University Course

ME 440
Intermediate Vibrations

University of Wisconsin, Madison
Fall 2017

My Class Notes

Nasser M. Abbasi

Fall 2017

Contents

Chapter 1

Introduction

I took this course in Fall 2017 to learn more about Vibration since it was a while since I studied this.

The instructor was very good and solved many problems in class which was very useful. All class notes were online. Exams were a little hard and time was short. There is closed notes portion and open notes portion in the exam. The grading was fair.

Links

1. class canvas site `https://canvas.wisc.edu/courses/57245` requires login.

1.1 syllabus

ME 440 – Intermediate Vibrations
Fall 2017

Time: 11 am – 12:15 pm (Tu, Th)

Location: ME 2108

Instructor: Andrew Mikkelson

Office: ME 1250

E-Mail: andrew.mikkelson@wisc.edu

Course Page: learnuw.wisc.edu

Office Hours:
Monday: 2 - 3 pm
Tuesday: 12:30 - 1:30 pm
Wednesday: 10:30 - 11:30 am
Thursday: 12:30 - 1pm
Other times by appointment (please email to arrange)

Text: S. S. Rao, Mechanical Vibrations, 2004 (4th edition). Text is optional.

Prerequisites: ME340

Catalog Description: Analytical methods for solution of typical vibratory and balancing problems encountered in engines and other mechanical systems. Special emphasis on dampers and absorbers.

Course Objectives:
The purpose of the course is to develop the skills needed to design and analyze mechanical systems in which vibration problems are typically encountered. These skills include analytical and numerical techniques (e.g., finite element methods) that allow the student to model the system, analyze the system performance and employ the necessary design changes. Emphasis is placed on developing a thorough understanding of how the changes in system parameters affect the system response.

Course Outcomes: Students must have the ability to:
1. Derive the equations of motion of single and multi-degree of freedom systems, using Newton's Laws and energy methods.
2. Determine the natural frequencies and mode shapes of single and multi-degree of freedom systems.
3. Evaluate the dynamic response of single and multi-degree of freedom systems under impulse loadings, harmonic loadings, and general periodic excitation.
4. Apply modal analysis and orthogonality conditions to establish the dynamic characteristics of multi-degree of freedom systems.
5. Generate finite element models of discrete systems to simulate the dynamic response to initial conditions and external excitations. (time permitting)

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ME 440 – Intermediate Vibrations
Fall 2017

Grades will be based on your performance on written homework and examinations. All homework and exam scores will be maintained on the Learn@UW course website. This will allow you to monitor your performance and see aggregate scores for the rest of the class, which can give you a continuous idea of your performance in relation to the rest of the class. Should you have questions about your score, please contact me. Policies regarding grading and turning in your homework:

1. *Score-related questions about homeworks and exams must be raised prior to the next class period after receiving the score.*
2. *If homework that you turned in appears not to be graded (missing) on the Learn@UW course website please point that out to me within one week after the return of the corresponding set of graded homeworks. It is a good practice to save your homeworks so that I will be able to update the grade to give you full credit for your work.*
3. *Please do not drop homework in my department mail box*
4. *Homework is due at the* <u>*beginning*</u> *of each lecture*
5. *One homework with the lowest score will be dropped when computing the final homework average*

Percentage participation to the final grade shall be distributed in the following manner:

Homework	=	**40%**
Exam I	=	**20%**
Exam II	=	**20%**
Exam III	=	**20%**
TOTAL		**100%**

Textbook reading assignments will be assigned prior to each class. You are asked to read the material, take notes and be prepared to participate in classroom activities. The Microsoft PowerPoint notes used in class will be posted online.

Homework: Problems will be assigned weekly during the semester and posted to LearnUW. All assigned homework will be collected at the beginning of class on the due date. No late homework will be accepted. Homework solutions should be *neat and well organized*. All necessary diagrams and calculations must be clearly shown.

Exams: The best way to prepare for exams is to participate in class, learn the fundamental concepts, and practice homework and example problems from lecture and the text.

Disability requests: I must hear from anyone who has a disability that may require some modification of seating, testing or other class requirements so that appropriate arrangements may be made. Please see me after class or during my office hours.

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Complaints: If you have a complaint regarding the course and if you are unsatisfied with the response of the instructor, then you should contact the Chair of the Department of Mechanical Engineering. The Chair's office is in ME 3650, and an appointment to see the Chair can be made by contacting the Department Office at 608 263-5372.

Campus Environment: Diversity is a source of strength, creativity, and innovation. All students in this course are expected to value the contributions of each person and respect the ways in which their identity, culture, background, experience, status, abilities, and opinion enrich our learning experience and university community. Disrespectful behavior or comments directed toward any group or individual will be addressed by the instructor.

Academic integrity: The Department of Mechanical Engineering takes Academic Integrity very seriously. According to state law, any instances of academic misconduct are reported to the UW Dean of Students. Once reported, the incident is retained in a permanent disciplinary file. This file may never see the light of day, or it may be released if you apply to graduate school, to medical school, to law school, for government clearance, for a visa, etc. As a result, even a minor infraction, such as plagiarism, copying a problem solution, or aid from an exam neighbor could have serious and permanent consequences.

Letter Grades: The grading scale listed below is a worst case scenario. At the end of semester letter grades may be curved up but they will not be curved down (i.e., A grade of 91% will guarantee you at least an AB, and might be an A). Final letter grades will be based on the total score accumulated on homework and exams throughout the semester using the following scale:

Score	Grade
≥92	A
88-92	AB
83-88	B
78-83	BC
70-78	C
60-70	D
< 60	F

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Tentative Schedule for ME 440
Intermediate Vibrations

Fall Semester 2017

TEXTBOOK: *Mechanical Vibrations*, 4th ed. by S. S. Rao (Optional)

COURSE INSTRUCTOR: Andrew Mikkelson, Rm. 1250 ME Bldg., andrew.mikkelson@wisc.edu

Date		Study Assignment	Topics Covered
Sept.	5	-	-
	7	1.1 – 1.6	Basic Concepts, Classifications, Procedures
Sept.	12	1.7 – 1.9	Spring, Mass, and Damping Elements
	14	1.10	Harmonic Motion, Complex Algebra, Fourier Series
Sept.	19	1.11	Fourier Series, Complex Representation
	21	2.1 – 2.2	Review of Single DOF Systems: Deriving EOMs
Sept.	26	2.2, 2.6	IVPs, Transient Response
	28	2.6	Coulomb Friction, Logarithmic decrement; Applications
Oct.	3	2.3	Pendulum Systems; Torsional Vibration; Energy Methods
	5	2.5	Energy Methods; Rayleigh's Method and Applications
Oct.	10		**Exam 1**
	12	3.1 – 3.5	Review of Single DOF Systems: Harmonic Excitation
Oct.	17	3.6 – 3.7	Harmonic Excitation: Rotating Unbalance, Design Problem Engine Mounts
	19	3.8 – 3.11	Harmonic Excitation: Base Excitation, Beating Phenomena
Oct.	24	4.1 – 4.3	Nonharmonic Excitation: General Periodic Excitation
	26	4.4	Nonharmonic Excitation: Impulsive Forces, Convolution Integral
Oct.	31	4.5 – 4.6	Nonharmonic Excitation: Convolution Integral, Superposition
Nov.	2		Impulse Loading – Response Spectrum, Dynamic Load Factor
Nov.	7	5.1 – 5.2	Two DOF Systems: Natural Frequencies and Mode Shapes
	9	5.3 – 5.4	Two DOF Systems: Natural Frequencies and Mode Shapes, MATLAB
Nov.	14		**Exam 2**
	16	5.4	Two DOF Systems: Coupling, Matrix Notation
Nov.	21	5.5	Two DOF Systems: Decoupling of EOMs, Principal Coordinates
	23	-	- No class – (Thanksgiving)
Nov.	28	6.8 – 6.10, 6.12	Modal Analysis: Natural Frequencies and Mode Shapes, MATLAB
	30	6.13	Modal Analysis: Free Response of Undamped and Underdamped Systems
Dec.	5	6.14 – 6.16	Multi-DOF Systems: Forced Response and Lumped Mass Modeling
	7	6.14 – 6.16, 6.7	Multi-DOF Systems: Lumped Mass Modeling, Lagrange's eqns
Dec.	12	6.7	**Exam 3**
	14	-	-
Dec.	23		**Festivus!**

*Note: We have 2 less class periods this semester as compared to the last time this class was offered. As a result, we will probably not complete all of the topics listed above.

Final Exam: N/A

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Chapter 2

HWs

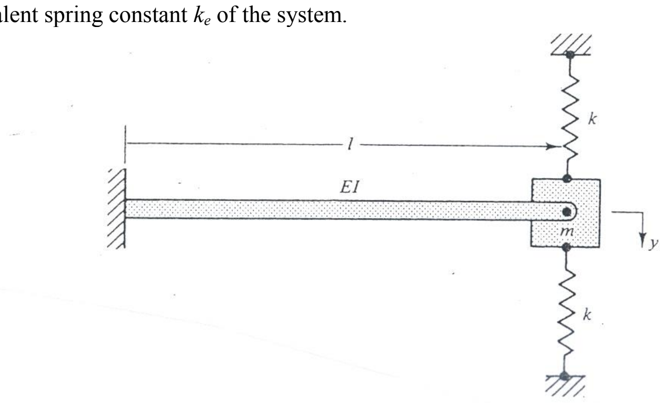
Local contents

2.1 HW1

2.1.1 Problem 1

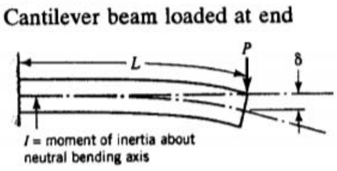
Problem 1

The mass m is pinned to the end of a cantilevered beam that has a bending stiffness factor of EI and a length of l. The spring constant of each of the two vertical springs is k. Determine the equivalent spring constant k_e of the system.

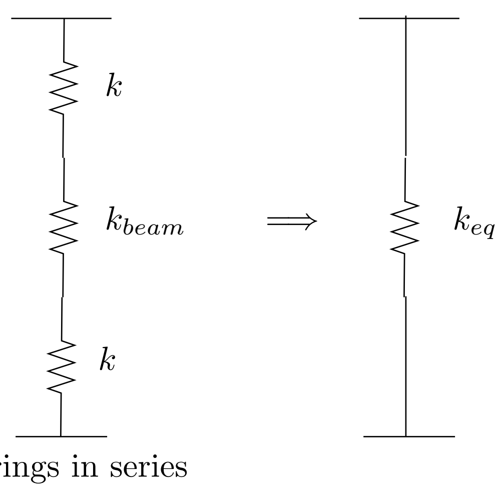


From tables we find that for cantilever beam loaded at end, the vertical deflection is $\delta = \frac{mL^3}{3EI}$, hence by definition $k_b = \frac{m}{\delta} = \frac{3EI}{L^3}$.

5. **Cantilever beam loaded at end** $\delta = \frac{PL^3}{3EI}$ $k = \frac{P}{\delta} = \frac{3EI}{L^3}$



Therefore, we can model the stiffness of the system as



Therefore

$$\begin{aligned} k_{eq} &= k + k_{beam} + k \\ &= 2k + k_{beam} \end{aligned}$$

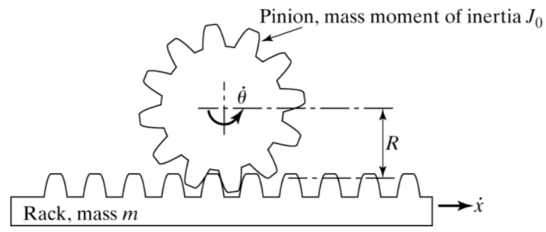
Since $k_b = \frac{3EI}{L^3}$ then the above becomes

$$k_{eq} = 2k + \frac{3EI}{L^3}$$

2.1.2 Problem 2

Problem 2

The pinion of the rack and pinion system shown below is free to rotate about its mass center but it can not translate in any direction. For this 1 degree-of-freedom system, find it's equivalent mass a) if the generalized coordinate that captures this degree of freedom is the angle θ, b) if the generalized coordinate that captures this degree of freedom is the horizontal displacement x of the rack.



2.1.2.1 Part (a)

Using energy method

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}J_{eq}\dot{\theta}_{eq}^2$$

But $\dot{\theta}_{eq} = \dot{\theta}$ for this part. And since $x = R\theta$ or $\dot{x} = R\dot{\theta}$, then the above becomes

$$\frac{1}{2}m\left(R\dot{\theta}\right)^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}J_{eq}\dot{\theta}^2$$

Simplifying gives

$$J_{eq} = mR^2 + J_0$$

2.1.2.2 Part (b)

Using energy method

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}m_{eq}\dot{x}_{eq}^2$$

But $\dot{x}_{eq} = \dot{x}$ for this part. And since $x = R\theta$ or $\dot{x} = R\dot{\theta}$, then $\dot{\theta} = \frac{\dot{x}}{R}$ and the above becomes

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\left(\frac{\dot{x}}{R}\right)^2 = \frac{1}{2}m_{eq}\dot{x}^2$$

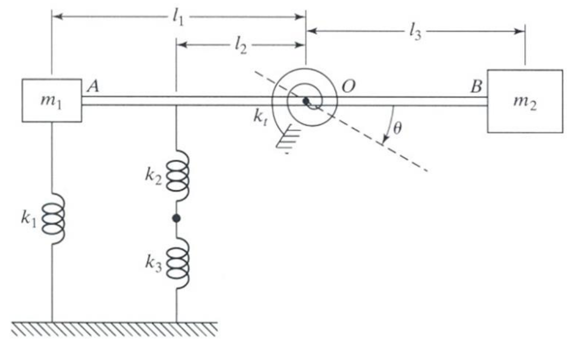
Simplifying gives

$$m_{eq} = m + \frac{J_0}{R^2}$$

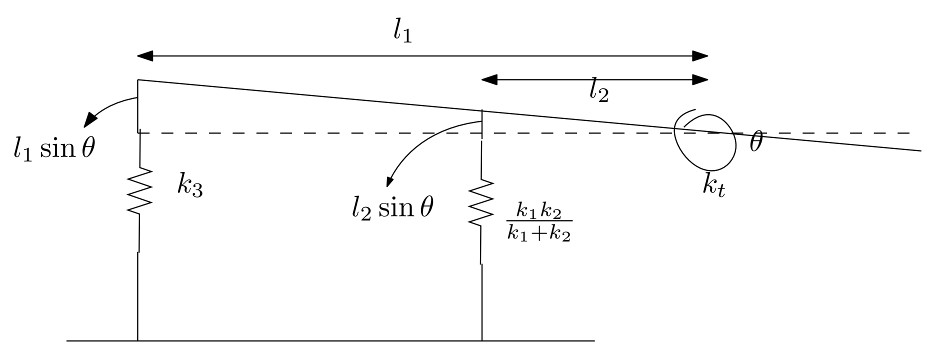
2.1.3 Problem 3

Problem 3

Find the equivalent spring constant and equivalent mass of the system shown below with regards to the θ degree of freedom shown in the figure. Assume that the bar *AOB* is rigid with negligible mass.



Assuming a small deflection as shown



2.1.3.1 Mass equivalent

The kinetic energy of the system is (assuming small angles)

$$\frac{1}{2}m_1\left(L_1\dot{\theta}\right)^2+\frac{1}{2}m_2\left(L_3\dot{\theta}\right)^2=\frac{1}{2}I_{eq}\dot{\theta}^2$$

Hence

$$m_1L_1^2+m_2L_3^2=I_{eq}$$

Where I_{eq} is the equivalent mass moment of inertia. The problem does not say where the equivalent mass should be located relative to the pivot point (where the torsional spring is located) so we can stop here. But assuming that distance was some $\bar{x}$, then we can write $I_{eq}=M_{eq}\bar{x}^2$ where equivalent mass is used as a point mass, and simplify the above more

$$\begin{aligned} m_1L_1^2+m_2L_3^2&=M_{eq}\bar{x}^2\\ M_{eq}&=\frac{m_1L_1^2+m_2L_3^2}{\bar{x}^2}\end{aligned}$$

2.1.3.2 Stiffness equivalent

Using potential energy method, where energy stored by a spring due to extension or compression is $\frac{1}{2}k\Delta^2$, then we see that the total energy using the above deformation is given by

$$\frac{1}{2}k_1\left(l_1\sin\theta\right)^2+\frac{1}{2}\left(\frac{k_3k_2}{k_3+k_2}\right)\left(l_2\sin\theta\right)^2+\frac{1}{2}k_t\theta^2=\frac{1}{2}k_{t,eq}\theta_{eq}^2$$

Where $\frac{k_3k_2}{k_3+k_2}$ is the equivalent stiffness of the springs k_2,k_3 since they are in series. The above assumes small angle θ, therefore we can simplify the above using $\sin\theta\approx\theta$, and obtain

$$\frac{1}{2}k_1\left(l_1\theta\right)^2+\frac{1}{2}\left(\frac{k_3k_2}{k_3+k_2}\right)\left(l_2\theta\right)^2+\frac{1}{2}k_t\theta^2=\frac{1}{2}k_{t,eq}\theta_{eq}^2$$

But here $\theta=\theta_{eq}$, therefore solving for $k_{t,eq}$ gives

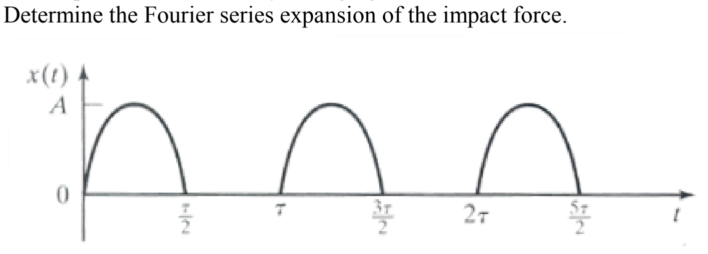
$$k_{t,eq}=k_1l_1^2+\left(\frac{k_3k_2}{k_3+k_2}\right)l_2^2+k_t$$

2.2 HW2

2.2.1 Problem 1

Problem 1

The impact force created by a forging hammer can be modeled as shown in the figure below. Determine the Fourier series expansion of the impact force.



Period is τ. This is not even and not odd. The first step is to determine the function $x(t)$. This is truncated sin. Therefore we see that, over first period

$$x(t)=\begin{cases} A\sin\left(\frac{2\pi}{\tau}t\right) & 0\leq t\leq\frac{\tau}{2}\\ 0 & \frac{\tau}{2}<t\leq\tau\end{cases}$$

This repeated over each period by shifting it. Now that we know $x(t)$ we can find a_0, a_n, b_n and plot the approximation for larger n

$$\begin{aligned}a_0&=\frac{1}{\frac{\tau}{2}}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}x(t)\,dt\\&=\frac{2}{\tau}\int_0^{\frac{\tau}{2}}x(t)\,dt\\&=\frac{2}{\tau}\int_0^{\frac{\tau}{2}}A\sin\left(\frac{2\pi}{\tau}t\right)dt\\&=-\frac{2}{\tau}\frac{A}{\frac{2\pi}{\tau}}\left[\cos\left(\frac{2\pi}{\tau}t\right)\right]_0^{\frac{\tau}{2}}\\&=-\frac{A}{\pi}\left[\cos\left(\frac{2\pi}{\tau}\frac{\tau}{2}\right)-1\right]\\&=-\frac{A}{\pi}\left[\cos(\pi)-1\right]\end{aligned}$$

Hence

$$a_0=\frac{2A}{\pi}$$

<u>Finding a_n</u>

$$\begin{aligned}a_n&=\frac{1}{\frac{\tau}{2}}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}x(t)\cos\left(\frac{2\pi}{\tau}nt\right)dt\\&=\frac{2}{\tau}\int_0^{\frac{\tau}{2}}x(t)\cos\left(\frac{2\pi}{\tau}nt\right)dt\\&=\frac{2}{\tau}\int_0^{\frac{\tau}{2}}A\sin\left(\frac{2\pi}{\tau}t\right)\cos\left(\frac{2\pi}{\tau}nt\right)dt\end{aligned}$$

But $\sin(u)\cos(v)=\frac{1}{2}(\sin(u+v)+\sin(u-v))$, therefore the above integral becomes

$$
\begin{aligned}
a_n &= \frac{2A}{\tau}\left(\frac{1}{2}\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}t+\frac{2\pi}{\tau}nt\right)dt + \frac{1}{2}\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}t-\frac{2\pi}{\tau}nt\right)dt\right)\\
&= \frac{A}{\tau}\left(\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right)dt + \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt\right) \qquad (1)
\end{aligned}
$$

The first integral above is

$$
\begin{aligned}
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right)dt &= -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)}\right]_0^{\frac{\tau}{2}}\\
&= \frac{-1}{\frac{2\pi}{\tau}(1+n)}\left[\cos\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right)-1\right]\\
&= \frac{-\tau}{2\pi(1+n)}\left[\cos(\pi(1+n))-1\right]
\end{aligned}
$$

For $n=1,3,5,\cdots$ the above becomes zero. For $n=2,4,6,\cdots$

$$
\begin{aligned}
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right)dt &= \frac{2\tau}{2\pi(1+n)}\\
&= \frac{\tau}{\pi(1+n)} \qquad n=2,4,6,\cdots \qquad (2)
\end{aligned}
$$

The second integral in (1) is

$$
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt = -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right]_0^{\frac{\tau}{2}}
$$

But this is undefined for $n=1$, since denominator is zero. Hence we need to handle $n=1$ first on its own. At $n=1$, since $\sin(0)=0$ then

$$
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt = 0 \qquad (3)
$$

For $n>1$

$$
\begin{aligned}
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt &= -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right]_0^{\frac{\tau}{2}}\\
&= \frac{-1}{\frac{2\pi}{\tau}(1-n)}\left[\cos\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right)-1\right]\\
&= \frac{-1}{\frac{2\pi}{\tau}(1-n)}\left[\cos(\pi(1-n))-1\right]\\
&= \frac{1}{\frac{2\pi}{\tau}(n-1)}\left[\cos(\pi(n-1))-1\right]
\end{aligned}
$$

For $n=2,4,6,\cdots$

$$
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt = \frac{-2}{\frac{2\pi}{\tau}(n-1)} = \frac{-\tau}{\pi(n-1)} \qquad (4)
$$

For $n = 3,5,7,\cdots$ the integral is zero. Using result in (2,3,4) in (1) gives final result

$$a_n = \begin{cases} \frac{A}{\tau}\left(\frac{\tau}{\pi(1+n)} + \frac{-\tau}{\pi(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{(n-1)-(1+n)}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{n-1-1-n}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{-2}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Finding b_n

$$\begin{aligned} b_n &= \frac{1}{\frac{\tau}{2}} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \sin\left(\frac{2\pi}{\tau} nt\right) dt \\ &= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} x(t) \sin\left(\frac{2\pi}{\tau} nt\right) dt \\ &= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} A \sin\left(\frac{2\pi}{\tau} t\right) \sin\left(\frac{2\pi}{\tau} nt\right) dt \end{aligned}$$

But $\sin(u)\sin(v) = \frac{1}{2}(\cos(u-v) - \cos(u+v))$, therefore the above integral becomes

$$\begin{aligned} a_n &= \frac{2A}{\tau}\left(\frac{1}{2}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t - \frac{2\pi}{\tau}nt\right) dt - \frac{1}{2}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t + \frac{2\pi}{\tau}nt\right) dt\right) \\ &= \frac{A}{\tau}\left(\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt - \int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right) dt\right) \end{aligned} \tag{6}$$

For the first integral

$$\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt = \left(\frac{\sin\frac{2\pi}{\tau}(1-n)t}{\frac{2\pi}{\tau}(1-n)}\right)_0^{\frac{\tau}{2}}$$

But this is undefined for $n = 1$, since denominator is zero. Hence we need to handle $n = 1$ first on its own. At $n = 1$, since $\cos(0) = 1$ then

$$\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt = \int_0^{\frac{\tau}{2}} dt = \frac{\tau}{2} \tag{7}$$

Now for $n > 1$

$$\begin{aligned}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right)dt &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right)_0^{\frac{\tau}{2}}\\ &= \frac{\tau}{2\pi(1-n)}\left(\sin\left(\frac{2\pi}{\tau}(1-n)t\right)\right)_0^{\frac{\tau}{2}}\\ &= \frac{\tau}{2\pi(1-n)}\left(\sin\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right)-0\right)\\ &= \frac{\tau}{2\pi(1-n)}\left(\sin\left(\pi(1-n)\right)-0\right)\end{aligned}$$

Which is zero for all n. For the second integral in (6)

$$\begin{aligned}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right) &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)}\right)_0^{\frac{\tau}{2}}\\ &= \frac{\tau}{2\pi(1+n)}\left(\sin\left(\frac{2\pi}{\tau}(1+n)t\right)\right)_0^{\frac{\tau}{2}}\\ &= \frac{\tau}{2\pi(1+n)}\left(\sin\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right)-0\right)\\ &= \frac{\tau}{2\pi(1+n)}\left(\sin\left(\pi(1+n)\right)-0\right)\end{aligned}$$

Which is zero for all n. Hence for b_n we have one term only

$$b_n = \begin{cases} \frac{A}{2} & n=1\\ 0 & n=2,3,\cdots\end{cases}$$

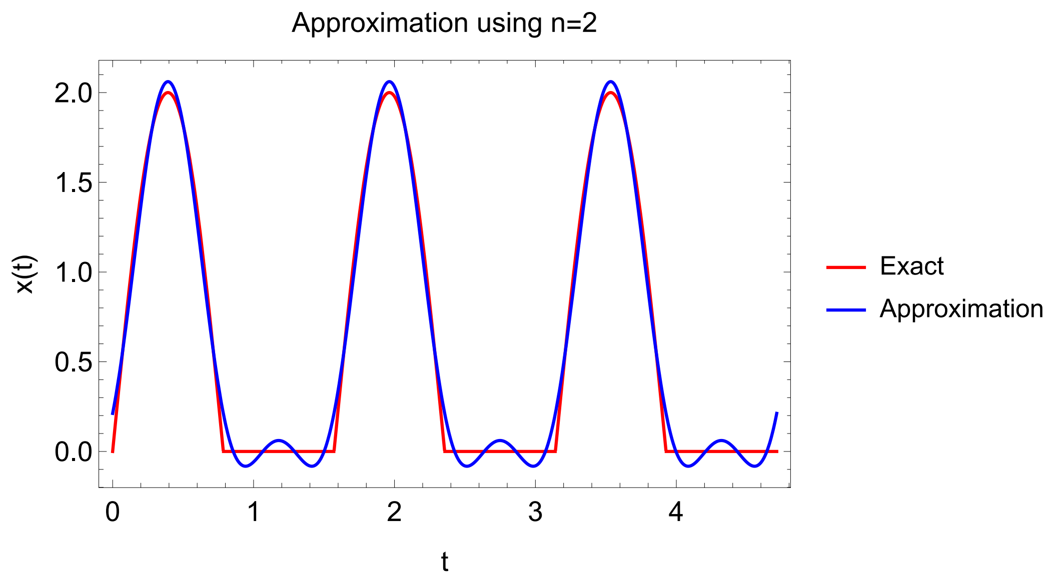
Therefore the Fourier series approximation is

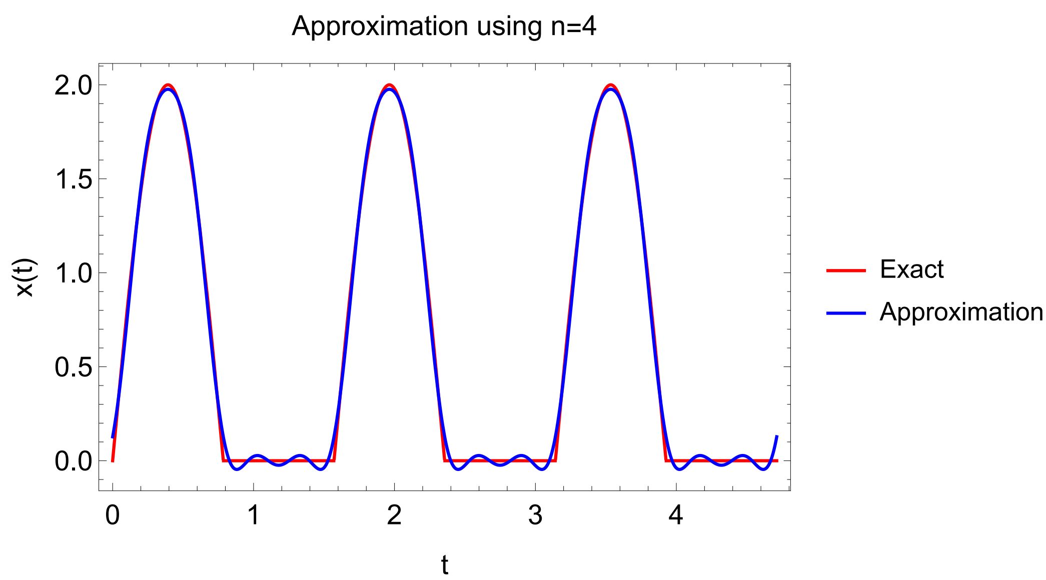
$$\begin{aligned}x(t) &= \overbrace{\frac{A}{\pi}}^{\frac{a_0}{2}} + \overbrace{\frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right)}^{b_1} + \sum_{n=2,4,6,\cdots}^{\infty} \overbrace{A\left(\frac{-2}{\pi(1+n)(n-1)}\right)}^{a_n}\cos\left(\frac{2\pi}{\tau}nt\right)\\ &= \frac{A}{\pi} + \frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{(1+n)(n-1)}\cos\left(\frac{2\pi}{\tau}nt\right)\end{aligned}$$

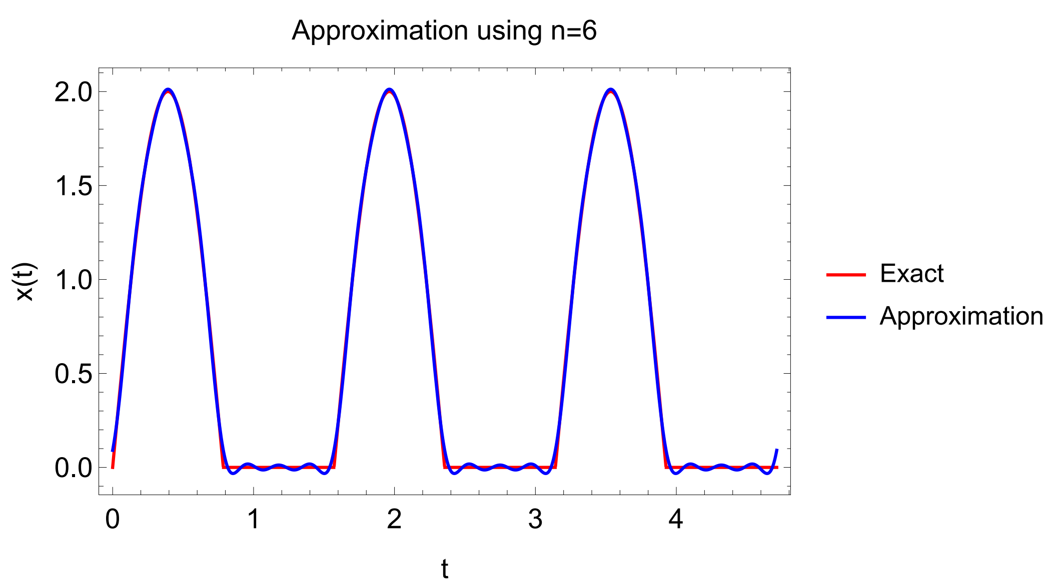
Therefore

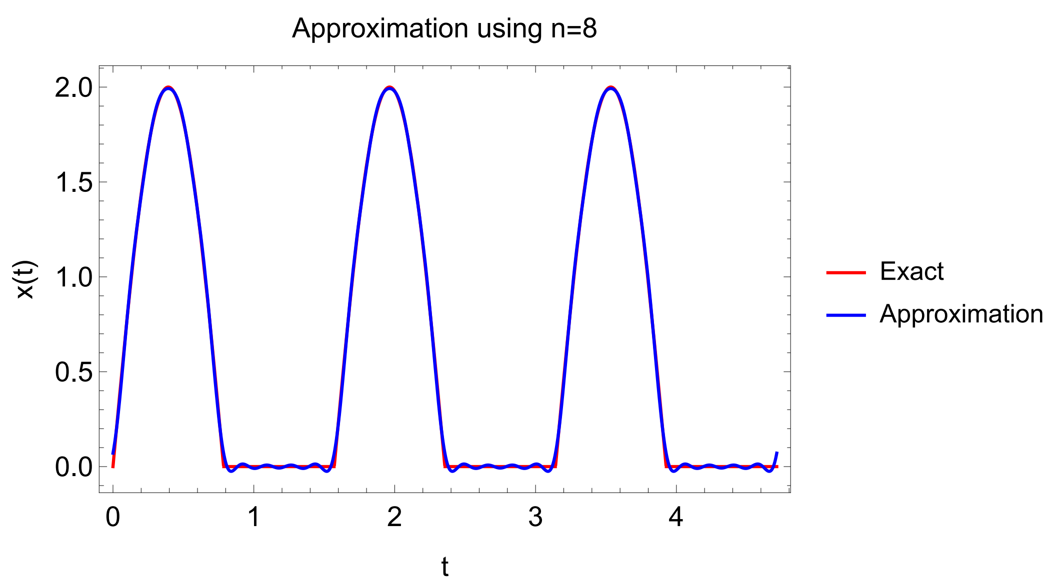
$$x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{(1+n)(n-1)}\cos\left(\frac{2\pi}{\tau}nt\right)$$

To verify this result, the following is a plot of increasing n, using $A = 2$ and $\tau = 1$ with the approximation superimposed on top of $x(t)$. We notice that small number of terms is needed in this case to obtain a good approximation.


Approximation using n=2
x(t)
t
Exact
Approximation


Approximation using n=4
x(t)
t
Exact
Approximation


Approximation using n=6
x(t)
t
Exact
Approximation




```
xApprox[t_, max_, A0_, period_] := A0/Pi + (A0/2)*Sin[2*(Pi/period)*t] -
   2*(A0/Pi)*
   Sum[(1/((1 + n)*(n - 1)))*Cos[2*(Pi/period)*n*t], {n, 2, max, 2}];

myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
    func /. val :> Mod[val - min, max - min] + min

f[t_] := Piecewise[{{A0*Sin[2*(Pi/period)*t], 0 < t < period/2}, {0,True}}]

maxTerms=2;
A0=2;
period=1/2 Pi;
p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
       xApprox[t,maxTerms,A0,period]},{t,0,3 period},
       PlotLegends->{"Exact","Approximation"},
       PlotStyle->{Red,Blue},
       Frame->True,
       FrameLabel->{{"x(t)",None},{"t","Approximation using n="<>ToString[
    maxTerms]]}},
       BaseStyle->14,ImageSize->400]
```

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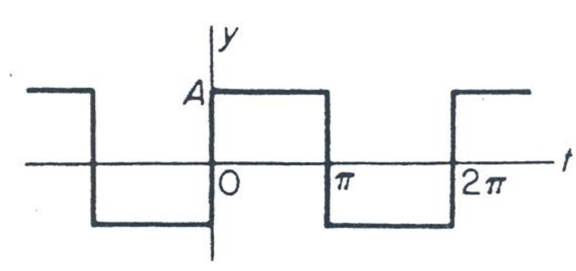
2.2.2 Problem 2

Problem 2

Determine the Complex Fourier series expansion for the periodic function y(t):



The function to approximate is defined as

$$y(t)=\begin{cases} A & 0\le t\le\pi \\ -A & \pi<t\le 2\pi\end{cases}$$

With period $\tau=2\pi$. This function is odd.

$$\begin{aligned}
c_n &= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} y(t)\,e^{-j\frac{2\pi}{\tau}nt}dt = \frac{1}{\tau}\int_0^{\tau} y(t)\,e^{-j\frac{2\pi}{\tau}nt}dt \\
&= \frac{1}{\tau}\left(\int_0^{\pi} Ae^{-j\frac{2\pi}{\tau}nt}dt - \int_{\pi}^{2\pi} Ae^{-j\frac{2\pi}{\tau}nt}dt\right) \\
&= \frac{A}{\tau}\left(\left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n}\right]_0^{\pi} - \left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n}\right]_{\pi}^{2\pi}\right) \\
&= \frac{A}{\tau}\left(\frac{-1}{j\frac{2\pi}{\tau}n}\left[e^{-j\frac{2\pi}{\tau}nt}\right]_0^{\pi} + \frac{1}{j\frac{2\pi}{\tau}n}\left[e^{-j\frac{2\pi}{\tau}nt}\right]_{\pi}^{2\pi}\right) \\
&= \frac{A}{\tau}\frac{\tau}{j2\pi n}\left(-\left[e^{-j\frac{2\pi}{\tau}nt}\right]_0^{\pi} + \left[e^{-j\frac{2\pi}{\tau}nt}\right]_{\pi}^{2\pi}\right)
\end{aligned}$$

But $\tau=2\pi$ and the above simplifies to

$$\begin{aligned}
c_n &= \frac{A}{j2\pi n}\left(-\left[e^{-jnt}\right]_0^{\pi} + \left[e^{-jnt}\right]_{\pi}^{2\pi}\right) \\
&= \frac{A}{j2\pi n}\left(\left[1-e^{-jn\pi}\right] + \left[e^{-j2n\pi}-e^{-jn\pi}\right]\right) \qquad (1)
\end{aligned}$$

But

$$\begin{aligned}
e^{-jn\pi} &= \cos n\pi - j\sin n\pi \\
&= \cos n\pi
\end{aligned}$$

And

$$\begin{aligned}
e^{-j2n\pi} &= \cos 2n\pi - j\sin 2n\pi \\
&= 1
\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
c_n &= \frac{A}{j2\pi n}\left([1-\cos n\pi] + [1-\cos n\pi]\right) \\
&= \frac{A}{j\pi n}(1-\cos n\pi)
\end{aligned}$$

For n odd $\cos n\pi=-1$ and the above becomes

$$c_n = \frac{2A}{j\pi n}$$

For n even $\cos n\pi = 1$ and $c_n=0$ in this case. Therefore the approximation is

$$\begin{aligned}
y(t) &\approx \sum_{n=\cdots-3,-1,1,3,\cdots}^{\infty} c_n e^{j2nt} \\
&= \frac{2A}{j\pi}\sum_{n=\cdots-3,-1,1,3,\cdots}^{\infty}\frac{1}{n}e^{j2nt} \qquad (2)
\end{aligned}$$

We can now obtain the standard form of the series if needed. $c_{-n}=c_n^*=\frac{2A}{-j\pi n}$ and hence

$$\begin{aligned} a_n &= c_n + c_{-n} \\ &= 0 \end{aligned}$$

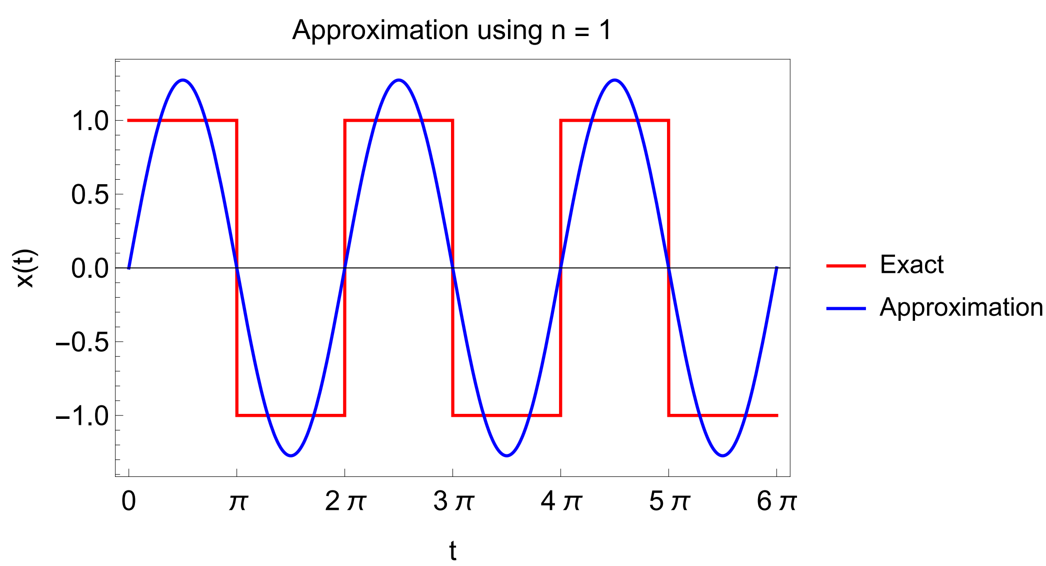
All $a_n = 0$, as expected, since this is an odd function.

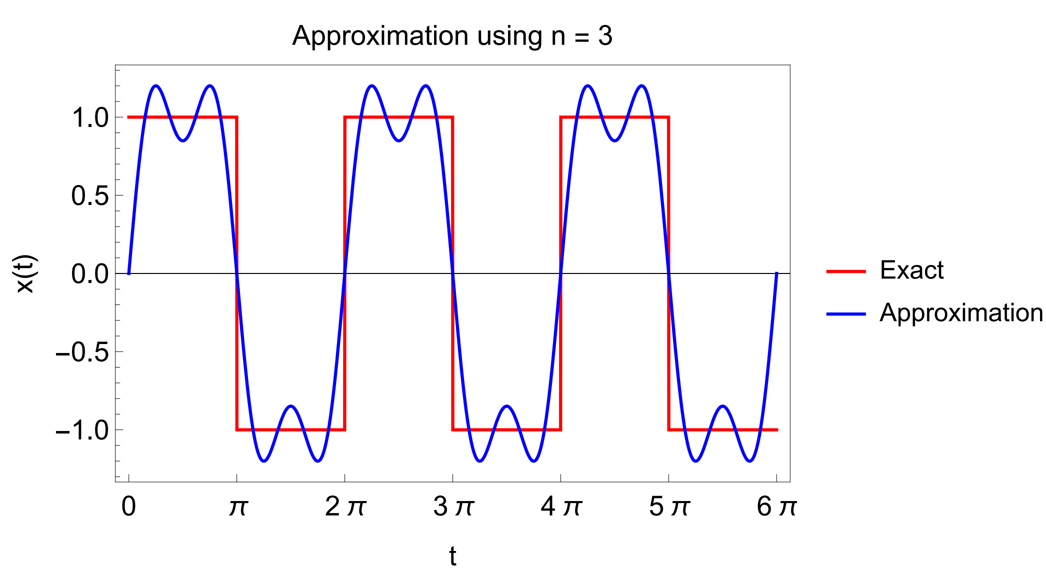
$$\begin{aligned} b_n &= j\left(c_n - c_{-n}\right) \\ &= j\left(\frac{2A}{j\pi n} - \frac{2A}{-j\pi n}\right) \\ &= j\left(\frac{4A}{j\pi n}\right) \\ &= \frac{4A}{\pi n} \end{aligned}$$

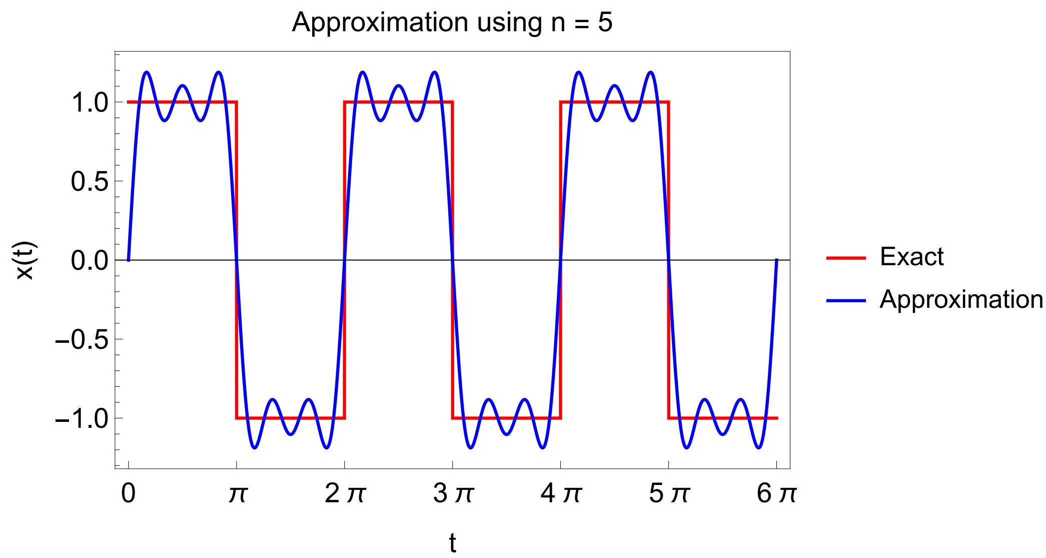
Hence

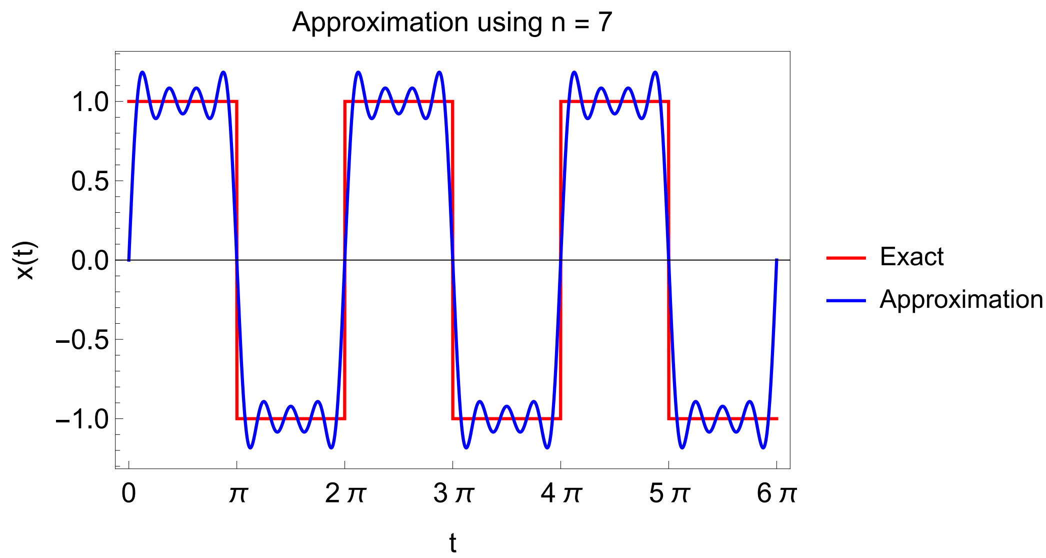
$$y\left(t\right) \approx \frac{4A}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \sin\left(nt\right) \tag{3}$$

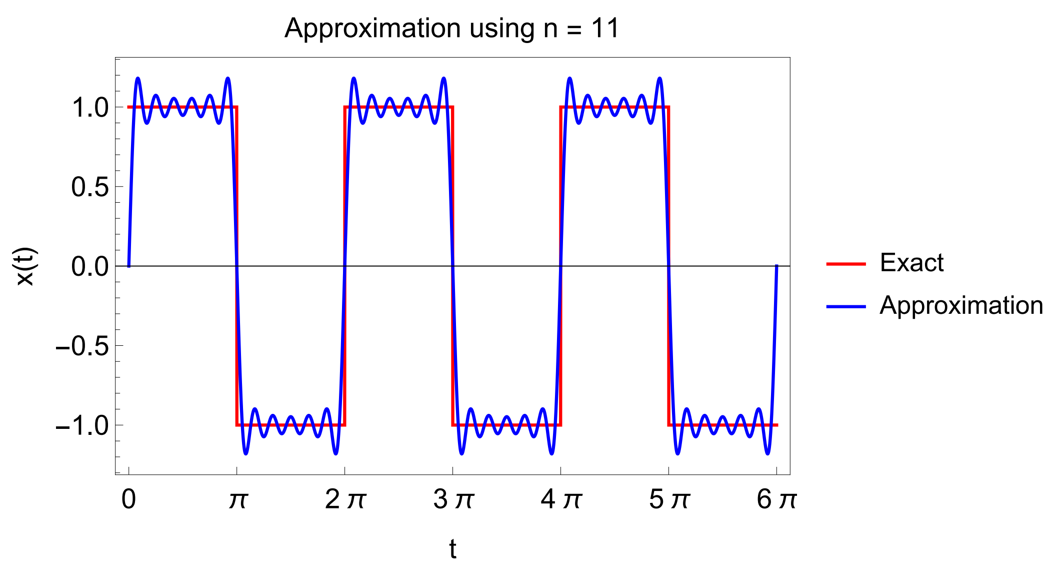
Both (2) and (3) are the same. (2) is complex form of (3). To see the approximation, here are some plots with increasing number of terms for $A = 1$

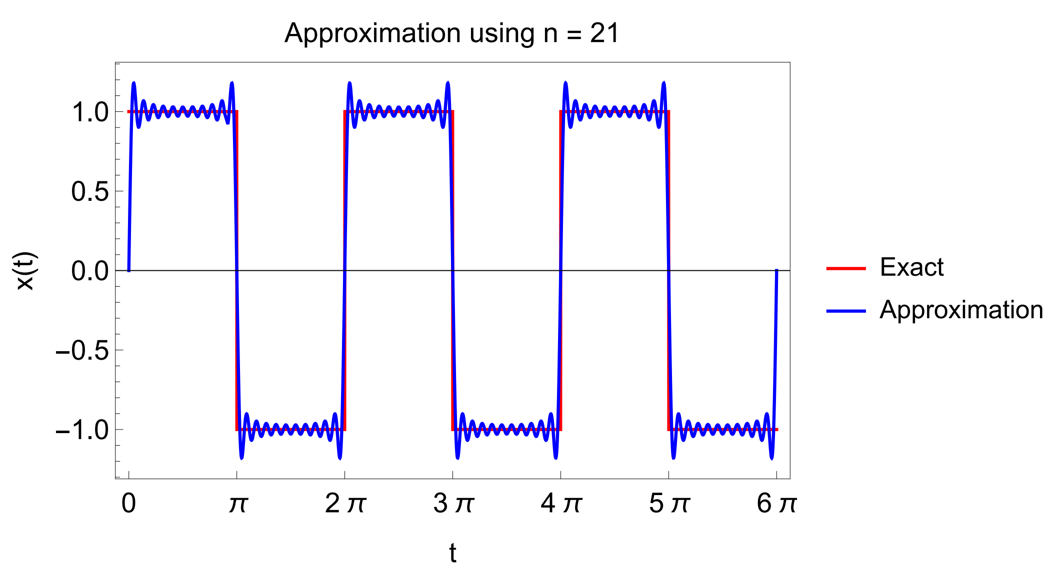





Approximation using n = 5
x(t)
1.0
0.5
0.0
−0.5
−1.0
0
π
2 π
3 π
4 π
5 π
6 π
t
Exact
Approximation








```

xApprox[t_, max_, A0_, period_] := 4 A0/Pi * Sum[(1/n)*Sin[n*t], {n, 1, max,
    2}];
myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
      func /. val :> Mod[val - min, max - min] + min

```

```
f[t_] := Piecewise[{{A0, 0 < t < period/2}, {-A0, True}}];

maxTerms=11;
A0=1;
period=2 Pi;
p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
    xApprox[t,maxTerms,A0,period]},{t,0,3 period},
    PlotLegends->{"Exact","Approximation"},
    PlotStyle->{Red,Blue},
    Frame->True,
    FrameLabel->{{"x(t)",None},{"t","Approximation using n = "<>ToString[
   maxTerms]}},
    BaseStyle->14,ImageSize->400,
    Exclusions->None,
    FrameTicks->{{Automatic,None},{Range[0,6 Pi,Pi],Automatic}}]
```

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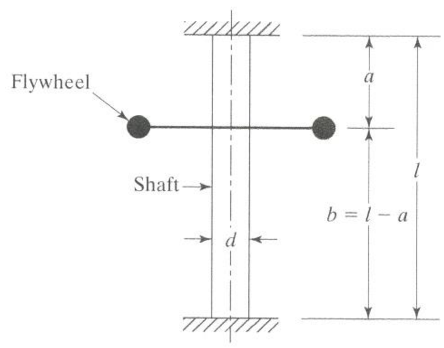
2.3 HW3

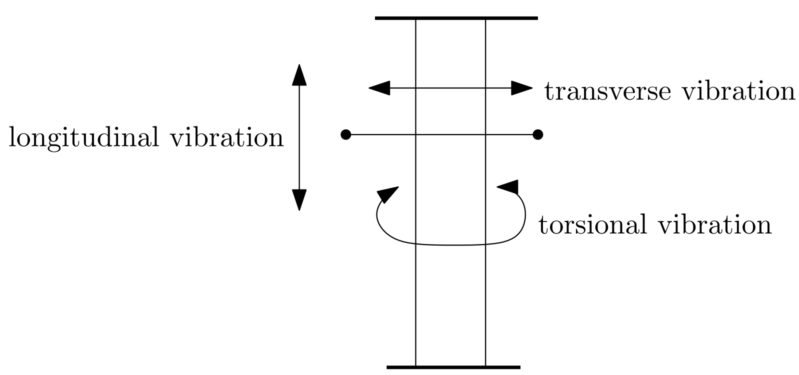
2.3.1 Problem 1

Problem 1

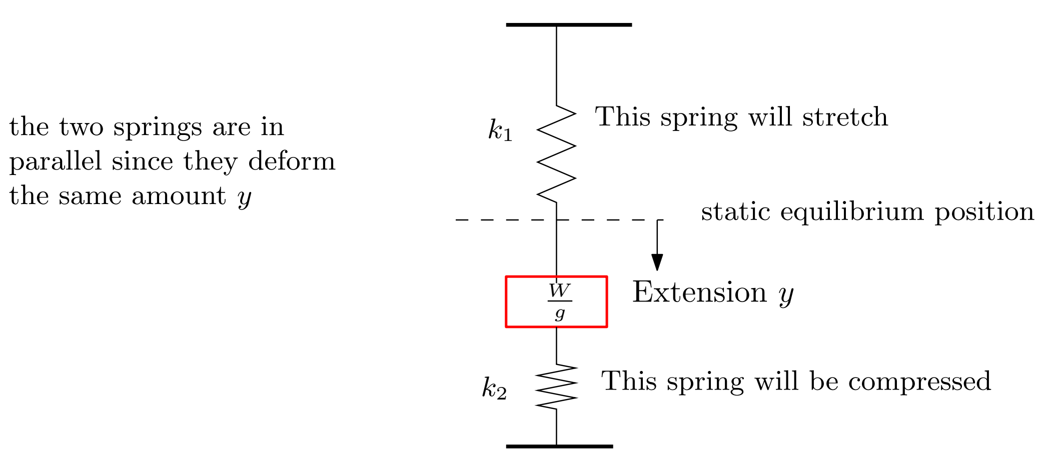
A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter *d* and length *l* and is fixed at both ends. The flywheel has a weight of *W* and a radius of gyration of *r*. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.



We need to find the natural frequency of vibration for the following cases



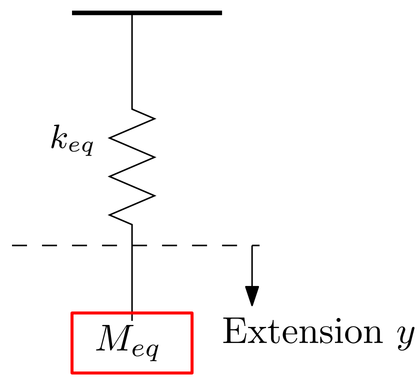
longitudinal In this mode the system can be modeled as the following



Since both springs are in parallel, then the equivalent spring stiffness is

$$k_{eq} = k_1 + k_2$$

The equivalent mass is just the mass of the flywheel $\frac{W}{g}$. Hence the overall system can now be modeled as follows



Which has the equation of motion

$$m_{eq}\ddot{y} + k_{eq}y = 0$$
$$\ddot{y} + \frac{k_{eq}}{m_{eq}}y = 0$$

Therefore

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

We now just need to determine $k_{eq} = k_1 + k_2$. But from mechanics of materials we know that $k_1 = \frac{AE}{a}$ and $k_2 = \frac{AE}{b}$. Therefore the above becomes

$$\omega_n = \sqrt{\frac{\frac{AE}{a} + \frac{AE}{b}}{\frac{W}{g}}}$$
$$= \sqrt{\frac{gAE}{W}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

<u>Transverse</u> In this mode the system can be modeled as beam with <u>fixed ends</u> with load W at distance a from one end and distance b from the other end. From tables, the stiffness coefficient in this case is given by

$$k_{eq} = 3EI\left(\frac{L}{ab}\right)^3$$

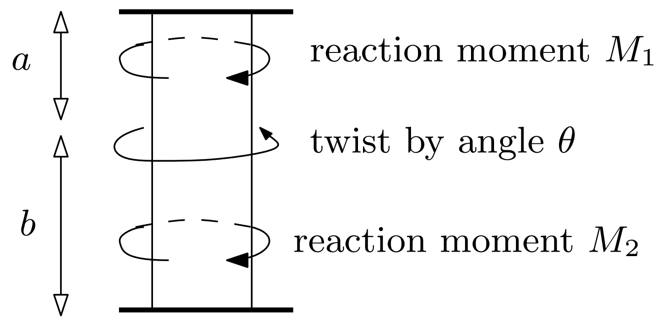
The equivalent mass remains as before which is just the mass of the flywheel $\frac{W}{g}$. Therefore, as above we find the natural frequency as

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

Or

$$\boxed{\omega_n = \sqrt{\frac{3gEI}{W}\left(\frac{L}{ab}\right)^3}}$$

<u>Torsional</u> In this mode, the flywheel is twisted by some degree θ, and therefore the top part of the beam and the bottom part of the beam will resist this twist by applying moment against the twist as shown in this diagram



From mechanics of materials, there is relation between the twisting angle and resisting torque by beam which is given by

$$M = \frac{GJ}{L}\theta$$

Where here θ is the twist angle (radians) and M is the torque (Nm) and L is length of beam and G is modulus of rigidity (N per m^2) and J is the second moment of area of the cross section (m^4) about its center. Therefore total moments is

$$\begin{aligned} M_1 + M_2 &= \frac{GJ}{a}\theta + \frac{GJ}{a}\theta \\ &= GJ\theta\left(\frac{1}{a} + \frac{1}{b}\right) \end{aligned}$$

Comparing the above to definition of stiffness which is $F = K\Delta$ but in this problem $\Delta \equiv \theta$ and $F \equiv (M_1 + M_2)$, then we see that the equivalent stiffness is

$$k_{eq} = GJ\left(\frac{1}{a} + \frac{1}{b}\right)$$

We now need the equivalent mass. In this case it is the <u>mass moment of inertia</u> of flywheel. We are given that radius of gyration is r, hence

$$m_{eq} = \frac{W}{g}r^2$$

We now have all the pieces needed to find ω_n

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_{eq}}{m_{eq}}} \\ &= \sqrt{\frac{GJ\left(\frac{1}{a} + \frac{1}{b}\right)}{\frac{W}{g}r^2}} \\ &= \sqrt{\frac{gGJ}{Wr^2}\left(\frac{1}{a} + \frac{1}{b}\right)} \end{aligned}$$

From tables, for circular bar of radius d, we see that $J = \frac{\pi}{32}d^4$. Hence the above becomes

$$\boxed{\omega_n = \sqrt{\frac{gG\pi d^4}{32Wr^2}\left(\frac{1}{a} + \frac{1}{b}\right)}}$$

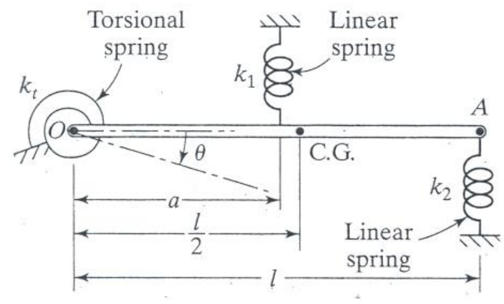
<u>Summary of results</u>

case	ω_n
longitudinal	$\sqrt{\frac{gAE}{W}\left(\frac{1}{a} + \frac{1}{b}\right)}$
Transverse	$\sqrt{\frac{g}{W}(3EI)\left(\frac{L}{ab}\right)^3}$
Torsional	$\sqrt{\frac{g}{W}\frac{G}{r^2}\frac{\pi d^4}{32}\left(\frac{1}{a} + \frac{1}{b}\right)}$

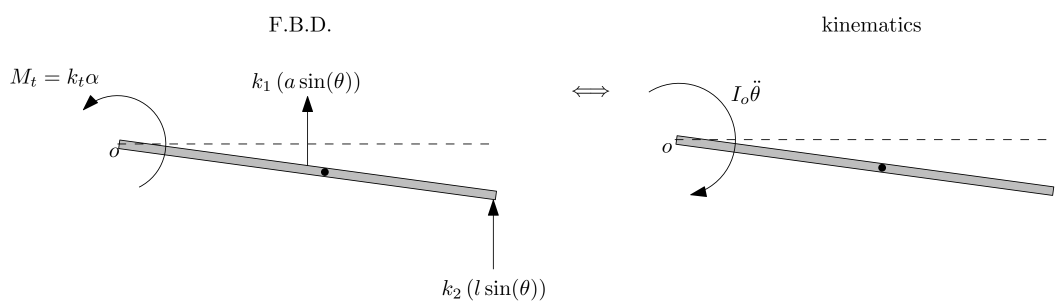
2.3.2 Problem 2

Problem 2

The uniform rigid bar OA of length L and mass m is pinned about point O. Using Newton's Second Law, find the equation of motion for the system using the generalized coordinate θ and also find the system's natural frequency.



The first step is to draw the free body diagram and the kinematic diagram



Taking moments about the joint O, noting that positive is anti-clockwise gives

$$k_t\theta + k_1 (a \sin\theta) a + k_2 (L \sin\theta) L = -I_o \ddot{\theta} \tag{1}$$

Using parallel axis theorem,

$$\begin{aligned} I_o &= I_{CG} + m\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{12} m L^2 + m\frac{L^2}{4} \\ &= \frac{1}{3} L^2 m \end{aligned}$$

Hence (1) becomes

$$\frac{1}{3} L^2 m \ddot{\theta} + k_t\theta + k_1 (a\sin\theta) a + k_2 (L\sin\theta) L = 0$$

For small angle approximation the above becomes (we have to apply small angle approximation in order to obtain the form that allows us to determine ω_n^2, since this only works for linear equations of motion).

$$\begin{aligned} \frac{1}{3} L^2 m \ddot{\theta} + k_t\theta + k_1 a^2\theta + k_2 L^2 \theta &= 0 \\ \frac{1}{3} L^2 m \ddot{\theta} + \theta\left(k_t + k_1 a^2 + k_2 L^2\right) &= 0 \\ \ddot{\theta} + \theta\frac{3\left(k_t + k_1 a^2 + k_2 L^2\right)}{mL^2} &= 0 \end{aligned}$$

Comparing the above to standard form of linearized $\ddot{\theta} + \omega_n^2 \theta = 0$ we see that the natural frequency (radians per second) is

$$\boxed{\omega_n = \sqrt{\frac{3\left(k_t + k_1 a^2 + k_2 L^2\right)}{mL^2}}}$$

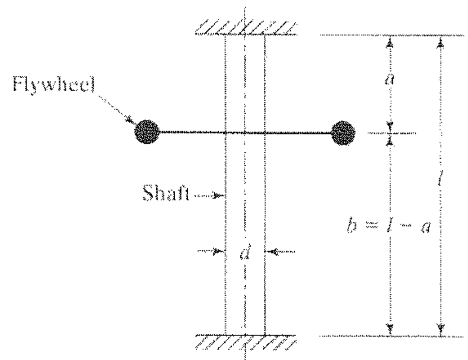
2.3.3 key solution version 1

ME 440
Intermediate Vibrations

Homework #3
due Thursday, October 5, 2017

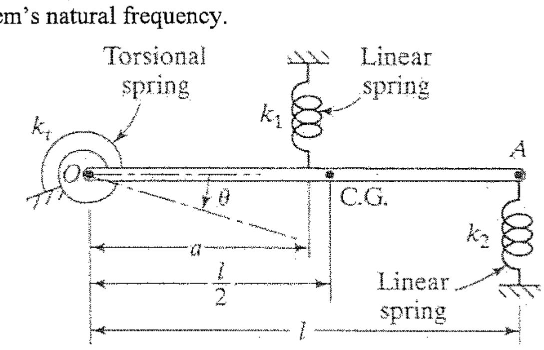
Problem 1

A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.



Problem 2

The uniform rigid bar OA of length L and mass m is pinned about point O. Using Newton's Second Law, find the equation of motion for the system using the generalized coordinate θ and also find the system's natural frequency.



Solution -

Transverse:

Deflection $y = \frac{Pb^2}{6EI\ell^3}\left[(2b-3\ell)a^3 + 3\ell(\ell-b)a^2\right]$

$$y = \frac{Pa^2b^2}{6EI\ell^3}\left[2ab - 3a\ell + 3\ell^2 - 3\ell b\right]$$

$$y = \frac{Pa^3b^3}{3EI(a+b)^3}$$

$$k_{eq} = \frac{P}{y} = \frac{3EI(a+b)^3}{a^3b^3} \qquad m_{eq} = \frac{W}{g}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{3EI(a+b)^3 g}{a^3b^3 W}} \qquad \text{where} \quad I = \frac{\pi d^4}{64}$$

Torsional:

$$k_{eq} = (k_1)_t + (k_2)_t$$

$$k_{eq} = \frac{GJ}{a} + \frac{GJ}{b} = \frac{GJ(a+b)}{ab}$$

$$\text{but} \quad J = \frac{\pi d^4}{32}$$

$$k_{eq} = \frac{G\pi d^4}{32}\frac{(a+b)}{ab}$$

$$m_{eq} = J_o = mk^2 = mr^2 \qquad r = \text{radius of gyration}$$

$$m_{eq} = \frac{W}{g}r^2$$

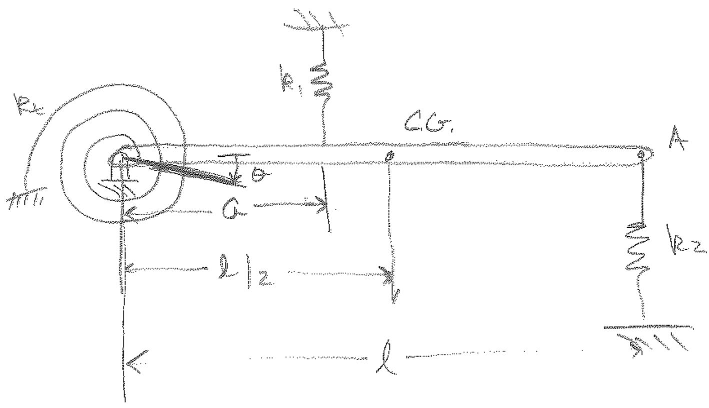
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{G\pi d^4(a+b)g}{32\,ab\,Wr^2}}$$

Longitudinal:

$$k_{eq} = k_1 + k_2 = \frac{AE}{a} + \frac{AE}{b} = \frac{AE(a+b)}{ab}$$

$$m_{eq} = W/g$$

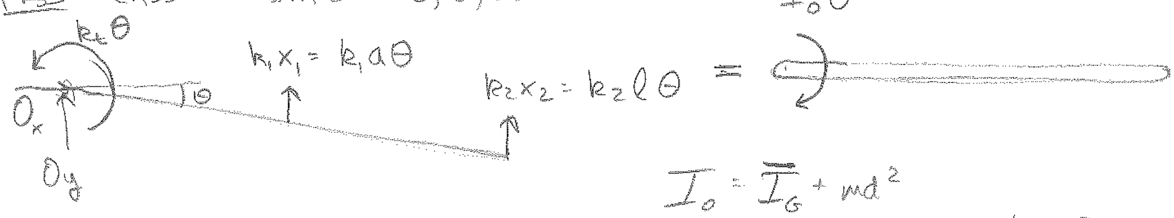
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{AE(a+b)g}{abW}} \qquad \text{where } A = \frac{\pi d^2}{4}$$



MASS OA = m

USE NEWTON'S 2ND LAW TO FIND E.O.M. AND ω_n

FBD (ASSUME SMALL + $\theta, \dot{\theta}, \ddot{\theta}$)



$$I_o = \bar{I}_G + md^2$$
$$= \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

(+) $\sum M_o = \sum M_{eff}$

$$k_t\theta + (k_1 a\theta)a + (k_2 l\theta)l = -I_o\ddot{\theta}$$

E.O.M.

$$\frac{1}{3}ml^2\ddot{\theta} + \left(k_2 + k_1a^2 + k_2l^2\right)\theta = 0$$

$$\omega_n = \sqrt{\frac{3(k_t + k_1a^2 + k_2l^2)}{ml^2}}$$

2.3.4 key solution version 2

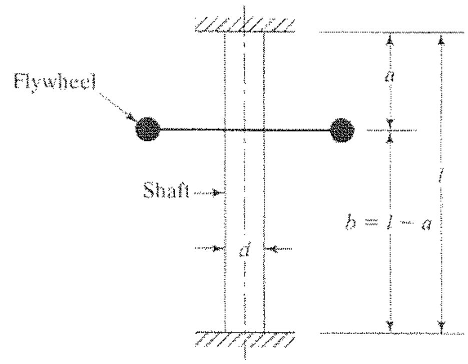
SOLUTIONS ORIGINAL

ME 440
Intermediate Vibrations

Homework #3
due Thursday, October 5, 2017

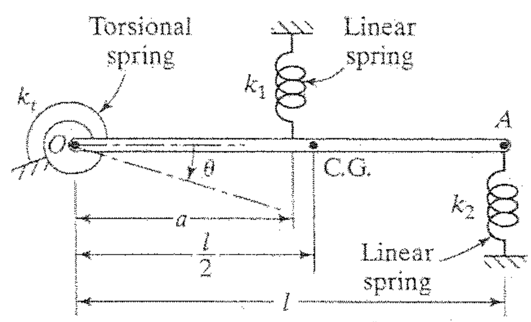
Problem 1

A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.



Problem 2

The uniform rigid bar OA of length L and mass m is pinned about point O. Using Newton's Second Law, find the equation of motion for the system using the generalized coordinate θ and also find the system's natural frequency.



TRANSVERSE (NO ROTATION) (OPTION 1)

FIXED - FIXED BEAM WITH LATERAL DISPLACEMENT

$$k = \frac{12EI}{\ell^3} \qquad k_a = \frac{12EI}{a^3} \qquad k_b = \frac{12EI}{b^3}$$

BOTH BEAMS HAVE THE SAME DEFLECTION/DEFORMATION ∴ PARALLEL

$$k_{eq} = k_a + k_b$$

$$= \frac{12EI}{a^3} + \frac{12EI}{b^3} = 12EI\left(\frac{1}{a^3} + \frac{1}{b^3}\right) = 12EI\left(\frac{a^3+b^3}{a^3b^3}\right)$$

$$=$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{12EI\,(a^3+b^3)\,g}{W\,a^3b^3}}$$

[ASSUMES NO ROTATION OF FLYWHEEL OCCURS]

FIXED - FIXED BEAM WITH NON CENTER TRANSVERSE POINT LOADS.

Solution -

Transverse (OPTION 2)

Deflection $y = \frac{Pb^2}{6EI\ell^3}\left[(2b-3\ell)a^3 + 3\ell(\ell-b)a^2\right]$

$$y = \frac{Pa^2b^2}{6EI\ell^3}\left[2ab - 3a\ell + 3\ell^2 - 3\ell b\right]$$

$$y = \frac{Pa^3b^3}{3EI(a+b)^3}$$

$$k_{eq} = \frac{P}{y} = \frac{3EI(a+b)^3}{a^3b^3} \qquad m_{eq} = \frac{W}{g}$$

$$\omega_n^{*} = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{3EI(a+b)^3 g}{a^3b^3W}} \qquad \text{where } I = \frac{\pi d^4}{64}$$

* THIS APPROACH ALLOWS ROTATION IN k_{eq} CALCULATION BUT NOT IN M_{eq}

Torsional:

$$k_{eq} = (k_1)_t + (k_2)_t$$

$$k_{eq} = \frac{GJ}{a} + \frac{GJ}{b} = \frac{GJ(a+b)}{ab}$$

$$\text{but } J = \frac{\pi d^4}{32}$$

$$k_{eq} = \frac{G\pi d^4}{32}\frac{(a+b)}{ab}$$

$$m_{eq} = J_o = mk^2 = mr^2 \qquad r = \text{radius of gyration}$$

$$m_{eq} = \frac{W}{g}r^2$$

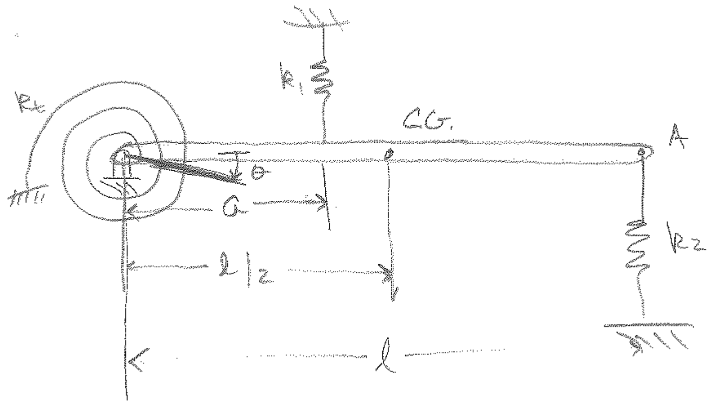
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{G\pi d^4(a+b)g}{32\,ab\,Wr^2}}$$

Longitudinal:

$$k_{eq} = k_1 + k_2 = \frac{AE}{a} + \frac{AE}{b} = \frac{AE(a+b)}{ab}$$

$$m_{eq} = W/g$$

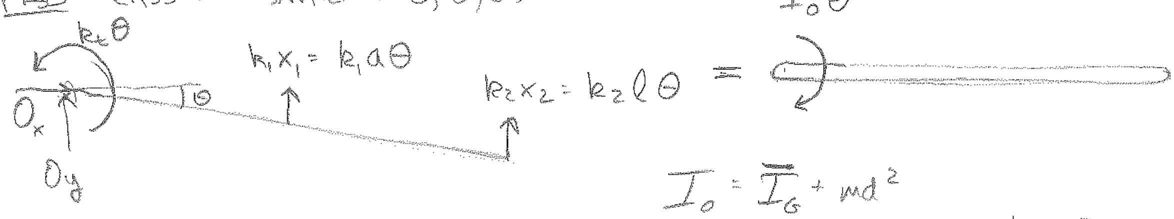
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{AE(a+b)g}{abW}} \qquad \text{where } A = \frac{\pi d^2}{4}$$



MASS OA = m

USE NEWTON'S 2ND LAW TO FIND E.O.M. AND ω_n

FBD (ASSUME SMALL + $\theta, \dot{\theta}, \ddot{\theta}$)



$$I_O = \bar{I}_G + md^2$$
$$= \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

1) $\Sigma M_O = \Sigma M_{eff}$

$$k_t\theta + (k_1 a\theta)a + (k_2 l\theta)l = -I_O\ddot{\theta}$$

E.O.M.

$$\frac{1}{3}ml^2\ddot{\theta} + (k_2 + k_1 a^2 + k_2 l^2)\ddot{\theta} = 0$$

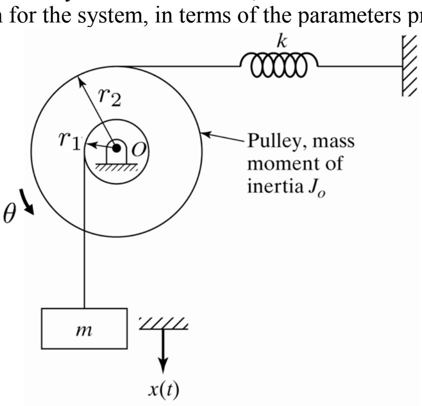
$$\omega_n = \sqrt{\frac{3(k_t + k_1 a^2 + k_2 l^2)}{ml^2}}$$

2.4 HW4

2.4.1 Problem 1

Problem 1

The pulley is in fixed axis rotation about Point O. Using energy concepts and θ as the generalized coordinate, determine
a) the natural frequency of the system shown below, and
b) the equation of motion for the system, in terms of the parameters provided.



2.4.1.1 Part a

Using Rayleigh method, we need to find $T_{\max}$ and $U_{\max}$ where T is the kinetic energy of the system and U is the potential energy and then solve for ω_n by setting $T_{\max} = U_{\max}$.

Kinetic energy is

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_o \dot{\theta}^2$$

But $x = r_1\theta$, therefore $\dot{x} = r_1\dot{\theta}$ and the above becomes

$$T = \frac{1}{2} m \left(r_1\dot{\theta}\right)^2 + \frac{1}{2} J_o \dot{\theta}^2 \tag{1}$$

And potential energy only comes from the spring, since we assume x is measured from static equilibrium. Hence

$$\begin{aligned} U &= \frac{1}{2} k x^2 \\ &= \frac{1}{2} k \left(r_2\theta\right)^2 \end{aligned} \tag{2}$$

To get ω_n into (1) and (2), we now assume that motion is harmonic, hence $\theta = \theta_{\max} \sin(\omega_n t)$, Therefore $\dot{\theta} = \theta_{\max}\omega_n \cos(\omega_n t)$ and rewriting (1,2) using these expressions results in

$$\begin{aligned} T &= \frac{1}{2} m \left(r_1\theta_{\max}\omega_n \cos(\omega_n t)\right)^2 + \frac{1}{2} J_o \left(\theta_{\max}\omega_n \cos(\omega_n t)\right)^2 \\ U &= \frac{1}{2} k \left(r_2 \left(\theta_{\max} \sin(\omega_n t)\right)\right)^2 \end{aligned}$$

Hence, maximum is when $\theta = \theta_{\max}$ and $\dot{\theta} = \theta_{\max}\omega_n$ and the above becomes

$$\begin{aligned} T_{\max} &= \frac{1}{2} m r_1^2 \theta_{\max}^2 \omega_n^2 + \frac{1}{2} J_o \theta_{\max}^2 \omega_n^2 \\ U_{\max} &= \frac{1}{2} k r_2^2 \theta_{\max}^2 \end{aligned}$$

Now

$$\begin{aligned} T_{\max} &= U_{\max} \\ \frac{1}{2} m r_1^2 \theta_{\max}^2 \omega_n^2 + \frac{1}{2} J_o \theta_{\max}^2 \omega_n^2 &= \frac{1}{2} k r_2^2 \theta_{\max}^2 \\ m r_1^2 \omega_n^2 + J_o \omega_n^2 &= k r_2^2 \end{aligned}$$

Hence

$$\omega_n^2 = \frac{kr_2^2}{mr_1^2 + J_o}$$

$$\omega_n = \sqrt{\frac{kr_2^2}{mr_1^2 + J_o}}$$

2.4.1.2 Part b

The equation of motion is given by

$$\frac{d}{dt}(T + U) = 0$$

We found T, U in part (a), therefore the above becomes

$$\frac{d}{dt}\left(\frac{1}{2}m\left(r_1\dot{\theta}\right)^2 + \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}k\left(r_2\theta\right)^2\right) = 0$$

$$mr_1^2\dot{\theta}\ddot{\theta} + J_o\dot{\theta}\ddot{\theta} + kr_2^2\theta\dot{\theta} = 0$$

For non trivial motion $\dot{\theta} \neq 0$ for all time, hence we can divide throughout by $\dot{\theta}$ and obtain

$$mr_1^2\ddot{\theta} + J_o\ddot{\theta} + kr_2^2\theta = 0$$

$$\ddot{\theta}\left(mr_1^2 + J_o\right) + kr_2^2\theta = 0$$

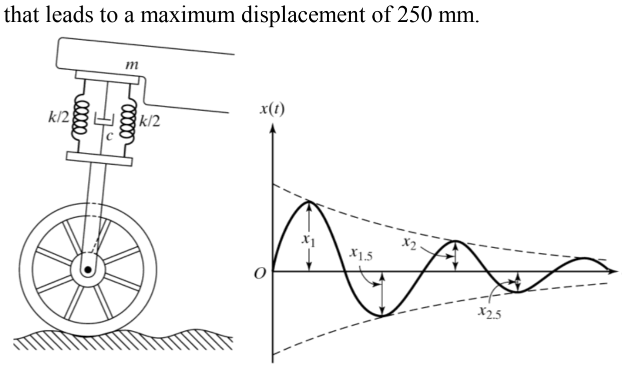
$$\ddot{\theta} + \frac{kr_2^2}{mr_1^2 + J_o}\theta = 0$$

The above is the equation of motion.

2.4.2 Problem 2

Problem 2

An underdamped shock absorber is to be designed for motorcycle of mass 200 kg. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as illustrated below. Determine the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 seconds and the amplitude x_1 is to be reduced to ¼ in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.



First part

The first step is to determine damping ratio ζ. This is done using logarithmic decrement. Since $X_{1.5} = \frac{1}{4}X_1$ and $X_2 = \frac{1}{4}X_{1.5}$ then

$$X_2 = \frac{1}{4}\left(\frac{1}{4}X_1\right)$$

$$= \frac{1}{16}X_1$$

Using

$$\frac{X_1}{X_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+t_2)}}$$

Where $t_2 = t_1 + \tau_d$ and τ_d is damped period. Therefore the above becomes

$$\frac{X_1}{\frac{1}{16}X_1} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+\tau_d)}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_1} e^{-\zeta\omega_n \tau_d}} = e^{\zeta\omega_n \tau_d}$$
$$\ln(16) = \zeta\omega_n\tau_d$$

Taking log of both sides gives

$$\ln(16) = \zeta\omega_n\tau_d \tag{1}$$

But

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

And (1) simplifies to

$$\ln(16) = \zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$
$$2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Squaring both sides and solving for ζ gives

$$(2.7726)^2\left(1-\zeta^2\right) = 4\pi^2\zeta^2$$
$$\zeta^2\left(4\pi^2 + 7.6873\right) = 7.6873$$
$$\zeta^2 = \frac{7.6873}{4\pi^2 + 7.6873}$$

Taking the positive root results in

$$\zeta = \sqrt{\frac{7.6873}{4\pi^2 + 7.6873}} = 0.40371$$

Now that ζ is know, ω_n can be found, since we are told that $\tau_d = 2$ seconds. Using

$$\tau_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Then solving for ω_n from the above gives

$$2 = \frac{2\pi}{\omega_n\sqrt{1-0.40371^2}}$$
$$\omega_n = \frac{\pi}{\sqrt{1-0.40371^2}} = 3.4339 \text{ rad/sec}$$

Now we are ready to find the stiffness coefficient k and damping coefficient c. Using

$$\zeta = \frac{c}{2\omega_n m}$$

Then

$$c = 2\zeta\omega_n m = 2(0.40371)(3.4339)(200) = 554.52 \text{ N-s/m}$$

But since

$$\omega_n^2 = \frac{k}{m}$$

Then k is now found

$$\begin{aligned} k &= \omega_n^2 m \\ &= (3.4339)^2 (200) \\ &= 2358.3 \text{ N/m} \end{aligned}$$

Second part

Maximum displacement occurs at time t_1 as given by (from textbook)

$$\sin \omega_d t_1 = \sqrt{1-\zeta^2}$$

Hence

$$\begin{aligned} \omega_d t_1 &= \arcsin\left(\sqrt{1-\zeta^2}\right) \\ t_1 &= \frac{1}{\omega_n \sqrt{1-\zeta^2}} \arcsin\left(\sqrt{1-\zeta^2}\right) \\ &= \frac{1}{3.4339\sqrt{1-0.40371^2}} \arcsin\left(\sqrt{1-0.40371^2}\right) \\ &= 0.36772 \text{ sec} \end{aligned}$$

Since

$$x(t) = Xe^{-\zeta\omega_n t} \sin(\omega_d t) \tag{2}$$

Then at maximum displacement, where $x = 0.25$ m, the above becomes

$$\begin{aligned} x_{\max}(t_1) &= Xe^{-\zeta\omega_n t_1} \sin(\omega_d t_1) \\ \frac{x_{\max} e^{\zeta\omega_n t_1}}{\sin(\omega_d t_1)} &= X \end{aligned}$$

Plug-in numerical values to solve for maximum displacement X gives

$$\begin{aligned} X &= \frac{0.25 \exp(0.40371 \times 3.4339 \times 0.36772)}{\sin\left(\left(3.4339\sqrt{1-0.40371^2}\right)(0.36772)\right)} \\ &= 0.45495 \text{ m} \end{aligned}$$

From (2), the velocity is found

$$\begin{aligned} \dot{x}(t) &= -\zeta\omega_n X e^{-\zeta\omega_n t} \sin(\omega_d t) + Xe^{-\zeta\omega_n t} \omega_d \cos(\omega_d t) \\ &= Xe^{-\zeta\omega_n t} (\omega_d \cos(\omega_d t) - \zeta\omega_n \sin(\omega_d t)) \end{aligned}$$

At $t = 0$ the above gives

$$\begin{aligned} \dot{x}(0) &= X\omega_d \\ &= X\left(\omega_n \sqrt{1-\zeta^2}\right) \end{aligned}$$

Plug-in in numerical values

$$\begin{aligned} \dot{x}(0) &= 0.45495\left(3.4339\sqrt{1-0.40371^2}\right) \\ &= 1.4293 \text{ m/s} \end{aligned}$$

2.4.3 key solution

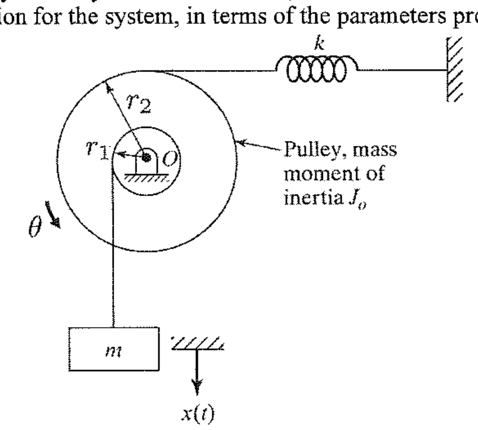
ME 440
Intermediate Vibrations

Homework #4 (2 problems)
due Friday, October 13, 2017

Problem 1

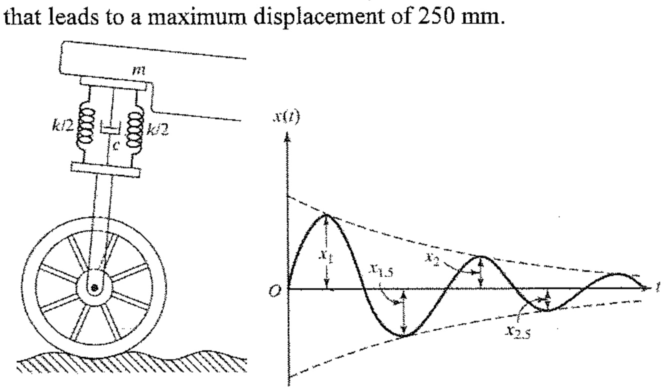
The pulley is in fixed axis rotation about Point O. Using energy concepts and θ as the generalized coordinate, determine

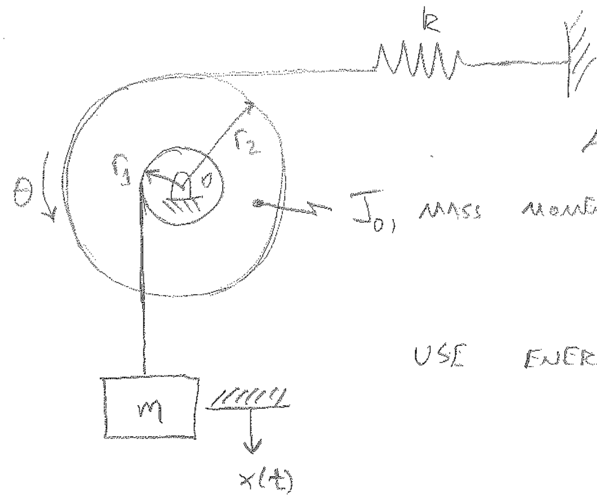
a) the natural frequency of the system shown below, and
b) the equation of motion for the system, in terms of the parameters provided.



Problem 2

An underdamped shock absorber is to be designed for motorcycle of mass 200 kg. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as illustrated below. Determine the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 seconds and the amplitude x_1 is to be reduced to ¼ in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.





ASSUME SMALL θ

J_{0}, MASS MOMENT OF INERTIA

USE ENERGY $(T_{MAX} = U_{MAX})$

$x(t) = r_1[\theta(t)] \qquad \dot{x} = r_1\dot{\theta} = v$

$$T_{MAX} = \left[\frac{1}{2} m v^2 + \frac{1}{2} I \omega^2\right]_{MAX}$$

$$= \frac{1}{2} m (r_1 \dot{\theta}_{MAX})^2 + \frac{1}{2} J_0 (\dot{\theta}_{MAX})^2 \qquad \text{EQN 1}$$

$$U_{MAX} = \left[\frac{1}{2} k \Delta^2\right]_{MAX} \qquad \Delta = \theta r_2$$

$$= \frac{1}{2} k (\theta_{MAX} r_2)^2 \qquad \text{EQN 2}$$

ASSUME

$\theta = \theta_{MAX} \sin \omega_n t$ THEN

$\dot{\theta} = \theta_{MAX} \omega_n \cos \omega_n t$ AND $\dot{\theta}_{MAX} = \theta_{MAX} \omega_n$ EQN 3

EQUATE

$$\frac{1}{2} m r_1^2 \dot{\theta}_{MAX}^2 + \frac{1}{2} J_0 \dot{\theta}_{MAX}^2 = \frac{1}{2} k r_2^2 \theta_{MAX}^2 \qquad \text{PLUG IN 3}$$

$$\frac{1}{2} m r_1^2 (\theta_{MAX} \omega_n)^2 + \frac{1}{2} J_0 (\theta_{MAX} \omega_n)^2 = \frac{1}{2} k r_2^2 \theta_{MAX}^2 \qquad \text{SOLVE FOR } \omega_n$$

$$\omega_n = \sqrt{\frac{k r_2^2}{J_0 + m r_1^2}}$$

PART b

$$\frac{d}{dt}(T+U) = 0$$

$$T + U = \frac{1}{2} m r_1^2 \dot{\theta}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} k r_2^2 \theta^2$$

$$\frac{d}{dt}(T+U) = 2\left(\frac{1}{2}\right) m r_1^2 \dot{\theta} \underbrace{\left[\frac{d\dot{\theta}}{dt}\right]}_{\ddot{\theta}} + 2\left(\frac{1}{2}\right) J_0 \dot{\theta} \underbrace{\left[\frac{d\dot{\theta}}{dt}\right]}_{\ddot{\theta}} + 2\left(\frac{1}{2}\right) k r_2^2 \theta \underbrace{\left[\frac{d\theta}{dt}\right]}_{\dot{\theta}} = 0$$

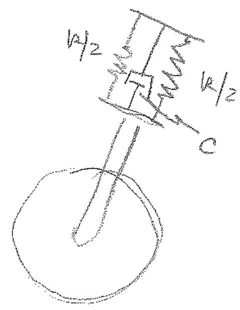
$$\left[2,\ 1/2,\ \dot{\theta} \text{ ALL CANCEL}\right]$$

$$m r_1^2 \ddot{\theta} + J_0 \ddot{\theta} + k r_2^2 \theta = 0$$

E.O.M.

$$\ddot{\theta}\left[m r_1^2 + J_0\right] + \theta\left[k r_2^2\right] = 0$$

UNDERDAMPED $m = 200kg$



- WANT $\tau_d = 2s$ AND AMPLITUDE TO BE REDUCED TO 1/4 IN ONE HALF CYCLE $\left(x_{1.5} = \frac{x_1}{4}\right)$
- FIND k AND c. AND $v_0 = \dot{x}_0$ SO MAX DISPLACEMENT IS 250mm

WE WANT $x_{1.5} = \frac{x_1}{4}$ AND $\tau_d = 2sec$

IF $x_{1.5} = \frac{x_1}{4}$ THEN $x_2 = x_{1.5}/4 = x_1/16$

CONSEQUENTLY

$$\ln\left(\frac{x_1}{x_2}\right) = \xi \omega_n \tau_d = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\ln(16) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = 2.7726$$

$$\xi^2\pi^2 = 1.922(1-\xi^2)$$

$$\xi^2 = 0.16298$$

$$\xi = 0.4037 \quad \text{OR} \quad \xi = \sqrt{\frac{\{\ln(x_1/x_2)\}^2}{\{4\pi^2 + \ln(x_1/x_2)\}^2}}$$

- NOW FIND ω_n

$$\omega_n = \frac{2\pi}{\tau_d\sqrt{1-\xi^2}} = \frac{2\pi}{2\sqrt{1-0.4037^2}} = 3.434 \frac{RAD}{SEC}$$

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_n^2 m = \left(3.434 \frac{RAD}{SEC}\right)^2 200kg = 2358 \frac{kg}{SEC^2}$$

$$\boxed{k = 2358 \frac{N}{m}}$$

$C = 2\xi\omega_n m = 2(0.4037)(3.434 \frac{\text{RAD}}{\text{SEC}})(200\,\text{kg})$ OR $C = 2\xi\sqrt{km}$

$C = 554.5 \frac{\text{kg}}{\text{SEC}} \Rightarrow \boxed{C = 554.5 \frac{N\cdot s}{m}}$

UNDERDAMPED FREE VIBRATION RESPONSE (SLIDE 120)

$$x(t) = e^{-\xi\omega_n t}\left[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right]$$

$$B_1 = x_0 \qquad B_2 = \frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d}$$

$$x_0 = x(0) = 0 \qquad \Rightarrow \qquad B_1 = 0$$

$$\dot{x}_0 = \dot{x}(0) = v_0 \qquad B_2 = \frac{\dot{x}_0}{\omega_d} = \frac{v_0}{\omega_d}$$

$$x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

TO FIND WHEN $x(t)$ IS A MAX, TAKE DERIVATIVE, THEN $=0$.

$$\frac{dx(t)}{dt} = \frac{\dot{x}_0}{\omega_d} e^{-\xi\omega_n t}\left[-\xi\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right] = 0$$

OR $-\xi\omega_n\sin\omega_d t + \omega_d\cos\omega_d t = 0$

$$\tan\omega_d t = \frac{\omega_d}{\xi\omega_n} = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n} \quad *$$

$$\tan\omega_d t = \frac{\sqrt{1-\xi^2}}{\xi} \Rightarrow \tan 3.434\sqrt{1-0.4037^2}^{\,*}\,t = \frac{\sqrt{1-0.4037^2}}{0.4037}$$

$\left[\text{NOTE } \omega_d = \omega_n\sqrt{1-\xi^2}\right]$

$t = 0.3677$ SEC, PLUG BACK INTO $x(t)$

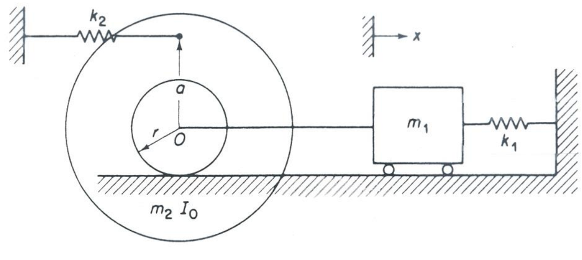
$$0.25\,m = \frac{\dot{x}_0 e^{-0.4037(3.434)(0.3677)}}{(3.434)\sqrt{1-0.4037^2}} \sin 3.434\sqrt{1-0.4037^2}\;0.3677$$

$\dot{x}_0 = 1.429$ m/s $= 1429$ mm/SEC

2.5 HW5

Problem 1

The stepped cylinder is connected to a spring of stiffness k_2 and an inextensible cable. The other end of the inextensible cable is attached to mass m_1. The stepped cylinder rolls without slip on the fixed surface. The mass m_1 rolls on 2 massless cylinders. Assume the system will be limited to small displacements. The total mass of the stepped cylinder is m_2 and it's mass moment of inertia about point O is I_0.



a) In preparation for using Newton's Second Law, sketch the free-body diagram(s) **and** inertial diagram for this system.
b) Using Newton's Laws exclusively, determine the differential equation of motion for small angular oscillations of the mass m_1 (in terms of the generalized coordinate x).

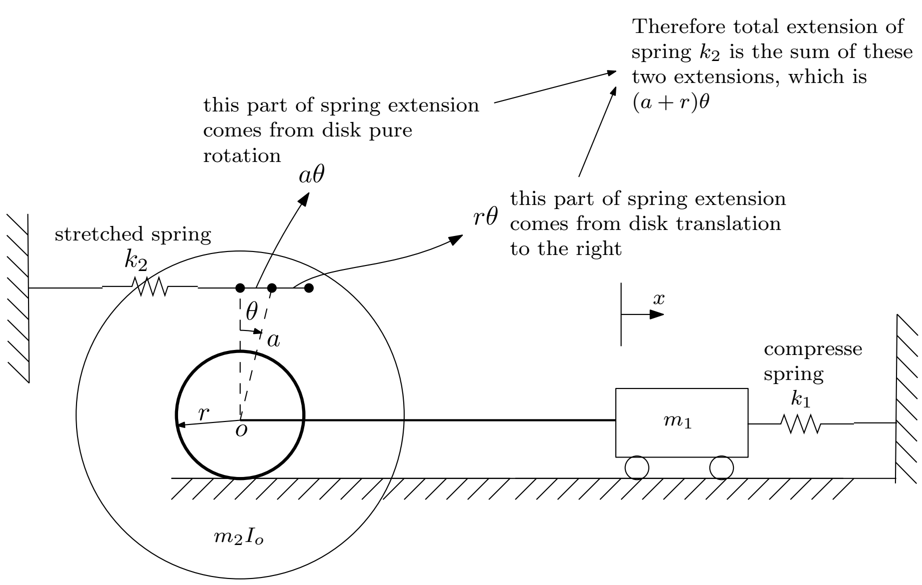
Problem 2

Repeat Problem 1 but use $T_{max} = U_{max}$ to find the natural frequency of the system.

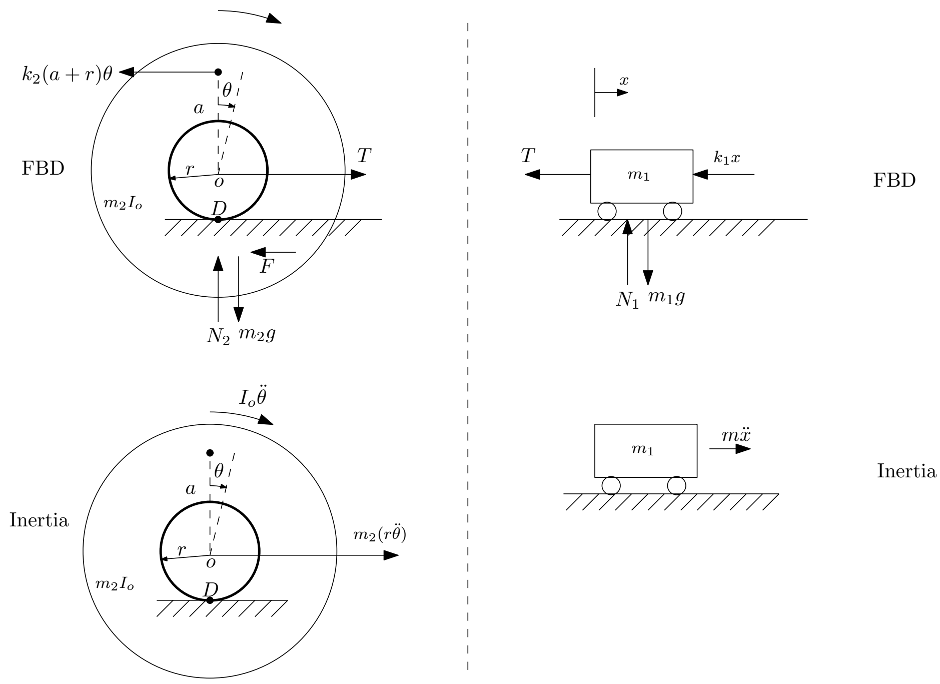
2.5.1 Problem 1

2.5.1.1 Part (A)

We start by assuming motion to the right, such that the small disk m_2 rotates clockwise as shown below. So the k_2 spring is stretched by amount $a\theta$ which come due to pure rotation, and it also stretch by $r\theta$ due to disk translation to the right at same time, therefore the spring k_1 will stretch by amount $(a + r)\theta$ and the k_1 spring will be compressed by amount x.



Based on the above, the following is the free body diagram for m_2 and m_1 and the corresponding kinematic diagrams. This assumes small angle θ and that springs remain straight.



2.5.1.2 Part (B)

Since cable is inextensible, then the constraint is that $x = r\theta$. Starting from the FBD for m_1

$$\begin{aligned}\sum F_x &= m_1\ddot{x}\\ -T - k_1 x &= m_1\ddot{x}\\ m_1\ddot{x} + k_1 x &= -T \end{aligned} \tag{1}$$

We do not need to resolve forces in vertical direction, since no motion is in that direction. To find T, which is the tension in cable, we go back to m_2 and find T.

We can do this part in two ways, either by taking moments around the instantaneous center of zero velocity which is point D at bottom of the small cylinder shown in the diagram, or we can take moments around the C.M. of the disk and then use another equation to solve for the friction F. We will show both methods, and that they give the same result.

Method one, using instantaneous center of zero velocity

Take moments around point D as shown in figure in order to not have to account for the friction force F and the N_2 force on m_2 and using positive as anti-clockwise gives

$$\begin{aligned}\sum M_D &= -I_D\ddot{\theta}\\ k_2(a+r)\theta(a+r) - Tr &= -\overbrace{\left(I_o + m_2 r^2\right)}^{\text{parallel axes}}\ddot{\theta}\\ T &= \frac{k_2(a+r)^2\theta + \left(I_o + m_2 r^2\right)\ddot{\theta}}{r}\end{aligned}$$

But due to constraint, then $\theta = \frac{x}{r}, \ddot{\theta} = \frac{\ddot{x}}{r}$. Hence the above can be written as

$$\begin{aligned}T &= \frac{k_2\frac{x}{r}(a+r)^2 + \left(I_o + m_2 r^2\right)\frac{\ddot{x}}{r}}{r}\\ &= \frac{xk_2(a+r)^2}{r^2} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2}\end{aligned} \tag{2}$$

Substituting (2) into (1) gives

$$m_1\ddot{x} + k_1 x = -\left(\frac{xk_2(a+r)^2}{r^2} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2}\right)$$

$$\begin{aligned}
m_1\ddot{x} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2} + k_1 x + \frac{xk_2(a+r)^2}{r^2} &= 0\\
\ddot{x}\left(m_1 + \frac{\left(I_o + m_2 r^2\right)}{r^2}\right) + x\left(k_1 + \frac{k_2(a+r)^2}{r^2}\right) &= 0\\
\ddot{x}\left(\frac{m_1 r^2 + \left(I_o + m_2 r^2\right)}{r^2}\right) + x\left(\frac{k_1 r^2 + k_2(a+r)^2}{r^2}\right) &= 0
\end{aligned}$$

Hence

$$\ddot{x}\left(m_1 r^2 + \left(I_o + m_2 r^2\right)\right) + x\left(k_1 r^2 + k_2(a+r)^2\right) = 0$$

In standard form

$$\ddot{x} + x\frac{k_1 r^2 + k_2(a+r)^2}{r^2(m_1 + m_2) + I_o} = 0 \tag{3}$$

Or

$$\ddot{x} + \omega_n^2 x = 0$$

Where

$$\omega_n^2 = \frac{r^2 k_1 + k_2(a+r)^2}{r^2(m_1 + m_2) + I_o}$$

Method two, moments around center of mass

Using this method. We start by taking moments around the center of mass of the disk m_2 and using positive as anti-clockwise gives

$$\begin{aligned}
\sum M_o &= -I_o\ddot{\theta}\\
(k_2(a+r)\theta)a - Fr &= -I_o\ddot{\theta}\\
F &= \frac{1}{r}\left(I_o\ddot{\theta} + (k_2(a+r)\theta)a\right)
\end{aligned} \tag{4}$$

Now resolving forces in the x direction for m_2, gives (with positive to the right)

$$\begin{aligned}
\sum F_x &= m_2 r\ddot{\theta}\\
T - k_2(a+r)\theta - F &= m_2 r\ddot{\theta}
\end{aligned} \tag{5}$$

Plugging (4) into (5) gives T

$$T - k_2(a+r)\theta - \frac{1}{r}\left(I_o\ddot{\theta} + (k_2(a+r)\theta)a\right) = m_2 r\ddot{\theta}$$

Solving for T gives

$$T = m_2 r\ddot{\theta} + \frac{1}{r}\left(I_o\ddot{\theta} + (k_2(a+r)\theta)a\right) + k_2(a+r)\theta$$

We now use the constraint that $x = r\theta$ to write everything in x. Hence $\theta = \frac{x}{r}, \ddot{\theta} = \frac{\ddot{x}}{r}$ and the above now becomes

$$\begin{aligned}
T &= m_2 r\frac{\ddot{x}}{r} + \frac{1}{r}\left(I_o\frac{\ddot{x}}{r} + \left(k_2(a+r)\frac{x}{r}\right)a\right) + k_2(a+r)\frac{x}{r}\\
&= m_2\ddot{x} + \frac{1}{r^2}\left(I_o\ddot{x} + (k_2(a+r)x)a\right) + k_2(a+r)\frac{x}{r}
\end{aligned}$$

Now that we found T, we go back to the equation of motion for m_1 in (1) and substitute the above into it, the result becomes

$$\begin{aligned}
m_1\ddot{x} + k_1 x &= -T\\
&= -\left(m_2\ddot{x} + \frac{1}{r^2}\left(I_o\ddot{x} + (k_2(a+r)x)a\right) + k_2(a+r)\frac{x}{r}\right)
\end{aligned}$$

Collecting terms

$$
\begin{aligned}
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+k_1x+\frac{1}{r^2}\left(\left(k_2\left(a+r\right)x\right)a\right)+k_2\left(a+r\right)\frac{x}{r}&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{1}{r^2}\left(k_2\left(a+r\right)a\right)+k_2\left(a+r\right)\frac{1}{r}\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[\left(a+r\right)a+r\left(a+r\right)\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[a^2+ra+ar+r^2\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[a^2+2ar+r^2\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left(a+r\right)^2\right)&=0
\end{aligned}
$$

Or

$$
\begin{aligned}
\ddot{x}\left(r^2\left(m_1+m_2\right)+I_o\right)+x\left(r^2k_1+k_2\left(a+r\right)^2\right)&=0\\
\ddot{x}+x\frac{r^2k_1+k_2\left(a+r\right)^2}{r^2\left(m_1+m_2\right)+I_o}&=0
\end{aligned}
$$

Which is the same equation of motion found in the first method.

2.5.2 Problem 2

In Rayleigh energy method, we ignore any friction, and assume motion is simple harmonic motion (which is valid, since there is no damping).

The Kinetic energy T of the system is (since disk rolls with no slip)

$$
T=\overbrace{\frac{1}{2}I_o\dot{\theta}^2+\frac{1}{2}m_2v_{cg}^2}^{\text{disk}}+\overbrace{\frac{1}{2}m_1\dot{x}^2}^{\text{cart}}
$$

But $v_{cg}=r\dot{\theta}$, hence the above becomes

$$
T=\frac{1}{2}I_o\dot{\theta}^2+\frac{1}{2}m_2\left(r\dot{\theta}\right)^2+\frac{1}{2}m_1\dot{x}^2
$$

But due to constraint, then $\theta=\frac{x}{r}$, then $\dot{\theta}=\frac{\dot{x}}{r}$ and the above becomes

$$
\begin{aligned}
T&=\frac{1}{2}I_o\left(\frac{\dot{x}}{r}\right)^2+\frac{1}{2}m_2\left(r\frac{\dot{x}}{r}\right)^2+\frac{1}{2}m_1\dot{x}^2\\
&=\frac{1}{2}I_o\frac{\dot{x}^2}{r^2}+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}m_1\dot{x}^2\\
&=\frac{1}{2}\dot{x}^2\left(\frac{I_o}{r^2}+m_2+m_1\right) \qquad (1)
\end{aligned}
$$

The potential energy is

$$
\begin{aligned}
U&=\frac{1}{2}k_2\left(\left(a+r\right)\theta\right)^2+\frac{1}{2}k_1x^2\\
&=\frac{1}{2}k_2\left(\left(a+r\right)\frac{x}{r}\right)^2+\frac{1}{2}k_1x^2\\
&=\frac{1}{2}k_2\left(a+r\right)^2\frac{x^2}{r^2}+\frac{1}{2}k_1x^2 \qquad (2)
\end{aligned}
$$

To find $T_{\max}$ and $U_{\max}$, we now assume m_1 undergoes simple harmonic motion given by $x\left(t\right)=X_{\max}\sin\left(\omega_nt\right)$. Hence $\dot{x}=X_{\max}\omega_n\cos\omega_nt$. Therefore

$$
\begin{aligned}
\dot{x}_{\max}&=X_{\max}\omega_n\\
x_{\max}&=X_{\max}
\end{aligned}
$$

Therefore using these into (1) and (2) gives

$$T_{\max} = \frac{1}{2}\left(\dot{x}_{\max}\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right)$$

$$U_{\max} = \frac{1}{2}k_2\left(a+r\right)^2\frac{x_{\max}^2}{r^2} + \frac{1}{2}k_1x_{\max}^2$$

Or

$$T_{\max} = \frac{1}{2}\left(X_{\max}\omega_n\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right)$$

$$U_{\max} = \frac{1}{2}X_{\max}^2\left(\frac{k_2\left(a+r\right)^2}{r^2} + k_1\right)$$

Hence

$$T_{\max} = U_{\max}$$

$$\frac{1}{2}\left(X_{\max}\omega_n\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right) = \frac{1}{2}X_{\max}^2\left(\frac{k_2\left(a+r\right)^2}{r^2} + k_1\right)$$

$$\omega_n^2\left(\frac{I_o}{r^2} + m_2 + m_1\right) = \frac{k_2\left(a+r\right)^2 + r^2k_1}{r^2}$$

Solving for ω_n^2

$$\omega_n^2 = \frac{k_2\left(a+r\right)^2 + r^2k_1}{I_o + r^2\left(m_2 + m_1\right)}$$

Therefore the equation of motion for m_2 is

$$\ddot{x} + \omega_n^2 x = 0$$

$$\ddot{x} + \frac{k_2\left(a+r\right)^2 + r^2k_1}{I_o + r^2\left(m_2 + m_1\right)}x = 0$$

Comparing this to the solution found in first problem, we see they are the same. The Rayleigh energy method was much simpler in this case. But we have to ignore any friction, and assume motion is harmonic, which is reasonable, since this is single degree of freedom system.

2.6 HW6

2.6.1 Problem 1

Problem 1

Download the ANSYS input file "*1DOF_spring_mass-problem_18p1.txt*" from Canvas and step through the ANSYS tutorial "*Intro to ANSYS modal analysis*" that is also posted to Canvas. Using the parameters defined in the text file, analytically determine the natural frequency of the 1 degree of freedom system. Show your work for this calculation and then compare the analytical and finite element results. And then answer the following questions:

a) Does ANSYS provide the frequency (*f*) or the circular frequency (ω)?
b) Can we verify the amplitude of displacement shown on Slide 10 of the "Intro to ANSYS modal analysis" slides? Why or why not?

The input file to ANSYS is given to us in plain text file as the following

```
/filnam, 1DOF_spring_mass
/title, 1 Degree of freedom spring mass example
/prep7
!element type
et,1,mass21                  !element type no.1 is mass21
et,2,combin14                  !element type no.2 is combination 14 (this is a spring element)
! model parameters
mass = 10                  ! mass of mass element
k = 10                           ! spring stiffness
initial_l = 2                  ! initial spring length (equilibrium length)
n_modes = 1                  ! number of modes wanted
!real constants
r,1,mass                  ! real constant set 1 is for the point mass
r,2,k,,,,,initial_l          ! real constant set 2 is for the spring
!create nodes
n,1,0,0,0                  ! Node 1 is at x=0, y=0, z=0
n,2,initial_l,0,0          ! Node 2 is at x=initial_l, y=0, z=0
!create elements
type,2                           ! specify element type of subsequently defined elements
real,2                           ! specify real constant set of subsequently defined elements
e,1,2                          ! define element to start at node 1 and end at node 2
type,1                           ! specify element type of subsequently defined elements
real,1                           ! specify real constant set of subsequently defined elements
e,2                          ! define element to be created at node 2
!displacement boundary conditions
nsel,s,loc,x,0                          !select node at x = 0
d,all,ux,0                          !displacement of selected node in x-dir is 0
d,all,uy,0                          !displacement of selected node in y-dir is 0
d,all,uz,0                          !displacement of selected node in z-dir is 0
nsel,s,loc,x,initial_l                  !select node at x = initial_l
d,all,uy,0                          !displacement of selected node in y-dir is 0
d,all,uz,0                          !displacement of selected node in z-dir is 0
allsel
finish
/solu                                  !select static load solution
antype,modal
modopt,lanb,n_modes
solve
finish
/post1
```

2.6.1.1 Part (1)

For a mass-spring system the equation of motion is

$$\ddot{x} + \omega_n^2 x = 0$$

Where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{10}} = 1$ rad/sec. Since $\omega_n = 2\pi f_n$, hence $f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} = 0.1592$ Hz. Therefore the frequency given by ANSYS is in Hz and not the circular frequency rad/sec.

2.6.1.2 Part (2)

Unable to verify this result. At first I thought ANSYS uses gravity and the spring is vertically connected, therefore the static displacement would be

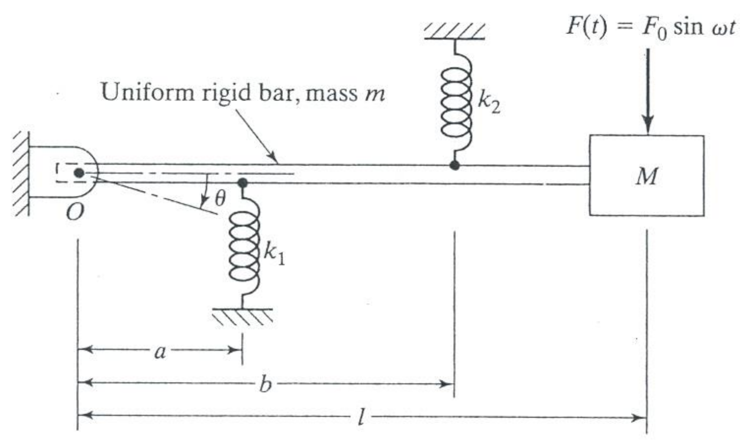
$$x_{st} = \frac{W}{k}$$

Where W is the weight attached to end of spring. But this gives $x_{st} = \frac{mg}{k} = \frac{10g}{10} = g$. And depending on units used (ANSYS do not use units and assumes that the input is using correct units), then value shown which is 0.316228 should be numerical value of g. But this would not be valid number using any units. Unable to find out how ANSYS came up with this value.

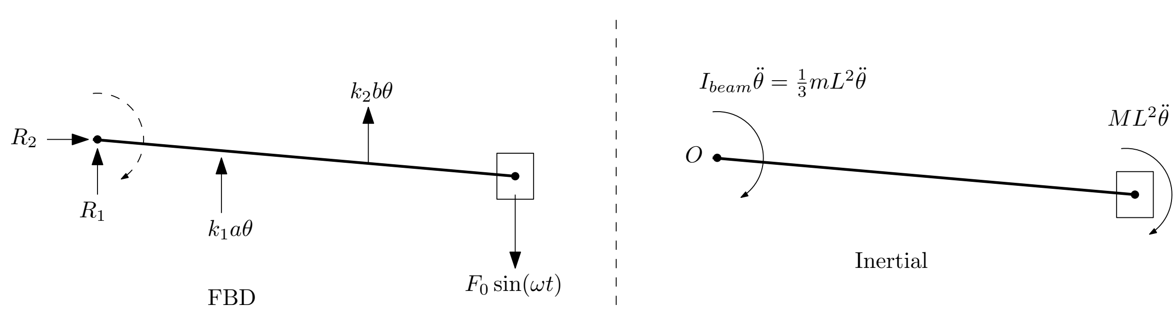
2.6.2 Problem 2

Problem 2

Derive the equation of motion and find the steady-state response $\{\theta(t)\}$ of the system shown below for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_o = 500$ N and $\omega = 1000$ rpm. Give the steady-state response in the simplest form possible.



The free body diagram and the inertial diagram are given below. It is assumed that motion is measured from equilibrium position with the mass already in attached to springs. Hence the weight of the beam do not show up in the FBD.



Taking moments around hinge at point o and using anti-clockwise as positive gives (assuming small angle θ)

$$k_1(a\theta)a + k_2(b\theta)b - F_0\sin(\omega t)L = -\left(\frac{1}{3}mL^2 + ML^2\right)\ddot{\theta}$$

$$\left(\frac{1}{3}mL^2 + ML^2\right)\ddot{\theta} + \theta\left(k_1a^2 + k_2b^2\right) = F_0\sin(\omega t)L$$

In standard form, the above becomes

$$m_{eq}\ddot{\theta} + k_{eq}\theta = F_0\sin\omega t$$

Where

$$\begin{aligned}\omega_n^2 &= \frac{k_{eq}}{m_{eq}}\\ &= \frac{k_1 a^2 + k_2 b^2}{L^2\left(\frac{1}{3}m + M\right)}\end{aligned}$$

This model is single degree of freedom system, undamped, with forced input. Hence we know its solution is given by

$$\theta(t) = \theta_h(t) + \theta_p(t)$$

Where $\theta_p(t)$ is particular solution and $\theta_h(t)$ is homogenous solution. We know that

$$\theta_h(t) = c_1 \cos \omega_n t + c_2 \sin \omega_n t$$

And assuming $\theta_p(t) = X \sin \omega t$. Now we need to check if $\omega \neq \omega_n$ so to decide on which solution to pick. Using the numerical values given

$$\begin{aligned}k_{eq} &= k_1 a^2 + k_2 b^2\\ &= (5000)(0.25)^2 + (5000)(0.5)^2\\ &= 1562.5 \text{ N/m}\end{aligned}$$

And

$$\begin{aligned}M_{eq} &= L^2\left(\frac{1}{3}m + M\right)\\ &= (1)^2\left(\left(\frac{1}{3}\right)(10) + 50\right)\\ &= 53.333 \text{ kg}\end{aligned}$$

Hence

$$\omega_n = \sqrt{\frac{k_{eq}}{M_{eq}}} = \sqrt{\frac{1562.5}{53.333}} = 5.413 \,\text{rad/sec}$$

But the forcing frequency is given as

$$\omega = 1000\left(\frac{2\pi}{rev}\right)\left(\frac{\min}{60}\right) = 1000\left(\frac{2\pi}{60}\right) = 104.72 \text{ rad/sec}$$

Hence $\underline{\omega \neq \omega_n}$. We also see $\omega > \omega_n$ which means $r > 1$ where $r = \frac{\omega}{\omega_n}$, so we also expect that particular solution displacement maximum displacement to be negative. Now we use the standard solution, which is

$$\theta_p(t) = X \sin \omega t$$

Where

$$\begin{aligned}X &= \frac{F_0}{k_{eq} - m_{eq}\omega^2}\\ &= \frac{F_0}{m_{eq}} \frac{1}{\frac{k_{eq}}{m_{eq}} - \omega^2}\\ &= \frac{F_0}{m_{eq}} \frac{1}{\omega_n^2 - \omega^2}\\ &= \frac{F_0}{m_{eq}\omega_n^2} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\\ &= \frac{F_0}{k_{eq}} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\end{aligned}$$

Calling $\frac{\omega}{\omega_n} = r$, which is the standard notation and since $\frac{F_0}{k_{eq}} = x_{st}$ the static deflection, then the above becomes

$$X = \frac{x_{st}}{1 - r^2}$$

We notice again, since $r > 1$ in this problem, then X is negative. It is <u>out of phase</u> with the forcing function. The particular solution can now be written as

$$\begin{aligned}\theta_p(t) &= X\sin\omega t\\ &= \frac{x_{st}}{1-r^2}\sin\omega t\end{aligned}$$

And the total solution is

$$\theta(t) = \overbrace{c_1\cos\omega_n t + c_2\sin\omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1-r^2}\sin\omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $\theta(0)=\theta_0, \dot{\theta}(0)=\dot{\theta}_0$, then (1) at $t=0$ becomes

$$\theta_0 = c_1$$

Hence solution becomes

$$\theta(t) = \theta_0\cos\omega_n t + c_2\sin\omega_n t + \frac{x_{st}}{1-r^2}\sin\omega t$$

Taking derivative

$$\theta'(t) = \omega_n\theta_0\sin\omega_n t + \omega_n c_2\cos\omega_n t + \omega\frac{x_{st}}{1-r^2}\cos\omega t$$

At $t=0$ the above becomes

$$\dot{\theta}_0 = \omega_n c_2 + \omega\frac{x_{st}}{1-r^2}$$

Hence

$$\begin{aligned}c_2 &= \frac{\dot{\theta}_0}{\omega_n} - \frac{\omega}{\omega_n}\frac{x_{st}}{1-r^2}\\ &= \frac{\dot{\theta}_0}{\omega_n} - \frac{r}{1-r^2}x_{st}\end{aligned}$$

Therefore the solution now becomes (again, this is for $\omega\neq\omega_n$)

$$\theta(t) = \overbrace{\theta_0\cos\omega_n t + \left(\frac{\dot{\theta}_0}{\omega_n} - \frac{r}{1-r^2}x_{st}\right)\sin\omega_n t}^{\text{homogeneous}} + \overbrace{\left(\frac{x_{st}}{1-r^2}\right)\sin\omega t}^{\text{particular}} \tag{2}$$

The problem now asks for steady state solution. It is not clear to me what is this meant to be, since there is no damping in the system, and hence the full solution remain for all time. Therefore, will show the full solution (using zero initial conditions) and will also show the particular solution.

This is a plot of the full solution, assuming that all initial conditions are zero. Therefore, this is a plot of this solution

$$\theta(t) = -\frac{F_0}{k_{eq}}\frac{r}{1-r^2}\sin\omega_n t + \left(\frac{F_0}{k_{eq}}\frac{1}{1-r^2}\right)\sin\omega t$$

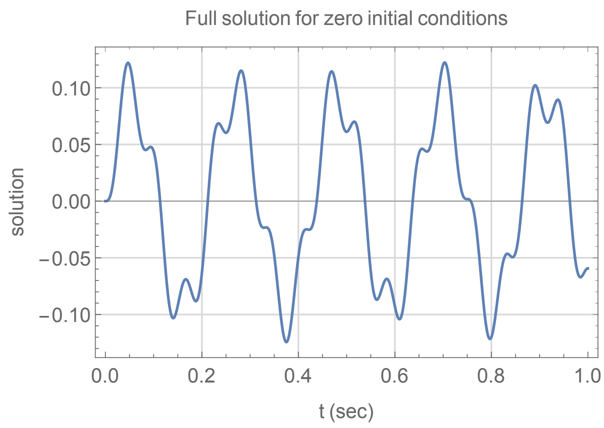
Obtained from (2) by setting $\theta_0=0, \dot{\theta}_0=0$

$$\begin{aligned}\theta(t) &= -\frac{500}{1562.5}\left(\frac{3.5744}{1-(3.5744)^2}\right)\sin(5.413t) + \left(\frac{500}{1562.5}\left(\frac{1}{1-(3.5744)^2}\right)\right)\sin(104.72t)\\ &= 0.09713\sin(5.413t) - 0.0272\sin(104.72t)\end{aligned}$$

Here is a plot of the full solution for the first 1 second

```
In[15]:= x[t_] := 0.09713 Sin[29.297 t] - 0.0272 Sin[104.72 t];
p = Plot[x[t], {t, 0, 1}, Frame → True,
  FrameLabel → {{"solution", None}, {"t (sec)", "Full solution for zero initial conditions"}},
  BaseStyle → 12, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[16]=



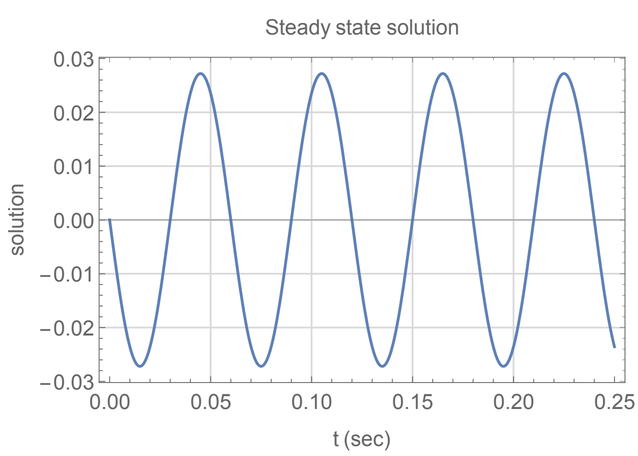
The particular solution (steady state?) is

$$
\begin{aligned}
\theta_p(t) &= \frac{x_{st}}{1-r^2}\sin\omega t \\
&= 0.0272\sin(104.72t)
\end{aligned}
$$

Here is a plot of the particular solution for the first 0.25 second

```
In[61]:= x[t_] := -0.0272 Sin[104.72 t];
p = Plot[x[t], {t, 0, .25}, Frame → True,
  FrameLabel → {{"solution", None}, {"t (sec)", "Steady state solution"}}, BaseStyle → 12,
  GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[62]=



2.6.3 Problem 3

Problem 3

A spring-mass system with $m = 10$ kg and $k = 5000$ N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

The equation of motion (assuming $\sin(\omega t)$ for the force) is[1]

$$m\ddot{x} + kx = F_0\sin(\omega t)$$

[1] The general solution changes depending on if the forcing function is sin or cos. But the particular solution is the same.

Where $k = 5000$ N/m, $m = 10$ kg, $F_0 = 250$ N. We know the solution to the above is given by (but we here have to assume that $\omega \neq \omega_n$)

$$x(t) = \overbrace{x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1 - r^2} x_{st} \right) \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\left(\frac{x_{st}}{1 - r^2} \right) \sin \omega t}^{\text{particular}}$$

Looking now at only the steady state solution (in this case, it is the particular solution) then we see that

$$x_{ss}(t) = \left(\frac{x_{st}}{1 - r^2} \right) \sin \omega t$$

Hence maximum is

$$x_{\max}(t) = \frac{x_{st}}{1 - r^2}$$

we are told that $x_{\max} = 0.1$ meter, and . But $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k}$. Therefore the above becomes

$$x_{\max} = \frac{F_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$$

In the above equation everything is known except for ω. Solving for ω gives

$$\begin{aligned} 1 - \left(\frac{\omega}{\omega_n} \right)^2 &= \frac{F_0}{k x_{\max}} \\ \left(\frac{\omega}{\omega_n} \right)^2 &= 1 - \frac{F_0}{k x_{\max}} \\ \omega^2 &= \left(1 - \frac{F_0}{k x_{\max}} \right) \omega_n^2 \end{aligned}$$

But $\omega_n = \sqrt{\frac{k}{m}}$, hence

$$\omega = \sqrt{\frac{k}{m}} \sqrt{\left(1 - \frac{F_0}{k x_{\max}} \right)}$$

Substituting numerical values

$$\begin{aligned} \omega &= \sqrt{\frac{5000}{10}} \sqrt{\left(1 - \frac{250}{(5000)(0.1)} \right)} \\ &= 22.361 \sqrt{0.5} \\ &= 15.812 \text{ rad/sec} \end{aligned}$$

ODE	solution
$m\ddot{x} + kx = F_0 \cos \omega t$	$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{x_{st}}{1-r^2} \cos \omega t$
$m\ddot{x} + kx = F_0 \sin \omega t$	$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st} \right) \sin \omega_n t + \frac{x_{st}}{1-r^2} \sin \omega t$

2.7 HW7

2.7.1 Problem 1

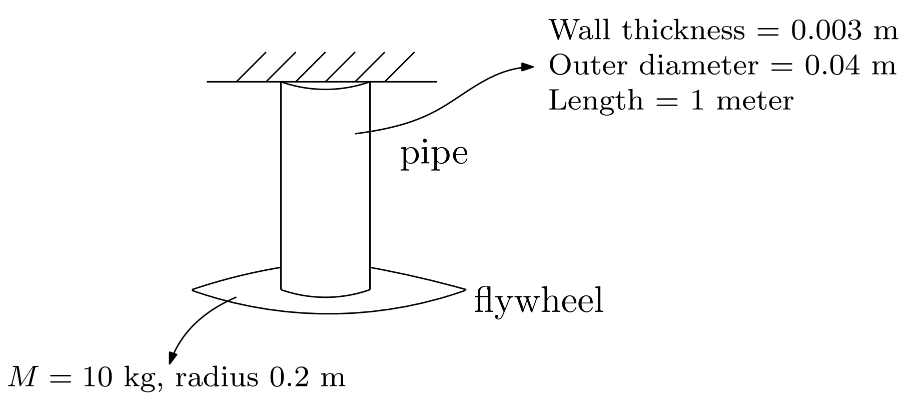
Problem 1

Download the ANSYS input file "*MODAL_pipe_flywheel.txt*" from Canvas, run this input file in ANSYS and go through the file line by line to figure out what the system parameters are for this modal analysis. (Hint: When viewing the mode shapes within ANSYS, try plotting all 3 displacements and all 3 rotations (1 at a time) available under the "Nodal Solu" / "DOF Solution" option; this should be helpful in determining the type of displacement associated with each specific frequency.

A) List the 4 frequencies from ANSYS and label each as longitudinal, transverse, or torsional.

B) Using the parameters defined in the text file, analytically determine 3 of the 4 natural frequencies of this system. Show ALL your work for these calculations and then compare the analytical and finite element frequencies in a table with % errors.

The following is diagram of the model of the problem to solve



The ANSYS APDL (input file) listing was provided to us to use and is given in the text file below for reference

```
/filnam, pipe_flywheel_modal
/title, Flywheel on torsional spring example
/prep7
!element type
et,1,mass21,,0,0  !element type no.1 is mass21 (",,,0" signifies
!that this is a 3-D mass with rotary inertia)
et,2,pipe288          !element type no.2 is pipe288 (this is a pipe element)
mp,ex,1,200e9          ! elastic modulus for steel is 200 GPa
mp,gxy,1,77.2e9          ! shear modulus for steel is 77.2 GPa
mp,prxy,1,0.295          ! poisson's ratio for steel is 0.295
! model parameters
mass = 10                         ! mass of flywheel (kg)
rad_f = 0.2                         ! outer radius of flywheel
izz = 0.5*mass*rad_f*rad_f        ! mass moment of inertia
outer_d = 0.04                        ! outer diameter of pipe (m)
wall_t = 0.003                        ! wall thickness of pipe (m)
pipe_l = 1                        ! pipe length (m)
n_modes = 10                        ! number of modes wanted
!real constants
r,1,mass,mass,mass,0.5*izz,izz,0.5*izz        ! real constant set 1 is for the mass21 element
sectype,1,pipe                ! section type 1 is "pipe"
secdata,outer_d,wall_t        ! section data for pipe is outer diameter and wall thickness
!create nodes
k,1,0,0,0        ! keypoint 1 is at x=0, y=0, z=0, this will be the fixed end of the pipe
k,2,0,-pipe_l,0        ! keypoint 2 is at x=0, y=-pipe_l, z=0, this will be the free
! end of the pipe with the flywheel
!create elements
```

```
type,2                 ! specify element type of subsequently defined elements
secnum,1         ! specify section type number of subsequently defined elements
l,1,2                 ! creates a line from keypoint 1 to keypoint 2
lesize,1,,,10         ! specifies that line 1 will consist of 10 elements when meshed
lmesh,1                 ! take line 1 and mesh it, resulting in elements representing the pipe
type,1                 ! specify element type of subsequently defined elements
real,1                 ! specify real constant set of subsequently defined elements
e,2                 ! create element to be created at node 2
nsel,all         ! selects all nodes
d,all,uz,0         ! sets the z displacements on selected nodes to be 0, thereby
! limiting our modal analysis to modes in the xy plane
d,all,rotx         ! sets the rotx displacments on selected nodes to be 0
!displacement boundary conditions
nsel,s,loc,y,0                 ! select node at x = 0
d,all,ux,0                 ! displacement of selected node in x-dir is 0
d,all,uy,0                 ! displacement of selected node in y-dir is 0
d,all,uz,0                 ! displacement of selected node in z-dir is 0
d,all,rotx,0                 ! rotations of selected node about x axis is 0
d,all,roty,0                 ! rotations of selected node about y axis is 0
d,all,rotz,0                 ! rotations of selected node about z axis is 0
allsel
finish
/solu                         !select static load solution
antype,modal
modopt,lanb,n_modes
solve
finish
/post1
```

2.7.1.1 Part 1

The following 4 modal frequencies were generated by ANSYS after running the above APDL file.

mode	Mode number	Frequency (Hz)
transverse	1	9.4438
torsional	2	34.272
transverse	3	111.41
longitudinal	4	420.31

The modal shapes were then plotted using ANSYS. They are given below for each mode

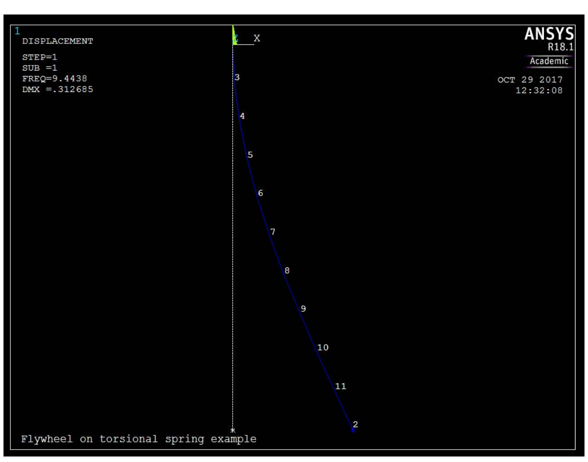


Figure 2.1: First mode: Transverseat 9.443 Hz

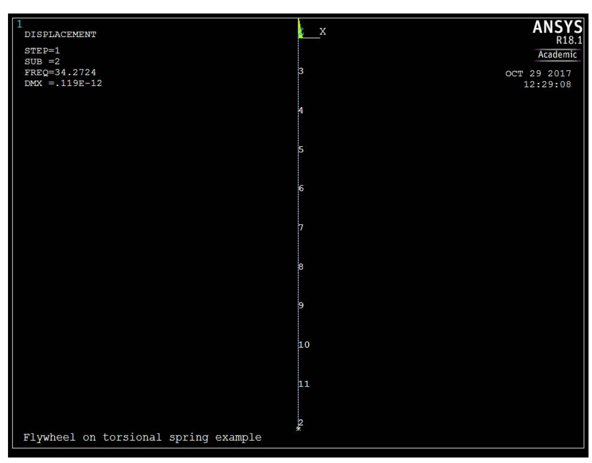


Figure 2.2: Second mode: Torsional at 34.2724 Hz

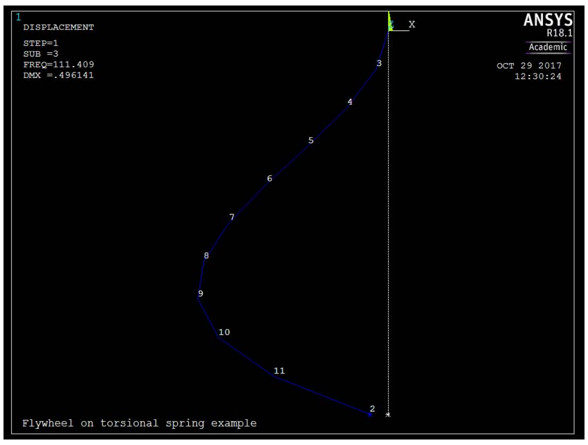


Figure 2.3: Third mode: Transverse at 111.408 Hz

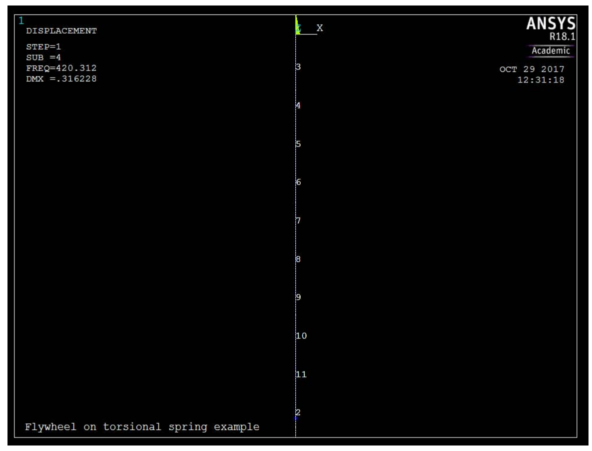


Figure 2.4: Fourth mode: longtitudinal at 420.312 Hz

The system parameters are

```
PARAMETER STATUS-           (     13 PARAMETERS DEFINED)
                  (INCLUDING       6 INTERNAL PARAMETERS)

NAME                              VALUE                        TYPE  DIMENSIONS
IZZ                              0.200000000                  SCALAR
MASS                              10.0000000                  SCALAR
N_MODES                           10.0000000                  SCALAR
OUTER_D                          4.000000000E-002             SCALAR
PIPE_L                            1.00000000                  SCALAR
RAD_F                            0.200000000                  SCALAR
WALL_T
```

Total U displacement by ANSYS for mode 1 is

```
PRINT U    NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     1
   FREQ=    9.4438      LOAD CASE=   0

  THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

    NODE       UX            UY           UZ           USUM
       1   0.0000       0.0000       0.0000       0.0000
       2  0.31268      -0.15719E-019  0.0000      0.31268
       3  0.45277E-002-0.15719E-020  0.0000      0.45277E-002
       4  0.17442E-001-0.31438E-020  0.0000      0.17442E-001
       5  0.37825E-001-0.47156E-020  0.0000      0.37825E-001
       6  0.64762E-001-0.62875E-020  0.0000      0.64762E-001
       7  0.97336E-001-0.78594E-020  0.0000      0.97336E-001
       8  0.13463      -0.94313E-020  0.0000      0.13463
       9  0.17573      -0.11003E-019  0.0000      0.17573
      10  0.21972      -0.12575E-019  0.0000      0.21972
      11  0.26567      -0.14147E-019  0.0000      0.26567

 MAXIMUM ABSOLUTE VALUES
 NODE          2             2            0             2
 VALUE   0.31268     -0.15719E-019  0.0000      0.31268
```

Total ROT displacement by ANSYS for mode 1 is

```
PRINT ROT  NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     1
   FREQ=    9.4438      LOAD CASE=   0
```

```
 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE       ROTX          ROTY          ROTZ          RSUM
      1    0.0000        0.0000        0.0000        0.0000
      2    0.0000       0.54021E-017 0.47205      0.47205
      3    0.0000       0.54021E-018 0.88444E-001 0.88444E-001
      4    0.0000       0.10804E-017 0.16772      0.16772
      5    0.0000       0.16206E-017 0.23784      0.23784
      6    0.0000       0.21609E-017 0.29879      0.29879
      7    0.0000       0.27011E-017 0.35058      0.35058
      8    0.0000       0.32413E-017 0.39320      0.39320
      9    0.0000       0.37815E-017 0.42666      0.42666
     10    0.0000       0.43217E-017 0.45095      0.45095
     11    0.0000       0.48619E-017 0.46608      0.46608

MAXIMUM ABSOLUTE VALUES
NODE          0             2             2             2
VALUE    0.0000       0.54021E-017 0.47205      0.47205
```

Total U displacement by ANSYS for mode 2 is

```
PRINT U    NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     2
  FREQ=    34.272      LOAD CASE=   0

 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE       UX            UY            UZ            USUM
      1    0.0000        0.0000        0.0000        0.0000
      2   0.11870E-012 0.57718E-018  0.0000      0.11870E-012
      3   0.17191E-014 0.57718E-019  0.0000      0.17191E-014
      4   0.66222E-014 0.11544E-018  0.0000      0.66222E-014
      5   0.14361E-013 0.17316E-018  0.0000      0.14361E-013
      6   0.24588E-013 0.23087E-018  0.0000      0.24588E-013
      7   0.36955E-013 0.28859E-018  0.0000      0.36955E-013
      8   0.51113E-013 0.34631E-018  0.0000      0.51113E-013
      9   0.66715E-013 0.40403E-018  0.0000      0.66715E-013
     10   0.83412E-013 0.46175E-018  0.0000      0.83412E-013
     11   0.10086E-012 0.51947E-018  0.0000      0.10086E-012

MAXIMUM ABSOLUTE VALUES
NODE          2             2             0             2
VALUE   0.11870E-012 0.57718E-018  0.0000      0.11870E-012
```

Total ROT displacement by ANSYS for mode 2 is

```
PRINT ROT  NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     2
  FREQ=    34.272      LOAD CASE=   0

 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE       ROTX          ROTY          ROTZ          RSUM
      1    0.0000        0.0000        0.0000        0.0000
      2    0.0000        2.2361       0.17917E-012  2.2361
      3    0.0000       0.22361       0.33581E-013 0.22361
      4    0.0000       0.44721       0.63680E-013 0.44721
      5    0.0000       0.67082       0.90299E-013 0.67082
      6    0.0000       0.89443       0.11344E-012 0.89443
      7    0.0000        1.1180       0.13309E-012  1.1180
      8    0.0000        1.3416       0.14927E-012  1.3416
      9    0.0000        1.5652       0.16197E-012  1.5652
     10    0.0000        1.7889       0.17118E-012  1.7889
     11    0.0000        2.0125       0.17691E-012  2.0125
```

```
 MAXIMUM ABSOLUTE VALUES
 NODE          0           2           2           2
 VALUE    0.0000      2.2361     0.17917E-012  2.2361
```

Total U displacement by ANSYS for mode 3 is

```
PRINT U    NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     3
   FREQ=    111.41      LOAD CASE=   0

  THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

    NODE       UX           UY           UZ           USUM
       1   0.0000       0.0000       0.0000       0.0000
       2 -0.47205E-001-0.10687E-014  0.0000      0.47205E-001
       3 -0.29904E-001-0.10687E-015  0.0000      0.29904E-001
       4 -0.10556     -0.21374E-015  0.0000      0.10556
       5 -0.20770     -0.32061E-015  0.0000      0.20770
       6 -0.31709     -0.42748E-015  0.0000      0.31709
       7 -0.41446     -0.53435E-015  0.0000      0.41446
       8 -0.48056     -0.64122E-015  0.0000      0.48056
       9 -0.49614     -0.74809E-015  0.0000      0.49614
      10 -0.44194     -0.85497E-015  0.0000      0.44194
      11 -0.29872     -0.96184E-015  0.0000      0.29872

 MAXIMUM ABSOLUTE VALUES
 NODE          9           2           0           9
 VALUE  -0.49614     -0.10687E-014  0.0000      0.49614
```

Total ROT displacement by ANSYS for mode 3 is

```
 PRINT ROT  NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     3
   FREQ=    111.41      LOAD CASE=   0

  THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

    NODE       ROTX         ROTY         ROTZ         RSUM
       1   0.0000       0.0000       0.0000       0.0000
       2   0.0000     -0.49528E-013  3.1268       3.1268
       3   0.0000     -0.49528E-014-0.55375      0.55375
       4   0.0000     -0.99056E-014-0.91496      0.91496
       5   0.0000     -0.14858E-013 -1.0836       1.0836
       6   0.0000     -0.19811E-013 -1.0598       1.0598
       7   0.0000     -0.24764E-013-0.84334      0.84334
       8   0.0000     -0.29717E-013-0.43438      0.43438
       9   0.0000     -0.34670E-013 0.16711      0.16711
      10   0.0000     -0.39622E-013 0.96115      0.96115
      11   0.0000     -0.44575E-013  1.9477       1.9477

 MAXIMUM ABSOLUTE VALUES
 NODE          0           2           2           2
 VALUE    0.0000     -0.49528E-013  3.1268       3.1268
```

Total U displacement by ANSYS for mode 4 is

```
PRINT U    NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     4
   FREQ=    420.31      LOAD CASE=   0
```

```
 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE       UX           UY           UZ           USUM
      1   0.0000       0.0000       0.0000        0.0000
      2 -0.14600E-009 0.31623       0.0000       0.31623
      3 -0.21370E-011 0.31623E-001  0.0000       0.31623E-001
      4 -0.82246E-011 0.63246E-001  0.0000       0.63246E-001
      5 -0.17820E-010 0.94868E-001  0.0000       0.94868E-001
      6 -0.30480E-010 0.12649       0.0000       0.12649
      7 -0.45762E-010 0.15811       0.0000       0.15811
      8 -0.63223E-010 0.18974       0.0000       0.18974
      9 -0.82420E-010 0.22136       0.0000       0.22136
     10 -0.10291E-009 0.25298       0.0000       0.25298
     11 -0.12425E-009 0.28460       0.0000       0.28460

 MAXIMUM ABSOLUTE VALUES
 NODE          2            2            0            2
```

And total ROT displacement by ANSYS for mode 4 is

```
PRINT ROT  NODAL SOLUTION PER NODE

  ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

  LOAD STEP=     1  SUBSTEP=     4
   FREQ=    420.31      LOAD CASE=   0

  THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

    NODE       ROTX         ROTY         ROTZ         RSUM
       1   0.0000       0.0000       0.0000       0.0000
       2   0.0000      0.15287E-011-0.21790E-009 0.21790E-009
       3   0.0000      0.15287E-012-0.41720E-010 0.41721E-010
       4   0.0000      0.30573E-012-0.79011E-010 0.79012E-010
       5   0.0000      0.45860E-012-0.11187E-009 0.11187E-009
       6   0.0000      0.61146E-012-0.14031E-009 0.14031E-009
       7   0.0000      0.76433E-012-0.16431E-009 0.16431E-009
       8   0.0000      0.91720E-012-0.18389E-009 0.18389E-009
       9   0.0000      0.10701E-011-0.19903E-009 0.19904E-009
      10   0.0000      0.12229E-011-0.20975E-009 0.20975E-009
      11   0.0000      0.13758E-011-0.21604E-009 0.21604E-009

 MAXIMUM ABSOLUTE VALUES
 NODE          0            2            2            2
 VALUE    0.0000      0.15287E-011-0.21790E-009 0.21790E-009
```

2.7.1.2 Part 2

To verify ANSYS solution, this was solved in two ways. By taking into account the mass m of the pipe and then by ignoring the mass m. Both hand solutions are given below. ANSYS do not take the mass of the pipe into account, since it was not told the density of the pipe material in the APDL input file. The first solution below is the recommend one to use to compare the ANSYS result against and it the method which gave more agreement with ANSYS result.

2.7.1.2.1 First solution. Not accounting for mass of pipe

Finding the longitudinal (axial) natural frequency.

Using $k_{eq} = \frac{AE}{L}$ where A is the cross sectional area of the pipe and L is the pipe length and using $m_{eq} = M$, then the longitudinal natural frequency is

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{AE}{LM}} \tag{1}$$

The cross sectional area of the pipe is

$$\begin{aligned}A &= \frac{\pi}{4}\left(D_o^2 - D_i^2\right)\\ &= \frac{\pi}{4}\left((0.04)^2 - (0.034)^2\right)\\ &= 3.4872\times 10^{-4}\text{ m}^2\end{aligned}$$

The length of pipe is 1 meter and $M = 10$ kg. Equation (1) becomes

$$\begin{aligned}\omega_n &= \sqrt{\frac{\left(3.4872\times 10^{-4}\right)\left(200\times 10^9\right)}{(1)(10)}}\\ &= 2640.9\text{ rad/sec}\end{aligned}$$

The cycle frequency is

$$\begin{aligned}f_n &= \frac{\omega_n}{2\pi}\\ &= \frac{2640.9}{2\pi}\\ &= 420.31\text{ Hz}\end{aligned}$$

ANSYS gives 420.31 Hz. So the error is 0.

Finding the torsional natural frequency.

Torsional stiffness k_t is

$$k_t = \frac{GJ}{L}$$

Where G is the shear modulus (given in handout), and J is the polar area moment of inertia of the cross section given by

$$\begin{aligned}J &= \frac{\pi}{32}\left(D_o^4 - D_i^4\right)\\ &= \frac{\pi}{32}\left(0.04^4 - 0.034^4\right)\\ &= 1.2013\times 10^{-7}\text{ m}^4\end{aligned}$$

Therefore

$$k_t = \frac{\left(77.2\times 10^9\right)\left(1.2013\times 10^{-7}\right)}{1} = 9274\text{ N-m per radian}$$

The equivalent mass is just the mass moment of inertia of the flywheel $\frac{1}{2}Mr_f^2$ (since the pipe assumed to have no mass). Hence the torsional frequency is

$$\begin{aligned}\omega &= \sqrt{\frac{k_t}{\frac{1}{2}Mr_f^2}}\\ &= \sqrt{\frac{9274}{\frac{1}{2}(10)(0.2)^2}}\\ &= 215.34\text{ rad/sec}\end{aligned}$$

Therefore the torsional frequency in Hz is

$$\begin{aligned}f &= \frac{215.34}{2\pi}\\ &= 34.272\text{ hz}\end{aligned}$$

ANSYS gives this as 34.272 Hz. The the error is 0.

Finding the transverse natural frequency:

Using $k = \frac{3EI}{L^3}$ and $M = 10$. Where I is the area moment of inertia given by

$$\begin{aligned}I &= \frac{\pi}{64}\left(D_o^4 - D_i^4\right)\\ &= \frac{\pi}{64}\left(0.04^4 - 0.034^4\right)\\ &= 6.0066\times 10^{-8}\text{ m}^4\end{aligned}$$

The transverse natural frequency is therefore

$$\begin{aligned}\omega &= \sqrt{\frac{3EI}{ML^3}} \\ &= \sqrt{\frac{3\left(200\times 10^9\right)\left(6.0066\times 10^{-8}\right)}{(10)\,(1)^3}} \\ &= 60.033 \text{ rad/sec}\end{aligned}$$

Hence

$$f_n = \frac{60.033}{2\pi} = 9.5545 \text{ Hz}$$

ANSYS gives 9.4438 for the first transverse natural frequency. Hence error is $\left(\frac{|9.4438-9.5545|}{9.4438}\right)100 = 1.1722\,\%$

Summary of results

mode	ANSYS result	Hand calculation	%error
First transverse	9.4438	9.5545	1.1722 %
First torsional	34.272	34.272	0%
First longitudinal (axial)	420.31	420.31	0%

All the analytical solutions gave exact agreement with ANSYS except for the transverse case. The transverse case uses stiffness $\frac{3EI}{L^3}$ due to load at end of fixed-free beam. This does not account for bending rotation in the beam. That is why ANSYS result is more accurate, as its finite elements account for the small bending associated with the transverse vibration. In the other two cases (Torsional and axial), there is no associated bending, hence the solutions agree.

2.7.1.2.2 Second solution. Accounting for mass of pipe

Finding the longitudinal natural frequency.

Following the example given in the textbook, at page 715, the (first) longitudinal natural frequency is found to be

$$\omega_1 = \frac{\alpha_1\sqrt{\frac{E}{\rho}}}{L} \tag{E.4}$$

Where α_1 is the (first) root of

$$\alpha \tan\alpha = \beta$$

Where β is the mass ratio $\beta = \frac{m}{M}$ where m is mass of pipe and M is end mass (flywheel). To find mass of pipe m, using steel density $\rho = 7800 \text{ kg/m}^3$, we first find the volume of the pipe.

Let D_i be the inner diameter and D_o the outer diameter. $D_o = 0.04$ meter and $D_i = 0.04 - 2\,(0.003) = 0.034$ meter , therefore the cross sectional area of the pipe is

$$\begin{aligned}A &= \frac{\pi}{4}\left(D_o^2 - D_i^2\right) \\ &= \frac{\pi}{4}\left((0.04)^2 - (0.034)^2\right) \\ &= 3.4872\times 10^{-4} \text{ m}^2\end{aligned}$$

And since length of pipe is 1 meter, the mass of pipe is

$$\begin{aligned}m &= \rho AL \\ &= (7800)\left(3.4872\times 10^{-4}\right)(1) \\ &= 2.72 \text{ kg.}\end{aligned}$$

The mass at the end is given as $M = 10$ kg. Therefore the mass ratio

$$\beta = \frac{m}{M} = \frac{2.72}{10} = 0.272$$

To find α_1 we now need to solve $\alpha_1 \tan\alpha_1 = 0.272$. This was solved numerical using root finder. The first root was found to be

$$\alpha_1 = 0.499$$

Therefore from equation E.4 in textbook (page 715)

$$\begin{aligned}\omega_1 &= \frac{\alpha_1 c}{L}\\ &= \frac{\alpha_1\sqrt{\frac{E}{\rho}}}{L}\\ &= \frac{(0.499)\sqrt{\frac{200\times10^9}{7800}}}{1}\\ &= 2526.8 \text{ rad/sec}\end{aligned}$$

Therefore

$$\begin{aligned}f_1 &= \frac{\omega_1}{2\pi}\\ &= \frac{2526.8}{2\pi}\\ &= 402.15 \text{ Hz}\end{aligned}$$

ANSYS gives 420.31 Hz. So the error is $\left(\frac{420.31-402.15}{420.31}\right)100 = 4.321\%$

Finding the torsional natural frequency.

k_t is

$$k_t = \frac{GJ}{L}$$

Where G is the shear modulus (given in handout), and J is the polar area moment of inertia of the cross section given by

$$\begin{aligned}J &= \frac{\pi}{32}\left(D_o^4 - D_i^4\right)\\ &= \frac{\pi}{32}\left(0.04^4 - 0.034^4\right)\\ &= 1.2013\times10^{-7} \text{ m}^4\end{aligned}$$

Hence

$$k_t = \frac{\left(77.2\times10^9\right)\left(1.2013\times10^{-7}\right)}{1} = 9274$$

To find equivalent mass, using kinetic energy method

$$\frac{1}{2}I_{flywheel}\dot{\theta}^2 + \frac{1}{2}I_{pipe}\dot{\theta}^2 = \frac{1}{2}I_{eq}\dot{\theta}^2 \tag{1}$$

For a hollow pipe, where now m is replaced by $\frac{1}{3}m$ from continuous system derivation.

$$\begin{aligned}I_{pipe} &= \frac{1}{2}\left(\frac{1}{3}m\right)\left(\left(\frac{D_o}{2}\right)^2 + \left(\frac{D_i}{2}\right)^2\right)\\ &= \frac{1}{24}m\left(D_o^2 + D_i^2\right)\end{aligned}$$

And for the flywheel, $I_{fly} = \frac{1}{2}Mr_f^2$ where $r_f = 0.2$ meter. Hence from (1)

$$\begin{aligned}I_{eq} &= \frac{1}{2}Mr_f^2 + \frac{1}{24}m\left(D_o^2 + D_i^2\right)\\ &= \frac{1}{2}(10)(0.2)^2 + \frac{1}{24}(2.72)\left(0.04^2 + 0.034^2\right)\\ &= 0.20031 \text{ kg-m}^2\end{aligned}$$

Hence the torsional frequency is

$$\begin{aligned}\omega &= \sqrt{\frac{k_t}{I_{eff}}}\\ &= \sqrt{\frac{9274}{0.20031}}\\ &= 215.17 \text{ rad/sec}\end{aligned}$$

Therefore the torsional frequency in Hz is

$$\begin{aligned}f &= \frac{215.17}{2\pi}\\ &= 34.245 \text{ hz}\end{aligned}$$

ANSYS gives this as 34.272 Hz. The the error is $\left(\frac{34.272-34.245}{34.272}\right)100 = 0.079\%$

Finding the transverse natural frequency:

From textbook, table 8.15 page 726, it gives for fixed-end beam the value $\beta_1 L = 1.875104$. But since there is a mass attach to the end in our problem, I did not know how to add this using the table.

So I used the other method we used before, which is the Rayleigh energy method, where we assume motion is simple harmonic motion. Taking the displacement as the transverse motion of the free end of the pipe (where the large mass is attached), measured from equilibrium then the kinetic energy is

$$T = \frac{1}{2}\overbrace{(M + 0.23m)}^{m_{eq}}\dot{x}^2$$

Where we added $0.23m$, where m is the mass of the pipe, since this is continuous mass. For the potential energy, we use the stiffness formula for the fixed-free beam which is $k = \frac{3EI}{L^3}$, hence

$$U = \frac{1}{2}kx^2$$

Now, assuming $x = X\sin\omega_n t$, then $\dot{x} = X\omega_n\cos\omega_n t$. Therefore when

$$U_{\max} = T_{\max}$$

We obtain

$$\begin{aligned}\frac{1}{2}\frac{3EI}{L^3}X^2 &= \frac{1}{2}(M + 0.23m)(X\omega_n)^2\\ \frac{3EI}{L^3} &= (M + 0.23m)\,\omega_n^2\\ \omega_n^2 &= \frac{3EI}{L^3(M + 0.23m)}\\ \omega_n &= \sqrt{\frac{3EI}{L^3(M + 0.23m)}} \qquad (1)\end{aligned}$$

Where I now is the area moment of inertia[2] is given by

$$\begin{aligned}I &= \frac{\pi}{64}\left(D_o^4 - D_i^4\right)\\ &= \frac{\pi}{64}\left(0.04^4 - 0.034^4\right)\\ &= 6.0066\times 10^{-8} \text{ m}^4\end{aligned}$$

And

$$\begin{aligned}M + 0.23m &= 10 + 0.23(2.72)\\ &= 10.626 \text{ kg}\end{aligned}$$

[2]Notice that the polar area moment of ineria has $\frac{1}{32}$ factor, while the area moment of interia, the factor is $\frac{1}{64}$

Substituting the numerical values in (1) gives

$$\begin{aligned}\omega_n &= \sqrt{\frac{3\left(200\times 10^9\right)\left(6.0066\times 10^{-8}\right)}{10.626}}\\ &= 58.239 \text{ rad/sec}\end{aligned}$$

Hence

$$f_n = \frac{58.239}{2\pi} = 9.269 \text{ Hz}$$

ANSYS gives 9.4438 for the first transverse natural frequency. Hence error is $\left(\frac{9.4438-9.269}{9.4438}\right)100 =$ 1.851 %

<u>Summary of results</u>

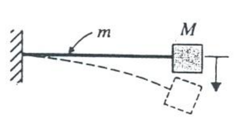
mode	ANSYS result	Hand calculation	%error
First transverse	9.4438	9.269	1.851 %
First torsional	34.272	34.245	0.079%
First longitudinal	420.31	402.15	4.321%

Comparing the above table to the first solution, it shows that ignoring the mass of the pipe gave result which agree with ANSYS result much better. This is because ANSYS did not take into the account the mass of the pipe. It will be interesting exercise to find how to change the APDL input file to make ANSYS account for the mass of the pipe and then compare the above results with ANSYS.

2.7.2 Problem 2

Problem 2

The signpost of a fast food restaurant consists of a hollow steel cylinder of height *h*, inside diameter *d*, and outside diameter *D*, fixed to the ground and carries a concentrated mass *M* at the top. It can be modeled as a single degree of freedom spring-mass-damper system with an equivalent viscous damping ratio of 0.1 for analyzing its transverse vibration characteristics under wind excitation. Assume the signpost mass (m) and concentrated mass (M) have an equivalent mass (m_{eq}) as defined below. (this equivalent mass equation was from a lecture example earlier in the semester). The specific weight (ρg) and the elastic modulus (E) of the steel are 76,500 N/m^3 and 207 GPa, respectively. For the density and viscosity of air, use 1.20 kg/m^3 and 1.80×10^{-5} N-s/m^2, respectively. For the remaining parameters, assume h = 10 m, D = 25 cm, d = 20 cm and M = 200 kg.
D



Cantilever beam of mass m carrying an end mass M

$m_{eq} = M + 0.23\, m$

Determine the following:
A) the natural frequency of transverse vibration of the signpost,
B) the wind velocity at which the signpost undergoes maximum steady-state displacement, and
C) the maximum wind induced steady-state displacement of the signpost.

2.7.2.1 Part A

The <u>first step</u> is to determine the natural frequency ω_n for the transverse vibration. Rayleigh energy method was used to find the transverse frequency. Taking the displacement as the transverse motion of the free end of the sigpost (where the large mass M is attached), measured from equilibrium, then the kinetic energy is

$$T = \frac{1}{2}\overbrace{(M+0.23m)}^{m_{eq}}\dot{x}^2$$

m is the mass of the sigpost. For the potential energy, the bending stiffness formula for the fixed-free beam with load at the end was used, which is

$$k = \frac{3EI}{L^3}$$

The potential energy is therefore

$$U = \frac{1}{2}kx^2$$

Assuming $x = X\sin\omega_n t$, then $\dot{x} = X\omega_n\cos\omega_n t$. Using

$$U_{\max} = T_{\max}$$

Then the above reduces to

$$\begin{aligned}
\frac{1}{2}\left(\frac{3EI}{L^3}\right)X^2 &= \frac{1}{2}(M+0.23m)(X\omega_n)^2\\
\frac{3EI}{L^3} &= (M+0.23m)\,\omega_n^2\\
\omega_n^2 &= \frac{3EI}{L^3(M+0.23m)}\\
\omega_n &= \sqrt{\frac{3EI}{L^3(M+0.23m)}} \qquad (1)
\end{aligned}$$

I is the area moment of inertia of the pipe cross section. Since $D_o = 0.25$ m and $D_i = 0.2$ m, then

$$\begin{aligned}
I &= \frac{\pi}{64}\left(D_o^4 - D_i^4\right)\\
&= \frac{\pi}{64}\left(0.25^4 - 0.2^4\right)\\
&= 1.1321\times10^{-4}\text{ m}^4
\end{aligned}$$

$M = 200$ kg, and $L = 10$ meter. Using $\rho_{steel}g = 76500$ N/m^3 and $E = 207\times10^9$ Pa. To find the mass m of the post, the cross sectional area is first found

$$\begin{aligned}
A &= \frac{\pi}{4}\left(D_o^2 - D_i^2\right)\\
&= \frac{\pi}{4}\left(0.25^2 - 0.2^2\right)\\
&= 0.017671\text{ m}^2
\end{aligned}$$

Hence the mass m is

$$\begin{aligned}
m &= \frac{\left(\rho_{steel}g\right)}{g}AL\\
&= \frac{76500}{9.81}(0.017671)(10)\\
&= 1378\text{ kg}
\end{aligned}$$

Substituting the numerical values in (1) gives

$$\begin{aligned}
\omega_n &= \sqrt{\frac{3EI}{L^3(M+0.23m)}}\\
&= \sqrt{\frac{3\left(207\times10^9\right)\left(1.1321\times10^{-4}\right)}{(10)^3(200+0.23(1378))}}\\
&= 11.662\text{ rad/sec}
\end{aligned}$$

Or

$$f_n = \frac{11.662}{2\pi}$$

Therefore

$$\boxed{f_n = 1.8561\text{ Hz}}$$

2.7.2.2 Part B

Maximum steady state displacement occurs at resonance. This is when the frequency of vortex shedding is the same as the natural frequency f_n of the post found above. Using Strouhal formula

$$v = \frac{f_n D_o}{S}$$

Where in the above v is the wind velocity and the vortex shedding frequency is set to be the natural frequency in order to obtain the maximum displacement. Assume $S = 0.21$ gives

$$v = \frac{(1.8561)(0.25)}{0.21}$$

Hence

$$\boxed{v = 2.2096 \text{ m/s}}$$

Checking Reynold number

$$\text{Re} = \frac{v D_o \rho_{air}}{\mu}$$

ρ_{air} is density of air and μ is viscosity of air. Using the numerical values given the above becomes

$$\begin{aligned}\text{Re} &= \frac{(2.2096)(0.25)(1.2)}{\left(1.8 \times 10^{-5}\right)} \\ &= 36827\end{aligned}$$

Since $400 \leq \text{Re} \leq 300000$ then the assumption of Strouhal $S = 0.21$ was valid.

2.7.2.3 Part C

The lateral force exerted by the wind on the sigpost is given by

$$\begin{aligned}F(t) &= \frac{1}{2} c \rho_{air} v^2 A \sin \omega t \\ &= F_0 \sin \omega t\end{aligned}$$

Where $c \approx 1$ for cylinder and v is the wind speed found in last part and A is the projected area $A = D_o L$. Hence

$$\begin{aligned}F_0 &= \frac{1}{2} c \rho_{air} v^2 A \\ &= \frac{1}{2}(1.2)(2.2096)^2(0.25)(10) \\ &= 7.3235 \text{ N}\end{aligned}$$

Using the steady state displacement formula for damped single degree of freedom system, which is

$$y_{ss} = \frac{F_0}{k} \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\xi r)^2}}$$

Where F_0 is total force from the wind over the whole span. Assuming this force acts at the end of a fixed-free beam (This is an over estimation. The wind force actually acts over the whole length of the sigpost, but it is now taken as acting on the end). Therefore $k = \frac{3EI}{L^3}$ can be used based on this. Since $r = 1$ (resonance) and $\xi = 0.1$, then y_{ss} is now evaluated

$$\begin{aligned}y_{ss} &= \frac{F_0}{k} \frac{1}{\sqrt{4\xi^2}} \\ &= \frac{F_0 L^3}{3EI} \frac{1}{2\xi} \\ &= \frac{(7.3235)(10)^3}{3\left(207 \times 10^9\right)\left(1.1321 \times 10^{-4}\right)} \frac{1}{2(0.1)} \\ &= 5.2085 \times 10^{-4} \text{ meter}\end{aligned}$$

Or

$$y_{ss} \approx 0.5 \text{ mm}$$

2.8 HW8

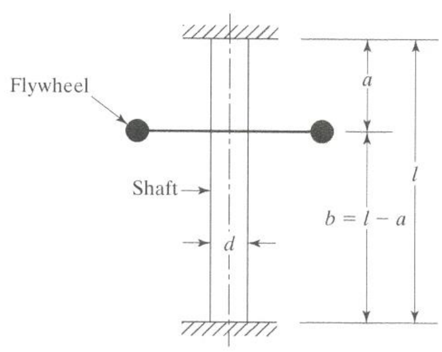
2.8.1 Problem 1

Problem 1

Download the ANSYS input file "*MODAL_pipe_flywheel.txt*" from HW7 on Canvas, run this input file in ANSYS and go through the file line by line to figure out what the system parameters are for this modal analysis. (Hint: When viewing the mode shapes within ANSYS, try plotting all 3 displacements and all 3 rotations (1 at a time) available under the "Nodal Solu" / "DOF Solution" option; this should be helpful in determining the type of displacement associated with each specific frequency.

A) Modify the *"MODAL_pipe_flywheel.txt"* file to use ANSYS to predict the natural frequencies and mode shapes for the problem listed below (NOTE: you should remember this problem from HW3).

A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter *d* and length *l* and is fixed at both ends. The flywheel has a weight of *W* and a radius of gyration of *r*. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system. For the parameters, assume that d = 1.2 in, a = 2 ft, b = 4 ft, W = 100 lbs and r = 16 in. (Assume the shaft is massless and the flywheel is rigid.)



For this problem, submit a hard copy of your modified .txt file and also create a table comparing the analytical and finite element frequencies (including % error) for the first longitudinal, first transverse and first torsional mode. Which mode has the most error? Which mode SHOULD have the most error? And why?

The APDL was modified to use solid pipe288 and put the mass element at the location as shown in the problem statement. The following are the four modes generated by ANSYS

set number	mode	frequency (Hz)
1	Torsion	10.437
2	First transverse (bending)	14.1815
3	Second transverse (bending)	35.384
4	First longitudinal (axial)	447.98

The following are the four plots showing the mode shapes for each of the above modes

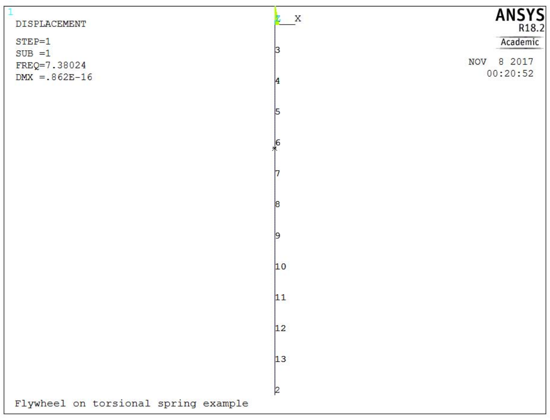


Figure 2.5: First mode: Torsion 7.3802 Hz

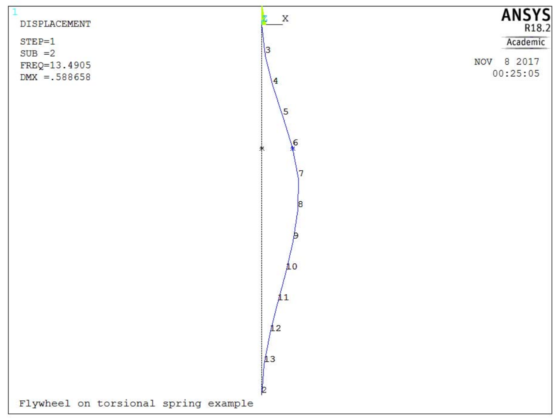


Figure 2.6: Second mode: Bending 13.491 Hz

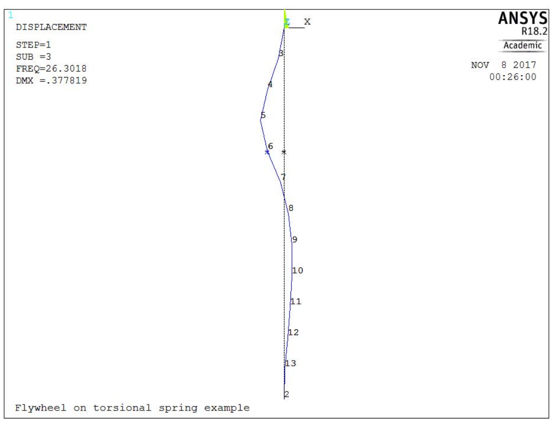


Figure 2.7: Third mode: Bending 26.302 Hz

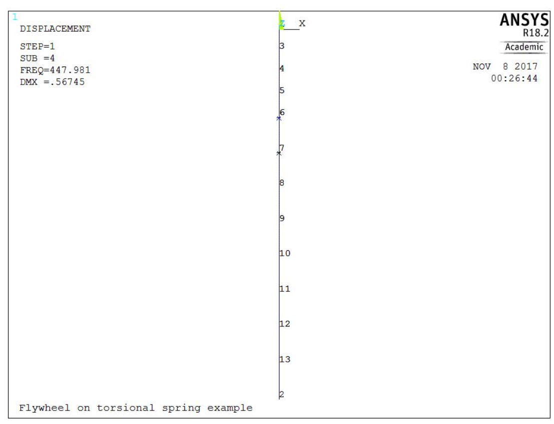


Figure 2.8: Fourth mode: Axial 447.98 Hz

The above result was next compared to the analytical result that was done in HW 3, by using the numerical value given in this problem. The numerical values for this problem are listed here

variable name	numerical value
L (length of pipe)	6 ft
a	2 ft
b	4 ft
d (diameter of pipe)	1.2 in $= \frac{1.2}{12} = 0.1$ ft
W (weight of flywheel)	100 lb
r (outer radius of flywheel)	16 in $= \frac{16}{12} = 1.3333$ ft
r_f (radius of gyration)	$\sqrt{\frac{r^2}{2}} = \sqrt{\frac{1.3333^2}{2}} = 0.94279$ ft
E (Elastic modulus of pipe material, steel)	29007547.546×144 psf (200 GPa)
G (shear modulus for pipe material, steel)	11196913.353×144 psf (7.2 GPa)
Poisson's ratio for steel	0.295
I area moment of inertia for pipe section	$\frac{\pi}{4}\left(\frac{d}{2}\right)^4 = 4.90874 \times 10^{-6}$ ft^4
$I_{flywheel}$ mass moment of inertial of flywheel	$\frac{W}{g} r_f^2 = 5.52105$ slug-ft^2

The above values were now used in the derivations from HW3 to obtain numerical values for the natural frequencies. The following are the results obtained (using analytical result from HW3 derivation)

mode	Analytical result	Numerical calculation ω_n in rad/sec	Hz
Torsion	$\omega_n=\sqrt{\frac{gG\pi d^4}{32Wr_f^2}\left(\frac{1}{a}+\frac{1}{b}\right)}$	$\sqrt{\frac{(32.2)(11196913.353\times 144)\pi(0.1)^4}{32(100)(0.94279)^2}\left(\frac{1}{2}+\frac{1}{4}\right)}=65.58$	10.437
bending (1)	$\omega_n=\sqrt{\frac{3gEI}{W}\left(\frac{L}{ab}\right)^3}$	$\sqrt{\frac{3(32.2)(29007547.546\times 144)\left(4.90874\times 10^{-6}\right)}{100}\left(\frac{6}{(2)(4)}\right)^3}=91.412$	14.549
axial	$\omega_n=\sqrt{\frac{gAE}{W}\left(\frac{1}{a}+\frac{1}{b}\right)}$	$\sqrt{\frac{(32.2)\left(\pi\left(\frac{0.1}{2}\right)^2\right)(29007547.546\times 144)}{100}\left(\frac{1}{2}+\frac{1}{4}\right)}=2814.8$	447.99

The following table compares the above analytical result with the ANSYS result shown earlier with the percentage error

mode	ANSYS result (Hz)	Analytical result (Hz)	error percentage
Torsion	10.437	10.437	0%
First bending	14.1815	14.549	$\left(\frac{14.1815-14.549}{14.1815}\right)\times 100=2.59\%$
First axial	447.98	447.99	$\frac{447.98-447.99}{447.98}\times 100=0.002\%$

The mode that has most error is the first bending (transverse) mode. This was the case also in HW7 ANSYS problem. ANSYS result is the more accurate one. The analytical result for this mode was derived The transverse case uses stiffness $3EI\left(\frac{L}{ab}\right)^3$ due to load at a distance from one end of fixed-free beam and b distance from the other end of the fixed beam. But this derivation does not account for any bending rotation in the beam as the ANSYS result would do.

2.8.1.1 Listing of modified APDL script

```
!-- Modified APDL script for HW 8, ME 440, Fall 2017
!

/filnam, pipe_flywheel_modal
/title, Flywheel on torsional spring example
/prep7


!-- give names for elements -----------
MASS_ELEMENT=1
PIPE_ELEMENT=2

!-- define the mass element ------------
ET,MASS_ELEMENT,mass21,,0,0  !element type no.1 is mass21 (",,,0" signifies
                             !that this is a 3-D mass with rotary inertia)
! model parameters for MASS_ELEMENT
mass        = (100/32.2)           ! mass of flywheel (lb)
r_wheel     =  (16/12)             ! radius of gyration (ft)
Iyy  = mass*(r_wheel*r_wheel)/2    ! mass moment of inertia
OUTER_DIAMETER  = (1.2/12)         ! outer diameter of pipe (ft)
wall_t = OUTER_DIAMETER/2-0.0001   ! Solid pipe! This gives warning
                                   ! but we can ignore it for now
SHAFT_LENGTH = 6    ! shaft length (ft)
n_modes      = 10   ! number of modes wanted, but ANSYS always gives 4

!real constants for MASS_ELEMENT
r,MASS_ELEMENT,mass,mass,mass,0.5*IYY,IYY,0.5*IYY

!-- define the shaft element as solid pipe -------
ET,PIPE_ELEMENT,pipe288

mp,ex,MASS_ELEMENT,29007547.546*144    !(200e9 SI) elastic modulus PSF
mp,gxy,MASS_ELEMENT,11196913.35276*144 !(77.2e9 SI) shear modulus PSF
mp,prxy,MASS_ELEMENT,0.295  ! poisson's ratio for steel is 0.295

KEYOPT,PIPE_ELEMENT,4,2     !Thick wall per ansys help

!SECTYPE, SECID, Type, Subtype, Name, REFINEKEY
!    Associates section type information with a section ID number.
sectype,1,pipe                ! section type 1 is "pipe"
```

```
secdata,OUTER_DIAMETER,wall_t ! section data for pipe is outer
                              ! diameter and wall thickness


!-- key points ---------
k,1,0,0,0    ! keypoint 1 is at x=0, y=0, z=0, one fixed end of pipe

! keypoint 2 where fluwheel is located
k,2,0,-SHAFT_LENGTH/2.0,0

k,3,0,-SHAFT_LENGTH,0  ! keypoint 3 is other end of the fixd pipe

!-- create elements -------------------
TYPE,PIPE_ELEMENT ! element type of subsequently defined elements.

!SECNUM, SECID
!   Sets the element section attribute pointer.
!   Defines the section ID number to be assigned to the
!   subsequently-defined elements Defaults to 1. See SECTYPE for more
!   information about the section ID number.

secnum,1 !specify section type number of subsequently defined elements


!-- create line ------------------
!L, P1, P2
!Defines a line between two keypoints.

L,1,3  ! creates ONE line from keypoint 1 to keypoint 3

!LESIZE, NL1, SIZE, ANGSIZ, NDIV, SPACE, KFORC, LAYER1, LAYER2, KYNDIV
!Specifies the divisions and spacing ratio on unmeshed lines.
!    NL1  Number of the line to be modified.
!    SIZE If NDIV is blank, SIZE is the division (element edge) length.
!       The number of divisions is automatically calculated from the
!       line length (rounded upward to next integer). If SIZE is zero
!       (or blank), use ANGSIZ or NDIV
!    ANGSIZ The division arc (in degrees) spanned by the element edge
!    NDIV If positive, NDIV is number of element divisions per line.

lesize,1,,,12  ! line 1 will consist of 12 elements when meshed

!LMESH, NL1, NL2, NINC  Generates nodes and line elements along lines
!       Mesh lines from NL1 to NL2

lmesh,ALL ! line 1 meshed, resulting in elements representing the pipe

!----------------------
type,MASS_ELEMENT  ! element type of subsequently defined elements
real,1             ! real constant set of subsequently defined element

!E, I, J, K, L, M, N, O, P
!Defines an element by node connectivity.
!  I   Number of node assigned to first nodal position (node I)
E,6   ! create element to be created at node 6

finish

/solu    !select static load solution

!-- Set the boundary conditions --------------
nsel,all        ! selects all nodes
d,all,uz,0      ! sets the z displacements on selected nodes to be 0
                ! limiting our modal analysis to modes in the xy plane
d,all,rotx      ! sets the rotx displacments on selected nodes to be 0

!displacement boundary conditions
! NSEL, Type, Item, Comp, VMIN, VMAX, VINC, KABS
!    Type S   Select a new set (default).
!    Item  LOC  X,Y,Z X,Y, or Z location in active coordinate system

nsel,S,NODE,,1    ! select node at x = 0
```

```
d,all,ux,0        ! displacement of selected node in x-dir is 0
d,all,uy,0        ! displacement of selected node in y-dir is 0
d,all,uz,0        ! displacement of selected node in z-dir is 0
d,all,rotx,0      ! rotations of selected node about x axis is 0
d,all,roty,0      ! rotations of selected node about y axis is 0
d,all,rotz,0      ! rotations of selected node about z axis is 0


nsel,A,NODE,,2    ! select node at x = -SHAFT_LENGTH
d,all,ux,0        ! displacement of selected node in x-dir is 0
d,all,uy,0        ! displacement of selected node in y-dir is 0
d,all,uz,0        ! displacement of selected node in z-dir is 0
d,all,rotx,0      ! rotations of selected node about x axis is 0
d,all,roty,0      ! rotations of selected node about y axis is 0
d,all,rotz,0      ! rotations of selected node about z axis is 0

allsel

antype,modal
modopt,lanb,20
solve
finish

/post1
```

2.8.2 Problem 2

Problem 2

A centrifugal pump, weighing 700 N and operating at 1000 rpm, is mounted on six springs of stiffness 6000 N/m each. Find the maximum permissible unbalance in order to limit the steady-state deflection to 5.0 mm peak-to-peak.

The first step is to determine the natural frequency of the system. Since the springs are in parallel then

$$k_{eq} = 6k$$

And the equivalent mass is $m_{eq} = \frac{W}{g}$ where $W = 700$ N. Hence

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{6k}{\frac{W}{g}}} = \sqrt{\frac{6\,(6000)}{\frac{700}{9.81}}} = 22.461 \text{ rad/sec}$$

Since this is undamped system, then the steady state solution (particular solution) is given by

$$y_p(t) = \frac{x_{st}}{\sqrt{\left(1-r^2\right)^2}} \cos\omega t \tag{1}$$

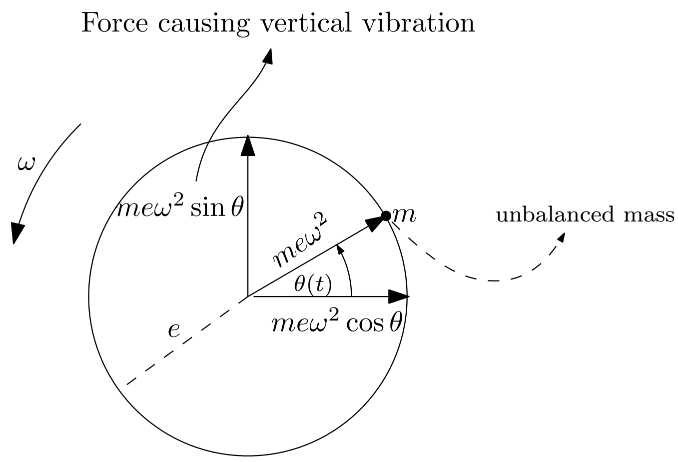
Where $r = \frac{\omega}{\omega_n}$ and ω is the driving frequency, which is

$$\omega = 1000\left(\frac{2\pi}{rev}\right)\left(\frac{\min}{60}\right) = 1000\left(\frac{2\pi}{60}\right) = 104.72 \text{ rad/sec}$$

From (1), we see that the maximum steady state response is

$$y_{ss} = \frac{x_{st}}{\sqrt{\left(1-r^2\right)^2}} \tag{2}$$

We now just need to determine x_{st} which is the static deflection. Let m_0 be the unbalanced mass which is spinning inside, and let e be the radius around the spin axis. Therefore, and assuming ω is constant, this mass will have only radial acceleration towards the center of $e\omega^2$ and therefore it will induce a centripetal force $m_0 e\omega^2$.



From the above we see that the vertical force is

$$F(t) = \overbrace{m_0 e\omega^2}^{F_0} \sin\theta(t)$$

Hence the static deflection is

$$x_{st} = \frac{F_0}{k_{eq}} = \frac{m_0 e\omega^2}{6k}$$

Substituting this into (2) gives

$$y_{ss} = \frac{\frac{m_0 e\omega^2}{6k}}{\sqrt{\left(1-r^2\right)^2}} = \frac{m_0 e\omega^2}{6k\sqrt{\left(1-r^2\right)^2}} \tag{3}$$

But r is

$$r = \frac{\omega}{\omega_n} = \frac{104.72}{22.461} = 4.6623$$

Since $r > 1$ then we now can simplify $\sqrt{\left(1-r^2\right)^2} = r^2 - 1$ and (3) becomes

$$y_{ss} = \frac{m_0 e\omega^2}{6k\left(r^2-1\right)}$$

Since we want to limit deflection to 5 mm peak to peak, then we want to limit $y_{ss} = 2.5$ mm (which is half of the peak-to-peak). The above equation becomes

$$\begin{aligned} 2.5\times 10^{-3} &= \frac{m_0 e\,(104.72)^2}{6\,(6000)\left(4.6625^2-1\right)} \\ &= \frac{m_0 e\,(104.72)^2}{36000\,(20.739)} \\ &= \frac{m_0 e\,(104.72)^2}{7.466\times 10^5} \end{aligned}$$

Solving for unbalance $m_0 e$ gives

$$m_0 e = \frac{\left(2.5\times 10^{-3}\right)\left(7.466\times 10^5\right)}{(104.72)^2}$$

Or

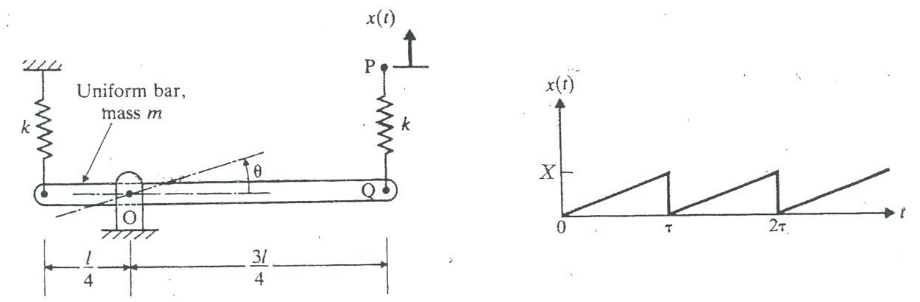
$$\boxed{m_0 e = 0.1702 \text{ kg-meter}}$$

This means to limit $m_0 e$ below this value in order to limit vibration to 5 mm, peak-to-peak.

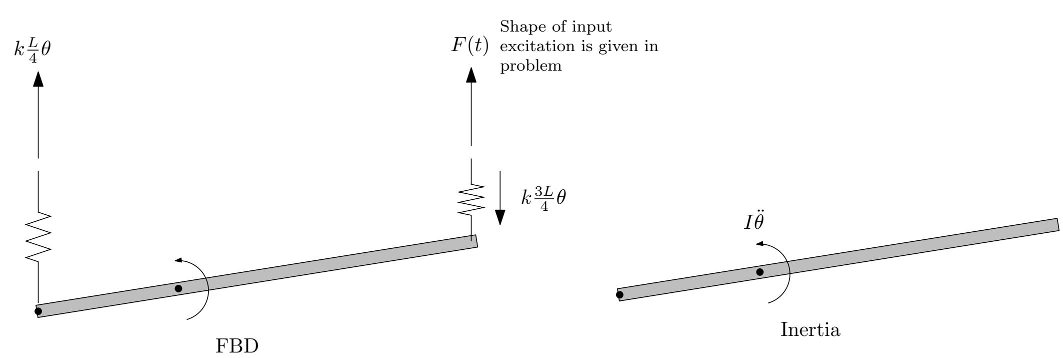
2.8.3 Problem 3

Problem 3

Determine the steady-state response of the system θ(t) due to the input excitation shown, using the system parameters given in the figure. (Use a trigonometric Fourier expansion of the input excitation.)



The first step is to make a FBD and corresponding inertia diagram Where it is assumed the left spring is in tension and the right side spring is in compression.



Taking moments around the pivot o where the bar is rotating around, and using anti-clockwise as positive gives (this assumes small angle approximation)

$$\sum M = I_o\ddot{\theta}$$

$$-k\left(\frac{L}{4}\theta\right)\frac{L}{4} - k\left(\frac{3L}{4}\theta\right)\frac{3L}{4} + kx(t)\left(\frac{3L}{4}\right) = I_o\ddot{\theta} \tag{1}$$

But I_o is the mass moment of inertia around o, which is

$$I_o = \overbrace{\frac{1}{12}mL^2}^{I_{cg}} + \overbrace{m\left(\frac{1}{4}L\right)^2}^{\text{parallel axis}} = \frac{7}{48}L^2m$$

Therefore the equation of motion (1) becomes

$$\frac{7}{48}L^2m\ddot{\theta} = -k\left(\frac{L^2}{16}\theta + \frac{9L^2}{16}\theta\right) + k\frac{3L}{4}x(t)$$

$$L^2m\ddot{\theta} + \theta\left(k\frac{10}{16}L^2\right)\frac{48}{7} = k\frac{48}{7}\left(\frac{3L}{4}\right)x(t)$$

$$m\ddot{\theta} + \theta\left(\frac{30}{7}k\right) = k\frac{36}{7}\frac{1}{L}x(t) \tag{2}$$

Therefore

$$\boxed{\omega_n = \sqrt{\frac{30}{7}\frac{k}{m}}}$$

We now need to expand $x(t)$ in Fourier series. $x(t)$ has period of τ. This is not even and not odd function.

$$x(t) = \frac{X}{\tau}t$$

Hence

$$\begin{aligned} a_0 &= \frac{1}{\frac{\tau}{2}}\int_0^\tau \frac{X}{\tau}t dt = \frac{2}{\tau}\frac{X}{\tau}\left(\frac{t^2}{2}\right)_0^\tau = \frac{X}{\tau^2}\tau^2 = X \\ a_n &= \frac{1}{\frac{\tau}{2}}\int_0^\tau \frac{X}{\tau}t\cos\left(\frac{2\pi}{\tau}nt\right)dt \\ &= \frac{2}{\tau}\frac{X}{\tau}\int_0^\tau t\cos\left(\frac{2\pi}{\tau}nt\right)dt \\ &= \frac{2}{\tau}\frac{X}{\tau}(0) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} b_n &= \frac{1}{\frac{\tau}{2}}\int_0^\tau \frac{X}{\tau}t\sin\left(\frac{2\pi}{\tau}nt\right)dt \\ &= \frac{2}{\tau}\frac{X}{\tau}\int_0^\tau t\sin\left(\frac{2\pi}{\tau}nt\right)dt \\ &= \frac{2}{\tau}\frac{X}{\tau}\left(-\frac{\tau^2}{2n\pi}\right) \\ &= -\frac{X}{n\pi} \end{aligned}$$

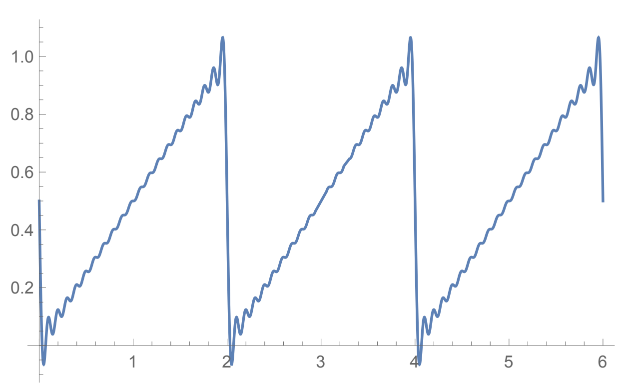
Hence

$$\begin{aligned} x(t) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{\tau}nt\right) \\ &\approx \frac{X}{2} - \frac{X}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi}{\tau}nt\right) \\ &\approx \frac{X}{2} - \frac{X}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi}{\tau}nt\right) \end{aligned}$$

To verify this solution, the above is plotted for number of terms to see if it will approximate the original $x(t)$.

```
In[120]:= ClearAll[x, n, t]
X0 = 1;
T0 = 2;
numberTerms = 20;
x[t_] = X0/2 - X0 / Pi Sum[1/n Sin[2 Pi / T0 n t], {n, 1, numberTerms}];
Plot[x[t], {t, 0, 3 T0}]
```

Out[125]=



Now we go back to the original equation of motion (2), and replace $x(t)$ by its Fourier series expansion

$$\begin{aligned}
m\ddot{\theta}+\theta\left(\frac{30}{7}k\right) &= k\frac{36}{7}\frac{1}{L}\left(\frac{X}{2}-\frac{X}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi}{\tau}nt\right)\right)\\
&= k\frac{18}{7}\frac{X}{L}-k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\left(\sin\left(\frac{2\pi}{\tau}t\right)+\frac{1}{2}\sin\left(\frac{2\pi}{\tau}2t\right)+\frac{1}{3}\sin\left(\frac{2\pi}{\tau}3t\right)+\cdots\right)\\
&= k\frac{18}{7}\frac{X}{L}-k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\left(\sin(\omega t)+\frac{1}{2}\sin(2\omega t)+\frac{1}{3}\sin(3\omega t)+\frac{1}{4}\sin(4\omega t)+\cdots\right) \qquad (3)
\end{aligned}$$

Linearity is now used to find the solution to the above by adding the the steady state response to each of the terms. The steady state response to the first term above, which is $\frac{18}{7}k\frac{X}{mL}$ is the steady state response to the ODE

$$m\ddot{\theta}+\theta\left(\frac{30}{7}k\right)=\left(\frac{18}{7}k\frac{X}{L}\right)$$

Which Is given by

$$y_{ss}=\left(k\frac{18}{7}\frac{X}{L}\right)\frac{1}{k_{eq}}$$

But $k_{eq}=\frac{30}{7}k$, therefore

$$\begin{aligned}
y_{ss} &= \left(\frac{18}{7}k\frac{X}{L}\right)\frac{7}{30k}\\
&= \frac{9}{15}\frac{X}{L}
\end{aligned}$$

This is the response to only the first term in (3). Now we do the same for each of the trig terms. But we only need to consider one general term. The ODE we will look at now is

$$\begin{aligned}
m\ddot{\theta}+\theta\left(\frac{30}{7}k\right) &= k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi}{\tau}nt\right)\\
&= k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\left(\sin\left(\frac{2\pi}{\tau}t\right)+\frac{1}{2}\sin\left(\frac{2\pi}{\tau}2t\right)+\frac{1}{3}\sin\left(\frac{2\pi}{\tau}3t\right)+\cdots\right)\\
&= k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\left(\sin(\omega t)+\frac{1}{2}\sin(2\omega t)+\frac{1}{3}\sin(2\omega t)+\cdots\right)
\end{aligned}$$

Considering one general term

$$\begin{aligned}
m\ddot{\theta}+\theta\left(\frac{30}{7}k\right) &= k\left(\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\frac{1}{n}\right)\sin(n\omega t)\\
&= F_0\sin(n\omega t) \qquad (4)
\end{aligned}$$

Where

$$\begin{aligned}
F_0 &= \left(k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\frac{1}{n}\right)\\
x_{st} &= \frac{F_0}{k_{eq}}\\
&= \frac{k\frac{1}{\pi}\frac{36}{7}\frac{X}{L}\frac{1}{n}}{\frac{30}{7}k}\\
&= \frac{6}{5\pi L}\frac{X}{n} \qquad (5)
\end{aligned}$$

We know the steady state (particular) solution for (4) is

$$\theta_{ss}(t)=\frac{x_{st}}{\left(1-(nr)^2\right)}\sin(n\omega t) \qquad (6)$$

Where r is

$$r=\frac{\omega}{\omega_n}=\frac{\frac{2\pi}{\tau}}{\sqrt{\frac{30}{7}\frac{k}{m}}}=\frac{2\pi}{\tau\sqrt{\frac{30}{7}\frac{k}{m}}} \qquad (7)$$

The above is the steady state response for the n^{th} term. So the total response is the sum of all these responses. Putting all this together, we now obtain the steady state solution as

$$\theta_{ss}(t) = k\frac{9X}{15kL} - \sum_{n=1}^{\infty} \frac{x_{st}}{\left(1-(nr)^2\right)} \sin(n\omega t) \tag{8}$$

Where x_{st} is given (5) and r is given by (7) and $\omega = \frac{2\pi}{\tau}$. To try verify the above, it is plotted using the following values $X = 1, L = 10$ meter,$k = 100$ N/m,$\tau = 3\,\text{sec}$ and $m = 5$ kg. This is the result (for 30 terms in Fourier sum)

```
Clear[n]
X0 = 1;
T0 = 3;
m0 = 5;
k0 = 100;
L0 = 10;
xst = 6/(5 Pi L0) X0/(k0 n);
wn = Sqrt[30/7 k0/m0];
w = (2 Pi)/T0;
r = w/wn;
y = (9 X0)/(15 k0 L0) - Sum[xst/(1 - (n r)^2) Sin[n w t], {n, 1, 30}];
Plot[y*180/(2 Pi), {t, 0, 5 T0}, Frame -> True, GridLines -> Automatic,
 GridLinesStyle -> LightGray, PlotStyle -> Red, ImageSize -> 500,
 FrameLabel -> {{"theta(t)  (degree)", None}, {"time  (sec)", "Steady state solution for prroblem 3"}},
 BaseStyle -> 14]
```

Out[123]=

2.8.4 HW 8 key solution

ORIGINAL

ME 440
Intermediate Vibrations

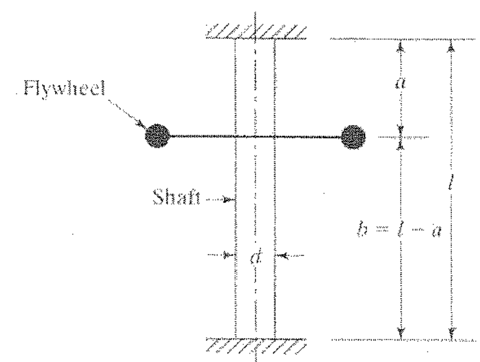
Homework #8 (3 problems)
due Thursday, November 9th, 2017

Problem 1

Download the ANSYS input file "*MODAL_pipe_flywheel.txt*" from HW7 on Canvas, run this input file in ANSYS and go through the file line by line to figure out what the system parameters are for this modal analysis. (Hint: When viewing the mode shapes within ANSYS, try plotting all 3 displacements and all 3 rotations (1 at a time) available under the "Nodal Solu" / "DOF Solution" option; this should be helpful in determining the type of displacement associated with each specific frequency.

A) Modify the *"MODAL_pipe_flywheel.txt"* file to use ANSYS to predict the natural frequencies and mode shapes for the problem listed below (NOTE: you should remember this problem from HW3).

A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system. For the parameters, assume that $d = 1.2$ in, $a = 2$ ft, $b = 4$ ft, $W = 100$ lbs and $r = 16$ in. (Assume the shaft is massless and the flywheel is rigid.)



For this problem, submit a hard copy of your modified .txt file and also create a table comparing the analytical and finite element frequencies (including % error) for the first longitudinal, first transverse and first torsional mode. Which mode has the most error? Which mode SHOULD have the most error? And why?

Problem 2

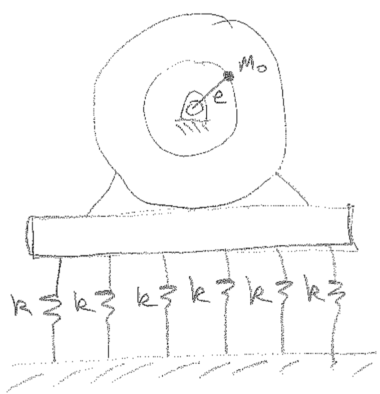
A centrifugal pump, weighing 700 N and operating at 1000 rpm, is mounted on six springs of stiffness 6000 N/m each. Find the maximum permissible unbalance in order to limit the steady-state deflection to 5.0 mm peak-to-peak.

CENTRIFUGAL PUMP $W = mg = 600N$

PUMP SPEED $= \omega = 1000$ RPM

MOUNTED ON 6 SPRINGS, $k_{ONE\ SPRING} = 6000 \frac{N}{m}$

FIND $(m_o e)_{MAX}$ TO LIMIT STEADY STATE DEFLECTION TO 5.0 mm PEAK-TO-PEAK



$$m = \frac{700N}{9.81 \frac{m}{s^2}} = 71.36 \text{ kg}$$

$$\omega = 1000 \frac{REV}{MIN}\left(\frac{1 MIN}{60 SEC}\right)\left(\frac{2\pi RAD}{1 REV}\right) = 104.72 \frac{RAD}{s}$$

$$k = 6\left(6000 \frac{N}{m}\right) = 36,000 \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{36,000 \frac{N}{m}}{71.36 \text{ kg}}} = 22.46 \frac{RAD}{s}$$

$$r = \frac{\omega}{\omega_n} = \frac{104.72 \text{ RAD/s}}{22.46 \text{ RAD/s}} = 4.6622 \qquad r^2 = 21.736$$

$$X = \frac{m_o e \omega^2}{\sqrt{(k - m\omega^2)^2 + (\underset{0}{c}\omega)^2}} = \frac{m_o e \omega^2}{\sqrt{(k - m\omega^2)^2}} = \frac{\frac{1}{m}(m_o e \omega^2)}{\sqrt{\left(\underset{\omega_n^2}{\frac{k}{m}} - \omega^2\right)^2}}$$

$$= \frac{\frac{1}{m}(m_o e \omega^2)\left(\frac{1}{\omega_n^2}\right)}{\sqrt{(\omega_n^2 - \omega^2)^2\left(\frac{1}{\omega_n^2}\right)}} = \frac{\frac{1}{m\omega_n^2}(m_o e \omega^2)}{\sqrt{\left(\underset{1}{\frac{\omega_n^2}{\omega_n^2}} - \underset{r^2}{\frac{\omega^2}{\omega_n^2}}\right)^2}} = \frac{\frac{1}{m\omega_n^2}(m_o e \omega^2)}{\sqrt{(1 - r^2)^2}}$$

IF $r > 1$, THEN $\sqrt{(1-r^2)^2} = r^2 - 1$

5.0mm PEAK-TO-PEAK EQUIVALENT TO AMPLITUDE OF 2.5mm = 0.0025 m

$$X = 0.0025\text{m} = \frac{\frac{1}{m\omega_n^2}(m_0 e \omega^2)}{r^2 - 1} \quad \text{SOLVE FOR } m_0 e$$

$$m_0 e = \frac{X(r^2-1)\, m\omega_n^2}{\omega^2} = \frac{(0.0025\text{m})(4.662^2 - 1)(71.36\text{kg})(22.46 \frac{\text{rad}}{\text{s}})^2}{(104.72 \text{ rad/s})^2}$$

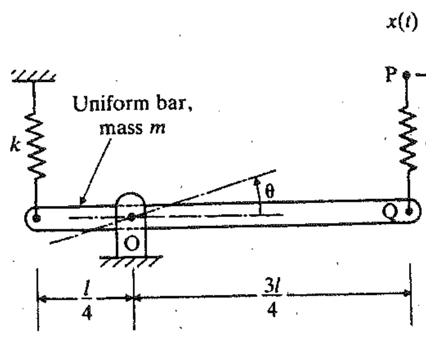
$$\boxed{m_0 e = 0.1702 \text{ kg}\cdot\text{m}}$$

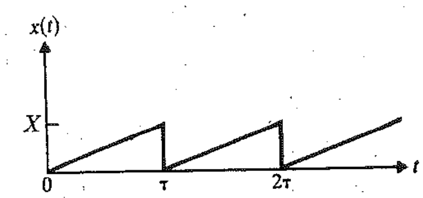
1/4

Homework Solution

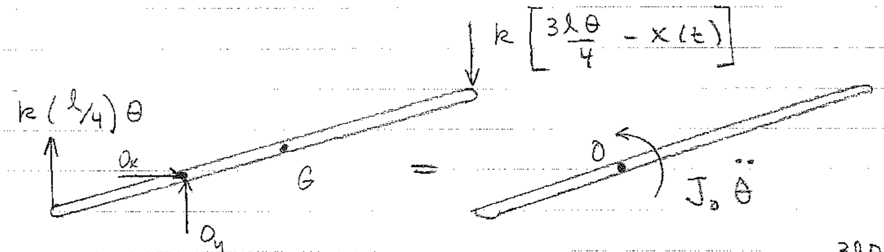
Determine the steady-state response of the system $\theta(t)$ due to the input excitation shown, using the system parameters given in the figure. (Use a Fourier expansion of the input excitation.)

Assume small oscillations –





Draw F.B.D's and determine EOM of the system:



Assume: $\frac{3l\theta}{4} > x(t)$

$$\circlearrowleft \Sigma M_o = (\Sigma M_o)_{eff}$$

$$-k(l/4)^2\theta - k\left[\frac{3l\theta}{4} - x(t)\right]\left(\frac{3l}{4}\right) = J_o\ddot{\theta}$$

$$J_o\ddot{\theta} + \frac{5}{8}kl^2\theta = \frac{3kl}{4}x(t)$$

But, $J_o = \frac{1}{12}ml^2 + m\left(l/4\right)^2 = \frac{7}{48}ml^2$

$$\boxed{\frac{7}{48}ml^2\ddot{\theta} + \frac{5}{8}kl^2\theta = \frac{3kl}{4}x(t)}$$

E.O.M

(3/4)

Finally,

$$x(t) = \frac{X}{2} - \frac{X}{\pi} \sum_{n=1,2,3..} \frac{1}{n} \sin \frac{2n\pi t}{\tau}$$

And the EOM becomes:

$$\frac{7}{48} m l^2 \ddot{\theta} + \frac{5}{8} k l^2 \theta = \frac{3kl}{4}\left[\frac{X}{2} - \frac{X}{\pi} \sum_n \frac{1}{n} \sin \frac{2n\pi t}{\tau}\right]$$

From previous handouts

Given: $m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$

Response: $x_p(t) = \frac{a_0}{2k_{eq}} + \sum_n \frac{a_n \cos(n\omega t - \phi_n) + b_n \sin(n\omega t - \phi_n)}{k\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}}$

where: $\phi_n = \tan^{-1} \frac{2\zeta n r}{1-n^2r^2}$ and $r = \frac{\omega}{\omega_n}$

Now, we need to compare the ODE of the system to the "form" previously established -

Comparing terms:

$a_n = 0$ $\quad b_n = \frac{-3kl}{4}\left(\frac{X}{n\pi}\right)$ $\quad \frac{a_0}{2k_{eq}} = \frac{3kl}{4}\left(\frac{X}{2}\right)\frac{8}{5kl^2} = \frac{3X}{5l}$

$\omega = \frac{2\pi}{\tau}$ $\quad k_{eq} = \frac{5}{8} k l^2$ $\quad \omega_n = \sqrt{\frac{5/8\, kl^2}{7/48\, ml^2}} = \sqrt{\frac{30k}{7m}}$ $\quad \zeta = 0$

(2/4)

<u>Express x(t) as a Fourier Series</u>

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{\tau} + b_n \sin\frac{2n\pi t}{\tau}\right)$$

(solve for a_n, b_n and a_0)

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos\frac{2n\pi t}{\tau}\,dt = \frac{2}{\tau}\int_0^{\tau}\left(X\,t/\tau\right)\cos\frac{2n\pi t}{\tau}\,dt$$

$$a_n = \frac{2X}{\tau^2}\int_0^{\tau} t\cos\frac{2n\pi t}{\tau}\,dt = \frac{2X}{\tau^2}\left[\frac{\tau^2}{(2n\pi)^2}\cos\frac{2n\pi t}{\tau}\right]_0^{\tau} + \frac{2X}{\tau^2}\left[\frac{\tau t}{2n\pi}\sin\frac{2n\pi t}{\tau}\right]_0^{\tau}$$

$$a_n = \frac{X}{2n^2\pi^2}\left[\cancelto{1}{\cos 2n\pi} - 1\right] + \left(\frac{2X}{\tau^2}\right)\frac{\tau^2}{2n\pi}\cancelto{0}{\sin 2n\pi}$$

$$\Rightarrow \boxed{a_n = 0}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin\frac{2n\pi t}{\tau}\,dt = \frac{2}{\tau}\int_0^{\tau}\frac{Xt}{\tau}\sin\frac{2n\pi t}{\tau}\,dt$$

$$b_n = \frac{2X}{\tau^2}\int_0^{\tau} t\sin\frac{2n\pi t}{\tau}\,dt = \frac{2X}{\tau^2}\left[\frac{\tau^2}{(2n\pi)^2}\sin\frac{2n\pi t}{\tau}\right]_0^{\tau} - \frac{2X}{\tau^2}\left[\frac{t\tau}{2n\pi}\cos\frac{2n\pi t}{\tau}\right]_0^{\tau}$$

$$b_n = \frac{X}{2n^2\pi^2}\left[\cancelto{0}{\sin 2n\pi} - 0\right] - \frac{X}{n\pi}\left[\cancelto{1}{\cos 2n\pi}\right] = -\frac{X}{n\pi}$$

$$\Rightarrow \boxed{b_n = -\frac{X}{n\pi}}$$

$$\frac{a_0}{2} = \frac{1}{\tau}\int_0^{\tau} x(t)\,dt = \frac{1}{\tau}\int_0^{\tau}\frac{Xt}{\tau}\,dt = \frac{X}{\tau^2}\int_0^{\tau} t\,dt = \frac{X}{\tau^2}\left[\frac{t^2}{2}\right]_0^{\tau} = \boxed{\frac{X}{2}}$$

4/4

Consequently, the response is:

$$\theta_p(t) = \frac{3X}{5l} + \sum_{n=1,2,3\ldots}^{\infty} \frac{-\frac{3klX}{4n\pi}\sin(n\omega t - \phi_n)}{\frac{5}{8}kl^2\sqrt{(1-n^2r^2)^2}}$$

$$\phi_n = \tan^{-1}\frac{2\zeta nr}{1-n^2r^2} = 0 \qquad r = \omega/\omega_n$$

and ω, ω_n and ζ are given above —

or

$$\theta_p(t) = \frac{3X}{5l} + \sum_{n=1,2,3\ldots}^{\infty} \frac{-6X \sin n\omega t}{5n\pi l\,(1-n^2r^2)}$$

where $r = \dfrac{\omega}{\omega_n}$ and $\omega_n = \sqrt{\dfrac{30k}{7m}}$

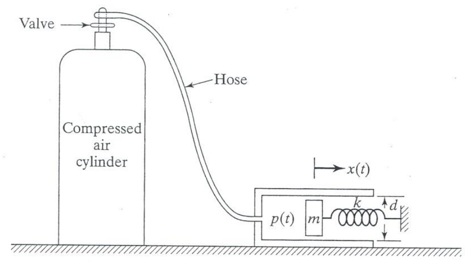
$\omega = 2\pi/\tau$

2.9 HW9

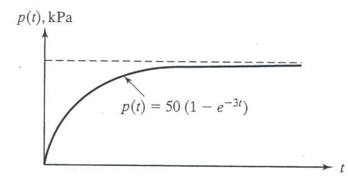
2.9.1 Problem 1

Problems 2/3 (due Friday, November 17th by 4pm)

A compressed air cylinder is connected to the spring-mass system shown in Figure (a) below. Due to a small leak in the valve, the pressure on the piston, *p(t)*, builds up as indicated in Figure (b) shown below. Assume $m = 10$ kg, $k = 1000$ N/m and $d = 0.1$ m and that all initial conditions are zero.



(a)



(b)

Solve for the complete response of the piston by using direct integration.

Since this is an undamped system, the equation of motion is

$$m\ddot{x} + kx = F(t)$$

Where $F(t) = Ap(t)$ and $p(t)$ is the pressure. Therefore

$$F(t) = \left(50 \times 10^3\right) A \left(1 - e^{-3t}\right)$$

The term 50×10^3 was added above because the units were given in kPa and need to convert them to Pa. The equation of motion becomes

$$\begin{aligned} m\ddot{x} + kx &= \left(50 \times 10^3\right) A \left(1 - e^{-3t}\right) \\ &= \left(50 \times 10^3\right) A - \left(50 \times 10^3\right) A e^{-3t} \end{aligned}$$

To simplify notations, let $\beta = \left(50 \times 10^3\right) A$. The above now becomes

$$m\ddot{x} + kx = \beta - \beta e^{-3t} \tag{1}$$

The solution to the above can be found by adding the two particular solutions of

$$m\ddot{x} + kx = \beta \tag{2}$$

And

$$m\ddot{x} + kx = -\beta e^{-3t} \tag{3}$$

To the homogeneous solution of $m\ddot{x} + kx = 0$. This can be done since the ODE is linear. The particular solution to (2) is found by assuming $x_p(t) = C_1$ where C_1 is some constant and substituting this into (1) and solving for C_1 gives $kC_1 = \beta$ or $C_1 = \frac{\beta}{k}$, hence

$$x_{p,1}(t) = \frac{\beta}{k} \tag{4A}$$

The particular solution to (2) is now found. From the lookup table, assuming $x_p(t) = C_1 e^{-3t}$

and substituting this into (2), and since $\dot{x}_p = -3C_1 e^{-3t}$ and $\ddot{x}_p = 9C_1 e^{-3t}$ gives

$$
\begin{aligned}
9mC_1 e^{-3t} + kC_1 e^{-3t} &= -\beta e^{-3t} \\
9mC_1 + kC_1 &= -\beta \\
C_1 &= \frac{-\beta}{9m+k}
\end{aligned}
$$

Therefore

$$
x_{p,2}(t) = \frac{-\beta}{9m+k} e^{-3t} \tag{4B}
$$

Now that the particular solutions are known (4A,4B), they are added to the homogeneous solution (which is known) and the complete solution for (1) is

$$
\begin{aligned}
x(t) &= \overbrace{A\cos\omega_n t + B\sin\omega_n t}^{x_h(t)} + \overbrace{x_{p,1}(t) + x_{p,2}(t)}^{x_p(t)} \\
&= A\cos\omega_n t + B\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m+k} e^{-3t} \qquad (5)
\end{aligned}
$$

Initial conditions are now applied to determine A, B. Since $x(0) = 0$ the above becomes

$$
\begin{aligned}
0 &= A + \frac{\beta}{k} - \frac{\beta}{9m+k} \\
A &= \frac{\beta}{9m+k} - \frac{\beta}{k}
\end{aligned}
$$

The solution (5) becomes

$$
x(t) = \left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\cos\omega_n t + B\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m+k} e^{-3t} \tag{6}
$$

Taking derivative of the above

$$
\dot{x}(t) = -\omega_n \left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\sin\omega_n t + \omega_n B\cos\omega_n t + 3\frac{\beta}{9m+k} e^{-3t}
$$

Since $\dot{x}(0) = 0$ then

$$
\begin{aligned}
0 &= \omega_n B + 3\frac{\beta}{9m+k} \\
B &= \frac{-3\beta}{(9m+k)\,\omega_n}
\end{aligned}
$$

Substituting this in (6) gives the final solution

$$
x(t) = \left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\cos\omega_n t - \frac{3\beta}{(9m+k)\,\omega_n}\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m+k} e^{-3t} \tag{7}
$$

Since

$$
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10
$$

And

$$
\begin{aligned}
\beta &= \left(50\times 10^3\right) A \\
&= \left(50\times 10^3\right) \pi \left(\frac{0.1}{2}\right)^2 \\
&= 392.70
\end{aligned}
$$

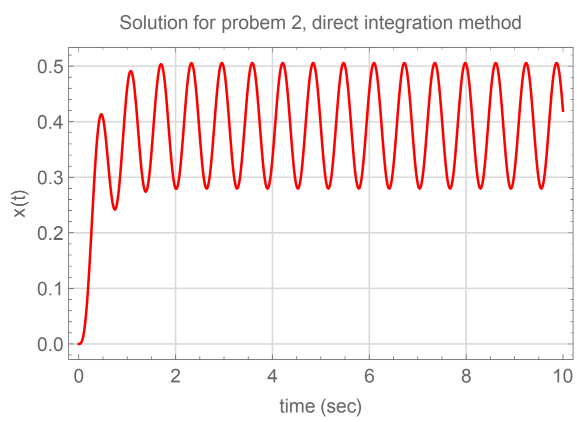
Then numerically, the solution (7) is

$$
\begin{aligned}
x(t) &= \left(\frac{392.70}{90+1000} - \frac{392.70}{1000}\right)\cos 10t - \frac{3\,(392.70)}{(90+1000)\,10}\sin 10t + \frac{392.70}{1000} - \frac{392.70}{90+1000} e^{-3t} \\
&= -0.032\cos 10t - 0.108\sin 10t + 0.393 - 0.360 e^{-3t}
\end{aligned}
$$

Below is a plot of the above to illustrate the solution for some arbitrary time t.

```
In[86]:= d = 0.1;
m = 10;
k = 1000;
wn = Sqrt[k / m];
A0 = Pi (d / 2) ^2;
beta = 50000 * Pi * (d / 2) ^2
Out[91]= 392.699

In[94]:= x[t_] := (beta/(9 m + k) - beta/k) Cos[wn t] - (3 beta)/((9 m + k) wn) Sin[wn t] + (beta/k - beta/(9 m + k)) Exp[-3 t];
Plot[x[t], {t, 0, 10}, Frame → True,
 FrameLabel → {{"x(t)", None}, {"time (sec)", "Solution for probem 2, direct integration method"}},
 GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, BaseStyle → 12]
```

Out[95]=



2.9.2 Problem 2

Problem 3

Set up both integrals (both options) for solving for the response of the piston by using Duhamel's integral. You do NOT need to complete either of the integrations.

The force on the piston is

$$F(t) = Ap(t)$$

Where A is the area of the piston which is $A = \pi\left(\frac{d}{2}\right)^2$. Since this is undamped system, the equation of motion is

$$m\ddot{x} + kx = F(t)$$

To solve using Duhamel integration, the impulse response $g(t) = \frac{1}{m\omega_n}\sin(\omega_n t)$ is used. The integration is done using the two options.

2.9.2.1 Option 1

$$\begin{aligned} x_{conv}(t) &= \int_0^t F(\tau)\, g(t-\tau)\, d\tau \\ &= \frac{A}{m\omega_n}\int_0^t p(t)\sin(\omega_n(t-\tau))\, d\tau \\ &= \frac{A}{m\omega_n}\int_0^t 50(1000)\left(1-e^{-3\tau}\right)\sin(\omega_n(t-\tau))\, d\tau \end{aligned}$$

Where $50(1000)$ is used since the units are in kPa. The above becomes

$$\begin{aligned} x_{conv}(t) &= \left(5\times 10^4\right)\frac{A}{m\omega_n}\int_0^t \left(1-e^{-3\tau}\right)\sin(\omega_n(t-\tau))\, d\tau \\ &= \left(5\times 10^4\right)\frac{A}{m\omega_n}\left(\int_0^t \sin(\omega_n(t-\tau))\, d\tau - \int_0^t e^{-3\tau}\sin(\omega_n(t-\tau))\, d\tau\right) \qquad (1) \end{aligned}$$

The first integral in (1) becomes

$$\begin{aligned}
\int_0^t \sin\left(\omega_n (t-\tau)\right) d\tau &= -\left(\frac{\cos\left(\omega_n (t-\tau)\right)}{-\omega_n}\right)_0^t \\
&= \frac{1}{\omega_n}\left(\cos\left(\omega_n (t-\tau)\right)\right)_0^t \\
&= \frac{1}{\omega_n}\left(\cos\left(\omega_n (t-t)\right) - \cos\left(\omega_n t\right)\right) \\
&= \frac{1}{\omega_n}\left(1 - \cos\left(\omega_n t\right)\right) \qquad (2)
\end{aligned}$$

The second integral in (1) is found using the handout integration tables

$$\int e^{ax}\sin(b+cx)\,dx = \frac{ae^{ax}\sin(b+cx)}{a^2+c^2} - \frac{ce^{ax}\cos(b+cx)}{a^2+c^2}$$

In this case $a=-3$ and $b=\omega_n t$ and $c=-\omega_n$. The above becomes after substitution

$$\begin{aligned}
\int_0^t e^{-3\tau}\sin\left(\omega_n (t-\tau)\right) d\tau &= \left(\frac{-3e^{-3\tau}\sin\left(\omega_n (t-\tau)\right)}{9+\omega_n^2} - \frac{-\omega_n e^{-3\tau}\cos\left(\omega_n (t-\tau)\right)}{9+\omega_n^2}\right)_0^t \\
&= \frac{1}{9+\omega_n^2}\left(-3e^{-3\tau}\sin\left(\omega_n (t-\tau)\right) + \omega_n e^{-3\tau}\cos\left(\omega_n (t-\tau)\right)\right)_0^t \\
&= \frac{1}{9+\omega_n^2}\left(\omega_n e^{-3t} - \left(-3\sin\left(\omega_n t\right) + \omega_n\cos\left(\omega_n t\right)\right)\right) \\
&= \frac{\omega_n e^{-3t} + 3\sin\left(\omega_n t\right) - \omega_n\cos\left(\omega_n t\right)}{9+\omega_n^2} \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives the final result

$$x_{conv}(t) = \left(5\times 10^4\right)\frac{A}{m\omega_n}\left(\frac{1}{\omega_n}\left(1-\cos\left(\omega_n t\right)\right) - \frac{\omega_n e^{-3t} + 3\sin\left(\omega_n t\right) - \omega_n\cos\left(\omega_n t\right)}{9+\omega_n^2}\right) \qquad (4)$$

Because initial conditions are zero the solution is

$$\begin{aligned}
x(t) &= x_h(t) + x_{cov(t)} \\
&= x_{cov(t)}
\end{aligned}$$

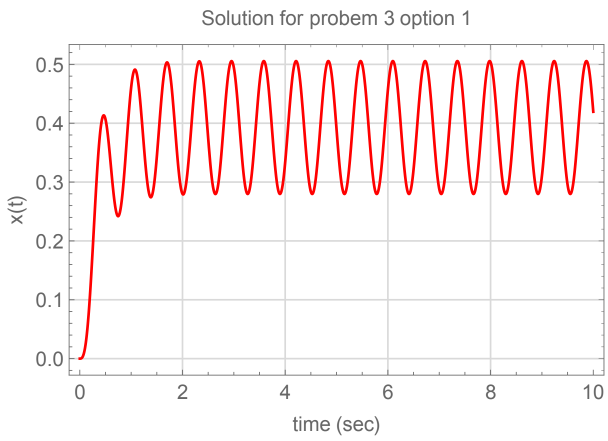
Substituting all the numerical values, and since $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ then (4) becomes

$$\begin{aligned}
x(t) &= \left(5\times 10^4\right)\frac{\pi\left(\frac{0.1}{2}\right)^2}{(10)(10)}\left(\frac{1}{10}\left(1-\cos(10t)\right) - \frac{10e^{-3t} + 3\sin(10t) - 10\cos(10t)}{109}\right) \\
&= 3.927\left(\frac{1}{10}\left(1-\cos(10t)\right) - \frac{10e^{-3t} + 3\sin(10t) - 10\cos(10t)}{109}\right) \\
&= 3.927\left(\frac{1}{10}\left(1-\cos(10t)\right) + \frac{10}{109}\cos(10t) - \frac{10}{109}e^{-3t} - \frac{3}{109}\sin(10t)\right) \\
&= 3.927\left(\frac{1}{10} - \frac{10}{109}e^{-3t} - \frac{3}{109}\sin(10t) - \frac{9}{1090}\cos(10t)\right)
\end{aligned}$$

This is a plot of the above, which agrees with plot from the direct integration method. This verifies the above result

```
In[102]:= d = 0.1;
m = 10;
k = 1000;
wn = Sqrt[k / m];
A0 = Pi (d / 2) ^2;

In[108]:= xconv[t_] := (50 * 1000 A0)/(m wn) (1/wn (1 - Cos[wn t]) - 1/(9 + wn^2) (wn Exp[-3 t] + 3 Sin[wn t] - wn Cos[wn t]))

In[110]:= Plot[xconv[t], {t, 0, 10}, Frame → True,
  FrameLabel → {{"x(t)", None}, {"time (sec)", "Solution for probem 3 option 1"}},
  GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, BaseStyle → 12]
```

Out[110]=



2.9.2.2 Option 2

$$\begin{aligned} x_{conv}(t) &= \int_0^t F(t-\tau)\, g(\tau)\, d\tau \\ &= \frac{A}{m\omega_n} \int_0^t p(t-\tau) \sin(\omega_n \tau)\, d\tau \\ &= \frac{A}{m\omega_n} \int_0^t 50\,(1000) \left(1 - e^{-3(t-\tau)}\right) \sin(\omega_n \tau)\, d\tau \end{aligned}$$

Where $50\,(1000)$ is used, since the units are in kPa. The above becomes

$$\begin{aligned} x_{conv}(t) &= \left(5 \times 10^4\right) \frac{A}{m\omega_n} \int_0^t \left(1 - e^{-3(t-\tau)}\right) \sin(\omega_n \tau)\, d\tau \\ &= \left(5 \times 10^4\right) \frac{A}{m\omega_n} \left(\int_0^t \sin(\omega_n \tau)\, d\tau - \int_0^t e^{-3(t-\tau)} \sin(\omega_n \tau)\, d\tau \right) \end{aligned} \tag{1}$$

The first integral in (1) is now evaluated

$$\begin{aligned} \int_0^t \sin(\omega_n \tau)\, d\tau &= -\frac{1}{\omega_n} \left(\cos(\omega_n \tau)\right)_0^t \\ &= \frac{-1}{\omega_n} \left(\cos(\omega_n t) - 1\right) \\ &= \frac{1}{\omega_n} \left(1 - \cos(\omega_n t)\right) \end{aligned} \tag{2}$$

The second integral in (1) is

$$\begin{aligned} \int_0^t e^{-3(t-\tau)} \sin(\omega_n \tau)\, d\tau &= \int_0^t e^{-3t+3\tau} \sin(\omega_n \tau)\, d\tau \\ &= \int_0^t e^{-3t} e^{3\tau} \sin(\omega_n \tau)\, d\tau \\ &= e^{-3t} \int_0^t e^{3\tau} \sin(\omega_n \tau)\, d\tau \end{aligned} \tag{3}$$

This integral is found using tables

$$\int e^{ax} \sin(bx)\, dx = \frac{e^{ax} \left(a \sin(bx) - b \cos(bx)\right)}{a^2 + b^2}$$

Where in this case $a = 3$ and $b = \omega_n$ Therefore (3) becomes

$$
\begin{aligned}
e^{-3t}\int_0^t e^{3\tau}\sin(\omega_n\tau)\,d\tau &= e^{-3t}\left(\frac{e^{3\tau}(3\sin(\omega_n\tau)-\omega_n\cos(\omega_n\tau))}{9+\omega_n^2}\right)_0^t \\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3\tau}(3\sin(\omega_n\tau)-\omega_n\cos(\omega_n\tau))\right)_0^t \\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3t}(3\sin(\omega_n t)-\omega_n\cos(\omega_n t))-(-\omega_n)\right) \\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3t}(3\sin(\omega_n t)-\omega_n\cos(\omega_n t))+\omega_n\right) \\
&= \frac{1}{9+\omega_n^2}\left(3\sin(\omega_n t)-\omega_n\cos(\omega_n t)+\omega_n e^{-3t}\right) \qquad (4)
\end{aligned}
$$

Substituting (2,4) into (1) gives the final result

$$
x_{conv}(t) = \left(5\times 10^4\right)\frac{A}{m\omega_n}\left(\frac{1}{\omega_n}(1-\cos(\omega_n t)) - \frac{3\sin(\omega_n t)-\omega_n\cos(\omega_n t)+\omega_n e^{-3t}}{9+\omega_n^2}\right) \qquad (5)
$$

Because initial conditions are zero then

$$
\begin{aligned}
x(t) &= x_h(t) + x_{cov(t)} \\
&= x_{cov(t)}
\end{aligned}
$$

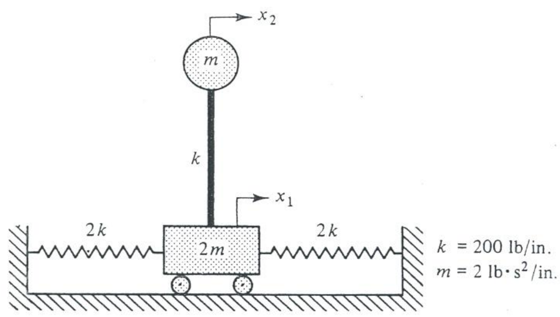
Comparing (5) above to equation (4) found using option (1) shows they are the same as expected.

2.10 HW10

Problem 1. Use Newton's Law to determine the equation of motion. Solve for the natural frequencies and mode shapes without using a computer (solve by hand). Use your hand written solution to write out the 2x2 modal matrix (normalized) and the 2x2 Ω matrix.

Problem 2. Solve for the natural frequencies and mode shapes using Matlab. (Include a screen shot of your Matlab output.)

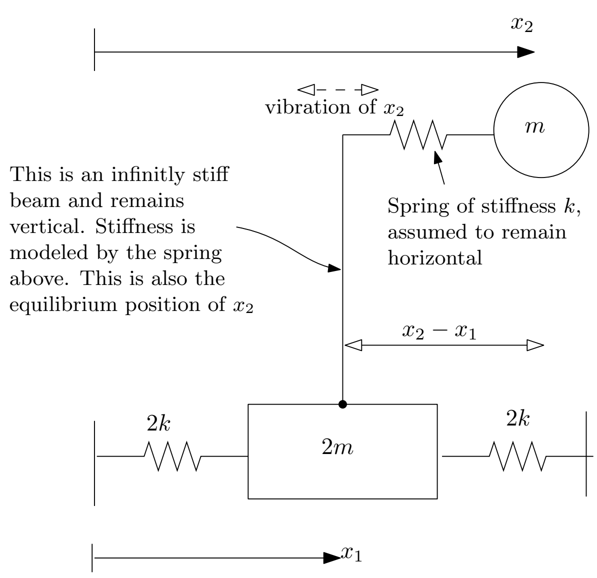
The sphere of mass m is attached to the end of a cantilevered beam that is fixed to a carriage of mass $2m$ as shown in the figure below. The generalized coordinates of the system are the absolute displacements x_1 and x_2 of the carriage and sphere, respectively. Determine (a) the mass and stiffness matrices of the system, and (b) the system's natural circular frequencies and modal matrix $[u]$ if k = 200 lb / in. and m = 2 lb·s^2 / in.



Partial answer: ω_2 = 16.68 rad/s

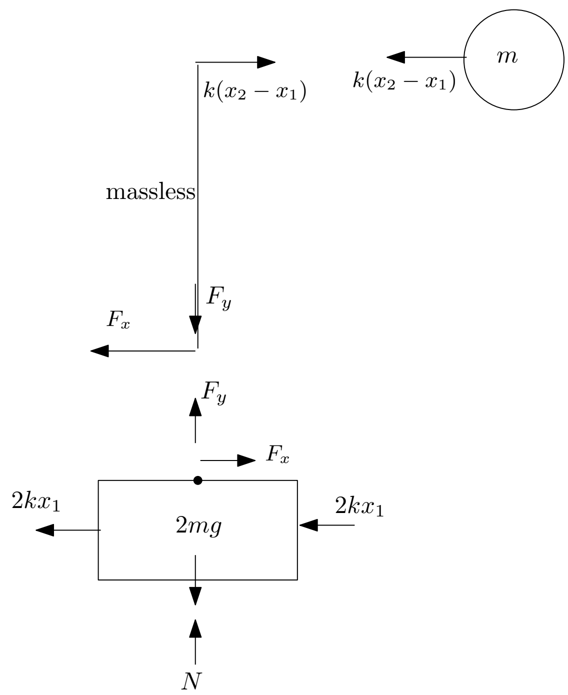
2.10.1 Problem 1

To make it easier to obtain the equation of motions, the top mass m is modeled as attached to spring of stiffness k which is in turn attached to an infinitely stiff vertical massless beam. This way the vibration of the mass m at the top can be more easily modeled.



Simplified model of the original system

Based on the above diagram, we now obtain the free body diagram as follows. In this, we assume that $x_2 > x_1$ and both as positive. Hence spring k attached to m is in tension.



The top mass m vibrates in horizontal direction only. Hence this assumes the spring will remain horizontal and we must assume that $x_2 - x_1$ remain small for this model to be realistic.

From this free body diagram we see now that the reaction force F_x is equal to $k(x_2 - x_1)$. (By resolving forces in the x direction for the massless beam).

Therefore

$$F_x = k(x_2 - x_1)$$

And the equation of motion for x_2 is

$$\begin{aligned} m\ddot{x}_2 &= -k(x_2 - x_1) \\ m\ddot{x}_2 + kx_2 - kx_1 &= 0 \end{aligned} \tag{1}$$

The equation of motion for the cart is

$$\begin{aligned} 2m\ddot{x}_1 &= -4kx_1 + F_x \\ 2m\ddot{x}_1 &= -4kx_1 + k(x_2 - x_1) \\ 2m\ddot{x}_1 + 5kx_1 - kx_2 &= 0 \end{aligned} \tag{2}$$

Writing (1) and (2) in matrix form

$$\boxed{\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 5k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Or

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1000 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The first step is to find the eigenvalues (which are the square of the natural frequency) for the system.

Let

$$\begin{aligned} A &= M^{-1}K \\ &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1000 & -200 \\ -200 & 200 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned}\begin{bmatrix}4 & 0\\0 & 2\end{bmatrix}^{-1} &= \frac{1}{\det(M)}\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}\\ &= \frac{1}{8}\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{4} & 0\\0 & \frac{1}{2}\end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}A &= \begin{bmatrix}\frac{1}{4} & 0\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1000 & -200\\-200 & 200\end{bmatrix}\\ &= \begin{bmatrix}250 & -50\\-100 & 100\end{bmatrix}\end{aligned}$$

Now we will find the eigenvalues of A (these will be the ω_n^2 values). To find the eigenvalues of A, we solve

$$\begin{aligned}\det([A] - \lambda[I]) &= 0\\ \det\left(\begin{bmatrix}250 & -50\\-100 & 100\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) &=\\ \begin{vmatrix}250-\lambda & -50\\-100 & 100-\lambda\end{vmatrix} &=\\ (250-\lambda)(100-\lambda) - 5000 &= 0\\ \lambda^2 - 350\lambda + 20\,000 &= 0\end{aligned}$$

Hence

$$\begin{aligned}\lambda &= \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}\\ &= \frac{350}{2} \pm \frac{\sqrt{350^2 - 4(20\,000)}}{2}\\ &= 175 \pm 103.08\\ &= \{71.92, 278.08\}\end{aligned}$$

Therefore, the eigenvalues are

$$\lambda = \omega_n^2 = \{71.92, 278.08\} \tag{3}$$

The natural frequencies of the system are the sqrt of the eigenvalues. Therefore

$$\begin{aligned}\omega_n &= \left\{\sqrt{71.92}, \sqrt{278.08}\right\}\\ &= \{8.4806, 16.676\}\end{aligned}$$

Hence

$$\begin{aligned}\omega_{n(1)} &= 8.4806 \text{ rad/sec}\\ \omega_{n(2)} &= 16.676 \text{ rad/sec}\end{aligned}$$

The next step is to find the eigenvectors. These are also called the shape vectors, or the u vectors. Each eigenvalue will generate one eigenvector. We need to solve

$$[A]\{u\} = \lambda\{u\}$$

For each eigenvalue, we find the corresponding eigenvector.

<u>For $\lambda = 71.92$,</u> we obtain the equation

$$\begin{bmatrix}250 & -50\\-100 & 100\end{bmatrix}\begin{Bmatrix}u_{11}\\u_{21}\end{Bmatrix} = 71.92\begin{Bmatrix}u_{11}\\u_{21}\end{Bmatrix}$$

From first equation

$$250u_{11} - 50u_{21} = 71.92u_{11}$$

We always let $u_{11} = 1$. Therefore

$$\begin{aligned} 250 - 50u_{21} &= 71.92 \\ u_{21} &= \frac{250 - 71.92}{50} \\ &= 3.5616 \end{aligned}$$

Therefore, the first eigenvector is

$$\vec{u}_1 = \begin{Bmatrix} 1 \\ 3.5616 \end{Bmatrix}$$

For $\lambda = 278.08$, we obtain the equation

$$\begin{bmatrix} 250 & -50 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = 278.08 \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix}$$

From first equation

$$250u_{12} - 50u_{22} = 278.08u_{12}$$

We always let $u_{12} = 1$. Hence

$$\begin{aligned} 250 - 50u_{22} &= 278.08 \\ u_{22} &= \frac{250 - 278.08}{50} \\ &= -0.561\,6 \end{aligned}$$

Therefore, the second eigenvector is

$$\vec{u}_2 = \begin{Bmatrix} 1 \\ -0.561\,6 \end{Bmatrix}$$

Therefore the modal matrix $[u]$ is

$$\boxed{u = \begin{bmatrix} 1 & 1 \\ 3.5616 & -0.5616 \end{bmatrix}}$$

And Ω matrix is

$$\begin{aligned} \Omega &= \begin{bmatrix} \omega_{n(1)}^2 & 0 \\ 0 & \omega_{n(2)}^2 \end{bmatrix} \\ &= \begin{bmatrix} 71.92 & 0 \\ 0 & 278.08 \end{bmatrix} \end{aligned}$$

And the system of equations written in principle coordinates q is

$$\{\ddot{q}\} + [\Omega]\{q\} = \{0\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} 71.92 & 0 \\ 0 & 278.08 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which is now decoupled. The solution in normal coordinates is

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= A_1 \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} \cos\left(\omega_{n(1)}t - \phi_1\right) + A_2 \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} \cos\left(\omega_{n(2)}t - \phi_2\right) \\ &= A_1 \begin{Bmatrix} 1 \\ 3.5616 \end{Bmatrix} \cos\left(8.481t - \phi_1\right) + A_2 \begin{Bmatrix} 1 \\ -0.561\,6 \end{Bmatrix} \cos\left(16.676t - \phi_2\right) \end{aligned}$$

2.10.1.1 Appendix

This is derivation of the same equations of motions using energy method. (In this example, this method is much simpler to use to find equation of motions). The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}(2m)\dot{x}_1^2$$

And the potential energy comes only from the springs, since we assumed the top mass m remain horizontal as it vibrates back and forth

$$U = \frac{1}{2}4kx_1^2 + \frac{1}{2}k\left(x_2 - x_1\right)^2$$

Therefore the Lagrangian is

$$\begin{aligned}\Gamma &= T - U \\ &= \frac{1}{2}m\dot{x}_2^2 + m\dot{x}_1^2 - \frac{1}{2}\left(4k\right)x_1^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2\end{aligned}$$

<u>EQM for x_1</u>

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\Gamma}{\dot{x}_1}\right) - \frac{\partial\Gamma}{x_1} &= 0 \\ \frac{d}{dt}\left(2m\dot{x}_1\right) - \left(-4kx_1 + k\left(x_2 - x_1\right)\right) &= 0 \\ 2m\ddot{x}_1 - \left(-4kx_1 + kx_2 - kx_1\right) &= 0 \\ 2m\ddot{x}_1 - \left(-5kx_1 + kx_2\right) &= 0 \\ 2m\ddot{x}_1 + 5kx_1 - kx_2 &= 0 \qquad (1)\end{aligned}$$

<u>EQM for x_2</u>

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\Gamma}{\dot{x}_2}\right) - \frac{\partial\Gamma}{x_2} &= 0 \\ \frac{d}{dt}\left(m\dot{x}_2\right) - \left(-k\left(x_2 - x_1\right)\right) &= 0 \\ m\ddot{x}_2 - \left(-kx_2 + kx_1\right) &= 0 \\ m\ddot{x}_2 + kx_2 - kx_1 &= 0 \qquad (2)\end{aligned}$$

In Matrix form (1,2) becomes

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 5k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Which is the same exact result obtained earlier.

2.10.2 Problem 2

The Matlab code is the following

```
%Solve HW 10, problem 2 using Matlab
%Nasser M. Abbasi, ME 440, Fall 2017
%see HW 10 for more details.

m = 2;
k = 200;

mass_mat = [2*m 0;
            0   m]

stiffness_mat = [5*k -k;
                 -k   k]

A_mat = inv(mass_mat) * stiffness_mat

[eig_vectors, eig_values] = eig(A_mat);

natural_frequencies = sqrt(diag( eig_values))

eig_vectors(:,1) = eig_vectors(:,1)/eig_vectors(1,1);
eig_vectors(:,2) = eig_vectors(:,2)/eig_vectors(1,2);

eig_vectors
```

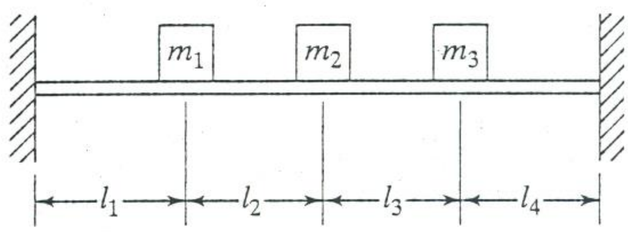
The output is

```

mass_mat =
     4     0
     0     2

stiffness_mat =
        1000        -200
        -200         200

A_mat =
   250   -50
  -100   100

natural_frequencies =
   16.6757
    8.4807

eig_vectors =
    1.0000    1.0000
   -0.5616    3.5616
```

2.10.3 Problem 3

Problem 3.

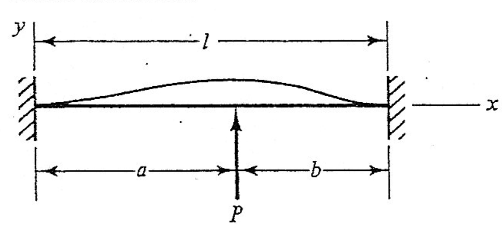
Determine the flexibility matrix of the uniform beam shown in the figure below. Disregard the mass of the beam compared to the concentrated masses fastened on the beam and assume the beam has a stiffness of EI and that all $l_i = l$.



Definitions For stiffness matrix $[K]$, element k_{ij} means: Apply unit displacement at location j and measure the force at location i. While for flexibility matrix $[a]$, its element a_{ij} means: Apply unit force at location j and measure the displacement at location i.

To solve this problem, this part of handout is used

Fixed-fixed beam*



(i)

$$y = \frac{Pb^2}{6EIl^3}\left[(2b-3l)x^3 + 3l(l-b)x^2\right] \qquad (x \le a)$$

$$y = \frac{Pb^2}{6EIl^3}\left[(2b-3l)x^3 + 3l(l-b)x^2 + \frac{l^3}{b^2}(x-a)^3\right] \qquad (x \ge a)$$

Since $[a]$ is symmetric, only lower triangle part needs to be found (or upper triangle).

$$\begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

To find a_{11}, a unit force is put at location m_1 and displacement at m_1 is measured. To find a_{21}, a unit force is put at location m_1 and displacement at m_2 is measured and so on. The formulas in the above hand out are used for this. To speed this process and make less error, a small function is written to do the computation. Here is the function and the result generated for $a_{11}, a_{21}, a_{32}, a_{22}, a_{32}, a_{33}$

Define the function to find a_ij

```
getFlexibility[x_, a_, b_] := Piecewise[{
    {b^2/(6 E0 I0 L0^3) ((2 b - 3 L0) x^3 + 3 L0 (L0 - b) x^2), x ≤ a},
    {b^2/(6 E0 I0 L0^3) ((2 b - 3 L0) x^3 + 3 L0 (L0 - b) x^2 + L0^3/b^2 (x - a)^3), x > a}}];
```

Call the function to find each element in lower triangle

In[43]:=
```
L0 = 4 L;
a = L; b = 3 L; x = L;
flex[1, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[45]= $\dfrac{9\,L^3}{64\,E0\,I0}$

In[48]:=
```
a = L; b = 3 L; x = 2 L;
flex[2, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[49]= $\dfrac{L^3}{6\,E0\,I0}$

In[50]:=
```
a = L; b = 3 L; x = 3 L;
flex[3, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[51]= $\dfrac{13\,L^3}{192\,E0\,I0}$

In[52]:=
```
a = 2 L; b = 2 L; x = 2 L;
flex[2, 2] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[53]= $\dfrac{L^3}{3\,E0\,I0}$

In[54]:=
```
a = 2 L; b = 2 L; x = 3 L;
flex[3, 2] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[55]= $\dfrac{L^3}{6\,E0\,I0}$

In[56]:=
```
a = 3 L; b = L; x = 3 L;
flex[3, 3] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[57]= $\dfrac{9\,L^3}{64\,E0\,I0}$

Therefore, using this result, the lower triangle is

$$\begin{bmatrix} \frac{9}{64} & & \\ \frac{1}{6} & \frac{1}{3} & \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix} \frac{L^3}{EI}$$

Hence by symmetry

$$[a] = \begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix} \frac{L^3}{EI}$$

Chapter 3

Study notes

Local contents

3.1 Solve slide 412

Reproduce flexibility matrix, slide 412

In[23]:=
```
EI = 86 * 10^6;
L0 = 120;
y[x_, a_, b_, L0_, pLocation_] := If[pLocation ≤ L0,
   Which[
     x ≤ a, 1/(12 EI) (3 b (1 - b^2/L0^2) x^2 - b/L0 (3 - b^2/L0^2) x^3),
     x ≥ L0, (-b a^2)/(4 EI L0) (x - L0)
   ],
   Which[
     x ≤ L0, a/(4 EI L0) (x^3 - L0 x^2),
     x ≥ L0, a/(4 EI L0) (x^3 - L0 x^2 - ((2 L0)/(3 a) + 1) (x - L0)^3)

   ]
  ];


L0 = 120; a = L0/2; b = L0/2; x = L0/2; pLocation = L0/2;
a11 = y[x, a, b, L0, pLocation];

L0 = 120; a = L0/2; b = L0/2; x = L0 + L0/2; pLocation = L0/2;
a21 = y[x, a, b, L0, pLocation];

L0 = 120; a = L0/2; x = L0/2; pLocation = L0/2 + L0;
a12 = y[x, a, b, L0, pLocation];

L0 = 120; a = L0/2; x = L0 + L0/2; pLocation = L0/2 + L0;
a22 = y[x, a, b, L0, pLocation];

a = {{a11, a12}, {a21, a22}};
MatrixForm[N[a]]
```

Out[67]//MatrixForm=

$$\begin{pmatrix} 0.00018314 & -0.000313953 \\ -0.000313953 & 0.00209302 \end{pmatrix}$$

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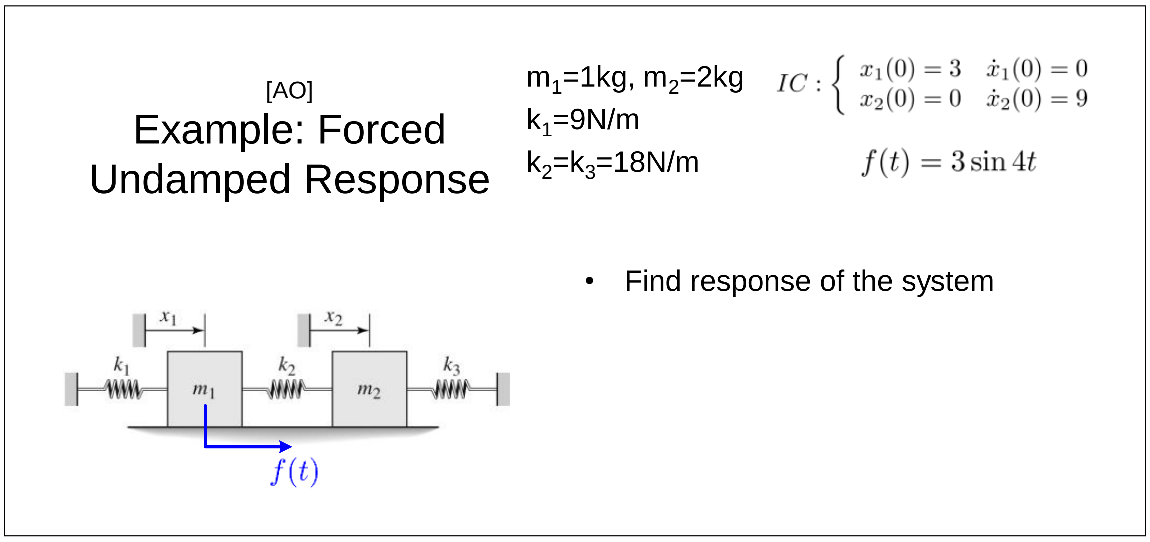
3.2 Solving slide 390 example



By inspection

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} = \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix}$$

And

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The system is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{1}$$

The above is solved using modal analysis in order to decouple the system. The first step is to determine the eigenvalues.

$$\begin{aligned} [A] &= [m]^{-1}[k] \\ &= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \\ &= \begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix} \end{aligned}$$

To find the eigenvalues of $[A]$ we solve $|A - \lambda I| = 0$ or

$$\begin{vmatrix} 27-\lambda & -18 \\ -9 & 18-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 45\lambda + 324 = 0$$

Hence

$$\lambda_1 = 9$$
$$\lambda_2 = 36$$

Which implies

$$\omega_{n(1)} = 3 \text{ rad/s}$$
$$\omega_{n(2)} = 9 \text{ rad/s}$$

Now we find the eigenvectors u_i or the shape vectors. For $\underline{\lambda_1 = 9}$

$$[A]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 9\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27u_1 - 18u_2 \\ -9u_1 + 18u_2 \end{bmatrix} = \begin{Bmatrix} 9u_1 \\ 9u_2 \end{Bmatrix}$$

Using first equation only gives

$$27u_1 - 18u_2 = 9u_1$$

We always normalized to $u_1 = 1$, hence the above gives

$$\begin{aligned} 27 - 18u_2 &= 9 \\ u_2 &= 1 \end{aligned}$$

Therefore the first eigenvector is

$$\vec{u}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

To find the second eigenvector. For $\underline{\lambda_2 = 36}$

$$[A]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda_2 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 36\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27u_1 - 18u_2 \\ -9u_1 + 18u_2 \end{bmatrix} = \begin{Bmatrix} 36u_1 \\ 35u_2 \end{Bmatrix}$$

Using first equation only gives

$$27u_1 - 18u_2 = 36u_1$$

We always normalized to $u_1 = 1$, hence the above gives

$$\begin{aligned} 27 - 18u_2 &= 36 \\ u_2 &= -\frac{1}{2} \end{aligned}$$

Therefore the second eigenvector is

$$\vec{u}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

Hence the <u>modal matrix</u> is

$$[u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Using the modal matrix, we can now decouple the original system given above in (1) which is

$$[m]\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + [k]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{2}$$

Let $\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [u]\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$, then the above becomes

$$[m][u]\begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [k][u]\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix}$$

Premultiplying both sides by $[u]^T$ gives

$$[u]^T [m] [u] \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [u]^T [k] [u] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [u]^T \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{4}$$

But

$$[u]^T [m] [u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

And

$$[u]^T [k] [u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix}$$

Then (4) becomes

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 3\sin 4t \end{Bmatrix}$$

Hence we obtain 2 ODEs

$$\begin{aligned} 3\ddot{q}_1(t) + 27q_1(t) &= 3\sin 4t \\ \frac{3}{2}\ddot{q}_2(t) + 54q_2(t) &= 3\sin 4t \end{aligned}$$

Or

$$\ddot{q}_1(t) + 9q_1(t) = \sin 4t \tag{5}$$

$$\ddot{q}_2(t) + 36q_2(t) = 2\sin 4t \tag{6}$$

Note There is a short cut to obtain the above (5,6) equations directly as follows. Starting with (2), we just write

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} \omega_{n(1)}^2 & 0 \\ 0 & \omega_{n(2)}^2 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [u]^{-1} [m]^{-1} \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix}$$

$$\begin{aligned} \begin{Bmatrix} \ddot{q}_1(t) + 9q_1(t) \\ \ddot{q}_2(t) + +36q_2(t) \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} \sin 4t \\ 2\sin 4t \end{Bmatrix} \end{aligned}$$

Which is the same as (5,6). This short cut just needs finding $[u]^{-1} [m]^{-1}$. Use this short cut for the exam.

Solving (5)

The homogeneous solution is

$$q_{1,h}(t) = A_1 \cos 3t + B_1 \sin 3t$$

And to find the particular solution, we guess $q_{1,p} = C\sin 4t$, hence $\dot{q}_{1,p} = 4C\cos 4t$ and $\ddot{q}_{1,p} = -16C\sin 4t$. Plug-in in (5) gives

$$\begin{aligned} -16C\sin 4t + 9(C\sin 4t) &= \sin 4t \\ -7C_1 \sin 4t &= \sin 4t \\ C_1 &= -\frac{1}{7} \end{aligned}$$

Hence $q_{1,p} = -\frac{1}{7}\sin 4t$ and the complete solution is

$$q_1(t) = A_1 \cos 3t + B_1 \sin 3t - \frac{1}{7}\sin 4t$$

Now we do the same to solve (6)

The homogeneous solution is

$$q_{2,h}(t) = A_2 \cos 6t + B_2 \sin 6t$$

And to find the particular solution, we guess $q_{2,p} = C\sin 4t$, hence $\dot{q}_{2,p} = 4C\cos 4t$ and

$\ddot{q}_{2,p} = -16C \sin 4t$. Plug-in in (6) gives

$$-16C \sin 4t + 36 (C \sin 4t) = 2 \sin 4t$$
$$20C_1 \sin 4t = 2 \sin 4t$$
$$C_1 = \frac{1}{10}$$

Hence $q_{2,p} = \frac{1}{10} \sin 4t$ and the complete solution is

$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t$$

Therefore the solution in principle coordinates is

$$q_1(t) = A_1 \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t \tag{5A}$$
$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t \tag{6A}$$

Since $\{x\} = [u]\{q\}$, then $\{q\} = [u]^{-1}\{x\}$. Therefore

$$\{q(0)\} = [u]^{-1}\{x(0)\}$$
$$\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$
$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

And

$$\{\dot{q}(0)\} = [u]^{-1}\{\dot{x}(0)\}$$
$$\begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{Bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}$$
$$= \begin{Bmatrix} 6 \\ -6 \end{Bmatrix}$$

Applying first initial conditions to (5A,6A) gives

$$1 = A_1$$
$$2 = A_2$$

Hence (5A,6A) becomes

$$q_1(t) = \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t \tag{5B}$$
$$q_2(t) = 2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t \tag{6B}$$

Taking derivatives

$$\dot{q}_1(t) = -3 \sin 3t + 3B_1 \cos 3t - \frac{4}{7} \cos 4t$$
$$\dot{q}_2(t) = -12 \sin 6t + 6B_2 \cos 6t + \frac{4}{10} \cos 4t$$

Applying the second initial conditions to the above gives

$$6 = 3B_1 - \frac{4}{7}$$
$$-6 = 6B_2 + \frac{4}{10}$$

Solving gives $B_1 = \frac{46}{21}, B_2 = -\frac{16}{15}$. Hence (5B,6B) become

$$q_1(t) = \cos 3t + \frac{46}{21}\sin 3t - \frac{1}{7}\sin 4t \tag{5C}$$

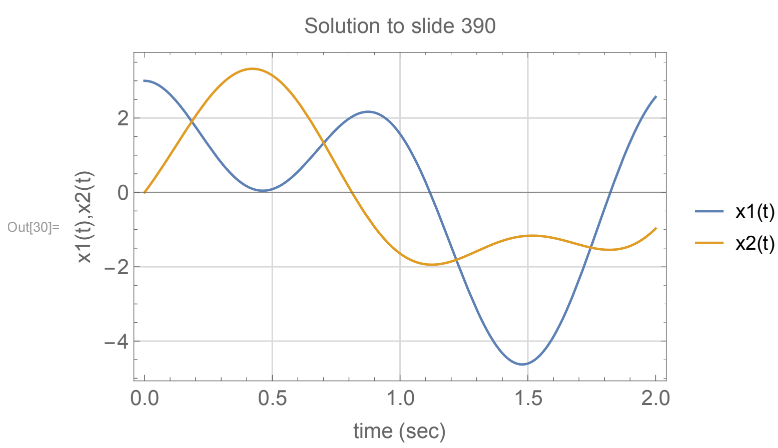
$$q_2(t) = 2\cos 6t - \frac{16}{15}\sin 6t + \frac{1}{10}\sin 4t \tag{6C}$$

The above is the solution in principle coordinates. Now we transform it back to normal coordinates. Since $\{x\} = [u]\left\{q\right\}$, then

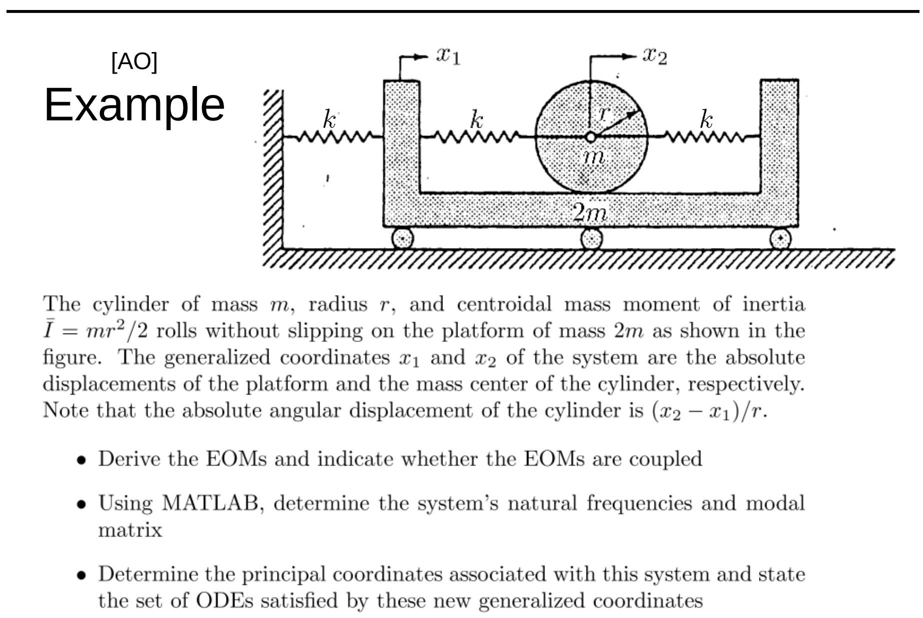
$$\begin{aligned}
\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \cos(3t) + \left(\frac{46}{21}\right)\sin(3t) - \left(\frac{1}{7}\right)\sin(4t) \\ 2\cos(6t) - \left(\frac{16}{15}\right)\sin(6t) + \frac{1}{10}\sin 4t \end{Bmatrix} \\
&= \begin{Bmatrix} \cos 3t + 2\cos 6t + \frac{46}{21}\sin 3t - \frac{3}{70}\sin 4t - \frac{16}{15}\sin 6t \\ \cos 3t - \cos 6t + \frac{46}{21}\sin 3t - \frac{27}{140}\sin 4t + \frac{8}{15}\sin 6t \end{Bmatrix}
\end{aligned}$$

The above is the final solution. Here is a plot of $x_1(t), x_2(t)$

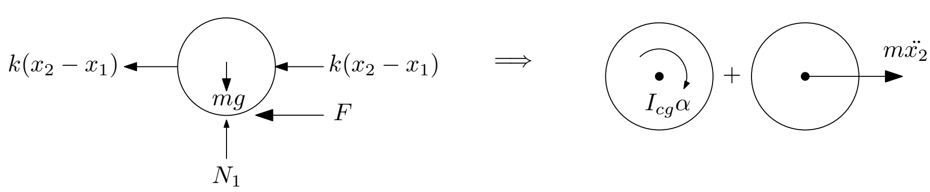
```
mySol = {x1[t] → Cos[3 t] + 2 Cos[6 t] + 46 / 21 Sin[3 t] - 3 / 70 Sin[4 t] - 16 / 15 Sin[6 t],
   x2[t] → Cos[3 t] - Cos[6 t] + 46 / 21 Sin[3 t] - 27 / 140 Sin[4 t] + 8 / 15 Sin[6 t]};

Plot[{ x1[t] /. mySol, x2[t] /. mySol}, {t, 0, 2}, Frame → True,
 GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"x1(t),x2(t)", None}, {"time (sec)", "Solution to slide 390"}},
 PlotLegends → {"x1(t)", "x2(t)"}, ImageSize → 400, BaseStyle → 14]
```

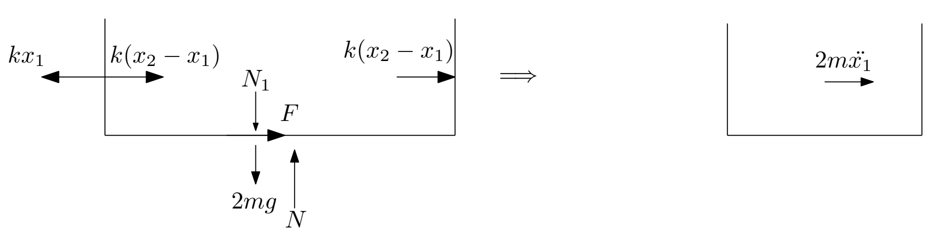


3.3 Solving slide 362 example



Assuming $x_2 > x_1, \dot{x}_2 > \dot{x}_1, \ddot{x}_2 > \ddot{x}_1$ and all are positive, the free body diagram for the cylinder and the cart is





Equation of motion for cylinder. $\sum F_x$

$$-2k(x_2 - x_1) - F = m\ddot{x}_2 \tag{1}$$

And taking moment around C.G. of cylinder, using anti-clock wise as positive

$$-Fr = -I_{cg}\alpha$$
$$Fr = I_{cg}\alpha$$

Since we assumed no slip, then $(\ddot{x}_2 - \ddot{x}_1) = \alpha r$ and the above becomes

$$Fr = I_{cg}\frac{(\ddot{x}_2 - \ddot{x}_1)}{r}$$
$$F = I_{cg}\frac{(\ddot{x}_2 - \ddot{x}_1)}{r^2}$$
$$= \frac{1}{2}mr^2\frac{(\ddot{x}_2 - \ddot{x}_1)}{r^2}$$
$$= \frac{1}{2}m(\ddot{x}_2 - \ddot{x}_1) \tag{2}$$

Using (2) in (1) gives EQM for x_2

$$m\ddot{x}_2 + 2k(x_2 - x_1) + \frac{1}{2}m(\ddot{x}_2 - \ddot{x}_1) = 0$$
$$\frac{3}{2}m\ddot{x}_2 - \frac{1}{2}m\ddot{x}_1 + 2kx_2 - 2kx_1 = 0 \tag{3}$$

For EQM for x_1, resolving forces in x direction gives

$$-kx_1 + 2k(x_2 - x_1) + F = 2m\ddot{x}_1$$

Using F found in (2) into the above gives

$$-kx_1 + 2k(x_2 - x_1) + \frac{1}{2}m(\ddot{x}_2 - \ddot{x}_1) = 2m\ddot{x}_1$$

Simplifying

$$2m\ddot{x}_1 - \frac{1}{2}m(\ddot{x}_2 - \ddot{x}_1) + kx_1 - 2k(x_2 - x_1) = 0$$
$$-\frac{1}{2}m\ddot{x}_2 + \frac{5}{2}m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0 \tag{4}$$

Writing (3,4) in matrix form gives (note. Using (4) for top row and then use (3) for second row)

$$\begin{bmatrix} \frac{5}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{3}{2}m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

If we had picked (3) for top row and then (4) for second row, the result will be

$$\begin{bmatrix} -\frac{1}{2}m & \frac{3}{2}m \\ \frac{5}{2}m & -\frac{1}{2}m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} -2k & 2k \\ 3k & -2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6}$$

Since So (5) and (6) are equivalent. To verify both (5) and (6) give the same eigenvalues, here is a check

```
In[49]:= (* eq 5*)
m = 1;
k = 1;
massMat = {{5/2 m, -1/2 m}, {-1/2 m, 3/2 m}};
kMat = {{3 k, -2 k}, {-2 k, 2 k}};
Amat = Inverse[massMat].kMat;
Sqrt[Eigenvalues[Amat]] // N

Out[54]= {1.353042756497228, 0.5586881437327312}

In[63]:= (* eq 6*)

SetOptions[$FrontEndSession, PrintPrecision → 16]
m = 1;
k = 1;
massMat = {{-1/2 m, 3/2 m}, {5/2 m, -1/2 m}};
kMat = {{-2 k, 2 k}, {3 k, -2 k}};
inv = Inverse[massMat];
Amat = (inv.kMat);
Sqrt[Eigenvalues[Amat]] // N

Out[70]= {1.353042756497228, 0.5586881437327312}
```

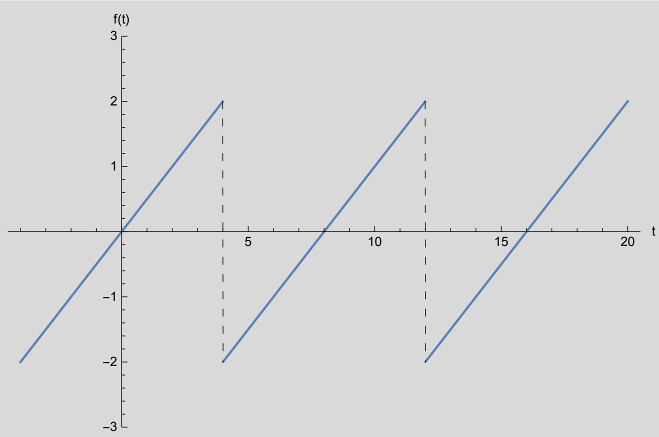
3.4 Solving example 2, lecture 4. ME 440 page 78

Solving example 2, lecture 4. ME 440 page 78

Plot the function

In[1]:=
```
A = 2;
T = 4;
myperiodic[func_, {val_Symbol, min_?NumericQ, max_?NumericQ}] :=
 func /. (val :→ Mod[val - min, max - min] + min)
f[t_] := A / T t;
Plot[myperiodic[f[t], {t, -T, T}] // Evaluate, {t, -T, 5 T},
 PlotRange → {Automatic, {-A - 1, A + 1}}, Exclusions → True,
 ExclusionsStyle → Dashing[Medium], AxesLabel → {"t", "f(t)"}, ImageSize → 450]
```

Out[5]=



Find a0,an,bn

In[6]:=
```
a0 = 1 / T Integrate[f[t], {t, -T, T}]
```

Out[6]=
```
0
```

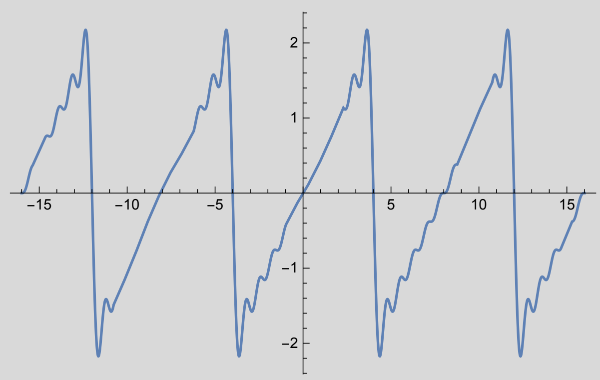
In[7]:=
```
an = 1 / T Integrate[f[t] Cos[2 Pi / (2 T) n t], {t, -T, T}]
```

Out[7]=
```
0
```

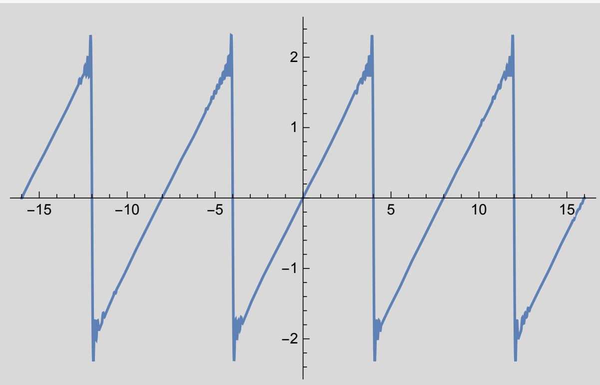
2 | *example_2.nb*

In[8]:=
```
bn = 1/T Integrate[f[t] Sin[2 Pi/(2 T) n t], {t, -T, T}];
b[n_] = Assuming[Element[n, Integers], Simplify[bn]]
```
Out[9]=
$$-\frac{4(-1)^n}{n\pi}$$

Plot approximation for n=10

In[10]:=
```
Plot[Sum[b[n] Sin[2 n Pi/(2 T) t], {n, 1, 10}], {t, -4 T, 4 T}]
```
Out[10]=


Plot approximation for n=50 to improve the approximation.

In[11]:=
```
Plot[Sum[b[n] Sin[2 n Pi/(2 T) t], {n, 1, 50}], {t, -4 T, 4 T}]
```
Out[11]=


Find Xn and Phase, where

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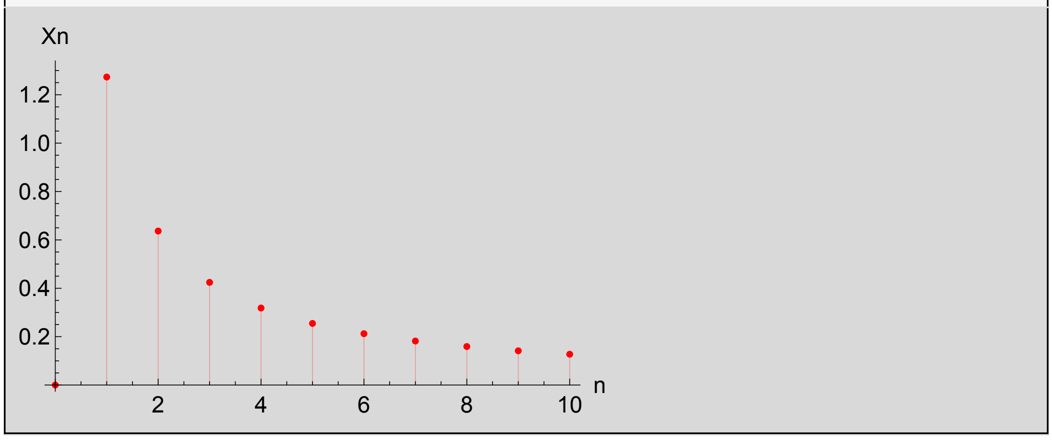
$X_n \cos(\omega_n t - \phi_n) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$

Where $X_n = \sqrt{a_n^2 + b_n^2}$ and $\Phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ and since $a_n = 0$ then phase is 90 degrees and $X_n = |\, b_n \,|$

In[12]:=

```
X[n_] := Abs[b[n]];
data = Join[{{0, 0}}, Table[{n, X[n]}, {n, 1, 10}]];
ListPlot[data, Filling → Axis, PlotStyle → Red, AxesLabel → {"n", "Xn"}, BaseStyle → 14]
```

Out[14]=



Verify using Mathematica build-in function

In[15]:=

```
data = Table[{i, Abs@FourierCoefficient[myperiodic[f[t], {t, -T, T}],
      t, i, FourierParameters → {1, 2 Pi / (2 T)}]}, {i, 0, 10}];
head = {"n", "|cn|"};
Grid[Insert[data, head, 1], Frame → All]
```

Out[17]=

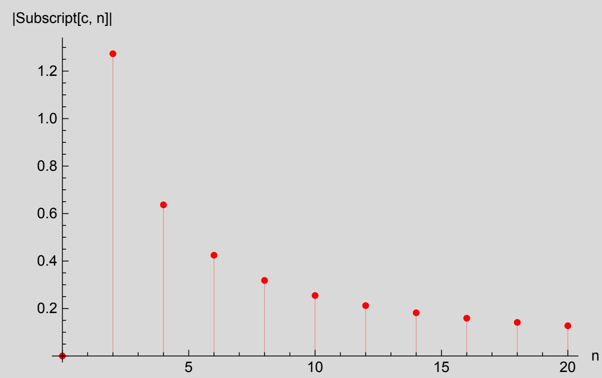
n	|cn|
0	0
1	$\frac{2}{\pi}$
2	$\frac{1}{\pi}$
3	$\frac{2}{3\pi}$
4	$\frac{1}{2\pi}$
5	$\frac{2}{5\pi}$
6	$\frac{1}{3\pi}$
7	$\frac{2}{7\pi}$
8	$\frac{1}{4\pi}$
9	$\frac{2}{9\pi}$
10	$\frac{1}{5\pi}$

4 | *example_2.nb*

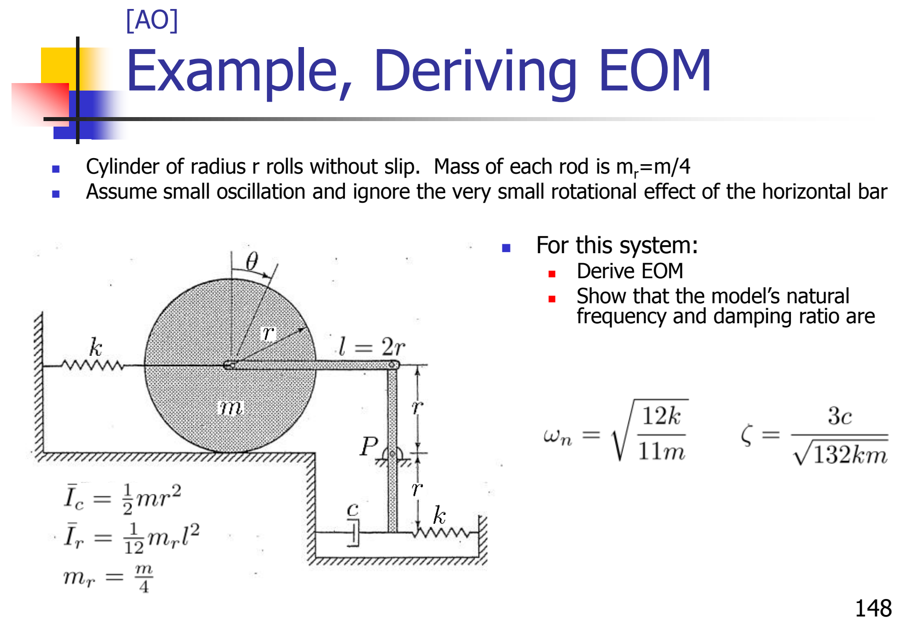
In[18]:=

```
mag = data;
mag[[All, 2]] = Map[Abs[#] &, data[[All, 2]]];
ListPlot[2 * mag, AxesOrigin → {0, 0}, Filling → Axis,
 PlotStyle → Red, AxesLabel → {"n", "|Subscript[c, n]|"}]
```

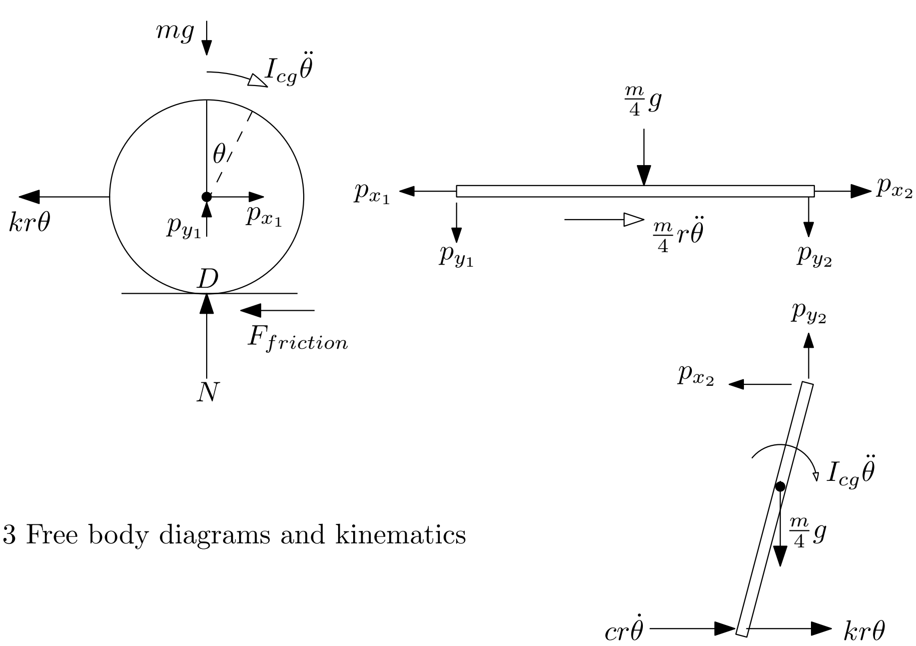
Out[20]=



3.5 Solving slide 148 example, lecture sept 28, 2017



We will solve this using 3 separate bodies. So there are three free body diagrams as shown below



3 Free body diagrams and kinematics

In this diagram, it is assumed the horizontal bar only moves in the x direction and this is all for small angle θ. Now we apply Newton laws to each body.

For disk, we apply $\tau = I_o\ddot{\theta}$ but using the point D on the figure to take moments around in order to get rid of the friction F and N terms. This gives (using counter clock wise as positive)

$$\begin{aligned}
(kr\theta)\, r - p_{x_1} r &= -I_o\ddot{\theta} \\
kr^2\theta - p_{x_1} r &= -\left(I_{cg} + mr^2\right)\ddot{\theta} \\
&= -\left(\frac{1}{2}mr^2 + mr^2\right)\ddot{\theta} \\
&= -\frac{3}{2}mr^2\ddot{\theta} \qquad (1)
\end{aligned}$$

We now move to the second body, which is the horizontal bar.

$$\sum F_x = m_{bar}\ddot{x}$$
$$-p_{x_1} + p_{x_2} = \frac{m}{4}r\ddot{\theta} \tag{2}$$

From (2) we solve for p_{x_1} and plug it into (1)

$$p_{x_1} = p_{x_2} - \frac{m}{4}r\ddot{\theta}$$

Hence (1) now becomes

$$\begin{aligned} kr^2\theta - \left(p_{x_2} - \frac{m}{4}r\ddot{\theta}\right)r &= -\frac{3}{2}mr^2\ddot{\theta} \\ kr^2\theta - p_{x_2}r &= -\left(\frac{3}{2}mr^2 + \frac{m}{4}r^2\right)\ddot{\theta} \\ &= -\frac{7}{4}mr^2\ddot{\theta} \end{aligned} \tag{3}$$

To find p_{x_2}, we use the third body, the vertical bar. Taking moments about C.G. of bar using counter clock wise as positive gives

$$\begin{aligned} \tau &= -I_{cg}\ddot{\theta} \\ (kr\theta)\,r\cos\theta + \left(cr\dot{\theta}\right)r\cos\theta + p_{x_2}r\cos\theta + p_{y_2}r\sin\theta &= -\frac{1}{12}\left(\frac{m}{4}\right)(2r)^2\ddot{\theta} \\ &= -\frac{1}{12}mr^2\ddot{\theta} \end{aligned}$$

For small angle the above becomes

$$kr^2\theta + cr^2\dot{\theta} + p_{x_2}r + p_{y_2}r\theta = -\frac{m}{12}r^2\ddot{\theta} \tag{4}$$

p_{y2} is now found from vertical balance of horizontal bar. Since it does not move vertically and assumed to only move horizontally, then

$$\sum F_y = 0$$
$$-p_{y_1} - p_{y_2} - \frac{m}{4}g = 0$$

Due to symmetry, $p_{y_1} = p_{y2}$ and the above becomes

$$\begin{aligned} -2p_{y_2} &= \frac{m}{4}g \\ p_{y_2} &= -\frac{m}{8}g \end{aligned}$$

Plugging this value for p_{y_2} into (4) and solving for p_{x_2} gives

$$\begin{aligned} kr^2\theta + cr^2\dot{\theta} + p_{x_2}r - \frac{m}{8}gr\theta &= -\frac{m}{12}r^2\ddot{\theta} \\ p_{x2} &= \frac{1}{r}\left(-\frac{m}{12}r^2\ddot{\theta} + \frac{m}{8}gr\theta - kr^2\theta - cr^2\dot{\theta}\right) \end{aligned}$$

Plugging the above into (3) gives the equation of motion for disk

$$\begin{aligned} kr^2\theta - \left(-\frac{m}{12}r^2\ddot{\theta} + \frac{m}{8}gr\theta - kr^2\theta - cr^2\dot{\theta}\right) &= -\frac{7}{4}mr^2\ddot{\theta} \\ kr^2\theta + \frac{m}{12}r^2\ddot{\theta} - \frac{m}{8}gr\theta + kr^2\theta + cr^2\dot{\theta} &= -\frac{7}{4}mr^2\ddot{\theta} \\ \theta\left(2kr^2 - \frac{m}{8}gr\right) + cr^2\dot{\theta} &= -\frac{7}{4}mr^2\ddot{\theta} - \frac{m}{12}r^2\ddot{\theta} \\ \frac{11}{6}mr^2\ddot{\theta} + cr^2\dot{\theta} + \theta\left(2kr^2 - \frac{m}{8}gr\right) &= 0 \end{aligned}$$

Or

$$\ddot{\theta} + \frac{6c}{11m}\dot{\theta} + \theta\left(\frac{12}{11}\frac{k}{m} - \frac{3}{44}\frac{g}{r}\right) = 0$$

Writing the above in the standard form $\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$ we see that

$$\omega_n^2 = \sqrt{\frac{12}{11}\frac{k}{m} - \frac{3}{44}\frac{g}{r}}$$

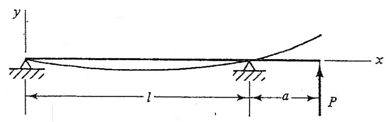
And

$$
\begin{aligned}
2\zeta\omega_n &= \frac{6c}{11m} \\
\zeta &= \frac{3c}{11m\omega_n} \\
&= \frac{3c}{11m\sqrt{\frac{12}{11}\frac{k}{m} - \frac{3}{44}\frac{g}{r}}} \\
&= \frac{3c}{\sqrt{132km - \frac{363}{44}\frac{gm^2}{r}}}
\end{aligned}
$$

3.6 Beam handouts

Pinned-pinned beam with overhang (P at $x = l + a$)*

$$k = \frac{P}{y|_{x=a+l}} = \frac{3EI}{a^2(a+l)}$$

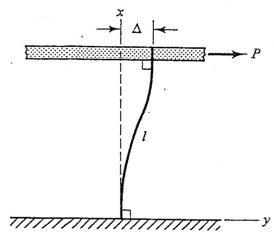


(g)

$$y = \frac{Pax}{6EIl}(x^2 - l^2) \qquad x \le l$$

$$y = \frac{P}{6EIl}[ax(x^2 - l^2) - (l + a)(x - l)^3] \qquad x \ge l$$

Fixed-fixed beam with lateral displacement



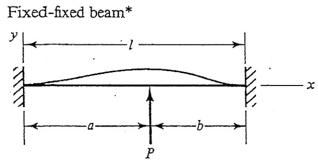
(h)

$$\Delta = \frac{Pl^3}{12EI}$$

$$k = \frac{12EI}{l^3}$$

$$y = \frac{P}{12EI}(3lx^2 - 2x^3)$$

Fixed-fixed beam*



(i)

$$k = \frac{P}{y|_{x=a}}$$

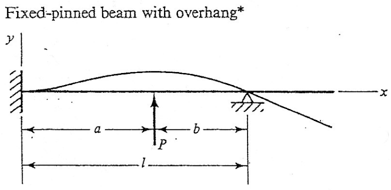
$$k|_{a=l/2} = \frac{192EI}{l^3}$$

$$K = 3EI\left(\frac{L}{ab}\right)^3$$

$$y = \frac{Pb^2}{6EIl^3}[(2b - 3l)x^3 + 3l(l - b)x^2] \qquad (x \le a)$$

$$y = \frac{Pb^2}{6EIl^3}\left[(2b - 3l)x^3 + 3l(l - b)x^2 + \frac{l^3}{b^2}(x - a)^3\right] \qquad (x \ge a)$$

Fixed-pinned beam with overhang*



(j)

$$k = \frac{P}{y|_{x=a}}$$

$$k|_{a=l/2} = \frac{768EI}{7l^3}$$

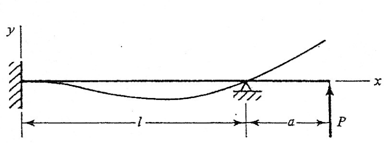
$$y = \frac{P}{12EI}\left[3b\left(1 - \frac{b^2}{l^2}\right)x^2 - \frac{b}{l}\left(3 - \frac{b^2}{l^2}\right)x^3\right] \qquad x \le a$$

$$y = \frac{P}{12EI}\left[3b\left(1 - \frac{b^2}{l^2}\right)x^2 - \frac{b}{l}\left(3 - \frac{b^2}{l^2}\right)x^3 + 2(x - a)^3\right] \qquad a \le x \le l$$

$$y = \frac{-pba^2}{4EIl}(x - l) \qquad x \ge l$$

Fixed-pinned beam with overhang (P at $x = l + a$)*

$$k = \frac{P}{y|_{x=l+a}} = \frac{12EI}{a^2(3l + 4a)}$$



(k)

$$y = \frac{Pa}{4EIl}(x^3 - lx^2) \qquad x \le l$$

$$y = \frac{Pa}{4EIl}\left[x^3 - lx^2 - \left(\frac{2l}{3a} + 1\right)(x - l)^3\right] \qquad x \ge l$$

* Axial extensions due to axial end constraints considered negligible.

Cantilevered Beam Slopes and Deflections

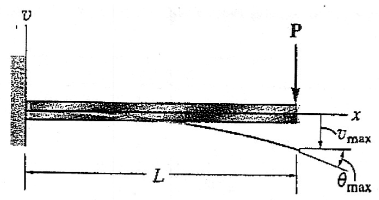
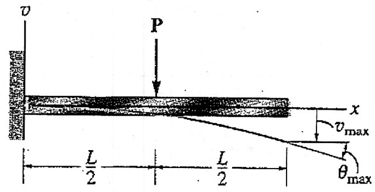
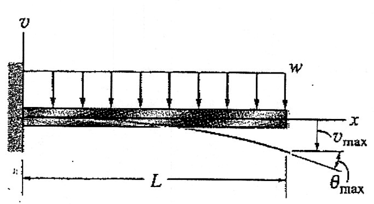
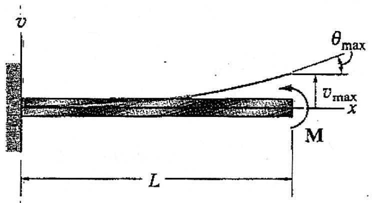
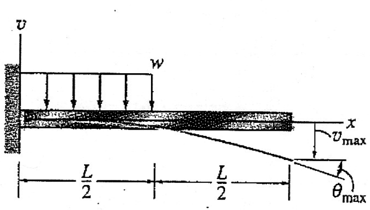
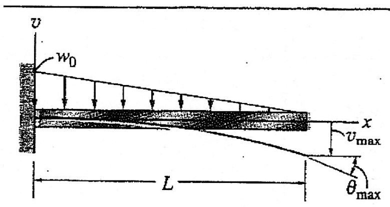
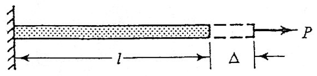
Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{6EI}(\frac{3}{2}L - x)$ $0 \le x \le L/2$ $v = \frac{-PL^2}{24EI}(3x - \frac{1}{2}L)$ $L/2 \le x \le L$
	$\theta_{max} = \frac{wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{ML}{EI}$	$v_{max} = \frac{ML^2}{2EI}$	$v = \frac{Mx^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 2Lx + \frac{3}{2}L^2)$ $0 \le x \le L/2$ $v = \frac{-wL^3}{192EI}(4x - L/2)$ $L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0L^3}{24EI}$	$v_{max} = \frac{-w_0L^4}{30EI}$	$v = \frac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

TABLE 2-1 SPRING CONSTANTS AND DEFLECTION EQUATIONS OF ELASTIC ELEMENTS

A = area of cross section
E = modulus of elasticity
I = area moment of inertia about neutral axis
G = modulus of rigidity
J = polar moment of inertia

Axial (rods, cables, etc.)

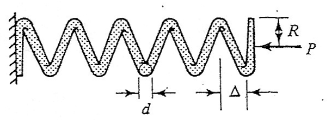


(a)

$$\Delta = \frac{Pl}{AE}$$

$$k = \frac{P}{\Delta} = \frac{AE}{l}$$

Coil spring



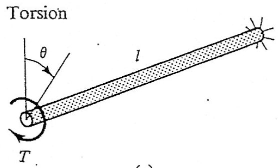
(b)

$$\Delta = \frac{64PnR^3}{Gd^4}$$

$$k = \frac{P}{\Delta} = \frac{Gd^4}{64nR^3}$$

n = number of active coils
R = mean helix radius

Torsion



(c)

$$\theta = \frac{Tl}{GJ}$$

$$k = \frac{T}{\theta} = \frac{GJ}{l}$$

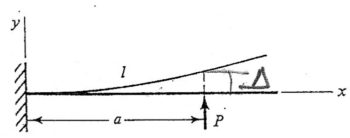
$$J = \frac{\pi d^4}{32} \quad (d = \text{dia.})$$

Polar area moment. Polar moment of inertia

$K = GJ\left(\frac{1}{a} + \frac{1}{b}\right)$

$F = K\Delta$

Cantilever beam



(d)

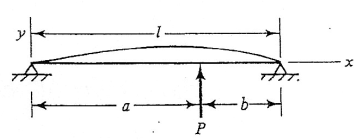
$$k = \frac{P}{y|_{x=a}}$$

$$k|_{a=l} = \frac{3EI}{l^3}$$

$$y = \frac{P}{6EI}(3ax^2 - x^3) \qquad x \le a$$

$$y = \frac{P}{6EI}(3a^2x - a^3) \qquad x \ge a$$

Simply supported beam (pinned-pinned)*



(e)

$$k = \frac{P}{y|_{x=a}}$$

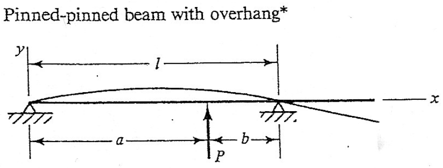
$$k|_{a=l/2} = \frac{48EI}{l^3}$$

$K = \frac{3EIL}{(ba)^2}$

$$y = \frac{Pbx}{6EIl}(l^2 - x^2 - b^2) \qquad x \le a$$

$$y = \frac{Pb}{6EIl}\left[(l^2 - b^2)x - x^3 + \frac{l}{b}(x - a)^3\right] \qquad x \ge a$$

Pinned-pinned beam with overhang*



(f)

$$y = \frac{Pa}{6EIl}(a^2 - l^2)(x - l) \qquad x \ge l$$

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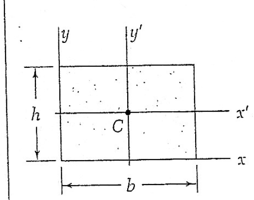
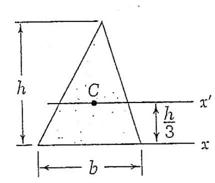
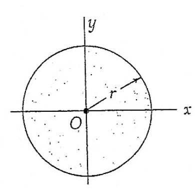
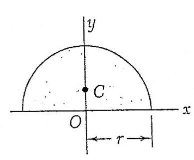
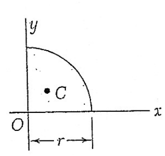
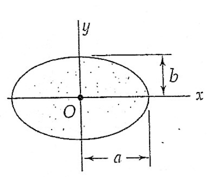
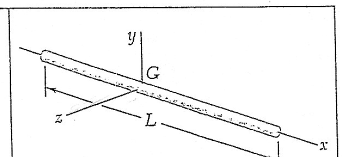
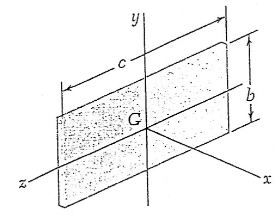
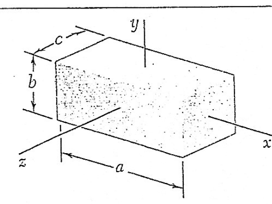
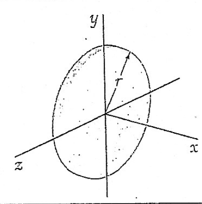
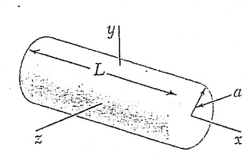
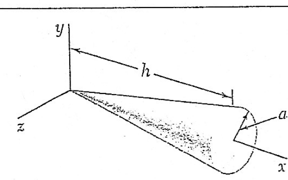
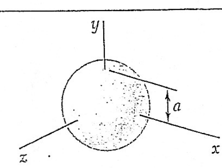
Moments of Inertia of Common Geometric Shapes

Rectangle $\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	
Triangle $\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	
Circle $\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	
Semicircle $I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$	
Quarter circle $I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$	
Ellipse $\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$	

Mass Moments of Inertia of Common Geometric Shapes

Slender rod $I_y = I_z = \frac{1}{12}mL^2$	
Thin rectangular plate $I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$	
Rectangular prism $I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}m(c^2 + a^2)$ $I_z = \frac{1}{12}m(a^2 + b^2)$	
Thin disk $I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$	
Circular cylinder $I_x = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$	
Circular cone $I_x = \frac{3}{10}ma^2$ $I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$	
Sphere $I_x = I_y = I_z = \frac{2}{5}ma^2$	

3.7 my cheat sheet

3.7.1 Solution to undamped forced harmonic

3.7.1.1 Input is $F_0 \cos \omega t$

$$m\ddot{x} + kx = F_0 \cos \omega t$$

This model is single degree of freedom system, undamped, with forced harmonice input. Its solution is given by

$$x(t) = x_h(t) + x_p(t)$$

Where $x_p(t)$ is particular solution and $x_h(t)$ is homogenous solution. We know that

$$x_h(t) = c_1 \cos \omega_n t + c_2 \sin \omega_n t$$

And assuming $x_p(t) = X \cos \omega t$ for the case $\underline{\omega \neq \omega_n}$ Pluggin this into the ODE, we find that

$$X = \frac{x_{st}}{1-r^2}$$

Where $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k_{eq}}$ the static deflection. Hence the solution becomes

$$x(t) = \overbrace{c_1 \cos \omega_n t + c_2 \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1-r^2} \cos \omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, then (1) at $t = 0$ becomes

$$x_0 = c_1 + \frac{x_{st}}{1-r^2}$$
$$c_1 = x_0 - \frac{x_{st}}{1-r^2}$$

Hence solution (1) now becomes

$$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right) \cos \omega_n t + c_2 \sin \omega_n t + \frac{x_{st}}{1-r^2} \cos \omega t$$

Taking derivative

$$\dot{x}(t) = -\omega_n \left(x_0 - \frac{x_{st}}{1-r^2}\right) \sin \omega_n t + c_2 \omega_n \cos \omega_n t - \omega \frac{x_{st}}{1-r^2} \sin \omega t$$

At $t = 0$ the above becomes

$$\dot{x}_0 = c_2 \omega_n$$
$$c_2 = \frac{\dot{x}_0}{\omega_n}$$

Therefore the solution now becomes (again, this is for $\omega \neq \omega_n$)

$$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{x_{st}}{1-r^2} \cos \omega t \tag{2}$$

3.7.1.2 Input is $F_0 \sin \omega t$

$$m\ddot{x} + kx = F_0 \sin \omega t$$

This model is single degree of freedom system, undamped, with forced harmonice input. Its solution is given by

$$x(t) = x_h(t) + x_p(t)$$

Where $x_p(t)$ is particular solution and $x_h(t)$ is homogenous solution. We know that

$$x_h(t) = c_1 \cos \omega_n t + c_2 \sin \omega_n t$$

And assuming $x_p(t) = X \sin \omega t$ for the case $\underline{\omega \neq \omega_n}$ Pluggin this into the ODE, we find that

$$X = \frac{x_{st}}{1-r^2}$$

Where $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k_{eq}}$ the static deflection. Hence the solution becomes

$$x(t) = \overbrace{c_1 \cos \omega_n t + c_2 \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1-r^2} \sin \omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, then (1) at $t = 0$ becomes

$$x_0 = c_1$$

Hence solution (1) now becomes

$$x(t) = x_0 \cos \omega_n t + c_2 \sin \omega_n t + \frac{x_{st}}{1-r^2} \sin \omega t$$

Taking derivative

$$\dot{x}(t) = -x_0 \sin \omega_n t + c_2 \omega_n \cos \omega_n t + \omega \frac{x_{st}}{1-r^2} \cos \omega t$$

At $t = 0$ the above becomes

$$\begin{aligned}
\dot{x}_0 &= c_2 \omega_n + \omega \frac{x_{st}}{1-r^2} \\
c_2 &= \frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{x_{st}}{1-r^2} \\
&= \frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st}
\end{aligned}$$

Therefore the solution now becomes (again, this is for $\omega \neq \omega_n$)

$$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st} \right) \sin \omega_n t + \frac{x_{st}}{1-r^2} \sin \omega t \tag{2}$$

Notice the difference in the solution. Here is summary

ODE	solution
$m\ddot{x} + kx = F_0 \cos \omega t$	$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \overbrace{\frac{x_{st}}{1-r^2} \cos \omega t}^{x_p}$
$m\ddot{x} + kx = F_0 \sin \omega t$	$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st}\right) \sin \omega_n t + \overbrace{\frac{x_{st}}{1-r^2} \sin \omega t}^{x_p}$

3.7.2 Solution to underdamped forced harmonic

ODE	particular solution only
$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$	$x_p(t) = \frac{a_0}{2} \frac{1}{k}$
$m\ddot{x} + c\dot{x} + kx = a_n \cos(n\omega t)$	$x_p(t) = \frac{a_n}{k} \frac{1}{\sqrt{\left(1-(nr)^2\right)^2 + (2\zeta nr)^2}} \cos\left(n\omega t - \phi_n\right)$
$m\ddot{x} + c\dot{x} + kx = b_n \sin(n\omega t)$	$x_p(t) = \frac{b_n}{k} \frac{1}{\sqrt{\left(1-(nr)^2\right)^2 + (2\zeta nr)^2}} \sin\left(n\omega t - \phi_n\right)$

Where

$$\begin{aligned}
r &= \frac{\omega}{\omega_n} \\
\phi_n &= \tan^{-1}\left(\frac{2\zeta nr}{1-(nr)^2}\right)
\end{aligned}$$

3.7.3 unit Impulse respones

For undamped system $m\ddot{x} + kx = \delta(t)$ the response (solution) is (notes calls these $g(t)$)

$$g(t) = \frac{1}{m\omega_n} \sin(\omega_n t)$$

And for an underdamped $m\ddot{x} + c\dot{x} + kx = \delta(t)$ the response is

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta \omega_n t} \sin(\omega_d t)$$

3.7.4 Duhamel Integral

For arbitray forcing function $F(t)$ which can be of any forum, the response of the system to $F(t)$, assuming the system was at rest is

$$x_{conv}(t) = \int_0^t F(\tau) g(t-\tau) d\tau$$

3.7.4.1 Some definitions

3.7.4.1.1 DLF Dynamic locad factor. $DLF = \frac{x(t)}{x_{st}}$. But we really only care for the maximum DLF. When the input is constant (step input), the $DLF_{\max} = 2$.

3.7.4.1.2 Response spectrum Plots the $\text{DLF}_{\max}$ on the y axis vs $\frac{t}{T}$ where T is the period of the system on the x axis. This is done for typical inputs such as unit step, triangle, half sine, etc...

cheat sheet ME440 written by Nasser M Abbasi

odd $f(-t)=-f(t)$ even $f(-t)=f(t)$

Fourier Series $f(t)\approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right)\sin\left(\frac{2\pi}{T}nt\right)$

$a_0=\frac{1}{T/2}\int_{-T/2}^{T/2} f(t)dt$; $a_n=\frac{1}{T/2}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi}{T}nt\right)dt \cdots$

$X\cos(\omega t-\theta)=A\cos(\omega t)+B\sin(\omega t)$

$X\cos(\omega t-\theta)=A\cos(\omega t)+B\cos(\omega t-\pi/2)$

$A=X\cos\theta$, $B=X\sin\theta$, $\theta=\tan^{-1}\frac{B}{A}$

$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots$; $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

$\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$ | $\sin u\sin v=\frac{1}{2}[\cos(u-v)-\cos(u+v)]$

$\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$ | $\cos u\cos v=\frac{1}{2}[\cos(u-v)+\cos(u+v)]$

$\sin 2x=2\sin x\cos x$ | $\sin u\cos v=\frac{1}{2}[\sin(u+v)+\sin(u-v)]$

$\cos 2x=1-2\sin^2 x$

complex F.S. $f(t)\sim\sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi}{T}nt}=\sum C_n e^{jn\omega t}$; $C_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\omega nt}dt$

$a_n=C_n+C_{-n}$

$b_n=j(C_n-C_{-n})$

$a_0=2C_0$

critical damping $x(t)=e^{-\omega_n t}(C_1+C_2 t)$. see slide 165

$C_1=x_0$

$C_2=\dot{x}_0+\omega_n x_0$

$C_{critical}=2m\omega_n$; $\xi=\frac{c}{C_{cr}}=\frac{c}{2m\omega_n}$

$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x=F$, $\omega_n=\sqrt{\frac{k}{m}}$, $2\xi\omega_n=\frac{c}{m}$ and $\xi=\frac{c}{2\sqrt{km}}$

see Undamped: $x(t)=x_0\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t$ or $x(t)=X\cos(\omega_n t-\theta)$

$X=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega_n}\right)^2}$; $\theta=\tan^{-1}\left(\frac{\dot{x}_0}{\omega_n x_0}\right)$

damped: $\xi>1$; $\xi=1$: $x(t)=e^{-\omega_n t}[x_0+(\dot{x}_0+\omega_n x_0)t]$

$\xi<1$: $x(t)=e^{-\xi\omega_n t}(A\cos(\omega_d t)+B\sin\omega_d t)$. $A=x_0$; $B=\frac{\dot{x}_0+\xi\omega_n x_0}{\omega_d}$

$\omega_d=\omega_n\sqrt{1-\xi^2}$ or $x(t)=Xe^{-\xi\omega_n t}\cos(\omega_d t-\phi)$; $X=\sqrt{A^2+B^2}$, $\phi=\tan^{-1}\left(\frac{B}{A}\right)$

Log decrement: $\ln\left(\frac{X_1}{X_2}\right)=\xi\omega_n\tau_d$ must be over one period

$=\frac{2\pi\xi}{\sqrt{1-\xi^2}}=\delta$. if more than one cycle is given Then

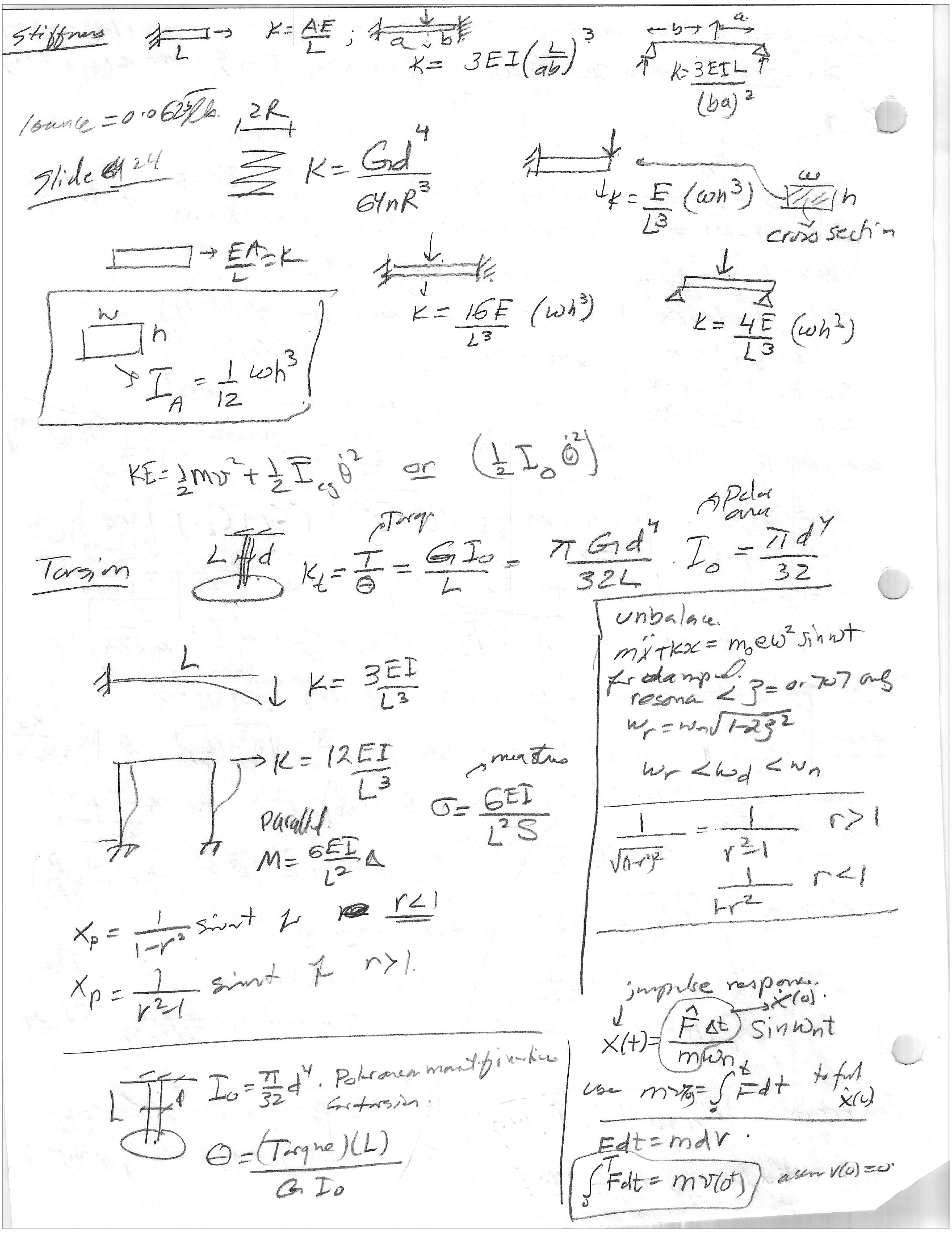
$\delta=\frac{1}{n}\ln\left(\frac{X_1}{X_{n+1}}\right)$ Find δ, then ξ to find ξ. Then find $\omega_n=\frac{2\pi}{\tau_d\sqrt{1-\xi^2}}$. Find ω_n

For rotation $C_{critical}=2I_0\omega_n$

For critical damping, $x_{max}=\frac{\dot{x}_0 e^{-1}}{\omega_n}$

For bending use $m=0.229$ column mass

for longitudinal use $\frac{1}{3}$ mass of spring



given $m\ddot{x}+Kx=f$.

(1) Find $A=M^{-1}K$. (2) Find λ_1,λ_2.

(3) find eigenvectors $u_1, u_2 \Rightarrow [u]$

(4) write $\{\ddot{q}\}+\begin{bmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{bmatrix}\{K\} = u^{-1}M^{-1}\{f\}$.

(5) solve these.

(6) let $x=[u]\{q\}$. $\Rightarrow q(0)=u^{-1}x(0)$.

(7) transfer back to normal coordinates.

for damped system. use slide 393.

$$[M]\ddot{q}+\beta[M]\Omega\dot{q}+M\Omega q=u^{T}f.$$

where $[M]=u^{T}mu$, and $\Omega=\begin{bmatrix}\omega_{n(1)}^2 & 0\\ 0 & \omega_{n(2)}^2\end{bmatrix}$

For Beam problems, easier to use Flexibility

For spring problems, easier to use stiffness

$a\lambda^2+b\lambda+c=0 \Rightarrow \lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ eigenvalue

One mile = 5280 ft.

one mile = 1609.34 meters

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1}=\frac{1}{\det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$