

HW 2, ME 440 Intermediate Vibration, Fall 2017

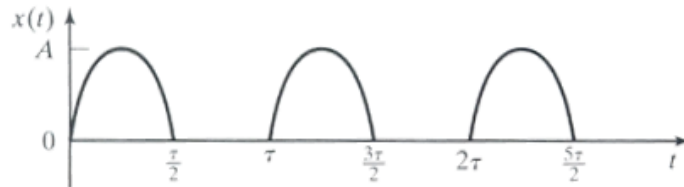
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0.1 Problem 1

Problem 1

The impact force created by a forging hammer can be modeled as shown in the figure below. Determine the Fourier series expansion of the impact force.



Period is τ . This is not even and not odd. The first step is to determine the function $x(t)$. This is truncated sin. Therefore we see that, over first period

$$x(t) = \begin{cases} A \sin\left(\frac{2\pi}{\tau}t\right) & 0 \leq t \leq \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < t \leq \tau \end{cases}$$

This repeated over each period by shifting it. Now that we know $x(t)$ we can find a_0, a_n, b_n and plot the approximation for larger n

$$\begin{aligned}
a_0 &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) dt \\
&= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} x(t) dt \\
&= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} A \sin\left(\frac{2\pi}{\tau} t\right) dt \\
&= -\frac{2}{\tau} \frac{A}{\frac{2\pi}{\tau}} \left[\cos\left(\frac{2\pi}{\tau} t\right) \right]_0^{\frac{\tau}{2}} \\
&= -\frac{A}{\pi} \left[\cos\left(\frac{2\pi}{\tau} \cdot \frac{\tau}{2}\right) - 1 \right] \\
&= -\frac{A}{\pi} [\cos(\pi) - 1]
\end{aligned}$$

Hence

$$a_0 = \frac{2A}{\pi}$$

Finding a_n

$$\begin{aligned}
a_n &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \cos\left(\frac{2\pi}{\tau} nt\right) dt \\
&= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} x(t) \cos\left(\frac{2\pi}{\tau} nt\right) dt \\
&= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} A \sin\left(\frac{2\pi}{\tau} t\right) \cos\left(\frac{2\pi}{\tau} nt\right) dt
\end{aligned}$$

But $\sin(u) \cos(v) = \frac{1}{2} (\sin(u+v) + \sin(u-v))$, therefore the above integral becomes

$$\begin{aligned}
a_n &= \frac{2A}{\tau} \left(\frac{1}{2} \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} t + \frac{2\pi}{\tau} nt\right) dt + \frac{1}{2} \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} t - \frac{2\pi}{\tau} nt\right) dt \right) \\
&= \frac{A}{\tau} \left(\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} (1+n)t\right) dt + \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau} (1-n)t\right) dt \right) \quad (1)
\end{aligned}$$

The first integral above is

$$\begin{aligned} \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right) dt &= -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)}\right]_0^{\frac{\tau}{2}} \\ &= \frac{-1}{\frac{2\pi}{\tau}(1+n)} \left[\cos\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right) - 1\right] \\ &= \frac{-\tau}{2\pi(1+n)} [\cos(\pi(1+n)) - 1] \end{aligned}$$

For $n = 1, 3, 5, \dots$ the above becomes zero. For $n = 2, 4, 6, \dots$

$$\begin{aligned} \int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1+n)t\right) dt &= \frac{2\tau}{2\pi(1+n)} \\ &= \frac{\tau}{\pi(1+n)} \quad n = 2, 4, 6, \dots \end{aligned} \tag{2}$$

The second integral in (1) is

$$\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt = -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right]_0^{\frac{\tau}{2}}$$

But this is undefined for $n = 1$, since denominator is zero. Hence we need to handle $n = 1$ first on its own. At $n = 1$, since $\sin(0) = 0$ then

$$\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt = 0 \tag{3}$$

For $n > 1$

$$\begin{aligned}
\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt &= -\left[\frac{\cos\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right]_0^{\frac{\tau}{2}} \\
&= \frac{-1}{\frac{2\pi}{\tau}(1-n)} \left[\cos\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right) - 1\right] \\
&= \frac{-1}{\frac{2\pi}{\tau}(1-n)} [\cos(\pi(1-n)) - 1] \\
&= \frac{1}{\frac{2\pi}{\tau}(n-1)} [\cos(\pi(n-1)) - 1]
\end{aligned}$$

For $n = 2, 4, 6, \dots$

$$\int_0^{\frac{\tau}{2}} \sin\left(\frac{2\pi}{\tau}(1-n)t\right) dt = \frac{-2}{\frac{2\pi}{\tau}(n-1)} = \frac{-\tau}{\pi(n-1)} \quad (4)$$

For $n = 3, 5, 7, \dots$ the integral is zero. Using result in (2,3,4) in (1) gives final result

$$a_n = \begin{cases} \frac{A}{\tau} \left(\frac{\tau}{\pi(1+n)} + \frac{-\tau}{\pi(n-1)} \right) & n = 2, 4, 6, \dots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A \left(\frac{(n-1)-(1+n)}{\pi(1+n)(n-1)} \right) & n = 2, 4, 6, \dots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A \left(\frac{n-1-1-n}{\pi(1+n)(n-1)} \right) & n = 2, 4, 6, \dots \\ 0 & \text{otherwise} \end{cases}$$

Or

$$a_n = \begin{cases} A \left(\frac{-2}{\pi(1+n)(n-1)} \right) & n = 2, 4, 6, \dots \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Finding b_n

$$\begin{aligned}
 b_n &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \sin\left(\frac{2\pi}{\tau}nt\right) dt \\
 &= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} x(t) \sin\left(\frac{2\pi}{\tau}nt\right) dt \\
 &= \frac{2}{\tau} \int_0^{\frac{\tau}{2}} A \sin\left(\frac{2\pi}{\tau}t\right) \sin\left(\frac{2\pi}{\tau}nt\right) dt
 \end{aligned}$$

But $\sin(u) \sin(v) = \frac{1}{2} (\cos(u-v) - \cos(u+v))$, therefore the above integral becomes

$$\begin{aligned}
 a_n &= \frac{2A}{\tau} \left(\frac{1}{2} \int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t - \frac{2\pi}{\tau}nt\right) dt - \frac{1}{2} \int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}t + \frac{2\pi}{\tau}nt\right) dt \right) \\
 &= \frac{A}{\tau} \left(\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt - \int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right) dt \right) \tag{6}
 \end{aligned}$$

For the first integral

$$\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt = \left(\frac{\sin \frac{2\pi}{\tau}(1-n)t}{\frac{2\pi}{\tau}(1-n)} \right)_0^{\frac{\tau}{2}}$$

But this is undefined for $n = 1$, since denominator is zero. Hence we need to handle $n = 1$ first on its own. At $n = 1$, since $\cos(0) = 1$ then

$$\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt = \int_0^{\frac{\tau}{2}} dt = \frac{\tau}{2} \tag{7}$$

Now for $n > 1$

$$\begin{aligned}
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1-n)t\right) dt &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1-n)t\right)}{\frac{2\pi}{\tau}(1-n)}\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1-n)} \left(\sin\left(\frac{2\pi}{\tau}(1-n)t\right)\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1-n)} \left(\sin\left(\frac{2\pi}{\tau}(1-n)\frac{\tau}{2}\right) - 0\right) \\
&= \frac{\tau}{2\pi(1-n)} (\sin(\pi(1-n)) - 0)
\end{aligned}$$

Which is zero for all n . For the second integral in (6)

$$\begin{aligned}
\int_0^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}(1+n)t\right) dt &= \left(\frac{\sin\left(\frac{2\pi}{\tau}(1+n)t\right)}{\frac{2\pi}{\tau}(1+n)}\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1+n)} \left(\sin\left(\frac{2\pi}{\tau}(1+n)t\right)\right)_0^{\frac{\tau}{2}} \\
&= \frac{\tau}{2\pi(1+n)} \left(\sin\left(\frac{2\pi}{\tau}(1+n)\frac{\tau}{2}\right) - 0\right) \\
&= \frac{\tau}{2\pi(1+n)} (\sin(\pi(1+n)) - 0)
\end{aligned}$$

Which is zero for all n . Hence for b_n we have one term only

$$b_n = \begin{cases} \frac{A}{2} & n = 1 \\ 0 & n = 2, 3, \dots \end{cases}$$

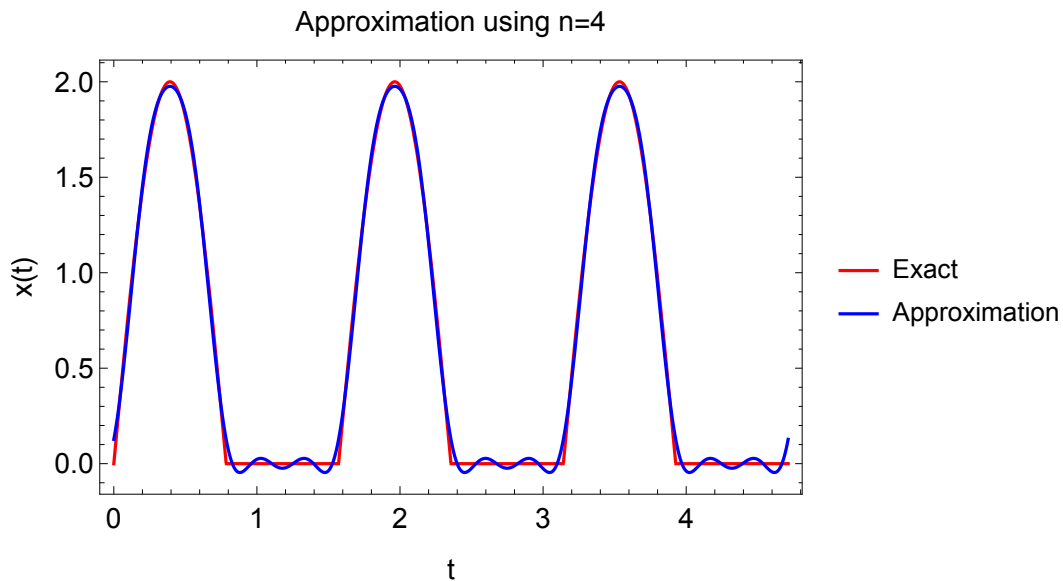
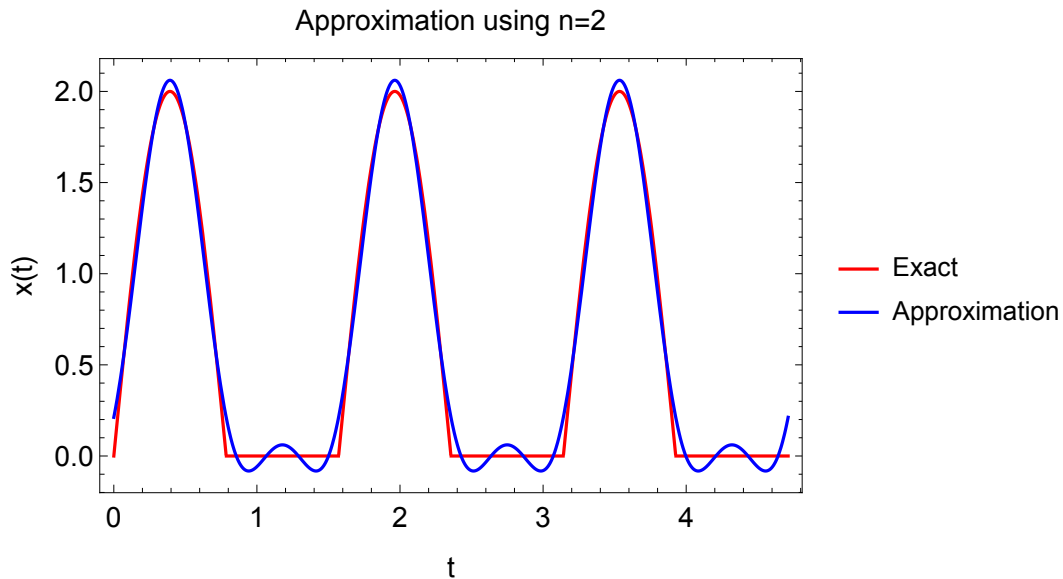
Therefore the Fourier series approximation is

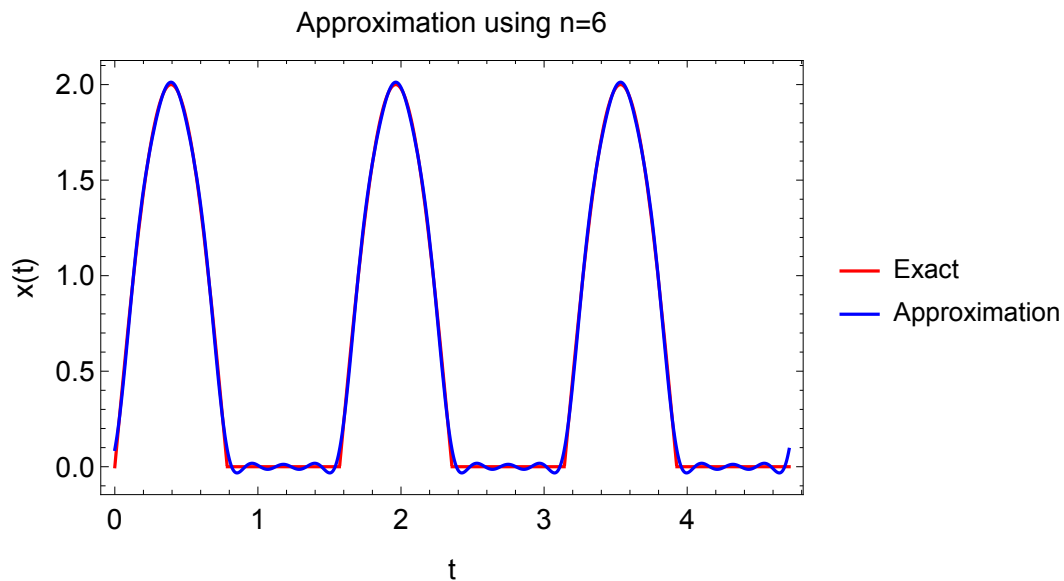
$$\begin{aligned}
x(t) &= \frac{\frac{a_0}{2}}{\pi} + \overbrace{\frac{A}{2} \sin\left(\frac{2\pi}{\tau}t\right)}^{b_1} + \sum_{n=2,4,6,\dots}^{\infty} \overbrace{A \left(\frac{-2}{\pi(1+n)(n-1)}\right)}^{a_n} \cos\left(\frac{2\pi}{\tau}nt\right) \\
&= \frac{A}{\pi} + \frac{A}{2} \sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{(1+n)(n-1)} \cos\left(\frac{2\pi}{\tau}nt\right)
\end{aligned}$$

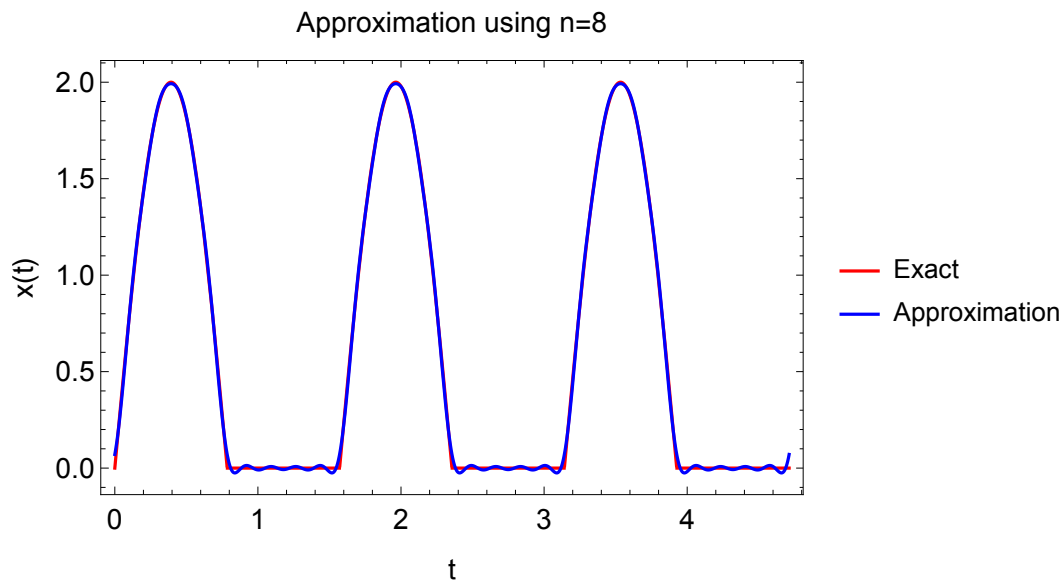
Therefore

$$x(t) = \frac{A}{\pi} + \frac{A}{2} \sin\left(\frac{2\pi}{\tau}t\right) - \frac{2A}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{(1+n)(n-1)} \cos\left(\frac{2\pi}{\tau}nt\right)$$

To verify this result, the following is a plot of increasing n , using $A = 2$ and $\tau = 1$ with the approximation superimposed on top of $x(t)$. We notice that small number of terms is needed in this case to obtain a good approximation.







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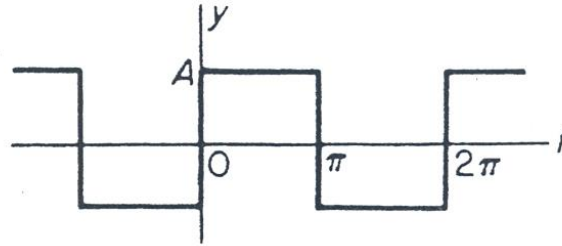
1
2 xApprox[t_, max_, A0_, period_] := A0/Pi + (A0/2)*Sin[2*(Pi/period)*t] -
3   2*(A0/Pi)*
4   Sum[(1/((1 + n)*(n - 1)))*Cos[2*(Pi/period)*n*t], {n, 2, max, 2}];
5
6 myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
7   func /. val -> Mod[val - min, max - min] + min
8
9 f[t_] := Piecewise[{{A0*Sin[2*(Pi/period)*t], 0 < t < period/2}, {0,True}}]
10
11 maxTerms=2;
12 A0=2;
13 period=1/2 Pi;
14 p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
15   xApprox[t,maxTerms,A0,period]},{t,0,3 period},
16   PlotLegends->{"Exact","Approximation"},
17   PlotStyle->{Red,Blue},
18   Frame->True,
19   FrameLabel->{{"x(t)",None},{t,"Approximation using n="<>ToString[
20   maxTerms]}}},
   BaseStyle->14,ImageSize->400]

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0.2 Problem 2

Problem 2

Determine the Complex Fourier series expansion for the periodic function $y(t)$:



The function to approximate is defined as

$$y(t) = \begin{cases} A & 0 \leq t \leq \pi \\ -A & \pi < t \leq 2\pi \end{cases}$$

With period $\tau = 2\pi$. This function is odd.

$$\begin{aligned} c_n &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} y(t) e^{-j\frac{2\pi}{\tau}nt} dt = \frac{1}{\tau} \int_0^{\tau} y(t) e^{-j\frac{2\pi}{\tau}nt} dt \\ &= \frac{1}{\tau} \left(\int_0^{\pi} A e^{-j\frac{2\pi}{\tau}nt} dt - \int_{\pi}^{2\pi} A e^{-j\frac{2\pi}{\tau}nt} dt \right) \\ &= \frac{A}{\tau} \left(\left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n} \right]_0^{\pi} - \left[\frac{e^{-j\frac{2\pi}{\tau}nt}}{-j\frac{2\pi}{\tau}n} \right]_{\pi}^{2\pi} \right) \\ &= \frac{A}{\tau} \left(\frac{-1}{j\frac{2\pi}{\tau}n} \left[e^{-j\frac{2\pi}{\tau}n\pi} \right]_0^{\pi} + \frac{1}{j\frac{2\pi}{\tau}n} \left[e^{-j\frac{2\pi}{\tau}n\pi} \right]_{\pi}^{2\pi} \right) \\ &= \frac{A}{\tau} \frac{\tau}{j2\pi n} \left(- \left[e^{-j\frac{2\pi}{\tau}nt} \right]_0^{\pi} + \left[e^{-j\frac{2\pi}{\tau}nt} \right]_{\pi}^{2\pi} \right) \end{aligned}$$

But $\tau = 2\pi$ and the above simplifies to

$$\begin{aligned}
 c_n &= \frac{A}{j2\pi n} \left(-[e^{-jnt}]_0^\pi + [e^{-jnt}]_\pi^{2\pi} \right) \\
 &= \frac{A}{j2\pi n} \left([1 - e^{-jn\pi}] + [e^{-j2n\pi} - e^{-jn\pi}] \right)
 \end{aligned} \tag{1}$$

But

$$\begin{aligned}
 e^{-jn\pi} &= \cos n\pi - j \sin n\pi \\
 &= \cos n\pi
 \end{aligned}$$

And

$$\begin{aligned}
 e^{-j2n\pi} &= \cos 2n\pi - j \sin 2n\pi \\
 &= 1
 \end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
 c_n &= \frac{A}{j2\pi n} ([1 - \cos n\pi] + [1 - \cos n\pi]) \\
 &= \frac{A}{j\pi n} (1 - \cos n\pi)
 \end{aligned}$$

For n odd $\cos n\pi = -1$ and the above becomes

$$c_n = \frac{2A}{j\pi n}$$

For n even $\cos n\pi = 1$ and $c_n = 0$ in this case. Therefore the approximation is

$$\begin{aligned}
 y(t) &\approx \sum_{n=\dots-3,-1,1,3,\dots}^{\infty} c_n e^{j2nt} \\
 &= \frac{2A}{j\pi} \sum_{n=\dots-3,-1,1,3,\dots}^{\infty} \frac{1}{n} e^{j2nt}
 \end{aligned} \tag{2}$$

We can now obtain the standard form of the series if needed. $c_{-n} = c_n^* = \frac{2A}{-j\pi n}$ and hence

$$\begin{aligned}
 a_n &= c_n + c_{-n} \\
 &= 0
 \end{aligned}$$

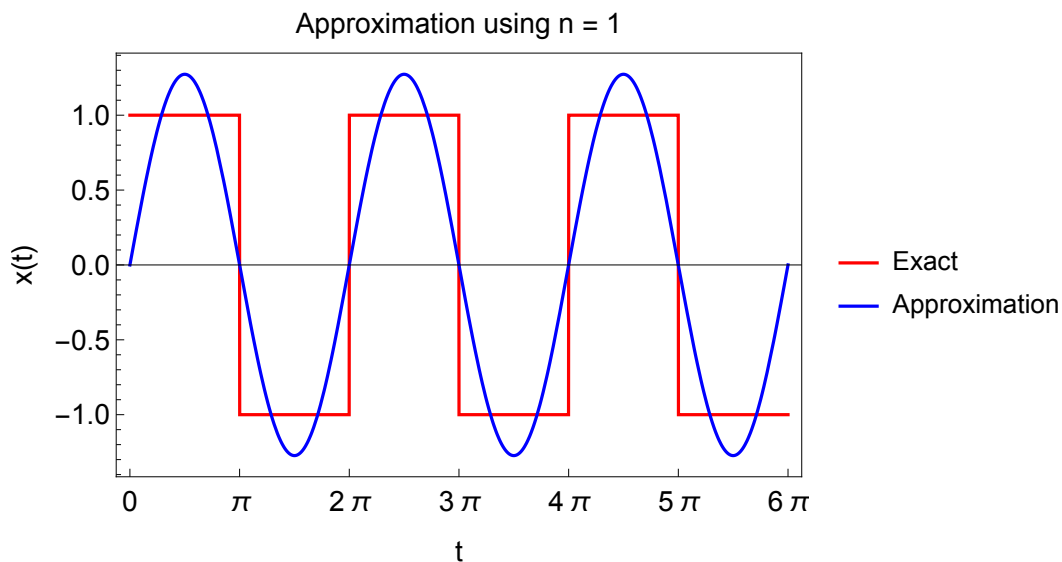
All $a_n = 0$, as expected, since this is an odd function.

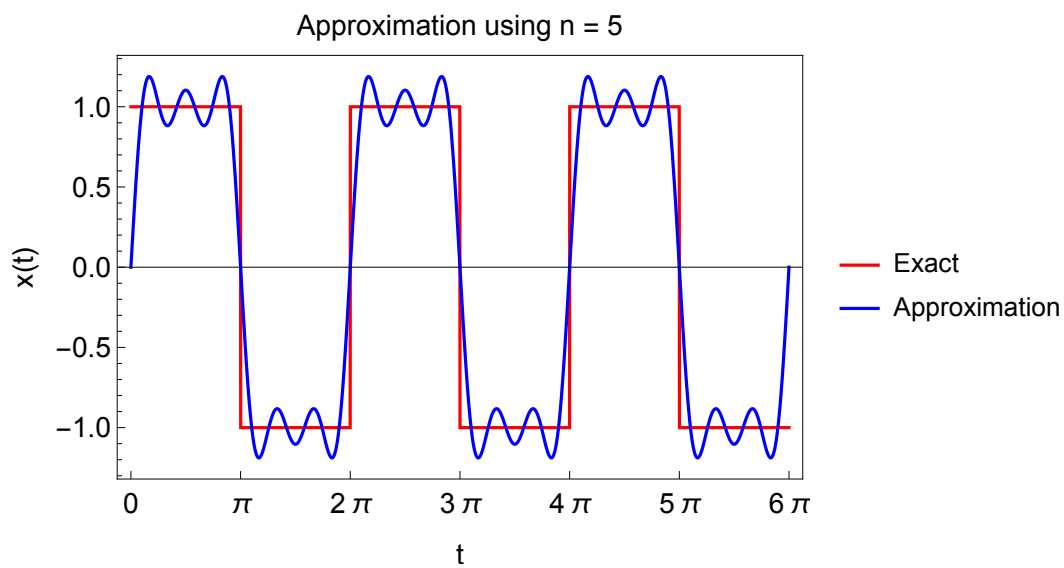
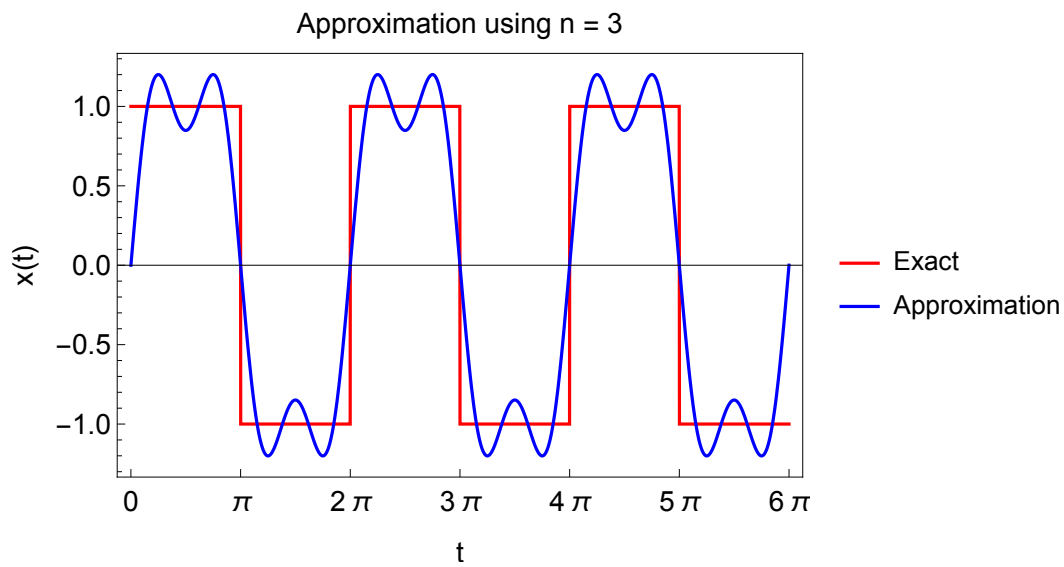
$$\begin{aligned}
 b_n &= j(c_n - c_{-n}) \\
 &= j\left(\frac{2A}{j\pi n} - \frac{2A}{-j\pi n}\right) \\
 &= j\left(\frac{4A}{j\pi n}\right) \\
 &= \frac{4A}{\pi n}
 \end{aligned}$$

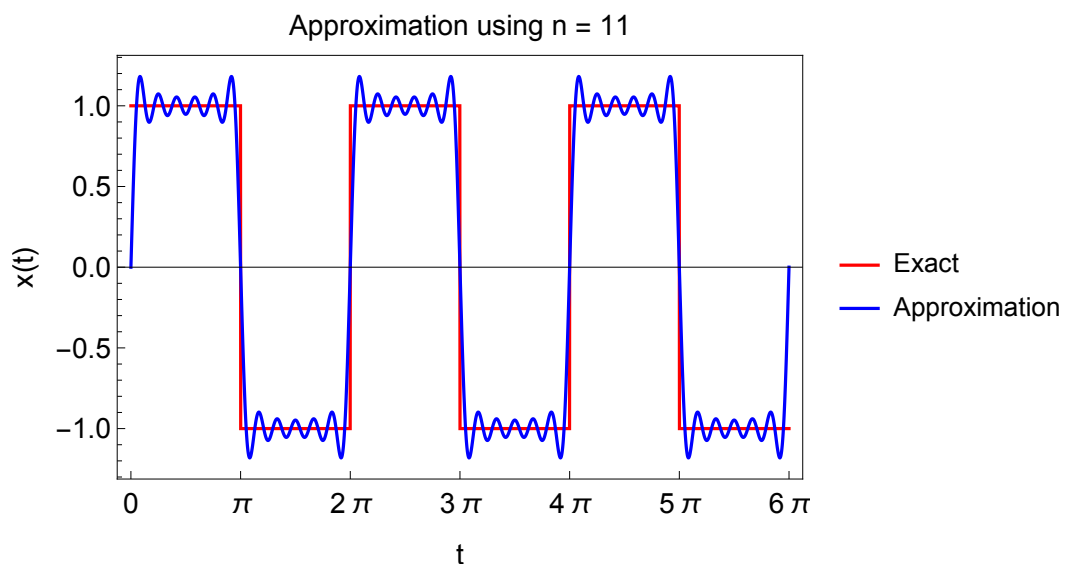
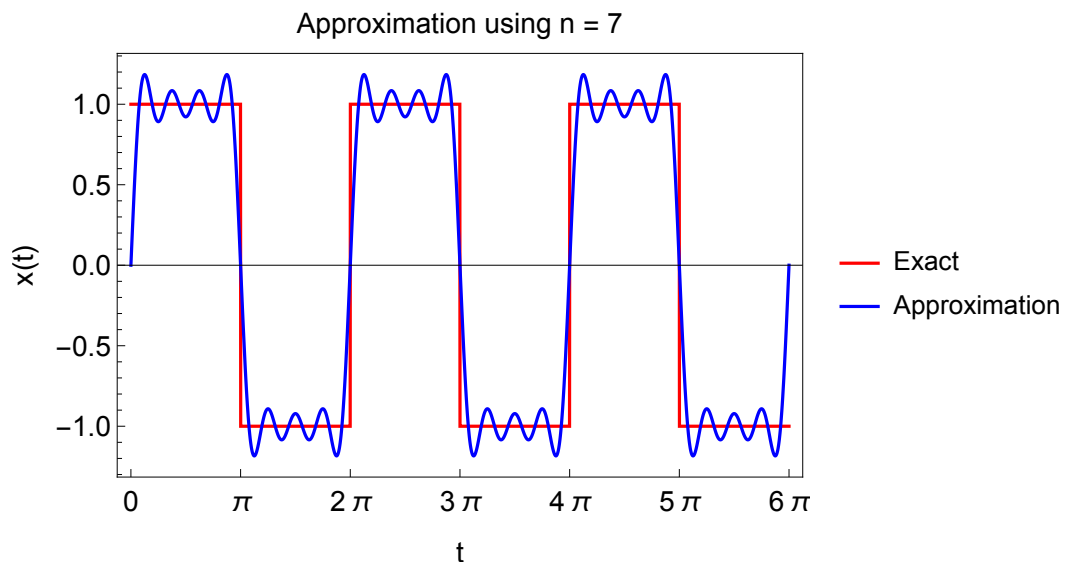
Hence

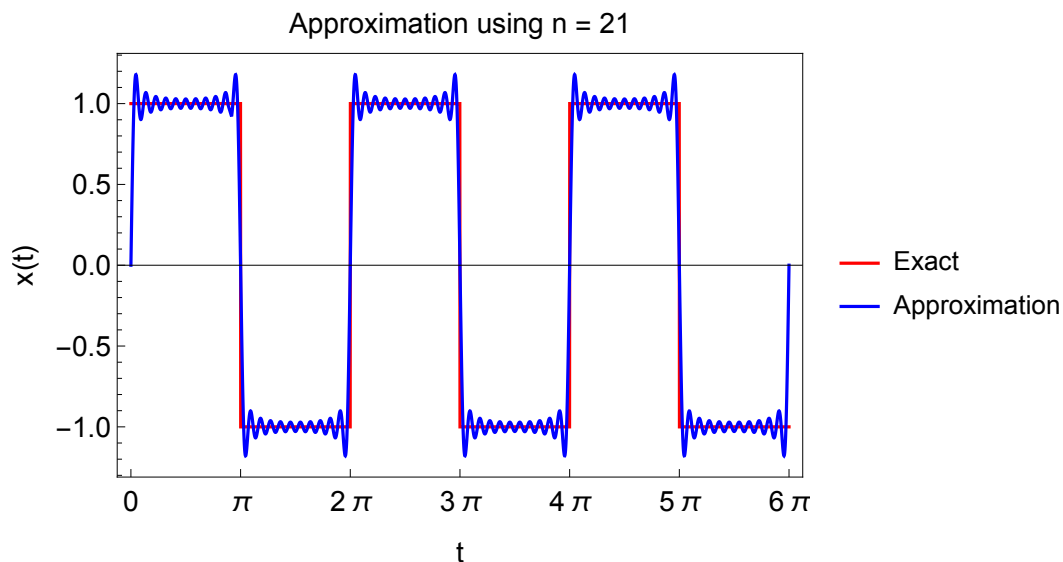
$$y(t) \approx \frac{4A}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin(nt) \quad (3)$$

Both (2) and (3) are the same. (2) is complex form of (3). To see the approximation, here are some plots with increasing number of terms for $A = 1$









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1
2 xApprox[t_, max_, A0_, period_] := 4 A0/Pi * Sum[(1/n)*Sin[n*t], {n, 1, max,
3     2}];
4 myperiodic[func_, {val_Symbol, (min_)?NumericQ, (max_)?NumericQ}] :=
5     func /. val -> Mod[val - min, max - min] + min
6 f[t_] := Piecewise[{{A0, 0 < t < period/2}, {-A0, True}}];
7
8 maxTerms=11;
9 A0=1;
10 period=2 Pi;
11 p=Plot[{Evaluate[myperiodic[f[t],{t,0,period}]],
12     xApprox[t,maxTerms,A0,period]},{t,0,3 period},
13     PlotLegends->{"Exact","Approximation"},
14     PlotStyle->{Red,Blue},
15     Frame->True,
16     FrameLabel->{{"x(t)",None},{t,"Approximation using n = "<>ToString[
17     maxTerms]}},
18     BaseStyle->14,ImageSize->400,
19     Exclusions->None,
20     FrameTicks->{{Automatic, None}, {Range[0, 6 Pi, Pi], Automatic}}]

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