University Course

EMA 523
Flight dynamics and control

University of Wisconsin, Madison
Spring 2014

My Class Notes

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Spring 2014

Contents

Chapter 1

Introduction

Took this course in spring 2014. Part of MSc. in Engineering Mechanics.

Instructor: professor Riccardo Bonazza

1.1 syllabus

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EMA 523
FLIGHT DYNAMICS AND CONTROL
SPRING 2014

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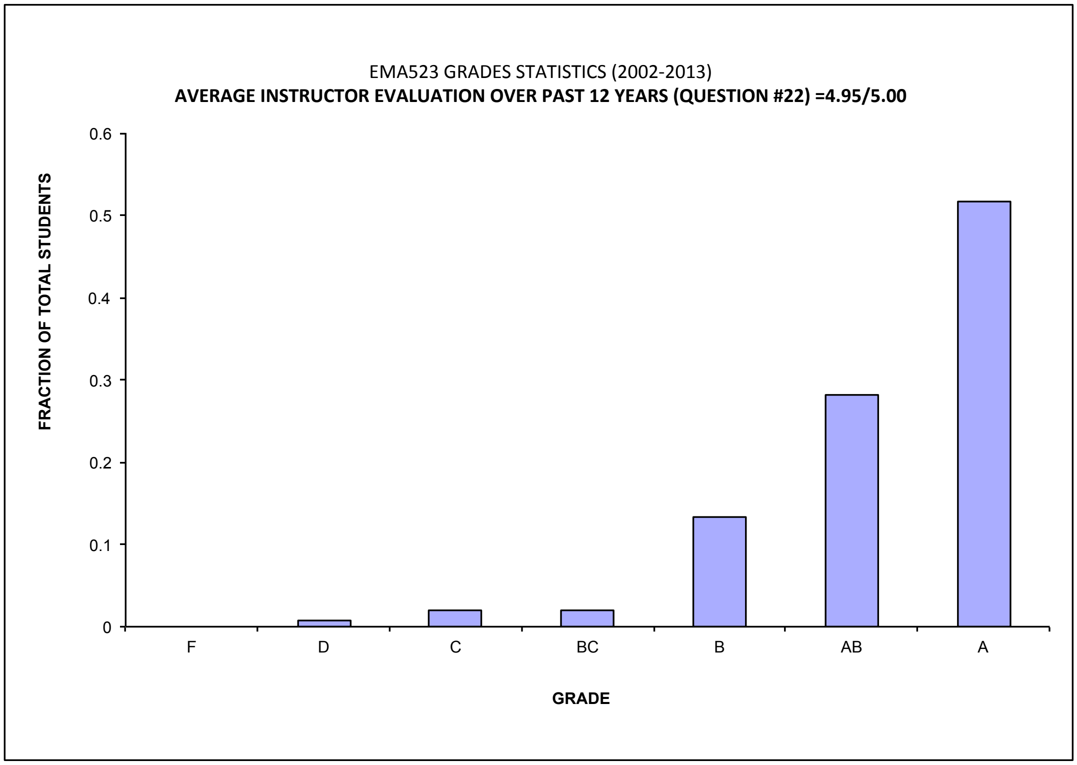
Textbook:
Etkin B. and Reid L.D.
Dynamics of flight, stability and control, 3rd ed
Wiley, 1996.

Grade: based on biweekly homework.

Tentative grading scheme:

95% - 100%:	A
91% - 94%:	AB
81% - 90%:	B
71% - 80%:	BC
61% - 70%:	C
51% - 60%:	D
0% - 50%:	F


EMA523 GRADES STATISTICS (2002-2013)
AVERAGE INSTRUCTOR EVALUATION OVER PAST 12 YEARS (QUESTION #22) =4.95/5.00
FRACTION OF TOTAL STUDENTS
0.6
0.5
0.4
0.3
0.2
0.1
0
F
D
C
BC
B
AB
A
GRADE

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Course Content

1. **Introduction (approximately 1 lecture)**
   (a) Definitions
   (b) Nomenclature
   (c) Reference systems
2. **Static Stability (approximately 8 lectures)**
   (a) Longitudinal stability
   (b) Pitch stiffness
   (c) Longitudinal control
   (d) Lateral stability
   (e) Lateral control
3. **Aircraft equations of motion (approximately 3.5 lectures)**
   (a) Full, non-linear equations
   (b) Euler's angles
   (c) Small disturbance linearization
   (d) Description of aerodynamic actions
4. **Stability derivatives (approximately 0.5 lectures)**
5. **Dynamic stability of uncontrolled motion (approximately 3 lectures)**
   (a) Review of differential equations
   (b) Longitudinal modes and their approximations
   (c) Lateral modes and their approximations
6. **Open-loop aircraft control (approximately 5 lectures)**
   (a) Review of linear systems, Laplace transform and control theory
   (b) Application to longitudinal control
   (c) Application to lateral control
7. **Closed-loop control (approximately 5 lectures)**
   (a) Review of closed-loop control theory and stability criteria
   (b) Application to longitudinal control
   (c) Application to lateral control
8. **Analysis of control loops using Matlab (approximately 3 lectures)**
   (a) Matlab algorithms
   (b) Using *Simulink*

1.2 links

1. moodle internal course site
2. aircraft center of gravity calculator. Aerodynamic Center (AC), Mean Aerodynamic Chord (MAC), Center of Gravity (CG), Neutral Point (NP) and Wing Area `http://adamone.rchomepage.com/cg_calc.htm`

Chapter 2

My typed HWs and key solutions

Local contents

2.1 HW1

2.1.1 Problem 1

1. Problem 2.1 in the course textbook.

NOTE: in Problem 2.1(c), for the half wing $\overline{y} \neq 0$ and for the full wing $\overline{y} = 0$ so you only need to evaluate $\overline{x}$ and $\overline{z}$. Refer to appendix C in the book and keep in mind that Eqs. (C.1,2) and (C.1,5) are written for the half-wing.

NOTE: On page 320, in the first line below Eq.(B1.2), the symbol is supposed to be $C_{L\alpha}$. On page 322, in Fig. B1.2, the label of the vertical axis is supposed to be $C_{L\alpha}/A$ and the left hand side of the equation in the inset of the same figure is also supposed to be $C_{L\alpha}/A$.

2.1 A subsonic transport aircraft has a tapered, untwisted sweptback wing with straight leading and trailing edges. The wing tips are straight and parallel to the root chord. In the following, use the data of Appendix C and assume that the airfoil section local aerodynamic center is at the $\frac{1}{4}$-chord point.

(a) Make an accurate three-view drawing of the wing chord plane.

(b) Calculate wing area S, aspect ratio A, taper ratio $\lambda = c_t/c_r$ and the mean aerodynamic chord $\bar{c}$.

(c) Calculate the location of the wing's mean aerodynamic center, and locate it and $\bar{c}$ on the side view of the wing (with dimensions). (Assume a uniform additional lift coefficient $C_{L_a} = C_L$.)

(d) The aircraft is to be operated with its most rearward CG position limited to 25 ft (7.62 m) aft of the apex of the wing. The distance between the wing and tail mean aerodynamic centers is $\bar{l}_t = 55$ ft (16.76 m). Estimate the tail area required to provide a control-fixed static margin of at least 0.05 at all times. Assume that $a_t = a_{wb}$ and $h_{n_w} = h_{n_{wb}}$. Ignore power plant effects and use $\partial\epsilon/\partial\alpha = 0.25$.

Geometric Data

Wing Span, b	150 ft (45.72 m)
Root Chord, c_r	25 ft (7.62 m)
Tip Chord, c_t	12 ft (3.66 m)
Leading edge sweep, Λ_0	26°
Dihedral angle, γ	4°

Figure 2.1: problem 1 description

2.1.1.1 Part (a)

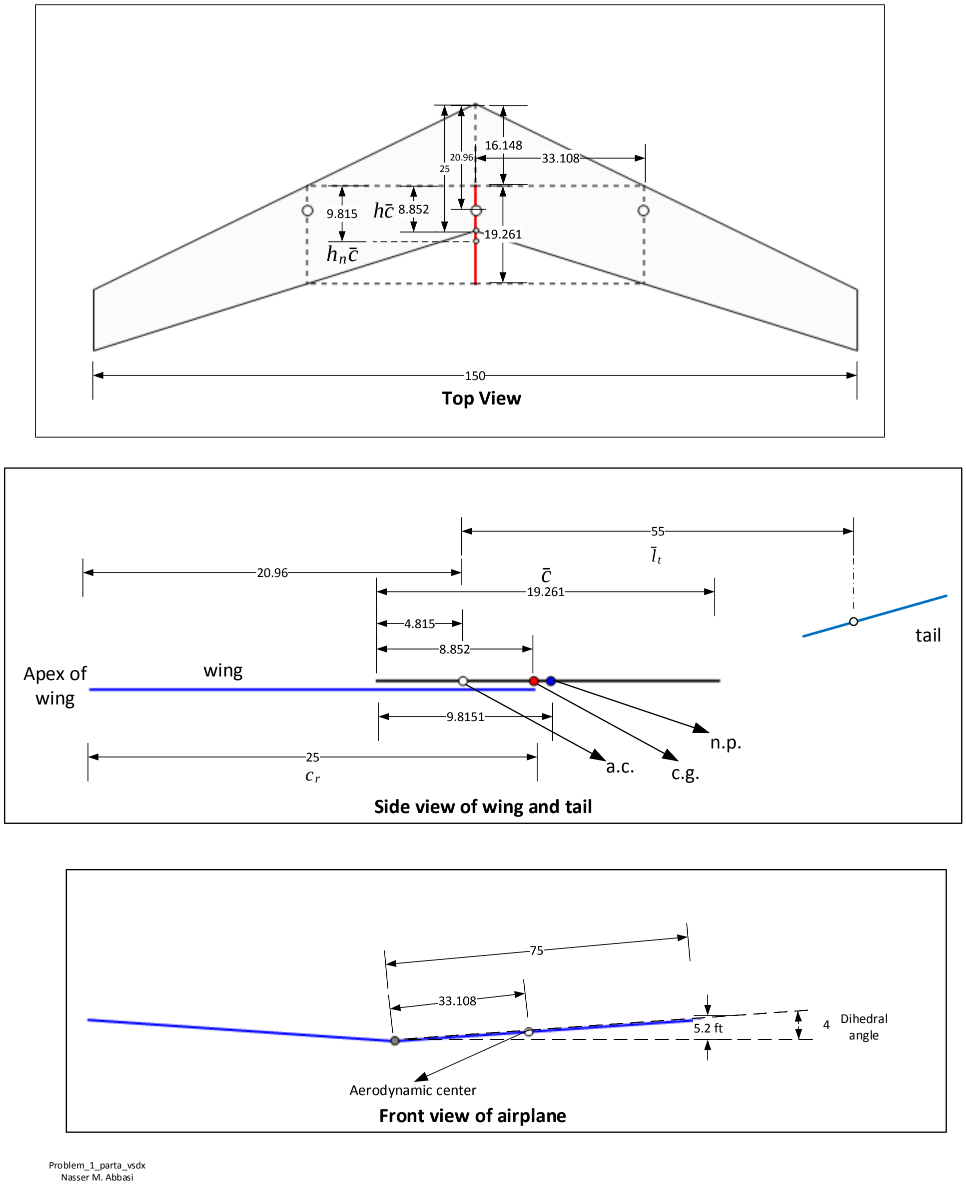


Figure 2.2: problem 1 part (a)

2.1.1.2 Part (b)

The following diagram shows the calculated aerodynamic dimensions of the wing using the three views.

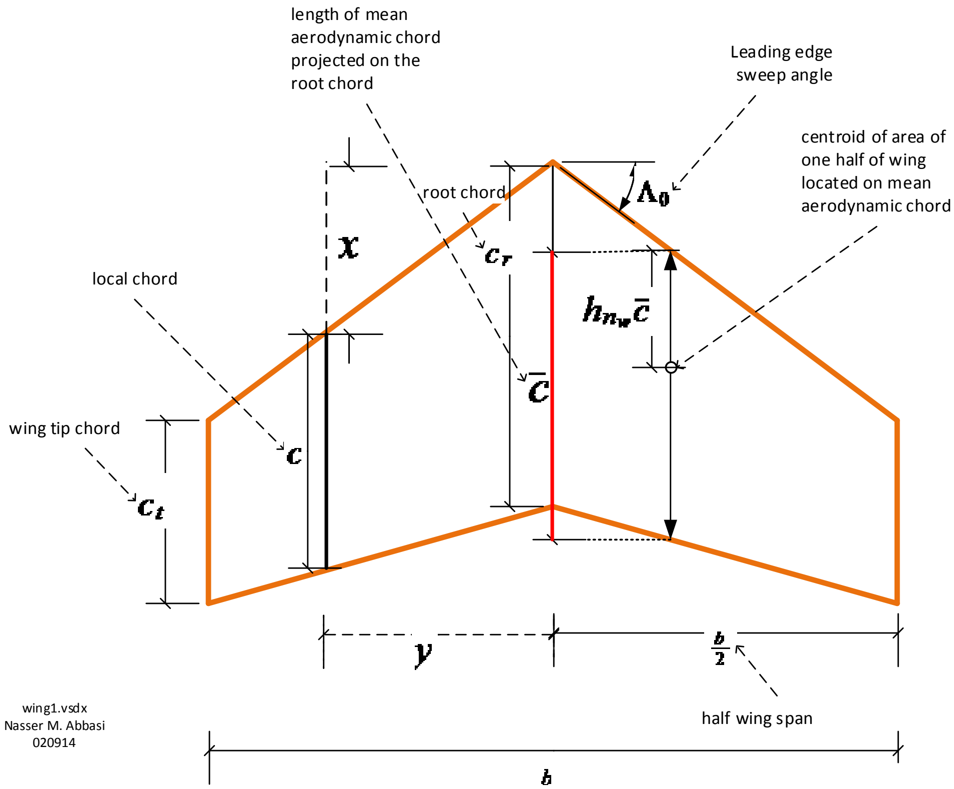


Figure 2.3: problem 1 part (b)

The wing area S is the mean of the root chord length and the wing tip chord multiplied by the wing span b, therefore

$$\begin{aligned} S &= \left(\frac{c_r + c_t}{2}\right) b \\ &= \left(\frac{25+12}{2}\right) 150 \\ &= 2775\,\text{ft}^2 \end{aligned}$$

The aspect ratio is

$$\mathscr{A} = \frac{b^2}{S} = \frac{150^2}{2775} = 8.108$$

The taper ratio λ is

$$\lambda = \frac{c_t}{c_r} = \frac{12}{25} = 0.48$$

To find the length of the mean aerodynamic chord $\bar{c}$, equation (C.3,3) in the textbook was used with $n = 0$. In using this equation, x, which is the distance from wing tip to the start of the $\bar{c}$ chord was found first using equation (C.3,1) as follows

$$\begin{aligned}x &= \left(\frac{b}{2}\right)\left(\frac{1}{3}\right)\frac{1+2\lambda}{1+\lambda}\tan(\Lambda_0)\\ &= \left(\frac{150}{2}\right)\left(\frac{1}{3}\right)\frac{1+2\times 0.48}{1+0.48}\tan(26^\circ)\\ &= 16.148\,\text{ft}\end{aligned}$$

Now equation (C3.3) was used since x is known

$$\begin{aligned}\frac{x}{\bar{c}} &= \frac{(1+2\lambda)(1+\lambda)}{8(1+\lambda+\lambda^2)}\mathscr{A}\tan(\Lambda_0)\\ \bar{c} &= \left(\frac{x}{\mathscr{A}\tan(\Lambda_0)}\right)\frac{8(1+\lambda+\lambda^2)}{(1+2\lambda)(1+\lambda)}\\ &= \frac{16.148}{8.108\tan(26^\circ)}\frac{8(1+0.48+0.48^2)}{(1+2\times 0.48)(1+0.48)}\\ &= 19.2613\,\text{ft}\end{aligned}$$

This value for $\bar{c}$ was verified using figure C.2 on page 261 based on the use of

$$\bar{c} = \frac{2}{3}c_r\frac{1+\lambda+\lambda^2}{1+\lambda}$$

The result matched that found using equation (C3.3) above.

2.1.1.3 Part(c)

The location of mean aerodynamic center on the full wing is given by the coordinates $(\bar{x},\bar{y},\bar{z})$ in the local frame of reference. For the full wing

$$\bar{y} = 0$$

And

$$\begin{aligned}\bar{x} &= x + \frac{1}{4}\bar{c}\\ &= 16.148 + \left(\frac{1}{4}\right)19.261\\ &= 20.964\,\text{ft}\end{aligned}$$

To obtain $\bar{z}$, (C.1,4) in appendix C was used

$$\bar{z} = \frac{2}{C_L S}\int_0^{\frac{b}{2}} C_{L_\alpha} cz\,dy \tag{C.1,4}$$

From the problem $C_{L_\alpha} = C_L$ as the lift coefficient is uniform. Therefore the above simplifies to

$$\bar{z} = \frac{2}{S}\int_0^{\frac{b}{2}} cz\,dy \tag{C.1,4}$$

The value for $c(y)$ in the above integral is given [1] by the following

$$c(y) = \frac{2s}{(1+\lambda)b}\left(1 - \frac{2(1-\lambda)}{b}y\right)$$

Given that $z(y) = y\tan(\Gamma)$ where Γ is the dihedral angle which is 4° and $S = 2775\,\text{ft}^2$ is the wing area, and $\lambda = 0.48$, (C.1,4) becomes

$$\begin{aligned}\bar{z} &= \frac{2}{S}\int_0^{\frac{b}{2}} \frac{2s}{(1+\lambda)b}\left(1 - \frac{2(1-\lambda)}{b}y\right) y\tan(\Gamma)\,dy \\ &= \frac{2}{2775}\int_0^{\frac{150}{2}} \frac{2(2775)}{(1+0.48)150}\left(1 - \frac{2(1-0.48)}{150}y\right) y\tan(4^\circ)\,dy \\ &= 2.315\,\text{ft}\end{aligned}$$

2.1.1.4 Appendix for part (c)

This section is extra as it finds the $\{\bar{x}, \bar{y}, \bar{z}\}$ for half wing, and not the full wing as the problems asks for. This was done to practice the use of appendix C integrals.

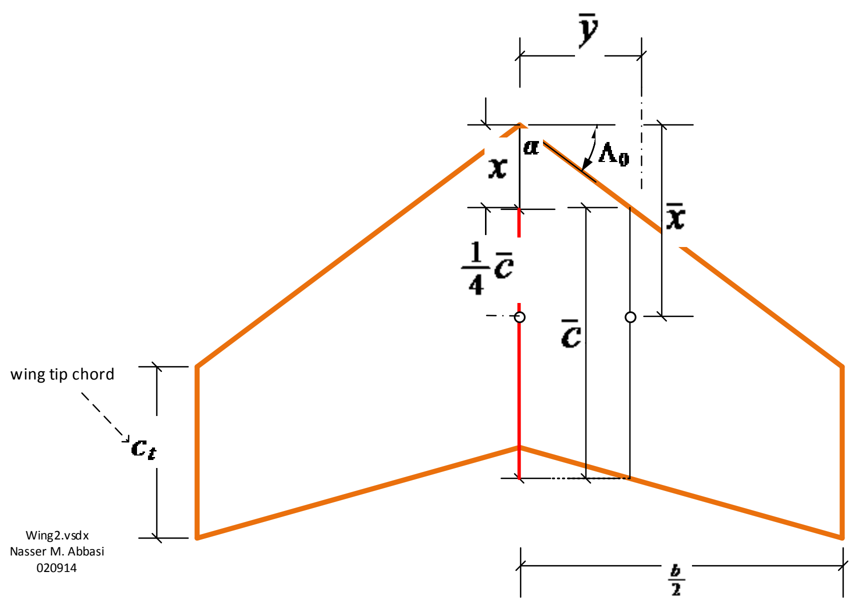


Figure 2.4: problem 1 part (e)

In finding $\{\bar{x}, \bar{y}, \bar{z}\}$, equations (C.1,2,3,4) in appendix C are used. The expression for c in these integrals is given by

$$c(y) = \frac{2s}{(1+\lambda)b}\left(1 - \frac{2(1-\lambda)}{b}y\right)$$

[1] http://en.wikipedia.org/wiki/Chord_aircraft

From (C1.2)

$$\begin{aligned}\overline{x} &= \frac{2}{C_L S}\int_0^{\frac{b}{2}} C_{L_\alpha} cx\, dy\\ &= \frac{2}{S}\int_0^{\frac{b}{2}} cx\, dy\end{aligned}$$

Where $x = \left(\frac{1}{4}\right)c + y\tan(\Gamma)$ as seen in the above diagram. substituting these in the above integral results in

$$\overline{x} = \frac{2}{S}\int_0^{\frac{b}{2}} \frac{2s}{(1+\lambda)\,b}\left(1-\frac{2\,(1-\lambda)}{b}y\right)\left(\frac{1}{4}\left(\frac{2s}{(1+\lambda)\,b}\left(1-\frac{2\,(1-\lambda)}{b}y\right)\right)+y\tan(\Gamma)\right)dy$$

Giving numerical values for all the variables in the above gives

$$\overline{x} = 20.9632\,\text{ft}$$

Similarly for $\overline{y}$

$$\begin{aligned}\overline{y} &= \frac{2}{C_L S}\int_0^{\frac{b}{2}} C_{L_\alpha} cy\, dy\\ &= \frac{2}{S}\int_0^{\frac{b}{2}} cy\, dy\\ &= \frac{2}{S}\int_0^{\frac{b}{2}} \frac{2s}{(1+\lambda)\,b}\left(1-\frac{2\,(1-\lambda)}{b}y\right)y\, dy\end{aligned}$$

Substituting numerical values for all the variables above gives

$$\overline{y} = 33.108\,\text{ft}$$

This value can also be found based on geometry using the above diagram as follows

$$\begin{aligned}\tan(\alpha) &= \frac{\overline{y}}{x}\\ \overline{y} &= x\tan(90^\circ-\Lambda_0)\\ &= 16.148\tan(90^\circ-26^\circ)\\ &= 33.108\,\text{ft}\end{aligned}$$

And finally for $\overline{z}$

$$\overline{z} = \frac{2}{S}\int_0^{\frac{b}{2}} cz\, dy$$

Where $z = y\tan(\Gamma)$ hence the above becomes

$$\overline{z} = \frac{2}{S}\int_0^{\frac{b}{2}} \left(1-\frac{2\,(1-\lambda)}{b}y\right)\left(y\tan\Gamma\right)dy$$

Substituting numerical values for all the variables above gives

$$\overline{z} = 2.315\,14\,\text{ft}$$

The diagram below was drawn to scale in Mathematica using the actual values found. This diagram shows the aerodynamic center for the full wing as well for the half wing.

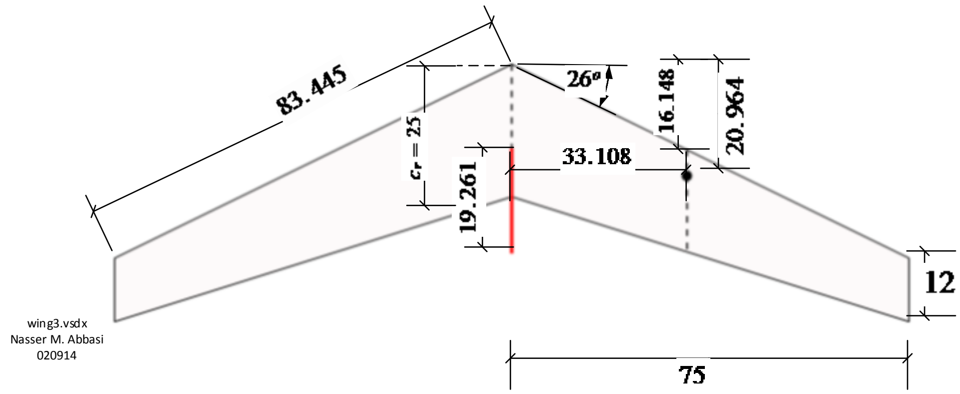


Figure 2.5: detailed wing dimensions

```
(*calculations used in the above*)
chordLength[y_, s_, b_, lambda_] := (2 s)/((1 + lambda) b) (1-(2(1 - lambda))/b y)
Clear[y];
s = 2775;
b = 150;
lambda = 0.48;
c = chordLength[y, s, b, lambda]
    25. (1 - 0.00693333 y)
cBar = 2/s Integrate[c^2, {y, 0, b/2}]
    19.2613
yBar = 2/s Integrate[c y, {y, 0, b/2}]
    33.1081
zBar = 2/s Integrate[c y Tan[4 Degree], {y, 0, b/2}]
    2.31514
xBar = 2/s Integrate[c ((1/4) c + y*Tan[26 Degree]) , {y, 0, b/2}]
    20.9632
```

2.1.1.5 Part(d)

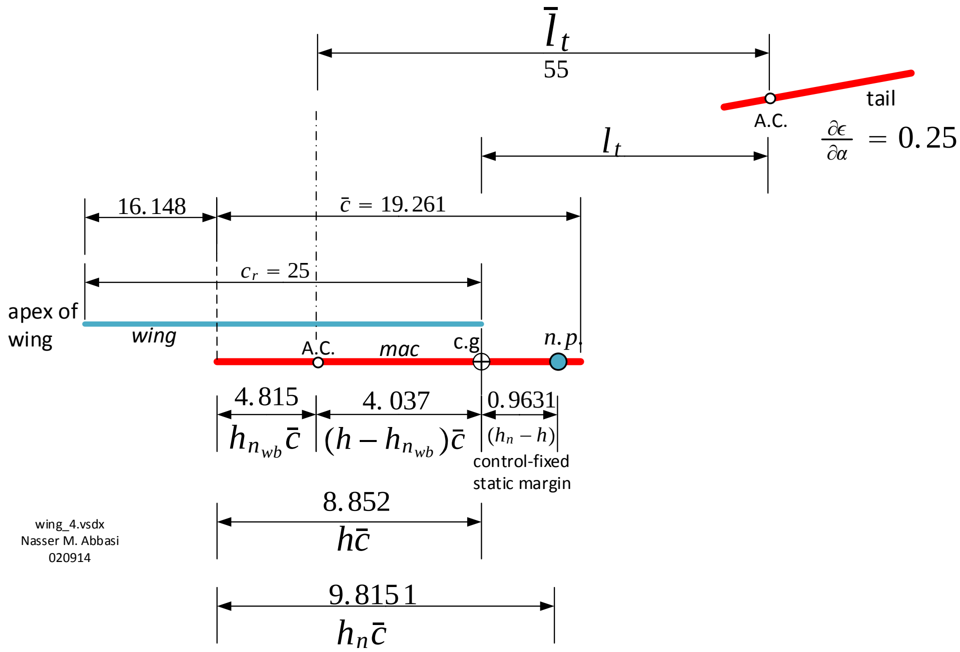


Figure 2.6: problem 1 part(d)

From the above diagram (not drawn to scale)

$$h_n = \frac{9.8151}{19.261} = 0.50958$$

From equation (2.3,23)[2]

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\overline{V}_H\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial\alpha} \tag{1}$$

Ignoring power plant effects and using a from (2.3,18) given by

$$a = a_{wb}\left(1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right) \tag{2}$$

Substituting (2) into (1) and using $\overline{V}_H = \frac{l_t}{\bar{c}}\frac{S_t}{S}$ results in

$$h_n = h_{n_{wb}} + \frac{a_t}{a_{wb}\left(1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)}\frac{l_t}{\bar{c}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)$$

Since $a_t = a_{wb}$ the above becomes

$$h_n = h_{n_{wb}} + \frac{S}{S + S_t\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)}\frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)$$

Substituting the following numerical values $S = 2775\,\text{ft}^2, \bar{c} = 19.2613\,\text{ft}, \bar{l}_t = 55\,\text{ft}, \frac{\partial\epsilon}{\partial\alpha} = 0.25$

[2] Etkin and Ried, Dynamic of Flight, stability and control, third edition. Page 32.

the above becomes

$$0.50958 = \frac{1}{4} + \frac{2775}{(2775 + 0.75S_t)} \frac{55}{19.2613} \frac{0.75S_t}{2775}$$

Solving for S_t gives the area of tail

$$S_t = 367\,\text{ft}^2$$

2.1.2 Problem 2

2.1.2.1 Part(d)

2. In the handouts on the course website, two expressions are given for the coefficient of pitching moment about the center of gravity:

$$C_m = \overline{C}_{m0} + C_{m_\alpha} \alpha_{wb}$$

with definitions for $\overline{C}_{m0}$ and C_{m_α}

and

$$C_m = C_{m0} + C_{m_\alpha} \alpha$$

with definitions for C_{m0}, C_{m_α}, and α (this latter in your notes). Show that the two expressions for C_{m_α} can be reduced to the same form.

Figure 2.7: problem 2 description

The expression for C_{m_α} in the first equation above is given by

$$C_{m_\alpha} = a_{wb}\left(h - h_{n_{wb}}\right) - a_t V_H \left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha} \tag{1}$$

While the expression for C_{m_α} in the second equation is given by

$$C_{m_\alpha} = a\left(h - h_{n_{wb}}\right) - a_t \overline{V}_H \left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha} \tag{2}$$

The above expressions are given in the class handout on page 32 and 34.

The problem asks to show that these two expression are the same. Starting from (2) in order to show it can be rewritten as (1). For this purpose, the following two definitions are used

$$a = a_{wb}\left(1 + \frac{a_t}{a_{wb}} \frac{S_t}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right) \tag{3}$$

$$\overline{V}_H = \frac{\bar{l}_t}{\bar{c}} \frac{S_t}{S}$$

Since $\bar{l}_t = l_t + \left(h - h_{n_{wb}}\right)\bar{c}$ the above becomes

$$
\begin{aligned}
\overline{V}_H &= \frac{l_t + \left(h - h_{n_{wb}}\right)\bar{c}}{\bar{c}}\frac{S_t}{S} \\
&= \frac{l_t}{\bar{c}}\frac{S_t}{S} + \left(h - h_{n_{wb}}\right)\frac{S_t}{S} \qquad (4)
\end{aligned}
$$

Substituting Eqs (3,4) into Eq (2) gives

$$
\begin{aligned}
C_{m_\alpha} &= a_{wb}\left(1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)\left(h - h_{n_{wb}}\right) - a_t\left[\frac{l_t}{\bar{c}}\frac{S_t}{S} + \left(h - h_{n_{wb}}\right)\frac{S_t}{S}\right]\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) + \frac{\partial C_{m_p}}{\partial\alpha} \\
&= \left(a_{wb} + a_t\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)\left(h - h_{n_{wb}}\right) - \left[a_t\frac{l_t}{\bar{c}}\frac{S_t}{S} + a_t\left(h - h_{n_{wb}}\right)\frac{S_t}{S}\right]\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) + \frac{\partial C_{m_p}}{\partial\alpha} \\
&= a_{wb}\left(h - h_{n_{wb}}\right) + \overbrace{a_t\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\left(h - h_{n_{wb}}\right)} - a_t\frac{l_t}{\bar{c}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) - \overbrace{a_t\left(h - h_{n_{wb}}\right)\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)} + \frac{\partial C_{m_p}}{\partial\alpha}
\end{aligned}
$$

The second term and the fourth term in the above cancel each others resulting in

$$
C_{m_\alpha} = a_{wb}\left(h - h_{n_{wb}}\right) - a_t\overbrace{\frac{l_t}{\bar{c}}\frac{S_t}{S}}^{V_H}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) + \frac{\partial C_{m_p}}{\partial\alpha}
$$

In the above $\frac{l_t}{\bar{c}}\frac{S_t}{S}$ is V_H hence the above becomes

$$
C_{m_\alpha} = a_{wb}\left(h - h_{n_{wb}}\right) - a_t V_H\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) + \frac{\partial C_{m_p}}{\partial\alpha}
$$

Comparing the above to (1), it can be seen it is same as (2).

2.1.3 Problem 3

2.5 The following data apply to a $\frac{1}{25}$ scale wind tunnel model of a transport airplane. The full-scale mass of the aircraft is 1,552.80 slugs (22,680 kg). Assume that the aerodynamic data can be applied at full-scale. For level unaccelerated flight at $V = 239$ knots (123 m/s) of the full-scale aircraft, under the assumption that propulsion effects can be ignored,

(a) Find the limits on tail angle i_t and CG position h imposed by the conditions $C_{m_0} > 0$ and $C_{m_\alpha} < 0$.

(b) For trimmed flight with $\delta_e = 0$, plot i_t vs. h for the aircraft and indicate where this line meets the boundaries of part (a).

Geometric Data

Wing area, S	1.50 ft^2 (0.139 m^2)
Wing mean aerodynamic chord, $\bar{c}$	6.145 in (15.61 cm)
$\bar{l}_t$	15.29 in (38.84 cm)
Tail area, S_t	0.368 ft^2 (0.0342 m^2)

Aerodynamic Data

a_{wb}	0.077/deg
a_t	0.064/deg
ϵ_0	0.72°
$\frac{\partial \epsilon}{\partial \alpha}$	0.30
$C_{m_{ac_{wb}}}$	−0.018
$h_{n_{wb}}$	0.25
ρ	2.377×10^{-3} slugs/ft^3 (1.225 kg/m^3)

Figure 2.8: problem 3 description

2.1.3.1 Part (a)

C_{m_0} is given by (2.3,22) on page 32 of the textbook.

$$C_{m_0} = C_{m_{ac_{wb}}} + a_t \overline{V}_H (\epsilon_0 + i_t) \left[1 - \frac{a_t}{a} \frac{S_t}{S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right] \tag{0}$$

Where (using SI units)

$$\begin{aligned} \overline{V}_H &= \frac{\bar{l}_t}{\bar{c}} \frac{S_t}{S} \\ &= \frac{38.84}{15.61} \frac{0.0342}{0.139} \\ &= 0.6122 \end{aligned}$$

And

$$\begin{aligned} a &= a_{wb}\left(1+\frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1-\frac{\partial\epsilon}{\partial\alpha}\right)\right) \\ &= 0.077\left(1+\frac{0.064}{0.077}\frac{0.0342}{0.139}(1-0.3)\right) \\ &= 0.088\,02\ \text{deg}^{-1} \end{aligned}$$

Using the numerical values given by (0) the above becomes

$$\begin{aligned} C_{m_0} &= -0.018+0.064\,(0.6122)\,(0.72+i_t)\left(1-\frac{0.064}{0.08802}\frac{0.0342}{0.139}(1-0.3)\right) \\ &= 0.03427\,i_t+0.006677 \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} C_{m_0} &> 0 \\ 0.03427\,i_t+0.006677 &> 0 \\ i_t &> \frac{-0.006677}{0.03427} \\ &> -0.194\,84^\circ \end{aligned}$$

C_{m_α} is given by

$$C_{m_\alpha} = a\left(h-h_{n_{wb}}\right) - a_t\overline{V}_H\left(1-\frac{\partial\epsilon}{\partial\alpha}\right) + \overbrace{\frac{\partial C_{mp}}{\partial\alpha}}^{0}$$

Hence

$$\begin{aligned} C_{m_\alpha} &= 0.08802\left(h-h_{n_{wb}}\right) - a_t\,(0.6122)\left(1-\frac{\partial\epsilon}{\partial\alpha}\right) \\ &= 0.08802\,(h-0.25) - 0.064\,(0.6122)\,(1-0.3) \end{aligned}$$

Therefore

$$C_{m_\alpha} = 0.08802\,h - 0.04943 \tag{2}$$

Hence

$$\begin{aligned} C_{m_\alpha} &< 0 \\ 0.08802h-0.04943 &< 0 \\ h &< \frac{0.04943}{0.08802} < 0.562 \end{aligned}$$

2.1.3.2 Part (b)

$$C_m = C_{m_0} + C_{m_\alpha}\alpha$$

But at trim $C_m = 0$, hence at trim the above becomes

$$C_{m_0} + C_{m_\alpha}\alpha_{trim} = 0 \tag{3}$$

We can find α_{trim} since $(C_L)_{trim} = a\alpha_{trim}$ and we know a which is C_{L_α} from part (a). Hence we just need to find C_L at trim. But

$$(C_L)_{trim} = \frac{L}{\frac{1}{2}\rho V^2 S} = \frac{W}{\frac{1}{2}\rho V^2 S}$$

where at trim the lift L is equal to the weight of the aircraft W. Therefore, since $\rho = 1.225\,\mathrm{kg\,m^{-3}}$, $V = 123\,\mathrm{m\,s^{-1}}$ and at trim $L = W = mg = 22680(9.8)$, and the scaled wing area is $S = (0.139)\,25^2 = 86.875\,\mathrm{m}^2$ then the above becomes

$$\begin{aligned}(C_L)_{trim} &= \frac{22680\,(9.8)}{\frac{1}{2}\,(1.225)\left(123^2\right)(86.875)} \\ &= 0.27609\end{aligned}$$

using $a = 0.088\,02^\circ$ which was found from part (a), the angle of attack at trim α_{trim} is now found

$$\alpha_{trim} = \frac{(C_L)_{trim}}{a} = \frac{0.2761}{0.08802} = 3.1367^\circ$$

Now that α_{trim} is found, then equation (3) is used to find the following equation

$$\begin{aligned} C_{m_0} + C_{m_\alpha}\alpha_{trim} &= 0 \\ \overbrace{(0.03427\,i_t + 0.006677)}^{C_{m_0}\text{ from part (a)}} + \overbrace{(0.08802h - 0.04943\,)}^{C_{m_\alpha}\text{ from part(a)}}3.1367 &= 0 \\ 0.27609h + 0.03427i_t - 0.14837 &= 0 \end{aligned}$$

Solving for i_t as a function of h gives

$$\begin{aligned} i_t &= \frac{0.1484 - 0.2761\,h}{0.0343} \\ &= 4.3294 - 8.0563\,h \end{aligned}$$

The following is a plot in a small region around $i_t = -0.19^\circ$ and $h = 0.56$

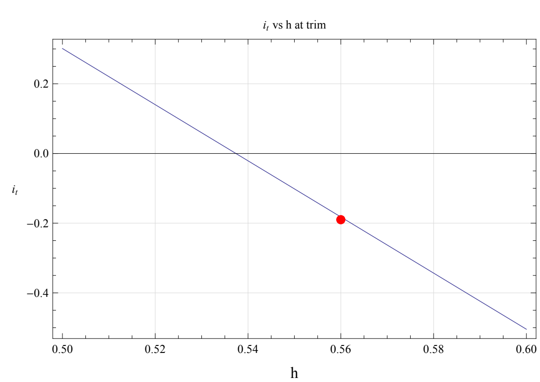


Figure 2.9: problem 3 part b

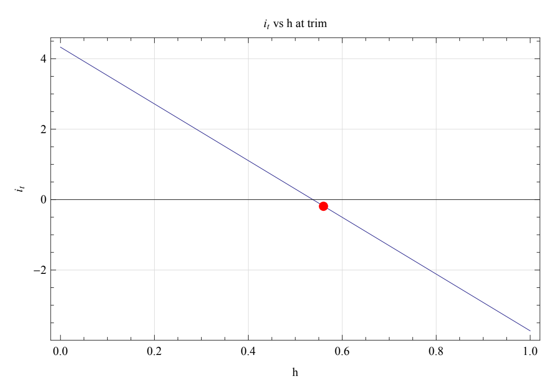


Figure 2.10: Plot for $h = 0 \cdots 1$ showing location $i_t = -0.19^\circ$ and $h = 0.56$

For static stability, $i_t > -0.19^\circ$ and $h < 0.562$ as was obtained above. This is the value of the above line to the *left* of the shown small point and *above* the point, which is the limit of static stability.

2.1.4 Problem 4

2.6* The McDonnell Douglas C-17 is a four-engined jet STOL transport airplane.

(a) Find A and $\bar{c}$ for the wing using the geometrical data and Appendix C.

(b) Use Appendix B to estimate a_w, the wing lift curve slope, assuming that $\beta = 1$ and $\kappa = 1$.

(c) If $a_t = 0.068/\text{deg}$ and $a_{wb} = a_w$, find the lift curve slope, a, of the aircraft. Assume $\frac{\partial \epsilon}{\partial \alpha} = \frac{2a_w}{\pi A}$ (with a_w expressed in rad^{-1}).

(d) Find C_{m_α} for the case where $l_t = \bar{l}_t = 92$ ft (28.04 m). Ignore propulsion effects.

*Problem courtesy of Professor E. K. Parks, University of Arizona.

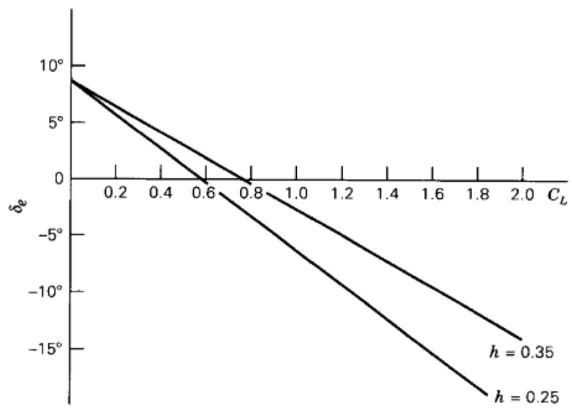


Figure 2.30 Trim data for Exercise 2.6.

(e) From the experimental curves of Figs. 2.29 and 2.30 and the given geometry, find $C_{m_{\delta_e}}$ and h_n. Find C_{m_α} for $h = 0.30$.

Geometric Data

Wing area, S	3,800 ft^2 (353.0 m^2)
Wing span, b	165 ft (50.29 m)
Root chord, c_r	37.3 ft (11.37 m)
Tip chord, c_t	8.8 ft (2.68 m)
$\frac{1}{4}$ chord line sweep, Λ	25°
$\frac{1}{2}$ chord line sweep, $\Lambda_{c/2}$	22°
Tail area, S_t	870 ft^2 (80.83 m^2)

Figure 2.11: problem 4 description

2.1.4.1 Part(a)

The aspect ratio is (SI units are used)

$$\mathscr{A} = \frac{b^2}{S} = \frac{50.29^2}{353} = 7.1645$$

The taper ratio λ is

$$\lambda = \frac{c_t}{c_r} = \frac{2.68}{11.37} = 0.23571$$

Using table C.1 in appendix C of the textbook, page 359

$$\begin{aligned}\bar{c} &= \frac{2}{3} c_r \frac{1 + \lambda + \lambda^2}{1 + \lambda} \\ &= \frac{2}{3}(11.37)\frac{1 + 0.236 + 0.236^2}{1 + 0.236} \\ &= 7.921\,\text{m}\end{aligned}$$

2.1.4.2 Part (b)

For this part, figure B.1-2 on page 322 was used. This figure is shown below

322 *Appendix B. Data for Estimating Aerodynamic Derivatives*

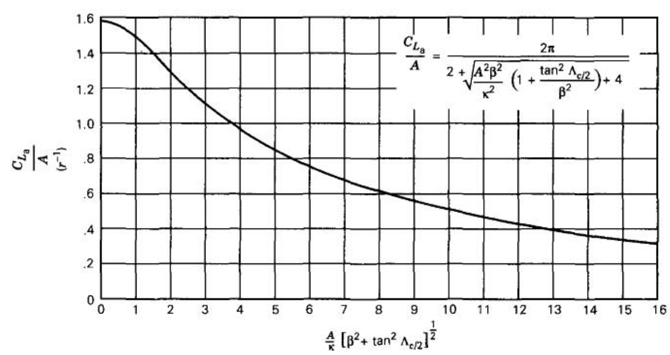


Figure B.1,2 Subsonic wing lift-curve slope.

$$a_w = C_{L_{w_\alpha}} = \frac{\partial C_{L_w}}{\partial \alpha}$$

In the above figure, β is the Prandtl-Glauert compressibility factor and $\kappa = \frac{\beta C_{l_\alpha}}{2\pi}$ where C_{l_α} is the 2D airfoil lift-curve slope and C_{L_w}.

The half chord sweep angle is $\Lambda_{\frac{1}{2}} = 22^\circ$. The problem asks to use the expression in the figure inset to find C_{L_α}

$$\frac{C_{L_\alpha}}{\mathscr{A}} = \frac{2\pi}{2 + \sqrt{\frac{\mathscr{A}^2\beta^2}{\kappa^2}\left(1 + \frac{\tan\left(\Lambda_{\frac{1}{2}}\right)^2}{\beta^2}\right) + 4}}$$

$$\frac{C_{L_\alpha}}{7.1645} = \frac{2\pi}{2 + \sqrt{\frac{\left(7.1645^2\right)\left(1^2\right)}{1^2}\left(1 + \frac{\tan\left(22^0\right)^2}{1^2}\right) + 4}}$$

$$= 0.63247$$

Hence

$$\begin{aligned} C_{L_\alpha} &= 7.165 \times 0.633 \\ &= 4.531\,\text{rad}^{-1} \\ &= 0.079\,\text{deg}^{-1} \end{aligned}$$

The angle C_L makes with the horizontal is $\arctan(4.531) = 1.354\,\text{rad} = 77.556^\circ$

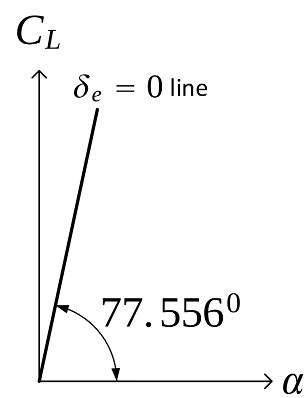


Figure 2.12: plot for problem 4 part b

Therefore

$$a_w = 4.531\,\text{rad}^{-1}$$

2.1.4.3 Part (c)

The lift curve slope of the aircraft a is given by

$$a = a_{wb}\left(1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right) \tag{1}$$

Using $a_t = 0.068\ \text{deg}^{-1}$, $a_{wb} = a_w = 0.079\ \text{deg}^{-1}$ and

$$\begin{aligned}\frac{\partial\epsilon}{\partial\alpha} &= \frac{2a_w}{\pi\mathscr{A}}\\ &= \frac{2(4.531)}{7.165\pi}\\ &= 0.403\end{aligned}$$

From Eq (1) we find

$$\begin{aligned}a &= 0.079\left(1+\left(\frac{0.068}{0.079}\right)\left(\frac{80.83}{353}\right)(1-0.403)\right)\\ &= 0.0883\,\text{deg}^{-1}\\ &= 5.059\,\text{rad}^{-1}\end{aligned}$$

2.1.4.4 Part(d)

$$C_{m_\alpha} = a\left(h-h_{n_{wb}}\right) - a_t\frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S}\left(1-\frac{\partial\epsilon}{\partial\alpha}\right)$$

The problem says that $\bar{l}_t = l_t$. This implies that the distance between aerodynamic center (a.c.) and the center of gravity (c.g.) of the aircraft is zero. This means $\left(h-h_{n_{wb}}\right) = 0$. Therefore

$$C_{m_\alpha} = -a_t\frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S}\left(1-\frac{\partial\epsilon}{\partial\alpha}\right)$$

Using $\bar{c} = 7.921\,\text{m}, \bar{l}_t = 28.04\,\text{m}, S_t = 80.83\,\text{m}^2, S = 353\,\text{m}^2$ and using $\frac{\partial\epsilon}{\partial\alpha} = 0.403$ and using $a_t = 0.068\,\text{deg}^{-1}$ found from part(c), then the above gives

$$\begin{aligned}C_{m_\alpha} &= -0.068\frac{28.04}{7.921}\frac{80.83}{353}(1-0.403)\\ &= -0.0329\,\text{deg}^{-1}\\ &= -1.885\,\text{rad}^{-1}\end{aligned}$$

Since $C_{m_\alpha} < 0$ then the airplane is statically stable.

2.1.4.5 Part(e)

From Fig 2.29, $\frac{\partial C_m}{\partial\delta_e}$ can be estimated using $C_m = 0.25$ and the corresponding line for $\delta_e = 0^\circ$ and using $C_m = 0.125$ and its corresponding line for $\delta_e = 5^\circ$. This gives

$$\begin{aligned}\frac{dC_m}{d\delta_e} &= \frac{0.25-0.125}{-0.50}\\ C_{m_{\delta_e}} &= -0.025\,\text{deg}^{-1}\end{aligned}$$

In solving for h_n, figure 2.30 was used. The slope for the $h = 0.35$ line is $-\frac{8}{0.78} = -10.256$ and the slope for the $h = 0.25$ line is $-\frac{8}{0.6} = -13.333$. Using

$$\left(\frac{d\delta_e}{dC_L}\right)_{trim} = -\frac{C_{L_\alpha}}{\Delta}\left(h - h_n\right) \tag{1}$$

Where $\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_\alpha}$. Evaluating Eq (1) for the two given values of h results in two equations

$$-10.256 = -\frac{C_{L_\alpha}}{\Delta}\left(0.35 - h_n\right) \tag{2}$$

$$-13.333 = -\frac{C_{L_\alpha}}{\Delta}\left(0.25 - h_n\right) \tag{3}$$

From (2) $\frac{C_{L_\alpha}}{\Delta} = \frac{10.256}{(0.35-h_n)}$, substituting this in (3) gives

$$\begin{aligned} -13.333 &= -\frac{10.256}{(0.35 - h_n)}\left(0.25 - h_n\right) \\ -12.25\left(0.35 - h_n\right) &= -9.494\left(0.25 - h_n\right) \\ h_n &= 0.6833 \end{aligned}$$

c_{m_α} at $h = 0.3$ is now found. Since

$$c_{m_\alpha} = a\left(h - h_n\right)$$

Where a is found from part(c) as $0.088\,296\,\mathrm{deg}^{-1}$ and $h_n = 0.6833$ therefore

$$\begin{aligned} c_{m_\alpha} &= 0.088296\left(0.3 - 0.6833\right) \\ &= -0.033\,84\,\mathrm{deg}^{-1} \\ &= -1.9389\,\mathrm{rad}^{-1} \end{aligned}$$

$c_{m_\alpha} < 0$ indicates static stability.

2.1.5 Problem 5

2.8* The following data were taken from a flight test of a PA-32R-300 Cherokee-6 airplane.

Altitude (ft)	Altitude (km)	V_E (mph)	V_E (m/s)	Mass (slugs)	Mass (kg)	i_t (deg)	x_{CG} (in)	x_{CG} (cm)
4540	1.384	91.0	40.7	113.4	1656	1.5	93.89	238.5
4560	1.390	109	48.7	113.0	1650	0	93.89	238.5
4700	1.433	126	56.3	112.9	1649	−1.0	93.89	238.5
4580	1.396	155	69.3	112.7	1646	−2.0	93.89	238.5
5320	1.622	89.0	39.8	100.4	1466	4.5	86.82	220.5
4620	1.408	105	46.9	100.2	1463	2.0	86.82	220.5
4740	1.445	123	55.0	100.0	1461	0.3	86.82	220.5
4900	1.494	151	67.5	99.84	1458	−1.0	86.82	220.5
4880	1.487	87.0	38.9	88.51	1293	7.2	80.43	204.3
4820	1.469	103	46.0	88.35	1290	3.5	80.43	204.3
4880	1.487	122	54.5	88.20	1288	1.5	80.43	204.3
4740	1.445	152	68.0	88.04	1286	0	80.43	204.3

The data were taken in trimmed level flight. x_{CG} is the distance of the CG aft of the nose of the aircraft. The aircraft has an all-moving tail and thus i_t is used instead of δ_e to trim the aircraft. The wing area is $S = 174.5$ ft^2 (16.21 m^2).

(a) Plot tail-setting angle, i_t, versus the lift coefficient of the aircraft for each of the three CG locations.

(b) Curve fit the data points in (a) with three straight lines having a common intercept (refer to Fig. 2.18).

(c) Use a graphical technique to find the location of the neutral point (controls fixed) relative to the nose of the aircraft (refer to Fig. 2.21).

Figure 2.13: problem 5 description

2.1.5.1 Part (a)

$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$ and at trim $L = mg$. Using $\rho = 1.225\,\mathrm{kg\,m^{-3}}$ and $S = 16.21\,\mathrm{m^2}$ Values for C_L for the different V_E values given in the table and the corresponding mass are calculated and plotted against the angle i_t

$C_{L_{trim}}$	i_t (degree)
0.986753	1.5
0.686694	0
0.5135	-1.
0.338299	-2
0.913492	4.5
0.656501	2
0.476718	0.3
0.315854	-1
0.843405	7.2
0.601743	3.5
0.428016	1.5
0.274511	0

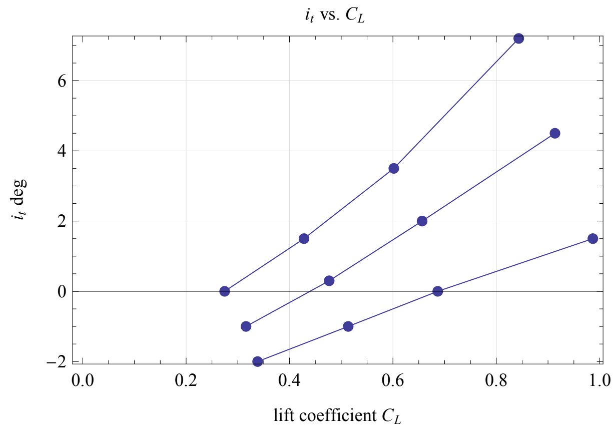


Figure 2.14: table and plot for part a problem 5

2.1.5.2 Part (b)

The three segments are first fitted each to a straight line giving the following plot. The fitted straight lines found by fitting[3] are $\{-3.78307 + 5.3984x, -4.02058 + 9.2621x, -3.76524 + 12.6932x\}$ where $y = i_t \approx -3.85$ is the intercept angle in degrees where $C_L = 0$ for all three lines

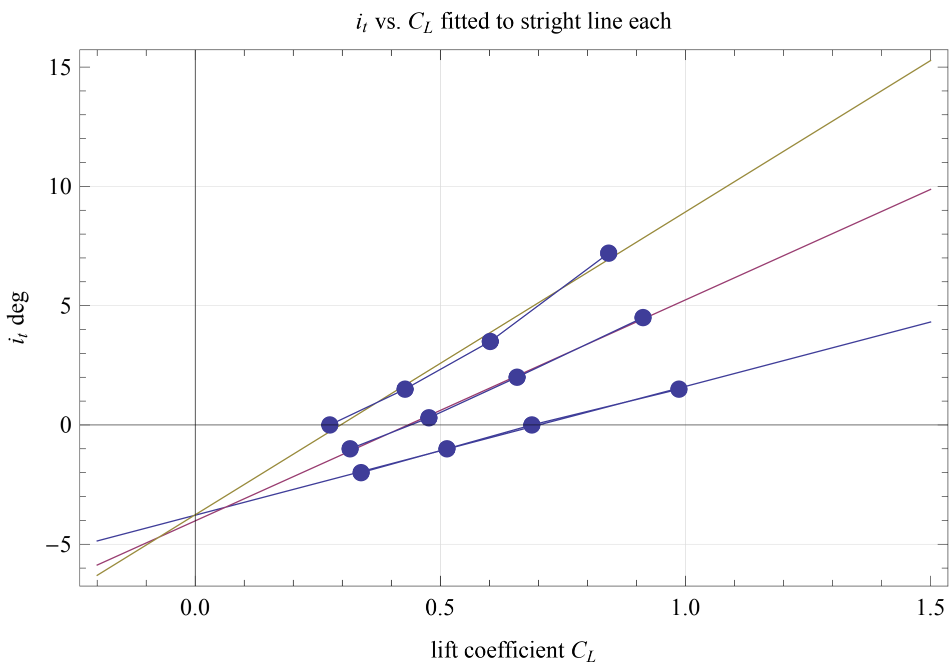


Figure 2.15: problem 5 part b plot

[3]using Mathematica line fit function applied to each set of data

2.1.5.3 Part (c)

It was not clear if one should use the speed effect here and plot $\frac{di_t}{dV_{trim}}$ against h to find the intersection or to use slope given by $\frac{di_t}{dC_{L_{trim}}}$ against h to find the intersection on the x-axis in order to determine h_n.

The second approach is used below since that is what figure 2.31 used which was referred to in the problem above. Therefore, the slope of each of the above lines is found for each h. This results in the following table

slope	h meter (c.g. measured from tip of aircraft)
5.4	2.385
9.26	2.205
12.69	2.043

The data above gives three points. They are plotted and the intersection with h axis is found. This intersection is h_n. Below is the plot of the data in the above table

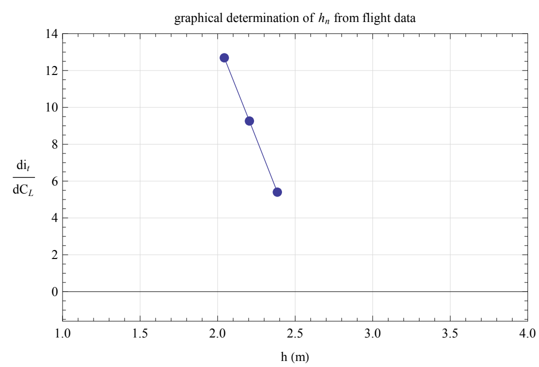


Figure 2.16: problem 5 part c (1)

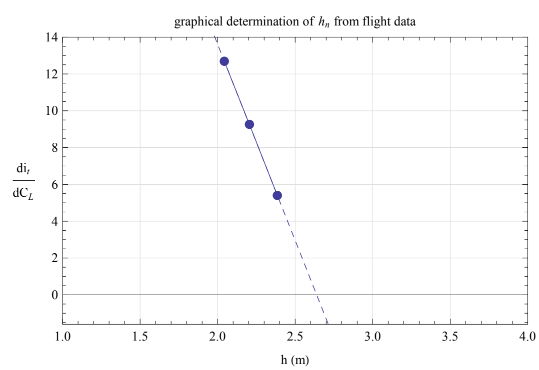
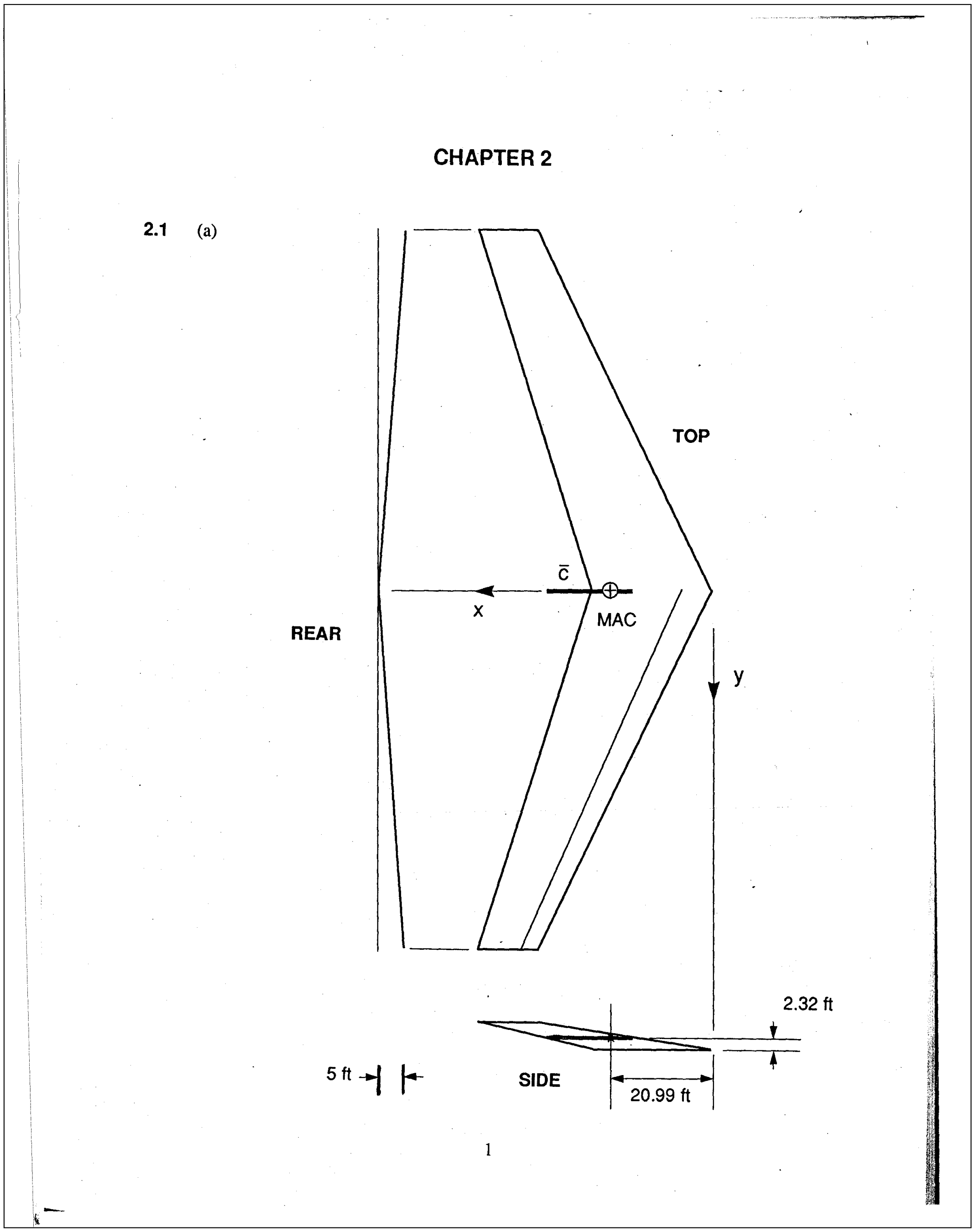


Figure 2.17: Line extended to the x-axis

From the above diagram

$$\begin{aligned} h_n &= 2.65\,\text{meter} \\ &= 265\,\text{cm} \\ &= 8.694\,\text{ft} \end{aligned}$$

2.1.6 HW 1 key solution



Chapter 2

(b) Each wing panel is a trapezoid, thus

$$S = 2\left[\frac{1}{2}(c_t + c_r)\frac{b}{2}\right]$$

$$= \frac{1}{2}[(12 + 25)\ 150]$$

$$= 2{,}775 \text{ ft}^2\ (257.81 \text{ m}^2)$$

From App. C, Sec. C.2

$$A = b^2/S$$

$$= (150)^2/2775$$

$$= 8.11$$

$$\lambda = c_t/c_r$$

$$= 12/25$$

$$= 0.48$$

From Table C.1

$$\bar{c} = \frac{2c_r}{3}\ \frac{1 + \lambda + \lambda^2}{1 + \lambda}$$

$$= \frac{2 \times 25}{3}\ \frac{1 + .48 + .48^2}{1 + .48}$$

$$\bar{c} = 19.26 \text{ ft } (5.87 \text{ m})$$

2

Chapter 2

(c) From Table C.1 for uniform C_{ℓ_a}

$$\bar{y} = \frac{b}{2}\,\frac{1+2\lambda}{3(1+\lambda)}$$

$$= \frac{150}{2}\,\frac{1+2\times .48}{3(1+.48)}$$

$$= 33.11 \text{ ft } (10.09 \text{ m})$$

for the right half wing. [$\bar{y} = 0$ for the complete wing (see Sec. C.1)].

From (C.1,5) for $n = \frac{1}{4}$

$$\bar{x} = \frac{c_r}{4} + \bar{y}\tan\Lambda_{1/4}$$

Use $\bar{y}$ = for the right half wing. From diagram 2.1 $\Lambda_{1/4} = 24°$. Thus

$$\bar{x} = \frac{25}{4} + 33.11\tan 24°$$

$$= 20.99 \text{ ft } (6.40 \text{ m})$$

From (C.1,4)

$$\bar{z} = \frac{2}{C_L S}\int_0^{b/2} C_{\ell_a} c z \, dy$$

but $C_{\ell_a} = C_L$ = constant at given α, thus

Chapter 2

$$\bar{z} = \frac{2}{S}\int_0^{b/2} czdy$$

Now for the right half wing

$$c(y) = 25 - \frac{13}{75}y = 25 - 0.173y$$

$$z(y) = y \tan 4^\circ = 0.0699y$$

$$\int czdy = \int (1.748y - 0.01209y^2)dy$$

$$= \frac{1.748y^2}{2} - \frac{0.01209y^3}{3} + K$$

$$\therefore \quad \bar{z} = \frac{2}{2775}\left[\frac{1.748}{2}y^2 - \frac{0.01209}{3}y^3\right]_0^{75}$$

$$= 2.32 \text{ ft } (0.707 \text{ m})$$

The 1/4-chord point of $\bar{c}$ is placed at $(\bar{x}, \bar{y}, \bar{z})$ as shown in diagram 2.1.

(d) From (2.3,6) the control-fixed static margin is

$$K_n = (h_n - h) \quad (1)$$

From (2.3,23) (ignoring propulsion effects)

4

Chapter 2

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) \tag{2}$$

From (2.2,10)

$$\bar{V}_H = \frac{\bar{\ell}_t S_t}{\bar{c}S} \tag{3}$$

From (2.3,18)

$$a = a_{wb}\left[1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right] \tag{4}$$

But we are given $a_t = a_{wb}$, thus (4) becomes

$$\frac{a}{a_t} = 1 + \frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) \tag{5}$$

From the given fact that $h_{n_{wb}} = h_{n_w}$, diagram 2.1, Fig. 2.12 and the results $\bar{x}$ of part (c), at the aft CG location

NOTE: $X_{CG} \neq \bar{c}h$
$\bar{X} \neq \bar{c}h_{n_{wb}}$
BUT $(X_{CG} - \bar{X}) = \bar{c}(h - h_{n_{wb}})$

$$(X_{CG} - \bar{X}) = (\bar{c}h - \bar{c}h_{n_{wb}}) = 25 - 20.99$$

or

$$(h - h_{n_{wb}}) = \frac{4.01}{19.26} = 0.208 \tag{6}$$

Since $K_n = 0.05$ at the aft CG location (where K_n is a minimum), therefore from (1) + (6)

5

Chapter 2

$$h_n - h_{n_{wb}} = 0.05 + 0.208$$
$$= 0.258 \quad (7)$$

From (2) and (7)

$$0.258 = \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) \quad (8)$$

From (3), (5) and (8)

$$0.258\left[1 + \frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right] = \frac{\bar{\ell}_t S_t}{\bar{c}S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)$$

or

$$0.258\left[1 + \frac{S_t}{2775}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right] = \frac{55S_t}{19.26 \times 2775}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) \quad (9)$$

For $\frac{\partial\varepsilon}{\partial\alpha} = 0.25$, (9) becomes

$$0.258 + 6.973 \times 10^{-5}S_t = 7.7180 \times 10^{-4}S_t$$

or

$$S_t = 367.5 \text{ ft}^2 \ (34.14 \text{ m}^2)$$

2.2 Assuming L = W

$$C_{L_w} = W \Big/ \left(\frac{1}{2}\rho V^2 S\right)$$

$$V = 180 \text{ m/s} = 590.6 \text{ ft/s}$$

6

2.

VERSION 1: $C_{m_\alpha} = a_{wb}(h - h_{n_{wb}}) - a_t V_H \left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha}$

VERSION 2: $C_{m_\alpha} = a(h - h_{n_{wb}}) - a_t \bar{V}_H \left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha}$

↑ NEGLECT

BY DEF.: $a \equiv a_{wb}\left[1 + \frac{a_t}{a_{wb}} \frac{S_t}{S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\right]$ ①

$\bar{V}_H \equiv V_H + (h - h_{n_{wb}}) \frac{S_t}{S}$ ②

PLUG ① & ② INTO VERSION 2:

$$C_{m_\alpha} = a_{wb}\left[1 + \frac{a_t}{a_{wb}} \frac{S_t}{S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\right](h - h_{n_{wb}}) +$$

$$- a_t\left[V_H + (h - h_{n_{wb}})\frac{S_t}{S}\right]\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) =$$

$$= a_{wb}(h - h_{n_{wb}}) - a_t V_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) = \text{VERSION 1}$$

Chapter 2

$$0 = a(h_n - h_{n_{wb}}) - a_t\bar{V}_H\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) + \frac{\partial C_{m_p}}{\partial\alpha}$$

or

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial\alpha} \qquad (2.3,23)$$

2.5 (a) From (2.3,21a) ignoring propulsion effects

$$C_{m_\alpha} = a(h - h_{n_{wb}}) - a_t\bar{V}_H\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right) \qquad (1)$$

From (2.3,22a) ignoring propulsion effects

$$C_{m_0} = C_{m_{ac_{wb}}} + a_t\bar{V}_H(\varepsilon_0 + i_t)\left[1 - \frac{a_t}{a}\frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right] \qquad (2)$$

From (2.2,10)

$$\bar{V}_H = \frac{\bar{\ell}_t S_t}{\bar{c} S} \qquad (3)$$

From (2.3,18)

$$a = a_{wb}\left[1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right] \qquad (4)$$

Evaluating (3)

12

Chapter 2

$$\bar{V}_H = \frac{15.29 \times .368}{6.145 \times 1.50} = 0.6104$$

evaluating (4)

$$a = .077\left[1 + \frac{.064}{.077} \times \frac{.368}{1.50}(1 - .3)\right]$$

$$= .0880/\text{deg}$$

Setting $C_{m_\alpha} < 0$ and evaluating (1)

$$C_{m_\alpha} = .088(h - .25) - .064 \times .6104(1 - .3) < 0$$

or

$$h < 0.5607$$

This is the controls-fixed pitch stiffness boundary. The CG must be forward of the point represented by h = 0.5607.

Setting $C_{m_o} > 0$ and evaluating (2)

$$C_{m_o} = -.018 + .064 \times .6104(.72 + i_t)\left[1 - \frac{.064}{.088} \times \frac{.368}{1.50} \times (1 - .3)\right] > 0$$

or

$$i_t > -0.193°$$

Chapter 2

The tail angle must be greater than -0.193° for the aircraft to be capable of trimmed flight with positive lift and positive pitch stiffness.

(b) For trimmed flight with $\delta_e = 0$, (2.3,20a) gives

$$C_m = C_{m_0} + C_{m_\alpha} \cdot \alpha = 0 \qquad (5)$$

In level unaccelerated flight

$$L = W = a \cdot \alpha \cdot \frac{1}{2} \rho V^2 S$$

thus $\alpha = 2W/(\rho V^2 S a)$

$$W = mg = 22{,}680 \times 9.81 = 222{,}491 \text{ N}$$

full scale $S = (25)^2 \times .139 \text{ m}^2$

$= 86.875 \text{ m}^2$

thus

$$\alpha = \frac{2 \times 222{,}491}{1.225 \times (123)^2 \times 86.875 \times .088}$$

$= 3.141°$

Combining (1), (2) and (5) with the numerical data gives

$$-.018 + .0342\,(.72 + i_t) + [.088(h - .25) - .0273]\,3.141 = 0$$

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or

$$.0342\, i_t + .276\, h = .1482$$

or

$$i_t = 4.33 - 8.07\, h \quad \text{deg.}$$

From diagram 2.5 it can be seen that as h moves rearward (h becomes larger) the plot of possible (h, i_t) hits the i_t boundary just at (0.560, -0.193°). Note that for this example C_{m_0} and C_{m_α} are both approximately zero at h = 0.560.

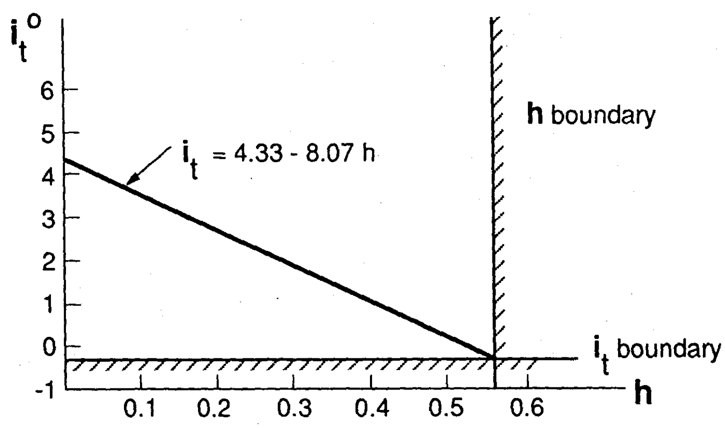

Diagram 2.5

2.6 (a) From App. C, Sec. C.2

$$A = b^2/S$$

$$= \frac{165^2}{3800} = 7.16$$

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$$\lambda = c_t/c_r$$

$$= 8.8/37.3 = 0.236$$

From Table C.1

$$\bar{c} = \frac{2c_r}{3}\,\frac{1+\lambda+\lambda^2}{1+\lambda}$$

$$= \frac{2\times 37.3}{3}\,\frac{1+.236+.236^2}{1+.236}$$

$$= 25.99 \text{ ft } (7.92 \text{ m})$$

(b) Use the equation included on Fig. B.1,2.

Here $\tan\Lambda_{c/2} = \tan 22^\circ$

$$= 0.404$$

The component

$$\sqrt{\frac{A^2\beta^2}{\kappa^2}\left((1+\frac{\tan^2\Lambda_{c/2}}{\beta^2})\right)+4}$$

$$= \left(7.16^2\times\frac{1}{1}\left(1+\frac{0.404^2}{1}\right)+4\right)^{1/2}$$

$$= 7.977$$

Thus

$$a_w = C_{L\alpha} = \frac{2\pi A}{2 + 7.977}$$

$$= \frac{2\pi \times 7.16}{9.977}$$

$$= 4.51/\text{rad}$$

(c)

$$\frac{\partial\varepsilon}{\partial\alpha} = \frac{2a_w}{\pi A}$$

$$= \frac{2 \times 4.51}{\pi \times 7.16}$$

$$= 0.40$$

$$a_t = 0.068/\text{deg} = 3.90/\text{rad}$$

From (2.3,18)

$$a = a_{wb}\left[1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial\varepsilon}{\partial\alpha}\right)\right]$$

$$= 4.51\left[1 + \frac{3.90}{4.51} \times \frac{870}{3800} \times (1 - .40)\right]$$

$$= 5.05/\text{rad}$$

Chapter 2

(d) If $\ell_t = \bar{\ell}_t$ then $V_H = \bar{V}_H$

From (2.2.11)

$$V_H = \bar{V}_H - \frac{S_t}{S}(h - h_{n_{wb}})$$

becomes for $V_H = \bar{V}_H$

$$h = h_{n_{wb}}$$

From (2.3,23) with $h_{n_{wb}} = h$

$$(h - h_n) = \frac{-a_t}{a}\bar{V}_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) \tag{1}$$

From (2.2,10)

$$\bar{V}_H = \frac{\bar{\ell}_t S_t}{\bar{c}S} \tag{2}$$

From (2.3,25c), (1) and (2)

$$C_{m_\alpha} = a(h - h_n) \tag{3}$$

$$= -a_t \bar{V}_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)$$

$$= \frac{-a_t \bar{\ell}_t S_t}{\bar{c} S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)$$

$$C_{m_\alpha} = \frac{-3.90 \times 92 \times 870 \times (1 - 0.40)}{25.99 \times 3800}$$

$$= -1.90/\text{rad}$$

(e) From the C_m vs C_L curves of Fig. 2.29 it can be seen that $C_{m_{\delta_c}}$ is independent of C_L and C_m; δ_c just shifts the lines by a constant $\times\ \delta_c$.

$$C_{m_{\delta_e}} = \frac{-0.56}{20}$$

$$= -0.028/\text{deg}$$

From the $\delta_{e_{trim}}$ vs $C_{L_{trim}}$ curves of Fig. 2.30

$$\frac{d\delta_{e_{trim}}}{dC_{L_{trim}}} = \frac{-22.8}{2} = -11.4^\circ \quad @\ h = .35$$

$$= \frac{-27.8}{1.85} = -15.03^\circ \ @\ h = .25$$

From (2.4,13c)

$$\frac{d\delta_{e_{trim}}}{dC_{L_{trim}}} = \frac{-a}{\det}(h - h_n)$$

or

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$$-\frac{\det}{a}\frac{d\delta_{e_{trim}}}{dC_{L_{trim}}} = (h - h_n)$$

Thus

$$(.35 - h_n) = 11.4\frac{\det}{a}$$

$$(.25 - h_n) = 15.03\frac{\det}{a}$$

or

$$.0877\, h_n = .0307 - \frac{\det}{a} \qquad (4)$$

$$.0665\, h_n = .0166 - \frac{\det}{a} \qquad (5)$$

Now from (4) and (5)

$$.0212\, h_n = .0141$$

or

$$h_n = 0.665$$

From (2.3,25c) and 'a' from part (c)

$$C_{m_\alpha} = a(h - h_n)$$

$$= 5.05(.3 - .665)$$

$$= -1.84/\text{rad}$$

Note that this value of C_{m_α} differs from that in part (d) because the CG location is different.

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$$\alpha_w = 0.133 + \frac{a_w}{a} i_t \text{ rad} \qquad (9)$$

From (1)

$$\frac{a_w}{a} = \left(2 - \frac{\partial\varepsilon}{\partial\alpha}\right)^{-1}$$

$$= (2 - .2)^{-1} = 0.556$$

thus

$$\alpha_w = 0.133 + .556\, i_t \text{ rad} \qquad (10)$$

Substitute (10) into (8)

$$\frac{L_w}{L_t} = \frac{0.133 + .556\, i_t}{0.8(.133 + .556\, i_t) - i_t}$$

$$= \frac{0.133 + 0.556\, i_t}{0.106 - 0.555\, i_t}$$

2.8 (a) $C_L = L / \frac{1}{2}\rho V^2 S$

$= L / \frac{1}{2}\rho_o V_E^2 S$

where ρ_o is sea level atmospheric density.

In level flight L = W, thus

$$C_L = W / \frac{1}{2}\rho_o V_E^2 S$$

Chapter 2

Use the Standard Atmosphere table from App. D to find $\rho_o = 2.3769 \times 10^{-3}$ slug/ft^3.

$$V_E \text{ (in mph)} \times 1.467 = V_E \text{ (in fps)}$$

$$S = 174.5 \text{ ft}^2$$

x_{CG}	altitude (ft)	V_E (fps)	W (lb)	$C_{L_{trim}}$	$i^o_{t_{trim}}$
93.89	4540	133.50	3651	0.99	1.5
	4560	159.90	3639	0.69	0
	4700	184.84	3635	0.51	-1.0
	4580	227.39	3629	0.34	-2.0
86.82	5320	130.56	3233	0.91	4.5
	4620	154.04	3226	0.66	2.0
	4740	180.44	3220	0.48	0.3
	4900	221.52	3215	0.32	-1.0
80.43	4880	127.63	2850	0.84	7.2
	4820	151.10	2845	0.60	3.5
	4880	178.97	2840	0.43	1.5
	4740	222.98	2835	0.27	0

(see diagram 2.8a)

(b) As shown in Fig. 2.18 $\delta_{c_{trim}}$ (here $i_{t_{trim}}$) vs $C_{L_{trim}}$ are straight lines for a given CG location. All the lines intersect at a common point on the $C_{L_{trim}} = 0$ axis. This was true for the present data (see diagram 2.8a)

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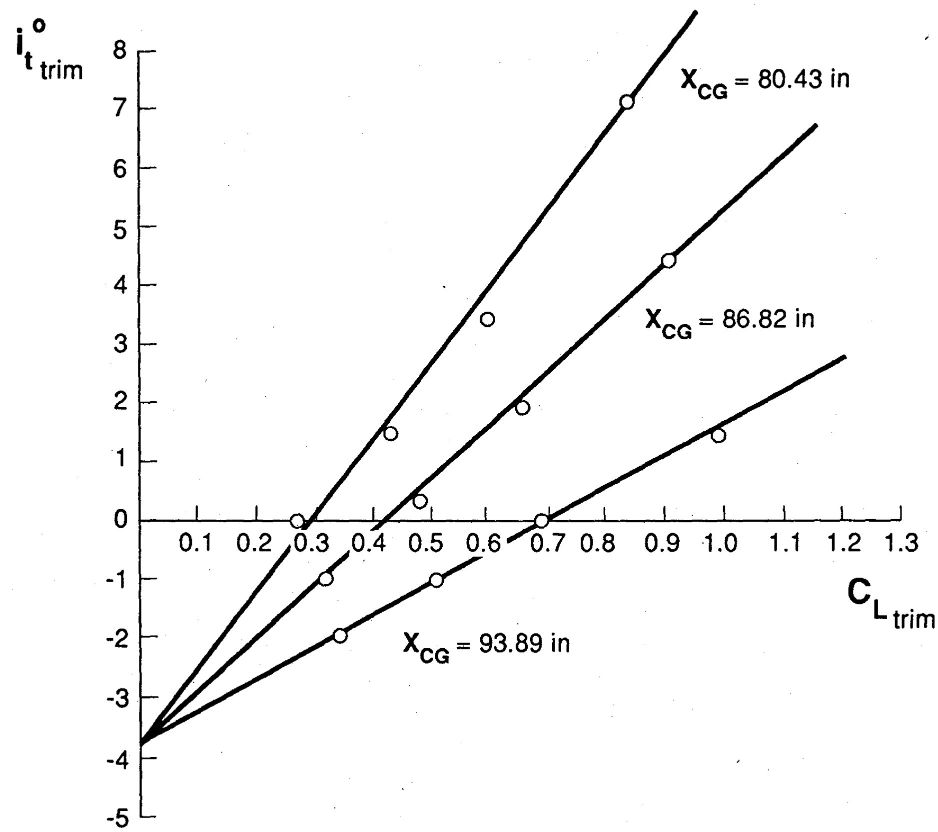

Diagram 2.8a

(c) From (2.4,29) and Fig. 2.21, when the CG is located at the neutral point then $\frac{\partial \delta_{e_{trim}}}{\partial C_{L_{trim}}} = 0 \left(\text{here we have } \frac{\partial i_{t_{trim}}}{\partial C_{L_{trim}}} = 0 \right)$. From our graph find the slopes for the 3 CG locations and make a plot like Fig. 2.21.

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x_{CG} (in)	$\frac{\partial i^o_{t\,trim}}{\partial C_{L_{trim}}}$
93.89	5.50
86.82	9.09
80.43	13.04

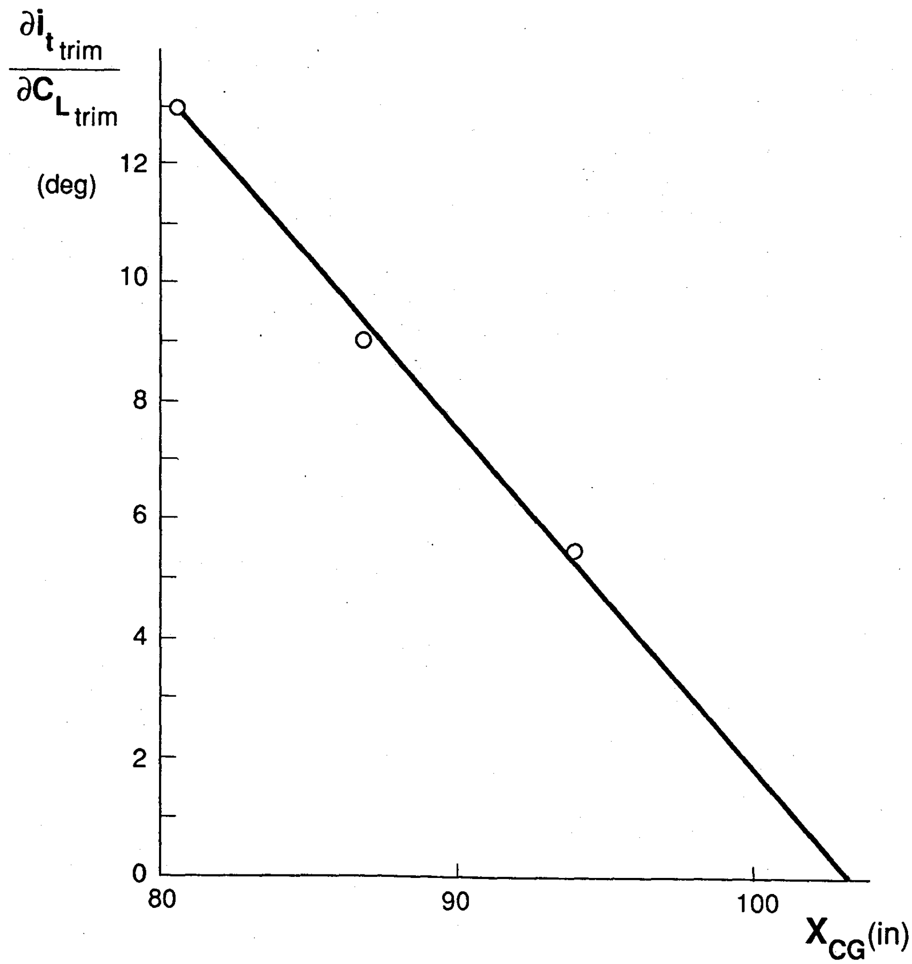

Diagram 2.8c

Since (2.4,29) indicates a linear variation with h a linear fit to the plot was made and the x intercept was (diagram 2.8c)

$$x_{CG} = 103.3 \text{ in}$$

as the location of the CG for $h = h_n$, i.e., the NP location.

2.9 (2.6,11b)

$$(h - h_n') = \frac{1}{a'}\left[a(h - h_n) - \frac{C_{m\delta_e} C_{he\alpha}}{b_2}\right]$$

(2.4,8b)

$$C_{m\delta_c} = -a_e \bar{V}_H + C_{L\delta_c}(h - h_{nwb})$$

Substitute (2.4,8b) into (2.6,11b)

$$(h - h_n') = \frac{a}{a'}(h - h_n) - \frac{C_{he\alpha}}{a'b_2} C_{L\delta_e}(h - h_{nwb}) + \frac{C_{he\alpha}}{a'b_2} a_e \bar{V}_H \qquad (1)$$

(2.6,4b) gives

$$a' = a - \frac{C_{L\delta_e} C_{he\alpha}}{b_2}$$

Use (2.6,4b) to replace "a" in "ah" on the right-hand side of (1) to obtain

$$(h - h_n') = h - \frac{1}{a'}\left(ah_n - \frac{C_{he\alpha}}{b_2} C_{L\delta_c} h_{nwb}\right) + \frac{C_{he\alpha} a_e \bar{V}_H}{a'b_2} \qquad (2)$$

2.2 HW2

2.2.1 Problem 1

4.6 Two airplanes are geometrically similar and have similar mass distributions. Airplane A has a span of 100 ft (30.48 m) and a weight of 100,000 lb (445,000 N). B has 150-ft (47.72 m) span and weighs 225,000 lb (1,001,250 N). Both fly at speeds low enough to neglect Mach number effects, and high enough to neglect Reynolds number effects.

When flying at 400 knots and 20,000-ft (6,096-m) altitude, airplane B has a spiral divergence (a lateral instability) that has a characteristic time of 20 seconds.

(a) At what speed and altitude will A be dynamically similar to B?

(b) What will be the characteristic time of the spiral divergence of A at that speed and altitude?

(c) What is the ratio of the C_L values for the two flight conditions?

Figure 2.18: problem 1 description

NOTE: plane B has a wing span of $150\,\text{ft} = 45.72\,\text{m}$ (the 47.72 value in the problem text is a mistake).

2.2.1.1 part a

Characteristic time (also called the time constant and given the letter τ, is the time it takes the system to reach[4] 63.2% of its final response. The textbook on page 115 says

> Two systems of the same class are dynamically similar when all the π functions of one are numerically equal to those of the other.

Therefore we need to find the π functions and equate them for each airplane to solve for the unknowns. The π functions are listed in page 115 as

$$\pi_1 = \frac{m}{\rho l}$$
$$\pi_2 = \frac{u_0 t}{l}$$
$$\pi_3 = \frac{u_0^2}{lg}$$

These are now derived using Buckingham's π theorem. We assume that the time constant τ is a function of all the system parameters u_0, ρ, m, l, g, t. The Mach number $\mathbf{M}$ and the Reylond's number $\mathbf{Re}$ are not used since the problem says that they can be ignored. The time constant is a function of the remaining system parameters

$$\tau = f(u_0, \rho, m, l, g)$$

In the above t in the RHS was not used as the time constant τ itself has units of time. The system still has 6 overall variables. Since there are 3 standard dimensions given by M, L, T

[4] $\tau = 1 - e^{-1} = 0.632121$

where M is mass and L is length and T is time, and since there are 6 variables, then we need to find $6 - 3 = 3$ independent groups, called π_1, π_2, π_3.

To initiate the process of using Buckingham's π theorem, we start by selecting 3 repeating variables out of original 6 variables to use for finding each one of the three π groups. Let us select $\left\{u_0, \rho, m\right\}$ as the three repeating variables. This choice is not unique, as we could have selected $\{u_0, m, l\}$ just as well or any other set of 3 independent variables out of the six variables to use as the three repeating variables.

The next step is to select one of non-repeating variables, one at a time, and for each one, we set up a system of equations and then match the dimensions. The non-repeating variables are $\left\{\tau, l, g\right\}$.

Starting with the first variable in the above set of the non-repeating variable, which is τ, and using $\approx$ to mean dimensionally equivalent, we set up the first equation as follows (remembering that the repeating variables are $\left\{u_0, \rho, m\right\}$)

$$
\begin{aligned}
\tau &\approx u_0^a \rho^b m^c && (1)\\
[T] &\approx [LT^{-1}]^a [ML^{-3}]^b [M]^c\\
[T] &\approx L^{(a-3b)} T^{-a} M^{b+c}
\end{aligned}
$$

Equating powers of similar dimensions gives 3 equations in 3 the three unknowns $\{a, b, c\}$ powers to solve for. Hence

$$
\begin{aligned}
[T] &\quad 1 = -a\\
[L] &\quad 0 = a - 3b\\
[M] &\quad 0 = b + c
\end{aligned}
$$

Solving the above gives $a = -1, b = -\frac{1}{3}, c = \frac{1}{3}$, therefore (1) becomes

$$
\begin{aligned}
\tau &\approx u_0^{-1} \rho^{-\frac{1}{3}} m^{\frac{1}{3}}\\
&\approx \left(\frac{m}{\rho}\right)^{\frac{1}{3}} \frac{1}{u_0}\\
&\approx \left(\frac{m}{\frac{m}{l^3}}\right)^{\frac{1}{3}} \frac{1}{u_0}\\
&\approx \frac{l}{u_0}
\end{aligned}
$$

Hence the first group is now found

$$
\boxed{\pi_1 = \frac{u_0 \tau}{l}}
$$

Now we find the second group π_2. The next non-repeating variable to use is l, therefore, as above, we set up an equation to solve for the powers

$$\begin{aligned} l &\approx u_0^a \rho^b m^c \qquad (2) \\ [L] &\approx [LT^{-1}]^a [ML^{-3}]^b [M]^c \\ [L] &\approx L^{(a-3b)} T^{-a} M^{b+c} \end{aligned}$$

Equating powers of similar dimensions gives 3 equations in 3 unknowns to solve for

$$\begin{aligned} [T] &\quad 0 = -a \\ [L] &\quad 1 = a - 3b \\ [M] &\quad 0 = b + c \end{aligned}$$

Solving the above gives $a = 0, b = -\frac{1}{3}, c = \frac{1}{3}$. Therefore (2) becomes

$$\begin{aligned} l &\approx \rho^{-\frac{1}{3}} m^{\frac{1}{3}} \\ &\approx \left(\frac{m}{\rho}\right)^{\frac{1}{3}} \end{aligned}$$

Hence

$$\boxed{\pi_2 = \frac{l^3 \rho}{m}}$$

The third and final group π_3 is found by selecting the last non-repeating variable which is g

$$\begin{aligned} g &\approx u_0^a \rho^b m^c \qquad (3) \\ [LT^{-2}] &\approx [LT^{-1}]^a [ML^{-3}]^b [M]^c \\ LT^{-2} &\approx L^{(a-3b)} T^{-a} M^{b+c} \end{aligned}$$

Equating powers of similar dimensions gives 3 equations in 3 unknowns to solve for

$$\begin{aligned} [T] &\quad -2 = -a \\ [L] &\quad 1 = a - 3b \\ [M] &\quad 0 = b + c \end{aligned}$$

Solving the above for a, b, c gives $a = 2, b = \frac{1}{3}, c = -\frac{1}{3}$, therefore

$$\begin{aligned} g &\approx \left(u_0^2 \rho^{\frac{1}{3}} m^{-\frac{1}{3}}\right) \\ &\approx \left(\frac{\rho}{m}\right)^{\frac{1}{3}} u_0^2 \\ &\approx \frac{u_0^2}{l} \end{aligned}$$

Hence

$$\boxed{\pi_3 = \frac{gl}{u_0^2}}$$

These are the 3 non-dimensional groups that should give the same numerical values for both airplanes if the two airplanes are to be dynamically similar.

For airplane B, we are given $(u_0)_B = 400$ knots, $l_B = 150$ ft, $m_B = 225000$ lb and altitude of B is 20000 ft. For airplane A $l_A = 100$ ft, $m_A = 100000$ lb. The problem asks to find $(u_0)_A$ and altitude of A.

The π groups use ρ (density of air) and not altitude, but appendix D in the textbook contains a table to convert from altitude to corresponding air density ρ at that level. Using this table, at altitude 20000 ft the air density is $\rho_B = 1.2673 \times 10^{-3}$ lb $\frac{\text{sec}^2}{\text{ft}^4}$. Now equating the three π groups for both airplanes gives 3 equations

$$
\begin{aligned}
(\pi_1)_A &= (\pi_1)_B \\
(\pi_2)_A &= (\pi_2)_B \\
(\pi_3)_A &= (\pi_3)_B
\end{aligned}
$$

Using the specific expression found for each π results in

$$
\begin{aligned}
\left(\frac{u_0\tau}{l}\right)_A &= \left(\frac{u_0\tau}{l}\right)_B \\
\left(\frac{l^3\rho}{m}\right)_A &= \left(\frac{l^3\rho}{m}\right)_B \\
\left(\frac{gl}{u_0^2}\right)_A &= \left(\frac{gl}{u_0^2}\right)_B
\end{aligned}
$$

For dynamic similarity, both sides of the equations must give the same value. Therefore we substitute the known numerical values for the parameters in these equations and solve for the unknowns

$$
\begin{aligned}
\left(\frac{u_0\tau}{100}\right)_A &= \left(\frac{400 \times 20}{150}\right)_B \\
\left(\frac{100^3\rho}{100000}\right)_A &= \left(\frac{150^3 \times 1.2673 \times 10^{-3}}{225000}\right)_B \\
\left(\frac{100g}{u_0^2}\right)_A &= \left(\frac{150g}{400^2}\right)_B
\end{aligned}
$$

Assuming g is the same for both planes, the above reduces to

$$
\left(\frac{u_0\tau}{100}\right)_A = 53.333 \tag{4}
$$

$$
10\rho_A = 0.01901 \tag{5}
$$

$$
\left(\frac{100}{u_0^2}\right)_A = 9.375 \times 10^{-4} \tag{6}
$$

From (6)

$$
\left(u_0^2\right)_A = \frac{100}{9.375 \times 10^{-4}} = 1.0667 \times 10^5
$$

Hence

$$(u_0)_A = \sqrt{1.0667 \times 10^5}$$
$$= \boxed{326.6 \text{ knot}}$$

To find the air density for airplane A, using (5) above gives

$$10\rho_A = 0.01901$$
$$\rho_A = \frac{0.01901}{10} = 1.901 \times 10^{-3} \text{ lb } \frac{\text{sec}^2}{\text{ft}^4}$$

From appendix D (assuming linear relation between each entries in each row in the table) we can interpolate the altitude for $\rho_A = 1.901 \times 10^{-3}$ lb $\frac{\text{sec}^2}{\text{ft}^4}$

$$\frac{7000}{y} = \frac{1.927 \times 10^{-3}}{1.901 \times 10^{-3}}$$
$$y = \frac{7000 \times 1.901 \times 10^{-3}}{1.927 \times 10^{-3}}$$
$$= \boxed{6905.6 \text{ ft}}$$

2.2.1.2 part b

Using the results from part (a) and from (4)

$$\left(\frac{u_0\tau}{100}\right)_A = 53.333$$

Where $(u_0)_A = 326.6$ knot found in part(a). Hence solving for τ gives

$$\left(\frac{326.6\tau}{100}\right)_A = 53.333$$
$$\tau_A = \frac{53.333 \times 100}{326.6}$$
$$= \boxed{16.330 \text{ sec}}$$

2.2.1.3 part c

$$(C_L)_A = \frac{L_A}{\frac{1}{2}\rho_A \left(u_0^2\right)_A S_A}$$

Assuming static equilibrium then the lift is the same as the airplane weight giving

$$(C_L)_A = \frac{m_A g}{\frac{1}{2}\rho_A \left(u_0^2\right)_A S_A}$$

And similarly for airplane B

$$(C_L)_B = \frac{m_B g}{\frac{1}{2}\rho_B \left(u_0^2\right)_B S_B}$$

Hence the ratio is

$$\frac{(C_L)_A}{(C_L)_B} = \frac{\frac{m_A g}{\frac{1}{2}\rho_A\left(u_0^2\right)_A S_A}}{\frac{m_B g}{\frac{1}{2}\rho_B\left(u_0^2\right)_B S_B}} = \frac{m_A \frac{1}{2}\rho_B \left(u_0^2\right)_B S_B}{m_B \frac{1}{2}\rho_A \left(u_0^2\right)_A S_A}$$

Substituting the numerical values found

$$\begin{aligned}\frac{(C_L)_A}{(C_L)_B} &= \frac{(100000)\,\frac{1}{2}\left(1.2673\times 10^{-3}\right)\left(400^2\right)S_B}{(225000)\,\frac{1}{2}\left(1.901\times 10^{-3}\right)\left(326.6^2\right)S_A}\\ &= 0.444\,43\frac{S_B}{S_A}\end{aligned}$$

The surface area of the airplanes is not given. But $\frac{S_B}{S_A}$ can be taken as similar to $\frac{l_B^2}{l_A^2}$ and now the above becomes

$$\begin{aligned}\frac{(C_L)_A}{(C_L)_B} &= 0.444436\frac{150^2}{100^2}\\ &= 0.99998\\ &\approx \boxed{1}\end{aligned}$$

2.2.2 Problem 2

Problem: Substitute the linear expressions for ΔZ and ΔM into the right side of (4.9,7c) and (4.9,8b) and solve the resulting equations to get the second and third components of (4.9,18).

Solution

The linearized form of Δz and ΔM are given in (4.9,17) in the textbook as

$$\Delta Z = Z_u \Delta u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + \Delta Z_c \quad (4.9,17\ (c))$$

$$\Delta M = M_u \Delta u + M_w w + M_{\dot{w}}\dot{w} + M_q q + \Delta M_c \quad (4.9,17\ (e))$$

(4.9,7 c) and (4.9,8 b) are given by

$$\dot{w} = \frac{\Delta Z}{m} - g\Delta\theta\sin\theta_0 + u_0 q \quad (4.9,7\ c)$$

$$\dot{q} = \frac{\Delta M}{I_y} \quad (4.9,8\ b)$$

Substituting (4.9,17 c) into (4.9,7 c) gives

$$\dot{w} = \frac{Z_u\Delta u}{m} + \frac{Z_w w}{m} + \frac{Z_{\dot{w}}\dot{w}}{m} + \frac{Z_q q}{m} + \frac{\Delta Z_c}{m} - g\Delta\theta\sin\theta_0 + u_0 q$$

Solving for $\dot{w}$

$$\begin{aligned}
\dot{w}-\frac{Z_{\dot{w}}\dot{w}}{m} &= \frac{Z_u\Delta u}{m}+\frac{Z_w w}{m}+\frac{Z_q q}{m}+\frac{\Delta Z_c}{m}-g\Delta\theta\sin\theta_0+u_0 q\\
\dot{w}\left(\frac{m-Z_{\dot{w}}}{m}\right) &= \frac{Z_u\Delta u}{m}+\frac{Z_w w}{m}+\frac{Z_q q}{m}+\frac{\Delta Z_c}{m}-g\Delta\theta\sin\theta_0+u_0 q\\
\dot{w} &= \frac{Z_u\Delta u}{m\left(\frac{m-Z_{\dot{w}}}{m}\right)}+\frac{Z_w w}{m\left(\frac{m-Z_{\dot{w}}}{m}\right)}+\frac{Z_q q}{m\left(\frac{m-Z_{\dot{w}}}{m}\right)}+\frac{\Delta Z_c}{m\left(\frac{m-Z_{\dot{w}}}{m}\right)}-\frac{g\Delta\theta\sin\theta_0}{\left(\frac{m-Z_{\dot{w}}}{m}\right)}+\frac{u_0 q}{\left(\frac{m-Z_{\dot{w}}}{m}\right)}\\
&= \frac{Z_u\Delta u}{m-Z_{\dot{w}}}+\frac{Z_w w}{m-Z_{\dot{w}}}+\frac{Z_q q}{m-Z_{\dot{w}}}+\frac{\Delta Z_c}{m-Z_{\dot{w}}}-\frac{g\Delta\theta\sin\theta_0}{m-Z_{\dot{w}}}m+\frac{u_0 q}{m-Z_{\dot{w}}}m\\
&= \left(\frac{Z_u}{m-Z_{\dot{w}}}\right)\Delta u+\left(\frac{Z_w}{m-Z_{\dot{w}}}\right)w+\left(\frac{Z_q+u_0 m}{m-Z_{\dot{w}}}\right)q-\left(\frac{gm\sin\theta_0}{m-Z_{\dot{w}}}\right)\Delta\theta+\frac{\Delta Z_c}{m-Z_{\dot{w}}} \qquad (1)
\end{aligned}$$

The above is the second component of (4.9.18) which will written in matrix form below as well. But now let us work on the second equation. Substituting (4.9,17 e) into (4.9,8 b) gives

$$\dot{q}=\frac{M_u}{I_y}\Delta u+\frac{M_w}{I_y}w+\frac{M_{\dot{w}}}{I_y}\dot{w}+\frac{M_q}{I_y}q+\frac{\Delta M_c}{I_y}$$

The above expression contains $\dot{w}$. But we found $\dot{w}$ above. Hence the above becomes

$$\begin{aligned}
\dot{q} &= \frac{M_u}{I_y}\Delta u+\frac{M_w}{I_y}w\\
&\quad+\frac{M_{\dot{w}}}{I_y}\left(\left(\frac{Z_u}{m-Z_{\dot{w}}}\right)\Delta u+\left(\frac{Z_w}{m-Z_{\dot{w}}}\right)w+\left(\frac{Z_q+u_0 m}{m-Z_{\dot{w}}}\right)q-\left(\frac{gm\sin\theta_0}{m-Z_{\dot{w}}}\right)\Delta\theta+\frac{\Delta Z_c}{m-Z_{\dot{w}}}\right)\\
&\quad+\frac{M_q}{I_y}q+\frac{\Delta M_c}{I_y}
\end{aligned}$$

Expanding and collecting terms

$$\begin{aligned}
\dot{q} &= \left(\frac{M_u}{I_y}+\frac{M_{\dot{w}}}{I_y}\left(\frac{Z_u}{m-Z_{\dot{w}}}\right)\right)\Delta u+\left(\frac{M_w}{I_y}+\frac{M_{\dot{w}}}{I_y}\left(\frac{Z_w}{m-Z_{\dot{w}}}\right)\right)w\\
&\quad+\left(\frac{M_q}{I_y}+\frac{M_{\dot{w}}}{I_y}\left(\frac{Z_q+u_0 m}{m-Z_{\dot{w}}}\right)\right)q-\frac{M_{\dot{w}}}{I_y}\left(\frac{gm\sin\theta_0}{m-Z_{\dot{w}}}\right)\Delta\theta+\frac{M_{\dot{w}}}{I_y}\frac{\Delta Z_c}{m-Z_{\dot{w}}}+\frac{\Delta M_c}{I_y}
\end{aligned}$$

Factoring out the I_y from each term gives

$$\begin{aligned}
\dot{q} &= \frac{1}{I_y}\left(M_u+\frac{M_{\dot{w}}Z_u}{m-Z_{\dot{w}}}\right)\Delta u+\frac{1}{I_y}\left(M_w+\frac{M_{\dot{w}}Z_w}{m-Z_{\dot{w}}}\right)w\\
&\quad+\frac{1}{I_y}\left(M_q+\frac{M_{\dot{w}}\left(Z_q+u_0 m\right)}{m-Z_{\dot{w}}}\right)q-\left(\frac{M_{\dot{w}}gm\sin\theta_0}{I_y\left(m-Z_{\dot{w}}\right)}\right)\Delta\theta+\frac{M_{\dot{w}}\Delta Z_c}{I_y\left(m-Z_{\dot{w}}\right)}+\frac{\Delta M_c}{I_y} \qquad (2)
\end{aligned}$$

The above is the third component of 4.9.18. Putting (1) and (2) into matrix form gives

$$\begin{pmatrix} \dot{w} \\ \dot{q} \end{pmatrix} = \left(\begin{array}{c|c|c|c} \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + u_0 m}{m - Z_{\dot{w}}} & -\dfrac{gm\sin\theta_0}{m - Z_{\dot{w}}} \\ \hline \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}\left(Z_q + u_0 m\right)}{m - Z_{\dot{w}}}\right] & -\dfrac{M_{\dot{w}} gm\sin\theta_0}{I_y\left(m - Z_{\dot{w}}\right)} \end{array}\right) \begin{pmatrix} \Delta u \\ w \\ q \\ \Delta\theta \end{pmatrix} + \begin{pmatrix} \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{M_{\dot{w}} \Delta Z_c}{I_y(m - Z_{\dot{w}})} + \frac{\Delta M_c}{I_y} \end{pmatrix}$$

2.2.3 Problem 3

4.10 A hovercraft in ground effect is acted on by the following aerodynamic forces, expressed as body frame components:

$$X = Y = 0$$
$$Z = -\text{mg} + Z_z z_E$$
$$L = L_\phi \phi; \quad M = M_\theta \theta; \quad N = 0$$

The body axes are principal axes, and the engine/rotor angular momentum is

$$\mathbf{h}'_B = [0 \quad 0 \quad H]^T$$

Derive a set of small-disturbance equations of motion.

HINTS: start from Eqs. 4.9,7, 8, 9 and replace/simplify as many terms as possible based on:

1. the expressions for X, Y, Z, L, M, and N given in the problem statement;
2. the fact that the axes are PRINCIPAL axes;
3. the angular momentum of the engine rotors also given in the statement (the effects of the rotors are described in Sec. 4.6, p.103). Specifically, refer to Eqs. 4.5,9, 4.6,2, and 4.9,3 to account correctly for the gyroscopic effects.

What the problem statement means is: $X_0 = 0$; $\Delta X = \Delta Y = \Delta N = 0$; $\Delta Z = Z_z \Delta z_E$;
Also assume $\phi_0 = \theta_0 = 0$.

Figure 2.19: problem 3 description

HINT: For HW2 Prob. 3 (#4.10 in the book): assume $\phi_0 = \theta_0 = 0$. u_0 is non-zero (the hovercraft is moving horizontally).

Given

$$\begin{aligned} X &= Y = 0 \\ Z &= -mg + Z_z z_E \\ L &= L_\phi \phi \\ M &= M_\theta \theta \\ N &= 0 \end{aligned}$$

The forces and moments and angles given above are illustrated in figure 2.21. Using the

Figure 2.20: Boeing Pelican ground effect vehicle. image thanks to http://www.aerospaceweb.org

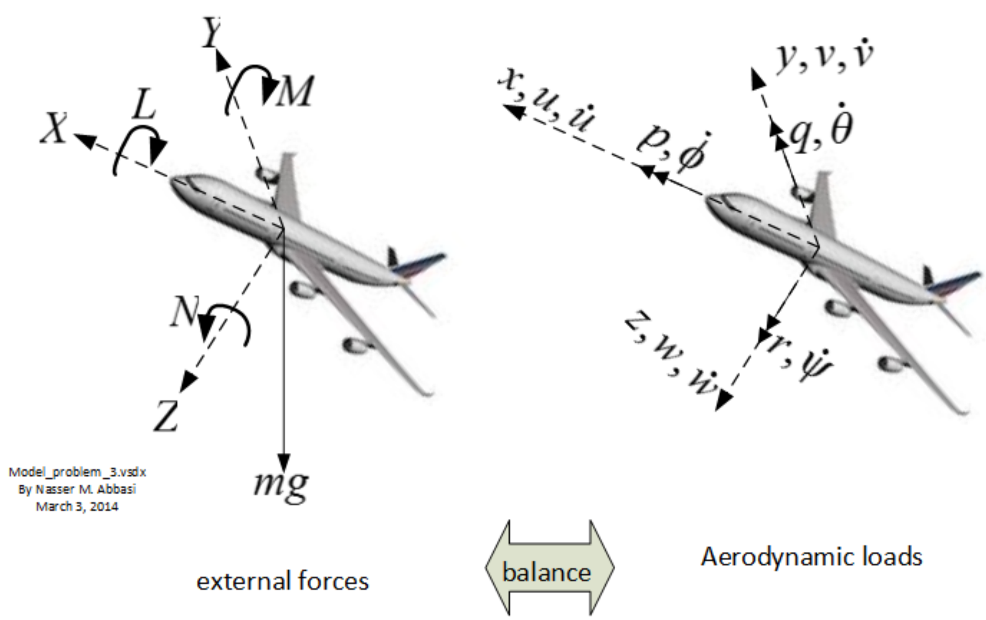

Figure 2.21: Balance of aerodynamic forces against external loads

hint, and starting from (4.9,7,8,9) in the textbook, for linear motion

$$\Delta\dot{u} = \frac{\Delta X}{m} - g\Delta\theta\cos\theta_0 \tag{4.9,7 (a)}$$

$$\dot{v} = \frac{\Delta Y}{m} + g\phi\cos\theta_0 - u_0 r \tag{4.9,7 (b)}$$

$$\dot{w} = \frac{\Delta Z}{m} - g\Delta\theta\sin\theta_0 + u_0 q \tag{4.9,7 (c)}$$

And for angular motion

$$\dot{p} = \frac{I_z\Delta L + I_{xz}\Delta N}{I_x I_z - I_{xz}^2} \tag{4.9,8 (a)}$$

$$\dot{q} = \frac{\Delta M}{I_y} \tag{4.9,8 (b)}$$

$$\dot{r} = \frac{I_{zx}\Delta L + I_x\Delta N}{I_x I_z - I_{zx}^2} \tag{4.9,8 (c)}$$

And Euler angles

$$\Delta\dot{\theta} = q \tag{4.9,9 (a)}$$

$$\dot{\phi} = p + r\tan\theta_0 \tag{4.9,9 (b)}$$

$$\dot{\psi} = r\sec\theta_0 \tag{4.9,9 (c)}$$

Starting with the set of (4.9,7) equations. Since $X = 0$ and $Y = 0$ therefore $\Delta X = 0$ and ΔY. We need to find ΔZ which by the problem statement $\Delta Z = Z_z\Delta z_E$. Also using the assumption given that $\theta_0 = \phi_0 = 0$ then the set of (4.9.7) equations reduces to

$$\Delta\dot{u} = -g\Delta\theta \tag{4.9,7 (a1)}$$

$$\dot{v} = g\phi - u_0 r \tag{4.9,7 (b1)}$$

$$\dot{w} = \frac{Z_z\Delta z_E}{m} + u_0 q \tag{4.9,7 (c1)}$$

Now considering the set of (4.9.8) equations. Since principal body axes is used, the off diagonal terms in the inertial matrix vanish which means $I_{xz} = I_{zx} = 0$. Also, since $N = 0$ then $\Delta N = 0$. This reduces (4.9.8) to

$$\dot{p} = \frac{\Delta L}{I_x} \tag{4.9,8 (a1)}$$

$$\dot{q} = \frac{\Delta M}{I_y} \tag{4.9,8 (b1)}$$

$$\dot{r} = 0 \tag{4.9,8 (c1)}$$

But in the above ΔL and ΔM do not yet have the gyroscopic effect. These need to be adjusted to add the gyroscopic effect before going further. Since $h'_B = [0, 0, H]^T$ then using (4.6.2) in order to find what terms to add to each moment, we write

$$L : qh'_z - rh'_y = qH$$

$$M : rh'_x - ph'_z = -pH$$

$$N : ph'_y - qh'_x = 0$$

Therefore, the modified moments are

$$L = L_\phi \phi + qH$$

and

$$M = M_\theta \theta - ph$$

Assuming $\Delta L \equiv L_\phi \phi$ and $\Delta M \equiv M_\theta \theta$ then the above two equations can be written as

$$L = \overbrace{\Delta L + qH}^{\Delta L'}$$

and

$$M = \overbrace{\Delta M - ph}^{\Delta M'}$$

We have to replace $\Delta L, \Delta M$, in (4.9.8 a1,b1,c1) above with these new corrected $\Delta L', \Delta M'$ giving

$$\dot{p} = \frac{\Delta L + qH}{I_x} \qquad (4.9,8\ (a2))$$

$$\dot{q} = \frac{\Delta M - ph}{I_y} \qquad (4.9,8\ (b2))$$

$$\dot{r} = 0 \qquad (4.9,8\ (c3))$$

The above now accounts for the gyroscopic effect. We now continue to the last three set of equations (4.9.9). Since $\theta_0 = 0$ these become

$$\Delta\dot{\theta} = q \qquad (4.9,9\ (a1))$$

$$\dot{\phi} = p \qquad (4.9,9\ (b1))$$

$$\dot{\psi} = r \qquad (4.9,9\ (c1))$$

Summary: The final set of equations are

$$\Delta\dot{u} = -g\Delta\theta \qquad (4.9,7\ (a1))$$

$$\dot{v} = g\phi - u_0 r \qquad (4.9,7\ (b1))$$

$$\dot{w} = \frac{Z_z \Delta z_E}{m} + u_0 q \qquad (4.9,7\ (c1))$$

$$\dot{p} = \frac{\Delta L + qH}{I_x} \qquad (4.9,8\ (a2))$$

$$\dot{q} = \frac{\Delta M - ph}{I_y} \qquad (4.9,8\ (b2))$$

$$\dot{r} = 0 \qquad (4.9,8\ (c3))$$

$$\Delta\dot{\theta} = q \qquad (4.9,9\ (a1))$$

$$\dot{\phi} = p \qquad (4.9,9\ (b1))$$

$$\dot{\psi} = r \qquad (4.9,9\ (c1))$$

2.2.4 Problem 4

4.12* An aircraft is performing a rolling pullup. At the instant of observation, the vehicle is at the bottom of a vertical circle of 2000 ft (610 m) radius moving at a constant speed of 500 fps (152 m/s) with wings horizontal. (See Fig. 3.1). At the same time the roll rate is constant at $p = 90^\circ \text{ s}^{-1}$. Given that

$$I_y - I_x = 300 \text{ slug ft}^2 \text{ } (406 \text{ kg m}^2) \quad \text{and} \quad I_z = 500 \text{ slug ft}^2 \text{ } (677 \text{ kg m}^2)$$

determine the moments required at this time to perform this maneuver. Assume that the axes are principal axes, with Cx horizontal. (You may assume constant Euler angle rates and $\psi = 0$.)

Figure 2.22: problem 4 description

HINT: the most useful equations are sets 4.7,2 and 4.7,3. In particular, to determine $\dot{r}$, it is best to differentiate (4.7,3)(c).

Solution

equations (4.7,2) and (4.7,3) from the textbook are

$$L = I_x\dot{p} - I_{zx}\dot{r} + qr\left(I_z - I_y\right) - I_{zx}pq + qh'_z - rh'_y \tag{4.7,2(a)}$$

$$M = I_y\dot{q} + rp\left(I_x - I_z\right) + I_{zx}\left(p^2 - r^2\right) + rh'_x - ph'_z \tag{4.7,2(b)}$$

$$N = I_z\dot{r} - I_{zx}\dot{p} + pq\left(I_y - I_x\right) + I_{zx}qr + ph'_y - qh'_x \tag{4.7,2(c)}$$

$$p = \dot{\phi} - \dot{\psi}\sin\theta \tag{4.7,3(a)}$$

$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \tag{4.7,3(b)}$$

$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \tag{4.7,3(c)}$$

Figure 2.23 which is figure 3.1 in the text book illustrating the problem with added annotations on it to help in solving the problem.

We start simplifying the above equations. Since body axes is used, then all the off diagonal moments of inertial are zero. Hence $I_{zx} = 0$. Ignoring the gyroscopic effect, then $h'_z = h'_y = h'_x = 0$. We are also told that $\psi = 0$, hence $\dot{\psi} = 0$. The above equations reduces to

$$L = I_x\dot{p} + qr\left(I_z - I_y\right) \tag{4.7,2(a1)}$$

$$M = I_y\dot{q} + rp\left(I_x - I_z\right) \tag{4.7,2(b1)}$$

$$N = I_z\dot{r} + pq\left(I_y - I_x\right) \tag{4.7,2(c1)}$$

$$p = \dot{\phi} \tag{4.7,3(a1)}$$

$$q = \dot{\theta}\cos\phi \tag{4.7,3(b1)}$$

$$r = -\dot{\theta}\sin\phi \tag{4.7,3(c1)}$$

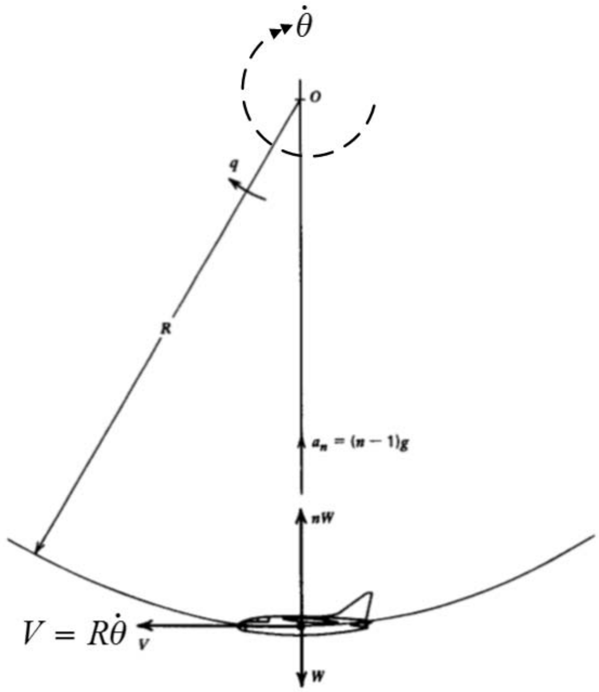

Figure 3.1 Airplane in a pull-up.

Figure 2.23: aircraft rolling pull up

Differentiating (4.7,3) gives

$$\dot{p} = \ddot{\phi} \tag{4.7,3(a2)}$$

$$\dot{q} = \ddot{\theta}\cos\phi + \dot{\theta}\dot{\phi}\sin\phi \tag{4.7,3(b2)}$$

$$\dot{r} = -\ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi \tag{4.7,3(c2)}$$

Since we are told that p is constant, then $\dot{p} = 0$. In addition, since the airplane is moving at constant speed, and since the radius of the vertical circle is constant this implies that the angular acceleration is zero or $\ddot{\theta} = 0$. The problem also says that the wings are horizontal at this moment of time, therefore the Euler angle $\phi = 0$ as can be seen from figure 2.24, which is figure 3.14 in the textbook.

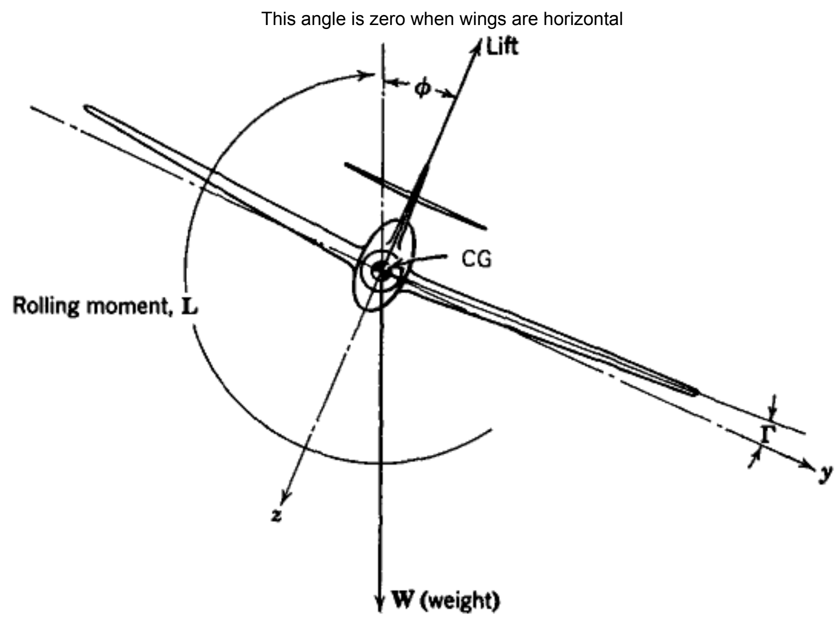

Figure 3.14 Rolled airplane.

Figure 2.24: aircraft rolled angle

Using the above then the set of (4.7) equations above reduces to

$$
\begin{aligned}
L &= I_x \dot{p} + qr\left(I_z - I_y\right) \\
M &= I_y \dot{q} + rp\left(I_x - I_z\right) \\
N &= I_z \dot{r} + pq\left(I_y - I_x\right) \\
p &= \dot{\phi} \\
q &= \dot{\theta} \\
r &= 0 \\
\dot{p} &= 0 \\
\dot{q} &= 0 \\
\dot{r} &= -\dot{\theta}\dot{\phi}
\end{aligned}
$$

The goal is to determine L, M, N. Substituting $\dot{p} = 0$ and $r = 0$ and $\dot{q} = 0$ in the first three equations above, the reduce more to

$$
\begin{aligned}
L &= 0 \\
M &= 0 \\
N &= -I_z \dot{\theta}\dot{\phi} + \dot{\phi}\dot{\theta}\left(I_y - I_x\right)
\end{aligned}
$$

The above are the moments needed to perform the rolling. Using the values given (SI) and using $V = R\dot{\theta}$ or $\dot{\theta} = \frac{V}{R}$ and using $p = \dot{\phi}$, the above becomes

$$
\begin{aligned}
L &= 0 \\
M &= 0 \\
N &= -I_z \frac{V}{R} p + p \frac{V}{R} \left(I_y - I_x \right)
\end{aligned}
$$

But $p = 90^0$ per sec, or $\frac{\pi}{2}$rad/sec. Substituting numerical values the above becomes

$$
\begin{aligned}
L &= 0 \\
M &= 0 \\
N &= -(677) \left(\frac{152}{610} \right) \frac{\pi}{2} + \frac{\pi}{2} \left(\frac{152}{610} \right) (406)
\end{aligned}
$$

or

$$
\begin{aligned}
L &= 0 \\
M &= 0 \\
N &= -106.07 \ \ \mathbf{Nm} \\
&= \boxed{-78.233 \, \text{foot-lb force}}
\end{aligned}
$$

2.2.5 Key solution

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$$\mathbf{h}_B = \mathbf{I}_B \boldsymbol{\omega}_B + \mathbf{h}'_B \tag{8}$$

From (4.5,5)

$$\mathbf{G}_B = \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_B \mathbf{h}_B \tag{9}$$

Thus from (8) and (9) the additional terms in the moment equations due to spinning rotors are (if we assume $\dot{\mathbf{h}}'_B = 0$)

$$\tilde{\boldsymbol{\omega}}\, h'_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} h'_x \\ h'_y \\ h'_z \end{bmatrix}$$

$$= \begin{bmatrix} qh'_z - rh'_y \\ rh'_x - ph'_z \\ ph'_y - qh'_x \end{bmatrix} \tag{10}$$

as given by (4.6,2).

4.6 In (4.10,3) let $\pi = T_s \frac{u_o}{\ell}$ where T_s is the characteristic time of the spiral divergence. Since we are told to ignore M and RN, consider the nondimensional combinations $\frac{m}{\rho \ell^3}$ and $\frac{u_o^2}{\ell g}$.

For dynamic similarity, these groupings must be the same for the two aircraft.

(a) Let $(\ )_A$ be aircraft A values. Equating the above two groupings for the two aircraft leads to

$$\frac{m_A}{\rho_A \ell_A^3} = \frac{m_B}{\rho_B \ell_B^3} \tag{1}$$

and

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$$\frac{u_{oA}^2}{\ell_A g} = \frac{u_{oB}^2}{\ell_B g} \qquad (2)$$

From App. D, at 20,000 ft

$$\rho_B = 1.2673 \times 10^{-3} \text{ slug/ft}^3$$

Thus from (1)

$$\rho_A = \frac{m_A}{m_B} \cdot \left(\frac{\ell_B}{\ell_A}\right)^3 \cdot \rho_B \qquad (3)$$

where

$$\frac{m_A}{m_B} = 100{,}000/225{,}000$$

$$= 0.444$$

$$\frac{\ell_B}{\ell_A} = \frac{150}{100}$$

$$= 1.5$$

Hence, from (3),

$$\rho_A = 0.444 \times (1.5)^3 \times 1.2673 \times 10^{-3}$$

$$= 1.899 \times 10^{-3} \text{ slug/ft}^3$$

From App. D the altitude corresponding to ρ_A is 7,500 ft.

From (2)

$$u_{oA}^2 = \frac{\ell_A}{\ell_B} u_{oB}^2 \qquad (4)$$

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Thus

$$u_{o_A}^2 = \frac{1}{1.5} \times (400)^2$$

or

$$u_{o_A} = 326.6 \text{ knots}$$

Thus A will be dynamically similar to B at 7,500 ft altitude and a speed of 326.6 knots.

(b) From $\pi = T_s \frac{u_o}{\ell}$, under conditions where the two aircraft are dynamically similar $\pi_A = \pi_B$.

Thus

$$T_{s_A} \frac{u_{o_A}}{\ell_A} = T_{s_B} \frac{u_{o_B}}{\ell_B} \qquad (5)$$

Hence

$$T_{s_A} = T_{s_B} \frac{u_{o_B}}{u_{o_A}} \cdot \frac{\ell_A}{\ell_B}$$

$$= 20 \times \frac{400}{326.6} \cdot \frac{100}{150}$$

$$= 16.33 \text{ seconds}$$

(c)

$$C_{L_A} = \frac{W_A}{\frac{1}{2} \rho_A u_{o_A}^2 S_A} \qquad (6)$$

$$C_{L_B} = \frac{W_B}{\frac{1}{2} \rho_B u_{o_B}^2 S_B} \qquad (7)$$

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Thus

$$\frac{C_{L_A}}{C_{L_B}} = \frac{W_A}{W_B} \cdot \frac{\rho_B}{\rho_A} \cdot \frac{u_{o_B}^2}{u_{o_A}^2} \cdot \frac{S_B}{S_A}$$

$$= \frac{100{,}000}{225{,}000} \cdot \frac{1.2673 \times 10^{-3}}{1.899 \times 10^{-3}} \cdot \left(\frac{400}{326.6}\right)^2 \left(\frac{150}{100}\right)^2$$

$$= 1$$

as expected, since C_L is also a nondimensional combination, and as such must also be the same for dynamic similarity.

4.7 From (4.9,17)

$$\Delta Z = Z_u \Delta u + Z_w w + Z_{\dot{w}} \dot{w} + Z_q q + \Delta Z_c \quad (1)$$

$$\Delta M = M_u \Delta u + M_w w + M_{\dot{w}} \dot{w} + M_q q + \Delta M_c \quad (2)$$

From (4.9,7c) and (4.9,8b)

$$\dot{w} = \frac{\Delta Z}{m} - g\Delta\theta \sin\theta_o + u_o q \quad (3)$$

$$\dot{q} = \frac{\Delta M}{I_y} \quad (4)$$

Substitute (1) and (2) into (3) and (4)

$$\dot{w} = \frac{Z_u}{m}\Delta u + \frac{Z_w w}{m} + \frac{Z_{\dot{w}}\dot{w}}{m} + \frac{Z_q q}{m} + \frac{\Delta Z_c}{m} - g\Delta\theta \sin\theta_o + u_o q \quad (5)$$

$$\dot{q} = \left[M_u \Delta u + M_w w + M_{\dot{w}} \dot{w} + M_q q + \Delta M_c\right] / I_y \quad (6)$$

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From (5)

$$\dot{w}(m - Z_{\dot{w}}) = Z_u\Delta u + Z_w w + (Zq + mu_o)q - mg\Delta\theta \sin\theta_o + \Delta Z_c \qquad (7)$$

and (7) is the second component of (4.9,18). Substitute (7) into (6) to eliminate $\dot{w}$

$$\dot{q}I_y = M_u\Delta u + \frac{M_{\dot{w}}Z_u\Delta u}{m - Z_{\dot{w}}} + M_w w + \frac{M_{\dot{w}}Z_w w}{m - Z_{\dot{w}}}$$

$$+ M_q q + M_{\dot{w}}\frac{(Zq + mu_o)q}{m - Z_{\dot{w}}} - \frac{M_{\dot{w}}mg\Delta\theta\sin\theta_o}{m - Z_{\dot{w}}}$$

$$+ \Delta M_c + M_{\dot{w}}\frac{\Delta Z_c}{m - Z_{\dot{w}}} \qquad (8)$$

and (8) is the third component of (4.9,18).

4.8 $\underline{X_q}$

Follow the method used in the text (Sec. 4.11) to generate Z_q

$$X_q = \left(\frac{\partial X}{\partial q}\right)_o \qquad (1)$$

where

$$X = C_x \frac{1}{2}\rho V^2 S \qquad (2)$$

Thus

$$X_q = \frac{1}{2}\rho u_o^2 S\left(\frac{\partial C_x}{\partial q}\right)_o \qquad (3)$$

since $V_o = u_o$.

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4.10 Assume that $\phi_o = \theta_o = 0$. Thus making the usual assumptions

$$Z_o = -mg + Z_z z_{E_o} \tag{1}$$

$$\Delta Z = Z_z \Delta z_E \tag{2}$$

$$\Delta X = \Delta Y = \Delta N = 0 \tag{3}$$

$$\Delta L = L_\phi \phi \tag{4}$$

$$\Delta M = M_\theta \theta \tag{5}$$

From the above and (4.9,7)

$$\Delta \dot{u} = -g\theta \tag{6}$$

$$\dot{v} = +g\phi - u_o r \tag{7}$$

$$\dot{w} = \frac{Z_z}{m} \Delta z_E + u_o q \tag{8}$$

From (4.9,10c)

$$\Delta \dot{z}_E = -u_o \theta + w \tag{9}$$

From (4.9,9)

$$\dot{\phi} = p \tag{10}$$

$$\dot{\theta} = q \tag{11}$$

$$\dot{\psi} = r \tag{12}$$

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When the moment equations of (4.7,2) are linearized to produce (4.9,3) the $\mathbf{h}_B'$ were dropped. In the present problem $h_x' = h_y' = 0$ but $h_z' = H$ and thus should be retained in (4.9,3). Also, since the body axes are principal axes it follows that $I_{zx} = 0$. Thus the linearized moment equations become (since $L_o = M_o = N_o = 0$)

$$L_\phi \phi = I_x \ddot{\phi} + qH \tag{13}$$

$$M_\theta \theta = I_y \ddot{\theta} - pH \tag{14}$$

$$0 = I_z \dot{r} \tag{15}$$

From (15) $\dot{r} = 0$, thus if we start up with $\psi = r = 0$ then they will remain equal to zero. Thus if this were true we could drop (12) and (15) and set $r = 0$ in (7).

4.11 Assume that **W** is the wind as seen at the CG of the aircraft. Next make the point approximation, that is, the aircraft is small compared with any spatial variations in the wind. This means that the wind at any value of time is uniform over the complete aircraft. Thus from (1.6,1)

$$\mathbf{V}^E = \mathbf{V} + \mathbf{W} \tag{1}$$

and the angular velocity $\boldsymbol{\omega}$ used in calculating aerodynamic forces and moments is unchanged from the text. Recall that the forces and moments also depend on **V**, the airspeed. From (4.2,15)

$$\mathbf{f}_E = m\dot{\mathbf{V}}_E^E \tag{2}$$

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4.12 For this problem

(a) $\psi = \dot{\psi} \equiv 0$ since it is a vertical circle.

(b) $\dot{\theta} = \frac{V}{R}$ for circular loop of radius R.

(c) At the bottom of the loop we are given $\phi = \theta = \psi = 0$.

From (4.7,3) it follows from the above that at the bottom of the loop:

$$
\begin{aligned}
p &= \dot{\phi} - \dot{\psi} \sin\theta \\
&= \dot{\phi} = \frac{\pi}{2} \text{ rad/s}
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
q &= \dot{\theta} \cos\phi + \dot{\psi} \cos\theta \sin\phi \\
&= \dot{\theta} = \frac{V}{R}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
r &= \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi \\
&= 0
\end{aligned} \tag{3}
$$

To evaluate the moment equations we need $\dot{p}, \dot{q}, \dot{r}$. Obtain these by differentiating (4.7,3) and using (a), (b), (c), (1), (2) and (3)

$$
\begin{aligned}
\dot{p} &= \ddot{\phi} \\
&= 0
\end{aligned} \tag{4}
$$

$$
\dot{q} = \ddot{\theta} \cos\phi - \dot{\theta}\dot{\phi} \sin\phi = 0 \tag{5}
$$

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Chapter 4

$$\dot{r} = -\ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi = -\dot{\theta}\dot{\phi} \quad (6)$$

From (4.7,2) and the above, and making use of $I_{zx} = 0$ and $\mathbf{h}' = \mathbf{0}$:

$$L = 0 \quad (7)$$

$$M = 0 \quad (8)$$

$$N = I_z\dot{r} + pq(I_y - I_x) \quad (9)$$

From (1), (2), (6) and (9)

$$N = \dot{\theta}\dot{\phi}[I_y - I_x - I_z]$$

$$= \frac{V}{R}\cdot\frac{\pi}{2}[I_y - I_x - I_z]$$

$$= \frac{500}{2000}\cdot\frac{\pi}{2}[300 - 500]$$

$$= -78.54 \quad \text{ft.lb} \quad (10)$$

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2.3 HW3

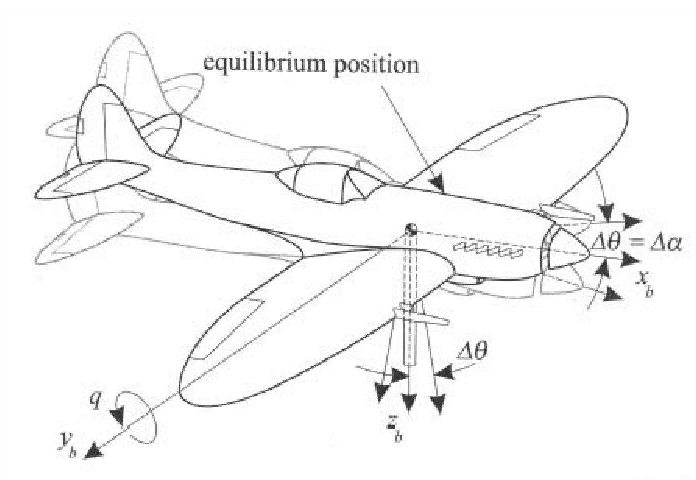
2.3.1 Problem 1

1. In a wind tunnel test, an airplane model is mounted so that it can only pivot about the body-fixed $y-$axis, as shown in the figure, so that the only possible departures from steady horizontal flight are in α and θ. This particular airplane motion is called "pure pitching motion".

a) Explain in plain English why, in this particular case, $\Delta\alpha = \Delta\theta$.

b) Starting from the general form of the equation of motion and eliminating all the quantities that do not change with time in this case, evaluate the eigenvalues of the motion, the period and the time to half. Feel free to use Maple (or any other package you may prefer) to carry out some of the symbolic manipulation but make sure to write clearly, by hand, what steps you are taking.

Hint: for part b), you need to reduce the full, 4×4 system of linear equations into a 2×2 system. This means that, of the original four longitudinal variables, only two are necessary to completely describe the present configuration (see the short-period approximation).



2.3.1.1 Part (1)

The angle θ is the kinematic Euler angle that measures the pitch of the airplane relative to the ground. The angle of attack α is an aerodynamic angle that measures the angle between the zero lift line and the velocity direction V of the airplane at any moment. The relation between θ and α is

$$\theta = \gamma + \alpha$$

Where γ is the angle between horizontal plane and the velocity vector V. This is illustrated in figure 2.25 from our text book on page 19. In trim conditions, the angle γ is constant (or zero). Therefore any change in θ must come from change in the angle of attack α. Hence

$$\Delta\theta = \Delta\alpha$$

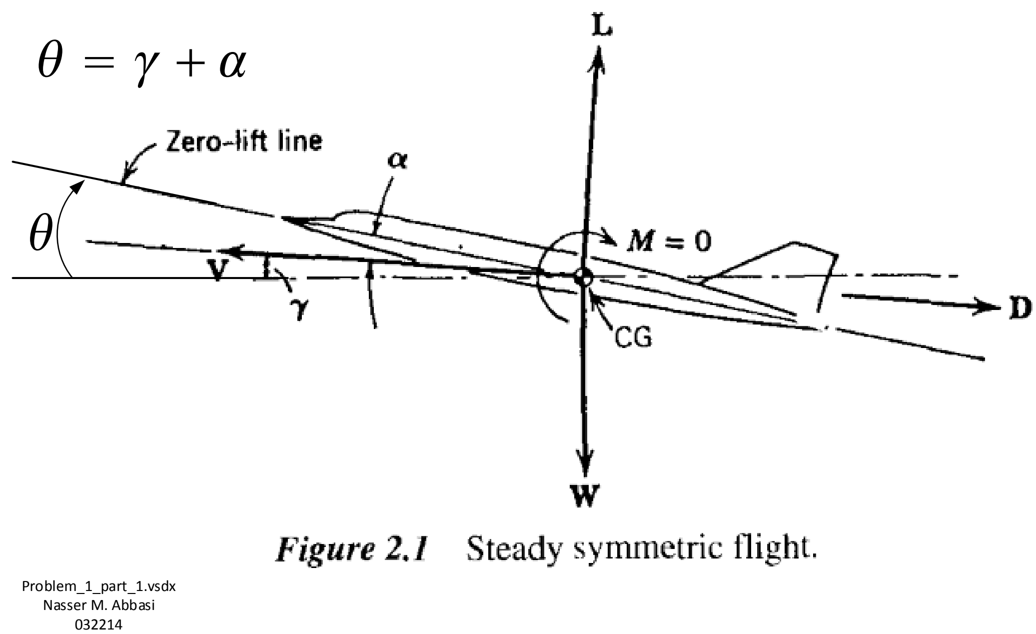


Figure 2.25: Relation between γ and θ and angle of attack α

2.3.1.2 Part(2)

Equation 4.9,18 in text book represents the longitudinal equation of motion using the four longitudinal variables $\{u, w, q, \theta\}$. In pure pitching mode, the two degrees of freedom are q and w (similar to short-period mode). The pure pitching mode is the short-period mode in the limit as $m \longrightarrow \infty$. From Mechanics of flight, Warren Phillips, page 873 :

Thus we see that as the dimensionless mass becomes large compared to the dimensionless moment of inertia, the short-period motion associated with free flight becomes pure pitching motion.

Therefore, to obtain the pure pitching mode, The short-period mode is used and then the limit is taken to obtain the pure pitching mode equations. The short-period mode is given by the following 2×2 system. (Equation 4.19,18 in the text book)

$$\begin{pmatrix} \dot{w} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + m u_o}{m - Z_{\dot{w}}} \\ \frac{1}{I_y}\left(M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right) & \frac{1}{I_y}\left(M_q + \frac{M_{\dot{w}}\left(Z_q + m u_o\right)}{m - Z_{\dot{w}}}\right) \end{pmatrix} \begin{pmatrix} w \\ q \end{pmatrix}$$

Taking the $\lim_{m \to \infty}$ results in

$$\begin{pmatrix} \dot{w} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & u_o \\ \frac{1}{I_y} M_w & \frac{1}{I_y}\left(M_q + M_{\dot{w}} u_o\right) \end{pmatrix} \begin{pmatrix} w \\ q \end{pmatrix}$$

The characteristic equation is found from

$$\begin{aligned}\det(A-\lambda I) &= 0\\ \begin{vmatrix} -\lambda & u_o \\ \frac{1}{I_y}M_w & \frac{1}{I_y}\left(M_q+M_{\dot{w}}u_o\right)-\lambda \end{vmatrix} &= 0\\ \lambda^2-\lambda\frac{1}{I_y}\left(M_q+M_{\dot{w}}u_o\right)-\frac{1}{I_y}M_w u_o &= 0 \qquad (1)\end{aligned}$$

Let the natural frequency

$$\boxed{\omega_n=\sqrt{-M_w\frac{u_o}{I_y}}}$$

And the damping ratio

$$\boxed{\zeta=\frac{-\left(M_q+M_{\dot{w}}u_o\right)}{I_y\sqrt{-M_w\frac{u_o}{I_y}}}}$$

Equation (1) can be written in standard form as

$$\lambda^2+2\zeta\omega_n\lambda+\omega_n^2=0$$

The eigenvalues are

$$\begin{aligned}\lambda &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &= \frac{1}{2I_y}\left(M_q+M_{\dot{w}}u_o\right)\pm\frac{1}{2I_y}\sqrt{\left(M_q+M_{\dot{w}}u_o\right)^2+4I_yM_wu_o}\\ &= \frac{1}{2I_y}\left(M_q+M_{\dot{w}}u_o\right)\pm\frac{i}{2I_y}\sqrt{-\left(M_q+M_{\dot{w}}u_o\right)^2-4I_yM_wu_o}\\ &= n\pm i\omega\end{aligned}$$

Where

$$\begin{aligned}n &= \frac{1}{2I_y}\left(M_q+M_{\dot{w}}u_o\right)\\ \omega &= \frac{1}{2I_y}\sqrt{-\left(M_q+M_{\dot{w}}u_o\right)^2-4I_yM_wu_o}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= n+i\omega\\ \lambda_2 &= n-i\omega\end{aligned}$$

The period T is given by

$$T = \frac{2\pi}{\omega} = \frac{4\pi I_y}{\sqrt{-\left(M_q + M_{\dot{w}}u_o\right)^2 - 4I_yM_wu_o}}$$

The time to double is

$$\begin{aligned} t &= \frac{0.693}{|n|} \\ &= \frac{\left(2I_y\right)0.693}{\left|M_q + M_{\dot{w}}u_o\right|} \\ &= I_y\frac{1.386}{\left|M_q + M_{\dot{w}}u_o\right|} \end{aligned}$$

2.3.2 Problem 2

6.3 The stability derivatives of a general aviation airplane are given in Table 7.2. The airplane weighs 2400 lb (10,675 N) and has a wing area of 160 ft^2 (14.9 m^2). The flight altitude is sea level. Calculate and plot the spiral stability criterion E as a function of speed ($0.15 < C_L < 1.7$) for values of $\theta_0 = -10°, 0°, 10°$.

Hint: refer to Eq. (6.8,6) in the book.

Solution

Table 7.2 in the text book is

Table 7.2
Nondimensional Derivatives—General Aviation Airplane (expressed in rad^{-1} and (rad/s)$^{-1}$)

	C_y	C_l	C_n
β	-0.14	$-0.0689 - 0.0917C_L$	$0.01326 + 0.017C_L^2$
$\hat{p}$	-0.039	-0.441	$-0.00109 - 0.0966C_L$
$\hat{r}$	0.165	$-0.0144 + 0.271C_L$	$-0.048 - 0.0238C_L^2$
δ_a	0	-0.0531	0.005
δ_r	0.117	0.0105	-0.0509

From equation 6.8,6 in the book

$$E = g\left[\left(C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}\right)\cos\theta_0 + \left(C_{l_p}C_{n_\beta} - C_{l_\beta}C_{n_p}\right)\sin\theta_0\right]$$

Using table 7.2, each of the above expressions are evaluated. Some are functions of C_L.

$$C_{l_\beta} = -0.0689 - 0.0917C_L$$
$$C_{n_r} = -0.048 - 0.0238C_L^2$$
$$C_{l_r} = -0.0144 + 0.271C_L$$
$$C_{n_\beta} = 0.01326 + 0.017C_L^2$$
$$C_{l_p} = -0.441$$
$$C_{n_p} = -0.00109 - 0.0966C_L$$

For each different value of θ_0, C_L is changed and new value for E is obtained. This was done three times, for $\theta_0 = -10^0, 0, +10^0$. The standard lift coefficient equation is used to obtain the corresponding value of speed for each C_L

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$$

At trim, $L = mg$, hence

$$C_L = \frac{mg}{\frac{1}{2}\rho V^2 S}$$

Or

$$V = \sqrt{\frac{2mg}{\rho S C_L}}$$

Substituting the numerical values given in the problem above (SI) gives

$$V = \sqrt{\frac{2(10675)}{\rho(14.9)C_L}}$$

The air density ρ is found from appendix D since the airplane is at see level. From appendix D

$$\rho = 2.3769 \times 10^{-3} \text{ lb } \frac{\sec^2}{ft^4}$$
$$= \boxed{1.225 \text{ kg/m}^3}$$

Hence

$$V = \sqrt{\frac{2(10675)}{(1.225)(14.9)}}\sqrt{\frac{1}{C_L}}$$
$$= 34.201\sqrt{\frac{1}{C_L}} \text{ m/sec}$$
$$= \boxed{112.18 \text{ ft/sec}}$$

These are the equations to plot E vs. V. For each C_L, V and E are calculated using the above. A small program was written to do this. The result is in figure 2.26

```
clbeta[cl_] := -0.0689 - 0.0917 cl;
cnr[cl_] := -0.048 - 0.0238 cl^2;
clr[cl_] := -0.0144 + 0.271 cl;
cnbeta[cl_] := 0.01326 + 0.017 cl^2;
clp = -0.441;
cnp[cl_] := -0.00109 - 0.0966 cl;
speed[cl_] := 34.201 Sqrt[1/cl];

spiralE[theta0_, cl_] := ( (clbeta[cl] cnr[cl] - clr[cl] cnbeta[cl])
      Cos[theta0] + (clp cnbeta[cl] - clbeta[cl] cnp[cl]) Sin[theta0]);

angles = {-10 Degree, 0, 10 Degree};

data = Table[{speed[cl], spiralE[#, cl]}, {cl, 0.15, 1.7, 0.1}] & /@ angles;

ListLinePlot[data, Joined -> True, PlotLegends -> angles, Frame -> True,
   FrameLabel -> {{"Stability criterion E", None}, {"airplane speed in m/sec",
   "Stability criterion E vs. speed for different angles theta"}},
   GridLines -> Automatic, GridLinesStyle -> LightGray,
   ImageSize -> 500, RotateLabel -> True, LabelStyle -> 18
]
```

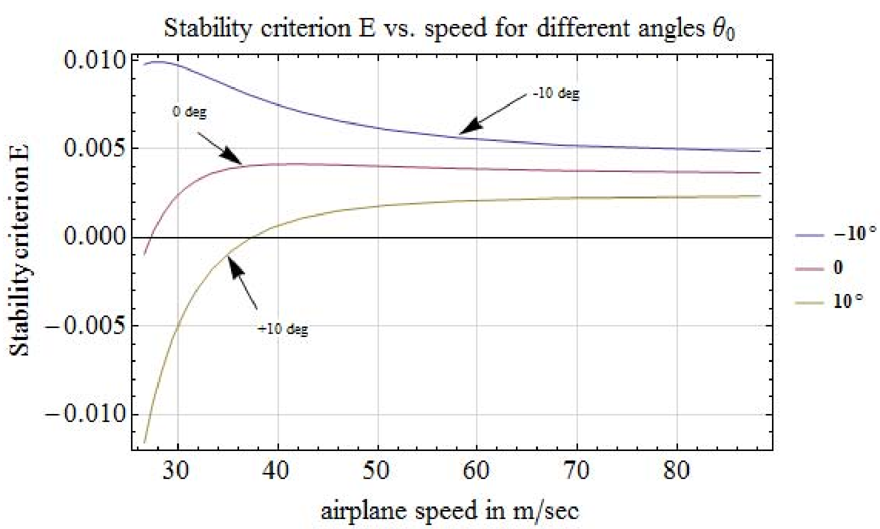


Figure 2.26: Stability criterion E vs. speed

2.3.2.1 Discussion of results

The plot in figure 2.26 shows that spiral mode can become unstable depending on the speed of the airplane.

The spiral mode characteristic equation was simplified to $D\lambda + E = 0$ as discussed in the textbook, page 193. E represents the static stability (as the case with the full, un-simplified characteristic equation). The condition for static stability is that E should remain positive. If E changes from positive to negative, which means a root has switched from negative to positive, then the response becomes unstable (diverges).

From the above plot, when the reference angle of climb θ_0 was larger than zero, this mode became unstable when the airplane slowed down to below critical value. The larger θ_0 became, the larger this critical value became.

For example, when $\theta_0 = 0$, this critical speed was about $27\,\mathrm{m\,s^{-1}}$, but when $\theta_0 = 10^\circ$, the critical speed was above $39\,\mathrm{m\,s^{-1}}$. When the reference angle of climb is negative, this mode remained stable for all the speed range shown since E was positive throughout.

2.3.3 Problem 3

6.5 Find the critical climb angle for spiral stability of the jet transport of Sec. 6.7. [Hint: start with (6.8,6)]. Having regard to its expected influence on the stability derivatives, state the effect on spiral stability in horizontal flight of increasing the wing dihedral angle.

3. Problem 6.5 in the textbook. For the first question: when calculating the critical climb angle θ_0 use the values of the stability derivatives listed in table 6.6.

For the second question: make sure to use the statement "in horizontal flight" to simplify Eq. (6.8,6). Refer to Appendices B.9 and B.11. For this plane, the aspect ratio is $A > 1$ (recall $A = b^2/S$). So, in App. B.9, use Eq (B.9,1). Of all the terms therein, the only one that matters in this problem is $\left(\frac{C_{l\beta}}{\Gamma} K_{M_\Gamma}\right)$.

(Note: there is a typo in Eq.(B.9,1). The last term within the square bracket is $\left(\frac{C_{\beta}}{C_L}\right)_A$. The A subscript should not be outside the square bracket.)

Look up the value of those terms in Figs. B.9,4 and B.9,5 and concentrate on their SIGN. Similarly, in App. B.11, use Eq. (B.11,1) and concentrate on the term $\left(\frac{\Delta C_{lr}}{\Gamma}\right)$. Look at its expression in Eq. (B.11,3) and concentrate on its SIGN.

Solution:

2.3.3.1 Part (1)

The first step is to evaluate E for the given jet using the stability derivatives in table 6.6.

Table 6.6
Nondimensional Derivatives—B747 Airplane

	C_y	C_l	C_n
β	−0.8771	−0.2797	0.1946
$\hat{p}$	0	−0.3295	−0.04073
$\hat{r}$	0	0.304	−0.2737

Where $E > 0$ implies the following (per equation 6.8,6 on page 194 of textbook)

$$\left(C_{l_\beta} C_{n_r} - C_{l_r} C_{n_\beta}\right)\cos\theta_0 + \left(C_{l_p} C_{n_\beta} - C_{l_\beta} C_{n_p}\right)\sin\theta_0 > 0 \tag{1}$$

The meaning of these coefficients is explained more in this table

C_l	The rolling moment coefficient $C_l = \frac{L}{\frac{1}{2}\rho V^2 S b}$ where L here is rolling moment and not lift
C_{l_β}	$\frac{\partial C_l}{\partial \beta}$ where β is the side slip angle
C_{l_r}	$\frac{\partial C_l}{\partial r}$ where r is yaw rate
C_{l_p}	$\frac{\partial C_l}{\partial p}$ rolling moment coefficient of propulsion units
C_n	Yawning moment coefficient $C_n = \frac{N}{\frac{1}{2}\rho V^2 S b}$ where N is the yawning moment
C_{n_r}	$\frac{\partial C_n}{\partial r}$ where r is yaw rate
C_{n_β}	$\frac{\partial C_n}{\partial \beta}$ where β is the side slip angle
C_{n_p}	$\frac{\partial C_n}{\partial p}$ Yawning moment coefficient of propulsion units

Using table 6.6 gives

$$C_{l_\beta} = -0.2797$$
$$C_{n_r} = -0.2737$$
$$C_{l_r} = 0.304$$
$$C_{n_\beta} = 0.1946$$
$$C_{l_p} = -0.3295$$
$$C_{n_p} = -0.04073$$

Substituting all these values in (6.8,6) gives

$$\left(C_{l_\beta} C_{n_r} - C_{l_r} C_{n_\beta}\right)\cos\theta_0 + \left(C_{l_p} C_{n_\beta} - C_{l_\beta} C_{n_p}\right)\sin\theta_0 > 0$$

Or

$$(-0.2797 \times -0.2737 - 0.304 \times 0.1946)\cos\theta_0 + (-0.3295 \times 0.1946 - (-0.2797) \times -0.04073)\sin\theta_0 > 0$$

Hence

$$\begin{aligned}1.7395\times10^{-2}\cos\theta_0-7.5513\times10^{-2}\sin\theta_0&>0\\1.7395\times10^{-2}-7.5513\times10^{-2}\tan\theta_0&>0\\\tan\theta_0&>\frac{1.7395\times10^{-2}}{7.5513\times10^{-2}}\\\tan\theta_0&>0.23036\end{aligned}$$

Therefore

$$\begin{aligned}\theta_0&>\tan^{-1}(0.23036)\\&>12.97^\circ\end{aligned}$$

The above implies that the climb angle has to be larger than 12.97° to insure that E remains positive and the jet remain statically stable in spiral mode at any speed.

2.3.3.2 Part (2)

E is now examined to see how it is depends on Γ (dihedral angle). The expression for E above does not have Γ in its as shown, but Γ comes into play when the coefficients are replaced by their expressions in appendix B.9 and B.11. Before doing this, the term multiplied by $\sin\theta_0$ are cancelled inthe above, since the jet is in a horizontal flight or $\theta_0=0$. $E>0$ now implies the following

$$\left(C_{l_\beta}C_{n_r}-C_{l_r}C_{n_\beta}\right)>0 \tag{2}$$

The appendix is now used to replace the expressions in the above. From appendix B.9 and for $A>1$

$$C_{l_\beta}=C_L\left[\left(\frac{C_{l_\beta}}{C_L}\right)_{\Lambda_{c/2}}K_{M_A}+\left(\frac{C_{l_\beta}}{C_L}\right)_A\right]+\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right) \tag{3}$$

The rest of the expression in B.9,1 was not used, since $\theta=0$. For C_{n_β}, looking at equation B.9.3 in the appendix B.9, shows it does not depend on Γ, therefore its current numerical value from the above table is used

$$C_{n_\beta}=0.1946 \tag{4}$$

Looking at C_{l_r}, and from appendix B.11

$$C_{l_r}=C_L\left(\frac{C_{L_r}}{C_L}\right)_{C_{L_M}=0}+\left(\frac{\Delta C_{l_r}}{\Gamma}\right)\Gamma \tag{5}$$

Where the last term was not used since $\theta=0$. Finally from B.11.4 one sees that C_{n_r} does not depend on Γ, therefore its current numerical value is used

$$C_{n_r}=-0.2737 \tag{6}$$

Substituting (3,4,5,6) into (2) results in

$$-0.2737\overbrace{\left(C_L\left[\left(\frac{C_{l_\beta}}{C_L}\right)_{\Lambda_{c/2}}K_{M_A}+\left(\frac{C_{l_\beta}}{C_L}\right)_A\right]+\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right)\right)}^{C_{l_\beta}}$$

$$-0.1946\overbrace{\left(C_L\left(\frac{C_{L_r}}{C_L}\right)_{C_{L_M}=0}+\left(\frac{\Delta C_{l_r}}{\Gamma}\right)\Gamma\right)}^{C_{l_r}}>0 \quad (7)$$

In the above, $\frac{\Delta C_{l_r}}{\Gamma}$ is found from (B.11,3) as

$$\frac{\Delta C_{l_r}}{\Gamma}=\frac{1}{12}\frac{\pi A\sin\Lambda_{c/4}}{A+4\cos\Lambda_{c/4}}$$

Hence (7) becomes

$$-0.2737\overbrace{\left(C_L\left[\left(\frac{C_{l_\beta}}{C_L}\right)_{\Lambda_{c/2}}K_{M_A}+\left(\frac{C_{l_\beta}}{C_L}\right)_A\right]+\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right)\right)}^{C_{l_\beta}}$$

$$-0.1946\overbrace{\left(C_L\left(\frac{C_{L_r}}{C_L}\right)_{C_{L_M}=0}+\left(\frac{1}{12}\frac{\pi A\sin\Lambda_{c/4}}{A+4\cos\Lambda_{c/4}}\right)\Gamma\right)}^{C_{l_r}}>0 \quad (8)$$

In order to determine what happens as Γ changes, it is assumed that all terms that do not involve Γ above are fixed for the time being and can be renamed to constants z_1 and z_2 for the purpose of finding the effect of changing Γ. Writing (8) as

$$-z_1\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right)-z_2\frac{1}{12}\frac{\pi A\sin\Lambda_{c/4}}{A+4\cos\Lambda_{c/4}}\Gamma>0 \quad (9)$$

This above can be simplified more since terms such as π and A are fixed and do not change with changing Γ. In addition, $\Lambda_{c/4}$ do not change with Γ. The above is simplified more resulting in

$$-\overbrace{z_1\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right)}^{C_{l_\beta}}-\overbrace{z_2\Gamma}^{C_{l_r}}>0 \quad (10)$$

The LHS above is required to remain positive for stability. The more positive it is, the more stable the system is. As Γ increases, then $C_{l_r}=z_2\Gamma$ will become more positive. But what happens to $\Gamma\left(\frac{C_{l_\beta}}{\Gamma}K_{M_\Gamma}\right)$ as Γ increases? To find what happens to $\frac{C_{l_\beta}}{\Gamma}$ and what happens to K_{M_Γ}, the hints given are used. From figure B.9,4 it is seen that $\frac{C_{l_\beta}}{\Gamma}$ is negative curve, as the y-axis is negative for positive aspect ratio (the x-axis). This was the case for all values of the taper ratios λ.

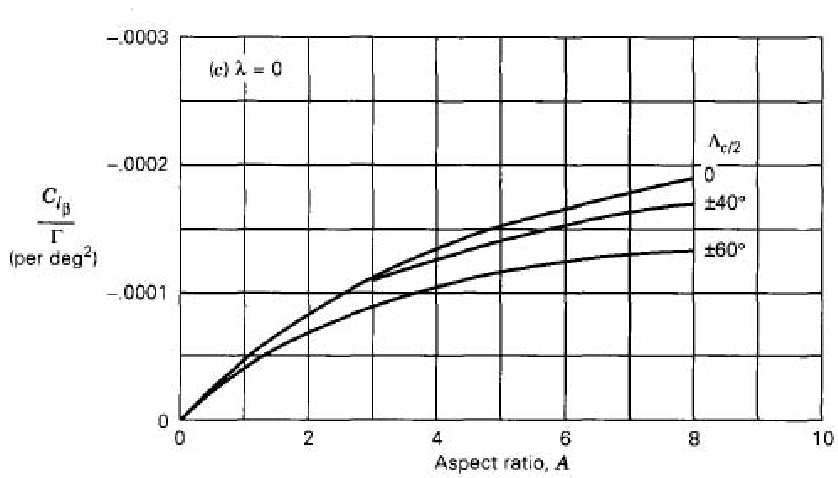


Figure B.9,4 Effect of uniform geometric dihedral on wing C_{l_β}.

Looking at figure B.9,5 shows that $K_{M_\Gamma} > 0$ for all ranges defined over $M \cos \Lambda_{c/2}$.

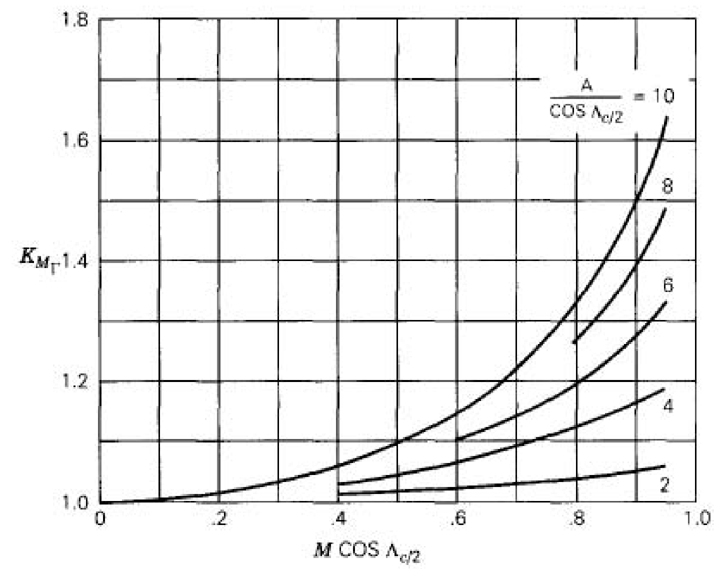


Figure B.9,5 Compressibility correction to dihedral effect on wing C_{l_β}.

This means $\left(\frac{C_{l_\beta}}{\Gamma} K_{M_\Gamma}\right)$ = (negative × positive) = negative. Therefore, as Γ increases, C_{l_β} becomes more negative (since Γ is positive). To see this more clearly, (10) is written as

$$-\overbrace{\Gamma\,(\text{negative})}^{C_{l_\beta}} - \overbrace{\Gamma\,(\text{positive})}^{C_{l_r}} > 0 \tag{11}$$

or

$$\overbrace{\Gamma\,(\text{positive})}^{C_{l_\beta}} + \overbrace{\Gamma\,(\text{negative})}^{C_{l_r}} > 0 \tag{12}$$

E has to be positive for spiral stability. Between the two terms above, C_{l_β} will cause E to be become more positive as Γ increases. While the second term C_{l_r} will have the opposite effect, it will reduce E and cause it to become negative.

It is assumed in all of this that Γ is a positive angle and remain positive.

2.3.3.3 Conclusion

For horizontal flight $\theta = 0$, as the Γ angle increases, its effect on C_{l_r} is to make the airplane become unstable in spiral mode, while Γ effect on C_{l_β} is to make the airplane become stable.

2.3.4 Problem 4

6.7 Using the stability derivatives given in Table 7.2 for a general aviation airplane, calculate the lateral modes *in the absence of gravity*. The relevant data are:

$W = 2400$ lb (10,675 N) $\quad I_x = 170$ slug·ft^2 (230 kg·m^2)

$S = 160$ ft^2 (14.9 m^2) $\quad I_z = 1{,}312$ slug·ft^2 (1,778 kg·m^2)

$b = 30$ ft (9.14 m) $\quad I_{zx} = 0$

$V = 150$ knots (77.3 m/s) $\quad \theta_0 = 0$

altitude = sea level

Compare the results with those for gravity present.

4. Problem 6.7 in the textbook. Although not stated quite clearly, you are required to find BOTH the eigenvalues AND the eigenvectors and to do so for BOTH the case WITH gravity AND the case WITHOUT it. Although unintuitive, in the absence of gravity the lift coefficient C_L=0. Use this in evaluating the stability derivative in Tab. 7.2.

Following the same approach we used in class: the eigenvector corresponding to each eigenvalue has four components; each component comes with an amplitude and a phase; arbitrarily pick an amplitude of 1 and a phase of 0 for any one of the components (here use r) and evaluate the 3 other amplitudes relative to A_r and the 3 other phases with respect to η_r. It's best to use the eig(A) MATLAB routine or similar. By all means, use the routine posted on the course website. This problem is quite long.

Solution method summary: It is required to write $\dot{x} = Ax$ for the lateral mode. This is equation 4.9,19 on page 113 of the textbook. These are expressed using dimensional derivatives $Y_v, Y_m \cdots$. Table 7.2 is used, and using non-dimensional derivatives $C_{y_\beta}, C_{l_\beta}, \cdots$, with table 4.5 on page 118, the numerical values in the dimensional matrix A are found. Having obtained $\dot{x} = Ax$ in numerical form, Matlab was used to obtain the eigenvalues and eigenvectors. The above is done for both $C_L = 0$ and $C_L \neq 0$. For the case of $C_L = 0$ or $g = 0$, the A matrix becomes 3×3 while for $g \neq 0$ the A matrix remained 4×4.

Solution:

Table 7.2 in the text book is

Table 7.2
Nondimensional Derivatives—General Aviation Airplane (expressed in rad^{-1} and (rad/s)$^{-1}$)

	C_y	C_l	C_n
β	−0.14	$-0.0689 - 0.0917C_L$	$0.01326 + 0.017C_L^2$
$\hat{p}$	−0.039	−0.441	$-0.00109 - 0.0966C_L$
$\hat{r}$	0.165	$-0.0144 + 0.271C_L$	$-0.048 - 0.0238C_L^2$
δ_a	0	−0.0531	0.005
δ_r	0.117	0.0105	−0.0509

Table 4.5, page 118 of the textbook is used to convert from dimensional to non-dimensional

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$

Equation 4.9.19 for lateral mode, in dimensional form is

$$\begin{pmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \left(\frac{L_v}{I'_x} + I'_{zx}N_v\right) & \left(\frac{L_p}{I'_x} + I'_{zx}N_p\right) & \left(\frac{L_r}{I'_x} + I'_{zx}N_r\right) & 0 \\ \left(I'_{zx}L_v + \frac{N_v}{I'_z}\right) & \left(I'_{zx}L_p + \frac{N_p}{I'_z}\right) & \left(I'_{zx}L_r + \frac{N_r}{I'_z}\right) & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix} \begin{pmatrix} v \\ p \\ r \\ \phi \end{pmatrix} \tag{4.9,19}$$

Where

$$\begin{aligned} I'_x &= \frac{\left(I_x I_z - I_{zx}^2\right)}{I_z} = \frac{(230 \times 1778 - 0)}{1778} = 230 \text{ kg m}^2 \\ I'_z &= \frac{\left(I_x I_z - I_{zx}^2\right)}{I_x} = \frac{\left(230 \times 1778 - I_{zx}^2\right)}{230} = 1778 \text{ kg m}^2 \\ I'_{zx} &= \frac{I_{zx}}{\left(I_x I_z - I_{xz}^2\right)} = 0 \end{aligned}$$

To find air density ρ, we are told the airplane is at see level. Hence using table in appendix

D then we find the corresponding air density at that altitude

$$\rho = 2.3769 \times 10^{-3} \text{ lb } \frac{\sec^2}{ft^4}$$
$$= 1.225 \text{ kg/m}^3$$

2.3.4.1 Gravity present case

In this case

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$$

But at trim, $L = mg$, hence

$$C_L = \frac{mg}{\frac{1}{2}\rho V^2 S} = \frac{10675}{\frac{1}{2}(1.225)(77.3)^2(14.9)} = \boxed{0.196}$$

Now the numerical values of the dimensional derivatives are calculated

non-dim.	value
C_{y_β}	-0.14
C_{y_p}	-0.039
C_{y_r}	0.165
C_{l_β}	$-0.0689 - 0.0917C_L = -0.0689 - 0.0917 \times 0.195\,76 = -0.08685$
C_{l_p}	-0.441
C_{l_r}	$-0.0144 + 0.271C_L = -0.0144 + 0.271 \times 0.195\,76 = 0.03865$
C_{n_β}	$0.01326 + 0.017C_L^2 = 0.01326 + 0.017 \times 0.195\,76^2 = 0.0139$
C_{n_p}	$-0.00109 - 0.0966C_L = -0.00109 - 0.0966 \times 0.195\,76 = -0.02$
C_{n_r}	$-0.048 - 0.0238C_L^2 = -0.048 - 0.0238 \times 0.195\,76^2 = -0.0489$

Hence

dim.	non-dim.	numerical equation	value
Y_v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}(1.225)(77.3)(14.9)(-0.14)$	-98.764
Y_p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}(1.225)(77.3)(9.14)(14.9)(-0.039)$	-125.73
Y_r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}(1.225)(77.3)(9.14)(14.9)(0.165)$	531.95
L_v	$\frac{1}{2}\rho u_0 S C_{l_\beta}$	$\frac{1}{2}(1.225)(77.3)(9.14)(14.9)\left(-8.685\,1 \times 10^{-2}\right)$	-560.01
L_p	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.441)$	-12995
L_r	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)\left(3.865\,1 \times 10^{-2}\right)$	1138.9
N_v	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$	$\frac{1}{2}(1.225)(77.3)(9.14)(14.9)\left(1.391\,1 \times 10^{-2}\right)$	89.700
N_p	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.02)$	-589.35
N_r	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)\left(-4.891\,2 \times 10^{-2}\right)$	-1441.3

Now that all the numerical values are calculated, 4.19,9 is written again in order to find a numerical A matrix in order to determine its eigenvalues.

Since $\theta = 0$, equation 4.19,9 becomes

$$\begin{pmatrix}\dot{v}\\ \dot{p}\\ \dot{r}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \frac{-98.764}{10675/9.81} & \frac{-125.73}{10675/9.81} & \left(\frac{531.95}{10675/9.81} - 77.3\right) & 9.81\\ \left(\frac{-560.01}{230} + I'_{zx}(89.7)\right) & \left(\frac{-12995}{230} + I'_{zx}(-589.35)\right) & \left(\frac{1138.9}{230} + I'_{zx}(-1441.3)\right) & 0\\ \left(I'_{zx}(-560.01) + \frac{89.700}{1778}\right) & \left(I'_{zx}(-12995) + \frac{-589.35}{1778}\right) & \left(I'_{zx}(1138.9) + \frac{-1441.3}{1778}\right) & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}v\\ p\\ r\\ \phi\end{pmatrix}$$

And since $I'_{zx} = 0$ the above reduces to

$$\begin{pmatrix}\dot{v}\\ \dot{p}\\ \dot{r}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} -0.0907610 & -0.11554 & -76.811 & 9.81\\ -2.4348 & -56.5 & 4.9517 & 0\\ 0.05045 & -0.33147 & -0.81063 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}v\\ p\\ r\\ \phi\end{pmatrix}$$

The above is in the form of $\dot{x} = Ax$. The eigenvalues of A are found using Matlab. They are

$$\begin{aligned}\lambda_{dutch} &= -0.4218 \pm 2.2873i\\ \lambda_{spiral} &= -0.0551\\ \lambda_{rolling} &= -56.5025\end{aligned}$$

The characteristic polynomial is

```
EDU>> syms x
EDU>> vpa(charpoly(A,x),6)
x^4 + 57.4014*x^3 + 56.2385*x^2 + 308.576*x + 16.832
```

Hence

$$\begin{aligned}p(\lambda) &= \lambda^4 + 57.4014\lambda^3 + 56.2385\lambda^2 + 308.576\lambda + 16.832\\ &= A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E\end{aligned}$$

Therefore

$$\begin{aligned}E &= 16.832 > 0\\ R &= D(BC - AD) - B^2E\\ &= 308.576(57.4014 \times 56.2385 - 1 \times 308.576) - \left(57.4014^2\right)(16.832)\\ &= 8.4546 \times 10^5 > 0\end{aligned}$$

Hence all modes are stable since both E and R are positive. Now that the eigenvalues are known, the system characteristic timing table is generated, as was done in the textbook on page 188. Here is the table for this airplane

Mode	name	$\lambda = n \pm \omega i$	period (sec) $\frac{2\pi}{\omega}$	t_{half} (s) $\frac{0.693}{\lvert n \rvert}$	N_{half} (cycles) $0.11\frac{\omega}{\lvert n \rvert}$
1	spiral	-0.0551	—	$\frac{0.693}{0.0551} = 12.577$	—
2	rolling	-56.5025	—	$\frac{0.693}{56.5025} = 0.012265$	—
3	dutch roll	$-0.4218 \pm 2.2873i$	$\frac{2\pi}{2.2873} = 2.7470$	$\frac{0.693}{0.4218} = 1.6430$	$0.11\frac{2.2873}{0.4218} = 0.59650$

The dutch roll is oscillatory, its characteristic transients is plotted below

```
w = 2.2873; n = -0.4218;
Plot[Exp[n t] (Sin[w t]), {t, 0, 10}, PlotRange -> {Automatic, {-1, 1}},
AxesOrigin -> {0, 0}, Frame -> True, GridLines -> Automatic,
GridLinesStyle -> LightGray, FrameLabel -> {{"pertrbation", None},
{"Time (sec)", "Dutch roll mode"}},
ImageSize -> 400]
```

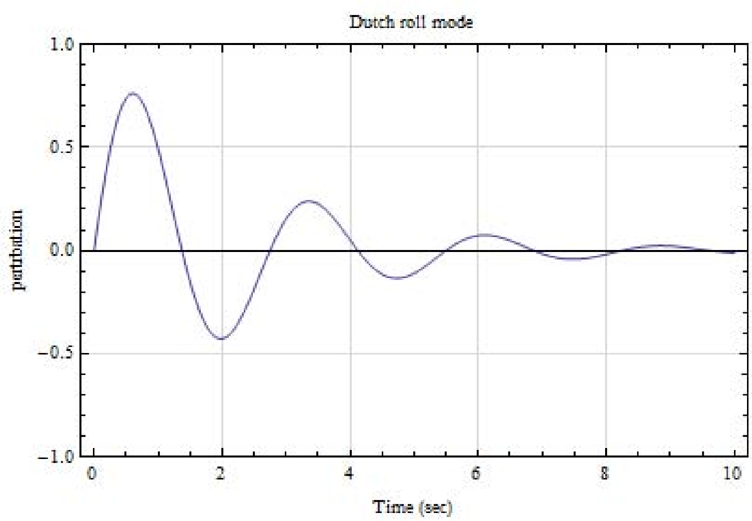


Figure 2.27: Dutch roll mode response for $C_L \neq 0$

The corresponding eigenvectors are now found in order to generate the eigenvector phase diagrams similar to figure 6.3, page 169 in the textbook. The solution will appear as follows

$$\begin{pmatrix} v \\ p \\ r \\ \phi \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} e^{\lambda_1 t} + \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix} e^{\lambda_2 t} + \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \end{pmatrix} e^{\lambda_3 t} + \begin{pmatrix} x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \end{pmatrix} e^{\lambda_4 t}$$

Where in the above, the vector $\begin{pmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{pmatrix}$ is the i^{th} eigenvector that corresponds to the i^{th} eigenvalue.
The above can also be written as

$$\boxed{\boldsymbol{x} = \boldsymbol{x}_1 e^{\lambda_1 t} + \boldsymbol{x}_2 e^{\lambda_2 t} + \boldsymbol{x}_3 e^{\lambda_3 t} + \boldsymbol{x}_4 e^{\lambda_4 t}}$$

Where the eigenvalues λ_i are known, but not the eigenvectors $\boldsymbol{x}_i$. By definition, an eigenvector x_i corresponding to eigenvalue λ_i can be found from

$$Ax_i = \lambda_i x_i$$
$$(A - \lambda_i I)\, x_i = 0$$

In this problem, Matlab was used to obtain the eigenvectors.

The problem asks to normalize the eigenvector using the third entry, which is r. Therefore, after finding the eigenvectors using the eig command, each entry in the eigenvector was divided by the third entry in the same eigenvector. A small function was written to automate this process for both $C_L = 0$ and $C_L \neq 0$. The function and its full output are listed in the appendix of this problem. Here is the result found. The entries in the eigenvector were made to be non-dimensional. This seems to be what was done in the textbook, page 189. Non-dimensional eigenvector was generated from the eigenvector returned by Matlab as follows:

To convert eigenvector $\begin{pmatrix} v \\ p \\ r \\ \Delta\phi \end{pmatrix}$ to non-dimensional form, we multiply elements as follows $\begin{pmatrix} \hat{v} \\ \hat{p} \\ \hat{r} \\ \Delta\hat{\phi} \end{pmatrix} = \begin{pmatrix} \frac{v}{u_0} \\ p\frac{b}{2u_0} \\ r\frac{b}{2u_0} \\ \Delta\phi \end{pmatrix}$. This was done for each eigenvector of each mode. The dutch mode has two complex conjugate eigenvalues and counts as one mode.

After the eigenvectors are found, polar form table for the eigenvectors is made, similar to table 6.4, page 168 of the textbook, then the vector phasor diagrams is drawn for the dutch mode.

The polar form of each eigenvector is summarized below

2.3.4.1.1 dimensional result

	dutch $\lambda_{3,4} = -0.4218 \pm 2.2873i$		spiral $\lambda_1 = -0.055268$		rolling $\lambda_2 = -56.5025$	
	magnitude	phase (deg)	magnitude	phase (deg)	magnitude	phase (deg)
v	35.874	80.158	12.115	0	2.2246	0
p	1.5436	−98.95	0.43488	180	168.36	0
r	1	0	1	0	1	0
$\Delta\phi$	0.66332	−199.39	7.868	0	2.9796	−180

2.3.4.1.2 non-dimensional result v, p, r are first made non-dimensional. Next, the ratio variable to r is found. The Matlab function below shows the the implementation details.

	dutch $\lambda_{3,4} = -0.4218 \pm 2.2873i$		spiral $\lambda_1 = -0.055268$		rolling $\lambda_2 = -56.5025$	
	amplitude	phase (deg)	amplitude	phase (deg)	amplitude	phase (deg)
v/r	7.849	80.158	2.651	0	0.4867	0
p/r	1.543	−98.95	0.4348	180	168.36	0
r/r	1	0	1	0	1	0
$\frac{\Delta\phi}{r}$	11.22	−199.39	133.09	0	50.4	−180
norm	13.33		133.09		176.43	

Figure 2.28 is the eigenvector diagram for the dutch mode

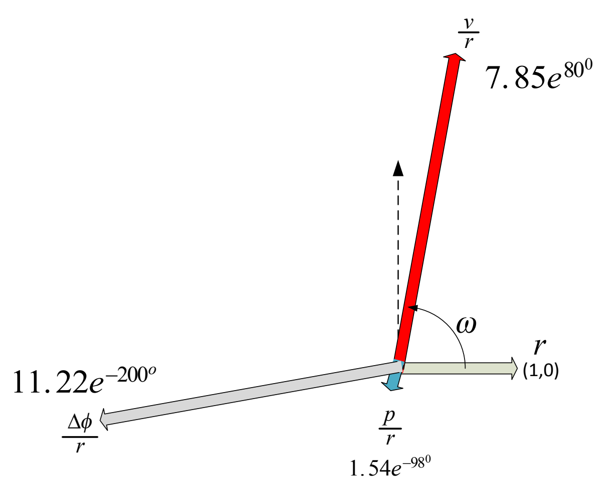


Figure 2.28: Dutch mode response for $C_L \neq 0$

2.3.4.2 No gravity case

The above calculations are now repeated for $C_L = 0$ case. Calculation of the numerical values of the dimensional derivatives gives

non-dim.	value
C_{y_β}	-0.14
C_{y_p}	-0.039
C_{y_r}	0.165
C_{l_β}	$-0.0689 - 0.0917C_L = -0.0689$
C_{l_p}	-0.441
C_{l_r}	$-0.0144 + 0.271C_L = -0.0144$
C_{n_β}	$0.01326 + 0.017C_L^2 = 0.01326$
C_{n_p}	$-0.00109 - 0.0966C_L = -0.00109$
C_{n_r}	$-0.048 - 0.0238C_L^2 = -0.048$

Hence

dim.	non-dim.	numerical equation	value
Y_v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}(1.225)(77.3)(14.9)(-0.14)$	-98.764
Y_p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}(1.225)(77.3)(9.14)(14.9)(-0.039)$	-125.73
Y_r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}(1.225)(77.3)(9.14)(14.9)(0.165)$	531.95
L_v	$\frac{1}{2}\rho u_0 S C_{l_\beta}$	$\frac{1}{2}(1.225)(77.3)(9.14)(14.9)(-0.0689)$	-444.26
L_p	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.441)$	-12995
L_r	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.0144)$	-424.32
N_v	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$	$\frac{1}{2}(1.225)(77.3)(9.14)(14.9)(0.01326)$	85.499
N_p	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.00109)$	-32.119
N_r	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$	$\frac{1}{4}(1.225)(77.3)\left(9.14^2\right)(14.9)(-0.048)$	-1441.3

With the above, equation 4.19,9 is written again in order to find a numerical A matrix to use to find its eigenvalues, and since $\theta = 0$ then 4.19,9 becomes (but remember to use a 3×3 matrix in this case, since the 4^{th} column is now zero column)

$$\begin{pmatrix}\dot{v}\\ \dot{p}\\ \dot{r}\end{pmatrix} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0\right)\\ \left(\frac{L_v}{I_x'} + I_{zx}'N_v\right) & \left(\frac{L_p}{I_x'} + I_{zx}'N_p\right) & \left(\frac{L_r}{I_x'} + I_{zx}'N_r\right)\\ \left(I_{zx}'L_v + \frac{N_v}{I_z'}\right) & \left(I_{zx}'L_p + \frac{N_p}{I_z'}\right) & \left(I_{zx}'L_r + \frac{N_r}{I_z'}\right)\end{pmatrix}$$

$$= \begin{pmatrix} \frac{-98.764}{10675/9.81} & \frac{-125.73}{10675/9.81} & \left(\frac{531.95}{10675/9.81} - 77.3\right)\\ \left(\frac{-444.26}{230} + I_{zx}'(85.499)\right) & \left(\frac{-12995}{230} + I_{zx}'(-32.119)\right) & \left(\frac{-424.32}{230} + I_{zx}'(-1441.3)\right)\\ \left(I_{zx}'(-444.26) + \frac{85.499}{1778}\right) & \left(I_{zx}'(-12995) + \frac{-32.119}{1778}\right) & \left(I_{zx}'(-424.32) + \frac{-1441.3}{1778}\right)\end{pmatrix}\begin{pmatrix}v\\ p\\ r\end{pmatrix}$$

And since $I'_{zx} = 0$ the above reduces to

$$\begin{pmatrix}\dot{v}\\ \dot{p}\\ \dot{r}\end{pmatrix}\begin{pmatrix}-0.0907761 & -0.11554 & -76.811\\ -1.9316 & -56.5 & -1.8449\\ 0.048087 & -0.018065 & -0.79551\end{pmatrix}\begin{pmatrix}v\\ p\\ r\end{pmatrix}$$

The above is in the form of $\dot{x} = Ax$. The eigenvalues of A (using eig command) are

$$\lambda_{dutch} = -0.44044 \pm 1.9015i$$
$$\lambda_{rolling} = -56.505$$

The characteristic polynomial is

```
EDU>> sym x
EDU>> vpa(charpoly(A,x),6)
x^3 + 57.3858*x^2 + 53.5831*x + 215.257
```

Hence

$$\begin{aligned} p(\lambda) &= 0x^4 + x^3 + 57.3858x^2 + 53.5831x + 215.257 \\ &= A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E \end{aligned}$$

Therefore

$$\begin{aligned} E &= 215.257 > 0 \\ R &= D(BC - AD) - B^2E \\ &= 53.5831(1 \times 57.3858) - \left(1^2\right)(215.257) \\ &= 2859.7 > 0 \end{aligned}$$

All modes are stable since both E and R are positive. Now that the eigenvalues are known, the characteristic times table is generated, as was done in the textbook on page 188.

Mode	name	$\lambda = n \pm \omega i$	period (sec) $\frac{2\pi}{\omega}$	t_{half} (s) $\frac{0.693}{\lvert n\rvert}$	N_{half} (cycles) $0.11\frac{\omega}{\lvert n\rvert}$
1	rolling	-56.505	—	$\frac{0.693}{56.505} = 0.01226$	—
2	dutch roll	$-0.44044 \pm 1.9015i$	$\frac{2\pi}{1.9015} = 3.3043$	$\frac{0.693}{0.44044} = 1.5734$	$0.11\frac{1.9015}{0.44044} = 0.4749$

The dutch roll is oscillatory. The characteristic transient is plotted below

```
w = 1.9015; n = -0.44044;
Plot[Exp[n t] (Sin[w t]), {t, 0, 10}, PlotRange -> {Automatic, {-1, 1}},
    AxesOrigin -> {0, 0}, Frame -> True, GridLines -> Automatic,
    GridLinesStyle -> LightGray, FrameLabel -> {{"pertrbation", None},
    {"Time (sec)", "Dutch roll mode"}},
    ImageSize -> 400
]
```

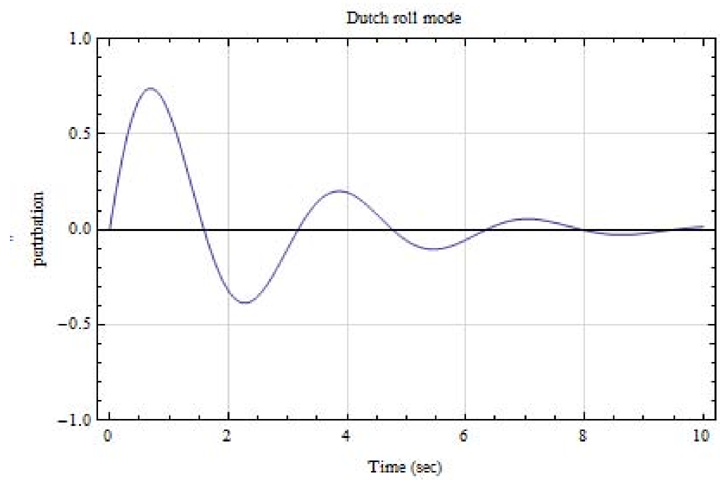


Figure 2.29: Dutch roll mode response for $C_L = 0$

Comparing the above to $g \neq 0$, it is seen that there is very little change in the dutch mode when $g = 0$.

Similar to first part, now that all the eigenvectors are found, the polar form table for the eigenvectors is made which is similar to table 6.4, page 168 of the textbook. This is followed by making the vector phasor diagrams for the dutch mode.

The polar form of each eigenvector is summarized below

2.3.4.2.1 dimensional result

	dutch $\lambda_{1,2} = -0.44044 \pm 1.9015i$		rolling $\lambda_3 = -56.505$	
	amplitude	phase (deg)	amplitude	phase (deg)
v	39.71	79.465	7.71	0
p	1.374	256.17	3104.4	0
r	1	0	1	0

2.3.4.2.2 non-dimensional result

	dutch $\lambda_{1,2} = -0.44044 \pm 1.9015i$		rolling $\lambda_2 = -56.505$	
	amplitude	phase (deg)	amplitude	phase (deg)
$\frac{v}{r}$	8.6893	79.465	1.69	0
$\frac{p}{r}$	1.3739	256.17	3104.4	0
$\frac{r}{r}$	1	0	1	0
norm	9.2688		3105.4	

Figure 2.30 shows the eigenvector diagram for the dutch mode for $g = 0$

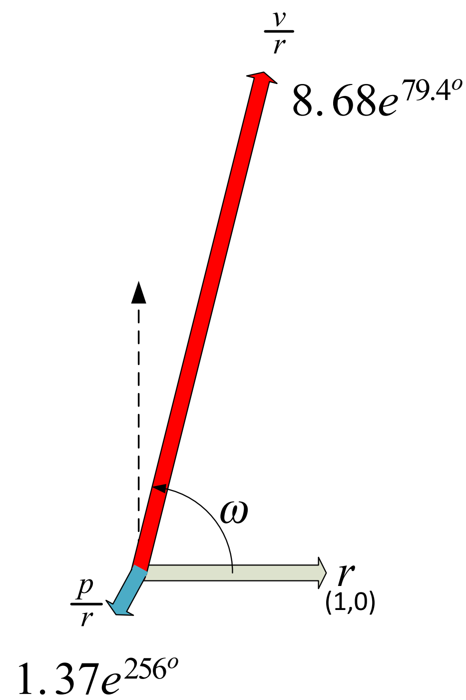


Figure 2.30: Dutch mode response for $C_L = 0$

2.3.4.3 Observation on results

The first observation is that, when gravity is absent, the spiral mode vanishes. There are only three modes when $g = 0$, and these are the dutch (complex conjugate eigenvalues) and rolling (real eigenvalue) and yaw r. Only the dutch and roll modes can be compared to each others for both cases. Spiral mode can not be compared since this mode does not exist when $g = 0$.

For the dutch mode (the oscillatory mode), There is no significant change that can be seen. So one can conclude that gravity has little effect on the dutch mode.

The final solution is now written for both cases, and plotted against each others to compare graphically. (this uses the ratios in the eigenvectors components taken from the non-dimensional results obtained above). The solution for $g \neq 0$ is

$$\begin{pmatrix} \frac{v}{r} \\ \frac{p}{r} \\ \frac{r}{r} \\ \phi_r \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix}_{spiral} e^{\lambda_1 t} + \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix}_{rolling} e^{\lambda_2 t} + \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \end{pmatrix}_{dutch} e^{\lambda_3 t} + \begin{pmatrix} x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \end{pmatrix}_{dutch} e^{\lambda_4 t}$$

$$= \begin{pmatrix} 2.651 \\ -0.4348 \\ 1 \\ 133.09 \end{pmatrix} e^{-0.055268t} + \begin{pmatrix} 0.4867 \\ 168.36 \\ 1 \\ -50.4 \end{pmatrix} e^{-56.5025t} +$$

$$e^{-0.4218t} \begin{pmatrix} 7.849\left(\cos\left(2.2873t + 80.15^o\right) + \sin\left(2.2873t + 80.15^o\right)\right) \\ 1.543\left(\cos\left(2.2873t - 98.95^o\right) + \sin\left(2.2873t - 98.95^o\right)\right) \\ \left(\cos 2.2873t + \sin 2.2873t\right) \\ 11.22\left(\cos\left(2.2873t - 199.39^o\right) + \sin\left(2.2873t - 199.39^o\right)\right) \end{pmatrix}$$

Hence

$$\begin{aligned} \frac{v}{r} &= 2.651e^{-0.055268t} + 0.4867e^{-56.5025t} \\ &\quad + 7.849e^{-0.4218t}\left(\cos\left(2.2873t + 80.15^\circ\right) + \sin\left(2.2873t + 80.15^\circ\right)\right) \\ \frac{p}{r} &= -0.4348e^{-0.055268t} + \overbrace{168.36e^{-56.5025t}}^{\text{compare to g=0 below}} \\ &\quad + 1.543e^{-0.4218t}\left(\cos\left(2.2873t - 98.95^\circ\right) + \sin\left(2.2873t - 98.95^\circ\right)\right) \\ \frac{r}{r} &= e^{-0.055268t} + e^{-56.5025t} + e^{-0.4218t}\left(\cos 2.2873t + \sin 2.2873t\right) \\ \frac{\phi}{r} &= 133.09e^{-0.055268t} - 50.4e^{-56.5025t} \\ &\quad + 11.22e^{-0.4218t}\left(\cos\left(2.2873t - 199.39^\circ\right) + \sin\left(2.2873t - 199.39^\circ\right)\right) \end{aligned}$$

The solution for $g = 0$ is

$$\begin{pmatrix} \frac{v}{r} \\ \frac{p}{r} \\ \frac{r}{r} \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}_{rolling} e^{\lambda_2 t} + \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}_{dutch} e^{\lambda_3 t} + \begin{pmatrix} x_{14} \\ x_{24} \\ x_{34} \end{pmatrix}_{dutch} e^{\lambda_4 t}$$

$$= \begin{pmatrix} 1.69 \\ 3104.4 \\ 1 \end{pmatrix} e^{-56.505t} + e^{-0.44044t} \begin{pmatrix} 8.6893\left(\cos\left(1.9015t + 79.465^o\right) + \sin\left(1.9015t + 79.465^o\right)\right) \\ 1.3739\left(\cos\left(1.9015t + 256.17^o\right) + \sin\left(1.9015t + 256.17^o\right)\right) \\ \left(\cos 1.9015t + \sin 1.9015t\right) \end{pmatrix}$$

Hence

$$\frac{v}{r} = 1.69e^{-56.505t} + 8.68939e^{-0.44044t}\left(\cos\left(1.9015t + 79.465^o\right) + \sin\left(1.9015t + 79.465^o\right)\right)$$

$$\frac{p}{r} = \overbrace{3104.4e^{-56.505t}}^{\text{larger but quickly damps}} + 1.3739e^{-0.44044t}\left(\cos\left(1.9015t\right) + \sin\left(1.9015t\right)\right)$$

$$\frac{r}{r} = e^{-56.505t} + e^{-0.44044t}\left(\cos 1.9015t + \sin 1.9015t\right)$$

From above, we see when $g = 0$ the contribution from rolling mode has much larger amplitude (3104 vs. 168.36). But this damps out very fast thanks to the large negative exponent on the exponential.

To compare the different modes for gravity and no gravity, each component from each solution is plotted against the similar component from the other solution. For example, $\left(\frac{v}{r}\right)_{g=0}$ and $\left(\frac{v}{r}\right)_{g\neq 0}$ are plotted on same plot. The same for $\left(\frac{p}{r}\right)_{g=0}$ and $\left(\frac{p}{r}\right)_{g\neq 0}$. Figure 2.31 shows the the result.

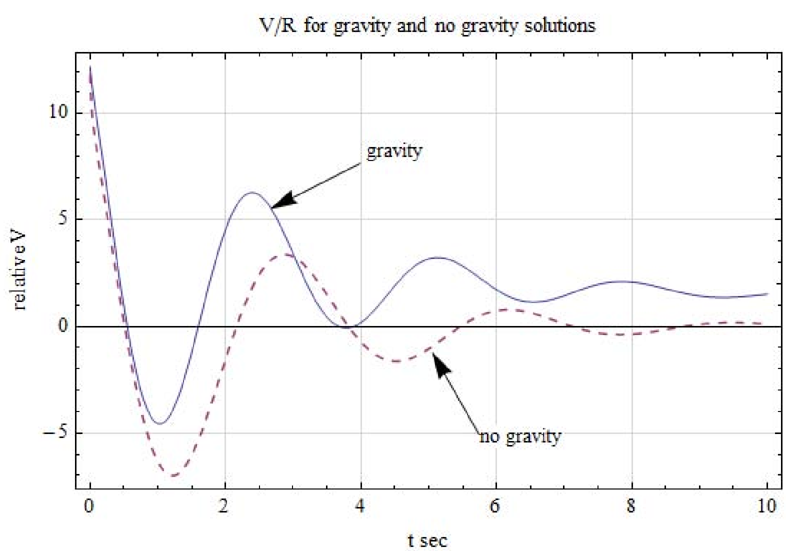


Figure 2.31: Comparing $\frac{v}{r}$ for gravity and no gravity

We see that there is little change in this part of the solution. Figure 2.32 shows the p/r result

Looking Figure 2.32, there does not seem to be difference as well. But that is because $\frac{p}{r}$ damped very quickly, even though it was much larger in the case when there is no gravity. We can see this more clearly if we zoom to a smaller time span, say for $t = 0.1$ seconds only. Figure 2.33 shows the result

This means that the rolling eigen mode will contribute much more to the $p(t)$ component of the total solution with no gravity when compared to with gravity. In other words, rolling becomes the more dominant motion with no gravity. But this only affect the solution for

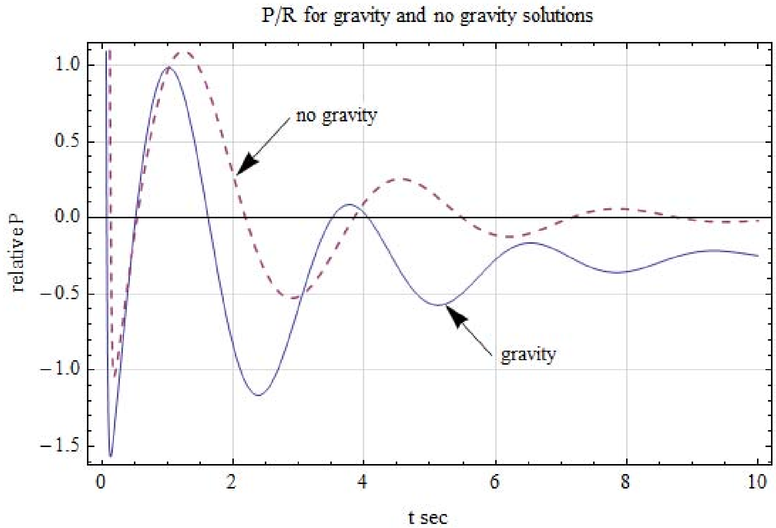


Figure 2.32: Comparing $\frac{p}{r}$ for gravity and no gravity

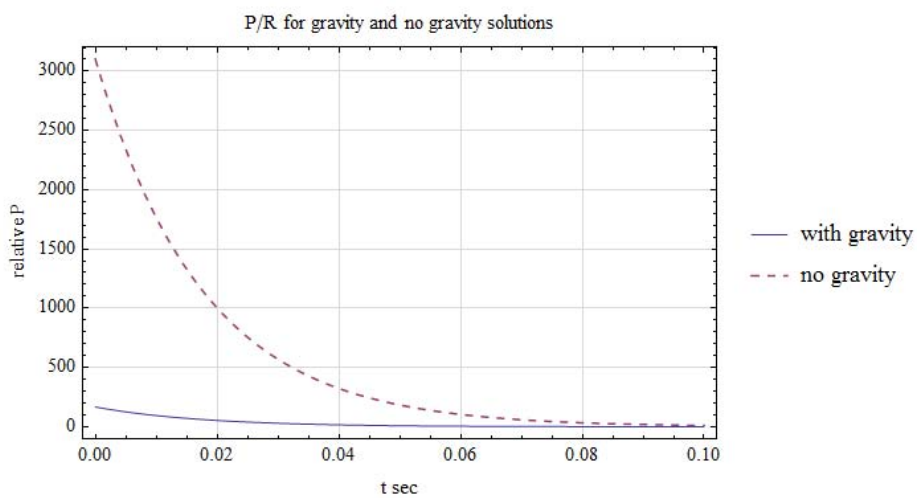


Figure 2.33: Comparing $\frac{v}{r}$ for gravity and no gravity

short amount of time. It will depend on other conditions and on the system being considered if this is important or not.

This also seems to imply that rolling control becomes more important in outer space since gravity becomes very weak. We do not need to worry about spiral mode control since it does not exist. Dutch mode is not affected by gravity.

2.3.4.4 Appendix for problem 4, source code and Matlab output

2.3.4.4.1 source code

```
function problem_4_matlab()
%problem_4_matlab()
%
% This program solves problem 4, HW3, EMA 523, Univ. Wisconsin, Madison
% by Nasser M. Abbasi
% March 29,2014
%
% supports lateral motion only. Does g=0 and g ~= 0
% This function build the matrix A as well as solve for the
% modal eigenvectors.
%
% supports SI and imperial units. Change configuration
% below.  set PROBLEM=2 to do the HW problem, and set
% PROBLEM=1 to solve the example in the book, page 189
% to verify code.
%

POROBLEM=2;  %to select which problem to do, 1 is the example in text
%else will do the HW3, problem 4 (the long one)

if POROBLEM==1
    %------------- problem 6.2 in text book
    RELATIVE_TERM=4;  %change if you want v,p,r,phi, in this order
    SI=false;  %change to true as needed
    g=32.2;
    W=636636;
    m=W/g;
    S=5500;
    b=195.7;
    Ix=0.183*10^8;
    Iz=0.497*10^8;
    Izx=-0.156*10^7;
    u0=774;
    CL=0.654;
    rao=0.0005909;
    Cyb=-0.8771;
    Clb=-0.2797;
    Cnb=0.1946;
    Cyp=0;
```

```
    Clp=-0.3295;
    Cnp=-0.04073;
    Cyr=0;
    Clr=0.304;
    Cnr=-0.2737;
    DUTCH=1;  %need a away to detect this automatically
    ROLL=3;
    SPIRAL=4;
else
    %---- HW problem 4
    RELATIVE_TERM=3; %problem asks to normalize by r, which is 3rd entry
    %do not know why we selected this while in
    %logintidudal the 4th entry, which is Euler angle phi
    %is used. Need to ask about this

    SI=false; %change to false if you want Imperial units
    if SI
        g=9.81;
        W=10675;
        m=W/g;
        S=14.9;
        b=9.14;
        Ix=230;
        Iz=1778;
        Izx=0;
        u0=77.3;
        rao=1.225;
    else
        g=32.2;
        W=2400;
        m=W/g;
        S=160;
        b=30;
        Ix=170;
        Iz=1312;
        Izx=0;
        u0=77.3*3.2808; %m/sec to ft/sec
        rao=2.3769*10^(-3);
    end

    CL=0; %W/(1/2*rao*u0^2*S)  %for part A or part B, change as needed
    %this below is from table 7.2 in text book
    Cyb=-0.14
    Clb=-0.0689-0.0917*CL
    Cnb=0.01326+0.017*CL^2
    Cyp=-0.039
    Clp=-0.441
```

```
    Cnp=-0.00109-0.0966*CL
    Cyr=0.165
    Clr=-0.0144+0.271*CL
    Cnr=-0.048-0.0238*CL^2
end

%common code to all problem, laterl motion only
%this computes the values to use for equation 4.9,19 in textbook
Yv= 1/2*rao*u0*S*Cyb
Yp= 1/4*rao*u0*b*S*Cyp
Yr= 1/4*rao*u0*b*S*Cyr

Lv= 1/2*rao*u0*b*S*Clb
Lp= 1/4*rao*u0*b^2*S*Clp
Lr= 1/4*rao*u0*b^2*S*Clr

Nv= 1/2*rao*u0*b*S*Cnb
Np= 1/4*rao*u0*b^2*S*Cnp
Nr= 1/4*rao*u0*b^2*S*Cnr

Ipx = (Ix*Iz-Izx^2)/Iz
Ipz = (Ix*Iz-Izx^2)/Ix
Ipzx = Izx/(Ix*Iz-Izx^2)
%build the A matrix. 4x4 for the lateral equation of motion in state space

if CL==0  % 3 by 3 now !
    DUTCH=1;  %tells which column in eig result is each mode.
    ROLL=3;   %need to automate this. Hardcoded for now
    A=[Yv/m                 Yp/m                 Yr/m-u0       ;
        Lv/Ipx+Ipzx*Nv       Lp/Ipx+Ipzx*Np       Lr/Ipx+Ipzx*Nr ;
        Ipzx*Lv+Nv/Ipz       Ipzx*Lp+Np/Ipz       Ipzx*Lr+Nr/Ipz]
else  % 4 by 4
    DUTCH=2;  %tells which column in eig result is each mode.
    ROLL=1;   %need to automate this. Hardcoded for now
    SPIRAL=4;
    A=[Yv/m                 Yp/m                 Yr/m-u0       g;
        Lv/Ipx+Ipzx*Nv       Lp/Ipx+Ipzx*Np       Lr/Ipx+Ipzx*Nr 0;
        Ipzx*Lv+Nv/Ipz       Ipzx*Lp+Np/Ipz       Ipzx*Lr+Nr/Ipz 0;
        0                    1                    0              0]
end

I=sqrt(-1);
[eigenvectors,eigenvalues] = eig(A);
fprintf('----------------- eigenvalues');
eigenvalues=diag(eigenvalues)

%display the char. polynomial
```

```
syms x;
vpa(charpoly(A,x),6)

format short g

fprintf('================> DIMENSIONAL\n');
fprintf('dutch mode\n');
dutch_mode=[abs(eigenvectors(:,DUTCH))/abs(eigenvectors(RELATIVE_TERM,DUTCH)), ...
 180/pi*((angle(eigenvectors(:,DUTCH))-angle(eigenvectors(RELATIVE_TERM,DUTCH))))]

fprintf('rolling mode\n');
rolling_mode=[abs(eigenvectors(:,ROLL))/abs(eigenvectors(RELATIVE_TERM,ROLL)),...
 180/pi*(angle(eigenvectors(:,ROLL))-angle(eigenvectors(RELATIVE_TERM,ROLL)))]

if CL ~= 0
  fprintf('spiral mode\n');
  spiral_mode=[abs(eigenvectors(:,SPIRAL))/abs(eigenvectors(RELATIVE_TERM,SPIRAL)),...
   180/pi*(angle(eigenvectors(:,SPIRAL))-angle(eigenvectors(RELATIVE_TERM,SPIRAL)))]
end

fprintf('================> NON-DIMENSIONAL\n');
%since we divided above to get the ratio, we now need to add the
%factors for non-dimensional. since v->v/u0 and p->p*b/(2 u0) and
%r->r*b/(2 u0) to do the non-dimensional, therefore   v/r and p/r and phi/r
%ratios above have to be adjusted now as follows (little algebra not shown)

dutch_mode(1,1)=dutch_mode(1,1)*2/b;
if CL ~= 0
    dutch_mode(4,1)=dutch_mode(4,1)*(2*u0)/b;
end

fprintf('dutch mode\n');
dutch_mode
fprintf('norm is %f\n',norm( dutch_mode(:,1)+exp(1i*pi/180*dutch_mode(:,2))));

if CL ~= 0
    spiral_mode(1,1)=spiral_mode(1,1)*2/b;
    spiral_mode(4,1)=spiral_mode(4,1)*(2*u0)/b;
    fprintf('spiral mode\n');
    spiral_mode
    fprintf('norm is %f\n',norm( spiral_mode(:,1)+exp(1i*pi/180*spiral_mode(:,2))));
end

rolling_mode(1,1)=rolling_mode(1,1)*2/b;
if CL ~= 0
    rolling_mode(4,1)=rolling_mode(4,1)*(2*u0)/b;
end
```

```

fprintf('rolling mode\n');
rolling_mode
fprintf('norm is %f\n',norm(rolling_mode(:,1)+exp(1i*pi/180*rolling_mode(:,2))));
end
```

2.3.4.5 output for C not zero SI units version

```
EDU>> problem_4_matlab
A =
-0.090761     -0.11555      -76.811         9.81
-2.4348         -56.5       4.9517             0
0.05045     -0.33146     -0.81062              0
0            1             0            0
----------------- eigenvalues
eigenvalues =
-56.502 +          0i
-0.42168 +     2.2886i
-0.42168 -     2.2886i
-0.055268 +          0i
x^4 + 57.4009*x^3 + 56.2364*x^2 + 308.924*x + 16.9114
=================> DIMENSIONAL
dutch_mode =
35.874       80.158
1.5436       -98.95
1            0
0.66332      -199.39
rolling_mode =
2.2246            0
168.36            0
1            0
2.9796          -180
spiral_mode =
12.115            0
0.43488          180
1            0
7.8686            0
=================> NON-DIMENSIONAL
dutch_mode =
7.8498       80.158
1.5436       -98.95
1            0
11.22      -199.39
norm is 13.338843
spiral_mode =
2.651            0
0.43488          180
1            0
133.09            0
norm is 134.160410
rolling_mode =
0.48679            0
```

```
168.36           0
1            0
50.4         -180
norm is 176.431860
```

2.3.4.6 output for $C_L = 0$ SI units version

```
EDU>> problem_4_matlab
A =
    -0.090761     -0.11555      -76.811
      -1.9316        -56.5      -1.8449
     0.048087    -0.018065     -0.79551
----------------- eigenvalues

     -0.44044 +     1.9015i
     -0.44044 -     1.9015i
      -56.505 +          0i

ans =
x^3 + 57.3858*x^2 + 53.5831*x + 215.257

=================> DIMENSIONAL
dutch_mode =

        39.71       79.465
       1.3739       256.17
            1            0

rolling_mode =

         7.72            0
       3104.4            0
            1            0

=================> NON-DIMENSIONAL
dutch_mode =

       8.6893       79.465
       1.3739       256.17
            1            0

norm is 9.268892

rolling_mode =
       1.6893            0
       3104.4            0
            1            0
norm is 3105.448784
```

2.3.4.7 output for CL not zero Imperial units version

```
EDU>> problem_4_matlab
A =
-0.09058      -0.3785         -252         32.2
-0.73948      -56.294       4.9505            0
0.015343     -0.33158     -0.80909            0
0             1            0            0
eigenvalues =
-56.297 +          0i
-0.42083 +      2.287i
-0.42083 -      2.287i
-0.05525 +          0i
x^4 + 57.1937*x^3 + 55.9473*x^2 + 307.346*x + 16.8197
=================> DIMENSIONAL
dutch_mode =
117.8       80.168
1.5451       -98.94
1            0
0.66446      -199.37
rolling_mode =
7.3191            0
167.68            0
1            0
2.9785          -180
spiral_mode =
39.741            0
0.43452          180
1            0
7.8647            0
=================> NON-DIMENSIONAL
dutch_mode =
7.8534       80.168
1.5451       -98.94
1            0
11.234      -199.37
norm is 13.351797
spiral_mode =
2.6494            0
0.43452          180
1            0
132.97            0
norm is 134.035132
rolling_mode =
0.48794            0
167.68            0
1            0
50.358          -180
norm is 175.773068
```

2.3.4.8 output for $C_L = 0$ Imperial units version

```
EDU>> problem_4_matlab
A =
     -0.09058      -0.3785         -252
     -0.58634      -56.294      -1.8382
     0.014621    -0.018029     -0.79393

----------------- eigenvalues
     -0.43956 +     1.8992i
     -0.43956 -     1.8992i
      -56.299 +          0i

x^3 + 57.1785*x^2 + 53.2939*x + 213.948
=================> DIMENSIONAL
dutch_mode =

       130.44       79.473
       1.3749       256.18
            1            0

rolling_mode =
       25.353            0
       3099.3            0
            1            0

=================> NON-DIMENSIONAL
dutch_mode =

        8.696       79.473
       1.3749       256.18
            1            0

norm is 9.275345

rolling_mode =

       1.6902            0
       3099.3            0
            1            0

norm is 3100.286548
```

2.3.5 Problem 5

6.8 Find the characteristic equation of the hovercraft of Exercise 4.10. Show that when it is statically unstable with both M_θ and L_ϕ positive it can be gyrostabilized (like a spinning top, i.e., solutions remain bounded) if H is large enough.

5. Problem 6.8 in the textbook. Hint: concentrate on the vector $\mathrm{x} = [\theta, q, \phi, p]^T$ and find the matrix A for the differential equation $\dot{\mathrm{x}} = \mathrm{Ax}$. Work in symbolic form and use Maple to solve the characteristic equation.

Notes: the matrix is quite sparse and the characteristic equation should boil down to a polynomial with only s^4 and s^2. Therefore you can solve symbolically for s^2 first and then for s. When solving for s^2, you'll come to a square root. The crux of the problem is to discuss the consequences of the sign of the argument of that square root and to relate the sign of the argument to the gyroscopic term.

solution

We are told in hint to use $x = \begin{pmatrix} \theta \\ q \\ \phi \\ p \end{pmatrix}$ as the state vector. Therefore we need to set up a state equation $\dot{x} = Ax$ that has this form

$$\begin{pmatrix} \dot{\theta} \\ \dot{q} \\ \dot{\phi} \\ \dot{p} \end{pmatrix} = A \begin{pmatrix} \theta \\ q \\ \phi \\ p \end{pmatrix} \tag{1}$$

So the question is, how to build the A matrix above? We have solved this same problem in HW 2, and in there we found expressions for $\{\dot{\theta}, \dot{q}, \dot{\phi}, \dot{p}\}$ and these will be used as is in this problem. These are the equations found in HW2 for this problem

$$\begin{aligned}
\dot{\theta} &= q \\
\dot{q} &= \frac{M - pH}{I_y} \\
\dot{\phi} &= p \\
\dot{p} &= \frac{L + qH}{I_x}
\end{aligned}$$

Where, from problem 4.10 statement, $M = M_\theta \theta$ and $L = L_\phi \phi$, hence the above becomes

$$
\begin{aligned}
\dot{\theta} &= q\\
\dot{q} &= \frac{M_\theta}{I_y}\theta - \frac{H}{I_y}p\\
\dot{\phi} &= p\\
\dot{p} &= \frac{L_\phi}{I_x}\phi + q\frac{H}{I_x}
\end{aligned}
$$

Substituting the above in (1) results in

$$
\begin{pmatrix}\dot{\theta}\\ \dot{q}\\ \dot{\phi}\\ \dot{p}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ \frac{M_\theta}{I_y} & 0 & 0 & -\frac{H}{I_y}\\ 0 & 0 & 0 & 1\\ 0 & \frac{H}{I_x} & \frac{L_\phi}{I_x} & 0\end{pmatrix}\begin{pmatrix}\theta\\ q\\ \phi\\ p\end{pmatrix} \tag{2}
$$

The characteristic equation can now be easily found from the definition

$$
|A - \lambda I| = 0
$$

$$
\begin{vmatrix} -\lambda & 1 & 0 & 0\\ \frac{M_\theta}{I_y} & -\lambda & 0 & -\frac{H}{I_y}\\ 0 & 0 & -\lambda & 1\\ 0 & \frac{H}{I_x} & \frac{L_\phi}{I_x} & -\lambda\end{vmatrix} = 0
$$

The determinant is found using CAS and this is the result

```
(mat = {{-s, 1, 0, 0},
        {M/Iy, -s, 0, -H/Iy},
        {0, 0, -s, 1},
        {0, H/Ix, L/Ix, -s}}) // MatrixForm
```

Out[16]//MatrixForm=

$$
\begin{pmatrix} -s & 1 & 0 & 0\\ \frac{M}{Iy} & -s & 0 & -\frac{H}{Iy}\\ 0 & 0 & -s & 1\\ 0 & \frac{H}{Ix} & \frac{L}{Ix} & -s\end{pmatrix}
$$

The characteristic equation is found

```
poly = Det[mat] == 0;
```

```
poly = Collect[poly, s]
```

Out[18]= $\frac{\text{L M}}{\text{Ix Iy}} + \frac{(\text{H}^2 - \text{Iy L} - \text{Ix M})\ \text{s}^2}{\text{Ix Iy}} + \text{s}^4 == 0$

Therefore

$$\boxed{p(\lambda) = \lambda^4 + \frac{H^2 - LI_y - MI_x}{I_x I_y}\lambda^2 + \frac{LM}{I_x I_y}}$$

This has 4 roots. They all should have negative real part for stability. Using the hint given, we set this as quadratic polynomial in say $r = s^2$ and solve for r first. Letting $s^2 = r$ in the above gives

$$p(r) = r^2 + \frac{H^2 - LI_y - MI_x}{I_x I_y} r + \frac{LM}{I_x I_y}$$

This can be solved using the quadratic equation

$$\begin{aligned} r = s^2 &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ s^2 &= -\frac{1}{2}\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) \pm \frac{1}{2}\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} \end{aligned}$$

$\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2$ is always positive. Let $A = \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)$, hence the above can be written as

$$\begin{aligned} s^2 &= -\frac{1}{2}A \pm \frac{1}{2}\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)} \\ &= \left\{ \overbrace{-\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)}}^{\text{case 1}}, \overbrace{-\frac{1}{2}A - \frac{1}{2}\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)}}^{\text{case 2}} \right\} \end{aligned}$$

We want to find out if the real part of s is negative or not to decide on stability. Considering each case at a time.

2.3.5.1 Case 1

$$s^2 = -\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)}$$

There are two sub-cases to consider under this case. If the expression $A^2 - 4\left(\frac{LM}{I_x I_y}\right)$ under the square root is positive or negative. Let $A^2 - 4\left(\frac{LM}{I_x I_y}\right) = B$ therefore

2.3.5.2 Case B negative

If $B < 0$ then

$$s^2 = -\frac{1}{2}A + \frac{1}{2}i\sqrt{|B|}$$

This is a complex number, say z. To make it easier to proceed, it is written in polar form as $|z|\, e^{i\arg(z)}$. Hence in polar form

$$s^2 = z = |z|\, e^{i\arg z}$$

Where $|z|$ is the magnitude of the complex root, which is always positive. Taking the square root of the complex root gives

$$\begin{aligned} s_{1,2} &= \pm\left(|z|\, e^{i\arg z}\right)^{\frac{1}{2}} \\ &= \pm\sqrt{|z|}e^{\frac{i}{2}\arg z} \\ &= \{+\sqrt{|z|}e^{\frac{i}{2}\arg z}, -\sqrt{z}e^{\frac{i}{2}\arg z}\} \end{aligned}$$

The real part of $+\sqrt{z}e^{\frac{i}{2}\arg z}$ is $\operatorname{Re}\left(\sqrt{z}\left[\cos\left(\frac{\arg z}{2}\right) + i\sin\left(\frac{\arg z}{2}\right)\right]\right) = \sqrt{z}\cos\left(\frac{\arg z}{2}\right)$ and the real part of the second root is $\operatorname{Re}\left(-\sqrt{z}\left[\cos\left(\frac{\arg z}{2}\right) + i\sin\left(\frac{\arg z}{2}\right)\right]\right) = -\sqrt{|z|}\cos\left(\frac{\arg z}{2}\right)$

These two complex roots have the following real parts

$$\begin{aligned} \operatorname{Re}(s_1) &= -\sqrt{|z|}\cos\left(\frac{\arg z}{2}\right) \\ \operatorname{Re}(s_2) &= \sqrt{|z|}\cos\left(\frac{\arg z}{2}\right) \end{aligned}$$

These real parts can not be both negative or both positive at the same time. If we assume $\cos\left(\frac{\arg z}{2}\right) > 0$ for some $\frac{\arg z}{2}$, then $\operatorname{Re}(s_1) < 0$ (since $\sqrt{|z|} > 0$ all the time, since $|z|$ is the magnitude of the complex root). But $\operatorname{Re}(s_2) > 0$ or unstable.

On the other hand, if we assume $\cos\left(\frac{\arg z}{2}\right) < 0$ then $\operatorname{Re}(s_2) < 0$ but now $\operatorname{Re}(s_1) > 0$, hence unstable. Therefore if $B < 0$ then the system is not stable as one of the roots must have positive real part. Since $B = A^2 - 4\left(\frac{LM}{I_x I_y}\right)$, this implies that $4\left(\frac{LM}{I_x I_y}\right) > A^2$. Since $A^2 = \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2$ then this condition means

$$\begin{aligned} 4\left(\frac{LM}{I_x I_y}\right) &> \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 \\ 4LM &> \frac{\left(H^2 - LI_y - MI_x\right)^2}{I_x I_y} \quad \text{(unstable)} \end{aligned}$$

Therefore, if $B < 0$, the system is always unstable as one of the roots is unstable.

2.3.5.3 Case B positive

If $B > 0$ then

$$\begin{aligned} s^2 &= -\frac{1}{2}A + \frac{1}{2}\sqrt{|B|} \\ &= \frac{1}{2}\left(\sqrt{|B|} - A\right) \end{aligned}$$

Now $\sqrt{|B|}$ is positive number. Let $\sqrt{|B|} = |D|$ then

$$s^2 = \frac{1}{2}\left(|D| - A\right)$$

Now we can take the square root

$$\begin{aligned} s &= \pm\sqrt{\frac{1}{2}}\sqrt{|D| - A} \\ &= \left\{\sqrt{\frac{1}{2}}\sqrt{|D| - A}, -\sqrt{\frac{1}{2}}\sqrt{|D| - A}\right\} \end{aligned}$$

If $(|D| - A) > 0$ then s_1 is real and positive, hence unstable. This means $|D| > A$ or $\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)} > A$ and since $A = \frac{H^2 - LI_y - MI_x}{I_x I_y}$ then the condition is

$$\begin{aligned} \sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} &> \frac{H^2 - LI_y - MI_x}{I_x I_y} \\ \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right) &> \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 \\ -4\left(\frac{LM}{I_x I_y}\right) &> 0 \\ \frac{LM}{I_x I_y} &< 0 \\ LM &< 0 \quad \text{(unstable)} \end{aligned}$$

But if $(|D| - A) < 0$ then s_1 and s_2 are pure imaginary numbers. Hence the system is marginally stable (real part is zero. Some books call this marginally unstable). This means

$|D| < A$ or $\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)} < A$ then the condition is

$$\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} < \frac{H^2 - LI_y - MI_x}{I_x I_y}$$
$$\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right) < \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2$$
$$-4\left(\frac{LM}{I_x I_y}\right) < 0$$
$$LM > 0 \quad \text{(marginally stable)}$$

We need repeat all the above for case 2

2.3.5.4 Case 2

$$s^2 = -\frac{1}{2}A - \frac{1}{2}\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)}$$

There is now 2 sub cases to consider. If the expression $\sqrt{A^2 - 4\left(\frac{LM}{I_x I_y}\right)}$under the square root is positive or negative. Let $A^2 - 4\left(\frac{LM}{I_x I_y}\right) = B$ therefore

2.3.5.5 Case negative

If $B < 0$ then

$$s^2 = -\frac{1}{2}A - \frac{1}{2}i\sqrt{|B|}$$

This is a complex number. To make it easier to proceed, it is written in polar form as $\left|s^2\right| e^{i\arg\left(s^2\right)}$. Hence in polar form

$$s^2 = \left|s^2\right| e^{i\arg\left(s^2\right)}$$

This is the same as case 1. Will just use that result. hence this is unstable condition

$$4\left(\frac{LM}{I_x I_y}\right) > \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2$$
$$4LM > \frac{\left(H^2 - LI_y - MI_x\right)^2}{I_x I_y} \quad \text{(unstable)}$$

Therefore, if $B < 0$, system unstable.

2.3.5.6 Case positive

If $B > 0$ then

$$\begin{aligned} s^2 &= -\frac{1}{2}A - \frac{1}{2}\sqrt{|B|} \\ &= -\frac{1}{2}\left(\sqrt{|B|} + A\right) \end{aligned}$$

Now $\sqrt{|B|}$ is positive number, since $B > 0$. Then

$$\begin{aligned} s &= \pm i\sqrt{\frac{1}{2}}\sqrt{\sqrt{|B|} + A} \\ &= \left\{ i\sqrt{\frac{1}{2}}\sqrt{\sqrt{|B|} + A}, -i\sqrt{\frac{1}{2}}\sqrt{\sqrt{|B|} + A} \right\} \end{aligned}$$

If $\sqrt{|B|} + A > 0$ then both s_1 and s_2 are pure imaginary and the system is marginally stable (real part is zero). This condition is $\sqrt{|B|} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) > 0$ or

$$\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) > 0 \qquad \text{(marginally stable)}$$

If $\sqrt{|B|} + A < 0$ then $s_1 = i\sqrt{\frac{1}{2}}i\left|\sqrt{\sqrt{|B|} + A}\right| = -\sqrt{\frac{1}{2}}\left|\sqrt{\sqrt{|B|} + A}\right|$ which is negative, hence s_1 is stable. and $s_2 = -i\sqrt{\frac{1}{2}}i\left|\sqrt{\sqrt{|B|} + A}\right| = \sqrt{\frac{1}{2}}\left|\sqrt{\sqrt{|B|} + A}\right|$ which is positive, hence s_2 is not stable.

This condition is $\sqrt{|B|} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) < 0$ or

$$\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) < 0 \qquad \text{(not stable)}$$

This covers all 4 cases. We now summarize the findings

2.3.5.7 Summary of results

condition	meaning
$LM > \frac{1}{4}\frac{\left(H^2 - LI_y - MI_x\right)^2}{I_x I_y}$	unstable
$LM < 0$	unstable
$LM > 0$	marginally stable
$\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) > 0$	marginally stable
$\sqrt{\left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right)^2 - 4\left(\frac{LM}{I_x I_y}\right)} + \left(\frac{H^2 - LI_y - MI_x}{I_x I_y}\right) < 0$	not stable

I do not think I did this correctly. $LM > \frac{1}{4}\frac{\left(H^2-LI_y-MI_x\right)^2}{I_xI_y}$ comes out as unstable. But $\frac{1}{4}\frac{\left(H^2-LI_y-MI_x\right)^2}{I_xI_y}$ is positive quantity, while $LM > 0$ came out as marginally stable. We can't have both cases be true.

2.3.6 Key solution

HW #3.

a) α IS ANGLE BETWEEN FREESTREAM AND x-AXIS

θ IS ANGLE BETWEEN x-AXIS AND x_E-DIRECTION (← GROUND)

HERE FREESTREAM IS ALWAYS ∥ x_E-DIRECTION

$\Rightarrow \alpha = \theta$

b) HERE $u = u_0 = \text{CONST}$

$\dot{u} = 0$

RECALL w REALLY MEANS $\hat{w} = \frac{w}{u_0} = \alpha$

$\Rightarrow w = \alpha = \Delta\theta$

$\dot{w} = \Delta\dot{\theta}$ ONLY NEED ONE EQUATION FOR ONE OF THEM

$$\Rightarrow \underline{x} = \begin{bmatrix} w \\ q \end{bmatrix} \qquad \underline{\dot{x}} = \begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + m u_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}\left(Z_q + m u_0\right)}{m - Z_{\dot{w}}}\right] \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

CHARACT. EQ. $\det\left(\lambda \underline{\underline{I}} - \underline{\underline{A}}\right) = 0$

$$\det \begin{bmatrix} \lambda - A_{11} & A_{12} \\ A_{21} & \lambda - A_{22} \end{bmatrix} = 0$$

$$(\lambda - A_{11})(\lambda - A_{22}) - A_{12}A_{21} = 0$$

$$\lambda^2 - \lambda(A_{11} + A_{22}) - A_{12}A_{21} = 0$$

$$\lambda = \frac{(A_{11}+A_{22}) \pm \sqrt{(A_{11}+A_{22})^2 + 4A_{12}A_{21}}}{2}$$

$$(A_{11}+A_{22}) = \frac{M_q}{I_y} + \frac{1}{m - Z_{\dot{w}}}\left(Z_w + \frac{M_{\dot{w}}(Z_q + m u_0)}{I_y}\right)$$

$$A_{12}A_{21} = \frac{1}{I_y}\left[M_w \frac{Z_q + m u_0}{m - Z_{\dot{w}}} + \frac{M_{\dot{w}} Z_w (Z_q + m u_0)}{(m - Z_{\dot{w}})^2}\right]$$

ASSUME $\Delta < 0$ INSIDE SQUARE ROOT

$$\Rightarrow \lambda_{1,2} = \mu \pm i\omega$$

$$\text{w/} \quad \omega = \frac{\sqrt{-(A_{11}+A_{22})^2 - 4A_{12}A_{21}}}{2}$$

$$\mu = \frac{A_{11}+A_{22}}{2}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{-(A_{11}+A_{22})^2 - 4A_{12}A_{21}}}$$

$$t_{HALF} = \frac{0.693}{|\mu|} = \frac{1.386}{-(A_{11}+A_{22})}$$

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where, from Table 4.1 and Section 6.2,

$$\frac{1}{t^*} = \frac{2u_o}{\bar{c}}$$

$$= \frac{2 \times 774}{27.31}$$

$$= 56.683 \quad (12)$$

The following table lists the results of applying (11) to Table 6.9 using the data from the above table.

Mode	ψ	$\hat{v}$	$\psi + \hat{v}$	$\frac{y_E}{u_o t^*}$
Spiral	0.997 +0i	-0.00119 +0i	0.9958 +0i	$7.735\times10^3 \angle 180°$
Roll	-0.0562 +0i	-0.0198 +0i	-0.0760 +0i	$7.659 \angle 0°$
Dutch Roll	-0.28162 +0.12716i	0.29110 -0.15543i	0.00948 -0.02827i	$1.78 \angle -163.5°$

6.3 From (6.8,6) for static spiral stability

$$E = (C_{\ell_\beta} C_{n_r} - C_{\ell_r} C_{n_\beta}) \cos\theta_o$$

$$+ (C_{\ell_p} C_{n_\beta} - C_{\ell_\beta} C_{n_p}) \sin\theta_o$$

$$> 0 \quad (1)$$

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Let

$$E = A\cos\theta_0 + B\sin\theta_0 \tag{2}$$

Also assume

$$L = W = \frac{1}{2}\rho V^2 S C_L$$

Thus

$$V = (2W/\rho S)^{1/2} C_L^{-1/2} \tag{3}$$

at sea level, from App. D

$$\rho = 2.3769 \times 10^{-3} \text{ slug/ft}^3$$

Thus

$$V = \left(\frac{2 \times 2400}{2.3769 \times 10^{-3} \times 160}\right)^{1/2} C_L^{-1/2}$$

$$= 112.35\, C_L^{-1/2} \text{ fps} \tag{4}$$

The results from applying (2) and (4) based on Table 7.2 are given below.

C_L	C_{ℓ_β}	C_{n_r}	C_{ℓ_r}	C_{n_β}	A
0.15	-.08266	-.04854	.02625	.01364	.003654
0.55	-.1193	-.05520	.1347	.01840	.004107
0.95	-.1560	-.06948	.2431	.02860	.003886
1.35	-.1927	-.09138	.3515	.04424	.002059
1.70	-.2248	-.1168	.4463	.06239	-.001588

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C_L	C_{ℓ_p}	C_{n_β}	C_{ℓ_β}	C_{n_p}	B
0.15	-.441	.01364	-.08266	-.01558	-.007303
0.55	-.441	.01840	-.1193	-.05422	-.01458
0.95	-.441	.02860	-.1560	-.09286	-.02710
1.35	-.441	.04424	-.1927	-.1315	-.04485
1.70	-.441	.06239	-.2248	-.1653	-.06467

V (fps)	C_L	E (-10°)	E (0°)	E (10°)
290	0.15	.004867	.003654	.002330
151	0.55	.006576	.004107	.001513
115	0.95	.008533	.003886	-.0008789
96.7	1.35	.009816	.002059	-.005760
86.2	1.70	.009666	-.001588	-.01279

See diagram 6.3 for plots of E vs. V.

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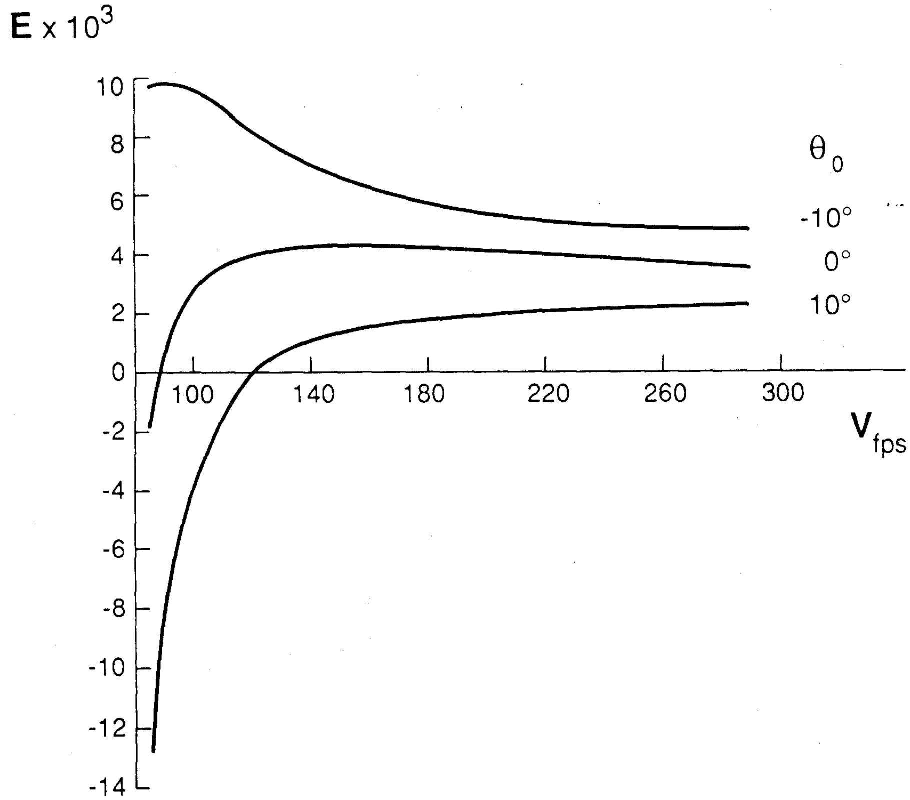

Diagram 6.3

6.4 Write the N^{th} order characteristic equation as

$$(\lambda - \lambda_N)(\lambda - \lambda_{N-1}) \ldots (\lambda - \lambda_1) = 0 \quad (1)$$

For N = 1 this becomes

$$(\lambda - \lambda_1) = 0 \quad (2)$$

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and the coefficient of λ^{N-1} or λ^0 is $-\lambda_1$. For N = 2 (1) becomes

$$(\lambda - \lambda_2)(\lambda - \lambda_1) = 0 \quad (3)$$

or

$$\lambda^2 + \lambda(-\lambda_1 - \lambda_2) + \lambda_1\lambda_2 = 0 \quad (4)$$

and the coefficient of λ^{N-1} or λ^1 is $-(\lambda_1 + \lambda_2)$. For N = n let the coefficient of λ^{n-1} be C_n.

Thus for N = n+1 (1) becomes

$$(\lambda - \lambda_{n+1})(\lambda^n + C_n\lambda^{n-1} + \dots) = 0 \quad (5)$$

or

$$\lambda^{n+1} + \lambda^n(C_n - \lambda_{n+1}) + \dots = 0 \quad (6)$$

and the coefficient of λ^{N-1} or λ^n is

$$C_{n+1} = C_n - \lambda_{n+1} \quad (7)$$

From (7) it follows that the coefficient of λ^{N-1} is the negative of the sum of the N roots of (1). Since these roots are either real or complex conjugate pairs it follows that this is also the negative of the sum of the real parts of the N roots of (1).

6.5 From (6.8,6) it is seen that the spiral is stable if E > 0. The critical climb angle θ_{o_c} causes E = 0. Combine (6.8,6) with the derivative data in Table 6.6 to obtain

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$$E = 0.0174 \cos \theta_0 - 0.07551 \sin \theta_0 \quad (1)$$

Set (1) equal to zero and solve for θ_{0c}, thus

$$\tan \theta_{0c} = 0.0174/0.07551$$

$$\theta_{0c} = 12.98^\circ = 0.2053 \text{ RAD} \quad (2)$$

Assume $\theta_0 = 0$ in horizontal flight, thus from (6.8,6)

Γ	↑	↓	
C_{ℓ_β}	↓	↑	STAB
C_{ℓ_r}	↑		DESTAB

$$E = C_{\ell_\beta} C_{n_r} - C_{\ell_r} C_{n_\beta} \quad (3)$$

The effect of Γ on the stability derivatives in (3) can be seen from App. B.9 and B.11. Only C_{ℓ_β} and C_{ℓ_r} are affected. For increasing Γ, C_{ℓ_β} becomes more negative and C_{ℓ_r} more positive. For $C_{n_r} = -0.2737$ and $C_{n_\beta} = 0.1946$ from Table 6.6, it follows that for increasing Γ, the C_{ℓ_β} effect is stabilizing and the C_{ℓ_r} effect is destabilizing (since $E > 0$ for stability). Based on the results of Exercise 5.3, the C_{ℓ_β} effect should normally dominate.

6.6 From (A.4,12) with $\phi = \psi = 0$ and θ small

$$\mathbf{L}_{BE} = \begin{bmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix} \quad (1)$$

Now

$$\mathbf{W}_B = \mathbf{L}_{BE} \mathbf{W}_E \quad (2)$$

From (1), (2) and (6.9,4)

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$$\mathbf{W}_B = \begin{bmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (W_0 + \Gamma z_E)$$

$$= \begin{bmatrix} 1 \\ 0 \\ \theta \end{bmatrix} (W_0 + \Gamma z_E)$$

$$\approx \begin{bmatrix} W_0 + \Gamma z_E \\ 0 \\ W_0\theta \end{bmatrix} \tag{3}$$

since $\Gamma\theta z_E$ is second order and can be dropped.

6.7 From (4.9,19) with $I_{zx} = \theta_0 = 0$ form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\delta_a \tag{1}$$

where

$$\mathbf{x} = [v \quad p \quad r \quad \phi]^T \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \frac{Y_r}{m} - u_0 & g \\ \frac{L_v}{I_x} & \frac{L_p}{I_x} & \frac{L_r}{I_x} & 0 \\ \frac{N_v}{I_z} & \frac{N_p}{I_z} & \frac{N_r}{I_z} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3}$$

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$$\mathbf{B} = \begin{bmatrix} \frac{Y_{\delta_a}}{m} \\ \frac{L_{\delta_a}}{I_x} \\ \frac{N_{\delta_a}}{I_z} \\ 0 \end{bmatrix} \qquad (4)$$

From Table 4.4

$$\begin{aligned}
Y_v &= \frac{1}{2}\rho u_0 S C_{y\beta} \\
Y_p &= \frac{1}{4}\rho u_0 b S C_{y_p} \\
Y_r &= \frac{1}{4}\rho u_0 b S C_{y_r} \\
L_v &= \frac{1}{2}\rho u_0 b S C_{\ell\beta} \\
L_p &= \frac{1}{4}\rho u_0 b^2 S C_{\ell_p} \\
L_r &= \frac{1}{4}\rho u_0 b^2 S C_{\ell_r} \qquad (5) \\
N_v &= \frac{1}{2}\rho u_0 b S C_{n\beta} \\
N_p &= \frac{1}{4}\rho u_0 b^2 S C_{n_p} \\
N_r &= \frac{1}{4}\rho u_0 b^2 S C_{n_r}
\end{aligned}$$

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$$Y_{\delta_a} = \frac{1}{2}\rho u_o^2 S C_{y\delta a}$$

$$L_{\delta_a} = \frac{1}{2}\rho u_o^2 b S C_{\ell\delta a}$$

$$N_{\delta_a} = \frac{1}{2}\rho u_o^2 b S C_{n\delta a}$$

From App. D $\rho_{\text{sea level}} = 2.3769\times10^{-3}$ slug/ft^3. Since L = W, thus

$$C_L = \frac{W}{\frac{1}{2}\rho u_o^2 S}$$

$$= \frac{2400}{\frac{1}{2}\times 2.3769\times 10^{-3}\times\left(\frac{77.3}{.3048}\right)^2\times 160} \qquad (6)$$

$$= 0.196$$

From Table 7.2 and (6)

$$C_{y\beta} = -0.14 \qquad (7)$$

$$C_{y_p} = -0.039$$

$$C_{y_r} = 0.165$$

$$C_{\ell\beta} = -0.0689 - 0.0917\, C_L$$

$$= -0.0869$$

$$C_{\ell_p} = -0.441$$

$$\begin{aligned} C_{\ell_r} &= -0.0144 + 0.271\ C_L \\ &= 0.0387 \end{aligned}$$

$$\begin{aligned} C_{n_\beta} &= 0.01326 + 0.017\ {C_L}^2 \\ &= 0.0139 \end{aligned}$$

$$\begin{aligned} C_{n_p} &= -0.00109 - 0.0966\ C_L \\ &= -0.0200 \end{aligned}$$

$$\begin{aligned} C_{n_r} &= -0.048 - 0.0238\ {C_L}^2 \\ &= -0.0489 \end{aligned}$$

[redacted]

$$\begin{aligned} \frac{1}{2}\rho u_o S &= \frac{1}{2} \times 2.3769 \times 10^{-3} \times \left(\frac{77.3}{.3048}\right) \times 160 \qquad (8) \\ &= 48.22 \end{aligned}$$

$$\begin{aligned} \frac{1}{4}\rho u_o b S &= \frac{30}{2} \times 48.22 \\ &= 723.3 \end{aligned}$$

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$$\frac{1}{2}\rho u_0 bS = 1{,}446.6$$

$$\frac{1}{4}\rho u_0 b^2 S = 21{,}701$$

$$\frac{1}{2}\rho u_0{}^2 bS = 366{,}903$$

From (5), (7) and (8)

$$Y_v = 48.22 \times (-0.14) = -6.751 \qquad (9)$$

$$Y_p = 723.3 \times (-0.039) = -28.209$$

$$Y_r = 723.3 \times (0.165) = 119.345$$

$$L_v = 1{,}446.6 \times (-0.0869) = -125.710$$

$$L_p = 21{,}701 \times (-0.441) = -9{,}570.14$$

$$L_r = 21{,}701 \times (0.0387) = 839.829$$

$$N_v = 1{,}446.6 \times (0.0139) = 20.108$$

$$N_p = 21{,}701 \times (-0.0200) = -434.020$$

$$N_r = 21{,}701 \times (-0.0489) = -1{,}061.18$$

[redacted]

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[redacted]

[redacted]

$$m = 2400/32.2 = 74.534 \text{ slugs}$$

$$I_x = 170 \text{ slug ft}^2 \qquad g = 32.2 \text{ ft/s}^2$$

$$I_z = 1{,}312 \text{ slug ft}^2 \qquad u_o = \frac{77.3}{.3048} = 253.6 \text{ fps}$$

From (3) and (9)

$$\mathbf{A} = \begin{bmatrix} -0.0906 & -0.378 & -252 & 32.2 \\ -0.739 & -56.3 & 4.94 & 0 \\ 0.0153 & -0.331 & -0.809 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

From (4) and (9)

$$\mathbf{B} = \begin{bmatrix} 0 \\ -115 \\ 1.40 \\ 0 \end{bmatrix} \tag{11}$$

NOW CAN FIND $\underline{\underline{A}}$'S EIGENVALUES AND EIGENVECTORS USING A MODIFIED VERSION OF THE UNCONTROLLED_LONGITUDINAL.M MATLAB SCRIPT.

NOTE THAT THE ELEMENTS OF $\underline{\underline{A}}$ WERE OBTAINED USING ENGLISH UNITS. THUS WE'LL HAVE TO DO THE SAME FOR THE NORMALIZATIONS.

HERE

$u_0 = 77.3 \text{ m/s} = 253.61 \text{ ft/s}$ (CRUISE SPEED)

$b = 30 \text{ ft}$ (WINGSPAN)

AND THE NORMALIZATIONS ARE

$$\hat{v} = \frac{v}{u_0} \qquad \hat{p} = \frac{p}{\frac{2u_0}{b}} \qquad \hat{r} = \frac{r}{\frac{2u_0}{b}} \qquad \hat{\phi} = \phi$$

```
hw4_prob3_w_gravity

A =

  -0.0906   -0.3780 -252.0000   32.2000
  -0.7390  -56.3000    4.9400         0
   0.0153   -0.3310   -0.8090         0
        0    1.0000         0         0


eigenvalues =

 -56.3028              0                   0                  0
        0       -0.4207 + 2.2839i          0                  0
        0              0          -0.4207 - 2.2839i           0
        0              0                   0           -0.0554


eigenvectors =

  -0.0435        -0.9999            -0.9999            0.9807
  -0.9989         0.0131 + 0.0002i   0.0131 - 0.0002i  -0.0107
  -0.0059        -0.0014 + 0.0084i  -0.0014 - 0.0084i   0.0246
   0.0177        -0.0009 - 0.0056i  -0.0009 + 0.0056i   0.1936


      Eigenvalue           Damping     Frequency

 -5.63e+01                 1.00e+00     5.63e+01
 -4.21e-01 + 2.28e+00i     1.81e-01     2.32e+00
 -4.21e-01 - 2.28e+00i     1.81e-01     2.32e+00
 -5.54e-02                 1.00e+00     5.54e-02

(Frequencies expressed in rad/TimeUnit)


period =

     Inf      Inf      Inf      Inf
     Inf   2.7510      Inf      Inf
     Inf      Inf  -2.7510      Inf
     Inf      Inf      Inf      Inf


thalf =

   0.0123     -Inf     -Inf     -Inf
     -Inf   1.6472     -Inf     -Inf
     -Inf     -Inf   1.6472     -Inf
     -Inf     -Inf     -Inf  12.5134
```

nhalf =

0	NaN	NaN	NaN
NaN	-0.5988	NaN	NaN
NaN	NaN	0.5988	NaN
NaN	NaN	NaN	0

Summary

r arbitrarily chosen with unit amplitude and zero phase

	Dimensional Ratios	Dimensionless Ratios	Phases
Roll			
v	7.3219	0.4881	0
p	167.9934	167.9934	0
r	1.0000	1.0000	0
phi	2.9837	50.4472	-180.0000
Dutch Roll			
v	117.9715	7.8648	80.1557
p	1.5462	1.5462	-98.9582
r	1.0000	1.0000	0
phi	0.6658	11.2568	-199.3952
Spiral			
v	39.8336	2.6556	0
p	0.4355	0.4355	180.0000
r	1.0000	1.0000	0
phi	7.8645	132.9684	0

Chapter 6

$$N_v(\lambda_3) = -309.33(-.33692 + 2.284i)(106.58 + 2.284i) \quad (20)$$
$$= 76{,}132\angle 279.62^\circ$$

$$N_p(\lambda_3) = -115(-.4207 + 2.284i)[(-.00097 + 2.284i)^2 + 1.2166^2]$$
$$= -115(-.4207 + 2.284i)(-3.7365 - .0044310i)$$
$$= 997.94\angle 100.50^\circ$$

$$N_r(\lambda_3) = 1.4(-.82245 + 2.284i)(.0764 + 2.284i)(83.064 + 2.284i)$$
$$= 645.38\angle 199.46^\circ$$

$$N_\phi(\lambda_3) = N_p(\lambda_3)/(-.4207 + 2.284i)$$
$$= 429.70\angle 0.06^\circ$$

From (20)

$$\frac{v}{r} = 117.96\angle 80.16^\circ \qquad \frac{\hat{v}}{\hat{r}} = 7.86\angle 80.16^\circ$$

$$\frac{p}{r} = 1.546\angle -98.96^\circ \qquad \frac{\hat{p}}{\hat{r}} = 1.546\angle -98.96^\circ \quad (21)$$

$$\frac{\phi}{r} = 0.666\angle -199.4^\circ \qquad \frac{\phi}{\hat{r}} = 11.26\angle -199.4^\circ$$

When gravity is absent $g = 0$ in (3). The variable ϕ can then be eliminated from the system equations by forming

$$\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x} + \mathbf{B}'\delta_a \quad (22)$$

where

$$\mathbf{x} = [v \quad p \quad r]^T \quad (23)$$

and $\mathbf{A}'$ is 3×3. Bank angle can be found from the auxiliary equation

115

Chapter 6

$$\phi = \int p dt + \phi_0 \tag{24}$$

Also, since gravity is now absent it follows that

$$L = W = mg = 0 \tag{25}$$

and

$$C_L = 0 \tag{26}$$

Thus from Table 7.2 the following derivatives which depend on C_L are altered

$$C_{\ell_\beta} = -0.0689 \tag{27}$$

$$C_{\ell_r} = -0.0144$$

$$C_{n_\beta} = 0.01326$$

$$C_{n_p} = -0.00109$$

$$C_{n_r} = -0.048$$

From (5), (8) and (27)

$$L_v = 1{,}446.6. \times (-0.0689) = -99.671 \tag{28}$$

$$L_r = 21{,}701 \times (-0.0144) = -312.494$$

Chapter 6

$$N_v = 1{,}446.6 \times (0.01326) = 19.182$$

$$N_p = 21{,}701 \times (-0.00109) = -23.654$$

$$N_r = 21{,}701 \times (-0.048) = -1{,}041.65$$

The modes of the system are determined from (22) and there are now only 3 distinct roots to the characteristic equation. In (22) (from (3), (9) and (28))

$$\mathbf{A}' = \begin{bmatrix} -0.0906 & -0.378 & -252 \\ -0.586 & -56.3 & -1.84 \\ 0.0146 & -0.0180 & -0.794 \end{bmatrix} \tag{29}$$

$$\mathbf{B}' = \begin{bmatrix} 0 \\ -115 \\ 1.40 \end{bmatrix} \tag{30}$$

The resulting eigenvalues are

$$\lambda_1 = -56.31 \qquad \text{(Roll)} \tag{31}$$

$$\lambda_{2,3} = -0.43961 \pm 1.8977i \qquad \text{(Dutch Roll)} \tag{32}$$

For $g = 0$, λ_1 and $\lambda_{2,3}$ look like λ_2 and $\lambda_{3,4}$ for $g = 32.2$ ft/s^2, and thus they were named accordingly. The numerators of the transfer functions $\bar{x}_i/\bar{\delta}_a$ are

```
hw4_prob3_no_gravity

A =

   -0.0906   -0.3780 -252.0000
   -0.5860  -56.3000   -1.8400
    0.0146   -0.0180   -0.7940


eigenvalues =

  -0.4396 + 1.8977i        0                  0
        0            -0.4396 - 1.8977i        0
        0                  0            -56.3054


eigenvectors =

   0.9999             0.9999             0.0082
  -0.0105 + 0.0006i  -0.0105 - 0.0006i   1.0000
   0.0014 - 0.0075i   0.0014 + 0.0075i   0.0003


       Eigenvalue          Damping     Frequency

 -4.40e-01 + 1.90e+00i     2.26e-01     1.95e+00
 -4.40e-01 - 1.90e+00i     2.26e-01     1.95e+00
 -5.63e+01                 1.00e+00     5.63e+01

(Frequencies expressed in rad/TimeUnit)


period =

    3.3109       Inf       Inf
       Inf   -3.3109       Inf
       Inf       Inf       Inf


thalf =

    1.5764      -Inf      -Inf
      -Inf    1.5764      -Inf
      -Inf      -Inf    0.0123


nhalf =

   -0.4761       NaN       NaN
       NaN    0.4761       NaN
       NaN       NaN         0
```

Summary

r arbitrarily chosen with unit amplitude and zero phase

Dimensional Ratios	Dimensionless Ratios	Phases
Roll		
1.0e+03 *		
v 0.0254	0.0017	0
p 3.1045	3.1045	0
r 0.0010	0.0010	0
Dutch Roll		
1.0e+03 *		
v 0.1305	0.0087	0.0795
p 0.0014	0.0014	0.2562
r 0.0010	0.0010	0

$\lambda_2 = -0.43961 + 1.8977i$ Dutch Roll Mode

$$N_v(\lambda_2) = -309.33(65.343 + 1.8977i) = 20{,}221\angle 181.66^\circ \quad (36)$$

$$N_p(\lambda_2) = -115[(.01389 + 1.8977i)^2 + 1.3228^2] = -115(-1.8513 + .05272i) = 212.99\angle -1.63^\circ$$

$$N_r(\lambda_2) = 1.4(-.34500 + 1.8977i)(57.335 + 1.8977i) = 154.91\angle 102.20^\circ$$

From (36)

$$\frac{v}{r} = 130.5\angle 79.46^\circ \qquad \frac{\hat{v}}{\hat{r}} = 8.70\angle 79.46^\circ$$

$$\frac{p}{r} = 1.375\angle -103.83^\circ \qquad \frac{\hat{p}}{\hat{r}} = 1.375\angle -103.83^\circ \quad (37)$$

The biggest effect of setting $g = 0$ was the disappearance of the spiral mode. Comparing (19) with (35) and (21) with (37) it can be seen that deleting gravity had no major impact on the Dutch roll mode but it did alter the roll mode (there is now relatively less yaw response).

6.8 From Exercise 4.10 the linearized equations for the hovercraft are

$$\Delta\dot{u} = -g\theta \quad (1)$$

$$\dot{v} = -g\phi - u_o r \qquad (2)$$

$$\dot{w} = \frac{Z_z}{m}\Delta z_E + u_o q \qquad (3)$$

$$\Delta\dot{z}_E = -u_o\theta + w \qquad (4)$$

$$\dot{\phi} = p \qquad (5)$$

$$\dot{\theta} = q \qquad (6)$$

$$\dot{\psi} = r \qquad (7)$$

$$\dot{p} = \frac{L_\phi}{I_x}\phi - \frac{H}{I_x}q \qquad (8)$$

$$\dot{q} = \frac{M_\theta}{I_y}\theta + \frac{H}{I_y}p \qquad (9)$$

$$\dot{r} = 0 \qquad (10)$$

From the above it can be seen that (5), (6), (8) and (9) represent a self-contained subset of equations containing all the parameters required to solve this exercise. Thus consider these 4 equations and represent them by

$$\mathbf{x} = [\theta \quad q \quad \phi \quad p]^T \qquad (11)$$

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \qquad (12)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M_\theta}{I_y} & 0 & 0 & \frac{H}{I_y} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{H}{I_x} & \frac{L_\phi}{I_x} & 0 \end{bmatrix} \qquad (13)$$

Thus the characteristic equation is

$$|\mathbf{I}\, s - \mathbf{A}| = 0 \qquad (14)$$

From (13) and (14) it follows that

$$\begin{vmatrix} s & -1 & 0 & 0 \\ -\frac{M_\theta}{I_y} & s & 0 & -\frac{H}{I_y} \\ 0 & 0 & s & -1 \\ 0 & \frac{H}{I_x} & -\frac{L_\phi}{I_x} & s \end{vmatrix} = 0 \qquad (15)$$

Expand (15) on the first row

$$s\begin{vmatrix} s & 0 & -\frac{H}{I_y} \\ 0 & s & -1 \\ \frac{H}{I_x} & -\frac{L_\phi}{I_x} & s \end{vmatrix} + \begin{vmatrix} -\frac{M_\theta}{I_y} & 0 & -\frac{H}{I_y} \\ 0 & s & -1 \\ 0 & -\frac{L_\phi}{I_x} & s \end{vmatrix} = 0 \qquad (16)$$

Expand the determinants on their second rows in (16)

$$s^2\begin{vmatrix} s & -\frac{H}{I_y} \\ \frac{H}{I_x} & s \end{vmatrix} + s\begin{vmatrix} s & 0 \\ \frac{H}{I_x} & -\frac{L_\phi}{I_x} \end{vmatrix} + s\begin{vmatrix} -\frac{M_\theta}{I_y} & -\frac{H}{I_y} \\ 0 & s \end{vmatrix} + \begin{vmatrix} -\frac{M_\theta}{I_y} & 0 \\ 0 & -\frac{L_\phi}{I_x} \end{vmatrix} = 0 \qquad (17)$$

and (17) becomes on expansion

$$s^2\left(s^2 + \frac{H^2}{I_x I_y}\right) - s^2\frac{L_\phi}{I_x} - s^2\frac{M_\theta}{I_y} + \frac{M_\theta L_\phi}{I_x I_y} = 0 \qquad (18)$$

or

$$s^4 + s^2\left(\frac{H^2}{I_x I_y} - \frac{L_\phi}{I_x} - \frac{M_\theta}{I_y}\right) + \frac{M_\theta L_\phi}{I_x I_y} = 0 \qquad (19)$$

Let (19) be represented by

$$s^4 + s^2 b + c = 0 \qquad (20)$$

Thus

$$s^2 = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} \qquad (21)$$

Consider the case where $\left(\frac{b}{2}\right)^2 < c$: thus

$$s^2 = -\frac{b}{2} \pm id \qquad (22)$$

where $d = \sqrt{c - \left(\frac{b}{2}\right)^2}$, and (22) is a complex number. Let (22) be represented by

$$s^2 = ae^{i\beta^+} \quad \text{and} \quad s^2 = ae^{i\beta^-} \qquad (23)$$

where "a" is real and positive.

Thus the solutions to (23) are

$$s_{1,2} = \pm\sqrt{a}\, e^{i(\beta^+/2)}$$
$$s_{3,4} = \pm\sqrt{a}\, e^{i(\beta^-/2)} \qquad (24)$$

and

$$\mathrm{Re}\{s_{1,2}\} = \pm\sqrt{a}\cos\frac{\beta^+}{2}$$
$$\mathrm{Re}\{s_{3,4}\} = \pm\sqrt{a}\cos\frac{\beta^-}{2} \qquad (25)$$

Since there are always positive real parts in (25) the system is unstable when $\left(\frac{b}{2}\right)^2 < c$ or

$$\left(H^2 - L_\phi I_y - M_\theta I_x\right)^2 < 4M_\theta L_\phi I_x I_y \qquad (26)$$

Consider the case where $\left(\frac{b}{2}\right)^2 > c$: thus s^2 is real with

$$s^2 = -\frac{b}{2} \pm e \quad (27)$$

where
$$e = \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Since c is given as positive (i.e., M_θ and L_ϕ positive) thus

and
$$0 < e < \left|\frac{b}{2}\right| \quad (28)$$

It follows from (27) and (28) that the polarity of s^2 is the same as that of (-b). Thus, if b is negative (b < 0). Then for f real

$$s^2 = f$$
$$f > 0 \quad (29)$$
$$s = \pm\sqrt{f}$$

and some solutions s_i to (27) will have positive real parts and the system is unstable.

If b is positive (b > 0) then

$$s^2 < 0 \quad (30)$$

and all solutions s_i to (27) will be pure imaginary, i.e., zero real parts, and the system will be neutrally stable and hence bounded.

From the above it is seen that to prevent instability requires

$$\left(\frac{b}{2}\right)^2 > c \quad (31)$$

and
$$b > 0 \quad (32)$$

For M_θ and $L_\phi > 0$ this can be achieved by increasing H to a suitable level. This can be seen from:

(31) becomes
$$(H^2 - L_\phi I_y - M_\theta I_x)^2 > 4M_\theta L_\phi I_x I_y \quad (33)$$

(32) becomes
$$H^2 - L_\phi I_y - M_\theta I_x > 0 \quad (34)$$

6.9 In the instant following the point in time when the headwind suddenly vanishes, the aircraft's inertial velocity $\mathbf{V}^E$ is unchanged and the airspeed is suddenly reduced in magnitude by an amount equal to the headwind speed. The governing equations following the removal of the headwind are those presented in Chapter 4 (the no-wind case). Thus for the given control settings the initial airspeed is too low and the aircraft will respond as if it had a negative Δu for its initial conditions. In general, the release of a dynamic system of linear differential equations from non-zero initial conditions will result in a response which is a linear combination of its modal responses. From the longitudinal modes described in Section 6.2 it can be seen that (see Fig. 6.3) Δu is almost absent from the short-period mode while it figures prominently in the phugoid mode. Thus any excitation of the short-period mode should be minimal, and since it is highly damped, it would soon disappear. On the other hand, the lightly damped phugoid mode should be strongly excited.

The steady state flight path angle without the headwind will be less steep than that in the presence of the headwind. If it is assumed that the headwind is larger than a small perturbation in u, then the initial response will be governed by the nonlinear flight equations. The initial drop in airspeed will cause the aircraft to lose lift and fall below the original flight path. This will be followed by an oscillation in flight path angle that will soon become dominated by the phugoid response as the response becomes that of the linearized equations. Note that flight path angle

2.4 HW4

2.4.1 Problem 1

1. Problem 7.4 in the textbook. Hint: write the $f(s)$ in the denominator (Eq. 7.7,11 (c)) in terms of the n and ω parameters of Eq. (7.3,10). Then manipulate the fraction until you can invert the transfer function using a table of Laplace transforms. Do not use Matlab. For once, it's good to do the inversion by hand.

7.4 In a test flight procedure, the airplane is brought to a condition of steady horizontal flight in quiet air. The elevator is then displaced rapidly through a small angle, held briefly, and then returned as rapidly to its original position. Assume that the resulting input can be treated as an impulse at $t = 0$ (see Sec. 7.3).

(a) Use the short period approximation (7.7,11*b*) to the transfer function for θ to derive a time domain solution for $\theta(t)$. Express the solution in terms of n, ω, b_0, and b_1.

(b) Assuming that θ and t can be determined very accurately from the flight test data, and hence that n and ω can be determined precisely, suggest how the experimental data could be used to determine b_0, b_1, c_0, and c_1. Note that if a_0 and a_1 could likewise be determined accurately, then the six equations (7.7,12) could in principle be used to solve for the six aerodynamic derivatives on the right side of the equations.

2.4.1.1 part (a)

Equation 7.7,11 b is

$$\frac{\Delta\theta(s)}{\delta_e(s)} = G_{\theta\delta_e} = \frac{b_1 s + b_0}{s f(s)} \tag{7.7,11 b}$$

Where $f(s) = s^2 + c_1 s + c_0$. Using the hint, from (7.3,10), page 211 in the textbook

$$\begin{aligned} f(s) &= s^2 + c_1 s + c_0 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \\ &= (s-n)^2 + \omega^2 \end{aligned} \tag{7.3,10}$$

Where $n = -\zeta\omega_n$ and $\omega = \omega_n\sqrt{\left(1-\zeta^2\right)}$. Substituting (7.3,10) into (7.7,11 b) gives

$$\frac{\Delta\theta(s)}{\delta_e(s)} = \frac{b_1 s + b_0}{s\left((s-n)^2+\omega^2\right)}$$

Since $\delta_e(t)$ is approximated as an impulse, its Laplace transform $\delta_e(s)$ is one. The above becomes

$$\Delta\theta(s) = \frac{b_1 s + b_0}{s\left((s-n)^2+\omega^2\right)} = \frac{b_1 s + b_0}{s\left(\underbrace{s^2 - 2ns + \left(n^2+\omega^2\right)}_{\text{polynomial in s}}\right)} \tag{1}$$

To find $\Delta\theta(t)$, the inverse Laplace of the RHS of (1) is found using partial fractions. The polynomial in the denominator of (1) is in the following form

$$\frac{1}{x\left(x^2+bx+c\right)}$$

From tables, the above has this partial fraction

$$\frac{A}{x}+\frac{Bx}{x^2+bx+c}+\frac{C}{x^2+bx+c}$$

Therefore (1) can be written as

$$\begin{aligned}
\frac{b_1 s+b_0}{s\left((s-n)^2+\omega^2\right)} &= \frac{A}{s}+\frac{Bs}{s^2-2ns+\left(n^2+\omega^2\right)}+\frac{C}{s^2-2ns+\left(n^2+\omega^2\right)} \\
&= \frac{A}{s}+\frac{Bs+C}{s^2-2ns+\left(n^2+\omega^2\right)} \\
&= \frac{A}{s}+\frac{Bs+C}{(s-n)^2+\omega^2} \qquad (2)
\end{aligned}$$

Expanding both sides and comparing powers of s of the numerators gives

$$\begin{aligned}
b_1 s+b_0 &= A\left((s-n)^2+\omega^2\right)+s(Bs+C) \\
&= A\left(\left(s^2+n^2-2ns\right)+\omega^2\right)+Bs^2+Cs \\
&= As^2-2Ans+A\left(n^2+\omega^2\right)+Bs^2+Cs \\
&= s^2(A+B)+s(-2An+C)+A\left(n^2+\omega^2\right)
\end{aligned}$$

Comparing powers of s results in

$$\begin{aligned}
b_0 &= A\left(n^2+\omega^2\right) \\
b_1 &= -2An+C \\
0 &= A+B
\end{aligned}$$

From the first equation above, $A=\frac{b_0}{n^2+\omega^2}$ and from the third equation $B=-A=-\frac{b_0}{n^2+\omega^2}$.

Using these, the second equation gives

$$\begin{aligned}
C &= b_1+2An \\
&= b_1+\frac{2b_0 n}{n^2+\omega^2} \\
&= \frac{b_1\left(n^2+\omega^2\right)+2b_0 n}{n^2+\omega^2}
\end{aligned}$$

Substituting the above values found for A, B, C into (2) results in

$$\begin{aligned}
\Delta\theta(s) &= \frac{A}{s} + \frac{Bs+C}{(s-n)^2+\omega^2} \\
&= \frac{b_0}{n^2+\omega^2}\frac{1}{s} + \frac{\frac{-b_0}{n^2+\omega^2}s + \frac{b_1\left(n^2+\omega^2\right)+2b_0 n}{n^2+\omega^2}}{(s-n)^2+\omega^2} \\
&= \frac{b_0}{n^2+\omega^2}\frac{1}{s} + \frac{1}{n^2+\omega^2}\frac{-b_0 s + b_1\left(n^2+\omega^2\right)+2b_0 n}{(s-n)^2+\omega^2} \\
&= \left(\frac{b_0}{n^2+\omega^2}\right)\frac{1}{s} - \frac{b_0}{\left(n^2+\omega^2\right)}\left(\frac{s-\frac{b_1}{b_0}\left(n^2+\omega^2\right)-2n}{(s-n)^2+\omega^2}\right)
\end{aligned}$$

Using Laplace transform tables[5] the expression $\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$ is seen as closest to the above. To use the above expression, the second term above is converted to match it. Rewriting $\Delta\theta(s)$ as

$$\Delta\theta(s) = \left(\frac{b_0}{n^2+\omega^2}\right)\frac{1}{s} - \frac{b_0}{\left(n^2+\omega^2\right)}\left(\frac{(s-n)-\frac{b_1}{b_0}\left(n^2+\omega^2\right)-n}{(s-n)^2+\omega^2}\right)$$

And breaking the second term on the RHS

$$\begin{aligned}
\Delta\theta(s) &= \left(\frac{b_0}{n^2+\omega^2}\right)\frac{1}{s} - \frac{b_0}{\left(n^2+\omega^2\right)}\left[\frac{(s-n)}{(s-n)^2+\omega^2} - \frac{\frac{b_1}{b_0}\left(n^2+\omega^2\right)+n}{(s-n)^2+\omega^2}\right] \\
&= \left(\frac{b_0}{n^2+\omega^2}\right)\frac{1}{s} - \frac{b_0}{\left(n^2+\omega^2\right)}\left[\frac{(s-n)}{(s-n)^2+\omega^2} - \frac{1}{\omega}\left(\frac{b_1}{b_0}\left(n^2+\omega^2\right)+n\right)\frac{\omega}{(s-n)^2+\omega^2}\right] \qquad (3)
\end{aligned}$$

Using $\frac{s-\alpha}{(s-\alpha)^2+\omega^2} \Rightarrow u(t)e^{\alpha t}\cos(\omega t)$ for the first part of the second term, and using $\frac{\omega}{(s-\alpha)^2+\omega^2} \Rightarrow u(t)e^{\alpha t}\sin\omega t$ for the second part of the second term above, and since $\frac{1}{s} \Rightarrow u(t)$ where $u(t)$ in the above is the unit step function which can be factored out of all the terms since it is a common term. Now the inverse Laplace of (3) can be written as

$$\Delta\theta(t) = \left(\frac{b_0}{n^2+\omega^2}\right)u(t) - \frac{b_0}{\left(n^2+\omega^2\right)}\left[e^{nt}\cos(\omega t) - \frac{1}{\omega}\left(\frac{b_1}{b_0}\left(n^2+\omega^2\right)+n\right)e^{nt}\sin(\omega t)\right]u(t)$$

Therefore the time domain is

$$\boxed{\Delta\theta(t) = \frac{b_0}{n^2+\omega^2} - \frac{e^{nt}b_0\cos(\omega t)}{\left(n^2+\omega^2\right)} + \frac{1}{\omega}\frac{b_1\left(n^2+\omega^2\right)+n}{\left(n^2+\omega^2\right)}e^{nt}\sin(\omega t)} \qquad (4)$$

For $t \geq 0$

[5] http://en.wikibooks.org/wiki/Engineering_Tables/Laplace_Transform_Table_2

2.4.1.2 Part (b)

c_1, c_0 are now found in terms of n, ω. This is by definition from

$$\begin{aligned} f(s) &= s^2 + c_1 s + c_0 \\ &= (s-n)^2 + \omega^2 \\ &= s^2 - 2ns + \left(\omega^2 + n^2\right) \end{aligned} \tag{7.3,10}$$

Comparing terms gives

$$\begin{aligned} c_1 &= -2n \\ c_0 &= \omega^2 + n^2 \end{aligned}$$

Since n, ω are known experimentally, using the above relation gives c_1, c_0.

To find b_0 and b_1, equation (4) in part (a) above is used, which is the solution as a function of time. n and ω are known, therefore equation (4) has two unknowns: b_1, b_0. This requires two equations to solve. From the data, using $\theta(t_1)$ and t_1 generates one instance of equation (4) which is the solution at the instance t_1. Another data set $\theta(t_2)$ and t_2 generates a second instance of the solution at t_2.

$$\begin{aligned} \Delta\theta_1(t_1) &= \frac{b_0}{n^2+\omega^2} - \frac{e^{nt_1} b_0 \cos(\omega t_1)}{\left(n^2+\omega^2\right)} + \frac{1}{\omega}\frac{b_1\left(n^2+\omega^2\right)+n}{\left(n^2+\omega^2\right)} e^{nt_1}\sin(\omega t_1) \\ \Delta\theta_2(t_2) &= \frac{b_0}{n^2+\omega^2} - \frac{e^{nt_2} b_0 \cos(\omega t_2)}{\left(n^2+\omega^2\right)} + \frac{1}{\omega}\frac{b_1\left(n^2+\omega^2\right)+n}{\left(n^2+\omega^2\right)} e^{nt_2}\sin(\omega t_2) \end{aligned}$$

These two equations can be solved numerically simultaneously for b_0, b_1. All the parameters in (4) are now known: $\{b_0, b_1, n, \omega\}$. The solution as given in (4) is now found and can be simulated or plotted as needed.

2.4.2 Problem 2

7.10 (a) Using the numerical data for the B747 example (Sec. 7.9), calculate the static gains for each of the eight responses that correspond to (7.9,5)—that is, the values of $|G(i\omega)_{ij}|$ for $\omega = 0$.

(b) Calculate the slopes of the high-frequency asymptotes for each of the eight frequency response amplitudes (express result in decades/decade).

(c) Assume that $\omega = 0$, that is, that a steady state exists in response to one of the controls being deflected, such that $\phi = 15°$. For each of the two controls—aileron and rudder—calculate the control angle, the sideslip angle, and the yaw rate r.

solution

The eight transfer functions are given in equation (7.9,5)

$$G_{v,\delta_a}(s) = \frac{N_{v,\delta a}(s)}{f(s)} = \frac{2.896s^2 + 6.542s + 0.622}{f(s)}$$
$$G_{v,\delta r}(s) = \frac{N_{v,\delta r}(s)}{f(s)} = \frac{5.642s^3 + 379.4s^2 + 167.9s - 5.934}{f(s)}$$
$$G_{p,\delta_a}(s) = \frac{N_{p,\delta a}(s)}{f(s)} = \frac{0.1431s^3 + 0.0273s^2 + 0.1102s}{f(s)}$$
$$G_{p,\delta r}(s) = \frac{N_{p,\delta r}(s)}{f(s)} = \frac{0.1144s^3 - 0.1997s^2 - 1.368s}{f(s)}$$
$$G_{r,\delta_a}(s) = \frac{N_{r,\delta a}(s)}{f(s)} = \frac{-0.003741s^3 - 0.002708s^2 - 0.0001394s + 0.004539}{f(s)}$$
$$G_{r,\delta r}(s) = \frac{N_{r,\delta r}(s)}{f(s)} = \frac{-0.4849s^3 - 0.2327s^2 - 0.009018s - 0.05647}{f(s)}$$
$$G_{\phi,\delta_a}(s) = \frac{N_{\phi,\delta a}(s)}{f(s)} = \frac{0.1431s^2 + 0.0273s + 0.1102}{f(s)}$$
$$G_{\phi,\delta r}(s) = \frac{N_{\phi v,\delta r}(s)}{f(s)} = \frac{0.1144s^2 - 0.1997s - 1.368}{f(s)}$$

Where

$$\boxed{f(s) = s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} \tag{6.7,2}$$

2.4.2.1 part (a)

To find the static gain, equation (7.4,5) is used, which says

$$k = \lim_{s\to 0} G_{ij}(s) \tag{7.5,4}$$

The above is applied to each of the transfer functions in (7.9,5)

$$K_{v,\delta_a} = \lim_{s\to 0} G_{v,\delta_a}(s) = \lim_{s\to 0} \frac{2.896s^2 + 6.542s + 0.622}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{0.622}{0.003682} = 168.93$$

$$K_{v,\delta r} = \lim_{s\to 0} G_{v,\delta r}(s) = \lim_{s\to 0} \frac{5.642s^3 + 379.4s^2 + 167.9s - 5.934}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{-5.934}{0.003682} = -1611.6$$

$$K_{p,\delta_a} = \lim_{s\to 0} G_{p,\delta_a}(s) = \lim_{s\to 0} \frac{0.1431s^3 + 0.0273s^2 + 0.1102s}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = 0$$

$$K_{p,\delta r} = \lim_{s\to 0} G_{p,\delta r}(s) = \lim_{s\to 0} \frac{0.1144s^3 - 0.1997s^2 - 1.368s}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = 0$$

$$K_{r,\delta_a} = \lim_{s\to 0} G_{r,\delta_a}(s) = \lim_{s\to 0} \frac{-0.003741s^3 - 0.002708s^2 - 0.0001394s + 0.004539}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{0.004539}{0.003682} = 1.2328$$

$$K_{r,\delta r} = \lim_{s\to 0} G_{r,\delta r}(s) = \lim_{s\to 0} \frac{-0.4849s^3 - 0.2327s^2 - 0.009018s - 0.05647}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{-0.05647}{0.003682} = -15.337$$

$$K_{\phi,\delta_a} = \lim_{s\to 0} G_{\phi,\delta_a}(s) = \lim_{s\to 0} \frac{0.1431s^2 + 0.0273s + 0.1102}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{0.1102}{0.003682} = 29.929$$

$$K_{\phi,\delta r} = \lim_{s\to 0} G_{\phi,\delta r}(s) = \lim_{s\to 0} \frac{0.1144s^2 - 0.1997s - 1.368}{s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682} = \frac{-1.368}{0.003682} = -371.54$$

2.4.2.2 part(b)

The slope for large frequency is determined for all the above transfer functions using the following method. The first step is to write the transfer function in a polynomial factored form (in Matlab, called zpk form). This results in the following form of the transfer function

$$G(s) = \frac{k(s+z_1)(s+z_2)\cdots(s+z_n)}{s\left(s+p_1\right)\left(s+p_2\right)\cdots\left(s+p_m\right)}$$

Where the z_i are the zeros of the numerator polynomial, and the p_i are the poles of the denominator polynomial.

s is now replaced by $j\omega$ and each factored term is converted to $\left(1+j\frac{\omega}{z_i}\right)$ in the numerator and to $\left(1+j\frac{\omega}{p_i}\right)$ in the denominator. This requires factoring out a z_i or p_i. This converts $G(s)$ to a standard form for corner frequency analysis in the bode plot.

$$\begin{aligned} G(s) &= \frac{k\frac{z_1 z_2\cdots z_n}{p_1 p_2\cdots p_m}\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)\cdots\left(1+j\frac{\omega}{z_n}\right)}{s\left(1+j\frac{\omega}{p_1}\right)\left(1+j\frac{\omega}{p_2}\right)\cdots\left(1+j\frac{\omega}{p_n}\right)} \\ &= \left(k\frac{z_1 z_2\cdots z_n}{p_1 p_2\cdots p_m}\right)\frac{\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)\cdots\left(1+j\frac{\omega}{z_n}\right)}{\left(j\omega\right)\left(1+j\frac{\omega}{p_1}\right)\left(1+j\frac{\omega}{p_2}\right)\cdots\left(1+j\frac{\omega}{p_n}\right)} \end{aligned}$$

In the above, z_i and p_i are corner frequencies. At a corner frequency the slope changes by +20 db/decade at each zero z_i, and changes by −20 db/decade for the each pole p_i.

Therefore when a corner frequency in the numerator is reached, the slope of the bode log magnitude increases by additional 20 db/decade and when a corner frequency p_i in the denominator is reached, the slope of the bode log magnitude decreases by 20 db/decade. The slope starts at −20 db/decade due to the $(j\omega)$ term in the denominator. This term has no corner frequency but it has slope of −20 db/decade.

Using this form of the transfer function, to find the slope for large frequency, 20 db/decade is added for each zero and 20 db/decade is subtracted for each pole. This is done for all corner frequencies until all frequencies are counted for. The final slope that results, is the slope needed, which is the slope at large frequency.

Since the number of zeros is the same as the degree of the numerator polynomial, and the number of poles is the same as the degree of the denominator polynomial, the difference between the degrees gives the final slope in db per decade. In this problem the number of poles is 4 for all the transfer functions since that is the common polynomial $f(s)$.

The final slope is converted to decade per decade since 20 db per decade is one decade per decade. The table below summarizes the result

Transfer function	zeros	poles	large frequency slope	slope (decade/decade)
$G_{v,\delta_a}(s)$	2	4	$(2-4)=-2 \Rightarrow -40$ db/dec	−2
$G_{v,\delta r}(s)$	3	4	$(3-4)=-1 \Rightarrow -20$ db/dec	−1
$G_{p,\delta_a}(s)$	3	4	$(3-4)=-1 \Rightarrow -20$ db/dec	−1
$G_{p,\delta r}(s)$	3	4	$(3-4)=-1 \Rightarrow -20$ db/dec	−1
$G_{r,\delta_a}(s)$	3	4	$(3-4)=-1 \Rightarrow -20$ db/dec	−1
$G_{r,\delta r}(s)$	3	4	$(3-4)=-1 \Rightarrow -20$ db/dec	−1
$G_{\phi,\delta_a}(s)$	2	4	$(2-4)=-2 \Rightarrow -40$ db/dec	−2
$G_{\phi,\delta r}(s)$	2	4	$(2-4)=-2 \Rightarrow -40$ db/dec	−2

2.4.2.3 Part (c)

To make it easier to answer this question, block diagrams are used for the transfer functions. The following diagram shows the 8 transfer functions in (7.9,5) in block diagrams. The input is the control signal, and the output is the result of multiplying the control signal by the transfer function.

To find the output from the transfer function, the input to the transfer function is multiplied by the transfer function itself. For example, looking at the last transfer function block diagram in the aileron transfer functions above, the following determines ϕ

$$\phi = \delta_a G_{\phi\delta_a}$$

The above from the definition of the transfer function, since $G_{\phi\delta_a} = \frac{output}{input} = \frac{\phi}{\delta_a}$. The input to the transfer function block is the control signal.

The problem says that $\omega = 0\,\text{rad}\,\text{s}^{-1}$, which is the steady state. But $k = \lim_{s\to 0} G_{ij}(s)$ from part (a). This means the transfer function in the above block diagram becomes the static gain. Each $G(s)$ is replaced by the corresponding gain found in part (a).

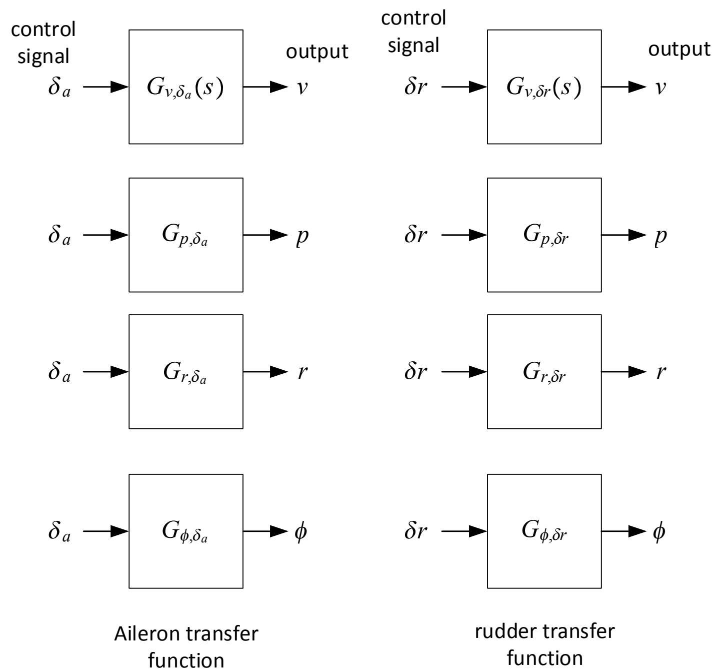

Figure 2.34: Block diagram view of lateral motion transfer functions

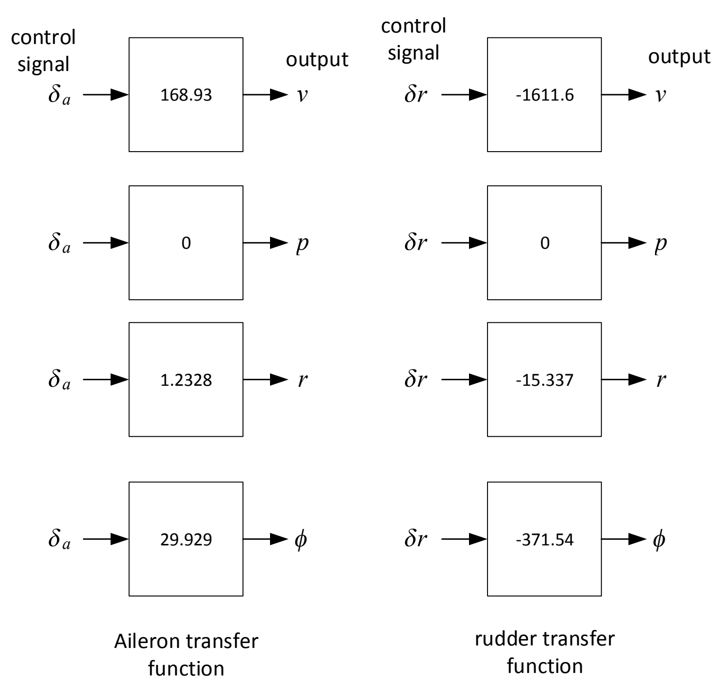

Figure 2.35: Block diagram view of lateral motion transfer functions with gain only

2.4.2.3.1 Aileron controls For the aileron control

$$\phi = \delta_a G_{\phi\delta_a}$$

using $\phi = 15^0 = 15\left(\frac{\pi}{180}\right) = 0.261\,80\,\text{rad}$ the above becomes

$$\begin{aligned}
0.261\,80 &= \delta_a \times 29.929 \\
\delta_a &= \frac{0.26180}{29.929} \\
&= 8.7474 \times 10^{-3} \text{ radian} \\
&= 8.7474 \times 10^{-3}\left(\frac{180}{\pi}\right) \\
&= \boxed{0.501\,19^\circ}
\end{aligned}$$

This is the aileron control angle that produces $\phi = 15^o$ in steady state. The problem now asks to find β, the side slip angle. The side slip angle is

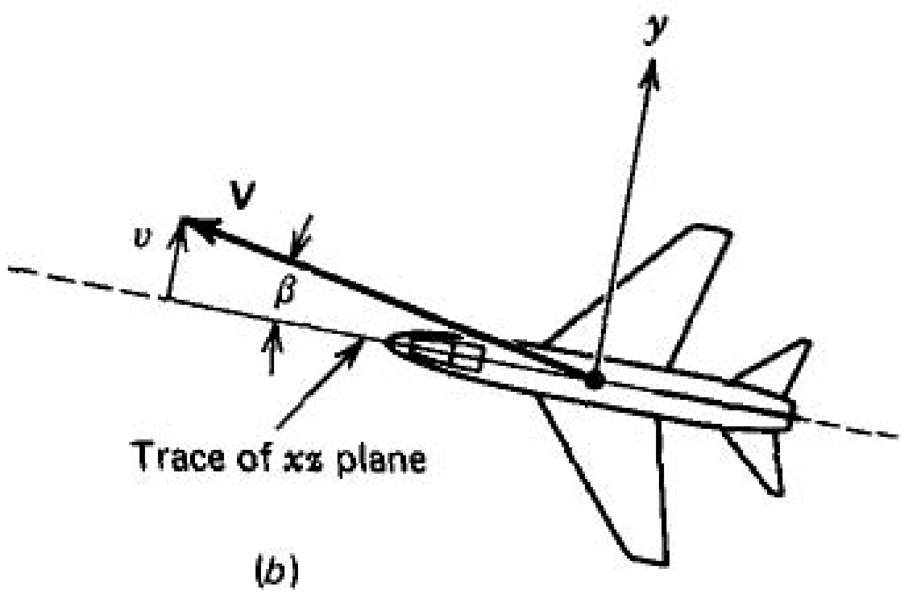

Figure 2.36: Showing side slip angle, From textbook

And given in equation (1.6,4) in the textbook as

$$\beta = \tan^{-1}\left(\frac{v}{V}\right) \tag{1.6,4}$$

Where v is the speed in the lateral direction and V is the magnitude of the velocity vector of the airplane. The example used in this problem is based on the same jet section 6.2, page 165 of the textbook, shown in the figure below.

6.2 Longitudinal Modes of a Jet Transport

The foregoing theory is now illustrated by applying it to the Boeing 747 transport. The needed geometrical and aerodynamic data for this airplane are given in Appendix E. The flight condition for this example is cruising in horizontal flight at approximately 40,000 ft at Mach number 0.8. Relevant data are as follows:

$W = 636{,}636$ lb (2.83176×10^6 N)	$S = 5500$ ft^2 (511.0 m^2)
$\bar{c} = 27.31$ ft (8.324 m)	$b = 195.7$ ft (59.64 m)
$I_x = 0.183 \times 10^8$ slug ft^2 (0.247×10^8 kg m^2)	$I_y = 0.331 \times 10^8$ slug ft^2 (0.449×10^8 kg m^2)
$I_z = 0.497 \times 10^8$ slug ft^2 (0.673×10^8 kg m^2)	$I_{zx} = -.156 \times 10^7$ slug ft^2 ($-.212 \times 10^7$ kg m^2)
$u_0 = 774$ fps (235.9 m/s) $\theta_0 = 0$	$\rho = 0.0005909$ slug/ft^3 (0.3045 kg/m^3)
$C_{L_0} = 0.654$	$C_{D_0} = 0.$

Figure 2.37: Airplane data used for problem 2

From the above $u_o = 774\,\text{ft s}^{-1}$. Using this in (1.6,4) gives

$$\beta = \tan^{-1}\left(\frac{v}{774}\right) \tag{2}$$

v is now found in order to find β. From the above transfer functions, using the first one gives v

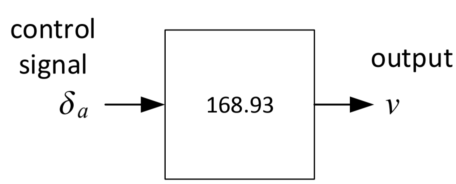

Figure 2.38: G_{r,δ_a} block diagram for problem 2

Therefore $v = \delta_a\,(168.93)$ but $\delta_a = 0.501\,19^{\circ}$ from above. Hence

$$\begin{aligned} v &= \left(0.50119^o \times \frac{\pi}{180}\right)(168.93) \\ &= 1.4777\,\text{ft s}^{-1} \end{aligned}$$

Substituting v from above in (2) gives

$$\begin{aligned}\beta &= \tan^{-1}\left(\frac{v}{774}\right)\\ &= \tan^{-1}\left(\frac{1.4777}{774}\right)\\ &= 0.0019\,\text{rad}\\ &= 0.1094^\circ\end{aligned}$$

To find the yaw rate r, the following transfer function is used

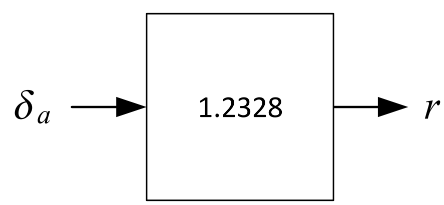

Figure 2.39: $G_{v,\delta a}$ block diagram for problem 2

Hence

$$\begin{aligned}r &= \delta_a\,(1.2328)\\ &= \left(0.50119^o \times \frac{\pi}{180}\right)(1.2328)\\ &= 0.010\,784\,\text{rad}\,\text{s}^{-1}\\ &= \boxed{0.617\,88\,^\circ\,\text{s}^{-1}}\end{aligned}$$

2.4.2.4 Rudder controls

The same process is repeated using the the rudder control blocks on the right side of the above figure. These are the blocks that takes δ_r as control signal.

For the rudder control

$$\phi = \delta_r G_{\phi\delta_r}$$

For $\phi = 15^0 = 15\left(\frac{\pi}{180}\right) = 0.2618\,\text{rad}$, the above becomes

$$\begin{aligned}0.2618 &= \delta_r \times (-371.54)\\ \delta_r &= \frac{0.2618}{-371.54}\\ &= -7.046\,3\times 10^{-4}\text{ radian}\\ &= -0.0404^o\end{aligned}$$

This is the rudder control angle that produces $\phi = 15^\circ$ in steady state. The problem now asks to find β, the side slip angle. Using

$$\beta = \tan^{-1}\left(\frac{v}{744}\right) \tag{2}$$

v is now found in order to find β. From the above blocks, using the block that output v from a rudder control signal gives

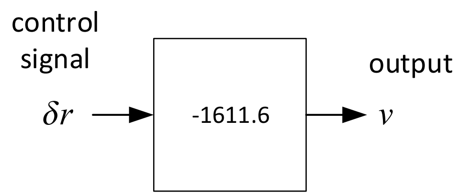

Figure 2.40: $G_{v,\delta r}$ block diagram for problem 2

Therefore $v = \delta_r(-1611.6)$. But $\delta_r = -0.0404^\circ$ from above. Hence

$$\begin{aligned} v &= \left(-0.0404^o \times \frac{\pi}{180}\right)(-1611.6) \\ &= 1.1364\,\text{ft s}^{-1} \end{aligned}$$

Substituting this v in (2) gives

$$\begin{aligned} \beta &= \tan^{-1}\left(\frac{v}{744}\right) \\ &= \tan^{-1}\left(\frac{1.1364}{744}\right) \\ &= 0.0015\,\text{rad} \\ &= \boxed{0.0841^\circ} \end{aligned}$$

To find the yaw rate r, the following block which output r for input δ_r is used

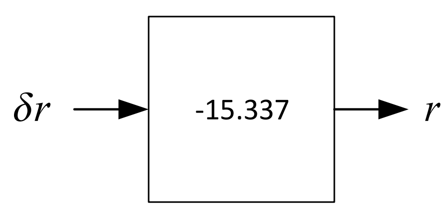

Figure 2.41: $G_{r,\delta r}$ block diagram for problem 2

Hence

$$
\begin{aligned}
r &= \delta_r (-15.337) \\
&= \left(-0.0404^\circ \times \frac{\pi}{180}\right)(-15.337) \\
&= 0.010\,814\,\mathrm{rad\,s^{-1}} \\
&= \boxed{0.6196\,^\circ\,\mathrm{s^{-1}}}
\end{aligned}
$$

2.4.3 problem 3

3. Problem 7.11 in the textbook. Hint, for part b): when a passenger is lifted from his seat, the value of the load factor n_z becomes . . .?

7.11 The elevator of the B747 airplane is oscillated at a frequency a little below that of the short-period mode.

(a) Use the results given in Fig. 7.18 to estimate the amplitude of the load factor if the elevator amplitude is 2°.

(b) What elevator amplitude would lift a passenger seated near the CG from the seat?

(c) What elevator amplitude would cause the load factor to reach the FAR Part 25 limit maneuvering value of 2.5?

solution:

2.4.3.1 Part (a)

The load factor n_z is the ratio of lift to weight

$$
n_z = \frac{L}{W} = \frac{-Z}{W} \tag{7.7,4}
$$

n_z is unity for straight horizontal steady flight[6]. The minus sign on the Z force is added since Z is positive downwards (in body coordinates of the airplane) while the lift L is upwards. The transfer function in figure 7.18 is defined as

$$
G_{\Delta n_z, \delta_e} = \frac{\Delta n_z}{\delta_e} \tag{1}
$$

In block transfer function diagram the above is

Where[7]

$$
n_z = n_{z_0} + \Delta n_z
$$

[6]Textbook, page 60 and page 230

[7]Thanks to hint from professor Bonazza for this relation.

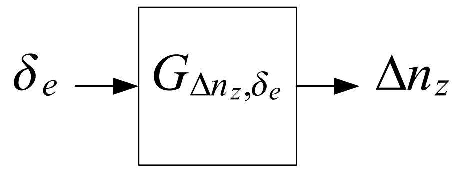

Figure 2.42: $G_{\Delta n_z,\delta e}$ block diagram for problem 3

n_{z_0} is the load factor n_z at trim defined as one. Hence the above becomes

$$\boxed{n_z = 1 + \Delta n_z} \tag{2}$$

Figure 7.18 shows that $\left|G_{\Delta n_z,\delta_e}\right| = 15$ when ω is close to the short term frequency.

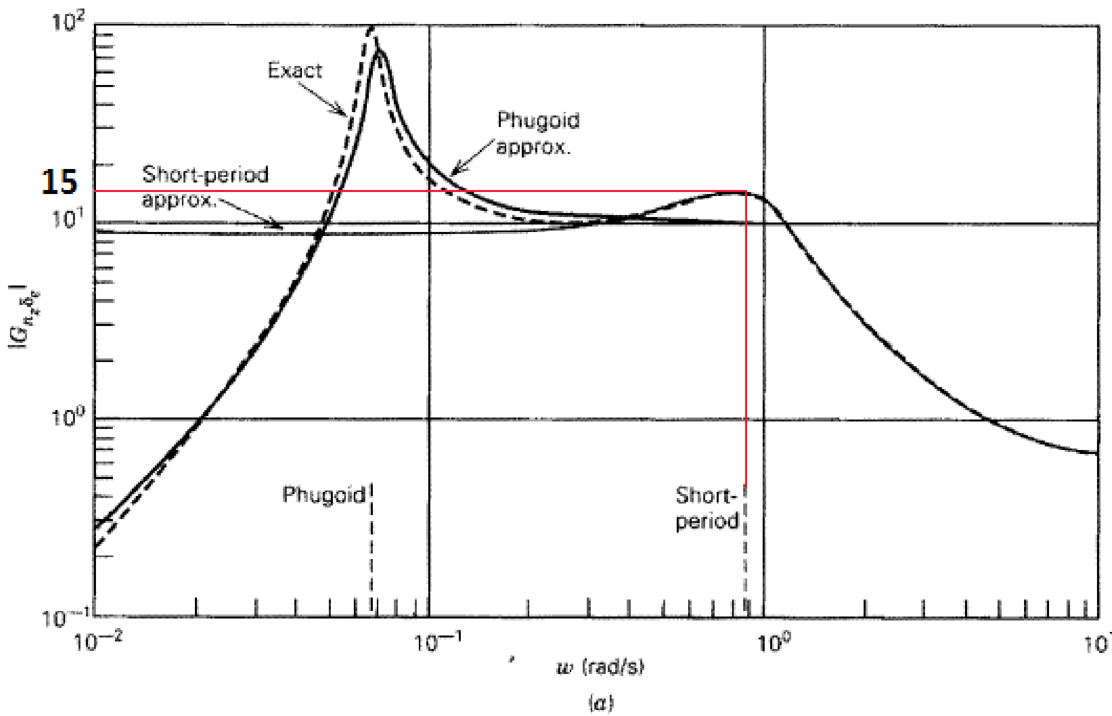

Figure 2.43: Figure 7.18 from textbook used for problem 3

Therefore (1) becomes

$$
\begin{aligned}
15 &= \frac{\Delta n_z}{2 \times \frac{\pi}{180}} \\
\Delta n_z &= 2 \times \frac{\pi}{180} \times 15 \\
&= 0.52360
\end{aligned}
$$

Using (2), the load factor n_z is now found

$$\begin{aligned} n_z &= 1 + 0.52360 \\ &= \boxed{1.5236} \end{aligned}$$

2.4.3.2 Part (b)

If a passenger at exactly the CG of the airplane floats up from the seat, it implies no external force Z acting down at the C.G. In other words, this is the same as saying the lift L is zero. So the airplane has only its weight W acting downwards (this is similar to having an airplane in free fall and moving with constant horizontal velocity as used to simulate being in outer space). When $L = 0$ then $n_z = 0$. From (2)

$$\begin{aligned} 0 &= 1 + \Delta n_z \\ \Delta n_z &= -1 \end{aligned}$$

And from (1)

$$\begin{aligned} G_{\Delta n_z, \delta_e} &= \frac{\Delta n_z}{\delta_e} \\ 15 &= \frac{-1}{\delta_e} \\ \delta_e &= \frac{-1}{15} = -0.066\,67\,\text{rad} \end{aligned}$$

Hence the elevator angle needed is

$$\boxed{\delta_e = -3.82^\circ}$$

2.4.3.3 Part (c)

For $n_z = 2.5$

$$\begin{aligned} 2.5 &= 1 + \Delta n_z \\ \Delta n_z &= 2.5 - 1 \\ &= 1.5 \end{aligned}$$

Hence from (1)

$$
\begin{aligned}
15 &= \frac{1.5}{\delta_e}\\
\delta_e &= \frac{1.5}{15}\\
&= 0.1 \text{ radian}
\end{aligned}
$$

Therefore

$$
\boxed{\delta_e = 5.7296^\circ}
$$

2.4.4 Problem 4

4. Modify the matlab routine open-loop_longitudinal.m (on the course website) to reproduce the plots of Fig. 7.21 describing the response of a B-747 to a step input in the throttle, $\delta_p = \frac{1}{6}$. **On a separate piece of paper, write** clearly how you determine the elements of the transfer function matrix, **for this specific problem**. In other words: how you use all the information available in the book and in the handouts, to calculate the appropriate transfer function matrix, using the fewest possible steps.

Then combine the original version of the routine and your modifications to study the response to simultaneous steps in elevator and throttle, $\delta_e = 1^\circ$ and $\delta_p = \frac{1}{6}$.

Next modify the routine into an open-loop_lateral.m and plot the response of the lateral variables to **separate** steps in δ_a and δ_r. Use $\delta_a = 6^\circ$ and $\delta_r = 3^\circ$.

Finally, use this routine to plot the response to **combined** steps in δ_a and δ_r. Study two distinct cases: ($\delta_a = 6^\circ$, $\delta_r = -3^\circ$) and ($\delta_a = 6^\circ$ and $\delta_r = 3^\circ$).

Make sure to plot all the responses over multiple ranges of time to observe all the modes present.

solution:

2.4.4.1 Summary of results

The summary of observations is given first. In longitudinal control, elevator action δ_e and thrust action δ_p can be applied separately from each others to achieve the expected response for each control. In lateral control, the rudder action δ_r and the aileron action δ_a have to be applied simultaneously to achieve the expected response for roll and yaw motion. For side slip rate (lateral speed v) rudder control δ_r was the primary control needed.

Once the transfer functions are found, all the required plots are generated using Matlab. Each plot generated has the Matlab code used to generate above it. In addition to the response plots, Bode plots were generated to verify the output with the textbook.

Table 2.1 gives a summary of the variables to control for each mode of motion.

type of motion	variable	meaning
Longitudinal	Δu	Velocity component in the x direction (cruise speed)
	w	Velocity component in the z direction
	q	Pitch rate
	$\Delta\theta$	pitch angle
Lateral	v	Side slip rate or side velocity.
	p	Roll rate
	r	Yaw rate
	$\Delta\phi$	Bank angle

Table 2.1: Summary of variables to control in longitudinal and lateral motion

2.4.4.1.1 results and observations found for longitudinal motion Table 2.2 summarizes the results and observations found for longitudinal motion.

2.4.4.1.2 results and observations found for lateral motion Table 2.3 summarizes the results and observations found for lateral motion when each control is applied separately.

Table 2.4 summarizes the results and observations found for lateral motion when both controls δ_p and δ_r are applied simultaneously.

2.4.4.2 open loop longitudinal responses

To obtain the transfer function matrix $G_{ij}(s)$ for the longitudinal motion, the following matrix is found

$$\boldsymbol{G}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} \tag{1}$$

Where A is the matrix for B-747 given on page 166, and B is given on page 229. The equation of motion for longitudinal motion becomes

$$\begin{pmatrix}\Delta\dot{u}\\ \dot{w}\\ \dot{q}\\ \Delta\dot{\theta}\end{pmatrix} = \overbrace{\begin{pmatrix}-0.006868 & 0.01395 & 0 & -32.2\\ -0.09055 & -0.3151 & 773.98 & 0\\ 0.000118 & -0.001026 & -0.4285 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}}^{A}\overbrace{\begin{pmatrix}\Delta u\\ w\\ q\\ \Delta\theta\end{pmatrix}}^{\text{output}} + \overbrace{\begin{pmatrix}-0.000187 & 9.66\\ -17.85 & 0\\ -1.158 & 0\\ 0 & 0\end{pmatrix}}^{B}\overbrace{\begin{Bmatrix}\delta_e\\ \delta_p\end{Bmatrix}}^{\text{control input}}$$

Once $G(s)$ is found, it will be a 4×2 matrix. $G(i,j)$ is the transfer function of the ratio of i^{th} output to the j^{th} input. For example, $G_{\Delta u,\delta_e}$ will be $G(1,1)$ which is a function of s in equation (1). To obtain all the transfer functions, equation (1) is evaluated. This can be done using

Control action	Numerator of planet transfer function	Observed response and comments
$\delta_e = 1^\circ$ elevator angle. positive is down. causes changes in pitch angle α which in steady state is meant to cause change in cruise speed only	N_{u,δ_e}	Increases cruise speed as expected, but too slowly. 10 minutes needed to reach steady state. All the response happens in phugoid mode. Large overshoot, with oscillations in response due to low damping in phugoid. Reference figure 7.19, figure 7.20 in textbook
	N_{α,δ_e}	Angle of attack responds fast, in short period mode, rapidly damped. 5 minutes to reach steady state with small residual seen present in steady state. Reference figure 7.19, figure 7.20 in textbook
	N_{γ,δ_e}	Response contained in phugoid, slow (10 minutes) with remaining γ angle residual remaining. Which implies a residual Δ_α residual exists as a result. Something that was supposed to be generated. Reference figure 7.19, figure 7.20 in textbook
$\delta_p = \frac{1}{6}$ Throttle (thrust). Causes climb up or down. In other words, this control action is used to cause a change Δ_θ. No change in Δu nor in angle of attack α should result if thrust lines pass through C.G.	N_{u,δ_p}	Δu Remained unchanged as expected but only after initial undesired oscillations. Took 20 minutes to damp out completely.
	N_{α,δ_p}	Remained unchanged as expected with very little oscillation. Damps out after 200 seconds.
	N_{γ,δ_e}	2.8° steady state response reached after 10 minutes. Large overshoot. Many oscillations before damping out.
$\delta_p + \delta_e$ Simultaneous effect of applying both controls at same time	$N_{u,(\delta_p+\delta_e)}$	Throttle action has almost no effect on speed Δu response, other than causing a small increase in overshoot and slight phase delay in oscillations compared to δ_e only control. Slow response as before.
	$N_{\alpha,(\delta_p+\delta_e)}$	Throttle action δ_p also had almost no effect on angle of attack response. Response followed very closely the δ_e response as described above.
	$N_{\gamma,(\delta_p+\delta_e)}$	In this case, the addition of elevator action was seen to have most effect on transient response of flight path angle. Steady state remained as with throttle action alone, but adding elevator action caused large overshoot compared to throttle only response. Also a phase shift was seen. Steady state was slow to be reached as with throttle only action.

Table 2.2: Results and observations for longitudinal motion

Control action	Numerator of plant transfer function	Observed response and comments
$\delta_r = 3^\circ$ Causes Yaw motion.	N_{v,δ_r}	Rudder affects initial side slip rate much more than aileron. After 10 minutes, v reached $-80\,\mathrm{ft\,s^{-1}}$ at steady state. High oscillatory response and faster response compared to aileron δ_a only.
	N_{p,δ_r}	2 minutes was needed to reach steady state of $-0.04\,\mathrm{rad\,s^{-1}}$, high overshoot, similar to aileron with oscillation. Aileron δ_a caused similar effect.
	N_{r,δ_r}	Took 10 minutes to reach steady state of $-0.8\,\mathrm{rad\,s^{-1}}$ compared to aileron δ_a case, which needed 20 minutes to cause only $-0.12\,\mathrm{rad\,s^{-1}}$ change. But small oscillation seen in the first minute of response.
	N_{ϕ,δ_r}	Much more effect on bank angle than aileron. In 6 minutes this action caused -19° change in bank angle. Smooth response.
$\delta_a = 6^\circ$ Causes roll motion	N_{v,δ_a}	After 10 minutes reached steady state of only $-18\,\mathrm{ft\,s^{-1}}$ compared to $-80\,\mathrm{ft\,s^{-1}}$ by rudder δ_r in the same amount of time. Side slip rate is seen to be more controlled by rudder alone.
	N_{p,δ_a}	2 minutes to reach steady state of $-0.006\,\mathrm{rad\,s^{-1}}$, high overshoot, similar to rudder with oscillation. But rudder 3° input caused much larger roll rate of $-0.04\,\mathrm{rad\,s^{-1}}$
	N_{r,δ_a}	Took 20 minutes to reach steady state of $-0.12\,\mathrm{rad\,s^{-1}}$, Smooth response, no oscillation. Rudder response was similar but rudder response reached steady state in half the time (10 minutes) and had much larger effect on yaw rate.
	N_{ϕ,δ_a}	Less affect on bank angle compared to rudder. After same amount of 10 minutes, cause only -3° change in bank angle compared to about -20° with the above rudder input.

Table 2.3: Results and observations for lateral motion. Controls applied separately

Control action	Numerator of plant transfer function	Observed response and comments
$\delta_r + \delta_a$ where $\delta_r = 3^\circ$ and $\delta_a = 6^\circ$	$N_{v,(\delta_r+\delta_a)}$	Combined side slip rate response followed the rudder only response. Aileron control response had small effect on final speed. Steady state reached in 10 minutes to about $-100\,\mathrm{ft\,s^{-1}}$ compared to $-80\,\mathrm{ft\,s^{-1}}$ with rudder alone.
	$N_{r,(\delta_r+\delta_a)}$	Yaw rate response is controlled mainly by Rudder. Aileron had little effect. Initial small oscillation was still present in combined response. Damped out after one minute.
	$N_{\phi,(\delta_r+\delta_a)}$	Combined response followed rudder response. Aileron effect on bank angle is minimal.
$\delta_r + \delta_a$ where $\delta_r = -3^\circ$ and $\delta_a = 6^\circ$	$N_{v,(\delta_r+\delta_a)}$	Combined side slip rate response followed the rudder only response. Aileron control response had small effect on final speed. Steady state was reached in 10 minutes to about $70\,\mathrm{ft\,s^{-1}}$ compared to $80\,\mathrm{ft\,s^{-1}}$ with rudder alone. Aileron effect has reduced final speed by $10\,\mathrm{ft\,s^{-1}}$
	$N_{r,(\delta_r+\delta_a)}$	Yaw rate response is controlled mainly by Rudder. Aileron had little effect. Initial small oscillation still present in combined response. Damped out after one minute.
	$N_{\phi,(\delta_r+\delta_a)}$	Combined response followed rudder response. Aileron effect on bank angle is minimal.

Table 2.4: Results and observations for lateral motion. Controls applied simultaneously

equation 7.2.7 on page 209

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{adj\,(s\boldsymbol{I}-\boldsymbol{A})}{\det\,(s\boldsymbol{I}-\boldsymbol{A})}$$

For this problem, finding the transfer functions is done using computer algebra. Here are the steps and the resulting $G(s)$ matrix.

```
A = {{-0.006868, 0.01395, 0, -32.2},
     {-0.09055, -0.3151, 773.98, 0},
     {0.0001187, -0.001026, -0.4285, 0},
     {0, 0, 1, 0}};
B = {{-0.00187, 9.66}, {-17.85, 0}, {-1.158, 0}, {0, 0}};
g = Inverse[s*IdentityMatrix[4] - A].B;
Map[Collect[Simplify@Numerator[#], s]/Denominator[#] &, g, {2}];
MatrixForm[%]
```

Which gives the following result

$$\begin{pmatrix}
\frac{11.1596+24.6778\,s-0.249147\,s^2-0.000187\,s^3}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4} & \frac{8.97534\,s+7.18318\,s^2+9.66\,s^3}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4}\\
\frac{-3.44462-6.20812\,s-904.04\,s^2-17.85\,s^3}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4} & \frac{0.512663\,s-0.874713\,s^2}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4}\\
\frac{-0.00387259\,s-0.354525\,s^2-1.158\,s^3}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4} & \frac{0.00125876\,s+0.00114664\,s^2}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4}\\
\frac{-0.00387259-0.354525\,s-1.158\,s^2}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4} & \frac{0.00125876+0.00114664\,s}{0.00419587+0.00946303\,s+0.935494\,s^2+0.750468\,s^3+s^4}
\end{pmatrix}$$

Using Matlab, the same procedure was also performed using symbolic toolbox as follows

```
A = [-0.006868 0.01395 0 -32.2;
    -0.09055 -0.3151 773.98 0;
    0.0001187 -0.001026 -0.4285 0;
    0, 0, 1, 0]

B = [-0.000187  9.66;
    -17.85  0;
    -1.158  0
    0, 0]

syms s;
G=inv(s*eye(4)-A)*B;
outPut={'u','w','q','theta'};
inPut={'del_e','del_p'};
for i=1:4
  for j=1:2
      [N,D] = numden(G(i,j));
      c=coeffs(D);
      fprintf('N(%s,%s) = %s',outPut{i},inPut{j},...
          char(vpa(simplify(N/c(end)),5)))
      fprintf('\n');
```

```
  end
end
```

The output generated is

```
N(u,del_e) = 24.678*s - 0.24915*s^2 - 0.000187*s^3 + 11.16
N(u,del_p) = 8.9753*s + 7.1832*s^2 + 9.66*s^3
N(w,del_e) = - 6.2081*s - 904.04*s^2 - 17.85*s^3 - 3.4446
N(w,del_p) = 0.51266*s - 0.87471*s^2
N(q,del_e) = -1.8808e-48*s*(1.885e47*s + 6.157e47*s^2 + 2.059e45)
N(q,del_p) = 6.5459e-23*s*(1.7517e19*s + 1.923e19)
N(theta,del_e) = - 0.35452*s - 1.158*s^2 - 0.0038726
N(theta,del_p) = 0.0011466*s + 0.0012588
```

2.4.4.2.1 Generating the transfer function when throttle δ_p is the input G_{ij} is now found to solve the problem. The input is the throttle δ_p which is $j = 2$. The output in figure 7.21 top plot is Δu, which is $i = 1$. Therefore

$$G_{12}(s) = G_{\Delta u,\delta_p} = \frac{\overbrace{-9.66s^3 + 7.18218s^2 + 8.97534s}^{N_{u,\delta_p}}}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \tag{2}$$

To obtain the transfer function for the second plot, the transfer function for the angle of attack is needed. But $\alpha = \frac{w}{u_0}$ where $u_0 = 774\,\mathrm{ft\,s^{-1}}$, which is the cruise speed. Therefore $N_{\alpha,\delta_p} = \frac{1}{u_0} N_{w,\delta_p}$. But what is N_{w,δ_p} ?. Since δ_p is the second input, then $j = 2$ and since w is the second output then $i = 2$, therefore

$$G_{w,\delta_p} = G_{2,2} = \frac{-0.874713s^2 + 0.512663s}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587}$$

Hence

$$\begin{aligned} G_{\alpha,\delta_p} &= \frac{G_{2,2}}{u_0} \\ &= \frac{1}{774} \frac{-0.874713s^2 + 0.512663s}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \\ &= \frac{0.0006624s - 0.00113s^2}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \end{aligned} \tag{3}$$

To obtain the result for the third plot in figure 7.21, a transfer function for γ is needed. Since $\gamma = \theta - \alpha$ then

$$G_{\gamma,\delta_p} = G_{\theta,\delta_p} - G_{\alpha,\delta_p} \tag{4}$$

But $G_{\theta,\delta_p} = G(4,2)$ since $\Delta\theta$ is the fourth output and δ_p is the second input. This gives

$$G_{\theta,\delta_p} = G(4,2) = \frac{0.00114664s + 0.0012587}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587}$$

Substituting the above in (4) results in

$$\begin{aligned} G_{\gamma,\delta_p} &= \frac{(0.00114664s + 0.00125876) - (0.00114664s + 0.00125876)}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \\ &= \frac{-0.00113s^2 + 0.0004843s + 0.001\,259}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \end{aligned} \tag{5}$$

The three transfer functions to generate figure 7.21 have been found. To summarize, they are

$$\begin{aligned} G_{\Delta u,\delta_p} &= \frac{-9.66s^3 + 7.18218s^2 + 8.97534s}{f(s)} \\ G_{\alpha,\delta_p} &= \frac{0.0006624s - 0.00113s^2}{f(s)} \\ G_{\gamma,\delta_p} &= \frac{-0.00113s^2 + 0.0004843s + 0.001\,259}{f(s)} \end{aligned}$$

Where $f(s) = s^4 + 0.750468s^3 + 0.935494s^2 + 0.009463s + 0.00419587$

2.4.4.2.2 Generating $G_{\Delta u,\delta_p}$ and $\delta_p = \frac{1}{6}$ response Matlab is used to generate figure 7.21 in the book. First the top plot showing the response to $\delta_p = \frac{1}{6}$ is given. The step response is found then multiplied by $\frac{1}{6}$ to obtain the result.

```
close all; clear all;
figure;
s=tf('s');
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
sys = tf((-9.66*s^3+7.18318*s^2+8.97534*s)/den);
t=0:.01:600;
u = step(sys,t);
subplot(2,1,1);
plot(t,u*1/6);

hold on;
plot([t(1) t(end)],[u(end)*1/6 u(end)*1/6],'r');
legend('response','steady state')
xlabel('Time (sec)');
ylabel('\Delta(u) (fps)');
title('speed response to throttle. Figure 7.21. First 10 minutes');
subplot(2,1,2);
t=0:.01:1200;
u = step(sys,t);
plot(t,u*1/6);
```

```
xlabel('Time (sec)');
ylabel('\Delta(u) (fps)');
title('speed response to throttle. First 20 minutes');
hold on;
plot([t(1) t(end)],[u(end)*1/6 u(end)*1/6],'r');
legend('response','steady state')
```

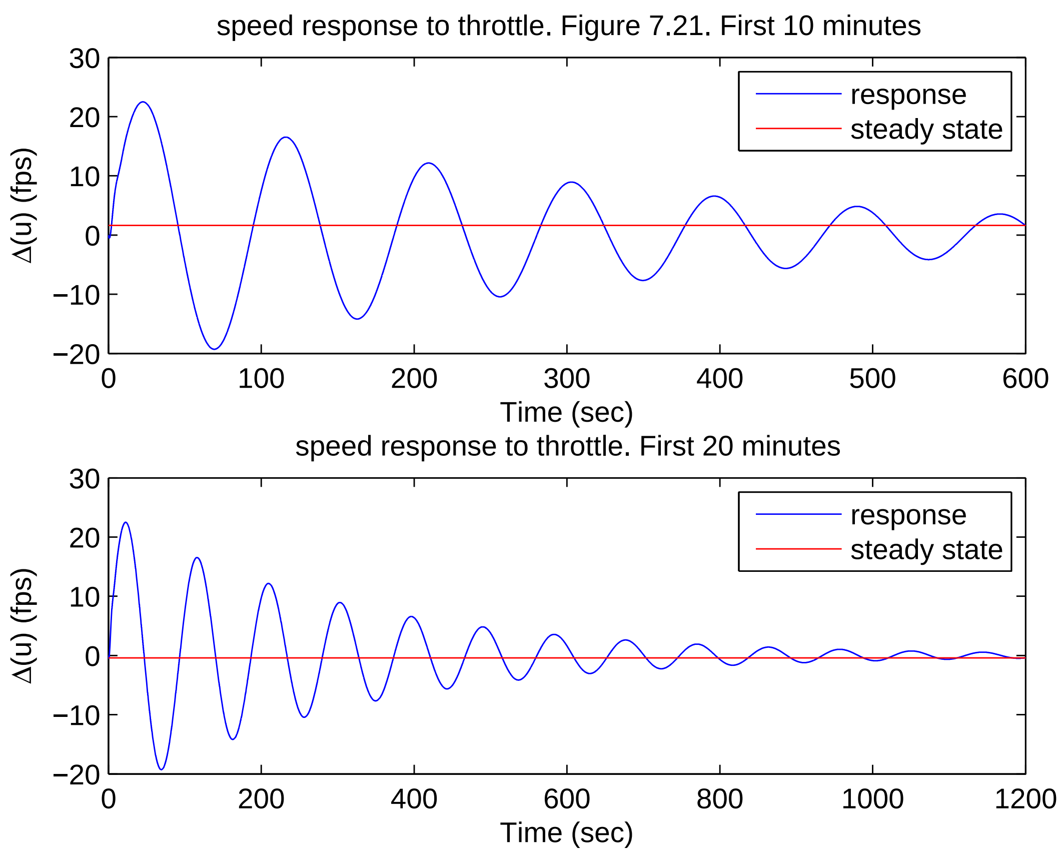

Figure 2.44: Speed response to throttle, open loop, longitudinal motion

2.4.4.2.3 Generating $G_{\Delta\alpha,\delta_p}$ and $\delta_p = \frac{1}{6}$ response Matlab is used to generate the second plot in figure 7.21 in the book. Using G_{α,δ_p} found above, the step response is found then multiplied by $\frac{1}{6}$ to obtain the result.

```
close all; clear all;
s=tf('s');
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
sys = tf((0.0006624*s-0.00113*s^2)/den);
[alpha,t] = step(sys);
plot(t,alpha*1/6);
xlim([0 600]);
ylim([-0.05 0.05]);
```

```
xlabel('Time (sec)');
ylabel('\alpha (rad)');
title('angle of attack response to throttle. Figure 7.21 reproduced');
```

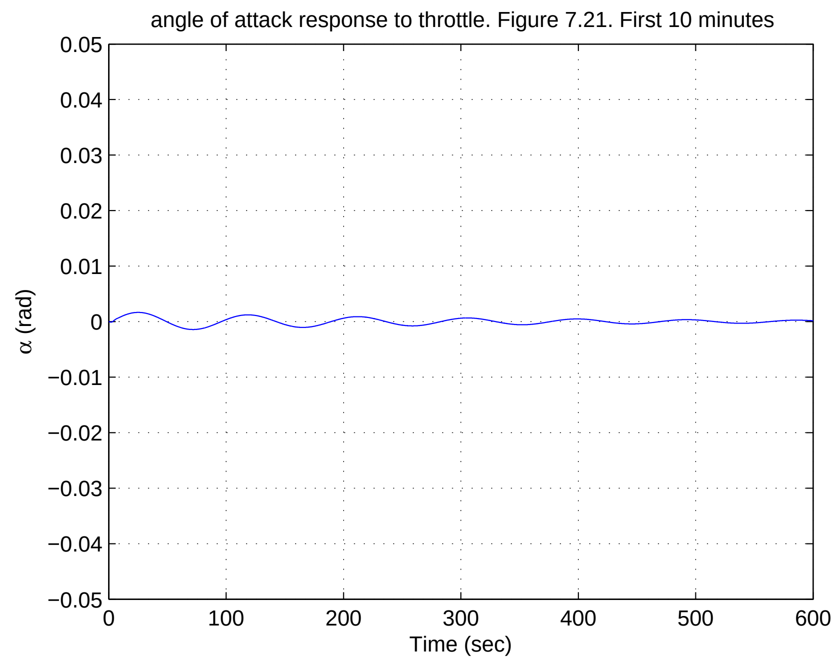

Figure 2.45: angle of attack α response to throttle, open loop, longitudinal motion

2.4.4.2.4 Generating $G_{\Delta\gamma,\delta_p}$ and $\delta_p = \frac{1}{6}$ response Matlab is used to generate the third plot in figure 7.21 in the book. Using G_{γ,δ_p} found above, the step response is found then multiplied by $\frac{1}{6}$ to obtain the result.

```
close all; clear all;
s=tf('s');
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
sys = tf((-0.00113*s^2+0.00048424*s+0.0012587)/den);
t=0:.1:300;
alpha= step(sys,t);
subplot(2,1,1);
plot(t,alpha*1/6);
ylim([0 0.2]);
xlabel('Time (sec)');
ylabel('\gamma (rad)');
title('flight path angle response to throttle. Figure 7.21. first 5 minutes');
hold on;
plot([t(1) t(end)],[alpha(end)*1/6 alpha(end)*1/6],'r');
legend('response','steady state')
subplot(2,1,2);
t=0:.1:1200;
```

```
alpha= step(sys,t);
plot(t,alpha*1/6);
ylim([0 0.2]);
xlabel('Time (sec)');
ylabel('\gamma (rad)');
title('flight path angle response to throttle. Figure 7.21. first 20 minutes');
hold on;
plot([t(1) t(end)],[alpha(end)*1/6 alpha(end)*1/6],'r');
legend('response','steady state')
```

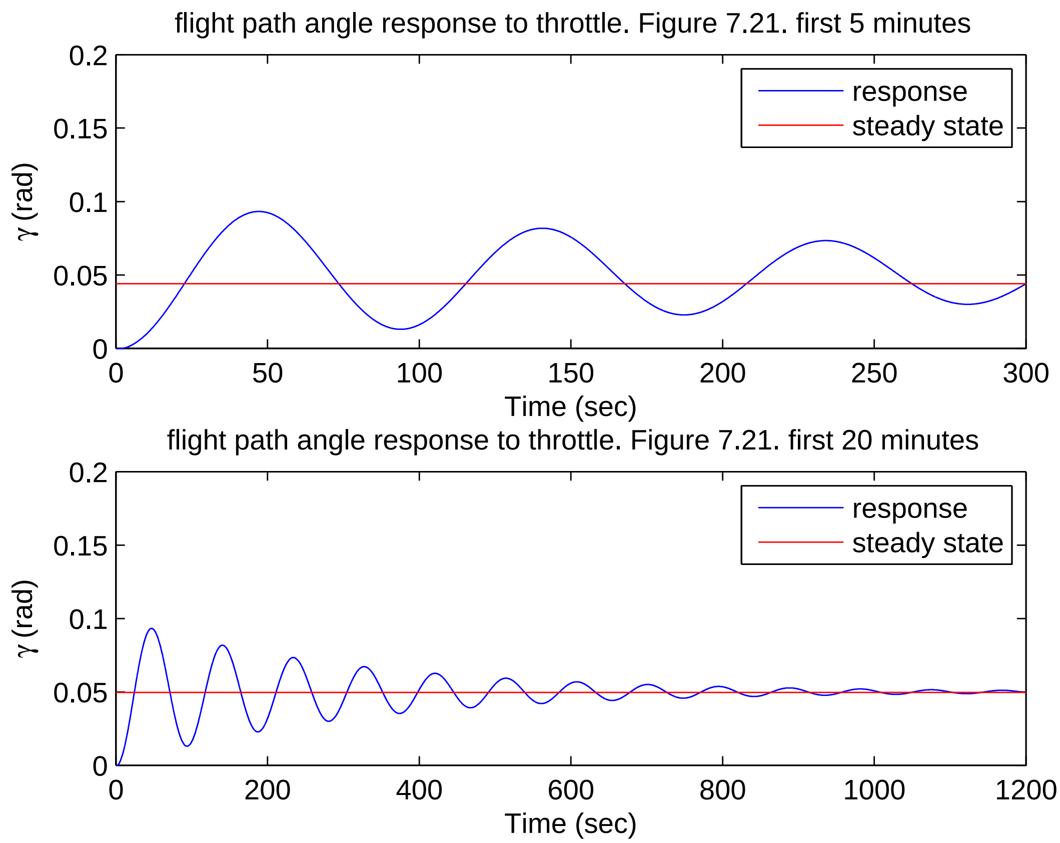

Figure 2.46: Flight path angle γ response to throttle, open loop, longitudinal motion

2.4.4.2.5 Generating the transfer function when throttle δ_e is the input Now $\delta_p = \frac{1}{6}$ and $\delta_e = 1^\circ$ are applied, still in open loop longitudinal motion. The responses are $\Delta u, \Delta\alpha$ and flight path angle γ. Since the system is linear, the input δ_p is applied and the output obtained, then the input δ_e is applied again, and its output obtained, then both outputs are added linearly (point wise). Now the transfer function for δ_e are found as above and the process is repeated, but this time the responses are added before making the final plot.

2.4.4.2.6 Generating $G_{\Delta u,\delta_e}$ G_{ij} are found to use to solve the problem. The input is the elevator angle δ_e which is $j = 1$. The output Δu, which is $i = 1$. Therefore $G_{11}(s)$ is the component selected from $\boldsymbol{G}(s)$

$$G_{11}(s) = G_{\Delta u,\delta_e} = \frac{\overbrace{-0.000187s^3 - 0.249147s^2 + 24.6778s + 11.1596}^{N_{u,\delta e}}}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587}$$

2.4.4.2.7 Generating $G_{\Delta\alpha,\delta_e}$ $\alpha = \frac{w}{u_0}$ where $u_0 = 774$ ft/sec, which is the cruise speed. Therefore $N_{\alpha,\delta_e} = \frac{1}{u_0}N_{w,\delta_e}$. But what is N_{w,δ_e} ?. Since δ_e is the first input, then $j = 1$ and since w is the second output then $i = 2$, therefore $G_{21}(s)$ is selected from $\boldsymbol{G}(s)$

$$G_{w,\delta_e} = G_{2,1} = \frac{-17.85s^3 - 904.04s^2 - 6.20812s - 3.44462}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587}$$

Hence

$$\begin{aligned} G_{\alpha,\delta_e} &= \frac{G_{w,\delta_e}}{u_0} \\ &= \frac{1}{774}\frac{-17.85s^3 - 904.04s^2 - 6.20812s - 3.44462}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \\ &= \frac{-0.023062s^3 - 1.168s^2 - 0.0080208s - 0.0044504}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \end{aligned}$$

2.4.4.2.8 Generating G_{γ,δ_e} The transfer function for γ is needed. But $\gamma = \theta - \alpha$, hence

$$G_{\gamma,\delta_e} = G_{\theta,\delta_e} - G_{\alpha,\delta_e}$$

But $G_{\theta,\delta_e} = G(4,1)$ since $\Delta\theta$ is the 4^{th} output and δ_e is the first input. Hence

$$G_{\theta,\delta_e} = G(4,1) = \frac{-1.158s^2 - 0.354525s - 0.00387259}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587}$$

Substituting this in (4) gives

$$\begin{aligned} G_{\gamma,\delta_e} &= \frac{\left(-1.158s^2 - 0.354525s - 0.00387259\right) - \left(-0.023062s^3 - 1.168s^2 - 0.0080208s - 0.0044504\right)}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \\ &= \frac{0.023062s^3 + 0.01s^2 - 0.3465s + 0.0005778}{s^4 + 0.750468s^3 + 0.935494s^2 + 0.00946303s + 0.00419587} \end{aligned}$$

2.4.4.3 Simultaneous response for elevator δ_e and throttle δ_p input

Using the transfer functions found above, now Matlab was used to generate the output.

2.4.4.3.1 Simultaneous response of speed Δ_u to combined δ_e and throttle δ_p The response to δ_e and δ_p are added to obtain the result using Matlab.

```
close all; clear all;
s = tf('s');
num_throttle = -9.66*s^3+7.18318*s^2+8.97534*s;
num_elevator = -0.000187*s^3-0.249147*s^2+24.6778*s+11.1596;
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
sys1 = tf(num_throttle/den);
t   = 0:.1:600;
subplot(2,1,1);
u_throttle = step(sys1,t);
plot(t,u_throttle*1/6,'-k');
sys2 = tf(num_elevator/den);
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
xlabel('Time (sec)');
ylabel('\Delta_u (fps)');
title('speed response to throttle and elevator combined, 10 minutes');
plot([t(1) t(end)],[(u_elevator(end)*pi/180 + u_throttle(end)*1/6 ),...
(u_elevator(end)*pi/180 + u_throttle(end)*1/6 )],'r');
legend('throttle','elevaltor','combined');
%
subplot(2,1,2);
t   = 0:.1:1200;
u_throttle = step(sys1,t);
plot(t,u_throttle*1/6,'-k');
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
xlabel('Time (sec)');
ylabel('\Delta_u (fps)');
title('speed response to throttle and elevator combined, 20 minutes');
plot([t(1) t(end)],[(u_elevator(end)*pi/180 + u_throttle(end)*1/6 ),...
(u_elevator(end)*pi/180 + u_throttle(end)*1/6 )],'r');
legend('throttle','elevaltor','combined');
```

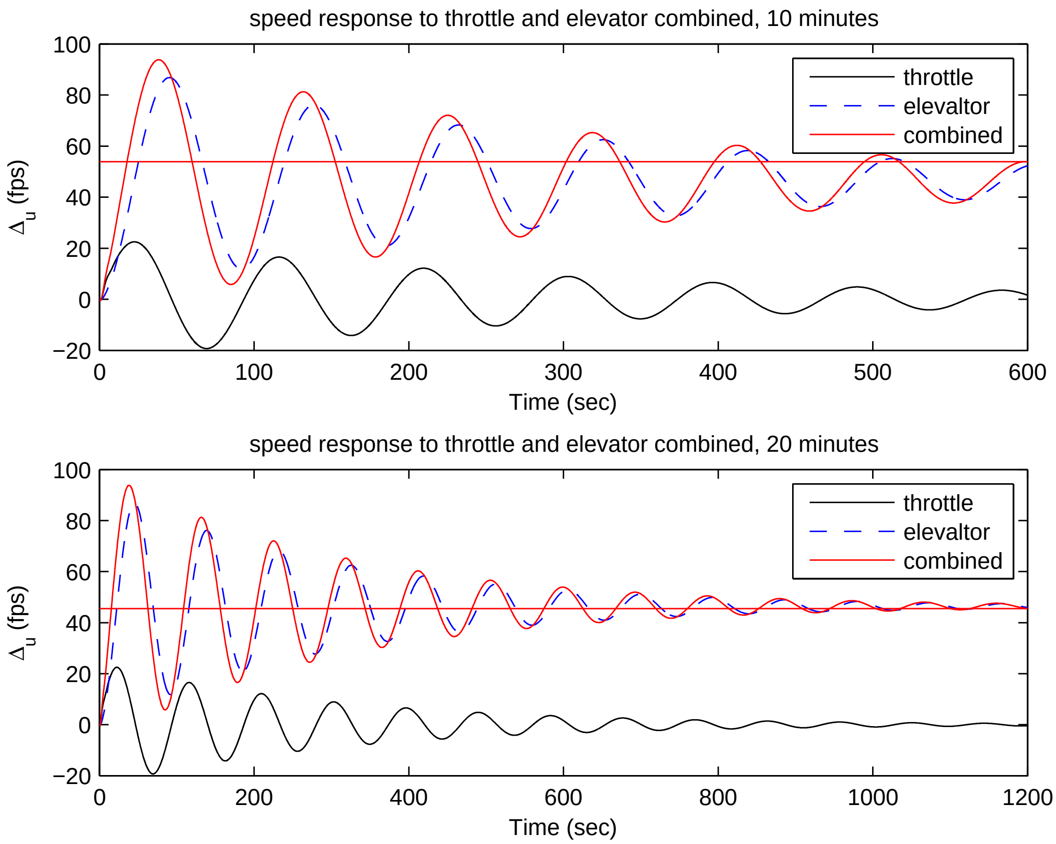

Figure 2.47: Speed response to throttle and elevator combined, open loop, longitudinal motion

2.4.4.3.2 Simultaneous response of angle of attack α to combined δ_e and throttle δ_p

The response to δ_e and δ_p are added to obtain the result using Matlab.

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_throttle = 0.0006624*s-0.00113*s^2;
num_elevator = -0.023062*s^3-1.168*s^2-0.0080208*s-0.0044505;
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
sys1 = tf(num_throttle/den);
t   = 0:.1:60;
u_throttle = step(sys1,t);
subplot(3,1,1);
plot(t,u_throttle*1/6,'-k');
sys2 = tf(num_elevator/den);
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
xlabel('Time (sec)');
ylabel('\alpha (rad)');
title('angle of attack response to throttle and elevator combined, first one minutes');
legend('throttle','elevaltor','combined');
%
subplot(3,1,2);
t   = 0:.1:300;
u_throttle = step(sys1,t);
plot(t,u_throttle*1/6,'-k');
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
xlabel('Time (sec)');
ylabel('\alpha (rad)');
title('angle of attack response to throttle and elevator combined, 3 minutes');
legend('throttle','elevaltor','combined');
%
subplot(3,1,3);
t   = 0:.1:600;
u_throttle = step(sys1,t);
plot(t,u_throttle*1/6,'-k');
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
xlabel('Time (sec)');
ylabel('\alpha (rad)');
title('angle of attack response to throttle and elevator combined, 10 minutes');
```

```
legend('throttle','elevaltor','combined');
plot([t(1) t(end)],[(u_elevator(end)*pi/180 + u_throttle(end)*1/6 ),...
(u_elevator(end)*pi/180 + u_throttle(end)*1/6 )],'r');
```

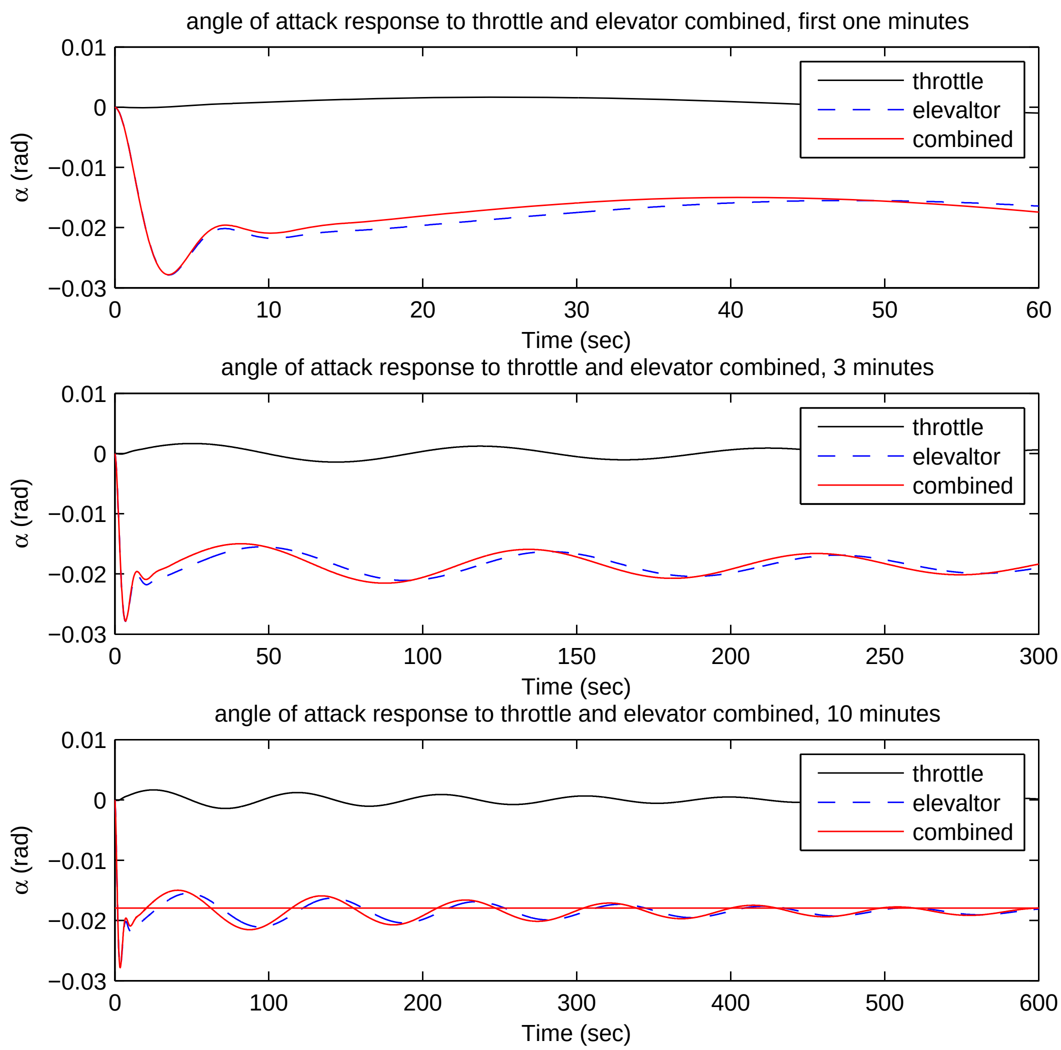

Figure 2.48: Angle of attack α response to throttle and elevator combined, open loop, longitudinal motion

2.4.4.3.3 Simultaneous response of flight path angle γ to combined δ_e and throttle δ_p

The response to δ_e and δ_p are added to obtain the result using Matlab.

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_throttle = -0.00113*s^2+0.00048424*s+0.0012587;
num_elevator = 0.023062*s^3+0.01*s^2-0.3465*s+0.0005778;
den = s^4+0.750468*s^3+0.935494*s^2+0.009463*s+0.00419587;
```

```
sys1 = tf(num_throttle/den);
t   = 0:.1:600;
u_throttle = step(sys1,t);
subplot(2,1,1);
plot(t,u_throttle*1/6,'k');
sys2 = tf(num_elevator/den);
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
ylim([-.1 .2]);
xlabel('Time (sec)');
ylabel('\gamma (rad)');
title('flight path angle response to throttle and elevator combined, 10 minutes');
legend('throttle','elevaltor','combined');
plot([t(1) t(end)],[(u_elevator(end)*pi/180 + u_throttle(end)*1/6 ),...
(u_elevator(end)*pi/180 + u_throttle(end)*1/6 )],'r');
subplot(2,1,2);
t   = 0:.1:1200;
u_throttle = step(sys1,t);
plot(t,u_throttle*1/6,'k');
u_elevator = step(sys2,t);
hold on;
plot(t,u_elevator*1*pi/180,'--');
plot(t,(u_elevator*1*pi/180 + u_throttle*1/6 ),'r');
ylim([-.1 .2]);
xlabel('Time (sec)');
ylabel('\gamma (rad)');
title('flight path angle response to throttle and elevator combined, 20 minutes');
legend('throttle','elevaltor','combined');
plot([t(1) t(end)],[(u_elevator(end)*pi/180 + u_throttle(end)*1/6 ),...
(u_elevator(end)*pi/180 + u_throttle(end)*1/6 )],'r');
```

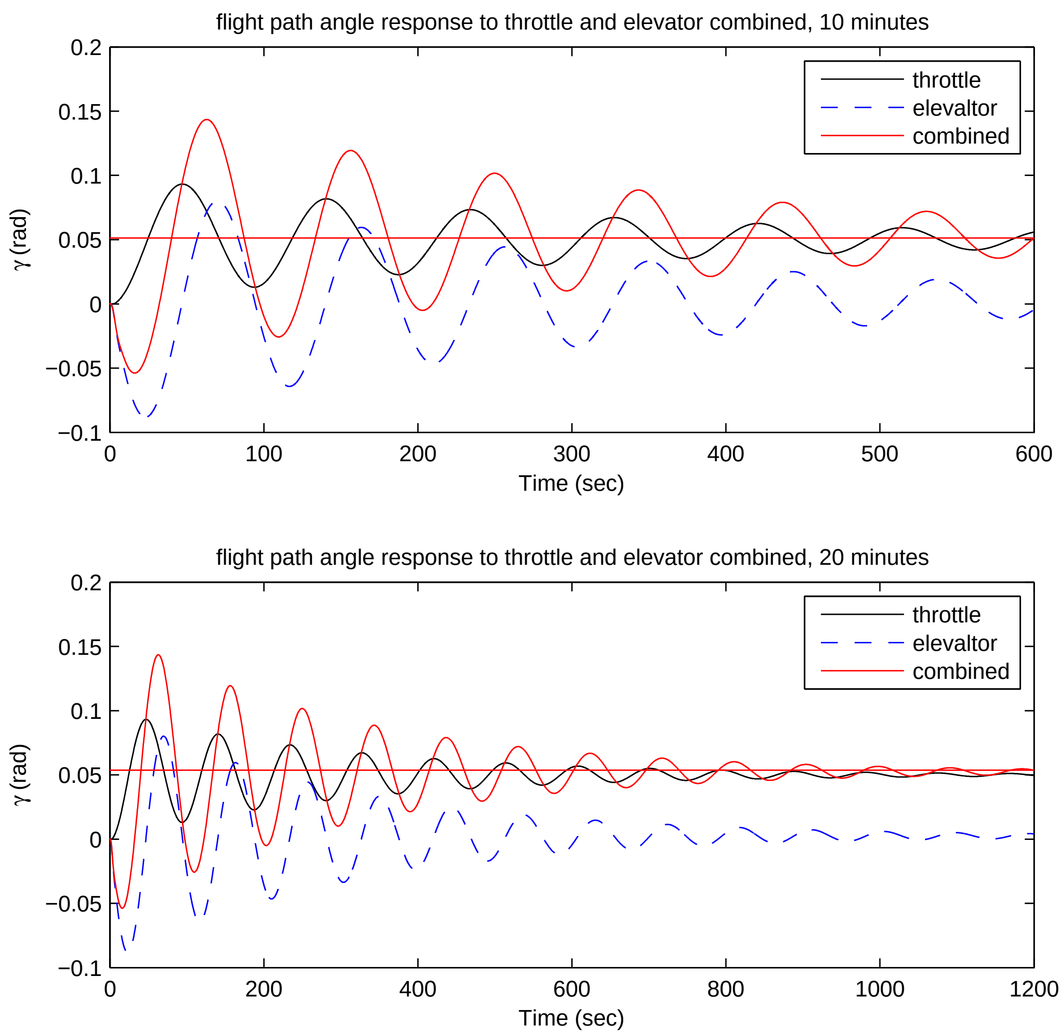

Figure 2.49: flight path angle γ response to throttle and elevator combined, open loop, longitudinal motion

2.4.4.4 open loop lateral motion responses

The first step is to generate $G(s)$ as was done in the above section for the longitudinal case. To obtain the transfer function matrix $G_{ij}(s)$ for the lateral motion, the following matrix needs to be found

$$\boldsymbol{G}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

Where A is the matrix for B-747 given on page 187, and B is given on page 224. The equation of motion for lateral motion becomes

$$\begin{pmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \Delta\dot{\phi} \end{pmatrix} = \overbrace{\begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} v \\ p \\ r \\ \Delta\phi \end{pmatrix}}^{\text{output}} + \overbrace{\begin{pmatrix} 0 & 5.642 \\ -0.1431 & 0.1144 \\ 0.003741 & -0.4859 \\ 0 & 0 \end{pmatrix}}^{B} \overbrace{\begin{Bmatrix} \delta_a \\ \delta_r \end{Bmatrix}}^{\text{control input}}$$

$G(s)$ is a 4×2 matrix. $G(i,j)$ is the transfer function of the ratio of i^{th} output to the j^{th} input.

For example, $G_{\Delta v,\delta_a}$ is $G(1,1)$ which is a function of s in equation (1). To obtain all the transfer functions, equation (1) is evaluated. This can be done using equation 7.2.7 on page 209 of the text.

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{adj(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}$$

For this problem, this was done using symbolic algebra using the following steps, and the resulting $G(s)$ matrix found is shown below

```
A = {{-0.0558, 0, -774, 32.2},
{-0.003865, -0.4342, 0.4136, 0},
{0.001086, -0.006112, -0.1458, 0},
{0, 1, 0, 0}};
B = {{0, 5.642}, {-0.1431, 0.1144}, {0.003741, -0.4859}, {0, 0}};
g = Inverse[s*IdentityMatrix[4] - A].B;
r = Map[Collect[Simplify@Numerator[#], s]/Denominator[#] &, g, {2}];
r2 = Map[Collect[Numerator[#], s]/Denominator[#] &, r, {2}];
MatrixForm[r2]
```

The output is

$$\begin{pmatrix} \frac{-0.621998-6.54202\,s-2.89553\,s^2}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} & \frac{-5.9341+167.893\,s+379.359\,s^2+5.642\,s^3}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} \\ \frac{-0.110171\,s-0.0273017\,s^2-0.1431\,s^3}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} & \frac{-1.36834\,s-0.199712\,s^2+0.1144\,s^3}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} \\ \frac{-0.00453851+0.000139442\,s+0.00270772\,s^2+0.003741\,s^3}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} & \frac{-0.0564712-0.00901786\,s-0.232663\,s^2-0.4859\,s^3}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} \\ \frac{-0.110171-0.0273017\,s-0.1431\,s^2}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} & \frac{-1.36834-0.199712\,s+0.1144\,s^2}{0.00368199+0.511384\,s+0.938762\,s^2+0.6358\,s^3+s^4} \end{pmatrix}$$

Figure 2.50: Transfer function matrix $G(s)$ for lateral motion

Using Matlab, the same procedure was done using syms as follows

```
A = [-0.0558, 0, -774, 32.2;
    -0.003865, -0.4342, 0.4136, 0;
    0.001086, -0.006112, -0.1458, 0;
    0, 1, 0, 0]

B = [0, 5.642;
    -0.1431, 0.1144;
    0.003741, -0.4859;
    0, 0]

syms s;
G=inv(s*eye(4)-A)*B;
outPut={'v','p','r','phi'};
inPut={'del_a','del_r'};
for i=1:4
  for j=1:2
      [N,D] = numden(G(i,j));
      c=coeffs(D);
      fprintf('N(%s,%s) = %s',outPut{i},inPut{j},...
           char(vpa(N/c(end),5)))
      fprintf('\n');
  end
end
```

And the output is

```
N(v,del_a) = - 6.542*s - 2.8955*s^2 - 0.622
N(v,del_r) = 167.89*s + 379.36*s^2 + 5.642*s^3 - 5.9341
N(p,del_a) = - 0.11017*s - 0.027302*s^2 - 0.1431*s^3
N(p,del_r) = 0.1144*s^3 - 0.19971*s^2 - 1.3683*s
N(r,del_a) = 0.00013944*s + 0.0027077*s^2 + 0.003741*s^3 - 0.0045385
N(r,del_r) = - 0.0090179*s - 0.23266*s^2 - 0.4859*s^3 - 0.056471
N(phi,del_a) = - 0.027302*s - 0.1431*s^2 - 0.11017
N(phi,del_r) = 0.1144*s^2 - 0.19971*s - 1.3683
```

2.4.4.4.1 Generating the transfer function when aileron $\delta_a = 6^\circ$ is the input Transfer function G_{v,δ_a}

G_{ij} is now found. The input is δ_a which is $j = 1$. The output is v, which is $i = 1$. Therefore

$$G_{11}(s) = G_{v,\delta_a} = \frac{\overbrace{-2.89553s^2 - 6.54202s - 0.621998}^{N_{v,\delta_a}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The side velocity v response to $\delta_a = 6^o$ is generated using Matlab

```
close all; clear all;
s = tf('s');
num_aileron = -2.89553*s^2-6.54202*s-0.621998;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
[v,t] = step(sys);
plot(t,v*6*pi/180);
xlim([0 600]);
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral velocity response, open loop, lateral motion');
grid
```

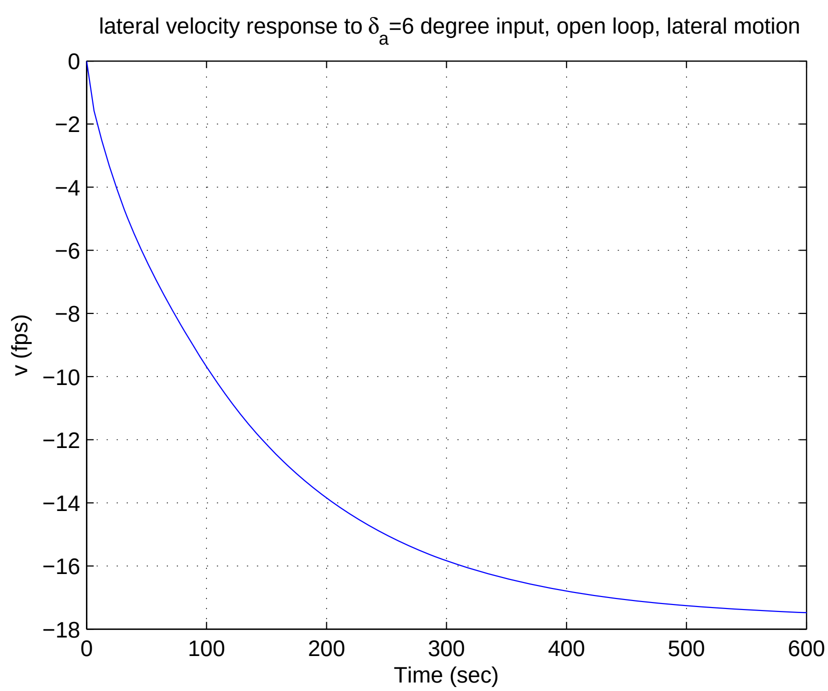

Figure 2.51: Lateral velocity v response to $\delta_a = 6^0$

The bode plot for G_{v,δ_a} is now generated. This is figure 7.27 on page 248 of the textbook.

```
close all; clear all;
```

```
s = tf('s');
num_aileron = -2.89553*s^2-6.54202*s-0.621998;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
bodeplot(sys,opts);
grid
```

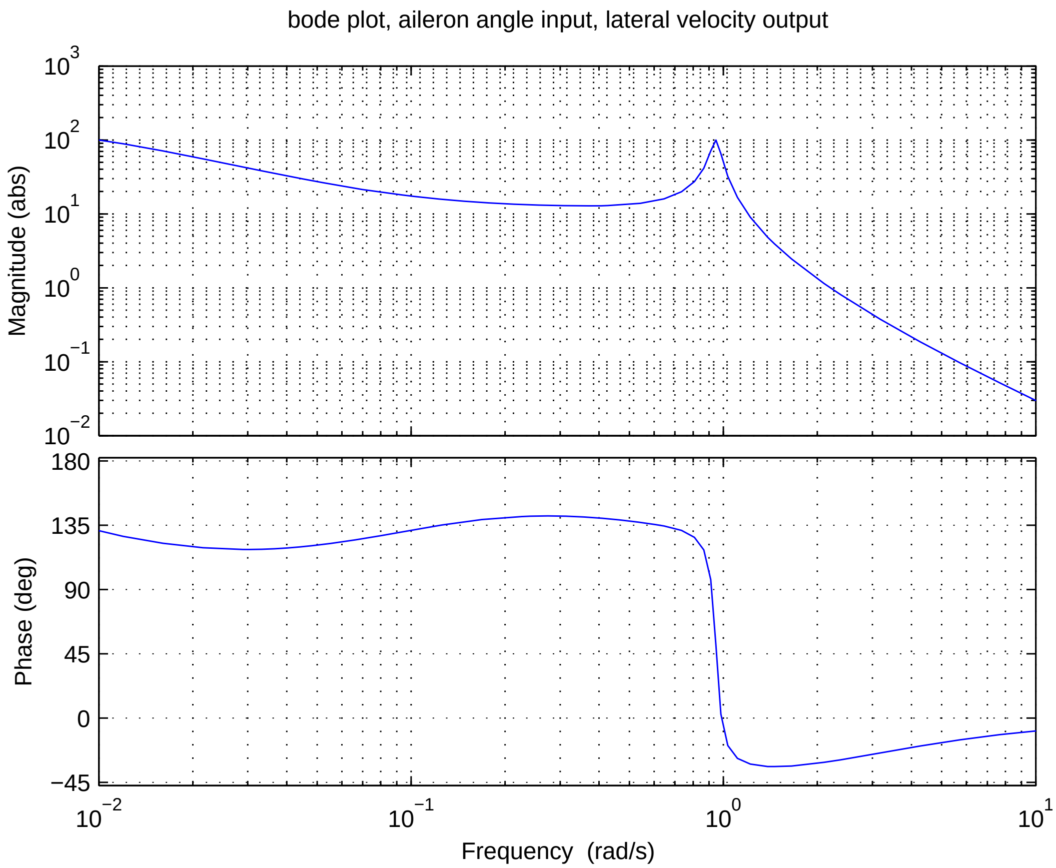

Figure 2.52: Bode plot of G_{v,δ_a}

Transfer function G_{p,δ_a}

The input now is δ_a which is $j = 1$. The output is p, which is $i = 2$. Therefore

$$G_{21}(s) = G_{p,\delta_a} = \frac{\overbrace{-0.1431s^3 - 0.0273017s^2 - 0.110171s}^{N_{p,\delta_a}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The roll rate p response to $\delta_a = 6^o$ is generated using Matlab

```
close all; clear all;
```

```
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_aileron = -0.1431*s^3-0.0273017*s^2-0.110171*s;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
t=0:.1:30;
p = step(sys,t);
subplot(2,1,1);
plot(t,p*6*pi/180);
xlabel('Time (sec)');
ylabel('p (rad/sec)');
title('roll rate (p) response to \delta_a=6 degree input, lateral motion, 30 seconds');
hold on;
plot([t(1) t(end)],[(p(end)*6*pi/180),(p(end)*6*pi/180)],'r');
legend('roll rate','steady state');
%
subplot(2,1,2);
t=0:.1:180;
p = step(sys,t);
plot(t,p*6*pi/180);
xlabel('Time (sec)');
ylabel('p (rad/sec)');
title('roll rate (p) response to \delta_a=6 degree input, lateral motion, 3 minutes');
hold on;
plot([t(1) t(end)],[(p(end)*6*pi/180),(p(end)*6*pi/180)],'r');
legend('roll rate','steady state');
```

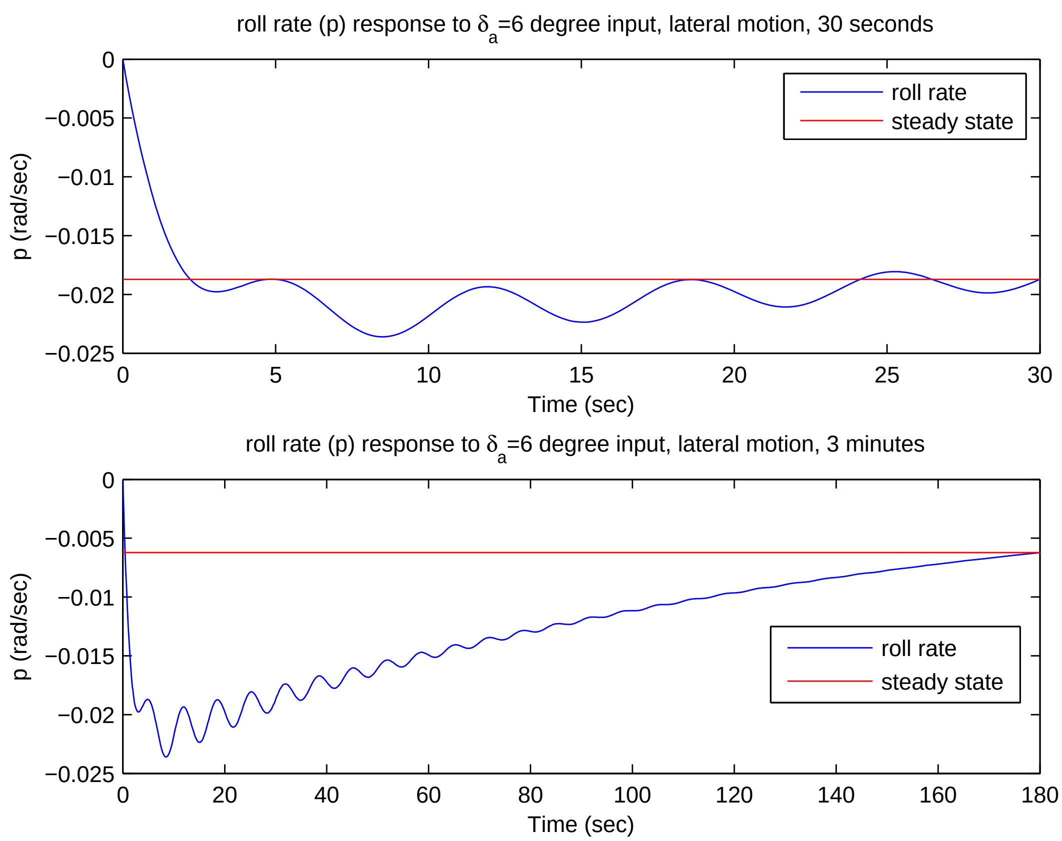

Figure 2.53: The roll rate p response to $\delta_a = 6^0$

The bode plot for G_{p,δ_a} is

```
close all; clear all;
s = tf('s');
num_aileron = -0.1431*s^3-0.0273017*s^2-0.110171*s;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function roll rate response to aileron input');
```

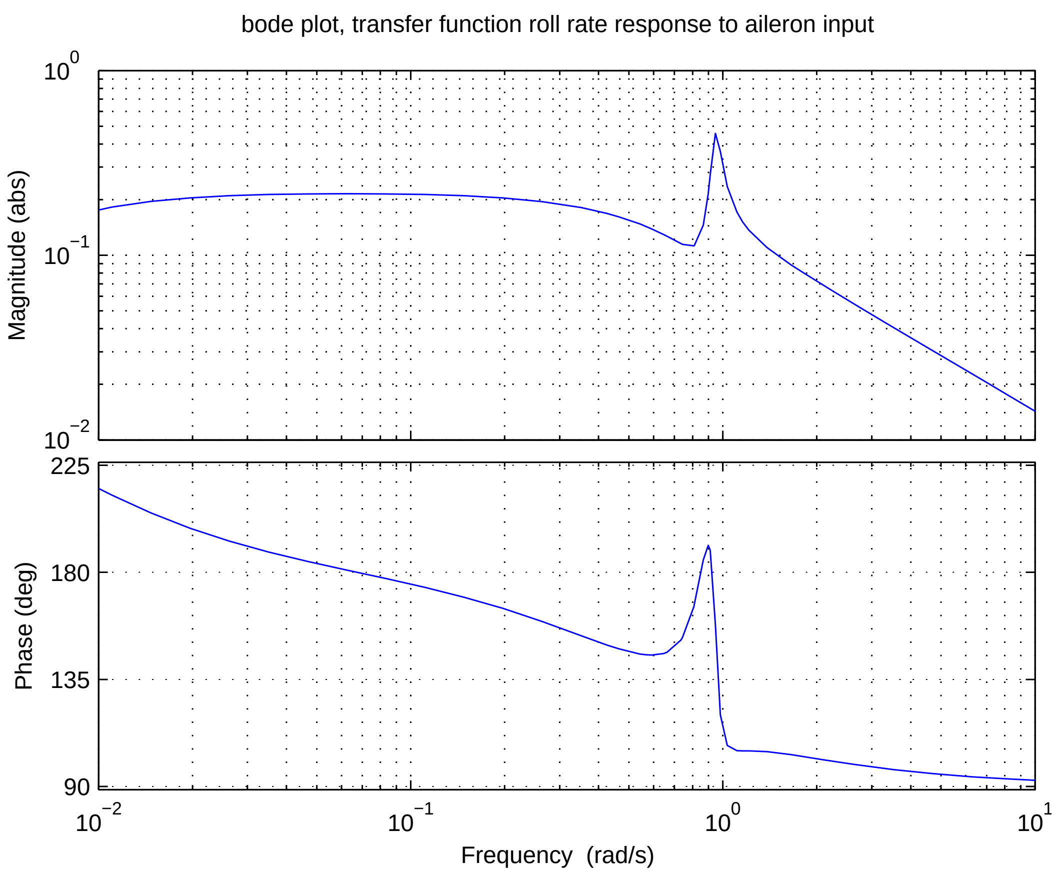

Figure 2.54: Bode plot of G_{p,δ_a}

Using transfer function G_{r,δ_a}

The input now is δ_a which is $j = 1$. The output is r which is $i = 3$. Therefore

$$G_{31}(s) = G_{r,\delta_a} = \frac{\overbrace{-0.003741s^3 + 0.00270772s^2 + 0.000139442s - 0.00453851}^{N_{r,\delta_a}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The yaw rate r response to $\delta_a = 6^o$ is generated using Matlab

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_aileron = -0.003741*s^3+0.00270772*s^2+0.000139442*s-0.00453851;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
t=0:.01:180;
r = step(sys,t);
subplot(2,1,1);
plot(t,r*6*pi/180);
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_a=6 degree input, lateral motion, 3 minutes');
hold on;
plot([t(1) t(end)],[(r(end)*6*pi/180),(r(end)*6*pi/180)],'r');
legend('yaw rate (r)','steady state');
grid
subplot(2,1,2);
t=0:.01:600;
r = step(sys,t);
plot(t,r*6*pi/180);
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_a=6 degree input, lateral motion, 10 minutes');
hold on;
plot([t(1) t(end)],[(r(end)*6*pi/180),(r(end)*6*pi/180)],'r');
legend('yaw rate (r)','steady state');
grid
```

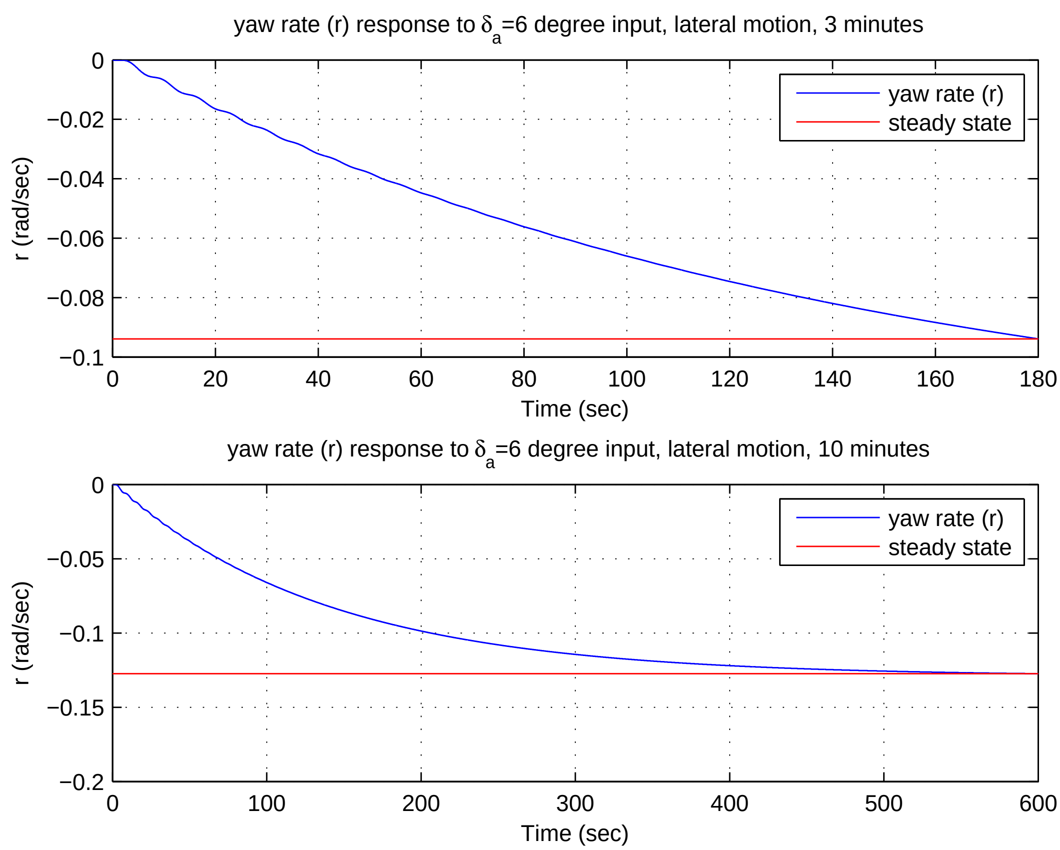

Figure 2.55: The yaw rate r response to $\delta_a = 6^0$

The bode plot for G_{r,δ_a} is now generated. This can be compared to figure 7.27(e) on page 250.

```
close all; clear all;
s = tf('s');
num_aileron = -0.003741*s^3+0.00270772*s^2+0.000139442*s-0.00453851;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function yaw rate (r) response to aileron input');
```

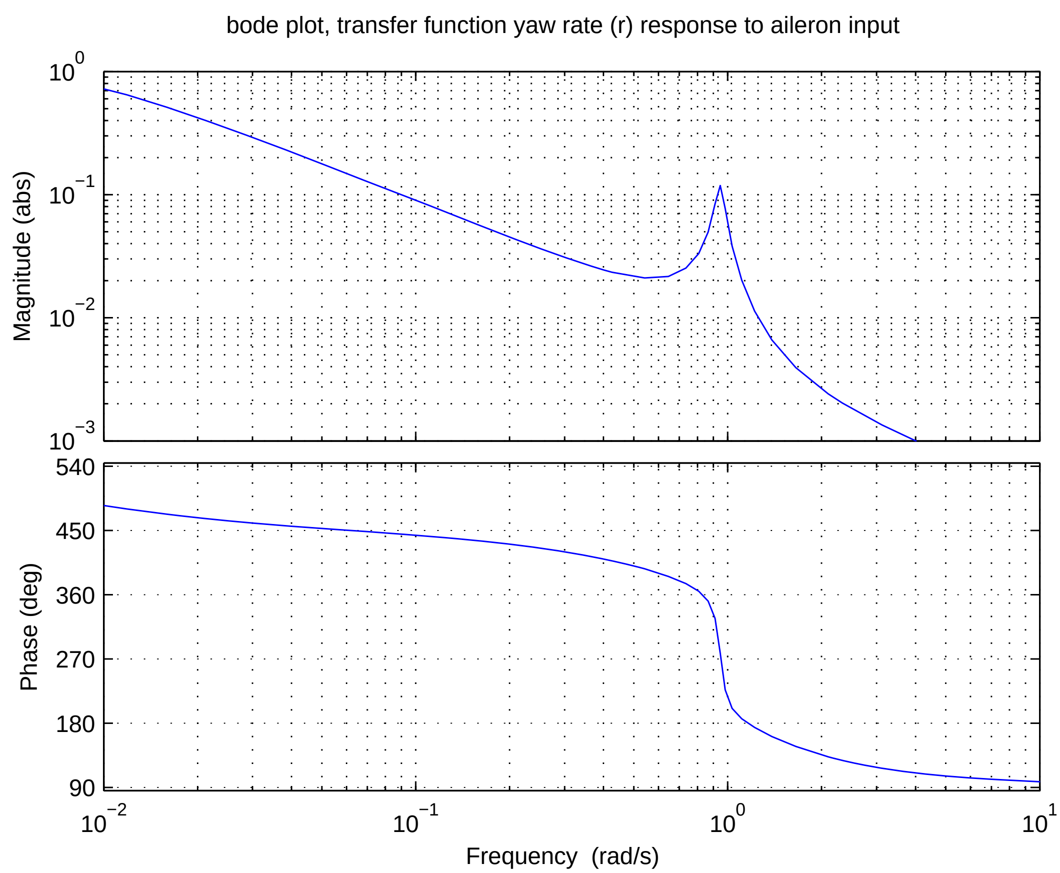

Figure 2.56: Bode plot of G_{r,δ_a}

Transfer function G_{ϕ,δ_a}

The input now is δ_a which is $j = 1$. The output is ϕ,which is $i = 4$. Therefore

$$G_{41}(s) = G_{\phi,\delta_a} = \frac{\overbrace{-0.1431s^2 - 0.0273017s - 0.110171}^{N_{\phi,\delta_a}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The Euler angle Φ response to $\delta_a = 6^\circ$ is found using Matlab

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num  = -0.1431*s^2-0.0273017*s-0.110172;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
t=0:.1:600;
r = step(sys,t);
plot(t,r*6*pi/180);
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Euler angle \Phi response to \delta_a=6 degree input, lateral motion, 10 minutes');
hold on;
plot([t(1) t(end)],[(r(end)*6*pi/180),(r(end)*6*pi/180)],'r');
legend('\Phi','steady state');
```

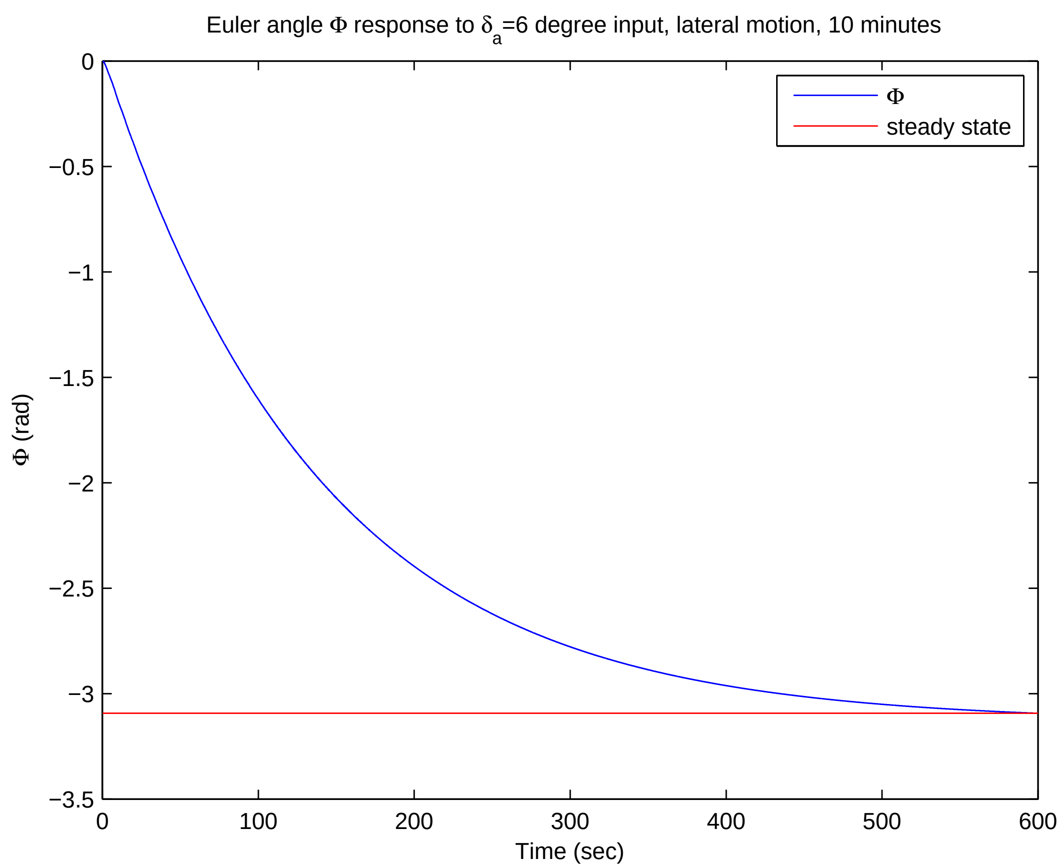

Figure 2.57: Euler angle Φ response to $\delta_a = 6^0$

The bode plot for G_{Φ,δ_a} is generated. This can be compared to figure 7.27(c) on page 249.

```
close all; clear all;
s = tf('s');
num_aileron = -0.1431*s^2-0.0273017*s-0.110172;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function Euler angle \Phi response to aileron input');
```

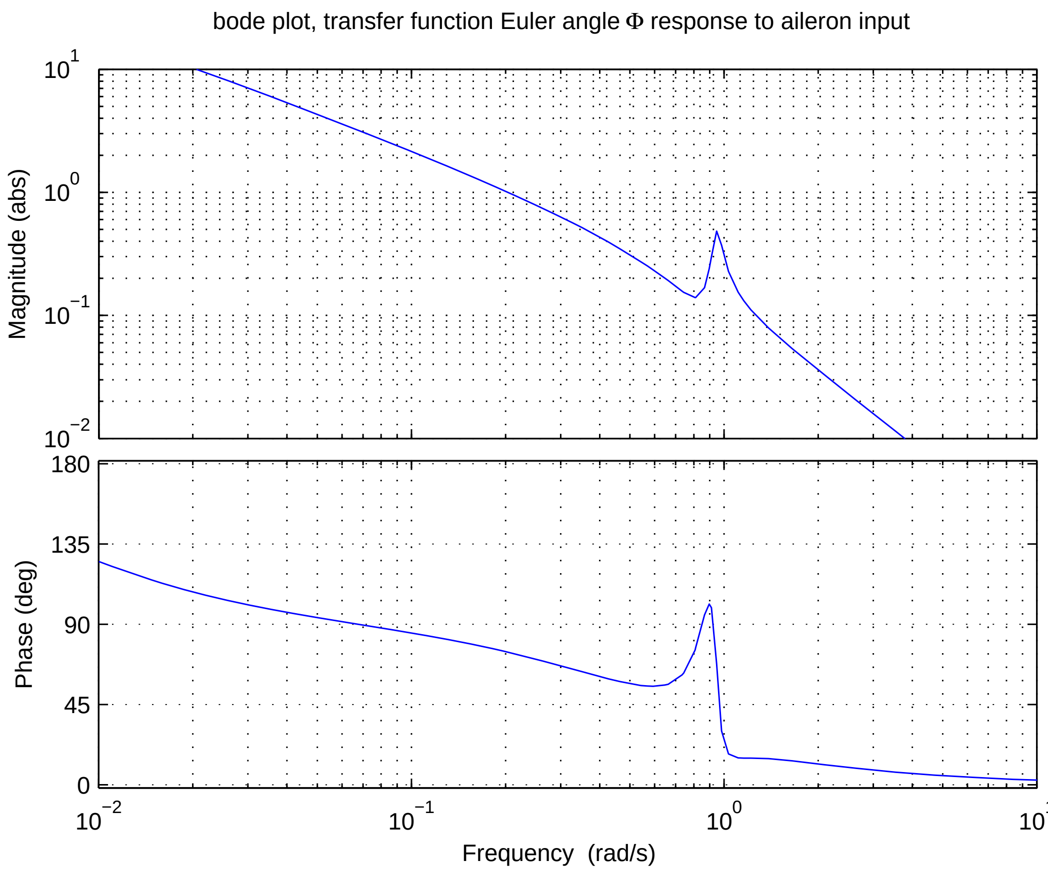

Figure 2.58: Bode plot of G_{Φ,δ_a}

2.4.4.4.2 Generating the transfer function when rudder $\delta_r = 3^o$ is the input

Transfer function $G_{v,\delta r}$

The input now is δ_r which is $j = 2$. The output is v which is $i = 1$. Therefore

$$G_{12}(s) = G_{v,\delta_r} = \frac{\overbrace{-5.642s^3 + 379.359s^2 + 167.893s - 5.9341}^{N_{v,\delta_r}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The lateral speed v response (side slip rate) to $\delta_r = 3^\circ$ is generated using Matlab

```
close all; clear all;
s = tf('s');
num  = -5.642*s^3+379.359*s^2+167.893*s-5.9341;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
t=0:.1:30;
v = step(sys,t);
subplot(2,1,1);
plot(t,v*3*pi/180);
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=3 degree input, first 1/2 minute');
hold on;
plot([t(1) t(end)],[(v(end)*3*pi/180),(v(end)*3*pi/180)],'r');
legend('lateral speed','steady state');
subplot(2,1,2);
t=0:.1:180;
v = step(sys,t);
plot(t,v*3*pi/180);
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=3 degree input, first 3 minutes');
hold on;
plot([t(1) t(end)],[(v(end)*3*pi/180),(v(end)*3*pi/180)],'r');
legend('lateral speed','steady state');
```

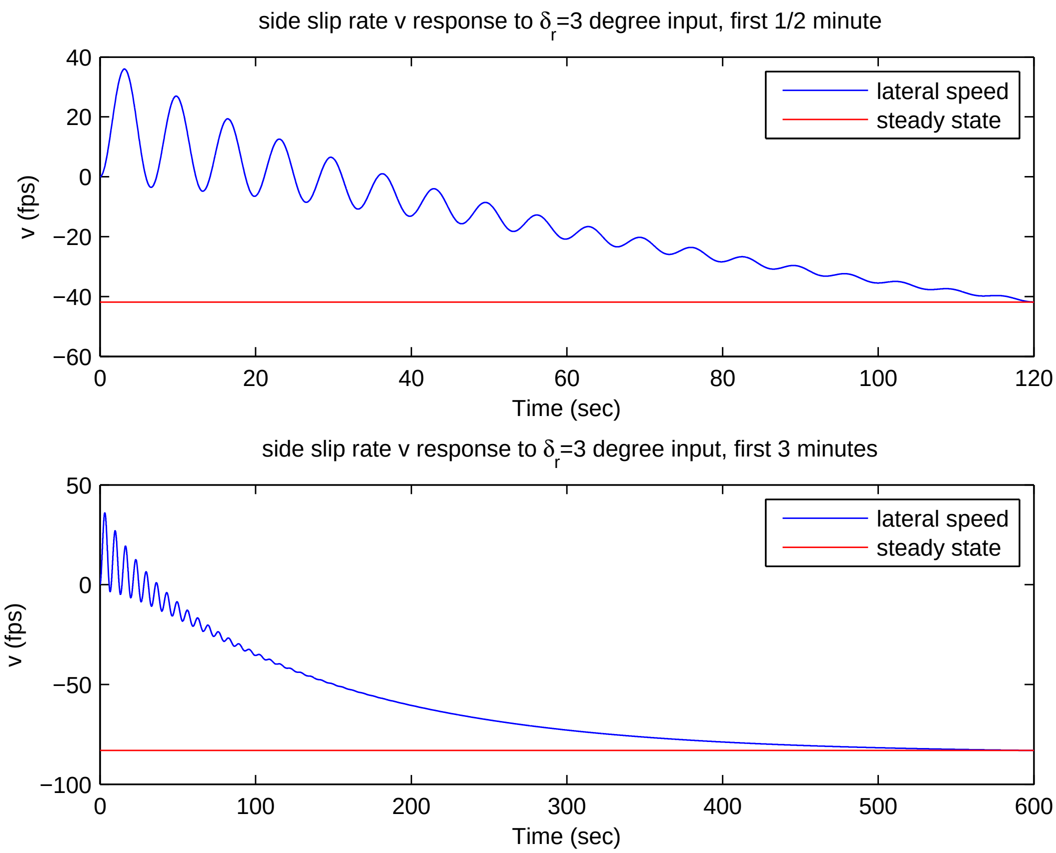

Figure 2.59: side slip rate v response to $\delta_r = 3^o$

Bode plot for G_{v,δ_r} is now generated. This can be compared to figure 7.26(a) on page 245.

```
close all; clear all;
s = tf('s');
num_aileron = -5.642*s^3+379.359*s^2+167.893*s-5.9341;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function lateral speed response to rudder input');
```

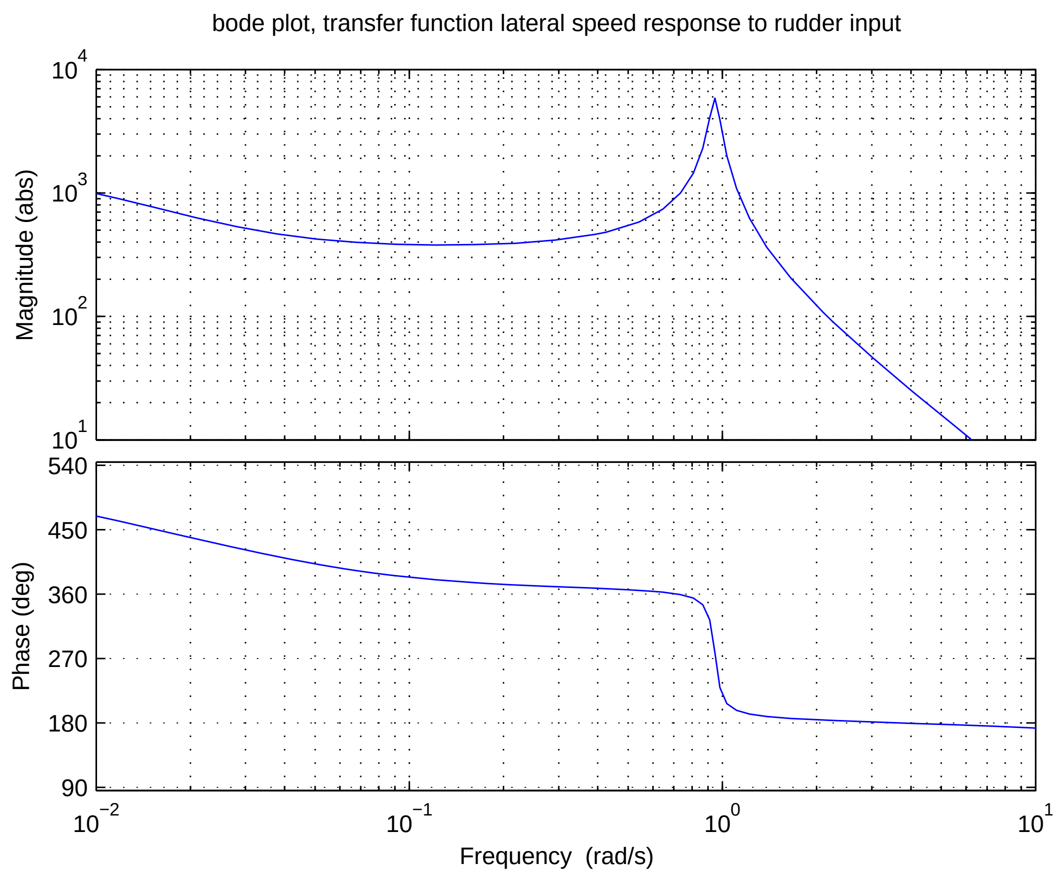

Figure 2.60: Bode plot of G_{v,δ_r}

Transfer function $G_{p,\delta r}$

The input now is δ_r which is $j = 2$. The output is p which is $i = 2$. Therefore

$$G_{22}(s) = G_{p,\delta_r} = \frac{\overbrace{-0.1144s^3 - 0.199712s^2 - 1.36834s}^{N_{p,\delta_r}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The roll rate p respones to $\delta_r = 3^o$ is generated using Matlab

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num  = -0.1144*s^3-0.199712*s^2-1.36834*s;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
t=0:.1:60;
p = step(sys,t);
subplot(2,1,1);
plot(t,p*3*pi/180);
xlabel('Time (sec)');
ylabel('p (rad/sec)');
title('roll rate (p) response to \delta_r=3 degree input, lateral motion, first one minute');
hold on;
plot([t(1) t(end)],[(p(end)*3*pi/180),(p(end)*3*pi/180)],'r');
legend('roll rate (p)','steady state');
subplot(2,1,2);
t=0:.1:180;
p = step(sys,t);
subplot(2,1,2);
plot(t,p*3*pi/180);
xlabel('Time (sec)');
ylabel('p (rad/sec)');
title('roll rate (p) response to \delta_r=3 degree input, lateral motion, first 3 minutes');
hold on;
plot([t(1) t(end)],[(p(end)*3*pi/180),(p(end)*3*pi/180)],'r');
legend('roll rate (p)','steady state');
```

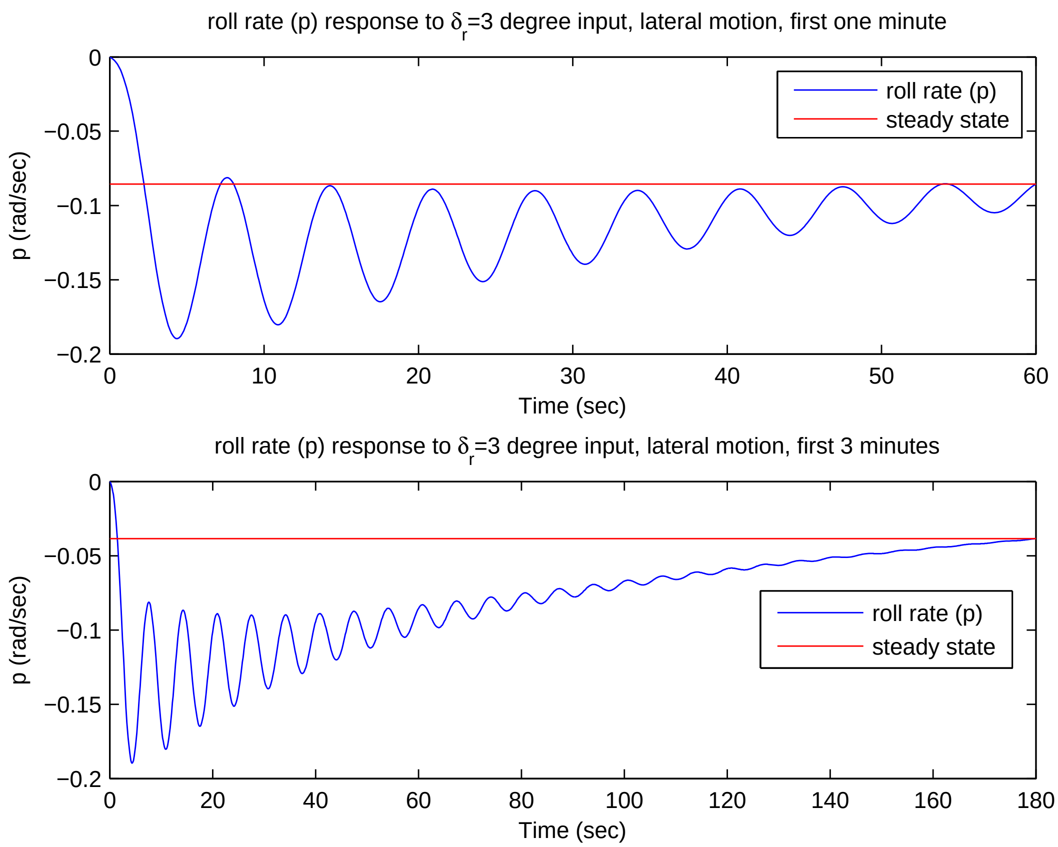

Figure 2.61: Roll rate p response to $\delta_r = 3^0$

Bode plot for G_{p,δ_r} is now generated

```
close all; clear all;
s = tf('s');
num_aileron = -0.1144*s^3-0.199712*s^2-1.36834*s;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function roll rate (p) response to rudder input');
```

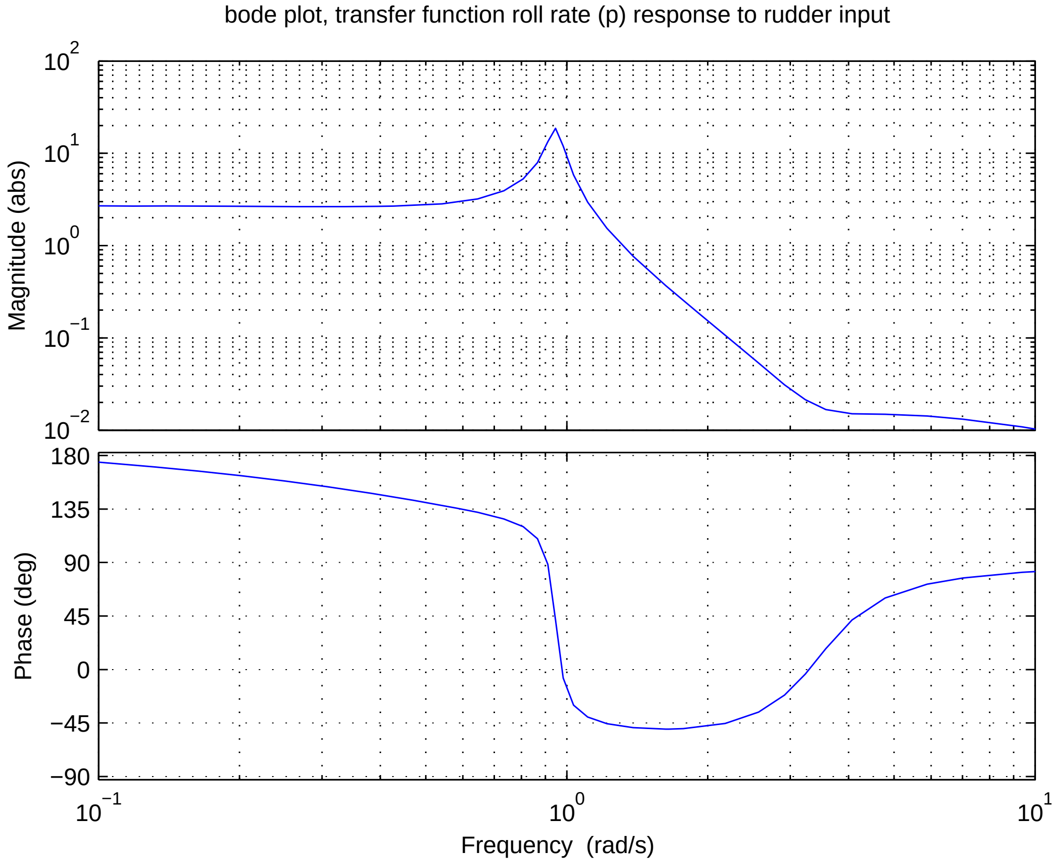

Figure 2.62: Bode plot of G_{p,δ_r}

Using transfer function $G_{r,\delta r}$

The input now is δ_r which is $j = 2$. The output is r which is $i = 3$. Therefore

$$G_{32}(s) = G_{r,\delta_r} = \frac{\overbrace{-0.4859s^3 - 0.232663s^2 - 0.00901786s - 0.0564712}^{N_{r,\delta_r}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The yaw rate r respones to $\delta_r = 3^o$ is generated using Matlab

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num  = -0.4859*s^3-0.232663*s^2-0.0090178*s-0.0564712;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
t=0:.01:600;
r = step(sys,t);
plot(t,r*3*pi/180);
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (p) response to \delta_r=3 degree input, open loop, lateral motion');
grid
```

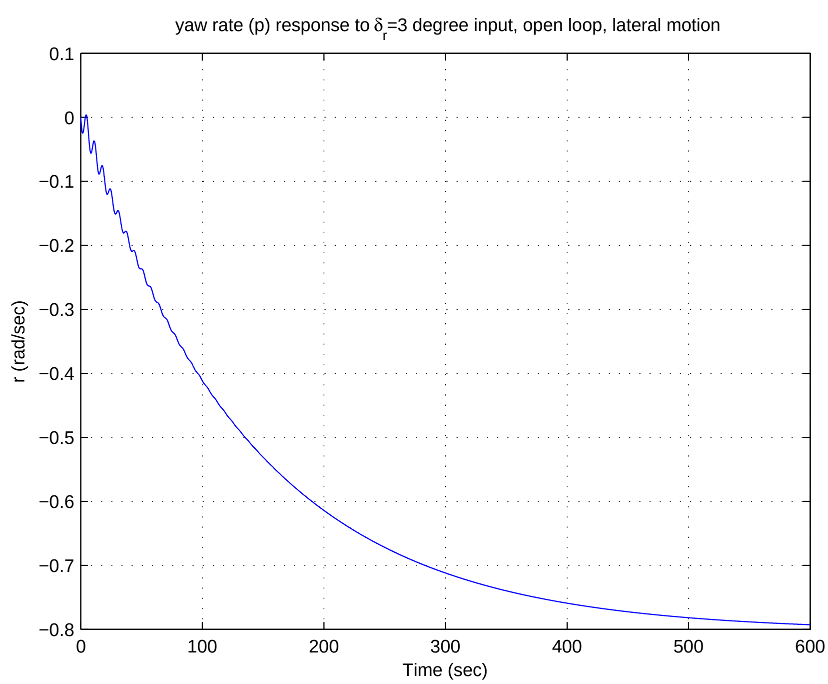

Figure 2.63: Yaw rate r response to $\delta_r = 3^0$ up to 600 seconds

The bode plot for G_{r,δ_r} is now generated.

```
close all; clear all;
s = tf('s');
num_aileron = -0.4859*s^3-0.232663*s^2-0.0090178*s-0.0564712;
```

```
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num_aileron/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
title('bode plot, transfer function yaw rate (r) response to rudder input');
```

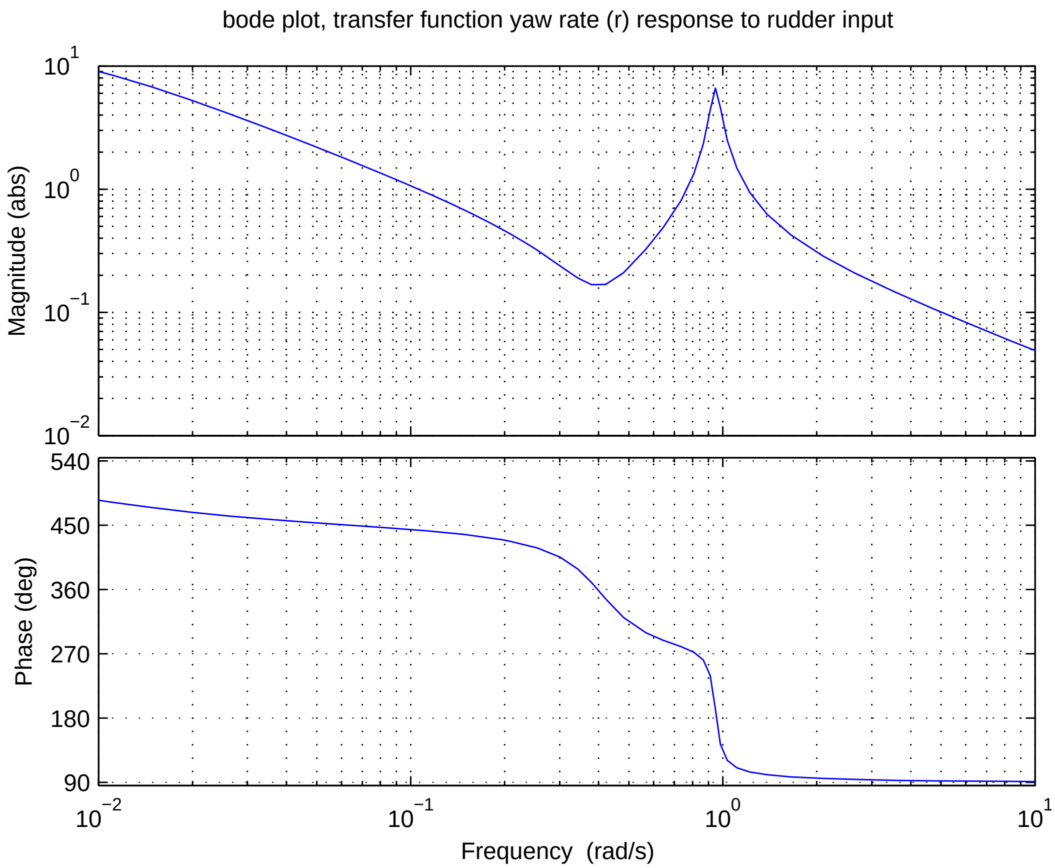

Figure 2.64: Bode plot of G_{r,δ_r}

Using transfer function G_{ϕ,δ_r}

The input now is δ_r which is $j = 2$. The output is ϕ which is $i = 4$. Therefore

$$G_{42}(s) = G_{\phi,\delta_r} = \frac{\overbrace{-0.1144s^2 - 0.199712s - 1.36834}^{N_{\phi,\delta_r}}}{s^4 + 0.6358s^3 + 0.938762s^2 + 0.511384s + 0.00368199}$$

The Euler angle Φ respones to $\delta_r = 3^o$ is generated using Matlab

```
close all; clear all;
```

```
s = tf('s');
num = -0.1144*s^2-0.199712*s-1.36834;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
[phi,t] = step(sys);
plot(t,phi*3*pi/180);
xlim([0 600]);
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Euler angle \Phi response to \delta_r=3 degree input, open loop, lateral motion');
grid
```

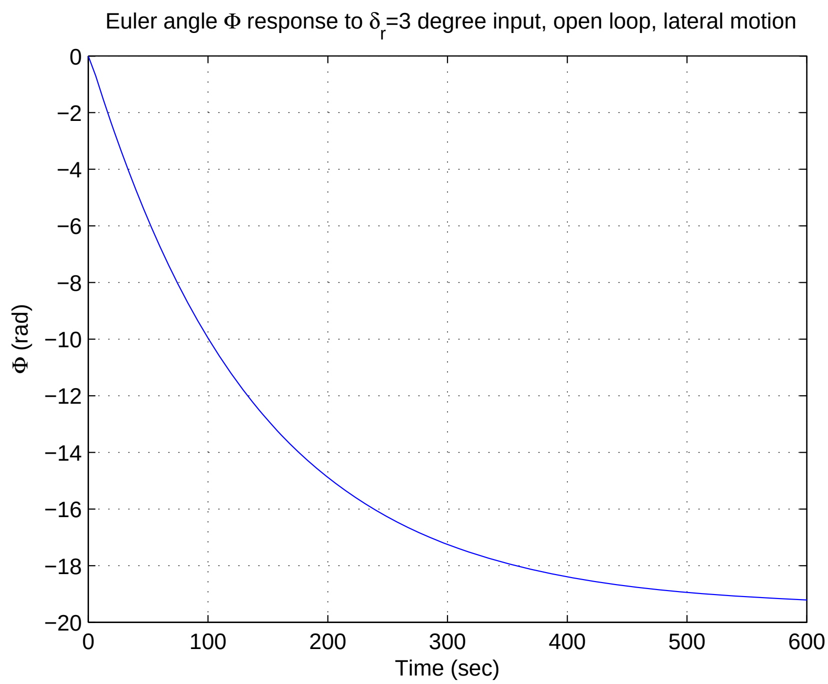

Figure 2.65: Euler Φ response to $\delta_r = 3^0$ up to 600 seconds

The bode plot for G_{Φ,δ_r} is now generated. This can be compared to figure 7.26(c) on page 246.

```
close all; clear all;
s = tf('s');
num = -0.1144*s^2-0.199712*s-1.36834;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys = tf(num/den);
opts = bodeoptions;
opts.MagUnits='abs';
opts.MagScale='log';
figure
bodeplot(sys,opts);
grid
```

```
title('bode plot, transfer function \Phi response to rudder input');
```

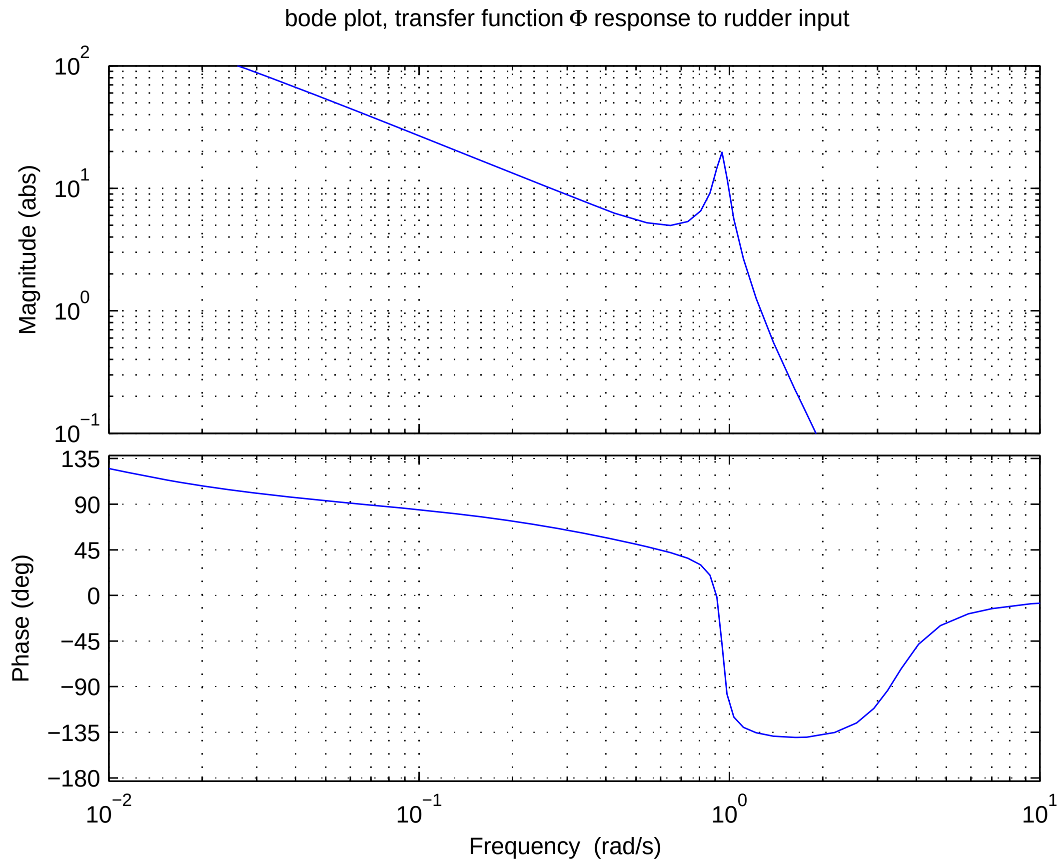

Figure 2.66: Bode plot of G_{Φ,δ_r}

2.4.4.4.3 Simultaneous response for aileron and rudder input ($\delta_a = 6^o, \delta_r = -3^o$) The transfer functions are found above. They are used to find the combined response. As was done for the longitudinal case, since the system is linear, the response to $\delta_a = 6^o$ is found and added to the response to $\delta_r = -3^\circ$ to obtain the combined response.

Lateral v response to combined ($\delta_a = 6^o, \delta_r = -3^o$)

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_a = -2.89553*s^2-6.54202*s-0.621998;
num_r = -5.642*s^3+379.359*s^2+167.893*s-5.9341;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.1:120;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-.k');
```

```
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'--');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=-3 and \delta_a=6 degree combined, 2 minutes');
legend('\delta_a','\delta_r','combined','steady state');
xlim([0 120])
%
subplot(2,1,2);
t=0:.1:300;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-.k');
hold on;
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'--');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=-3 and \delta_a=6 degree combined, 5 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180-yr(end)*3*pi/180),...
(ya(end)*6*pi/180-yr(end)*3*pi/180)],'--r');
legend('\delta_a','\delta_r','combined','steady state');
xlim([0 300])
```

Yaw rate r response to combined ($\delta_a = 6^o, \delta_r = -3^o$)

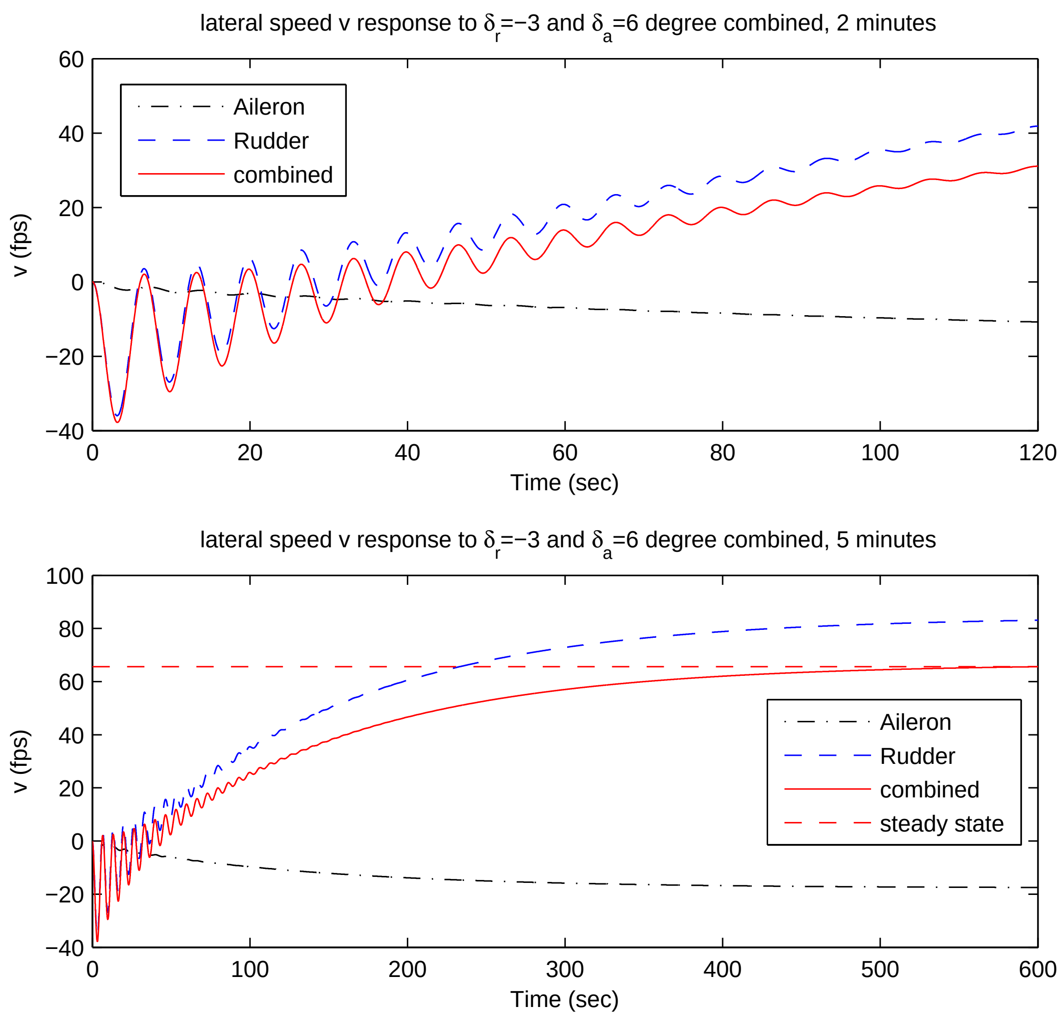

Figure 2.67: side speed v response to $\delta_r = -3^o$ and $\delta_a = 6^o$ combined

```
close all; clear all;
close all; clear all;
s = tf('s');
num_a = -0.003741*s^3+0.00270772*s^2+0.000139442*s-0.00453851;
num_r  = -0.4859*s^3-0.232663*s^2-0.0090178*s-0.0564712;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.1:30;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'-');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_r=-3 and \delta_a=6 degree combined, 30 seconds');
legend('\delta_a','\delta_r','combined');
%
subplot(2,1,2);
t=0:.1:300;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'-');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_r=-3 and \delta_a=6 degree combined, 5 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180-yr(end)*3*pi/180),...
(ya(end)*6*pi/180-yr(end)*3*pi/180)],'--r');
legend('\delta_a','\delta_r','combined','steady state');
```

Euler angle Φ response to combined ($\delta_a = 6^o, \delta_r = -3^o$)

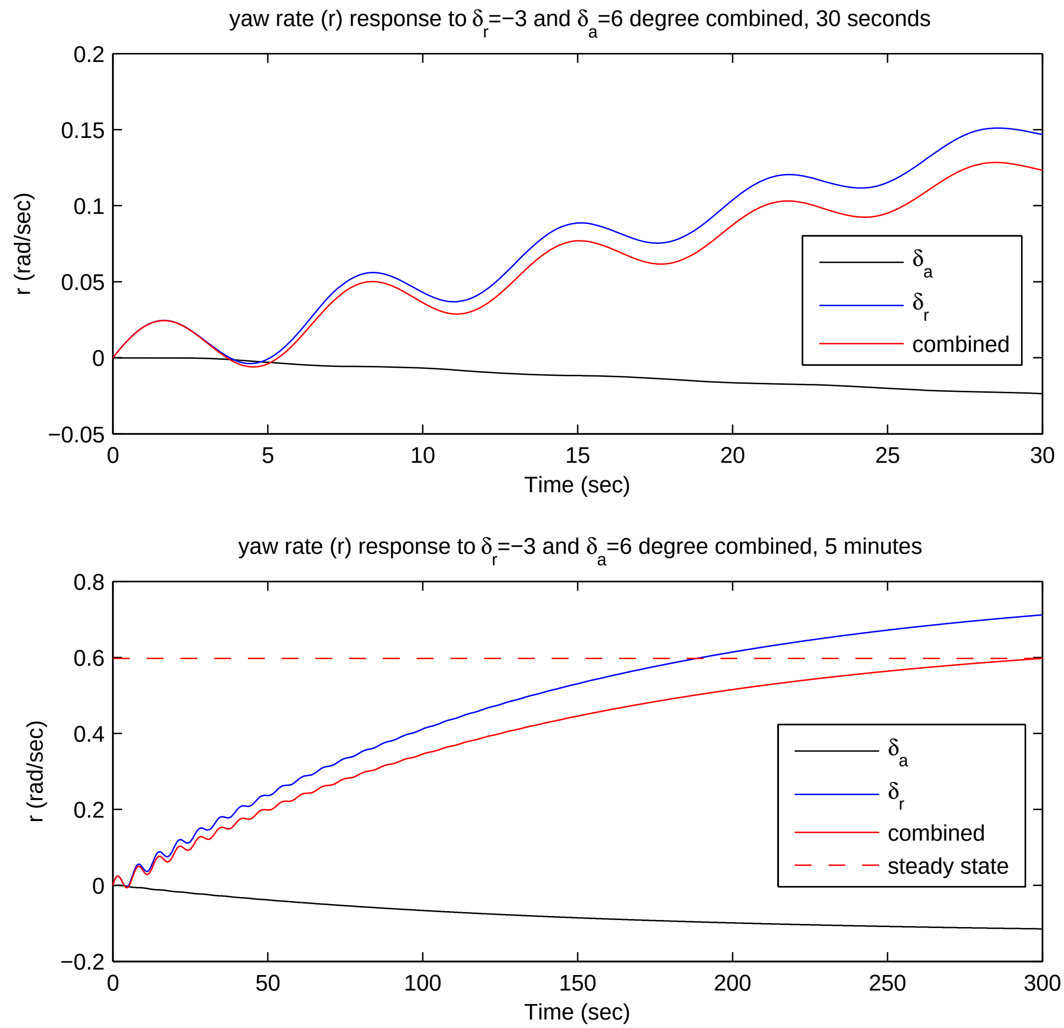

Figure 2.68: Yaw rate r response to $\delta_r = -3^o$ and $\delta_a = 6^o$ combined

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_a  = -0.1431*s^2-0.0273017*s-0.110172;
num_r = -0.1144*s^2-0.199712*s-1.36834;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.1:180;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'-');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Roll angle \Phi response to \delta_r=-3 and \delta_a=6 degree combined, 3 minutes');
legend('Aileron','Rudder','combined');
%
subplot(2,1,2);
t=0:.1:480;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-k');
hold on;
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'-');
plot(t,ya*6*pi/180-yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Roll angle \Phi response to \delta_r=-3 and \delta_a=6 degree combined, 7 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180-yr(end)*3*pi/180),...
(ya(end)*6*pi/180-yr(end)*3*pi/180)],'--r');
legend('Aileron','Rudder','combined','steady state');
xlim([0 480]);
```

2.4.4.4.4 Simultaneous response for aileron and rudder input ($\delta_a = 6^o, \delta_r = +3^o$) The transfer functions are found above. They are used to obtain the combined response. As was done for the longitudinal case, since the system is linear, the response to $\delta_a = 6^0$ was added to the response to $\delta_r = 3^o$ to obtain the combined response.

Lateral v response to combined ($\delta_a = 6^o, \delta_r = +3^o$)

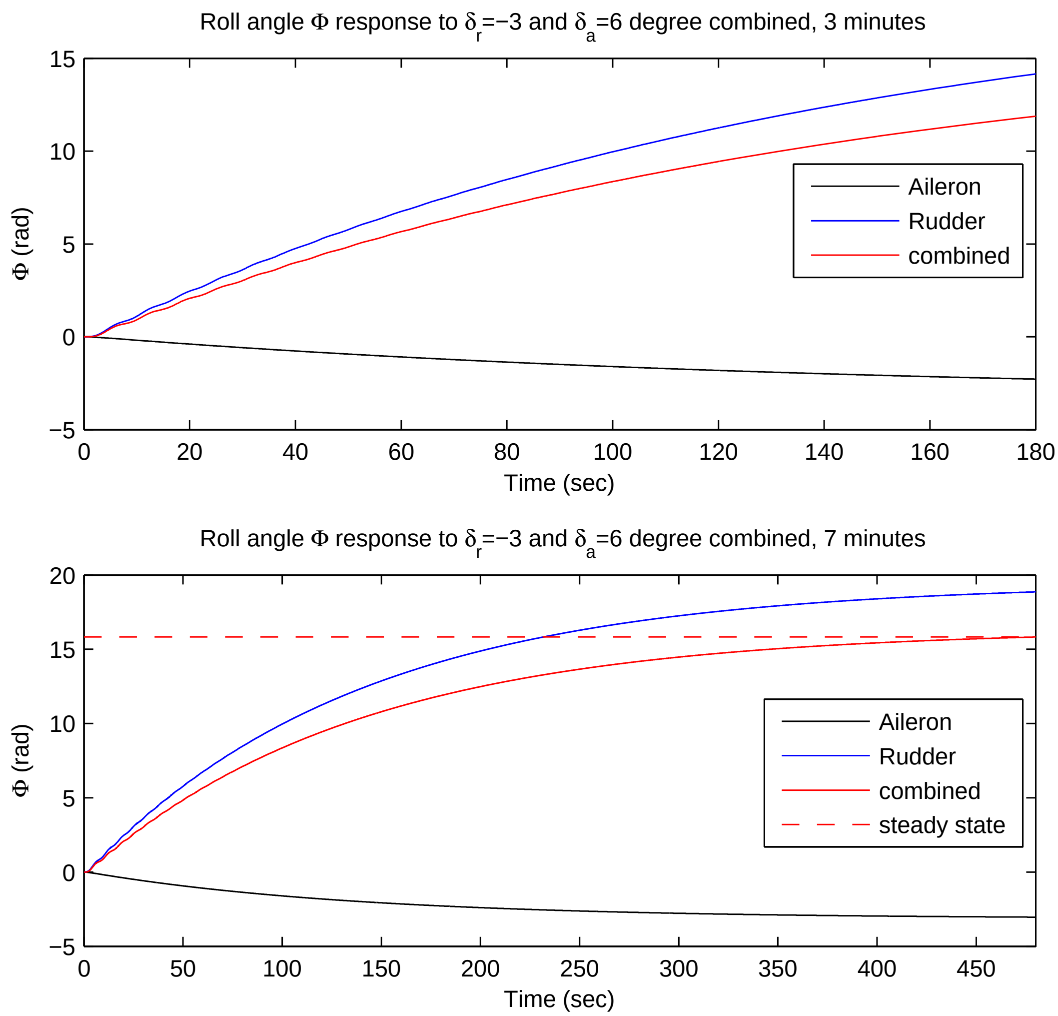

Figure 2.69: Euler angle Φ response to $\delta_r = -3^o$ and $\delta_a = 6^o$ combined

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_a = -2.89553*s^2-6.54202*s-0.621998;
num_r = -5.642*s^3+379.359*s^2+167.893*s-5.9341;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.01:120;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-.k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,yr*3*pi/180,'--');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=-3 and \delta_a=6 degree combined, 2 minutes');
legend('Aileron','Rudder','combined');
%
subplot(2,1,2);
t=0:.01:600;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-.k');
hold on;
yr = step(sys2,t);
plot(t,yr*3*pi/180,'--');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('v (fps)');
title('lateral speed v response to \delta_r=-3 and \delta_a=6 degree combined, 5 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180+yr(end)*3*pi/180),...
(ya(end)*6*pi/180+yr(end)*3*pi/180)],'--r');
legend('Aileron','Rudder','combined','steady state');
```

Yaw rate r response to combined ($\delta_a = 6^o, \delta_r = +3^o$)

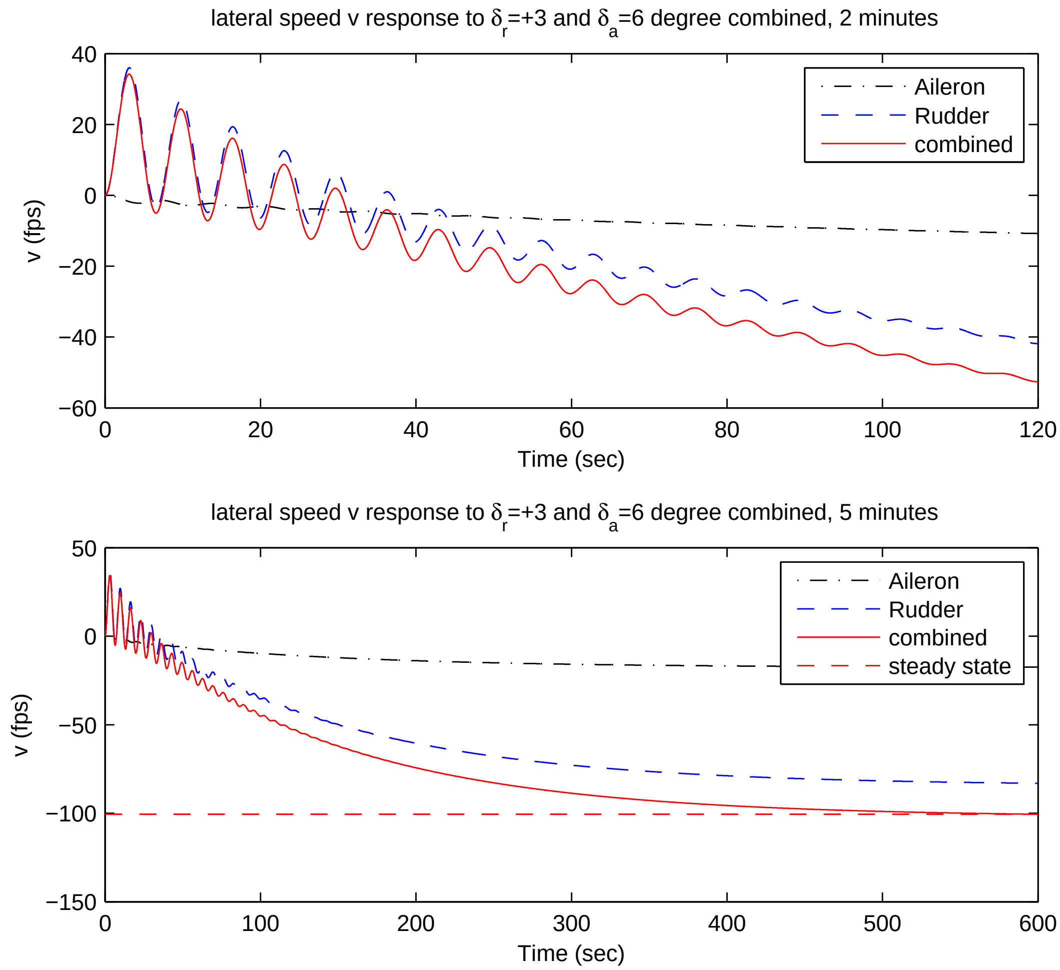

Figure 2.70: Lateral speed v response to $\delta_r = +3^o$ and $\delta_a = 6^o$ combined

```
close all; clear all;
s = tf('s');
num_a = -0.003741*s^3+0.00270772*s^2+0.000139442*s-0.00453851;
num_r  = -0.4859*s^3-0.232663*s^2-0.0090178*s-0.0564712;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.1:30;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,yr*3*pi/180,'-');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_r=+3 and \delta_a=6 degree combined, 30 seconds');
legend('Aileron','Rudder','combined');
%
subplot(2,1,2);
t=0:.1:300;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,yr*3*pi/180,'-');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('r (rad/sec)');
title('yaw rate (r) response to \delta_r=+3 and \delta_a=6 degree combined, 5 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180+yr(end)*3*pi/180),...
(ya(end)*6*pi/180+yr(end)*3*pi/180)],'--r');
legend('Aileron','Rudder','combined','steady state');
```

Euler angle Φ response to combined ($\delta_a = 6^o, \delta_r = +3^o$)

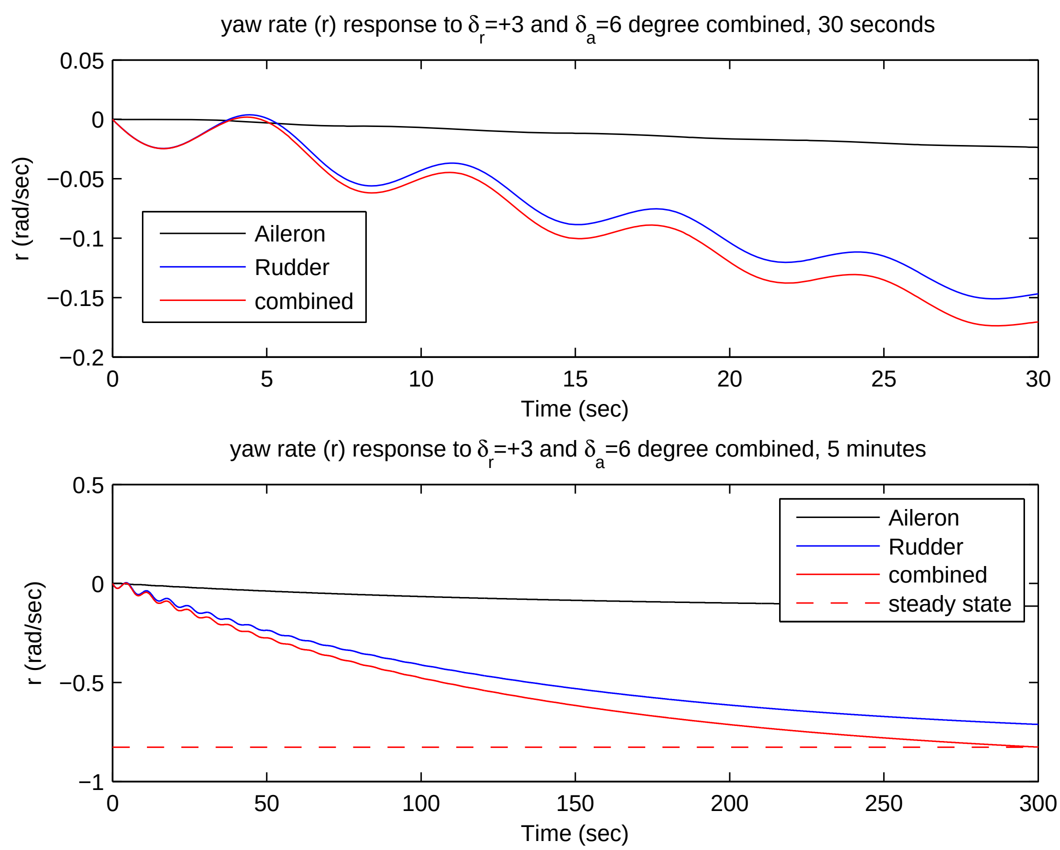

Figure 2.71: Yaw rate r response to $\delta_r = +3^o$ and $\delta_a = 6^o$ combined

```
close all; clear all;
set(0,'DefaultAxesFontSize',8)
s = tf('s');
num_a  = -0.1431*s^2-0.0273017*s-0.110172;
num_r = -0.1144*s^2-0.199712*s-1.36834;
den = s^4+0.6358*s^3+0.938762*s^2+0.511384*s+0.00368199;
sys1 = tf(num_a/den);
t=0:.1:180;
ya = step(sys1,t);
subplot(2,1,1);
plot(t,ya*6*pi/180,'-k');
hold on;
sys2 = tf(num_r/den);
yr = step(sys2,t);
plot(t,-yr*3*pi/180,'-');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Roll angle \Phi response to \delta_r=+3 and \delta_a=6 degree combined, 3 minutes');
legend('Aileron','Rudder','combined');
%
subplot(2,1,2);
t=0:.1:480;
ya = step(sys1,t);
plot(t,ya*6*pi/180,'-k');
hold on;
yr = step(sys2,t);
plot(t,yr*3*pi/180,'-');
plot(t,ya*6*pi/180+yr*3*pi/180,'r');
xlabel('Time (sec)');
ylabel('\Phi (rad)');
title('Roll angle \Phi response to \delta_r=-3 and \delta_a=6 degree combined, 7 minutes');
plot([t(1) t(end)],[(ya(end)*6*pi/180+yr(end)*3*pi/180),...
(ya(end)*6*pi/180+yr(end)*3*pi/180)],'--r');
legend('Aileron','Rudder','combined','steady state');
xlim([0 480]);
```

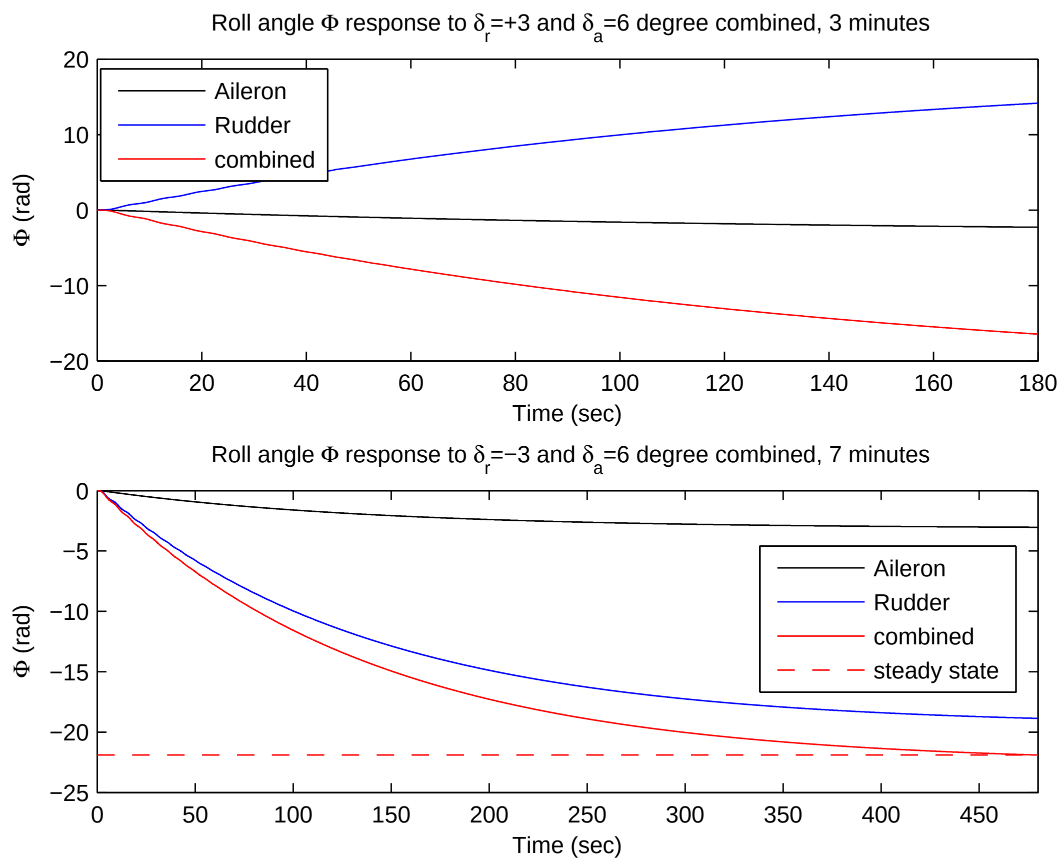

Figure 2.72: Euler angle Φ response to $\delta_r = +3^o$ and $\delta_a = 6^o$ combined

2.4.5 HW 4 key solution

Chapter 7

7.3 When the throttle is first opened, thrust will be greater than drag and the airspeed will increase. Thus you will progress up the drag polar towards P. When P is reached, thrust will equal drag and the airspeed will no longer change. P is a stable equilibrium point.

7.4 From (7.7,11b)

$$G_{\theta\delta_e} = \frac{b_1 s + b_o}{s(s^2 + c_1 s + c_o)} \tag{1}$$

(a) Expressing (1) in terms of $\lambda_{1,2} = n \pm i\omega$

$$G_{\theta\delta_e} = \frac{b_1 s + b_o}{s(s - \lambda_1)(s - \lambda_2)}$$

$$= \frac{b_1 s + b_o}{s[(s-n)^2 + \omega^2]} \tag{2}$$

For $\delta_e = \delta(t)$, it follows from Table A.1

$$\bar{\delta}_e = 1 \tag{3}$$

and

$$\bar{\theta} = G_{\theta\delta_e} \cdot \bar{\delta}_e$$

$$= G_{\theta\delta_e} \tag{4}$$

Expand $G_{\theta\delta_e}$ using partial fractions

$$\bar{\theta} = \frac{b_1}{\omega} \cdot \frac{\omega}{(s-n)^2 + \omega^2} + \frac{b_o}{s[(s-n)^2 + \omega^2]} \tag{5}$$

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Chapter 7

The second term in (5) can be expanded as follows:

$$\frac{b_0}{s[(s-n)^2+\omega^2]} = b_0\left[\frac{A}{s} + \frac{Bs+C}{(s-n)^2+\omega^2}\right] \tag{6}$$

The numerator of the part of (6) inside the square brackets is

$$As^2 - 2nAs + (n^2+\omega^2)A + Bs^2 + Cs \tag{7}$$

and equating this to 1 in order to satisfy (6) results in

$$A + B = 0 \tag{8}$$

$$C - 2nA = 0 \tag{9}$$

$$(n^2+\omega^2)A = 1 \tag{10}$$

From (10)

$$A = (n^2+\omega^2)^{-1} \tag{11}$$

From (8) and (11)

$$B = -(n^2+\omega^2)^{-1} \tag{12}$$

From (9) and (11)

$$C = 2n(n^2+\omega^2)^{-1} \tag{13}$$

From (5), (6), (11), (12) and (13)

$$\bar{\theta} = \frac{b_1}{\omega} \cdot \frac{\omega}{(s-n)^2 + \omega^2} + \frac{b_o}{(n^2+\omega^2)} \cdot \frac{1}{s}$$

$$+ b_o \left\{ \frac{B(s-n) + (Bn + C)}{(s-n)^2 + \omega^2} \right\}$$

$$= \frac{b_1}{\omega} \left\{ \frac{\omega}{(s-n)^2 + \omega^2} \right\} + \frac{b_o}{(n^2+\omega^2)} \cdot \frac{1}{s}$$

$$- \frac{b_o}{(n^2+\omega^2)} \left\{ \frac{(s-n)}{(s-n)^2 + \omega^2} \right\}$$

$$+ \frac{b_o \cdot n}{(n^2+\omega^2)\,\omega} \cdot \frac{\omega}{(s-n)^2 + \omega^2} \tag{14}$$

From (14) and Table A.1 (3, 13 and 14)

$$\theta(t) = e^{nt} \sin \omega t \left[\frac{b_1}{\omega} + \frac{b_o \cdot n}{(n^2+\omega^2)\,\omega} \right] + \frac{b_o}{(n^2+\omega^2)} \left[1 - e^{nt} \cos \omega t \right] \tag{15}$$

(b) b_o and b_1 can be found by using (15) at two different times t_1 and t_2 (since $\theta(t_1)$ and $\theta(t_2)$ are known) and then solving the two equations for the two unknowns b_o and b_1. c_o and c_1 can be found by using (1) and (2). By expanding and equating the two denominators it follows that

$$s^2 + c_1 s + c_o = s^2 - 2ns + (n^2 + \omega^2) \tag{16}$$

for all s. Thus equating the coefficients of the same powers in s

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$$c_1 = -2n \quad (17)$$

$$c_o = n^2 + \omega^2 \quad (18)$$

where n and ω are known.

7.5 From Table 4.5

$$Y_v = \frac{1}{2}\rho u_o S C_{y\beta} \quad (1)$$

$$L_v = \frac{1}{2}\rho u_o b S C_{\ell\beta}$$

$$N_v = \frac{1}{2}\rho u_o b S C_{n\beta}$$

$$Y_{\delta_r} = \frac{1}{2}\rho u_o^2 S C_{y\delta_r}$$

$$L_{\delta_a} = \frac{1}{2}\rho u_o^2 b S C_{\ell\delta_a}$$

$$L_{\delta_r} = \frac{1}{2}\rho u_o^2 b S C_{\ell\delta_r}$$

$$N_{\delta_a} = \frac{1}{2}\rho u_o^2 b S C_{n\delta_a}$$

$$N_{\delta_r} = \frac{1}{2}\rho u_o^2 b S C_{n\delta_r}$$

Thus the condition that must be satisfied is

$$\det A \neq 0 \tag{6}$$

7.10 (a) From (6.7,2)

$$f(s) = s^4 + .6358s^3 + .9388s^2 + .5114s + .003682 \tag{1}$$

In (1) replace s by $i\omega$ with $\omega = 0$, thus

$$f(0) = .003682 \tag{2}$$

In (7.9,5) the static gain case is also given by $N_{ij}(0)$ which is the constant term in the expression. Thus the static gains $G_{ij}(0) = N_{ij}(0)/f(0)$ are given by

$$G_{v\delta_a}(0) = .6220/.003682 = 168.9 \tag{3}$$

$$G_{v\delta_r}(0) = -5.934/.003682 = -1612$$

$$G_{p\delta_a}(0) = 0$$

$$G_{p\delta_r}(0) = 0$$

$$G_{r\delta_a}(0) = .004539/.003682 = 1.233$$

$$G_{r\delta_r}(0) = -.05647/.003682 = -15.34$$

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$$G_{\phi\delta_a}(0) = .1102/.003682 = 29.93$$

$$G_{\phi\delta_r}(0) = -1.368/.003682 = -371.5$$

(b) As $\omega \to \infty$, $G_{ij}(i\omega)$ can be simplified by keeping only the dominant high order terms in each of $N_{ij}(i\omega)$ and $f(i\omega)$. From (1)

$$\lim_{\omega \to \infty} f(i\omega) = \omega^4 \qquad (4)$$

From (7.9,5) as $\omega \to \infty$ in $N_{ij}(i\omega)$ only the first term need be kept. Thus

$$\lim_{\omega \to \infty} |N_{ij}(i\omega)| = a_{ij}\, \omega^n \qquad (5)$$

where n is the largest index of ω in $N_{ij}(i\omega)$. Thus

$$\lim_{\omega \to \infty} |G_{ij}(i\omega)| = a_{ij}\, \omega^{(n-4)} \qquad (6)$$

and the slope of (6) in decades per decade can be determined by taking the $\log_{10}$ of the right-hand side

$$\log_{10} a_{ij} + (n - 4)\log_{10}\omega \qquad (7)$$

The desired slope is given by (n - 4) decades/decade. From (7.9,5)

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Transfer Function	Slope (decades/decade)
$G_{v\delta_a}$	-2
$G_{v\delta_r}$	-1
$G_{p\delta_a}$	-1
$G_{p\delta_r}$	-1
$G_{r\delta_a}$	-1
$G_{r\delta_r}$	-1
$G_{\phi\delta_a}$	-2
$G_{\phi\delta_r}$	-2

(c) Consider the case with δ_a deflection only. From part (a)

$$G_{\phi\delta_a}(0) \doteq 29.93$$

$$= \phi/\delta_a \qquad (8)$$

Thus for $\phi = 15°$

$$\delta_a = 15/29.93$$
$$= 0.501° \qquad (9)$$

To determine β just find v from (a)

$$G_{v\delta_a}(0) = 168.9$$
$$= v/\delta_a \qquad (10)$$

Hence

$$v = 168.9\ \delta_a$$

(with δ_a in rad)

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But $\beta \approx v/u_o$ (u_o = 774 fps from Sec. 6.2) and thus for $\delta_a = 0.501°$

$$\beta = \frac{168.9}{774}\left(\frac{0.501}{57.3}\right)57.3$$

$$= 0.109° \qquad (11)$$

The yaw rate can be found from

$$G_{r\delta_a}(0) = 1.233$$

$$= r/\delta_a \qquad (12)$$

Hence for $\delta_a = 0.501°$

$$r = 1.233 \times \left(\frac{0.501}{57.3}\right)57.3$$

$$= 0.618 \text{ deg/s} \qquad (13)$$

Consider the case with δ_r deflection only. From part (a)

$$G_{\phi\delta_r}(0) = -371.5$$

$$= \phi/\delta_r \qquad (14)$$

Thus for $\phi = 15°$

$$\delta_r = -15/371.5$$

$$= -0.0404° \qquad (15)$$

From (a)

$$G_{v\delta_r}(0) = -1612$$

$$= v/\delta_r \qquad (16)$$

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Hence

$$v = -1612\delta_r$$

and for $\delta_r = -0.0404°$

$$\beta = \frac{1612}{774}\left(\frac{0.0404}{57.3}\right)57.3$$

$$= 0.0841° \quad (17)$$

From (a)

$$G_{r\delta_r}(0) = -15.34$$

$$= r/\delta_r \quad (18)$$

Hence for $\delta_r = -0.0404°$

$$r = 15.34 \times \left(\frac{0.0404}{57.3}\right)57.3$$

$$= 0.620 \text{ deg/s} \quad (19)$$

7.11 (a) From Fig. 7.18a it is found that at the frequency of the short-period mode

$$|G_{n_z\delta_e}| = 13.5 \quad (1)$$

Thus for a sinusoidal δ_e of amplitude $|\delta_e|$ at that frequency, the amplitude $|\Delta n_z|$ of the response Δn_z (about $n_{z_0} = 1$) is (for $|\delta_e|$ in rad)

$$|\Delta n_z| = 13.5 \times |\delta_e|$$

$$= 13.5 \times \frac{2}{57.3}$$

$$= 0.471 \quad (2)$$

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and $$n_z = 1 + 0.471 \sin(\omega_{sp} t) \tag{3}$$

(b) Since n_z is measured at the CG and must reach a value of $n_z = 0$ for a passenger at the CG to be lifted from his seat, it follows that Δn_z must reach a value of -1. This is achieved by (see (2))

$$|\delta_c| = (1/13.5) \times 57.3$$
$$= 4.24^\circ \tag{4}$$

(c) For n_z to reach a peak value of 2.5 would require Δn_z to reach a peak of 1.5. From (2)

$$|\delta_c| = (1.5/13.5) \times 57.3$$
$$= 6.37^\circ \tag{5}$$

PROB. 4

WE'RE LOOKING FOR $\underline{G} = \begin{Bmatrix} G_{u\delta_p} \\ G_{w\delta_p} \\ G_{q\delta_p} \\ G_{\theta\delta_p} \end{Bmatrix} = \begin{Bmatrix} \frac{N_{u\delta_p}(s)}{D(s)} \\ \frac{N_{w\delta_p}(s)}{D(s)} \\ \frac{N_{q\delta_p}(s)}{D(s)} \\ \frac{N_{\theta\delta_p}(s)}{D(s)} \end{Bmatrix}$

RECALL $\underline{\underline{G}} = \left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1} \underline{\underline{B}}$

w/ $\underline{\underline{B}} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix}$

HERE ONLY USE $\underline{B} = \begin{Bmatrix} B_{12} \\ B_{22} \\ B_{32} \\ B_{42} \end{Bmatrix} = \begin{Bmatrix} 9.66 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

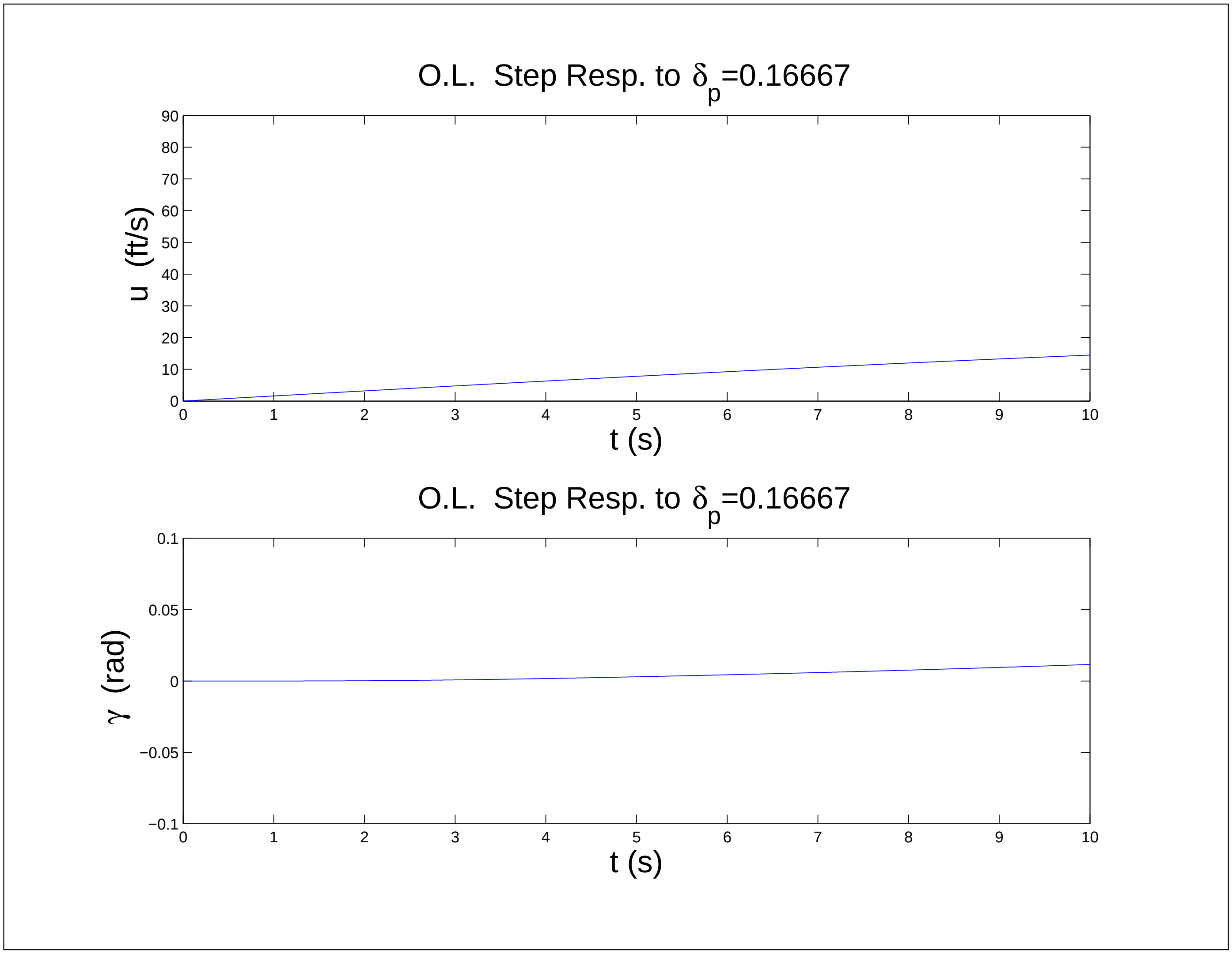
IN HANDOUTS, WE HAVE $\underline{G} = \begin{Bmatrix} G_{u\delta_e} \\ G_{w\delta_e} \\ G_{q\delta_e} \\ G_{\theta\delta_e} \end{Bmatrix} = \begin{Bmatrix} \frac{N_{u\delta_e}(s)}{D(s)} \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix}$

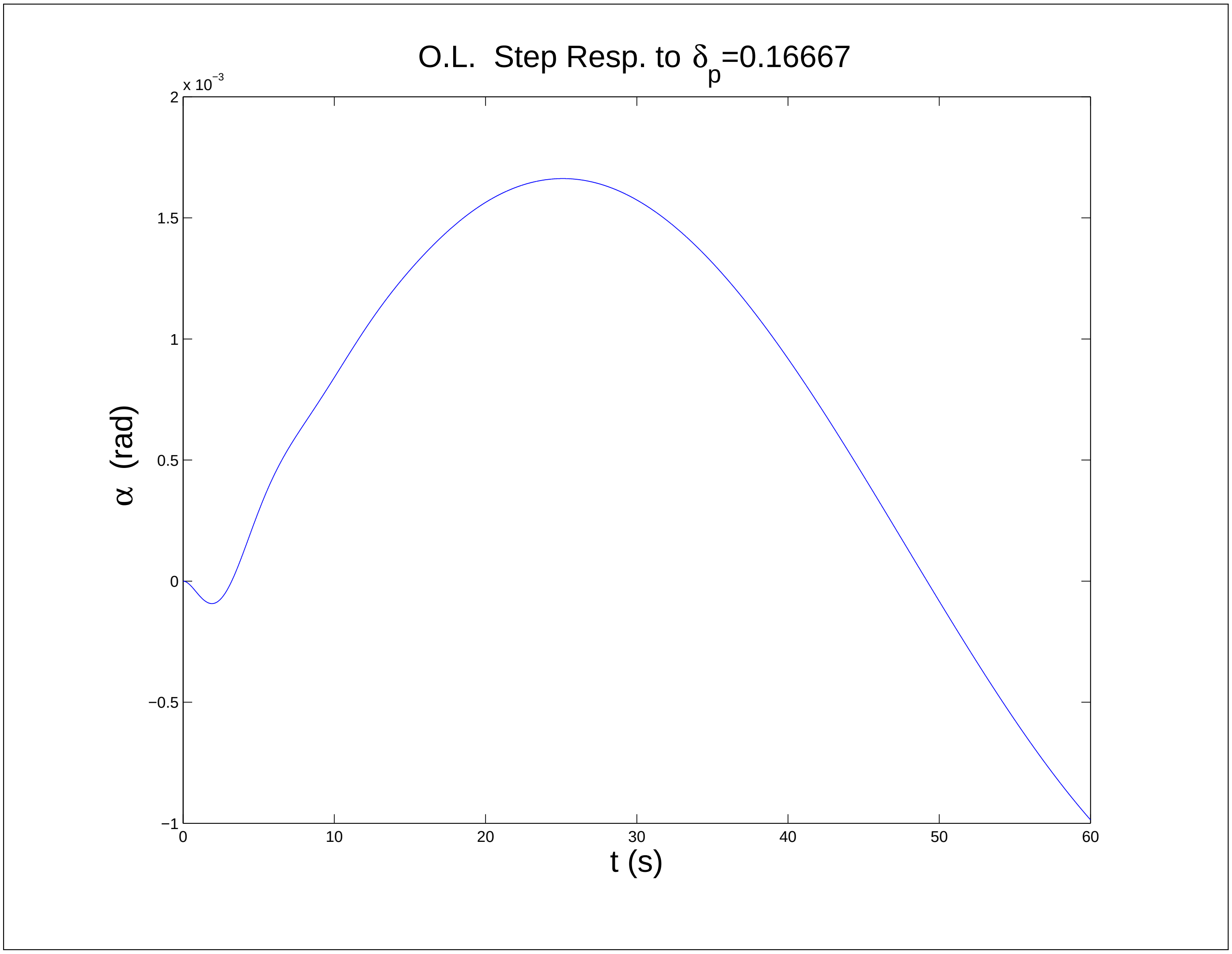
AND AT THE 2nd STEP OF THE EVALUATION WE READ

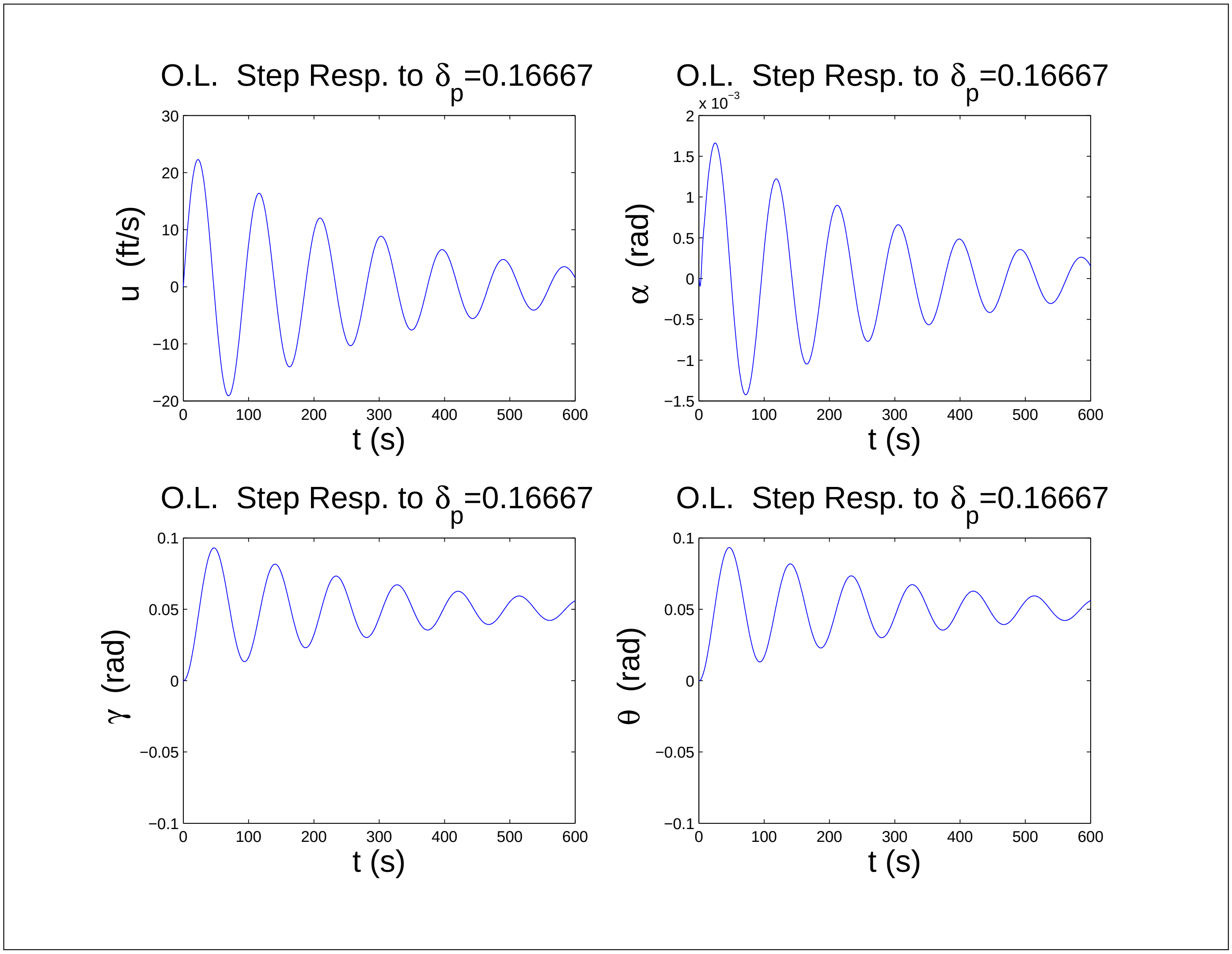
(SEE p. 78 OF THE HANDOUT ON COURSE WEBSITE)

$$N_{u\delta e}(s) = -0.000187\, s\left(s^2 + 0.743 s + 0.932\right) +$$
$$-17.8\left(0.0140 s^2 + 0.00599 s + 0.0332\right) +$$
$$+1.16\left(21.4 s + 10.1\right)$$

HERE, SINCE THE ONLY ELEMENT OF $\underline{B} \neq 0$ IS THE FIRST ONE, ALL WE HAVE TO DO IS USE THE FIRST TERM OF EACH OF THE $N_{u\delta e}$, $N_{w\delta e}$, $N_{q\delta e}$, $N_{\theta\delta e}$ AND REPLACE THE FACTOR -0.000187 W/ 9.66


O.L. Step Resp. to δp=0.16667
u (ft/s)
t (s)
O.L. Step Resp. to δp=0.16667
γ (rad)
t (s)


O.L. Step Resp. to δ_p=0.16667
x 10^-3
2
1.5
1
0.5
0
−0.5
−1
α (rad)
0
10
20
30
40
50
60
t (s)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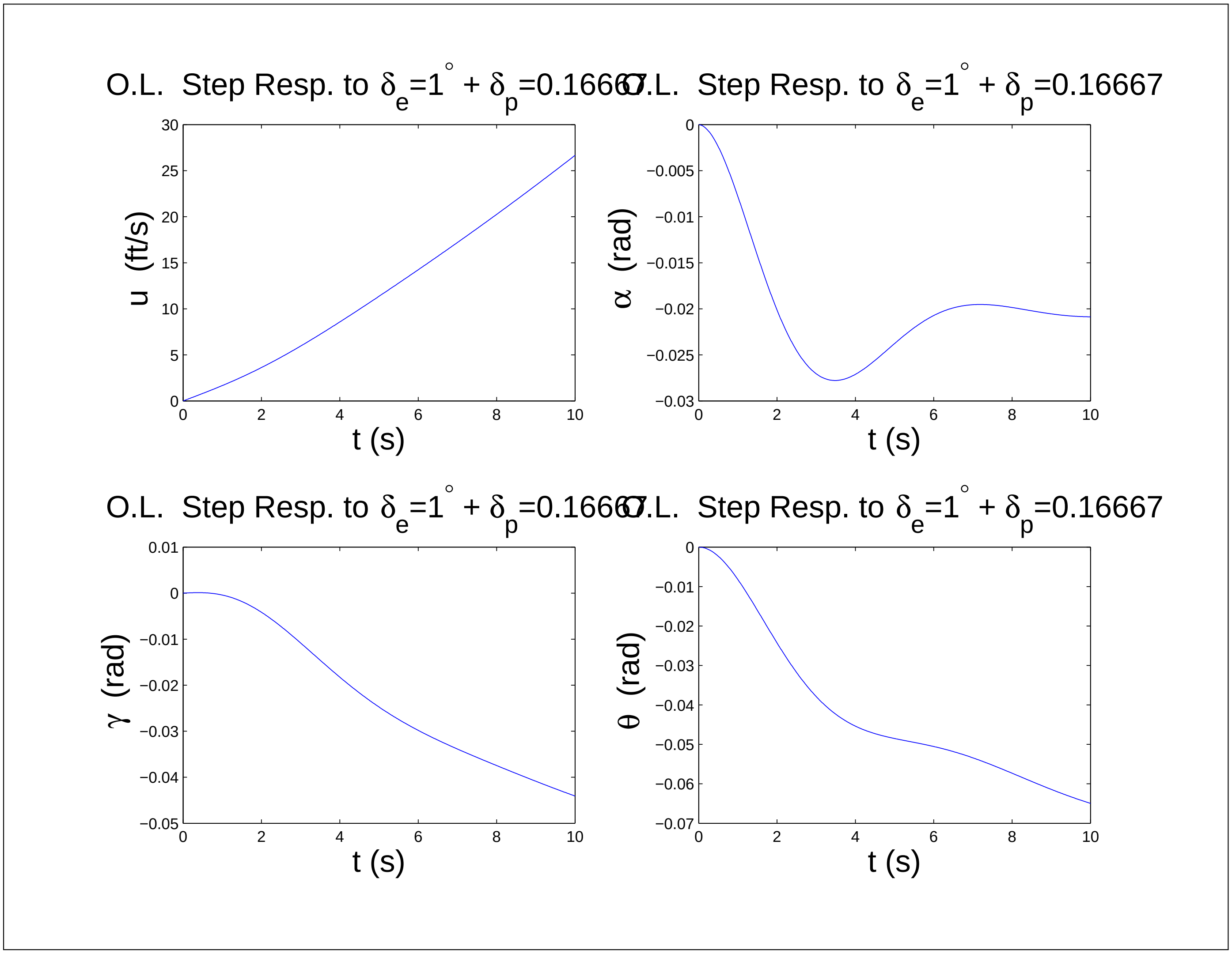
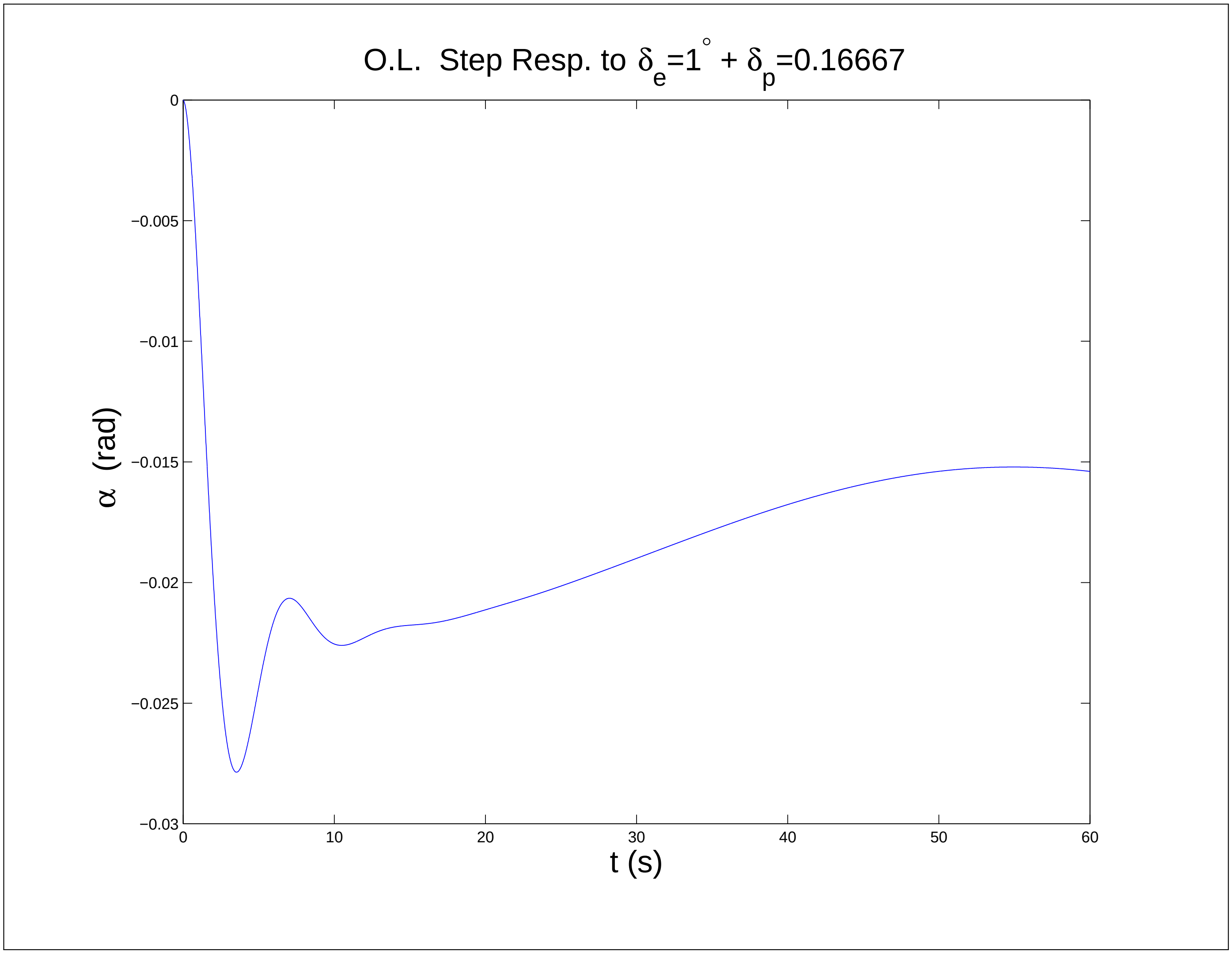
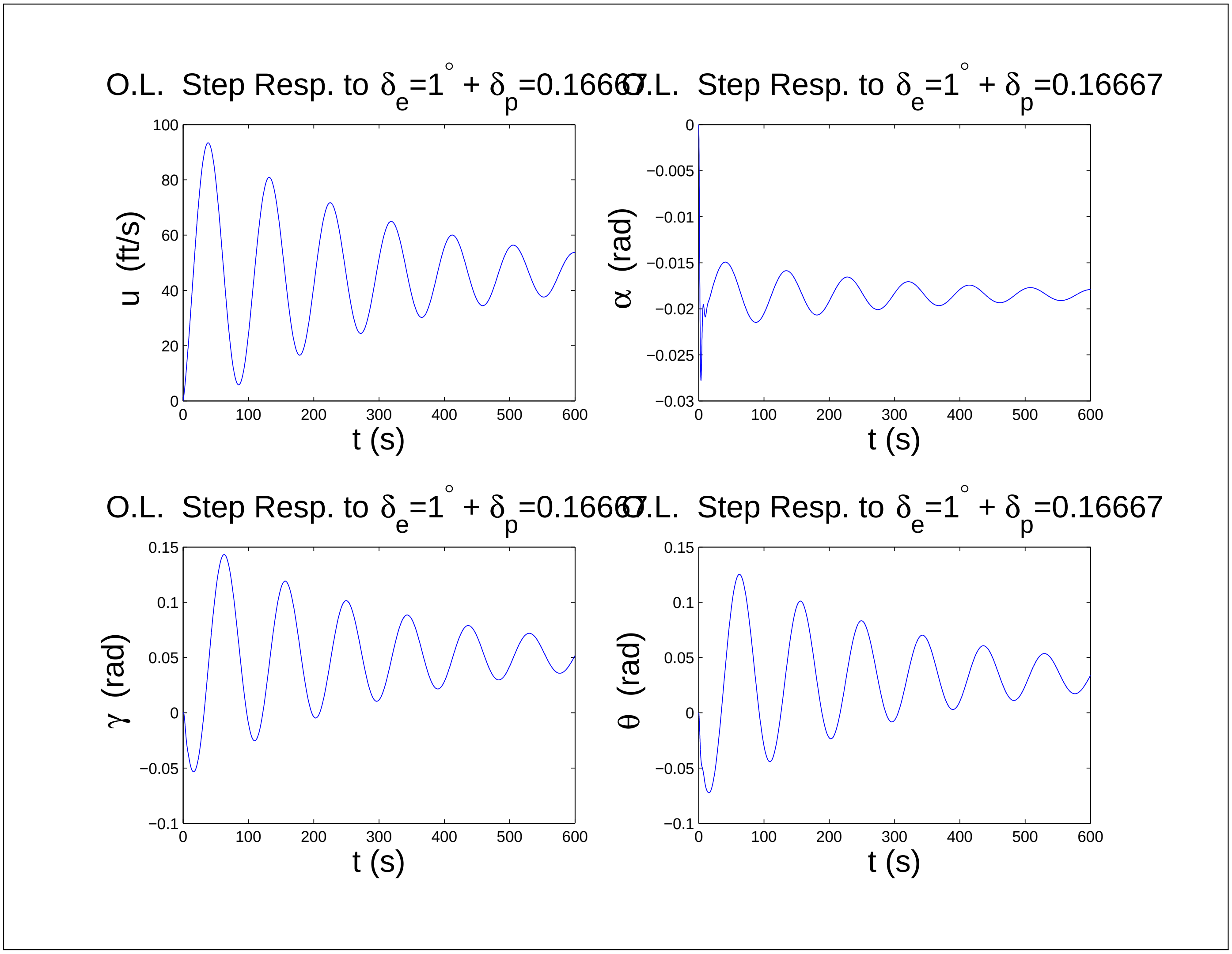
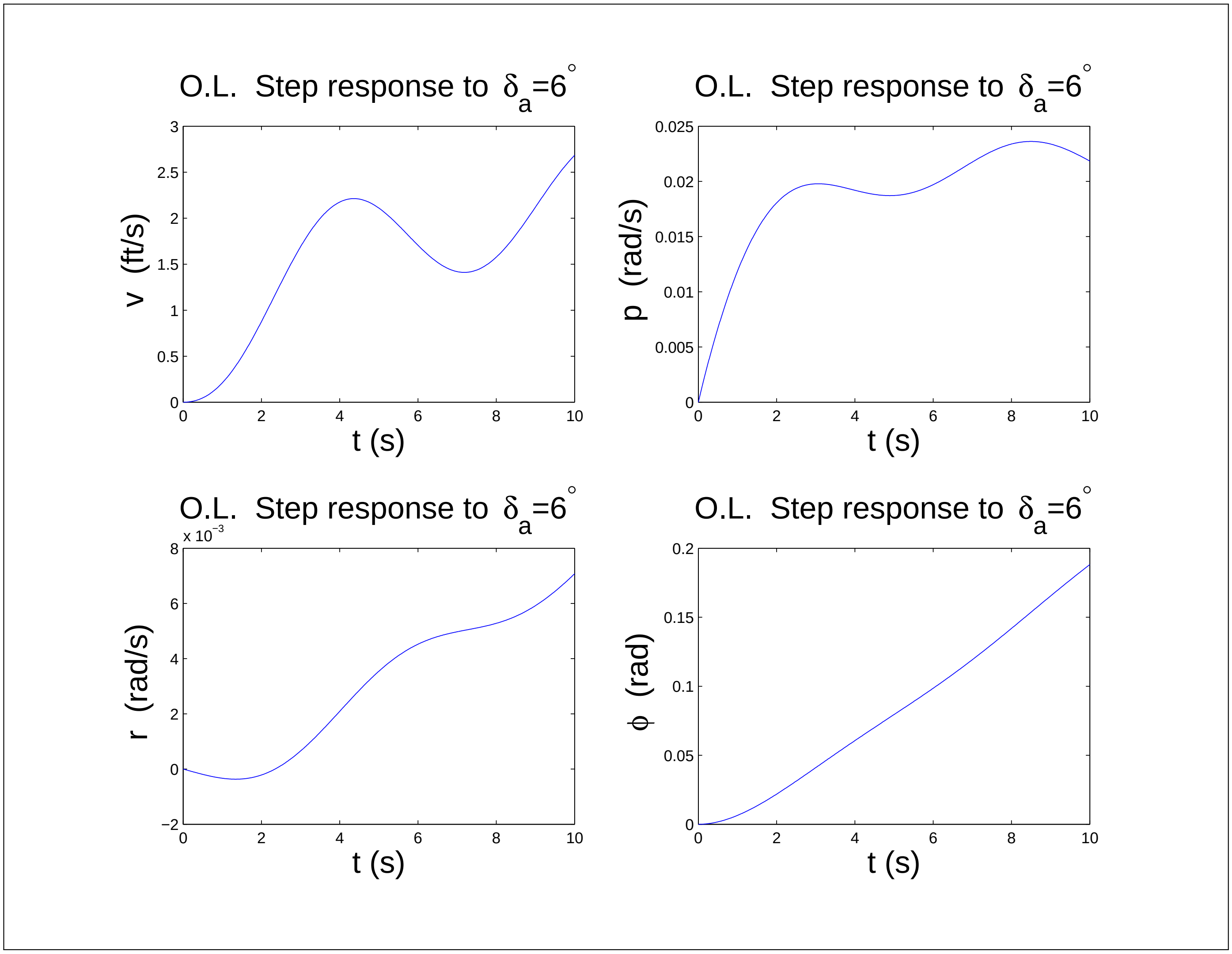
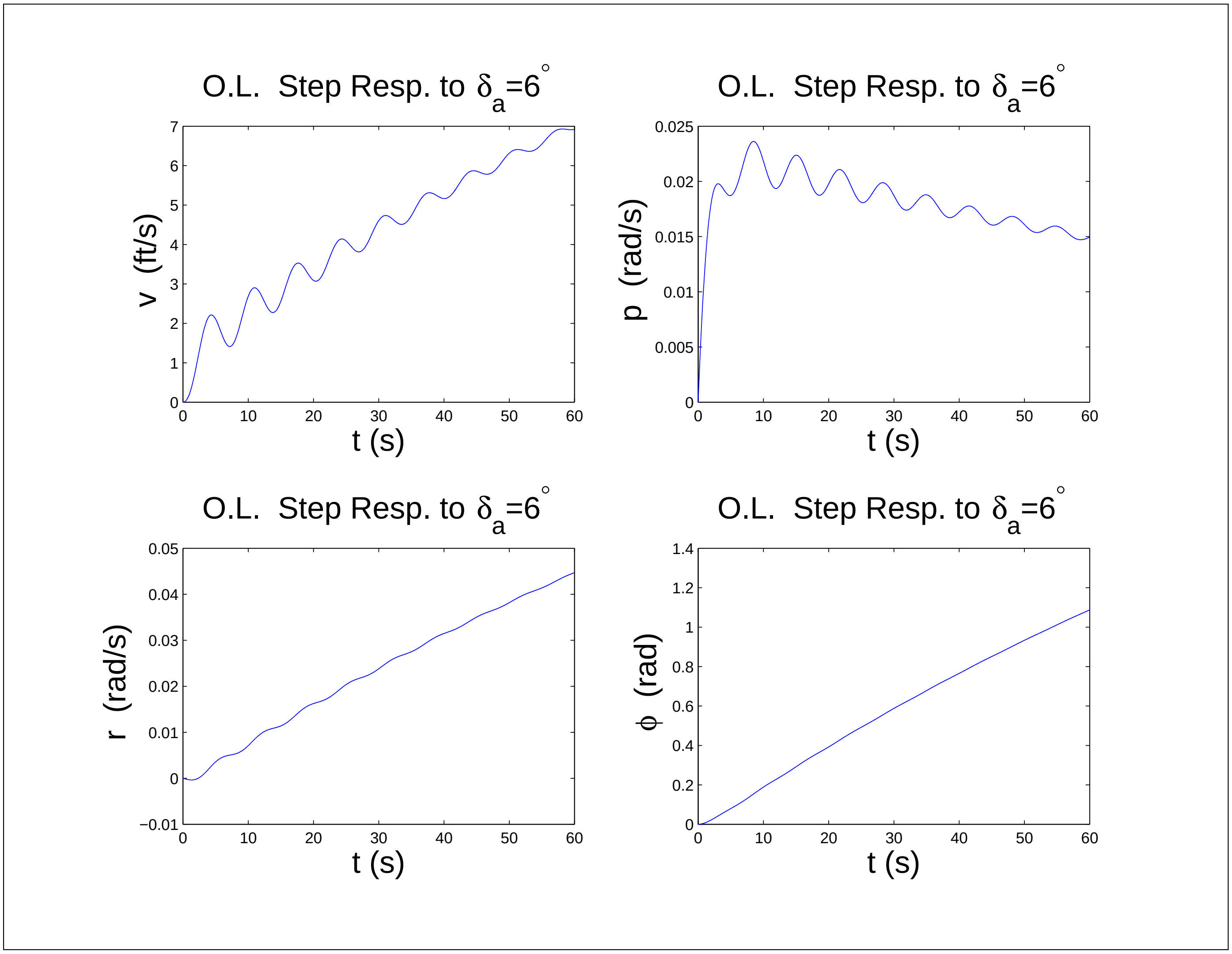
u (ft/s)
t (s)
O.L. Step Resp. to δp=0.16667
x 10−3
α (rad)
t (s)
O.L. Step Resp. to δp=0.16667
γ (rad)
t (s)
O.L. Step Resp. to δp=0.16667
θ (rad)
t (s)


O.L. Step Resp. to δe=1° + δp=0.16667
u (ft/s)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
α (rad)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
γ (rad)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
θ (rad)
t (s)


O.L. Step Resp. to δe=1° + δp=0.16667
α (rad)
t (s)
0
−0.005
−0.01
−0.015
−0.02
−0.025
−0.03
0
10
20
30
40
50
60


O.L. Step Resp. to δe=1° + δp=0.16667
u (ft/s)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
α (rad)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
γ (rad)
t (s)
O.L. Step Resp. to δe=1° + δp=0.16667
θ (rad)
t (s)


O.L. Step response to δa=6°
v (ft/s)
t (s)
O.L. Step response to δa=6°
p (rad/s)
t (s)
O.L. Step response to δa=6°
x 10−3
r (rad/s)
t (s)
O.L. Step response to δa=6°
ϕ (rad)
t (s)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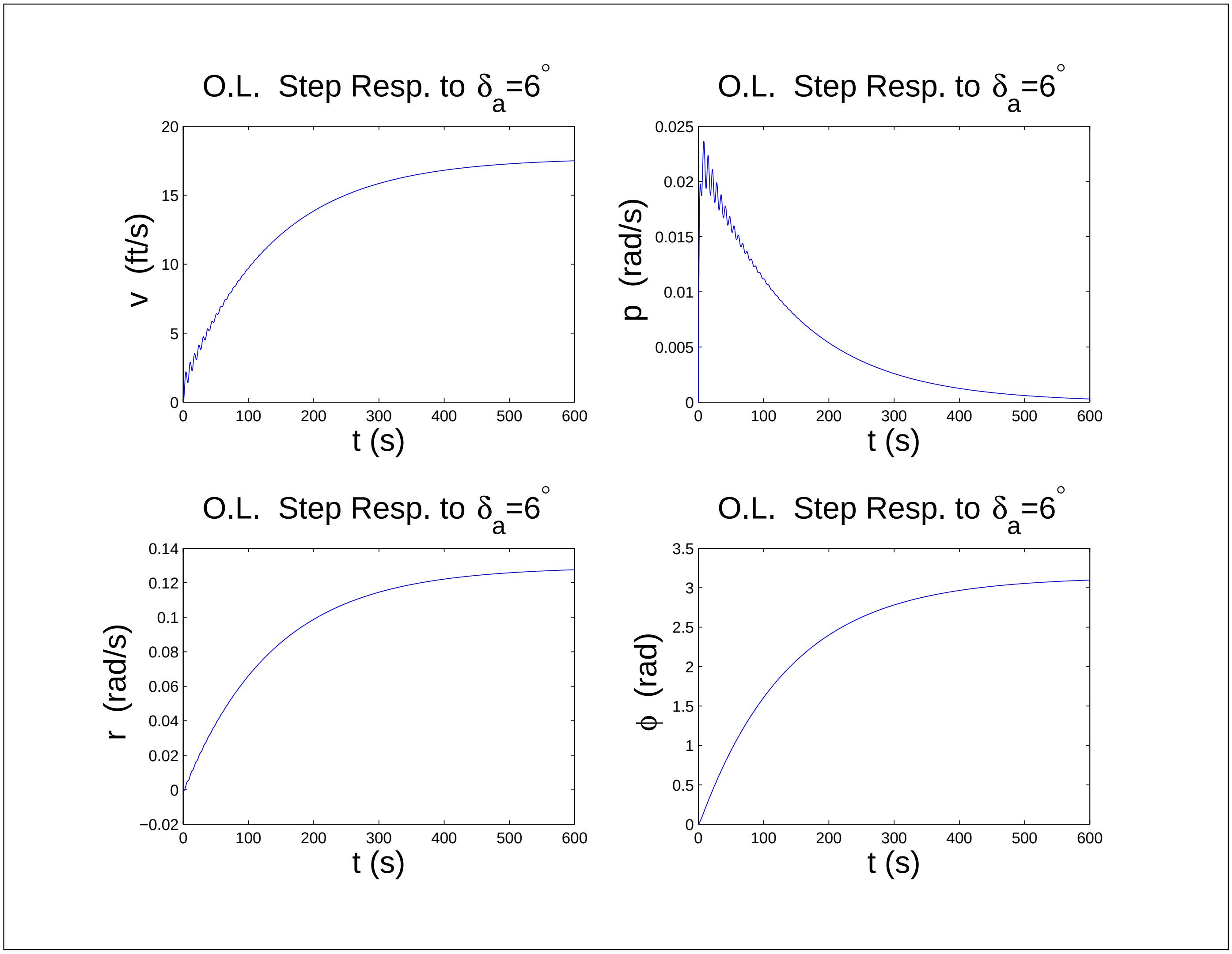
v (ft/s)
t (s)
O.L. Step Resp. to δa=6°
p (rad/s)
t (s)
O.L. Step Resp. to δa=6°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6°
ϕ (rad)
t (s)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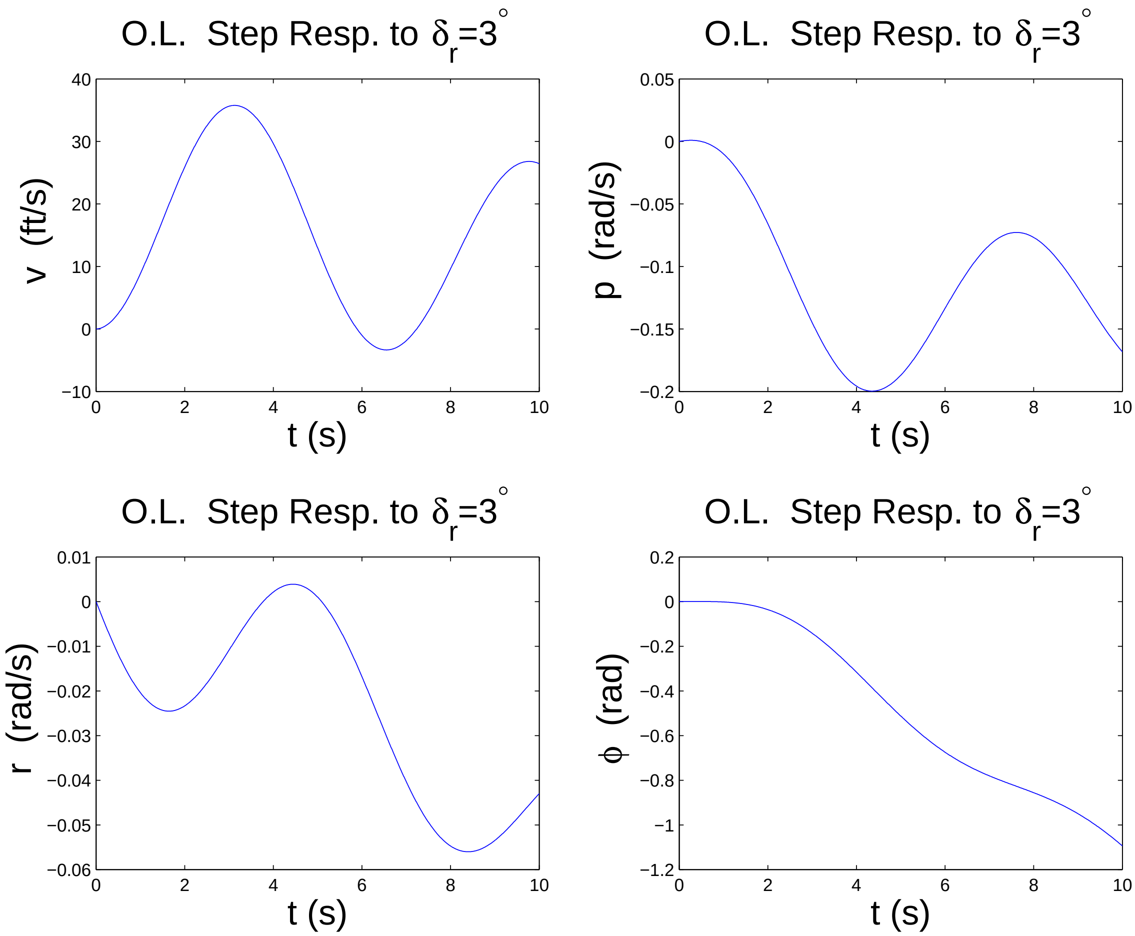
v (ft/s)
t (s)
O.L. Step Resp. to δa=6°
p (rad/s)
t (s)
O.L. Step Resp. to δa=6°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6°
φ (rad)
t (s)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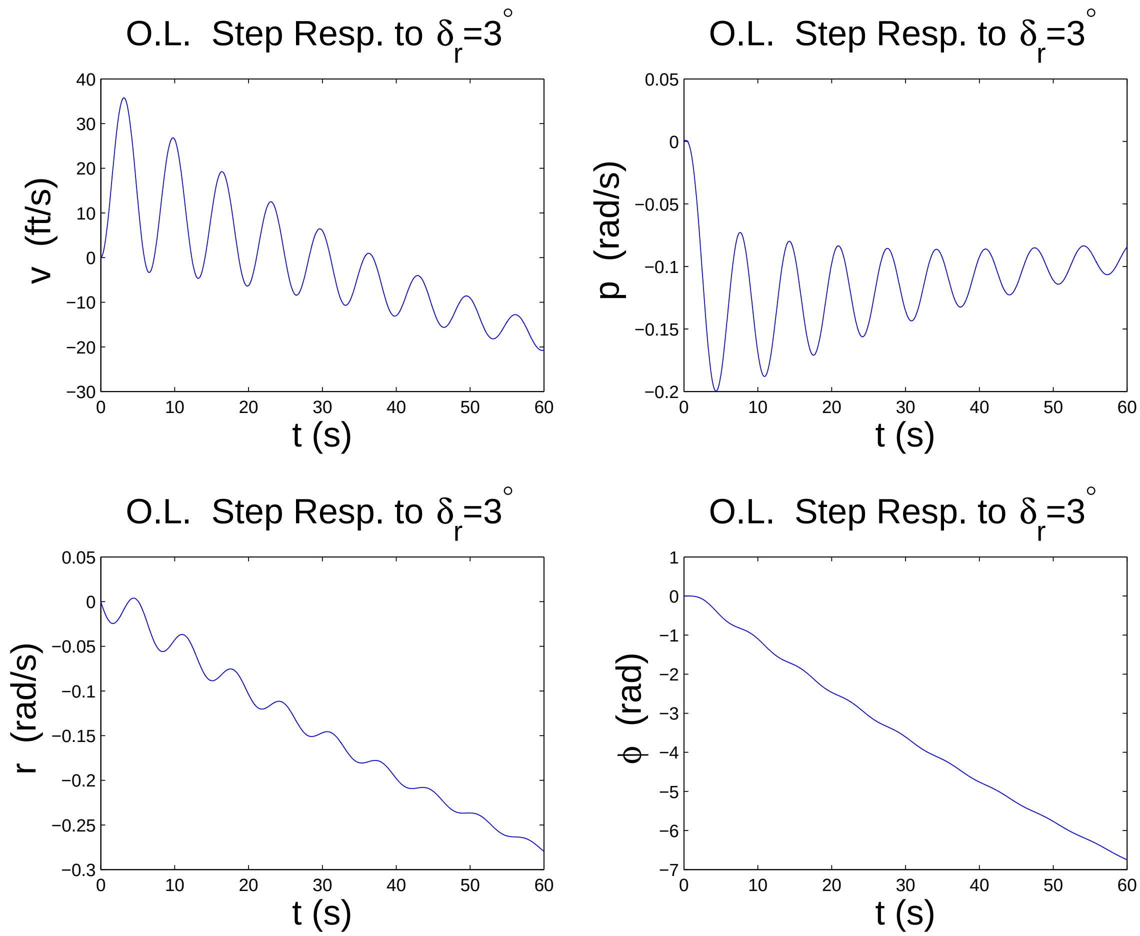
v (ft/s)
t (s)
O.L. Step Resp. to δr=3°
p (rad/s)
t (s)
O.L. Step Resp. to δr=3°
r (rad/s)
t (s)
O.L. Step Resp. to δr=3°
φ (rad)
t (s)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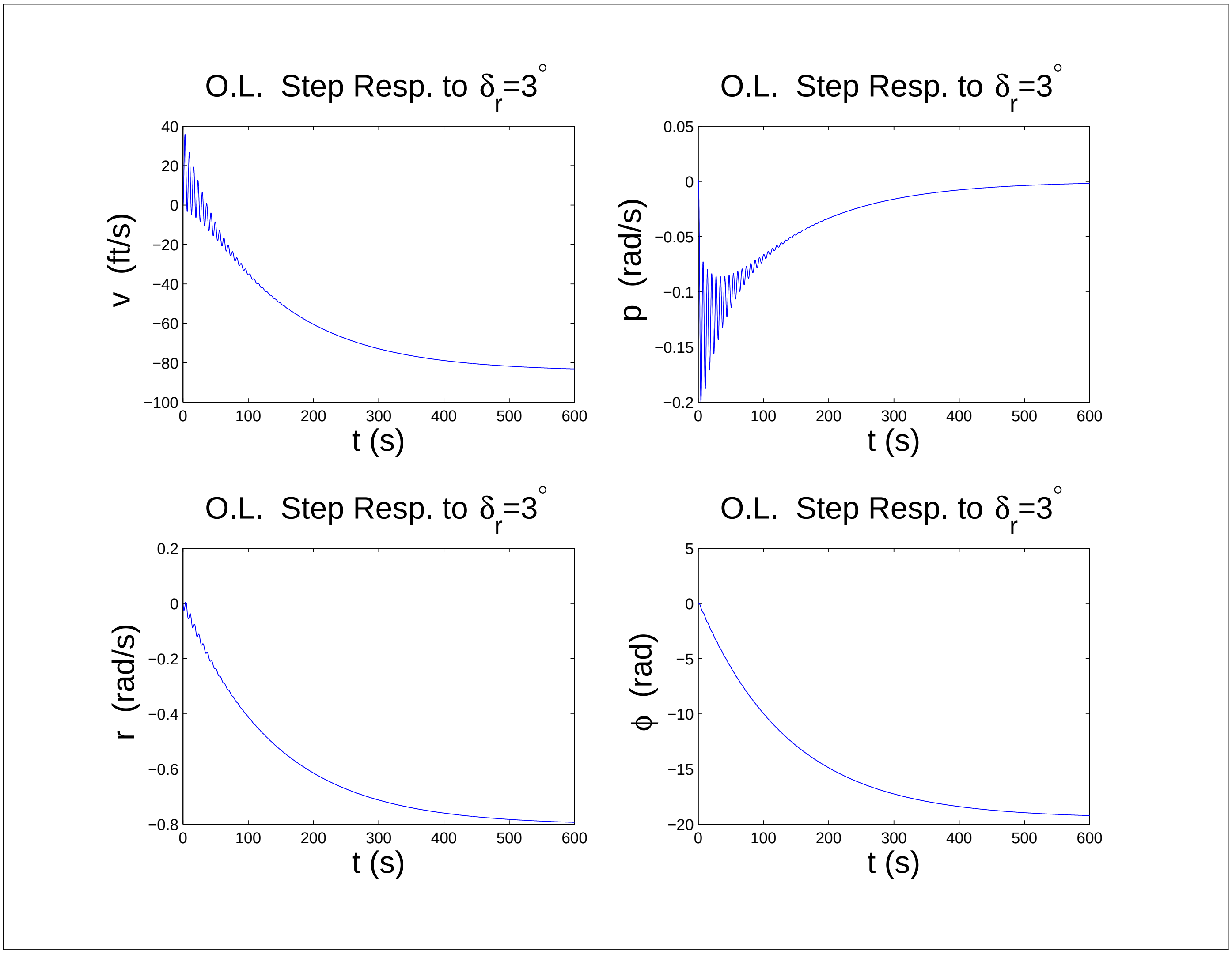
v (ft/s)
t (s)
O.L. Step Resp. to δr=3°
p (rad/s)
t (s)
O.L. Step Resp. to δr=3°
r (rad/s)
t (s)
O.L. Step Resp. to δr=3°
φ (rad)
t (s)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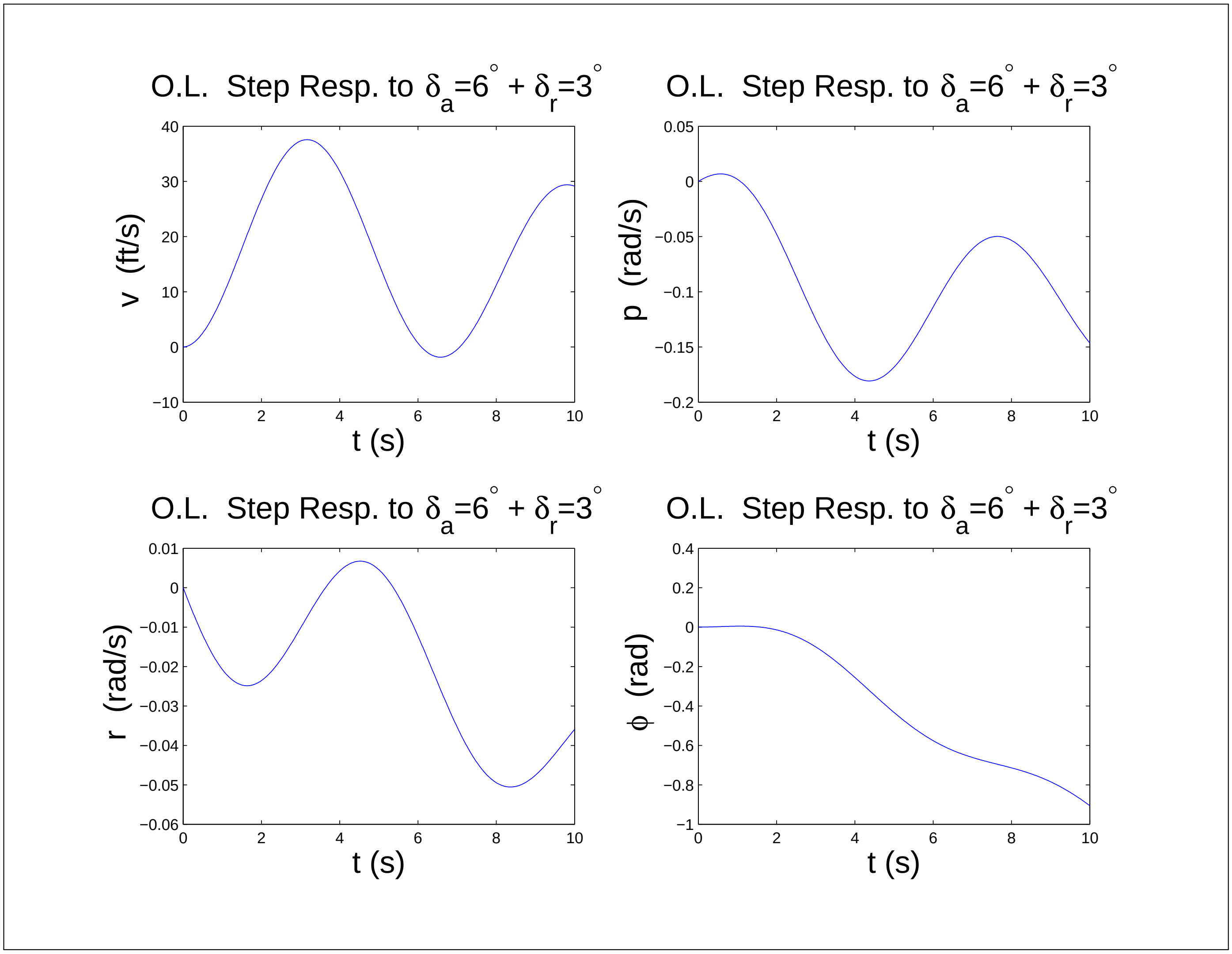
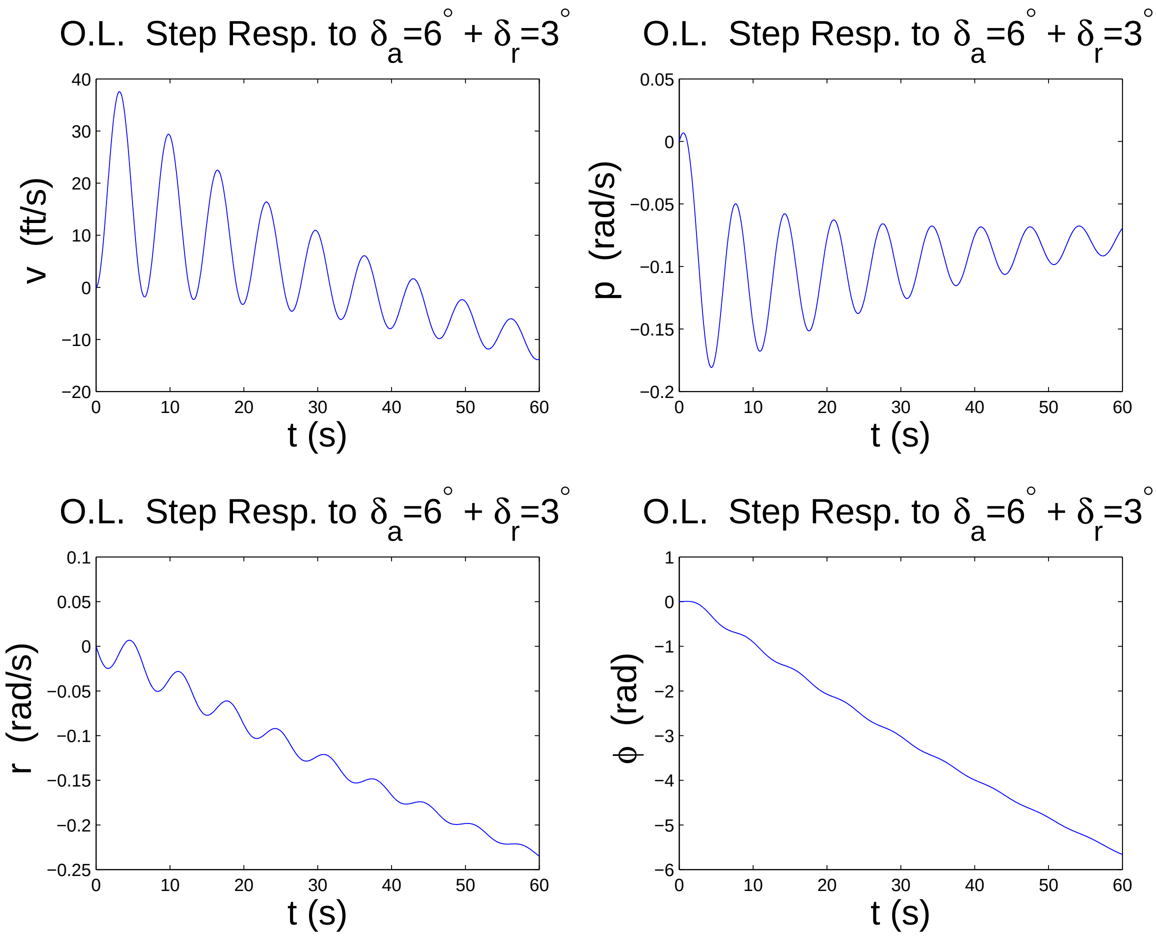
v (ft/s)
t (s)
O.L. Step Resp. to δ_r=3°
p (rad/s)
t (s)
O.L. Step Resp. to δ_r=3°
r (rad/s)
t (s)
O.L. Step Resp. to δ_r=3°
ϕ (rad)
t (s)


O.L. Step Resp. to δa=6° + δr=3°
v (ft/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
φ (rad)
t (s)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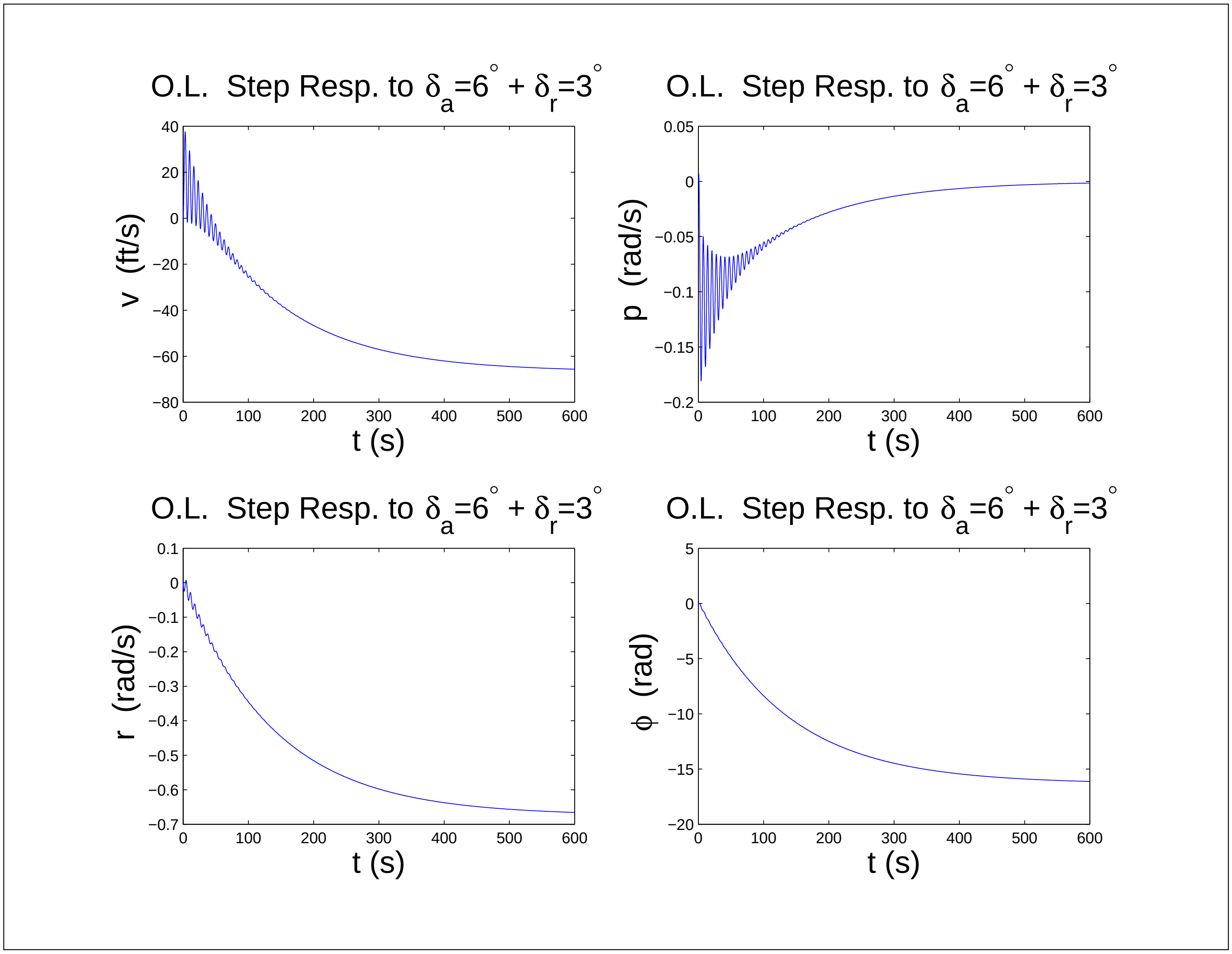
t (s)
O.L. Step Resp. to δa=6° + δr=3°
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
ϕ (rad)
t (s)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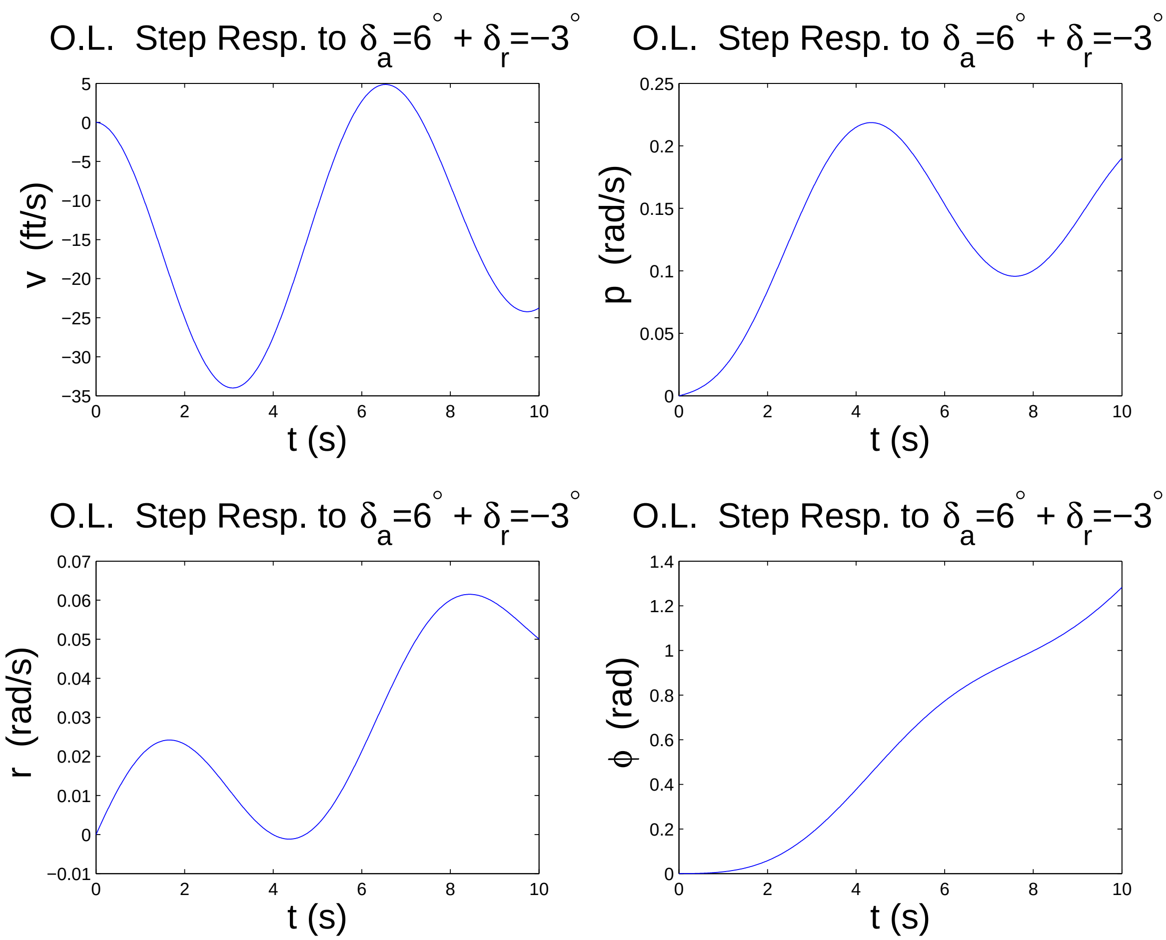
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=3°
φ (rad)
t (s)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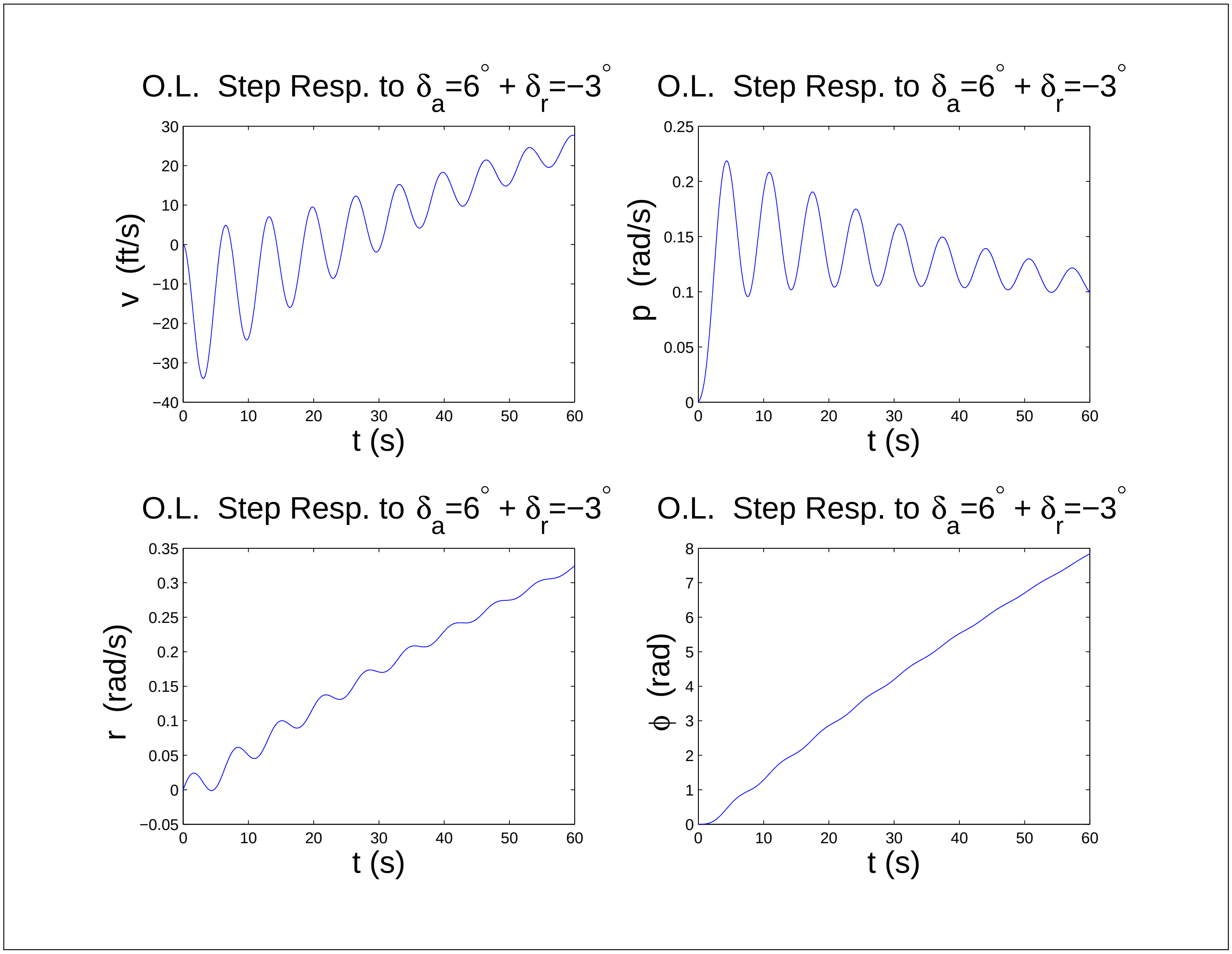
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
φ (rad)
t (s)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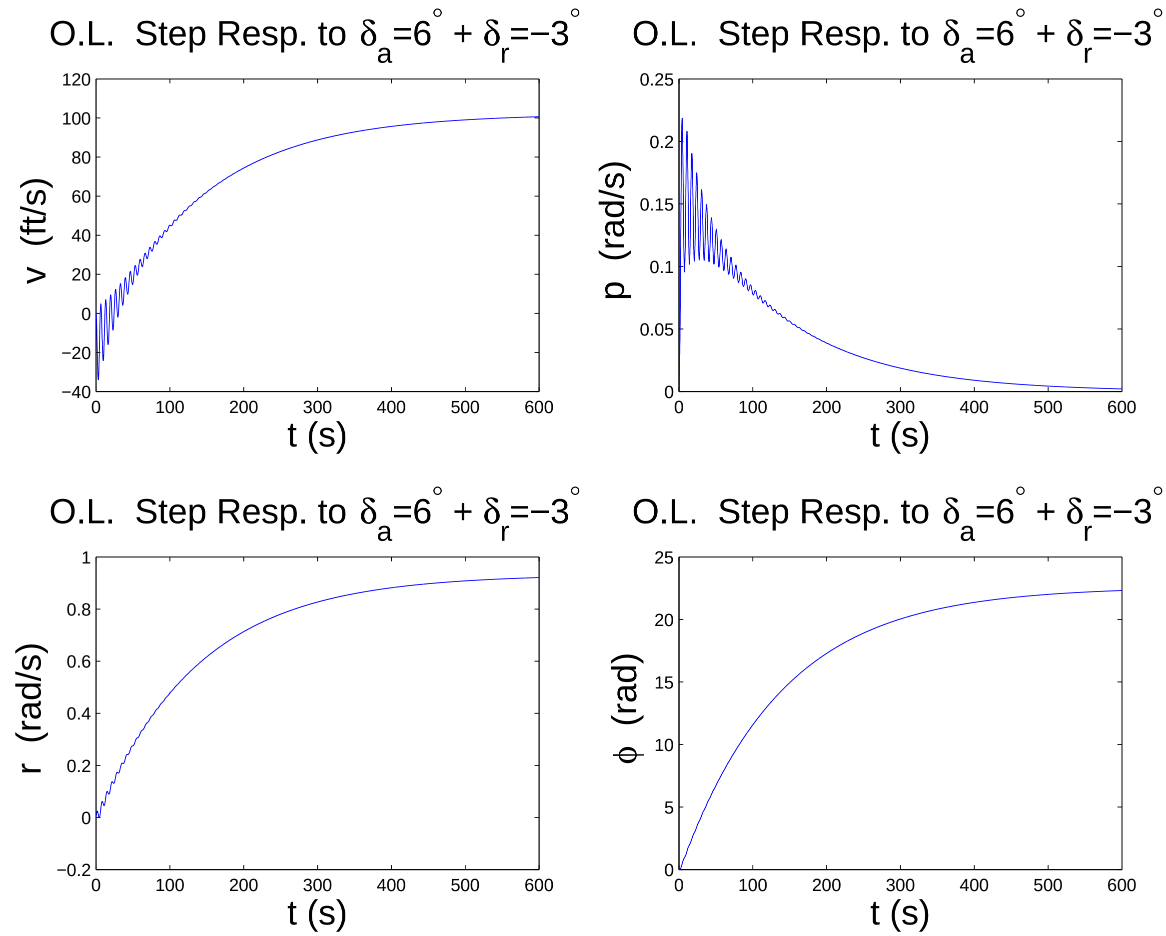
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
ϕ (rad)
t (s)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t (s)
O.L. Step Resp. to δa=6° + δr=−3°
p (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
r (rad/s)
t (s)
O.L. Step Resp. to δa=6° + δr=−3°
φ (rad)
t (s)

2.5 HW5

2.5.1 Problem 1

1. Problem 8.2. Hints: in part a), use the open-loop transfer function you used in Chap.7 (it is repeated in Eq.(8.3) or you can form it using Eqs.(6.2,2) and (7.7,2)).
In part b), start with the schematic of Fig.8.5. Write an expression for $\overset{\circ}{G}_{\theta\theta_c} \equiv \frac{\theta}{\theta_c}$. Develop an expression for the error, $e(s)$, in terms of $\overset{\circ}{G}_{\theta\theta_c}(s)$ and $\theta_c(s)$ and evaluate $e_{ss} \equiv \lim_{t\to\infty} e(t)$.

8.2 (a) What is the steady state θ that results from a steady $\Delta\delta_e = 5°$ for the jet transport of Sec. 8.3?

(b) For the closed-loop response to a unit step input in Sec. 8.3, with $J = k_2$, derive an expression for the steady-state error e_{ss} as a function of k_2. (Hint: start with (8.3,1)).

(c) Calculate the value of k_2 needed to keep $e_{ss} < 0.1°$ for $\theta_c = 5°$.

(d) For the value of k_2 found in (c) what is the elevator angle at $t = 0^+$ when θ_c is a step input of 5°? Comment on the practicality of using k_2 alone to reduce e_{ss}.

Solution

2.5.1.1 Part (a)

To find the steady state $\theta(t)$, the final value theorem will be used

$$\lim_{t\to\infty}\theta(t) = \lim_{s\to0} s\theta(s) \tag{1}$$

But

$$\theta(s) = \Delta\delta_e(s)\, G_{\theta,\delta_e}(s) \tag{2}$$

Substituting (2) in (1) gives

$$\lim_{t\to\infty}\theta(t) = \lim_{s\to0} s\Delta\delta_e(s)\, G_{\theta,\delta_e}(s) \tag{3}$$

Using the hint given, the open loop transfer function $G_{\theta,\delta_e}$is used, which is given in (8.3,3) on page 267 in the text as

$$\boxed{G_{\theta,\delta_e}(s) = \frac{-\left(1.158s^2 + 0.3545s + 0.003873\right)}{s^4 + 0.750468s^3 + 0.935494s^2 + 9.453025\times10^{-3}s + 4.195875\times10^{-3}}} \tag{(8.3,3)}$$

Since $\Delta\delta_e = 5^o$, then $\mathscr{L}(\Delta\delta_e) = \frac{1}{s}\Delta\delta_e$ and (3) becomes

$$\begin{aligned}\lim_{t\to\infty}\theta(t) &= \lim_{s\to0} s\left(\frac{1}{s}\Delta\delta_e\right)\frac{-\left(1.158s^2 + 0.3545s + 0.003873\right)}{s^4 + 0.750468s^3 + 0.935494s^2 + 9.453025\times10^{-3}s + 4.195875\times10^{-3}} \qquad (4)\\ &= \Delta\delta_e\frac{-(0.003873)}{4.195875\times10^{-3}}\\ &= 5\left(\frac{-0.003873}{4.195875\times10^{-3}}\right)\\ &= \boxed{-4.6152^o}\end{aligned}$$

2.5.1.2 Part(b)

Starting from (8.3,1) on page 266 of the textbook

$$\begin{aligned}\theta(s) &= \theta_c(s)\,\mathring{G}_{\theta,\delta_e} \\ &= \theta_c(s)\,\frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}\end{aligned} \tag{5}$$

Since the error by definition is given by

$$e(s) = \theta_c(s) - \theta(s) \tag{6}$$

Then using (6) and (5) results in

$$\begin{aligned}e(s) &= \theta_c(s) - \theta_c(s)\,\frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}} \\ &= \theta_c(s)\left(1-\frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}\right)\end{aligned} \tag{7}$$

But $\theta_c(s)$ is step input, whose Laplace transform is $\frac{1}{s}$, hence

$$e(s) = \frac{1}{s}\left(\frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}-1\right)$$

Since $J = k_2$ then

$$e(s) = \frac{1}{s}\left(1-\frac{k_2G_{\theta,\delta_e}}{1+k_2G_{\theta,\delta_e}}\right)$$

Using final value theorem gives

$$
\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} s e(s)\\
&= \lim_{s\to 0} s\frac{1}{s}\left(1-\frac{k_2 G_{\theta,\delta_e}}{1+k_2 G_{\theta,\delta_e}}\right)\\
&= \lim_{s\to 0}\left(1-\frac{k_2 G_{\theta,\delta_e}}{1+k_2 G_{\theta,\delta_e}}\right)\\
&= \lim_{s\to 0} 1 - \lim_{s\to 0}\frac{k_2 G_{\theta,\delta_e}}{1+k_2 G_{\theta,\delta_e}}\\
&= 1-\left(\lim_{s\to 0}\frac{k_2\frac{-\left(1.158s^2+0.3545s+0.003873\right)}{s^4+0.750468s^3+0.935494s^2+9.453025\times 10^{-3}s+4.195875\times 10^{-3}}}{1+k_2\frac{-\left(1.158s^2+0.3545s+0.003873\right)}{s^4+0.750468s^3+0.935494s^2+9.453025\times 10^{-3}s+4.195875\times 10^{-3}}}\right)\\
&= 1-\frac{k_2\frac{-(0.003873)}{4.195875\times 10^{-3}}}{1+k_2\frac{-(0.003873)}{4.195875\times 10^{-3}}}\\
&= 1-\frac{-0.923\,05 k_2}{1-0.923\,05 k_2}\\
&= \frac{(1-0.923\,05k_2)+0.923\,05k_2}{1-0.92305k_2}\\
&= \frac{1}{1-0.92305k_2}
\end{aligned}
$$

Which simplifies to

$$
\boxed{\lim_{t\to\infty} e(t) = \frac{1}{1-0.9231\,k_2}}
$$

This is a plot showing the steady state error $e(\infty)$ as function of k_2 as k_2 is changed from 0 to -50

```
ess[k2_] := 1/(1 - 0.932305 k2);
Plot[Evaluate@ess[k2], {k2, -50, 0}, Frame -> True,
    FrameLabel -> {
    {"ess", None},
    {"k2", "steady state error as function of k2"}},
    BaseStyle -> FontSize -> 18
]
```

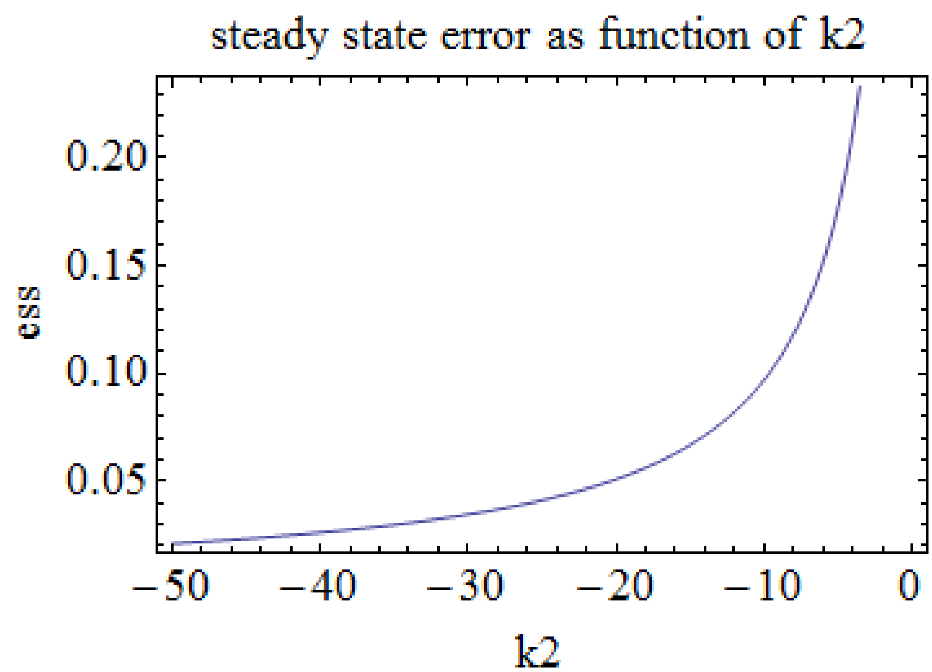

Figure 2.73: Steady state error as function of k_2, problem 1

2.5.1.3 Part(c)

From (7) above

$$e(s) = \theta_c(s)\left(1 - \frac{JG_{\theta,\delta_e}}{1 + JG_{\theta,\delta_e}}\right)$$

When $\theta_c(t) = 5^0$ then $\theta_c(s) = \frac{5}{s}$ and using final value theorem, with requirement that $e_{ss} < 0.1$ then

$$5\left(\frac{1}{1 - 0.92305k_2}\right) < 0.1$$

Hence

$$\begin{aligned} 5 &< 0.1 - 0.092305\,k_2 \\ 4.9 &< -0.092305\,k_2 \\ k_2 &< -\frac{4.9}{0.092305} \\ k_2 &< \boxed{-53.085} \end{aligned}$$

Hence k_2 has to be kept below −53.085 for the steady state error to be less than 0.1^o when $\theta_c(s) = 5^o$

2.5.1.4 Part(d)

From figure 8.5, in textbook δ_e is the elevator angle (output from the controller and the

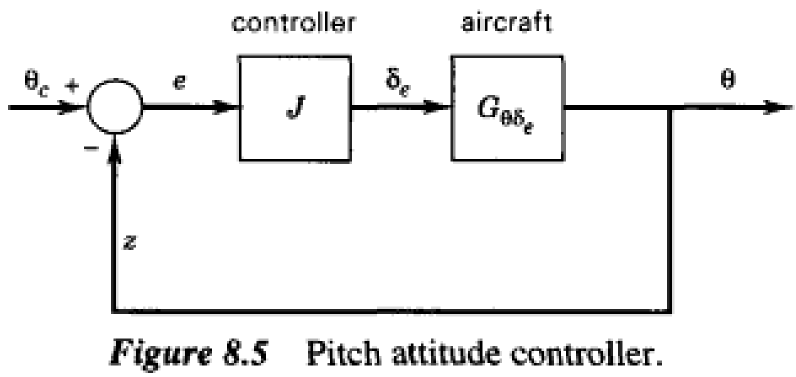

Figure 2.74: Figure 8.5 from text, pitch attitude controller, problem 1

input to $G_{\theta\delta_e}$. The controller is now $J = k_2$ where $k_2 = -53.085$. Hence

$$\delta_e = e(s) k_2 \tag{8}$$

But $e(s)$ is given in (7). Hence (8) becomes

$$\delta_e(s) = \theta_c(s)\left(1 - \frac{k_2 G_{\theta,\delta_e}}{1 + k_2 G_{\theta,\delta_e}}\right) k_2$$

Since $\theta_c(s) = \frac{5}{s}$ and $k_2 = -53.085$ the above becomes

$$\delta_e(s) = -\frac{5}{s}\left(1 - \frac{(-53.085)\, G_{\theta,\delta_e}}{1 - (53.085)\, G_{\theta,\delta_e}}\right)(53.085) \tag{9}$$

Using initial value theorem

$$\lim_{t\to 0} \delta_e(t) = \lim_{s\to\infty} s\delta_e(s)$$

Applying this to (9) gives

$$\begin{aligned} \delta_e(t=0) &= -\lim_{s\to\infty} 5\left(1 - \frac{(-53.085)\, G_{\theta,\delta_e}}{1 - (53.085)\, G_{\theta,\delta_e}}\right)(53.085) \qquad (10) \\ &= -\lim_{s\to\infty} 265.43\left(1 - \frac{(-53.085)\, G_{\theta,\delta_e}}{1 - (53.085)\, G_{\theta,\delta_e}}\right) \end{aligned}$$

Since $G_{\theta,\delta_e} = \frac{-\left(1.158s^2 + 0.3545s + 0.003873\right)}{s^4 + 0.750468s^3 + 0.935494s^2 + 9.453025\times 10^{-3}s + 4.195875\times 10^{-3}}$ then, by dividing numerator and denominator by s^4 and then taking the limit, it is clear that

$$\lim_{s\to\infty} G_{\theta,\delta_e} = 0$$

Therefore (10) reduces to

$$\boxed{\delta_e(t=0) = -265.43^o}$$

2.5.2 Problem 2

2. Problem 8.3. Hint: In part b), you are to find the $\lim_{t\to\infty} \delta_e(t)$ for the case $J(s) = 0.5\left(1+s+\frac{1}{s}\right)$ and for a step in θ_c.

8.3 (a) With respect to Fig. 8.5, write out the transfer function for the elevator angle response to θ_c input.

(b) Calculate the steady-state response for the case of Fig. 8.7*c*.

Solution

2.5.2.1 Part(a)

Figure 8.5 from the textbook is

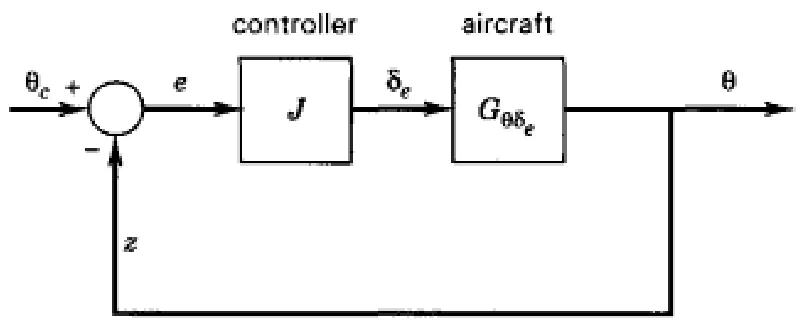

Figure 8.5 Pitch attitude controller.

Figure 2.75: Figure 8.5 from text, pitch attitude controller, problem 2

We need to find transfer function $\frac{\delta_e}{\theta_c}$. From the above diagram we see that

$$\delta_e(s) = e(s) J(s) \tag{1}$$

Where $e(s)$ was found in problem 1 above in equation (7) as

$$e(s) = \theta_c(s)\left(1 - \frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}\right)$$

Hence (1) becomes

$$\delta_e(s) = J\theta_c(s)\left(1 - \frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}\right)$$

$$\frac{\delta_e}{\theta_c} = G_{\delta_e,\theta_c} = J\left(1 - \frac{JG_{\theta,\delta_e}}{1+JG_{\theta,\delta_e}}\right)$$

$$G_{\delta_e,\theta_c} = J\left(\frac{1}{1+JG_{\theta,\delta_e}}\right)$$

$$= \frac{J}{1+JG_{\theta,\delta_e}} \tag{2}$$

Where

$$\boxed{G_{\theta,\delta_e} = \frac{-\left(1.158s^2 + 0.3545s + 0.003873\right)}{s^4 + 0.750468s^3 + 0.935494s^2 + 9.453025\times10^{-3}s + 4.195875\times10^{-3}}}$$

2.5.2.2 Part(b)

Using the hint, let $J = 0.5\left(1 + s + \frac{1}{s}\right)$ and apply the final value theorem to obtain the steady state $\delta_e(\infty)$ when $\theta_c(s) = \frac{1}{s}$ (step input).

Hence (2) becomes

$$\frac{\delta_e}{\theta_c} = \frac{J}{1+JG_{\theta,\delta_e}}$$

$$\delta_e(s) = \left(\frac{1}{s}\right)\frac{0.5\left(1+s+\frac{1}{s}\right)}{1+0.5\left(1+s+\frac{1}{s}\right)G_{\theta,\delta_e}}$$

Therefore

$$\delta_e(\infty) = \lim_{s\to0}\frac{0.5\left(1+s+\frac{1}{s}\right)}{1+0.5\left(1+s+\frac{1}{s}\right)G_{\theta,\delta_e}}$$

To simplify the above, the numerator and denominator are multiplied by s

$$\delta_e(\infty) = \lim_{s\to0}\frac{0.5\left(s+s^2+1\right)}{s+0.5\left(s+s^2+1\right)G_{\theta,\delta_e}}$$

Now the limit is taken, and noting that $\lim_{s\to0} G_{\theta,\delta_e} = \frac{-(0.003873)}{4.195875\times10^{-3}}$ results in

$$\delta_e(\infty) = \frac{0.5}{0.5(1)\frac{-(0.003873)}{4.195875\times10^{-3}}}$$

$$= \boxed{-1.0834^o}$$

2.5.3 Problem 3

3. Problem 8.9. Note: the constant r_c in the expression for r_{ss} is the magnitude of the step in $r_c(t)$.

8.9 (a) Prove that in the yaw damper with washout, the steady-state yaw rate for a step in r_c is independent of the washout time constant and is given by

$$r_{ss} = r_c J(0) G_{r\delta_r}(0)$$

(b) Prove that if the washout filter is in the forward path, instead of the feedback path, then

$$r_{ss} = 0$$

regardless of the washout time constant.

Solution

2.5.3.1 Part(a)

The transfer function diagram for the Yaw damper is shown on figure 8.21, page 288 in the textbook Where

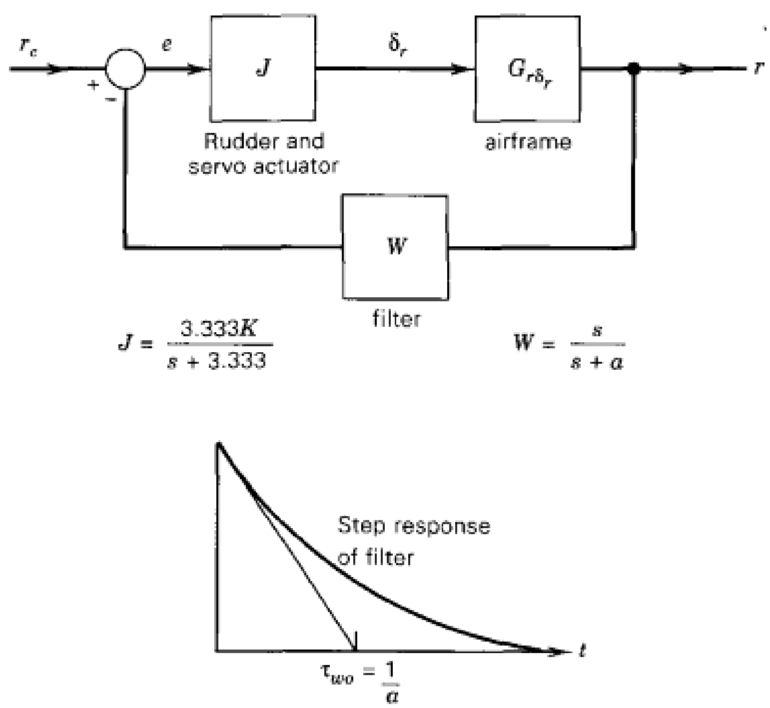

Figure 8.21 Yaw damper.

Figure 2.76: figure 8.21 from text book, yaw damper

$$r(s) = r_c(s)\,\mathring{G}_{r,\delta_r} \tag{1}$$

The closed loop $\mathring{G}_{r,\delta_r}$ is

$$\mathring{G}_{r,\delta_r} = \frac{JG_{r,\delta_r}}{1 + WJG_{r,\delta_r}}$$

To obtain $r_{ss} = r(\infty)$ the final value theorem is used. Here $r_c(s) = r_c \frac{1}{s}$ where r_c on the right side is now the magnitude of the step input (per the hint given). Equation (1) becomes

$$\begin{aligned} r_{ss} &= \lim_{s\to 0} s \frac{r_c}{s} \mathring{G}_{r,\delta_r} \\ &= \lim_{s\to 0} \frac{r_c JG_{r,\delta_r}}{1 + WJG_{r,\delta_r}} \end{aligned}$$

Using $W = \frac{s}{s+a}$ the above becomes

$$\begin{aligned} r_{ss} &= \lim_{s\to 0} \frac{r_c JG_{r,\delta_r}}{1 + \frac{s}{s+a} JG_{r,\delta_r}} \\ &= r_c J(0) G_{r,\delta_r}(0) \end{aligned}$$

Since $\lim_{s\to 0} \frac{s}{s+a} = 0$ then the above reduces to

$$\boxed{r_{ss} = r_c J(0) G_{r,\delta_r}(0)}$$

Since the expression for r_{ss} does not contain the time constant $\frac{1}{a}$ in it, (it does not contain a at all), therefore r_{ss} does not depend on the time constant of the washout filter.

2.5.3.2 Part(b)

Putting the washout filter in the forward path instead of in feedback, then

$$\mathring{G}_{r,\delta_r} = \frac{WJG_{r,\delta_r}}{1 + WJG_{r,\delta_r}}$$

Following what was done in part (a), to obtain $r_{ss} = r(\infty)$ the final value theorem is used. Here $r_c(s) = r_c \frac{1}{s}$ where r_c on the right side is the magnitude of the step input (per hint above). Equation (1) becomes

$$\begin{aligned} r_{ss} &= \lim_{s\to 0} s \frac{r_c}{s} \mathring{G}_{r,\delta_r} \\ &= \lim_{s\to 0} \frac{r_c WJG_{r,\delta_r}}{1 + WJG_{r,\delta_r}} \end{aligned}$$

Using $W = \frac{s}{s+a}$ the above becomes

$$r_{ss} = \lim_{s\to 0} \frac{r_c \frac{s}{s+a} JG_{r,\delta_r}}{1 + \frac{s}{s+a} JG_{r,\delta_r}}$$

Since $\lim_{s\to 0}\frac{s}{s+a}=0$ then the above becomes

$$
\begin{aligned}
r_{ss} &= \frac{0}{1+0}\\
&= 0
\end{aligned}
$$

Hence

$$\lim_{t\to\infty} r_c(t) = 0$$

Regardless of what a is.

2.5.4 Problem 4

4. Following the same procedure shown in class, develop expressions for $\mathring{G}_{\delta_a\phi_c}$ and $\mathring{G}_{v\phi_c}$. Any quantity developed in class can be used with no need to rederive it.

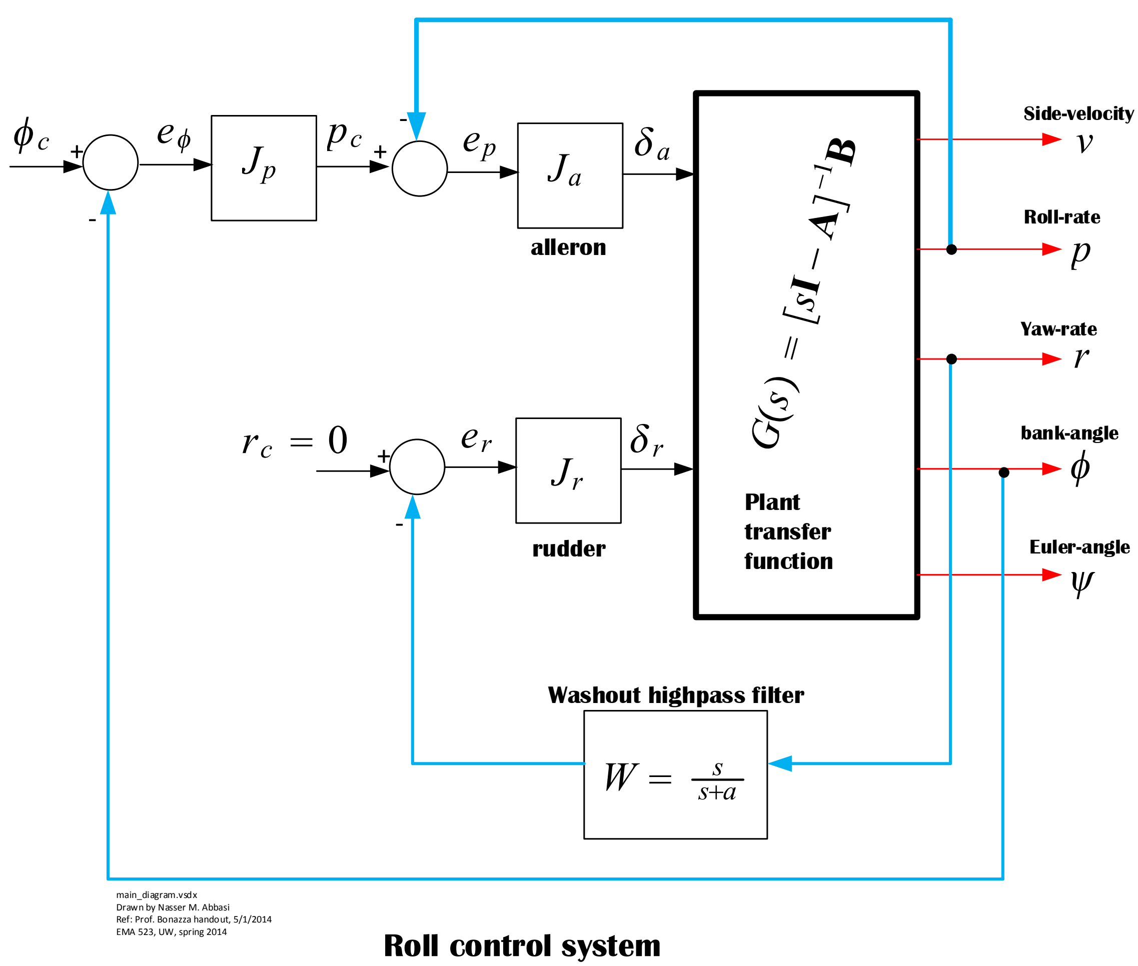

Figure 2.77: Roll control system, nonaugmented, problem 4

2.5.4.1 Part(a)

We need to obtain $\mathring{G}_{\delta_a\phi_c}$. From the above diagram we see that

$$\begin{aligned}\delta_a &= e_p J_a \\ &= \left(p_c - p\right) J_a\end{aligned}$$

But $p_c = e_\phi J_p = \left(\phi_c - \phi\right) J_p$, hence the above becomes

$$\delta_a = \left(\left(\phi_c - \phi\right) J_p - p\right) J_a$$

Since $p = G_{p\delta_r}\delta r + G_{p\delta_a}\delta a$, the above becomes

$$\delta_a = \left(\left(\phi_c - \phi\right) J_p - \left(G_{p\delta_r}\delta r + G_{p\delta_a}\delta_a\right)\right) J_a$$

From lecture 5/1/2014 in class, $\delta r = B\delta_a$ where $B = \frac{-WJ_r G_{r\delta a}}{1+WJ_r G_{r\delta r}}$. Therefore

$$\delta_a = \left(\left(\phi_c - \phi\right) J_p - \left(G_{p\delta_r} B\delta_a + G_{p\delta_a}\delta_a\right)\right) J_a$$

Also from lecture 5/1/2014 in class, $\phi = \phi_c \mathring{G}_{\phi\phi_c}$, and the above reduces to

$$\begin{aligned}\delta_a &= \left(\left(\phi_c - \phi_c \mathring{G}_{\phi\phi_c}\right) J_p - \left(G_{p\delta_r} B\delta_a + G_{p\delta_a}\delta_a\right)\right) J_a \\ &= \phi_c J_p - \phi_c \mathring{G}_{\phi\phi_c} J_p - G_{p\delta_r} B\delta_a J_a - G_{p\delta_a}\delta_a J_a \\ \delta_a\left(1 + G_{p\delta_r} B J_a + G_{p\delta_a} J_a\right) &= \phi_c\left(J_p\left(1 - \mathring{G}_{\phi\phi_c}\right)\right)\end{aligned}$$

Therefore

$$\mathring{G}_{\delta_a\phi_c} = \frac{\delta_a}{\phi_c} = \frac{J_p\left(1 - \mathring{G}_{\phi\phi_c}\right)}{1 + G_{p\delta_r} B J_a + G_{p\delta_a} J_a}$$

2.5.4.2 Part(b)

From the diagram above

$$v = G_{v\delta_r}\delta r + G_{v\delta_a}\delta_a$$

But $\delta r = B\delta_a$ where $B = \frac{-WJ_r G_{r\delta a}}{1+WJ_r G_{r\delta r}}$ hence

$$\begin{aligned}v &= G_{v\delta_r} B\delta a + G_{v\delta_a}\delta_a \\ &= \left(G_{v\delta_r} B + G_{v\delta_a}\right)\delta_a\end{aligned}$$

From part (a), we found $\delta_a = \phi_c \mathring{G}_{\delta_a\phi_c}$, where $\mathring{G}_{\delta_a\phi_c} = \frac{J_p\left(1-\mathring{G}_{\phi\phi_c}\right)}{1+G_{p\delta_r}BJ_a+G_{p\delta_a}J_a}$. The above becomes

$$v = \left(G_{v\delta_r} B + G_{v\delta_a}\right)\phi_c \mathring{G}_{\delta_a\phi_c}$$

Therefore

$$\mathring{G}_{v\phi_c} = \frac{v}{\phi_c} = \left(G_{v\delta_r} B + G_{v\delta_a}\right)\mathring{G}_{\delta_a\phi_c}$$

2.5.5 Problem 5

5. Problem 8.11. Hints: modify the schematic of Fig. 8.26 as described in the problem statement (heeding the hint given there). In the problem statement, "write out the augmented system differential equation" means write something like $\dot{\mathrm{z}} = \mathbf{P}\mathrm{z} + \mathrm{Q}\psi_c$ where z, $\mathbf{P}$ and Q are modified versions of the same vectors and matrices seen in the example in class. Specifically:

add a new summator point and a new reference value ψ_c;

then $\psi_c - \psi = e_\psi$;

e_ψ now enters a gain block that turns it into ϕ_c . Now you're back to the system we studied in class.

The new state vector, z, is the same as before except that this time it includes ψ (place it after ϕ), so it has 8 components. Then vector Q must have 8 components and matrix $\mathbf{P}$ becomes an 8×8.

Write a relationship between ψ_c and ϕ_c

In the original equation $\dot{\mathrm{z}} = \mathbf{P}\mathrm{z} + \mathrm{Q}\phi_c$, replace ϕ_c with an expression involving ψ_c.

Remember that $\dot{\psi} = r \sec\theta_0$ and assume $\theta_0 = 0$.

Now rearrange matrix $\mathbf{P}$ and vector Q to account for the changes you just made.

Once done with that, modify the Matlab and/or Simulink scripts in the course website to include the additions you made to the control loop and generate plots of the responses of all the variables studied in class to an impulse and a step in the new input variable. For the gains, I recommend: a value of 2.5 between e_ψ and ϕ_c and a value of 1.5 between e_ϕ and p_c.

Especially with the Simulink script, it is easy to experiment with different inputs. Comment on the results you obtained, especially on the response of ψ and r to step inputs in ψ_c and r_c. **Make sure to attach your Matlab script and/or your Simulink schematic.**

8.11 **Add an outer loop to the system of Fig. 8.26 to control the heading angle ψ. Draw a new block diagram and write out the augmented system differential equation. (Hint: design the loop to command a bank angle proportional to heading error.)**

Figure 2.78: problem 5 description

Solution:

2.5.5.1 Generating the augmented system

From class notes on may 1, 2014, the following was derived

$$\{\dot{z}\} = \boldsymbol{P}\{z\} + \{Q\}\phi_c$$

Where for $\theta_0 = 0$

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\delta}_a \\ \dot{\delta}_r \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & g & b_{11} & b_{12} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & b_{21} & b_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 & b_{31} & b_{32} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_a}{\tau_a} & 0 & -\frac{k_a k_p}{\tau_a} & -\frac{1}{\tau_a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_r}{\tau_r}a_{31} & \frac{k_r}{\tau_r}a_{32} & \frac{k_r}{\tau_r}a_{33} & 0 & \frac{k_r}{\tau_r}b_{31} & \left(\frac{k_r b_{32}}{\tau_r} - \frac{1}{\tau_r \tau_{w0}}\right) & -\left(\frac{1}{\tau_r} + \frac{1}{\tau_{w0}}\right) \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \delta_a \\ \delta_r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{k_a k_p}{\tau_a} \\ 0 \\ 0 \end{bmatrix} \phi_c$$

The values for a_{ij} in the above are those from lateral equations of motion equation 4.9,19 on page 111 in the textbook

$$\begin{bmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_o\right) & g\cos\theta_0 \\ \left(\frac{L_v}{I'_x} + I'_{zx}N_v\right) & \left(\frac{L_p}{I'_x} + I'_{zx}N_p\right) & \left(\frac{L_r}{I'_x} + I'_{zx}N_r\right) & 0 \\ \left(I'_{zx}L_v + \frac{N_v}{I'_z}\right) & \left(I'_{zx}L_p + \frac{N_p}{I'_z}\right) & \left(I'_{zx}L_r + \frac{N_r}{I'_z}\right) & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix}$$

Figure 2.79: Details of A matrix from $x'(t) = Ax(t) + Bu(t)$ for problem 5

And the b_{ij} are from the B matrix (4×2) from equation 7.9,3 on page 244 We are now ready to augment the above system. Since $e_\psi = \psi_c - \psi$ and $e_\psi k = \phi_c$ where k is the gain shown in the above diagram feeding to ϕ_c, then

$$\phi_c = k\left(\psi_c - \psi\right) \tag{1}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{Y_{\delta_a}}{m} & \dfrac{Y_{\delta_r}}{m} \\ \dfrac{L_{\delta_a}}{I'_x} + I'_{zx}N_{\delta_a} & \dfrac{L_{\delta_r}}{I'_x} + I'_{zx}N_{\delta_r} \\ I'_{zx}L_{\delta_a} + \dfrac{N_{\delta_a}}{I'_z} & I'_{zx}L_{\delta_r} + \dfrac{N_{\delta_r}}{I'_z} \\ 0 & 0 \end{bmatrix}$$

Figure 2.80: Details of B matrix from $x'(t) = Ax(t) + Bu(t)$ for problem 5

Equation (3) in the notes from 5/1/2014, needs to be modified. It was

$$\dot{\delta}_a = -\frac{k_a k_p}{\tau_a}\phi - \frac{k_a}{\tau_a}p - \frac{\delta_a}{\tau_a} + \frac{k_a k_p}{\tau_a}\phi_c \tag{3}$$

Using (1) and (3) results in

$$\begin{aligned}\dot{\delta}_a &= -\frac{k_a k_p}{\tau_a}\phi - \frac{k_a}{\tau_a}p - \frac{\delta_a}{\tau_a} + \frac{k_a k_p}{\tau_a}k\left(\psi_c - \psi\right) \\ &= -\frac{k_a k_p}{\tau_a}\phi - \frac{k_a}{\tau_a}p - \frac{\delta_a}{\tau_a} - \frac{k_a k_p}{\tau_a}k\psi + \frac{k_a k_p}{\tau_a}k\psi_c\end{aligned}$$

Given that $\dot{\psi} = r\sec\theta_0 = r$ since $\theta_0 = 0$. In Laplace domain this results in $\psi(s) = \frac{1}{s}r$ The new augmented system becomes

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{\delta}_a \\ \dot{\delta}_r \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & g & 0 & b_{11} & b_{12} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & b_{21} & b_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & b_{31} & b_{32} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_a}{\tau_a} & 0 & -\frac{k_a k_p}{\tau_a} & -\frac{k_a k_p}{\tau_a}k & -\frac{1}{\tau_a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_r}{\tau_r}a_{31} & \frac{k_r}{\tau_r}a_{32} & \frac{k_r}{\tau_r}a_{33} & 0 & 0 & \frac{k_r}{\tau_r}b_{31} & \left(\frac{k_r b_{32}}{\tau_r} - \frac{1}{\tau_r\tau_{wo}}\right) & -\left(\frac{1}{\tau_r} + \frac{1}{\tau_{wo}}\right) \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \\ \delta_a \\ \delta_r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{k_a k_p}{\tau_a}k \\ 0 \\ 0 \end{bmatrix} \psi_c$$

The above is the new system $\{\dot{z}\} = \boldsymbol{P}\{z\} + \{Q\}\phi_c$ where $\boldsymbol{P}$ now is an 8×8 matrix and Q is an

8×1 vector. A new state ψ was added and the input now is ψ_c instead of ϕ_c. The closed loop transfer function is

$$\boxed{\mathring{G} = [s\boldsymbol{I} - \boldsymbol{P}]^{-1} Q}$$

For the controller, the following values will be used $k = 2.5$ and $W(s) = \frac{s}{s+\frac{1}{\tau_{wo}}}$, where $\tau_{wo} = 4$, and $J_r(s) = \frac{k_r \frac{1}{\tau_r}}{s+\frac{1}{\tau_r}}$ where $k_r = -1.6$ and $\tau_r = 0.3$, hence $J_r(s) = \frac{(1/0.3)1.6}{s+(1/0.3)}$ and $J_p(s) = k_p = 1.5$ and $J_a = \frac{k_a \frac{1}{\tau_a}}{s+\frac{1}{\tau_a}}$ where $k_a = -1$ and $\tau_a = 0.15$. Hence $J_a = \frac{\frac{-1}{0.15}}{s+\frac{1}{0.15}}$. To summarize

$$\begin{aligned}
k &= 2.5 \\
W(s) &= \frac{s}{s+\frac{1}{4}} \\
J_r(s) &= \frac{-(1/0.3)\,1.6}{s+(1/0.3)} = \frac{-5.3333}{s+3.3333} \\
J_a &= \frac{-\frac{1}{0.15}}{s+\frac{1}{0.15}} \frac{-6.6667}{s+6.6667}
\end{aligned}$$

Now that all the controllers are known and the new augmented system is shown above, Matlab was used to obtain the response due to an impulse and step in the new input ψ_c. The structure of A, B, C and D matrices is as follows

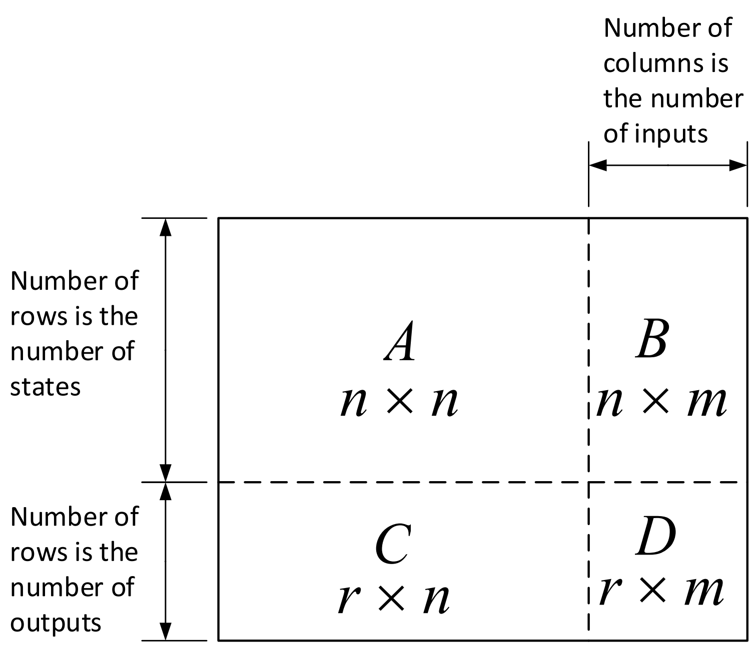

Figure 2.81: State space matrices dimensions

And the augmented roll controller becomes

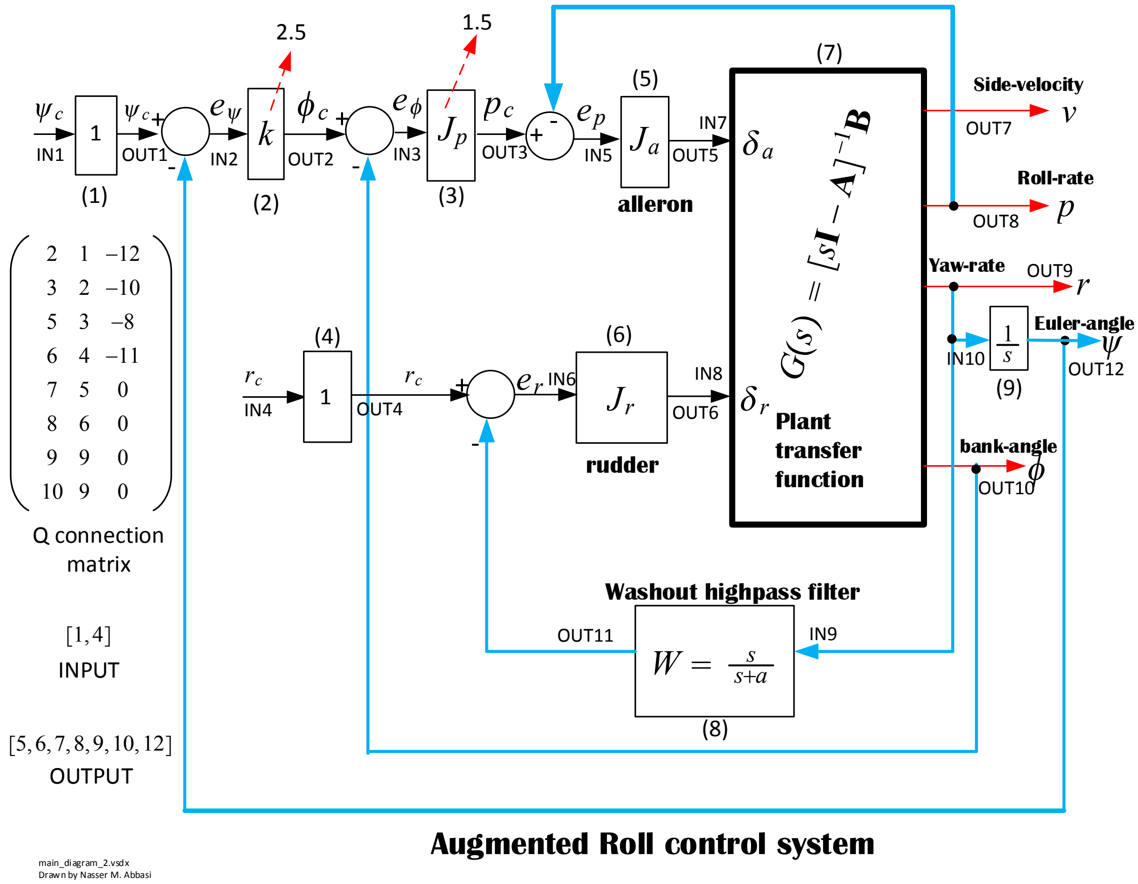

Figure 2.82: Roll control system, augmented, problem 5

2.5.5.2 A,B,C,D generated by Matlab

These are the numerical value of the matrices A,B,C,D generated by Matlab after connecting the system

```
sys=connect(sysa,Q,inputs,outputs)
a=sys.A
b=sys.B
c=sys.C
d=sys.D

a =
   -6.6667         0         0   -1.0000         0   -1.5000         0   -3.7500
         0   -3.3333         0         0   -1.0000         0    0.2500         0
         0  -30.0907   -0.0558         0 -774.0000   32.2000         0         0
    0.9540   -0.6101   -0.0039   -0.4342    0.4136         0         0         0
   -0.0249    2.5915    0.0011   -0.0061   -0.1458         0         0         0
```

```
         0         0         0    1.0000         0         0         0         0
         0         0         0         0    1.0000         0   -0.2500         0
         0         0         0         0    1.0000         0         0         0

b =
    3.7500         0
         0    1.0000
         0         0
         0         0
         0         0
         0         0
         0         0
         0         0

c =
   -6.6667         0         0         0         0         0         0         0
         0   -5.3333         0         0         0         0         0         0
         0         0    1.0000         0         0         0         0         0
         0         0         0    1.0000         0         0         0         0
         0         0         0         0    1.0000         0         0         0
         0         0         0         0         0    1.0000         0         0
         0         0         0         0         0         0         0    1.0000
d =
     0     0
     0     0
     0     0
     0     0
     0     0
     0     0
     0     0
```

2.5.5.3 Generating the responses

Four different inputs are used, and for each input, seven outputs were plotted.

The inputs are: 15^0 step input in ψ_c and one radian angle impulse in ψ_c. For each of these two inputs the responses $\beta, p, r, \phi, \psi, \delta_a, \delta_r$ were plotted.

Next, a step input r_c of amplitude 1^o per second, and an impulse r_c of 1^o per second are used, and for each of these inputs, the responses $\beta, p, r, \phi, \psi, \delta_a, \delta_r$ were plotted.

There are 28 different plots generated. Special attention is given to the response ψ to the 15^o step input ψ_c and to the response r to the 1^o per second step input r_c.

Final conclusion is given below at the end after showing the responses obtained.

2.5.5.4 Response for one radian impulse in ψ_c

All variables subside to negligible level, including ψ which had a residual value in the non-augmented system when ϕ_c was used as input instead of ψ_c here.

All state variables had good damped oscillatory decay as well. The aileron angle was larger than the case with the non-augmented system, reaching almost 50 degrees before damping down. The rudder angle went to 2 degrees which is twice as much as with the non-augmented system in the text book at page 293. Variables decay to negligible level in about 15 seconds, similar to the non-augmented system, except for ψ which needed about 30 seconds.

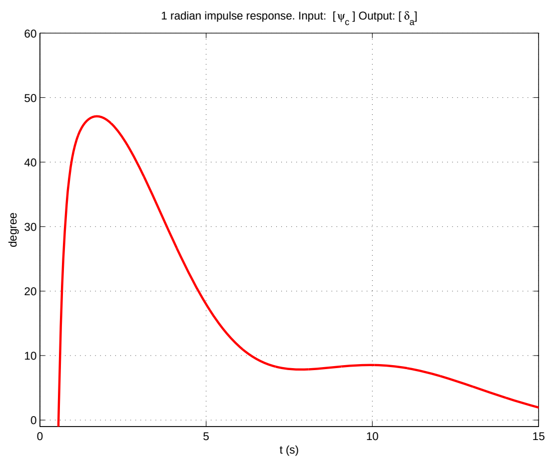

Figure 2.83: Impulse response. Input ψ_c, output δ_a

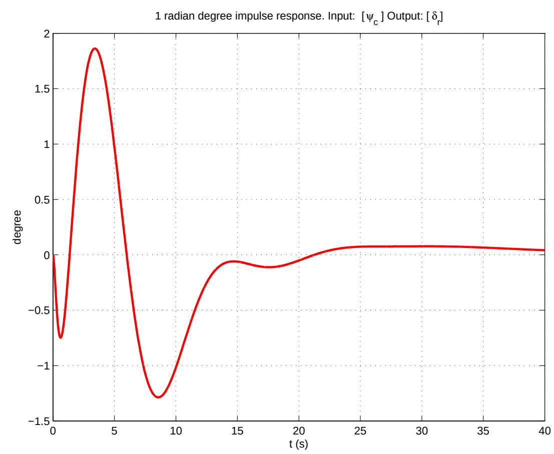

Figure 2.84: Impulse response. Input ψ_c, output δ_r

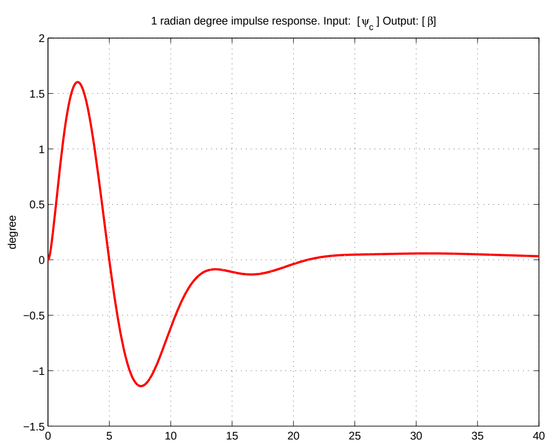

Figure 2.85: Impulse response. Input ψ_c, output β

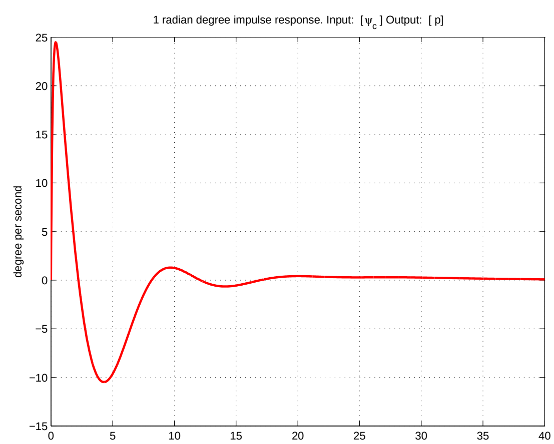

Figure 2.86: Impulse response. Input ψ_c, output p

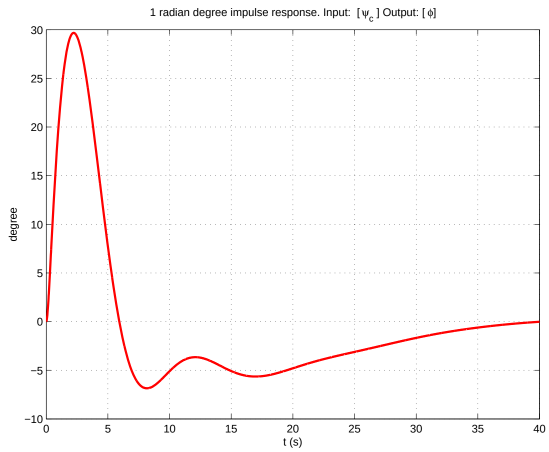

Figure 2.87: Impulse response. Input ψ_c output ϕ

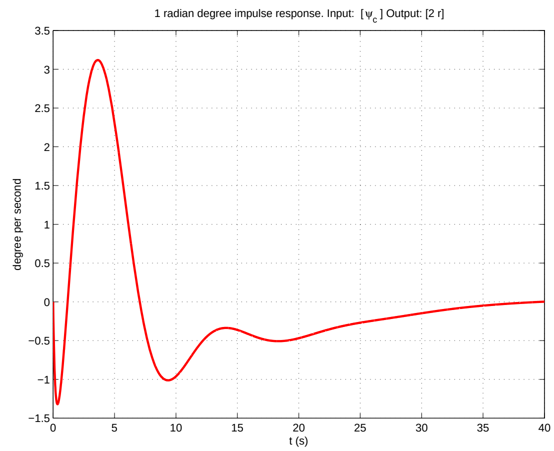

Figure 2.88: Impulse response. Input ψ_c, output r

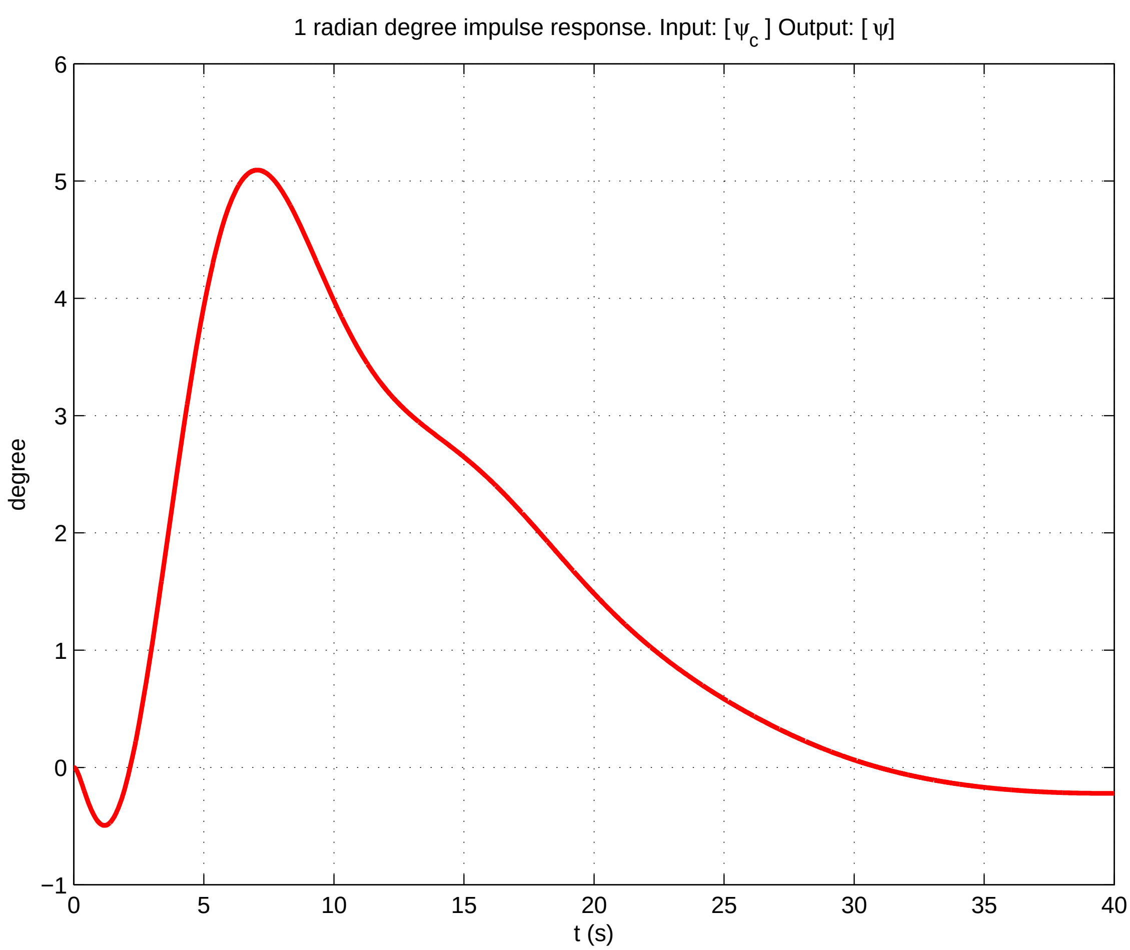

Figure 2.89: Impulse response. Input ψ_c, output ψ

2.5.5.5 Response for one degree per second impulse in r_c

All variables here also subsided to negligible level in about 15 seconds, except for ϕ and ψ which needed 40 seconds.

All state variables had good damped oscillatory decay as well. The aileron angle reached only 5 degrees before damping down.

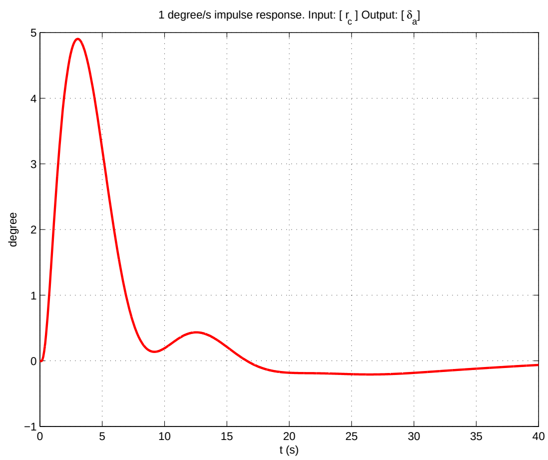

Figure 2.90: Impulse response. Input r_c, output δ_a

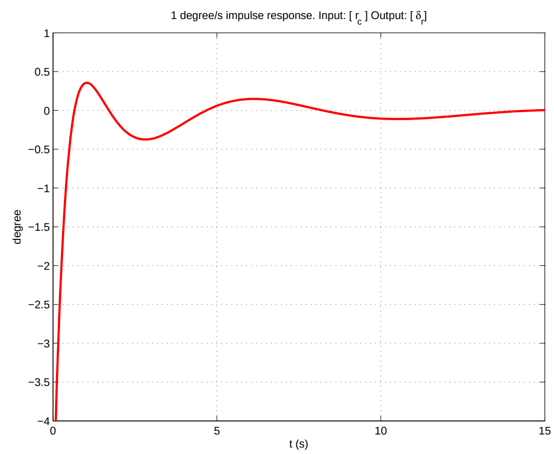

Figure 2.91: Impulse response. Input rc_c, output δ_r

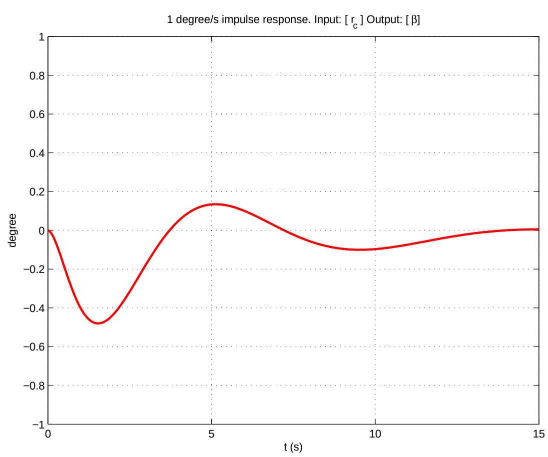

Figure 2.92: Impulse response. Input rc_c, output β

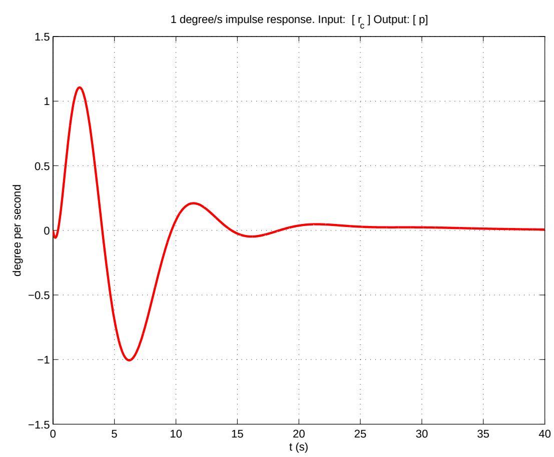

Figure 2.93: Impulse response. Input rc_c, output p

2.5.5.6 Response for 15 degrees step input in ψ_c

Aileron angle took 40 seconds to damped to zero and had large initial oscillation (-50 degrees). Rudder angle reached 1.2 degrees before damping.

The yaw rate r did not residual value as the case was with the non-augmented system when

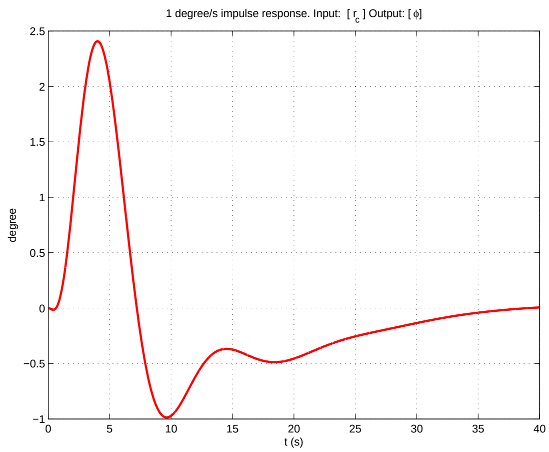

Figure 2.94: Impulse response. Input rc_c output ϕ

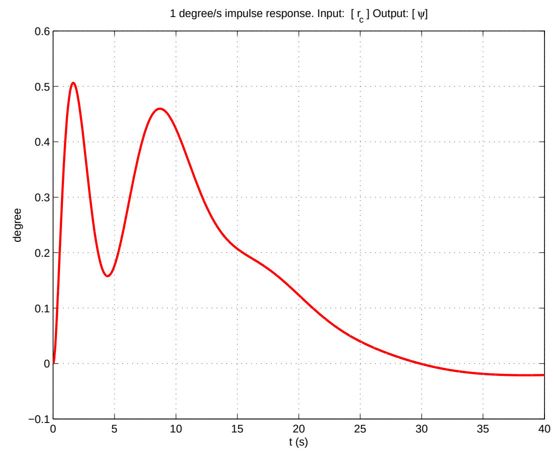

Figure 2.95: Impulse response. Input rc_c, output ψ

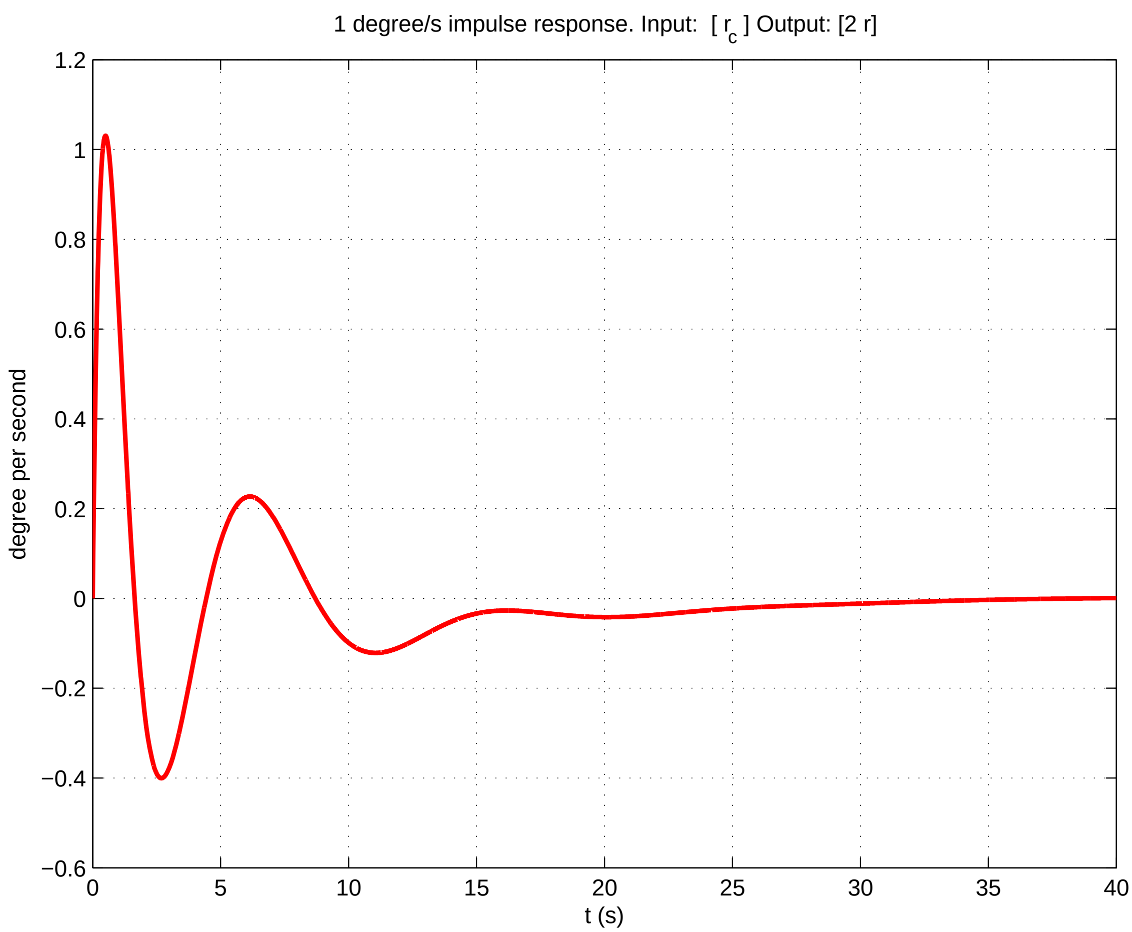

Figure 2.96: Impulse response. Input rc_c, output r

roll command was used as can be seen in figure 8.28, page 294 in the text. Here we see r damping down to almost zero in 40 seconds.

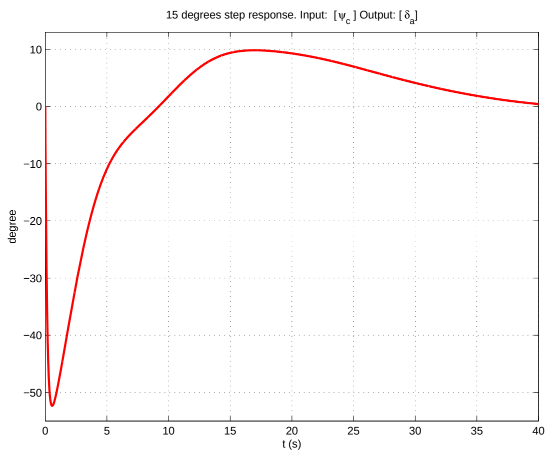

Figure 2.97: Step response. Input ψ_c, output δ_a

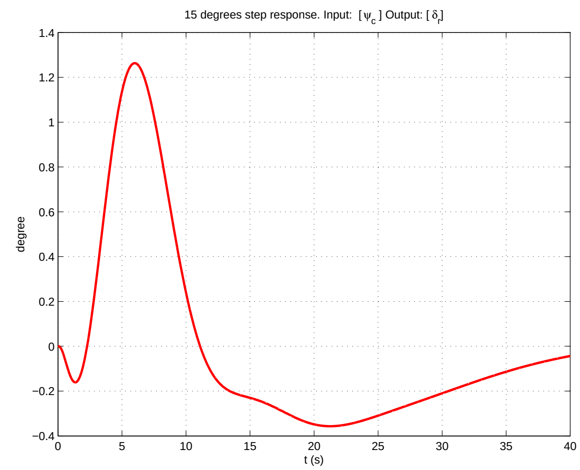

Figure 2.98: Step response. Input ψ_c, output δ_r

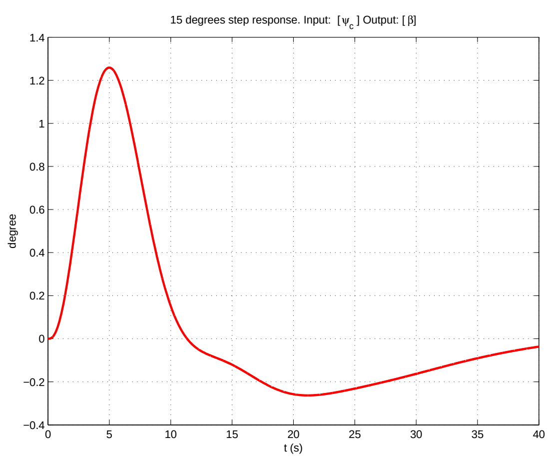

Figure 2.99: Step response. Input ψ_c, output β

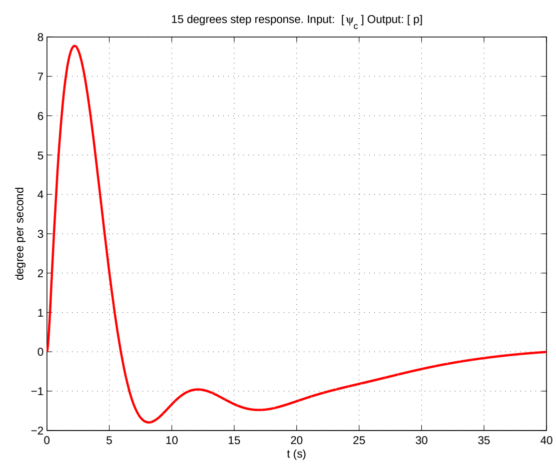

Figure 2.100: Step response. Input ψ_c, output p

2.5.5.7 Response for one degree per second step input in r_c

Aileron angle reach 20 degrees steady state, while rudder was -1.6 degrees when the reference command is set to one degree per second yaw rate r_c.

Rudder angle had more oscillation than aileron but both reached steady state in 40 seconds. The roll rate p damped to zero in 40 seconds.

State variable yaw rate r did not track r_c in this case. The augmented system could not control Yaw rate as it did not follow the step input r_c as is discussed more below.

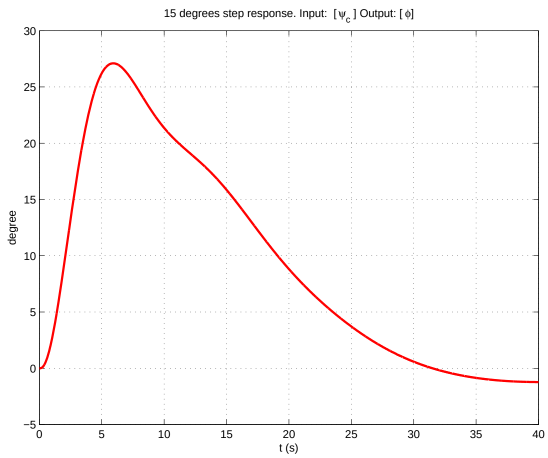

Figure 2.101: Step response. Input ψ_c output ϕ

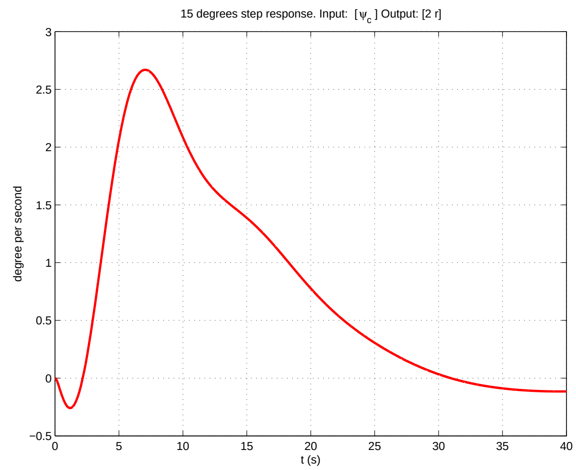

Figure 2.102: Step response. Input ψ_c, output r

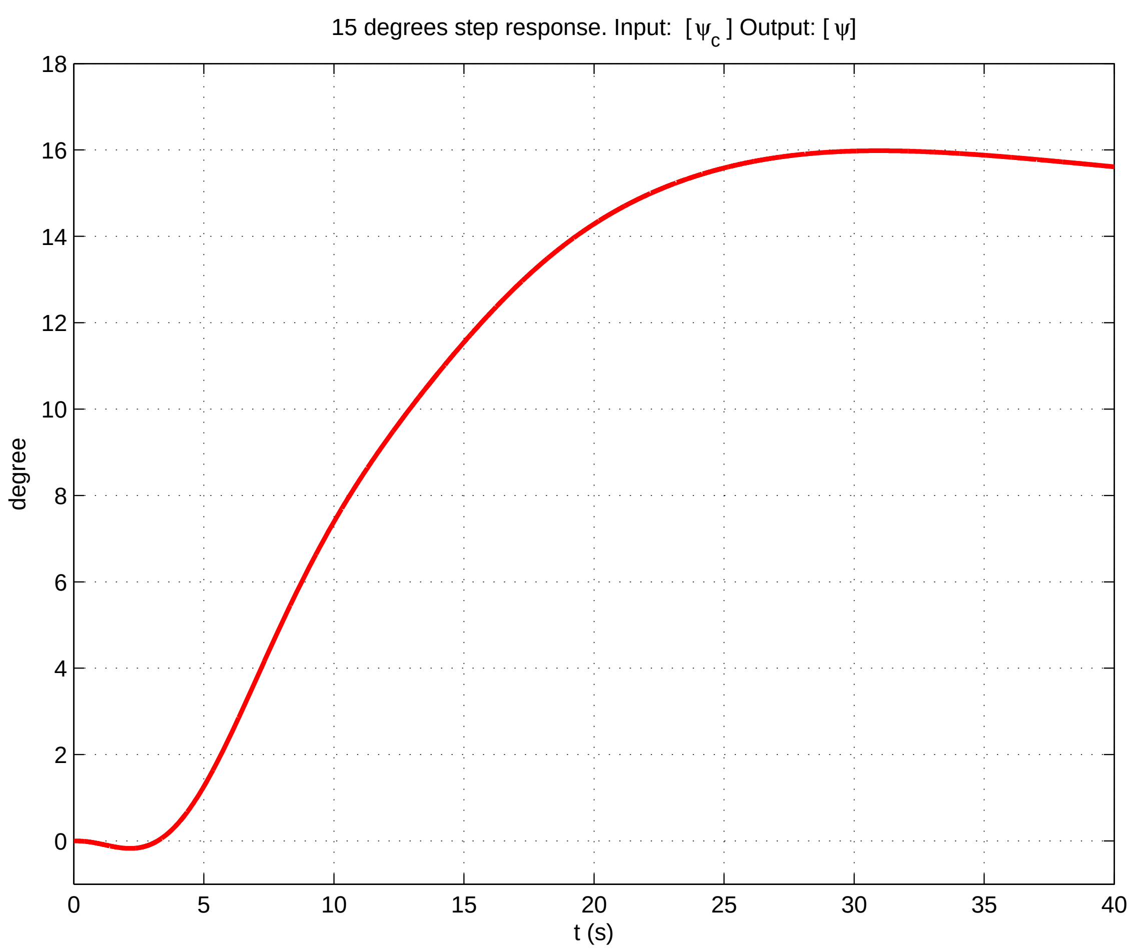

Figure 2.103: Step response. Input ψ_c, output ψ

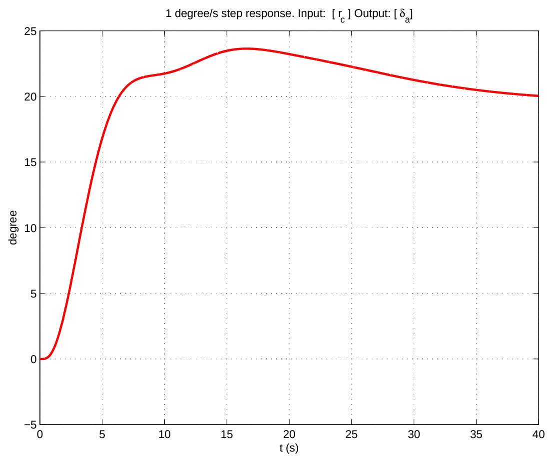

Figure 2.104: Step response. Input r_c, output δ_a

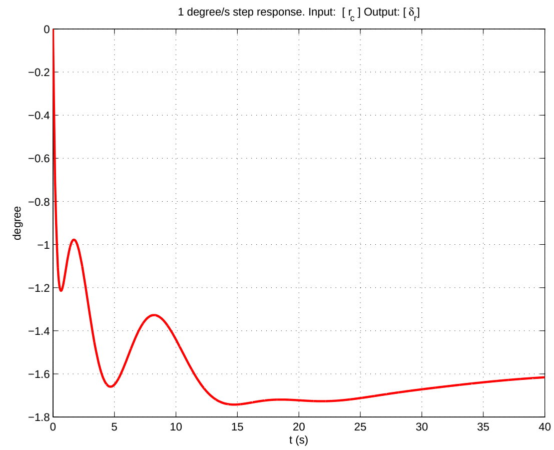

Figure 2.105: Step response. Input r_c, output δ_r

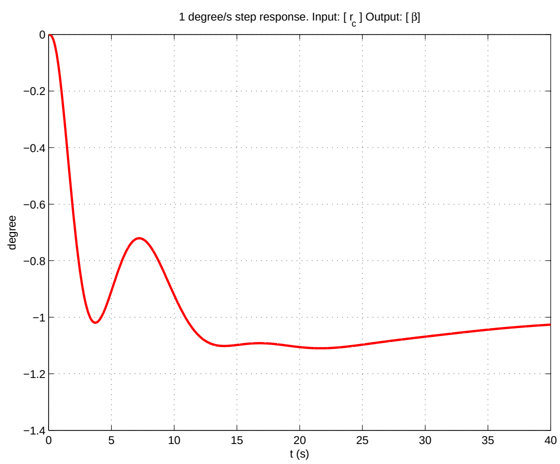

Figure 2.106: Step response. Input r_c, output β

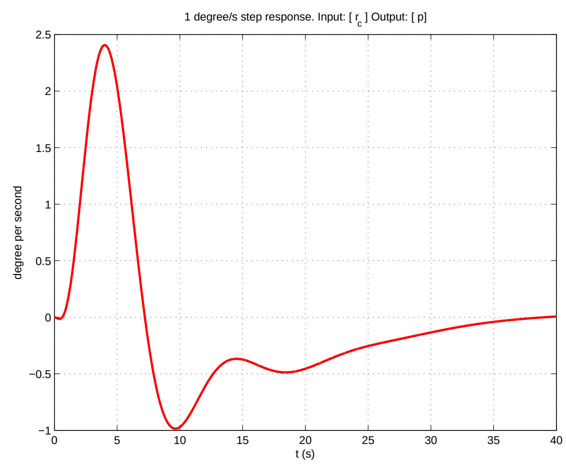

Figure 2.107: Step response. Input r_c, output p

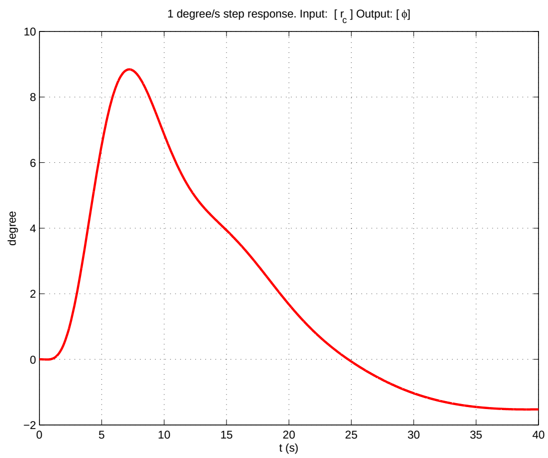

Figure 2.108: Step response. Input r_c output ϕ

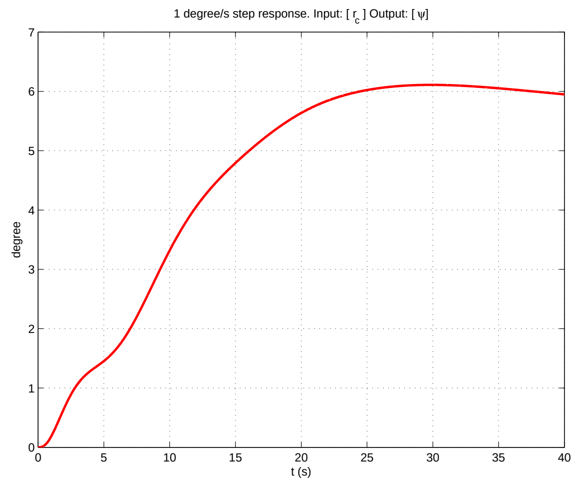

Figure 2.109: Step response. Input r_c, output ψ

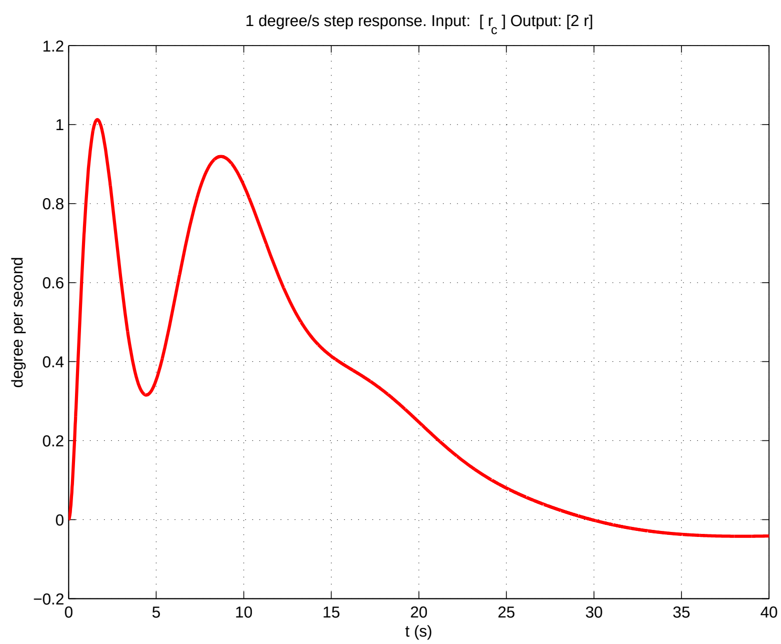

Figure 2.110: Step response. Input r_c, output r

2.5.5.8 Conclusion

Looking at the ψ step response of 15^o in ψ_c given in figure 2.103, we can see that the response ψ was good to the step input. After 30 seconds, it had amplitude of 16 degrees, and at 40 seconds it was close to the 15 degrees reference input. There was no oscillation and almost no overshoot (about 1 degree overshoot).

However, Looking at the r step response of 1^o per second in r_c given in figure 2.110, the r step response was not as good as the case was when using the nonaugmented system.

There was similar oscillation initially in the response r, but after 10 seconds, the response failed to reach one degree per second, and it actually went to zero instead, as can be seen in the following figure.

This shows the augmented system is not suitable for controlling r.

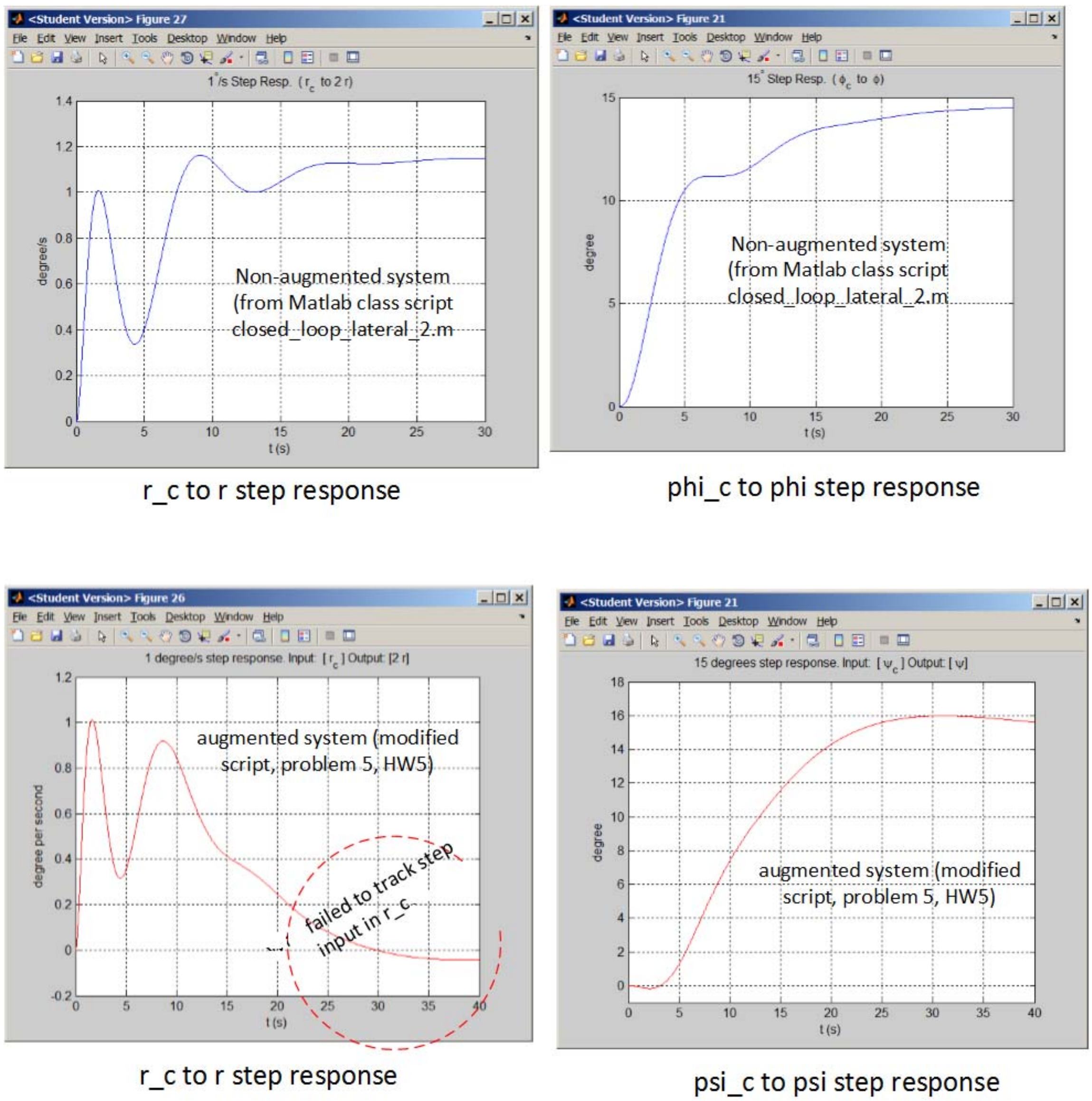

Figure 2.111: Showing augmented system is not suitable to tracking r_c

2.5.5.9 Source code listing

```
%This script is modified version of EMA 523 class script to solve
%problem 5, HW5.
%step input in \psi_c and r_c and impulse in \psi_c and r_c
%are given. For each, response in v,p,r,\phi,\psi,del_a and del_r is
%plotted.
% EMA 523 Univ. Wisconsin, Madison, spring 2014
% modified by Nasser M. Abbasi

close all;
clear all;

v_cruise = 774;%cruise speed in fps
max_time = 40;
T        = 0:0.01:max_time; %time interval for plotting

%set up matrices%%%%%%%%%%%%%%%%%
A=[-0.0558 0.0 -774.0 32.20;
    -0.003865 -0.4342 0.4136 0.0;
    0.001086 -0.006112 -0.1458 0.0;
    0.0 1.0 0.0 0.0]

B=[0.0 5.642;
    -0.1431 0.1144;
    0.003741 -0.4859;
    0.0 0.0]

C=[1 0 0 0;
   0 1 0 0;
   0 0 1 0;
   0 0 0 1]

D=[0 0;
   0 0;
   0 0;
   0 0]

%----------- set up string arrays and cells
states='v p r \phi'
inputs='aileron rudder';
outputs='v p r \phi';

states_st={' v ';' p ';' r ';' \phi '};
inputs_st={' aileron ';' rudder '};
outputs_st=states_st;

%---------- display all matrices
```

```
printsys(A,B,C,D,inputs, outputs,states);

%------- SYS1
[A1,B1,C1,D1]=zp2ss(0,0,1)  %between zeta_c and zeta_c
sys1=ss(A1,B1,C1,D1,'inputname','\psi_c','outputname','\psi_c');

%-------- SYS2
[A2,B2,C2,D2]=zp2ss(0,0,2.5)  %between error_zeta and phi_c
sys2=ss(A2,B2,C2,D2,'inputname','e_\psi','outputname','\phi_c');

%-------- SYS 3, Roll-rate controller (Jp=SYS3)
K_p=1.5
[A3,B3,C3,D3]=zp2ss(0,0,K_p)
sys3=ss(A3,B3,C3,D3,'inputname','e_\phi','outputname','p_c')

%--------- SYS4
[A4,B4,C4,D4]=zp2ss(0,0,1)
sys4=ss(A4,B4,C4,D4,'inputname','r_c','outputname','r_c')

%--------- SYS 5, Aileron controller (Ja=SYS5)
tau_a  = 0.15
K_a    = -1.0
den_ja = [1 1./tau_a]
num_ja = [K_a/tau_a]
[A5,B5,C5,D5] = tf2ss(num_ja,den_ja)
sys5 = ss(A5,B5,C5,D5,'inputname','e_p','outputname','\delta_a')

%---------- SYS 6, Rudder controller (Jr=SYS6)
tau_r   = 0.3
K_r     = -1.6
den_jr  = [1 1./tau_r]
num_jr  = [K_r/tau_r]
[A6,B6,C6,D6] = tf2ss(num_jr,den_jr)
sys6 = ss(A6,B6,C6,D6,'inputname','e_r','outputname','\delta_r')

%---------- SYS 7
sys7 = ss(A,B,C,D,'inputname',{'\delta_a' '\delta_r'}, 'outputname',{'v' 'p' 'r' '\phi'})

%---------- SYS 8, Washout
tau_wo = 4.0
[A8,B8,C8,D8] = zp2ss(0,-1/tau_wo,1)
sys8=ss(A8,B8,C8,D8,'inputname','r','outputname','r_f')

%---------- SYS 9, Integrator
den_int = [1 0]
num_int = [1]
[A9,B9,C9,D9] = tf2ss(num_int,den_int)
```

```
sys9 = ss(A9,B9,C9,D9,'inputname','r','outputname','\psi')

sysa=append(sys1,sys2,sys3,sys4,sys5,sys6,sys7,sys8,sys9)

Q=[2 1 -12
   3 2 -10
   5 3 -8
   6 4 -11
   7 5 0
   8 6 0
   9 9 0
   10 9 0];

inputs=[1 4];
outputs=[5 6 7 8 9 10 12];

sys=connect(sysa,Q,inputs,outputs)

a=sys.A
b=sys.B
c=sys.C
d=sys.D

%set up string arrays and cells%%%%%%%%%
states_1='v p r \phi \psi';
inputs_1='\psi_c r_c';
outputs_1='\delta_a \delta_r v p r \phi \psi';

states_st_1={' \delta_a ';' \delta_r ';' \beta ';' p ';' r ';' \phi ';' \psi '};
inputs_st_1={' \psi_c ';' r_c '};
outputs_st_1=states_st_1;
%set up string arrays and cells%%%%%%%%%

% IMPULSE IN PSI_C
figure;
plot(T,impulse(a,b,c(1,:),d(1,:),1,T)*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 radian impulse response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(1))], ']'));
axis([0 15 -1 60]);
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_psi_to_del_a', 'epsc');

figure;
plot(T,impulse(a,b,c(2,:),d(2,:),1,T)*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 radian degree impulse response. Input:  [',...
```

```
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(2))], ']'));
%axis([0 max_time -0.05 0.05])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_psi_to_del_r', 'epsc');


figure;
plot(T,impulse(a,b,c(3,:),d(3,:),1,T)/v_cruise*180/pi,'r','LineWidth',2);%*0.262/v_cruise);
title(strcat('1 radian degree impulse response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(3))], ']'));
%axis([0 max_time -0.03 0.05])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_psi_to_beta', 'epsc');


figure;
plot(T,impulse(a,b,c(4,:),d(4,:),1,T)*180/pi,'r','LineWidth',2);%*0.262*2);
title(strcat('1 radian degree impulse response. Input:  [',...
    [char(inputs_st_1(1)), '] Output:  [', char(outputs_st_1(4))], ']'));
%axis([0 max_time -0.3 0.5])
xlabel('t (s)')
ylabel('degree per second')
grid;
saveas(gcf, 'impulse_psi_to_p', 'epsc');

figure;
plot(T,impulse(a,b,c(5,:),d(5,:),1,T)*2*180/pi,'r','LineWidth',2);%*0.262*2);
title(strcat('1 radian degree impulse response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [2', char(outputs_st_1(5))], ']'));
%axis([0 max_time -0.05 0.05])
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'impulse_psi_to_r', 'epsc');

figure;
plot(T,impulse(a,b,c(6,:),d(6,:),1,T)*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 radian degree impulse response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(6))], ']'));
%axis([0 max_time -0.3 0.6])
xlabel('t (s)');
ylabel('degree');
grid;
```

```
saveas(gcf, 'impulse_psi_to_phi', 'epsc');

figure;
plot(T,impulse(a,b,c(7,:),d(7,:),1,T)*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 radian degree impulse response. Input: [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(7))], ']'));
%axis([0 max_time -0.02 0.1])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_psi_to_psi', 'epsc');

%------------------------------
% IMPULSE IN r_C
figure;
plot(T,impulse(a,b,c(1,:),d(1,:),2,T)*0.0175*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 degree/s impulse response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(1))], ']'));
%axis([0 max_time -0.3 0.5])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_rc_to_del_a', 'epsc');


figure;
plot(T,impulse(a,b,c(2,:),d(2,:),2,T)*0.0175*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 degree/s impulse response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(2))], ']'));
axis([0 15 -4 1])
xlabel('t (s)')
ylabel('degree')
grid;
saveas(gcf, 'impulse_rc_to_del_r', 'epsc');

figure;
plot(T,impulse(a,b,c(3,:),d(3,:),2,T)*0.0175/v_cruise*180/pi,'r','LineWidth',2);%*0.262/v_cruise)
title(strcat('1 degree/s impulse response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(3))], ']'));
axis([0 15 -1 1])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_rc_to_beta', 'epsc');


figure;
```

```
plot(T,impulse(a,b,c(4,:),d(4,:),2,T)*0.0175*180/pi,'r','LineWidth',2);%*0.262*2);
title(strcat('1 degree/s impulse response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(4))], ']'));
%axis([0 max_time -0.3 0.5])
xlabel('t (s)')
ylabel('degree per second')
grid;
saveas(gcf, 'impulse_rc_to_p', 'epsc');


figure;
plot(T,impulse(a,b,c(5,:),d(5,:),2,T)*2*0.0175*180/pi,'r','LineWidth',2);%*0.262*2);
title(strcat('1 degree/s impulse response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [2', char(outputs_st_1(5))], ']'));
%axis([0 max_time -0.05 0.05])
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'impulse_rc_to_r', 'epsc');


figure;
plot(T,impulse(a,b,c(6,:),d(6,:),2,T)*0.0175*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 degree/s impulse response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(6))], ']'));
%axis([0 max_time -0.3 0.5])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_rc_to_phi', 'epsc');


figure;
plot(T,impulse(a,b,c(7,:),d(7,:),2,T)*0.0175*180/pi,'r','LineWidth',2);%*0.262);
title(strcat('1 degree/s impulse response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(7))], ']'));
%axis([0 max_time -0.05 0.05])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'impulse_rc_to_psi', 'epsc');



%--------------------------15 degree STEP IN PSI_C

figure
```

```
plot(T,step(a,b,c(1,:),d(1,:),1,T)*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(1))], ']'));
axis([0 max_time -55 13]);
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_psi_to_del_a', 'epsc');

figure;
plot(T,step(a,b,c(2,:),d(2,:),1,T)*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(2))], ']'));
%axis([0 max_time -0.022 0.1])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_psi_to_del_r', 'epsc');


figure;
plot(T,step(a,b,c(3,:),d(3,:),1,T)*0.262/v_cruise *180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(3))], ']'));
%axis([0 max_time -0.022 0.1])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_psi_to_beta', 'epsc');


figure;
plot(T,step(a,b,c(4,:),d(4,:),1,T)*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(4))], ']'));
%axis([0 max_time -0.5 0.5])
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'step_psi_to_p', 'epsc');


figure;
plot(T,step(a,b,c(5,:),d(5,:),1,T)*2*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [2', char(outputs_st_1(5))], ']'));
%axis([0 max_time -0.022 0.1])
```

```
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'step_psi_to_r', 'epsc');


figure;
plot(T,step(a,b,c(6,:),d(6,:),1,T)*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(6))], ']'));
%axis([0 max_time -0.1 0.6])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_psi_to_phi', 'epsc');


figure;
plot(T,step(a,b,c(7,:),d(7,:),1,T)*0.262*180/pi,'r','LineWidth',2);
title(strcat('15 degrees step response. Input:  [',...
    [char(inputs_st_1(1)), '] Output: [', char(outputs_st_1(7))], ']'));
axis([0 max_time -1 18])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_psi_to_psi', 'epsc');


%-----------------------------------------
%STEP INPUT 1 deg/sec in r_c

figure
plot(T,step(a,b,c(1,:),d(1,:),2,T)*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(1))], ']'));
%axis([0 max_time -0.5 0.5])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_rc_to_dela', 'epsc');


figure;
plot(T,step(a,b,c(2,:),d(2,:),2,T)*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(2))], ']'));
%axis([0 max_time -0.022 0.1])
```

```
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_rc_to_delr', 'epsc');


figure;
plot(T,step(a,b,c(3,:),d(3,:),2,T)*0.0175/v_cruise*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(3))], ']'));
%axis([0 max_time -0.022 0.1])
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_rc_to_beta', 'epsc');


figure;
plot(T,step(a,b,c(4,:),d(4,:),2,T)*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(4))], ']'));
%axis([0 max_time -0.5 0.5])
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'step_rc_to_p', 'epsc');


figure;
plot(T,step(a,b,c(5,:),d(5,:),2,T)*2*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [2', char(outputs_st_1(5))], ']'));
%axis([0 max_time -0.022 0.1])
xlabel('t (s)');
ylabel('degree per second');
grid;
saveas(gcf, 'step_rc_to_r', 'epsc');


figure;
plot(T,step(a,b,c(6,:),d(6,:),2,T)*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input:  [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(6))], ']'));
%axis([0 max_time -0.5 0.5])
xlabel('t (s)');
ylabel('degree');
grid;
```

```
saveas(gcf, 'step_rc_to_phi', 'epsc');


figure;
plot(T,step(a,b,c(7,:),d(7,:),2,T)*0.0175*180/pi,'r','LineWidth',2);
title(strcat('1 degree/s step response. Input: [',...
    [char(inputs_st_1(2)), '] Output: [', char(outputs_st_1(7))], ']'));
%axis([0 max_time -0.022 0.15]);
xlabel('t (s)');
ylabel('degree');
grid;
saveas(gcf, 'step_rc_to_psi', 'epsc');
```

2.5.6 Problem 6

6. Use Simulink to model the speed controller of Fig. 8.5 in the book, reproducing Fig. 8.7. **Note:** with the parameters given in the book, you will not be able to precisely reproduce Fig. 8.7(a); you will be able to reproduce Figs. 8.7 (b) and (c) exactly.

Then use Simulink to model the speed controller of Fig. 8.8, reproducing Figs. 8.12, 8.13, 8.14, and 8.15. **Note:** you determined the elements of the **G** matrix in HW 4, Prob. 4. **Make sure to**

use the Simulink handout on the website for tips and to attach your Simulink schematic. **Note:** to model the $J = 0.005(3s - 1)$ actuator of Figs. 8.8 and 8.11 in Simulink, I used a PID box and I entered values of I=0 and N=10000. This essentially sets the last ratio in Simulink's representation of a PID to 1 and therefore delivers a proportional-derivative block. I am sure there are better ways to do this but I couldn't find any.

Solution:

2.5.6.1 Pitch attitude controller

The following diagram illustrates the system that we need to implement in simulink. It is figure 8.5 in the text, page 266, which is a pitch attitude controller. The three different controllers are implemented in simulink. A scope was used to show the responses in order to reproduce figure 8.7 in the textbook (page 268). These are the resulting plots showing the simulink model used for each.

2.5.6.2 Speed controller

For this part, the speed controller given by figure 8.8, page 270 is implemented in simulink.

Using the exact equations, the aircraft A, B, C, D state space matrices are defined in the Matlab workspace before starting simulink. This was done since a state space control block was used for the aircraft model directly in simulink instead of using transfer functions. This lead to a much simpler model in simulink. The matrices A, B, C, D longitudinal motion are the following

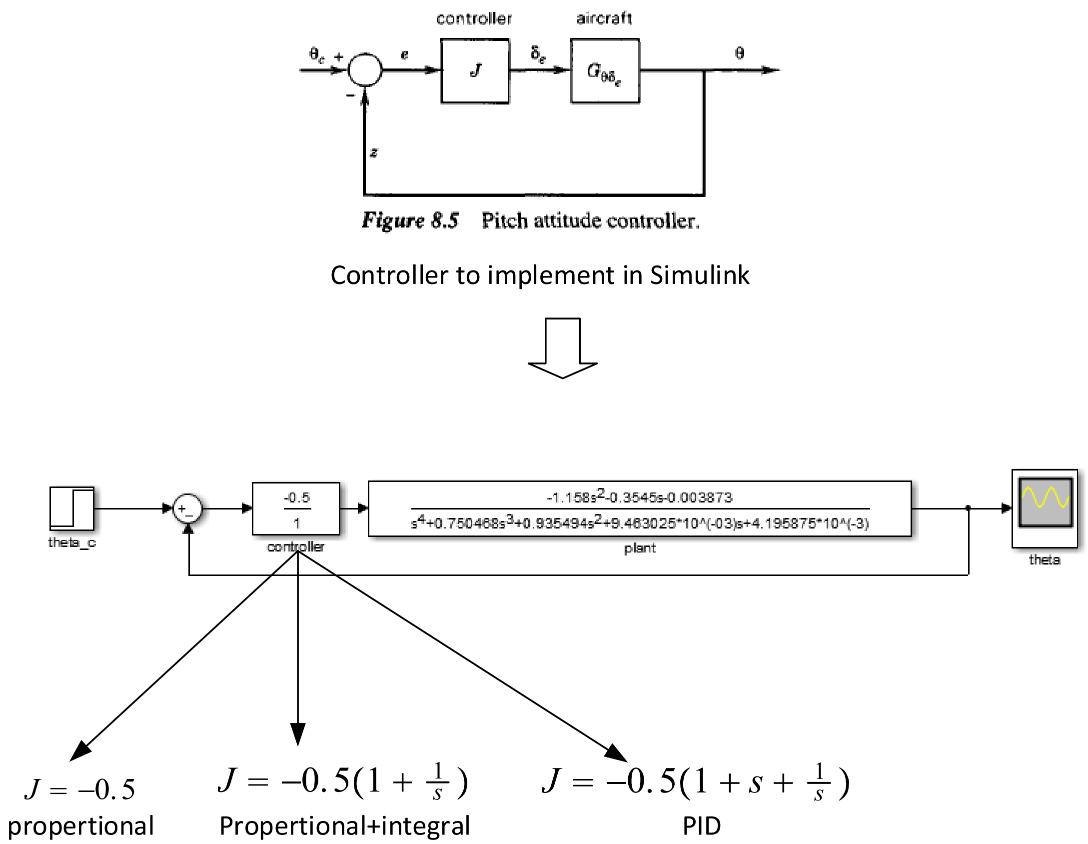

Figure 2.112: pitch attitude controller for problem 6, showing the three type of controllers to implement in simulink

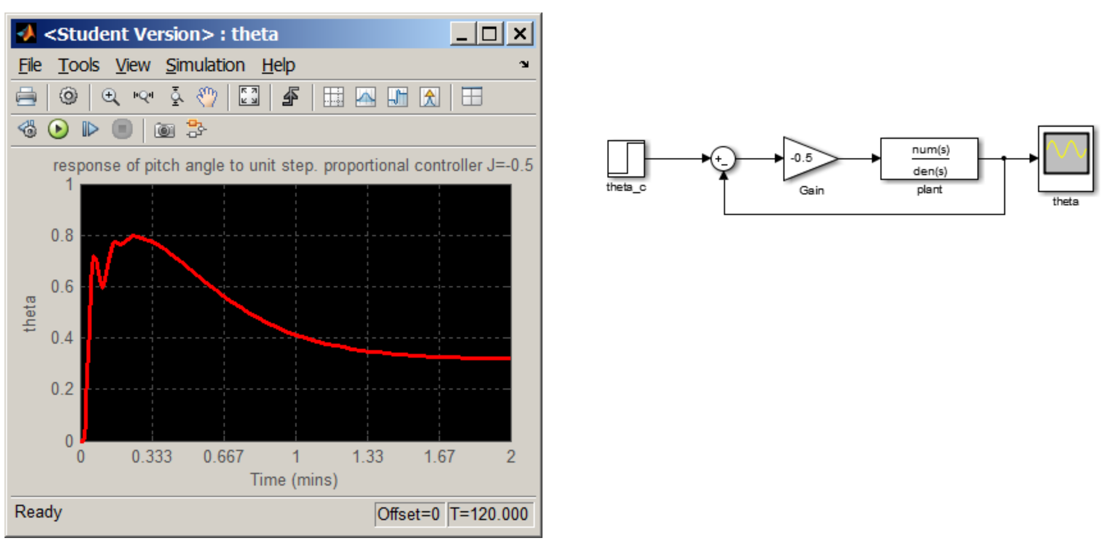

Figure 2.113: producing figure 8.7(a) for problem 6

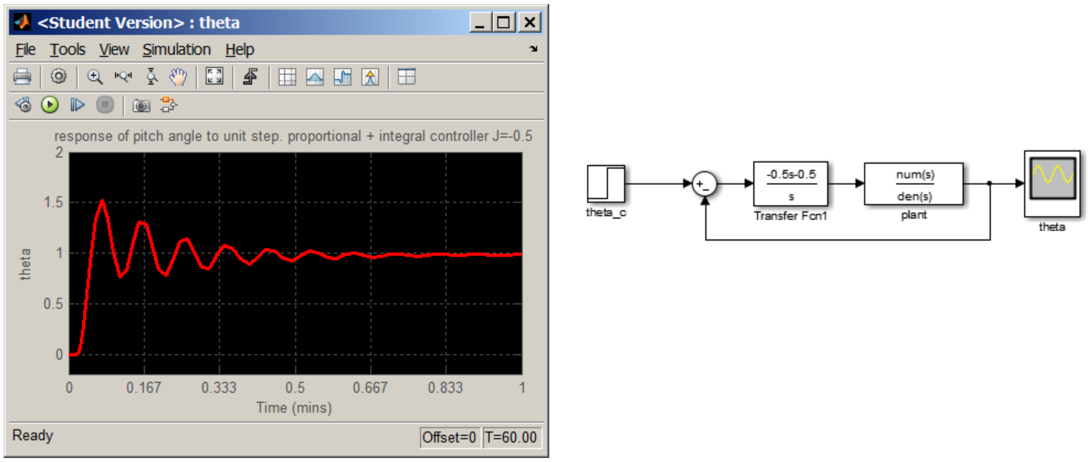

Figure 2.114: producing figure 8.7(b) for problem 6

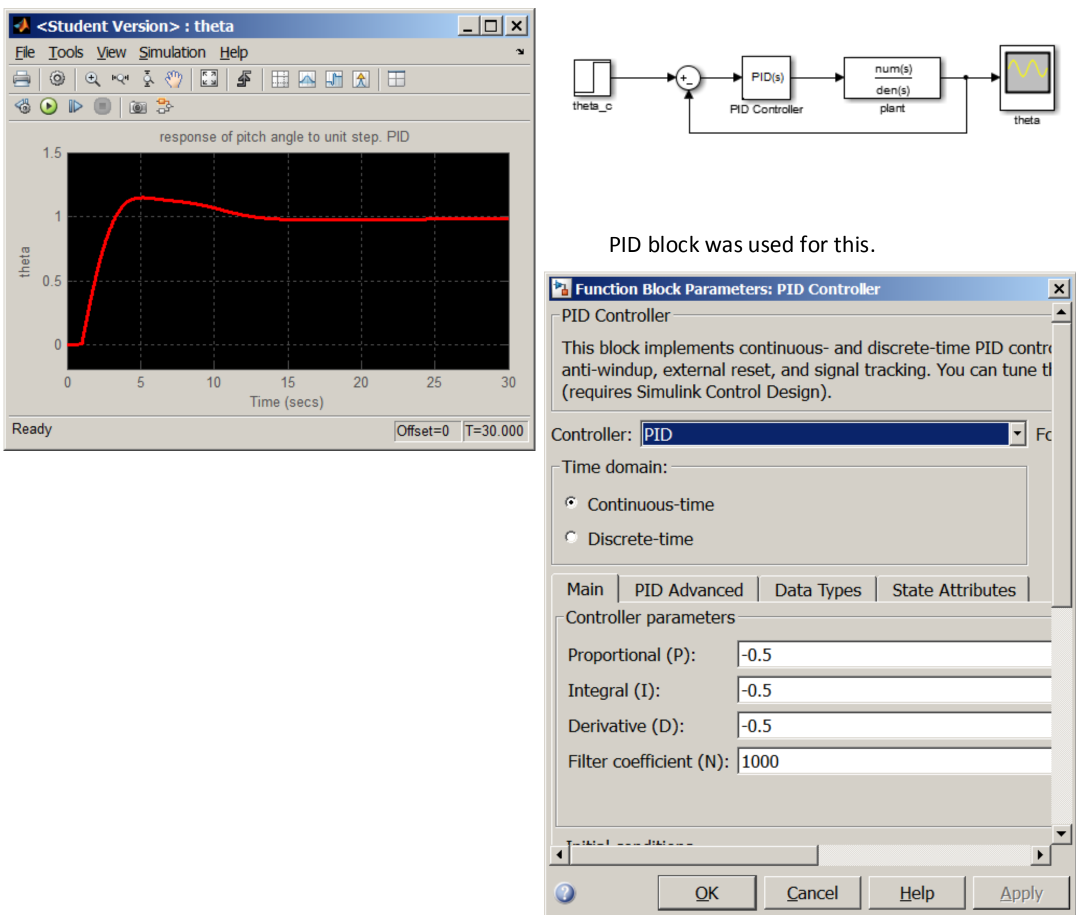

Figure 2.115: producing figure 8.7(c) for problem 6

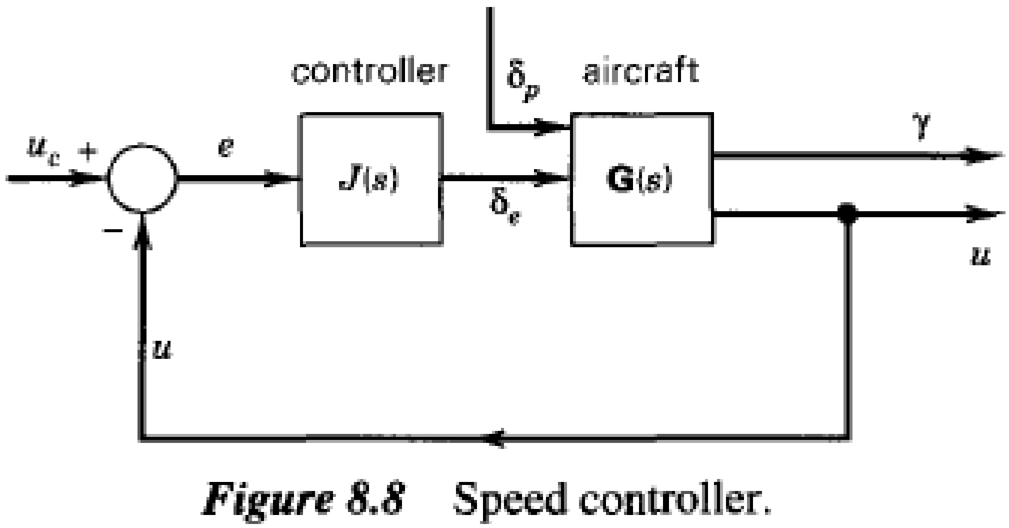

Figure 2.116: figure 8.8, speed controller for problem 6

```
A=[-0.006868   0.01395     0         -32.20;
   -0.09055   -0.3151      773.98     0.0;
   0.0001187  -0.001026   -0.4285     0.0;
   0.0         0.0         1          0.0];

B=[-0.000187 9.66;-17.85 0;-1.158 0; 0 0]

 C=[1 0 0 0;
   0 1 0 0;
   0 0 1 0;
   0 0 0 1]

D=[0 0;
   0 0;
   0 0;
   0 0]
```

2.5.6.3 Generating figure 8.12, speed response

Figure 8.12 was reproduced using controller $J = 0.005(3s+1)$ which was implemented using PID block. $\delta_p = \frac{-1}{6}$ was implemented using step input with amplitude of $\frac{-1}{6}$. The following shows the resulting plot with the model used.

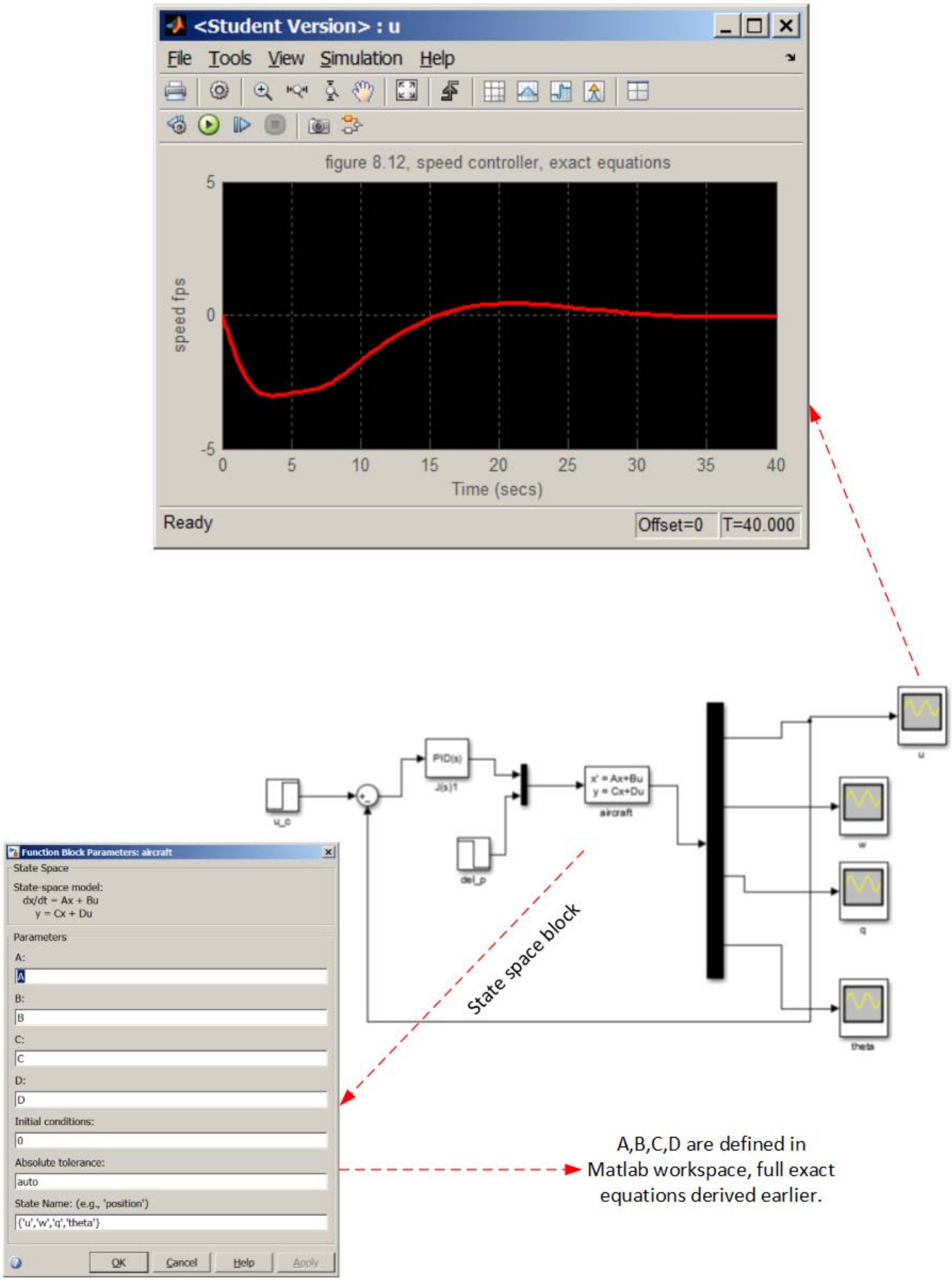

Figure 2.117: producing figure 8.12, speed controller for problem 6

2.5.6.4 Figure 8.13, γ response

To reproduce figure 8.13, we first note that $\gamma = \theta - \alpha$ where α is the angle of attack found from $\alpha = \frac{w}{u_0}$ where $u_0 = 774$ fps (the cruise speed). Therefore, the model was adjusted to find γ according to the above. Here is the simulink model and the figure reproduced.

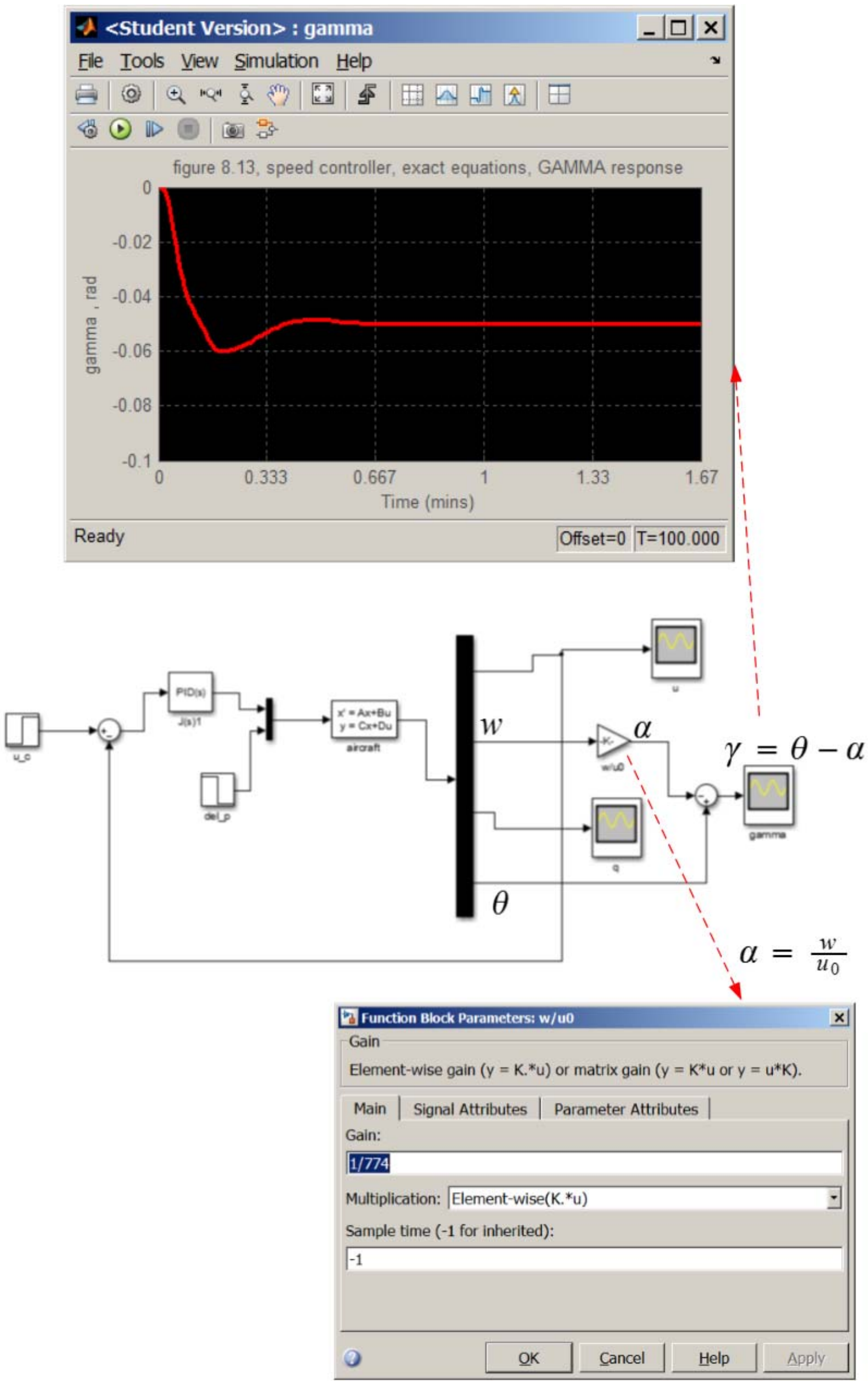

Figure 2.118: producing figure 8.13, speed controller for problem 6

2.5.6.5 Figure 8.14

To produce this figure, a scope as added after the controller to capture the value of the elevator angle δ_e feeding into the aircraft as input. The y axes scale was changed to have units of degrees instead of radians by using a gain block with gain $\frac{180}{\pi}$. Here is the result.

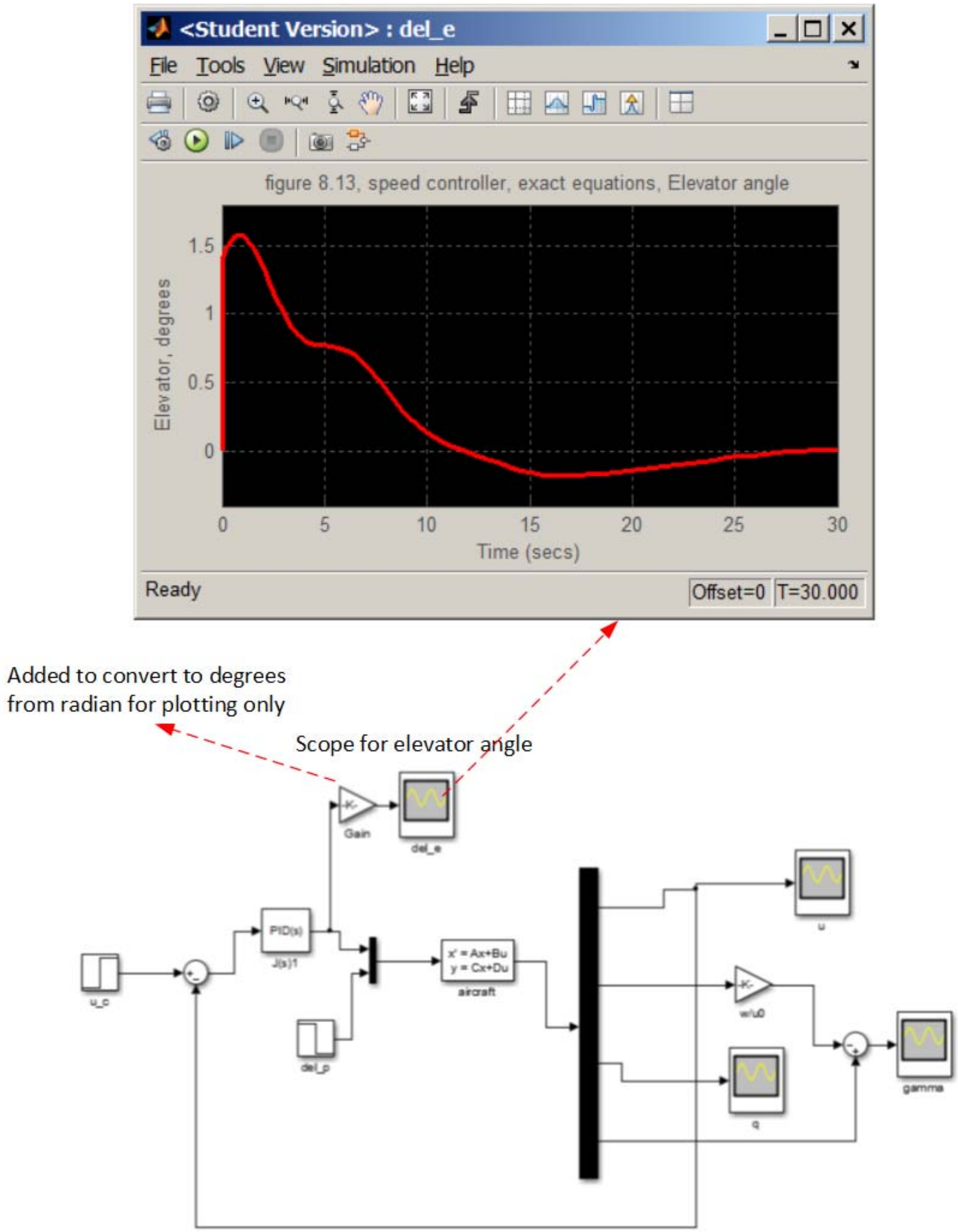

Figure 2.119: producing figure 8.14, speed controller for problem 6

2.5.6.6 Figure 8.15

To produce this figure, which shows the resulting angle of attack, a scope as added after α was calculated using $\alpha = \frac{w}{u_0}$. Here is the result.

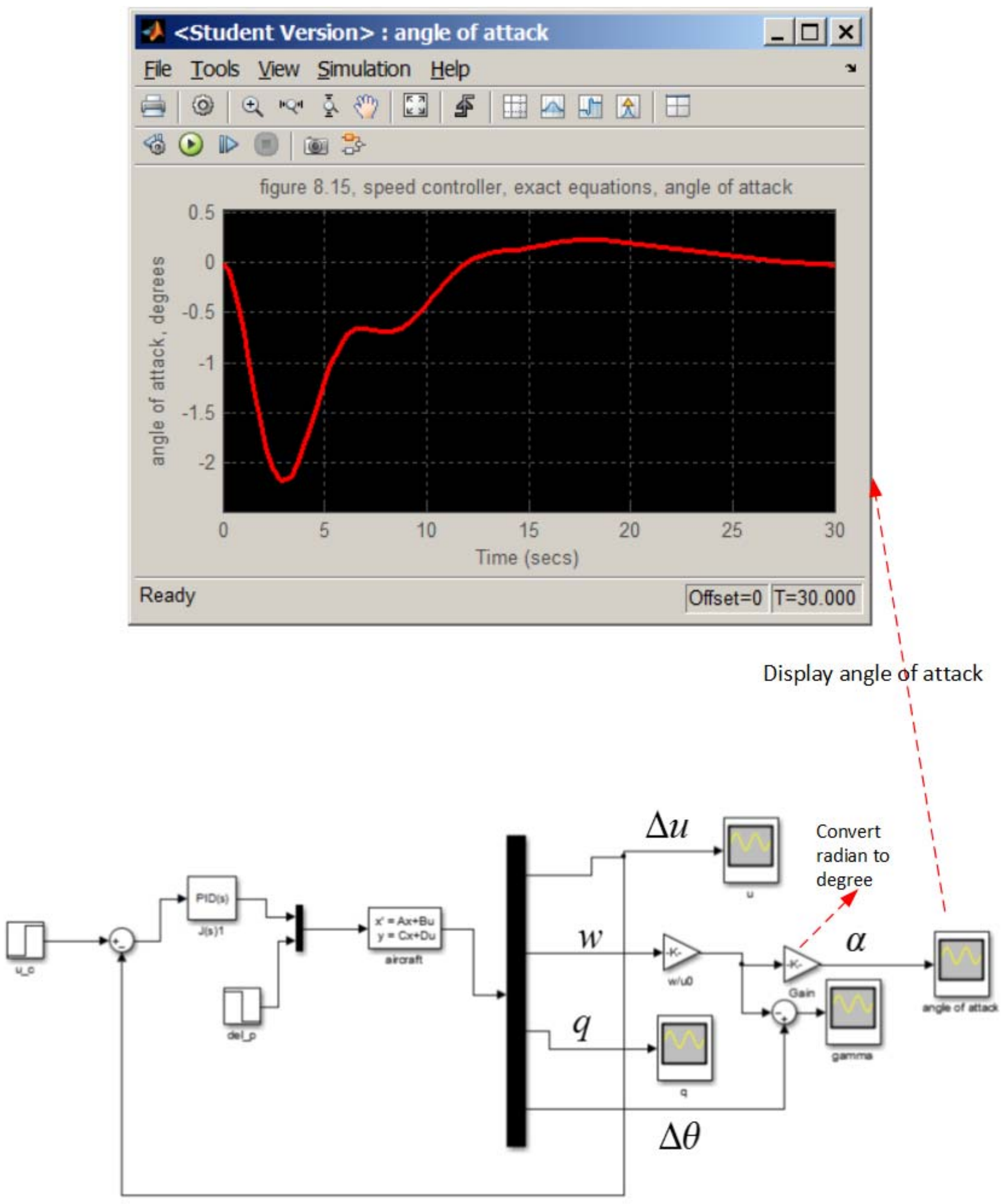

Figure 2.120: producing figure 8.15, speed controller for problem 6