omega x^{\frac{1}{4}}\end{aligned}$$

So, lets check the assumption. Since $S_0'' = \omega \frac{1}{4} x^{-\frac{3}{4}}$. Therefore (ignoring all multiplicative constants)

$$\begin{aligned}S_0'' &\overset{?}{\lll} (S_0')^2 \\ \frac{1}{x^{\frac{3}{4}}} &\overset{?}{\lll} x^{\frac{1}{2}}\end{aligned}$$

No. Did not check out. Since when $x \to 0^+$ the LHS blow up while the RHS goes to zero. So in this example, this "rule of thumb" did not work out, and it is the other way around. Assuming I did not make a mistake, when the book says "it is usually true", it will good to know under what conditions is this true or why did the book say this?

This will help in solving these problem.

4.2 note on Airy added 1/31/2017

Note on using CAS for working with book AiryAI/AiryBI terms

CAS such as Mathematica or Maple can actually handle very large numbers. It is possible to calculate individual terms of the Airy series given by the book (but no sum it). Calculating individual terms, up to 10 million terms for x=1,100 and x=10000 is shown below as illustration. It takes few minutes to do each term

define small function to calculate one term

```
bookAi[x_Integer, n_Integer] :=
```

$$3^{\frac{-2}{3}} \frac{x^{3n}}{9^n \text{Factorial}[n]\ \text{Gamma}[n+2/3]} - 3^{\frac{-4}{3}} \frac{x^{3n+1}}{9^n \text{Factorial}[n]\ \text{Gamma}[n+4/3]};$$

This is for x=1, N=10,000,000

```
In[3]:= bookAi[1, 10000000] // N
```

Out[3]= $5.7642115 \times 10^{-140856542}$

This is for x=100, N=10,000,000

```
In[4]:= bookAi[100, 10000000] // N
```

Out[4]= $5.7583009 \times 10^{-80856542}$

This is for x=10,000 and N=10,000,000

```
In[5]:= Timing[bookAi[10000, 10000000] // N]
```

Out[5]= $\{582.3829332,\ 5.167240 \times 10^{-20856542}\}$

The above took 582 seconds to complete! The largest number it can handle on my PC is

```
In[6]:= $MaxNumber
```

Out[6]= $1.605216761933662 \times 10^{1355718576299609}$

Let compare the above to Gamma[10,000,000]

```
In[8]:= Gamma[10000000.0]
```

Out[8]= $1.2024234 \times 10^{65657052}$

```
In[9]:= Gamma[10000000.0] < $MaxNumber
```

Out[9]= True

We see that Gamma[10,000,000] is much less than the largest number it can handle. Here is Gamma for 10 billion

In[10]:=
```
Gamma[10 000 000 000.0]
```

Out[10]= $2.3258\times10^{95\,657\,055\,176}$

And 10 billion Factorial

In[11]:=
```
Factorial[10 000 000 000.0]
```

Out[11]= $2.3258\times10^{95\,657\,055\,186}$

So CAS can handle these terms. But not the complete series summation as given in the book

4.3 note p1 added 2/3/2017

Current rules I am using in simplifications are

1. $S_0' \ggg S_1'$
2. $S_0'S_1' \ggg \left(S_1'\right)^2$
3. $S_0 \ggg S_1$
4. $(S_0)^n \ggg S_0^{(n)}$ (the first is power, the second is derivative order).

Verify the above are valid for $x \to 0^+$ and well for $x \to \infty$. What can we say about (S') compared to $(S')^2$?

4.3.1 problem (a), page 88

$$y'' = \frac{1}{x^5}y$$

Irregular singular point at $x \to 0^+$. Let $y = e^{S_0(x)}$ and the above becomes

$$\begin{aligned}
y(x) &= e^{S_0(x)} \\
y'(x) &= S_0'e^S \\
y'' &= S_0''e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\
&= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0}
\end{aligned}$$

Substituting back into $\frac{d^2}{dx^2}y = x^{-5}y$ gives

$$\left(S_0'' + \left(S_0'\right)^2\right) e^S = x^{-5}e^{S_0(x)}$$
$$S_0'' + \left(S_0'\right)^2 = x^{-5}$$

Before solving for S_0, we can do one more simplification. Using the approximation that $\left(S_0'\right)^2 \ggg S_0''$ for $x \to x_0$, the above becomes

$$\left(S_0'\right)^2 \sim x^{-5}$$

Now we are ready to solve for S_0

$$
\begin{aligned}
S_0' &\sim \omega x^{-\frac{5}{2}} \\
S_0 &\sim \omega \int x^{-\frac{5}{2}} dx \\
&\sim \omega \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} \\
&\sim -\frac{2}{3}\omega x^{-\frac{3}{2}}
\end{aligned}
$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$
\begin{aligned}
y'(x) &= (S_0(x) + S_1(x))' e^{S_0+S_1} \\
y''(x) &= \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}
\end{aligned}
$$

Using the above, the ODE $\frac{d^2}{dx^2}y = x^{-5}y$ now becomes

$$
\begin{aligned}
\left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1} &\sim x^{-5}e^{S_0+S_1} \\
\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)'' &\sim x^{-5} \\
\left(S_0' + S_1'\right)^2 + S_0'' + S_1'' &\sim x^{-5} \\
\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' &\sim x^{-5}
\end{aligned}
$$

But $S_0' = \omega x^{-\frac{5}{2}}$, found before, hence $\left(S_0'\right)^2 = x^{-5}$ and the above simplifies to

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

Using approximation $S_0'S_1' \ggg \left(S_1'\right)^2$ the above simplifies to

$$2S_0'S_1' + S_0'' + S_1'' = 0$$

Finally, using approximation $S_0'' \ggg S_1''$, the above becomes

$$
\begin{aligned}
2S_0'S_1' + S_0'' &= 0 \\
S_1' &\sim -\frac{S_0''}{2S_0'} \\
S_1 &\sim -\frac{1}{2}\ln S_0' + c \\
S_1' &\sim -\frac{1}{2}\ln x^{-\frac{5}{2}} + c \\
S_1' &\sim \frac{5}{4}\ln x + c
\end{aligned}
$$

Hence, the leading behavior is

$$\begin{aligned}
y(x) &= e^{S_0(x)+S_1(x)} \\
&= \exp\left(-\frac{2}{3}\omega x^{-\frac{3}{2}} + \frac{5}{4}\ln x + c\right) \\
&= c x^{\frac{5}{4}} \exp\left(-\omega \frac{2}{3} x^{-\frac{3}{2}}\right) \qquad (1)
\end{aligned}$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim c Q^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt\right)$$

In this case, $n = 2$, since the ODE $y'' = x^{-5} y$ is second order. Here we have $Q(x) = x^{-5}$, therefore, plug-in into the above gives

$$\begin{aligned}
y(x) &\sim c\left(x^{-5}\right)^{\frac{1-2}{4}} \exp\left(\omega \int^x \left(t^{-5}\right)^{\frac{1}{2}} dt\right) \\
&\sim c\left(x^{-5}\right)^{\frac{-1}{4}} \exp\left(\omega \int^x t^{-\frac{5}{2}} dt\right) \\
&\sim c x^{\frac{5}{4}} \exp\left(\omega\left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right)\right) \\
&\sim c x^{\frac{5}{4}} \exp\left(-\omega \frac{2}{3} x^{-\frac{3}{2}}\right) \qquad (2)
\end{aligned}$$

Comparing (1) and (2), we see they are the same.

4.3.2 problem (b), page 88

$$y''' = xy$$

Irregular singular point at $x \to +\infty$. Let $y = e^{S_0(x)}$ and the above becomes

$$\begin{aligned}
y(x) &= e^{S_0(x)} \\
y'(x) &= S_0' e^{S_0} \\
y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\
&= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \\
y''' &= \left(S_0'' + \left(S_0'\right)^2\right)' e^{S_0} + \left(S_0'' + \left(S_0'\right)^2\right) S_0' e^{S_0} \\
&= \left(S_0''' + 2 S_0' S_0''\right) e^{S_0} + \left(S_0' S_0'' + \left(S_0'\right)^3\right) e^{S_0} \\
&= \left(S_0''' + 3 S_0' S_0'' + \left(S_0'\right)^3\right) e^{S_0}
\end{aligned}$$

Substituting back into $y''' = xy$ gives

$$\left(S_0''' + 3S_0'S_0'' + \left(S_0'\right)^3\right)e^{S_0} = xe^{S_0(x)}$$
$$S_0''' + 3S_0'S_0'' + \left(S_0'\right)^3 = x$$

Before solving for S_0, we can do one more simplification. Using the approximation that $\left(S_0'\right)^3 \ggg S_0'''$ for $x \to x_0$, the above becomes

$$3S_0'S_0'' + \left(S_0'\right)^3 \sim x$$

In addition, since $S_0' \ggg S_0''$ then we can use the approximation $\left(S_0'\right)^3 \ggg S_0'S_0''$ and the above becomes

$$\begin{aligned}
\left(S_0'\right)^3 &\sim x \\
S_0' &\sim \omega x^{\frac{1}{3}} \\
S_0 &\sim \omega \int x^{\frac{1}{3}} dx \\
S_0 &\sim \omega \frac{3}{4} x^{\frac{4}{3}}
\end{aligned}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$\begin{aligned}
y'(x) &= \left(S_0(x) + S_1(x)\right)' e^{S_0+S_1} \\
y''(x) &= \left(\left(S_0 + S_1\right)'\right)^2 e^{S_0+S_1} + \left(S_0 + S_1\right)'' e^{S_0+S_1} \\
&= \left(S_0' + S_1'\right)^2 e^{S_0+S_1} + \left(S_0'' + S_1''\right) e^{S_0+S_1} \\
&= \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1'\right) e^{S_0+S_1} + \left(S_0'' + S_1''\right) e^{S_0+S_1} \\
&= \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right) e^{S_0+S_1}
\end{aligned}$$

We can take the third derivative

$$
\begin{aligned}
y'''(x) &\sim \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)' e^{S_0+S_1} \\
&+ \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)\left(S_0+S_1\right)' e^{S_0+S_1} \\
\\
&\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1} \\
&+ \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)\left(S_0'+S_1'\right) e^{S_0+S} \\
\\
&\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1} \\
&+ \left(\left(S_0'\right)^3 + S_0'\left(S_1'\right)^2 + 2\left(S_0'\right)^2 S_1' + S_0'S_0'' + S_0'S_1''\right) e^{S_0+S_1} \\
&+ \left(S_1'\left(S_0'\right)^2 + \left(S_1'\right)^3 + 2S_0'\left(S_1'\right)^2 + S_1'S_0'' + S_1'S_1''\right) e^{S_0+S_1} \\
\\
&\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1} \\
&+ \left[\left(S_0'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + S_0'S_0'' + S_0'S_1'' + \left(S_1'\right)^3 + S_1'S_0'' + S_1'S_1''\right] e^{S_0+S_1} \\
&\sim \left(3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1''' + \left(S_0'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + \left(S_1'\right)^3\right) e^{S_0+S_1} \\
&\sim \left(\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1}
\end{aligned}
$$

Lets go ahead and plug-in this into the ODE

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1''' \sim x$$

Now we do some simplification. $\left(S_0'\right)^3 \ggg S_0'''$ and $\left(S_1'\right)^3 \ggg S_1'''$, hence above becomes

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' \sim x$$

Also, since $S_0'' \ggg S_1''$ then $3S_0'S_0'' \ggg 3S_0'S_1''$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' \sim x$$

Also, since $\left(S_0'\right)^2 \ggg S_0''$ then $3\left(S_0'\right)^2 S_1' \ggg 3S_0''S_1'$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' \sim x$$

Also since $S_1' \ggg S_1''$ then $3\left(S_0'\right)^2 S_1' \ggg 3S_1'S_1''$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' \sim x$$

But $S_0' \sim x^{\frac{1}{3}}$ hence$\left(S_0'\right)^3 \sim x$ and the above simplies to

$$\left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

Using $3S_0'\left(S_1'\right)^2 \ggg \left(S_1'\right)^3$ since $S_0' \ggg S_1'$ then

$$3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

Using $3\left(S_0'\right)^2 S_1' \ggg S_0'\left(S_1'\right)^2$ since $\left(S_0'\right)^2 \ggg S_0'$ then

$$3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

No more simplification. We are ready to solve for S_1.

$$\begin{aligned} S_1' &\sim \frac{-S_0'S_0''}{\left(S_0'\right)^2} \\ &\sim \frac{-S_0''}{S_0'} \end{aligned}$$

Hence

$$\begin{aligned} S_1 &\sim -\int \frac{S_0''}{S_0'} dx \\ &\sim -\ln S_0' + c \end{aligned}$$

Since $S_0' \sim x^{\frac{1}{3}}$then the above becomes

$$\begin{aligned} S_1 &\sim -\ln x^{\frac{1}{3}} + c \\ S_1 &\sim -\frac{1}{3}\ln x + c \end{aligned}$$

Hence, the leading behavior is

$$\begin{aligned} y(x) &= e^{S_0(x)+S_1(x)} \\ &= \exp\left(\omega\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{3}\ln x + c\right) \\ &= cx^{\frac{-1}{3}}\exp\left(\omega\frac{3}{4}x^{\frac{4}{3}}\right) \end{aligned} \tag{1}$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim cQ^{\frac{1-n}{2n}}\exp\left(\omega\int^x Q(t)^{\frac{1}{n}}\,dt\right)$$

In this case, $n = 3$, since the ODE $y''' = xy$ is third order. Here we have $Q(x) = x$, therefore, plug-in into the above gives

$$
\begin{aligned}
y(x) &\sim c(x)^{\frac{1-3}{6}} \exp\left(\omega \int^{x} (t)^{\frac{1}{3}} \, dt\right) \\
&\sim c x^{\frac{-1}{3}} \exp\left(\omega \frac{4}{3} x^{\frac{4}{3}}\right)
\end{aligned}
$$

Comparing (1) and (2), we see they are the same.

4.4 note added 1/31/2017

This note solves in details the ODE

$$x^3 y''(x) = y(x)$$

Using asymptoes method using what is called the dominant balance submethod where it is assumed that $y(x) = e^{S(x)}$.

4.4.1 Solution

$x = 0$ is an irregular singular point. The solution is assumeed to be $y(x) = e^{S(x)}$. Therefore $y' = S' e^{S(x)}$ and $y'' = S'' e^{S(x)} + (S')^2 e^{S(x)}$ and the given ODE becomes

$$x^3 \left(S'' + (S')^2\right) = 1 \tag{1}$$

Assuming that

$$S'(x) \sim c x^{\alpha}$$

Hence $S'' \sim c\alpha x^{\alpha-1}$. and (1) becomes

$$\begin{aligned} x^3 \left(c\alpha x^{\alpha-1} + (cx^{\alpha})^2\right) &\sim 1 \\ c\alpha x^{\alpha+2} + c^2 x^{2\alpha+3} &\sim 1 \end{aligned}$$

Term $c\alpha x^{\alpha+2} \ggg c^2 x^{2\alpha+3}$, hence we set $\alpha = \frac{-3}{2}$ to remove the subdominant term. Therefore the above becomes, after substituting for the found α

$$\begin{aligned} \overbrace{\frac{-3}{2} c x^{\frac{1}{2}}}^{x\to 0} + c^2 &\sim 1 \\ c^2 &= 1 \end{aligned}$$

Therefore $c = \pm 1$. The result so far is $S'(x) \sim c x^{\frac{-3}{2}}$. Now another term is added. Let

$$S'(x) \sim c x^{\frac{-3}{2}} + A(x)$$

Now we will try to find $A(x)$. Hence $S''(x) \sim \frac{-3}{2} c x^{\frac{-5}{2}} + A'$ and $x^3 \left(S'' + (S')^2\right) = 1$ now becomes

$$\begin{aligned} x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + \left(c x^{\frac{-3}{2}} + A\right)^2\right) &\sim 1 \\ x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + c^2 x^{-3} + A^2 + 2Acx^{\frac{-3}{2}}\right) &\sim 1 \\ \left(\frac{-3}{2} c x^{\frac{1}{2}} + x^3 A' + c^2 + x^3 A^2 + 2Acx^{\frac{3}{2}}\right) &\sim 1 \end{aligned}$$

Since $c^2 = 1$ from the above, then

$$\frac{-3}{2} c x^{\frac{1}{2}} + x^3 A' + x^3 A^2 + 2Acx^{\frac{3}{2}} \sim 0$$

Dominant balance says to keep dominant term (but now looking at those terms in A only). From the above, since $A \ggg A^2$ and $A \ggg A'$ then from the above, we can cross out A^2 and A' resulting in

$$\frac{-3}{2}cx^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} \sim 0$$

Hence we just need to find A to balance the above

$$\begin{aligned}
\frac{-3}{2}cx^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} &\sim 0\\
2Acx^{\frac{3}{2}} &\sim \frac{3}{2}cx^{\frac{1}{2}}\\
A &\sim \frac{3}{4x}
\end{aligned}$$

We found $A(x)$ for the second term. Therefore, so far we have

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x}$$

Or

$$S(x) = -2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x + C_0$$

But C_0 can be dropped (subdominant to $\ln x$ when $x \to 0$) and so far then we can write the solution as

$$\begin{aligned}
y(x) &= e^{S(x)}W(x)\\
&= e^{S(x)}\sum_{n=0}^{\infty} a_n x^{nr}\\
&= \exp\left(-2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x\right)\sum_{n=0}^{\infty} a_n x^{nr}\\
&= e^{-2cx^{\frac{-1}{2}}} x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{nr}\\
&= e^{-\frac{2c}{\sqrt{x}}}\sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}\\
&= e^{\pm\frac{2}{\sqrt{x}}}\sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}
\end{aligned}$$

Since $c = \pm 1$. We can now try adding one more term to $S(x)$. Let

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x} + B(x)$$

Hence

$$S'' = \frac{-3}{2}cx^{\frac{-5}{2}} - \frac{3}{4x^2} + B'(x)$$

And $x^3\left(S''+(S')^2\right)\sim 1$ now becomes

$$x^3\left(\left(\frac{-3}{2}cx^{\frac{-5}{2}}-\frac{3}{4x^2}+B'(x)\right)+\left(cx^{\frac{-3}{2}}+\frac{3}{4x}+B(x)\right)^2\right)\sim 1$$

$$x^3\left(\frac{c^2}{x^3}-\frac{3}{16}x^{-2}+2cBx^{-\frac{3}{2}}+\frac{3}{2}Bx^{-1}+B^2+B'\right)\sim 1$$

$$\left(\overset{=1}{\overbrace{c^2}}-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2+x^3B^2+x^3B'\right)\sim 1$$

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2+x^3B^2+x^3B'\sim 0$$

From the above, since $B(x)\ggg B^2(x)$ and $B(x)\ggg B'(x)$ and for small x, then we can cross out terms with B^2 and B' from above, and we are left with

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2\backsim 0$$

Between $2cBx^{\frac{3}{2}}$ and $\frac{3}{2}Bx^2$, for small x, then $2cBx^{\frac{3}{2}}\ggg\frac{3}{2}Bx^2$, so we can cross out $\frac{3}{2}Bx^2$ from above

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}\backsim 0$$

$$2cBx^{\frac{3}{2}}\backsim\frac{3}{16}x$$

$$B\backsim\frac{3}{32c}x^{-\frac{1}{2}}$$

We found $B(x)$, Hence now we have

$$S'(x)=cx^{\frac{-3}{2}}+\frac{3}{4x}+\frac{3}{32c}x^{-\frac{1}{2}}$$

Or

$$S(x)=-2cx^{\frac{-1}{2}}+\frac{3}{4}\ln x+\frac{3}{16c}x^{\frac{1}{2}}+C_1$$

But C_1 can be dropped (subdominant to $\ln x$ when $x\to 0$) and so far then we can write the solution as

$$\begin{aligned}
y(x)&=e^{S(x)}W(x)\\
&=e^{S(x)}\sum_{n=0}^{\infty}a_nx^{nr}\\
&=\exp\left(-2cx^{\frac{-1}{2}}+\frac{3}{4}\ln x+\frac{3}{16c}x^{\frac{1}{2}}\right)\sum_{n=0}^{\infty}a_nx^{nr}\\
&=e^{-2cx^{\frac{-1}{2}}+\frac{3}{16c}x^{\frac{1}{2}}}x^{\frac{3}{4}}\sum_{n=0}^{\infty}a_nx^{nr}\\
&=e^{-2cx^{\frac{-1}{2}}+\frac{3}{16c}x^{\frac{1}{2}}}\sum_{n=0}^{\infty}a_nx^{nr+\frac{3}{4}}
\end{aligned}$$

For $c = 1$

$$y_1(x) = e^{-2x^{\frac{-1}{2}}+\frac{3}{16}x^{\frac{1}{2}}}\sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}$$

For $c = -1$

$$y_2(x) = e^{2x^{\frac{-1}{2}}-\frac{3}{16}x^{\frac{1}{2}}}\sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}$$

Hence

$$y(x) \sim Ay_1(x) + By_2(x)$$

Reference

1. Page 80-82 Bender and Orszag textbook.
2. Lecture notes, Lecture 5, Tuesday Janurary 31, 2017. EP 548, University of Wisconsin, Madison by Professor Smith.
3. Lecture notes from http://www.damtp.cam.ac.uk/

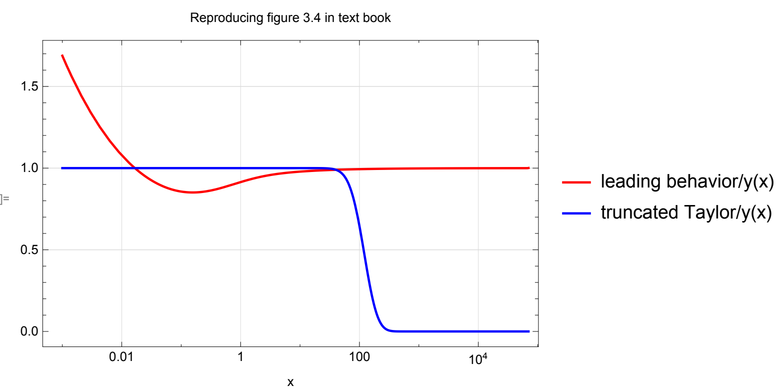
4.5 note p3 figure 3.4 in text (page 92) reproduced in color

figure 3.4 in text (page 92) was not too clear (it is black and white in my text book that I am using). So I reproduced it in color. This type of plot is needed to check what happens with the improvement in approximation for problem 3.42 (a) as more terms are added to leading behavior.

It is done in Wolfram Mathematica. Used 300 terms to find y(x). We see the red line, which is leading behavior/y(x) ratio going to 1 for large x as would be expected.

In[1]:=
```
leading[x] := 1/2 Pi^(-1/2) x^(-1/4) Exp[2 x^(1/2)];
y[x_, max_] := Sum[x^n/(Factorial[n]^2), {n, 0, max}];
LogLinearPlot[Evaluate[{leading[x]/y[x, 300], y[x, 10]/y[x, 300]}], {x, 0.001, 70000},
 PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 PlotLegends → {"leading behavior/y(x)", "truncated Taylor/y(x)"}, FrameLabel →
  {{None, None}, {"x", "Reproducing figure 3.4 in text book"}}, PlotStyle → {Red, Blue}]
```

Out[3]=



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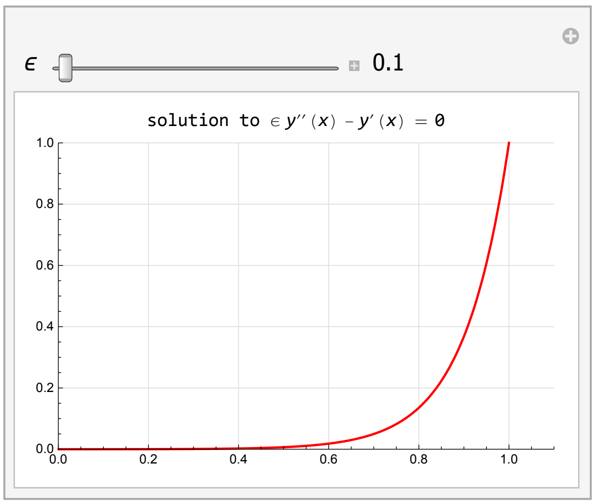
4.6 note p4 added 2/15/17 animations of the solutions to ODE from lecture 2/9/17 for small parameter

These are small animations of the solutions to the 2 ODE's from lecture 2/9/17 for the small parameter.
Nasser M. Abbasi
2/20/2017, EP 548, Univ. Wisconsin Madison.

```
Manipulate[
 Labeled[
  Plot[(-1 + e^(x/z))/(-1 + e^(1/z)), {x, 0, 2}, PlotRange → {{0, 1.1}, {0, 1}},
   GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red],
  Row[{"solution to ", TraditionalForm[ϵ y''[x] - y'[x] == 0]}], Top],
 {{z, 0.1, "ϵ"}, .1, 0.001, -0.001, Appearance → "Labeled"}
]
```



For eps*y''+y=0 ode, we notice global variations showing up in frequency as well as in amplitude as eps become very small

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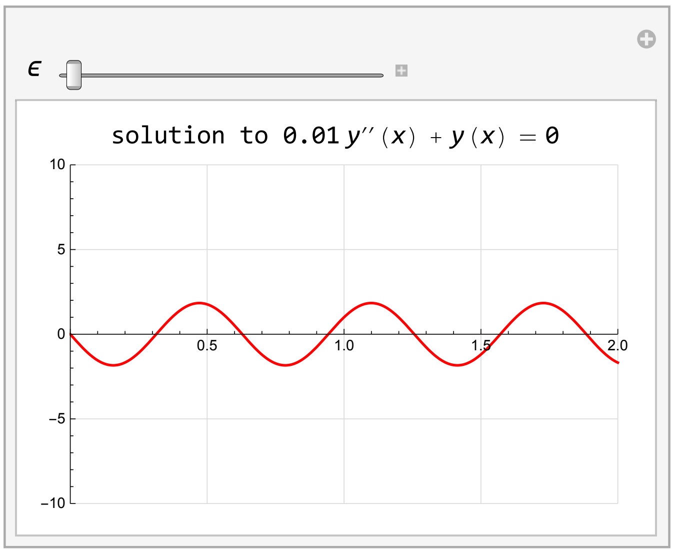
```
Manipulate[
 Labeled[
  Plot[Sin[x/Sqrt[z]]/Sin[1/Sqrt[z]], {x, 0, 2}, PlotRange → {{0, 2}, {-10, 10}},
   GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red],
  Style[Row[{"solution to ", TraditionalForm[z y''[x] + y[x] == 0]}], 16], Top],
 {{z, 0.01, "ϵ"}, 0.01, 0.0001, -0.0001}
]
```



First animation

second animation

4.7 note p5 added 2/15/17 animation figure 9.5 in text page 434

Animation of figure 9.5

Nasser M. Abbasi, Feb 2017, EP 548 course

Animation of figure 9.5 in text page 434, compare higher order boundary layer solutions

This is an animation of the percentage error between boundary layer solution and the exact solution for $\epsilon y'' + (1+x)y' + y = 0$ for different orders. Orders are given in text up y0,y1,y2,y3. We see the error is smaller when order increases as what one would expect.

The percentage error is largest around x=0.1 than other regions. Why? Location of matching between yin and yout. Due
to approximation made when doing matching? But all boundary layer solutions underestimate the exact solution since error is negative.

The red color is the most accurate. 3rd order.

First animation

In[45]:=

```
Clear["Global`*"];
yout[x_] := 2 / (1 + x);
yin[x_, eps_] :=
  Plot[2 - Exp[-x / eps], {x, 0, 1}, AxesOrigin → {0, 1}, PlotStyle → Black];
p1 = Plot[yout[x], {x, 0, 1}, PlotStyle → Blue];
combine[x_, eps_] := Plot[yout[x] - Exp[-x / eps], {x, 0, 1}, PlotStyle → {Thick, Red},
   AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray];
Animate[
 Grid[{{TraditionalForm[
     NumberForm[eps, {Infinity, 4}] HoldForm[y''[x] + (1 + x) y'[x] + y[x] == 0]]},
   {Show[
     Legended[combine[x, eps], Style["combinbed solution", Red]],
     Legended[p1, Style["Outer solution", Blue]],
     Legended[yin[x, eps], Style["Inner solution", Black]]
     , PlotRange → {{0, 1}, {1, 2}}, ImageSize → 400]
   }}], {eps, 0.001, .1, .00001}]
```

Printed by Wolfram Mathematica Student Edition

second animation, y0 vs. exact

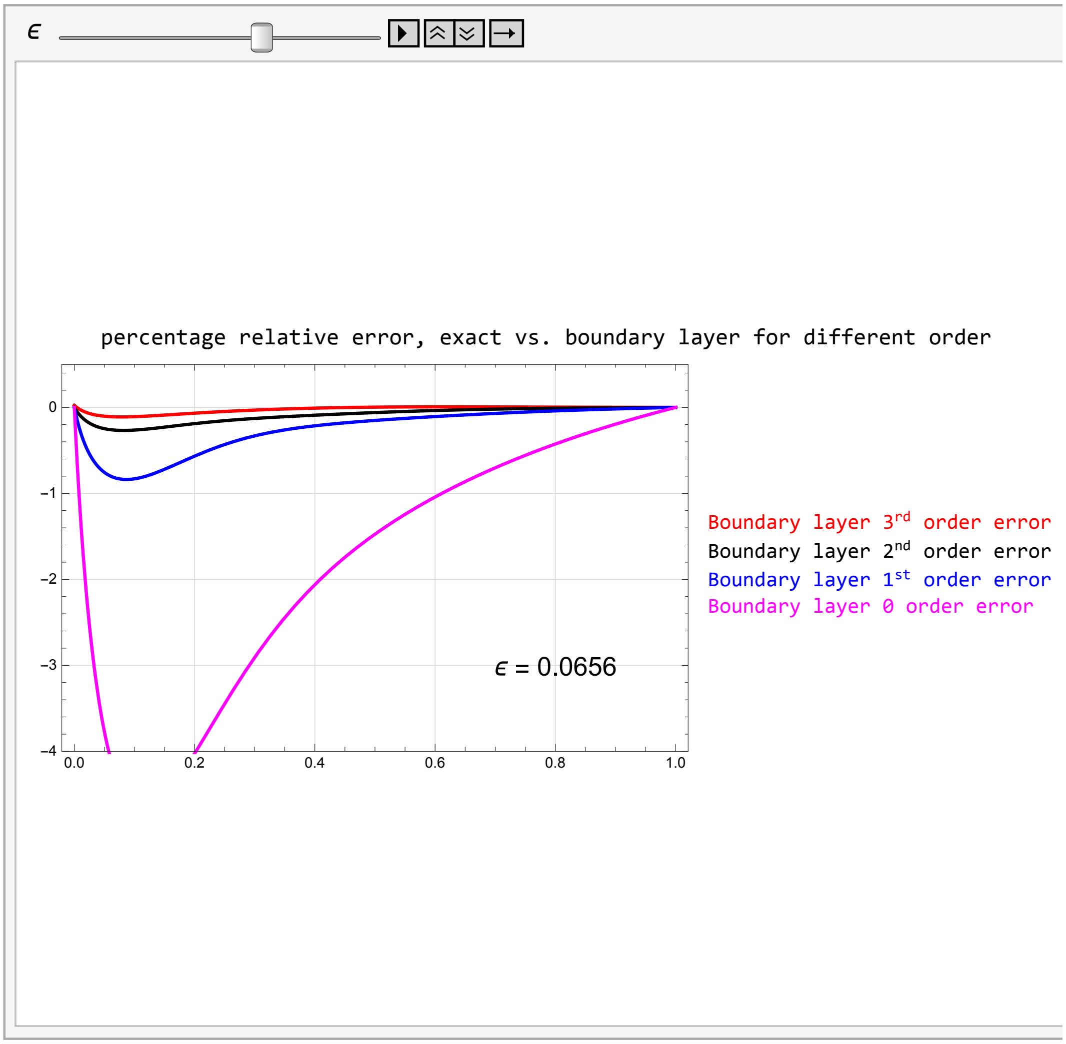
In[51]:=
```
Clear["Global`*"];
yout[x_] := 2 / (1 + x);
sol =
  y[x] /. First@DSolve[{ϵ y''[x] + (1 + x) y'[x] + y[x] == 0, y[0] == 1, y[1] == 1}, y[x], x];
exact[x_, ϵ_] := Evaluate@sol;
yin[x_, eps_] :=
  Plot[2 - Exp[-x / eps], {x, 0, 1}, AxesOrigin → {0, 1}, PlotStyle → Black];
p1 = Plot[yout[x], {x, 0, 1}, PlotStyle → Blue];
combine[x_, eps_] := Plot[yout[x] - Exp[-x / eps], {x, 0, 1}, PlotStyle → {Thick, Red},
   AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray];
Animate[
 Grid[{
   {TraditionalForm[
     NumberForm[ϵ, {Infinity, 4}] HoldForm[y''[x] + (1 + x) y'[x] + y[x] == 0]]},
   {Row[{"exact solution ", TraditionalForm[sol]}]},
   {Show[
     Legended[combine[x, ϵ], Style["Boundary layer solution, zero order", Red]],
     Legended[Plot[Evaluate@exact[x, ϵ], {x, 0, 1}, AxesOrigin → {0, 0}, PlotStyle → Blue],
      Style["Exact analytical solution", Blue]]
     , PlotRange → {{0, 1}, {1, 2}}, ImageSize → 400]
   }}], {ϵ, 0.0001, .1, .00001}]
```

add more terms

In[60]:=
```
Clear["Global`*"];
sol =
  y[x] /. First@DSolve[{ϵ y''[x] + (1 + x) y'[x] + y[x] == 0, y[0] == 1, y[1] == 1}, y[x], x];
exact[x_, ϵ_] := Evaluate@sol;
boundary0[x_, eps_] := (2/(1 + x) - Exp[-x / eps]);
boundary1[x_, eps_] :=
  (2/(1 + x) - Exp[-x / eps]) + eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]);
boundary2[x_, eps_] := (2/(1 + x) - Exp[-x / eps]) +
   eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]) +
   eps^2 (6/(1 + x)^5 - 1/(2 (1 + x)^3) - 1/(4 (1 + x)) - (1/8 (x / eps)^4 - 3/4 (x / eps)^2 + 21/4) Exp[-x / eps]);
boundary3[x_, eps_] := (2/(1 + x) - Exp[-x / eps]) +
   eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]) +
   eps^2 (6/(1 + x)^5 - 1/(2 (1 + x)^3) - 1/(4 (1 + x)) - (1/8 (x / eps)^4 - 3/4 (x / eps)^2 + 21/4) Exp[-x / eps]) +
   eps^3 (30/(1 + x)^7 - 3/(2 (1 + x)^5) - 1/(4 (1 + x)^3) - 5/(16 (1 + x)) +
     (1/48 (x / eps)^6 - 3/16 (x / eps)^4 + 21/8 (x / eps)^2 - 1949/72) Exp[-x / eps]);
```

In[67]:=
```
Animate[
  ex = exact[x, ϵ];
  Grid[{
    {Style[
      "percentage relative error, exact vs. boundary layer for different order", 14]},
    {Show[
      Legended[Plot[100 * ((boundary3[x, ϵ] - ex)/ex), {x, 0, 1},
        PlotStyle → {Thick, Red}, AxesOrigin → {0, 1}, GridLines → Automatic,
        GridLinesStyle → LightGray, PlotRange → {Automatic, {-4, .5}}, Frame → True,
        Epilog → Text[Style[Row[{"ϵ = ", NumberForm[ϵ, {Infinity, 4}]}], 16], {.8, -3}]],
       Style["Boundary layer 3rd order error", Red]],
      Legended[Plot[100 * ((boundary2[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Black},
        AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
       Style["Boundary layer 2nd order error", Black]],
      Legended[Plot[100 * ((boundary1[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Blue},
        AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
       Style["Boundary layer 1st order error", Blue]],
      Legended[Plot[100 * ((boundary0[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Magenta},
        AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
       Style["Boundary layer 0 order error", Magenta]]
      , PlotRange → {{0, 1}, {-4, .5}}, ImageSize → 400]
    }}], {ϵ, 0.001, .1, .0001}]
```

Out[67]=


ϵ
percentage relative error, exact vs. boundary layer for different order
0
−1
−2
−3
−4
0.0
0.2
0.4
0.6
0.8
1.0
ϵ = 0.0656
Boundary layer 3rd order error
Boundary layer 2nd order error
Boundary layer 1st order error
Boundary layer 0 order error

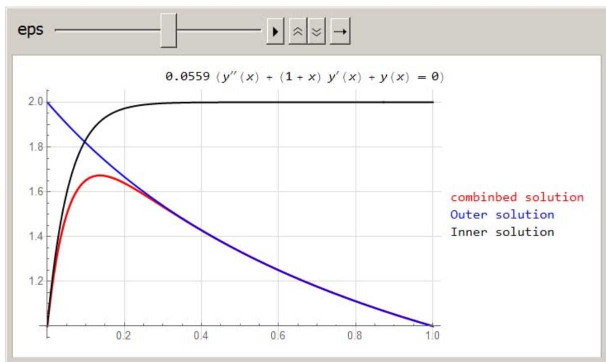
Animation of figure 9.5 in text page 434, compare higher order boundary layer solutions

This is an animation of the percentage error between boundary layer solution and the exact solution for $\epsilon y'' + (1+x)y' + y = 0$ for different orders. Orders are given in text up $y0, y1, y2, y3$. We see the error is smaller when the order increases as what one would expect.

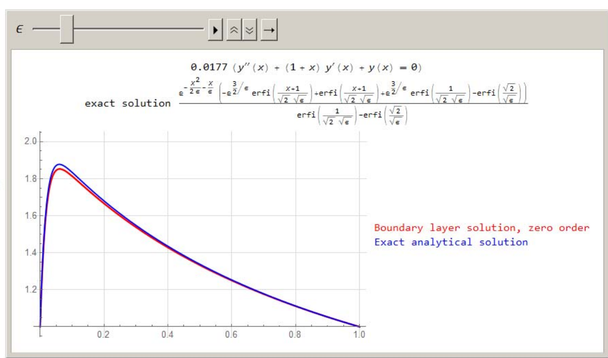
The percentage error is largest around $x = 0.1$ than other regions. Why? Location of matching is between y_{in} and y_{out}. Due to approximation made when doing matching? But all boundary layer solutions underestimate the exact solution since error is negative.

The red color is the most accurate. 3rd order.

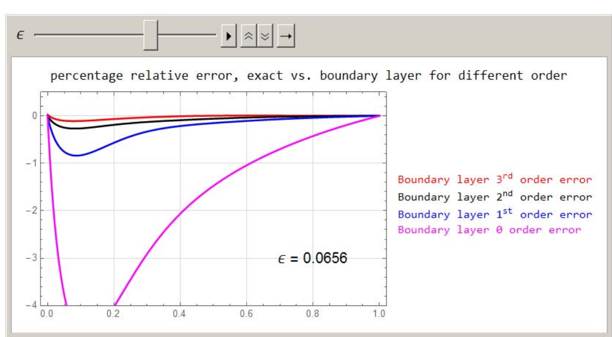
First animation



second animation



third animation



4.8 note p7 added 2/15/17, boundary layer problem solved in details

This note solves

$$\begin{aligned}\varepsilon y''(x) + (1+x)\,y'(x) + y(x) &= 0\\ y(0) &= 1\\ y(1) &= 1\end{aligned}$$

Where ε is small parameter, using boundary layer theory.

4.8.0.1 Solution

Since $(1+x) > 0$ in the domain, we expect boundary layer to be on the left. Let $y_{out}(x)$ be the solution in the outer region. Starting with $y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n$ and substituting back into the ODE gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \cdots\right) + (1+x)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

$O(1)$ terms

Collecting all terms with zero powers of ε

$$(1+x)\,y_0' + y_0 = 0$$

The above is solved using the right side conditions, since this is where the outer region is located. Solving the above using $y_0(1) = 1$ gives

$$y_0^{out}(x) = \frac{2}{1+x}$$

Now we need to find $y_{in}(x)$. To do this, we convert the ODE using transformation $\xi = \frac{x}{\varepsilon}$. Hence $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{dy}{d\xi}\frac{1}{\varepsilon}$. Hence the operator $\frac{d}{dx} \equiv \frac{1}{\varepsilon}\frac{d}{d\xi}$. This means the operator $\frac{d^2}{dx^2} \equiv \left(\frac{1}{\varepsilon}\frac{d}{d\xi}\right)\left(\frac{1}{\varepsilon}\frac{d}{d\xi}\right) = \frac{1}{\varepsilon^2}\frac{d^2}{d\xi^2}$. The ODE becomes

$$\begin{aligned}\varepsilon\frac{1}{\varepsilon^2}\frac{d^2 y(\xi)}{d\xi^2} + (1+\xi\varepsilon)\frac{1}{\varepsilon}\frac{dy(\xi)}{d\xi} + y(\xi) &= 0\\ \frac{1}{\varepsilon}y'' + \left(\frac{1}{\varepsilon} + \xi\right)y' + y &= 0\end{aligned}$$

Plugging $y(\xi) = \sum_{n=0}^{\infty} \varepsilon^n y(\xi)_n$ into the above gives

$$\begin{aligned}\frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\frac{1}{\varepsilon} + \xi\right)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) &= 0\\ \frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \frac{1}{\varepsilon}\left(y_0' + \varepsilon y_1' + \cdots\right) + \xi\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) &= 0\end{aligned}$$

Collecting all terms with smallest power of ε , which is ε^{-1} in this case, gives

$$\frac{1}{\varepsilon}y_0'' + \frac{1}{\varepsilon}y_0' = 0$$
$$y_0'' + y_0' = 0$$

Let $z = y_0'$, the above becomes

$$z' + z = 0$$
$$d\left(e^{\xi}z\right) = 0$$
$$e^{\xi}z = c$$
$$z = ce^{-\xi}$$

Hence $y_0'(\xi) = ce^{-\xi}$. Integrating

$$y_0^{in}(\xi) = -ce^{-\xi} + c_1 \tag{1A}$$

Since c is arbitrary constant, the negative sign can be removed, giving

$$y_0^{in}(\xi) = ce^{-\xi} + c_1 \tag{1A}$$

This is the lowest order solution for the inner $y^{in}(\xi)$. We have two boundary conditions, but we can only use the left side one, where $y_{in}(\xi)$ lives. Hence using $y_0(\xi = 0) = 1$, the above becomes

$$1 = c + c_1$$
$$c_1 = 1 - c$$

The solution (1A) becomes

$$y_0(\xi) = ce^{-\xi} + (1 - c)$$
$$= 1 + c\left(e^{-\xi} - 1\right)$$

Let $c = A_0$ to match the book notation.

$$y_0(\xi) = 1 + A_0\left(e^{-\xi} - 1\right)$$

To find A_0, we match $y_0^{in}(\xi)$ with $y_0^{out}(x)$

$$\lim_{\xi\to\infty} 1 + A_0\left(e^{-\xi} - 1\right) = \lim_{x\to 0^+} \frac{2}{1+x}$$
$$1 - A_0 = 2$$
$$A_0 = -1$$

Hence

$$y_0^{in}(\xi) = 2 - e^{-\xi}$$

<u>$O(\varepsilon)$ terms</u>

We now repeat the process to find $y_1^{in}(\xi)$ and $y_1^{out}(x)$. Starting <u>with $y^{out}(x)$</u>

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \cdots\right) + (1 + x)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

Collecting all terms with ε^1 now

$$\begin{aligned}\varepsilon y_0'' + (1+x)\,\varepsilon y_1' + \varepsilon y_1 &= 0\\ y_0'' + (1+x)\,y_1' + y_1 &= 0\end{aligned}$$

But we know $y_0 = \frac{2}{1+x}$, from above. Hence $y_0'' = \frac{4}{(1+x)^3}$ and the above becomes

$$\begin{aligned}(1+x)\,y_1' + y_1 &= -\frac{4}{(1+x)^3}\\ y_1' + \frac{y_1}{1+x} &= -\frac{4}{(1+x)^4}\end{aligned}$$

Integrating factor $\mu = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$ and the above becomes

$$\begin{aligned}\frac{d}{dx}\left(\mu y_1\right) &= -\mu\frac{4}{(1+x)^4}\\ \frac{d}{dx}\left((1+x)\,y_1\right) &= -\frac{4}{(1+x)^3}\end{aligned}$$

Integrating

$$\begin{aligned}(1+x)\,y_1 &= -\int \frac{4}{(1+x)^3}dx + c\\ &= \frac{2}{(1+x)^2} + c\end{aligned}$$

Hence

$$y_1(x) = \frac{2}{(1+x)^3} + \frac{c}{1+x}$$

Applying $y(1) = 0$ (notice the boundary condition now becomes $y(1) = 0$ and not $y(1) = 1$, since we have already used $y(1) = 1$ to find leading order). From now on, all boundary conditions will be $y(1) = 0$.

$$\begin{aligned}0 &= \frac{2}{(1+1)^3} + \frac{c}{1+1}\\ c &= -\frac{1}{2}\end{aligned}$$

Hence

$$y_1(x) = \frac{2}{(1+x)^3} - \frac{1}{2}\frac{1}{(1+x)}$$

Now we need to find $y_1^{in}(\xi)$. To do this, starting from

$$\frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\frac{1}{\varepsilon} + \xi\right)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

$$\frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \frac{1}{\varepsilon}\left(y_0' + \varepsilon y_1' + \cdots\right) + \xi\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

But now collecting all terms with $O(1)$ order, (last time, we collected terms with $O\left(\varepsilon^{-1}\right)$).

$$y_1'' + y_1' + \xi y_0' + y_0 = 0$$
$$y_1'' + y_1' = -\xi y_0' - y_0 \tag{1}$$

But we found y_0^{in} earlier which was

$$y_0^{in}(\xi) = 1 + A_0\left(e^{-\xi} - 1\right)$$

Hence $y_0' = -A_0 e^{-\xi}$ and the ODE (1) becomes

$$y_1'' + y_1' = \xi A_0 e^{-\xi} - \left(1 + A_0\left(e^{-\xi} - 1\right)\right)$$

We need to solve this with boundary conditions $y_1(0) = 0$. (again, notice change in B.C. as was mentioned above). The solution is

$$\begin{aligned} y_1(\xi) &= -\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) + A_1\left(1 - e^{-\xi}\right) \\ &= -\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) - A_1\left(e^{-\xi} - 1\right) \end{aligned}$$

Since A_1 is arbitrary constant, and to match the book, we can call $A_2 = -A_1$ and then rename A_2 back to A_1 and obtain

$$y_1(\xi) = -\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) + A_1\left(e^{-\xi} - 1\right)$$

This is to be able to follow the book. Therefore, this is what we have so far

$$\begin{aligned} y_{out} &= y_0^{out} + \varepsilon y_1^{out} \\ &= \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) \end{aligned}$$

And

$$\begin{aligned} y^{in}(\xi) &= y_0^{in} + \varepsilon y_1^{in} \\ &= \left(1 + A_0\left(e^{-\xi} - 1\right)\right) + \varepsilon\left(-\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) + A_1\left(e^{-\xi} - 1\right)\right) \\ &= A_0 e^{-\xi} - \xi\varepsilon - \varepsilon A_1 - A_0 + \varepsilon A_1 e^{-\xi} + \xi\varepsilon A_0 - \frac{1}{2}\xi^2 \varepsilon A_0 e^{-\xi} + 1 \end{aligned}$$

To find A_0, A_1, we match y_{in} with y_{out}, therefore

$$\lim_{\xi\to\infty} y_{in} = \lim_{x\to 0} y_{out}$$

Or

$$\lim_{\xi\to\infty}\left(A_0 e^{-\xi} - \xi\varepsilon - \varepsilon A_1 - A_0 + \varepsilon A_1 e^{-\xi} + \xi\varepsilon A_0 - \frac{1}{2}\xi^2\varepsilon A_0 e^{-\xi} + 1\right) = \lim_{x\to 0}\frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right)$$

Which simplifies to

$$-\xi\varepsilon - \varepsilon A_1 - A_0 + \xi\varepsilon A_0 + 1 = \lim_{x\to 0}\frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right)$$

It is easier now to convert the LHS to use x instead of ξ so we can compare. Since $\xi = \frac{x}{\varepsilon}$, then the above becomes

$$-x - \varepsilon A_1 - A_0 + xA_0 + 1 = \lim_{x\to 0} \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right)$$

Using Taylor series on the RHS

$$\begin{aligned}1 - A_0 - x + A_0 x + A_1\varepsilon = \lim_{x\to 0} 2\left(1 - x + x^2 + \cdots\right)\\ + 2\varepsilon\left(1 - x + x^2 + \cdots\right)\left(1 - x + x^2 + \cdots\right)\left(1 - x + x^2 + \cdots\right) - \frac{\varepsilon}{2}\left(1 - x + x^2 + \cdots\right)\end{aligned}$$

Since we have terms on the the LHS of only $O(1), O(x), O(\varepsilon)$, then we need to keep at least terms with $O(1), O(x), O(\varepsilon)$ on the RHS and drop terms with $O\left(x^2\right), O(\varepsilon x), O\left(\varepsilon^2\right)$ to be able to do the matching. So in the above, RHS simplifies to

$$\begin{aligned}-x - \varepsilon A_1 - A_0 + xA_0 + 1 &= 2(1-x) + 2\varepsilon - \frac{\varepsilon}{2}\\ -x - \varepsilon A_1 - A_0 + xA_0 + 1 &= 2 - 2x + 2\varepsilon - \frac{\varepsilon}{2}\\ -\varepsilon A_1 - A_0 + x(A_0 - 1) + 1 &= 2 - 2x + \frac{3}{2}\varepsilon\end{aligned}$$

Comparing, we see that

$$\begin{aligned}A_0 - 1 &= -2\\ A_0 &= -1\end{aligned}$$

We notice this is the same A_0 we found for the lowest order. This is how it should always come out. If we get different value, it means we made mistake. We could also match $-A_0 + 1 = 2$ which gives $A_0 = -1$ as well. Finally

$$\begin{aligned}-\varepsilon A_1 &= \frac{3}{2}\varepsilon\\ A_1 &= -\frac{3}{2}\end{aligned}$$

So we have used matching to find all the constants for y_{in}. Here is the final solution so far

$$\begin{aligned}y_{out}(x) &= \overbrace{\frac{2}{1+x}}^{y_0} + \varepsilon\overbrace{\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right)}^{y_1}\\ y_{in}(\xi) &= \overbrace{1 + A_0\left(e^{-\xi} - 1\right)}^{y_0} + \varepsilon\overbrace{\left(-\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) + A_1\left(e^{-\xi} - 1\right)\right)}^{y_1}\\ &= 1 - \left(e^{-\xi} - 1\right) + \varepsilon\left(-\xi - \left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) - \frac{3}{2}\left(e^{-\xi} - 1\right)\right)\\ &= \frac{3}{2}\varepsilon - e^{-\xi} - 2\xi\varepsilon - \frac{3}{2}\varepsilon e^{-\xi} + \frac{1}{2}\xi^2\varepsilon e^{-\xi} + 2\end{aligned}$$

In terms of x, since Since $\xi = \frac{x}{\varepsilon}$ the above becomes

$$\begin{aligned} y_{in}(x) &= \frac{3}{2}\varepsilon - e^{-\frac{x}{\varepsilon}} - 2x - \frac{3}{2}\varepsilon e^{-\frac{x}{\varepsilon}} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\frac{x}{\varepsilon}} + 2 \\ &= 2 - 2x + \frac{3}{2}\varepsilon + e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) \end{aligned}$$

Hence

$$y_{uniform} = y_{in} + y_{out} - y_{match}$$

Where

$$\begin{aligned} y_{match} &= \lim_{\xi\to\infty} y_{in} \\ &= 2 - 2x + \frac{3}{2}\varepsilon \end{aligned}$$

Hence

$$\begin{aligned} y_{uniform} &= 2 - 2x + \frac{3}{2}\varepsilon + e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) + \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) - \left(2 - 2x + \frac{3}{2}\varepsilon\right) \\ &= e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) + \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) \\ &= \left(\frac{2}{1+x} - e^{-\frac{x}{\varepsilon}} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\frac{x}{\varepsilon}}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\frac{x}{\varepsilon}}\right) \end{aligned}$$

Which is the same as

$$\begin{aligned} y_{uniform} &= \left(\frac{2}{1+x} - e^{-\xi} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\xi}\right) \\ &= \left(\frac{2}{1+x} - e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\xi} + \frac{1}{2}\xi^2 e^{-\xi}\right) \\ &= \left(\frac{2}{1+x} - e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} + \left(\frac{1}{2}\xi^2 - \frac{3}{2}\right)e^{-\xi}\right) \qquad (1) \end{aligned}$$

Comparing (1) above, with book result in first line of 9.3.16, page 433, we see the same result.

4.8.0.2 References

1. Advanced Mathematica methods, Bender and Orszag. Chapter 9.
2. Lecture notes. Feb 16, 2017. By Professor Smith. University of Wisconsin. NE 548

4.9 note p9. added 2/24/17, asking question about problem in book

How did the book, on page 429, near the end, arrive at $A_1 = -e$? I am not able to see it. This is what I tried. The book does things little different than what we did in class. The book does not do

$$\lim_{\xi\to\infty} y^{in} \sim \lim_{x\to 0} y^{out}$$

But instead, book replaces x in the $y^{out}(x)$ solution already obtained, with $\xi\varepsilon$, and rewrites $y^{out}(x)$, which is what equation (9.2.14) is. So following this, I am trying to verify the book result for $A_1 = -e$, but do not see how. Using the book notation, of using X in place of ξ, we have

$$Y_1(X) = (A_1 + A_0)\left(1 - e^{-X}\right) - eX$$

Which is the equation in the book just below 9.2.14. The goal now is to find A_1. Book already found $A_0 = e$ earlier. So we write

$$\begin{aligned}
\lim_{X\to\infty} \overbrace{(A_1 + A_0)\left(1 - e^{-X}\right) - eX}^{Y_1(X)} &\sim y^{out}(x) \\
\lim_{X\to\infty} (A_1 + A_0)\left(1 - e^{-X}\right) - eX &\sim e\left(1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \cdots\right)
\end{aligned}$$

So far so good. But now the book says "comparing $Y_1(x)$ when $X \to \infty$ with the second term in 9.2.14 gives $A_1 = -e$". But how? If we take $X \to \infty$ on the LHS above, we get

$$\lim_{X\to\infty} (A_1 + A_0) - eX \sim e\left(1 - \varepsilon X + \frac{\varepsilon X^2}{2!} - \cdots\right)$$

But $A_0 = e$, so

$$\begin{aligned}
\lim_{X\to\infty} A_1 + e - eX &\sim e - e\varepsilon X + e\frac{\varepsilon X^2}{2!} - \cdots \\
\lim_{X\to\infty} A_1 - eX &\sim -e\varepsilon X + e\frac{\varepsilon X^2}{2!} - \cdots
\end{aligned}$$

How does the above says that $A_1 = -e$? If we move $-eX$ to the right sides, it becomes

$$\begin{aligned}
A_1 &\sim eX - e\varepsilon X + e\frac{\varepsilon X^2}{2!} - \cdots \\
A_1 &\sim e(X - \varepsilon X) + e\frac{\varepsilon X^2}{2!} - \cdots
\end{aligned}$$

I do not see how $A_1 = -e$. Does any one see how to get $A_1 = -e$?

Let redo this using the class method

$$
\begin{aligned}
\lim_{X\to\infty} \overbrace{(A_1 + A_0)\left(1 - e^{-X}\right) - eX}^{Y_1(X)} &\sim \lim_{x\to 0} y^{out}(x) \\
\lim_{X\to\infty} (A_1 + A_0)\left(1 - e^{-X}\right) - eX &\sim \lim_{x\to 0} e\left(1 - x + \frac{x^2}{2!} - \cdots\right) \\
\lim_{X\to\infty} A_1 + e - eX &\sim e \\
\lim_{X\to\infty} A_1 &\sim eX
\end{aligned}
$$

How does the above says that $A_1 = -e$? and what happend to the $\lim_{X\to\infty}$ of X which remains there?

4.10 note p11. added 3/7/17, Showing that scaling for normalization is same for any n

This small computation verifies that normalization constant for the S-L from lecture 3/2/2017 for making all eigenfunction orthonormal is the same for each n. Its numerical value is 0.16627. Here is the table generated for $n = 1, 2, \ldots, 6$.

```
In[42]:= ClearAll[n, c, y, m];
lam[n_] := n^2 π^2 (3/(7 π^3))^2;
y[n_, x_] := c/(x + π) Sin[Sqrt[lam[n]] ((x + π)^3/3 - π^3/3)];
data = Table[{m, r = c /. Last@Solve[∫_0^π (y[m, x])^2 (x + π)^4 dx == 1, c]; r, N@r}, {m, 1, 6}];
Grid[Join[{{"n", "c(n)", "numerical c(n)"}}, data], Frame → All]
```

Out[46]=

n	c(n)	numerical c(n)
1	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627
2	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627
3	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627
4	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627
5	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627
6	$\frac{\sqrt{\frac{6}{7}}}{\pi^{3/2}}$	0.16627

4.11 note p12. added 3/8/17, comparing exact solution to WKB solution using Mathematica

Comparing Exact to WKB solution for ODE in lecture 3/2/2017

by Nasser M. Abbasi EP 548, Spring 2017.

This note shows how to obtain exact solution for the ODE given in lecture 3/2/2017, EP 548, and to compare it to the WKB solution for different modes. This shows that the WKB becomes very close to the exact solution for higher modes.

Obtain the exact solution, in terms of BesselJ functions

In[16]:=
```
ClearAll[n, c, y, m, lam];
lam[n_] := (9 n^2 / (49 Pi^4)); (*eigenvalues from WKB solution*)
c = Sqrt[6 / (7 Pi^3)]; (*normalization value found for WKB*)

y[n_, x_] := c 1/(Pi + x) Sin[n (x^3 + 3 x^2 Pi + 3 Pi^2 x) / (7 Pi^2)];
(*WKB solution found*)
```

Find exact solution

In[18]:=
```
ode = y''[x] + lam (x + Pi)^4 y[x] == 0;
(solExact = y[x] /. First@DSolve[{ode, y[0] == 0}, y[x], x]) // TraditionalForm
```

Out[19]//TraditionalForm=

$$\frac{1}{\sqrt[6]{6}\,\sqrt[24]{\text{lam}}\,J_{\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)} c_1\,\Gamma\left(\frac{5}{6}\right)\sqrt[8]{\text{lam}\,(x+\pi)^4}\left(J_{\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)J_{-\frac{1}{6}}\left(\frac{\left(\text{lam}\,x^4+4\,\text{lam}\,\pi\,x^3+6\,\text{lam}\,\pi^2\,x^2+4\,\text{lam}\,\pi^3\,x+\text{lam}\,\pi^4\right)^{3/4}}{3\,\sqrt[4]{\text{lam}}}\right)- J_{-\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)J_{\frac{1}{6}}\left(\frac{\left(\text{lam}\,x^4+4\,\text{lam}\,\pi\,x^3+6\,\text{lam}\,\pi^2\,x^2+4\,\text{lam}\,\pi^3\,x+\text{lam}\,\pi^4\right)^{3/4}}{3\,\sqrt[4]{\text{lam}}}\right)\right)$$

Make function which normalizes the exact solution eigenfunctions and plot each mode eigenfunction with the WKB on the same plot

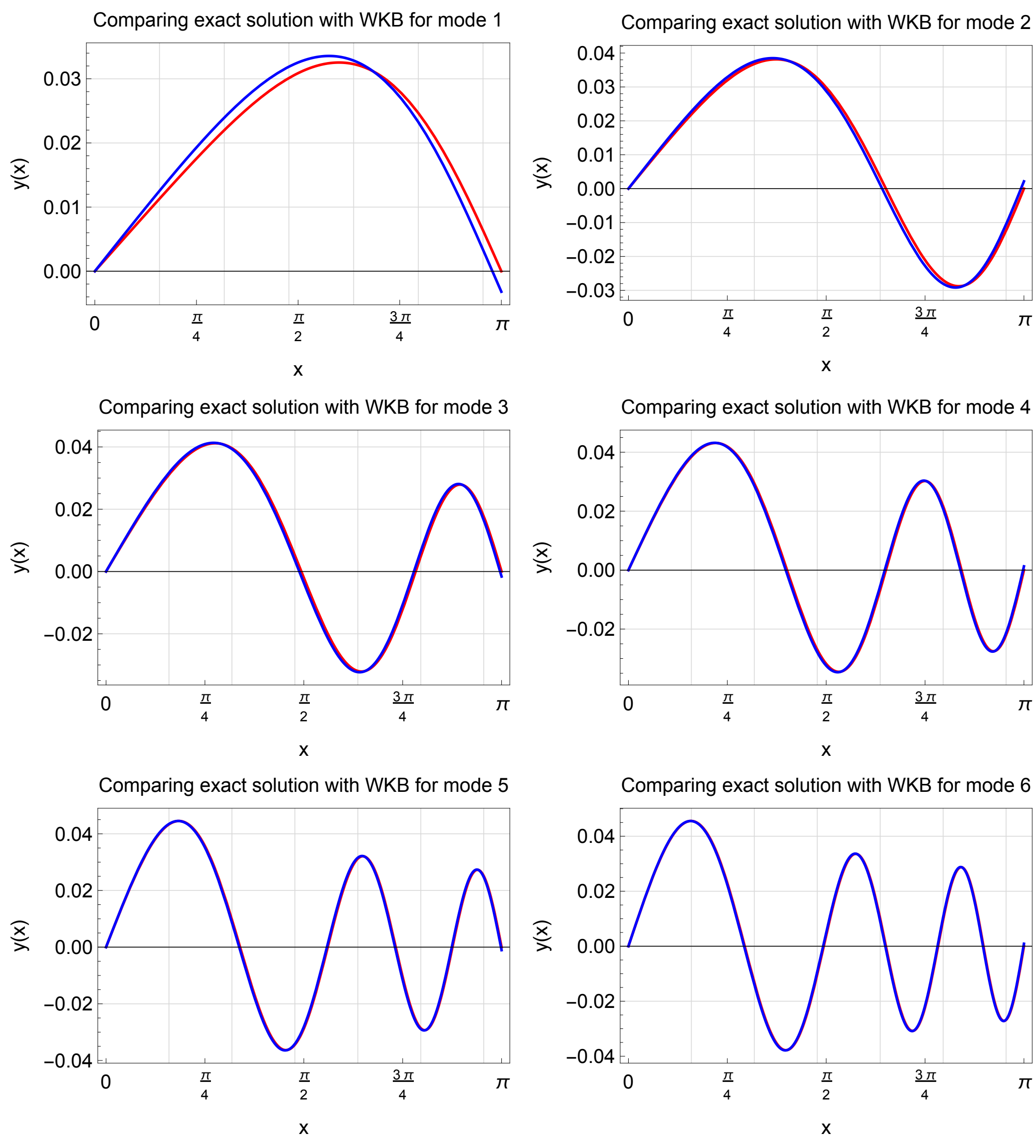
```
In[65]:= compare[modeNumber_] :=
  Module[{solExact1, int, cFromExact, eigenvalueFromHandSolution, flip},
   eigenvalueFromHandSolution = lam[modeNumber];
   solExact1 = solExact /. lam → eigenvalueFromHandSolution;
   int = Integrate[solExact1^2 * (x + Pi)^4, {x, 0, Pi}];
   cFromExact = First@NSolve[int == 1, C[1]];
   solExact1 = solExact1 /. cFromExact;
   If[modeNumber > 5, flip = -1, flip = 1];
   Plot[ {y[modeNumber, x], flip * solExact1}, {x, 0, Pi},
    PlotStyle → {Red, Blue}, Frame -> True, FrameLabel → {{"y(x)", None},
      {"x", Row[{"Comparing exact solution with WKB for mode ", modeNumber}]}},
    GridLines → Automatic, GridLinesStyle → LightGray, BaseStyle → 12, ImageSize → 310,
    FrameTicks → {{Automatic, None}, {{0, Pi/4, Pi/2, 3/4 Pi, Pi}, None}}
   ]
  ];
```

Generate 4 plots, for mode 1, up to 6

These plots show that after mode 5 or 6, the two eigenfunctions are almost exact

In[66]:=
```
plots = Table[compare[n], {n, 6}];
Grid[Partition[plots, 2]]
```

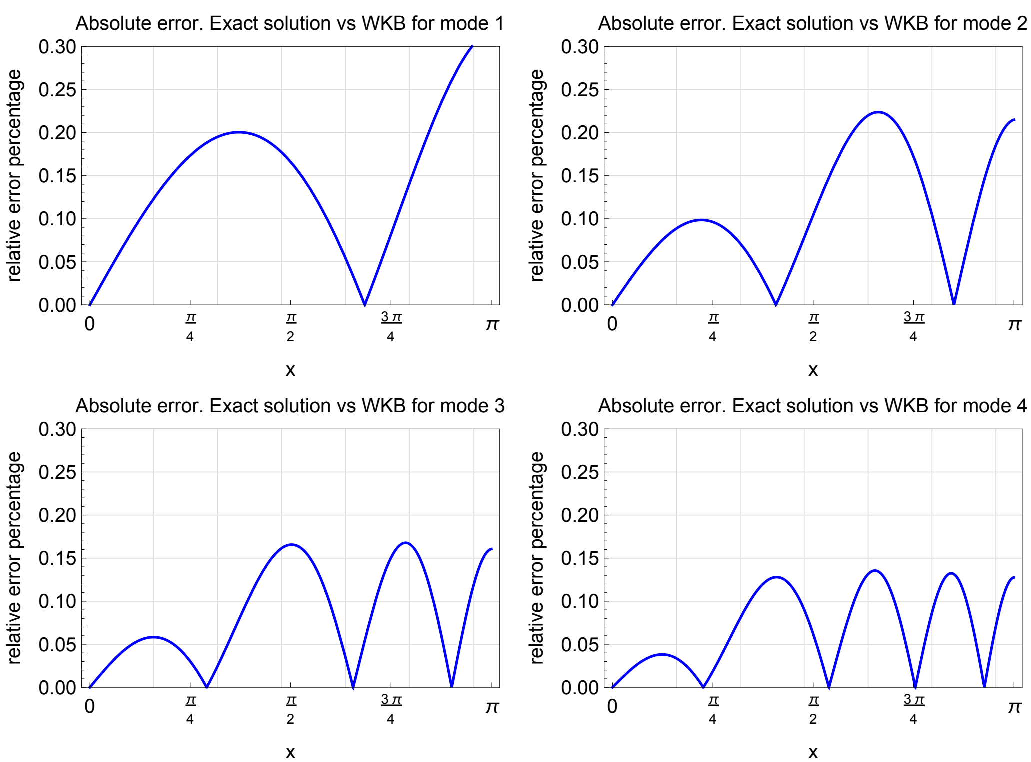
Out[67]=



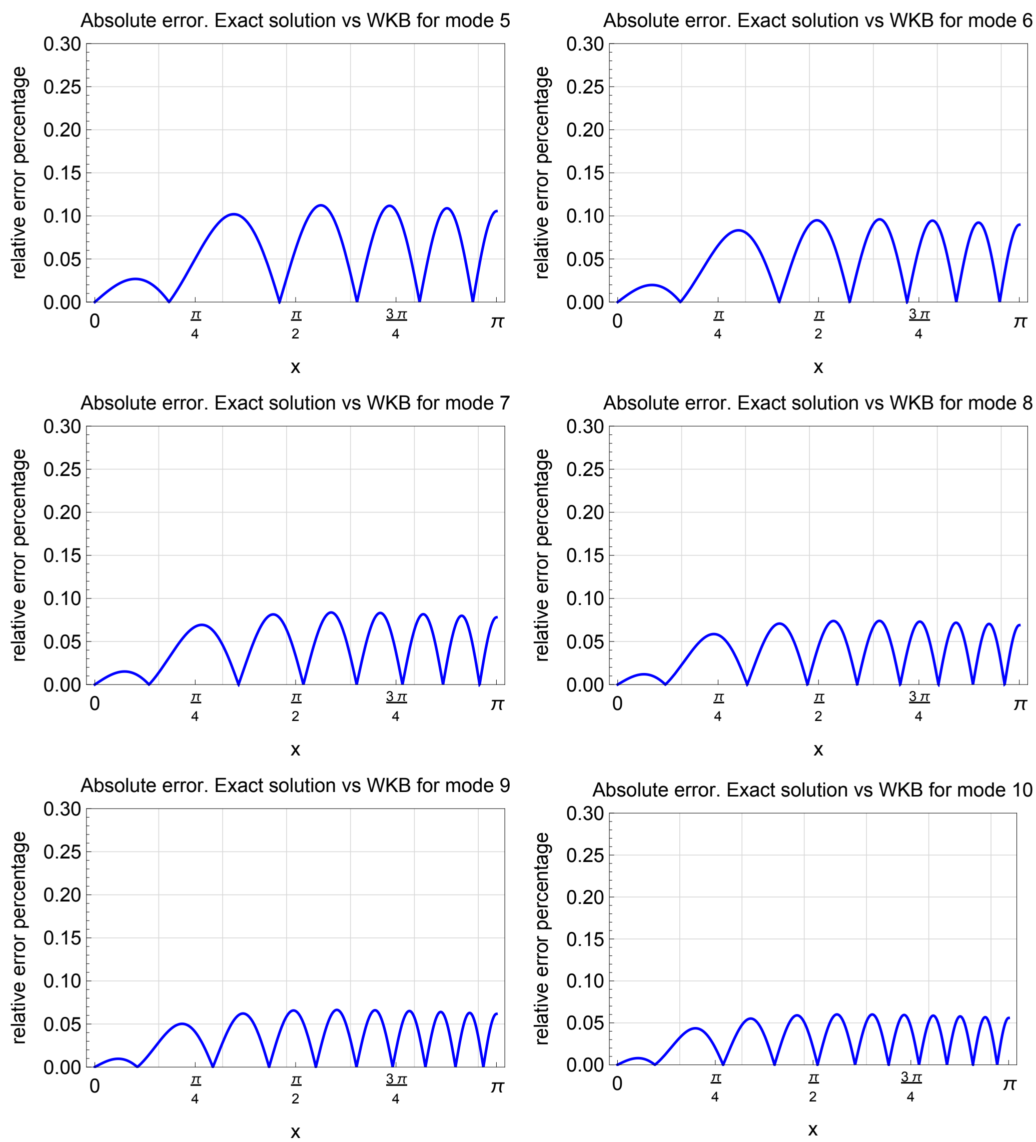
Generate the above again, but now using relative error between the exact and WKB for each mode, to make it more clear

In[68]:=
```
compareError[modeNumber_] :=
  Module[{solExact1, eigenvalueFromHandSolution, int, cFromExact, flip},
   eigenvalueFromHandSolution = lam[modeNumber];
   solExact1 = solExact /. lam → eigenvalueFromHandSolution;
   int = Integrate[solExact1^2 * (x + Pi)^4, {x, 0, Pi}];
   cFromExact = First@NSolve[int == 1, C[1]];
   solExact1 = solExact1 /. cFromExact;
   If[modeNumber > 5, flip = -1, flip = 1];
   Plot[ 100 * Abs[(flip * solExact1 - y[modeNumber, x])], {x, 0, Pi}, PlotStyle →
     {Red, Blue}, Frame -> True, FrameLabel → {{"relative error percentage", None},
      {"x", Row[{"Absolute error. Exact solution vs WKB for mode ", modeNumber}]}},
    GridLines → Automatic, GridLinesStyle → LightGray, BaseStyle → 12, ImageSize → 310,
    FrameTicks → {{Automatic, None}, {{0, Pi/4, Pi/2, 3/4 Pi, Pi}, None}},
    PlotRange → {Automatic, {0, 0.3}}
   ]
  ];
```

In[69]:=
```
plots = Table[compareError[n], {n, 10}]; (*let do 10 modes*)
Grid[Partition[plots, 2]]
```



Out[69]=


Absolute error. Exact solution vs WKB for mode 5
Absolute error. Exact solution vs WKB for mode 6
Absolute error. Exact solution vs WKB for mode 7
Absolute error. Exact solution vs WKB for mode 8
Absolute error. Exact solution vs WKB for mode 9
Absolute error. Exact solution vs WKB for mode 10
relative error percentage
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
π/4
π/2
3π/4
π
x

4.12 Convert ODE to Liouville form

Handy image to remember. Thanks to http://people.uncw.edu/hermanr/mat463/ODEBook/Book/SL.pdf

to turn it into Sturm-Liouville form.

In summary,

Equation (6.1),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{6.7}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{6.8}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)}\,dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{6.9}$$

4.13 note p13. added 3/15/17, solving $\epsilon^2 y''(x) = (a + x + x^2)y(x)$ in Maple

Define the ODE

```
> assume(a>0 and 'real');
  ode:=epsilon^2*diff(y(x),x$2)=(x^2+a*x)*y(x);
```

$$ode := \epsilon^2 \left(\frac{d^2}{dx^2} y(x)\right) = \left(a{\sim}\, x + x^2\right) y(x)$$

Solve without giving any B.C.

```
> sol:=dsolve(ode,y(x));
```

$$sol := y(x) = \_C1 \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 4\,\epsilon}{\epsilon}\right], \left[\frac{1}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}} + \_C2\,(a{\sim} + 2\,x) \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 12\,\epsilon}{\epsilon}\right], \left[\frac{3}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}}$$

Solve with one B.C. at infinity given

```
> sol:=dsolve({ode,y(infinity)=0},y(x));
```

$$sol := y(x) = \lim_{\_a \to \infty} \left(-\left(\_C2\,(a{\sim} + 2\,\_a) \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 12\,\epsilon}{\epsilon}\right], \left[\frac{3}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,\_a)^2}{\epsilon}\right) \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 4\,\epsilon}{\epsilon}\right], \left[\frac{1}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}}\right) \Big/ \left(\operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 4\,\epsilon}{\epsilon}\right], \left[\frac{1}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,\_a)^2}{\epsilon}\right)\right)\right) + \_C2\,(a{\sim} + 2\,x) \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 12\,\epsilon}{\epsilon}\right], \left[\frac{3}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}}$$

Now solve giving B.C. at -infinity

```
> sol:=dsolve({ode,y(-infinity)=0},y(x));
```

$$sol := y(x) = \lim_{\_a \to \infty} \left(\left(\operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 12\,\epsilon}{\epsilon}\right], \left[\frac{3}{2}\right], \frac{1}{4}\,\frac{(-a{\sim} + 2\,\_a)^2}{\epsilon}\right) (-a{\sim} + 2\,\_a)\,\_C2 \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 4\,\epsilon}{\epsilon}\right], \left[\frac{1}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}}\right) \Big/ \left(\operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 4\,\epsilon}{\epsilon}\right], \left[\frac{1}{2}\right], \frac{1}{4}\,\frac{(-a{\sim} + 2\,\_a)^2}{\epsilon}\right)\right)\right) + \_C2\,(a{\sim} + 2\,x) \operatorname{hypergeom}\left(\left[-\frac{1}{16}\,\frac{a{\sim}^2 - 12\,\epsilon}{\epsilon}\right], \left[\frac{3}{2}\right], \frac{1}{4}\,\frac{(a{\sim} + 2\,x)^2}{\epsilon}\right) e^{-\frac{1}{2}\,\frac{x\,(a{\sim}+x)}{\epsilon}}$$

Now solve by giving both B.C.at both ends

```
> sol:=dsolve({ode,y(infinity)=0,y(-infinity)=0},y(x));
```

$$sol := y(x) = 0$$

Chapter 5

Exams

Local contents

5.1 Exam 1

5.1.1 problem 3.26 (page 139)

Problem Perform local analysis solution to $(x-1)\,y'' - xy' + y = 0$ at $x = 1$. Use the result of this analysis to prove that a Taylor series expansion of any solution about $x = 0$ has an infinite radius of convergence. Find the exact solution by summing the series.

solution

Writing the ODE in standard form

$$y''(x) + a(x)\,y'(x) + b(x)\,y(x) = 0 \tag{1}$$

$$y'' - \frac{x}{(x-1)}y' + \frac{1}{(x-1)}y = 0 \tag{2}$$

Where $a(x) = \frac{-x}{(x-1)}, b(x) = \frac{1}{(x-1)}$. The above shows that $x = 1$ is singular point for both $a(x)$ and $b(x)$. The next step is to classify the type of the singular point. Is it regular singular point or irregular singular point?

$$\begin{aligned}\lim_{x\to 1}(x-1)\,a(x) &= \lim_{x\to 1}(x-1)\frac{-x}{(x-1)}\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\lim_{x\to 1}(x-1)^2\,b(x) &= \lim_{x\to 1}(x-1)^2\frac{1}{(x-1)}\\ &= 0\end{aligned}$$

Because the limit exist, then $x = 1$ is a regular singular point. Therefore solution is assumed to be a Frobenius power series given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}$$

Substituting this in the original ODE $(x-1)\,y'' - xy' + y = 0$ gives

$$\begin{aligned}y'(x) &= \sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r-1}\\ y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n(x-1)^{n+r-2}\end{aligned}$$

In order to move the $(x-1)$ inside the summation, the original ODE $(x-1)\,y'' - xy' + y = 0$ is first rewritten as

$$(x-1)\,y'' - (x-1)\,y' - y' + y = 0 \tag{3}$$

Substituting the Frobenius series into the above gives

$$\begin{aligned}
(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n(x-1)^{n+r-2}\\
-(x-1)\sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r-1}\\
-\sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r-1}\\
+\sum_{n=0}^{\infty}a_n(x-1)^{n+r}=0
\end{aligned}$$

Or

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n(x-1)^{n+r-1}\\
-\sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r}\\
-\sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r-1}\\
+\sum_{n=0}^{\infty}a_n(x-1)^{n+r}=0
\end{aligned}$$

Adjusting all powers of $(x-1)$ to be the same by rewriting exponents and summation indices gives

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n(x-1)^{n+r-1}\\
-\sum_{n=1}^{\infty}(n+r-1)\,a_{n-1}(x-1)^{n+r-1}\\
-\sum_{n=0}^{\infty}(n+r)\,a_n(x-1)^{n+r-1}\\
+\sum_{n=1}^{\infty}a_{n-1}(x-1)^{n+r-1}=0
\end{aligned}$$

Collecting terms with same powers in $(x-1)$ simplifies the above to

$$\sum_{n=0}^{\infty}\left((n+r)(n+r-1)-(n+r)\right)a_n(x-1)^{n+r-1}-\sum_{n=1}^{\infty}(n+r-2)\,a_{n-1}(x-1)^{n+r-1}=0 \tag{4}$$

Setting $n=0$ gives the indicial equation

$$\begin{aligned}
\left((n+r)(n+r-1)-(n+r)\right)a_0&=0\\
\left((r)(r-1)-r\right)a_0&=0
\end{aligned}$$

Since $a_0\neq 0$ then the indicial equation is

$$\begin{aligned}
(r)(r-1)-r&=0\\
r^2-2r&=0\\
r(r-2)&=0
\end{aligned}$$

The roots of the indicial equation are therefore

$$r_1 = 2$$
$$r_2 = 0$$

Each one of these roots generates a solution to the ODE. The next step is to find the solution $y_1(x)$ associated with $r = 2$. (The largest root is used first). Using $r = 2$ in equation (4) gives

$$\sum_{n=0}^{\infty}((n+2)(n+1)-(n+2))\,a_n(x-1)^{n+1} - \sum_{n=1}^{\infty} n a_{n-1}(x-1)^{n+1} = 0$$
$$\sum_{n=0}^{\infty} n(n+2)\,a_n(x-1)^{n+1} - \sum_{n=1}^{\infty} n a_{n-1}(x-1)^{n+1} = 0 \tag{5}$$

At $n \geq 1$, the recursive relation is found and used to generate the coefficients of the Frobenius power series

$$n(n+2)\,a_n - n a_{n-1} = 0$$
$$a_n = \frac{n}{n(n+2)}a_{n-1}$$

Few terms are now generated to see the pattern of the series and to determine the closed form. For $n = 1$

$$a_1 = \frac{1}{3}a_0$$

For $n = 2$

$$a_2 = \frac{2}{2(2+2)}a_1 = \frac{2}{8}\frac{1}{3}a_0 = \frac{1}{12}a_0$$

For $n = 3$

$$a_3 = \frac{3}{3(3+2)}a_2 = \frac{3}{15}\frac{1}{12}a_0 = \frac{1}{60}a_0$$

For $n = 4$

$$a_4 = \frac{4}{4(4+2)}a_3 = \frac{1}{6}\frac{1}{60}a_0 = \frac{1}{360}a_0$$

And so on. From the above, the first solution becomes

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n(x-1)^{n+2} \\
&= a_0(x-1)^2 + a_1(x-1)^3 + a_2(x-1)^4 + a_3(x-1)^4 + a_4(x-1)^5 + \cdots \\
&= (x-1)^2\left(a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots\right) \\
&= (x-1)^2\left(a_0 + \frac{1}{3}a_0(x-1) + \frac{1}{12}a_0(x-1)^2 + \frac{1}{60}a_0(x-1)^3 + \frac{1}{360}a_0(x-1)^4 + \cdots\right) \\
&= a_0(x-1)^2\left(1 + \frac{1}{3}(x-1) + \frac{1}{12}(x-1)^2 + \frac{1}{60}(x-1)^3 + \frac{1}{360}(x-1)^4 + \cdots\right) \qquad (6)
\end{aligned}$$

To find closed form solution to $y_1(x)$, Taylor series expansion of e^x around $x = 1$ is found first

$$\begin{aligned} e^x &\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4 + \frac{e}{5!}(x-1)^5 + \cdots \\ &\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e}{120}(x-1)^5 + \cdots \\ &\approx e\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right) \end{aligned}$$

Multiplying the above by 2 gives

$$2e^x \approx e\left(2 + 2(x-1) + (x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \frac{1}{60}(x-1)^5 + \cdots\right)$$

Factoring $(x-1)^2$ from the RHS results in

$$2e^x \approx e\left(2 + 2(x-1) + (x-1)^2\left(1 + \frac{1}{3}(x-1) + \frac{1}{12}(x-1)^2 + \frac{1}{60}(x-1)^3 + \cdots\right)\right) \tag{6A}$$

Comparing the above result with the solution $y_1(x)$ in (6), shows that the (6A) can be written in terms of $y_1(x)$ as

$$2e^x = e\left(2 + 2(x-1) + (x-1)^2\left(\frac{y_1(x)}{a_0(x-1)^2}\right)\right)$$

Therefore

$$\begin{aligned} 2e^x &= e\left(2 + 2(x-1) + \frac{y_1(x)}{a_0}\right) \\ 2e^{x-1} &= 2 + 2(x-1) + \frac{y_1(x)}{a_0} \\ 2e^{x-1} - 2 - 2(x-1) &= \frac{y_1(x)}{a_0} \end{aligned}$$

Solving for $y_1(x)$

$$\begin{aligned} y_1(x) &= a_0\left(2e^{x-1} - 2 - 2(x-1)\right) \\ &= a_0\left(2e^{x-1} - 2 - 2x + 2\right) \\ &= a_0\left(2e^{x-1} - 2x\right) \\ &= \frac{2a_0}{e}e^x - 2a_0x \end{aligned}$$

Let $\frac{2a_0}{e} = C_1$ and $-2a_0 = C_2$, then the above solution can be written as

$$y_1(x) = C_1e^x + C_2x$$

Now that $y_1(x)$ is found, which is the solution associated with $r = 2$, the next step is to find the second solution $y_2(x)$ associated with $r = 0$. Since $r_2 - r_1 = 2$ is an integer, the solution can be either case $II(b)(i)$ or case $II(b)(ii)$ as given in the text book at page 72.

From equation (3.3.9) at page 72 of the text, using $N = 2$ since $N = r_2 - r_1$ and where $p(x) = -x$ and $q(x) = 1$ in this problem by comparing our ODE with the standard ODE in (3.3.2) at page 70 given by

$$y'' + \frac{p(x)}{(x-x_0)}y' + \frac{q(x)}{(x-x_0)^2}y = 0$$

Expanding $p(x), q(x)$ in Taylor series

$$p(x) = \sum_{n=0}^{\infty} p_n (x-1)^n$$
$$q(x) = \sum_{n=0}^{\infty} q_n (x-1)^n$$

Since $p(x) = -x$ in our ODE, then $p_0 = -1$ and $p_1 = -1$ and all other terms are zero. For $q(x)$, which is just 1 in our ODE, then $q_0 = 1$ and all other terms are zero. Hence

$$\begin{aligned} p_0 &= -1 \\ p_1 &= -1 \\ q_0 &= 1 \\ N &= 2 \\ r &= 0 \end{aligned}$$

The above values are now used to evaluate RHS of 3.3.9 in order to find which case it is. (book uses α for r)

$$0a_N = -\sum_{k=0}^{N-1} \left[(r+k) p_{N-k} + q_{N-k} \right] a_k \tag{3.3.9}$$

Since $N = 2$ the above becomes

$$0a_2 = -\sum_{k=0}^{1} \left[(r+k) p_{2-k} + q_{2-k} \right] a_k$$

Using $r = 0$, since this is the second root, gives

$$\begin{aligned} 0a_2 &= -\sum_{k=0}^{1} \left(kp_{2-k} + q_{2-k} \right) a_k \\ &= -\left(\left(0p_{2-0} + q_{2-0} \right) a_0 + \left(p_{2-1} + q_{2-1} \right) a_1 \right) \\ &= -\left(\left(0p_2 + q_2 \right) a_0 + \left(p_1 + q_1 \right) a_1 \right) \\ &= -\left(0 + q_2 \right) a_0 - \left(p_1 + q_1 \right) a_1 \end{aligned}$$

Since $q_2 = 0, p_1 = -1, q_1 = 1$, therefore

$$\begin{aligned} 0a_2 &= -(0+0) a_0 - (-1+1) a_1 \\ &= 0 \end{aligned}$$

The above shows that this is case $II(b)(ii)$, because the right side of 3.3.9 is zero. This means the second solution $y_2(x)$ is also a Fronbenius series. If the above was not zero, the method of reduction of order would be used to find second solution.

Assuming $y_2(x) = \sum b_n (x-1)^{n+r}$, and since $r = 0$, therefore

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x-1)^n$$

Following the same method used to find the first solution, this series is now used in the ODE to determine b_n.

$$y_2'(x) = \sum_{n=0}^{\infty} nb_n (x-1)^{n-1} = \sum_{n=1}^{\infty} nb_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n$$

$$y_2''(x) = \sum_{n=0}^{\infty} n(n+1) b_{n+1} (x-1)^{n-1} = \sum_{n=1}^{\infty} n(n+1) b_{n+1} (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) b_{n+2} (x-1)^n$$

The ODE $(x-1)y'' - (x-1)y' - y' + y = 0$ now becomes

$$\begin{aligned}(x-1)\sum_{n=0}^{\infty} (n+1)(n+2) b_{n+2} (x-1)^n \\ -(x-1)\sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n \\ -\sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n \\ +\sum_{n=0}^{\infty} b_n (x-1)^n = 0\end{aligned}$$

Or

$$\begin{aligned}\sum_{n=0}^{\infty} (n+1)(n+2) b_{n+2} (x-1)^{n+1} \\ -\sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^{n+1} \\ -\sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n \\ +\sum_{n=0}^{\infty} b_n (x-1)^n = 0\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} (n)(n+1) b_{n+1} (x-1)^n - \sum_{n=1}^{\infty} nb_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n + \sum_{n=0}^{\infty} b_n (x-1)^n = 0$$

$n = 0$ gives

$$\begin{aligned}-(n+1) b_{n+1} + b_n &= 0 \\ -b_1 + b_0 &= 0 \\ b_1 &= b_0\end{aligned}$$

$n \geq 1$ generates the recursive relation to find all remaining b_n coefficients

$$\begin{aligned}(n)(n+1)b_{n+1} - nb_n - (n+1)b_{n+1} + b_n &= 0\\(n)(n+1)b_{n+1} - (n+1)b_{n+1} &= nb_n - b_n\\b_{n+1}((n)(n+1) - (n+1)) &= b_n(n-1)\\b_{n+1} &= b_n\frac{(n-1)}{(n)(n+1)-(n+1)}\end{aligned}$$

Therefore the recursive relation is

$$b_{n+1} = \frac{b_n}{n+1}$$

Few terms are generated to see the pattern and to find the closed form solution for $y_2(x)$. For $n = 1$

$$b_2 = b_1\frac{1}{2} = \frac{1}{2}b_0$$

For $n = 2$

$$b_3 = \frac{b_2}{3} = \frac{1}{3}\frac{1}{2}b_0 = \frac{1}{6}b_0$$

For $n = 3$

$$b_4 = \frac{b_3}{3+1} = \frac{1}{4}\frac{1}{6}b_0 = \frac{1}{24}b_0$$

For $n = 4$

$$b_5 = \frac{b_4}{4+1} = \frac{1}{5}\frac{1}{24}b_0 = \frac{1}{120}b_0$$

And so on. Therefore, the second solution is

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n(x-1)^n\\&= b_0 + b_1(x-1) + b_2(x-1)^2 + \cdots\\&= b_0 + b_0(x-1) + \frac{1}{2}b_0(x-1)^2 + \frac{1}{6}b_0(x-1)^3 + \frac{1}{24}b_0(x-1)^4 + \frac{1}{120}b_0(x-1)^5 + \cdots\\&= b_0\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right) \qquad (7A)\end{aligned}$$

The Taylor series for e^x around $x = 1$ is

$$\begin{aligned}e^x &\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e}{120}(x-1)^5 + \cdots\\&\approx e\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right) \qquad (7B)\end{aligned}$$

Comparing (7A) with (7B) shows that the second solution closed form is

$$y_2(x) = b_0\frac{e^x}{e}$$

Let $\frac{b_0}{e}$ be some constant, say C_3, the second solution above becomes

$$y_2(x) = C_3e^x$$

Both solutions $y_1(x), y_2(x)$ have now been found. The final solution is

$$\begin{aligned} y(x) &= y_1(x) + y_2(x) \\ &= \overbrace{C_1 e^x + C_2 x}^{y_1(x)} + \overbrace{C_3 e^x}^{y_2(x)} \\ &= C_4 e^x + C_2 x \end{aligned}$$

Hence, the exact solution is

$$y(x) = Ae^x + Bx \tag{7}$$

Where A, B are constants to be found from initial conditions if given. Above solution is now verified by substituting it back to original ODE

$$\begin{aligned} y' &= Ae^x + B \\ y'' &= Ae^x \end{aligned}$$

Substituting these into $(x-1)y'' - xy' + y = 0$ gives

$$\begin{aligned} (x-1)Ae^x - x(Ae^x + B) + Ae^x + Bx &= 0 \\ xAe^x - Ae^x - xAe^x - xB + Ae^x + Bx &= 0 \\ -Ae^x - xB + Ae^x + Bx &= 0 \\ 0 &= 0 \end{aligned}$$

To answer the final part of the question, the above solution (7) is analytic around $x = 0$ with infinite radius of convergence since $\exp(\cdot)$ is analytic everywhere. Writing the solution as

$$y(x) = \left(A \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) + Bx$$

The function x have infinite radius of convergence, since it is its own series. And the exponential function has infinite radius of convergence as known, verified by using standard ratio test

$$A \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = A \lim_{n\to\infty} \left|\frac{x^{n+1} n!}{(n+1)! x^n}\right| = A \lim_{n\to\infty} \left|\frac{xn!}{(n+1)!}\right| = A \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$$

For any x. Since the ratio is less than 1, then the solution $y(x)$ expanded around $x = 0$ has an infinite radius of convergence.

5.1.2 problem 9.8 (page 480)

Problem Use boundary layer to find uniform approximation with error of order $O\left(\varepsilon^2\right)$ for the problem $\varepsilon y'' + y' + y = 0$ with $y(0) = e, y(1) = 1$. Compare your solution to exact solution. Plot the solution for some values of ε.

solution

$$\varepsilon y'' + y' + y = 0 \tag{1}$$

Since $a(x) = 1 > 0$, then a boundary layer is expected at the left side, near $x = 0$. Matching will fail if this was not the case. Starting with the outer solution near $x = 1$. Let

$$y^{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

Substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + \left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0 \tag{2}$$

Collecting powers of $O\left(\varepsilon^0\right)$ results in the ODE

$$\begin{aligned}
y_0' &\sim -y_0 \\
\frac{dy_0}{y_0} &\sim -dx \\
\ln|y_0| &\sim -x + C_1 \\
y_0^{out}(x) &\sim C_1 e^{-x} + O(\varepsilon) \qquad (3)
\end{aligned}$$

C_1 is found from boundary conditions $y(1) = 1$. Equation (3) gives

$$\begin{aligned}
1 &= C_1 e^{-1} \\
C_1 &= e
\end{aligned}$$

Hence solution (3) becomes

$$y_0^{out}(x) \sim e^{1-x}$$

$y_1^{out}(x)$ is now found. Using (2) and collecting terms of $O\left(\varepsilon^1\right)$ gives the ODE

$$y_1' + y_1 \sim -y_0'' \tag{4}$$

But

$$\begin{aligned}
y_0'(x) &= -e^{1-x} \\
y_0''(x) &= e^{1-x}
\end{aligned}$$

Using the above in the RHS of (4) gives

$$y_1' + y_1 \sim -e^{1-x}$$

The integrating factor is e^x, hence the above becomes

$$\frac{d}{dx}\left(y_1 e^x\right) \sim -e^x e^{1-x}$$
$$\frac{d}{dx}\left(y_1 e^x\right) \sim -e$$

Integrating both sides gives

$$y_1 e^x \sim -ex + C_2$$
$$y_1^{out}(x) \sim -xe^{1-x} + C_2 e^{-x} \tag{5}$$

Applying boundary conditions $y(1) = 0$ to the above gives

$$0 = -1 + C_2 e^{-1}$$
$$C_2 = e$$

Hence the solution in (5) becomes

$$y_1^{out}(x) \sim -xe^{1-x} + e^{1-x}$$
$$\sim (1-x)\, e^{1-x}$$

Therefore the outer solution is

$$y^{out}(x) = y_0 + \varepsilon y_1$$
$$= e^{1-x} + \varepsilon (1-x)\, e^{1-x} \tag{6}$$

Now the boundary layer (inner) solution $y^{in}(x)$ near $x = 0$ is found. Let $\xi = \frac{x}{\varepsilon^p}$ be the inner variable. The original ODE is expressed using this new variable, and p is found. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. The differential operator is $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$ therefore

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx}$$
$$= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)$$
$$= \varepsilon^{-2p}\frac{d^2}{d\xi^2}$$

Hence $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and $\varepsilon y'' + y' + y = 0$ becomes

$$\varepsilon\left(\varepsilon^{-2p}\frac{d^2y}{d\xi^2}\right) + \varepsilon^{-p}\frac{dy}{d\xi} + y = 0$$
$$\varepsilon^{1-2p}y'' + \varepsilon^{-p}y' + y = 0 \tag{7A}$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, balance gives $1 - 2p = -p$ or

$$\boxed{p = 1}$$

The ODE (7A) becomes

$$\varepsilon^{-1}y'' + \varepsilon^{-1}y' + y = 0 \tag{7}$$

Assuming that solution is

$$y_{in}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting the above into (7) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{8}$$

Collecting terms with $O\left(\varepsilon^{-1}\right)$ gives the first order ODE to solve

$$y_0'' \sim -y_0'$$

Let $z = y_0'$, the above becomes

$$\begin{aligned} z' &\sim -z \\ \frac{dz}{z} &\sim -d\xi \\ \ln|z| &\sim -\xi + C_4 \\ z &\sim C_4 e^{-\xi} \end{aligned}$$

Hence

$$y_0' \sim C_4 e^{-\xi}$$

Integrating

$$\begin{aligned} y_0^{in}(\xi) &\sim C_4 \int e^{-\xi} d\xi + C_5 \\ &\sim -C_4 e^{-\xi} + C_5 \end{aligned} \tag{9}$$

Applying boundary conditions $y(0) = e$ gives

$$\begin{aligned} e &= -C_4 + C_5 \\ C_5 &= e + C_4 \end{aligned}$$

Equation (9) becomes

$$\begin{aligned} y_0^{in}(\xi) &\sim -C_4 e^{-\xi} + e + C_4 \\ &\sim C_4\left(1 - e^{-\xi}\right) + e \end{aligned} \tag{10}$$

The next leading order $y_1^{in}(\xi)$ is found from (8) by collecting terms in $O\left(\varepsilon^0\right)$, which results in the ODE

$$y_1'' + y_1' \sim -y_0$$

Since $y_0^{in}(\xi) \sim C_4\left(1 - e^{-\xi}\right) + e$, therefore $y_0' \sim C_4 e^{-\xi}$ and the above becomes

$$y_1'' + y_1' \sim -C_4 e^{-\xi}$$

The homogenous solution is found first, then method of undetermined coefficients is used to find particular solution. The homogenous ODE is

$$y_{1,h}'' \sim y_{1,h}'$$

This was solved above for y_0^{in}, and the solution is

$$y_{1,h} \sim -C_5 e^{-\xi} + C_6$$

To find the particular solution, let $y_{1,p} \sim A\xi e^{-\xi}$, where ξ was added since $e^{-\xi}$ shows up in the homogenous solution. Hence

$$\begin{aligned} y_{1,p}' &\sim Ae^{-\xi} - A\xi e^{-\xi} \\ y_{1,p}'' &\sim -Ae^{-\xi} - \left(Ae^{-\xi} - A\xi e^{-\xi}\right) \\ &\sim -2Ae^{-\xi} + A\xi e^{-\xi} \end{aligned}$$

Substituting these in the ODE $y_{1,p}'' + y_{1,p}' \sim -C_4 e^{-\xi}$ results in

$$\begin{aligned} -2Ae^{-\xi} + A\xi e^{-\xi} + Ae^{-\xi} - A\xi e^{-\xi} &\sim -C_4 e^{-\xi} \\ -A &= -C_4 \\ A &= C_4 \end{aligned}$$

Therefore the particular solution is

$$y_{1,p} \sim C_4 \xi e^{-\xi}$$

And therefore the complete solution is

$$\begin{aligned} y_1^{in}(\xi) &\sim y_{1,h} + y_{1,p} \\ &\sim -C_5 e^{-\xi} + C_6 + C_4 \xi e^{-\xi} \end{aligned}$$

Applying boundary conditions $y(0) = 0$ to the above gives

$$\begin{aligned} 0 &= -C_5 + C_6 \\ C_6 &= C_5 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y_1^{in}(\xi) &\sim -C_5 e^{-\xi} + C_5 + C_4 \xi e^{-\xi} \\ &\sim C_5\left(1 - e^{-\xi}\right) + C_4 \xi e^{-\xi} \end{aligned} \tag{11}$$

The complete inner solution now becomes

$$\begin{aligned} y^{in}(\xi) &\sim y_0^{in} + \varepsilon y_1^{in} \\ &\sim C_4\left(1 - e^{-\xi}\right) + e + \varepsilon\left(C_5\left(1 - e^{-\xi}\right) + C_4 \xi e^{-\xi}\right) \end{aligned} \tag{12}$$

There are two constants that need to be determined in the above from matching with the outer solution.

$$\begin{aligned} \lim_{\xi\to\infty} y^{in}(\xi) &\sim \lim_{x\to 0} y^{out}(x) \\ \lim_{\xi\to\infty} C_4\left(1 - e^{-\xi}\right) + e + \varepsilon\left(C_5\left(1 - e^{-\xi}\right) + C_4 \xi e^{-\xi}\right) &\sim \lim_{x\to 0} e^{1-x} + \varepsilon(1-x)e^{1-x} \\ C_4 + e + \varepsilon C_5 &\sim e + \varepsilon e \end{aligned}$$

The above shows that

$$\begin{aligned} C_5 &= e \\ C_4 + e &= e \\ C_4 &= 0 \end{aligned}$$

This gives the boundary layer solution $y^{in}(\xi)$ as

$$\begin{aligned} y^{in}(\xi) &\sim e + \varepsilon e\left(1 - e^{-\xi}\right) \\ &\sim e\left(1 + \varepsilon\left(1 - e^{-\xi}\right)\right) \end{aligned}$$

In terms of x, since $\xi = \frac{x}{\varepsilon}$, the above can be written as

$$y^{in}(x) \sim e\left(1 + \varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right)\right)$$

The uniform solution is therefore

$$\begin{aligned} y_{\text{uniform}}(x) &\sim y^{in}(x) + y^{out}(x) - y_{match} \\ &\sim \overbrace{e\left(1 + \varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right)\right)}^{y^{in}} + \overbrace{e^{1-x} + \varepsilon(1-x)e^{1-x}}^{y^{out}} - (e + \varepsilon e) \\ &\sim e + e\varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right) + e^{1-x} + (\varepsilon - \varepsilon x)e^{1-x} - (e + \varepsilon e) \\ &\sim e + e\varepsilon - \varepsilon e^{1-\frac{x}{\varepsilon}} + e^{1-x} + \varepsilon e^{1-x} - \varepsilon x e^{1-x} - e - \varepsilon e \\ &\sim -\varepsilon e^{1-\frac{x}{\varepsilon}} + e^{1-x} + \varepsilon e^{1-x} - \varepsilon x e^{1-x} \\ &\sim e^{1-x}\left(-\varepsilon e^{-\frac{1}{\varepsilon}} + 1 + \varepsilon - \varepsilon x\right) \end{aligned}$$

Or

$$y_{\text{uniform}}(x) \sim e^{1-x}\left(1 + \varepsilon\left(1 - x - e^{-\frac{1}{\varepsilon}}\right)\right)$$

With error $O\left(\varepsilon^2\right)$.

The above solution is now compared to the exact solution of $\varepsilon y'' + y' + y = 0$ with $y(0) = e, y(1) = 1$. Since this is a homogenous second order ODE with constant coefficient, it is easily solved using characteristic equation.

$$\varepsilon\lambda^2 + \lambda + 1 = 0$$

The roots are

$$\begin{aligned} \lambda &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1}{2\varepsilon} \pm \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} \\ &= Ae^{\left(\frac{-1}{2\varepsilon} + \frac{\sqrt{1-4\varepsilon}}{2\varepsilon}\right)x} + Be^{\left(\frac{-1}{2\varepsilon} - \frac{\sqrt{1-4\varepsilon}}{2\varepsilon}\right)x} \\ &= Ae^{\frac{-x}{2\varepsilon}}e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + Be^{\frac{-x}{2\varepsilon}}e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x} \\ &= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + Be^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) \end{aligned}$$

Applying first boundary conditions $y(0)=e$ to the above gives

$$e = A + B$$
$$B = e - A$$

Hence the solution becomes

$$\begin{aligned} y(x) &= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + (e-A)\,e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)\\ &= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - Ae^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)\\ &= e^{\frac{-x}{2\varepsilon}}\left(A\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) \qquad (13)\end{aligned}$$

Applying second boundary conditions $y(1)=1$ gives

$$\begin{aligned} 1 &= e^{\frac{-1}{2\varepsilon}}\left(A\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}\right)\\ e^{\frac{1}{2\varepsilon}} &= A\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}\\ A &= \frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}} \qquad (14)\end{aligned}$$

Substituting this into (13) results in

$$y^{\text{exact}}(x) = e^{\frac{-x}{2\varepsilon}}\left(A\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)$$

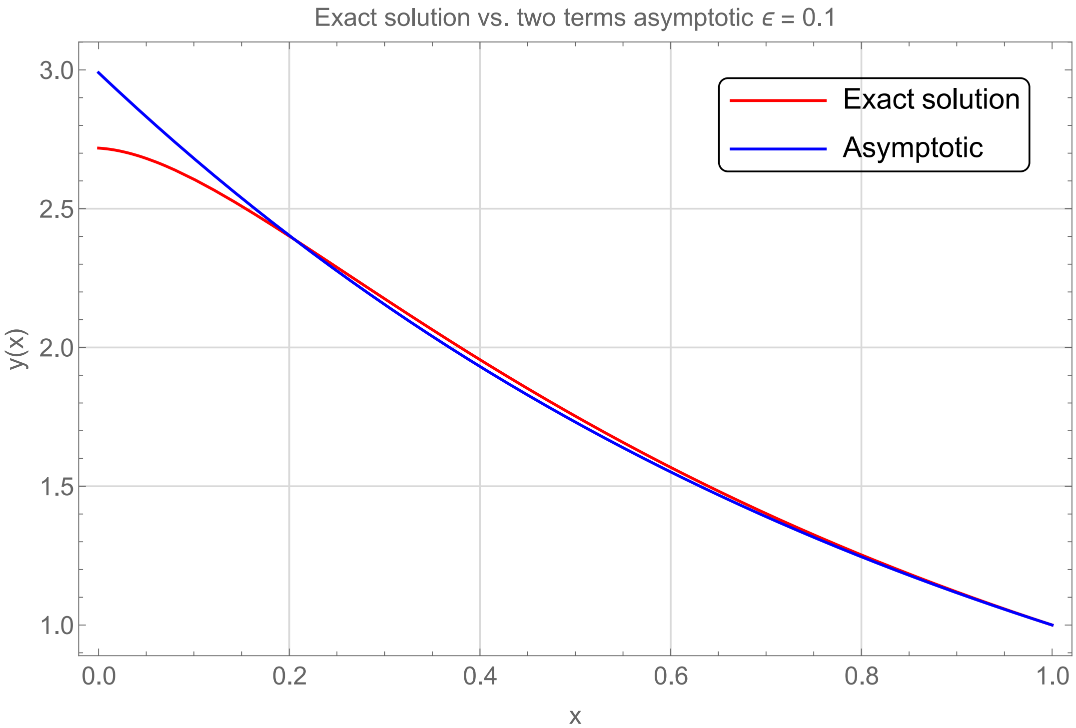
Where A is given in (14). hence

$$y^{\text{exact}}(x) = e^{\frac{-x}{2\varepsilon}}\left(\left(\frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}\right)}\right)\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)$$

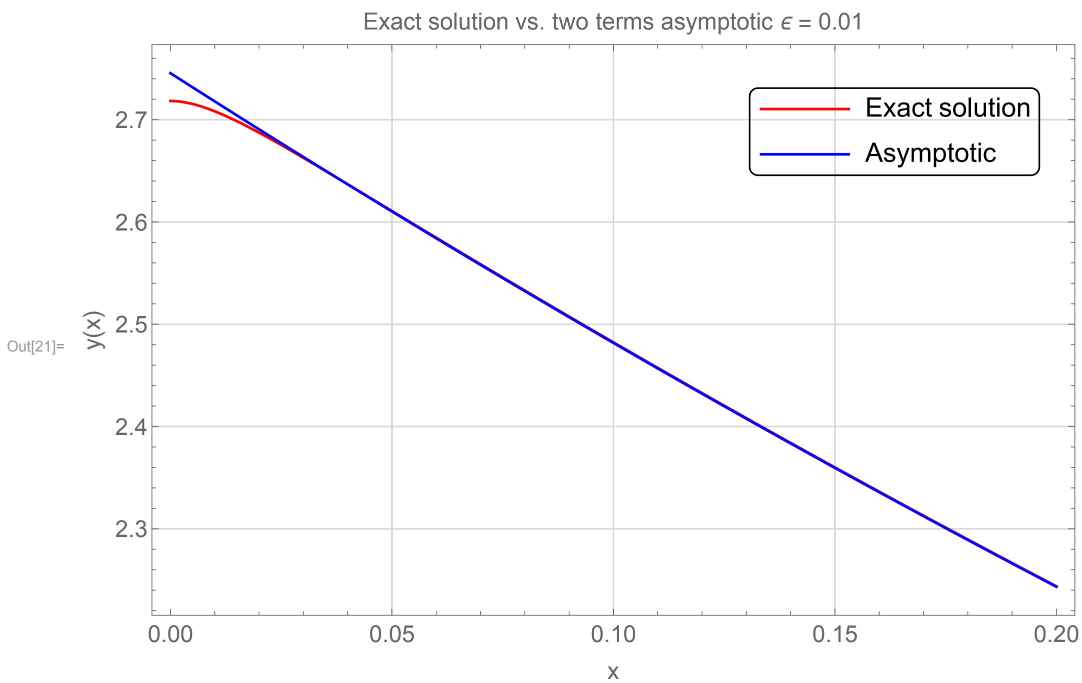
In summary

exact solution	asymptotic solution
$e^{\frac{-x}{2\varepsilon}}\left(\left(\frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}\right)}\right)\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)$	$e^{1-x} + \varepsilon e^{1-x}\left(1 - x - e^{-\frac{1}{\varepsilon}}\right) + O\left(\varepsilon^2\right)$

The following plot compares the exact solution with the asymptotic solution for $\varepsilon = 0.1$



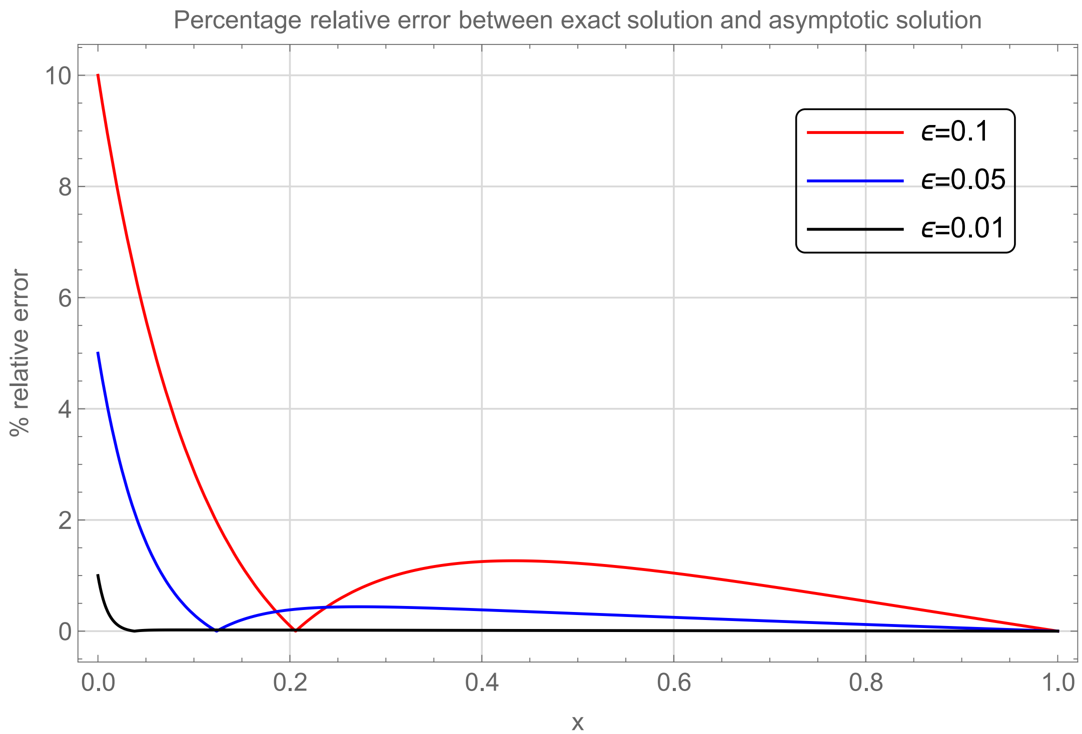
The following plot compares the exact solution with the asymptotic solution for $\varepsilon = 0.01$. The difference was too small to notice in this case, the plot below is zoomed to be near $x = 0$



At $\varepsilon = 0.001$, the difference between the exact and the asymptotic solution was not noticeable. Therefore, to better compare the solutions, the following plot shows the relative percentage error given by

$$100\left|\frac{y^{\text{exact}} - y^{\text{uniform}}}{y^{\text{exact}}}\right|\%$$

For different ε.



<u>Some observations</u>: The above plot shows more clearly how the difference between the exact solution and the asymptotic solution became smaller as ε became smaller. The plot also shows that the boundary layer near $x = 0$ is becoming more narrow are ε becomes smaller as expected. It also shows that the relative error is smaller in the outer region than in the boundary layer region. For example, for $\varepsilon = 0.05$, the largest percentage error in the outer region was less than 1%, while in the boundary layer, very near $x = 0$, the error grows to about 5%. Another observation is that at the matching location, the relative error goes down to zero. One also notices that the matching location drifts towards $x = 0$ as ε becomes smaller because the boundary layer is becoming more narrow. The following table summarizes these observations.

ε	% error near $x = 0$	apparent width of boundary layer
0.1	10	0.2
0.05	5	0.12
0.01	1	0.02

5.1.3 problem 3

Problem (a) Find physical optics approximation to the eigenvalue and eigenfunctions of the Sturm-Liouville problem are $\lambda \to \infty$

$$
\begin{aligned}
-y'' &= \lambda (\sin (x) + 1)^2 y \\
y(0) &= 0 \\
y(\pi) &= 0
\end{aligned}
$$

(b) What is the integral relation necessary to make the eigenfunctions orthonormal? For some reasonable choice of scaling coefficient (give the value), plot the eigenfunctions for $n = 5, n = 20$.

(c) Estimate how large λ should be for the relative error of less than 0.1%

solution

5.1.3.1 Part a

Writing the ODE as

$$y'' + \lambda (\sin (x) + 1)^2 y = 0$$

Let[1]

$$\boxed{\lambda = \frac{1}{\varepsilon^2}}$$

Then the given ODE becomes

$$\varepsilon^2 y''(x) + (\sin (x) + 1)^2 y(x) = 0 \tag{1}$$

Physical optics approximation is obtained when $\lambda \to \infty$ which implies $\varepsilon \to 0^+$. Since the ODE is linear and the highest derivative is now multiplied by a very small parameter ε, WKB can therefore be used to solve it. WKB starts by assuming that the solution has the form

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right) \qquad \delta \to 0$$

Therefore, taking derivatives and substituting back in the ODE results in

$$
\begin{aligned}
y'(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n'(x)\right) \\
y''(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n'(x)\right)^2 + \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n''(x)\right)
\end{aligned}
$$

[1] $\lambda = \frac{1}{\varepsilon}$ could also be used. But the book uses ε^2.

Substituting these into (1) and canceling the exponential terms gives

$$\begin{aligned}
\varepsilon^2\left(\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2+\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\right) &\sim -(\sin(x)+1)^2\\
\frac{\varepsilon^2}{\delta^2}\left(S_0'+\delta S_1'+\cdots\right)\left(S_0'+\delta S_1'+\cdots\right)+\frac{\varepsilon^2}{\delta}\left(S_0''+\delta S_1''+\cdots\right) &\sim -(\sin(x)+1)^2\\
\frac{\varepsilon^2}{\delta^2}\left(\left(S_0'\right)^2+\delta\left(2S_1'S_0'\right)+\cdots\right)+\frac{\varepsilon^2}{\delta}\left(S_0''+\delta S_1''+\cdots\right) &\sim -(\sin(x)+1)^2\\
\left(\frac{\varepsilon^2}{\delta^2}\left(S_0'\right)^2+\frac{2\varepsilon^2}{\delta}S_1'S_0'+\cdots\right)+\left(\frac{\varepsilon^2}{\delta}S_0''+\varepsilon^2 S_1''+\cdots\right) &\sim -(\sin(x)+1)^2 \qquad (2)
\end{aligned}$$

The largest term in the left side is $\frac{\varepsilon^2}{\delta^2}\left(S_0'\right)^2$. By dominant balance, this term has the same order of magnitude as the right side $-(\sin(x)+1)^2$. This implies that δ^2 is proportional to ε^2. For simplicity (following the book) δ can be taken as equal to ε

$$\delta=\varepsilon$$

Using the above in equation (2) results in

$$\left(\left(S_0'\right)^2+2\varepsilon S_1'S_0'+\cdots\right)+\left(\varepsilon S_0''+\varepsilon^2 S_1''+\cdots\right)\sim -(\sin(x)+1)^2$$

Balance of $O(1)$ gives

$$\left(S_0'\right)^2\sim -(\sin(x)+1)^2 \qquad (3)$$

Balance of $O(\varepsilon)$ gives

$$2S_1'S_0'\sim -S_0'' \qquad (4)$$

Equation (3) is solved first in order to find $S_0(x)$.

$$S_0'\sim \pm i(\sin(x)+1)$$

Hence

$$\begin{aligned}
S_0(x) &\sim \pm i\int_0^x(\sin(t)+1)\,dt+C^{\pm}\\
&\sim \pm i(t-\cos(t))_0^x+C^{\pm}\\
&\sim \pm i(1+x-\cos(x))+C^{\pm} \qquad (5)
\end{aligned}$$

$S_1(x)$ is now found from (4) and using $S_0''=\pm i\cos(x)$ gives

$$\begin{aligned}
S_1' &\sim -\frac{1}{2}\frac{S_0''}{S_0'}\\
&\sim -\frac{1}{2}\frac{\pm i\cos(x)}{\pm i(\sin(x)+1)}\\
&\sim -\frac{1}{2}\frac{\cos(x)}{(\sin(x)+1)}
\end{aligned}$$

Hence the solution is

$$S_1(x)\sim -\frac{1}{2}\ln(1+\sin(x)) \qquad (6)$$

Having found $S_0(x)$ and $S_1(x)$, the leading behavior is now obtained from

$$\begin{aligned} y(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \\ &\sim \exp\left(\frac{1}{\varepsilon}\left(S_0(x)+\varepsilon S_1(x)\right)+\cdots\right) \\ &\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)+\cdots\right) \end{aligned}$$

The leading behavior is only the first two terms (called physical optics approximation in WKB), therefore

$$\begin{aligned} y(x) &\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)\right) \\ &\sim \exp\left(\pm\frac{i}{\varepsilon}\left(1+x-\cos(x)\right)+C^{\pm}-\frac{1}{2}\ln\left(1+\sin(x)\right)\right) \\ &\sim \frac{1}{\sqrt{1+\sin x}}\exp\left(\pm\frac{i}{\varepsilon}\left(1+x-\cos(x)\right)+C^{\pm}\right) \end{aligned}$$

Which can be written as

$$y(x) \sim \frac{C}{\sqrt{1+\sin x}}\exp\left(\frac{i}{\varepsilon}\left(1+x-\cos(x)\right)\right)-\frac{C}{\sqrt{1+\sin x}}\exp\left(\frac{-i}{\varepsilon}\left(1+x-\cos(x)\right)\right)$$

In terms of $\sin$ and $\cos$ the above becomes (using the standard Euler relation simplifications)

$$y(x) \sim \frac{A}{\sqrt{1+\sin x}}\cos\left(\frac{1}{\varepsilon}\left(1+x-\cos(x)\right)\right)+\frac{B}{\sqrt{1+\sin x}}\sin\left(\frac{1}{\varepsilon}\left(1+x-\cos(x)\right)\right)$$

Where A,B are the new constants. But $\lambda=\frac{1}{\varepsilon^2}$, and the above becomes

$$y(x) \sim \frac{A}{\sqrt{1+\sin x}}\cos\left(\sqrt{\lambda}\left(1+x-\cos(x)\right)\right)+\frac{B}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda}\left(1+x-\cos(x)\right)\right) \tag{7}$$

Boundary conditions are now applied to determine A,B.

$$\begin{aligned} y(0) &= 0 \\ y(\pi) &= 0 \end{aligned}$$

First B.C. applied to (7) gives (where now $\sim$ is replaced by $=$ for notation simplicity)

$$\begin{aligned} 0 &= A\cos\left(\sqrt{\lambda}\left(1-\cos(0)\right)\right)+B\sin\left(\sqrt{\lambda}\left(1-\cos(0)\right)\right) \\ 0 &= A\cos(0)+B\sin(0) \\ 0 &= A \end{aligned}$$

Hence solution (7) becomes

$$y(x) \sim \frac{B}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda}\left(1+x-\cos(x)\right)\right)$$

Applying the second B.C. $y(\pi) = 0$ to the above results in

$$
\begin{aligned}
0 &= \frac{B}{\sqrt{1+\sin\pi}} \sin\left(\sqrt{\lambda}\left(1+\pi-\cos(\pi)\right)\right) \\
0 &= B\sin\left(\sqrt{\lambda}\left(1+\pi+1\right)\right) \\
&= B\sin\left(\left(2+\pi\right)\sqrt{\lambda}\right)
\end{aligned}
$$

Hence, non-trivial solution implies that

$$
\begin{aligned}
\left(2+\pi\right)\sqrt{\lambda_n} &= n\pi \qquad n = 1,2,3,\cdots \\
\sqrt{\lambda_n} &= \frac{n\pi}{2+\pi}
\end{aligned}
$$

The eigenvalues are

$$
\lambda_n = \frac{n^2\pi^2}{\left(2+\pi\right)^2} \qquad n = 1,2,3,\cdots
$$

Hence $\lambda_n \approx n^2$ for large n. The eigenfunctions are

$$
y_n(x) \sim \frac{B_n}{\sqrt{1+\sin x}} \sin\left(\sqrt{\lambda_n}\left(1+x-\cos(x)\right)\right) \qquad n = 1,2,3,\cdots
$$

The solution is therefore a linear combination of the eigenfunctions

$$
\begin{aligned}
y(x) &\sim \sum_{n=1}^{\infty} y_n(x) \\
&\sim \sum_{n=1}^{\infty} \frac{B_n}{\sqrt{1+\sin x}} \sin\left(\sqrt{\lambda_n}\left(1+x-\cos(x)\right)\right) \qquad \text{(7A)}
\end{aligned}
$$

This solution becomes more accurate for large λ or large n.

5.1.3.2 Part b

For normalization, the requirement is that

$$
\int_0^{\pi} y_n^2(x) \overbrace{\left(\sin(x)+1\right)^2}^{\text{weight}} dx = 1
$$

Substituting the eigenfunction $y_n(x)$ solution obtained in first part in the above results in

$$
\int_0^{\pi} \left(\frac{B_n}{\sqrt{1+\sin x}} \sin\left(\sqrt{\lambda_n}\left(1+x-\cos(x)\right)\right)\right)^2 \left(\sin(x)+1\right)^2 dx \sim 1
$$

The above is now solved for constant B_n. The constant B_n will the same for each n for normalization. Therefore any n can be used for the purpose of finding the scaling constant.

Selecting $n=1$ in the above gives

$$\int_0^\pi \left(\frac{B}{\sqrt{1+\sin x}}\sin\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\right)^2 \left(\sin\left(x\right)+1\right)^2 dx \sim 1$$
$$B^2\int_0^\pi \frac{1}{1+\sin x}\sin^2\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\left(\sin\left(x\right)+1\right)^2 dx \sim 1$$
$$B^2\int_0^\pi \sin^2\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\left(\sin\left(x\right)+1\right) dx \sim 1 \tag{8}$$

Letting $u=\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)$, then

$$\frac{du}{dx}=\frac{\pi}{2+\pi}\left(1+\sin\left(x\right)\right)$$

When $x=0$, then $u=\frac{\pi}{2+\pi}\left(1+0-\cos\left(0\right)\right)=0$ and when $x=\pi$ then $u=\frac{\pi}{2+\pi}\left(1+\pi-\cos\left(\pi\right)\right)=\frac{\pi}{2+\pi}\left(2+\pi\right)=\pi$, hence (8) becomes

$$B^2\int_0^\pi \sin^2\left(u\right)\frac{2+\pi}{\pi}\frac{du}{dx}dx=1$$
$$\frac{2+\pi}{\pi}B^2\int_0^\pi \sin^2\left(u\right)du=1$$

But $\sin^2\left(u\right)=\frac{1}{2}-\frac{1}{2}\cos 2u$, therefore the above becomes

$$\frac{2+\pi}{\pi}B^2\int_0^\pi\left(\frac{1}{2}-\frac{1}{2}\cos 2u\right)du=1$$
$$\frac{1}{2}\frac{2+\pi}{\pi}B^2\left(u-\frac{\sin 2u}{2}\right)_0^\pi=1$$
$$\frac{2+\pi}{2\pi}B^2\left(\left(\pi-\frac{\sin 2\pi}{2}\right)-\left(0-\frac{\sin 0}{2}\right)\right)=1$$
$$\frac{2+\pi}{2\pi}B^2\pi=1$$
$$B^2=\frac{2}{2+\pi}$$

Therefore

$$B=\sqrt{\frac{2}{\pi+2}}$$
$$=0.62369$$

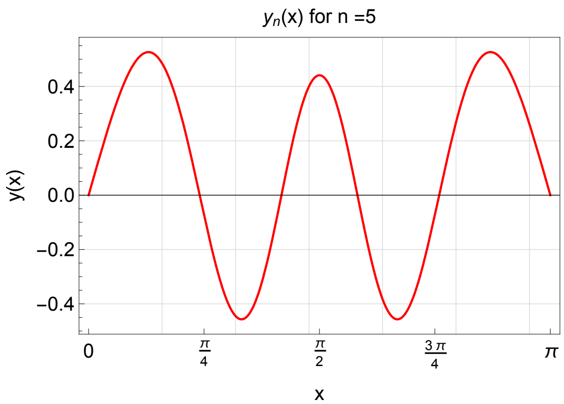
Using the above for each B_n in the solution obtained for the eigenfunctions in (7A), and pulling this scaling constant out of the sum results in

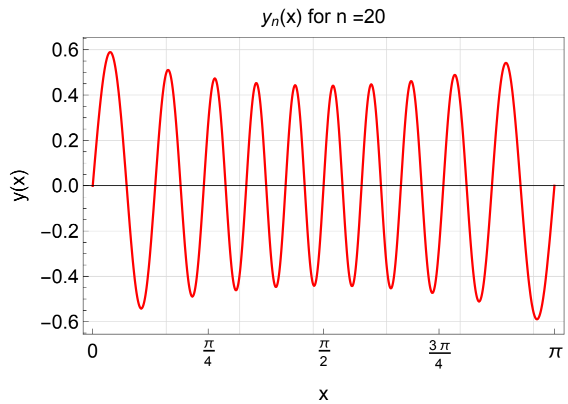
$$y^{\text{normalized}}\sim\sqrt{\frac{2}{\pi+2}}\sum_{n=1}^\infty\frac{1}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda_n}\left(1+x-\cos\left(x\right)\right)\right) \tag{9}$$

Where

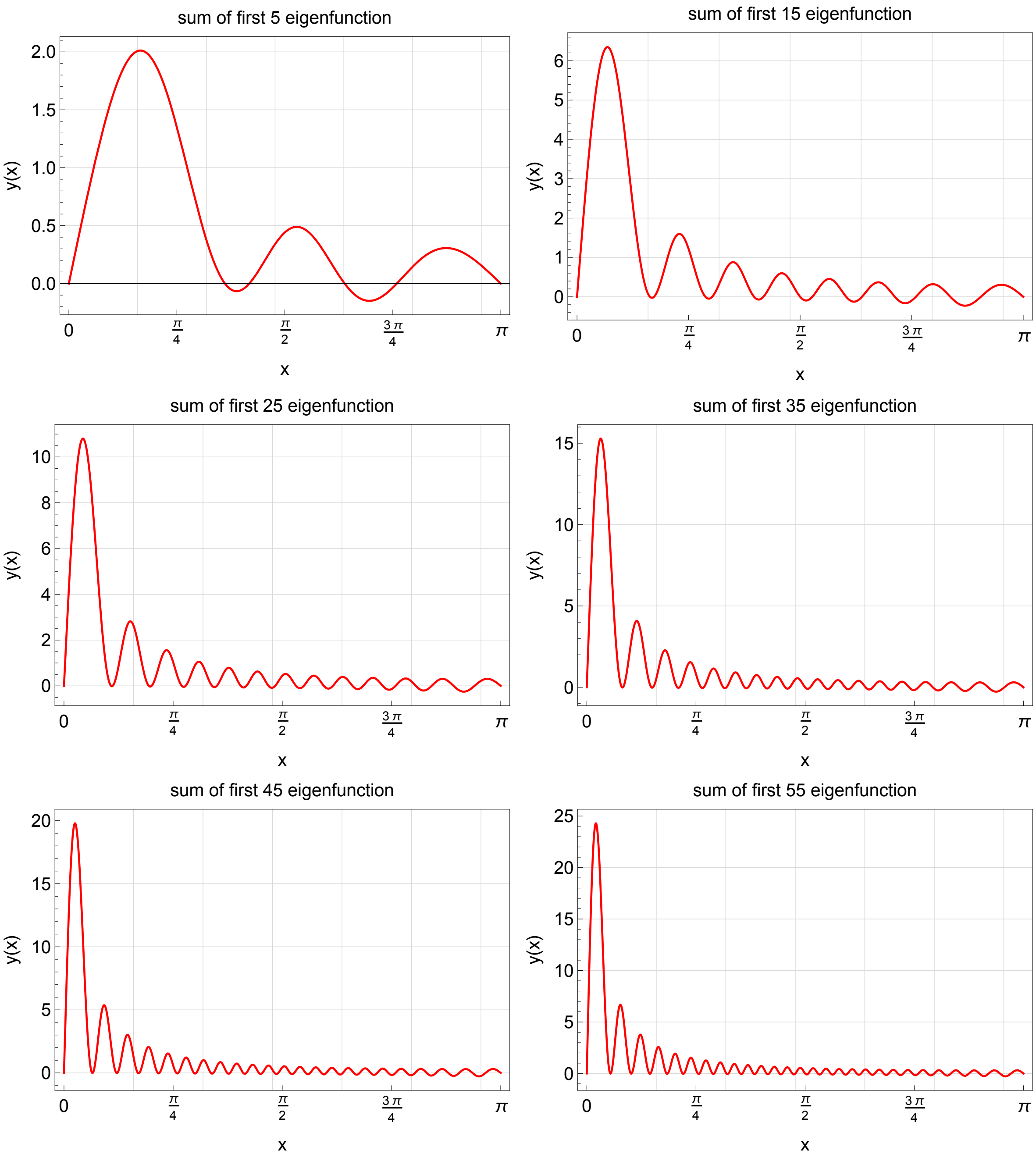
$$\sqrt{\lambda_n}=\frac{n\pi}{2+\pi}\qquad n=1,2,3,\cdots$$

The following are plots for the normalized $y_n\left(x\right)$ for n values it asks to show.





The following shows the $y(x)$ as more eigenfunctions are added up to 55.


sum of first 5 eigenfunction
y(x)
x
sum of first 15 eigenfunction
y(x)
x
sum of first 25 eigenfunction
y(x)
x
sum of first 35 eigenfunction
y(x)
x
sum of first 45 eigenfunction
y(x)
x
sum of first 55 eigenfunction
y(x)
x

5.1.3.3 Part c

Since approximate solution is

$$\begin{aligned}
y(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta\to 0\\
&\sim \exp\left(\frac{1}{\delta}S_0(x)+S_1(x)+\delta S_2(x)+\cdots\right) \qquad (1)
\end{aligned}$$

And the physical optics approximation includes the first two terms in the series above, then the relative error between physical optics and exact solution is given by $\delta S_2(x)$. But $\delta=\varepsilon$. Hence (1) becomes

$$y(x)\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)+\varepsilon S_2(x)+\cdots\right)$$

Hence the relative error must be such that

$$|\varepsilon S_2(x)|_{\max}\leq 0.001 \qquad (1A)$$

Now $S_2(x)$ is found. From (2) in part(a)

$$\frac{\varepsilon^2}{\delta^2}\left(S_0'+\delta S_1'+\delta^2 S_2'+\cdots\right)\left(S_0'+\delta S_1'+\delta^2 S_2'+\cdots\right)+\frac{\varepsilon^2}{\delta}\left(S_0''+\delta S_1''+\delta^2 S_2''+\cdots\right)\sim-(\sin(x)+1)^2$$

$$\frac{\varepsilon^2}{\delta^2}\left(\left(S_0'\right)^2+\delta\left(2S_1'S_0'\right)+\delta^2\left(2S_0'S_2'+\left(S_1'\right)^2\right)+\cdots\right)+\frac{\varepsilon^2}{\delta}\left(S_0''+\delta S_1''+\delta^2 S_2''+\cdots\right)\sim-(\sin(x)+1)^2$$

$$\left(\left(S_0'\right)^2+\varepsilon\left(2S_1'S_0'\right)+\varepsilon^2\left(2S_0'S_2'+\left(S_1'\right)^2\right)+\cdots\right)+\left(\varepsilon S_0''+\varepsilon^2 S_1''+\varepsilon^3 S_2''+\cdots\right)\sim-(\sin(x)+1)^2$$

A balance on $O\left(\varepsilon^2\right)$ gives the ODE to solve to find S_2

$$2S_0'S_2'\sim-\left(S_1'\right)^2-S_1'' \qquad (2)$$

But

$$\begin{aligned}
S_0' &\sim \pm i(1+\sin(x))\\
\left(S_1'\right)^2 &\sim \left(-\frac{1}{2}\frac{\cos(x)}{(\sin(x)+1)}\right)^2\\
&\sim \frac{1}{4}\frac{\cos^2(x)}{(1+\sin(x))^2}\\
S_1'' &\sim -\frac{1}{2}\frac{d}{dx}\left(\frac{\cos(x)}{(1+\sin(x))}\right)\\
&\sim \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)
\end{aligned}$$

Hence (2) becomes

$$
\begin{aligned}
2S_0'S_2' &\sim -\left(S_1'\right)^2 - S_1'' \\
S_2' &\sim -\frac{\left(\left(S_1'\right)^2 + S_1''\right)}{2S_0'} \\
&\sim -\frac{\left(\frac{1}{4}\frac{\cos^2(x)}{(1+\sin(x))^2} + \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)\right)}{\pm 2i\left(1+\sin\left(x\right)\right)} \\
&\sim \pm\frac{i\left(\frac{1}{4}\frac{\cos^2(x)}{(\sin(x)+1)^2} + \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)\right)}{2\left(\sin\left(x\right)+1\right)} \\
&\sim \pm\frac{i\frac{1}{4}\left(\frac{\cos^2(x)+2(1+\sin(x))}{(\sin(x)+1)^2}\right)}{2\left(\sin\left(x\right)+1\right)} \\
&\sim \pm\frac{i}{8}\frac{\cos^2\left(x\right) + 2\left(1+\sin\left(x\right)\right)}{\left(1+\sin\left(x\right)\right)^3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_2\left(x\right) &\sim \pm\frac{i}{8}\int_0^x \frac{\cos^2\left(t\right) + 2\left(1+\sin\left(t\right)\right)}{\left(1+\sin\left(t\right)\right)^3}dt \\
&\sim \pm\frac{i}{8}\left(\int_0^x \frac{\cos^2\left(t\right)}{\left(1+\sin\left(t\right)\right)^3}dt + 2\int_0^x \frac{1}{\left(1+\sin\left(t\right)\right)^2}dt\right) \qquad (3)
\end{aligned}
$$

To do $\int_0^x \frac{\cos^2(t)}{(1+\sin(t))^3}dt$, I used a lookup integration rule from tables which says $\int \cos^p\left(t\right)\left(a+\sin t\right)^m dt = \frac{1}{(a)(m)}\cos^{p+1}\left(t\right)\left(a+\sin t\right)^m$, therefore using this rule the integral becomes, where now $m = -3, p = 2, a = 1$,

$$
\begin{aligned}
\int_0^x \frac{\cos^2 t}{\left(1+\sin t\right)^3}dt &= \frac{1}{-3}\left(\frac{\cos^3 t}{\left(1+\sin t\right)^3}\right)_0^x \\
&= \frac{1}{-3}\left(\frac{\cos^3 x}{\left(1+\sin x\right)^3} - 1\right) \\
&= \frac{1}{3}\left(1 - \frac{\cos^3 x}{\left(1+\sin x\right)^3}\right)
\end{aligned}
$$

And for $\int \frac{1}{(1+\sin(x))^2}dx$, half angle substitution can be used. I do not know what other substitution to use. Using CAS for little help on this, I get

$$
\begin{aligned}
\int_0^x \frac{1}{\left(1+\sin t\right)^2}dt &= \left(-\frac{\cos t}{3\left(1+\sin t\right)^2} - \frac{1}{3}\frac{\cos t}{1+\sin t}\right)_0^x \\
&= \left(-\frac{\cos x}{3\left(1+\sin x\right)^2} - \frac{1}{3}\frac{\cos x}{1+\sin x}\right) - \left(-\frac{1}{3} - \frac{1}{3}\right) \\
&= \frac{2}{3} - \frac{\cos x}{3\left(1+\sin x\right)^2} - \frac{1}{3}\frac{\cos x}{1+\sin x}
\end{aligned}
$$

Hence from (3)

$$
\begin{aligned}
S_2(x) &\sim \pm\frac{i}{8}\left(\frac{1}{3}\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}\right)+2\left(\frac{2}{3}-\frac{\cos x}{3(1+\sin x)^2}-\frac{1}{3}\frac{\cos x}{1+\sin x}\right)\right)\\
&\sim \pm\frac{i}{8}\left(\frac{1}{3}-\frac{1}{3}\frac{\cos^3(x)}{(1+\sin(x))^3}+\frac{4}{3}-\frac{2\cos x}{3(1+\sin x)^2}-\frac{2}{3}\frac{\cos x}{1+\sin x}\right)\\
&\sim \pm\frac{i}{24}\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)
\end{aligned}
$$

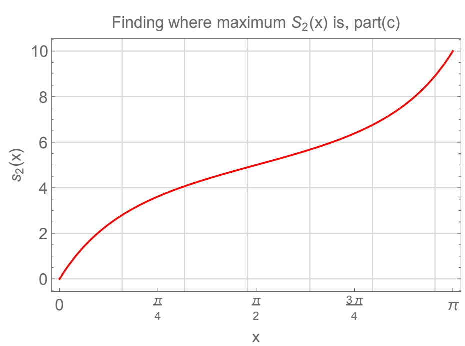
Therefore, from (1A)

$$
\begin{aligned}
|\varepsilon S_2(x)|_{\max} &\leq 0.001\\
\left|\varepsilon\frac{i}{24}\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.001\\
\frac{1}{24}\left|\varepsilon\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.001\\
\left|\varepsilon\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.024 \qquad (2)
\end{aligned}
$$

The maximum value of $\left(1-\frac{\cos^3(x)}{(1+\sin(x))^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)$ between $x=0$ and $x=\pi$ is now found and used to find ε. A plot of the above shows the maximum is maximum at the end, at $x=\pi$ (Taking the derivative and setting it to zero to determine where the maximum is can also be used).

```
In[324]:= myResult = (1 - Cos[x]^3/(1 + Sin[x])^3 + 4 - 2 Cos[x]/(1 + Sin[x])^2 - 2 Cos[x]/(1 + Sin[x]));
Plot[myResult, {x, 0, Pi}, PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel → {{"S2(x)", None}, {"x", "Finding where maximum S2(x) is, part(c)"}}, PlotStyle → Red, BaseStyle → 14,
  FrameTicks → {Automatic, {{0, Pi/4, Pi/2, 3/4 Pi, Pi}, None}}, ImageSize → 400]
```

Out[325]=



Therefore, at $x=\pi$

$$
\begin{aligned}
\left(1-\frac{\cos^3 x}{(1+\sin x)^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)_{x=\pi} &= \left(1-\frac{\cos^3(\pi)}{(1+\sin\pi)^3}+4-\frac{2\cos\pi}{(1+\sin\pi)^2}-\frac{2\cos\pi}{1+\sin\pi}\right)\\
&= 10
\end{aligned}
$$

Hence (2) becomes

$$
\begin{aligned}
10\varepsilon &\leq 0.024 \\
\varepsilon &\leq 0.0024
\end{aligned}
$$

But since $\lambda = \frac{1}{\varepsilon^2}$ the above becomes

$$
\begin{aligned}
\frac{1}{\sqrt{\lambda}} &\leq 0.0024 \\
\sqrt{\lambda} &\geq \frac{1}{0.0024} \\
\sqrt{\lambda} &\geq 416.67
\end{aligned}
$$

Hence

$$
\boxed{\lambda \geq 17351.1}
$$

To find which mode this corresponds to, since $\lambda_n = \frac{n^2\pi^2}{(2+\pi)^2}$, then need to solve for n

$$
\begin{aligned}
17351.1 &= \frac{n^2\pi^2}{(2+\pi)^2} \\
n^2\pi^2 &= (17351.1)(2+\pi)^2 \\
n &= \sqrt{\frac{(17351.1)(2+\pi)^2}{\pi^2}} \\
&= 215.58
\end{aligned}
$$

Hence the next largest integer is used

$$
\boxed{n = 216}
$$

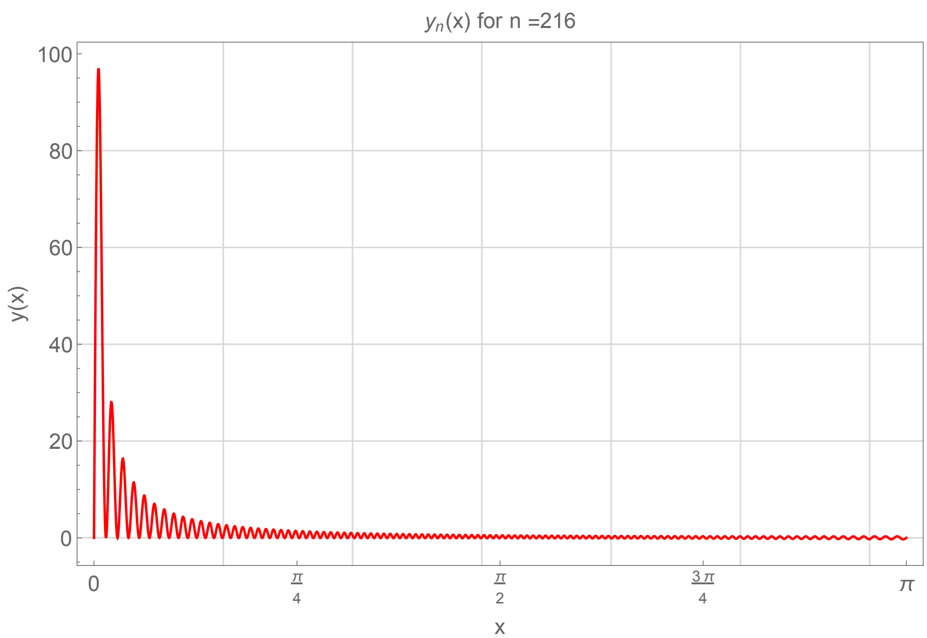
To have relative error less than 0.1% compared to exact solution. Therefore using the result obtained in (9) in part (b) the normalized solution needed is

$$
y^{\text{normalized}} \sim \sqrt{\frac{2}{\pi+2}} \sum_{n=1}^{216} \frac{1}{\sqrt{1+\sin x}} \sin\left(\frac{n\pi}{2+\pi}(1+x-\cos(x))\right)
$$

The following is a plot of the above solution adding all the first 216 modes for illustration.

```
In[42]:= ClearAll[x, n, lam]
mySol[x_, max_] := Sqrt[2/(Pi + 2)] Sum[1/Sqrt[1 + Sin[x]] Sin[(n Pi)/(2 + Pi) (1 + x - Cos[x])], {n, 1, max}];

In[46]:= p[n_] := Plot[mySol[x, n], {x, 0, Pi}, PlotRange → All, Frame → True,
  FrameLabel → {{"y(x)", None}, {"x", Row[{"yn(x) for n =", n}]}}, BaseStyle → 14, GridLines → Automatic,
  GridLinesStyle → LightGray, ImageSize → 600, PlotStyle → Red,
  FrameTicks → { {Automatic, None}, {{0, Pi / 4, Pi / 2, 3 / 4 Pi, Pi}, None}}, PlotRange → All]

In[47]:= p[216]
```

Out[47]=



5.2 Exam 2

5.2.1 problem 3

3. Here we study the competing effects of nonlinearity and diffusion in the context of Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{3a}$$

which is the simplest model equation for diffusive waves in fluid dynamics. It can be solved exactly using the Cole-Hopf transformation

$$u = -2\nu\frac{\phi_x}{\phi} \tag{3b}$$

as follows (with 2 steps to achieve the transformation (3b)).

(a) Let $u = \psi_x$ (where the subscript denotes partial differentiation) and integrate once with respect to x.

(b) Let $\psi = -2\nu\ln(\phi)$ to get the diffusion equation for ϕ.

(c) Solve for ϕ with $\phi(x,0) = \Phi(x)$, $-\infty < x < \infty$. In your integral expression for ϕ, use dummy variable η to facilitate the remaining parts below.

(d) Show that

$$\Phi(x) = \exp\left[\frac{-1}{2\nu}\int_{x_o}^{x} F(\alpha)d\alpha\right]$$

where $u(x,0) = F(x)$, with x_o arbitrary which we will take to be positive for convenience below ($x_o > 0$).

(e) Write your expression for $\phi(x,t)$ in terms of

$$f(\eta,x,t) = \int_{x_o}^{\eta} F(\alpha)d\alpha + \frac{(x-\eta)^2}{2t}.$$

(f) Find $\phi_x(x,t)$ and then use equation (3b) to find $u(x,t)$.

5.2.1.1 Part (a)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let

$$\begin{aligned} u &= -2\nu\frac{\phi_x}{x} \\ &= \frac{\partial}{\partial x}\left(-2\nu\ln\phi\right) \end{aligned} \tag{1A}$$

5.2.1.2 Part(b)

Let

$$\psi = -2\nu\ln\phi \tag{2}$$

Hence (1A) becomes

$$u = \frac{\partial}{\partial x}\psi$$

We now substitute the above back into (1) noting first that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \psi}{\partial x}$$

Interchanging the order gives

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\frac{\partial \psi}{\partial t} \\ &= \frac{\partial}{\partial x}\psi_t \end{aligned}$$

And

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x} \\ &= \psi_{xx} \end{aligned}$$

And

$$\frac{\partial^2 u}{\partial x^2} = \psi_{xxx}$$

Hence the original PDE (1) now can be written in term of ψ as the new dependent variable as

$$\frac{\partial}{\partial x}\psi_t + \psi_x\left(\psi_{xx}\right) = \nu\psi_{xxx} \tag{3}$$

But

$$\psi_x\left(\psi_{xx}\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right)$$

Using the above in (3), then (3) becomes

$$
\begin{aligned}
\frac{\partial}{\partial x}\psi_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right) &= \nu\psi_{xxx} \\
\frac{\partial}{\partial x}\psi_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right) - \nu\frac{\partial}{\partial x}\left(\psi_{xx}\right) &= 0 \\
\frac{\partial}{\partial x}\left(\psi_t + \frac{1}{2}\psi_x^2 - \nu\psi_{xx}\right) &= 0
\end{aligned}
$$

Therefore

$$
\psi_t + \frac{1}{2}\psi_x^2 - \nu\psi_{xx} = 0 \tag{4}
$$

But from (2) $\psi = -2\nu\ln\phi$, then using this in (4), we now rewrite (4) in terms of ϕ

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(-2\nu\ln\phi\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}\left(-2\nu\ln\phi\right)\right)^2 - \nu\frac{\partial^2}{\partial x^2}\left(-2\nu\ln\phi\right) &= 0 \\
\left(-2\nu\frac{\phi_t}{\phi}\right) + \frac{1}{2}\left(-2\nu\frac{\phi_x}{\phi}\right)^2 - \nu\frac{\partial}{\partial x}\left(-2\nu\frac{\phi_x}{\phi}\right) &= 0 \\
-2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\frac{\partial}{\partial x}\left(\frac{\phi_x}{\phi}\right) &= 0
\end{aligned}
$$

But $\frac{\partial}{\partial x}\left(\frac{\phi_x}{\phi}\right) = \frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2}$, hence the above becomes

$$
\begin{aligned}
-2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\left(\frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2}\right) &= 0 \\
-2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\frac{\phi_{xx}}{\phi} - 2\nu^2\frac{\phi_x^2}{\phi^2} &= 0 \\
-2\nu\frac{\phi_t}{\phi} + 2\nu^2\frac{\phi_{xx}}{\phi} &= 0 \\
-\frac{\phi_t}{\phi} + \nu\frac{\phi_{xx}}{\phi} &= 0
\end{aligned}
$$

Since $\phi \neq 0$ identically, then the above simplifies to the heat PDE

$$
\begin{aligned}
\phi_t &= \nu\phi_{xx} \\
\phi(x,0) &= \Phi(x) \\
-\infty &< x < \infty
\end{aligned} \tag{5}
$$

5.2.1.3 Part (c)

Now we solve (5) for $\phi(x,t)$ and then convert the solution back to $u(x,t)$ using the Cole-Hopf transformation. This infinite domain heat PDE has known solution (as $\phi(\pm\infty,t)$ is bounded which is

$$
\phi(x,t) = \int_{-\infty}^{\infty}\Phi(\eta)\frac{1}{\sqrt{4\pi\upsilon t}}\exp\left(\frac{-\left(x-\eta\right)^2}{4\upsilon t}\right)d\eta \tag{6}
$$

5.2.1.4 Part(d)

Now

$$\Phi(x) = \phi(x,0) \tag{7}$$

But since $u(x,t) = \frac{\partial}{\partial x}\left(-2\nu \ln \phi\right)$, then integrating

$$\begin{aligned}
\int_{x_0}^{x} u(\alpha,t)\,d\alpha &= -2\nu \ln \phi \\
\ln \phi &= \frac{-1}{2\nu}\int_{x_0}^{x} u(\alpha,t)\,d\alpha \\
\phi(x,t) &= \exp\left(\frac{-1}{2\nu}\int_{x_0}^{x} u(\alpha,t)\,d\alpha\right)
\end{aligned}$$

Hence at $t=0$ the above becomes

$$\begin{aligned}
\phi(x,0) &= \exp\left(\frac{-1}{2\nu}\int_{x_0}^{x} u(\alpha,0)\,d\alpha\right) \\
&= \exp\left(\frac{-1}{2\nu}\int_{x_0}^{x} F(\alpha)\,d\alpha\right)
\end{aligned}$$

Where $F(x) = u(x,0)$. Hence from the above, comparing it to (6) we see that

$$\Phi(x) = \exp\left(\frac{-1}{2\nu}\int_{x_0}^{x} F(\alpha)\,d\alpha\right) \tag{8}$$

5.2.1.5 Part(e)

From (6), we found $\phi(x,t) = \int_{-\infty}^{\infty} \Phi(\eta)\frac{1}{\sqrt{4\pi\upsilon t}}\exp\left(\frac{-(x-\eta)^2}{4\upsilon t}\right)d\eta$. Plugging (8) into this expression gives

$$\begin{aligned}
\phi(x,t) &= \frac{1}{\sqrt{4\pi\upsilon t}}\int_{-\infty}^{\infty}\exp\left(\frac{-1}{2\nu}\int_{x_0}^{\eta}F(\alpha)\,d\alpha\right)\exp\left(\frac{-\left(x-\eta\right)^2}{4\upsilon t}\right)d\eta \\
&= \frac{1}{\sqrt{4\pi\upsilon t}}\int_{-\infty}^{\infty}\exp\left(\frac{-1}{2\nu}\int_{x_0}^{\eta}F(\alpha)\,d\alpha - \frac{\left(x-\eta\right)^2}{4\upsilon t}\right)d\eta \\
&= \frac{1}{\sqrt{4\pi\upsilon t}}\int_{-\infty}^{\infty}\exp\left(\frac{-1}{2\nu}\left[\int_{x_0}^{\eta}F(\alpha)\,d\alpha + \frac{\left(x-\eta\right)^2}{2t}\right]\right)d\eta
\end{aligned} \tag{9}$$

Let

$$\int_{x_0}^{\eta}F(\alpha)\,d\alpha + \frac{\left(x-\eta\right)^2}{2t} = f\left(\eta,x,t\right)$$

Hence (9) becomes

$$\phi(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}e^{\frac{-f(\eta,x,t)}{2\nu}}d\eta \tag{10}$$

5.2.1.6 Part(f)

From (10)

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}\frac{\partial}{\partial x}\left(e^{\frac{-f(\eta,x,t)}{2\nu}}\right)d\eta\\&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}\left(\frac{\partial}{\partial x}f\left(\eta,x,t\right)e^{\frac{-f(\eta,x,t)}{2\nu}}\right)d\eta\\&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}\left(\frac{\partial}{\partial x}\left[\int_{x_0}^{\eta}F(\alpha)\,d\alpha+\frac{\left(x-\eta\right)^2}{2t}\right]e^{\frac{-f(\eta,x,t)}{2\nu}}\right)d\eta\end{aligned}$$

Using Leibniz integral rule the above simplifies to

$$\frac{\partial\phi}{\partial x}=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}\left(\frac{\left(x-\eta\right)}{t}e^{\frac{-f(\eta,x,t)}{2\nu}}\right)d\eta \tag{11}$$

But

$$u=-2\nu\frac{\phi_x}{\phi}$$

Hence, using (10) and (11) in the above gives

$$u=-2\nu\frac{\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}\left(\frac{(x-\eta)}{t}e^{\frac{-f(\eta,x,t)}{2\nu}}\right)d\eta}{\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\upsilon t}}e^{\frac{-f(\eta,x,t)}{2\nu}}d\eta}$$

Hence the solution is

$$u(x,t)=-2\nu\frac{\int_{-\infty}^{\infty}\frac{(x-\eta)}{t}e^{\frac{-f(\eta,x,t)}{2\nu}}d\eta}{\int_{-\infty}^{\infty}e^{\frac{-f(\eta,x,t)}{2\nu}}d\eta}$$

Where

$$f\left(\eta,x,t\right)=\int_{x_0}^{\eta}F(\alpha)\,d\alpha+\frac{\left(x-\eta\right)^2}{2t}$$

5.2.2 problem 4

4. (a) Use the Method of Images to solve

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad 0 \leq x \leq L, \quad t \geq 0$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = A$$

(b) For $A = 0$, compare your expression for the solution in (a) to the eigenfunction solution.

5.2.2.1 Part(a)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \upsilon \frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ 0 &\leq x \leq L \\ t &\geq 0 \end{aligned} \tag{1}$$

Initial conditions are

$$u(x,0) = f(x)$$

Boundary conditions are

$$\begin{aligned} \frac{\partial u(0,t)}{\partial x} &= 0 \\ u(L,t) &= A \end{aligned}$$

Multiplying both sides of (1) by $G(x,t;x_0,t_0)$ and integrating over the domain gives (where in the following G is used instead of $G(x,t;x_0,t_0)$ for simplicity).

$$\int_{x=0}^{L}\int_{t=0}^{\infty} Gu_t \; dtdx = \int_{x=0}^{L}\int_{t=0}^{\infty} \upsilon u_{xx} G \; dtdx + \int_{x=0}^{L}\int_{t=0}^{\infty} QG \; dtdx \tag{1}$$

For the integral on the LHS, we apply integration by parts once to move the time derivative from u to G

$$\int_{x=0}^{L}\int_{t=0}^{\infty} Gu_t \; dtdx = \int_{x=0}^{L} [uG]_{t=0}^{\infty} dx - \int_{x=0}^{L}\int_{t=0}^{\infty} G_t u \; dtdx \tag{1A}$$

And the first integral in the RHS of (1) gives, after doing integration by parts two times on it

$$\begin{aligned}\int_{x=0}^{L}\int_{t=0}^{\infty} k u_{xx} G\, dtdx &= \int_{t=0}^{\infty} [u_x G]_{x=0}^{L}\, dt - \int_{x=0}^{L}\int_{t=0}^{\infty} v u_x G_x\, dtdx\\ &= \int_{t=0}^{\infty} [u_x G]_{x=0}^{L}\, dt - \left(\int_{t=0}^{\infty} [u G_x]_{x=0}^{L}\, dt - \int_{x=0}^{L}\int_{t=0}^{\infty} v u G_{xx}\, dtdx\right)\\ &= \int_{t=0}^{\infty} \left([u_x G]_{x=0}^{L} - [u G_x]_{x=0}^{L}\right) dt + \int_{x=0}^{L}\int_{t=0}^{\infty} v u G_{xx}\, dtdx\\ &= \int_{t=0}^{\infty} [u_x G - u G_x]_{x=0}^{L}\, dt + \int_{x=0}^{L}\int_{t=0}^{\infty} v u G_{xx}\, dtdx\\ &= -\int_{t=0}^{\infty} [u G_x - u_x G]_{x=0}^{L}\, dt + \int_{x=0}^{L}\int_{t=0}^{\infty} v u G_{xx}\, dtdx \qquad (1B)\end{aligned}$$

Substituting (1A) and (1B) back into (1) results in

$$\int_{x=0}^{L} [uG]_{t=0}^{\infty}\, dx - \int_{x=0}^{L}\int_{t=0}^{\infty} G_t u\, dtdx = \int_{t=0}^{\infty} [u_x G - u G_x]_{x=0}^{L}\, dt + \int_{x=0}^{L}\int_{t=0}^{\infty} v u G_{xx}\, dtdx + \int_{x=0}^{L}\int_{t=0}^{\infty} GQ\, dtdx$$

Or

$$\int_{x=0}^{L}\int_{t=0}^{\infty} -G_t u - v u G_{xx}\, dtdx = -\int_{x=0}^{L} [uG]_{t=0}^{\infty}\, dx - \int_{t=0}^{\infty} [u G_x - u_x G]_{x=0}^{L}\, dt + \int_{x=0}^{L}\int_{t=0}^{\infty} GQ\, dtdx \qquad (2)$$

We now want to choose $G(x,t;x_0,t_0)$ such that

$$\begin{aligned}-G_t u - v u G_{xx} &= \delta(x-x_0)\,\delta(t-t_0)\\ -G_t u &= v u G_{xx} + \delta(x-x_0)\,\delta(t-t_0) \qquad (3)\end{aligned}$$

This way, the LHS of (2) becomes just $u(x_0,t_0)$. Hence (2) now (after the above choice of G) reduces to

$$u(x_0,t_0) = -\int_{x=0}^{L} [uG]_{t=0}^{\infty}\, dx - \int_{t=0}^{t_0} [u G_x - u_x G]_{x=0}^{L}\, dt + \int_{x=0}^{L}\int_{t=0}^{t_0} GQ\, dtdx \qquad (4)$$

We now need to find the Green function which satisfies (3). But (3) is equivalent to solution of problem

$$\begin{aligned}-G_t u &= v u G_{xx}\\ G(x,0) &= \delta(x-x_0)\,\delta(t-t_0)\\ -\infty &< x < \infty\\ G(x,t;x_0,t_0) &= 0 \qquad t > t_0\\ G(\pm\infty,t;x_0,t_0) &= 0\\ G(x,t_0;x_0,t_0) &= \delta(x-x_0)\end{aligned}$$

The above problem has a known fundamental solution which we found before, but for the forward heat PDE instead of the reverse heat PDE as it is now. The fundamental solution to the forward heat PDE is

$$G(x,t)=\frac{1}{\sqrt{4\pi v(t-t_0)}}\exp\left(\frac{-(x-x_0)^2}{4v(t-t_0)}\right)\qquad 0\leq t_0\leq t$$

Therefore, for the reverse heat PDE the above becomes

$$G(x,t)=\frac{1}{\sqrt{4\pi v(t_0-t)}}\exp\left(\frac{-(x-x_0)^2}{4v(t_0-t)}\right)\qquad 0\leq t\leq t_0 \tag{5}$$

We now go back to (4) and try to evaluate all terms in the RHS. Starting with the first term $\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx$. Since $G(x,\infty;x_0,t_0)=0$ then the upper limit is zero. But at lower limit $t=0$ we are given that $u(x,0)=f(x)$, hence this term becomes

$$\begin{aligned}\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx&=\int_{x=0}^{L}-u(x,0)\,G(x,0)\,dx\\&=\int_{x=0}^{L}-f(x)\,G(x,0)\,dx\end{aligned}$$

Now looking at the second term in RHS of (4), we expand it and find

$$[uG_x-u_xG]_{x=0}^{L}=(u(L,t)\,G_x(L,t)-u_x(L,t)\,G(L,t))-(u(0,t)\,G_x(0,t)-u_x(0,t)\,G(0,t))$$

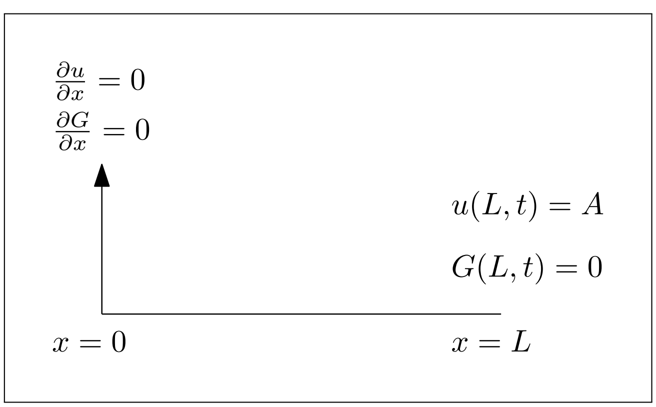
We are told that $u_x(0,t)=0$ and $u(L,t)=A$, then the above becomes

$$[uG_x-u_xG]_{x=0}^{L}=AG_x(L,t)-u_x(L,t)\,G(L,t)-u(0,t)\,G_x(0,t) \tag{5A}$$

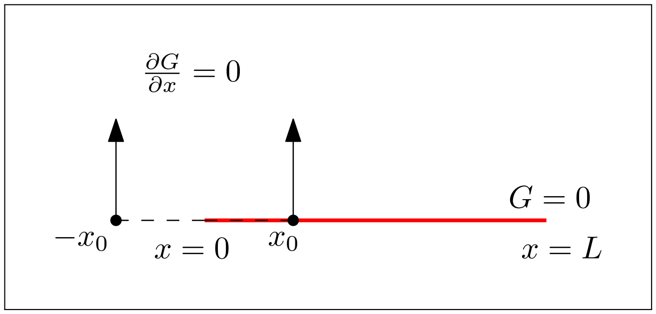
There are still two terms above we do not know. We do not know $u_x(L,t)$ and we also do not know $u(0,t)$. If we can configure, using method of images, such that $G(L,t)=0$ and $G_x(0,t)=0$ then we can get rid of these two terms and end up only with $[uG_x-u_xG]_{x=0}^{\infty}=AG_x(L,t)$ which we can evaluate once we know what $G(x,t)$ is.

This means we need to put images on both sides of the boundaries such that to force $G(L,t)=0$ and also $G_x(0,t)=0$.

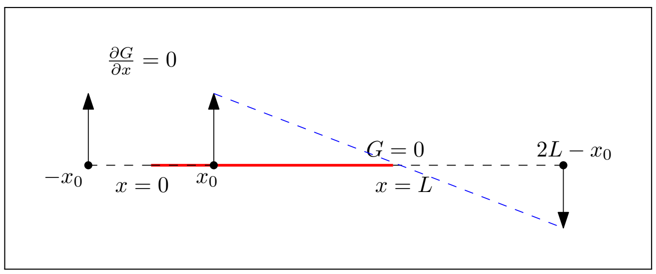
We see that this result agrees what we always did, which is, If the prescribed boundary conditions on u are such that $u=A$, then we want $G=0$ there. And if it is $\frac{\partial u}{\partial x}=A$, then we want $\frac{\partial G}{\partial x}=0$ there. And this is what we conclude here also from the above. In other words, the boundary conditions on Green functions are always the homogeneous version of the boundary conditions given on u.



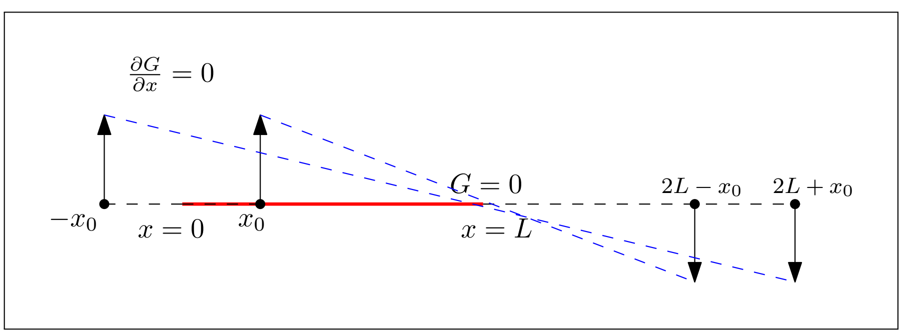
To force $G_x(0,t) = 0$, we need to put same sign images on both sides of $x = 0$. So we end up with this



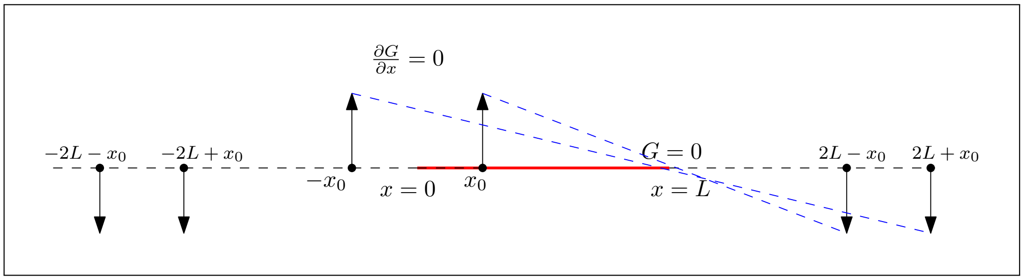
The above makes $G_x(0,t) = 0$ at $x = 0$. Now we want to make $G = 0$ at $x = L$. Then we update the above and put a negative image at $x = 2L - x_0$ to the right of $x = L$ as follows



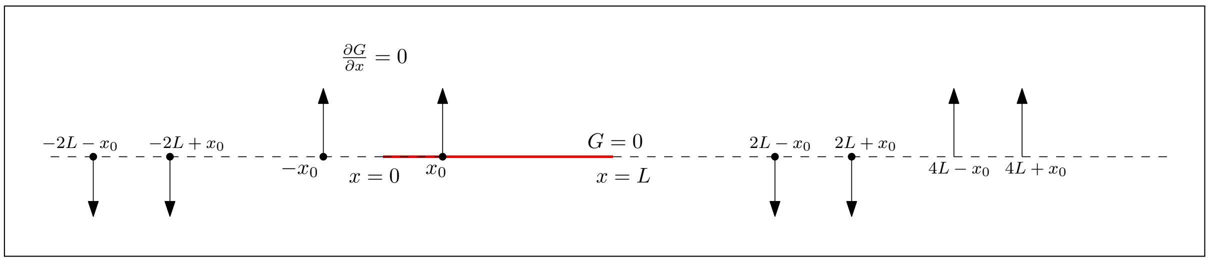
But now we see that the image at $x = -x_0$ has affected condition of $G = 0$ at $x = L$ and will make it not zero as we wanted. So to counter effect this, we have to add another negative image at distance $x = 2L + x_0$ to cancel the effect of the image at $x = -x_0$. We end up with this setup



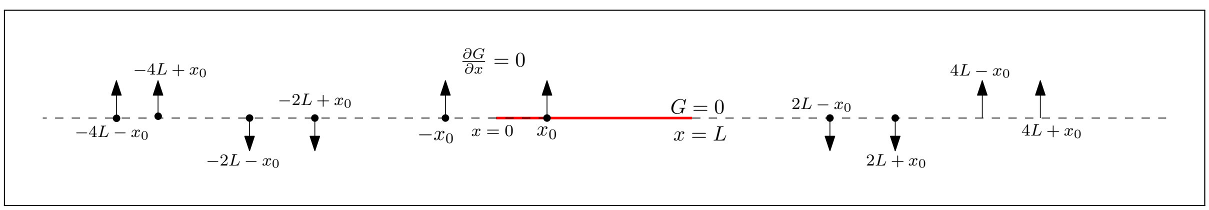
But now we see that the two negative images we added to the right will no longer make $G_x(0,t) = 0$, so we need to counter effect this by adding two negative images to the left side to keep $G_x(0,t) = 0$. So we end up with



But now we see that by putting these two images on the left, we no longer have $G=0$ at $x=L$. So to counter effect this, we have to put copies of these 2 images on the right again but with positive sign, as follows



But now these two images on the right, no longer keep $G_x(0,t)=0$, so we have to put same sign images to the left, as follows



And so on. This continues for infinite number of images. Therefore we see from the above, for the positive images, we have the following sum

$$
\begin{aligned}
\sqrt{4\pi v(t_0-t)}G(x,t;x_0,t_0) &= \exp\left(\frac{-(x-x_0)^2}{4v(t_0-t)}\right)+\exp\left(\frac{-(x+x_0)^2}{4v(t_0-t)}\right)+\exp\left(\frac{-(x-(4L-x_0))^2}{4v(t_0-t)}\right)\\
&+\exp\left(\frac{-(x-(-4L+x_0))^2}{4v(t_0-t)}\right)+\exp\left(\frac{-(x-(4L+x_0))^2}{4v(t_0-t)}\right)\\
&+\exp\left(\frac{-(x-(-4L-x_0))^2}{4v(t_0-t)}\right)+\cdots
\end{aligned}
$$

Or

$$
G(x,t;x_0,t_0)=\frac{1}{\sqrt{4\pi v(t_0-t)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x-(4nL-x_0))^2}{4v(t_0-t)}\right)+\exp\left(\frac{-(x-(-4nL+x_0))^2}{4v(t_0-t)}\right)\right) \tag{6}
$$

The above takes care of the positive images. For negative images, we have this sum of images

$$\begin{aligned}\sqrt{4\pi v\left(t_0-t\right)}G\left(x,t;x_0,t_0\right) &= \exp\left(\frac{-\left(x-\left(2L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-2L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(2L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-2L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(6L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-6L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(6L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-6L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\cdots\end{aligned}$$

Or

$$\sqrt{4\pi v\left(t_0-t\right)}G\left(x,t;x_0,t_0\right)=-\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right) \tag{7}$$

Hence the Green function to use is (6)+(7) which gives

$$\begin{aligned}G\left(x,t;x_0,t_0\right) &= \frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(4nL-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-4nL+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right)\\ &-\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right)\end{aligned} \tag{7A}$$

Using the above Green function, we go back to 5A and finally are able to simplify it

$$\left[uG_x-u_xG\right]_{x=0}^{L}=AG_x\left(L,t\right)-u_x\left(L,t\right)G\left(L,t\right)-u\left(0,t\right)G_x\left(0,t\right)$$

The above becomes now (with the images in place as above)

$$\left[uG_x-u_xG\right]_{x=0}^{L}=A\frac{\partial G\left(L,t;x_0,t_0\right)}{\partial x} \tag{8}$$

Since now we know what $G\left(x,t;x_0,t_0\right)$, from (7), we can evaluate its derivative w.r.t. x. (broken up, so it fits on one page)

$$\begin{aligned}\frac{\partial G\left(x,t;x_0,t_0\right)}{\partial x} &= \frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\sum_{n=-\infty}^{\infty}\frac{-\left(x-\left(4nL-x_0\right)\right)}{2v\left(t_0-t\right)}\exp\left(\frac{-\left(x-\left(4nL-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\sum_{n=-\infty}^{\infty}\frac{-\left(x-\left(-4nL+x_0\right)\right)}{2v\left(t_0-t\right)}\exp\left(\frac{-\left(x-\left(-4nL+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &-\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\sum_{n=-\infty}^{\infty}\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)}{2v\left(t_0-t\right)}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &-\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\sum_{n=-\infty}^{\infty}\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)}{2v\left(t_0-t\right)}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\end{aligned}$$

At $x = L$, the above derivative becomes

$$\begin{aligned}\frac{\partial G(L,t;x_0,t_0)}{\partial x} &= \frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(4nL-x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-(4nL-x_0))^2}{4v(t_0-t)}\right)\\ &+\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(-4nL+x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-(-4nL+x_0))^2}{4v(t_0-t)}\right)\\ &-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L+x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-((4n-2)L+x_0))^2}{4v(t_0-t)}\right)\\ &-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L-x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-((4n-2)L-x_0))^2}{4v(t_0-t)}\right)\end{aligned}\tag{9}$$

From (4), we now collect all terms into the solution

$$u(x_0,t_0) = -\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx - \int_{t=0}^{t_0}[uG_x-u_xG]_{x=0}^{L}dt + \int_{x=0}^{L}\int_{t=0}^{t_0}GQ\,dtdx \tag{4}$$

We found $\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx = \int_{x=0}^{L}-f(x)G(x,0)\,dx$ and now we know what G is. Hence we can find $G(x,0;x_0,t_0)$. It is, from (7A)

$$\begin{aligned}G(x,0;x_0,t_0) &= \frac{1}{\sqrt{4\pi v t_0}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x-(4nL-x_0))^2}{4vt_0}\right)+\exp\left(\frac{-(x-(-4nL+x_0))^2}{4vt_0}\right)\right)\\ &-\frac{1}{\sqrt{4\pi v t_0}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x-((4n-2)L+x_0))^2}{4vt_0}\right)+\exp\left(\frac{-(x-((4n-2)L-x_0))^2}{4vt_0}\right)\right)\end{aligned}\tag{10}$$

And we now know what $[uG_x-u_xG]_{x=0}^{L}$ is. It is $A\frac{\partial G(L,t;x_0,t_0)}{\partial x}$. Hence (4) becomes

$$u(x_0,t_0) = \int_{x=0}^{L}f(x)G(x,0;x_0,t_0)\,dx-\int_{t=0}^{t_0}A\frac{\partial G(L,t;x_0,t_0)}{\partial x}dt+\int_{x=0}^{L}\int_{t=0}^{t_0}G(x,t;x_0,t_0)Q(x,t)\,\,dtdx$$

Changing the roles of x_0, t_0

$$u(x,t) = \int_{x_0=0}^{L}f(x_0)G(x_0,t_0;x,0)\,dx_0-\int_{t=0}^{t}A\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}dt_0+\int_{x_0=0}^{L}\int_{t_0=0}^{t}G(x_0,t_0;x,t)Q(x_0,t_0)\,\,dt_0dx_0 \tag{11}$$

This completes the solution.

Summary

The solution is

$$u(x,t) = \int_{x_0=0}^{L}f(x_0)G(x_0,t_0;x,0)\,dx_0-\int_{t=0}^{t}A\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}dt_0+\int_{x_0=0}^{L}\int_{t_0=0}^{t}G(x_0,t_0;x,t)Q(x_0,t_0)\,\,dt_0dx_0$$

Where $G(x_0,t_0;x,0)$ is given in (10) (after changing roles of parameters):

$$G(x_0,t_0;x,0)=\frac{1}{\sqrt{4\pi vt}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-(4nL-x))^2}{4vt}\right)+\exp\left(\frac{-(x_0-(-4nL+x))^2}{4vt}\right)\right)$$
$$-\frac{1}{\sqrt{4\pi vt}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-((4n-2)L+x))^2}{4vt}\right)+\exp\left(\frac{-(x_0-((4n-2)L-x))^2}{4vt}\right)\right) \tag{10A}$$

and $\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}$ is given in (9) (after also changing roles of parameters):

$$\begin{aligned}\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}&=\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(4nL-x))}{2v(t-t_0)}\exp\left(\frac{-(L-(4nL-x))^2}{4v(t-t_0)}\right)\\&+\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(-4nL+x))}{2v(t-t_0)}\exp\left(\frac{-(L-(-4nL+x))^2}{4v(t-t_0)}\right)\\&-\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L+x))}{2v(t-t_0)}\exp\left(\frac{-(L-((4n-2)L+x))^2}{4v(t-t_0)}\right)\\&-\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L-x))}{2v(t-t_0)}\exp\left(\frac{-(L-((4n-2)L-x))^2}{4v(t-t_0)}\right)\end{aligned} \tag{9A}$$

and $G(x_0,t_0;x,t)$ is given in (7A), but with roles changed as well to become

$$G(x_0,t_0;x,t)=\frac{1}{\sqrt{4\pi v(t-t_0)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-(4nL-x))^2}{4v(t-t_0)}\right)+\exp\left(\frac{-(x_0-(-4nL+x))^2}{4v(t-t_0)}\right)\right) \tag{7AA}$$
$$-\frac{1}{\sqrt{4\pi v(t-t_0)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-((4n-2)L+x))^2}{4v(t-t_0)}\right)+\exp\left(\frac{-(x_0-((4n-2)L-x))^2}{4v(t-t_0)}\right)\right)$$

5.2.2.2 Part (b)

When $A=0$, the solution in part (a) becomes

$$u(x,t)=\int_{x_0=0}^{L}f(x_0)\,G(x_0,t_0;x,0)\,dx_0+\int_{x_0=0}^{L}\int_{t_0=0}^{t}G(x_0,t_0;x,t)\,Q\;dt_0dx_0$$

Where $G(x_0,t_0;x,0)$ is given in (10A) in part (a), and $G(x_0,t_0;x,t)$ is given in (7AA) in part (a). Now we find the eigenfunction solution for this problem order to compare it with the above green function images solution. Since $A=0$ then the PDE now becomes

$$\begin{aligned}\frac{\partial u}{\partial t}&=v\frac{\partial^2 u}{\partial x^2}+Q(x,t)\\0&\leq x\leq L\\t&\geq 0\end{aligned} \tag{1}$$

Initial conditions

$$u(x,0)=f(x)$$

Boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0$$
$$u(L,t) = 0$$

Since boundary conditions has now become homogenous (thanks for $A = 0$), we can use separation of variables to find the eigenfunctions, and then use eigenfunction expansion. Let the solution be

$$u_n(x,t) = a_n(t)\,\phi_n(x)$$
$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \tag{1A}$$

Where $\phi_n(t)$ are eigenfunctions for the associated homogeneous PDE $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}$ which can be found from separation of variables. To find $\phi_n(x)$, we start by separation of variables. Let $u(x,t) = X(x)T(t)$ and we plug this solution back to the PDE to obtain

$$XT' = vX''T$$
$$\frac{1}{v}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence the spatial ODE is $\frac{X''}{X} = -\lambda$ or $X'' + \lambda X = 0$ with boundary conditions

$$X'(0) = 0$$
$$X(L) = 0$$

case $\underline{\lambda = 0}$ The solution is $X = c_1 + c_2 x$. Hence $X' = c_2$. Therefore $c_2 = 0$. Hence $X = c_1 = 0$. Trivial solution. So $\underline{\lambda = 0 \text{ is not possible}}$.

case $\underline{\lambda > 0}$ The solution is $X = c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right)$. and $X' = -\sqrt{\lambda}c_1\sin\lambda x + c_2\sqrt{\lambda}\cos\lambda x$. From first B.C. at $x = 0$ we find $0 = c_2\sqrt{\lambda}$, hence $c_2 = 0$ and the solution becomes $X = c_1\cos\left(\sqrt{\lambda}x\right)$. At $x = L$, we have $0 = c_1\cos\left(\sqrt{\lambda}L\right)$ which leads to $\sqrt{\lambda}L = n\frac{\pi}{2}$ for $n = 1,3,5,\cdots$ or

$$\sqrt{\lambda_n} = \left(\frac{2n-1}{2}\right)\frac{\pi}{L} \qquad n = 1,2,3,\cdots$$
$$\lambda_n = \left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 \qquad n = 1,1,2,3,\cdots$$

Hence the $X_n(x)$ solution is

$$X_n(x) = c_n\cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$

The time ODE is now solved using the above eigenvalues. (we really do not need to do this part, since $A_n(t)$ will be solved for later, and $A_n(t)$ will contain all the time dependent parts, including those that come from $Q(x,t)$, but for completion, this is done)

$$\begin{aligned}\frac{1}{v}\frac{T'}{T} &= -\lambda_n\\ T'+v\lambda_n T &= 0\\ \frac{dT}{T} &= -v\lambda_n dt\\ \ln|T| &= -v\lambda_n t + C\\ T &= C_n e^{v\lambda_n t}\end{aligned}$$

Hence the solution to the homogenous PDE is

$$\begin{aligned}u_n(x,t) &= X_n T_n\\ &= c_n\cos\left(\sqrt{\lambda_n}x\right)e^{v\lambda_n t}\end{aligned}$$

Where constants of integration are merged into one. Therefore

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} X_n T_n\\ &= \sum_{n=1}^{\infty} c_n\cos\left(\sqrt{\lambda_n}x\right)e^{v\lambda_n t} \qquad (2)\end{aligned}$$

From the above we see that

$$\boxed{\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right)}$$

Using this, we now write the solution to $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ using eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(x) \qquad (3)$$

Where now $A_n(t)$ will have all the time dependent terms from $Q(x,t)$ as well from the time solution from the homogenous PDE $e^{v\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 t}$ part. We will solve for $A_n(t)$ now.

In this below, we will expand $Q(x,t)$ using these eigenfunctions (we can do this, since the eigenfunctions are basis for the whole solution space which the forcing function is in as well). We plug-in (3) back into the PDE, and since boundary conditions are now homogenous, then term by term differentiation is justified. The PDE $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ now becomes

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} A_n(t)\phi_n(x) = v\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} A_n(t)\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x) \qquad (4)$$

Where $\sum_{n=1}^{\infty} q_n(t)\phi_n(x)$ is the eigenfunction expansion of $Q(x,t)$. To find $q_n(t)$ we apply orthogonality as follows

$$\begin{aligned}
Q(x,t) &= \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x) \\
\int_0^L \phi_m(x)\,Q(x,t)\,dx &= \int_0^L \left(\sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)\right)\phi_m(x)\,dx \\
\int_0^L \phi_m(x)\,Q(x,t)\,dx &= \sum_{n=1}^{\infty} \int_0^L q_n(t)\,\phi_n(x)\,\phi_m(x)\,dx \\
&= \int_0^L q_m(t)\,\phi_m^2(x)\,dx \\
&= q_m(t)\int_0^L \cos^2\left(\sqrt{\lambda_n}x\right)dx
\end{aligned}$$

But

$$\int_0^L \cos^2\left(\sqrt{\lambda_n}x\right)dx = \frac{L}{2}$$

Hence

$$q_n(t) = \frac{2}{L}\int_0^L \phi_n(x)\,Q(x,t)\,dx \tag{5}$$

Now that we found $q_n(t)$, we go back to (4) and simplifies it more

$$\begin{aligned}
\sum_{n=1}^{\infty} A_n'(t)\,\phi_n(x) &= v\sum_{n=1}^{\infty} A_n(t)\,\phi_n''(x) + \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x) \\
A_n'(t)\,\phi_n(x) &= vA_n(t)\,\phi_n''(x) + q_n(t)\,\phi_n(x)
\end{aligned}$$

But since $\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right)$ then

$$\phi_n'(x) = -\left(\sqrt{\lambda_n}\right)\sin\left(\sqrt{\lambda_n}x\right)$$

And

$$\begin{aligned}
\phi_n''(x) &= -\lambda_n\cos\left(\sqrt{\lambda_n}x\right) \\
&= -\lambda_n\phi_n(x)
\end{aligned}$$

Hence the above ODE becomes

$$A_n'\phi_n = -vA_n\lambda_n\phi_n + q_n\phi_n$$

Canceling the eigenfunction $\phi_n(x)$ (since not zero) gives

$$A_n'(t) + vA_n(t)\,\lambda_n = q_n(t) \tag{6}$$

We now solve this for $A_n(t)$. Integrating factor is

$$\begin{aligned}
\mu &= \exp\left(\int v\lambda_n dt\right) \\
&= e^{\lambda_n v t}
\end{aligned}$$

Hence (6) becomes

$$\begin{aligned}\frac{d}{dt}\left(\mu A_n(t)\right) &= \mu q_n(t)\\ e^{\lambda_n v t}A_n(t) &= \int_0^t e^{\lambda_n v s} q_n(s)\,ds + C\\ A_n(t) &= e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds + Ce^{-\lambda_n v t} \qquad (7)\end{aligned}$$

Now that we found $A_n(t)$, then the solution (3) becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(e^{-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vt}\int_0^t e^{\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vs} q_n(s)\,ds + Ce^{-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vt}\right)\phi_n(x) \qquad (8)$$

At $t=0$, we are given that $u(x,0)=f(x)$, hence the above becomes

$$f(x) = \sum_{n=1}^{\infty} C\phi_n(x)$$

To find C, we apply orthogonality again, which gives

$$\begin{aligned}\int_0^L f(x)\phi_m(x)\,dx &= \sum_{n=1}^{\infty}\int_0^L C\phi_n(x)\phi_m(x)\,dx\\ &= C\int_0^L \phi_m^2(x)\,dx\\ \int_0^L f(x)\phi_m(x)\,dx &= \frac{L}{2}C\\ C &= \frac{2}{L}\int_0^L f(x)\phi_n(x)\,dx\end{aligned}$$

Now that we found C, then the solution in (8) is complete. It is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} A_n(t)\phi_n(x)\\ &= \sum_{n=1}^{\infty}\left(e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds + \frac{2}{L}e^{-\lambda_n v t}\int_0^L f(x)\phi_n(x)\,dx\right)\phi_n(x)\end{aligned}$$

Or

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds\right)\\ &\quad + \frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^L f(x)\phi_n(x)\,dx\right)\end{aligned}$$

Summary

The eigenfunction solution is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds\right)\\ &\quad + \frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^L f(x)\phi_n(x)\,dx\right)\end{aligned}$$

Where

$$\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$
$$\lambda_n = \left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 \qquad n = 1,1,2,3,\cdots$$

And

$$q_n(t) = \frac{2}{L}\int_0^L \phi_n(x)\,Q(x,t)\,dx$$

To compare the eigenfunction expansion solution and the Green function solution, we see the following mapping of the two solutions

$$\overbrace{\int_0^L f(x_0)\,G(x_0,t_0;x,0)\,dx_0}^{\frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n \upsilon t}\int_0^L f(x)\phi_n(x)dx\right)} + \overbrace{\int_0^L\int_0^t G(x_0,t_0;x,t)\,Q(x_0,t_0)\;dt_0dx_0}^{\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n \upsilon t}\int_0^t e^{\lambda_n \upsilon s}q_n(s)ds\right)}$$

Where the top expression is the eigenfunction expansion and the bottom expression is the Green function solution using method of images. Where $G(x_0,t_0;x,t)$ in above contains the infinite sums of the images. So the Green function solution contains integrals and inside these integrals are the infinite sums. While the eigenfunction expansions contains two infinite sums, but inside the sums we see the integrals. So summing over the images seems to be equivalent to the operation of summing over eigenfunctions. These two solutions in the limit should of course give the same result (unless I made a mistake somewhere). In this example, I found method of eigenfunction expansion easier, since getting the images in correct locations and sign was tricky to get right.

5.2.3 problem 5

5. This problem is a simple model for diffraction of light passing through infinitesimally small slits separated by a distance $2a$.

Solve the diffraction equation

$$\frac{\partial u}{\partial t}=\frac{i\lambda}{4\pi}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial source $u(x,0)=f(x)=\delta(x-a)+\delta(x+a),\ a>0$.

Show that the solution $u(x,t)$ oscillates wildly, but that the intensity $|u(x,t)|^2$ is well-behaved. The intensity $|u(x,t)|^2$ shows that the diffraction pattern at a distance t consists of a series of alternating bright and dark fringes with period $\lambda t/(2a)$.

$$\frac{\partial u(x,t)}{\partial t}=\frac{i\lambda}{4\pi}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$
$$u(x,0)=f(x)=\delta(x-a)+\delta(x+a)$$

I will Use Fourier transform to solve this, since this is for $-\infty<x<\infty$ and the solution $u(x,t)$ is assumed bounded at $\pm\infty$ (or goes to zero there), hence $u(x,t)$ is square integrable and therefore we can assume it has a Fourier transform.

Let $U(\xi,t)$ be the spatial part only Fourier transform of $u(x,t)$. Using the Fourier transform pairs defined as

$$U(\xi,t)=\mathscr{F}(u(x,t))=\int_{-\infty}^{\infty}u(x,t)e^{-i2\pi x\xi}dx$$
$$u(x,t)=\mathscr{F}^{-1}(U(\xi,t))=\int_{-\infty}^{\infty}U(\xi,t)e^{i2\pi x\xi}d\xi$$

Therefore, by Fourier transform properties of derivatives

$$\mathscr{F}\left(\frac{\partial u(x,t)}{\partial x}\right)=(2\pi i\xi)U(\xi,t)$$
$$\mathscr{F}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)=(2\pi i\xi)^2U(\xi,t) \tag{2}$$

And

$$\mathscr{F}\left(\frac{\partial u(x,t)}{\partial t}\right)=\frac{\partial U(\xi,t)}{\partial t} \tag{3}$$

Where in (3), we just need to take time derivative of $U(\xi,t)$ since the transform is applied only to the space part. Now we take the Fourier transform of the given PDE and using (2,3) relations we obtain the PDE but now in Fourier space.

$$\begin{aligned}\frac{\partial U(\xi,t)}{\partial t} &= \left(\frac{i\lambda}{4\pi}\right)(2\pi i\xi)^2 U(\xi,t)\\ &= -\left(\frac{i\lambda}{4\pi}\right)4\pi^2\xi^2 U(\xi,t)\\ &= \left(-i\lambda\pi\xi^2\right)U(\xi,t) \qquad (4)\end{aligned}$$

Equation (4) can now be easily solved for $U(\xi,t)$ since it is separable.

$$\frac{\partial U(\xi,t)}{U(\xi,t)} = \left(-i\lambda\pi\xi^2\right)\partial t$$

Integrating

$$\begin{aligned}\ln|U(\xi,t)| &= \left(-i\lambda\pi\xi^2\right)t + C\\ U(\xi,t) &= U(\xi,0)\,e^{\left(-i\lambda\pi\xi^2\right)t}\end{aligned}$$

Where $U(\xi,0)$ is the Fourier transform of $u(x,0)$, the initial conditions, which is $f(x)$ and is given in the problem. To go back to spatial domain, we now need to do the inverse Fourier transform. By applying the convolution theorem, we know that multiplication in Fourier domain is convolution in spatial domain, therefore

$$\mathscr{F}^{-1}(U(\xi,t)) = \mathscr{F}^{-1}(U(\xi,0)) \circledast \mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right) \qquad (5)$$

But

$$\begin{aligned}\mathscr{F}^{-1}(U(\xi,t)) &= u(x,t)\\ \mathscr{F}^{-1}(U(\xi,0)) &= f(x)\end{aligned}$$

And

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right) = \int_{-\infty}^{\infty} e^{-i\lambda\pi\xi^2 t} e^{2i\pi x\xi}\, d\xi \qquad (5A)$$

Hence (5) becomes

$$u(x,t) = f(x) \circledast \mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right)$$

Here, I used Mathematica to help me with the above integral (5A) as I could not find it in tables so far[2]. Here is the result

Find inverse Fourier transform, for problem 5, NE 548

```
In[13]:= InverseFourierTransform[ Exp[-I lam Pi z^2 t], z, x, FourierParameters -> {1,  -2 * Pi}]
```

Out[13]= $\dfrac{e^{\frac{i\pi x^2}{\text{lam}\, t}}}{\sqrt{2\pi}\sqrt{i\,\text{lam}\, t}}$

[2]Trying to do this integral by hand also, but so far having some difficulty..

Therefore, from Mathematica, we see that

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right) = \frac{e^{\frac{i\pi x^2}{\lambda t}}}{\sqrt{2\pi}\sqrt{i}\sqrt{\lambda t}} \tag{6}$$

But[3]

$$\begin{aligned}\sqrt{i} &= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\ &= e^{i\frac{\pi}{4}}\end{aligned}$$

Hence (6) becomes

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right) = \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} e^{i\frac{\pi x^2}{\lambda t}} \tag{7}$$

Now we are ready to do the convolution in (5A) since we know everything in the RHS, hence

$$\begin{aligned}u(x,t) &= f(x) \circledast \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} e^{i\frac{\pi x^2}{\lambda t}} \\ &= \left(\delta(x-a) + \delta(x+a)\right) \circledast \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} e^{i\frac{\pi x^2}{\lambda t}} \end{aligned} \tag{8}$$

Applying convolution integral on (8), which says that

$$\begin{aligned}f(x) &= g_1(x) \circledast g_2(x) \\ &= \int_{-\infty}^{\infty} g_1(z)\, g_2(x-z)\, dz\end{aligned}$$

Therefore (8) becomes

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} \left(\delta(z-a) + \delta(z+a)\right) \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} e^{i\frac{\pi(x-z)^2}{\lambda t}} dz \\ &= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} \left(\delta(z-a) + \delta(z+a)\right) e^{i\frac{\pi(x-z)^2}{\lambda t}} dz \\ &= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} \left(\int_{-\infty}^{\infty} \delta(z-a)\, e^{i\frac{\pi(x-z)^2}{\lambda t}} dz + \int_{-\infty}^{\infty} \delta(z+a)\, e^{i\frac{\pi(x-z)^2}{\lambda t}} dz\right)\end{aligned}$$

But an integral with delta function inside it, is just the integrand evaluated where the delta argument become zero which is at $z = a$ and $z = -a$ in the above. (This is called the sifting property). Hence the above integrals are now easily found and we obtain the solution

$$u(x,t) = \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}} \left(\exp\left(i\frac{\pi(x-a)^2}{\lambda t}\right) + \exp\left(i\frac{\pi(x+a)^2}{\lambda t}\right)\right)$$

[3]Taking the positive root only.

The above is the solution we need. But we can simplify it more by using Euler relation.

$$
\begin{aligned}
u(x,t) &= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(i\frac{\pi\left(x^2+a^2-2xa\right)}{\lambda t}\right)+\exp\left(i\frac{\pi\left(x^2+a^2+2ax\right)}{\lambda t}\right)\right)\\
&= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)\exp\left(\frac{-i2\pi xa}{\lambda t}\right)+\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)\exp\left(\frac{i2\pi ax}{\lambda t}\right)\right)
\end{aligned}
$$

Taking $\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)$ as common factor outside results in

$$
\begin{aligned}
u(x,t) &= \frac{\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(i\frac{2\pi ax}{\lambda t}\right)+\exp\left(-i\frac{2\pi xa}{\lambda t}\right)\right)\\
&= \frac{2\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\frac{\exp\left(i\frac{2\pi ax}{\lambda t}\right)+\exp\left(-i\frac{2\pi xa}{\lambda t}\right)}{2}\right)\\
&= \frac{2\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)\\
&= \frac{\sqrt{2}\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}-i\frac{\pi}{4}\right)}{\sqrt{\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)\\
&= \frac{\sqrt{2}\exp\left(i\left(\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\right)}{\sqrt{\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)
\end{aligned}
$$

Hence the final solution is

$$
\boxed{u(x,t) = \sqrt{\frac{2}{\pi\lambda t}}\exp\left(i\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)} \tag{9}
$$

Hence the real part of the solution is

$$
\Re\left(u(x,t)\right) = \sqrt{\frac{2}{\pi\lambda t}}\cos\left(\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)
$$

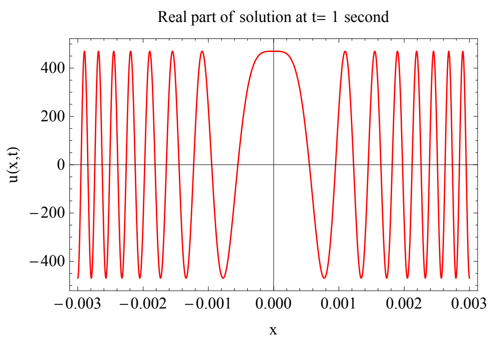
And the imaginary part of the solution is

$$
\Im\left(u(x,t)\right) = \sqrt{\frac{2}{\pi\lambda t}}\sin\left(\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)
$$

The $\frac{\pi}{4}$ is just a phase shift. Here is a plot of the Real and Imaginary parts of the solution, using for $\lambda = 600\times 10^{-9} meter, a = 1000\times 10^{-9} meter$ at $t = 1$ second

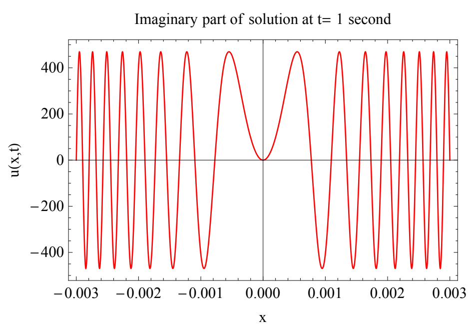
```
In[15]:= a = 1000 * 10^(-9);
         λ = 600 * 10^(-9);
         u[x_, t_] := Sqrt[2/(π λ t)] Exp[I (π (x^2 + a^2))/(λ t) - π/4] Cos[(2 π a)/λ x/t]

In[59]:= Clear[t];
         Plot[Re@u[x, 1], {x, -3000 a, 3000 a}, Frame -> True,
          FrameLabel → {{"u(x,t)", None}, {"x", "Real part of solution at t= 1 second"}}, BaseStyle → 12,
          PlotStyle → Red, PlotTheme → "Classic"]
```

Out[60]=




```
In[61]:= Plot[Im@u[x, 1], {x, -3000 a, 3000 a}, Frame -> True,
          FrameLabel → {{"u(x,t)", None}, {"x", "Imaginary part of solution at t= 1 second"}}, BaseStyle → 12,
          PlotStyle → Red, PlotTheme → "Classic"]
```

Out[61]=

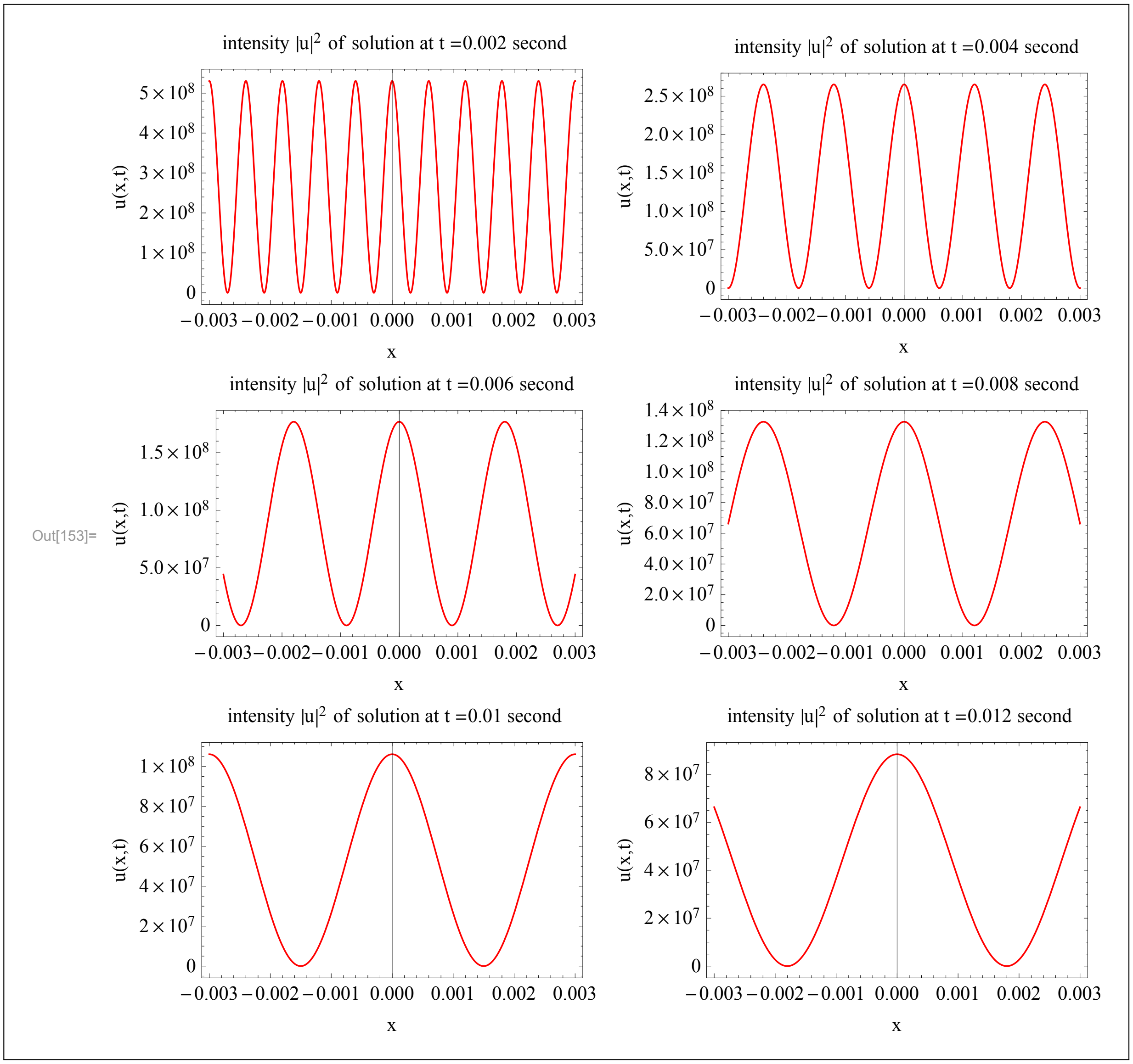


We see the rapid oscillations as distance goes away from the origin. This is due to the x^2 term making the radial frequency value increase quickly with x. We now plot the $|u(x,t)|^2$. Looking at solution in (9), and since complex exponential is ± 1, then the amplitude is governed by $\sqrt{\frac{2}{\pi\lambda t}}\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)$ part of the solution. Hence

$$\boxed{|u(x,t)|^2 = \frac{2}{\pi\lambda t}\cos^2\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)}$$

These plots show the intensity at different time values. We see from these plots, that the intensity is well behaved in that it does not have the same rapid oscillations seen in the $u(x,t)$ solution plots.

```
In[154]:= a = 1000 * 10^(-9);
λ = 600 * 10^(-9);
intensity[x_, t_] := 2/(π λ t) (Cos[(2 π a)/λ x/t])^2
p =
  Plot[intensity[x, #], {x, -3000 a, 3000 a}, Frame -> True,
    FrameLabel → {{"u(x,t)", None}, {"x", Row[{" intensity |u|^2 of solution at t =", #, " second"}]}},
    BaseStyle → 12, PlotStyle → Red, PlotTheme → "Classic", ImageSize → 300] & /@ Range[0.002, 0.012, 0.002];
```



Now Comparing argument to cosine in above to standard form in order to find the period:

$$\frac{2\pi a}{\lambda}\frac{x}{t} = 2\pi f t$$

Where f is now in hertz, then when $x = t$, we get by comparing terms that

$$\frac{2\pi a}{\lambda}\frac{1}{t} = 2\pi f$$
$$\frac{a}{\lambda}\frac{1}{t} = f$$

But $f = \frac{1}{T}$ where T is the period in seconds. Hence $\frac{a}{\lambda}\frac{1}{t} = \frac{1}{T}$ or

$$\boxed{T = \frac{\lambda t}{a}}$$

So period on intensity is $\frac{\lambda t}{a}$ at $x = t$ (why problem statement is saying period is $\frac{\lambda t}{2a}$?).

5.2.4 problem 2 (optional)

2. Consider the 1D heat equation in a semi-infinite domain:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0$$

with boundary conditions: $u(0,t) = \exp(-i\omega t)$ and $u(x,t)$ bounded as $x \to \infty$. In order to construct a real forcing, we need both positive and negative real values of ω. Consider that this forcing has been and will be applied for all time. This "pure boundary value problem" could be an idealization of heating the surface of the earth by the sun (periodic forcing). One could then ask, how far beneath the surface of the earth do the periodic fluctuations of the heat propagate?

(a) Consider solutions of the form $u(x,t;\omega) = \exp(ikx)\exp(-i\omega t)$. Find a single expression for k as a function of (given) ω real, $sgn(\omega)$ and ν real.

Write $u(x,t;\omega)$ as a function of (given) ω real, $sgn(\omega)$ and ν real. To obtain the most general solution by superposition, one would next integrate over all values of ω, $-\infty < \omega < \infty$ (do not do this).

(b) The basic solution can be written as $u(x,t;\omega) = \exp(-i\omega t)\exp(-\sigma x)\exp(i\sigma\ sgn(\omega)\ x)$. Find σ in terms of $|\omega|$ and ν.

(c) Make an estimate for the propagation depth of daily temperature fluctuations.

5.2.4.1 Part (a)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial x^2} \qquad (1) \\ u(0,t) &= e^{-i\omega t} \end{aligned}$$

And $x \geq 0, u(\infty,t)$ bounded. Let

$$u(x,t) = e^{ikx}e^{-i\omega t}$$

Hence

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= -i\omega e^{ikx}e^{-i\omega t} \\ &= -i\omega u(x,t) \qquad (2) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= ike^{ikx}e^{-i\omega t} \\ \frac{\partial^2 u(x,t)}{\partial x^2} &= -k^2 e^{ikx}e^{-i\omega t} \\ &= -k^2 u(x,t) \qquad (3) \end{aligned}$$

Substituting (2,3) into (1) gives

$$-i\omega u(x,t) = -\nu k^2 u(x,t)$$

Since $u_{\omega}(x,t)$ can not be identically zero (trivial solution), then the above simplifies to

$$-i\omega = -\nu k^2$$

Or

$$\boxed{k^2 = \frac{i\omega}{\nu}} \tag{4}$$

Writing

$$\omega = \operatorname{sgn}(\omega)\,|\omega|$$

Where

$$\operatorname{sgn}(\omega) = \begin{cases} +1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases}$$

Then (4) becomes

$$\begin{aligned} k^2 &= \frac{i\ \operatorname{sgn}(\omega)\,|\omega|}{\nu} \\ k &= \pm\frac{\sqrt{i}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}} \end{aligned}$$

Since

$$\sqrt{i} = i^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$$

Hence k can be written as

$$k = \pm\frac{e^{i\frac{\pi}{4}}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

case A. Let start with the positive root hence

$$k = \frac{e^{i\frac{\pi}{4}}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

Case (A1) $\omega < 0$ then the above becomes

$$\begin{aligned} k &= \frac{ie^{i\frac{\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}} = \frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}} = \frac{e^{i\frac{3\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}} \\ &= \left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}} \\ &= \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}} \\ &= \left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right) \end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}i - \frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}ix\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)$ for large x. This term will decay for large x since $-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}$ is negative (assuming $\nu > 0$ always). Hence positive root hence worked OK when $\omega < 0$. Now we check if it works OK also when $\omega > 0$

<u>case A2</u> When $\omega > 0$ then k now becomes

$$\begin{aligned}
k &= \frac{e^{i\frac{\pi}{4}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \frac{e^{i\frac{\pi}{4}}\sqrt{\omega}}{\sqrt{\nu}}\\
&= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{\sqrt{\omega}}{\sqrt{\nu}}\\
&= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\omega}}{\sqrt{\nu}}\\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}
\end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} - \frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)\exp\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)$ for large x.

This term will decay for large x since $-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}$ is negative (assuming $\nu > 0$ always). Hence positive root hence worked OK when $\omega > 0$ as well.

Let check what happens if we use the negative root.

<u>case B</u>. negative root hence

$$k = -\frac{e^{i\frac{\pi}{4}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

<u>Case (A1)</u> $\omega < 0$ then the above becomes

$$\begin{aligned}
k &= -\frac{ie^{i\frac{\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}} = -\frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}} = -\frac{e^{i\frac{3\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)\\
&= \left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} - i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)
\end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} - i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + \frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)\exp\left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}ix\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)$ for large x. This term will blow up for large x since $+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}$ is positive (assuming $\nu > 0$ always). Hence we reject the case of negative sign on k. And pick

$$\begin{aligned}
k &= \frac{e^{i\frac{\pi}{4}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u(x,t;\omega) &= \exp(ikx)\exp(-i\omega t) \\
&= \exp\left(i\left(\frac{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t) \\
&= \exp\left(i\left(\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}+i\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}\right)x\right)\exp(-i\omega t) \\
&= \exp\left(i\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}x-\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}\,x\right)\exp(-i\omega t)
\end{aligned}$$

Hence

$$\boxed{u(x,t;\omega)=\exp\left(\frac{-\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)}$$

The general solution $u(x,t)$ is therefore the integral over all ω, hence

$$\begin{aligned}
u(x,t) &= \int_{\omega=-\infty}^{\infty} u(x,t;\omega)\ d\omega \\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(\frac{i\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)\ d\omega \\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\left[\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x-\omega t\right]\right)\ d\omega \\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(ix\left[\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}-\omega\frac{t}{x}\right]\right)\ d\omega
\end{aligned}$$

5.2.4.2 Part (b)

From part (a), we found that

$$\boxed{u(x,t;\omega)=\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)} \tag{1}$$

comparing the above to expression given in problem which is

$$u(x,t)=\exp(-\sigma x)\ \exp(i\sigma\operatorname{sgn}(\omega)x)\ \exp(-i\omega t) \tag{2}$$

Therefore, by comparing $\exp(-\sigma x)$ to $\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)$ we see that

$$\boxed{\sigma=\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}} \tag{3}$$

5.2.4.3 part (c)

Using the solution found in part (a)

$$u(x,t;\omega)=\exp(-\sigma x)\ \exp(i\sigma\operatorname{sgn}(\omega)x)\ \exp(-i\omega t)$$

To find numerical estimate, assuming $\omega > 0$ for now

$$\begin{aligned}
u(x,t;\omega) &= \exp(-\sigma x)\ \exp(i\sigma x)\ \exp(-i\omega t)\\
&= e^{-\sigma x}(\cos\sigma x + i\sin\sigma x)(\cos\omega t - i\sin\omega t)\\
&= e^{-\sigma x}(\cos\sigma x\cos\omega t - i\sin\omega t\cos\sigma x + i\cos\omega t\sin\sigma x + \sin\sigma x\sin\omega t)\\
&= e^{-\sigma x}(\cos(\sigma x)\cos(\omega t) + \sin(\sigma x)\sin(\omega t) + i(\cos(\omega t)\sin(\sigma x) - \sin(\omega t)\cos(\sigma x)))\\
&= e^{-\sigma x}(\cos(t\omega - x\sigma) - i\sin(t\omega - x\sigma))
\end{aligned}$$

Hence will evaluate

$$\begin{aligned}
\operatorname{Re}(u(x,t;\omega)) &= e^{-\sigma x}\operatorname{Re}(\cos(t\omega - x\sigma) - i\sin(t\omega - x\sigma))\\
&= e^{-\sigma x}\cos(t\omega - x\sigma)
\end{aligned}$$

I assume here it is asking for numerical estimate. We only need to determine numerical estimate for σ. For ω, using the period $T = 24$ hrs or $T = 86\,400$ seconds, then $\omega = \frac{2\pi}{T}$ is now found. Then we need to determine ν, which is thermal diffusivity for earth crust. There does not seem to be an agreed on value for this and this value also changed with depth inside the earth crust. The value I found that seem mentioned more is 1.2×10^{-6} meter^2 per second. Hence

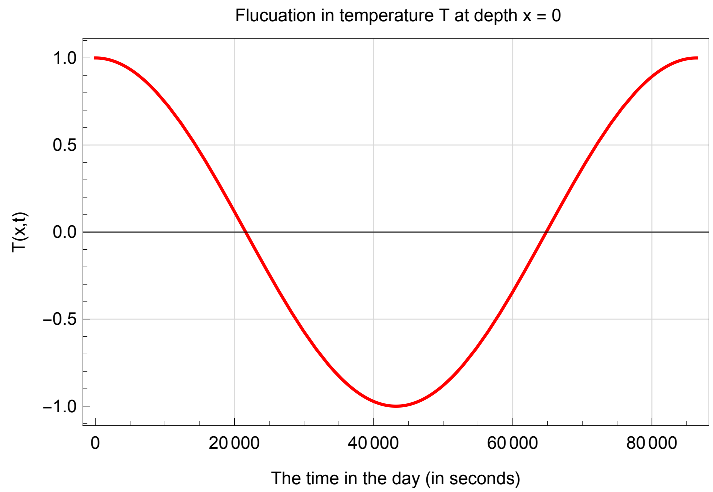
$$\begin{aligned}
\sigma &= \frac{\sqrt{\frac{2\pi}{86\,400}}}{\sqrt{2\left(1.2\times10^{-6}\right)}}\\
&= 5.505 \text{ per meter}
\end{aligned}$$

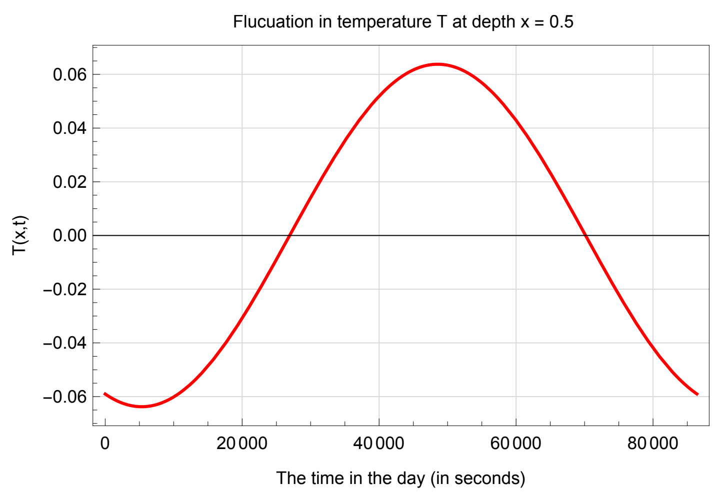
Therefore

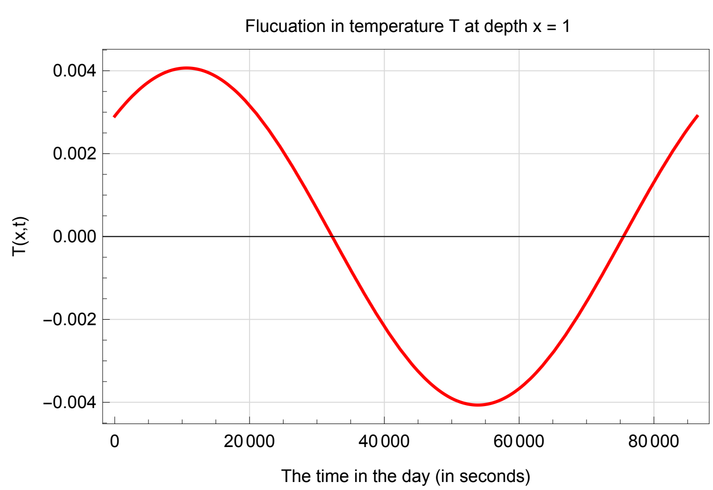
$$\operatorname{Re}(u(x,t;\omega)) = e^{-5.505x}\cos(t\omega - 5.505x)$$

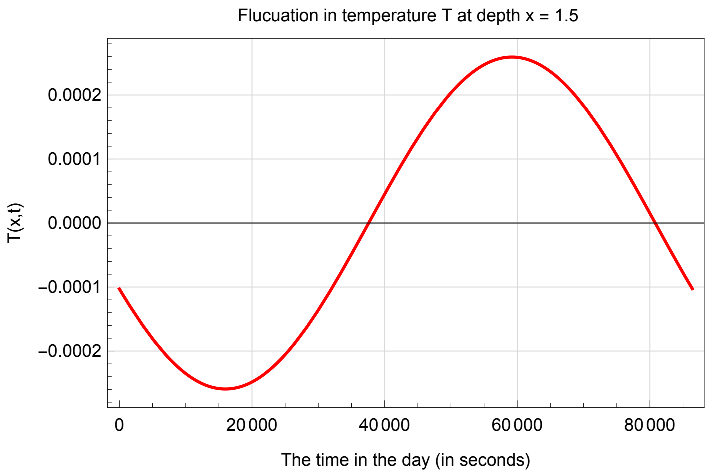
Using $\omega = \frac{2\pi}{86\,400} = 7.29\times10^{-5}$ rad/sec. Now we can use the above to estimate fluctuation of heat over 24 hrs period. But we need to fix x for each case. Here $x = 0$ means on the earth surface and x say 10, means at depth 10 meters and so on as I understand that x is starts at 0 at surface or earth and increases as we go lower into the earth crust. Plotting at the above for $x = 0, 0.5, 1, 1.5, 2$ I see that when $x > 2$ then maximum value of $e^{-5.505x}\cos(t\omega - 5.505x)$ is almost zero. This seems to indicate a range of heat reach is about little more than 2 meters below the surface of earth.

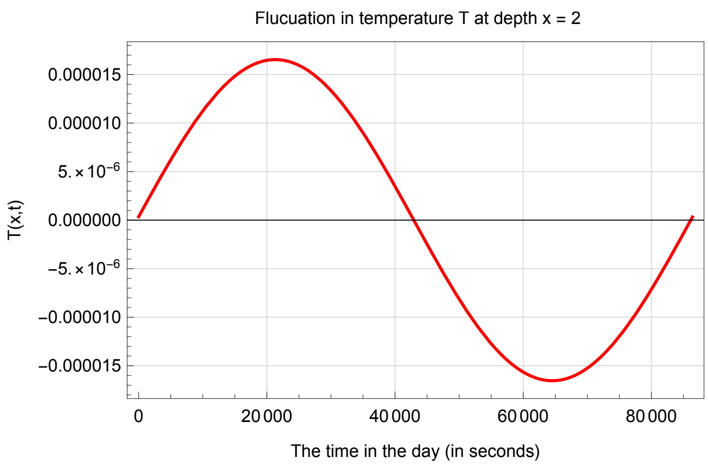
This is a plot of the fluctuation in temperature at different x each in separate plot, then later a plot is given that combines them all.

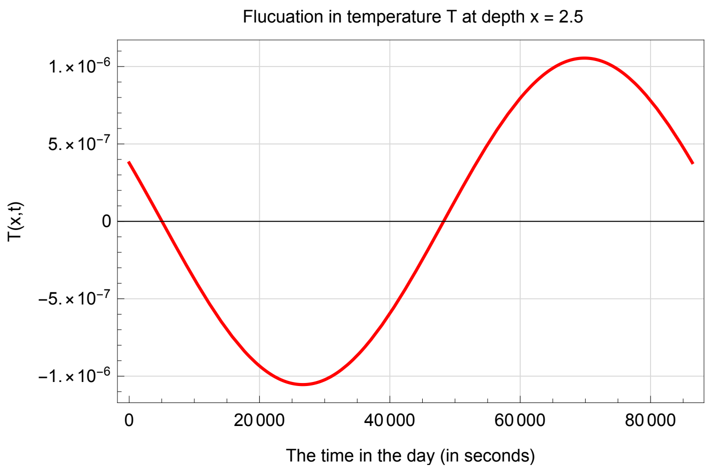

Flucuation in temperature T at depth x = 0
1.0
0.5
0.0
−0.5
−1.0
T(x,t)
0
20 000
40 000
60 000
80 000
The time in the day (in seconds)


Flucuation in temperature T at depth x = 0.5
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
T(x,t)
0
20 000
40 000
60 000
80 000
The time in the day (in seconds)

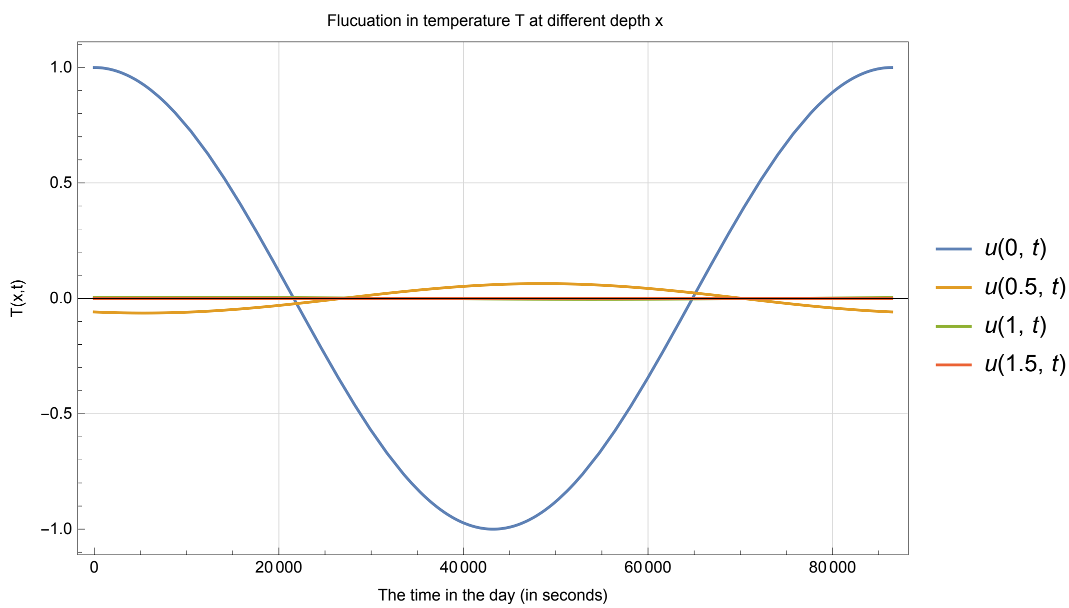

Flucuation in temperature T at depth x = 1
0.004
0.002
0.000
−0.002
−0.004
T(x,t)
0
20 000
40 000
60 000
80 000
The time in the day (in seconds)







This plot better show the difference per depth, as it combines all the plots into one.


Flucuation in temperature T at different depth x
1.0
0.5
0.0
−0.5
−1.0
T(x,t)
0
20000
40000
60000
80000
The time in the day (in seconds)
u(0, t)
u(0.5, t)
u(1, t)
u(1.5, t)