University Course

Math 319
Techniques in Ordinary Differential Equations

University of Wisconsin, Madison

Fall 2016

My Class Notes

Nasser M. Abbasi

Fall 2016

Contents

Chapter 1

Introduction

Local contents

1.1 links

1. Professor Minh-Binh Tran web page
2. class web page http://www.math.wisc.edu/ minhbinh/Math319.htm
3. TA web site
4. canvas.wisc.edu Needs login

1.2 syllabus

Math 319: Techniques in Ordinary Differential Equations

Instructor: Minh-Binh Tran

- Office: B312 Sterling Hall
- email: minhbinh@math.wisc.edu
- WWW home page: http://www.math.wisc.edu/~minhbinh with homework and exam information)

Office hours: Mondays 9:50 - 10:50 a.m, Fridays 13:10 - 14:10 p.m.
TAs Office hours:
Ramos, Eric: Wednesdays 3:30-5:30 p.m. and 1 more hour by appointments.
Enkhtaivan, Enkhzaya: Tuesdays and Thursdays 9-10 a.m. and 1 more hour by appointments.
Textbook: Elementary Differential Equations and Boundary Value Problems, Boyce and DiPrima, 10th Ed.
Sylabus: We will cover the following material from the book:
Chapter 1. Introduction
Chapter 2. First Order Differential Equations
Chapter 3. Second Order Differential Equations
Chapter 5. Series Solutions of Second Order Linear Equations
Chapter 6. The Laplace Transform
Chapter 7. Systems of First Order Linear Equations

Course website: Available through Learn@UW.

Homework:

- There will be 10-11 homeworks.
- The lowest homework score will be dropped.
- Homework is assigned weekly on Friday and collected at the beginning of lecture the following Friday.
- Rules for homework submission: All homework should be written clearly.
  - You are required to prepare your homework assignments on your own (but are allowed to work on the problems with others). Writings up the solution on your own is a way to ensure you understand all of the relevant concepts completely.
  - All homework should be submitted in hard copy before the start of lecture on the day it is due (hand in to me or slide under the door of my office).
  - Consulting solutions from prior years or using collections of solutions found on the internet is not allowed in the course.
  - No credit for copied or unexcused late homework. Valid excuses for late homework are illness or family emergency.

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Chapter 2

HWs

Local contents

2.1 HW1

Note on plots: Some of these problems requires plotting. These were done both by hand and also by the computer but only the computer version of the plot was included.

2.1.1 Section 1.2 problem 1

Solve each of the following and plot the solution for different y_0 values.

2.1.1.1 part a

$\frac{dy}{dt} = -y + 5, y(0) = y_0$

$$\frac{dy}{dt} + y = 5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 1, g(t) = 5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

The Integrating factor is $e^{\int dt} = e^t$. Multiplying both sides by e^t gives

$$\frac{d}{dt}\left(ye^t\right) = 5e^t$$

Integrating

$$\begin{aligned} ye^t &= 5\int e^t dt + c \\ &= 5e^t + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{-t} \tag{1}$$

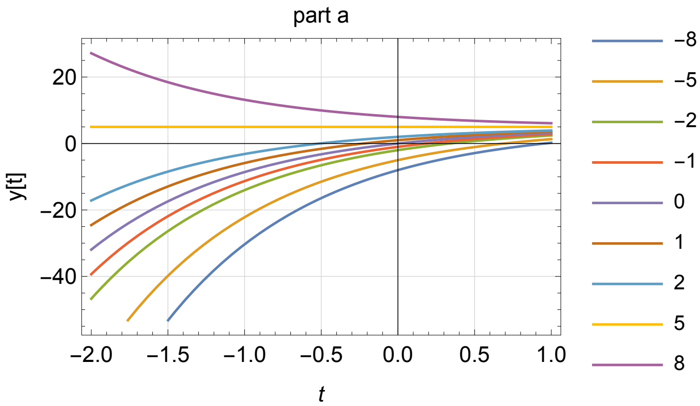
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{-t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 5$.The following plot gives the solution $y(t)$ for few values of y_0



2.1.1.2 part b

$\frac{dy}{dt} = -2y + 5, y(0) = y_0$

$$\frac{dy}{dt} + 2y = 5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 2, g(t) = 5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = 5e^{2t}$$

Integrating

$$\begin{aligned} ye^{2t} &= 5\int e^{2t}dt + c \\ &= \frac{5}{2}e^{2t} + c \end{aligned}$$

Hence

$$y(t) = \frac{5}{2} + ce^{-2t} \tag{1}$$

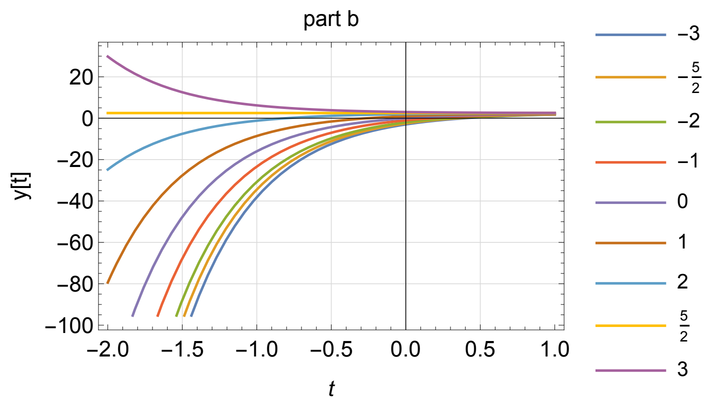
Applying initial conditions gives

$$\begin{aligned} y_0 &= \frac{5}{2} + c \\ c &= y_0 - \frac{5}{2} \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 2.5 + \left(y_0 - 2.5\right)e^{-2t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 2.5$. The following plot gives the solution $y(t)$ for few values of y_0



2.1.1.3 part c

$\frac{dy}{dt} = -2y + 10, y(0) = y_0$

$$\frac{dy}{dt} + 2y = 10$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 2, g(t) = 10$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = 10e^{2t}$$

Integrating

$$\begin{aligned} ye^{2t} &= 10\int e^{2t}dt + c \\ &= 5e^{2t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{-2t} \tag{1}$$

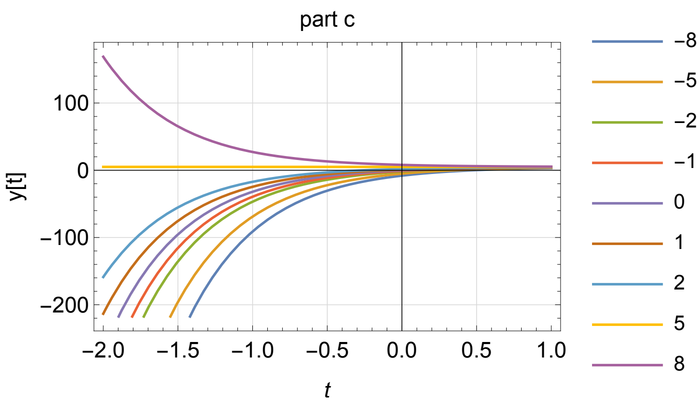
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{-2t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 5$. The following plot gives the solution $y(t)$ for few values of y_0



Discussion of differences In all solutions the term with e^{-t} and e^{-2t} in it will vanish as $t \to +\infty$. Hence for $t > 0$ all solution approach a constant value as $t \to \infty$, which is 5 for part (a) and (c) and 2.5 for part (b). Since part(b,c) has e^{-2t} term, these will approach the asymptote faster (converges faster) than part (a) which has e^{-t} term.

2.1.2 Section 1.2, problem 2

2.1.2.1 part (a)

$\frac{dy}{dt} = y - 5, y(0) = y_0$

$$\frac{dy}{dt} - y = -5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -1, g(t) = -5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-\int dt} = e^{-t}$. Multiplying both sides by e^{-t} gives

$$\frac{d}{dt}\left(ye^{-t}\right) = -5e^{-t}$$

Integrating

$$\begin{aligned} ye^{-t} &= -5\int e^{-t}dt + c \\ &= 5e^{-t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{t} \tag{1}$$

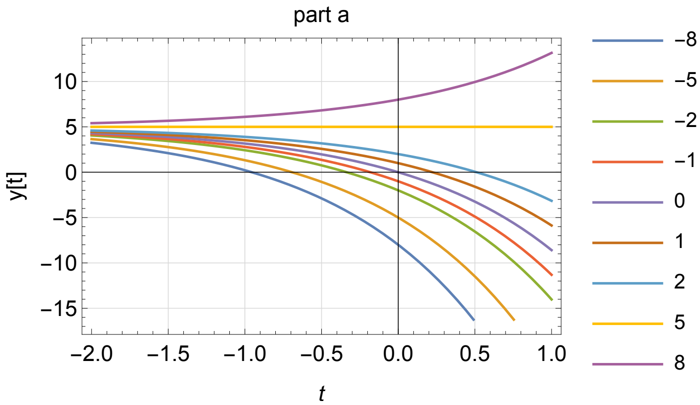
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



2.1.2.2 part (b)

$\frac{dy}{dt} = 2y - 5, y(0) = y_0$

$$\frac{dy}{dt} - 2y = -5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = -5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-2\int dt} = e^{-2t}$. Multiplying both sides by e^{-2t} gives

$$\frac{d}{dt}\left(ye^{-2t}\right) = -5e^{-2t}$$

Integrating

$$\begin{aligned} ye^{-2t} &= -5\int e^{-2t}dt + c \\ &= 2.5e^{-2t} + c \end{aligned}$$

Hence

$$y(t) = 2.5 + ce^{2t} \tag{1}$$

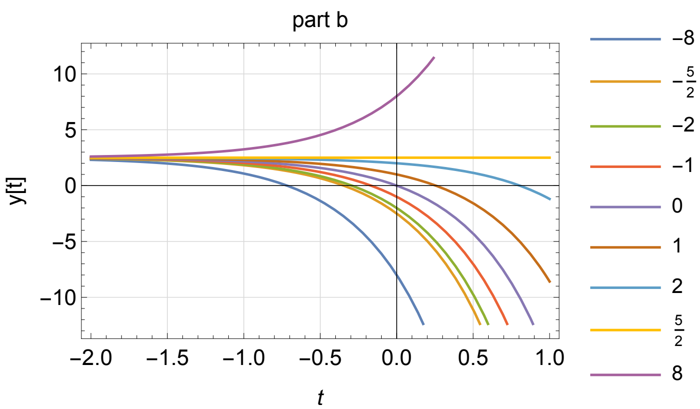
Applying initial conditions gives

$$\begin{aligned} y_0 &= 2.5 + c \\ c &= y_0 - 2.5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 2.5 + \left(y_0 - 2.5\right)e^{2t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



2.1.2.3 part (c)

$\frac{dy}{dt} = 2y - 10, y(0) = y_0$

$$\frac{dy}{dt} - 2y = -10$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = -10$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-2\int dt} = e^{-2t}$. Multiplying both sides by e^{-2t} gives

$$\frac{d}{dt}\left(ye^{-2t}\right) = -10e^{-2t}$$

Integrating

$$\begin{aligned} ye^{-2t} &= -10\int e^{-2t}dt + c \\ &= 5e^{-2t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{2t} \tag{1}$$

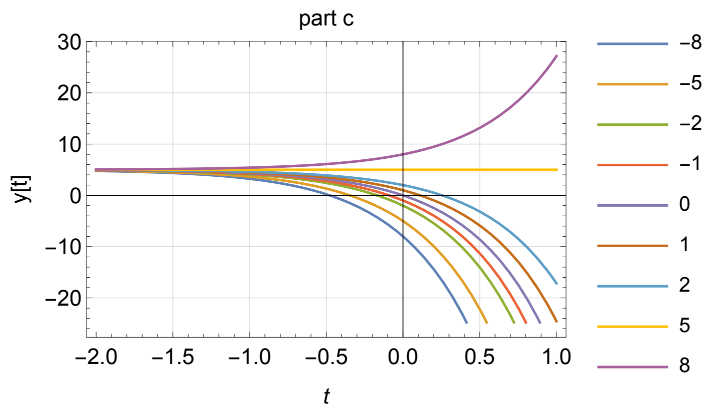
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{2t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



Discussion of differences In all solutions the term with e^t and e^{2t} in it will vanish as $t \to -\infty$. Hence for $t < 0$ all solution approach a constant value as $t \to -\infty$, which is 5 for part (a) and (c) and 2.5 for part (b). Since part(b,c) has e^{2t} term, these will diverge faster for large t than part (a) which has e^t term.

2.1.3 Section 1.3, problem 7

In each of the problems below, verify that each given function is the solution to the ODE

$y'' - y = 0; y_1(t) = e^t; y_2(t) = \cosh(t)$.

For $y_1(t)$, taking derivatives of y_1 gives $y_1' = e^t, y_1'' = e^t$. Substituting into the ODE gives

$$e^t - e^t = 0$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For $y_2(t)$, taking derivatives of y_2 gives $y_2' = \sinh(t), y_2'' = \cosh(t)$. Substituting into the ODE gives

$$\cosh(t) - \cosh(t) = 0$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

2.1.4 Section 1.3, problem 8

$y'' + 2y' - 3y = 0; y_1(t) = e^{-3t}; y_2(t) = e^t$.

For $y_1(t)$: Taking derivatives of y_1 gives $y_1' = -3e^{-3t}, y_1'' = 9e^{-3t}$. Substituting into the ODE gives

$$\begin{aligned} 9e^{-3t} + 2\left(-3e^{-3t}\right) - 3\left(e^{-3t}\right) &= 9e^{-3t} - 6e^{-3t} - 3e^{-3t} \\ &= 0 \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For $y_2(t)$: Taking derivatives of y_2 gives $y_2' = e^t, y_2'' = e^t$. Substituting into the ODE gives

$$e^t + 2e^t - 3e^t = 0$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

2.1.5 Section 1.3, problem 9

$ty' - y = t^2; y_1(t) = 3t + t^2$

Taking derivative of y_1 gives $y_1' = 3 + 2t$. Substituting into the ODE gives

$$\begin{aligned} t(3+2t) - \left(3t + t^2\right) &= 3t + 2t^2 - 3t - t^2 \\ &= t^2 \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

2.1.6 Section 1.3, problem 10

$y^{(4)} + 4y''' + 3y = t; y_1(t) = \frac{t}{3}; y_2(t) = e^{-t} + \frac{t}{3}$

For y_1: Taking derivative of y_1 gives $y_1' = \frac{1}{3}, y_1'' = 0, y_1''' = 0, y_1^{(4)} = 0$. Substituting into the ODE gives

$$0 + 0 + 3\left(\frac{t}{3}\right) = t$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For y_2: Taking derivatives of y_2 gives $y_2' = -e^{-t} + \frac{1}{3}, y_2'' = e^{-t}, y_2''' = -e^{-t}, y_2^{(4)} = e^{-t}$ .Substituting into the ODE gives

$$\begin{aligned} e^{-t} - 4e^{-t} + 3\left(e^{-t} + \frac{t}{3}\right) &= e^{-t} - 4e^{-t} + 3e^{-t} + t \\ &= t \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

2.1.7 Section 1.3, problem 15

Determine the value of r for which the given ODE has solution in the form $y = e^{rt}$

$$y' + 2y = 0$$

Assume the solution is of the form Ae^{rt} where A is arbitrary constant. Substituting this into the ODE gives

$$Are^{rt} + 2Ae^{rt} = 0$$

Since $e^{rt} \neq 0$ and $A \neq 0$ (else trivial solution), then dividing thought by Ae^{rt} gives

$$r + 2 = 0$$

Hence

$$\boxed{r = -2}$$

The solution is

$$y(t) = Ae^{-2t}$$

2.1.8 Section 1.3, problem 16

Determine the value of r for which the given ODE has solution in the form $y = e^{rt}$

$$y'' - y = 0$$

Assume the solution is of the form Ae^{rt} where A is arbitrary constant. Substituting this into the ODE gives

$$Ar^2e^{rt} - Ae^{rt} = 0$$

Since $e^{rt} \neq 0$ and $A \neq 0$ (else trivial solution), then dividing thought by Ae^{rt} gives

$$r^2 - 1 = 0$$

Hence

$$\boxed{r = \pm 1}$$

The solution is

$$y(t) = c_1e^{-t} + c_2e^{t}$$

2.1.9 Section 2.1 problem 1

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' + 3y = t + e^{-2t}$

2.1.9.1 Part (a)

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 3, g(t) = t + e^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique.

First the ODE is written such that y' is on one side, and everything else on the other side.

$$\begin{aligned} y' &= -3y + t + e^{-2t} \\ &= f\left(t, y\right) \end{aligned}$$

Global view: For fixed y, as $t \to \infty, y' \to \infty$ and for $t \to -\infty, y' \to \infty$. At $t = 0, y' = -3y + 1$.

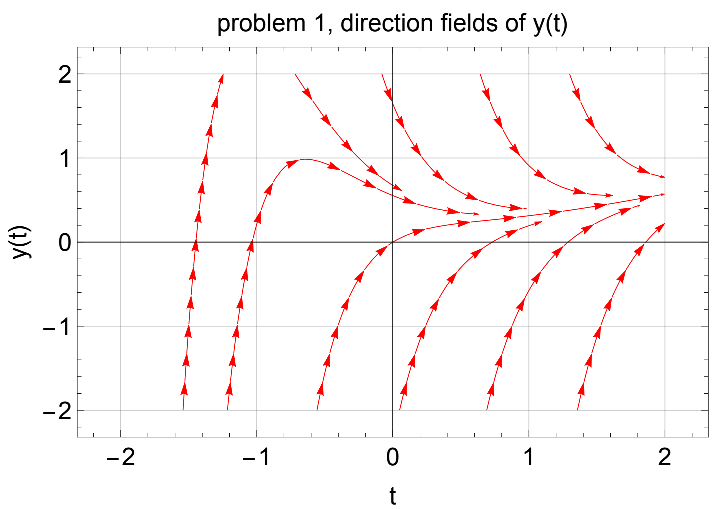
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t=-1$	$t=0$	$t=1$
$y=-1$	$y'=3-1+e^2\approx 9$	$y'=3+0+e^0=4$	$y'=3+1+e^{-2}\approx 4$
$y=0$	$y'=-1+e^2\approx 6$	$y'=0+e^0=1$	$y'=0+1+e^{-2}\approx 1$
$y=1$	$y'=-3-1+e^2\approx 3$	$y'=-3+0+e^0=-2$	$y'=-3+1+e^{-2}\approx -2$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



2.1.9.2 Part (b)

The solutions for large positive t appear to approach an asymptote straight line with positive slope. This is confirmed by next part.

2.1.9.3 Part (c)

$$y'+3y=t+e^{-2t}$$

This is first order, linear ODE of the form $y'+p(t)y=g(t)$ where $p(t)=3, g(t)=t+e^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^{3t}, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^{3t}y\right)=te^{3t}+e^{t}$$

Integrating

$$\begin{aligned} e^{3t}y &= \int te^{3t}dt + \int e^{t}dt + c \\ &= e^{3t}\left(\frac{t}{3} - \frac{1}{9}\right) + e^{t} + c \end{aligned}$$

Therefore

$$y = \left(\frac{t}{3} - \frac{1}{9}\right) + e^{-2t} + ce^{-3t} \qquad t \in \Re$$

For large positive t, the term $\frac{t}{3}$ dominates and $y(t) \approx \frac{1}{3}t$. Hence the solution as $t \to \infty$ approaches asymptote line with slope $\frac{1}{3}$. For $t \to -\infty$ the solution grows exponentially in the negative half plane. The sign of c determines which direction the solution grows to since e^{-3t} increases faster than e^{-2t} for negative t.

2.1.10 Section 2.1 problem 2

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' - 2y = te^{-2t}$

2.1.10.1 Part (a)

First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = 2y + te^{-2t}$$

Global view: As $t \to \infty, y' \to \infty$ and as $t \to -\infty, y' \to \infty$. And $y' = 0$ at point $t = 0, y = -\frac{1}{2}$.

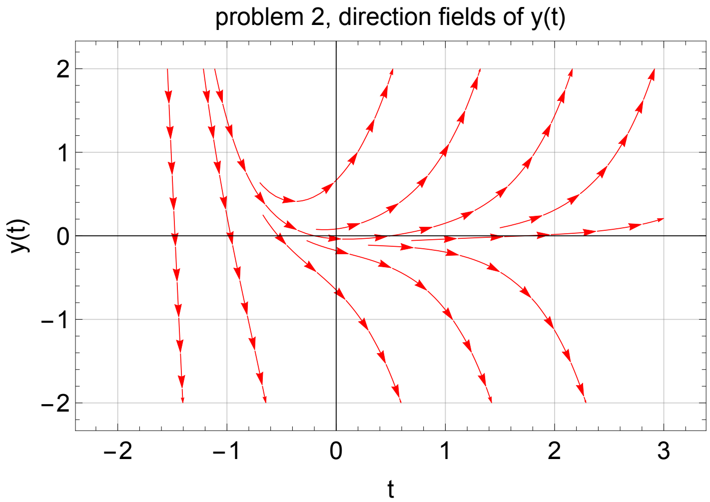
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t=-1$	$t=0$	$t=1$
$y=-1$	$y' = -2 - e^2 \approx -9$	$y' = -2 + 0 = -2$	$y' = -2 + e^{-2} \approx -1.8$
$y=0$	$y' = -e^{2t} \approx -7$	$y' = 0$	$y' = 0 + e^{-2} \approx 0.1$
$y=1$	$y' = 2 - e^2 \approx -5$	$y' = 2$	$y' = 2 + e^{-2} \approx 2.1$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



2.1.10.2 Part (b)

The solutions for large positive t appear to grow exponentially. This is confirmed by next part.

2.1.10.3 Part (c)

$$y' - 2y = te^{-2t}$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = te^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^{-2t}, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^{-2t}y\right) = te^{-4t}$$

Integrating

$$\begin{aligned} e^{-2t}y &= \int te^{-4t}dt + c \\ &= e^{-4t}\left(-\frac{t}{4} - \frac{1}{16}\right) + c \end{aligned}$$

Hence

$$y = e^{-2t}\left(-\frac{t}{4} - \frac{1}{16}\right) + ce^{2t}$$

For large positive t, the term $e^{-2t}\left(\frac{t}{4} - \frac{1}{16}\right) \to 0$ and what is left is $e^{2t}c$ which grows exponentially.

$$\lim_{t\to\infty} y(t) = ce^2t$$

For large negative t, the solution grows exponentially in the negative half plane.

2.1.11 Section 2.1 problem 3

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' + y = te^{-t} + 1$

2.1.11.1 Part (a)

First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = -y + te^{-t} + 1$$

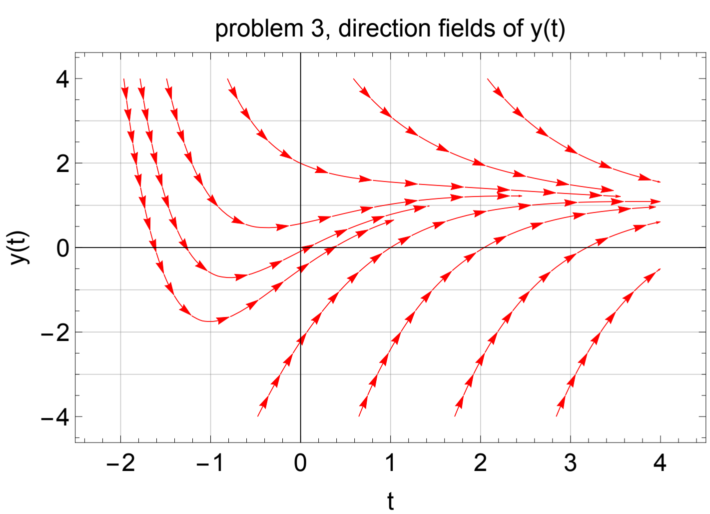
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t = -1$	$t = 0$	$t = 1$
$y = -1$	$y' = 1 - e^{t} + 1 \approx -0.7$	$y' = 1 + 1 = 2$	$y' = 1 + e^{-1} + 1 \approx 2.3$
$y = 0$	$y' = 0 - e^{t} + 1 \approx -1.7$	$y' = 1$	$y' = 0 + e^{-1} + 1 \approx 1.3$
$y = 1$	$y' = -1 - e^{1} + 1 \approx -0.7$	$y' = -1 + 1 = 0$	$y' = 1 + e^{-1} + 1 \approx 0.3$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



2.1.11.2 Part (b)

The solutions for large positive t appear to approach an asymptote line $y(t) = 1$. This is confirmed by next part.

2.1.11.3 Part (c)

$$y' + y = te^{-t} + 1$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 1, g(t) = te^{-2t} + 1$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^t, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^t y\right) = t + e^t$$

Integrating

$$\begin{aligned} e^t y &= \int t dt + \int e^t dt + c \\ &= \frac{1}{2}t^2 + e^t + c \end{aligned}$$

Hence

$$\begin{aligned} y &= \frac{1}{2}t^2 e^{-t} + 1 + ce^{-t} \\ &= e^{-t}\left(\frac{1}{2}t^2 + c\right) + 1 \end{aligned}$$

For large positive t, the term $e^{-t}\left(\frac{1}{2}t^2 + c\right) \to 0$ and what is left is 1. Hence the solution as $t \to \infty$ approaches asymptote line $y(t) = 1$.

$$\lim_{t\to\infty} y(t) = 1$$

2.1.12 Section 2.1 problem 4

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' + \frac{y}{t} = 3\cos 2t$ for $t > 0$

2.1.12.1 Part (a)

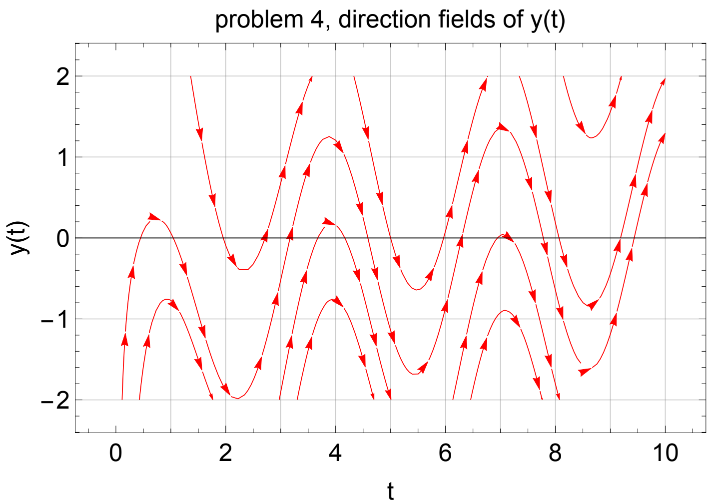
First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = -\frac{y}{t} + 3\cos 2t$$

For each value of $y = \{-1, 0, 1\}$ and for each $t = \{1, 2, 3\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t=1$	$t=2$	$t=3$
$y=-1$	$y'=1+3\cos 2\approx -0.25$	$y'=\frac{1}{2}+3\cos 4\approx -1.5$	$y'=\frac{1}{3}+3\cos 6\approx 3.2$
$y=0$	$y'=3\cos 2\approx -1.25$	$y'=0+3\cos 4\approx -2$	$y'=0+3\cos 6\approx 2.9$
$y=1$	$y'=-1+3\cos 2\approx -2.25$	$y'=-\frac{1}{2}+3\cos 4=-2.5$	$y'=-\frac{1}{3}+3\cos 2t\approx 2.5$

A sketch was now made by hand as well using the computer. The computer version is given below.



2.1.12.2 Part (b)

The solutions for large positive t appear to oscillate, but it is hard to see that from the few points above, as more points is needed and only after using the computer plot and solving it did this become more clear.

2.1.12.3 Part (c)

$$y'+\frac{y}{t}=3\cos 2t$$

This is first order, linear ODE of the form $y'+p(t)y=g(t)$ where $p(t)=\frac{1}{t}, g(t)=3\cos 2t$. $p(t)$ is singular at $t=0$ (not continuous at that point) while $g(t)$ is continuous on the whole real line, then by theorem 1, a solution exists and is unique only if the initial condition is not at $t_0=0$. The solution found is valid on an interval that excludes $t=0$ but includes t_0. The problem says to solve this on $t>0$ which bypasses $t=0$. Now the ODE is solved.

Integrating factor is $e^{\int \frac{1}{t}dt}=e^{\ln t}=t$, and multiplying both sides by this results in

$$\frac{d}{dt}\left(ty\right)=3t\cos 2t$$

Integrating

$$ty=3\int t\cos(2t)\,dt+c \tag{1}$$

Using $\int udv = uv - \int vdv$, let $u = t, dv = \cos(2t) \rightarrow du = 1, v = \frac{1}{2}\sin(2t)$, hence

$$\begin{aligned}\int t\cos(2t)\,dt &= \frac{1}{2}t\sin(2t) - \int \frac{1}{2}\sin(2t)\,dt\\ &= \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t)\end{aligned}$$

Equation (1) becomes

$$\begin{aligned}ty &= 3\left(\frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t)\right) + c\\ y &= \frac{3}{2}\sin(2t) + \frac{3}{4}\frac{\cos(2t)}{t} + \frac{c}{t} \qquad t > 0\end{aligned}$$

In the limit as $t \rightarrow \infty$ the terms $\frac{c}{t} \rightarrow 0$ and $\frac{\cos(2t)}{t} \rightarrow 0$, therefore

$$\lim_{t\rightarrow\infty} y(t) = \frac{3}{2}\sin(2t)$$

Hence the solution is sinusoidal at large t.

2.1.13 Section 2.1 problem 13

Find the solution to the given initial value problem. $y' - y = 2te^{2t}$ with $y(0) = 1$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -1, g(t) = 2te^{2t}$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique.

Integrating factor is $e^{\int -dt} = e^{-t}$. Multiplying both sides by this results in

$$\frac{d}{dt}\left(ye^{-t}\right) = 2te^{t}$$

Integrating

$$ye^{-t} = 2\int te^{t}dt + c \tag{1}$$

Using $\int udv = uv - \int vdv$, let $u = t, dv = e^{t} \rightarrow du = 1, v = e^{t}$, hence

$$\begin{aligned}\int te^{t}dt &= te^{t} - \int e^{t}dt\\ &= te^{t} - e^{t}\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}ye^{-t} &= 2\left(te^{t} - e^{t}\right) + c\\ y &= 2\left(te^{2t} - e^{2t}\right) + ce^{t}\\ &= 2e^{2t}(t-1) + ce^{t}\end{aligned}$$

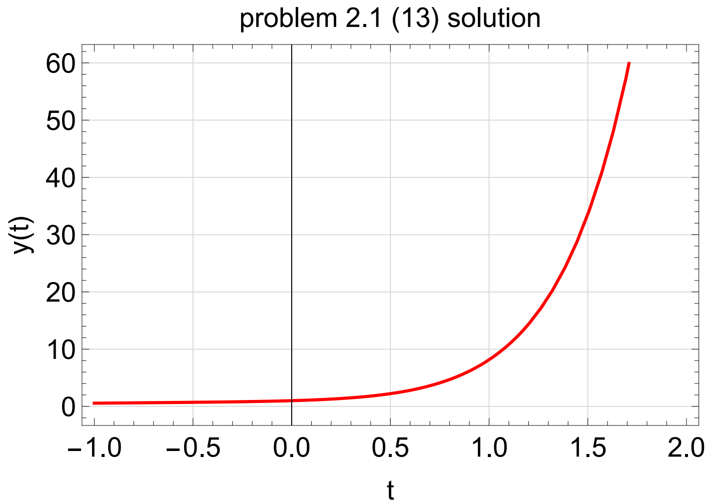
Applying initial conditions gives

$$\begin{aligned}1 &= 2e^{0}(0-1) + ce^{0}\\ 1 &= -2 + c\\ c &= 3\end{aligned}$$

Hence the general solution is

$$y = 2e^{2t}(t-1) + 3e^{t} \qquad t \in \Re$$

Here is a plot of the solution



2.1.14 Section 2.1 problem 14

Find the solution to the given initial value problem. $y' + 2y = te^{-2t}$ with $y(1) = 0$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 2, g(t) = te^{2t}$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1 a solution exists and is unique.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = t$$

Integrating

$$\begin{aligned} ye^{2t} &= \frac{1}{2}t^2 + c \\ y &= \frac{1}{2}t^2e^{-2t} + ce^{-2t} \end{aligned} \tag{1}$$

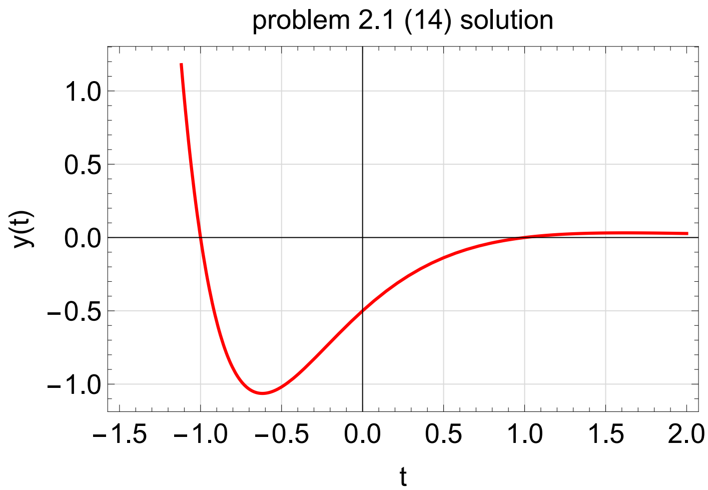
Applying initial conditions

$$\begin{aligned} 0 &= \frac{1}{2}e^{-2} + ce^{-2} \\ c &= -\frac{1}{2} \end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned} y &= \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t} \\ &= \frac{1}{2}e^{-2t}\left(t^2 - 1\right) \qquad t \in \Re \end{aligned}$$

Here is a plot of the solution



2.1.15 Section 2.1, problem 29

Consider $y' + \frac{1}{4}y = 3 + 2\cos(2t)$ with $y(0) = 0$. (a) find the solution and describe its behavior for large t. (b) Determine t for which the solution first intersects the line $y = 12$.

2.1.15.1 Part (a)

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = \frac{1}{4}, g(t) = 3 + 2\cos(2t)$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1 a solution exists and is unique.

Integrating factor is $e^{\frac{1}{4}\int dt} = e^{\frac{t}{4}}$. Multiplying both sides by $e^{\frac{1}{4}t}$ gives

$$\frac{d}{dt}\left(ye^{\frac{t}{4}}\right) = 3e^{\frac{t}{4}} + 2e^{\frac{t}{4}}\cos 2t$$

Integrating

$$ye^{\frac{t}{4}} = 3\int e^{\frac{t}{4}}dt + 2\int e^{\frac{t}{4}}\cos(2t)\,dt + c \tag{1}$$

$\int e^{\frac{t}{4}}dt = 4e^{\frac{t}{4}}$. For the second integral, integration by parts is used. Using $\int udv = uv - \int vdv$, let $u = \cos(2t), dv = e^{\frac{t}{4}} \rightarrow du = -2\sin(2t), v = 4e^{\frac{t}{4}}$, hence

$$\begin{aligned} I &= \int e^{\frac{t}{4}}\cos(2t)\,dt \\ &= 4\cos(2t)\,e^{\frac{t}{4}} - \int(-2\sin(2t))\,4e^{\frac{t}{4}}dt \\ &= 4\cos(2t)\,e^{\frac{t}{4}} + 8\int \sin(2t)\,e^{\frac{t}{4}}dt \end{aligned}$$

Applying integration by parts again on $\int \sin(2t)\,e^{\frac{t}{4}}dt$. Let $u = \sin(2t), dv = e^{\frac{t}{4}}$, hence $du =$

$2\cos(2t), v = 4e^{\frac{t}{4}}$. Therefore the above becomes

$$
\begin{aligned}
I &= 4\cos(2t)\, e^{\frac{t}{4}} + 8\left(4\sin(2t)\, e^{\frac{t}{4}} - \int 2\cos(2t)\, 4e^{\frac{t}{4}} dt\right) \\
&= 4\cos(2t)\, e^{\frac{t}{4}} + 32\sin(2t)\, e^{\frac{t}{4}} - 64\int \cos(2t)\, e^{\frac{t}{4}} dt
\end{aligned}
$$

But $I = \int e^{\frac{t}{4}} \cos(2t)\, dt$, hence the above is

$$
I = 4\cos(2t)\, e^{\frac{t}{4}} + 32\sin(2t)\, e^{\frac{t}{4}} - 64I
$$

Solving for I

$$
\begin{aligned}
65I &= 4\cos(2t)\, e^{\frac{t}{4}} + 32\sin(2t)\, e^{\frac{t}{4}} \\
I &= \frac{4}{65}\cos(2t)\, e^{\frac{t}{4}} + \frac{32}{65}\sin(2t)\, e^{\frac{t}{4}}
\end{aligned}
$$

Putting these results back into (1) gives

$$
\begin{aligned}
ye^{\frac{t}{4}} &= 3\int e^{\frac{t}{4}} dt + 2\int e^{\frac{t}{4}} \cos(2t)\, dt + c \\
&= 3\left(4e^{\frac{t}{4}}\right) + 2\left(\frac{4}{65}\cos(2t)\, e^{\frac{t}{4}} + \frac{32}{65}\sin(2t)\, e^{\frac{t}{4}}\right) + c
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= 12 + 2\left(\frac{4}{65}\cos(2t) + \frac{32}{65}\sin(2t)\right) + ce^{-4t} \\
&= 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) + ce^{-4t}
\end{aligned}
$$

Applying initial conditions

$$
\begin{aligned}
0 &= 12 + \frac{8}{65}\cos(0) + \frac{64}{65}\sin(0) + ce^{0} \\
&= 12 + \frac{8}{65} + c \\
c &= -12 - \frac{8}{65} \\
&= -\frac{788}{65}
\end{aligned}
$$

Hence the general solution is

$$
y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t} \qquad t \in \Re \tag{2}
$$

As t becomes very large, the term $\frac{788}{65}e^{-4t} \to 0$ and the solution only contains sinusoidal.

$$
\lim_{t\to\infty} y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t)
$$

2.1.15.2 Part (b)

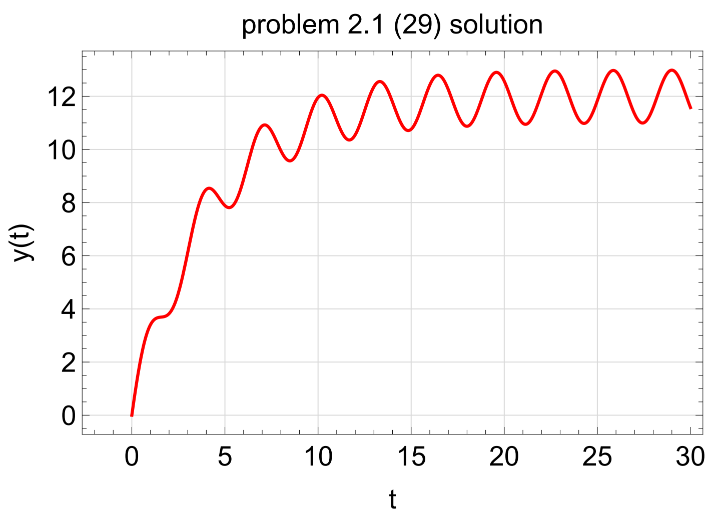
Solving for t when $y = 12$ results in

$$12 = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t}$$
$$0 = \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t}$$

It is not clear what method is supposed to be used to solve the above for t since it is non-linear. So $y(t)$ was first plotted and by inspection $y(t)$ cross the line $y = 12$ at about $t = 10$. Then using computer root finding with search starting at $t = 10$ the required value of t was found to be

$$\boxed{t = 10.0658}$$

Here is a plot of the solution given in (2)



2.2 HW2

2.2.1 Section 2.2 problem 1

Solve $y' = \frac{x^2}{y}$

This is <u>first order non-linear ODE</u>. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere except at the line $y = 0$. Now the ODE is solved by separation

$$
\begin{aligned}
y\frac{dy}{dx} &= x^2 \\
ydy &= x^2 dx
\end{aligned}
$$

Integrating

$$
\begin{aligned}
\int ydy &= \int x^2 dx \\
\frac{y^2}{2} &= \frac{x^3}{3} + c
\end{aligned}
$$

Since initial conditions is not gives, the solution is left in implicit form (as mentioned in discussion class, Thursday Sept. 29, 2016)

$$
\boxed{y^2 = \frac{2}{3}x^3 + c_0 \qquad y \neq 0}
$$

2.2.2 Section 2.2 problem 2

Solve $y' = \frac{x^2}{y\left(1+x^3\right)}$

This is <u>first order non-linear ODE</u>. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere except at line $y = 0$ and at line $x = -1$. Now the ODE is solved by separation

$$
\begin{aligned}
y\frac{dy}{dx} &= \frac{x^2}{\left(1+x^3\right)} \\
ydy &= \frac{x^2}{\left(1+x^3\right)} dx
\end{aligned}
$$

Integrating

$$
\int ydy = \int \frac{x^2}{\left(1+x^3\right)} dx
$$

To integrate $\int \frac{x^2}{\left(1+x^3\right)} dx$ let $u = 1 + x^3$, hence $\frac{du}{dx} = 3x^2$. Therefore the integral becomes

$\int \frac{x^2}{u}\frac{du}{3x^2} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|1+x^3\right|$. Hence the above becomes

$$\frac{y^2}{2} = \frac{1}{3}\ln\left|1+x^3\right| + c$$
$$y^2 = \frac{2}{3}\ln\left|1+x^3\right| + c_1$$

Since initial condition is not gives, the solution is left in implicit form

$$\boxed{y^2 = \frac{2}{3}\ln\left|1+x^3\right| + c_1 \qquad y \neq 0, x \neq -1}$$

2.2.3 Section 2.2 problem 3

Solve $y' = -y^2 \sin x$

This is first order non-linear ODE. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere and $\frac{\partial f}{\partial y} = -2y\sin(x)$ is also continuous everywhere but unbounded at $y = -\infty$. This is separable, assuming $y \neq 0$ and dividing by y^2 the ODE becomes

$$\frac{1}{y^2}\frac{dy}{dx} = -\sin(x)$$
$$\frac{dy}{y^2} = -\sin()\, xdx$$

Integrating

$$\int \frac{dy}{y^2} = -\int \sin(x)\, dx$$
$$-\frac{1}{y} = \cos(x) + c$$
$$\frac{1}{y} = -\cos(x) + c_1$$

Therefore the solution is

$$\boxed{y(x) = \frac{1}{c_1 - \cos(x)} \qquad y \neq 0}$$

The reason for $y \neq 0$ was the assumption to divide by y^2 above. Another solution is

$$\boxed{y(x) = 0}$$

2.2.4 Section 2.2 problem 4

Solve $y' = \frac{3x^2-1}{3+2y}$

This is first order non-linear ODE. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere except at $3+2y=0$ or $y=-\frac{3}{2}$. Now the ODE is solved by separation.

$$\begin{aligned}\left(3+2y\right)\frac{dy}{dx} &= 3x^2-1\\ \left(3+2y\right)dy &= \left(3x^2-1\right)dx\end{aligned}$$

Integrating

$$\begin{aligned}\int\left(3+2y\right)dy &= \int\left(3x^2-1\right)dx\\ y^2+3y &= x^3-x+c\end{aligned}$$

Complete the square

$$y^2+3y+\left(\frac{3}{2}\right)^2 = x^3-x+c+\left(\frac{3}{2}\right)^2$$

Since initial condition is not gives, the solution is left in implicit form.

$$\boxed{\left(y+\frac{3}{2}\right)^2 = x^3-x+c_0 \qquad y\neq-\frac{3}{2}}$$

2.2.5 Section 2.2 problem 5

$y' = \cos^2(x)\cos^2\left(2y\right)$

This is first order non-linear ODE. In the form $y'=f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \cos^2(x)\,2\cos\left(2y\right)\left(-2\sin\left(2y\right)\right)\\ &= -4\cos^2(x)\cos\left(2y\right)\sin\left(2y\right)\end{aligned}$$

Which is continuous everywhere and bounded. Hence a solution exist and is unique. Now the ODE is solved by separation.

Case $\cos^2\left(2y\right)\neq 0$

To divide by $\cos^2\left(2y\right)$, then for $\cos^2\left(2y\right)\neq 0$ or $\cos\left(2y\right)\neq 0$ or $2y\neq\left(n+\frac{1}{2}\right)\pi$ or $y\neq\left(n+\frac{1}{2}\right)\frac{\pi}{4}$ for all integers.

$$\begin{aligned}\frac{1}{\cos^2\left(2y\right)}\frac{dy}{dx} &= \cos^2(x)\\ \int\frac{dy}{\cos^2\left(2y\right)} &= \int\cos^2(x)\,dx \qquad (1)\end{aligned}$$

Now $\int \frac{dy}{\cos^2(2y)} = \frac{1}{2}\tan\left(2y\right)$ and

$$
\begin{aligned}
\int \cos^2(x)\,dx &= \int \frac{1+\cos(2x)}{2}dx \\
&= \int \left(\frac{1}{2} + \frac{\cos(2x)}{2}\right) dx \\
&= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + c_1 \\
&= \frac{x}{2} + \frac{\sin(2x)}{4} + c_1
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
\frac{1}{2}\tan\left(2y\right) &= \frac{x}{2} + \frac{\sin(2x)}{4} + c_1 \\
\tan\left(2y\right) &= x + \frac{1}{2}\sin(2x) + c
\end{aligned}
$$

Since initial condition is not gives, the solution is left in implicit form.

Case $\cos^2\left(2y\right) = 0$

This is when $\cos\left(2y\right) = 0$ or $2y = \left(n + \frac{1}{2}\right)\pi$ or $y = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$ for all integers. In this case the solution is

$$
y = \left(n + \frac{1}{2}\right)\frac{\pi}{2}
$$

Summary of solution $y(x)$

$$
\begin{cases}
\tan\left(2y\right) = x + \frac{1}{2}\sin(2x) + c & \cos^2\left(2y\right) \neq 0 \\
\left(n + \frac{1}{2}\right)\frac{\pi}{2} & \cos^2\left(2y\right) = 0
\end{cases}
$$

2.2.6 Section 2.2 problem 6

Solve $xy' = \left(1 - y^2\right)^{\frac{1}{2}}$

This is nonlinear first order of the form $y' = f\left(x, y\right)$ where $f\left(x, y\right) = \frac{\left(1-y^2\right)^{\frac{1}{2}}}{x}$. This is continuous everywhere except at $x = 0$. ODE is solved by separation.

Case $1 - y^2 \neq 0$

Or $y^2 \neq 1$ or $y \neq \pm 1$, then dividing by $\left(1-y^2\right)^{\frac{1}{2}}$ and integrating

$$\int \frac{dy}{\left(1-y^2\right)^{\frac{1}{2}}} = \int \frac{dx}{x}$$
$$\arcsin\left(y\right) = \ln|x| + c$$
$$y(x) = \sin(\ln|x| + c)$$

Hence the solution is

$$y(x) = \sin(\ln|x| + c) \qquad y \neq \pm 1, x \neq 0$$

Case $1-y^2 = 0$

Then

$$y(x) = \pm 1$$

Summary of solutions

$$\begin{cases} y(x) = \sin(\ln|x| + c) & y \neq \pm 1, x \neq 0 \\ y(x) = \pm 1 & x \neq 0 \end{cases}$$

2.2.7 Section 2.2 problem 7

Solve $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$

This is non-linear first order ODE of the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere except at y which is the solution of $e^y + y = 0$. Using a computer, this is $y_c = -0.567143\cdots$. The ODE is solved by separation

$$\int \left(y + e^y\right) dy = \int \left(x - e^{-x}\right) dx$$
$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + c$$

Hence the solution is given by

$$y^2 + 2e^y - x^2 - 2e^{-x} = c_1 \qquad y \neq y_c$$

2.2.8 Section 2.2 problem 8

Solve $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

This is non-linear first order ODE of the form $y' = f\left(x,y\right)$ where $f\left(x,y\right)$ is continuous everywhere except at $y = \pm 1$. The ODE is solved by separation

$$\left(1 + y^2\right) dy = x^2 dx$$

$$y + \frac{y^3}{3} = \frac{x^3}{3} + c_1$$

Hence the solution is given by

$$y^3 + 3y - x^3 = c \qquad y = \pm 1$$

Since initial condition is not gives, the solution is left in implicit form.

2.2.9 Section 2.3 problem 1

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

To reduce confusion, let x be the substance which causes the concentration in the die. Let $Q(t)$ be the mass (normally called the amount, but saying mass is more clear than saying amount) of x at time t. Hence $Q(0) = 200g$ since initial concentration was $1[g/L]$ and the volume is $200[L]$.

The goal is to find an ODE that describes how $Q(t)$ changes in time. That is, how the mass of x in the tank changes in time. Knowing the mass of x at any time in the tank, gives the concentration also, since the tank volume is fixed at $200[L]$. So the concentration can always be found using $\frac{Q(t)}{200}$. Using

$$\frac{dQ}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of x moving into the tank, i.e. how many grams of x is being poured in per minute, which is zero, since fresh water is moving in. R_{out} is rate of x moving out, i.e. how many grams of x is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{Q(t)}{200} \frac{[\text{gram}]}{[\text{L}]} \times 2\left[\frac{\text{L}}{\text{min}}\right] \\ &= \frac{2}{200} Q(t) \frac{[\text{gram}]}{[\text{min}]} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\frac{dQ}{dt} &= 0 - 100Q(t)\\ &= -100Q(t)\end{aligned}$$

Solving the ODE, for $Q(t) \neq 0$

$$\begin{aligned}\frac{dQ}{Q} &= -100dt\\ \ln|Q| &= -100t + c\end{aligned}$$

Since Q represent mass, it can not be negative, then there is no need to use $|Q|$.

$$\begin{aligned}\ln Q &= -100t + c\\ Q(t) &= Ae^{-100t}\end{aligned}$$

At $t = 0, Q(0) = 200[g]$, hence $A = 200$ from the above. The solution becomes

$$Q(t) = 200e^{-100t}$$

Since initial Q was $200[g]$ then 1% of that is 2. Solving for time gives

$$\begin{aligned}2 &= 200e^{-100t_0}\\ 0.01 &= e^{-100t_0}\\ \ln(0.01) &= -100t_0\end{aligned}$$

Solving on the computer gives

$$\boxed{t_0 = 460.517[\text{min}]}$$

Hence it takes 460.517 minutes for the mass of x to reach 1% of its original amount of 200 gram. This is also the same amount of time for the concentration of x to reach 1% of its original amount of $1\left[g/L\right]$. It is easier to work with mass in the ODE, and then convert to concentration when needed.

2.2.10 Section 2.3 problem 2

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Let $y(t)$ be the mass of salt at time t in the tank in grams. Hence $y(0) = 0$ since tank initially contains pure water. The goal is to find an ODE that describes how $y(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many grams of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= \gamma \left[\frac{\text{gram}}{\text{L}}\right] \times 2 \left[\frac{\text{L}}{\text{min}}\right] \\ &= 2\gamma \left[\frac{\text{gram}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many grams of salt is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{y(t)}{120} \frac{[\text{gram}]}{[\text{L}]} \times 2 \left[\frac{\text{L}}{\text{min}}\right] \\ &= \frac{1}{60} y(t) \left[\frac{\text{gram}}{\text{min}}\right] \end{aligned}$$

Hence (1) becomes

$$\frac{dy(t)}{dt} = 2\gamma - \frac{1}{60} y(t)$$

With $y(0) = 0$. The ODE is linear and first order, of the form $y' + p(t) y = g(t)$ with $p(t) = \frac{1}{60}$ and $g(t) = 2\gamma$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique.

$$y' + \frac{1}{60} y = 2\gamma$$

Integrating factor is $e^{\int \frac{1}{60} dt} = e^{\frac{1}{60} t}$, therefore

$$\frac{d}{dt}\left(e^{\frac{1}{60} t} y\right) = 2\gamma e^{\frac{1}{60} t}$$

Integrating

$$\begin{aligned} e^{\frac{1}{60} t} y &= 2\gamma \int e^{\frac{1}{60} t} dt \\ &= 2\gamma \frac{e^{\frac{1}{60} t}}{\frac{1}{60}} + c \\ &= 120\gamma e^{\frac{1}{60} t} + c \end{aligned}$$

Hence

$$y(t) = 120\gamma + c^{\frac{-t}{60}}$$

In the above, $y(t)$ is the mass of salt in grams in the tank at time t. Hence the concentration of salt in the tank at time t can always be found by dividing $y(t)$ by the volume of the tank. In the limit, as $t \to \infty$ then from above

$$\lim_{t \to \infty} y(t) = 120\gamma$$

2.2.11 Section 2.3 problem 3

3. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

This problem is solved in two stages. The first ODE is used to find what the amount of salt in the tank will be after 10 minutes. Then a new ODE is set up, with this value as its initial conditions, in order to find the amount of salt in the tank after an additional 10 minutes.

First 10 minutes

Let $y_1(t)$ be the mass of salt at time t in the tank in lbs. Hence $y_1(0) = 0$ since tank initially contains pure water. The goal is to find an ODE that describes how $y_1(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy_1}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= \frac{1}{2}\left[\frac{\text{lb}}{\text{gallon}}\right] \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= 1\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many grams of salt is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{y_1(t)}{100}\frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= \frac{1}{50}y_1(t)\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

Hence (1) becomes

$$\frac{dy_1(t)}{dt} = 1 - \frac{1}{50}y_1(t)$$

With $y_1(0) = 0$. The ODE is linear and first order, of the form $y' + p(t)y = g(t)$ with $p(t) = \frac{1}{50}$ and $g(t) = 1$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique.

$$y_1' + \frac{1}{50}y_1 = 1$$

Integrating factor is $e^{\int \frac{1}{50}dt} = e^{\frac{1}{50}t}$, therefore

$$\frac{d}{dt}\left(e^{\frac{1}{50}t}y_1\right) = e^{\frac{1}{50}t}$$

Integrating

$$\begin{aligned} e^{\frac{1}{50}t}y &= \int e^{\frac{1}{50}t}dt \\ &= 50e^{\frac{1}{50}t} + c \end{aligned}$$

Hence

$$y_1(t) = 50 + c^{\frac{-t}{50}}$$

To find c, from initial conditions

$$\begin{aligned} 0 &= y_1(0) \\ &= 50 + c \\ c &= -50 \end{aligned}$$

Hence the solution to the first phase is

$$\begin{aligned} y_1(t) &= 50 - 50^{\frac{-t}{50}} \\ &= 50\left(1 - e^{\frac{-t}{50}}\right) \end{aligned}$$

After $t = 10$ minutes

$$y_1(10) = 50\left(1 - e^{\frac{-1}{5}}\right)$$

The above value is now used as initial conditions for new problem. The new problem will use $t = 0$ as initial time for simplicity, but it is understood that 10 minutes has already elapsed in global scale.

Second phase

Let $y_2(t)$ be the mass of salt at time t in the tank in grams. Hence

$$\begin{aligned} y_2(0) &= y_1(10) \\ &= 50\left(1 - e^{\frac{-1}{5}}\right) \end{aligned}$$

From phase one above, this is the amount of salt in lbs in the tank at this moment. The goal is to find an ODE that describes how $y_2(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy_2}{dt} = R_{in} - R_{out} \tag{2}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute. But now $R_{in} = 0$ since fresh water is poured in. And R_{out} is rate of salt moving out, i.e. how many lbs of salt is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{y_2(t)}{100}\frac{[\text{lb}]}{[\text{gallon}]}\times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= \frac{1}{50}y_2(t)\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \frac{dy_2(t)}{dt} &= 0 - \frac{1}{50}y_2(t) \\ &= -\frac{1}{50}y_2(t) \end{aligned}$$

The ODE is linear and first order, of the form $y' + p(t)y = g(t)$ with $p(t) = \frac{1}{50}$ and $g(t) = 0$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique. This is separable.

$$\begin{aligned} \frac{dy_2}{y_2} &= -\frac{1}{50}dt \\ \ln|y_2| &= -\frac{t}{50} + c_1 \\ y_2(t) &= ce^{\frac{-t}{50}} \end{aligned} \tag{3}$$

To find c, from initial conditions $y_2(0) = 50\left(1 - e^{\frac{-1}{5}}\right)$, hence

$$50\left(1 - e^{\frac{-1}{5}}\right) = c$$

Hence the solution (3) to the second phase is

$$y_2(t) = 50\left(1 - e^{\frac{-1}{5}}\right)e^{\frac{-t}{50}}$$

After $t = 10$ minutes (which will be 20 in global scale)

$$\begin{aligned} y_2(10) &= 50\left(1 - e^{\frac{-1}{5}}\right)e^{\frac{-1}{5}} \\ &= 7.4205 \text{ lbs} \end{aligned}$$

Therefore after 20 minutes from the global initial time (or 10 minutes from the start of the second phase), the mass of salt in tank is 7.4205 lbs. Therefore the concentration at the same moment, if needed, will be $\frac{7.4205}{100}\left[\frac{\text{lbs}}{\text{gallon}}\right] = 0.074\left[\frac{\text{lbs}}{\text{gallon}}\right]$.

2.2.12 Section 2.3 problem 4

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Let $y(t)$ be the mass of salt at time t in the tank in lbs. Hence $y(0) = 100$ since tank initially contains that much salt. The goal is to find an ODE that describes how $y(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= 1\left[\frac{\text{lb}}{\text{gallon}}\right] \times 3\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= 3\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many lbs of salt is leaving the tank per minute. This is found as follows

$$R_{out} = \frac{y(t)}{V(t)} \frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \tag{2}$$

Where $V(t)$ is the volume of the whole mixture at time t. This is different from earlier problems where volume was constant. This is because in this problem the rate of pouring into the tank is larger than the rate of flow out of the tank. The volume at time t can easily be found as

$$\begin{aligned} V(t) &= 200\left[\text{gallon}\right] + 3\left[\frac{\text{gallon}}{\text{min}}\right] t\,[\text{min}] - 2\left[\frac{\text{gallon}}{\text{min}}\right] t[\text{min} \\ &= (200 + t)\left[\text{gallon}\right] \end{aligned}$$

This means at any time t, there will be $200 + t$ gallons of mixture in the tank. This value is now used in (2) above to complete the solution. Note that the tank will overflow when $200 + t = 500$ since 500 is the maximum size of the tank. Going back to (2) now it becomes

$$\begin{aligned} R_{out} &= \frac{y(t)}{200 + t} \frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= \frac{2y(t)}{200 + t} \end{aligned}$$

Therefore (1) becomes

$$y' = 3 - \frac{2y}{200+t}$$

This is linear ODE of first order of the form $y' + p(t)y = g(t)$ where $p(t) = \frac{2}{200+t}$ and $g(t)$. Both are continuous for $t \geq 0$ hence there will be a unique solution for $t \geq 0$. Now the ODE is solved using an integration factor

$$\frac{dy}{dt} + \frac{2y}{200+t} = 3$$

The integrating factor is $e^{2\int \frac{1}{200+t}dt}$. To evaluate $\int \frac{1}{200+t}dt$, let $u = 200+t$, hence $\frac{du}{dt} = 1$ and the integral becomes $\int \frac{1}{u}du = \ln|u|$ Therefore $\int \frac{1}{200+t}dt = \ln|200+t|$ and the integrating factor is $e^{2\ln|200+t|} = |200+t|^2 = (200+t)^2$. Therefore now that the integrating is found, the solution can be written as

$$\frac{d}{dt}\left(y(200+t)^2\right) = 3(200+t)^2$$

Integrating both sides gives

$$y(200+t)^2 = 3\int (200+t)^2\,dt$$

Let $u = 200+t$, then $\frac{du}{dt} = t$, hence $\int (200+t)^2\,dt = \int u^2 du = \frac{u^3}{3} + c_1 = \frac{(200+t)^3}{3} + c_1$. Therefore the above becomes

$$\begin{aligned} y(200+t)^2 &= 3\left(\frac{(200+t)^3}{3} + c_1\right) \\ &= (200+t)^3 + c \end{aligned}$$

Solving for $y(t)$ gives

$$\boxed{y(t) = (200+t) + \frac{c}{(200+t)^2}} \tag{3}$$

Now c is found from initial conditions. Given that $y(0) = 100$, then from the above

$$\begin{aligned} 100 &= 200 + \frac{c}{(200)^2} \\ &= 200 + \frac{c}{40000} \\ c &= (-100)(40000) \\ &= -4 \times 10^6 \end{aligned}$$

Therefore the solution (3) becomes

$$y(t) = (200+t) - \frac{4\times 10^6}{(200+t)^2} \tag{4}$$

Now the above ODE is only valid until the tank overflows. This value of time is found by solving $200+t=500$ for t, which gives $t=300$. Hence (4) becomes

$$\boxed{y(t)=(200+t)-\frac{4\times 10^6}{(200+t)^2} \qquad 0\leq t\leq 300} \tag{5}$$

At $t=300$ minutes, the mass of salt in lbs is therefore $y(300)$ which is

$$\begin{aligned} y(300) &= (200+300)-\frac{4\times 10^6}{(200+300)^2} \\ &= 484 \ \text{[lbs]} \end{aligned}$$

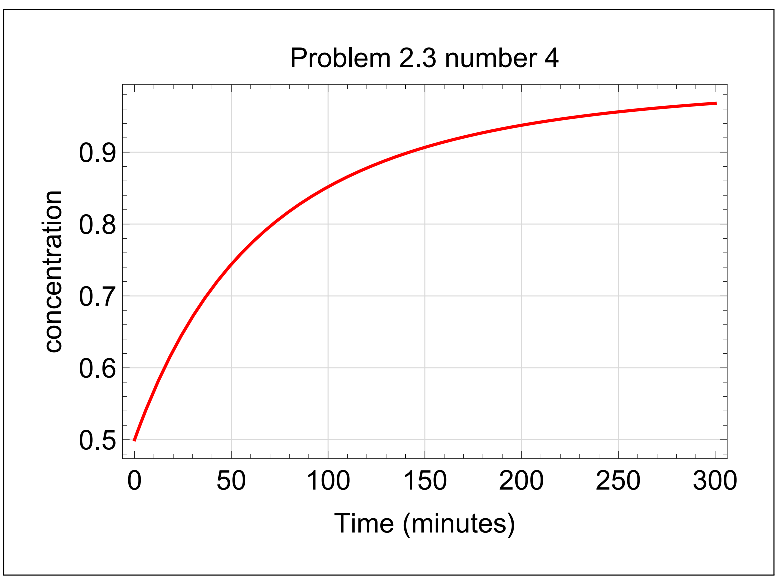
And since the volume now is 500 gallons, then the concentration at time $t=300$ minutes is

$$\frac{484}{500}\left[\frac{\text{lbs}}{\text{gallon}}\right]=0.968\left[\frac{\text{lbs}}{\text{gallon}}\right]$$

If the tank had infinite capacity, then using the solution found in (5) and dividing by current volume, which was found before to be $200+t$ and then taking the limit $t\to\infty$ gives the answer. Let $\rho(t)$ be now the concentration in $\left[\frac{\text{lbs}}{\text{gallon}}\right]$ at any time t. Then

$$\begin{aligned} \rho(t) &= \frac{(200+t)-\frac{4\times 10^6}{(200+t)^2}}{V(t)}=\frac{(200+t)-\frac{4\times 10^6}{(200+t)^2}}{200+t} \\ &= 1-\frac{4\times 10^6}{200+t} \end{aligned}$$

As $t\to\infty$ then $\rho(t)\to 1$. Therefore at 300 minutes the concentration is 96.8% of the theoretical limit. The following is a plot of $\rho(t)$ as function of time. At $t=0$ the concentration is 0.5 since this is the initial condition.



2.2.13 Section 2.4 problem 1

Determine an interval which the given initial value problem is valid. $(t-3)y'+\ln(t)y=2t$ with $y(1)=2$.

This is linear first order ODE. In standard form it becomes $y' + \frac{\ln(t)}{t-3}y = \frac{2t}{t-3}$, and comparing to $y' + p(t)y = g(t)$ then

$$p(t) = \frac{\ln(t)}{t-3}$$
$$g(t) = \frac{2t}{t-3}$$

$p(t)$ is not not continuous at $t = 3$ and also at $t = 0$ since $\ln(0) = -\infty$. $g(t)$ is not continuous at $t = 3$. Therefore the region must include initial point, which is $t = 1$ but not include $t = 3$ nor $t = 0$. Hence

$$0 < t < 3$$

And for forward only ODE the region is

$$1 \leq t < 3$$

2.2.14 Section 2.4 problem 2

Determine an interval which the given initial value problem is valid. $t(t-4)y' + y = 2t$ with $y(2) = 1$.

This is linear first order ODE. In standard form it becomes $y' + \frac{1}{t(t-4)}y = \frac{2}{(t-4)}$, and comparing to $y' + p(t)y = g(t)$ then

$$p(t) = \frac{1}{t(t-4)}$$
$$g(t) = \frac{2}{(t-4)}$$

$p(t)$ is not continuous at $t = 0$ and $t = 4$ while $g(t)$ is not continuous at $t = 4$. Therefore the region must include initial point, which is $t = 2$ but not include $t = 4$ nor $t = 0$. Hence

$$0 < t < 4$$

And for forward only ODE the region is

$$2 \leq t < 4$$

2.2.15 Section 2.4 problem 3

Determine an interval which the given initial value problem is valid. $y' + \tan(t)y = \sin(t)$ with $y(\pi) = 0$.

This is linear first order ODE. Comparing to $y' + p(t)y = g(t)$ then

$$p(t) = \tan(t)$$
$$g(t) = \sin(t)$$

$g(t)$ is continuous everywhere but $p(t)$ is not continuous at $\left\{\cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots\right\}$ therefore the region must be between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ since the initial point π is inside this region. Hence

$$\frac{\pi}{2} < t < 1.5\pi$$

2.2.16 Section 2.4 problem 4

Determine an interval which the given initial value problem is valid. $\left(4 - t^2\right) y' + 2ty = 3t^2$ with $y(-3) = 1$.

This is linear first order ODE. In standard form it becomes $y' + \frac{2t}{\left(4-t^2\right)} y = \frac{3t^2}{\left(4-t^2\right)}$, and comparing to $y' + p(t) y = g(t)$ then

$$p(t) = \frac{2t}{\left(4 - t^2\right)}$$
$$g(t) = \frac{3t^2}{\left(4 - t^2\right)}$$

$p(t)$ is not not continuous at $t^2 = 4$ or $t = \pm 2$ and the same for $g(t)$. Therefore the region must include initial point, which is $t = -3$ but not include $t = \pm 2$. Hence

$$-\infty < t < -2$$

And for forward only ODE the region is

$$-3 \leq t < -2$$

2.2.17 Section 2.4 problem 5

Determine an interval which the given initial value problem is valid. $\left(4 - t^2\right) y' + 2ty = 3t^2$ with $y(1) = -3$.

This is linear first order ODE. In standard form it becomes $y' + \frac{2t}{\left(4-t^2\right)} y = \frac{3t^2}{\left(4-t^2\right)}$, and comparing to $y' + p(t) y = g(t)$ then

$$p(t) = \frac{2t}{\left(4 - t^2\right)}$$
$$g(t) = \frac{3t^2}{\left(4 - t^2\right)}$$

$p(t)$ is not not continuous at $t^2 = 4$ or $t = \pm 2$ and the same for $g(t)$. Therefore the region must include initial point, which is $t = 1$ but not include $t = \pm 2$. Hence

$$-2 < t < 2$$

And for forward only ODE the region is

$$1 \leq t < 2$$

2.2.18 Section 2.4 problem 6

Determine an interval which the given initial value problem is valid. $\ln(t) y' + y = \frac{1}{\tan(t)}$ with $y(2) = 3$.

This is linear first order ODE. In standard form it becomes $y' + \frac{1}{\ln(t)} y = \frac{1}{\tan(t)\ln(t)}$, and comparing to $y' + p(t) y = g(t)$ then

$$p(t) = \frac{1}{\ln(t)}$$
$$g(t) = \frac{1}{\tan(t)\ln(t)}$$

When $t = 1$ then $\ln(t) = 0$ and $p(t)$ becomes unbounded. And since for real t then t must remain positive, else $\ln(t)$ becomes complex. Then $p(t)$ says that $t \geq 0$ and $t \neq 1$. Looking at $g(t)$ then $\tan(t) = 0$ when $t = \{\cdots, -\pi, \pi, \cdots\}$ hence the region that includes initial point $t_0 = 2$ must be inside these. Therefore the singular points are $t = 1, -\pi, \pi$ and $t \geq 0$. Putting all these together, the region is

$$1 < t < \pi$$

And for forward only ODE the region is

$$2 \leq t < \pi$$

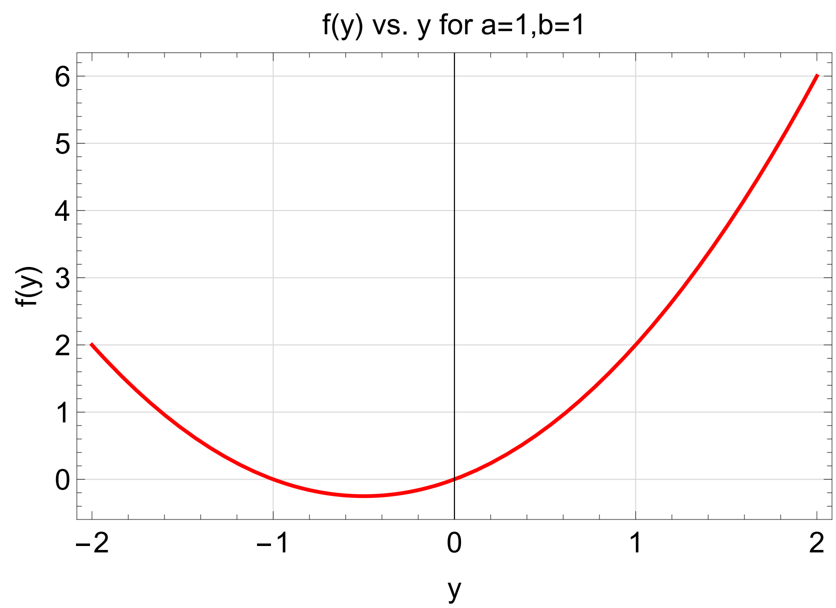
2.3 HW3

2.3.1 Section 2.5 problem 1

Sketch the graph of $f(y)$ vs. y and determine critical points and classify each as stable or not stable. $\frac{dy}{dt} = ay + by^2; a > 0, b > 0, y_0 \geq 0$

$$f(y) = ay + by^2$$

The following is sketch of $f(y)$ for $a = 1, b = 1$. Or $f(y) = y + y^2$



The critical points are solution of

$$f(y) = 0$$
$$y(a + by) = 0$$

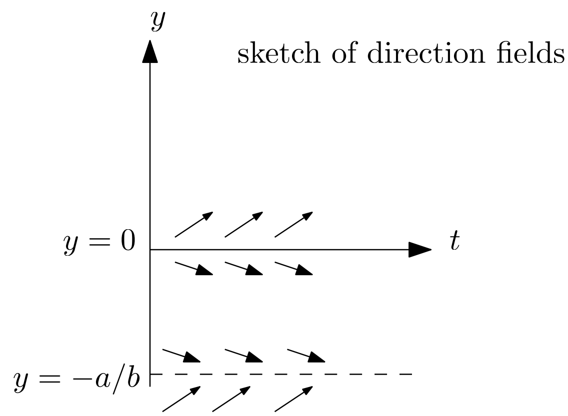
Therefore the critical points are

$$y_1 = 0$$
$$y_2 = \frac{-a}{b} \qquad \text{(not in domain)}$$

Or

$$y_1 = 0$$
$$y_2 = -1$$

Notice that since $a > 0, b > 0$, then $y_2 < 0$. Here is sketch of the direction field.



Therefore $y = 0$ is not stable, and $y = \frac{-a}{b}$ is stable. However, since $y_0 \geq 0$, then $y = \frac{-a}{b}$ will not be reached. Per discussion, only lines above y_0 are to be considered. In the following problem, since $-\infty < y < \infty$, then all lines will be considered. This is the only difference between this problem and the next one.

2.3.1.1 Appendix

This is extra. The problem is also solved to determined which is the stable and which is the unstable critical points. But using direction field as above, is simpler method. The ODE is $\frac{dy}{dt} = ay + by^2$. This is separable

$$\frac{dy}{y\left(a+by\right)} = dx$$

Integrating

$$\int \frac{dy}{y\left(a+by\right)} = \int dx \tag{1}$$

For the integral $\int \frac{dy}{y\left(a+by\right)}$, partial fractions is used to split it. Let $\frac{A}{y} + \frac{B}{a+by} = \frac{1}{y\left(a+by\right)}$, therefore

$$A\left(a+by\right) + By = 1$$
$$Aa + y\left(Ab + B\right) = 1$$

Hence comparing terms, gives

$$Ab + B = 0$$
$$Aa = 1$$

Solving for A, B, gives

$$A = \frac{1}{a}$$
$$B = -\frac{b}{a}$$

Hence the integral becomes

$$\begin{aligned}\int\frac{dy}{y\left(a+by\right)} &= \frac{1}{a}\int\frac{1}{y}dy-\frac{b}{a}\int\frac{dy}{a+by}\\ &= \frac{1}{a}\ln\left|y\right|-\frac{b}{a}\int\frac{dy}{a+by}\end{aligned}$$

Let $u=a+by,\rightarrow du=bdy$, hence $\int\frac{dy}{a+by}=\frac{1}{b}\int\frac{du}{u}=\frac{1}{b}\ln|u|=\frac{1}{b}\ln\left|a+by\right|$ and the above becomes

$$\begin{aligned}\int\frac{dy}{y\left(a+by\right)} &= \frac{1}{a}\ln\left|y\right|-\frac{b}{a}\frac{1}{b}\ln\left|a+by\right|\\ &= \frac{1}{a}\ln\left|y\right|-\frac{1}{a}\ln\left|a+by\right|\\ &= \frac{1}{a}\left(\ln\left|y\right|-\ln\left|a+by\right|\right)\\ &= \frac{1}{a}\ln\left|\frac{y}{a+by}\right|\end{aligned}$$

Hence (1) becomes

$$\frac{1}{a}\ln\left|\frac{y}{a+by}\right|=x+c$$

Where c is constant of integration. Therefore

$$\ln\left|\frac{y}{a+by}\right|=ax+ac$$

Let $ac=c_0$ a new constant. Then

$$\begin{aligned}\ln\left|\frac{y}{a+by}\right| &= ax+c_0\\ \left|\frac{y}{a+by}\right| &= e^{ax+c_0}\\ \frac{y}{a+by} &= C_0e^{ax}\end{aligned}$$

Solving for y

$$\begin{aligned}y &= aC_0e^{ax}+byC_0e^{ax}\\ y\left(1-bC_0e^{ax}\right) &= aC_0e^{ax}\\ y &= \frac{aC_0e^{ax}}{\left(1-bC_0e^{ax}\right)}\end{aligned}$$

$$\lim_{x\rightarrow\infty}y=\lim_{x\rightarrow\infty}\frac{aC_0}{\frac{1}{e^{ax}}-bC_0}$$

Since $a > 0$ then $e^{ax} \to \infty$ as $x \to \infty$ and the above simplifies to

$$\begin{aligned}\lim_{x\to\infty} y &= \frac{aC_0}{-bC_0}\\ &= -\frac{a}{b}\end{aligned}$$

Since the limit goes to the point $-\frac{a}{b}$ then this point is stable equilibrium and the point $y = 0$ is not stable.

2.3.2 Section 2.5 problem 2

Sketch the graph of $f(y)$ vs. y and determine critical points and classify each as stable or not stable. $\frac{dy}{dt} = ay + by^2; a > 0, b > 0, -\infty < y_0 < \infty$

$$f(y) = ay + by^2$$

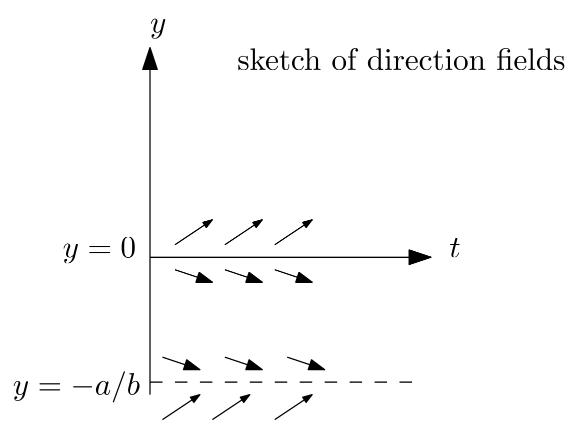
This is the same problem as above, with same direction field. But now the phase line will include both critical points. The critical points are from above

$$\begin{aligned}y_1 &= 0\\ y_2 &= \frac{-a}{b}\end{aligned}$$

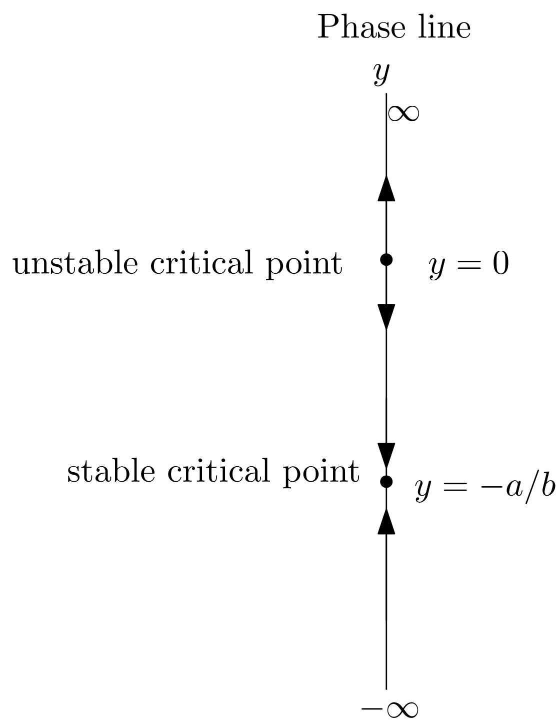
Or

$$\begin{aligned}y_1 &= 0\\ y_2 &= -1\end{aligned}$$

For $a = 1, b = 1$. Here is sketch of the direction field.



Therefore $y = 0$ is not stable, and $y = \frac{-a}{b}$ is stable. The following is the phase line for this problem



2.3.3 Section 2.6 problem 1

Determine if $(2x+3)+\left(2y-2\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{(2x+3)}^{M(x,y)}+\overbrace{\left(2y-2\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=0$$
$$\frac{\partial N}{\partial x}=0$$

Therefore, it is exact. Before solving, it is always best to apply singular point analysis on $f\left(x,y\right)$ in order to determined if the solution is unique or not. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-(2x+3)}{\left(2y-2\right)}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y=1$. Now the ODE is solved. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=2x+3 \tag{1}$$
$$\frac{\partial\Psi}{\partial y}=N=2y-2 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int 2x + 3dx$$
$$\Psi = x^2 + 3x + f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = f'\left(y\right)$$

Comparing the above to (2) shows that $f'\left(y\right) = 2y - 2$. By integrating $f\left(y\right)$ is found to be

$$f\left(y\right) = y^2 - 2y + c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y\left(x\right)\right)$

$$\Psi\left(x, y\left(x\right)\right) = x^2 + 3x + \left(y^2 - 2y + c\right)$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^2 + 3x + \left(y^2 - 2y + c\right) = c_1$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$\boxed{x^2 + 3x + y^2 - 2y = c_0 \qquad y \neq 1} \tag{4}$$

2.3.4 Section 2.6 problem 2

Determine if $\left(2x + 4y\right) + \left(2x - 2y\right)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\left(2x + 4y\right)}^{M(x,y)} + \overbrace{\left(2x - 2y\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = 4$$
$$\frac{\partial N}{\partial x} = 2$$

Therefore the <u>ODE is not exact</u>.

2.3.5 Section 2.6 problem 3

Determine if $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\left(3x^2 - 2xy + 2\right)}^{M(x,y)} + \overbrace{\left(6y^2 - x^2 + 3\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = -2x$$
$$\frac{\partial N}{\partial x} = -2x$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx} = f\left(x, y\right) = \frac{-\left(3x^2-2xy+2\right)}{\left(6y^2-x^2+3\right)}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y = \pm\sqrt{\frac{1}{6}x^2 - \frac{1}{2}}$. Now the ODE is solved. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = 3x^2 - 2xy + 2 \tag{1}$$
$$\frac{\partial \Psi}{\partial y} = N = 6y^2 - x^2 + 3 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \left(3x^2 - 2xy + 2\right) dx$$
$$\Psi = x^3 - x^2 y + 2x + f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = -x^2 + f'\left(y\right)$$

Equating the above to (2) gives

$$-x^2 + f'\left(y\right) = 6y^2 - x^2 + 3$$
$$f'\left(y\right) = 6y^2 + 3$$

Integrating the above w.r.t. y gives

$$f\left(y\right) = 2y^3 + 3y + c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y\left(x\right)\right)$

$$\Psi\left(x, y\left(x\right)\right) = x^3 - x^2 y + 2x + 2y^3 + 3y + c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^3 - x^2 y + 2x + 2y^3 + 3y + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y = c_o}$$

The above solution is valid only for $y \neq \pm\sqrt{\frac{1}{6}x^2 - \frac{1}{2}}$

2.3.6 Section 2.6 problem 4

Determine if $\left(2xy^2+2y\right)+\left(2x^2y+2x\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(2xy^2+2y\right)}^{M(x,y)}+\overbrace{\left(2x^2y+2x\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=4xy+2$$
$$\frac{\partial N}{\partial x}=4xy+2$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-\left(2xy^2+2y\right)}{\left(2x^2y+2x\right)}$ shows that this is non-linear first order and applying theorem 2, shows that $f\left(x\right)$ is not continuous at $y=\frac{-1}{x}$ for $x\neq 0$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=2xy^2+2y \tag{1}$$
$$\frac{\partial\Psi}{\partial y}=N=2x^2y+2x \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\Psi}{\partial x}dx=\int\left(2xy^2+2y\right)dx$$
$$\Psi=x^2y^2+2yx+f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial\Psi}{\partial y}=2x^2y+2x+f'\left(y\right)$$

Equating the above to (2) gives

$$2x^2y+2x+f'\left(y\right)=2x^2y+2x$$
$$f'\left(y\right)=0$$

Integrating the above w.r.t. y gives

$$f\left(y\right)=c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y\left(x\right)\right)$

$$\Psi\left(x,y\left(x\right)\right)=x^2y^2+2yx+c$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^2y^2+2yx+c=c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$x^2y^2 + 2yx = c_o \qquad y \neq \frac{-1}{x}, x \neq 0$$

2.3.7 Section 2.6 problem 5

Determine if $\frac{dy}{dx} = \frac{-(ax+by)}{bx+cy}$ is exact and solve if so.

$$\overbrace{(ax+by)}^{M(x,y)} + \overbrace{(bx+cy)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = b$$
$$\frac{\partial N}{\partial x} = b$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx} = f(x,y) = \frac{-(ax+by)}{bx+cy}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y = \frac{-bx}{c}$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = ax + by \tag{1}$$
$$\frac{\partial \Psi}{\partial y} = N = bx + cy \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int (ax+by)\, dx$$
$$\Psi = \frac{a}{2}x^2 + byx + f(y) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = bx + f'(y)$$

Equating the above to (2) gives

$$bx + f'(y) = bx + cy$$
$$f'(y) = cy$$

Integrating the above w.r.t. y gives

$$f(y) = \frac{1}{2}cy^2 + k$$

Where k is constant. Substituting $f(y)$ back into (3) gives $\Psi(x, y(x))$

$$\Psi(x, y(x)) = \frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 + k$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = k_1$, where k_1 is some constant. Therefore the above can be written as

$$\frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 + k = k_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$\begin{aligned}\frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 &= k_o \\ ax^2 + 2byx + cy^2 &= 2k_o = k_2\end{aligned}$$

Summary of solution

$$ax^2 + 2byx + cy^2 = k_2 \qquad y \neq \frac{-bx}{c}$$

2.3.8 Section 2.6 problem 6

Determine if $\frac{dy}{dx} = \frac{-(ax-by)}{bx-cy}$ is exact and solve if so.

$$\overbrace{(ax - by)}^{M(x,y)} + \overbrace{(bx - cy)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= -b \\ \frac{\partial N}{\partial x} &= b\end{aligned}$$

Hence the ODE is not exact.

2.3.9 Section 2.6 problem 7

Determine if $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{(e^x \sin y - 2y \sin x)}^{M(x,y)} + \overbrace{(e^x \cos y + 2\cos x)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= e^x \cos y - 2\sin x \\ \frac{\partial N}{\partial x} &= e^x \cos y - 2\sin x\end{aligned}$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-\left(e^{x}\sin y-2y\sin x\right)}{e^{x}\cos y+2\cos x}$ shows that this is non-linear first order and applying theorem 2, shows that $f\left(x\right)$ is not continuous at $y=\arccos\left(\frac{-2\cos x}{e^{x}}\right)$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=e^{x}\sin y-2y\sin x \tag{1}$$

$$\frac{\partial\Psi}{\partial y}=N=e^{x}\cos y+2\cos x \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\Psi}{\partial x}dx=\int\left(e^{x}\sin y-2y\sin x\right)dx$$

$$\Psi=e^{x}\sin y+2y\cos x+f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial\Psi}{\partial y}=e^{x}\cos y+2\cos x+f'\left(y\right)$$

Equating the above to (2) gives

$$e^{x}\cos y+2\cos x+f'\left(y\right)=e^{x}\cos y+2\cos x$$

$$f'\left(y\right)=0$$

Hence

$$f\left(y\right)=c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y\left(x\right)\right)$

$$\Psi\left(x,y\left(x\right)\right)=e^{x}\sin y+2y\cos x+c$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_{1}$, where c_{1} is some constant. Therefore the above can be written as

$$e^{x}\sin y+2y\cos x+c=c_{1}$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$e^{x}\sin y+2y\cos x=c_{o}\qquad y\neq\arccos\left(\frac{-2\cos x}{e^{x}}\right)$$

Since c_{0} is constant, then $c_{0}=0$ is allowed value. This implies $e^{x}\sin y+2y\cos x=0$ is allowed, which means $y\left(x\right)=0$ is solution also, since when $y=0$ then $e^{x}\sin\left(0\right)+2\left(0\right)\cos x$ gives zero. Hence a second solution is

$$y\left(x\right)=0$$

Summary

$$\begin{cases} e^{x}\sin y+2y\cos x=c_{o} & y\neq\arccos\left(\frac{-2\cos x}{e^{x}}\right)\\ y\left(x\right)=0 & c_{0}=0\end{cases}$$

2.3.10 Section 2.6 problem 8

Determine if $\left(e^x\sin y+3y\right)-\left(3x-e^x\sin y\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(e^x\sin y+3y\right)}^{M(x,y)}+\overbrace{\left(-3x+e^x\sin y\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=e^x\sin y+3\\ \frac{\partial N}{\partial x}&=-3+e^x\sin y\end{aligned}$$

Hence the <u>ODE is not exact</u>

2.3.11 Section 2.6 problem 9

Determine if $\left(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x\right)+\left(xe^{xy}\cos 2x-3\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x\right)}^{M(x,y)}+\overbrace{\left(xe^{xy}\cos 2x-3\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=e^{xy}\cos 2x+yxe^{xy}\cos 2x-2xe^{xy}\sin 2x\\ \frac{\partial N}{\partial x}&=e^{xy}\cos 2x+xye^{xy}\cos 2x-2xe^{xy}\sin 2x\end{aligned}$$

Hence the <u>ODE is exact</u>. Now the ODE is solved. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x \tag{1}$$

$$\frac{\partial\Psi}{\partial y}=N=xe^{xy}\cos 2x-3 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\Psi}{\partial x}dx=\int\left(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x\right)dx$$

$$\Psi=y\int e^{xy}\cos 2x dx-2\int e^{xy}\sin 2x dx+2\int x dx+f\left(y\right) \tag{3}$$

Let

$$I=\int e^{xy}\cos 2x dx$$

Using integration by parts. Let $u = \cos 2x, dv = e^{xy} \rightarrow du = -2\sin(2x), v = \frac{e^{xy}}{y}$, hence

$$\begin{aligned} I &= uv - \int v du \\ &= \frac{e^{xy}}{y}\cos 2x + \frac{2}{y}\int e^{xy}\sin 2x dx \end{aligned}$$

Applying integration by parts again to $\int e^{xy}\sin 2x dx$, where now $u = \sin 2x, dv = e^{xy} \rightarrow du = 2\cos(2x), v = \frac{e^{xy}}{y}$ Therefore the above becomes

$$\begin{aligned} I &= \frac{e^{xy}}{y}\cos 2x + \frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x - \int \frac{e^{xy}}{y} 2\cos 2x dx\right) \\ &= \frac{e^{xy}}{y}\cos 2x + \frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x - \frac{2}{y}\int e^{xy}\cos 2x dx\right) \end{aligned}$$

But $I = \int e^{xy}\cos 2x dx$ and the above becomes

$$I = \frac{e^{xy}}{y}\cos 2x + \frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x - \frac{2}{y}I\right)$$

Solving for I

$$\begin{aligned} I &= \frac{e^{xy}}{y}\cos 2x + \frac{2e^{xy}}{y^2}\sin 2x - \frac{4}{y^2}I \\ I + \frac{4}{y^2}I &= \frac{e^{xy}}{y}\cos 2x + \frac{2e^{xy}}{y^2}\sin 2x \\ I\left(\frac{y^2+4}{y^2}\right) &= \frac{e^{xy}}{y}\cos 2x + \frac{2e^{xy}}{y^2}\sin 2x \\ I &= \frac{y^2}{y^2+4}\frac{e^{xy}}{y}\cos 2x + \frac{y^2}{y^2+4}\frac{2e^{xy}}{y^2}\sin 2x \end{aligned}$$

Therefore

$$\int e^{xy}\cos 2x dx = \frac{ye^{xy}}{y^2+4}\cos 2x + \frac{2e^{xy}}{y^2+4}\sin 2x \tag{4}$$

Similarly $I = \int e^{xy}\sin 2x dx$ is solve by integration by parts. Let $ev = e^{xy}, u = \sin 2x \rightarrow du = 2\cos 2x, v = \frac{e^{xy}}{y}$, hence

$$I = \frac{e^{xy}}{y}\sin 2x - \frac{2}{y}\int e^{xy}\cos 2x dx$$

For $\int e^{xy}\cos 2x dx$, let $u = \cos 2x, dv = e^{xy} \rightarrow du = -2\sin 2x, v = \frac{e^{xy}}{y}$ and the above becomes

$$I = \frac{e^{xy}}{y}\sin 2x - \frac{2}{y}\left(\frac{e^{xy}}{y}\cos 2x + \frac{2}{y}\int e^{xy}\sin 2x dx\right)$$

But $\int e^{xy}\sin 2x dx = I$ and the above becomes

$$I = \frac{e^{xy}}{y}\sin 2x - \frac{2}{y}\left(\frac{e^{xy}}{y}\cos 2x + \frac{2}{y}I\right)$$

Solving for I

$$\begin{aligned}
I &= \frac{e^{xy}}{y}\sin 2x - \left(\frac{2e^{xy}}{y^2}\cos 2x + \frac{4}{y^2}I\right)\\
I + \frac{4}{y^2}I &= \frac{e^{xy}}{y}\sin 2x - \frac{2e^{xy}}{y^2}\cos 2x\\
I\left(\frac{y^2+4}{y^2}\right) &= \frac{e^{xy}}{y}\sin 2x - \frac{2e^{xy}}{y^2}\cos 2x\\
I &= \frac{y^2}{y^2+4}\frac{e^{xy}}{y}\sin 2x - \frac{y^2}{y^2+4}\frac{2e^{xy}}{y^2}\cos 2x
\end{aligned}$$

Hence

$$\int e^{xy}\sin 2x dx = \frac{ye^{xy}}{y^2+4}\sin 2x - \frac{2e^{xy}}{y^2+4}\cos 2x \tag{5}$$

Substituting (4,5) into (3) gives

$$\Psi = y\left(\frac{ye^{xy}}{y^2+4}\cos 2x + \frac{2e^{xy}}{y^2+4}\sin 2x\right) - 2\left(\frac{ye^{xy}}{y^2+4}\sin 2x - \frac{2e^{xy}}{y^2+4}\cos 2x\right) + x^2 + f\left(y\right)$$

Simplifying

$$\begin{aligned}
\Psi &= \frac{y^2e^{xy}}{y^2+4}\cos 2x + \frac{2ye^{xy}}{y^2+4}\sin 2x - \frac{2ye^{xy}}{y^2+4}\sin 2x + \frac{4e^{xy}}{y^2+4}\cos 2x + x^2 + f\left(y\right)\\
&= \frac{y^2e^{xy}}{y^2+4}\cos 2x + \frac{4e^{xy}}{y^2+4}\cos 2x + x^2 + f\left(y\right)\\
&= e^{xy}\cos\left(2x\right)\left(\frac{y^2}{y^2+4} + \frac{4}{y^2+4}\right) + x^2 + f\left(y\right)\\
&= e^{xy}\cos\left(2x\right)\left(\frac{4+y^2}{y^2+4}\right) + x^2 + f\left(y\right)
\end{aligned}$$

Therefore

$$\Psi = e^{xy}\cos\left(2x\right) + x^2 + f\left(y\right) \tag{6}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = xe^{xy}\cos\left(2x\right) + f'\left(y\right)$$

Equating the above to (2) gives

$$\begin{aligned}
xe^{xy}\cos\left(2x\right) + f'\left(y\right) &= xe^{xy}\cos 2x - 3\\
f'\left(y\right) &= -3
\end{aligned}$$

Hence

$$f\left(y\right) = -3y + c$$

Where c is constant. Substituting $f\left(y\right)$ back into (6) gives

$$\Psi\left(x, y\left(x\right)\right) = e^{xy}\cos\left(2x\right) + x^2 - 3y + c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$e^{xy}\cos(2x) + x^2 - 3y + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$e^{xy}\cos(2x) + x^2 - 3y = c_o$$

2.3.12 Section 2.6 problem 10

Determine if $\left(\frac{y}{x} + 6x\right) + (\ln x - 2)\frac{dy}{dx} = 0; x > 0$ is exact and solve if so.

$$\overbrace{\left(\frac{y}{x} + 6x\right)}^{M(x,y)} + \overbrace{(\ln x - 2)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = \frac{1}{x}$$
$$\frac{\partial N}{\partial x} = \frac{1}{x}$$

Hence the <u>ODE is exact</u>. Writing the ODE as $\frac{dy}{dx} = f(x,y) = \frac{-\left(\frac{y}{x}+6x\right)}{\ln x - 2}$ shows that this is <u>non-linear first order</u> and applying theorem 2, shows that $f(x)$ is not continuous at $x = e^2$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = \frac{y}{x} + 6x \tag{1}$$
$$\frac{\partial \Psi}{\partial y} = N = \ln x - 2 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \left(\frac{y}{x} + 6x\right) dx$$
$$\Psi = y\ln(x) + 3x^2 + f(y) \tag{3}$$

No need to use $\ln|x|$ since the problem said that $x > 0$. Therefore

$$\frac{\partial \Psi}{\partial y} = \ln(x) + f'(y)$$

Equating the above to (2) gives

$$\ln(x) + f'(y) = \ln(x) - 2$$
$$f'(y) = -2$$

Hence

$$f(y) = -2y + c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y\left(x\right)\right)$

$$\Psi\left(x,y\left(x\right)\right)=y\ln\left(x\right)+3x^2-2y+c$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_1$, where c_1 is some constant. Therefore the above can be written as

$$y\ln\left(x\right)+3x^2-2y+c=c_1$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$y\ln\left(x\right)+3x^2-2y=c_o \qquad x>0;x\neq e^2$$

2.3.13 Section 2.6 problem 11

Determine if $\left(x\ln\left(y\right)+xy\right)+\left(y\ln\left(x\right)+xy\right)\frac{dy}{dx}=0;x>0;y>0$ is exact and solve if so.

$$\overbrace{\left(x\ln\left(y\right)+xy\right)}^{M(x,y)}+\overbrace{\left(y\ln\left(x\right)+xy\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=\frac{x}{y}+x=\frac{x\left(1+y\right)}{y}$$
$$\frac{\partial N}{\partial x}=\frac{y}{x}+y=\frac{y\left(1+x\right)}{x}$$

Hence this ODE is <u>not exact</u>.

2.3.14 Section 2.6 problem 12

Determine if $\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}+\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}}^{M(x,y)}+\overbrace{\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=\frac{-3}{2}\frac{x}{\left(x^2+y^2\right)^{\frac{5}{2}}}\left(2y\right)=\frac{-3xy}{\left(x^2+y^2\right)^{\frac{5}{2}}}$$
$$\frac{\partial N}{\partial x}=\frac{-3}{2}\frac{y}{\left(x^2+y^2\right)^{\frac{5}{2}}}\left(2x\right)=\frac{-3xy}{\left(x^2+y^2\right)^{\frac{5}{2}}}$$

Hence ODE is exact. Writing the ODE as $\frac{dy}{dx} = f\left(x,y\right) = \frac{-\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}}{\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}} = \frac{-x}{y}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y = 0$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = \frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} \tag{1}$$

$$\frac{\partial \Psi}{\partial y} = N = \frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} dx$$

Let $u = x^2 + y^2$, then $\frac{du}{dx} = 2x$. Substituting this into $\int \frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} dx$ gives

$$\begin{aligned}
\int \frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} dx &= \int \frac{x}{u^{\frac{3}{2}}} \frac{du}{2x} \\
&= \frac{1}{2} \int u^{\frac{-3}{2}} du \\
&= \frac{1}{2} \frac{u^{\frac{-1}{2}}}{\frac{-1}{2}} + f\left(y\right) \\
&= -\frac{1}{u^{\frac{1}{2}}} + f\left(y\right) \\
&= -\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}} + f\left(y\right)
\end{aligned}$$

Hence

$$\Psi = -\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}} + f\left(y\right) \tag{3}$$

Therefore

$$\begin{aligned}
\frac{\partial \Psi}{\partial y} &= \frac{1}{2} \left(x^2+y^2\right)^{-\frac{3}{2}} \left(2y\right) + f'\left(y\right) \\
&= \frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}} + f'\left(y\right)
\end{aligned}$$

Equating the above to (2) gives

$$\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}+f'\left(y\right)=\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$
$$f'\left(y\right)=0$$

Hence

$$f\left(y\right)=c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y\left(x\right)\right)$

$$\Psi\left(x,y\left(x\right)\right)=-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}+c$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_1$, where c_1 is some constant. Therefore the above can be written as

$$-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}+c=c_1$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}=c_o$$
$$\left(x^2+y^2\right)^{\frac{1}{2}}=-\frac{1}{c_0}=c_2$$

2.4 HW4

2.4.1 Section 2.6 problem 19

Question Show that $x^2y^3 + x\left(1+y^2\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu\left(x,y\right) = \frac{1}{xy^3}$ and then solve.

Solution The first step is to apply theorem two and also check where the ODE is singular. Writing it as

$$\frac{dy}{dx} = f\left(x,y\right) = \frac{-x^2y^3}{x\left(1+y^2\right)}$$

This is non-linear first order ODE. There is a pole at $x=0$. From theorem two, this says that unique solution is not guaranteed to exist since the first condition which says that $f\left(x,y\right)$ must be continuous, was not satisfied. Now the ODE is solved.

$$\overbrace{x^2y^3}^{M} + \overbrace{x\left(1+y^2\right)}^{N}y' = 0$$

Hence

$$M\left(x,y\right) = x^2y^3$$
$$N\left(x,y\right) = x\left(1+y^2\right)$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\frac{\partial M}{\partial y} = 3x^2y^2$$
$$\frac{\partial N}{\partial x} = 1+y^2$$

The above shows that that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\left(\mu x^2y^3\right) + \mu x\left(1+y^2\right)y' = 0$$
$$\frac{x^2y^3}{xy^3} + \frac{1}{xy^3}x\left(1+y^2\right)y' = 0$$
$$x + \frac{1}{y^3}\left(1+y^2\right)y' = 0$$

Now $\bar{M} = x$ and $\bar{N} = \frac{1}{y^3}\left(1+y^2\right)$. Checking that the new $\bar{M},\bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = 0$$
$$\frac{\partial \bar{N}}{\partial x} = 0$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = x \tag{1}$$

$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = \frac{1}{y^3}\left(1+y^2\right) \tag{2}$$

Integrating (1) w.r.t x gives

$$\Psi = \frac{1}{2}x^2 + f\left(y\right) \tag{3}$$

$$\frac{\partial \Psi}{\partial y} = f'\left(y\right)$$

Comparing the above to (2) in order to solve for $f'\left(y\right)$ gives

$$f'\left(y\right) = \frac{1+y^2}{y^3}$$

$$f\left(y\right) = \int \frac{1+y^2}{y^3} dy + c \tag{4}$$

We need now to solve $\int \frac{1+y^2}{y^3} dy$

$$\begin{aligned}
\int \frac{1+y^2}{y^3} dy &= \int \frac{1}{y^3} dy + \int \frac{y^2}{y^3} dy \\
&= -\frac{1}{2y^2} + \int \frac{1}{y} dy \\
&= -\frac{1}{2y^2} + \ln\left|y\right|
\end{aligned}$$

Using the above solution in (4) gives

$$f\left(y\right) = -\frac{1}{2y^2} + \ln\left|y\right| + c$$

Using the above in (3) gives

$$\Psi = \frac{1}{2}x^2 - \frac{1}{2y^2} + \ln\left|y\right| + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln\left|y\right| = C \qquad x \neq 0$$

Checking $y = 0$ as solution, shows that putting $y = 0$ in $f\left(x,y\right) = \frac{-x^2y^3}{x\left(1+y^2\right)} = 0$. Hence $y = 0$ is also a solution.

Summary The solutions are

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln|y| = C \qquad x \neq 0, y \neq 0$$

$$y = 0 \qquad x \neq 0$$

2.4.2 Section 2.6 problem 20

Question Show that $\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu(x,y) = ye^x$ and then solve.

Solution First we will check where the ODE is singular. Writing it as

$$\frac{dy}{dx} = f(x,y) = \frac{\frac{\sin y}{y} - 2e^{-x}\sin x}{\frac{\cos y + 2e^{-x}\cos x}{y}}$$

This is non-linear first order ODE. We see a pole at $y = 0$. Hence $y \neq 0$. From theorem two, this says that that unique solution is not guaranteed since first condition which says that $f(x,y)$ must be continuous, was not satisfied.

$$M(x,y) = \frac{\sin y}{y} - 2e^{-x}\sin x$$

$$N(x,y) = \frac{\cos y + 2e^{-x}\cos x}{y}$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\frac{\partial M}{\partial y} = \ln y \sin y + \frac{1}{y}\cos y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\cos y + \frac{1}{y}2e^{-x}\cos x\right) = \frac{-1}{y}2e^{-x}\cos x - \frac{1}{y}2e^{-x}\sin x = \frac{-2e^{-x}}{y}(\cos x + \sin x)$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\mu\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) + \mu\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)y' = 0$$

$$ye^x\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) + ye^x\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)y' = 0$$

$$\left(e^x\sin y - 2y\sin x\right) + \left(e^x\cos y + 2\cos x\right)y' = 0$$

Now

$$\bar{M} = e^x\sin y - 2y\sin x$$

$$\bar{N} = e^x\cos y + 2\cos x$$

Checking now the new $\bar{M},\bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = e^x \cos y - 2\sin x$$
$$\frac{\partial \bar{N}}{\partial x} = e^x \cos y - 2\sin x$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = e^x \sin y - 2y \sin x \tag{1}$$
$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = e^x \cos y + 2\cos x \tag{2}$$

Integrating (1) w.r.t x gives

$$\Psi = e^x \sin y + 2y\cos x + f\left(y\right) \tag{3}$$
$$\frac{\partial \Psi}{\partial y} = e^x \cos y + 2\cos x + f'\left(y\right)$$

Comparing the above to (2) in order to solve for $f'\left(y\right)$ gives

$$e^x \cos y + 2\cos x + f'\left(y\right) = e^x \cos y + 2\cos x$$
$$f'\left(y\right) = 0$$
$$f\left(y\right) = c \tag{4}$$

Substituting the above into (3) gives

$$\Psi = e^x \sin y + 2y\cos x + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$e^x \sin y + 2y\cos x = C \qquad y \neq 0$$

2.4.3 Section 2.6 problem 21

Question Show that $y + \left(2x - ye^y\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu\left(x,y\right) = y$ and then solve.

Solution

$$M\left(x,y\right) = y$$
$$N\left(x,y\right) = 2x - ye^y$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\frac{\partial M}{\partial y} = 1$$
$$\frac{\partial N}{\partial x} = 2$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\mu y + \mu \left(2x - ye^{y}\right) y' = 0$$
$$y^2 + \left(2xy - y^2 e^{y}\right) y' = 0$$

Now

$$\bar{M} = y^2$$
$$\bar{N} = 2xy - y^2 e^{y}$$

Checking now the new $\bar{M}, \bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = 2y$$
$$\frac{\partial \bar{N}}{\partial x} = 2y$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = y^2 \tag{1}$$
$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = 2xy - y^2 e^{y} \tag{2}$$

Integrating (1) w.r.t x gives

$$\Psi = y^2 x + f\left(y\right) \tag{3}$$
$$\frac{\partial \Psi}{\partial y} = 2yx + f'\left(y\right)$$

Comparing the above to (2) in order to solve for $f'\left(y\right)$ gives

$$2yx + f'\left(y\right) = 2xy - y^2 e^{y}$$
$$f'\left(y\right) = -y^2 e^{y}$$
$$f\left(y\right) = -\int y^2 e^{y} dy + c \tag{4}$$

The integral $\int y^2 e^{y} dy$ can be found using integration by parts. Let $u = y^2, dv = e^{y} \to du = 2y, v = e^{y}$, therefore

$$\int y^2 e^{y} dy = \int u dv$$
$$= uv - \int v du$$
$$= y^2 e^{y} - 2\int y e^{y} dy$$

Applying integration by parts again to $\int y e^{y} dy$, where now $u = y, dv = e^{y} \to du = 1, v = e^{y}$, the above becomes

$$\begin{aligned}\int y^2 e^y dy &= y^2 e^y - 2\left(ye^y - \int e^y dy\right)\\ &= y^2 e^y - 2\left(ye^y - e^y\right)\\ &= y^2 e^y - 2ye^y + 2e^y\\ &= e^y\left(y^2 - 2y + 2\right)\end{aligned}$$

Therefore from (4)

$$f\left(y\right) = -e^y\left(y^2 - 2y + 2\right) + c$$

Substituting the above into (3) gives

$$\Psi = y^2 x - e^y\left(y^2 - 2y + 2\right) + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$y^2 x - e^y\left(y^2 - 2y + 2\right) = C$$

2.4.4 Section 2.6 problem 22

Question Show that $(x+2)\sin y + \left(x\cos y\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu\left(x,y\right) = xe^x$ and then solve.

Solution

$$\begin{aligned}M\left(x,y\right) &= (x+2)\sin y\\ N\left(x,y\right) &= x\cos y\end{aligned}$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\begin{aligned}\frac{\partial M}{\partial y} &= (x+2)\cos y\\ \frac{\partial N}{\partial x} &= \cos y\end{aligned}$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\begin{aligned}\mu(x+2)\sin y + \mu\left(x\cos y\right)y' &= 0\\ xe^x(x+2)\sin y + xe^x\left(x\cos y\right)y' &= 0\end{aligned}$$

Now

$$\begin{aligned}\bar{M} &= \left(x^2 e^x + 2xe^x\right)\sin y\\ \bar{N} &= x^2 e^x\cos y\end{aligned}$$

Checking now the new $\bar{M}, \bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = \left(x^2 e^x + 2xe^x\right)\cos y$$
$$\frac{\partial \bar{N}}{\partial x} = 2xe^x \cos y + x^2 e^x \cos y = \left(x^2 e^x + 2xe^x\right)\cos y$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = \left(x^2 e^x + 2xe^x\right)\sin y \tag{1}$$
$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = x^2 e^x \cos y \tag{2}$$

Integrating (2) w.r.t y as it is simpler than integrating (1) w.r.t. x, gives

$$\Psi = \int x^2 e^x \cos y dy = x^2 e^x \sin y + f\left(x\right) \tag{3}$$
$$\frac{\partial \Psi}{\partial x} = 2xe^x \sin y + x^2 e^x \sin y + f'\left(x\right)$$

Comparing the above to (1) in order to solve for $f'\left(x\right)$ gives

$$2xe^x \sin y + x^2 e^x \sin y + f'\left(x\right) = \left(x^2 e^x + 2xe^x\right)\sin y$$
$$f'\left(x\right) = 0$$
$$f\left(x\right) = c \tag{4}$$

Substituting the above into (3) gives

$$\Psi = x^2 e^x \sin y + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore $\Psi = c_1$ and the above simplifies to, after collecting all constants to one

$$x^2 e^x \sin y = C$$

2.4.5 Section 2.6 problem 23

Question Show that if $\frac{N_x - M_y}{M} = Q$ where Q is function of y only, then $M + Ny' = 0$ has integrating factor of form $\mu\left(y\right) = e^{\int Q\left(y\right)dt}$

Solution Given the differential equation

$$M\left(x,y\right) + N\left(x,y\right)\frac{dy\left(x\right)}{dx} = 0$$

Multiplying by $\mu\left(y\right)$ results in

$$\mu M + \mu N y' = 0$$

The above is exact if

$$\frac{\partial\left(\mu M\right)}{\partial y} = \frac{\partial\left(\mu N\right)}{\partial x}$$

Performing the above, taking into account that μ depends on y only, results in

$$\frac{d\mu}{dy}M+\mu\frac{\partial M}{\partial y}=\mu\frac{\partial N}{\partial x}$$

The above is first order ODE in μ

$$\frac{d\mu}{dy}M=\mu\frac{\partial N}{\partial x}-\mu\frac{\partial M}{\partial y}$$

$$\frac{d\mu}{dy}=\mu\left(\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}\right)$$

Let $Q=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$. If Q depends on y only, then the above ODE is separable. Hence

$$\frac{d\mu}{dy}=\mu Q\left(y\right)$$

$$\frac{d\mu}{\mu}=Q\left(y\right)dy$$

Integrating both sides gives

$$\ln\left|\mu\right|=\int Q\left(y\right)dy+C$$

$$\left|\mu\right|=e^{\int Q(y)dy+C}$$

$$\mu\left(y\right)=Ae^{\int Q(y)dy}$$

Where A is some constant, which can be taken to be 1 leading to the result required to show. The above procedure works only when $Q=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ happened to be function of y only. This complete the proof.

2.4.6 Section 2.6 problem 24

Question Show that if $\frac{N_x-M_y}{xM-yN}=R$ where R is function of xy only, then $M+Ny'=0$ has integrating factor of form $\mu\left(x,y\right)$. Find the general formula for μ.

Solution Given the differential equation

$$M\left(x,y\right)+N\left(x,y\right)\frac{dy\left(x\right)}{dx}=0$$

Let $\mu\left(t\right)$ where $t=xy$. Multiplying the above with $\mu\left(t\right)$ gives

$$\mu\left(t\right)M\left(x,y\right)+\mu\left(t\right)N\left(x,y\right)\frac{dy\left(x\right)}{dx}=0$$

The above is exact when

$$\frac{\partial\mu M}{\partial y}=\frac{\partial\mu N}{\partial x}$$

Hence

$$\frac{\partial\mu}{\partial y}M+\mu\frac{\partial M}{\partial y}=\frac{\partial\mu}{\partial x}N+\mu\frac{\partial N}{\partial x} \tag{1}$$

However,

$$\frac{\partial\mu}{\partial y}=\frac{d\mu}{dt}\frac{\partial t}{dy}=\frac{d\mu}{dt}x \tag{2}$$

And

$$\frac{\partial\mu}{\partial x}=\frac{d\mu}{dt}\frac{\partial t}{dx}=\frac{d\mu}{dt}y \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}\frac{d\mu}{dt}xM+\mu\frac{\partial M}{\partial y}&=\frac{d\mu}{dt}yN+\mu\frac{\partial N}{\partial x}\\ \frac{d\mu}{dt}\left(xM-yN\right)&=\mu\frac{\partial N}{\partial x}-\mu\frac{\partial M}{\partial y}\\ \frac{d\mu\left(t\right)}{dt}&=\mu\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}\end{aligned}$$

In the above, μ depends on t only, where t is function of xy only. If $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}$ depends on t only, then the above can be considered a separable first order ODE in μ. Let $R\left(t\right)=\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}$ and the above can be written as

$$\frac{d\mu\left(t\right)}{dt}=\mu R\left(t\right)$$

Since separable, then

$$\begin{aligned}\frac{d\mu\left(t\right)}{\mu}&=R\left(t\right)dt\\ \int\frac{d\mu}{\mu}&=\int Rdt\\ \ln\left|\mu\right|&=\int Rdt+C\\ \left|\mu\right|&=e^{\int Rdt+C}\\ \mu&=Ae^{\int Rdt}\end{aligned}$$

Where A is constant of integration which can be taken to be 1. Hence $\mu=e^{\int Rdt}$. This works only if R is function of t only.

2.4.7 Section 2.7 problem 20

20. **Convergence of Euler's Method.** It can be shown that under suitable conditions on f, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y),\ y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \qquad y(t_0) = y_0.$$

(a) Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b) Using the Euler formula, show that

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, \ldots.$$

(c) Noting that $y_1 = (1+h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1+h)^n (y_0 - t_0) + t_n \tag{i}$$

for each positive integer n.

(d) Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.

Hint: $\lim_{n\to\infty}(1 + a/n)^n = e^a$.

2.4.7.1 part a

$$\begin{aligned} y' &= 1 - t + y \\ y(t_0) &= y_0 \end{aligned}$$

This is linear first order ODE. Writing it as $y' - y = 1 - t$, then the integrating factor is $\mu = e^{-\int dt} = e^{-t}$ and the ODE becomes

$$\frac{d}{dt}\left(ye^{-t}\right) = e^{-t}(1-t)$$

Integrating both sides

$$\begin{aligned} ye^{-t} &= \int e^{-t}(1-t)\,dt + c \\ &= \int e^{-t}dt - \int te^{-t}dt + c \end{aligned} \tag{1}$$

But $\int te^{-t}dt = \int udv$ where $u = t, dv = e^{-t} \rightarrow du = 1, v = -e^{-t}$, hence

$$\begin{aligned}\int te^{-t}dt &= uv - \int vdu \\ &= -te^{-t} + \int e^{-t}dt \\ &= -te^{-t} - e^{-t}\end{aligned}$$

Putting this result in (1) gives

$$\begin{aligned}ye^{-t} &= -e^{-t} - \left(-te^{-t} - e^{-t}\right) + c \\ &= -e^{-t} + te^{-t} + e^{-t} + c \\ &= te^{-t} + c\end{aligned}$$

Therefore solving for y gives

$$y = t + ce^{t} \tag{2}$$

The constant c is now found from initial conditions.

$$\begin{aligned}y_0 &= t_0 + ce^{t_0} \\ c &= \left(y_0 - t_0\right) e^{-t_0}\end{aligned}$$

Substituting c found back into (2) gives the final solution

$$\begin{aligned}y &= t + \left(y_0 - t_0\right) e^{-t_0}e^{t} \\ &= \left(y_0 - t_0\right) e^{t-t_0} + t \qquad (3)\end{aligned}$$

2.4.7.2 Part b

Euler formula is

$$y_k = hf\left(t_{k-1}, y_{k-1}\right) + y_{k-1} \qquad k = 1, 2, 3, \cdots \tag{1}$$

Where in this problem $f\left(t_{k-1}, y_{k-1}\right)$ is the RHS of $y' = 1 - t + y$ but evaluated at t_{k-1}. Hence

$$f\left(t_{k-1}, y_{k-1}\right) = 1 - t_{k-1} + y_{k-1}$$

Substituting this into (1) gives

$$\begin{aligned}y_k &= h\left(1 - t_{k-1} + y_{k-1}\right) + y_{k-1} \\ &= h - ht_{k-1} + hy_{k-1} + y_{k-1} \\ &= (1 + h)\, y_{k-1} + h - ht_{k-1} \qquad k = 1, 2, 3, \cdots\end{aligned}$$

Which is the required formula asked to derive.

2.4.7.3 Part c

The formula given $y_1 = (1+h)\left(y_0 - t_0\right) + t_1$ can be found as follows. Since

$$
\begin{aligned}
y_1 &= y_0 + hf\left(t_0, y_0\right)\\
&= y_0 + h\left(1 - t_0 + y_0\right)\\
&= y_0 + h - ht_1 + hy_0
\end{aligned}
$$

Adding $t_0 - t_0$ to the above will not changed anything, hence

$$y_1 = y_0 + h - ht_1 + hy_0 + t_0 - t_0$$

But $t_1 = t_0 + h$ by definition, hence the above becomes, by replacing $t_0 + h$ above with t_1

$$y_1 = y_0 + t_1 - ht_1 + hy_0 - t_0$$

Simplifying

$$
\begin{aligned}
y_1 &= \left(y_0 - t_0\right) + h\left(y_0 - t_0\right) + t_1\\
&= (1+h)\left(y_0 - t_0\right) + t_1
\end{aligned}
$$

Now the question will be answered. Need to show that $y_n = (1+h)^n\left(y_0 - t_0\right) + t_n$ is true, using induction. This is true for $k = 1$ as shown above. Now assuming it is true for k, we then need to show it is true for $k+1$.

By assumption, it is true for k, hence

$$y_k = (1+h)^k\left(y_0 - t_0\right) + t_k \tag{1}$$

But using Euler formula

$$
\begin{aligned}
y_{k+1} &= y_k + hf\left(t_k, y_k\right)\\
&= y_k + h\left(1 - t_k + y_k\right) \qquad (2)
\end{aligned}
$$

Substituting (1) into RHS of (2)

$$
\begin{aligned}
y_{k+1} &= \left((1+h)^k\left(y_0 - t_0\right) + t_k\right) + h\left(1 - t_k + \left((1+h)^k\left(y_0 - t_0\right) + t_k\right)\right)\\
&= (1+h)^k\left(y_0 - t_0\right) + t_k + h - ht_k + h\left((1+h)^k\left(y_0 - t_0\right) + t_k\right)\\
&= (1+h)^k\left(y_0 - t_0\right) + t_k + h - ht_k + h(1+h)^k\left(y_0 - t_0\right) + ht_k\\
&= (1+h)^k\left(y_0 - t_0\right) + t_k + h + h(1+h)^k\left(y_0 - t_0\right)
\end{aligned}
$$

But $t_k + h = t_{k+1}$ by definition, hence

$$
\begin{aligned}
y_{k+1} &= (1+h)^k\left(y_0 - t_0\right) + t_{k+1} + h(1+h)^k\left(y_0 - t_0\right)\\
&= (1+h)^k\left(y_0 - t_0\right)(1+h) + t_{k+1}\\
&= (1+h)^{k+1}\left(y_0 - t_0\right) + t_{k+1}
\end{aligned}
$$

The above shows it is true for $k+1$ given it is true for k. Therefore, it is true for any positive integer n.

2.4.7.4 Part d

Using

$$y_n = (1+h)^n \left(y_0 - t_0\right) + t_n$$

Replacing $h = \frac{t_n - t_0}{n}$ in the above gives

$$y_n = \left(1 + \left(\frac{t_n - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + t_n$$

Taking the limit

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \left(\frac{t_n - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + \lim_{n\to\infty} t_n$$

But $\lim_{n\to\infty} t_n = t$, hence replacing all t_n with t in the above gives

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \left(\frac{t - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + t$$

Using hint that $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$ the above simplifies to

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n \\ &= e^{(t-t_0)} \left(y_0 - t_0\right) + t \end{aligned}$$

Which is the analytical solution found in part (a).

2.4.8 Section 3.1 problem 1

Find the general solution to $y'' + 2y' - 3y = 0$.

This is second order, linear, constant coefficient ODE. Letting $y = e^{rt}$ and replacing this into the ODE gives

$$e^{rt}\left(r^2 + 2r - 3\right) = 0$$

Since $e^{rt} \neq 0$, the above reduces to what is called the characteristic equation of the ODE

$$\boxed{r^2 + 2r - 3 = 0}$$

Which can be written as $(r-1)(r+3) = 0$. Hence $r_1 = 1, r_2 = -3$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{-3t}$$

2.4.9 Section 3.1 problem 2

Find the general solution to $y'' + 3y' + 2y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{r^2 + 3r + 2 = 0}$$

Which can be written as $(r+1)(r+2) = 0$. Hence $r_1 = -1, r_2 = -2$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

2.4.10 Section 3.1 problem 3

Find the general solution to $6y'' - y' - y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{6r^2 - r - 1 = 0}$$

Hence $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1-4(6)(-1)}}{12} = \frac{1 \pm \sqrt{1+24}}{12} = \frac{1 \pm 5}{12}$. Hence $r_1 = \frac{1}{2}, r_2 = \frac{-1}{3}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\frac{1}{2} t} + c_2 e^{\frac{-1}{3} t}$$

2.4.11 Section 3.1 problem 4

Find the general solution to $2y'' - 3y' + y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{2r^2 - 3r + 1 = 0}$$

Hence $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{3 \pm \sqrt{9-4(2)(1)}}{4} = \frac{3 \pm 1}{4}$. Hence $r_1 = 1, r_2 = \frac{1}{2}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{\frac{1}{2} t}$$

2.4.12 Section 3.1 problem 5

Find the general solution to $y'' + 5y' = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{r^2 + 5r = 0}$$

Which can be written as $r(r+5) = 0$, hence $r_1 = 0, r_2 = -5$.Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 + c_2 e^{-5t}$$

2.4.13 Section 3.1 problem 6

Find the general solution to $4y'' - 9y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{4r^2 - 9 = 0}$$

Therefore $r^2 = \frac{9}{4}$ or $r = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$. Hence $r_1 = \frac{3}{2}, r_2 = -\frac{3}{2}$.Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}$$

2.4.14 Section 3.1 problem 7

Find the general solution to $y'' - 9y' + 9y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE is

$$\boxed{r^2 - 9r + 9 = 0}$$

Hence $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{9\pm\sqrt{81-4(1)(9)}}{2} = \frac{9\pm\sqrt{81-36}}{2} = \frac{9\pm\sqrt{45}}{2} = \frac{9\pm 3\sqrt{5}}{2}$. Hence $r_1 = \frac{9+3\sqrt{5}}{2}, r_2 = \frac{9-3\sqrt{5}}{2}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\frac{9+3\sqrt{5}}{2}t} + c_2 e^{\frac{9-3\sqrt{5}}{2}t}$$

2.4.15 Section 3.1 problem 8

Find the general solution to $y'' - 2y' - 2y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE is

$$\boxed{r^2 - 2r - 2 = 0}$$

Hence $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-4(1)(-2)}}{2} = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm\sqrt{12}}{2} = \frac{2\pm2\sqrt{3}}{2} = 1 \pm \sqrt{3}$. Hence $r_1 = 1 + \sqrt{3}, r_2 = 1 - \sqrt{3}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where c_1, c_2 are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\left(1+\sqrt{3}\right)t} + c_2 e^{\left(1-\sqrt{3}\right)t}$$

2.5 HW5

2.5.1 Section 3.1 problem 9

Find the solution to $y'' + y' - 2y = 0; y(0) = 1, y'(0) = 1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 + r - 2 = 0$$
$$(r+2)(r-1) = 0$$

Hence the roots are $r_1 = -2, r_2 = 1$. Roots are real and distinct. The two solutions are

$$y_1 = e^{-2t}$$
$$y_2 = e^t$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-2t} + c_2 e^t \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 1$) to the general solution gives

$$1 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t) = -2c_1 e^{-2t} + c_2 e^t$. Applying second initial condition to this results in

$$1 = -2c_1 + c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1 = 1 - c_2$. Substituting this into (2) gives

$$\begin{aligned} 1 &= -2(1 - c_2) + c_2 \\ &= -2 + 2c_2 + c_2 \\ &= -2 + 3c_2 \end{aligned}$$

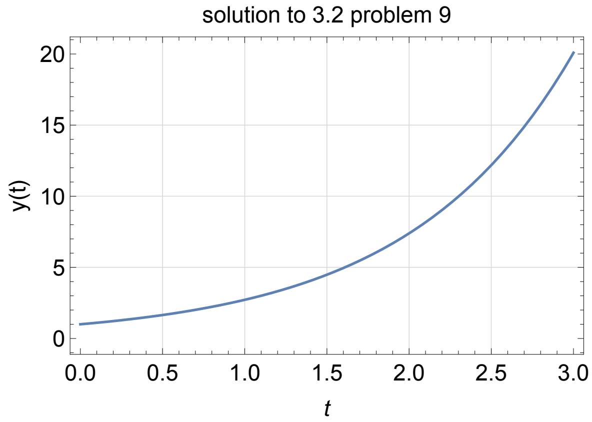
Hence $c_2 = \frac{1+2}{3} = 1$. Therefore $c_1 = 1 - 1 = 0$. Hence

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these back into the general solution gives

$$y(t) = e^t$$

Since the solution is exponential, it will grow in time and blows up. Here is sketch of the solution.



2.5.2 Section 3.1 problem 10

Find the solution to $y''+4y'+3y=0; y(0)=2, y'(0)=-1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y=e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} r^2+4r+3&=0\\ (r+3)(r+1)&=0 \end{aligned}$$

Hence the roots are $r_1=-3, r_2=-1$. Roots are real and distinct. The two solutions are

$$\begin{aligned} y_1&=e^{-3t}\\ y_2&=e^{-t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y&=c_1y_1+c_2y_2\\ &=c_1e^{-3t}+c_2e^{-t} \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0)=2$) to the general solution gives

$$2=c_1+c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t)=-3c_1e^{-3t}-c_2e^{-t}$. Applying second initial condition to this results in

$$-1=-3c_1-c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1=2-c_2$. Substituting this into (2) gives

$$\begin{aligned} -1&=-3(2-c_2)-c_2\\ &=-6+3c_2-c_2\\ &=-6+2c_2 \end{aligned}$$

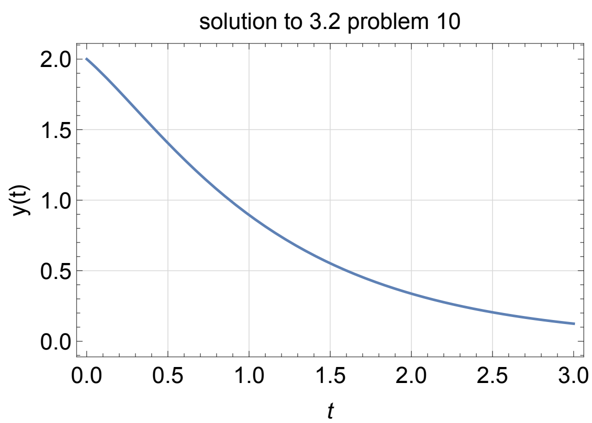
Hence $c_2 = \frac{-1+6}{2} = 2.5$. Therefore $c_1 = 2 - 2.5 = 0.5$. Hence

$$c_1 = 0.5$$
$$c_2 = 2.5$$

Substituting these back into the general solution gives

$$y(t) = 0.5e^{-3t} + 2.5e^{-t}$$

At t becomes large, both solutions decay to zero. So we expect the general solution to go to zero very fast. Here is a sketch.



2.5.3 Section 3.1 problem 11

Find the solution to $6y'' - 5y' + y = 0; y(0) = 4, y'(0) = 0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$6r^2 - 5r + 1 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 25 - (4)(6) = 1$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$\begin{aligned} r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{5}{12} \pm \frac{1}{12} \end{aligned}$$

Hence the roots are $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$. Roots are real and distinct. The two solutions are

$$y_1 = e^{\frac{1}{2}t}$$
$$y_2 = e^{\frac{1}{3}t}$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{3}t} \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 4$) to the general solution gives

$$4 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t) = \frac{1}{2}c_1 e^{\frac{1}{2}t} + \frac{1}{3}c_2 e^{\frac{1}{3}t}$. Applying second initial condition to this results in

$$0 = \frac{1}{2}c_1 + \frac{1}{3}c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1 = 4 - c_2$. Substituting this into (2) gives

$$\begin{aligned} 0 &= \frac{1}{2}(4 - c_2) + \frac{1}{3}c_2 \\ &= 2 - \frac{1}{2}c_2 + \frac{1}{3}c_2 \\ &= 2 - \frac{1}{6}c_2 \end{aligned}$$

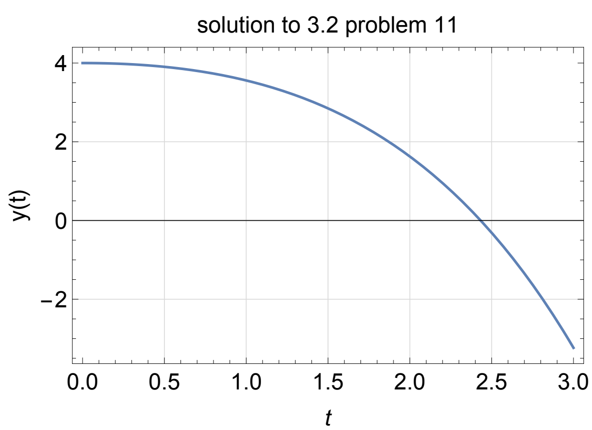
Hence $c_2 = 12$. Therefore $c_1 = 4 - 12 = -8$. Hence

$$\begin{aligned} c_1 &= -8 \\ c_2 &= 12 \end{aligned}$$

Substituting these back into the general solution gives

$$y(t) = -8e^{\frac{1}{2}t} + 12e^{\frac{1}{3}t}$$

Since $e^{\frac{1}{2}t}$ grows faster than $e^{\frac{1}{3}t}$ and since $e^{\frac{1}{2}t}$ has negative coefficient, then the solution will go to $-\infty$ as t increases. Here is sketch of the solution



2.5.4 Section 3.1 problem 12

Find the solution to $y'' + 3y' = 0; y(0) = -2, y'(0) = 3$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 + 3r = 0$$
$$r(r+3) = 0$$

Hence the roots are $r_1 = 0, r_2 = -3$. Roots are real and distinct. The two solutions are

$$y_1 = 1$$
$$y_2 = e^{-3t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 + c_2 e^{-3t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(0) = -2)$ to the general solution gives

$$-2 = c_1 + c_2 \tag{1}$$

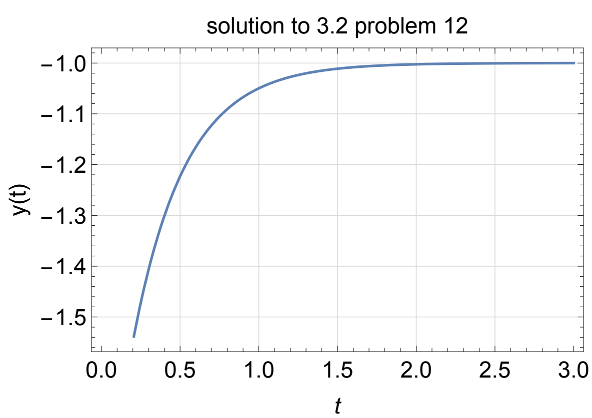
Taking time derivative of the general solution gives $y'(t) = -3c_2 e^{-3t}$. Applying second initial condition to this results in

$$3 = -3c_2 \tag{2}$$

Hence $c_2 = -1$. Therefore $c_1 = -1$. Substituting these back into the general solution gives

$$y(t) = -1 - e^{-3t}$$

As $t \to \infty$, the term $e^{-3t} \to 0$ and we are left with -1. Hence $\lim_{t-\infty} y(t) = -1$. Here is sketch of the solution



2.5.5 Section 3.1 problem 13

Find the solution to $y'' + 5y' + 3y = 0; y(0) = 1, y'(0) = 0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 + 5r + 3 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 25 - (4)(3) = 13$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$\begin{aligned} r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5}{2} \pm \frac{\sqrt{13}}{2} \end{aligned}$$

Hence the roots are $r_1 = \frac{-5}{2} + \frac{\sqrt{13}}{2}, r_2 = \frac{-5}{2} - \frac{\sqrt{13}}{2}$. The two solutions are

$$\begin{aligned} y_1 &= e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} \\ y_2 &= e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + c_2 e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 1$) to the general solution gives

$$1 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1 \left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right) e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + c_2 \left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right) e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t}$$

Applying second initial condition to this results in

$$0 = c_1 \left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right) + c_2 \left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right) \tag{2}$$

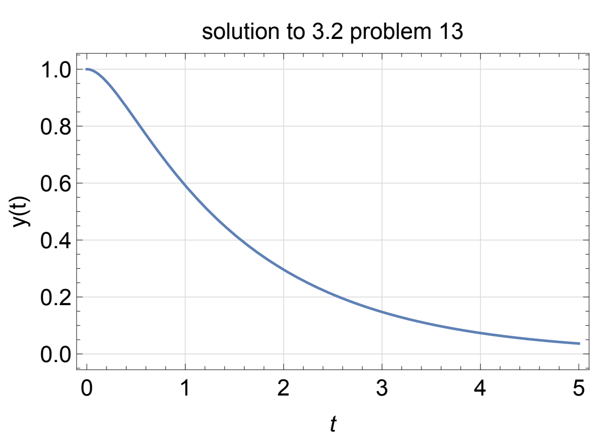
From (1), $c_1 = 1 - c_2$ and from (2)

$$\begin{aligned}
0 &= (1-c_2)\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)+c_2\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)\\
&= \frac{-5}{2}+\frac{\sqrt{13}}{2}+\frac{5}{2}c_2-\frac{\sqrt{13}}{2}c_2-\frac{5}{2}c_2-\frac{\sqrt{13}}{2}c_2\\
&= -\frac{5}{2}+\frac{\sqrt{13}}{2}-\sqrt{13}c_2\\
c_2 &= \frac{-5}{2\sqrt{13}}+\frac{1}{2}\\
&= \frac{-5+\sqrt{13}}{2\sqrt{13}}
\end{aligned}$$

Therefore $c_1 = 1 + \frac{5-\sqrt{13}}{2\sqrt{13}}$ and the solution becomes

$$\begin{aligned}
y(t) &= \left(1+\frac{5-\sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\left(\frac{-5+\sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\\
&= e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\frac{5-\sqrt{13}}{2\sqrt{13}}e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\left(\frac{-5+\sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\\
&= e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\frac{5\sqrt{13}-13}{26}e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\left(\frac{-5\sqrt{13}+13}{26}\right)e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\\
&= \frac{1}{26}\left(26e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\left(5\sqrt{13}-13\right)e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+\left(-5\sqrt{13}+13\right)e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\right)\\
&= \frac{1}{26}\left(26e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+5\sqrt{13}e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}-13e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}-5\sqrt{13}e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}+13e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\right)\\
&= \frac{1}{26}\left(13e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}+5\sqrt{13}e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t}-5\sqrt{13}e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}+13e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}\right)
\end{aligned}$$

Here is sketch of the solution showing that $y \to 0$ as $t \to \infty$



2.5.6 Section 3.1 problem 14

Find the solution to $2y'' + y' - 4y = 0; y(0) = 0, y'(0) = 1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$2r^2 + r - 4 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 1 - (4)(2)(-4) = 33$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$\begin{aligned} r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1}{4} \pm \frac{\sqrt{33}}{4} \end{aligned}$$

Hence the roots are $r_1 = \frac{1}{4} + \frac{\sqrt{33}}{4}, r_2 = \frac{1}{4} - \frac{\sqrt{33}}{4}$. The two solutions are

$$\begin{aligned} y_1 &= e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} \\ y_2 &= e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} + c_2 e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(0) = 0)$ to the general solution gives

$$0 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) e^{\left(\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) e^{\left(\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

Applying second initial condition to this results in

$$1 = c_1 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) \tag{2}$$

From (1), $c_1 = -c_2$ and from (2)

$$\begin{aligned} 1 &= -c_2\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)+c_2\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right) \\ &= \frac{1}{4}c_2-\frac{\sqrt{33}}{4}c_2-\frac{1}{4}c_2-\frac{\sqrt{33}}{4}c_2 \\ &= \frac{-\sqrt{33}}{2}c_2 \\ c_2 &= \frac{-2}{\sqrt{33}} \end{aligned}$$

Therefore $c_1 = \frac{2}{\sqrt{33}}$ and the solution becomes

$$y = \frac{2}{\sqrt{33}}e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} - \frac{2}{\sqrt{33}}e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

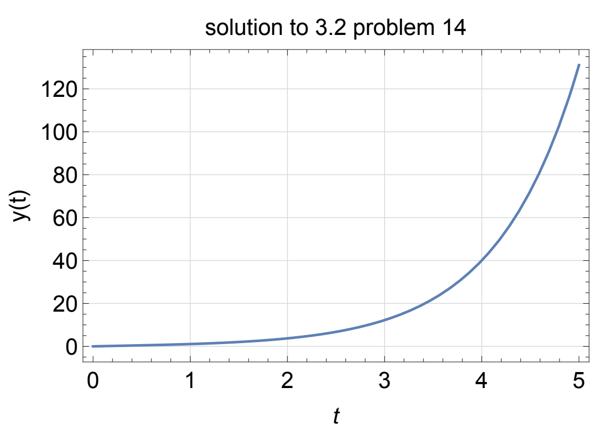
Since $-\frac{1}{4}+\frac{\sqrt{33}}{4} = 1.186$ and $-\frac{1}{4}-\frac{\sqrt{33}}{4} = -1.686$ then the above can be written as

$$y = \frac{2}{\sqrt{33}}e^{1.186t} - \frac{2}{\sqrt{33}}e^{-1.186t}$$

Then we see that as $t\to\infty$ the second term $e^{-1.186t}\to 0$ and we are left with $e^{1.186t}$ which will go to ∞ for large t. Hence

$$\lim_{t\to\infty} y(t) = \infty$$

Here is sketch of the solution



2.5.7 Section 3.1 problem 15

Find the solution to $y''+8y'-9y=0; y(1)=1, y'(1)=0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y=e^{rt}$ into the ODE and simplifying, giving

$$r^2 + 8r - 9 = 0$$
$$(r-1)(r+9) = 0$$

Hence the roots are $r_1 = 1, r_2 = -9$. The two solutions are

$$y_1 = e^t$$
$$y_2 = e^{-9t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^t + c_2 e^{-9t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(1) = 1)$ to the general solution gives

$$1 = c_1 e^1 + c_2 e^{-9} \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1 e^t - 9c_2 e^{-9t}$$

Applying second initial condition to this results in

$$0 = c_1 e^1 - 9c_2 e^{-9} \tag{2}$$

From (1), $c_1 = \frac{1 - c_2 e^{-9}}{e^1} = e^{-1} - c_2 e^{-10}$ and from (2)

$$\begin{aligned} 0 &= \left(e^{-1} - c_2 e^{-10}\right) e^1 - 9c_2 e^{-9} \\ &= 1 - c_2 e^{-9} - 9c_2 e^{-9} \\ &= 1 + c_2\left(-e^{-9} - 9e^{-9}\right) \\ 0 &= 1 + c_2\left(-10e^{-9}\right) \end{aligned}$$

Hence

$$c_2 = \frac{1}{10} e^9$$

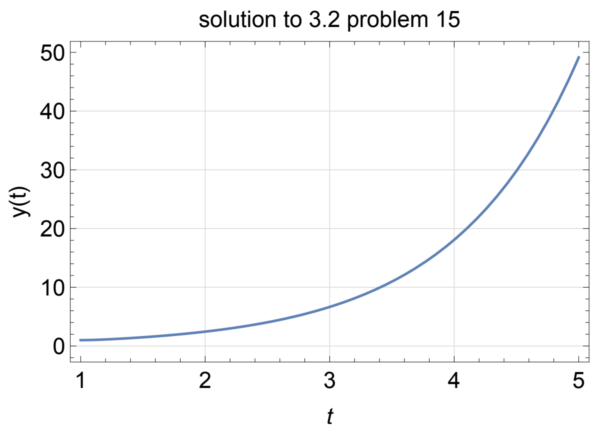
Therefore $c_1 = e^{-1} - c_2 e^{-10} = e^{-1} - \frac{1}{10} e^9 e^{-10} = e^{-1} - \frac{1}{10} e^{-1} = \frac{9}{10} e^{-1}$ and the solution becomes

$$\begin{aligned} y &= \frac{9}{10} e^{-1} e^t + \frac{1}{10} e^9 e^{-9t} \\ &= \frac{9}{10} e^{t-1} + \frac{1}{10} e^{9-9t} \end{aligned}$$

Then we see that as $t \to \infty$ the second term $e^{9-9t} \to 0$ and we are left with e^{t-1} which will go to ∞ for large t. Hence

$$\lim_{t\to\infty} y(t) = \infty$$

Here is sketch of the solution.



2.5.8 Section 3.1 problem 16

Find the solution to $4y'' - y = 0; y(-2) = 1, y'(-2) = -1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 - 1 = 0$$

Hence the roots are $r_1 = \pm\frac{1}{2}$. The two solutions are

$$y_1 = e^{\frac{1}{2}t}$$
$$y_2 = e^{-\frac{1}{2}t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(-2) = 1)$ to the general solution gives

$$1 = c_1 e^{-1} + c_2 e \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} - \frac{1}{2} c_2 e^{-\frac{1}{2}t}$$

Applying second initial condition to this results in

$$-1 = \frac{1}{2} c_1 e^{-1} - \frac{1}{2} c_2 e \tag{2}$$

From (1), $c_1 = \frac{1-c_2 e}{e^{-1}} = e - c_2 e^2$ and from (2)

$$\begin{aligned}
-1 &= \frac{1}{2}\left(e - c_2 e^2\right) e^{-1} - \frac{1}{2} c_2 e \\
&= \frac{1}{2} - \frac{1}{2} c_2 e - \frac{1}{2} c_2 e \\
&= \frac{1}{2} - c_2 e
\end{aligned}$$

Hence

$$c_2 = \frac{1}{2} e^{-1} + e^{-1} = \frac{3}{2} e^{-1}$$

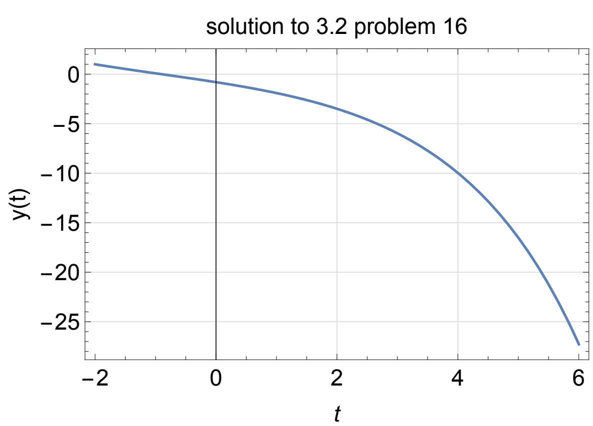
Therefore $c_1 = e - \left(\frac{3}{2} e^{-1}\right) e^2 = e - \frac{3}{2} e = -\frac{1}{2} e$ and the solution becomes

$$\begin{aligned}
y &= c_1 e^{\frac{1}{2} t} + c_2 e^{-\frac{1}{2} t} \\
&= -\frac{1}{2} e e^{\frac{1}{2} t} + \frac{3}{2} e^{-1} e^{\frac{-1}{2} t} \\
&= -\frac{1}{2} e^{1+\frac{t}{2}} + \frac{3}{2} e^{-1-\frac{t}{2}}
\end{aligned}$$

Then we see that as $t \to \infty$ the second term $e^{-1-\frac{t}{2}} \to 0$ and we are left with $-\frac{1}{2} e^{1+\frac{t}{2}}$ which will go to $-\infty$ for large t. Hence

$$\lim_{t\to\infty} y(t) = -\infty$$

Here is sketch of the solution.



2.5.9 Section 3.2 problem 1

Find the Wronskian of the given pair of functions $e^{2t}, e^{-\frac{3t}{2}}$

<u>solution</u>

We are given $y_1(t) = e^{2t}, y_2(t) = e^{\frac{-3}{2}t}$, hence by definition, the Wronskian is

$$
\begin{aligned}
W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\
&= \begin{vmatrix} e^{2t} & e^{-\frac{3t}{2}} \\ 2e^{2t} & -\frac{2}{3}e^{-\frac{3t}{2}} \end{vmatrix} \\
&= \frac{-3}{2}e^{\frac{t}{2}} - 2e^{\frac{t}{2}} \\
&= \frac{-7}{2}e^{\frac{t}{2}}
\end{aligned}
$$

2.5.10 Section 3.2 problem 2

Find the Wronskian of the given pair of functions $\cos t, \sin t$

solution

We are given $y_1(t) = \cos t, y_2(t) = \sin t$, hence by definition, the Wronskian is

$$
\begin{aligned}
W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\
&= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\
&= \cos^2 t + \sin^2 t \\
&= 1
\end{aligned}
$$

2.5.11 Section 3.2 problem 3

Find the Wronskian of the given pair of functions e^{-2t}, te^{-2t}

solution

We are given $y_1(t) = e^{-2t}, y_2(t) = te^{-2t}$, hence by definition, the Wronskian is

$$
\begin{aligned}
W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\
&= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} \\
&= \left(e^{-2t}\right)\left(e^{-2t} - 2te^{-2t}\right) + 2e^{-2t}te^{-2t} \\
&= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\
&= e^{-4t}
\end{aligned}
$$

2.5.12 Section 3.2 problem 4

Find the Wronskian of the given pair of functions x, xe^x

solution

We are given $y_1(x) = x, y_2(x) = xe^x$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= (x)(e^x + xe^x) - xe^x \\ &= xe^x + x^2 e^x - xe^x \\ &= x^2 e^x \end{aligned}$$

2.5.13 Section 3.2 problem 5

Find the Wronskian of the given pair of functions $e^t \sin t, e^t \cos t$

solution

We are given $y_1(t) = e^t \sin t, y_2(t) = e^t \cos t$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} \\ &= \left(e^t \sin t\right)\left(e^t \cos t - e^t \sin t\right) - e^t \cos t \left(e^t \sin t + e^t \cos t\right) \\ &= e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \cos t \sin t - e^{2t} \cos^2 t \\ &= -e^{2t} \sin^2 t - e^{2t} \cos^2 t \\ &= -2e^{2t}\left(\sin^2 t + \cos^2 t\right) \\ &= -2e^{2t} \end{aligned}$$

2.5.14 Section 3.2 problem 6

Find the Wronskian of the given pair of functions $\cos^2 \theta, 1 + \cos 2\theta$

solution

We are given $y_1(\theta) = \cos^2\theta, y_2(\theta) = 1 + \cos 2\theta$, hence by definition, the Wronskian is

$$
\begin{aligned}
W &= \begin{vmatrix} y_1(\theta) & y_2(\theta) \\ y_1'(\theta) & y_2'(\theta) \end{vmatrix} \\
&= \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ -2\cos\theta\sin\theta & -2\sin 2\theta \end{vmatrix} \\
&= -2\cos^2\theta\sin 2\theta - (1+\cos 2\theta)(-2\cos\theta\sin\theta) \\
&= -2\cos^2\theta\sin 2\theta - (-2\cos\theta\sin\theta - 2\cos\theta\sin\theta\cos 2\theta) \\
&= -2\cos^2\theta\sin 2\theta + 2\cos\theta\sin\theta + 2\cos\theta\sin\theta\cos 2\theta
\end{aligned}
$$

Using $\cos 2\theta = 2\cos^2\theta - 1$ And $\sin 2\theta = 2\sin\theta\cos\theta$ the above becomes

$$
\begin{aligned}
W &= -2\cos^2\theta(2\sin\theta\cos\theta) + 2\cos\theta\sin\theta + 2\cos\theta\sin\theta\left(2\cos^2\theta - 1\right) \\
&= -4\cos^3\theta\sin\theta + 2\cos\theta\sin\theta + 4\cos^3\theta\sin\theta - 2\cos\theta\sin\theta \\
&= -4\cos^3\theta\sin\theta + 4\cos^3\theta\sin\theta \\
&= 0
\end{aligned}
$$

We could also see that $W = 0$ more directly, by noticing that $y_1 = \cos^2\theta = 1 - \sin^2\theta$ and since $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ then

$$
\begin{aligned}
y_1 &= \cos^2\theta \\
&= 1 - \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) \\
&= \frac{1}{2} + \frac{1}{2}\cos 2\theta \\
&= \frac{1}{2}(1 + \cos 2\theta)
\end{aligned}
$$

Therefore, $y_1 = \frac{1}{2}y_2$. Hence y_2 is just a scaled version of y_1 and so these are two solutions are not linearly independent functions, (parallel to each others in vector space view) and so we expect that the Wronskian to be zero.

2.6 HW6

2.6.1 Section 3.4 problem 1

Find the general solution of $y'' - 2y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)(r-1) &= 0 \end{aligned}$$

Hence $r = 1$ double root. Therefore the two solutions are

$$\begin{aligned} y_1 &= e^t \\ y_2 &= te^t \end{aligned}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^t + c_2 t e^t$$

2.6.2 Section 3.4 problem 2

Find the general solution of $9y'' + 6y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} 9r^2 + 6r + 1 &= 0 \\ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-6 + \sqrt{36 - 36}}{18} = -\frac{1}{3} \end{aligned}$$

Hence $r = -\frac{1}{3}$ double root. Therefore the two solutions are

$$\begin{aligned} y_1 &= e^{\frac{-1}{3}t} \\ y_2 &= te^{\frac{-1}{3}t} \end{aligned}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{-1}{3}t} + c_2 t e^{\frac{-1}{3}t}$$

2.6.3 Section 3.4 problem 3

Find the general solution of $4y'' - 4y' - 3y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 - 4r - 3 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 + 48}}{8} = \frac{4 \pm 8}{8} = \frac{1 \pm 2}{2} = \frac{1}{2} \pm 1$$

Hence $r_1 = \frac{3}{2}, r_2 = -\frac{1}{2}$. Therefore the two solutions are

$$y_1 = e^{\frac{3}{2}t}$$
$$y_2 = e^{-\frac{1}{2}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{1}{2}t}$$

2.6.4 Section 3.4 problem 4

Find the general solution of $4y'' + 12y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 + 12r + 9 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{144 - 144}}{8} = \frac{-3}{2}$$

Hence $r = \frac{-3}{2}$ double root. Therefore the two solutions are

$$y_1 = e^{\frac{-3}{2}t}$$
$$y_2 = te^{\frac{-3}{2}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{-3}{2}t} + c_2 t e^{\frac{-3}{2}t}$$

2.6.5 Section 3.4 problem 5

Find the general solution of $y'' - 2y' + 10y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 - 2r + 10 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

Hence $r_1 = 1 + 3i, r_2 = 1 - 3i$. Therefore the two solutions are

$$y_1 = e^{(1+3i)t} = e^t e^{i3t}$$
$$y_2 = e^t e^{i3t}$$

And the general solution is linear combination of the above solutions, the complex exponential can be converted to trig functions $\cos, \sin$ using the standard Euler identities, resulting in

$$y = e^t \left(c_1 \cos 3t + c_2 \sin 3t\right)$$

2.6.6 Section 3.4 problem 6

Find the general solution of $y'' - 6y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 - 6r + 9 = 0$$
$$(r - 3)^2 = 0$$

Hence $r = 3$. Double root. Therefore the two solutions are

$$y_1 = e^{3t}$$
$$y_2 = te^{3t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{3t} + c_2 te^{3t}$$

2.6.7 Section 3.4 problem 7

Find the general solution of $4y'' + 17y' + 4y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 + 17r + 4 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-17 \pm \sqrt{289 - 64}}{8} = \frac{-17 \pm \sqrt{225}}{8} = \frac{-17 \pm 15}{8}$$

Hence $r_1 = \frac{-17-15}{8} = -4, r_2 = \frac{-17+15}{8} = -\frac{1}{4}$. Therefore the two solutions are

$$y_1 = e^{-4t}$$
$$y_2 = e^{-\frac{1}{4}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{-4t} + c_2 e^{-\frac{1}{4}t}$$

2.6.8 Section 3.4 problem 8

Find the general solution of $16y'' + 24y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$16r^2 + 24r + 9 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-24 \pm \sqrt{576 - 4(16)(9)}}{32} = \frac{-24}{32} = -\frac{3}{4}$$

Hence $r = -\frac{3}{4}$. Double root. Therefore the two solutions are

$$y_1 = e^{-\frac{3}{4}t}$$
$$y_2 = te^{-\frac{3}{4}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{-\frac{3}{4}t} + c_2 te^{-\frac{3}{4}t}$$

2.6.9 Section 3.4 problem 9

Find the general solution of $25y'' - 20y' + 4y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$25r^2 - 20r + 4 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{20 \pm \sqrt{400 - 4(25)(4)}}{50} = \frac{20}{50} = \frac{2}{5}$$

Hence $r = \frac{2}{5}$. Double root. Therefore the two solutions are

$$y_1 = e^{\frac{2}{5}t}$$
$$y_2 = te^{\frac{2}{5}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{2}{5}t} + c_2 t e^{\frac{2}{5}t}$$

2.6.10 Section 3.4 problem 10

Find the general solution of $2y'' + 2y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$2r^2 + 2r + 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{4} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = \frac{-1}{2} \pm \frac{i}{2}$$

Hence $r_1 = \frac{-1}{2} + \frac{i}{2}, r_2 = \frac{-1}{2} - \frac{i}{2}$. Therefore the two solutions are

$$y_1 = e^{\left(\frac{-1}{2} - \frac{i}{2}\right)t} = e^{\frac{-1}{2}t} e^{\frac{-i}{2}t}$$

$$y_2 = e^{\left(\frac{-1}{2} + \frac{i}{2}\right)t} = e^{\frac{-1}{2}t} e^{\frac{i}{2}t}$$

And the general solution is linear combination of the above solutions, the complex exponential can be converted to trig functions $\cos, \sin$ using the standard Euler identities, resulting in

$$y = e^{\frac{-1}{2}t}\left(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}\right)$$

2.6.11 Section 3.5 problem 1

Find the general solution of $y'' - 2y' - 3y = 3e^{2t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' - 2y' - 3y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$r^2 - 2r - 3 = 0$$

$$(r+1)(r-3) = 0$$

Hence $r_1 = -1, r_2 = 3$. Therefore the two solution are

$$y_1 = e^{-t}$$
$$y_2 = e^{3t}$$

And the homogeneous solution is linear combination of the above solutions

$$y_h = c_1 e^{-t} + c_2 e^{3t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' - 2y' - 3y = 3e^{2t}$$

We guess $y_p = Ae^{2t}$. Hence

$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

Substituting this into the original ODE in order to solve for A gives

$$4Ae^{2t} - 2\left(2Ae^{2t}\right) - 3\left(Ae^{2t}\right) = 3e^{2t}$$
$$-3Ae^{2t} = 3e^{2t}$$

Hence $A = -1$ and therefore

$$y_p = -e^{2t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{3t} - e^{2t} \end{aligned}$$

2.6.12 Section 3.5 problem 2

Find the general solution of $y'' + 2y' + 5y = 3\sin 2t$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + 2y' + 5y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

From before, we know the solution is of the form

$$y_h = e^{-t}\left(c_1 \cos 2t + \sin 2t\right)$$

Where

$$y_1 = e^{-t} \cos 2t$$
$$y_2 = e^{-t} \sin 2t$$

Finding y_p

Now we need to find one particular solution to

$$y'' + 2y' + 5y = 3 \sin 2t$$

We guess $y_p = A \cos 2t + B \sin 2t$ hence

$$y_p' = -2A \sin 2t + 2B \cos 2t$$
$$y_p'' = -4A \cos 2t - 4B \sin 2t$$

Substituting these back into the original ODE in order to solve for A, B gives

$$\begin{aligned} y_p'' + 2y_p' + 5y_p &= 3 \sin 2t \\ -4A \cos 2t - 4B \sin 2t + 2\left(-2A \sin 2t + 2B \cos 2t\right) + 5\left(A \cos 2t + B \sin 2t\right) &= 3 \sin 2t \\ -4A \cos 2t - 4B \sin 2t - 4A \sin 2t + 4B \cos 2t + 5A \cos 2t + 5B \sin 2t &= 3 \sin 2t \\ (A + 4B) \cos 2t + (B - 4A) \sin 2t &= 3 \sin 2t \end{aligned}$$

Hence

$$A + 4B = 0$$
$$B - 4A = 3$$

From first equation, $A = -4B$, and the second equation becomes $B - 4(-4B) = 3$ or $B + 16B = 3$ or $B = \frac{3}{17}$, hence $A = \frac{-12}{17}$, therefore

$$y_p = \frac{-12}{17} \cos 2t + \frac{3}{17} \sin 2t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= e^{-t}\left(c_1 \cos 2t + \sin 2t\right) - \frac{12}{17} \cos 2t + \frac{3}{17} \sin 2t \end{aligned}$$

2.6.13 Section 3.5 problem 3

Find the general solution of $y'' - y' - 2y = -2t + 4t^2$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' - y' - 2y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 - r - 2 &= 0 \\ (r+1)(r-2) &= 0 \end{aligned}$$

Hence $r_1 = -1, r_2 = 2$ and therefore

$$y_h = c_1 e^{-t} + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' - y' - 2y = -2t + 4t^2$$

We guess $y_p = A_0 + A_1 t + A_2 t^2$. Therefore

$$\begin{aligned} y_p' &= A_1 + 2A_2 t \\ y_p'' &= 2A_2 \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} 2A_2 - (A_1 + 2A_2 t) - 2\left(A_0 + A_1 t + A_2 t^2\right) &= -2t + 4t^2 \\ t^0 (2A_2 - A_1 - 2A_0) + t(-2A_2 - 2A_1) + t^2(-2A_2) &= -2t + 4t^2 \end{aligned}$$

Hence

$$\begin{aligned} 2A_2 - A_1 - 2A_0 &= 0 \\ -2A_2 - 2A_1 &= -2 \\ -2A_2 &= 4 \end{aligned}$$

From the last equation, $A_2 = -2$, and from the second equation $A_1 = \frac{-2+2(-2)}{-2} = 3$ and from the first equation $2(-2) - 3 - 2A_0 = 0$ hence $A_0 = \frac{4+3}{-2} = -\frac{7}{2}$, Therefore

$$\begin{aligned} y_p &= A_0 + A_1 t + A_2 t^2 \\ &= -\frac{7}{2} + 3t - 2t^2 \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{7}{2} + 3t - 2t^2 \end{aligned}$$

2.6.14 Section 3.5 problem 4

Find the general solution of $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + y' - 6y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 + r - 6 &= 0 \\ (r+3)(r-2) &= 0 \end{aligned}$$

Hence $r_1 = -3, r_2 = 2$ and therefore

$$y_h = c_1 e^{-3t} + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$$

We guess $y_p = Ae^{3t} + Be^{-2t}$. Therefore

$$\begin{aligned} y_p' &= 3Ae^{3t} - 2Be^{-2t} \\ y_p'' &= 9Ae^{3t} + 4Be^{-2t} \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} y_p'' + y_p' - 6y_p &= 12e^{3t} + 12e^{-2t} \\ 9Ae^{3t} + 4Be^{-2t} + 3Ae^{3t} - 2Be^{-2t} - 6\left(Ae^{3t} + Be^{-2t}\right) &= 12e^{3t} + 12e^{-2t} \\ e^{3t}(9A + 3A - 6A) + e^{-2t}(4B - 2B - 6B) &= 12e^{3t} + 12e^{-2t} \\ 6Ae^{3t} - 4Be^{-2t} &= 12e^{3t} + 12e^{-2t} \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} A &= 2 \\ B &= -3 \end{aligned}$$

Hence

$$y_p = 2e^{3t} - 3e^{-2t}$$

And the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{-3t} + c_2 e^{2t} + 2e^{3t} - 3e^{-2t}
\end{aligned}
$$

2.6.15 Section 3.5 problem 5

Find the general solution of $y'' - 2y' - 3y = -3te^{-t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$
y'' - 2y' - 3y = 0
$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$
\begin{aligned}
r^2 - 2r - 3 &= 0 \\
(r-3)(r+1) &= 0
\end{aligned}
$$

Hence $r_1 = 3, r_2 = -1$ and therefore

$$
y_h = c_1 e^{3t} + c_2 e^{-t}
$$

Finding y_p

Now we need to find one particular solution to

$$
y'' - 2y' - 3y = -3te^{-t}
$$

Guess for t is $A_0 + B_0 t$ and the guess for e^{-t} is Cte^{-t} (where we multiplied by t since e^{-t} shows up in the homogenous solution. Therefore the product is

$$
\begin{aligned}
y_p &= (A_0 + B_0 t)\, Cte^{-t} \\
&= A_0 Cte^{-t} + CB_0 t^2 e^{-t}
\end{aligned}
$$

Let $A_0 C = A, CB_0 = B$, and the above becomes

$$
\begin{aligned}
y_p &= Ate^{-t} + Bt^2 e^{-t} \\
&= (A + Bt)\, te^{-t}
\end{aligned}
$$

Substituting these back into the ODE and solving for A, B gives $B = \frac{3}{8}$ and $A = \frac{3}{16}$, hence

$$
\begin{aligned}
y_p &= \left(At + Bt^2\right) e^{-t} \\
&= \left(\frac{3}{16} t + \frac{3}{8} t^2\right) e^{-t}
\end{aligned}
$$

And the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{16}t + \frac{3}{8}t^2\right) e^{-t} \end{aligned}$$

2.6.16 Section 3.5 problem 6

Find the general solution of $y'' + 2y' = 3 + 4\sin 2t$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + 2y' = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 + 2r &= 0 \\ r(r+2) &= 0 \end{aligned}$$

Hence $r_1 = 0, r_2 = -2$ and therefore

$$y_h = c_1 + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' + 2y' = 3 + 4\sin 2t$$

Guess that $y_p = At + B\cos 2t + C\sin 2t$, hence

$$\begin{aligned} y_p' &= A - 2B\sin 2t + 2C\cos 2t \\ y_p'' &= -4B\cos 2t - 4C\sin 2t \end{aligned}$$

Substituting back into

$$\begin{aligned} y_p'' + 2y_p' &= 3 + 4\sin 2t \\ -4B\cos 2t - 4C\sin 2t + 2(A - 2B\sin 2t + 2C\cos 2t) &= 3 + 4\sin 2t \\ -4B\cos 2t - 4C\sin 2t + 2A - 4B\sin 2t + 4C\cos 2t &= 3 + 4\sin 2t \\ (-4B + 4C)\cos 2t + 2A + (-4C - 4B)\sin 2t &= 3 + 4\sin 2t \end{aligned}$$

Hence

$$\begin{aligned} 2A &= 3 \\ 1 &= -C - B \\ 0 &= -B + C \end{aligned}$$

From first equation, $A = \frac{3}{2}$, From third equation, $B = C$ and from the second equation $1 = -2B$ or $B = \frac{-1}{2}$, hence $C = \frac{-1}{2}$, and the particular solution is

$$y_p = \frac{3}{2}t + \frac{-1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

Hence the complete solution is

$$y = c_1 + c_2 e^{2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

2.7 HW7

2.7.1 Section 3.6 problem 1

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' - 5y' + 6y = 2e^t$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - 5y' + 6y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 - 5r + 6 = 0$ or $(r-3)(r-2) = 0$. Therefore the roots are $r_1 = 3, r_2 = 2$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= e^{3t} \\ y_2 &= e^{2t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{3t} + c_2 e^{2t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = 2e^{5t} - 3e^{5t} = -e^{5t}$$

Letting $g(t) = 2e^t$ therefore the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W} dt = -\int \frac{e^{2t} 2e^t}{-e^{5t}} dt = 2\int \frac{e^{3t}}{e^{5t}} dt = 2\int e^{-2t} dt = 2\left[\frac{e^{-2t}}{-2}\right] = -e^{-2t}$$

And

$$u_2(t) = \int \frac{y_1(t) g(t)}{W} dt = \int \frac{e^{3t} 2e^t}{-e^{5t}} dt = -2\int \frac{e^{4t}}{e^{5t}} dt = -2\int e^{-t} dt = -2\left[\frac{e^{-t}}{-1}\right] = 2e^{-t}$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-e^{-2t}\right) e^{3t} + 2e^{-t} e^{2t} \\ &= -e^t + 2e^t \\ &= e^t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{3t} + c_2 e^{2t} + e^t \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t)$ in the problem, particular solution is assumed to be

$$y_p = Ae^t$$

Hence

$$\begin{aligned} y_p' &= Ae^t \\ y_p'' &= Ae^t \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' - 5y_p' + 6y_p &= 2e^t \\ Ae^t - 5Ae^t + 6Ae^t &= 2e^t \end{aligned}$$

Dividing by $e^t \neq 0$ gives

$$\begin{aligned} A - 5A + 6A &= 2 \\ 2A &= 2 \\ A &= 1 \end{aligned}$$

Therefore

$$y_p = e^t$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

2.7.2 Section 3.6 problem 2

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' - y' - 2y = 2e^{-t}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - y' - 2y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to

compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 - r - 2 = 0$ or $(r+1)(r-2) = 0$. Therefore the roots are $r_1 = -1, r_2 = 2$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= e^{-t} \\ y_2 &= e^{2t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-t} + c_2 e^{2t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 2e^t + e^t = 3e^t$$

Letting $g(t) = 2e^{-t}$ therefore the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W} dt = -\int \frac{e^{2t} 2e^{-t}}{3e^t} dt = -\frac{2}{3}\int \frac{e^t}{e^t} dt = -\frac{2}{3} t$$

And

$$u_2(t) = \int \frac{y_1(t) g(t)}{W} dt = \int \frac{e^{-t} 2e^{-t}}{3e^t} dt = \frac{2}{3}\int \frac{e^{-2t}}{e^t} dt = \frac{2}{3}\int e^{-3t} dt = \frac{2}{3}\left[\frac{e^{-3t}}{-3}\right] = -\frac{2}{9} e^{-3t}$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-\frac{2}{3} t\right) e^{-t} - \frac{2}{9} e^{-3t} e^{2t} \\ &= -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t} \end{aligned}$$

We notice something here. The extra term $-\frac{2}{9} e^{-t}$ above is constant times one of the fundamental solutions (one of the solutions to the homogenous equation), which is y_1 in this case found earlier. But adding a multiple of a fundamental solution to a particular solution gives another particular solution. So the term $-\frac{2}{9} e^{-t}$ will be merged with the term from the homogenous solution. Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t} \end{aligned}$$

We can now combine $\frac{2}{9}e^{-t}$ that shows up from the particular solution with the c_1e^{-t} term from the homogenous solution, since c_1 is arbitrary constant, which simplifies the above to

$$\begin{aligned} y &= y_h + y_p \\ &= c_1e^{-t} + c_2e^{2t} - \frac{2}{3}te^{-t} \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t)$ in the problem, and since e^{-t} is already one of the fundamental solutions, then particular solution is assumed to be

$$y_p = Ate^{-t}$$

Hence

$$\begin{aligned} y_p' &= A\left(e^{-t} - te^{-t}\right) \\ y_p'' &= A\left(-e^{-t} - e^{-t} + te^{-t}\right) \\ &= A\left(-2e^{-t} + te^{-t}\right) \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' - y_p' - 2y_p &= 2e^{-t} \\ A\left(-2e^{-t} + te^{-t}\right) - A\left(e^{-t} - te^{-t}\right) - 2Ate^{-t} &= 2e^{-t} \end{aligned}$$

Dividing by $e^{-t} \neq 0$ gives

$$\begin{aligned} A(-2+t) - A(1-t) - 2At &= 2 \\ t(A + A - 2A) - 2A - A &= 2 \\ -3A &= 2 \\ A &= \frac{-2}{3} \end{aligned}$$

Therefore

$$y_p = \frac{-2}{3}te^{-t}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

2.7.3 Section 3.6 problem 3

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' + 2y' + y = 3e^{-t}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 2y' + y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to

compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 2r + 1 = 0$ or $(r+1)(r+1) = 0$, Therefore the roots are duplicate $r_1 = -1$. Hence the two fundamental solutions are

$$
\begin{aligned}
y_1 &= e^{-t} \\
y_2 &= te^{-t}
\end{aligned}
$$

And the homogenous solution is therefore given by

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 e^{-t} + c_2 te^{-t}
\end{aligned}
$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$
\begin{aligned}
W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} \\
&= \left(e^{-t}\right)\left(e^{-t} - te^{-t}\right) + \left(te^{-t}\right)\left(e^{-t}\right) \\
&= e^{-2t} - te^{-2t} + te^{-2t} \\
&= e^{-2t}
\end{aligned}
$$

Letting $g(t) = 3e^{-t}$ therefore the particular solution is

$$
y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)
$$

Where

$$
u_1(t) = -\int \frac{y_2(t) g(t)}{W} dt = -\int \frac{te^{-t}\left(3e^{-t}\right)}{e^{-2t}} dt = -3\int t dt = -\frac{3}{2}t^2
$$

And

$$
u_2(t) = \int \frac{y_1(t) g(t)}{W} dt = \int \frac{e^{-t}\left(3e^{-t}\right)}{e^{-2t}} dt = 3\int dt = 3t
$$

Hence the particular solution becomes

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \left(-\frac{3}{2}t^2\right) e^{-t} + 3t\left(te^{-t}\right) \\
&= -\frac{3}{2}t^2 e^{-t} + 3t^2 e^{-t} \\
&= \frac{3}{2}t^2 e^{-t}
\end{aligned}
$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 t e^{-t} + \frac{3}{2} t^2 e^{-t} \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t) = 3e^{-t}$ in the problem, we want to try e^{-t} but since e^{-t} is already one of the fundamental solutions, we then look at te^{-t} but this is also one fundamental solutions, then we look for $t^2 e^{-t}$. Hence

$$y_p = A t^2 e^{-t}$$

Hence

$$\begin{aligned} y_p' &= A\left(2te^{-t} - t^2 e^{-t}\right) \\ y_p'' &= A\left(2e^{-t} - 2te^{-t} - \left(2te^{-t} - t^2 e^{-t}\right)\right) \\ &= A\left(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2 e^{-t}\right) \\ &= A\left(2e^{-t} - 4te^{-t} + t^2 e^{-t}\right) \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' + 2y_p' + y_p &= 3e^{-t} \\ A\left(2e^{-t} - 4te^{-t} + t^2 e^{-t}\right) + 2A\left(2te^{-t} - t^2 e^{-t}\right) + At^2 e^{-t} &= 3e^{-t} \end{aligned}$$

Dividing by $e^{-t} \neq 0$ gives

$$\begin{aligned} A\left(2 - 4t + t^2\right) + 2A\left(2t - t^2\right) + At^2 &= 3 \\ t(-4A + 4A) + t^2(A - 2A + A) + 2A &= 3 \\ A &= \frac{3}{2} \end{aligned}$$

Therefore

$$y_p = \frac{3}{2} t e^{-t}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

2.7.4 Section 3.6 problem 4

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $4y'' - 4y' + y = 16e^{\frac{t}{2}}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $4y'' - 4y' + y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to

compare with.

Finding y_h

The first step is to put the ODE in standard form, with the coefficient of y'' being one. Hence it becomes

$$y''-y'+\frac{1}{4}y=4e^{\frac{t}{2}}$$

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2-r+\frac{1}{4}=0$ or $\left(r-\frac{1}{2}\right)\left(r-\frac{1}{2}\right)=0$, Therefore the roots are duplicate $r=\frac{1}{2}$. Hence the two fundamental solutions are

$$\begin{aligned}y_1&=e^{\frac{1}{2}t}\\y_2&=te^{\frac{1}{2}t}\end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned}y_h&=c_1y_1+c_2y_2\\&=c_1e^{\frac{1}{2}t}+c_2te^{\frac{1}{2}t}\end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$\begin{aligned}W(t)&=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}=\begin{vmatrix}e^{\frac{1}{2}t}&te^{\frac{1}{2}t}\\\frac{1}{2}e^{\frac{1}{2}t}&e^{\frac{1}{2}t}+\frac{1}{2}te^{\frac{1}{2}t}\end{vmatrix}\\&=\left(e^{\frac{1}{2}t}\right)\left(e^{\frac{1}{2}t}+\frac{1}{2}te^{\frac{1}{2}t}\right)-\left(te^{\frac{1}{2}t}\right)\left(\frac{1}{2}e^{\frac{1}{2}t}\right)\\&=e^t+\frac{1}{2}te^t-\frac{1}{2}te^t\\&=e^t\end{aligned}$$

Letting $g(t)=4e^{\frac{t}{2}}$ therefore the particular solution is

$$y_p(t)=u_1(t)\,y_1(t)+u_2(t)\,y_2(t)$$

Where

$$u_1(t)=-\int\frac{y_2(t)\,g(t)}{W}dt=-\int\frac{te^{\frac{1}{2}t}\left(4e^{\frac{t}{2}}\right)}{e^t}dt=-4\int t\,dt=-2t^2$$

And

$$u_2(t)=\int\frac{y_1(t)\,g(t)}{W}dt=\int\frac{e^{\frac{1}{2}t}\left(4e^{\frac{t}{2}}\right)}{e^t}dt=4\int dt=4t$$

Hence the particular solution becomes

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \left(-2t^2\right) e^{\frac{1}{2}t} + 4t\left(te^{\frac{1}{2}t}\right) \\
&= -2t^2 e^{\frac{1}{2}t} + 4t^2 e^{\frac{1}{2}t} \\
&= 2t^2 e^{\frac{1}{2}t}
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t} + 2t^2 e^{\frac{1}{2}t}
\end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t) = 4e^{\frac{t}{2}}$ in the problem, we want to try $e^{\frac{t}{2}}$ but since $e^{\frac{t}{2}}$ is already one of the fundamental solutions, we then look at $te^{\frac{t}{2}}$ but this is also one fundamental solutions, then we look for $t^2 e^{\frac{t}{2}}$. Hence

$$y_p = At^2 e^{\frac{t}{2}}$$

Hence

$$\begin{aligned}
y_p' &= A\left(2te^{\frac{t}{2}} + \frac{1}{2}t^2 e^{\frac{t}{2}}\right) \\
y_p'' &= A\left(2e^{\frac{t}{2}} + te^{\frac{t}{2}} + te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right) \\
&= A\left(2e^{\frac{t}{2}} + 2te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right)
\end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned}
y_p'' - y_p' + \frac{1}{4}y_p &= 4e^{\frac{t}{2}} \\
A\left(2e^{\frac{t}{2}} + 2te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right) - A\left(2te^{\frac{t}{2}} + \frac{1}{2}t^2 e^{\frac{t}{2}}\right) + \frac{1}{4}At^2 e^{\frac{t}{2}} &= 4e^{\frac{t}{2}}
\end{aligned}$$

Dividing by $e^{\frac{t}{2}} \neq 0$ gives

$$\begin{aligned}
A\left(2 + 2t + \frac{1}{4}t^2\right) - A\left(2t + \frac{1}{2}t^2\right) + \frac{1}{4}At^2 &= 4 \\
t(2A - 2A) + t^2\left(\frac{1}{4}A - \frac{1}{2}A + \frac{1}{4}A\right) + 2A &= 4 \\
A &= 2
\end{aligned}$$

Therefore

$$y_p = 2t^2 e^{\frac{t}{2}}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

2.7.5 Section 3.6 problem 5

Find the general solution of $y'' + y = \tan t$ for $0 < t < \frac{\pi}{2}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 1 = 0$ or $r = \pm i$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= \cos t \\ y_2 &= \sin t \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos t + c_2 \sin t \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

Let $g(t) = \tan t$, therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \frac{\sin t \tan t}{1} dt = -\int \sin t \frac{\sin t}{\cos t} dt = -\int \frac{\sin^2 t}{\cos t} dt \\ &= -\int \frac{1 - \cos^2 t}{\cos t} dt = \int \frac{\cos^2 t - 1}{\cos t} dt = \int \cos t - \frac{1}{\cos t} dt \\ &= \int \cos t dt - \int \frac{1}{\cos t} dt \\ &= \sin t - \int \sec t dt \\ &= \sin t - \ln(\sec(t) + \tan(t)) \end{aligned}$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)} dt = \int \frac{\cos t \tan t}{1} dt = \int \cos t \frac{\sin t}{\cos t}\, dt = \int \sin t\, dt = -\cos t$$

Hence the particular solution becomes

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= (\sin t - \ln(\sec(t) + \tan(t)))\cos t + (-\cos t)\sin t \\
&= -\cos(t)\ln(\sec(t) + \tan(t))
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1 \cos t + c_2 \sin t - \cos(t)\ln(\sec(t) + \tan(t))
\end{aligned}$$

2.7.6 Section 3.6 problem 6

Find the general solution of $y'' + 9y = 9\sec^2 3t$ for $0 < t < \frac{\pi}{6}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 9y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 9 = 0$ or $r = \pm 3i$. Hence the two fundamental solutions are

$$\begin{aligned}
y_1 &= \cos 3t \\
y_2 &= \sin 3t
\end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 \cos 3t + c_2 \sin 3t
\end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 t + 3\sin^2 t = 3$$

Let $g(t) = \frac{9}{\cos^2 3t}$, therefore the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W(t)} dt = -\int \frac{9\sin(3t)}{3\cos^2(3t)} dt = -3\int \frac{\sin(3t)}{\cos^2(3t)} dt$$

Let $u = \cos(3t)$, hence $\frac{du}{dt} = -3\sin 3t \to dt = \frac{du}{-3\sin 3t}$ and the above integral becomes

$$u_1(t) = -3\int \frac{\sin(3t)}{u^2}\frac{du}{-3\sin 3t} = \int \frac{1}{u^2}du = \frac{-1}{u} = \frac{-1}{\cos 3t} = -\sec(3t)$$

And

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}dt = \int \frac{9\cos 3t}{3\cos^2(3t)}dt = 3\int \frac{1}{\cos(3t)}\,dt = 3\int \sec(3t)\;dt = \ln(\sec(3t) + \tan(3t))$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= -\sec(3t)\cos 3t + \ln(\sec(3t) + \tan(3t))\sin 3t \\ &= -1 + \ln(\sec(t) + \tan(t))\sin 3t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\cos 3t + c_2\sin 3t - 1 + \sin 3t\ln(\sec(t) + \tan(t)) \end{aligned}$$

2.7.7 Section 3.6 problem 7

Find the general solution of $y'' + 4y' + 4y = t^{-2}e^{-2t}$ for $t > 0$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 4y' + 4y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 4r + 4 = 0$ or $(r+2)(r+2) = 0$. Hence double root $r = -2$ and the fundamental solutions are

$$\begin{aligned} y_1 &= e^{-2t} \\ y_2 &= te^{-2t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1y_1 + c_2y_2 \\ &= c_1e^{-2t} + c_2te^{-2t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}\left(e^{-2t} - 2te^{-2t}\right) + 2e^{-2t}\left(te^{-2t}\right)$$
$$= e^{-4t} - 2te^{-4t} + 2te^{-4t}$$
$$= e^{-4t}$$

Let $g(t) = t^{-2}e^{-2t}$, therefore the particular solution is

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(t)}dt = -\int \frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = -\int t^{-1}dt = -\ln|t|$$

And

$$u_2(t) = \int \frac{y_1(t)\,g(t)}{W(t)}dt = \int \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = \int t^{-2}dt = -\frac{1}{t}$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= -\ln|t|\,e^{-2t} - \frac{1}{t}te^{-2t} \\ &= -e^{-2t}\ln|t| - e^{-2t} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln|t| - e^{-2t} \end{aligned}$$

We can combine e^{-2t} that shows up from the particular solution with the c_1e^{-2t} term from the homogenous solution, since c_1 is arbitrary constant, which simplifies the above to

$$y = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln|t|$$

2.7.8 Section 3.6 problem 8

Find the general solution of $y'' + 4y = 3\frac{1}{\sin 2t}$ for $0 < t < \frac{\pi}{2}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 4y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by

$r^2 + 4 = 0$ or $r = \pm 2i$. The fundamental solutions are

$$\begin{aligned} y_1 &= \cos 2t \\ y_2 &= \sin 2t \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos 2t + c_2 \sin 2t \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

Let $g(t) = \frac{3}{\sin 2t}$, therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \frac{\sin(2t)\, 3}{2\sin 2t} dt = -\frac{3}{2}\int dt = \frac{-3}{2} t$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)} dt = \int \frac{\cos(2t)\, 3}{2\sin 2t} dt = \frac{3}{2}\int \frac{\cos(2t)}{\sin(2t)} dt$$

Let $u = \sin 2t \rightarrow du = 2\cos 2t dt$ and the above integral becomes

$$u_2(t) = \frac{3}{2}\int \frac{\cos(2t)}{u} \frac{du}{2\cos 2t} = \frac{3}{4}\int \frac{1}{u} du = \frac{3}{4}\ln|u| = \frac{3}{4}\ln|\sin 2t|$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{-3}{2} t \cos 2t + \frac{3}{4}\ln|\sin 2t| \sin 2t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2} t \cos 2t + \frac{3}{4}\sin(2t)\ln|\sin 2t| \end{aligned}$$

2.8 Quizz

1. Let $y_1, y_2, ..., y_n$ be differentiable (real-valued) solutions of the following system of differential equations

$$\begin{aligned}\frac{dy_1}{dt} &= a_{11}y_1 + \cdots + a_{1n}y_n,\\ \frac{dy_2}{dt} &= a_{21}y_1 + \cdots + a_{2n}y_n,\\ &\cdots\\ \frac{dy_n}{dt} &= a_{n1}y_1 + \cdots + a_{nn}y_n,\end{aligned}$$

for some constant $a_{ij} > 0$. Suppose that

$$y_i(t) \to 0,$$

as $t \to \infty, \forall i = 1, \cdots, n$. Are the functions $y_1, y_2, ..., y_n$ necessarily linearly dependent?

What we know (given): We have state space representation of a system in the form $Y' = AY$, where $y_1, y_2, \cdots, y_n$ are the states, and we are told the system goes to stable equilibrium $Y = 0$, as $t \to \infty$ when starting from any initial point in the n dimensions state space. The original system is described by a single n^{th} degree one differential equation, and is broken down to n first order differential equation. These are $y_1', y_2', \cdots, y_n'$. The system is coupled, since each $y_i'(t)$ depends on all other $y_i(t)$.

Solution The only way I can see to answer this question in concrete way, is to resort to using the Wronskian. Writing down the Wronskian $W(t)$ of the functions $y_1(t), y_2(t), \cdots, y_n(t)$ we obtain

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

In the limit, as $t \to \infty$, since we are told $y_i \to 0$, then the above becomes

$$\lim_{t\to\infty} W(t) = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Since, at least, one row becomes all zero, then the determinant above is zero (from linear algebra). Therefore

$$\lim_{t\to\infty} W(t) = 0$$

We could conclude now that $y_1, y_2, \cdots, y_n$ are therefore linearly dependent functions since we found that $W(t) = 0$ at some point. However, the Wronskian being zero at some point does not necessarily imply that the functions are linearly dependent. So the Wronskian test is not conclusive when it gives zero when evaluated at one point, and we need another test to do. The following are the important facts about using the Wronskian

1. If $W(t) \neq 0$ at any point t (in the interval of interest) $\Rightarrow y_1, y_2, \cdots, y_n$ are linearly independent (in that interval).

2. If $y_1, y_2, \cdots, y_n$ are analytic (differentiable) functions and linearly dependent (in the interval of interest) $\Rightarrow W(t) = 0$ at every point t in the interval.

3. If $W(t) = 0$ at every point t (in the interval of interest) and $y_1(t), y_2(t), \cdots, y_n(r)$ are all analytic functions $\Rightarrow y_1(t), y_2(t), \cdots, y_n(r)$ are linearly dependent in that interval.

4. If $W(t) = 0$ at one point t (or at countable number of points) in the interval of interest $\Rightarrow$ test is not conclusive.

The above are results from Linear algebra. We see from the above, that $W(t) = 0$ at $t = \infty$ does not imply that the functions are necessarily linearly dependent. In this case, we would use a different test if we are given the functions, by writing

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

And then we would try to find constants $c_1, c_2, \cdots, c_n$, not all zero, which would satisfy the above. If we can find such constants, only then we can conclude that $y_1, y_2, \cdots, y_n$ are linearly dependent since If the functions are linearly independent, then $c_i = 0$ will be the only possible solution.

In conclusion The functions $y_1(t), y_2(t), \cdots, y_n(t)$ are not necessarily linearly dependent, even though $W(t) = 0$ in the limit as $t \to \infty$.

2.9 Example problem from lecture Nov 30, 2016

This is complete solution of class example (example 2). Math 319, lecture Nov. 30. 2016.

Solve the differential equation

$$\begin{aligned} 2y''(t) + y'(t) + 2y(t) &= g(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

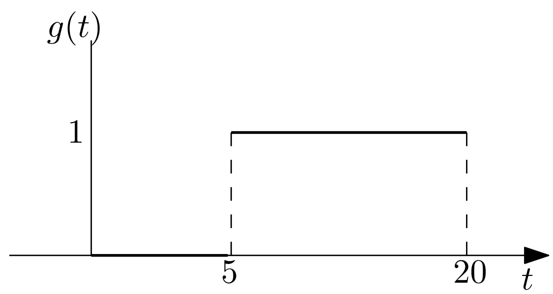
Where

$$g(t) = \begin{cases} 1 & 5 \leq t < 20 \\ 0 & \text{otherwise} \end{cases}$$

Using Laplace transform method.

Solution

The first step is to find the Laplace transform of the forcing function $g(t)$. The function $g(t)$ is



We now write $g(t)$ in terms of the unit step function $u_c(t)$ defined as $u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$ as follows

$$g(t) = u_5(t) - u_{20}(t) \tag{1}$$

Now we use the property that

$$\mathscr{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs}\mathscr{L}\left\{f(t)\right\}$$

To obtain the Laplace transform of $g(t)$ in (1) as follows

$$\begin{aligned} \mathscr{L}\left\{g(t)\right\} &= \mathscr{L}\{u_5(t)\} - \mathscr{L}\{u_{20}(t)\} \\ &= e^{-5s}\mathscr{L}\{1\} - e^{-20s}\mathscr{L}\{1\} \end{aligned}$$

But $\mathscr{L}\{1\} = \frac{1}{s}$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{g(t)\right\} &= e^{-5s}\frac{1}{s} - e^{-20s}\frac{1}{s}\\ &= \frac{e^{-5s} - e^{-20s}}{s}\end{aligned}$$

Now that we found $\mathscr{L}\left\{g(t)\right\}$, we go back to the original ODE and take the Laplace transform of the ODE, which results in

$$\mathscr{L}\left\{2y''(t)\right\} + \mathscr{L}\left\{y'(t)\right\} + \mathscr{L}\left\{2y(t)\right\} = \mathscr{L}\left\{g(t)\right\}$$

Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$, then the above becomes

$$2\left\{s^2Y(s) - sy(0) - y'(0)\right\} + \left\{sY(s) - y(0)\right\} + 2Y(s) = \mathscr{L}\left\{g(t)\right\}$$

But $y(0) = y'(0) = 0$ and the above reduces to

$$2s^2Y(s) + sY(s) + 2Y(s) = \frac{e^{-5s} - e^{-20s}}{s}$$

Solving for $Y(s)$ gives

$$\begin{aligned}Y(s) &= \frac{e^{-5s} - e^{-10s}}{s\left(2s^2 + s + 2\right)}\\ &= \left(\frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} - \frac{e^{-20s}}{s\left(2s^2 + s + 2\right)}\right) \qquad (2)\end{aligned}$$

We now need to find the inverse Laplace transform of $Y(s)$. Looking at $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$,the first step is to use the property

$$u_c(t) f(t-c) \overset{\mathscr{L}}{\Longleftrightarrow} e^{-cs}F(s)$$

Comparing the expressions, we see that

$$u_5(t) f(t-c) \Longleftrightarrow \frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} \qquad (3)$$

Where

$$f(t) \Longleftrightarrow \frac{1}{s\left(2s^2 + s + 2\right)} \qquad (4)$$

Therefore, we just need to find inverse Laplace transform of $\frac{1}{s\left(2s^2+s+2\right)}$. Using partial fractions

$$\frac{1}{s\left(2s^2 + s + 2\right)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2} \qquad (5)$$

$$\begin{aligned}1 &= A\left(2s^2 + s + 2\right) + (Bs + C)s\\ 1 &= 2As^2 + As + 2A + Bs^2 + Cs\\ 1 &= 2A + s(A + C) + s^2(2A + B)\end{aligned}$$

Therefore

$$\begin{aligned} A &= \frac{1}{2} \\ A + C &= 0 \\ 2A + B &= 0 \end{aligned}$$

Hence from the second equation $C = -\frac{1}{2}$, and from the third equation $B = -1$ Therefore (5) becomes

$$\begin{aligned} \frac{1}{s\left(2s^2+s+2\right)} &= \frac{1}{2}\frac{1}{s} + \frac{-s-\frac{1}{2}}{2s^2+s+2} \\ &= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{-1-2s}{2s^2+s+2} \\ &= \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2} \end{aligned} \tag{5A}$$

The first term above is easy, we know that

$$\frac{1}{2}\frac{1}{s} \Longleftrightarrow \frac{1}{2} \tag{6}$$

Now we will find inverse Laplace transform of second term in (5A) $\frac{s}{2s^2+s+2}$. For this we start by completing the squares in the denominator. Let

$$\begin{aligned} 2s^2+s+2 &= a(s+b)^2 + c \\ &= a\left(s^2+b^2+2bs\right) + c \\ &= as^2 + ab^2 + 2bas + c \end{aligned}$$

Hence $a = 2, 2ab = 1$ or $b = \frac{1}{4}$ and $ab^2 + c = 2$, hence $c = 2 - 2\left(\frac{1}{4}\right)^2 = 2 - 2\left(\frac{1}{16}\right) = 2 - \frac{1}{8} = \frac{15}{8}$, Therefore

$$2s^2+s+2 = 2\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}$$

We now re-write second term in (5A), which is $\frac{s}{2s^2+s+2}$ as $\frac{s}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}}$. We did this because we wanted this to be in the form $\frac{s}{s^2+a}$, therefore

$$\frac{s}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}} = \frac{1}{2}\frac{s}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

Now we let $\tilde{s} = s + \frac{1}{4}$, therefore the above becomes

$$\frac{1}{2}\frac{\tilde{s}-\frac{1}{4}}{\tilde{s}^2+\frac{15}{16}} = \frac{1}{2}\left(\frac{\tilde{s}}{\tilde{s}^2+\frac{15}{16}} - \frac{1}{4}\frac{1}{\tilde{s}^2+\frac{15}{16}}\right) \tag{7}$$

Using $\frac{s}{s^2+a^2} \Leftrightarrow \cos(at)$ then

$$\frac{\tilde{s}}{\tilde{s}^2+\frac{15}{16}} \Leftrightarrow e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right)$$

The reason for $e^{-\frac{t}{4}}$ being there, is because we evaluated $F(s)$ at $F\left(s+\frac{1}{4}\right)$. This used the shift property

$$F(s+a) = e^{-at}f(t)$$

Therefore $F\left(s+\frac{1}{4}\right) = e^{-\frac{t}{4}}f(t)$. Now we do the second term in (7). Since $\frac{1}{\tilde{s}^2+\frac{15}{16}} = \frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}}$, then, now using $\frac{a}{s^2+a^2} \Leftrightarrow \sin(at)$ we obtain

$$\frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}} \Leftrightarrow \frac{1}{\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right)$$

And we remember to add $e^{-\frac{t}{4}}$ again, due to the shift in s. Therefore (7) becomes

$$\begin{aligned}
\frac{s}{2s^2+s+2} &\Leftrightarrow \frac{1}{2}\left(e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right) - \frac{1}{4}e^{-\frac{t}{4}}\frac{1}{\sqrt{\frac{15}{16}}}\sin\left(\sqrt{\frac{15}{16}}t\right)\right) \\
&= \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) \qquad (8)
\end{aligned}$$

This complete the second term in (5A). Now we will do the third term in (5A) which is $\frac{1}{2s^2+s+2}$ which is

$$\begin{aligned}
\frac{1}{2s^2+s+2} &= \frac{1}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}} \\
&= \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \\
&= \frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} &\Longleftrightarrow \frac{1}{2\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right) \\
&= \frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right) \qquad (9)
\end{aligned}$$

Now we put all the results back together.

$$
\begin{aligned}
\frac{1}{s\left(2s^2+s+2\right)} &= \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2} \\
&\Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) - \frac{1}{2}\left(\frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right)
\end{aligned}
$$

We can simplify this more

$$
\begin{aligned}
\frac{1}{s\left(2s^2+s+2\right)} &\Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right) + 2\sin\left(\frac{\sqrt{15}}{4}t\right)\right) \\
&= \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) + \sin\left(\frac{\sqrt{15}}{4}t\right)\right)
\end{aligned}
$$

Using this back in (2), where we want to evaluates $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$, gives

$$
\frac{e^{-5s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_5(t)\, f(t-5)
$$

Where

$$
f(t-5) = \frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)
$$

The above complete the first term in (2). The second term in (2) is the same, but the delay now is 20 instead of 5. Hence

$$
\frac{e^{-20s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_{20}(t)\, f(t-20)
$$

With the same function $f(t)$ found above. Therefore, the final inverse transform now is

$$
\begin{aligned}
y(t) &\Longleftrightarrow \left(\frac{e^{-5s}}{s\left(2s^2+s+2\right)} - \frac{e^{-20s}}{s\left(2s^2+s+2\right)}\right) \\
&= \left(u_5(t)\, f(t-5) - u_{20}(t)\, f(t-20)\right)
\end{aligned}
$$

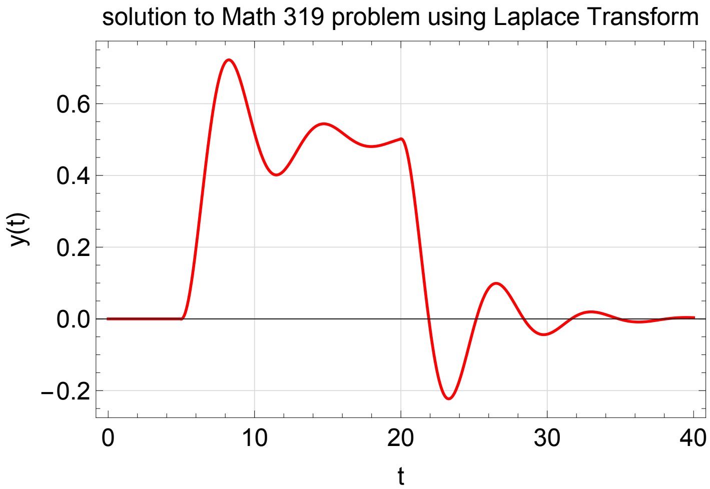
Where

$$
f(t-20) = \frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right)
$$

This complete the solution. The final solution is

$$
\begin{aligned}
y(t) = u_5(t)&\left(\frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)\right) \\
- u_{20}(t)&\left(\frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right)\right)
\end{aligned}
$$

Here is a plot of the above solution



References

1. Lecture notes Nov. 30, 2016 by Professor Minh-Binh Tran. Math dept. Univ. Of Wisconsin Madison.

2. Wikipedia web page on Laplace transform properties.

2.10 HW8

2.10.1 Section 6.1 problem 7

Find Laplace Transform of $f(t) = \cosh(bt)$

<u>solution</u> Since $\cosh(bt) = \frac{e^{bt}+e^{-bt}}{2}$ then

$$\begin{aligned}\mathscr{L}\cosh(bt) &= \frac{1}{2}\mathscr{L}\left(e^{bt}+e^{-bt}\right)\\ &= \frac{1}{2}\left(\mathscr{L}e^{bt}+\mathscr{L}e^{-bt}\right)\end{aligned}$$

But

$$\mathscr{L}e^{bt} = \frac{1}{s-b}$$

For $s > b$ and

$$\mathscr{L}e^{bt} = \frac{1}{s-b}$$

For $s < b$. Hence

$$\begin{aligned}\mathscr{L}\cosh(bt) &= \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right)\\ &= \frac{s^2}{s^2-b^2}\end{aligned}$$

For $s > |b|$

2.10.2 Section 6.1 problem 8

Find Laplace Transform of $f(t) = \sinh(bt)$

<u>solution</u> Since $\sinh(bt) = \frac{e^{bt}-e^{-bt}}{2}$ then

$$\begin{aligned}\mathscr{L}\sinh(bt) &= \frac{1}{2}\mathscr{L}\left(e^{bt}-e^{-bt}\right)\\ &= \frac{1}{2}\left(\mathscr{L}e^{bt}-\mathscr{L}e^{-bt}\right)\end{aligned}$$

But, as we found in the last problem

$$\mathscr{L}e^{bt} = \frac{1}{s-b} \qquad s > b$$

And

$$\mathscr{L}e^{-bt} = \frac{1}{s+b} \qquad s < b$$

Therefore

$$\begin{aligned}\mathscr{L}\sinh(bt) &= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) \qquad s > b; s < b \\ &= \frac{b}{s^2 - b^2} \qquad s > |b|\end{aligned}$$

2.10.3 Section 6.1 problem 9

Find Laplace Transform of $f(t) = e^{at}\cosh(bt)$

solution Using the property that

$$e^{at} f(t) \Longleftrightarrow F(s-a)$$

Where $f(t) = \cosh(bt)$ now. We already found above that $\cosh(bt) \Longleftrightarrow \frac{s}{s^2-b^2}$, for $s > |b|$. In other words, $F(s) = \frac{s}{s^2-b^2}$, therefore

$$e^{at}\cosh(bt) \Longleftrightarrow \frac{(s-a)}{(s-a)^2 - b^2} \qquad s - a > |b|$$

2.10.4 Section 6.1 problem 10

Find Laplace Transform of $f(t) = e^{at}\sinh(bt)$

solution Using the property that

$$e^{at} f(t) \Longleftrightarrow F(s-a)$$

Where $f(t) = \sinh(bt)$ now. We already found above that $\sinh(bt) \Longleftrightarrow \frac{b}{s^2-b^2}$, for $s > |b|$. In other words, $F(s) = \frac{b}{s^2-b^2}$, therefore

$$e^{at}\sinh(bt) \Longleftrightarrow \frac{b}{(s-a)^2 - b^2} \qquad s - a > |b|$$

2.10.5 Section 6.2 problem 17

Use Laplace transform to solve $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$ for $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1$

Solution Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\} - 4\mathscr{L}\left\{y'''\right\} + 6\mathscr{L}\left\{y''\right\} - 4\mathscr{L}\left\{y'\right\} + \mathscr{L}\left\{y\right\} = 0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\} = Y(s)$ then

$$\begin{aligned}\mathscr{L}\left\{y^{(4)}\right\} &= s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) \\ &= s^4 Y(s) - s^3(0) - s^2(1) - s(0) - 1 \\ &= s^4 Y(s) - s^2 - 1\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y'''\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\ &= s^3Y(s) - s^2(0) - s(1) - 0\\ &= s^3Y(s) - s\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y''\right\} &= s^2Y(s) - sy(0) - y'(0)\\ &= s^2Y(s) - s(0) - 1\\ &= s^2Y(s) - 1\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y'\right\} &= sY(s) - y(0)\\ &= sY(s)\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(s^4Y(s) - s^2 - 1\right) - 4\left(s^3Y(s) - s\right) + 6\left(s^2Y(s) - 1\right) - 4\left(sY(s)\right) + Y(s) &= 0\\ Y(s)\left(s^4 - 4s^3 + 6s^2 - 4s + 1\right) - s^2 - 1 + 4s - 6 &= 0\end{aligned}$$

Therefore

$$\begin{aligned}Y(s) &= \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}\\ &= \frac{s^2 - 4s + 7}{(s-1)^4}\\ &= \frac{s^2}{(s-1)^4} - \frac{4s}{(s-1)^4} + \frac{7}{(s-1)^4} \qquad (2)\end{aligned}$$

But

$$\begin{aligned}\frac{s^2}{(s-1)^4} &= \frac{(s-1)^2 - 1 + 2s}{(s-1)^4}\\ &= \frac{(s-1)^2}{(s-1)^4} - \frac{1}{(s-1)^4} + 2\frac{(s-1)+1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} - \frac{1}{(s-1)^4} + 2\frac{(s-1)}{(s-1)^4} + 2\frac{1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} - \frac{1}{(s-1)^4} + 2\frac{1}{(s-1)^3} + 2\frac{1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{(s-1)^4}\end{aligned}$$

And

$$\begin{aligned}\frac{4s}{(s-1)^4} &= 4\frac{(s-1)+1}{(s-1)^4} \\ &= 4\frac{(s-1)}{(s-1)^4} + 4\frac{1}{(s-1)^4} \\ &= \frac{4}{(s-1)^3} + \frac{4}{(s-1)^4}\end{aligned}$$

Therefore (2) becomes

$$\begin{aligned}Y(s) &= \left(\frac{1}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{(s-1)^4}\right) - \left(\frac{4}{(s-1)^3} + \frac{4}{(s-1)^4}\right) + \frac{7}{(s-1)^4} \\ &= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4} \qquad (3)\end{aligned}$$

Now using property the shift property of $F(s)$ together with

$$\begin{aligned}\frac{1}{s^2} &\Longleftrightarrow t \\ \frac{1}{s^3} &\Longleftrightarrow \frac{t^2}{2} \\ \frac{1}{s^4} &\Longleftrightarrow \frac{t^3}{6}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(s-1)^2} &\Longleftrightarrow e^t t \\ \frac{1}{(s-1)^3} &\Longleftrightarrow e^t \frac{t^2}{2} \\ \frac{1}{(s-1)^4} &\Longleftrightarrow e^t \frac{t^3}{6}\end{aligned}$$

And (3) becomes

$$\begin{aligned}\frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4} &\Longleftrightarrow e^t t - 2\left(e^t\frac{t^2}{2}\right) + 4\left(e^t\frac{t^3}{6}\right) \\ &= e^t t - e^t t^2 + \frac{2}{3}e^t t^3\end{aligned}$$

Hence

$$y(t) = e^t\left(t - t^2 + \frac{2}{3}t^3\right)$$

2.10.6 Section 6.2 problem 18

Use Laplace transform to solve $y^{(4)} - y = 0$ for $y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0$

Solution Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\} - \mathscr{L}\left\{y\right\} = 0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\} = Y(s)$ then

$$\begin{aligned}\mathscr{L}\left\{y^{(4)}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\\ &= s^4Y(s) - s^3(1) - s^2(0) - s(1) - 0\\ &= s^4Y(s) - s^3 - s\end{aligned}$$

Hence (1) becomes

$$s^4Y(s) - s^3 - s - Y(s) = 0$$

Solving for $Y(s)$ gives

$$\begin{aligned}Y(s) &= \frac{s^3+s}{s^4-1}\\ &= \frac{s\left(s^2+1\right)}{s^4-1}\\ &= \frac{s\left(s^2+1\right)}{\left(s^2-1\right)\left(s^2+1\right)}\\ &= \frac{s}{s^2-1}\end{aligned}$$

But, Hence above becomes, where $a=1$

$$\frac{s}{s^2-1} \Longleftrightarrow \cosh(t)$$

Hence

$$y(t) = \cosh(at)$$

2.10.7 Section 6.2 problem 19

Use Laplace transform to solve $y^{(4)} - 4y = 0$ for $y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0$

Solution Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\} - 4\mathscr{L}\left\{y\right\} = 0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\} = Y(s)$ then

$$\begin{aligned}\mathscr{L}\left\{y^{(4)}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\\ &= s^4Y(s) - s^3(1) - s^2(0) - s(-2) - 0\\ &= s^4Y(s) - s^3 + 2s\end{aligned}$$

Hence (1) becomes

$$s^4Y(s) - s^3 + 2s - 4Y(s) = 0$$

Solving for $Y(s)$ gives

$$\begin{aligned}
Y(s) &= \frac{s^3-2s}{s^4-4}\\
&= \frac{s^3-2s}{\left(s^2-2\right)\left(s^2+2\right)}\\
&= \frac{s\left(s^2-2\right)}{\left(s^2-2\right)\left(s^2+2\right)}\\
&= \frac{s}{\left(s^2+2\right)}
\end{aligned}$$

Using $\cos(at) \iff \frac{s}{s^2+a^2}$, the above becomes, where $a=\sqrt{2}$

$$\frac{s}{\left(s^2+2\right)} \iff \cos\left(\sqrt{2}t\right)$$

Hence

$$y(t) = \cos\left(\sqrt{2}t\right)$$

2.10.8 Section 6.2 problem 20

Use Laplace transform to solve $y''+\omega^2 y = \cos 2t$; $\omega^2 \neq 4$; $y(0)=1, y'(0)=0$

Solution Let $Y(s)=\mathscr{L}\left\{y(t)\right\}$. Taking Laplace transform of the ODE, and using $\cos(at) \iff \frac{s}{s^2+a^2}$ gives

$$s^2Y(s) - sy(0) - y'(0) + \omega^2 Y(s) = \frac{s}{s^2+4} \tag{1}$$

Applying initial conditions

$$s^2Y(s) - s + \omega^2 Y(s) = \frac{s}{s^2+4}$$

Solving for $Y(s)$

$$\begin{aligned}
Y(s)\left(s^2+\omega^2\right) - s &= \frac{s}{s^2+4}\\
Y(s) &= \frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} + \frac{s}{\left(s^2+\omega^2\right)} \qquad (2)
\end{aligned}$$

But

$$\begin{aligned}
\frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} &= \frac{As+B}{\left(s^2+4\right)} + \frac{Cs+D}{\left(s^2+\omega^2\right)}\\
s &= (As+B)\left(s^2+\omega^2\right) + (Cs+D)\left(s^2+4\right)\\
s &= 4D + As^3 + Bs^2 + Cs^3 + B\omega^2 + s^2D + 4Cs + As\omega^2\\
s &= \left(4D+B\omega^2\right) + s\left(4C+A\omega^2\right) + s^2(B+D) + s^3(A+C)
\end{aligned}$$

Hence

$$\begin{aligned} 4D + B\omega^2 &= 0 \\ 4C + A\omega^2 &= 1 \\ B + D &= 0 \\ A + C &= 0 \end{aligned}$$

Equation (2,4) gives $A = \frac{1}{\omega^2-4}, C = \frac{1}{4-\omega^2}$ and (1,3) gives $B = 0, D = 0$. Hence

$$\frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} = \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{1}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)}$$

Therefore (2) becomes

$$\begin{aligned} Y(s) &= \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{1}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} + \frac{s}{\left(s^2+\omega^2\right)} \\ &= \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{5-\omega^2}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} \end{aligned}$$

Using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$, the above becomes

$$\begin{aligned} \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{5-\omega^2}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} &\Longleftrightarrow \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{5-\omega^2}{4-\omega^2}\right)\cos(\omega t) \\ &= \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t) \end{aligned}$$

Hence

$$\begin{aligned} y(t) &= \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t) \\ &= \frac{\left(\omega^2-5\right)\cos(\omega t) + \cos(2t)}{\omega^2-4} \end{aligned}$$

2.10.9 Section 6.2 problem 21

Use Laplace transform to solve $y'' - 2y' + 2y = \cos t;\ y(0) = 1, y'(0) = 0$

Solution Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$. Taking Laplace transform of the ODE, and using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$ gives

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{s}{s^2+1} \tag{1}$$

Applying initial conditions

$$s^2 Y(s) - s - 2\left(sY(s) - 1\right) + 2Y(s) = \frac{s}{s^2+1}$$

Solving for $Y(s)$

$$\begin{aligned} s^2Y(s) - s - 2sY(s) + 2 + 2Y(s) &= \frac{s}{s^2+1} \\ Y(s)\left(s^2 - 2s + 2\right) - s + 2 &= \frac{s}{s^2+1} \\ Y(s) &= \frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} + \frac{s}{\left(s^2-2s+2\right)} - \frac{2}{\left(s^2-2s+2\right)} \qquad (2) \end{aligned}$$

But

$$\begin{aligned} \frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{As+B}{\left(s^2+1\right)} + \frac{Cs+D}{s^2-2s+2} \\ s &= (As+B)\left(s^2-2s+2\right) + (Cs+D)\left(s^2+1\right) \\ s &= 2B + D - 2As^2 + As^3 + Bs^2 + Cs^3 + s^2D + 2As - 2Bs + Cs \\ s &= (2B+D) + s(2A - 2B + C) + s^2(-2A + B + D) + s^3(A + C) \end{aligned}$$

Hence

$$\begin{aligned} 2B + D &= 0 \\ 2A - 2B + C &= 1 \\ -2A + B + D &= 0 \\ A + C &= 0 \end{aligned}$$

Solving gives $A = \frac{1}{5}, B = -\frac{2}{5}, C = -\frac{1}{5}, D = \frac{4}{5}$, hence

$$\begin{aligned} \frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{s-2}{\left(s^2+1\right)} - \frac{1}{5}\frac{s-4}{s^2-2s+2} \\ &= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{s^2-2s+2} + \frac{4}{5}\frac{1}{s^2-2s+2} \qquad (3) \end{aligned}$$

Completing the squares for

$$\begin{aligned} s^2 - 2s + 2 &= a(s+b)^2 + d \\ &= a\left(s^2 + b^2 + 2bs\right) + d \\ &= as^2 + ab^2 + 2abs + d \end{aligned}$$

Hence $a = 1, 2ab = -2, \left(ab^2 + d\right) = 2$, hence $b = -1, d = 1$, hence

$$s^2 - 2s + 2 = (s-1)^2 + 1$$

Hence (3) becomes

$$
\begin{aligned}
\frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{(s-1)^2+1} + \frac{4}{5}\frac{1}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)+1}{(s-1)^2+1} + \frac{4}{5}\frac{1}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{1}{5}\frac{1}{(s-1)^2+1} + \frac{4}{5}\frac{1}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}
\end{aligned}
$$

Therefore (2) becomes

$$
\begin{aligned}
Y(s) &= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} + \frac{s}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} + \frac{(s-1)+1}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} + \frac{(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{4}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1}
\end{aligned}
$$

Using $\cos(at) \iff \frac{s}{s^2+a^2}, \sin(at) \iff \frac{a}{s^2+a^2}$ and the shift property of Laplace transform, then

$$
\begin{aligned}
\frac{1}{5}\frac{s}{s^2+1} &\iff \frac{1}{5}\cos(t) \\
\frac{2}{5}\frac{1}{s^2+1} &\iff \frac{2}{5}\sin(t) \\
\frac{4}{5}\frac{(s-1)}{(s-1)^2+1} &\iff \frac{4}{5}e^t\cos t \\
\frac{2}{5}\frac{1}{(s-1)^2+1} &\iff \frac{8}{5}e^t\sin t
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(t) &= \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t \\
& \frac{1}{5}\left(\cos t - 2\sin t + 4e^t\cos t - 2e^t\sin t\right)
\end{aligned}
$$

2.10.10 Section 6.2 problem 22

Use Laplace transform to solve $y'' - 2y' + 2y = e^{-t}; y(0) = 0, y'(0) = 1$

<u>Solution</u> Let $Y(s) = \mathscr{L}\{y(t)\}$. Taking Laplace transform of the ODE, and using $e^{-t} \iff \frac{1}{s+1}$ gives

$$
\left(s^2Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1}{s+1} \tag{1}
$$

Applying initial conditions gives

$$s^2 Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solving for $Y(s)$

$$\begin{aligned} Y(s)\left(s^2 - 2s + 2\right) - 1 &= \frac{1}{s+1} \\ Y(s) &= \frac{1}{(s+1)\left(s^2-2s+2\right)} + \frac{1}{s^2-2s+2} \end{aligned} \tag{2}$$

But

$$\begin{aligned} \frac{1}{(s+1)\left(s^2-2s+2\right)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} \\ 1 &= A\left(s^2-2s+2\right) + (Bs+C)(s+1) \\ 1 &= 2A + C + As^2 + Bs^2 - 2As + Bs + Cs \\ 1 &= (2A+C) + s(-2A+B+C) + s^2(A+B) \end{aligned}$$

Hence

$$\begin{aligned} 1 &= 2A + C \\ 0 &= -2A + B + C \\ 0 &= A + B \end{aligned}$$

Solving gives $A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{3}{5}$, therefore

$$\begin{aligned} \frac{1}{(s+1)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{1}{s+1} + \frac{-\frac{1}{5}s + \frac{3}{5}}{s^2-2s+2} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{s^2-2s+2} + +\frac{3}{5}\frac{1}{s^2-2s+2} \end{aligned}$$

Completing the square for $s^2 - 2s + 2$ which was done in last problem, gives $(s-1)^2 + 1$, hence the above becomes

$$\begin{aligned} \frac{1}{(s+1)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{(s-1)^2+1} + +\frac{3}{5}\frac{1}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)+1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{1}{5}\frac{1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{2}{5}\frac{1}{(s-1)^2+1} \end{aligned}$$

Therefore (2) becomes

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

Using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}, \sin(at) \Longleftrightarrow \frac{a}{s^2+a^2}$ and the shift property of Laplace transform, then

$$
\begin{aligned}
\frac{1}{5}\frac{1}{s+1} &\Longleftrightarrow \frac{1}{5}e^{-t} \\
\frac{1}{5}\frac{(s-1)}{(s-1)^2+1} &\Longleftrightarrow \frac{1}{5}e^t\cos t \\
\frac{2}{5}\frac{1}{(s-1)^2+1} &\Longleftrightarrow \frac{2}{5}e^t\sin t \\
\frac{1}{(s-1)^2+1} &\Longleftrightarrow e^t\sin t
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(t) &= \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos t + \frac{2}{5}e^t\sin t + e^t\sin t \\
&= \frac{1}{5}\left(e^{-t} - e^t\cos t + 7e^t\sin t\right)
\end{aligned}
$$

2.10.11 Section 6.2 problem 23

Use Laplace transform to solve $y'' + 2y' + y = 4e^{-t}; y(0) = 2, y'(0) = -1$

Solution Let $Y(s) = \mathscr{L}\{y(t)\}$. Taking Laplace transform of the ODE, and using $e^{-t} \Longleftrightarrow \frac{1}{s+1}$ gives

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{4}{s+1} \tag{1}$$

Applying initial conditions gives

$$\left(s^2Y(s) - 2s + 1\right) + 2\left(sY(s) - 2\right) + Y(s) = \frac{4}{s+1}$$

Solving for $Y(s)$

$$
\begin{aligned}
Y(s)\left(s^2+2s+1\right) - 2s + 1 - 4 &= \frac{4}{s+1} \\
Y(s)\left(s^2+2s+1\right) &= \frac{4}{s+1} + 2s - 1 + 4 \\
Y(s) &= \frac{4}{(s+1)\left(s^2+2s+1\right)} + \frac{2s}{\left(s^2+2s+1\right)} - \frac{1}{\left(s^2+2s+1\right)} + \frac{4}{\left(s^2+2s+1\right)}
\end{aligned}
$$

But $\left(s^2+2s+1\right) = (s+1)^2$, hence

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s}{(s+1)^2} - \frac{1}{(s+1)^2} + \frac{4}{(s+1)^2} \tag{2}$$

But

$$\begin{aligned}\frac{2s}{(s+1)^2} &= 2\frac{s+1-1}{(s+1)^2}\\ &= 2\frac{(s+1)}{(s+1)^2} - 2\frac{1}{(s+1)^2}\\ &= 2\frac{1}{s+1} - 2\frac{1}{(s+1)^2}\end{aligned}$$

Hence (2) becomes

$$Y(s) = \frac{4}{(s+1)^3} + 2\frac{1}{s+1} - 2\frac{1}{(s+1)^2} - \frac{1}{(s+1)^2} + \frac{4}{(s+1)^2} \tag{3}$$

We now ready to do the inversion. Since $\frac{1}{s^3} \Longleftrightarrow \frac{t^2}{2}$ and $\frac{1}{s^2} \Longleftrightarrow t$ and $\frac{1}{s} \Longleftrightarrow 1$ and using the shift property $e^{at} f(t) \Longleftrightarrow F(s-a)$, then using these into (3) gives

$$\begin{aligned}\frac{4}{(s+1)^3} &\Longleftrightarrow 4e^{-t}\left(\frac{t^2}{2}\right)\\ 2\frac{1}{s+1} &\Longleftrightarrow 2e^{-t}\\ 2\frac{1}{(s+1)^2} &\Longleftrightarrow 2e^{-t}t\\ \frac{1}{(s+1)^2} &\Longleftrightarrow e^{-t}t\\ \frac{4}{(s+1)^2} &\Longleftrightarrow 4e^{-t}t\end{aligned}$$

Now (3) becomes

$$\begin{aligned}Y(s) &\Longleftrightarrow 4e^{-t}\left(\frac{t^2}{2}\right) + 2e^{-t} - 2e^{-t}t - e^{-t}t + 4e^{-t}t\\ &= e^{-t}\left(2t^2 + 2 - 2t - t + 4t\right)\\ &= e^{-t}\left(2t^2 + t + 2\right)\end{aligned}$$

2.10.12 Section 6.3 problem 25

Suppose that $F(s) = \mathscr{L}\left\{f(t)\right\}$ exists for $s > a \geq 0$.

1. Show that if c is positive constant then $\mathscr{L}\left\{f(ct)\right\} = \frac{1}{c}F\left(\frac{s}{c}\right)$ for $s > ca$
2. Show that if k is positive constant then $\mathscr{L}^{-1}\left\{F(ks)\right\} = \frac{1}{k}f\left(\frac{t}{k}\right)$
3. Show that if a, b are constants with $a > 0$ then $\mathscr{L}^{-1}\left\{F(as+b)\right\} = \frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)$

Solution

2.10.12.1 Part (a)

From definition,

$$\mathscr{L}\left\{f(ct)\right\} = \int_0^\infty f(ct)\, e^{-st} dt$$

Let $ct = \tau$, then when $t = 0, \tau = 0$ and when $t = \infty, \tau = \infty$, and $c = \frac{d\tau}{dt}$. Hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{f(ct)\right\} &= \int_0^\infty f(\tau)\, e^{-s\left(\frac{\tau}{c}\right)} \frac{d\tau}{c} \\ &= \frac{1}{c}\int_0^\infty f(\tau)\, e^{-\tau\left(\frac{s}{c}\right)} d\tau\end{aligned}$$

We see from above that $\mathscr{L}\left\{f(ct)\right\}$ is $\frac{1}{c}F\left(\frac{s}{c}\right)$.Now we look at the conditions which makes the above integral converges. Let

$$\left|f(\tau)\, e^{-\tau\left(\frac{s}{c}\right)}\right| \leq k\left|e^{at}e^{-\tau\left(\frac{s}{c}\right)}\right|$$

Where k is some constant. Then

$$\begin{aligned}\int_0^\infty f(t)\, e^{-t\left(\frac{s}{c}\right)} dt &\leq k\int_0^\infty e^{at}e^{-t\left(\frac{s}{c}\right)} dt \\ &= k\int_0^\infty e^{-t\left(\frac{s}{c}-a\right)} dt\end{aligned}$$

But $\int_0^\infty e^{-t\left(\frac{s}{c}-a\right)} d\tau$ converges if $\frac{s}{c} - a > 0$ or

$$s > ca$$

Hence this is the condition for $\int_0^\infty f(t)\, e^{-t\left(\frac{s}{c}\right)} dt$ to converge. Which is what we required to show.

2.10.12.2 Part (b)

From definition

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} &= \frac{1}{k}\mathscr{L}\left\{f\left(\frac{t}{k}\right)\right\} \\ &= \frac{1}{k}\int_0^\infty f\left(\frac{t}{k}\right) e^{-st} dt\end{aligned}$$

Let $\frac{t}{k} = \tau$. When $t = 0, \tau = 0$ and when $t = \infty, \tau = \infty$. $\frac{dt}{d\tau} = k$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} &= \frac{1}{k}\int_0^\infty f(\tau)\, e^{-s(k\tau)}\,(k\,d\tau) \\ &= \int_0^\infty f(\tau)\, e^{-\tau(sk)} d\tau\end{aligned}$$

We see from above that $\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\}$ is $F(sk)$. In other words, $\mathscr{L}^{-1}\left\{F(ks)\right\} = \frac{1}{k}f\left(\frac{t}{k}\right)$.

2.10.12.3 Part (c)

From definition

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\} &= \frac{1}{a}\mathscr{L}\left\{e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\}\\ &= \frac{1}{a}\int_0^\infty e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)e^{-st}dt\end{aligned}$$

Let $\frac{t}{a}=\tau$, at $t=0,\tau=0$ and at $t=\infty,\tau=\infty$. And $\frac{dt}{d\tau}=a$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\} &= \frac{1}{a}\int_0^\infty e^{\frac{-b(a\tau)}{a}}f(\tau)\,e^{-s(a\tau)}\,(ad\tau)\\ &= \int_0^\infty e^{-b\tau}f(\tau)\,e^{-\tau(sa)}d\tau\\ &= \int_0^\infty f(\tau)\,e^{-\tau(sa+b)}d\tau\end{aligned}$$

We see from the above, that $\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\}=F(sa+b)$. Now we look at the conditions which makes the above integral converges. Let

$$\left|f(\tau)\,e^{-t(sa+b)}\right| \le k\left|e^{at}e^{-t(sa+b)}\right|$$

Where k is some constant. Then

$$\begin{aligned}\int_0^\infty f(t)\,e^{-t(sa+b)}dt &\le k\int_0^\infty e^{at}e^{-t(sa+b)}dt\\ &= k\int_0^\infty e^{-t(sa+b-a)}dt\end{aligned}$$

But $\int_0^\infty e^{-t(sa+b-a)}dt$ converges if $sa+b-a>0$ or $sa>a-b$ or $s>1-\frac{b}{a}$

2.10.13 Section 6.3 problem 26

Find inverse Laplace transform of $F(s)=\frac{2^{n+1}n!}{s^{n+1}}$

<u>Solution</u>

We know from tables that

$$\frac{n!}{s^{n+1}} \Longleftrightarrow t^n$$

Hence

$$\begin{aligned}2^{n+1}\frac{n!}{s^{n+1}} &\Longleftrightarrow 2^{n+1}t^n\\ &= 2\,(2t)^n\end{aligned}$$

2.10.14 Section 6.3 problem 27

Find inverse Laplace transform of $F(s)=\frac{2s+1}{4s^2+4s+5}$

Solution

$$F(s)=\frac{2s}{4s^2+4s+5}+\frac{1}{4s^2+4s+5}$$

But $4s^2+4s+5=4\left(s+\frac{1}{2}\right)^2+4$, hence

$$\begin{aligned}
F(s) &= \frac{2s}{4\left(s+\frac{1}{2}\right)^2+4}+\frac{1}{4\left(s+\frac{1}{2}\right)^2+4}\\
&= \frac{s}{2\left(s+\frac{1}{2}\right)^2+2}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\\
&= \frac{1}{2}\frac{s}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\\
&= \frac{1}{2}\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\\
&= \frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}-\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\\
&= \frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} \qquad (1)
\end{aligned}$$

Now we ready to do the inversion. Using $e^{-at}f(t)\Longleftrightarrow F(s+a)$ and using $\sin(at)\Longleftrightarrow\frac{a}{s^2+a^2}$, and $\cos(at)\Longleftrightarrow\frac{s}{s^2+a^2}$then

$$\frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}\Longleftrightarrow\frac{1}{2}e^{-\frac{1}{2}t}\cos(t)$$

Hence

$$f(t)=\frac{1}{2}e^{-\frac{1}{2}t}\cos(t)$$

2.10.15 Section 6.3 problem 28

Find inverse Laplace transform of $F(s)=\frac{1}{9s^2-12s+3}$

Solution

$$\frac{1}{9s^2-12s+3}=\frac{1}{9}\frac{1}{s^2-\frac{4}{3}s+\frac{1}{3}}=\frac{1}{9}\frac{1}{(s-1)\left(s-\frac{1}{3}\right)}$$

But

$$\frac{1}{(s-1)\left(s-\frac{1}{3}\right)} = \frac{A}{s-1} + \frac{B}{s-\frac{1}{3}}$$

$$A = \left(\frac{1}{\left(s-\frac{1}{3}\right)}\right)_{s=1} = \frac{3}{2}$$

$$B = \left(\frac{1}{(s-1)}\right)_{s=\frac{1}{3}} = -\frac{3}{2}$$

Hence

$$\frac{1}{9s^2-12s+3} = \frac{1}{9}\left(\frac{3}{2}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s-\frac{1}{3}}\right) \tag{1}$$

Using

$$e^{at} \Longleftrightarrow \frac{1}{s-a}$$

Then (1) becomes

$$\begin{aligned}\frac{1}{9s^2-12s+3} &\Longleftrightarrow \frac{1}{9}\left(\frac{3}{2}e^t - \frac{3}{2}e^{\frac{1}{3}t}\right)\\ &= \frac{1}{6}e^t - \frac{1}{6}e^{\frac{1}{3}t}\\ &= \frac{1}{6}\left(e^t - e^{\frac{1}{3}t}\right)\end{aligned}$$

2.10.16 Section 6.3 problem 29

Find inverse Laplace transform of $F(s) = \frac{e^2 e^{-4s}}{2s-1}$

<u>solution</u>

$$F(s) = \frac{e^2}{2}\frac{e^{-4s}}{s-\frac{1}{2}}$$

Using

$$u_c(t) f(t-c) \Longleftrightarrow e^{-cs}F(s) \tag{1}$$

Since

$$\frac{1}{s-\frac{1}{2}} \Longleftrightarrow e^{\frac{1}{2}t}$$

Then using (1)

$$e^{-4s}\frac{1}{s-\frac{1}{2}} \Longleftrightarrow u_4(t)\, e^{\frac{1}{2}(t-4)}$$

Hence

$$\begin{aligned}\frac{e^2}{2}\frac{e^{-4s}}{s-\frac{1}{2}} &\Longleftrightarrow \frac{e^2}{2}u_4(t)\,e^{\frac{1}{2}(t-4)}\\&= \frac{1}{2}u_4(t)\,e^{\frac{1}{2}(t-4)+2}\\&= \frac{1}{2}u_4(t)\,e^{\frac{1}{2}t-2+2}\\&= \frac{1}{2}u_4(t)\,e^{\frac{t}{2}}\end{aligned}$$

Therefore

$$f(t) = \frac{1}{2}u_4(t)\,e^{\frac{t}{2}}$$

ps. Book answer is wrong. It gives

$$f(t) = \frac{1}{2}u_4\left(\frac{t}{2}\right)e^{\frac{t}{2}}$$

2.10.17 Section 6.3 problem 30

Find Laplace transform of $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$

solution

Writing $f(t)$ in terms of Heaviside step function gives

$$f(t) = u_0(t) - u_1(t)$$

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\begin{aligned}\mathscr{L}\{u_0(t)\} &= e^{-0s}\frac{1}{s} = \frac{1}{s}\\\mathscr{L}\{u_1(t)\} &= e^{-s}\frac{1}{s}\end{aligned}$$

Hence

$$\begin{aligned}\mathscr{L}\{u_0(t) - u_1(t)\} &= \frac{1}{s} - e^{-s}\frac{1}{s}\\&= \frac{1}{s}\left(1 - e^{-s}\right) \qquad s > 0\end{aligned}$$

2.10.18 Section 6.3 problem 31

Find Laplace transform of $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \\ 1 & 2 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$

solution

Writing $f(t)$ in terms of Heaviside step function gives

$$f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$$

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

But $f(t) = 1$ in this case. Hence $F(s) = \frac{1}{s}$. Therefore

$$\begin{aligned} f(t) &\Longleftrightarrow \frac{1}{s}e^{0s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} \\ &= \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s}\right) \qquad s > 0 \end{aligned}$$

2.10.19 Section 6.3 problem 32

Find Laplace transform of $f(t) = 1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)$

solution

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\begin{aligned} \mathscr{L}\left\{1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} &= \mathscr{L}\{1\} + \mathscr{L}\left\{\sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} \\ &= \frac{1}{s} + \sum_{k=1}^{2n+1}(-1)^k \frac{1}{s}e^{-ks} \\ &= \sum_{k=0}^{2n+1}(-1)^k \frac{1}{s}e^{-ks} \\ &= \frac{1}{s}\sum_{k=0}^{2n+1}\left(-e^{-s}\right)^k \end{aligned}$$

Since $|e^{-s}| < 1$ the sum converges. Using $\sum_0^N a_n = \left(\frac{1-r^{N+1}}{1-r}\right)$. Where $|r| < 1$. So the answer is

$$\mathscr{L}\left\{1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)\right\} = \frac{1}{s}\left(\frac{1-(-e^{-s})^{2n+2}}{1-(-e^{-s})}\right)$$
$$= \frac{1}{s}\left(\frac{1-(-e)^{-(2n+2)s}}{1+e^{-s}}\right)$$

Since $2n+2$ is even then

$$\mathscr{L}\left\{1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)\right\} = \frac{1}{s}\left(\frac{1+e^{-(2n+2)s}}{1+e^{-s}}\right) \qquad s > 0$$

2.10.20 Section 6.3 problem 33

Find Laplace transform of $f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$

solution

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\mathscr{L}\left\{1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)\right\} = \mathscr{L}\{1\} + \mathscr{L}\left\{\sum_{k=1}^{\infty} (-1)^k u_k(t)\right\}$$
$$= \frac{1}{s} + \sum_{k=1}^{\infty} (-1)^k \frac{1}{s} e^{-ks}$$
$$= \frac{1}{s} + \frac{1}{s}\sum_{k=1}^{\infty} (-1)^k e^{-ks}$$
$$= \frac{1}{s} + \frac{1}{s}\sum_{k=1}^{\infty} (-e^{-s})^k$$

But

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r} \qquad |r| < 1$$

Since $s > 0$ then$|e^{-s}| < 1$. So the answer is

$$\frac{1}{s} + \frac{1}{s}\frac{-e^{-s}}{1-(-e^{-s})} = \frac{1}{s} - \frac{1}{s}\frac{e^{-s}}{1+e^{-s}}$$
$$= \frac{1+e^{-s}-e^{-s}}{s(1+e^{-s})}$$
$$= \frac{1}{s(1+e^{-s})} \qquad s > 0$$

2.10.21 Section 6.4 problem 21

21. Consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

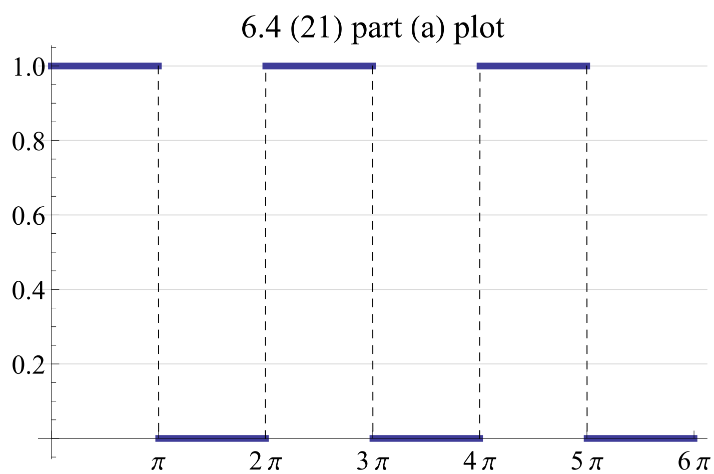
where

$$g(t) = u_0(t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t).$$

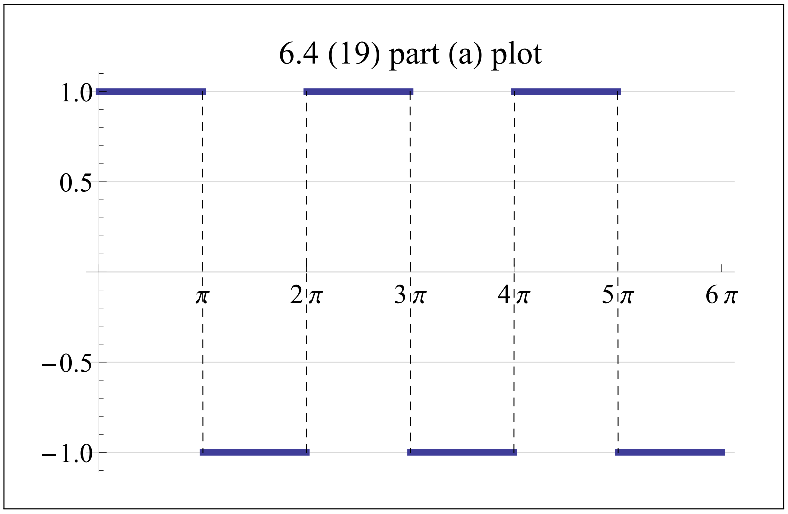
(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).
(b) Find the solution of the initial value problem.
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

2.10.21.1 Part (a)

A plot of part (a) is the following



And a plot of part(a) for problem 19 is the following



We see the effect of having a 2 inside the sum. It extends the step $u_c(t)$ function to negative side.

2.10.21.2 Part (b)

The easy way to do this, is to solve for each input term separately, and then add all the solutions, since this is a linear ODE. Once we solve for the first 2-3 terms, we will see the pattern to use for the overall solution. Since the input $g(t)$ is $u_0(t) + \sum_{k=1}^{\infty} (-1)^k u_{k\pi}(t)$, we will first first the response to $u_0(t)$, then for $-u_\pi(t)$ then for $+u_{2\pi}(t)$, and so on, and add them.

When the input is $u_0(t)$, then its Laplace transform is $\frac{1}{s}$, Hence, taking Laplace transform of the ODE gives (where now $Y(s) = \mathscr{L}\left(y(t)\right)$)

$$\left(s^2 Y(s) - sy(0) + y'(0)\right) + Y(s) = \frac{1}{s}$$

Applying initial conditions

$$s^2 Y(s) + Y(s) = \frac{1}{s}$$

Solving for $Y_0(s)$ (called it $Y_0(s)$ since the input is $u_0(t)$)

$$\begin{aligned} Y_0(s) &= \frac{1}{s\left(s^2+1\right)} \\ &= \frac{1}{s} - \frac{s}{s^2+1} \end{aligned}$$

Hence

$$y_0(t) = 1 - \cos t$$

We now do the next input, which is $-u_\pi(t)$, which has Laplace transform of $-\frac{e^{-\pi s}}{s}$, therefore, following what we did above, we obtain now

$$\begin{aligned} Y_\pi(s) &= \frac{-e^{-\pi s}}{s\left(s^2+1\right)} \\ &= -e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \end{aligned}$$

The effect of $e^{-\pi s}$ is to cause delay in time. Hence the the inverse Laplace transform of the above is the same as $y_0(t)$ but with delay

$$y_\pi(t) = -u_\pi(t)\left(1-\cos(t-\pi)\right)$$

Similarly, when the input is $+u_{2\pi}(t)$, which which has Laplace transform of $\frac{e^{-2\pi s}}{s}$, therefore, following what we did above, we obtain now

$$\begin{aligned} Y_\pi(s) &= \frac{e^{-2\pi s}}{s\left(s^2+1\right)} \\ &= e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \end{aligned}$$

The effect of $e^{-2\pi s}$ is to cause delay in time. Hence the the inverse Laplace transform of the above is the same as $y_0(t)$ but with now with delay of 2π, therefore

$$y_{2\pi}(t) = +u_{2\pi}(t)\left(1-\cos(t-2\pi)\right)$$

And so on. We see that if we add all the responses, we obtain

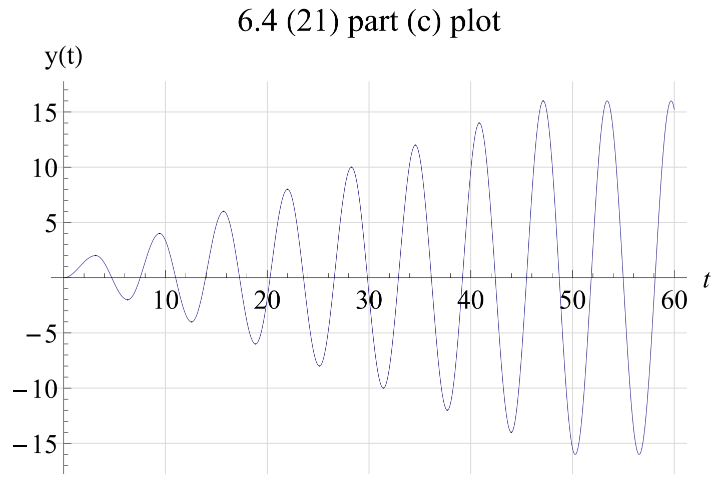
$$\begin{aligned} y(t) &= y_0(t) + y_\pi(t) + y_{2\pi}(t) + \cdots \\ &= (1-\cos t) - u_\pi(t)\left(1-\cos(t-\pi)\right) + u_{2\pi}(t)\left(1-\cos(t-2\pi)\right) - \cdots \end{aligned}$$

Or

$$y(t) = (1-\cos t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t)\left(1-\cos(t-k\pi)\right) \tag{1}$$

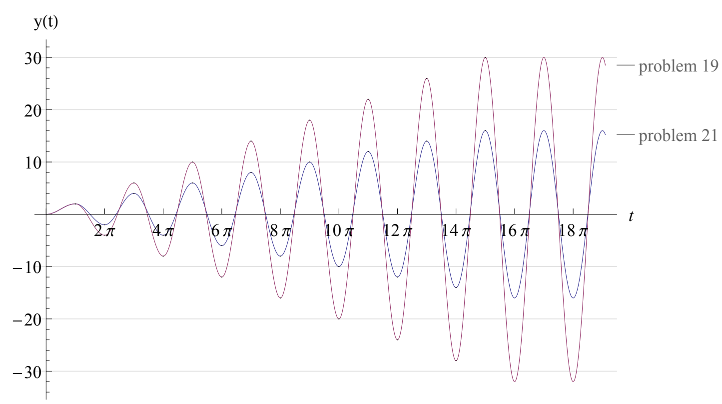
2.10.21.3 Part (c)

This is a plot of (1) for $n = 15$



We see the solution growing rapidly, they settling down after about $t = 50$ to sinusoidal wave at amplitude of about ±15. This shows the system reached steady state at around $t = 50$.

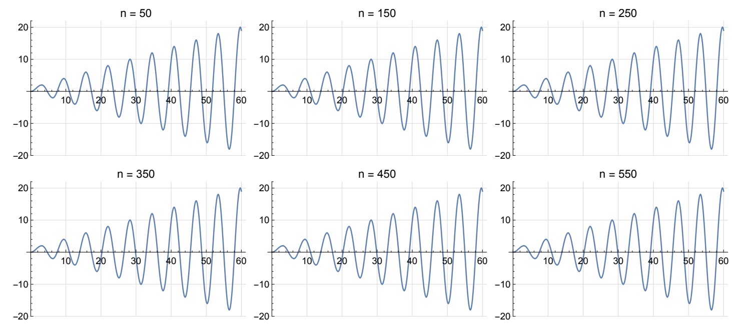
To compare it with problem 19 solution, I used the solution for 19 given in the book, and plotted both solution on top of each others. Also for up to $t = 60$. Here is the result



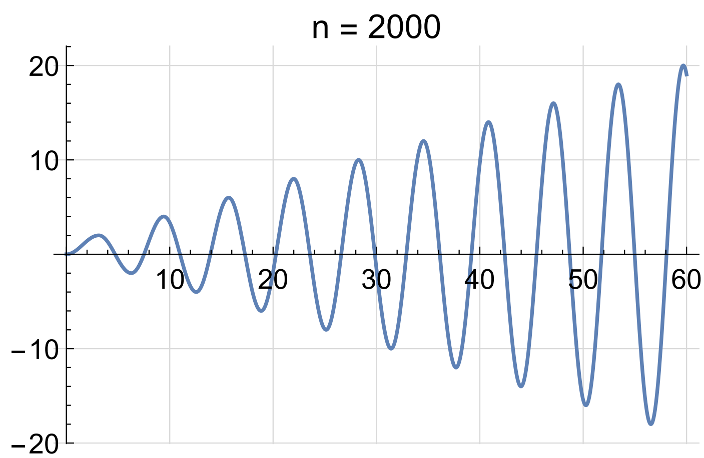
We see that problem 19 output follows the same pattern (since same frequency is used), but with double the amplitude. This is due to the 2 factor used in problem 19 compared to this problem.

2.10.21.4 Part (d)

At first, I tried it with $n = 50, 150, 250, 350, 450, 550$. I can not see any noticeable change in the plot. Here is the result.



Even at $n = 2000$ there was no change to be noticed.



This shows additional input in the form of shifted unit steps, do not change the steady state solution.

2.11 HW9

2.11.1 Section 6.6 problem 1

Question: Establish

1. $f \circledast g = g \circledast f$
2. $f \circledast \left(g_1 + g_2\right) = f \circledast g_1 + f \circledast g_2$
3. $f \circledast \left(g \circledast h\right) = \left(f \circledast g\right) \circledast h$

2.11.1.1 Part (a)

From definition

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$

Let $u = t - \tau$, hence $\frac{du}{d\tau} = -1$. When $\tau = -\infty \rightarrow u = +\infty$ and when $\tau = +\infty \rightarrow u = -\infty$, hence the above becomes

$$f(t) \circledast g(t) = \int_{+\infty}^{-\infty} f(u)\, g(t-u)\,(-du)$$

Pulling the minus sign outside and changing the integration limits

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} g(t-u)\, f(u)\, du$$

But since u is arbitrary, we can relabel u as τ in the above. Hence the above RHS can be written as

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau$$

But $\int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau = g(t) \circledast f(t)$, hence

$$f(t) \circledast g(t) = g(t) \circledast f(t)$$

QED.

2.11.1.2 Part (b)

From definition

$$f(t) \circledast \left(g_1(t) + g_2(t)\right) = \int_{-\infty}^{\infty} f(t-\tau)\left(g_1(\tau) + g_1(\tau)\right) d\tau$$

By linearity of the integral operation, we can break the integral above

$$\int_{-\infty}^{\infty} f(t-\tau)\left(g_1(\tau) + g_1(\tau)\right) d\tau = \int_{-\infty}^{\infty} f(t-\tau)\, g_1(\tau)\, d\tau + \int_{-\infty}^{\infty} f(t-\tau)\, g_1(\tau)\, d\tau$$

But $\int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) d\tau = f(t) \circledast g_1(t)$ and $\int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) d\tau = f(t) \circledast g_2(t)$, hence the above becomes

$$\int_{-\infty}^{\infty} f(t-\tau) \left(g_1(\tau) + g_1(\tau)\right) d\tau = \left(f(t) \circledast g_1(t)\right) + \left(f(t) \circledast g_2(t)\right)$$

Therefore

$$f(t) \circledast \left(g_1(t) + g_2(t)\right) = \left(f(t) \circledast g_1(t)\right) + \left(f(t) \circledast g_2(t)\right)$$

QED.

2.11.1.3 Part (c)

From definition

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} \left(f \circledast g\right)(\tau) h(t-\tau) d\tau \\
&= \int_{\Re} \left[\int_{\Re} f(\tau_1) g(\tau - \tau_1) d\tau_1\right] h(t-\tau) d\tau \\
&= \int_{\Re} \int_{\Re} f(\tau_1) g(\tau - \tau_1) h(t-\tau) d\tau_1 d\tau
\end{aligned}$$

By Fubini, we can change order of integration

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} \int_{\Re} f(\tau_1) g(\tau - \tau_1) h(t-\tau) d\tau d\tau_1 \\
&= \int_{\Re} f(\tau_1) \left[\int_{\Re} g(\tau - \tau_1) h(t-\tau) d\tau\right] d\tau_1
\end{aligned}$$

By translation, if we add τ_1 to τ for both functions in the inner integral above, we obtain

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} f(\tau_1) \left[\int_{\Re} g((\tau + \tau_1) - \tau_1) h(t - (\tau + \tau_1)) d\tau\right] d\tau_1 \\
&= \int_{\Re} f(\tau_1) \left[\int_{\Re} g(\tau) h((t - \tau_1) - \tau) d\tau\right] d\tau_1
\end{aligned}$$

But now we see that inner integral is $\int_{\Re} g(\tau) h((t-\tau_1) - \tau) d\tau = \left(g \circledast h\right)(t - \tau_1)$, hence the above becomes

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} f(\tau_1) \left(g \circledast h\right)(t - \tau_1) d\tau_1 \\
&= \left(f \circledast \left(g \circledast h\right)\right)(t)
\end{aligned}$$

QED

2.11.2 Section 6.6 problem 2

Find an example showing $\left(f \circledast 1\right)(t)$ need not be equal to $f(t)$

Solution Let $f(t) = e^t$, hence

$$\begin{aligned}\left(f \circledast 1\right)(t) &= \int_0^t f(t-\tau) \times 1 d\tau \\ &= \int_0^t e^{(t-\tau)} d\tau \\ &= \left[\frac{e^{(t-\tau)}}{-1}\right]_{\tau=0}^{\tau=t} \\ &= -\left[e^{(t-t)} - e^{(t-0)}\right] \\ &= -\left[e^0 - e^t\right] \\ &= -\left(1 - e^t\right) \\ &= e^t - 1\end{aligned}$$

Which is not the same as e^t. QED

2.11.3 Section 6.6 problem 3

Show that $\left(f \circledast f\right)(t)$ is not necessarily non-negative, using $f(t) = \sin(t)$

Solution From definition

$$\left(f \circledast f\right)(t) = \int_0^t \sin(\tau)\sin(t-\tau)\, d\tau$$

Using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ on the integrand gives

$$\begin{aligned}\left(f \circledast f\right)(t) &= \int_0^t \frac{1}{2}(\cos(\tau-(t-\tau)) - \cos(\tau+(t-\tau)))\, d\tau \\ &= \frac{1}{2}\int_0^t \cos(\tau-(t-\tau))\, d\tau - \frac{1}{2}\int_0^t \cos(t)\, d\tau \\ &= \frac{1}{2}\int_0^t \cos(2\tau-t)\, d\tau - \frac{1}{2}\int_0^t \cos(t)\, d\tau\end{aligned}$$

For the second integral above, since it is w.r.t τ, then we can pull $\cos(t)$ outside, which gives

$$\begin{aligned}\left(f \circledast f\right)(t) &= \frac{1}{2}\left(\frac{\sin(2\tau-t)}{2}\right)_{\tau=0}^{\tau=t} - \frac{1}{2}\cos(t)\int_0^t d\tau \\ &= \frac{1}{4}(\sin(2t-t) - \sin(-t)) - \frac{1}{2}t\cos t \\ &= \frac{1}{4}(\sin(t) + \sin(t)) - \frac{1}{2}t\cos t \\ &= \frac{1}{2}\sin t - \frac{1}{2}t\cos t\end{aligned}$$

Let $t=2\pi$ then

$$\begin{aligned}\left(f\circledast f\right)(t) &= 0-\frac{1}{2}(2\pi)\\ &= -\pi\end{aligned}$$

Which is negative. Hence we showed that $\left(f\circledast f\right)(t)$ can be negative at some t. QED.

2.11.4 Section 6.6 problem 4

Find Laplace transform of $f(t)=\int_0^t (t-\tau)^2\cos(2\tau)\,d\tau$

Solution We see that

$$f(t)=t^2\circledast\cos(2t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\}=\mathscr{L}\left\{t^2\right\}\mathscr{L}\{\cos(2t)\}$$

But $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$ and $\mathscr{L}\{\cos(2t)\}=\frac{s}{s^2+4}$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} &= \left(\frac{2}{s^3}\right)\left(\frac{s}{s^2+4}\right)\\ &= \frac{2}{s^2}\frac{1}{s^2+4}\end{aligned}$$

2.11.5 Section 6.6 problem 5

Find Laplace transform of $f(t)=\int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau$

Solution We see that

$$f(t)=e^{-t}\circledast\sin(t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\}=\mathscr{L}\left\{e^{-t}\right\}\mathscr{L}\{\sin(t)\}$$

But $\mathscr{L}\left\{e^{-t}\right\}=\frac{1}{s+1}$ and $\mathscr{L}\{\sin(t)\}=\frac{1}{s^2+1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\}=\frac{1}{(s+1)\left(s^2+1\right)}$$

2.11.6 Section 6.6 problem 6

Find Laplace transform of $f(t)=\int_0^t (t-\tau)\,e^{\tau}d\tau$

Solution We see that

$$f(t)=t\circledast e^t$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\{t\}\,\mathscr{L}\left\{e^t\right\}$$

But $\mathscr{L}\{t\} = \frac{1}{s^2}$ and $\mathscr{L}\left\{e^t\right\} = \frac{1}{s-1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

2.11.7 Section 6.6 problem 7

Find Laplace transform of $f(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$

Solution We see that

$$f(t) = \sin(t) \circledast \cos(t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\{\sin t\}\,\mathscr{L}\{\cos t\}$$

But $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$ and $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\} = \left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)$$

2.11.8 Section 6.6 problem 8

Find the inverse Laplace transform of $F(s) = \frac{1}{s^4\left(s^2+1\right)}$ using convolution theorem.

Solution We see that

$$\begin{aligned} F(s) &= \frac{1}{s^4}\frac{1}{s^2+1} \\ &= \mathscr{L}\left(\frac{t^3}{6}\right)\mathscr{L}(\sin t) \end{aligned}$$

Hence, using convolution theorem

$$\begin{aligned} f(t) &= \frac{t^3}{6} \circledast \sin t \\ &= \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \; d\tau \end{aligned}$$

Integrate by parts. $\int udv = uv - \int vdu$. Let $u = (t-\tau)^3, dv = \sin\tau \rightarrow du = -3(t-\tau)^2, v = -\cos\tau$,

hence

$$
\begin{aligned}
\frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(-\left[(t-\tau)^3\cos\tau\right]_0^t - \int_0^t -3(t-\tau)^2(-\cos\tau)\, d\tau\right) \\
&= \frac{1}{6}\left(-\left[(t-t)^3\cos t - (t-0)^3\cos 0\right] - 3\int_0^t (t-\tau)^2(\cos\tau)\, d\tau\right) \\
&= \frac{1}{6}\left(-\left[0-t^3\right] - 3\int_0^t (t-\tau)^2(\cos\tau)\, d\tau\right) \\
&= \frac{1}{6}\left(t^3 - 3\int_0^t (t-\tau)^2(\cos\tau)\, d\tau\right)
\end{aligned}
$$

Integrate by parts. Let $u=(t-\tau)^2, dv=\cos\tau \rightarrow du=-2(t-\tau), v=\sin\tau$, hence

$$
\begin{aligned}
\frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(t^3 - 3\left[\left((t-\tau)^2\sin\tau\right)_0^t - \int_0^t -2(t-\tau)\sin\tau d\tau\right]\right) \\
&= \frac{1}{6}\left(t^3 - 3\left[\left((t-t)^2\sin t - (t-0)^2\sin 0\right)_0^t + 2\int_0^t (t-\tau)\sin\tau d\tau\right]\right) \\
&= \frac{1}{6}\left(t^3 - 3\left[0 + 2\int_0^t (t-\tau)\sin\tau d\tau\right]\right) \\
&= \frac{1}{6}\left(t^3 - 6\int_0^t (t-\tau)\sin\tau d\tau\right)
\end{aligned}
$$

Integrate by parts. Let $u=(t-\tau), dv=\sin\tau \rightarrow du=-1, v=-\cos\tau$, hence above becomes

$$
\begin{aligned}
\frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(t^3 - 6\left[\left(-(t-\tau)\cos\tau\right)_0^t - \int_0^t \cos\tau d\tau\right]\right) \\
&= \frac{1}{6}\left(t^3 - 6\left[-\left((t-t)\cos t - (t-0)\cos 0\right) - (\sin\tau)_0^t\right]\right) \\
&= \frac{1}{6}\left(t^3 - 6\left[-(0-t) - \sin t\right]\right) \\
&= \frac{1}{6}\left(t^3 - 6(t-\sin t)\right) \\
&= \frac{1}{6}\left(t^3 - 6t + 6\sin t\right)
\end{aligned}
$$

Hence

$$
f(t) = \frac{1}{6}\left(t^3 - 6t + 6\sin t\right)
$$

2.11.9 Section 6.6 problem 9

Find the inverse Laplace transform of $F(s)=\frac{s}{(s+1)(s^2+4)}$ using convolution theorem.

Solution We see that

$$
\begin{aligned}
F(s) &= \frac{1}{s+1}\frac{s}{s^2+4} \\
&= \mathscr{L}\left(e^{-t}\right)\mathscr{L}(\cos 2t)
\end{aligned}
$$

Hence, using convolution theorem

$$\begin{aligned} f(t) &= e^{-t} \circledast \cos 2t \\ &= \int_0^t e^{-(t-\tau)} \cos 2\tau \; d\tau \end{aligned}$$

Integrate by parts. $\int u dv = uv - \int v du$. Let $u = \cos 2\tau, dv = e^{-(t-\tau)} \rightarrow du = -2\sin 2\tau, v = e^{-(t-\tau)}$, hence

$$\begin{aligned} \int_0^t e^{-(t-\tau)} \cos 2\tau \; d\tau &= \left(\cos 2\tau e^{-(t-\tau)}\right)_0^t - \int_0^t e^{-(t-\tau)} (-2\sin 2\tau)\, d\tau \\ &= \left(\cos 2t e^{-(t-t)} - \cos 0 e^{-(t-0)}\right) + 2\int_0^t e^{-(t-\tau)} \sin 2\tau d\tau \\ &= \left(\cos 2t - e^{-t}\right) + 2\int_0^t e^{-(t-\tau)} \sin 2\tau d\tau \end{aligned}$$

Integrate by parts. Let $u = \sin 2\tau, dv = e^{-(t-\tau)} \rightarrow du = 2\cos 2\tau, v = e^{-(t-\tau)}$, hence

$$\begin{aligned} \int_0^t e^{-(t-\tau)} \cos 2\tau \; d\tau &= \left(\cos 2t - e^{-t}\right) + 2\left[\left(\sin 2\tau e^{-(t-\tau)}\right)_0^t - \int_0^t e^{-(t-\tau)} 2\cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\left[\left(\sin 2t e^{-(t-t)} - 0\right) - 2\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\left[\sin 2t - 2\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\sin 2t - 4\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau \end{aligned}$$

Hence

$$\begin{aligned} \int_0^t e^{-(t-\tau)} \cos 2\tau \; d\tau + 4\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \cos 2t - e^{-t} + 2\sin 2t \\ 5\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \cos 2t - e^{-t} + 2\sin 2t \\ \int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \frac{1}{5}\left(\cos 2t - e^{-t} + 2\sin 2t\right) \end{aligned}$$

Therefore

$$f(t) = \frac{1}{5}\left(\cos 2t - e^{-t} + 2\sin 2t\right)$$

2.11.10 Section 6.6 problem 10

Find the inverse Laplace transform of $F(s) = \frac{1}{(s+1)^2(s^2+4)}$ using convolution theorem.

Solution We see that

$$F(s) = \frac{1}{(s+1)^2}\frac{1}{s^2+4} = \mathscr{L}\left(te^{-t}\right)\mathscr{L}\left(\frac{1}{2}\sin 2t\right)$$

Hence, using convolution theorem

$$\begin{aligned} f(t) &= te^{-t} \circledast \frac{1}{2}\sin 2t \\ &= \frac{1}{2}\int_0^t (t-\tau)\, e^{-(t-\tau)} \sin 2\tau \\ &= \frac{1}{2}\int_0^t te^{-(t-\tau)} \sin 2\tau \, d\tau - \frac{1}{2}\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau \end{aligned}$$

The first integral is

$$\int_0^t te^{-(t-\tau)} \sin 2\tau \, d\tau = t\int_0^t e^{-(t-\tau)} \sin 2\tau \, d\tau$$

This is similar to one we did in problem 10 but now we have $\sin 2\tau$. Using integration by parts again as before gives

$$\begin{aligned} t\int_0^t e^{-(t-\tau)} \sin 2\tau \, d\tau &= t\left(\frac{1}{5}\left(2e^{-t} - 2\cos 2t + \sin 2t\right)\right) \\ &= \frac{t}{5}\left(2e^{-t} - 2\cos 2t + \sin 2t\right) \end{aligned}$$

Now we need to evaluate the second integral $\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau$. This can also be done using integration by part. But I used CAS here, the result is

$$\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau = \frac{1}{25}\left(-4e^{-t} + (4-10t)\cos 2t + (3+5t)\sin 2t\right)$$

Therefore

$$\begin{aligned} f(t) &= \frac{1}{2}\frac{t}{5}\left(2e^{-t} - 2\cos(2t) + \sin(2t)\right) - \frac{1}{2}\frac{1}{25}\left(-4e^{-t} + (4-10t)\cos(2t) + (3+5t)\sin(2t)\right) \\ &= \frac{2}{25}e^{-t} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t + \frac{1}{5}te^{-t} \end{aligned}$$

2.11.11 Section 6.6 problem 11

Find the inverse Laplace transform of $F(s) = \frac{G(s)}{s^2+1}$ using convolution theorem.

Solution We see that

$$F(s) = G(s)\frac{1}{s^2+1} = G(s)\mathscr{L}(\sin t)$$

Hence, using convolution theorem

$$f(t) = g(t) \circledast \sin t = \int_0^t \sin(t-\tau)\, g(\tau)\, d\tau$$

Or

$$f(t) = \int_0^t g(t-\tau)\sin(\tau)\,d\tau$$

Chapter 3

Discussion

Local contents

3.1 week 3

319 Discussion Week 3

September 20, 2016

1. Draw a direction field for the given differential equation. Determine the behavior of y as $t \to \infty$. If this behavior depends on a chosen initial condition, describe the dependency.
   (a) $y' = y^2$.
   (b) $y' = y(y-6)$.
2. Determine the values of r for which e^{rt} is a solution to $y'' - y' - 2e^{rt} = 0$.
3. Solve the following differential equations. If initial conditions are provided, solve the resulting initial value problem.
   (a) $y' - y = 2te^{2t}$.
   (b) $y' + y = 5\sin(2t)$.
   (c) $ty' + 2y = \sin(t), y(\pi/2) = 1$.
   (d) $y' = y(1-y), y(0) = 2$.

3.2 week 4

319 Discussion Week 4

September 27, 2016

1. Solve the given differential equation.
   (a) $y' = \sin^2(t)\cos^2(y)$;
   (b) $y' = \frac{2x}{1+2y}$;
   (c) $xy' = 1 + y^2$.
2. Consider a tank used in certain hydrodynamics. After one experiment the tank contains 500 L of a dye solution with a concentration of 2 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 5 L/min, the well stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 50% of its original value.
3. Determine an interval in which the solution of the given initial value problem is certain to exist.
   (a) $y' + \sin(t)y = \cos(t), y(0) = 0$;
   (b) $t(t-1)y' = y, y(1/2) = 1$;
   (c) $\log(t)y' + y = \tan(t), y(2) = 5$.

1

3.3 week 5

319 Discussion Week 5

October 4, 2016

1. Compute the following partial derivatives.
   (a) $\partial_y(x^3 + 2x + e^x)$.
   (b) $\partial_x(3ye^x + y^2x^3)$.
   (c) $\partial_y(\cos(x) + 3\log(x + y))$.

2. The following problems involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical points, and classify each one as asymptotically stable or unstable. In each case, draw the phase line.
   (a) $dy/dt = y + y^3$, $y_0 \leq 0$.
   (b) $dy/dt = e^y - 1$
   (c) $dy/dt = y(y - 1)(y - 2), y_0 > 0$.

3. Determine whether each of the following equations is exact. If it is exact, find the solution.
   (a) $(x\log(y) + xy) + (y\log(x) + xy)y' = 0$
   (b) $(9x^2 + y - 1) - (4y - x)y' = 0$
   (c) $(e^x\cos(y) + 3y + 2x) + (3x - e^x\sin(y) + 2y)y'$.

3.4 week 7

319 Discussion Week 7

October 18, 2016

1. In the following problems, show that the given equation is not exact, but becomes exact when multiplied by the given integrating factor μ. Solve the resulting equation.
   (a) $y + (2x - ye^y)y' = 0, \mu = y$;
   (b) $x^2y^3 + x(1 + y^2)y' = 0, \mu = \frac{1}{xy^3}$.

2. Use Euler's method to find approximate values of the solution to $y' = y^2 + t, y(0) = 1$ at $t = .5$ with $h = .1$

3. In each of the following problems, find the general solution to the given equation:
   (a) $y'' + 3y' - 10y = 0$;
   (b) $y'' + y' - y = 0$.

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3.5 week 8

319 Discussion Week 8

November 1, 2016

1. In the following problems, find the solution to the given initial value problem. Describe the behavior of the solution as t increases.
   (a) $2y'' - 3y' + y = 0; y(0) = 2, y'(0) = 1/2$.
   (b) $y'' - y' - 2y = 0; y(0) = -1, y'(0) = 2$.
   (c) $y'' + 3y' = 0; y(0) = -2, y'(0) = 3$.
2. In each of the following problems, compute the Wronskian of the given pair of functions.
   (a) x^2, xe^x.
   (b) $\tan(t), \cos(t)$.
   (c) $\log(x), x^3$.
   (d) $e^x \sin(x), e^x \cos(x)$.

3.6 week 9

319 Discussion Week 9

November 1, 2016

1. In the following problems, find the general solution to the given differential equation.

   (a) $y'' - 6y' + 9y = 0$;

   (b) $y'' + 4y' + 4y = 0$;

   (c) $4y'' + 4y' + y = 0$.

2. In the following problems, find the general solution to the given differential equation.

   (a) $y'' - y' - 2y = t - 1$;

   (b) $y'' - 2y' - 3y = 3e^{5t}$;

   (c) $y'' + 2y' + 5y = 3\sin(2t)$;

   (d) $y'' + 2y' = 3 + 4\sin(2t)$.

3.7 week 10

319 Discussion Week 10

November 9, 2016

1. In the following problems, find a particular solution to the given equation using the method of variation of parameters, as well as the method of undetermined coefficients.
   (a) $4y'' - 4y' + y = 16e^{t/2}$;
   (b) $y'' + 2y' + y = 3e^{-t}$;
   (c) $y'' + 2y' + y = te^{-t}$.
2. In the following problems, find the general solution to the given differential equation.
   (a) $y'' + y = \tan(t), 0 < t < \pi/2$;
   (b) $y'' + 4y = 3\csc(2t), 0 < t < \pi$.

3.8 week 11

319 Discussion Week 11

November 15, 2016

1. In the following problems, find the general solution to the given differential equation.

   (a) $y'' + 4y = 3\csc(2t), 0 < t < \pi$;

   (b) $y'' + 2y' = 3 + 4\sin(2t)$;

   (c) $y'' + 2y' + y = te^{-t}$;

   (d) $y'' + 16y = g(t)$, where $g(t)$ is an arbitrary continuous function;

   (e) $t^2y'' - 2y = 3t^2 - 1, t > 0$, given that t^2 and t^{-1} are solutions to the equation $t^2y'' - 2y = 0$.

3.9 week 13

319 Discussion Week 12

November 28, 2016

1. Compute the Laplace transform of the following functions:
   (a) $\cosh(bt)$, where b is some constant;
   (b) $\sinh(bt)$, where b is some constant;
   (c) t;
   (d) $t\cosh(bt)$, where b is some constant;
   (e) $\cos(bt)$, where b is some constant (Hint: $\cos(bt) = \frac{e^{bit}+e^{-bit}}{2}$);
   (f) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty. \end{cases}$

2. Compute the inverse Laplace transform of the following functions:
   (a) $\frac{3}{s^2+4}$;
   (b) $\frac{2s+1}{s^2-2s+2}$;
   (c) $\frac{4}{(s-1)^3}$.

3.10 week 14

319 DISCUSSION WEEK 14

(1) In the following problems, express the given function in terms of the functions $u_c(t)$, then compute its Laplace transformation.

(a) $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ 2, & t \geq 2; \end{cases}$

(b) $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 3, & 1 \leq t < 2 \\ 0, & t \geq 2; \end{cases}$

(c) $f(t) = \begin{cases} t^2, & 0 \leq t < 2 \\ 1, & t \geq 2. \end{cases}$

(2) Solve the given initial value problem using the Laplace transformation.

(a) $y'' + 2y' + y = 4e^{-t}, y(0) = 2, y'(0) = -1;$

(b) $y'' - 2y' + 2y = \cos(t), y(0) = 1, y'(0) = 0;$

(c) $y'' + y = \begin{cases} t, & 0 \leq t < 1 \\ 0 & t \geq 1. \end{cases}, y(0) = y'(0) = 0$

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Chapter 4

Exams

Local contents

4.1 Final exam, Dec. 22, 2016