HW3 Physics 311 Mechanics

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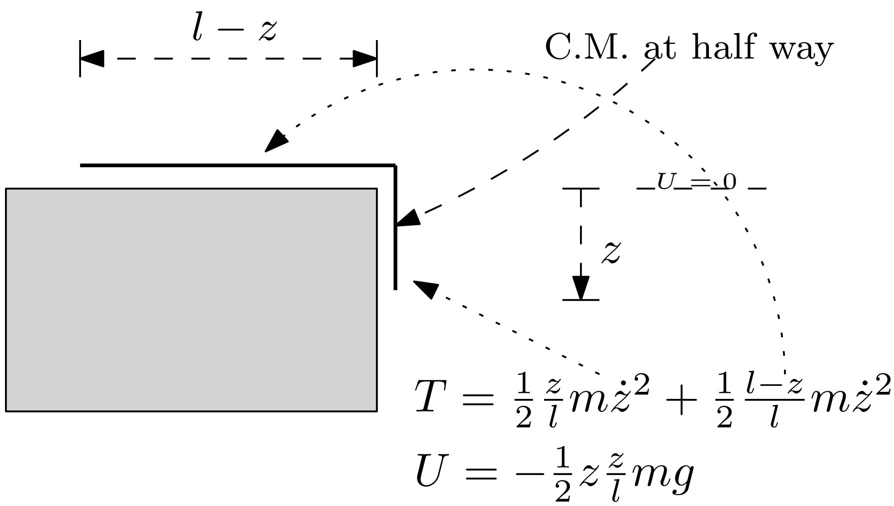
Contents

0.1 Problem 1

1. (5 points)
A uniform rope of total mass m and total length l lies on a table, with a length z hanging over the edge. Find the differential equation of motion.

SOLUTION



The top portion of the rope moves with same speed as the hanging portion. Hence z is used to describe the motion as the generalized coordinate. From the above

$$U=-\left(\frac{1}{2}z\right)\left(\frac{z}{l}\right)mg=-\frac{1}{2}\left(\frac{z^2}{l}\right)mg$$

$$T=\frac{1}{2}\left(\frac{z}{l}\right)m\dot{z}^2+\frac{1}{2}\left(\frac{l-z}{l}\right)m\dot{z}^2=\frac{1}{2}m\dot{z}^2$$

In finding U we used $\frac{1}{2}$ since the center of mass of the hanging part is half way over the length. So the potential energy is taken from the center of mass. In the above, $\dot{z}$ is used for both parts of the rope, since both parts move with same speed. Applying Lagrangian equations gives

$$\begin{aligned}L&=T-U\\&=\frac{1}{2}m\dot{z}^2+\frac{1}{2}\left(\frac{z^2}{l}\right)mg\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial L}{\partial z}&=\frac{z}{l}mg\\\frac{d}{dt}\frac{\partial L}{\partial \dot{z}}&=m\ddot{z}\end{aligned}$$

And therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$
$$m\ddot{z} - \frac{z}{l}mg = 0$$
$$\ddot{z} = \frac{z}{l}g$$

When $z = 0$ then the acceleration is zero as expected. When $z = \frac{l}{2}$ then $\ddot{z} = \frac{1}{2}g$ and when $z = l$ then $\ddot{z} = g$ as expected since in this case the rope will all be falling down on its own weight due to gravity and should have g as the acceleration.

0.2 Problem 2

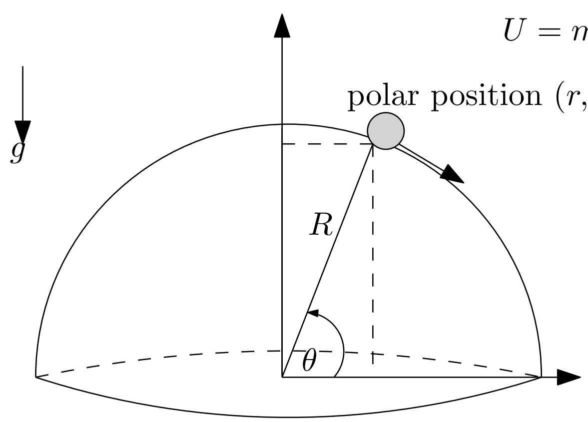
2. (10 points)
A particle of mass m perched on top of a smooth hemisphere of radius R is disturbed slightly, so that it begins to slide down the side. Use Lagrange multipliers to find the normal force of constraint exerted by the hemisphere on the particle and determine the angle relative to the vertical at which it leaves the hemisphere.

SOLUTION

constraint $f(r,\theta) = r - R = 0$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$U = mgr\sin\theta$



Generalized coordinates used r,θ

There are two coordinates r,θ (polar) and one constraint

$$f(r,\theta) = r - R = 0 \tag{1}$$

Now we set up the equations of motion for m

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\\
U &= mgr\sin\theta\\
L &= T - U\\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta
\end{aligned}
$$

Hence the Euler-Lagrangian equations are

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} + \lambda\frac{\partial f}{\partial r} = 0 \tag{2}
$$

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{3}
$$

But

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r}\\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)\\
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - mg\sin\theta\\
\frac{\partial L}{\partial \theta} &= -mgr\cos\theta\\
\frac{\partial f}{\partial r} &= 1\\
\frac{\partial f}{\partial \theta} &= 0
\end{aligned}
$$

Hence (2) becomes

$$
m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta + \lambda = 0 \tag{4}
$$

And (3) becomes

$$
m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + mgr\cos\theta = 0
$$

$$
r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\cos\theta = 0 \tag{5}
$$

We now need to solve (1,4,5) for λ. Now we have to apply the constrain that $r = R$ in the above to be able to solve (4,5) equations. Therefore, (4,5) becomes

$$
-mR\dot{\theta}^2 + mg\cos\theta + \lambda = 0 \tag{4A}
$$

$$
R\ddot{\theta} + g\cos\theta = 0 \tag{5A}
$$

Where (4A,5A) were obtained from (4,5) by replacing $r = R$ and $\dot{r} = 0$ and $\ddot{r} = 0$ since we are using that $r = R$ which is constant (the radius).

From (5A) we see that this can be integrated giving

$$R\dot{\theta}^2 + 2g\sin\theta + c = 0 \tag{6}$$

Where c is constant. Since if we differentiate the above with time, we obtain

$$\begin{aligned} 2R\dot{\theta}\ddot{\theta} + 2g\dot{\theta}\cos\theta &= 0 \\ R\ddot{\theta} + g\cos\theta &= 0 \end{aligned}$$

Which is the same as (5A). Therefore from (6) we find $\dot{\theta}^2$ to use in (4A). Hence from (6)

$$\dot{\theta}^2 = -2\frac{g}{R}\sin\theta + c$$

To find c we use initial conditions. At $t = 0$, $\theta = 90^0$ and $\dot{\theta}(0) = 0$ hence

$$c = 2\frac{g}{R}$$

Therefore

$$\begin{aligned} \dot{\theta}^2 &= -2\frac{g}{R}\sin\theta + 2\frac{g}{R} \\ &= 2\frac{g}{R}(1 - \sin\theta) \end{aligned}$$

Plugging the above into (4A) in order to find λ gives

$$\begin{aligned} -mR\left(2\frac{g}{R}(1-\sin\theta)\right) + mg\sin\theta + \lambda &= 0 \\ \lambda &= m\left(2g(1-\sin\theta)\right) - mg\sin\theta \\ \lambda &= 2mg - 2mg\sin\theta - mg\sin\theta \\ &= mg(2 - 3\sin\theta) \end{aligned}$$

Now that we found λ ,we can find the constraint force in the radial direction

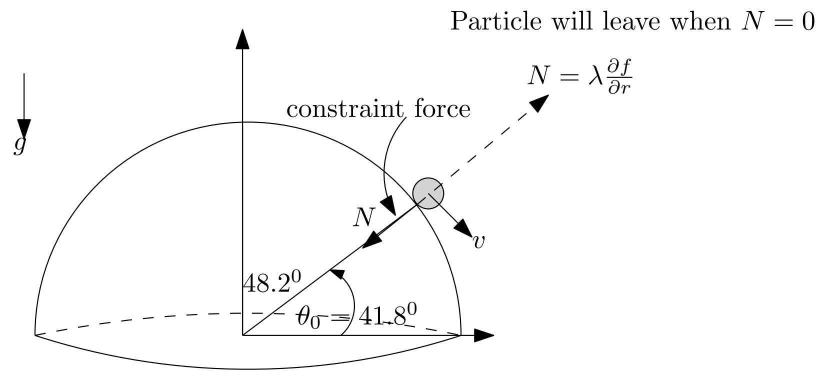
$$\begin{aligned} N &= \lambda\frac{\partial f}{\partial r} \\ &= mg(2 - 3\sin\theta) \end{aligned}$$

The particle will leave when $N = 0$ which will happen when

$$\begin{aligned} 2 - 3\sin\theta &= 0 \\ \theta &= \sin^{-1}\left(\frac{2}{3}\right) \\ &= 41.8^0 \end{aligned}$$

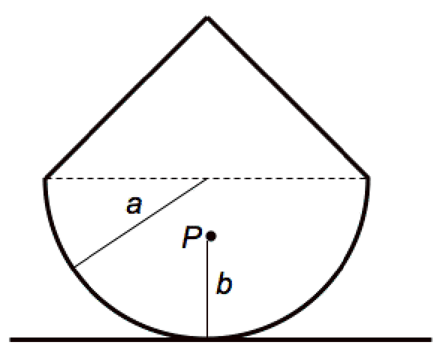
Therefore, the angle from the vertical is

$$90 - 41.8 = 48.2^0$$

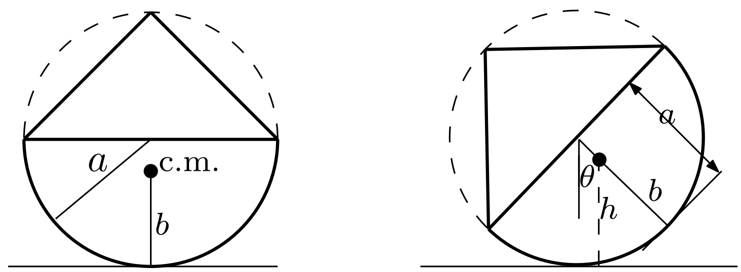


0.3 Problem 3

3. (10 points)
Consider the object shown in the figure below, which has a half-sphere of radius a as the bottom part and a cone on top. The center of mass (P) is at a distance b from the ground when the object is standing upright. Let I be the moment of inertia. Find the frequency of small oscillations if the object is disturbed slightly from its upright position. What happens if $a = b$ or $b > a$?



SOLUTION



$$h = a - (a - b)\cos\theta$$
$$U = mgh = mg(a - (a - b)\cos\theta$$

From the above, we see that the center of mass has height above the ground level after rotation of

$$h = a - (a - b)\cos\theta$$

Taking the ground state as the floor, the potential energy in this state is

$$\begin{aligned} U &= mgh \\ &= mg\,(a - (a - b)\cos\theta) \end{aligned}$$

And the kinetic energy

$$T = \frac{1}{2} I\dot{\theta}^2$$

Hence the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} I\dot{\theta}^2 - mg\,(a - (a - b)\cos\theta) \end{aligned}$$

Therefore the equation of motion is

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= 0 \\ I\ddot{\theta} - \frac{\partial}{\partial\theta}\left(\frac{1}{2} I\dot{\theta}^2 - mg\,(a - (a - b)\cos\theta)\right) &= 0 \\ I\ddot{\theta} + \frac{\partial}{\partial\theta} mg\,(a - (a - b)\cos\theta) &= 0 \\ I\ddot{\theta} - \frac{\partial}{\partial\theta} mg\,(a - b)\cos\theta &= 0 \\ I\ddot{\theta} + mg\,(a - b)\sin\theta &= 0 \end{aligned}$$

For small $\theta, \sin\theta \simeq \theta$, hence the above becomes

$$\ddot{\theta} + \frac{mg\,(a - b)}{I}\theta = 0$$

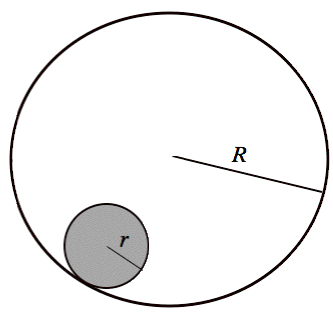
Therefore the natural angular frequency is

$$\boxed{\omega_n = \sqrt{\frac{mg(a-b)}{I}}}$$

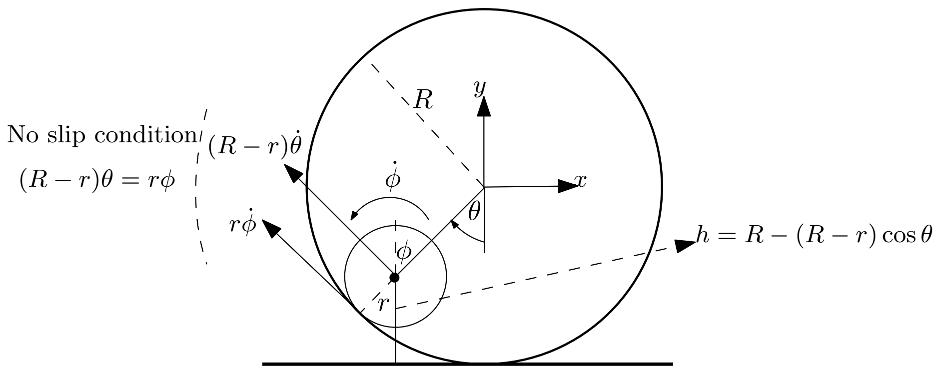
When $a = b$ then $\omega_n = 0$ and the mass do not oscillate but remain at the new positions. When $b > a$ then ω_n is complex valued. This is not possible, as the natural frequency must be real. So center of mass can not be in the upper half.

0.4 Problem 4

4. (15 points)
A sphere of radius r, mass m, and moment of inertia $I = \frac{2}{5}mr^2$ is contrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R (which does not move). Let the z-direction go along the axis of the cylinder.
(1) Determine the Lagrangian, the equations of motion, and the period for *small* oscillations. Ignore a possible motion in the z-direction.
(2) Determine the Lagrangian in the more general case where the motion in the z-direction is included. Describe the motion in the z-direction.



SOLUTION



2 generalized coordinates θ, ϕ but constraint reduces this to one coordinate θ

Part (1): There are two coordinates are θ, ϕ, but due to dependency between them (no slip) then this reduces the degree of freedom by one, and there is one generalized coordinate θ. The constraints of no slip means

$$f\left(\theta,\phi\right) = (R-r)\,\theta - r\phi = 0$$

Which means the center of the small disk move in speed the same as the point of the disk that moves on the edge of the larger cylinder as shown in the figure above.

$$T = \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left((R-r)\,\dot{\theta}\right)^2$$
$$U = mgh = mg\,(R-(R-r)\cos\theta)$$

Using $I = \frac{2}{5}mr^2$ and using $\dot{\phi} = \frac{(R-r)}{r}\dot{\theta}$ from the constraint conditions, then T becomes

$$\begin{aligned} T &= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{(R-r)}{r}\dot{\theta}\right)^2 + \frac{1}{2}m\left((R-r)\,\dot{\theta}\right)^2 \\ &= \frac{1}{5}m\,(R-r)^2\,\dot{\theta}^2 + \frac{1}{2}m\,(R-r)^2\,\dot{\theta}^2 \\ &= \frac{7}{10}m\,(R-r)^2\,\dot{\theta}^2 \end{aligned}$$

Hence

$$\begin{aligned} L &= T-U \\ &= \frac{7}{10}m\,(R-r)^2\,\dot{\theta}^2 - mg\,(R-(R-r)\cos\theta) \end{aligned}$$

And

$$\frac{\partial L}{\partial \theta} = -mg\,(R-r)\sin\theta$$
$$\frac{\partial L}{\partial \dot{\theta}} = \frac{7}{5}m\,(R-r)^2\,\dot{\theta}$$

Therefore the equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{7}{5}m\,(R-r)^2\,\ddot{\theta} + mg\,(R-r)\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{\frac{7}{5}\,(R-r)}\sin\theta = 0$$

For small angle

$$\ddot{\theta} + \frac{5g}{7\,(R-r)}\theta = 0$$

The frequency of oscillation is

$$\omega_n = \sqrt{\frac{5g}{7\,(R-r)}}$$

Using $\omega_n = \frac{2\pi}{T}$ then the period of oscillation is

$$T = \frac{2\pi}{\sqrt{\frac{5g}{7(R-r)}}} = 2\pi\sqrt{\frac{7\,(R-r)}{5g}}$$

Part (2):

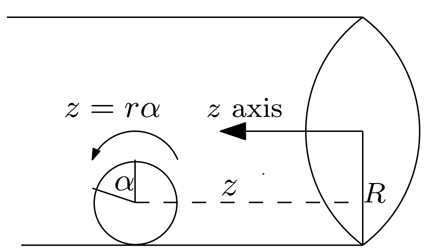
There are now two generalized coordinates, θ and z. The sphere now rotates in 2 angular motions, $\dot{\phi}$ which is the same as it did in part 1, and in addition, it rotate with angular motion, $\dot{\alpha}$ which is rolling down the z axis. The new constraint is that

$$f_1\,(\alpha,z) = z - r\alpha = 0 \tag{1}$$

So that no slip occurs in the z direction. This is in additional of the original no slip condition which is

$$f_2\left(\theta,\phi\right) = (R-r)\,\theta - r\phi = 0 \tag{2}$$

The following diagram illustrates this



The sphere is now distance z away from the origin. There is new constraint now as shown

Now there are translation kinetic energy in the z direction as well as new rotational kinetic energy due to spin α. Therefore

$$
\begin{aligned}
T &= \overbrace{\frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left((R-r)\dot{\theta}\right)^2}^{\text{part(1)}} + \overbrace{\frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\alpha}^2}^{\text{due to moving in z}} \\
U &= mgh = mg\left(R-(R-r)\cos\theta\right)
\end{aligned}
$$

Notice that the potential energy do not change, since it depends only on the height above the ground. Using $I = \frac{2}{5}mr^2$ and from constraints (1,2) then T becomes

$$
\begin{aligned}
T &= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\overbrace{\left(\frac{(R-r)}{r}\dot{\theta}\right)^2}^{\dot{\phi}^2} + \frac{1}{2}m\left((R-r)\dot{\theta}\right)^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\overbrace{\left(\frac{\dot{z}}{r}\right)}^{\dot{\alpha}^2}{}^2 \\
&= \left(\frac{1}{5}mr^2\right)\frac{(R-r)}{r^2}\dot{\theta}^2 + \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \left(\frac{1}{5}mr^2\right)\frac{\dot{z}^2}{r^2} \\
&= \frac{7}{10}m(R-r)\dot{\theta}^2 + \frac{7}{10}m\dot{z}^2
\end{aligned}
$$

Hence the Lagrangian is

$$
\begin{aligned}
L &= T - U \\
&= \frac{7}{10}m(R-r)\dot{\theta}^2 + \frac{7}{10}m\dot{z}^2 - mg\left(R-(R-r)\cos\theta\right)
\end{aligned}
$$

This part only now asks for motion in z direction. Hence

$$
\begin{aligned}
\frac{\partial L}{\partial z} &= 0 \\
\frac{\partial L}{\partial \dot{z}} &= \frac{7}{5}m\dot{z}
\end{aligned}
$$

Since $\frac{\partial L}{\partial z} = 0$ then

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0
$$

Hence $\frac{\partial L}{\partial \dot{z}}$ is the integral of motion. Or

$$
\frac{7}{5}m\ddot{z} = 0
$$

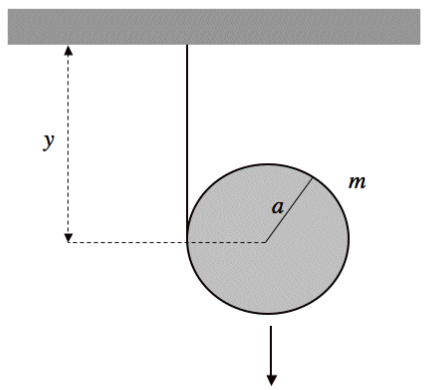
or

$$
\begin{aligned}
\ddot{z} &= 0 \\
\dot{z} &= c
\end{aligned}
$$

Where c is constant. This means the sphere rolls down the z axis at constant speed.

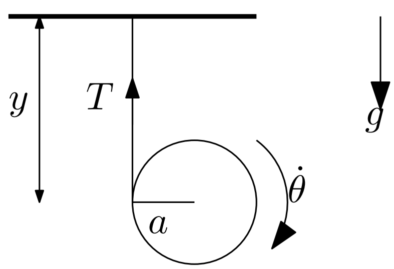
0.5 Problem 5

5. (10 points)
Consider a disc of mass m and radius a that has a string wrapped around it with one end attached to a fixed support and allowed to fall with the string unwinding as it falls. (This is essentially a yo-yo with the string attached to a finger held motionless as a fixed support.) Find the equation of motion of the disc.



SOLUTION

This is first solved using energy method, then solved using Newton method.



constraint: $\dot{y}a = \dot{\theta}$

Energy method

Constraint is $f\left(y,\theta\right)=y-a\theta=0$. Hence $\dot{\theta}=\frac{\dot{y}}{a}$

$$
\begin{aligned}
U &= -mgy\\
T &= \frac{1}{2}I\dot{\theta}^2+\frac{1}{2}m\dot{y}^2\\
&= \frac{1}{2}I\left(\frac{\dot{y}}{a}\right)^2+\frac{1}{2}m\dot{y}^2\\
&= \frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{\dot{y}}{a}\right)^2+\frac{1}{2}m\dot{y}^2\\
&= \frac{1}{4}m\dot{y}^2+\frac{1}{2}m\dot{y}^2\\
&= \frac{3}{4}m\dot{y}^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
L &= T-U\\
&= \frac{3}{4}m\dot{y}^2+mgy
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\partial L}{\partial y} &= mg\\
\frac{\partial L}{\partial \dot{y}} &= \frac{3}{2}m\dot{y}\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{3}{2}m\ddot{y}
\end{aligned}
$$

And the equation of motion becomes

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}-\frac{\partial L}{\partial y} &= 0\\
\frac{3}{2}m\ddot{y}-mg &= 0\\
\ddot{y} &= \frac{2}{3}g
\end{aligned}
$$

Newton method

Using Newton method, this can be solved as follows. The linear equation of motion is (positive is taken downwards)

$$
\begin{aligned}
F &= m\ddot{y}\\
-T+mg &= m\ddot{y} \qquad (1)
\end{aligned}
$$

And the angular equation of motion is given by

$$
Ta = I\ddot{\theta} \qquad (2)
$$

Due to constraint $f\left(y,\theta\right)=y-a\theta=0$, then

$$\frac{\ddot{y}}{a}=\ddot{\theta}$$

Using the above in (2) gives

$$\begin{aligned} Ta &= I\frac{\ddot{y}}{a} \\ T &= I\frac{\ddot{y}}{a^2} \end{aligned} \tag{3}$$

Replacing T in (1) with the T found in (3) results in

$$\begin{aligned} m\ddot{y} &= -I\frac{\ddot{y}}{a^2}+mg \\ \ddot{y}\left(m+\frac{I}{a^2}\right) &= mg \\ \ddot{y} &= \frac{mg}{m+\frac{I}{a^2}} \end{aligned}$$

But $I=\frac{1}{2}ma^2$ then the above becomes

$$\begin{aligned} \ddot{y} &= \frac{mg}{m+\frac{\frac{1}{2}ma^2}{a^2}} \\ &= \frac{g}{1+\frac{1}{2}} \\ &= \frac{2}{3}g \end{aligned}$$

Which is the same (as would be expected) using the energy method