University Course

CEE 744
Advanced Structural Dynamics

University of Wisconsin, Madison
spring 2013

My Class Notes

Nasser M. Abbasi

spring 2013

Contents

Chapter 1

Introduction

Local contents

1.1 syllabus

DYNAMICS OF STRUCTURES

CEE 744

Description:

3 credits, Introduction to basics of dynamics: lumped mass dynamics with various loading functions to develop the dynamic equations of motion, dynamics of multi-degree of freedom systems, dynamic analysis of structural systems, introduction to earthquake engineering,

Reference:

Dynamics of Structures, Anil K. Chopra, Prentice-Hall

Tentative course schedule:

(We start by looking at dynamics of a lump of mass - because actual buildings are generally modelled for structural analysis by using discreet degrees of freedom. -In CEE440 or in programs such asRISA, STAAD-II or SAP and ETABS a structure is modelled by defining “nodes” and “elements”. The nodes have discreet degrees of freedom in movement and the elements are like springs that connect nodes together. *Each single degree of freedom can be considered as a “lump of mass” that has a unique degree of freedom in movement*.)

Dynamic equilibrium of unloaded lumped mass

Dynamics of lumped mass under harmonic loading

Dynamics of lumped mass under random dynamic loadings

(Now with the equations developed for solving how a lump of mass will move - we consider an entire structure to just be a bunch of lumps of mass connected together by spring elements. For each lump we have the equations describing how it moves. *We can combine those equations to define how the whole structure moves.)*

Multi-degree of freedom systems - equations of motion

Analysis of structural systems - natural frequencies
response under loading

Introduction to earthquake engineering

CLASS GUIDE

Problems:

Completed home problems are a required part of this course. Problem due date will vary depending on the length of the assignment. **The due dates will be absolutely definite - 50% maximum credit for late assignments!** *All problems must be completed by the end of the semester.*

Problem layout: I have to read your assignments so I like to see them in a form which is easy to follow.

1. Reserve the right margin of each sheet for comments to me describing what you are trying to do. (Similar to providing comments in a computer program.)

2. Results of particular steps or conclusions should be boxed to set them off from regular calculations.

3. Use only 8-1/2 by 11 paper, lined, unlined or the green structural grid paper.

Reading:

I generally cover material in lectures which I feel is the material you should understand. Many of the lectures will be based on material that is very well presented in the reference and you should consult references for detailed further information if desired. Many lectures will be based on material which is not in the reference.

Grading:

Grading will be proportionally based upon the following:

Final exam...30%
homework..70%

Text:

The following chapters/sections or topics will be studied from the Chopra reference.

Chapter 1: introduction and equation of motion
Chapter 2: response in free vibration
Chapter 3: response to harmonic loading, Parts A and B, response to periodic loads, Part D
Chapter 4: response to impulse, pulse
Chapter 5: time stepping methods
Chapter 6: response spectrum concept
Chapter 8: generalized systems
Chapter 9: multi-dof equation of motion
Chapter 10: natural freqs and mode shapes
Chapter 11: modal analysis

1.2 Links

1. public course web page
2. internal course web page

Chapter 2

HWs

Local contents

2.1 HW1

Local contents

2.1.1 Problem description

CEE 744

Homework assn #1

Use the data from the measured plastic beam free vibration response to estimate the natural frequency (cycles/second) and damping of the beam The data is in the file titled: “free vibr.txt”.

Format of data:

- first column is date of test
- second column is time of test
- third column is output of accelerometer in Volts

To convert the data (volts) to accelerations you need to divide by the calibration factor of 0.500volts/g. That will provide output in acceleration as a fraction of g. For output in in/sec^2 multiply by $386in/sec^2/g$.

Find:

1. natural frequency in cycles/sec, list the number of peaks used,
2. damping based on two successive peaks using the first formula based on the log of the ratio of the peaks,
3. damping based on two successive peaks using the formula derived from the series expansion of the exponential,
4. the damping based more than 2 successive peaks, using the final formula with an interval of “m” peaks.

DUE: in class on Thursday, Feb 7

2.1.2 Solution

HW1 CEE 744 Spring 2013

Nasser M. Abbasi, Feb 2, 2013

Reading the data from file and ready it for processing

In[9]:=
```
SetDirectory[NotebookDirectory[]];
Clear[data, yy, y, t];
data = Import["free_vibr.txt", "Elements"]
```
Out[11]=
```
{Data, Lines, Plaintext, String, Words}
```

In[12]:=
```
data = StringSplit[Import["free_vibr.txt", "Lines"]];
Dimensions[data]
```
Out[13]=
```
{8192, 3}
```

Show 3 lines of data

In[14]:=
```
data[[1 ;; 3]] // TableForm
```
Out[14]//TableForm=
```
1/22/2013    12:52:00.987959    -1.171216E-1
1/22/2013    12:52:00.988936    -1.152905E-1
1/22/2013    12:52:00.989912    -1.183423E-1
```

pull out the time and the voltage columns

In[15]:=
```
filteredData = Transpose[{ToExpression[Part[StringSplit[#, ":"], 3]] & /@ data[[All, 2]],
    Internal`StringToDouble[#] & /@ data[[All, 3]]}];
Dimensions[filteredData]
```
Out[16]=
```
{8192, 2}
```

Show 3 lines of the above result

In[17]:=
```
filteredData[[1 ;; 3]] // TableForm
```
Out[17]//TableForm=
```
0.987959    -0.117122
0.988936    -0.115291
0.989912    -0.118342
```

Filter the data

Make the data start at zero

In[18]:=
```
filteredData[[All, 1]] = filteredData[[All, 1]] - filteredData[[1, 1]];
```

Normalize the data by subtracting the mean

In[19]:=
```
mean = Mean[filteredData[[All, 2]]]
```
Out[19]=
```
-0.00600079
```

In[20]:=
```
filteredData[[All, 2]] = filteredData[[All, 2]] - mean;
```

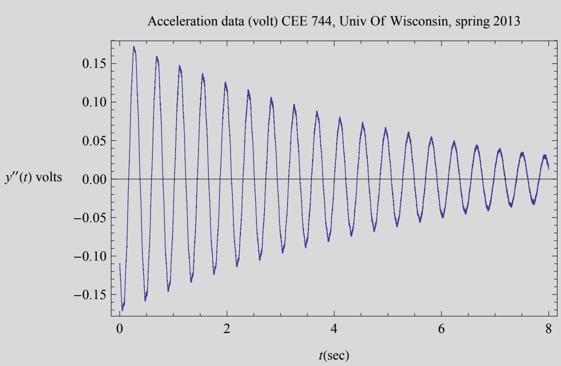
In[21]:=
```
filteredData[[1 ;; 3]] // TableForm
```
Out[21]//TableForm=

0.	-0.111121
0.000977	-0.10929
0.001953	-0.112342

Plot the data before analysis

first in raw data as volts

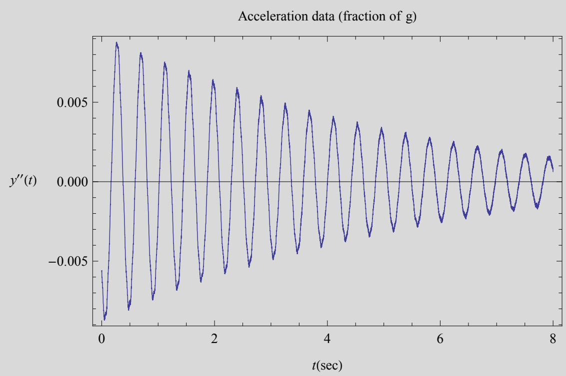
In[22]:=
```
ListLinePlot[filteredData, Frame → True,
 FrameLabel → {{Row[{y''[t], " volts"}], None}, {Row[{t, "(sec)"}],
    "Acceleration data (volt) CEE 744, Univ Of Wisconsin, spring 2013"}},
 RotateLabel → False, GridLines → Automatic, GridLinesStyle → LightGray]
```
Out[22]=



Convert to fractions of g

In[23]:=
```
filteredData[[All, 2]] = filteredData[[All, 2]] * 0.5 / 9.81;
```

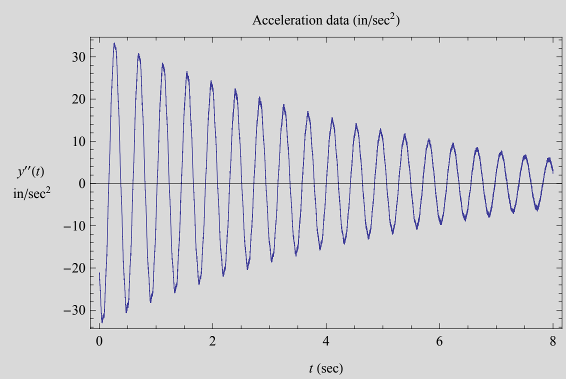
In[24]:=
```
ListLinePlot[filteredData, Frame → True, FrameLabel →
  {{y''[t], None}, {Row[{t, "(sec)"}], "Acceleration data (fraction of g)"}},
 RotateLabel → False, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[24]=


Convert to inches per second^2

In[25]:=
```
filteredData[[All, 2]] = filteredData[[All, 2]] * 386 * 9.81;
```

In[26]:=
```
ListLinePlot[filteredData, Frame → True,
 FrameLabel → {{Column[{y''[t], " in/sec²"}, Alignment → Center], None},
   {Row[{t, " (sec)"}], "Acceleration data (in/sec²)"}},
 RotateLabel → False, GridLines → Automatic, GridLinesStyle → LightGray]
```

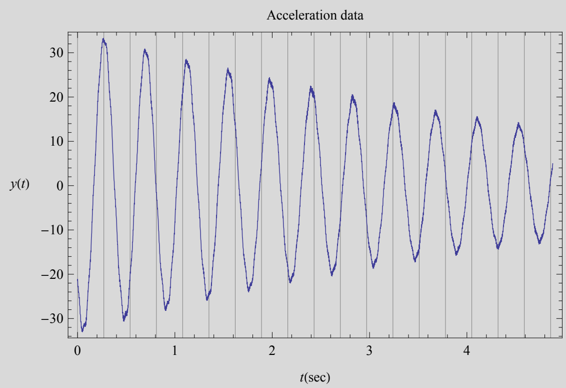
Out[26]=


Plot few values (3 seconds)

In[27]:=

```
ListLinePlot[filteredData[[1 ;; 5000]], Frame → True,
 FrameLabel → {{y[t], None}, {Row[{t, "(sec)"}], "Acceleration data"}},
 RotateLabel → False, GridLines → {Range[0, 5, .27], None},
 GridLinesStyle → Gray, Axes → None]
```

Out[27]=



Find the natural frequency

From the above plot, Using the first 8 peaks, the period is **11*0.27=2.97/7 =0.424 seconds**. Hence the frequency is 2.35 Hz

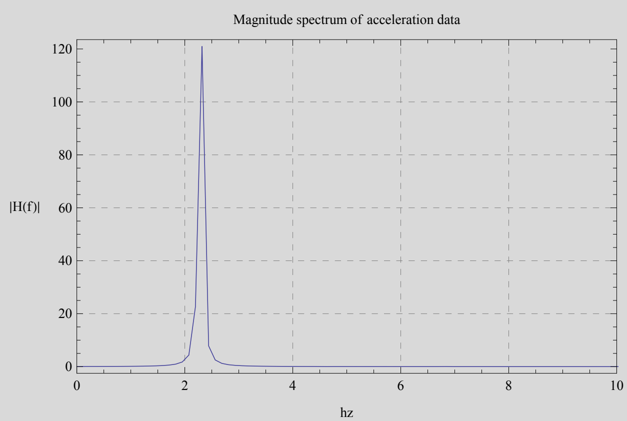
Finding the natural frequency using Fourier transform to obtain the spectrum

In[28]:=

```
py = Fourier[filteredData[[All, 2]] , FourierParameters → {1, -1}];
nSamples = Length[filteredData[[All, 2]]];
nUniquePts = Ceiling[(nSamples + 1) / 2];
py = py[[1 ;; nUniquePts]];
py = Abs[py];
py = py / nSamples;
py = py^2;

If[OddQ[nSamples], py[[2 ;; -1]] = 2 * py[[2 ;; -1]], py[[2 ;; -2]] = 2 * py[[2 ;; -2]]];
fs = 1000;
f = N[(Range[0, nUniquePts - 1] fs) / nSamples];
ListPlot[Transpose[{f, py}], Joined → True,
 FrameLabel → {{"|H(f)|", None}, {"hz", "Magnitude spectrum of acceleration data"}},
 ImageSize → 400, Frame → True, RotateLabel → False, GridLines → Automatic,
 GridLinesStyle → Dashed, PlotRange → {{0, 10}, All}]
```

Out[38]=



We see from the above that $f = 2.3$ cycles per second. Here is a zoom in view

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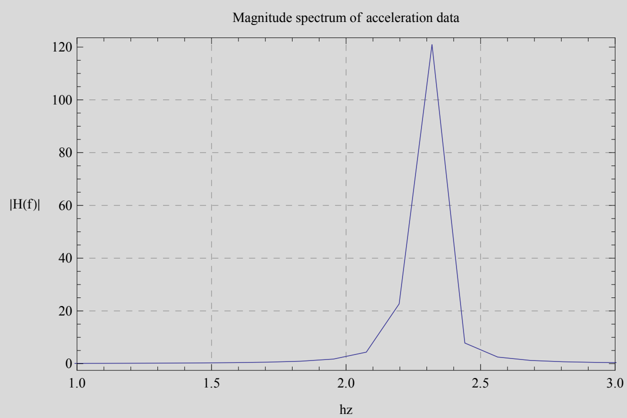
In[39]:=

```
ListPlot[Transpose[{f, py}], Joined → True,
 FrameLabel → {{"|H(f)|", None}, {"hz", "Magnitude spectrum of acceleration data"}},
 ImageSize → 400, Frame → True, RotateLabel → False, GridLines → Automatic,
 GridLinesStyle → Dashed, PlotRange → {{1, 3}, All}]
```

Out[39]=



We see that the **above result matches that we obtained by counting the peaks from the plot directly**. But using the spectrum would be a better method to use.

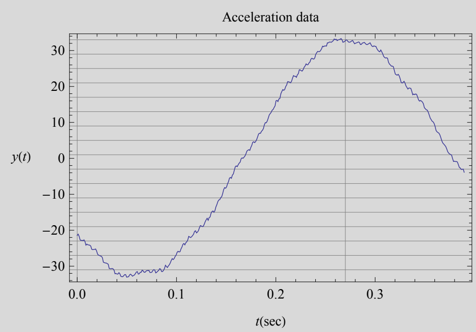
Finding the damping ζ

We first need to generate a list of say 10 peak values of y''(t) and the corresponding time. From the plot we see that the first positive peak is located at time 0.27 seconds. Hence we start from that point and look for a value at each sample point that is 1/f away from it. The data is available such that the separation in time between each data point is one milliseconds. First here is the plot showing the initial phase

In[40]:=

```
to = 400;
ListLinePlot[filteredData[[1 ;; to]], Frame → True,
 FrameLabel → {{y[t], None}, {Row[{t, "(sec)"}], "Acceleration data"}},
 RotateLabel → False, GridLines → {Range[0, to / 1000, .27], Range[-35, 35, 4]},
 GridLinesStyle → Gray, Axes → None, ImageSize → 300]
```

Out[41]=



Here is a list of the first 10 peaks

In[42]:=

```
period = 1 / 2.3;
initial = 0.27;
scale = 1000; (*one sample per millisecond*)
peaks = Table[ Flatten[
    {n + 1, Part[filteredData[[ Round[(initial + n * period) * scale]]]]}], {n, 0, 9}];
TableForm[peaks, TableHeadings → {None, {"peak #", "time", "peak"}}]
```

Out[46]//TableForm=

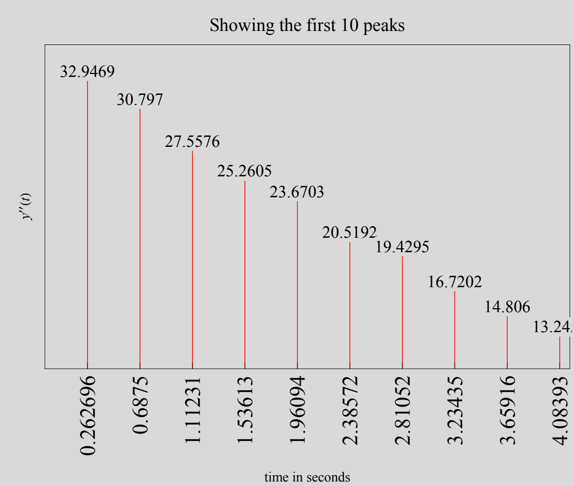
peak #	time	peak
1	0.262696	32.9469
2	0.6875	30.797
3	1.11231	27.5576
4	1.53613	25.2605
5	1.96094	23.6703
6	2.38572	20.5192
7	2.81052	19.4295
8	3.23435	16.7202
9	3.65916	14.806
10	4.08393	13.2452

Plot the above peaks to verify

In[47]:=

```
ticks = {{None, None}, {{#, Style[Rotate[#, 90 Degree], 14]} & /@peaks[[ All, 2]], None}};
ListPlot[peaks[[ All, {2, 3}]], Filling → Axis, FillingStyle → Red,
 Frame → True, FrameTicks → ticks, ImageMargins → 30, Epilog → MapThread[
   Text[Style[#2, 11], {#1, #2}] &, {peaks[[ All, 2]], peaks[[ All, 3]]}], FrameLabel →
  {{y''[t], None}, {"time in seconds", Style["Showing the first 10 peaks", 12]}},
 PlotRange → {Automatic, {10, 35}}]
```

Out[48]=



Damping based on two successive peaks using the first formula

The formula to use here is $\ln\left(\frac{y_1}{y_2}\right) = 2\pi\,\xi$. Therefore, using the first 2 values we found above we obtain

In[49]:=

```
y1 = peaks[[ 1, 3]];
y2 = peaks[[ 2, 3]];
ξ = Log[y1/y2] / (2 π)
```

Out[51]=

```
0.0107394
```

Hence this shows that $\boxed{\xi = 1.074\,\%}$

Damping based on two successive peaks using the series expansion

The formula to use here is $\frac{y_1}{y_2} = 1 + 2\pi\xi$ therefore using the first 2 peaks we obtain

In[55]:=
```
y1 = peaks[[1, 3]];
y2 = peaks[[2, 3]];
ξ = 1/(2 π) (y1 - y2)/y2
```

Out[57]=
```
0.01111
```

This shows that $\boxed{\xi = 1.111\,\%}$

Damping based more than 2 successive peaks, using the final formula with an interval of "m" peaks

Here we use the formula $\frac{y_1}{y_{1+m}} = 1 + 2\pi m\zeta$ where m is a number we can change. Using $m = 5$ for example gives

In[58]:=
```
y1 = peaks[[1, 3]];
m = 5;
y2 = peaks[[1 + m, 3]];
ξ = 1/(2 m π) (y1/y2 - 1)
```

Out[61]=
```
0.0192788
```

Hence using $m = 5$ gives $\boxed{\xi = 1.93\,\%}$

Trying for $m = 9$ gives

In[62]:=
```
y1 = peaks[[1, 3]];
m = 9;
y2 = peaks[[1 + m, 3]];
ξ = (y1/y2 - 1)/(2 m π)
```

Out[65]=
```
0.0263042
```

Hence using $m = 9$ gives $\boxed{\xi = 2.63\,\%}$

Finding number of cycles to have the amplitude decay by 1/2

Using $\ln(2) = 2\, m\, \pi\, \zeta\, \frac{\omega}{\omega_d}$ we can estimate m the number of cycles for the amplitude to decay by half. We use ξ=0.0107394 from above since that is the ξ value found from the same formula. Hence

In[66]:=

```
ξ = 0.0107394;
m = Log[2] Sqrt[1 - ξ^2] / (2 π ξ)
```

Out[67]=

```
10.2717
```

This shows that it takes 10 cycles for the amplitude to decay by half. Looking again at the plots, this is verified

Applet to analyze the data allowing different formulas to be selected and different values for M

This is a small applet to help analyze this data. It allows you to select the formula to determine ξ and also select m for the final formula. For each formula used, the corresponding value of number of cycles for the first peak to decay by half is computed.

```
Manipulate[
 Module[{dataPlot},

  dataPlot = ListPlot[
    filteredData[[1 ;; tscale]],
    Joined → True,
    Frame → True,
    FrameLabel → {
      {None, None}, {Row[{t, " (sec)"}], "Acceleration data y''(t) (in/sec^2)"}},
    GridLines → Automatic,
    GridLinesStyle → LightGray,
    ImageSize → {250},
    ImageMargins → 0,
    ImagePadding → {{20, 5}, {40, 20}}];

  Grid[{
    {Row[{ξ, " = ", padIt2[ 100 * findZeta[formula, mm], {5, 4}], " %"}],
     Row[{"frequency ", " = ", 2.3, " Hz"}]},
    {dataPlot, spectrum},
    {peaksPlot, tbl}
   }, Frame → All, Alignment → Center]
 ],

 Grid[
  {
   {
```

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```
    Row[{Style["m ", 12],
      Manipulator[Dynamic[mm, {mm = #} &], {1, 8, 1}, ImageSize → Tiny],
      Style[Dynamic@padIt2[mm, 1], 11]
     }], SpanFromLeft
   },

   {
    Row[{Style["ξ formula", 11],
      PopupMenu[Dynamic[formula, {formula = #} &],
       {
        1 → Row[{Style["first method ", Bold],
           Style[TraditionalForm[Log[y_m/y_(m+1)] == 2 ξ π], 10]}],
        2 → Row[{Style["series method ", Bold], Style[
            TraditionalForm[y_m/y_(m+1) == 1 + 2 ξ π], 10]}],
        3 → Row[{Style["m method ", Bold], Style[TraditionalForm[
             y_1/y_(1+m) == 1 + 2 ξ m π], 10]}],
       }
       , ImageSize -> All
      ]
     }
    ],
    ,
    Row[{Style["time scale ", 12],
      Manipulator[Dynamic[tscale, {tscale = #} &],
       {1, 8192, 1}, ImageSize → Tiny], Spacer[5],
      Style[Dynamic@padIt2[tscale, 4], 11], Spacer[5], "ms"
     }]
   }
  }, Alignment → Left
 ],

 {{mm, 1}, None},
 {{formula, 3}, None},
 {{tscale, 4000}, None},
 SynchronousUpdating → True,
 ControlPlacement → Top,
 Alignment → Center,
 SynchronousInitialization → True,
 ContinuousAction → True,
 AutorunSequencing → Automatic,
 TrackedSymbols → {mm, formula, tscale},
```

```
Initialization :→ {

  SetDirectory[NotebookDirectory[]];
  data = Import["free_vibr.txt", "Elements"];
  data = StringSplit[Import["free_vibr.txt", "Lines"]];

  filteredData =
   Transpose[{ToExpression[Part[StringSplit[#, ":"], 3]] & /@ data[[All, 2]],
     Internal`StringToDouble[#] & /@ data[[All, 3]]}];

  filteredData[[All, 1]] = filteredData[[All, 1]] - filteredData[[1, 1]];
  mean = Mean[filteredData[[All, 2]]];
  filteredData[[All, 2]] = filteredData[[All, 2]] - mean;
  filteredData[[All, 2]] = filteredData[[All, 2]] * 0.5 / 9.81;
  filteredData[[All, 2]] = filteredData[[All, 2]] * 386 * 9.81;

  py = Fourier[filteredData[[All, 2]] , FourierParameters → {1, -1}];
  nSamples = Length[filteredData[[All, 2]]];
  nUniquePts = Ceiling[(nSamples + 1) / 2];
  py = py[[1 ;; nUniquePts]];
  py = Abs[py];
  py = py / nSamples;
  py = py^2;

  If[OddQ[nSamples],
   py[[2 ;; -1]] = 2 * py[[2 ;; -1]], py[[2 ;; -2]] = 2 * py[[2 ;; -2]]];
  fs = 1000;
  f = N[(Range[0, nUniquePts - 1] fs) / nSamples];

  spectrum = ListPlot[Transpose[{f, py}], Joined → True, FrameLabel →
     {{"|H(f)|", None}, {"hz", "Magnitude spectrum of acceleration data"}},
    Frame → True, RotateLabel → False, GridLines → Automatic, GridLinesStyle → Dashed,
    PlotRange → {{0, 10}, All}, ImageSize → {250}, ImageMargins → 0];


  period = 1 / 2.3;
  initial = 0.27;
  scale = 1000;  (*one sample per millisecond*)
  peaks = Table[  Flatten[
     {n + 1, Part[filteredData[[ Round[(initial + n * period) * scale]]]]}], {n, 0, 9}];
  tbl = TableForm[peaks[[1 ;; 10]], TableHeadings → {None, {"#", "time", "peak"}}];

  ticks = {{None, None}, {{#, Style[Rotate[padIt2[#, {3, 2}], 90 Degree], 14]} & /@
      peaks[[ 1 ;; 10, 2]], None}};
  peaksPlot = ListPlot[peaks[[ 1 ;; 10, {2, 3}]],
    Filling → Axis,
    FillingStyle → Red,
    Frame → True,
    FrameTicks → ticks,
    ImageMargins → 0,
    Epilog → MapThread[Text[Style[padIt2[#2, {3, 1}], 10], {#1, #2}, {0, -1}] &,
      {peaks[[ 1 ;; 10, 2]], peaks[[ 1 ;; 10, 3]]}], FrameLabel →
```

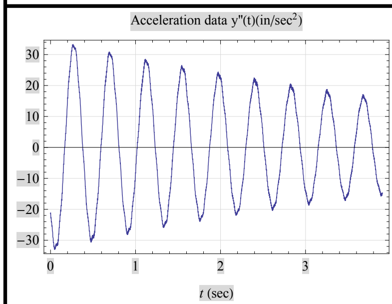
```
    {{y''[t], None}, {"time in seconds", Style["Showing the first 10 peaks", 12]}},
   PlotRange → {Automatic, {10, 39}},
   ImageSize → {250},
   ImagePadding → {{20, 5}, {50, 20}}];

 (*--------------------------------------------*)
 padIt1[v_, f_List] := AccountingForm[Chop[v],
   f, NumberSigns → {"-", "+"}, NumberPadding → {"0", "0"}, SignPadding → True];
 (*--------------------------------------------*)
 padIt2[v_, f_List] := AccountingForm[Chop[v],
   f, NumberSigns → {"", ""}, NumberPadding → {"0", "0"}, SignPadding → True];
 (*--------------------------------------------*)
 padIt2[v_, f_Integer] := AccountingForm[Chop[v],
   f, NumberSigns → {"", ""}, NumberPadding → {"0", "0"}, SignPadding → True];

 findZeta[formula_, m_] := Module[{y1, y2},
   Which[formula == 1,
    y1 = peaks[[ m, 3]];
    y2 = peaks[[ m + 1, 3]];
    Log[y1/y2]/(2 π),
    formula == 2,
    y1 = peaks[[ m, 3]];
    y2 = peaks[[ m + 1, 3]];
    1/(2 π) (y1 - y2)/y2,
    formula == 3,
    y1 = peaks[[ 1, 3]];
    y2 = peaks[[ 1 + m, 3]];
    1/(2 m π) (y1/y2 - 1)
   ]
  ];

 }
]
```

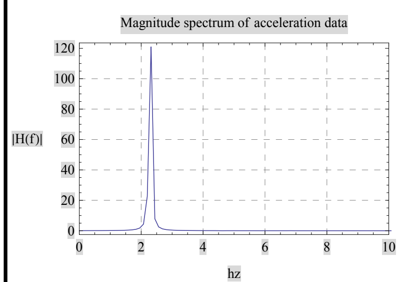
m 1

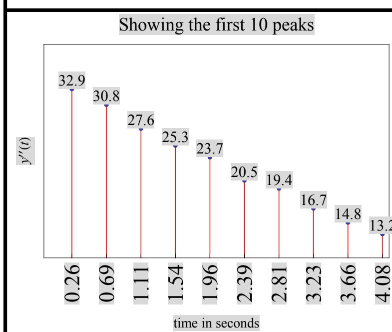
ξ formula **m method** $\frac{y_1}{y_{11.2717}} = 1.69311$ time scale 4000 ms

Out[68]=

0.0107394 = 1.1110 %	frequency = 2.3 Hz







#	time	peak
1	0.262696	32.9469
2	0.6875	30.797
3	1.11231	27.5576
4	1.53613	25.2605
5	1.96094	23.6703
6	2.38572	20.5192
7	2.81052	19.4295
8	3.23435	16.7202
9	3.65916	14.806
10	4.08393	13.2452

2.1.3 Mathematica applet

To run the applet, the following zip file needs to be downloaded to your PC. Then extracting it will make a folder with the applet file in it along with a text file that contains the acceleration data. Now just double click on the .CDF file there and this will start the applet.

You might see a warning message at the top first time you start the applet. Simply click on the button at the top right corner to enable dynamics. That is all. This message is harmless. Here is the zip file `HWs/HW1/CEE_744_applet.zip`

Here is the data file `HWs/HW1/free_vibr.txt`

2.2 HW2 Generalized single degree of freedom system applied to wind tower

1. Excel file that contains the final result table `turbine_tower_RESULT.xlsx`
2. Mathematica simulation using CDF is available on this web page. The demo is titled `Generalized Single Degree Of Freedom Method` (you can search for it on the page since its link can change with time)

This is the original Excel file used to load data from `HWs/HW2/turbine_tower_prob_ORIGINAL.xlsx`

2.2.1 Problem description

Using different shape functions an estimate of the natural frequency for the wind tower was found using the method of generalized single degree of freedom for each method.

The following table summarizes the results obtained. For each shape function the following items are calculated: Effective mass M_e, effective stiffness $K_e = K_{fe} + K_{ge}$,effective flexural stiffness K_{fe}, effective geometric stiffness K_{ge}, The ratio $\frac{M}{M_e}$and the natural frequency f in Hz.

The rows of the table below are listed from the lowest to the largest natural frequency found.

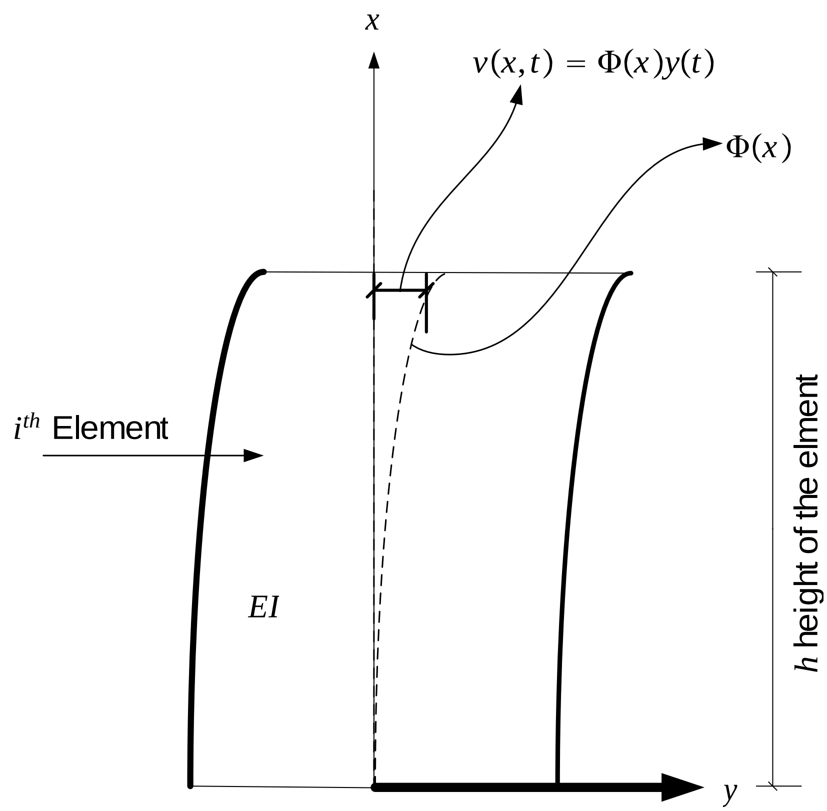
The shape function that produces the lowest natural frequency will be the one to select as the closest approximation to the real solution. The actual mass is 404171 Kg.

An Excel worksheet is also available on my web page for this HW for the lowest natural frequency case.

shape function $\Phi(x)$	M_e kg)	K_e	Flexural K_e (N/m)	Geometric K_{ge} (N/m)	$\frac{M_e}{M}$	f (H
$\frac{x^2}{L^2}$	$159,636$	$383,031$	$393,520$	-10489	39.49%	0.246
$1-\cos\left(\frac{\pi x}{2L}\right)$	$164,157$	$431,388$	$441,587$	-10198	40.62	0.258
$\frac{2Lx^2-x^3}{2L^3}$	$165,830$	$472,453$	$482,548$	-10095	41.03	0.268
first mode	$168,445$	$543,282$	$553,333$	-10051	41.68	0.285
$\frac{6L^2x^2-4Lx^3+x^4}{3L^4}$	$169,764$	$595,562$	$605,586$	-10024	42	0.298
2nd mode	$185,852$	$14,443,032$	$14,509,551$	-66519	45.98	1.403
3rd mode	$192,575$	$100,304,976$	$100,475,002$	-170026	47.65	3.632
4th mode	$195,562$	$371,956,138$	$372,284,973$	-328835	48.386	6.941

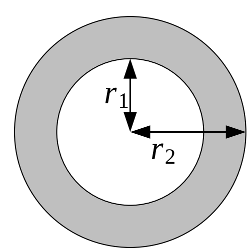
The shape functions above indicated by the mode, are the mode shape function for a beam with fixed-free boundary conditions obtained from table 8.1 from reference [1].

The following diagram describes the computation done at each element of the wind tower



The effective flexural stiffness is $K_{fe} = \sum_i EI_i M_i \theta_i$

Where θ_i angle increment given by $\Phi_i'' h$
M_i is the bending moment given by $\Phi_i'' y_i$
Where $I_i = \frac{\pi}{4}(r_2^4 - r_1^4)$
Hence $K_{fe} = \sum_i EI_i (\Phi_i'')^2 h$



Effective mass $M_e = \sum_i m_i \Phi_i^2$

And effective geometric stiffness is $K_{ge} = \sum_i (\Phi_i')^2 \bar{m}_i g h$

where $\bar{m}$ is the accumlative mass from all the top elements
and g is 9.81 meter/sec^2

2.2.2 Conclusions

The lowest approximate natural frequency found is $\boxed{0.2465 \text{ Hz}}$ for the shape function $\frac{x^2}{L^2}$. The effective mass to actual mass ratio for this case was 39.49%

The higher the natural frequency became as the shape function is changed, this ratio also increased. At $f = 6.941$ Hz, this ratio became almost 50%.

An applet was written to simulate the result allowing one to select different shape functions and observe the result.

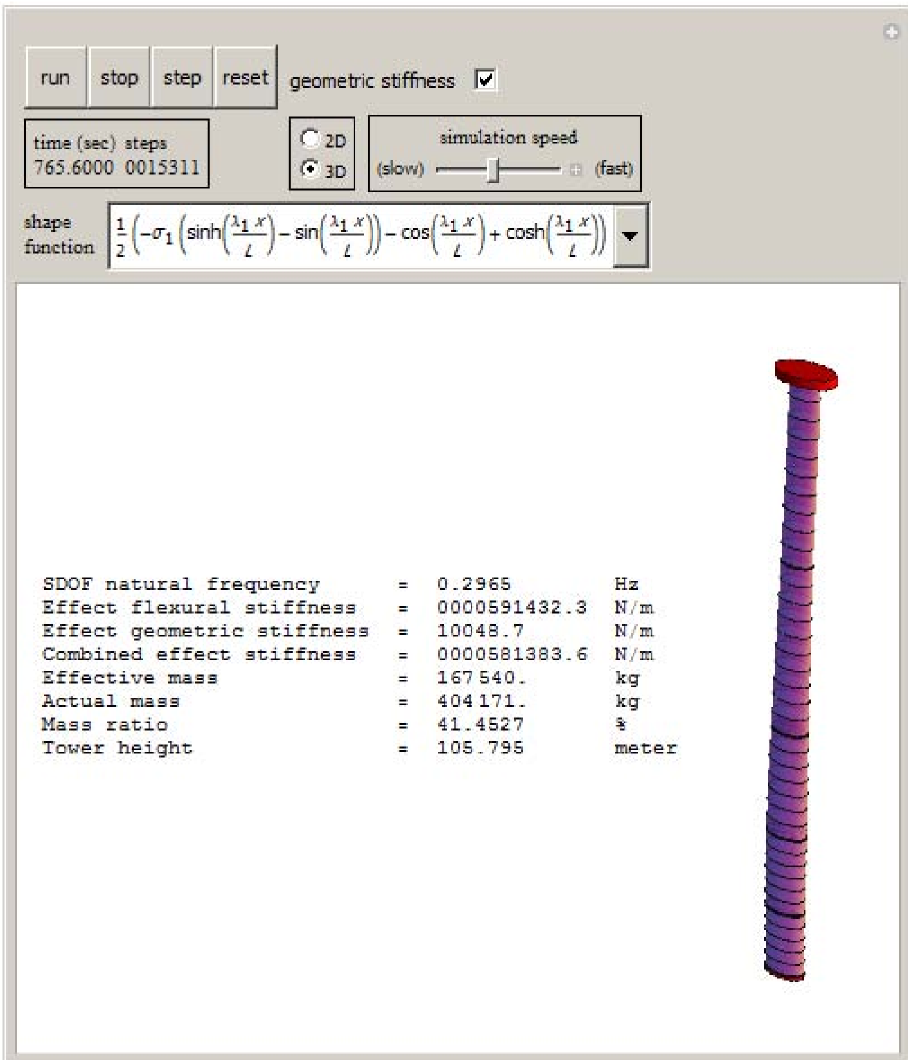


Figure 2.1: Mathematica demonstration

This table shows the final computation result for the case that gave the lowest natural frequency

#	height	T (m)	D (m)	mass (kg)	E (GPa)	geometric Stiffness (N/m)	flexural Stiffness (N/m)	effective stiffness (N/m)	current height (m)	shape Function	curveture	angle	I (m^4)	effective mass (kg)
1	1.34	0.	0.	130 000.	2.1×10^{11}	0	0	0	106.815	1.	0	0	0	130 000.
2	0.295	0.121	2.8	2374.83	2.1×10^{11}	65.4782	1742.69	1677.21	105.475	0.975067	0.000175293	0.0000517115	0.915479	2257.88
3	2.3	0.015	2.8	2386.14	2.1×10^{11}	516.807	1888.5	1371.7	105.18	0.969621	0.000175293	0.000403175	0.127245	2243.36
4	2.94	0.015	2.822	3062.18	2.1×10^{11}	646.4	2471.66	1825.26	102.88	0.927678	0.000175293	0.000515363	0.130284	2635.27
5	2.94	0.015	2.844	3086.28	2.1×10^{11}	623.643	2530.23	1906.59	99.94	0.875415	0.000175293	0.000515363	0.133371	2365.18
6	2.935	0.015	2.868	3106.17	2.1×10^{11}	599.42	2590.76	1991.34	97.	0.824668	0.000175293	0.000514486	0.136794	2112.43
7	2.935	0.015	2.89	3131.32	2.1×10^{11}	575.951	2651.15	2075.2	94.065	0.775518	0.000175293	0.000514486	0.139983	1883.26
8	2.935	0.015	2.912	3155.37	2.1×10^{11}	552.162	2712.48	2160.32	91.13	0.727877	0.000175293	0.000514486	0.143222	1671.73
9	2.935	0.016	2.934	3390.22	2.1×10^{11}	528.833	2956.7	2427.87	88.195	0.681747	0.000175293	0.000514486	0.156116	1575.7
10	2.93	0.017	2.956	3621.95	2.1×10^{11}	505.006	3204.35	2699.34	85.26	0.637127	0.000175293	0.00051361	0.169481	1470.26
11	2.93	0.018	2.978	3862.51	2.1×10^{11}	482.455	3466.1	2983.65	82.33	0.594089	0.000175293	0.00051361	0.183326	1363.24
12	2.925	0.019	3.	4099.09	2.1×10^{11}	459.353	3730.72	3271.37	79.4	0.552556	0.000175293	0.000512733	0.197659	1251.53
13	0.28	0.18	3.	3529.65	2.1×10^{11}	41.6633	2875.77	2834.1	76.475	0.512595	0.000175293	0.0000490821	1.59164	927.428
14	2.885	0.02	3.052	4307.75	2.1×10^{11}	437.018	4075.61	3638.59	76.195	0.508848	0.000175293	0.000505721	0.218926	1115.39
15	2.885	0.02	3.124	4396.6	2.1×10^{11}	414.825	4372.9	3958.07	73.31	0.471045	0.000175293	0.000505721	0.234895	975.529
16	2.88	0.021	3.196	4715.07	2.1×10^{11}	392.306	4905.37	4513.07	70.425	0.4347	0.000175293	0.000504845	0.263955	890.978
17	2.88	0.021	3.268	4823.23	2.1×10^{11}	370.427	5246.71	4876.28	67.545	0.399873	0.000175293	0.000504845	0.282322	771.226
18	2.88	0.022	3.34	5164.63	2.1×10^{11}	348.886	5865.08	5516.19	64.665	0.3665	0.000175293	0.000504845	0.315596	693.725
19	2.875	0.022	3.412	5268.77	2.1×10^{11}	326.664	6244.36	5917.7	61.785	0.334581	0.000175293	0.000503968	0.336589	589.81
20	2.875	0.022	3.484	5381.87	2.1×10^{11}	305.063	6650.73	6345.67	58.91	0.304168	0.000175293	0.000503968	0.358494	497.921
21	2.87	0.023	3.556	5733.12	2.1×10^{11}	283.321	7376.82	7093.5	56.035	0.275204	0.000175293	0.000503092	0.398325	434.21
22	2.87	0.023	3.628	5851.16	2.1×10^{11}	262.196	7837.06	7574.87	53.165	0.247735	0.000175293	0.000503092	0.423176	359.101
23	2.86	0.023	3.7	5947.94	2.1×10^{11}	240.32	8287.09	8046.77	50.295	0.22171	0.000175293	0.000501339	0.449041	292.373
24	0.33	0.23	3.7	6540.68	2.1×10^{11}	25.3972	8071.31	8045.92	47.435	0.197212	0.000175293	0.0000578468	3.79036	254.384
25	2.71	0.024	3.76	5986.44	2.1×10^{11}	211.099	8594.67	8383.57	47.105	0.194478	0.000175293	0.000475045	0.491484	226.416
26	2.71	0.024	3.825	6087.4	2.1×10^{11}	192.408	9051.11	8858.7	44.395	0.172744	0.000175293	0.000475045	0.517585	181.652
27	2.71	0.024	3.89	6192.4	2.1×10^{11}	174.03	9523.42	9349.39	41.685	0.152298	0.000175293	0.000475045	0.544595	143.631
28	2.705	0.025	3.955	6546.	2.1×10^{11}	155.911	10 401.9	10 246.	38.975	0.13314	0.000175293	0.000474169	0.59593	116.036
29	2.705	0.025	4.02	6655.17	2.1×10^{11}	138.59	10 926.6	10 788.	36.27	0.1153	0.000175293	0.000474169	0.62599	88.475
30	2.705	0.025	4.085	6764.34	2.1×10^{11}	121.796	11 468.6	11 346.8	33.565	0.0987436	0.000175293	0.000474169	0.657044	65.9543
31	2.685	0.026	4.15	7093.36	2.1×10^{11}	104.928	12 408.	12 303.1	30.86	0.0834694	0.000175293	0.000470663	0.716154	49.4204
32	0.36	0.24	4.15	8389.62	2.1×10^{11}	12.0884	13 136.7	13 124.6	28.175	0.0695766	0.000175293	0.0000631056	5.65503	40.6134
33	2.41	0.026	4.15	6417.42	2.1×10^{11}	80.6735	11 137.1	11 056.5	27.815	0.06781	0.000175293	0.000422457	0.716154	29.5086
34	2.41	0.027	4.15	6662.63	2.1×10^{11}	68.8614	11 557.1	11 488.3	25.405	0.0565684	0.000175293	0.000422457	0.74316	21.3203
35	2.41	0.028	4.15	6907.72	2.1×10^{11}	57.743	11 976.5	11 918.7	22.995	0.0463449	0.000175293	0.000422457	0.770126	14.8368
36	2.41	0.029	4.15	7152.69	2.1×10^{11}	47.3746	12 395.2	12 347.9	20.585	0.0371396	0.000175293	0.000422457	0.797053	9.86606
37	2.405	0.029	4.15	7137.85	2.1×10^{11}	37.7092	12 369.5	12 331.8	18.175	0.0289524	0.000175293	0.000421581	0.797053	5.98324
38	2.405	0.03	4.15	7382.19	2.1×10^{11}	29.0553	12 786.8	12 757.7	15.77	0.0217971	0.000175293	0.000421581	0.82394	3.50738
39	0.44	0.39	4.15	16 023.5	2.1×10^{11}	4.00782	23 363.5	23 359.5	13.365	0.0156557	0.000175293	0.0000771291	8.22878	3.92739
40	2.4	0.031	4.15	7610.56	2.1×10^{11}	20.905	13 176.	13 155.1	12.925	0.0146419	0.000175293	0.000420704	0.850788	1.63158
41	2.4	0.032	4.15	7854.15	2.1×10^{11}	14.177	13 591.2	13 577.	10.525	0.00970912	0.000175293	0.000420704	0.877597	0.740387
42	2.395	0.034	4.15	8323.61	2.1×10^{11}	8.62937	14 389.7	14 381.	8.125	0.00578605	0.000175293	0.000419828	0.931096	0.278661
43	2.395	0.06	4.15	14 595.9	2.1×10^{11}	4.46481	24 919.	24 914.5	5.73	0.00287769	0.000175293	0.000419828	1.61241	0.120871
44	2.395	0.06	4.15	14 595.9	2.1×10^{11}	1.57107	24 919.	24 917.4	3.335	0.000974826	0.000175293	0.000419828	1.61241	0.0138703
45	0.24	0.4	4.15	8940.34	2.1×10^{11}	0.0127931	12 974.4	12 974.4	0.94	0.0000774446	0.000175293	0.0000420704	8.37774	0.000053621
46	0.7	0.055	4.15	3915.31	2.1×10^{11}	0.0208944	6700.56	6700.54	0.7	0.0000429469	0.000175293	0.000122705	1.48341	7.22154×10

Figure 2.2: Final table

2.2.3 References

1. Formulas for Natural Frequency and Mode Shape, Robert D. Blevins
2. Dynamics of structures by Ray W. Clough and Joseph Penzien.
3. Structural Dynamics, 5th edition by Mario Paz and William Leigh.
4. Professor Oliva class lecture notes, CEE 744, structural dynamics, spring 2013, University of Wisconsin, Madison.
5. `http://en.wikipedia.org/wiki/List_of_moment_of_areas`

2.3 HW3

Local contents

2.3.1 Problem description

CEE 744

Periodic response analysis:

Develop a solution for the response felt by a driver crossing a bridge similar to the south beltline highway viaduct. For fun, we'll look at how different cars and drivers respond to the trip.

The first part of the problem setup is in a MathCad file you can download from the class web site. Finish the MathCad sheet for the response solution.

Provide the following summary of information on your solution:

1. natural period and damped period of your car
2. the time to cross one span (period of loading)
3. time to cross the bridge (duration of loading)
4. at least the first four "a_n" values from your representation of the load
5. the peak *relative* displacement of the driver
6. the peak total displacement of the driver
7. the number of "a" values you decided to use in your solution

Submit: 1) your MathCad solution electronically to the class web site
2) the summary data above on a paper sheet
3) a plot of the series representation of the load on a sheet
4) a plot of relative displacement vs time on a sheet
5) a plot of the sum of steady state solutions for relative displacement on a sheet
6) a plot of your transient solution on a sheet
8) a plot of the total car displacement with the bridge shape on a sheet

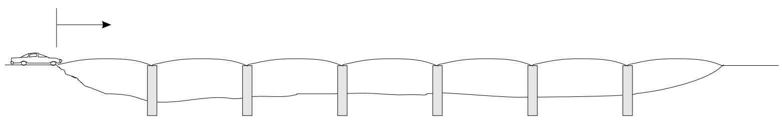
Driver	speed	car weight	shocks (% damp)	springs (lb/ft)
Nasser	30	1500	75	2400
Sam	50	1500	75	2400
Jeffryd	80	1500	75	2400
Tom	50	2300	75	2400
Moon	50	1500	75	3600
Ian	50	2200	50	3600
Henrik	80	1500	50	2000
Derek	80	2200	50	3600
Brad	50	1200	40	2000

2.3.2 Mathcad initial calculations

Analysis of motion in a car crossing south beltline bridge

A vehicle crossing the south beltline bridges experiences vertical dynamic vibration due to the residual camber in the bridge. Determine the extent of vertical motion that will occur. (Insert your values in the highlighted regions.)

x(t) = location of the car



Bridge data: span length = 70 ft. upward camber = 2.5 inches

$\lambda := 70\cdot ft$

$\Delta := \frac{2.5}{12}\cdot ft$

Car data:

$weight := 1800lb$

$speed := 60mph$

$\xi := 0.75$

$k := 5000\frac{lb}{ft}$

$m := \frac{weight}{32.2}\cdot\frac{sec^2}{ft}$ $\qquad m = 55.901\frac{lb\cdot s^2}{ft}$

$\omega_n := \sqrt{\frac{k}{m}}$ $\qquad \omega_n = 9.458\cdot\frac{rad}{sec}$

$f_n := \frac{\omega_n}{2\cdot\pi}$ $\qquad f_n = 1.505\frac{1}{s}$

$T_n := \frac{1}{f_n}$ $\qquad T_n = 0.664\,s$

$\omega_d := \omega_n\cdot\sqrt{1-\xi^2}$ $\qquad \omega_d = 6.256\cdot\frac{rad}{sec}$

$sp := speed$ $\qquad sp = 88\frac{ft}{s}$

PART #1: Define load and convert to a series form

Loading is as if ground is moving up and down under car. This is like an EQ load. Define the ground movement and acceleration.

-the car travels through 1/2 cycle in one span:

$$y_g(t) := \Delta \cdot \sin\left(\pi \cdot \frac{x}{\lambda}\right)$$ for 0<x<span

-the car location "x" is dependent on speed and time:

$x := sp \cdot t$

$$y_g(t) := \Delta \cdot \sin\left(\pi \cdot \frac{sp}{\lambda} \cdot t\right)$$

$$acc_g(t) := -\Delta \cdot \left(\frac{\pi \cdot sp}{\lambda}\right)^2 \cdot \sin\left(\pi \cdot \frac{sp}{\lambda} \cdot t\right)$$

-the span length/speed = time to cross one span,

$T_p := \frac{\lambda}{sp}$ $T_p = 0.795\,s$ $\beta\varepsilon\tau\alpha := \frac{T_n}{T_p}$ $\beta\varepsilon\tau\alpha = 0.835$

$$acc_g(t) := -\Delta \cdot \left(\frac{\pi}{T_p}\right)^2 \cdot \sin\left(\frac{\pi}{T_p} \cdot t\right)$$

rounded time, more than one period:

$T_a := 1.2s$

Then the load in one span (0<t<Tp):

$$P_a(t) := m \cdot \Delta \cdot \left(\frac{\pi}{T_p}\right)^2 \cdot \sin\left(\frac{\pi}{T_p} \cdot t\right)$$

steps in analysis:

$st := \frac{T_a}{.01s}$ $st = 120$

We need to convert this load to a periodic form that works for any point in time, until the vehicle is off of the bridge.

end of load:

$T_{max} := 7 \cdot T_p$ $T_{max} = 5.568\,s$

$T_{max} =$!

<u>Convert the load to a series - Fourier Transform:</u>

$$P_a(t) := m \cdot \Delta \cdot \left(\frac{\pi}{T_p}\right)^2 \cdot \sin\left(\frac{\pi}{T_p} \cdot t\right)$$ with a period of Tp, $P_o := m \cdot \Delta \cdot \left(\frac{\pi}{T_p}\right)^2$

$P_o = 181.654\ \text{lb}$

$$P_f(t) := a_o + \sum_{n=1}^{\infty}\left(a_n \cdot \cos\left(2\cdot\pi\cdot n\cdot\frac{t}{T_p}\right)\right) + \sum_{n=1}^{\infty}\left(b_n \cdot \sin\left(2\cdot\pi\cdot n\cdot\frac{t}{T_p}\right)\right)$$

where:
$$a_o := \frac{1}{T_p}\cdot\int_0^{T_p} P_o\cdot\sin\left(\pi\cdot\frac{t}{T_p}\right)dt$$
$a_o = 115.644\ \text{lb}$

and: $n := 1, 2 .. 10$

$$a_n := 2\cdot\frac{P_o}{T_p}\cdot\int_0^{T_p}\sin\left(\pi\cdot\frac{t}{T_p}\right)\cdot\cos\left(2\cdot\pi\cdot n\cdot\frac{t}{T_p}\right)dt$$

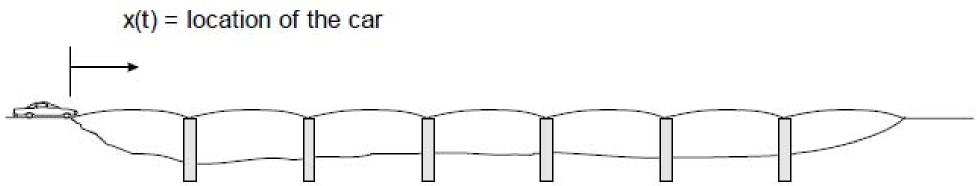
$a_n =$

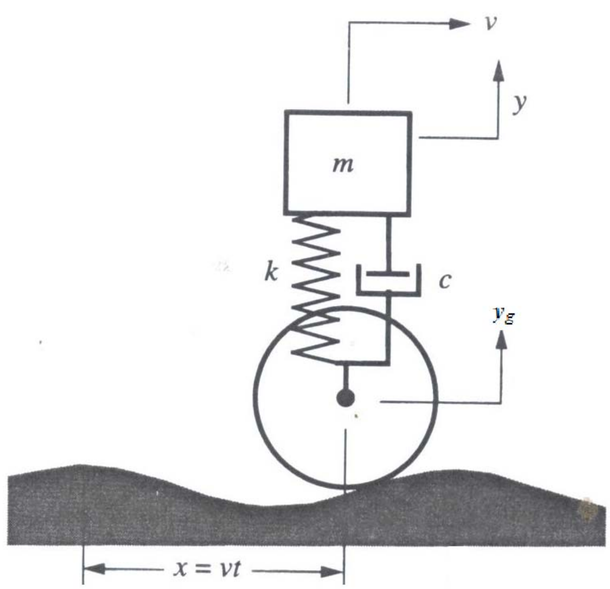
-77.096	lb
-15.419	
-6.608	
-3.671	
-2.336	
-1.617	
-1.186	
-0.907	
-0.716	
-0.58	

theoretically all "b" = zero,

$a_0 := a_o$ $\quad a_0 = 115.644\ \text{lb}$

2.3.3 Mathematical model





The equation of motion of the car is

$$my'' + cy' + ky = cy'_g + ky_g$$

Let $y - y_g = u$ which is the distance between m and the ground. Hence the equation of motion now becomes

$$\begin{aligned} m\left(u'' + y''_g\right) + c\left(u' + y'_g\right) + k\left(u + y_g\right) &= cy'_g + ky_g \\ mu'' + cu' + ku &= -my''_g \end{aligned}$$

2.3.4 Summary of results found

2.3.4.1 Bridge data

	imperial	SI
span length λ	70 ft	$70 \times 0.3048 = 21.336$ meter
upward camber Δ	$2.5'' = 0.208\,33$ft	$2.5 \times 0.0254 = 0.0635$ meter

2.3.4.2 Car data

	imperial
mass of car	$\frac{1500}{32.2} = 46.584\ \frac{lb.s^2}{ft}$
speed of car	30 mile/hr= 44.0 ft/sec
critical damping ratio is ζ	0.75
spring constant k	2400 lb/ft
natural frequency $\omega_n = \sqrt{\frac{k}{m}}$	$\sqrt{\frac{2400}{46.584}} = 7.1777$ rad/sec
natural frequency $f_n = \frac{\omega_n}{2\pi}$	$\frac{7.1777}{2\pi} = 1.1424$ Hz
natural period $T_n = \frac{1}{f_n}$	$\frac{1}{1.1424} = 0.87535$ sec
natural damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$	$7.1777\sqrt{1-0.75^2} = 4.7476$ rad/sec
natural damped frequency $f_d = \frac{\omega_d}{2\pi}$	$\frac{4.7476}{2\pi} = 0.7556$ Hz
T_p time to driver over one span $= \frac{\lambda}{v}$	1.591 sec
T_a time to cross the bridge (duration of loading)	$7 \times 1.591 = 11.137$ sec

2.3.4.3 Results

a_n values found for up to $n = 10$

a_n distribution

n	a[n]
0	24.0897
1	-16.0598
2	-3.21196
3	-1.37655
4	-0.764752
5	-0.48666
6	-0.336919
7	-0.247074
8	-0.188939
9	-0.149162
10	-0.12075

Peak relative displacement of the driver

Maximum relative displacement was 0.24 inch and it occurred during transient phase.

Peak total displacement of the driver

0.165 inch + 2.5 inch =2.665 inch and it occurred during steady state phase at multiples of half the period T_p while on the bridge.

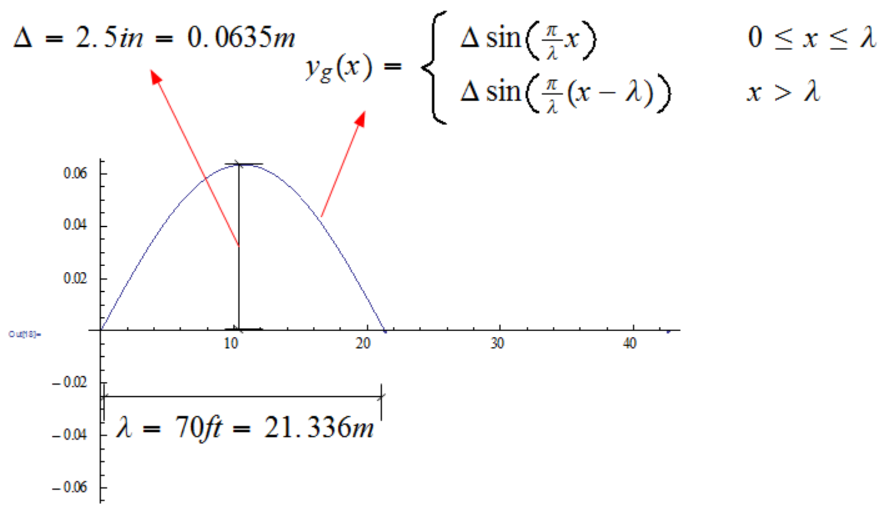
Number of a_n terms used

In addition to a_0 term, the next 5 terms are used for a total of 6 terms.

2.3.5 Analysis

2.3.5.1 Generate load equation

The equation of the ground is shown in this diagram



Description of one span and equation of road

Therefore, the equation of span is

$$y_g(x) = \Delta\sin\left(\tfrac{\pi}{\lambda}x\right) \qquad 0 \leq x \leq \lambda$$

Hence, we convert it to be a function of time using $x = vt$, hence

$$\begin{aligned} y_g(t) &= \Delta\sin\left(\tfrac{\pi}{\lambda}vt\right) \qquad 0 \leq t \leq \tfrac{\lambda}{v} \\ &= \Delta\sin(\varpi t) \qquad 0 \leq t \leq T_p \\ &= \Delta\sin\left(\tfrac{\pi}{T_p}t\right) \qquad 0 \leq t \leq T_p \end{aligned}$$

Where in the above $\varpi = \frac{\pi}{T_p}$ is the fundamental frequency of the ground motion. Hence $y_g'(t) = \Delta\frac{\pi}{T_p}\cos\left(\frac{\pi}{T_p}t\right)$ and

$$y_g''(t) = -\Delta\left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right)$$

And

$$\beta = \frac{T_n}{T_p} = \frac{0.875\,35}{1.591} = 0.550\,19$$

Then load in one span $0 < t < T_p$ is

$$P_a(t) = m\Delta\left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right)$$

Let

$$
\begin{aligned}
P_o &= m\Delta\left(\frac{\pi}{T_p}\right)^2 = (46.584)\,(0.208\,33)\left(\frac{\pi}{1.591}\right)^2 \\
&= 37.840 \text{ lb}
\end{aligned}
$$

Then the load becomes

$$
P_a(t) = P_o \sin\left(\frac{\pi}{T_p}t\right) \tag{2.1}
$$

2.3.5.2 Convert load to Fourier series

Now we need to convert Eq 2.1 to Fourier series[1]. Let $\tilde{P}_a(t)$ be the Fourier series approximation to $P_a(t)$, hence

$$
\begin{aligned}
\tilde{P}_a(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T_p}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi}{T_p}t\right) \\
a_0 &= \frac{1}{T_p}\int_0^{T_p} P_a(t)\,dt \\
a_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
b_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\sin\left(n\frac{2\pi}{T_p}t\right)dt
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_0 &= \frac{1}{T_p}\int_0^{T_p} P_a(t)\,dt = \frac{P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)dt = \frac{P_o}{T_p}\left(\frac{-\cos\left(\frac{\pi}{T_p}t\right)}{\frac{\pi}{T_p}}\right)_0^{T_p} = -\frac{P_o}{\pi}\left(\cos(\pi) - 1\right) \\
&= \frac{2P_o}{\pi} = \frac{2\,(37.840)}{\pi} \\
&= 24.090 \text{ lb}
\end{aligned}
$$

[1]The Fourier series can also be found using complex form. This was done in the appendix.

And

$$\begin{aligned}
a_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
&= \frac{2P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
&= \frac{4P_o}{\pi-4n^2\pi}\cos(n\pi)^2
\end{aligned}$$

But $\cos(n\pi)^2 = 1$ Hence

$$a_n = \frac{4P_o}{\pi-4n^2\pi}$$

and

$$\begin{aligned}
b_n &= \frac{1}{T_p}\int_0^{T_p} P_a(t)\sin\left(2\pi n\frac{t}{T_p}\right)dt \\
&= \frac{2P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
&= \frac{2P_o}{\pi-4n^2\pi}\sin(2n\pi)
\end{aligned}$$

But $\sin(2n\pi) = 0$ for all integer n, hence $b_n = 0$.Therefore

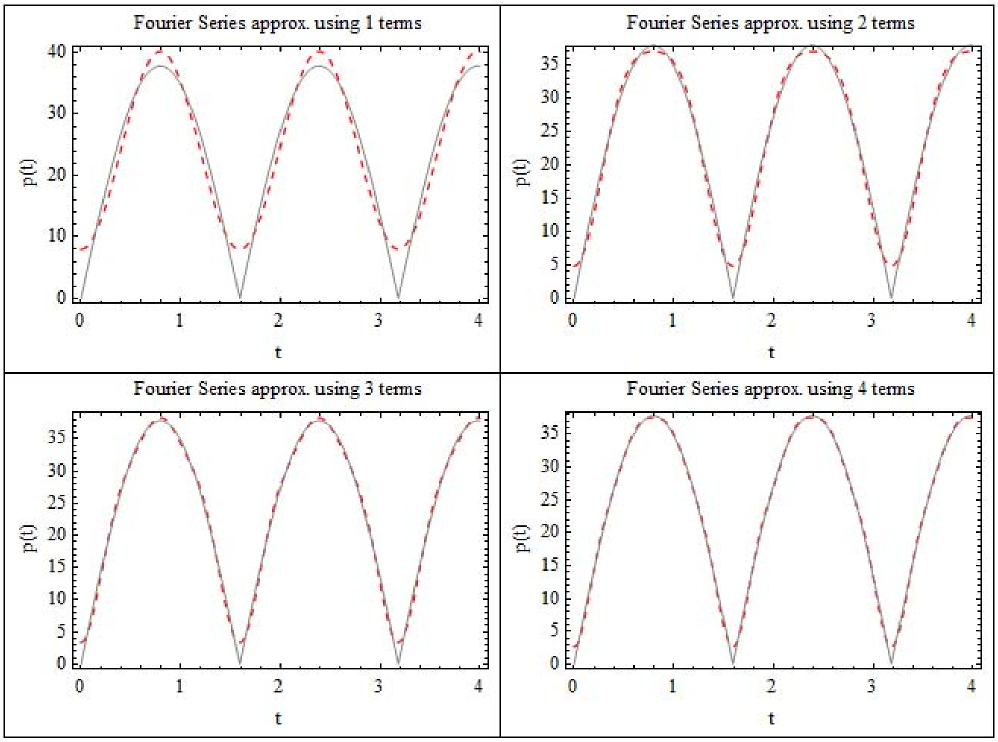
$$\begin{aligned}
\tilde{P}_a(t) &= a_0 + \sum_{n=1}^{\infty} a_n\cos\left(2\pi n\frac{t}{T_p}\right) \\
&= a_0 + \sum_{n=1}^{\infty}\frac{4P_o}{\pi-4n^2\pi}\cos\left(2\pi n\frac{t}{T_p}\right)
\end{aligned}$$

Using the numerical values found, we obtain

$$\tilde{P}_a(t) = 24.0897 + \sum_{n=1}^{\infty}\frac{4(37.840)}{\pi-4n^2\pi}\cos\left(2\pi n\frac{t}{1.591}\right)$$

2.3.5.3 Plot of load and its Fourier series approximation

This plot below shows $P_a(t)$ and its Fourier series approximation $\tilde{P}_a(t)$ as more terms are added. This was plotted for $t = 0 \cdots 5\,\text{sec}$. This was done to verify that the Fourier series approximation is correct before going to the next stage of the analysis. The actual calculations used the first 6 terms of a_n.



2.3.5.4 Finding the steady state response

The equation of motion of the car is

$$my'' + cy' + ky = cy_g' + ky_g$$

Let $y - y_g = u$ which is the distance between m and the ground. Hence the equation of motion now becomes

$$\begin{aligned} m\left(u'' + y_g''\right) + c\left(u' + y_g'\right) + k\left(u + y_g\right) &= cy_g' + ky_g \\ mu'' + cu' + ku &= -my_g'' \end{aligned} \tag{2.2}$$

Hence Eq 2.2 becomes

$$
\begin{aligned}
mu'' + cu' + ku &= m\Delta \left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right) \\
&= P_a(t) \\
&= \sum_{n=0}^{\infty} a_n \cos\left(2\pi n \frac{t}{T_p}\right) \\
&= \operatorname{Re}\left\{\sum_{n=0}^{\infty} a_n e^{in\varpi t}\right\}
\end{aligned}
$$

Where $\varpi = \frac{2\pi}{T_p}$ is the fundamental loading harmonic. Let $u_{ss}(n) = \operatorname{Re}\{U_n e^{in\varpi t}\}$ be the response due to the n term in the loading function. Hence the equation of motion now becomes

$$
\begin{aligned}
m \operatorname{Re}\left\{\sum_{n=0}^{\infty} - n^2\varpi^2 U_n e^{in\varpi t}\right\} + c \operatorname{Re}\left\{\sum_{n=0}^{\infty} i\varpi n U_n e^{in\varpi t}\right\} + k \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\} &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} a_n e^{in\varpi t}\right\} \\
\left(-n^2\varpi^2 m + cin\varpi + k\right) U_n &= a_n \\
U_n &= \frac{a_n}{-n^2\varpi^2 m + cin\varpi + k} \\
&= \frac{a_n}{k} \frac{1}{(1 - n^2 r^2) + 2i\zeta n r}
\end{aligned}
$$

Hence the transfer function is

$$
\begin{aligned}
\left(-n^2\varpi^2 m + cin\varpi + k\right) U_n &= a_n \\
U_n &= \frac{a_n}{-n^2\varpi^2 m + cin\varpi + k} \\
&= \frac{a_n}{k} \frac{1}{(1 - n^2 r^2) + 2i\zeta n r}
\end{aligned}
$$

Therefore, steady state response is

$$
\begin{aligned}
y_{ss}(t) &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\} \\
&= \operatorname{Re}\left\{\sum_{n=0}^{\infty} \frac{a_n}{k} \frac{1}{(1 - n^2 r^2) + 2i\zeta n r} e^{in\varpi t}\right\} \\
&= \operatorname{Re}\left\{\sum_{n=0}^{\infty} \overbrace{\frac{a_n}{k} D(\zeta, r, n)}^{U_n} e^{in\varpi t}\right\} \qquad (2.3)
\end{aligned}
$$

Where $D(\zeta, r, n)$ is the n^{th} harmonic dynamic magnification factor

$$|D(\zeta, r, n)| = \sqrt{\frac{1}{(1-n^2r^2)^2 + (2\zeta nr)^2}}$$

and

$$\arg D(\zeta, r, n) = -\tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$$

In the above,

$$r = \frac{\varpi}{\omega_{nat}} = \frac{\frac{2\pi}{T_p}}{\omega_{nat}} = \frac{\frac{2\pi}{1.591}}{7.1777} = \boxed{0.550\,2}$$

This is a list of the magnitude of U_n for different n value to examine the contribution of each harmonic to the steady state response.

Distribution of U_{ss} harmonics

n	U_n	$\lvert U_n \rvert$	phase(U_n) degree
0	0.0100374	0.0100374	0.
1	-0.00170524 + 0.00366837 i	0.00404534	-114.931
2	0.0000312072 + 0.000443912 i	0.000445007	-85.9787
3	0.0000425907 + 0.000111137 i	0.000119018	-69.0317
4	0.000024122 + 0.0000376552 i	0.000044719	-57.3563
5	0.0000134002 + 0.0000153014 i	0.0000203396	-48.7897
6	$7.7639\times10^{-6} + 7.05943\times10^{-6}$ i	0.0000104935	-42.2791
7	$4.72164\times10^{-6} + 3.58384\times10^{-6}$ i	5.92772×10^{-6}	-37.1994
8	$3.00344\times10^{-6} + 1.96149\times10^{-6}$ i	3.58722×10^{-6}	-33.1478
9	$1.98761\times10^{-6} + 1.14081\times10^{-6}$ i	2.29173×10^{-6}	-29.8542
10	$1.36131\times10^{-6} + 6.97572\times10^{-7}$ i	1.52963×10^{-6}	-27.1318

2.3.5.5 Find the transient solution

From the steady state solution $u_{ss}(t)$ we found above, we now find $u_{ss}(0)$ and $u'_{ss}(0)$ these are the initial conditions, but in opposite sign, that the transient solution have to satisfy.

From above, we found the steady state solution to be

$$y_{ss}(t) = \text{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\}$$

Hence

$$y'_{ss}(t) = \text{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n e^{in\varpi t}\right\}$$

At time $t=0$ the above becomes

$$y_{ss}(0) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n\right\}$$
$$y'_{ss}(0) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n\right\}$$

Now we need to decide on how many harmonics to use in order to determine $y_{ss}(0)$ and $y'_{ss}(0)$. From above we see that after $\boxed{n=5}$ then a_n became very small. Hence we will use up to $n=5$ to find the initial conditions from the above 2 equations.

$$\begin{aligned} y_{ss}(0) &= \operatorname{Re}\left\{\sum_{n=0}^{5} U_n\right\} = \operatorname{Re}\left\{\sum_{n=0}^{5} \frac{a_n}{k}\frac{1}{(1-n^2r^2)+2i\zeta nr}\right\} \\ &= 0.0084435 \text{ ft} = 0.101322 \text{ inch} \end{aligned}$$

and for the initial velocity we obtain

$$\begin{aligned} y'_{ss}(0) &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n\right\} \\ &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi \frac{a_n}{k}\frac{1}{(1-n^2r^2)+2i\zeta nr}\right\} \\ &= -0.020207 \text{ ft/sec} = -0.242484 \text{ inch/sec} \end{aligned}$$

Now the transient solution for damped system is given by

$$u_{tr}(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

with

$$u_{tr}(0) = -0.0084435$$
$$u'_{tr}(0) = +0.020207$$

Hence

$$A = u_{tr}(0) = \boxed{-0.0084435}$$

Taking derivative of $u_{tr}(t)$ gives

$$u'_{tr}(t) = \zeta\omega_n e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right)$$

Hence at $t=0$ we obtain

$$u'_{tr}(0) = \zeta\omega_n A + B\omega_d$$
$$B = \frac{u'_{tr}(0) - \zeta\omega_n A}{\omega_d}$$

But $u'_{tr}(0) = +0.020207$ ft/sec, $A = -0.0084435$ ft, $\zeta = 0.75$,$\omega_d = 4.7476$ rad/sec,$\omega_n = 7.1777$ rad/sec, hence

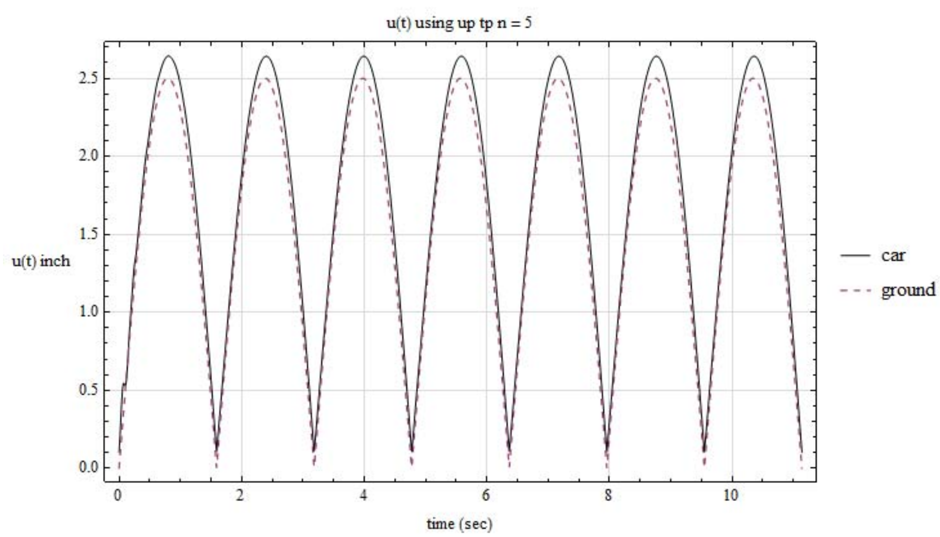
$$\begin{aligned} B &= \frac{0.020207 - 0.75 \times 7.1777 \times (-0.0084435)}{4.7476} \\ &= 0.01383 \end{aligned}$$

Therefore

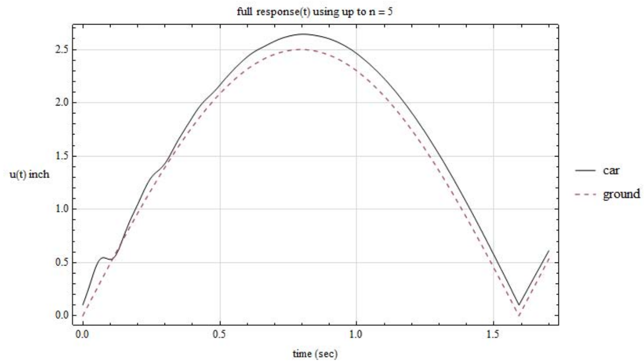
$$\begin{aligned} u_{tr}(t) &= e^{-\zeta\omega_n t}\left(-0.0084435\cos\omega_d t + 0.01383\sin\omega_d t\right) \\ &= e^{-0.75(7.1777)t}\left(-0.0084435\cos\left(4.7476t\right) + 0.01383\sin 4.7476t\right) \end{aligned}$$

This solution is now added to the steady state solution.

2.3.5.6 Plot of the absolute total displacement with the bridge for both steady state and transient combined



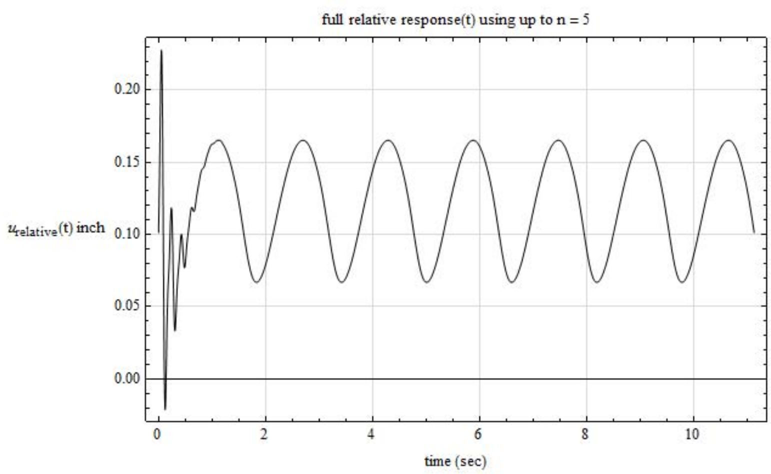
Zooming on the first 1.8 seconds shows more clearly the effect of transient solution

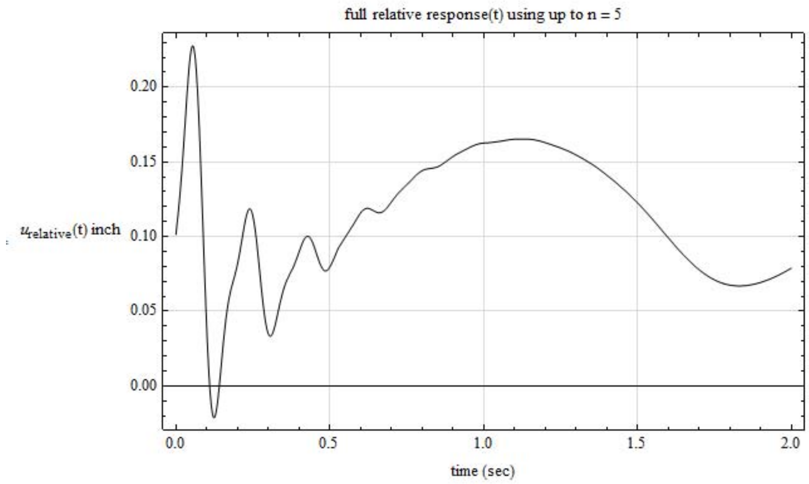


The transient solution effect vanishes after about 1.5 second.

2.3.5.7 Plotting the full relative solution

To better see the solution obtained, we plot the relative displacement. This is the displacement felt by the passenger. First the solution is shown for the whole time to cross the bridge, then we zoom to the first 2 seconds to better see the transient solution





From the above we see that the maximum relative displacement is about 0.24 inch and it occurs during transient phase. During steady state, the maximum relative displacement is about 0.165 inch

2.3.6 Appendix

2.3.6.1 Finding Fourier series approximation using complex form

The Fourier series approximation can also be found using the complex representation. This is the derivation using this method which gives the same result as was found earlier.

$$\tilde{P}_a(t) = \frac{1}{2}Y_0 + \operatorname{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

Where

$$\begin{aligned} Y_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\, e^{-in\varpi t}\, dt \\ &= \frac{2P_o}{T_p}\int_0^{T_p} \sin\left(\frac{\pi}{T_p}t\right) e^{-in\varpi t}\, dt \end{aligned} \tag{2.4}$$

Integration by parts, $\int u dv = uv - \int v du$, let $u = \sin\left(\frac{\pi}{T_p}t\right)$, hence $du = \frac{\pi}{T_p}\cos\left(\frac{\pi}{T_p}t\right)$ and $v = \frac{e^{-in\varpi t}}{-in\varpi}$, therefore the above becomes

$$\begin{aligned} Y_n &= \frac{2P_o}{T_p}\left(\left[\sin\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}\right]_0^{T_p} - \int_0^{T_p}\frac{\pi}{T_p}\cos\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}dt\right) \\ &= \frac{2P_o}{T_p}\left(\sin\left(\frac{\pi}{T_p}T_p\right)\frac{ie^{-in\varpi T_p}}{n\varpi} - \frac{i}{2n}\int_0^{T_p}\cos\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right) \\ &= -\frac{iP_o}{nT_p}\int_0^{T_p}\cos\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt \end{aligned} \tag{2.5}$$

Now integrate by parts again where now $\int u dv = uv - \int v du$, let $u = \cos\left(\frac{\pi}{T_p}t\right)$, hence

$du = -\frac{\pi}{T_p}\sin\left(\frac{\pi}{T_p}t\right)$ and $v = \frac{e^{-in\varpi t}}{-in\varpi}$, therefore Eq 2.5 becomes

$$\begin{aligned}
Y_n &= -\frac{iP_o}{nT_p}\left[\left(\cos\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}\right)_0^{T_p} - \int_0^{T_p} -\frac{\pi}{T_p}\sin\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}dt\right]\\
&= -\frac{iP_o}{nT_p}\left[\left(\cos\left(\frac{\pi}{T_p}T_p\right)\frac{e^{-in\frac{2\pi}{T_p}T_p}}{-in\varpi} - \frac{1}{-in\varpi}\right) + \frac{i}{n2}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right]\\
&= -\frac{iP_o}{nT_p}\left[\left(-i\frac{e^{-in2\pi}}{n\varpi} - \frac{i}{n\varpi}\right) + \frac{i}{n2}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right]\\
&= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) + \frac{\Delta}{2n^2T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt
\end{aligned} \tag{2.6}$$

Now we see that the term $\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt$ has repeated again. This term is the same as what we started with in Eq 2.4, therefore, we write

$$\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt = \frac{T_p}{2\Delta}Y_n$$

and replace this term back into Eq 2.6, hence it becomes

$$\begin{aligned}
Y_n &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) + \frac{\Delta}{2n^2T_p}\frac{T_p}{2\Delta}Y_n\\
&= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) + \frac{1}{2^2n^2}Y_n\\
Y_n - \frac{1}{2^2n^2}Y_n &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right)\\
Y_n\left(1-\frac{1}{(2n)^2}\right) &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right)\\
Y_n &= -\frac{2P_o\left(e^{-in2\pi}+1\right)}{\pi\left((2n)^2-1\right)} = \frac{2P_o\left(e^{-in2\pi}+1\right)}{\pi-\pi(2n)^2}\\
&= \frac{4P_o}{\pi(1-4n^2)}
\end{aligned}$$

And

$$
\begin{aligned}
Y_0 &= \frac{2}{T_p}\int_0^{T_p} P_o \sin\left(\frac{\pi}{T_p}t\right)dt = \frac{2P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)dt = \frac{2P_o}{T_p}\left(\frac{-\cos\left(\frac{\pi}{T_p}t\right)}{\frac{\pi}{T_p}}\right)_0^{T_p} = -\frac{2P_o}{\pi}\left(\cos\left(\frac{\pi}{T_p}T_p\right)-1\right)\\
&= -\frac{2P_o}{\pi}\left(-1-1\right)\\
&= \frac{4P_o}{\pi}
\end{aligned}
$$

Therefore, the Fourier series approximation for ground motion is now

$$
\begin{aligned}
\tilde{P}_a\left(t\right) &= \frac{1}{2}Y_0 + \operatorname{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)\\
&= \frac{4P_o}{2\pi} + \operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{4P_o}{\pi\left(1-4n^2\right)}e^{in\varpi t}\right)\\
&= \frac{2P_o}{\pi} + \operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{4P_o}{\pi\left(1-4n^2\right)}e^{in\varpi t}\right)
\end{aligned}
$$

We see that we obtained the same result using the classical Fourier series form.

Chapter 3

Final project. Dynamic Analysis of the Elizabeth Ashman Bridge

3.1 Solution

3.1.1 Introduction

current database for the bridge, in the format of SDB SAP2000 1.5 version is SBD file

Results of each step are given in separate section. Each section has two parts, the first shows the results and the second describes the methods and analysis performed to obtain the results.

3.1.2 step one. Displacements at joints S15L, S07L and 21

3.1.2.1 Results

Joint	U1 ft	U2 ft	U3 ft	R1 rad	R2 rad	R3 rad
S07L	0.000179	-0.003174	0.021538	-0.000119	0.000098	4.253E-06
S15L	0.000035	-0.003104	-0.032437	-0.000216	-0.001357	0.000029

Joint	U1 ft	U2 ft	U3 ft	R1 rad	R2 rad	R3 rad
21	0.007568	-0.002749	-0.024066	-0.000011	0.001533	-0.000120

Table 3.1: Displacements at joint 21

3.1.2.2 Method used

Problem description is

> Find the deflections of the arch portion of the bridge at node or joint S15L and S07L when a 10k downward point load is applied at joint S15L. Also find the displacements of joint 21 on the ramp when a 10k downward load is applied at that joint.

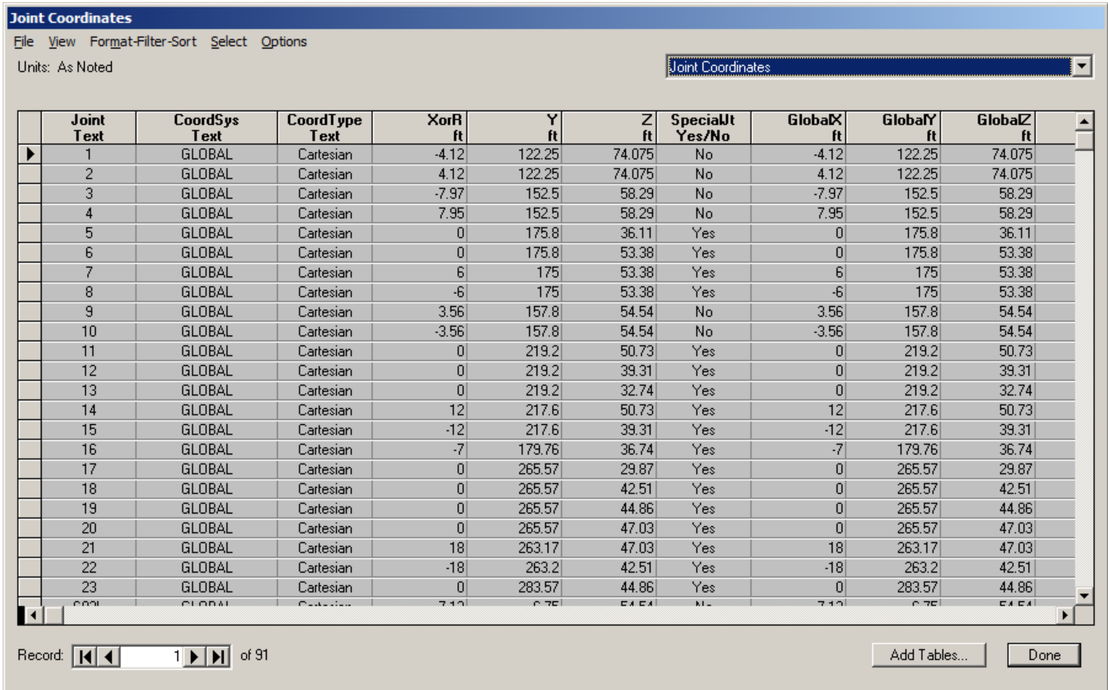
There are the steps performed

1. The original bridge database was not complete. The missing joints were first added. After opening the database, the `XZ` view was selected. This is needed as it was found it is not possible to add a point in the default 3D view.
2. Clicked on the `Draw Special joint` icon located on the left edge of the window. This is the small blue square in version 15 of SAP2000.
3. Clicked on an empty area on the screen to add a point.
4. Right clicked on the added point again to bring up a pop-up menu dialogue that was used for data entry of given coordinates.
5. Filled the coordinates and the labels as given in the PDF file.
6. Made sure that the menu item in the `JOINT COORDINATES` called `SPECIAL Jt (User Def)` is labeled `YES`. If this is labeled `NO` then this procedure did not work and the point was not added.
7. Clicked `UPDATE DISPLAY` then clicked `OK`.
8. Verified that the points were added by selecting `DISPLAY->SHOW TABLES` then using the pop-up menu and searched `Joint Coordinates`
9. Figure 3.1 shows part of the joints coordinates table after completing the above steps. Partial listing of joints is shown below

SAP2000 v15.0.1 5/2/13 22:18:29
Table: Joint Coordinates

Joint	CoordSys	CoordType	XorR ft	Y ft	Z ft	SpecialJt	GlobalX ft	GlobalY ft	GlobalZ ft
1	GLOBAL	Cartesian	-4.1200	122.2500	74.0750	No	-4.1200	122.2500	74.0750
2	GLOBAL	Cartesian	4.1200	122.2500	74.0750	No	4.1200	122.2500	74.0750
3	GLOBAL	Cartesian	-7.9700	152.5000	58.2900	No	-7.9700	152.5000	58.2900
4	GLOBAL	Cartesian	7.9500	152.5000	58.2900	No	7.9500	152.5000	58.2900
5	GLOBAL	Cartesian	0.0000	175.8000	36.1100	Yes	0.0000	175.8000	36.1100
6	GLOBAL	Cartesian	0.0000	175.8000	53.3800	Yes	0.0000	175.8000	53.3800
7	GLOBAL	Cartesian	6.0000	175.0000	53.3800	Yes	6.0000	175.0000	53.3800
8	GLOBAL	Cartesian	-6.0000	175.0000	53.3800	Yes	-6.0000	175.0000	53.3800
9	GLOBAL	Cartesian	3.5600	157.8000	54.5400	No	3.5600	157.8000	54.5400
10	GLOBAL	Cartesian	-3.5600	157.8000	54.5400	No	-3.5600	157.8000	54.5400
11	GLOBAL	Cartesian	0.0000	219.2000	50.7300	Yes	0.0000	219.2000	50.7300
12	GLOBAL	Cartesian	0.0000	219.2000	39.3100	Yes	0.0000	219.2000	39.3100
13	GLOBAL	Cartesian	0.0000	219.2000	32.7400	Yes	0.0000	219.2000	32.7400
14	GLOBAL	Cartesian	12.0000	217.6000	50.7300	Yes	12.0000	217.6000	50.7300
15	GLOBAL	Cartesian	-12.0000	217.6000	39.3100	Yes	-12.0000	217.6000	39.3100
16	GLOBAL	Cartesian	-7.0000	179.7600	36.7400	Yes	-7.0000	179.7600	36.7400
17	GLOBAL	Cartesian	0.0000	265.5700	29.8700	Yes	0.0000	265.5700	29.8700
18	GLOBAL	Cartesian	0.0000	265.5700	42.5100	Yes	0.0000	265.5700	42.5100
19	GLOBAL	Cartesian	0.0000	265.5700	44.8600	Yes	0.0000	265.5700	44.8600
20	GLOBAL	Cartesian	0.0000	265.5700	47.0300	Yes	0.0000	265.5700	47.0300
21	GLOBAL	Cartesian	18.0000	263.1700	47.0300	Yes	18.0000	263.1700	47.0300
22	GLOBAL	Cartesian	-18.0000	263.2000	42.5100	Yes	-18.0000	263.2000	42.5100
23	GLOBAL	Cartesian	0.0000	283.5700	44.8600	Yes	0.0000	283.5700	44.8600

10. Connected the joints added above to the bridge in order to establish the ramp. Figure 3.2 is screen shot showing the ramp connected to bridge. RBEAM elements are used.



Joint Text	CoordSys Text	CoordType Text	XorR ft	Y ft	Z ft	SpecialJt Yes/No	GlobalX ft	GlobalY ft	GlobalZ ft
1	GLOBAL	Cartesian	-4.12	122.25	74.075	No	-4.12	122.25	74.075
2	GLOBAL	Cartesian	4.12	122.25	74.075	No	4.12	122.25	74.075
3	GLOBAL	Cartesian	-7.97	152.5	58.29	No	-7.97	152.5	58.29
4	GLOBAL	Cartesian	7.95	152.5	58.29	No	7.95	152.5	58.29
5	GLOBAL	Cartesian	0	175.8	36.11	Yes	0	175.8	36.11
6	GLOBAL	Cartesian	0	175.8	53.38	Yes	0	175.8	53.38
7	GLOBAL	Cartesian	6	175	53.38	Yes	6	175	53.38
8	GLOBAL	Cartesian	-6	175	53.38	Yes	-6	175	53.38
9	GLOBAL	Cartesian	3.56	157.8	54.54	No	3.56	157.8	54.54
10	GLOBAL	Cartesian	-3.56	157.8	54.54	No	-3.56	157.8	54.54
11	GLOBAL	Cartesian	0	219.2	50.73	Yes	0	219.2	50.73
12	GLOBAL	Cartesian	0	219.2	39.31	Yes	0	219.2	39.31
13	GLOBAL	Cartesian	0	219.2	32.74	Yes	0	219.2	32.74
14	GLOBAL	Cartesian	12	217.6	50.73	Yes	12	217.6	50.73
15	GLOBAL	Cartesian	-12	217.6	39.31	Yes	-12	217.6	39.31
16	GLOBAL	Cartesian	-7	179.76	36.74	Yes	-7	179.76	36.74
17	GLOBAL	Cartesian	0	265.57	29.87	Yes	0	265.57	29.87
18	GLOBAL	Cartesian	0	265.57	42.51	Yes	0	265.57	42.51
19	GLOBAL	Cartesian	0	265.57	44.86	Yes	0	265.57	44.86
20	GLOBAL	Cartesian	0	265.57	47.03	Yes	0	265.57	47.03
21	GLOBAL	Cartesian	18	263.17	47.03	Yes	18	263.17	47.03
22	GLOBAL	Cartesian	-18	263.2	42.51	Yes	-18	263.2	42.51
23	GLOBAL	Cartesian	0	283.57	44.86	Yes	0	283.57	44.86

Figure 3.1: Adding missing joints to bridge database

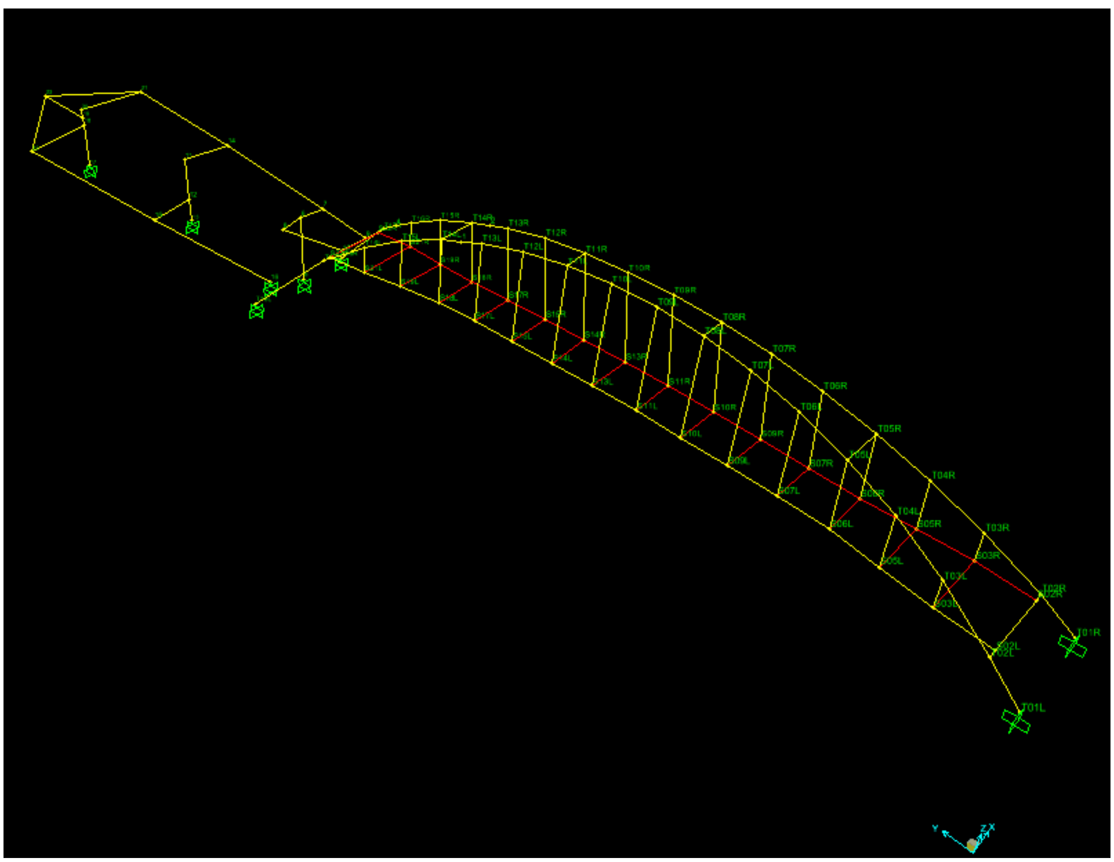


Figure 3.2: connected ramp to bridge using RBEAMS

11. Before adding the 10 kips downwards load, a load pattern is defined. Selected `DEFINE->LOAD PATTERNS` and added new load pattern called `S15L` of type `DEAD` with self weight multiplier 0.

12. 10 kips downwards load at joint `S15L` was added. This was done by clicking on the joint and right clicking again. Using the pop up menu that appeared the value minus 10 was entered. Minus sign was used since load is downwards. The load pattern selected was `S15L`. Figure 3.3 shows the result.

13. Clicked on `RUN ANALYSIS`. In the `set load case to run` case `S15L` was the only one selected. All other load cases, including `DEAD` was not selected. This was done to obtain result due to vertical load only. Model was locked now. After run was completed, clicked on `DISPLAY->SHOW TABLES->JOINT DISPACEMENTS` and located nodes `S15L` and `S07L` to find the node displacements. Figure 3.4 shows the result of this step In addition a listing from the table is shown below

```
SAP2000 v15.0.1    5/3/13 1:35:52
Table:  Joint Displacements
 Joint OutputCase  CaseType      U1          U2          U3          R1         R2         R3
                                 ft          ft          ft        Radians    Radians    Radians

 S07L    S15L    LinStatic    0.000179  -0.003174   0.021538  -0.000119   0.000098  4.253E-
06
 S15L    S15L    LinStatic    0.000035  -0.003104  -0.032437  -0.000216  -0.001357  0.000029
```

14. Before adding the 10 kips downwards load to node 21, a load pattern is defined for use. Selected `DEFINE->LOAD PATTERNS` and added new load pattern called `node21` of type `DEAD` with self weight multiplier 0.

15. 10 kips downwards load at joint `21` was now added. This was done by clicking on the joint and right clicking aging. Using the pop-up menu that appeared the value minus 20 was entered. Minus sign was used since load is downwards. The load pattern selected was `node20`. Figure 3.5 shows this step.

16. Clicked on `RUN ANALYSIS`. In the `setload case to run` case `node21` was the only one selected. All other load cases, including `DEAD` was not selected. This was done to obtain result due to vertical load only. Model was locked now. After run was completed, clicked on `DISPLAY->SHOW TABLES->JOINT DISPACEMENTS` and located nodes `21` to find the node displacements. Figure 3.6 shows the result. Listing from the table is shown below

```
SAP2000 v15.0.1    5/3/13 2:33:04
Table:  Joint Displacements
 Joint  OutputCase  CaseType      U1          U2          U3          R1         R2         R3
                                  ft          ft          ft        Radians    Radians    Radians

  21      node21    LinStatic   0.007568  -0.002749  -0.024066  -0.000011 0.001533  -
0.000120
```

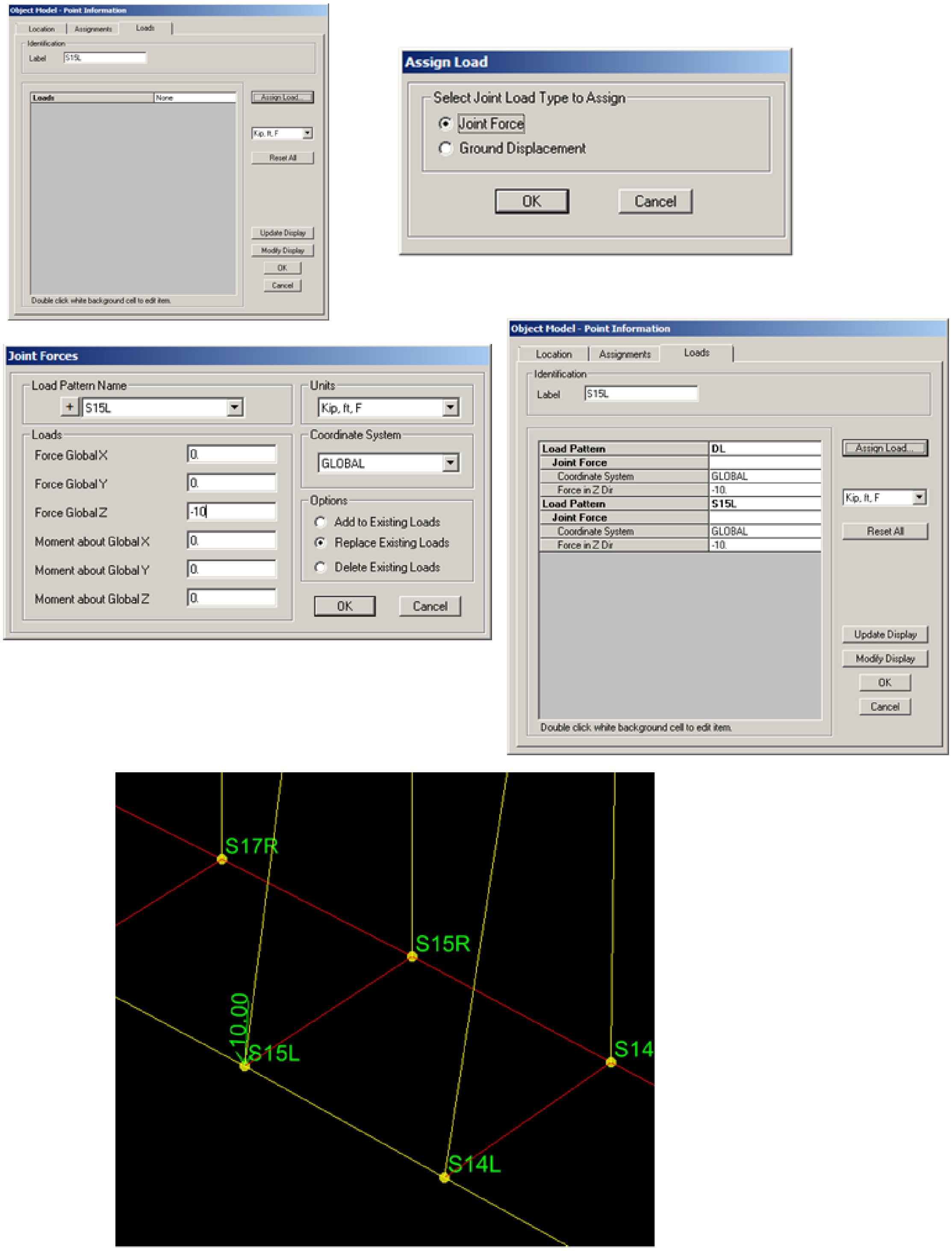


Figure 3.3: adding vertical load pattern for step one use

Joint Displacements

File View Format-Filter-Sort Select Options

Units: As Noted

Filter: Joint = 'S07L'

Joint Displacements

Joint Text	OutputCase Text	CaseType Text	U1 ft	U2 ft	U3 ft	R1 Radians	R2 Radians	R3 Radians
S07L	S15L	LinStatic	0.000179	-0.003174	0.021538	-0.000119	0.000098	0.000004253

Joint Displacements

File View Format-Filter-Sort Select Options

Units: As Noted

Filter: Joint = 'S15L'

Joint Displacements

Joint Text	OutputCase Text	CaseType Text	U1 ft	U2 ft	U3 ft	R1 Radians	R2 Radians	R3 Radians
S15L	S15L	LinStatic	0.000035	-0.003104	-0.032437	-0.000216	-0.001357	0.000029

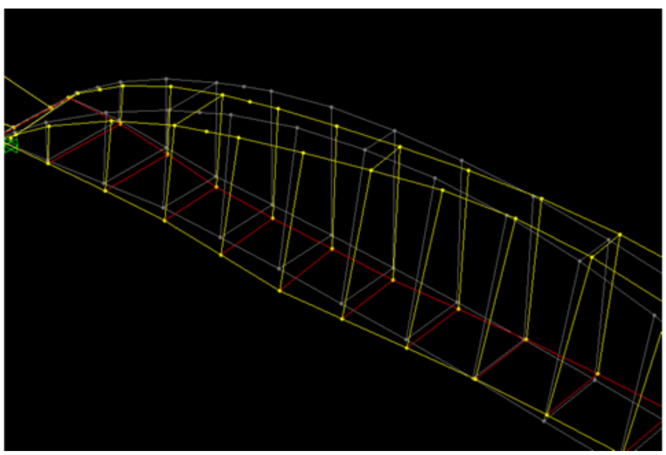


Figure 3.4: adding vertical load to joint S15L

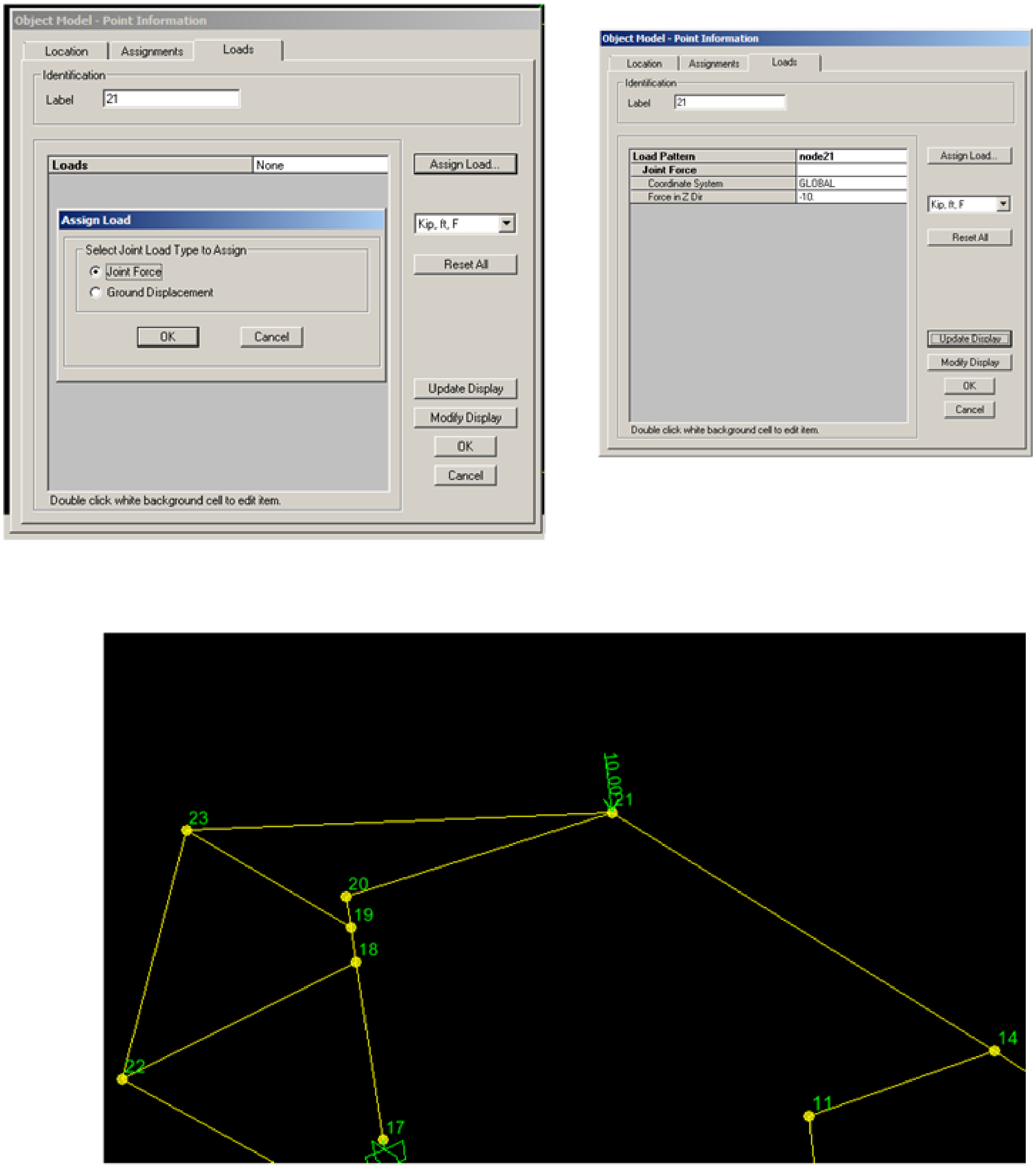
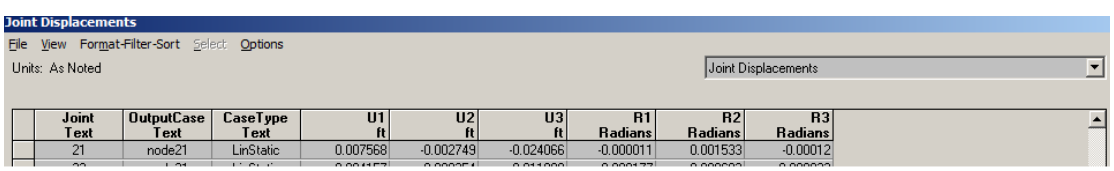


Figure 3.5: adding vertical load pattern for step one use



Joint Text	OutputCase Text	CaseType Text	U1 ft	U2 ft	U3 ft	R1 Radians	R2 Radians	R3 Radians
21	node21	LinStatic	0.007568	-0.002749	-0.024066	-0.000011	0.001533	-0.00012

Figure 3.6: adding vertical load to joint 21

3.1.3 Step two, period and damping calculations

3.1.3.1 Results

The result is shown in table 3.2

Natural period T (sec)	Natural frequency f_n (hz)	critical damping ratio ζ
0.5	2.0	0.0014%

Table 3.2: Period and damping

3.1.3.2 Method used

This is the problem description

> Two people jogging across the bridge created the vertical acceleration records shown below. Each set of pulses is when the joggers were running, in between they stopped. Once they stopped it is as if the bridge had an initial displacement and velocity and then decayed in free vibration. Using the enlarged portion of the record estimate - the natural period of the structure and the

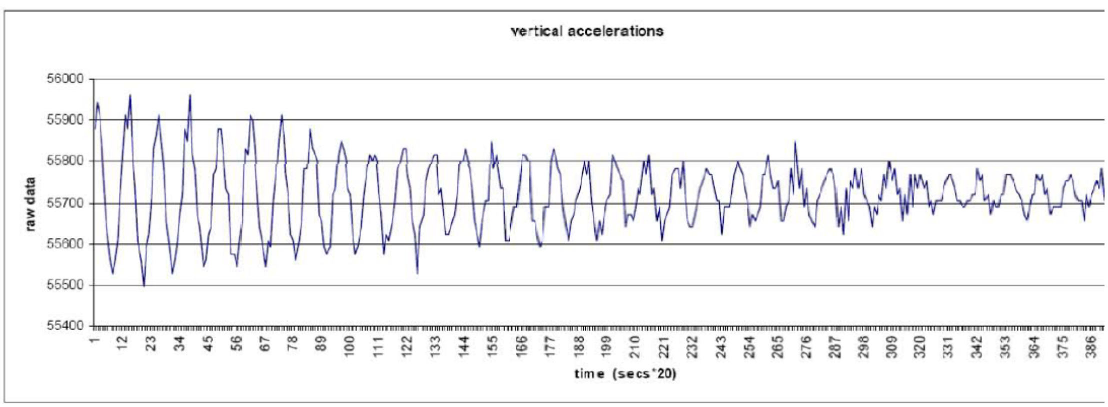


Figure 3.7: vertical acceleration time records

The above profile can be used as free the vibration profile. The method of logarithmic decrement was used to obtain the natural period and ζ (damping critical coefficient). Figure 3.8 shows a closer zoom view of the above plot in order to estimate the period. It shows the natural period to be around 10 division.

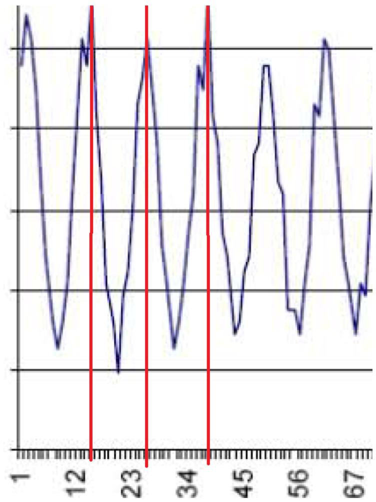


Figure 3.8: zoomed view on the vertical acceleration time records

The units used are `sec*20`, therefore natural period is $T = \frac{10}{20} = 0.5$ sec. Hence natural frequency is $f = 2$ hz.

To obtain the damping ζ, a number of methods can be used. The more accurate methods uses more peaks. Using $N = 35$ as number of peaks and using method of series expansion ζ can be found. From the above plot the value of first peak is 55940 and value of peak number 35 was found to be 55770. Hence

$$\begin{aligned}\frac{y_0}{y_0+N} &= 1 + 2\pi N\zeta \\ \zeta &= \frac{1}{35(2\pi)}\frac{55940-55770}{55770} \\ &= 1.3861\times 10^{-5} \\ &= 0.0014\%\end{aligned}$$

3.1.4 Step three. Modal analysis

3.1.4.1 Results

The following are the modal analysis results. Mode 3 has period 0.426531 seconds and natural frequency 2.3445 hz.

```
SAP2000 v15.0.1    5/3/13 3:29:56
Table:  Modal Periods And Frequencies
```

OutputCase	StepType	StepNum	Period Sec	Frequency Cyc/sec	CircFreq rad/sec	Eigenvalue rad2/sec2
Modal	Mode	1.000000	0.486993	2.0534E+00	1.2902E+01	1.6646E+02
Modal	Mode	2.000000	0.435780	2.2947E+00	1.4418E+01	2.0789E+02
Modal	Mode	3.000000	0.426531	2.3445E+00	1.4731E+01	2.1700E+02

Modal	Mode	4.000000	0.352227	2.8391E+00	1.7838E+01	3.1821E+02
Modal	Mode	5.000000	0.321345	3.1119E+00	1.9553E+01	3.8231E+02
Modal	Mode	6.000000	0.268232	3.7281E+00	2.3424E+01	5.4871E+02
Modal	Mode	7.000000	0.258425	3.8696E+00	2.4313E+01	5.9114E+02
Modal	Mode	8.000000	0.249385	4.0099E+00	2.5195E+01	6.3477E+02

In this description, reference is made to different view angles. Figure 3.48 shows the axis orientation used by SAP2000.

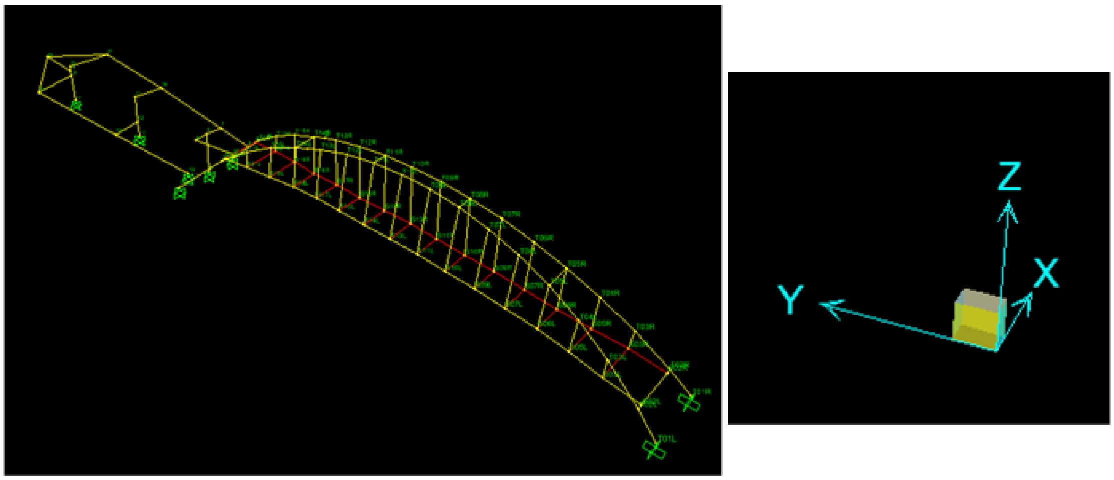


Figure 3.9: 3D axis orientation used

The maximum stress at the base of the column (label 11) in the ramp was also found for each mode. This was done using SAP2000 v15.1 which has this added feature. The following diagrams give stress S11 for each mode.

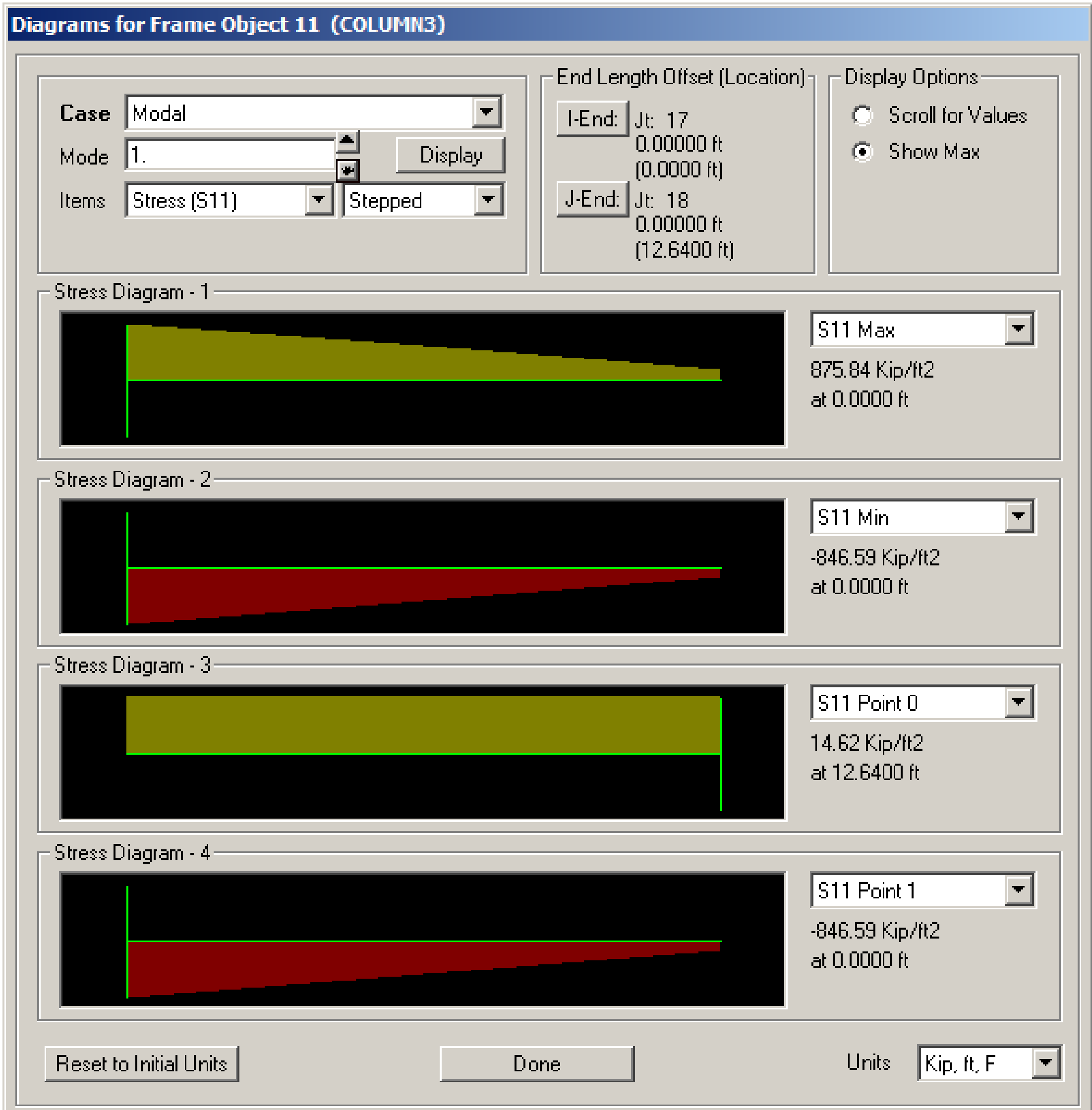


Figure 3.10: Stress at base of column, mode 1

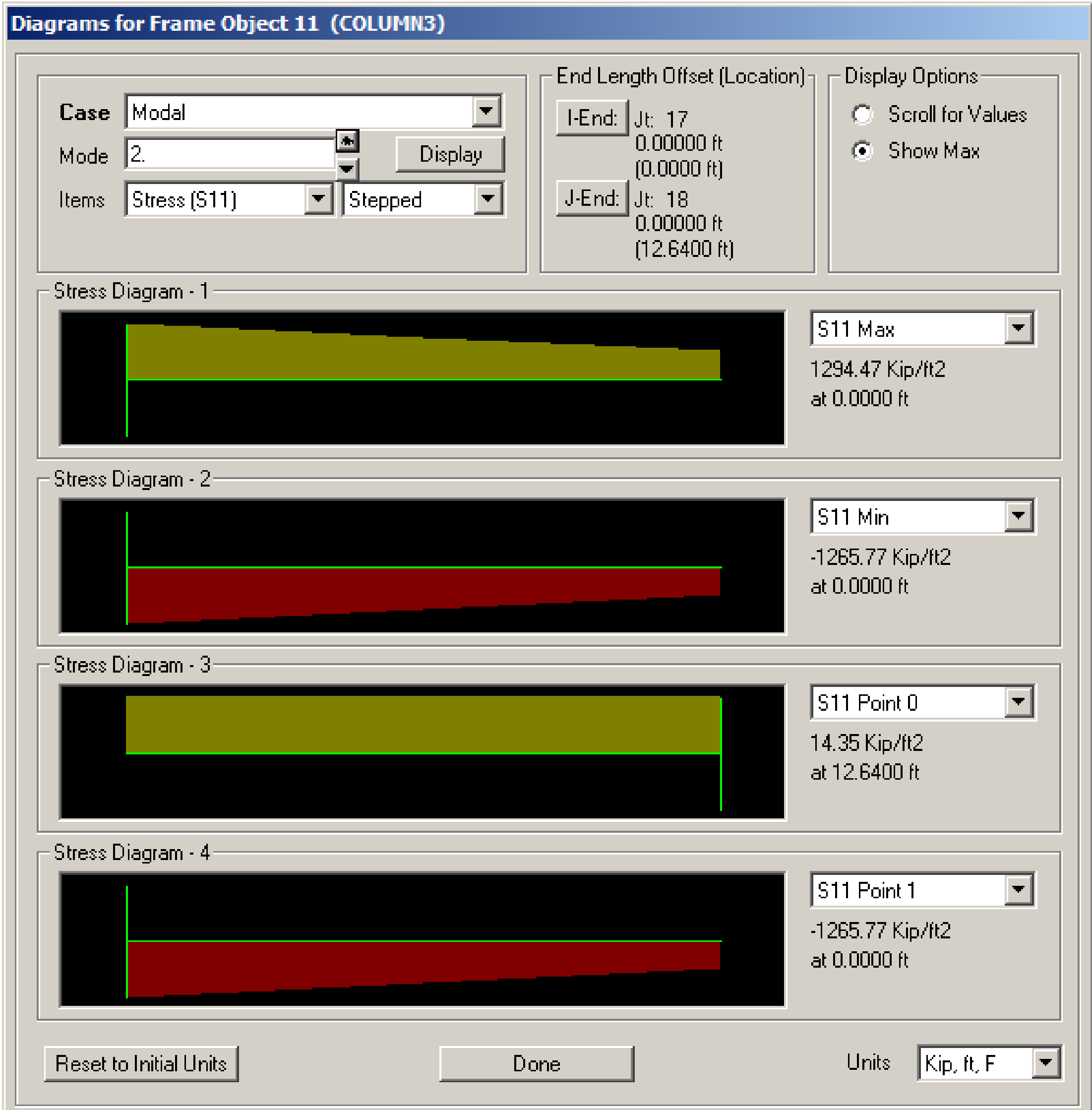


Figure 3.11: Stress at base of column, mode 2

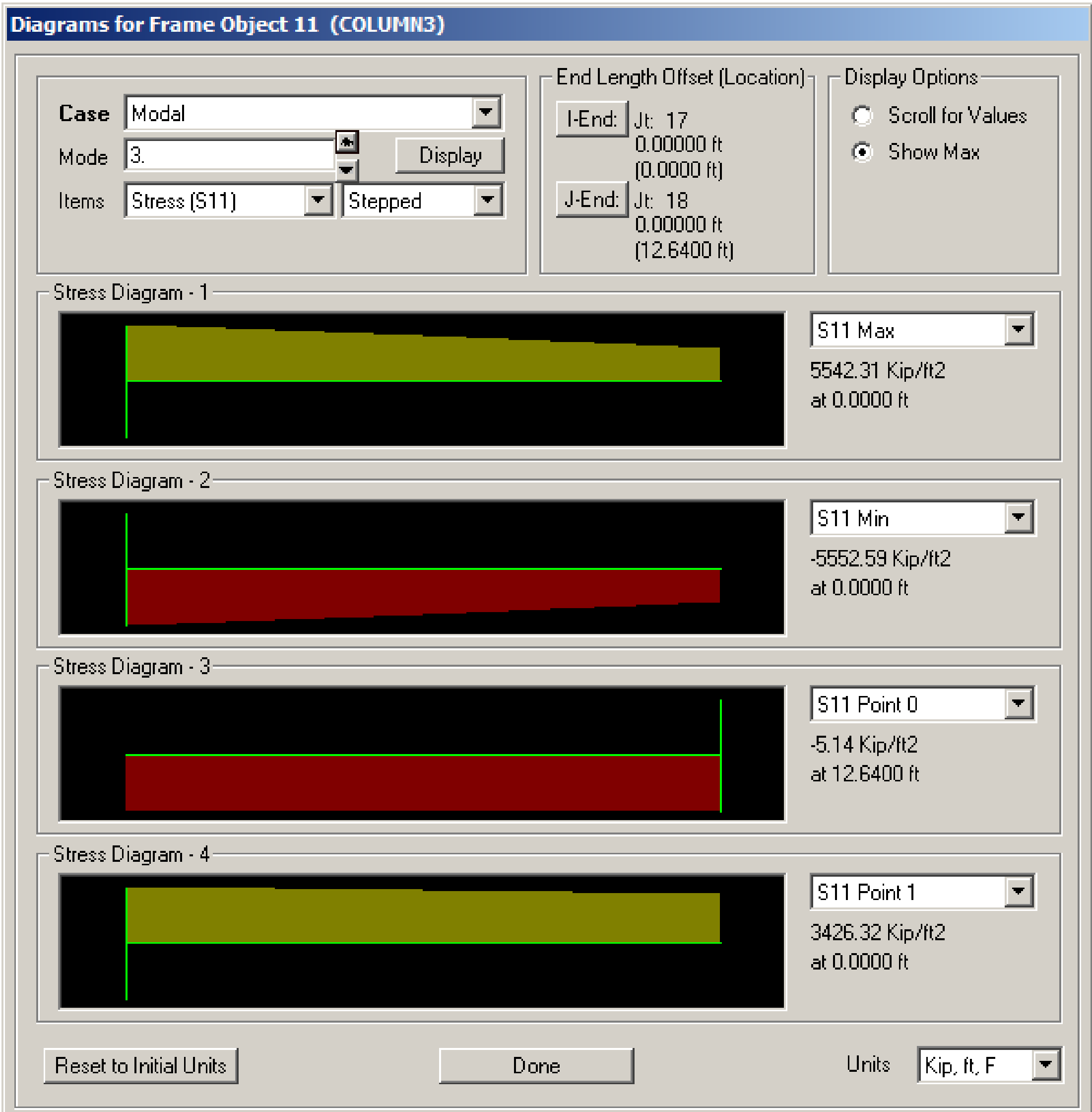


Figure 3.12: Stress at base of column, mode 3

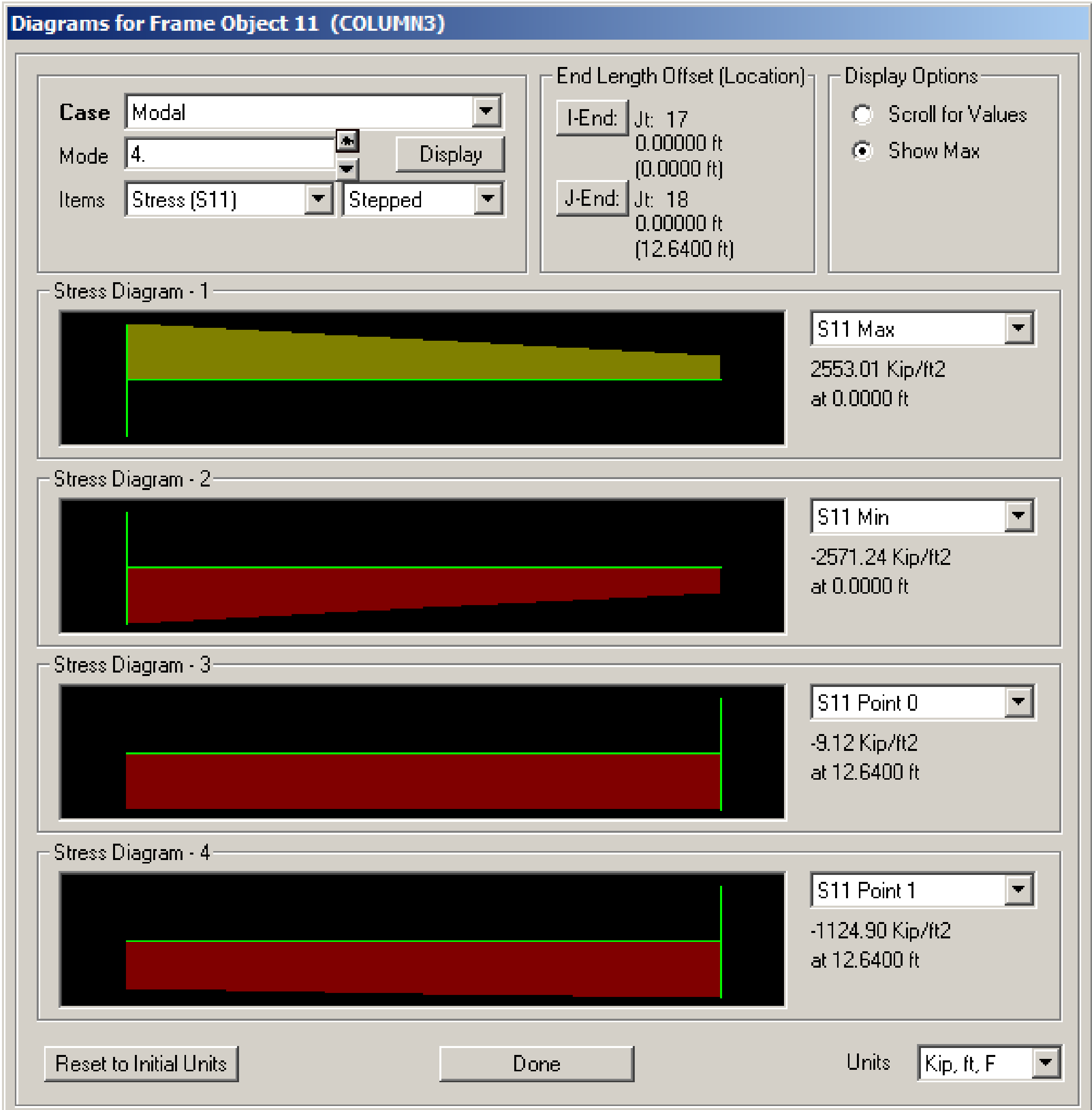


Figure 3.13: Stress at base of column, mode 4

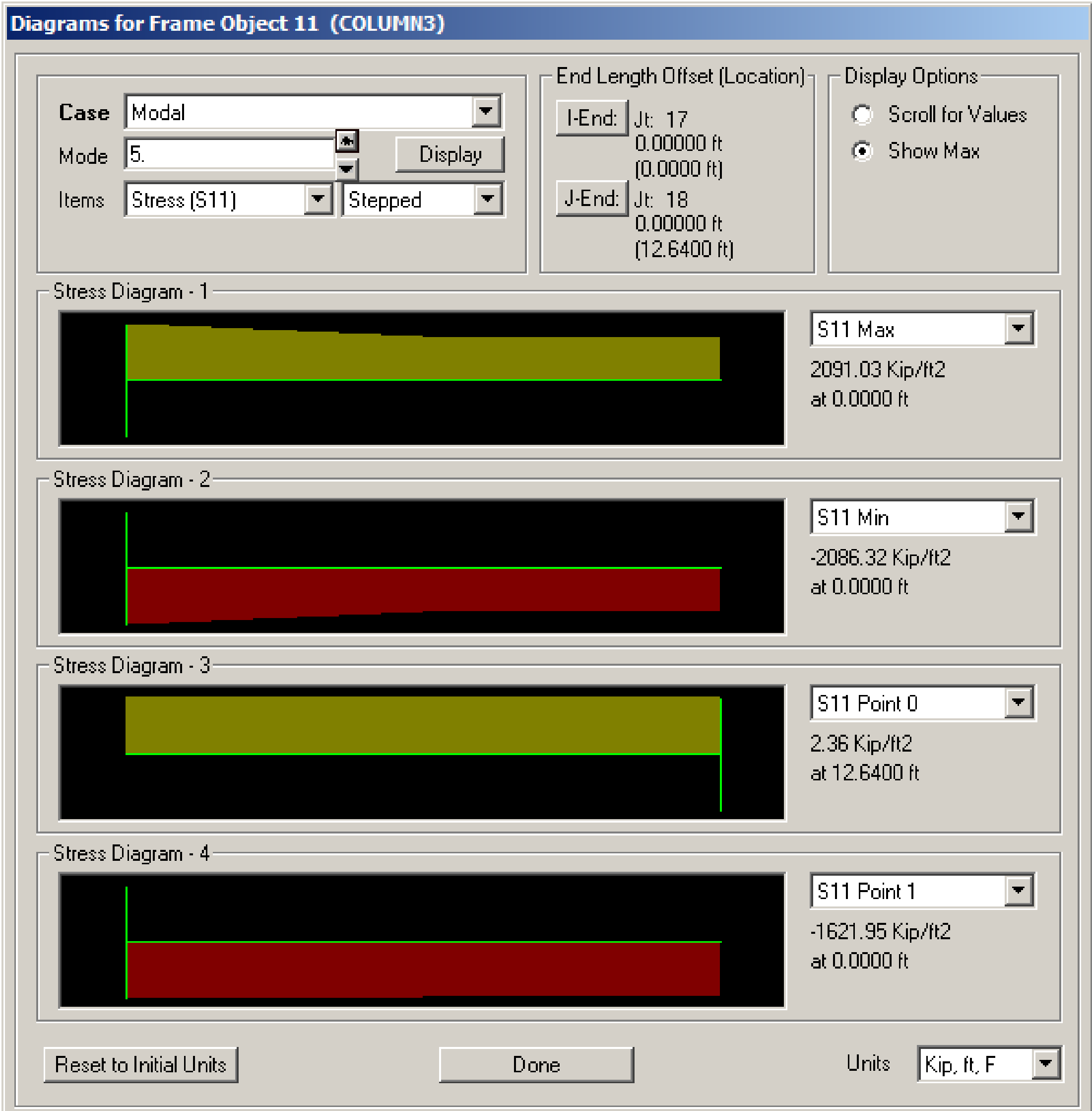


Figure 3.14: Stress at base of column, mode 5

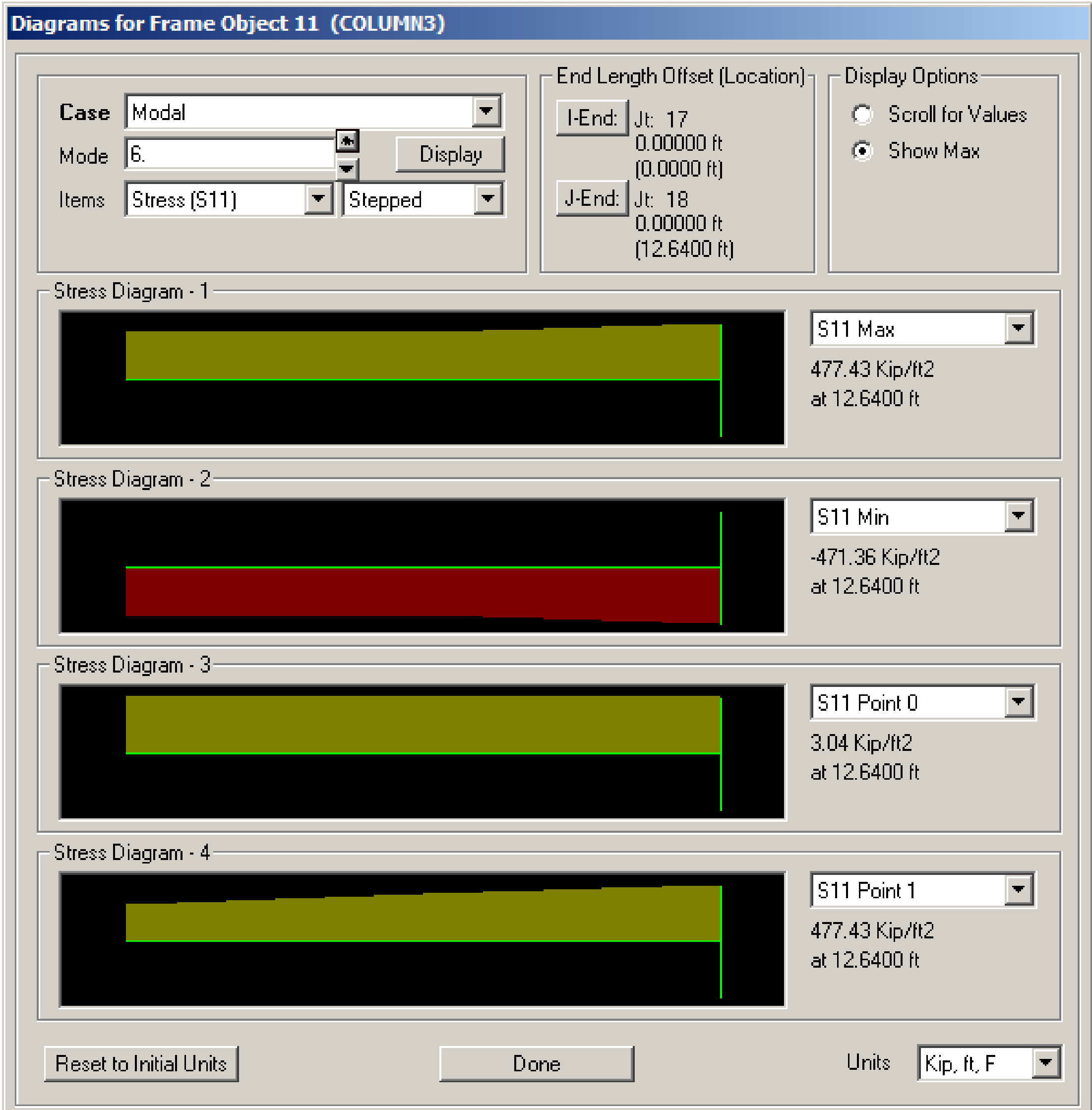


Figure 3.15: Stress at base of column, mode 6

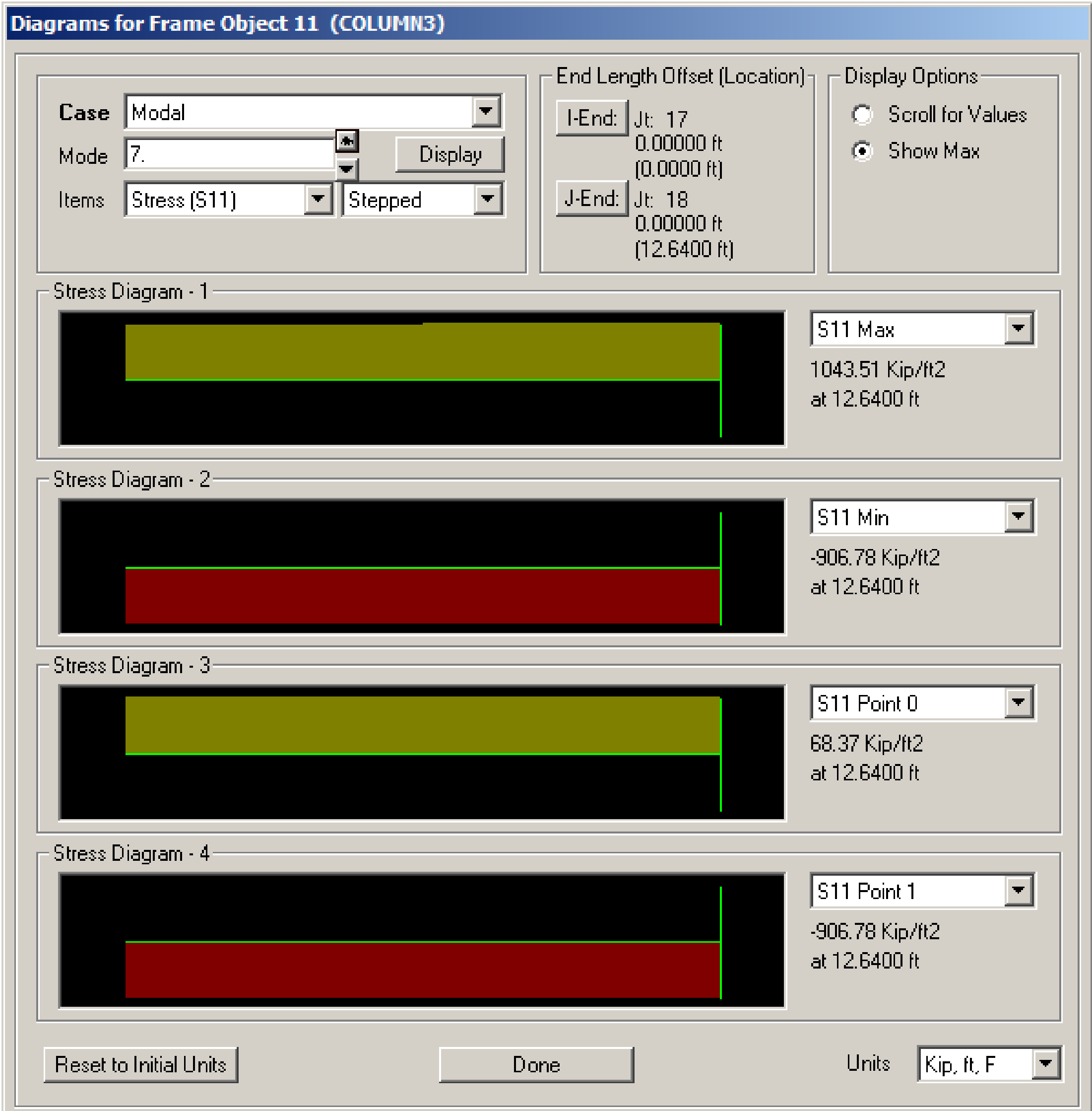


Figure 3.16: Stress at base of column, mode 7

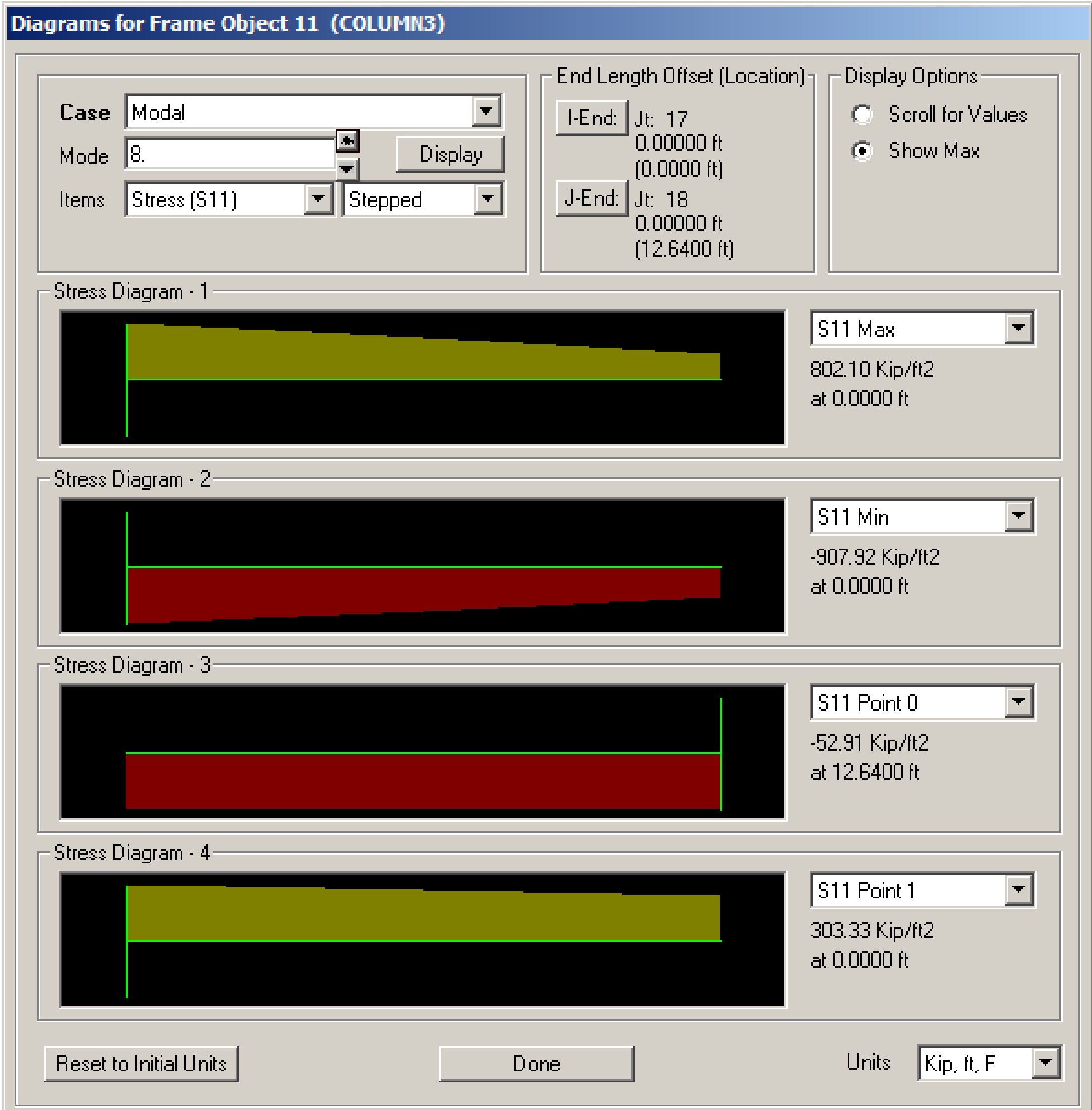


Figure 3.17: Stress at base of column, mode 8

3.1.5 Step four. Solving for response under simulated marching band

3.1.5.1 Results

The nodes to find the displacements for are marked and given in figure ??.

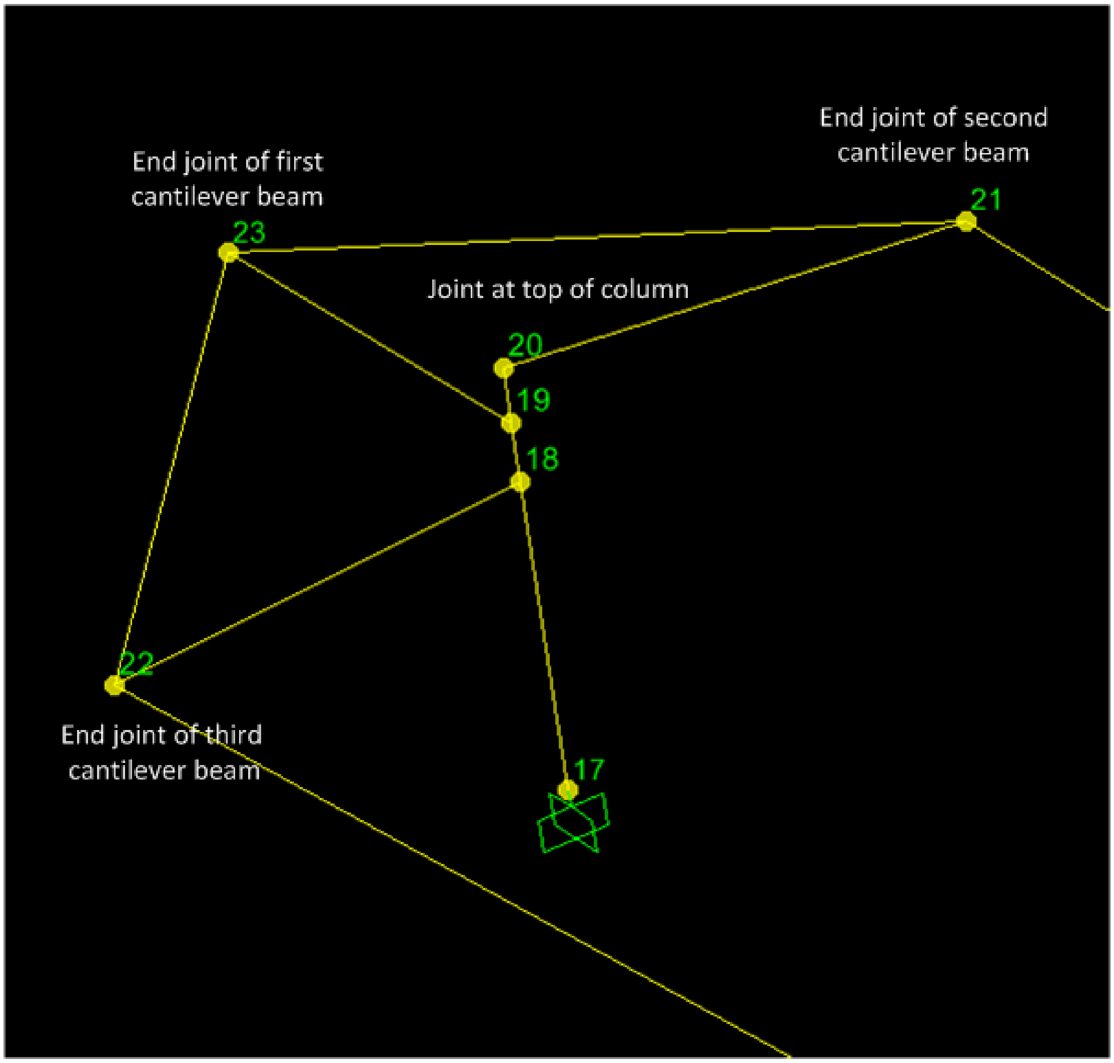


Figure 3.18: node locations for cantilever beams

The result is shown below. The labels for local axes for joints are shown below, and are the same as the global axes. This is from SAP2000 help section

> `By default, the joint local 1-2-3 coordinate system is identical to the global X-Y-Z coordinate system`

Therefore, `U1` is in the X direction, and `U2` in the Y direction, and `U3` is the vertical displacement.

```
SAP2000 v15.0.1    5/4/13 1:02:04
Table:  Joint Displacements
 Joint  OutputCase  StepType     U1          U2          U3          R1          R2          R
                                 ft          ft          ft        Radians     Radians     Rad
  20       COMO        Max     0.146711    0.019285   -0.000479    0.001667    0.015544    0.00
  20       COMO        Min    -0.141382   -0.017992   -0.000676   -0.001788   -0.013209   -0.00

  21       COMO        Max     0.144476    0.034315    0.262294    0.002986    0.022603    0.00
```

```
 21      COMO      Min    -0.139764   -0.037636   -0.375865    0.000383   -0.015478   -0.0012

 22      COMO      Max     0.082805    0.009682    0.236030    0.003108    0.014103   -0.0000
 22      COMO      Min    -0.083333   -0.005028   -0.305074   -0.000501   -0.018799   -0.0003

 23      COMO      Max     0.123308    0.015499    0.013802    0.001111    0.015690    0.0005
 23      COMO      Min    -0.123593   -0.014890   -0.049072   -0.002825   -0.014603   -0.0004
```

Figure 3.19 shows screen shot of the deformed part of the ramp with the above joints marked on the diagram showing the relative displacement for better illustration.

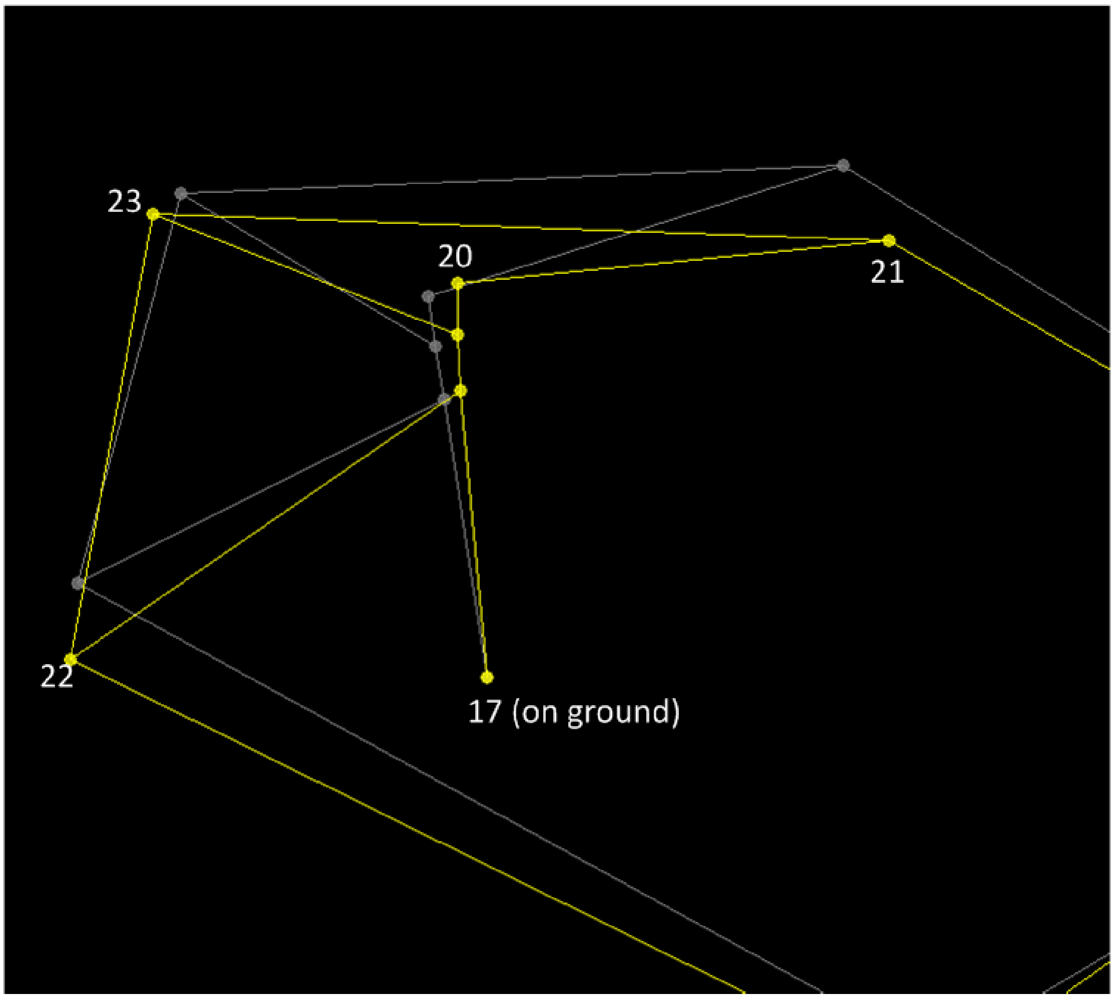


Figure 3.19: relative displacements of joints on ramp

The following text file contains the result for all nodes. `step_4_beam_result.txt`

In addition, below are plots of nodal displacements of node 20, on top of column labeled 11 on the ramp (this is the column being analyzed for stress). This plot shows that it took about 20 seconds for dynamic loading to settle down.

This means after 20 second of the marching band moving into the ramp, the ramp vibration reached steady state, therefore, the ramp is now vibrating at the same forcing frequency and transient response of the ramp has completed.

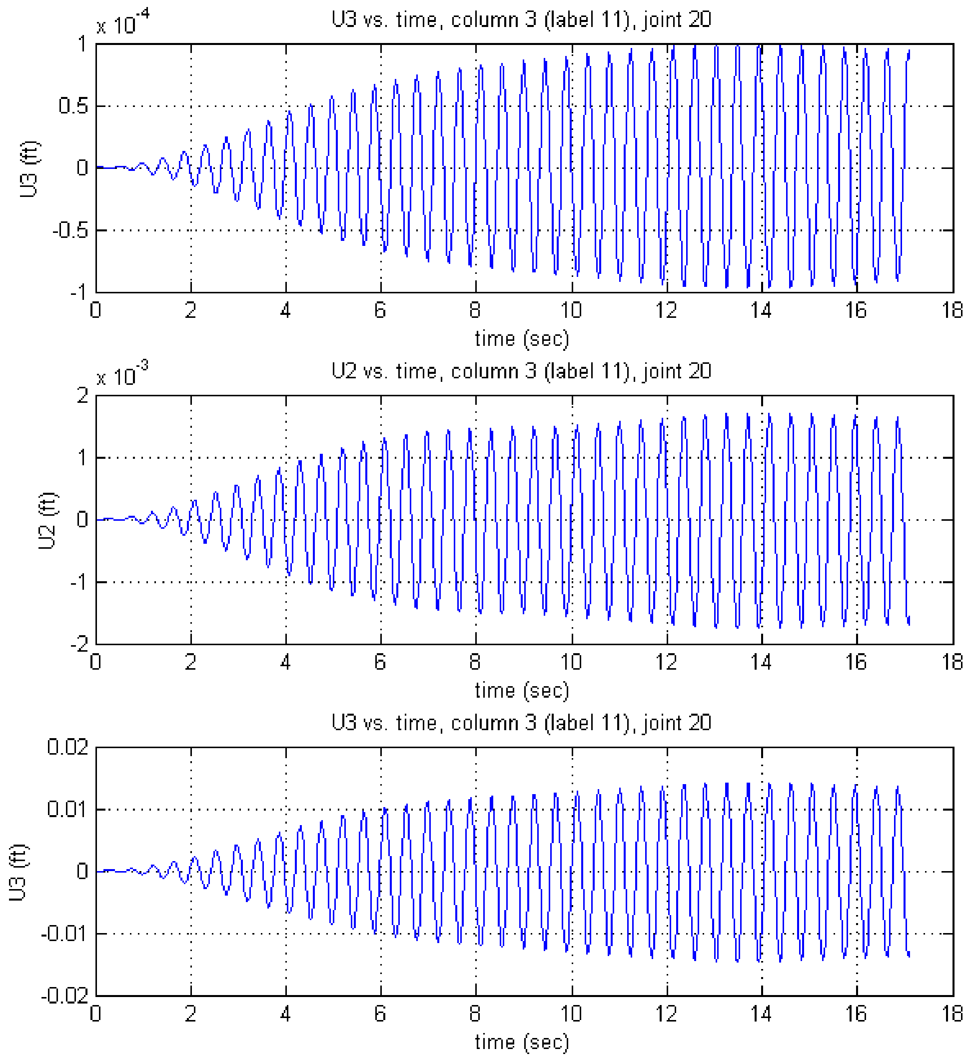


Figure 3.20: Displacement of node 20 on ramp column 3 during dynamic response

This is a plot the total axial load P on the column for the first 20 seconds.

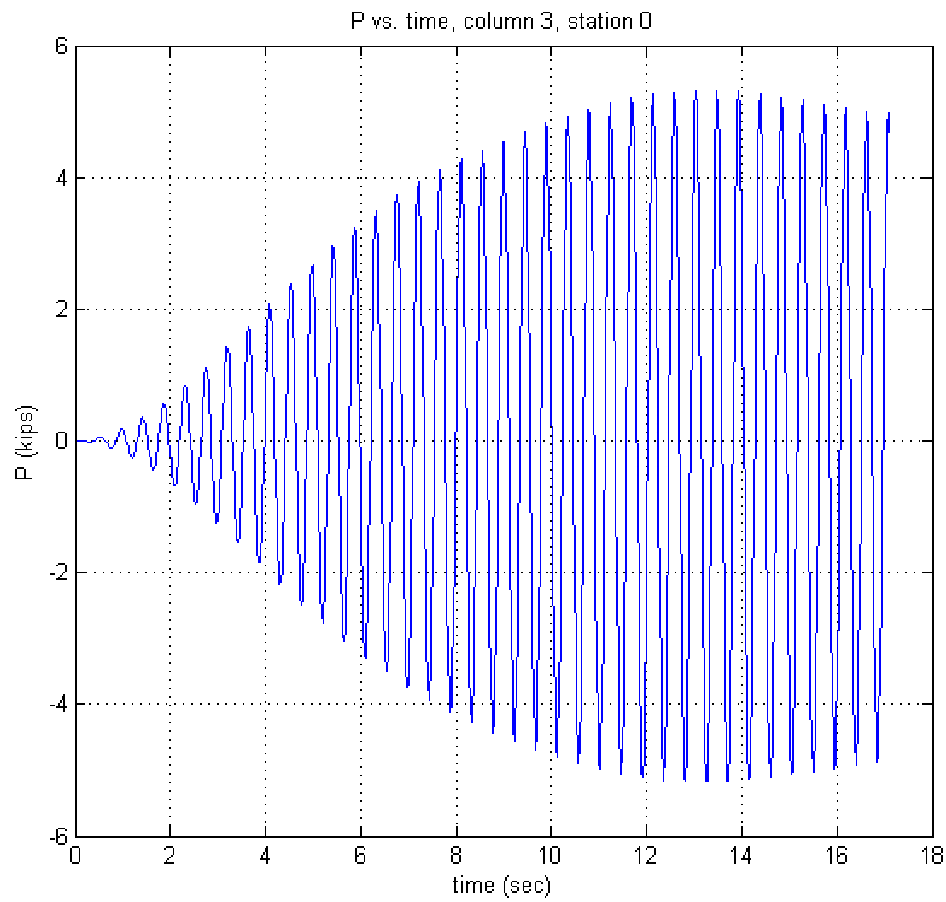


Figure 3.21: Axial load P variation in column during during dynamic excitation

This is movie of the first 20 seconds of the bridge vibration during marching band motion.

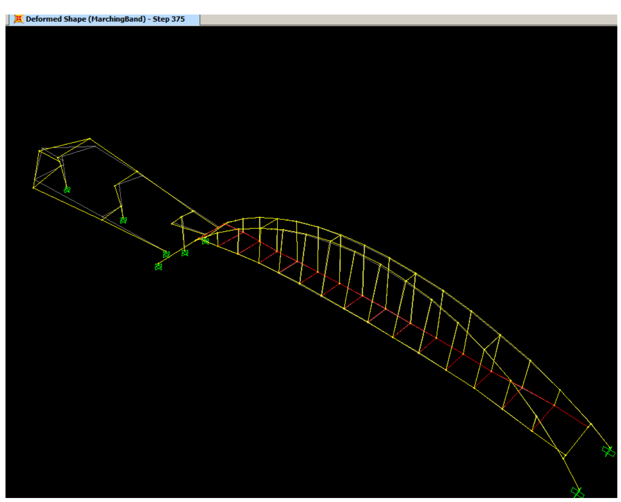


Figure 3.22: movie of first 20 seconds during marching band motion

Node displacement for joint 20 under marching band (time history) is given below. The output is in this file `node_20_final_displacement.txt`

This is partial listing of the table from SAP2000.

```
SAP2000 v15.0.1    5/3/13 5:24:47
Table:  Joint Displacements

 Joint  OutputCase   CaseType    StepType  StepNum      U1          U2          U3          R1          R2          R3
                                                        ft          ft          ft        Radians     Radians     Radians

   20 MarchingBand  LinModHist     Time    0.000000   0.000000    0.000000    0.000000    0.000000    0.000000    0.000000
   20 MarchingBand  LinModHist     Time    0.021400  -3.745E-07   6.625E-08   2.712E-10  -7.568E-09  -3.795E-08   2.215E-09
   20 MarchingBand  LinModHist     Time    0.042800  -2.849E-06   5.016E-07   2.063E-09  -5.715E-08  -2.887E-07   1.677E-08
   20 MarchingBand  LinModHist     Time    0.064200  -8.717E-06   1.522E-06   6.310E-09  -1.726E-07  -8.828E-07   5.090E-08
   20 MarchingBand  LinModHist     Time    0.085600   -0.000018   3.074E-06   1.289E-08  -3.463E-07  -1.804E-06   1.028E-07
```

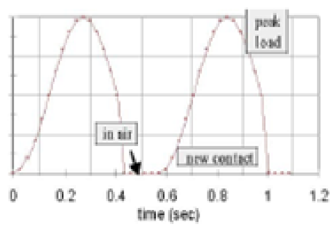
3.1.5.2 Method

Description of the problem is given below

4. Solve for the displacement response of the structure under the simulated marching band load shown below. Determine the peak displacements at the ends of the cantilever beams extending from the far north column and the peak displacements at the column top.

This step in the analysis is to estimate the possible response of the bridge to a marching band crossing and marching in step. This will involve placing a dynamic vertical loading on the bridge floor and solving for the elastic response of the structure.

The bridge will be considered to be under full dead load, with a reduced uniform live load of 40 psf (160 lb. person over 4 sq. ft. of floor), and a superimposed 30 psf varying live load. The 30 psf multiplies the following variation. The assumed variation (but not amplitude) in the live load is shown below and is intended to represent the effect of stepping down and then removing weight from the floor. So, when the load variation is -1 and multiplies the varying live load of 30psf, combined with the static live load of 40 psf, the resulting live load decreases to 10psf.



dynamic load variation

period = .542 seconds

The vertical live load (dynamic) will only be applied to half the ramp: from the arch to the turn-around. The dynamic load will be applied where you see green shading (see the Figure). Static vertical live load will be applied over the entire bridge floor. A file that provides the time variation of dynamic load, as shown above, will be on the web site – titled "load.dat". This time variation will have to be multiplied by the 30 psf amplitude in the SAP program. The "time history" load function option is used in SAP to define this load.

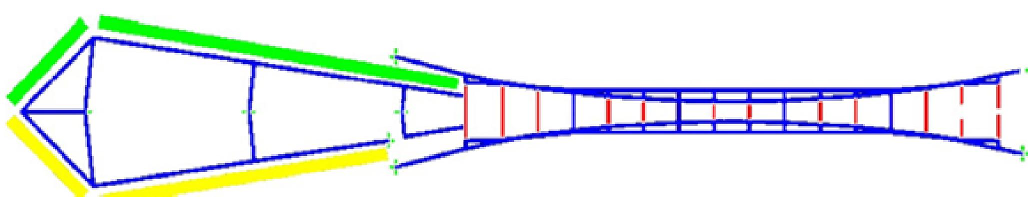


Figure 3.23: Description of step 4, solving for response under dynamic marching band

The following are the steps performed

1. Load patterns are first defined. In SAP2000, a load case uses a load pattern. Hence a load pattern must first be be defined. Load pattern tells SAP where the loads are while a load cases tells SAP how to apply a specific load pattern, for example, either statically or dynamically and also tells SAP how to perform the analysis, for example, either using modal or direct integration.

   Figure 3.24 shows the relation between load patterns and load cases as used in SAP2000.

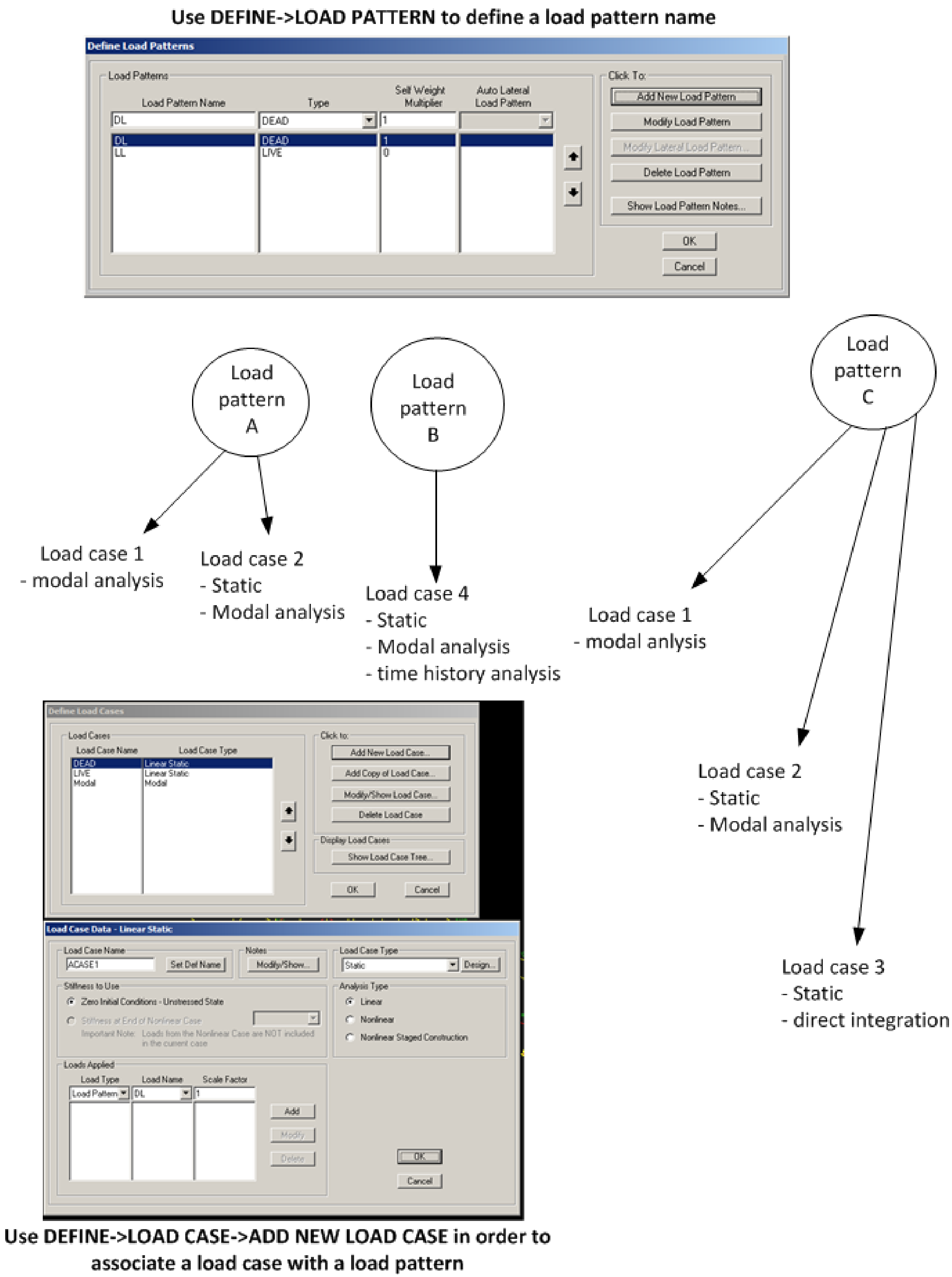


Figure 3.24: Relation between load pattern and load case

The first load pattern is live load. This is the load of people on the bridge and is present all the time. The bridge is 10 ft wide, and the problem says to use 40 lb per square feet, or 400 lb per linear feet.

Selected `DEFINE->LOAD PATTERNS` and wrote LL in the `Load Pattern Name` window. selected `LIVE` as type, and set self weight multiplier to 0 then clicked `Add New Load Pattern`. Figure 3.25 shows this step.

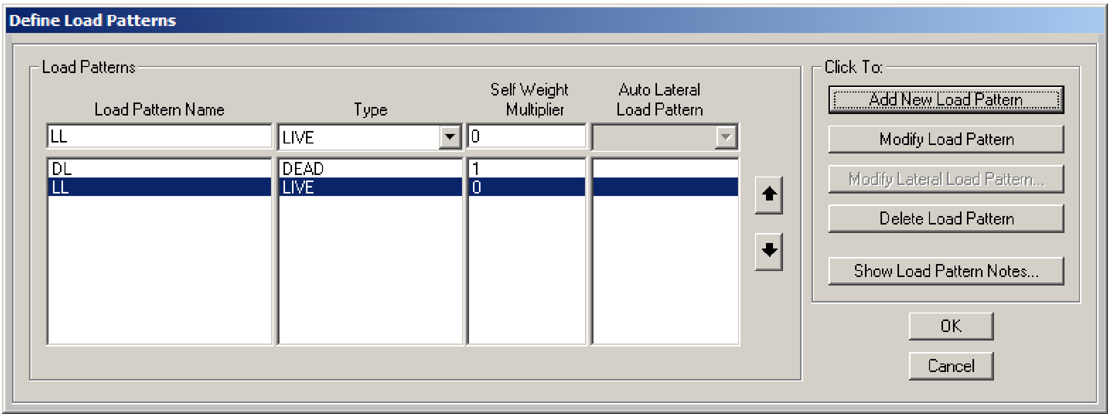


Figure 3.25: Defining live load pattern LL

2. Defined a new load pattern similar to the above called `DYNALOAD` of type `LIVE` and also a self weight of zero.
3. Selected the floor of the bridge using `SELECT->PROPERTIES->AREA SECTIONS->FLOOR`. Added load `LL` using `ASSIGN->AREA LOADS->UNIFORM(SHELL)` and selected `LL` for load pattern. Used 0.04 for the load amount. This is 40 psf. (or 400 lb per linear ft, since the bridge is 10 ft wide). Figure 3.26 shows this step.

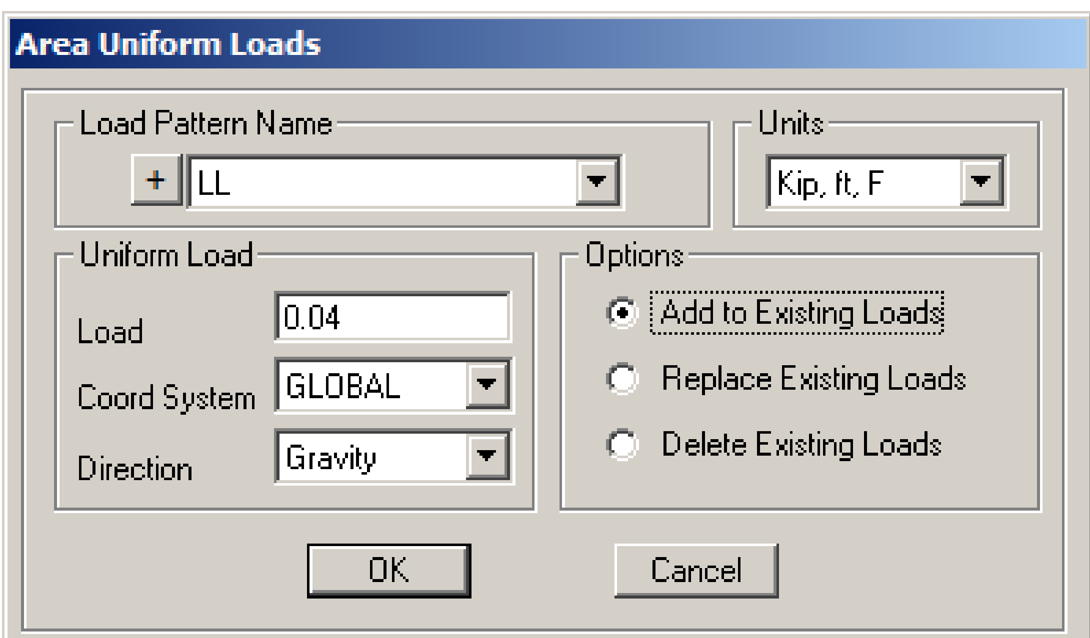


Figure 3.26: Adding live load to bridge floors

4. added 400 lb per linear ft also to on the ramp. `SELECT->PROPERTIES->FRAME SECTIONS->RBEAM` and as the ramp is selected clicked `ASSIGN->FRAME LOAD->DISTRIBUTED LOAD` and entered 400 (lb per linear ft). Load pattern `LL` was used. Figure 3.27 shows this step.

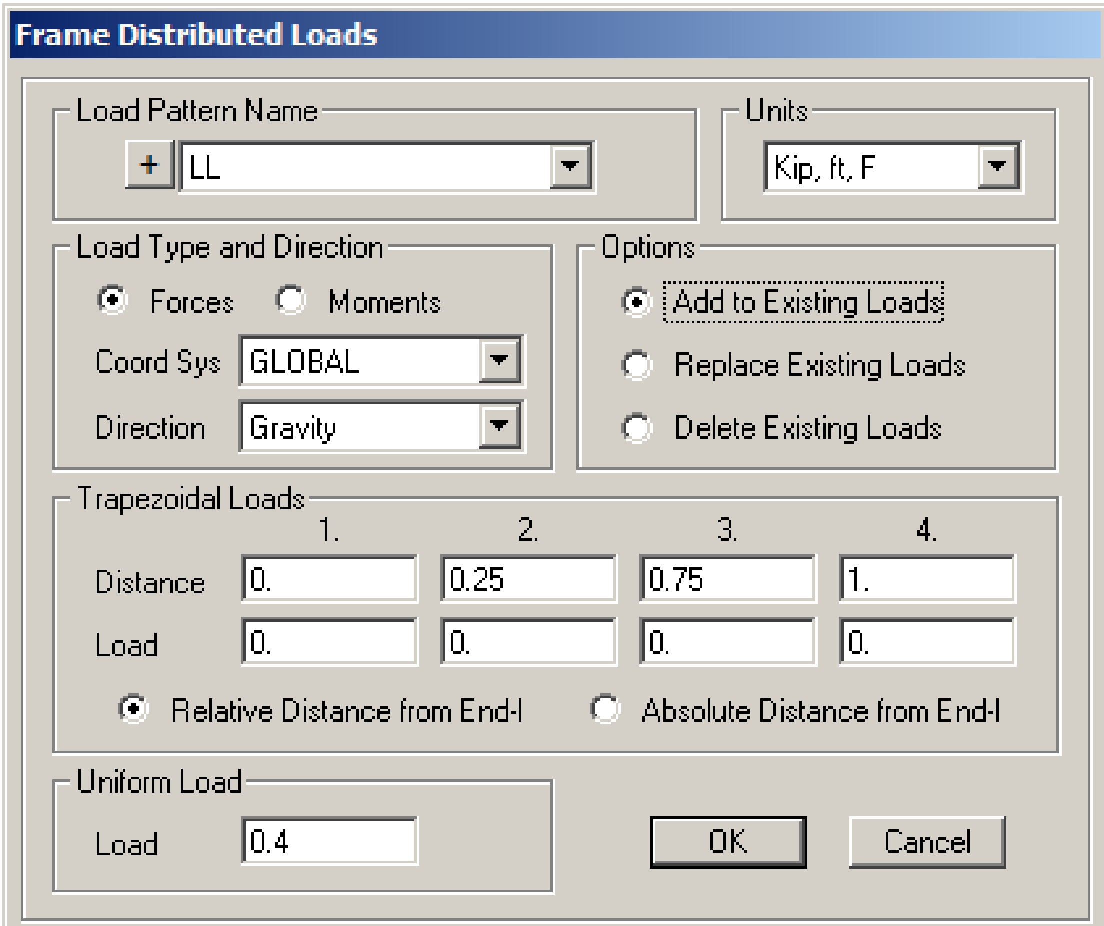


Figure 3.27: Adding LL load to ramp RBEAMs

5. Added 10 kips per linear ft as distributed load on the first 4 RBEAMS on the right side of the ramp. Selected `DYNALOAD` as the load definition. Figure 3.28 shows this step.

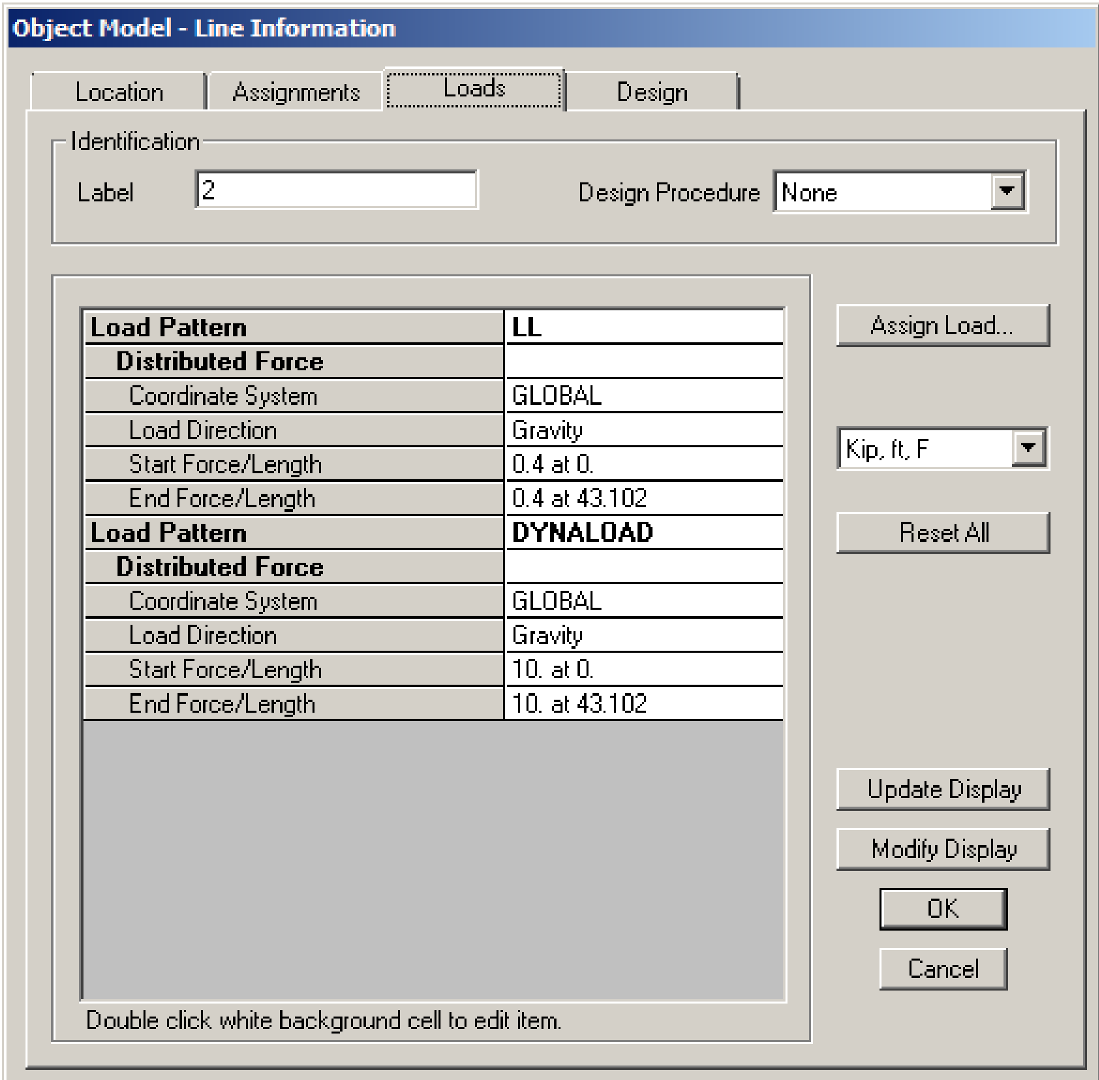


Figure 3.28: Adding 10 kips load on right side of RAMP

6. Using the menu, selected `DEFINE->FUNCTIONS->TIME HISTORY` then selected `From file` and clicked on `Add New Function...` and gave it name and used the browser to locate the text file that contains the time history. The time history file was downloaded from the class web site.

   Set `VALUES AT EQUAL INTERVALS` to 0.0214. Figure 3.29 shows this step.

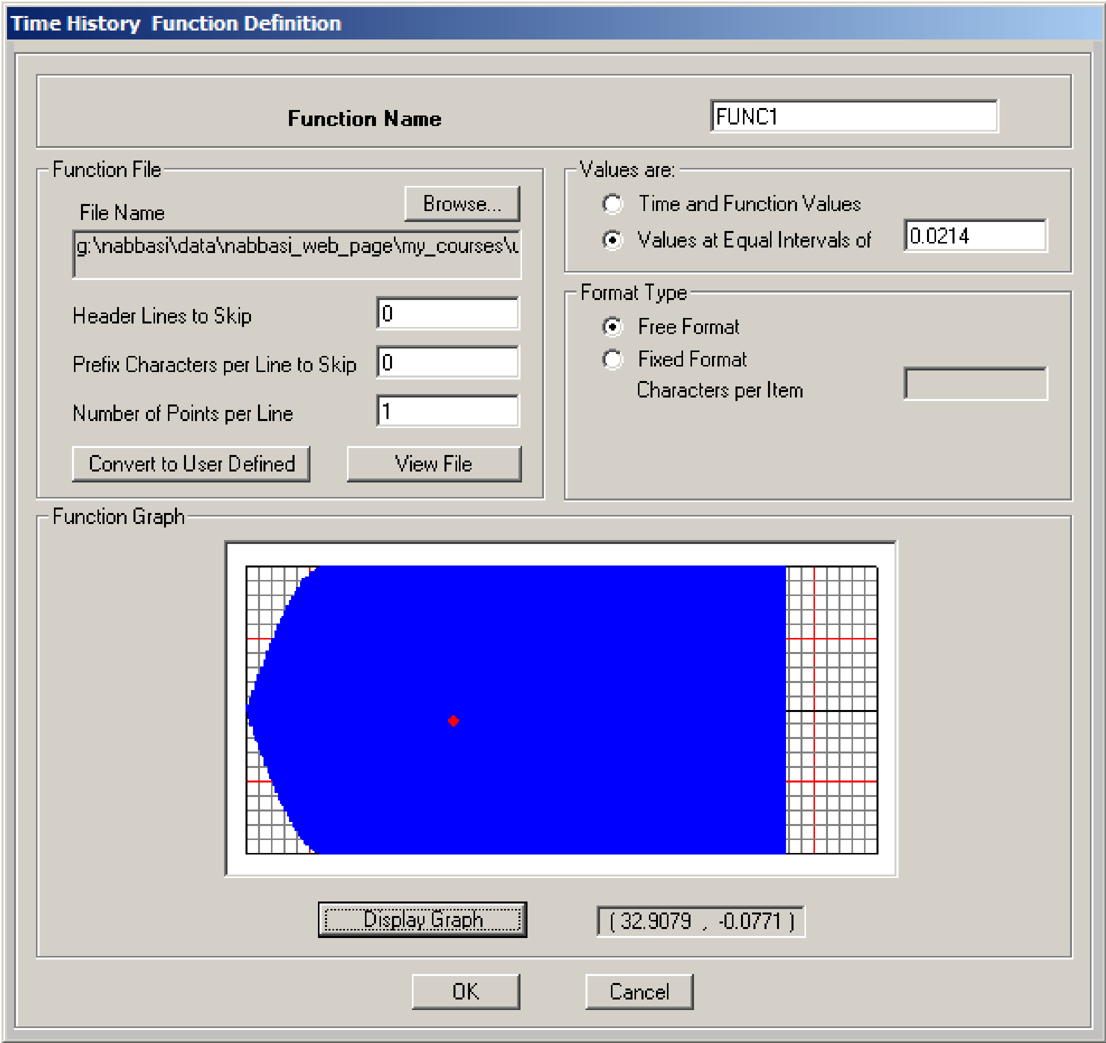


Figure 3.29: Adding time history function

7. Defined `MODAL` load case. Selected `EIGN VECTOR` and not `RITZ` Figure 3.30 shows this step.
8. Defined load case `MarchingBand` to use for time history loading to simulate the marching band on the ramp. Selected `DYNALOAD` as load pattern. Made sure to change the scale to 0.03. Figure 3.31 shows this step.

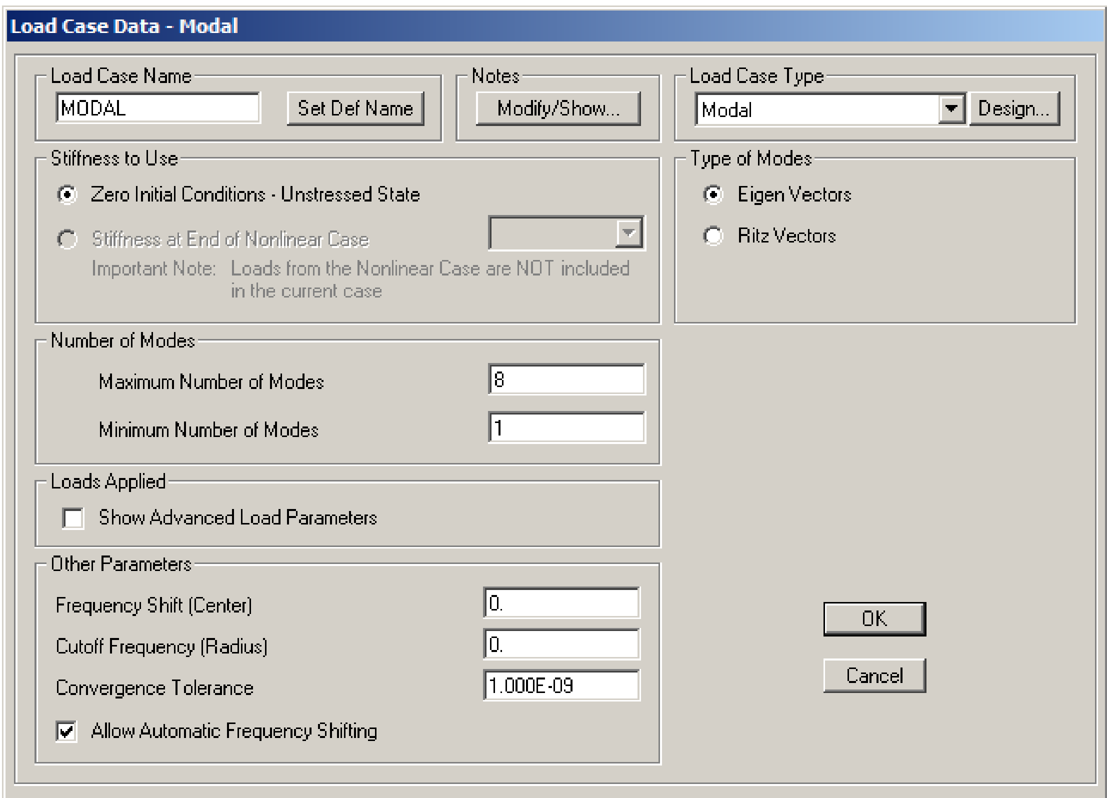


Figure 3.30: Adding MODAL load case

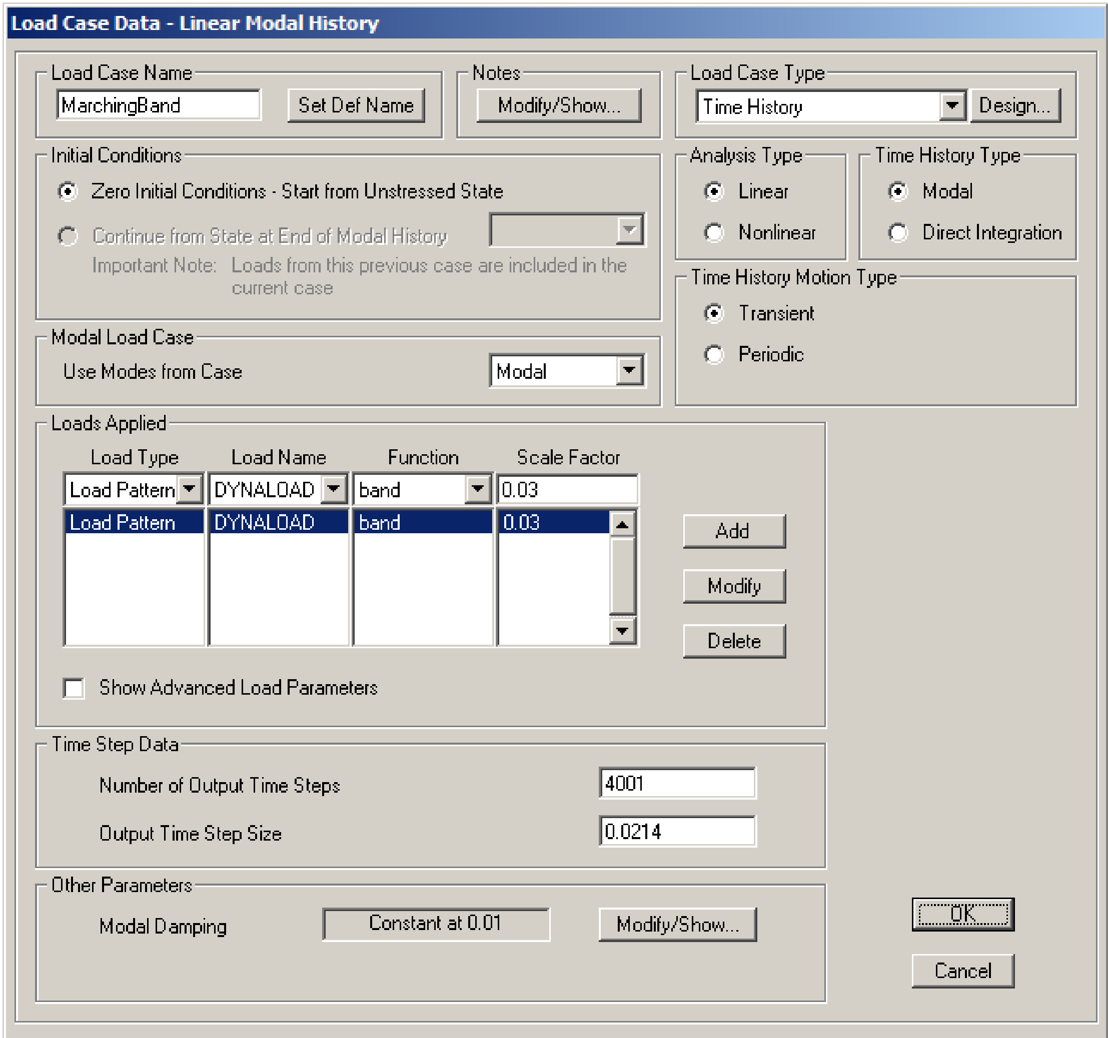


Figure 3.31: defining marching band dynamic load case

9. Defined a `COMBINATION` load case called `COMO` as shown in Figure 3.32

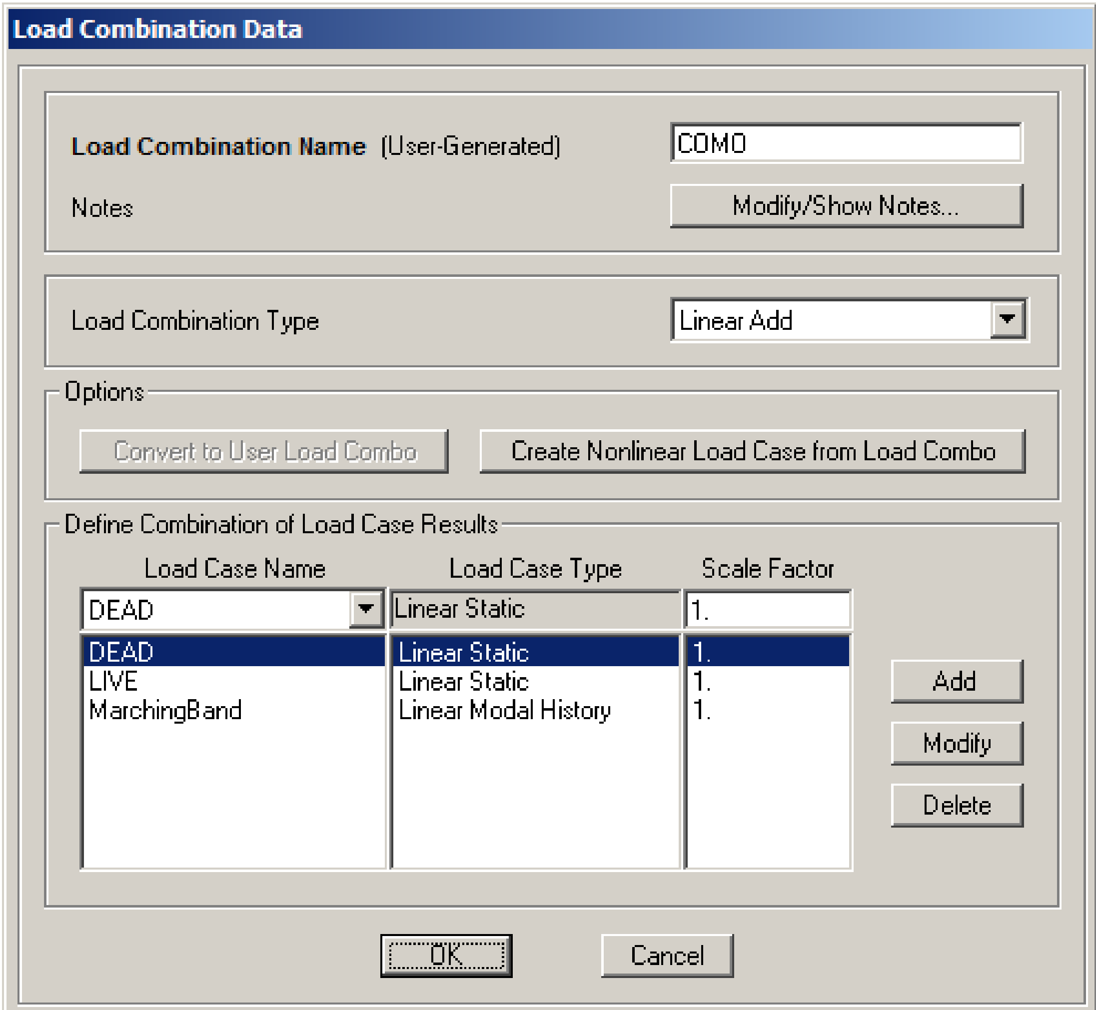


Figure 3.32: defining combination load case

10. Modified mass and weight property of `RBEAM` by changing property modifier mass to 2.1762 and property modifier weight to 2.1748 as shown in Figure 3.33

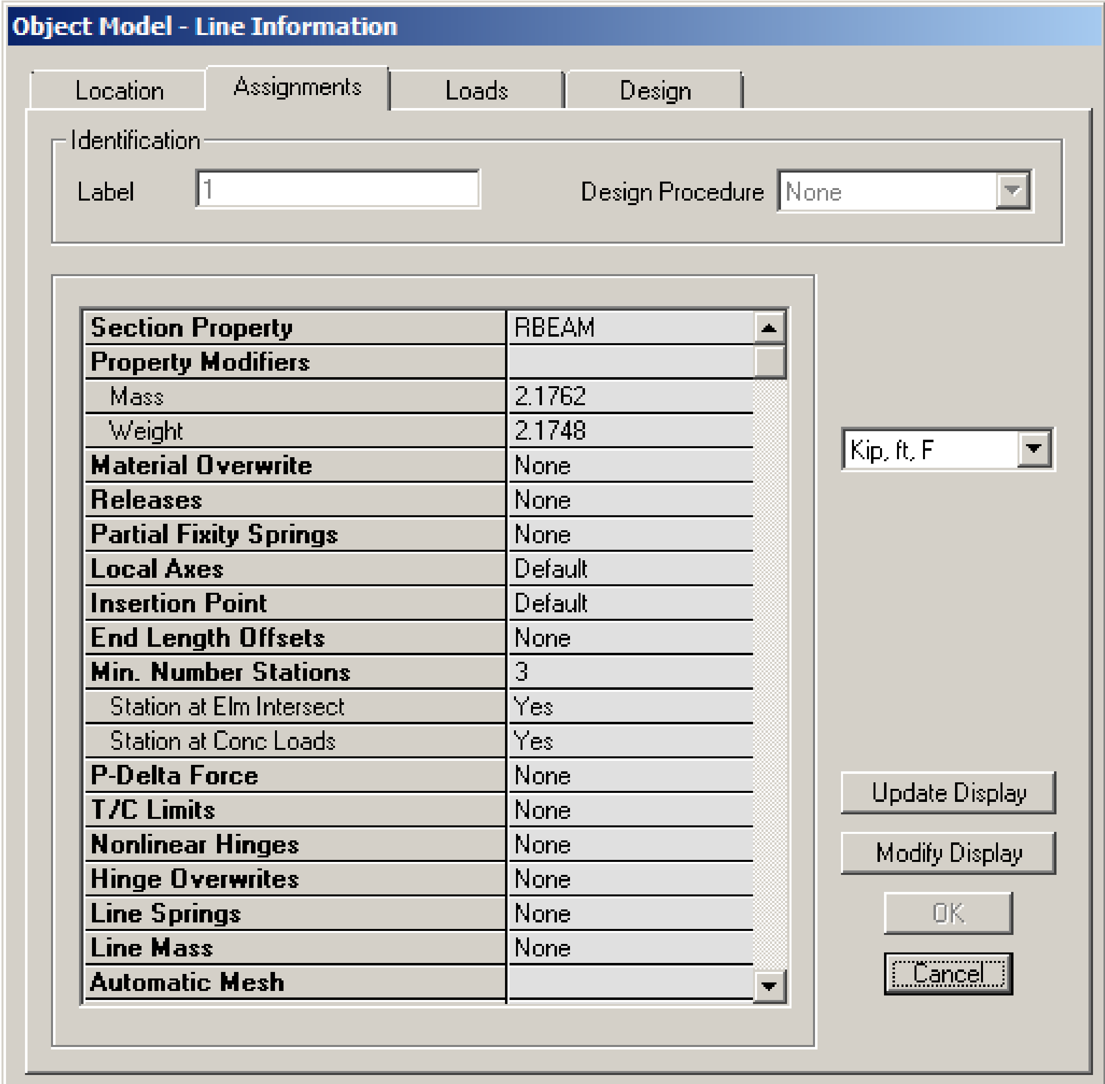


Figure 3.33: modified section property RBEAM

11. `PEAK DISPLACEMENT` at end of cantilever beams extending from far north column are found. These are the sections called `CANT3`. The first beam is from node 20 to 21, the second beam from node 23 to 19, and the third beam from node 22 to 18.

    Clicked on `run` and selected all cases to run. When run was completed, clicked on `Display->Tables` and clicked on `Select load cases...` and selected `COMO`. Then selected `ANALYSIS RESULTS` followed by `Joint Output->Displacements->Table`.

    Searched the table of `joint Displacements` for the 3 beams given above.

12. Wrote a Matlab script to plot the time history displacement for node 20 under marching band motion is in this file `sap_post_processes.m`

3.1.6 Step five. Stress results

3.1.6.1 Results

In this step, peak stress calculations at the bottom of came column under the peak marching band are made. A Matlab script was written to do the computation based on result obtained from SAP tables.

Maximum tensile and compressive stress due to marching band load only was first found. Then the stress due to dead and live load was added as a separate step. The final result is show on table 3.3

load case	max compressive stress (kip/sq inch)	max tensile stress (kip/sq inch)
marching band (4001 steps)	-44.125	45.24
dead load	-1.3812	
live load	-0.519	
combined	-46.02	45.24

Table 3.3: Stress calculation result for step 5

Figure 3.34 shows variation of stress during the 85 seconds of the time history of the marching band.

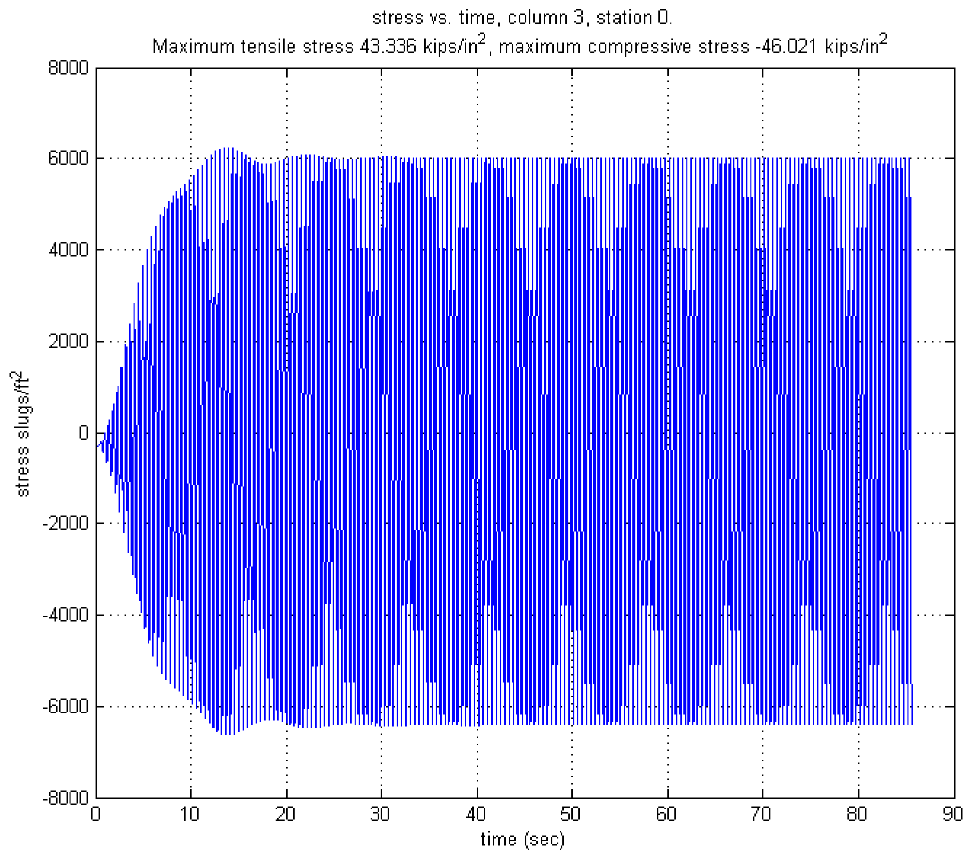


Figure 3.34: Plot of stress vs. time during dynamic loading

3.1.6.2 additional results

Additional analysis was done using SAP2000 V15.1 which allows one to visually examine stress diagrams. By selecting this `Show stress` and selecting this column and point 17 (which is station 0) which is the base of the column, the following diagrams are obtained for different measures at this location. However, these results are obtained before changing the section module of the column to the one we are asked to used in this project. Hence the results shown are not the same found above due to this. These are left here for reference and illustration of this SAP2000 feature.

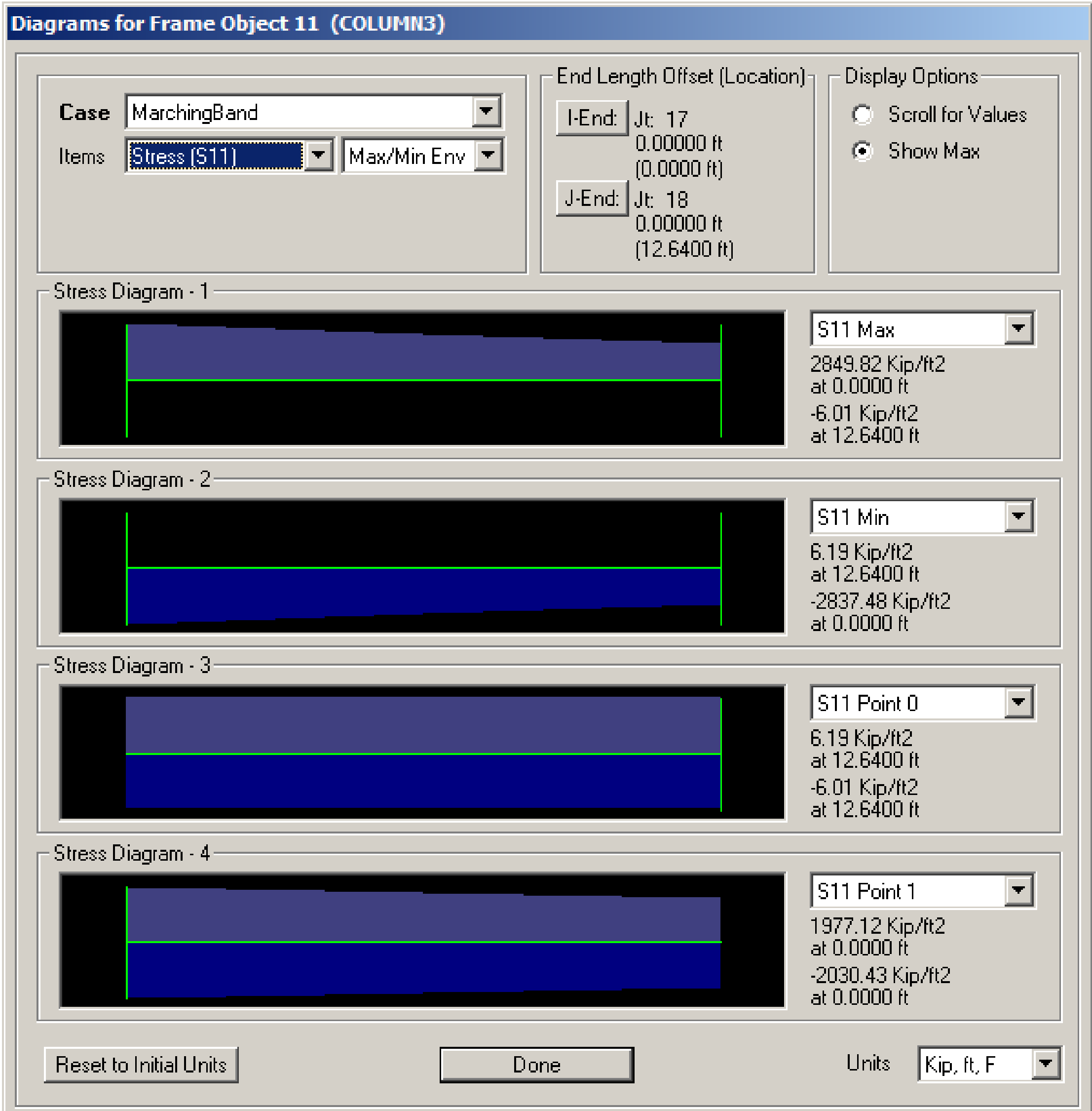


Figure 3.35: max/min of S11 stress at base of column, Marching band case

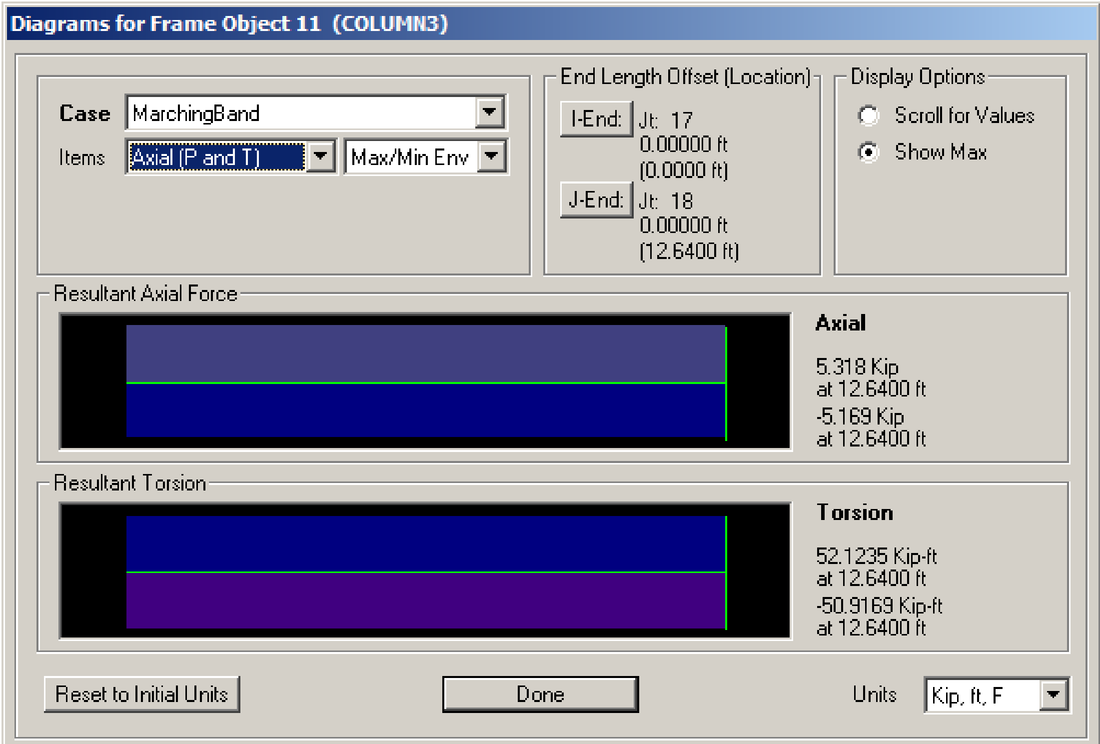


Figure 3.36: Max/min of axial load at base of column, Marching band case

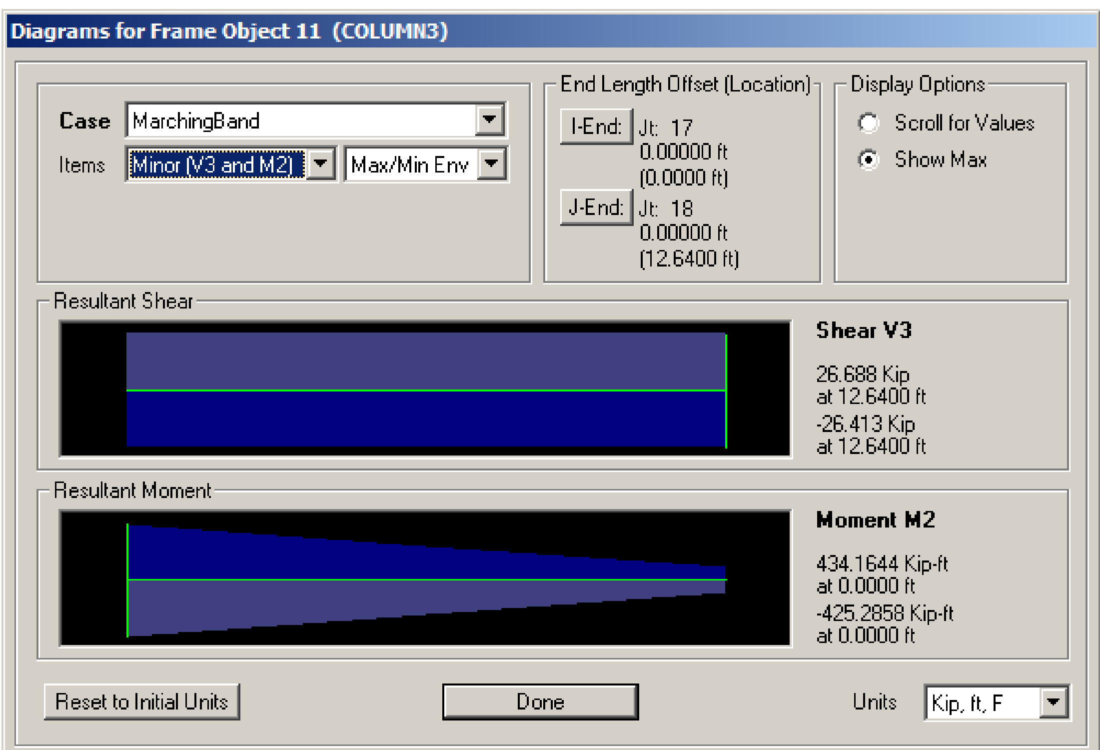


Figure 3.37: Max/min M_{22} at base of column, Marching band case

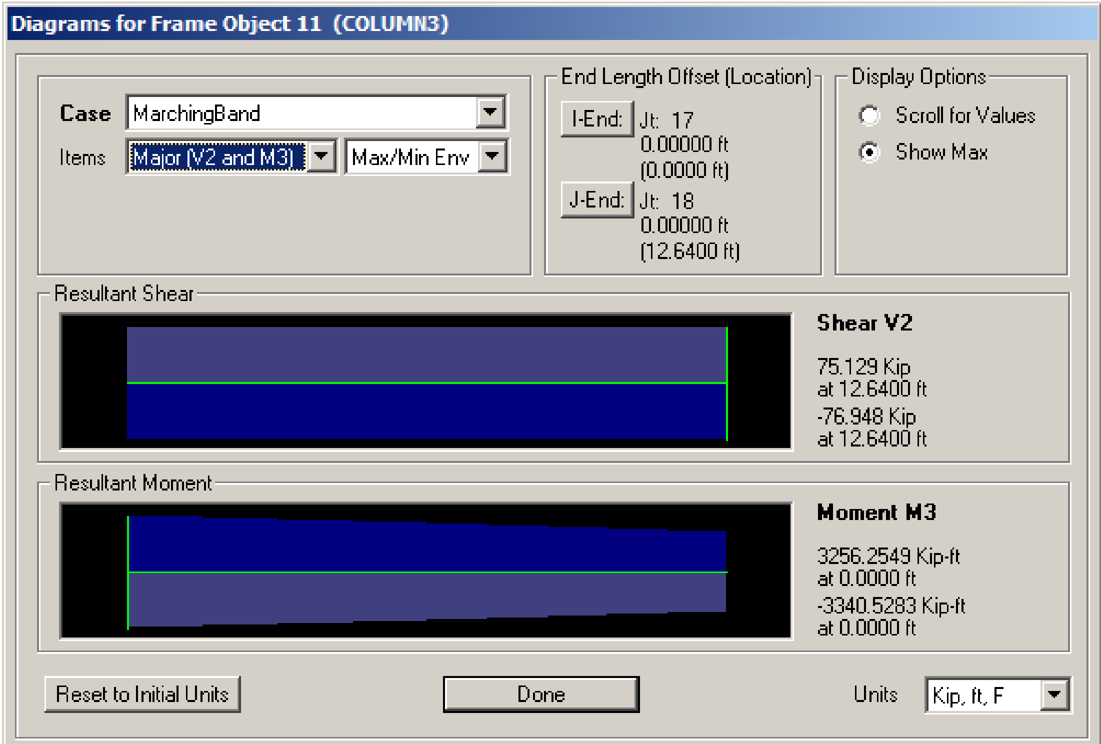


Figure 3.38: Max/min M_{33} at base of column, Marching band case

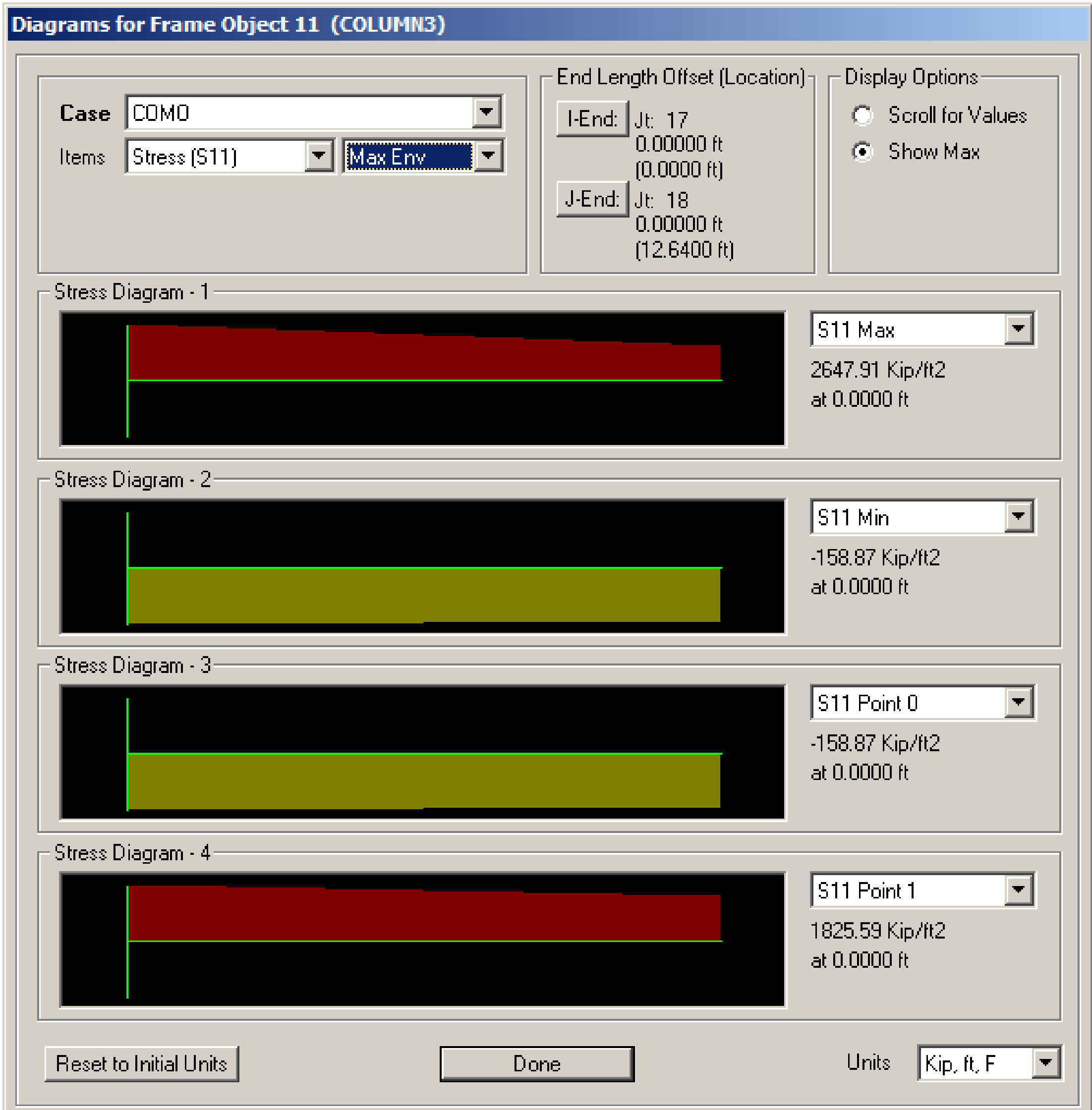


Figure 3.39: Stress S11 at base of column, Combination test case

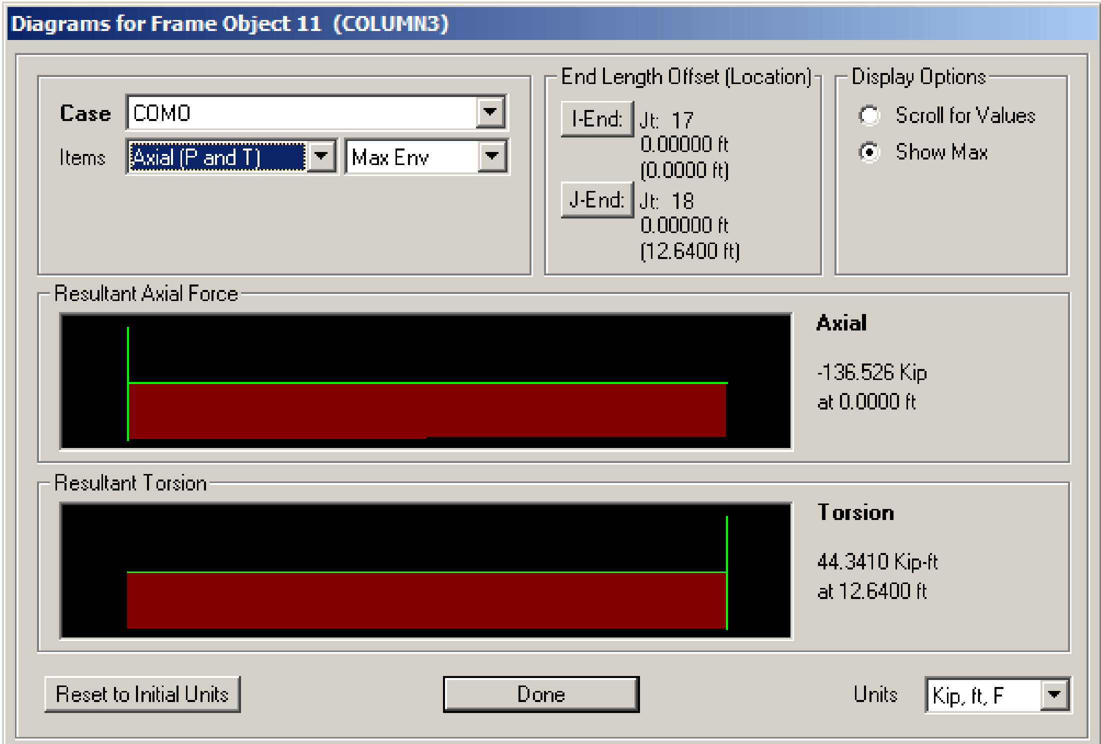


Figure 3.40: Axial load at base of column, Combination test case

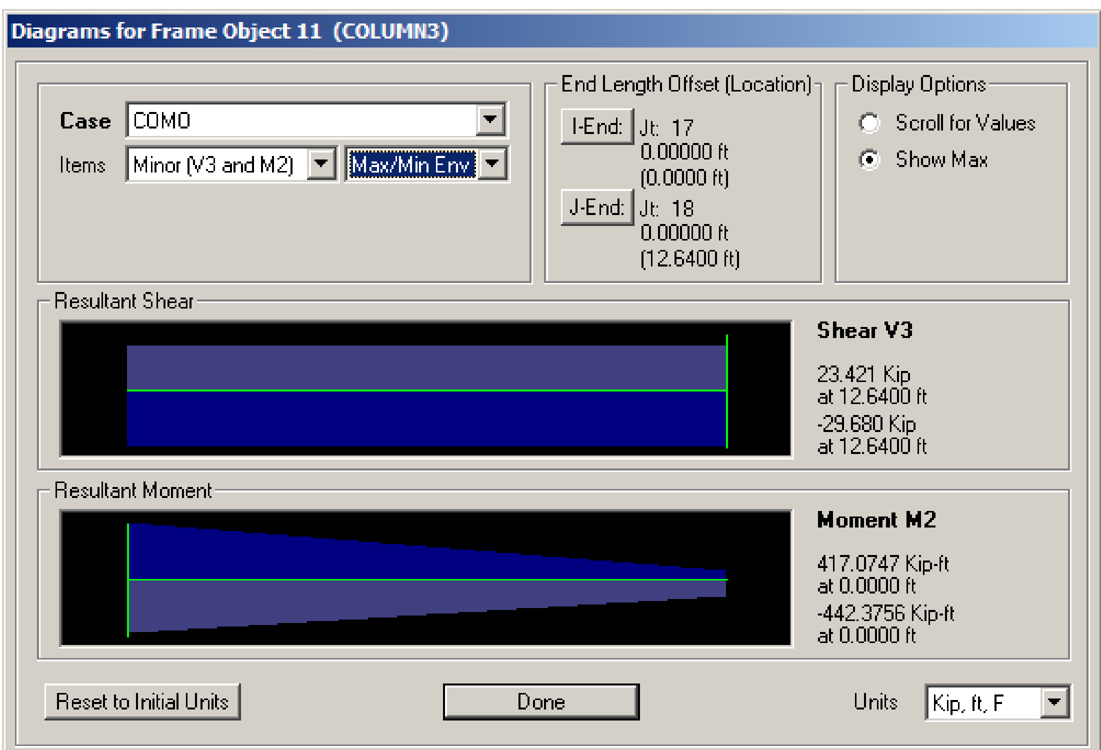


Figure 3.41: Max/min M_{22} at base of column, Combination test case

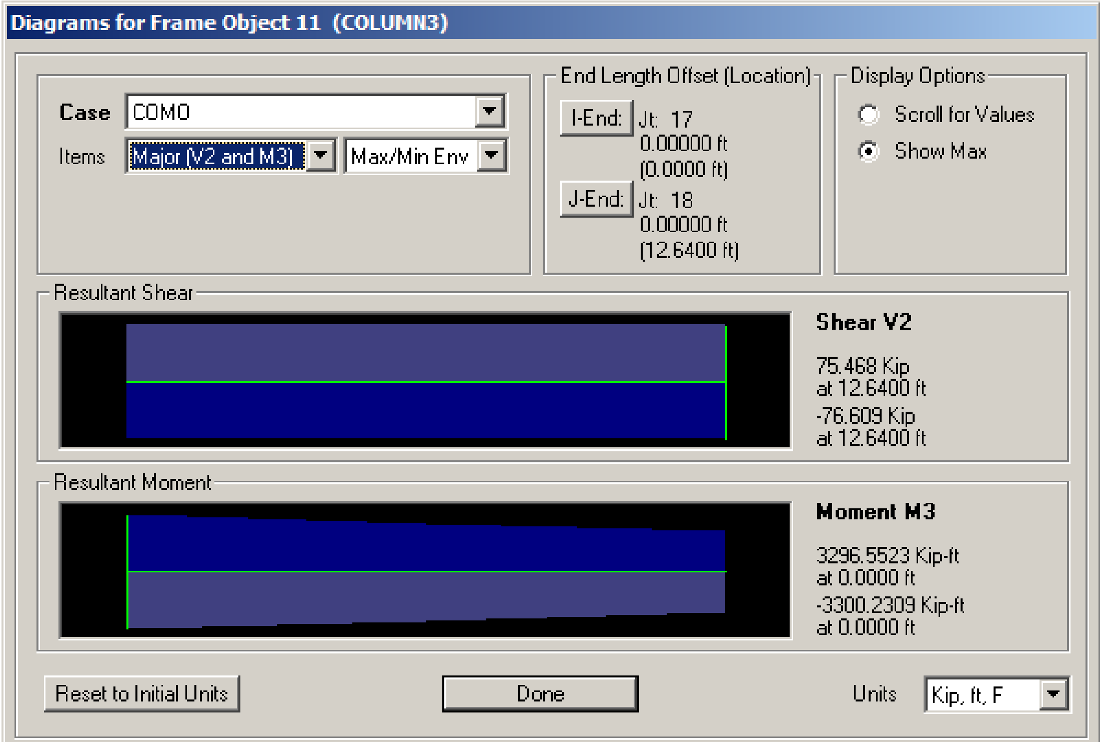


Figure 3.42: Max/min M_{33} at base of column, Combination test case

3.1.6.3 Method

1. Selected run with all load cases.
2. Selected `Display-Show Tables-Analysis Results-Element Output-Frame Output-Element Forces Modify/Show Options..` was used to make sure the envelope option is not selected and that the `step-by-step` option is selected under the `Modal History Results`. Also made sure that the load case `MarchingBand` and `COMO` are the only ones selected.
3. Waited for table to build. This took about 30 minutes. Then used the table filter to select column 11 and station 0 (this is the bottom of the column).
4. Saved the table to a text file to process using Matlab. Here is the text file that contains the results. `final_station_zero_forces.txt`
5. Now obtained the stress due to dead load and dynamic load. This was done by running the analysis again and now selecting LIVE and DEAD load cases and using the envelope. The result is in this file `final_load_result_DEAD_and_LIVE.txt`

```
SAP2000 v15.0.1    5/8/13 2:08:08
Table:  Element Forces - Frames

 Frame     Station  OutputCase   CaseType          P          V2          V3          T          M2          M3      S11Max   PtS11Max    x2S11Max   x3S1
                ft                                Kip         Kip         Kip     Kip-ft      Kip-ft      Kip-ft     Kip/ft2                    ft

    11      0.0000        DEAD  LinStatic   -101.634       0.257      -3.227    -6.3522    -19.2532     26.2485      -81.17          2   -
0.50000     0.50000     -155.36          3    0.50000    -0.50000        11-1     0.0000
    11      0.0000        LIVE  LinStatic    -40.210       0.082      -0.040    -1.4303      2.1635     14.0489      -34.51          1   -
0.50000    -0.50000      -59.07          4    0.50000     0.50000        11-1     0.0000
```

6. Ran the Matlab script and obtained the maximum stress. The area for the column cross section is 0.8594 square ft. The matlab script is in this file `stress_calc.m`

7. Calculation used for stress is based on the following formula $\sigma = \frac{P}{A} \pm \frac{M_{22}}{0.536} \pm \frac{M_{33}}{0.586}$ Where A is the section area of the beam and M_{22} and M_{33} are the internal bending moments at the base of the column obtained from SAP2000 finite elements results. Final stress was converted from kip per sq ft to kip per sq inch by dividing by 144.

3.1.7 Appendix

3.1.7.1 SAP2000 definitions used in this report

These below are obtained from SAP2000 help sections.

Sign Convention

Normal Axis 3

Local axis 3 is always normal to the plane of the shell element. This axis is directed towards you when the path **j1-j2-j3** appears counter-clockwise. For quadrilateral elements, the element plane is defined by the vectors that connect the mid-points of the two pairs of opposite sides.

Default Orientation

The default orientation of the local 1 and 2 axes is determined by the relationship between the local 3 axis and the global Z axis:

- The local 3-2 plane is taken to be vertical, i.e., parallel to the Z axis
- The local 2 axis is taken to have an upward (+Z) sense unless the element is horizontal, in which case the local 2 axis is taken along the global +Y direction
- The local 1 axis is horizontal, i.e., it lies in the X-Y plane

The element is considered to be horizontal if the sine of the angle between the local 3 axis and the Z axis is less than 10^{-3}.

The local 2 axis makes the same angle with the vertical axis as the local 3 axis makes with the horizontal plane. This means that the local 2 axis points vertically upward for vertical elements.

Figure 3.43: SAP2000 local axis signs

3.1.7.1.1 Local axis signs

Frame Element Internal Forces Output Conventions

The frame internal forces are:

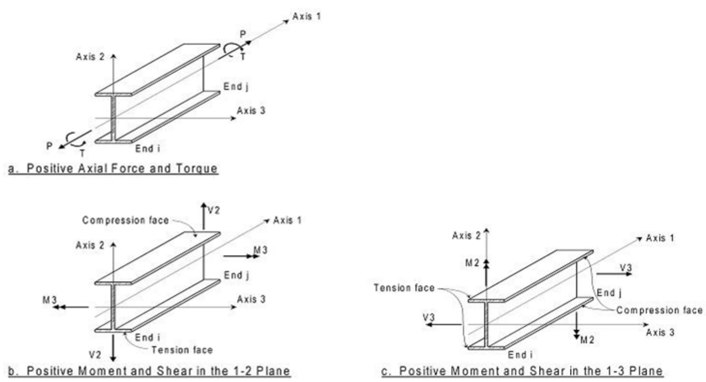
- P, the axial force
- V2, the shear force in the 1-2 plane
- V3, the shear force in the 1-3 plane
- T, the axial torque (about the 1-axis)
- M2, the bending moment in the 1-3 plane (about the 2-axis)
- M3, the bending moment in the 1-2 plane (about the 3-axis)

These internal forces and moments are present at every cross-section along the length of the frame.

For each load pattern and load combination the frame internal forces and moments are computed and reported at each frame output station.

For frame output displayed in a tabular form on the computer screen, printed to a printer or printed to a file, the locations of the output stations are identified by the absolute distance to the station measured from the i-end of the frame.

The sign convention for frame internal forces is illustrated in the figure below. This sign convention can be described by defining the concept of positive and negative faces of an object. Consider a section cut through the object in the 2-3 plane. At this section the positive 1 face is the face whose outward normal (arrow that is perpendicular to the section and pointing away from the section) is in the positive local 1 direction. At this same section the negative 1 face is one whose outward normal is in the negative local 1 direction. The positive 2 and 3 faces are those faces with outward normals in the positive local 2 and 3 directions, respectively, from the neutral axis. Note the following about the frame internal forces:



- Positive internal forces (P, V2 and V3) and positive axial torque (T) acting on a positive 1 face are oriented in the positive direction of the corresponding object local coordinate axis. For example, when V2 acting on a positive 1 face is positive, it is oriented in the direction of the positive local 2-axis.
- Positive internal forces (P, V2 and V3) and positive axial torque (T) acting on a negative 1 face are oriented in the negative direction of the corresponding object local coordinate axis. For example, when V2 acting on a negative 1 face is positive, it is oriented in the direction of the negative local 2-axis.
- Positive M2 bending moments cause compression on the positive 3 face and tension on the negative 3 face.
- Positive M3 bending moments cause compression on the positive 2 face and tension on the negative 2 face.
- When end offsets along the length of the frame are present, the internal forces and moments are output at the faces of the supports rather than the ends of the object. No output is produced within the end offset length.

Figure 3.44: SAP2000 Frame element internal forces output convention

3.1.7.1.2 Frame element internal forces output convention

Frame Axial Stress S11 for Display

Frame axial stresses can be displayed using the **SAP2000 > Display > Show Forces/Stresses > Frames/Cables/Tendons** command. Additional detail can be obtained by right-clicking on a frame, cable, or tendon object while displaying frame, cable, or tendon forces or stresses.

The axial stress S11 is the tension or compression stress that exists at every material point in the cross-section due to the combined effects of axial force P and the bending moments M2 and M3. The stress is reported and can be displayed at selected stress points that depend on the shape of the cross-section:

- I-sections, T-sections, Rectangles, Tubes, Channels, and Angles - at all corners where the maximum stresses could occur.
- Circles and Pipes - at eight points on the circumference.
- Section Designer, General Sections, and all other shapes - at the four corners of the rectangular bounding box for the section.
- Nonprismatic sections - computed as above from the interpolated shape, if the shape type is the same at both ends of the frame segment; if the shape type is not the same at both ends, then zero stress is reported.
- For all shapes except the Tube and Pipe, stresses are also computed at the centroid of the section.
- For cables and tendon, the stress is computed only at the centroid.

Tensile stress is reported as positive, and compressive stress is negative, regardless of the type of material. Stresses are computed from P, M2, and M3 for the base material of the section, with no account for modular ratio or nonlinear behavior in frame hinges. The displayed stresses are computed as analysis results and are independent of the stresses used for design, which may depend on the type of material and the design code.

For plotting the axial stress S11, any of the following options can be chosen:

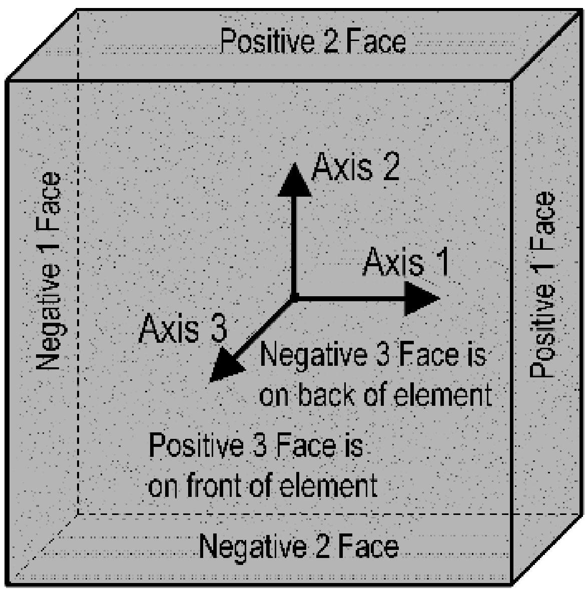
- **Stress S11 At Point** plots the axial stress at the chosen stress point in the cross section for all frame, cables, and tendons. Point 0 is the centroid of the section, and exists for all section types except the Tube and Pipe. Cables and tendons report only stress at Point 0. For all frame sections, the number of stress points in addition to the centroid may vary from 2 to 8, depending on the shape type. Zero stress will be plotted for frame, cable, or tendon objects that do not report at the selected stress point.
- **S11 Max** plots the maximum stress taken over all stress points at each station. When an enveloping load case or combination is displayed, S11 Max plots the maximum stress for the envelope maximum and the maximum stress for the envelope minimum.
- **S11 min** plots the minimum stress taken over all stress points at each station. When an enveloping load case or combination is displayed, S11 Min plots the minimum stress for the envelope maximum and the minimum stress for the envelope minimum.
- **S11 Max/Min** plots both the maximum and minimum stress taken over all stress points at each station. When a single-valued load case or combination result is displayed, S11 Max/Min displays either the maximum stress or the minimum stress, whichever has the larger absolute value. When an enveloping load case or

Figure 3.45: SAP2000 S11 description

3.1.7.1.3 SAP2000 S11 description (stress calculations)

Shell Element Internal Forces/Stresses Output Convention

The six faces of a shell element are defined as the positive 1 face, negative 1 face, positive 2 face, negative 2 face, positive 3 face and negative 3 face as shown in the figure below. In this definition the numbers 1, 2 and 3 correspond to the local axes of the shell element. The positive 1 face of the element is the face that is perpendicular to the 1-axis of the element whose outward normal (pointing away from the element) is in the positive 1-axis direction. The negative 1 face of the element is a face that is perpendicular to the 1-axis of the element whose outward normal (pointing away from the element) is in the negative 1-axis direction. The other faces have similar definitions.



Note that the positive 3 face is sometimes called the top of the shell element in SAP2000, particularly in the output, and the negative 3 face is called the bottom of the shell element.

Shell Element Internal Forces

The shell element internal forces, like stresses, act throughout the element. They are present at every point on the midsurface of the shell element. SAP reports values for the shell internal forces at the element nodes. It is important to note that the internal forces are reported as forces and moments per unit of in-plane length.

The basic shell element forces and moments are identified as F11, F22, F12, M11, M22, M12, V13 and V23. You might

Figure 3.46: SAP2000 shell element internal forces/stresses output convention

3.1.7.1.4 SAP2000 shell element internal forces/stresses output convention

3.1.7.2 references

1. Lecture notes given by professor Michael G. Oliva, college of engineering, dept. of civil engineering. CEE 744 structural dynamics, spring 2013.
2. SAP2000 The modeling and analysis of human-induced vibrations due to footfalls or another type of impact.
3. Structural vibrations which result from human footfalls may be modeled in ETABS using modal time-history analysis
4. Description of joints in SAP2000 https://wiki.csiberkeley.com/display/kb/Joint

These below are documents that describe the project itself and SAP 2000 guide and the original SAP model we obtained to start from.

1. Problem statment for Elizabeth Ashman Bridge `CEE744Ashman2013.pdf`
2. Original SAP 2000 data file `ashdynstat_original.sdb`
3. SAP 2000 GUIDE `SAPGuide.pdf`

3.2 animations

The following are the first few vibration modes of the Elizabeth Ashman Bridge, generated using SAP 2000 software.

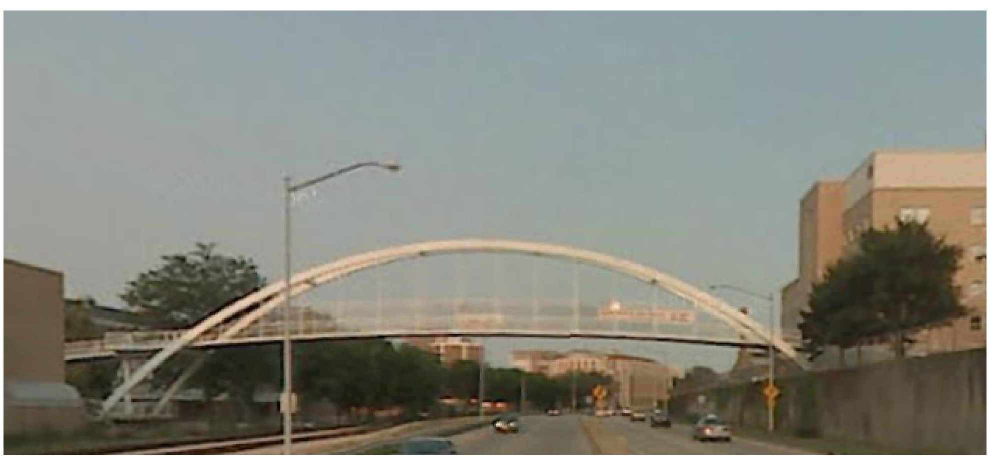


Figure 3.47: Picture of Elizabeth Ashman Bridge, located near UW Madison.

Figure 3.48 shows the axis orientation used by SAP2000.

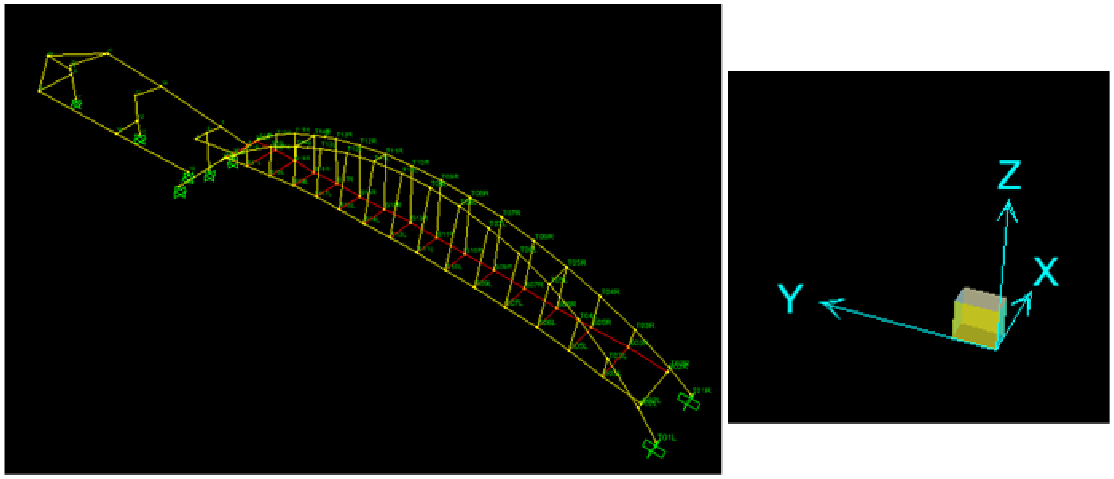


Figure 3.48: 3D axis orientation used

The following table is description of each mode. Clicking on the image plays an animated gif file of that mode.

mode 1	Bridge main body vibrates sinusoidally in the YZ plan with almost a full sin wave being described along the full length of the bridge. Ramp shows little vibration. Little motion in X direction. HTML version contains the animation.
mode 2	Similar to mode 1 but with larger amplitudes. Bridge vibration remained in the YZ plan. Ramp remains with little motion HTML version contains the animation.
mode 3	This is the ramp torsion mode. Ramp shows large twisting motion around the Y axis. Main bridge body now vibrates sideways moving in the X axes direction. The top of the bridge is tilting sidways more than the floor. HTML version contains the animation.
mode 4	Larger twists on the main bridge. Twist is around the Z axis where one half of the bridge swings to one side and the other half to the opposite side. Ramp has less torsion compared to mode 3. HTML version contains the animation.
mode 5	Both ramp and bridge now show large vibration. On the bridge, more twisting vibration are seen around the Z axis going through the middle of the bridge. Little vibration in the XY plan (up and down). Most of vibration is sideways. The half of the bridge connected to the ramp is vibrating in opposite direction to the ramp (out of phase with ramp). HTML version contains the animation.
mode 6	On the bridge, larger torsion vibration around the Z axis in the middle of the bridge. Ramp appears to vibrate less than in mode 5. HTML version contains the animation.
mode 7	Bridge floor vibration now in the YZ plane (vertically up and down) with larger vertical amplitude in the middle of the bridge. Almost two full sin wave can be seen across the full span of the bridge. Ramp appears to vibrate much less than it did in mode 7. HTML version contains the animation.
mode 8	Bridge has large torsional motion around the Y axis (Axis along its length). Bridge almost closes on itself near the top. The part of the Ramp attached to the bridge moves in phase with the bridge motion. HTML version contains the animation.

Table 3.4: Description of each mode

Chapter 4

Notes

lecture notes CEE 744, Spring 2013

Nasser M. Abbasi

Spring 2013 Compiled on August 15, 2022 at 4:43am [public]

4.1 Lecture Thursday April 4, 2013

Multidegree freedom system, free vibration, no damping

$$[m]\{\ddot{v}(t)\}+[k]\{v(t)\}=\{0\} \tag{4.1}$$

Assume

$$\{v(t)\}=\{\hat{v}\}\sin(\omega t+\theta)$$

where $\{\hat{v}\}$ is an amplitude vector of constants. Acts like shape function. Hence $\{\ddot{v}(t)\}=-\omega^2\{\hat{v}\}\sin(\omega t+\theta)$. Substituting into Eq 4.1

$$\begin{aligned}[m]\left(-\omega^2\{\hat{v}\}\sin(\omega t+\theta)\right)+[k]\{\hat{v}\}\sin(\omega t+\theta)&=\{0\}\\ \left(-[m]\,\omega^2+[k]\right)\{\hat{v}\}&=\{0\}\end{aligned}$$

This is an eigenvalue problem. Hence

$$\det\left([k]-[m]\,\omega^2\right)=0$$

We obtain n unique eigenvalues ω_i and corresponding n independent mode shapes $\{\hat{v}\}_i$

$$\left(-[m]\,\omega_i^2+[k]\right)\{\hat{v}\}_i=\{0\}$$

Where $\{\hat{v}\}_i=\begin{Bmatrix}1\\v_2\\\vdots\\v_n\end{Bmatrix}_i$ or $\{\hat{v}\}_i=\begin{Bmatrix}\varphi_1\\\varphi_2\\\vdots\\\varphi_n\end{Bmatrix}_i$. Hence for each ω_i we get different shape function vector $\{\hat{v}\}_i$. Let the mode shape matrix $[\Phi]$ be

$$\begin{aligned}[\Phi]&=\left[\{\hat{v}\}_1,\{\hat{v}\}_2,\cdots,\{\hat{v}\}_n\right]\\ &=\begin{Bmatrix}\varphi_{11}&\varphi_{12}&\cdots&\varphi_{1n}\\\varphi_{21}&\varphi_{22}&\cdots&\varphi_{2n}\\\vdots&\vdots&\ddots&\vdots\\\varphi_{n1}&\varphi_{n2}&\cdots&\varphi_{nn}\end{Bmatrix}\end{aligned}$$

4.2 Lecture Tuesday April 9, 2013

Modal decoupling

Instructions on using my dynamic response analysis program

Accessing dynamic response modeling app

Nasser Abbasi

- First you need to have the CDF Player installed, This can be obtained from the Wolfram-Mathematica site at: http://www.wolfram.com/cdf-player/

- Then go to Nasser's web site and launch the app:
http://www.12000.org/my_notes/mma_demos/single_degree_of_freedom_responses/index.htm

Move down past the introduction – to the rectangular processing area.

- input values are set in the upper left hand side box and include:
    - damping c
    - stiffness K
    - initial displacement u(0)
    - initial velocity v(0)
    - mass m
    - amplitude of a loading function F (set to 0 for free vibration case)
- additional input in the second box from top:
    - beta ratio for a harmonic loading function
    - check bullet if you want a step load function
- at top of plot on right hand side you can select the kind of plot you wish to see, "excitation with response" might be a good choice