

# HW 7, ME 440 Intermediate Vibration, Fall 2017

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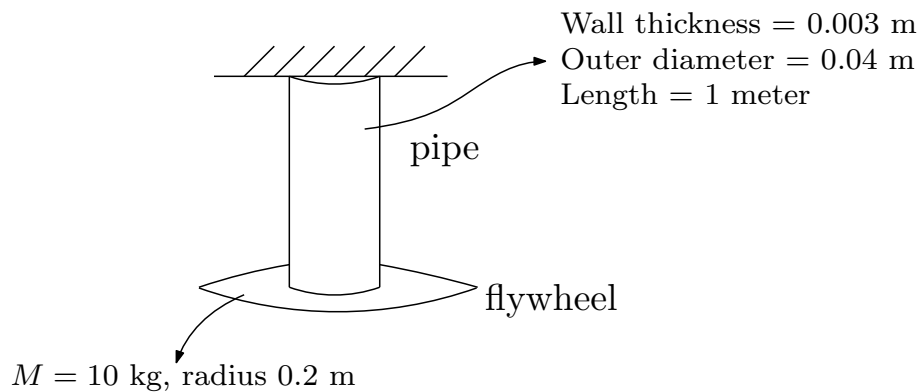
## 0.1 Problem 1

### Problem 1

Download the ANSYS input file “*MODAL\_pipe\_flywheel.txt*” from Canvas, run this input file in ANSYS and go through the file line by line to figure out what the system parameters are for this modal analysis. (Hint: When viewing the mode shapes within ANSYS, try plotting all 3 displacements and all 3 rotations (1 at a time) available under the “Nodal Solu” / “DOF Solution” option; this should be helpful in determining the type of displacement associated with each specific frequency.

- A) List the 4 frequencies from ANSYS and label each as longitudinal, transverse, or torsional.
- B) Using the parameters defined in the text file, analytically determine 3 of the 4 natural frequencies of this system. Show ALL your work for these calculations and then compare the analytical and finite element frequencies in a table with % errors.

The following is diagram of the model of the problem to solve



The ANSYS APDL (input file) listing was provided to us to use and is given in the text file below for reference

```

/filnam, pipe_flywheel_modal
/title, Flywheel on torsional spring example
/prep7
!element type
et,1,mass21,,0,0 !element type no.1 is mass21 (",,0" signifies
!that this is a 3-D mass with rotary inertia)
et,2,pipe288      !element type no.2 is pipe288 (this is a pipe element)
mp,ex,1,200e9     ! elastic modulus for steel is 200 GPa
mp,gxy,1,77.2e9   ! shear modulus for steel is 77.2 GPa
mp,prxy,1,0.295   ! poisson's ratio for steel is 0.295
! model parameters
mass = 10          ! mass of flywheel (kg)
rad_f = 0.2        ! outer radius of flywheel
izz = 0.5*mass*rad_f*rad_f ! mass moment of inertia
outer_d = 0.04     ! outer diameter of pipe (m)
wall_t = 0.003     ! wall thickness of pipe (m)
pipe_l = 1         ! pipe length (m)
n_modes = 10      ! number of modes wanted
!real constants
r,1,mass,mass,mass,0.5*izz,izz,0.5*izz ! real constant set 1 is for the mass21 element
sectype,1,pipe     ! section type 1 is "pipe"
secdata,outer_d,wall_t ! section data for pipe is outer diameter and wall thickness
!create nodes
k,1,0,0,0         ! keypoint 1 is at x=0, y=0, z=0, this will be the fixed end of the pipe
k,2,0,-pipe_l,0   ! keypoint 2 is at x=0, y=-pipe_l, z=0, this will be the free
! end of the pipe with the flywheel
!create elements
type,2            ! specify element type of subsequently defined elements
secnum,1         ! specify section type number of subsequently defined elements
l,1,2            ! creates a line from keypoint 1 to keypoint 2
lesize,1,,10    ! specifies that line 1 will consist of 10 elements when meshed
lmesh,1         ! take line 1 and mesh it, resulting in elements representing the pipe
type,1          ! specify element type of subsequently defined elements
real,1          ! specify real constant set of subsequently defined elements
e,2            ! create element to be created at node 2
nselect,all     ! selects all nodes
d,all,uz,0     ! sets the z displacements on selected nodes to be 0, thereby
! limiting our modal analysis to modes in the xy plane
d,all,rotx     ! sets the rotx displacements on selected nodes to be 0
!displacement boundary conditions
nselect,s,loc,y,0 ! select node at x = 0
d,all,ux,0     ! displacement of selected node in x-dir is 0
d,all,uy,0     ! displacement of selected node in y-dir is 0
d,all,uz,0     ! displacement of selected node in z-dir is 0
d,all,rotx,0   ! rotations of selected node about x axis is 0
d,all,roty,0   ! rotations of selected node about y axis is 0
d,all,rotz,0   ! rotations of selected node about z axis is 0
allselect
finish
/solu          !select static load solution
antype,modal

```

```

modopt,lanb,n_modes
solve
finish
/post1

```

### 0.1.1 Part 1

The following 4 modal frequencies were generated by ANSYS after running the above APDL file.

mode	Mode number	Frequency (Hz)
transverse	1	9.4438
torsional	2	34.272
transverse	3	111.41
longitudinal	4	420.31

The modal shapes were then plotted using ANSYS. They are given below for each mode

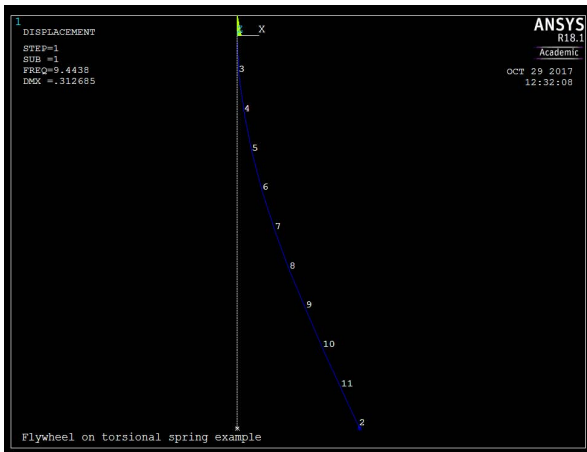


Figure 1: First mode: Transverse at 9.443 Hz

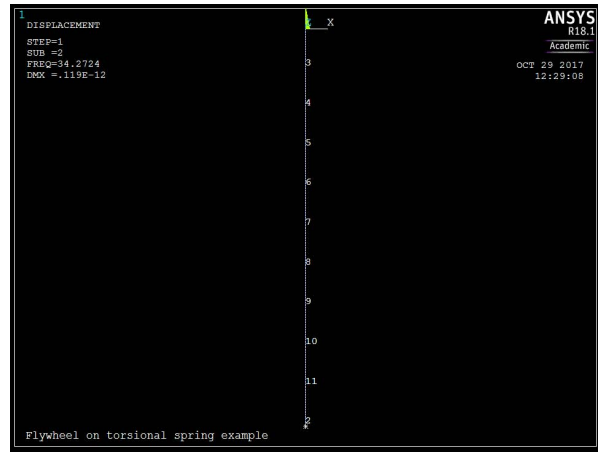


Figure 2: Second mode: Torsional at 34.2724 Hz

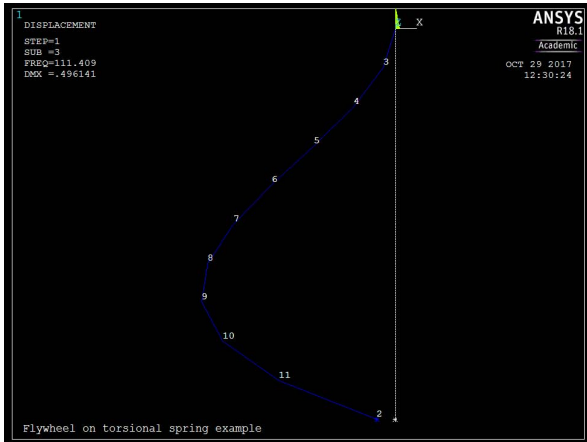


Figure 3: Third mode: Transverse at 111.408 Hz

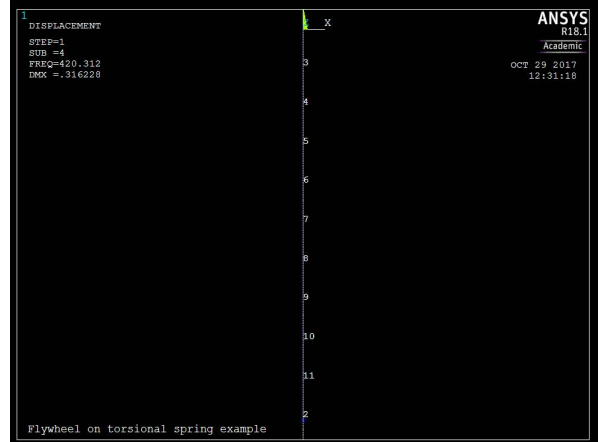


Figure 4: Fourth mode: longitudinal at 420.312 Hz

The system parameters are

PARAMETER STATUS-		( 13 PARAMETERS DEFINED)	
(INCLUDING		6 INTERNAL PARAMETERS)	
NAME	VALUE	TYPE	DIMENSIONS
IZZ	0.200000000	SCALAR	
MASS	10.0000000	SCALAR	
N_MODES	10.0000000	SCALAR	
OUTER_D	4.000000000E-002	SCALAR	
PIPE_L	1.00000000	SCALAR	
RAD_F	0.200000000	SCALAR	
WALL_T			

Total U displacement by ANSYS for mode 1 is

```
PRINT U      NODAL SOLUTION PER NODE

***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

LOAD STEP=      1  SUBSTEP=      1
FREQ=      9.4438      LOAD CASE=      0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM
```

NODE	UX	UY	UZ	USUM
1	0.0000	0.0000	0.0000	0.0000
2	0.31268	-0.15719E-019	0.0000	0.31268
3	0.45277E-002	-0.15719E-020	0.0000	0.45277E-002
4	0.17442E-001	-0.31438E-020	0.0000	0.17442E-001
5	0.37825E-001	-0.47156E-020	0.0000	0.37825E-001
6	0.64762E-001	-0.62875E-020	0.0000	0.64762E-001
7	0.97336E-001	-0.78594E-020	0.0000	0.97336E-001
8	0.13463	-0.94313E-020	0.0000	0.13463

9	0.17573	-0.11003E-019	0.0000	0.17573
10	0.21972	-0.12575E-019	0.0000	0.21972
11	0.26567	-0.14147E-019	0.0000	0.26567

## MAXIMUM ABSOLUTE VALUES

NODE	2	2	0	2
VALUE	0.31268	-0.15719E-019	0.0000	0.31268

Total ROT displacement by ANSYS for mode 1 is

PRINT ROT NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 1  
 FREQ= 9.4438 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	ROTX	ROTY	ROTZ	RSUM
1	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.54021E-017	0.47205	0.47205
3	0.0000	0.54021E-018	0.88444E-001	0.88444E-001
4	0.0000	0.10804E-017	0.16772	0.16772
5	0.0000	0.16206E-017	0.23784	0.23784
6	0.0000	0.21609E-017	0.29879	0.29879
7	0.0000	0.27011E-017	0.35058	0.35058
8	0.0000	0.32413E-017	0.39320	0.39320
9	0.0000	0.37815E-017	0.42666	0.42666
10	0.0000	0.43217E-017	0.45095	0.45095
11	0.0000	0.48619E-017	0.46608	0.46608

## MAXIMUM ABSOLUTE VALUES

NODE	0	2	2	2
VALUE	0.0000	0.54021E-017	0.47205	0.47205

Total U displacement by ANSYS for mode 2 is

PRINT U NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 2  
 FREQ= 34.272 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	UX	UY	UZ	USUM
1	0.0000	0.0000	0.0000	0.0000
2	0.11870E-012	0.57718E-018	0.0000	0.11870E-012
3	0.17191E-014	0.57718E-019	0.0000	0.17191E-014
4	0.66222E-014	0.11544E-018	0.0000	0.66222E-014

5	0.14361E-013	0.17316E-018	0.0000	0.14361E-013
6	0.24588E-013	0.23087E-018	0.0000	0.24588E-013
7	0.36955E-013	0.28859E-018	0.0000	0.36955E-013
8	0.51113E-013	0.34631E-018	0.0000	0.51113E-013
9	0.66715E-013	0.40403E-018	0.0000	0.66715E-013
10	0.83412E-013	0.46175E-018	0.0000	0.83412E-013
11	0.10086E-012	0.51947E-018	0.0000	0.10086E-012

## MAXIMUM ABSOLUTE VALUES

NODE	2	2	0	2
VALUE	0.11870E-012	0.57718E-018	0.0000	0.11870E-012

Total ROT displacement by ANSYS for mode 2 is

PRINT ROT NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 2  
 FREQ= 34.272 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	ROTX	ROTY	ROTZ	RSUM
1	0.0000	0.0000	0.0000	0.0000
2	0.0000	2.2361	0.17917E-012	2.2361
3	0.0000	0.22361	0.33581E-013	0.22361
4	0.0000	0.44721	0.63680E-013	0.44721
5	0.0000	0.67082	0.90299E-013	0.67082
6	0.0000	0.89443	0.11344E-012	0.89443
7	0.0000	1.1180	0.13309E-012	1.1180
8	0.0000	1.3416	0.14927E-012	1.3416
9	0.0000	1.5652	0.16197E-012	1.5652
10	0.0000	1.7889	0.17118E-012	1.7889
11	0.0000	2.0125	0.17691E-012	2.0125

## MAXIMUM ABSOLUTE VALUES

NODE	0	2	2	2
VALUE	0.0000	2.2361	0.17917E-012	2.2361

Total U displacement by ANSYS for mode 3 is

PRINT U NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 3  
 FREQ= 111.41 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	UX	UY	UZ	USUM
------	----	----	----	------

1	0.0000	0.0000	0.0000	0.0000
2	-0.47205E-001	-0.10687E-014	0.0000	0.47205E-001
3	-0.29904E-001	-0.10687E-015	0.0000	0.29904E-001
4	-0.10556	-0.21374E-015	0.0000	0.10556
5	-0.20770	-0.32061E-015	0.0000	0.20770
6	-0.31709	-0.42748E-015	0.0000	0.31709
7	-0.41446	-0.53435E-015	0.0000	0.41446
8	-0.48056	-0.64122E-015	0.0000	0.48056
9	-0.49614	-0.74809E-015	0.0000	0.49614
10	-0.44194	-0.85497E-015	0.0000	0.44194
11	-0.29872	-0.96184E-015	0.0000	0.29872

## MAXIMUM ABSOLUTE VALUES

NODE	9	2	0	9
VALUE	-0.49614	-0.10687E-014	0.0000	0.49614

Total ROT displacement by ANSYS for mode 3 is

PRINT ROT NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 3  
 FREQ= 111.41 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	ROTX	ROTY	ROTZ	RSUM
1	0.0000	0.0000	0.0000	0.0000
2	0.0000	-0.49528E-013	3.1268	3.1268
3	0.0000	-0.49528E-014	-0.55375	0.55375
4	0.0000	-0.99056E-014	-0.91496	0.91496
5	0.0000	-0.14858E-013	-1.0836	1.0836
6	0.0000	-0.19811E-013	-1.0598	1.0598
7	0.0000	-0.24764E-013	-0.84334	0.84334
8	0.0000	-0.29717E-013	-0.43438	0.43438
9	0.0000	-0.34670E-013	0.16711	0.16711
10	0.0000	-0.39622E-013	0.96115	0.96115
11	0.0000	-0.44575E-013	1.9477	1.9477

## MAXIMUM ABSOLUTE VALUES

NODE	0	2	2	2
VALUE	0.0000	-0.49528E-013	3.1268	3.1268

Total U displacement by ANSYS for mode 4 is

PRINT U NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 4  
 FREQ= 420.31 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	UX	UY	UZ	USUM
1	0.0000	0.0000	0.0000	0.0000
2	-0.14600E-009	0.31623	0.0000	0.31623
3	-0.21370E-011	0.31623E-001	0.0000	0.31623E-001
4	-0.82246E-011	0.63246E-001	0.0000	0.63246E-001
5	-0.17820E-010	0.94868E-001	0.0000	0.94868E-001
6	-0.30480E-010	0.12649	0.0000	0.12649
7	-0.45762E-010	0.15811	0.0000	0.15811
8	-0.63223E-010	0.18974	0.0000	0.18974
9	-0.82420E-010	0.22136	0.0000	0.22136
10	-0.10291E-009	0.25298	0.0000	0.25298
11	-0.12425E-009	0.28460	0.0000	0.28460

MAXIMUM ABSOLUTE VALUES				
NODE	2	2	0	2

And total ROT displacement by ANSYS for mode 4 is

PRINT ROT NODAL SOLUTION PER NODE

\*\*\*\*\* POST1 NODAL DEGREE OF FREEDOM LISTING \*\*\*\*\*

LOAD STEP= 1 SUBSTEP= 4  
 FREQ= 420.31 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	ROTX	ROTY	ROTZ	RSUM
1	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.15287E-011-0.21790E-009	0.21790E-009	0.21790E-009
3	0.0000	0.15287E-012-0.41720E-010	0.41721E-010	0.41721E-010
4	0.0000	0.30573E-012-0.79011E-010	0.79012E-010	0.79012E-010
5	0.0000	0.45860E-012-0.11187E-009	0.11187E-009	0.11187E-009
6	0.0000	0.61146E-012-0.14031E-009	0.14031E-009	0.14031E-009
7	0.0000	0.76433E-012-0.16431E-009	0.16431E-009	0.16431E-009
8	0.0000	0.91720E-012-0.18389E-009	0.18389E-009	0.18389E-009
9	0.0000	0.10701E-011-0.19903E-009	0.19904E-009	0.19904E-009
10	0.0000	0.12229E-011-0.20975E-009	0.20975E-009	0.20975E-009
11	0.0000	0.13758E-011-0.21604E-009	0.21604E-009	0.21604E-009

MAXIMUM ABSOLUTE VALUES				
NODE	0	2	2	2
VALUE	0.0000	0.15287E-011-0.21790E-009	0.21790E-009	0.21790E-009

### 0.1.2 Part 2

To verify ANSYS solution, this was solved in two ways. By taking into account the mass  $m$  of the pipe and then by ignoring the mass  $m$ . Both hand solutions are given below. ANSYS



do not take the mass of the pipe into account, since it was not told the density of the pipe material in the APDL input file. The first solution below is the recommend one to use to compare the ANSYS result against and it the method which gave more agreement with ANSYS result.

### **First solution. Not accounting for mass of pipe**

#### Finding the longitudinal (axial) natural frequency.

Using  $k_{eq} = \frac{AE}{L}$  where  $A$  is the cross sectional area of the pipe and  $L$  is the pipe length and using  $m_{eq} = M$ , then the longitudinal natural frequency is

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{AE}{LM}} \quad (1)$$

The cross sectional area of the pipe is

$$\begin{aligned} A &= \frac{\pi}{4} (D_o^2 - D_i^2) \\ &= \frac{\pi}{4} ((0.04)^2 - (0.034)^2) \\ &= 3.4872 \times 10^{-4} \text{ m}^2 \end{aligned}$$

The length of pipe is 1 meter and  $M = 10$  kg. Equation (1) becomes

$$\begin{aligned} \omega_n &= \sqrt{\frac{(3.4872 \times 10^{-4}) (200 \times 10^9)}{(1)(10)}} \\ &= 2640.9 \text{ rad/sec} \end{aligned}$$

The cycle frequency is

$$\begin{aligned} f_n &= \frac{\omega_n}{2\pi} \\ &= \frac{2640.9}{2\pi} \\ &= 420.31 \text{ Hz} \end{aligned}$$

ANSYS gives 420.31 Hz. So the error is 0.

#### Finding the torsional natural frequency.

Torsional stiffness  $k_t$  is

$$k_t = \frac{GJ}{L}$$

Where  $G$  is the shear modulus (given in handout), and  $J$  is the polar area moment of inertia

of the cross section given by

$$\begin{aligned} J &= \frac{\pi}{32} (D_o^4 - D_i^4) \\ &= \frac{\pi}{32} (0.04^4 - 0.034^4) \\ &= 1.2013 \times 10^{-7} \text{ m}^4 \end{aligned}$$

Therefore

$$k_t = \frac{(77.2 \times 10^9) (1.2013 \times 10^{-7})}{1} = 9274 \text{ N-m per radian}$$

The equivalent mass is just the mass moment of inertia of the flywheel  $\frac{1}{2}Mr_f^2$  (since the pipe assumed to have no mass). Hence the torsional frequency is

$$\begin{aligned} \omega &= \sqrt{\frac{k_t}{\frac{1}{2}Mr_f^2}} \\ &= \sqrt{\frac{9274}{\frac{1}{2}(10)(0.2)^2}} \\ &= 215.34 \text{ rad/sec} \end{aligned}$$

Therefore the torsional frequency in Hz is

$$\begin{aligned} f &= \frac{215.34}{2\pi} \\ &= 34.272 \text{ hz} \end{aligned}$$

ANSYS gives this as 34.272 Hz. The the error is 0.

Finding the transverse natural frequency:

Using  $k = \frac{3EI}{L^3}$  and  $M = 10$ . Where  $I$  is the area moment of inertia given by

$$\begin{aligned} I &= \frac{\pi}{64} (D_o^4 - D_i^4) \\ &= \frac{\pi}{64} (0.04^4 - 0.034^4) \\ &= 6.0066 \times 10^{-8} \text{ m}^4 \end{aligned}$$

The transverse natural frequency is therefore

$$\begin{aligned} \omega &= \sqrt{\frac{3EI}{ML^3}} \\ &= \sqrt{\frac{3(200 \times 10^9)(6.0066 \times 10^{-8})}{(10)(1)^3}} \\ &= 60.033 \text{ rad/sec} \end{aligned}$$

Hence

$$f_n = \frac{60.033}{2\pi} = 9.5545 \text{ Hz}$$

ANSYS gives 9.4438 for the first transverse natural frequency. Hence error is  $\left(\frac{|9.4438-9.5545|}{9.4438}\right)100 = 1.1722\%$

### Summary of results

mode	ANSYS result	Hand calculation	%error
First transverse	9.4438	9.5545	1.1722 %
First torsional	34.272	34.272	0%
First longitudinal (axial)	420.31	420.31	0%

All the analytical solutions gave exact agreement with ANSYS except for the transverse case. The transverse case uses stiffness  $\frac{3EI}{L^3}$  due to load at end of fixed-free beam. This does not account for bending rotation in the beam. That is why ANSYS result is more accurate, as its finite elements account for the small bending associated with the transverse vibration. In the other two cases (Torsional and axial), there is no associated bending, hence the solutions agree.

### **Second solution. Accounting for mass of pipe**

#### Finding the longitudinal natural frequency.

Following the example given in the textbook, at page 715, the (first) longitudinal natural frequency is found to be

$$\omega_1 = \frac{\alpha_1 \sqrt{\frac{E}{\rho}}}{L} \quad (\text{E.4})$$

Where  $\alpha_1$  is the (first) root of

$$\alpha \tan \alpha = \beta$$

Where  $\beta$  is the mass ratio  $\beta = \frac{m}{M}$  where  $m$  is mass of pipe and  $M$  is end mass (flywheel). To find mass of pipe  $m$ , using steel density  $\rho = 7800 \text{ kg/m}^3$ , we first find the volume of the pipe.

Let  $D_i$  be the inner diameter and  $D_o$  the outer diameter.  $D_o = 0.04$  meter and  $D_i = 0.04 - 2(0.003) = 0.034$  meter, therefore the cross sectional area of the pipe is

$$\begin{aligned} A &= \frac{\pi}{4} (D_o^2 - D_i^2) \\ &= \frac{\pi}{4} ((0.04)^2 - (0.034)^2) \\ &= 3.4872 \times 10^{-4} \text{ m}^2 \end{aligned}$$

And since length of pipe is 1 meter, the mass of pipe is

$$\begin{aligned} m &= \rho AL \\ &= (7800) (3.4872 \times 10^{-4}) (1) \\ &= 2.72 \text{ kg.} \end{aligned}$$

The mass at the end is given as  $M = 10$  kg. Therefore the mass ratio

$$\beta = \frac{m}{M} = \frac{2.72}{10} = 0.272$$

To find  $\alpha_1$  we now need to solve  $\alpha_1 \tan \alpha_1 = 0.272$ . This was solved numerical using root finder. The first root was found to be

$$\alpha_1 = 0.499$$

Therefore from equation E.4 in textbook (page 715)

$$\begin{aligned} \omega_1 &= \frac{\alpha_1 c}{L} \\ &= \frac{\alpha_1 \sqrt{\frac{E}{\rho}}}{L} \\ &= \frac{(0.499) \sqrt{\frac{200 \times 10^9}{7800}}}{1} \\ &= 2526.8 \text{ rad/sec} \end{aligned}$$

Therefore

$$\begin{aligned} f_1 &= \frac{\omega_1}{2\pi} \\ &= \frac{2526.8}{2\pi} \\ &= 402.15 \text{ Hz} \end{aligned}$$

ANSYS gives 420.31 Hz. So the error is  $\left(\frac{420.31-402.15}{420.31}\right)100 = 4.321\%$

Finding the torsional natural frequency.

$k_t$  is

$$k_t = \frac{GJ}{L}$$

Where  $G$  is the shear modulus (given in handout), and  $J$  is the polar area moment of inertia of the cross section given by

$$\begin{aligned} J &= \frac{\pi}{32} (D_o^4 - D_i^4) \\ &= \frac{\pi}{32} (0.04^4 - 0.034^4) \\ &= 1.2013 \times 10^{-7} \text{ m}^4 \end{aligned}$$

Hence

$$k_t = \frac{(77.2 \times 10^9)(1.2013 \times 10^{-7})}{1} = 9274$$

To find equivalent mass, using kinetic energy method

$$\frac{1}{2} I_{\text{flywheel}} \dot{\theta}^2 + \frac{1}{2} I_{\text{pipe}} \dot{\theta}^2 = \frac{1}{2} I_{\text{eq}} \dot{\theta}^2 \quad (1)$$

For a hollow pipe, where now  $m$  is replaced by  $\frac{1}{3}m$  from continuous system derivation.

$$\begin{aligned} I_{pipe} &= \frac{1}{2} \left( \frac{1}{3}m \right) \left( \left( \frac{D_o}{2} \right)^2 + \left( \frac{D_i}{2} \right)^2 \right) \\ &= \frac{1}{24} m (D_o^2 + D_i^2) \end{aligned}$$

And for the flywheel,  $I_{fly} = \frac{1}{2}Mr_f^2$  where  $r_f = 0.2$  meter. Hence from (1)

$$\begin{aligned} I_{eq} &= \frac{1}{2}Mr_f^2 + \frac{1}{24}m(D_o^2 + D_i^2) \\ &= \frac{1}{2}(10)(0.2)^2 + \frac{1}{24}(2.72)(0.04^2 + 0.034^2) \\ &= 0.20031 \text{ kg-m}^2 \end{aligned}$$

Hence the torsional frequency is

$$\begin{aligned} \omega &= \sqrt{\frac{k_t}{I_{eff}}} \\ &= \sqrt{\frac{9274}{0.20031}} \\ &= 215.17 \text{ rad/sec} \end{aligned}$$

Therefore the torsional frequency in Hz is

$$\begin{aligned} f &= \frac{215.17}{2\pi} \\ &= 34.245 \text{ hz} \end{aligned}$$

ANSYS gives this as 34.272 Hz. The the error is  $\left( \frac{34.272-34.245}{34.272} \right) 100 = 0.079\%$

Finding the transverse natural frequency:

From textbook, table 8.15 page 726, it gives for fixed-end beam the value  $\beta_1 L = 1.875104$ . But since there is a mass attach to the end in our problem, I did not know how to add this using the table.

So I used the other method we used before, which is the Rayleigh energy method, where we assume motion is simple harmonic motion. Taking the displacement as the transverse motion of the free end of the pipe (where the large mass is attached), measured from equilibrium then the kinetic energy is

$$T = \frac{1}{2} \overbrace{(M + 0.23m)}^{m_{eq}} \dot{x}^2$$

Where we added  $0.23m$ , where  $m$  is the mass of the pipe, since this is continuous mass. For the potential energy, we use the stiffness formula for the fixed-free beam which is  $k = \frac{3EI}{L^3}$ , hence

$$U = \frac{1}{2}kx^2$$

Now, assuming  $x = X \sin \omega_n t$ , then  $\dot{x} = X \omega_n \cos \omega_n t$ . Therefore when

$$U_{\max} = T_{\max}$$

We obtain

$$\begin{aligned} \frac{1}{2} \frac{3EI}{L^3} X^2 &= \frac{1}{2} (M + 0.23m) (X \omega_n)^2 \\ \frac{3EI}{L^3} &= (M + 0.23m) \omega_n^2 \\ \omega_n^2 &= \frac{3EI}{L^3 (M + 0.23m)} \\ \omega_n &= \sqrt{\frac{3EI}{L^3 (M + 0.23m)}} \end{aligned} \quad (1)$$

Where  $I$  now is the area moment of inertia<sup>1</sup> is given by

$$\begin{aligned} I &= \frac{\pi}{64} (D_o^4 - D_i^4) \\ &= \frac{\pi}{64} (0.04^4 - 0.034^4) \\ &= 6.0066 \times 10^{-8} \text{ m}^4 \end{aligned}$$

And

$$\begin{aligned} M + 0.23m &= 10 + 0.23(2.72) \\ &= 10.626 \text{ kg} \end{aligned}$$

Substituting the numerical values in (1) gives

$$\begin{aligned} \omega_n &= \sqrt{\frac{3(200 \times 10^9)(6.0066 \times 10^{-8})}{10.626}} \\ &= 58.239 \text{ rad/sec} \end{aligned}$$

Hence

$$f_n = \frac{58.239}{2\pi} = 9.269 \text{ Hz}$$

ANSYS gives 9.4438 for the first transverse natural frequency. Hence error is  $\left(\frac{9.4438-9.269}{9.4438}\right)100 = 1.851\%$

### Summary of results

mode	ANSYS result	Hand calculation	%error
First transverse	9.4438	9.269	1.851 %
First torsional	34.272	34.245	0.079%
First longitudinal	420.31	402.15	4.321%

<sup>1</sup>Notice that the polar area moment of inertia has  $\frac{1}{32}$  factor, while the area moment of inertia, the factor is  $\frac{1}{64}$

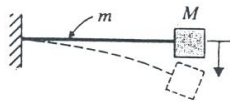
Comparing the above table to the first solution, it shows that ignoring the mass of the pipe gave result which agree with ANSYS result much better. This is because ANSYS did not take into the account the mass of the pipe. It will be interesting exercise to find how to change the APDL input file to make ANSYS account for the mass of the pipe and then compare the above results with ANSYS.

## 0.2 Problem 2

### Problem 2

The signpost of a fast food restaurant consists of a hollow steel cylinder of height  $h$ , inside diameter  $d$ , and outside diameter  $D$ , fixed to the ground and carries a concentrated mass  $M$  at the top. It can be modeled as a single degree of freedom spring-mass-damper system with an equivalent viscous damping ratio of 0.1 for analyzing its transverse vibration characteristics under wind excitation. Assume the signpost mass ( $m$ ) and concentrated mass ( $M$ ) have an equivalent mass ( $m_{eq}$ ) as defined below. (this equivalent mass equation was from a lecture example earlier in the semester). The specific weight ( $\rho g$ ) and the elastic modulus ( $E$ ) of the steel are  $76,500 \text{ N/m}^3$  and  $207 \text{ GPa}$ , respectively. For the density and viscosity of air, use  $1.20 \text{ kg/m}^3$  and  $1.80 \times 10^{-5} \text{ N-s/m}^2$ , respectively. For the remaining parameters, assume  $h = 10 \text{ m}$ ,  $D = 25 \text{ cm}$ ,  $d = 20 \text{ cm}$  and  $M = 200 \text{ kg}$ .

D



Cantilever beam of mass  $m$   
carrying an end mass  $M$

$$m_{eq} = M + 0.23 m$$

Determine the following:

- the natural frequency of transverse vibration of the signpost,
- the wind velocity at which the signpost undergoes maximum steady-state displacement, and
- the maximum wind induced steady-state displacement of the signpost.

### 0.2.1 Part A

The first step is to determine the natural frequency  $\omega_n$  for the transverse vibration. Rayleigh energy method was used to find the transverse frequency. Taking the displacement as the transverse motion of the free end of the signpost (where the large mass  $M$  is attached), measured from equilibrium, then the kinetic energy is

$$T = \frac{1}{2} \overbrace{(M + 0.23m)}^{m_{eq}} \dot{x}^2$$

$m$  is the mass of the signpost. For the potential energy, the bending stiffness formula for the fixed-free beam with load at the end was used, which is

$$k = \frac{3EI}{L^3}$$

The potential energy is therefore

$$U = \frac{1}{2}kx^2$$

Assuming  $x = X \sin \omega_n t$ , then  $\dot{x} = X\omega_n \cos \omega_n t$ . Using

$$U_{\max} = T_{\max}$$

Then the above reduces to

$$\begin{aligned} \frac{1}{2} \left( \frac{3EI}{L^3} \right) X^2 &= \frac{1}{2} (M + 0.23m) (X\omega_n)^2 \\ \frac{3EI}{L^3} &= (M + 0.23m) \omega_n^2 \\ \omega_n^2 &= \frac{3EI}{L^3 (M + 0.23m)} \\ \omega_n &= \sqrt{\frac{3EI}{L^3 (M + 0.23m)}} \end{aligned} \quad (1)$$

$I$  is the area moment of inertia of the pipe cross section. Since  $D_o = 0.25$  m and  $D_i = 0.2$  m, then

$$\begin{aligned} I &= \frac{\pi}{64} (D_o^4 - D_i^4) \\ &= \frac{\pi}{64} (0.25^4 - 0.2^4) \\ &= 1.1321 \times 10^{-4} \text{ m}^4 \end{aligned}$$

$M = 200$  kg, and  $L = 10$  meter. Using  $\rho_{steel}g = 76500$  N/m<sup>3</sup> and  $E = 207 \times 10^9$  Pa. To find the mass  $m$  of the post, the cross sectional area is first found

$$\begin{aligned} A &= \frac{\pi}{4} (D_o^2 - D_i^2) \\ &= \frac{\pi}{4} (0.25^2 - 0.2^2) \\ &= 0.017671 \text{ m}^2 \end{aligned}$$

Hence the mass  $m$  is

$$\begin{aligned} m &= \frac{(\rho_{steel}g)}{g} AL \\ &= \frac{76500}{9.81} (0.017671) (10) \\ &= 1378 \text{ kg} \end{aligned}$$



Substituting the numerical values in (1) gives

$$\begin{aligned}\omega_n &= \sqrt{\frac{3EI}{L^3(M + 0.23m)}} \\ &= \sqrt{\frac{3(207 \times 10^9)(1.1321 \times 10^{-4})}{(10)^3(200 + 0.23(1378))}} \\ &= 11.662 \text{ rad/sec}\end{aligned}$$

Or

$$f_n = \frac{11.662}{2\pi}$$

Therefore

$$f_n = 1.8561 \text{ Hz}$$

### 0.2.2 Part B

Maximum steady state displacement occurs at resonance. This is when the frequency of vortex shedding is the same as the natural frequency  $f_n$  of the post found above. Using Strouhal formula

$$v = \frac{f_n D_o}{S}$$

Where in the above  $v$  is the wind velocity and the vortex shedding frequency is set to be the natural frequency in order to obtain the maximum displacement. Assume  $S = 0.21$  gives

$$v = \frac{(1.8561)(0.25)}{0.21}$$

Hence

$$v = 2.2096 \text{ m/s}$$

### Checking Reynold number

$$\text{Re} = \frac{v D_o \rho_{air}}{\mu}$$

$\rho_{air}$  is density of air and  $\mu$  is viscosity of air. Using the numerical values given the above becomes

$$\begin{aligned}\text{Re} &= \frac{(2.2096)(0.25)(1.2)}{(1.8 \times 10^{-5})} \\ &= 36827\end{aligned}$$

Since  $400 \leq \text{Re} \leq 300000$  then the assumption of Strouhal  $S = 0.21$  was valid.

### 0.2.3 Part C

The lateral force exerted by the wind on the sigpost is given by

$$\begin{aligned} F(t) &= \frac{1}{2} c \rho_{air} v^2 A \sin \omega t \\ &= F_0 \sin \omega t \end{aligned}$$

Where  $c \approx 1$  for cylinder and  $v$  is the wind speed found in last part and  $A$  is the projected area  $A = D_0 L$ . Hence

$$\begin{aligned} F_0 &= \frac{1}{2} c \rho_{air} v^2 A \\ &= \frac{1}{2} (1.2) (2.2096)^2 (0.25) (10) \\ &= 7.3235 \text{ N} \end{aligned}$$

Using the steady state displacement formula for damped single degree of freedom system, which is

$$y_{ss} = \frac{F_0}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

Where  $F_0$  is total force from the wind over the whole span. Assuming this force acts at the end of a fixed-free beam (This is an over estimation. The wind force actually acts over the whole length of the sigpost, but it is now taken as acting on the end). Therefore  $k = \frac{3EI}{L^3}$  can be used based on this. Since  $r = 1$  (resonance) and  $\xi = 0.1$ , then  $y_{ss}$  is now evaluated

$$\begin{aligned} y_{ss} &= \frac{F_0}{k} \frac{1}{\sqrt{4\xi^2}} \\ &= \frac{F_0 L^3}{3EI} \frac{1}{2\xi} \\ &= \frac{(7.3235) (10)^3}{3 (207 \times 10^9) (1.1321 \times 10^{-4})} \frac{1}{2(0.1)} \\ &= 5.2085 \times 10^{-4} \text{ meter} \end{aligned}$$

Or

$$y_{ss} \approx 0.5 \text{ mm}$$