HW 2, ME 440 Intermediate Vibration, Fall 2017

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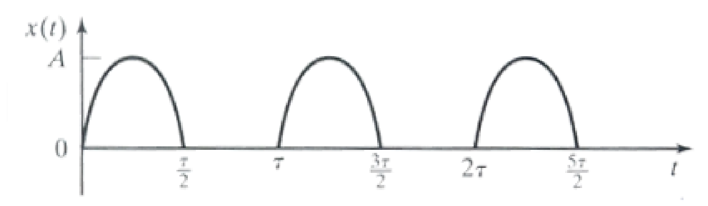
0.1 Problem 1

Problem 1

The impact force created by a forging hammer can be modeled as shown in the figure below. Determine the Fourier series expansion of the impact force.



Period is τ. This is not even and not odd. The first step is to determine the function $x(t)$. This is truncated $\sin$. Therefore we see that, over first period

$$x(t) = \begin{cases} A\sin\left(\frac{2\pi}{\tau}t\right) & 0 \leq t \leq \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < t \leq \tau \end{cases}$$

This repeated over each period by shifting it. Now that we know $x(t)$ we can find a_0, a_n, b_n and plot the approximation for larger n

$$\begin{aligned} a_0 &= \frac{1}{\frac{\tau}{2}} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)\,dt \\ &= \frac{2}{\tau}\int_0^{\frac{\tau}{2}} x(t)\,dt \\ &= \frac{2}{\tau}\int_0^{\frac{\tau}{2}} A\sin\left(\frac{2\pi}{\tau}t\right)dt \\ &= -\frac{2}{\tau}\frac{A}{\frac{2\pi}{\tau}}\left[\cos\left(\frac{2\pi}{\tau}t\right)\right]_0^{\frac{\tau}{2}} \\ &= -\frac{A}{\pi}\left[\cos\left(\frac{2\pi}{\tau}\frac{\tau}{2}\right)-1\right] \\ &= -\frac{A}{\pi}\left[\cos(\pi)-1\right] \end{aligned}$$

Hence

$$a_0 = \frac{2A}{\pi}$$

<u>Finding a_n</u>

$$
\begin{aligned}
a_n &= \frac{1}{\frac{\tau}{2}} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \cos\left(\frac{2\pi}{\tau} n t\right) dt \\
&= \frac{2}{\tau} \int_{0}^{\frac{\tau}{2}} x(t) \cos\left(\frac{2\pi}{\tau} n t\right) dt \\
&= \frac{2}{\tau} \int_{0}^{\frac{\tau}{2}} A \sin\left(\frac{2\pi}{\tau} t\right) \cos\left(\frac{2\pi}{\tau} n t\right) dt
\end{aligned}
$$

But $\sin(u)\cos(v) = \frac{1}{2}(\sin(u+v) + \sin(u-v))$, therefore the above integral becomes

$$
\begin{aligned}
a_n &= \frac{2A}{\tau} \left(\frac{1}{2} \int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} t + \frac{2\pi}{\tau} n t\right) dt + \frac{1}{2} \int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} t - \frac{2\pi}{\tau} n t\right) dt \right) \\
&= \frac{A}{\tau} \left(\int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} (1+n) t\right) dt + \int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} (1-n) t\right) dt \right) \qquad (1)
\end{aligned}
$$

The first integral above is

$$
\begin{aligned}
\int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right) dt &= -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)} \right]_{0}^{\frac{\tau}{2}} \\
&= \frac{-1}{\frac{2\pi}{\tau}(1+n)} \left[\cos\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right) - 1 \right] \\
&= \frac{-\tau}{2\pi(1+n)} \left[\cos(\pi(1+n)) - 1 \right]
\end{aligned}
$$

For $n = 1, 3, 5, \cdots$ the above becomes zero. For $n = 2, 4, 6, \cdots$

$$
\begin{aligned}
\int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right) dt &= \frac{2\tau}{2\pi(1+n)} \\
&= \frac{\tau}{\pi(1+n)} \qquad n = 2, 4, 6, \cdots \qquad (2)
\end{aligned}
$$

The second integral in (1) is

$$
\int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt = -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)} \right]_{0}^{\frac{\tau}{2}}
$$

But this is undefined for $n = 1$, since denominator is zero. Hence we need to handle $n = 1$ first on its own. At $n = 1$, since $\sin(0) = 0$ then

$$
\int_{0}^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt = 0 \qquad (3)
$$

For $n > 1$

$$\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt = -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right]_0^{\frac{\tau}{2}}$$
$$= \frac{-1}{\frac{2\pi}{\tau}(1-n)}\left[\cos\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right)-1\right]$$
$$= \frac{-1}{\frac{2\pi}{\tau}(1-n)}\left[\cos\left(\pi(1-n)\right)-1\right]$$
$$= \frac{1}{\frac{2\pi}{\tau}(n-1)}\left[\cos\left(\pi(n-1)\right)-1\right]$$

For $n = 2,4,6,\cdots$

$$\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right)dt = \frac{-2}{\frac{2\pi}{\tau}(n-1)} = \frac{-\tau}{\pi(n-1)} \tag{4}$$

For $n = 3,5,7,\cdots$ the integral is zero. Using result in (2,3,4) in (1) gives final result

$$a_n = \begin{cases} \frac{A}{\tau}\left(\frac{\tau}{\pi(1+n)} + \frac{-\tau}{\pi(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{(n-1)-(1+n)}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{n-1-1-n}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A\left(\frac{-2}{\pi(1+n)(n-1)}\right) & n = 2,4,6,\cdots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Finding b_n

$$b_n = \frac{1}{\frac{\tau}{2}}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)\sin\left(\frac{2\pi}{\tau}nt\right)dt$$
$$= \frac{2}{\tau}\int_0^{\frac{\tau}{2}} x(t)\sin\left(\frac{2\pi}{\tau}nt\right)dt$$
$$= \frac{2}{\tau}\int_0^{\frac{\tau}{2}} A\sin\left(\frac{2\pi}{\tau}t\right)\sin\left(\frac{2\pi}{\tau}nt\right)dt$$

But $\sin(u)\sin(v) = \frac{1}{2}(\cos(u-v) - \cos(u+v))$, therefore the above integral becomes

$$
\begin{aligned}
a_n &= \frac{2A}{\tau}\left(\frac{1}{2}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t - \frac{2\pi}{\tau}nt\right)dt - \frac{1}{2}\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t + \frac{2\pi}{\tau}nt\right)dt\right) \\
&= \frac{A}{\tau}\left(\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right)dt - \int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right)dt\right) \qquad (6)
\end{aligned}
$$

For the first integral

$$
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right)dt = \left(\frac{\sin\frac{2\pi}{\tau}(1-n)t}{\frac{2\pi}{\tau}(1-n)}\right)_0^{\frac{\tau}{2}}
$$

But this is undefined for $n=1$, since denominator is zero. Hence we need to handle $n=1$ first on its own. At $n=1$, since $\cos(0)=1$ then

$$
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right)dt = \int_0^{\frac{\tau}{2}} dt = \frac{\tau}{2} \qquad (7)
$$

Now for $n>1$

$$
\begin{aligned}
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right)dt &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1-n)}\left(\sin\left(\frac{2\pi}{\tau}(1-n)t\right)\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1-n)}\left(\sin\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right)-0\right) \\
&= \frac{\tau}{2\pi(1-n)}\left(\sin(\pi(1-n))-0\right)
\end{aligned}
$$

Which is <u>zero for all n</u>. For the second integral in (6)

$$
\begin{aligned}
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right) &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)}\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1+n)}\left(\sin\left(\frac{2\pi}{\tau}(1+n)t\right)\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1+n)}\left(\sin\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right)-0\right) \\
&= \frac{\tau}{2\pi(1+n)}\left(\sin(\pi(1+n))-0\right)
\end{aligned}
$$

Which is <u>zero for all n</u>. Hence for b_n we have one term only

$$
b_n = \begin{cases} \frac{A}{2} & n=1 \\ 0 & n=2,3,\cdots \end{cases}
$$

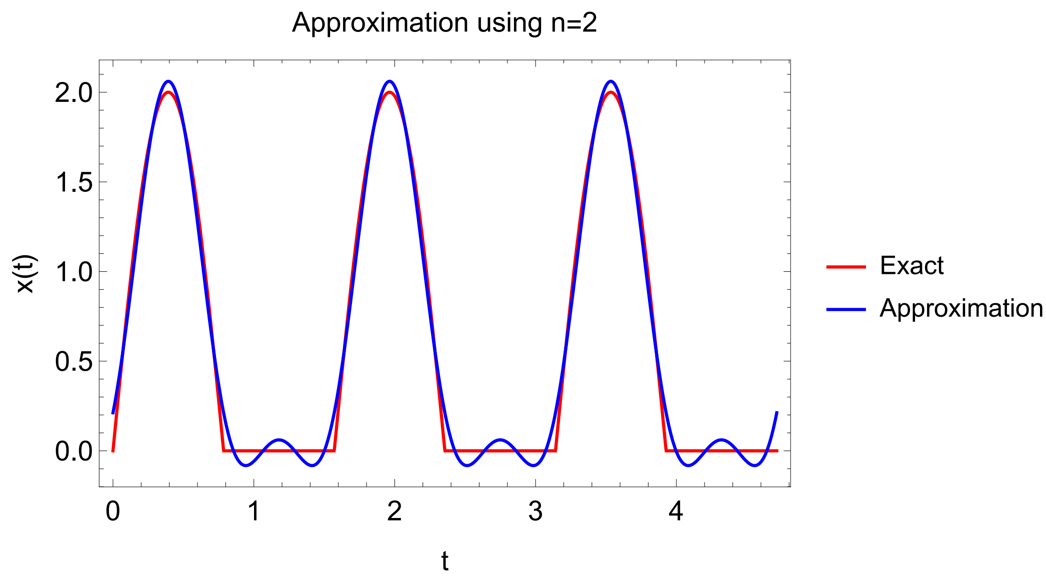
Therefore the Fourier series approximation is

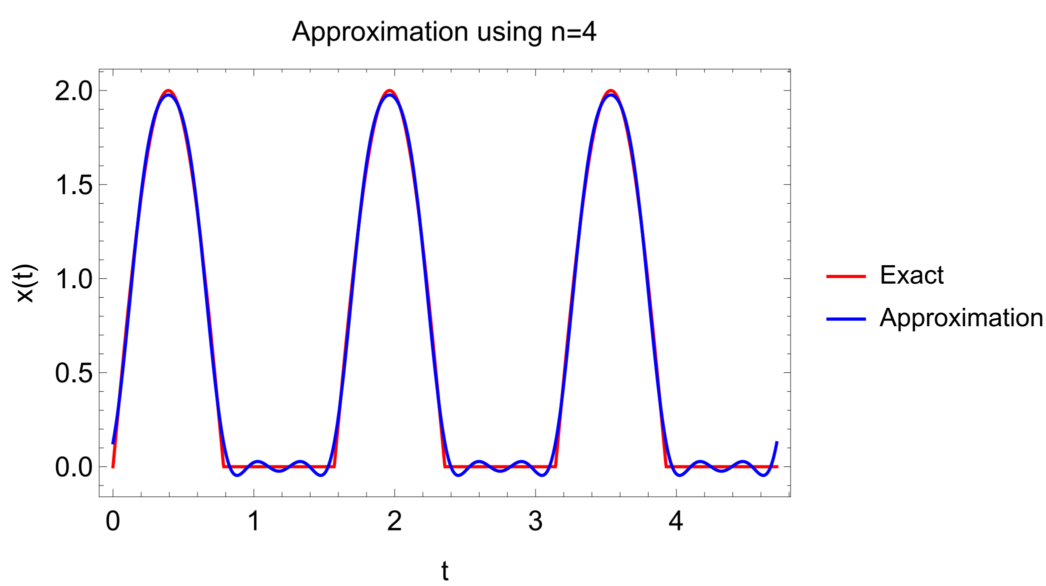
$$
\begin{aligned}
x(t) &= \overbrace{\frac{A}{\pi}}^{\frac{a_0}{2}} + \overbrace{\frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right)}^{b_1} + \sum_{n=2,4,6,\cdots}^{\infty} \overbrace{A\left(\frac{-2}{\pi(1+n)(n-1)}\right)}^{a_n}\cos\left(\frac{2\pi}{\tau}nt\right)\\
&= \frac{A}{\pi} + \frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{(1+n)(n-1)}\cos\left(\frac{2\pi}{\tau}nt\right)
\end{aligned}
$$

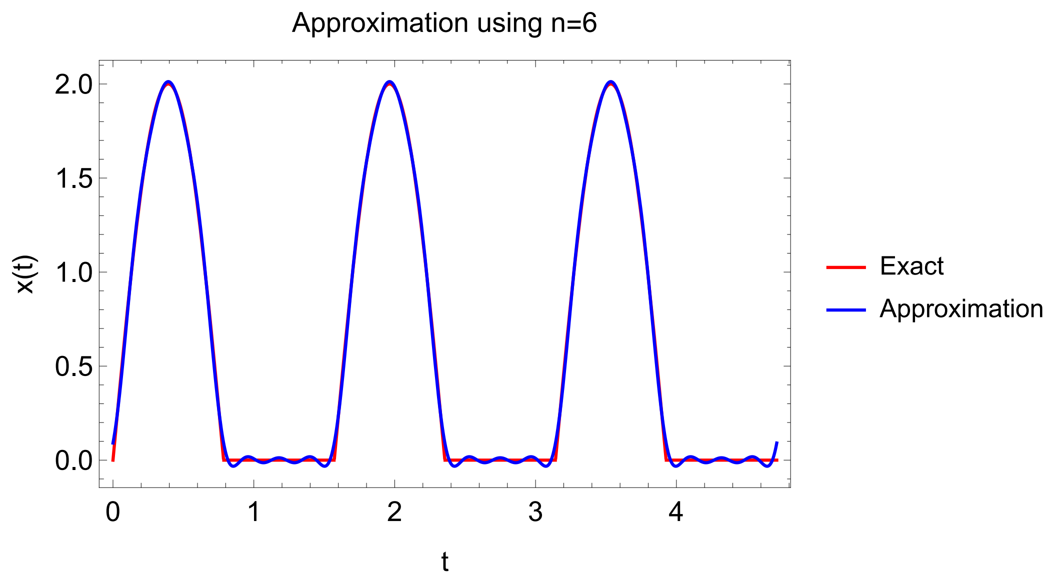
Therefore

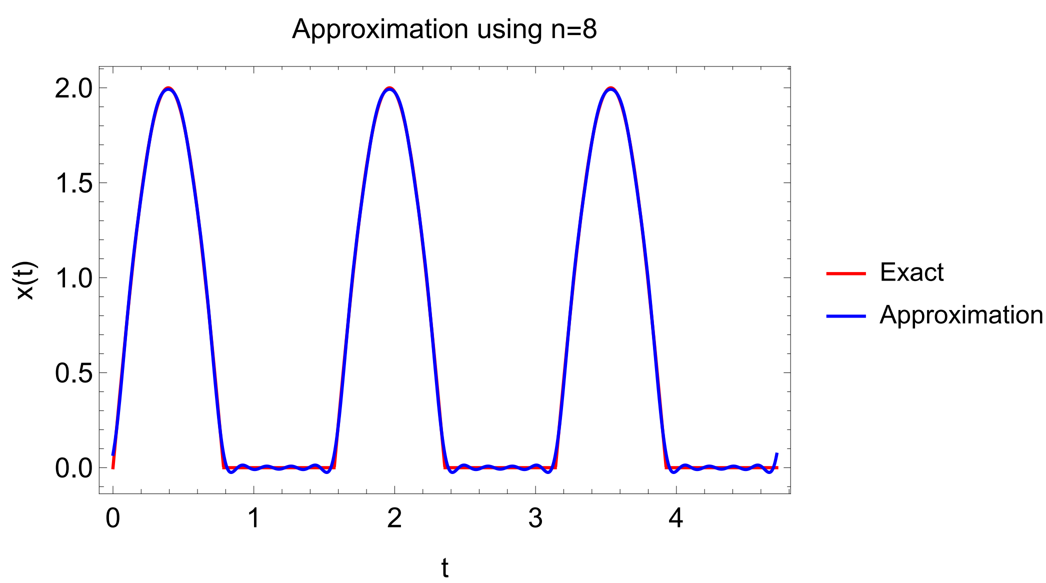
$$
x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{(1+n)(n-1)}\cos\left(\frac{2\pi}{\tau}nt\right)
$$

To verify this result, the following is a plot of increasing n, using $A = 2$ and $\tau = 1$ with the approximation superimposed on top of $x(t)$. We notice that small number of terms is needed in this case to obtain a good approximation.










```

xApprox[t_, max_, A0_, period_] := A0/Pi + (A0/2)*Sin[2*(Pi/period)*t] -
   2*(A0/Pi)*
   Sum[(1/((1 + n)*(n - 1)))*Cos[2*(Pi/period)*n*t], {n, 2, max, 2}];

myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
    func /. val :> Mod[val - min, max - min] + min

f[t_] := Piecewise[{{A0*Sin[2*(Pi/period)*t], 0 < t < period/2}, {0,True}}]

maxTerms=2;
A0=2;
period=1/2 Pi;
p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
       xApprox[t,maxTerms,A0,period]},{t,0,3 period},
       PlotLegends->{"Exact","Approximation"},
       PlotStyle->{Red,Blue},
       Frame->True,
       FrameLabel->{{"x(t)",None},{"t","Approximation using n="<>ToString[maxTerms]}},
       BaseStyle->14,ImageSize->400]
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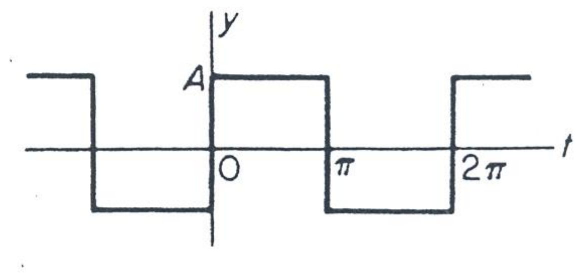
0.2 Problem 2

Problem 2

Determine the Complex Fourier series expansion for the periodic function y(t):



The function to approximate is defined as

$$y(t) = \begin{cases} A & 0 \leq t \leq \pi \\ -A & \pi < t \leq 2\pi \end{cases}$$

With period $\tau = 2\pi$. This function is odd.

$$\begin{aligned} c_n &= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} y(t)\, e^{-j\frac{2\pi}{\tau}nt} dt = \frac{1}{\tau}\int_{0}^{\tau} y(t)\, e^{-j\frac{2\pi}{\tau}nt} dt \\ &= \frac{1}{\tau}\left(\int_0^{\pi} A e^{-j\frac{2\pi}{\tau}nt} dt - \int_{\pi}^{2\pi} A e^{-j\frac{2\pi}{\tau}nt} dt\right) \\ &= \frac{A}{\tau}\left(\left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n}\right]_0^{\pi} - \left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n}\right]_{\pi}^{2\pi}\right) \\ &= \frac{A}{\tau}\left(\frac{-1}{j\frac{2\pi}{\tau}n}\left[e^{-j\frac{2\pi}{\tau}nt}\right]_0^{\pi} + \frac{1}{j\frac{2\pi}{\tau}n}\left[e^{-j\frac{2\pi}{\tau}nt}\right]_{\pi}^{2\pi}\right) \\ &= \frac{A}{\tau}\frac{\tau}{j2\pi n}\left(-\left[e^{-j\frac{2\pi}{\tau}nt}\right]_0^{\pi} + \left[e^{-j\frac{2\pi}{\tau}nt}\right]_{\pi}^{2\pi}\right) \end{aligned}$$

But $\tau = 2\pi$ and the above simplifies to

$$\begin{aligned} c_n &= \frac{A}{j2\pi n}\left(-\left[e^{-jnt}\right]_0^{\pi} + \left[e^{-jnt}\right]_{\pi}^{2\pi}\right) \\ &= \frac{A}{j2\pi n}\left(\left[1 - e^{-jn\pi}\right] + \left[e^{-j2n\pi} - e^{-jn\pi}\right]\right) \end{aligned} \tag{1}$$

But

$$\begin{aligned} e^{-jn\pi} &= \cos n\pi - j\sin n\pi \\ &= \cos n\pi \end{aligned}$$

And

$$\begin{aligned} e^{-j2n\pi} &= \cos 2n\pi - j\sin 2n\pi \\ &= 1 \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} c_n &= \frac{A}{j2\pi n}\left([1-\cos n\pi]+[1-\cos n\pi]\right) \\ &= \frac{A}{j\pi n}\left(1-\cos n\pi\right) \end{aligned}$$

For n odd $\cos n\pi = -1$ and the above becomes

$$c_n = \frac{2A}{j\pi n}$$

For n even $\cos n\pi = 1$ and $c_n = 0$ in this case. Therefore the approximation is

$$\begin{aligned} y(t) &\approx \sum_{n=\cdots-3,-1,1,3,\cdots}^{\infty} c_n e^{j2nt} \\ &= \frac{2A}{j\pi} \sum_{n=\cdots-3,-1,1,3,\cdots}^{\infty} \frac{1}{n} e^{j2nt} \end{aligned} \tag{2}$$

We can now obtain the standard form of the series if needed. $c_{-n} = c_n^* = \frac{2A}{-j\pi n}$ and hence

$$\begin{aligned} a_n &= c_n + c_{-n} \\ &= 0 \end{aligned}$$

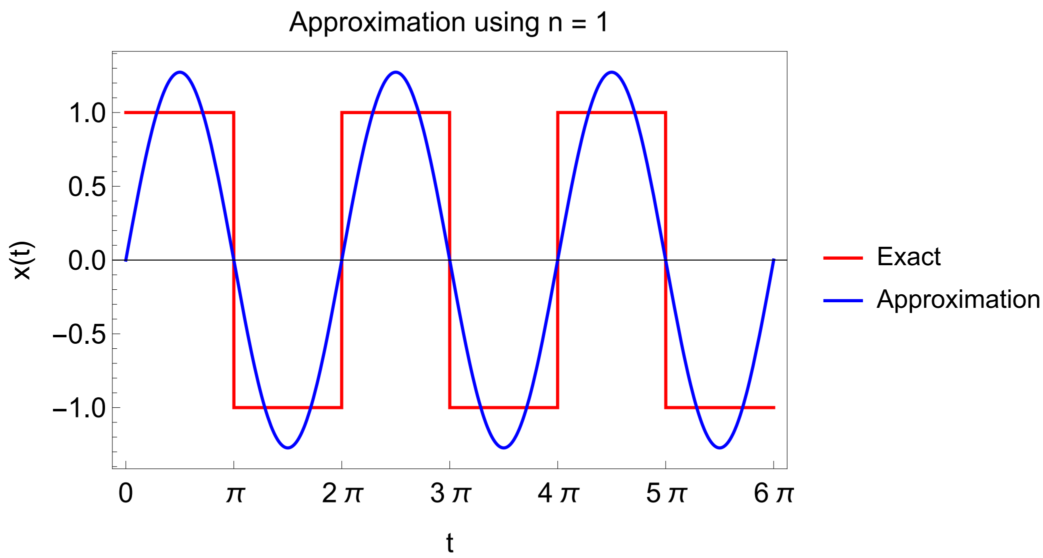
All $a_n = 0$, as expected, since this is an odd function.

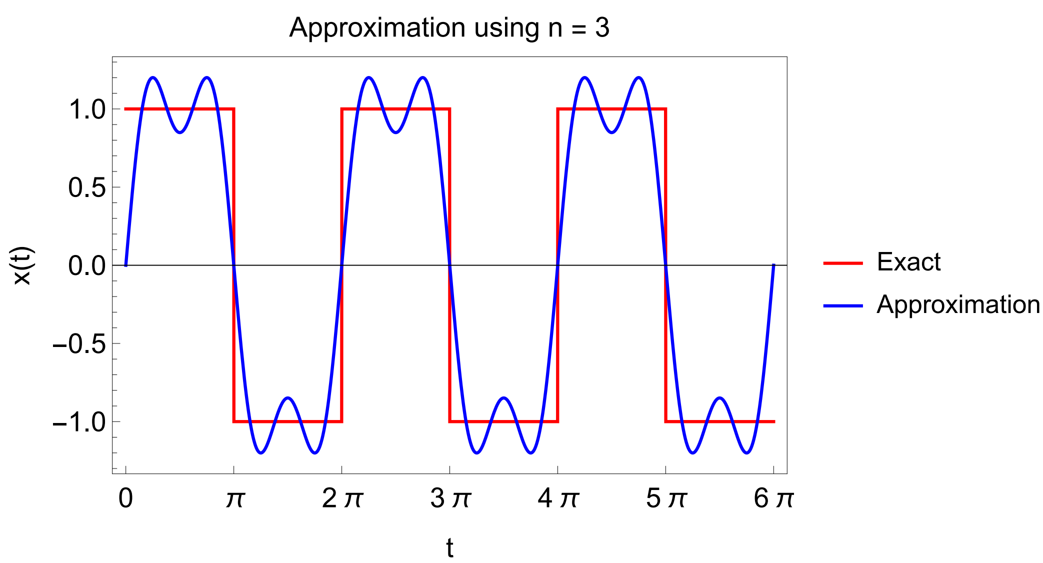
$$\begin{aligned} b_n &= j\left(c_n - c_{-n}\right) \\ &= j\left(\frac{2A}{j\pi n} - \frac{2A}{-j\pi n}\right) \\ &= j\left(\frac{4A}{j\pi n}\right) \\ &= \frac{4A}{\pi n} \end{aligned}$$

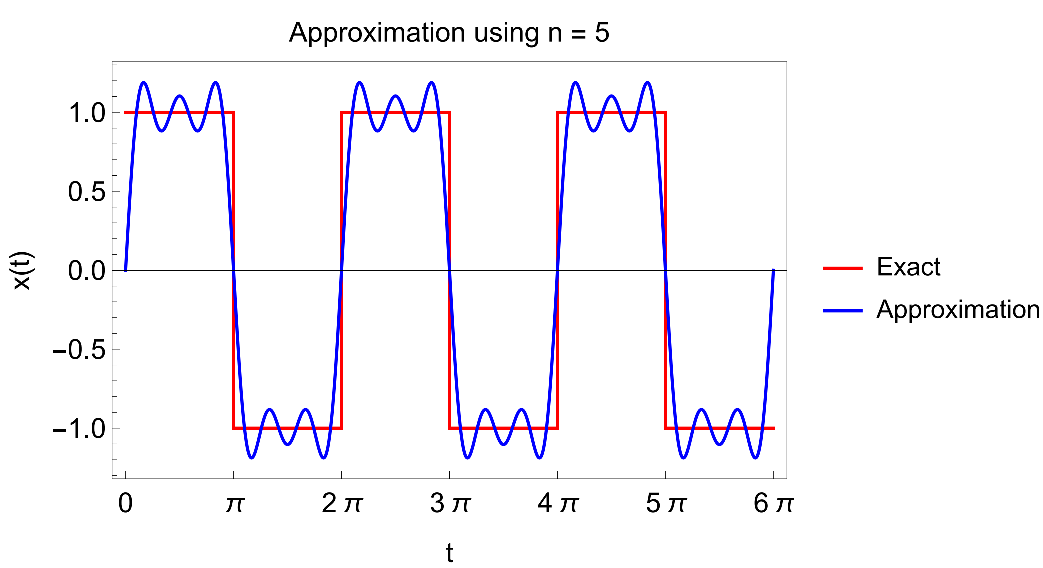
Hence

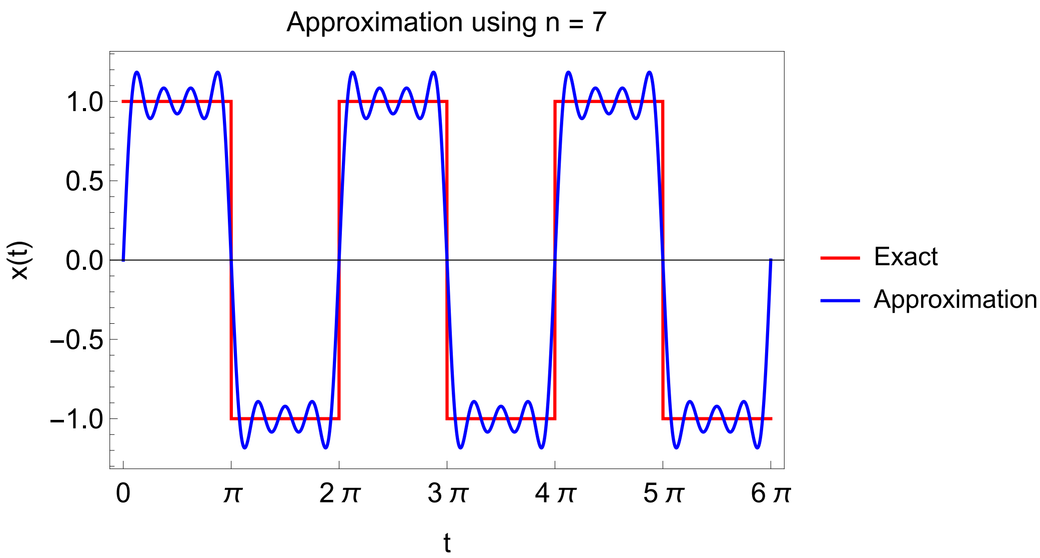
$$y(t) \approx \frac{4A}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \sin(nt) \tag{3}$$

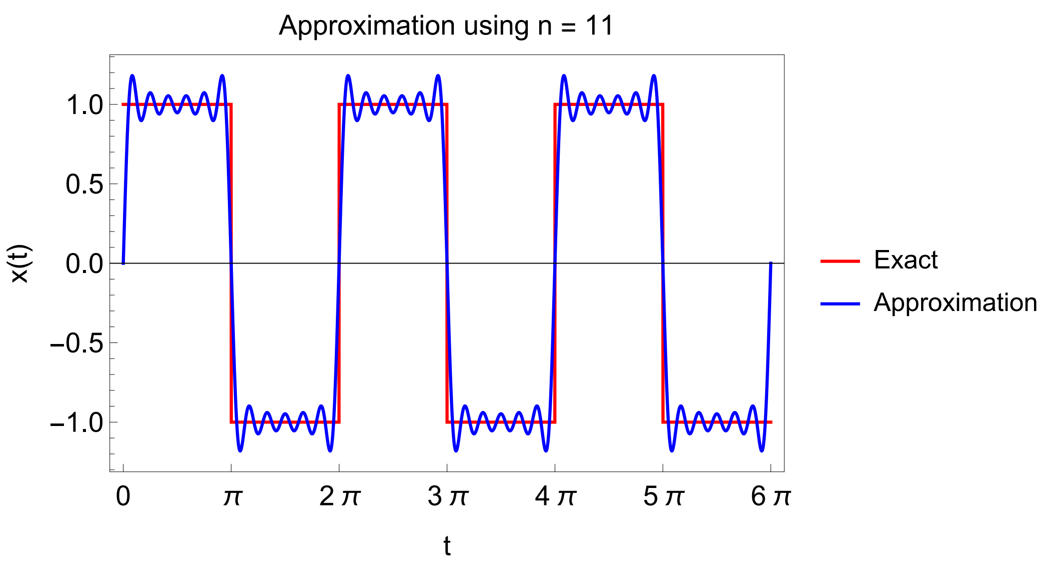
Both (2) and (3) are the same. (2) is complex form of (3). To see the approximation, here are some plots with increasing number of terms for $A = 1$

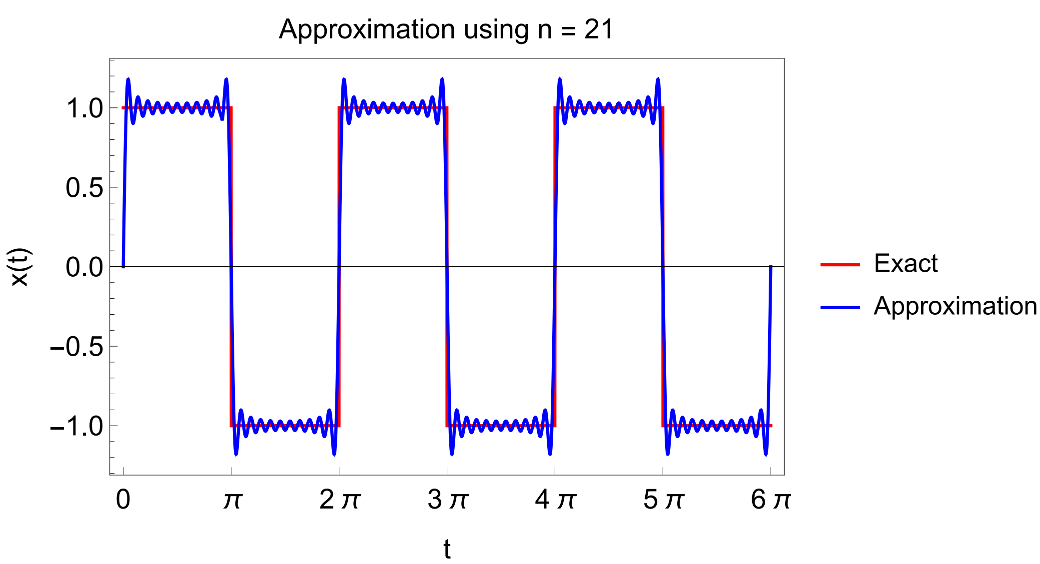

Approximation using n = 1
x(t)
1.0
0.5
0.0
−0.5
−1.0
0
π
2 π
3 π
4 π
5 π
6 π
t
Exact
Approximation


Approximation using n = 3
x(t)
1.0
0.5
0.0
−0.5
−1.0
0
π
2 π
3 π
4 π
5 π
6 π
t
Exact
Approximation


Approximation using n = 5
x(t)
1.0
0.5
0.0
−0.5
−1.0
0
π
2 π
3 π
4 π
5 π
6 π
t
Exact
Approximation








```

xApprox[t_, max_, A0_, period_] := 4 A0/Pi * Sum[(1/n)*Sin[n*t], {n, 1, max, 2}];
myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
      func /. val :> Mod[val - min, max - min] + min

f[t_] := Piecewise[{{A0, 0 < t < period/2}, {-A0, True}}];
```

```

maxTerms=11;
A0=1;
period=2 Pi;
p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
    xApprox[t,maxTerms,A0,period]},{t,0,3 period},
    PlotLegends->{"Exact","Approximation"},
    PlotStyle->{Red,Blue},
    Frame->True,
    FrameLabel->{{"x(t)",None},{"t","Approximation using n = "<>ToString[
   maxTerms]}},
    BaseStyle->14,ImageSize->400,
    Exclusions->None,
    FrameTicks->{{Automatic,None},{Range[0,6 Pi,Pi],Automatic}}]
```

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