University Course

Physics 321
Mechanics

University of Wisconsin, Madison
Fall 2015

My Class Notes

Nasser M. Abbasi

Fall 2015

Contents

Chapter 1

Introduction

Local contents

Took this course in Spring 2015 to learn a little about mechanics.

Instructor: professor Stefan Westerhoff Office hrs: During the fall semester, office hours are Wednesdays 1:30 to 3:30 PM

class web page learn UW

1.1 Syllubus

Mechanics

Physics 311

Fall 2015

Classes:	Monday, Wednesday, Friday 11:00 am - 11:50 am
	Van Hise 494
Discussion Section:	Session 1 (DIS 303): Thursdays 1:20 pm - 2:10 pm
	Chamberlin Hall 2108
	Session 2 (DIS 301): Thursdays 2:25 pm - 3:15 pm
	Van Vleck B235
Instructor:	Stefan Westerhoff
E-mail:	stefan.westerhoff@wisc.edu
Office:	Chamberlin Hall, Room 4209
Office Hours:	Wednesdays 1:30 pm - 3:30 pm, or by appointment
	(no office hours on Sep. 9 & Oct. 14)
TA:	James Hanson
E-mail:	jehanson2@wisc.edu
Textbook:	S.T. Thornton, J.B. Marion,
	Classical Dynamics of Particles and Systems,
	5$^{\text{th}}$ Edition, Brooks/Cole, 2004

Homework

Homework is assigned each Friday to be handed in 7 days later in class. Teamwork is encouraged in solving the homework problems, but the write-up must be entirely your own work. Homework and exam solutions will be posted on the course page which is accessible via Learn@UW.

Examinations and Grades

There will be two in-class midterms and a final exam. Final grades will be based on the midterms (20 % each), the final exam (40 %), and the homework (20 %).

Other Helpful Books

(1) L.D. Landau & E.M. Lifshitz, *Mechanics*, 3$^{\text{rd}}$ ed., Butterworth-Heinemann, 1976

(2) V. Barger & M. Olsson, *Classical Mechanics: A Modern Perspective*, McGraw-Hill, 1973

Class Schedule

	1. Newtonian Mechanics.
Sep 2, 4	—– Introduction. Newton's Laws.
	2. Lagrangian Mechanics.
Sep 9	—– Motivation. Principle of Least Action.
Sep 11	—– Euler-Lagrange Equations.
Sep 14	—– Lagrange Equations of Motion.
Sep 16, 18	—– Conservation Laws. Mechanical Similarity.
Sep 21	—– Lagrange Multipliers.
	3. Oscillations.
Sep 23	—– Equilibrium. Free Oscillations in One Dimension.
Sep 25, 28	—– Damped Oscillations. Phase Space.
Sep 30, Oct 2	—– Forced Oscillations.
Oct 5	—– Nonlinear Oscillations.
	4. Gravitation and Central Force Motion.
Oct 7	—– Gravitational Fields.
Oct 9, 12	—– Tidal Forces.
Oct 14	*Midterm 1*
Oct 16	—– Two-Body-Problem.
Oct 19, 21, 23	—– Kepler's Laws. Stability of Orbits.
	5. Systems of Particles.
Oct 26, 28	—– Elastic Collisions. Inelastic Collisions.
Oct 30	—– Motion of Bodies with Variable Mass.
Nov 2, 4	—– Scattering in a Central Force Field.
	6. Noninertial Reference Frames.
Nov 6	—– Langrangian and Equations of Motion in Noninertial Frames.
Nov 9, 11	—– Motion Relative to the Earth. Foucault Pendulum.
	7. Motion of a Rigid Body.
Nov 13	—– Rigid Bodies.
Nov 16	*Midterm 2*
Nov 18	—– Inertia Tensor.
Nov 20	—– Principal Axis Transformation. Parallel-Axis Theorem.
Nov 23, 25	—– Equations of Motion. Euler Angles.
Nov 30	—– Symmetric Top in a Gravitational Field. Stability of Rotation.
	8. Coupled Oscillations.
Dec 2	—– Two Coupled Harmonic Oscillators.
Dec 4, 7	—– Systems with N Degrees of Freedom.
	9. Hamiltonian Dynamics.
Dec 9, 11	—– The Canonical Equations.
Dec 14	—– Virial Theorem.
Dec 17	*Final Exam* (5:05 pm - 7:05 pm, room TBA).

Chapter 2

Lecture notes

Local contents

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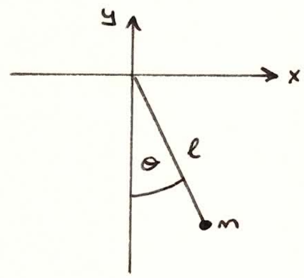
2.7 Lagrange Multipliers

In a couple of examples, we have used constraints to reduce the number of coordinates.

Example: Simple pendulum



coordinates x, y constraint: $x^2 + y^2 - \ell^2 = 0$

If the equations of constraint are of the form

$$f_j(q_i, t) = 0, \qquad \begin{array}{ll} i = 1, 2, \dots N & \text{, N particles} \\ j = 1, 2, \dots m & \text{, m equations of constraint} \end{array}$$

the constraints are called holonomic.

If a system is subject to holonomic constraints, there is always a set of proper coordinates in terms of which the equations of motion are free from explicit reference to the constraints.

In the example:

$$x = \ell \sin\theta$$
$$y = -\ell \cos\theta$$

$\Rightarrow$ use θ as the (only) coordinate in the calculation, and the equations of motion have the constraint "built in" and we do not have to worry about the constraint

2.1 Dec. 3, 2015, symmetric top notes

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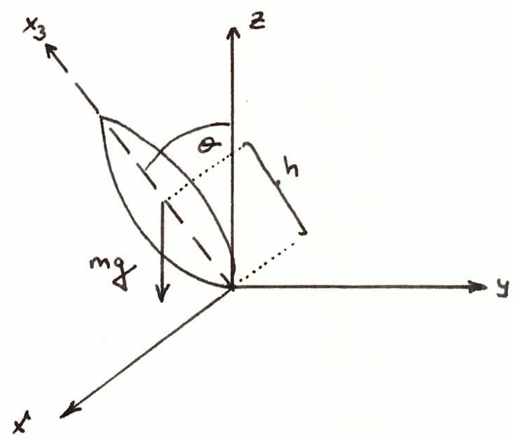
7.6 Symmetric Top in a Gravitational Field

$I_1 = I_2 \neq I_3$

Lowest point is fixed



Lagrangian: $L = T - U$ $\quad U = mgh\cos\theta$

$$T = \frac{1}{2} I_i \omega_i^2$$

$$= \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

$$\omega_1^2 = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2$$

$$= \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \cos^2\psi$$

$$\omega_2^2 = (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)^2$$

$$= \dot\phi^2 \sin^2\theta \cos^2\psi - 2\dot\phi\dot\theta \sin\theta \cos\psi \sin\psi + \dot\theta^2 \sin^2\psi$$

$$\Rightarrow \omega_1^2 + \omega_2^2 = \dot\phi^2 \sin^2\theta + \dot\theta^2$$

$$\omega_3^2 = (\dot\phi \cos\theta + \dot\psi)^2$$

Chapter 3

Exams

Local contents

3.1 first midterm

3.1.1 practice exam

3.1.1.1 questions

Mechanics
Physics 311
Fall 2012
Midterm 1 (October 5, 2012)

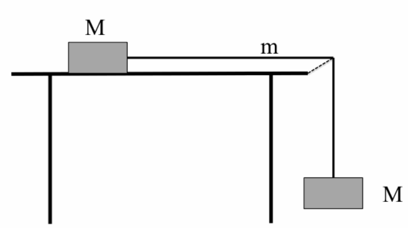
There are 50 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way. For partial credit you must show your work. The exam is closed book, but you are allowed to bring one letter size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

Problem 1 (15 points)

Two blocks of equal mass M are connected by a cord of length l. One block is placed on a smooth horizontal table, the other block hangs over the edge. The cord is heavy and has a total mass m.

(a) (1 point) How many generalized coordinates are needed to describe the system?
(b) (6 points) Determine the Lagrangian of this system.
(c) (6 points) From the Lagrangian, obtain the differential equation(s) governing the motion of the system.
(d) (2 points) Find the acceleration of the blocks in the special case that the mass of the cord can be neglected ($m = 0$).



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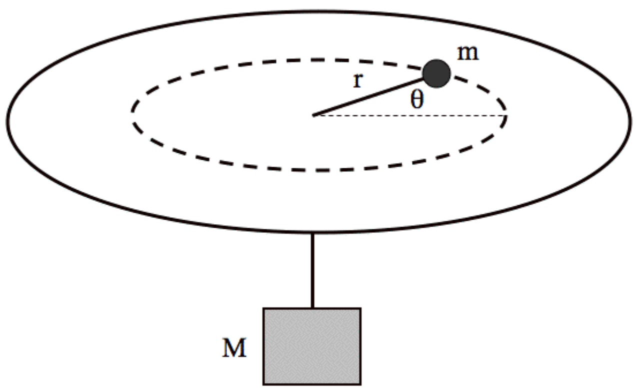
Problem 2 (15 points)

An object of mass m slides on a horizontal, friction-free table. A light, inextensible string, which passes through a small hole in the table, attaches the mass to a second body of mass M. The second body hangs below the table as shown below.

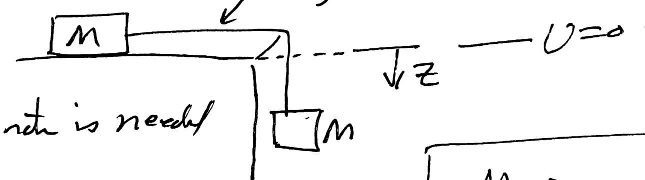
(a) (1 point) How many generalized coordinates are needed to describe the system?
(b) (4 points) Determine the Lagrangian of the system.
(c) (5 points) Determine the differential equation(s) governing the motion of the system.
(d) (3 points) For the special case that r is constant, solve the resulting equation(s) and interpret your results.
(e) (2 points) What are the integrals of motion for this system?



3.1.1.2 my solution to practice exam

my solution for practice exam 1. Physics 311.
by Nasser M. Abbasi

(1)



(a) one coordinate is needed

(b) $U = -Mg\left(\frac{z}{\cancel{2}}\right) - \underbrace{m\left(\frac{z}{l}\right)\left(\frac{z}{2}\right)g}_{\text{P.E. for hanging part of wire}} = \boxed{-Mg\frac{z}{\cancel{2}} - \frac{mz^2}{2l}g}$

$$T = \overbrace{\tfrac{1}{2}M\dot{z}^2 + \tfrac{1}{2}M\dot{z}^2}^{\text{both block move same speed}} + \underbrace{\tfrac{1}{2}m\dot{z}^2}_{\text{K.E. for wire}} = M\dot{z}^2 + \tfrac{1}{2}m\dot{z}^2$$

so $\boxed{L = T - U = \cancel{M}\dot{z}^2\left(M + \frac{m}{2}\right) + Mg\frac{z}{\cancel{2}} + m\frac{z^2}{2l}g}$

(c) $\frac{\partial L}{\partial z} = M\frac{g}{\cancel{2}} + \frac{mz}{l}g$

$\frac{\partial L}{\partial \dot{z}} = 2\dot{z}\left(M + \frac{m}{2}\right)$

$\Rightarrow \boxed{2\ddot{z}\left(M + \frac{m}{2}\right) - \left(M\frac{g}{\cancel{2}} + \frac{mz}{l}g\right) = 0}$

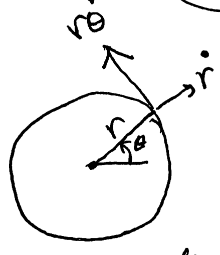
$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$

(d) From (c). $\ddot{z} = \dfrac{g\left(\frac{M}{\cancel{2}} + \frac{mz}{l}\right)}{2\left(M + \frac{m}{2}\right)}$. When $m=0$ we obtain

$$\ddot{z} = \frac{g\frac{M}{\cancel{2}}}{2M} = \boxed{\frac{g}{2}}$$

problem 2

Topview:



velocity diagram

$$\Rightarrow v^2 = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2$$

$$\boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2} \leftarrow \text{velocity of small } m$$

large M has same radial speed, since string do not stretch.

(a) need 2 generalized coordinates, $\boxed{r, \theta}$

(b) $U = -Mg(l-r)$ assuming string has total length $= l$.

$$T = \frac{1}{2}mv^2 + \frac{1}{2}M\dot{r}^2$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}M\dot{r}^2$$

so $$\boxed{L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}M\dot{r}^2 + Mg(l-r)}$$

(c) For r $\quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - Mg$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + M\dot{r} = \dot{r}(M+m).$$

so $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \Rightarrow \boxed{\ddot{r}(M+m) = mr\dot{\theta}^2 - Mg}$ (1)

for θ $\quad \frac{\partial L}{\partial \theta} = 0. \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

so $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \ddot{\theta}mr^2 = 0$ or $\boxed{\ddot{\theta} = 0}$ (2)

(d) when r constant, then (1) becomes $0 = mr\dot{\theta}^2 - Mg = 0$

$\dot{\theta}^2 = \frac{Mg}{mr}$ or $\dot{\theta} = \pm\sqrt{\frac{Mg}{mr}} \Rightarrow \boxed{\theta = \pm\sqrt{\frac{Mg}{mr}}\,t + \text{Constant}}$

when r constant, then $\dot{\theta}$ is also constant.

(e) From (2), integral of motion is $\frac{\partial L}{\partial \dot{\theta}} = \text{constant} \Rightarrow mr^2\dot{\theta} = \text{constant}$ angular momentum.

3.1.1.3 key solution to practice exam

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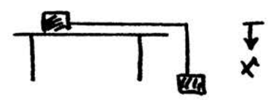
Mechanics

Physics 311 - Fall 2012

Midterm 1 - Solutions

Problem 1:



(a) one coordinate describes the system, for example x (as shown)

(b)

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2$$

$$= M\dot{x}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -Mgx - \frac{m}{\ell} x g \frac{x}{2}$$

fraction of cord mass hanging over the edge: $\frac{m}{\ell} \cdot x$

center of mass is at $\frac{x}{2}$ for the overhanging part

$$\Rightarrow \boxed{L = \left(M + \frac{m}{2}\right)\dot{x}^2 + Mgx + \frac{mg}{2\ell} x^2}$$

(c)

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad Mg + \frac{mg}{\ell} x - 2\left(M + \frac{m}{2}\right)\ddot{x} = 0$$

$$\Rightarrow \quad (2M + m)\ddot{x} = \frac{g}{\ell}(M\ell + mx)$$

$$\Leftrightarrow \quad \boxed{\ddot{x} = \frac{g}{\ell}\left(\frac{M\ell + mx}{2M + m}\right)}$$

(d) for $m = 0$, the acceleration of the blocks is

$$\ddot{x} = \frac{g}{\ell}\frac{M\ell}{2M} \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{g}{2}}$$

2

Problem 2:

(a) Since the object can move in r and θ, we need two generalized coordinates

(b) $T = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$

$U = Mgr$ ($U=0$ for $r=0$)

$$\Rightarrow \boxed{L = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - Mgr}$$

(c) in θ: $\frac{\partial L}{\partial \theta} = 0 \Rightarrow$ Lagrangian does not explicitely depend on θ, so there is an integral of the motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 = \frac{d}{dt}\underbrace{(mr^2\dot{\theta})}_{\text{angular momentum}}$$

$mr^2\dot{\theta} = \text{const.}$

in r: $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$

$$\Rightarrow mr\dot{\theta}^2 - Mg - (m+M)\ddot{r} = 0$$

$$\Leftrightarrow \boxed{(m+M)\ddot{r} - mr\dot{\theta}^2 + Mg = 0}$$

(d) $r = \text{const.} \Rightarrow \ddot{r} = 0 \Rightarrow mr\dot{\theta}^2 = Mg$

so $\dot{\theta} = \omega = \text{const}'.$

$\Rightarrow mr\omega^2 = Mg$

or $\boxed{\frac{mv^2}{r} = Mg}$

so Mg is the force responsible for the centripetal acceleration

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(e) there are two integrals of the motion,

(i) $mr^2\dot{\theta}$ (angular momentum)

(ii) mechanical energy $T+U$, since the Lagrangian does not explicitely depend on time

3.1.2 Review to first midterm

By James Hanson

Problem 1

Consider a small ball of radius s and moment of inertia I rolling off of a sphere of radius R. At what angle does the ball leave the surface of the sphere if it is gently displaced from the top (i.e. total energy is equal to potential energy of a stationary ball at the top)?

A lot of the new difficulty of this problem (relative to the particle sliding off of a sphere) comes from setting up the constraints correctly.

The Lagrangian (without constraints) is given by:

$$\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}I\dot{\phi}^2-mgr\cos\theta$$

The distance from the center of the big sphere to the center of the small sphere is $R+s$, so the natural constraint for that is $r=R+s$. We also need to constrain the rolling of the ball relative to the motion of the ball along the sphere. The simplest constraint that will work is setting the arclength along the ball to the arclength along the surface of the sphere, i.e. $R\theta=s\phi$ (I was being overly cautious when I said that this wouldn't work in discussion). So the constraint function is given by $\lambda_1(r-R-s)+\lambda_2(R\theta-s\phi)$ and now our equations of motion become:

$$m\ddot{r}-mr\dot{\theta}^2+mg\cos\theta+\lambda_1=0$$

$$m(r^2\ddot{\theta}+2r\dot{r}\dot{\theta})+mgr\sin\theta+\lambda_2 R=0$$

$$I\ddot{\phi}-\lambda_2 s=0$$

With some substitutions from the constraint equations and their time derivatives we can reduce this to:

$$-m(R+s)\dot{\theta}^2+mg\cos\theta+\lambda_1=0$$

$$m(R+s)^2\ddot{\theta}+mg(R+s)\sin\theta+(\frac{R}{s})^2I\ddot{\theta}=0$$

And to completely solve this problem we need to use conservation of energy. The Hamiltonian (total energy) of the system is given by

$$\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}I\dot{\phi}^2+mgr\cos\theta=\frac{1}{2}m(R+s)^2\dot{\theta}^2+\frac{1}{2}I(\frac{R}{s}\dot{\theta})^2+mg(R+s)\cos\theta$$

And since the ball has been 'gently pushed' from the top of the sphere we have

$$\left[\frac{1}{2}m(R+s)^2+\frac{1}{2}I(\frac{R}{s})^2\right]\dot{\theta}^2+mg(R+s)\cos\theta=mg(R+s)$$

1

The ball will leave the surface of the sphere when the constraint force (corresponding to the normal force) that keeps the radius fixed changes signs, i.e. when $\lambda_1 = 0$. So we have

$$m(R+s)\dot{\theta}^2 = mg\cos\theta$$

Putting these two together we have

$$\left[\frac{1}{2}m(R+s)^2 + \frac{1}{2}I(\frac{R}{s})^2\right]\frac{g\cos\theta}{R+s} + mg(R+s)\cos\theta = mg(R+s)$$

$$\left[m(R+s)^2 + I(\frac{R}{s})^2 + 2m(R+s)^2\right]\cos\theta = 2m(R+s)^2$$

$$\cos\theta = \frac{2m(R+s)^2}{3m(R+s)^2 + I(\frac{R}{s})^2}$$

which you can see reduces to $\frac{2}{3}$ when $I = 0$, consistent with the simpler version of the problem.

2

Problem 2

The Lagrangian of a free particle in a magnetic field is given by $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q(A_x\dot{x} + A_y\dot{y})$, where A is the magnetic vector potential (whose curl is the magnetic field). Consider the field given by $A_x = \alpha y$, $A_y = 0$. Find the equations of motion and solve them. Find an integral of motion that is not energy and confirm that it is conserved.

The Lagrangian in this case is given by

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q\alpha y\dot{x}$$

So the equations of motion are given by

$$m\ddot{x} + q\alpha\dot{y} = 0$$

$$m\ddot{y} - q\alpha\dot{x} = 0$$

Let $\beta = \frac{q\alpha}{m}$ and note that we have $\dddot{x} + \beta\ddot{y} = 0$ and therefore

$$\dddot{x} + \beta^2\dot{x} = 0$$

Which is the equation of a harmonic oscillator in $\dot{x}$. The same equation can be derived for $\dot{y}$, so we know the solution must have the form

$$\dot{x} = A\cos(\beta t + \phi)$$

$$\dot{y} = B\cos(\beta t + \psi)$$

Plugging these into the original equations constrains A, ϕ, B, and ψ relative to each other. Assume without loss of generality that $\phi = 0$, then you can show that the solution must be of the form

$$\dot{x} = A\cos\beta t$$

$$\dot{y} = A\sin\beta t$$

So integrating gives the full solution:

$$x = x_0 + \frac{A}{\beta}\sin\beta t$$

$$y = y_0 - \frac{A}{\beta}\cos\beta t$$

For the integral of motion notice that the Lagrangian has no x dependence, therefore the corresponding generalized momentum $\frac{\partial L}{\partial \dot{x}}$ must be conserved.

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + q\alpha y$$

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Plugging in the solution we got gives

$$mA\cos\beta t + q\alpha(y_0 - \frac{A}{\beta}\cos\beta t) = q\alpha y_0$$

which is in fact a conserved quantity. There actually is an analogous generalized momentum for y but it is less obvious why it should be conserved.

Problem 4

Consider an anharmonic (or nonlinear) spring with potential energy $V = \frac{1}{2}kr^2 + \frac{1}{4}\alpha r^4$ $(k, \alpha > 0)$ spinning at some fixed angular frequency ω_0 with a mass at the end. What are the equilibrium positions of the system as a function of ω_0 and which equilibria are stable?

The coordinates in this problem are given by

$$x = r\cos\omega_0 t$$

$$y = r\sin\omega_0 t$$

with derivatives

$$\dot{x} = \dot{r}\cos\omega_0 t - r\omega_0\sin\omega_0 t$$

$$\dot{y} = \dot{r}\sin\omega_0 t + r\omega_0\cos\omega_0 t$$

So our kinetic energy is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\omega_0^2)$$

$$\boxed{U = \frac{1}{2}kr^2 + \frac{1}{4}\alpha r^4}$$

And our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\omega_0^2) - \frac{1}{2}kr^2 - \frac{1}{4}\alpha r^4$$

Giving equation of motion

$$m\ddot{r} = -(k - \omega_0^2)r - \alpha r^3$$

This is at equilibrium when $\ddot{r} = 0$ or in other words $(k - \omega_0^2)r + \alpha r^3 = 0$. This is always solved by $r = 0$, but it is also solved by $r = \pm\sqrt{\frac{\omega_0^2 - k}{\alpha}}$. If $\omega_0^2 < k$ then these solutions are imaginary and unphysical. Although it's a little unusal relative to polar coordinates the way we set up the coordinate system allows negative r, so both of the equilibria are physical once $k < \omega_0^2$, although they look very similar. The stability of the equilibrium is determined by the derivative of the force as a function of position, which is $\frac{\partial}{\partial r}(-(k - \omega_0^2)r - \alpha r^3) = -(k - \omega_0^2) - 3\alpha r^2$. At $r = 0$ this is negative (and therefore stable) when $\omega_0^2 < k$ and positive (and therefore unstable) when $k < \omega_0^2$. At the other two equilibria we have $-(k - \omega_0^2) - 3\alpha\frac{\omega_0^2 - k}{\alpha} = 2(k - \omega_0^2)$. So these equilibria are stable only if the $r = 0$ equilibrium is unstable, i.e. when $\omega_0^2 < k$.

For the critical $\omega_0^2 = k$ case we have $m\ddot{r} = -\alpha r^3$ for the equations of motion. The second derivative test fails to determine stability, since it gives 0, so we need to consider the fourth derivative of the energy (the third derivative of the force) which is -6α, which is always negative and therefore stable.

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Problem 3

Consider a double pendulum (i.e. a rod attached to another rod by a hinge) with both rods the same length ℓ, where the inner rod is constrained to rotate at a fixed angular velocity ω_0. What is the frequency of small oscillations of the system if there is no gravity?

While it would be possible with constraints it would be simpler to set this problem up directly in terms of the coordinates. The coordinates are given by (where θ is the angle of the second pendulum relative to some fixed vertical axis)

$$x = \ell(\cos\omega_0 t + \cos\theta)$$

$$y = \ell(\sin\omega_0 t + \sin\theta)$$

The time derivatives of these are

$$\dot{x} = -\ell(\omega_0 \sin\omega_0 t + \dot{\theta}\sin\theta)$$

$$\dot{y} = \ell(\omega_0 \cos\omega_0 t + \dot{\theta}\cos\theta)$$

So our kinetic energy is (using the trig identity $\sin a \sin b + \cos a \cos b = \cos(a-b)$)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \omega_0^2 + 2\omega_0\dot{\theta}\cos(\theta - \omega_0 t))$$

And there is no potential energy since the system is somewhere where there's no gravity (like space). So now we have the equation of motion is

$$m\ell^2\ddot{\theta} - 2m\ell^2\omega_0\sin(\theta-\omega_0 t)(\dot{\theta}-\omega_0) + 2m\ell^2\omega_0\dot{\theta}\sin(\theta-\omega_0 t) = m\ell^2\ddot{\theta} + 2m\ell^2\omega_0^2\sin(\theta-\omega_0 t) = 0$$

Now since we're free to change coordinate systems, a more transparent coordiante system would be $\phi = \theta - \omega_0 t$, $\dot{\phi} = \dot{\theta} - \omega_0$, $\ddot{\phi} = \ddot{\theta}$. In these coordinate we have

$$m\ell^2\ddot{\phi} + 2m\ell^2\omega_0^2\sin\phi = 0$$

Which we know from experience is the equation of motion of a pendulum. In particular in the small ϕ approximation this becomes

$$m\ell^2\ddot{\phi} + 2m\ell^2\omega_0^2\phi = 0$$

$$\ddot{\phi} + 2\omega_0^2\phi = 0$$

So the frequency of small oscillations is given by $\omega = \sqrt{2}\omega_0$. This form makes sense in terms of dimensional analysis. We could have figured out at the beginning that that answer needed to be of the form $\omega = \#\omega_0$ for some fixed number $\#$.

5

3.1.3 First midterm

First midterm was hard. We only had only 50 minutes, 2 large problems with many parts each.

3.1.3.1 questions

Mechanics
Physics 311
Fall 2015
Midterm 1 (October 14, 2015)

There are 50 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way and **explain your answers.** Just providing the final answer is not sufficient - you must explain how you got there! For partial credit, you must show your work.

The exam is closed book, but you are allowed to bring one letter size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

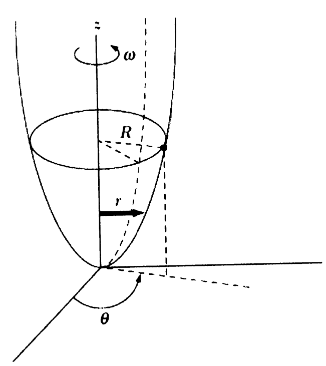
Problem 1 (15 points)

A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$, where c is a constant. The wire rotates with angular velocity ω about the vertical symmetry axis and is placed in a uniform gravitational field g parallel to the axis of rotation.

(1) (6 points) Find the Lagrangian for the bead using r as generalized coordinate.

(2) (6 points) Find the differential equation of motion.

(3) (3 points) Find the value of c that allows the bead to rotate in a circle of radius R with constant angular velocity ω.



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Problem 2 (15 points)

A simple pendulum of mass m and length l is attached to a mass M that is free to move in a single dimension along a frictionless horizontal surface.

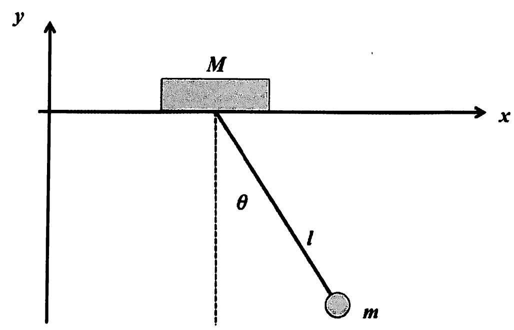
(1) (5 points) Find the Lagrangian of the system.

(2) (4 points) From the Lagrangian, obtain the differential equation(s) governing the motion of the system.

(3) (2 points) What are the integrals of motion for this system?

(4) (2 points) Determine the motion of the pendulum in the limit $M \gg m$.

(5) (2 points) How do we need to move M so that the pendulum hangs "motionless" at some constant angle θ_c? Determine θ_c.



3.1.3.2 key solution to first midterm

1

Mechanics

Physics 311 - Fall 2015

Midterm 1 - Solutions

Problem 1

(1)
$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + 4c^2 r^2 \dot{r}^2\right)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = cr^2 \Rightarrow \dot{z} = 2cr\dot{r}$$

$$U = mgz = mgcr^2$$ and with $\dot{\theta} = \omega$

so
$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2 + r^2\omega^2 + 4c^2 r^2 \dot{r}^2\right) - mgcr^2}$$

(2)
$$\frac{\partial L}{\partial r} = m\left(r\omega^2 + 4c^2 r \dot{r}^2\right) - 2mgcr$$

$$\frac{\partial L}{\partial \dot{r}} = m\left(\dot{r} + 4c^2 r^2 \dot{r}\right)$$

$\Rightarrow$ equation of motion $\dfrac{\partial L}{\partial r} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} = 0$

$$\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r \dot{r}^2 - r\omega^2 - 4c^2 r \dot{r}^2 + 2gcr = 0$$

$$\Rightarrow \boxed{\ddot{r}\left(1 + 4c^2 r^2\right) + \dot{r}^2\, 4c^2 r + r\left(2gc - \omega^2\right) = 0}$$

2

(3) $r = R$, $\dot{r} = \ddot{r} = 0$

$$\Rightarrow \quad 2gc - \omega^2 = 0 \quad \Leftrightarrow \quad \boxed{c = \frac{\omega^2}{2g}}$$

<u>Problem 2</u>

(1) set $U = 0$ at $x = 0$

choose x and θ as generalized coordinates

$T_M = \frac{1}{2} M \dot{x}^2$

$U_M = 0$

coordinates of m are $x' = x + \ell \sin\theta$

$\dot{x}' = \dot{x} + \ell\dot{\theta}\cos\theta$

$y' = -\ell\cos\theta$

$\dot{y}' = \ell\dot{\theta}\sin\theta$

$$T_m = \frac{1}{2} m \left[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + \ell^2\dot{\theta}^2\sin^2\theta \right]$$
$$= \frac{1}{2} m \left(\dot{x}^2 + \ell^2\dot{\theta}^2\cos^2\theta + 2\dot{x}\dot{\theta}\ell\cos\theta + \ell^2\dot{\theta}^2\sin^2\theta \right)$$
$$= \frac{1}{2} m \left(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}\ell\cos\theta \right)$$

$U_m = mg y' = -mg\ell\cos\theta$

so
$$\boxed{L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m \left(\ell^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}\ell\cos\theta \right) + mg\ell\cos\theta}$$

3

(2) equations of motion

in x $$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left[(M+m)\dot{x} + m\ell\dot{\theta}\cos\theta \right] = 0$$

$$\Rightarrow \quad \boxed{(M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = 0} \qquad (1)$$

in θ $$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\Rightarrow \quad -m\dot{x}\ell\dot{\theta}\sin\theta - mg\ell\sin\theta = m\ell^2\ddot{\theta} + m\ddot{x}\ell\cos\theta - m\dot{x}\ell\sin\theta\,\dot{\theta}$$

$$\Leftrightarrow \quad \boxed{\ddot{\theta} + \frac{\ddot{x}}{\ell}\cos\theta + \frac{g}{\ell}\sin\theta = 0} \qquad (2)$$

(3) Lagrangian is independent of x and t, so there two integrals of the motion : total energy $T + U = \text{const.}$ and the total momentum in x, $(M+m)\dot{x} + m\ell\dot{\theta}\cos\theta = \text{const.}$

(4) use equation (1) : $$\ddot{x} + \frac{m}{M+m}\ell\ddot{\theta}\cos\theta - \frac{m}{M+m}\ell\dot{\theta}^2\sin\theta = 0$$

$m \ll M$ $\quad \ddot{x} = 0$

$\Rightarrow$ M does not move (if it didn't at $t = 0$)

so equation (2) becomes $\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0$

(simple pendulum)

(5) for a motionless pendulum, $\ddot{\theta} = 0$,

so $\ddot{x} = -g \tan\theta$

$\Rightarrow$ we need to move M with constant acceleration $\ddot{x}$;

then the pendulum will hang motionless at

$\theta_c = \operatorname{atan}\left(-\frac{\ddot{x}}{g}\right)$.

3.2 second midterm

3.2.1 practice exam

3.2.1.1 questions

Mechanics
Physics 311
Fall 2012
Midterm 2 (November 16, 2012)

There are 50 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way. For partial credit you must show your work. The exam is closed book, but you are allowed to bring one letter size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

Some formulae:

$$U(r) = -\frac{\alpha}{r} \qquad \frac{p}{r} = 1 + e\cos\theta \qquad e = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}}$$

$$p = \frac{\ell^2}{m\alpha} \qquad |E| = \frac{\alpha}{2a} \qquad T^2 = \frac{4\pi^2}{G(m_1+m_2)}\,a^3$$

Problem 1 (15 points)

A moving particle of mass m_1 collides elastically with a target particle of mass m_2 which is initially at rest. If the collision is head-on, show that the incident particle loses a fraction $4m/M$ of its original kinetic energy, where m is the reduced mass and $M = m_1 + m_2$.

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Problem 2 (15 points)

Two spacecraft (A and B) are in circular orbit about the Earth, traveling in the same plane in the same directional sense. Spacecraft A is in low Earth orbit and spacecraft B is in geosynchronous orbit. The astronauts on board spacecraft A want to meet those on spacecraft B. To do so, the astronauts on A must fire their propulsion rocket and change the speed of A from v_1 to v_2 when spacecraft B is in the right place in its orbit for each spacecraft to reach the rendezvous point at apogee at the same time (see figure).

(a) (8 points) Show that the required speed boost for spacecraft A is

$$\frac{v_2}{v_1} = \sqrt{\frac{2\,r_B}{r_A + r_B}} \quad ,$$

where r_A and r_B are the radii of the initial circular orbits of the two spacecraft.

(b) (5 points) Show that the time T it takes spacecraft A to reach apogee is

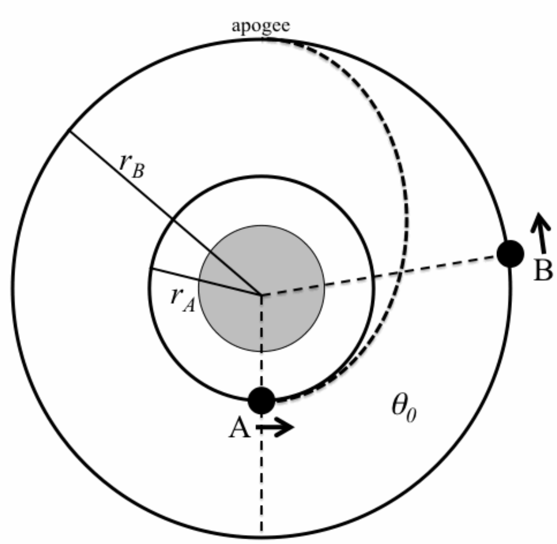
$$T = \frac{\pi}{\sqrt{GM}} \left(\frac{r_A + r_B}{2} \right)^{\frac{3}{2}} \quad ,$$

where M is the mass of the Earth. What approximations did you make?

(c) (2 points) Show that in order for A and B to meet at apogee,

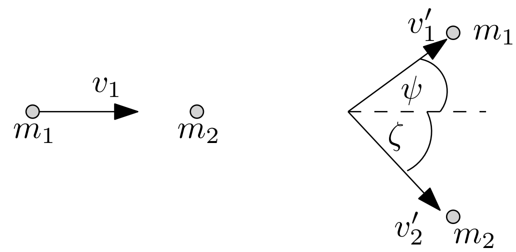
$$\theta_0 = 180^\circ \left(1 - \frac{T}{12} \right) \quad ,$$

where T is in hours.



3.2.1.2 my solution to practice exam

3.2.1.2.1 Problem 1 SOLUTION:



Before collision After collision

Conservation of Linear momentum gives

$$m_1 v_1 = m_1 v_1' + m_2 v_2' \tag{1}$$

Conservation of energy gives

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \left(v_1'\right)^2 + \frac{1}{2} m_2 \left(v_2'\right)^2 + Q$$

But since this is elastic collision, then $Q = 0$. Hence the above becomes

$$m_1 v_1^2 = m_1 \left(v_1'\right)^2 + m_2 \left(v_2'\right)^2 \tag{2}$$

The goal now is to eliminate v_2 from (1) and (2) and solve for v_1' in terms of v_1 to be able to answer the question. Let $\frac{m_2}{m_1} = \gamma$, then (1,2) can be written as

$$v_1 = v_1' + \gamma v_2' \tag{A1}$$

$$v_1^2 = \left(v_1'\right)^2 + \gamma \left(v_2'\right)^2 \tag{A2}$$

We now move the m_1 terms to one side,

$$v_1 - v_1' = \gamma v_2' \tag{C1}$$

$$v_1^2 - \left(v_1'\right)^2 = \gamma \left(v_2'\right)^2 \tag{C2}$$

Dividing (2) by (1), using long division (this step is tricky, must be careful), gives

$$\begin{aligned} \frac{v_1^2 - \left(v_1'\right)^2}{v_1 - v_1'} &= v_2' \\ v_1 + v_1' &= v_2' \end{aligned} \tag{3}$$

We now replace v_2' in (C1) with what (3) giving

$$\begin{aligned} v_1 - v_1' &= \gamma \left(v_1 + v_1'\right) \\ v_1 - \gamma v_1 &= \gamma v_1' + v_1' \\ v_1 \left(1 - \gamma\right) &= v_1' \left(1 + \gamma\right) \\ v_1' &= v_1 \frac{\left(1 - \gamma\right)}{\left(1 + \gamma\right)} \end{aligned} \tag{4}$$

We achieved our goal of finding v_1' in terms of v_1. Now to answer the question. The question is asking to find

$$\Delta = \frac{T_1 - T_1'}{T_1} \tag{5}$$

Which is the fraction of kinetic energy loss of m_1. So now we calculate the above, and see if it gives the answer we are asked to show.

$$\begin{aligned}
\Delta &= \frac{\frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_1\left(v_1'\right)^2}{\frac{1}{2}m_1 v_1^2} \\
&= \frac{v_1^2 - \left(v_1'\right)^2}{v_1^2}
\end{aligned}$$

Using (4) into the above gives

$$\Delta = \frac{v_1^2 - \left(v_1 \frac{(1-\gamma)}{(1+\gamma)}\right)^2}{v_1^2}$$

But $\gamma = \frac{m_2}{m_1}$, expanding the above gives

$$\begin{aligned}
\Delta &= \frac{v_1^2 - \left(v_1 \frac{\left(1-\frac{m_2}{m_1}\right)}{\left(1+\frac{m_2}{m_1}\right)}\right)^2}{v_1^2} \\
&= \frac{v_1^2 - v_1^2 \frac{\left(1-\frac{m_2}{m_1}\right)^2}{\left(1+\frac{m_2}{m_1}\right)^2}}{v_1^2} \\
&= 1 - \frac{\left(1 - \frac{m_2}{m_1}\right)^2}{\left(1 + \frac{m_2}{m_1}\right)^2} \\
&= 1 - \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2} \\
&= \frac{(m_1 + m_2)^2 - (m_1 - m_2)^2}{(m_1 + m_2)^2} \\
&= \frac{\left(m_1^2 + m_2^2 + 2m_1 m_2\right) - \left(m_1^2 + m_2^2 - 2m_1 m_2\right)}{(m_1 + m_2)^2}
\end{aligned}$$

Simplifying

$$\begin{aligned}
\Delta &= \frac{4m_1 m_2}{(m_1 + m_2)^2} \\
&= 4\frac{m_1 m_2}{(m_1 + m_2)}\frac{1}{(m_1 + m_2)}
\end{aligned}$$

But $m=\frac{m_1 m_2}{m_1+m_2}$ which is the reduced mass, and $M=m_1+m_2$. So the above becomes

$$\Delta=\frac{4m}{M}$$

Which is the result we are asked to show.

3.2.1.2.2 Problem 2 SOLUTION:

Part(a)

Let v_1 be the speed in the lower circular orbit. Let v_2 be the speed at the perigee just after speed boost. Let $GM\equiv\mu$. Since

$$v_1=\sqrt{\frac{\mu}{r_A}}$$
$$v_2=\sqrt{\mu\left(\frac{2}{r_A}-\frac{1}{a}\right)}$$

Where $a=\frac{r_A+r_B}{2}$, then $\frac{v_2}{v_2}$ can now be evaluated

$$\begin{aligned}
\frac{v_2}{v_1} &= \frac{\sqrt{\mu\left(\frac{2}{r_A}-\frac{1}{a}\right)}}{\sqrt{\frac{\mu}{r_A}}}\\
&= \sqrt{\frac{\mu\left(\frac{2}{r_A}-\frac{1}{\frac{r_A+r_B}{2}}\right)}{\frac{\mu}{r_A}}}\\
&= \sqrt{r_A\left(\frac{2}{r_A}-\frac{2}{r_A+r_B}\right)}\\
&= \sqrt{2-\frac{2r_A}{r_A+r_B}}\\
&= \sqrt{\frac{2\left(r_A+r_B\right)-2r_A}{r_A+r_B}}\\
&= \sqrt{\frac{2r_B}{r_A+r_B}}
\end{aligned}$$

Part(b)

Using the period for an ellipse given in the formulas and dividing this by half, since we are looking for half the period, then

$$
\begin{aligned}
T_p &= \pi\sqrt{\frac{a^3}{G\left(m_1 + M_{earth}\right)}} \\
&= \pi\sqrt{\frac{\left(\frac{r_A+r_B}{2}\right)^3}{G\left(m_1+M_e\right)}}
\end{aligned}
$$

Assuming the mass of the satellite (m_1) is much smaller than M_{earth}, then the above becomes

$$
T_p = \frac{\pi}{\sqrt{GM_e}}\sqrt{\left(\frac{r_A+r_B}{2}\right)^3}
$$

Part(c)

The time it takes B to travel one circle (2π) is

$$
T_c = 2\pi\sqrt{\frac{r_B^3}{GM}}
$$

Therefore, the angle B travels during T_p is found by the equating the ratios

$$
\frac{2\pi}{\alpha} \Leftrightarrow \frac{2\pi\sqrt{\frac{r_B^3}{GM}}}{T_p}
$$

But $\theta_0 = \pi - \alpha$ (assuming the diagram given, where α is the angle between B and the apogee, while θ is the angle between B and the perigee). Therefore we use the above to solve for θ_0

$$
\begin{aligned}
\frac{2\pi}{\pi-\theta_0} &= \frac{2\pi\sqrt{\frac{r_B^3}{GM}}}{T_p} \\
\frac{T_p}{\pi-\theta_0} &= \sqrt{\frac{r_B^3}{GM}} \\
\pi-\theta_0 &= T_p\sqrt{\frac{GM}{r_B^3}} \\
\theta_0 &= \pi - T_p\sqrt{\frac{GM}{r_B^3}} \\
&= \pi - \frac{\pi}{\sqrt{GM_e}}\sqrt{\left(\frac{r_A+r_B}{2}\right)^3}\sqrt{\frac{GM}{r_B^3}}
\end{aligned}
$$

Therefore

$$
\theta_0 = \pi\left(1-\sqrt{\left(\frac{r_A+r_B}{2r_B}\right)^3}\right)
$$

3.2.1.3 key solution to practice exam

1

Mechanics
Physics 311 - Fall 2012
Midterm 2 - Solutions

1. conservation of momentum $m_1 v_1 = m_1 v_1' + m_2 v_2'$

" " energy $\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

$$\frac{T_1 - T_1'}{T_1} = \frac{\frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_1 v_1'^2}{\frac{1}{2} m_1 v_1^2} = \frac{m_2 v_2'^2}{m_1 v_1^2}$$

find v_2' :

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \left(v_1 - \frac{m_2}{m_1} v_2'\right)^2 + \frac{1}{2} m_2 v_2'^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_1 \left(\frac{m_2}{m_1} v_2'\right)^2$$

$$- m_1 v_1 \frac{m_2}{m_1} v_2' + \frac{1}{2} m_2 v_2'^2$$

$$\Leftrightarrow \quad 0 = -m_2 v_1 v_2' + \frac{1}{2} m_2 \left(1 + \frac{m_2}{m_1}\right) v_2'^2$$

so $$v_2' = \frac{2 v_1 m_1}{m_1 + m_2}$$

$$\frac{T_1 - T_1'}{T_1} = \frac{m_2}{m_1} \frac{v_2'^2}{v_1^2} = \frac{m_2}{m_1} \frac{4 m_1^2}{(m_1 + m_2)^2}$$

so with $M = m_1 + m_2$

and $m = \frac{m_1 m_2}{M}$ $\qquad \frac{T_1 - T_1'}{T_1} = \frac{4 m}{M}$

2

2. (a) first, we need the velocity of A in LEO

$$r_A = p_{LEO} = \frac{\ell_{LEO}^2}{m\alpha}$$

and with $\ell_{LEO} = m r_A v_1$,

$$r_A = \frac{m^2 r_A^2 v_1^2}{m\alpha} \quad \Rightarrow \quad v_1^2 = \frac{\alpha}{m r_A}$$

next, we need the velocity of A on the elliptical transfer orbit at perigee

$$r_A = \frac{p_e}{1+e} = \frac{\ell_e^2}{m\alpha}\,\frac{1}{1+e}$$

and with $\ell_e = m r_A v_2$,

$$r_A = \frac{m^2 r_A^2 v_2^2}{m\alpha}\,\frac{1}{1+e} \quad \Rightarrow \quad v_2^2 = \frac{\alpha}{m r_A}(1+e)$$

so $\quad \dfrac{v_2^2}{v_1^2} = 1+e$

for eccentricity e, use $\quad 2a = r_A + r_B \qquad r_B = (1+e)a$

$$\Rightarrow \quad 1+e = \frac{r_B}{a} = \frac{2r_B}{r_A + r_B}$$

So $\quad \dfrac{v_2}{v_1} = \sqrt{\dfrac{2r_B}{r_A + r_B}}$

3

(b) the time of transfer is a half-period of the elliptical transfer orbit

Kepler 2 $$T_A^2 = \frac{4\pi^2}{GM} a^3$$ (with the approximation $M + m_A \simeq M$)

so $$\tau = \frac{1}{2} T_A = \frac{1}{2} \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

$$= \frac{\pi}{\sqrt{GM}} \left(\frac{r_A + r_B}{2} \right)^{3/2}$$

(c) B is in geosynchronous orbit, so $\omega = \frac{360°}{24h}$

$$\theta_0 = 180° - \omega T$$

$$= 180° - \frac{360° T}{24h}$$

$$= 180° \left(1 - \frac{T}{12h} \right)$$

3.2.2 Review Problems by TA

3.2.2.1 questions

311 Midterm 2 Review

November 12, 2015

1) (a) Since we often visualize precesssing orbits as elliptical orbits with a rotating apogee (really this is only an approximation) it's natural to wonder what kind of force gives these orbits precisely. Show that the force law that gives rise to orbits of the form

$$r(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\beta\theta)}$$

(Note that this gives an apsidal angle of $\frac{\pi}{\beta}$) is of the form

$$F(r) = -\frac{k}{r^2} - \frac{c}{r^3}$$

(b) Newton originally considered this problem to analyze precessing orbits. Show that not only is it true that $r(\theta) = r_0(\beta\theta)$ where $r_0(t)$ is an ordinary gravitational orbit, but in fact $r(t) = r_0(t)$ and $\theta(t) = \beta\theta_0(t)$ where $r(t)$ and $\theta(t)$ are the trajectory as a function of time of the precessing orbit and $r_0(t)$ and $\theta_0(t)$ are the trajetory as a function of time of an ordinary orbit (you can work backwards, starting from the trajectory and showing that its acceleration corresponds to the force law found in part (a). This fact is not actually special to gravitational orbits; Newton showed that you can speed up the angular velocity of an arbitrary orbit in an arbitrary central potential by adding a carefully chosen $\frac{1}{r^3}$ force).

2) (a) Suppose that you are in a spaceship that is trapped at the center of a uniform spherical cloud of dust with density ρ and radius R. What is the escape velocity of this configuration (i.e. what is the minimum velocity you would need, starting from the center of the cloud, to escape to infinity)?

(b) You do not have enough fuel to escape the cloud, but you have managed to achieve a circular orbit of radius r_0. You see a derelict spaceship that may have more fuel at a larger circular orbit (still inside the cloud) of radius r_1. What δv do you need to achieve an elliptical transfer orbit from radius r_0 to radius r_1? (Note that gravity inside the cloud is of the form $F(r) = -kr$ for some constant k so orbits are centered ellipses.)

(c) Show that the energy of the transfer orbit is the average of the energies of the two circular orbits.

3) You encounter a strange central force with potential

$$V(r) = k(r - a)(4r^2 - 11ar + 9a^2)$$

For what radii are circular orbits stable? Is the circular orbit with radius $r = a$ stable? Why or why not? You may have to graph the effective potential to answer this question.

4) (a) Two spheres of mass m_1 and m_2 and radius r_1 and r_2 start off at rest in space a distance of d apart (center to center). Determine their speeds and positions when they collide.

(b) The two spheres are chemically reactive and explode a little bit. Determine the coefficient of restitution (> 1) necessary for the objects to achieve escape velocity.

3.2.2.2 key solution to review problems

311 Midterm 2 Review Solutions

November 14, 2015

1) (a) We need the Binet equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{mr^2}{\ell^2}F(r)$$

We have that the orbit is

$$\frac{1}{r(\theta)}=\frac{1+\varepsilon\cos(\beta\theta)}{a(1-\varepsilon^2)}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r(\theta)}\right)=\frac{-\beta^2\varepsilon\cos(\beta\theta)}{a(1-\varepsilon^2)}$$

So we get

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{\beta^2}{a(1-\varepsilon^2)}+(1-\beta^2)\frac{1+\varepsilon\cos(\beta\theta)}{a(1-\varepsilon^2)}$$

$$=\frac{\beta^2}{a(1-\varepsilon^2)}+(1-\beta^2)\frac{1}{r}$$

And

$$F(r)=-\frac{\beta^2\ell^2}{a(1-\varepsilon^2)mr^2}-\frac{(1-\beta^2)\ell^2}{mr^3}$$

Now since $\theta_0=\beta\theta$ (there was a typo in the problem) we have $\beta^2\ell^2=\ell_0^2$ where ℓ_0 is the angular momentum of the original we have

$$F(r)=-\frac{k}{r^2}+(1-\beta^{-2})\frac{\ell_0^2}{mr^3}$$

1

(Typically this theorem is stated in terms of β^{-1} rather than β.)

(b) For an ordinary particle in a gravitational potential we have

$$L_0 = \frac{1}{2}m(\dot{r}_0^2 + r_0^2\dot{\theta}_0^2) + \frac{k}{r}$$

The equations of motion are

$$m\ddot{r}_0 - mr_0\dot{\theta}_0^2 + \frac{k}{r_0^2} = 0$$

$$mr_0^2\ddot{\theta}_0 + 2mr_0\dot{r}_0\dot{\theta}_0 = 0$$

For the precessing orbit we have

$$L = \frac{1}{2}m(\dot{r}_0^2 + r_0^2\dot{\theta}_0^2) + \frac{k}{r} - (1-\beta^{-2})\frac{\ell_0^2}{2mr^2}$$

The equations of motion are

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} - (1-\beta^{-2})\frac{\ell_0^2}{mr^3} = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$$

Making the substitution $r_0 \to r$ and $\theta_0 \to \beta\theta$ clearly leaves the second equation of motion unchanged so we only need to check the first equation of motion. The first equation of motion for both orbits with the effective potential is

$$m\ddot{r}_0 - \frac{\ell_0^2}{mr_0^3} + \frac{k}{r_0^2} = 0$$

$$m\ddot{r} - \frac{\ell^2}{mr^3} + \frac{k}{r^2} - (1-\beta^{-2})\frac{\ell_0^2}{mr^3} = 0$$

And then since $\beta^2\ell^2 = \ell_0^2$ we get immediately

$$m\ddot{r} + \frac{k}{r^2} - \frac{\ell_0^2}{mr^3} + \beta^{-2}\frac{\ell_0^2}{mr^3} - \beta^{-2}\frac{\ell_0^2}{mr^3} = 0$$

2

$$m\ddot{r} - \frac{\ell_0^2}{mr^3} + \frac{k}{r^2} = 0$$

Confirming the solution.

2) (a) We know that the gravitational force on an object near a spherical mass of radius R and uniform density ρ is given by

$$F(r) = -\frac{GmM(r)}{r^2} = -\frac{4\pi Gm\rho}{3r^2}\begin{cases} r^3 & r < R \\ R^3 & R < r \end{cases}$$

So to get the potential energy (and then the escape velocity) we need to integrate:

$$U(r) = -\int_r^{\infty} F(r)dr$$

$$= \frac{2}{3}\pi Gm\rho \begin{cases} r^2 - 3R^2 & r < R \\ -2\frac{R^3}{r} & R < r \end{cases}$$

Which tells us that the potential energy at $r = 0$ (with $U(\infty) = 0$) is

$$U(0) = -2\pi Gm\rho R^2$$

So escape velocity is the velocity which gives a kinetic energy equal to $-U(0)$ or more specifically

$$v = \sqrt{4\pi G\rho}R$$

(b) The easiest way to do this problem is to remember that the equations of motion for a particle in an $F(r) = -k_s r$ force separates in Cartesian coordinates ($m\ddot{x} = -k_s x$, $m\ddot{y} = -k_s y$). So the centered elliptical orbit of a particle in such a central force has x and y coordinates that are just oscillatory:

$$x = x_0 \cos\omega t$$

3

$$y = y_0 \sin \omega t$$

(where we chose our coordinate system so that the sine and cosine would be simple.) A transfer orbit in this case would be of the form

$$x = r_0 \cos \omega t$$

$$y = r_1 \sin \omega t$$

So that the perigee/semi-minor axis (these are the same thing when $F(r) = -k_s r$) is the radius of the smaller orbit and the apogee/semi-major axis is the radius of the larger orbit. The y velocity in the circular orbit ($y = r_0 \sin \omega t$) at the perigee is

$$v_y = r_0 \omega = r_0 \sqrt{\frac{4}{3}\pi G\rho}$$

And the y velocity in the transfer orbit at the perigee is

$$v_y' = r_1 \omega = r_1 \sqrt{\frac{4}{3}\pi G\rho}$$

So we get just

$$\delta v = \delta r \sqrt{\frac{4}{3}\pi G\rho}$$

(c) The total energy (setting $U(0) = 0$ now for convienence, but this doesn't change that the transfer orbit energy is the average of the two circular orbit energies) of the smaller circular orbit is

$$E_0 = 2m\omega^2 r_0^2 = \frac{8}{3}\pi G m\rho r_0^2$$

and likewise

$$E_1 = 2m\omega^2 r_1^2 = \frac{8}{3}\pi G m\rho r_1^2$$

For the transfer orbit notice that $E = T + V = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}ky^2$

4

And since $x_t = x_0$ and $y_t = y_1$ and for a circular orbit $\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_y^2 + \frac{1}{2}ky^2 = \frac{1}{2}E$ we get the required result

$$E_t = \frac{4}{3}\pi G m\rho(r_0^2 + r_1^2)$$

3) To check stability we need to look at

$$\frac{3}{r} + \frac{V''(r)}{V'(r)} = \frac{3}{r} - \frac{3(5a-4r)}{10a^2 - 15ar + 6r^2}$$

$= \frac{30(a-r)^2}{r(10a^2-15ar+6r^2)}$

The polynomial $10a^2 - 15ar + 6r^2$ is always positive for positive a and r (the easiest way to see this without plotting is to calculate the minimum value for a fixed a).

So we have that circular orbits are stable except at $r = a$ where the quantity exactly vanishes. The stability test is inconclusive. To look at the effective potential for the $r = a$ orbit we need to figure out the angular momentum. The force is given by

$$F(r) = -U'(r) = -2k(10a^2 - 15ar + 6r^2)$$

So at $r = a$

$$F(a) = -2ka^2$$

So we have an angular velocity given by $m\dot{\theta}^2 a = 2ka^2$ and the angular momentum is

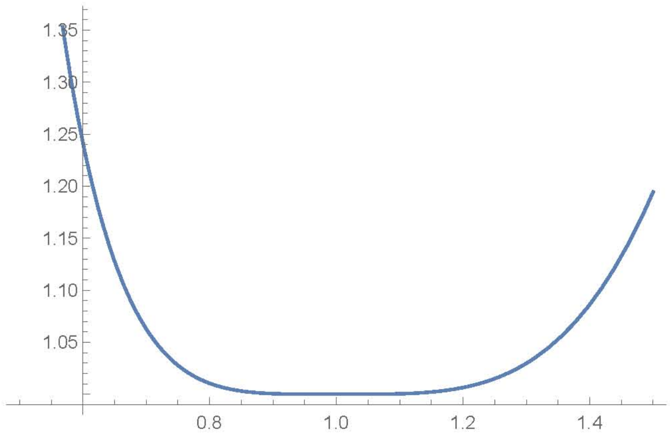
$$\ell^2 = m^2 r^4 \dot{\theta}^2 = 2kma^5$$

So the effective potential is

$$U_{eff}(r) = k(r-a)(4r^2 - 11ar + 9a^2) + k\frac{a^5}{r^2}$$

(As a sanity check note that the units are consistent.) Plotting this (with $a = 1$) and looking near a we have

5



Which looks very flat at a (which is to be expected since the second derivative is 0), but still clearly should correspond to a stable orbit, which it does, since the 4th derivative is positive:

$$U^{(4)}_{eff}(r) = 120k\frac{a^5}{r^6} > 0$$

4) You can determine the speeds with conservation of energy and momentum and you can determine the position with the fact that the center of mass is stationary (which is a consequence of conservation of momentum):

$$E_0 = -\frac{Gm_1m_2}{d}$$

$$E_1 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{r_1+r_2}$$

$$P_0 = 0$$

$$P_1 = m_1v_1 + m_2v_2$$

6

Which can be solved to give

$$v_1 = \pm\sqrt{\frac{2m_2}{m_1(m_1+m_2)}\Delta U}$$

$$v_2 = \mp\sqrt{\frac{2m_1}{m_2(m_1+m_2)}\Delta U}$$

where $\Delta U = Gm_1m_2\left(\frac{1}{r_1+r_2} - \frac{1}{d}\right)$

(I was lazy an solved this with Mathematica, but it is doable. A pro tip is that once you have one of v_1 or v_2 you can immediately get the other by noting that the problem is symmetric between m_1 and m_2, so you just need to flip the labels and the sign.) The sign of the velocities doesn't really matter, other than the fact that they need to be in the opposite direction.

The center of mass is located at $\frac{m_2}{m_1+m_2}d$ away from m_1's initial position towards m_2. When they collide there is a total distance of $r_1 + r_2$ between them, so the center of mass is located $\frac{m_2}{m_1+m_2}(r_1+r_2)$ away from m_1's final position towards m_2, so m_1 is $\frac{m_2}{m_1+m_2}(d-r_1-r_2)$ away from its initial position towards m_2 and by symmetry m_2 is $\frac{m_1}{m_1+m_2}(d-r_1-r_2)$ away from its initial position towards m_1. As a sanity check note that the total distance traveled by both spheres is $\frac{m_2}{m_1+m_2}(d-r_1-r_2)+\frac{m_1}{m_1+m_2}(d-r_1-r_2) = d-r_1-r_2$, which makes sense. If $m_2 \gg m_1$ we get that m_1 travels $d-r_1-r_2$ and m_2 doesn't move, which also makes sense.

(b) By conservation of momentum we have

$$m_1v_1' = -m_2v_2'$$

In order for both objects to escape (by the constancy of the center of mass, if one object escapes the other must as well) the total energy needs to be 0. So we have

$$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 - \frac{Gm_1m_2}{r_1+r_2} = 0$$

Solving these two equations gives

$$v_1' = \mp\sqrt{\frac{-2m_2U}{m_1(m_1+m_2)}}$$

$$v_2' = \pm\sqrt{\frac{-2m_1U}{m_2(m_1+m_2)}}$$

7

where $U = -\frac{Gm_1 m_2}{r_1+r_2}$. I chose the signs so that the solution would make sense relative to the sign choice in the first part (if you choose v_1 and v_1' to both be positive that corresponds to m_1 and m_2 somehow shooting *past each other* after the explosion.)

The coefficient of restitution is the ratio of the relative speed before and after the collision. So in this case we get

$$\epsilon = \frac{\left|\sqrt{\frac{2m_2}{m_1(m_1+m_2)}\Delta U} - \sqrt{\frac{2m_1}{m_2(m_1+m_2)}\Delta U}\right|}{\left|\sqrt{\frac{-2m_1 U}{m_2(m_1+m_2)}} - \sqrt{\frac{-2m_2 U}{m_1(m_1+m_2)}}\right|}$$

$$= \frac{\left|\sqrt{\frac{m_2}{m_1}} - \sqrt{\frac{m_1}{m_2}}\right|\sqrt{\Delta U}}{\left|\sqrt{\frac{m_1}{m_2}} - \sqrt{\frac{m_2}{m_1}}\right|\sqrt{-U}}$$

$$= \frac{\sqrt{Gm_1 m_2\left(\frac{1}{r_1+r_2} - \frac{1}{d}\right)}}{\sqrt{\frac{Gm_1 m_2}{r_1+r_2}}}$$

$$= \sqrt{\frac{d}{d-r_1-r_2}} > 1$$

8

3.2.3 Exam, Nov 16, 2015

3.2.3.1 questions

Mechanics
Physics 311
Fall 2015
Midterm 2 (November 16, 2015)

There are 50 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way and **explain your answers.** Just providing the final answer is not sufficient - you must explain how you got there! For partial credit, you must show your work.

The exam is closed book, but you are allowed to bring one letter size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

Problem 1 (15 points)

A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (The mass of the carbon nucleus is about 12 times the mass of the neutron.)

...continued on next page...

Problem 2 (15 points)

The orbit of a particle of mass m in a central force field $F(r)$ is a circle passing through the origin,

$$r(\theta) = r_0 \cos\theta \qquad \theta \in [-\pi/2, \pi/2] \ ,$$

where r is the distance from the center of the force, θ is the angular displacement, and r_0 is the distance from the center of the force at $\theta = 0$, i.e., the diameter of the circle.

(1) (5 points) Using the equation of the orbit

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{\ell^2}F(r) \ ,$$

where ℓ is the magnitude of the conserved angular momentum, show that the central force $F(r)$ varies like the inverse of the fifth power of r according to

$$F(r) = -\frac{2r_0^2\ell^2}{m}\frac{1}{r^5} \ .$$

(2) (5 points) Find the potential energy $U(r)$ corresponding to $F(r)$, write the total mechanical energy of the particle, and define the effective potential $U_{eff}(r)$. Sketch the shape of $U_{eff}(r)$.

(3) (5 points) Does an inverse fifth-power force law allow stable circular orbits about the force center? Argue qualitatively based on the sketch of $U_{eff}(r)$ in (2), but also perform the calculation.

3.2.3.2 key solution

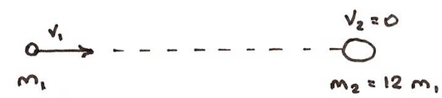
1

Mechanics

Physics 311 - Fall 2015

Midterm 2 - Solutions

Problem 1



momentum conservation $\quad m_1 V_1 = m_1 v_1' + m_2 V_2' \quad$ (i)

energy conservation $\quad \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 V_2'^2 \quad$ (ii)

the fraction of kinetic energy transfered to the carbon nucleus is

$$\frac{T_2'}{T_1} = \frac{m_2 V_2'^2}{m_1 v_1^2} \quad , \text{ so we need } \frac{V_2'}{V_1}$$

$$(i) \Rightarrow \quad v_1' = v_1 - \frac{m_2}{m_1} V_2'$$

$$\Rightarrow \quad m_1 V_1^2 = m_1 \left(v_1 - \frac{m_2}{m_1} V_2'\right)^2 + m_2 V_2'^2$$

$$\Leftrightarrow \quad 0 = m_1 \left(\frac{m_2}{m_1}\right)^2 V_2'^2 - 2 m_1 v_1 \frac{m_2}{m_1} V_2' + m_2 V_2'^2$$

$$\Leftrightarrow \quad 0 = \frac{m_2}{m_1} V_2'^2 - 2 v_1 V_2' + V_2'^2$$

so

$$V_2' = \frac{2 m_1}{m_1 + m_2} V_1$$

so

$$\frac{T_2'}{T_1} = \frac{m_2}{m_1} \frac{4 m_1^2}{(m_1 + m_2)^2} = \frac{4 m_1 m_2}{(m_1 + m_2)^2}$$

$$= \frac{4 \cdot 12}{13^2} \simeq \underline{\underline{0.3}}$$

$\boxed{\Rightarrow \text{ 30\% of the energy is transfered to the carbon}}$

2

Problem 2

(1)
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d^2}{d\theta^2}\left(\frac{1}{r_0 \cos\theta}\right)$$
$$= \frac{1}{r_0}\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos^2\theta}\right)$$
$$= \frac{1}{r_0}\frac{\cos^3\theta + \sin^2\theta\, 2\cos\theta}{\cos^4\theta}$$
$$= \frac{1}{r_0}\frac{2\sin^2\theta + \cos^2\theta}{\cos^3\theta} = \frac{1}{r_0}\frac{2-\cos^2\theta}{\cos^3\theta}$$

so
$$\frac{1}{r_0}\frac{2-\cos^2\theta}{\cos^3\theta} + \frac{1}{r_0\cos\theta} = -\frac{mr^2}{\ell^2}F(r)$$
$$\Rightarrow \frac{2-\cos^2\theta+\cos^2\theta}{r_0\cos^3\theta} = -\frac{mr^2}{\ell^2}F(r)$$
$$\Rightarrow F(r) = -\frac{\ell^2}{mr^2}\frac{2}{r_0\cos^3\theta}$$
$$= -\frac{\ell^2}{mr^2}\frac{2r_0^2}{r^3} = -\frac{2r_0^2\ell^2}{m}\frac{1}{r^5}$$

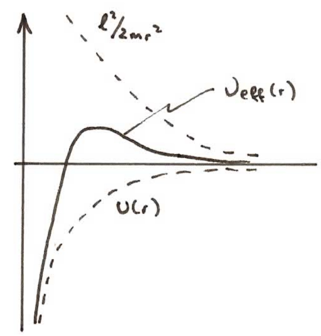
(2)
$$U(r) = -\int F(r)\,dr$$
$$= -\frac{1}{4}\frac{2r_0^2\ell^2}{m}\frac{1}{r^4} + U_0$$
$$= -\frac{r_0^2\ell^2}{2m}\frac{1}{r^4} \qquad (\text{setting } U_0 = 0)$$

3

so $E = \frac{1}{2} m \dot{\vec{r}}^2 + U(r)$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{\ell^2}{2mr^2} + U(r)}_{U_{eff}}$$

$$\Rightarrow \boxed{U_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{r_0^2 \ell^2}{2mr^4}}$$



(3) the graph of $U_{eff}(r)$ shows an extremum, but it is a maximum and therefore <u>unstable</u>

$$\frac{\partial U_{eff}}{\partial r} = -\frac{\ell^2}{mr^3} + \frac{2 r_0^2 \ell^2}{mr^5}$$

$$= \frac{\ell^2}{mr^3}\left(-1 + \frac{2r_0^2}{r^2}\right) \stackrel{!}{=} 0 \qquad \text{so} \quad r^2 = 2r_0^2$$

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r^2 = 2r_0^2} = \left.\frac{3\ell^2}{mr^4} - \frac{10 r_0^2 \ell^2}{mr^6}\right|_{r^2 = 2r_0^2}$$

$$= \frac{3\ell^2}{m\,4r_0^4} - \frac{10 r_0^2 \ell^2}{m\,8r_0^6}$$

$$= \left(\frac{3}{4} - \frac{5}{4}\right)\frac{\ell^2}{mr_0^4} = -\frac{1}{2}\frac{\ell^2}{mr_0^4} < 0$$

$\leadsto$ the potential admits a circular orbit, but it is <u>not</u> stable

3.3 Finals

3.3.1 practice exam

3.3.1.1 questions

Mechanics
Physics 311
Fall 2012
Final Exam (December 17, 2012)

There are 120 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way. For partial credit you must show your work. The exam is closed book, but you are allowed to bring one letter-size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

$ax^2 + bx + c = 0$ has solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Euler Equations:
$I_1\,\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\,\omega_3 = \tau_1$
$I_2\,\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\,\omega_1 = \tau_2$
$I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\,\omega_2 = \tau_3$

(Page 1 of 4)

Problem 1 (15 points)

A chain of mass m and length L rests with $(1-\alpha)L$ of its length on a table top and αL of its length hanging over the smooth edge. The coefficient of friction of the tabletop is μ.

(1) (5 points) What is the maximum value, α_c, for which the chain remains stationary?
(2) (10 points) If α is larger than α_c, when released the chain will slide off the table. What is the velocity of the chain when the last link leaves the table?

Hint: to calculate the final velocity in (2), you can use the work-energy theorem: if one or more external forces act upon a body causing its kinetic energy to change by ΔT, then the work done by the net force is equal to ΔT.

Problem 2 (15 points)

A particle moves with velocity v_0 on a horizontal plane on the surface of the Earth. Show by explicitly solving the equations of motion in the non-inertial frame that the particle will move in a circle and that the radius of the circle is

$$R = \frac{v_0}{2\,\omega_z} \quad ,$$

where ω_z is the vertical component of the Earth's angular velocity vector $\vec{\omega}$. You may neglect centrifugal forces.

Problem 3 (15 points)

A frisbee is thrown into the air in such a way that it has a small wobble. Air friction exerts a torque $-c\vec{\omega}$, where c is a constant, on the rotation of the frisbee. Let x_3 be the symmetry axis of the frisbee (see below).

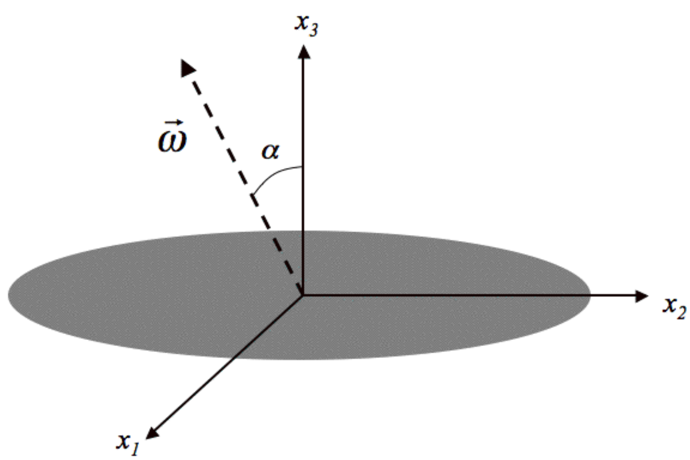
(1) (5 points) Use Euler's equations to show that ω_3, the component of $\vec{\omega}$ in the direction of the symmetry axis, decreases exponentially with time.
(2) (10 points) Show that the angle α between the symmetry axis and $\vec{\omega}$ decreases with time if I_3 is larger than $I = I_1 = I_2$. This is the reason why frisbees work so well: air friction diminishes the wobble for a flat (frisbee-shaped) object.

Hint: For (2), express α in terms of

$$\tan\alpha = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3}$$

and use Euler's equations to find a solution for $\omega_1^2 + \omega_2^2$. Together with the solution for ω_3 from part (1), this should give

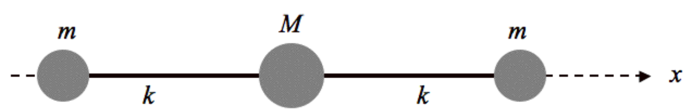
$$\tan\alpha = (\tan\alpha_0)\, e^{-c\,t\left(\frac{1}{I} - \frac{1}{I_3}\right)} \quad .$$



Problem 4 (15 points)

Consider the simple model for the carbon dioxide molecule CO_2 shown below. Two end particles of mass m are bound to the central particle of mass M via a potential function that is equivalent to two springs with spring constant k. Consider motion in one dimension only, along the x-axis.

(1) (5 points) Determine the Lagrangian of the system.
(2) (5 points) Find the eigenfrequencies of the system.
(3) (5 points) Find the eigenvectors and describe the normal mode motion; i.e., find the relative amplitudes of oscillations for the three masses for each normal mode.



3.3.1.2 key solution

1

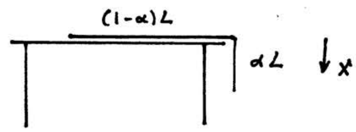
Mechanics

Physics 311 - Fall 2012

Final Exam - Solutions

Problem 1

(1)



$$\underbrace{\left(\frac{m}{L}\alpha_c L\right) g}_{\text{gravity}} - \underbrace{\underbrace{\left(\frac{m}{L}(1-\alpha_c)L\right) g}_{\text{normal force}}\mu}_{\text{friction}} = 0$$

$$\Rightarrow \quad \alpha_c - (1-\alpha_c)\mu = 0$$

$$\Leftrightarrow \quad \boxed{\alpha_c = \frac{\mu}{1+\mu}}$$

(2) Work-energy theorem: $\Delta T = \frac{1}{2} m v_f^2 = \int_{\alpha L}^{L} F(x)\,dx$

$$F(x) = \frac{m}{L} x g - \frac{m}{L}(L-x) g \mu$$

$$= \frac{m}{L} x g (1+\mu) - m g \mu$$

2

$$\int_{\alpha L}^{L} F\,dx = \left[\frac{1}{2}\frac{m}{L} g(1+\mu)x^2\right]_{\alpha L}^{L} - \left[mg\mu x\right]_{\alpha L}^{L}$$

$$= \frac{1}{2} mg(1+\mu) L(1-\alpha^2) - mg\mu L(1-\alpha)$$

$$\overset{!}{=} \frac{1}{2} m v_f^2$$

$$\Rightarrow \boxed{v_f = \sqrt{g(1+\mu)(1-\alpha^2)L - 2g\mu(1-\alpha)L}}$$

Problem 2

equation of motion (without centrifugal term)

$$m\frac{d\vec{v}}{dt} = 2m\,\vec{v}\times\vec{\omega}$$

here, $\vec{v} = v_x\,\hat{x} + v_y\,\hat{y}$

so
$$\dot{v}_x = 2\,v_y\omega_z \qquad (1)$$
$$\dot{v}_y = -2\,v_x\omega_z \qquad (2)$$

solve by adding $(1) + i(2)$

$$\Rightarrow (\dot{v}_x + i\dot{v}_y) = 2\,\omega_z(v_y - i v_x)$$
$$= -2\,i\,\omega_z(v_x + i v_y)$$

set $v = v_x + i v_y$

$$\Rightarrow \dot{v} + 2i\omega_z v = 0$$

3

$\Rightarrow \quad v(t) = v_0 \, e^{-2i\omega_z t}$

so

$$V_x(t) = V_0 \cos(2\omega_z t)$$
$$V_y(t) = -V_0 \sin(2\omega_z t)$$

Solve for $x(t)$ and $y(t)$

$$x = x_0 + \frac{V_0}{2\omega_z} \sin(2\omega_z t)$$
$$y = y_0 + \frac{V_0}{2\omega_z} \cos(2\omega_z t)$$

$\Rightarrow$ the object moves in a circle with radius R, where

$$R^2 = (x-x_0)^2 + (y-y_0)^2 = \left(\frac{V_0}{2\omega_z}\right)^2$$

$$\Rightarrow \quad \boxed{R = \frac{V_0}{2\omega_z}}$$

4

Problem 3

(1) the third Euler equation gives

$$I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 = \tau_3$$

so with $I_1 = I_2$ and $\tau_3 = -c\,\omega_3$

$$I_3\,\dot{\omega}_3 = -c\,\omega_3$$

try $\omega_3 = a\,e^{-bt}$ $\Rightarrow$ $-I_3\,b = -c$ $\Rightarrow$ $b = \frac{c}{I_3}$

$\Rightarrow$ $\omega_3 = \omega_0\, e^{-\frac{c}{I_3}t}$

(2) the other two Euler equations give

$$I\,\dot{\omega}_1 - (I - I_3)\,\omega_2\omega_3 = -c\,\omega_1$$
$$I\,\dot{\omega}_2 - (I_3 - I)\,\omega_3\omega_1 = -c\,\omega_2$$

eliminate ω_3

$$I\,\dot{\omega}_1\omega_1 - (I - I_3)\,\omega_1\omega_2\omega_3 = -c\,\omega_1^2$$
$$I\,\dot{\omega}_2\omega_2 - (I_3 - I)\,\omega_1\omega_2\omega_3 = -c\,\omega_2^2$$

add: $I\,(\dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2) = -c\,(\omega_1^2 + \omega_2^2)$

$\Leftrightarrow$ $\frac{I}{2}\,\frac{d}{dt}(\omega_1^2 + \omega_2^2) = -c\,(\omega_1^2 + \omega_2^2)$

5

this is a similar differential equation as in (1), so

$$(\omega_1^2+\omega_2^2) = A\, e^{-\frac{2c}{I}t}$$

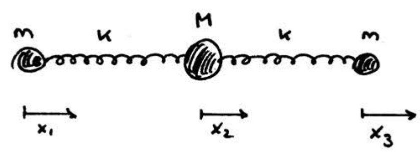
$$\Rightarrow \quad \tan\alpha = \frac{\sqrt{A}\, e^{-\frac{c}{I}t}}{\omega_0\, e^{-\frac{c}{I_3}t}} = (\tan\alpha_0)\, e^{-ct\left(\frac{1}{I}-\frac{1}{I_3}\right)}$$

so for this to decay exponentially, we need $\frac{1}{I} > \frac{1}{I_3}$

$$\Rightarrow \boxed{I_3 > I}$$

Problem 4

(1)



Lagrangian: $L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_3^2 + \frac{M}{2}\dot{x}_2^2$

$$- \underbrace{\left[\tfrac{1}{2}k(x_2-x_1)^2 + \tfrac{1}{2}k(x_3-x_2)^2\right]}_{=U}$$

(2) $m_{11} = m$

$m_{33} = m$

$m_{22} = M$

$m_{12} = m_{13} = m_{23} = 0$

6

$$A_{11} = \left(\frac{\partial^2 U}{\partial x_1^2}\right)_0 = k \qquad A_{12} = \left(\frac{\partial^2 U}{\partial x_1 \partial x_2}\right)_0 = -k$$

$$A_{22} = \left(\frac{\partial^2 U}{\partial x_2^2}\right)_0 = 2k \qquad A_{13} = \left(\frac{\partial^2 U}{\partial x_1 \partial x_3}\right)_0 = 0$$

$$A_{33} = \left(\frac{\partial^2 U}{\partial x_3^2}\right)_0 = k \qquad A_{23} = \left(\frac{\partial^2 U}{\partial x_2 \partial x_3}\right)_0 = -k$$

$$\Rightarrow \quad \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (k - m\omega^2)^2 (2k - M\omega^2) - k^2 (k - m\omega^2) - k^2 (k - m\omega^2) = 0$$

$$\Leftrightarrow \quad (k - m\omega^2) \left[(k - m\omega^2)(2k - M\omega^2) - 2k^2 \right] = 0$$

$$\Leftrightarrow \quad (k - m\omega^2) \left[\cancel{2k^2} - Mk\omega^2 - 2mk\omega^2 + mM\omega^4 - \cancel{2k^2} \right] = 0$$

$$\Leftrightarrow \quad \omega^2 (k - m\omega^2)(mM\omega^2 - Mk - 2mk) = 0$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \omega_1 = 0 \\ \omega_2 = \sqrt{\frac{k}{m}} \\ \omega_3 = \sqrt{\frac{k}{m} + \frac{2k}{M}} \end{array}}$$

7

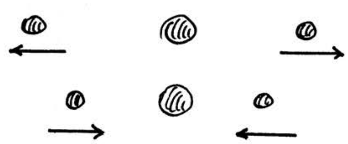
(3) (i) $\omega_1 = 0$ no oscillation, just translation of the system as a whole

(ii) $\omega_2 = \sqrt{\frac{k}{m}}$

$$\left(k - m\frac{k}{m}\right) a_{12} - k\, a_{22} = 0 \quad \Rightarrow \boxed{a_{22} = 0}$$

$$-k\, a_{12} - k\, a_{32} = 0 \quad \Rightarrow \boxed{a_{12} = -a_{32}}$$

$\Rightarrow$ the center particle is at rest and the two end particles oscillate in <u>opposite</u> directions with the <u>same</u> amplitude



(iii) $\omega_3 = \sqrt{\frac{k}{m} + \frac{2k}{M}}$

$$\left[k - m\left(\frac{k}{m} + \frac{2k}{M}\right)\right] a_{13} - k\, a_{23} = 0$$

$$\Rightarrow \boxed{a_{23} = -2\frac{m}{M} a_{13}}$$

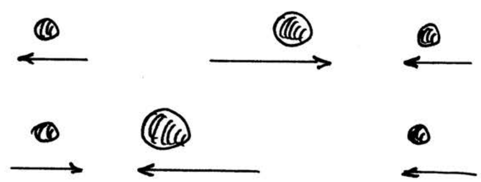
$$-k\, a_{13} + \left[2k - M\left(\frac{k}{m} + \frac{2k}{M}\right)\right] a_{23} - k\, a_{33} = 0$$

$$\Rightarrow \quad -a_{13} - \frac{M}{m} a_{23} = a_{33}$$

$$\Rightarrow \quad -a_{13} + 2\, a_{23} = a_{33} \quad \Rightarrow \boxed{a_{13} = a_{33}}$$

8

⇒ the two end particles oscillate in phase at the same amplitude while the center particle oscillates oppositely with a different amplitude



3.3.2 Offical finals, 2015

3.3.2.1 questions

Mechanics
Physics 311
Fall 2015
Final Exam (December 17, 2012)

There are 120 minutes permitted for the complete examination. Do not discuss the exam at any time. Answer the questions in a transparent way and **explain your answers.** Just providing the final answer is not sufficient - you must explain how you got there! For partial credit, you must show your work.

The exam is closed book, but you are allowed to bring one letter size note sheet which must be an original copy (no Xeroxes) in your handwriting. Calculators, computers, cellphones, or any other electronic devices are not permitted.

Good luck !

$ax^2+bx+c=0$ has solutions $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$
$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

Euler Equations:
$I_1\,\dot{\omega}_1-(I_2-I_3)\,\omega_2\,\omega_3=\tau_1$
$I_2\,\dot{\omega}_2-(I_3-I_1)\,\omega_3\,\omega_1=\tau_2$
$I_3\,\dot{\omega}_3-(I_1-I_2)\,\omega_1\,\omega_2=\tau_3$

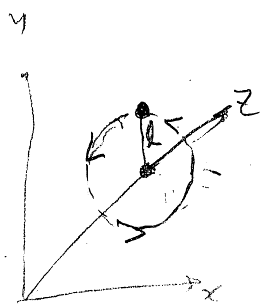
(Page 1 of 4)

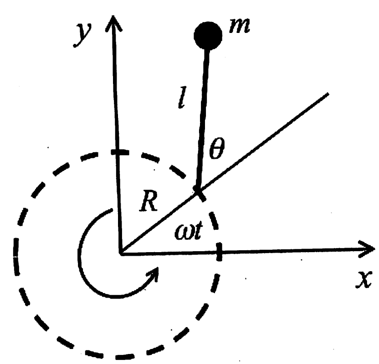
Problem 1 (15 points)

You are charged with designing a pendulum clock for use on a gravity-free spacecraft. The mechanism is a simple pendulum, i.e., a mass m at the end of a massless rod of length l hung from a pivot, about which it can swing in a plane. To provide artificial gravity, the pivot is forced to rotate at a frequency ω in a circle of radius R in the same plane as the pendulum arm (see Figure). angular velocity rad/sec

(1) (8 points) Determine the Lagrangian of the system.

(2) (7 points) Determine the equation of motion and show that the motion of this pendulum is identical to the motion of a simple pendulum in a uniform gravitational field. What is the strength of this field?





Problem 2 (15 points)

Consider a bucket of radius R that is spinning with a constant angular velocity ω about the symmetry axis, i.e., the vertical axis through the center of the bucket. Determine the shape of the surface of the water in the bucket by deriving an equation which describes the shape as a function of r, the distance from the center of the bucket.

Problem 3 (15 points)

check

A rigid body is undergoing force-free rotation about one of its principal axes. In class, we showed that in the case that all principal axes are distinct and $I_3 > I_2 > I_1$, rotation about the x_1- and x_3-axes is stable, but rotation about the x_2-axis is not. Now consider the case that two of the moments of inertia are equal, $I_1 = I_2$.

(1) (13 points) Is the rotation about the corresponding axes x_1 and x_2 stable or unstable? To check this, apply a small pertubation to the rotation, for example

$$\vec{\omega} = \omega_1 \hat{x}_1 + \lambda \hat{x}_2 + \mu \hat{x}_3$$

for the rotation about x_1, where $\lambda(t)$ and $\mu(t)$ are small quantities. Find the solution for λ and μ as a function of time. Do a similar calculation for the rotation about x_2.

(2) (2 points) Does the answer depend on whether I_3 is greater or less than $I_1 = I_2$?

$\bar{\omega} = \lambda x_1 + \omega_2 \hat{x} + \mu \hat{x}_3$

$\cos\theta = 1 - \frac{\theta}{2}$

Problem 4 (15 points)

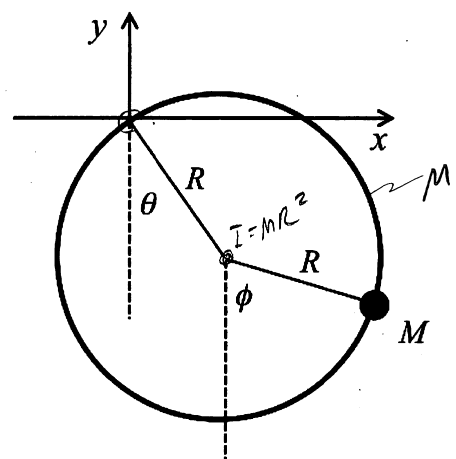
A thin hoop of radius R and mass M is suspended from a single point and oscillates in its own plane. A point-like mass M is constrained to move along the hoop. The moment of inertia of the hoop for rotations about the center of mass is $I_{CM} = MR^2$.

(1) (5 points) Consider small oscillations and determine the Lagrangian of the system. Neglect all terms of order higher than quadratic in small quantities (θ, ϕ, ...).

(2) (5 points) Show that the two eigenfrequencies are

$$\omega_1 = \sqrt{\frac{2g}{R}}, \qquad \omega_2 = \sqrt{\frac{g}{2R}} \ .$$

(3) (5 points) Determine the amplitude ratios for the two normal modes and describe the oscillation of the system for these modes. Identify the symmetric and the antisymmetic mode.



3.3.2.2 key solution

1

Mechanics

Physics 311 - Fall 2015

Final Exam - Solutions

Problem 1

(1) the x- and y- coordinates of mass m are given by

$$x = R\cos\omega t + \ell\cos(\theta + \omega t)$$
$$y = R\sin\omega t + \ell\sin(\theta + \omega t)$$

so

$$\dot{x} = -R\omega\sin\omega t + \ell(\dot\theta + \omega)\sin(\theta + \omega t)$$
$$\dot{y} = R\omega\cos\omega t + \ell(\dot\theta + \omega)\cos(\theta + \omega t)$$

$$\Rightarrow \quad L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\Big\{\omega^2R^2\sin^2\omega t + \ell^2(\dot\theta+\omega)^2\sin^2(\theta+\omega t) - 2R\omega\ell(\dot\theta+\omega)\sin\omega t\sin(\theta+\omega t) + \omega^2R^2\cos^2\omega t + \ell^2(\dot\theta+\omega)^2\cos^2(\theta+\omega t) + 2R\omega\ell(\dot\theta+\omega)\cos\omega t\cos(\theta+\omega t)\Big\}$$

$$= \frac{1}{2}m\Big\{\omega^2R^2 + \ell^2(\dot\theta+\omega)^2 + 2R\ell\omega(\dot\theta+\omega)\big[\sin\omega t\sin(\theta+\omega t) + \cos\omega t\cos(\theta+\omega t)\big]\Big\}$$

2

now use

$$\sin\omega t \sin(\theta+\omega t) + \cos\omega t \cos(\theta+\omega t)$$

$$= \sin\omega t \sin\theta \cos\omega t + \sin\omega t \cos\theta \sin\omega t$$

$$+ \cos\omega t \cos\theta \cos\omega t - \cos\omega t \sin\theta \sin\omega t$$

$$= \cos\theta \, (\sin^2\omega t + \cos^2\omega t) = \cos\theta$$

$$\Rightarrow \quad \boxed{L = \frac{1}{2} m \left[\omega^2 R^2 + \ell^2 (\dot\theta + \omega)^2 + 2R\ell\omega(\dot\theta+\omega)\cos\theta \right]}$$

(2)

$$\frac{\partial L}{\partial \theta} = -2R\ell\omega(\dot\theta+\omega)\sin\theta$$

$$\frac{\partial L}{\partial \dot\theta} = m\ell^2(\dot\theta+\omega) + mR\ell\omega\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m\ell^2\ddot\theta - mR\ell\omega\sin\theta\,\dot\theta$$

$$\Rightarrow \quad m\ell^2\ddot\theta - mR\ell\omega\sin\theta\,\dot\theta + mR\ell\omega(\dot\theta+\omega)\sin\theta = 0$$

$$\Leftrightarrow \quad \boxed{\ddot\theta + \frac{R\omega^2}{\ell}\sin\theta = 0}$$

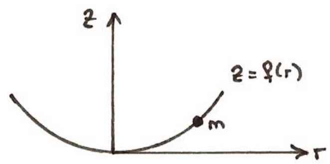
compare to simple pendulum: $\ddot\theta + \frac{g}{\ell}\sin\theta = 0$

↳ same equation, same solution!

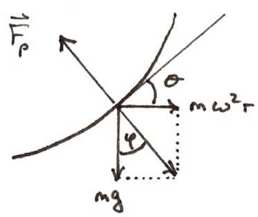
↳ the equivalent g is $\frac{g}{\ell} = \frac{R\omega^2}{\ell} \Rightarrow \boxed{g = R\omega^2}$

3

Problem 2



consider a small mass m on the surface of the water; in the rotating frame, the mass is at rest



the sum of the gravitational and centrifugal force must be normal to the surface; it is equal (but opposite) to the force due to the pressure gradient ($\vec{F}_p$)

$$\theta = \varphi$$

$$\tan\varphi = \frac{m\omega^2 r}{mg} = \frac{\omega^2 r}{g}$$

also, $\tan\varphi = \frac{dz}{dr}$ $\Rightarrow$ $\frac{dz}{dr} = \frac{\omega^2 r}{g}$

$$\Rightarrow \boxed{z = \frac{1}{2}\frac{\omega^2}{g} r^2 + \text{const.}}$$

the shape is parabolic!

4

Problem 3

(1) $\vec{\omega} = \omega_1 \hat{x} + \lambda \hat{x}_2 + \mu \hat{x}_3$ rotation about x_1

$$\Rightarrow \quad (I_2 - I_3)\lambda\mu - I_1 \dot{\omega}_1 = 0 \qquad (1)$$
$$(I_3 - I_1)\mu\omega_1 - I_2 \dot{\lambda} = 0 \qquad (2)$$
$$(I_1 - I_2)\lambda\omega_1 - I_3 \dot{\mu} = 0 \qquad (3)$$

Since $\lambda\mu$ is very small, we neglect it in (1), so

$$I_1 \dot{\omega}_1 = 0 \Rightarrow \omega_1 = \text{const.}$$

now $I_1 = I_2$, so (3) becomes $\dot{\mu} = 0 \Rightarrow \boxed{\mu = \text{const.}}$

(2) $\Rightarrow$ $$\dot{\lambda} = \underbrace{\frac{I_3 - I_1}{I_1}\omega_1 \mu}_{= \text{const.}}$$

$$\Rightarrow \boxed{\lambda(t) = \left(\frac{I_3 - I_1}{I_1}\omega_1 \mu\right) t + C}$$

the perturbation increases linearly with time, so the rotation about the x_1-axis is unstable

now
$\vec{\omega} = \lambda \hat{x} + \omega_2 \hat{y} + \mu \hat{x}_3$ rotation about x_2

5

$$(I_2 - I_3)\,\omega_2 \mu - I_1 \dot{\lambda} = 0 \qquad (1)$$
$$(I_3 - I_1)\,\lambda \mu - I_2 \dot{\omega}_2 = 0 \qquad (2)$$
$$(I_1 - I_2)\,\lambda \omega_2 - I_3 \dot{\mu} = 0 \qquad (3)$$

so $\dot{\omega}_2 = 0 \Rightarrow \omega_2 = \text{const}$

$\dot{\mu} = 0 \Rightarrow \boxed{\mu = \text{const.}}$

and $\boxed{\lambda(t) = \left(\frac{I_1 - I_3}{I_1}\,\omega_2 \mu\right) t + c'}$

Λ rotation about x_2 is also unstable

(2) the result is independent of whether I_3 is greater or Less than $I_1 = I_2$

6

Problem 4

(1) the origin of the coordinate system is the fixed point on the hoop

the coordinates of the point-like mass M are

$$x = R\sin\theta + R\sin\phi \simeq R\theta + R\phi$$

$$y = -R\cos\theta - R\cos\phi$$

$$\simeq -R\left(1-\frac{\theta^2}{2}+1-\frac{\phi^2}{2}\right) = R\left(\frac{\theta^2}{2}+\frac{\phi^2}{2}-2\right)$$

using $\sin\theta \simeq \theta$, $\cos\theta \simeq 1-\frac{\theta^2}{2}$, neglecting all terms higher than quadratic

$$T = T_{hoop} + T_{mass}$$

$$= \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2}M(\dot{x}^2+\dot{y}^2)$$

with

$$I = I_{CM} + MR^2 = 2MR^2$$

$$\Rightarrow \quad T = \frac{1}{2}(2MR^2)\dot{\theta}^2 + \frac{1}{2}MR^2\left(\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi}\right)$$

+ terms not quadratic in θ, ϕ etc.

$$= \frac{3}{2}MR^2\dot{\theta}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + MR^2\dot{\theta}\dot{\phi}$$

$$U = U_{hoop} + U_{mass}$$

$$= -MgR\cos\theta - MgR(\cos\theta+\cos\phi)$$

$$= -MgR(2\cos\theta+\cos\phi)$$

7

So $U \simeq MgR\,(\theta^2 + \frac{1}{2}\phi^2) + \text{const.}$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{L} = \tfrac{3}{2} MR^2 \dot\theta^2 + \tfrac{1}{2} MR^2 \dot\phi^2 + MR^2 \dot\theta \dot\phi \\ - MgR\,(\theta^2 + \tfrac{1}{2}\phi^2) \end{aligned}}$$

(2) now compare to $T = \frac{1}{2} m_{jk}\, \dot q_j \dot q_k$

$$\Rightarrow \quad m_{11} = 3MR^2 \qquad m_{12} = m_{21} = MR^2$$
$$m_{22} = MR^2$$

and use

$$A_{11} = \left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta_0} = 2MgR$$

$$A_{22} = \left.\frac{\partial^2 U}{\partial \phi^2}\right|_{\phi_0} = MgR$$

$$A_{12} = A_{21} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2MgR - 3MR^2\omega^2 & -MR^2\omega^2 \\ -MR^2\omega^2 & MgR - MR^2\omega^2 \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow \quad \left(2\frac{g}{R} - 3\omega^2\right)\left(\frac{g}{R} - \omega^2\right) - \omega^4 = 0$$

$$\Leftrightarrow \quad 2\frac{g^2}{R^2} - 2\frac{g}{R}\omega^2 - 3\frac{g}{R}\omega^2 + 3\omega^4 - \omega^4 = 0$$

$$\Leftrightarrow \quad \omega^4 - \frac{5}{2}\frac{g}{R}\omega^2 + \frac{g^2}{R^2} = 0$$

8

So $$\omega^2 = \frac{5}{4}\frac{g}{R} \pm \sqrt{\frac{25}{16}\frac{g^2}{R^2} - \frac{g^2}{R^2}}$$

$$= \frac{5}{4}\frac{g}{R} \pm \frac{3}{4}\frac{g}{R}$$

So $\boxed{\omega_1 = \sqrt{\frac{2g}{R}}}$ $\boxed{\omega_2 = \sqrt{\frac{g}{2R}}}$

(3) for ω_1 $$\left(2MgR - 3MR^2\frac{2g}{R}\right)a_{11} - MR^2\frac{2g}{R}a_{21} = 0$$

$$\Leftrightarrow \quad -4MgR \quad a_{11} - 2MgR\, a_{21} = 0$$

$$\Leftrightarrow \quad \boxed{a_{21} = -2a_{11}}$$

↳ the center of mass of the hoop and mass M are on opposite sides of the vertical through the pivot point; the oscillation amplitude of mass M is twice that of the hoop's CM

→ antisymmetric mode

for ω_2 $$\left(2MgR - 3MR^2\frac{g}{2R}\right)a_{12} - MR^2\frac{g}{2R}a_{22} = 0$$

$$\Leftrightarrow \quad \frac{1}{2}MgR\ a_{12} - \frac{1}{2}MgR^2\ a_{22} = 0$$

$$\Leftrightarrow \quad \boxed{a_{12} = a_{22}}$$

↳ the amplitude of θ and ϕ are the same and are also in the same direction; in this mode, the pivot point, the mass M and the hoop's CM are always on a straight line

→ symmetric mode

Chapter 4

HWs

4.1 HW 1

4.1.1 Problem 1

1. (5 points)
   A particle is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and calculate the time required for the particle to return to its starting point. Find the time for $v_0 = 2.4\,\mathrm{m/s}$ and $\alpha = 26°$.

SOLUTION

The vertical component of motion is only considered since that is the component that changes due to the action of gravity.

The equation of motion in the vertical y direction is given by $F = ma$. Hence

$$\begin{aligned} my'' &= -mg \\ y'' &= -g \end{aligned}$$

Integrating once gives

$$y' - y'(0) = -gt$$

Where $y'(0) = v_0 \sin(\alpha)$. The time for the particle to reach a final velocity of zero in the vertical direction is now find by solving the above for t

$$y'_{\text{final}} = y'(0) - gt$$

Where $y'_{\text{final}} = 0$. Solving the above for the time t gives

$$0 = v_0 \sin\alpha - gt$$
$$t = \frac{v_0 \sin\alpha}{g}$$

Hence the total time to reach back to its starting point is twice the above time, which is

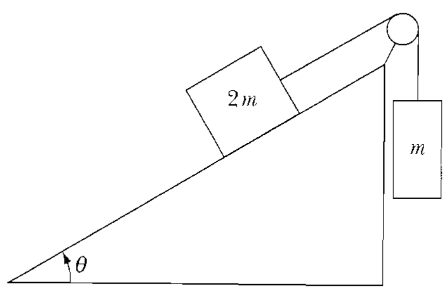
$$\boxed{\text{total time} = 2\left(\frac{v_0 \sin\alpha}{g}\right)}$$

For $\alpha = 20$ degree and $v_0 = 2.4\frac{m}{s}$ and $g = 9.81\frac{m}{s^2}$, the total time is found from

$$\text{total time} = 2\left(\frac{2.4 \sin 20^o}{9.81}\right)$$
$$= \boxed{0.167\,\text{second}}$$

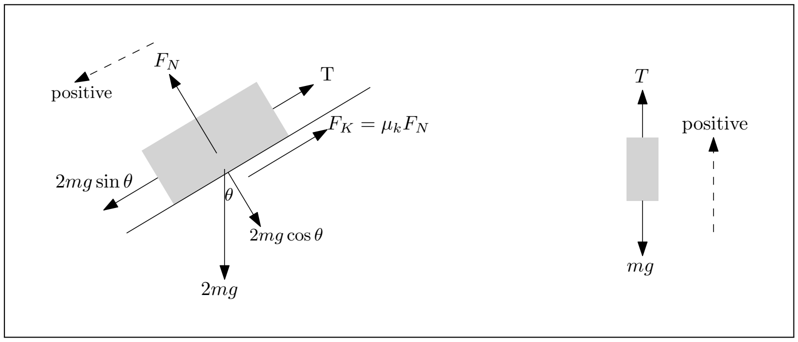
4.1.2 Problem 2

2. (10 points)
Two blocks of unequal mass are connected by a string over an ideal pulley (whose mass is negligible and that rotates with negligible friction). If the coefficient of kinetic friction is μ_k, what angle θ allows the mass to move at a constant speed?



SOLUTION

The free body diagram is shown below for each mass.



The acceleration of each body is the same. Let this acceleration be a. From the above free body diagram of the $2m$ body the equation of motion is now derived (using positive direction as show)

$$\sum F = 2ma$$
$$2mg\sin\theta - \mu_k F_N - T = 2ma$$

Using $F_N = 2mg\cos\theta$ the above becomes

$$2mg\sin\theta - \mu_k 2mg\cos\theta - T = 2ma \tag{1}$$

The tension T in the string is found from the free body diagram of the smaller hanging mass since the tension T is same. From the free body diagram of the small mass the equation of motion is

$$\sum F = ma$$
$$-mg + T = ma$$

Hence

$$T = m\left(a + g\right)$$

Substituting T in (1) gives

$$2mg\sin\theta - \mu_k 2mg\cos\theta - m\left(a + g\right) = (2m)\,a$$
$$2g\sin\theta - \mu_k 2g\cos\theta - g = 3a$$

Therefore

$$a = \frac{2}{3}\left(g\sin\theta - \mu_k g\cos\theta - \frac{1}{2}g\right)$$

For constant speed, $a = 0$ at some angle θ_c. The above reduces to

$$\frac{2}{3}\left(g\sin\theta_c - \mu_k g\cos\theta_c - \frac{1}{2}g\right) = 0$$
$$\sin\theta_c - \mu_k\cos\theta_c - \frac{1}{2} = 0$$
$$\sin\theta_c - \mu_k\cos\theta_c = \frac{1}{2} \tag{2}$$

To solve this, the following identity is used

$$R\sin\left(\theta_c + \alpha\right) = R\left(\sin\theta_c\cos\alpha + \cos\theta_c\sin\alpha\right) \tag{3}$$

Comparing the RHS of (3) with the LHS of (2) gives

$$R\cos\alpha = 1 \tag{4}$$

$$R\sin\alpha = -\mu_k \tag{5}$$

Dividing (5) by (4) gives $\tan\alpha = -\mu_k$ or

$$\alpha = \tan^{-1}\left(-\mu_k\right) = -\tan^{-1}\left(\mu_k\right)$$

Squaring (4) and (5) and adding gives

$$\begin{aligned} R^2\cos^2\alpha + R^2\sin^2\alpha &= 1+\mu_k^2 \\ R &= \sqrt{1+\mu_k^2} \end{aligned}$$

Therefore the equation $R\sin(\theta+\alpha) = \frac{1}{2}$ becomes

$$\begin{aligned} \sqrt{1+\mu_k^2}\sin(\theta_c+\alpha) &= \frac{1}{2} \\ \sin\left(\theta_c - \tan^{-1}\left(\mu_k\right)\right) &= \frac{1}{2\sqrt{1+\mu_k^2}} \\ \theta_c - \tan^{-1}\left(\mu_k\right) &= \sin^{-1}\left(\frac{1}{2\sqrt{1+\mu_k^2}}\right) \end{aligned}$$

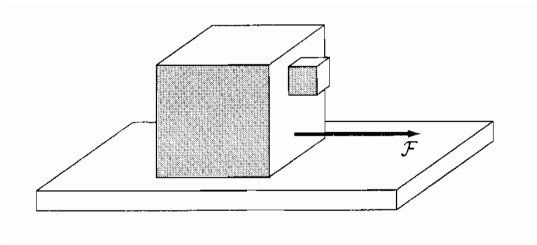
Therefore

$$\theta_c = \sin^{-1}\left(\frac{1}{2\sqrt{1+\mu_k^2}}\right) + \tan^{-1}\left(\mu_k\right)$$

For the case of no friction, where $\mu_k = 0$ the above gives $\theta_c = \sin^{-1}\left(\frac{1}{2}\right) = 30^0$. As μ_k increases, the angle θ_c will increase. (in the limit, as $\mu_k \to \infty, \theta_c \to 90^0$). This is a plot showing how the angle changes as μ_k increases.

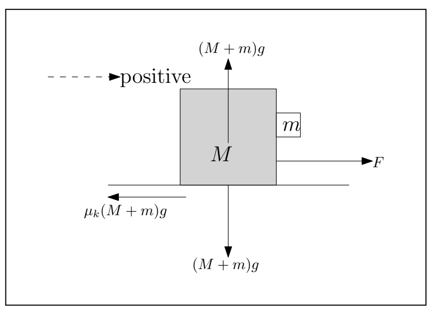
4.1.3 Problem 3

3. (10 points)
A small box of mass m is in contact with a large box of mass M as shown in the picture. A force $\vec{F}$ pushes on the large box. Because of friction, the small box will not fall if $\vec{F}$ is large enough. How large does $\vec{F}$ need to be? Take into account *all* frictional forces and assume that the coefficients of friction at all surfaces are μ_s for static and μ_k for kinetic friction.



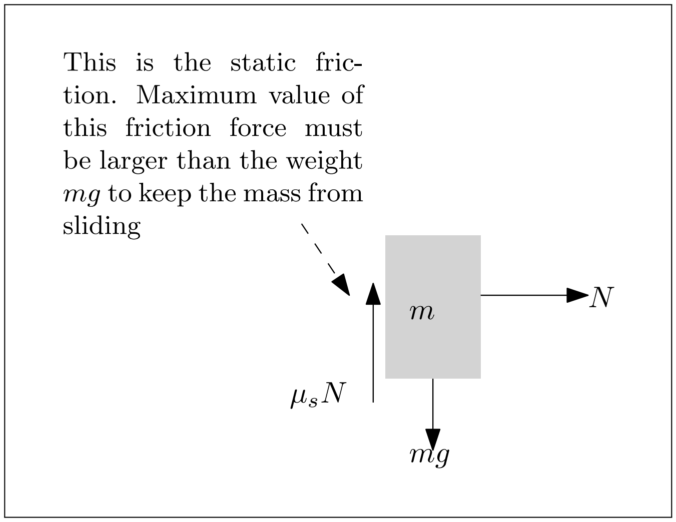
SOLUTION

Looking at the case where the small mass m is not moving (not sliding down the side), and considering both $M+m$ as one body. Let the horizontal acceleration of both bodies be a



$$
\begin{aligned}
\sum F_x &= (M+m)a \\
F - \mu_k(M+m)g &= (M+m)a \\
a &= \frac{F - \mu_k(M+m)g}{(M+m)} \qquad (1)
\end{aligned}
$$

The small mass m is now considered. The static friction force between m and M has to be larger than the weight mg so that m does not move and fall. This implies $f_{s_{\max}} = \mu_s N$ must be larger than the weight mg



This implies the following condition is required

$$\mu_s N \geq mg \tag{2}$$

Where N is the normal force on m. But

$$ma = N$$

From (1) we find

$$N = m\left(\frac{F - \mu_k(M+m)g}{(M+m)}\right)$$

Therfore (2) becomes

$$\mu_s m\left(\frac{F - \mu_k(M+m)g}{(M+m)}\right) \geq mg$$

Hence

$$\begin{aligned} F - \mu_k(M+m)g &\geq \frac{g}{\mu_s}(M+m) \\ F &\geq \frac{g}{\mu_s}(M+m) + \mu_k(M+m)g \\ &\geq (M+m)g\left(\frac{1}{\mu_s} + \mu_k\right) \end{aligned}$$

Hence

$$\boxed{F \geq (M+m)g\left(\frac{1+\mu_s\mu_k}{\mu_s}\right)}$$

4.1.4 Problem 4

4. (5 points)
Show that the terminal velocity of a falling object is given by

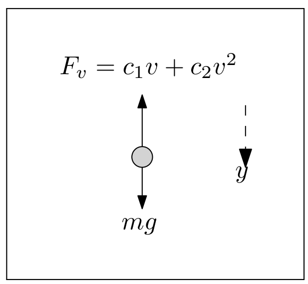
$$v_t = \left[\left(\frac{mg}{c_2}\right) + \left(\frac{c_1}{2\,c_2}\right)^2\right]^{\frac{1}{2}} - \left(\frac{c_1}{2\,c_2}\right)$$

if the drag force F_v has *both* a linear *and* quadratic term in v:

$$F_v = c_1\,v + c_2\,v^2 \quad .$$

SOLUTION

From the free body diagram



The equation of motion is

$$\sum F_y = my''$$
$$mg - (c_1 y' + c_2(y'^2)) = my''$$

At the terminal velocity the body is not accelerating. Setting $y'' = 0$ in the above gives an equation to solve for the terminal velocity (where now y' is written as v_t)

$$mg - (c_1 v_t + c_2 v_t^2) = 0$$
$$c_2 v_t^2 + c_1 v_t - mg = 0$$

This is a quadratic equation in v_t, hence the roots are given by

$$\begin{aligned} v_t &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-c_1}{2c_2} \pm \frac{\sqrt{c_1^2 + 4c_2 mg}}{2c_2} \\ &= \frac{-c_1}{2c_2} \pm \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}} \end{aligned}$$

Since the terminal velocity v_t has to be positive as indicated in the diagram above, then the solution is the positive root given by

$$\boxed{v_t = \frac{-c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}}}$$

4.1.5 Problem 5

5. (10 points)
   A projectile is fired with an initial velocity $v_0 = 500\,\text{m/s}$ in a direction making an angle $\alpha = 30°$ with the horizontal. We want to study the effect of air resistance on the range of the projectile. Assume that the drag force has the form $F_v = k\,m\,v$, where m and v are the mass and velocity of the projectile and k is a constant.

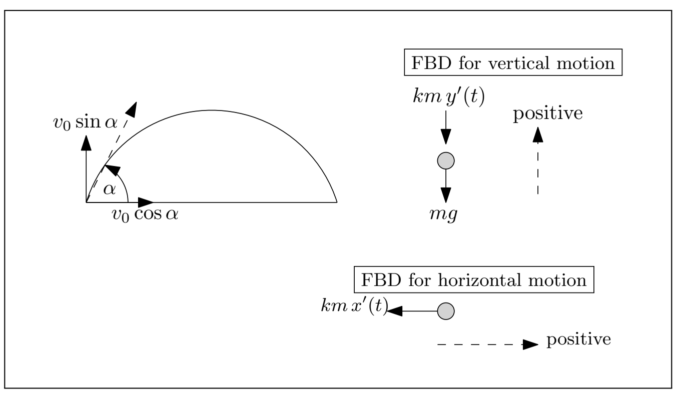
   (1) Solve the equations of motions and determine the time T required for the full trajectory.

   (2) Use a computer to draw the trajectories of the projectile for $k = 0$ (no air resistance), $k = 0.001$, $k = 0.01$ and $k = 0.1$. From your plots, estimate roughly the range for the different k.

SOLUTION

4.1.5.1 Part (1)

The following is the free body diagram used to solve this problem.



In the vertical direction, with positive taken upwards as shown in the diagram, the equation of motion is given by

$$\begin{aligned}\sum F_y &= my'' \\ -mg - kmy' &= my'' \\ y'' + ky' &= -g \qquad (1)\end{aligned}$$

In the horizontal direction, the equation of motion is

$$\begin{aligned}\sum F_x &= mx'' \\ -kmx' &= mx'' \qquad (2)\end{aligned}$$

The initial conditions for equation of motion in the vertical direction are $y(0) = 0, y'(0) = v_0 \sin\alpha$ and the initial conditions for the equation of motion in the horizontal direction are $x(0) = 0, x'(0) = v_0 \cos\alpha$.

Equation (1) is now solved. The characteristic equation is $\lambda^2 + k\lambda = 0$ or $\lambda(\lambda + k) = 0$, hence the roots are $\lambda = 0, \lambda = -k$, and therefore the homogeneous solution is

$$y_h(t) = A + Be^{-kt}$$

The particular solution is now found. Let $y_p(t) = ct$ where c is some constant. Substituting this into (1) gives

$$\begin{aligned} kc &= -g \\ c &= \frac{-g}{k} \end{aligned}$$

Hence the particular solution is $y_p(t) = -\frac{g}{k}t$, and the complete solution in the y direction is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \left(A + Be^{-kt}\right) - \frac{g}{k}t \end{aligned}$$

The initial conditions are now applied to determine the constants A, B. (Initial conditions must be used in the complete solution and not the homogeneous solution). When $t = 0$, $y(0) = 0$ and the above gives

$$A = -B$$

Since $y'(t) = -Bke^{-kt} - \frac{g}{k}$ and since $y'(0) = v_0 \sin\alpha$, then at $t = 0$

$$\begin{aligned} v_0 \sin\alpha &= -Bk - \frac{g}{k} \\ B &= -\left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right) \end{aligned}$$

Using values for the constants A, B, the complete solution for equation of motion in the vertical direction becomes

$$\begin{aligned} y(t) &= \left(A + Be^{-kt}\right) - \frac{g}{k}t \\ &= \left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right) - \left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t \end{aligned}$$

Hence

$$y(t) = \left(\frac{g + kv_0 \sin\alpha}{k^2}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \tag{3}$$

The duration time T is now found by solving for $y = 0$ from (3). Hence

$$\begin{aligned} 0 &= \left(\frac{g + kv_0 \sin\alpha}{k^2}\right)\left(1 - e^{-kT}\right) - \frac{g}{k}T \\ T &= \left(\frac{g + kv_0 \sin\alpha}{gk}\right)\left(1 - e^{-kT}\right) \end{aligned} \tag{4}$$

An analytical solution based on perturbation method for this is given in the text book at page 67 as

$$\boxed{T \simeq \frac{2v_0 \sin \alpha}{g}\left(1 - \frac{kv_0 \sin \alpha}{3g}\right)}$$

However in this solution equation (4) was solved numerically instead for T for the numerical values given in this problem, and the results are summarized on the following table

k	T (sec)
0.001	50.5427
0.01	47.2597
0.1	34.3395

The equation of motion in the x direction is now solved. This equation is given above in (2) as $x'' + kx' = 0$. The characteristic equation is $\lambda^2 + k\lambda = 0$ or $\lambda(\lambda + k) = 0$, hence the roots are $\lambda = 0, \lambda = -k$, and therefore, the homogeneous solution is

$$x_h(t) = A + Be^{-kt}$$

Since there is no forcing function, the complete solution is the same

$$x(t) = A + Be^{-kt} \tag{5}$$

The constants are found from the initial conditions. At $t = 0$

$$\begin{aligned} 0 &= A + B \\ A &= -B \end{aligned}$$

Since $x'(t) = -Bke^{-kt}$, then at $t = 0$

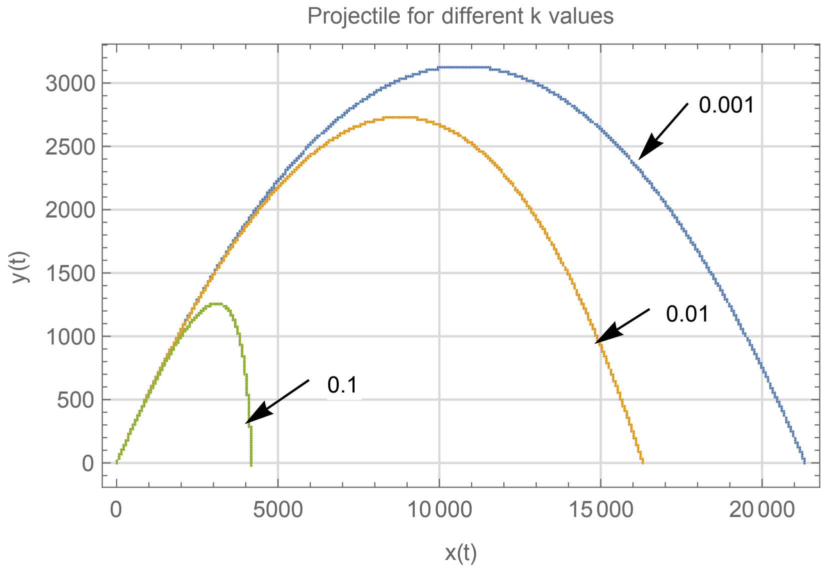
$$\begin{aligned} v_0 \cos \alpha &= -Bk \\ B &= \frac{-v_0 \cos \alpha}{k} \end{aligned}$$

Substituting the above values for A, B into (4) gives the solution for the motion in the horizontal direction

$$x(t) = \frac{v_0 \cos \alpha}{k}\left(1 - e^{-kt}\right) \tag{6}$$

4.1.5.2 Part (2)

The following shows the projectile path for each different k value.



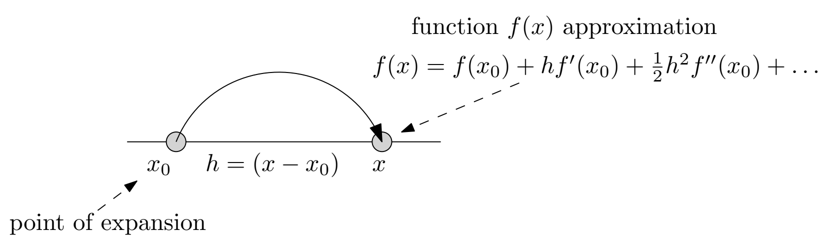
From the above, an estimate of the range for each k is given in the following table

k	range (meters)
0.001	21500
0.01	16500
0.1	4100

4.1.6 Problem 6

6. (10 points)
Find the Taylor series expansion of
(1) $f(x) = \cos x$ about $x = 0$,
(2) $f(x) = \cosh x$ about $x = 0$,
(3) $f(x) = \ln x$ about $x = 2$,
(4) $f(x) = \frac{1}{x^2}$ about $x = -1$,
(5) $f(x) = \sqrt{1+x}$ about $x = 0$.
Check out *Appendix A* of Thornton/Marion if you are unfamiliar with Taylor expansions.

SOLUTION



4.1.6.1 Part (1)

$f(x) = \cos(x)$ **about** $x = 0$

$$\begin{aligned} f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\ &= \cos(0) + x(-\sin(0)) + \frac{1}{2}x^2(-\cos(0)) + \frac{1}{6}x^3 \sin(0) + \frac{1}{24}x^4 \cos(0) + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \end{aligned}$$

4.1.6.2 Part (2)

$f(x) = \cosh(x)$ **about** $x = 0$

$$\begin{aligned} f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\ &= \cosh(0) + x(\sinh(0)) + \frac{1}{2}x^2(\cosh(0)) + \frac{1}{6}x^3 \sinh(0) + \frac{1}{24}x^4 \cosh(0) + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots \end{aligned}$$

4.1.6.3 Part(3)

$f(x) = \ln(x)$ **about** $x = 2$

$$\begin{aligned} f(x) &\simeq f(2) + hf'(2) + \frac{1}{2}h^2 f''(2) + \frac{1}{3!}h^3 f'''(2) + \frac{1}{4!}h^4 f^{(4)}(2) + \cdots \\ &= \ln(2) + (x-2)\left(\frac{1}{x}\right)_{x=2} + \frac{1}{2}(x-2)^2\left(\frac{-1}{x^2}\right)_{x=2} + \frac{1}{6}(x-2)^3\left(\frac{2}{x^3}\right)_{x=2} + \frac{1}{24}(x-2)^4\left(\frac{-6}{x^4}\right)_{x=2} + \cdots \\ &= \ln(2) + \frac{x-2}{2} - \frac{1}{2}\frac{(x-2)^2}{4} + \frac{1}{3}\frac{(x-2)^3}{8} - \frac{1}{4}\frac{(x-2)^4}{16} + \cdots \\ &= \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + \cdots \end{aligned}$$

4.1.6.4 Part(4)

$f(x) = \frac{1}{x^2}$ **about** $x = -1$

$$\begin{aligned} f(x) &\simeq f(-1) + hf'(-1) + \frac{1}{2}h^2 f''(-1) + \frac{1}{3!}h^3 f'''(-1) + \frac{1}{4!}h^4 f^{(4)}(-1) + \cdots \\ &= 1 + (x+1)\left(\frac{-2}{x^3}\right)_{x=-1} + \frac{1}{2}(x+1)^2\left(\frac{6}{x^4}\right)_{x=-1} + \frac{1}{6}(x+1)^3\left(\frac{-24}{x^5}\right)_{x=-1} + \frac{1}{24}(x+1)^4\left(\frac{120}{x^6}\right)_{x=-1} + \cdots \\ &= 1 + (x+1)\left(\frac{-2}{-1}\right) + \frac{1}{2}(x+1)^2\left(\frac{6}{1}\right) + \frac{1}{6}(x+1)^3\left(\frac{-24}{-1}\right) + \frac{1}{24}(x+1)^4\left(\frac{120}{1}\right) + \cdots \\ &= 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + 5(x+1)^4 + \cdots \end{aligned}$$

4.1.6.5 Part(5)

$f(x) = \sqrt{1+x}$ about $x = 0$

$$
\begin{aligned}
f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\
&= 1 + x\left(\frac{1}{2\sqrt{1+x}}\right)_{x=0} + \frac{1}{2}x^2\left(\frac{-1}{4(1+x)^{\frac{3}{2}}}\right)_{x=0} + \frac{1}{6}x^3\left(\frac{3}{8(1+x)^{\frac{5}{2}}}\right)_{x=0} + \frac{1}{24}x^4\left(\frac{-15}{16(1+x)^{\frac{7}{2}}}\right)_{x=0} + \cdots \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots
\end{aligned}
$$

4.1.7 HW 1 key solution

1

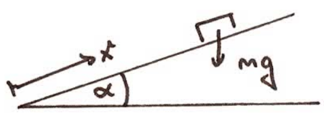
Mechanics

Physics 311 - Fall 2015

Homework Set 1 - Solutions

Problem 1

positive x-direction up the slope, starting at bottom



forces in x: $m\ddot{x} = -mg\sin\alpha$

$$\dot{x} = -g\sin\alpha\, t + v_0$$

$$x = -\frac{1}{2} g \sin\alpha\, t^2 + v_0 t$$

$x = 0$ for $t = 0$ (start) and $t = t_r$ (return), so

$$\frac{1}{2} g \sin\alpha\, t_r = v_0$$

$$\Rightarrow \quad t_r = \frac{2 v_0}{g \sin\alpha}$$

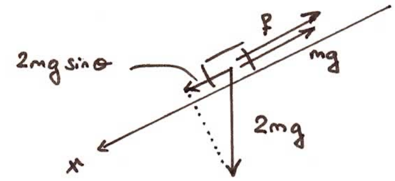
so for $\alpha = 26°$ and $v_0 = 2.4\,\frac{m}{s}$, $\quad t_r = \frac{2 \cdot 2.4\, m/s}{9.8\, m/s^2 \sin 26°}$

$$= \underline{\underline{1.1\ s}}$$

2

Problem 2

x-axis down the slope ; forces on mass $(2m)$:



$$f = \mu_k F_N = \mu_k \, 2mg \cos\theta$$

$$\Sigma F_x = 2mg\sin\theta - f - mg = m\ddot{x} = 0 \quad \text{for } \ddot{x} = 0$$

(constant speed)

So

$$2mg\sin\theta - \mu_k 2mg\cos\theta - mg = 0$$

$$\Rightarrow \quad 2(\sin\theta - \mu_k\cos\theta) = 1$$

Solve for θ :

$$2\left(\sin\theta - \mu_k\sqrt{1-\sin^2\theta}\right) = 1$$

$$\Leftrightarrow \quad \sin\theta - \frac{1}{2} = \mu_k\sqrt{1-\sin^2\theta}$$

So

$$\sin^2\theta - \sin\theta + \frac{1}{4} = \mu_k^2(1-\sin^2\theta)$$

$$\Leftrightarrow \quad (1+\mu_k)^2\sin^2\theta - \sin\theta + \left(\frac{1}{4} - \mu_k^2\right) = 0$$

So

$$\sin\theta = \frac{1 \pm \sqrt{1 - 4(1+\mu_k^2)(\frac{1}{4}-\mu_k^2)}}{2(1+\mu_k^2)}$$

$$= \frac{1 \pm \sqrt{1-(1-4\mu_k^2+\mu_k^2-4\mu_k^4)}}{2(1+\mu_k^2)}$$

3

$$\Rightarrow \qquad \sin\theta = \frac{1 \pm \sqrt{4\mu_k^4 + 3\mu_k^2}}{2(1+\mu_k^2)}$$

or

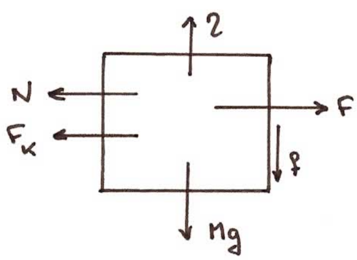
$$\boxed{\sin\theta = \frac{1 + \mu_k\sqrt{4\mu_k^2 + 3}}{2(1+\mu_k^2)}}$$

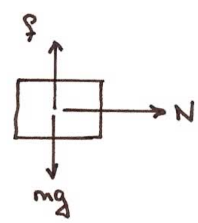
note: only the "+" solution satisfies

$$2(\sin\theta - \mu_k\cos\theta) = 1$$

Problem 3

free-body diagram for each body





weight Mg downward; M is in contact with the floor, so there is a normal force η and friction F_K

weight mg downward; m is in contact with M, so there could be a normal force N; if there is a normal force N, there is a frictional force f

4

M is in contact with m ; the large box pushes the small box to the right with a normal force N , so the small box pushes the large box to the left with force N (Newton 3) ; the large box pushes the small box upward with force f , so the small box pushes the large box downward with f (Newton 3 again).

<u>body M</u>, x- and y-direction

$$F - N - F_k = M\ddot{x}$$

$$\eta - Mg - f = M\ddot{y}_M = 0$$

$$F_k = \mu_k \eta$$

<u>body m</u> , x- and y-direction

$$N = m\ddot{x}$$

$$f - mg = m\ddot{y}_m = 0$$

$$f = \mu_s N$$

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$$\Rightarrow \quad F - N - \mu_k \eta = M\ddot{x} \qquad (1)$$

$$\eta - Mg - \mu_s N = 0 \qquad (2)$$

$$N = m\ddot{x} \qquad (3)$$

$$\mu_s N - mg = 0 \qquad (4)$$

$$(4) \Rightarrow \quad N = \frac{mg}{\mu_s}$$

$$(3) \Rightarrow \quad \ddot{x} = \frac{N}{m} = \frac{mg}{\mu_s m} = \frac{g}{\mu_s}$$

$$(2) \Rightarrow \quad \eta = Mg + \mu_s N = Mg + mg = (M+m)g$$

$$\Rightarrow \quad F = N + \mu_k \eta + M\ddot{x}$$

$$= \frac{mg}{\mu_s} + \mu_k (M+m)g + M\frac{g}{\mu_s}$$

$$\Rightarrow \quad \boxed{F = (M+m)\left(\frac{1}{\mu_s} + \mu_k\right)g}$$

<u>note</u>: the <u>smaller</u> μ_s, the <u>larger</u> the force
the <u>larger</u> μ_k, the <u>larger</u> the force $\Rightarrow$ makes sense!

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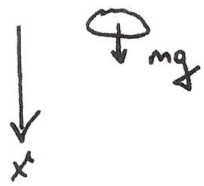
Problem 4



$$m\ddot{x} = mg - c_1 v - c_2 v^2$$

$$m\frac{dv}{dt} = mg - c_1 v - c_2 v^2$$

when $v = v_t$, $\frac{dv}{dt} = 0$

$\Rightarrow$ $mg - c_1 v_t - c_2 v_t^2 = 0$

so

$$v_t = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2 mg}}{2c_2}$$

$$= -\left(\frac{c_1}{2c_2}\right) \pm \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}}$$

with x going downwards, v_t should be positive, so only

$$v_t = -\left(\frac{c_1}{2c_2}\right) + \left(\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}\right)^{1/2}$$

is a solution. ∎

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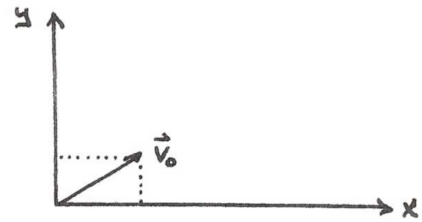
Problem 5



$$\vec{V}_0 = V_0 \cos\theta\ \hat{x} + V_0 \sin\theta\ \hat{y}$$

(1)

X-direction:

$$m\ddot{x} = -kmẋ \quad \Rightarrow \quad \frac{dv}{dt} = -kv$$

so $\frac{dv}{v} = -kt \qquad \int \frac{dv}{v} = -k\int dt$

$$\Rightarrow \ln v = -kt + C_1$$

$$\Rightarrow \quad V = C_2 e^{-kt}$$

initial condition: $\vec{V} = \vec{V}_0$ at $t=0$, so $V = V_0 \cos\theta$ at $t=0$

$$\Rightarrow \quad V = V_0 \cos\theta\ e^{-kt}$$

and $\quad X(t) = \frac{V_0 \cos\theta}{(-k)} e^{-kt} + C_3$

initial condition: $X(t=0) = 0$

$$\Rightarrow \quad X(t) = \frac{V_0 \cos\theta}{k}\left(1 - e^{-kt}\right)$$

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y-direction: $m\ddot{y} = -km\dot{y} - mg$

Solved in class: $V = -\frac{g}{k} + \frac{kV_0\sin\theta + g}{k} e^{-kt}$

$\Rightarrow \quad y(t) = -\frac{g}{k} t + \frac{kv_0\sin\theta + g}{k^2}\left(1 - e^{-kt}\right)$

for the total time, $y=0$

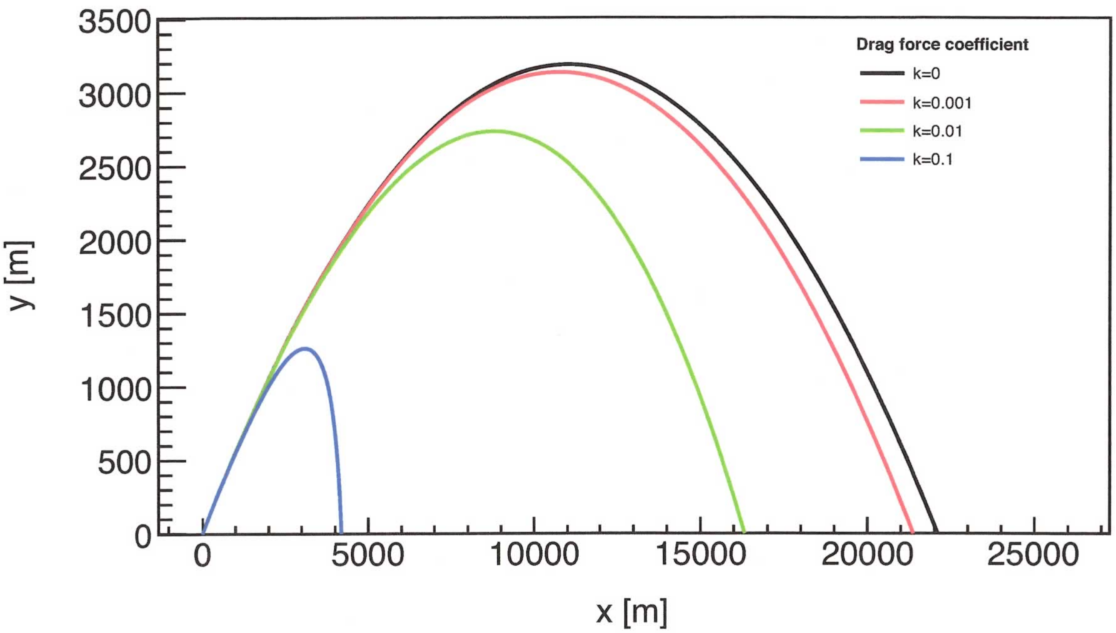
$\Rightarrow \quad T = \frac{kv_0\sin\theta + g}{gk}\left(1 - e^{-kT}\right)$

can't be solved for T ↯

(2)

ranges (from plots on the next page)

$k=0$	$R =$	22 km
0.001	$\simeq$	21.3 km
0.01	$\simeq$	16.2 km
0.1	$\simeq$	4.1 km



Drag force coefficient
k=0
k=0.001
k=0.01
k=0.1
y [m]
x [m]
3500
3000
2500
2000
1500
1000
500
0
0
5000
10000
15000
20000
25000

10

Problem 6

Taylor series of $f(x)$ near a:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

or
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(1) $f(x) = \cos x$ about $a=0$

$$\begin{array}{ll} f(x) = \cos x & f(0) = 1 \\ f'(x) = -\sin x & f'(0) = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f''''(x) = \cos x & f''''(0) = 1 \quad \text{etc.} \end{array}$$

$$\Rightarrow \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

(2) $f(x) = \cosh x$ about $a=0$

$$\begin{array}{ll} f(x) = \cosh x & f(0) = 1 \\ f'(x) = \sinh x & f'(0) = 0 \\ f''(x) = \cosh x & f''(0) = 1 \quad \text{etc.} \end{array}$$

$$\Rightarrow \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

(3) $f(x) = \ln x$ about $a = 2$

$$f(x) = \ln x \qquad f(2) = \ln 2$$
$$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$$
$$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{2^2}$$
$$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{2}{2^3}$$
$$f''''(x) = -\frac{2\cdot 3}{x^4} \qquad f''''(2) = -\frac{2\cdot 3}{2^4}$$

$$\Rightarrow \quad \ln x \Big|_2 = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(n-1)!}{n!\;2^n}(x-2)^n$$

(4) $f(x) = \frac{1}{x^2}$ about $a = -1$

$$f(x) = \frac{1}{x^2} \qquad f(-1) = 1$$
$$f'(x) = -\frac{2}{x^3} \qquad f'(-1) = 2$$
$$f''(x) = \frac{2\cdot 3}{x^4} \qquad f''(-1) = 2\cdot 3$$
$$f'''(x) = -\frac{2\cdot 3\cdot 4}{x^5} \qquad f'''(-1) = 2\cdot 3\cdot 4$$

$$\Rightarrow \quad \frac{1}{x^2}\Big|_{-1} = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n$$

12

(5) $f(x) = \sqrt{1+x}$ about $a = 0$

$$f(x) = \sqrt{1+x} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2}\frac{1}{(1+x)^{1/2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}\frac{1}{(1+x)^{3/2}} \qquad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}\frac{1}{(1+x)^{5/2}} \qquad f'''(0) = \frac{3}{8}$$

$$f''''(x) = -\frac{15}{16}\frac{1}{(1+x)^{7/2}} \qquad f''''(0) = -\frac{15}{16}$$

$$\Rightarrow \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$$

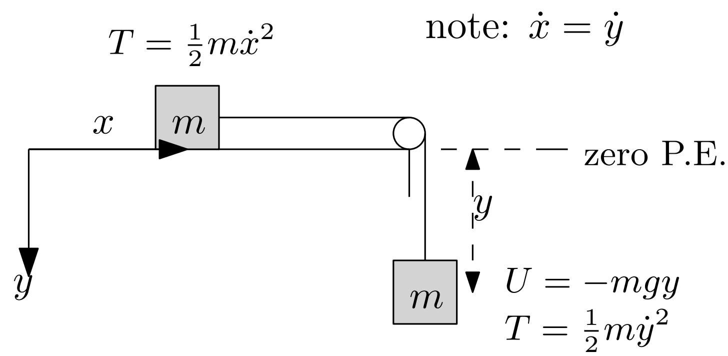
$$\left(\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^n \right)$$

4.2 HW 2

4.2.1 Problem 1

1. (5 points)
Two blocks of equal mass m are connected by an extensionless uniform string of length l. One block is placed on a smooth horizontal table, the other block hangs over the edge, the string passing over a frictionless pulley. Determine the Lagrangian of the system and find the acceleration of the blocks, assuming the mass of the string is negligible.

SOLUTION



$$L = T - U$$

Where U is the potential energy of the whole system and T is the kinetic energy of the whole system. The two masses will have the same speed since the string does not stretch. This means $\dot{x} = \dot{y}$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

Since $\dot{x} = \dot{y}$, we can write the above as

$$T = m\dot{y}^2$$

The potential energy U, using zero as the level shown in the above diagram is

$$U = -mgy$$

Hence the Lagrangian is $L = T - U$ or

$$\boxed{L = m\dot{y}^2 + mgy}$$

To find equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$$

But $\frac{\partial L}{\partial y} = mg$ and $\frac{d}{dt}\frac{\partial L}{\dot{y}} = \frac{d}{dt}\left(2m\dot{y}\right) = 2m\ddot{y}$, hence the above becomes

$$2m\ddot{y} - mg = 0$$

Or

$$\boxed{\ddot{y} = \frac{g}{2}}$$

This is an acceleration in the downward direction as down was taken positive as shown in the diagram. Since both masses move with same acceleration (magnitude is the same, but direction is ofcourse is as shown in the diagram), then the acceleration of the top mass is also the same

$$\boxed{\ddot{x} = \frac{g}{2}}$$

4.2.2 Problem 2

> 2. (5 points)
> Use the Euler-Lagrange equation to show that the shortest path between two points in a plane is a straight line. *Hint:* An element of length in a plane is $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$.

SOLUTION

$$ds = \sqrt{dx^2 + dy^2}$$

Therefore we want to minimize

$$J = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{1 + \left(y'\right)^2} dx$$

Hence

$$f = \sqrt{1 + \left(y'\right)^2}$$

And the Euler Lagrangian equation is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, but

$$\begin{aligned}
\frac{\partial f}{\partial y} &= 0 \\
\frac{\partial f}{\partial y'} &= \frac{1}{2}\frac{2y'}{\sqrt{1 + \left(y'\right)^2}}
\end{aligned}$$

And since $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$ then this mean that $\frac{\partial f}{\partial y'}=c$ where c is some constant. Hence

$$\frac{1}{2}\frac{2y'}{\sqrt{1+\left(y'\right)^2}}=c$$
$$y'=c\sqrt{1+\left(y'\right)^2}$$

Squaring both sides

$$\left(y'\right)^2=c_1\left(1+\left(y'\right)^2\right)$$

Where c_1 is new constant. Hence

$$\left(y'\right)^2=c_1+c_1\left(y'\right)^2$$
$$\left(y'\right)^2=\frac{c_1}{1-c_1}=c_2$$

Where c_2 is new constant. Therefore

$$y'=\pm c_3$$

Where c_3 is new constant. So the above says that $\frac{dy}{dx}$ is constant. In other words, a line, since line has constant slope. The solution to the above is

$$y=m\pm c_3x$$

Where m is some constant and c_3 is the slope. This is the equation of a line.

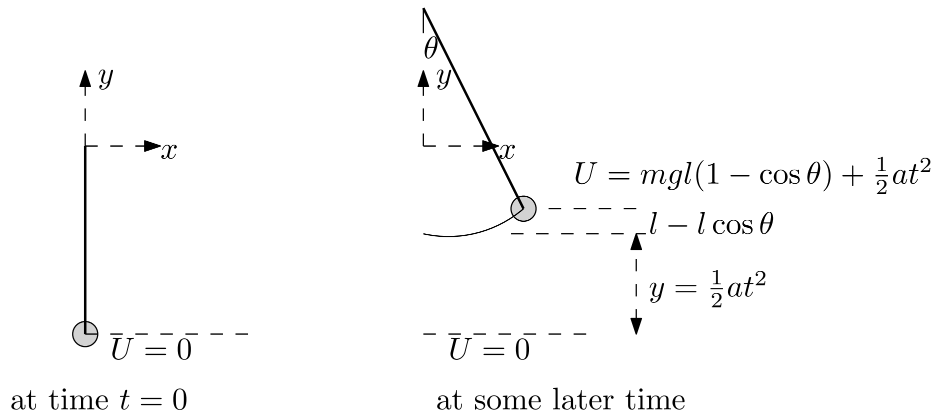
4.2.3 Problem 3

3. (10 points)
The point of support of a simple pendulum is being elevated at a constant acceleration a. Use Lagrange's method to find the differential equation of motion and show that for small oscillations, the period T of the pendulum is

$$T=2\pi\sqrt{\frac{l}{g+a}}\quad .$$

SOLUTION

The coordinate system is as shown below. $U=0$ is taken when the pendulum is hanging in the vertical position before the base starts moving upwards.



Therefore,

$$U = mgl\,(1-\cos\theta) + \frac{1}{2}at^2$$

Where $y = \frac{1}{2}at^2$ is the distance the pendulum moves upwards in time t since it has constant acceleration. We now need to obtain the kinetic energy. Resolving the velocity of the pendulum bob in the horizontal and in the vertical direction gives

$$\begin{aligned}\dot{x} &= l\dot{\theta}\cos\theta\\ \dot{y} &= l\dot{\theta}\sin\theta + at\end{aligned}$$

Therefore

$$\begin{aligned}v^2 &= \dot{x}^2 + \dot{y}^2\\ &= \left(l\dot{\theta}\right)^2\cos^2\theta + \left(l\dot{\theta}\right)^2\sin^2\theta + a^2t^2 + 2atl\dot{\theta}\sin\theta\\ &= l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\end{aligned}$$

Hence

$$\begin{aligned}T &= \frac{1}{2}mv^2\\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right)\end{aligned}$$

Now that U and T are determined, the Lagrangian L is computed

$$\begin{aligned}L &= T - U\\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right) - mgl\,(1-\cos\theta) + \frac{1}{2}at^2\end{aligned}$$

Hence

$$\frac{\partial L}{\partial\theta} = matl\dot{\theta}\cos\theta - mgl\sin\theta$$

And

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + matl\sin\theta$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + mal\sin\theta + \dot{\theta}matl\cos\theta$$

Therefore the Euler Lagrangian equation is

$$\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 \\ matl\dot{\theta}\cos\theta - mgl\sin\theta - \left(ml^2\ddot{\theta} + mal\sin\theta + \dot{\theta}matl\cos\theta\right) &= 0 \\ -mgl\sin\theta - ml^2\ddot{\theta} - mal\sin\theta &= 0 \end{aligned}$$

Hence

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{a}{l}\sin\theta = 0}$$

For small oscillations $\sin\theta \approx \theta$ and the above becomes

$$\ddot{\theta} + \theta\left(\frac{g+a}{l}\right) = 0$$

Which is now in the form $\ddot{\theta} + \omega_n^2\theta = 0$ where $\omega_n = \frac{2\pi}{T}$ is the undamped natural radian frequency, and T is the period of oscillation in seconds. hence

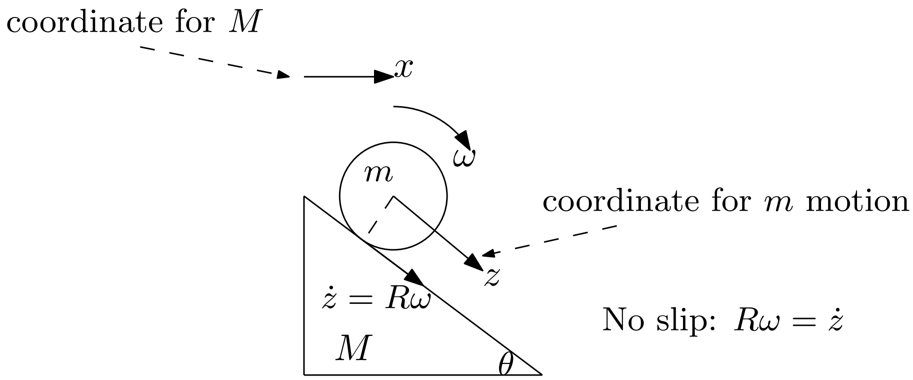
$$\begin{aligned} T &= \frac{2\pi}{\omega_n} \\ &= \frac{2\pi}{\sqrt{\left(\frac{g+a}{l}\right)}} \\ &= 2\pi\sqrt{\left(\frac{l}{g+a}\right)} \end{aligned}$$

4.2.4 Problem 4

4. (10 points)
A ball of mass m, radius R, and moment of inertia $I = \frac{2}{5}mR^2$ rolls down a moveable wedge of mass M without slipping. The angle of the wedge is θ and it is free to slide without friction on a smooth horizontal surface. Find the acceleration of the wedge.

SOLUTION

There are 2 generalized coordinates in this problem. One for the motion of center of mass of m and one for the motion of the wedge M itself. The positive directions are taken as shown in this diagram



2 generalized coordinates: x and z

The first step is to determine the kinetic energy T and potential energy U of the whole system. For mass M

$$T_M = \frac{1}{2}M\dot{x}^2$$

For the rolling mass m since it has both rotational motion and translation motion then

$$T_m = \frac{1}{2}m\left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2\right] + \frac{1}{2}I\omega^2 \tag{1}$$

Where in the above the term $(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2$ is the translation velocity of the rolling mass. Since the motion is without slip, then we can now relate ω to z using

$$R\omega = \dot{z}$$

Hence (1) becomes

$$T_m = \frac{1}{2}m\left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2\right] + \frac{1}{2}I\left(\frac{\dot{z}}{R}\right)^2$$

But $I = \frac{2}{5}R^2 m$, hence the above reduces to

$$T_m = \frac{1}{2}m\left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2\right] + \frac{1}{5}m\dot{z}^2$$

Now that the overall T is found from

$$\begin{aligned}
T &= T_M + T_m \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2\right] + \frac{1}{5}m\dot{z}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + \dot{z}^2\cos^2\theta + 2\dot{x}\dot{z}\cos\theta + \dot{z}^2\sin^2\theta\right] + \frac{1}{5}m\dot{z}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z}\cos\theta\right) + \frac{1}{5}m\dot{z}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{z}\cos\theta + \frac{7}{10}m\dot{z}^2
\end{aligned}$$

Now we find U. The potential energy comes from the rolling mass losing U as it moves down. Assuming zero U is at top of the wedge, the distance it moves it $z\sin\theta$. Hence

$$U = -mgz\sin\theta$$

Now the Lagrangian is found $L = T - U$, hence

$$\boxed{L = \left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{z}\cos\theta + \frac{7}{10}m\dot{z}^2\right) + mgz\sin\theta}$$

Let us find the equation of motion for m, which has acceleration $\ddot{z}$ first, then find the equation of motion for M which is the required acceleration $\ddot{x}$

$$\begin{aligned}\frac{\partial L}{\partial z} &= mg\cos\theta\\ \frac{\partial L}{\partial \dot{z}} &= m\dot{x}\cos\theta + \frac{7}{5}m\dot{z}\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} &= m\ddot{x}\cos\theta + \frac{7}{5}m\ddot{z}\end{aligned}$$

Therefore, using Euler-Lagrangian equation

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} &= 0\\ m\ddot{x}\cos\theta + \frac{7}{5}m\ddot{z} - mg\cos\theta &= 0\end{aligned}$$

Hence

$$\boxed{\ddot{z} = \frac{5}{7}\left(g\sin\theta - \ddot{x}\cos\theta\right)} \tag{2}$$

We now apply Euler-Lagrangian equation to find $\ddot{x}$

$$\begin{aligned}\frac{\partial L}{\partial x} &= 0\\ \frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m\dot{x} + m\dot{z}\cos\theta\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x} + m\ddot{x} + m\ddot{z}\cos\theta\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0\\ M\ddot{x} + m\ddot{x} + m\ddot{z}\cos\theta &= 0\\ \ddot{x}\left(M+m\right) &= -m\ddot{z}\cos\theta\end{aligned}$$

But we found $\ddot{z}$ earlier. Hence using (2) into the above gives

$$\begin{aligned}
\ddot{x}(M+m) &= -m\frac{5}{7}\left(g\sin\theta - \ddot{x}\cos\theta\right)\cos\theta \\
\ddot{x}(M+m) &= \frac{5}{7}m\ddot{x}\cos^2\theta - \frac{5}{7}mg\sin\theta\cos\theta \\
\ddot{x}(M+m) - \frac{5}{7}m\ddot{x}\cos^2\theta &= -m\frac{5}{7}g\sin\theta\cos\theta \\
\ddot{x}\left((M+m) - \frac{5}{7}m\cos^2\theta\right) &= -\frac{5}{7}mg\sin\theta\cos\theta \\
\ddot{x} &= \frac{-\frac{5}{7}mg\sin\theta\cos\theta}{\left((M+m) - \frac{5}{7}m\cos^2\theta\right)} \\
&= \frac{-5mg\sin\theta\cos\theta}{7(M+m) - 5m\cos^2\theta}
\end{aligned}$$

Hence

$$\boxed{\ddot{x} = \frac{5g\sin\theta\cos\theta}{5\cos^2\theta - 7\left(\frac{M+m}{m}\right)}}$$

4.2.5 Problem 5

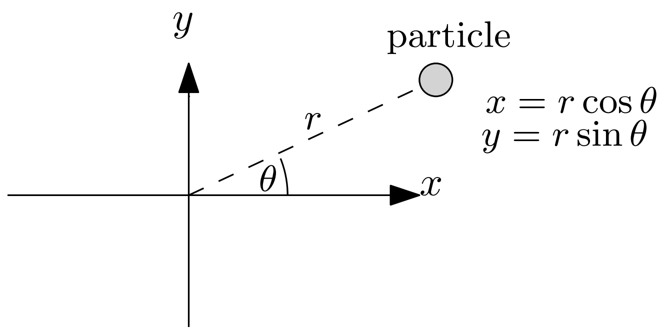
5. (10 points)
Use Lagrange's equations to determine the equations of motion of a particle constrained to move in a plane in a central force field. Show that the angular momentum of the particle is conserved.

SOLUTION

In a central force field, the force on the particle depends only on the magnitude of the direct distance r between the particle and the center of the force. Let the force be located at the origin, then the force on the particle depends only on the magnitude of the position vector $\boldsymbol{r}$ of the particle and not on the angular position of the particle.

$$\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$$

Where $\hat{\boldsymbol{r}}$ is a unit vector pointing in the direction of the force. If the force $\boldsymbol{F}$ causes the distance r between the particle and the origin (where the source of force is assumed) to become smaller, then this force is attractive and it is assigned a negative sign. There are 2 degrees of freedom, hence there are two generalized coordinates. It is easier to use polar coordinates (r,θ) where r is the distance of the particle from the origin, and θ is the angle from the x axis



The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

But $x = r\cos\theta$, hence $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $y = r\sin\theta$, hence $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$, therefore

$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2 \\
&= \left(\dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta\right) + \left(\dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\right) \\
&= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta \\
&= \dot{r}^2 + r^2\dot{\theta}^2
\end{aligned}$$

Hence in polar coordinates

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

And

$$U(r) = V(r)$$

Therefore the Lagrangian

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r} \\
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r}
\end{aligned}$$

Hence the equation of motion for the linear (radial) coordinate r is

$$\left(mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r}\right) - m\ddot{r} = 0$$
$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r}$$

But $-\frac{\partial V(r)}{\partial r} = f(r)$ then

$$\boxed{m\ddot{r} = mr\dot{\theta}^2 + f(r)} \tag{1}$$

Now the equation of motion in the θ coordinate is found.

$$\frac{\partial L}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$$

Hence, since $\frac{\partial L}{\partial \theta} = 0$ then $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ or

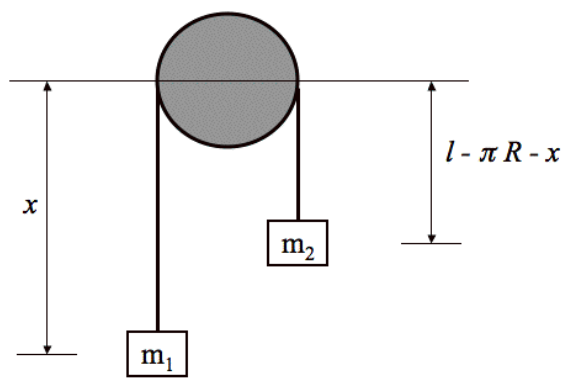
$$\boxed{mr^2\dot{\theta} = \text{constant}} \tag{2}$$

Therefore (2) shows that the angular momentum $I\omega$ is conserved (where I is mr^2, the moment of inertia). This is called the integral of motion.

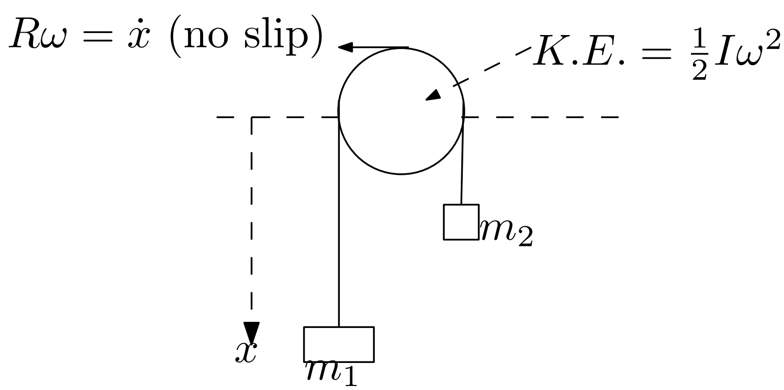
4.2.6 Problem 6

6. (10 points)
Atwood's machine consists of two weights of mass m_1 and m_2 connected by an ideal massless string of length l that passes over a frictionless pulley of radius R and moment of inertia I. Show that the acceleration of the system is

$$\ddot{x} = \frac{(m_1 - m_2)\, g}{m_1 + m_2 + I/R^2} \quad .$$



SOLUTION



Since both masses will move with same speed $\dot{x}$, then the total kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I\omega^2$$

Assuming no slip, we can relate ω to $\dot{x}$ using $R\omega = \dot{x}$, hence the above becomes

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) \end{aligned}$$

Using $U = 0$ as the level shown where the pulley is located, then

$$V = -m_1xg - m_2\left(l - \pi R - x\right)g$$

Hence the Lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) - \left(-m_1x - m_2\left(l - \pi R - x\right)\right)g \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) + \left(m_1x + m_2l - m_2\pi R - xm_2\right)g \end{aligned}$$

Hence

$$\frac{\partial L}{\partial x} = \left(m_1 - m_2\right)g$$

And

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \dot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \ddot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \ddot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) - \left(m_1 - m_2\right)g &= 0 \end{aligned}$$

Therefore

$$\boxed{\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}}$$

4.2.7 HW 2 key solution

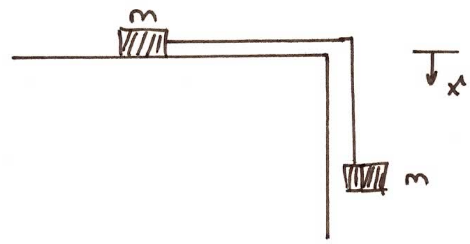
1

Mechanics

Physics 311 - Fall 2015

Homework Set 2 - Solutions

Problem 1



$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 = m \dot{x}^2 \qquad U = -mgx$$

$$\therefore \qquad L = m\dot{x}^2 + mgx$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad mg - 2m\ddot{x} = 0$$

$$\Rightarrow \quad \boxed{\ddot{x} = \frac{g}{2}}$$

Problem 2

The arc length between two points (x_1, y_1) and (x_2, y_2) is

$$L = \int ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

with $y' = \frac{dy}{dx}$

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A f in Euler-Lagrange equation is $f = \sqrt{1+y'^2}$

so $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\left[\frac{y'}{(1+y'^2)^{1/2}}\right]$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{y'}{(1+y'^2)^{1/2}}\right] = 0$$

or $$\frac{y'}{\sqrt{1+y'^2}} = c \quad \Rightarrow \quad y'^2 = c^2(1+y'^2)$$

$$\Leftrightarrow \quad y'^2(1-c^2) = c^2$$

$$\text{so} \quad y' = \frac{c}{\sqrt{1-c^2}} = a$$

integration gives $y = ax + b$, which is a straight line

$\blacksquare$

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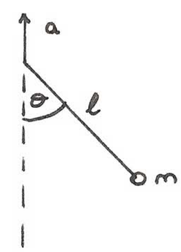
Problem 3



coordinates x, y of mass m

$$x = l \sin\theta$$
$$y = \frac{1}{2} a t^2 - l \cos\theta$$

($y=0$ at start of upward motion)

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \qquad \dot{x} = l\dot{\theta}\cos\theta$$
$$\dot{y} = at + l\dot{\theta}\sin\theta$$

$$= \frac{1}{2} m \left(l^2\dot{\theta}^2\cos^2\theta + a^2t^2 + l^2\dot{\theta}^2\sin^2\theta + 2atl\dot{\theta}\sin\theta\right)$$

$$= \frac{1}{2} m \left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right)$$

$$U = mgy = mg\left(\frac{1}{2}at^2 - l\cos\theta\right)$$

$$\Rightarrow \quad L = T - U = \frac{1}{2} m \left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right) + mg\left(l\cos\theta - \frac{1}{2}at^2\right)$$

Lagrange's equation for θ:

$$\frac{\partial L}{\partial \theta} = matl\dot{\theta}\cos\theta - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + matl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mal\sin\theta + matl\dot{\theta}\cos\theta$$

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So

$$m a t \ell \dot{\theta} \cos\theta - m g \ell \sin\theta - m \ell^2 \ddot{\theta} - m a \ell \sin\theta - m a t \ell \dot{\theta} \cos\theta = 0$$

$$\iff \quad \ddot{\theta} + \frac{a+g}{\ell} \sin\theta = 0$$

For small oscillations, $\sin\theta \simeq \theta$, so

$$\ddot{\theta} + \frac{a+g}{\ell} \theta = 0 ,$$

So $\omega^2 = \dfrac{a+g}{\ell}$ and $T = \dfrac{2\pi}{\omega} = 2\pi \sqrt{\dfrac{\ell}{a+g}}$ $\blacksquare$

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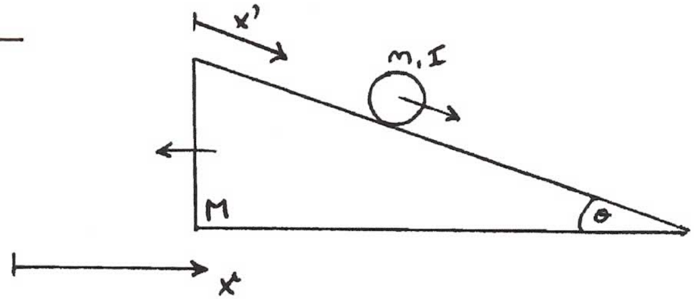
Problem 4



Velocity of the ball: $\vec{v} = \dot{x}\,\hat{x} + \dot{x}'\,\hat{\theta}$

where $\hat{\theta}$ points down the plane

$\Rightarrow \quad \vec{v}^{\,2} = \dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta$

For T, we also need the kinetic energy of the rotation, $\frac{I\omega^2}{2}$

$$\Rightarrow \quad T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right) + \frac{1}{2}I\omega^2$$

$\omega = \frac{\dot{x}'}{R}$, $I = \frac{2}{5}mR^2 \quad \Rightarrow \quad \frac{1}{2}I\omega^2 = \frac{1}{5}m\dot{x}'^2$

$U = -mg\,x'\sin\theta$ for $U=0$ at the initial position of the ball

$$\text{so} \quad L = \frac{1}{2}m\left[\dot{x}^2 + \frac{7}{5}\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right] + \frac{1}{2}M\dot{x}^2 + mg\,x'\sin\theta$$

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i. Lagrange equation for x' :

$$\frac{\partial L}{\partial x'} = mg\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}'} = \frac{7}{5} m \dot{x}' + m\dot{x}\cos\theta$$

so $$mg\sin\theta - \frac{7}{5} m\ddot{x}' - m\ddot{x}\cos\theta = 0$$

$$\Leftrightarrow \quad \ddot{x}' = \frac{5}{7}(g\sin\theta - \ddot{x}\cos\theta) \qquad (1)$$

ii. Lagrange equation for x :

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{x}'\cos\theta + M\dot{x}$$

so $$(m+M)\ddot{x} + m\ddot{x}'\cos\theta = 0 \qquad (2)$$

use (1)

$$\Rightarrow \quad (m+M)\ddot{x} + m\frac{5}{7}(g\sin\theta - \ddot{x}\cos\theta)\cos\theta = 0$$

$$\Leftrightarrow \quad 7\ddot{x}\left[(m+M) - m\,5\cos^2\theta\right] = -5mg\sin\theta\cos\theta$$

$$\Leftrightarrow \quad \boxed{\ddot{x} = \frac{5mg\sin\theta\cos\theta}{5m\cos^2\theta - 7(m+M)}}$$

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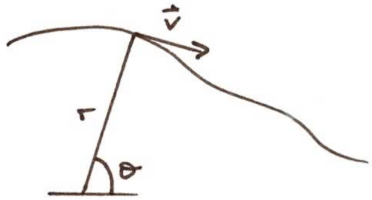
Problem 5

$z = 0$ (movement in a plane)

$\Rightarrow$ two generalized coordinates r, θ $\quad \omega = \frac{d\theta}{dt} = \dot{\theta}$



movement can be in r and θ

$\wedge$ $\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$

$\vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$

$\mathcal{L} = T - U \qquad T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$U = U(r)$ central force field

$\Rightarrow \quad \mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$

$\mathcal{L}$ does not depend explicitely on θ, so $\frac{\partial \mathcal{L}}{\partial \theta} = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0$$

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta}$ is an integral of the motion

$mr^2\dot{\theta}$ is the angular momentum of the system and is conserved

Since $\quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \qquad$ ■

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Problem 6

(1) Lagrangian: $T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \omega^2$

and with $\omega = \frac{V}{R} = \frac{\dot{x}}{R}$

$$T = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2$$

$$U = -m_1 g x - m_2 g (\ell - \pi R - x)$$

($U = 0$ for $x = 0$)

$$\Rightarrow \quad L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2) g x + m_2 g (\ell - \pi R)$$

(2) equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}\right) = \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x}$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2) g$$

$$\Rightarrow \quad \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2) g$$

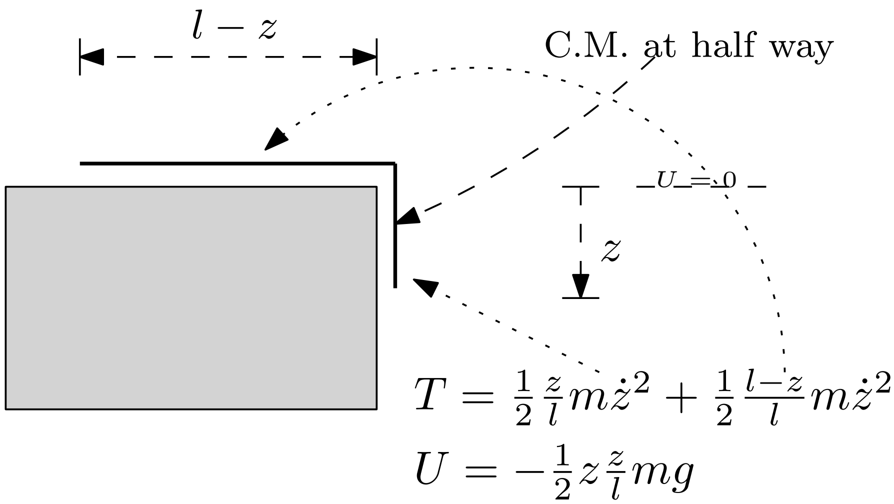
$$\Rightarrow \quad \boxed{\ddot{x} = \frac{(m_1 - m_2) g}{m_1 + m_2 + I/R^2}}$$

4.3 HW 3

4.3.1 Problem 1

1. (5 points)
A uniform rope of total mass m and total length l lies on a table, with a length z hanging over the edge. Find the differential equation of motion.

SOLUTION



The top portion of the rope moves with same speed as the hanging portion. Hence z is used to describe the motion as the generalized coordinate. From the above

$$U=-\left(\frac{1}{2}z\right)\left(\frac{z}{l}\right)mg=-\frac{1}{2}\left(\frac{z^2}{l}\right)mg$$
$$T=\frac{1}{2}\left(\frac{z}{l}\right)m\dot{z}^2+\frac{1}{2}\left(\frac{l-z}{l}\right)m\dot{z}^2=\frac{1}{2}m\dot{z}^2$$

In finding U we used $\frac{1}{2}$ since the center of mass of the hanging part is half way over the length. So the potential energy is taken from the center of mass. In the above, $\dot{z}$ is used for both parts of the rope, since both parts move with same speed. Applying Lagrangian equations gives

$$\begin{aligned}L&=T-U\\&=\frac{1}{2}m\dot{z}^2+\frac{1}{2}\left(\frac{z^2}{l}\right)mg\end{aligned}$$

Hence

$$\frac{\partial L}{\partial z} = \frac{z}{l} mg$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z}$$

And therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$
$$m\ddot{z} - \frac{z}{l} mg = 0$$
$$\ddot{z} = \frac{z}{l} g$$

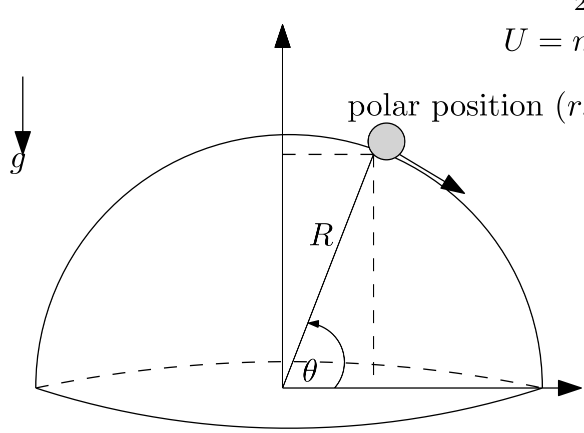
When $z = 0$ then the acceleration is zero as expected. When $z = \frac{l}{2}$ then $\ddot{z} = \frac{1}{2}g$ and when $z = l$ then $\ddot{z} = g$ as expected since in this case the rope will all be falling down on its own weight due to gravity and should have g as the acceleration.

4.3.2 Problem 2

2. (10 points)
A particle of mass m perched on top of a smooth hemisphere of radius R is disturbed slightly, so that it begins to slide down the side. Use Lagrange multipliers to find the normal force of constraint exerted by the hemisphere on the particle and determine the angle relative to the vertical at which it leaves the hemisphere.

SOLUTION

constraint $f(r,\theta) = r - R = 0$
$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$
$U = mgr\sin\theta$



Generalized coordinates used r,θ

There are two coordinates r,θ (polar) and one constraint

$$f(r,\theta) = r - R = 0 \tag{1}$$

Now we set up the equations of motion for m

$$\begin{aligned}
T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\\
U &= mgr\sin\theta\\
L &= T - U\\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta
\end{aligned}$$

Hence the Euler-Lagrangian equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} + \lambda\frac{\partial f}{\partial r} = 0 \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{3}$$

But

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r}\\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)\\
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - mg\sin\theta\\
\frac{\partial L}{\partial \theta} &= -mgr\cos\theta\\
\frac{\partial f}{\partial r} &= 1\\
\frac{\partial f}{\partial \theta} &= 0
\end{aligned}$$

Hence (2) becomes

$$m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta + \lambda = 0 \tag{4}$$

And (3) becomes

$$m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + mgr\cos\theta = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\cos\theta = 0 \tag{5}$$

We now need to solve (1,4,5) for λ. Now we have to apply the constrain that $r = R$ in the above to be able to solve (4,5) equations. Therefore, (4,5) becomes

$$-mR\dot{\theta}^2 + mg\cos\theta + \lambda = 0 \tag{4A}$$

$$R\ddot{\theta} + g\cos\theta = 0 \tag{5A}$$

Where (4A,5A) were obtained from (4,5) by replacing $r = R$ and $\dot{r} = 0$ and $\ddot{r} = 0$ since we are using that $r = R$ which is constant (the radius).

From (5A) we see that this can be integrated giving

$$R\dot{\theta}^2 + 2g\sin\theta + c = 0 \tag{6}$$

Where c is constant. Since if we differentiate the above with time, we obtain

$$\begin{aligned} 2R\dot{\theta}\ddot{\theta} + 2g\dot{\theta}\cos\theta &= 0 \\ R\ddot{\theta} + g\cos\theta &= 0 \end{aligned}$$

Which is the same as (5A). Therefore from (6) we find $\dot{\theta}^2$ to use in (4A). Hence from (6)

$$\dot{\theta}^2 = -2\frac{g}{R}\sin\theta + c$$

To find c we use initial conditions. At $t = 0$, $\theta = 90^0$ and $\dot{\theta}(0) = 0$ hence

$$c = 2\frac{g}{R}$$

Therefore

$$\begin{aligned} \dot{\theta}^2 &= -2\frac{g}{R}\sin\theta + 2\frac{g}{R} \\ &= 2\frac{g}{R}(1 - \sin\theta) \end{aligned}$$

Plugging the above into (4A) in order to find λ gives

$$\begin{aligned} -mR\left(2\frac{g}{R}(1 - \sin\theta)\right) + mg\sin\theta + \lambda &= 0 \\ \lambda &= m\left(2g(1 - \sin\theta)\right) - mg\sin\theta \\ \lambda &= 2mg - 2mg\sin\theta - mg\sin\theta \\ &= mg(2 - 3\sin\theta) \end{aligned}$$

Now that we found λ ,we can find the constraint force in the radial direction

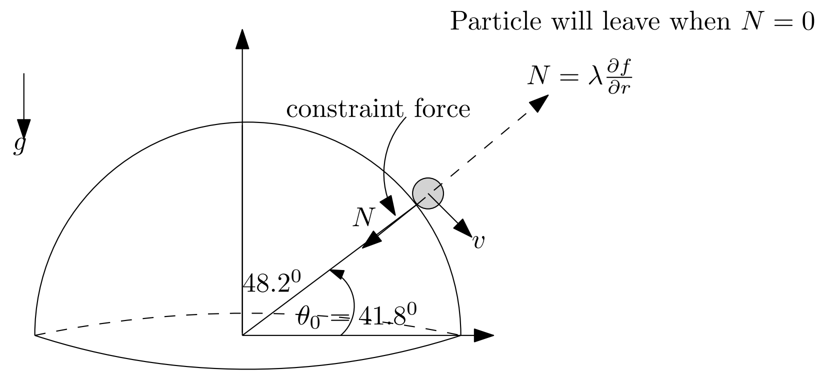
$$\begin{aligned} N &= \lambda\frac{\partial f}{\partial r} \\ &= mg(2 - 3\sin\theta) \end{aligned}$$

The particle will leave when $N = 0$ which will happen when

$$\begin{aligned} 2 - 3\sin\theta &= 0 \\ \theta &= \sin^{-1}\left(\frac{2}{3}\right) \\ &= 41.8^0 \end{aligned}$$

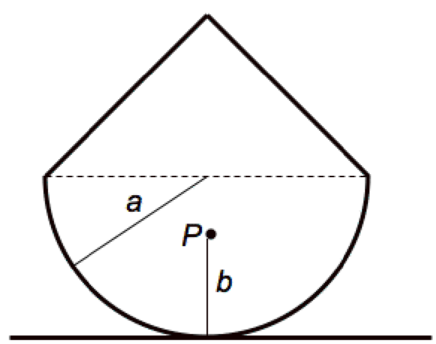
Therefore, the angle from the vertical is

$$90 - 41.8 = 48.2^0$$

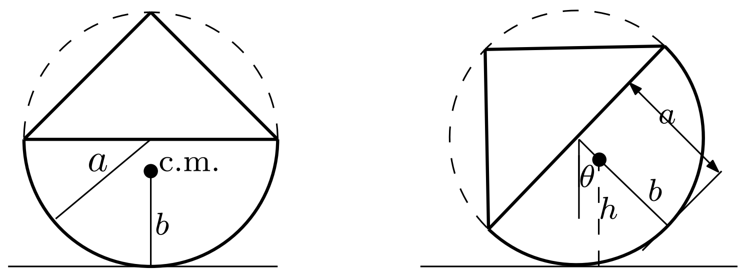


4.3.3 Problem 3

3. (10 points)
   Consider the object shown in the figure below, which has a half-sphere of radius a as the bottom part and a cone on top. The center of mass (P) is at a distance b from the ground when the object is standing upright. Let I be the moment of inertia. Find the frequency of small oscillations if the object is disturbed slightly from its upright position. What happens if $a = b$ or $b > a$?



SOLUTION



$$h = a - (a-b)\cos\theta$$
$$U = mgh = mg(a - (a-b)\cos\theta$$

From the above, we see that the center of mass has height above the ground level after rotation of

$$h = a - (a-b)\cos\theta$$

Taking the ground state as the floor, the potential energy in this state is

$$\begin{aligned} U &= mgh \\ &= mg\left(a - (a-b)\cos\theta\right) \end{aligned}$$

And the kinetic energy

$$T = \frac{1}{2} I\dot{\theta}^2$$

Hence the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} I\dot{\theta}^2 - mg\left(a - (a-b)\cos\theta\right) \end{aligned}$$

Therefore the equation of motion is

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ I\ddot{\theta} - \frac{\partial}{\partial \theta}\left(\frac{1}{2} I\dot{\theta}^2 - mg\left(a - (a-b)\cos\theta\right)\right) &= 0 \\ I\ddot{\theta} + \frac{\partial}{\partial \theta} mg\left(a - (a-b)\cos\theta\right) &= 0 \\ I\ddot{\theta} - \frac{\partial}{\partial \theta} mg\left(a-b\right)\cos\theta &= 0 \\ I\ddot{\theta} + mg\left(a-b\right)\sin\theta &= 0 \end{aligned}$$

For small $\theta, \sin\theta \simeq \theta$, hence the above becomes

$$\ddot{\theta} + \frac{mg\left(a-b\right)}{I}\theta = 0$$

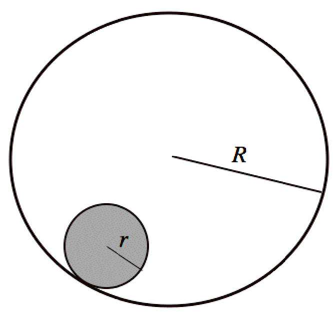
Therefore the natural angular frequency is

$$\boxed{\omega_n = \sqrt{\frac{mg(a-b)}{I}}}$$

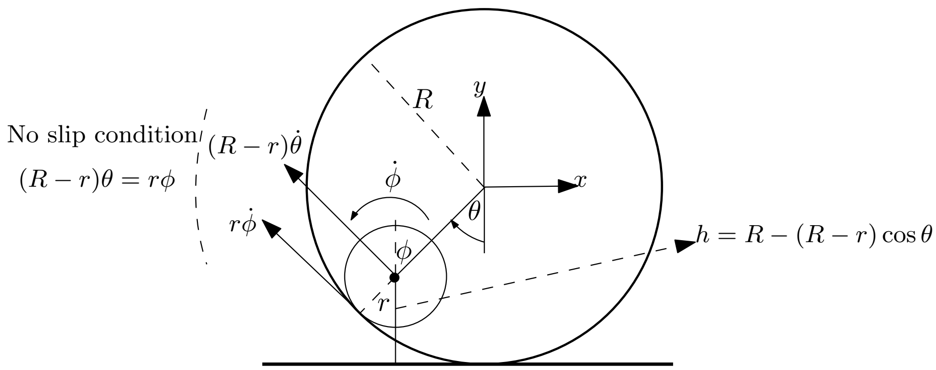
When $a = b$ then $\omega_n = 0$ and the mass do not oscillate but remain at the new positions. When $b > a$ then ω_n is complex valued. This is not possible, as the natural frequency must be real. So center of mass can not be in the upper half.

4.3.4 Problem 4

4. (15 points)
A sphere of radius r, mass m, and moment of inertia $I = \frac{2}{5}mr^2$ is contrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R (which does not move). Let the z-direction go along the axis of the cylinder.
(1) Determine the Lagrangian, the equations of motion, and the period for *small* oscillations. Ignore a possible motion in the z-direction.
(2) Determine the Lagrangian in the more general case where the motion in the z-direction is included. Describe the motion in the z-direction.



SOLUTION



2 generalized coordinates θ, ϕ but constraint reduces this to one coordinate θ

Part (1): There are two coordinates are θ,ϕ, but due to dependency between them (no slip) then this reduces the degree of freedom by one, and there is one generalized coordinate θ. The constraints of no slip means

$$f\left(\theta,\phi\right) = (R-r)\,\theta - r\phi = 0$$

Which means the center of the small disk move in speed the same as the point of the disk that moves on the edge of the larger cylinder as shown in the figure above.

$$\begin{aligned} T &= \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left((R-r)\,\dot{\theta}\right)^2 \\ U &= mgh = mg\left(R-(R-r)\cos\theta\right) \end{aligned}$$

Using $I=\frac{2}{5}mr^2$ and using $\dot{\phi}=\frac{(R-r)}{r}\dot{\theta}$ from the constraint conditions, then T becomes

$$\begin{aligned} T &= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{(R-r)}{r}\dot{\theta}\right)^2 + \frac{1}{2}m\left((R-r)\,\dot{\theta}\right)^2 \\ &= \frac{1}{5}m\left(R-r\right)^2\dot{\theta}^2 + \frac{1}{2}m\left(R-r\right)^2\dot{\theta}^2 \\ &= \frac{7}{10}m\left(R-r\right)^2\dot{\theta}^2 \end{aligned}$$

Hence

$$\begin{aligned} L &= T-U \\ &= \frac{7}{10}m\left(R-r\right)^2\dot{\theta}^2 - mg\left(R-(R-r)\cos\theta\right) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial L}{\partial\theta} &= -mg\left(R-r\right)\sin\theta \\ \frac{\partial L}{\partial\dot{\theta}} &= \frac{7}{5}m\left(R-r\right)^2\dot{\theta} \end{aligned}$$

Therefore the equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}=0$$

$$\frac{7}{5}m\left(R-r\right)^{2}\ddot{\theta}+mg\left(R-r\right)\sin\theta=0$$

$$\ddot{\theta}+\frac{g}{\frac{7}{5}\left(R-r\right)}\sin\theta=0$$

For small angle

$$\ddot{\theta}+\frac{5g}{7\left(R-r\right)}\theta=0$$

The frequency of oscillation is

$$\omega_{n}=\sqrt{\frac{5g}{7\left(R-r\right)}}$$

Using $\omega_n=\frac{2\pi}{T}$ then the period of oscillation is

$$T=\frac{2\pi}{\sqrt{\frac{5g}{7(R-r)}}}=2\pi\sqrt{\frac{7\left(R-r\right)}{5g}}$$

Part (2):

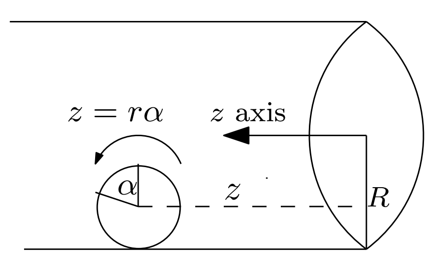
There are now two generalized coordinates, θ and z. The sphere now rotates in 2 angular motions, $\dot{\phi}$ which is the same as it did in part 1, and in addition, it rotate with angular motion, $\dot{\alpha}$ which is rolling down the z axis. The new constraint is that

$$f_{1}\left(\alpha,z\right)=z-r\alpha=0 \tag{1}$$

So that no slip occurs in the z direction. This is in additional of the original no slip condition which is

$$f_{2}\left(\theta,\phi\right)=\left(R-r\right)\theta-r\phi=0 \tag{2}$$

The following diagram illustrates this



The sphere is now distance z away from the origin. There is new constraint now as shown

Now there are translation kinetic energy in the z direction as well as new rotational kinetic energy due to spin α. Therefore

$$
\begin{aligned}
T &= \overbrace{\frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left((R-r)\dot{\theta}\right)^2}^{\text{part}(1)} + \overbrace{\frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\alpha}^2}^{\text{due to moving in z}} \\
U &= mgh = mg\left(R-(R-r)\cos\theta\right)
\end{aligned}
$$

Notice that the potential energy do not change, since it depends only on the height above the ground. Using $I = \frac{2}{5}mr^2$ and from constraints (1,2) then T becomes

$$
\begin{aligned}
T &= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\overbrace{\left(\frac{(R-r)}{r}\dot{\theta}\right)^2}^{\dot{\phi}^2} + \frac{1}{2}m\left((R-r)\dot{\theta}\right)^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\overbrace{\left(\frac{\dot{z}}{r}\right)^2}^{\dot{\alpha}^2} \\
&= \left(\frac{1}{5}mr^2\right)\frac{(R-r)}{r^2}\dot{\theta}^2 + \frac{1}{2}m\left(R-r\right)^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \left(\frac{1}{5}mr^2\right)\frac{\dot{z}^2}{r^2} \\
&= \frac{7}{10}m\left(R-r\right)\dot{\theta}^2 + \frac{7}{10}m\dot{z}^2
\end{aligned}
$$

Hence the Lagrangian is

$$
\begin{aligned}
L &= T - U \\
&= \frac{7}{10}m\left(R-r\right)\dot{\theta}^2 + \frac{7}{10}m\dot{z}^2 - mg\left(R-(R-r)\cos\theta\right)
\end{aligned}
$$

This part only now asks for motion in z direction. Hence

$$
\begin{aligned}
\frac{\partial L}{\partial z} &= 0 \\
\frac{\partial L}{\partial \dot{z}} &= \frac{7}{5}m\dot{z}
\end{aligned}
$$

Since $\frac{\partial L}{\partial z} = 0$ then

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0
$$

Hence $\frac{\partial L}{\partial \dot{z}}$ is the integral of motion. Or

$$
\frac{7}{5}m\ddot{z} = 0
$$

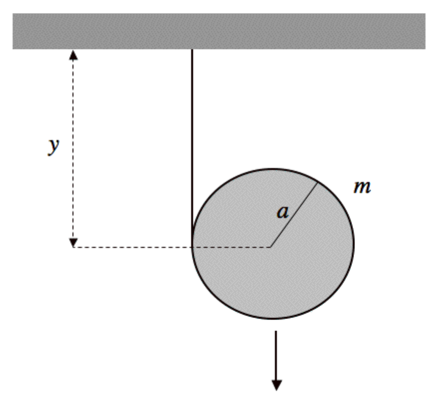
or

$$
\begin{aligned}
\ddot{z} &= 0 \\
\dot{z} &= c
\end{aligned}
$$

Where c is constant. This means the sphere rolls down the z axis at constant speed.

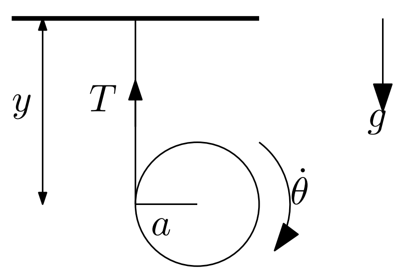
4.3.5 Problem 5

5. (10 points)
Consider a disc of mass m and radius a that has a string wrapped around it with one end attached to a fixed support and allowed to fall with the string unwinding as it falls. (This is essentially a yo-yo with the string attached to a finger held motionless as a fixed support.) Find the equation of motion of the disc.



SOLUTION

This is first solved using energy method, then solved using Newton method.



constraint: $\dot{y}a = \dot{\theta}$

Energy method

Constraint is $f\left(y,\theta\right) = y - a\theta = 0$. Hence $\dot{\theta} = \frac{\dot{y}}{a}$

$$
\begin{aligned}
U &= -mgy\\
T &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{y}^2\\
&= \frac{1}{2}I\left(\frac{\dot{y}}{a}\right)^2 + \frac{1}{2}m\dot{y}^2\\
&= \frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{\dot{y}}{a}\right)^2 + \frac{1}{2}m\dot{y}^2\\
&= \frac{1}{4}m\dot{y}^2 + \frac{1}{2}m\dot{y}^2\\
&= \frac{3}{4}m\dot{y}^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
L &= T - U\\
&= \frac{3}{4}m\dot{y}^2 + mgy
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\partial L}{\partial y} &= mg\\
\frac{\partial L}{\partial \dot{y}} &= \frac{3}{2}m\dot{y}\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{3}{2}m\ddot{y}
\end{aligned}
$$

And the equation of motion becomes

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= 0\\
\frac{3}{2}m\ddot{y} - mg &= 0\\
\ddot{y} &= \frac{2}{3}g
\end{aligned}
$$

Newton method

Using Newton method, this can be solved as follows. The linear equation of motion is (positive is taken downwards)

$$
\begin{aligned}
F &= m\ddot{y}\\
-T + mg &= m\ddot{y} \qquad (1)
\end{aligned}
$$

And the angular equation of motion is given by

$$
Ta = I\ddot{\theta} \qquad (2)
$$

Due to constraint $f\left(y,\theta\right) = y - a\theta = 0$, then

$$\frac{\ddot{y}}{a} = \ddot{\theta}$$

Using the above in (2) gives

$$\begin{aligned} Ta &= I\frac{\ddot{y}}{a} \\ T &= I\frac{\ddot{y}}{a^2} \end{aligned} \tag{3}$$

Replacing T in (1) with the T found in (3) results in

$$\begin{aligned} m\ddot{y} &= -I\frac{\ddot{y}}{a^2} + mg \\ \ddot{y}\left(m + \frac{I}{a^2}\right) &= mg \\ \ddot{y} &= \frac{mg}{m + \frac{I}{a^2}} \end{aligned}$$

But $I = \frac{1}{2}ma^2$ then the above becomes

$$\begin{aligned} \ddot{y} &= \frac{mg}{m + \frac{\frac{1}{2}ma^2}{a^2}} \\ &= \frac{g}{1 + \frac{1}{2}} \\ &= \frac{2}{3}g \end{aligned}$$

Which is the same (as would be expected) using the energy method

4.3.6 HW 3 key solution

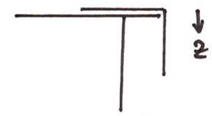
1

Mechanics

Physics 311 - Fall 2011

Homework Set 3 - Solutions

Problem 1



fraction of mass hanging over the edge : $\frac{z}{\ell} m$

position of center of mass of this fraction : $z/2$

$$\Rightarrow \quad U = -\frac{z}{\ell} m \frac{z}{2}$$

$$T = \frac{1}{2} m \dot{z}^2$$

$$\Rightarrow \quad L = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} \frac{z^2}{\ell} m g$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 \quad \Rightarrow \quad \frac{z}{\ell} m g - m \ddot{z} = 0$$

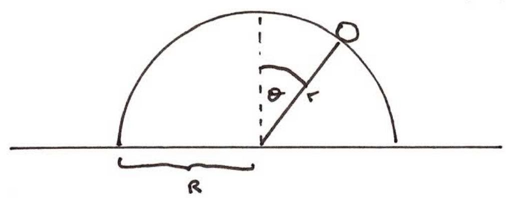
$$\Rightarrow \quad \boxed{\ddot{z} = \frac{g}{\ell} z}$$

Problem 2

mass m on hemisphere of radius R

equation of constraint :

$$f(r, \theta) = r - R = 0$$



2

$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right)$ $\quad U = mgr\cos\theta$

($U=0$ at bottom)

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda \frac{\partial f}{\partial r} = 0 \qquad \frac{\partial f}{\partial r} = 1$$

$$\Rightarrow \quad mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad \frac{\partial f}{\partial \theta} = 0$$

$$\Rightarrow \quad mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

with $r=R$, $\quad mR\dot{\theta}^2 - mg\cos\theta + \lambda = 0$

$$\Rightarrow \quad \lambda = mg\cos\theta - mR\dot{\theta}^2$$

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

$$\Rightarrow \quad \ddot{\theta} = \frac{g}{R}\sin\theta$$

now use $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$

$$\Rightarrow \quad \dot{\theta}\, d\dot{\theta} = \frac{g}{R}\sin\theta\, d\theta$$

so $\quad \frac{1}{2}\dot{\theta}^2 = -\frac{g}{R}\cos\theta + C$

but at $\theta = 0$, $\dot{\theta} = 0$,

so $\quad C = g/R$

$$\Rightarrow \qquad \frac{1}{2}\dot{\theta}^2 = -\frac{g}{R}\cos\theta + \frac{g}{R}$$

so

$$\lambda = mg\cos\theta - mR\left(-\frac{2g}{R}\cos\theta + \frac{2g}{R}\right)$$

$$= 3mg\cos\theta - 2mg$$

$$= mg\,(3\cos\theta - 2)$$

The particle leaves the hemisphere for $\lambda \to 0$, so

$$3\cos\theta_0 - 2 = 0$$

$$\Rightarrow \quad \cos\theta_0 = 2/3 \qquad \Rightarrow \quad \theta_0 = a\cos 2/3$$

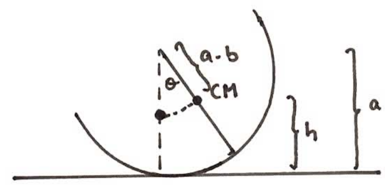
$$\simeq \underline{\underline{48.2^\circ}}$$

4

Problem 3

the moment of inertia about the center of mass is given as I;

however, the object does <u>not</u> rotate about the center of mass, it rotates about the point of contact with the table (in first approximation for <u>small</u> angles)



$h = a - (a-b)\cos\theta$

So $\quad T = \frac{1}{2} I' \dot{\theta}^2$, where I' is the moment of inertia about the contact point

$$I' = mb^2 + I \qquad \text{(parallel axis theorem)}$$

So $\quad T = \frac{1}{2} m (b\dot{\theta})^2 + \frac{1}{2} I \dot{\theta}^2 \qquad U = mg\left[a - (a-b)\cos\theta\right]$

$$L = \frac{1}{2} m (b\dot{\theta})^2 + \frac{1}{2} I \dot{\theta}^2 - mg\left[a - (a-b)\cos\theta\right]$$
$$= \frac{1}{2}(mb^2 + I)\dot{\theta}^2 - mga + mg(a-b)\cos\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad -mg(a-b)\sin\theta - (mb^2 + I)\ddot{\theta} = 0$$
$$\Leftrightarrow \quad \ddot{\theta} + \frac{mg(a-b)}{mb^2 + I}\sin\theta = 0$$

for small oscillations, $\sin\theta \simeq \theta \quad \Rightarrow \quad \boxed{\omega = \sqrt{\frac{mg(a-b)}{mb^2 + I}}}$

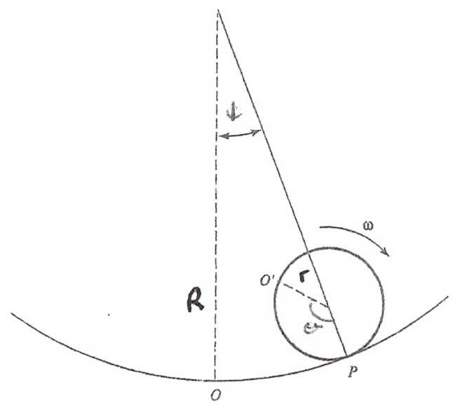
for $a = b$, $\omega = 0$ (no oscillation)

for $a < b$, the object turns over (unstable equilibrium)

5

Problem 4

define two angles θ and ϕ to describe the motion



ϕ angle between the line connecting the centers of the cylinders and the sphere and the vertical

θ angle that the contact point P and the point O' on the circumference make about the center of the sphere (O' makes contact with the lowest point of the cylinder)

Velocity of center-of-mass:

$$V_{CM} = \omega r = \dot{\theta} r$$

rolling: $= (R-r)\dot{\phi} \quad \Rightarrow \quad \dot{\theta} r = (R-r)\dot{\phi}$

constraint

(1) no z-motion:

$$T = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \dot{\theta}^2 \qquad I = \frac{2}{5} m R^2$$

$$= \frac{1}{2} m (R-r)^2 \dot{\phi}^2 + \frac{1}{5} m r^2 \dot{\theta}^2$$

$$= \frac{1}{2} m (R-r)^2 \dot{\phi}^2 + \frac{1}{5} m (R-r)^2 \dot{\phi}^2$$

$$= \frac{7}{10} m (R-r)^2 \dot{\phi}^2$$

$$U = -mg (R-r) \cos\phi \qquad (U=0 \text{ at equator})$$

$$\Rightarrow \quad L = \frac{7}{10} m (R-r)^2 \dot{\phi}^2 + mg (R-r) \cos\phi$$

6

equation of motion $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$

so $-mg(R-r)\sin\phi - \frac{7}{5}m(R-r)^2\ddot{\phi} = 0$

$\Leftrightarrow \quad \ddot{\phi} + \frac{5}{7}\frac{g}{R-r}\sin\phi = 0$

Small angles: $\sin\phi \simeq \phi \quad \Rightarrow \quad \ddot{\phi} + \frac{5}{7}\frac{g}{R-r}\phi = 0$

so $\omega = \sqrt{\frac{5}{7}\frac{g}{R-r}}$

$\Rightarrow$ period $\boxed{P = 2\pi\sqrt{\frac{7}{5}\frac{R-r}{g}}}$

(2) include z-motion:

ψ angle of rotation along z, constraint $z = r\psi$

T becomes $T = \frac{1}{2}m\left[(R-r)^2\dot{\phi}^2 + \dot{z}^2\right] + \frac{1}{2}I\left[\dot{\theta}^2 + \dot{\psi}^2\right]$

$= \frac{1}{2}m\left[(R-r)^2\dot{\phi}^2 + \dot{z}^2\right] + \frac{1}{5}m\left[(R-r)^2\dot{\phi}^2 + \dot{z}^2\right]$

$= \frac{7}{10}m\left[(R-r)^2\dot{\phi}^2 + \dot{z}^2\right]$

$\Rightarrow \quad L = \frac{7}{10}m\left[(R-r)^2\dot{\phi}^2 + \dot{z}^2\right] + mg(R-r)\cos\phi$

$\frac{\partial L}{\partial z} = 0$, so $\ddot{z} = 0$, $\boxed{\dot{z} = \text{const.}}$

$\rightarrow$ the movement in z has a <u>constant</u> velocity.

7

Problem 5

(1) Lagrangian: $T = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$

with $I = \frac{1}{2} ma^2$

constraint: $y = a\theta$

$$\Rightarrow \quad T = \frac{1}{2} m\dot{y}^2 + \frac{1}{4} m a^2 \left(\frac{\dot{y}}{a}\right)^2$$

$$= \frac{3}{4} m\dot{y}^2$$

$$U = -mgy \qquad (U=0 \text{ for } y=0)$$

$$\Rightarrow \quad L = \frac{3}{4} m\dot{y}^2 + mgy$$

(2) equation of motion: $\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$

$$\Rightarrow \quad mg - \frac{d}{dt}\left(\frac{3}{2} m\dot{y}\right) = 0$$

$$\Rightarrow \quad \boxed{\ddot{y} = \frac{2}{3} g}$$

so $\boxed{\ddot{\theta} = \frac{2}{3}\frac{g}{a}}$

(both equations can be integrated easily)

4.4 HW 4

4.4.1 Problem 1

1. (5 points)
The damping factor λ of a spring suspension system is one-tenth the critical value. Let ω_0 be the undamped frequency. Find (i) the resonant frequency, (ii) the quality factor Q, (iii) the phase angle Φ when the system is driven at frequency $\omega = \omega_0/2$, and (iv) the steady-state amplitude at this frequency.

SOLUTION:

Note that $\lambda_{critical} = \omega_0$. We are told that $\lambda = 0.1\omega_0$ in this problem.

4.4.1.1 part(1)

The resonant frequency (for this case of under-damped) occurs when the steady state amplitude is maximum

$$b = \frac{\frac{f}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}}$$

This happens when the denominator is *minimum*. Taking derivative of the denominator w.r.t. ω and setting the result to zero gives

$$\begin{aligned}
\frac{d}{d\omega}\left(\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2\right) &= 0 \\
2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 8\lambda^2\omega &= 0 \\
8\lambda^2\omega + 4\omega^3 - 4\omega\omega_0^2 &= 0 \\
2\lambda^2 + \omega^2 - \omega_0^2 &= 0 \\
\omega^2 &= \omega_0^2 - 2\lambda^2
\end{aligned}$$

Taking the positive root (since ω must be positive) gives

$$\omega = \sqrt{\omega_0^2 - 2\lambda^2}$$

When $\lambda = 0.1\omega_0$ the above becomes

$$\begin{aligned}
\omega &= \sqrt{\omega_0^2 - 2\left(\frac{1}{10}\omega_0\right)^2} \\
&= \sqrt{\frac{98}{100}\omega_0^2} \\
&= 0.98995\omega_0 \text{ rad/sec}
\end{aligned}$$

4.4.1.2 part(2)

Quality factor Q is defined as

$$\begin{aligned} Q &= \frac{\omega_d}{2\lambda} \\ &= \frac{\sqrt{\omega_0^2 - \lambda^2}}{2\lambda} \\ &= \frac{\sqrt{\omega_0^2 - (0.1\omega_0)^2}}{2(0.1\omega_0)} \\ &= \frac{\omega_0\sqrt{1-0.1^2}}{0.2\omega_0} \\ &= \frac{\sqrt{1-0.1^2}}{0.2} \end{aligned}$$

Therefore

$$\boxed{Q = 4.975}$$

4.4.1.3 Part(3)

Given

$$x''(t) + 2\lambda x' + \omega_0^2 x = \frac{f}{m} e^{i\omega t} \tag{1}$$

Assuming the particular solution is $x_p(t) = Be^{i\omega t}$ where $B = be^{i\phi}$ is the complex amplitude and b is the amplitude and ϕ is the phase of B. We want to find the phase. Plugging $x_p(t)$ into (1) and simplifying gives

$$B = \frac{\frac{f}{m}}{\omega_0^2 - \omega^2 + 2\lambda i\omega}$$

Hence

$$\begin{aligned} \phi &= 0 - \tan^{-1}\left(\frac{2\lambda\omega}{\omega_0^2 - \omega^2}\right) \\ &= \tan^{-1}\left(\frac{-2\lambda\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}$$

Since $\lambda = 0.1\omega_0$ and $\omega = \frac{\omega_0}{2}$ the above becomes

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{-2(0.1\omega_0)\frac{\omega_0}{2}}{\omega_0^2 - \left(\frac{\omega_0}{2}\right)^2}\right) \\ &= \tan^{-1}(-0.13333) \\ &= -0.13255 \text{ rad} \end{aligned}$$

4.4.1.4 Part(4)

The steady state amplitude is b from above, which is found as follows

$$b^2 = BB^*$$

Where B^* is the complex conjugate of $B = \frac{\frac{f}{m}}{\omega_0^2-\omega^2+2\lambda i\omega}$. Therefore

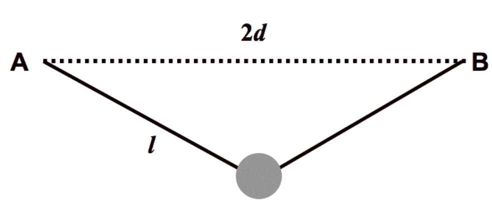
$$\begin{aligned}
b &= \frac{\frac{f}{m}}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\lambda^2\omega^2}} \\
&= \frac{f}{m}\frac{1}{\sqrt{\left(\omega_0^2-\left(\frac{\omega_0}{2}\right)^2\right)^2+4\left(0.1\omega_0\right)^2\left(\frac{\omega_0}{2}\right)^2}} \\
&= \frac{f}{m}\frac{1}{\sqrt{0.572\,5\omega_0^4}} \\
&= 1.3216\frac{f}{m\omega_0^2}
\end{aligned}$$

But $m\omega_0^2 = k$, the stiffness, hence the above is

$$\boxed{b = 1.3216\frac{f}{k}}$$

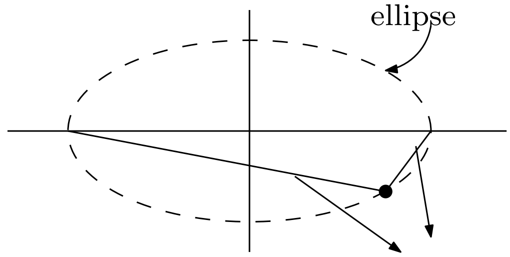
4.4.2 Problem 2

2. (10 points)
A string of length $2l$ is suspended at points A and B located on a horizontal line. The distance between A and B is $2d$, with $d < l$. A small, heavy bead can slide on the string without friction. Find the period of the small-amplitude oscillations of the bead in the vertical plane containing the suspension points.
Hint: The trajectory of the bead is a section of an ellipse (why?). Move the origin to the equilibrium point and use a Taylor expansion to get an approximate expression for the trajectory around the equilibrium point. Apply Lagrange.

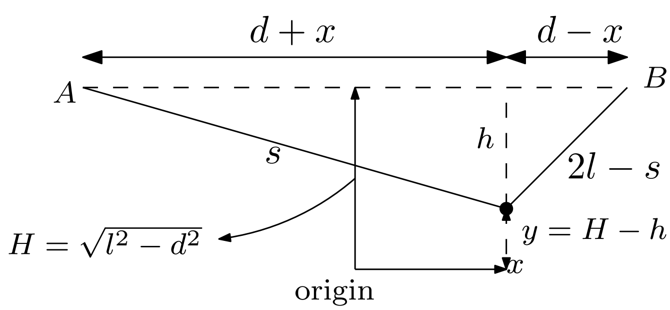


SOLUTION:

The locus the bead describes is an ellipse, since in an ellipse the total distance from any point on it to the points A, B is always the same



To obtain the potential energy, we move the bead a little from the origin and find how much the bead moved above the origin, as shown in the following diagram



$$
\begin{aligned}
s^2 &= h^2 + (d+x)^2 \\
(2l-s)^2 &= h^2 + (d-x)^2
\end{aligned}
$$

From the above, we see that, by applying pythagoras triangle theorem to the left and to the right triangles, we obtain two equations which we solve for h in order to obtain the potential energy

$$
\begin{aligned}
s^2 &= h^2 + (d+x)^2 \\
(2l-s)^2 &= h^2 + (d-x)^2
\end{aligned}
$$

Solving for h gives

$$
h = \sqrt{1 - \frac{d^2}{l^2}}\sqrt{l^2 - x^2}
$$

Therefore

$$
\begin{aligned}
y &= H - h \\
&= H - \sqrt{1 - \frac{d^2}{l^2}}\sqrt{l^2 - x^2}
\end{aligned}
$$

Hence

$$\begin{aligned}U &= mgy\\ &= mg\left(H-\sqrt{1-\frac{d^2}{l^2}}\sqrt{l^2-x^2}\right)\end{aligned}$$

The kinetic energy is

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)$$

Therefore the Lagrangian is

$$\begin{aligned}L &= T-U\\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mg\left(H-\sqrt{1-\frac{d^2}{l^2}}\sqrt{l^2-x^2}\right)\end{aligned}$$

The equation of motion in the x coordinate is now found. From

$$\begin{aligned}\frac{\partial L}{\partial x} &= \frac{1}{2}mg\sqrt{1-\frac{d^2}{l^2}}\frac{(-2x)}{\sqrt{l^2-x^2}}\\ &= -mg\sqrt{1-\frac{d^2}{l^2}}\frac{x}{\sqrt{l^2-x^2}}\end{aligned}$$

And

$$\frac{d}{dt}\frac{\partial L}{\partial x}=m\ddot{x}$$

Applying Euler-Lagrangian equation gives

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial x}-\frac{\partial L}{\partial x} &= 0\\ \ddot{x}+g\sqrt{1-\frac{d^2}{l^2}}\frac{x}{\sqrt{l^2-x^2}} &= 0\end{aligned}$$

For very small x, we drop the x^2 term and the above reduces to

$$\ddot{x}+g\sqrt{1-\frac{d^2}{l^2}}\frac{x}{l}=0$$

Hence the undamped natural frequency is

$$\omega_0^2=\frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}$$

or

$$\omega_0=\sqrt{\frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}}$$

The period of small oscillation is therefore

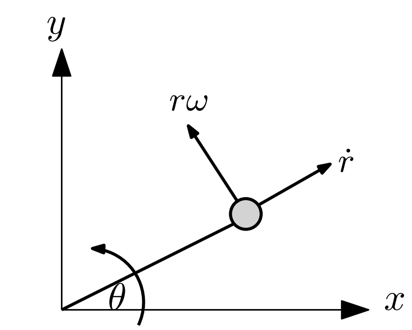
$$
\begin{aligned}
T &= \frac{2\pi}{\omega_0} \\
&= 2\pi \frac{1}{\sqrt{\frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}}}
\end{aligned}
$$

4.4.3 Problem 3

> 3. (10 points)
> A rod of length L rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially at the stationary end of the rod. It is given a slight push so that its initial speed along the rod is ωL. Find the time it takes the bead to reach the other end of the rod.

4.4.3.1 SOLUTION method one

The velocity of the particle is as shown in the following diagram



velocity diagram

$$
\begin{aligned}
\omega &= \dot{\theta} \\
V_x &= \dot{r}\cos\theta - r\omega\sin\theta \\
V_y &= \dot{r}\sin\theta + r\omega\cos\theta
\end{aligned}
$$

There is no potential energy, and the Lagrangian only comes from kinetic energy.

$$
\begin{aligned}
v^2 &= V_x^2 + V_y^2 \\
&= (\dot{r}\cos\theta - r\omega\sin\theta)^2 + (\dot{r}\sin\theta + r\omega\cos\theta)^2
\end{aligned}
$$

Exapnding and simplifying gives

$$v^2 = \dot{r}^2 + r^2\omega^2$$

Hence

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right)$$

And the equation of motion in the radial r direction is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}-\frac{\partial L}{\partial r}=0$$

$$\frac{d}{dt}m\dot{r}-mr\omega^{2}=0$$

Hence the equation of motion is

$$\boxed{\ddot{r}-r\omega^{2}=0} \tag{1}$$

The roots of the characteristic equation are $\pm\omega$, hence the solution is

$$r(t)=c_{1}e^{\omega t}+c_{2}e^{-\omega t}$$

At $t=0$, $r(0)=0$ and $\dot{r}(t)=L\omega$. Using these we can find c_1,c_2.

$$0=c_{1}+c_{2} \tag{2}$$

But $\dot{r}(t)=\omega c_{1}e^{\omega t}-\omega c_{2}e^{-\omega t}$ and at $t=0$ this becomes

$$L\omega=\omega c_{1}-\omega c_{2} \tag{3}$$

From (2,3) we solve for c_1,c_2. From (2), $c_1=-c_2$ and (3) becomes

$$L\omega=-\omega c_{2}-\omega c_{2}$$

$$c_{2}=\frac{L\omega}{-2\omega}=\frac{-1}{2}L$$

Hence $c_1=\frac{1}{2}L$ and the solution is

$$\begin{aligned} r(t)&=c_{1}e^{\omega t}+c_{2}e^{-\omega t}\\ &=\frac{1}{2}Le^{\omega t}-\frac{1}{2}Le^{-\omega t}\\ &=L\left(\frac{e^{\omega t}-e^{-\omega t}}{2}\right)\end{aligned}$$

Or

$$\boxed{r(t)=L(\sinh\omega t)}$$

To find the time it takes to reach end of rod, we solve for t_p from

$$L=L\left(\sinh\omega t_{p}\right)$$

$$1=\sinh\omega t_{p}$$

Hence

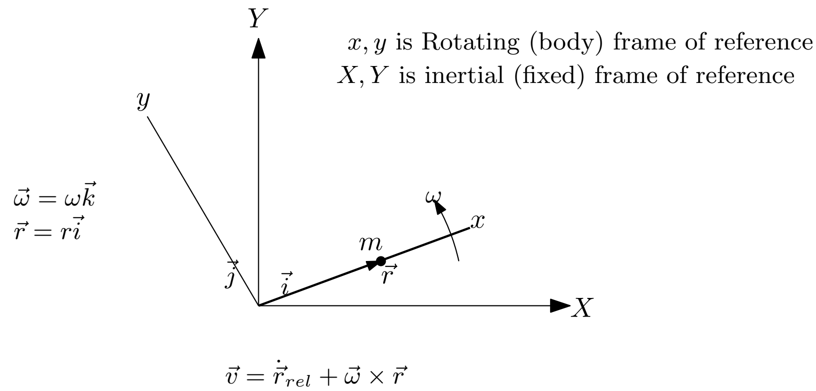
$$\begin{aligned}\omega t_{p}&=\sinh^{-1}(1)\\ &=0.88137\end{aligned}$$

Therefore

$$\boxed{t_{p}=\frac{0.88137}{\omega}\text{ sec}}$$

4.4.3.2 another solution

Let the local coordinate frame rotate with the bar, where the bar is oriented along the x axis of the local body coordinate frame as shown below.



The position vector of the particle is $\boldsymbol{r} = \boldsymbol{i}r$ where $\boldsymbol{i}$ is unit vector along the x axis. Taking time derivative, and using the rotating vector time derivative rule which says that $\frac{dA}{dt} = \left(\frac{dA}{dt}\right)_{relative} + \omega \times \boldsymbol{A}$ where ω is the angular velocity of the rotating frame then

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_{rel} + \omega \times \boldsymbol{r} \tag{1}$$

To find the acceleration of the particle, we take time derivative one more time

$$\frac{d}{dt}\dot{\boldsymbol{r}} = \frac{d}{dt}\left(\dot{\boldsymbol{r}}_{rel}\right) + \dot{\omega} \times \boldsymbol{r} + \omega \times \dot{\boldsymbol{r}}$$

But $\frac{d}{dt}\left(\dot{\boldsymbol{r}}_{rel}\right) = \ddot{\boldsymbol{r}}_{rel} + \omega \times \dot{\boldsymbol{r}}_{rel}$ by applying the rule of time derivative of rotating vector again. Therefore the above equation becomes

$$\frac{d}{dt}\dot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_{rel} + \omega \times \dot{\boldsymbol{r}}_{rel} + \dot{\omega} \times \boldsymbol{r} + \omega \times \dot{\boldsymbol{r}}$$

Replacing $\dot{\boldsymbol{r}}$ in the above from its value in (1) gives

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_{rel} + \omega \times \dot{\boldsymbol{r}}_{rel} + \dot{\omega} \times \boldsymbol{r} + \omega \times \left(\dot{\boldsymbol{r}}_{rel} + \omega \times \boldsymbol{r}\right) \\
&= \ddot{\boldsymbol{r}}_{rel} + \omega \times \dot{\boldsymbol{r}}_{rel} + \dot{\omega} \times \boldsymbol{r} + \omega \times \dot{\boldsymbol{r}}_{rel} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}\right) \\
&= \ddot{\boldsymbol{r}}_{rel} + 2\left(\omega \times \dot{\boldsymbol{r}}_{rel}\right) + \dot{\omega} \times \boldsymbol{r} + \omega \times \left(\omega \times \boldsymbol{r}\right)
\end{aligned}$$

But ω is constant (bar rotate with constant angular speed), hence the term $\dot{\omega}$ above is zero, and the above reduces to

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_{rel} + 2\left(\omega \times \dot{\boldsymbol{r}}_{rel}\right) + \omega \times \left(\omega \times \boldsymbol{r}\right) \tag{2}$$

The above is the acceleration of the particle as seen in the inertial frame. Now we calculate this acceleration by preforming the vector operations above, noting that $\boldsymbol{r} = \boldsymbol{i}r, \omega = \boldsymbol{k}\omega$,

hence (2) becomes

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \boldsymbol{i}\ddot{r}_{rel} + 2\left(\boldsymbol{k}\omega \times \boldsymbol{i}\dot{r}_{rel}\right) + \boldsymbol{k}\omega \times \left(\boldsymbol{k}\omega \times \boldsymbol{i}r\right) \\
&= \boldsymbol{i}\ddot{r}_{rel} + 2\left(\boldsymbol{j}\omega\dot{r}_{rel}\right) + \boldsymbol{k}\omega \times \left(\boldsymbol{j}\omega r\right) \\
&= \boldsymbol{i}\ddot{r}_{rel} + 2\left(\boldsymbol{j}\omega\dot{r}_{rel}\right) - \boldsymbol{i}\omega^2 r \\
&= \boldsymbol{i}\left(\ddot{r}_{rel} - \omega^2 r\right) + \boldsymbol{j}\left(2\omega\dot{r}_{rel}\right)
\end{aligned}$$

The particle has an acceleration along x axis and an acceleration along y axis. We are interested in the acceleration along x since this is where the rod is oriented along. The scalar version of the acceleration in the x direction is

$$a_x = \ddot{r}_{rel} - \omega^2 r$$

Using $F_x = ma_x$ and since $F_x = 0$ (there is no force on the particle) then the equation of motion along the bar (x axis) is

$$\ddot{r}_{rel} - \omega^2 r = 0$$

The roots of the characteristic equation is $\pm\omega$, hence the solution is

$$r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

At $t = 0$, $r(0) = 0$ and $\dot{r}(t) = L\omega$. Using these we can find c_1, c_2.

$$0 = c_1 + c_2 \tag{3}$$

But $\dot{r}(t) = \omega c_1 e^{\omega t} - \omega c_2 e^{-\omega t}$ and at $t = 0$ this becomes

$$L\omega = \omega c_1 - \omega c_2 \tag{4}$$

From (3,4) we solve for c_1, c_2. From (3), $c_1 = -c_2$ and (4) becomes

$$\begin{aligned}
L\omega &= -\omega c_2 - \omega c_2 \\
c_2 &= \frac{L\omega}{-2\omega} = \frac{-1}{2}L
\end{aligned}$$

Hence $c_1 = \frac{1}{2}L$ and the solution is

$$\begin{aligned}
r(t) &= c_1 e^{\omega t} + c_2 e^{-\omega t} \\
&= \frac{1}{2}Le^{\omega t} - \frac{1}{2}Le^{-\omega t} \\
&= L\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) \\
&= L\left(\sinh \omega t\right)
\end{aligned}$$

To find the time it takes to reach end of rod, we solve for t_p from

$$\begin{aligned}
L &= L\left(\sinh \omega t_p\right) \\
1 &= \sinh \omega t_p
\end{aligned}$$

Hence

$$\begin{aligned}
\omega t_p &= \sinh^{-1}(1) \\
&= 0.88137
\end{aligned}$$

Therefore

$$\boxed{t_p = \frac{0.88137}{\omega} \text{ sec}}$$

4.4.4 Problem 4

4. (10 points)
Consider a harmonic oscillator with $\omega_0 = 0.5\,\mathrm{s}^{-1}$. Let $x_0 = 1.0\,\mathrm{m}$ be the initial amplitude at $t = 0$ and assume that the oscillator is released with zero initial velocity. Use a computer to plot the phase-space plot ($\dot{x}$ versus x) for the following damping coefficients λ.
(1) $\lambda = 0.05\,\mathrm{s}^{-1}$ (weak damping)
(2) $\lambda = 0.25\,\mathrm{s}^{-1}$ (strong damping)
(3) $\lambda = \omega_0$ (critical damping).

SOLUTION:

Starting with the equation of motion for damped oscillator

$$x'' + 2\lambda x' + \omega_0^2 x = 0$$

The solution for cases 1,2 (both are underdamped) is

$$x = e^{-\lambda t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{1}$$

Where $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$. While the solution for case (3), the critical damped case is

$$x = (A + tB)\,e^{-\lambda t} \tag{2}$$

For (1) above, at $t = 0$ we obtain

$$1 = A$$

Hence (1) becomes $x = e^{-\lambda t}(\cos\omega_d t + B\sin\omega_d t)$, and taking derivative gives

$$\dot{x} = -\lambda e^{-\lambda t}\left(\cos\omega_d t + B\sin\omega_d t\right) + e^{-\lambda t}\left(-\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right)$$

At $t = 0$ we have

$$0 = -\lambda + B\omega_d$$
$$B = \frac{\lambda}{\omega_d}$$

Hence the complete solution for (1) is

$$x = e^{-\lambda t}\left(\cos\omega_d t + \frac{\lambda}{\omega_d}\sin\omega_d t\right) \tag{3}$$

$$\dot{x} = -\lambda x + e^{-\lambda t}\left(-\omega_d\sin\omega_d t + \lambda\cos\omega_d t\right) \tag{4}$$

Now we find the solution for (2), the critical damped case. At $t = 0$

$$1 = A$$

Hence (2) becomes $x = (1 + tB)\, e^{-\lambda t}$, and taking derivative gives

$$\dot{x} = Be^{-\lambda t} - \lambda\,(1 + tB)\, e^{-\lambda t}$$

At $t = 0$

$$\begin{aligned} 0 &= B - \lambda \\ B &= \lambda \end{aligned}$$

Hence the solution to (2) becomes

$$x = (1 + \lambda t)\, e^{-\lambda t} \tag{5}$$

$$\dot{x} = \lambda e^{-\lambda t} - \lambda\,(1 + \lambda t)\, e^{-\lambda t} \tag{6}$$

Now that the solutions are found, we plot the phase space using the computer, using parametric plot command

4.4.4.1 case (1)

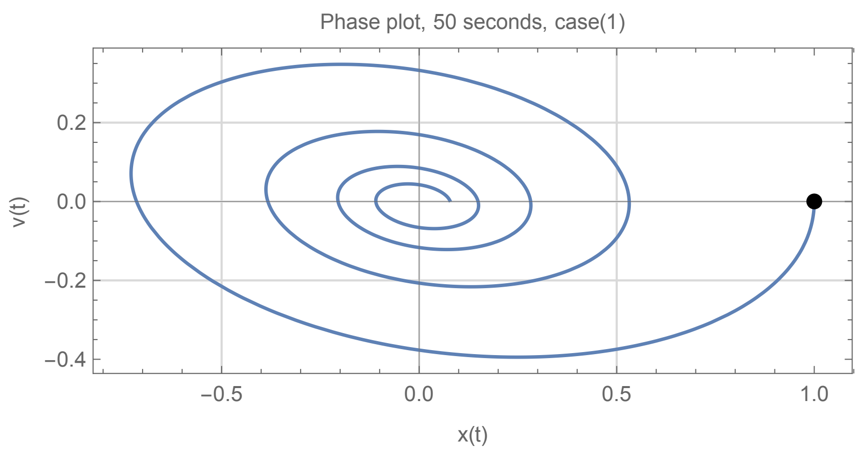
For $\lambda = 0.05$, and $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{0.5^2 - 0.05^2} = 0.4975$, then equations (3,4) become

$$x = e^{-0.05t}\,(\cos 0.4975t + 0.1005 \sin 0.4975t) \tag{3A}$$

$$\dot{x} = -0.05x + e^{-0.05t}\,(-0.4975 \sin 0.4975t + 0.05 \cos 0.4975t) \tag{4A}$$

Here is the plot generated, showing starting point $(1, 0)$ with the code used

Out[59]=




```
am = 0.05;
wn = 0.5;
wd = Sqrt[wn^2 - lam^2];
x = Exp[-lam t] (Cos[wd t] + lam/wd Sin[wd t]);
y = -lam x + Exp[-lam t] (-wd Sin[wd t] + lam Cos[lam t]);
ParametricPlot[{x, y}, {t, 0, 50}, Frame -> True,
 GridLines -> Automatic, GridLinesStyle -> LightGray,
 FrameLabel -> {{"v(t)", None}, {"x(t)",
    "Phase plot, 50 seconds, case(1)"}}, Epilog -> Disk[{1, 0}, .02],
 ImageSize -> 400]
```

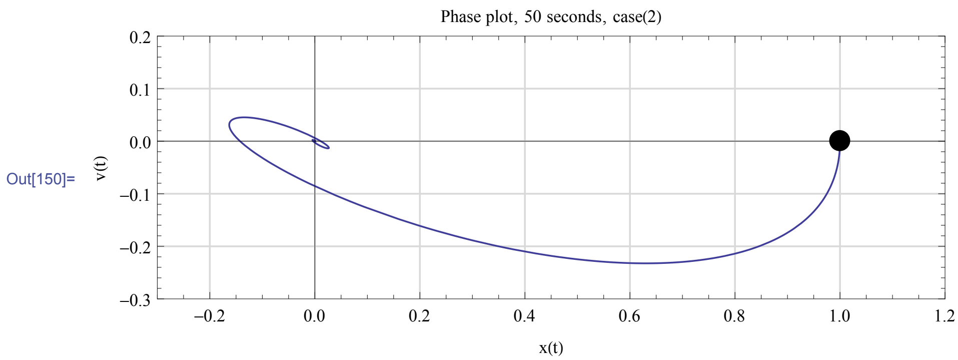
4.4.4.2 case (2)

For $\lambda = 0.25$, and $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{0.5^2 - 0.25^2} = 0.433$, equations (3,4) become

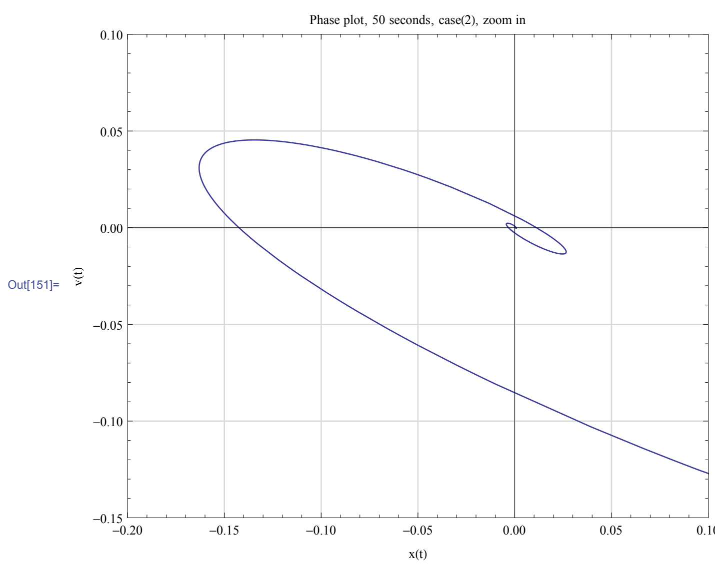
$$x = e^{-0.25t}\left(\cos 0.433t + 0.5774\sin 0.433t\right) \tag{3A}$$

$$\dot{x} = -0.05x + e^{-0.25t}\left(-0.433\sin 0.433t + 0.05\cos 0.433t\right) \tag{4A}$$

Here is the plot generated where the starting point was $(1,0)$



This below is a zoomed in version of the above close to the origin



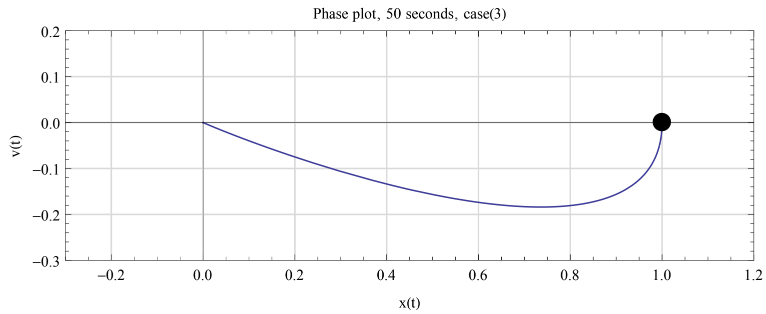
4.4.4.3 case(3)

For this case, equations (5,6) are used. For $\lambda = 0.5$, equations (5,6) become

$$x = (1 + 0.5t)\,e^{-0.5t} \tag{5A}$$

$$\dot{x} = 0.5e^{-0.5t} - 0.5\,(1 + 0.5t)\,e^{-0.5t} \tag{6A}$$

Here is the plot generated, showing starting point $(1,0)$ with the code used

Out[155]=



```
lam = 0.5;
x = (1 + lam*t) Exp[-lam t];
y = lam*Exp[-lam t] - lam*(1 + lam t) Exp[- lam t]
ParametricPlot[{x, y}, {t, 0, 30}, Frame -> True,
 GridLines -> Automatic, GridLinesStyle -> LightGray,
 FrameLabel -> {{"v(t)", None}, {"x(t)",
    "Phase plot, 50 seconds, case(3)"}}, Epilog -> Disk[{1, 0}, .02],
 ImageSize -> 500, PlotRange -> {{-.3, 1.2}, {-.3, .2}},
 PlotTheme -> "Classic"]
```

4.4.5 Problem 5

5. (15 points)
A damped harmonic oscillator has a period of free oscillation (with no damping) of $T_0 = 1.0\,\mathrm{s}$. The oscillator is initially displaced by an amount $x_0 = 0.1\,\mathrm{m}$ and released with zero initial velocity.
(1) Consider the case that the oscillator is critically damped. Determine the displacement x as a function of time and use a computer program to plot $x(t)$ for $0 \leq t \leq 2\,\mathrm{s}$.
(2) Now consider the case that the system is overdamped. Determine the displacement as a function of time and use a computer program to plot $x(t)$ for damping coefficients (i) $\lambda = 2.2\,\pi\mathrm{s}^{-1}$, (ii) $\lambda = 4\,\pi\mathrm{s}^{-1}$, and (iii) $\lambda = 10\,\pi\mathrm{s}^{-1}$ for $0 \leq t \leq 2\,\mathrm{s}$. Compare to the critically damped case.
(3) Now consider the case that the system is underdamped. Determine the displacement as a function of time and use a computer program to plot $x(t)$ for damping coefficients (i) $\lambda = 5.0\,\mathrm{s}^{-1}$, (ii) $\lambda = 1.0\,\mathrm{s}^{-1}$, and (iii) $\lambda = 0.1\,\mathrm{s}^{-1}$ for $0 \leq t \leq 2\,\mathrm{s}$. Compare to the critically damped case.

SOLUTION:

Since $\omega_0 = \frac{2\pi}{T_0}$, then $\omega_0 = \frac{2\pi}{1} = 2\pi$.

4.4.5.1 Part (1)

For critical damping $\lambda = \omega_0$ and the solution is

$$x(t) = (A + Bt)\,e^{-\lambda t} \tag{1}$$
$$\dot{x}(t) = Be^{-\lambda t} - \lambda(A + Bt)\,e^{-\lambda t} \tag{2}$$

Initial conditions are now used to find A, B. At $t = 0$, $x(0) = x_0 = 0.1$. From (1) we obtain

$$x_0 = A$$

And since $\dot{x}(0) = 0$, then from (2)

$$\begin{aligned} 0 &= B - \lambda A \\ B &= \lambda A \\ &= \lambda x_0 \end{aligned}$$

Putting values found for A, B, back into (1) gives

$$x(t) = (x_0 + \lambda x_0 t)\,e^{-\lambda t}$$

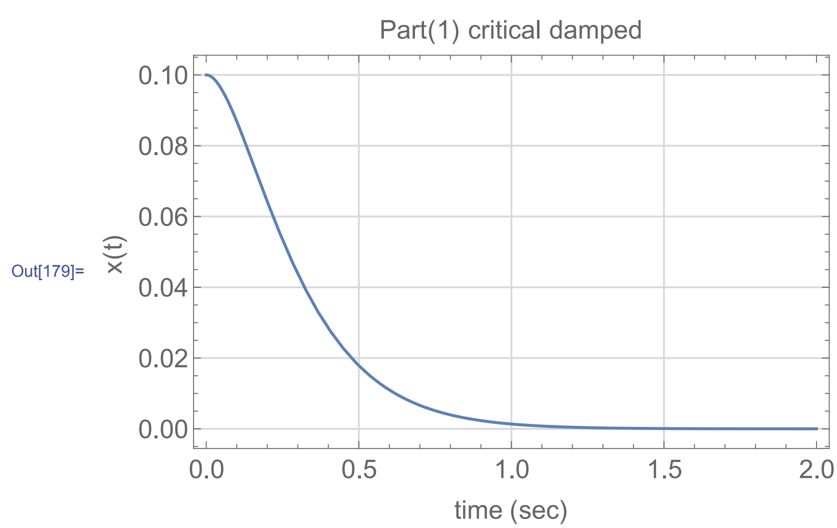
Since this is critical damping, then $\lambda = \omega_0 = 2\pi$, hence

$$x(t) = (x_0 + 2\pi x_0 t)\,e^{-2\pi t}$$

Finally, since $x_0 = 0.1$ meter, then

$$x(t) = \left(\frac{1}{10} + \frac{2\pi}{10}t\right)e^{-2\pi t}$$

A plot of the above for $0 \le t \le 2s$ is given below



4.4.5.2 Part(2)

For overdamped, $\lambda > \omega_0$ the two roots of the characteristic polynomial are real, hence no oscillation occur. The solution is given by

$$x(t) = Ae^{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} + Be^{\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)t} \tag{1}$$

A, B are found from initial conditions. When $t = 0$ the above becomes

$$x_0 = A + B \tag{2}$$

Taking derivative of (1) gives

$$\dot{x}(t) = A\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)e^{\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)t} + B\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)e^{\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)t}$$

At $t = 0$ the above becomes

$$0 = \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)A + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)B \tag{3}$$

We have two equations (2,3) which we solve for A, B. From (2), $A = x_0 - B$, and (3) becomes

$$\begin{aligned}
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)(x_0 - B) + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)B \\
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)x_0 - B\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)B \\
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)x_0 - 2B\sqrt{\lambda^2 - \omega_0^2} \\
B &= \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)x_0}{2\sqrt{\lambda^2 - \omega_0^2}} \qquad (4)
\end{aligned}$$

Using B found in (4) then (3) now gives A as

$$\begin{aligned}
A &= x_0 - B \\
&= x_0 - \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)x_0}{2\sqrt{\lambda^2 - \omega_0^2}} \\
&= x_0\left(1 - \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)}{2\sqrt{\lambda^2 - \omega_0^2}}\right) \\
&= x_0\left(\frac{\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right)
\end{aligned}$$

Hence the complete solution from (1) becomes

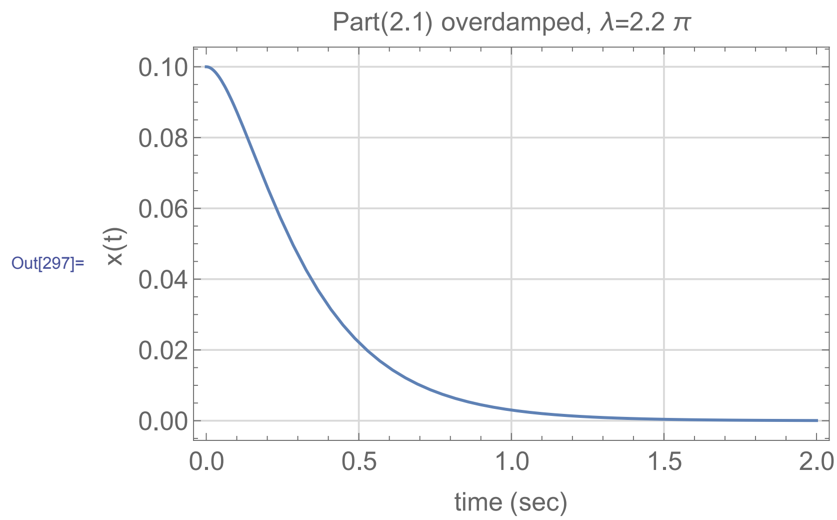
$$x(t) = x_0\left(\frac{\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right)e^{\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)t} + x_0\left(\frac{-\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right)e^{\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)t} \tag{5}$$

The above is now used for each case below to plot the solution..

4.4.5.2.1 case (i) $\lambda = 2.2\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

$$
\begin{aligned}
x(t) &= 0.1\left(\frac{2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}}{2\sqrt{(2.2\pi)^2 - (2\pi)^2}}\right) e^{\left(-2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}\right)t} + 0.1\left(\frac{-2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}}{2\sqrt{(2.2\pi)^2 - (2\pi)^2}}\right) e^{\left(-2.2\pi - \sqrt{(2.2\pi)^2 - (2\pi)^2}\right)t} \\
&= 0.17e^{-4.032\,2t} - 0.07\,e^{-9.791t}
\end{aligned}
$$

A plot of the above for $0 \le t \le 2s$ is given below

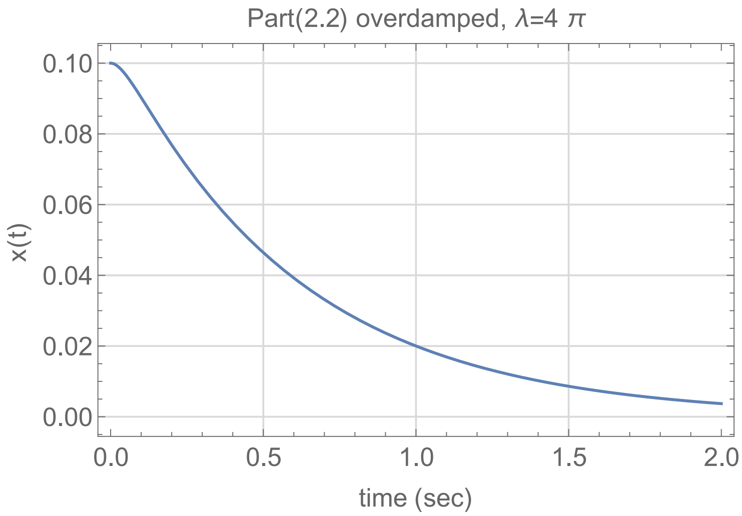


4.4.5.2.2 case (ii) $\lambda = 4\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

$$
\begin{aligned}
x(t) &= 0.1\left(\frac{4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}}{2\sqrt{(4\pi)^2 - (2\pi)^2}}\right) e^{\left(-4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}\right)t} + 0.1\left(\frac{-4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}}{2\sqrt{(4\pi)^2 - (2\pi)^2}}\right) e^{\left(-4\pi - \sqrt{(4\pi)^2 - (2\pi)^2}\right)t} \\
&= 0.1077e^{-1.6836t} - 0.00774e^{-23.449t}
\end{aligned}
$$

A plot of the above for $0 \le t \le 2s$ is given below

Out[310]=

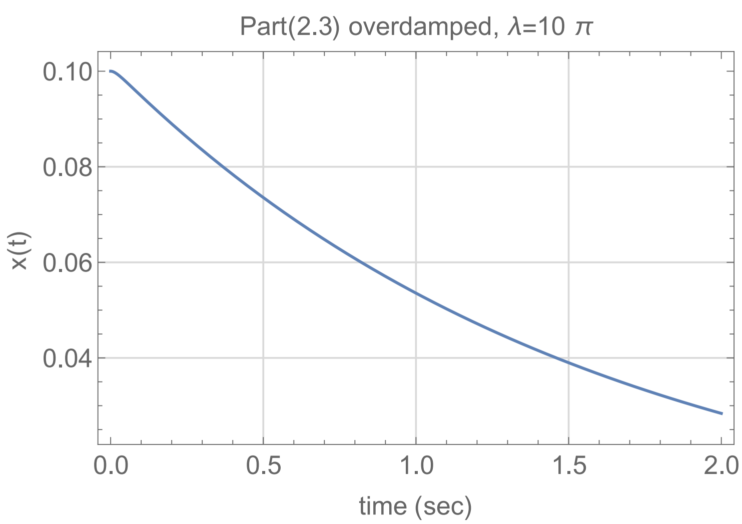


4.4.5.2.3 case (iii) $\lambda = 10\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

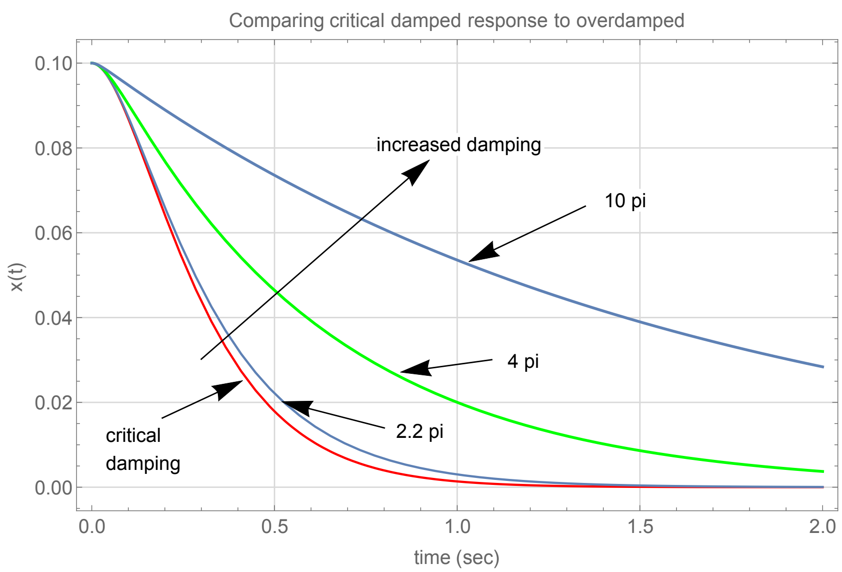
$$x(t) = 0.1\left(\frac{10\pi + \sqrt{(10\pi)^2 - (2\pi)^2}}{2\sqrt{(10\pi)^2 - (2\pi)^2}}\right)e^{\left(-10\pi+\sqrt{(10\pi)^2-(2\pi)^2}\right)t} + 0.1\left(\frac{-10\pi + \sqrt{(10\pi)^2 - (2\pi)^2}}{2\sqrt{(10\pi)^2 - (2\pi)^2}}\right)e^{\left(-10\pi-\sqrt{(10\pi)^2-(2\pi)^2}\right)t}$$
$$= 0.101\, e^{-0.634\,73t} - 0.001034e^{-62.197t}$$

A plot of the above for $0 \le t \le 2s$ is given below

Out[314]=



To compare to the critical damped case, the above three plots are plotted on the same figure against the critical damped case in order to get a better picture and be able to compare the results



From the above we see that critical damping has the fastest decay of the response $x(t)$. As the damping increases, it takes longer for the response to decay.

4.4.5.3 Part (3)

For the underdamped case, the solution is given by

$$x(t) = e^{-\lambda t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{1}$$

Where $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$ and A, B are constant of integration that can be found from initial conditions. And

$$\dot{x}(t) = -\lambda e^{-\lambda t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + e^{-\lambda t}\left(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right) \tag{2}$$

Applying initial conditions $x(0) = x_0$ then (1) becomes

$$x_0 = A$$

Applying initial conditions $\dot{x}(0) = 0$ then (2) becomes

$$0 = -\lambda x_0 + B\omega_d$$

$$B = \frac{\lambda x_0}{\omega_d}$$

Replacing A, B back into the solution (1) gives the solution

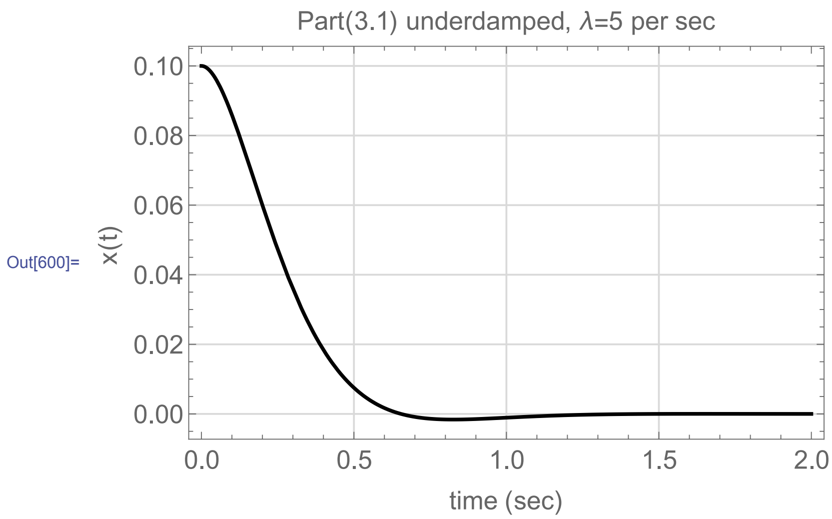
$$x(t) = e^{-\lambda t}\left(x_0\cos\omega_d t + \frac{\lambda x_0}{\omega_d}\sin\omega_d t\right) \tag{3}$$

We now use the above solution for the rest of the problem

4.4.5.3.1 case (i) $\lambda = 5s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 5^2} = 3.8051$ and (3) becomes

$$\begin{aligned} x(t) &= e^{-5t}\left(0.1\cos(3.8051t) + \frac{(5)(0.1)}{3.8051}\sin(3.8051t)\right) \\ &= e^{-5t}(0.1\cos(3.8051t) + 0.1314\sin(3.8051t)) \end{aligned}$$

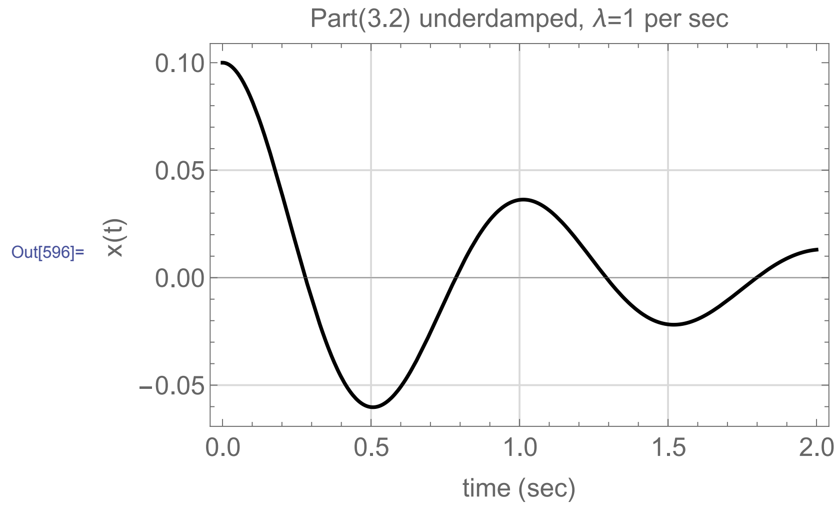
A plot of the above solution $x(t)$ for $0 \leq t \leq 2s$ is given below



4.4.5.3.2 case (ii) $\lambda = 1s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 1^2} = 6.2031$ and (3) becomes

$$\begin{aligned} x(t) &= e^{-t}\left(0.1\cos(6.2031t) + \frac{(1)(0.1)}{6.2031}\sin(6.2031t)\right) \\ &= e^{-t}(0.1\cos(6.2031t) + 0.016\sin(6.2031t)) \end{aligned}$$

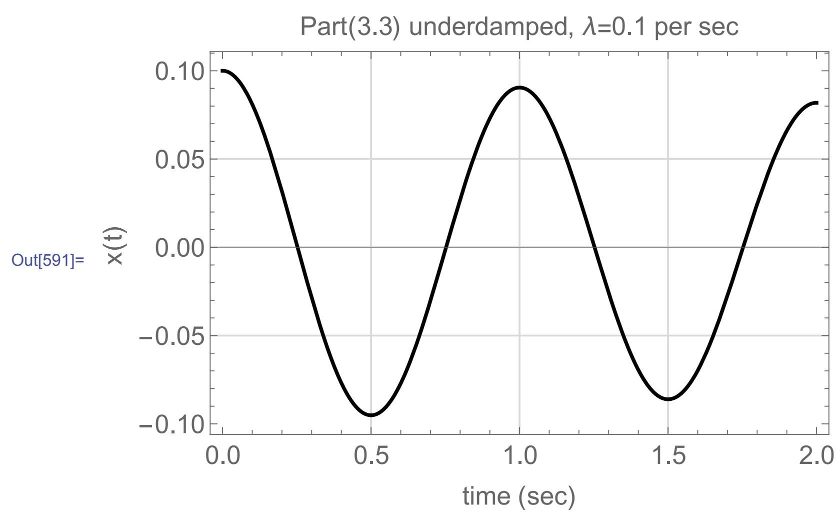
A plot of the above solution $x(t)$ for $0 \leq t \leq 2s$ is given below



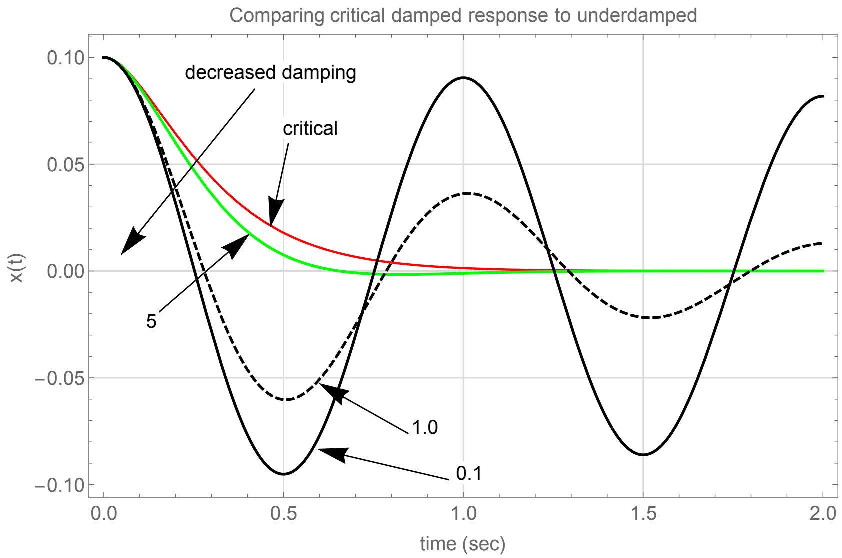
4.4.5.3.3 case(iii) $\lambda = 0.1s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 0.1^2} = 6.2824$ and (3) becomes

$$
\begin{aligned}
x(t) &= e^{-0.1t}\left(0.1\cos(6.2824t) + \frac{(0.1)(0.1)}{6.2824}\sin(6.2824t)\right) \\
&= e^{-0.1t}(0.1\cos(6.2824t) + 0.001592\sin(6.2824t))
\end{aligned}
$$

A plot of the above solution $x(t)$ for $0 \le t \le 2s$ is given below



To compare to the critical damped case, the above 3 plots are now plotted on the same figure against the critical damped case in order to get a better picture and be able to compare the results



As the damping becomes smaller, more oscillation occur. The case for $\lambda = 5s^{-1}$ had the smallest oscillation.

4.4.6 HW 4 key solution

1

Mechanics

Physics 311 - Fall 2015

Homework Set 4 - Solutions

Problem 1

(i) $\beta = \frac{1}{10}\beta_{crit} = \frac{\omega_0}{10}$

$$\omega_r = \sqrt{\omega_0^2 - 2\left(\frac{\omega_0}{10}\right)^2} = \omega_0\sqrt{0.98} = \underline{\underline{0.99\,\omega_0}}$$

(ii) system is weakly damped, so $Q \simeq \frac{\omega_0}{2\beta} = \frac{\omega_0}{2\left(\frac{\omega_0}{10}\right)} = \underline{\underline{5}}$

(iii)
$$\phi = \operatorname{atan}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$
$$= \operatorname{atan}\left(\frac{2\,\omega_0/10\;\omega_0/2}{\omega_0^2-\left(\frac{\omega_0}{2}\right)^2}\right) = \operatorname{atan}\left(\frac{1/10}{3/4}\right)$$
$$= \operatorname{atan}\left(\frac{2}{15}\right) = \underline{\underline{7.6^\circ}}$$

(iv)
$$A(\omega) = \frac{F_0}{m}\,\frac{1}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}}$$
$$\rightsquigarrow\quad A\left(\frac{\omega_0}{2}\right) = \frac{F_0}{m}\left\{\left[\omega_0^2-\left(\frac{\omega_0}{2}\right)^2\right]^2 + 4\left(\frac{\omega_0}{10}\right)^2\left(\frac{\omega_0}{2}\right)^2\right\}^{-1/2}$$
$$= \frac{F}{m}\,\frac{1}{9/16 + 1/100}\,\frac{1}{\omega_0^2} = \underline{\underline{1.322\,\frac{F_0}{m\omega_0^2}}}$$

(so 1.322 times the steady-state amplitude for zero driving frequency)

2

Problem 2

the sum of the distances of the bead to points A and B is $2\ell = $ const. $\Rightarrow$ ellipse

semi-major axis ℓ

semi-minor axis $\sqrt{\ell^2 - d^2}$

$$\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2 - d^2} = 1$$

equilibrium at $x = 0$, $y = -\sqrt{\ell^2 - d^2}$ $\Big\}$ for a system with $(0,0)$ at the center of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = \left(1 - \frac{x^2}{a^2}\right) b^2$$

so $$y = -b\sqrt{1 - x^2/a^2} \simeq -b\left(1 - \frac{x^2}{2a^2}\right)$$

consider lowest point!

so $$y = -b + \frac{bx^2}{2a^2}$$

y is (to first order) parabolic about the equilibrium

now move the origin to the equilibrium point:

$$y = b\,\frac{x^2}{2a^2}$$

in the following, neglect all terms that are more than second order in small quantities, for example

$$\dot{y}^2 = \left(b\,\frac{2x\dot{x}}{2a^2}\right)^2 \approx 0$$

3

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \simeq \frac{1}{2} m \dot{x}^2$$

$$U = mgy = mgb\frac{x^2}{2a^2}$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - mg\frac{bx^2}{2a^2}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad -\frac{2mgb}{2a^2} x - m\ddot{x} = 0$$

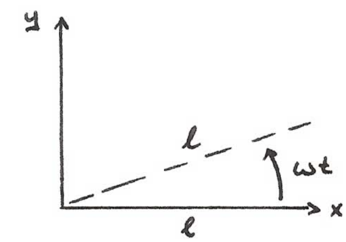
$$\Leftrightarrow \quad \ddot{x} + \frac{gb}{a^2} x = 0$$

so $\omega^2 = \frac{gb}{a^2}$ or $\omega^2 = \frac{g\sqrt{\ell^2 - d^2}}{\ell^2}$

$$\Rightarrow \quad \boxed{T = 2\pi \frac{\ell}{\sqrt{g\sqrt{\ell^2 - d^2}}}}$$

4

<u>Problem 3</u>



Constraint :

$\theta - \omega t = 0$

$T = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2)$ $\quad U = 0$ on the plane

$= L$

$$\Rightarrow \quad \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0$$

$$\text{so} \quad m r \dot\theta^2 - m \ddot r = 0$$

$$\Rightarrow \quad \ddot r - \omega^2 r = 0$$

ansatz : $\quad r = A e^{\omega t} + B e^{-\omega t}$

$\dot r = \omega A e^{\omega t} - \omega B e^{-\omega t}$

initial conditions : $\quad r(0) = 0 \qquad \dot r(0) = \omega L$

$\Rightarrow \quad A + B = 0 \quad \Leftrightarrow \quad B = -A$

$\omega A - \omega B = \omega \ell \quad \Rightarrow \quad A - B = \ell$

$\Rightarrow \quad 2A = \ell$

$\Leftrightarrow \quad A = \ell/2$

$$\Rightarrow \quad r(t) = \frac{\ell}{2}\left(e^{\omega t} - e^{-\omega t}\right) = \ell \sinh(\omega t)$$

time to reach the end at $r = \ell \qquad r(T) = \ell$

$$\Rightarrow \quad \ell = \ell \sinh(\omega T) \quad \Leftrightarrow \quad T = \frac{1}{\omega}\,\mathrm{asinh}(1)$$

$$\Rightarrow \quad \boxed{T = \frac{0.88}{\omega}}$$

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Problem 4

we need $x(t)$ and $\dot{x}(t)$ for the underdamped and critically damped case

(i) underdamped:

$$x(t) = A e^{-\lambda t} \sin(\omega t + \phi_0)$$

$$\dot{x}(t) = -A e^{-\lambda t}\left[\lambda \sin(\omega t+\phi_0) - \omega \cos(\omega t+\phi_0)\right]$$

now we $x(0) = x_0 = A \sin\phi_0 \Rightarrow A = \dfrac{x_0}{\sin\phi_0}$

$$\dot{x}(0) = 0 = -A\lambda \sin\phi_0 + A\omega\cos\phi_0$$

$$\Leftrightarrow \quad \lambda \sin\phi_0 = \omega\cos\phi_0$$

$$\Leftrightarrow \quad \tan\phi_0 = \frac{\omega}{\lambda}$$

so $x(t) = \dfrac{x_0}{\sin\phi_0} e^{-\lambda t} \sin(\omega t + \phi_0)$

with $\phi_0 = \operatorname{atan}\left(\dfrac{\omega}{\lambda}\right)$

$$\dot{x}(t) = -\frac{x_0}{\sin\phi_0}\lambda e^{-\lambda t}\sin(\omega t+\phi_0) + \frac{x_0}{\sin\phi_0}\omega e^{-\lambda t}\cos(\omega t+\phi_0)$$

$$= \frac{x_0}{\sin\phi_0} e^{-\lambda t}\left[\omega\cos(\omega t+\phi_0) - \lambda\sin(\omega t+\phi_0)\right]$$

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(ii) Critically damped

$$x(t) = (A + Bt)\, e^{-\lambda t}$$

$$\dot{x}(t) = -\lambda (A + Bt)\, e^{-\lambda t} + B e^{-\lambda t}$$

use $x(0) = x_0 \Rightarrow A = x_0$

$$\dot{x}(0) = 0 = -\lambda A + B \Rightarrow B = \lambda A = \lambda x_0$$

so

$$x(t) = x_0 (1 + \lambda t)\, e^{-\lambda t}$$

$$\dot{x}(t) = -\lambda (x_0 + \lambda x_0 t)\, e^{-\lambda t} + \lambda x_0 e^{-\lambda t}$$

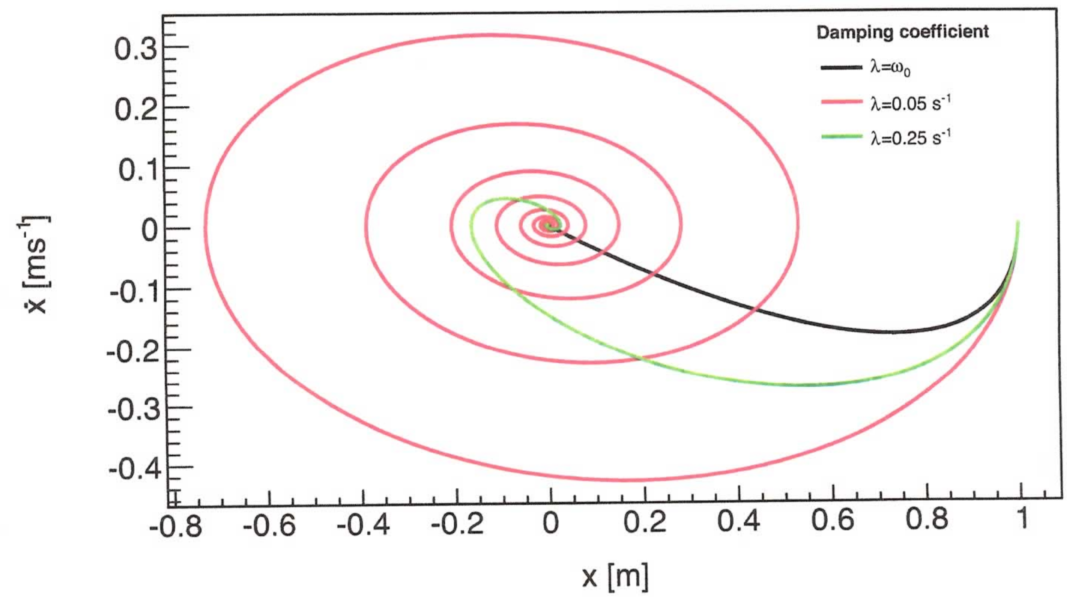
$$= -x_0 \lambda^2 t\, e^{-\lambda t}$$

plots for $\dot{x}$ versus x on the following pages

(1) $\lambda = 0.05\ s^{-1}$ } underdamped

(2) $\lambda = 0.25\ s^{-1}$ }

(3) $\lambda = \omega_0 = 0.5 s^{-1}$ critically damped


Damping coefficient
λ=ω0
λ=0.05 s-1
λ=0.25 s-1
ẋ [ms-1]
x [m]
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

8

Problem 5

equations for $x(t)$ for the underdamped and

Critically damped case were derived in (4) for the case

$x(t=0) = x_0$ and $\dot{x}(t=0) = 0$

We still need the overdamped case:

$$x(t) = A\, e^{-\lambda t + \alpha t} + B\, e^{-\lambda t - \alpha t}$$

$$\alpha = \sqrt{\lambda^2 - \omega_0^2}$$

$$\dot{x}(t) = -A(\lambda - \alpha)\, e^{-\lambda t + \alpha t} - B(\lambda + \alpha)\, e^{-\lambda t - \alpha t}$$

use

$$x(0) = x_0 = A + B \Rightarrow B = x_0 - A$$

$$\dot{x}(0) = 0 = -A(\lambda - \alpha) - B(\lambda + \alpha)$$

$$\Rightarrow B = -A\,\frac{\lambda - \alpha}{\lambda + \alpha}$$

So

$$x_0 - A = -A\,\frac{\lambda - \alpha}{\lambda + \alpha} \Leftrightarrow x_0 = A\left(1 - \frac{\lambda - \alpha}{\lambda + \alpha}\right)$$

$$\Leftrightarrow A = x_0\,\frac{\lambda + \alpha}{2\alpha}$$

and

$$B = x_0\left(1 - \frac{\lambda + \alpha}{2\alpha}\right)$$

$$= -x_0\,\frac{\lambda - \alpha}{2\alpha}$$

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$$\Rightarrow \quad x(t) = x_0 \left\{ \frac{\lambda+\alpha}{2\alpha} e^{-\lambda t + \alpha t} - \frac{\lambda-\alpha}{2\alpha} e^{-\lambda t - \alpha t} \right\}$$

$$= \frac{x_0}{2\alpha} e^{-\lambda t} \left\{ (\lambda+\alpha) e^{\alpha t} - (\lambda-\alpha) e^{-\alpha t} \right\}$$

$$= \frac{x_0}{\alpha} e^{-\lambda t} \left\{ \frac{\lambda}{2} \left(e^{\alpha t} - e^{-\alpha t}\right) + \frac{\alpha}{2} \left(e^{\alpha t} + e^{-\alpha t}\right) \right\}$$

$$= \frac{x_0}{\alpha} e^{-\lambda t} \left\{ \lambda \sinh\alpha t + \alpha \cosh\alpha t \right\}$$

in summary, the three cases are

(1) <u>Critically damped</u>

$$x(t) = x_0 \, (1 + \lambda t) \, e^{-\lambda t}$$

(2) <u>Overdamped</u>

$$x(t) = x_0 \left(\frac{\lambda}{\alpha} \sinh\alpha t + \cosh\alpha t \right) e^{-\lambda t}$$

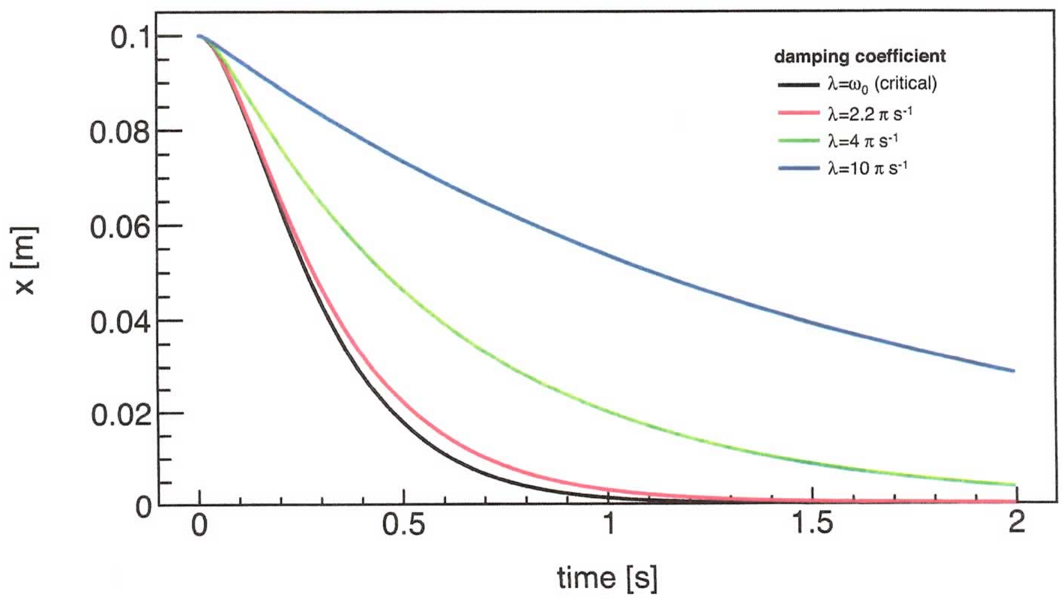
with $\alpha^2 = \lambda^2 - \omega_0^2$

(3) <u>underdamped</u>

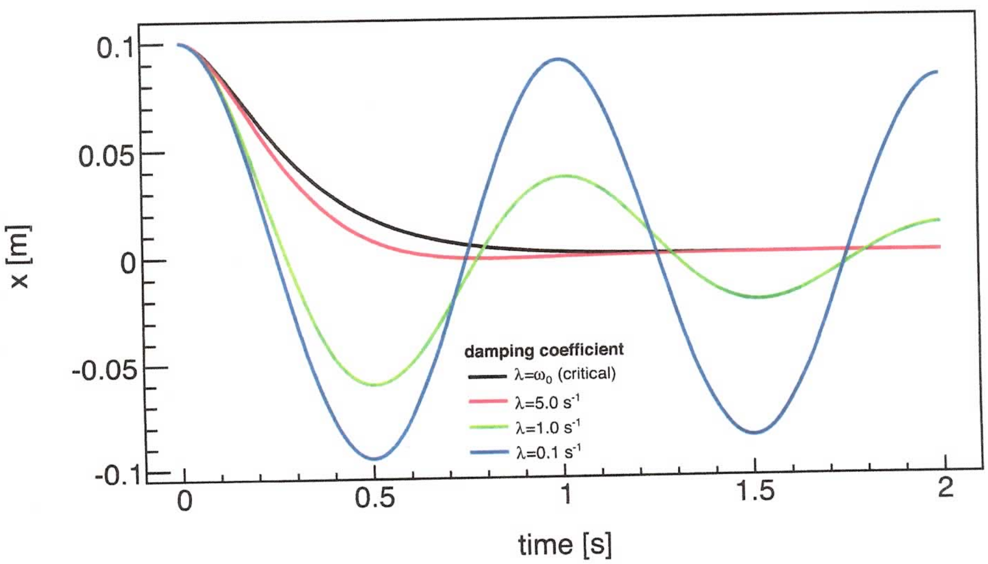
$$x(t) = \frac{x_0}{\sin\phi_0} \sin(\omega t + \phi_0) \, e^{-\lambda t}$$

with $\omega^2 = \omega_0^2 - \lambda^2$

and $\phi_0 = \operatorname{atan} \frac{\omega}{\lambda}$


damping coefficient
λ=ω0 (critical)
λ=2.2 π s-1
λ=4 π s-1
λ=10 π s-1
x [m]
time [s]
0.1
0.08
0.06
0.04
0.02
0
0.5
1
1.5
2

11

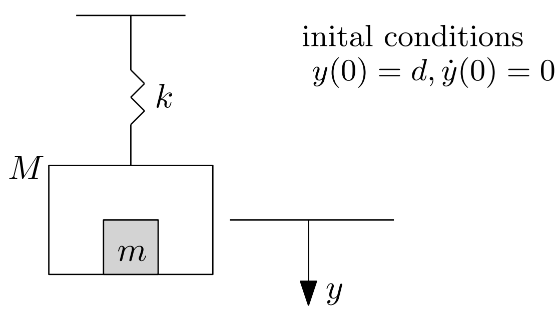

0.1
0.05
0
-0.05
-0.1
x [m]
0
0.5
1
1.5
2
time [s]
damping coefficient
λ=ω0 (critical)
λ=5.0 s-1
λ=1.0 s-1
λ=0.1 s-1

4.5 HW 5

4.5.1 Problem 1

1. (5 points)
A spring of spring constant k supports a box of mass M, which contains a block of mass m. If the system is pulled downward a distance d from the equilibrium position and then released, it starts to oscillate. For what value of d does the block just begin to leave the bottom of the box at the top of the vertical oscillations?

SOLUTION:



The block of mass m will leave the floor of the box when the vertical acceleration is large enough to match the gravity acceleration g. The equation of motion of the overall system is given by

$$y'' + \omega_0^2 y = 0 \tag{1}$$

Where ω_0 is the undamped natural frequency

$$\omega_0 = \sqrt{\frac{k}{M+m}}$$

The solution to (1) is

$$y = A\cos\omega_0 t + B\sin\omega_0 t \tag{2}$$

Initial conditions are used to find A, B. Since at $t = 0$, $y(0) = d$, then from (2) we find

$$A = d$$

Taking derivative of (2) gives

$$y' = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t \tag{3}$$

At $t = 0$, $y'(0) = 0$, this gives $B = 0$. Therefore the full solution (2) becomes

$$y = d\cos\omega_0 t$$

The acceleration is now found as

$$y' = -\omega_0 d \sin \omega_0 t$$
$$y'' = -\omega_0^2 d \cos \omega_0 t$$

The period is $T_p = \frac{2\pi}{\omega_0}$. After one T_p from release the box will be the top. Therefore, the acceleration at that moment is

$$\begin{aligned} y''\left(T_p\right) &= -\omega_0^2 d \cos \omega_0 T_p \\ &= -\omega_0^2 d \cos 2\pi \\ &= \omega_0^2 d \end{aligned}$$

The condition for m to just leave the floor of the box is when the above acceleration is the same as g.

$$\begin{aligned} \omega_0^2 d &= g \\ d &= \frac{g}{\omega_0^2} \end{aligned}$$

Therefore

$$\boxed{d = \frac{g}{k}(M+m)}$$

4.5.2 Problem 2

2. (15 points)
(1) Show that the Fourier series of a periodic square wave is

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + ...\right] \ ,$$

where

$$\begin{aligned} f(t) &= +1 \quad \text{for} \quad 0 < \omega\text{t} < \pi, \quad 2\pi < \omega\text{t} < 3\pi, ... \\ f(t) &= -1 \quad \text{for} \quad \pi < \omega\text{t} < 2\pi, \quad 3\pi < \omega\text{t} < 4\pi, ... \ . \end{aligned}$$

(2) Use the result from above to find the steady-state motion of a damped harmonic oscillator that is driven by a periodic square-wave force of amplitude F_0. In particular, find the relative amplitudes of the first three terms, A_1, A_3, and A_5, of the response function $x(t)$ in the case that the third harmonic 3ω of the driving frequency coincides with the frequency ω_0 of the undamped oscillator. Assume a quality factor of $Q = 100$.

SOLUTION:

4.5.2.1 Part (1)

The function $f(t)$ is an odd function, therefore we only need to evaluate b_n terms. To more clearly see the period, the definition of $f(t)$ is written as

$$f(t)=\begin{cases} +1 & 0<t<\frac{\pi}{\omega},\cdots \\ -1 & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega},\cdots \end{cases}$$

Therefore the period is

$$T_p=\frac{2\pi}{\omega}$$

Finding b_n

$$\begin{aligned}
b_n &= \frac{1}{\frac{T_p}{2}}\int_0^{T_p} f(t)\sin(n\omega t)\,dt \\
&= \frac{2}{\frac{2\pi}{\omega}}\left(\int_0^{\frac{\pi}{\omega}}(+1)\sin(n\omega t)\,dt+\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}(-1)\sin(n\omega t)\,dt\right) \\
&= \frac{\omega}{\pi}\left(\int_0^{\frac{\pi}{\omega}}\sin(n\omega t)\,dt-\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}\sin(n\omega t)\,dt\right) \\
&= \frac{\omega}{\pi}\left(\left[-\frac{\cos(n\omega t)}{n\omega}\right]_0^{\frac{\pi}{\omega}}-\left[-\frac{\cos(n\omega t)}{n\omega}\right]_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}\right) \\
&= \frac{\omega}{\pi}\left(-\frac{1}{n\omega}\left[\cos(n\omega t)\right]_0^{\frac{\pi}{\omega}}+\frac{1}{n\omega}\left[\cos(n\omega t)\right]_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}\right) \\
&= \frac{1}{n\pi}\left(-\left[\cos\left(n\omega\frac{\pi}{\omega}\right)-\cos(0)\right]+\left[\cos\left(n\omega\frac{2\pi}{\omega}\right)-\cos\left(n\omega\frac{\pi}{\omega}\right)\right]\right) \\
&= \frac{1}{n\pi}\left(-\left[\cos(n\pi)-1\right]+\left[\cos(2n\pi)-\cos(n\pi)\right]\right) \\
&= \frac{1}{n\pi}\left(-\cos(n\pi)+1+\cos(2n\pi)-\cos(n\pi)\right) \\
&= \frac{1}{n\pi}\left(-2\cos(n\pi)+\overbrace{\cos(2n\pi)}^{1}+1\right) \\
&= \frac{2}{n\pi}\left(1-\cos(n\pi)\right)
\end{aligned}$$

And since n is an integer, then $\cos(n\pi)=(-1)^n$ and the above reduces to

$$b_n=\frac{2}{n\pi}\left(1-(-1)^n\right)$$

Therefore

$$b_n=\begin{cases} \frac{4}{n\pi} & n=1,3,5,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} f(t) &= \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin(\omega n t) \\ &= \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi} \sin(\omega n t) \end{aligned}$$

Writing down few terms to see the sequence

$$f(t) = \frac{4}{\pi}\left\{\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \cdots\right\}$$

4.5.2.2 Part (2)

When the system is driven by the above periodic square wave of amplitude F_0, the steady state response is the sum to the response of each harmonic in the Fourier series expansion of the forcing function. Since the steady state response of a second order system to $F_n \sin(n\omega t)$ is given by

$$y_n(t) = \frac{F_n/m}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda_n^2 (n\omega)^2}} \sin(n\omega t + \delta_n)$$

Where the phase δ_n is defined as

$$\delta_n = \tan^{-1} \frac{-2\lambda(n\omega)}{\omega_0^2 - (n\omega)^2}$$

Then the steady state response to $f(t) = \sum_{n=1,3,5,\cdots}^{\infty} F_0 \frac{4}{n\pi} \sin(\omega n t)$ is given by

$$\begin{aligned} y_{ss}(t) &= \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi} \frac{F_0/m}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda^2 (n\omega)^2}} \sin(n\omega t + \delta_n) \qquad (1) \\ &= \frac{4F_0}{\pi m} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \frac{\sin(n\omega t + \delta_n)}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda^2 (n\omega)^2}} \end{aligned}$$

Looking at the first three responses gives

$$y_{ss}(t) = \frac{4F_0}{\pi m}\left\{\frac{\sin(\omega t + \delta_1)}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}} + \frac{1}{3}\frac{\sin(3\omega t + \delta_3)}{\sqrt{\left(\omega_0^2 - (3\omega)^2\right)^2 + 4\lambda^2 (3\omega)^2}} + \frac{1}{5}\frac{\sin(5\omega t + \delta_5)}{\sqrt{\left(\omega_0^2 - (5\omega)^2\right)^2 + 4\lambda^2 (5\omega)^2}} + \cdots\right\} \qquad (2)$$

We are told that $3\omega = \omega_0$ or $\omega = \frac{1}{3}\omega_0$ and in addition, using using $Q = \frac{\omega_0}{2\lambda}$ we find

$$\begin{aligned} 100 &= \frac{\omega_0}{2\lambda} \\ \lambda &= \frac{\omega_0}{200} \end{aligned}$$

Using this λ and given value of ω then the phase δ_n becomes

$$\begin{aligned}\delta_n &= \tan^{-1}\frac{-2\lambda\left(n\omega\right)}{\omega_0^2-\left(n\omega\right)^2}\\ &= \tan^{-1}\frac{-2\left(\frac{\omega_0}{200}\right)\left(n\frac{\omega_0}{3}\right)}{\omega_0^2-\left(n\frac{\omega_0}{3}\right)^2}\\ &= \tan^{-1}\frac{3n}{100n^2-900}\end{aligned}$$

Using the above phase in (2) gives[1]

$$\begin{aligned}y_{ss}\left(t\right) &= \frac{4F_0}{\pi m}\left\{\frac{\sin\left(\frac{\omega_0}{3}t+\tan^{-1}\frac{-3}{800}\right)}{\sqrt{\left(\omega_0^2-\left(\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)^2\left(\frac{\omega_0}{3}\right)^2}}+\frac{\frac{1}{3}\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\sqrt{\left(\omega_0^2-\left(3\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)^2\left(3\frac{\omega_0}{3}\right)^2}}+\frac{\frac{1}{5}\sin\left(5\frac{\omega_0}{3}t+\tan^{-1}\frac{3}{320}\right.}{\sqrt{\left(\omega_0^2-\left(5\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)^2}}\right.\\ &= \frac{4}{\pi}\frac{F_0}{m}\left\{\frac{\sin\left(\frac{\omega_0}{3}t-\tan^{-1}\frac{3}{800}\right)}{\sqrt{\frac{640\,009}{810\,000}\omega_0^4}}+\frac{1}{3}\frac{\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\sqrt{\frac{1}{10\,000}\omega_0^4}}+\frac{1}{5}\frac{\sin\left(5\frac{\omega_0}{3}t+\tan^{-1}\frac{3}{320}\right)}{\sqrt{\frac{102\,409}{32\,400}\omega_0^4}}+\cdots\right\}\\ &= \frac{4}{\pi}\frac{F_0}{m}\left\{1.125\frac{\sin\left(0.333\omega_0 t-\tan^{-1}\frac{3}{800}\right)}{\omega_0^2}+33.333\frac{\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\omega_0^2}+0.11249\frac{\sin\left(1.6667\omega_0 t+\tan^{-1}\frac{3}{320}\right)}{\omega_0^2}+\cdots\right\}\end{aligned}$$

The relative amplitudes of A_1, A_3, A_5 are given by

$$\{1.125, 33.333, 0.11249\}$$

We see that the third harmonic ($n=3$) has the largest amplitude, since this is where $3\omega=\omega_0$. In normalized size, dividing all amplitudes by the smallest amplitude gives

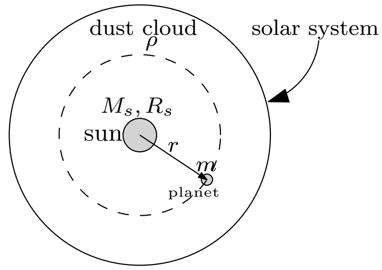
$$\{A_1, A_3, A_5\}_{normalized}=\{10, 296, 1\}$$

4.5.3 Problem 3

3. (5 points)
If the solar system were imbedded in a uniform dust cloud of density ρ, what would be the force on a planet a distance r from the center of the Sun?

SOLUTION:

[1] The third harmonic $n=3$ has $\frac{\pi}{2}$ phase since $\tan^{-1}(\infty)=\frac{\pi}{2}$



The total force on the planet m is due to the mass inside the region centered at the center of the sun. The mass outside can be ignored since its effect cancels out. Let the radius of the sun be R_{sun}, then the total mass that pulls the planet toward the center of the solar system is given by

$$M_{total} = M_{sun} + \frac{4}{3}\pi\left(r^3 - R_{sun}^3\right)\rho$$

The force on the planet is therefore

$$\begin{aligned}\bar{F} &= -\frac{GM_{total}m}{r^2}\hat{r}\\ &= -\frac{G\left(M_{sun} + \frac{4}{3}\pi\left(r^3 - R_{sun}^3\right)\rho\right)m}{r^2}\hat{r}\end{aligned}$$

Where $\hat{r}$ is a unit vector pointing from the sun towards the planet m and G is the gravitational constant and ρ is the cloud density.

4.5.4 Problem 4

4. (10 points)
(1) What is the speed (in km/s) for a satellite in a low-lying orbit close to Earth? Assume that the radius of the satellite's orbit is roughly equal to the Earth's radius.
(2) Show that the radius for a circular orbit of a synchronous (24-h) Earth satellite is about 6.6 Earth radii.
(3) The distance to the Moon is about 60.3 Earth radii. From this, calculate the length of the sidereal month (the period of the Moon's orbital revolution).

SOLUTION:

4.5.4.1 Part (1)

The force on the satellite is $mr_e\omega^2$ where r_e is taken as the earth radius since this is low-lying orbit. Therefore

$$\frac{GM_em}{r_e^2} = mr_e\omega^2$$

But $v = r_e\omega$ where v is the satellite speed we want to find. Hence $\omega^2 = \frac{v^2}{r_e^2}$ and the above becomes

$$\begin{aligned}
\frac{GM_e}{r_e^2} &= r_e\frac{v^2}{r_e^2}\\
v &= \sqrt{\frac{GM_e}{r_e}}\\
&= \sqrt{\frac{\left(6.67408\times10^{-11}\right)\left(5.972\times10^{24}\right)}{6.371\times10^{6}}}\\
&= 7909.6 \text{ meter/sec}\\
&= 7.9 \text{ km/sec}
\end{aligned}$$

4.5.4.2 Part (2)

Let the radius of the satellite orbit be r. Using

$$\begin{aligned}
\frac{GM_e m}{r^2} &= mr\omega^2\\
r &= \left(\frac{GM_e}{\omega^2}\right)^{\frac{1}{3}}
\end{aligned}$$

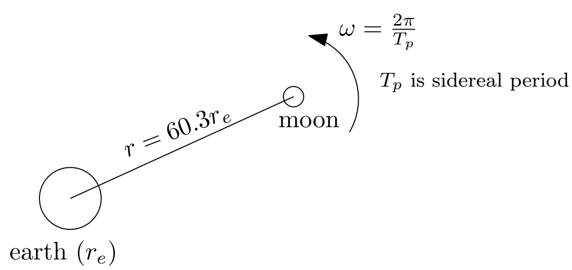
where $\omega = \frac{2\pi}{T_p}$ where T_p is the period of the satellite. But for synchronous satellite, this period is 24 hrs. Hence the above becomes

$$\begin{aligned}
r &= \left(\frac{GM_e}{\left(\frac{2\pi}{T_p}\right)^2}\right)^{\frac{1}{3}}\\
&= \left(\frac{\left(6.67408\times10^{-11}\right)\left(5.972\times10^{24}\right)}{\left(\frac{2\pi}{24(60)(60)}\right)^2}\right)^{\frac{1}{3}}\\
&= 4.224\times10^{7} \text{ meter}
\end{aligned}$$

But radius of earth is $r_e = 6.371\times10^6$ meters. Hence

$$\frac{r}{r_e} = \frac{4.224\times10^7}{6.371\times10^6} = 6.63$$

4.5.4.3 Part (3)



From

$$
\begin{aligned}
\frac{GM_e m}{r^2} &= mr\omega^2 \\
\frac{GM_e}{r^3} &= \omega^2 \\
\frac{GM_e}{r^3} &= \left(\frac{2\pi}{T_p}\right)^2
\end{aligned}
$$

We solve for T_p, hence

$$
\begin{aligned}
\frac{2\pi}{T_p} &= \sqrt{\frac{GM_e}{r^3}} \\
T_p &= \frac{2\pi}{\sqrt{\frac{GM_e}{r^3}}} = \frac{2\pi}{\sqrt{\frac{(6.67408\times10^{-11})(5.972\times10^{24})}{((60.3)(6.371\times10^6))^3}}} \\
&= 2.369\,8\times10^6 \text{ sec}
\end{aligned}
$$

Therefore, in days, the above becomes

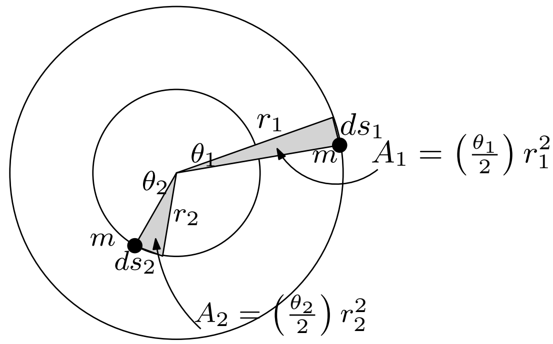
$$
\begin{aligned}
T_p &= \frac{2.369\,8\times10^6}{(24)(60)(60)} \\
&= 27.428 \text{ days}
\end{aligned}
$$

4.5.5 Problem 5

5. (15 points)
(1) A particle is subject to an attractive force $f(r)$, where r is the distance between the particle and the center of the force. Find $f(r)$ if all circular orbits are to have identical areal velocities.
(2) The orbit of a particle moving in a central field is a circle passing through the origin, $r = r_0\cos(\theta)$. Show that the force law is inverse-fifth power.

SOLUTION:

4.5.5.1 Part(1)



From the above diagram, where we have two particles of same mass m in two circular orbits. The area of each sector is given by

$$A = \frac{\theta}{2} r^2$$

The time rate of each sector area is

$$\frac{dA_1}{dt} = \frac{\dot{\theta}_1}{2} r_1^2 \tag{1}$$

Similarly

$$\frac{dA_2}{dt} = \frac{\dot{\theta}_2}{2} r_2^2 \tag{2}$$

Since we have a central force, then this force attracts each mass with a force given by $f = mr\dot{\theta}^2$. Therefore $f_{r_1} = mr_1\dot{\theta}_1^2$, Similarly $f_{r_2} = mr_2\dot{\theta}_2^2$. Substituting for $\dot{\theta}$ from these expressions back into (1) and (2) gives

$$\frac{dA_1}{dt} = \sqrt{\frac{f_1}{mr_1}} \frac{r_1^2}{2} \tag{1B}$$

Similarly

$$\frac{dA_2}{dt} = \sqrt{\frac{f_1}{mr_1}} \frac{r_1^2}{2} \tag{2B}$$

We are told the areal speeds are the same, therefore equating the above gives

$$\frac{dA_1}{dt} = \frac{dA_2}{dt}$$
$$\sqrt{\frac{f_1}{mr_1}}\frac{r_1^2}{2} = \sqrt{\frac{f_2}{mr_2}}\frac{r_1^2}{2}$$
$$\frac{f_1}{mr_1}\frac{r_1^4}{4} = \frac{f_2}{mr_2}\frac{r_2^2}{4}$$
$$f_1 r_1^3 = f_2 r_2^3$$

Hence

$$\frac{f_{r_1}}{f_{r_2}} = \frac{r_2^3}{r_1^3}$$

This says that, since we using the same mass, that the force $f(r)$ on a mass is inversely proportional to the cube of the mass distance from the center. To see this more clearly, let $r_1 = 1$ then

$$f_{r_2} = \frac{1}{r_2^3} f_{r_1}$$

So if we move the mass from $r_1 = 1$ to say 3 times as far to $r_2 = 3$, then the force on the same mass becomes $\frac{1}{27}$ smaller than it was.

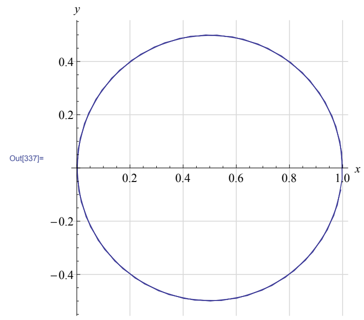
4.5.5.2 Part(2)

The orbit first is plotted as follows

```
Clear[r0, r]
r0 = 1;
r[angle_] := r0 Cos[angle]
xyData = Table[{r[a] Cos[a], r[a] Sin[a]}, {a, 0, 2 Pi, .1}];

ListLinePlot[xyData, GridLines -> Automatic,
 GridLinesStyle -> LightGray, AxesOrigin -> {0, 0},
 AxesLabel -> {x, y}, BaseStyle -> 14, PlotTheme -> "Classic",
 AspectRatio -> Automatic]
```

Which produces the following plot



Using 8.21 in textbook, page 293

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu r^2}{l^2}F(r) \tag{1}$$

Where μ is the reduces mass, l is the angular momentum and $F(r)$ is the force we are solving for. Since $r = r_0 \cos\theta$ then

$$\begin{aligned}
\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) &= \frac{d}{d\theta}\left(\frac{d}{d\theta}\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{d}{d\theta}\frac{1}{r_0\cos\theta}\right)\\
&= \frac{d}{d\theta}\left(\frac{(-1)(-\sin\theta)}{r_0\cos^2\theta}\right)\\
&= \frac{d}{d\theta}\left(\frac{\sin\theta}{r_0\cos^2\theta}\right)\\
&= \left(\frac{\cos\theta}{r_0\cos^2\theta}+\frac{2\sin^2\theta}{r_0\cos^3\theta}\right)\\
&= \left(\frac{1}{r_0\cos\theta}+\frac{2\sin^2\theta}{r_0\cos^3\theta}\right) \qquad (2)
\end{aligned}$$

But from $r = r_0\cos\theta$ we see that $\cos\theta = \frac{r}{r_0}$ and $\sin^2\theta = 1-\cos^2\theta = 1-\left(\frac{r}{r_0}\right)^2$, hence (2) becomes

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \left(\frac{1}{r_0\left(\frac{r}{r_0}\right)} + \frac{2\left(1-\left(\frac{r}{r_0}\right)^2\right)}{r_0\left(\frac{r}{r_0}\right)^3}\right)$$
$$= \left(\frac{1}{r} + \frac{2\left(1-\frac{r^2}{r_0^2}\right)}{\frac{r^3}{r_0^2}}\right)$$
$$= \left(\frac{1}{r} + \frac{2-\frac{2r^2}{r_0^2}}{\frac{r^3}{r_0^2}}\right)$$
$$= \left(\frac{1}{r} + \frac{2r_0^2 - 2r^2}{r^3}\right)$$
$$= \frac{r^2 + 2r_0^2 - 2r^2}{r^3}$$
$$= \frac{2r_0^2 - r^2}{r^3}$$

Therefore (1) becomes

$$\frac{2r_0^2 - r^2}{r^3} + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)$$
$$\frac{2r_0^2 - r^2 + r^2}{r^3} = -\frac{\mu r^2}{l^2}F(r)$$

Solving for $F(r)$

$$F(r) = -\frac{2l^2 r_0^2}{\mu r^5}$$
$$= -\left(\frac{2l^2 r_0^2}{\mu}\right)\frac{1}{r^5}$$

The above shows that the force is an inverse fifth power.

4.5.6 HW 5 key solution

1

Mechanics

Physics 311 - Fall 2015

Homework Set 5 - Solutions

Problem 1

for the system of the two masses,

$$-kx = (m+M)\ddot{x}$$

$$\Rightarrow \ddot{x} = -\frac{kx}{m+M}$$

position and acceleration for m are the same:

$$x_m = x$$

$$\ddot{x}_m = \ddot{x} = -\frac{kx}{m+M}$$

at $x = -d$, the total force on m (if m is just leaving the bottom of the box) is mg

$$\Rightarrow g = -\frac{k}{m+M}(-d) = \frac{kd}{m+M}$$

$$\Leftrightarrow \boxed{d = \frac{(m+M)g}{k}}$$

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Problem 2

(1) $f(t) = \sum_n C_n e^{in\omega t}$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega t}\, dt \qquad T = \frac{2\pi}{\omega}$$

so

$$C_n = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\, e^{-in\omega t}\, dt$$

$$= \frac{\omega}{2\pi}\left\{ \int_{-\pi/\omega}^{0} (-1)\, e^{-in\omega t}\, dt + \int_{0}^{\pi/\omega} e^{-in\omega t}\, dt \right\}$$

$$= \frac{\omega}{2\pi}\left\{ \frac{1}{in\omega} e^{-in\omega t}\Big|_{-\pi/\omega}^{0} - \frac{1}{in\omega} e^{-in\omega t}\Big|_{0}^{\pi/\omega} \right\}$$

$$= \frac{1}{2\pi i n}\left\{ 1 - e^{in\pi} - e^{-in\pi} + 1 \right\}$$

for n even, $e^{in\pi} = e^{-in\pi} = 1$, so $C_n = 0$

for n odd, $e^{in\pi} = e^{-in\pi} = -1$, so

$$C_n = \frac{2}{\pi i n} \qquad n = \pm 1, \pm 3$$

$$\Rightarrow \quad f(t) = \sum_n \frac{2}{\pi i n} e^{in\omega t} \qquad n = \pm 1, \pm 3, \ldots$$

$$= \sum_n \frac{4}{n\pi} \frac{1}{2i}\left(e^{in\omega t} - e^{-in\omega t}\right) \qquad n = 1, 3, 5, \ldots$$

$$= \sum_n \frac{4}{\pi} \frac{1}{n} \sin(n\omega t)$$

3

so

$$f(t) = \frac{4}{\pi}\left\{ \sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \dots \right\}$$

(2) in steady state,

$$x(t) = \sum_n A_n e^{i(n\omega t - \phi_n)}$$

where $A_n = \dfrac{F_n}{m} \dfrac{1}{\left[(\omega_0^2 - n^2\omega^2)^2 + 4\lambda^2 n^2 \omega^2 \right]^{1/2}}$

from part (1), $F_n = \dfrac{4F_0}{n\pi}$ $\quad n = 1, 3, 5, \dots$

$\omega_0 = 3\omega$

$$Q = 100 \simeq \frac{\omega_0}{2\lambda} \Rightarrow \lambda = \frac{\omega_0}{200}, \text{ so } \lambda^2 = \frac{9\omega^2}{40{,}000}$$

(i)

$$A_1 = \frac{4F_0}{m\pi} \frac{1}{\left[(9\omega^2 - \omega^2)^2 + 4\,\frac{9\omega^4}{40000} \right]^{1/2}}$$

$$\simeq \frac{4F_0}{m\pi}\frac{1}{8\omega^2} = \frac{F_0}{2m\pi\omega^2}$$

(ii)

$$A_3 = \frac{4F_0}{3m\pi} \frac{1}{\left[(9\omega^2 - 9\omega^2)^2 + 4\,\frac{81\omega^4}{40000} \right]^{1/2}}$$

$$= \frac{4F_0}{3m\pi}\frac{1}{18\omega^2/200} = \frac{400F_0}{27m\pi\omega^2}$$

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(iii)
$$A_5 = \frac{4F_0}{5m\pi} \frac{1}{\left[(9\omega^2 - 25\omega^2)^2 + 4\,\frac{9 \cdot 25\,\omega^4}{40\,000} \right]^{1/2}}$$

$$\approx \frac{4F_0}{5m\pi\omega^2}\,\frac{1}{16} = \frac{1}{20}\,\frac{F_0}{m\pi\omega^2}$$

$$\Rightarrow \quad A_1 : A_3 : A_5 = \underline{\underline{1 : \underset{\substack{\uparrow \\ \text{resonance}}}{29.6} : 0.1}}$$

Problem 3

$$F(r) = \underset{\substack{\uparrow \\ \text{Sun}}}{F_s} + \underset{\substack{\uparrow \\ \text{dust}}}{F_d}$$

$$F_s = -\frac{GMm}{r^2}$$

M = mass of Sun
m = mass of planet
M_d = mass of dust

$$F_d = -\frac{GM_d m}{r^2}$$

the net effect of the dust outside the planet's radius is zero;
the effect of the dust inside the planet's radius is that of a mass M_d at the center (= Sun's position)

$$M_d = \frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \boxed{F(r) = -\frac{GMm}{r^2} - \frac{4}{3}\pi\rho m G r}$$

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Problem 4

for circular motion $\quad r = \frac{\ell^2}{\alpha m} \qquad \ell = m r^2 \dot{\theta} = m r v$

$\alpha = G M_E m_s$

$M_E \gg m_s \Rightarrow m \simeq m_s$

So

$$r = \frac{r^2 m_s^2 v^2}{G m_s M_E m_s} \quad \Rightarrow \quad v^2 = \frac{G M_E}{r}$$

(1) $\quad G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

$M_E = 5.97 \cdot 10^{24}\, kg \qquad R_E = 6.4 \cdot 10^6\, m$

$$\Rightarrow \quad v = \sqrt{\frac{G M_E}{R_E}} = \sqrt{\frac{6.67 \cdot 10^{-11}\, m^3/kg s^2 \; 5.97 \cdot 10^{24}\, kg}{6.4 \cdot 10^6\, m}}$$

$$= \underline{\underline{7.9\, km/s}}$$

(2) $\quad T = \frac{2\pi r}{v} \qquad v = \sqrt{\frac{GM}{r}}$

$$\text{so} \quad T = \frac{2\pi r^{3/2}}{\sqrt{G M_E}} \quad \Rightarrow \quad r = \left\{ \frac{T^2 G M_E}{4\pi^2} \right\}^{1/3}$$

$$r = \left\{ \frac{(24\, h \cdot 3600\, s/h)^2 \; 6.67 \cdot 10^{-11}\, m^3/kg s^2 \; 5.97 \cdot 10^{24}\, kg}{4\pi^2} \right\}^{1/3}$$

$$= 42.2 \cdot 10^3\, km \simeq \underline{\underline{6.6\, R_E}}$$

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(3)
$$\tau = \frac{2\pi\, r^{3/2}}{\sqrt{GM}}$$

$$= \frac{2\pi\,(60 \cdot 6.4 \cdot 10^{6}\,m)^{3/2}}{\sqrt{6.67 \cdot 10^{-11}\,m^3/kg\,s^2 \;\; 5.97 \cdot 10^{24}\,kg}}$$

$$= 2\,369 \cdot 10^{3}\,s \;\simeq\; \underline{\underline{27.4\,d}}$$

<u>Problem 5</u>

(1) for a circular orbit , $f(r) = -mr\dot{\theta}^2$

($f(r)$ has to provide the centripetal force)

areal velocity $A = \frac{1}{2} r\,(r\theta) = \frac{1}{2} r^2 \theta$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \text{const.}$$

$$\Rightarrow \quad f(r) = -mr\dot{\theta}^2 = -mr\left(\frac{2\dot{A}}{r^2}\right)^2$$

$$\Leftrightarrow \quad \boxed{f(r) = -\frac{4m\dot{A}^2}{r^3}} \qquad \dot{A} = \text{const.}$$

↳ an inverse r-cube law gives identical areal velocities

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(2) $r = r_0 \cos\theta$

use $\dfrac{d^2}{d\theta^2}\left(\dfrac{1}{r}\right) + \dfrac{1}{r} = -\dfrac{m}{\ell^2} r^2 F(r)$

$$\Rightarrow \quad F(r) = -\frac{\ell^2}{m}\frac{1}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{\ell^2}{m}\frac{1}{r^3}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{r_0 \cos\theta}\right) = \frac{\sin\theta}{r_0\cos^2\theta}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{r_0\cos\theta} + \frac{2\sin^2\theta}{r_0\cos^3\theta}$$

$$= \frac{1}{r_0\cos\theta}\left(1 + \frac{2-2\cos^2\theta}{\cos^2\theta}\right)$$

$$= \frac{1}{r_0\cos\theta}\left(\frac{2}{\cos^2\theta} - 1\right)$$

$$= \frac{1}{r}\left(\frac{2r_0^2}{r^2} - 1\right)$$

so $F(r) = -\dfrac{\ell^2}{m}\dfrac{1}{r^2}\;\dfrac{1}{r}\left(\dfrac{2r_0^2}{r^2} - 1\right) - \dfrac{\ell^2}{m}\dfrac{1}{r^3}$

$$\boxed{F(r) = -\frac{2r_0^2\ell^2}{mr^5}}$$

4.6 HW 6

4.6.1 Problem 1

1. (5 points)
An Earth satellite has a speed of 28,070 km/h when it is at its perigee of 220 km above Earth's surface. Find the apogee distance, its speed at apogee, and its period of revolution.

SOLUTION:

From the vis-viva relation

$$v_{perigee} = \sqrt{\frac{\alpha}{m}\left(\frac{2}{r_p} - \frac{1}{a}\right)} \tag{1}$$

Where m is the reduced mass and $\alpha = GM_{earth}m_{satt}$, which reduces to GM_{earth} and known constant called the Standard gravitational parameter which for earth is (From table)

$$\frac{\alpha}{m} = 398600 \text{ km}^3/s^2$$

And

$$\begin{aligned} r_p &= 220 + 6378 \\ &= 6598 \text{ km} \end{aligned}$$

Where 6378 is the equatorial radius of earth. And $v_{perigee} = 28070$ km/h. Therefore, we use (1) to solve for a, the length of the semimajor axes of the elliptical orbit of the satellite around the earth. From (1), by squaring both sides

$$\begin{aligned} v_p^2 &= \frac{\alpha}{m}\left(\frac{2}{r_p} - \frac{1}{a}\right) \\ \left(\frac{28070}{60 \times 60}\right)^2 &= 398600\left(\frac{2}{220 + 6378} - \frac{1}{a}\right) \end{aligned}$$

Solving for a gives

$$a = 6640 \text{ km}$$

Hence the apogee distance is

$$2a = 13280 \text{ km}$$

We can also find

$$\begin{aligned} r_a &= 2a - r_p \\ &= 13280 - 6598 \\ &= 6682 \text{ km} \end{aligned}$$

When the satellite is at the apogee, it will be above the earth at height of

$$\begin{aligned}h_a &= r_a - r_{earth}\\ &= 6682 - 6378\\ &= 304 \text{ km}\end{aligned}$$

The period T is given by

$$\begin{aligned}T &= 2\pi\sqrt{\frac{a^3}{\frac{\alpha}{m}}}\\ &= 2\pi\sqrt{\frac{6640^3}{398600}}\\ &= 5385 \text{ sec}\\ &= \frac{5385}{60\times 60} = 1.496 \text{ hr}\end{aligned}$$

4.6.2 Problem 2

> 2. (5 points)
> A spacecraft is in circular orbit 200 km above Earth's surface. What minimum velocity kick must be applied to let the spacecraft escape from Earth's influence? What is the spacecraft's escape trajectory with respect to Earth?

SOLUTION:

The total energy is

$$E = \frac{1}{2}m\dot{r}^2 + U_{effective}$$

The escape velocity is when $U_{effective} = 0$, therefore

$$0 = -U + \frac{l^2}{2mr^2}$$

But angular momentum $l = mrv$ and $U = \frac{GM_e m}{r}$, hence the above becomes

$$\begin{aligned}0 &= -\frac{GM_e m}{r} + \frac{m^2 r^2 v^2}{2mr^2}\\ &= -\frac{GM_e m}{r} + \frac{mv^2}{2}\\ &= -\frac{GM_e}{r} + \frac{v^2}{2} \qquad (1)\end{aligned}$$

Now we are given that the satellite was at $r = 200 + 6378 = 6578$ km (this is r_p for the new orbit as well). Using $GM_e = 398600 \text{ km}^3/s^2$ from tables then we solve now for v in (1), which will be the new velocity. Hence

$$0 = -\frac{398600}{6578} + \frac{v^2}{2}$$
$$v = 11.009 \text{ km/sec}$$

Before this, the spacecraft was in circular orbit. So its speed was

$$\begin{aligned} v_c &= \sqrt{\frac{\alpha}{m}\frac{1}{r}} \\ &= \sqrt{\frac{398600}{6578}} \\ &= 7.784 \text{ km/sec} \end{aligned}$$

The difference is the minimum speed kick needed, which is

$$11.009 - 7.784 = 3.225 \text{ km/sec}$$

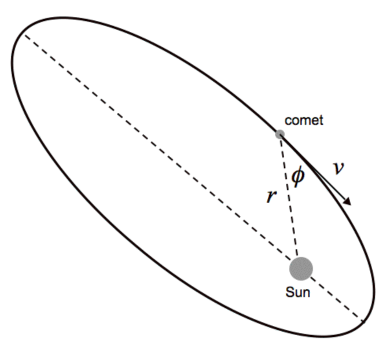
This orbit is ***parabolic*** since $U_{effective} = 0$ as seen on the $U_{effective}$ vs. r graph. parabolic is the first orbit beyond elliptic that do not contain turn points. The next orbit is hyperbolic.

4.6.3 Problem 3

3. (15 points)
A comet is observed to have a speed v when it is at a distance r from the Sun. Its direction of motion makes an angle ϕ with the radius vector from the Sun.

(1) Find the eccentricity of the comet's orbit.

(2) If the velocity of the comet is expressed as q times the Earth's velocity and its distance to the Sun as d astronomical units, show that the orbit of the comet is hyperbolic, parabolic, or elliptic, depending on whether the quantity q^2d is greater than, equal to, or less than 2, respectively.



SOLUTION:

4.6.3.1 Part (1)

Eccentricity is defined as (for all conic sections)

$$e = \sqrt{1 + \frac{2El^2}{m\alpha^2}} \tag{1}$$

Where $\alpha = GM_{sun}m$ and l is the angular momentum

$$\begin{aligned} l &= m\left|\boldsymbol{r}\times\boldsymbol{v}\right| \\ &= mrv\sin\phi \end{aligned}$$

Therefore (1) becomes

$$e = \sqrt{1 + \frac{2E\left(rv\sin\phi\right)^2}{m\left(GM_{sun}\right)^2}}$$

The energy of the comet is given by $E = \frac{1}{2}mv^2 - \frac{GM_{sun}m}{r}$, then the above becomes

$$\begin{aligned} e &= \sqrt{1 + \frac{2\left(\frac{1}{2}mv^2 - \frac{GM_{sun}m}{r}\right)\left(rv\sin\phi\right)^2}{m\left(GM_{sun}\right)^2}} \\ &= \sqrt{1 + \left(\frac{2\left(\frac{1}{2}mv^2 - \frac{GM_{sun}m}{r}\right)}{m}\right)\left(\frac{rv\sin\phi}{GM_{sun}}\right)^2} \\ &= \sqrt{1 + \left(v^2 - \frac{2GM_{sun}}{r}\right)\left(\frac{rv\sin\phi}{GM_{sun}}\right)^2} \end{aligned}$$

4.6.3.2 Part (2)

Let $v = qv_e$ where v_e is earth velocity around the sun and let $r = dr_e$ where r_e is the astronomical unit (the distance between the earth and sun) then result of part (1) becomes

$$e = \sqrt{1 + \left(\left(qv_e\right)^2 - \frac{2GM_{sun}}{dr_e}\right)\left(\frac{dr_e qv_e\sin\phi}{GM_{sun}}\right)^2} \tag{2}$$

Looking at the earth/sun system, we know that

$$\begin{aligned} \frac{GM_{sun}m_{earth}}{r_e^2} &= \frac{m_{earth}v_e^2}{r_e} \\ \frac{GM_{sun}}{r_e} &= v_e^2 \\ GM_{sun} &= r_e v_e^2 \end{aligned}$$

Replacing GM_{sun} in (2) by the above result gives

$$\begin{aligned}
e &= \sqrt{1+\left(\left(qv_e\right)^2-\frac{2r_ev_e^2}{dr_e}\right)\left(\frac{dr_eqv_e\sin\phi}{r_ev_e^2}\right)^2} \\
&= \sqrt{1+\left(\left(qv_e\right)^2-\frac{2v_e^2}{d}\right)\left(\frac{dq\sin\phi}{v_e}\right)^2} \\
&= \sqrt{1+\left(q^2-\frac{2}{d}\right)\left(dq\sin\phi\right)^2} \\
&= \sqrt{1+\left(\frac{q^2d-2}{d}\right)\left(dq\sin\phi\right)^2}
\end{aligned}$$

We are now ready to answer the final part. If $q^2d=2$ then $e=1$ which means it is parabolic. If $q^2d>2$ then $\left(\frac{q^2d-2}{d}\right)$ is positive and the expression inside $\sqrt{.}$ is larger than one, and hence $e>1$, which means the orbit is hyperbolic. Finally, if $q^2d<2$ then $\left(\frac{q^2d-2}{d}\right)$ is negative, and the expression inside $\sqrt{.}$ is less than one, which means $e<1$ and hence the orbit is elliptic.

4.6.4 Problem 4

4. (10 points)
If the minimum and maximum velocities of a moon rotating around a planet are $v_{min}=v-v_0$ and $v_{max}=v+v_0$, show that the eccentricity is given by

$$e=\frac{v_0}{v}\quad .$$

SOLUTION:

The angular momentum l is constant. At perigee, where the speed is maximum, we have

$$l_p = mv_{\max}r_p$$

And at apogee, where the speed is minimum, we have

$$l_a = mv_{\min}r_a$$

Since l is constant, then

$$\begin{aligned}
mv_{\max}r_p &= mv_{\min}r_a \\
v_{\max}r_p &= v_{\min}r_a \qquad (1)
\end{aligned}$$

But

$$\begin{aligned}
r_a &= a\left(1+e\right) \\
r_p &= a\left(1-e\right)
\end{aligned}$$

Hence (1) becomes

$$
\begin{aligned}
v_{\max} a (1-e) &= v_{\min} a (1+e) \\
v_{\max} (1-e) &= v_{\min} (1+e) \\
v_{\max} - e v_{\max} &= v_{\min} + e v_{\min} \\
v_{\max} - v_{\min} &= e (v_{\min} + v_{\max}) \\
e &= \frac{v_{\max} - v_{\min}}{v_{\min} + v_{\max}}
\end{aligned}
$$

Replacing $v_{\max} = v + v_0$ and $v_{\min} = v - v_0$ gives

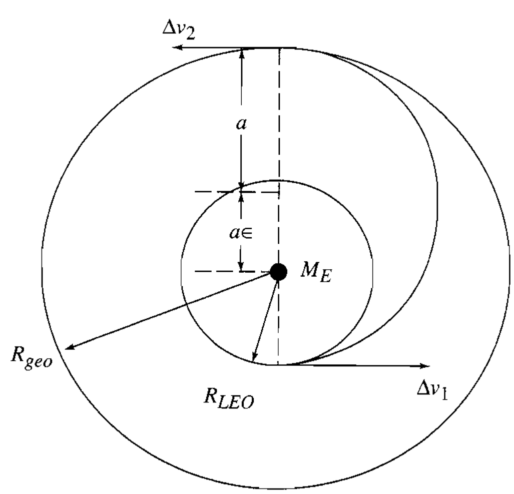
$$
\begin{aligned}
e &= \frac{(v+v_0) - (v-v_0)}{(v+v_0) + (v-v_0)} \\
&= \frac{2v_0}{2v} \\
&= \frac{v_0}{v}
\end{aligned}
$$

4.6.5 Problem 5

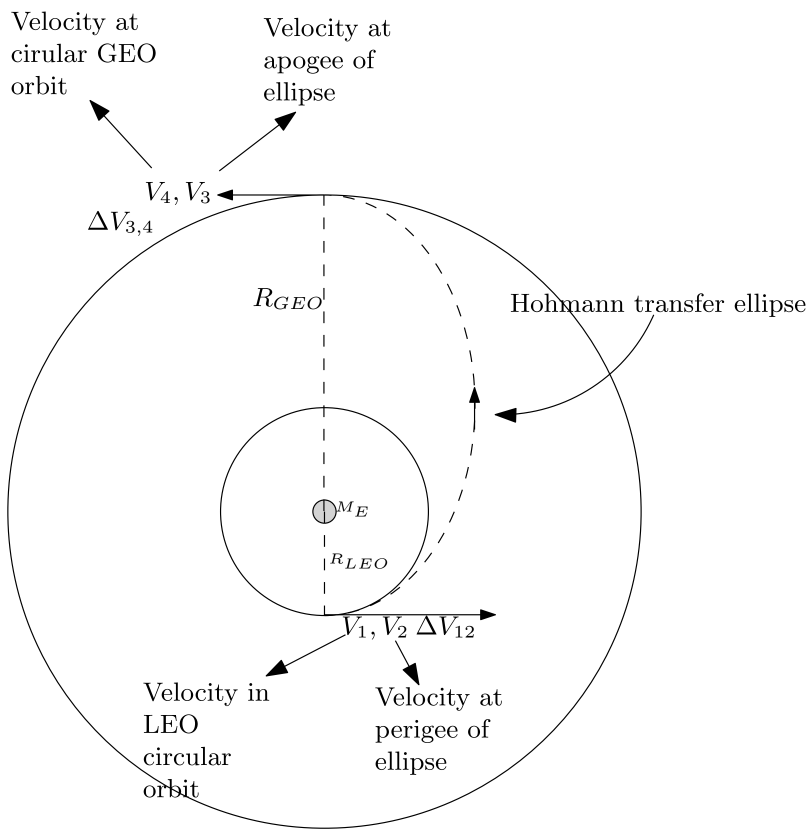
5. (15 points)
When a spacecraft is placed into geosynchronous orbit, it is first launched, along with a propulsion stage, into a near circular low Earth orbit (LEO) using a booster rocket. Then the propulsion stage is fired and the spacecraft is transferred to an elliptical "transfer" orbit designed to take it to geosynchronous altitude at orbital apogee. At apogee, the propulsion stage is fired again to take it out of the elliptical orbit back into a circular (now geosynchronous) orbit.

(1) Calculate the required velocity boost Δv_1 to move the satellite from its circular low Earth orbit into the elliptical transfer orbit.

(2) Calculate the required velocity boost Δv_2 to move the satellite from the elliptical transfer orbit into the geosynchronous circular orbit.



SOLUTION:



4.6.5.1 Part (1)

In this calculation, the standard symbol μ is used for GM_{earth} which is the Standard gravitational parameter (in class, we used $\frac{\alpha}{m}$ for this same parameter). For earth

$$\mu = 398600 \text{ km}^3/s^2$$

The first step is to find a for the transfer ellipse. This is given by

$$a = \frac{R_{LEO} + R_{GEO}}{2}$$

Next, we first find V_1, which is velocity in the LEO circular orbit just before initial kick to V_2. Since this is circular, the speed is given by

$$V_1 = \sqrt{\frac{\mu}{R_{LEO}}}$$

Next step is to find V_2, which is the speed at the perigee of the ellipse (the transfer orbit). This is given by the standard vis-viva relation

$$V_2 = \sqrt{\mu \left(\frac{2}{R_{LEO}} - \frac{1}{a} \right)} \tag{1}$$

Where $R_{LEO} = r_{perigee}$ for the ellipse. Now that we found V_2 and V_1, then

$$\begin{aligned}\Delta V_{12} &= V_2 - V_1\\ &= \sqrt{\mu\left(\frac{2}{R_{LEO}} - \frac{1}{a}\right)} - \sqrt{\frac{\mu}{R_{LEO}}}\end{aligned}$$

4.6.5.2 Part (2)

When at the apogee of the transfer ellipse, the speed is given by

$$V_3 = \sqrt{\mu\left(\frac{2}{R_{GEO}} - \frac{1}{a}\right)}$$

We now want to be of GEO circular orbit, hence

$$V_4 = \sqrt{\frac{\mu}{R_{GEO}}}$$

And therefore, the speed boost is

$$\begin{aligned}\Delta V_{34} &= V_4 - V_3\\ &= \sqrt{\frac{\mu}{R_{GEO}}} - \sqrt{\mu\left(\frac{2}{R_{GEO}} - \frac{1}{a}\right)}\end{aligned}$$

4.6.6 HW 6 key solution

1

Mechanics

Physics 311 - Fall 2015

Homework Set 6 - Solutions

Problem 1

$r_a = 2a - r_p.$

r_a = distance at apogee

r_p = " " perigee

to get the semimajor axis a, use $E = -\frac{GMm}{2a}$

M = Earth's mass

m = satellite mass

$$\Rightarrow \quad E = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = -\frac{GMm}{2a}$$

$$\Leftrightarrow \quad -\frac{GM}{a} = v_p^2 - \frac{2GM}{r_p}$$

$$\Leftrightarrow \quad \frac{1}{a} = \frac{2}{r_p} - \frac{v_p^2}{GM}$$

$$\Leftrightarrow \quad a = \frac{GMr_p}{2GM - v_p^2 r_p}$$

so

$$r_a = \frac{2GMr_p}{2GM - v_p^2 r_p} - r_p$$

$$= r_p \left(\frac{2GM - 2GM + v_p^2 r_p}{2GM - v_p^2 r_p} \right)$$

$$= r_p \left(\frac{v_p^2 r_p}{2GM - v_p^2 r_p} \right)$$

2

now use $M = 5.976 \cdot 10^{24}$ kg
$G = 6.67 \cdot 10^{-11}$ Nm^{2}kg^{-2}

$$r_p = (6.371 + 0.220) \cdot 10^6 \text{ m} = 6.591 \cdot 10^6 \text{ m}$$

$$v_p = \frac{28070 \text{ km}}{3600 \text{ s}} = 7.797 \cdot 10^3 \text{ m/s}$$

$\Rightarrow \quad r_a = 1.011 \, r_p = 6.664 \cdot 10^6$ m

or $\simeq \underline{\underline{293 \text{ km}}}$ above the Earth's surface

Speed at apogee : use conservation of angular momentum,

$$m r_a v_a = m r_p v_p$$

$$\Rightarrow \quad v_a = \frac{r_p}{r_a} v_p = \underline{\underline{27765 \text{ km/h}}}$$

period from Kepler 3

$$T^2 = \frac{4\pi^2}{GM} a^3 = \frac{4\pi^2}{GM} \left(\frac{GMr_p}{2GM - v_p^2 r_p} \right)^3$$

so

$$T = 5.368 \cdot 10^3 \text{ s} = \underline{\underline{1.49 \text{ h}}}$$

3

Problem 2

to just escape from Earth, the velocity kick must be such that the final total energy is zero:

$$0 = \frac{1}{2} m V_2^2 - \frac{GMm}{r} = 0$$

V_2 = velocity after kick
M = Earth's mass
m = satellite mass

$$\Rightarrow \quad V_2 = \sqrt{\frac{2GM}{r}}$$

$r = (6.371 + 0.2)\,10^6\,m$
$= 6.571 \cdot 10^6\,m$
$M = 5.976 \cdot 10^{24}\,kg$

$$= \sqrt{\frac{2 \cdot 6.67 \cdot 10^{-11}\,Nm^2 kg^{-2}\; 5.976 \cdot 10^{24}\,kg}{6.571 \cdot 10^6\,m}}$$

$$= 11.01 \cdot 10^3\,\frac{m}{s}$$

for a circular orbit, $V_1 = \sqrt{\frac{GM}{r}}$

$$= 7.79 \cdot 10^3\,m/s$$

$\therefore$ to escape from Earth, a velocity boost of $\underline{\underline{3.2\,km/s}}$ must be applied

Since $E=0$, the escape trajectory is a parabola

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Problem 3

(1) eccentricity $e = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}}$

so we need ℓ and E

$$\ell = m\,|\vec{r} \times \vec{v}| = m r v \sin\phi$$

$$E = \frac{1}{2} m v^2 - G\frac{mM}{r}$$

M = mass of Sun
m = mass of satellite
$\alpha = GMm$

So $\frac{2E}{m} = v^2 - \frac{2GM}{r}$

and $\frac{\ell^2}{\alpha^2} = \left(\frac{m r v \sin\phi}{GmM}\right)^2 = \left(\frac{r v \sin\phi}{GM}\right)^2$

$$\Rightarrow \quad \boxed{e = \sqrt{1 + \left(v^2 - \frac{2GM}{r}\right)\left(\frac{r v \sin\phi}{GM}\right)^2}}$$

(2) use $q = \frac{v}{V_e}$ and $d = \frac{r}{a_e}$

assuming a circular orbit for the Earth,

$$\frac{p}{a_e} = 1 \quad \Rightarrow \quad p = a_e = \frac{\ell^2}{m_e\alpha} = \frac{(m_e a_e V_e)^2}{m_e G m_e M}$$

$$\Leftrightarrow \quad a_e = \frac{a_e^2 V_e^2}{GM}$$

$$\Leftrightarrow \quad a_e V_e^2 = GM$$

5

$$\Rightarrow \quad e^2 = 1 + \left(q^2 v_e^2 - \frac{2GM}{a_e d}\right)\left(\frac{a_e d\, v_e q \sin\phi}{GM}\right)^2$$

$$= 1 + \left(q^2 v_e^2 - \frac{2 v_e^2}{d}\right)\left(\frac{d q \sin\phi}{v_e}\right)^2$$

$$= 1 + \left(q^2 - \frac{2}{d}\right)\left(d q \sin\phi\right)^2$$

so $$e = \sqrt{1 + \left(q^2 - \frac{2}{d}\right)\left(d q \sin\phi\right)^2}$$

Orbit is	hyperbolic	for	$e > 1$
	parabolic		$e = 1$
	elliptic		$e < 1$

$e > 1$ for $q^2 - \frac{2}{d} > 0$ or $q^2 d > 2$

$e = 1$ for $q^2 - \frac{2}{d} = 0$ or $q^2 d = 2$

$e < 1$ for $q^2 - \frac{2}{d} < 0$ or $q^2 d < 2$ ∎

6

Problem 4

use conservation of momentum

$$(V-V_0)\, m\, r_{max} = (V+V_0)\, m\, r_{min}$$

$$\Rightarrow \quad \frac{r_{max}}{r_{min}} = \frac{V+V_0}{V-V_0}$$

also $\quad a = \dfrac{r_{min}+r_{max}}{2} \quad$ and $\quad r_{max} = a\,(1+e)$

$$\Rightarrow \quad 1+e = \frac{r_{max}}{a} = \frac{2\, r_{max}}{r_{min}+r_{max}}$$

so

$$e = \frac{2\, r_{max}}{\frac{V-V_0}{V+V_0} r_{max} + r_{max}} - 1$$

$$= \frac{2}{\frac{V-V_0+V+V_0}{V+V_0}} - 1$$

$$= \frac{V+V_0}{V} - 1 = \underline{\underline{\frac{V_0}{V}}}$$

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Problem 5

(1) Since R_{LEO} and R_{geo} are the perigee and apogee distances of the elliptical transfer orbit,

$$R_{LEO} = a(1-e)$$
$$R_{geo} = a(1+e)$$

$$R_{LEO} + R_{geo} = 2a$$

We need (i) the velocity at perigee for the elliptical transfer orbit and (ii) the velocity of the satellite in circular LEO

(i) $$E = -\frac{GMm}{2a} = \frac{1}{2} m v_p^2 - \frac{GMm}{a(1-e)}$$

$$\Rightarrow \quad v_p^2 = \frac{GM}{a}\left(\frac{2}{1-e} - 1\right) \qquad \begin{array}{l} M = \text{Earth's mass} \\ m = \text{satellite mass} \end{array}$$

$$= \frac{GM}{a} \frac{1+e}{1-e}$$

$$= \frac{2GM}{R_{LEO} + R_{geo}} \frac{R_{geo}}{R_{LEO}}$$

(ii) for the circular LEO, $\frac{p}{R_{LEO}} = 1$

$$\Rightarrow \quad p = R_{LEO} = \frac{\ell^2}{m\alpha} = \frac{m^2 R_{LEO}^2 V_{LEO}^2}{m\, GmM}$$

$$\Leftrightarrow \quad V_{LEO}^2 = \frac{GM}{R_{LEO}}$$

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so $\Delta V_1 = V_p - V_{LEO}$

$$= \sqrt{\frac{2GM}{R_{LEO}+R_{geo}} \frac{R_{geo}}{R_{LEO}}} - \sqrt{\frac{GM}{R_{LEO}}}$$

$$\boxed{\Delta V_1 = \sqrt{\frac{GM}{R_{LEO}}} \left\{ \sqrt{\frac{2R_{geo}}{R_{LEO}+R_{geo}}} - 1 \right\}}$$

(2)
$$\epsilon = -\frac{GMm}{2a} = \frac{1}{2} m V_a^2 - \frac{GMm}{a(1+e)}$$

$$\Rightarrow \quad V_a^2 = \frac{GM}{a}\left(\frac{2}{1+e} - 1\right)$$

$$= \frac{GM}{a} \frac{1-e}{1+e}$$

$$= \frac{2GM}{R_{LEO}+R_{geo}} \frac{R_{LEO}}{R_{geo}}$$

for circle, $\quad V_{geo}^2 = \dfrac{GM}{R_{geo}}$

$$\boxed{\Delta V_2 = V_{geo} - V_a = \sqrt{\frac{GM}{R_{geo}}} \left\{ 1 - \sqrt{\frac{2R_{LEO}}{R_{LEO}+R_{geo}}} \right\}}$$

4.7 HW 7

4.7.1 Problem 1

1. (10 points)
If a problem involves forces that cannot be derived from a potential (for example frictional forces), Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \quad ,$$

where the Q_i are the generalized forces not derivable from a potential. The Q_i are defined through

$$Q_i = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} \quad .$$

Use this formalism for the following example.

A particle of mass m moves in a plane under the influence of a central force of potential $U(r)$ and also of a linear viscous drag $-mk(d\vec{r}/dt)$. Set up Lagrange's equations of motion and show that the angular momentum decays exponentially.

SOLUTION:

Using polar coordinates. The position vector of the particle is

$$\vec{r} = r\hat{r} + r\theta\hat{\theta} \tag{1}$$

We now find the Lagrangian

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ U &= V(r) \\ L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \end{aligned}$$

Since we are asked about the angular momentum part, we will just find the equation of motion for the θ generalized coordinates.

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \end{aligned}$$

Hence the EQM is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = Q_\theta$$

Where Q_θ is the generalized force corresponding to generalized coordinate θ. From (1)

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta}$$

Hence

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\end{aligned}$$

Therefore, the drag force can be written as

$$\begin{aligned}\vec{F} &= -mk\frac{d\vec{r}}{dt}\\ &= -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \qquad (2)\end{aligned}$$

Applying the definition of $Q_\theta = \vec{F}\cdot\frac{\partial\vec{r}}{\partial\theta}$ gives

$$\begin{aligned}Q_\theta &= -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)\cdot\frac{\partial}{\partial\theta}\left(r\hat{r} + r\theta\hat{\theta}\right)\\ &= -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)\cdot\left(r\hat{\theta}\right)\\ &= -mkr^2\dot{\theta} \qquad (3)\end{aligned}$$

Now that we found Q_θ, the EQM is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mkr^2\dot{\theta}$$

We notice the same term on both sides (but for a constant k). The above is the same as

$$\frac{d}{dt}(Z) = -kZ$$

The solution must be exponential $Z = e^{-kt} + C$ where C is some constant. This means

$$mr^2\dot{\theta} = e^{-kt} + C$$

But $mr^2\dot{\theta}$ is the angular momentum. Hence, for positive k, the angular momentum decays exponentially with time.

4.7.2 Problem 2

2. (10 points)
In the lecture, we derived a formula for the percentage increase in speed necessary to transfer a spacecraft from low Earth orbit of radius r_0 to an elliptical orbit with the Moon at the apogee at distance r_1.

(1) Find the fractional change in the apogee $\delta r_1/r_1$ as a function of a small fractional change in the ratio of required perigee speed v_0 to circular orbit speed v_c, $\delta(v_0/v_c)/(v_0/v_c)$.

(2) If the speed ratio is 1% too great, by how much would the spacecraft miss the Moon?

SOLUTION:

4.7.2.1 Part (1)

From class notes, we found

$$\frac{v_o}{v_c} = \sqrt{\frac{2r_1}{r_1 + r_o}} = \sqrt{\frac{2}{1 + \frac{r_o}{r_1}}}$$

Where v_c is the velocity in the circular orbit just before speed boost, and v_o is the speed at the perigee of the ellipse just after the speed boost, and r_0 is the perigee distance and r_1 is the apogee distance. We need to find $\frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)}$. To make the calculation easier, let $\frac{v_o}{v_c} = z$. Then we have

$$z = \left(\frac{2}{1 + \frac{r_o}{r_1}}\right)^{\frac{1}{2}}$$

Hence

$$\frac{\delta z}{\delta r_1} = \frac{1}{2} \frac{1}{\left(\frac{2}{1+\frac{r_o}{r_1}}\right)^{\frac{1}{2}}} \frac{\delta}{\delta r_1} \left(\frac{2}{1 + \frac{r_o}{r_1}}\right)$$

But $\left(\frac{2}{1+\frac{r_o}{r_1}}\right)^{\frac{1}{2}} = z$ so the above becomes

$$\begin{aligned}
\frac{\delta z}{\delta r_1} &= \frac{1}{2}\frac{1}{z}\frac{\delta}{\delta r_1}\left(\frac{2}{1 + \frac{r_o}{r_1}}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\frac{\delta}{\delta r_1}\left(1 + \frac{r_o}{r_1}\right)^{-1}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\,(-1)\left(1 + \frac{r_o}{r_1}\right)^{-2} \frac{\delta}{\delta r_1}\left(\frac{r_o}{r_1}\right)\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\,(-1)\left(1 + \frac{r_o}{r_1}\right)^{-2} (-r_o)\, r_1^{-2}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(\frac{2}{\left(1 + \frac{r_o}{r_1}\right)^2} \frac{r_o}{r_1^2}\right)
\end{aligned}$$

Since $\frac{2}{\left(1+\frac{r_o}{r_1}\right)} = z^2$ the above simplifies to

$$
\begin{aligned}
\frac{\delta z}{\delta r_1} &= \frac{1}{2}\frac{1}{z}\left(z^2 \frac{1}{\left(1+\frac{r_o}{r_1}\right)}\frac{r_o}{r_1^2}\right)\\
&= \frac{1}{2} z \frac{r_o}{r_1^2\left(1+\frac{r_o}{r_1}\right)}\\
&= \frac{1}{2} z \frac{r_o}{r_1\left(r_1+r_o\right)}
\end{aligned}
$$

We want to find $\frac{\delta z}{z}$, therefore the above can be written as

$$
\frac{\delta z}{z} = \frac{\delta r_1}{r_1}\frac{1}{2}\frac{r_o}{(r_1+r_o)}
$$

Or in terms of $\frac{\delta r_1}{r_1}$ the above becomes

$$
\frac{\delta r_1}{r_1} = \frac{\delta z}{z}\left(2\frac{(r_1+r_o)}{r_o}\right)
$$

Since $z = \frac{v_o}{v_c}$, the reduces to

$$
\boxed{\frac{\delta r_1}{r_1} = \frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)}\left(2\frac{(r_1+r_o)}{r_o}\right)}
$$

4.7.2.2 Part (2)

For $\frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)} = 0.01$ then

$$
\frac{\delta r_1}{r_1} = 0.01\left(2\frac{(r_1+r_o)}{r_o}\right)
$$

Using $r_0 = \frac{1}{60}r_1$ in the above gives

$$
\begin{aligned}
\frac{\delta r_1}{r_1} &= 0.01\left(2\frac{\left(r_1+\frac{1}{60}r_1\right)}{\frac{1}{60}r_1}\right)\\
&= 1.22
\end{aligned}
$$

This means that δr_1 is 22% of r_1. The spacecraft will miss the moon by 22% of r_1. (This seems like a big miss for such small speed boost error)

4.7.3 Problem 3

3. (10 points)
A particle of mass m moves in a circular orbit of radius $r = a$ under the influence of the central attractive force $F(r) = -c\exp(-br)/r^2$, where c and b are positive constants.

(1) What is the effective potential energy in terms of r and the angular momentum ℓ? (Your answer may contain an integral.)

(2) Write down the Lagrangian of the system. Derive the equation of motion.

(3) For what values of b will this orbit be stable?

(4) Find the apsidal angle Ψ for nearly circular orbits in this field.

SOLUTION:

4.7.3.1 Part (1)

One way to find $U_{eff}(r)$ is to find the Largrangian L and pick the terms in it that have r without time derivative in them.

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

To find $U(r)$, since we are given $f(r)$ and since $f(r) = -\frac{\partial U(r)}{\partial r}$, then

$$\begin{aligned} U(r) &= -\int f(r)\,dr \\ &= \int \frac{ce^{-rb}}{r^2}dr \end{aligned}$$

Hence

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \int \frac{ce^{-rb}}{r^2}dr \end{aligned}$$

Hence

$$U_{eff}(r) = \frac{1}{2}mr^2\dot{\theta}^2 - \int \frac{ce^{-rb}}{r^2}dr$$

In terms of $l = mr^2\dot{\theta}$, the above can be written as

$$U_{eff}(r) = \frac{1}{2}l\dot{\theta} - \int \frac{ce^{-rb}}{r^2}dr$$

Or, it can also be written, as done in class notes, as

$$U_{eff}(r) = \frac{1}{2}\frac{l^2}{mr^2} - \int \frac{ce^{-rb}}{r^2}dr$$

4.7.3.2 Part (2)

$$L=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2-\int\frac{ce^{-rb}}{r^2}dr$$

Hence

$$\begin{aligned}\frac{\partial L}{\partial r}&=mr\dot{\theta}^2-\frac{ce^{-rb}}{r^2}\\ \frac{\partial L}{\partial\dot{r}}&=m\dot{r}\end{aligned}$$

The equation of motion for r is

$$\begin{aligned}m\ddot{r}-\left(mr\dot{\theta}^2-\frac{ce^{-rb}}{r^2}\right)&=0\\ m\ddot{r}-mr\dot{\theta}^2+\frac{ce^{-rb}}{r^2}&=0\\ m\ddot{r}-mr\dot{\theta}^2&=F(r)\end{aligned}$$

Written in terms of angular momentum, since $\dot{\theta}=\frac{l}{mr^2}$ (integral of motion) where l is the angular momentum, the above becomes

$$m\ddot{r}-\frac{l^2}{mr^3}=F(r)\tag{1}$$

For θ,

$$\begin{aligned}\frac{\partial L}{\partial\theta}&=0\\ \frac{\partial L}{\partial\dot{\theta}}&=mr^2\dot{\theta}\end{aligned}$$

The equation of motion for θ is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right)=C$$

Where C is some constant. The full EQM for θ is

$$\begin{aligned}m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right)&=0\\ r^2\ddot{\theta}+2r\dot{r}\dot{\theta}&=0\end{aligned}$$

4.7.3.3 Part (3)

To check for stability, since this is circular orbit, the radius is constant, say a. Then we perturb it by replacing a by $x+a$ where $x\ll a$ in the equation of motion $m\ddot{r}-\frac{l^2}{mr^3}=F(r)$ and it becomes

$$\begin{aligned}m\ddot{x}-\frac{l^2}{m(x+a)^3}&=F(x+a)\\ m\ddot{x}&=\frac{l^2(x+a)^{-3}}{m}+F(a+x)\end{aligned}$$

Since $x \ll a$, we expand $(x+a)^{-3}$ in Binomial and obtain

$$\begin{aligned} m\ddot{x} &= \frac{l^2}{ma^3}\left(1+\frac{x}{a}\right)^{-3} + F(a+x) \\ &\approx \frac{l^2}{ma^3}\left(1-\frac{3x}{a}+\cdots\right) + \overbrace{F(a)+xF'(a)+\cdots}^{\text{Taylor expansion}} \end{aligned}$$

Since circular orbit, then $\ddot{r}=0$ and the EQM motion becomes $-\frac{l^2}{ma^3}=F(a)$. Using this to replace $\frac{l^2}{ma^3}$ with in the above expression we find

$$\begin{aligned} m\ddot{x} &\approx -F(a)\left(1-\frac{3x}{a}\right) + F(a) + xF'(a) \\ &= -F(a) + F(a)\frac{3x}{a} + F(a) + xF'(a) \\ &= F(a)\frac{3x}{a} + xF'(a) \end{aligned}$$

Hence

$$\begin{aligned} m\ddot{x} + \left(-F(a)\frac{3x}{a} - xF'(a)\right) &= 0 \\ m\ddot{x} + \left(-\frac{3}{a}F(a) - F'(a)\right)x &= 0 \end{aligned}$$

This perturbation motion is stable if $\left(-\frac{3}{a}F(a)-F'(a)\right) > 0$. But $F(a) = -\frac{ce^{-ba}}{a}$ and $F'(a) = \frac{ce^{-ab}}{a^2}+\frac{bce^{-ab}}{a}$, hence

$$\begin{aligned} \Delta &= -\frac{3}{a}F(a) - F'(a) \\ &= -\frac{3}{a}\left(-\frac{ce^{-ba}}{a}\right) - \left(\frac{ce^{-ab}}{a^2}+\frac{bce^{-ab}}{a}\right) \end{aligned}$$

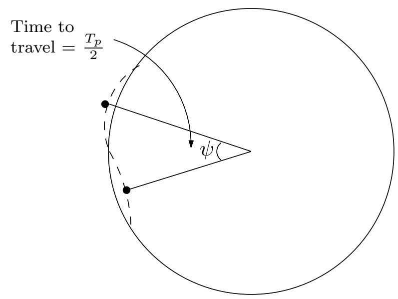
We want the above to be positive for stability. Simplifying gives

$$\begin{aligned} \Delta &= \frac{3ce^{-ba}}{a^2} - \frac{ce^{-ab}}{a^2} - \frac{bce^{-ab}}{a} \\ &= \frac{2ce^{-ba}}{a^2} - \frac{bce^{-ab}}{a} \\ &= \frac{2ce^{-ba} - abce^{-ab}}{a^2} \\ &= \frac{ce^{-ba}}{a^2}(2-ab) \end{aligned}$$

Therefore, we want $(2-ab)>0$ or $2>ab$ or

$$\boxed{b < \frac{2}{a}}$$

4.7.3.4 Part (4)



The angle ψ is found from

$$\psi = \frac{T_p}{2}\dot{\theta} \tag{1}$$

Where T_p is the period of oscillation due to the perturbation from the exact circular orbit, and $\dot{\theta}$ is the angular velocity on the circular orbit. But

$$\dot{\theta} \approx \frac{l}{ma^2} \tag{2}$$

But from part(3) we found that

$$\begin{aligned} -\frac{l^2}{ma^3} &= F(a) \\ l &= \sqrt{-F(a)\,ma^3} \end{aligned}$$

Therefore (2) becomes

$$\begin{aligned} \dot{\theta} &\approx \frac{1}{ma^2}\sqrt{-F(a)\,ma^3} \\ &= \sqrt{\frac{-F(a)}{ma}} \end{aligned}$$

We now find T_p. Since the perturbation equation of motion, from part (3) is $m\ddot{x}+\left(-\frac{3}{a}F(a)-F'(a)\right)x = 0$, which is of the form

$$\ddot{x} + \overbrace{\left(\frac{-\frac{3}{a}F(a)-F'(a)}{m}\right)}^{\omega_0^2} x = 0$$

Then, the natural frequency is $\omega = \sqrt{\frac{\left(-\frac{3}{a}F(a)-F'(a)\right)}{m}}$, therefore

$$\frac{2\pi}{T_p} = \sqrt{\frac{-\frac{3}{a}F(a) - F'(a)}{m}}$$
$$T_p = 2\pi\sqrt{\frac{m}{-\frac{3}{a}F(a) - F'(a)}}$$

Equation (1) now becomes

$$\begin{aligned}
\psi &= \frac{T_p}{2}\dot{\theta} \\
&= \pi\sqrt{\frac{m}{-\frac{3}{a}F(a) - F'(a)}}\sqrt{\frac{-F(a)}{ma}} \\
&= \pi\sqrt{\frac{-F(a)}{-3F(a) - aF'(a)}} \\
&= \pi\sqrt{\frac{F(a)}{3F(a) + aF'(a)}}
\end{aligned}$$

But $F(a) = -\frac{ce^{-ba}}{a^2}$ and $F'(a) = \frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}$ then the above becomes

$$\begin{aligned}
\psi &= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{3F(a) + aF'(a)}} \\
&= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{3\left(-\frac{ce^{-ba}}{a^2}\right) + a\left(\frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}\right)}} \\
&= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{-3\frac{ce^{-ba}}{a^2} + \left(\frac{ce^{-ab}+abce^{-ab}}{a}\right)}} \\
&= \pi\sqrt{\frac{-ce^{-ba}}{-3ce^{-ba} + \left(ace^{-ab} + a^2bce^{-ab}\right)}} \\
&= \pi\sqrt{\frac{-1}{-3 + a + a^2b}}
\end{aligned}$$

Hence

$$\boxed{\psi = \pi\sqrt{\frac{1}{3-a(1+ab)}}}$$

4.7.4 Problem 4

> 4. (10 points)
> A ball is dropped from a height h onto a horizontal pavement. If the coefficient of restitution is ϵ, show that the total vertical distance the ball goes before the rebounds end is $h(1+\epsilon^2)/(1-\epsilon^2)$. What is the total length of time that the ball bounces?

SOLUTION:

The first time the ball falls from height h it will have speed of $v_1 = \sqrt{2gh}$ just before hitting the platform, which is found using

$$mgh = \frac{1}{2}mv_1^2$$

On bouncing back, it will have speed of $v_1' = \varepsilon\sqrt{2gh}$. It will then travel up a distance of $h_1 = \varepsilon^2 h$ which is found by solving for $h_1$from

$$mgh_1 = \frac{1}{2}m\left(v_1'\right)^2$$

The second time it it falls back it will have speed of $v_2 = \varepsilon\sqrt{2gh_1}$. When it bounces back up, it will have speed $v_2' = \varepsilon^2\sqrt{2gh_1}$ and now it will travel up a distance of $h_2 = \varepsilon^4 h$ which is found by solving for h_2 from

$$mgh_2 = \frac{1}{2}m\left(v_2'\right)^2$$

This process will continue until the ball stops. We see that the distance travelled at each bouncing is

$$\Delta = \left\{h, 2\varepsilon^2 h, 2\varepsilon^4 h, 2\varepsilon^6 h, \cdots, 2\varepsilon^{2n} h\right\}$$

We added 2 to each bounce after the first one to count for going up and then coming down the same distance. The first time it will only have one h. We now can calculate total distance travelled Δ as

$$\begin{aligned}\Delta &= h + 2\varepsilon^2 h + 2\varepsilon^4 h + \cdots \\ &= h\left(1 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots\right)\end{aligned}$$

The above can be written as

$$\Delta = h\left(2 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots\right) - h \tag{1}$$

But since $\varepsilon \leq 1$ the series sum is

$$2 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots = 2\sum_{n=0}^{\infty}\varepsilon^{2n} = 2\frac{1}{1-\varepsilon^2}$$

Therefore (1) becomes

$$\begin{aligned}\Delta &= \frac{2h}{1-\varepsilon^2} - h\\ &= \frac{2h - h\left(1-\varepsilon^2\right)}{1-\varepsilon^2}\\ &= \frac{2h - h + h\varepsilon^2}{1-\varepsilon^2}\end{aligned}$$

Hence total distance is

$$\boxed{\frac{h\left(1+\varepsilon^2\right)}{1-\varepsilon^2}}$$

To find the total time of all ball bounces, we need to find the time it takes to travel in each bounce. The time it takes to fall distance h is $\sqrt{\frac{2h}{g}}$, using the information we found about each h_i from above, we now set up the sequence of times we we did for distances

$$\Delta_{time} = \left\{\sqrt{\frac{2h}{g}}, 2\sqrt{\frac{2\varepsilon^2 h}{g}}, 2\sqrt{\frac{2\varepsilon^4 h}{g}}, 2\sqrt{\frac{2\varepsilon^6 h}{g}}, \cdots\right\}$$

Adding the times gives

$$\begin{aligned}\Delta &= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2\varepsilon^2 h}{g}} + 2\sqrt{\frac{2\varepsilon^4 h}{g}} + 2\sqrt{\frac{2\varepsilon^6 h}{g}}\\ &= \sqrt{\frac{2h}{g}}\left(1 + 2\varepsilon + 2\varepsilon^2 + 2\varepsilon^3 + 2\varepsilon^4 \cdots\right)\\ &= \sqrt{\frac{2h}{g}}\left(2 + 2\varepsilon + 2\varepsilon^2 + 2\varepsilon^3 + 2\varepsilon^4 \cdots\right) - \sqrt{\frac{2h}{g}}\\ &= \sqrt{\frac{2h}{g}} \sum_{n=0}^{\infty} 2\varepsilon^n - \sqrt{\frac{2h}{g}}\end{aligned}$$

But $2\sum_{n=0}^{\infty} \varepsilon^n = 2\frac{1}{1-\varepsilon}$, hence the above becomes

$$\begin{aligned}\Delta &= \sqrt{\frac{2h}{g}}\frac{2}{1-\varepsilon} - \sqrt{\frac{2h}{g}}\\ &= \sqrt{\frac{2h}{g}}\left(\frac{2}{1-\varepsilon} - 1\right)\\ &= \sqrt{\frac{2h}{g}}\left(\frac{2-(1-\varepsilon)}{1-\varepsilon}\right)\end{aligned}$$

Hence total time is

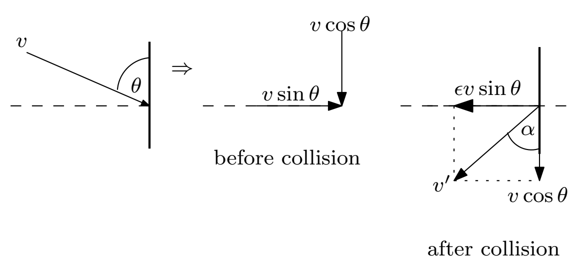
$$\boxed{\sqrt{\frac{2h}{g}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}$$

4.7.5 Problem 5

5. (10 points)
A particle of mass m strikes a wall at an angle θ with respect to the normal. The collision is inelastic with coefficient of restitution ϵ. Find the rebound angle of the particle after collision with the wall.

SOLUTION:

First we make a diagram showing the geometry involved



We resolve the incoming velocity into its x, y components and apply conservation of linear momentum to each part. The vertical component remain the same after collision since it is parallel to the wall. Hence

$$v'_y = v_y = v\cos\theta$$

While the x component will change to

$$v'_x = \varepsilon v_x = \varepsilon v\sin\theta$$

By definition of ε. Therefore we see that after collision

$$\begin{aligned}\tan\alpha &= \frac{\varepsilon v\sin\theta}{v\cos\theta}\\ &= \varepsilon\tan\theta\end{aligned}$$

Hence

$$\boxed{\alpha = \arctan\left(\varepsilon\tan\theta\right)}$$

4.7.6 HW 7 key solution

1

Mechanics

Physics 311 - Fall 2015

Homework Set 7 - Solutions

Problem 1

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r)$$

generalized coordinates r, θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad \text{with } Q_i = \vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i}$$

here, $\vec{F} = -m k \frac{d\vec{r}}{dt}$

in polar coordinates, $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

and $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$

so $\vec{F} = -mk\frac{dr}{dt}\hat{r} - mkr\frac{d\theta}{dt}\hat{\theta}$

$$\Rightarrow \quad Q_r = F_r = -mk\frac{dr}{dt} \qquad Q_\theta = rF_\theta = -mkr^2\frac{d\theta}{dt}$$

Euler-Lagrange equations

in θ: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mkr^2\dot{\theta}$ with $\ell = mr^2\dot{\theta}$

so with $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$$\frac{d}{dt}\ell = -k\ell \quad \Rightarrow \quad \frac{d\ell}{\ell} = -k\,dt \quad \Rightarrow \quad \boxed{\ell = \ell_0 e^{-kt}}$$

2

Problem 2

(1) percentage increase necessary to transfer a spacecraft from Low Earth orbit to the Moon is

$$\frac{v_0}{v_c} = \sqrt{\frac{2r_1}{r_1 + r_0}} = \sqrt{\frac{2}{1 + r_0/r_1}}$$

v_c = speed on circular orbit of radius r_0 (LEO)

v_0 = speed needed to be on elliptical path with Moon at apogee

we need $\frac{\delta r_1}{r_1}$ as a function of $\frac{\delta(v_0/v_c)}{(v_0/v_c)}$

first, calculate $$\frac{d(v_0/v_c)}{dr_1} = \frac{1}{2(v_0/v_c)} \frac{(-2)}{(1+r_0/r_1)^2}\left(-\frac{r_0}{r_1^2}\right)$$

so $$\frac{\frac{d(v_0/v_c)}{(v_0/v_c)}}{\frac{dr_1}{r_1}} = \frac{1}{2(v_0/v_c)^2} r_1 \frac{2r_0}{(1+r_0/r_1)^2}\frac{1}{r_1^2}$$

$$= \frac{1}{2}\frac{1}{(1+r_0/r_1)}\frac{r_0}{r_1}$$

so $$\frac{\delta r_1}{r_1} = 2\left(1+\frac{r_0}{r_1}\right)\frac{r_1}{r_0}\frac{\delta(v_0/v_c)}{(v_0/v_c)}$$

and with $r_0 = R_E$ and $r_1 = 60 R_E$, $\frac{r_0}{r_1} \approx 0$,

so $$\boxed{\frac{\delta r_1}{r_1} \simeq 2\frac{r_1}{r_0}\frac{\delta(v_0/v_c)}{(v_0/v_c)}}$$

(2) $\frac{\delta r_1}{r_1} \simeq 2 \cdot 60 \cdot 1\% = 120\%$!! (of course this means that the above approximation as a differential has broken down)

3

Problem 3

(1)
$$U(r) = -\int F(r)\,dr$$
$$= c\int \frac{e^{-br}}{r^2}\,dr$$

$$\epsilon = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + U(r)$$
$$= \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

$$\Rightarrow \boxed{U_{eff} = \frac{\ell^2}{2mr^2} + c\int \frac{e^{-br}}{r^2}\,dr}$$

(2)
$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r)$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial U(r)}{\partial r}$$
$$= mr\dot{\theta}^2 + F(r)$$
$$= mr\dot{\theta}^2 - c\,\frac{e^{-br}}{r^2}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + c\,\frac{e^{-br}}{r^2} = 0$$

$$\text{or} \quad \boxed{m\ddot{r} - \frac{\ell^2}{mr^3} + c\,\frac{e^{-br}}{r^2} = 0}$$

4

(3) condition for stable orbit (from class)

$$F(a) + \frac{a}{3} F'(a) < 0$$

So
$$-c\,\frac{e^{-ba}}{a^2} + \frac{a}{3}\,c\,\frac{e^{-ba}}{a^2}\left(b + \frac{2}{a}\right) < 0$$

$$\Rightarrow \quad -c\,\frac{e^{-ba}}{3a^2} + c\,\frac{b\,e^{-ba}}{3a} < 0$$

$$\Leftrightarrow \quad -\frac{1}{a} + b < 0 \qquad \text{or} \qquad \boxed{\frac{1}{a} > b}$$

(4)
$$2\psi = \pi\left[3 + a\,\frac{F'(a)}{F(a)}\right]^{-1/2}$$

$$= \pi\left[3 + a\,\frac{c e^{-ba}\left(b + \frac{2}{a}\right)}{a^2\left(-c\frac{e^{-ba}}{a^2}\right)}\right]^{-1/2}$$

$$= \pi\left[3 - a\left(b + \frac{2}{a}\right)\right]^{-1/2}$$

$$= \pi\left[3 - (ab + 2)\right]^{-1/2}$$

$$= \pi\left[1 - ab\right]^{-1/2}$$

So
$$\boxed{2\psi = \frac{\pi}{\sqrt{1 - ab}}}$$

5

Problem 4

velocity with which the ball reaches the floor the first time : $\frac{1}{2} m v^2 = mgh$

$\Rightarrow \quad v^2 = 2gh$

velocity after the bounce : $\frac{v'}{v} = \varepsilon$

height after first bounce : $h' = \frac{v'^2}{2g} = \frac{\varepsilon^2 v^2}{2g} = \varepsilon^2 h$

velocity after the second bounce : $\frac{v''}{v'} = \varepsilon$

height after second bounce : $h'' = \frac{v''^2}{2g} = \frac{\varepsilon^2 v'^2}{2g}$

$= \frac{\varepsilon^4 v^2}{2g} = \varepsilon^4 h$

$\Rightarrow$ total distance

$$d = h + 2\varepsilon^2 h + 2\varepsilon^4 h + \ldots$$

$$= h \left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^{2n}\right)$$

now use $\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$ for $|r| < 1$

$$\Rightarrow \quad d = h \left(-1 + \frac{2}{1-\varepsilon^2}\right)$$

$$= h \frac{-1 + \varepsilon^2 + 2}{1 - \varepsilon^2} = \frac{1+\varepsilon^2}{1-\varepsilon^2} h$$

6

for the total time :

first fall $\quad \frac{1}{2} g t^2 = h \Rightarrow t = \sqrt{\frac{2h}{g}}$

fall from h' $\quad t' = \sqrt{\frac{2h'}{g}} = \sqrt{\frac{2\varepsilon^2 h}{g}} = \varepsilon t$

fall from h'' $\quad t'' = \sqrt{\frac{2\varepsilon^4 h}{g}} = \varepsilon^2 t$

$$\Rightarrow \quad t_{tot} = t + 2\varepsilon t + 2\varepsilon^2 t + \dots$$

$$= \sqrt{\frac{2h}{g}} \left(1 + 2\varepsilon + 2\varepsilon^2 + \dots\right)$$

$$= \sqrt{\frac{2h}{g}} \left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^n\right)$$

$$= \sqrt{\frac{2h}{g}} \left(-1 + \frac{2}{1-\varepsilon}\right)$$

$$= \sqrt{\frac{2h}{g}} \frac{-1+\varepsilon+2}{1-\varepsilon}$$

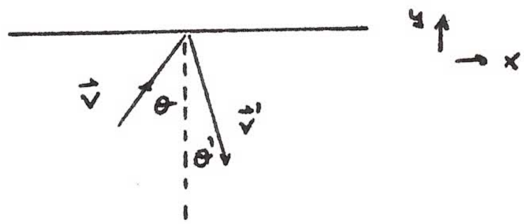
$$\Rightarrow \quad \boxed{t_{tot} = \sqrt{\frac{2h}{g}} \frac{1+\varepsilon}{1-\varepsilon}}$$

check : $\varepsilon = 1$ (elastic)
$t \to \infty$ ✓

$\varepsilon = 0$ (totally inelastic)
$t = \sqrt{\frac{2h}{g}}$ (mass sticks to surface) ✓

7

Problem 5



component of velocity along the wall is unchanged

$$v_x' = v \sin\theta$$

$$\varepsilon = \frac{|v_y'|}{|v_y|} = \frac{|v_y'|}{v\cos\theta}$$ (wall does not move)

$$\Rightarrow \quad |v_y'| = \varepsilon v \cos\theta$$

So

$$v' = \left[v^2 \sin^2\theta + \varepsilon^2 v^2 \cos^2\theta \right]^{1/2}$$

$$\Leftrightarrow \quad \boxed{v' = v \sqrt{\sin^2\theta + \varepsilon^2 \cos^2\theta}}$$

and $\tan\theta' = \dfrac{v\sin\theta}{\varepsilon v \cos\theta} \quad \Rightarrow \quad \boxed{\theta' = \operatorname{atan}\left[\frac{1}{\varepsilon}\tan\theta\right]}$

4.8 HW 8

4.8.1 Problem 1

1. (15 points)
Consider the case where a fixed force center scatters a particle of mass m according to an inverse-cube force law $F(r) = k/r^3$. If the initial velocity of m is v, show that the differential cross section is

$$\sigma(\theta) = \frac{k\,\pi^2\,(\pi-\theta)}{m\,v^2\,\theta^2\,(2\pi-\theta)^2\,\sin\theta} \quad .$$

SOLUTION:

Starting from

$$\theta_0(b) = \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2Em}{l^2} - \frac{2mU}{l^2} - \frac{1}{r^2}}} \tag{1}$$

But

$$l = b\sqrt{2mE}$$
$$l^2 = b^2(2mE)$$

Hence (1) becomes

$$\begin{aligned}\theta_0(b) &= \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{U}{b^2E} - \frac{1}{r^2}}} \\ &= \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{U}{E} - \frac{b^2}{r^2}}} dr \end{aligned} \tag{1A}$$

In this problem, since $F(r) = \frac{k}{r^3}$, therefore since $F(r) = -\nabla U$

$$\begin{aligned}U(r) &= -\int \frac{k}{r^3} dr \\ &= \frac{k}{2r^2}\end{aligned}$$

Then (1A) becomes

$$\theta_0(b) = \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{k}{2r^2E} - \frac{b^2}{r^2}}} dr \tag{1B}$$

Let $z = \frac{1}{r}$ then $\frac{dr}{dz} = -\frac{1}{z^2}$. When $r = \infty$ then $z = 0$ and when $r = r_{\min}$ then $z = \frac{1}{r_{\min}}$. Now we need to find $r_{\min}$. We know that when $E = U_{effective}$ then $r = r_{\min}$. But

$$\begin{aligned} U_{effective} &= \frac{l^2}{2mr^2} + U(r) \\ &= \frac{l^2}{2mr^2} + \frac{k}{2r^2} \end{aligned}$$

Hence

$$\begin{aligned} E &= U_{effective} \\ &= \frac{l^2}{2mr_{\min}^2} + \frac{k}{2r_{\min}^2} \\ &= \frac{l^2 + mk}{2mr_{\min}^2} \end{aligned}$$

Solving for $r_{\min}$

$$\begin{aligned} r_{\min}^2 &= \frac{l^2 + mk}{2mE} \\ &= \frac{l^2}{2mE} + \frac{k}{2E} \end{aligned} \tag{2}$$

But $l^2 = b^2(2mE)$ then (2) becomes

$$\begin{aligned} r_{\min}^2 &= \frac{b^2(2mE)}{2mE} + \frac{k}{2E} \\ &= b^2 + \frac{k}{2E} \end{aligned}$$

Therefore

$$r_{\min} = \sqrt{b^2 + \frac{k}{2E}} \tag{3}$$

Now we can finish the limits of integration in (1B). When $r = r_{\min}$ then $z = \frac{1}{r_{\min}} = \frac{1}{\sqrt{b^2 - \frac{k}{2E}}}$, now (1B) becomes (where we now replace r^2 by $\frac{1}{z^2}$)

$$\begin{aligned} \theta_0(b) &= \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{k}{2r^2E} - \frac{b^2}{r^2}}} dr \\ &= \int_{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}}^{0} \frac{z^2 b}{\sqrt{1 - \frac{kz^2}{2E} - b^2z^2}} \left(-\frac{1}{z^2}dz\right) \\ &= b\int_0^{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}} \frac{1}{\sqrt{1 - \frac{kz^2}{2E} - b^2z^2}} dz \\ &= b\int_0^{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}} \frac{dz}{\sqrt{1 - z^2\left(\frac{k}{2E} + b^2\right)}} \end{aligned}$$

Using CAS, it gives $\int \frac{dz}{\sqrt{1-az^2}} = \frac{1}{\sqrt{a}} \sin^{-1}\left(z\sqrt{a}\right)$. Using this result above, where $a = \left(\frac{k}{2E} + b^2\right)$ gives

$$\begin{aligned}
\theta_0(b) &= \frac{b}{\sqrt{\frac{k}{2E}+b^2}} \left(\sin^{-1}\left(z\sqrt{\frac{k}{2E}+b^2}\right)\Bigg|_0^{\frac{1}{\sqrt{b^2+\frac{k}{2E}}}} \right) \\
&= \frac{b}{\sqrt{\frac{k}{2E}+b^2}} \left[\sin^{-1}\left(\frac{1}{\sqrt{b^2+\frac{k}{2E}}} \sqrt{\frac{k}{2E}+b^2} \right) - \sin^{-1}(0) \right] \\
&= \frac{b}{\sqrt{\frac{k}{2E}+b^2}} \left[\sin^{-1}(1) - 0 \right] \\
&= \frac{b}{\sqrt{\frac{k}{2E}+b^2}} \frac{\pi}{2}
\end{aligned}$$

Now we solve for b. Squaring both sides

$$\theta_0^2 = \frac{b^2}{\frac{k}{2E}+b^2} \frac{\pi^2}{4}$$

Using $E = \frac{1}{2}mv^2$ then

$$\begin{aligned}
\theta_0^2 &= \frac{b^2}{\left(\frac{k}{mv^2}+b^2\right)} \frac{\pi^2}{4} \\
4\theta_0^2 \left(\frac{k}{mv^2}+b^2\right) &= b^2\pi^2 \\
\frac{k4\theta_0^2}{mv^2} + 4\theta_0^2 b^2 - b^2\pi^2 &= 0 \\
b^2\left(4\theta_0^2 - \pi^2\right) &= -\frac{k4\theta_0^2}{mv^2} \\
b^2\left(\pi^2 - 4\theta_0^2\right) &= \frac{k4\theta_0^2}{mv^2} \\
b^2 &= \frac{k4\theta_0^2}{mv^2\left(\pi^2 - 4\theta_0^2\right)} \\
b &= \frac{2\theta_0}{v} \sqrt{\frac{k}{m\left(\pi^2 - 4\theta_0^2\right)}} \qquad (4)
\end{aligned}$$

But $\theta_0(b) = \frac{\pi}{2} - \frac{\theta_s}{2}$, where θ_s is the scattering angle. Therefore the above becomes

$$
\begin{aligned}
b &= \frac{2\left(\frac{\pi}{2} - \frac{\theta_s}{2}\right)}{v}\sqrt{\frac{k}{m\left(\pi^2 - 4\left(\frac{\pi}{2} - \frac{\theta_s}{2}\right)^2\right)}} \\
b &= \frac{\pi - \theta_s}{v}\sqrt{\frac{k}{m\left(\pi^2 - \left(\theta_s^2 - 2\pi\theta_s + \pi^2\right)\right)}} \\
b &= \frac{\pi - \theta_s}{v}\sqrt{\frac{k}{m\left(2\pi\theta_s - \theta_s^2\right)}} \qquad (5)
\end{aligned}
$$

Now we are ready to find $\sigma(\theta_s)$

$$
\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|
$$

From (5)

$$
\frac{db}{d\theta_s} = -\frac{\pi^2\sqrt{\frac{k}{m\left(2\pi\theta_s - \theta_s^2\right)}}}{v\left(2\pi\theta_s - \theta_s^2\right)}
$$

Therefore

$$
\begin{aligned}
\sigma(\theta_s) &= \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| \\
&= \frac{\frac{\pi-\theta_s}{v}\sqrt{\frac{k}{m\left(2\pi\theta_s-\theta_s^2\right)}}}{\sin\theta_s}\frac{\pi^2\sqrt{\frac{k}{m\left(2\pi\theta_s-\theta_s^2\right)}}}{v\left(2\pi\theta_s - \theta_s^2\right)} \\
&= \frac{\frac{\pi-\theta_s}{v}\frac{k}{m\left(2\pi\theta_s-\theta_s^2\right)}}{\sin\theta_s}\frac{\pi^2}{v\left(2\pi\theta_s - \theta_s^2\right)} \\
&= \frac{(\pi - \theta_s)k}{mv\sin\theta_s}\frac{\pi^2}{v\left(2\pi\theta_s - \theta_s^2\right)^2} \\
&= \frac{k\pi^2(\pi - \theta_s)}{mv^2\sin\theta_s\left(2\pi\theta_s - \theta_s^2\right)^2}
\end{aligned}
$$

Or

$$
\sigma(\theta_s) = \frac{k\pi^2(\pi - \theta_s)}{mv^2\theta_s^2(2\pi - \theta_s)^2\sin\theta_s}
$$

Hard problem. Time taken to solve: 6 hrs.

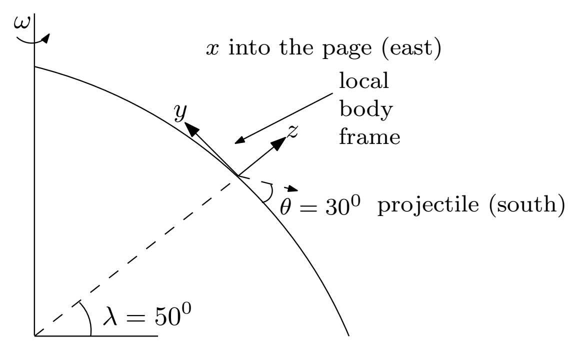
4.8.2 Problem 2

2. (10 points)
(1) A warship fires a projectile due South at a southern latitude of 50°. The shells are fired at 37° elevation with a speed of 800 ms^{-1}. Neglecting air resistance, calculate by how much the shells will miss their target and in what direction.

(2) A batter hits a baseball a distance of 200 ft in a roughly flat trajectory. Should he take the Coriolis force into account? Neglect air resistance, assume the elevation angle is 15°, and the location is Yankee Stadium (or Wrigley Field, if you prefer).

SOLUTION:

4.8.2.1 part (1)



Using

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(\dot{z}_0 \cos\lambda - \dot{y}_0 \sin\lambda\right) + \dot{x}_0 t + x_0 \\
y &= \dot{y}_0 t - \omega t^2 \dot{x}_0 \sin\lambda + y_0 \\
z &= \dot{z}_0 t - \frac{1}{2} g t^2 + \omega t^2 \dot{x}_0 \cos\lambda + z_0
\end{aligned} \tag{1}
$$

Where $\{\dot{x}_0, \dot{y}_0, \dot{z}_0\}$ are the initial speeds in each of the body frame directions and $\{x_0, y_0, z_0\}$ are the initial position of the projectile at $t = 0$. Let $v_0 = 800\ m/s^2$ and $\theta = 37^0$. We are given that

$$
\begin{aligned}
\dot{y}_0 &= -v_0 \cos\theta \\
\dot{z}_0 &= v_0 \sin\theta \\
\dot{x}_0 &= 0
\end{aligned}
$$

The minus sign for $\dot{y}_0$ above was added since the direction is south, which is negative y direction for the local frame. And we are given that $x_0 = y_0 = z_0 = 0$. Substituting these in

(1) gives (where $\lambda = 50^0$)

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(v_0 \sin\theta \cos\lambda + v_0 \cos\theta \sin\lambda\right) \\
y &= -\left(v_0 \cos\theta\right) t \\
z &= \left(v_0 \sin\theta\right) t - \frac{1}{2} g t^2
\end{aligned} \tag{2}
$$

The drift due to the Coriolis force is found from the x component. The projectile will drift west (to the right direction of its motion) since it is moving south. We can now calculate this x drift. We know that $\omega = 7.3 \times 10^{-5}$ rad/sec (rotation speed of earth), so we just need to find time of flight t. From

$$
\begin{aligned}
\dot{z} &= \dot{z}_0 - gt \\
&= v_0 \sin\theta - gt
\end{aligned}
$$

The projectile time up (when $\dot{z}$ first becomes zero) is $t = \frac{v_0 \sin\theta}{g} = \frac{800 \sin\left(37\left(\frac{\pi}{180}\right)\right)}{9.81} \approx 50$ sec. Hence total time of flight is twice this which is $t_f = 100$ sec. Now we use this time in the x equation in (2) above

$$
\begin{aligned}
x &= \frac{1}{3}\left(7.3 \times 10^{-5}\right)(9.81)(100)^3 \cos\left(50^0\right) - \left(7.3 \times 10^{-5}\right)(100)^2 \left(800 \sin 37^0 \cos 50^0 + 800 \cos 37^0 \sin 50^0\right) \\
&= -532
\end{aligned}
$$

So it will drift by about 532 meter to the west (since negative sign). In the above $g = 9.81$ was used. This does not include all the terms such as the centrifugal acceleration. But $9.81 \frac{m}{s^2}$ is good approximation for this problem.

4.8.2.2 part (2)

Taking Latitude as 42^0 (New York). Therefore $\lambda = 42^0$ and $\theta = 15^0$. Initial conditions are

$$
\begin{aligned}
\dot{y}_0 &= V_0 \cos\theta \\
\dot{z}_0 &= V_0 \sin\theta \\
\dot{x}_0 &= 0
\end{aligned}
$$

Where V_0 is the initial speed the ball was hit with (which we do not know yet), and $x_0 = y_0 = z_0 = 0$. Using

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(\dot{z}_0 \cos\lambda - \dot{y}_0 \sin\lambda\right) + \dot{x}_0 t + x_0 \\
y &= \dot{y}_0 t - \omega t^2 \dot{x}_0 \sin\lambda + y_0 \\
z &= \dot{z}_0 t - \frac{1}{2} g t^2 + \omega t^2 \dot{x}_0 \cos\lambda + z_0
\end{aligned} \tag{1}
$$

Then applying initial conditions the above reduces to

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(V_0 \sin\theta \cos\lambda - V_0 \cos\theta \sin\lambda\right) \\
y &= \left(V_0 \cos\theta\right) t \\
z &= \left(V_0 \sin\theta\right) t - \frac{1}{2} g t^2
\end{aligned} \tag{2}
$$

From $y\left(t_f\right) = \left(V_0 \cos\theta\right) t_f$ then, since we are told that $y\left(t_f\right) = 200$ ft,

$$
200\,(0.3048) = \left(V_0 \cos\theta\right) t_f \tag{3}
$$

Where t_f is time of flight. But time of flight is also found

$$
\begin{aligned}
\dot{z} &= \dot{z}_0 - gt \\
&= V_0 \sin\theta - gt
\end{aligned}
$$

And solving for $\dot{z} = 0$, which gives $\frac{V_0 \sin\theta}{g}$. So time of flight is twice this or

$$
t_f = \frac{2V_0 \sin\theta}{g}
$$

Substituting the above into (3) to solve for V_0 gives

$$
\begin{aligned}
200\,(0.3048) &= \left(V_0 \cos\theta\right) \frac{2V_0 \sin\theta}{g} \\
60.96 &= \frac{2}{9.81} V_0^2 \left(\cos 15^0\right)\left(\sin 15^0\right) \\
V_0^2 &= \frac{(60.96)\,(9.81)}{2 \cos 15^0 \sin 15^0} \\
&= 1196.0
\end{aligned}
$$

Hence

$$
V_0 = 34.583 \quad \text{m/s}
$$

Now we can go back and solve for time of flight t_f. From

$$
\begin{aligned}
200\,(0.3048) &= \left(V_0 \cos\theta\right) t_f \\
t_f &= \frac{200\,(0.3048)}{34.583\ \left(\cos 15^0\right)} \\
&= 1.825 \text{ sec}
\end{aligned}
$$

Using (2) we solve for x, the drift due to Coriolis forces.

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(V_0 \sin\theta \cos\lambda - V_0 \cos\theta \sin\lambda\right) \\
&= \frac{1}{3}\left(7.3 \times 10^{-5}\right)(9.81)\,(1.825)^3 \cos 42^0 - \left(7.3 \times 10^{-5}\right)(1.825)^2 \left(34.58 \sin 15^0 \cos 42^0 + 34.58 \cos 15^0 \sin 42^0\right) \\
&= 4.897 \times 10^{-3} \text{ meter}
\end{aligned}
$$

So the ball will drift about $5mm$. This is too small and the ball player can therefore ignore Coriolis forces when hitting the ball.

4.8.3 Problem 3

3. (5 points)
A bullet is fired straight up with initial speed v_0. Show that the bullet will hit the ground *west* of the initial point of upward motion by an amount $4\,\omega\,v_0^3\,\cos\lambda/(3\,g^2)$, where λ is the latitude and ω is the angular velocity of Earth's rotation. Ignore air resistance.

SOLUTION:

Initial conditions are

$$
\begin{aligned}
\dot{y}_0 &= 0\\
\dot{z}_0 &= v_0\\
\dot{x}_0 &= 0
\end{aligned}
$$

And $x_0 = y_0 = z_0 = 0$. Using

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(\dot{z}_0 \cos\lambda - \dot{y}_0 \sin\lambda\right) + \dot{x}_0 t + x_0\\
y &= \dot{y}_0 t - \omega t^2 \dot{x}_0 \sin\lambda + y_0\\
z &= \dot{z}_0 t - \frac{1}{2} g t^2 + \omega t^2 \dot{x}_0 \cos\lambda + z_0
\end{aligned}
\tag{1}
$$

The reduce to (using initial conditions) to

$$
\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 v_0 \cos\lambda\\
y &= 0\\
z &= v_0 t - \frac{1}{2} g t^2
\end{aligned}
\tag{2}
$$

To find time of flight of bullet (going up and then down again), from $\dot{z} = v_0 - gt$, we solve for $\dot{z} = 0$, which gives $t = \frac{v_0}{g}$. So time of flight is twice this amount

$$
t_f = \frac{2v_0}{g} \text{ sec}
$$

To find the amount x the bullet moves during this time, we use (2) and solve for x

$$\begin{aligned}
x\left(t_f\right) &= \frac{1}{3}\omega g t_f^3 \cos\lambda - \omega t_f^2 v_0 \cos\lambda \\
&= \frac{1}{3}\omega g \left(\frac{2v_0}{g}\right)^3 \cos\lambda - \omega \left(\frac{2v_0}{g}\right)^2 v_0 \cos\lambda \\
&= \frac{1}{3}\omega \frac{8v_0^3}{g^2} \cos\lambda - \omega \frac{4v_0^3}{g^2} \cos\lambda \\
&= \left(\frac{8}{3} - 4\right)\left(\omega \frac{v_0^3}{g^2} \cos\lambda\right) \\
&= -\frac{4}{3}\omega \frac{v_0^3}{g^2} \cos\lambda
\end{aligned}$$

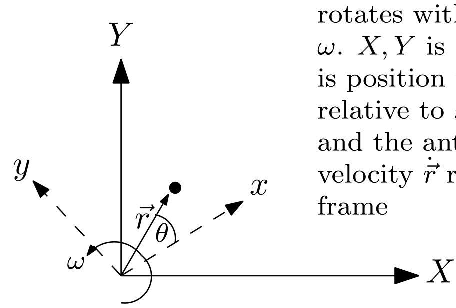
This means when it lands again, the bullet will be $-\frac{4}{3}\omega\frac{v_0^3}{g^2}\cos\lambda$ meters relative to the original point it was fired from (the origin of the local body frame). Since the sign is negative, it means it is west.

4.8.4 Problem 4

4. (10 points)
A bug crawls with constant speed in a circular path of radius b on a phonograph turntable rotating with constant angular speed ω. The bug's path is concentric with the center of the turntable. If the bug's mass is m and the coefficient of static friction for the bug on the table is μ, how fast (relative to the turntable) can the bug crawl before it starts to slip if it goes (1) in the direction of rotation and (2) opposite to the direction of rotation?

SOLUTION:

4.8.4.1 Part(1)



x, y is local frame that rotates with angular velocity ω. X, Y is inertial frame. $\vec{r}$ is position vector of ant relative to x, y local frame, and the ant is moving with velocity $\dot{\vec{r}}$ relative to its local frame

When Ant is moving in direction of rotation:

$$
\begin{aligned}
\vec{r} &= b\cos\theta\vec{i} + b\sin\theta\vec{j}\\
\vec{v} &= \vec{v}_{rel} + \vec{\omega}\times\vec{r} \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\vec{v}_{rel} &= \frac{d}{dt}\vec{r}\\
&= -b\dot{\theta}\sin\theta\vec{i} + b\dot{\theta}\cos\theta\vec{j}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{\omega}\times\vec{r} &= \omega\vec{k}\times\left(b\cos\theta\vec{i} + b\sin\theta\vec{j}\right)\\
&= b\omega\cos\theta\vec{j} - b\omega\sin\theta\vec{i}
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
\vec{v} &= \left(-b\dot{\theta}\sin\theta\vec{i} + b\dot{\theta}\cos\theta\vec{j}\right) + \left(b\omega\cos\theta\vec{j} - b\omega\sin\theta\vec{i}\right)\\
&= \vec{i}\left(-b\dot{\theta}\sin\theta - b\omega\sin\theta\right) + \vec{j}\left(b\dot{\theta}\cos\theta + b\omega\cos\theta\right)
\end{aligned}
$$

The above is the velocity of the ant, in the inertial frame, using local body unit vector $\vec{i},\vec{j}$. Now we find the ant acceleration, given by

$$
\vec{a} = \vec{a}_{rel} + 2\left(\omega\vec{k}\times\vec{v}_{rel}\right) + \left(\dot{\omega}\vec{k}\times\vec{r}\right) + \omega\vec{k}\times\left(\vec{\omega}\times\vec{r}\right)
$$

But $\dot{\omega} = 0$ since disk has constant ω then

$$
\vec{a} = \vec{a}_{rel} + 2\left(\omega\vec{k}\times\vec{v}_{rel}\right) + \omega\vec{k}\times\left(\vec{\omega}\times\vec{r}\right) \qquad (1)
$$

But

$$
\begin{aligned}
\vec{a}_{rel} &= \frac{d}{dt}\vec{v}_{rel}\\
&= \vec{i}\left(-b\ddot{\theta}\sin\theta - b\dot{\theta}^2\cos\theta\right) + \vec{j}\left(b\ddot{\theta}\cos\theta - b\dot{\theta}^2\sin\theta\right)
\end{aligned}
$$

Since Bug moves with constant speed, then $\ddot{\theta} = 0$ and the above becomes

$$
\vec{a}_{rel} = \vec{i}\left(-b\dot{\theta}^2\cos\theta\right) + \vec{j}\left(-b\dot{\theta}^2\sin\theta\right)
$$

Now the Coriolis term $2\left(\vec{\omega}\times\vec{v}_{rel}\right)$ is found

$$
\begin{aligned}
2\left(\vec{\omega}\times\vec{v}_{rel}\right) &= 2\left(\omega\vec{k}\times\left(-b\dot{\theta}\sin\theta\vec{i} + b\dot{\theta}\cos\theta\vec{j}\right)\right)\\
&= 2\left(-\omega b\dot{\theta}\sin\theta\vec{j} - b\omega\dot{\theta}\cos\theta\vec{i}\right)
\end{aligned}
$$

Now the $\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$ is found

$$
\begin{aligned}
\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) &= \omega\vec{k}\times\left(b\omega\cos\theta\vec{j} - b\omega\sin\theta\vec{i}\right)\\
&= -b\omega^2\cos\theta\vec{i} - b\omega^2\sin\theta\vec{j}
\end{aligned}
$$

Hence (1) becomes

$$\begin{aligned}
\vec{a} &= \vec{a}_{rel} + 2\left(\omega\vec{k}\times\vec{v}_{rel}\right) + \omega\vec{k}\times\left(\vec{\omega}\times\vec{r}\right)\\
&= \vec{i}\left(-b\dot{\theta}^2\cos\theta\right) + \vec{j}\left(-b\dot{\theta}^2\sin\theta\right) + 2\left(-\omega b\dot{\theta}\sin\theta\vec{j} - b\omega\dot{\theta}\cos\theta\vec{i}\right) - b\omega^2\cos\theta\vec{i} - b\omega^2\sin\theta\vec{j}\\
&= \vec{i}\left(-b\dot{\theta}^2\cos\theta - 2b\omega\dot{\theta}\cos\theta - b\omega^2\cos\theta\right) + \vec{j}\left(-b\dot{\theta}^2\sin\theta - 2\omega b\dot{\theta}\sin\theta - b\omega^2\sin\theta\right)
\end{aligned}$$

Since this is valid for all time, lets take snap shot when $\theta = 0$, which gives

$$\vec{a} = \vec{i}\left(-b\dot{\theta}^2 - 2b\omega\dot{\theta} - b\omega^2\right)$$

So when $\theta = 0$, the ant acceleration (as seen in inertial frame) is towards the center of the disk with the above magnitude. If the ant speed is V then $V = b\dot{\theta}$ and the above can be re-written in terms of V as

$$\vec{a} = -\vec{i}\left(\frac{V^2}{b} + 2V\omega + b\omega^2\right)$$

The ant will starts to slip, when the force preventing it from sliding radially in the outer direction equals the centrifugal force $m\left(\frac{V^2}{b} + 3V\omega + b\omega^2\right)$Hence

$$\mu mg = m\left(\frac{V^2}{b} + 2V\omega + b\omega^2\right)$$

$$\begin{aligned}
\frac{V^2}{b} + 2V\omega + b\omega^2 - \mu g &= 0\\
V^2 + 2Vb\omega - \left(\mu bg + b^2\omega^2\right) &= 0
\end{aligned}$$

This is quadratic in V, hence

$$\begin{aligned}
V &= \frac{-2b\omega}{2} \pm \frac{1}{2}\sqrt{4b^2\omega^2 + 4\left(-\mu bg + b^2\omega^2\right)}\\
&= -b\omega \pm \sqrt{b^2\omega^2 - \mu bg + b^2\omega^2}\\
&= -b\omega \pm \sqrt{2b^2\omega^2 - \mu bg}
\end{aligned}$$

Since $V > 0$ then

$$\begin{aligned}
V &= -b\omega + b\omega\sqrt{2 - \frac{\mu g}{b\omega^2}}\\
&= b\omega\left(\sqrt{2 - \frac{\mu g}{b\omega^2}} - 1\right)
\end{aligned}$$

4.8.4.2 Part(2)

When Ant is moving the opposite direction of rotation, then the Coriolis term $2\left(\omega\vec{k}\times\vec{v}_{rel}\right)$ will have the opposite sign from the above. Then means the final answer will be

$$\vec{a} = -\vec{i}\left(\frac{V^2}{b} - 2V\omega + b\omega^2\right)$$

Which means

$$
\begin{aligned}
V &= \frac{2b\omega}{2} \pm \frac{1}{2}\sqrt{4b^2\omega^2 + 4\left(-\mu bg + b^2\omega^2\right)} \\
&= b\omega \pm \sqrt{b^2\omega^2 - \mu bg + b^2\omega^2} \\
&= b\omega \pm \sqrt{2b^2\omega^2 - \mu bg}
\end{aligned}
$$

Or

$$
\begin{aligned}
V &= b\omega + b\omega\sqrt{2 - \frac{\mu g}{b\omega^2}} \\
&= b\omega\left(\sqrt{2 - \frac{\mu g}{b\omega^2}} + 1\right)
\end{aligned}
$$

4.8.5 Problem 5

5. (10 points)
(1) Show that the small angular deviation ϵ of a plumb line from the true vertical (toward the center of the Earth) at a point on Earth's surface is

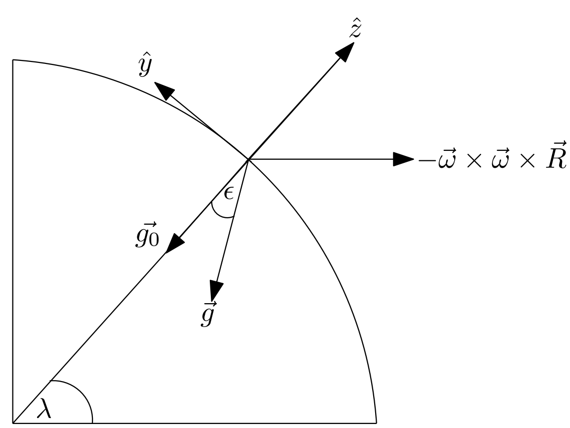
$$
\epsilon = \frac{R\,\omega^2\,\sin\lambda\,\cos\lambda}{g_0 - R\,\omega^2\cos^2\lambda}\quad ,
$$

where g_0 is the acceleration due to gravity, λ is the latitude, and R is the radius of the Earth.

(2) Use a computer to plot ϵ as a function of latitude. At what latitude do we observe the largest deviation, and how large is it?

SOLUTION:

4.8.5.1 Part(1)



$$\vec{g} = \vec{g}_0 - \vec{\omega} \times \vec{\omega} \times \vec{R}$$

Using $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ the above becomes

$$\begin{aligned}\vec{g} &= \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega} \cdot \vec{R}\right) - (\vec{\omega} \cdot \vec{\omega})\vec{R}\right) \\ &= \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega} \cdot \vec{R}\right) - \omega^2\vec{R}\right)\end{aligned}$$

Then using

$$\vec{g} \times \vec{g}_0 = gg_0 (\sin\varepsilon)\vec{n} \tag{1}$$

Where $\vec{n}$ is perpendicular to plane of $\vec{g}, \vec{g}_0$ which is $\hat{x}$ in this case. Then the LHS of the above is

$$\begin{aligned}\vec{g} \times \vec{g}_0 &= \left[\vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega} \cdot \vec{R}\right) - \omega^2\vec{R}\right)\right] \times \vec{g}_0 \\ &= \vec{g}_0 \times \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega} \cdot \vec{R}\right) \times \vec{g}_0\right) + \left(\omega^2\vec{R} \times \vec{g}_0\right)\end{aligned}$$

But $\vec{R} \times \vec{g}_0 = 0$ since they are in same direction, also $\vec{g}_0 \times \vec{g}_0 = 0$ and the above becomes

$$\vec{g} \times \vec{g}_0 = -\vec{\omega}\left(\vec{\omega} \cdot \vec{R}\right) \times \vec{g}_0 \tag{2}$$

But

$$\vec{\omega} \cdot \vec{R} = \omega R \cos\left(\frac{\pi}{2} - \lambda\right)$$

Therefore (2) becomes

$$\vec{g} \times \vec{g}_0 = -\omega R \cos\left(\frac{\pi}{2} - \lambda\right)\vec{\omega} \times \vec{g}_0$$

But $\vec{\omega} \times \vec{g}_0 = -\omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right)\hat{x}$, hence the above becomes

$$\vec{g} \times \vec{g}_0 = \omega R \cos\left(\frac{\pi}{2} - \lambda\right)\omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right)\hat{x}$$

Now we go back to (1) and apply the definition, therefore

$$\omega R \cos\left(\frac{\pi}{2} - \lambda\right)\omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right)\hat{x} = gg_0 (\sin\varepsilon)\hat{x}$$

Or

$$\begin{aligned}\omega R \cos\left(\frac{\pi}{2} - \lambda\right)\omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right) &= gg_0 (\sin\varepsilon) \\ \sin\varepsilon &= \frac{\omega R \cos\left(\frac{\pi}{2} - \lambda\right)\omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right)}{gg_0} \\ &= \frac{R\omega^2 \cos\left(\frac{\pi}{2} - \lambda\right)\sin\left(\frac{\pi}{2} - \lambda\right)}{g}\end{aligned}$$

But $\sin\left(\frac{\pi}{2} - \lambda\right) = \cos\lambda$ and $\cos\left(\frac{\pi}{2} - \lambda\right) = \sin\lambda$ hence the above becomes

$$\sin\varepsilon = \frac{R\omega^2 \sin\lambda \cos\lambda}{g} \tag{3}$$

To find $g = |\vec{g}|$, since $\vec{g} = \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)$, then taking dot product gives

$$
\begin{aligned}
|\vec{g}| &= \vec{g}\cdot\vec{g} \\
&= \left[\vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)\right]\cdot\left[\vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)\right] \\
&= g_0^2 - 2\vec{g}_0\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right) + \overbrace{\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)}^{\text{ignore. All } \omega^4 \text{ powers. too small}} \\
&\approx g_0^2 - 2\vec{g}_0\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right) \\
&= g_0^2 - \left(-2g_0\hat{z}\right)\cdot\left(\left(\omega\cos\lambda\hat{y} + \omega\sin\lambda\hat{z}\right)\left(\omega R\cos\left(\frac{\pi}{2} - \lambda\right)\right) - \omega^2 R\hat{z}\right) \\
&= g_0^2 - \left(-2g_0\hat{z}\right)\cdot\left(\left(\omega\cos\lambda\hat{y} + \omega\sin\lambda\hat{z}\right)\left(\omega R\sin\lambda\right) - \omega^2 R\hat{z}\right) \\
&= g_0^2 - \left(-2g_0\hat{z}\right)\cdot\left(\omega^2 R\sin\lambda\cos\lambda\hat{y} + \left(\omega^2 R\sin^2\lambda - \omega^2 R\right)\hat{z}\right) \\
&= g_0^2 - \left(-2g_0\left(\omega^2 R\sin^2\lambda - \omega^2 R\right)\right) \\
&= g_0^2 + 2g_0\omega^2 R\sin^2\lambda - 2g_0\omega^2 R \\
&= g_0^2 + 2g_0\omega^2 R\left(1 - \cos^2\lambda\right) - 2g_0\omega^2 R \\
&= g_0^2 + 2g_0\omega^2 R - 2g_0\omega^2 R\cos^2\lambda - 2g_0\omega^2 R \\
&= g_0^2 - 2g_0\omega^2 R\cos^2\lambda
\end{aligned}
$$

Therefore (3) becomes

$$\sin\varepsilon = \frac{R\omega^2\sin\lambda\cos\lambda}{g_0^2 - 2g_0\omega^2 R\cos^2\lambda}$$

Since ε is small, then $\sin\varepsilon \approx \varepsilon$, therefore

$$\varepsilon \approx \frac{R\omega^2\sin\lambda\cos\lambda}{g_0^2 - 2g_0\omega^2 R\cos^2\lambda}$$

The solutions has an extra g_0 in the denominator. I am not sure why. I will what is given for part(2) to plot it.

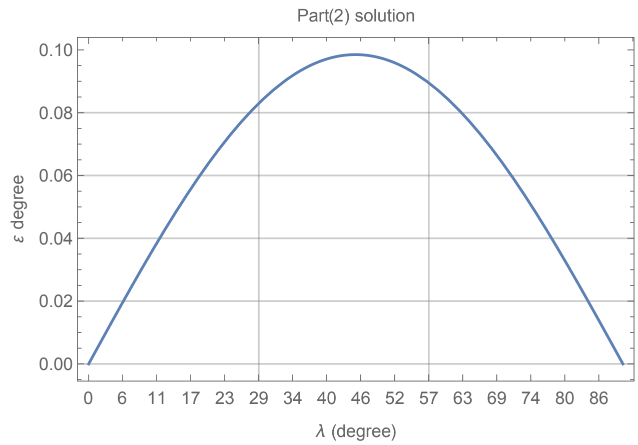
4.8.5.2 Part(2)

This plot shows the maximum ε is at $\lambda = 45^0$. Here is the code used and the plot generated

```
R0 = 6371*10^3; (*earth radius*)
omega = 7.27*10^(-5); (*earth rotation*)
g0 = 9.81;
e[lam_] := (R0 omega^2 Sin[lam] Cos[lam])/(g0 - R0 omega^2 Cos[lam]^2)*180/Pi;
newTicks[min_, max_] := Table[{i, Round[i*180/Pi]}, {i, 0, Pi/2, .1}];

Plot[e[lam], {lam, 0, Pi/2}, Frame -> True,
 FrameLabel -> {{"\[CurlyEpsilon] degree", None}, {"\[Lambda] (degree)",
    "Part(2) solution"}}, GridLines -> Automatic,
 FrameTicks -> {{Automatic, Automatic}, {newTicks, Automatic}}]
```

Out[80]=



4.8.6 HW 8 key solution

Mechanics

Physics 311 - Fall 2015

Homework Set 8 - Solutions

Problem 1

$$F(r) = \frac{k}{r^3} \qquad U(r) = -\int F(r)\,dr = \frac{k}{2r^2}$$

from the lecture:

$$\theta_0 = \int_{r_{min}}^{\infty} \frac{b\;dr}{r^2\sqrt{\left(1-\frac{U}{E}\right)-\frac{b^2}{r^2}}}$$

r_{min} is the distance of closest approach; at r_{min},

$E = U_{eff}$, so

$$E = \frac{k}{2r_{min}^2} + \frac{\ell^2}{2mr_{min}^2}$$

$$\Rightarrow \quad r_{min}^2 = \frac{k + \ell^2/m}{2E}$$

and with $\quad \ell^2 = b^2\,(2mE)\;, \qquad r_{min}^2 = \frac{k}{2E} + b^2$

now substitute $\quad z = \frac{1}{r} \quad , \quad \frac{dz}{dr} = -\frac{1}{r^2}$

2

$$\Rightarrow \quad \Theta_0(b) = \int_0^{z_{max}} \frac{b\ dz}{\sqrt{1 - \frac{U}{E} - b^2 z^2}} \qquad z_{max} = \left(\frac{u}{2E} + b^2\right)^{-1/2}$$

$$= \int_0^{z_{max}} \frac{b\ dz}{\sqrt{1 - \left[b^2 + \frac{u}{2E}\right] z^2}}$$

$$= \frac{b}{\sqrt{b^2 + u/2E}} \int_0^{z_{max}} \frac{1}{\sqrt{(b^2 + u/2E)^{-1} - z^2}}$$

now use $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = a\sin \frac{x}{a} + \text{const.}$

$$\Rightarrow \quad = \frac{b}{\sqrt{b^2 + \frac{K}{2E}}} \ a\sin \frac{z}{\left(b^2 + \frac{K}{2E}\right)^{-1/2}} \Bigg|_0^{\left(\frac{K}{2E} + b^2\right)^{-1/2}}$$

$$= \frac{b}{\sqrt{b^2 + K/2E}} \left(a\sin 1 - a\sin 0 \right)$$

$$= \frac{\pi b}{2\sqrt{b^2 + \frac{K}{2E}}}$$

we need $b(\theta)$, so we solve for b :

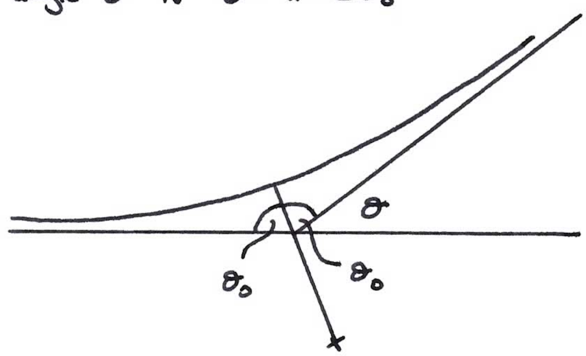
$$\Theta_0^2 \left(4 b^2 + \frac{2u}{E}\right) = \pi^2 b^2$$

3

$$b^2\left(4\theta_0^2-\pi^2\right)=-\frac{2k}{E}\theta_0^2$$

$$\Rightarrow \quad b=\sqrt{\frac{k}{2E}}\frac{2\theta_0}{\sqrt{\pi^2-4\theta_0^2}}$$

the scattering angle θ is $\theta=\pi-2\theta_0$



so $\theta_0=\frac{1}{2}(\pi-\theta)$

$$\Rightarrow \quad b=\sqrt{\frac{k}{2E}}\frac{\pi-\theta}{\sqrt{\pi^2-(\pi-\theta)^2}}$$

$$=\sqrt{\frac{k}{2E}}\frac{\pi-\theta}{\sqrt{\theta(2\pi-\theta)}}$$

now use $\sigma(\theta)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$

$$\Rightarrow \quad \sigma(\theta)=\sqrt{\frac{k}{2E}}\frac{\pi-\theta}{A\sin\theta}\left|\frac{-A-(\pi-\theta)\frac{1}{2}\frac{2\pi-2\theta}{A}}{A^2}\right|$$

where $A=\sqrt{\theta(2\pi-\theta)}$

$$\Rightarrow \quad \sigma(\theta) = \frac{k\,(\pi-\theta)}{2E\sin\theta\;A} \quad \frac{A\left(1+\frac{(\pi-\theta)^2}{\theta(2\pi-\theta)}\right)}{A^2}$$

$$= \frac{k\,(\pi-\theta)}{2E\sin\theta} \quad \frac{\theta(2\pi-\theta)+\pi^2-2\pi\theta+\theta^2}{\theta^2(2\pi-\theta)^2}$$

$$= \frac{k\,(\pi-\theta)}{2E\sin\theta} \quad \frac{\pi^2}{\theta^2(2\pi-\theta)^2}$$

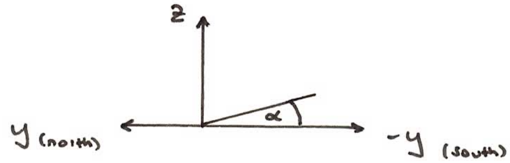
and with $E = \frac{1}{2}mv^2$

$$\boxed{\sigma(\theta) = \frac{k\,\pi^2(\pi-\theta)}{m\,v^2\,\theta^2(2\pi-\theta)^2\sin\theta}}$$

5

Problem 2

(1)



$$\dot{y}_0 = -V_0 \cos\alpha$$

$$\dot{z}_0 = V \sin\alpha$$

$$x_0 = y_0 = z_0 = 0$$

so

$$z = V_0 t \sin\alpha - \frac{1}{2} g t^2$$

$\Rightarrow$ time to reach $z=0$ $\quad t = \dfrac{2 V_0 \sin\alpha}{g}$

$$x' = \frac{1}{3} \omega g t^3 \cos\lambda - \omega t^2 \left(V_0 \sin\alpha \cos\lambda + V_0 \cos\alpha \sin\lambda\right)$$

$$= \frac{1}{3} \omega g t^3 \cos\lambda - \omega t^2 V_0 \sin(\alpha + \lambda)$$

$$= \omega t^2 \left(\frac{1}{3} g t \cos\lambda - V_0 \sin(\alpha + \lambda)\right)$$

now

$$t = \frac{2 \cdot 800\ ^m/_s \sin 37^\circ}{9.8\ ^m/_{s^2}} = 98.3\,s$$

$$x' = \frac{2\pi}{24 \cdot 3600\,s} (98.3\,s)^2 \left[\frac{1}{3}\, 9.8 \frac{m}{s}\, 98.3\,s \cos(-50^\circ) - 800 \frac{m}{s} \sin(37^\circ - 50^\circ)\right]$$

$$= \underline{\underline{272\ m}} \quad \text{(east)}$$

6

(2) total deflection of a projectile is

$$\Delta = \omega \frac{h^2}{V_0} \sin \lambda$$

V_0 is the initial speed and h is the range of the baseball

$\lambda = 41°$ (Yankee stadium)

the horizontal range in an <u>inertial</u> reference frame is

$$h = V_0 \sin\alpha \cdot t$$

with t from $y_0 = -\frac{1}{2} g t^2 + V_0 \cos\alpha \, t \Rightarrow t = \frac{2 V_0 \cos\alpha}{g}$

$$\Rightarrow h = \frac{V_0^2}{g} 2 \sin\alpha \cos\alpha$$

$$\Leftrightarrow h = \frac{V_0^2 \sin 2\alpha}{g}$$ neglecting air resistance

So $$\Delta = \frac{\omega \sin\lambda}{V_0} h^2 = \omega \sin\lambda \, h^2 \sqrt{\frac{\sin 2\alpha}{g h}}$$

$$= \omega \sin\lambda \sqrt{\frac{\sin 2\alpha}{g} h^3}$$

$h = 200 \text{ ft} = 200 \cdot 0.3048 \text{ m} = 61.0 \text{ m}$

$$\Rightarrow \Delta = 7.27 \cdot 10^{-5} \text{ s}^{-1} \sin 41° \sqrt{\frac{\sin 30°}{9.8 \text{ ms}^{-2}} (61.0 \text{ m})^3}$$

$$= \underline{\underline{0.5 \text{ cm}}}$$

not a big deal..

1

Problem 3

use the general equations of motion from class, with initial conditions

$$x_0 = y_0 = z_0 = 0$$

$$\dot{x}_0 = \dot{y}_0 = 0$$

$$\dot{z} = v_0$$

$$\Rightarrow \quad x(t) = \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 v_0 \cos\lambda$$

$$y(t) = 0$$

$$z(t) = -\frac{1}{2} g t^2 + v_0 t$$

when the bullet hits the ground $z(t)=0$, so

$$t = \frac{2v_0}{g}$$

$$\Rightarrow \quad x(t) = \frac{1}{3}\omega g \left(\frac{8v_0^3}{g^3}\right)\cos\lambda - \omega\left(\frac{4v_0^2}{g^2}\right) v_0 \cos\lambda$$

$$= \frac{\omega v_0^3}{g^2}\cos\lambda\left(\frac{8}{3} - 4\right)$$

$$= -\frac{4}{3}\frac{\omega v_0^3}{g^2}\cos\lambda$$

in the northern hemisphere, $x(t) < 0$, and the bullet thus lands <u>west</u> of the initial point

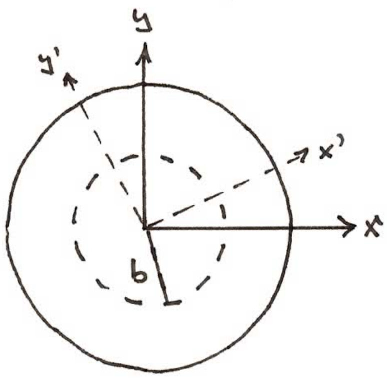
Problem 4

in a non-inertial frame, the equation of motion is

$$\vec{F} - 2m\,\vec{\omega}\times\vec{v}' - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}') = m\vec{a}'$$

here,
$$\vec{a}' = -\frac{\vec{v}'^2}{b}\,\hat{r}$$

$$\vec{\omega} = \omega\,\hat{z} \qquad \vec{v}' = v'\,\hat{\theta}$$
$$\vec{r} = b\,\hat{r}$$



so
$$\vec{\omega}\times(\vec{\omega}\times\vec{r}') = -b\,\omega^2\,\hat{r}$$
$$\vec{\omega}\times\vec{v}' = -\omega v'\,\hat{r}$$

$$\Rightarrow \vec{a} = \vec{a}' + 2\,\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$\Rightarrow \vec{a} = -\frac{v'^2}{b}\,\hat{r} - 2\omega v'\,\hat{r} - b\omega^2\,\hat{r}$$

for no slipping $\quad |\vec{F}| < \mu_s m g$

Let v_s' be the velocity for which slipping starts;
then

$$\frac{v_s'^2}{b} + 2\omega v_s' + b\omega^2 = \mu_s g$$

so $\quad v_s'^2 + 2\omega b\, v_s' + b^2\omega^2 - b\mu_s g = 0$

$$\Rightarrow \quad v_s' = -\omega b \pm \sqrt{\omega^2 b^2 - b^2\omega^2 + b\mu_s g}$$

v_s' is defined positive, so $\quad \boxed{v_s' = -\omega b + \sqrt{b\mu_s g}}$

if the bug crawls opposite to the direction of rotation, $\vec{v}' = -v'\hat{\theta}$,

so

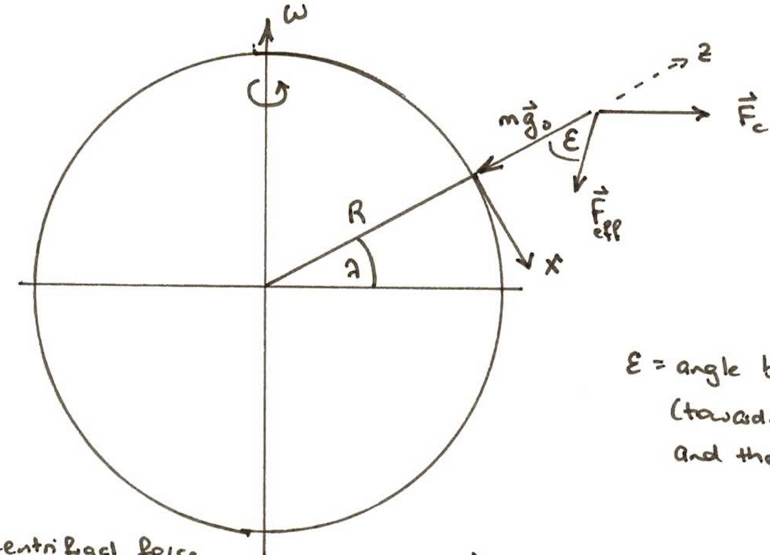
$$\vec{\omega} \times \vec{v}' = \omega v' \hat{r}$$

$$\Rightarrow \quad \vec{a} = -\frac{v'^2}{b}\hat{r} + \omega v' \hat{r} - b\omega^2 \hat{r}$$

so $\quad v_s'^2 - 2\omega b\, v_s' + b^2\omega^2 - b\mu_s g = 0$

$$\Rightarrow \quad \boxed{v_s' = \omega b + \sqrt{b\mu_s g}}$$

Problem 5



ε = angle between $\vec{g}_0$ (towards Earth's center) and the plumb line

$\vec{F}_c$ = centrifugal force

$\vec{F}_{eff}$ is the direction of the plumb line

$$\vec{F}_c = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{F}_{eff} = -m g_0\,\hat{z} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{r} = \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix} \qquad \vec{\omega} = \begin{pmatrix} -\omega\cos\lambda \\ 0 \\ \omega\sin\lambda \end{pmatrix}$$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & R \end{vmatrix} = R\omega\cos\lambda\ \hat{y}$$

$$\vec{F}_{eff} = -m g_0\,\hat{z} - m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & R\omega\cos\lambda & 0 \end{vmatrix}$$

$$\vec{F}_{eff} = -m g_0 \hat{z} - m(-\omega^2 \cos^2\lambda\, R\, \hat{z} - R\omega^2 \cos\lambda \sin\lambda\, \hat{x})$$

$$= (m R \omega^2 \cos\lambda \sin\lambda)\hat{x} + (m\omega^2 \cos^2\lambda\, R - m g_0)\hat{z}$$

now use

$$\tan\varepsilon = \frac{|(F_{eff})_x|}{|(F_{eff})_z|}$$

$$= \frac{R\omega^2 \sin\lambda \cos\lambda}{g_0 - R\omega^2\cos^2\lambda}$$

and since ε is small, $\tan\varepsilon \simeq \varepsilon$,

so

$$\boxed{\varepsilon = \frac{R\omega^2 \sin\lambda \cos\lambda}{g_0 - R\omega^2 \cos^2\lambda}}$$

$R = 6.4 \cdot 10^6\ m$

$\omega = 7.3 \cdot 10^{-5}\ s^{-1}$

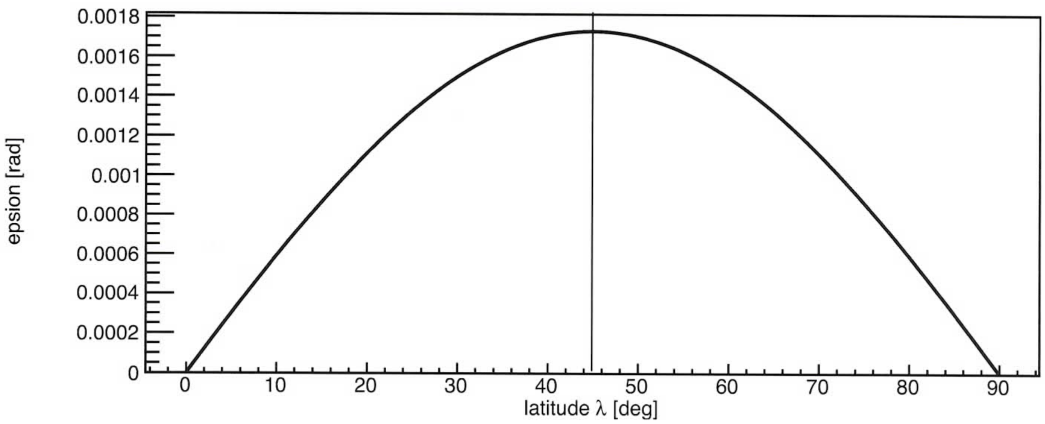
$g_0 = 9.8\ m/s^2$

the function is plotted on the next page; it has a maximum at $\underline{\underline{\lambda = 45^\circ}}$

at $\lambda = 45^\circ$,
$$\varepsilon = \frac{6.4\cdot 10^6\ (7.3\cdot 10^{-5})^2 \cos 45^\circ \sin 45^\circ}{9.8 - 6.4\cdot 10^6\ (7.3\cdot 10^{-5})^2 \cos^2 45^\circ}$$

$$= 0.0017 \text{ rad}$$

$$\simeq \underline{\underline{0.1^\circ}}$$


0.0018
0.0016
0.0014
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
epsion [rad]
0
10
20
30
40
50
60
70
80
90
latitude λ [deg]

4.9 HW 9

4.9.1 Problem 1

1. (5 points)
A rigid body of arbitrary shape rotates freely under zero torque. Use Euler's equations to show that the rotational kinetic energy and the magnitude of the angular momentum are constant.

SOLUTION:

Euler solid body rotation equations are

$$(I_2 - I_3)\,\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \tag{1}$$
$$(I_3 - I_1)\,\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \tag{2}$$
$$(I_1 - I_2)\,\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \tag{3}$$

Where I_1, I_2, I_3 are the body moments of inertia around the principal axes. Multiplying both sides of (1) by $I_1\omega_1$ and both sides of (2) by $I_2\omega_2$ and both sides of (3) by $I_3\omega_3$ gives

$$\omega_1\omega_2\omega_3 I_1 I_2 - \omega_1\omega_2\omega_3 I_1 I_3 - I_1^2\omega_1\dot{\omega}_1 = 0 \tag{1A}$$
$$\omega_1\omega_2\omega_3 I_2 I_3 - \omega_1\omega_2\omega_3 I_1 I_2 - I_2^2\omega_2\dot{\omega}_2 = 0 \tag{2A}$$
$$\omega_1\omega_2\omega_3 I_1 I_3 - \omega_1\omega_2\omega_3 I_2 I_3 - I_3^2\omega_3\dot{\omega}_3 = 0 \tag{3A}$$

Adding (1A,2A,3A) gives (lots of terms cancel, that has $\omega_1\omega_2\omega_3$ in them)

$$I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 = 0 \tag{4}$$

But (4) is the same thing as

$$\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 = 0$$

where $\boldsymbol{L}$ is the angular momentum vector

$$\boldsymbol{L} = \{I_1\omega_1, I_2\omega_2, I_3\omega_3\}$$

Hence

$$\boldsymbol{L}^2 = \boldsymbol{L}\cdot\boldsymbol{L} = \left\{I_1^2\omega_1^2, I_2^2\omega_2^2, I_3^2\omega_3^2\right\}$$

Therefore, and since the $I's$ are constant, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 &= \frac{1}{2}\left\{2I_1^2\omega_1\dot{\omega}_1, 2I_2^2\omega_2\dot{\omega}_2, 2I_3^2\omega_3\dot{\omega}_3\right\}\\ &= \left\{I_1^2\omega_1\dot{\omega}_1, I_2^2\omega_2\dot{\omega}_2, I_3^2\omega_3\dot{\omega}_3\right\} \qquad (5)\end{aligned}$$

Comparing (5) and (4), we see they are the same. This means that $\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 = 0$ or $\boldsymbol{L}^2$ is a constant. Which implies $\boldsymbol{L}$ or the angular momentum is a constant vector.

To show that rotational kinetic energy is constant, we need to show that $\frac{1}{2}(\omega\cdot\boldsymbol{L})$ (which is the kinetic energy) is constant, where $\boldsymbol{\omega} = \{\omega_1, \omega_2, \omega_3\}$ is the angular velocity vector. But

$$\frac{1}{2}\frac{d}{dt}(\omega \cdot \boldsymbol{L}) = \frac{1}{2}\left(\dot{\omega} \cdot \boldsymbol{L} + \omega \cdot \dot{\boldsymbol{L}}\right)$$

But we found that $\dot{\boldsymbol{L}} = 0$ since $\boldsymbol{L}$ is constant. Hence the above becomes

$$\frac{1}{2}\frac{d}{dt}(\omega \cdot \boldsymbol{L}) = \frac{1}{2}\,\dot{\omega} \cdot \boldsymbol{L} \tag{6}$$

If we can show that $\dot{\omega} \cdot \boldsymbol{L} = 0$ then we are done. To do this, we go back to Euler equations (1,2,3) and now instead of multiplying by $I_i\omega_i$ as before, we now multiply by just ω_i each equation. This gives

$$\omega_1\omega_2\omega_3 I_2 - \omega_1\omega_2\omega_3 I_3 - I_1\omega_1\dot{\omega}_1 = 0 \tag{1C}$$
$$\omega_1\omega_2\omega_3 I_3 - \omega_1\omega_2\omega_3 I_1 - I_2\omega_2\dot{\omega}_2 = 0 \tag{2C}$$
$$\omega_1\omega_2\omega_3 I_1 - \omega_1\omega_2\omega_3 I_2 - I_3\omega_3\dot{\omega}_3 = 0 \tag{3C}$$

Adding gives (lots of terms cancel, that has $\omega_1\omega_2\omega_3$ in them)

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0 \tag{7}$$

But the above is the same as (6), with a factor of $\frac{1}{2}$. This means $\dot{\omega} \cdot \boldsymbol{L} = 0$ or $\frac{d}{dt}(\omega \cdot \boldsymbol{L}) = 0$ or that the rotational kinetic energy is constant. Which is what we are asked to show.

4.9.2 Problem 2

2. (10 points)
   A uniform block of mass m and dimensions a by $2a$ by $3a$ spins about a long diagonal with angular velocity $\vec{\omega}$.

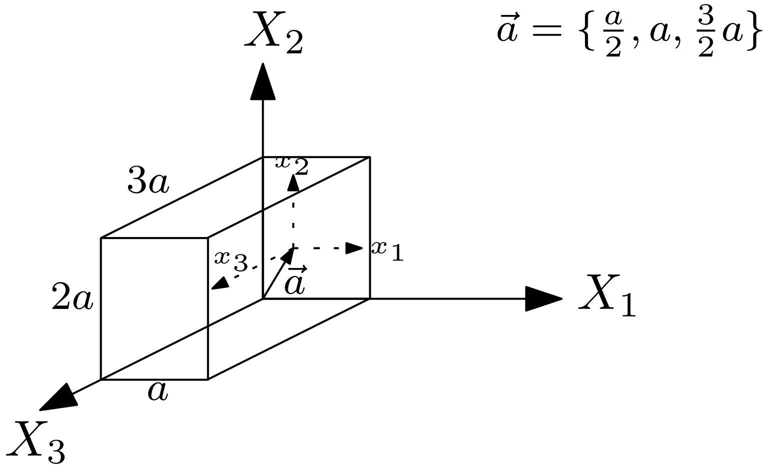
   (1) Using a coordinate system with the origin at the center of the block, calculate the inertia tensor.

   (2) Find the kinetic energy.

   (3) Find the angle between the angular velocity $\vec{\omega}$ and the angular momentum $\vec{L}$.

   (4) Find the magnitude of the torque that must be exerted on the block if $\vec{\omega}$ is constant.

SOLUTION:



4.9.2.1 Part (1)

We first find I (called J for now) around the origin of the inertial frame X_1, X_2, X_3 then use parallel axes theorem to find I at the center of the cube at $a = \left\{\frac{1}{2}a, a, \frac{3}{2}a\right\}$. The volume of the cube is $a(2a)(3a) = 6a^3$.

$$
\begin{aligned}
J_{11} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_2^2 + X_3^2\right) \\
&= \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 X_2^2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[a(3a) \int_0^{2a} dX_2 X_2^2\right] + \rho \left[a(2a) \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[a(3a) \left(\frac{X_2^3}{3}\right)_0^{2a}\right] + \rho \left[a(2a) \left(\frac{X_3^3}{3}\right)_0^{3a}\right] \\
&= \rho \left[3a^2 \frac{(2a)^3}{3}\right] + \rho \left[2a^2 \frac{(3a)^3}{3}\right] \\
&= \rho \left[3a^2 \frac{8a^3}{3}\right] + \rho \left[2a^2 \frac{27a^3}{3}\right] \\
&= \rho 8a^5 + \rho \frac{54a^5}{3} \\
&= 26a^5 \rho \\
&= \frac{26}{6} a^2 \left(6a^3 \rho\right) \\
&= \frac{13}{3} M a^2
\end{aligned}
$$

And

$$\begin{aligned}
J_{12} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1 X_2\right) \\
&= -\rho \int_0^a X_1 dX_1 \int_0^{2a} X_2 dX_2 \int_0^{3a} dX_3 \\
&= -\rho \left(\frac{X_1^2}{2}\right)_0^a \left(\frac{X_2^2}{2}\right)_0^{2a} 3a \\
&= -\rho \left(\frac{a^2}{2}\right)\left(\frac{4a^2}{2}\right) 3a \\
&= -3a^5 \rho \\
&= -\frac{3}{6} a^2 \left(6a^3 \rho\right) \\
&= -\frac{1}{2} M a^2
\end{aligned}$$

And

$$\begin{aligned}
J_{13} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1 X_3\right) \\
&= -\rho \int_0^a X_1 dX_1 \int_0^{2a} X_2 \int_0^{3a} X_3 dX_3 \\
&= -\rho \left(\frac{X_1^2}{2}\right)_0^a 2a \left(\frac{X_3^2}{2}\right)_0^{3a} \\
&= -\rho \frac{a^2}{2} 2a \frac{9a^2}{2} \\
&= -\frac{9}{2} a^5 \rho \\
&= -\frac{9}{2\left(6\right)} a^2 \left(6a^3 \rho\right) \\
&= -\frac{3}{4} M a^2
\end{aligned}$$

And $J_{21} = J_{12}$ and

$$
\begin{aligned}
J_{22} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1^2 + X_3^2\right) \\
&= \rho \left[\int_0^a X_1^2 dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[\left(\frac{X_1^3}{3}\right)_0^a (2a)(3a)\right] + \rho \left[a(2a)\left(\frac{X_3^3}{3}\right)_0^{3a}\right] \\
&= \rho \left[\frac{a^3}{3}(2a)(3a)\right] + \rho \left[a(2a)\frac{(3a)^3}{3}\right] \\
&= \rho \left[\frac{6a^5}{3}\right] + \rho \left[2a^2 \frac{27a^3}{3}\right] \\
&= \rho 2a^5 + 18a^5 \rho \\
&= 20a^5 \rho \\
&= \frac{20}{6} a^2 \left(6a^3 \rho\right) \\
&= M \frac{20}{6} a^2
\end{aligned}
$$

And

$$
\begin{aligned}
J_{23} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_2 X_3\right) \\
&= -\rho \int_0^a X_1 \int_0^{2a} X_2 dX_2 \int_0^{3a} X_3 dX_3 \\
&= -\rho a \left(\frac{X_2^2}{2}\right)_0^{2a} \left(\frac{X_3^2}{2}\right)_0^{3a} \\
&= -\rho a \left(\frac{4a^2}{2}\right)\left(\frac{9a^2}{2}\right) \\
&= -9a^5 \rho \\
&= -\frac{9}{6} a^2 \left(6a^3 \rho\right) \\
&= -\frac{9}{6} M a^2
\end{aligned}
$$

And $J_{31} = J_{13}$ and $J_{32} = J_{23}$ and

$$
\begin{aligned}
J_{33} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1^2 + X_2^2\right) \\
&= \rho \left[\int_0^a X_1^2 dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} X_2^2 dX_2 \int_0^{3a} dX_3\right] \\
&= \rho \left[\left(\frac{X_1^3}{3}\right)_0^a (2a)(3a)\right] + \rho \left[a \left(\frac{X_2^3}{3}\right)_0^{2a} 3a\right] \\
&= \rho \left[\frac{a^3}{3}(2a)(3a)\right] + \rho \left[a \left(\frac{8a^3}{3}\right) 3a\right] \\
&= \rho 2a^5 + \rho 8a^5 \\
&= 10a^5\rho \\
&= \frac{10}{6} a^2 \left(6a^3\rho\right) \\
&= M\frac{10}{6}a^2
\end{aligned}
$$

Therefore

$$
J = Ma^2 \begin{pmatrix} \frac{13}{3} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{20}{6} & -\frac{9}{6} \\ -\frac{3}{4} & -\frac{9}{6} & \frac{10}{6} \end{pmatrix}
$$

We now find I around the center of the cube where the position vector of the center is $\vec{a} = \left\{\frac{1}{2}a, a, \frac{3}{2}a\right\}$. Therefore

$$
\begin{aligned}
I_{11} &= J_{11} - M\left(\vec{a}^2 - a_1^2\right) \\
&= Ma^2\frac{13}{3} - M\left(a_2^2 + a_3^2\right) \\
&= Ma^2\frac{13}{3} - M\left(a^2 + \left(\frac{3}{2}a\right)^2\right) \\
&= \frac{13}{12}Ma^2
\end{aligned}
$$

And

$$
\begin{aligned}
I_{12} &= J_{12} - M\left(-a_1 a_2\right) \\
&= -Ma^2\frac{1}{2} - M\left(-\left(\frac{1}{2}a\right)a\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}
I_{13} &= J_{13} - M\left(-a_1 a_3\right) \\
&= -Ma^2\frac{3}{4} - M\left(-\left(\frac{1}{2}a\right)\frac{3}{2}a\right) \\
&= 0
\end{aligned}$$

And $I_{21} = I_{12}$ And

$$\begin{aligned}
I_{22} &= J_{22} - M\left(\vec{a}^2 - a_2^2\right) \\
&= Ma^2\frac{20}{6} - M\left(a_1^2 + a_3^2\right) \\
&= Ma^2\frac{20}{6} - M\left(\left(\frac{1}{2}a\right)^2 + \left(\frac{3}{2}a\right)^2\right) \\
&= \frac{5}{6}Ma^2
\end{aligned}$$

And

$$\begin{aligned}
I_{23} &= J_{23} - M\left(-a_2 a_3\right) \\
&= -Ma^2\frac{9}{6} - M\left(-(a)\frac{3}{2}a\right) \\
&= 0
\end{aligned}$$

And $I_{31} = I_{31}$ and $I_{32} = I_{23}$ and

$$\begin{aligned}
I_{33} &= J_{33} - M\left(\vec{a}^2 - a_3^2\right) \\
&= Ma^2\frac{10}{6} - M\left(a_1^2 + a_2^2\right) \\
&= Ma^2\frac{10}{6} - M\left(\left(\frac{1}{2}a\right)^2 + a^2\right) \\
&= \frac{5}{12}Ma^2
\end{aligned}$$

Therefore the moment of inertia tensor around the center of mass is

$$I = Ma^2\begin{pmatrix} \frac{13}{12} & 0 & 0 \\ 0 & \frac{10}{12} & 0 \\ 0 & 0 & \frac{5}{12} \end{pmatrix}$$

4.9.2.2 Part(2)

The kinetic energy is $\frac{1}{2}\omega \cdot L$ where $\omega = \{\omega_1, \omega_2, \omega_3\}$ and

$$
\begin{aligned}
L &= I\omega \\
&= Ma^2 \begin{pmatrix} \frac{13}{12} & 0 & 0 \\ 0 & \frac{10}{12} & 0 \\ 0 & 0 & \frac{5}{12} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{12} Ma^2 \omega_1 \\ \frac{10}{12} Ma^2 \omega_2 \\ \frac{5}{12} Ma^2 \omega_3 \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
T &= \frac{1}{2}\omega \cdot L = \frac{1}{2}\left(\frac{13}{12} Ma^2 \omega_1^2 + \frac{10}{12} Ma^2 \omega_2^2 + \frac{5}{12} Ma^2 \omega_3^2\right) \\
&= \frac{1}{24} Ma^2 \left(13\omega_1^2 + 10\omega_2^2 + 5\omega_3^2\right)
\end{aligned}
$$

Since body is rotating around the long diagonal. The long diagonal has length $\sqrt{a^2 + (2a)^2 + (3a)^2} = \sqrt{14}a$, therefore

$$
\omega = \frac{\omega}{\sqrt{14}a} \{a, 2a, 3a\} = \frac{\omega}{\sqrt{14}} \{1, 2, 3\}
$$

and the above becomes

$$
\begin{aligned}
T &= \frac{1}{24} Ma^2 \omega^2 \left(\frac{13}{14} + 10\left(\frac{4}{14}\right) + 5\left(\frac{9}{14}\right)\right) \\
&= \frac{7}{24} Ma^2 \omega^2
\end{aligned}
$$

4.9.2.3 Part(3)

Using

$$
\begin{aligned}
\omega \cdot L &= |\omega|\,|L| \cos\theta \\
\cos\theta &= \frac{\omega \cdot L}{|\omega|\,|L|} \\
&= \frac{\frac{14}{24} M a^2 \omega^2}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\sqrt{\left(\frac{13}{12} M a^2 \omega_1\right)^2 + \left(\frac{10}{12} M a^2 \omega_2\right)^2 + \left(\frac{5}{12} M a^2 \omega_3\right)^2}} \\
&= \frac{\frac{14}{24} M a^2 \omega^2}{\sqrt{\left(\frac{\omega}{\sqrt{14}}\right)^2 + \left(\frac{2\omega}{\sqrt{14}}\right)^2 + \left(\frac{3\omega}{\sqrt{14}}\right)^2}\sqrt{\left(\frac{13}{12} M a^2 \frac{\omega}{\sqrt{14}}\right)^2 + \left(\frac{10}{12} M a^2 \frac{2\omega}{\sqrt{14}}\right)^2 + \left(\frac{5}{12} M a^2 \frac{3\omega}{\sqrt{14}}\right)^2}} \\
&= \frac{\frac{14}{24} M a^2 \omega^2}{\sqrt{\omega^2}\sqrt{\frac{397}{1008} M^2 a^4 \omega^2}} \\
&= \frac{\frac{14}{24}}{\sqrt{\frac{397}{1008}}} \\
&= 0.92951
\end{aligned}
$$

Hence

$$
\boxed{\theta = 21.64^0}
$$

4.9.2.4 Part(4)

Since

$$
\begin{aligned}
\tau_{external} &= \frac{d}{dt}\left(\boldsymbol{L}\right)_{inertial} \\
&= \frac{d}{dt}\left(\boldsymbol{L}\right)_{body} + \omega \times \boldsymbol{L}
\end{aligned}
$$

But $\frac{d}{dt}(\boldsymbol{L})_{body} = 0$ since $\boldsymbol{L} = I\omega$ and I is constant and ω is constant. Therefore

$$
\begin{aligned}
\tau &= \omega \times \boldsymbol{L} \\
&= \omega \times I\omega \\
&= \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \times \begin{pmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \\
&= \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \times \begin{pmatrix}I_1\omega_1\\ I_2\omega_2\\ I_3\omega_3\end{pmatrix} \\
&= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \omega_1 & \omega_2 & \omega_3\\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3\end{vmatrix} \\
&= \boldsymbol{i}\left(I_3\omega_2\omega_3 - I_2\omega_2\omega_3\right) - \boldsymbol{j}\left(I_3\omega_3\omega_1 - I_1\omega_1\omega_3\right) + \boldsymbol{k}\left(I_2\omega_2\omega_1 - I_1\omega_1\omega_2\right) \\
&= \begin{pmatrix}\omega_2\omega_3\left(I_3 - I_2\right)\\ \omega_3\omega_1\left(I_1 - I_3\right)\\ \omega_2\omega_1\left(I_2 - I_1\right)\end{pmatrix}
\end{aligned}
$$

The above are Euler equations for constant ω, and could have been written down directly from Euler equations by setting all the $\dot{\omega}_i = 0$ also.

Now, since $\omega = \frac{\omega}{\sqrt{14}}\{1,2,3\}$ and $I_1 = \frac{13}{12}Ma^2, I_2 = \frac{10}{12}Ma^2, I_3 = \frac{5}{12}Ma^2$, Therefore the above torque becomes

$$
\begin{aligned}
\tau &= \frac{\omega^2}{14}Ma^2\begin{pmatrix}6\left(\frac{5}{12} - \frac{10}{12}\right)\\ 3\left(\frac{13}{12} - \frac{5}{12}\right)\\ 2\left(\frac{10}{12} - \frac{13}{12}\right)\end{pmatrix} \\
&= \frac{\omega^2}{14}Ma^2\begin{pmatrix}-\frac{5}{2}\\ 2\\ -\frac{1}{2}\end{pmatrix} \\
&= \omega^2 Ma^2\begin{pmatrix}-\frac{5}{28}\\ \frac{1}{7}\\ -\frac{1}{28}\end{pmatrix} \\
&= \omega^2 Ma^2\begin{pmatrix}-0.1786\\ 0.1429\\ -0.0357\end{pmatrix}
\end{aligned}
$$

Units check: $\frac{1}{T^2}ML^2 = [N][L]$ units of torque. OK. The above is the external torque exerted on the block.

4.9.3 Problem 3

3. (10 points)
Consider a simple top consisting of a heavy circular disc of mass m and radius a mounted at the center of a thin rod of mass $m/2$ and length a. The top is set spinning at a rate S with the axis at an angle 45° with the vertical.

(1) Show that there are two possible values of the precession rate $\dot{\phi}$ such that the top precesses steadily at a constant value of $\theta = 45°$.

(2) Calculate the numerical values for $\dot{\phi}$ if $S = 900\,\text{rpm}$ and $a = 10\,\text{cm}$.

(3) If a top is set spinning sufficiently fast and is started in a vertical position, the axis remains steady in the upright position. This is called a "sleeping top." How fast must the top spin to sleep in the vertical position?

SOLUTION:

4.9.3.1 Part (1)

Starting with the Euler equations for Gyroscope precession, equations 9.71. in textbook, page 371, Analytical mechanics, 6th edition, by Fowles and Cassiday

$$
\begin{aligned}
Mgl\sin\theta &= I_x\ddot{\theta} + I_z S\dot{\phi}\sin\theta - I_y\dot{\phi}^2\cos\theta\sin\theta \\
0 &= I_y\frac{d}{dt}\left(\dot{\phi}\sin\theta\right) - I_z S\dot{\theta} + I_x\dot{\theta}\dot{\phi}\cos\theta \qquad (1)\\
0 &= I_z\dot{S}
\end{aligned}
$$

Where the spin of the disk S around its own z body axis is

$$S = \dot{\psi} + \dot{\phi}\cos\theta$$

Instead of drawing this again, which would take sometime, I am showing the diagram from the book above, page 371 for illustration

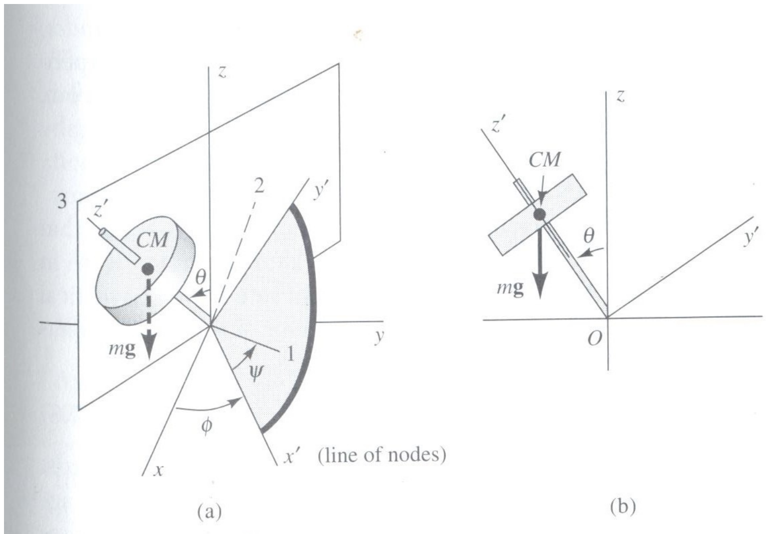


Figure 9.7.1 The simple gyroscope.

In (1), the length l is the distance from center of mass of the combined disc and rod, to the origin of the inertial frame. This will be $l=\frac{a}{2}$. M is the total mass of both the disc and the rod, which will be $M=\frac{3}{2}m$.

We are told that $\theta(t)$ is constant. Hence $\ddot{\theta}=0$ and first equation in (1) becomes

$$
\begin{aligned}
Mgl\sin\theta &= I_z S\dot{\phi}\sin\theta - I_y\dot{\phi}^2\cos\theta\sin\theta\\
Mgl &= I_z S\dot{\phi} - I_y\dot{\phi}^2\cos\theta
\end{aligned}
$$

This is quadratic in $\dot{\phi}$. Solving gives

$$
\begin{aligned}
I_y\dot{\phi}^2\cos\theta - I_z S\dot{\phi} + Mgl = 0&\\
\dot{\phi} &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{I_z S\pm\sqrt{I_z^2S^2-4I_y\cos\theta Mgl}}{2\cos\theta I_y} \qquad (2)
\end{aligned}
$$

The only thing left is to calculate I_z and I_y for the disc and the rod about the mass center, then use parallel axes theorem to move this to the pivot, which is the origin of the inertial frame.

Due to symmetry, the center of mass for both disk and rod is located distance $\frac{a}{2}$ from pivot. Hence $l=\frac{a}{2}$. For the disc, its moment of inertial around the spin axes at its center of mass is

$$
(I_z)_{disk} = m\frac{a^2}{2}
$$

And along the y axis $I_y=m\frac{a^2}{4}$. Since the distance of the center of mass from the pivot is $\frac{a}{2}$, we need to adjust I_y by this distance using parallel axes. Hence

$$\begin{aligned}\left(I_y\right)_{disk} &= m\frac{a^2}{4}+m\left(\frac{a}{2}\right)^2\\ &= \frac{1}{2}a^2m\end{aligned}$$

For the rod, it only has moment of inertial around y at the end of the rod. From tables $\left(I_y\right)_{rod} = \left(\frac{m}{2}\right)\left(\frac{a^2}{3}\right)$. Therefore

$$\begin{aligned}I_z &= m\frac{a^2}{2}\\ I_y &= \left(I_y\right)_{disk}+\left(I_y\right)_{rod} = \frac{1}{2}a^2m+\frac{m}{2}\frac{a^2}{3}\\ &= \frac{2}{3}a^2m\end{aligned}$$

From (2), and using $\theta = 45^0$ we find, using $M = m+\frac{m}{2} = \frac{3}{2}m$ and $l = \frac{a}{2}$

$$\dot{\phi} = \frac{I_zS \pm \sqrt{I_z^2S^2 - 4I_y\cos\theta Mgl}}{2\cos\theta I_y} \tag{3}$$

4.9.3.2 Part(2)

For $\theta = 45^0$ and $S = 900$ rpm, which is 94.248 rad/sec. $a = 0.1$ meter and $l = \frac{a}{2} = 0.05$ meter (3) becomes

$$\begin{aligned}\dot{\phi} &= \frac{\left(m\frac{a^2}{2}\right)(94.248) \pm \sqrt{\left(m\frac{a^2}{2}\right)^2(94.248)^2 - 4\left(\frac{2}{3}a^2m\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}m\right)(9.8)(0.05)}}{2\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{2}{3}a^2m\right)}\\ &= \frac{\left(m\frac{(0.1)^2}{2}\right)(94.248) \pm m\sqrt{\left(\frac{(0.1)^2}{2}\right)^2(94.248)^2 - 4\left(\frac{2}{3}(0.1)^2\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}\right)(9.8)(0.05)}}{2\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{2}{3}(0.1)^2m\right)}\\ &= \frac{3}{4}\frac{\left(\frac{(0.1)^2}{2}\right)(94.248)}{\cos\left(45\left(\frac{\pi}{180}\right)\right)(0.1)^2} \pm \frac{3}{4}\frac{\sqrt{\left(\frac{(0.1)^2}{2}\right)^2(94.248)^2 - 4\left(\frac{2}{3}(0.1)^2\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}\right)(9.8)(0.05)}}{\cos\left(45\left(\frac{\pi}{180}\right)\right)(0.1)^2}\\ &= 49.983 \pm 48.398 \text{ rad/sec}\end{aligned}$$

Or

$$\boxed{\dot{\phi} = 939.47 \text{ or } 15.13 \text{ rpm}}$$

4.9.3.3 Part(3)

From (2) above, repeated below

$$\dot{\phi} = \frac{I_z S \pm \sqrt{I_z^2 S^2 - 4I_y \cos\theta Mgl}}{2\cos\theta I_y}$$

Since $\dot{\phi}$ must be real, then $I_z^2 S^2 - 4I_y \cos\theta Mgl$ must be either positive or zero.

$$S^2 - 4I_y \cos\theta Mgl \geq 0$$

$$S^2 \geq \frac{4I_y \cos\theta Mgl}{I_z^2}$$

For $\theta = 0$ the above becomes

$$S^2 \geq \frac{4I_y Mgl}{I_z^2}$$

The above is the condition on spin speed S for keeping $\theta = 0$. Hence

$$\begin{aligned}
S^2 &\geq \frac{4\left(\frac{2}{3}a^2 m\right)\left(\frac{3}{2}m\right)(9.8)\,l}{\left(m\frac{a^2}{2}\right)^2} \\
&\geq \frac{156.8}{a^2} l \\
&\geq \frac{156.8}{(0.1)^2}(0.05) \\
&\geq 784
\end{aligned}$$

Therefore

$$\begin{aligned}
S &\geq \sqrt{784} \\
&\geq 28 \text{ rad/sec}
\end{aligned}$$

Or

$$\boxed{S \geq 267.31 \text{ RPM}}$$

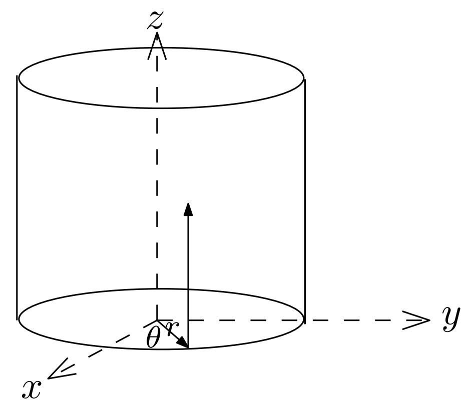
4.9.4 Problem 4

4. (10 points)
Determine the principal moments of inertia and the corresponding principle axes about the center of mass of a homogeneous circular cone of height h and radius R. (You might find it easier to calculate the moments in a reference frame with the origin at the apex first, and then transform to the center of mass system.)

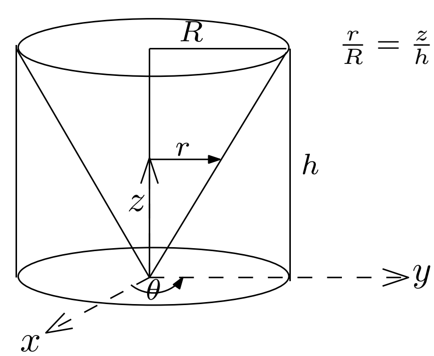
SOLUTION:

4.9.4.1 Solution using Cylindrical coordinates

Will show the solution using Cylindrical coordinates. Then later will also show the solution using Cartesian coordinates. Using Cylindrical coordinates



The limits of volume integration will be from $z = 0 \cdots h$ and $\theta = 0 \cdots 2\pi$. For r, it depends on z. Since $\frac{r}{R} = \frac{z}{h}$, then $r = \frac{R}{h}z$, therefore the limit for $r = 0 \cdots \frac{R}{h}z$. This is when the tip of the cone at the origin as follows



The density is $\rho = \frac{3M}{\pi R^2 h}$. The center of mass is $\frac{h}{4}$ distance away from the base or $\frac{3}{4}h$ from the tip. The moment of inertia is found at the origin (which is the tip of the cone also), then moved to the center of mass using parallel axes theorem. We know from Cartesian coordinates that the inertia matrix is found using

$$J = \rho \int\int\int \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} dzdydx$$

Therefore, in cylindrical coordinates this becomes, after using the mapping $x = r\cos\theta, y =$

$r\sin\theta, z = z$

$$J = \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z} \begin{pmatrix} r^2\sin^2\theta + z^2 & -r^2\cos\theta\sin\theta & -r\cos\theta z \\ -r\cos\theta z & r^2\cos^2\theta + z^2 & -r\sin\theta z \\ -r\cos\theta z & -r\sin\theta z & r^2 \end{pmatrix} rdrd\theta dz$$

Due to symmetry, the off diagonal elements will be zero. So we only have to perform the following integration

$$J = \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z} \begin{pmatrix} r^2\sin^2\theta + z^2 & 0 & 0 \\ 0 & r^2\cos^2\theta + z^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} rdrd\theta dz$$

For J_{11} we find

$$\begin{aligned}
J_{11} &= \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z}\left(r^2\sin^2\theta + z^2\right) rdrd\theta dz \\
&= \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z}\left(r^2\sin^2\theta\right) rdrd\theta dz + \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z} z^2 rdrd\theta dz \\
&= \rho\int_0^h dz\int_0^{2\pi} d\theta\left(\int_0^{\frac{R}{h}z}\left(r^3\sin^2\theta\right)dr\right) + \rho\int_0^h z^2dz\int_0^{2\pi} d\theta\left(\int_0^{\frac{R}{h}z} rdr\right) \\
&= \rho\int_0^h dz\int_0^{2\pi}\sin^2\theta d\theta\left[\frac{r^4}{4}\right]_0^{\frac{R}{h}z} + \rho\int_0^h z^2dz\int_0^{2\pi} d\theta\left[\frac{r^2}{2}\right]_0^{\frac{R}{h}z} \\
&= \frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz\int_0^{2\pi}\sin^2\theta d\theta + \frac{\rho}{2}\frac{R^2}{h^2}\int_0^h z^4dz\int_0^{2\pi} d\theta \\
&= \frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} + \frac{\rho}{2}\frac{R^2}{h^2}2\pi\int_0^h z^4dz \\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz + \frac{\rho}{2}\frac{R^2}{h^2}2\pi\left[\frac{z^5}{5}\right]_0^h \\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\left[\frac{z^5}{5}\right]_0^h + \rho\frac{R^2}{h^2}\pi\frac{h^5}{5} \\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\frac{h^5}{5} + \rho R^2\pi\frac{h^3}{5} \\
&= \pi\frac{\rho}{20}R^4h + \rho R^2\pi\frac{h^3}{5}
\end{aligned}$$

Using $\rho = \frac{3M}{\pi R^2 h}$ the above becomes

$$\begin{aligned}
J_{11} &= \frac{3M}{\pi R^2h}\pi\frac{1}{20}R^4h + \frac{3M}{\pi R^2h}R^2\pi\frac{h^3}{5} \\
&= \frac{3M}{20}R^2 + \frac{3M}{5}h^2
\end{aligned}$$

For J_{22} it will be the same as the above, since the only difference is $\cos^2\theta$ instead of $\sin^2\theta$

in the integrand. Therefore

$$J_{22} = \frac{3M}{20}R^2 + \frac{3M}{5}h^2$$

For the final entry (the easy one) we have

$$\begin{aligned}
J_{33} &= \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}z} r^2 r dr d\theta dz \\
&= \rho \int_0^h \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^{\frac{R}{h}z} d\theta dz \\
&= \frac{\rho}{4}\frac{R^4}{h^4} \int_0^h z^4 dz \int_0^{2\pi} d\theta \\
&= \frac{\rho}{4}\frac{R^4}{h^4} 2\pi \int_0^h z^4 dz \\
&= \frac{\rho}{4}\frac{R^4}{h^4} 2\pi \left[\frac{z^5}{5}\right]_0^h \\
&= \frac{\rho}{20}\frac{R^4}{h^4} 2\pi h^5
\end{aligned}$$

Using $\rho = \frac{3M}{\pi R^2 h}$ the above becomes

$$\begin{aligned}
J_{33} &= \frac{3M}{\pi R^2 h}\frac{1}{20}\frac{R^4}{h^4} 2\pi h^5 \\
&= \frac{6}{20} MR^2
\end{aligned}$$

Therefore

$$J = \begin{pmatrix} \frac{3M}{20}R^2 + \frac{3M}{5}h^2 & 0 & 0 \\ 0 & \frac{3M}{20}R^2 + \frac{3M}{5}h^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

Using $I_{ij} = I_{ij}^{cm} + M\left(a^2\delta_{ij} - a_i a_j\right)$, we now find I. The vector from the origin to the center of mass is $a = \left\{0, 0, \frac{3}{4}h\right\}$, hence

$$\begin{aligned}
I_{11} &= \left(\frac{3M}{20}R^2 + \frac{3M}{5}h^2\right) - M\left(\left(\frac{3}{4}h\right)^2 - \left(0^2\right)\right) \\
&= \frac{3M}{20}R^2 + \frac{3M}{5}h^2 - M\left(\frac{3}{4}h\right)^2 \\
&= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2
\end{aligned}$$

And

$$I_{22} = I_{11}$$

And

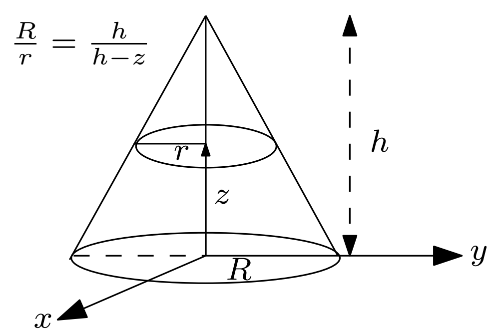
$$\begin{aligned} I_{33} &= \frac{3}{10}MR^2 - M\left(\left(\frac{3}{4}h\right)^2 - \left(\frac{3}{4}h\right)^2\right) \\ &= \frac{3}{10}MR^2 \end{aligned}$$

Therefore the final inertial matrix around the center of the mass of the cone is

$$I = M\begin{pmatrix} \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 & 0 \\ 0 & \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 \\ 0 & 0 & \frac{3}{10}R^2 \end{pmatrix}$$

4.9.4.2 Solution using Cartesian coordinates

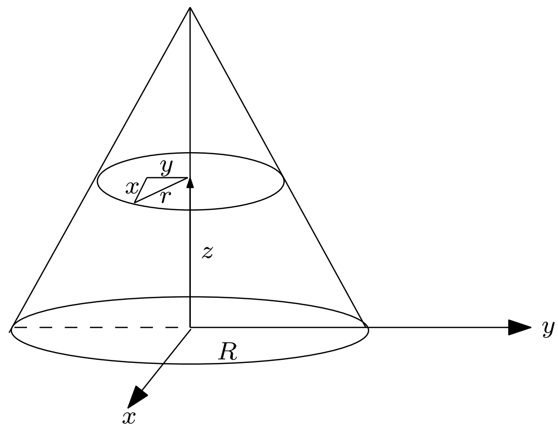
Will find mass moment of inertia tensor at center of base of cone, then use parallel axes to move it to the center of mass of cone.



We basically want to perform this integral

$$J = \rho\int_{z=0}^{z=h}\int_{y=y(z_{\min})}^{y=y(z_{\max})}\int_{x=x(y_{\min})}^{x=x(y_{\max})}\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dzdydx$$

The limit on z is easy. It is from $z = 0$ to $z = h$. Now at specific z, we need to know the limit on y. The radius r at some z distance from the origin is $r = \frac{R(h-z)}{h}$ as shown above, which is by proportions. Therefore the limit of integration for y is from $y = -r$ to $+r$. Now we need to find the limit on x. At some specific y distance from origin, we see from the following diagram



We see from the above that $x^2 = r^2 - y^2$ but $r = \frac{R(h-z)}{h}$, hence the limit on x is from $-\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}$ to $+\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}$. Now that we found all the limits, the integration is

$$J = \rho \int_0^h \int_{-\frac{R(h-z)}{h}}^{\frac{R(h-z)}{h}} \int_{-\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}}^{\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} dzdydx$$

Where $\rho = \frac{3M}{\pi R^2 h}$.Using computer algebra software to do the integration (too messy by hand), the above gives

$$J = \begin{pmatrix} \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 & 0 & 0 \\ 0 & \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

Now we use parallel axis to find I at center of mass. The center of mass is at $\vec{a} = \left\{0, 0, \frac{1}{4}h\right\}$, hence

$$\begin{aligned} I_{11} &= J_{11} - M\left(\vec{a}^2 - a_1^2\right) \\ &= \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 - M\left(\frac{1}{4}h\right)^2 \\ &= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2 \end{aligned}$$

And

$$\begin{aligned} I_{12} &= J_{12} - M(-a_1 a_2) \\ &= 0 - M(0) \\ &= 0 \end{aligned}$$

And

$$
\begin{aligned}
I_{13} &= J_{13} - M\left(-a_1 a_3\right)\\
&= -Ma^2\frac{3}{4} - M\left(-\left(\frac{1}{2}a\right)\frac{3}{2}a\right)\\
&= 0
\end{aligned}
$$

And $I_{21} = I_{12}$ And

$$
\begin{aligned}
I_{22} &= J_{22} - M\left(\vec{a}^2 - a_2^2\right)\\
&= \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 - M\left(\frac{1}{4}h\right)^2\\
&= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2
\end{aligned}
$$

And

$$
\begin{aligned}
I_{23} &= J_{23} - M\left(-a_2 a_3\right)\\
&= 0 - M\left(0\right)\\
&= 0
\end{aligned}
$$

And $I_{31} = I_{31}$ and $I_{32} = I_{23}$ and

$$
\begin{aligned}
I_{33} &= J_{33} - M\left(\vec{a}^2 - a_3^2\right)\\
&= \frac{3}{10}MR^2 - M\left(\left(\frac{1}{4}h\right)^2 - \left(\frac{1}{4}h\right)^2\right)\\
&= \frac{3}{10}MR^2
\end{aligned}
$$

Therefore the moment of inertia tensor around the center of mass

$$
I = M\begin{pmatrix} \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 & 0\\ 0 & \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0\\ 0 & 0 & \frac{3}{10}R^2 \end{pmatrix}
$$

Which is the same as using Cylindrical coordinates (as would be expected).

4.9.5 Problem 5

5. (15 points)
A homogeneous slab of thickness a is placed on top of a fixed cylinder of radius R whose axis is horizontal (as in the Figure below).

(1) Determine the Lagrangian of the system.

(2) Derive the equations of motion and determine the frequency of *small* oscillations.

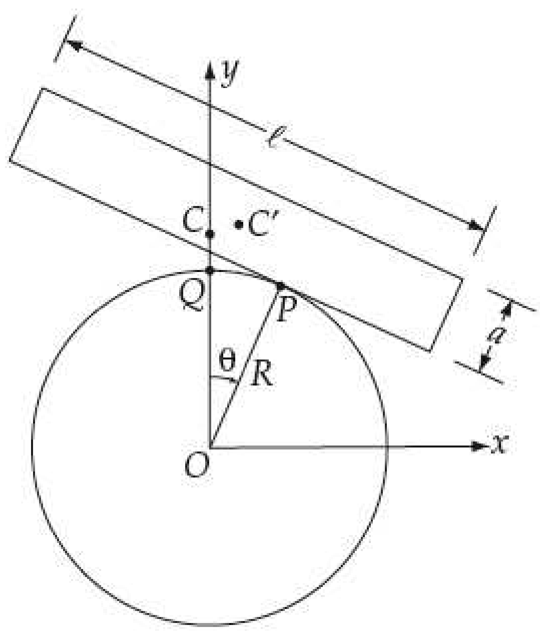
(3) Show that the condition for stable equilibrium of the slab, assuming no slipping, is $R > a/2$.

(4) Use a computer to plot the potential energy U as a function of the angular displacement θ for a slab of mass $M = 1\,\text{kg}$ and

(a) $R = 20\,\text{cm}$ and $a = 5\,\text{cm}$, and

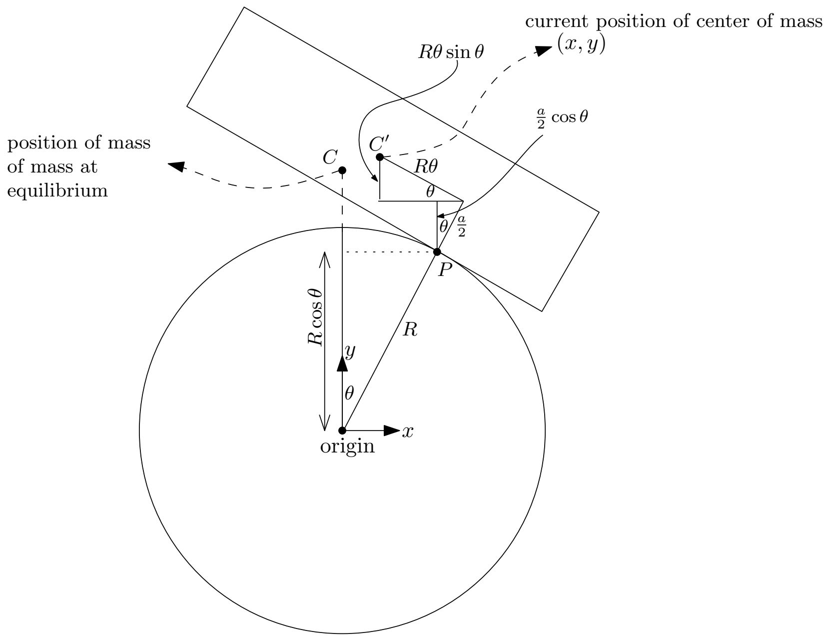
(b) $R = 10\,\text{cm}$ and $a = 30\,\text{cm}$.

(5) Show that the potential energy $U(\theta)$ has a minimum at $\theta = 0$ for $R > a/2$, but not for $R < a/2$.



SOLUTION:

4.9.5.1 Part (1)



The system has three degrees of freedom (x, y, θ). But they are not independent. Because if we know $\theta(t)$, we can find $x(t)$ and $y(t)$ (for small angle approximation) as shown below in equations (1) and (2).

The cylinder itself does not move or rotate. Only the slab has rotational and translational motion. When the slab center of mass at C it is in equilibrium. When the slab center of mass at point C' the location of the center of mass is (x, y), where from the diagram above we see that (for small angle θ)

$$x = \left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta \tag{1}$$

$$y = \left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta \tag{2}$$

The distance from C' to O which is the zero reference for potential energy is therefore (assuming mass of slab is M)

$$\begin{aligned} U &= Mgy \\ &= Mg\left(R\theta\sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right) \end{aligned}$$

Let the moment of inertial of the slab around the axis of rotation be I therefore

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) \tag{3}$$

Now, we write $\dot{x}^2 + \dot{y}^2$ above in terms of θ using (1) and (2). (Initially I did not know if we should do this or not. So I left the original solution as an appendix in case that was how we are supposed to do it). Using this method below, we find only one equation of motion, not three as in the solution in the appendix.

$$
\begin{aligned}
\dot{x} &= \left(R+\frac{a}{2}\right)\dot{\theta}\cos\theta-\left(R\dot{\theta}\cos\theta+R\theta\dot{\theta}\sin\theta\right)\\
\dot{y} &= -\left(R+\frac{a}{2}\right)\dot{\theta}\sin\theta+\left(R\dot{\theta}\sin\theta+R\theta\dot{\theta}\cos\theta\right)
\end{aligned}
$$

Hence (using CAS for simplification) we find

$$
\dot{x}^2=\frac{1}{4}\dot{\theta}^2\left(a\cos\theta+2R\theta\sin\theta\right)^2
$$

Similarly for $\dot{y}^2$ we find

$$
\dot{y}^2=\frac{1}{4}\dot{\theta}^2\left(a\sin\theta-2R\theta\cos\theta\right)^2
$$

Hence (3) becomes

$$
T=\frac{1}{2}I\dot{\theta}^2+\frac{1}{8}M\dot{\theta}^2\left(\left(a\cos\theta+2R\theta\sin\theta\right)^2+\left(a\sin\theta-2R\theta\cos\theta\right)^2\right)
$$

And the Lagrangian is

$$
\begin{aligned}
L &= T-U\\
&= \frac{1}{2}I\dot{\theta}^2+\frac{1}{8}M\dot{\theta}^2\left(\left(a\cos\theta+2R\theta\sin\theta\right)^2+\left(a\sin\theta-2R\theta\cos\theta\right)^2\right)-Mg\left(R\theta\sin\theta+\left(\frac{a}{2}+R\right)\cos\theta\right)
\end{aligned}
$$

4.9.5.2 Part(2)

$$
\begin{aligned}
\frac{\partial L}{\partial\theta} &= \frac{1}{2}M\left(ga\sin\theta+2R\theta\left(-g\cos\theta+R\dot{\theta}^2\right)\right)\\
\frac{\partial L}{\partial\dot{\theta}} &= \frac{1}{4}\left(4I+a^2M+4MR^2\theta^2\right)\dot{\theta}\\
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} &= 2MR^2\theta\dot{\theta}^2+\frac{1}{4}\left(4I+a^2M+4MR^2\theta^2\right)\ddot{\theta}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta} &= 0\\
I\ddot{\theta}+\frac{1}{4}M\left(a^2+4R^2\theta^2\right)\ddot{\theta}-\frac{1}{2}agM\sin\theta+MR\theta\left(g\cos\theta+R\dot{\theta}^2\right) &= 0
\end{aligned}
$$

For small angles, we use $\sin\theta\approx\theta$ and $\cos\theta\approx 1$, $\dot{\theta}^2\approx 0$ and $\theta^2\approx 0$. The above becomes

$$
\begin{aligned}
I\ddot{\theta}+\frac{1}{4}Ma^2\ddot{\theta}-\frac{1}{2}agM\theta+MR\theta g &= 0\\
\ddot{\theta}\left(I+\frac{1}{4}Ma^2\right)+\theta\left(MRg-\frac{1}{2}agM\right) &= 0\\
\ddot{\theta}+\frac{Mg\left(R-\frac{1}{2}a\right)}{\left(I+\frac{1}{4}a^2M\right)}\theta &= 0
\end{aligned}
$$

The above is now in the form $\ddot{\theta}+\omega_0^2\theta=0$, therefore the natural frequency is

$$\omega_0=\sqrt{\frac{Mg\left(R-\frac{1}{2}a\right)}{\left(I+\frac{1}{4}a^2M\right)}}$$

4.9.5.3 Part(3)

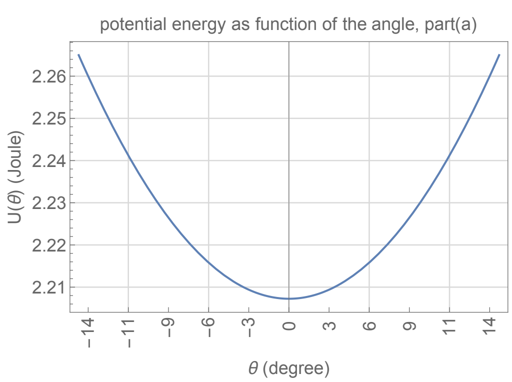
For stable equilibrium, we need $\frac{Mg\left(R-\frac{1}{2}a\right)}{\left(I+\frac{1}{4}a^2M\right)}>0$ in order to obtain an oscillator (simple harmonic motion), otherwise the solution will contain pure exponential term and it will blow up. Hence we need

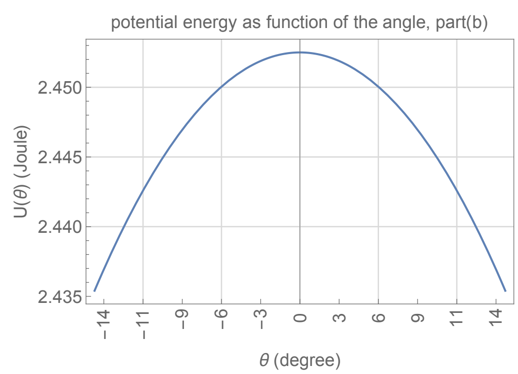
$$\begin{aligned}Mg\left(R-\frac{1}{2}a\right)&>0\\R-\frac{1}{2}a&>0\\R&>\frac{1}{2}a\end{aligned}$$

4.9.5.4 Part(4)

Here is a plot of $Mg\left(R\theta\sin\theta+\left(\frac{a}{2}+R\right)\cos\theta\right)$, for small angle, using $M=1$kg. For parts (a) and (b)

Out[319]=





We see from the above, that in part(b), where $R<\frac{a}{2}$, the potential energy at $\theta=0$ is not minimum. This implies $\theta=0$ is not a stable equilibrium. While in part(a) it is stable.

4.9.5.5 Part(5)

$$U(\theta)=Mg\left(R\theta\sin\theta+\left(\frac{a}{2}+R\right)\cos\theta\right)$$

Hence to find where the minimum is

$$U'(\theta) = gR\theta\cos\theta - \frac{1}{2}ga\sin\theta$$

Setting this to zero and for small angle we obtain

$$0 = gR\theta - \frac{1}{2}ga\theta$$
$$0 = \theta g\left(R - \frac{1}{2}a\right)$$

This implies $\theta = 0$ is where the minimum potential energy is. We know this is stable equilibrium. Therefore we expect $U''(\theta = 0)$ to be positive for a local minimum (from calculus). We now check the condition for this.

$$U''(\theta) = -\frac{1}{2}g\left((a - 2R)\cos\theta + 2R\theta\sin\theta\right)$$

At $\theta = 0$ we obtain

$$U''(\theta = 0) = -\frac{1}{2}g(a - 2R)$$

For the above to be positive, then

$$a - 2R < 0$$
$$2R > a$$
$$R > \frac{a}{2}$$

The above is the condition for having stable equilibrium at $\theta = 0$. If $R < \frac{a}{2}$, then at $\theta = 0$ the slab will not be stable, which is not we have shown in part(3).

4.9.5.6 Appendix. Second Solution of problem 5

4.9.5.6.1 Part(1) In this solution, we find three equations of motion.

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right)$$

Hence the Lagrangian is

$$\begin{aligned}
L &= T - U \\
&= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) - Mg\left(R\theta\sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right)
\end{aligned}$$

4.9.5.6.2 Part(2) For θ

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= -Mg\left(R\left(\sin\theta + \theta\cos\theta\right) - \left(\frac{a}{2} + R\right)\sin\theta\right) \\
\frac{\partial L}{\partial \dot{\theta}} &= I\dot{\theta} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= I\ddot{\theta}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\
I\ddot{\theta} + Mg\left(R\left(\sin\theta + \theta\cos\theta\right) - \left(\frac{a}{2} + R\right)\sin\theta\right) &= 0
\end{aligned}$$

For small angles, we use $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and the above becomes

$$\begin{aligned}
I\ddot{\theta} + Mg\left(2R\theta - \left(\frac{a}{2} + R\right)\theta\right) &= 0 \\
I\ddot{\theta} + Mg\left(R - \frac{1}{2}a\right)\theta &= 0 \\
\ddot{\theta} + \frac{Mg\left(R - \frac{1}{2}a\right)}{I}\theta &= 0
\end{aligned}$$

The above is now in the form $\ddot{\theta} + \omega_0^2\theta = 0$, therefore the natural frequency is

$$\omega_0 = \sqrt{\frac{Mg\left(R - \frac{1}{2}a\right)}{I}}$$

For x, we have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \dot{x}} &= M\dot{x} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x}
\end{aligned}$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$
$$M\ddot{x} = 0$$

For y we also obtain

$$M\ddot{y} = 0$$

The rest follows as first solution above and will not be repeated.

4.9.6 HW 9 key solution

1

Mechanics

Physics 311 - Fall 2015

Homework Set 9 - Solutions

Problem 1

kinetic energy of rotation:

$$T_{rot} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

need to show that $\frac{d}{dt}T_{rot} = 0$,

so $\frac{d}{dt}\left\{\frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)\right\}$

$$= I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3$$

from Euler:

$$I_1\omega_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_1\omega_2\omega_3$$
$$I_2\omega_2\dot{\omega}_2 = (I_3 - I_1)\,\omega_1\omega_2\omega_3$$
$$I_3\omega_3\dot{\omega}_3 = (I_1 - I_2)\,\omega_1\omega_2\omega_3$$

$$\Rightarrow \frac{d}{dt}T_{rot} = \omega_1\omega_2\omega_3\left\{(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2)\right\}$$
$$= 0$$

2

magnitude of angular momentum

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega^2 + I_3^2\omega_3^2$$

need to show that $\frac{d}{dt}L^2 = 0$

so

$$\frac{d}{dt}\left\{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right\}$$

$$= 2I_1^2\omega_1\dot{\omega}_1 + 2I_2^2\omega_2\dot{\omega}_2 + 2I_3^2\omega_3\dot{\omega}_3$$

from Euler:

$$I_1^2\omega_1\dot{\omega}_1 = I_1(I_2-I_3)\omega_1\omega_2\omega_3$$

$$I_2^2\omega_2\dot{\omega}_2 = I_2(I_3-I_1)\omega_1\omega_2\omega_3$$

$$I_3^2\omega_3\dot{\omega}_3 = I_3(I_1-I_2)\omega_1\omega_2\omega_3$$

$$\Rightarrow \frac{d}{dt}L^2 = 2\omega_1\omega_2\omega_3\left\{I_1(I_2-I_3) + I_2(I_3-I_1) + I_3(I_1-I_2)\right\}$$

$$= 0$$

∎

3

Problem 2

Start with the general case of a block of mass m and dimensions a, b, and c

$$I_1 = \rho \int_{-\frac{a}{2}}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (x_2^2 + x_3^2)\, dx_1\, dx_2\, dx_3$$

$$= \rho\, a \int_{-b/2}^{b/2} dx_2 \int_{-c/2}^{c/2} dx_3\ (x_2^2 + x_3^2)$$

$$= \rho\, a \int_{-b/2}^{b/2} \left[x_2^2 x_3 + \frac{1}{3} x_3^3 \right]_{-c/2}^{c/2} dx_2$$

$$= \rho\, a \int_{-b/2}^{b/2} \left(c\, x_2^2 + \frac{2}{3} \left(\frac{c}{2}\right)^3 \right) dx_2$$

$$= \rho\, a \left[\frac{c}{3} x_2^3 + \frac{2}{3}\left(\frac{c}{2}\right)^3 x_2 \right]_{-b/2}^{b/2}$$

$$= \rho\, a \left[\frac{2}{3} c \left(\frac{b}{2}\right)^3 + \frac{2}{3} b \left(\frac{c}{2}\right)^3 \right]$$

$$= \rho\, a\, b\, c \left[\frac{2}{3} \frac{b^2}{8} + \frac{2}{3} \frac{c^2}{8} \right] = \frac{1}{12} \rho\, a b c \left[b^2 + c^2 \right]$$

$$= \frac{m}{12} \left[b^2 + c^2 \right]$$

(1) $$I_1 = \frac{m}{12} \left[(2a)^2 + (3a)^2 \right] = \frac{13}{12} m a^2$$

$$I_2 = \frac{m}{12} \left[a^2 + (3a)^2 \right] = \frac{10}{12} m a^2$$

$$I_3 = \frac{m}{12} \left[a^2 + (2a)^2 \right] = \frac{5}{12} m a^2$$

4

$$\Rightarrow \quad \overleftrightarrow{I} = m a^2 \begin{Bmatrix} \frac{13}{12} & 0 & 0 \\ 0 & \frac{10}{12} & 0 \\ 0 & 0 & \frac{5}{12} \end{Bmatrix}$$

(2) $\vec{\omega} = \omega \hat{n}$, where $\hat{n}$ points along the diagonal

$$\hat{n} = \frac{1}{\sqrt{14}} \left(\hat{x}_1 + 2\hat{x}_2 + 3\hat{x}_3 \right)$$

$$\vec{\omega} = \frac{\omega}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$T = \frac{1}{2} \vec{\omega} \overleftrightarrow{I} \vec{\omega}$$

$$= \frac{1}{2} m a^2 \left\{ \frac{13}{12} \frac{\omega^2}{14} + \frac{10}{12} \frac{4\omega^2}{14} + \frac{5}{12} \frac{9\omega^2}{14} \right\}$$

$$= \underline{\underline{\frac{7}{24} m a^2 \omega^2}}$$

(3) $\vec{L} = \overleftrightarrow{I} \vec{\omega}$

$$\Rightarrow \quad \vec{L} = \hat{x}_1 \left(\frac{13}{12} m a^2 \frac{\omega}{\sqrt{14}} \right) + \hat{x}_2 \left(\frac{10}{12} m a^2 \frac{2\omega}{\sqrt{14}} \right) + \hat{x}_3 \left(\frac{5}{12} m a^2 \frac{3\omega}{\sqrt{14}} \right)$$

$$= \frac{m a^2 \omega}{12 \sqrt{14}} \begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

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$$\cos\theta = \frac{\vec{\omega}\cdot\vec{L}}{|\vec{\omega}||\vec{L}|} = \frac{1\cdot 13 + 2\cdot 20 + 3\cdot 15}{\sqrt{1^2+2^2+3^2}\sqrt{13^2+20^2+15^2}}$$

$$= 0.9295 \qquad \Rightarrow \quad \underline{\underline{\theta = 21.6^\circ}}$$

(4)

$$\tau_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3$$

$$= 0 - \frac{ma^2}{12}(10-5)\frac{\omega^2}{14}\,2\cdot 3$$

$$= -\frac{5}{28}ma^2\omega^2$$

$$\tau_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1$$

$$= 0 - \frac{ma^2}{12}(5-13)\frac{\omega^2}{14}\,3$$

$$= \frac{1}{7}ma^2\omega^2$$

$$\tau_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2$$

$$= 0 - \frac{ma^2}{12}(13-10)\frac{\omega^2}{14}\,2$$

$$= -\frac{1}{28}ma^2\omega^2$$

$$\underline{\underline{\vec{\tau} = \frac{ma^2\omega^2}{28}\begin{pmatrix}-5\\4\\-1\end{pmatrix}}}$$

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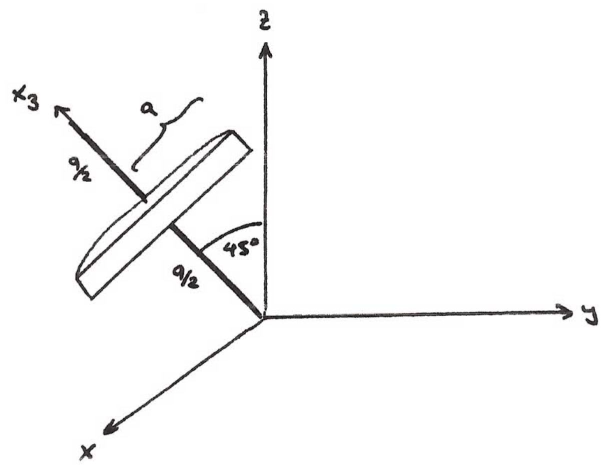
Problem 4



(1)

need I_3 and $I_1 = I_2$:

$I_3 = \frac{1}{2} m a^2$ (disk about the symmetry axis)

$I_1 = I_{1_{disk}} + I_{1_{rod}}$

$I_{1_{rod}} = \left(\frac{m}{2}\right)\frac{a^2}{3}$ (about one end)

$I'_{1_{disk}} = \frac{1}{4} m a^2$ through the center of mass, in the plane of the disk

$\Rightarrow \quad I_{1_{disk}} = \frac{1}{4} m a^2 + m\left(\frac{a}{2}\right)^2$

$= \frac{1}{2} m a^2$

$\Rightarrow I_1 = \frac{1}{2} m a^2 + \frac{1}{6} m a^2 = \frac{2}{3} m a^2$

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from class : $\dot{\phi} = \dfrac{\beta}{I_1 \sin^2\theta}$

for $\theta = \text{const.} = \theta_0$, β has two solutions :

$$\beta = \frac{L_3 \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4Mgh I_1 \cos\theta_0}{L_3^2}}\right]$$

$$\Rightarrow \boxed{\dot{\phi} = \frac{I_3\omega_3}{2 I_1 \cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{I_3^2\omega_3^2}}\right)}$$

using $L_3 = I_3\omega_3$

$M = m + \frac{m}{2}$ (mass of top)

(2) $S = \omega_3 = 900 \text{ rpm} = 900 \cdot \frac{2\pi}{60 s} = 94.2 \, s^{-1}$

$\theta_0 = 45°$

$$\frac{I_3}{I_1} = \frac{\frac{1}{2}ma^2}{\frac{2}{3}ma^2} = \frac{3}{4}$$

$$\frac{Mh}{I_3} = \frac{(m+\frac{1}{2}m)\frac{1}{2}a}{\frac{1}{2}ma^2} = \frac{3}{2a}$$

$$\Rightarrow \dot{\phi} = \frac{1}{2}\left(\frac{3}{4}\right)\sqrt{2}\,\omega_3\left\{1 \pm \sqrt{1 - 4g\left(\frac{3}{2a}\right)\left(\frac{4}{3}\right)\sqrt{\frac{1}{2}}\,\frac{1}{\omega_3^2}}\right\}$$

$$= 477.3 \text{ rpm } (1 \pm 0.9683)$$

$$\Rightarrow \dot{\phi}_{slow} = \underline{\underline{15.1 \text{ rpm}}} \qquad \dot{\phi}_{fast} = \underline{\underline{939.5 \text{ rpm}}}$$

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(3) a sleeping top has $\theta_0 = \text{const.} = 0°$

Since $\dot{\Phi}$ must be real,

$$I_3^2 \omega_3^2 \geq 4 M g h I_1 \qquad (\cos\theta_0 = 0)$$

$$\Rightarrow \quad \omega_3 \geq \frac{2}{I_3} \sqrt{M g h I_1} = 2 \sqrt{\frac{M g h I_1}{I_3^2}}$$

So

$$\omega_3 \geq 2 \sqrt{\left(\frac{3}{2a}\right) g \frac{4}{3}}$$

$$= 28\, s^{-1} = \frac{28}{s} \cdot \frac{60\, s}{2\pi} \text{ rpm} = \underline{\underline{267.4 \text{ rpm}}}$$

∴ the top must spin with more than $\underline{267.4}$ rpm !

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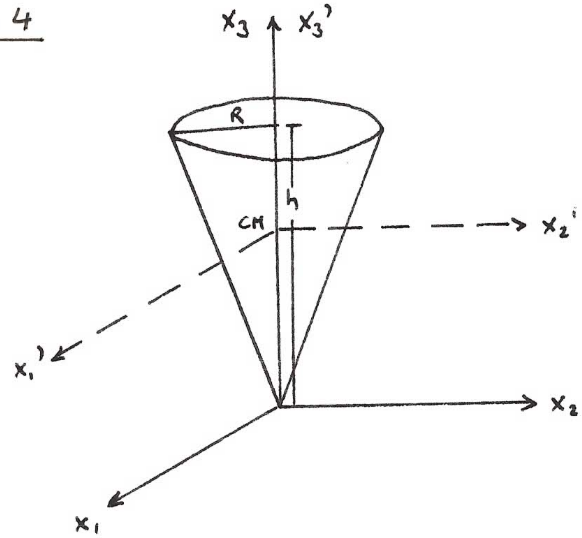
Problem 4



from symmetry $\Rightarrow$ $I_1 = I_2 \neq I_3$

with the above choice of axes, $I_{ij} = 0$ for $i \neq j$, so

I_{ii} are the principal moments I_i

We calculate the I_i for the system $x_1 x_2 x_3$ (easier!) and then make a parallel-axis transformation to $x_1' x_2' x_3'$

$$I_1 = I_2 = \frac{I_1 + I_2}{2} = \frac{\rho}{2} \int (2x_3^2 + x_1^2 + x_2^2)\, dV$$

$$= \frac{\rho}{2} \int_0^{2\pi} d\phi \int_0^h dz \int_0^{\frac{R}{h}z} (r^2 + 2z^2)\, r\, dr$$

$$= \pi \rho \int_0^h dz \left[\frac{1}{4} r^4 + z^2 r^2 \right]_0^{\frac{R}{h}z}$$

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$$= \pi \rho \int_0^h \left[\frac{1}{4} \frac{R^4}{h^4} z^4 + \frac{R^2}{h^2} z^4 \right] dz$$

$$= \pi \rho \left[\frac{1}{20} \frac{R^4}{h^4} z^5 + \frac{1}{5} \frac{R^2}{h^2} z^5 \right]_0^h$$

$$= \pi \rho \left[\frac{1}{20} R^4 h + \frac{1}{5} R^2 h^3 \right]$$

now use $\rho = \frac{M}{V}$ and $V = \frac{1}{3} \pi R^2 h \Rightarrow \pi \rho = \frac{3M}{R^2 h}$

$$\Rightarrow \quad I_1 = I_2 = \frac{3}{20} M \left(R^2 + 4 h^2 \right)$$

also $I_3 = \rho \int (x_1^2 + x_2^2)\, dV = \rho \int r^2\, r\, dr\, d\phi\, dz$

$$= 2\pi \rho \int_0^h dz \int_0^{\frac{R}{h} z} r^3\, dr$$

$$= 2\pi \rho \int_0^h \left[\frac{1}{4} r^4 \right]_0^{\frac{R}{h} z} dz$$

$$= 2\pi \rho \int_0^h \frac{1}{4} \frac{R^4}{h^4} z^4\, dz$$

$$= 2\pi \rho \left[\frac{1}{20} \frac{R^4}{h^4} z^5 \right]_0^h$$

$$= \frac{1}{10} \pi \rho R^4 h$$

$$= \frac{3}{10} M R^2$$

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to transform to $x_1' x_2' x_3'$, we need the position of the center of mass

because of the symmetry, the center of mass is on the x_3'-axis;

Let $(0,0,z_0)$ be the center of mass, with

$$z_0 = \frac{\int x_3' \, dV}{\int dV}$$

$$\int x_3' \, dV = \int_0^{2\pi} d\phi \int_0^h z \, dz \int_0^{\frac{R}{h}z} r \, dr$$

$$= 2\pi \int_0^h z \left[\frac{1}{2} r^2\right]_0^{\frac{R}{h}z} dz$$

$$= 2\pi \int_0^h z \, \frac{1}{2} \frac{R^2}{h^2} z^2 \, dz$$

$$= \frac{R^2}{h^2} \pi \left[\frac{1}{4} z^4\right]_0^h = \frac{1}{4} \pi R^2 h^2$$

$$\int dV = V = \frac{1}{3} \pi R^2 h \qquad \Rightarrow \quad z_0 = \frac{\frac{1}{4}\pi R^2 h^2}{\frac{1}{3}\pi R^2 h}$$

$$= \frac{3}{4} h$$

$\therefore$ center of mass is at $(0, 0, \frac{3}{4} h)$

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now use parallel-axis theorem

$$I_{ij}' = I_{ij} - M\left[a^2 \delta_{ij} - a_i a_j\right]$$

with $a_1 = a_2 = 0$, $a_3 = \frac{3}{4}h$

$$\Rightarrow \quad I_1' = I_1 - \frac{9}{16}Mh^2 = \frac{3}{20}M\left(R^2 + 4h^2\right) - \frac{9}{16}Mh^2$$

$$= \frac{3}{20}MR^2 + \left(\frac{12}{20} - \frac{9}{16}\right)Mh^2$$

$$= \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right)$$

$$I_2' = I_2 - \frac{9}{16}Mh^2 = \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right)$$

$$I_3' = I_3 - \frac{9}{16}Mh^2 + \frac{9}{16}Mh^2 = I_3 \qquad \text{(makes sense!)}$$

$$= \frac{3}{10}MR^2$$

$$\Rightarrow \quad \boxed{\begin{aligned} I_1' = I_2' &= \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) \\ I_3' &= \frac{3}{10}MR^2 \end{aligned}}$$

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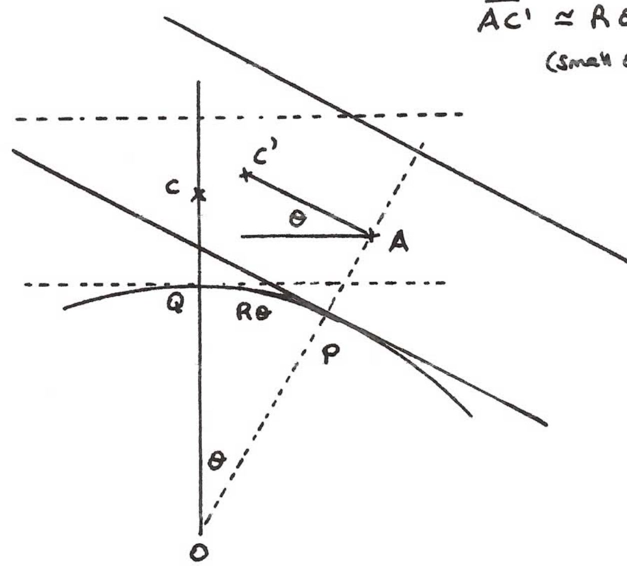
Problem 5

the figure shows the slab rotated through an angle θ from its equilibrium position; at equilibrium, the contact point is Q and the center of mass is C; after rotation, the contact point is P and the center of mass is C'

(1) for the Lagrangian, we need the x and y position of the center of mass

$\overline{AC'} \simeq R\theta$
(small θ)



the coordinates of C' are $\vec{r} = \vec{OA} + \vec{AC'}$,

so

$$x = \left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta$$

$$y = \left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta$$

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So $\dot{x} = \left[\left(R + \frac{a}{2}\right) \cos\theta - R\cos\theta + R\theta\sin\theta \right] \dot{\theta}$

$= \left(\frac{a}{2} \cos\theta + R\theta \sin\theta \right) \dot{\theta}$

$\dot{y} = \left[-\left(R + \frac{a}{2}\right) \sin\theta + R\sin\theta + R\theta\cos\theta \right] \dot{\theta}$

$= \left(-\frac{a}{2} \sin\theta + R\theta \cos\theta \right) \dot{\theta}$

$\Rightarrow \quad \dot{x}^2 + \dot{y}^2 = \left(\frac{a^2}{4} + R^2\theta^2 \right) \dot{\theta}^2$

$\Rightarrow$ kinetic energy $T = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$

potential energy $U = Mgy$

$= Mg \left[\left(R + \frac{a}{2}\right) \cos\theta + R\theta \sin\theta \right]$

So the Lagrangian is

$$\boxed{L = \frac{1}{2} \dot{\theta}^2 \left[M \left(\frac{a^2}{4} + R^2\theta^2 \right) + I \right] - Mg \left[\left(R + \frac{a}{2}\right) \cos\theta + R\theta \sin\theta \right]}$$

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(2) equation of motion $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0$

$$\frac{\partial L}{\partial \theta} = \dot\theta^2 MR^2\theta + Mg\left(R+\frac{a}{2}\right)\sin\theta - MgR\sin\theta - MgR\theta\cos\theta$$

$$= \dot\theta^2 MR^2\theta + Mg\frac{a}{2}\sin\theta - MgR\theta\cos\theta$$

$$\frac{\partial L}{\partial \dot\theta} = \dot\theta\left[M\left(\frac{a^2}{4}+R^2\theta^2\right)+I\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \ddot\theta\left[M\left(\frac{a^2}{4}+R^2\theta^2\right)+I\right] + \dot\theta^2\, 2MR^2\theta$$

$$\Rightarrow \quad \dot\theta^2\theta MR^2 + Mg\frac{a}{2}\sin\theta - MgR\theta\cos\theta - \ddot\theta\left[M\left(\frac{a^2}{4}+R^2\theta^2\right)+I\right] - 2\dot\theta^2\theta MR^2 = 0$$

$$\Leftrightarrow \quad \left[M\left(\frac{a^2}{4}+R^2\theta^2\right)+I\right]\ddot\theta + MR^2\theta\dot\theta^2 - Mg\left[\frac{a}{2}\sin\theta - R\theta\cos\theta\right] = 0$$

for small oscillations, $\theta^2 \ll \theta$, $\dot\theta^2 \ll \dot\theta$, $\sin\theta \simeq \theta$

so

$$\left(M\frac{a^2}{4}+I\right)\ddot\theta + Mg\left(R-\frac{a}{2}\right)\theta = 0$$

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$$\Rightarrow \quad \ddot{\theta} + \frac{Mg\left(R-\frac{a}{2}\right)}{\frac{Ma^2}{4}+I}\,\theta = 0$$

so the frequency for small oscillations is

$$\boxed{\omega = \sqrt{\frac{Mg\left(R-\frac{a}{2}\right)}{\frac{Ma^2}{4}+I}}}$$

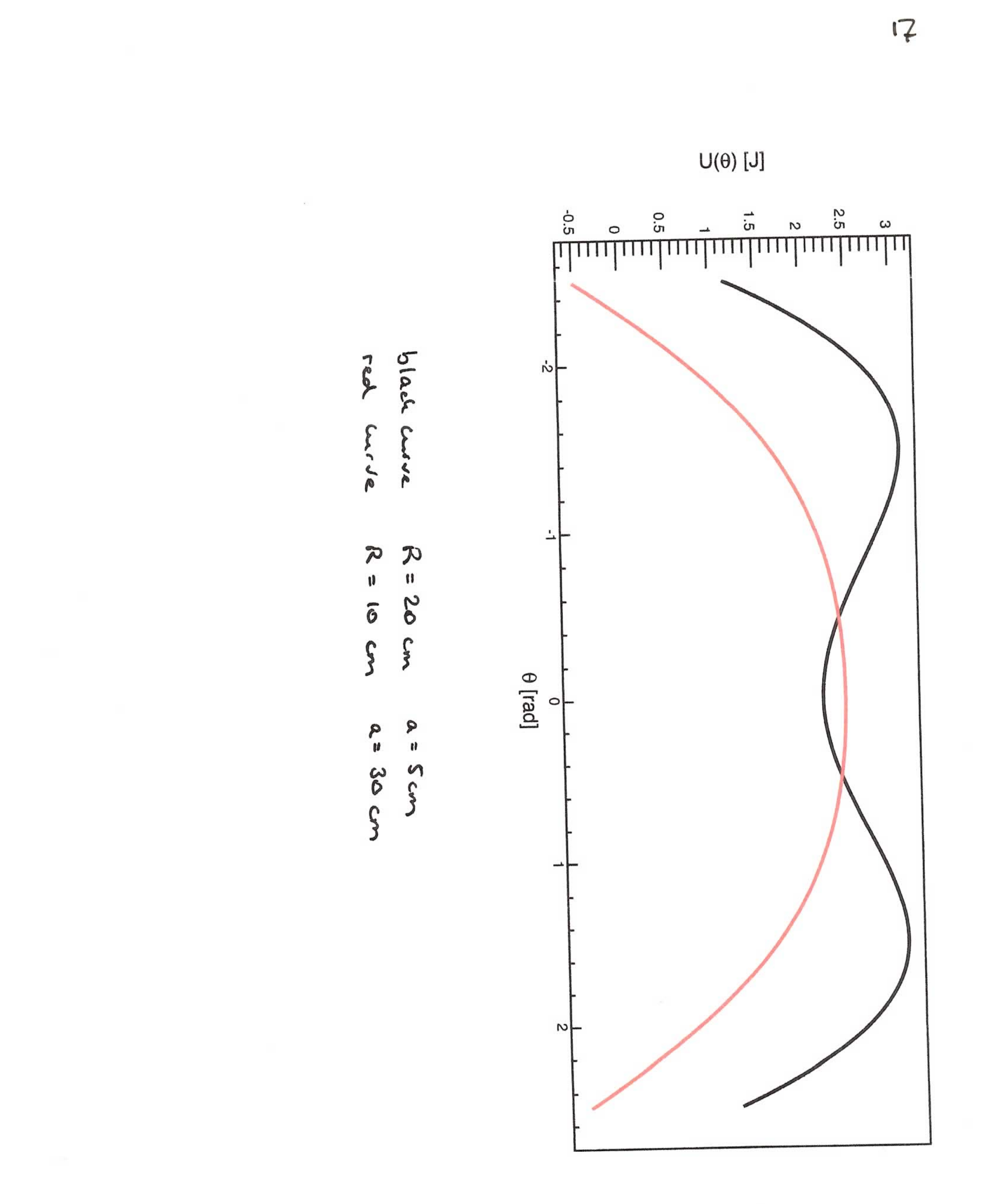
(3) the system is <u>stable</u> for oscillations around $\theta = 0$ if

$$\frac{Mg\left(R-\frac{a}{2}\right)}{\frac{Ma^2}{4}+I} > 0$$

$$\Rightarrow \quad \boxed{R > \frac{a}{2}}$$

(4) potential energy is

$$U(\theta) = Mg\left[\left(R+\frac{a}{2}\right)\cos\theta + R\,\theta\sin\theta\right]$$


17
U(θ) [J]
θ [rad]
black curve R = 20 cm a = 5 cm
red curve R = 10 cm a = 30 cm

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(5) $$\frac{\partial U}{\partial \theta} = Mg\left[-\frac{a}{2}\sin\theta + R\theta\cos\theta\right]$$

$$\frac{\partial^2 U}{\partial \theta^2} = Mg\left[-\frac{a}{2}\cos\theta + R\cos\theta - R\theta\sin\theta\right]$$

So $$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=0} = Mg\left(R - \frac{a}{2}\right)$$

So $\frac{\partial^2 U}{\partial \theta^2} > 0$ if $R > \frac{a}{2}$ $\blacksquare$

4.10 HW 10

4.10.1 Problem 1

1. (10 points)
   Show that the total energy associated with each normal mode of oscillation is separately conserved.

SOLUTION:

The motion in each normal mode is de-coupled from each other mode. Each motion is a simple harmonic motion in terms of normal coordinates, and reduces to second order differential equation of the form

$$\ddot{\eta}_i + \omega_i^2 \eta_i = 0 \tag{1}$$

Where i ranges over the number of modes. The number of modes is equal to the number of independent degrees of freedoms in the system. Each mode oscillates at frequency ω_i. Since this is a simple harmonic motion, its energy is given by

$$E_i = \frac{1}{2} m_i \dot{\eta}_i^2 + \frac{1}{2} k_i \eta_i^2 \tag{2}$$

Where k_i is the effective stiffness of the mode and $\omega_i^2 = \frac{k_i}{m_i}$. Therefore $k_i = m_i \omega_i^2$.

To show that E is conserved, we need to show that $\frac{\partial E}{\partial t} = 0$. Hence from (2)

$$\frac{\partial E_i}{\partial t} = m_i \dot{\eta}_i \ddot{\eta}_i + \left(m_i \omega_i^2\right) \eta_i \dot{\eta}_i$$

But from (1) we see that $\ddot{\eta}_i = -\omega_i^2 \eta_i$. Substituting into the above gives

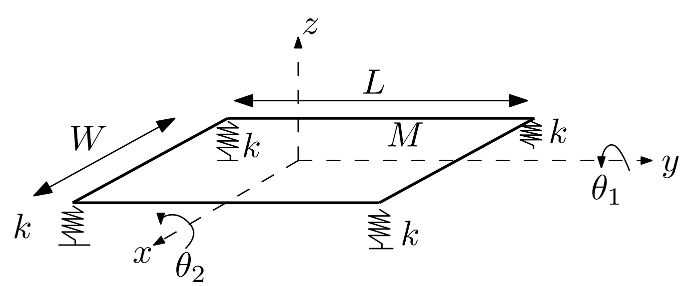
$$\begin{aligned}\frac{\partial E_i}{\partial t} &= m_i \dot{\eta}_i \left(-\omega_i^2 \eta_i\right) + \left(m_i \omega_i^2\right) \eta_i \dot{\eta}_i \\ &= 0\end{aligned}$$

Therefore energy in each mode is constant.

4.10.2 Problem 2

2. (10 points)
A uniform horizontal rectangular plate of mass M, length L, and width W rests with its corners on four similar vertical springs with spring constant k. Assume that the center of mass of the plate is restricted to move along a vertical line. Find the normal modes of vibration and prove that their frequencies are in the ratio $1 : \sqrt{3} : \sqrt{3}$. (This problem is simpler if you decide beforehand what the normal modes are and then use the appropriate generalized coordinates so that the equations of motion are decoupled from the start.)

SOLUTION:



degrees of freedom: z, θ_1, θ_2

Kinetic energy is

$$T = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

Where I_1 is moment of inertia of plate around axis y, and I_2 is moment of inertia of plate around axis x. These are (from tables) :

$$I_1 = \frac{1}{12}MW^2$$
$$I_2 = \frac{1}{12}ML^2$$

The potential energy is

$$\begin{aligned} U &= 4\left(\frac{1}{2}Kz^2\right) + 4\left(\frac{1}{2}K\left(\frac{W}{2}\theta_1\right)^2\right) + 4\left(\frac{1}{2}K\left(\frac{L}{2}\theta_2\right)^2\right) \\ &= 2Kz^2 + 2K\left(\frac{W}{2}\theta_1\right)^2 + 2K\left(\frac{L}{2}\theta_2\right)^2 \\ &= 2Kz^2 + \frac{1}{2}KW^2\theta_1^2 + \frac{1}{2}KL^2\theta_2^2 \end{aligned}$$

Where small angle approximation is used in the above. Hence the Lagrangian is

$$\begin{aligned}L &= T - U\\ &= \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - 2Kz^2 - \frac{1}{2}KW^2\theta_1^2 - \frac{1}{2}KL^2\theta_2^2\end{aligned}$$

Equation of motion for z

$$\begin{aligned}\frac{\partial L}{\partial z} &= -4Kz\\ \frac{\partial L}{\partial \dot{z}} &= M\dot{z}\end{aligned}$$

Hence

$$M\ddot{z} + 4Kz = 0$$

Equation of motion for θ_1

$$\begin{aligned}\frac{\partial L}{\partial \theta_1} &= -KW^2\theta_1\\ \frac{\partial L}{\partial \dot{\theta}_1} &= I_1\dot{\theta}_1\end{aligned}$$

Hence

$$I_1\ddot{\theta}_1 + KW^2\theta_1 = 0$$

Similarly, we find

$$I_2\ddot{\theta}_2 + KL^2\theta_2 = 0$$

Therefore

$$[M]\,\ddot{\boldsymbol{q}} + [K]\,\boldsymbol{q} = 0$$

$$\begin{pmatrix} M & 0 & 0\\ 0 & I_1 & 0\\ 0 & 0 & I_1\end{pmatrix}\begin{pmatrix}\ddot{z}\\ \ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} + \begin{pmatrix} 4K & 0 & 0\\ 0 & KW^2 & 0\\ 0 & 0 & KL^2\end{pmatrix}\begin{pmatrix} z\\ \theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Which leads to

$$\det\begin{pmatrix} 4K - M\omega^2 & 0 & 0\\ 0 & KW^2 - I_1\omega^2 & 0\\ 0 & 0 & KL^2 - I_2\omega^2\end{pmatrix} = 0$$

$$\begin{aligned}4K^3L^2W^2 - MK^2L^2\omega^2W^2 - 4I_1K^2L^2\omega^2 - 4I_2K^2\omega^2W^2 + MI_1KL^2\omega^4 + MI_2K\omega^4W^2 + 4I_1I_2K\omega^4 - MI_1I_2\omega^6 &= 0\\ \left(KL^2 - \omega^2 I_2\right)\left(KW^2 - \omega^2 I_1\right)\left(M\omega^2 - 4K\right) &= 0\end{aligned}$$

Therefore

$$\omega_1 = \sqrt{\frac{KL^2}{I_2}}$$
$$\omega_2 = \sqrt{\frac{KW^2}{I_1}}$$
$$\omega_3 = \sqrt{\frac{4K}{M}}$$

Using $I_1 = \frac{1}{12}MW^2, I_2 = \frac{1}{12}ML^2$, the above become

$$\omega_1 = \sqrt{12\frac{KL^2}{ML^2}} = 2\sqrt{3\frac{K}{M}}$$
$$\omega_2 = \sqrt{12\frac{KW^2}{MW^2}} = 2\sqrt{3\frac{K}{M}}$$
$$\omega_3 = \sqrt{\frac{4K}{M}} = 2\sqrt{\frac{K}{M}}$$

Hence $\frac{\omega_1}{\omega_2} = \frac{1}{1}, \frac{\omega_1}{\omega_3} = \sqrt{3}, \frac{\omega_2}{\omega_3} = \sqrt{3}$. Therefore

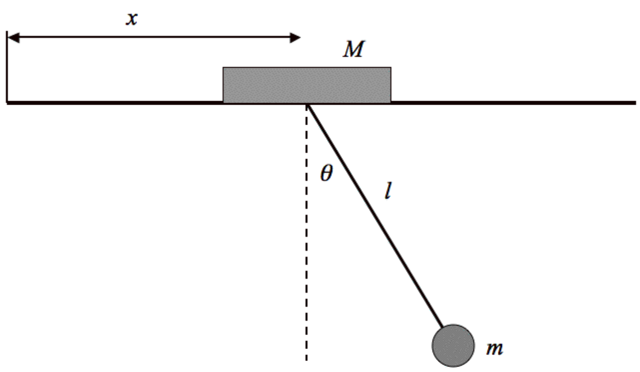
$$\omega_1 : \omega_2 : \omega_3 = 1 : 1 : \sqrt{3}$$

Or

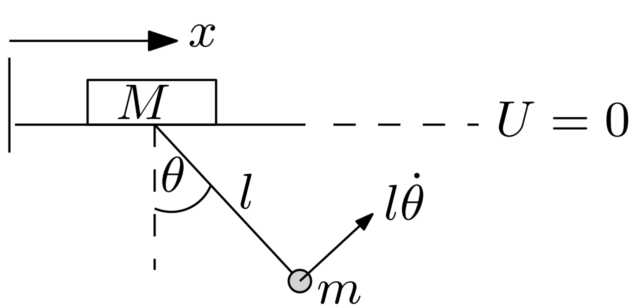
$$\omega_1 : \omega_2 : \omega_3 = \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}} : 1$$

4.10.3 Problem 3

3. (15 points)
A pendulum of mass m and length l is attached to a support of mass M that can move on a frictionless horizontal track as shown on the figure below. Find the normal frequencies and the normal modes of (small) oscillations. Sketch the normal modes.



SOLUTION:



Kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2\right) \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta\right) \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)
\end{aligned}
$$

And potential energy is

$$
U = -mgl\cos\theta
$$

Hence the Lagrangian

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta
\end{aligned}
$$

Now we find equations of motions. For θ

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= -m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2}m\left(2\dot{x}l\cos\theta + 2l^2\dot{\theta}\right) \\
&= m\left(\dot{x}l\cos\theta + l^2\dot{\theta}\right) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= m\left(\ddot{x}l\cos\theta - \dot{x}l\dot{\theta}\sin\theta + l^2\ddot{\theta}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\
m\left(\ddot{x}l\cos\theta - \dot{x}l\dot{\theta}\sin\theta + l^2\ddot{\theta}\right) + m\dot{x}l\dot{\theta}\sin\theta + mgl\sin\theta &= 0 \\
m\ddot{x}l\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta &= 0 \qquad (1)
\end{aligned}
$$

Now we find equation of motion for x

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m\left(\dot{x} + l\dot{\theta}\cos\theta\right) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\
M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) &= 0 \\
\ddot{x}\left(M+m\right) + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= 0 \qquad (2)
\end{aligned}
$$

Now we can write them in matrix form $[M]\ddot{\boldsymbol{q}} + [K]\boldsymbol{q} = 0$, from (1) and (2) we obtain, after using small angle approximation $\cos\theta \approx 1, \sin\theta \approx \theta$ and also $\dot{\theta}^2 \approx 0$

$$
\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

Now assuming solution is $\boldsymbol{q}(t) = \boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$
\begin{pmatrix} -\omega^2\left(M+m\right) & -\omega^2 ml \\ -\omega^2 ml & mgl - ml^2\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)
$$

These have non-trivial solution when

$$\det\begin{pmatrix}-\omega^2(M+m) & -\omega^2 ml\\ -\omega^2 ml & mgl-ml^2\omega^2\end{pmatrix}=0$$
$$Ml^2m\omega^4-glm^2\omega^2-Mglm\omega^2=0$$
$$\omega^2\left(Ml^2m\omega^2-glm^2-Mglm\right)=0$$

Hence $\omega=0$ is one eigenvalue and $\omega=\sqrt{\frac{g}{l}\frac{m+M}{M}}$ is another.

$$\boxed{\omega_1=0}$$
$$\boxed{\omega_2=\sqrt{\frac{g}{l}\frac{(M+m)}{M}}}$$

Now that we found ω_i we go back to (1) to find corresponding eigenvectors. For ω_1, (1) becomes

$$\begin{pmatrix}0&0\\0&mgl\end{pmatrix}\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Hence from the second equation above

$$0a_{11}+mgla_{21}=0$$

So a_{11} can be any value, and $a_{21}=0$. So the following is a valid first eigenvector

$$\boldsymbol{a}_1=\begin{pmatrix}a_{11}\\0\end{pmatrix}$$

For ω_2 (1) becomes

$$\begin{pmatrix}-\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m) & -\left(\frac{g}{l}\frac{(M+m)}{M}\right)ml\\ -\left(\frac{g}{l}\frac{(M+m)}{M}\right)ml & mgl-ml^2\left(\frac{g}{l}\frac{(M+m)}{M}\right)\end{pmatrix}\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

From first equation we find

$$-\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m)a_{12}-\left(\frac{g}{l}\frac{(M+m)}{M}\right)mla_{22}=0$$
$$(M+m)a_{12}+mla_{22}=0$$

Hence $a_{12}=-\frac{ml}{(M+m)}a_{22}$. So the following is a valid second eigenvector

$$\boldsymbol{a}_2=\begin{pmatrix}-\frac{ml}{(M+m)}a_{22}\\a_{22}\end{pmatrix}$$

Therefore

$$x=a_{11}\eta_1+a_{12}\eta_2$$
$$\theta=a_{21}\eta_1+a_{22}\eta_2$$

Where η_i are the normal coordinates. Using relation found earlier, then

$$x = a_{11}\eta_1 \tag{2}$$
$$\theta = -\frac{ml}{(M+m)}a_{22}\eta_1 + a_{22}\eta_2 \tag{3}$$

Hence from (2)

$$\eta_1 = -\frac{x}{a_{11}}$$

And now (3) can be written as

$$\theta = -\frac{ml}{(M+m)}a_{22}\frac{x}{a_{11}} + a_{22}\eta_2$$

Therefore

$$\eta_2 = \frac{\theta}{a_{22}} + \frac{1}{a_{11}}\frac{mlx}{(M+m)}$$

To sketch the mode shapes. Looking at $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$ and $\boldsymbol{a}_2 = \begin{pmatrix} -\frac{ml}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$ and normalizing we can write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{ml}{(M+m)} \\ 1 \end{pmatrix}$$

So in the first mode shape, the mass M moves with the pendulum fixed to it in the same orientation all the time. So the whole system just slides along x with $\theta = 0$ all the time. In the second mode, x move by $\frac{-ml}{(M+m)}$ factor to θ motion. For example, for $M \ll m$, then mode 2 is $\begin{pmatrix} -l \\ 1 \end{pmatrix}$, hence antisymmetric mode. If $M = m$ then we get $\begin{pmatrix} -\frac{l}{2} \\ 1 \end{pmatrix}$ antisymmetric, but now the ratio changes. So the second mode shape is antisymmetric, but the ratio depends on the ratio of m to M.

4.10.3.1 Appendix to problem 3

This is extra and can be ignored if needed. I was not sure if we should use $s = l\theta$ as the generalized coordinate instead of θ in order to make all the coordinates of same units. So this is repeat of the above, but using $s = l\theta$ transformation. Starting with equations of motion

$$\ddot{x}(M+m) + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0$$
$$m\ddot{\theta} + m\ddot{x}\frac{\cos\theta}{l} + m\frac{g}{l}\sin\theta = 0$$

Will now use $s = l\theta$ transformation, and use s as the second degree of freedom, which is the small distance the pendulum mass swings by. This is so that both x and s has same units of length to make it easier to work with the shape functions. Hence the equations of motions become

$$\ddot{x}(M+m) + ml\frac{\ddot{s}}{l}\cos\left(\frac{s}{l}\right) - ml\frac{\dot{s}^2}{l^2}\sin\left(\frac{s}{l}\right) = 0$$

$$m\frac{\ddot{s}}{l} + m\ddot{x}\frac{\cos\left(\frac{s}{l}\right)}{l} + m\frac{g}{l}\sin\left(\frac{s}{l}\right) = 0$$

We first apply small angle approximation, which implies $\cos\frac{s}{l} \to 1, \sin\left(\frac{s}{l}\right) \to \frac{s}{l}$ and also $\frac{\dot{s}^2}{l^2} \to 0$, therefore the equations of motions becomes

$$\ddot{x}(M+m) + m\ddot{s} = 0$$

$$m\frac{\ddot{s}}{l} + m\ddot{x}\frac{1}{l} + m\frac{g}{l}\frac{s}{l} = 0$$

And now we write the matrix form

$$\begin{pmatrix} M+m & m \\ m & m \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & m\frac{g}{l} \end{pmatrix}\begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now assuming solution is $\boldsymbol{q}(t) = \boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$\begin{pmatrix} -\omega^2(M+m) & -\omega^2 m \\ -\omega^2 m & m\frac{g}{l} - m\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

These have non-trivial solution when

$$\det\begin{pmatrix} -\omega^2(M+m) & -\omega^2 m \\ -\omega^2 m & m\frac{g}{l} - m\omega^2 \end{pmatrix} = 0$$

$$-\frac{1}{l}\left(gm^2\omega^2 - Mlm\omega^4 + Mgm\omega^2\right) = 0$$

$$\omega^2\left(\frac{gm^2}{l} - Mm\omega^2 + M\frac{g}{l}m\right) = 0$$

$$\omega^2\left(M\omega^2 - \left(\frac{g}{l}(m+M)\right)\right) = 0$$

Hence $\omega = 0$ is one eigenvalue and $\omega = \sqrt{\frac{g}{l}\frac{(M+m)}{M}}$ is another.

$$\boxed{\omega_1 = 0}$$

$$\boxed{\omega_2 = \sqrt{\frac{g}{l}\frac{(M+m)}{M}}}$$

Now that we found ω_i we go back to (1) to find corresponding eigenvectors. For ω_1, (1) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & m\frac{g}{l} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0a_{11} + m\frac{g}{l}a_{21} = 0$$

Hence from the second equation above

$$0a_{11} + m\frac{g}{l}a_{21} = 0$$

So a_{11} can be any value, and $a_{21} = 0$. So the following is a valid first eigenvector

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$$

For ω_2 (1) becomes

$$\begin{pmatrix} -\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m) & -\left(\frac{g}{l}\frac{(M+m)}{M}\right)m \\ -\left(\frac{g}{l}\frac{(M+m)}{M}\right)m & m\frac{g}{l} - m\left(\frac{g}{l}\frac{(M+m)}{M}\right) \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find

$$\begin{aligned} -\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m)\,a_{12} - \left(\frac{g}{l}\frac{(M+m)}{M}\right)ma_{22} &= 0 \\ (M+m)\,a_{12} + ma_{22} &= 0 \end{aligned}$$

Hence $a_{12} = -\frac{m}{(M+m)}a_{22}$. So the following is a valid second eigenvector

$$\boldsymbol{a}_2 = \begin{pmatrix} -\frac{m}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$$

Therefore

$$\begin{aligned} x &= a_{11}\eta_1 + a_{12}\eta_2 \\ \theta &= a_{12}\eta_1 + a_{22}\eta_2 \end{aligned}$$

Where η_i are the normal coordinates. Using relation found earlier, then

$$x = a_{11}\eta_1 \tag{2}$$

$$\theta = -\frac{m}{(M+m)}a_{22}\eta_1 + a_{22}\eta_2 \tag{3}$$

Hence from (2)

$$\eta_1 = -\frac{x}{a_{11}}$$

And now (3) can be written as

$$\theta = -\frac{m}{(M+m)}a_{22}\frac{x}{a_{11}} + a_{22}\eta_2$$

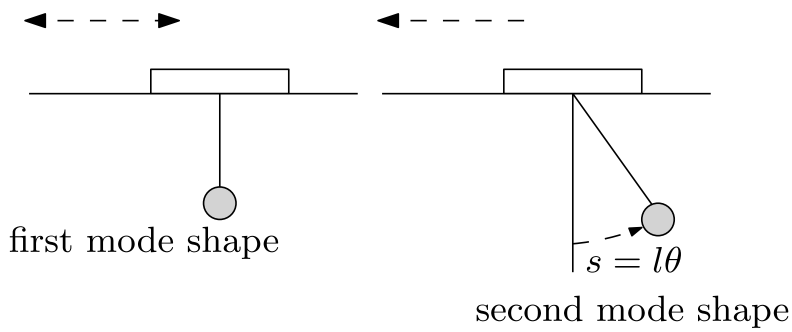
Therefore

$$\eta_2 = \frac{\theta}{a_{22}} + \frac{mx}{(M+m)}\frac{1}{a_{11}}$$

To sketch the mode shapes. Looking at $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$ and $\boldsymbol{a}_2 = \begin{pmatrix} -\frac{m}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$ and normalizing we can write

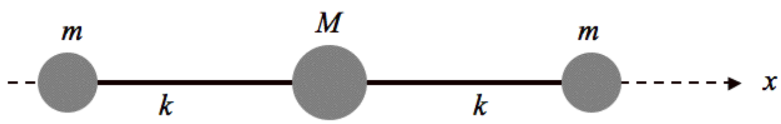
$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-\frac{m}{(M+m)}\\1\end{pmatrix}$$

So in the first mode shape, the mass M moves with the pendulum fixed to it in the same orientation all the time. So the whole system just slides along x with $\theta = 0$ all the time. In the second mode, x move by $\frac{-m}{(M+m)}$ factor to θ motion. For example, for $M \ll m$, then mode 2 is $\begin{pmatrix}-1\\1\end{pmatrix}$, hence antisymmetric mode. If $M = m$ then we get $\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}$ antisymmetric, but now the ratio changes. So the second mode shape is antisymmetric, but the ratio depends on the ratio of m to M.

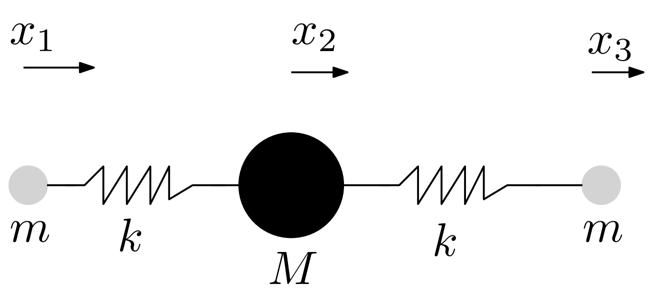


4.10.4 Problem 4

4. (15 points)
Consider the simple model for the carbon dioxide molecule CO_2 shown below. Two end particles of mass m are bound to the central particle M via a potential function that is equivalent to two springs with spring constant k. Consider motion in one dimension only, along the x-axis. Find the normal frequencies and the normal modes. Make a rough sketch of the normal modes.



SOLUTION:



Kinetic energy

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

Potential energy

$$U = \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2$$

Hence the Lagrangian

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2 - \frac{1}{2}k\left(x_3 - x_2\right)^2
\end{aligned}
$$

EQM for x_1

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= k\left(x_2 - x_1\right) \\
\frac{\partial L}{\partial \dot{x}_1} &= m\dot{x}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} &= m\ddot{x}_1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
m\ddot{x}_1 - k\left(x_2 - x_1\right) &= 0 \\
m\ddot{x}_1 + kx_1 - kx_2 &= 0 \qquad (1)
\end{aligned}
$$

EQM for x_2

$$
\begin{aligned}
\frac{\partial L}{\partial x_2} &= -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) \\
\frac{\partial L}{\partial \dot{x}_2} &= M\dot{x}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} &= M\ddot{x}_2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M\ddot{x}_2 + k\left(x_2 - x_1\right) - k\left(x_3 - x_2\right) &= 0 \\
M\ddot{x}_2 + kx_2 - kx_1 - kx_3 + kx_2 &= 0 \\
M\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 &= 0 \qquad (2)
\end{aligned}
$$

EQM for x_3

$$
\begin{aligned}
\frac{\partial L}{\partial x_3} &= -k\left(x_3 - x_2\right) \\
\frac{\partial L}{\partial \dot{x}_3} &= m\dot{x}_3 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_3} &= m\ddot{x}_3
\end{aligned}
$$

Therefore

$$
\begin{aligned}
m\ddot{x}_3 + k\left(x_3 - x_2\right) &= 0 \\
m\ddot{x}_3 + kx_3 - kx_2 &= 0 \qquad (3)
\end{aligned}
$$

Now we can write equations (1,2,3) in matrix form $[M]\ddot{\boldsymbol{q}} + [K]\boldsymbol{q} = 0$ to obtain

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now assuming solution is $\boldsymbol{q}(t) = \boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

These have non-trivial solution when

$$\det\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} = 0$$

$$\omega^2\left(k - m\omega^2\right)\left(-Mm\omega^2 + Mk + 2km\right) = 0$$

Hence we have 3 normal frequencies. One of them is zero.

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= \sqrt{\frac{k}{m}} \\ \omega_3 &= \sqrt{k\frac{M + 2m}{Mm}} \end{aligned}$$

For each normal frequency, there is a corresponding eigen shape vector. Now we find these eigen shapes. For ω_1, and from (4)

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} ka_1 - ka_2 + 0a_3 &= 0 \\ -ka_1 + 2ka_2 - ka_3 &= 0 \\ 0a_1 - ka_2 + ka_3 &= 0 \end{aligned}$$

Or

$$\begin{aligned} a_1 - a_2 &= 0 \\ -a_1 + 2a_2 - a_3 &= 0 \\ -a_2 + a_3 &= 0 \end{aligned}$$

Hence $a_1 = a_2$ and $a_2 = a_3$. So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is first eigenvector. Now we find the second one for ω_2.

From (4) and using $\omega = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} k - m\frac{k}{m} & -k & 0 \\ -k & 2k - M\frac{k}{m} & -k \\ 0 & -k & k - m\frac{k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k - M\frac{k}{m} & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} -ka_2 &= 0 \\ -ka_1 + \left(2k - M\frac{k}{m}\right)a_2 - ka_3 &= 0 \\ -ka_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} a_2 &= 0 \\ -a_1 + a_2\left(2 - \frac{M}{m}\right) - a_3 &= 0 \\ a_2 &= 0 \end{aligned}$$

hence $a_2 = 0$ and $a_1 = -a_3$. So $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is second eigenvector. Now we find the third one for ω_3.

From (4) and using $\omega = \sqrt{k\frac{M+2m}{Mm}}$

$$\begin{pmatrix} k - m\left(k\frac{M+2m}{Mm}\right) & -k & 0 \\ -k & 2k - M\left(k\frac{M+2m}{Mm}\right) & -k \\ 0 & -k & k - m\left(k\frac{M+2m}{Mm}\right) \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k - k\frac{M+2m}{M} & -k & 0 \\ -k & 2k - k\frac{M+2m}{m} & -k \\ 0 & -k & k - k\frac{M+2m}{M} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

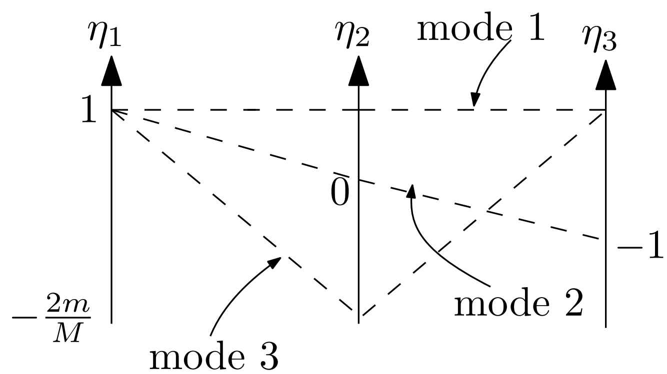
Hence

$$\begin{aligned} k\left(1 - \frac{M+2m}{M}\right)a_1 - ka_2 &= 0 \\ -ka_1 + k\left(2 - \frac{M+2m}{m}\right)a_2 - ka_3 &= 0 \\ -ka_2 + k\left(1 - \frac{M+2m}{M}\right)a_3 &= 0 \end{aligned}$$

Or

$$\left(1-\frac{M+2m}{M}\right)a_1-a_2=0$$
$$-a_1+\left(2-\frac{M+2m}{m}\right)a_2-a_3=0$$
$$-a_2+\left(1-\frac{M+2m}{M}\right)a_3=0$$

Solution is: $a_1=a_3,a_2=-\frac{2}{M}ma_3$ So $\begin{pmatrix}1\\-\frac{2m}{M}\\1\end{pmatrix}$ is third eigevector. To sketch the mode shapes, will use the following diagram



4.10.5 HW 10 key solution

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Mechanics
Physics 311 - Fall 2015
Homework Set 10 - Solutions

Problem 1

total energy of the r-th normal mode:

$$\mathcal{E}_r = T_r + U_r$$
$$= \frac{1}{2}\dot{\eta}_r^2 + \frac{1}{2}\omega_r^2\eta_r^2$$

with $\eta_r = \beta_r e^{i\omega_r t}$ $\qquad \dot{\eta}_r = i\omega_r\beta_r e^{i\omega_r t}$

but: need the real part of η_r and $\dot{\eta}_r$

(note that β is complex, so $\beta_r = \mu_r + i\nu_r$)

$$\mathrm{Re}\{\eta_r\} = \mathrm{Re}\{(\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\}$$
$$= \mu_r\cos\omega_r t - \nu_r\sin\omega_r t$$

$$\mathrm{Re}\{\dot{\eta}_r\} = \mathrm{Re}\{i\omega_r(\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\}$$
$$= -\omega_r\nu_r\cos\omega_r t - \omega_r\mu_r\sin\omega_r t$$

$$\Rightarrow \quad \mathcal{E}_r = \frac{1}{2}(-\omega_r\nu_r\cos\omega_r t - \omega_r\mu_r\sin\omega_r t)^2$$
$$+ \frac{1}{2}\omega_r^2(\mu_r\cos\omega_r t - \nu_r\sin\omega_r t)^2$$

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$$\Leftrightarrow \quad E_r = \frac{1}{2}\omega_r^2 \left[\nu_r^2 \cos^2\omega_r t + \mu_r^2 \sin^2\omega_r t + 2\mu_r\nu_r \cos\omega_r t \sin\omega_r t + \mu_r^2 \cos^2\omega_r t + \nu_r^2 \sin^2\omega_r t - 2\mu_r\nu_r \cos\omega_r t \sin\omega_r t \right]$$

$$= \frac{1}{2}\omega_r^2 \left[\mu_r^2 + \nu_r^2\right]$$

or

$$\boxed{E_r = \frac{1}{2}\omega_r^2 |\beta_r|^2}$$

$\therefore$ energy of each normal mode is separately conserved ∎

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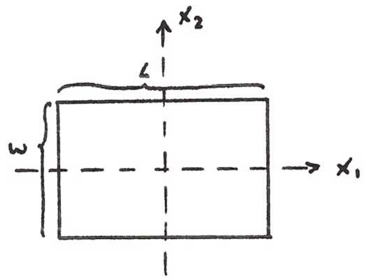
Problem 2



Symmetry implies that the three normal modes are:

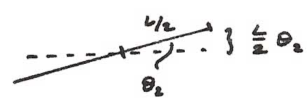
(i) the plate moves up and down and does not rotate
(ii) the plate rotates about the line x_2, the center of mass is fixed
(iii) the plate rotates about the line x_1 the center of mass is fixed

case (i) $\quad T = \frac{1}{2} M \left(\frac{dx_3}{dt}\right)^2 \qquad U = \frac{1}{2}(4k)\, x_3^2$

$$\Rightarrow \quad M\ddot{x}_3 + 4k\, x_3 = 0 \quad \Rightarrow \quad \boxed{\omega_1 = \sqrt{\frac{4k}{M}}}$$

case (ii) $\quad T = \frac{1}{2} I_2 (\dot{\theta}_2)^2 \qquad U = \frac{1}{2}(4k)\left(\frac{L}{2}\theta_2\right)^2$

for small oscillations



$$I_2 = w\rho \int_{-L/2}^{L/2} x_1^2 \, dx_1$$

$$= w\rho \, \frac{2}{3} \, \frac{L^3}{8} = \frac{1}{12} M L^2 \qquad \text{with } M = wL\rho$$

$$\Rightarrow \quad I_2 \, \ddot{\theta}_2 + 4k \frac{L^2}{4} \theta_2 = 0$$

$$\Leftrightarrow \quad \ddot{\theta}_2 + \frac{12k}{M} \theta_2 = 0 \quad \Rightarrow \quad \boxed{\omega_2 = \sqrt{\frac{12k}{M}}}$$

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case (iii) $T = \frac{1}{2} I_1 (\dot{\theta}_1)^2$ $U = \frac{1}{2}(4k)\left(\frac{W}{2}\theta_1\right)^2$

$I_1 = \frac{1}{12} M W^2$

so (as in (ii)) $\ddot{\theta}_1 + \frac{12k}{M}\theta_1 = 0 \Rightarrow \boxed{\omega_3 = \sqrt{\frac{12k}{M}}}$

So the frequency ratio is

$\omega_1 : \omega_2 : \omega_3 = 1 : \sqrt{3} : \sqrt{3}$ VII

$\omega_2 = \omega_3 \Rightarrow$ degeneracy !

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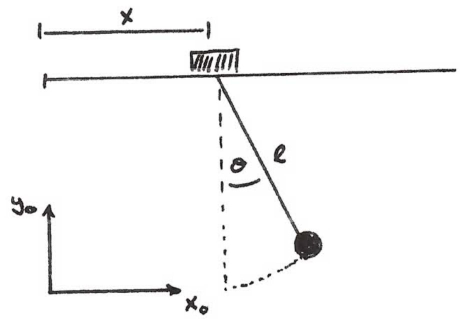
Problem 3

generalized coordinates x, $s = \ell\theta$



in the inertial frame x_0, y_0:

$$x_0 = x + \ell \sin\theta$$
$$y_0 = \ell(1-\cos\theta)$$

$$\dot{x}_0 = \dot{x} + \ell\dot{\theta}\cos\theta$$
$$\dot{y}_0 = \ell\dot{\theta}\sin\theta$$

$$\Rightarrow \quad T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_0^2 + \dot{y}_0^2)$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (\ell\dot{\theta}\sin\theta)^2\right]$$

$$U = mgy_0 = mg\ell(1-\cos\theta)$$

Small oscillations:

$$T \simeq \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\dot{s})^2$$
$$U = mg\ell\left(1 - 1 + \frac{\theta^2}{2}\right) = \frac{1}{2}mg\ell\theta^2 = \frac{m}{2\ell}s^2 g$$

(disregarding all terms θ^4, $\dot{\theta}^2\theta^2$, ... etc)

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So $A_{11} = \left(\frac{\partial^2 U}{\partial x^2}\right)_0 = 0$ $\qquad A_{12} = A_{21} = \left(\frac{\partial^2 U}{\partial x \partial s}\right) = 0$

$A_{22} = \left(\frac{\partial^2 U}{\partial s^2}\right)_0 = \frac{mg}{\ell}$

and $m_{11} = m + M$ $\qquad m_{12} = m_{21} = m$

$m_{22} = m$

$$\Rightarrow \quad \begin{vmatrix} -\omega^2 (M+m) & -m\omega^2 \\ -m\omega^2 & \frac{mg}{\ell} - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad -\omega^2 (M+m)\left(\frac{mg}{\ell} - m\omega^2\right) - m^2\omega^4 = 0$$

$$\Rightarrow \quad \boxed{\omega_1^2 = 0} \quad \text{and} \quad -(M+m)\left(\frac{mg}{\ell} - m\omega_2^2\right) = m^2\omega_2^2$$

$$\Leftrightarrow \quad -\frac{Mmg}{\ell} + Mm\omega_2^2 - \frac{m^2 g}{\ell} + \cancel{m^2\omega_2^2} = \cancel{m^2\omega_2^2}$$

$$\Leftrightarrow \quad \boxed{\omega_2^2 = \frac{g}{\ell}\,\frac{M+m}{M}}$$

$\therefore$ eigenvector for $\omega_1^2 = 0$ $\qquad \begin{pmatrix} 0 & 0 \\ 0 & mg/\ell \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$

$$\Rightarrow \quad a_{11} = a_{21} = 0$$

Λ this mode is <u>not</u> an oscillation :

and $\boxed{\begin{aligned} &\theta = 0 \\ &x = A_1 t + A_2 \end{aligned}}$ $\Big\}$ pure <u>translation</u>!

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ii. eigenvector for $\omega_2^2 = \frac{g}{\ell}\,\frac{M+m}{M}$

$$\begin{pmatrix} -\frac{g}{\ell}\frac{(M+m)^2}{M} & -m\frac{g}{\ell}\frac{M+m}{M} \\ -m\frac{g}{\ell}\frac{M+m}{M} & \frac{mg}{\ell} - m\frac{g}{\ell}\frac{M+m}{M} \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$$

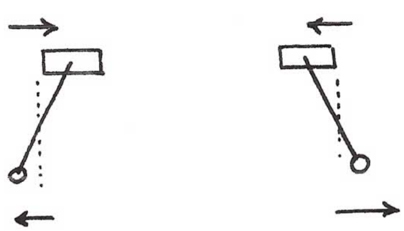
$$\Rightarrow \quad -\frac{g}{\ell}\frac{M+m}{M}\,a_{12} - \frac{mg}{\ell M}\,a_{22} = 0$$

$$\Leftrightarrow \quad (M+m)\,a_{12} = -m\,a_{22}$$

$$\Leftrightarrow \quad a_{12} = -\frac{m}{M+m}\,a_{22}$$

So
$$\boxed{\begin{aligned} x &= A\,e^{i\omega_2 t} \\ s &= -A\,\frac{M+m}{m}\,e^{i\omega_2 t} \end{aligned}}$$

pendulum oscillates with different amplitude than M, both are 180° out of phase



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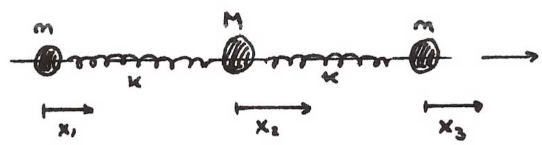
Problem 4

The Lagrangian of the system is

$$L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_3^2 + \frac{M}{2}\dot{x}_2^2$$
$$- \left(\frac{k}{2}(x_2 - x_1)^2 + \frac{k}{2}(x_3 - x_2)^2 \right)$$



$m_{11} = m = m_{33}$

$m_{22} = M$

$m_{12} = 0$

$m_{31} = 0$

$m_{32} = 0$

$$A_{11} = \left(\frac{\partial^2 U}{\partial x_1^2}\right)_0 = k$$

$$A_{22} = \left(\frac{\partial^2 U}{\partial x_2^2}\right)_0 = 2k$$

$$A_{33} = \left(\frac{\partial^2 U}{\partial x_3^2}\right) = k$$

$$A_{12} = \left(\frac{\partial^2 U}{\partial x_1 \partial x_2}\right)_0 = -k$$

$$A_{13} = \left(\frac{\partial^2 U}{\partial x_1 \partial x_3}\right)_0 = 0$$

$$A_{23} = \left(\frac{\partial^2 U}{\partial x_2 \partial x_3}\right)_0 = -k$$

$$\Rightarrow \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

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$$\Rightarrow \quad (K - m\omega^2)^2 (2K - M\omega^2) - K^2 (K - m\omega^2) - K^2 (K - m\omega^2) = 0$$

$$\Leftrightarrow \quad (K - m\omega^2) \left[(K - m\omega^2)(2K - M\omega^2) - 2K^2 \right] = 0$$

$$\Leftrightarrow \quad (K - m\omega^2) \left[2K^2 - MK\omega^2 - 2mK\omega^2 + mM\omega^4 - 2K^2 \right] = 0$$

$$\Leftrightarrow \quad \omega^2 (K - m\omega^2)(mM\omega^2 - MK - 2mK) = 0$$

$$\Rightarrow \quad \boxed{\omega_1 = 0} \qquad \boxed{\omega_2 = \sqrt{\frac{K}{m}}} \qquad \boxed{\omega_3 = \sqrt{\frac{K}{m} + \frac{2K}{M}}}$$

(i) $\omega_1 = 0$ no oscillation, just translation of the system as a whole

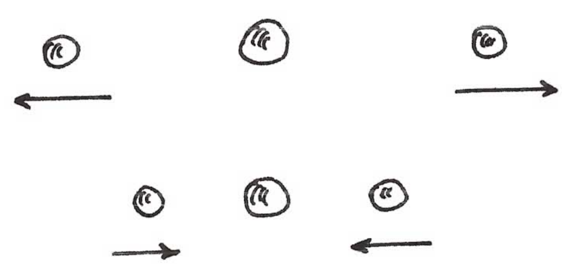
$$a_{11} = a_{21} = a_{31} = 0 \qquad x_1(t) = A_1 t + A_2 \quad \text{etc.}$$

(ii) $\omega_2 = \sqrt{\frac{K}{m}}$

$$\left(K - m\frac{K}{m}\right) a_{12} - K a_{22} = 0 \Rightarrow a_{22} = 0$$

$$-K a_{12} - K a_{32} = 0 \Rightarrow a_{12} = -a_{32}$$

the center particle is at rest and the two end particles vibrate in opposite directions with the same amplitude



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(iii) $\omega_3 = \sqrt{\frac{k}{m} + \frac{2k}{M}}$

$$\left[k - m\left(\frac{k}{m} + \frac{2k}{M}\right)\right] a_{13} - k\, a_{23} = 0$$

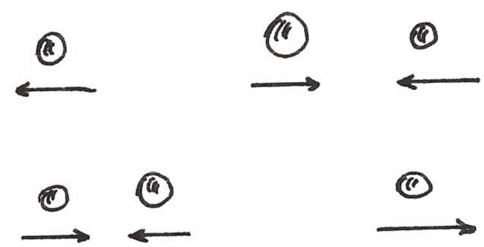
$$\Rightarrow \quad a_{23} = -2\,\frac{m}{M}\, a_{13}$$

$$-k\, a_{13} + \left[2k - M\left(\frac{k}{m} + \frac{2k}{M}\right)\right] a_{23} - k\, a_{33} = 0$$

$$\Rightarrow \quad -a_{13} + 2\, a_{13} = a_{33}$$

$$\Leftrightarrow \quad a_{33} = a_{13}$$

the two end particles oscillate in phase at the same amplitude while the center particle oscillates oppositely with a different amplitude



4.11 HW For honors extra credit only

Mechanics
Physics 311
Fall 2015
Problems for Honors Credit (10/23/15, due 12/4/15)

You will need the help of a computer to find the solutions and to produce plots of the results. Teamwork is encouraged in solving the problems.

The Restricted Three-Body Problem and the 5 Lagrange Points

1. Read Chapter 7.4 from *Analytical Mechanics* by Fowles and Cassiday. A copy of the chapter is attached.
2. Determine the coordinates of the five Lagrange points L_1 to L_5 for the Earth-Moon system. Describe the behavior of the effective potential function in the neighborhood of these points.
3. Show by explicit calculation that the gradient of the effective potential function vanishes at L_4 and L_5.

Chapter 5

Study and cheat sheets

5.1 Note added Nov 12, 2015

Looking at Example 5.3, textbook page 190, Physics 311.

Nasser M. Abbasi

Define dΦ

In[81]:=
```
Clear[x, r, a, ρ, G0, m];
dphi = -ρ G0 / Sqrt[1 + (r / a) ^2 - 2 r / a Cos[x]]
```

Out[82]=

$$-\frac{G0\,\rho}{\sqrt{1+\frac{r^2}{a^2}-\frac{2\,r\,\mathrm{Cos}[x]}{a}}}$$

Integrate it over 0 to 2π

In[83]:=
```
u = Int[dphi, x]
```

Out[83]=

$$-\left(\left(2\,G0\,\rho\,\sqrt{\frac{a^2\left(1+\frac{r^2}{a^2}-\frac{2\,r\,\mathrm{Cos}[x]}{a}\right)}{(a-r)^2}}\;\mathrm{EllipticF}\left[\frac{x}{2},-\frac{4\,a\,r}{(a-r)^2}\right]\right)\Big/\left(\sqrt{1+\frac{r^2}{a^2}-\frac{2\,r\,\mathrm{Cos}[x]}{a}}\right)\right)$$

Evaluate it over the limit

In[84]:=
```
U0 = m ((u /. x → 2 Pi) - (u /. x → 0))
```

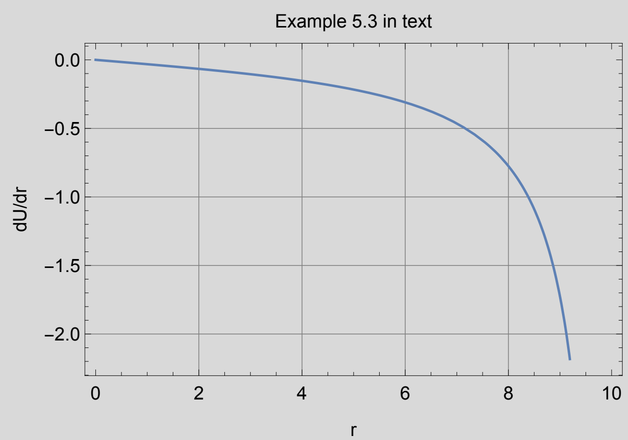
Out[84]=

$$-\frac{4\,G0\,m\,\sqrt{\frac{a^2\left(1-\frac{2\,r}{a}+\frac{r^2}{a^2}\right)}{(a-r)^2}}\;\rho\,\mathrm{EllipticK}\left[-\frac{4\,a\,r}{(a-r)^2}\right]}{\sqrt{1-\frac{2\,r}{a}+\frac{r^2}{a^2}}}$$

Find dU/dr and plot it for $r = 0$ to a, and see where it is zero. These will be the equilibrium points. Give "a" some value to plot

In[85]:=
```
Plot[Evaluate[D[U0, r] /. {a → 10, ρ -> 1, G0 → 1, m → 1}],
 {r, 0, 10}, GridLines → Automatic, GridLinesStyle → Gray, Frame → True,
 FrameLabel → {{"dU/dr", None}, {"r", "Example 5.3 in text"}},
 BaseStyle → 12, ImageSize → 400]
```

Out[85]=



We see from above that du/dr is zero only at r=0. Also r=0 is not a stable point. (as shown in text).

Find $\frac{d^2 U}{dr^2}$ at $r = 0$ to verify the text book result

In[86]:=
```
Limit[D[U0, {r, 2}] /. ρ → M/(2 Pi a), r → 0]
```

Out[86]=

$$-\frac{G0\, m\, M}{2\, a^3}$$