HW2, Math 322, Fall 2016

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Contents

0.1 Summary table

For $1D$ bar

Left	Right	$\lambda = 0$	$\lambda > 0$	$u(x,t)$
$u(0) = 0$	$u(L) = 0$	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$\frac{\partial u(L)}{\partial x} = 0$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = 0$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n} x\right)$	$\sum_{n=1,3,5\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$u(0) = 0$	$u(L) + \frac{\partial u(L)}{\partial x} = 0$	$\lambda_0 = 0$ $X_0 = A_0$	$\tan\left(\sqrt{\lambda_n} L\right) = -\lambda_n$ $X_\lambda = B_\lambda \sin\left(\sqrt{\lambda_n} x\right)$	$A_0 + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$
$\frac{\partial u(0)}{\partial x} = 0$	$\frac{\partial u(L)}{\partial x} = 0$	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n} x\right)$	$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$

For periodic conditions $u(-L) = u(L)$ and $\frac{\partial u(-L)}{\partial x} = \frac{\partial u(L)}{\partial x}$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$$

$$u(x,t) = \overbrace{a_0}^{\lambda=0} + \overbrace{\sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t} + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}}^{\lambda>0}$$

<u>Note on notation</u> When using separation of variables $T(t)$ is used for the time function and $X(x), R(r), \Theta(\theta)$ etc. for the spatial functions. This notation is more common in other books and easier to work with as the dependent variable $T, X, \cdots$ and the independent variable $t, x, \cdots$ are easier to match (one is upper case and is one lower case) and this produces less symbols to remember and less chance of mixing wrong letters.

0.2 section 2.3.1 (problem 1)

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

* (a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

* (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$

* (e) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$

* (f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

0.2.1 part (a)

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{1}$$

Let

$$u(t,r) = T(t)R(r)$$

Then

$$\frac{\partial u}{\partial t} = T'(t)R(r)$$

And

$$\begin{aligned}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\\ &= TR'(r) + rTR''(r)\end{aligned}$$

Hence (1) becomes

$$\frac{1}{k}T'(t)R(r) = \frac{1}{r}\left(TR'(r) + rTR''(r)\right)$$

<u>Note</u> From now on $T'(t)$ is written as just T' and similarly for $R'(r) = R'$ and $R''(r) = R''$ to simplify notations and make it easier and more clear to read. The above is reduced to

$$\frac{1}{k}T'R = \frac{1}{r}TR' + TR''$$

Dividing throughout [1] by $T(t)R(r)$ gives

$$\frac{1}{k}\frac{T'}{T} = \frac{1}{r}\frac{R'}{R} + \frac{R''}{R}$$

Since each side in the above depends on a different independent variable and both are equal to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T} = \frac{1}{r}\frac{R'}{R} + \frac{R''}{R} = -\lambda$$

The following differential equations are obtained

$$\begin{aligned}T' + \lambda kT &= 0\\ rR'' + R' + r\lambda R &= 0\end{aligned}$$

In expanded form, the above is

$$\begin{aligned}\frac{dT}{dt} + \lambda kT(t) &= 0\\ r\frac{d^2R}{dr^2} + \frac{dR}{dt} + r\lambda R(r) &= 0\end{aligned}$$

[1] $T(t)R(r)$ can not be zero, as this would imply that either $T(t) = 0$ or $R(r) = 0$ or both are zero, in which case there is only the trivial solution.

0.2.2 Part (b)

$$\frac{1}{k}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\frac{v_0}{k}\frac{\partial u}{\partial x} \tag{1}$$

Let

$$u(x,t)=TX$$

Then

$$\frac{\partial u}{\partial t}=T'X$$

And

$$\frac{\partial u}{\partial x}=X'T$$
$$\frac{\partial^2 u}{\partial x^2}=X''T$$

Substituting these in (1) gives

$$\frac{1}{k}T'X=X''T-\frac{v_0}{k}X'T$$

Dividing throughout by $TX\neq 0$ gives

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}-\frac{v_0}{k}\frac{X'}{X}$$

Since each side in the above depends on a different independent variable and both are equal to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}-\frac{v_0}{k}\frac{X'}{X}=-\lambda$$

The following differential equations are obtained

$$T'+\lambda kT=0$$
$$X''-\frac{v_0}{k}X'+\lambda X=0$$

The above in expanded form is

$$\frac{dT}{dt}+\lambda kT(t)=0$$
$$\frac{d^2X}{dx^2}-\frac{v_0}{k}\frac{dX}{dx}+\lambda X(x)=0$$

0.2.3 Part (d)

$$\frac{1}{k}\frac{\partial u}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \tag{1}$$

Let

$$u(t,r)\equiv TR$$

Then

$$\frac{\partial u}{\partial t}=T'R$$

And

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)&=2r\frac{\partial u}{\partial r}+r^2\frac{\partial^2 u}{\partial r^2}\\&=2rTR'+r^2TR''\end{aligned}$$

Substituting these in (1) gives

$$\begin{aligned}\frac{1}{k}T'R&=\frac{1}{r^2}\left(2rTR'+r^2TR''\right)\\&=\frac{2}{r}TR'+TR''\end{aligned}$$

Dividing throughout by $TR\neq 0$ gives

$$\frac{1}{k}\frac{T'}{T}=\frac{2}{r}\frac{R'}{R}+\frac{R''}{R}$$

Since each side in the above depends on a different independent variable and both are equal to each others, then each side is equal to the same constant, say $-\lambda$. Therefore

$$\frac{1}{k}\frac{T'}{T} = \frac{2}{r}\frac{R'}{R} + \frac{R''}{R} = -\lambda$$

The following differential equations are obtained

$$T' + \lambda k T = 0$$
$$rR'' + 2R' + \lambda r R = 0$$

The above in expanded form is

$$\frac{dT}{dt} + \lambda k T(t) = 0$$
$$r\frac{d^2R}{dr^2} + 2\frac{dR}{dr} + \lambda r R(r) = 0$$

0.3 section 2.3.2 (problem 2)

2.3.2. **Consider the differential equation**

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ **and** $\phi(\pi) = 0$

***(b)** $\phi(0) = 0$ **and** $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$ **(If necessary, see Sec. 2.4.1.)**

***(d)** $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ **and** $\phi(L) = 0$

***(f)** $\phi(a) = 0$ **and** $\phi(b) = 0$ **(You may assume that $\lambda > 0$.)**

(g) $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(L) + \phi(L) = 0$ **(If necessary, see Sec. 5.8.)**

0.3.1 Part (d)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
$$\phi(0) = 0$$
$$\frac{d\phi}{dx}(L) = 0$$

Substituting an assumed solution of the form $\phi = Ae^{rx}$ in the above ODE and simplifying gives the characteristic equation

$$r^2 + \lambda = 0$$
$$r^2 = -\lambda$$
$$r = \pm\sqrt{-\lambda}$$

Assuming λ is real. The following cases are considered.

<u>case $\lambda < 0$</u> In this case, $-\lambda$ and also $\sqrt{-\lambda}$, are positive. Hence both roots $\pm\sqrt{-\lambda}$ are real and positive. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. Therefore the solution is

$$\phi(x) = Ae^{sx} + Be^{-sx}$$
$$\frac{d\phi}{dx} = Ase^{sx} - Bse^{-sx}$$

Applying the first boundary conditions (B.C.) gives

$$
\begin{aligned}
0 &= \phi(0) \\
&= A + B
\end{aligned}
$$

Applying the second B.C. gives

$$
\begin{aligned}
0 &= \frac{d\phi}{dx}(L) \\
&= As - Bs \\
&= s(A - B) \\
&= A - B
\end{aligned}
$$

The last step above was after dividing by s since $s \neq 0$. Therefore, the following two equations are solved for A, B

$$
\begin{aligned}
0 &= A + B \\
0 &= A - B
\end{aligned}
$$

The second equation implies $A = B$ and the first gives $2A = 0$ or $A = 0$. Hence $B = 0$. Therefore the only solution is the trivial solution $\phi(x) = 0$. $\underline{\lambda < 0 \text{ is not an eigenvalue.}}$

$\underline{\text{case } \lambda = 0}$ In this case the ODE becomes

$$
\frac{d^2\phi}{dx^2} = 0
$$

The solution is

$$
\begin{aligned}
\phi(x) &= Ax + B \\
\frac{d\phi}{dx} &= A
\end{aligned}
$$

Applying the first B.C. gives

$$
\begin{aligned}
0 &= \phi(0) \\
&= B
\end{aligned}
$$

Applying the second B.C. gives

$$
\begin{aligned}
0 &= \frac{d\phi}{dx}(L) \\
&= A
\end{aligned}
$$

Hence A, B are both zero in this case as well and the only solution is the trivial one $\phi(x) = 0$. $\underline{\lambda = 0 \text{ is not an eigenvalue.}}$

$\underline{\text{case } \lambda > 0}$ In this case, $-\lambda$ is negative, therefore the roots are both complex.

$$
r = \pm i\sqrt{\lambda}
$$

Hence the solution is

$$
\phi(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x}
$$

Which can be writing in terms of $\cos, \sin$ using Euler identity as

$$
\phi(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)
$$

Applying first B.C. gives

$$
\begin{aligned}
0 &= \phi(0) \\
&= A\cos(0) + B\sin(0) \\
0 &= A
\end{aligned}
$$

The solution now is $\phi(x) = B\sin\left(\sqrt{\lambda}x\right)$. Hence

$$
\frac{d\phi}{dx} = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)
$$

Applying the second B.C. gives

$$\begin{aligned} 0 &= \frac{d\phi}{dx}(L) \\ &= \sqrt{\lambda} B \cos\left(\sqrt{\lambda} L\right) \\ &= \sqrt{\lambda} B \cos\left(\sqrt{\lambda} L\right) \end{aligned}$$

Since $\lambda \neq 0$ then either $B = 0$ or $\cos\left(\sqrt{\lambda}L\right) = 0$. But $B = 0$ gives trivial solution, therefore

$$\cos\left(\sqrt{\lambda}L\right) = 0$$

This implies

$$\sqrt{\lambda}L = \frac{n\pi}{2} \qquad n = 1, 3, 5, \cdots$$

In other words, for <u>all positive odd integers</u>. $n < 0$ can not be used since λ is assumed positive.

$$\boxed{\lambda = \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 3, 5, \cdots}$$

The eigenfunctions associated with these eigenvalues are

$$\phi_n(x) = B_n \sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1, 3, 5, \cdots$$

0.3.2 Part (f)

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(a) &= 0 \\ \phi(b) &= 0 \end{aligned}$$

It is easier to solve this if one boundary condition was at $x = 0$. (So that one constant drops out). Let $\tau = x - a$ and the ODE becomes (where now the independent variable is τ)

$$\frac{d^2\phi(\tau)}{d\tau^2} + \lambda\phi(\tau) = 0 \tag{1}$$

With the new boundary conditions $\phi(0) = 0$ and $\phi(b-a) = 0$. Assuming the solution is $\phi = Ae^{r\tau}$, the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Assuming λ is real and also assuming $\lambda > 0$ (per the problem statement) then $-\lambda$ is negative, and both roots are complex.

$$r = \pm i\sqrt{\lambda}$$

This gives the solution

$$\phi(\tau) = A\cos\left(\sqrt{\lambda}\tau\right) + B\sin\left(\sqrt{\lambda}\tau\right)$$

Applying first B.C.

$$\begin{aligned} 0 &= \phi(0) \\ &= A\cos 0 + B\sin 0 \\ &= A \end{aligned}$$

Therefore the solution is $\phi(\tau) = B\sin\left(\sqrt{\lambda}\tau\right)$. Applying the second B.C.

$$\begin{aligned} 0 &= \phi(b-a) \\ &= B\sin\left(\sqrt{\lambda}(b-a)\right) \end{aligned}$$

$B=0$ leads to trivial solution. Choosing $\sin\left(\sqrt{\lambda}\,(b-a)\right)=0$ gives

$$\begin{aligned}\sqrt{\lambda_n}\,(b-a) &= n\pi\\ \sqrt{\lambda_n} &= \frac{n\pi}{(b-a)} \qquad n=1,2,3\cdots\end{aligned}$$

Or

$$\boxed{\lambda_n=\left(\frac{n\pi}{b-a}\right)^2 \qquad n=1,2,3,\cdots}$$

The eigenfunctions associated with these eigenvalue are

$$\begin{aligned}\phi_n(\tau) &= B_n\sin\left(\sqrt{\lambda_n}\tau\right)\\ &= B_n\sin\left(\frac{n\pi}{(b-a)}\tau\right)\end{aligned}$$

Transforming back to x

$$\phi_n(x)=B_n\sin\left(\frac{n\pi}{(b-a)}(x-a)\right)$$

0.3.3 Part (g)

$$\begin{aligned}\frac{d^2\phi}{dx^2}+\lambda\phi &= 0\\ \phi(0) &= 0\\ \frac{d\phi}{dx}(L)+\phi(L) &= 0\end{aligned}$$

Assuming solution is $\phi=Ae^{rx}$, the characteristic equation is

$$\begin{aligned}r^2+\lambda &= 0\\ r^2 &= -\lambda\\ r &= \pm\sqrt{-\lambda}\end{aligned}$$

The following cases are considered.

<u>case $\lambda<0$</u> In this case $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda}=s$$

Where $s>0$. This gives the solution

$$\phi(x)=A_0e^{sx}+B_0e^{-sx}$$

Which can be manipulated using $\sinh(sx)=\frac{e^{sx}-e^{-sx}}{2}$, $\cosh(sx)=\frac{e^{sx}+e^{-sx}}{2}$ to the following

$$\phi(x)=A\cosh(sx)+B\sinh(sx)$$

Where A,B above are new constants. Applying the left boundary condition gives

$$\begin{aligned}0 &= \phi(0)\\ &= A\end{aligned}$$

The solution becomes $\phi(x)=B\sinh(sx)$ and hence $\frac{d\phi}{dx}=s\cosh(sx)$. Applying the right boundary conditions gives

$$\begin{aligned}0 &= \phi(L)+\frac{d\phi}{dx}(L)\\ &= B\sinh(sL)+Bs\cosh(sL)\\ &= B(\sinh(sL)+s\cosh(sL))\end{aligned}$$

But $B=0$ leads to trivial solution, therefore the other option is that

$$\sinh(sL)+s\cosh(sL)=0$$

But the above is

$$\tanh(sL)=-s$$

Since it was assumed that $s>0$ then the RHS in the above is a negative quantity. However the $\tanh$ function is positive for positive argument and negative for negative argument.

The above implies then that $sL < 0$. Which is invalid since it was assumed $s > 0$ and L is the length of the bar. Hence $B = 0$ is the only choice, and this leads to trivial solution. $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case, the ODE becomes

$$\frac{d^2\phi}{dx^2} = 0$$

The solution is

$$\phi(x) = c_1 x + c_2$$

Applying left B.C. gives

$$\begin{aligned} 0 &= \phi(0) \\ &= c_2 \end{aligned}$$

The solution becomes $\phi(x) = c_1 x$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \phi(L) + \frac{d\phi}{dx}(L) \\ &= c_1 L + c_1 \\ &= c_1 (1 + L) \end{aligned}$$

Since $c_1 = 0$ leads to trivial solution, then $1 + L = 0$ is the only other choice. But this invalid since $L > 0$ (length of the bar). Hence $c_1 = 0$ and this leads to trivial solution. $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

This implies that $-\lambda$ is negative, and therefore the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

This gives the solution

$$\begin{aligned} \phi(x) &= Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} \\ &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying first B.C. gives

$$\begin{aligned} \phi(0) = 0 &= A\cos(0) + B\sin(0) \\ 0 &= A \end{aligned}$$

The solution becomes $\phi(x) = B\sin\left(\sqrt{\lambda}x\right)$ and

$$\frac{d\phi}{dx} = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)$$

Applying the second B.C.

$$\begin{aligned} 0 &= \frac{d\phi}{dx}(L) + \phi(L) \\ &= \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right) + B\sin\left(\sqrt{\lambda}L\right) \qquad (1) \end{aligned}$$

Dividing (1) by $\cos\left(\sqrt{\lambda}L\right)$, which can not be zero, because if $\cos\left(\sqrt{\lambda}L\right) = 0$, then $B\sin\left(\sqrt{\lambda}L\right) = 0$ from above, and this means the trivial solution, results in

$$B\left(\sqrt{\lambda} + \tan\left(\sqrt{\lambda}L\right)\right) = 0$$

But $B \neq 0$, else the solution is trivial. Therefore

$$\tan\left(\sqrt{\lambda}L\right) = -\sqrt{\lambda}$$

The eigenvalue λ is given by the solution to the above nonlinear equation. The text book, in section 5.4, page 196 gives the following approximate (asymptotic) solution which becomes accurate only for large n and not used here

$$\sqrt{\lambda_n} \sim \frac{\pi}{L}\left(n - \frac{1}{2}\right)$$

Therefore the eigenfunction is

$$\phi_\lambda(x) = B\sin\left(\sqrt{\lambda}x\right)$$

Where λ is the solution to $\tan\left(\sqrt{\lambda}L\right) = -\sqrt{\lambda}$.

0.4 section 2.3.3 (problem 3)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

(b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

* (c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

(d) $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

0.4.1 Part (b)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = T(t)X(x)$, and the PDE becomes

$$\frac{1}{k}T'X = X''T$$

Dividing by $XT \neq 0$

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$ where λ is assumed to be real.

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + k\lambda T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Starting with the space ODE equation (2), with corresponding boundary conditions $X(0) = 0, X(L) = 0$. Assuming the solution is $X(x) = e^{rx}$, Then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

The following cases are considered.

<u>case $\lambda < 0$</u> In this case, $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. This gives the solution

$$X(x) = A\cosh(sx) + B\sinh(sx)$$

Applying the left B.C. $X(0)=0$ gives

$$
\begin{aligned}
0 &= A\cosh(0) + B\sinh(0) \\
&= A
\end{aligned}
$$

The solution becomes $X(x) = B\sinh(sx)$. Applying the right B.C. $u(L,t)=0$ gives

$$
0 = B\sinh(sL)
$$

We want $B \neq 0$ (else trivial solution). This means $\sinh(sL)$ must be zero. But $\sinh(sL)$ is zero only when its argument is zero. This means either $L=0$ which is not possible or $\lambda = 0$, but we assumed $\lambda \neq 0$ in this case, therefore we run out of options to satisfy this case. Hence $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The ODE becomes

$$
\frac{d^2X}{dx^2} = 0
$$

The solution is

$$
X(x) = c_1 x + c_2
$$

Applying left boundary conditions $X(0)=0$ gives

$$
\begin{aligned}
0 &= X(0) \\
&= c_2
\end{aligned}
$$

Hence the solution becomes $X(x) = c_1 x$. Applying the right B.C. gives

$$
\begin{aligned}
0 &= X(L) \\
&= c_1 L
\end{aligned}
$$

Hence $c_1 = 0$. Hence trivial solution. $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

Hence $-\lambda$ is negative, and the roots are both complex.

$$
r = \pm i\sqrt{\lambda}
$$

The solution is

$$
X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)
$$

The boundary conditions are now applied. The first B.C. $X(0)=0$ gives

$$
\begin{aligned}
0 &= A\cos(0) + B\sin(0) \\
&= A
\end{aligned}
$$

The ODE becomes $X(x) = B\sin\left(\sqrt{\lambda}x\right)$. Applying the second B.C. gives

$$
0 = B\sin\left(\sqrt{\lambda}L\right)
$$

$B \neq 0$ else the solution is trivial. Therefore taking

$$
\begin{aligned}
\sin\left(\sqrt{\lambda}L\right) &= 0 \\
\sqrt{\lambda_n}L &= n\pi \qquad n = 1,2,3,\cdots
\end{aligned}
$$

Hence eigenvalues are

$$
\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1,2,3,\cdots
$$

The eigenfunctions associated with these eigenvalues are

$$
X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right)
$$

The time domain ODE is now solved. $T' + k\lambda_n T = 0$ has the solution

$$
T_n(t) = e^{-k\lambda_n t}
$$

For the same set of eigenvalues. Notice that there is no need to add a new constant in the above as it will be absorbed in the B_n when combined in the following step below. The solution to the PDE becomes

$$u_n(x,t) = T_n(t)\,X_n(x)$$

But for linear system the sum of eigenfunctions is also a solution, therefore

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

Initial conditions are now applied. Setting $t = 0$, the above becomes

$$u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

As the series is unique, the terms coefficients must match for those shown only, and all other B_n terms vanish. This means that by comparing terms

$$3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) = B_1 \sin\left(\frac{\pi x}{L}\right) + B_3 \sin\left(\frac{3\pi}{L}x\right)$$

Therefore

$$\begin{aligned} B_1 &= 3 \\ B_3 &= -1 \end{aligned}$$

And all other $B_n = 0$. The solution is

$$u(x,t) = 3\sin\left(\frac{\pi}{L}x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

0.4.2 Part (d)

Part (b) found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The new initial conditions are now applied.

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Where

$$f(x) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$$

Multiplying both sides of (1) by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating over the domain gives

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = \int_0^L \left[\sum_{n=1}^{\infty} B_n \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\right] dx$$

Interchanging the order of integration and summation

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = \sum_{n=1}^{\infty} \left[B_n \left(\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx\right)\right]$$

But $\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) dx = 0$ for $n \neq m$, hence only one term survives

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) f(x)\,dx = B_m \int_0^L \sin^2\left(\frac{m\pi}{L}x\right) dx$$

Renaming m back to n and since $\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}$ the above becomes

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx = \frac{L}{2}B_n$$

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx + \int_{\frac{L}{2}}^L \sin\left(\frac{n\pi}{L}x\right) f(x)\,dx\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) dx + 2\int_{\frac{L}{2}}^L \sin\left(\frac{n\pi}{L}x\right) dx\right) \\
&= \frac{2}{L}\left(\left.\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\right|_0^{\frac{L}{2}} + 2\left.\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\right|_{\frac{L}{2}}^L\right) \\
&= \frac{2}{n\pi}\left(\left(-\cos\left(\frac{n\pi}{L}x\right)\right)_0^{\frac{L}{2}} + 2\left(-\cos\left(\frac{n\pi}{L}x\right)\right)_{\frac{L}{2}}^L\right) \\
&= \frac{2}{n\pi}\left(\left[-\cos\left(\frac{n\pi}{L}\frac{L}{2}\right) + \cos(0)\right] + 2\left[-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right]\right) \\
&= \frac{2}{n\pi}\left(-\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi) + 2\cos\left(\frac{n\pi}{2}\right)\right) \\
&= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi)\right)
\end{aligned}$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

With

$$\begin{aligned}
B_n &= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) + 1\right) \\
&= \frac{2}{n\pi}\left(1 - 2(-1)^n + \cos\left(\frac{n\pi}{2}\right)\right)
\end{aligned}$$

0.5 section 2.3.4 (problem 4)

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = 0, u(L,t) = 0$, and $u(x,0) = f(x)$.

***(a) What is the total heat energy in the rod as a function of time?**

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

***(c) What relationship should exist between parts (a) and (b)?**

0.5.1 Part (a)

By definition the total heat energy is

$$E = \int_V \rho c u(x,t)\,dv$$

Assuming constant cross section area A, the above becomes (assuming all thermal properties are constant)

$$E = \int_0^L \rho c u(x,t) A\,dx$$

But $u(x,t)$ was found to be

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

For these boundary conditions from problem 2.3.3. Where B_n was found from initial conditions. Substituting the solution found into the energy equation gives

$$\begin{aligned}
E &= \rho c A \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right) dx \\
&= \rho c A \sum_{n=1}^{\infty} \left(B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L \sin\left(\frac{n\pi}{L} x\right) dx \right) \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \left(\frac{-\cos\left(\frac{n\pi}{L} x\right)}{\frac{n\pi}{L}} \right)_0^L \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{L}{n\pi} \left(-\cos\left(\frac{n\pi}{L} L\right) + \cos(0) \right) \\
&= \rho c A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{L}{n\pi} (1 - \cos(n\pi)) \\
&= \frac{L\rho c A}{\pi} \sum_{n=1}^{\infty} \left[\frac{B_n}{n} (1 - \cos(n\pi)) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right]
\end{aligned}$$

0.5.2 Part (b)

By definition, the flux is the amount of heat flow per unit time per unit area. Assuming the area is A, then heat flow at $x = 0$ into the rod per unit time (call it $H(x)$) is

$$\begin{aligned}
H|_{x=0} &= A\, \phi|_{x=0} \\
&= -Ak \left.\frac{\partial u}{\partial x}\right|_{x=0}
\end{aligned}$$

Similarly, heat flow at $x = L$ out of the rod per unit time is

$$\begin{aligned}
H|_{x=L} &= A\, \phi|_{x=L} \\
&= -Ak \left.\frac{\partial u}{\partial x}\right|_{x=L}
\end{aligned}$$

To obtain heat flow at $x = 0$ leaving the rod, the sign is changed and it becomes $Ak \left.\frac{\partial u}{\partial x}\right|_{x=0}$.

Since $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ then

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Then at $x = 0$ then heat flow leaving of the rod becomes

$$Ak \left.\frac{\partial u}{\partial x}\right|_{x=0} = Ak \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

And at $x = L$, the heat flow out of the bar

$$\begin{aligned}
-Ak \left.\frac{\partial u}{\partial x}\right|_{x=L} &= -Ak \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L} L\right) e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \\
&= -Ak \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos(n\pi) e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \\
&= -Ak \sum_{n=1}^{\infty} (-1)^n B_n \frac{n\pi}{L} e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}
\end{aligned}$$

0.5.3 Part (c)

Total E inside the bar at time t is given by initial energy $E_{t=0}$ and time integral of flow of heat energy into the bar. Since from part (a)

$$E = L\frac{\rho c A}{\pi} \sum_{n=1}^{\infty} \frac{B_n}{n} e^{-k\left(\frac{n\pi}{L}\right)^2 t} (1 - \cos(n\pi))$$

Then initial energy is

$$E_{t=0} = L\frac{\rho c A}{\pi} \sum_{n=1}^{\infty} \frac{B_n}{n} (1 - \cos(n\pi))$$

And total heat flow <u>into</u> the rod (per unit time) is $\left(-Ak\left.\frac{\partial u}{\partial x}\right|_{x=0}+Ak\left.\frac{\partial u}{\partial x}\right|_{x=L}\right)$, therefore

$$\begin{aligned}
L\frac{\rho c A}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(1-\cos\left(n\pi\right)\right) &= \int_0^t\left(-Ak\left.\frac{\partial u}{\partial x}\right|_{x=0}+Ak\left.\frac{\partial u}{\partial x}\right|_{x=L}\right)dx\\
&= Ak\int_0^t\left(\frac{\partial u\left(L\right)}{\partial x}-\frac{\partial u\left(0\right)}{\partial x}\right)dx
\end{aligned}$$

But

$$\begin{aligned}
\frac{\partial u\left(L\right)}{\partial x}-\frac{\partial u\left(0\right)}{\partial x} &= \frac{\pi}{L}\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2 t}-\frac{\pi}{L}\sum_{n=1}^{\infty}nB_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\\
&= \frac{\pi}{L}\left(\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2 t}-\sum_{n=1}^{\infty}nB_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)
\end{aligned}$$

Hence

$$\frac{L\rho c A}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{n}\exp^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(1-\cos\left(n\pi\right)\right)=\frac{Ak\pi}{L}\int_0^t\left(\sum_{n=1}^{\infty}nB_n\left(-1\right)^n e^{-k\left(\frac{n\pi}{L}\right)^2 t}-\sum_{n=1}^{\infty}nB_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)dx$$

0.6 section 2.3.5 (problem 5)

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n>0, m>0.$$

Use the trigonometric identity

$$\sin a\sin b=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right].$$

$$I=\int_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Considering first the case $m=n$. The integral becomes

$$I=\int_0^L\sin^2\left(\frac{n\pi x}{L}\right)dx=\frac{L}{2}$$

For the case where $n\neq m$, using

$$\sin a\sin b=\frac{1}{2}\left(\cos\left(a-b\right)-\cos\left(a+b\right)\right)$$

The integral I becomes [2]

$$\begin{aligned}
I &= \frac{1}{2}\int_0^L\cos\left(\frac{n\pi x}{L}-\frac{m\pi x}{L}\right)-\cos\left(\frac{n\pi x}{L}+\frac{m\pi x}{L}\right)dx\\
&= \frac{1}{2}\int_0^L\cos\left(\frac{\pi x\left(n-m\right)}{L}\right)-\cos\left(\frac{\pi x\left(n+m\right)}{L}\right)dx\\
&= \frac{1}{2}\left(\frac{\sin\left(\frac{\pi x(n-m)}{L}\right)}{\frac{\pi(n-m)}{L}}\right)_0^L-\frac{1}{2}\left(\frac{\sin\left(\frac{\pi x(n+m)}{L}\right)}{\frac{\pi(n+m)}{L}}\right)_0^L\\
&= \frac{L}{2\pi\left(n-m\right)}\left(\sin\left(\frac{\pi x\left(n-m\right)}{L}\right)\right)_0^L-\frac{L}{2\pi\left(n+m\right)}\left(\sin\left(\frac{\pi x\left(n+m\right)}{L}\right)\right)_0^L \qquad (1)
\end{aligned}$$

But

$$\left(\sin\left(\frac{\pi x\left(n-m\right)}{L}\right)\right)_0^L=\sin\left(\pi\left(n-m\right)\right)-\sin\left(0\right)$$

[2] Note that the term $(n-m)$ showing in the denominator is not a problem now, since this is the case where $n\neq m$.

And since $n-m$ is integer, then $\sin(\pi(n-m))=0$, therefore $\left(\sin\left(\frac{\pi x(n-m)}{L}\right)\right)_0^L=0$. Similarly

$$\left(\sin\left(\frac{\pi x(n+m)}{L}\right)\right)_0^L=\sin(\pi(n+m))-\sin(0)$$

Since $n+m$ is integer then $\sin(\pi(n+m))=0$ and $\left(\sin\left(\frac{\pi x(n+m)}{L}\right)\right)_0^L=0$. Therefore

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\begin{cases}\frac{L}{2} & n=m\\ 0 & \text{otherwise}\end{cases}$$

0.7 section 2.3.7 (problem 6)

2.3.7. **Consider the following boundary value problem (if necessary, see Sec. 2.4.1):**

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\quad\text{with}\quad\frac{\partial u}{\partial x}(0,t)=0,\ \frac{\partial u}{\partial x}(L,t)=0,\quad\text{and}\quad u(x,0)=f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x,t)=A_0+\sum_{n=1}^{\infty}A_n e^{-\lambda_n kt}\cos\frac{n\pi x}{L}.\Big]$$

What is λ_n?

0.7.1 part (a)

This PDE describes how temperature u changes in a rod of length L as a function of time t and location x. The left and right end are insulated, so no heat escapes from these boundaries. Initially at $t=0$, the temperature distribution in the rod is described by the function $f(x)$.

0.7.2 Part (b)

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t)=T(t)X(x)$, then the PDE becomes

$$\frac{1}{k}T'X=X''T$$

Dividing by $XT\neq 0$

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{k}\frac{T'}{T}=\frac{X''}{X}=-\lambda$$

The two ODE's generated are

$$T'+k\lambda T=0 \tag{1}$$
$$X''+\lambda X=0 \tag{2}$$

Starting with the space ODE equation (2), with corresponding boundary conditions $\frac{dX}{dx}(0) = 0, \frac{dX}{dx}(L) = 0$. Assuming the solution is $X(x) = e^{rx}$, Then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

The following cases are considered.

case $\lambda < 0$ In this case, $-\lambda$ and also $\sqrt{-\lambda}$ are positive. Hence the roots $\pm\sqrt{-\lambda}$ are both real. Let

$$\sqrt{-\lambda} = s$$

Where $s > 0$. This gives the solution

$$\begin{aligned} X(x) &= A\cosh(sx) + B\sinh(sx) \\ \frac{dX}{dx} &= A\sinh(sx) + B\cosh(sx) \end{aligned}$$

Applying the left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\cosh(0) \\ &= B \end{aligned}$$

The solution becomes $X(x) = A\cosh(sx)$ and hence $\frac{dX}{dx} = A\sinh(sx)$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(L) \\ &= A\sinh(sL) \end{aligned}$$

$A = 0$ result in trivial solution. Therefore assuming $\sinh(sL) = 0$ implies $sL = 0$ which is not valid since $s > 0$ and $L \neq 0$. Hence only trivial solution results from this case. $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The ODE becomes

$$\frac{d^2X}{dx^2} = 0$$

The solution is

$$\begin{aligned} X(x) &= c_1 x + c_2 \\ \frac{dX}{dx} &= c_1 \end{aligned}$$

Applying left boundary conditions gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= c_1 \end{aligned}$$

Hence the solution becomes $X(x) = c_2$. Therefore $\frac{dX}{dx} = 0$. Applying the right B.C. provides no information.

Therefore this case leads to the solution $X(x) = c_2$. Associated with this one eigenvalue, the time equation becomes $\frac{dT_0}{dt} = 0$ hence T_0 is constant, say α. Hence the solution $u_0(x,t)$ associated with this $\lambda = 0$ is

$$\begin{aligned} u_0(x,t) &= X_0 T_0 \\ &= c_2\alpha \\ &= A_0 \end{aligned}$$

where constant $c_2\alpha$ was renamed to A_0 to indicate it is associated with $\lambda = 0$. $\lambda = 0$ is an eigenvalue.

case $\lambda > 0$

Hence $-\lambda$ is negative, and the roots are both complex.

$$r = \pm i\sqrt{\lambda}$$

The solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ \frac{dX}{dx} &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying the left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\sqrt{\lambda}\cos(0) \\ &= B\sqrt{\lambda} \end{aligned}$$

Therefore $B = 0$ as $\lambda > 0$. The solution becomes $X(x) = A\cos\left(\sqrt{\lambda}x\right)$ and $\frac{dX}{dx} = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$. Applying the right B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(L) \\ &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right) \end{aligned}$$

$A = 0$ gives a trivial solution. Selecting $\sin\left(\sqrt{\lambda}L\right) = 0$ gives

$$\sqrt{\lambda}L = n\pi \qquad n = 1, 2, 3, \cdots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Therefore the space solution is

$$X_n(x) = A_n\cos\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \cdots$$

The time solution is found by solving

$$\frac{dT_n}{dt} + k\lambda_n T_n = 0$$

This has the solution

$$\begin{aligned} T_n(t) &= e^{-k\lambda_n t} \\ &= e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \cdots \end{aligned}$$

For the same set of eigenvalues. Notice that no need to add a constant here, since it will be absorbed in the A_n when combined in the following step below. Since for $\lambda = 0$ the time solution was found to be constant, and for $\lambda > 0$ the time solution is $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$, then no time solution will grow with time. Time solutions always decay with time as the exponent $-k\left(\frac{n\pi}{L}\right)^2 t$ is negative quantity. The solution to the PDE for $\lambda > 0$ is

$$u_n(x,t) = T_n(t)X_n(x) \qquad n = 0, 1, 2, 3, \cdots$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned} u(x,t) &= u_{\lambda=0}(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

0.7.3 Part c

From the solution found above, setting $t = 0$ gives

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)$$

Therefore, $f(x)$ must satisfy the above

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

0.7.4 Part d

Multiplying both sides with $\cos\left(\frac{m\pi}{L}x\right)$ where in this problem $m = 0, 1, 2, \cdots$ (since there was an eigenvalue associated with $\lambda = 0$), and integrating over the domain gives

$$\begin{aligned}
\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \cos\left(\frac{m\pi}{L}x\right)\left(A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\right)dx \\
&= \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \int \cos\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \int_0^L \sum_{n=1}^{\infty} A_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx
\end{aligned}$$

Interchanging the order of summation and integration

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx \tag{1}$$

case $m = 0$

When $m = 0$ then $\cos\left(\frac{m\pi}{L}x\right) = 1$ and the above simplifies to

$$\int_0^L f(x)\,dx = \int_0^L A_0 dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = 0$ and the above becomes

$$\begin{aligned}
\int_0^L f(x)\,dx &= \int_0^L A_0 dx \\
&= A_0 L
\end{aligned}$$

Therefore

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

case $m > 0$

From (1), one term survives in the integration when only $n = m$, hence

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_0\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + A_m\int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx = 0$ and the above becomes

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_m\frac{L}{2}$$

Therefore

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

For $n = 1, 2, 3, \cdots$

0.7.5 Part (e)

The solution was found to be

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

In the limit as $t \to \infty$ the term $e^{-k\left(\frac{n\pi}{L}\right)^2 t} \to 0$. What is left is A_0. But $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ from above. This quantity is the average of the initial temperature.

0.8 section 2.3.8 (problem 7)

***2.3.8.** **Consider**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem [$u(x,0) = f(x)$] if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

0.8.1 part (a)

Equilibrium is at steady state, which implies $\frac{\partial u}{\partial t} = 0$ and the PDE becomes an ODE, since $u \equiv u(x)$ at steady state. Hence

$$\frac{d^2 u}{dx^2} - \frac{\alpha}{k}u = 0$$

The characteristic equation is $r^2 = \frac{\alpha}{k}$ or $r = \pm\sqrt{\frac{\alpha}{k}}$. Since $\alpha > 0$ and $k > 0$ then the roots are real, and the solution is

$$u = A_0 e^{\sqrt{\frac{\alpha}{k}}x} + B_0 e^{-\sqrt{\frac{\alpha}{k}}x}$$

This can be rewritten as

$$u(x) = A\cosh\left(\sqrt{\frac{\alpha}{k}}x\right) + B\sinh\left(\sqrt{\frac{\alpha}{k}}x\right)$$

Applying left B.C. gives

$$\begin{aligned} 0 &= u(0) \\ &= A\cosh(0) \\ &= A \end{aligned}$$

The solution becomes $u(x) = B\sinh\left(\sqrt{\frac{\alpha}{k}}x\right)$. Applying the right boundary condition gives

$$\begin{aligned} 0 &= u(L) \\ &= B\sinh\left(\sqrt{\frac{\alpha}{k}}L\right) \end{aligned}$$

$B = 0$ leads to trivial solution. Setting $\sinh\left(\sqrt{\frac{\alpha}{k}}L\right) = 0$ implies $\sqrt{\frac{\alpha}{k}}L = 0$. But this is not possible since $L \neq 0$. Hence the only solution possible is

$$\boxed{u(x) = 0}$$

0.8.2 Part (b)

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} - \alpha u \\ \frac{\partial u}{\partial t} + \alpha u &= k\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

Assuming $u(x,t) = X(x)T(t)$ and substituting in the above gives

$$XT' + \alpha XT = kTX''$$

Dividing by $kXT \neq 0$

$$\frac{T'}{kT} + \frac{\alpha}{k} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{k}\frac{T'}{T}+\frac{\alpha}{k}=\frac{X''}{X}=-\lambda$$

The two ODE's are

$$\begin{aligned}\frac{1}{k}\frac{T'}{T}+\frac{\alpha}{k}&=-\lambda\\ \frac{X''}{X}&=-\lambda\end{aligned}$$

Or

$$\begin{aligned}T'+(\alpha+\lambda k)\,T&=0\\ X''+\lambda X&=0\end{aligned}$$

The solution to the space ODE is the familiar (where $\lambda>0$ is only possible case, As found in problem 2.3.3, part d. Since it has the same B.C.)

$$X_n=B_n\sin\left(\frac{n\pi}{L}x\right)\qquad n=1,2,3,\cdots$$

Where $\lambda_n=\left(\frac{n\pi}{L}\right)^2$. The time ODE is now solved.

$$\frac{dT_n}{dt}+(\alpha+\lambda_n k)\,T_n=0$$

This has the solution

$$\begin{aligned}T_n(t)&=e^{-(\alpha+\lambda_n k)t}\\ &=e^{-\alpha t}e^{-\left(\frac{n\pi}{L}\right)^2kt}\end{aligned}$$

For the same eigenvalues. Notice that no need to add a constant here, since it will be absorbed in the B_n when combined in the following step below. Therefore the solution to the PDE is

$$u_n(x,t)=T_n(t)\,X_n(x)$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}u_n(x,t)\\ &=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-\alpha t}e^{-\left(\frac{n\pi}{L}\right)^2kt}\\ &=e^{-\alpha t}\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2kt}\end{aligned}$$

Where $e^{-\alpha t}$ was moved outside since it does not depend on n. From initial condition

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality of $\sin$ as before to find B_n results in

$$B_n=\frac{2}{L}\int_0^L\sin\left(\frac{n\pi}{L}x\right)f(x)\,dx$$

Hence the solution becomes

$$u(x,t)=\frac{2}{L}e^{-\alpha t}\left(\sum_{n=1}^{\infty}\left[\int_0^L\sin\left(\frac{n\pi}{L}x\right)f(x)\,dx\right]\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2kt}\right)$$

Hence it is clear that in the limit as t becomes large $u(x,t)\rightarrow 0$ since the sum is multiplied by $e^{-\alpha t}$ and $\alpha>0$

$$\lim_{t\rightarrow\infty}u(x,t)=0$$

This agrees with part (a)

0.9 section 2.3.10 (problem 8)

2.3.10. For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta$, implies that

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.44}$$

In this exercise we generalize this to n-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A}-\gamma\boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A}\cdot\boldsymbol{B}/\boldsymbol{B}\cdot\boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

*(c) Generalize (2.3.44) to functions. [*Hint*: Let $\boldsymbol{A}\cdot\boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]

$$\left|\bar{A}-\gamma\bar{B}\right|^2 = \left(\bar{A}-\gamma\bar{B}\right)\cdot\left(\bar{A}-\gamma\bar{B}\right)$$

Since $\left|\bar{A}-\gamma\bar{B}\right|^2 \ge 0$ then

$$\left(\bar{A}-\gamma\bar{B}\right)\cdot\left(\bar{A}-\gamma\bar{B}\right) \ge 0$$

Expanding

$$\left(\bar{A}\cdot\bar{A}\right) - \gamma\left(\bar{A}\cdot\bar{B}\right) - \gamma\left(\bar{B}\cdot\bar{A}\right) + \gamma^2\left(\bar{B}\cdot\bar{B}\right) \ge 0$$

But $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$, hence

$$\left(\bar{A}\cdot\bar{A}\right) - 2\gamma\left(\bar{A}\cdot\bar{B}\right) + \gamma^2\left(\bar{B}\cdot\bar{B}\right) \ge 0$$

Using the definition of $\gamma = \frac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{B}}$ into the above gives

$$\left(\bar{A}\cdot\bar{A}\right) - 2\frac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{B}}\left(\bar{A}\cdot\bar{B}\right) + \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\left(\bar{B}\cdot\bar{B}\right)^2}\left(\bar{B}\cdot\bar{B}\right) \ge 0$$

$$\left(\bar{A}\cdot\bar{A}\right) - 2\frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} + \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} \ge 0$$

$$\left(\bar{A}\cdot\bar{A}\right) - \frac{\left(\bar{A}\cdot\bar{B}\right)^2}{\bar{B}\cdot\bar{B}} \ge 0$$

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) - \left(\bar{A}\cdot\bar{B}\right)^2 \ge 0$$

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) \ge \left(\bar{A}\cdot\bar{B}\right)^2$$

But $\left(\bar{A}\cdot\bar{B}\right)^2 = \left|\bar{A}\cdot B\right|^2$ since $\bar{A}\cdot\bar{B}$ is just a number. The above becomes

$$\left(\bar{A}\cdot\bar{A}\right)\left(\bar{B}\cdot\bar{B}\right) \ge \left|\bar{A}\cdot B\right|^2$$

And $\bar{A}\cdot\bar{A} = \left|\bar{A}\right|^2$ and $\left(\bar{B}\cdot\bar{B}\right) = \left|\bar{B}\right|^2$ by definition as well. Therefore the above becomes

$$\left|\bar{A}\cdot B\right|^2 \le \left|\bar{A}\right|^2\left|\bar{B}\right|^2$$

Taking square root gives

$$\left|\bar{A}\cdot B\right| \le \left|\bar{A}\right|\left|\bar{B}\right|$$

Which is Schwarz's inequality.

0.9.1 Part b

From the norm definition

$$\left|\bar{A}\right| = \sqrt{\sum x^2 + y^2 + z^2}$$

Then

$$\left(\bar{A}\cdot\bar{A}\right) = \left|\bar{A}\right|^2 = \sum x^2 + y^2 + z^2$$

Hence

$$\left|\bar{A}\right|^2 = \sum_{n=1}^{\infty} a_n^2$$
$$\left|\bar{B}\right|^2 = \sum_{n=1}^{\infty} b_n^2$$

And

$$\bar{A}\cdot\bar{B} = \sum_{n=1}^{\infty} a_n b_n$$

Therefore the inequality can be written as

$$\left(\bar{A}\cdot\bar{B}\right)^2 \leq \left|\bar{A}\right|^2 \left|\bar{B}\right|^2$$
$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leq \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right)$$

0.9.2 Part c

Using $\bar{A}\cdot\bar{B}$ for functions to mean $\int_0^L A(x)B(x)\,dx$ then inequality for functions becomes

$$\left(\int_0^L A(x)B(x)\,dx\right)^2 \leq \left(\int_0^L A^2(x)\,dx\right)\left(\int_0^L B^2(x)\,dx\right)$$

0.10 section 2.4.1 (problem 9)

***2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to**

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0.$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ **(b)** $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\frac{\pi x}{L}$ **(d)** $u(x,0) = -3\cos\frac{8\pi x}{L}$

The same boundary conditions was encountered in problem 2.3.7, therefore the solution used here starts from the same general solution already found, which is

$$\lambda_0 = 0$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$
$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

0.10.1 Part (b)

$$u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$$

Comparing terms with the general solution at $t = 0$ which is

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

results in

$$A_0 = 6$$
$$A_3 = 4$$

And all other $A_n = 0$. Hence the solution is

$$u(x,t) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

0.10.2 Part (c)

$$u(x,0) = -2\sin\frac{\pi x}{L}$$

Hence

$$-2\sin\frac{\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) \tag{1}$$

Multiplying both sides of (1) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\begin{aligned}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \left(A_0\cos\left(\frac{m\pi}{L}x\right) + \cos\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\right)dx \\ &= \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \int_0^L \sum_{n=1}^{\infty} A_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx\end{aligned}$$

Interchanging the order of integration and summation

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

<u>Case $m = 0$</u>

The above becomes

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)dx = \int_0^L A_0 dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = 0$ hence

$$\begin{aligned}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)dx &= \int_0^L A_0 dx \\ A_0 L &= -2\int_0^L \sin\left(\frac{\pi x}{L}\right)dx \\ A_0 L &= -2\left(-\frac{\cos\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}}\right)_0^L \\ &= -\frac{2L}{\pi}\left(-\cos\left(\frac{\pi L}{L}\right) + \cos\left(\frac{\pi 0}{L}\right)\right) \\ &= -\frac{2L}{\pi}\left(-(-1)+1\right) \\ &= -\frac{4L}{\pi}\end{aligned}$$

Hence

$$\boxed{A_0 = \frac{-4}{\pi}}$$

<u>Case $m > 0$</u>

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

One term survives the summation resulting in

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = \frac{-4}{\pi}\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + A_m \int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx$$

But $\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx = 0$ and $\int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}$, therefore

$$\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx = A_m\frac{L}{2}$$

$$A_n = \frac{-4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

But

$$\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{-L\left(1+\cos(n\pi)\right)}{\pi\left(n^2-1\right)}$$

Therefore

$$\begin{aligned} A_n &= 4\frac{\left(1+\cos(n\pi)\right)}{\pi\left(n^2-1\right)} \\ &= 4\frac{(-1)^n+1}{\pi\left(n^2-1\right)} \qquad n = 1,2,3,\cdots \end{aligned}$$

Hence the solution becomes

$$u(x,t) = \frac{-4}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{\left(n^2-1\right)}\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

0.11 section 2.4.2 (problem 10)

***2.4.2. Solve**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = T(t)X(x)$, then the PDE becomes

$$\frac{1}{\kappa}T'X = X''T$$

Dividing by XT

$$\frac{1}{\kappa}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is real.

$$\frac{1}{\kappa}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + k\lambda T = 0 \tag{1}$$

$$X'' + \lambda X = 0 \tag{2}$$

Per problem statement, $\lambda \geq 0$, so only two cases needs to be examined.

<u>Case $\lambda = 0$</u>

The space equation becomes $X'' = 0$ with the solution

$$X = Ax + b$$

Hence left B.C. implies $X'(0) = 0$ or $A = 0$. Therefore the solution becomes $X = b$. The right B.C. implies $X(L) = 0$ or $b = 0$. Therefore this leads to $X = 0$ as the only solution.

This results in trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

Starting with the space ODE, the solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ \frac{dX}{dx} &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Left B.C. gives

$$\begin{aligned} 0 &= \frac{dX}{dx}(0) \\ &= B\sqrt{\lambda} \end{aligned}$$

Hence $B = 0$ since it is assumed $\lambda \neq 0$ and $\lambda > 0$. Solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$

Applying right B.C. gives

$$\begin{aligned} 0 &= X(L) \\ &= A\cos\left(\sqrt{\lambda}L\right) \end{aligned}$$

$A = 0$ leads to trivial solution. Therefore $\cos\left(\sqrt{\lambda}L\right) = 0$ or

$$\begin{aligned} \sqrt{\lambda} &= \frac{n\pi}{2L} \qquad n = 1,3,5,\cdots \\ &= \frac{(2n-1)\pi}{2L} \qquad n = 1,2,3\cdots \end{aligned}$$

Hence

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1,3,5,\cdots \\ &= \frac{(2n-1)^2\pi^2}{4L^2} \qquad n = 1,2,3\cdots \end{aligned}$$

Therefore

$$X_n(x) = A_n\cos\left(\frac{n\pi}{2L}x\right) \qquad n = 1,3,5,\cdots$$

And the corresponding time solution

$$T_n = e^{-k\left(\frac{n\pi}{2L}\right)^2 t} \qquad n = 1,3,5,\cdots$$

Hence

$$\begin{aligned} u_n(x,t) &= X_n T_n \\ u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} A_n\cos\left(\frac{n\pi}{2L}x\right)e^{-k\left(\frac{n\pi}{2L}\right)^2 t} \\ &= \sum_{n=1}^{\infty} A_n\cos\left(\frac{(2n-1)\pi}{2L}x\right)e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t} \end{aligned}$$

From initial conditions

$$f(x) = \sum_{n=1,3,5,\cdots}^{\infty} A_n\cos\left(\frac{n\pi}{2L}x\right)$$

Multiplying both sides by $\cos\left(\frac{m\pi}{2L}x\right)$ and integrating

$$\int_0^L f(x)\cos\left(\frac{m\pi}{2L}x\right)dx = \int\left(\sum_{n=1,3,5,\cdots}^{\infty} A_n\cos\left(\frac{m\pi}{2L}x\right)\cos\left(\frac{n\pi}{2L}x\right)\right)dx$$

Interchanging order of summation and integration and applying orthogonality results in

$$\begin{aligned} \int_0^L f(x)\cos\left(\frac{m\pi}{2L}x\right)dx &= A_m\frac{L}{2} \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{2L}x\right)dx \end{aligned}$$

Therefore the solution is

$$u(x,t)=\frac{2}{L}\sum_{n=1,3,5,\cdots}^{\infty}\left[\int_0^L f(x)\cos\left(\frac{n\pi}{2L}x\right)dx\right]\cos\left(\frac{n\pi}{2L}x\right)e^{-k\left(\frac{n\pi}{2L}\right)^2 t}$$

or

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\cos\left(\frac{(2n-1)\pi}{2L}x\right)dx\right]\cos\left(\frac{(2n-1)\pi}{2L}x\right)e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

0.12 section 2.4.3 (problem 11)

***2.4.3. Solve the eigenvalue problem**

$$\frac{d^2\phi}{dx^2}=-\lambda\phi$$

subject to

$$\phi(0)=\phi(2\pi)\quad\text{and}\quad\frac{d\phi}{dx}(0)=\frac{d\phi}{dx}(2\pi).$$

$$\begin{aligned}\frac{d\phi^2}{dx^2}+\lambda\phi&=0\\ \phi(0)&=\phi(2\pi)\\ \frac{d\phi}{dx}(0)&=\frac{d\phi}{dx}(2\pi)\end{aligned}$$

First solution using transformation

Let $\tau=x-\pi$, hence the above system becomes

$$\begin{aligned}\frac{d\phi^2}{d\tau^2}+\lambda\phi&=0\\ \phi(-\pi)&=\phi(\pi)\\ \frac{d\phi}{d\tau}(-\pi)&=\frac{d\phi}{d\tau}(\pi)\end{aligned}$$

The characteristic equation is $r^2+\lambda=0$ or $r=\pm\sqrt{-\lambda}$. Assuming λ is real. There are three cases to consider.

Case $\lambda<0$

Let $s=\sqrt{-\lambda}>0$

$$\begin{aligned}\phi(\tau)&=c_1\cosh(s\tau)+c_2\sinh(s\tau)\\ \phi'(\tau)&=sc_1\sinh(s\tau)+sc_2\cosh(s\tau)\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}\phi(-\pi)&=\phi(\pi)\\ c_1\cosh(s\pi)-c_2\sinh(s\pi)&=c_1\cosh(s\pi)+c_2\sinh(s\pi)\\ 2c_2\sinh(s\pi)&=0\\ c_2\sinh(s\pi)&=0\end{aligned}\tag{1}$$

Applying second B.C. gives

$$\begin{aligned}\phi'(-\pi)&=\phi'(\pi)\\ -sc_1\sinh(s\pi)+sc_2\cosh(s\pi)&=sc_1\sinh(s\pi)+sc_2\cosh(s\pi)\\ 2c_1\sinh(s\pi)&=0\\ c_1\sinh(s\pi)&=0\end{aligned}\tag{2}$$

Since $\sinh(s\pi)$ is zero only for $s\pi=0$ and $s\pi$ is not zero because $s>0$. Then the only other option is that both $c_1 = 0$ and $c_2 = 0$ in order to satisfy equations (1)(2). Hence trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$

The space equation becomes $\frac{d\phi^2}{d\tau^2} = 0$ with the solution $\phi(\tau) = A\tau + B$. Applying the first B.C. gives

$$\begin{aligned}\phi(-\pi) &= \phi(\pi)\\ -A\pi + B &= A\pi + B\\ 0 &= 2A\pi\end{aligned}$$

Hence $A = 0$. The solution becomes $\phi(\tau) = B$. And $\phi'(\tau) = 0$. The second B.C. just gives $0 = 0$. Therefore the solution is

$$\phi(\tau) = C$$

Where C is any constant. Hence $\lambda = 0$ is an eigenvalue.

Case $\lambda > 0$

$$\begin{aligned}\phi(\tau) &= c_1 \cos\left(\sqrt{\lambda}\tau\right) + c_2 \sin\left(\sqrt{\lambda}\tau\right)\\ \phi'(\tau) &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\tau\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\tau\right)\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}\phi(-\pi) &= \phi(\pi)\\ c_1\cos\left(\sqrt{\lambda}\pi\right) - c_2\sin\left(\sqrt{\lambda}\pi\right) &= c_1\cos\left(\sqrt{\lambda}\pi\right) + c_2\sin\left(\sqrt{\lambda}\pi\right)\\ 2c_2\sin\left(\sqrt{\lambda}\pi\right) &= 0\\ c_2\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}\phi'(-\pi) &= \phi'(\pi)\\ c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\ 2c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= 0\\ c_1\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (2)\end{aligned}$$

Both (3) and (2) can be satisfied for non-zero $\sqrt{\lambda}\pi$. The trivial solution is avoided. Therefore the eigenvalues are

$$\begin{aligned}\sin\left(\sqrt{\lambda}\pi\right) &= 0\\ \sqrt{\lambda_n}\pi &= n\pi \qquad n = 1,2,3,\cdots\\ \lambda_n &= n^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

Hence the corresponding eigenfunctions are

$$\left\{\cos\left(\sqrt{\lambda_n}\tau\right), \sin\left(\sqrt{\lambda_n}\tau\right)\right\} = \{\cos(n\tau), \sin(n\tau)\}$$

Transforming back to x using $\tau = x - \pi$

$$\{\cos(n(x-\pi)), \sin(n(x-\pi))\} = \{\cos(nx - n\pi), \sin(nx - n\pi)\}$$

But $\cos(x-\pi) = -\cos x$ and $\sin(x-\pi) = -\sin x$, hence the eigenfunctions are

$$\{-\cos(nx), -\sin(nx)\}$$

The signs of negative on an eigenfunction (or eigenvector) do not affect it being such as this is just a multiplication by −1. Hence the above is the same as saying the eigenfunctions are

$$\{\cos(nx), \sin(nx)\}$$

Summary

	eigenfunctions
$\lambda = 0$	arbitrary constant
$\lambda > 0$	$\{\cos(nx), \sin(nx)\}$ for $n = 1, 2, 3 \cdots$

Second solution without transformation

(note: Using transformation as shown above seems to be easier method than this below).

The characteristic equation is $r^2 + \lambda = 0$ or $r = \pm\sqrt{-\lambda}$. Assuming λ is real. There are three cases to consider.

Case $\lambda < 0$

In this case $-\lambda$ is positive and the roots are both real. Assuming $\sqrt{-\lambda} = s$ where $s > 0$, then the solution is

$$\begin{aligned} \phi(x) &= Ae^{sx} + Be^{-sx} \\ \phi'(x) &= Ase^{sx} - Bse^{-sx} \end{aligned}$$

First B.C. gives

$$\begin{aligned} \phi(0) &= \phi(2\pi) \\ A + B &= Ae^{2s\pi} + Be^{-2s\pi} \\ A\left(1 - e^{2s\pi}\right) + B\left(1 - e^{-2s\pi}\right) &= 0 \end{aligned} \tag{1}$$

The second B.C. gives

$$\begin{aligned} \phi'(0) &= \phi'(2\pi) \\ As - Bs &= Ase^{2s\pi} - Bse^{-2s\pi} \\ A\left(1 - e^{2s\pi}\right) + B\left(-1 + e^{-2s\pi}\right) &= 0 \end{aligned} \tag{2}$$

After dividing by s since $s \neq 0$. Now a 2 by 2 system is setup from (1),(2)

$$\begin{pmatrix} \left(1 - e^{2s\pi}\right) & \left(1 - e^{-2s\pi}\right) \\ \left(1 - e^{2s\pi}\right) & \left(-1 + e^{-2s\pi}\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since this is $Mx = b$ with $b = 0$ then for non-trivial solution $|M|$ must be zero. Checking the determinant to see if it is zero or not:

$$\begin{aligned} \begin{vmatrix} \left(1 - e^{2s\pi}\right) & \left(1 - e^{-2s\pi}\right) \\ \left(1 - se^{2s\pi}\right) & \left(-1 + se^{-2s\pi}\right) \end{vmatrix} &= \left(1 - e^{2s\pi}\right)\left(-1 + e^{-2s\pi}\right) - \left(1 - e^{-2s\pi}\right)\left(1 - e^{2s\pi}\right) \\ &= \left(-1 + e^{-2s\pi} + e^{2s\pi} - 1\right) - \left(1 - e^{2s\pi} - e^{-2s\pi} + 1\right) \\ &= -1 + e^{-2s\pi} + e^{2s\pi} - 1 - 1 + e^{2s\pi} + e^{-2s\pi} - 1 \\ &= -4 + 2e^{2s\pi} + 2e^{-2s\pi} \\ &= -4 + 2\left(e^{2s\pi} + e^{-2s\pi}\right) \\ &= -4 + 4\cosh(2s\pi) \end{aligned}$$

Hence for the determinant to be zero (so that non-trivial solution exist) then $-4 + 4\cosh(2s\pi) = 0$ or $\cosh(2s\pi) = 1$ which has the solution $2s\pi = 0$. Which means $s = 0$. But the assumption was that $s > 0$. This implies only a trivial solution exist and $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The space equation becomes $\frac{d\phi^2}{dx^2} = 0$ with the solution $\phi(x) = Ax + B$. Applying the first

B.C. gives

$$
\begin{aligned}
B &= 2A\pi + B \\
0 &= 2A\pi
\end{aligned}
$$

Hence $A = 0$. The solution becomes $\phi(x) = B$. And $\phi'(x) = 0$. The second B.C. just gives $0 = 0$. Therefore the solution is

$$
\phi(x) = C
$$

Where C is any constant. <u>Hence $\lambda = 0$ is an eigenvalue.</u>

<u>Case $\lambda > 0$</u>

In this case the solution is

$$
\begin{aligned}
\phi(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\
\phi'(x) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)
\end{aligned}
$$

Applying first B.C. gives

$$
\begin{aligned}
\phi(0) &= \phi(2\pi) \\
A &= A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right) \\
A\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) - B\sin\left(2\pi\sqrt{\lambda}\right) &= 0
\end{aligned}
$$

Applying second B.C. gives

$$
\begin{aligned}
\phi'(0) &= \phi'(2\pi) \\
B\sqrt{\lambda} &= -A\sqrt{\lambda}\sin\left(2\pi\sqrt{\lambda}\right) + B\sqrt{\lambda}\cos\left(2\pi\sqrt{\lambda}\right) \\
A\sqrt{\lambda}\sin\left(2\pi\sqrt{\lambda}\right) + B\left(\sqrt{\lambda} - \sqrt{\lambda}\cos\left(2\pi\sqrt{\lambda}\right)\right) &= 0 \\
A\sin\left(2\pi\sqrt{\lambda}\right) + B\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) &= 0
\end{aligned}
$$

Therefore

$$
\begin{pmatrix} 1-\cos\left(2\pi\sqrt{\lambda}\right) & -\sin\left(2\pi\sqrt{\lambda}\right) \\ \sin\left(2\pi\sqrt{\lambda}\right) & 1-\cos\left(2\pi\sqrt{\lambda}\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}
$$

Setting $|M| = 0$ to obtain the eigenvalues gives

$$
\begin{aligned}
\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right)\left(1-\cos\left(2\pi\sqrt{\lambda}\right)\right) + \sin\left(2\pi\sqrt{\lambda}\right)\sin\left(2\pi\sqrt{\lambda}\right) &= 0 \\
1-\cos\left(2\pi\sqrt{\lambda}\right) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
\cos\left(2\pi\sqrt{\lambda}\right) &= 1 \\
2\pi\sqrt{\lambda_n} &= n\pi \qquad n = 2,4,\cdots \\
\sqrt{\lambda_n} &= \frac{n}{2} \qquad n = 2,4,\cdots
\end{aligned}
$$

Or

$$
\begin{aligned}
\sqrt{\lambda_n} &= n \qquad n = 1,2,3,\cdots \\
\lambda_n &= n^2 \qquad n = 1,2,3,\cdots
\end{aligned}
$$

Therefore the eigenfunctions are

$$
\phi_n(x) = \{\cos(nx), \sin(nx)\}
$$

Summary

	eigenfunctions
$\lambda = 0$	arbitrary constant
$\lambda > 0$	$\{\cos(nx), \sin(nx)\}$ for $n = 1,2,3\cdots$

0.13 section 2.4.6 (problem 12)

2.4.6. Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) Directly from the equilibrium problem (see Sec. 1.4)

(b) By computing the limit as $t \to \infty$ of the time-dependent problem

The PDE for the thin circular ring is

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\\ u(-L,t) &= u(L,t)\\ \frac{\partial u(-L,t)}{\partial t} &= \frac{\partial u(L,t)}{\partial t}\\ u(x,0) &= f(x)\end{aligned}$$

0.13.1 Part (a)

At equilibrium $\frac{\partial u}{\partial t} = 0$ and the PDE becomes

$$0 = \frac{\partial^2 u}{\partial x^2}$$

As it now has one independent variable, it becomes the following ODE to solve

$$\begin{aligned}\frac{d^2 u(x)}{dx^2} &= 0\\ u(-L) &= u(L)\\ \frac{du}{dx}(-L) &= \frac{du}{dx}(L)\end{aligned}$$

Solution to $\frac{d^2u}{dx^2} = 0$ is

$$u(x) = c_1 x + c_2$$

Where c_1, c_2 are arbitrary constants. From the first B.C.

$$\begin{aligned}u(-L) &= u(L)\\ -c_1 L + c_2 &= c_1 L + c_2\\ 2c_1 L &= 0\\ c_1 &= 0\end{aligned}$$

Hence the solution becomes

$$u(x) = c_2$$

The second B.C. adds nothing as it results in $0 = 0$. Hence the solution at equilibrium is

$$\boxed{u(x) = c_2}$$

This means at equilibrium the temperature in the ring reaches a constant value.

0.13.2 Part (b)

The time dependent solution was derived in problem 2.4.3 and also in section 2.4, page 62 in the book, given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi x}{L}\right)^2 t} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi x}{L}\right)^2 t}$$

As $t \to \infty$ the terms $e^{-k\left(\frac{n\pi x}{L}\right)^2 t} \to 0$ and the above reduces to

$$u(x,\infty) = a_0$$

Since a_0 is constant, this is the same result found in part (a).