University Course

Math 322
Applied Mathematical Analysis

University of Wisconsin, Madison
Fall 2016

My Class Notes

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Fall 2016

Contents

Chapter 1

Introduction

Local contents

1.1 links

1. piazza page `http://piazza.com/wisc/fall2016/math322` requires login
2. Professor Leslie Smith page Office hrs: MW from 1:15-2:15 in Van Vleck 825.

1.2 syllabus

Math 322 - 001 Syllabus
Applied Mathematical Analysis (Introduction to Partial Differential Equations)
MWF 12:05-12:55 in Van Hise 115

Textbook: *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Richard Haberman, 5th Edition, Pearson.

Pre-requisites: Math 319; Math 321; recommended: Math 340 or Math 320.

Professor: Leslie Smith, Departments of Mathematics and Engineering Physics, Office Hours in Van Vleck 825 MW 1:15-2:15, lsmith@math.wisc.edu,
http://www.math.wisc.edu/~lsmith.

Midterm Exams: There will be two in-class exams: **Monday October 10 and Monday November 14**. Please plan accordingly. Each exam is 25% of the final grade.

Final Exam: Saturday December 17, 5:05 - 7:05 PM, 35% of grade.

Piazza: There will be a Piazza course page to facilitate peer-group discussions. Please consider this resource mainly as a discussion among students. The instructor will check in a few times per week. **Piazza Sign-Up Page:** piazza.com/wisc/fall2016/math322

Piazza Course Page: piazza.com/wisc/fall2016/math322/home

Weekly Problem Sets: Homework is **due at the beginning of class**, normally on Friday. Homework problems will be selected from the book, and will be available on-line at www.math.wisc.edu/~lsmith approximately one week prior to the due date.

Please write your name clearly on each homework set, stapled please! Unstapled homework will not be accepted.

Grading of Homework: A grader will grade a subset of the homework problems given out each week, with some points also given for completeness. The homework scores will count for 15% of the grade. The lowest homework score will be dropped.

Late Policy: Homework turned in after the beginning of class will be considered late and will be graded at 80% credit. Late homework will be accepted until 5 PM on the due date (no credit thereafter, no exceptions). The policy is intended to keep everyone as current as possible.

Please email the instructor directly **before the due time/day** to make arrangements regarding late homework submission.

Expectations In Class: You are required to come to class. Some classes may involve student participation such as discussion, group work, student presentation of material, etc.

If you should need to miss a class for any reason, please let me know ahead of time, and make sure that you get notes and other important information from a classmate.

No cell phones, ipods, computers or other electronic devices may be used in class. In particular, please refrain from texting during class.

Find my mistakes in class, get brownie points!

Expectations Outside of Class: In order to fully understand the material and do well in the course, it is vital that you stay on top of your reading and homework assignments. The six hours (minimum) of work outside class includes (but is not limited to) reading the texts (before and after the material is covered in lecture), completing/writing homework problems, and reviewing for exams. In addition, be prepared to work additional problems as needed, to formulate coherent questions for me and for your classmates, and to prepare material for discussion and or student presentation.

Grading Scale for Final Grade: 92-100 A, 89-91 AB, 82-88 B, 79-81 BC, 70-78 C, 60-69 D, 59 and below F

Course description: This is a first course in Partial Differential Equations. We will focus on the physical phenomena represented by three canonical equations – the Heat Equation, Laplace's Equation and the Wave Equation– and learn the mathematical solution techniques. A basic starting point for these linear equations is Separation of Variables, and we will learn how to construct Eigenfunction Solutions, starting in one space dimension and then in two and three dimensions. More advanced topics include Green's function solutions, Fourier Transform solutions, and the Method of Characteristics.

Course outline: The course covers most of the material in Chapters 1-5, and selected material from Chapters 7-10, 12 (time permitting). The topics are listed below with corresponding chapter.

Chapter 1: The Heat Equation

Chapter 2: Method of Separation of Variables

Chapter 3: Fourier Series

Chapter 4: Wave Equation: Vibrating Strings and Membranes

Chapter 5: Sturm-Liouville Eigenvalue Problems

Chapter 7: Higher-Dimensional Partial Differential Equations

Chapter 8: Non-homogeneous Problems

Chapter 9: Green's Functions for Time Independent Problems

Chapter 10: Infinite Domain Problems: Fourier Transform Solutions of Partial Differential Equations

Chapter 12: The Method of Characteristics

Chapter 2

HWs

Local contents

2.1 summary pf HWs

HW	description
1	chapter one. Steady state. Heat flux. Total heat energy
2	chapter 2.4,2.5. separation of variables from selected PDE's. Finding eigevalues for different homogeneous B.C. Heat PDE in 1D with initial conditions, homogeneous B.C. Total heat energy
3	chapter 2.5 Laplace on rectangle Laplace on quarter circle Laplace inside circular annulus backward heat PDE is not well posed. Drag force zero for uniform flow past cylinder Circulation around cylinder
4	chapters 2.5,3.2, 3.4 Fourier series, even and odd extensions Heat PDE 1D, source with homogenous B.C.
5	chapters 3.5,3.6,4.2,4.4 Fourier series, even and odd extensions Heat PDE 1D, source with homogenous B.C. Find Fourier series for x^m , Complex Fourier series Derive vibrating string wave equation. Wave equation with damping. Derive conservation of energy for vibrating string.
6	chapter 5.3,5.5 Wave equation Sturm-Liouville DE, more eigenvalue Sturm-Liouville, self adjoint Show that eigenfunctions are orthogonal. More S-L problems
7	chapter 5.6,5.9 Rayleigh quotient to find upper bound on lowest eigenvalue for S-L ODE Show eigenvalue is positive for S-L Estimating large eigenvalues for S-L with different boundary conditions Sketch eigenfunctions for $y'' + \lambda(1 + x)y = 0$
8	chapter 5.10, 7.3, 7.4 How many terms needed for Fourier series for $f(x) = 1$? Find formula for infinite series using Parseval's equality. More on Parseval's equality Solve Wave equation with homogeneous B.C. Solve Laplace in 3D, seperation of variables. Show that $\lambda \geq 0$ using Rayleigh quotient. Derive Green formula

9	Chapter 8.2,8.3,8.4,8.5 Heat PDE 1D with zero source and non-homogeneous BC. Using $u = v + u_E$ where u_E is equilibrium solution. (Use u_E if source is zero. Heat PDE 1D with source and non-homogeneous BC. Using $u = v + u_r$ where u_r is reference solution (only needs to satisfy BC) since source is not zero. Solve heat PDE inside circle. No source, non-homogeneous BC, use u_E. Solve heat PDE in 1D with time dependent k Solve heat PDE inside circle. Source, homogeneous BC. Solve heat PDE 1D. Source and non-homogeneous BC. use u_r Solve heat PDE 1D. Source and non-homogeneous BC without using u_r. Solve wave equation, 1D with source and homogeneous BC. Solve wave equation, 2D membrane. With source and fixed boundaries.
10	Hand problems, not from text Solve ODE's using two sided Green function with different boundary conditions
11	Hand problems, not from text Solve Laplace using method of images. Different boundary conditions
12	chapter 12.2 Wave equation in 1D, such as $\frac{\partial y}{\partial t} - 3\frac{\partial y}{\partial x} = 0$ using method of charaterstics. More wave in 1D with source. $\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = e^{2x}$

2.2 HW 1

Reference table used in HW

$\vec{\phi}$	flux (class uses $\vec{q}$)	vector field. thermal energy per unit time per unit area. $\boxed{\frac{M}{T^3}}$
$\vec{\phi} \cdot \hat{n}$	flux	Flux component that is outward normal to the surface $\boxed{\frac{M}{T^3}}$
Q	heat source	heat energy generated per unit volume per unit time. $\boxed{\frac{M}{LT^3}}$
e		thermal energy density. Scalar field. $\boxed{\frac{M}{LT^2}}$
ρ	density	mass density of material which heat flows in. $\boxed{\frac{M}{L^3}}$
c	specific heat	energy to raise temp. of unit mass by one degree Kelvin. $\boxed{\frac{L^2}{T^2 k^o}}$
k_0	Thermal conductivity	Used in flux equation $q = -k_0 \nabla u$, where u is temperature. $\boxed{\frac{ML}{T^3 k^0}}$
κ	Thermal diffusivity	Used in heat equation $\frac{\partial u}{\partial t} = \kappa \nabla u + \tilde{Q}$. Where $\kappa = \frac{k_0}{\rho c}$, u is temperature.
	conservation of energy	$\frac{d}{dt}\int_V e(x,t)\,dv = \int_A \bar{q} \cdot (-\hat{n})\,dA + \int_V Q dv$. Each term has units $\boxed{\frac{ML^2}{T^3}}$
	Fourier law	$\vec{\phi} = -k_0 \bar{\nabla} u$. Relates flux to temperature gradient.
∇	Divergence operator	A vector operator. $\bar{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$

2.2.1 Problem 1 (1.5.2)

***1.5.2. For conduction of thermal energy, the heat flux vector is $\phi = -K_0\nabla u$. If in addition the molecules move at an average velocity V, a process called convection, then briefly explain why $\phi = -K_0\nabla u + c\rho u V$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).**

Fourier law is used to relate the flux to the temperature u by $\phi = -k_0 \frac{\partial u}{\partial x}$ for 1D or $\vec{\phi} = -k_0 \bar{\nabla} u$ in general.

In addition to conduction, there is convection present. This implies there is physical material mass flowing out of the control volume carrying thermal energy with it in addition to the process of conduction. Hence the flux is adjusted by this extra amount of thermal energy motion. The amount of mass that flows out of the surface per unit time per unit area is $\left(\bar{v}\rho\right) \equiv \left[\frac{L}{T}\frac{M}{L^3}\right] = \left[\frac{M}{T}\frac{1}{L^2}\right]$. Where $\rho \equiv \left[\frac{M}{L^3}\right]$ is the mass density of the material and $\bar{v} \equiv \left[\frac{L}{T}\right]$ is velocity vector of material flow at the surface.

Amount of thermal energy that $\left(\bar{v}\rho\right)$ contains is given by $\left(\bar{v}\rho\right) cu$ where c is the specific heat and u is the temperature. Therefore $\left(\bar{v}\rho\right) cu$ is the additional flux due to convection part. Total flux becomes

$$\vec{\phi} = -k_0 \bar{\nabla} u + \bar{v}\rho c u \tag{1}$$

Starting from first principles. Using conservation of thermal energy given by

$$\frac{\partial e}{\partial t} = -\left(\bar{\nabla} \cdot \vec{\phi}\right)$$

Where e is thermal energy density in the control volume. In this problem $Q = 0$ (no energy source). The integral form of the above is

$$\frac{d}{dt}\int_V e\left(\bar{x}, t\right) dV = \int_S \vec{\phi} \cdot \left(-\hat{n}\right) dA$$

The dot product with the unit normal vector $\hat{n}$ was added to indicate the normal component of $\vec{\phi}$ at the surface. Since $e\left(\bar{x}, t\right) = \rho c u$ and by using divergence theorem the above is written as

$$\frac{d}{dt}\int_V \rho c u dV = \int_V \bar{\nabla} \cdot \left(-\vec{\phi}\right) dV$$

Using (1) in the above and moving the time derivative inside the integral (which becomes partial derivative) results in

$$\int_V \rho c \frac{\partial u}{\partial t} dV = \int_V \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v}\rho c u\right) dV$$

Moving all terms under one integral sign

$$\int_V \left[\rho c \frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v} \rho c u\right)\right] dV = 0$$

Since this is zero for all control volumes, therefore the integrand is zero

$$\rho c \frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v} \rho c u\right) = 0$$

Assuming $\kappa = \frac{k_0}{\rho c}$, the above simplifies to

$$\boxed{\frac{\partial}{\partial t} u = \kappa \nabla^2 u - \bar{\nabla} \cdot (\bar{v} u)} \tag{2}$$

Applying to (2) the property of divergence of the product of scalar and a vector given by

$$\bar{\nabla} \cdot (\bar{v} u) = u \left(\bar{\nabla} \cdot \bar{v}\right) + \bar{v} \cdot \left(\bar{\nabla} u\right)$$

Equation (2) becomes

$$\boxed{\frac{\partial u}{\partial t} = \kappa \nabla^2 u - \left(u \left(\bar{\nabla} \cdot \bar{v}\right) + \bar{v} \cdot \left(\bar{\nabla} u\right)\right)}$$

2.2.2 Problem 2 (1.5.3)

1.5.3. **Consider the polar coordinates**

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

(a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$, and $\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$.

(b) Show that $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ and $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$.

(c) Using the chain rule, show that $\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}$ and hence $\nabla u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}$.

(d) If $A = A_r\hat{r} + A_\theta\hat{\theta}$, show that $\nabla\cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta)$, since $\partial\hat{r}/\partial\theta = \hat{\theta}$ and $\partial\hat{\theta}/\partial\theta = -\hat{r}$ follows from part (b).

(e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

$$x = r\cos\theta \tag{1}$$
$$y = r\sin\theta \tag{2}$$

2.2.2.1 part (a)

since $r^2 = x^2 + y^2$ then taking derivative w.r.t. x

$$\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \\ \frac{\partial r}{\partial x} &= \frac{x}{r} \\ &= \frac{r\cos\theta}{r} \\ &= \cos\theta \end{aligned} \tag{3}$$

And taking derivative w.r.t. y

$$\begin{aligned} 2r\frac{\partial r}{\partial y} &= 2y \\ \frac{\partial r}{\partial y} &= \frac{y}{r} \\ &= \frac{r\sin\theta}{r} \\ &= \sin\theta \end{aligned} \tag{4}$$

Now taking derivative w.r.t. y of (2) gives

$$1 = \frac{\partial r}{\partial y}\sin\theta + r\frac{\partial \sin\theta}{\partial y}$$

From (4) $\frac{\partial r}{\partial y} = \sin\theta$ and $\frac{\partial \sin\theta}{\partial y} = \cos\theta\left(\frac{\partial\theta}{\partial y}\right)$. Therefore the above becomes

$$\begin{aligned} 1 &= \sin^2\theta + r\cos\theta\left(\frac{\partial\theta}{\partial y}\right) \\ \frac{\partial\theta}{\partial y} &= \frac{1-\sin^2\theta}{r\cos\theta} \\ &= \frac{\cos^2\theta}{r\cos\theta} \end{aligned}$$

Hence

$$\boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}}$$

Similarly, taking derivative w.r.t. x of (1) gives

$$1 = \frac{\partial r}{\partial x}\cos\theta + r\frac{\partial \cos\theta}{\partial x}$$

From (3), $\frac{\partial r}{\partial x} = \cos\theta$ and $\frac{\partial \cos\theta}{\partial x} = -\sin\theta\left(\frac{\partial\theta}{\partial x}\right)$, Therefore the above becomes

$$
\begin{aligned}
1 &= \cos^2\theta - r\sin\theta\left(\frac{\partial\theta}{\partial x}\right)\\
\frac{\partial\theta}{\partial x} &= \frac{1-\cos^2\theta}{r\sin\theta}\\
&= \frac{\sin^2\theta}{r\sin\theta}
\end{aligned}
$$

Hence

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{\sin\theta}{r}}$$

2.2.2.2 Part (b)

By definition of unit vector

$$
\begin{aligned}
\hat{r} &= \frac{\bar{r}}{|r|} = \frac{(|r|\cos\theta)\,\hat{\imath} + (|r|\sin\theta)\,\hat{\jmath}}{|r|}\\
&= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}
\end{aligned}
$$

To find $\hat{\theta}$, two relations are used.$\left\|\hat{\theta}\right\| = 1$ by definite of unit vector. Also $\hat{\theta}\cdot\hat{r} = 0$ since these are orthogonal vectors (basis vectors). Assuming that $\hat{\theta} = c_1\hat{\imath} + c_2\hat{\jmath}$, the two equations generated are

$$\left\|\hat{\theta}\right\| = 1 = c_1^2 + c_2^2 \tag{1}$$

$$\hat{\theta}\cdot\hat{r} = 0 = \left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right)\cdot\left(c_1\hat{\imath} + c_2\hat{\jmath}\right) = c_1\cos\theta + c_2\sin\theta \tag{2}$$

From (2), $c_1 = \frac{-c_2\sin\theta}{\cos\theta}$. Substituting this into (1) gives

$$
\begin{aligned}
1 &= \left(\frac{-c_2\sin\theta}{\cos\theta}\right)^2 + c_2^2\\
&= \frac{c_2^2\sin^2\theta}{\cos^2\theta} + c_2^2
\end{aligned}
$$

Solving for c_2 gives

$$
\begin{aligned}
\cos^2\theta &= c_2^2\left(\sin^2\theta + \cos^2\theta\right)\\
c_2 &= \cos\theta
\end{aligned}
$$

Since c_2 is now known, c_1 is found from (2)

$$
\begin{aligned}
0 &= c_1\cos\theta + c_2\sin\theta\\
0 &= c_1\cos\theta + (\cos\theta)\sin\theta\\
c_1 &= \frac{-(\cos\theta)\sin\theta}{\cos\theta}
\end{aligned}
$$

Hence $c_1 = -\sin\theta$. Therefore

$$\boxed{\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}}$$

2.2.2.3 Part (c)

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial x}\hat{\jmath} \tag{1}$$

Since $x \equiv x(r,\theta), y \equiv y(r,\theta)$, then

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\end{aligned}$$

Equation (1) becomes

$$\nabla = \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\hat{\imath} + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\hat{\jmath}$$

Using result found in (a), the above becomes

$$\nabla = \left(\frac{\partial}{\partial r}\cos\theta + \frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right)\hat{\imath} + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{\cos\theta}{r}\right)\hat{\jmath}$$

Collecting on $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ gives

$$\begin{aligned}\nabla &= \frac{\partial}{\partial r}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\hat{\imath} + \frac{\cos\theta}{r}\hat{\jmath}\right)\\ &= \frac{\partial}{\partial r}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right)\end{aligned}$$

Using result from (b), the above simplifies to

$$\boxed{\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}}$$

Hence

$$\nabla u = \left(\hat{r}\frac{\partial}{\partial r}u + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}u\right)$$

2.2.2.4 Part (d)

$$\begin{aligned}\bar{A} &= A_r\hat{r} + A_\theta\hat{\theta}\\ \nabla &= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

Hence

$$\begin{aligned}\nabla\cdot\bar{A} &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)\cdot\left(A_r\hat{r} + A_\theta\hat{\theta}\right)\\ &= \left(\hat{r}\frac{\partial}{\partial r}\cdot A_r\hat{r}\right) + \left(\hat{r}\frac{\partial}{\partial r}\cdot A_\theta\hat{\theta}\right) + \left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\cdot A_r\hat{r}\right) + \left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\cdot A_\theta\hat{\theta}\right) \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\hat{r}\frac{\partial}{\partial r}\cdot A_r\hat{r} &= \hat{r}\frac{\partial}{\partial r}\left(A_r\hat{r}\right)\\ &= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r}+A_r\frac{\partial\hat{r}}{\partial r}\right)\\ &= \frac{\partial A_r}{\partial r}\left(\hat{r}\cdot\hat{r}\right)+A_r\left(\hat{r}\cdot\frac{\partial\hat{r}}{\partial r}\right)\\ &= \frac{\partial A_r}{\partial r}\left(1\right)+A_r\left(0\right)\\ &= \frac{\partial A_r}{\partial r} \qquad (2)\end{aligned}$$

And

$$\begin{aligned}\hat{r}\frac{\partial}{\partial r}\cdot A_\theta\hat{\theta} &= \hat{r}\frac{\partial}{\partial r}\left(A_\theta\hat{\theta}\right)\\ &= \hat{r}\cdot\left(\frac{\partial A_\theta}{\partial r}\hat{\theta}+A_\theta\frac{\partial\hat{\theta}}{\partial r}\right)\\ &= \frac{\partial A_\theta}{\partial r}\left(\hat{r}\cdot\hat{\theta}\right)+A_\theta\left(\hat{r}\cdot\frac{\partial\hat{\theta}}{\partial r}\right)\\ &= \frac{\partial A_\theta}{\partial r}\left(0\right)+A_\theta\left(0\right)=0 \qquad (3)\end{aligned}$$

And

$$\begin{aligned}\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_r\hat{r} &= \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(A_r\hat{r}\right)\\ &= \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial A_r}{\partial\theta}\hat{r}+A_r\frac{\partial\hat{r}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_r}{\partial\theta}\left(\hat{\theta}\cdot\hat{r}\right)+\frac{1}{r}A_r\left(\hat{\theta}\cdot\frac{\partial\hat{r}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_r}{\partial\theta}\left(0\right)+\frac{1}{r}A_r\left(\hat{\theta}\cdot\hat{\theta}\right)\end{aligned}$$

Since $\frac{\partial\hat{r}}{\partial\theta}=\hat{\theta}$. Therefore

$$\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_r\hat{r}=\frac{1}{r}A_r \qquad (4)$$

And finally

$$
\begin{aligned}
\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_\theta\hat{\theta} &= \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(A_\theta\hat{\theta}\right) \\
&= \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial A_\theta}{\partial\theta}\hat{\theta}+A_\theta\frac{\partial\hat{\theta}}{\partial\theta}\right) \\
&= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}\left(\hat{\theta}\cdot\hat{\theta}\right)+\frac{1}{r}A_\theta\left(\hat{\theta}\cdot\frac{\partial\hat{\theta}}{\partial\theta}\right) \\
&= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}(1)+\frac{1}{r}A_\theta\left(\hat{\theta}\cdot(-\hat{r})\right) \\
&= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}(1)+\frac{1}{r}A_\theta(0) \\
&= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}
\end{aligned} \tag{5}
$$

Substituting (2,3,4,5) into (1) gives

$$
\begin{aligned}
\nabla\cdot\bar{A} &= \frac{\partial A_r}{\partial r}+0+\frac{1}{r}A_r+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta} \\
&= \frac{1}{r}A_r+\frac{\partial A_r}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}
\end{aligned}
$$

Add since $\frac{\partial}{\partial r}(rA_r)=A_r+r\frac{\partial A_r}{\partial r}$, the above can also be written as

$$
\begin{aligned}
\nabla\cdot\bar{A} &= \frac{1}{r}\left(A_r+r\frac{\partial A_r}{\partial r}\right)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}
\end{aligned}
$$

2.2.2.5 Part (e)

From part (c), it was found that

$$
\nabla=\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}
$$

But

$$
\begin{aligned}
\nabla^2 &= \nabla\cdot\nabla \\
&= \left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)
\end{aligned}
$$

Using result of part (d), which says that $\nabla\cdot\bar{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}$, the above becomes (where now $A_r\equiv\frac{\partial}{\partial r}, A_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}$)

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}
\end{aligned}
$$

Hence

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

2.2.3 Problem 3 (1.5.4)

1.5.4. **Using Exercise 1.5.3(a) and the chain rule for partial derivatives, derive the special case of Exercise 1.5.3(e) if $u(r)$ only.**

Let $u \equiv u(r)$. From problem 2 part (a) it was found that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \frac{\partial r}{\partial x} &= \cos\theta \\ \frac{\partial r}{\partial y} &= \sin\theta \\ \frac{\partial \theta}{\partial y} &= \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial x} &= \frac{-\sin\theta}{r} \end{aligned}$$

And

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1}$$

But

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\theta\right) \\
&= \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial r}\right)\cos\theta + \frac{\partial u}{\partial r}\frac{\partial \cos\theta}{\partial x} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial r}\left(-\sin\theta\frac{\partial \theta}{\partial x}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\cos\theta\right)\cos\theta + \frac{\partial u}{\partial r}\left(-\sin\theta\left(\frac{-\sin\theta}{r}\right)\right) \\
&= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial r} \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\sin\theta\right) \\
&= \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial r}\right)\sin\theta + \frac{\partial u}{\partial r}\frac{\partial \sin\theta}{\partial y} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y}\right)\sin\theta + \frac{\partial u}{\partial r}\left(\cos\theta\frac{\partial \theta}{\partial y}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta\right)\sin\theta + \frac{\partial u}{\partial r}\left(\cos\theta\left(\frac{\cos\theta}{r}\right)\right) \\
&= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{1}{r}\cos^2\theta\frac{\partial u}{\partial r} \qquad (3)
\end{aligned}$$

Substituting (2),(3) into (1) gives

$$\begin{aligned}
\nabla^2 u &= \left(\frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial r}\right) + \left(\frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{1}{r}\cos^2\theta\frac{\partial u}{\partial r}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\left(\sin^2\theta\frac{\partial u}{\partial r} + \cos^2\theta\frac{\partial u}{\partial r}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}
\end{aligned}$$

Which can be written as

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}$$

Which is the special case of problem 2(e) $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ when $u \equiv u(r)$ only.

2.2.4 Problem 4 (1.5.5)

1.5.5. Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r\, dr$.

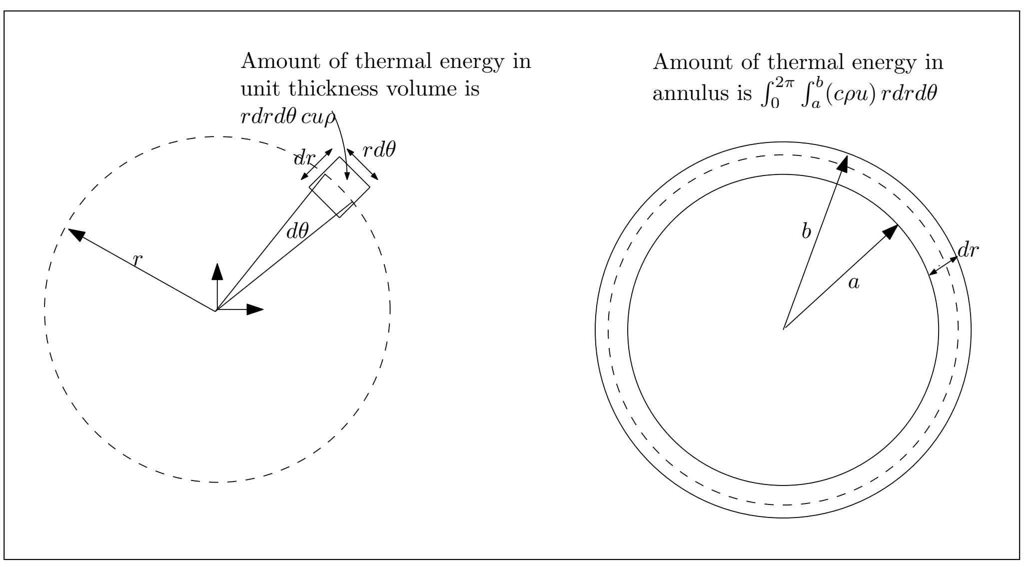
(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \partial u/\partial r \mid_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

2.2.4.1 Part(a)

Considering the thermal energy in a annulus as shown



Integrating gives total thermal energy

$$\begin{aligned}\int_0^{2\pi}\int_a^b \left(c\rho u\right) r dr d\theta &= \int_0^{2\pi} d\theta \int_a^b \left(c\rho u\right) r dr\\ &= 2\pi \int_a^b \left(c\rho u\right) r dr\end{aligned}$$

2.2.4.2 Part (b)

Using Fourier law,

$$\begin{aligned}\vec{\phi} &= -k_0 \bar{\nabla} u\\ &= -k_0\left(\hat{r}\frac{\partial u}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\end{aligned}$$

Since symmetric in θ, then $\frac{\partial u}{\partial \theta} = 0$ and the above reduces to

$$\vec{\phi} = -k_0 \hat{r}\frac{\partial u}{\partial r}$$

Hence the heat flow per unit time through surface at $r = b$ is

$$\int_0^{2\pi} \vec{\phi}\cdot(-\hat{n})\, ds$$
$$\int_0^{2\pi} \left(-k_0\hat{r}\frac{\partial u}{\partial r}\right)\cdot(\hat{n})\, r d\theta$$

But $\hat{n} = \hat{r}$ since radial unit vector. The above becomes

$$\int_0^{2\pi} -k_0\frac{\partial u}{\partial r} r d\theta = -\left(2\pi k_0\right) r\frac{\partial u}{\partial r}$$

At $r = b$ the above becomes

$$-\left(2\pi k_0\right) b \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

Similarly at $r = a$

$$-\left(2\pi k_0\right) a \left.\frac{\partial u}{\partial r}\right|_{r=a}$$

2.2.4.3 Part (c)

Applying that the rate of time change of total energy equal to flux through the boundaries gives

$$\begin{aligned}\frac{d}{dt}\left(2\pi\int_a^b \left(c\rho u\right) r dr\right) &= -\left(2\pi k_0\right) a \left.\frac{\partial u}{\partial r}\right|_{r=a} + \left(2\pi k_0\right) b \left.\frac{\partial u}{\partial r}\right|_{r=b}\\ &= 2\pi k_0 \int_a^b \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) dr\end{aligned}$$

Moving $\frac{d}{dt}$ inside the first integral, it become partial

$$2\pi\int_a^b\left(c\rho\frac{\partial u}{\partial t}\right)rdr=2\pi k_0\int_a^b\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)dr$$

Moving everything under one integral

$$\int_a^b\left[\left(c\rho\frac{\partial u}{\partial t}\right)r-k_0\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]dr=0$$

Hence, since this is valid for any annulus, then the integrand is zero

$$\left(c\rho\frac{\partial u}{\partial t}\right)r-k_0\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=0$$

$$\frac{\partial u}{\partial t}=\frac{k_0}{c\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

Hence

$$\boxed{\frac{\partial u}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}$$

Where $\kappa=\frac{k_0}{c\rho}$.

2.2.5 Problem 5 (1.5.6)

1.5.6. Modify Exercise 1.5.5 if the thermal properties depend on r.

The earlier problem is now repeated but in this problem $c\equiv c(r), k_0\equiv k_0(r)$ and $\rho\equiv\rho(r)$. These are the thermal properties in the problem.

2.2.5.1 Part(a)

$$\int_0^{2\pi}\int_a^b\left(c(r)\rho(r)u\right)rdrd\theta=\int_0^{2\pi}d\theta\int_a^b\left(c(r)\rho(r)u\right)rdr$$
$$=2\pi\int_a^b\left(c(r)\rho(r)u\right)rdr$$

2.2.5.2 Part (b)

$$\vec{\phi}=-k_0(r)\hat{r}\frac{\partial u}{\partial r}$$

The heat flow per unit time through surface at r is therefore

$$\int_0^{2\pi} \vec{\phi} \cdot (\hat{n})\, ds = \int_0^{2\pi} \left(-k_0(r)\, \hat{r} \frac{\partial u}{\partial r} \right) \cdot (\hat{n})\, r d\theta$$

But $\hat{n} = \hat{r}$ since radial therefore

$$\int_0^{2\pi} -k_0(r) \frac{\partial u}{\partial r} r d\theta = -(2\pi k_0(r))\, r \frac{\partial u}{\partial r}$$

At $r = b$ the above becomes

$$-\left(2\pi\, k_0 \big|_{r=b} \right) b \left. \frac{\partial u}{\partial r} \right|_{r=b}$$

Similarly at $r = a$

$$-\left(2\pi\, k_0 \big|_{r=a} \right) a \left. \frac{\partial u}{\partial r} \right|_{r=a}$$

2.2.5.3 Part (c)

Applying that the rate of time change of total energy equal to flux through the boundaries gives

$$\begin{aligned} \frac{d}{dt} \left(2\pi \int_a^b \left(c(r)\, \rho(r)\, u \right) r dr \right) &= -\left(2\pi\, k_0 \big|_{r=a} \right) a \left. \frac{\partial u}{\partial r} \right|_{r=a} + \left(2\pi\, k_0 \big|_{r=b} \right) b \left. \frac{\partial u}{\partial r} \right|_{r=b} \\ &= 2\pi \int_a^b \frac{\partial}{\partial r} \left(k_0(r)\, r \frac{\partial u}{\partial r} \right) dr \end{aligned}$$

Moving $\frac{d}{dt}$ inside the first integral, it become partial

$$2\pi \int_a^b \left(c(r)\, \rho(r) \frac{\partial u}{\partial t} \right) r dr = 2\pi \int_a^b \frac{\partial}{\partial r} \left(k_0(r)\, r \frac{\partial u}{\partial r} \right) dr$$

Moving everything under one integral

$$\int_a^b \left[\left(c(r)\, \rho(r) \frac{\partial u}{\partial t} \right) r - \frac{\partial}{\partial r} \left(k_0(r)\, r \frac{\partial u}{\partial r} \right) \right] dr = 0$$

Since this is valid for any annulus then the integrand is zero

$$\left(c(r)\, \rho(r) \frac{\partial u}{\partial t} \right) r - \frac{\partial}{\partial r} \left(k_0(r)\, r \frac{\partial u}{\partial r} \right) = 0$$

Therefore, the heat equation when the thermal properties depends on r becomes

$$\boxed{\frac{\partial u(r,t)}{\partial t} = \frac{1}{\rho(r) c(r)} \frac{1}{r} \frac{\partial}{\partial r} \left(k_0(r)\, r \frac{\partial u(r,t)}{\partial r} \right)}$$

2.2.6 Problem 6 (1.5.9)

1.5.9. Determine the equilibrium temperature distribution inside a circular annulus ($r_1 \leq r \leq r_2$):

***(a) if the outer radius is at temperature T_2 and the inner at T_1**

(b) if the outer radius is insulated and the inner radius is at temperature T_1

2.2.6.1 Part (a)

The heat equation is $\frac{\partial u}{\partial t} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$. At steady state $\frac{\partial u}{\partial t} = 0$. And since circular region, symmetry in θ is assumed and therefore temperature u depends only on r only. This means $u(r_0)$ is the same at any angle θ for that specific r_0. This becomes a second order ODE

$$\begin{aligned}\frac{\kappa}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) &= 0\\ \frac{\kappa}{r}\left(\frac{du}{dr}+r\frac{d^2u}{dr^2}\right) &= 0\\ \frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr} &= 0\end{aligned}$$

Since $\frac{\kappa}{r} \neq 0$. Assuming $\frac{du}{dr} = v(r)$, the above becomes

$$\begin{aligned}\frac{dv}{dr}+\frac{1}{r}v &= 0\\ \frac{dv}{dr} &= -\frac{1}{r}v\\ \frac{dv}{v} &= -\frac{dr}{r}\end{aligned}$$

Integrating

$$\begin{aligned}\ln v &= -\ln r + c_1\\ v &= e^{-\ln r + c_1}\\ &= c_2 e^{-\ln r}\\ &= c_2\frac{1}{r}\end{aligned}$$

Where $c_2 = e^{c_1}$. Since $\frac{du}{dr} = v$, then

$$\begin{aligned}\frac{du}{dr} &= c_2\frac{1}{r}\\ du &= c_2\frac{1}{r}dr\end{aligned}$$

Integrating

$$u(r) = c_2 \ln r + c_3$$

When $r = r_1, u = T_1$, and when $r = r_2, u = T_2$, therefore

$$\begin{aligned} T_1 &= c_2 \ln r_1 + c_3 \\ T_2 &= c_2 \ln r_2 + c_3 \end{aligned}$$

From first equation, $c_3 = T_1 - c_2 \ln r_1$. Substituting in second equation gives

$$\begin{aligned} T_2 &= c_2 \ln r_2 + T_1 - c_2 \ln r_1 \\ &= c_2 (\ln r_2 - \ln r_1) + T_1 \end{aligned}$$

Therefore

$$c_2 = \frac{T_2 - T_1}{\ln r_2 - \ln r_1}$$

Hence $c_3 = T_1 - \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r_1$. Therefore the steady state solution becomes

$$\begin{aligned} u(r) &= c_2 \ln r + c_3 \\ &= \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r + T_1 - \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r_1 \\ &= T_1 + \frac{(T_2 - T_1) \ln r - (T_2 - T_1) \ln r_1}{\ln r_2 - \ln r_1} \\ &= T_1 + \frac{(T_2 - T_1)(\ln r - \ln r_1)}{\ln r_2 - \ln r_1} \\ &= T_1 + (T_2 - T_1) \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \end{aligned}$$

Hence

$$\boxed{u(r) = T_1 + (T_2 - T_1) \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}}$$

2.2.6.2 Part (b)

Insulated condition implies $\frac{du}{dr} = 0$. So the above is repeated, but this new boundary condition is now used at r_2. Starting from the general solution found in part (a)

$$u(r) = c_2 \ln r + c_3$$

When $r = r_1, u = T_1$ and when $r = r_2, \frac{du}{dr} = 0$. But $\frac{du}{dr} = \frac{c_2}{r}$. Hence $r = r_2$ gives $\frac{c_2}{r_2} = 0$ or $c_2 = 0$. Therefore the solution is

$$u(r) = c_3$$

When $r = r_1$, $u = T_1$, hence $c_3 = T_1$. The solution becomes

$$\boxed{u(r) = T_1}$$

The temperature is T_1 everywhere. This makes sense as this is steady state, and no heat escapes to the outside.

2.2.7 Problem 7 (1.5.10)

1.5.10. Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at temperature T_0.

Last problem found the solution to the heat equation in polar coordinates with symmetry in θ to be

$$u(r) = c_2 \ln r + c_3$$

c_2 must be zero since at $r = 0$ the temperature must be finite. The solution becomes

$$u(r) = c_3$$

Applying the boundary conditions at $r = r_0$

$$T_0 = c_3$$

Therefore,

$$u(r) = T_0$$

The temperature everywhere is the the same as on the edge.

2.2.8 Problem 8 (1.5.11)

***1.5.11.** Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad a < r < b$$

subject to

$$u(r,0) = f(r), \quad \frac{\partial u}{\partial r}(a,t) = \beta, \text{ and } \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

For equilibrium the total rate of heat flow at $r = a$ should be the same as at $r = b$. Circumference at $r = a$ is $2\pi a$ and total rate of flow at $r = a$ is given by β. Hence total heat flow rate at

$r = a$ is given by

$$(2\pi a) \left.\frac{\partial u}{\partial r}\right|_{r=a} = 2\pi a\beta$$

Similarly, total heat flow rate at $r = b$ is given by

$$(2\pi b) \left.\frac{\partial u}{\partial r}\right|_{r=b} = 2\pi b$$

Therefore $2\pi a\beta = 2\pi a$ or

$$\beta = \frac{a}{b}$$

2.2.9 Problem 9 (1.5.12)

1.5.12. Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2\, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0\, \partial u/\partial r\,|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

2.2.9.1 Part (a)

Total heat energy is, by definition

$$E = \int_V c\rho u dv \tag{1}$$

Volume v of sphere of radius r is $v = \frac{4}{3}\pi r^3$. Hence

$$\frac{dv}{dr} = 4\pi r^2$$
$$dv = 4\pi r^2 dr$$

Equation (1) becomes, where now the r limits are from a to b

$$\begin{aligned} E &= \int_a^b c\rho u \left(4\pi r^2 dr\right) \\ &= 4\pi \int_a^b c\rho u r^2 dr \end{aligned}$$

2.2.9.2 Part (b)

By definition, the flux at $r = b$ is

$$\phi_b = -k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

The above is per unit area. At $r = b$, the surface area of the sphere is $4\pi b^2$. Therefore, the total energy per unit time is $\phi_b \left(4\pi b^2\right)$ or

$$-4\pi b^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

Similarly for $r = a$.

2.2.9.3 Part(c)

By conservation of thermal energy

$$\begin{aligned}
\frac{d}{dt}E &= -4\pi a^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=a} + 4\pi b^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b} \\
\frac{d}{dt}\left(4\pi \int_a^b c\rho u r^2 dr\right) &= 4\pi k_0 \int_a^b \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr \\
\int_a^b c\rho \frac{\partial u}{\partial t} r^2 dr &= k_0 \int_a^b \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr
\end{aligned}$$

Moving everything into one integral

$$\int_a^b \left[c\rho \frac{\partial u}{\partial t} r^2 - k_0 \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right] dr = 0$$

Since this is valid for any limits the integrand must be zero

$$\begin{aligned}
c\rho \frac{\partial u}{\partial t} r^2 - k_0 \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) &= 0 \\
\frac{\partial u}{\partial t} &= \frac{k_0}{c\rho} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)
\end{aligned}$$

Therefore

$$\boxed{\frac{\partial u}{\partial t} = \frac{\kappa}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)}$$

Where $\kappa = \frac{k_0}{c\rho}$

2.2.10 Problem 10 (1.5.13)

***1.5.13. Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0° (see Exercise 1.5.12).**

The heat equation is $\frac{\partial u}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$. For steady state $\frac{\partial u}{\partial t} = 0$ and assuming symmetry in θ, the heat equation becomes an ODE in r

$$\begin{aligned}\frac{\kappa}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) &= 0\\ \frac{d}{dr}\left(r^2\frac{du}{dr}\right) &= 0\\ 2r\frac{du}{dr} + r^2\frac{d^2u}{dr^2} &= 0\end{aligned}$$

For $r \neq 0$

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0$$

Let $\frac{du}{dr} = v(r)$, hence

$$\begin{aligned}r\frac{dv}{dr} + 2v &= 0\\ \frac{dv}{dr} &= -\frac{2v}{r}\\ \frac{dv}{v} &= -2\frac{dr}{r}\end{aligned}$$

Integrating

$$\begin{aligned}\ln v &= -2\ln r + c\\ v &= e^{-2\ln r + c}\\ &= c_1 e^{-2\ln r}\\ &= c_1\frac{1}{r^2}\end{aligned}$$

Therefore, since $\frac{du}{dr} = v(r)$ then

$$\begin{aligned}\frac{du}{dr} &= c_1\frac{1}{r^2}\\ du &= c_1\frac{dr}{r^2}\end{aligned}$$

Integrating

$$\boxed{u(r) = \frac{-c_1}{r} + c_2}$$

When $r = 1$, $u = 0$ and when $r = 4, u = 80$, hence

$$0 = -c_1 + c_2$$
$$80 = \frac{-c_1}{4} + c_2$$

From first equation, $c_1 = c_2$, and from second equation $80 = \frac{-c_1}{4} + c_1$, hence $\frac{3}{4}c_1 = 80$ or $c_1 = \frac{(4)(80)}{3} = \frac{320}{3}$. Therefore, the general solution becomes

$$u(r) = -\frac{320}{3}\frac{1}{r} + \frac{320}{3}$$

or

$$\boxed{u(r) = \frac{320}{3}\left(1 - \frac{1}{r}\right)}$$

2.3 HW 2

2.3.1 Summary table

For $1D$ bar

Left	Right	$\lambda = 0$	$\lambda > 0$	$u(x,t)$
$u(0) = 0$	$u(L) = 0$	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$\frac{\partial u(L)}{\partial x} = 0$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = 0$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1,3,5\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$u(L) + \frac{\partial u(L)}{\partial x} = 0$	$\lambda_0 = 0$ $X_0 = A_0$	$\tan\left(\sqrt{\lambda_n} L\right) = -\lambda_n$ $X_\lambda = B_\lambda \sin\left(\sqrt{\lambda_n} x\right)$	$A_0 + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$\frac{\partial u(L)}{\partial x} = 0$	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n} x\right)$	$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$

For periodic conditions $u(-L) = u(L)$ and $\frac{\partial u(-L)}{\partial x} = \frac{\partial u(L)}{\partial x}$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$$

$$u(x,t) = \overbrace{a_0}^{\lambda=0} + \overbrace{\sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t} + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}}^{\lambda>0}$$

Note on notation When using separation of variables $T(t)$ is used for the time function and $X(x), R(r), \Theta(\theta)$ etc. for the spatial functions. This notation is more common in other books and easier to work with as the dependent variable $T, X, \cdots$ and the independent variable $t, x, \cdots$ are easier to match (one is upper case and is one lower case) and this produces less symbols to remember and less chance of mixing wrong letters.

2.3.2 section 2.3.1 (problem 1)

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

* (a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

* (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$

* (e) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$

* (f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

2.3.2.1 part (a)

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{1}$$

Let

$$u(t,r) = T(t)R(r)$$

Then

$$\frac{\partial u}{\partial t} = T'(t)R(r)$$

And

$$\begin{aligned}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\\ &= TR'(r) + rTR''(r)\end{aligned}$$

Hence (1) becomes

$$\frac{1}{k}T'(t)R(r) = \frac{1}{r}\left(TR'(r) + rTR''(r)\right)$$

Note From now on $T'(t)$ is written as just T' and similarly for $R'(r) = R'$ and $R''(r) = R''$ to simplify notations and make it easier and more clear to read. The above is reduced to

$$\frac{1}{k}T'R = \frac{1}{r}TR' + TR''$$

Dividing throughout [1] by $T(t)R(r)$ gives

$$\frac{1}{k}\frac{T'}{T} = \frac{1}{r}\frac{R'}{R} + \frac{R''}{R}$$

Since each side in the above depends on a different independent variable and both are equal

[1] $T(t)R(r)$ can not be zero, as this would imply that either $T(t) = 0$ or $R(r) = 0$ or both are zero, in which case there is only the trivial solution.

to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T}=\frac{1}{r}\frac{R'}{R}+\frac{R''}{R}=-\lambda$$

The following differential equations are obtained

$$\begin{aligned} T'+\lambda kT &= 0\\ rR''+R'+r\lambda R &= 0 \end{aligned}$$

In expanded form, the above is

$$\begin{aligned} \frac{dT}{dt}+\lambda kT(t) &= 0\\ r\frac{d^2R}{dr^2}+\frac{dR}{dt}+r\lambda R(r) &= 0 \end{aligned}$$

2.3.2.2 Part (b)

$$\frac{1}{k}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\frac{v_0}{k}\frac{\partial u}{\partial x} \tag{1}$$

Let

$$u(x,t)=TX$$

Then

$$\frac{\partial u}{\partial t}=T'X$$

And

$$\begin{aligned} \frac{\partial u}{\partial x} &= X'T\\ \frac{\partial^2 u}{\partial x^2} &= X''T \end{aligned}$$

Substituting these in (1) gives

$$\frac{1}{k}T'X=X''T-\frac{v_0}{k}X'T$$

Dividing throughout by $TX\neq 0$ gives

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}-\frac{v_0}{k}\frac{X'}{X}$$

Since each side in the above depends on a different independent variable and both are equal to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}-\frac{v_0}{k}\frac{X'}{X}=-\lambda$$

The following differential equations are obtained

$$\begin{aligned} T'+\lambda kT &= 0\\ X''-\frac{v_0}{k}X'+\lambda X &= 0 \end{aligned}$$

The above in expanded form is

$$\begin{aligned}\frac{dT}{dt}+\lambda kT(t)&=0\\ \frac{d^2X}{dx^2}-\frac{v_0}{k}\frac{dX}{dx}+\lambda X(x)&=0\end{aligned}$$

2.3.2.3 Part (d)

$$\frac{1}{k}\frac{\partial u}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\tag{1}$$

Let

$$u(t,r)\equiv TR$$

Then

$$\frac{\partial u}{\partial t}=T'R$$

And

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)&=2r\frac{\partial u}{\partial r}+r^2\frac{\partial^2 u}{\partial r^2}\\ &=2rTR'+r^2TR''\end{aligned}$$

Substituting these in (1) gives

$$\begin{aligned}\frac{1}{k}T'R&=\frac{1}{r^2}\left(2rTR'+r^2TR''\right)\\ &=\frac{2}{r}TR'+TR''\end{aligned}$$

Dividing throughout by $TR\neq 0$ gives

$$\frac{1}{k}\frac{T'}{T}=\frac{2}{r}\frac{R'}{R}+\frac{R''}{R}$$

Since each side in the above depends on a different independent variable and both are equal to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T}=\frac{2}{r}\frac{R'}{R}+\frac{R''}{R}=-\lambda$$

The following differential equations are obtained

$$\begin{aligned}T'+\lambda kT&=0\\ rR''+2R'+\lambda rR&=0\end{aligned}$$

The above in expanded form is

$$\begin{aligned}\frac{dT}{dt}+\lambda kT(t)&=0\\ r\frac{d^2R}{dr^2}+2\frac{dR}{dr}+\lambda rR(r)&=0\end{aligned}$$

2.3.3 section 2.3.2 (problem 2)

2.3.2. **Consider the differential equation**

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ **and** $\phi(\pi) = 0$

***(b)** $\phi(0) = 0$ **and** $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$ **(If necessary, see Sec. 2.4.1.)**

***(d)** $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ **and** $\phi(L) = 0$

***(f)** $\phi(a) = 0$ **and** $\phi(b) = 0$ **(You may assume that $\lambda > 0$.)**

(g) $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) + \phi(L) = 0$ **(If necessary, see Sec. 5.8.)**

2.3.3.1 Part (d)

$$\begin{aligned}\frac{d^2\phi}{dx^2} + \lambda\phi &= 0\\ \phi(0) &= 0\\ \frac{d\phi}{dx}(L) &= 0\end{aligned}$$

Substituting an assumed solution of the form $\phi = Ae^{rx}$ in the above ODE and simplifying gives the characteristic equation

$$\begin{aligned}r^2 + \lambda &= 0\\ r^2 &= -\lambda\\ r &= \pm\sqrt{-\lambda}\end{aligned}$$

Assuming λ is real. The following cases are considered.

case $\lambda < 0$ In this case, $-\lambda$ and also $\sqrt{-\lambda}$, are positive. Hence both roots $\pm\sqrt{-\lambda}$ are real and positive. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. Therefore the solution is

$$\phi(x) = Ae^{sx} + Be^{-sx}$$
$$\frac{d\phi}{dx} = Ase^{sx} - Bse^{-sx}$$

Applying the first boundary conditions (B.C.) gives

$$\begin{aligned} 0 &= \phi(0) \\ &= A + B \end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned} 0 &= \frac{d\phi}{dx}(L) \\ &= As - Bs \\ &= s(A - B) \\ &= A - B \end{aligned}$$

The last step above was after dividing by s since $s \neq 0$. Therefore, the following two equations are solved for A, B

$$\begin{aligned} 0 &= A + B \\ 0 &= A - B \end{aligned}$$

The second equation implies $A = B$ and the first gives $2A = 0$ or $A = 0$. Hence $B = 0$. Therefore the only solution is the trivial solution $\phi(x) = 0$. $\underline{\lambda < 0 \text{ is not an eigenvalue.}}$

$\underline{\text{case } \lambda = 0}$ In this case the ODE becomes

$$\frac{d^2\phi}{dx^2} = 0$$

The solution is

$$\phi(x) = Ax + B$$
$$\frac{d\phi}{dx} = A$$

Applying the first B.C. gives

$$\begin{aligned} 0 &= \phi(0) \\ &= B \end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned} 0 &= \frac{d\phi}{dx}(L) \\ &= A \end{aligned}$$

Hence A, B are both zero in this case as well and the only solution is the trivial one $\phi(x) = 0$. $\underline{\lambda = 0 \text{ is not an eigenvalue.}}$

$\underline{\text{case } \lambda > 0}$ In this case, $-\lambda$ is negative, therefore the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

Hence the solution is

$$\phi(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x}$$

Which can be writing in terms of $\cos, \sin$ using Euler identity as

$$\phi(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Applying first B.C. gives

$$\begin{aligned}
0 &= \phi(0) \\
&= A\cos(0) + B\sin(0) \\
0 &= A
\end{aligned}$$

The solution now is $\phi(x) = B\sin\left(\sqrt{\lambda}x\right)$. Hence

$$\frac{d\phi}{dx} = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)$$

Applying the second B.C. gives

$$\begin{aligned}
0 &= \frac{d\phi}{dx}(L) \\
&= \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right) \\
&= \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right)
\end{aligned}$$

Since $\lambda \neq 0$ then either $B = 0$ or $\cos\left(\sqrt{\lambda}L\right) = 0$. But $B = 0$ gives trivial solution, therefore

$$\cos\left(\sqrt{\lambda}L\right) = 0$$

This implies

$$\sqrt{\lambda}L = \frac{n\pi}{2} \qquad n = 1,3,5,\cdots$$

In other words, for <u>all positive odd integers</u>. $n < 0$ can not be used since λ is assumed positive.

$$\boxed{\lambda = \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1,3,5,\cdots}$$

The eigenfunctions associated with these eigenvalues are

$$\phi_n(x) = B_n\sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1,3,5,\cdots$$

2.3.3.2 Part (f)

$$\begin{aligned}
\frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\
\phi(a) &= 0 \\
\phi(b) &= 0
\end{aligned}$$

It is easier to solve this if one boundary condition was at $x = 0$. (So that one constant drops out). Let $\tau = x - a$ and the ODE becomes (where now the independent variable is τ)

$$\frac{d^2\phi(\tau)}{d\tau^2} + \lambda\phi(\tau) = 0 \tag{1}$$

With the new boundary conditions $\phi(0) = 0$ and $\phi(b-a) = 0$. Assuming the solution is $\phi = Ae^{r\tau}$, the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Assuming λ is real and also assuming $\lambda > 0$ (per the problem statement) then $-\lambda$ is negative, and both roots are complex.

$$r = \pm i\sqrt{\lambda}$$

This gives the solution

$$\phi(\tau) = A\cos\left(\sqrt{\lambda}\tau\right) + B\sin\left(\sqrt{\lambda}\tau\right)$$

Applying first B.C.

$$\begin{aligned} 0 &= \phi(0) \\ &= A\cos 0 + B\sin 0 \\ &= A \end{aligned}$$

Therefore the solution is $\phi(\tau) = B\sin\left(\sqrt{\lambda}\tau\right)$. Applying the second B.C.

$$\begin{aligned} 0 &= \phi(b-a) \\ &= B\sin\left(\sqrt{\lambda}(b-a)\right) \end{aligned}$$

$B = 0$ leads to trivial solution. Choosing $\sin\left(\sqrt{\lambda}(b-a)\right) = 0$ gives

$$\begin{aligned} \sqrt{\lambda_n}(b-a) &= n\pi \\ \sqrt{\lambda_n} &= \frac{n\pi}{(b-a)} \qquad n = 1, 2, 3\cdots \end{aligned}$$

Or

$$\boxed{\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \qquad n = 1, 2, 3, \cdots}$$

The eigenfunctions associated with these eigenvalue are

$$\begin{aligned} \phi_n(\tau) &= B_n\sin\left(\sqrt{\lambda_n}\tau\right) \\ &= B_n\sin\left(\frac{n\pi}{(b-a)}\tau\right) \end{aligned}$$

Transforming back to x

$$\phi_n(x) = B_n\sin\left(\frac{n\pi}{(b-a)}(x-a)\right)$$

2.3.3.3 Part (g)

$$\begin{aligned}\frac{d^2\phi}{dx^2}+\lambda\phi&=0\\ \phi(0)&=0\\ \frac{d\phi}{dx}(L)+\phi(L)&=0\end{aligned}$$

Assuming solution is $\phi=Ae^{rx}$, the characteristic equation is

$$\begin{aligned}r^2+\lambda&=0\\ r^2&=-\lambda\\ r&=\pm\sqrt{-\lambda}\end{aligned}$$

The following cases are considered.

case $\lambda<0$ In this case $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda}=s$$

Where $s>0$. This gives the solution

$$\phi(x)=A_0e^{sx}+B_0e^{-sx}$$

Which can be manipulated using $\sinh(sx)=\frac{e^{sx}-e^{-sx}}{2},\cosh(sx)=\frac{e^{sx}+e^{-sx}}{2}$ to the following

$$\phi(x)=A\cosh(sx)+B\sinh(sx)$$

Where A,B above are new constants. Applying the left boundary condition gives

$$\begin{aligned}0&=\phi(0)\\ &=A\end{aligned}$$

The solution becomes $\phi(x)=B\sinh(sx)$ and hence $\frac{d\phi}{dx}=s\cosh(sx)$. Applying the right boundary conditions gives

$$\begin{aligned}0&=\phi(L)+\frac{d\phi}{dx}(L)\\ &=B\sinh(sL)+Bs\cosh(sL)\\ &=B(\sinh(sL)+s\cosh(sL))\end{aligned}$$

But $B=0$ leads to trivial solution, therefore the other option is that

$$\sinh(sL)+s\cosh(sL)=0$$

But the above is

$$\tanh(sL)=-s$$

Since it was assumed that $s>0$ then the RHS in the above is a negative quantity. However the $\tanh$ function is positive for positive argument and negative for negative argument. The above implies then that $sL<0$. Which is invalid since it was assumed $s>0$ and L is the length of the bar. Hence $B=0$ is the only choice, and this leads to trivial solution. $\lambda<0$ is not an eigenvalue.

<u>case $\lambda = 0$</u>

In this case, the ODE becomes

$$\frac{d^2\phi}{dx^2} = 0$$

The solution is

$$\phi(x) = c_1 x + c_2$$

Applying left B.C. gives

$$\begin{aligned} 0 &= \phi(0) \\ &= c_2 \end{aligned}$$

The solution becomes $\phi(x) = c_1 x$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \phi(L) + \frac{d\phi}{dx}(L) \\ &= c_1 L + c_1 \\ &= c_1(1 + L) \end{aligned}$$

Since $c_1 = 0$ leads to trivial solution, then $1 + L = 0$ is the only other choice. But this invalid since $L > 0$ (length of the bar). Hence $c_1 = 0$ and this leads to trivial solution. <u>$\lambda = 0$ is not an eigenvalue.</u>

<u>case $\lambda > 0$</u>

This implies that $-\lambda$ is negative, and therefore the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

This gives the solution

$$\begin{aligned} \phi(x) &= Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} \\ &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying first B.C. gives

$$\begin{aligned} \phi(0) = 0 &= A\cos(0) + B\sin(0) \\ 0 &= A \end{aligned}$$

The solution becomes $\phi(x) = B\sin\left(\sqrt{\lambda}x\right)$ and

$$\frac{d\phi}{dx} = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)$$

Applying the second B.C.

$$\begin{aligned} 0 &= \frac{d\phi}{dx}(L) + \phi(L) \\ &= \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right) + B\sin\left(\sqrt{\lambda}L\right) \qquad (1) \end{aligned}$$

Dividing (1) by $\cos\left(\sqrt{\lambda}L\right)$, which can not be zero, because if $\cos\left(\sqrt{\lambda}L\right) = 0$, then $B\sin\left(\sqrt{\lambda}L\right) =$

0 from above, and this means the trivial solution, results in

$$B\left(\sqrt{\lambda}+\tan\left(\sqrt{\lambda}L\right)\right)=0$$

But $B\neq 0$, else the solution is trivial. Therefore

$$\tan\left(\sqrt{\lambda}L\right)=-\sqrt{\lambda}$$

The eigenvalue λ is given by the solution to the above nonlinear equation. The text book, in section 5.4, page 196 gives the following approximate (asymptotic) solution which becomes accurate only for large n and not used here

$$\sqrt{\lambda_n}\sim\frac{\pi}{L}\left(n-\frac{1}{2}\right)$$

Therefore the eigenfunction is

$$\phi_\lambda(x)=B\sin\left(\sqrt{\lambda}x\right)$$

Where λ is the solution to $\tan\left(\sqrt{\lambda}L\right)=-\sqrt{\lambda}$.

2.3.4 section 2.3.3 (problem 3)

2.3.3. **Consider the heat equation**

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t)=0 \quad \text{and} \quad u(L,t)=0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0)=6\sin\frac{9\pi x}{L}$ (b) $u(x,0)=3\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}$

* (c) $u(x,0)=2\cos\frac{3\pi x}{L}$ (d) $u(x,0)=\begin{cases}1 & 0<x\le L/2\\ 2 & L/2<x<L\end{cases}$

2.3.4.1 Part (b)

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t)=T(t)X(x)$, and the PDE becomes

$$\frac{1}{k}T'X=X''T$$

Dividing by $XT \neq 0$

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$ where λ is assumed to be real.

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + k\lambda T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Starting with the space ODE equation (2), with corresponding boundary conditions $X(0) = 0, X(L) = 0$. Assuming the solution is $X(x) = e^{rx}$, Then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

The following cases are considered.

<u>case $\lambda < 0$</u> In this case, $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. This gives the solution

$$X(x) = A\cosh(sx) + B\sinh(sx)$$

Applying the left B.C. $X(0) = 0$ gives

$$\begin{aligned} 0 &= A\cosh(0) + B\sinh(0) \\ &= A \end{aligned}$$

The solution becomes $X(x) = B\sinh(sx)$. Applying the right B.C. $u(L,t) = 0$ gives

$$0 = B\sinh(sL)$$

We want $B \neq 0$ (else trivial solution). This means $\sinh(sL)$ must be zero. But $\sinh(sL)$ is zero only when its argument is zero. This means either $L = 0$ which is not possible or $\lambda = 0$, but we assumed $\lambda \neq 0$ in this case, therefore we run out of options to satisfy this case. Hence <u>$\lambda < 0$ is not an eigenvalue.</u>

<u>case $\lambda = 0$</u>

The ODE becomes

$$\frac{d^2X}{dx^2} = 0$$

The solution is

$$X(x) = c_1 x + c_2$$

Applying left boundary conditions $X(0) = 0$ gives

$$\begin{aligned} 0 &= X(0) \\ &= c_2 \end{aligned}$$

Hence the solution becomes $X(x) = c_1 x$. Applying the right B.C. gives

$$\begin{aligned} 0 &= X(L) \\ &= c_1 L \end{aligned}$$

Hence $c_1 = 0$. Hence trivial solution. $\underline{\lambda = 0 \text{ is not an eigenvalue}}$.

<u>case $\lambda > 0$</u>

Hence $-\lambda$ is negative, and the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

The solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

The boundary conditions are now applied. The first B.C. $X(0) = 0$ gives

$$\begin{aligned} 0 &= A\cos(0) + B\sin(0) \\ &= A \end{aligned}$$

The ODE becomes $X(x) = B\sin\left(\sqrt{\lambda}x\right)$. Applying the second B.C. gives

$$0 = B\sin\left(\sqrt{\lambda}L\right)$$

$B \neq 0$ else the solution is trivial. Therefore taking

$$\begin{aligned} \sin\left(\sqrt{\lambda}L\right) &= 0 \\ \sqrt{\lambda_n}L &= n\pi \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Hence eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3, \cdots$$

The eigenfunctions associated with these eigenvalues are

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right)$$

The time domain ODE is now solved. $T' + k\lambda_n T = 0$ has the solution

$$T_n(t) = e^{-k\lambda_n t}$$

For the same set of eigenvalues. Notice that there is no need to add a new constant in the above as it will be absorbed in the B_n when combined in the following step below. The solution to the PDE becomes

$$u_n(x,t) = T_n(t) X_n(x)$$

But for linear system the sum of eigenfunctions is also a solution, therefore

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

Initial conditions are now applied. Setting $t=0$, the above becomes

$$u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

As the series is unique, the terms coefficients must match for those shown only, and all other B_n terms vanish. This means that by comparing terms

$$3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) = B_1 \sin\left(\frac{\pi x}{L}\right) + B_3 \sin\left(\frac{3\pi}{L}x\right)$$

Therefore

$$\begin{aligned} B_1 &= 3 \\ B_3 &= -1 \end{aligned}$$

And all other $B_n = 0$. The solution is

$$u(x,t) = 3\sin\left(\frac{\pi}{L}x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

2.3.4.2 Part (d)

Part (b) found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The new initial conditions are now applied.

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Where

$$f(x) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$$

Multiplying both sides of (1) by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating over the domain gives

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = \int_0^L \left[\sum_{n=1}^{\infty} B_n \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\right] dx$$

Interchanging the order of integration and summation

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = \sum_{n=1}^{\infty} \left[B_n \left(\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx\right)\right]$$

But $\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$ for $n \neq m$, hence only one term survives

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = B_m \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$

Renaming m back to n and since $\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}$ the above becomes

$$\begin{aligned}
\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx &= \frac{L}{2}B_n \\
B_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx + \int_{\frac{L}{2}}^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right)dx + 2\int_{\frac{L}{2}}^L \sin\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{2}{L}\left(\left.\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\right|_0^{\frac{L}{2}} + 2\left.\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\right|_{\frac{L}{2}}^L\right) \\
&= \frac{2}{n\pi}\left(\left(-\cos\left(\frac{n\pi}{L}x\right)\right)_0^{\frac{L}{2}} + 2\left(-\cos\left(\frac{n\pi}{L}x\right)\right)_{\frac{L}{2}}^L\right) \\
&= \frac{2}{n\pi}\left(\left[-\cos\left(\frac{n\pi}{L}\frac{L}{2}\right) + \cos(0)\right] + 2\left[-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right]\right) \\
&= \frac{2}{n\pi}\left(-\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi) + 2\cos\left(\frac{n\pi}{2}\right)\right) \\
&= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi)\right)
\end{aligned}$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

With

$$\begin{aligned}
B_n &= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) + 1\right) \\
&= \frac{2}{n\pi}\left(1 - 2(-1)^n + \cos\left(\frac{n\pi}{2}\right)\right)
\end{aligned}$$

2.3.5 section 2.3.4 (problem 4)

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = 0, u(L,t) = 0$, and $u(x,0) = f(x)$.

***(a) What is the total heat energy in the rod as a function of time?**

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

***(c) What relationship should exist between parts (a) and (b)?**

2.3.5.1 Part (a)

By definition the total heat energy is

$$E = \int_V \rho c u(x,t)\, dv$$

Assuming constant cross section area A, the above becomes (assuming all thermal properties are constant)

$$E = \int_0^L \rho c u(x,t)\, A dx$$

But $u(x,t)$ was found to be

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

For these boundary conditions from problem 2.3.3. Where B_n was found from initial conditions. Substituting the solution found into the energy equation gives

$$\begin{aligned}
E &= \rho c A \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right) dx \\
&= \rho c A \sum_{n=1}^{\infty} \left(B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx\right) \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \left(\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\right)_0^L \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{L}{n\pi}\left(-\cos\left(\frac{n\pi}{L}L\right) + \cos(0)\right) \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{L}{n\pi}\left(1 - \cos(n\pi)\right) \\
&= \frac{L\rho c A}{\pi} \sum_{n=1}^{\infty} \left[\frac{B_n}{n}\left(1 - \cos(n\pi)\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right]
\end{aligned}$$

2.3.5.2 Part (b)

By definition, the flux is the amount of heat flow per unit time per unit area. Assuming the area is A, then heat flow at $x=0$ into the rod per unit time (call it $H(x)$) is

$$\begin{aligned} H|_{x=0} &= A\,\phi\big|_{x=0} \\ &= -Ak\,\frac{\partial u}{\partial x}\bigg|_{x=0} \end{aligned}$$

Similarly, heat flow at $x=L$ out of the rod per unit time is

$$\begin{aligned} H|_{x=L} &= A\,\phi\big|_{x=L} \\ &= -Ak\,\frac{\partial u}{\partial x}\bigg|_{x=L} \end{aligned}$$

To obtain heat flow at $x=0$ <u>leaving</u> the rod, the sign is changed and it becomes $Ak\,\frac{\partial u}{\partial x}\Big|_{x=0}$.

Since $u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ then

$$\frac{\partial u}{\partial x}=\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Then at $x=0$ then heat flow leaving of the rod becomes

$$Ak\,\frac{\partial u}{\partial x}\bigg|_{x=0}=Ak\sum_{n=1}^{\infty}\frac{n\pi}{L}B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

And at $x=L$, the heat flow out of the bar

$$\begin{aligned} -Ak\,\frac{\partial u}{\partial x}\bigg|_{x=L} &= -Ak\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}L\right)e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \\ &= -Ak\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}\cos(n\pi)\,e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \\ &= -Ak\sum_{n=1}^{\infty}(-1)^n B_n\frac{n\pi}{L}e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

2.3.5.3 Part (c)

Total E inside the bar at time t is given by initial energy $E_{t=0}$ and time integral of flow of heat energy into the bar. Since from part (a)

$$E=L\frac{\rho cA}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(1-\cos(n\pi)\right)$$

Then initial energy is

$$E_{t=0}=L\frac{\rho cA}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}\left(1-\cos(n\pi)\right)$$

And total heat flow <u>into</u> the rod (per unit time) is $\left(-Ak\left.\frac{\partial u}{\partial x}\right|_{x=0}+Ak\left.\frac{\partial u}{\partial x}\right|_{x=L}\right)$, therefore

$$\begin{aligned}
L\frac{\rho cA}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}e^{-k\left(\frac{n\pi}{L}\right)^2t}\left(1-\cos\left(n\pi\right)\right) &= \int_0^t\left(-Ak\left.\frac{\partial u}{\partial x}\right|_{x=0}+Ak\left.\frac{\partial u}{\partial x}\right|_{x=L}\right)dx \\
&= Ak\int_0^t\left(\frac{\partial u\left(L\right)}{\partial x}-\frac{\partial u\left(0\right)}{\partial x}\right)dx
\end{aligned}$$

But

$$\begin{aligned}
\frac{\partial u\left(L\right)}{\partial x}-\frac{\partial u\left(0\right)}{\partial x} &= \frac{\pi}{L}\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2t}-\frac{\pi}{L}\sum_{n=1}^{\infty}nB_ne^{-k\left(\frac{n\pi}{L}\right)^2t} \\
&= \frac{\pi}{L}\left(\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2t}-\sum_{n=1}^{\infty}nB_ne^{-k\left(\frac{n\pi}{L}\right)^2t}\right)
\end{aligned}$$

Hence

$$\frac{L\rho cA}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}\exp^{-k\left(\frac{n\pi}{L}\right)^2t}\left(1-\cos\left(n\pi\right)\right)=\frac{Ak\pi}{L}\int_0^t\left(\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2t}-\sum_{n=1}^{\infty}nB_ne^{-k\left(\frac{n\pi}{L}\right)^2t}\right)dx$$

2.3.6 section 2.3.5 (problem 5)

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n>0, m>0.$$

Use the trigonometric identity

$$\sin a\sin b=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right].$$

$$I=\int_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Considering first the case $m=n$. The integral becomes

$$I=\int_0^L\sin^2\left(\frac{n\pi x}{L}\right)dx=\frac{L}{2}$$

For the case where $n\neq m$, using

$$\sin a\sin b=\frac{1}{2}\left(\cos\left(a-b\right)-\cos\left(a+b\right)\right)$$

The integral I becomes [2]

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^L \cos\left(\frac{n\pi x}{L}-\frac{m\pi x}{L}\right)-\cos\left(\frac{n\pi x}{L}+\frac{m\pi x}{L}\right)dx \\
&= \frac{1}{2}\int_0^L \cos\left(\frac{\pi x\,(n-m)}{L}\right)-\cos\left(\frac{\pi x\,(n+m)}{L}\right)dx \\
&= \frac{1}{2}\left(\frac{\sin\left(\frac{\pi x(n-m)}{L}\right)}{\frac{\pi(n-m)}{L}}\right)_0^L-\frac{1}{2}\left(\frac{\sin\left(\frac{\pi x(n+m)}{L}\right)}{\frac{\pi(n+m)}{L}}\right)_0^L \\
&= \frac{L}{2\pi\,(n-m)}\left(\sin\left(\frac{\pi x\,(n-m)}{L}\right)\right)_0^L-\frac{L}{2\pi\,(n+m)}\left(\sin\left(\frac{\pi x\,(n+m)}{L}\right)\right)_0^L \qquad (1)
\end{aligned}
$$

But

$$
\left(\sin\left(\frac{\pi x\,(n-m)}{L}\right)\right)_0^L=\sin\left(\pi\,(n-m)\right)-\sin\left(0\right)
$$

And since $n-m$ is integer, then $\sin\left(\pi\,(n-m)\right)=0$, therefore $\left(\sin\left(\frac{\pi x(n-m)}{L}\right)\right)_0^L=0$. Similarly

$$
\left(\sin\left(\frac{\pi x\,(n+m)}{L}\right)\right)_0^L=\sin\left(\pi\,(n+m)\right)-\sin\left(0\right)
$$

Since $n+m$ is integer then $\sin\left(\pi\,(n+m)\right)=0$ and $\left(\sin\left(\frac{\pi x(n+m)}{L}\right)\right)_0^L=0$. Therefore

$$
\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\begin{cases}\frac{L}{2} & n=m\\ 0 & \text{otherwise}\end{cases}
$$

[2] Note that the term $(n-m)$ showing in the denominator is not a problem now, since this is the case where $n\neq m$.

2.3.7 section 2.3.7 (problem 6)

2.3.7. Consider the following boundary value problem (if necessary, see Sec. 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \cos\frac{n\pi x}{L}.\Bigg]$$

What is λ_n?

2.3.7.1 part (a)

This PDE describes how temperature u changes in a rod of length L as a function of time t and location x. The left and right end are insulated, so no heat escapes from these boundaries. Initially at $t = 0$, the temperature distribution in the rod is described by the function $f(x)$.

2.3.7.2 Part (b)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = T(t)X(x)$, then the PDE becomes

$$\frac{1}{k}T'X = X''T$$

Dividing by $XT \neq 0$

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's generated are

$$T' + k\lambda T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Starting with the space ODE equation (2), with corresponding boundary conditions $\frac{dX}{dx}(0) = 0, \frac{dX}{dx}(L) = 0$. Assuming the solution is $X(x) = e^{rx}$, Then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

The following cases are considered.

case $\lambda < 0$ In this case, $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. This gives the solution

$$\begin{aligned} X(x) &= A\cosh(sx) + B\sinh(sx) \\ \frac{dX}{dx} &= A\sinh(sx) + B\cosh(sx) \end{aligned}$$

Applying the left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\cosh(0) \\ &= B \end{aligned}$$

The solution becomes $X(x) = A\cosh(sx)$ and hence $\frac{dX}{dx} = A\sinh(sx)$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(L) \\ &= A\sinh(sL) \end{aligned}$$

$A = 0$ result in trivial solution. Therefore assuming $\sinh(sL) = 0$ implies $sL = 0$ which is not valid since $s > 0$ and $L \neq 0$. Hence only trivial solution results from this case. $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The ODE becomes

$$\frac{d^2X}{dx^2} = 0$$

The solution is

$$\begin{aligned} X(x) &= c_1 x + c_2 \\ \frac{dX}{dx} &= c_1 \end{aligned}$$

Applying left boundary conditions gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= c_1 \end{aligned}$$

Hence the solution becomes $X(x) = c_2$. Therefore $\frac{dX}{dx} = 0$. Applying the right B.C. provides no information.

Therefore this case leads to the solution $X(x) = c_2$. Associated with this one eigenvalue, the time equation becomes $\frac{dT_0}{dt} = 0$ hence T_0 is constant, say α. Hence the solution $u_0(x,t)$ associated with this $\lambda = 0$ is

$$\begin{aligned} u_0(x,t) &= X_0 T_0 \\ &= c_2 \alpha \\ &= A_0 \end{aligned}$$

where constant $c_2\alpha$ was renamed to A_0 to indicate it is associated with $\lambda = 0$. $\lambda = 0$ is an eigenvalue.

case $\lambda > 0$

Hence $-\lambda$ is negative, and the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

The solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ \frac{dX}{dx} &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying the left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\sqrt{\lambda}\cos(0) \\ &= B\sqrt{\lambda} \end{aligned}$$

Therefore $B = 0$ as $\lambda > 0$. The solution becomes $X(x) = A\cos\left(\sqrt{\lambda}x\right)$ and $\frac{dX}{dx} = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(L) \\ &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right) \end{aligned}$$

$A = 0$ gives a trivial solution. Selecting $\sin\left(\sqrt{\lambda}L\right) = 0$ gives

$$\sqrt{\lambda}L = n\pi \qquad n = 1,2,3,\cdots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$

Therefore the space solution is

$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots$$

The time solution is found by solving

$$\frac{dT_n}{dt} + k\lambda_n T_n = 0$$

This has the solution

$$\begin{aligned} T_n(t) &= e^{-k\lambda_n t} \\ &= e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1,2,3,\cdots \end{aligned}$$

For the same set of eigenvalues. Notice that no need to add a constant here, since it will be absorbed in the A_n when combined in the following step below. Since for $\lambda = 0$ the time solution was found to be constant, and for $\lambda > 0$ the time solution is $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$, then no time solution will grow with time. Time solutions always decay with time as the exponent $-k\left(\frac{n\pi}{L}\right)^2 t$ is negative quantity. The solution to the PDE for $\lambda > 0$ is

$$u_n(x,t) = T_n(t) X_n(x) \qquad n = 0,1,2,3,\cdots$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned} u(x,t) &= u_{\lambda=0}(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

2.3.7.3 Part c

From the solution found above, setting $t = 0$ gives

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

Therefore, $f(x)$ must satisfy the above

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

2.3.7.4 Part d

Multiplying both sides with $\cos\left(\frac{m\pi}{L}x\right)$ where in this problem $m = 0, 1, 2, \cdots$ (since there was an eigenvalue associated with $\lambda = 0$), and integrating over the domain gives

$$\begin{aligned}
\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \cos\left(\frac{m\pi}{L}x\right)\left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\right)dx \\
&= \int_0^L A_0 \cos\left(\frac{m\pi}{L}x\right)dx + \int \cos\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)dx \\
&= \int_0^L A_0 \cos\left(\frac{m\pi}{L}x\right)dx + \int_0^L \sum_{n=1}^{\infty} A_n \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx
\end{aligned}$$

Interchanging the order of summation and integration

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0 \cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx \tag{1}$$

<u>case $m = 0$</u>

When $m = 0$ then $\cos\left(\frac{m\pi}{L}x\right) = 1$ and the above simplifies to

$$\int_0^L f(x)\,dx = \int_0^L A_0 dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = 0$ and the above becomes

$$\begin{aligned}
\int_0^L f(x)\,dx &= \int_0^L A_0 dx \\
&= A_0 L
\end{aligned}$$

Therefore

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

<u>case $m > 0$</u>

From (1), one term survives in the integration when only $n = m$, hence

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_0 \int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + A_m \int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx = 0$ and the above becomes

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_m \frac{L}{2}$$

Therefore

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

For $n = 1, 2, 3, \cdots$

2.3.7.5 Part (e)

The solution was found to be

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

In the limit as $t \to \infty$ the term $e^{-k\left(\frac{n\pi}{L}\right)^2 t} \to 0$. What is left is A_0. But $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ from above. This quantity is the average of the initial temperature.

2.3.8 section 2.3.8 (problem 7)

***2.3.8.** **Consider**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem [$u(x,0) = f(x)$] if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

2.3.8.1 part (a)

Equilibrium is at steady state, which implies $\frac{\partial u}{\partial t} = 0$ and the PDE becomes an ODE, since $u \equiv u(x)$ at steady state. Hence

$$\frac{d^2u}{dx^2} - \frac{\alpha}{k}u = 0$$

The characteristic equation is $r^2 = \frac{\alpha}{k}$ or $r = \pm\sqrt{\frac{\alpha}{k}}$. Since $\alpha > 0$ and $k > 0$ then the roots are real, and the solution is

$$u = A_0 e^{\sqrt{\frac{\alpha}{k}}x} + B_0 e^{-\sqrt{\frac{\alpha}{k}}x}$$

This can be rewritten as

$$u(x) = A\cosh\left(\sqrt{\frac{\alpha}{k}}x\right) + B\sinh\left(\sqrt{\frac{\alpha}{k}}x\right)$$

Applying left B.C. gives

$$\begin{aligned} 0 &= u(0) \\ &= A\cosh(0) \\ &= A \end{aligned}$$

The solution becomes $u(x) = B\sinh\left(\sqrt{\frac{\alpha}{k}}x\right)$. Applying the right boundary condition gives

$$\begin{aligned} 0 &= u(L) \\ &= B\sinh\left(\sqrt{\frac{\alpha}{k}}L\right) \end{aligned}$$

$B = 0$ leads to trivial solution. Setting $\sinh\left(\sqrt{\frac{\alpha}{k}}L\right) = 0$ implies $\sqrt{\frac{\alpha}{k}}L = 0$. But this is not possible since $L \neq 0$. Hence the only solution possible is

$$\boxed{u(x) = 0}$$

2.3.8.2 Part (b)

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} - \alpha u \\ \frac{\partial u}{\partial t} + \alpha u &= k\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

Assuming $u(x,t) = X(x)T(t)$ and substituting in the above gives

$$XT' + \alpha XT = kTX''$$

Dividing by $kXT \neq 0$

$$\frac{T'}{kT} + \frac{\alpha}{k} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{k}\frac{T'}{T} + \frac{\alpha}{k} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$\begin{aligned} \frac{1}{k}\frac{T'}{T} + \frac{\alpha}{k} &= -\lambda \\ \frac{X''}{X} &= -\lambda \end{aligned}$$

Or

$$\begin{aligned} T' + (\alpha + \lambda k)T &= 0 \\ X'' + \lambda X &= 0 \end{aligned}$$

The solution to the space ODE is the familiar (where $\lambda > 0$ is only possible case, As found in problem 2.3.3, part d. Since it has the same B.C.)

$$X_n = B_n \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. The time ODE is now solved.

$$\frac{dT_n}{dt} + (\alpha + \lambda_n k)T_n = 0$$

This has the solution

$$\begin{aligned}T_n(t) &= e^{-(\alpha+\lambda_n k)t}\\ &= e^{-\alpha t}e^{-\left(\frac{n\pi}{L}\right)^2 kt}\end{aligned}$$

For the same eigenvalues. Notice that no need to add a constant here, since it will be absorbed in the B_n when combined in the following step below. Therefore the solution to the PDE is

$$u_n(x,t) = T_n(t)\,X_n(x)$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t)\\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha t}e^{-\left(\frac{n\pi}{L}\right)^2 kt}\\ &= e^{-\alpha t}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}\end{aligned}$$

Where $e^{-\alpha t}$ was moved outside since it does not depend on n. From initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality of $\sin$ as before to find B_n results in

$$B_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx$$

Hence the solution becomes

$$u(x,t) = \frac{2}{L}e^{-\alpha t}\left(\sum_{n=1}^{\infty}\left[\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx\right]\sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}\right)$$

Hence it is clear that in the limit as t becomes large $u(x,t) \to 0$ since the sum is multiplied by $e^{-\alpha t}$ and $\alpha > 0$

$$\lim_{t\to\infty} u(x,t) = 0$$

This agrees with part (a)

2.3.9 section 2.3.10 (problem 8)

2.3.10. For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta$, implies that

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.44}$$

In this exercise we generalize this to n-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A}-\gamma\boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A}\cdot\boldsymbol{B}/\boldsymbol{B}\cdot\boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

*(c) Generalize (2.3.44) to functions. [*Hint*: Let $\boldsymbol{A}\cdot\boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]

$$\left|\bar{A}-\gamma\bar{B}\right|^2 = \left(\bar{A}-\gamma\bar{B}\right)\cdot\left(\bar{A}-\gamma\bar{B}\right)$$

Since $\left|\bar{A}-\gamma\bar{B}\right|^2 \geq 0$ then

$$\left(\bar{A}-\gamma\bar{B}\right)\cdot\left(\bar{A}-\gamma\bar{B}\right) \geq 0$$

Expanding

$$\left(\bar{A}\cdot\bar{A}\right) - \gamma\left(\bar{A}\cdot\bar{B}\right) - \gamma\left(\bar{B}\cdot\bar{A}\right) + \gamma^2\left(\bar{B}\cdot\bar{B}\right) \geq 0$$

But $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$, hence

$$\left(\bar{A}\cdot\bar{A}\right) - 2\gamma\left(\bar{A}\cdot\bar{B}\right) + \gamma^2\left(\bar{B}\cdot\bar{B}\right) \geq 0$$

Using the definition of $\gamma = \frac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{B}}$ into the above gives

$$\left(\bar{A}\cdot\bar{A}\right) - 2\frac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{B}}\left(\bar{A}\cdot\bar{B}\right) + \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\left(\bar{B}\cdot\bar{B}\right)^2}\left(\bar{B}\cdot\bar{B}\right) \geq 0$$

$$\left(\bar{A}\cdot\bar{A}\right) - 2\frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} + \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} \geq 0$$

$$\left(\bar{A}\cdot\bar{A}\right) - \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} \geq 0$$

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) - \left(\bar{A}\cdot\bar{B}\right)^2 \geq 0$$

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) \geq \left(\bar{A}\cdot\bar{B}\right)^2$$

But $\left(\bar{A}\cdot\bar{B}\right)^2 = \left|\bar{A}\cdot B\right|^2$ since $\bar{A}\cdot\bar{B}$ is just a number. The above becomes

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) \geq \left|\bar{A}\cdot B\right|^2$$

And $\bar{A}\cdot\bar{A}=\left|\bar{A}\right|^2$ and $\left(\bar{B}\cdot\bar{B}\right)=\left|\bar{B}\right|^2$ by definition as well. Therefore the above becomes

$$\left|\bar{A}\cdot B\right|^2\leq\left|\bar{A}\right|^2\left|\bar{B}\right|^2$$

Taking square root gives

$$\left|\bar{A}\cdot B\right|\leq\left|\bar{A}\right|\left|\bar{B}\right|$$

Which is Schwarz's inequality.

2.3.9.1 Part b

From the norm definition

$$\left|\bar{A}\right|=\sqrt{\sum x^2+y^2+z^2}$$

Then

$$\left(\bar{A}\cdot\bar{A}\right)=\left|\bar{A}\right|^2=\sum x^2+y^2+z^2$$

Hence

$$\left|\bar{A}\right|^2=\sum_{n=1}^{\infty}a_n^2$$
$$\left|\bar{B}\right|^2=\sum_{n=1}^{\infty}b_n^2$$

And

$$\bar{A}\cdot\bar{B}=\sum_{n=1}^{\infty}a_nb_n$$

Therefore the inequality can be written as

$$\left(\bar{A}\cdot\bar{B}\right)^2\leq\left|\bar{A}\right|^2\left|\bar{B}\right|^2$$
$$\left(\sum_{n=1}^{\infty}a_nb_n\right)^2\leq\left(\sum_{n=1}^{\infty}a_n^2\right)\left(\sum_{n=1}^{\infty}b_n^2\right)$$

2.3.9.2 Part c

Using $\bar{A}\cdot\bar{B}$ for functions to mean $\int_0^L A(x)B(x)\,dx$ then inequality for functions becomes

$$\left(\int_0^L A(x)B(x)\,dx\right)^2\leq\left(\int_0^L A^2(x)\,dx\right)\left(\int_0^L B^2(x)\,dx\right)$$

2.3.10 section 2.4.1 (problem 9)

***2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to**

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0.$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ **(b)** $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\frac{\pi x}{L}$ **(d)** $u(x,0) = -3\cos\frac{8\pi x}{L}$

The same boundary conditions was encountered in problem 2.3.7, therefore the solution used here starts from the same general solution already found, which is

$$\lambda_0 = 0$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$
$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

2.3.10.1 Part (b)

$$u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$$

Comparing terms with the general solution at $t = 0$ which is

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

results in

$$A_0 = 6$$
$$A_3 = 4$$

And all other $A_n = 0$. Hence the solution is

$$u(x,t) = 6 + 4\cos\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

2.3.10.2 Part (c)

$$u(x,0) = -2\sin\frac{\pi x}{L}$$

Hence

$$-2\sin\frac{\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) \tag{1}$$

Multiplying both sides of (1) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\begin{aligned}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \left(A_0\cos\left(\frac{m\pi}{L}x\right) + \cos\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\right)dx \\ &= \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \int_0^L \sum_{n=1}^{\infty} A_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx\end{aligned}$$

Interchanging the order of integration and summation

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

<u>Case $m = 0$</u>

The above becomes

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)dx = \int_0^L A_0 dx + \sum_{n=1}^{\infty} A_n\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = 0$ hence

$$\begin{aligned}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)dx &= \int_0^L A_0 dx \\ A_0 L &= -2\int_0^L \sin\left(\frac{\pi x}{L}\right)dx \\ A_0 L &= -2\left(-\frac{\cos\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}}\right)_0^L \\ &= -\frac{2L}{\pi}\left(-\cos\left(\frac{\pi L}{L}\right) + \cos\left(\frac{\pi 0}{L}\right)\right) \\ &= -\frac{2L}{\pi}\left(-(-1)+1\right) \\ &= -\frac{4L}{\pi}\end{aligned}$$

Hence

$$\boxed{A_0 = \frac{-4}{\pi}}$$

<u>Case $m > 0$</u>

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

One term survives the summation resulting in

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \frac{-4}{\pi}\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + A_m\int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx = 0$ and $\int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}$, therefore

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = A_m\frac{L}{2}$$

$$A_n = \frac{-4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

But

$$\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{-L\left(1+\cos\left(n\pi\right)\right)}{\pi\left(n^2-1\right)}$$

Therefore

$$\begin{aligned} A_n &= 4\frac{\left(1+\cos\left(n\pi\right)\right)}{\pi\left(n^2-1\right)} \\ &= 4\frac{(-1)^n+1}{\pi\left(n^2-1\right)} \qquad n=1,2,3,\cdots \end{aligned}$$

Hence the solution becomes

$$u\left(x,t\right) = \frac{-4}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{\left(n^2-1\right)}\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

2.3.11 section 2.4.2 (problem 10)

***2.4.2.** **Solve**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = T(t)X(x)$, then the PDE becomes

$$\frac{1}{\kappa}T'X = X''T$$

Dividing by XT

$$\frac{1}{\kappa}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is real.

$$\frac{1}{\kappa}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + k\lambda T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Per problem statement, $\lambda \geq 0$, so only two cases needs to be examined.

Case $\lambda = 0$

The space equation becomes $X'' = 0$ with the solution

$$X = Ax + b$$

Hence left B.C. implies $X'(0) = 0$ or $A = 0$. Therefore the solution becomes $X = b$. The right B.C. implies $X(L) = 0$ or $b = 0$. Therefore this leads to $X = 0$ as the only solution. This results in trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

Starting with the space ODE, the solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ \frac{dX}{dx} &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\sqrt{\lambda} \end{aligned}$$

Hence $B = 0$ since it is assumed $\lambda \neq 0$ and $\lambda > 0$. Solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$

Applying right B.C. gives

$$\begin{aligned} 0 &= X(L) \\ &= A\cos\left(\sqrt{\lambda}L\right) \end{aligned}$$

$A=0$ leads to trivial solution. Therefore $\cos\left(\sqrt{\lambda}L\right)=0$ or

$$\begin{aligned}\sqrt{\lambda} &= \frac{n\pi}{2L} \qquad n=1,3,5,\cdots\\ &= \frac{(2n-1)\,\pi}{2L} \qquad n=1,2,3\cdots\end{aligned}$$

Hence

$$\begin{aligned}\lambda_n &= \left(\frac{n\pi}{2L}\right)^2 \qquad n=1,3,5,\cdots\\ &= \frac{(2n-1)^2\,\pi^2}{4L^2} \qquad n=1,2,3\cdots\end{aligned}$$

Therefore

$$X_n(x) = A_n \cos\left(\frac{n\pi}{2L}x\right) \qquad n=1,3,5,\cdots$$

And the corresponding time solution

$$T_n = e^{-k\left(\frac{n\pi}{2L}\right)^2 t} \qquad n=1,3,5,\cdots$$

Hence

$$\begin{aligned}u_n(x,t) &= X_n T_n\\ u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi}{2L}x\right) e^{-k\left(\frac{n\pi}{2L}\right)^2 t}\\ &= \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\,\pi}{2L}x\right) e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}\end{aligned}$$

From initial conditions

$$f(x) = \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi}{2L}x\right)$$

Multiplying both sides by $\cos\left(\frac{m\pi}{2L}x\right)$ and integrating

$$\int_0^L f(x)\cos\left(\frac{m\pi}{2L}x\right)dx = \int\left(\sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{m\pi}{2L}x\right)\cos\left(\frac{n\pi}{2L}x\right)\right)dx$$

Interchanging order of summation and integration and applying orthogonality results in

$$\begin{aligned}\int_0^L f(x)\cos\left(\frac{m\pi}{2L}x\right)dx &= A_m\frac{L}{2}\\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{2L}x\right)dx\end{aligned}$$

Therefore the solution is

$$u(x,t) = \frac{2}{L}\sum_{n=1,3,5,\cdots}^{\infty}\left[\int_0^L f(x)\cos\left(\frac{n\pi}{2L}x\right)dx\right]\cos\left(\frac{n\pi}{2L}x\right)e^{-k\left(\frac{n\pi}{2L}\right)^2 t}$$

or

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\cos\left(\frac{(2n-1)\pi}{2L}x\right)dx\right]\cos\left(\frac{(2n-1)\pi}{2L}x\right)e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

2.3.12 section 2.4.3 (problem 11)

***2.4.3. Solve the eigenvalue problem**

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

$$\begin{aligned}\frac{d\phi^2}{dx^2} + \lambda\phi &= 0\\ \phi(0) &= \phi(2\pi)\\ \frac{d\phi}{dx}(0) &= \frac{d\phi}{dx}(2\pi)\end{aligned}$$

First solution using transformation

Let $\tau = x - \pi$, hence the above system becomes

$$\begin{aligned}\frac{d\phi^2}{d\tau^2} + \lambda\phi &= 0\\ \phi(-\pi) &= \phi(\pi)\\ \frac{d\phi}{d\tau}(-\pi) &= \frac{d\phi}{d\tau}(\pi)\end{aligned}$$

The characteristic equation is $r^2 + \lambda = 0$ or $r = \pm\sqrt{-\lambda}$. Assuming λ is real. There are three cases to consider.

Case $\lambda < 0$

Let $s = \sqrt{-\lambda} > 0$

$$\begin{aligned}\phi(\tau) &= c_1\cosh(s\tau) + c_2\sinh(s\tau)\\ \phi'(\tau) &= sc_1\sinh(s\tau) + sc_2\cosh(s\tau)\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}
\phi(-\pi) &= \phi(\pi) \\
c_1 \cosh(s\pi) - c_2 \sinh(s\pi) &= c_1 \cosh(s\pi) + c_2 \sinh(s\pi) \\
2c_2 \sinh(s\pi) &= 0 \\
c_2 \sinh(s\pi) &= 0 \qquad (1)
\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}
\phi'(-\pi) &= \phi'(\pi) \\
-sc_1 \sinh(s\pi) + sc_2 \cosh(s\pi) &= sc_1 \sinh(s\pi) + sc_2 \cosh(s\pi) \\
2c_1 \sinh(s\pi) &= 0 \\
c_1 \sinh(s\pi) &= 0 \qquad (2)
\end{aligned}$$

Since $\sinh(s\pi)$ is zero only for $s\pi = 0$ and $s\pi$ is not zero because $s > 0$. Then the only other option is that both $c_1 = 0$ and $c_2 = 0$ in order to satisfy equations (1)(2). Hence trivial solution. <u>Hence $\lambda < 0$ is not an eigenvalue</u>.

<u>Case $\lambda = 0$</u>

The space equation becomes $\frac{d\phi^2}{d\tau^2} = 0$ with the solution $\phi(\tau) = A\tau + B$. Applying the first B.C. gives

$$\begin{aligned}
\phi(-\pi) &= \phi(\pi) \\
-A\pi + B &= A\pi + B \\
0 &= 2A\pi
\end{aligned}$$

Hence $A = 0$. The solution becomes $\phi(\tau) = B$. And $\phi'(\tau) = 0$. The second B.C. just gives $0 = 0$. Therefore the solution is

$$\phi(\tau) = C$$

Where C is any constant. <u>Hence $\lambda = 0$ is an eigenvalue</u>.

<u>Case $\lambda > 0$</u>

$$\begin{aligned}
\phi(\tau) &= c_1 \cos\left(\sqrt{\lambda}\tau\right) + c_2 \sin\left(\sqrt{\lambda}\tau\right) \\
\phi'(\tau) &= -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}\tau\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}\tau\right)
\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}
\phi(-\pi) &= \phi(\pi) \\
c_1 \cos\left(\sqrt{\lambda}\pi\right) - c_2 \sin\left(\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\
2c_2 \sin\left(\sqrt{\lambda}\pi\right) &= 0 \\
c_2 \sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)
\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}
\phi'(-\pi) &= \phi'(\pi) \\
c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\
2c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= 0 \\
c_1\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (2)
\end{aligned}$$

Both (3) and (2) can be satisfied for non-zero $\sqrt{\lambda}\pi$. The trivial solution is avoided. Therefore the eigenvalues are

$$\begin{aligned}
\sin\left(\sqrt{\lambda}\pi\right) &= 0 \\
\sqrt{\lambda_n}\pi &= n\pi \qquad n = 1,2,3,\cdots \\
\lambda_n &= n^2 \qquad n = 1,2,3,\cdots
\end{aligned}$$

Hence the corresponding eigenfunctions are

$$\left\{\cos\left(\sqrt{\lambda_n}\tau\right), \sin\left(\sqrt{\lambda_n}\tau\right)\right\} = \{\cos(n\tau), \sin(n\tau)\}$$

Transforming back to x using $\tau = x - \pi$

$$\{\cos(n(x-\pi)), \sin(n(x-\pi))\} = \{\cos(nx - n\pi), \sin(nx - n\pi)\}$$

But $\cos(x-\pi) = -\cos x$ and $\sin(x-\pi) = -\sin x$, hence the eigenfunctions are

$$\{-\cos(nx), -\sin(nx)\}$$

The signs of negative on an eigenfunction (or eigenvector) do not affect it being such as this is just a multiplication by -1. Hence the above is the same as saying the eigenfunctions are

$$\{\cos(nx), \sin(nx)\}$$

Summary

	eigenfunctions
$\lambda = 0$	arbitrary constant
$\lambda > 0$	$\{\cos(nx), \sin(nx)\}$ for $n = 1,2,3\cdots$

Second solution without transformation

(note: Using transformation as shown above seems to be easier method than this below).

The characteristic equation is $r^2 + \lambda = 0$ or $r = \pm\sqrt{-\lambda}$. Assuming λ is real. There are three cases to consider.

<u>Case $\lambda < 0$</u>

In this case $-\lambda$ is positive and the roots are both real. Assuming $\sqrt{-\lambda} = s$ where $s > 0$, then the solution is

$$\begin{aligned}
\phi(x) &= Ae^{sx} + Be^{-sx} \\
\phi'(x) &= Ase^{sx} - Bse^{-sx}
\end{aligned}$$

First B.C. gives

$$\begin{aligned}
\phi(0) &= \phi(2\pi) \\
A + B &= Ae^{2s\pi} + Be^{-2s\pi} \\
A\left(1 - e^{2s\pi}\right) + B\left(1 - e^{-2s\pi}\right) &= 0 \qquad (1)
\end{aligned}$$

The second B.C. gives

$$\begin{aligned}
\phi'(0) &= \phi'(2\pi) \\
As - Bs &= Ase^{2s\pi} - Bse^{-2s\pi} \\
A\left(1 - e^{2s\pi}\right) + B\left(-1 + e^{-2s\pi}\right) &= 0 \qquad (2)
\end{aligned}$$

After dividing by s since $s \neq 0$. Now a 2 by 2 system is setup from (1),(2)

$$\begin{pmatrix} \left(1 - e^{2s\pi}\right) & \left(1 - e^{-2s\pi}\right) \\ \left(1 - e^{2s\pi}\right) & \left(-1 + e^{-2s\pi}\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since this is $Mx = b$ with $b = 0$ then for non-trivial solution $|M|$ must be zero. Checking the determinant to see if it is zero or not:

$$\begin{aligned}
\begin{vmatrix} \left(1 - e^{2s\pi}\right) & \left(1 - e^{-2s\pi}\right) \\ \left(1 - se^{2s\pi}\right) & \left(-1 + se^{-2s\pi}\right) \end{vmatrix} &= \left(1 - e^{2s\pi}\right)\left(-1 + e^{-2s\pi}\right) - \left(1 - e^{-2s\pi}\right)\left(1 - e^{2s\pi}\right) \\
&= \left(-1 + e^{-2s\pi} + e^{2s\pi} - 1\right) - \left(1 - e^{2s\pi} - e^{-2s\pi} + 1\right) \\
&= -1 + e^{-2s\pi} + e^{2s\pi} - 1 - 1 + e^{2s\pi} + e^{-2s\pi} - 1 \\
&= -4 + 2e^{2s\pi} + 2e^{-2s\pi} \\
&= -4 + 2\left(e^{2s\pi} + e^{-2s\pi}\right) \\
&= -4 + 4\cosh(2s\pi)
\end{aligned}$$

Hence for the determinant to be zero (so that non-trivial solution exist) then $-4+4\cosh(2s\pi) = 0$ or $\cosh(2s\pi) = 1$ which has the solution $2s\pi = 0$. Which means $s = 0$. But the assumption was that $s > 0$. This implies only a trivial solution exist and <u>$\lambda < 0$ is not an eigenvalue.</u>

<u>case $\lambda = 0$</u>

The space equation becomes $\frac{d\phi^2}{dx^2} = 0$ with the solution $\phi(x) = Ax + B$. Applying the first B.C.

gives

$$\begin{aligned} B &= 2A\pi + B \\ 0 &= 2A\pi \end{aligned}$$

Hence $A = 0$. The solution becomes $\phi(x) = B$. And $\phi'(x) = 0$. The second B.C. just gives $0 = 0$. Therefore the solution is

$$\phi(x) = C$$

Where C is any constant. <u>Hence $\lambda = 0$ is an eigenvalue.</u>

<u>Case $\lambda > 0$</u>

In this case the solution is

$$\begin{aligned} \phi(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ \phi'(x) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying first B.C. gives

$$\begin{aligned} \phi(0) &= \phi(2\pi) \\ A &= A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right) \\ A\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) - B\sin\left(2\pi\sqrt{\lambda}\right) &= 0 \end{aligned}$$

Applying second B.C. gives

$$\begin{aligned} \phi'(0) &= \phi'(2\pi) \\ B\sqrt{\lambda} &= -A\sqrt{\lambda}\sin\left(2\pi\sqrt{\lambda}\right) + B\sqrt{\lambda}\cos\left(2\pi\sqrt{\lambda}\right) \\ A\sqrt{\lambda}\sin\left(2\pi\sqrt{\lambda}\right) + B\left(\sqrt{\lambda} - \sqrt{\lambda}\cos\left(2\pi\sqrt{\lambda}\right)\right) &= 0 \\ A\sin\left(2\pi\sqrt{\lambda}\right) + B\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) &= 0 \end{aligned}$$

Therefore

$$\begin{pmatrix} 1-\cos\left(2\pi\sqrt{\lambda}\right) & -\sin\left(2\pi\sqrt{\lambda}\right) \\ \sin\left(2\pi\sqrt{\lambda}\right) & 1-\cos\left(2\pi\sqrt{\lambda}\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

Setting $|M| = 0$ to obtain the eigenvalues gives

$$\begin{aligned} \left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right)\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) + \sin\left(2\pi\sqrt{\lambda}\right)\sin\left(2\pi\sqrt{\lambda}\right) &= 0 \\ 1-\cos\left(2\pi\sqrt{\lambda}\right) &= 0 \end{aligned}$$

Hence

$$\cos\left(2\pi\sqrt{\lambda}\right) = 1$$
$$2\pi\sqrt{\lambda_n} = n\pi \qquad n = 2, 4, \cdots$$
$$\sqrt{\lambda_n} = \frac{n}{2} \qquad n = 2, 4, \cdots$$

Or

$$\sqrt{\lambda_n} = n \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Therefore the eigenfunctions are

$$\phi_n(x) = \{\cos(nx), \sin(nx)\}$$

Summary

	eigenfunctions
$\lambda = 0$	arbitrary constant
$\lambda > 0$	$\{\cos(nx), \sin(nx)\}$ for $n = 1, 2, 3 \cdots$

2.3.13 section 2.4.6 (problem 12)

2.4.6. **Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:**

(a) **Directly from the equilibrium problem (see Sec. 1.4)**

(b) **By computing the limit as $t \to \infty$ of the time-dependent problem**

The PDE for the thin circular ring is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(-L, t) = u(L, t)$$
$$\frac{\partial u(-L, t)}{\partial t} = \frac{\partial u(L, t)}{\partial t}$$
$$u(x, 0) = f(x)$$

2.3.13.1 Part (a)

At equilibrium $\frac{\partial u}{\partial t} = 0$ and the PDE becomes

$$0 = \frac{\partial^2 u}{\partial x^2}$$

As it now has one independent variable, it becomes the following ODE to solve

$$\begin{aligned}\frac{d^2u(x)}{dx^2} &= 0\\ u(-L) &= u(L)\\ \frac{du}{dx}(-L) &= \frac{du}{dx}(L)\end{aligned}$$

Solution to $\frac{d^2u}{dx^2} = 0$ is

$$u(x) = c_1x + c_2$$

Where c_1, c_2 are arbitrary constants. From the first B.C.

$$\begin{aligned}u(-L) &= u(L)\\ -c_1L + c_2 &= c_1L + c_2\\ 2c_1L &= 0\\ c_1 &= 0\end{aligned}$$

Hence the solution becomes

$$u(x) = c_2$$

The second B.C. adds nothing as it results in $0 = 0$. Hence the solution at equilibrium is

$$\boxed{u(x) = c_2}$$

This means at equilibrium the temperature in the ring reaches a constant value.

2.3.13.2 Part (b)

The time dependent solution was derived in problem 2.4.3 and also in section 2.4, page 62 in the book, given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi x}{L}\right)^2 t} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi x}{L}\right)^2 t}$$

As $t \to \infty$ the terms $e^{-k\left(\frac{n\pi x}{L}\right)^2 t} \to 0$ and the above reduces to

$$u(x,\infty) = a_0$$

Since a_0 is constant, this is the same result found in part (a).

2.4 HW 3

2.4.1 Problem 2.5.1(e) (problem 1)

2.5.1. Solve Laplace's equation inside a rectangle $0 \leq x \leq L,\ 0 \leq y \leq H$, with the following boundary conditions:

*(a)	$\frac{\partial u}{\partial x}(0,y) = 0,$	$\frac{\partial u}{\partial x}(L,y) = 0,$	$u(x,0) = 0,$	$u(x,H) = f(x)$
(b)	$\frac{\partial u}{\partial x}(0,y) = g(y),$	$\frac{\partial u}{\partial x}(L,y) = 0,$	$u(x,0) = 0,$	$u(x,H) = 0$
*(c)	$\frac{\partial u}{\partial x}(0,y) = 0,$	$u(L,y) = g(y),$	$u(x,0) = 0,$	$u(x,H) = 0$
(d)	$u(0,y) = g(y),$	$u(L,y) = 0,$	$\frac{\partial u}{\partial y}(x,0) = 0,$	$u(x,H) = 0$
*(e)	$u(0,y) = 0,$	$u(L,y) = 0,$	$u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0,$	$u(x,H) = f(x)$

Let $u\left(x,y\right) = X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Each side depends on different independent variable and they are equal, therefore they must be equal to same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm\lambda$$

Since the boundary conditions along the x direction are the homogeneous ones, $-\lambda$ is selected in the above. Two ODE's (1,2) are obtained as follows

$$X'' + \lambda X = 0 \tag{1}$$

With the boundary conditions

$$X(0) = 0$$
$$X(L) = 0$$

And

$$Y'' - \lambda Y = 0 \tag{2}$$

With the boundary conditions

$$Y(0) = Y'(0)$$
$$Y(H) = f(x)$$

In all these cases λ will turn out to be positive. This is shown for this problem only and not be repeated again. The solution to (1) is

$$X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

<u>Case $\lambda < 0$</u>

$$X = A\cosh\left(\sqrt{\lambda}x\right) + B\sinh\left(\sqrt{\lambda}x\right)$$

At $x = 0$, the above gives $0 = A$. Hence $X = B\sinh\left(\sqrt{\lambda}x\right)$. At $x = L$ this gives $X = B\sinh\left(\sqrt{\lambda}L\right)$. But $\sinh\left(\sqrt{\lambda}L\right) = 0$ only at 0 and $\sqrt{\lambda}L \neq 0$, therefore $B = 0$ and this leads to trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

<u>Case $\lambda = 0$</u>

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = L$, $B = 0$. Hence the trivial solution. $\lambda = 0$ is not an eigenvalue.

<u>Case $\lambda > 0$</u>

Solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\lambda}x\right)$. At $x = L$

$$0 = B\sin\left(\sqrt{\lambda}L\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}L\right) = 0$ or $\sqrt{\lambda}L = n\pi$ where $n = 1, 2, 3, \cdots$, therefore

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Eigenfunctions are

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \cdots \tag{3}$$

For the Y ODE, the solution is

$$\begin{aligned} Y_n &= C_n\cosh\left(\frac{n\pi}{L}y\right) + D_n\sinh\left(\frac{n\pi}{L}y\right) \\ Y_n' &= C_n\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}y\right) + D_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) \end{aligned}$$

Applying B.C. at $y = 0$ gives

$$\begin{aligned} Y(0) &= Y'(0) \\ C_n\cosh(0) &= D_n\frac{n\pi}{L}\cosh(0) \\ C_n &= D_n\frac{n\pi}{L} \end{aligned}$$

The eigenfunctions Y_n are

$$\begin{aligned} Y_n &= D_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + D_n\sinh\left(\frac{n\pi}{L}y\right) \\ &= D_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right) \end{aligned}$$

Now the complete solution is produced

$$\begin{aligned} u_n\left(x,y\right) &= Y_n X_n \\ &= D_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)B_n\sin\left(\frac{n\pi}{L}x\right) \end{aligned}$$

Let $D_nB_n = B_n$ since a constant. (no need to make up a new symbol).

$$u_n\left(x,y\right) = B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Sum of eigenfunctions is the solution, hence

$$u\left(x,y\right) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

The nonhomogeneous boundary condition is now resolved. At $y=H$

$$u\left(x,H\right) = f\left(x\right)$$

Therefore

$$f\left(x\right) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\begin{aligned} \int_0^L f\left(x\right)\sin\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= B_m\left(\frac{m\pi}{L}\cosh\left(\frac{m\pi}{L}H\right)+\sinh\left(\frac{m\pi}{L}H\right)\right)\frac{L}{2} \end{aligned}$$

Hence

$$B_n = \frac{2}{L}\frac{\int_0^L f\left(x\right)\sin\left(\frac{n\pi}{L}x\right)dx}{\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)} \tag{4}$$

This completes the solution. In summary

$$u\left(x,y\right) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

With B_n given by (4). The following are some plots of the solution above for different $f\left(x\right)$.

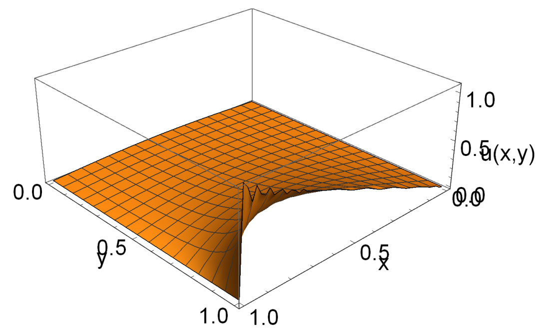


Figure 2.1: Solution using $f(x) = x, L = 1, H = 1$

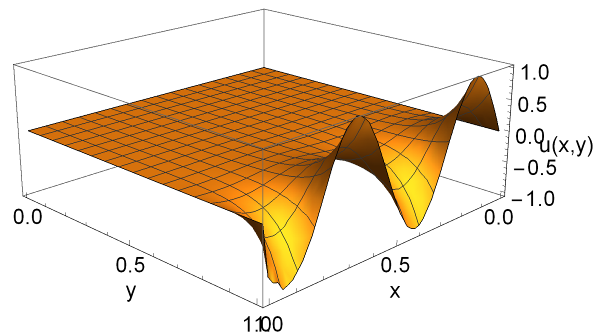


Figure 2.2: Solution using $f(x) = \sin(12x), L = 1, H = 1$

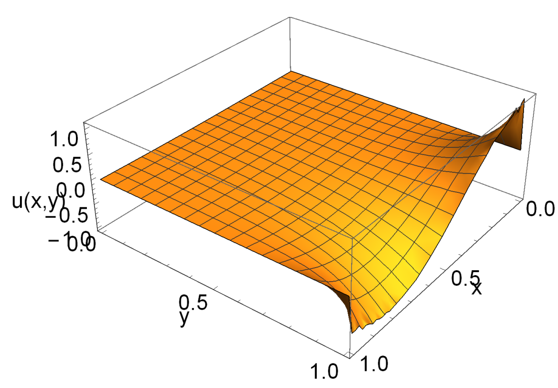


Figure 2.3: Solution using $f(x) = \cos(4x), L = 1, H = 1$

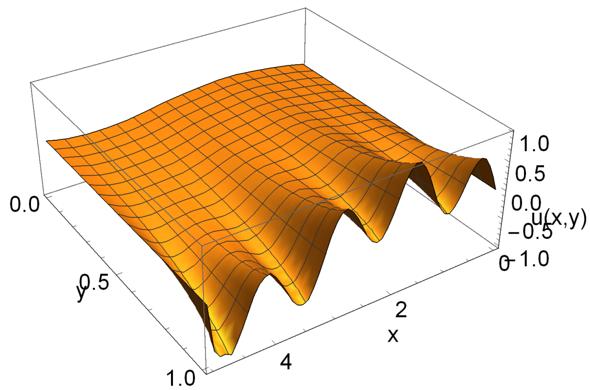


Figure 2.4: Solution using $f(x) = \sin(3x)\cos(2x), L = 5, H = 1$

2.4.2 Problem 2.5.2 (problem 2)

2.5.2. Consider $u(x, y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad \frac{\partial u}{\partial y}(x, 0) = 0$$
$$\frac{\partial u}{\partial x}(L, y) = 0 \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

*(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a).

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x, y, 0) = g(x, y).$$

2.4.2.1 part (a)

At steady state, there will be no heat energy flowing across the boundaries. Which implies the flux is zero. Three of the boundaries are already insulated and hence the flux is zero at those boundaries as given. Therefore, the flux should also be zero at the top boundary at steady state.

By definition, the flux is $\bar{\phi} = -k\bar{\nabla}u \cdot \hat{n}$. (Direction of flux vector is from hot to cold). At the top boundary, this becomes

$$\phi = -k\frac{\partial u}{\partial y}(x,H) \tag{1}$$

Therefore, For the condition of a solution, total flux on the boundary is zero, or

$$\int_0^L \phi dx = 0$$

Using (1) in the above gives

$$-k\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx = 0$$
$$\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx = 0$$

But $\frac{\partial u}{\partial y}(x,H) = f(x)$ and the above becomes

$$\boxed{\int_0^L f(x)\,dx = 0}$$

2.4.2.2 Part (b)

Using separation of variables results in the following two ODE's

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(L) = 0$$

And

$$Y'' - \lambda Y = 0$$
$$Y'(0) = 0$$
$$Y'(L) = f(x)$$

The solution to the $X(x)$ ODE has been obtained before as

$$X_n = A_0 + A_n\cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$
$$X_n = A_n\cos\left(\sqrt{\lambda_n}x\right) \qquad n = 0,1,2,3,\cdots \tag{1}$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. In this ODE $\lambda = 0$ is applicable as well as $\lambda > 0$. (As found in last HW).

Now the $Y(y)$ ODE is solved (for same set of eigenvalues). For $\lambda = 0$ the ODE becomes $Y'' = 0$ and solution is $Y = Cy + D$. Hence $Y' = C$ and since $Y'(0) = 0$ then $C = 0$. Hence the solution is $Y = C_0$, where C_0 is some new constant. For $\lambda > 0$, the solution is

$$Y_n = C_n\cosh\left(\sqrt{\lambda_n}y\right) + D_n\sinh\left(\sqrt{\lambda_n}y\right) \qquad n = 1,2,3,\cdots$$
$$Y_n' = C_n\sqrt{\lambda_n}\sinh\left(\sqrt{\lambda_n}y\right) + D_n\sqrt{\lambda_n}\cosh\left(\sqrt{\lambda_n}y\right)$$

At $y=0$

$$\begin{aligned} 0 &= Y_n'(0) \\ &= D_n\sqrt{\lambda_n} \qquad n=1,2,3,\cdots \end{aligned}$$

Since $\lambda_n>0$ for $n=1,2,3,\cdots$ then $D_n=0$ and the $Y\left(y\right)$ solution becomes

$$\begin{aligned} Y_n &= C_0 + C_n\cosh\left(\sqrt{\lambda_n}y\right) \qquad n=1,2,3,\cdots \\ Y_n &= C_n\cosh\left(\sqrt{\lambda_n}y\right) \qquad n=0,1,2,3,\cdots \end{aligned} \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} u_n\left(x,y\right) &= X_nY_n \\ &= A_n\cos\left(\sqrt{\lambda_n}x\right)C_n\cosh\left(\sqrt{\lambda_n}y\right) \qquad n=0,1,2,3,\cdots \\ &= A_n\cos\left(\sqrt{\lambda_n}x\right)\cosh\left(\sqrt{\lambda_n}y\right) \qquad n=0,1,2,3,\cdots \end{aligned}$$

Where A_nC_n above was combined and renamed to A_n (No need to add new symbol). Hence by superposition the solution becomes

$$u\left(x,y\right) = \sum_{n=0}^{\infty} A_n\cos\left(\sqrt{\lambda_n}x\right)\cosh\left(\sqrt{\lambda_n}y\right)$$

Since $\lambda_0=0$ and $\cos\left(\sqrt{\lambda_0}x\right)\cosh\left(\sqrt{\lambda_0}y\right)=1$, the above can be also be written as

$$u\left(x,y\right) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right) \tag{3}$$

At $y=H$, it is given that $\frac{\partial u}{\partial y}(x,H)=f(x)$. But

$$\frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}y\right)$$

At $y=H$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}H\right) \tag{4}$$

To <u>verify part (a)</u> by integrating both sides

$$\begin{aligned} \int_0^L f(x)\,dx &= \int_0^L \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}H\right)dx \\ &= \sum_{n=1}^{\infty} A_n\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}H\right)\int_0^L\cos\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

But $\int_0^L\cos\left(\frac{n\pi}{L}x\right)dx=0$, hence

$$\int_0^L f(x)\,dx = 0$$

The verification is completed. Now back to (4) and multiplying by $\cos\left(\frac{m\pi}{L}x\right)$ and integrating

$$\begin{aligned}
\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\sqrt{\lambda_n}\sinh\left(\frac{n\pi}{L}H\right)dx \\
&= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L}H\right)\int_0^L \cos\left(\frac{n\pi}{L}x\right)\sqrt{\lambda_n}dx \\
&= A_m \sinh\left(\frac{m\pi}{L}H\right)\frac{L}{2}
\end{aligned}$$

Hence

$$A_n = \frac{2}{L}\frac{\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}H\right)} \qquad n = 1,2,3,\cdots$$

Therefore the solution now becomes (from (3))

$$u(x,y) = A_0 + \sum_{n=1}^{\infty}\left(\frac{2}{L}\frac{\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}H\right)}\right)\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)$$

Only A_0 remains to be found. This is done in next part.

2.4.2.3 Part (c)

Since at steady state, total energy is the same as initial energy. Initial temperature is given as $g(x,y)$, therefore initial thermal energy is found by integrating over the whole domain. This is 2D, therefore

$$\int\int \rho c g(x,y)\,dA = \rho c\int_0^L\int_0^H g(x,y)\,dydx$$

Setting the above to $\rho c\int_0^L\int_0^H u(x,y)\,dydx$ found in last part, gives one equation with one unknown, which is A_0 to solve for. Hence

$$\begin{aligned}
\rho c\int_0^L\int_0^H g(x,y)\,dydx &= \rho c\int_0^L\int_0^H A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx \\
\int_0^L\int_0^H g(x,y)\,dydx &= \int_0^L\int_0^H A_0 dydx + \int_0^L\int_0^H \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx \\
\int_0^L\int_0^H g(x,y)\,dydx &= A_0 HL + \sum_{n=1}^{\infty} A_n\int_0^L\int_0^H \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx \qquad (5)
\end{aligned}$$

But

$$\int_0^L\int_0^H \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx = \int_0^H \cosh\left(\frac{n\pi}{L}y\right)\left(\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx\right)dy$$

Where $\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = 0$. Hence the whole sum vanish. Therefore (5) reduces to

$$\int_0^L \int_0^H g\left(x,y\right)dydx = A_0 HL$$

$$A_0 = \frac{1}{HL}\int_0^L \int_0^H g\left(x,y\right)dydx$$

Summary The complete solution is

$$u\left(x,y\right) = \left(\frac{1}{HL}\int_0^L \int_0^H g\left(x,y\right)dydx\right) + \sum_{n=1}^{\infty}\left(\frac{2}{L}\frac{\int_0^L f\left(x\right)\cos\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}H\right)}\right)\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)$$

The following are some plots of the solution.

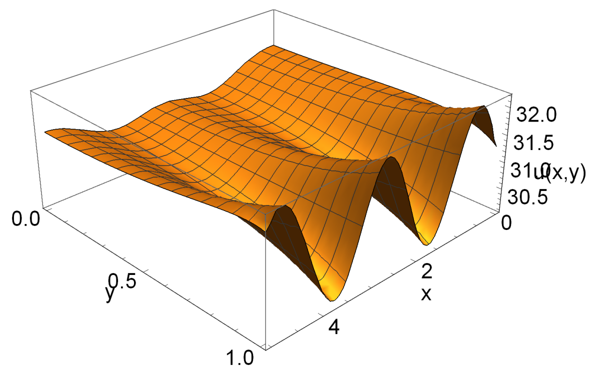


Figure 2.5: Solution using $g(x,y) = xy, f(x) = \sin(3x), L = 5, H = 1$

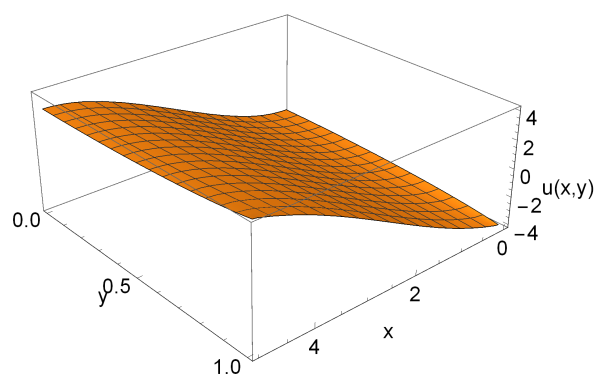


Figure 2.6: Solution using $g(x,y) = \sin(y)\cos(xy), f(x) = x, L = 5, H = 1$

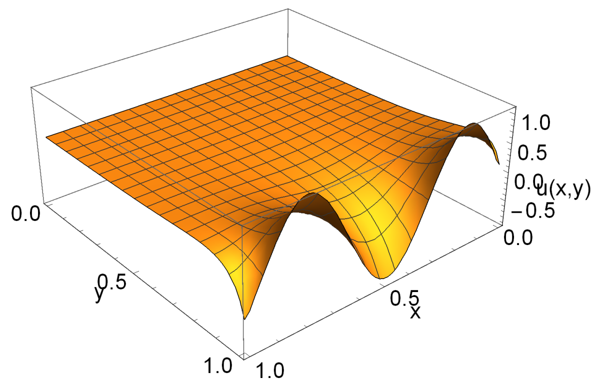


Figure 2.7: Solution using $g(x,y) = y\sin(y)\cos(xy), f(x) = \sin(10x), L = 1, H = 1$

2.4.3 Problem 2.5.5(c,d) (problem 3)

2.5.5. Solve Laplace's equation inside the quarter-circle of radius 1 ($0 \leq \theta \leq \pi/2$, $0 \leq r \leq 1$) subject to the boundary conditions

* (a) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $u\left(r,\frac{\pi}{2}\right) = 0$, $u(1,\theta) = f(\theta)$

(b) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $\frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0$, $u(1,\theta) = f(\theta)$

* (c) $u(r,0) = 0$, $u\left(r,\frac{\pi}{2}\right) = 0$, $\frac{\partial u}{\partial r}(1,\theta) = f(\theta)$

(d) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $\frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0$, $\frac{\partial u}{\partial r}(1,\theta) = g(\theta)$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta)\, d\theta = 0$. Explain this condition physically.

2.4.3.1 Part c

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With boundary conditions

$$\begin{aligned} u(r,0) &= 0 \\ u\left(r,\frac{\pi}{2}\right) &= 0 \\ u(1,\theta) &= f(\theta) \end{aligned} \tag{B}$$

Assuming the solution can be written as

$$u(r,\theta) = R(r)\Theta(\theta)$$

And substituting this assumed solution back into the (A) gives

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta \neq 0$ gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. This gives

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ \Theta(0) &= 0 \\ \Theta\left(\frac{\pi}{2}\right) &= 0 \end{aligned} \tag{1}$$

And

$$\begin{aligned} r^2R'' + rR' - \lambda R &= 0 \\ |R(0)| &< \infty \end{aligned} \tag{2}$$

Starting with (1). Consider the Case $\lambda < 0$. The solution in this case will be

$$\Theta = A\cosh\left(\sqrt{\lambda}\theta\right) + B\sinh\left(\sqrt{\lambda}\theta\right)$$

Applying first B.C. gives $A = 0$. The solution becomes $\Theta = B\sinh\left(\sqrt{\lambda}\theta\right)$. Applying second B.C. gives

$$0 = B\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

But $\sinh$ is zero only when $\sqrt{\lambda}\frac{\pi}{2} = 0$ which is not the case here. Therefore $B = 0$ and hence trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$ The ODE becomes $\Theta'' = 0$ with solution $\Theta = A\theta + B$. First B.C. gives $0 = B$. The solution becomes $\Theta = A\theta$. Second B.C. gives $0 = A\frac{\pi}{2}$, hence $A = 0$ and trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$ The ODE becomes $\Theta'' + \lambda\Theta = 0$ with solution

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$

The first B.C. gives $0 = A$. The solution becomes

$$\Theta = B\sin\left(\sqrt{\lambda}\theta\right)$$

And the second B.C. gives

$$0 = B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$ or $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ for $n = 1,2,3,\cdots$. Hence the eigenvalues are

$$\begin{aligned}
\sqrt{\lambda_n} &= 2n \\
\lambda_n &= 4n^2 \qquad n = 1,2,3,\cdots
\end{aligned}$$

And the eigenfunctions are

$$\boxed{\Theta_n(\theta) = B_n \sin(2n\theta) \qquad n = 1,2,3,\cdots} \tag{3}$$

<u>Now the R ODE is solved</u>. There is one case to consider, which is $\lambda > 0$ based on the above. The ODE is

$$\begin{aligned}
r^2R'' + rR' - \lambda_n R &= 0 \\
r^2R'' + rR' - 4n^2R &= 0 \qquad n = 1,2,3,\cdots
\end{aligned}$$

This is Euler ODE. Let $R(r) = r^p$. Then $R' = pr^{p-1}$ and $R'' = p\left(p-1\right)r^{p-2}$. This gives

$$\begin{aligned}
r^2\left(p\left(p-1\right)r^{p-2}\right) + r\left(pr^{p-1}\right) - 4n^2r^p &= 0 \\
\left(\left(p^2-p\right)r^p\right) + pr^p - 4n^2r^p &= 0 \\
r^pp^2 - pr^p + pr^p - 4n^2r^p &= 0 \\
p^2 - 4n^2 &= 0 \\
p &= \pm 2n
\end{aligned}$$

Hence the solution is

$$R(r) = Cr^{2n} + D\frac{1}{r^{2n}}$$

Applying the condition that $|R(0)| < \infty$ implies $D = 0$, and the solution becomes

$$R_n(r) = C_n r^{2n} \qquad n = 1,2,3,\cdots \tag{4}$$

Using (3,4) the solution $u_n(r,\theta)$ is

$$\begin{aligned}
u_n(r,\theta) &= R_n\Theta_n \\
&= C_n r^{2n} B_n \sin(2n\theta) \\
&= B_n r^{2n} \sin(2n\theta)
\end{aligned}$$

Where C_nB_n was combined into one constant B_n. (No need to introduce new symbol). The final solution is

$$\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty} u_n(r,\theta) \\
&= \sum_{n=1}^{\infty} B_n r^{2n} \sin(2n\theta)
\end{aligned}$$

Now the nonhomogeneous condition is applied to find B_n.

$$\frac{\partial}{\partial r}u(r,\theta)=\sum_{n=1}^{\infty}B_n(2n)r^{2n-1}\sin(2n\theta)$$

Hence $\frac{\partial}{\partial r}u(1,\theta)=f(\theta)$ becomes

$$f(\theta)=\sum_{n=1}^{\infty}2B_n n\sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating gives

$$\begin{aligned}\int_0^{\frac{\pi}{2}}f(\theta)\sin(2m\theta)\,d\theta&=\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sum_{n=1}^{\infty}2B_n n\sin(2n\theta)\,d\theta\\&=\sum_{n=1}^{\infty}2nB_n\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sin(2n\theta)\,d\theta\end{aligned}\tag{5}$$

When $n=m$ then

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sin(2n\theta)\,d\theta&=\int_0^{\frac{\pi}{2}}\sin^2(2n\theta)\,d\theta\\&=\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}-\frac{1}{2}\cos 4n\theta\right)d\theta\\&=\frac{1}{2}[\theta]_0^{\frac{\pi}{2}}-\frac{1}{2}\left[\frac{\sin 4n\theta}{4n}\right]_0^{\frac{\pi}{2}}\\&=\frac{\pi}{4}-\left(\frac{1}{8n}\left(\sin\frac{4n}{2}\pi\right)-\sin(0)\right)\end{aligned}$$

And since n is integer, then $\sin\frac{4n}{2}\pi=\sin 2n\pi=0$ and the above becomes $\frac{\pi}{4}$.

Now for the case when $n\neq m$ using $\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$ then

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sin(2n\theta)\,d\theta&=\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos(2m\theta-2n\theta)-\cos(2m\theta+2n\theta))\,d\theta\\&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2m\theta-2n\theta)\,d\theta-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2m\theta+2n\theta)\,d\theta\\&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos((2m-2n)\theta)\,d\theta-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos((2m+2n)\theta)\,d\theta\\&=\frac{1}{2}\left[\frac{\sin((2m-2n)\theta)}{(2m-2n)}\right]_0^{\frac{\pi}{2}}-\frac{1}{2}\left[\frac{\sin((2m+2n)\theta)}{(2m+2n)}\right]_0^{\frac{\pi}{2}}\\&=\frac{1}{4(m-n)}[\sin((2m-2n)\theta)]_0^{\frac{\pi}{2}}-\frac{1}{4(m+n)}[\sin((2m+2n)\theta)]_0^{\frac{\pi}{2}}\\&=\frac{1}{4(m-n)}\left[\sin\left((2m-2n)\frac{\pi}{2}\right)-0\right]-\frac{1}{4(m+n)}\left[\sin\left((2m+2n)\frac{\pi}{2}\right)-0\right]\end{aligned}$$

Since $2m-2n\frac{\pi}{2}=\pi(m-n)$ which is integer multiple of π and also $(2m+2n)\frac{\pi}{2}$ is integer multiple of π then the whole term above becomes zero. Therefore (5) becomes

$$\int_0^{\frac{\pi}{2}} f(\theta)\sin(2m\theta)\,d\theta = 2mB_m\frac{\pi}{4}$$

Hence

$$\boxed{B_n = \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} f(\theta)\sin(2n\theta)\,d\theta}$$

Summary: the final solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n\left(r^{2n}\sin(2n\theta)\right)$$
$$B_n = \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} f(\theta)\sin(2n\theta)\,d\theta$$

The following are some plots of the solution

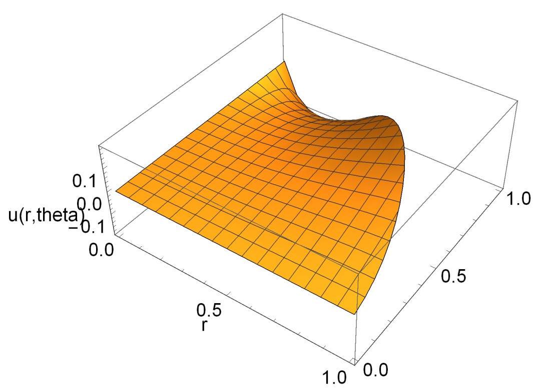


Figure 2.8: Solution using $f(\theta) = \theta\sin(3\theta)$

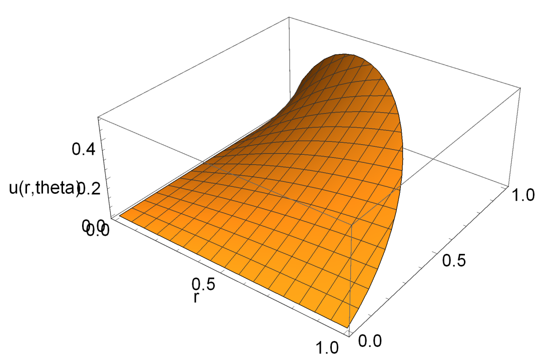


Figure 2.9: Solution using $f(\theta) = \theta$

2.4.3.2 Part (d)

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0$$

With boundary conditions

$$\begin{aligned}u(r,0)&=0\\ u\left(r,\frac{\pi}{2}\right)&=0\\ u(1,\theta)&=f(\theta)\end{aligned}$$

Assuming the solution is

$$u(r,\theta)=R(r)\Theta(\theta)$$

Substituting this back into the PDE gives

$$r^2R''\Theta+rR'\Theta+R\Theta''=0$$

Dividing by $R\Theta\neq 0$ gives

$$\begin{aligned}r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}&=0\\ r^2\frac{R''}{R}+r\frac{R'}{R}&=-\frac{\Theta''}{\Theta}\end{aligned}$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

This results in two ODE's with the following boundary conditions

$$\begin{aligned}\Theta''+\lambda\Theta&=0\\ \Theta'(0)&=0\\ \Theta'\left(\frac{\pi}{2}\right)&=0\end{aligned}\tag{1}$$

And

$$\begin{aligned}r^2R''+rR'-\lambda R&=0\\ |R(0)|&<\infty\end{aligned}\tag{2}$$

Starting with (1). Consider <u>Case $\lambda<0$</u> The solution will be

$$\Theta=A\cosh\left(\sqrt{\lambda}\theta\right)+B\sinh\left(\sqrt{\lambda}\theta\right)$$

And

$$\Theta'=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\theta\right)+B\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\theta\right)$$

Applying first B.C. gives $0=B\sqrt{\lambda}$, therefore $B=0$ and the solution becomes $A\cosh\left(\sqrt{\lambda}\theta\right)$ and $\Theta'=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\theta\right)$. Applying second B.C. gives $0=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)$. But $\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)\neq 0$

since $\lambda \neq 0$, therefore $A = 0$ and the trivial solution results. Hence $\underline{\lambda < 0 \text{ is not an eigenvalue}}$.

$\underline{\text{Case } \lambda = 0}$ The ODE becomes

$$\Theta'' = 0$$

With solution

$$\Theta = A\theta + B$$

And $\Theta' = A$. First B.C. gives $0 = A$. Hence $\Theta = B$. Second B.C. produces no result and the solution is constant. Hence

$$\boxed{\Theta = C_0}$$

Where C_0 is constant. Therefore $\underline{\lambda = 0 \text{ is an eigenvalue}}$.

$\underline{\text{Case } \lambda > 0}$ The ODE becomes $\Theta'' + \lambda\Theta = 0$ with solution

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$
$$\Theta' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)$$

The first B.C. gives $0 = B\sqrt{\lambda}$ or $B = 0$. The solution becomes

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right)$$

And $\Theta' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right)$. The second B.C. gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$ or $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ for $n = 1, 2, 3, \cdots$. Hence the eigenvalues are

$$\begin{aligned} \sqrt{\lambda_n} &= 2n \\ \lambda_n &= 4n^2 \qquad n = 1, 2, 3, \cdots \end{aligned}$$

And the eigenfunction is

$$\boxed{\Theta_n(\theta) = A_n\cos(2n\theta) \qquad n = 1, 2, 3, \cdots} \tag{3}$$

Now $\underline{\text{the } R \text{ ODE is solved}}$. The ODE is

$$r^2R'' + rR' - \lambda R = 0$$

$\underline{\text{Case } \lambda = 0}$

The ODE becomes $r^2R'' + rR' = 0$. Let $v(r) = R'(r)$ and the ODE becomes

$$r^2v' + rv = 0$$

Dividing by $r \neq 0$

$$v'(r) + \frac{1}{r}v(r) = 0$$

Using integrating factor $e^{\int \frac{1}{r} dr} = e^{\ln r} = r$. Hence

$$\frac{d}{dr}(rv) = 0$$

Hence

$$\begin{aligned} rv &= A \\ v(r) &= \frac{A}{r} \end{aligned}$$

But since $v(r) = R'(r)$ then $R' = \frac{c_1}{r}$. The solution to this ODE Is

$$R(r) = \int \frac{A}{r} dr + B$$

Therefore, <u>for $\lambda = 0$ the solution is</u>

$$\boxed{R(r) = A \ln |r| + B \qquad r \neq 0}$$

Since

$$\lim_{r \to 0} |R(r)| < \infty$$

Then $A = 0$ and the solution is just a constant

$$R(r) = B_0$$

<u>Case $\lambda > 0$</u> The ODE is

$$r^2 R'' + rR' - 4n^2 R = 0 \qquad n = 1, 2, 3, \cdots$$

The Let $R(r) = r^p$. Then $R' = pr^{p-1}$ and $R'' = p(p-1) r^{p-2}$. This gives

$$\begin{aligned} r^2 \left(p (p-1) r^{p-2}\right) + r \left(pr^{p-1}\right) - 4n^2 r^p &= 0 \\ \left(\left(p^2 - p\right) r^p\right) + pr^p - 4n^2 r^p &= 0 \\ r^p p^2 - pr^p + pr^p - 4n^2 r^p &= 0 \\ p^2 - 4n^2 &= 0 \\ p &= \pm 2n \end{aligned}$$

Hence the solution is

$$R(r) = Cr^{2n} + D\frac{1}{r^{2n}}$$

The condition that

$$\lim_{r \to 0} |R(r)| < \infty$$

Implies $D = 0$, Hence the solution becomes

$$R_n(r) = C_n r^2 n \qquad n = 1, 2, 3, \cdots \tag{4}$$

Now the solutions are combined. For $\lambda = 0$ the solution is

$$u_0(r, \theta) = C_0 B_0$$

Which can be combined to one constant B_0. Hence

$$\boxed{u_0 = B_0} \tag{5}$$

And for $\lambda > 0$ the solution is

$$\begin{aligned}u_n(r,\theta) &= R_n\Theta_n\\ &= C_n r^{2n}\left(A_n\cos(2n\theta)\right)\\ &= B_n r^{2n}\cos(2n\theta)\end{aligned}$$

Where C_nA_n are combined into one constant B_n. Hence

$$u_n(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n}\cos(2n\theta) \tag{6}$$

Equation (5) and (6) can be combined into one this now includes eigenfunctions for both $\lambda = 0$ and $\lambda > 0$

$$u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{2n}\cos(2n\theta) \tag{7}$$

Where B_0 represent the products of the eigenfunctions for R and Θ for $\lambda = 0$. Now the nonhomogeneous condition is applied to find B_n.

$$\frac{\partial}{\partial r}u(r,\theta) = \sum_{n=1}^{\infty} B_n(2n)\,r^{2n-1}\cos(2n\theta)$$

Hence $\frac{\partial}{\partial r}u(1,\theta) = g(\theta)$ becomes

$$g(\theta) = \sum_{n=1}^{\infty} 2B_n n\cos(2n\theta) \tag{8}$$

Multiplying by $\cos(2m\theta)$ and integrating gives

$$\begin{aligned}\int_0^{\frac{\pi}{2}} g(\theta)\cos(2m\theta)\,d\theta &= \int_0^{\frac{\pi}{2}}\cos(2m\theta)\sum_{n=1}^{\infty}2B_n n\cos(2n\theta)\,d\theta\\ &= \sum_{n=1}^{\infty}2nB_n\int_0^{\frac{\pi}{2}}\cos(2m\theta)\cos(2n\theta)\,d\theta \end{aligned}\tag{9}$$

As in the last part, the integral on right gives $\frac{\pi}{4}$ when $n = m$ and zero otherwise, hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} g(\theta)\cos(2n\theta)\,d\theta &= 2nB_n\frac{\pi}{4}\\ B_n &= \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} g(\theta)\cos(2n\theta)\,d\theta \qquad n = 1,2,3,\cdots\end{aligned}$$

Therefore the final solution is from (7) and (9)

$$\begin{aligned}u(r,\theta) &= B_0 + \sum_{n=1}^{\infty} B_n r^{2n}\cos(2n\theta)\\ &= B_0 + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}\int_0^{\frac{\pi}{2}} g(\theta)\cos(2m\theta)\,d\theta\right)r^{2n}\cos(2n\theta)\end{aligned}$$

The unknown constant B_0 can be found if given the initial temperature as was done in problem 2.5.2 part (c). To answer the last part. Using (8) and integrating

$$\begin{aligned}\int_0^{\frac{\pi}{2}} g(\theta)\,d\theta &= \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} 2nB_n \cos(2n\theta)\,d\theta\\ &= \sum_{n=1}^{\infty} 2nB_n \int_0^{\frac{\pi}{2}} \cos(2n\theta)\,d\theta\end{aligned}$$

But

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(2n\theta)\,d\theta &= \left[\frac{\sin(2n\theta)}{2n}\right]_0^{\frac{\pi}{2}}\\ &= \frac{1}{2n}\left(\sin\frac{2n}{2}\pi - 0\right)\\ &= \frac{1}{2n}(\sin n\pi - 0)\\ &= 0\end{aligned}$$

Since n is an integer. This condition physically means the same as in part (b) problem 2.5.2. Which is, since at steady state the flux must be zero on all boundaries, and $g(\theta)$ represents the flux over the surface of the quarter circle, then the integral of the flux must be zero. This means there is no thermal energy flowing across the boundary.

2.4.4 Problem 2.5.8(b) (problem 4)

2.5.8. Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions

*** (a) $u(a,\theta) = f(\theta)$, $u(b,\theta) = g(\theta)$**

(b) $\frac{\partial u}{\partial r}(a,\theta) = 0$, $u(b,\theta) = g(\theta)$

(c) $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$, $\frac{\partial u}{\partial r}(b,\theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With

$$\begin{aligned}\frac{\partial u}{\partial r}(a,\theta) &= 0\\ u(b,\theta) &= g(\theta)\end{aligned} \tag{B}$$

Assuming the solution can be written as

$$u(r,\theta) = R(r)\Theta(\theta)$$

And substituting this assumed solution back into the (A) gives

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta$ gives

$$\begin{aligned} r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta} \end{aligned}$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. This results in

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \qquad (1) \\ \Theta(-\pi) &= \Theta(\pi) \\ \Theta'(-\pi) &= \Theta'(\pi) \end{aligned}$$

And

$$\begin{aligned} r^2R'' + rR' - \lambda R &= 0 \qquad (2) \\ R'(a) &= 0 \end{aligned}$$

Starting with (1) <u>Case $\lambda < 0$</u> The solution is

$$\Theta(\theta) = A\cosh\left(\sqrt{\lambda}\theta\right) + B\sinh\left(\sqrt{\lambda}\theta\right)$$

First B.C. gives

$$\begin{aligned} \Theta(-\pi) &= \Theta(\pi) \\ A\cosh\left(-\sqrt{\lambda}\pi\right) + B\sinh\left(-\sqrt{\lambda}\pi\right) &= A\cosh\left(\sqrt{\lambda}\pi\right) + B\sinh\left(\sqrt{\lambda}\pi\right) \\ A\cosh\left(\sqrt{\lambda}\pi\right) - B\sinh\left(\sqrt{\lambda}\pi\right) &= A\cosh\left(\sqrt{\lambda}\pi\right) + B\sinh\left(\sqrt{\lambda}\pi\right) \\ 2B\sinh\left(\sqrt{\lambda}\pi\right) &= 0 \end{aligned}$$

But $\sinh\left(\sqrt{\lambda}\pi\right) = 0$ only at zero and $\lambda \neq 0$, hence $B = 0$ and the solution becomes

$$\begin{aligned} \Theta(\theta) &= A\cosh\left(\sqrt{\lambda}\theta\right) \\ \Theta'(\theta) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\theta\right) \end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned}
\Theta'(-\pi) &= \Theta'(\pi) \\
A\sqrt{\lambda}\cosh\left(-\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) \\
2A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) &= 0
\end{aligned}$$

But $\cosh\left(\sqrt{\lambda}\pi\right) \neq 0$ hence $A = 0$. Therefore trivial solution and $\underline{\lambda < 0 \text{ is not an eigenvalue}}$.

$\underline{\text{Case } \lambda = 0}$ The solution is $\Theta = A\theta + B$. Applying the first B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi) \\
-A\pi + B &= \pi A + B \\
2\pi A &= 0 \\
A &= 0
\end{aligned}$$

And the solution becomes $\Theta = B_0$. A constant. Hence $\underline{\lambda = 0 \text{ is an eigenvalue}}$.

$\underline{\text{Case } \lambda > 0}$

The solution becomes

$$\begin{aligned}
\Theta &= A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right) \\
\Theta' &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)
\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi) \\
A\cos\left(-\sqrt{\lambda}\pi\right) + B\sin\left(-\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) \\
A\cos\left(\sqrt{\lambda}\pi\right) - B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) \\
2B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)
\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}
\Theta'(-\pi) &= \Theta'(\pi) \\
-A\sqrt{\lambda}\sin\left(-\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(-\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \\
2A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (4)
\end{aligned}$$

Equations (3,4) can be both zero only if $A = B = 0$ which gives trivial solution, or when

$\sin\left(\sqrt{\lambda}\pi\right) = 0$. Therefore taking $\sin\left(\sqrt{\lambda}\pi\right) = 0$ gives a non-trivial solution. Hence

$$\begin{aligned} \sqrt{\lambda}\pi &= n\pi \qquad n = 1,2,3,\cdots \\ \lambda_n &= n^2 \qquad n = 1,2,3,\cdots \end{aligned}$$

Hence the solution for Θ is

$$\Theta = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \tag{5}$$

<u>Now the R equation is solved</u>

The <u>case for $\lambda = 0$</u> gives

$$\begin{aligned} r^2 R'' + rR' &= 0 \\ R'' + \frac{1}{r} R' &= 0 \qquad r \neq 0 \end{aligned}$$

As was done in last problem, the solution to this is

$$R(r) = A \ln|r| + C$$

Since $r > 0$ no need to keep worrying about $|r|$ and is removed for simplicity. Applying the B.C. gives

$$R' = A\frac{1}{r}$$

Evaluating at $r = a$ gives

$$0 = A\frac{1}{a}$$

Hence $A = 0$, and the solution becomes

$$R(r) = C_0$$

Which is a constant.

<u>Case $\lambda > 0$</u> The ODE in this case is

$$r^2 R'' + rR' - n^2 R = 0 \qquad n = 1,2,3,\cdots$$

Let $R = r^p$, the above becomes

$$\begin{aligned} r^2 p\left(p-1\right) r^{p-2} + rpr^{p-1} - n^2 r^p &= 0 \\ p\left(p-1\right) r^p + pr^p - n^2 r^p &= 0 \\ p\left(p-1\right) + p - n^2 &= 0 \\ p^2 &= n^2 \\ p &= \pm n \end{aligned}$$

Hence the solution is

$$R_n(r) = Cr^n + D\frac{1}{r^n} \qquad n = 1,2,3,\cdots$$

Applying the boundary condition $R'(a) = 0$ gives

$$\begin{aligned} R_n'(r) &= nC_n r^{n-1} - nD_n \frac{1}{r^{n+1}} \\ 0 &= R_n'(a) \\ &= nC_n a^{n-1} - nD_n \frac{1}{a^{n+1}} \\ &= nC_n a^{2n} - nD_n \\ &= C_n a^{2n} - D_n \\ D_n &= C_n a^{2n} \end{aligned}$$

The solution becomes

$$\begin{aligned} R_n(r) &= C_n r^n + C_n a^{2n} \frac{1}{r^n} \qquad n = 1, 2, 3, \cdots \\ &= C_n \left(r^n + \frac{a^{2n}}{r^n} \right) \end{aligned}$$

Hence the complete solution for $R(r)$ is

$$R(r) = C_0 + \sum_{n=1}^{\infty} C_n \left(r^n + \frac{a^{2n}}{r^n} \right) \tag{6}$$

Using (5),(6) gives

$$\begin{aligned} u_n(r,\theta) &= R_n \Theta_n \\ u(r,\theta) &= \left[C_0 + \sum_{n=1}^{\infty} C_n \left(r^n + \frac{a^{2n}}{r^n} \right) \right] \left[A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \right] \\ &= D_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) C_n \left(r^n + \frac{a^{2n}}{r^n} \right) + \sum_{n=1}^{\infty} B_n \sin(n\theta) C_n \left(r^n + \frac{a^{2n}}{r^n} \right) \end{aligned}$$

Where $D_0 = C_0 A_0$. To simplify more, $A_n C_n$ is combined to A_n and $B_n C_n$ is combined to B_n. The full solution is

$$u(r,\theta) = D_0 + \sum_{n=1}^{\infty} A_n \left(r^n + \frac{a^{2n}}{r^n} \right) \cos(n\theta) + \sum_{n=1}^{\infty} B_n \left(r^n + \frac{a^{2n}}{r^n} \right) \sin(n\theta)$$

The final nonhomogeneous B.C. is applied.

$$\begin{aligned} u(b,\theta) &= g(\theta) \\ g(\theta) &= D_0 + \sum_{n=1}^{\infty} A_n \left(b^n + \frac{a^{2n}}{b^n} \right) \cos(n\theta) + \sum_{n=1}^{\infty} B_n \left(b^n + \frac{a^{2n}}{b^n} \right) \sin(n\theta) \end{aligned}$$

For $n = 0$, integrating both sides give

$$\begin{aligned} \int_{-\pi}^{\pi} g(\theta)\, d\theta &= \int_{-\pi}^{\pi} D_0 d\theta \\ D_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta \end{aligned}$$

For $n>0$, multiplying both sides by $\cos(m\theta)$ and integrating gives

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\,d\theta &= \int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} A_n\left(b^n+\frac{a^{2n}}{b^n}\right)\cos(m\theta)\cos(n\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} B_n\left(b^n+\frac{a^{2n}}{b^n}\right)\cos(m\theta)\sin(n\theta)\,d\theta
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\,d\theta &= \int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta \\
&\quad + \sum_{n=1}^{\infty} A_n\left(b^n+\frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta \\
&\quad + \sum_{n=1}^{\infty} B_n\left(b^n+\frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta \qquad (7)
\end{aligned}$$

But

$$\begin{aligned}
\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta &= \pi \qquad n=m\neq 0 \\
\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta &= 0 \qquad n\neq m
\end{aligned}$$

And

$$\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta = 0$$

And

$$\int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta = 0$$

Then (7) becomes

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta &= \pi A_n\left(b^n+\frac{a^{2n}}{b^n}\right) \\
A_n &= \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta}{b^n+\frac{a^{2n}}{b^n}} \qquad (8)
\end{aligned}$$

Again, multiplying both sides by $\sin(m\theta)$ and integrating gives

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta &= \int_{-\pi}^{\pi} D_0\sin(m\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} A_n\left(b^n+\frac{a^{2n}}{b^n}\right)\sin(m\theta)\cos(n\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} B_n\left(b^n+\frac{a^{2n}}{b^n}\right)\sin(m\theta)\sin(n\theta)\,d\theta
\end{aligned}$$

Hence

$$\begin{aligned}\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta = \int_{-\pi}^{\pi} D_0 \sin(m\theta)\,d\theta \\ + \sum_{n=1}^{\infty} A_n\left(b^n+\frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\sin(m\theta)\cos(n\theta)\,d\theta \\ + \sum_{n=1}^{\infty} B_n\left(b^n+\frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta \end{aligned} \tag{9}$$

But

$$\int_{-\pi}^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta = \pi \qquad n=m\neq 0$$
$$\int_{-\pi}^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta = 0 \qquad n\neq m$$

And

$$\int_{-\pi}^{\pi}\sin(m\theta)\cos(n\theta)\,d\theta = 0$$

And

$$\int_{-\pi}^{\pi} D_0\sin(m\theta)\,d\theta = 0$$

Then (9) becomes

$$\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta = \pi B_n\left(b^n+\frac{a^{2n}}{b^n}\right)$$
$$B_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta}{b^n+\frac{a^{2n}}{b^n}}$$

This complete the solution. Summary

$$u(r,\theta) = D_0 + \sum_{n=1}^{\infty} A_n\left(r^n+\frac{a^{2n}}{r^n}\right)\cos(n\theta) + \sum_{n=1}^{\infty} B_n\left(r^n+\frac{a^{2n}}{r^n}\right)\sin(n\theta)$$
$$D_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta$$
$$A_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta}{b^n+\frac{a^{2n}}{b^n}}$$
$$B_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta}{b^n+\frac{a^{2n}}{b^n}}$$

The following are some plots of the solution.

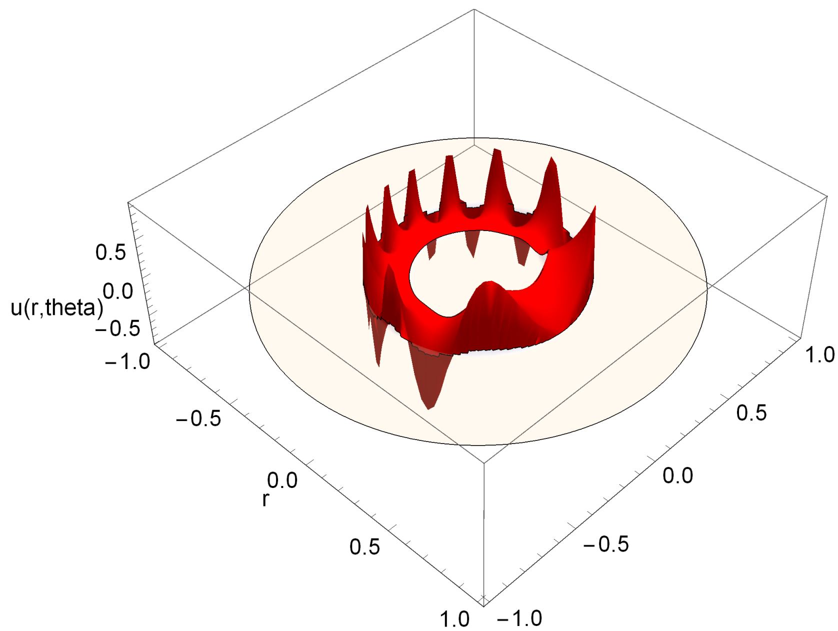


Figure 2.10: Solution using $f(\theta) = \sin(3\theta^2), a = 0.3, b = 0.5$

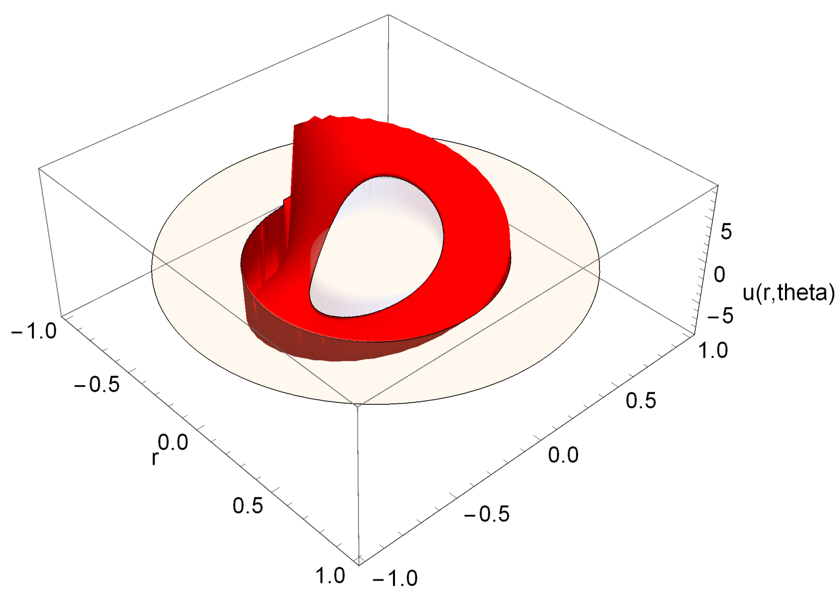


Figure 2.11: Solution using $f(\theta) = 3\theta, a = 0.3, b = 0.6$

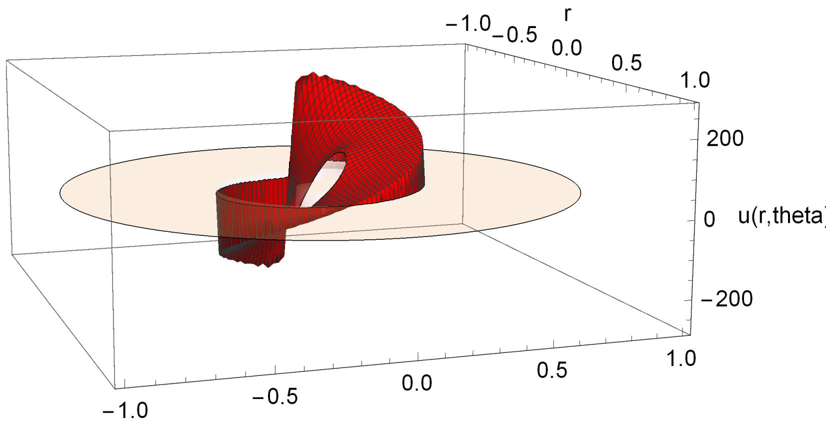


Figure 2.12: Solution using $f(\theta) = 100\theta, a = 0.1, b = 0.4$

2.4.5 Problem 2.5.14 (problem 5)

2.5.14. Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$, is *not* well posed. [*Hint:* Show that if the data are changed an arbitrarily small amount, for example,

$$f(x) \to f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$$

for large n, then the solution $u(x,t)$ changes by a large amount.]

$$\begin{aligned}\frac{-1}{k}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}\\ u(0,t) &= 0\\ u(L,t) &= 0\\ u(x,0) &= f(x)\end{aligned}$$

Assume $u(x,t) = XT$. Hence the PDE becomes

$$\begin{aligned}-\frac{1}{k}T'X &= X''T\\ -\frac{1}{k}\frac{T'}{T} &= \frac{X''}{X}\end{aligned}$$

Hence, forλ real

$$-\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The space ODE was solved before. Only positive eigenvalues exist. The solution is

$$X(x) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right)$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

The time ODE becomes

$$T_n' = \lambda_n T_n$$
$$T_n' - \lambda_n T_n = 0$$

With solution

$$T_n(t) = A_n e^{\lambda_n t}$$
$$T(t) = \sum_{n=1}^{\infty} A_n e^{\lambda_n t}$$

For the same eigenvalues. Therefore the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) e^{\left(\frac{n\pi}{L}\right)^2 t} \tag{1}$$

Where $C_n = A_n B_n$. Applying initial conditions gives

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right)$$

Multiplying by $\sin\left(\frac{m\pi}{L} x\right)$ and integrating results in

$$\begin{aligned}
\int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx &= \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) dx \\
&= \sum_{n=1}^{\infty} C_n \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx \\
&= C_m \frac{L}{2}
\end{aligned}$$

Therefore

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

The solution (1) becomes

$$u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \right) \left(\sin\left(\frac{n\pi}{L} x\right) e^{\left(\frac{n\pi}{L}\right)^2 t} \right) \tag{2}$$

Assuming initial data is changed to $f(x) + \frac{1}{n} \sin\left(\frac{n\pi}{L} x\right)$ then

$$f(x) + \frac{1}{m} \sin\left(\frac{m\pi}{L} x\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{1}{m}\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx=\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{1}{m}\frac{L}{2}=C_m\frac{L}{2}$$

$$C_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{1}{n}$$

Therefore, the new solution is

$$\begin{aligned}\tilde{u}(x,t)&=\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{1}{n}\right)\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}\\&=\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}+\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}\\&=\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}\end{aligned}$$

But $\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}=u(x,t)$, therefore the above can be written as

$$\tilde{u}(x,t)=u(x,t)+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2t}$$

For large n, the difference between initial data $f(x)$ and $f(x)+\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)$ is very small, since $\frac{1}{n}\to 0$. However, the effect in the solution above, due to the presence of $e^{\left(\frac{n\pi}{L}\right)^2t}$ is that $\frac{1}{n}e^{\left(\frac{n\pi}{L}\right)^2t}$ increases now for large n, since the exponential is to the positive power, and it grows at a faster rate than $\frac{1}{n}$ grows small as n increases, with the net effect that the produce blow up for large n. This is because the power of the exponential is positive and not negative is normally would be the case. Also by looking at the series of $e^{\left(\frac{n\pi}{L}\right)^2t}$ which is $1+\left(\frac{n\pi}{L}\right)^4\frac{t^2}{2}+\left(\frac{n\pi}{L}\right)^6\frac{t^3}{3!}+\cdots$, then $\frac{1}{n}e^{\left(\frac{n\pi}{L}\right)^2t}$ expands to $\frac{1}{n}+\frac{1}{n}\left(\frac{n\pi}{L}\right)^4\frac{t^2}{2}+\frac{1}{n}\left(\frac{n\pi}{L}\right)^6\frac{t^3}{3!}+\cdots$ which becomes very large for large n. In the normal PDE case, the above solution would have instead been the following

$$\tilde{u}(x,t)=u(x,t)+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2t}$$

And now as $n\to\infty$ then $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2t}\to 0$ as well. Notice that $\sin\left(\frac{n\pi}{L}x\right)$ term is not important for this analysis, as its value oscillates between -1 and $+1$.

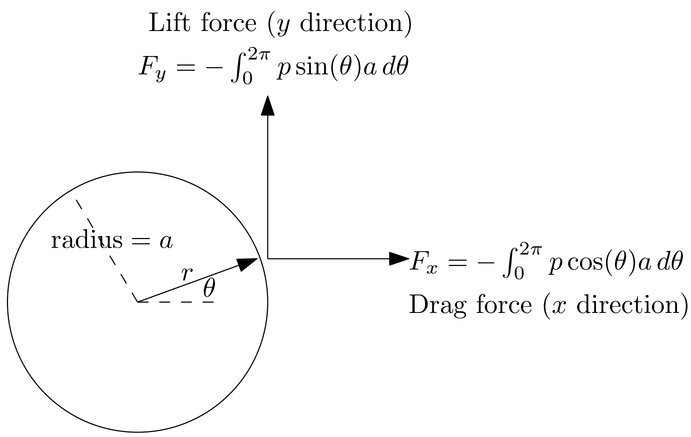
2.4.6 Problem 2.5.22 (problem 6)

2.5.22. **Show the drag force is zero for a uniform flow past a cylinder including circulation.**

The force exerted by the fluid on the cylinder is given by equation 2.5.56, page 77 of the text as

$$\bar{F} = -\int_0^{2\pi} p \left\langle \cos\theta, \sin\theta \right\rangle a d\theta$$

Where a is the cylinder radius, p is the fluid pressure. This vector has 2 components. The x component is the drag force and the y component is the left force as illustrated by this diagram.



Therefore the drag force (per unit length) is

$$F_x = -\int_0^{2\pi} p\cos\theta a d\theta \tag{1}$$

Now the pressure p needs to be determined in order to compute the above. The fluid pressure p is related to fluid flow velocity by the Bernoulli condition

$$p + \frac{1}{2}\rho \left|\bar{u}\right|^2 = C \tag{2}$$

Where C is some constant and ρ is fluid density and $\bar{u}$ is the flow velocity vector. Hence in order to find p, the fluid velocity is needed. But the fluid velocity is given by

$$\begin{aligned}\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta}\\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{r} - \frac{\partial\Psi}{\partial r}\hat{\theta}\end{aligned}$$

Since the radial component of the fluid velocity is zero at the surface if the cylinder (This is one of the boundary conditions used to derive the solution), then only the tangential component comes into play. Hence $\left|\bar{u}\right| = \left|-\frac{\partial\Psi}{\partial r}\right|$ but

$$\Psi(r,\theta) = c_1 \ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Therefore

$$\frac{\partial\Psi}{\partial r} = \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

And hence

$$\begin{aligned}|\bar{u}| &= \left|-\frac{\partial\Psi}{\partial r}\right| \\ &= \left|-\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right|\end{aligned}$$

At the surface $r = a$, hence

$$|\bar{u}| = \left|-\frac{c_1}{a} + 2u_0\sin\theta\right|$$

Substituting this into (2) in order to solve for pressure p gives

$$p + \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2 = C$$

$$p = C - \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2$$

Substituting the above into (1) in order to solve for the drag gives

$$F_x = -\int_0^{2\pi}\left[C - \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\right]\cos\theta a d\theta$$

The above is the quantity that needs to be shown to be zero.

$$F_x = -aC\int_0^{2\pi}\cos\theta d\theta - \frac{a}{2}\rho\int_0^{2\pi}\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\cos\theta d\theta$$

But $\int_0^{2\pi}\cos\theta d\theta = 0$ hence the above simplifies to

$$\begin{aligned}F_x &= -\frac{a}{2}\rho\int_0^{2\pi}\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\cos\theta d\theta \\ &= -\frac{a}{2}\rho\int_0^{2\pi}\frac{c_1^2}{a^2}\cos\theta + 4u_0^2\sin^2\theta\cos\theta - 4\frac{c_1}{a}u_0\sin\theta\cos\theta d\theta \\ &= -\frac{a}{2}\rho\left[\frac{c_1^2}{a^2}\int_0^{2\pi}\cos\theta d\theta + 4u_0^2\int_0^{2\pi}\sin^2\theta\cos\theta d\theta - 4\frac{c_1}{a}u_0\int_0^{2\pi}\sin\theta\cos\theta d\theta\right]\end{aligned}$$

But $\int_0^{2\pi}\cos\theta d\theta = 0$ and $\int_0^{2\pi}\sin\theta\cos\theta d\theta = 0$ hence the above reduces to

$$F_x = -4a\rho u_0^2\int_0^{2\pi}\sin^2\theta\cos\theta d\theta$$

But $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and the above becomes

$$\begin{aligned}F_x &= -4a\rho u_0^2\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\cos\theta d\theta \\ &= -4a\rho u_0^2\left(\frac{1}{2}\int_0^{2\pi}\cos\theta d\theta - \frac{1}{2}\int_0^{2\pi}\cos(2\theta)\cos\theta d\theta\right)\end{aligned}$$

But $\int_0^{2\pi} \cos\theta d\theta = 0$ and by orthogonality of $\cos$ function $\int_0^{2\pi} \cos(2\theta)\cos(\theta)\,d\theta = 0$ as well. Therefore the above reduces to

$$F_x = 0$$

The drag force (x component of the force exerted by fluid on the cylinder) is zero just outside the surface of the surface of the cylinder. Which is what the question asks to show.

2.4.7 Problem 2.5.24 (problem 7)

2.5.24. Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

Introduction. The stream velocity $\bar{u}$ in Cartesian coordinates is

$$\begin{aligned}\bar{u} &= u\hat{\imath} + v\hat{\jmath}\\ &= \frac{\partial\Psi}{\partial y}\hat{\imath} - \frac{\partial\Psi}{\partial x}\hat{\jmath}\end{aligned} \tag{1}$$

Where Ψ is the stream function which satisfies Laplace PDE in 2D $\nabla^2\Psi = 0$. In Polar coordinates the above becomes

$$\begin{aligned}\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta}\\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{r} - \frac{\partial\Psi}{\partial r}\hat{\theta}\end{aligned} \tag{2}$$

The solution to $\nabla^2\Psi = 0$ was found under the following conditions

1. When r very large, or in other words, when too far away from the cylinder or the wing, the flow lines are horizontal only. This means at $r = \infty$ the y component of $\bar{u}$ in (1) is zero. This means $\frac{\partial\Psi(x,y)}{\partial x} = 0$. Therefore $\Psi(x,y) = u_0 y$ where u_0 is some constant. In polar coordinates this implies $\underline{\Psi(r,\theta) = u_0 r\sin\theta}$, since $y = r\sin\theta$.

2. The second condition is that radial component of $\bar{u}$ is zero. In other words, $\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = 0$ when $r = a$, where a is the radius of the cylinder.

3. In addition to the above two main condition, there is a condition that $\Psi = 0$ at $r = 0$

Using the above three conditions, the solution to $\nabla^2\Psi = 0$ was derived in lecture Sept. 30, 2016, to be

$$\Psi(r,\theta) = c_1\ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Using the above solution, the velocity $\bar{u}$ can now be found using the definition in (2) as follows

$$\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta$$
$$\frac{\partial\Psi}{\partial r} = \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

Hence, in polar coordinates

$$\boxed{\bar{u} = \left(\tfrac{1}{r}u_0\left(r - \tfrac{a^2}{r}\right)\cos\theta\right)\hat{r} - \left(\tfrac{c_1}{r} + u_0\left(1 + \tfrac{a^2}{r^2}\right)\sin\theta\right)\hat{\theta}} \tag{3}$$

Now the question posed can be answered. The circulation is given by

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$

But from (3) $u_\theta = -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)$, therefore the above becomes

$$\Gamma = \int_0^{2\pi} -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right) r d\theta$$

At $r = a$ the above simplifies to

$$\begin{aligned}
\Gamma &= \int_0^{2\pi} -\left(\frac{c_1}{a} + 2u_0\sin\theta\right) a d\theta \\
&= \int_0^{2\pi} -c_1 - 2au_0\sin\theta d\theta \\
&= -\int_0^{2\pi} c_1 d\theta - 2au_0\int_0^{2\pi}\sin\theta d\theta
\end{aligned}$$

But $\int_0^{2\pi}\sin\theta d\theta = 0$, hence

$$\begin{aligned}
\Gamma &= -c_1\int_0^{2\pi} d\theta \\
&= -2c_1\pi
\end{aligned}$$

Since $\Gamma < 0$, then $\underline{c_1 > 0}$. Now that c_1 is known to be positive, then the velocity is calculated at $\theta = \frac{-\pi}{2}$ and then at $\theta = \frac{+\pi}{2}$ to see which is larger. Since this is calculated at $r = a$, then the radial velocity is zero and only u_θ needs to be evaluated in (3).

<u>At $\theta = \frac{-\pi}{2}$</u>

$$\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{-\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\right)
\end{aligned}$$

At $r = a$

$$
\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} - 2u_0\right) \\
&= -\frac{c_1}{a} + 2u_0 \qquad (4)
\end{aligned}
$$

<u>At $\theta = \frac{+\pi}{2}$</u>

$$
\begin{aligned}
u_{\left(\frac{+\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\right)
\end{aligned}
$$

At $r = a$

$$
\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} + 2u_0\right) \\
&= -\frac{c_1}{a} - 2u_0 \qquad (5)
\end{aligned}
$$

Comparing (4),(5), and since $c_1 > 0$, then the <u>magnitude</u> of u_θ at $\frac{\pi}{2}$ is larger than the magnitude of u_θ at $\frac{-\pi}{2}$. Which implies the stream flows faster above the cylinder than below it.

2.5 HW 4

2.5.1 Problem 2.5.24

> **2.5.24.** Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

Introduction. The stream velocity $\bar{u}$ in Cartesian coordinates is

$$\begin{aligned}\bar{u} &= u\hat{\imath} + v\hat{\jmath} \\ &= \frac{\partial\Psi}{\partial y}\hat{\imath} - \frac{\partial\Psi}{\partial x}\hat{\jmath} \qquad (1)\end{aligned}$$

Where Ψ is the stream function which satisfies Laplace PDE in 2D $\nabla^2\Psi = 0$. In Polar coordinates the above becomes

$$\begin{aligned}\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta} \\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{r} - \frac{\partial\Psi}{\partial r}\hat{\theta} \qquad (2)\end{aligned}$$

The solution to $\nabla^2\Psi = 0$ was found under the following conditions

1. When r very large, or in other words, when too far away from the cylinder or the wing, the flow lines are horizontal only. This means at $r = \infty$ the y component of $\bar{u}$ in (1) is zero. This means $\frac{\partial\Psi(x,y)}{\partial x} = 0$. Therefore $\Psi(x,y) = u_0 y$ where u_0 is some constant. In polar coordinates this implies $\underline{\Psi(r,\theta) = u_0 r\sin\theta}$, since $y = r\sin\theta$.
2. The second condition is that radial component of $\bar{u}$ is zero. In other words, $\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = 0$ when $r = a$, where a is the radius of the cylinder.
3. In addition to the above two main condition, there is a condition that $\Psi = 0$ at $r = 0$

Using the above three conditions, the solution to $\nabla^2\Psi = 0$ was derived in lecture Sept. 30, 2016, to be

$$\Psi(r,\theta) = c_1\ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Using the above solution, the velocity $\bar{u}$ can now be found using the definition in (2) as follows

$$\begin{aligned}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} &= \frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta \\ \frac{\partial\Psi}{\partial r} &= \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\end{aligned}$$

Hence, in polar coordinates

$$\boxed{\bar{u} = \left(\frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta\right)\hat{r} - \left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)\hat{\theta}} \tag{3}$$

Now the question posed can be answered. The circulation is given by

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$

But from (3) $u_\theta = -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)$, therefore the above becomes

$$\Gamma = \int_0^{2\pi} -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right) r d\theta$$

At $r = a$ the above simplifies to

$$\begin{aligned}
\Gamma &= \int_0^{2\pi} -\left(\frac{c_1}{a} + 2u_0\sin\theta\right) a d\theta \\
&= \int_0^{2\pi} -c_1 - 2au_0\sin\theta d\theta \\
&= -\int_0^{2\pi} c_1 d\theta - 2au_0\int_0^{2\pi}\sin\theta d\theta
\end{aligned}$$

But $\int_0^{2\pi}\sin\theta d\theta = 0$, hence

$$\begin{aligned}
\Gamma &= -c_1\int_0^{2\pi} d\theta \\
&= -2c_1\pi
\end{aligned}$$

Since $\Gamma < 0$, then $\underline{c_1 > 0}$. Now that c_1 is known to be positive, then the velocity is calculated at $\theta = \frac{-\pi}{2}$ and then at $\theta = \frac{+\pi}{2}$ to see which is larger. Since this is calculated at $r = a$, then the radial velocity is zero and only u_θ needs to be evaluated in (3).

$\underline{\text{At } \theta = \frac{-\pi}{2}}$

$$\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{-\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\right)
\end{aligned}$$

At $r = a$

$$\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} - 2u_0\right) \\
&= -\frac{c_1}{a} + 2u_0
\end{aligned} \tag{4}$$

At $\theta = \frac{+\pi}{2}$

$$
\begin{aligned}
u_{\left(\frac{+\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\
&= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\right)
\end{aligned}
$$

At $r = a$

$$
\begin{aligned}
u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} + 2u_0\right) \\
&= -\frac{c_1}{a} - 2u_0 \qquad (5)
\end{aligned}
$$

Comparing (4),(5), and since $c_1 > 0$, then the magnitude of u_θ at $\frac{\pi}{2}$ is larger than the magnitude of u_θ at $\frac{-\pi}{2}$. Which implies the stream flows faster above the cylinder than below it.

2.5.2 Problem 3.2.2 (b,d)

3.2.2. **For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:**

* (a) $f(x) = x$

(b) $f(x) = e^{-x}$

* (c) $f(x) = \sin \frac{\pi x}{L}$

(d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$

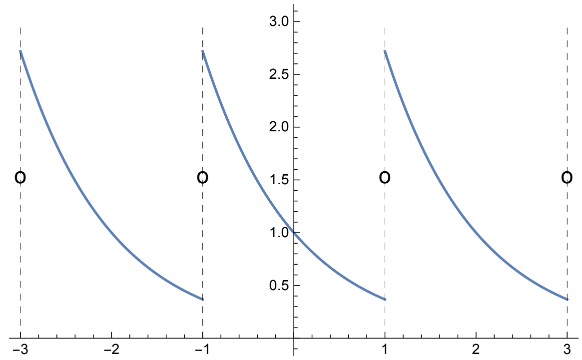
(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$

* (f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

2.5.2.1 Part b

The following is sketch of periodic extension of e^{-x} from $x = -L \cdots L$ (for $L = 1$) for illustration. The function will converge to e^{-x} between $x = -L \cdots L$ and between $x = -3L \cdots -L$ and between $x = L \cdots 3L$ and so on. But at the jump discontinuities which occurs at $x = \cdots, -3L, -L, L, 3L, \cdots$ it will converge to the average shown as small circles in the sketch.



By definitions,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx$$
$$a_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\cos\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$
$$b_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\sin\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$

The period here is $T = 2L$, therefore the above becomes

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

These are now evaluated for $f(x) = e^{-x}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}dx = \frac{1}{2L}\left(\frac{e^{-x}}{-1}\right)_{-L}^{L} = \frac{-1}{2L}\left(e^{-x}\right)_{-L}^{L} = \frac{-1}{2L}\left(e^{-L} - e^{L}\right) = \frac{e^{L} - e^{-L}}{2L}$$

Now a_n is found

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$$

This can be done using integration by parts. $\int u\,dv = uv - \int v\,du$. Let

$$I = \int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$$

and $u=\cos\left(n\frac{\pi}{L}x\right), dv=e^{-x}, \rightarrow du=-\frac{n\pi}{L}\sin\left(n\frac{\pi}{L}x\right), v=-e^{-x}$, therefore

$$\begin{aligned}
I &= [uv]_{-L}^{L}-\int_{-L}^{L}vdu\\
&= \left[-e^{-x}\cos\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\\
&= \left[-e^{-L}\cos\left(n\frac{\pi}{L}L\right)+e^{L}\cos\left(n\frac{\pi}{L}(-L)\right)\right]-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

Applying integration by parts again to $\int e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx$ where now $u=\sin\left(n\frac{\pi}{L}x\right), dv=e^{-x}\rightarrow du=\frac{n\pi}{L}\cos\left(n\frac{\pi}{L}x\right), v=-e^{-x}$, hence the above becomes

$$\begin{aligned}
I &= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(uv-\int vdu\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(\overbrace{\left[-e^{-x}\sin\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}}^{0}+\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\left(\frac{n\pi}{L}\right)^2\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

But $\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx=I$ and the above becomes

$$I=-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)-\left(\frac{n\pi}{L}\right)^2 I$$

Simplifying and solving for I

$$\begin{aligned}
I+\left(\frac{n\pi}{L}\right)^2 I &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(1+\left(\frac{n\pi}{L}\right)^2\right) &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(\frac{L^2+n^2\pi^2}{L^2}\right) &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I &= \left(\frac{L^2}{L^2+n^2\pi^2}\right)\cos(n\pi)\left(e^{L}-e^{-L}\right)
\end{aligned}$$

Hence a_n becomes

$$a_n=\frac{1}{L}\left(\frac{L^2}{L^2+n^2\pi^2}\right)\cos(n\pi)\left(e^{L}-e^{-L}\right)$$

But $\cos(n\pi)=-1^n$ hence

$$a_n=(-1)^n\left(\frac{L}{n^2\pi^2+L^2}\right)\left(e^{L}-e^{-L}\right)$$

Similarly for b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx$$

This can be done using integration by parts. $\int udv = uv - \int vdu$. Let

$$I = \int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx$$

and $u = \sin\left(n\frac{\pi}{L}x\right), dv = e^{-x}, \rightarrow du = \frac{n\pi}{L}\cos\left(n\frac{\pi}{L}x\right), v = -e^{-x}$, therefore

$$\begin{aligned}
I &= [uv]_{-L}^{L} - \int_{-L}^{L} vdu \\
&= \overbrace{\left[-e^{-x}\sin\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}}^{0} + \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx \\
&= \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

Applying integration by parts again to $\int e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$ where now $u = \cos\left(n\frac{\pi}{L}x\right), dv = e^{-x} \rightarrow du = \frac{-n\pi}{L}\sin\left(n\frac{\pi}{L}x\right), v = -e^{-x}$, hence the above becomes

$$\begin{aligned}
I &= \frac{n\pi}{L}\left(uv - \int vdu\right) \\
&= \frac{n\pi}{L}\left(\left[-e^{-x}\cos\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L} - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{n\pi}{L}\left(-e^{-L}\cos\left(n\frac{\pi}{L}L\right) + e^{L}\cos\left(n\frac{\pi}{L}L\right) - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{n\pi}{L}\left(\cos(n\pi)\left(e^{L} - e^{-L}\right) - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right)
\end{aligned}$$

But $\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx = I$ and the above becomes

$$I = \frac{n\pi}{L}\left(\cos(n\pi)\left(e^{L} - e^{-L}\right) - \frac{n\pi}{L}I\right)$$

Simplifying and solving for I

$$\begin{aligned}
I &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L} - e^{-L}\right) - \left(\frac{n\pi}{L}\right)^2 I \\
I + \left(\frac{n\pi}{L}\right)^2 I &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L} - e^{-L}\right) \\
I\left(1 + \left(\frac{n\pi}{L}\right)^2\right) &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L} - e^{-L}\right) \\
I\left(\frac{L^2 + n^2\pi^2}{L^2}\right) &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L} - e^{-L}\right) \\
I &= \left(\frac{L^2}{L^2 + n^2\pi^2}\right)\frac{n\pi}{L}\cos(n\pi)\left(e^{L} - e^{-L}\right)
\end{aligned}$$

Hence b_n becomes

$$\begin{aligned}b_n &= \frac{1}{L}\left(\frac{L^2}{L^2+n^2\pi^2}\right)\frac{n\pi}{L}\cos(n\pi)\left(e^L-e^{-L}\right)\\&=\left(\frac{n\pi}{L^2+n^2\pi^2}\right)\cos(n\pi)\left(e^L-e^{-L}\right)\end{aligned}$$

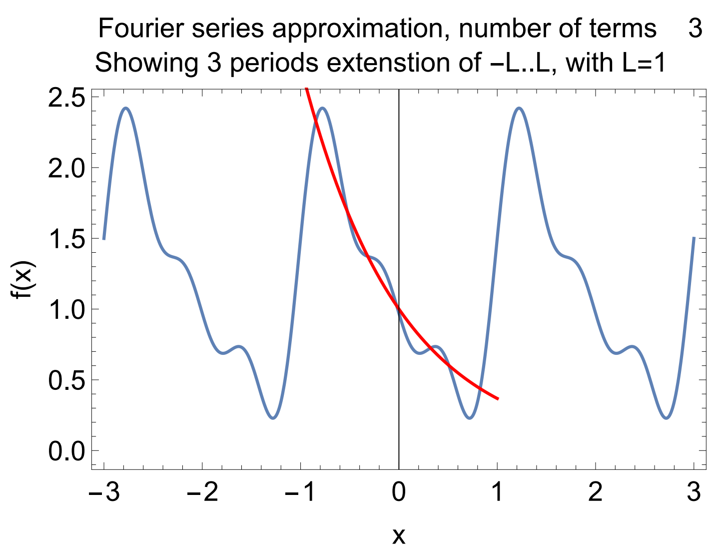
But $\cos(n\pi) = -1^n$ hence

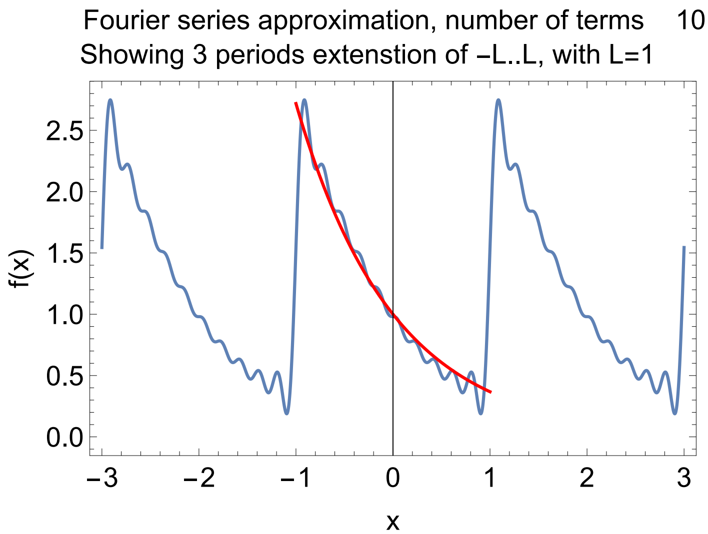
$$b_n = (-1)^n\left(\frac{n\pi}{L^2+n^2\pi^2}\right)\left(e^L-e^{-L}\right)$$

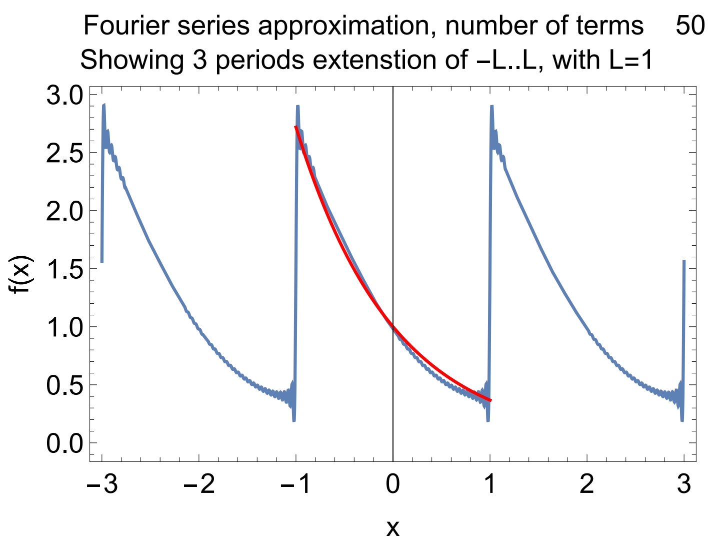
<u>Summary</u>

$$\begin{aligned}a_0 &= \frac{e^L-e^{-L}}{2L}\\a_n &= (-1)^n\left(\frac{L}{n^2\pi^2+L^2}\right)\left(e^L-e^{-L}\right)\\b_n &= (-1)^n\left(\frac{n\pi}{L^2+n^2\pi^2}\right)\left(e^L-e^{-L}\right)\\f(x) &\approx a_0+\sum_{n=1}^{\infty}a_n\cos\left(n\left(\frac{2\pi}{T}\right)x\right)+b_n\sin\left(n\left(\frac{2\pi}{T}\right)x\right)\\&\approx a_0+\sum_{n=1}^{\infty}a_n\cos\left(n\frac{\pi}{L}x\right)+b_n\sin\left(n\frac{\pi}{L}x\right)\end{aligned}$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.

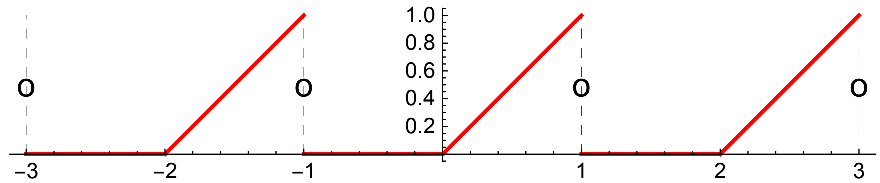






2.5.2.2 Part d

The following is sketch of periodic extension of $f(x)$ from $x = -L \cdots L$ (for $L = 1$) for illustration. The function will converge to $f(x)$ between $x = -L \cdots L$ and between $x = -3L \cdots -L$ and between $x = L \cdots 3L$ and so on. But at the jump discontinuities which occurs at $x = \cdots, -3L, -L, L, 3L, \cdots$ it will converge to the average $\frac{1}{2}$ shown as small circles in the sketch.



Showing 3 periods extenstion of f(x) between -L..L, with L=1

By definitions,

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx\\
a_n &= \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\cos\left(n\left(\frac{2\pi}{T}\right)x\right)dx\\
b_n &= \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\sin\left(n\left(\frac{2\pi}{T}\right)x\right)dx
\end{aligned}
$$

The period here is $T = 2L$, therefore the above becomes

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx
\end{aligned}
$$

These are now evaluated for given $f(x)$

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\\
&= \frac{1}{2L}\left(\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\right)\\
&= \frac{1}{2L}\left(0 + \int_{0}^{L} x dx\right)\\
&= \frac{1}{2L}\left(\frac{x^2}{2}\right)_0^L\\
&= \frac{L}{4}
\end{aligned}
$$

Now a_n is found

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx\\
&= \frac{1}{L}\left(\int_{-L}^{0} f(x)\cos\left(n\frac{\pi}{L}x\right)dx + \int_{0}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{1}{L}\int_{0}^{L} x\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}
$$

Integration by parts. Let $u = x, du = 1, dv = \cos\left(n\frac{\pi}{L}x\right), v = \frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}$, hence the above becomes

$$
\begin{aligned}
a_n &= \frac{1}{L}\left(\overbrace{\left(\frac{n\pi}{L}x\sin\left(n\frac{\pi}{L}x\right)\right)_0^L}^{0} - \int_0^L \frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}dx\right)\\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\int_0^L \sin\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\left(\frac{-\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right)_0^L\right)\\
&= \frac{1}{L}\left(\left(\frac{L}{n\pi}\right)^2\cos\left(n\frac{\pi}{L}x\right)_0^L\right)\\
&= \frac{L}{n^2\pi^2}\cos\left(n\frac{\pi}{L}x\right)_0^L\\
&= \frac{L}{n^2\pi^2}\left[\cos\left(n\frac{\pi}{L}L\right)-1\right]\\
&= \frac{L}{n^2\pi^2}\left[-1^n-1\right]
\end{aligned}
$$

Now b_n is found

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx\\
&= \frac{1}{L}\left(\int_{-L}^0 f(x)\sin\left(n\frac{\pi}{L}x\right)dx + \int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{1}{L}\int_0^L x\sin\left(n\frac{\pi}{L}x\right)dx
\end{aligned}
$$

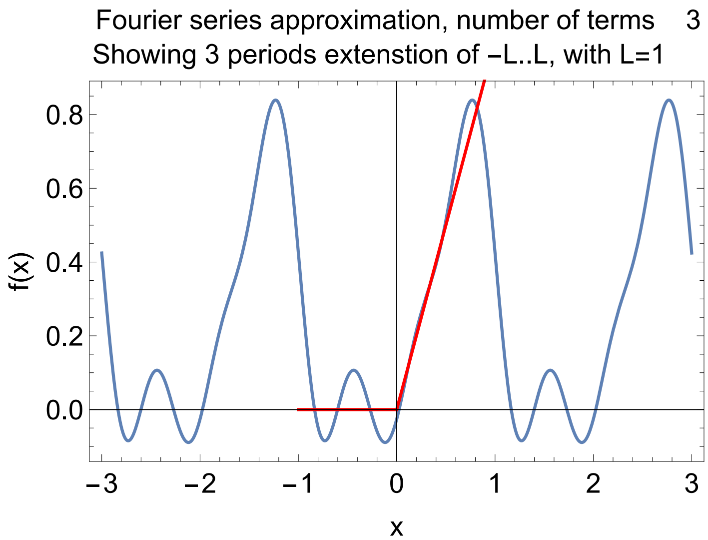
Integration by parts. Let $u = x, du = 1, dv = \sin\left(n\frac{\pi}{L}x\right), v = \frac{-\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}$, hence the above becomes

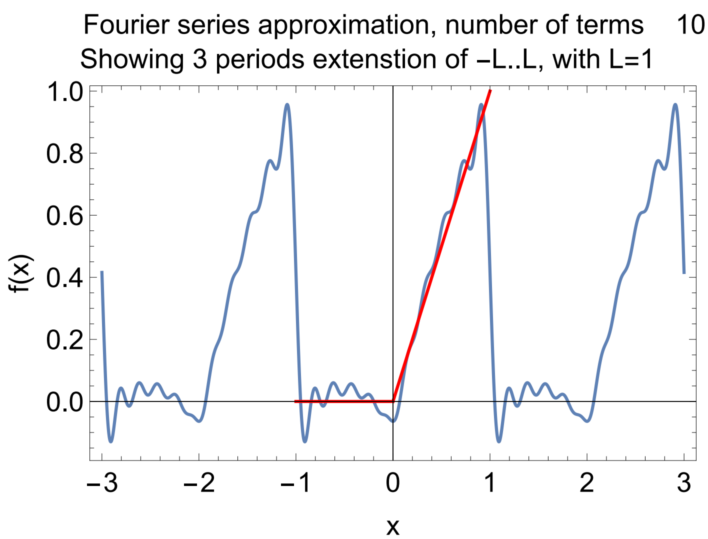
$$
\begin{aligned}
b_n &= \frac{1}{L}\left(\left(-\frac{L}{n\pi}x\cos\left(n\frac{\pi}{L}x\right)\right)_0^L + \int_0^L \frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}dx\right)\\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\left(L\cos\left(n\frac{\pi}{L}L\right)-0\right)+\frac{L}{n\pi}\int_0^L\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{1}{L}\left(-\frac{L^2}{n\pi}(-1)^n + \frac{L}{n\pi}\overbrace{\left[\frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_0^L}^{0}\right)\\
&= \frac{L}{n\pi}\left(-(-1)^n\right)\\
&= (-1)^{n+1}\frac{L}{n\pi}
\end{aligned}
$$

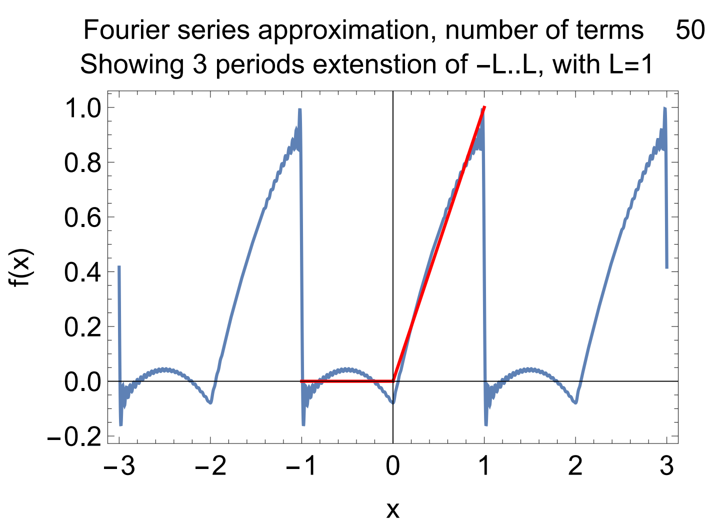
<u>Summary</u>

$$
\begin{aligned}
a_0 &= \frac{L}{4}\\
a_n &= \frac{L}{n^2\pi^2}\left[-1^n-1\right]\\
b_n &= (-1)^{n+1}\frac{L}{n\pi}\\
f(x) &\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\left(\frac{2\pi}{T}\right)x\right) + b_n\sin\left(n\left(\frac{2\pi}{T}\right)x\right)\\
&\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\frac{\pi}{L}x\right) + b_n\sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.


Fourier series approximation, number of terms 3
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3


Fourier series approximation, number of terms 10
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x
1.0
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3


Fourier series approximation, number of terms 50
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−3
−2
−1
0
1
2
3

2.5.3 Problem 3.2.4

> **3.2.4.** **Suppose that $f(x)$ is piecewise smooth. What value does the Fourier series of $f(x)$ converge to at the endpoint $x = -L$? at $x = L$?**

It will converge to the average value of the function at the end points after making periodic extensions of the function. Specifically, at $x = -L$ the Fourier series will converge to

$$\frac{1}{2}\left(f(-L) + f(L)\right)$$

And at $x = L$ it will converge to

$$\frac{1}{2}\left(f(L) + f(-L)\right)$$

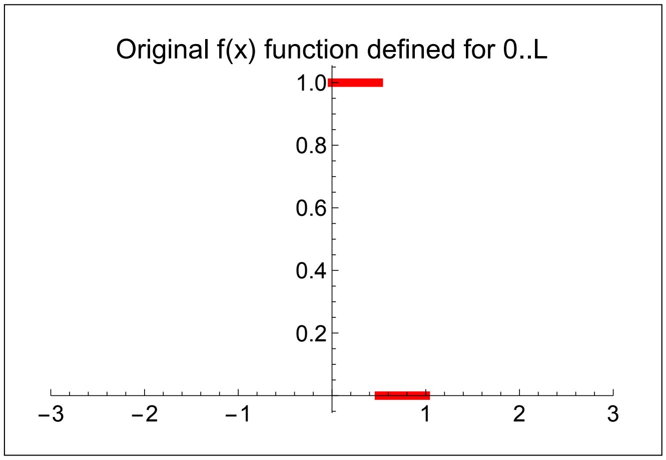
Notice that if $f(L)$ has same value as $f(-L)$, then there will not be a jump discontinuity when periodic extension are made, and the above formula simply gives the value of the function at either end, since it is the same value.

2.5.4 Problem 3.3.2 (d)

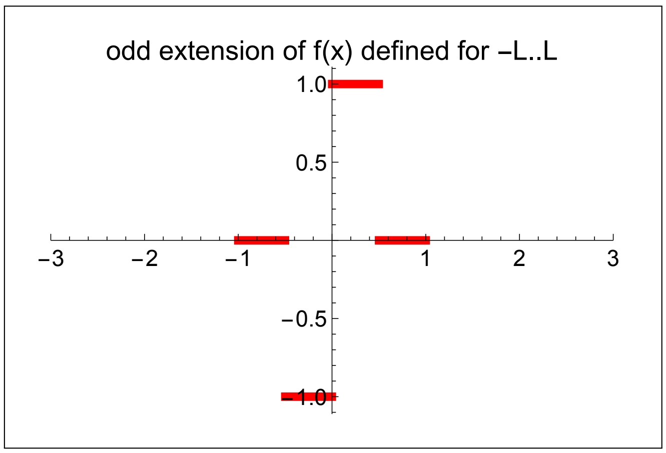
> **3.3.2.** **For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.**
>
> **(a)** $f(x) = \cos \pi x/L$ [Verify formula (3.3.13).]
>
> **(b)** $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$
>
> **(c)** $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$
>
> ***(d)** $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

$$f(x) = \begin{cases} 1 & x < \frac{L}{2} \\ 0 & x > \frac{L}{2} \end{cases}$$

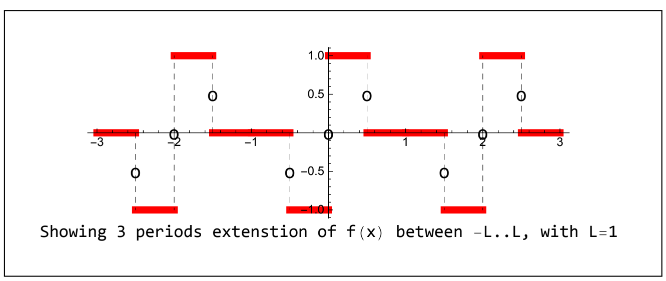
The first step is to sketch $f(x)$ over $0 \cdots L$. This is the result for $L = 1$ as an example.



The second step is to make an odd extension of $f(x)$ over $-L \cdots L$. This is the result.



The third step is to extend the above as periodic function with period $2L$ (as normally would be done) and mark the average value at the jump discontinuities. This is the result



Now the Fourier sin series is found for the above function. Since the function $f(x)$ is odd, then only b_n will exist

$$
\begin{aligned}
f(x) &\approx \sum_{n=1}^{\infty} b_n \sin\left(n\left(\frac{2\pi}{2L}\right)x\right) \\
&\approx \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\left(\frac{2\pi}{2L}\right)x\right)dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

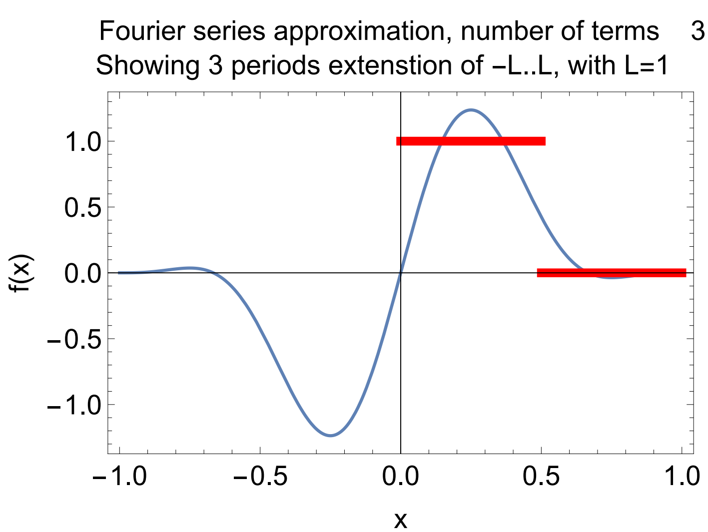
Since $f(x)\sin\left(n\frac{\pi}{L}x\right)$ is even, then the above becomes

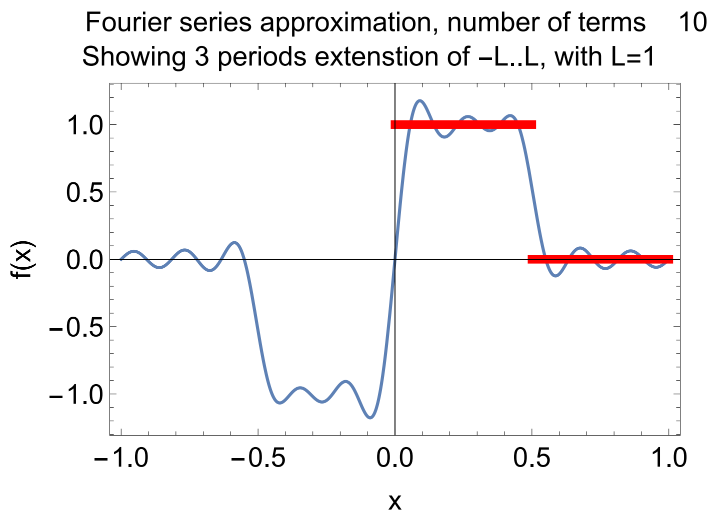
$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx \\
&= \frac{2}{L}\left(\int_0^{L/2} 1\times\sin\left(n\frac{\pi}{L}x\right)dx + \int_0^{L/2} 0\times\sin\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{2}{L}\int_0^{L/2}\sin\left(n\frac{\pi}{L}x\right)dx \\
&= \frac{2}{L}\left[-\frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_0^{L/2} \\
&= \frac{-2}{n\pi}\left[\cos\left(n\frac{\pi}{L}x\right)\right]_0^{L/2} \\
&= \frac{-2}{n\pi}\left[\cos\left(n\frac{\pi}{L}\frac{L}{2}\right)-1\right] \\
&= \frac{-2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-1\right] \\
&= \frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]
\end{aligned}$$

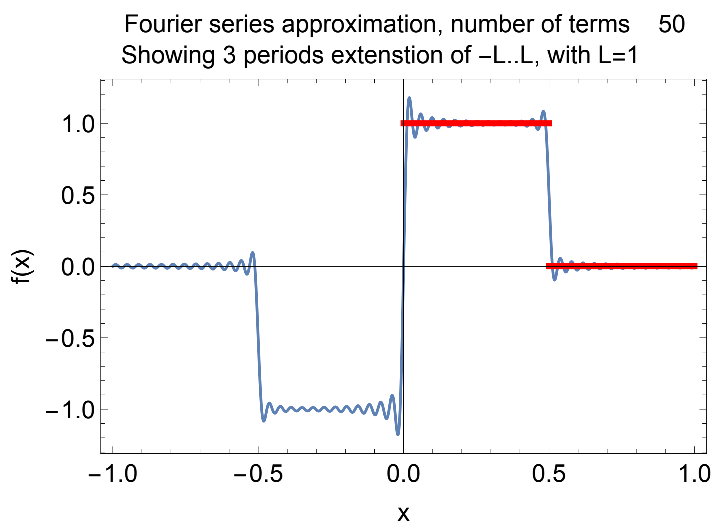
Therefore

$$f(x) \approx \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(n\frac{\pi}{L}x\right)$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.







2.5.5 Problem 3.3.3 (b)

3.3.3. For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(a) $f(x) = \cos \pi x/L$ [Use formula (3.3.13).]

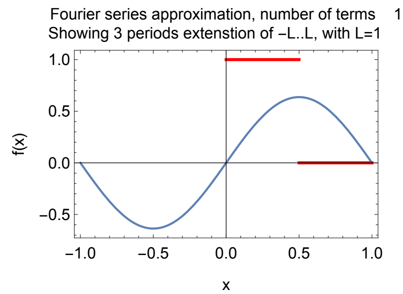
(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

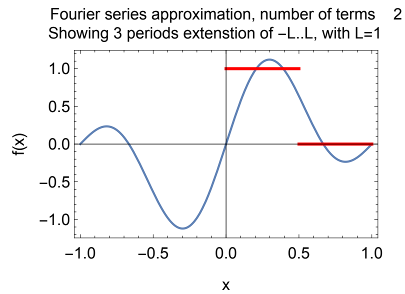
(c) $f(x) = x$ [Use formula (3.3.12).]

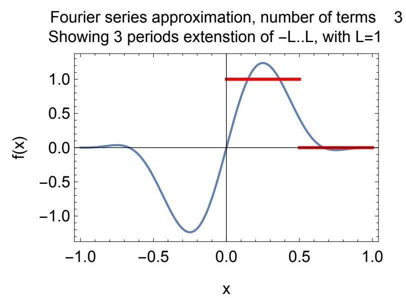
This is the same problem as 3.3.2 part (d). But it asks to plot for $n = 1$ and $n = 2$ in the sum. The sketch of the Fourier sin series was done above in solving 3.3.2 part(d) and will not be repeated again. From above, it was found that

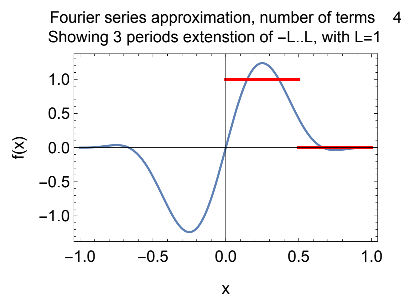
$$f(x) \approx \sum_{n=1}^{\infty} B_n \sin\left(n \frac{\pi}{L} x\right)$$

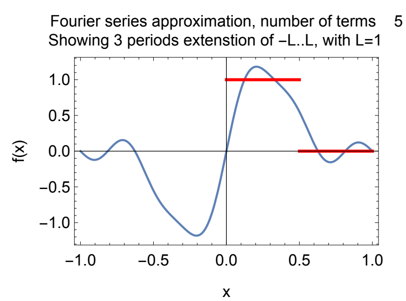
Where $B_n = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$. The following is the plot for $n = 1 \cdots 10$.

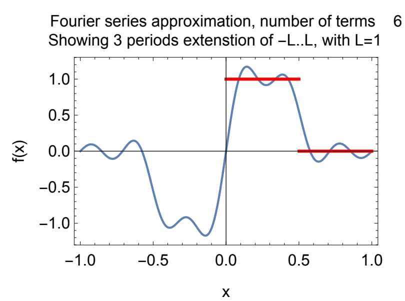

Fourier series approximation, number of terms 1
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x

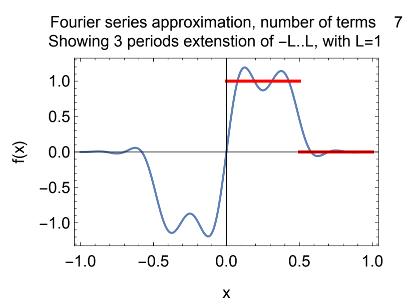

Fourier series approximation, number of terms 2
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x

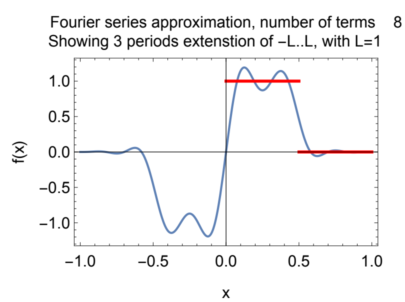

Fourier series approximation, number of terms 3
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x

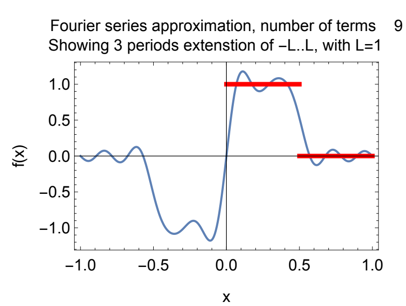

Fourier series approximation, number of terms 4
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x

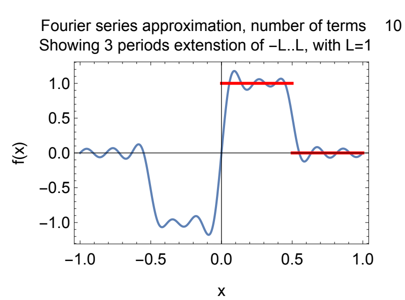

Fourier series approximation, number of terms 5
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x


Fourier series approximation, number of terms 6
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x


Fourier series approximation, number of terms 7
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x


Fourier series approximation, number of terms 8
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x


Fourier series approximation, number of terms 9
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x


Fourier series approximation, number of terms 10
Showing 3 periods extenstion of −L..L, with L=1
f(x)
x

2.5.6 Problem 3.3.8

3.3.8. (a) Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

2.5.6.1 Part (a)

The even extension of $f(x)$ is

$$\begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

But the even part of $f(x)$ is

$$\frac{1}{2}\left(f(x) + f(-x)\right)$$

2.5.6.2 Part (b)

The odd extension of $f(x)$ is

$$\begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

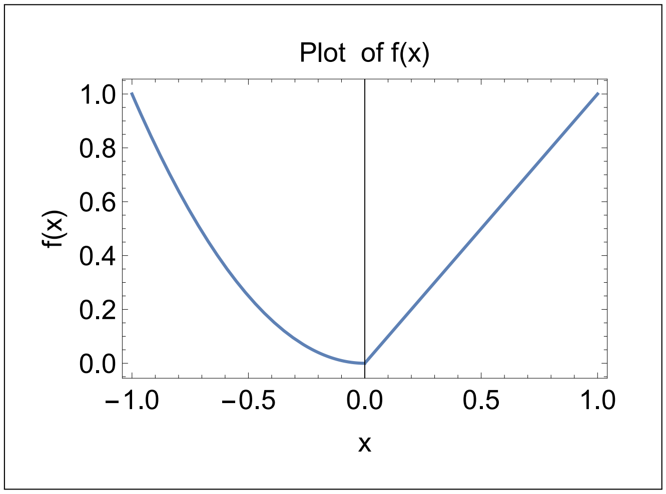
While the odd part of $f(x)$ is

$$\frac{1}{2}\left(f(x) - f(-x)\right)$$

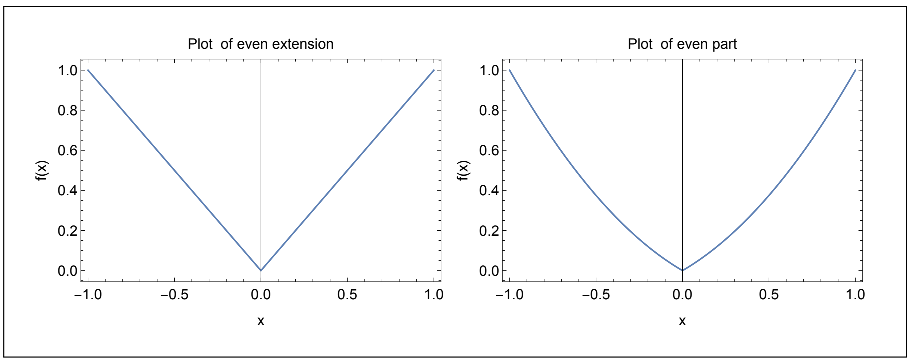
2.5.6.3 Part (c)

First a plot of $f(x)$ is given

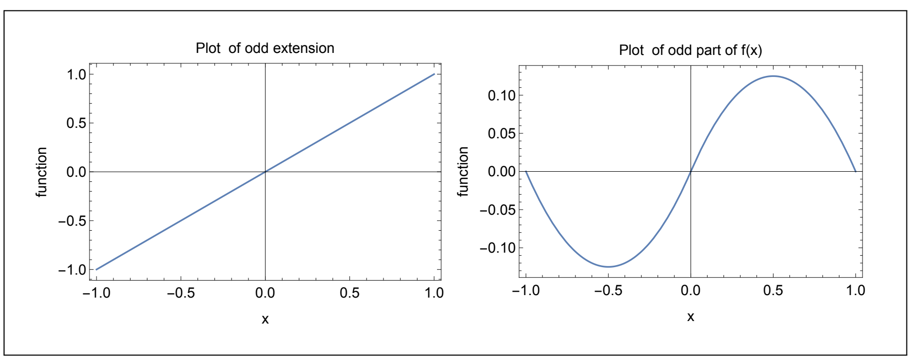
$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0 \end{cases}$$



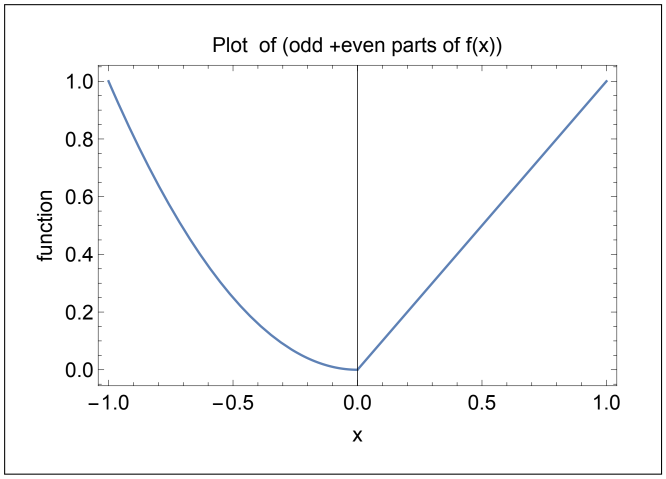
A plot of even extension and the even part for $f(x)$ Is given below



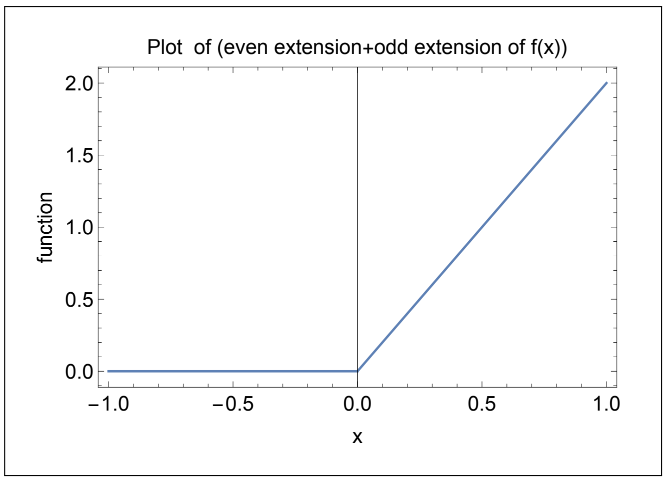
A plot of odd extension and the odd part is given below



Adding the even part and the odd part gives back the original function



Plot of adding the even extension and the odd extension is below



2.5.7 Problem 3.4.3

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and df/dx is piecewise smooth.

*(a) Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

2.5.7.1 Part (a)

Fourier sin series of $f'(x)$ is given by, assuming period is $-L \cdots L$

$$f'(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

Where

$$b_n = \frac{2}{L}\int_0^L f'(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

Applying integration by parts. Let $f'(x) = dv, u = \sin\left(n\frac{\pi}{L}x\right)$, then $v = f(x), du = \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)$. Since $v = f(x)$ has has jump discontinuity at x_0 as described, and assuming $x_0 > 0$, then, and using $\sin\left(n\frac{\pi}{L}x\right) = 0$ at $x = L$

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L u dv \\
&= \frac{2}{L}\left[\left([uv]_0^{x_0^-} + [uv]_{x_0^+}^L\right) - \int_0^L v du\right] \\
&= \frac{2}{L}\left(\left[\sin\left(n\frac{\pi}{L}x\right)f(x)\right]_0^{x_0^-} + \left[\sin\left(n\frac{\pi}{L}x\right)f(x)\right]_{x_0^+}^L - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{2}{L}\left(\sin\left(n\frac{\pi}{L}x_0^-\right)f\left(x_0^-\right) - \sin\left(n\frac{\pi}{L}x_0^+ f\left(x_0^+\right)\right) - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \qquad (1)
\end{aligned}$$

In the above, $\sin\left(n\frac{\pi}{L}x\right) = 0$ and at $x = L$ was used. But

$$\begin{aligned}
f\left(x_0^-\right) &= \alpha \\
f\left(x_0^+\right) &= \beta
\end{aligned}$$

And since sin is continuous, then $\sin\left(n\frac{\pi}{L}x_0^-\right) = \sin\left(n\frac{\pi}{L}x_0^+\right) = \sin\left(n\frac{\pi}{L}x_0\right)$. Equation (1) simplifies to

$$b_n = \frac{2}{L}\left(\left(\alpha - \beta\right)\sin\left(n\frac{\pi}{L}x_0\right) - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \qquad (2)$$

<u>On the other hand</u>, the Fourier cosine series for $f(x)$ is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right)$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

Therefore $\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx = \frac{L}{2}a_n$. Substituting this into (2) gives

$$\begin{aligned}
b_n &= \frac{2}{L}\left(\left(\alpha - \beta\right)\sin\left(n\frac{\pi}{L}x_0\right) - \frac{n\pi}{L}\left(\frac{L}{2}a_n\right)\right) \\
&= \frac{2}{L}\left(\alpha - \beta\right)\sin\left(n\frac{\pi}{L}x_0\right) - \frac{2}{L}\frac{n\pi}{L}\left(\frac{L}{2}a_n\right)
\end{aligned}$$

Hence

$$\boxed{b_n = \frac{2}{L}\sin\left(n\frac{\pi}{L}x_0\right)\left(\alpha - \beta\right) - \frac{n\pi}{L}a_n} \qquad (3)$$

Summary the Fourier $\sin$ series of $f'(x)$ is

$$f'(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

With b_n given by (3). The above is in terms of a_n, which is the Fourier cosine series of $f(x)$, which is what required to show. In addition, the $\cos$ series of $f(x)$ can also be written in terms of $\sin$ series of $f'(x)$. From (3), solving for a_n

$$a_n = \frac{L}{n\pi}b_n - \frac{2}{n\pi}\sin\left(n\frac{\pi}{L}x_0\right)(\alpha-\beta)$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{\pi}b_n - \frac{2}{\pi}\sin\left(n\frac{\pi}{L}x_0\right)(\alpha-\beta)\right)\cos\left(\frac{n\pi}{L}x\right)$$

This shows more clearly that the Fourier series of $f(x)$ has order of convergence in a_n as $\frac{1}{n}$ as expected.

2.5.7.2 Part (b)

Fourier $\cos$ series of $f'(x)$ is given by, assuming period is $-L\cdots L$

$$f'(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right)$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f'(x)\,dx \\
&= \frac{1}{L}\left(\int_0^{x_0^-} f'(x)\,dx + \int_{x_0^+}^{L} f'(x)\,dx\right) \\
&= \frac{1}{L}\left(\left[f(x)\right]_0^{x_0^-} + \left[f(x)\right]_{x_0^+}^{L}\right) \\
&= \frac{1}{L}\left(\left[\alpha - f(0)\right] + \left[f(L) - \beta\right]\right) \\
&= \frac{(\alpha-\beta)}{L} + \frac{f(0)+f(L)}{L}
\end{aligned}$$

And for $n > 0$

$$a_n = \frac{2}{L}\int_0^L f'(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

Applying integration by parts. Let $f'(x) = dv, u = \cos\left(n\frac{\pi}{L}x\right)$, then $v = f(x), du = \frac{-n\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$. Since $v = f(x)$ has has jump discontinuity at x_0 as described, then

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L u dv \\
&= \frac{2}{L}\left[\left([uv]_0^{x_0^-} + [uv]_{x_0^+}^L\right) - \int_0^L v du\right] \\
&= \frac{2}{L}\left(\left[\cos\left(n\frac{\pi}{L}x\right) f(x)\right]_0^{x_0^-} + \left[\cos\left(n\frac{\pi}{L}x\right) f(x)\right]_{x_0^+}^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) dx\right) \\
&= \frac{2}{L}\left(\cos\left(n\frac{\pi}{L}x_0^-\right) f\left(x_0^-\right) - f(0) + \cos(n\pi) f(L) - \cos\left(n\frac{\pi}{L}x_0^+\right) f\left(x_0^+\right) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) dx\right)
\end{aligned} \tag{1}$$

But

$$\begin{aligned}
f\left(x_0^-\right) &= \alpha \\
f\left(x_0^+\right) &= \beta
\end{aligned}$$

And since $\cos$ is continuous, then $\cos\left(n\frac{\pi}{L}x_0^-\right) = \cos\left(n\frac{\pi}{L}x_0^+\right) = \cos\left(n\frac{\pi}{L}x_0\right)$, therefore (1) becomes

$$a_n = \frac{2}{L}\left(\cos(n\pi) f(L) - f(0) + \cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) dx\right) \tag{2}$$

<u>On the other hand</u>, the Fourier *sin* series for $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

Where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right) dx$$

Therefore $\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right) dx = \frac{L}{2}b_n$. Substituting this into (2) gives

$$\begin{aligned}
a_n &= \frac{2}{L}\left(\cos(n\pi) f(L) - f(0) + \cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}\frac{L}{2}b_n\right) \\
&= \frac{2}{L}\cos(n\pi) f(L) - \frac{2}{L}f(0) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{2}{L}\frac{n\pi}{2}b_n \\
&= \frac{2}{L}\left(\left(-1^n\right) f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}b_n
\end{aligned}$$

Hence

$$\boxed{a_n = \frac{2}{L}\left(\left(-1^n\right) f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}b_n} \tag{3}$$

Summary the Fourier cos series of $f'(x)$ is

$$
\begin{aligned}
f'(x) &\sim \sum_{n=0}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right)\\
a_0 &= \frac{\left(\alpha-\beta\right)}{L} + \frac{f(0)+f(L)}{L}\\
a_n &= \frac{2}{L}\left((-1^n)\, f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)\left(\alpha-\beta\right) + \frac{n\pi}{L}b_n
\end{aligned}
$$

The above is in terms of b_n, which is the Fourier *sin* series of $f(x)$, which is what required to show.

2.5.8 Problem 3.4.9

***3.4.9 Consider the heat equation with a known source $q(x,t)$:**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \text{with} \quad u(0,t)=0 \quad \text{and} \quad u(L,t)=0.$$

Assume that $q(x,t)$ (for each $t>0$) is a piecewise smooth function of x. Also assume that u and $\partial u/\partial x$ are continuous functions of x (for $t>0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}.$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

The PDE is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \tag{1}$$

Since the boundary conditions are homogenous Dirichlet conditions, then the solution can be written down as

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(n\frac{\pi}{L}x\right)$$

Since the solution is assumed to be continuous with continuous derivative, then term by term differentiation is allowed w.r.t. x

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} n\frac{\pi}{L} b_n(t)\cos\left(n\frac{\pi}{L}x\right)\\
\frac{\partial^2 u}{\partial x^2} &= -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) \qquad (2)
\end{aligned}
$$

Also using assumption that $\frac{\partial u}{\partial t}$ is smooth, then

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) \tag{3}$$

Substituting (2,3) into (1) gives

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) = -k \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(n\frac{\pi}{L}x\right) + q(x,t) \tag{4}$$

Expanding $q(x,t)$ as Fourier $\sin$ series in x. Hence

$$q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

Where now $q_n(t)$ are time dependent given by (by orthogonality)

$$q_n(t) = \frac{2}{L} \int_0^L q(x,t) \sin\left(\frac{n\pi}{L}x\right)$$

Hence (4) becomes

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) = -\sum_{n=1}^{\infty} k \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{\infty} q(t)_n \sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality the above reduces to one term only

$$\frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) = -k \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(n\frac{\pi}{L}x\right) + q(t)_n \sin\left(\frac{n\pi}{L}x\right)$$

Dividing by $\sin\left(n\frac{\pi}{L}x\right) \neq 0$

$$\frac{db_n(t)}{dt} = -k \left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)$$

$$\frac{db_n(t)}{dt} + k \left(\frac{n\pi}{L}\right)^2 B_n(t) = q_n(t) \tag{5}$$

The above is the ODE that needs to be solved for $b_n(t)$. It is first order inhomogeneous ODE. The question asks to stop here.

2.5.9 Problem 3.4.11

3.4.11. Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

The PDE is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x) \tag{1}$$

Since the boundary conditions are homogenous Dirichlet conditions, then the solution can be written down as

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(n\frac{\pi}{L}x\right)$$

Since the solution is assumed to be continuous with continuous derivative, then term by term differentiation is allowed w.r.t. x

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} n\frac{\pi}{L} b_n(t) \cos\left(n\frac{\pi}{L}x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(n\frac{\pi}{L}x\right) \tag{2}$$

Also using assumption that $\frac{\partial u}{\partial t}$ is smooth, then

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) \tag{3}$$

Substituting (2,3) into (1) gives

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \sin\left(n\frac{\pi}{L}x\right) = -k\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(n\frac{\pi}{L}x\right) + g(x) \tag{4}$$

Using hint given in the problem, which is to expand $g(x)$ as Fourier sin series. Hence

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right)$$

Where

$$g_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)$$

Hence (4) becomes

$$\sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -\sum_{n=1}^{\infty}k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{\infty}g_n\sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality the above reduces to one term only

$$\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + g_n\sin\left(\frac{n\pi}{L}x\right)$$

Dividing by $\sin\left(n\frac{\pi}{L}x\right) \neq 0$

$$\frac{db_n(t)}{dt} = -k\left(\frac{n\pi}{L}\right)^2 b_n(t) + g_n$$

$$\frac{db_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n(t) = g_n \tag{5}$$

This is of the form $y' + ay = g_n$, where $a = k\left(\frac{n\pi}{L}\right)^2$. This is solved using an integration factor $\mu = e^{at}$, where $\frac{d}{dt}\left(e^{at}y\right) = e^{at}g_n$, giving the solution

$$y(t) = \frac{1}{\mu}\int \mu g_n dt + \frac{c}{\mu}$$

Hence the solution to (5) is

$$\begin{aligned}
b_n(t)e^{k\left(\frac{n\pi}{L}\right)^2 t} &= \int e^{k\left(\frac{n\pi}{L}\right)^2 t} g_n dt + c\\
b_n(t)e^{k\left(\frac{n\pi}{L}\right)^2 t} &= \frac{L^2 e^{k\left(\frac{n\pi}{L}\right)^2 t}}{kn^2\pi^2} g_n + c\\
b_n(t) &= \frac{L^2}{kn^2\pi^2} g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}
\end{aligned}$$

Where c above is constant of integration. Hence the solution becomes

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\sin\left(n\frac{\pi}{L}x\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{L^2}{kn^2\pi^2}g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)\sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

At $t = 0$, $u(x,0) = f(x)$, therefore

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{L^2}{kn^2\pi^2}g_n + c\right)\sin\left(n\frac{\pi}{L}x\right)$$

Therefore

$$\frac{L^2}{kn^2\pi^2}g_n + c = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

Solving for c gives

$$c = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx - \frac{L^2}{kn^2\pi^2}g_n$$

This completes the solution. Everything is now known. Summary

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\sin\left(n\frac{\pi}{L}x\right)\\
b_n(t) &= \left(\frac{L^2}{kn^2\pi^2}g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)\\
g_n &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\\
c &= \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx - \frac{L^2}{kn^2\pi^2}g_n
\end{aligned}$$

2.6 HW 5

2.6.1 Problem 3.5.2

3.5.2. **(a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of x^2.**
(b) From part (a), determine the Fourier sine series of x^3.

2.6.1.1 Part a

Equation 3.3.11, page 100 is the Fourier sin series of x

$$x=\sum_{n=1}^{\infty}B_n\sin\left(n\frac{\pi}{L}x\right)\qquad -L<x<L \tag{3.3.11}$$

Where

$$B_n=\frac{2L}{n\pi}(-1)^{n+1} \tag{3.3.12}$$

The goal is to find the Fourier $\cos$ series of x^2. Since $\int_0^x t\,dt=\frac{x^2}{2}$, then $x^2=2\int_0^x t\,dt$. Hence from 3.3.11

$$x^2=2\int_0^x\left[\sum_{n=1}^{\infty}B_n\sin\left(n\frac{\pi}{L}t\right)\right]dt$$

Interchanging the order of summation and integration the above becomes

$$\begin{aligned}
x^2&=2\sum_{n=1}^{\infty}\left(B_n\int_0^x\sin\left(n\frac{\pi}{L}t\right)dt\right)\\
&=2\sum_{n=1}^{\infty}B_n\left(\frac{-\cos\left(n\frac{\pi}{L}t\right)}{n\frac{\pi}{L}}\right)_0^x\\
&=\sum_{n=1}^{\infty}\frac{-2L}{n\pi}B_n\left[\cos\left(n\frac{\pi}{L}t\right)\right]_0^x\\
&=\sum_{n=1}^{\infty}\frac{-2L}{n\pi}B_n\left[\cos\left(n\frac{\pi}{L}x\right)-1\right]\\
&=\sum_{n=1}^{\infty}\left(\frac{-2L}{n\pi}B_n\cos\left(n\frac{\pi}{L}x\right)+\frac{2L}{n\pi}B_n\right)\\
&=\sum_{n=1}^{\infty}\frac{-2L}{n\pi}B_n\cos\left(n\frac{\pi}{L}x\right)+\sum_{n=1}^{\infty}B_n\frac{2L}{n\pi}
\end{aligned} \tag{1}$$

But a Fourier $\cos$ series has the form

$$x^2=A_0+\sum_{n=1}^{\infty}A_n\cos\left(n\frac{\pi}{L}x\right) \tag{2}$$

Comparing (1) and (2) gives

$$A_n = \frac{-2L}{n\pi} B_n$$

Using 3.3.12 for B_n the above becomes

$$\begin{aligned} A_n &= \frac{-2L}{n\pi} \frac{2L}{n\pi} (-1)^{n+1} \\ &= (-1)^n \left(\frac{2L}{n\pi}\right)^2 \end{aligned}$$

And

$$\begin{aligned} A_0 &= \sum_{n=1}^{\infty} B_n \frac{2L}{n\pi} \\ &= \sum_{n=1}^{\infty} \left(\frac{2L}{n\pi} (-1)^{n+1}\right) \frac{2L}{n\pi} \\ &= \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \end{aligned}$$

But $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}$, hence the above becomes

$$\begin{aligned} A_0 &= \frac{4L^2}{\pi^2} \frac{\pi^2}{12} \\ &= \frac{L^2}{3} \end{aligned}$$

<u>Summary</u> The Fourier $\cos$ series of x^2 is

$$\begin{aligned} x^2 &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{\pi}{L}x\right) \\ &= \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \cos\left(n\frac{\pi}{L}x\right) \end{aligned}$$

2.6.1.2 Part (b)

Since

$$x^3 = 3\int_0^x t^2 dt$$

Then, using result from part (a) for Fourier $\cos$ series of t^2 results in

$$\begin{aligned}
x^3 &= 3\int_0^x \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{\pi}{L}t\right)\right] dt \\
&= 3\int_0^x \frac{L^2}{3} dt + 3\int_0^x \sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \cos\left(n\frac{\pi}{L}t\right) dt \\
&= L^2 (t)_0^x + 3\sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \int_0^x \cos\left(n\frac{\pi}{L}t\right) dt \\
&= L^2 x + 3\sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \left[\frac{\sin\left(n\frac{\pi}{L}t\right)}{n\frac{\pi}{L}}\right]_0^x \\
&= L^2 x + 3\sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \left[\sin\left(n\frac{\pi}{L}t\right)\right]_0^x \\
&= L^2 x + (3\cdot 4)\sum_{n=1}^{\infty} (-1)^n \left(\frac{L}{n\pi}\right)^3 \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

Using 3.3.11 which is $x = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right)$, with $B_n = \frac{2L}{n\pi}(-1)^{n+1}$ the above becomes

$$x^3 = L^2 \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin\left(n\frac{\pi}{L}x\right) + (3\cdot 4)\sum_{n=1}^{\infty} (-1)^n \left(\frac{L}{n\pi}\right)^3 \sin\left(n\frac{\pi}{L}x\right)$$

Combining all above terms

$$x^3 = \sum_{n=1}^{\infty} \left[L^2 \frac{2L}{n\pi} (-1)^{n+1} + (3\cdot 4)(-1)^n \left(\frac{L}{n\pi}\right)^3\right] \sin\left(n\frac{\pi}{L}x\right)$$

Will try to simplify more to obtain B_n

$$\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} (-1)^n \frac{L^3}{n\pi} \left[-2 + (3\cdot 4)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{2L^3}{n\pi} \left[-1 + (3\times 2)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

Comparing the above to the standard Fourier $\sin$ series $x^3 = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right)$ then the above is the required $\sin$ series for x^3 with

$$B_n = (-1)^n \frac{2L^3}{n\pi} \left[-1 + (3\times 2)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right)$$

Expressing the above using B_n from x^1 to help find recursive relation for next problem.

Will now use the notation ${}^iB_n$ to mean the B_n for x^i. Then since ${}^1B_n = \frac{2L}{n\pi}(-1)^{n+1} = (-1)^n \left(-\frac{2L}{n\pi}\right)$

for x, then, using ${}^3B_n$ as the B_n for x^3, the series for x^3 can be written

$$
\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} (-1)^n L^2 \left[-\frac{2L}{n\pi} + 6\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^1B_n + 6\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Where now

$$
{}^3B_n = (-1)^n L^2 \left[B_n^1 + 6\left(2\frac{L}{n^2\pi^2}\right)\right]
$$

The above will help in the next problem in order to find recursive relation.

2.6.2 Problem 3.5.3

3.5.3. **Generalize Exercise 3.5.2, in order to derive the Fourier sine series of x^m, m odd.**

Result from Last problem showed that

$$
\begin{aligned}
x &= \sum_{n=1}^{\infty} B_n^1 \sin\left(n\frac{\pi}{L}x\right) \\
{}^1B_n &= (-1)^n \left(-\frac{2L}{n\pi}\right)
\end{aligned}
$$

And

$$
x^3 = \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^1B_n + (3\times 2)\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right)
$$

This suggests that

$$
\begin{aligned}
x^5 &= \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^3B_n + (5\times 4\times 3\times 2)\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right) \\
{}^3B_n &= (-1)^n L^2 \left[{}^1B_n + 6\left(2\frac{L}{n^2\pi^2}\right)\right]
\end{aligned}
$$

And in general

$$
x^m = \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^{m-2}B_n + m!\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right)
$$

Where

$$
{}^{m-2}B_n = (-1)^n L^2 \left[{}^{m-4}B_n + (m-2)!\left(2\frac{L}{n^2\pi^2}\right)\right]
$$

The above is a recursive definition to find x^m Fourier series for m odd.

2.6.3 Problem 3.5.7

***3.5.7. Evaluate**

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots$$

using (3.5.6).

Equation 3.5.6 is

$$\frac{x^2}{2}=\frac{L}{2}x-\frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L}+\frac{\sin\frac{3\pi x}{L}}{3^3}+\frac{\sin\frac{5\pi x}{L}}{5^3}+\frac{\sin\frac{7\pi x}{L}}{7^3}+\cdots\right) \tag{3.5.6}$$

Letting $x=\frac{L}{2}$ in (3.5.6) gives

$$\begin{aligned}\frac{L^2}{8}&=\frac{L^2}{4}-\frac{4L^2}{\pi^3}\left(\sin\frac{\pi\frac{L}{2}}{L}+\frac{\sin\frac{3\pi\frac{L}{2}}{L}}{3^3}+\frac{\sin\frac{5\pi\frac{L}{2}}{L}}{5^3}+\frac{\sin\frac{7\pi\frac{L}{2}}{L}}{7^3}+\cdots\right)\\&=\frac{L^2}{4}-\frac{4L^2}{\pi^3}\left(\sin\frac{\pi}{2}+\frac{\sin 3\frac{\pi}{2}}{3^3}+\frac{\sin 5\frac{\pi}{2}}{5^3}+\frac{\sin 7\frac{\pi}{2}}{7^3}+\cdots\right)\\&=\frac{L^2}{4}-\frac{4L^2}{\pi^3}\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\cdots\right)\end{aligned}$$

Hence

$$\begin{aligned}\frac{L^2}{8}-\frac{L^2}{4}&=-\frac{4L^2}{\pi^3}\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\cdots\right)\\-\frac{L^2}{8}&=-\frac{4L^2}{\pi^3}\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\cdots\right)\\\frac{\pi^3}{4\times 8}&=\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\cdots\right)\end{aligned}$$

Or

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\cdots$$

2.6.4 Problem 3.6.1

***3.6.1.** **Consider**

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta. \end{cases}$$

Assume that $x_0 > -L$ and $x_0 + \Delta < L$. Determine the complex Fourier coefficients c_n.

The function defined above is the Dirac delta function. (in the limit, as $\Delta \to 0$). Now

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\frac{\pi}{L}x} dx \\ &= \frac{1}{2L}\int_{x_0}^{x_0+\Delta} \frac{1}{\Delta} e^{in\frac{\pi}{L}x} dx \\ &= \frac{1}{2L}\frac{1}{\Delta}\left[\frac{e^{in\frac{\pi}{L}x}}{in\frac{\pi}{L}}\right]_{x_0}^{x_0+\Delta} \\ &= \frac{1}{2L}\frac{L}{\Delta i n\pi}\left[e^{in\frac{\pi}{L}x}\right]_{x_0}^{x_0+\Delta} \\ &= \frac{1}{i2n\Delta\pi}\left(e^{in\frac{\pi}{L}(x_0+\Delta)} - e^{in\frac{\pi}{L}x_0}\right) \end{aligned}$$

Since $\frac{e^{iz}-e^{-iz}}{2i} = \sin z$. The denominator above has $2i$ in it. Factoring out $e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}$ from the above gives

$$\begin{aligned} c_n &= \frac{1}{i2n\Delta\pi} e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}\left(e^{in\frac{\pi}{L}\frac{\Delta}{2}} - e^{-in\frac{\pi}{L}\frac{\Delta}{2}}\right) \\ &= \frac{1}{n\Delta\pi} e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}\frac{\left(e^{in\frac{\pi}{L}\frac{\Delta}{2}} - e^{-in\frac{\pi}{L}\frac{\Delta}{2}}\right)}{i2} \end{aligned}$$

Now the form is $\sin(z)$ is obtained, hence it can be written as

$$c_n = \frac{e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}}{n\Delta\pi}\sin\left(n\frac{\pi}{L}\frac{\Delta}{2}\right)$$

Or

$$c_n = \frac{\cos\left(n\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)\right) + i\sin\left(n\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)\right)}{\Delta n\pi}\sin\left(n\frac{\pi}{L}\frac{\Delta}{2}\right)$$

2.6.5 Problem 4.2.1

4.2.1. **(a)** Using Equation (4.2.7), compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(O) = 0$ and $u(L) = 0$.

(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies (4.2.9).

2.6.5.1 Part (a)

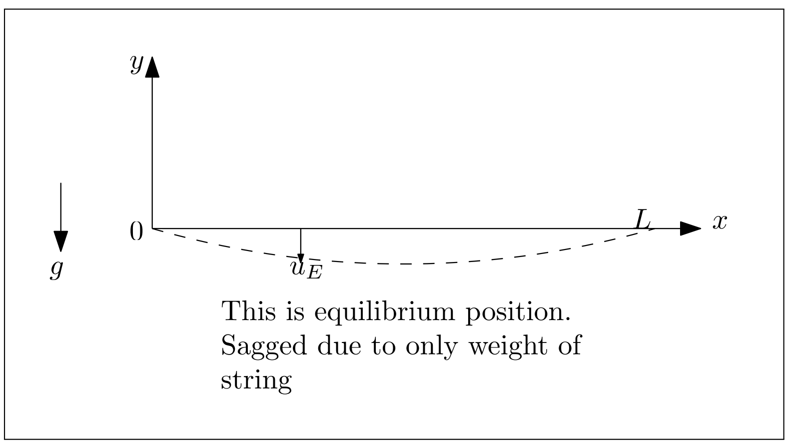
Equation 4.2.7 is

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho(x) \tag{4.2.7}$$

Replacing $Q(x,t)$ by $-g$

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - g\rho(x)$$

At equilibrium, the string is sagged but is not moving.



Therefore $\frac{\partial^2 u_E}{\partial t^2} = 0$. The above becomes

$$0 = T_0 \frac{\partial^2 u_E}{\partial x^2} - g\rho(x)$$

This is now partial differential equation in only x. It becomes an ODE

$$\frac{d^2 u_E}{dx^2} = \frac{g\rho(x)}{T_0}$$

With boundary conditions $u_E(0) = 0, u_E(L) = 0$. By double integration the solution is found. Integrating once gives

$$\frac{du_E}{dx} = \int_0^x \frac{g\rho(s)}{T_0} ds + c_1$$

Integrating again

$$\begin{aligned} u_E &= \int_0^x \left(\int_0^s \frac{g\rho(z)}{T_0} dz + c_1 \right) ds + c_2 \\ &= \int_0^x \left(\int_0^s \frac{g\rho(z)}{T_0} dz \right) ds + \int_0^x c_1 ds + c_2 \\ &= \frac{g}{T_0} \int_0^x \int_0^s \rho(z)\, dz ds + c_1 x + c_2 \end{aligned} \tag{1}$$

Equation (1) is the solution. Applying B.C. to find c_1, c_2. At $x = 0$ the above gives

$$0 = c_2$$

The solution (1) becomes

$$u_E = \frac{g}{T_0} \int_0^x \int_0^s \rho(z)\, dz ds + c_1 x \tag{2}$$

And at $x = L$ the above becomes

$$\begin{aligned} 0 &= \frac{g}{T_0} \int_0^L \int_0^s \rho(z)\, dz ds + c_1 L \\ c_1 &= \frac{-g}{LT_0} \int_0^L \int_0^s \rho(z)\, dz ds \end{aligned}$$

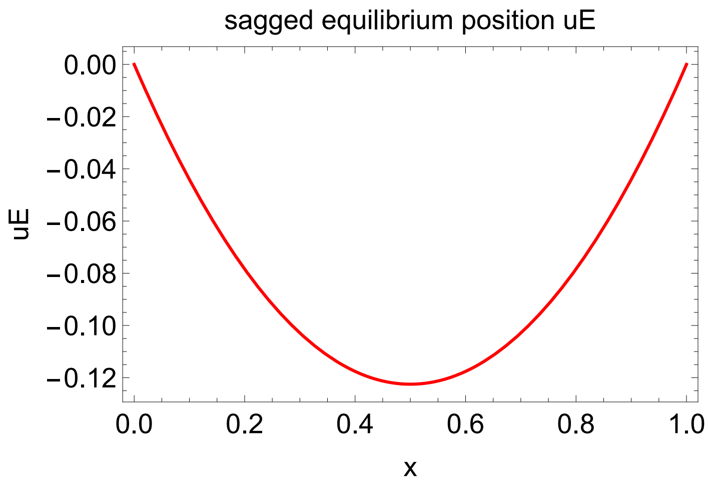
Substituting this into (2) gives the final solution

$$u_E = \frac{g}{T_0} \int_0^x \left(\int_0^s \rho(z)\, dz \right) ds + \left(\frac{-g}{LT_0} \int_0^L \left(\int_0^s \rho(z)\, dz \right) ds \right) x \tag{3}$$

If the density was constant, (3) reduces to

$$\begin{aligned} u_E &= \frac{g\rho}{T_0} \int_0^x s ds + \left(\frac{-g\rho}{LT_0} \int_0^L s ds \right) x \\ &= \frac{g\rho}{T_0} \frac{x^2}{2} - \frac{g\rho}{LT_0} \frac{L^2}{2} x \\ &= \frac{g\rho}{T_0} \left(\frac{x^2}{2} - \frac{L}{2} x \right) \end{aligned}$$

Here is a plot of the above function for $g = 9.8, L = 1, T_0 = 1, \rho = 0.1$ for verification.



2.6.5.2 Part (b)

Equation 4.2.9 is

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho(x)} \frac{\partial^2 u}{\partial x^2} \tag{4.2.9}$$

Since

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho(x) \tag{1}$$

And

$$\rho(x) \frac{\partial^2 u_E}{\partial t^2} = T_0 \frac{\partial^2 u_E}{\partial x^2} + Q(x,t)\,\rho(x) \tag{2}$$

Then by subtracting (2) from (1)

$$\begin{aligned}
\rho(x) \frac{\partial^2 u}{\partial t^2} - \rho(x) \frac{\partial^2 u_E}{\partial t^2} &= T_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho(x) - T_0 \frac{\partial^2 u_E}{\partial x^2} - Q(x,t)\,\rho(x) \\
\rho(x) \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_E}{\partial t^2} \right) &= T_0 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} \right)
\end{aligned}$$

Since $v(x,t) = u(x,t) - u_E(x,t)$ then $\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_E}{\partial t^2}$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2}$, therefore the above equation becomes

$$\begin{aligned}
\rho(x) \frac{\partial^2 v}{\partial t^2} &= T_0 \frac{\partial^2 v}{\partial x^2} \\
\frac{\partial^2 v}{\partial t^2} &= \frac{T_0}{\rho(x)} \frac{\partial^2 v}{\partial x^2} \\
&= c^2 \frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

Which is 4.2.9. QED.

2.6.6 Problem 4.2.5

4.2.5. Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density ρ_0, you may assume the tension T_0 is constant, and you may assume small displacements (with small slopes).

Let us consider a small segment of the string of length Δx from x to $x+\Delta x$. The mass of this segment is $\rho\Delta x$, where ρ is density of the string per unit length, assumed here to be constant. Let the angle that the string makes with the horizontal at x and at $x+\Delta x$ be $\theta(x,t)$ and $\theta(x+\Delta x,t)$ respectively. Since we are only interested in the vertical displacement $u(x,t)$ of the string, the vertical force on this segment consists of two parts: Its weight (acting downwards) and the net tension resolved in the vertical direction. Let the total vertical force be F_y. Therefore

$$F_y = \overbrace{-\rho\Delta x g}^{\text{weight}} + \overbrace{\left(T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)\right)}^{\text{net tension on segment in vertical direction}}$$

Applying Newton's second law in the vertical direction $F_y = ma_y$ where $a_y = \frac{\partial^2 u(x,t)}{\partial t^2}$ and $m=\rho\Delta x$, gives the equation of motion of the string segment in the vertical direction

$$\rho\Delta x\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho\Delta x g + \left(T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)\right)$$

Dividing both sides by Δx

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\left(T(x+\Delta x)\sin\theta(x+\Delta x,t) - T(x)\sin\theta(x,t)\right)}{\Delta x}$$

Taking the limit $\Delta x \to 0$

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right)$$

Assuming small angles then $\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$, then we can replace $\sin\theta$ in the above with $\frac{\partial u}{\partial x}$ giving

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u(x,t)}{\partial x}\right)$$

Assuming tension $T(x,t)$ is constant, say T_0 then the above becomes

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + T_0\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\right)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{T_0}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} - \rho g$$

Setting $\frac{T_0}{\rho} = c^2$ then the above becomes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \rho g$$

Note: In the above g (gravity acceleration) was used instead of $Q(x,t)$ as in the book to represent the body forces. In other words, the above can also be written as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \rho Q(x,t)$$

This is the required PDE, assuming constant density, constant tension, small angles and small vertical displacement.

2.6.7 Problem 4.4.1

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

***(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?**

***(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.**

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

2.6.7.1 Part (a)

The natural frequencies of vibrating string of length L with fixed ends, is given by equation 4.4.11 in the book, which is the solution to the string wave equation

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\frac{\pi}{L}x\right)\left(A_n \cos\left(n\frac{\pi c}{L}t\right) + B_n \sin\left(n\frac{\pi c}{L}t\right)\right)$$

The frequency of the time solution part of the PDE is given by the arguments of eigenfucntions $A_n \cos\left(n\frac{\pi c}{L}t\right) + B_n \sin\left(n\frac{\pi c}{L}t\right)$. Therefore $n\frac{\pi c}{L}$ represents the circular frequency ω_n. Comparing general form of $\cos \omega t$ with $\cos\left(n\frac{\pi c}{L}t\right)$ we see that each mode n has circular frequency given by

$$\omega_n \equiv n\frac{\pi c}{L}$$

For $n = 1, 2, 3, \cdots$. In cycles per seconds (Hertz), and since $\omega = 2\pi f$, then $2\pi f = n\frac{\pi c}{L}$. Solving for f gives

$$\begin{aligned} f_n &= n\frac{\pi c}{2\pi L} \\ &= n\frac{c}{2L} \end{aligned}$$

Where $c = \sqrt{\frac{T_0}{\rho_0}}$ in all of the above.

2.6.7.2 Part (b)

Equation 4.4.11 above was for a string with fixed ends. Now the B.C. are different, so we need to solve the spatial equation again to find the new eigenvalues. Starting with $u = X(x)T(t)$ and substituting this in the PDE $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ with $0 < x < H$ gives

$$\begin{aligned} T''X &= c^2 T X'' \\ \frac{1}{c^2}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where both sides are set equal to some constant $-\lambda$. We now obtain the two ODE's to solve. The spatial ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(H) &= 0 \end{aligned}$$

And the time ODE is

$$T'' + \lambda c^2 T = 0$$

The eigenvalues will always be positive for the wave equation. Taking $\lambda > 0$ the solution to the space ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Applying first B.C. gives

$$0 = A$$

Hence $X(x) = B\sin\left(\sqrt{\lambda}x\right)$ and $X'(x) = -B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$. Applying second B.C. gives

$$0 = -B\sqrt{\lambda}\cos\left(\sqrt{\lambda}H\right)$$

Therefore for non-trivial solution, we want $\sqrt{\lambda}H = \frac{n}{2}\pi$ for $n = 1, 3, 5, \cdots$ or written another way

$$\sqrt{\lambda}H = \left(n - \frac{1}{2}\right)\pi \qquad n = 1, 2, 3, \cdots$$

Therefore

$$\lambda_n = \left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right)^2 \qquad n = 1, 2, 3, \cdots$$

These are the eigenvalues. Now that we know what λ_n is, we go back to the solution found before, which is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\sqrt{\lambda_n}x\right)\left(A_n\cos\left(\sqrt{\lambda_n}ct\right) + B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)$$

And see now that the circular frequency ω_n is given by

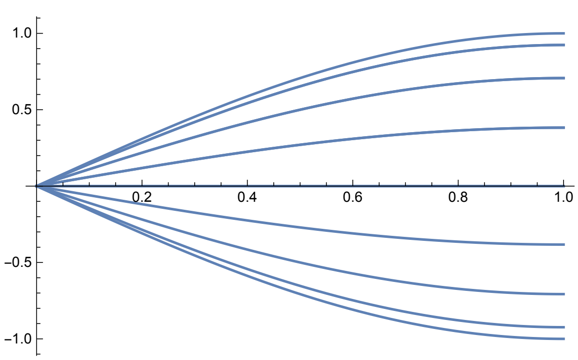
$$\begin{aligned}\omega_n &= \sqrt{\lambda_n}c\\ &= \frac{\left(n-\frac{1}{2}\right)\pi}{H}c \qquad n = 1,2,3,\cdots\end{aligned}$$

In cycles per second, since $\omega = 2\pi f$ then

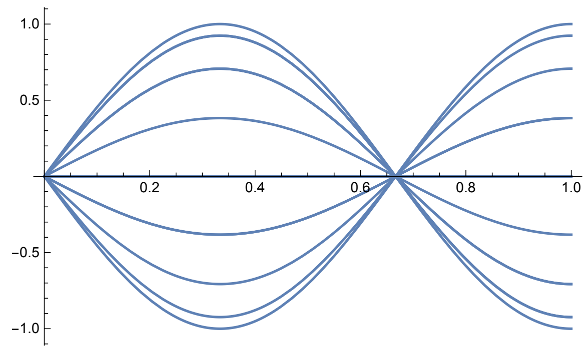
$$\begin{aligned}2\pi f_n &= \frac{\left(n-\frac{1}{2}\right)\pi}{H}c\\ f_n &= \frac{\left(n-\frac{1}{2}\right)}{2H}c \qquad n = 1,2,3,\cdots\end{aligned}$$

The following are plots for $n = 1,2,3,4,5$ for $t = 0\cdots 3$ seconds by small time increments.

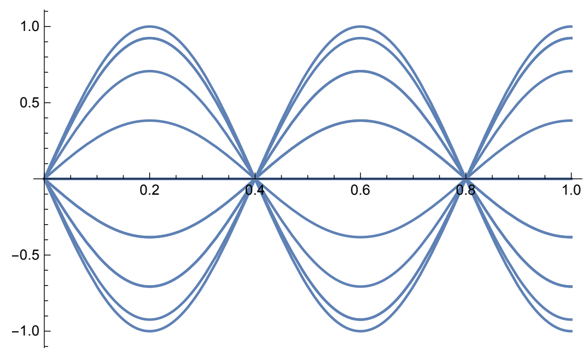
```
(*solution for HW 5, problem 4.4.1*)
f[x_, n_, t_] := Module[{H0 = 1, c = 1, lam},
lam = ((n - 1/2) Pi/H0);
Sin[lam x] (Sin[lam c t])
] ;
Table[Plot[f[x, 1, t], {x, 0, 1}, AxesOrigin -> {0, 0}], {t, 0,3, .25}];
p = Labeled[Show[
```



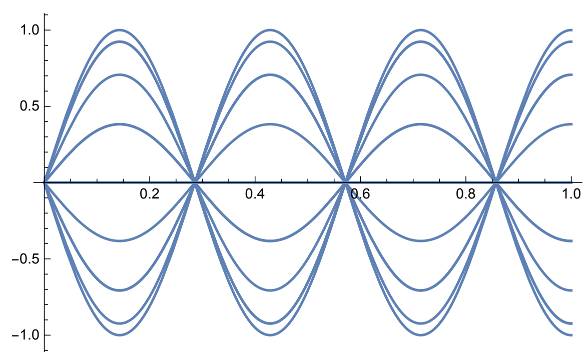
Mode n=1 vibration, from t=0 to t=3 seconds



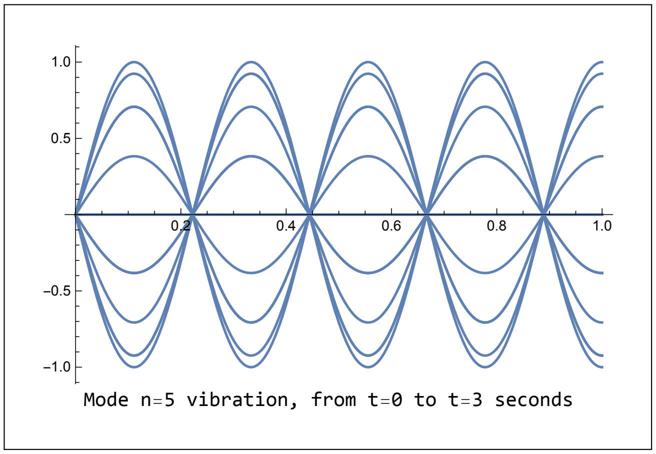
Mode n=2 vibration, from t=0 to t=3 seconds



Mode n=3 vibration, from t=0 to t=3 seconds



Mode n=4 vibration, from t=0 to t=3 seconds



2.6.7.3 Part (c)

For part (a), the harmonics had circular frequency $\omega_n = \frac{n\pi}{L}c$. Hence for odd n, these will generate

$$\frac{\pi}{L}c, 3\frac{\pi}{L}c, 5\frac{\pi}{L}c, 7\frac{\pi}{L}c, \cdots \tag{1}$$

For part (b), $\omega_n = \frac{\left(n-\frac{1}{2}\right)\pi}{H}c$. When $H = \frac{L}{2}$, this becomes $\omega_n = \frac{2\left(n-\frac{1}{2}\right)\pi}{L}c$. Looking at the first few modes gives

$$\begin{aligned}&\frac{2\left(1-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(2-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(3-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(4-\frac{1}{2}\right)\pi}{L}c, \cdots\\ &\frac{\pi}{L}c, \frac{3\pi}{L}c, \frac{5\pi}{L}c, \frac{7\pi}{L}c, \cdots\end{aligned} \tag{2}$$

Comparing (1) and (2) we see they are the same. Which is what we asked to show.

2.6.8 Problem 4.4.3

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small $(\beta^2 < 4\pi^2 \rho_0 T_0 / L^2)$.

2.6.8.1 Part (a)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

The term $-\beta \frac{\partial u}{\partial t}$ is the force that acts on the spring segment due to damping. This is the Viscous damping force which is proportional to speed, where β represents viscous damping coefficient. This damping force always opposes the direction of the motion. Hence if $\frac{\partial u}{\partial t} > 0$ then $-\beta \frac{\partial u}{\partial t}$ should come out to be negative. This occurs if $\beta > 0$. On the other hand, if $\frac{\partial u}{\partial t} < 0$ then $-\beta \frac{\partial u}{\partial t}$ should now be positive. Which means again that β must be positive quantity. Hence only case were the damping force always opposes the motion of the string is when $\beta > 0$.

2.6.8.2 Part (b)

Starting with $u = X(x)T(t)$ and substituting this in the above PDE with $0 < x < L$ gives

$$\rho_0 T'' X = T_0 T X'' - \beta T' X$$
$$\frac{\rho_0}{T_0} \frac{T''}{T} + \frac{\beta}{T_0} \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence we obtain two ODE's. The space ODE is

$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X(L) = 0$$

And the time ODE is

$$\begin{aligned}T'' + c^2\beta T' + c^2\lambda T &= 0\\ T(0) &= f(x)\\ T'(0) &= g(x)\end{aligned}$$

The eigenvalues will always be positive for the wave equation. Hence taking $\lambda > 0$ the solution to the space ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Applying first B.C. gives

$$0 = A$$

Hence $X = B\sin\left(\sqrt{\lambda}x\right)$. Applying the second B.C. gives

$$0 = B\sin\left(\sqrt{\lambda}L\right)$$

Therefore

$$\begin{aligned}\sqrt{\lambda}L &= n\pi \qquad n = 1,2,3,\cdots\\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

Hence the space solution is

$$X = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Now we solve the time ODE. This is second order ODE, linear, with constant coefficients.

$$\begin{aligned}\frac{\rho_0}{T_0}\frac{T''}{T} + \frac{\beta}{T_0}\frac{T'}{T} &= -\lambda\\ \frac{\rho_0}{T_0}T'' + \frac{\beta}{T_0}T' + \lambda T &= 0\\ T'' + \frac{\beta}{\rho_0}T' + \frac{T_0}{\rho_0}\lambda T &= 0\end{aligned}$$

Where in the above $\lambda \equiv \lambda_n$ for $n = 1,2,3,\cdots$. The characteristic equation is $r^2 + c^2\beta r + c^2\lambda = 0$. The roots are found from the quadratic formula

$$\begin{aligned}r_{1,2} &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\\ &= \frac{-\frac{\beta}{\rho_0} \pm \sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\lambda}}{2}\\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\lambda}\end{aligned}$$

Replacing $\lambda = \left(\frac{n\pi}{L}\right)^2$, gives

$$\begin{aligned} r_{1,2} &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2} \\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\frac{n^2\pi^2}{L^2}} \\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2\rho_0}\sqrt{\beta^2 - n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right)} \end{aligned}$$

We are told that $\beta^2 < 4\rho_0 T_0 \frac{\pi^2}{L^2}$, what this means is that $\beta^2 - n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right) < 0$, since $n^2 > 0$. This means we will get complex roots. Let

$$\Delta = n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right) - \beta^2$$

Hence the roots can now be written as

$$r_{1,2} = -\frac{\beta}{2\rho_0} \pm \frac{i\sqrt{\Delta}}{2\rho_0}$$

Therefore the time solution is

$$T_n(t) = e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right) + B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)$$

This is sinusoidal damped oscillation. Therefore

$$T(t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right) + B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right) \tag{2}$$

Combining (1) and (2), gives the total solution

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right) + B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right) \tag{3}$$

Where b_n constants for space ODE merged with the constants A_n, B_n for the time solution. Now we are ready to find A_n, B_n from initial conditions. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)A_n$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \int_0^L\sum_{n=1}^{\infty}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)A_n dx$$

Changing the order of integration and summation

$$\begin{aligned} \int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= A_m\frac{L}{2} \end{aligned}$$

Hence

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

To find B_n, we first take time derivative of the solution above in (3) which gives

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(-\frac{\sqrt{\Delta}}{2\rho_0}A_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\frac{\sqrt{\Delta}}{2\rho_0}\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)\\ &\quad -\frac{\beta}{2\rho_0}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)\end{aligned}$$

At $t=0$, using the second initial condition gives

$$g(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)B_n\frac{\sqrt{\Delta}}{2\rho_0}-\frac{\beta}{2\rho_0}A_n\sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx = \int_0^L\sum_{n=1}^{\infty}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)B_n\frac{\sqrt{\Delta}}{2\rho_0}dx-\sum_{n=1}^{\infty}\frac{\beta}{2\rho_0}A_n\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)$$

Changing the order of integration and summation

$$\begin{aligned}\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \sum_{n=1}^{\infty}B_n\frac{\sqrt{\Delta}}{2\rho_0}\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx-\sum_{n=1}^{\infty}\frac{\beta}{2\rho_0}A_n\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\\ &= B_m\frac{\sqrt{\Delta}}{2\rho_0}\frac{L}{2}-\frac{\beta}{2\rho_0}A_n\frac{L}{2}\\ &= \frac{L}{2}\left(B_m\frac{\sqrt{\Delta}}{2\rho_0}-\frac{\beta}{2\rho_0}A_n\right)\end{aligned}$$

Hence

$$\begin{aligned}B_m\frac{\sqrt{\Delta}}{2\rho_0}-\frac{\beta}{2\rho_0}A_n &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx\\ B_m &= \left(\frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{\beta}{2\rho_0}A_n\right)\frac{2\rho_0}{\sqrt{\Delta}}\end{aligned}$$

This completes the solution. Summary of solution

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)\\ A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\\ B_n &= \left(\frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{\beta}{2\rho_0}A_n\right)\frac{2\rho_0}{\sqrt{\Delta}}\\ \Delta &= n^2\left(4\rho_0T_0\frac{\pi^2}{L^2}\right)-\beta^2\end{aligned}$$

2.6.9 Problem 4.4.9

4.4.9 From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

$$E = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

Hence

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\frac{d}{dt}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

Moving $\frac{d}{dt}$ inside the integral, it becomes partial derivative

$$\frac{dE}{dt} = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{1}$$

But

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} = 2\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right) \tag{2}$$

And

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x\partial t}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2}\int_0^L 2\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right)dx + \frac{c^2}{2}\int_0^L 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= \int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx\end{aligned}$$

But $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ then the above becomes

$$\begin{aligned}\frac{dE}{dt} &= \int_0^L \left(\frac{\partial u}{\partial t}\left[c^2\frac{\partial^2 u}{\partial x^2}\right]\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\right)dx \end{aligned} \tag{4}$$

But since the integrand in (4) can also be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x\partial t}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}$$

Then (4) becomes

$$\begin{aligned}\frac{dE}{dt}&=c^2\int_0^L\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)dx\\&=c^2\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)_0^L\end{aligned}$$

Which is what we are asked to show. QED.

2.7 HW 6

2.7.1 Problem 5.3.2

5.3.2. Consider

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u + \beta\frac{\partial u}{\partial t}.$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

2.7.1.1 Part (a)

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial t^2} + \alpha u + \beta\frac{\partial u}{\partial t}$$

The PDE equation represents the vertical displacement $u(x,t)$ of the string as a function of time and horizontal position. This is 1D wave equation. The term $\beta\frac{\partial u}{\partial t}$ represents the damping force (can be due to motion of the string in air or fluid). The damping coefficient β must be negative to make $\beta\frac{\partial u}{\partial t}$ opposite to direction of motion. Damping force is proportional to velocity and acts opposite to direction of motion.

The term αu represents the stiffness in the system. This is a restoring force, and acts also opposite to direction of motion and is proportional to current displacement from equilibrium position. Hence $\alpha < 0$ also.

2.7.1.2 Part (b)

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$\rho T''X = T_0X''T + \alpha XT + \beta T'X$$

Dividing by $XT \neq 0$

$$\rho\frac{T''}{T} = T_0\frac{X''}{X} + \alpha + \beta\frac{T'}{T}$$
$$\rho\frac{T''}{T} - \beta\frac{T'}{T} = T_0\frac{X''}{X} + \alpha$$

To make each side depends on one variable only, we move $\rho(x), \beta(x)$ to the right side since these depends on x. Then dividing by $\rho(x)$ gives

$$\frac{T''}{T} - \frac{\beta}{\rho}\frac{T'}{T} = T_0\frac{X''}{\rho X} + \frac{\alpha}{\rho}$$

If $\frac{\beta(x)}{\rho(x)} = c$ is constant, then we see the equations have now been separated, since $\frac{\beta(x)}{\rho(x)}$ do not depend on x any more and the above becomes

$$\frac{T''}{T} - c\frac{T'}{T} = T_0\frac{X''}{\rho X} + \frac{\alpha(x)}{\rho(x)}$$

Now we can say that both side is equal to some constant $-\lambda$ giving the two ODE's

$$\begin{aligned}\frac{T''}{T} - c\frac{T'}{T} &= -\lambda\\ T_0\frac{X''}{\rho X} + \frac{\alpha}{\rho} &= -\lambda\end{aligned}$$

Or

$$\begin{aligned}T'' - cT' + \lambda T &= 0\\ X'' + X\left(\frac{\alpha}{T_0} + \lambda\frac{\rho}{T_0}\right) &= 0\end{aligned}$$

2.7.1.3 Part (c)

From above, the spatial ODE is

$$X'' + X\left(\frac{\alpha}{T_0} + \lambda\frac{\rho}{T_0}\right) = 0 \tag{1}$$

Comparing to regular Sturm Liouville (RSL) form, which is

$$\begin{aligned}\frac{d}{dx}\left(pX'\right) + qX + \lambda\sigma X &= 0\\ pX'' + p'X' + \left(q + \lambda\sigma\right)X &= 0 \qquad (2)\end{aligned}$$

Comparing (1) and (2) we see that

$$\begin{aligned}p &= 1\\ q &= \frac{\alpha}{T_0}\\ \sigma &= \frac{\rho}{T_0}\end{aligned}$$

To solve the time ODE $T'' - cT' + \lambda T = 0$, since this is second order linear with constant coefficients, then the characteristic equation is

$$\begin{aligned}r^2 - cr + \lambda = 0\\ r &= \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}\\ &= \frac{c}{2} \pm \frac{\sqrt{c^2 - 4\lambda}}{2}\end{aligned}$$

Hence the two solutions are

$$T_1(t) = e^{\left(\frac{c}{2}+\frac{\sqrt{c^2-4\lambda}}{2}\right)t}$$
$$T_2(t) = e^{\left(\frac{c}{2}-\frac{\sqrt{c^2-4\lambda}}{2}\right)t}$$

The general solution is linear combination of the above two solution, therefore final solution is

$$T(t) = c_1 e^{\left(\frac{c}{2}+\frac{\sqrt{c^2-4\lambda}}{2}\right)t} + c_2 e^{\left(\frac{c}{2}-\frac{\sqrt{c^2-4\lambda}}{2}\right)t}$$

Where c_1, c_2 are arbitrary constants of integration.

2.7.2 Problem 5.3.3

***5.3.3. Consider the non-Sturm-Liouville differential equation**

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + \left(\lambda\beta(x) + \gamma(x)\right)\phi = 0$$

Multiplying by $H(x)$ gives

$$H(x)\phi''(x) + H(x)\alpha(x)\phi'(x) + H(x)\left(\lambda\beta(x) + \gamma(x)\right)\phi = 0 \tag{1}$$

Comparing (1) to Sturm Liouville form, which is

$$\frac{d}{dx}\left(p\phi'\right) + q\phi + \lambda\sigma\phi = 0$$
$$p(x)\phi''(x) + p'(x)\phi'(x) + \left(q + \lambda\sigma\right)\phi(x) = 0 \tag{2}$$

Then we need to satisfy

$$H(x) = P(x)$$
$$H(x)\alpha(x) = P'(x)$$

Therefore, by combining the above, we obtain one ODE equation to solve for $H(x)$

$$H'(x) = H(x)\alpha(x)$$

This is first order separable ODE. $\frac{H'}{H}=\alpha$ or $\ln|H|=\int\alpha dx+c$ or

$$H=Ae^{\int\alpha(x)dx}$$

Where A is some constant. By comparing (1),(2) again, we see that

$$q+\lambda\sigma=\lambda\beta(x)H(x)+\gamma(x)H(x)$$

Summary of solution

$$\begin{aligned}\sigma(x)&=\beta(x)H(x)\\ q(x)&=\gamma(x)H(x)\\ P(x)&=H(x)\\ H(x)&=Ae^{\int\alpha(x)dx}\end{aligned}$$

QED

2.7.3 Problem 5.3.9

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2}+x\frac{d\phi}{dx}+\lambda\phi=0\quad\text{with}\quad\phi(1)=0,\quad\text{and}\quad\phi(b)=0.\quad(5.3.10)$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that $\lambda\geq0$.

***(c)** Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda=0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the nth eigenfunction has $n-1$ zeros.

$$\begin{aligned}x^2\phi''+x\phi'+\lambda\phi&=0 \qquad (1)\\ \phi(1)&=0\\ \phi(b)&=0\end{aligned}$$

2.7.3.1 Part (a)

Multiplying (1) by $\frac{1}{x}$ where $x\neq0$ gives

$$x\phi''+\phi'+\frac{\lambda}{x}\phi=0 \qquad (2)$$

Comparing (2) to Sturm-Liouville form

$$p\phi'' + p'\phi' + \left(q + \lambda\sigma\right)\phi = 0 \tag{3}$$

Then

$$\begin{aligned} p &= x \\ q &= 0 \\ \sigma &= \frac{1}{x} \end{aligned}$$

And since the given boundary conditions also satisfy the Sturm-Liouville boundary conditions, then (2) is a regular Sturm-Liouville ODE.

2.7.3.2 Part(b)

Using equation 5.3.8 in page 160 of text (called Raleigh quotient), which applies to regular Sturm-Liouville ODE, which relates the eigenvalues to the eigenfunctions

$$\begin{aligned} \lambda &= \frac{-\left[p\phi\phi'\right]_{x=1}^{x=b} + \int_1^b p\left(\phi'\right)^2 - q\phi^2 dx}{\int_1^b \phi^2\sigma dx} \qquad (5.3.8) \\ &= \frac{-\left[p(b)\phi(b)\phi'(b) - p(1)\phi(1)\phi'(b)\right] + \int_1^b p\left(\phi'\right)^2 - q\phi^2 dx}{\int_1^b \phi^2\sigma dx} \end{aligned}$$

Using $p = x, q = 0, \sigma = \frac{1}{x}$ and using $\phi(1) = 0, \phi(b) = 0$, then the above simplifies to

$$\lambda = \frac{-\int_1^b p\left(\phi'\right)^2 dx}{\int_1^b \frac{\phi^2}{x} dx}$$

The integrands in the numerator and denominator can not be negative, since they are squared quantities, and also since $x > 0$ as the domain starts from $x = 1$, then RHS above can not be negative. This means the eigenvalue λ can not be negative. It can only be $\lambda \geq 0$. QED.

2.7.3.3 Part(c)

The possible values of $\lambda > 0$ are determined by trying to solve the ODE and seeing which λ produces non-trivial solutions given the boundary conditions. The ODE to solve is (1) above. Here it is again

$$x^2\phi'' + x\phi' + \lambda\phi = 0 \tag{1}$$

We know $\lambda \geq 0$, so we do not need to check for negative λ.

<u>Case $\lambda = 0$</u>.

Equation (1) becomes

$$
\begin{aligned}
x^2\phi'' + x\phi' &= 0 \\
x\phi'' + \phi' &= 0 \\
\frac{d}{dx}\left(x\phi'\right) &= 0
\end{aligned}
$$

Hence $x\phi' = c_1$ where c_1 is constant. Therefore $\frac{d}{dx}\phi = \frac{c_1}{x}$ or

$$
\begin{aligned}
\phi &= c_1 \int \frac{1}{x} dx + c_2 \\
&= c_1 \ln|x| + c_2
\end{aligned}
$$

At $x = 1$, $\phi(1) = 0$, hence

$$
0 = c_1 \ln(1) + c_2
$$

But $\ln(1) = 0$, therefore $\underline{c_2 = 0}$. The solution now becomes

$$
\phi = c_1 \ln|x|
$$

At the right end, $x = b, \phi(b) = 0$, therefore

$$
0 = c_1 \ln b
$$

But since $b > 1$ the above implies that $c_1 = 0$. This gives trivial solution. Therefore $\underline{\lambda = 0 \text{ is not an eigenvalue.}}$

$\underline{\text{Case } \lambda > 0}$

$$
x^2\phi'' + x\phi' + \lambda\phi = 0
$$

This is non-constant coefficients, linear, second order ODE. Let $\phi(x) = x^p$. Equation (1) becomes

$$
\begin{aligned}
x^2 p\left(p-1\right) x^{p-2} + xpx^{p-1} + \lambda x^p &= 0 \\
p\left(p-1\right) x^p + px^p + \lambda x^p &= 0
\end{aligned}
$$

Dividing by $x^p \neq 0$ gives the characteristic equation

$$
\begin{aligned}
p\left(p-1\right) + p + \lambda &= 0 \\
p^2 - p + p + \lambda &= 0 \\
p^2 &= -\lambda
\end{aligned}
$$

Since $\lambda \geq 0$ then p is complex. Therefore the roots are

$$
p = \pm i\sqrt{\lambda}
$$

Therefore the two solutions (eigenfunctions) are

$$
\begin{aligned}
\phi_1(x) &= x^{i\sqrt{\lambda}} \\
\phi_2(x) &= x^{-i\sqrt{\lambda}}
\end{aligned}
$$

To more easily use standard form of solution, the standard trick is to rewrite these solution in exponential form

$$\phi_1(x) = e^{i\sqrt{\lambda}\ln x}$$
$$\phi_2(x) = e^{-i\sqrt{\lambda}\ln x}$$

The general solution to (1) is linear combination of these two solutions, therefore

$$\phi(x) = c_1 e^{i\sqrt{\lambda}\ln x} + c_2 e^{-i\sqrt{\lambda}\ln x} \tag{2}$$

Since $\lambda > 0$ then the above can be written using trig functions as

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda}\ln x\right) + c_2 \sin\left(\sqrt{\lambda}\ln x\right)$$

We are now ready to check for allowed values of λ by applying B.C's. The first B.C. gives

$$\begin{aligned}0 &= c_1 \cos\left(\sqrt{\lambda}\ln 1\right) + c_2 \sin\left(\sqrt{\lambda}\ln 1\right)\\ &= c_1 \cos(0) + c_2 \sin(0)\\ &= c_1\end{aligned}$$

Hence the solution now simplifies to

$$\phi(x) = c_2 \sin\left(\sqrt{\lambda}\ln x\right)$$

Applying the second B.C. gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\ln b\right)$$

For non-trivial solution we want

$$\begin{aligned}\sqrt{\lambda}\ln b &= n\pi \qquad n = 1,2,3,\cdots\\ \sqrt{\lambda} &= \frac{n\pi}{\ln b}\\ \lambda_n &= \left(\frac{n\pi}{\ln b}\right)^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

Therefore, there are infinite numbers of eigenvalues. The smallest is when $n = 1$ given by

$$\lambda_1 = \left(\frac{\pi}{\ln b}\right)^2$$

2.7.3.4 Part (d)

From Equation 5.3.6, page 159 in textbook, the eigenfunction are orthogonal with weight function $\sigma(x)$

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \qquad n \neq m$$

In this problem, the weight $\sigma = \frac{1}{x}$ and the solution (eigenfuctions) were found above to be

$$\phi_n(x) = \sin\left(\sqrt{\lambda_n}\ln x\right)$$

Now we can verify the orthogonality

$$\int_1^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = \int_{x=1}^{x=b} \sin\left(\frac{n\pi}{\ln b}\ln x\right)\sin\left(\frac{m\pi}{\ln b}\ln x\right)\frac{1}{x}dx$$

Using the substitution $z = \ln x$, then $\frac{dz}{dx} = \frac{1}{x}$. When $x = 1, z = \ln 1 = 0$ and when $x = b, z = \ln b$, then the above integral becomes

$$\begin{aligned} I &= \int_{z=0}^{z=\ln b} \sin\left(\frac{n\pi}{\ln b}z\right)\sin\left(\frac{m\pi}{\ln b}z\right)\frac{dz}{dx}dx \\ &= \int_0^{\ln b} \sin\left(\frac{n\pi}{\ln b}z\right)\sin\left(\frac{m\pi}{\ln b}z\right)dz \end{aligned}$$

But $\sin\left(\frac{n\pi}{\ln b}z\right)$ and $\sin\left(\frac{m\pi}{\ln b}z\right)$ are orthogonal functions (now with weight 1). Hence the above gives 0 when $n \neq m$ using standard orthogonality of the $\sin$ functions we used before many times. QED.

2.7.3.5 Part(e)

The n^{th} eigenfunction is

$$\phi_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

Here, the zeros are inside the interval, not counting the end points $x = 1$ and $x = b$.

$$\left.\left(\frac{n\pi}{\ln b}\ln x\right)\right|_{x=1} = \left(\frac{n\pi}{\ln b}0\right) = 0$$

And

$$\begin{aligned} \left.\left(\frac{n\pi}{\ln b}\ln x\right)\right|_{x=b} &= \frac{n\pi}{\ln b}\ln b \\ &= n\pi \end{aligned}$$

Hence for $n = 1$, The domain of $\phi_1(x)$ is $0 \cdots \pi$. And there are no zeros inside this for $\sin$ function not counting the end points. For $n = 2$, the domain is $0 \cdots 2\pi$ and $\sin$ has one zero inside this (at π), not counting end points. And for $n = 3$, the domain is $0 \cdots 3\pi$ and $\sin$ has two zeros inside this (at $\pi, 2\pi$), not counting end points. And so on. Hence $\phi_n(x)$ has $n - 1$ zeros not counting the end points.

2.7.4 Problem 5.5.1 (b,d,g)

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b=0$$

since then $\int_a^b [uL(v)-vL(u)]\ dx=0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) $\phi(0)=0$ and $\phi(L)=0$

(b) $\frac{d\phi}{dx}(0)=0$ and $\phi(L)=0$

(c) $\frac{d\phi}{dx}(0)-h\phi(0)=0$ and $\frac{d\phi}{dx}(L)=0$

(d) $\phi(a)=\phi(b)$ and $p(a)\frac{d\phi}{dx}(a)=p(b)\frac{d\phi}{dx}(b)$

(e) $\phi(a)=\phi(b)$ and $\frac{d\phi}{dx}(a)=\frac{d\phi}{dx}(b)$ [self-adjoint only if $p(a)=p(b)$]

(f) $\phi(L)=0$ and [in the situation in which $p(0)=0$] $\phi(0)$ bounded and $\lim_{x\to 0}p(x)\frac{d\phi}{dx}=0$

*(g) Under what conditions is the following self-adjoint (if p is constant)?

$$\begin{aligned}\phi(L)+\alpha\phi(0)+\beta\tfrac{d\phi}{dx}(0)&=0\\ \tfrac{d\phi}{dx}(L)+\gamma\phi(0)+\delta\tfrac{d\phi}{dx}(0)&=0\end{aligned}$$

The Sturm-Liouville ODE is

$$\frac{d}{dx}\left(p\phi'\right)+q\phi=-\lambda\sigma\phi$$

Or in operator form, defining $L\equiv\frac{d}{dx}\left(p\frac{d}{dx}\right)+q$, becomes

$$L\left[\phi\right]=-\lambda\sigma\phi$$

The operator L is self adjoined when

$$\int_a^b uL\left[v\right]dx=\int_a^b vL\left[u\right]dx$$

For the above to work out, we need to show that

$$p\left(uv'-vu'\right)\Big|_a^b=0$$

And this is what we will do now.

2.7.4.1 Part(b)

Here $a = 0$ and $b = L$.

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_a^b &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L \\ &= \left[p(L)\left(u(L)\frac{dv}{dx}(L) - v(L)\frac{du}{dx}(L)\right) - p(0)\left(u(0)\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0)\right)\right] \end{aligned}$$

Substituting $u(L) = v(L) = 0$ and $\frac{dv}{dx}(0) = \frac{du}{dx}(0) = 0$ into the above (since there are the B.C. given) gives

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_a^b &= \left[p(L)\left(0 \times \frac{dv}{dx}(L) - 0 \times \frac{du}{dx}(L)\right) - p(0)\left(u(0) \times 0 - v(0) \times 0\right)\right] \\ &= [0 - 0] \\ &= 0 \end{aligned}$$

2.7.4.2 Part (d)

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_a^b &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_b^a \\ &= \left[p(a)\left(u(a)v'(a) - v(a)u'(a)\right) - p(b)\left(u(b)v'(b) - v(b)u'(b)\right)\right] \\ &= p(a)u(a)v'(a) - p(a)v(a)u'(a) - p(b)u(b)v'(b) + p(b)v(b)u'(b) \qquad (1) \end{aligned}$$

We are given that $u(a) = u(b)$ and $v(a) = v(b)$ and $p(a)u'(a) = p(b)u'(b)$ and $p(a)v'(a) = p(b)v'(b)$.

We start by replacing $u(a)$ by $u(a)$ and replacing $v(a)$ by $v(b)$ in (1), this gives

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_a^b &= p(a)u(b)v'(a) - p(a)v(b)u'(a) - p(b)u(b)v'(b) + p(b)v(b)u'(b) \\ &= u(b)\left(p(a)v'(a) - p(b)v'(b)\right) + v(b)\left(p(b)u'(b) - p(a)u'(a)\right) \end{aligned}$$

Now using $p(a)u'(a) = p(b)u'(b)$ and $p(a)v'(a) = p(b)v'(b)$ in the above gives

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_a^b &= u(b)\left(p(b)v'(b) - p(b)v'(b)\right) + v(b)\left(p(b)u'(b) - p(b)u'(b)\right) \\ &= u(b)(0) + v(b)(0) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

2.7.4.3 Part (g)

p is constant. Hence

$$\begin{aligned} p\left(uv' - vu'\right)\Big|_0^L &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L \\ &= p\left[\left(u(L)v'(L) - v(L)u'(L)\right) - \left(u(0)v'(0) - v(0)u'(0)\right)\right] \qquad (1) \end{aligned}$$

We are given that

$$u(L) + \alpha u(0) + \beta u'(0) = 0 \tag{2}$$
$$u'(L) + \gamma u(0) + \delta u'(0) = 0 \tag{3}$$

And

$$v(L) + \alpha v(0) + \beta v'(0) = 0 \tag{4}$$
$$v'(L) + \gamma v(0) + \delta v'(0) = 0 \tag{5}$$

From (2),

$$u(L) = -\alpha u(0) - \beta u'(0)$$

From (3)

$$u'(L) = -\gamma u(0) - \delta u'(0)$$

From (4)

$$v(L) = -\alpha v(0) - \beta v'(0)$$

From (5)

$$v'(L) = -\gamma v(0) - \delta v'(0)$$

Using these 4 relations in equation (1) gives (where p is removed out, since it is constant, to simplify the equations)

$$\begin{aligned}
(uv' - vu')\big|_0^L &= u(L)\,v'(L) - v(L)\,u'(L) - u(0)\,v'(0) + v(0)\,u'(0) \\
&= \left(-\alpha u(0) - \beta u'(0)\right)\left(-\gamma v(0) - \delta v'(0)\right) \\
&- \left(-\alpha v(0) - \beta v'(0)\right)\left(-\gamma u(0) - \delta u'(0)\right) \\
&- u(0)\,v'(0) + v(0)\,u'(0)
\end{aligned}$$

Simplifying

$$\begin{aligned}
(uv' - vu')\big|_0^L &= \alpha u(0)\,\gamma v(0) + \alpha u(0)\,\delta v'(0) + \beta u'(0)\,\gamma v(0) + \beta u'(0)\,\delta v'(0) \\
&- \left(\alpha v(0)\,\gamma u(0) + \alpha v(0)\,\delta u'(0) + \beta v'(0)\,\gamma u(0) + \beta v'(0)\,\delta u'(0)\right) \\
&- u(0)\,v'(0) + v(0)\,u'(0) \\
&= \alpha u(0)\,\gamma v(0) + \alpha u(0)\,\delta v'(0) + \beta u'(0)\,\gamma v(0) + \beta u'(0)\,\delta v'(0) \\
&- \alpha v(0)\,\gamma u(0) - \alpha v(0)\,\delta u'(0) - \beta v'(0)\,\gamma u(0) - \beta v'(0)\,\delta u'(0) - u(0)\,v'(0) + v(0)\,u'(0)
\end{aligned}$$

Collecting

$$\begin{aligned}
(uv' - vu')\big|_0^L &= \alpha\delta\left(u(0)\,v'(0) - v(0)\,u'(0)\right) \\
&+ \beta\delta\left(u'(0)\,v'(0) - v'(0)\,u'(0)\right) \\
&+ \alpha\gamma\left(u(0)\,v(0) - v(0)\,u(0)\right) \\
&+ \beta\gamma\left(u'(0)\,v(0) - v'(0)\,u(0)\right) \\
&- u(0)\,v'(0) + v(0)\,u'(0) \\
&= \alpha\delta\left(u(0)\,v'(0) - v(0)\,u'(0)\right) + \beta\gamma\left(u'(0)\,v(0) - v'(0)\,u(0)\right) - \left(u(0)\,v'(0) - v(0)\,u'(0)\right) \\
&= \alpha\delta\left(u(0)\,v'(0) - v(0)\,u'(0)\right) - \beta\gamma\left(v'(0)\,u(0) - u'(0)\,v(0)\right) - \left(u(0)\,v'(0) - v(0)\,u'(0)\right)
\end{aligned}$$

Let $u(0)v'(0) - v(0)u'(0) = \Delta$ then we see that the above is just

$$\begin{aligned}(uv' - vu')\big|_0^L &= \alpha\delta(\Delta) - \beta\gamma(\Delta) - (\Delta)\\ &= \Delta\left(\alpha\delta - \beta\gamma - 1\right)\end{aligned}$$

Hence, for $(uv' - vu')\big|_0^L = 0$, we need

$$\alpha\delta - \beta\gamma - 1 = 0$$

2.7.5 Problem 5.5.3

5.5.3. **Consider the eigenvalue problem $L(\phi) = -\lambda\sigma(x)\phi$, subject to a given set of homogeneous boundary conditions. Suppose that**

$$\int_a^b [uL(v) - vL(u)]\ dx = 0$$

for all functions u and v satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

We are given that

$$\int_a^b uL[v] - vL[u]\,dx = 0 \tag{1}$$

But

$$L[v] = -\lambda_v\sigma(x)v \tag{2}$$

$$L[u] = -\lambda_u\sigma(x)u \tag{3}$$

Where $\sigma(x)$ is the weight function of the corresponding Sturm-Liouville ODE that u, v are its solution eigenfunctions. Substituting (2,3) into (1) gives

$$\begin{aligned}\int_a^b u(-\lambda_v\sigma(x)v) - v(-\lambda_u\sigma(x)u)\,dx &= 0\\ \int_a^b -\lambda_v\sigma(x)uv + \lambda_u\sigma(x)uvdx &= 0\\ (\lambda_u - \lambda_v)\int_a^b \sigma(x)uvdx &= 0\end{aligned}$$

Since u, v are different eigenfunctions, then the $\lambda_u - \lambda_v \neq 0$ as these are different eigenvalues. (There is one eigenfunction corresponding to each eigenvalue). Therefore the above says that

$$\int_a^b \sigma(x)u(x)v(x)\,dx = 0$$

Hence different eigenfunctions $u(x), v(x)$ are orthogonal to each others. The weight is $\sigma(x)$.

2.7.6 Problem 5.5.8

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\ dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\ dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned} \phi(0) &= 0 & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 & \frac{d^2\phi}{dx^2}(1) &= 0. \end{aligned}$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\ dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

$$L = \frac{d^4}{dx^4}$$

2.7.6.1 Part (a)

$$\begin{aligned} uL[v] - vL[u] &= u\frac{d^4 v}{dx^4} - v\frac{d^4 u}{dx^4} \\ &= uv^{(4)} - vu^{(4)} \end{aligned}$$

We want to obtain expression of form $\frac{d}{dx}()$ such that it comes out to be $uv^{(4)} - vu^{(4)}$. If we can do this, then it is exact differential. Now, since

$$\frac{d}{dx}(uv''' - u'v'') = u'v''' + uv^{(4)} - u''v'' - u'v''' \tag{1}$$

And

$$\frac{d}{dx}\left(vu''' - v'u''\right) = v'u''' + vu^{(4)} - v''u'' - v'u''' \tag{2}$$

Then (1)-(2) gives

$$\begin{aligned}
\frac{d}{dx}\left(uv''' - u'v''\right) - \frac{d}{dx}\left(vu''' - v'u''\right) &= \left(u'v''' + uv^{(4)} - u''v'' - u'v'''\right) - \left(v'u''' + vu^{(4)} - v''u'' - v'u'''\right) \\
&= u'v''' + uv^{(4)} - u''v'' - u'v''' - v'u''' - vu^{(4)} + v''u'' + v'u''' \\
&= uv^{(4)} - vu^{(4)}
\end{aligned}$$

Hence we found that

$$\begin{aligned}
\frac{d}{dx}\left(uv''' - u'v'' - vu''' + v'u''\right) &= uv^{(4)} - vu^{(4)} \\
&= uL[v] - vL[u]
\end{aligned}$$

Therefore $uL[v] - vL[u]$ is exact differential.

2.7.6.2 Part (b)

$$\begin{aligned}
I &= \int_a^b uL[v] - vL[u]\,dx \\
&= \int_a^b \frac{d}{dx}\left(uv''' - u'v'' - vu''' + v'u''\right) dx \\
&= \left. uv''' - u'v'' - vu''' + v'u''\right|_a^b \\
&= u(b)\,v'''(b) - u'(b)\,v''(b) - v(b)\,u'''(b) + v'(b)\,u''(b) \\
&\quad - \left(u(a)\,v'''(a) - u'(a)\,v''(a) - v(a)\,u'''(a) + v'(a)\,u''(a)\right)
\end{aligned}$$

Or

$I = u(b)\,v'''(b) - u'(b)\,v''(b) - v(b)\,u'''(b) + v'(b)\,u''(b) - u(a)\,v'''(a) + u'(a)\,v''(a) + v(a)\,u'''(a) - v'(a)\,u''(a)$

2.7.6.3 Part (c)

From part(b),

$$I = \int_0^1 uL[v] - vL[u]\,dx = \left. uv''' - u'v'' - vu''' + v'u''\right|_0^1 \tag{1}$$

Since we are given that

$$\begin{aligned}
\phi(0) &= 0 \\
\phi'(0) &= 0 \\
\phi(1) &= 0 \\
\phi''(1) &= 0
\end{aligned}$$

The above will give

$$
\begin{aligned}
u(0) &= v(0) = 0\\
u'(0) &= v'(0) = 0\\
u(1) &= v(1) = 0\\
u''(1) &= v''(1) = 0
\end{aligned}
$$

Substituting these into (1) gives

$$
\begin{aligned}
\int_0^1 uL[v] - vL[u]\,dx &= u(1)v'''(1) - u'(1)v''(1) - v(1)u'''(1) + v'(1)u''(1)\\
&\quad - u(0)v'''(0) + u'(0)v''(0) + v(0)u'''(0) - v'(0)u''(0)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^1 uL[v] - vL[u]\,dx &= (0\times v'''(1)) - 0 - (0\times u'''(1)) + 0 - (0\times v'''(0)) + 0 + (0\times u'''(0)) - 0\\
&= 0
\end{aligned}
$$

2.7.6.4 Part (d)

Any boundary conditions which makes $uv''' - u'v'' - vu''' + v'u''|_0^1 = 0$ will do. For example,

$$
\begin{aligned}
\phi(0) &= 0\\
\phi'(0) &= 0\\
\phi(1) &= 0\\
\phi'(1) &= 0
\end{aligned}
$$

The above will give

$$
\begin{aligned}
u(0) &= v(0) = 0\\
u'(0) &= v'(0) = 0\\
u(1) &= v(1) = 0\\
u'(1) &= v'(1) = 0
\end{aligned}
$$

Substituting these into (1) gives

$$
\begin{aligned}
\int_0^1 uL[v] - vL[u]\,dx &= u(1)v'''(1) - u'(1)v''(1) - v(1)u'''(1) + v'(1)u''(1)\\
&\quad - u(0)v'''(0) + u'(0)v''(0) + v(0)u'''(0) - v'(0)u''(0)\\
&= (0\times v'''(1)) - (0\times v''(1)) - (0\times u'''(1)) + (0\times u''(1))\\
&\quad - (0\times v'''(0)) + (0\times v''(0)) + (0\times u'''(0)) - (0\times u''(0))\\
&= 0
\end{aligned}
$$

2.7.6.5 Part (e)

Given

$$\frac{d^4}{dx^4}\phi + \lambda e^x \phi = 0$$

Therefore

$$L\left[\phi\right] = -\lambda e^x \phi$$

Therefore, for eigenfunctions u, v we have

$$\begin{aligned} L[u] &= -\lambda_u e^x u \\ L[v] &= -\lambda_v e^x v \end{aligned}$$

Where λ_u, λ_v are the eigenvalues associated with eigenfunctions u, v and they are not the same. Hence now we can write

$$\begin{aligned} 0 &= \int_0^1 uL[v] - vL[u]\, dx \\ &= \int_0^1 u(-\lambda_v e^x v) - v(-\lambda_u e^x u)\, dx \\ &= \int_0^1 -\lambda_v e^x uv + \lambda_u e^x uv dx \\ &= \int_0^1 (\lambda_u - \lambda_v)(e^x uv)\, dx \\ &= (\lambda_u - \lambda_v)\int_0^1 (e^x uv)\, dx \end{aligned}$$

Since $\lambda_u - \lambda_v \neq 0$ then

$$\int_0^1 (e^x uv)\, dx = 0$$

Hence u, v are orthogonal to each others with weight function e^x.

2.7.7 Problem 5.5.10

5.5.10. (a) Show that (5.5.22) yields (5.5.23) if at least one of the boundary conditions is of the regular Sturm-Liouville type.

(b) Do part (a) if one boundary condition is of the singular type.

2.7.7.1 Part(a)

Equation 5.5.22 is

$$p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = \text{constant} \tag{5.5.22}$$

Looking at boundary conditions at one end, say at $x = a$ (left end), and let the boundary conditions there be

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0$$

Therefore for eigenfunctions ϕ_1, ϕ_2 we obtain

$$\beta_1\phi_1(a) + \beta_2\phi_1'(a) = 0 \tag{1}$$

$$\beta_1\phi_2(a) + \beta_2\phi_2'(a) = 0 \tag{2}$$

From (1),

$$\phi_1'(a) = -\frac{\beta_1}{\beta_2}\phi_1(a) \tag{3}$$

From (2)

$$\phi_2'(a) = -\frac{\beta_1}{\beta_2}\phi_2(a) \tag{4}$$

Substituting (3,4) into $\phi_1\phi_2' - \phi_2\phi_1'$ gives, at end point a, the following

$$\begin{aligned}
\phi_1(a)\phi_2'(a) - \phi_2(a)\phi_1'(a) &= \phi_1(a)\left(-\frac{\beta_1}{\beta_2}\phi_2(a)\right) - \phi_2(a)\left(-\frac{\beta_1}{\beta_2}\phi_1(a)\right) \\
&= -\frac{\beta_1}{\beta_2}\phi_2(a)\phi_1(a) + \frac{\beta_1}{\beta_2}\phi_2(a)\phi_1(a) \\
&= 0
\end{aligned}$$

In the above, we evaluated $\phi_1\phi_2' - \phi_2\phi_1'$ at one end point, and found it to be zero. But $\phi_1\phi_2' - \phi_2\phi_1'$ is the Wronskian $W(x)$. It is known that if $W(x) = 0$ at just one point, then it is zero at all points in the range. Hence we conclude that

$$\phi_1\phi_2' - \phi_2\phi_1' = 0$$

For all x. This also means the eigenfunctions ϕ_1, ϕ_2 are linearly dependent. This gives equation 5.5.23. QED.

2.7.7.2 Part(b)

Equation 5.5.22 is

$$p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = \text{constant} \tag{5.5.22}$$

At one end, say end $x = a$, is where the singularity exist. This means $p(a) = 0$. Now to show that $p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = 0$ at $x = a$, we just need to show that $\phi_1\phi_2' - \phi_2\phi_1'$ is bounded. Since in that case, we will have $0 \times A = 0$, where A is some value which is $\phi_1\phi_2' - \phi_2\phi_1'$. But boundary conditions at $x = 1$ must be $\phi(a) < \infty$ and also $\phi'(a) < \infty$. This is always the case at the end where $p = 0$.

Then let $\phi(a) = c_1$ and $\phi'(a) = c_2$, where c_1, c_2 are some constants. Then we write

$$\begin{aligned}
\phi_1(a) &= c_1 \\
\phi_1'(a) &= c_2 \\
\phi_2(a) &= c_1 \\
\phi_2'(a) &= c_2
\end{aligned}$$

Hence it follows immediately that

$$\begin{aligned}
\phi_1\phi_2' - \phi_2\phi_1' &= c_1c_2 - c_2c_1 \\
&= 0
\end{aligned}$$

Hence we showed that $\phi_1\phi_2' - \phi_2\phi_1'$ is bounded. Then $p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = 0$. QED.

2.8 HW 7

2.8.1 Problem 5.6.1 (a)

5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

(a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ **with** $\frac{d\phi}{dx}(0) = 0$ **and** $\phi(1) = 0$

(b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ **with** $\frac{d\phi}{dx}(0) = 0$ **and** $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$

***(c)** $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ **with** $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(1) + \phi(1) = 0$ **(See Exercise 5.8.10.)**

2.8.1.1 part (a)

$$
\begin{aligned}
\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi &= 0 \\
\phi'(0) &= 0 \\
\phi(1) &= 0
\end{aligned}
$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2} - x^2\phi = -\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2} + p'\frac{d\phi}{dx} + q\phi = -\lambda\sigma\phi$$

Shows that

$$
\begin{aligned}
p &= 1 \\
q &= -x^2 \\
\sigma &= 1
\end{aligned}
$$

Now the Rayleigh quotient is

$$\lambda = \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dx}{\int_0^1 \sigma\phi^2 dx}$$

Substituting known values, and since $\phi'(0) = 0, \phi(1) = 0$ the above simplifies to

$$\lambda = \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx}$$

Now we can say that

$$\lambda_{\min} = \lambda_1 \leq \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx} \tag{1}$$

We now need a trial solution ϕ_{trial} to use in the above, which needs only to satisfy boundary conditions to use to estimate lowest $\lambda_{\min}$. The simplest such function will do. The boundary conditions are $\phi'(0) = 0, \phi(1) = 0$. We see for example that $\phi_{trial}(x) = x^2 - 1$ works, since $\phi'_{trial}(x) = 2x$, and $\phi'_{trial}(0) = 0$ and $\phi_{trial}(1) = 1 - 1 = 0$. So will use this in (1)

$$\begin{aligned}
\lambda_{\min} = \lambda_1 &\leq \frac{\int_0^1 (2x)^2 + x^2\left(x^2-1\right)^2 dx}{\int_0^1 \left(x^2-1\right)^2 dx} \\
&= \frac{\int_0^1 (2x)^2 + x^2\left(x^4 - 2x^2 + 1\right) dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 4x^2 + x^6 - 2x^4 + x^2 dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 3x^2 + x^6 dx}{\int_0^1 x^4 - 2x^2 + 1 dx} \\
&= \frac{\left(x^3 + \frac{1}{7}x^7\right)_0^1}{\left(\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right)_0^1} = \frac{\left(1 + \frac{1}{7}\right)}{\left(\frac{1}{5} - \frac{2}{3} + 1\right)} \\
&= \frac{15}{7} \\
&= 2.1429
\end{aligned}$$

Hence

$$\lambda_1 \leq 2.1429$$

2.8.2 Problem 5.6.2

5.6.2. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

$$\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi = 0$$
$$\phi'(0) = 0$$
$$\phi'(1) = 0$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2} - x^2\phi = -\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2} + p'\frac{d\phi}{dx} + q\phi = -\lambda\sigma\phi$$

Shows that

$$p = 1$$
$$q = -x^2$$
$$\sigma = 1$$

Now the Rayleigh quotient is

$$\lambda = \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dx}{\int_0^1 \sigma\phi^2 dx}$$

Substituting known values, and since $\phi'(0) = 0, \phi(1) = 0$ the above simplifies to

$$\lambda = \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx}$$

Since eigenfunction ϕ can not be identically zero, the denominator in the above expression can only be positive, since the integrand is positive. So we need now to consider the numerator term only:

$$\int_0^1 \left(\phi'\right)^2 dx + \int_0^1 x^2\phi^2 dx$$

For the second term, again, this can only be positive since ϕ can not be zero. For the first term, there are two cases. If ϕ' zero or not. If it is not zero, then the term is positive and we are done. This means $\lambda > 0$. if $\phi' = 0$ then $\int_0^1 \left(\phi'\right)^2 dx = 0$ and also conclude $\lambda > 0$ thanks to the second term $\int_0^1 x^2\phi^2$ being positive. So we conclude that λ can only be positive.

2.8.3 Problem 5.6.4

<u>Problem</u>

Consider eigenvalue problem $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda r\phi, 0 < r < 1$ subject to B.C. $\left|\phi(0)\right| < \infty$ (you may also assume $\frac{d\phi}{dr}$ bounded). And $\frac{d\phi}{dr}(1) = 0$. (a) prove that $\lambda \geq 0$. (b) Solve the boundary value problem. You may assume eigenfunctions are known. Derive coefficients using orthogonality.

Notice: Correction was made to problem per class email. Book said to show that $\lambda > 0$ which is error changed to $\lambda \geq 0$.

2.8.3.1 Part (a)

From the problem we see that $p = r, q = 0, \sigma = r$. The Rayleigh quotient is

$$\begin{aligned}\lambda &= \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dr}{\int_0^1 \sigma\phi^2 dr} \\ &= \frac{-\left(r\phi\phi'\right)_0^1 + \int_0^1 r\left(\phi'\right)^2 dr}{\int_0^1 r\phi^2 dr} \qquad (1)\end{aligned}$$

The term $-\left(r\phi\phi'\right)_0^1$ expands to

$$-\left((1)\,\phi(1)\,\phi'(1) - (0)\,\phi(0)\,\phi'(0)\right)$$

Since $\phi'(1) = 0$, the above is zero and Equation(1) reduces to

$$\lambda = \frac{\int_0^1 r\left(\phi'\right)^2 dr}{\int_0^1 r\phi^2 dr}$$

The denominator above can only be positive, as an eigenfunction ϕ can not be identically zero. For the numerator, we have to consider two cases.

case 1 If $\phi' \neq 0$ then we are done. The numerator is positive and we conclude that $\lambda > 0$.

case 2 If $\phi' = 0$ then ϕ is constant and this means $\lambda = 0$ is possible hence $\lambda \geq 0$. Now we need to show ϕ being constant is also possible. Since $\phi'(1) = 0$, then for $\phi' = 0$ to be true everywhere, it should also be $\phi'(0) = 0$ which means $\phi(0)$ is some constant. We are told that $\left|\phi(0)\right| < \infty$. Hence means $\phi(0)$ is constant is possible value (since bounded). Hence $\phi' = 0$ is possible.

Therefore $\lambda \geq 0$. QED.

2.8.3.2 Part (b)

The ODE is

$$\begin{aligned} r\phi'' + \phi' + \lambda r\phi &= 0 \qquad 0 < r < 1 \qquad (1) \\ \left|\phi(0)\right| &< \infty \\ \phi'(1) &= 0 \end{aligned}$$

In standard form the ODE is $\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0$. This shows that $r = 0$ is a regular singular point. Therefore we try

$$\phi(r)=\sum_{n=0}^{\infty}a_n r^{n+\alpha}$$

Hence

$$\begin{aligned}\phi'(r)&=\sum_{n=0}^{\infty}(n+\alpha)\,a_n r^{n+\alpha-1}\\ \phi''(r)&=\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\,a_n r^{n+\alpha-2}\end{aligned}$$

Substituting back into the ODE gives

$$\begin{aligned}r\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\,a_n r^{n+\alpha-2}+\sum_{n=0}^{\infty}(n+\alpha)\,a_n r^{n+\alpha-1}+\lambda r\sum_{n=0}^{\infty}a_n r^{n+\alpha}&=0\\ \sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\,a_n r^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)\,a_n r^{n+\alpha-1}+\lambda\sum_{n=0}^{\infty}a_n r^{n+\alpha+1}&=0\end{aligned}$$

To make all powers of r the same, we subtract 2 from the power of last term, and add 2 to the index, resulting in

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\,a_n r^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)\,a_n r^{n+\alpha-1}+\lambda\sum_{n=2}^{\infty}a_{n-2}r^{n+\alpha-1}=0$$

For $n=0$ we obtain

$$\begin{aligned}(\alpha)(\alpha-1)\,a_0 r^{\alpha-1}+(\alpha)\,a_0 r^{\alpha-1}&=0\\ (\alpha)(\alpha-1)\,a_0+(\alpha)\,a_0&=0\\ a_0\left(\alpha^2-\alpha+\alpha\right)&=0\\ a_0\alpha^2&=0\end{aligned}$$

But $a_0\neq 0$ (we always enforce this condition in power series solution), which implies

$$\boxed{\alpha=0}$$

Now we look at $n=1$, which gives

$$\begin{aligned}(1)(1-1)\,a_1 r^{n+\alpha-1}+(1)\,a_1 r^{n+\alpha-1}&=0\\ a_1&=0\end{aligned}$$

For $n\geq 2$, now all terms join in, and we get a recursive relation

$$\begin{aligned}(n)(n-1)\,a_n r^{n-1}+(n)\,a_n r^{n-1}+\lambda a_{n-2}r^{n-1}&=0\\ (n)(n-1)\,a_n+(n)\,a_n+\lambda a_{n-2}&=0\\ a_n&=\frac{-\lambda a_{n-2}}{(n)(n-1)+n}\\ &=\frac{-\lambda}{n^2}a_{n-2}\end{aligned}$$

For example, for $n=2$, we get

$$a_2=\frac{-\lambda}{2^2}a_0$$

All odd powers of n result in $a_n = 0$. For $n = 4$

$$a_4 = \frac{-\lambda}{4^2}a_2 = \frac{-\lambda}{4^2}\left(\frac{-\lambda}{2^2}a_0\right) = \frac{\lambda^2}{\left(2^2\right)\left(4^2\right)}a_0$$

And for $n = 6$

$$a_6 = \frac{-\lambda}{6^2}a_4 = \frac{-\lambda}{6^2}\frac{\lambda^2}{\left(2^2\right)\left(4^2\right)}a_0 = \frac{-\lambda^3}{\left(2^2\right)\left(4^2\right)\left(6^2\right)}a_0$$

And so on. The series is

$$\begin{aligned}\phi(r) &= \sum_{n=0}^{\infty} a_n r^n \\ &= a_0 + 0 + a_2 r^2 + 0 + a_4 r^4 + 0 + a_6 r^6 + \cdots \\ &= a_0 - \frac{\lambda r^2}{2^2}a_0 + \frac{\lambda^2 r^4}{\left(2^2\right)\left(4^2\right)}a_0 - \frac{\lambda^3 r^6}{\left(2^2\right)\left(4^2\right)\left(6^2\right)}a_0 + \cdots \\ &= a_0\left(1 - \frac{\left(\sqrt{\lambda}r\right)^2}{2^2} + \frac{\left(\sqrt{\lambda}r\right)^4}{\left(2^2\right)\left(4^2\right)} - \frac{\left(\sqrt{\lambda}r\right)^6}{\left(2^2\right)\left(4^2\right)\left(6^2\right)} + \cdots\right) \qquad (2)\end{aligned}$$

From tables, Bessel function of first kind of order zero, has series expansion given by

$$\begin{aligned}J_o(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n} \\ &= 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{(2)^2}\left(\frac{z}{2}\right)^4 - \frac{1}{((2)(3))^2}\left(\frac{z}{2}\right)^6 + \cdots \\ &= 1 - \frac{z^2}{2^2} + \frac{1}{2^2 4^2}z^4 - \frac{1}{2^2 3^2 2^6}z^6 + \cdots \\ &= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \cdots \qquad (3)\end{aligned}$$

By comparing (2),(3) we see a match between $J_o(z)$ and $\phi(r)$, if we let $z = \sqrt{\lambda}r$ we conclude that

$$\boxed{\phi_1(r) = a_0 J_0\left(\sqrt{\lambda}r\right)}$$

We can now normalized the above eigenfunction so that $a_0 = 1$ as mentioned in class. But it is not needed. The above is the first solution. We now need second solution. For repeated roots, the second solution will be

$$\phi_2(r) = \phi_1(r)\ln(r) + r^{\alpha}\sum_{n=0}^{\infty} b_n r^n$$

But $\alpha = 0$, hence

$$\phi_2(r) = \phi_1(r)\ln(r) + \sum_{n=0}^{\infty} b_n r^n$$

Hence the solution is

$$\phi(r) = c_1\phi_1(r) + c_2\phi_2(r)$$

Since $\phi(0)$ is bounded, then $c_2 = 0$ (since $\ln(0)$ not bounded at zero), and the solution becomes (where a_0 is now absorbed with the constant c_1)

$$
\begin{aligned}
\phi(r) &= c_1\phi_1(r) \\
&= cJ_0\left(\sqrt{\lambda}r\right)
\end{aligned}
$$

The boundedness condition has eliminated the second solution altogether. Now we apply the second boundary conditions $\phi'(1) = 0$ to find allowed eigenvalues. Since

$$
\phi'(r) = -cJ_1\left(\sqrt{\lambda}r\right)
$$

Then $\phi'(1) = 0$ implies

$$
0 = -cJ_1\left(\sqrt{\lambda}\right)
$$

The zeros of this are the values of $\sqrt{\lambda}$. Using the computer, these are the first few such values.

$$
\begin{aligned}
\sqrt{\lambda_1} &= 3.8317 \\
\sqrt{\lambda_2} &= 7.01559 \\
\sqrt{\lambda_3} &= 10.1735 \\
&\vdots
\end{aligned}
$$

or

$$
\begin{aligned}
\lambda_1 &= 14.682 \\
\lambda_2 &= 49.219 \\
\lambda_3 &= 103.5 \\
&\vdots
\end{aligned}
$$

Hence

$$
\begin{aligned}
\phi_n(r) &= c_nJ_0\left(\sqrt{\lambda_n}r\right) \qquad (4) \\
\phi(r) &= \sum_{n=1}^{\infty}\phi_n(r) \\
&= \sum_{n=1}^{\infty}c_nJ_0\left(\sqrt{\lambda_n}r\right)
\end{aligned}
$$

To find c_n, we use orthogonality. Per class discussion, we can now assume this problem was part of initial value problem, and that at $t = 0$ we had initial condition of $f(r)$, therefore, we now write

$$
f(r) = \sum_{n=1}^{\infty}c_nJ_0\left(\sqrt{\lambda_n}r\right)
$$

Multiplying both sides by $J_0\left(\sqrt{\lambda_m}r\right)\sigma$ and integrating gives (where $\sigma = r$)

$$\begin{aligned}\int_0^1 \sum_{n=1}^{\infty} f(r) J_0\left(\sqrt{\lambda_m}r\right) rdr &= \sum_{n=1}^{\infty} \int_0^1 c_n J_0\left(\sqrt{\lambda_m}r\right) J_0\left(\sqrt{\lambda_n}r\right) rdr \\ &= \int_0^1 c_m J_0^2\left(\sqrt{\lambda_m}r\right) rdr \\ &= c_m \int_0^1 J_0^2\left(\sqrt{\lambda_m}r\right) rdr \\ &= c_m \Omega\end{aligned}$$

Where Ω is some constant. Therefore

$$c_n = \frac{\int_0^1 \sum_{n=1}^{\infty} f(r) J_0\left(\sqrt{\lambda_m}r\right) rdr}{\Omega}$$

And $\phi(r)$

$$\sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n}r\right)$$

With the eigenvalues given as above, which have to be computed for each n using the computer.

2.8.4 Problem 5.9.1 (b)

5.9.1. **Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for**

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$

*(b) $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$

(c) $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) + h\phi(L) = 0$

From textbook, equation 5.9.8, we are given that for large λ

$$\begin{aligned}\phi(x) &\approx (\sigma p)^{\frac{-1}{4}} \exp\left(\pm i\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \\ &= (\sigma p)^{\frac{-1}{4}}\left(c_1 \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) + c_2 \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right) \qquad (1)\end{aligned}$$

Where c_1, c_2 are the two constants of integration since this is second order ODE. For $\phi(0) = 0$, the integral $\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt = \int_0^0 \sqrt{\frac{\sigma(t)}{p(t)}}dt = 0$ and the above becomes

$$\begin{aligned}
0 &= \phi(0) \\
&= (\sigma p)^{\frac{-1}{4}} (c_1 \cos(0) + c_2 \sin(0)) \\
&= c_1 (\sigma p)^{\frac{-1}{4}}
\end{aligned}$$

Hence $c_1 = 0$ and (1) reduces to

$$\phi(x) = c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Hence

$$\begin{aligned}
\phi'(x) &= c_2 (\sigma p)^{\frac{-1}{4}} \cos\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\frac{d}{dx}\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \\
&= c_2 (\sigma p)^{\frac{-1}{4}} \cos\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\sqrt{\lambda}\sqrt{\frac{\sigma(x)}{p(x)}}\right) \\
&= c_2 (\sigma p)^{\frac{-1}{4}} \sqrt{\frac{\lambda\sigma}{p}} \cos\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)
\end{aligned}$$

Since $\phi'(L) = 0$ then

$$0 = c_2 (\sigma p)^{\frac{-1}{4}} \sqrt{\frac{\lambda\sigma}{p}} \cos\left(\sqrt{\lambda} \int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Which means, for non-trivial solution, that

$$\sqrt{\lambda_n} \int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt = \left(n - \frac{1}{2}\right)\pi$$

Therefore, for large λ, (i.e. large n) the estimate is

$$\begin{aligned}
\sqrt{\lambda_n} &= \frac{\left(n - \frac{1}{2}\right)\pi}{\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt} \\
\lambda_n &= \left(\frac{\left(n - \frac{1}{2}\right)\pi}{\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt}\right)^2
\end{aligned}$$

2.8.5 Problem 5.9.2

5.9.2. Consider

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for λ large. Take into account amplitude and period variations.

$$\phi'' + \lambda(1+x)\phi = 0$$

Comparing the above to Sturm-Liouville form

$$\left(p\phi'\right)' + q\phi = -\lambda\sigma\phi$$

Shows that

$$\begin{aligned} p &= 1 \\ q &= 0 \\ \sigma &= 1+x \end{aligned}$$

Now, from textbook, equation 5.9.8, we are given that for large λ

$$\begin{aligned} \phi(x) &\approx \left(\sigma p\right)^{\frac{-1}{4}} \exp\left(\pm i\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \\ &= \left(\sigma p\right)^{\frac{-1}{4}} \left(c_1 \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) + c_2 \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right) \end{aligned} \tag{1}$$

Where c_1, c_2 are the two constants of integration since this is second order ODE. For $\phi(0) = 0$, the integral $\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt = \int_0^0 \sqrt{\frac{\sigma(t)}{p(t)}}dt = 0$ and the above becomes

$$\begin{aligned} 0 &= \phi(0) \\ &= \left(\sigma p\right)^{\frac{-1}{4}} \left(c_1 \cos(0) + c_2 \sin(0)\right) \\ &= c_1 \left(\sigma p\right)^{\frac{-1}{4}} \end{aligned}$$

Hence $c_1 = 0$ and (1) reduces to

$$\phi(x) = c_2 \left(\sigma p\right)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Applying the second boundary condition $\phi(1) = 0$ on the above gives

$$\begin{aligned} 0 &= \phi(1) \\ &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}} dt\right) \end{aligned}$$

Hence for non-trivial solution we want, for large positive integer n

$$\begin{aligned} \sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}} dt &= n\pi \\ \sqrt{\lambda} &= \frac{n\pi}{\int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}} dt} \\ &= \frac{n\pi}{\int_0^1 \sqrt{1+t} dt} \end{aligned}$$

But $\int_0^1 \sqrt{1+t} dt = 1.21895$, hence

$$\begin{aligned} \sqrt{\lambda} &= \frac{n\pi}{1.21895} = 2.5773n \\ \lambda &= 6.6424n^2 \end{aligned}$$

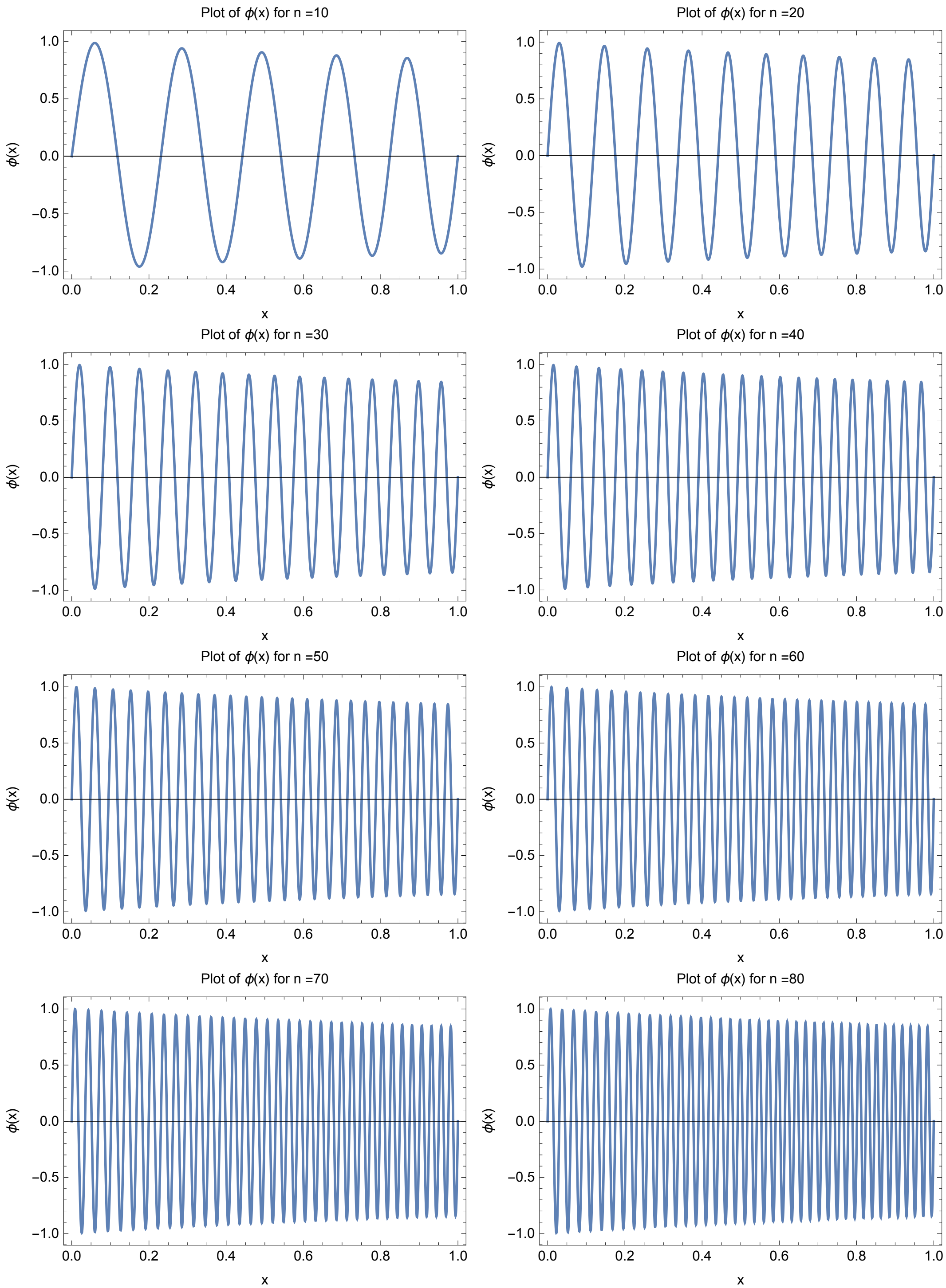
Therefore, solution for large λ is

$$\begin{aligned} \phi(x) &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}} dt\right) \\ &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}} dt\right) \\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{1+t} dt\right) \\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right) \end{aligned}$$

To plot this, let us assume $c_2 = 1$ (we have no information given to find c_2). What value of n to use? Will use different values of n in increasing order. So the following is plot of

$$\phi(x) = (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right)$$

For $x = 0 \cdots 1$ and for $n = 10, 20, 30, \cdots, 80$.


Plot of ϕ(x) for n =10
Plot of ϕ(x) for n =20
Plot of ϕ(x) for n =30
Plot of ϕ(x) for n =40
Plot of ϕ(x) for n =50
Plot of ϕ(x) for n =60
Plot of ϕ(x) for n =70
Plot of ϕ(x) for n =80
ϕ(x)
x

2.9 HW 8

2.9.1 Problem 5.10.1

5.10.1. Consider the Fourier sine series for $f(x) = 1$ on the interval $0 \leq x \leq L$. How many terms in the series should be kept so that the mean-square error is 1% of $\int_0^L f^2\sigma\, dx$?

The Fourier sin series of $f(x) = 1$ on $0 \leq x \leq L$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Where

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\left(\int_{-L}^{0}(-1)\sin\left(\frac{n\pi}{L}x\right)dx + \int_{0}^{L}(+1)\sin\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\left(-\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{-L}^{0} + \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{0}^{L}\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}x\right)\right]_{-L}^{0} - \frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}x\right)\right]_{0}^{L}\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[\cos(0) - \cos(n\pi)\right] - \frac{L}{n\pi}\left[\cos(n\pi) - \cos(0)\right]\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[1 - \cos(n\pi)\right] - \frac{L}{n\pi}\left[\cos(n\pi) - 1\right]\right)
\end{aligned}$$

We see that $b_n = 0$ for $n = 2,4,6,\cdots$, and b_n odd for $n = 1,3,5,\cdots$ so we can simplify the above to be

$$\begin{aligned}
b_n &= \frac{1}{L}\left(\frac{L}{n\pi}\left[1-(-1)\right] - \frac{L}{n\pi}\left[-1-1\right]\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}[2] - \frac{L}{n\pi}[-2]\right) \\
&= \frac{1}{L}\left(\frac{4L}{n\pi}\right) \\
&= \frac{4}{n\pi}
\end{aligned}$$

Equation (1) becomes

$$f(x) \sim \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{L}x\right) \tag{2}$$

mean-square error is, from textbook, page 213, is given by equation 5.10.11

$$E = \int_0^L f^2(x)\,\sigma(x)\,dx - \sum_{n=1,3,5,\cdots}^{\infty} \alpha_n^2 \int_0^L \phi_n^2 \sigma(x)\,dx \tag{5.10.11}$$

In this problem, $\phi_n = \sin\left(\frac{n\pi}{L}x\right)$ and $\alpha_n = a_n = \frac{4}{n\pi}$. The above equation becomes

$$\begin{aligned} E &= \int_0^L f^2(x)\,\sigma(x)\,dx - \sum_{n=1,3,5,\cdots}^{\infty} \left(\frac{4}{n\pi}\right)^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma(x)\,dx \\ &= \int_0^L f^2(x)\,\sigma dx - \sum_{n=1,3,5,\cdots}^{\infty} \frac{16}{n^2\pi^2} \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma dx \end{aligned}$$

For $\sigma = 1$ we know that

$$\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma dx = \frac{L}{2}$$

Hence E becomes

$$E = \int_0^L f^2(x)\,\sigma dx - \sum_{n=1,3,5,\cdots}^{\infty} \frac{16}{n^2\pi^2}\frac{L}{2}$$

But $\int_0^L f^2(x)\,\sigma dx$ for $\sigma = 1$ is just $\int_0^L 1^2 dx = L$, and the above becomes

$$\begin{aligned} E &= L - \frac{L}{2}\frac{16}{\pi^2} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2} \\ &= L - \frac{8L}{\pi^2} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2} \end{aligned}$$

We need to find N so that $E = 0.01L$. The above becomes

$$0.01L = L - \frac{8L}{\pi^2} \sum_{n=1,3,5,\cdots}^{N} \frac{1}{n^2}$$

We need now to solve for N in the above

$$\begin{aligned} 0.01L - L &= -\frac{8L}{\pi^2} \sum_{n=1,3,5,\cdots}^{N} \frac{1}{n^2} \\ 0.99L\left(\frac{\pi^2}{8L}\right) &= \sum_{n=1,3,5,\cdots}^{N} \frac{1}{n^2} \\ 1.2214 &= \sum_{n=1,3,5,\cdots}^{N} \frac{1}{n^2} \end{aligned}$$

A small Mathematica program written which prints the RHS sum for each n, and was visually checked when it reached 1.2214, here is the result

```
In[53]:= data = Table[{i, Sum[1/n^2, {n, 1, i, 2}]}, {i, 1, 50, 2}] // N;
Grid[Join[{{"n", "sum"}}, data], Frame → All]
```

Out[54]=

n	sum
1.	1.
3.	1.11111
5.	1.15111
7.	1.17152
9.	1.18386
11.	1.19213
13.	1.19805
15.	1.20249
17.	1.20595
19.	1.20872
21.	1.21099
23.	1.21288
25.	1.21448
27.	1.21585
29.	1.21704
31.	1.21808
33.	1.219
35.	1.21982
37.	1.22055
39.	1.2212
41.	1.2218
43.	1.22234
45.	1.22283
47.	1.22329
49.	1.2237

Counting the number of terms needed to reach 1.2214, we see there are 21 terms (21 rows in the table, since only odd entries are counted, the table above skips the even n values in the sum since these are all zero).

2.9.2 Problem 5.10.2 (b)

5.10.2. Obtain a formula for an infinite series using Parseval's equality applied to the

(a) Fourier sine series of $f(x) = 1$ on the interval $0 \le x \le L$

*(b) Fourier cosine series of $f(x) = x$ on the interval $0 \le x \le L$

(c) Fourier sine series of $f(x) = x$ on the interval $0 \le x \le L$

Parseval's equality is given by equation 5.10.14, page 214 in textbook

$$\int_a^b f^2 \sigma dx = \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx \tag{5.10.14}$$

The books uses α_n instead of a_n, but it is the same, these are the coefficients in the Fourier series for $f(x)$. We now need to find the cosine Fourier series for $f(x) = x$. This is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\left[\int_{-L}^{0}(-x)\cos\left(\frac{n\pi}{L}x\right)dx + \int_0^L (+x)\cos\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{1}{L}\left[-\int_{-L}^{0} x\cos\left(\frac{n\pi}{L}x\right)dx + \int_0^L x\cos\left(\frac{n\pi}{L}x\right)dx\right]
\end{aligned}$$

But

$$\begin{aligned}
\int_{-L}^{0} x\cos\left(\frac{n\pi}{L}x\right)dx &= -\frac{-1+(-1)^n}{n^2\pi^2}L^2 \\
\int_0^L x\cos\left(\frac{n\pi}{L}x\right)dx &= \frac{-1+(-1)^n}{n^2\pi^2}L^2
\end{aligned}$$

Hence

$$\begin{aligned}
a_n &= \frac{1}{L}\left[2\frac{-1+(-1)^n}{n^2\pi^2}L^2\right] \\
&= \frac{2L}{\pi^2}\frac{\left(-1+(-1)^n\right)}{n^2}
\end{aligned}$$

Looking at few terms to see the pattern

$$a_n = \frac{2L}{\pi^2}\left\{\frac{-2}{1}, 0, \frac{-2}{3^2}, 0, \frac{-2}{5^2}, \cdots\right\}$$

Therefore, we can write a_n as

$$a_n = \frac{-4L}{\pi^2 n^2} \qquad n = 1, 3, 5, \cdots$$

And

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\\
&= \frac{1}{2L}\left[\int_{-L}^{0}(-x)\,dx + \int_{0}^{L}(+x)\,dx\right]\\
&= \frac{1}{2L}\left[-\left[\frac{x^2}{2}\right]_{-L}^{0} + \left[\frac{x^2}{2}\right]_{0}^{L}\right]\\
&= \frac{1}{2L}\left[-\left[0-\frac{L^2}{2}\right] + \left[\frac{L^2}{2}-0\right]\right]\\
&= \frac{1}{2L}\left[\frac{L^2}{2}+\frac{L^2}{2}\right]\\
&= \frac{L}{2}
\end{aligned}
$$

Hence the Fourier series is

$$
f(x) = \frac{L}{2} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{-4L}{\pi^2 n^2}\cos\left(\frac{n\pi}{L}x\right)
$$

We now go back to equation 5.10.14 (but need to add a_0 to it, since there is this extra term with cosine Fourier series)

$$
\begin{aligned}
\int_a^b f^2\sigma dx &= a_0^2\int_a^b 1^2 dx + \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2\sigma dx\\
\int_0^L x^2 dx &= \left(\frac{L}{2}\right)^2\int_0^L dx + \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{-4L}{\pi^2 n^2}\right)^2\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx\\
\left[\frac{x^3}{3}\right]_0^L &= \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{16L^2}{\pi^4 n^4}\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx
\end{aligned}
$$

Since $\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}$ the above becomes

$$
\begin{aligned}
\frac{L^3}{3} &= \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{16L^2}{\pi^4 n^4}\frac{L}{2}\\
\frac{L^3}{3} &= \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{8L^3}{\pi^4 n^4}\\
\frac{L^3}{3} &= \frac{L^3}{4} + \frac{8L^3}{\pi^4}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4}
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
\frac{1}{3} &= \frac{1}{4} + \frac{8}{\pi^4} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^4} \\
\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^4} &= \left(\frac{1}{3} - \frac{1}{4}\right)\frac{\pi^4}{8} \\
\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{96}
\end{aligned}
$$

Hence

$$
\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots
$$

Which agrees with the book solution given in back of book.

2.9.3 Problem 5.10.6

5.10.6. Assuming that the operations of summation and integration can be interchanged, show that if

$$
f = \sum \alpha_n \phi_n \quad \text{and} \quad g = \sum \beta_n \phi_n,
$$

then for normalized eigenfunctions

$$
\int_a^b f g \sigma \, dx = \sum_{n=1}^{\infty} \alpha_n \beta_n,
$$

a generalization of Parseval's equality.

$$
\begin{aligned}
\int_a^b f g \sigma dx &= \int_a^b \left(\sum_{n=1}^{\infty} \alpha_n \phi_n\right)\left(\sum_{n=1}^{\infty} \beta_n \phi_n\right) \sigma dx \\
&= \int_a^b \left(\alpha_1\phi_1 + \alpha_2\phi_2 + \cdots\right)\left(\beta_1\phi_1 + \beta_2\phi_2 + \cdots\right) \sigma dx \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\left(\alpha_1\phi_1 + \alpha_2\phi_2 + \cdots\right)\left(\beta_1\phi_1 + \beta_2\phi_2 + \cdots\right) &= \alpha_1\beta_1\phi_1^2 + \alpha_1\beta_2\phi_1\phi_2 + \alpha_1\beta_3\phi_1\phi_3 + \cdots \\
&\quad + \alpha_2\beta_1\phi_2\phi_1 + \alpha_2\beta_2\phi_2^2 + \alpha_2\beta_3\phi_1\phi_3 + \cdots \\
&\quad + \alpha_3\beta_1\phi_3\phi_1 + \alpha_3\beta_2\phi_3\phi_2 + \alpha_3\beta_3\phi_3^2 + \cdots \\
&\quad \vdots
\end{aligned}
$$

Which means when expanding the product of the two series, only the terms on the diagonal (the terms with $\alpha_i\beta_j\phi_i\phi_j$ with $i = j$) will survive. This due to orthogonality. To show this more clearly, we put the above expansion back into the integral (1) and break up the integral into sum of integrals

$$\begin{aligned}\int_a^b fg\sigma dx = &\int_a^b \alpha_1\beta_1\phi_1^2\sigma dx + \int_a^b \alpha_1\beta_2\phi_1\phi_2\sigma dx + \int_a^b \alpha_1\beta_3\phi_1\phi_3\sigma dx + \cdots\\ &+ \int_a^b \alpha_2\beta_1\phi_2\phi_1\sigma dx + \int_a^b \alpha_2\beta_2\phi_2^2\sigma dx + \int_a^b \alpha_2\beta_3\phi_1\phi_3\sigma dx + \cdots\\ &+ \int_a^b \alpha_3\beta_1\phi_3\phi_1\sigma dx + \int_a^b \alpha_3\beta_2\phi_3\phi_2\sigma dx + \int_a^b \alpha_3\beta_3\phi_3^2\sigma dx + \cdots\\ &\vdots\end{aligned}$$

The above simplifies to

$$\int_a^b fg\sigma dx = \int_a^b \alpha_1\beta_1\phi_1^2\sigma dx + \int_a^b \alpha_2\beta_2\phi_2^2\sigma dx + \int_a^b \alpha_3\beta_3\phi_3^2\sigma dx + \cdots \int_a^b \alpha_n\beta_n\phi_n^2\sigma dx + \cdots$$

Since all other terms vanish due to orthogonality of eigenfunctions. The above simplifies to

$$\begin{aligned}\int_a^b fg\sigma dx &= \sum_{n=1}^{\infty}\int_a^b \alpha_n\beta_n\phi_n^2\sigma dx\\ &= \sum_{n=1}^{\infty}\left(\alpha_n\beta_n\int_a^b \phi_n^2\sigma dx\right)\end{aligned}$$

Because the eigenfunctions are normalized, then $\int_a^b \phi_n^2\sigma dx = 1$ and the above reduces to the result needed

$$\int_a^b fg\sigma\, dx = \sum_{n=1}^{\infty}\alpha_n\beta_n$$

2.9.4 Problem 7.3.4

7.3.4. **Consider the wave equation for a vibrating rectangular membrane ($0 < x < L,\ 0 < y < H$)**

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial conditions

$$u(x,y,0) = 0 \text{ and } \frac{\partial u}{\partial t}(x,y,0) = f(x,y).$$

Solve the initial value problem if

(a) $u(0,y,t) = 0,\quad u(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$

* (b) $\frac{\partial u}{\partial x}(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$

2.9.4.1 part(a)

Let $u = X(x)Y(y)T(t)$. Substituting this back into the PDE gives

$$T''XY = c^2 (X''YT + Y''XT)$$

Dividing by $XYT \neq 0$ gives

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Since left side depends on t only and right side depends on (x,y) only, then both must be equal to some constant, say $-\lambda$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

We obtain the following

$$\begin{aligned} T'' + c^2\lambda T &= 0 \\ \frac{X''}{X} &= -\lambda - \frac{Y''}{Y} \end{aligned}$$

Again, looking at the second ODE above, we see that the left side depends on x only, and the right side on y only. Then they must be equal to some constant, say $-\mu$ and we obtain

$$\frac{X''}{X} = \left(-\lambda - \frac{Y''}{Y}\right) = -\mu$$

Which results in two ODE's. The first is

$$\begin{aligned} X'' + \mu X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

And the second is

$$\begin{aligned} \lambda + \frac{Y''}{Y} &= \mu \\ Y'' &= Y\mu - \lambda Y \\ Y'' + Y(\lambda - \mu) &= 0 \end{aligned}$$

With B.C.

$$\begin{aligned} Y'(0) &= 0 \\ Y'(H) &= 0 \end{aligned}$$

Starting with the X ODE since it is simpler, the solution is

$$X = c_1 \cos\left(\sqrt{\mu}x\right) + c_2 \sin\left(\sqrt{\mu}x\right)$$

Applying $X(0) = 0$ gives

$$0 = c_1$$

Hence solution is

$$X = c_2 \sin\left(\sqrt{\mu}x\right)$$

Applying $X(L) = 0$ gives

$$0 = c_2 \sin\left(\sqrt{\mu} L\right)$$

For non-trivial solution

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

And the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

We now solve the Y ODE.

$$Y'' + \left(\lambda - \mu_n\right) Y = 0$$

Assuming that $\left(\lambda - \mu\right) > 0$ for all λ, μ, (we know this is the only case, since only positive $\lambda - \mu_n$ will be possible when B.C. are homogeneous Dirichlet). Then, for $\left(\lambda - \mu\right) > 0$, the solution is

$$\begin{aligned}
Y\left(y\right) &= c_1 \cos\left(\sqrt{\lambda - \mu_n} y\right) + c_2 \sin\left(\sqrt{\lambda - \mu_n} y\right) \\
Y' &= -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} y\right) + c_2 \sqrt{\lambda - \mu_n} \cos\left(\sqrt{\lambda - \mu_n} y\right)
\end{aligned}$$

Applying B.C. $Y'(0) = 0$ the above becomes

$$0 = c_2 \sqrt{\lambda - \mu_n}$$

Hence $c_2 = 0$ and the solution becomes

$$\begin{aligned}
Y &= c_1 \cos\left(\sqrt{\lambda - \mu_n} y\right) \\
Y' &= -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} y\right)
\end{aligned}$$

Applying second B.C. $Y'(H) = 0$ gives

$$0 = -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} H\right)$$

For non-trivial solution we want

$$\begin{aligned}
\sin\left(\sqrt{\lambda - \mu_n} H\right) &= 0 \\
\sqrt{\lambda_{nm} - \mu_n} &= m\frac{\pi}{H} \\
\lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \\
\lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2 + \mu_n \qquad m = 0, 1, 2, \cdots
\end{aligned}$$

Hence the eigenfunctions are

$$Y_{nm} = \cos\left(m\frac{\pi}{H} y\right) \qquad m = 0, 1, 2, \cdots, n = 1, 2, 3, \cdots$$

For each n, m, we find solution of $T'' + c^2 \lambda_{nm} T = 0$.The solution is

$$T_{nm}(t) = A_{nm} \cos\left(c\sqrt{\lambda_{nm}} t\right) + B_{nm} \sin\left(c\sqrt{\lambda_{nm}} t\right)$$

Putting all these results together gives

$$
\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}\left(t\right) X_{nm}\left(x\right) Y_{nm}\left(y\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}\cos\left(c\sqrt{\lambda_{n,m}}t\right)+B_{nm}\sin\left(c\sqrt{\lambda_{n,m}}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}
$$

We now apply initial conditions to find A_{nm}, B_{nm}. At $t=0$

$$
\begin{aligned}
u\left(x,y,0\right) &= 0\\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}
$$

Hence

$$
A_{nm} = 0
$$

And the solution becomes

$$
u\left(x,y,t\right) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

Taking derivative of the solution w.r.t. time t gives

$$
\frac{\partial}{\partial t}u\left(x,y,t\right) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

At $t=0$ the above becomes

$$
\alpha\left(x,y\right) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

Multiplying both sides by $\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$ and integrating gives

$$
\begin{aligned}
\int_0^L\int_0^H \alpha\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\,dxdy &= c\sqrt{\lambda_{nm}}B_{nm}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\int_0^L\int_0^H \sin^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)\,dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\int_0^L\int_0^H \sin^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)\,dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right)
\end{aligned}
$$

Hence

$$
B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
$$

Summary of solution

$$
\begin{aligned}
X_n(x) &= \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots \\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots \\
Y_{nm}(y) &= \cos\left(m\frac{\pi}{H}y\right) \qquad m = 0,1,2,\cdots \\
\lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \qquad m = 0,1,2,\cdots, n = 1,2,3,\cdots \\
T_{nm}(t) &= B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \\
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right) \\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
\end{aligned}
$$

2.9.4.2 Part (b)

In this case we have

$$
\begin{aligned}
X'' + \mu X &= 0 \\
X'(0) &= 0 \\
X'(L) &= 0
\end{aligned}
$$

And the second spatial ODE is

$$
\begin{aligned}
\lambda + \frac{Y''}{Y} &= \mu \\
Y'' &= Y\mu - \lambda Y \\
Y'' + Y(\lambda - \mu) &= 0
\end{aligned}
$$

With B.C.

$$
\begin{aligned}
Y'(0) &= 0 \\
Y'(H) &= 0
\end{aligned}
$$

Starting with the X ODE. The solution is

$$
\begin{aligned}
X &= c_1\cos\left(\sqrt{\mu}x\right) + c_2\sin\left(\sqrt{\mu}x\right) \\
X' &= -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}x\right) + c_2\sqrt{\mu}\cos\left(\sqrt{\mu}x\right)
\end{aligned}
$$

First B.C. gives

$$
0 = c_2\sqrt{\mu}
$$

Hence $c_2 = 0$ and the solution becomes

$$
\begin{aligned}
X &= c_1\cos\left(\sqrt{\mu}x\right) \\
X' &= -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}x\right)
\end{aligned}
$$

Second B.C. gives

$$0 = -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}L\right)$$

Hence

$$\begin{aligned}\sqrt{\mu}L &= n\pi\\ \mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 0,1,2,\cdots\end{aligned}$$

Now for the Y solution. This is the same as part (a).

$$\begin{aligned}Y_{nm}\left(y\right) &= \cos\left(m\frac{\pi}{H}y\right)\\ \lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2\\ \lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2 + \mu_n\\ &= \left(m\frac{\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad m = 0,1,2,\cdots,n = 0,1,2,\cdots\end{aligned}$$

For each n,m, we find solution of $T'' + c^2\lambda_{nm}T = 0$.When $n = 0, m = 0$, $\lambda_{nm} = 0$ and the ODE becomes

$$T'' = 0$$

With solution

$$T = At + B$$

And total solution is

$$\begin{aligned}u\left(x,y,t\right) &= T_{nm}\left(t\right)X_{nm}\left(x\right)Y_{nm}\left(y\right)\\ &= T_{00}\left(t\right)X_{00}\left(x\right)Y_{00}\left(y\right)\\ &= \left(At + B\right)\end{aligned}$$

Since $X_{00}\left(x\right) = 1$ and $Y_{00}\left(y\right) = 1$. Applying initial conditions gives

$$u\left(x,y,0\right) = 0 = B$$

Therefore the solution is $u\left(x,y,t\right) = At$. Applying second initial conditions gives

$$A = \alpha\left(x,y\right)$$

Hence the time solution for $n = m = 0$ is

$$T_{00} = t\alpha\left(x,y\right)$$

For each n,m, other than $n = m = 0$, the time solution of $T'' + c^2\lambda_{nm}T = 0$ is

$$T_{nm}\left(t\right) = A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)$$

Putting all these results together, we obtain

$$
\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_{nm}\left(t\right) X_{nm}\left(x\right) Y_{nm}\left(y\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right]\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}
$$

The difference in part(b) from part(a), is that the space solutions eigenfunctions are now all cosine instead of cosine and sine. When the eigenfunction is $\cos$ the sum starts from zero. When eigenfunction is $\sin$ the sum starts from 1. Now initial conditions are applied as in part (a).

$$
u\left(x,y,0\right) = 0 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

Hence $A_{nm} = 0$. And the solution becomes

$$
u\left(x,y,t\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

Taking derivative of the solution w.r.t. time

$$
\frac{\partial}{\partial t}u\left(x,y,t\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

At $t = 0$ the above becomes

$$
\alpha\left(x,y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
$$

Multiplying both sides by $\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$ and integrating gives

$$
\begin{aligned}
\int_0^L\int_0^H \alpha\left(x,y\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy &= c\sqrt{\lambda_{nm}}B_{nm}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^L\int_0^H \cos^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right)
\end{aligned}
$$

Hence

$$
B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
$$

Summary of solution

$$
\begin{aligned}
X_n(x) &= \cos\left(\frac{n\pi}{L}x\right) \qquad n=0,1,2,\cdots\\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n=0,1,2,\cdots\\
Y_{nm}(y) &= \cos\left(m\frac{\pi}{H}y\right) \qquad m=0,1,2,\cdots\\
\lambda_{nm}-\mu_n &= \left(m\frac{\pi}{H}\right)^2 \qquad m=0,1,2,\cdots,n=0,1,2,\cdots\\
T_{nm}(t) &= \begin{cases} t\alpha(x,y) & n=m=0\\ B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) & \text{otherwise}\end{cases}\\
u(x,y,t) &= \begin{cases} t\alpha(x,y) & n=m=0\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right) & \text{otherwise}\end{cases}\\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
\end{aligned}
$$

2.9.4.3 Part (c)

Same problem, but using the following boundary conditions

$$
\begin{aligned}
u(0,y,t) &= 0\\
u(L,y,t) &= 0\\
u(x,0,t) &= 0\\
u(x,H,t) &= 0
\end{aligned}
$$

Since the boundary conditions are homogeneous Dirichlet then the $X(x)$ ODE solution is

$$
\begin{aligned}
X_n &= \sin\left(\frac{n\pi}{L}x\right)\\
\mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,3,\cdots
\end{aligned}
$$

And $Y(y)$ ODE solution is

$$
\begin{aligned}
Y_{nm}(y) &= \sin\left(\frac{m\pi}{H}y\right)\\
\lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2+\mu_n\\
&= \left(m\frac{\pi}{H}\right)^2+\left(\frac{n\pi}{L}\right)^2 \qquad m=1,2,3,\cdots,n=1,2,3,\cdots
\end{aligned}
$$

And the time solution is

$$
T_{nm}(t) = A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \qquad m=1,2,3,\cdots,n=1,2,3,\cdots
$$

Hence the total solution is

$$\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}\left(t\right) X_{nm}\left(x\right) Y_{nm}\left(y\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
\end{aligned}$$

At $t=0$

$$0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$$

Hence $A_{nm}=0$ and the solution becomes

$$u\left(x,y,t\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$$

Taking derivative

$$\frac{\partial}{\partial t}u\left(x,y,t\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}c\sqrt{\lambda_{nm}}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$$

At $t=0$

$$\alpha\left(x,y\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}c\sqrt{\lambda_{nm}}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$$

Therefore, using orthogonality in 2D, we find

$$B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)dxdy$$

Summary of solution

$$\begin{aligned}
X_n\left(x\right) &= \sin\left(\frac{n\pi}{L}x\right) \qquad n=1,2,3,\cdots \\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,3,\cdots \\
Y_{nm}\left(y\right) &= \cos\left(m\frac{\pi}{H}y\right) \qquad m=1,2,3,\cdots \\
\lambda_{nm}-\mu_n &= \left(m\frac{\pi}{H}\right)^2 \qquad m=1,2,3,\cdots,n=1,2,3,\cdots \\
T_{nm}\left(t\right) &= B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \\
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(m\frac{\pi}{H}y\right) \\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)dxdy
\end{aligned}$$

2.9.5 Problem 7.3.6

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$\begin{aligned} \frac{\partial}{\partial z} u(x, y, 0) &= 0 \\ u(x, y, H) &= f(x, y) \end{aligned}$$

and $u = 0$ on the "lateral" sides.

(a) Separate the z-variable in general.

*(b) Solve for $u(x, y, z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

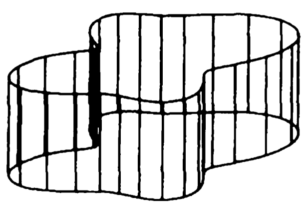


Figure 7.3.3

2.9.5.1 Part (a)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Let $u = XYZ$ where $X \equiv X(x), Y \equiv Y(y), Z \equiv Z(z)$. Substituting this back in the above gives

$$X''YZ + Y''XZ + Z''XY = 0$$

Dividing by $XYZ \neq 0$ gives

$$\begin{aligned} \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= 0 \\ \frac{X''}{X} + \frac{Y''}{Y} &= -\frac{Z''}{Z} \end{aligned}$$

Since the left side depends on x, y only and the right side depends on z only and they are equal, they must both be the same constant. Say $-\lambda$, and we write

$$\frac{X''}{X}+\frac{Y''}{Y}=-\lambda \tag{1}$$

$$\frac{Z''}{Z}=\lambda$$

The problem asks to separate the z variable, then the ODE for this variable is

$$Z''-\lambda Z=0 \tag{2}$$

With boundary conditions

$$\begin{aligned} Z'(0)&=0\\ Z(H)&=f\left(x,y\right)\end{aligned}$$

2.9.5.2 Part(b)

We will continue separation from part(a). From (1) in part (a)

$$\frac{X''}{X}+\frac{Y''}{Y}=-\lambda$$

We now need to separate X,Y. Therefore

$$\frac{X''}{X}=-\lambda-\frac{Y''}{Y}$$

As the left side depends on x only and right side depends on y only and both are equal, then they are equal to some constant, say $-\mu$

$$\begin{aligned}\frac{X''}{X}&=-\mu\\ -\lambda-\frac{Y''}{Y}&=-\mu\end{aligned}$$

The x ODE becomes

$$X''+\mu X=0 \tag{1}$$

$$\begin{aligned} X(0)&=0\\ X(L)&=0\end{aligned}$$

And the y ODE becomes

$$-\frac{Y''}{Y}=-\mu+\lambda$$

$$Y''+\left(\lambda-\mu\right)Y=0 \tag{2}$$

With B.C.

$$\begin{aligned} Y(0)&=0\\ Y(W)&=0\end{aligned}$$

Now that we have the three ODE's we start solving them. Starting with the x ODE (1). The solution is

$$
\begin{aligned}
X_n &= \sin\left(\frac{n\pi}{L}x\right)\\
\mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots
\end{aligned}
$$

For each n there is solution for the y ODE

$$
\begin{aligned}
Y_{nm} &= \sin\left(\frac{m\pi}{W}y\right)\\
\lambda_{nm} - \mu_n &= \left(\frac{m\pi}{W}\right)^2 \qquad m = 1,2,3,\cdots
\end{aligned}
$$

Or

$$
\lambda_{nm} = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots
$$

And for each n and for each m there is a solution for the z ODE we found in part (a), which is

$$
\begin{aligned}
Z'' - \lambda_{nm} Z &= 0\\
Z'(0) &= 0
\end{aligned}
$$

The solution is, since $\lambda_{nm} > 0$ is

$$
\begin{aligned}
Z &= c_1 \cosh\left(\sqrt{\lambda_{nm}}z\right) + c_2 \sinh\left(\sqrt{\lambda_{nm}}z\right)\\
Z' &= c_1\sqrt{\lambda_{nm}} \sinh\left(\sqrt{\lambda_{nm}}z\right) + c_2\sqrt{\lambda_{nm}} \cosh\left(\sqrt{\lambda_{nm}}z\right)
\end{aligned}
$$

Applying B.C. $Z'(0) = 0$ gives

$$
0 = c_2\sqrt{\lambda_{nm}}
$$

Hence $c_2 = 0$ and the solution becomes

$$
Z = c_{nm}\cosh\left(\sqrt{\lambda_{nm}}z\right)
$$

Putting all these solutions together, we obtain

$$
u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}z\right)
$$

Only now we apply the last boundary condition $u(x,y,H) = f(x,y)$ to find c_{nm}.

$$
f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}H\right)
$$

Applying 2D orthogonality gives

$$
\begin{aligned}
\int_0^L\int_0^W f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy &= c_{nm}\cosh\left(\sqrt{\lambda_{nm}}H\right)\int_0^L\int_0^W \sin^2\left(\frac{n\pi}{L}x\right)\sin^2\left(\frac{m\pi}{W}y\right)dxdy\\
&= c_{nm}\cosh\left(\sqrt{\lambda_{nm}}H\right)\left(\frac{L}{2}\right)\left(\frac{W}{2}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
c_{nm} &= \frac{\int_0^L \int_0^W f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy}{\cosh\left(\sqrt{\lambda_{nm}}H\right)\left(\frac{L}{2}\right)\left(\frac{W}{2}\right)} \\
&= \frac{4}{LW\cosh\left(\sqrt{\lambda_{nm}}H\right)}\int_0^L\int_0^W f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy
\end{aligned}
$$

Summary of solution

$$
\begin{aligned}
X_n &= \sin\left(\frac{n\pi}{L}x\right) \\
Y_{nm} &= \sin\left(\frac{m\pi}{W}y\right) \\
\lambda_{nm} &= \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots \\
u(x,y,z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}z\right) \\
c_{nm} &= \frac{4}{LW\cosh\left(\sqrt{\lambda_{nm}}H\right)}\int_0^L\int_0^W f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy
\end{aligned}
$$

2.9.6 Problem 7.4.2

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

Equation 7.4.7 is

$$
\begin{aligned}
\nabla^2\phi + \lambda\phi &= 0 \\
\phi(0,y) &= 0 \\
\phi(L,y) &= 0 \\
\phi(x,0) &= 0 \\
\phi(x,H) &= 0
\end{aligned}
$$

And 7.4.6 is

$$
\lambda = \frac{-\oint \phi\nabla\phi\cdot\hat{n}ds + \iint_R |\nabla\phi|^2 dxdy}{\iint_R \phi^2 dxdy}
$$

$\oint \phi\nabla\phi\cdot\hat{n}ds = 0$ as we are told $\phi = 0$ on the boundary and this integration is for the boundary only. Hence λ simplifies to

$$\lambda = \frac{\iint\limits_R \left|\nabla\phi\right|^2 dxdy}{\iint\limits_R \phi^2 dxdy}$$

The numerator can not be negative, since the integrand $\left|\nabla\phi\right|^2$ is not negative. Similarly, the denominator has positive integrand, because ϕ can not be identically zero, as it is an eigenfunction. Hence we conclude that $\lambda \geq 0$.

2.9.7 Problem 7.4.3

7.4.3. If necessary, see Sec. 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u)\cdot \hat{n}\, ds$.

(b) From part (a), derive (7.4.5).

2.9.7.1 part (a)

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v \tag{1}$$
$$\nabla\cdot(v\nabla u) = v\nabla^2 u + \nabla v\cdot\nabla u \tag{2}$$

Equation (1)-(2) leads to

$$\begin{aligned}\nabla\cdot(u\nabla v) - \nabla\cdot(v\nabla u) &= \left(u\nabla^2 v + \nabla u\cdot\nabla v\right) - \left(v\nabla^2 u + \nabla v\cdot\nabla u\right)\\ \nabla\cdot(u\nabla v - v\nabla u) &= u\nabla^2 v - v\nabla^2 u + \nabla u\cdot\nabla v - \nabla v\cdot\nabla u\end{aligned}$$

But $\nabla u\cdot\nabla v = \nabla v\cdot\nabla u$ so the above reduces to

$$\nabla\cdot(u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

Therefore

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \iint \nabla\cdot(u\nabla v - v\nabla u)\, dxdy \tag{3}$$

But the RHS of the above is of the form $\iint (\nabla\cdot A)\, dxdy$ where $A = (u\nabla v - v\nabla u)$ here. Which we can apply divergence theorem on it and obtain $\oint (A\cdot\hat{n})\, ds$. Therefore, using divergence theorem on the RHS of (3), then (3) can be written as

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \oint (u\nabla v - v\nabla u)\cdot\hat{n} ds$$

Which is what is required to show.

2.9.7.2 Part(b)

Equation 7.4.5 is

$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2} dxdy = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \tag{7.4.5}$$

From part (a), we found

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \oint \left(u\nabla v - v\nabla u\right)\cdot \hat{n} ds \tag{1}$$

But we know that, since both u, v satisfy the multidimensional eigenvalue problem on same domain, then

$$\nabla^2 v + \lambda_v v = 0 \tag{2}$$

$$\beta_1 v + \beta_2 \left(\nabla v \cdot \hat{n}\right) = 0 \tag{3}$$

And similarly

$$\nabla^2 u + \lambda_u u = 0 \tag{4}$$

$$\beta_1 u + \beta_2 \left(\nabla u \cdot \hat{n}\right) = 0 \tag{5}$$

Now we will use (2,3,4,5) into (1) to obtain 7.4.5. From (2), we see that $\nabla^2 v = -\lambda_v v$ and from (4) $\nabla^2 u = -\lambda_u u$ and from (3) $\nabla v \cdot \hat{n} = -\frac{\beta_1}{\beta_2} v$ and from (5) $\nabla u \cdot \hat{n} = -\frac{\beta_1}{\beta_2} u$. Substituting all of these back into (1) gives

$$\begin{aligned}
\iint \left(u\left(-\lambda_v v\right) - v\left(-\lambda_u u\right)\right)\ dxdy &= \oint u\left(\nabla v \cdot \hat{n}\right) - v\left(\nabla u \cdot \hat{n}\right) ds \\
\iint \left(-\lambda_v uv + \lambda_u vu\right)\ dxdy &= \oint u\left(-\frac{\beta_1}{\beta_2} v\right) - v\left(-\frac{\beta_1}{\beta_2} u\right) ds \\
\iint \left(\lambda_u - \lambda_v\right) uv\ dxdy &= \oint \frac{\beta_1}{\beta_2}\left[-uv + uv\right] ds \\
\left(\lambda_u - \lambda_v\right)\iint uv\ dxdy &= 0 \qquad (6)
\end{aligned}$$

We now use (6) the above to show that 7.4.5 is correct. In (6), if we replace $u = \phi_{\lambda_1}, v = \phi_{\lambda 1}$ and $\lambda_u = \lambda_1, \lambda_v = \lambda_2$ then (6) becomes

$$\left(\lambda_1 - \lambda_2\right)\iint \left(\phi_{\lambda_1}\phi_{\lambda 1}\right)\ dxdy = 0$$

We see now that for $\lambda_1 \neq \lambda_2$, then $\iint \left(\phi_{\lambda_1}\phi_{\lambda 1}\right)\ dxdy = 0$. Which is what we asked to show.

2.10 HW 9

2.10.1 Problem 8.2.1 (a,b)

8.2.1. **Solve the heat equation with time-independent sources and boundary conditions**

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x)\\ u(x,0) &= f(x)\end{aligned}$$

if an equilibrium solution exists. Analyze the limits as $t \to \infty$. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions (but do not solve). Assume

* (a)	$Q(x) = 0$,	$u(0,t) = A$,	$\frac{\partial u}{\partial x}(L,t) = B$
(b)	$Q(x) = 0$,	$\frac{\partial u}{\partial x}(0,t) = 0$,	$\frac{\partial u}{\partial x}(L,t) = B \neq 0$
(c)	$Q(x) = 0$,	$\frac{\partial u}{\partial x}(0,t) = A \neq 0$,	$\frac{\partial u}{\partial x}(L,t) = A$
* (d)	$Q(x) = k$,	$u(0,t) = A$,	$u(L,t) = B$
(e)	$Q(x) = k$,	$\frac{\partial u}{\partial x}(0,t) = 0$,	$\frac{\partial u}{\partial x}(L,t) = 0$
(f)	$Q(x) = \sin\frac{2\pi x}{L}$,	$\frac{\partial u}{\partial x}(0,t) = 0$,	$\frac{\partial u}{\partial x}(L,t) = 0$

2.10.1.1 Part (a)

Let

$$u(x,t) = v(x,t) + u_E(x) \tag{1}$$

Since $Q(x)$ in this problem is zero, we can look for $u_E(x)$ which is the steady state solution that satisfies the non-homogenous boundary conditions. (If Q was present, and if it also was time dependent, then we replace $u_E(x)$ by $r(x,t)$ which becomes a reference function that only needs to satisfy the non-homogenous boundary conditions and not the PDE itself at steady state. In (1) $v(x,t)$ satisfies the PDE itself but with homogenous boundary conditions. The first step is to find $u_E(x)$. We use the equilibrium solution in this case. At equilibrium $\frac{\partial u_E(x,t)}{\partial t} = 0$ and hence the solution is given $\frac{d^2 u_E}{\partial x^2} = 0$ or

$$\boxed{u_E(x) = c_1 x + c_2}$$

At $x = 0, u_E(x) = A$, Hence

$$c_2 = A$$

And solution becomes $u_E(x) = c_1 x + A$. at $x = L, \frac{\partial u_E(x)}{\partial x} = c_1 = B$, Therefore

$$\boxed{u_E(x) = Bx + A}$$

Now we plug-in (1) into the original PDE, this gives

$$\frac{\partial v(x,t)}{\partial t} = k\left(\frac{\partial^2 v(x,t)}{\partial x} + \frac{\partial^2 u_E(x)}{\partial x}\right)$$

But $\frac{\partial^2 u_E(x)}{\partial x} = 0$, hence we need to solve

$$\frac{\partial v(x,t)}{\partial t} = k\frac{\partial^2 v(x,t)}{\partial x}$$

for $v(x,t) = u(x,t) - u_E(x)$ with homogenous boundary conditions $v(0,t) = 0$, $\frac{\partial v(L,t)}{\partial t} = 0$ and initial conditions

$$\begin{aligned} v(x,0) &= u(x,0) - u_E(x) \\ &= f(x) - (Bx + A) \end{aligned}$$

This PDE we already solved before in earlier HW's and we know that it has the following solution

$$\begin{aligned} v(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t} \\ \lambda_n &= \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1,3,5,\cdots \end{aligned} \tag{2}$$

With b_n found from orthogonality using initial conditions $v(x,0) = f(x) - (Bx + A)$

$$\begin{aligned} v(x,0) &= \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin\left(\sqrt{\lambda_n}x\right) \\ \int_0^L \left(f(x) - (Bx + A)\right)\sin\left(\sqrt{\lambda_m}x\right)dx &= \int_0^L \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin\left(\sqrt{\lambda_n}x\right)\sin\left(\sqrt{\lambda_m}x\right)dx \\ \int_0^L \left(f(x) - (Bx + A)\right)\sin\left(\sqrt{\lambda_m}x\right)dx &= b_m\frac{L}{2} \end{aligned}$$

Hence

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - (Bx + A)\right)\sin\left(\sqrt{\lambda_n}x\right)dx \qquad n = 1,3,5,\cdots \tag{3}$$

Therefore, from (1) the solution is

$$u(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t} + \overbrace{Bx + A}^{u_E(x)}$$

With b_n given by (3) and eigenvalues λ_n given by (2).

2.10.1.2 Part (b)

Let

$$u(x,t) = v(x,t) + r(x) \tag{1}$$

Since $Q(x)$ in this problem is zero, we can look for $r(x)$, since unique equilibrium solution is not possible due to both boundary conditions being insulated. The idea is that, if we can find u_E then we use that, else we switch to reference function $r(x)$ which only needs to satisfy the non-homogenous boundary condition $\frac{\partial u_E(L)}{\partial x} = 0$ but does not have to satisfy equilibrium solution. Let

$$r(x) = c_1 x + c_2 x^2$$
$$\frac{\partial r}{\partial x} = c_1 + 2c_2 x$$

At $x = 0$, second equation above reduces to

$$0 = c_1$$

Hence $r(x) = c_2 x^2$. Now $\frac{\partial r}{\partial x} = 2c_2 x$. At $x = L$, this gives $2c_2 L = B$ or $c_2 = \frac{B}{2L}$, therefore

$$r(x) = \frac{B}{2L} x^2$$

The above satisfies the non-homogenous B.C. at the right, and also satisfies the homogenous B.C. at the left. Now we plug-in (1) into the original PDE, this gives

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &= k\left(\frac{\partial^2 v(x,t)}{\partial x} + \frac{\partial^2 u_E(x)}{\partial x}\right)\\
\frac{\partial v(x,t)}{\partial t} &= k\left(\frac{\partial^2 v(x,t)}{\partial x} + \frac{B}{L}\right)\\
&= k\frac{\partial^2 v(x,t)}{\partial x} + k\frac{B}{L}
\end{aligned}$$

Hence

$$\frac{\partial v(x,t)}{\partial t} = k\frac{\partial^2 v(x,t)}{\partial x} + \frac{kB}{L}$$

We now treat $k\frac{B}{L}$ as forcing function. So the above can be written as

$$\frac{\partial v(x,t)}{\partial t} = k\frac{\partial^2 v(x,t)}{\partial x} + Q \tag{2}$$

The above is now solved using eigenfunction expansion, since no steady state equilibrium solution exist. Let

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(x) \tag{3}$$

Where the index starts from zero, since there is a zero eigenvalue, due to B.C. being Neumann. $\phi_n(x)$ are the eigenfunctions of the corresponding homogenous PDE $\frac{\partial v(x,t)}{\partial t} = k\frac{\partial^2 v(x,t)}{\partial x}$ with homogenous BC $\frac{\partial v(0,t)}{\partial t} = 0, \frac{\partial v(L,t)}{\partial t} = 0$. This we solved before. The eigenfunctions are

$$\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$$

With eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 0,1,2,\cdots$$

Notice that $\lambda_0 = 0$. Substituting (3) into (2) gives

$$\sum_{n=0}^{\infty} a_n'(t)\,\phi_n(x) = \left(k\sum_{n=0}^{\infty} a_n(t)\,\frac{d^2\phi_n(x)}{dx^2}\right) + Q$$

Term by term differentiation is justified, since $v(x,t)$ and $\phi_n(x)$ both solve the same homogenous B.C. problem. Since $\frac{d^2\phi_n(x)}{dx^2} = -\lambda_n\phi_n(x)$ the above equation reduces to

$$\sum_{n=0}^{\infty} a_n'(t)\,\phi_n(x) = \left(-k\sum_{n=0}^{\infty} a_n(t)\,\lambda_n\phi_n(x)\right) + Q$$

Now we expand Q, which gives

$$\sum_{n=0}^{\infty} a_n'(t)\,\phi_n(x) = -k\sum_{n=0}^{\infty} a_n(t)\,\lambda_n\phi_n(x) + \sum_{n=0}^{\infty} q_n\phi_n(x)$$

By orthogonality

$$a_n'(t) + ka_n(t)\,\lambda_n = q_n$$

<u>case $n = 0$</u>

$$a_0'(t) + ka_0(t)\,\lambda_0 = q_0$$

But $\lambda_0 = 0$

$$a_0'(t) = q_0$$

But since $Q = \frac{kB}{L}$ is constant, then $\frac{kB}{L} = \sum_{n=0}^{\infty} q_n\phi_n(x)$ implies that $\frac{kB}{L} = q_0\phi_0(x)$. But $\phi_0(x) = 1$ for this problem. Hence $q_0 = \frac{kB}{L}$ and the ODE becomes

$$a_0'(t) = \frac{kB}{L}$$

Hence

$$a_0(t) = \frac{kB}{L}t + c_1$$

<u>case $n > 0$</u>

$$a_n'(t) + ka_n(t)\,\lambda_n = q_n$$

Since all $q_n = 0$ for $n > 0$ the above becomes

$$a_n'(t) + ka_n(t)\,\lambda_n = 0$$

Integrating factor is $\mu = e^{k\lambda_n t}$. Hence $\frac{d}{dt}\left(a_n(t)\,e^{k\lambda_n t}\right) = 0$ or

$$a_n(t) = c_2 e^{-k\lambda_n t}$$

Therefore the solution from (3) becomes

$$v(x,t) = \frac{kB}{L}t + c_1 + c_2\sum_{n=1}^{\infty} e^{-k\lambda_n t}\cos\left(\sqrt{\lambda_n}x\right) \tag{4}$$

Now we find the initial conditions on $v(x,t)$. Since $u(x,0)=v(x,0)+r(x)$ then

$$v(x,0)=f(x)-\frac{B}{2L}x^2$$

Hence equation (4) at $t=0$ becomes

$$f(x)-\frac{B}{2L}x^2=c_1+c_2\sum_{n=1}^{\infty}\cos\left(\sqrt{\lambda_n}x\right)$$

We now find c_1,c_2 by orthogonality.

case $n=0$

$$\int_0^L\left(f(x)-\frac{B}{2L}x^2\right)\cos\left(\sqrt{\lambda_0}x\right)dx=\int_0^L c_1\cos\left(\sqrt{\lambda_0}x\right)dx$$

But $\lambda_0=0$

$$\begin{aligned}\int_0^L\left(f(x)-\frac{B}{2L}x^2\right)dx&=\int_0^L c_1dx\\ \int_0^L\left(f(x)-\frac{B}{2L}x^2\right)dx&=c_1L\\ c_1&=\frac{1}{L}\int_0^L\left(f(x)-\frac{B}{2L}x^2\right)dx\end{aligned}$$

case $n>0$

$$\begin{aligned}\int_0^L\left(f(x)-\frac{B}{2L}x^2\right)\cos\left(\sqrt{\lambda_m}x\right)dx&=\int_0^L c_2\sum_{n=1}^{\infty}\cos\left(\sqrt{\lambda_n}x\right)\cos\left(\sqrt{\lambda_m}x\right)dx\\ &=c_2\frac{L}{2}\\ c_2&=\frac{2}{L}\int_0^L\left(f(x)-\frac{B}{2L}x^2\right)\cos\left(\sqrt{\lambda_n}x\right)dx\end{aligned}$$

Therefore the solution for $v(x,t)$ is now complete from (4). Hence

$$\begin{aligned}u(x,t)&=v(x,t)+r(x)\\ &=\frac{kB}{L}t+c_1+\left(c_2\sum_{n=1}^{\infty}e^{-k\lambda_nt}\cos\left(\sqrt{\lambda_n}x\right)\right)+\frac{B}{2L}x^2\end{aligned}$$

Where c_1,c_2 are given by above result. This completes the solution.

2.10.2 Problem 8.2.2 (a,d)

8.2.2. Consider the heat equation with time-dependent sources and boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(x,0) &= f(x).\end{aligned}$$

Reduce the problem to one with homogeneous boundary conditions if

* (a)	$\frac{\partial u}{\partial x}(0,t) = A(t)$	and	$\frac{\partial u}{\partial x}(L,t) = B(t)$
(b)	$u(0,t) = A(t)$	and	$\frac{\partial u}{\partial x}(L,t) = B(t)$
* (c)	$\frac{\partial u}{\partial x}(0,t) = A(t)$	and	$u(L,t) = B(t)$
(d)	$u(0,t) = 0$	and	$\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$
(e)	$\frac{\partial u}{\partial x}(0,t) = 0$	and	$\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$

2.10.2.1 Part (a)

Let

$$u(x,t) = v(x,t) + r(x,t) \tag{1}$$

Since the problem has time dependent source function $Q(x,t)$ then $r(x,t)$ is now a reference function that only needs to satisfy the non-homogenous boundary conditions which in this problem are at both ends and $v(x,t)$ has homogenous boundary conditions. The first step is to find $r(x,t)$. Let

$$r(x,t) = c_1(t)\,x + c_2(t)\,x^2$$

Then

$$\frac{\partial r(x,t)}{\partial x} = c_1(t) + 2c_2(t)\,x$$

At $x = 0$

$$A(t) = c_1(t)$$

And at $x = L$

$$\begin{aligned}B(t) &= c_1(t) + 2c_2(t)\,L\\ c_2(t) &= \frac{B(t) - c_1(t)}{2L}\end{aligned}$$

Solving for c_1, c_2 gives

$$r(x,t) = A(t)\,x + \left(\frac{B(t) - A(t)}{2L}\right)x^2 \tag{2}$$

Replacing (1) into the original PDE $u_t = ku_{xx} + Q(x,t)$ gives

$$\frac{\partial}{\partial t}(v(x,t) - r(x,t)) = k\frac{\partial^2}{\partial x}(v(x,t) - r(x,t)) + Q(x,t)$$
$$\frac{\partial v}{\partial t} - \frac{\partial r}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - k\frac{\partial^2 r}{\partial x^2} + Q(x,t)$$

But $\frac{\partial^2 r}{\partial x^2} = \frac{B(t)-A(t)}{L}$, hence the above reduces to

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - k\frac{B(t) - A(t)}{L} + \frac{\partial r}{\partial t} \tag{3}$$

Let

$$\tilde{Q}(x,t) = Q(x,t) + \frac{\partial r}{\partial t} - k\frac{B(t) - A(t)}{L}$$

then (3) becomes

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \tilde{Q}(x,t)$$

The above PDE now has homogenous boundary conditions

$$v_t(0,t) = 0$$
$$v_t(L,t) = 0$$

And initial condition is

$$\begin{aligned} v(x,0) &= u(x,0) - r(x,0) \\ &= f(x) - \left(A(0)x + \left(\frac{B(0) - A(0)}{2L}\right)x^2\right) \end{aligned}$$

The problem does not ask us to solve it. So will stop here.

2.10.2.2 Part (d)

Let

$$u(x,t) = v(x,t) + r(x,t) \tag{1}$$

Since the problem has time dependent source function $Q(x,t)$ then $r(x,t)$ is now a reference function that only needs to satisfy the non-homogenous boundary conditions which in this problem are at both ends and $v(x,t)$ has homogenous boundary conditions. The boundary condition $r(x,t)$ need to satisfy is

$$\begin{aligned} \frac{\partial r}{\partial x}(L,t) + hr(L,t) - hB(t) &= 0 \\ r(0,t) &= 0 \end{aligned} \tag{2}$$

Let

$$r(x,t) = c_1(t)x + c_2(t)$$

Since $r(0,t)=0$ then $c_2=0$. Now we use the right side non-homogenous B.C. to solve for c_1. Plugging the above into the right side B.C. gives

$$\begin{aligned} c_1 + hc_1 L - hB(t) &= 0 \\ c_1 &= \frac{hB(t)}{1+hL} \end{aligned}$$

Hence

$$\boxed{r(x,t) = \frac{hB(t)}{1+hL}x} \tag{3}$$

The rest is very similar to what we did in part (a). Replacing (1) into the original PDE $\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x} + Q(x,t)$ gives

$$\begin{aligned} \frac{\partial}{\partial t}\left(v(x,t) - r(x,t)\right) &= k\frac{\partial^2}{\partial x}\left(v(x,t) - r(x,t)\right) + Q(x,t) \\ \frac{\partial v}{\partial t} - \frac{\partial r}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} - k\frac{\partial^2 r}{\partial x^2} + Q(x,t) \end{aligned}$$

But $\frac{\partial^2 r}{\partial x^2} = 0$ hence the above reduces to

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) + \frac{\partial r}{\partial t} \tag{4}$$

Let

$$\tilde{Q}(x,t) = Q(x,t) + \frac{\partial r}{\partial t}$$

Then (4) becomes

$$\boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \tilde{Q}(x,t)}$$

The above PDE now has homogeneous boundary conditions

$$\begin{aligned} v(0,t) &= 0 \\ \frac{\partial v(L,t)}{\partial t} &= 0 \end{aligned}$$

And initial condition is

$$\begin{aligned} v(x,0) &= u(x,0) - r(x,0) \\ &= f(x) - \frac{hB(0)}{1+hL}x \end{aligned}$$

The problem does not ask us to solve it. So will stop here.

2.10.3 Problem 8.2.5

Analyze the limit as $t \to \infty$.

8.2.5. Solve the initial value problem for a two-dimensional heat equation inside a circle (of radius a) with time-independent boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\nabla^2 u\\ u(a,\theta,t) &= g(\theta)\\ u(r,\theta,0) &= f(r,\theta).\end{aligned}$$

$$\begin{aligned}\frac{\partial u(r,\theta,t)}{\partial t} &= k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)\\ |u(0,\theta,t)| &< \infty\\ u(a,\theta,t) &= g(\theta)\\ u(r,-\pi,t) &= u(r,\pi,t)\\ \frac{\partial u}{\partial \theta}(r,-\pi,t) &= \frac{\partial u}{\partial \theta}(r,\pi,t)\end{aligned}$$

With initial conditions $u(r,\theta,0)=f(r,\theta)$. Since the boundary conditions are not homogenous, and since there are no time dependent sources, then in this case we look for $u_E(r,\theta)$ which is solution at steady state which needs to satisfy the nonhomogeneous B.C., where $u(r,\theta,t)=v(r,\theta,t)+u_E(r,\theta)$ and $v(r,\theta,t)$ solves the PDE but with homogenous B.C. Therefore, we need to find equilibrium solution for Laplace PDE on disk, that only needs to satisfy the nonhomogeneous B.C.

$$\begin{aligned}\nabla^2 u_E &= 0\\ \frac{\partial^2 u_E}{\partial r^2}+\frac{1}{r}\frac{\partial u_E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_E}{\partial \theta^2} &= 0\end{aligned}$$

With boundary condition

$$\begin{aligned}|u_E(0,\theta)| &< \theta\\ u_E(a,\theta) &= g(\theta)\\ u_E(r,-\pi) &= u_E(r,\pi)\\ \frac{\partial u_E}{\partial \theta}(r,-\pi) &= \frac{\partial u_E}{\partial \theta}(r,\pi)\end{aligned}$$

But this PDE we have already solved before. But to practice, will solve it again. Let

$$\boxed{u_E(r,\theta)=R(r)\Theta(\theta)}$$

Where $R(r)$ is the solution in radial dimension and $\Theta(\theta)$ is solution in angular dimension. Substituting $u_E(r,\theta)$ in the PDE gives

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}\Theta''R=0$$

Dividing by $R(r)\Phi(\theta)$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=0$$
$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}$$

Hence each side is equal to constant, say λ and we obtain

$$r^2\frac{R''}{R}+r\frac{R'}{R}=\lambda$$
$$-\frac{\Theta''}{\Theta}=\lambda$$

Or

$$r^2R''+rR'-\lambda R=0 \tag{1}$$
$$\Theta''+\lambda\Theta=0 \tag{2}$$

We start with Φ ODE. The boundary conditions on (3) are

$$\Theta(-\pi)=\Theta(\pi)$$
$$\frac{\partial\Theta}{\partial\theta}(-\pi)=\frac{\partial\Theta}{\partial\theta}(\pi)$$

case $\lambda=0$ The solution is $\Phi=c_1\theta+c_2$. Hence we obtain, from first initial conditions

$$-\pi c_1+c_2=\pi c_1+c_2$$
$$c_1=0$$

Second boundary conditions just says that $c_2=c_2$, so any constant will do. Hence $\lambda=0$ is an eigenvalue with constant being eigenfunction.

case $\lambda>0$ The solution is

$$\Theta(\theta)=c_1\cos\sqrt{\lambda}\theta+c_2\sin\sqrt{\lambda}\theta$$

The first boundary conditions gives

$$c_1\cos\left(-\sqrt{\lambda}\pi\right)+c_2\sin\left(-\sqrt{\lambda}\pi\right)=c_1\cos\left(\sqrt{\lambda}\pi\right)+c_2\sin\left(\sqrt{\lambda}\pi\right)$$
$$c_1\cos\left(\sqrt{\lambda}\pi\right)-c_2\sin\left(\sqrt{\lambda}\pi\right)=c_1\cos\left(\sqrt{\lambda}\pi\right)+c_2\sin\left(\sqrt{\lambda}\pi\right)$$
$$2c_2\sin\left(\sqrt{\lambda}\pi\right)=0 \tag{3}$$

From second boundary conditions we obtain

$$\Theta'(\theta)=-\sqrt{\lambda}c_1\sin\sqrt{\lambda}\theta+c_2\sqrt{\lambda}\cos\sqrt{\lambda}\theta$$

Therefore

$$\begin{aligned}
-\sqrt{\lambda}c_1\sin\left(-\sqrt{\lambda}\pi\right)+c_2\sqrt{\lambda}\cos\left(-\sqrt{\lambda}\pi\right) &= -\sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}\pi\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\
\sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}\pi\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -\sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}\pi\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\
\sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}\pi\right) &= -\sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}\pi\right)\\
2c_1\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (4)
\end{aligned}$$

Both (3) and (4) are satisfied if

$$\begin{aligned}
\sqrt{\lambda}\pi &= n\pi \qquad n=1,2,3,\cdots\\
\lambda &= n^2 \qquad n=1,2,3,\cdots
\end{aligned}$$

Therefore

$$\Theta_n(\theta)=\overbrace{\tilde{A}_0}^{\lambda=0}+\sum_{n=1}^{\infty}\tilde{A}_n\cos(n\theta)+\tilde{B}_n\sin(n\theta) \tag{5}$$

I put tilde on top of these constants, so not confuse them with constants used for $v(r,\theta,t)$ found later below. Now we go back to the R ODE (2) given by $r^2R''+rR'-\lambda_nR=0$ and solve it. This is Euler PDE whose solution is found by substituting $R(r)=r^{\alpha}$. The solution comes out to be (Lecture 9)

$$R_n(r)=c_0+\sum_{n=1}^{\infty}c_nr^n \tag{6}$$

Combining (5,6) we now find u_E as

$$\begin{aligned}
u_{E_n}(r,\theta) &= R_n(r)\Theta_n(\theta)\\
u_E(r,\theta) &= \tilde{A}_0+\sum_{n=1}^{\infty}\tilde{A}_n\cos(n\theta)r^n+\tilde{B}_n\sin(n\theta)r^n\\
&= \sum_{n=0}^{\infty}\tilde{A}_n\cos(n\theta)r^n+\sum_{n=1}^{\infty}\tilde{B}_n\sin(n\theta)r^n \qquad (7)
\end{aligned}$$

Where c_0 was combined with A_0. Now the above equilibrium solution needs to satisfy the non-homogenous B.C. $u_E(a,\theta)=g(\theta)$. Using orthogonality on (7) to find A_n,B_n gives

$$\begin{aligned}
g(\theta) &= \sum_{n=0}^{\infty}\tilde{A}_n\cos(n\theta)a^n+\sum_{n=1}^{\infty}\tilde{B}_n\sin(n\theta)a^n\\
\int_0^{2\pi}g(\theta)\cos(n'\theta)d\theta &= \int_0^{2\pi}\sum_{n=0}^{\infty}\tilde{A}_n\cos(n\theta)\cos(n'\theta)a^nd\theta+\int_0^{2\pi}\sum_{n=1}^{\infty}\tilde{B}_n\sin(n\theta)\cos(n'\theta)a^nd\theta\\
&= \sum_{n=0}^{\infty}\int_0^{2\pi}\tilde{A}_n\cos(n\theta)\cos(n'\theta)a^nd\theta+\sum_{n=0}^{\infty}\overbrace{\int_0^{2\pi}\tilde{B}_n\sin(n\theta)\cos(n'\theta)a^nd\theta}^{0}\\
&= \tilde{A}_{n'}\int_0^{2\pi}\cos^2(n'\theta)a^nd\theta
\end{aligned}$$

For $n = 0$

$$\int_0^{2\pi} g(\theta)\,d\theta = \tilde{A}_0 \int_0^{2\pi} d\theta$$
$$\tilde{A}_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta$$

For $n > 0$

$$\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta = \tilde{A}_n \int_0^{2\pi} \cos^2(n\theta)\,a^n d\theta$$
$$\tilde{A}_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta$$

Similarly, we apply orthogonality to find $\tilde{B}_n$ which gives (for $n > 0$ only)

$$\tilde{B}_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,d\theta$$

Therefore, we have found $u_E(r,\theta)$ completely now. It is given by

$$u_E(r,\theta) = \tilde{A}_0 + \sum_{n=1}^{\infty} \tilde{A}_n \cos(n\theta)\,r^n + \tilde{B}_n \sin(n\theta)\,r^n$$
$$\tilde{A}_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta$$
$$\tilde{A}_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta$$
$$\tilde{B}_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,d\theta$$

The above satisfies the non-homogenous B.C. $u_E(a,\theta) = g(\theta)$. Now, since $u(r,\theta,t) = v(r,\theta,t) + u_E(r,\theta)$, then we need to solve now for $v(r,\theta,t)$ specified by

$$\frac{\partial v(r,\theta,t)}{\partial t} = k\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right) \tag{8}$$
$$|v(0,\theta,t)| < \theta$$
$$v(a,\theta,t) = 0$$
$$v(r,-\pi,t) = v(r,\pi,t)$$
$$\frac{\partial v}{\partial \theta}(r,-\pi,t) = \frac{\partial v}{\partial \theta}(r,\pi,t)$$

Let $v(r,\theta,t) = R(r)\Theta(\theta)T(t)$. Substituting into (8) gives

$$T'R\Theta = k\left(R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}\Theta''RT\right)$$

Dividing by $R(r)\Theta(\theta)T(t) \neq 0$ gives

$$\frac{1}{k}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

Let first separation constant be $-\lambda$, hence the above becomes

$$\frac{1}{k}\frac{T'}{T} = -\lambda$$
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

Or

$$T' + \lambda k T = 0$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = -\frac{\Theta''}{\Theta}$$

We now separate the second equation above using μ giving

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = \mu$$
$$-\frac{\Theta''}{\Theta} = \mu$$

Or

$$R'' + \frac{1}{r}R' + R\left(\lambda - \frac{\mu}{r^2}\right) = 0 \tag{9}$$
$$\Theta'' + \mu\Theta = 0 \tag{10}$$

Equation (9) is Sturm-Liouville ODE with boundary conditions $R(a) = 0$ and bounded at $r = 0$ and (10) has periodic boundary conditions as was solved above. The solution to (10) is given in (5) above, no change for this part.

$$\begin{aligned}\Theta_n(\theta) &= \overbrace{A_0}^{\lambda=0} + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta)\\ &= \sum_{n=0}^{\infty} A_n\cos(n\theta) + \sum_{n=1}^{\infty} B_n\sin(n\theta)\end{aligned} \tag{11}$$

Therefore (9) becomes $R'' + \frac{1}{r}R' + R\left(\lambda - \frac{n^2}{r^2}\right) = 0$ with $n = 0,1,2,\cdots$. We found the solution to this Sturm-Liouville before, it is given by

$$R_{nm}(r) = J_n\left(\sqrt{\lambda_{nm}}r\right) \qquad n = 0,1,2,\cdots, m = 1,2,3,\cdots \tag{12}$$

Where $\sqrt{\lambda_{nm}} = \frac{a}{z_{nm}}$ where a is the radius of the disk and z_{nm} is the m^{th} zero of the Bessel function of order n. This is found numerically. We now just need to find the time solution from $T' + \lambda_{nm}kT = 0$. This has solution

$$T_{nm}(t) = e^{-\sqrt{k\lambda_{nm}}t} \tag{13}$$

Now we combine (11,12,13) to find solution for $v(r,\theta,t)$

$$v_{nm}(r,\theta,t) = \Theta_n(\theta)R_{nm}(r)T_{nm}(t)$$
$$v(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n\cos(n\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)e^{-\sqrt{k\lambda_{nm}}t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n\sin(n\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)e^{-\sqrt{k\lambda_{nm}}t} \tag{14}$$

We now need to find A_n, B_n, which is found from initial conditions on $v(r,\theta,0)$ which is given by

$$
\begin{aligned}
v(r,\theta,0) &= u(r,\theta,0) - u_E(r,\theta) \\
&= f(r,\theta) - u_E(r,\theta)
\end{aligned}
$$

Hence from (14), at $t=0$

$$
f(r,\theta) - u_E(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos(n\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin(n\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) \tag{15}
$$

For each n, inside the m sum, $\cos(n\theta)$ and $\sin(n\theta)$ will be constant. So we need to apply orthogonality twice in order to remove both sums. Multiplying (15) by $\cos(n'\theta)$ and integrating gives

$$
\begin{aligned}
\int_{-\pi}^{\pi}\left(f(r,\theta) - u_E(r,\theta)\right)\cos(n'\theta)\,d\theta &= \int_{-\pi}^{\pi}\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} A_n J_n\left(\sqrt{\lambda_{nm}}r\right)\right)\cos(n\theta)\cos(n'\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} B_n J_n\left(\sqrt{\lambda_{nm}}r\right)\right)\sin(n\theta)\cos(n'\theta)
\end{aligned}
$$

The second sum in the RHS above goes to zero due to $\int_{-\pi}^{\pi}\sin(n\theta)\cos(n'\theta)\,d\theta$ and we end up with

$$
\int_{-\pi}^{\pi}\left(f(r,\theta) - u_E(r,\theta)\right)\cos(n\theta)\,d\theta = A_n\int_{-\pi}^{\pi}\cos^2(n\theta)\sum_{m=1}^{\infty} J_n\left(\sqrt{\lambda_{nm}}r\right)d\theta
$$

We now apply orthogonality again, but on Bessel functions and remember to add the weight r. The above becomes

$$
\begin{aligned}
\int_0^a\int_{-\pi}^{\pi}\left(f(r,\theta) - u_E(r,\theta)\right)\cos(n\theta) J_n\left(\sqrt{\lambda_{nm'}}r\right) r\,d\theta dr &= A_n\int_0^a\int_{-\pi}^{\pi}\cos^2(n\theta)\sum_{m=1}^{\infty} J_n\left(\sqrt{\lambda_{nm}}r\right) J_n\left(\sqrt{\lambda_{nm'}}r\right) r\,d\theta dr \\
&= A_n\int_0^a\int_{-\pi}^{\pi}\cos^2(n\theta) J_n^2\left(\sqrt{\lambda_{nm'}}r\right) r\,d\theta dr
\end{aligned}
$$

Hence

$$
A_n = \frac{\int_0^a\int_{-\pi}^{\pi}\left(f(r,\theta) - u_E(r,\theta)\right)\cos(n\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) r\,d\theta dr}{\int_0^a\int_{-\pi}^{\pi}\cos^2(n\theta) J_n^2\left(\sqrt{\lambda_{nm}}r\right) r\,d\theta dr} \qquad n=0,1,2,\cdots, m=1,2,3,\cdots
$$

We will repeat the same thing to find B_n. The only difference now is to use $\sin n\theta$. repeating these steps gives

$$
B_n = \frac{\int_0^a\int_{-\pi}^{\pi}\left(f(r,\theta) - u_E(r,\theta)\right)\sin(n\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) r\,d\theta dr}{\int_0^a\int_{-\pi}^{\pi}\sin^2(n\theta) J_n^2\left(\sqrt{\lambda_{nm}}r\right) r\,d\theta dr} \qquad n=0,1,2,\cdots, m=1,2,3,\cdots
$$

This complete the solution.

Summary of solution

$$\begin{aligned} u(r,\theta,t) &= v(r,\theta,t) + u_E(r,\theta) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos(n\theta) J_n\left(\sqrt{\lambda_{nm}} r\right) e^{-\sqrt{k\lambda_{nm}}t} + \\ &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin(n\theta) J_n\left(\sqrt{\lambda_{nm}} r\right) e^{-\sqrt{k\lambda_{nm}}t} + u_E(r,\theta) \end{aligned}$$

Where

$$\begin{aligned} u_E(r,\theta) &= \tilde{A}_0 + \sum_{n=1}^{\infty} \tilde{A}_n \cos(n\theta) r^n + \tilde{B}_n \sin(n\theta) r^n \\ \tilde{A}_0 &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta \\ \tilde{A}_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\, d\theta \\ \tilde{B}_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\, d\theta \end{aligned}$$

And

$$A_n = \frac{\int_0^a \int_{-\pi}^{\pi} \left(f(r,\theta) - u_E(r,\theta)\right)\cos(n\theta) J_n\left(\sqrt{\lambda_{nm}} r\right) r d\theta dr}{\int_0^a \int_{-\pi}^{\pi} \cos^2(n\theta) J_n^2\left(\sqrt{\lambda_{nm}} r\right) r d\theta dr} \qquad n = 0,1,2,\cdots, m = 1,2,3,\cdots$$

And

$$B_n = \frac{\int_0^a \int_{-\pi}^{\pi} \left(f(r,\theta) - u_E(r,\theta)\right)\sin(n\theta) J_n\left(\sqrt{\lambda_{nm}} r\right) r d\theta dr}{\int_0^a \int_{-\pi}^{\pi} \sin^2(n\theta) J_n^2\left(\sqrt{\lambda_{nm}} r\right) r d\theta dr} \qquad n = 0,1,2,\cdots, m = 1,2,3,\cdots$$

Where $\sqrt{\lambda_{nm}} = \frac{a}{z_{nm}}$ where a is the radius of the disk and z_{nm} is the m^{th} zero of the Bessel function of order n.

2.10.4 Problem 8.3.3

Problem Solve the initial value problem

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + qu + f(x,t) \tag{1}$$

Where c, ρ, K_0, q are functions of x only, subject to conditions $u(0,t) = 0, u(L,t) = 0, u(x,0) = g(x)$. Assume that eigenfunctions are know. Hint: let $L = \frac{d}{dx}\left(K_0 \frac{d}{dx}\right) + q$

solution

Because this problem has homogeneous B.C. but has time dependent source (i.e. non-homogenous in the PDE itself), then we will use the method of eigenfunction expansion. In this method, we first need to find the eigenfunctions $\phi_n(x)$ of the associated PDE without the source being present. Then use these $\phi_n(x)$ to expand the source $f(x,t)$ as generalized

Fourier series. We now switch to the associated homogenous PDE in order to find the eigenfunctions. This the same as above, but without the source term.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{c\rho}\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)+\frac{q}{c\rho}u \qquad (2)\\ u(0,t) &= 0\\ u(L,t) &= 0\\ u(x,0) &= g(x) \end{aligned}$$

We are told to assume the eigenfunctions $\phi_n(x)$ are known. But it is better to do this explicitly, also needed to find the weight. Let $u = X(x)T(t)$. Then (2) becomes

$$T'X = \frac{1}{c\rho}K_0'X'T + \frac{1}{c\rho}K_0X''T + \frac{q}{c\rho}XT$$

Dividing by XT gives

$$\frac{T'}{T} = \frac{1}{c\rho}K_0'\frac{X'}{X} + \frac{1}{c\rho}K_0\frac{X''}{X} + \frac{q}{c\rho}$$

As the right side depends on x only, and the left side depends on t only, we can now separate them. Using $-\lambda$ as separation constant gives

$$T' + \lambda T = 0$$

And for the x part

$$\begin{aligned} \frac{1}{c\rho}K_0'\frac{X'}{X} + \frac{1}{c\rho}K_0\frac{X''}{X} + \frac{q}{c\rho} &= -\lambda\\ K_0'X' + K_0X'' + qX &= -\lambda c\rho X \qquad (2A)\\ (K_0X')' + qX &= -\lambda c\rho X \end{aligned}$$

We now see this is Sturm-Liouville ODE, with

$$\begin{aligned} p &= K_0\\ q &\equiv q\\ \sigma &= c\rho \end{aligned}$$

And

$$\begin{aligned} L[X] &= \frac{d}{dx}\left(K_0\frac{dX}{dx}\right)+qX\\ L &\equiv \frac{d}{dx}\left(K_0\frac{dX}{dx}\right)+q \end{aligned}$$

Where

$$L[X] = -\lambda c\rho X$$

The solution to S-L, with homogeneous B.C. is given as

$$X(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$$

When we plug-in this back into (2), and incorporate the time solution from $T' + \lambda_n T = 0$, we end up with solution for (2) as

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x) \tag{3}$$

Where now the Fourier coefficients became time dependent. We now substitute this back into the original PDE (1) with the source present (the nonhomogeneous PDE) and obtain

$$c\rho \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\, L\left[\phi_n(x)\right] + f(x,t) \tag{4}$$

We now expand $f(x,t)$ using same eigenfunctions found from the homogeneous PDE solution (we can do this, since eigenfunctions found from Sturm-Liouville can be used to expand any piecewise continuous function). Let

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\, \phi_n(x) \tag{5}$$

Hence (4) becomes

$$c\rho \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\, L\left[\phi_n(x)\right] + \sum_{n=1}^{\infty} f_n(t)\, \phi_n(x) \tag{6}$$

But from above, we know that $L\left[\phi_n(x)\right] = -\lambda_n c\rho\phi_n(x)$, hence (6) becomes

$$\begin{aligned}
c\rho \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x) &= -c\rho \sum_{n=1}^{\infty} \lambda_n a_n(t)\, \phi_n(x) + \sum_{n=1}^{\infty} f_n(t)\, \phi_n(x) \\
\sum_{n=1}^{\infty} c\rho a_n'(t)\, \phi_n(x) + c\rho\lambda_n a_n(t)\, \phi_n(x) &= \sum_{n=1}^{\infty} f_n(t)\, \phi_n(x) \\
\sum_{n=1}^{\infty} \left(a_n'(t) + \lambda_n a_n(t)\right) c\rho\phi_n(x) &= \sum_{n=1}^{\infty} f_n(t)\, \phi_n(x)
\end{aligned}$$

By orthogonality, (weight is $c\rho$) then from the above we obtain

$$a_n'(t) + \lambda_n a_n(t) = f_n(t)$$

The solution to the above is

$$a_n(t) = e^{-\lambda_n t} \int_0^t f_n(s)\, e^{\lambda_n s} ds + c e^{-\lambda_n t}$$

To find constant of integration c in the above, we use initial conditions. At $t = 0$

$$c = a_n(0)$$

Hence the solution becomes

$$\begin{aligned}
a_n(t) &= e^{-\lambda_n t} \int_0^t f_n(s)\, e^{\lambda_n s} ds + a_n(0)\, e^{-\lambda_n t} \\
&= e^{-\lambda_n t} \left(a_n(0) + \int_0^t f_n(s)\, e^{\lambda_n s} ds \right)
\end{aligned}$$

To find $a_n(0)$, from (3), putting $t = 0$ gives

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\, \phi_n(x)$$

Applying orthogonality

$$\int_0^L g(x)\,\phi_n(x)\,dx = a_n(0)\int_0^L \phi_n^2(x)\,c\rho dx$$

$$a_n(0) = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,c\rho dx}$$

And finally, to find $f_n(t)$, which is the generalized Fourier coefficient of the expansion of the source in (5) above, we also use orthogonality

$$\int_0^L f(x,t)\,\phi_n(x)\,dx = f_n(t)\int_0^L \phi_n^2(x)\,c\rho dx$$

$$f_n(t) = \frac{\int_0^L f(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,c\rho dx}$$

Summary of solution

The solution to $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + qu + f(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

Where $a_n(t)$ is the solution to

$$a_n'(t) + \lambda_n a_n(t) = f_n(t)$$

Given by

$$a_n(t) = e^{-\lambda_n c\rho t}\left(a_n(0) + \int_0^t f_n(s)\,e^{\lambda_n c\rho s} ds\right)$$

Where

$$f_n(t) = \frac{\int_0^L f(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,c\rho dx}$$

And

$$a_n(0) = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,c\rho dx}$$

2.10.5 Problem 8.3.5

***8.3.5. Solve**

$$\frac{\partial u}{\partial t} = k\nabla^2 u + f(r,t)$$

inside the circle ($r < a$) with $u = 0$ at $r = a$ and initially $u = 0$.

Since this problem has homogeneous B.C. but has time dependent source (i.e. non-homogenous in the PDE itself), then we will use the method of eigenfunction expansion. In this method, we first find the eigenfunctions $\phi_n(x)$ of the associated homogenous PDE without the source being present. Then use these $\phi_n(x)$ to expand the source $f(x,t)$ as generalized Fourier series. We now switch to the associated homogenous PDE in order to find the eigenfunctions. $u \equiv u(r,t)$. There is no θ. Hence

$$\frac{\partial u(r,t)}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

$$u(a,t) = 0$$

$$|u(0,t)| < \infty$$

$$u(r,0) = 0$$

We need to solve the above in order to find the eigenfunctions $\phi_n(r)$. Let $u = R(r)T(t)$. Substituting this back into (1) gives

$$T'R = k\left(R''T + \frac{1}{r}R'T\right)$$

Dividing by RT

$$\frac{1}{k}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$$

Let separation constant be $-\lambda$. We obtain

$$T' + k\lambda T = 0$$

And

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$$

$$R'' + \frac{1}{r}R' = -\lambda R$$

$$rR'' + R' + \lambda rR = 0$$

This is a singular Sturm-Liouville ODE. Standard form is

$$(rR')' = -\lambda rR$$

Hence

$$p = r$$
$$q = 0$$
$$\sigma = r$$

We solved $R'' + \frac{1}{r}R' + \lambda R = 0$ before. The solution is

$$R_n(r) = J_0\left(\sqrt{\lambda_n}r\right)$$

Where $\sqrt{\lambda_n}$ is found by solving $J_0\left(\sqrt{\lambda_n}a\right) = 0$. Now that we know what the eigenfunctions are, then we write

$$u(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0\left(\sqrt{\lambda_n}r\right) \tag{2}$$

Where $a_n(t)$ is function of time since it includes the time solution in it. Now we use the above in the original PDE with the source in it

$$\frac{\partial u(r,t)}{\partial t} = k\nabla^2 u + f(r,t) \tag{3}$$

Where $\nabla^2 u = -\lambda u$. Substituting (2) into (3), and using $f(r,t) = \sum_{n=1}^{\infty} f_n(t) J_0\left(\sqrt{\lambda_n}r\right)$ gives

$$\sum_{n=1}^{\infty} a_n'(t) J_0\left(\sqrt{\lambda_n}r\right) = -k\sum_{n=1}^{\infty} \lambda_n a_n(t) J_0\left(\sqrt{\lambda_n}r\right) + \sum_{n=1}^{\infty} f_n(t) J_0\left(\sqrt{\lambda_n}r\right)$$

$$\sum_{n=1}^{\infty} \left(a_n'(t) + k\lambda_n a_n(t)\right) J_0\left(\sqrt{\lambda_n}r\right) = \sum_{n=1}^{\infty} f_n(t) J_0\left(\sqrt{\lambda_n}r\right)$$

Applying orthogonality, the above simplifies to

$$a_n'(t) + k\lambda_n a_n(t) = f_n(t)$$

The solution is

$$a_n(t) = e^{-k\lambda_n t}\int_0^t f_n(s) e^{k\lambda_n s} ds + ce^{-k\lambda_n t}$$

To find constant of integration c in the above, we use initial conditions. At $t = 0$

$$c = a_n(0)$$

Hence the solution becomes

$$\begin{aligned} a_n(t) &= e^{-k\lambda_n t}\int_0^t f_n(s) e^{k\lambda_n s} ds + a_n(0) e^{-k\lambda_n t} \\ &= e^{-k\lambda_n t}\left(a_n(0) + \int_0^t f_n(s) e^{k\lambda_n s} ds\right) \end{aligned}$$

To find $a_n(0)$, from (2), putting $t = 0$ gives

$$0 = \sum_{n=1}^{\infty} a_n(0) J_0\left(\sqrt{\lambda_n}r\right)$$

Hence $a_n(0) = 0$. Therefore $a_n(t)$ becomes.

$$a_n(t) = e^{-k\lambda_n t}\int_0^t f_n(s) e^{k\lambda_n s} ds$$

And finally, to find $f_n(t)$, which is the generalized Fourier coefficient of the expansion of the source in (3) above, we also use orthogonality

$$\int_0^a f(r,t) J_0\left(\sqrt{\lambda_n} r\right) r dr = f_n(t) \int_0^a J_0^2\left(\sqrt{\lambda_n} r\right) r dr$$

$$f_n(t) = \frac{\int_0^a f(r,t) J_0\left(\sqrt{\lambda_n} r\right) r dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n} r\right) r dr}$$

Summary of solution

The solution to $\frac{\partial u(r,t)}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + f(r,t)$ is given by

$$u(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0\left(\sqrt{\lambda_n} r\right)$$

Where $a_n(t)$ is the solution to

$$a_n'(t) + k\lambda_n a_n(t) = f_n(t)$$

Given by

$$a_n(t) = e^{-k\lambda_n t} \int_0^t f_n(s) e^{k\lambda_n s} ds$$

Where

$$f_n(t) = \frac{\int_0^a f(r,t) J_0\left(\sqrt{\lambda_n} r\right) r dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n} r\right) r dr}$$

2.10.6 Problem 8.3.6

8.3.6. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 5x\, e^{-2t}$$

subject to $u(0,t) = 1, u(\pi,t) = 0$, and $u(x,0) = 0$.

This problem has nonhomogeneous B.C. and non-homogenous in the PDE itself (source present). First step is to use reference function to remove the nonhomogeneous B.C. then use the method of eigenfunction expansion on the resulting problem.

Let

$$r(x) = c_1 x + c_2$$

At $x = 0, r(x) = 1$, hence $1 = c_2$ and at $x = \pi, r(x) = 0$, hence $0 = c_1\pi + 1$ or $c_1 = -\frac{1}{\pi}$, hence

$$r(x) = 1 - \frac{x}{\pi}$$

Therefore

$$u(x,t) = v(x,t) + r(x)$$

Where $v(x,t)$ solution for the given PDE but with homogeneous B.C., therefore

$$\begin{aligned}\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + e^{-2t}\sin 5x \qquad (1)\\ v(0,t) &= 0\\ v(\pi,t) &= 0\\ v(x,0) &= u(x,0) - r(x) = 0 - \left(1 - \frac{x}{\pi}\right) = \frac{x}{\pi} - 1\end{aligned}$$

We now solve (1). This is homogeneous in the PDE itself. To solve, we first solve the nonhomogeneous PDE in order to find the eigenfunctions. Hence we need to solve

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2}$$

This has solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \qquad (2)$$

With

$$\begin{aligned}\phi_n(x) &= \sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3\cdots\\ \lambda_n &= n^2 \qquad n = 1,2,3\cdots\end{aligned}$$

Plug-in (2) back into (1) gives

$$\begin{aligned}\sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) &= \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) + e^{-2t}\sin 5x\\ &= \sum_{n=1}^{\infty} a_n(t)\,\frac{\partial^2}{\partial x^2}\phi_n(x) + e^{-2t}\sin 5x\end{aligned}$$

But $\frac{\partial^2}{\partial x^2}\phi_n(x) = -\lambda_n\phi_n = -n\phi_n$, hence the above becomes

$$\begin{aligned}\sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) + n^2 a_n(t)\,\phi_n(x) &= e^{-2t}\sin 5x\\ \sum_{n=1}^{\infty}\left(a_n'(t) + n^2 a_n(t)\right)\sin(nx) &= e^{-2t}\sin 5x\end{aligned}$$

Therefore, since Fourier series expansion is unique, we can compare coefficients and obtain

$$a_n'(t) + n^2 a_n(t) = \begin{cases} e^{-2t} & n = 5\\ 0 & n \neq 5\end{cases}$$

<u>For the case $n = 5$</u>

$$\begin{aligned}
a_5'(t) + 25a_5(t) &= e^{-2t} \\
\frac{d}{dt}\left(a_5(t)\,e^{25t}\right) &= e^{23t} \\
a_5(t)\,e^{25t} &= \int e^{23t}dt + c \\
&= \frac{e^{23t}}{23} + c
\end{aligned}$$

Hence

$$a_5(t) = \frac{e^{-2t}}{23} + ce^{-25t}$$

At $t = 0$, $a_5(0) = \frac{1}{23} + c$, hence

$$c = a_5(0) - \frac{1}{23}$$

And the solution becomes

$$a_5(t) = \frac{1}{23}e^{-2t} + \left(a_5(0) - \frac{1}{23}\right)e^{-25t}$$

<u>For the case $n \neq 5$</u>

$$\begin{aligned}
a_n'(t) + n^2 a_n(t) &= 0 \\
\frac{d}{dt}\left(a_n(t)\,e^{n^2t}\right) &= 0 \\
a_n(t)\,e^{n^2t} &= c \\
a_n(t) &= ce^{-n^2t}
\end{aligned}$$

At $t = 0$, $a_n(0) = c$, hence

$$a_n(t) = a_n(0)\,e^{-nt}$$

Therefore

$$a_n(t) = \begin{cases} \frac{1}{23}e^{-2t} + \left(a_5(0) - \frac{1}{23}\right)e^{-25t} & n = 5 \\ a_n(0)\,e^{-n^2t} & n \neq 5 \end{cases}$$

To find $a_n(0)$ we use orthogonality. Since $u(x,t) = v(x,t) + r(x)$, then

$$u(x,t) = \left(\sum_{n=1}^{\infty} a_n(t)\sin(nx)\right) + \left(1 - \frac{x}{\pi}\right)$$

And at $t = 0$ the above becomes

$$0 = \left(\sum_{n=1}^{\infty} a_n(0)\sin(nx)\right) + \left(1 - \frac{x}{\pi}\right)$$

Or

$$\frac{x}{\pi} - 1 = \sum_{n=1}^{\infty} a_n(0)\sin(nx)$$

Applying orthogonality

$$\int_0^\pi \left(\frac{x}{\pi}-1\right)\sin(n'x)\,dx = a_{n'}(0)\int_0^\pi \sin^2(n'x)\,dx$$

Therefore

$$\begin{aligned}
a_n(0) &= \frac{\int_0^\pi \left(\frac{x}{\pi}-1\right)\sin(nx)\,dx}{\frac{\pi}{2}}\\
&= \frac{2}{\pi}\int_0^\pi \left(\frac{x}{\pi}-1\right)\sin(nx)\,dx\\
&= \frac{2}{\pi}\left[-\int_0^\pi \sin(nx)\,dx + \frac{1}{\pi}\int_0^\pi x\sin(nx)\,dx\right]\\
&= \frac{2}{\pi}\left[-\left(\frac{-\cos(nx)}{n}\right)_0^\pi + \frac{1}{\pi}\left(\frac{\sin(nx)}{n^2}-\frac{x\cos(nx)}{n}\right)_0^\pi\right]\\
&= \frac{2}{\pi}\left[\left(\frac{\cos(n\pi)}{n}-\frac{1}{n}\right)+\frac{1}{\pi}\left(\left(\frac{\sin(n\pi)}{n^2}-\frac{\pi\cos(n\pi)}{n}\right)-\left(\frac{\sin(0)}{n^2}-\frac{0\cos(0)}{n}\right)\right)\right]\\
&= \frac{2}{\pi}\left[\left(\frac{-1^n}{n}-\frac{1}{n}\right)+\frac{1}{\pi}\left(0-\frac{\pi(-1)^n}{n}\right)\right]\\
&= \frac{2}{\pi}\left[\frac{(-1)^n}{n}-\frac{1}{n}-\frac{(-1)^n}{n}\right]\\
&= \frac{-2}{n\pi}
\end{aligned}$$

Therefore $a_5(0)=\frac{-2}{5\pi}$. Hence

$$a_n(t) = \begin{cases} \frac{1}{23}e^{-2t}+\left(\frac{-2}{5\pi}-\frac{1}{23}\right)e^{-25t} & n=5\\ \frac{-2}{n\pi}e^{-n^2t} & n\neq 5\end{cases}$$

Where

$$\begin{aligned}
u(x,t) &= v(x,t)+r(x)\\
&= \left(\sum_{n=1}^{\infty} a_n(t)\sin(nx)\right)+\left(1-\frac{x}{\pi}\right)
\end{aligned}$$

2.10.7 Problem 8.4.1 (b)

8.4.1. In these exercises, do not make a reduction to homogeneous boundary conditions. Solve the initial value problem for the heat equation with time-dependent sources

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ u(x,0) &= f(x) \end{aligned}$$

subject to the following boundary conditions:

(a) $u(0,t) = A(t)$, $\frac{\partial u}{\partial x}(L,t) = B(t)$
* (b) $\frac{\partial u}{\partial x}(0,t) = A(t)$, $\frac{\partial u}{\partial x}(L,t) = B(t)$

Let

$$u(x,t) \sim \sum_{n=0}^{\infty} b_n(t)\,\phi_n(x) \tag{1}$$

Where in this problem $\phi_n(x)$ are the eigenfunctions of the corresponding homogenous PDE, which due to having both sides insulated, we know they are given by $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ where now $n = 0,1,2,\cdots$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. That is why the sum above starts from zero and not one. We now substitute (1) back into the given PDE, but remember not to do term-by-term differentiation on the spatial terms.

$$\sum_{n=0}^{\infty} b_n'(t)\,\phi_n(x) = k\frac{\partial^2 x}{\partial u^2} + Q(x,t)$$

But $Q(x,t) \sim \sum_{i=0}^{\infty} q_n(t)\,\phi_n(x)$ so the above becomes

$$\sum_{n=0}^{\infty} b_n'(t)\,\phi_n(x) = k\frac{\partial^2 u}{\partial x^2} + \sum_{n=0}^{\infty} q_n(t)\,\phi_n(x)$$

Multiplying both sides by $\phi_m(x)$ and integrating

$$\int_0^L \sum_{n=0}^{\infty} b_n'(t)\,\phi_n(x)\,\phi_m(x)\,dx = \int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_m(x)\,dx + \int_0^L \sum_{n=0}^{\infty} q_n(t)\,\phi_n(x)\,\phi_m(x)\,dx$$

Applying orthogonality

$$b_n'(t)\int_0^L \phi_n^2(x)\,dx = \int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx + q_n(t)\int_0^L \phi_n^2(x)\,dx$$

Dividing both sides by $\int_0^L \phi_n^2(x)\,dx$ gives

$$b_n'(t) = \frac{k\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} + q_n(t) \tag{1A}$$

We now use Green's formula to simplify $\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx$. We rewrite $\frac{\partial^2 u}{\partial x^2} \equiv L[u]$ and let $\phi_n(x) \equiv v$, then

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \int_0^L vL[u]\,dx$$

But we know from Green's formula that

$$\int_0^L (vL[u] - uL[v])\,dx = p\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)_0^L$$

In this problem $p = 1$, so we solve for $\int_0^L vL[u]\,dx$ (which is really all what we want) from the above and obtain

$$\begin{aligned}\int_0^L vL[u]\,dx - \int_0^L uL[v]\,dx &= \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)_0^L\\ \int_0^L vL[u]\,dx &= \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)_0^L + \int_0^L uL[v]\,dx\end{aligned}$$

Since we said $\phi_n(x) \equiv v$, then we replace these back into the above to make it more explicit

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \left(\phi_n(x)\frac{du}{dx} - u\frac{d\phi_n(x)}{dx}\right)_0^L + \int_0^L uL\left[\phi_n(x)\right]dx$$

But $L\left[\phi_n(x)\right] = -\lambda_n\phi_n(x)$ and above becomes

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \left(\phi_n(x)\frac{du}{dx} - u\frac{d\phi_n(x)}{dx}\right)_0^L - \lambda_n\int_0^L u\phi_n(x)\,dx \tag{2}$$

We are now ready to substitute boundary conditions. In this problem we know that

$$\begin{aligned}\frac{du}{dx}(L,t) &= B(t)\\ \frac{d\phi_n(L,t)}{dx} &= \frac{d}{dx}\cos\left(\frac{n\pi}{L}x\right)_{x=L} = -\frac{n\pi}{L}\sin\left(\frac{n\pi}{L}x\right)_{x=L} = 0\\ \phi_n(L,t) &= \cos\left(\frac{n\pi}{L}x\right)_{x=L} = \cos(n\pi) = (-1)^n\\ \frac{d\phi_n(0,t)}{dx} &= \frac{d}{dx}\cos\left(\frac{n\pi}{L}x\right)_{x=0} = 0\\ \phi_n(0,t) &= \cos\left(\frac{n\pi}{L}x\right)_{x=0} = 1\\ \frac{du}{dx}(0,t) &= A(t)\end{aligned}$$

Now we have all the information to evaluate (2)

$$\begin{aligned}\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = {} & \left(\phi_n(L)\frac{du}{dx}(L) - u(L)\frac{d\phi_n(L)}{dx}\right) - \left(\phi_n(0)\frac{du}{dx}(0) - u(0)\frac{d\phi_n(0)}{dx}\right)\\ & - \lambda_n\int_0^L u\phi_n(x)\,dx\end{aligned}$$

Which becomes

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \left((-1)^n B(t) - 0\right) - (A(t) - 0) - \lambda_n \int_0^L u\phi_n(x)\,dx$$

$$= (-1)^n B(t) - A(t) - \lambda_n \int_0^L u\phi_n(x)\,dx \tag{3}$$

Now we need to sort out the $\int_0^L u\phi_n(x)\,dx$ term above, since $u(x,t)$ is unknown, so we can't leave the above as is. But we know from $u(x,t) \sim \sum_{n=0}^{\infty} b_n(t)\phi_n(x)$ that $b_n(t) = \frac{\int_0^L u\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}$ by orthogonality. Hence $\int_0^L u\phi_n(x)\,dx = b_n(t)\int_0^L \phi_n^2(x)\,dx$. Using this in (3), we finally found the result for $\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx$

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = (-1)^n B(t) - A(t) - \lambda_n b_n(t)\int_0^L \phi_n^2(x)\,dx$$

But $\int_0^L \phi_n^2(x)\,dx = \int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}$ hence

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = (-1)^n B(t) - A(t) - \lambda_n b_n(t)\frac{L}{2} \tag{4}$$

Substituting the above in (1A) gives

$$b_n'(t) = \frac{k\left((-1)^n B(t) - A(t) - \lambda_n b_n(t)\frac{L}{2}\right)}{\frac{L}{2}} + q_n(t)$$

$$b_n'(t) = \frac{2}{L}k\left((-1)^n B(t) - A(t) - \lambda_n b_n(t)\frac{L}{2}\right) + q_n(t)$$

$$= \frac{2}{L}k\left((-1)^n B(t) - A(t)\right) - k\lambda_n b_n(t) + q_n(t)$$

Or

$$b_n'(t) + k\lambda_n b_n(t) = q_n(t) + \frac{2}{L}k\left((-1)^n B(t) - A(t)\right)$$

Now that we found the differential equation for $b_n(t)$ we solve it. The integrating factor is $\mu = e^{k\lambda_n t}$, hence the solution is

$$\frac{d}{dt}\left(\mu b_n(t)\right) = \mu q_n(t) + \mu\frac{2}{L}k\left((-1)^n B(t) - A(t)\right)$$

Integrating

$$\mu b_n(t) = \int \mu q_n(t)\,dt + \int \mu\frac{2}{L}k\left((-1)^n B(t) - A(t)\right)dt + c$$

Or

$$b_n(t) = e^{-k\lambda_n t}\int e^{k\lambda_n t}q_n(t)\,dt + \int e^{k\lambda_n t}\frac{2}{L}k\left((-1)^n B(t) - A(t)\right)dt + ce^{-k\lambda_n t}$$

The constant of integration c is $b_n(0)$, therefore

$$b_n(t) = e^{-k\lambda_n t}\int e^{k\lambda_n t}q_n(t)\,dt + \int e^{k\lambda_n t}\frac{2}{L}k\left((-1)^n B(t) - A(t)\right)dt + b_n(0)\,e^{-k\lambda_n t}$$

The above could also be written as

$$b_n(t) = e^{-k\lambda_n t}\int_0^t e^{k\lambda_n s} q_n(s)\,ds + \int_0^t e^{k\lambda_n s}\frac{2}{L}k\left((-1)^n B(s) - A(s)\right)ds + b_n(0)\,e^{-k\lambda_n t}$$

Now that we found $b_n(t)$, the last step is to determine $b_n(0)$. This is done from initial conditions

$$u(x,0) \sim \sum_{n=0}^{\infty} b_n(0)\,\phi_n(x)$$

By orthogonality

$$b_n(0) = \frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

This complete the solution. Summary of result

The solution is

$$u(x,t) \sim \sum_{n=0}^{\infty} b_n(t)\,\phi_n(x)$$

Where

$$b_n(t) = e^{-k\lambda_n t}\int_0^t e^{k\lambda_n s} q_n(s)\,ds + \int_0^t e^{k\lambda_n s}\frac{2}{L}k\left((-1)^n B(s) - A(s)\right)ds + b_n(0)\,e^{-k\lambda_n t}$$

Where

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

And

$$q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\cos\left(\frac{n\pi}{L}x\right)dx$$

And

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 0,1,2,3,\cdots$$

2.10.8 Problem 8.4.3

8.4.3. Consider

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x,t)$$

$$u(x,0) = g(x) \qquad \begin{aligned} u(0,t) &= \alpha(t) \\ u(L,t) &= \beta(t). \end{aligned}$$

Assume that the eigenfunctions $\phi_n(x)$ of the related homogeneous problem are known.

(a) Solve without reducing to a problem with homogeneous boundary conditions.

(b) Solve by first reducing to a problem with homogeneous boundary conditions.

2.10.8.1 Part (a)

From problem 8.3.3, we found the eigenfunctions $\phi_n(x)$ from the Sturm-Liouville to have weight

$$\sigma = c\rho$$

Let

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$$

Substituting the above in the PDE gives

$$\sigma \sum_{i=1}^{\infty} b_n'(t)\,\phi_n(x) = L[u] + f(x,t)$$

Where $L = \frac{\partial}{\partial x}\left(K_0 \frac{\partial}{\partial x}\right) + q$. Following same procedure using Green's formula on page 35, we obtain

$$\sigma\frac{db_n(t)}{dt} + k\lambda_n b_n(t) = f_n(t) + \frac{k\sqrt{\lambda_n}\left(\alpha(t) - (-1)^n \beta(t)\right)}{\int_0^L \phi_n^2(x)\,\sigma dx} \tag{1}$$

Where

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x)$$

$$f_n(t) = \frac{\int_0^L f(x,t)\,\phi_n(x)\,\sigma dx}{\int_0^L \phi_n^2(x)\,\sigma dx}$$

The solution to (1) is found using integrating factor.

$$\frac{db_n(t)}{dt}+\left(\frac{\lambda_n}{\sigma}\right)b_n(t)=\frac{1}{\sigma}f_n(t)+\frac{\frac{k}{\sigma}\sqrt{\lambda_n}\left(\alpha(t)-(-1)^n\beta(t)\right)}{\int_0^L\phi_n^2(x)\,\sigma dx}$$

Hence $\mu=e^{\frac{\lambda_n}{c\rho}t}$ and the solution becomes

$$b_n(t)=e^{-\frac{\lambda_n}{\sigma}t}\left(\frac{1}{\sigma}\int e^{\frac{\lambda_n}{\sigma}t}f_n(t)\,dt+\frac{\frac{k}{\sigma}\sqrt{\lambda_n}}{\int_0^L\phi_n^2(x)\,\sigma dx}\int e^{\frac{\lambda_n}{\sigma}t}\left(\alpha(t)-(-1)^n\beta(t)\right)dt\right)+ce^{-\frac{\lambda_n}{\sigma}t}$$

Where c is found from

$$b_n(0)=c$$

And $b_n(0)$ is found from initial conditions

$$g(x)=\sum_{n=1}^{\infty}b_n(0)\,\phi_n(x)$$

$$b_n(0)=\frac{\int_0^L g(x)\,\phi_n(x)\,\sigma dx}{\int_0^L\phi_n^2(x)\,\sigma dx}$$

This complete the solution. <u>Summary</u>

Solution is

$$u(x,t)\sim\sum_{n=1}^{\infty}b_n(t)\,\phi_n(x)$$

Where

$$b_n(t)=e^{-\frac{\lambda_n}{\sigma}t}\left(\frac{1}{\sigma}\int e^{\frac{\lambda_n}{c\rho}t}f_n(t)\,dt+\frac{\frac{k}{\sigma}\sqrt{\lambda_n}}{\int_0^L\phi_n^2(x)\,\sigma dx}\int e^{\frac{\lambda_n}{\sigma}t}\left(\alpha(t)-(-1)^n\beta(t)\right)dt\right)+b_n(0)\,e^{-\frac{\lambda_n}{\sigma}t}$$

$$b_n(0)=\frac{\int_0^L g(x)\,\phi_n(x)\,\sigma dx}{\int_0^L\phi_n^2(x)\,\sigma dx}$$

$$\sigma=c\rho$$

2.10.8.2 Part (b)

The first step is to obtain a reference function $r(x,t)$ where $u(x,t)=v(x,t)+r(x,t)$. The reference function only needs to satisfy the nonhomogeneous B.C.

We see that

$$r(x,t)=\alpha(t)+\frac{\beta(t)-\alpha(t)}{L}x$$

does the job. Now we solve the following PDE

$$
\begin{aligned}
c\rho\frac{\partial v}{\partial t} &= \frac{\partial}{\partial x}\left(K_0\frac{\partial v}{\partial x}\right) + q(x)\,v + f(x,t)\\
v(0,t) &= 0\\
v(\pi,t) &= 0\\
v(x,0) &= g(x) - \left(\alpha(0) + \frac{\beta(0)-\alpha(0)}{L}x\right)
\end{aligned}
$$

Using Green's formula, starting with

$$v(x,t) = \sum_{i=1}^{\infty} b_n(t)\,\phi_n(x)$$

Where we used = instead of ∼ above now, since both $v(x,t)$ and $\phi_n(x)$ satisfy the homogenous B.C., and where $b_n(t)$ satisfies the ODE

$$\sigma\frac{db_n(t)}{dt} + \lambda_n b_n(t) = f_n(t) \tag{1}$$

Where $\sigma = c\rho$ and

$$
\begin{aligned}
f(x,t) &= \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x)\\
f_n(t) &= \frac{\int_0^L f(x,t)\,\phi_n(x)\,\sigma dx}{\int_0^L \phi_n^2(x)\,\sigma dx}
\end{aligned}
$$

The solution to (1) is found using integrating factor $\mu = e^{\frac{\lambda_n}{\sigma}t}$, hence

$$b_n(t) = e^{-\frac{\lambda_n}{\sigma}t}\frac{1}{\sigma}\int e^{\frac{\lambda_n}{\sigma}t} f_n(t)\,dt + b_n(0)\,e^{-\frac{\lambda_n}{\sigma}t}$$

And $b_n(0)$ is found from initial conditions $v(x,0)$

$$
\begin{aligned}
g(x) - \left(\alpha(0) + \frac{\beta(0)-\alpha(0)}{L}x\right) &= \sum_{i=1}^{\infty} b_n(0)\,\phi_n(x)\\
b_n(0) &= \frac{\int_0^L g(x) - \left(\alpha(0) + \frac{\beta(0)-\alpha(0)}{L}x\right)\phi_n(x)\,\sigma dx}{\int_0^L \phi_n^2(x)\,\sigma dx}
\end{aligned}
$$

This complete the solution. Summary

Solution is given by

$$
\begin{aligned}
u(x,t) &= \left(\sum_{i=1}^{\infty} b_n(t)\,\phi_n(x)\right) + r(x,t)\\
&= \left(\sum_{i=1}^{\infty} b_n(t)\,\phi_n(x)\right) + \alpha(t) + \frac{\beta(t)-\alpha(t)}{L}x
\end{aligned}
$$

Where

$$b_n(t) = e^{-\frac{\lambda_n}{\sigma}t}\frac{1}{\sigma}\int e^{\frac{\lambda_n}{\sigma}t} f_n(t)\,dt + b_n(0)\,e^{-\frac{\lambda_n}{\sigma}t}$$

And

$$b_n(0) = \frac{\int_0^L g(x) - \left(\alpha(0) + \frac{\beta(0)-\alpha(0)}{L}x\right)\phi_n(x)\,\sigma dx}{\int_0^L \phi_n^2(x)\,\sigma dx}$$

And

$$f_n(t) = \frac{\int_0^L f(x,t)\,\phi_n(x)\,\sigma dx}{\int_0^L \phi_n^2(x)\,\sigma dx}$$

Where $\sigma = c\rho$

2.10.9 Problem 8.5.2

8.5.2. **Consider a vibrating string with time-dependent forcing:**

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ u(0,t) = 0 \quad & \quad u(x,0) = f(x) \\ u(L,t) = 0 \quad & \quad \frac{\partial u}{\partial t}(x,0) = 0. \end{aligned}$$

(a) Solve the initial value problem.

***(b) Solve the initial value problem if $Q(x,t) = g(x)\cos\omega t$. For what values of ω does resonance occur?**

2.10.9.1 Part (a)

Let

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x)$$

Where we used = instead of $\sim$ above, since the PDE given has homogeneous B.C. We know that $\phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ for $n = 1,2,3,\cdots$ where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Substituting the above in the given PDE gives

$$\sum_{n=1}^{\infty} A_n''(t)\,\phi_n(x) = c^2\sum_{n=1}^{\infty} A_n(t)\frac{d^2\phi_n(x)}{dx^2} + Q(x,t)$$

But $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$, hence the above becomes

$$\sum_{n=1}^{\infty} A_n''(t)\,\phi_n(x) = c^2\sum_{n=1}^{\infty} A_n(t)\frac{d^2\phi_n(x)}{dx^2} + \sum_{n=1}^{\infty} g_n(t)\,\phi_n(x)$$

But $\frac{d^2\phi_n(x)}{dx^2} = -\lambda_n\phi_n(x)$, hence

$$\sum_{n=1}^{\infty} A_n''(t)\,\phi_n(x) = -c^2\sum_{n=1}^{\infty}\lambda_n A_n(t)\,\phi_n(x) + \sum_{n=1}^{\infty} g_n(t)\,\phi_n(x)$$

Multiplying both sides by $\phi_m(x)$ and integrating gives

$$\int_0^L \sum_{n=1}^{\infty} A_n''(t)\,\phi_m(x)\,\phi_n(x)\,dx = -c^2 \int_0^L \sum_{n=1}^{\infty} \lambda_n A_n(t)\,\phi_m(x)\,\phi_n(x)\,dx + \int_0^L \sum_{n=1}^{\infty} g_n(t)\,\phi_m(x)\,\phi_n(x)\,dx$$

$$A_n''(t) \int_0^L \phi_n^2(x)\,dx = -c^2\lambda_n A_n(t) \int_0^L \phi_n^2(x)\,dx + g_n(t) \int_0^L \phi_n^2(x)\,dx$$

Hence

$$A_n''(t) + c^2\lambda_n A_n(t) = g_n(t)$$

Now we solve the above ODE. Let solution be

$$A_n(t) = A_n^h(t) + A_n^p(t)$$

Which is the sum of the homogenous and particular solutions. The homogenous solution is

$$A_n^h(t) = c_{1_n} \cos\left(c\sqrt{\lambda_n}t\right) + c_{2_n} \sin\left(c\sqrt{\lambda_n}t\right)$$

And the particular solution depends on $q_n(t)$. Once we find $q_n(t)$, we plug-in everything back into $u(x,t) = \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x)$ and then use initial conditions to find c_{1_n}, c_{2_n}, the two constant of integrations. We will do this in the second part.

2.10.9.2 Part (b)

Now we are given that $Q(x,t) = g(x)\cos(\omega t)$. Hence

$$g_n(t) = \frac{\int_0^L Q(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = \frac{\cos(\omega t)\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = \cos(\omega t)\,\gamma_n$$

Where

$$\gamma_n = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

is constant that depends on n. Now we use the above in result found in part (a)

$$A_n''(t) + c^2\lambda_n A_n(t) = \gamma_n \cos(\omega t) \tag{1}$$

We know the homogenous solution from part (a).

$$A_n^h(t) = c_{1_n} \cos\left(c\sqrt{\lambda_n}t\right) + c_{2_n} \sin\left(c\sqrt{\lambda_n}t\right)$$

We now need to find the particular solution. Will solve using method of undetermined coefficients.

<u>Case 1</u> $\omega \neq c\sqrt{\lambda_n}$ (no resonance)

We can now guess

$$A_n^p(t) = z_1 \cos(\omega t) + z_2 \sin(\omega t)$$

Plugging this back into (1) gives

$$
\begin{aligned}
(z_1 \cos(\omega t) + z_2 \sin(\omega t))'' + c^2\lambda_n (z_1 \cos(\omega t) + z_2 \sin(\omega t)) &= \gamma_n \cos(\omega t) \\
(-\omega z_1 \sin(\omega t) + \omega z_2 \cos(\omega t))' + c^2\lambda_n (z_1 \cos(\omega t) + z_2 \sin(\omega t)) &= \gamma_n \cos(\omega t) \\
-\omega^2 z_1 \cos(\omega t) - \omega^2 z_2 \sin(\omega t) + c^2\lambda_n (z_1 \cos(\omega t) + z_2 \sin(\omega t)) &= \gamma_n \cos(\omega t)
\end{aligned}
$$

Collecting terms

$$
\cos(\omega t)\left(-\omega^2 z_1 + c^2\lambda_n z_1\right) + \sin(\omega t)\left(-\omega^2 z_2 + c^2\lambda_n z_2\right) = \gamma_n \cos(\omega t)
$$

Therefore we obtain two equations in two unknowns

$$
\begin{aligned}
-\omega^2 z_1 + c^2\lambda_n z_1 &= \gamma_n \\
-\omega^2 z_2 + c^2\lambda_n z_2 &= 0
\end{aligned}
$$

From the second equation, $z_2 = 0$ and from the first equation

$$
\begin{aligned}
z_1\left(c^2\lambda_n - \omega^2\right) &= \gamma_n \\
z_1 &= \frac{\gamma_n}{c^2\lambda_n - \omega^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
A_n^p(t) &= z_1 \cos(\omega t) + z_2 \sin(\omega t) \\
&= \frac{\gamma_n}{c^2\lambda_n - \omega^2} \cos(\omega t)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
A_n(t) &= A_n^h(t) + A_n^p(t) \\
&= c_{1_n} \cos\left(c\sqrt{\lambda_n} t\right) + c_{2_n} \sin\left(c\sqrt{\lambda_n} t\right) + \frac{\gamma_n}{c^2\lambda_n - \omega^2} \cos(\omega t)
\end{aligned}
$$

Now we need to find c_{1_n}, c_{2_n}. Since

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x) \\
&= \sum_{n=1}^{\infty} \left(c_{1_n} \cos\left(c\sqrt{\lambda_n} t\right) + c_{2_n} \sin\left(c\sqrt{\lambda_n} t\right) + \frac{\gamma_n}{c^2\lambda_n - \omega^2} \cos(\omega t)\right) \sin\left(\frac{n\pi}{L} x\right)
\end{aligned}
$$

At $t = 0$ the above becomes

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \left(c_{1_n} + \frac{\gamma_n}{c^2\lambda_n - \omega^2}\right) \sin\left(\frac{n\pi}{L} x\right) \\
&= \sum_{n=1}^{\infty} c_{1_n} \sin\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} \frac{\gamma_n}{c^2\lambda_n - \omega^2} \sin\left(\frac{n\pi}{L} x\right)
\end{aligned}
$$

Applying orthogonality

$$
\int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx = \int_0^L \sum_{n=1}^{\infty} c_{1_n} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right) dx + \int_0^L \sum_{n=1}^{\infty} \frac{\gamma_n}{c^2\lambda_n - \omega^2} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right) dx
$$

$$
\int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx = c_{1_n} \int_0^L \sin^2\left(\frac{n\pi}{L} x\right) dx + \frac{\gamma_n}{c^2\lambda_n - \omega^2} \int_0^L \sin^2\left(\frac{n\pi}{L} x\right) dx
$$

Rearranging

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx - \frac{\gamma_n}{c^2\lambda_n-\omega^2}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = c_{1_n}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx$$

$$\begin{aligned} c_{1_n} &= \frac{\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx}{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx} - \frac{\gamma_n}{c^2\lambda_n-\omega^2} \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx - \frac{\gamma_n}{c^2\lambda_n-\omega^2} \end{aligned}$$

We now need to find c_{2_n}. For this we need to differentiate the solution once.

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-c\sqrt{\lambda_n}c_{1_n}\sin\left(c\sqrt{\lambda_n}t\right) + c\sqrt{\lambda_n}c_{2_n}\cos\left(c\sqrt{\lambda_n}t\right) - \frac{\gamma_n}{c^2\lambda_n-\omega^2}\omega\sin(\omega t)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Applying initial conditions $\frac{\partial u(x,0)}{\partial t}=0$ gives

$$0 = \sum_{n=1}^{\infty} c\sqrt{\lambda_n}c_{2_n}\sin\left(\frac{n\pi}{L}x\right)$$

Hence

$$c_{2_n} = 0$$

Therefore the final solution is

$$A_n(t) = c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right) + \frac{\gamma_n}{c^2\lambda_n-\omega^2}\cos(\omega t)$$

And

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

Where

$$c_{1_n} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx - \frac{\gamma_n}{c^2\lambda_n-\omega^2}$$

<u>Case 2</u> $\omega = c\sqrt{\lambda_n}$ <u>Resonance case</u>. Now we can't guess $A_n^p(t) = z_1\cos(\omega t) + z_2\sin(\omega t)$ so we have to use

$$A_n^p(t) = z_1 t\cos(\omega t) + z_2 t\sin(\omega t)$$

Substituting this in $A_n''(t) + c^2\lambda_n A_n(t) = \gamma_n\cos(\omega t)$ gives

$$\left(z_1 t\cos(\omega t) + z_2 t\sin(\omega t)\right)'' + c^2\lambda_n\left(z_1 t\cos(\omega t) + z_2 t\sin(\omega t)\right) = \gamma_n\cos(\omega t) \tag{2}$$

But

$$\begin{aligned} \left(z_1 t\cos(\omega t) + z_2 t\sin(\omega t)\right)'' &= \left(z_1\cos(\omega t) - z_1\omega t\sin(\omega t) + z_2\sin(\omega t) + z_2\omega t\cos(\omega t)\right)' \\ &= -z_1\omega\sin(\omega t) - \left(z_1\omega\sin(\omega t) + z_1\omega^2 t\cos(\omega t)\right) \\ &\quad + z_2\omega\cos(\omega t) + \left(z_2\omega\cos(\omega t) - z_2\omega^2 t\sin(\omega t)\right) \\ &= -2z_1\omega\sin(\omega t) - z_1\omega^2 t\cos(\omega t) + 2z_2\omega\cos(\omega t) - z_2\omega^2 t\sin(\omega t) \end{aligned}$$

Hence (2) becomes

$-2z_1\omega\sin(\omega t)-z_1\omega^2 t\cos(\omega t)+2z_2\omega\cos(\omega t)-z_2\omega^2 t\sin(\omega t)+c^2\lambda_n(z_1 t\cos(\omega t)+z_2 t\sin(\omega t))=\gamma_n\cos(\omega t)$

Comparing coefficients we see that $2z_2\omega=\gamma_n$ or

$$z_2=\frac{\gamma_n}{2\omega}$$

And $z_1=0$. Therefore

$$A_n^p(t)=\frac{\gamma_n}{2\omega}t\sin(\omega t)$$

Therefore

$$\begin{aligned}A_n(t)&=A_n^h(t)+A_n^p(t)\\&=c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right)+c_{2_n}\sin\left(c\sqrt{\lambda_n}t\right)+\frac{\gamma_n}{2c\sqrt{\lambda_n}}t\sin(\omega t)\end{aligned}$$

We now can find c_{1_n},c_{2_n} from initial conditions.

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}A_n(t)\phi_n(x)\\&=\sum_{n=1}^{\infty}\left(c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right)+c_{2_n}\sin\left(c\sqrt{\lambda_n}t\right)+\frac{\gamma_n}{2c\sqrt{\lambda_n}}t\sin(\omega t)\right)\sin\left(\frac{n\pi}{L}x\right)\end{aligned}\tag{4}$$

At $t=0$

$$\begin{aligned}f(x)&=\sum_{n=1}^{\infty}c_{1_n}\sin\left(\frac{n\pi}{L}x\right)\\c_{1n}&=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\end{aligned}$$

Taking time derivative of (4) and setting it to zero will give c_{2n}. Since initial speed is zero then $c_{2_n}=0$. Hence

$$A_n(t)=c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right)+\frac{\gamma_n}{2c\sqrt{\lambda_n}}t\sin(\omega t)$$

This completes the solution.

Summary of solution

The solution is given by

$$u(x,t)=\sum_{n=1}^{\infty}A_n(t)\phi_n(x)$$

Case $\omega\neq c\sqrt{\lambda_n}$

$$A_n(t)=c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right)+\frac{\gamma_n}{c^2\lambda_n-\omega^2}\cos(\omega t)$$

And

$$c_{1_n}=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx-\frac{\gamma_n}{c^2\lambda_n-\omega^2}$$

And

$$\gamma_n = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

And $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3,$

Case $\omega = c\sqrt{\lambda_n}$ (resonance)

$$A_n(t) = c_{1_n}\cos\left(c\sqrt{\lambda_n}t\right) + \frac{\gamma_n}{2c\sqrt{\lambda_n}}t\sin(\omega t)$$

And

$$c_{1_n} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

2.10.10 Problem 8.5.5 (b)

8.5.5. Solve the initial value problem for a membrane with time-dependent forcing and fixed boundaries ($u = 0$),

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(x, y, t),$$

$$u(x, y, 0) = f(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

if the membrane is

(a) a rectangle ($0 < x < L, 0 < y < H$)
(b) a circle ($r < a$)
***(c) a semicircle ($0 < \theta < \pi, r < a$)**
(d) a circular annulus ($a < r < b$)

The solution to the corresponding homogeneous PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2$$

Is

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_n(t) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n(t) J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)$$

Where λ_{nm} are found by solving roots of $J_n\left(\sqrt{\lambda_{nm}}a\right) = 0$. To make things simpler, we will write

$$u(r,\theta,t)=\sum_i a_i(t)\,\Phi_i(r,\theta)$$

Where the above means the double sum of all eigenvalues λ_i. So $\Phi_i(r,\theta)$ represents $J_n\left(\sqrt{\lambda_{nm}}r\right)\{\cos(n\theta),\sin(\theta$ combined. So double sum is implied everywhere. Given this, we now expand the source term

$$Q(r,\theta,t)=\sum_i q_i(t)\,\Phi_i(r,\theta)$$

And the original PDE becomes

$$\sum_i a_i''(t)\,\Phi(\lambda_i)=c^2\sum_i a_i(t)\,\nabla^2(\Phi_i(r,\theta))+\sum_i q_i(t)\,\Phi_i(r,\theta) \tag{1}$$

But

$$\nabla^2(\Phi_i(r,\theta))=-\lambda_i\Phi_i(r,\theta)$$

Hence (1) becomes

$$\begin{aligned}\sum_i a_i''(t)\,\Phi_i(r,\theta)+c^2\lambda_i a_i(t)\,\Phi_i(r,\theta)&=\sum_i q_i(t)\,\Phi_i(r,\theta)\\ \sum_i\left(a_i''(t)+c^2\lambda_i a_i(t)\right)\Phi_i(r,\theta)&=\sum_i q_i(t)\,\Phi_i(r,\theta)\end{aligned}$$

Applying orthogonality gives

$$a_i''(t)+c^2\lambda_i a_i(t)=q_i(t)$$

Where

$$q_i(t)=\frac{\int_0^a\int_{-\pi}^{\pi}Q(r,\theta,t)\,\Phi_i(r,\theta)\,rdrd\theta}{\int_0^a\int_{-\pi}^{\pi}\Phi_i^2(r,\theta)\,rdrd\theta}$$

The solution to the homogenous ODE is

$$a_i^h(t)=A_i\cos\left(c\sqrt{\lambda_i}t\right)+B_i\sin\left(c\sqrt{\lambda_i}t\right)$$

And the particular solution is found if we know what $Q(r,\theta,t)$ and hence $q_i(t)$. For now, lets call the particular solution as $a_i^p(t)$. Hence the solution for $a_i(t)$ is

$$a_i(t)=A_i\cos\left(c\sqrt{\lambda_i}t\right)+B_i\sin\left(c\sqrt{\lambda_i}t\right)+a_i^p(t)$$

Plugging the above into the $u(r,\theta,t)=\sum_i a_i(t)\,\Phi_i(r,\theta)$, gives

$$u(r,\theta,t)=\sum_i\left(A_i\cos\left(c\sqrt{\lambda_i}t\right)+B_i\sin\left(c\sqrt{\lambda_i}t\right)+a_i^p(t)\right)\Phi_i(r,\theta) \tag{2}$$

We now find A_i,B_i from initial conditions. At $t=0$

$$f(r,\theta)=\sum_i\left(A_i+a_i^p(0)\right)\Phi_i(r,\theta)$$

Applying orthogonality

$$\int_0^a \int_{-\pi}^{\pi} f(r,\theta)\,\Phi_j(r,\theta)\,rdrd\theta = \int_0^a \int_{-\pi}^{\pi} \sum_i \left(A_i + a_i^p(0)\right)\Phi_i(r,\theta)\,\Phi_j(r,\theta)\,rdrd\theta$$

$$\int_0^a \int_{-\pi}^{\pi} f(r,\theta)\,\Phi_j(r,\theta)\,rdrd\theta = \left(A_j + a_j^p(0)\right)\int_0^a \int_{-\pi}^{\pi} \Phi_j^2(r,\theta)\,rdrd\theta$$

$$\left(A_i + a_i^p(0)\right) = \frac{\int_0^a \int_{-\pi}^{\pi} f(r,\theta)\,\Phi_i(r,\theta)\,rdrd\theta}{\int_0^a \int_{-\pi}^{\pi} \Phi_i^2(r,\theta)\,rdrd\theta}$$

Taking time derivative of (2)

$$\frac{\partial u(r,\theta,t)}{\partial t} = \sum_i \left(-A_i c\sqrt{\lambda_i}\sin\left(c\sqrt{\lambda_i}t\right) + c\sqrt{\lambda_i}B_i\cos\left(c\sqrt{\lambda_i}t\right) + \frac{da_i^p(t)}{dt}\right)\Phi_i(r,\theta)$$

At $t = 0$

$$0 = \sum_i \left(c\sqrt{\lambda_i}B_i + \frac{da_i^p(0)}{dt}\right)\Phi_i(r,\theta)$$

Hence $B_i = 0$. Therefore the final solution is

$$u(r,\theta,t) = \sum_i \left(A_i\cos\left(c\sqrt{\lambda_i}t\right) + a_i^p(t)\right)\Phi_i(r,\theta)$$

Where

$$\left(A_i + a_i^p(0)\right) = \frac{\int_0^a \int_{-\pi}^{\pi} f(r,\theta)\,\Phi_i(r,\theta)\,rdrd\theta}{\int_0^a \int_{-\pi}^{\pi} \Phi_i^2(r,\theta)\,rdrd\theta}$$

This complete the solution.

2.11 HW 10

Math 322 (Smith): Problem Set 10

Due Wednesday Dec. 7, 2016

1-4) For the following problems, determine a representation of the solution in terms of a symmetric Green's function. Use appropriate homogeneous boundary conditions for the Green's function. Show that the boundary terms can also be understood using homogeneous solutions of the differential equation.

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1, \quad u(0) = A, \quad \frac{du}{dx}(1) = B \tag{1}$$

$$\frac{d^2u}{dx^2} + u = f(x), \quad 0 < x < L, \quad u(0) = A, \quad u(L) = B, \quad L \neq n\pi \tag{2}$$

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < L, \quad u(0) = A, \quad \frac{du}{dx}(L) + hu(L) = 0 \tag{3}$$

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 1 \tag{4}$$

2.11.1 Problem 1

version 040417

$$\frac{d^2u}{dx^2} = f(x); 0 < x < L; u(0) = A; \frac{du}{dx}(1) = B$$

Note: I used L for the length instead of one. Will replace L by one at the very end. This makes it more clear. Compare the above to the standard form (Sturm-Liouville)

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f(x)$$

Therefore it becomes

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right) = -f(x)$$

$$p(x) = 1$$

Green function is $G(x, x_0)$ (will use x_0 which is what the book uses, instead of a, as x_0 is more clear). Green function is the solution to

$$\frac{d^2G(x,x_0)}{dx^2} = \delta(x-x_0)$$
$$G(0,x_0) = 0$$
$$\frac{dG(L,x_0)}{dx} = 0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2} = 0$ for $x \neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2} = 0$, which is a linear function in this case, is broken into two regions

$$G(x,x_0) = \begin{cases} A_1x + A_2 & 0 < x < x_0 \\ B_1x + B_2 & x_0 < x < L \end{cases}$$

The first solution, using $G(0,x_0) = 0$ gives $A_2 = 0$ and the second solution using $\frac{dG(1,x_0)}{dx} = 0$ gives $B_1 = 0$, hence the above reduces to

$$G(x,x_0) = \begin{cases} A_1x & x < x_0 \\ B_2 & x_0 < x \end{cases} \tag{1}$$

We are left with constants to A_1, B_2 to find. The continuity condition at $x = x_0$ gives

$$A_1x_0 = B_2 \tag{2}$$

The jump discontinuity of the derivative of $G(x,x_0)$ at $x = x_0$, gives the final equation

$$\left(\frac{d}{dx}G(x,x_0)\right)_{x_0<x} - \left(\frac{d}{dx}G(x,x_0)\right)_{x<x_0} = \frac{-1}{p(x_0)} = -1 \tag{2A}$$

Since $p(x) = 1$ in this problem. But

$$\frac{dG(x,x_0)}{dx} = \begin{cases} A_1 & x < x_0 \\ 0 & x_0 < x \end{cases} \tag{3}$$

Hence (2A) becomes

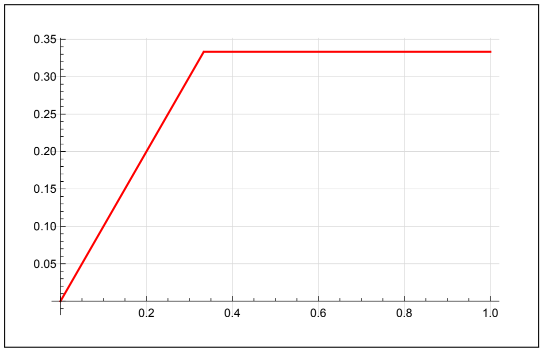
$$0 - (A_1) = -1$$

Therefore

$$A_1 = 1 \tag{4}$$

Solving (2,4) gives $B_2 = x_0$. Hence the Green function is, from (1)

$$G(x,x_0) = \begin{cases} x & x < x_0 \\ x_0 & x_0 < x \end{cases}$$



And $\frac{dG(x,x_0)}{dx_0}$ where now derivative is w.r.t. x_0, is

$$\frac{dG(x,x_0)}{dx_0} = \begin{cases} 0 & x < x_0 \\ 1 & x_0 < x \end{cases}$$

We now have all the information needed to evaluate the solution to the original ODE.

$$u(x) = \overbrace{\int_0^x G(x,x_0) f(x_0) \, dx_0}^{\text{particular solution}} + \overbrace{\left[p(x_0) G(x,x_0) \frac{du(x_0)}{dx_0} - p(x_0) u(x_0) \frac{dG(x,x_0)}{dx_0} \right]_{x_0=0}^{x_0=L}}^{\text{boundary terms}}$$

Since $p(x_0) = 1$ then

$$y(x) = \int_0^x G(x,x_0) f(x_0) \, dx_0 + \left[G(x,x_0) \frac{du(x_0)}{dx_0} - u(x_0) \frac{dG(x,x_0)}{dx_0} \right]_{x_0=0}^{x_0=L}$$

Let $u_h = \left[G(x,x_0) \frac{du(x_0)}{dx_0} - u(x_0) \frac{dG(x,x_0)}{dx_0} \right]_{x_0=0}^{x_0=L}$, hence

$$u_h = G(x,L) \frac{du(x_0)}{dx_0}(L) - u(L) \frac{dG(x,x_0)}{dx_0}(L) - G(x,0) \frac{du(x_0)}{dx_0}(0) + u(0) \frac{dG(x,x_0)}{dx_0}(0)$$

But

$$\begin{aligned} G(x,L) &= x \\ \frac{du(x_0)}{dx_0}(L) &= B \\ \frac{dG(x,x_0)}{dx_0}(L) &= 0 \\ G(x,0) &= 0 \\ \frac{dG(x,x_0)}{dx_0}(0) &= 1 \\ u(0) &= A \end{aligned}$$

Then

$$u_h = xB + A$$

We see that the boundary terms are linear in x, which is expected as the fundamental solutions for the homogenous solution as linear. The complete solution is

$$\begin{aligned}
y(x) &= \int_0^L G(x,x_0)\, f(x_0)\, dx_0 + (xB + A) \\
&= \int_0^L G(x,x_0)\, f(x_0)\, dx_0 + xB + A \\
&= \int_0^x x_0 f(x_0)\, dx_0 + \int_x^L x f(x_0)\, dx_0 + xB + A
\end{aligned}$$

For example, if $f(x) = x$, or $f(x_0) = x_0$ then (but remeber, we have to use $-f(x)$ since we are using S-L form)

$$\begin{aligned}
y(x) &== \int_0^x G(x,x_0)\,(-f(x_0))dx_0 + \int_x^1 G(x,x_0)\,(-f(x_0))dx_0 + (xB + A) \\
&= -\int_0^x x_0 x_0 dx_0 - \int_x^1 x x_0 dx_0 + xB + A \\
&= -\left(\frac{x_0^3}{3}\right)_0^x - x\left(\frac{x_0^2}{2}\right)_x^1 + xB + A \\
&= -\left(\frac{x^3}{3}\right) - x\left(\frac{1}{2} - \frac{x^2}{2}\right) + xB + A \\
&= A - \frac{1}{2}x + Bx + \frac{1}{6}x^3
\end{aligned}$$

To verify the result, this was solved directly, with $f(x) = x$, giving same answer as above.

```
In[39]:= DSolve[{u''[x] == x, u[0] == A0, u'[1] == B0}, u[x], x]
Out[39]= {{u[x] → 1/6 (6 A0 - 3 x + 6 B0 x + x^3)}}

In[40]:= Expand[%]
Out[40]= {{u[x] → A0 - x/2 + B0 x + x^3/6}}
```

And if $f(x) = x^2$, or $f(x_0) = x_0^2$, then

$$\begin{aligned}
y(x) &= \int_0^x x_0 \left(-f(x_0)\right) dx_0 + \int_x^L x \left(-f(x_0)\right) dx_0 + xB + A \\
&= -\int_0^x x_0 x_0^2 dx_0 - x\int_x^1 x_0^2 dx_0 + xB + A \\
&= -\left(\frac{x_0^4}{4}\right)_0^x - x\left(\frac{x_0^3}{3}\right)_x^1 + xB + A \\
&= -\left(\frac{x^4}{4}\right) - x\left(\frac{1}{3} - \frac{x^3}{3}\right) + xB + A \\
&= A - \frac{1}{3}x + Bx + \frac{1}{12}x^4
\end{aligned}$$

To verify the result, this was solved directly, with $f(x) = x^2$, giving same answer as above.

```
In[41]:= DSolve[{u''[x] == x^2, u[0] == A0, u'[1] == B0}, u[x], x]
```

$$\text{Out[41]= } \left\{\left\{u[x] \to \frac{1}{12}\left(12\,A0 - 4\,x + 12\,B0\,x + x^4\right)\right\}\right\}$$

```
In[42]:= Expand[%]
```

$$\text{Out[42]= } \left\{\left\{u[x] \to A0 - \frac{x}{3} + B0\,x + \frac{x^4}{12}\right\}\right\}$$

This shows the benefit of Green function. Once we know $G(x, x_0)$, then changing the source term, requires only convolution to find the new solution, instead of solving the ODE again as normally done.

2.11.2 Problem 2

$$\frac{d^2u}{dx^2} + u = f(x); 0 < x < L; u(0) = A; u(L) = B; L \neq n\pi$$

Solution

Compare the above to the standard form

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f(x)$$
$$-pu'' + qu = f(x)$$

Therefore

$$p(x) = -1$$

Green function is the solution to

$$\frac{d^2G(x, x_0)}{dx^2} + G(x, x_0) = \delta(x - x_0)$$
$$G(0, x_0) = 0$$
$$G(L, x_0) = 0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2} = 0$ for $x \neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2} + G(x, x0) = 0$, is broken into two regions

$$G(x, x_0) = \begin{cases} A_1 \cos x + A_2 \sin x & x < x_0 \\ B_1 \cos x + B_2 \sin x & x_0 < x \end{cases}$$

The first boundary condition on the left gives $A_1 = 0$. Second boundary conditions on the right gives

$$B_1 \cos L + B_2 \sin L = 0$$
$$B_1 = -B_2 \frac{\sin L}{\cos L}$$

Hence the solution now looks like

$$G(x,x_0)=\begin{cases} A_2\sin x & x<x_0\\ -B_2\frac{\sin L}{\cos L}\cos x+B_2\sin x & x_0<x\end{cases}$$

But

$$-B_2\frac{\sin L}{\cos L}\cos x+B_2\sin x=\frac{B_2}{\cos L}(\sin x\cos L-\cos x\sin L)$$

Using trig identity $\sin(a-b)=\sin a\cos b-\cos a\sin b$, the above can be written as $\frac{B_2}{\cos L}\sin(x-L)$, hence the solution becomes

$$G(x,x_0)=\begin{cases} A_2\sin x & x<x_0\\ \frac{B_2}{\cos L}\sin(x-L) & x_0<x\end{cases} \tag{1}$$

Continuity at x_0 gives

$$A_2\sin x_0=\frac{B_2}{\cos L}\sin(x_0-L) \tag{2}$$

And jump discontinuity on derivative of G gives

$$\begin{aligned}\frac{B_2}{\cos L}\cos(x_0-L)-A_2\cos x_0&=-\frac{1}{p(x)}=1\\ \frac{B_2}{\cos L}\cos(x_0-L)-A_2\cos x_0&=1 \qquad (3)\end{aligned}$$

Now we need to solve (2,3) for A_2,B_2 to obtain the final solution for $G(x,x_0)$. From (2),

$$A_2=\frac{B_2}{\cos L\sin x_0}\sin(x_0-L) \tag{4}$$

Plug into (3)

$$\begin{aligned}\frac{B_2}{\cos L}\cos(x_0-L)-\frac{B_2}{\cos L\sin x_0}\sin(x_0-L)\cos x_0&=1\\ \frac{B_2}{\cos L}\cos(x_0-L)-\frac{B_2}{\cos L}\sin(x_0-L)\frac{\cos x_0}{\sin x_0}&=1\\ B_2\cos(x_0-L)-B_2\sin(x_0-L)\frac{\cos x_0}{\sin x_0}&=\cos L\\ B_2\left(\cos(x_0-L)-\sin(x_0-L)\frac{\cos x_0}{\sin x_0}\right)&=\cos L\\ B_2(\sin x_0\cos(x_0-L)-\cos x_0\sin(x_0-L))&=\cos L\sin x_0\end{aligned}$$

But using trig identity $\sin(a-b)=\sin a\cos b-\cos a\sin b$ we can write above as

$$\begin{aligned}B_2(\sin(x_0-(x_0-L)))&=\cos L\sin x_0\\ B_2\sin L&=\cos L\sin x_0\\ B_2&=\frac{\cos L\sin x_0}{\sin L}\end{aligned}$$

Now that we found B_2, we go back and find A_2 from (4)

$$\begin{aligned} A_2 &= \frac{\cos L \sin x_0}{\sin L} \frac{1}{\cos L \sin x_0} \sin(x_0 - L) \\ &= \frac{\sin(x_0 - L)}{\sin L} \end{aligned}$$

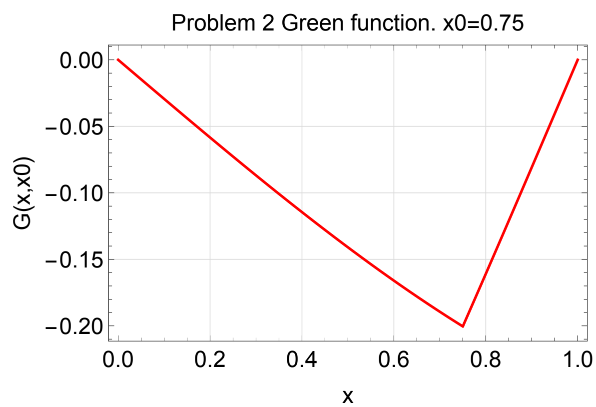
Therefore Green function is, from (1)

$$G(x, x_0) = \begin{cases} \frac{\sin(x_0 - L)}{\sin L} \sin x & x < x_0 \\ \frac{\cos L \sin x_0}{\sin L} \frac{1}{\cos L} \sin(x - L) & x_0 < x \end{cases}$$

Or

$$G(x, x_0) = \begin{cases} \frac{\sin(x_0 - L)}{\sin L} \sin x & x < x_0 \\ \frac{\sin x_0}{\sin L} \sin(x - L) & x_0 < x \end{cases} \tag{5}$$

It is symmetrical. Here is a plot of $G(x, x_0)$ for some arbitrary x_0 located at $x = 0.75$ for $L = 1$.



Now comes the hard part. We need to find the solution using

$$y(x) = \overbrace{\int_0^x G(x, x_0) f(x_0) \, dx_0}^{\text{particular solution}} + \overbrace{\left[p(x_0) G(x, x_0) \frac{du(x_0)}{dx_0} - p(x_0) u(x_0) \frac{dG(x, x_0)}{dx_0} \right]_{x_0=0}^{x_0=L}}^{\text{boundary terms}} \tag{6}$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (5), we find

$$\frac{dG(x, x_0)}{dx_0} = \begin{cases} \frac{\cos(x_0 - L)}{\sin L} \sin x & x < x_0 \\ \frac{\cos x_0}{\sin L} \sin(x - L) & x_0 < x \end{cases} \tag{7}$$

Now we plug everything in (6). But remember that $G(0, x_0) = 0, G(L, x_0) = 0, u(0) = A, u(L) =$

B. Hence

$$\begin{aligned}
\Delta &= \left[p\left(x_0\right) G\left(x,x_0\right) \frac{du\left(x_0\right)}{dx_0} - p\left(x_0\right) u\left(x_0\right) \frac{dG\left(x,x_0\right)}{dx_0} \right]_{x_0=0}^{x_0=L} \\
&= G\left(x,L\right) \frac{du\left(L\right)}{dx_0} - u\left(L\right) \frac{dG\left(x,L\right)}{dx_0} - G\left(x,0\right) \frac{du\left(0\right)}{dx_0} + u\left(0\right) \frac{dG\left(x,0\right)}{dx_0} \\
&= 0 - \left(B\right) \left(\frac{\cos\left(x_0 - L\right)}{\sin L} \sin x \right)_{x_0=L} - 0 + \left(A\right) \left(\frac{\cos x_0}{\sin L} \sin\left(x-L\right) \right)_{x_0=0} \\
&= -\left(B\right) \left(\frac{1}{\sin L} \sin x \right) + \left(A\right) \left(\frac{\sin\left(x-L\right)}{\sin L} \right) \\
&= -B \frac{\sin x}{\sin L} + A \frac{\sin\left(x-L\right)}{\sin L}
\end{aligned}$$

But $p = -1$, hence the above becomes

$$\begin{aligned}
\Delta &= -G\left(x,L\right) \frac{du\left(L\right)}{dx_0} + u\left(L\right) \frac{dG\left(x,L\right)}{dx_0} + G\left(x,0\right) \frac{du\left(0\right)}{dx_0} - u\left(0\right) \frac{dG\left(x,0\right)}{dx_0} \\
&= 0 + \left(B\right) \left(\frac{\cos\left(x_0 - L\right)}{\sin L} \sin x \right)_{x_0=L} + 0 - \left(A\right) \left(\frac{\cos x_0}{\sin L} \sin\left(x-L\right) \right)_{x_0=0} \\
&= +\left(B\right) \left(\frac{1}{\sin L} \sin x \right) - \left(A\right) \left(\frac{\sin\left(x-L\right)}{\sin L} \right) \\
&= B \frac{\sin x}{\sin L} - A \frac{\sin\left(x-L\right)}{\sin L}
\end{aligned}$$

We see that the boundary terms are linear combination of sin and cosine in x, which is expected as the fundamental solutions for the homogenous solution as linear combination of sin and cosine in x as was found initially above. Equation (6) becomes

$$y\left(x\right) = \int_0^x G\left(x,x_0\right) f\left(x_0\right) dx_0 + B \frac{\sin x}{\sin L} - A \frac{\sin\left(x-L\right)}{\sin L} \tag{8}$$

Now we can do the integration part. Therefore

$$\int_0^x G\left(x,x_0\right) f\left(x_0\right) dx_0 = \int_0^x \left(\frac{\sin\left(x-L\right)}{\sin L} \sin x_0 \right) f\left(x_0\right) dx_0 + \int_x^L \left(\frac{\sin x}{\sin L} \sin\left(x_0 - L\right) \right) f\left(x_0\right) dx_0$$

We can test the solution to see if it correct. Let $f(x)=x$ or $f(x_0)=x_0$, hence

$$\begin{aligned}\int_0^x G(x,x_0)f(x_0)\,dx_0 &= \int_0^x x_0\left(\frac{\sin(x-L)}{\sin L}\sin x_0\right)dx_0+\int_x^L x_0\left(\frac{\sin x}{\sin L}\sin(x_0-L)\right)dx_0\\
&=\frac{\sin(x-L)}{\sin L}\int_0^x x_0\sin x_0 dx_0+\frac{\sin x}{\sin L}\int_x^L x_0\sin(x_0-L)\,dx_0\\
&=\frac{\sin(x-L)}{\sin L}(-x\cos x+\sin x)+\frac{\sin x}{\sin L}(-L+x\cos(x-L)-\sin(x-L))\\
&=\frac{-x\cos x\sin(x-L)}{\sin L}+\frac{\sin x\sin(x-L)}{\sin L}-L\frac{\sin x}{\sin L}+\frac{x\cos(x-L)\sin x}{\sin L}-\frac{\sin x\sin(x-L)}{\sin L}\\
&=\frac{-x\cos x\sin(x-L)}{\sin L}-L\frac{\sin x}{\sin L}+\frac{x\cos(x-L)\sin x}{\sin L}\\
&=\frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L))\end{aligned}$$

Hence the solution is

$$\begin{aligned}u(x)&=\frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L))+\left(B\frac{\sin x}{\sin L}-A\frac{\sin(x-L)}{\sin L}\right)\\
&=\frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L)+B\sin x-A\sin(x-L))\end{aligned}$$

To verify, the problem is solved directly using CAS, and solution above using Green function was compared, same answer confirmed.

```
In[81]:= f = x;
L0 = 1;
A0 = 1;
B0 = 2;
computerSolution = u[x] /. First@DSolve[{u''[x] + u[x] == f, u[0] == A0, u[L0] == B0}, u[x], x];
mySolUsingGreenFunction = 1 / Sin[L0] (-L0 Sin[x] + x Cos[x - L0] Sin[x] - x Cos[x] Sin[x - L0] + B0 Sin[x] - A0 Sin[x - L0]);
Simplify[computerSolution - mySolUsingGreenFunction]

Out[87]= 0
```

2.11.3 Problem 3

$$\frac{d^2u}{dx^2}=f(x);0<x<L;u(0)=A;\frac{du}{dx}(L)+hu(L)=0$$

Solution

Compare the above to the standard form

$$\begin{aligned}-\frac{d}{dx}\left(p\frac{du}{dx}\right)&=f(x)\\-pu''&=f(x)\end{aligned}$$

Therefore

$$p(x)=-1$$

Green function is the solution to

$$\frac{d^2G(x,x_0)}{dx^2} = \delta(x - x_0)$$
$$G(0,x_0) = 0$$
$$\frac{d}{dx}G(L,x_0) + hG(L,x_0) = 0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2} = 0$ for $x \neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2} = 0$, is broken into two regions

$$G(x,x_0) = \begin{cases} A_1x + A_2 & x < x_0 \\ B_1x + B_2 & x_0 < x \end{cases}$$

The first boundary condition on the left gives $A_2 = 0$. Second boundary conditions on the right gives

$$B_1 + h(B_1L + B_2) = 0$$
$$B_1(1 + hL) = -hB_2$$
$$B_1 = \frac{-hB_2}{1 + hL}$$

Hence the solution now looks like

$$G(x,x_0) = \begin{cases} A_1x & x < x_0 \\ \left(\frac{-hB_2}{1+hL}\right)x + B_2 & x_0 < x \end{cases}$$

But

$$\left(\frac{-hB_2}{1+hL}\right)x + B_2 = \left(\frac{-hB_2}{1+hL}\right)x + \frac{B_2(1+hL)}{1+hL}$$
$$= \frac{B_2(1 + hL - hx)}{1 + hL}$$

Hence

$$G(x,x_0) = \begin{cases} A_1x & x < x_0 \\ \frac{B_2(1+hL-hx)}{1+hL} & x_0 < x \end{cases} \tag{1}$$

Continuity at x_0 gives

$$A_1x_0 = \frac{B_2(1 + hL - hx_0)}{1 + hL} \tag{2}$$

And jump discontinuity on derivative of G gives

$$\frac{B_2}{\cos L}\cos(x_0 - L) - A_2\cos x_0 = -\frac{1}{p(x)} = 1$$
$$\frac{-hB_2}{1 + hL} - A_1 = -\frac{1}{p(x)} = 1 \tag{3}$$

We solve (2,3) for A_1, B_2. From (3)

$$
\begin{aligned}
\frac{-hB_2}{1+hL} - A_1 &= 1 \\
A_1 &= \frac{-hB_2}{1+hL} - 1 \\
&= \frac{-hB_2}{1+hL} - \frac{(1+hL)}{1+hL} \\
&= \frac{-hB_2 - 1 - hL}{1+hL}
\end{aligned}
$$

Substituting in (2)

$$
\begin{aligned}
\frac{-hB_2 - 1 - hL}{1+hL} x_0 &= \frac{B_2 (1 + hL - hx_0)}{1+hL} \\
(-hB_2 - 1 - hL) x_0 &= B_2 (1 + hL - hx_0) \\
-hB_2 x_0 - x_0 - hLx_0 &= B_2 + hLB_2 - hx_0 B_2 \\
B_2 (-hx_0 - 1 - hL + hx_0) &= x_0 + hLx_0 \\
B_2 &= \frac{(1+hL) x_0}{-1 - hL} \\
&= \frac{-(1+hL)}{1+hL} x_0 \\
&= -x_0
\end{aligned}
$$

Hence

$$
\begin{aligned}
A_1 &= \frac{-hB_2 - 1 - hL}{1+hL} \\
&= \frac{-h(-x_0) - 1 - hL}{1+hL} \\
&= \frac{hx_0 - 1 - hL}{1+hL} \\
&= \frac{hx_0}{1+hL} - \frac{1+hL}{1+hL} \\
&= \frac{hx_0}{1+hL} - 1
\end{aligned}
$$

Therefore (1) becomes

$$
G(x, x_0) = \begin{cases} \left(\frac{hx_0}{1+hL} - 1\right) x & x < x_0 \\ \frac{-x_0(1+hL-hx)}{1+hL} & x_0 < x \end{cases} \tag{1}
$$

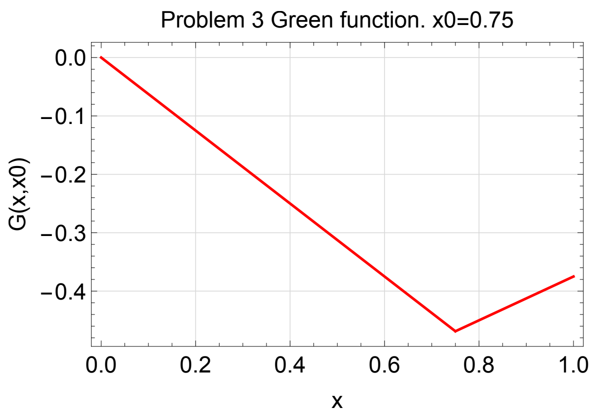
But

$$
\begin{aligned}
\frac{-x_0 (1 + hL - hx)}{1+hL} &= x_0 \frac{(-1 - hL + hx)}{1+hL} \\
&= x_0 \left(\frac{hx}{1+hL} - \frac{1+hL}{1+hL} \right) \\
&= x_0 \left(\frac{hx}{1+hL} - 1 \right)
\end{aligned}
$$

Hence (1) becomes

$$G(x,x_0)=\begin{cases}\left(\frac{hx_0}{1+hL}-1\right)x & x<x_0\\ \left(\frac{hx}{1+hL}-1\right)x_0 & x_0<x\end{cases} \tag{2}$$

We see they are symmetrical in x, x_0. Here is a plot of $G(x,x_0)$ for some arbitrary x_0 located at $x=0.75, h=1$, for $L=1$.



We need to find the solution using

$$y(x)=\int_0^x G(x,x_0)\,f(x_0)\,dx_0+p(x_0)\left[G(x,x_0)\frac{du(x_0)}{dx_0}-u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L} \tag{3}$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (2), we find

$$\frac{dG(x,x_0)}{dx_0}=\begin{cases}\frac{hx}{1+hL} & x<x_0\\ \frac{hx}{1+hL}-1 & x_0<x\end{cases} \tag{4}$$

Now we plug everything in (3). But remember that $G(0,x_0)=0, \frac{dG(L,x_0)}{dx}=-hG(L,x_0), u(0)=A, \frac{du(L)}{dx}=-hu(L)$. Hence

$$\begin{aligned}\Delta&=\left[G(x,x_0)\frac{du(x_0)}{dx_0}-u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}\\
&=G(x,L)\frac{du(L)}{dx_0}-u(L)\frac{dG(x,L)}{dx_0}-G(x,0)\frac{du(0)}{dx_0}+u(0)\frac{dG(x,0)}{dx_0}\\
&=G(x,L)(-hu(L))-u(L)(-hG(x_0,L))-0+(A)\left(\frac{dG(x,0)}{dx_0}\right)\\
&=\overbrace{-G(x,L)hu(L)+u(L)hG(x,L)}^{0}+A\left(\frac{hx}{1+hL}-1\right)_{x_0=0}\\
&=A\left(\frac{hx}{1+hL}-1\right)\end{aligned}$$

Now we do the integration, From (3), and since $p = -1$ then we obtain

$$
\begin{aligned}
y(x) &= \int_0^x G(x,x_0) f(x_0)\,dx_0 - A\left(\frac{hx}{1+hL}-1\right)\\
&= \overbrace{\int_0^x G(x,x_0) f(x_0)\,dx_0 + \int_x^L G(x,x_0) f(x_0)\,dx_0}^{\text{particular solution}} \overbrace{- A\left(\frac{hx}{1+hL}-1\right)}^{\text{boundary terms}}
\end{aligned}
$$

We see that the boundary terms are linear combination x, which is expected as the fundamental solutions for the homogenous solution as linear in x as was found initially above. Plugging values From (3) for $G(x,x_0)$ for each region into the above gives

$$
y(x) = \int_0^x \left(\frac{hx}{1+hL}-1\right) x_0 f(x_0)\,dx_0 + \int_x^L \left(\frac{hx_0}{1+hL}-1\right) x f(x_0)\,dx_0 - A\left(\frac{hx}{1+hL}-1\right)
$$

This completes the solution. Now we should test it. Let $f(x) = x$ or $f(x_0) = x_0$ and compare to direction solution. The above becomes

$$
\begin{aligned}
y(x) &= \left(\frac{hx}{1+hL}-1\right)\int_0^x x_0^2 dx_0 + x\int_x^L \left(\frac{hx_0}{1+hL}-1\right) x_0 dx_0 - A\left(\frac{hx}{1+hL}-1\right)\\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x_0^3}{3}\right)_0^x + x\int_x^L \left(\frac{hx_0^2}{1+hL}-x_0\right) dx_0 - A\left(\frac{hx}{1+hL}-1\right)\\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x^3}{3}\right) + x\left(\frac{h}{1+hL}\frac{x_0^3}{3}-\frac{x_0^2}{2}\right)_x^L - A\left(\frac{hx}{1+hL}-1\right)\\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x^3}{3}\right) + x\left(\frac{h}{1+hL}\frac{L^3}{3}-\frac{L^2}{2}-\frac{h}{1+hL}\frac{x^3}{3}+\frac{x^2}{2}\right) - A\left(\frac{hx}{1+hL}-1\right)\\
&= \frac{1}{6(1+Lh)}\left(-hL^3x - 3L^2x + hLx^3 + x^3\right) - A\left(\frac{hx}{1+hL}-1\right)\\
&= \frac{1}{6(1+Lh)}\left(x^3(1+hL) - x\left(hL^3+3L^2\right)\right) + A\left(1-\frac{hx}{1+hL}\right)\\
&= \frac{1}{6(1+Lh)}\left(x^3(1+hL) - xL^2(hL+3)\right) + A\left(\frac{1+h(L-x)}{1+hL}\right)
\end{aligned}
$$

To verify, the problem is solved directly, and solution above using Green function was compared, same answer confirmed.

```
In[152]:= ClearAll[x, L0, y, A0, h]
f = x;
computerSolution = u[x] /. First@DSolve[{u''[x] == f, u[0] == A0, u'[L0] + h u[L0] == 0}, u[x], x];
mySolUsingGreenFunction = 1/(6 (1 + L0 h)) (x^3 (1 + h L0) - x L0^2 (h L0 + 3)) + A0 (1 + h (L0 - x))/(1 + h L0);
Simplify[computerSolution - mySolUsingGreenFunction]

Out[156]= 0
```

2.11.4 Problem 4

$$u'' + 2u' + u = f(x);0 < x < L; u(0) = 0; u(L) = 1$$

Solution

Since the coefficient on u' is 2, then the Integrating factor is $\mu(x) = e^{\int 2dx} = e^{2x}$. Multiplying the ODE by $\mu(x)$ gives

$$\begin{aligned} e^{2x}u'' + 2e^{2x}u' + e^{2x}u &= e^{2x}f(x) \\ \frac{d}{dx}\left(e^{2x}u'\right) + e^{2x}u &= e^{2x}f(x) \end{aligned}$$

To keep the solution consistent with the class notes, we now multiply both sides by -1 in order to obtain the same form as used in class notes. Hence our ODE is

$$-\frac{d}{dx}\left(e^{2x}u'\right) - e^{2x}u = -e^{2x}f(x)$$

We now see from above that

$$p(x) = e^{2x}$$

Once we found $p(x)$, we now find the Green function. The Green function is the solution to

$$\begin{aligned} \frac{d^2G(x,x_0)}{dx^2} + 2\frac{dG(x,x_0)}{dx} + G(x,x_0) &= \delta(x-x_0) \\ G(0,x_0) &= 0 \\ G(L,x_0) &= 0 \end{aligned}$$

Where x_0 is the location of the impulse. We first need to find fundamental solutions to the homogeneous ODE. The solution to $u'' + 2u' + u = 0$ is found by characteristic method. $r^2 + 2r + 1 = 0$, hence $(r+1)^2 = 0$. Therefore the roots are $r = -1$, double root. Hence the fundamental solutions are

$$\begin{aligned} u_1 &= e^{-x} \\ u_2 &= xe^{-x} \end{aligned}$$

Therefore

$$\begin{aligned} G(x,x_0) &= \begin{cases} A_1u_1 + A_2u_2 & x < x_0 \\ B_1u_1 + B_2u_2 & x > x_0 \end{cases} \\ &= \begin{cases} A_1e^{-x} + A_2xe^{-x} & x < x_0 \\ B_1e^{-x} + B_2xe^{-x} & x > x_0 \end{cases} \end{aligned}$$

The first boundary condition on the left end gives $A_1 = 0$ from the first region. The second B.C. on the right end, gives

$$\begin{aligned} B_1e^{-L} + B_2Le^{-L} &= 0 \\ B_1 &= -\frac{B_2Le^{-L}}{e^{-L}} = -B_2L \end{aligned}$$

Hence the above solution now reduces to

$$G(x,x_0)=\begin{cases} A_2xe^{-x} & x<x_0\\ -B_2Le^{-x}+B_2xe^{-x} & x>x_0\end{cases}$$

Simplifying $-B_2Le^{-x}+B_2xe^{-x}=B_2(x-L)e^{-x}$, the above can be written as

$$G(x,x_0)=\begin{cases} A_2xe^{-x} & x<x_0\\ B_2(x-L)e^{-x} & x>x_0\end{cases}\tag{1}$$

Continuity at x_0 gives

$$\begin{aligned} A_2x_0e^{-x_0} &= B_2(x_0-L)e^{-x_0}\\ A_2x_0 &= B_2(x_0-L) \end{aligned}\tag{2}$$

And jump discontinuity on derivative of G gives

$$\frac{d}{dx}G(x,x_0)=\begin{cases} A_2\left(e^{-x}-xe^{-x}\right) & x<x_0\\ B_2(1-x+L)e^{-x} & x>x_0\end{cases}$$

Hence (important note: we use $\frac{-1}{p(x_0)}$ below and not $\frac{1}{p(x_0)}$ because we started with $\frac{-d}{dx}\left(p\frac{dy}{dx}\right)+\cdots$ instead of $+\frac{d}{dx}\left(p\frac{dy}{dx}\right)+\cdots$)

$$B_2(1-x_0+L)e^{-x_0}-A_2\left(e^{-x_0}-x_0e^{-x_0}\right)=\frac{-1}{p(x_0)}=\frac{-1}{e^{2x_0}}=-e^{-2x_0}$$

Dividing by e^{-x_0} to simplify gives

$$B_2(1-x_0+L)-A_2(1-x_0)=-e^{-x_0}\tag{3}$$

We solve (2,3) for A_1,B_2. From (3)

$$B_2=\frac{-e^{-x_0}+A_2(1-x_0)}{1-x_0+L}\tag{4}$$

Substituting in (2)

$$\begin{aligned} A_2x_0 &= \frac{-e^{-x_0}+A_2(1-x_0)}{1-x_0+L}(x_0-L)\\ A_2x_0(1-x_0+L) &= -e^{-x_0}(x_0-L)+A_2(1-x_0)(x_0-L)\\ A_2\left(x_0(1-x_0+L)-(1-x_0)(x_0-L)\right) &= -e^{-x_0}(x_0-L)\\ A_2 &= \frac{-e^{-x_0}(x_0-L)}{x_0(1-x_0+L)-(1-x_0)(x_0-L)}\\ &= \frac{1}{L}e^{-x_0}(L-x_0) \end{aligned}$$

Hence, from (4)

$$\begin{aligned} B_2 &= \frac{-e^{-x_0}+\left(\frac{1}{L}e^{-x_0}(L-x_0)\right)(1-x_0)}{1-x_0+L}\\ &= -\frac{1}{L}x_0e^{-x_0} \end{aligned}$$

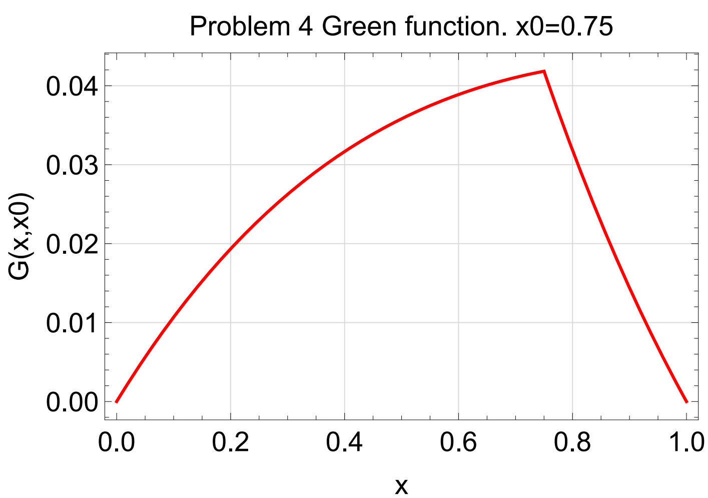
Therefore the solution (1) becomes

$$G(x,x_0)=\begin{cases}\frac{1}{L}e^{-x_0}(L-x_0)\,xe^{-x} & x<x_0\\ \frac{1}{L}x_0e^{-x_0}(L-x)\,e^{-x} & x>x_0\end{cases} \tag{5}$$

Or

$$G(x,x_0)=\begin{cases}\frac{(L-x_0)}{L}xe^{-x_0-x} & x<x_0\\ \frac{(L-x)}{L}x_0e^{-x_0-x} & x>x_0\end{cases} \tag{5}$$

We see they are symmetrical in x, x_0. Here is a plot of $G(x,x_0)$ for some arbitrary x_0 located at $x=0.75$ for $L=1$.



We now need to find the solution using

$$y(x)=\overbrace{\int_0^x G(x,x_0)\,f(x_0)\,dx_0}^{\text{particular solution}}+\overbrace{\left[p(x_0)\,G(x,x_0)\,\frac{du(x_0)}{dx_0}-p(x_0)\,u(x_0)\,\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}}^{\text{homogeneous solution/boundary terms}} \tag{6}$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (5), we find

$$\frac{dG(x,x_0)}{dx_0}=\begin{cases}-\frac{xe^{-x-x_0}}{L}-\frac{xe^{-x-x_0}(L-x_0)}{L} & x<x_0\\ \frac{e^{-x-x_0}(L-x)}{L}-\frac{(L-x)x_0e^{-x-x_0}}{L} & x>x_0\end{cases} \tag{7}$$

Now we plug everything in (3). But remember that $G(x,0)=0, G(x,L)=0, u(0)=0, u(L)=$

$1, p(x) = e^{2x}$. The following is the result of the homogeneous part

$$
\begin{aligned}
\Delta &= \left[p(x_0)\,G(x,x_0)\,\frac{du(x_0)}{dx_0} - p(x_0)\,u(x_0)\,\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}\\
&= \left(e^{2L}\right)G(x,L)\,\frac{du(L)}{dx_0} - \left(e^{2L}\right)u(L)\,\frac{dG(x,L)}{dx_0} - \left(e^{2(0)}\right)G(x,0)\,\frac{du(0)}{dx_0} + \left(e^{2(0)}\right)u(0)\,\frac{dG(x,0)}{dx_0}\\
&= 0 - e^{2L}(1)\overbrace{\left(-\frac{xe^{-x-x_0}}{L} - \frac{xe^{-x-x_0}(L-x_0)}{L}\right)_{x_0=L}}^{x<x_0 \text{ branch from (7)}} - 0 + (0)\\
&= -e^{2L}\left(-\frac{xe^{-x-L}}{L} - \frac{xe^{-x-L}(L-L)}{L}\right)\\
&= e^{2L}\left(\frac{xe^{-x-L}}{L}\right)\\
&= \frac{xe^{L-x}}{L}
\end{aligned}
$$

Now we complete the integration, From (3)

$$
\begin{aligned}
u(x) &= \overbrace{\int_0^x G(x,x_0)\,f(x_0)\,dx_0 +}^{\text{particular}} \overbrace{\left(\frac{xe^{L-x}}{L}\right)}^{\text{homogeneous}}\\
&= \int_0^x G(x,x_0)\,f(x_0)\,dx_0 + \int_x^L G(x,x_0)\,f(x_0)\,dx_0 + \frac{xe^{L-x}}{L}
\end{aligned}
$$

Plug-in in values From (5) $G(x,x_0)$ for each region,

$$
u(x) = \int_0^x \overbrace{\left(\frac{(L-x)}{L}x_0e^{-x_0-x}\right)}^{\text{from } x>x_0 \text{ branch in (5)}} g(x_0)\,dx_0 + \int_x^L \overbrace{\left(\frac{(L-x_0)}{L}xe^{-x_0-x}\right)}^{\text{from } x<x_0 \text{ branch in (5)}} g(x_0)\,dx_0 + \frac{1}{L}xe^{-x+L}
$$

This completes the solution. Now we should test it. Let $f(x) = x$ or $f(x_0) = x_0$. But since we multiplied by $-e^{2x}$ (integrating factor) at start, we should now use $g(x_0) = -e^{2x_0}x_0$ as $f(x_0)$ below.The above becomes

$$
\begin{aligned}
u(x) &= \int_0^x \left(\frac{(L-x)}{L}x_0e^{-x_0-x}\right)\left(-e^{2x_0}x_0\right)dx_0 + \int_x^L \left(\frac{(L-x_0)}{L}xe^{-x_0-x}\right)\left(-e^{2x_0}x_0\right)dx_0 + \frac{xe^{L-x}}{L}\\
&= -\frac{(L-x)\,e^{-x}}{L}\int_0^x x_0^2e^{x_0}dx_0 - \frac{xe^{-x}}{L}\int_x^L (L-x_0)\,x_0e^{x_0}dx_0 + \frac{xe^{L-x}}{L} \qquad (8)
\end{aligned}
$$

But

$$
\int_0^x x_0^2e^{x_0}dx_0 = -2 + e^x\left(2 + x^2 - 2x\right)
$$

$$
\int_x^L e^{x_0}(L-x_0)\,x_0dx_0 = e^L(L-2) + e^x\left(2 + L - 2x - Lx + x^2\right)
$$

Hence (8) becomes

$$
u(x) = -\frac{(L-x)\,e^{-x}}{L}\left(-2 + e^x\left(2 + x^2 - 2x\right)\right) - \frac{xe^{-x}}{L}\left(e^L(L-2) + e^x\left(2 + L - 2x - Lx + x^2\right)\right) + \frac{xe^{L-x}}{L}
$$

Which can be simplified to

$$u(x) = \frac{1}{L} e^{-x} \left(\left(3e^{L} - 2 \right) x + L \left(2 + e^{x} (x-2) - x e^{L} \right) \right)$$

For $L = 1$, the above becomes

$$u(x) = x - 2\frac{x}{e^x} + \frac{2}{e^x} + 2x\frac{e}{e^x} - 2$$

Verification

To verify the above, a plot of the solution was compare to Mathematica result. Here is plot of the result. My solution gives exact plot as Mathematica.

```
In[773]:=
        ClearAll[L, x, f]
        f = x;
        computerSolution = u[x] /. First@DSolve[{u''[x] + 2 u'[x] + u[x] == f, u[0] == 0, u[L] == 1}, u[x], x]
```

Out[775]= $\frac{e^{-x} \left(2 L - 2 e^{x} L - 2 x + 3 e^{L} x - e^{L} L x + e^{x} L x\right)}{L}$

```
In[778]:= mysolution = - (L - x) Exp[-x] / L (-2 + Exp[x] (2 + x^2 - 2 x)) - x Exp[-x] / L (Exp[L] (L - 2) + Exp[x] (2 + L - 2 x - L x + x^2)) + x / L Exp[L - x];
          Simplify[mysolution]
```

Out[779]= $\frac{e^{-x} \left(\left(-2 + 3 e^{L}\right) x + L \left(2 + e^{x} (-2 + x) - e^{L} x\right)\right)}{L}$

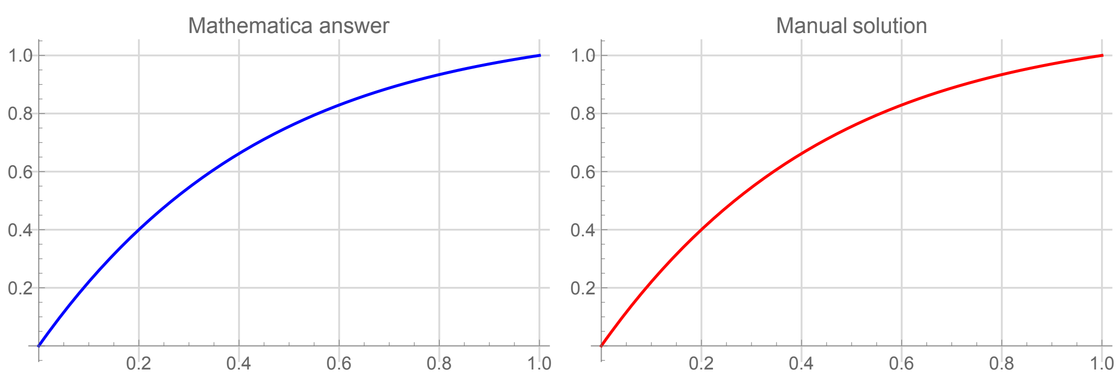
```
In[780]:= (mysolution - computerSolution) // Simplify
```

Out[780]= 0

```
In[781]:= L = 1;
          p1 = Plot[computerSolution, {x, 0, 1}, PlotRange → All, ImageSize → 300, PlotLabel → "Mathematica answer", PlotStyle → Blue, GridLines → Automatic,
              GridLinesStyle → LightGray];

In[783]:= p2 = Plot[mysolution, {x, 0, 1}, PlotRange → All, ImageSize → 300, PlotLabel → "Manual solution, Green function method", PlotStyle → {Red},
              GridLines → Automatic, GridLinesStyle → LightGray];
```

Out[575]=



2.12 HW 11

Math 322 Homework 11

Due Wednesday Dec. 14, 2016

1. (a) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1a}$$

in a semi-infinite 2D domain with boundary condition $u(x, 0) = h(x)$.
(b) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1b}$$

in a semi-infinite 2D domain with boundary condition $\partial u(x, 0)/\partial y = h(x)$.
(c) (a) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1c}$$

in a semi-infinite 3D domain with boundary condition $\partial u(x, 0, z)/\partial y = h(x, z)$.

2. Using the method of images, solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{2}$$

in the 2D domain $x \geq 0$, $y \geq 0$ with boundary conditions $u(0, y) = g(y)$ and $u(x, 0) = h(x)$.

The following are the general steps used in all the problems below :

1. Image points were placed to satisfy homogenous boundary conditions for Green function using the solution for infinite domain.
2. The Green formula was applied to determine the particular solution and the boundary terms.
3. Derivative of Green function was found and used in the result found above.
4. The role of $\vec{x}_0, \vec{x}$ was reversed in the final expression to express the final result as $u\left(\vec{x}\right)$ instead of $u\left(\vec{x}_0\right)$.

2.12.1 Problem 1

2.12.1.1 Part (a)

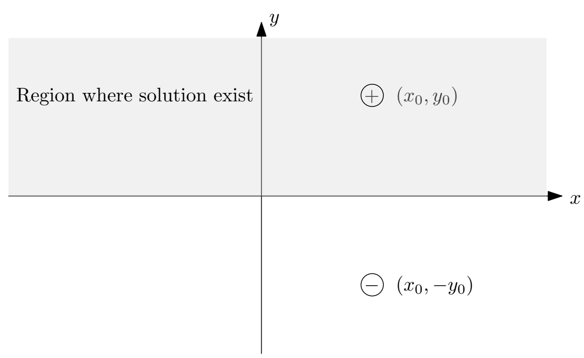
Green function on infinite domain, which is the solution to

$$\nabla^2 G\left(\vec{x}, \vec{x}_0\right) = \delta\left(\vec{x} - \vec{x}_0\right)$$

Is given by

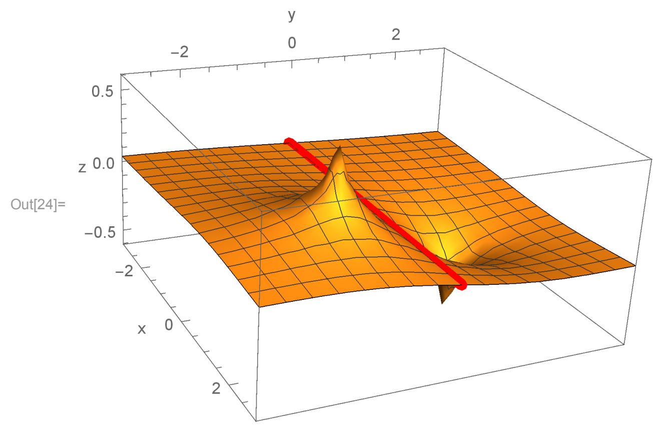
$$\begin{aligned}
G_\infty\left(\vec{x},\vec{x}_0\right) &= \frac{1}{2\pi}\ln\left(r\right)\\
&= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right)\\
&= \frac{1}{2\pi}\ln\left(\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2}\right)\\
&= \frac{1}{4\pi}\ln\left(\left(x-x_0\right)^2+\left(y-y_0\right)^2\right)
\end{aligned}$$

By placing a negative impulse at location $\vec{x}_0^* = \left(x_0,-y_0\right)$, the Green function for semi-infinite domain is obtained



$$\begin{aligned}
G\left(\vec{x},\vec{x}_0\right) &= \frac{1}{2\pi}\ln\left(r_1\right)-\frac{1}{2\pi}\ln\left(r_2\right)\\
&= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right)-\frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0^*\right|\right)\\
&= \frac{1}{4\pi}\left(\ln\left(\left(x-x_0\right)^2+\left(y-y_0\right)^2\right)-\ln\left(\left(x-x_0\right)^2+\left(y+y_0\right)^2\right)\right)\\
&= \frac{1}{4\pi}\ln\frac{\left(x-x_0\right)^2+\left(y-y_0\right)^2}{\left(x-x_0\right)^2+\left(y+y_0\right)^2} \qquad (1)
\end{aligned}$$

The following is 3D plot of the above Green function, showing the image impulse and showing that $G=0$ at the line $y=0$ (marked as red)



The Green function in (1) is now used to solve $\nabla^2 u(\vec{x}) = f(\vec{x})$, with $u(x,0) = h(x)$. Starting with Green formula for 2D

$$\begin{aligned}\iint u(\vec{x})\nabla^2 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla^2 u(\vec{x})\,dA &= \oint \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot \hat{n}\,ds\\ &= \oint \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot\left(-\hat{\jmath}\right)\,ds\\ &= \oint \left(G(\vec{x},\vec{x}_0)\nabla u(\vec{x}) - u(\vec{x})\nabla G(\vec{x},\vec{x}_0)\right)\cdot\hat{\jmath}\,ds\\ &= \int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx\end{aligned}$$

But $\nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x},\vec{x}_0)$ and $\nabla^2 u(\vec{x}) = f(\vec{x})$, therefore the above becomes

$$\iint u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\,dA - \iint G(\vec{x},\vec{x}_0)f(\vec{x})\,dA = \int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx$$

Since $\int\int u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\,dA = u(\vec{x}_0)$, the above reduces to

$$u(\vec{x}_0) - \iint G(\vec{x},\vec{x}_0)f(\vec{x})\,dA = \int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{G(\vec{x},\vec{x}_0)}{dy}\right)_{y=0} dx$$

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0)f(\vec{x})\;dA + \int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx \tag{2}$$

And since $G(\vec{x},\vec{x}_0) = 0$ at $y = 0$, therefore

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0)f(\vec{x})\;dA + \int_{-\infty}^{\infty}\left(-u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx$$

And since $u\left(\vec{x}\right) = h(x)$ at $y = 0$, then

$$u\left(\vec{x}_0\right) = \iint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right)\, dA - \int_{-\infty}^{\infty} h(x) \left(\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0} dx \tag{3}$$

$\left(\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy}\right)_{y=0}$ is now evaluated to complete the solution. Using $G\left(\vec{x},\vec{x}_0\right)$ in equation (1), therefore

$$\begin{aligned}
\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy} &= \frac{1}{4\pi}\frac{d}{dy}\left(\ln\left((x-x_0)^2 + \left(y-y_0\right)^2\right) - \ln\left((x-x_0)^2 + \left(y+y_0\right)^2\right)\right) \\
&= \frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x-x_0)^2 + \left(y-y_0\right)^2} - \frac{2\left(y+y_0\right)}{(x-x_0)^2 + \left(y+y_0\right)^2}\right)
\end{aligned}$$

Evaluating the above at $y = 0$ gives

$$\begin{aligned}
\left(\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy}\right)_{y=0} &= \frac{1}{4\pi}\left(\frac{-2y_0}{(x-x_0)^2 + y_0^2} - \frac{2y_0}{(x-x_0)^2 + y_0^2}\right) \\
&= \frac{-1}{\pi}\left(\frac{y_0}{(x-x_0)^2 + y_0^2}\right)
\end{aligned}$$

Replacing the above into (3) gives

$$u\left(\vec{x}_0\right) = \iint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dA + \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{(x-x_0)^2 + y_0^2}\, dx$$

Using the expression for $G\left(\vec{x},\vec{x}_0\right)$ from (1), the above result becomes

$$u\left(x_0,y_0\right) = \frac{1}{4\pi}\int_{x=-\infty}^{\infty}\int_{y=0}^{\infty} \ln\frac{(x-x_0)^2 + \left(y-y_0\right)^2}{(x-x_0)^2 + \left(y+y_0\right)^2} f\left(x,y\right) dy dx + \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{(x-x_0)^2 + y_0^2}\, dx$$

And finally, order of $\vec{x},\vec{x}_0$ is reversed giving

$$u\left(x,y\right) = \frac{1}{4\pi}\int_{x_0=-\infty}^{\infty}\int_{y_0=0}^{\infty} \ln\frac{(x_0-x)^2 + \left(y_0-y\right)^2}{(x_0-x)^2 + \left(y_0+y\right)^2} f\left(x_0,y_0\right) dy_0 dx_0 + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0-x)^2 + y^2}\, dx_0$$

2.12.1.2 Part (b)

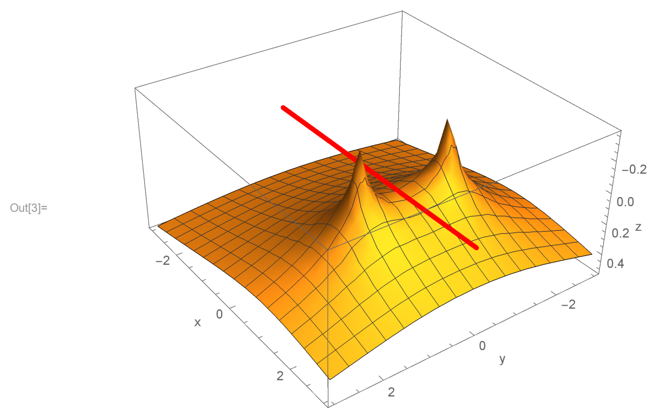
This is similar to part (a), and the image is placed on the same location as shown above, but now the boundary conditions are different. Starting from equation (2) in part (a)

$$u\left(\vec{x}_0\right) = \iint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dA + \int_{-\infty}^{\infty} \left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(\vec{x}\right)}{\partial y} - u\left(\vec{x}\right)\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0} dx \tag{1}$$

But now $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ and not $G(\vec{x},\vec{x}_0) = 0$ as in part (a). This means the image is a positive impulse and not negative as in part (a). Therefore $G(\vec{x},\vec{x}_0)$ becomes the following

$$\begin{aligned} G(\vec{x},\vec{x}_0) &= \frac{1}{2\pi}\ln(r_1) + \frac{1}{2\pi}\ln(r_2) \\ &= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right) + \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0^*\right|\right) \\ &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2 + (y-y_0)^2\right) + \ln\left((x-x_0)^2 + (y+y_0)^2\right)\right) \end{aligned} \tag{2}$$

The following is 3D plot of the above Green function, showing that showing that $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ (marked as red)



Green function, semi-infinite 2D, x0 at (1,1) and image at (1,-1) part(b)

Since $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ then (1) becomes

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA + \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y}\right)_{y=0} dx$$

But $\frac{\partial u(\vec{x})}{\partial y} = h(x)$ at $y = 0$, hence the above reduces to

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA + \int_{-\infty}^{\infty} G(\vec{x},\vec{x}_0)_{y=0}\, h(x)\ dx \tag{3}$$

Evaluating $G(\vec{x},\vec{x}_0)$ at $y = 0$ gives

$$\begin{aligned} G(\vec{x},\vec{x}_0)_{y_0=0} &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2 + (y-y_0)^2\right) + \ln\left((x-x_0)^2 + (y+y_0)^2\right)\right)_{y=0} \\ &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2 + y_0^2\right) + \ln\left((x-x_0)^2 + y_0^2\right)\right) \\ &= \frac{1}{4\pi}\ln\left(\left((x-x_0)^2 + y_0^2\right)^2\right) \\ &= \frac{1}{2\pi}\ln\left((x-x_0)^2 + y_0^2\right) \end{aligned}$$

Substituting the above in RHS of (3) gives

$$u\left(\vec{x}_0\right)=\int_{x=-\infty}^{\infty}\int_{y=0}^{\infty}G\left(\vec{x},\vec{x}_0\right)f\left(x,y\right)dydx+\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left((x-x_0)^2+y_0^2\right)h\left(x\right)\ dx$$

Reversing the role of $\vec{x}_0,\vec{x}$ gives

$$u\left(\vec{x}\right)=\int_{x_0=-\infty}^{\infty}\int_{y_0=0}^{\infty}G\left(\vec{x},\vec{x}_0\right)f\left(x_0,y_0\right)dy_0dx_0+\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left((x_0-x)^2+y^2\right)h\left(x_0\right)\ dx_0$$

2.12.1.3 Part (c)

In infinite 3D domain, the Green function for Poisson PDE is given by

$$G\left(\vec{x},\vec{x}_0\right)=\frac{-1}{4\pi r}$$

Where r is given by

$$r=\sqrt{(x-x_0)^2+\left(y-y_0\right)^2+(z-z_0)^2}$$

$\vec{x}_0=\left(x_0,y_0,z_0\right)$ is the location of the impulse. Since $\frac{\partial G}{\partial y}=0$, then the same sign impulse is located at $x_0^*=\left(x_0,-y_0,z_0\right)$, and the Green function becomes

$$\begin{aligned}G\left(\vec{x},\vec{x}_0\right)&=\frac{-1}{4\pi r_0}-\frac{1}{4\pi r_0^*}\\&=\frac{1}{4\pi}\left(-\frac{1}{\sqrt{(x-x_0)^2+\left(y-y_0\right)^2+(z-z_0)^2}}-\frac{1}{\sqrt{(x-x_0)^2+\left(y+y_0\right)^2+(z-z_0)^2}}\right)\end{aligned}\tag{1}$$

Using Green formula in 3D gives

$$\begin{aligned}\iiint u\left(\vec{x}\right)\nabla^2G\left(\vec{x},\vec{x}_0\right)-G\left(\vec{x},\vec{x}_0\right)\nabla^2u\left(\vec{x}\right)dV&=\iint\left(u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right)-G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right)\right)\cdot\hat{n}\ dxdz\\&=\iint\left(u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right)-G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right)\right)\cdot\left(-\hat{\jmath}\right)\ dxdz\\&=\iint\left(G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right)-u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right)\right)\cdot\hat{\jmath}\ dxdz\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(x,y,z\right)}{\partial y}-u\left(\vec{x}\right)\frac{\partial G\left(x,y,z,\vec{x}_0\right)}{\partial y}\right)_{y=0}dxdz\end{aligned}$$

But $\nabla^2G\left(\vec{x},\vec{x}_0\right)=\delta\left(\vec{x},\vec{x}_0\right)$ and $\nabla^2u\left(\vec{x}\right)=f\left(\vec{x}\right)$, and the above becomes

$$\iiint u\left(\vec{x}\right)\delta\left(\vec{x},\vec{x}_0\right)dV-\iiint G\left(\vec{x},\vec{x}_0\right)f\left(\vec{x}\right)dv=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(\vec{x}\right)}{\partial y}-u\left(\vec{x}\right)\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0}dxdz$$

But $\iiint u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\,dV = u(\vec{x}_0)$, hence

$$u(\vec{x}_0) - \iiint G(\vec{x},\vec{x}_0)\,f(\vec{x})\,dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dxdz$$

Rearranging

$$u(\vec{x}_0) = \iiint G(\vec{x},\vec{x}_0)\,f(\vec{x})\,dV + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dxdz \tag{2}$$

But $\left(\frac{\partial u(\vec{x})}{\partial y}\right)_{y=0} = h(x,z)$ and we impose $\left(\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} = 0$, therefore the above becomes

$$u(\vec{x}_0) = \iiint G(\vec{x},\vec{x}_0)\,f(\vec{x})\,dV + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{x},\vec{x}_0)_{y=0}\,h(x,z)\;dxdz \tag{3}$$

Evaluating $G(\vec{x},\vec{x}_0)_{y=0}$ gives

$$\begin{aligned}
G(\vec{x},\vec{x}_0)_{y=0} &= \frac{1}{4\pi}\left(-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2+(z-z_0)^2}}\right)_{y=0} \\
&= \frac{1}{4\pi}\left(\frac{-1}{\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}}\right) \\
&= -\frac{1}{2\pi\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}}
\end{aligned}$$

Using the above in (3) results in

$$u(\vec{x}_0) = \iiint G(\vec{x},\vec{x}_0)\,f(\vec{x})\,dV - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}}h(x,z)\;dxdz$$

And finally reversing the role of $\vec{x}_0, \vec{x}$ gives the final answer

$$u(\vec{x}) = \iiint G(\vec{x},\vec{x}_0)\,f(\vec{x}_0)\,dV_0 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{(x_0-x)^2+y^2+(z_0-z)^2}}h(x_0,z_0)\;dx_0dz_0$$

2.12.2 Problem 2

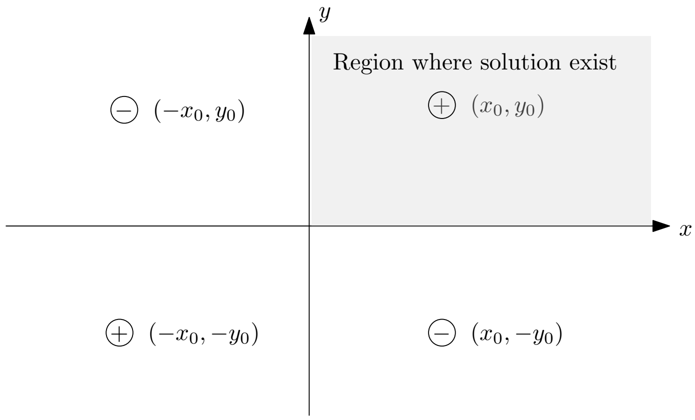
Green function in 2D on infinite domain, which is the solution to

$$\nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0)$$

Is given by

$$G_\infty(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln(r)$$

A negative impulse is placed $\vec{x}_1 = (x_0, -y_0)$ and another negative impulse at $\vec{x}_2 = (-x_0, y_0)$ and positive one at $\vec{x}_3 = (-x_0, -y_0)$. The following is a diagram showing the placement of images.



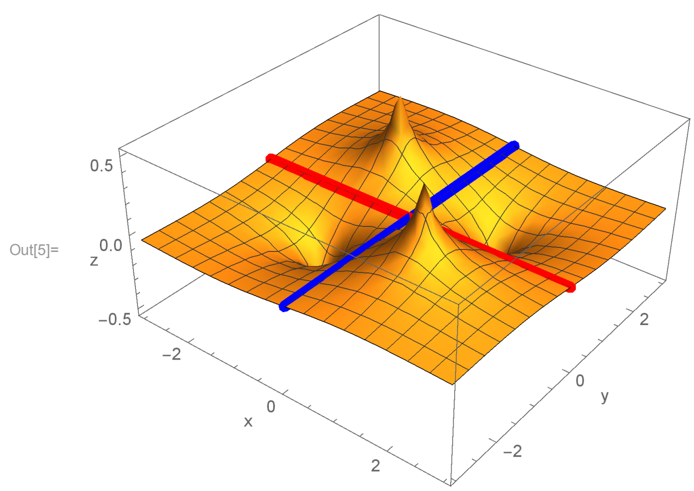
The resulting Green function becomes

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi}\ln(r) - \frac{1}{2\pi}\ln(r_1) - \frac{1}{2\pi}\ln(r_2) + \frac{1}{2\pi}\ln(r_3)$$

Or

$$\begin{aligned} G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi}\ln\left((x-x_0)^2 + (y-y_0)^2\right) - \frac{1}{4\pi}\ln\left((x-x_0)^2 + (y+y_0)^2\right) \\ - \frac{1}{4\pi}\ln\left((x+x_0)^2 + (y-y_0)^2\right) + \frac{1}{4\pi}\ln\left((x+x_0)^2 + (y+y_0)^2\right) \end{aligned} \tag{1}$$

The following is 3D plot of the above Green function, showing the image impulse and showing that $G = 0$ at the line $y = 0$ and also at line $x = 0$. (Lines marked as red and blue)



Now that the Green function is found, it is used to solve $\nabla^2 u(\vec{x}) = f(\vec{x})$, with $u(x,0) = h(x)$,

$u(0,y) = g(y)$ .Starting with Green formula for 2D

$$\iint u(\vec{x})\nabla^2 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla^2 u(\vec{x})\, dA = \oint_{s_1} \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot\hat{n}\, ds + \oint_{s_2} \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot\hat{n}\, ds$$

To simplify the notation, from now on, G is used of $G(\vec{x},\vec{x}_0)$, and also u instead of $u(\vec{x})$. The line s_1 in the above is the line $x>0, y=0$ and s_2 is the line $x=0, y>0$. Therefore the above becomes

$$\iint u\nabla^2 G - G\nabla^2 u\, dA = \oint_{s_1} (u\nabla G - G\nabla u)\cdot(-\hat{j})\, ds + \oint_{s_2} (u\nabla G - G\nabla u)\cdot(-\hat{\imath})\, ds$$

Or

$$\iint u\nabla^2 G\, dA - \iint G\nabla^2 u\, dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

But $\nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x},\vec{x}_0)$ and $\nabla^2 u(\vec{x}) = f(\vec{x})$, hence the above reduces to

$$\iint u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\, dA - \iint Gf(\vec{x})\, dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

But $\iint u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\, dA = u(\vec{x}_0)$ therefore

$$u(\vec{x}_0) - \iint Gf(\vec{x})\, dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Or

$$u(\vec{x}_0) = \iint Gf(\vec{x})\, dA + \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Since $G(\vec{x},\vec{x}_0) = 0$ at $y=0$, and $G(\vec{x},\vec{x}_0) = 0$ at $x=0$, the above becomes

$$u(\vec{x}_0) = \iint Gf(\vec{x})\, dA - \int_0^\infty \left(u\frac{\partial G}{\partial y}\right)_{y=0} dx - \int_0^\infty \left(u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Since $u(\vec{x}) = h(x)$ at $y=0$ and $u(\vec{x}) = g(y)$ at $x=0$ then

$$u(x_0,y_0) = \iint Gf(\vec{x})\, dA - \int_0^\infty h(x)\left(\frac{\partial G}{\partial y}\right)_{y=0} dx - \int_0^\infty g(y)\left(\frac{\partial G}{\partial x}\right)_{x=0} dy \tag{2}$$

$\left(\frac{dG}{dy}\right)_{y=0}$ and $\left(\frac{\partial G}{\partial x}\right)_{x=0}$ are now evaluated to complete the solution. Using $G(\vec{x},\vec{x}_0)$ in equation

(1) gives

$$\begin{aligned}\frac{\partial G}{\partial y} &= \frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x-x_0)^2+\left(y-y_0\right)^2}\right)-\frac{1}{4\pi}\left(\frac{2\left(y+y_0\right)}{(x-x_0)^2+\left(y+y_0\right)^2}\right)\\ &\quad -\frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x+x_0)^2+\left(y-y_0\right)^2}\right)+\frac{1}{4\pi}\left(\frac{2\left(y+y_0\right)}{(x+x_0)^2+\left(y+y_0\right)^2}\right)\end{aligned}$$

Evaluating the above at $y=0$ results in

$$\begin{aligned}\left(\frac{\partial G}{\partial y}\right)_{y=0} &= \frac{1}{4\pi}\left(\frac{-2y_0}{(x-x_0)^2+y_0^2}\right)-\frac{1}{4\pi}\left(\frac{2y_0}{(x-x_0)^2+y_0^2}\right)\\ &\quad -\frac{1}{4\pi}\left(\frac{-2y_0}{(x+x_0)^2+y_0^2}\right)+\frac{1}{4\pi}\left(\frac{2y_0}{(x+x_0)^2+y_0^2}\right)\end{aligned}$$

Or

$$\left(\frac{\partial G}{\partial y}\right)_{y=0} = \frac{y_0}{\pi}\left(\frac{1}{(x+x_0)^2+y_0^2}-\frac{1}{(x-x_0)^2+y_0^2}\right) \tag{3}$$

Finding $\frac{\partial G}{\partial x}$ gives

$$\begin{aligned}\frac{\partial G}{\partial x} &= \frac{1}{4\pi}\left(\frac{2\left(x-x_0\right)}{(x-x_0)^2+\left(y-y_0\right)^2}\right)-\frac{1}{4\pi}\left(\frac{2\left(x-x_0\right)}{(x-x_0)^2+\left(y+y_0\right)^2}\right)\\ &\quad -\frac{1}{4\pi}\left(\frac{2\left(x+x_0\right)}{(x+x_0)^2+\left(y-y_0\right)^2}\right)+\frac{1}{4\pi}\left(\frac{2\left(x+x_0\right)}{(x+x_0)^2+\left(y+y_0\right)^2}\right)\end{aligned}$$

Evaluating the above at $x=0$ results in

$$\begin{aligned}\left(\frac{\partial G}{\partial x}\right)_{x=0} &= \frac{1}{4\pi}\left(\frac{-2x_0}{x_0^2+\left(y-y_0\right)^2}\right)-\frac{1}{4\pi}\left(\frac{-2x_0}{x_0^2+\left(y+y_0\right)^2}\right)\\ &\quad -\frac{1}{4\pi}\left(\frac{2x_0}{x_0^2+\left(y-y_0\right)^2}\right)+\frac{1}{4\pi}\left(\frac{2x_0}{x_0^2+\left(y+y_0\right)^2}\right)\end{aligned}$$

Or

$$\begin{aligned}\left(\frac{\partial G}{\partial x_0}\right)_{x=0} &= \frac{1}{\pi}\left(\frac{x_0}{x_0^2+\left(y+y_0\right)^2}\right)-\frac{1}{\pi}\left(\frac{x_0}{x_0^2+\left(y-y_0\right)^2}\right)\\ &= \frac{x_0}{\pi}\left(\frac{1}{x_0^2+\left(y+y_0\right)^2}-\frac{1}{x_0^2+\left(y-y_0\right)^2}\right) \end{aligned}\tag{4}$$

Substituting (3,4) into (2) gives the final answer

$$
\begin{aligned}
u\left(x_0,y_0\right) = & \int_0^\infty\int_0^\infty G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right)\ dxdy \\
& -\frac{y_0}{\pi}\int_0^\infty h\left(x\right)\left(\frac{1}{\left(x+x_0\right)^2+y_0^2}-\frac{1}{\left(x-x_0\right)^2+y_0^2}\right)\ dx \\
& -\frac{x_0}{\pi}\int_0^\infty g\left(y\right)\left(\frac{1}{x_0^2+\left(y+y_0\right)^2}-\frac{1}{x_0^2+\left(y-y_0\right)^2}\right)\ dy
\end{aligned}
$$

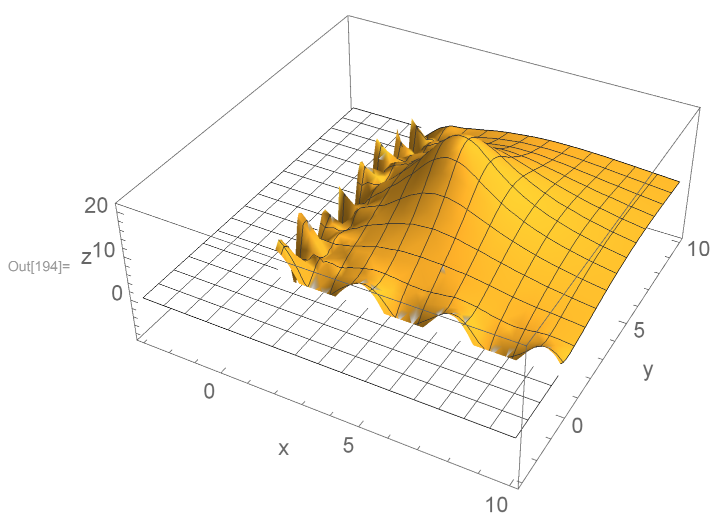
Reversing the role of $\vec{x},\vec{x}_0$ gives

$$
\begin{aligned}
u\left(x,y\right) = & \int_0^\infty\int_0^\infty G\left(\vec{x},\vec{x}_0\right) f\left(x_0,y_0\right)\ dx_0dy_0 \\
& -\frac{y}{\pi}\int_0^\infty h\left(x_0\right)\left(\frac{1}{\left(x_0+x\right)^2+y^2}-\frac{1}{\left(x_0-x\right)^2+y^2}\right)\ dx_0 \\
& -\frac{x}{\pi}\int_0^\infty g\left(y_0\right)\left(\frac{1}{x^2+\left(y_0+y\right)^2}-\frac{1}{x^2+\left(y_0-y\right)^2}\right)\ dy_0
\end{aligned}
$$

Where $G\left(\vec{x},\vec{x}_0\right)$ is given by equation (1). This complete the solution.

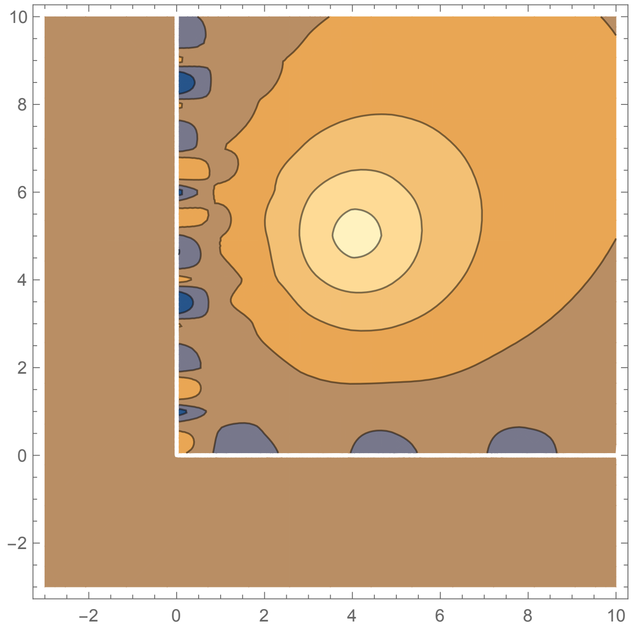
The following is 3D plot of the solution (for small area is first quadrant) generated using Mathematica using

$$
\begin{aligned}
f\left(x\right) &= -20e^{-(x-4)^2-\left(y-5\right)^2} \\
g\left(y\right) &= 10\sin\left(5y\right) \\
h\left(x\right) &= 5\cos\left(2x\right)
\end{aligned}
$$



This is a contour plot of the above solution

Out[186]=



2.13 HW 12

2.13.1 Problem 12.2.1

Show that the wave equation can be considered as the following system of two coupled first-order PDE

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w \tag{1}$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \tag{2}$$

Answer

The wave PDE in 1D is $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$. Taking time derivative of equation (1) gives (assuming c is constant)

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial w}{\partial t} \tag{3}$$

Taking space derivative of equation (1) gives (assuming c is constant)

$$\frac{\partial^2 u}{\partial t \partial x} - c\frac{\partial^2 u}{\partial x^2} = \frac{\partial w}{\partial x} \tag{4}$$

Multiplying (4) by c

$$c\frac{\partial^2 u}{\partial t \partial x} - c^2\frac{\partial^2 u}{\partial x^2} = c\frac{\partial w}{\partial x} \tag{5}$$

Adding (3)+(5) gives

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial t \partial x} - c^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} \\ \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} \end{aligned}$$

But the RHS of the above is zero, since it is equation (2). Therefore the above reduces to

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Which is the wave PDE.

2.13.2 Problem 12.2.2

Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0 \tag{1}$$

with $w(x,0) = \cos x$

Answer

Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = -3$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 0$. Therefore, $w(x(t), t)$ is constant. At time $t = 0$, we are given that

$$w(x(0), t) = \cos x(0) \qquad t = 0 \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt} = -3$, which has the solution $x = x(0) - 3t$. Hence $x(0) = x + 3t$. Therefore (3) becomes

$$w(x(t), t) = \cos(x + 3t)$$

2.13.3 Problem 12.2.3

Solve

$$\frac{\partial w}{\partial t} + 4\frac{\partial w}{\partial x} = 0 \tag{1}$$

with $w(0, t) = \sin 3t$

Answer

Let

$$w \equiv w(x, t(x))$$

Hence

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\frac{dt}{dx} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dt}{dx} = \frac{1}{4}$ in (2), then we obtain (1). Hence we conclude that $\frac{dw}{dx} = 0$. Therefore, $w(x, t(x))$ is constant. At $x = 0$, we are given that

$$w(x, t(0)) = \sin(3t(0)) \qquad x = 0 \tag{3}$$

We just now need to determine $t(0)$. This is found from $\frac{dt}{dx} = \frac{1}{4}$, which has the solution $t(x) = t(0) + \frac{1}{4}x$. Hence $t(0) = t(x) - \frac{1}{4}x$. Therefore (3) becomes

$$\begin{aligned} w(x, t(x)) &= \sin\left(3\left(t(x) - \frac{1}{4}x\right)\right) \\ &= \sin\left(3t - \frac{3}{4}x\right) \end{aligned}$$

2.13.4 Problem 12.2.4

Solve

$$\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}=0 \tag{1}$$

with $c>0$ and

$$\begin{aligned} w(x,0) &= f(x) \qquad x>0\\ w(0,t) &= h(t) \qquad t>0 \end{aligned}$$

Answer

Let

$$w\equiv w(x(t),t)$$

Hence

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt}=c$ in (2), then we obtain (1). Hence we conclude that $\frac{dw}{dt}=0$. Therefore, $w(x(t),t)$ is constant. At $t=0$, we are given that

$$w(x(t),t)=f(x(0)) \qquad t=0 \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt}=c$, which has the solution $x(t)=x(0)+ct$. Hence $x(0)=x(t)-ct$. Therefore (3) becomes

$$w(x,t)=f(x-ct)$$

This is valid for $x>ct$. We now start all over again, and look at Let

$$w\equiv w(x,t(x))$$

Hence

$$\frac{dw}{dx}=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial t}\frac{dt}{dx} \tag{4}$$

Comparing (4) and (1), we see that if we let $\frac{dt}{dx}=\frac{1}{c}$ in (4), then we obtain (1). Hence we conclude that $\frac{dw}{dx}=0$. Therefore, $w(x,t(x))$ is constant. At $x=0$, we are given that

$$w(x,t(x))=h(t(0)) \qquad x=0 \tag{5}$$

We just now need to determine $t(0)$. This is found from $\frac{dt}{dx}=\frac{1}{c}$, which has the solution $t(x)=t(0)+\frac{1}{c}x$. Hence $t(0)=t(x)-\frac{1}{c}x$. Therefore (5) becomes

$$w(x,t)=h\left(t-\frac{1}{c}x\right)$$

Valid for $t>\frac{x}{c}$ or $x<ct$. Therefore, the solution is

$$w(x,t)=\begin{cases} f(x-ct) & x>ct\\ h\left(t-\frac{1}{c}x\right) & x<ct \end{cases}$$

2.13.5 Problem 12.2.5

2.13.5.1 Part (a)

Solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x} \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = c$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = e^{2x}$. Hence

$$w = w(0) + te^{2x}$$

At $t = 0$, $w(0) = f(x(0))$, hence

$$w = f(x(0)) + te^{2x} \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt} = c$, which has the solution $x = x(0) + ct$. Hence $x(0) = x - ct$. Therefore (3) becomes

$$w(x(t), t) = f(x - ct) + te^{2x}$$

2.13.5.2 Part (b)

Solve

$$\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1 \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = x$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 1$. Hence

$$w = w(0) + t$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0)) + t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = x$, the solution is $\ln|x| = t + x(0)$ or $x = x(0)e^t$. Hence $x(0) = xe^{-t}$ and the above becomes

$$w = f\left(xe^{-t}\right) + t$$

2.13.5.3 Part (c)

Solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1 \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = t$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 1$. Hence

$$w = w(0) + t$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0)) + t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = t$, the solution is $x = x(0) + \frac{t^2}{2}$. Hence $x(0) = x - \frac{t^2}{2}$ and the above becomes

$$w = f\left(x - \frac{t^2}{2}\right) + t$$

2.13.5.4 Part (d)

Solve

$$\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = 3t$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = w$. Hence

$$\begin{aligned}\ln|w| &= w(0) + t \\ w &= w(0)e^t\end{aligned}$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0))e^t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = 3t$, the solution is $x = x(0) + \frac{3t^2}{2}$. Hence $x(0) = x - \frac{3t^2}{2}$ and the above becomes

$$w = f\left(x - \frac{3t^2}{2}\right)e^t$$

Chapter 3

study notes

3.1 Heat PDE inside disk

We only did steady state. i.e. $u_t = 0$. Hence using polar coordinates the dependent variable is $u(r,\theta)$. No time dependency. The heat PDE becomes

$$\nabla^2 u(r,\theta) = 0 \tag{1}$$
$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2}$$
$$u_{rr} + \frac{1}{r} u_r + u_{\theta\theta} = 0 \tag{3}$$

With $0 < r < a$ and $0 < \theta < 2\pi$. The boundary conditions are

$$\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ \frac{\partial u(r,-\pi)}{\partial \theta} &= \frac{\partial u(r,\pi)}{\partial \theta} \\ |u(0,\theta)| &< \infty \\ u(a,\theta) &= f(\theta) \end{aligned}$$

Solution is

$$\begin{aligned} R_0(r) &= c_1 \qquad \lambda = 0 \\ R_n(r) &= c_2 r^n \qquad \lambda > 0 \\ \Theta_n(\theta) &= \{\cos(n\theta), \sin(n\theta)\} \qquad n \geq 0 \end{aligned}$$

Hence solution is

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n\cos(n\theta)+B_n r^n\sin(n\theta)$$

3.1.1 Mean value principle (steady state, heat PDE, disk)

Temperature at center of any disk is the average of all points on the disk boundary

3.1.2 Maximum value principle (steady state, heat PDE, disk)

Temperature inside the disk can not be the maximum of all points. Proof by contradiction, using the mean value.

3.1.3 Minimum value principle (steady state, heat PDE, disk)

Temperature inside the disk can not be the minimum of all points. Proof by contradiction, using the mean value.

Maximum/Minimum principle can be used to proof well possdness and uniqueness of Laplace PDE.

Solvability conditions $\nabla^2 u=0$ implies total thermal energy in any closed region is constant. This implies total flux is zero, or $\int_\Omega \nabla u.\hat{n}=0$. i.e. no heat flow across boundaries.

3.1.4 Heat PDE Outside disk

$$\nabla^2\Psi=0$$

Boundary conditions,

$$\begin{aligned}\Psi(\infty,\theta)&=u_0 y=u_0 r\sin\theta\\ \Psi(a,\theta)&=0\end{aligned}$$

Solution is

$$\Psi(r,\theta)=c_1\ln\left(\frac{r}{a}\right)+u_0\left(\frac{r^2-a^2}{r}\right)\sin\theta$$

Note, when $r=a$, $\Psi(a,\theta)=0$. Use

$$\begin{aligned}u_x&=\frac{\partial\Psi}{\partial y}\\ u_y&=-\frac{\partial\Psi}{\partial x}\end{aligned}$$

where u_x, v_x are horizontal and vertical components of fluid velocity in Cartesian coordinates. Also

$$
\begin{aligned}
u_r &= \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \\
u_\theta &= -\frac{\partial \Psi}{\partial r} = -\frac{c_1}{r} - u_0\left(\frac{r^2-a^2}{r^2}\right)\sin\theta
\end{aligned}
$$

For radial and angular components of the fluid velocity in polar coordinates. Circulation is

$$
\int_0^{2\pi} u_\theta r d\theta = -2\pi c_1
$$

Bernoulli relation

$$
\begin{aligned}
p + \frac{1}{2}\rho\left(u_\theta^2 + u_r^2\right) &= c \\
p + \frac{1}{2}\rho u_\theta^2 &= c \qquad \text{at } r = a
\end{aligned}
$$

Lift is

$$
\begin{aligned}
f_y &= -a\int_0^{2\pi} p\sin\theta d\theta \\
&= a\rho\int_0^{2\pi}\left(-\frac{c_1}{r} - u_0\left(\frac{r^2-a^2}{r^2}\right)\sin\theta\right)^2\sin\theta d\theta
\end{aligned}
$$

Negative circulations, means velocity above disk is higher than below. This means lower pressure above, hence lift.

3.2 Easy way to get the signs for Newton's cooling law.

Newton's cooling law says

$$\overbrace{(-k_0 \nabla T)}^{\text{flux } \vec{q}} \cdot \left(-\vec{n}\right) \propto (T_\Omega - T_{bath})$$
$$(-k_0 \nabla T) \cdot \left(-\vec{n}\right) = -H\,(T_\Omega - T_{bath})$$

Where T_Ω is the temperature of the surface of body and T_{bath} is the temperature of the outside.

In the above the proportionality constant $H > 0$ always. So $-H$ is always a negative number.

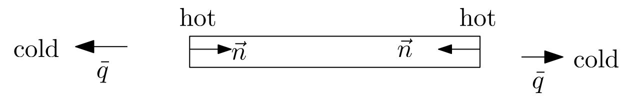
The above works in all cases. 1D, 2D and 3D and in any configuration. Here is how to use it. Direction of flux vector $\vec{q} = -k_0 \nabla T$ is always from hot to cold. We start by drawing $\vec{q}$. Now we compare the direction $\vec{q}$ of to the direction of the reverse of outer normal $\vec{n}$ to the surface. In other words, we compare the direction of $\vec{q}$ to the inner normal (not the outer normal), since we are looking at $-\vec{n}$.

These two vectors are always parallel. They could be in either same direction or in reverse directions.

If direction of $\vec{q}$ and the inner normal are in the same direction, then the sign on the left is positive. i.e. $(-k_0 \nabla T) \cdot \left(-\vec{n}\right)$ is a positive quantity (since $\cos(0) = 1$).

If the direction of $\vec{q}$ and inner normal are in the opposite direction, then the sign of $(-k_0 \nabla T) \cdot \left(-\vec{n}\right)$ is negative (since $\cos\left(180^0\right) = -1$).

Hence always use $(-k_0 \nabla T) \cdot \left(-\vec{n}\right) = -H\,(T_\Omega - T_{bath})$ for $H > 0$. Examples are given below.



We see that $\vec{q}$ and the inner normal are in opposite direction, hence sign in the left is negative. Therefore

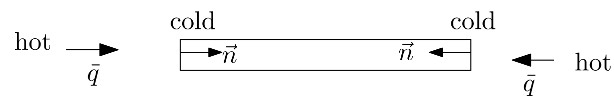
$$
\begin{aligned}
\text{-ve} &= (-H)(T_{\text{hot}} - T_{\text{cold}})\\
\text{-ve} &= (\text{-ve})(+\text{ve})\\
\text{-ve} &= \text{-ve}
\end{aligned}
$$

Signs OK.

We see that $\vec{q}$ and the inner normal are in opposite direction, hence sign in the left is negative. Therefore

$$
\begin{aligned}
\text{-ve} &= (-H)(T_{\text{hot}} - T_{\text{cold}})\\
\text{-ve} &= (\text{-ve})(+\text{ve})\\
\text{-ve} &= \text{-ve}
\end{aligned}
$$

Signs OK.



We see that $\vec{q}$ and the inner normal are in same direction, hence sign in the left is positive. Therefore

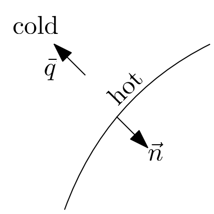
$$
\begin{aligned}
+\text{ve} &= (-H)(T_{\text{cold}} - T_{\text{hot}})\\
+\text{ve} &= (\text{-ve})(\text{-ve})\\
+\text{ve} &= +\text{ve}
\end{aligned}
$$

Signs OK.

We see that $\vec{q}$ and the inner normal are in same direction, hence sign in the left is positive. Therefore

$$
\begin{aligned}
+\text{ve} &= (-H)(T_{\text{cold}} - T_{\text{hot}})\\
+\text{ve} &= (\text{-ve})(\text{-ve})\\
+\text{ve} &= +\text{ve}
\end{aligned}
$$

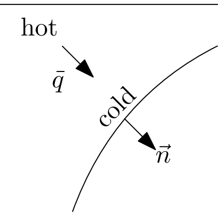
Signs OK.



We see that $\vec{q}$ and the inner normal are in opposite direction, hence sign in the left is negative. Therefore

$$
\begin{aligned}
\text{-ve} &= (-H)(T_{\text{hot}} - T_{\text{cold}})\\
\text{-ve} &= (\text{-ve})(+\text{ve})\\
\text{-ve} &= \text{-ve}
\end{aligned}
$$

Signs OK.



We see that $\vec{q}$ and the inner normal are in same direction, hence sign in the left is positive. Therefore

$$
\begin{aligned}
+\text{ve} &= (-H)(T_{\text{cold}} - T_{\text{hot}})\\
+\text{ve} &= (\text{-ve})(\text{-ve})\\
+\text{ve} &= +\text{ve}
\end{aligned}
$$

Signs OK.