

**University Course**

**ECE 332**  
**Feedback control systems**

**University of Wisconsin, Madison**  
**Fall 2015**

My Class Notes

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Fall 2015



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# Chapter 1

## Introduction

Took this course in Fall 2015.

Instructor: professor B Ross Barmish Office Hours: Wednesday 1:00-2:30 PM

## 1.1 Syllabus

### ECE 332 – Handout Organization

- *Course Organization*

Lectures: B. R. Barmish (3613 Engineering Hall)

E-mail: barmish@engr.wisc.edu

Office Hours: Wednesday 1:00-2:30 PM

- *No Official Course Textbook*

I will draw on material from the following textbooks on Reserve:

B. C. Kuo and F. Golnaraghi, *Automatic Control Systems*, John Wiley and Sons, Ninth Edition, New York.

R. C. Dorf and R. H. Bishop, *Modern Control Systems*, Prentice Hall, Eleventh Edition, New York.

J. J. DiStefano, A. R. Stubberud and I. J. Williams, *Feedback and Control Systems*, Schaum's Outline Series, McGraw-Hill, New York.

Given the classical nature of the material, there are also hundreds of sources on the web providing coverage of the ECE 332 topics with many illustrative examples demonstrating the theory covered in class.

- *Course Grading Components*

Test 1: 25%; Tuesday, September 29, 2015

Test 2: 30%; Thursday, November 12, 2015

Test 3: 35%; Thursday, December 10, 2015

Homework: 10% (Total of 7-10 Assignments)

Instructor Discretion: Maximum 10% in any category

- *Cancellations and Makeup Classes*

No lectures on Thursday October 1, Tuesday November 3 and Tuesday December 15; no office hours on Wednesday September 30.

Makeup or Review Classes: Scheduled for 6 PM on Wednesday September 23, Wednesday November 18 and Wednesday December 9.

- *Additional Points*

Course Announcements: via e-mail

Homework: E = excellent; S = satisfactory; U = unsatisfactory

Matlab/Simulink: Both used heavily in course

**ECE 332 – Handout Overview**

**Catalog Data:** Modelling of continuous systems; computer-aided solution to systems problems; feedback control systems; stability, frequency response and transient response using root locus; frequency domain and state variable methods.

**Prerequisites:** ECE 330 or consent of instructor.

**No Required Textbook:** I have a number of books on reserve. In previous offerings of this course, I have used:

B. C. Kuo and F. Golnaraghi *Automatic Control Systems*, John Wiley and Sons.

**Instructor:** Professor B. Ross Barmish, ECE Department

**Goals:** This junior/senior level course develops the fundamentals associated with the analysis, design and simulation of automatic control systems.

**Prerequisites by Topic:**

1. Linear differential equations with constant coefficients
2. Laplace transforms and transfer functions for linear systems
3. Elementary matrix manipulations (such as determinant and inverse)
4. Adequate familiarity with computers and use of various packages; the specific package used in this course is Matlab/Simulink

**Topics:**

1. Modelling of dynamic systems in a control context
2. Block diagrams, signal flow graphs and Mason's Rule
3. Feedback in a sensitivity, linearization, disturbance context
4. Steady state behavior of feedback systems
5. Time response with emphasis on second order systems
6. Stability analysis: criteria of Routh, Nyquist and Kharitonov
7. The root locus and its variants
8. Frequency response: Bode analysis of feedback systems
9. Compensator design methods: PID, lead/lag and root locus methods
10. Closed loop considerations: frequency response and Nichol's plot

**Computer Usage:**

Extensive use of Simulink and Matlab on weekly homework





# Chapter 2

## Class notes

### 2.1 Summary table

These are my notes taken during lectures. Any errors in these notes, then all blames to me and not to the instructor.

date	day	event	Topic
1. Sept 3, 2015	Thursday	First class	Introduction, Laplace transforms
2. Sept 8, 2015	Tuesday		Closed loop and Simulink review
3. Sept 10, 2015	Thursday	HW 1	Steady state and transient responses, different feedback loops
4. Sept 15, 2015	Tuesday		Performance specs, steady state error
5. Sept 17, 2015	Thursday		MIMO matrix of transfer functions, MIMO in a feedback loop.
6. Sept 22, 2015	Tuesday	HW 2	Armature controlled DC motor with MIMO Example. signal graph
7. Sept 23, 2015	Wednesday	make up	Start of signal graph and Mason rule. Mason gain
8. Sept 24, 2015	Thursday	HW3	Mason example, benefits of feedback, non-linear systems
9. Sept 29, 2015	Tuesday	Exam 1	First exam
10. Oct. 1, 2015	Thursday	No class	
11. Oct. 6, 2015	Tuesday	Sensitivity	Sensitivity of transfer function with change of parameters
12. Oct. 8, 2015	Thursday	Noise rejection	Design controls to reject noise and disturbances using feedback
13. Oct. 13, 2015	Tuesday	HW4. Second order	Noise rejection, second order systems, dominant pole method

14. Oct. 15, 2015	Thursday	Second order	More second order, Overshoot and resonance calculations.
15. Oct 20, 2015	Tuesday	Design for $k$	Feedback using user specified time specs for response
16. Oct 22, 2015	Thursday	Stability/Ruth	Routh stability table examples, stability, examples
17. Oct 27, 2015	Tuesday	Root Locus	Starting on root locus, how $K$ affect poles locations
18. Oct 29, 2015	Thursday	Root Locus	More root locus, more lemmas, up to lemma 4
19. Nov. 3, 2015	Tuesday	No lecture	
20. Nov 5, 2015	Thursday	Root Locus	Finished Root Locus, all 9 lemmas. Examples given
21. Nov 10, 2015	Tuesday	Extension Root Locus	Review mid term 2, started extension root locus (will be on final)
22. Nov 12, 2015	Thursday	Exam 2	Hard exam
23. Nov 17, 2015	Tuesday	Starting Nyquist	Started Nyquist. What it does and how to make Nyquist path
24. Nov 19, 2015	Thursday	More Nyquist	More Nyquist. 2 examples. Handout given
25. Nov 24, 2015	Tuesday	More Nyquist	More Nyquist. Using for stability, gain and phase margins
26. Nov 26, 2015	Thursday	thanks giving	No class
27. Dec 1, 2015	Tuesday	Starting Bode	Start Bode frequency analysis
28. Dec 4, 2015	Thursday	More Bode	More Bode analysis and examples
29. Dec 8, 2015	Tuesday	Bode gain and phase	Effects of delay on Bode, gain and phase margins.
30. Dec 10, 2015	Thursday	Final exam	Finals

## **2.2 Lecture 1, Thursday Sept. 3, 2015**

Introduction to Laplace transform, handouts, syllabus overview

## 2.3 Lecture 2, Tuesday Sept. 8, 2015, Closed loop and Simulink review

Discussion on Laplace transform. We want to be able to switch from  $t$  to  $s$  domain and back. For tough ones, use tables. In Matlab use syms. Examples shown how to use syms in Matlab and obtain the Laplace and inverse Laplace transforms. Use of expand and simplify commands.

Closed loop can be used to improve performance. An example given of a mass spring damper. Steps of solution

1. Model the system
2. Convert to Laplace domain
3. analyze in Laplace domain
4. Convert result to time domain

Control engineers need a model to analyze. Example is  $mx'' + cx' + kx = u(t)$ . We can always find transfer function TF assuming system is at rest initially. Need to know basic relation  $\mathcal{L}\left\{\frac{d^k y}{dt^k}\right\} = s^k Y(s)$  assuming all initial conditions are zero. Back to the above equation. Take Laplace transform we obtain

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + cs + k} \quad (1)$$

We see if the open loop does what we want. If it does, no need for feedback. If open loop response is not good, then we use feedback to improve the response.

Behavior of open loop: Let  $u(t) = 1$  (unit step). What is  $y(t)$ ? From (1)

$$Y(s) = G(s) U(s)$$

But  $\mathcal{L}\{1\} = \frac{1}{s}$  then the above becomes

$$Y(s) = \frac{1}{ms^2 + cs + k} \frac{1}{s}$$

To get  $y(t)$  we need to inverse Laplace the above. Let  $m = 1, c = 10, k = 1$  then

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 10s + 1} \frac{1}{s} \right\}$$

There are 3 possibility of behavior of this system depending on roots of the denominator. Real roots implies  $y(t)$  involves only exponential. If the roots are complex, the result contain harmonics. The over solution will look like  $e^{-2t} \cos(\dots)$ .

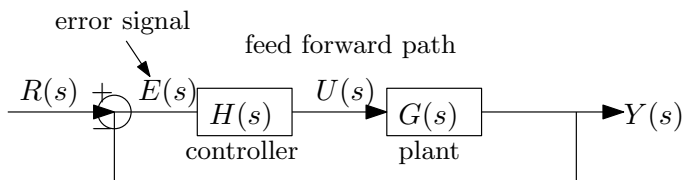
**Reader:** For the above example, find  $y(t)$  for  $c = 1, c = 10$ . Use partial fractions to find the inverse Laplace transform, then sketch  $y(t)$  for each.

Now overview was given on using Simulink. Examples of using basic blocks explained.

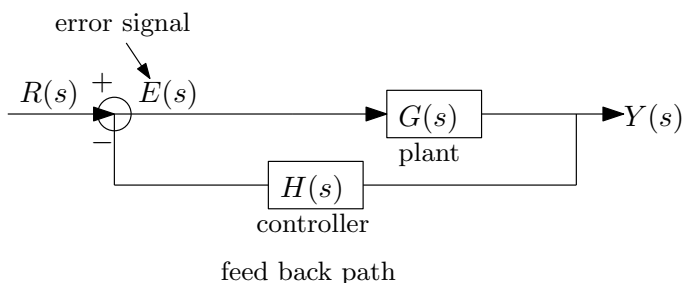
Classical responses of system shown.

## 2.4 Lecture 3, Thursday Sept. 10, 2015, Steady state and transient responses

Open loop vs. closed loop. In closed loop we add a controller to obtain desired response.



Another possibility is to put the controller in the feedback path



Now we will find the closed loop TF for the feed forward path configuration.

$$Y(s) = E(s) H(s) G(s)$$

$$E(s) = R(s) - Y(s)$$

Substituting the second equation in the first gives

$$\begin{aligned} Y(s) &= (R(s) - Y(s)) H(s) G(s) \\ &= R(s) H(s) G(s) - Y(s) H(s) G(s) \end{aligned}$$

Hence

$$\begin{aligned} Y(s) (1 + H(s) G(s)) &= R(s) H(s) G(s) \\ \frac{Y(s)}{R(s)} &= \frac{H(s) G(s)}{1 + H(s) G(s)} \end{aligned}$$

**Reader:** Do the above for the feedback path configuration. We should get  $\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + H(s)G(s)}$

**Reader:** For the feedforward case, find the error transfer function  $\frac{E(s)}{R(s)}$

Working with the feedforward case.  $E = R - Y$ . We want  $E = 0$  for tracking. Most common choices of  $H(s)$  are

1.  $H = k$  a constant. Called proportional or pure gain.

2.  $H = k_1 + \frac{k_2}{s}$  The second term is an integrator. This is called PI controller.
3.  $H = k_1 + \frac{k_2}{s} + sk_3$ . This is called PID. Note that  $s$  is derivative. So if there is lots of noise in the  $E$  signal, this will cause problems since derivative of noise generated large signal (large actuating signal). So it is safe to use an integrator, but not always safe to use derivative, unless we know that the error signal will always be smooth.

For example, let  $G(s) = \frac{1}{s+2}$  with a unit step  $\left(\frac{1}{s}\right)$  input. Then in the openloop,  $Y(s) = \frac{1}{s+2} \frac{1}{s}$  or  $y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$ . Now we close the loop, using the feedforward configuration and add a pure gain controller  $k$ . Hence

$$\begin{aligned} \frac{Y}{R} &= \frac{HG}{1+HG} \\ &= \frac{k \frac{1}{s+2}}{1 + k \frac{1}{s+2}} \\ &= \frac{k}{s + (2+k)} \end{aligned}$$

Now  $R(s) = \frac{1}{s}$ , hence

$$Y(s) = \frac{1}{s} \frac{k}{s + (2+k)}$$

And

$$y(t) = \overbrace{\frac{k}{k+2}}^{\text{steady state response}} - \overbrace{\frac{k}{k+2} e^{-t(k+2)}}^{\text{transient response}}$$

By design, we want  $y(t)$  to track  $r(t)$  which is unit step in this case. Also we want the transient response to go away quickly. In this example, only when  $k$  very large do we approach a steady state close to one. But in practice having very large gain is not good due to sensitivity problems. (will talk about sensitivity later in the course). Also large  $k$  might lead to actuating signal that can not be satisfied. Note also, no matter how large  $k$  is, we can't obtain perfect tracking. Some application might require perfect tracking.

**Reader:** Redo the analysis above using integrator only controller. i.e.  $H(s) = \frac{k}{s}$  and see if  $y(t)$  will now track the input at steady state.

Final value theorem: Suppose  $F(s) = \frac{N(s)}{D(s)}$  is stable, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

## 2.5 Lecture 4, Tuesday Sept. 15, 2015, Performance specs, steady state error

Watch today for email on HW1 solution and HW2. See this you tube on steady state error ??

Today we will spend more time on final value theorem. Then talk about typical specs for control system.

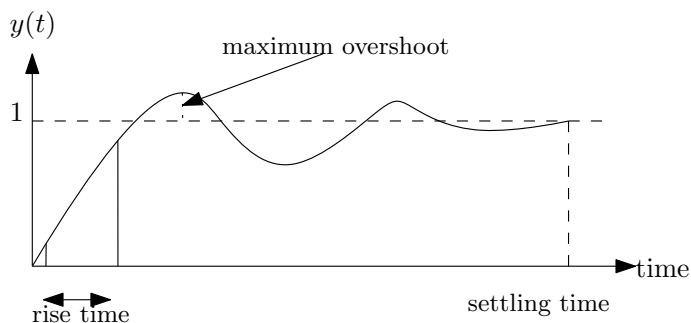
F.V.T. is important for tracking. Given  $F(s)$ , does  $f(t)$  have a final value? ie. does  $\lim_{t \rightarrow \infty} f(t)$  exist? Sometimes infinity is allowed as final value. But many times we do not have a final value, such as for period signals such as  $\cos(t)$ .

When can we apply F.V.T. ? In practice,  $F(s)$  should be stable. This means the poles should be in the left half plane. But we allow one pole to be at the origin. Example  $\frac{1}{s+1} \Leftrightarrow e^{-t}$ . How about  $\frac{1}{s+1} \frac{1}{s}$ ? Final value still exist.

**Reader** Consider  $F(s) = \frac{1}{s^2} \frac{1}{s+1}$

When  $F(s)$  is stable, then  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$ .

Specification of control system: Consider classical response



Some specifications are settling time, and rise time (time to go from 10% to 90% of the step input) and amount of overshoot. Other specifications are given in terms of damping and sensitivity and steady state error. Steady state error is important and we will talk more about it today.

How to measure quality of tracking? Study  $E(s)$ . What steady state error we accept depends on the application. The error transfer function is  $\frac{E(s)}{R(s)} = \frac{1}{1+H(s)G(s)}$ . Now consider a step input, hence  $R(s) = \frac{1}{s}$  and the error becomes

$$E(s) = \frac{1}{s} \frac{1}{1+H(s)G(s)}$$

Assuming F.V.T. applies (i.e.  $E(s)$  is stable) then

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + H(s)G(s)} \\ &= \frac{1}{1 + H(0)G(0)}\end{aligned}$$

Example, if  $G(s) = \frac{1}{s^2 + 2s + 4}$  then  $\lim_{t \rightarrow \infty} e(t) = \frac{4}{5}$  which is not good. We want this to be zero. So for  $\frac{1}{1 + H(0)G(0)}$  to be zero, we want  $H(0)G(0)$  to be very large. This means  $H(s)$  should have  $\frac{1}{s}$  in it as a factor. This means an integrator. Hence an integrator in  $H(s)$  guarantees that error goes to zero when the input is step.

**Reader** For the mass spring damper,  $G = \frac{1}{ms^2 + cs + k}$  design  $H(s)$  leading to zero steady state error for step command. Use  $\frac{k}{s}$  in  $H(s)$ .

What if the input is ramp? which is  $\frac{1}{s^2}$ ? Then

$$\begin{aligned}E(s) &= \frac{1}{s^2} \frac{1}{1 + H(s)G(s)} \\ \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \frac{1}{1 + H(s)G(s)} \\ &= \lim_{s \rightarrow 0} \frac{1}{sH(s)G(s)}\end{aligned}$$

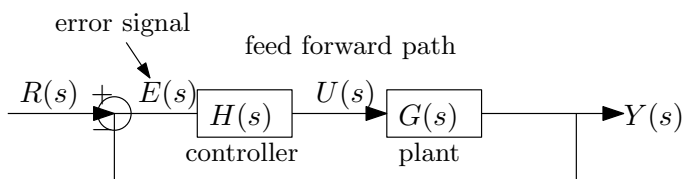
So now we need  $H(s)$  to have  $\frac{1}{s^2}$  factor in it, so that it becomes very large at  $s = 0$  and cause the error to go to zero. This means 2 integrator in series. This means to track  $r(t) = t^k$  we need  $k + 1$  integrators in  $H(s)$  to get zero error at steady state.

Finally for any signal  $r(t) = \sum_{i=0}^k a_i t^i$ , we need  $k + 1$  integrators, since the largest term is the only term that needs to be satisfied, due to linearity of the system. So given a complicated polynomial, we look at the largest power and this tells us how many integrators we need for zero steady state error.

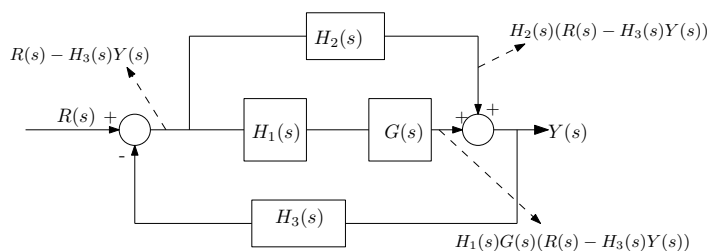


## 2.6 Lecture 5, Thursday Sept. 17, 2015, MIMO matrix of transfer functions

Today lecture on more complicated systems. So far we talked about SISO in the most common configuration



But systems come in more complicated forms. We need to reformulate complicated systems this the above common configuration form to be able to analyze them. There can be multiple inputs and multiple outputs as well, more blocks, loops, etc.. To illustrate, given this system



How to find  $\frac{Y(s)}{R(s)}$ ? We can go to first principles. Use signal analysis. The goal is to find  $\frac{Y(s)}{R(s)}$  without all the intermediate variables. i.e we want  $\frac{Y(s)}{R(s)}$  to be a function of only the shown blocks transfer functions  $H_1, H_2, H_3, G$ . Hence

$$Y = H_2 (R - H_3 Y) + H_1 G (R - H_3 Y)$$

Solve for  $Y$  in terms of  $R$  from the above

$$\begin{aligned} Y &= H_2 R - H_2 H_3 Y + H_1 G R - H_1 G H_3 Y \\ Y (1 + H_2 H_3 + H_1 H_3 G) &= R (H_2 + H_1 G) \end{aligned}$$

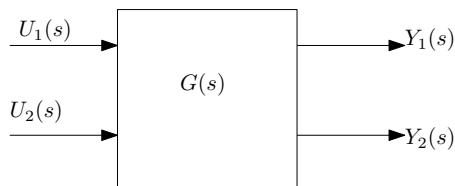
Hence

$$\begin{aligned} \frac{Y}{R} &= \frac{H_2 + H_1 G}{1 + H_2 H_3 + H_1 H_3 G} \\ &= \frac{H_2 + H_1 G}{1 + H_3 (H_2 + H_1 G)} \end{aligned}$$

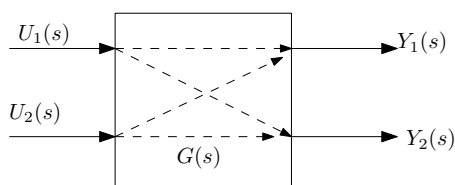
But for more complicated systems, with more loops and inner blocks, this process can become more complicated and one can make mistakes. We need a more systematic way. Next lecture we will look at Mason formula to do the above.

For the rest of the lecture we will look at multiple input, multiple output (MIMO). Motivation example, is an electric circuit with say 2 input ports (voltages  $V_1, V_2$ ) and 2 output ports,

way  $V_3, V_4$ . In block diagram we draw



In the above  $G(s)$  will now be a matrix of  $G_{ij}(s)$  transfer functions. The above is the open loop block diagram of a 2 input/2 outputs system. Internally, there can be cross coupling. Meaning, one input can affect all of some of the other outputs. Like this



So we need a transfer function  $G_{ij}(s)$  from each input  $U_j$  to each output  $V_i$ . So need a total of 4 transfer functions in this case.

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

Expanding

$$Y_1(s) = G_{11}(s) U_1(s) + G_{12}(s) U_2(s)$$

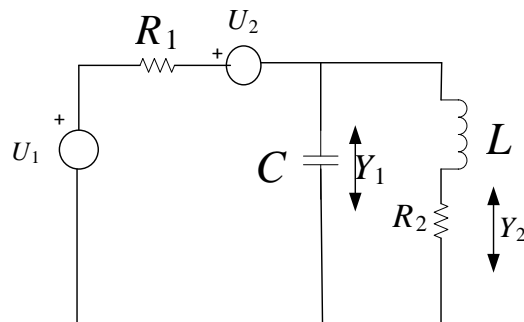
$$Y_2(s) = G_{21}(s) U_1(s) + G_{22}(s) U_2(s)$$

Suppose we want to find  $G_{11}(s)$  only. How to do this?

$$G_{11}(s) = \frac{Y_1(s)}{U_1(s)}_{U_2=0}$$

In practice, this is done by shorting the input  $U_2$  (i.e making the input  $U_2$  zero) and then supplying the input  $U_1$  only and then measuring the output at port  $Y_2$ .

**Reader** Create transfer function model for this circuit



In the above, there are 2 input voltages  $U_1, U_2$  and 2 outputs  $Y_1, Y_2$ . Notice in the above we can not use the impedance method and use the voltage divider as in the first problem in HW1. We need to setup 2 loop equations and solve. Another possibility is to short  $U_1$  and then solve the circuit without  $U_1$  and then short  $U_2$  input and then solve the circuit again.

### Reader solution

We first set up the 2 loops, and obtain these 2 equations

$$u_1 - u_2 = I_1 \left( R_1 + \frac{1}{Cs} \right) - I_2 \frac{1}{Cs} \quad (1)$$

$$0 = I_2 \left( \frac{1}{Cs} + Ls + R_2 \right) - I_1 \frac{1}{Cs} \quad (2)$$

And the output equations are

$$Y_1 = (I_1 - I_2) \frac{1}{Cs}$$

$$Y_2 = I_2 R_2$$

Solving (1,2) for  $I_1, I_2$  gives

$$I_1 = \frac{1 + Cs(R_2 + Ls)(u_1 - u_2)}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))}$$

$$I_2 = \frac{u_1 - u_2}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))}$$

Using these in the output equations gives

$$Y_1 = \left( \frac{1 + Cs(R_2 + Ls)(u_1 - u_2)}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} - \frac{u_1 - u_2}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} \right) \frac{1}{Cs}$$

$$Y_2 = \frac{u_1 - u_2}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} R_2$$

These can be written in matrix form as

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{R_2 + Ls}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} & \frac{R_2 + Ls}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} \\ \frac{R_2}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} & \frac{R_2}{R_2 + Ls + R_1(1 + Cs(R_2 + Ls))} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

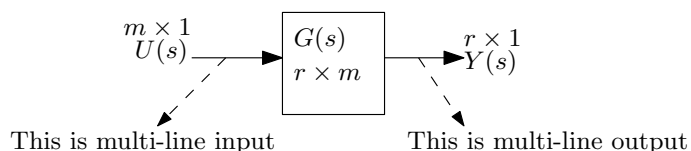
More generally, given  $m$  inputs and  $r$  outputs then

$$\overline{Y} = \overline{G} \overline{U}$$

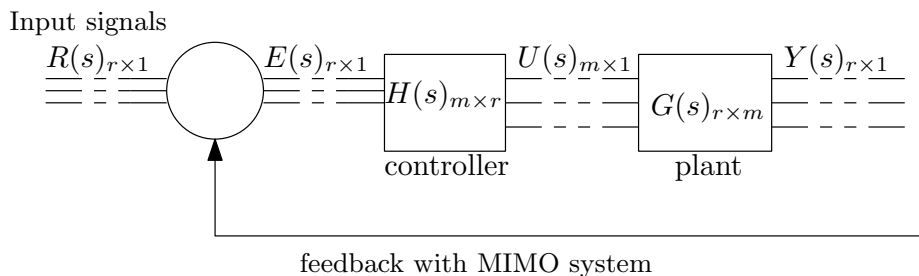
So  $G$  is an  $r \times m$  matrix.

$$G_{ij}(s) = \frac{Y_j(s)}{U_i(s)} \quad U_k=0 \text{ for } k \neq i$$

Now we are ready to take a MIMO open loop block and imbed it in a feedback loop as we did with SISO. The process is the same, but now we have to be careful with order since these are now matrices and not scalars. Given the system



Now we want to add a controller as before. Hence we obtain



Say we want  $Y_i$  to track input  $R_i$ . So now we want the closed loop transfer function as we did with SISO, but now we have to do it using vectors and matrices. So order is important. As before we write

$$Y(s) = \overbrace{(I + G(s)H(s))^{-1} G(s)H(s)}^{\text{closed loop T.F. matrix}} R(s)$$

In the above, the closed loop transfer function is  $(I + G(s)H(s))^{-1}$  where  $I$  is the identity matrix or size  $r \times r$

$$\widehat{Y}(s) = \left( I + \overline{G(s)} \overline{H(s)} \right)^{-1} \overline{G(s)} \overline{H(s)} \overline{R}(s)$$

Example, given  $G(s) = \begin{pmatrix} 1 & \frac{1}{s} \\ \frac{2}{s+1} & 1 \end{pmatrix}$ ,  $H(s) = \begin{pmatrix} -2 & -1 \\ -3 & \frac{1}{s} \end{pmatrix}$  find closed loop transfer function  $(I + GH)^{-1} GH$

First,

$$\begin{aligned} GH &= \begin{pmatrix} 1 & \frac{1}{s} \\ \frac{2}{s+1} & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ -3 & \frac{1}{s} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{s} - 2 & \frac{1}{s^2} - 1 \\ -\frac{4}{s+1} - 3 & \frac{1}{s} - \frac{2}{s+1} \end{pmatrix} \end{aligned}$$

Then

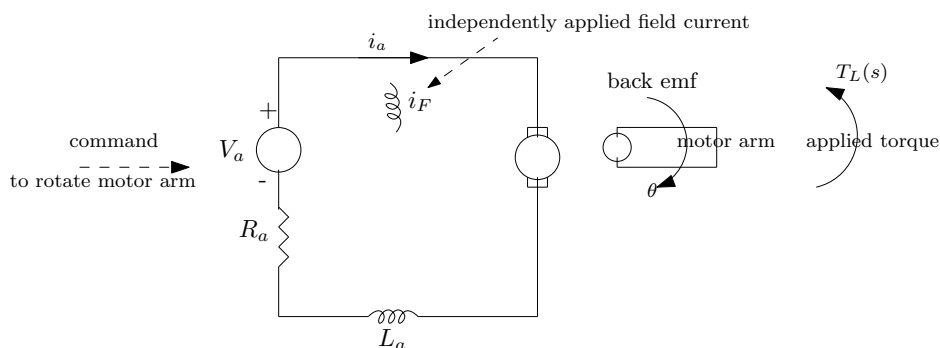
$$\begin{aligned}
(I + GH)^{-1} GH &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{3}{s+1} - 2 & \frac{1}{s^2} - 1 \\ -\frac{\frac{s}{4}}{s+1} - 3 & \frac{1}{s} - \frac{2}{s+1} \end{pmatrix} \right)^{-1} \begin{pmatrix} -\frac{3}{s+1} - 2 & \frac{1}{s^2} - 1 \\ -\frac{\frac{s}{4}}{s+1} - 3 & \frac{1}{s} - \frac{2}{s+1} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{s^3+s}{4s^3+10s^2-2s-4} & \frac{1}{4s^3+10s^2-2s-4} (-s^3 - s^2 + s + 1) \\ -\frac{3s^3+7s^2}{4s^3+10s^2-2s-4} & \frac{1}{3s^2+7s} (3s^3 + 7s^2) \frac{s^2+4s+3}{4s^3+10s^2-2s-4} \end{pmatrix} \begin{pmatrix} -\frac{3}{s+1} - 2 & \frac{1}{s^2} - 1 \\ -\frac{\frac{s}{4}}{s+1} - 3 & \frac{1}{s} - \frac{2}{s+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2(-2s^3-5s^2+s+2)} (-5s^3 - 10s^2 + s + 4) & -\frac{1}{2} (s-1) \frac{(s+1)^2}{-2s^3-5s^2+s+2} \\ -\frac{1}{2} s^2 \frac{3s+7}{-2s^3-5s^2+s+2} & -\frac{1}{2} \frac{3s^3+6s^2-5s-4}{-2s^3-5s^2+s+2} \end{pmatrix} \\
&= \frac{1}{-2s^3 - 5s^2 + s + 2} \begin{pmatrix} \frac{1}{2} (-5s^3 - 10s^2 + s + 4) & -\frac{1}{2} (s-1)(s+1)^2 \\ -\frac{1}{2} s^2 (3s+7) & -\frac{1}{2} (3s^3 + 6s^2 - 5s - 4) \end{pmatrix}
\end{aligned}$$

Verify using Matlab syms.

## 2.7 Lecture 6, Tuesday Sept. 22, 2015, Armature controlled DC motor with MIMO

Reminder, class tomorrow at 6 pm. No class Oct. 1, 2015. Test on sept 29. Exam everything up to and including MIMO.

**Consolidating example** We will cover main points in class so far, from modeling, to finding T.F. to building block diagrams and MIMO. The example is to model Armature controlled DC motor. From physical system to differential equations.



Applying kirchoff voltage rule on the circuit gives

$$V_a(t) = R_a i_a + L_a \frac{di_a}{dt} + k_1 \omega \quad (1)$$

Where  $k_1 \omega$  is the backemf induced by the rotating arm and  $\omega$  is the angular velocity  $\frac{d\theta}{dt}$  of the motor arm. In addition, we have a mechanical relation between the applied torque  $t_L$  and  $i_a$

$$J \frac{d\omega}{dt} = k_2 i_a - t_L \quad (2)$$

Finally

$$t_L = k_L i_a \quad (3)$$

The above are the three equations needed. Let  $k_1 = k_2 = k_m$ . Taking Laplace transform of each gives

$$V_a(s) = R_a I_a(s) + sL_a I_a(s) + sk_m \theta(s) \quad (1A)$$

$$sJW(s) = k_2 I_a(s) - T_L(s) \quad (2A)$$

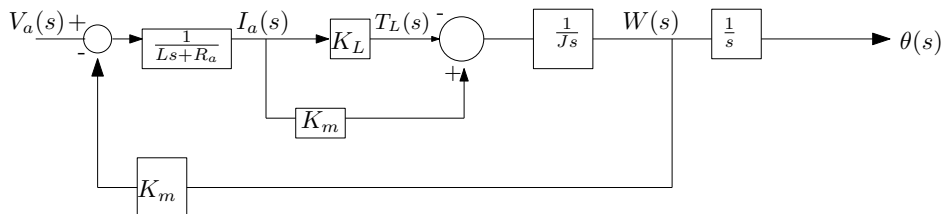
$$T_L(s) = k_L I_a(s) \quad (3A)$$

Notice that  $W(s)$  is the Laplace transform of  $\omega$  and that  $\omega = \frac{d\theta}{dt}$  or  $W(s) = s\theta(s)$ , hence  $\frac{W(s)}{s} = \theta(s)$ . Now we build the block diagram from the above three equations. The input is  $V_a(s)$  and the output is  $\theta(s)$ . From the above we find

$$I_a(s) = \frac{V_a(s) - k_m W(s)}{R_a + sL}$$

$$W(s) = \frac{k_m I_a(s) - T_L(s)}{Js}$$

And the block diagram is



**Reader:** Find the transfer function  $\frac{\theta(s)}{V_a(s)}$ . Consider step input. Find steady state  $\theta(\infty)$  for step input. Does it go to one? Note, every RLC circuit is stable circuit. Called passive circuit.

**Reader answer**

I get this

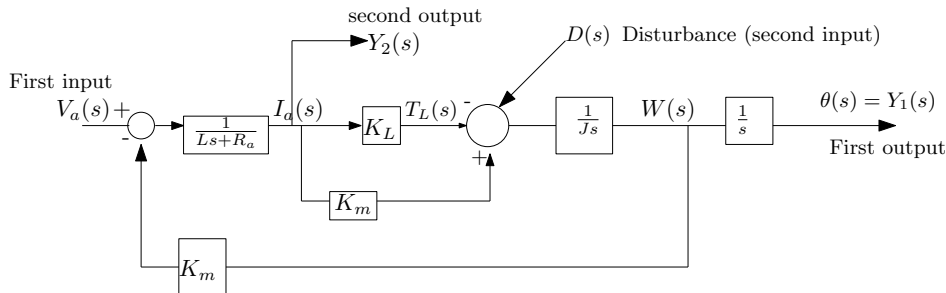
$$E = R(s) - s\theta k_m$$

$$\theta = E(s) \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right)$$

Hence

$$\begin{aligned}
 \theta &= (R(s) - s\theta k_m) \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right) \\
 \theta &= R(s) \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right) - s\theta k_m \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right) \\
 \theta \left( 1 + s k_m \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right) \right) &= R(s) \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right) \\
 \frac{\theta(s)}{R(s)} &= \frac{\left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right)}{1 + s k_m \left( \frac{k_m - k_L}{(R_a + sL) s^2 J} \right)} \\
 &= \frac{k_m - k_L}{s^2 J (R_a + sL) + s k_m (k_m - k_L)} \\
 &= \frac{k_m - k_L}{s^3 J L + s^2 J R_a + s k_m^2 - s k_m k_L} \\
 &= \frac{(k_m - k_L)}{J L} \frac{1}{s^3 + s^2 \frac{R_a}{L} + s \left( \frac{k_m^2 - k_m k_L}{J L} \right)}
 \end{aligned}$$

What if we have another output of interest? say  $I_a(s)$  as output? And what if we have another input, a disturbance  $d(t)$  as shown



**Reader** Find the MIMO transfer function

$$\begin{pmatrix} \theta(s) \\ I_a(s) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} V_a(s) \\ D(s) \end{pmatrix}$$

Where  $T_{11}$  is the transfer function from  $V_a(s)$  to  $\theta(s)$ , and  $T_{12}$  is the transfer function from  $\theta(s)$  to  $D(s)$  and  $T_{21}$  is the transfer function from  $V_a(s)$  to  $I_a(s)$  and  $T_{22}$  is the transfer function from  $D(s)$  to  $I_a(s)$ .

$$\theta(s) = T_{11} V_a(s) + T_{12} D(s)$$

$$I_a(s) = T_{21} V_a(s) + T_{22} D(s)$$



Hence

$$T_{11} = \frac{\theta(s)}{V_a(s)}$$

$$T_{12} = \frac{\theta(s)}{D(s)}$$

$$T_{21} = \frac{I_a(s)}{V_a(s)}$$

$$T_{22} = \frac{I_a(s)}{D(s)}$$

We now start on signal graph. First we convert block diagram to signal graph. The block become a branch, and the variable become a node.

Then we will start on Mason rule, which uses the signal graph to obtain the transfer function.

## 2.8 Lecture 7, Wednesday Sept. 23, 2015, Start of signal graph and Mason rule

6PM lecture. Makeup lecture.

On Monday there will be extra office hrs. Exam on Tuesday.

Example. Lets say we have  $X_1, X_2$  as variables, and  $U$  as input and  $Y$  as output. Then given

$$\begin{aligned} X_1 + \alpha X_2 &= U \\ \beta X_1 - 3X_2 &= 3U \\ Y &= X_1 + 2X_2 \end{aligned}$$

The goal is to solve for  $Y$  in terms of  $U$  without all the variables  $X_1, X_2$  involved. This can be solved of course using algebra:

```
>> clear all
>> syms X1 X2 U Y alpha beta
>> eq1=X1+alpha*X2==U;
>> eq2=beta*X1-3*X2==3*U;
>> eq3=Y==X1+2*X2;
>> [X1,X2]=solve(eq1,eq2,X1,X2)

X1 =
(3*(U + U*alpha))/(alpha*beta + 3)
X2 =
-(3*U - U*beta)/(alpha*beta + 3)

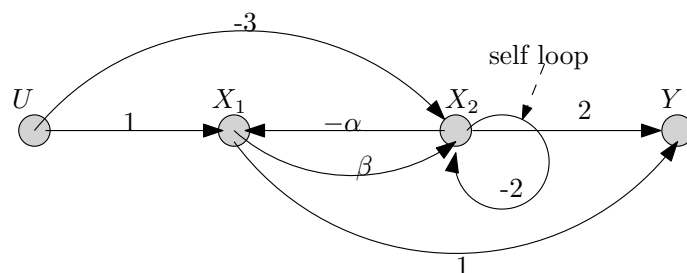
>> subs(Y)
>> pretty(ans)

3 (U + U alpha)   (3 U - U beta) 2
-----
alpha beta + 3   alpha beta + 3
```

Using Mason method, we first rewrite the equations so that the variables are on the LHS. In the above, this becomes

$$\begin{aligned} X_1 &= U - \alpha X_2 \\ X_2 &= \beta X_1 - 2X_2 - 3U \end{aligned}$$

Next, we first set up a signal graph. Each variable becomes a node, like this



Note: At each node, all incoming branch gains are added. We now setup the loop gains. A loop must not visit a node more than once. There are two loops here. The gains on them are  $\{-\alpha\beta, -2\}$ . Next, we find all the forward paths from  $U$  to  $Y$ . A forward path must not visit same node more than once. There are 4 forward paths. The gains on each are

$$M_1 = (-3)(2) = -6$$

$$M_2 = (1)(\beta)(2) = 2\beta$$

$$M_3 = (1)(1) = 1$$

$$M_4 = (-3)(-\alpha)(1) = 3\alpha$$

Now we defined Mason delta  $\Delta$

$$\Delta = 1 - \sum \text{loop gains} + \sum \text{loop gains 2 at times} - \sum \text{loop gains 3 at times} \dots$$

In the above, when looking for loop gains 2 at times, the loops must not be sharing a node. Same for loop gains 3 at times and higher sums. In our example, this gives

$$\begin{aligned} \Delta &= 1 - \sum \text{loop gains} \\ &= 1 - (-\alpha\beta - 2) \\ &= 3 + \alpha\beta \end{aligned}$$

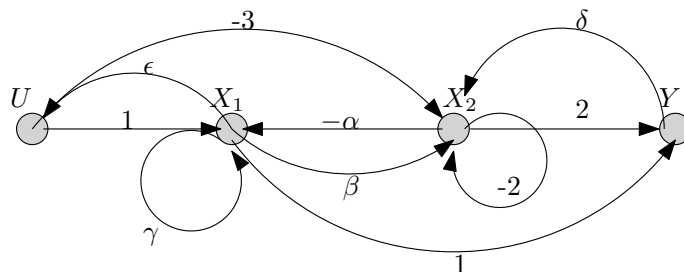
Finally, we define  $\Delta_i$ , which is Mason  $\Delta$  but with the forward path  $M_i$  removed from the graph. There are 4 forward paths in this problem, so there are  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ . Each time we remove a forward path, we find  $\Delta$  again using the above Mason rule method. In this problem we see that

$$\begin{aligned} \Delta_1 &= 1 \\ \Delta_2 &= 1 \\ \Delta_3 &= 1 - (-2) = 3 \\ \Delta_4 &= 1 \end{aligned}$$

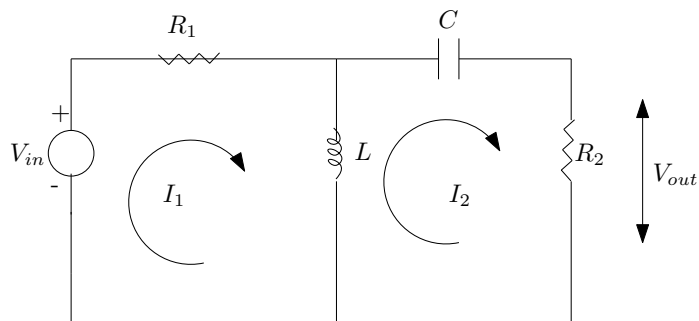
Finally, we apply the Mason gain formula

$$\begin{aligned}\frac{Y}{U} &= \frac{\sum_{i=1}^4 M_i \Delta_1}{\Delta} \\ &= \frac{(-6)(1) + (2\beta)(1) + (1)(3) + (3\alpha)(1)}{3 + \alpha\beta} \\ &= \frac{\alpha + 2\beta - 3}{3 + \alpha\beta}\end{aligned}$$

**Reader** Find  $\frac{Y}{U}$  for this graph



Second example. Here we take a circuit and obtain the equations, then use signal graph in order to use Mason rule to obtain the transfer function



Solving the circuit loops gives (all in Laplace domain)

$$(R_1 + sL)I_1 - I_2Ls - V_{in}(s) = 0$$

$$\left(R_2 + \frac{1}{Cs}\right)I_2 + LsI_2 - I_1Ls = 0$$

$$V_{out}(s) = R_2I_2$$

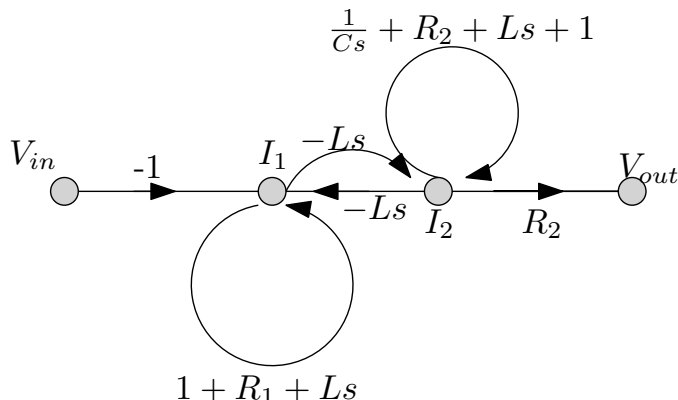
Now the variables are  $I_1, I_2$ , so we need to have these on the LHS. To do this, do this trick: Add  $I_1$  to each side of the first equation, and add  $I_2$  to each side of the second equation, this

gives

$$I_1 = (R_1 + sL) I_1 - I_2 Ls - V_{in}(s) + I_1$$

$$I_2 = I_2 + \left(R_2 + \frac{1}{Cs}\right) I_2 + Ls I_2 - I_1 Ls$$

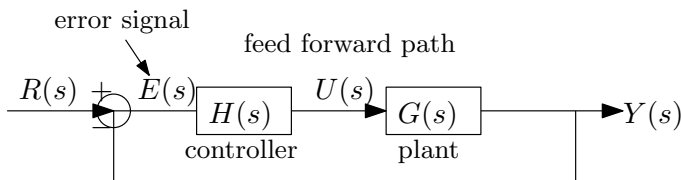
Now set up the signal graph



**Reader:** Find  $\frac{V_{out}}{V_{in}}$  for the above.

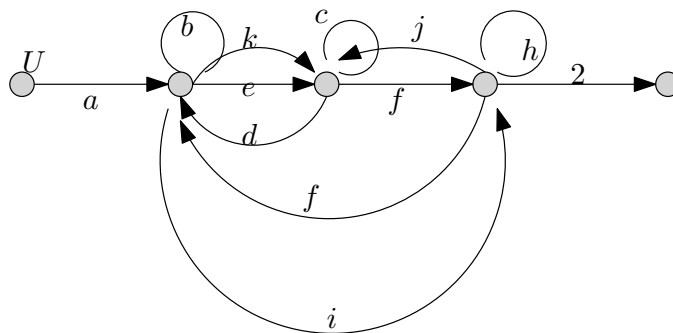
$$\begin{aligned} \frac{V_{out}}{V_{in}} &= \frac{\sum_{i=1}^1 M_i \Delta_i}{1 - \sum \text{one at time} + \sum 2 \text{ at times}} \\ &= \frac{(-1)(-Ls)(R_2)}{1 - \sum (R_1 + Ls + 1) + \left(\frac{1}{Cs} + R_2 + Ls + 1\right) + \sum (R_1 + Ls + 1) \left(\frac{1}{Cs} + R_2 + Ls + 1\right)} \\ &= \frac{LsR_2}{1 - \left(R_1 + R_2 + \frac{1}{Cs} + 2Ls + 2\right) + (R_1 + Ls + 1) \left(R_2 + \frac{1}{Cs} + Ls + 1\right)} \\ &= \frac{LsR_2}{\frac{1}{Cs} (R_1 + Ls) (CLs^2 + CR_2s + 1)} \end{aligned}$$

Now we take a block diagram and convert to signal graph. Given



We know that  $\frac{Y(s)}{R(s)} = \frac{H(s)G(s)}{1+H(s)G(s)}$  and that  $\frac{E(s)}{R(s)} = \frac{1}{1+H(s)G(s)}$ . Use Mason to show the above.

Reader Find  $\frac{Y}{U}$  for this



### 2.8.1 MIMO Practice problems

#### ECE 332 – Handout MIMO Practice Problems

**Problem 1:** A system with inputs  $u_1, u_2$  outputs  $y_1, y_2$  and intermediate states  $x_1, x_2$  and  $x_3$  is described by the differential equations

$$\dot{x}_1 = x_2 - x_3 + u_1 + 3u_2;$$

$$\dot{x}_2 = -3x_1 + x_3 - 4u_2;$$

$$\dot{x}_3 = -2x_2 + u_1;$$

$$y_1 = 2x_1 - x_3 + u_1;$$

$$y_2 = x_1 + x_2 - u_2.$$

For this MIMO system, find the associated open loop transfer function matrix  $G(s)$ . Express each entry of  $G(s)$  as a quotient of polynomials with numerator and denominator factored, if possible.

**Problem 2:** (a) Consider a MIMO  $2 \times 2$  controller  $H(s)$  connected in a classical unity feedback configuration to the system  $G(s)$  in Problem 1. With

$$H(s) = \begin{bmatrix} 0 & 1 \\ 1/s & 0 \end{bmatrix},$$

use syms in Matlab to find the closed loop transfer function matrix  $T(s)$ . You may wish to check your solution by calculating by hand.

(b) Is the closed loop stable? Explain.

## Handout MIMO Problem Solutions

Q1) Take  $\mathcal{L}$  with zero initial conditions

$$sX_1(s) = X_2(s) - X_3(s) + U_1(s) + 3U_2(s)$$

$$sX_2(s) = -3X_1(s) + X_3(s) - 4U_2(s)$$

$$sX_3(s) = -2X_2(s) + U_1(s)$$

Solve (algebra) and obtain with  $\phi(s) = s^2 + 5s + 6$

$$X_1(s) = \frac{1}{\phi(s)} [(s^2 - s + 3)U_1(s) + (3s^2 - 4s - 2)U_2(s)]$$

$$X_2(s) = \frac{1}{\phi(s)} [(3 - 2s)U_1(s) - (4s^2 + 9s)U_2(s)]$$

$$X_3(s) = \frac{1}{\phi(s)} [(s^2 + 9)U_1(s) + (8s + 18)U_2(s)]$$

Now  $Y_1(s) = +2X_1(s) - X_3(s) + U_1(s)$

$$Y_2(s) = X_1(s) + X_2(s) - U_2(s)$$

We now substitute  $X_i(s)$  into  $Y_1(s)$ ,  $Y_2(s)$  and after some algebra obtain

$$Y_1(s) = \frac{s^2 + 3}{s^2 - s + 6} U_1(s) + \frac{2(3s - 11)}{s^2 - s + 6}$$

1

$$Y_2(s) = \frac{s^2 - 3s + 6}{s^3 + 5s + 6} U_1(s) + \frac{(s^3 + s^2 + 18s + 8)}{s^3 + 5s + 6} U_2(s)$$

$$\text{Hence } G(s) = \begin{bmatrix} \frac{s^2 + 3}{s^2 - s + 6} & \frac{2(3s - 11)}{s^2 - s + 6} \\ \frac{s^2 - 3s + 6}{s^3 + 5s + 6} & -\frac{(s^3 + s^2 + 18s + 8)}{s^3 + 5s + 6} \end{bmatrix}$$

The CLTF matrix is

$$T(s) = [\mathbf{I} + G(s)H(s)]^{-1} G(s)H(s)$$

Subst.  $G(s)$ ,  $H(s)$ , we run Matlab code (syms) and obtain

$$T(s) = \frac{1}{s^4 + 2s^3 + 10s^2 + 19s - 40} \begin{bmatrix} s^3 + 8s^2 + 7s - 40 & s^2 + 3s + 3 \\ -(s^3 + s^2 + 18s + 8) & 2s^3 - s^2 + 29s - 18 \end{bmatrix}$$

(b) Notice denominator poly above  $s^4 + 2s^3 + 10s^2 + 19s - 40$   
 Finding roots of this poly, use roots([1 2 10 19 -40]) in Matlab,  
 and observe one root is 1.15 and the rest have real part < 0  
 Hence system is unstable.



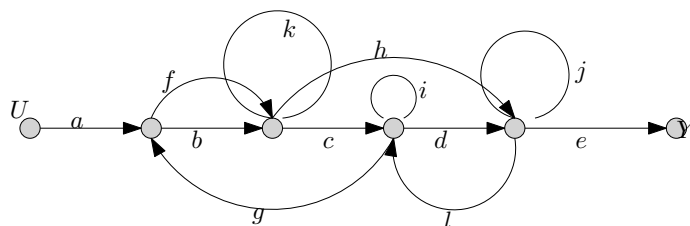
## 2.9 Lecture 8, Thursday Sept. 24, 2015, Mason, benefits of feedback, non-linear systems

Exam 1 on Tuesday Sept. 29, 2015. Closed book, closed notes. There will be office hours Monday 1-3 pm.

Keywords for exam:

1. Modeling, basic circuit or spring mass damper.
2. Block diagrams. Go from model to block diagram. Laplace and transfer function. For example, given a mass/spring, find the ODE and use Laplace to find the transfer function
3. Block diagram, open loop vs. closed loop. Classic unity feedback
4. Know basic Laplace and inverse Laplace
5. Given TF, and  $R(s)$  find  $Y(s)$
6. Steady state error. Know when to use F.V.T. This is related to tracking. The more complicated the signal, the more integrated we need. This is called the integrator principle.
7. MIMO basics. Matrix transfer function. Watch out for order here.
8. 4 or 5 questions.

Now for one more Mason problem. We use Mason any time we want to find a transfer function. Find  $\frac{Y(s)}{U(s)}$  for this signal graph



There are 4 forward paths from  $U$  to  $Y$ , here they are, with the associated Mason  $\Delta$

$$M_1 = abcde, \Delta_1 = 1$$

$$M_2 = afhe, \Delta_2 = 1 - i$$

$$M_3 = afcde, \Delta_3 = 1$$

$$M_4 = abhe, \Delta_4 = 1 - i$$

We need  $\frac{Y}{U} = \frac{\sum_{i=1}^4 M_i \Delta_i}{\Delta}$ . The loops are  $\{k, j, i, bcg, dl, fhl, g, bhl, g, fcg\}$ , hence

$$\Delta = 1 - (k + i + j + dk + fhl\ g + bhl\ g + fcg) + (ki + kj + ij + kdl + jbcg + jfcg) - (kij)$$

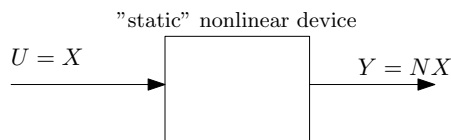
Hence

$$\begin{aligned} \frac{Y}{U} &= \frac{\sum_{i=1}^4 M_i \Delta_i}{\Delta} \\ &= \frac{abcde(1) + afhe(1-i) + afcde(1) + abhe(1-i)}{1 - (k + i + j + dk + fhl\ g + bhl\ g + fcg) + (ki + kj + ij + kdl + jbcg + jfcg) - (kij)} \end{aligned}$$

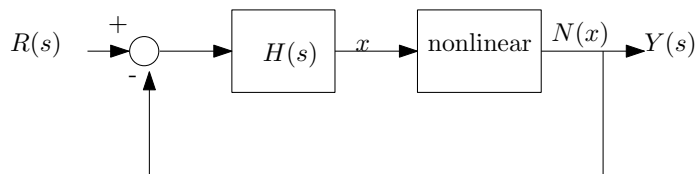
Next topic we will start on is the benefits of feedback. So far we talked about tracking only. Other benefits are

1. Linearization
2. Sensitivity
3. Disturbances

We can use feedback to pre compensate a nonlinear system to make it approximately linear. Given a non-linear device, say diode, with input  $U = X$  which represent voltage and output  $Y$  which is nonlinear function of the input such as  $Y = NX$  can we use feedback to make the output closed to linear?



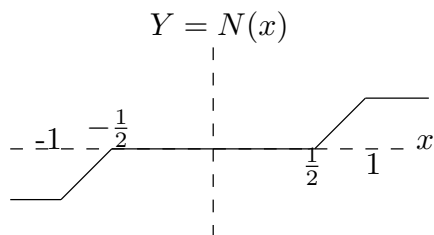
Warning: When the system is non-linear, we can not use transfer functions and can not use Laplace. These are only for linear systems. Transfer functions and Laplace transforms are used only when the system is linear. So how do we analyze non-linear system? We use time domain. For example, if  $Y = X^2$



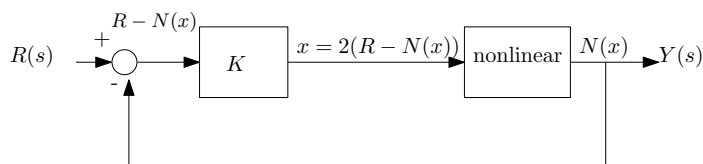
These are not transfer functions

Closed loop is nonlinear. We need relation between  $R(s)$  and  $Y(s)$

There are two type of nonlinearity, saturations and dead-zone. Dead zone is an area where the input is not yet sufficient to cause any output to be generated, it might be a threshold for the device to start operating. Here is a typical output from a non linear device



**Reader** For this open loop, if  $x = 2 \sin t$ , sketch  $y(t)$ . Another example



We know

$$Y = N(x) \tag{1}$$

and

$$x = 2(R - Y)$$

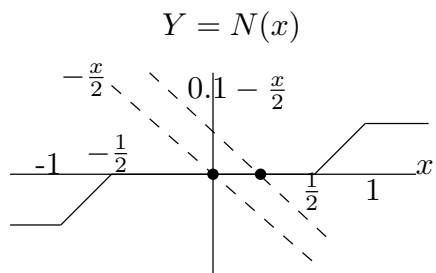
hence  $2Y = 2R - x$  and

$$Y = R - \frac{x}{2} \tag{2}$$

(1) and (2) must both hold. For each  $R$  input, we solve (1,2) for  $Y, x$  and plot them. For example, for  $R = \{0, 0.1, 0.2, \dots\}$  for each  $R(i)$  we solve for  $Y(i)$

$R$	$Y = R - \frac{x}{2}$
0	$0 - \frac{x}{2}$
0.1	$0.1 - \frac{x}{2}$
0.2	$0.2 - \frac{x}{2}$
$\vdots$	$\vdots$

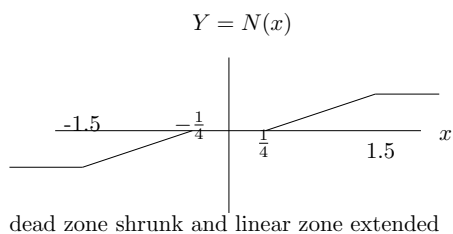
For each line in the above, such as  $-\frac{x}{2}, 0.1 - \frac{x}{2}, \dots$ , we now draw this line on top of the original  $Y(x)$  plot, and see where this line intersect with the original  $Y(x)$ . The point of intersection is the new value of  $Y$ . This is done for each entry of  $R$ , so we obtain



So we obtain this table

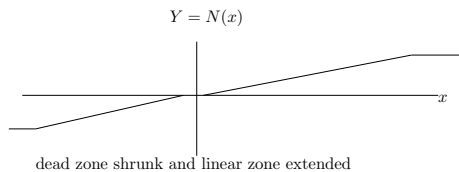
$R$	$Y = R - \frac{x}{2}$
0	0
0.1	0
0.2	0
$\vdots$	$\vdots$
big	.3
$\vdots$	1

**Reader:** calculate and obtain closed loop. We might obtain this



We see that dead zone has shrunk (good) and linear region increased (good).

**Reader:** Redo with  $k = 10$ . For large  $K$  we should obtain



## **2.10 Lecture 9, Tuesday Sept. 29, 2015, First exam**

First exam

## **2.11 Lecture 10, Thursday Oct. 1, 2015, No lecture**

No lecture today

## 2.12 Lecture 11, Tuesday Oct. 6, 2015, Sensitivity of transfer function

Today lecture on sensitivity. Definition of sensitivity: percentage change in magnitude of transfer function  $T(s)$  per one percent change of parameter  $\alpha$  in the transfer function. We normally make this parameter be  $\alpha$ . This could be  $R$  (resistance) or  $C$  (capacitance) and so on. We call this  $S_{\alpha}^T$  which is read as the sensitivity of the  $T(s)$  with respect to changes in  $\alpha$ .

Therefore

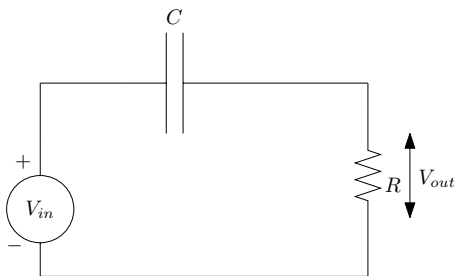
$$\begin{aligned} S_{\alpha}^T &= \frac{\frac{\Delta T}{T}}{\frac{\Delta \alpha}{\alpha}} \\ &= \frac{\Delta T}{\Delta \alpha} \frac{\alpha}{T} \\ &= \frac{dT}{d\alpha} \frac{\alpha}{T} \end{aligned}$$

We then have to evaluate  $S_{\alpha}^T$  at the nominal value of the parameter  $\alpha = \alpha_0$ . Therefore

$$S_{\alpha=\alpha_0}^T = \frac{dT}{d\alpha} \frac{\alpha}{T} \Big|_{\alpha=\alpha_0}$$

$\alpha_0$  is given numerical value. It is meant to be the value that the parameter  $\alpha$  fluctuate around and will be given in the problem to use.

For example, given this circuit



Let  $R$  be the parameter that will change and let amount of change be  $\Delta R$  and we want to find the sensitivity of change in the transfer function  $T(s) = \frac{V_{out}(s)}{V_{in}(s)}$  to changes in  $R$ .

We know that

$$T(s) = \frac{RCs}{1 + RCs}$$

Say that  $R = R_0 + \Delta R$  where  $R_0$  is the nominal value of resistance  $R$  and  $\Delta R$  is the amount of variation it has. Hence (1) becomes

$$T(s) = \frac{(R_0 + \Delta R)Cs}{1 + (R_0 + \Delta R)Cs}$$

To find  $S_R^T$  we let  $R = \alpha$  and apply the definition  $S_\alpha^T = \frac{dT}{d\alpha} \frac{\alpha}{T}$  at  $\alpha = \alpha_0$ . Hence

$$\begin{aligned} S_\alpha^T &= \frac{dT}{d\alpha} \frac{\alpha}{T} \\ &= \frac{d}{d\alpha} \left( \frac{\alpha Cs}{1 + \alpha Cs} \right) \frac{\alpha}{\frac{\alpha Cs}{1 + \alpha Cs}} \end{aligned}$$

Assume  $C = 1$  and assume nominal value of  $R$  is 1 also. This means  $\alpha_0 = 1$ . The above becomes

$$\begin{aligned} S_\alpha^T &= \frac{d}{d\alpha} \left( \frac{\alpha s}{1 + \alpha s} \right) \frac{\alpha}{\frac{\alpha s}{1 + \alpha s}} = \frac{s}{(s\alpha + 1)^2} \frac{1 + \alpha s}{s} \\ &= \frac{1}{s\alpha + 1} \end{aligned}$$

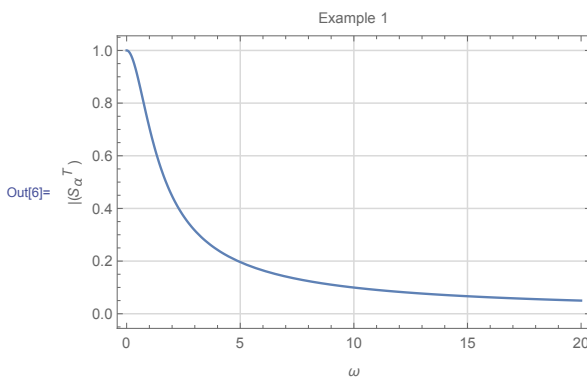
Evaluate at  $\alpha = \alpha_0 = 1$  then

$$S_{\alpha=\alpha_0}^T = \frac{1}{s + 1}$$

Next step is to replace  $s = j\omega$  and plot the magnitude in frequency domain

$$\begin{aligned} S_\alpha^T &= \frac{1}{j\omega + 1} \\ |S_\alpha^T| &= \frac{1}{\sqrt{1 + \omega^2}} \end{aligned}$$

A plot is

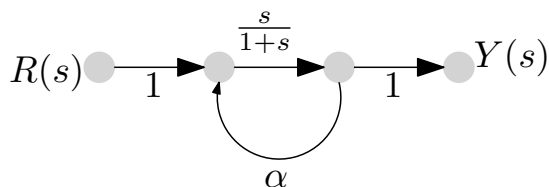


Examples below shows to calculate  $S_\alpha^T$  for difference parameters.

### 2.12.0.1 Example 1

Given the signal graph





$$T(s) = \frac{\frac{s}{1+s}}{1 - \alpha \left( \frac{s}{s+1} \right)}$$

$$= \frac{s}{1 + s(1 - \alpha)}$$

hence

$$S_{\alpha}^T = \frac{\alpha dT}{T d\alpha} = \frac{\alpha}{\frac{s}{1+(s-\alpha)}} \left( \frac{-(-s)s}{((1-\alpha)s+1)^2} \right)$$

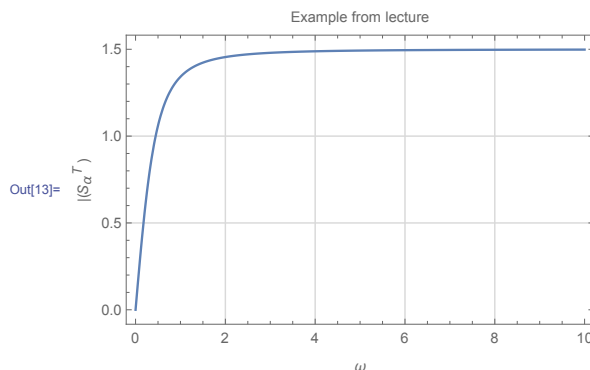
$$= \frac{\alpha s}{((1-\alpha)s+1)}$$

Let  $s = j\omega$  and let  $\alpha = \alpha_0 = 3$  then above becomes

$$S_{\alpha}^T = \frac{3j\omega}{1 - 2j\omega}$$

$$|S_{\alpha}^T| = \frac{3\omega}{\sqrt{1 + 4\omega^2}}$$

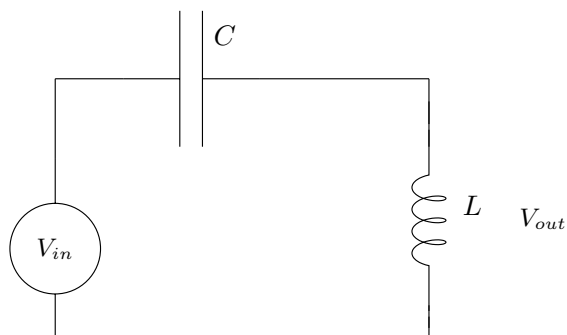
The plot is



The above curve give the percentage of change in  $T$  when  $\alpha$  changes by one percentage. We just need to determine the magnitude plot, we normally do not worry above phase when doing sensitivity analysis. The above plots says that  $T(s)$  is not sensitive to changes in  $\alpha$  when the frequency is near DC, and as  $\omega$  increases, the sensitivity increases. For 1% change in  $\alpha$ , at high  $\omega$ , the magnitude of  $T(s)$  changes by 1.5%

### 2.12.1 Example 2

Given the circuit



Let us see how  $T(s) = \frac{V_{out}(s)}{V_{in}(s)}$  changes when  $L$  changes. So we make  $L$  as our  $\alpha$  here. The nominal  $\alpha_0 = 1$  and we also take  $C = 1$ .

$$T(s) = \frac{Ls}{Ls + \frac{1}{Cs}} = \frac{\alpha s}{\alpha s + \frac{1}{s}} = \frac{\alpha s^2}{\alpha s^2 + 1}$$

Hence

$$S_{\alpha}^T = \frac{\alpha dT}{T d\alpha} = \frac{1}{\alpha s^2 + 1}$$

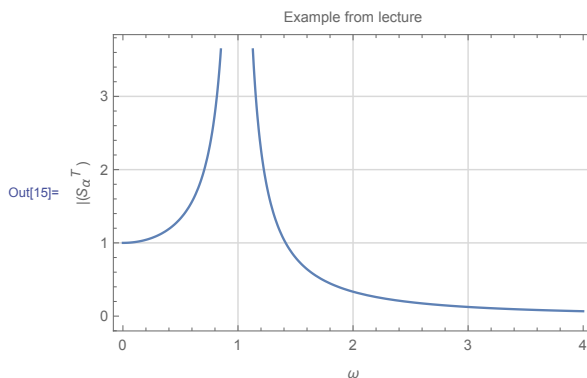
Let  $s = j\omega$   $\alpha = \alpha_0 = 1$  and the above becomes

$$S_{\alpha}^T = \frac{1}{1 - \omega^2}$$

Hence

$$|S_{\alpha}^T| = \frac{1}{|1 - \omega^2|}$$

Notice that at  $\omega = 1$  there is resonance. Here is the plot

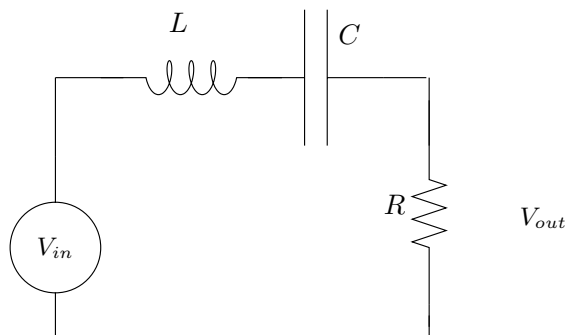


The above says that when  $\omega$  near 1, then the transfer function is very sensitive to changes in  $L$ . For 1% change in  $L$ , the magnitude of the transfer function become very large at that frequency. This can cause problems, so we need to avoid getting close to  $\omega = 1$  and must

stay above it for safe operations.

### 2.12.2 Example 3

Given this circuit



Now we will find the  $S_\alpha^T$  for each parameter in the circuit. These are  $R, L, C$ , each time we fix all the parameters, except the one in interest, and call that one  $\alpha$  and repeat the steps we did in the earlier examples.

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + \frac{1}{Cs} + Ls} = \frac{RCs}{RCs + LCs^2 + 1}$$

Let  $\alpha = R$ , and let  $C = 1, L = 1$  and let  $\alpha_0 = 1$  as well. Hence

$$T(s) = \frac{\alpha s}{s^2 + \alpha s + 1}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha}{T} \frac{dT}{d\alpha} \\ &= \frac{1 + s^2}{s^2 + \alpha s + 1} \end{aligned}$$

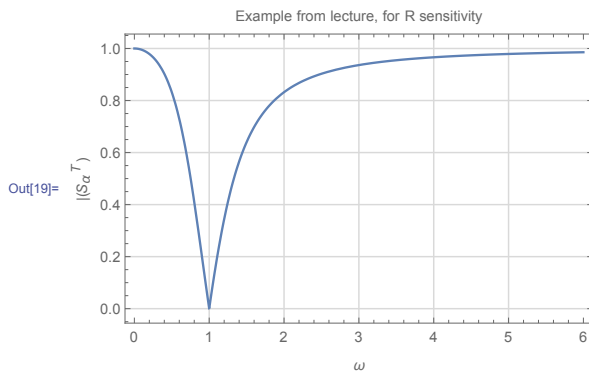
We now switch to  $\omega$  domain

$$\begin{aligned} S_\alpha^T &= \frac{1 - \omega^2}{1 - \omega^2 + \alpha j\omega} \Big|_{\alpha=1} \\ &= \frac{1 - \omega^2}{1 - \omega^2 + j\omega} \end{aligned}$$

Hence

$$|S_\alpha^T| = \frac{|1 - \omega^2|}{\sqrt{(1 - \omega^2)^2 + \omega^2}}$$

Be careful to use  $|1 - \omega^2|$  above and not just  $1 - \omega^2$  since these are norms. Plotting the above gives



We see from above that  $T(s)$  is least sensitive to changes in  $R$  when  $\omega = 1$  and that the maximum change is 1%

We now repeat the above, but for  $C$ . Hence now  $\alpha = C$ , and  $L = 1, R = 1$ . Therefore

$$\begin{aligned} T(s) &= \frac{RCs}{RCs + LCs^2 + 1} \\ &= \frac{\alpha s}{\alpha s + s^2 + 1} \end{aligned}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha}{T} \frac{dT}{d\alpha} \\ &= \frac{1}{\alpha s^2 + \alpha s + 1} \end{aligned}$$

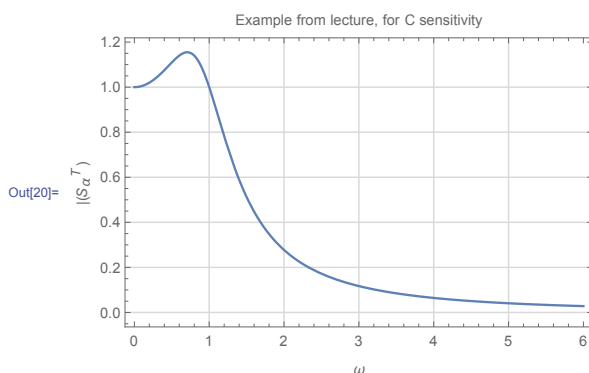
We now switch to  $\omega$  domain and set  $\alpha = 1$  which gives

$$S_\alpha^T = \frac{1}{1 - \omega^2 + j\omega}$$

Hence

$$|S_\alpha^T| = \frac{1}{\sqrt{(1 - \omega^2)^2 + \omega^2}}$$

Plotting the above gives



The maximum occurs near  $\omega = 0.7$ . Finally, we now look at sensitivity against changes in  $L$ .

Hence now  $\alpha = L$ , and  $C = 1, R = 1$ . Therefore

$$\begin{aligned} T(s) &= \frac{RCs}{RCs + LCs^2 + 1} \\ &= \frac{s}{\alpha s + s^2 + 1} \end{aligned}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha}{T} \frac{dT}{d\alpha} \\ &= \frac{-\alpha s^2}{\alpha s^2 + \alpha s + 1} \end{aligned}$$

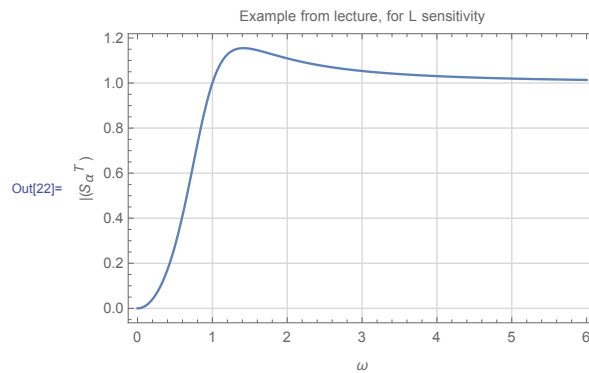
We now switch to  $\omega$  domain and set  $\alpha = 1$  which gives

$$S_\alpha^T = \frac{-(-\omega^2)}{1 - \omega^2 + j\omega}$$

Hence

$$|S_\alpha^T| = \frac{\omega^2}{\sqrt{(1 - \omega^2)^2 + \omega^2}}$$

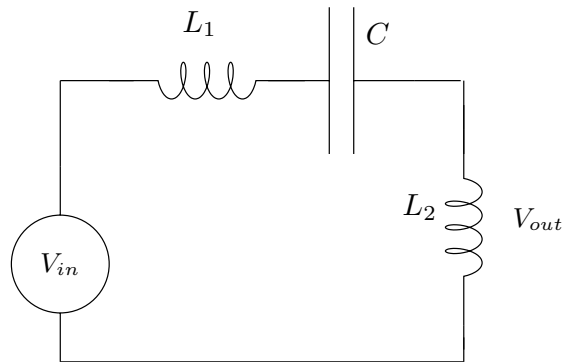
Plotting the above gives



The maximum occurs near  $\omega = 1.14$ .

### 2.12.3 Example 4

Given this circuit



Let the nominal values be  $C = 1, L_1 = 1$  and  $L_2 = 1$ . We will find  $S_\alpha^T$  for each of these parameters now one at a time as in the above example.

$$T(s) = \frac{L_2 s}{(L_1 + L_2)s + \frac{1}{Cs}} = \frac{L_2 C s^2}{(L_1 + L_2) C s^2 + 1}$$

When  $\alpha = L_1$  then (after putting  $C = 1, L_2 = 1$ ) the above becomes

$$T(s) = \frac{s^2}{(\alpha + 1)s^2 + 1}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha}{T} \frac{dT}{d\alpha} \\ &= \frac{-\alpha s^2}{(\alpha + 1)s^2 + 1} \end{aligned}$$

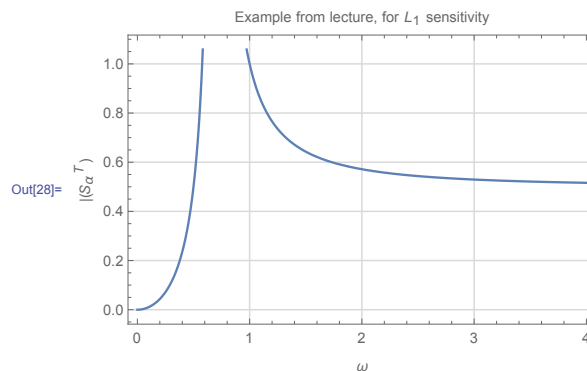
We now switch to  $\omega$  domain and set  $\alpha = 1$  which gives

$$S_\alpha^T = \frac{\omega^2}{1 - 2\omega^2}$$

Hence

$$|S_\alpha^T| = \frac{\omega^2}{|1 - 2\omega^2|}$$

The plot is



We see that  $|S_\alpha^T|$  blows up at  $\omega = \frac{1}{\sqrt{2}}$  and at  $\omega = 1, |S_\alpha^T| = 1$ . We now consider  $\alpha = L_2$  then (after putting  $C = 1, L_1 = 1$ ) the transfer function becomes

$$T(s) = \frac{\alpha s^2}{(1 + \alpha)s^2 + 1}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha dT}{T d\alpha} \\ &= \frac{s^2 + 1}{(\alpha + 1)s^2 + 1} \end{aligned}$$

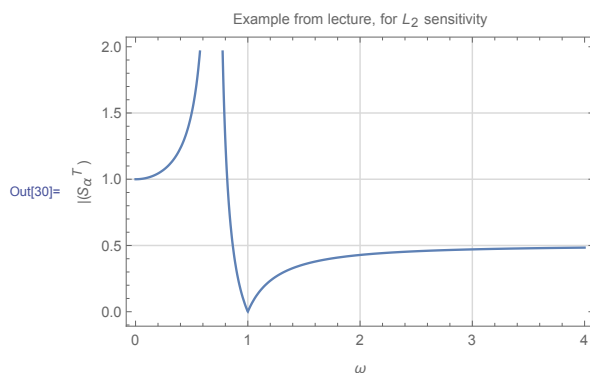
We now switch to  $\omega$  domain and set  $\alpha = 1$  which gives

$$S_\alpha^T = \frac{1 - \omega^2}{1 - 2\omega^2}$$

Hence

$$|S_\alpha^T| = \frac{|1 - \omega^2|}{|1 - 2\omega^2|}$$

The plot is



We see that now at low frequency  $T(s)$  is sensitive to  $L_2$  while it is not sensitive to changes in  $L_1$ . Finally, looking at  $\alpha = C$  then (after putting  $L_2 = 1, L_1 = 1$ ) the transfer function becomes

$$T(s) = \frac{\alpha s^2}{2\alpha s^2 + 1}$$

Hence

$$\begin{aligned} S_\alpha^T &= \frac{\alpha dT}{T d\alpha} \\ &= \frac{1}{2\alpha s^2 + 1} \end{aligned}$$

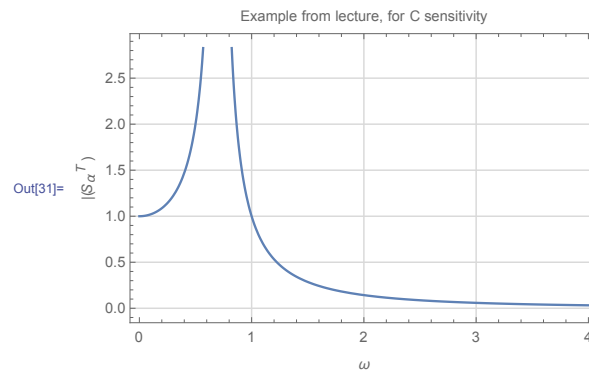
We now switch to  $\omega$  domain and set  $\alpha = 1$  which gives

$$S_\alpha^T = \frac{1}{1 - 2\omega^2}$$

Hence

$$|S_{\alpha}^T| = \frac{1}{|1 - 2\omega^2|}$$

The plot is



We see that  $|S_{\alpha}^T|$  blows up at  $\omega = \frac{1}{\sqrt{2}}$ .

In conclusion, we can use these plots to determine how each component affect the transfer function. We do not want the transfer function to be sensitive to changes in components. If we know the range of operating frequencies, we can now know which components can cause most problems and may be spend more money to buy better quality component for that specific one.

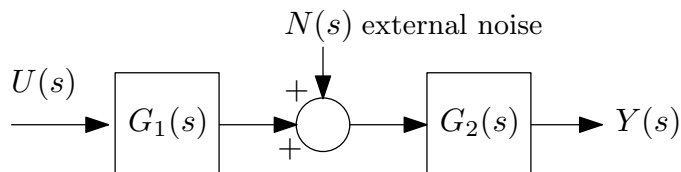


## 2.13 Lecture 12. Thursday Oct 8, 2015, control to reject noise and disturbances

First midterm and second HW returned. Review of midterm results given.

In this lecture we will continue to talk about benefits of feedback (see Lecture 8, Thursday Sept. 24, 2015). We talked about sensitivity and reducing nonlinearity in the plant, now we will talk about the third benefit which is noise or disturbance rejection.

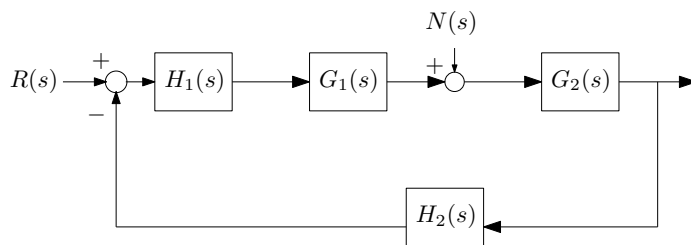
Classical setup is the following



There are many physical examples that can be represented using the above. For example, the above can be a communication channel with noise affecting the data transmission in the channel. The above is written as

$$\begin{aligned} Y(s) &= \left. \frac{Y(s)}{U(s)} \right|_{N=0} U(s) + \left. \frac{Y(s)}{N(s)} \right|_{U=0} N(s) \\ &= G_1 G_2 U + G_2 N \end{aligned}$$

Classical approach to reducing the disturbance effect is to setup the feedback as follows



The design problem is now to pick the appropriate  $H_1(s)$  and  $H_2(s)$  to reduce  $N(s)$  effect on the system. Let us study the closed loop transfer function

$$\begin{aligned} Y(s) &= \left. \frac{Y(s)}{R(s)} \right|_{N=0} R(s) + \left. \frac{Y(s)}{N(s)} \right|_{R=0} N(s) \\ &= \frac{G_1 G_2 H_1}{1 + G_1 G_2 H_1 H_2} R(s) + \frac{G_2}{1 + G_1 G_2 H_1 H_2} N(s) \end{aligned}$$

Pick  $H_2$  first. Let  $H_2 = \frac{1}{G_1 G_2}$  then the above becomes

$$Y(s) = \frac{G_1 G_2 H_1}{1 + H_1} R(s) + \frac{G_2}{1 + H_1} N(s)$$

And let  $H_1 = \alpha$  where  $\alpha$  is a very large gain value. Then above reduces to

$$Y(s) = \frac{G_1 G_2 \alpha}{1 + \alpha} R(s) + \frac{G_2}{1 + \alpha} N(s)$$

$$\lim_{\alpha \rightarrow \infty} Y(s) = G_1 G_2 R(s)$$

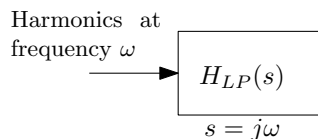
Which is good. This is what we wanted. So  $N(s)$  effect has not been eliminated. But this method has the following disadvantages

1. If  $G_1 G_2 = \frac{1}{1+s}$  then  $H_2$  becomes  $\frac{1}{G_1 G_2} = s + 1$ . This is not good. We normally do not want to have differentiators in the loop as they cause problems we talked about early in the course.
2. Another problem. Lets say  $G_1 G_2 = \frac{1+s}{s^2+2s+1}$ . Then  $H_2 = \frac{s^2+2s+1}{1+s}$ , and we still have the same problem as above since after long division, we see this is still  $1 + s$ . There was a hidden differentiator in there. In general, if the numerator has degree less than the denominator in  $G_1 G_2$  then  $H_2$  that results will have a differentiator.

How to fix the above? The fix is to introduce a low pass filter, called  $H_{LP}(s)$ . So that instead of using  $H_2(s)$  we use  $H_2(s)H_{LP}(s)$ . Low pass filter attenuate high frequency noise. The simplest low pass filter is

$$H_{LP}(s) = \frac{1}{(\epsilon s + 1)^k}$$

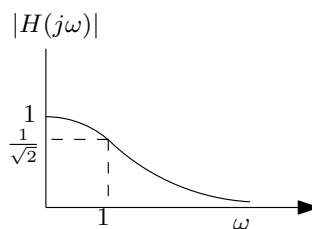
Where  $k$  is an integer specified by the designer and  $\epsilon > 0$ . Now we introduce frequency. This is done by letting  $s = j\omega$ . Imaging we have this system



$H_{LP}(j\omega)$  is called the frequency response. It is complex valued. Has magnitude and phase.

Example:  $\epsilon = 1, k = 1$  then  $H_{LP}(j\omega) = \frac{1}{1+j\omega}$  and  $|H_{LP}| = \frac{1}{\sqrt{1+\omega^2}}$  and phase  $\angle H_{LP} = 0 - \tan^{-1} \omega$

Plotting the magnitude  $|H_{LP}|$  gives

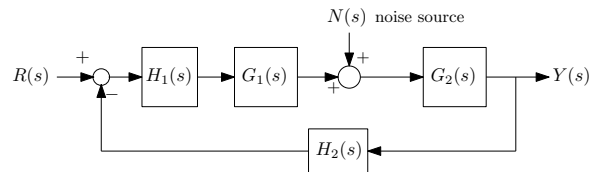


So back to using  $H_{LP}$  in our original problem, which is noise rejection. As we said, we now will use  $H_2 H_{LP}$  in place of  $H_2$ . Does  $H_{LP}$  mess up the cancellation of  $G_1 G_2$  as we had before?

It depends on the noise type. For low frequency noise, then  $|H_{LP}|$  will be close to 1 and hence  $H_2 H_{LP}$  will remain very close to  $H_2$ . But if the noise is high frequency, then  $|H_{LP}|$  is much smaller than one, and hence  $H_2 H_{LP}$  will be much smaller than original  $H_2$ . For example, if  $\omega = 1$ , then  $H_2$  is attenuated by about 30%. Next time, we will build more on this topic.

## 2.14 Lecture 13. Tuesday Oct 13, 2015, Noise rejection, second order systems

Today we will finish noise attenuation, then start on second order systems. The classical method of noise attenuating is based on this feedback system block diagram

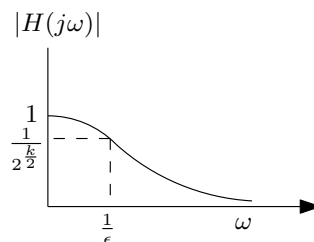


We often have systems where noise or disturbance comes in between the input and the output. Without  $N(s)$  we would have perfect open loop. The classical approach to noise attenuation is as shown in the above diagram, which is to add  $H_1(s)$  and  $H_2(s)$  with the idea to reduce the effect of  $N(s)$  while at the same time to preserve  $R(s)$  input signal and not affect it. For  $H_1(s)$  we use large pure gain  $\alpha$ , this is for attenuation. For  $H_2(s)$ , we start with what is called the inversion method, which is to use  $H_2(s) = \frac{1}{G_1 G_2}$ . As discussed in last lecture, this method looks good in math, but not good in practice, since  $H_2(s)$  becomes improper transfer function. Now we will explain a more practical method, which is to introduce a low pass filter  $H_{LP} = \frac{1}{(\varepsilon s + 1)^k}$  which will reject noise frequency and also make  $H_2(s)$  become proper. We will use  $H_2(s) = \frac{1}{G_1 G_2} H_{LP}(s)$  instead of just  $H_2(s) = \frac{1}{G_1 G_2}$  as before.

We need to pick  $\varepsilon, k$ . Both are positive. To design for  $H_{LP}(s)$  we need to know something about  $N(s)$ . We need to know the frequency content of  $N(s)$  so we can design  $H_{LP}(s)$  to block most of frequency content of  $N(s)$  while allowing all the content of  $R(s)$  to pass through. We assume  $R(s)$  frequency is all in the passband of the low pass filter. This is done in frequency domain.

$$|H_{LP}(j\omega)| = \left| \frac{1}{(\varepsilon j\omega + 1)^k} \right| = \frac{1}{|\varepsilon j\omega + 1|^k} = \frac{1}{(\sqrt{\varepsilon^2 \omega^2 + 1})^k} = \frac{1}{(\varepsilon^2 \omega^2 + 1)^{\frac{k}{2}}}$$

The plot of  $|H_{LP}(j\omega)|$  might now look like this



We can make the filter closed to desired by boosting  $k$  and decreasing  $\varepsilon$ .

## Second order systems

We will now start on second order systems. We want to study transient response. So far, we said nothing about transient response. Final value theorem give the steady state response (when it exists, if the system is stable) but not what happens in between. The system could have undesired transient response before getting to the steady state. For example, we could want to send the response to zero very quickly, but this can cause bad transient response.

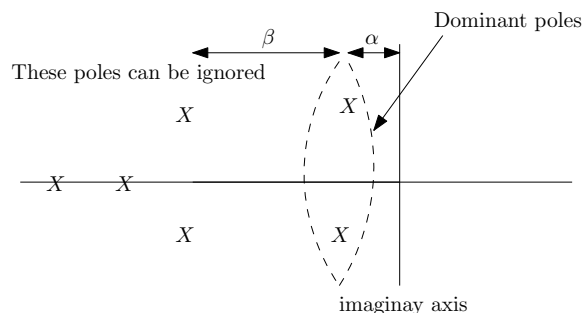
Why consider only second order systems?

1. Many physical systems are second order system
2. Many systems can be well approximated by second order system, using the method of dominant poles.
3. Math is much simplified when using second order system than higher order

When we design, say RLC circuit we get second order system. Same for mass spring damper. When the system is higher order, we use dominant pole method to approximate the system to second order. But after approximate to second order and doing the analysis on the second order, we should go back and simulate the original higher order system numerically (say using simulink) and compare the second order approximation with the full order system to make sure the approximation used produces close enough results.

## Dominant pole method

Imagine 6th order system. We can ignore poles much further away from the imaginary axis, since these indicate modes that attenuate very fast



In many practical systems,  $\beta \gg \alpha$  and the poles further to the left can be ignored since these are modes which disappear very quickly. So we are left with the two dominant poles  $s_{1,2} = -\alpha \pm j\omega$ . Generic second order system is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Where  $\omega_n$  is the natural frequency and  $\xi$  is the damping ratio. Consider a unit step input

$R(s) = \frac{1}{s}$ . For practical system,  $0 < \xi < 1$ .

$$Y(s) = G(s)R(s) \\ = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \frac{1}{s}$$

The inverse Laplace transform of the above is

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi) \quad (1)$$

To plot the above in Matlab, here is small code

```
t=0:.1:45;
z=0.707;
wn=.2;
y=@(t,z,wn) 1- exp(-z*wn*t)/sqrt(1-z^2).*sin(wn*sqrt(1-z^2)*t+acos(z));
plot(t,y(t,z,wn));
```

To use Matlab step() command, here is small code

```
z=0.707;
wn=.2;
s=tf('s');
sys= wn^2/(s^2+2*z*wn*s+wn^2);
step(sys);
```

In (1),  $\omega_d$  is the damped natural frequency given by  $\omega_d = \omega_n \sqrt{1-\xi^2}$  and  $\phi = \cos^{-1} \xi$ . (In our textbook,  $\phi$  was defined as  $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{-\xi}$ , but this seems strange to me. I will use the more common definition of  $\sin \phi = \sqrt{1-\xi^2}$  and  $\cos \phi = \xi$  as used by Nise text and other, hence  $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$  from now on).

What if we are not given a standard second order system transfer function such as  $G(s) = \frac{25}{s^2+5s+10}$ , we can convert this to standard by doing  $G(s) = \frac{\left(\frac{25}{10}\right)10}{s^2+5s+10} = 2.5 \frac{10}{s^2+5s+10}$  and now apply the result to  $\frac{10}{s^2+5s+10}$  and then scale the output by 2.5.

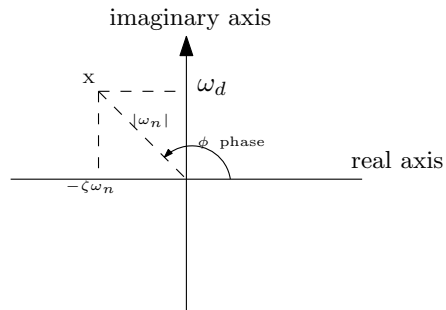
For undamped case,  $\xi = 0$ , the response is pure harmonics with no damping. The harmonics have  $\omega_n$  frequencies. We will now look at poles and zeros of  $G(s)$  in complex domain. Poles of  $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s + \omega_n^2}$  are

$$s_{1,2} = -\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2 - 1)} \\ = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

For underdamped,  $\xi < 1$  the above can be written as

$$\begin{aligned} s_{1,2} &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \\ &= -\xi\omega_n \pm j\omega_d \end{aligned}$$

The following diagram shows the main components on the  $s$  space plot

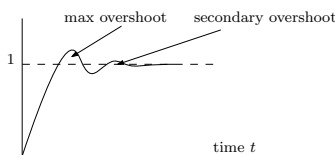


$$\phi = \tan^{-1} \frac{\omega_d}{\xi\omega_n}$$

$\phi$  is called the damped phase.

## 2.15 Lecture 14. Thursday Oct 15, 2015, More second order, Overshoot and resonance

Will now determine the maximum overshoot called  $OS_{\max}$  as shown in this diagram



We will use  $\frac{dy}{dt} = 0$  to find  $y_{\max}$ . Since  $\frac{dy}{dt} = 0$  will generate many solutions, we will take the first one.

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$$

Then

$$\begin{aligned} \frac{dy}{dt} = 0 &= \xi\omega_n \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \omega_d \cos(\omega_d t + \phi) \\ 0 &= \xi \sin(\omega_d t + \phi) - \sqrt{1-\xi^2} \cos(\omega_d t + \phi) \end{aligned} \quad (1)$$

To solve this, since  $\cos \phi = \xi$  and  $\sin \phi = \sqrt{1-\xi^2}$

then (1) becomes

$$0 = \cos \phi \sin(\omega_d t + \phi) - \sin \phi \cos(\omega_d t + \phi)$$

Using  $\sin(A-B) = \cos A \sin B - \sin A \cos B$  the above can be written as (using  $A = \phi$ )

$$0 = \sin(\phi - (\omega_d t + \phi))$$

Hence

$$\sin(\omega_d t) = 0$$

The solution is  $\omega_d = \omega_n \sqrt{1-\xi^2} t = k\pi$  for  $k = 0, 1, 2, \dots$ . We pick  $k = 1$  since this is the first one after  $t = 0$ , hence

$$\begin{aligned} \omega_n \sqrt{1-\xi^2} t_{\max} &= \pi \\ t_{\max} &= \frac{\pi}{\omega_n \sqrt{1-\xi^2}} \\ &= \frac{\pi}{\omega_d} \end{aligned}$$

To find  $y_{\max}(t) = y(t_{\max})$ , we plug the above  $t_{\max}$  back in the original solution which is



$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$ . Hence

$$\begin{aligned} OS_{\max} &= y(t_{\max}) - 1 \\ &= \left(1 - \frac{e^{-\xi\omega_n t_{\max}}}{\sqrt{1-\xi^2}} \sin(\omega_d t_{\max} + \phi)\right) - 1 \end{aligned} \quad (2)$$

**Reader:** Show that the above reduces to

$$OS_{\max} = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

Reader solution: substitute  $t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$  in (2) gives

$$\begin{aligned} OS_{\max} &= -\frac{e^{-\xi\omega_n \left(\frac{\pi}{\omega_n \sqrt{1-\xi^2}}\right)}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \left(\frac{\pi}{\omega_d}\right) + \phi\right) \\ &= -\frac{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \phi) \end{aligned}$$

But  $\sin(\pi + \phi) = -\sin\phi$  which is  $-\sqrt{1-\xi^2}$ , hence the above becomes

$$\begin{aligned} OS_{\max} &= -\frac{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \left(-\sqrt{1-\xi^2}\right) \\ &= e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \end{aligned}$$

Notice the overshoot do not depend on  $\omega_n$ . It only depends on damping. There are two ways to change damping. Either change the system itself, or add a controller to compensate.

Second main property of second order system is resonance. This arises in the frequency context. When the frequency the system is operating at is close to the natural frequency of the system. We are now interested in  $|G(j\omega)|$  vs.  $\omega$ . We will call the resonance frequency  $\omega_r$  and  $|G(j\omega_r)| = M_r$ . From

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \\ |G(j\omega)| &= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \end{aligned}$$

To find where this is maximum,

$$\frac{d}{d\omega} |G(j\omega)| = 0$$

To simplify, we will instead use  $|G(j\omega)|^2$  to get rid of the square root of the denominator

giving

$$|G(j\omega)| = \frac{\omega_n^4}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}$$

Then the maximum is where the denominator is minimum. Hence

$$\begin{aligned} \frac{d}{d\omega} \left( (\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2 \right) &= 0 \\ 2(\omega_n^2 - \omega^2)2\omega + 8\xi^2\omega_n^2\omega &= 0 \\ \omega_r &= \omega_n\sqrt{1 - 2\xi^2} \end{aligned}$$

So the above  $\omega_r$  is where  $G(j\omega)$  is maximum. To find  $M_r$  we plug-in is  $\omega_r$  in place of  $\omega$  in  $|G(j\omega)|$ .

**Reader:**

Show that

$$M_r = |G(j\omega_r)| = \frac{1}{2\xi\sqrt{1 - \xi^2}}$$

Reader answer:

From

$$|G(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}}$$

Replacing  $\omega$  in the above by  $\omega_r = \omega_n\sqrt{1 - 2\xi^2}$  and working out the algebra gives  $|G(j\omega = \omega_r)| = \frac{1}{2\xi\sqrt{1 - \xi^2}}$ . To verify, here is small Matlab code

```
syms wn w z positive
assume(z>0&z<1)
```

```
wr = wn*sqrt(1-2*z^2);
G_mag = wn^2/sqrt((wn^2-w^2)^2 + 4*z^2*wn^2*w^2)
G_mag = simplify(subs(G_mag, w , wr))
```

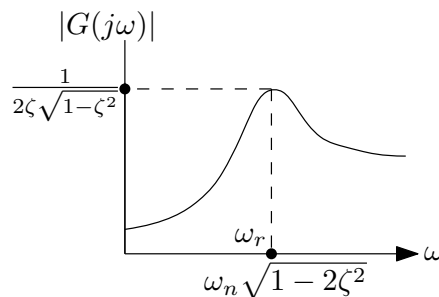
$$1/(2*z*(1 - z^2)^(1/2))$$

## 2.16 Lecture 15. Tuesday Oct 20, 2015, Feedback using user specified time specs

We have the standard second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The magnitude of the frequency response, for  $0 < \zeta < 1$  would look something like



As  $\zeta$  becomes smaller (system less damped) then at resonance, the amplitude of the steady state response will increase, since  $M_r = |G(j\omega_r)| = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Second order system typical concerns are

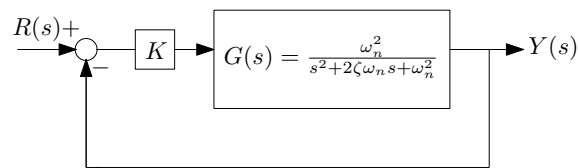
1. Overshoot
2. Resonance
3. rise time  $t_r$ , which is the time for the response to go (first time) from 10% to 90% of its steady state final value.
4. Settling time  $t_s$ , which is the time for the response to reach and *stay* within certain band/range around the final value. Typically within 2% of the final value (steady state).

What to do when these requirements (some or all) do not meet our requirements? We add a controller and use feedback. Today we will talk about adding a controller (pure gain  $k$ ) and feedback to address the problem of designing to meet the requirement of specification made on amount of overshoot while keeping the final steady state value within some acceptable limit.

Lets take the open loop  $G(s)$  and say we have  $\zeta = 0.01$ , then the overshoot

$$\begin{aligned} y_{\max} &= 1 + OS_{\max} \\ &= 1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \\ &= 1 + e^{\frac{-\pi(0.01)}{\sqrt{1-(0.01)^2}}} \\ &= 1.9691 \end{aligned}$$

This is almost two times the steady state final value which is one in this example. This is not good (it depends on the application as well). We will now start on **the overshoot design problem**. We add feedback and pure gain controller



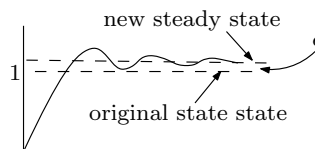
We will create the user specs. We want  $y_{\max}^{new} \leq \gamma$ . For example,  $\gamma = 1.2$ . Remember that when adding controller  $k$  and using feedback, the final value will no longer be one. Since with  $k$  present,  $y_{ss}^{new}$  will no longer be one (as mentioned above), so we now need another specification which says by how much the new steady state response can deviate from the original steady state response (which is one in this example). So we write

$$|y_{ss}^{new} - y_{ss}| \leq \epsilon$$

Or, since  $y_{ss} = 1$  (for this example only, since the input is step), then the above becomes

$$|y_{ss}^{new} - 1| \leq \epsilon$$

For example, if  $\epsilon = 0.1$ , then the above says that the new steady state response (final value) should remain within 10% of the steady state response before adding the feedback and the controller.



So now we have two specifications to meet by using feedback with the controller in place. We need to see if we can find  $k$  which meets both the above design requirements. Again, the design requirements are

1.  $y_{\max}^{new} \leq \gamma$
2.  $|y_{ss}^{new} - 1| \leq \epsilon$

Once we add the feedback and the controller, we obtain a closed loop transfer function

$$\begin{aligned}
 T(s) &= \frac{KG(s)}{1 + KG(s)} \\
 &= \frac{K \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}{1 + \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}} \\
 &= \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2(1 + K)} \tag{1}
 \end{aligned}$$

So the new  $T(s)$  has different natural frequency, given by (will call  $\bar{\omega}_n, \bar{\zeta}$  the new  $\omega_n$  and the new  $\zeta$ ).

$$\bar{\omega}_n^2 = \omega_n^2(1 + K)$$

or

$$\bar{\omega}_n = \omega_n \sqrt{1 + K} \tag{2}$$

Re-scaling (1) gives

$$T(s) = \frac{\frac{K\omega_n^2}{\bar{\omega}_n^2} \bar{\omega}_n^2}{s^2 + 2 \frac{\xi\omega_n}{\bar{\zeta}\bar{\omega}_n} \bar{\zeta}\bar{\omega}_n s + \bar{\omega}_n^2}$$

We want  $\frac{\xi\omega_n}{\bar{\zeta}\bar{\omega}_n} = 1$  to obtain the same form of the standard second order system. Hence, using (2)

$$\begin{aligned}
 \frac{\xi\omega_n}{\bar{\zeta}\bar{\omega}_n} &= 1 \\
 \frac{\xi\omega_n}{\bar{\zeta}(\omega_n\sqrt{1+K})} &= 1 \tag{2.1} \\
 \frac{\xi}{\bar{\zeta}\sqrt{1+K}} &= 1 \\
 \bar{\zeta} &= \frac{\xi}{\sqrt{1+K}} \tag{3}
 \end{aligned}$$

And we call  $\frac{K\omega_n^2}{\bar{\omega}_n^2}$  as  $\Delta$  which is scaling term. Hence

$$\Delta = \frac{K\omega_n^2}{\omega_n^2(1+K)} = \frac{K}{1+K} \tag{4}$$

Therefore,  $T(s)$  can now be written in standard second order system as

$$T(s) = \left( \frac{K}{1+K} \right) \frac{\bar{\omega}_n^2}{s^2 + 2\bar{\zeta}\bar{\omega}_n s + \bar{\omega}_n^2}$$

Where  $\bar{\zeta}$  is given by (3) and  $\bar{\omega}_n$  is given by (2). Summary is below before we go to next stage and design for  $K$

$$T(s) = \left( \frac{K}{1+K} \right) \frac{\bar{\omega}_n^2}{s^2 + 2\bar{\zeta}\bar{\omega}_n s + \bar{\omega}_n^2}$$

$$\bar{\zeta} = \frac{\xi}{\sqrt{1+K}}$$

$$\bar{\omega}_n = \omega_n \sqrt{1+K}$$

Now we find to find  $K$  which meets  $y_{\max}^{new} \leq \gamma$  and  $|y_{ss}^{new} - 1| \leq \varepsilon$  at the same time. We start  $y_{\max}^{new} \leq \gamma$ . Since we know that

$$y_{\max}^{new} = 1 + e^{\frac{-\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}}}$$

Then, since system is linear, then  $y_{\max}^{new}$  is just just the constant  $\left( \frac{K}{1+K} \right)$  times the above, or

$$y_{\max}^{new} = \left( \frac{K}{1+K} \right) \left( 1 + e^{\frac{-\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}}} \right)$$

So our requirement becomes

$$\frac{K}{1+K} \left( 1 + e^{\frac{-\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}}} \right) \leq \gamma$$

$$e^{\frac{-\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}}} \leq \gamma \frac{1+K}{K} - 1$$

$$\leq \frac{\gamma(1+K) - K}{K}$$

Taking natural logs gives

$$\frac{-\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}} \leq \ln \left( \frac{\gamma(1+K) - K}{K} \right)$$

By multiplying both sides by  $-1$ , this will change the inequality sign from  $\leq$  to  $\geq$  and the above becomes

$$\frac{\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}} \geq -\ln \left( \frac{\gamma(1+K) - K}{K} \right)$$

$$\frac{\pi\bar{\zeta}}{\sqrt{1-\bar{\zeta}^2}} \geq \ln \left( \frac{K}{\gamma(1+K) - K} \right)$$

Moving all terms to one sides gives

$$1 \geq \ln \left( \frac{K}{\gamma(1+K) - K} \right) \frac{\sqrt{1-\bar{\zeta}^2}}{\pi\bar{\zeta}}$$

Or, same as above

$$\ln \left( \frac{K}{\gamma(1+K) - K} \right) \frac{\sqrt{1-\bar{\zeta}^2}}{\pi\bar{\zeta}} \leq 1 \quad (5)$$

The above complete the specification for  $y_{\max}^{new} \leq \gamma$ . We now work on the second specification

$|y_{ss}^{new} - 1| \leq \varepsilon$ . Since the new system is scaled by  $\frac{K}{1+K}$ , then  $y_{ss}^{new} = \frac{K}{1+K}y_{ss}$  but  $y_{ss} = 1$ , then  $y_{ss}^{new} = \frac{K}{1+K}$ . Therefore, this requirement says

$$\left| \frac{K}{1+K} - 1 \right| \leq \varepsilon$$

Since  $\frac{K}{1+K} < 1$  then  $\left| \frac{K}{1+K} - 1 \right| = 1 - \frac{K}{1+K}$  and the above becomes

$$\begin{aligned} 1 - \frac{K}{1+K} &\leq \varepsilon \\ \frac{(1+K) - K}{1+K} &\leq \varepsilon \\ \frac{1}{1+K} &\leq \varepsilon \end{aligned}$$

Therefore we now have two specifications ready for design. They are

1. (A)  $F(K) = \ln\left(\frac{K}{\gamma(1+K)-K}\right) \frac{\sqrt{1-\bar{\zeta}^2}}{\pi\bar{\zeta}} \leq 1$  which comes from  $y_{\max}^{new} \leq \gamma$
2. (B)  $\frac{1}{1+K} \leq \varepsilon$  which comes from  $|y_{ss}^{new} - 1| \leq \varepsilon$

We now start the design for finding  $K$ . Suppose that user specification is that  $\varepsilon = 0.1$  and  $\gamma = 1.2$ . Assume also that  $\zeta = 0.1$ . We start with (B) above<sup>1</sup>.

$$\begin{aligned} \frac{1}{1+K} &\leq \varepsilon \\ \frac{1}{1+K} &\leq 0.1 \\ 1+K &\geq 10 \\ K &\geq 9 \end{aligned}$$

We now work on (A). Recall that  $\bar{\zeta} = \frac{\varepsilon}{\sqrt{1+K}} = \frac{0.1}{\sqrt{1+K}}$ , hence

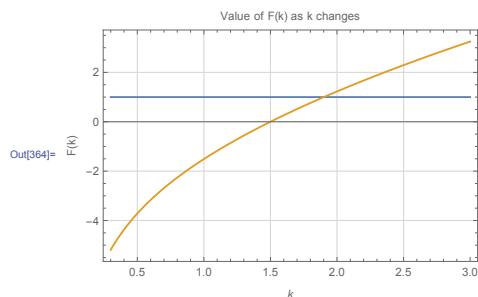
$$\begin{aligned} \ln\left(\frac{K}{\gamma(1+K)-K}\right) \frac{\sqrt{1-\bar{\zeta}^2}}{\pi\bar{\zeta}} &\leq 1 \\ \ln\left(\frac{K}{1.2(1+K)-K}\right) \frac{\sqrt{1-\frac{(0.1)^2}{(1+K)}}}{\pi\frac{0.1}{\sqrt{1+K}}} &\leq 1 \\ \ln\left(\frac{K}{0.2K+1.2}\right) \frac{1}{0.1\pi} \sqrt{(1+K)-(0.1)^2} &\leq 1 \\ \ln\left(\frac{K}{0.2K+1.2}\right) \frac{10}{\pi} \sqrt{K+0.99} &\leq 1 \end{aligned}$$

**Reader:** Plot  $F(K) = \frac{10}{\pi} \sqrt{K+0.99} \ln\left(\frac{K}{0.2K+1.2}\right)$

<sup>1</sup>Note that  $K < -1$  is also a solution, but we are looking for positive gain. Also  $K < -1$  do not work with the second constraint below.

Here is a plot of  $F(K)$  above

```
f[k_] := Log[k/(0.2 k + 1.2)] 10/Pi Sqrt[k + 0.99]
Plot[{1, f[k]}, {k, .3, 3}, Frame -> True, GridLines -> Automatic,
  GridLineStyle -> LightGray,
  FrameLabel -> {"F(k)", None}, {k, "Value of F(k) as k changes"}]
```



We see from the above, that for  $F(K) \leq 1$  the largest  $K$  is around  $K = 1.9$ .

```
Reduce[f[k] <= 1 && k > 0, k, Reals]
0 < k <= 1.90086
```

But our requirement for  $|y_{ss}^{new} - 1| \leq \varepsilon$  said we needed  $K \geq 9$ . This means we are not able to meet user specifications to find  $K$  which satisfies both A and B at the same time.

**Reader:** How much does the  $\gamma$  specs has to be relaxed so that we can find  $K$  with  $\varepsilon = 0.1$  kept the same as above?

**Reader:** How much does the  $\varepsilon$  specs has to be relaxed so that we can find  $K$  with  $\gamma = 1.2$  kept the same as above?

**Reader:** With  $\zeta = 0.1$ , find the region in the  $(\varepsilon, \gamma)$  space for which a spec meeting  $K$  exist.



## 2.17 Lecture 16. Thursday Oct 22, 2015, Routh stability

Today lecture is on stability and how to use Routh table to check for stability. We will use BIBO stability. BIBO stable system is one which has bounded output for all time, when the input is also bounded for all time. Analytically, a system can be determined if it is stable from the convolution definition of system response given by

$$y(t) = \int_0^t r(t-\tau)g(\tau) d\tau \quad (1)$$

Where  $r(t)$  is the input, and  $g(t)$  is the impulse response of the system. For bounded input, which means  $|r(t)| \leq B$  where  $B$  is some constant that do not depend on time, the output  $y(t)$  magnitude can be now found from (1) as follows

$$\begin{aligned} |y(t)| &= \left| \int_0^t r(t-\tau)g(\tau) d\tau \right| \\ &\leq \int_0^t |r(t-\tau)g(\tau)| d\tau \\ &= \int_0^t |r(t-\tau)||g(\tau)| d\tau \\ &\leq B \int_0^t |g(\tau)| d\tau \end{aligned}$$

Therefore, for  $|y(t)|$  to be bounded, which means  $|y(t)| \leq C$  where  $C$  is some constant, then we need  $\int_0^t |g(\tau)| d\tau \leq \infty$ . This means that a system is BIBO is  $\int_0^\infty |g(t)| dt \leq \infty$  where  $g(t)$  is the impulse response of the system. The following are examples how to use the above to determine BIBO stability.

**Example:** Given  $G(s) = \frac{1}{s-1}$  check if it is BIBO stable. The impulse response is  $g(t) = e^t$ . Hence  $\int_0^\infty |e^t| dt = \int_0^\infty e^t dt = e^t \Big|_0^\infty = \infty$  so this is not BIBO stable.

We can also see that this is not stable, since it has one pole in the RHS.

**Example:** Given  $G(s) = \frac{1}{s+1}$  check if it is BIBO stable. The impulse response is  $g(t) = e^{-t}$ . Hence  $\int_0^\infty |e^{-t}| dt = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_0^\infty = 1$  so this is BIBO stable.

We can also see that this is stable since it has no poles in the RHS.

Therefore, as long as a system has no poles in the RHS, then it is BIBO stable. One way to check if there are poles in the RHS and how many there are, without actually solving for the roots or without doing the above integration, is to use Routh-Hurwitz table. We will look at three cases. When the first column in the table has no zeros (classical case), and when the first column has a zero, and when a whole row in the table has zeros, and see how to handle each case. We will do this using three examples of each case.

**Example 1:** Given  $G(s) = \frac{N(s)}{D(s)}$  where  $D(s) = 3s^4 + 10s^3 + 5s^2 + 5s + 2$ . We set up Routh table as follows

$s^4$	3	5	2
$s^3$	10	5	
$s^2$	3.5	2	
$s^1$	$-\frac{5}{7}$	0	
$s^0$	2	0	

Looking at the first numerical column (i.e. second column in the table above), we see there are two sign changes. This means there are 2 poles in the RHS. Which also means this system is not BIBO stable.

**Reader:** Check the roots using Matlab using roots command and verify the above.

**Example 2:** Given  $G(s) = \frac{N(s)}{D(s)}$  where  $D(s) = 4s^4 + 10s^3 + 5s^2 + 12.5s + 5$ . We set up Routh table as follows

$s^4$	4	5	5
$s^3$	10	12.5	
$s^2$	0	5	

Since we have zero at the pivot, then we change it with  $\varepsilon$  and continue as follows

$s^4$	4	5	5
$s^3$	10	12.5	
$s^2$	$\varepsilon$	5	
$s^1$	$12.5 - \frac{50}{\varepsilon}$		
$s^0$	5		

We now take the limit as  $\varepsilon \rightarrow 0$  from above, and see that a sign change between the third and fourth row and then another sign change from the fourth to the fifth row. So this is not stable system.

**Example 3:** Given  $G(s) = \frac{N(s)}{D(s)}$  where  $D(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 20s^2 + 16s + 16$ . We set up Routh table as follows

$s^6$	1	8	20	16
$s^5$	2	12	16	
$s^4$	2	12	16	
$s^3$	0	0		

Since we have row of zeros. To handle this, we take the polynomial from the row above, which is  $A(s) = 2s^4 + 12s^2 + 16$  and take its derivative, giving  $A'(s) = 8s^3 + 24s$ , and use this to replace the row of zeros, so we end up with

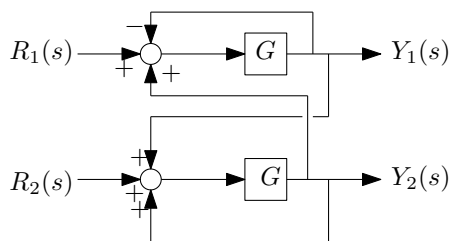
$s^6$	1	8	20	16
$s^5$	2	12	16	
$s^4$	2	12	16	
$s^3$	8	24		

And now we continue as before

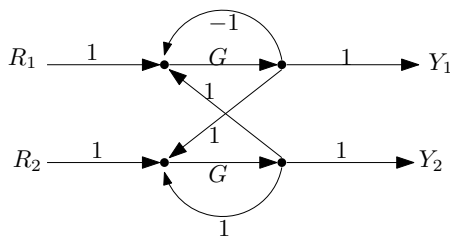
$s^6$	1	8	20	16
$s^5$	2	12	16	
$s^4$	2	12	16	
$s^3$	8	24		
$s^2$	6	16		
$s$	$\frac{8}{3}$			
$s^0$	16			

So there is no sign change, so this is stable.

**Another example.** Given this system



The above is called master/slave controller design. Where  $G = \frac{K}{(s+1)(s+2)}$  and we want to find if the transfer function from any input to any output is BIBO stable or not. We use Mason rule to obtain the denominator, which is Mason  $\Delta$ . The signal graph is



Mason delta is

$$\Delta = 1 - (-G + G + G^2) + (-GG) = 1 - 2G^2$$

Hence for  $G = \frac{K}{(s+1)(s+2)}$  then  $\Delta = s^4 + 6s^3 + 13s^2 + 12s + 4 - 2K^2$  and now we setup Routh table

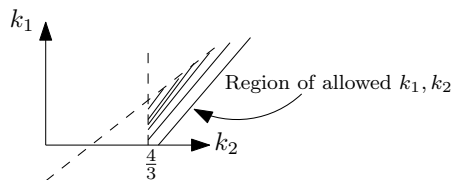
$s^4$	1	13	$4 - 2k^2$
$s^3$	6	12	
$s^2$	11	$4 - 2k^2$	
$s^1$	$\frac{108+12k^2}{11}$		
$s^0$	$4 - 2k^2$		

Therefore for no sign change we need  $\frac{108+12k^2}{11} > 0$  which is always true, and we want also  $4 - 2k^2 > 0$  or  $|k| < \sqrt{2}$  as the condition for stability.

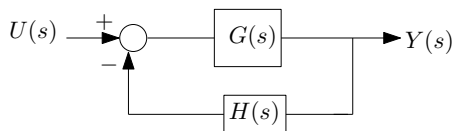
Final example. For  $G(s) = \frac{1}{s^4+3s^3+k_2s^2+4s+k_1}$  find conditions for stability.

$s^4$	1	$k_2$	$k_1$
$s^3$	3	4	
$s^2$	$\frac{3k_2-4}{3}$	$k_1$	
$s^1$	$\frac{4\left(\frac{3k_2-4}{3}\right)-3k_1}{\frac{3k_2-4}{3}}$		
$s^0$	$k_1$		

The condition is  $\frac{3k_2-3}{3} > 0$  or  $k_2 > \frac{4}{3}$  and  $k_1 > 0$ . We see the region of stability to be the following

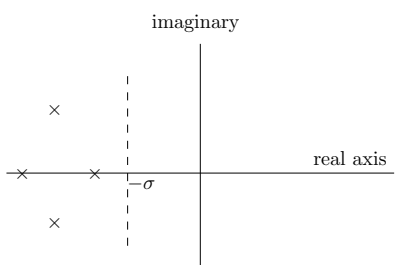


**Reader** For  $G(s) = \frac{1}{s^2(s+1)(s+2)}$  and  $H(s) = 2s + k$  find  $k$  for stable system in the following

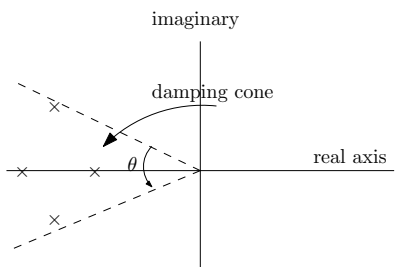


## 2.18 Lecture 17, Tuesday Oct 27, 2015, Starting on root locus

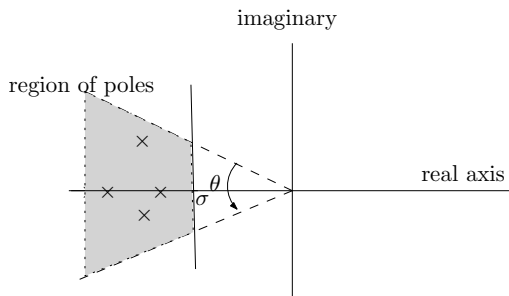
Suppose we want to know how does the controller affect pole locations? Routh table just tells us if the system is stable or not and how many poles are in the RHS, but it does not tell us how the poles behave as the gain  $K$  changes. Suppose our controller  $H(s)$  is a function of  $(K_1, K_2, \dots)$  where we the designer set the values of these  $K_i$  for example, selecting  $K_1$  for overshoot specification as we did before. Now we will talk about how to select  $K$  for other purpose, which is pole locations of the closed loop system. Closed loop pole location gives us many information about the system, and pole locations are informative about behavior of closed loop response. For example



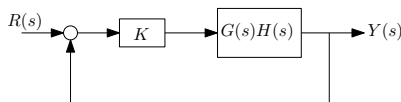
The above diagram tell us about the speed of the response or speed of decay.  $e^{-\sigma t}$  is indicator of speed to decay of response. If we have large  $\sigma$  then the system will go to its final value (steady state much faster). There is also what is called the damping cone. This is the cone where the poles lie inside.



The angle  $\theta$  above is informative about damping. Large damping implies small angle  $\theta$ . Suppose we have combination of needs: speed of decay of response and low damping, hence we have region where we want the poles be located.



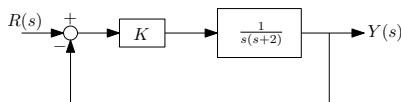
So many performance specs (but not all) can be found by location of poles. Suppose that the open loop pole locations are not desired, and we want to move them to desired region. We use feedback with controller  $K$  such that these locations are moved to the desired location in the closed loop poles. Root locus is such method to allow us to do this in systematic way (computer aided design) rather by trial and error. The classical setup for root locus design is this



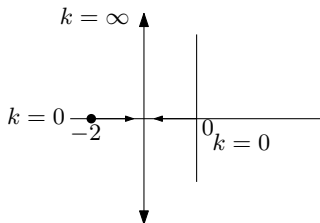
Suppose we designed controller  $H(s)$  and are not happy with the pole locations. i.e.  $G(s)H(s)$  are given. We need to select  $K \geq 0$  to move the poles to desired location. We view  $G(s)H(s)$  above as the open loop system.

Root locus is the locus of the closed loop in the complex plane obtained by changing  $K$  from 0 to  $\infty$ .

A common sense approach to use root locus. Example:

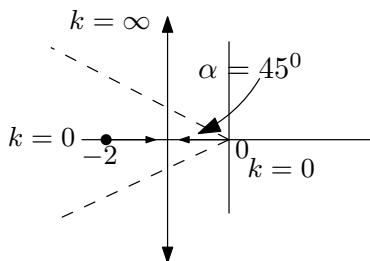


The open loop poles are at  $s = 0$  and  $s = -2$ . The closed loop is  $T(s) = \frac{K}{s(s+2)+k} = \frac{K}{s^2+2s+K}$  and the pole location are  $s = -1 \pm \sqrt{1-k}$ . Now we increase  $K$  from 0 and see how the locus of the poles change. We get this



So for large  $k$  the poles will move outside the cone of interest.

**Reader:** How large can  $K$  be to satisfy damping constraint of  $\alpha = 45^\circ$ ?



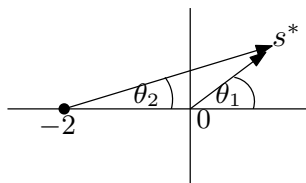
Formal root locus. Will develop 9 lemmas. Each lemma gives more information about the locus. We begin with  $G(s)H(s)$  which is the open loop. Write it as  $\frac{N(s)}{D(s)}$  where it is proper (degree of numerator ( $m$ )  $\leq$  degree of denominator ( $n$ )). Closed loop is  $T(s) = \frac{KGH}{1+KGH}$ . To find closed loop poles, we write

$$\begin{aligned} 1 + KGH &= 0 \\ 1 + K \frac{N(s)}{D(s)} &= 0 \\ D(s) + KN(s) &= 0 \end{aligned}$$

Observer that, for  $K \neq 0$ , the above have  $n$  poles.

**Lemma 1** root locus has  $n$  branches for  $K > 0$ .

Root locus (R.L.) geometry: Central idea. A point  $s$  is on R.L. if  $1 + KGH = 0$  for some  $K$  value.  $GH = -\frac{1}{K}$ . Hence the phase of  $GH$  is  $\pi$ . And the corresponding magnitude of  $K$  is  $\left| \frac{1}{GH} \right|$ . So to decide if point  $s$  is on the R.L. quickly, look at the angle. For example, for open loop poles  $s = -2$  and  $s = 0$ , and suppose to want to know if some point  $s^*$  is on the R.L., then we draw this



Then

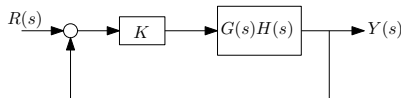
$$\angle GH = -\theta_1 - \theta_2 \neq \pi$$

Then  $s^*$  is not on the root locus path.

## 2.19 Lecture 18, Thursday Oct 29, 2015, More root locus

Reminder: No lecture Tuesday Nov 3. Makeup lecture on Wed. Nov 18 at 6 pm.

Will continue root locus. We are using the classical setup



We want to study the behavior of closed loop poles as  $K$  changes from zero to infinity. There is a geometric condition which is the driver behind root locus. It is the angle condition. This condition says that a point  $s^*$  is on R.L. iff phase of open loop  $G(s)H(s)$  evaluated at  $s^*$  is  $\pi$

$$\angle G(s)H(s)|_{s=s^*} = \pi$$

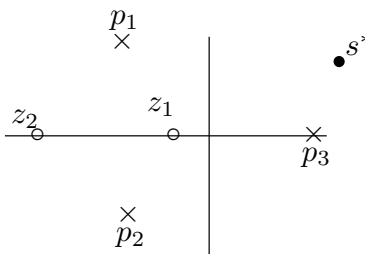
When the point  $s^*$  is on the R.L., then the corresponding gain is (notice, this is valid only after we decided the point  $s^*$  is on R.L.)

$$k = \frac{1}{|G(s)H(s)|_{s=s^*}}$$

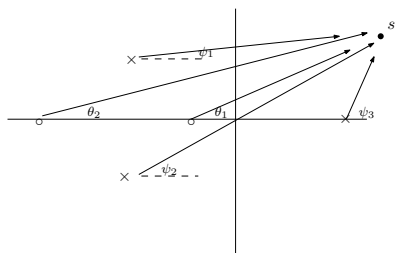
Let us now look at  $G(s)H(s)$  with two zeros and three poles.

$$G(s)H(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)}$$

We want to know if point  $s^*$  is on R.L.



How to know if  $s^*$  on R.L.? We check the angle  $\angle G(s)H(s)|_{s=s^*}$  and see if it  $\pi$ . We do this graphically, like this





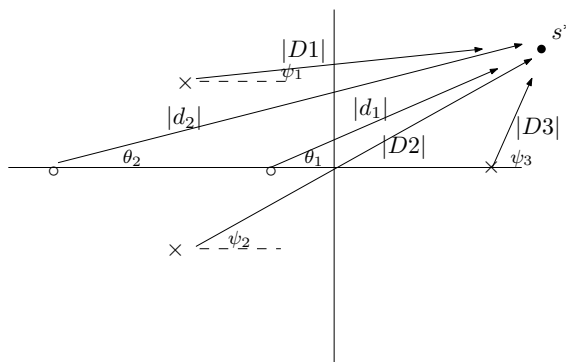
We look at the angle each pole and zero makes with the point  $s^*$ , then

$$\angle G(s)H(s)|_{s=s^*} = (\theta_1 + \theta_2) - (\psi_1 + \psi_2 + \psi_3)$$

i.e. we add the zero angles, and subtract the poles angles. Angle is always measured counter clock wise from the horizontal line as shown. If the above adds to  $\pi$  then the point  $s^*$  is on R.L. The gain  $K$  in this case is

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{\frac{|(s-z_1)(s-z_2)|}{|(s-p_1)(s-p_2)(s-p_3)|}} = \frac{|(s-p_1)(s-p_2)(s-p_3)|}{|(s-z_1)(s-z_2)|} = \frac{D_1 D_2 D_3}{d_1 d_2}$$

Where  $D_i$  is the size of the vector from each pole  $p_i$  to  $s^*$  and  $d$  is the length of the vector from each zero  $z_i$  to  $s^*$



Back to the 9 lemmas. When used together, enable us to draw R.L. for any system.

**Lemma 1:** R.L. has  $n$  branches, where  $n$  is number of open loop  $GH$  poles.

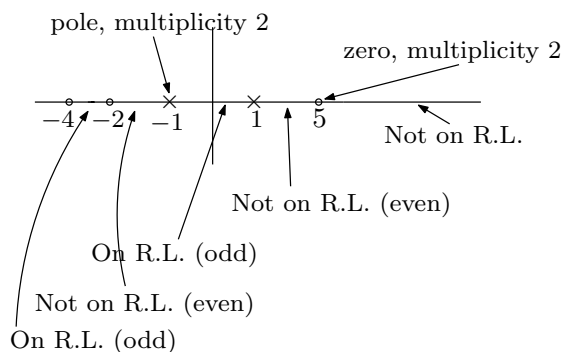
**Lemma 2:** We talk about very small  $K$  and very large  $K$ . When  $K$  is very small, R.L. is at the open loop poles. This means R.L. always starts from the open loop pole locations. When  $K$  is very large, R.L. is at open loop zeros. For example  $GH = \frac{(s+1)(s-2)}{(s+4)^5}$ , the open loop has poles at  $s = -4$ , this is where R.L. starts from. Since it has zero at  $s = -1$  and zero at  $s = 2$ , then this is where 2 of the branches will end up at. This also means for large gain  $K$  the closed loop is not stable, since it ends up at  $s = 2$ .

Since  $n = 5$  in the above, so we have 5 branches. Two of these end up at  $s = -1$  and  $s = 2$ . What about the other three? Those will end up infinity. And we need to check if there will be stable or not.

**Lemma 3** (called the number criterion) Which real axis points are on the R.L. ? Since complex poles lead to angle cancellations when point  $s^*$  being checked for is on the real axis, then only poles and zeros of the open loop decide if a point  $s^*$  is on R.L. or not. A real axis point  $s^*$  is on R.L. iff the number of real axis poles and zeros (taking multiplicity into account) found as we travel towards  $s^*$  from the right side is odd.

Example:  $GH = \frac{(s+2)(s-5)^2(s^2+8s+20)}{(s+1)^2(s^3-1)(s^2+s+1)}$ . R.L. has 7 branches since  $n = 7$  from lemma 1. From

lemma 2, we know R.L. starts from open loop poles and end up at open loop zeros. Now we find which part of the real axis are on R.L. We use lemma 3. We just need to mark the open loop pole and zeros that are on the real axis for this. All complex poles and zero have no effect. Hence the diagram is



This system is unstable at high gain  $K$  and also unstable to low gain  $K$ . There are 2 branches that do not end up at zeros since  $n = 7$  and  $m = 5$ . Next we need to check what happens when  $k \rightarrow \infty$  after counting for the open loop zeros, since we know what happens in this case. How does phase behave when  $k \rightarrow \infty$ ? When  $K$  is very large, we can approximate  $GH = \frac{N(s)}{D(s)} \approx \frac{R_1 S^m}{R_2 S^n} = \frac{R_1 e^{jm\theta}}{R_1 e^{jn\theta}} = A e^{j(m-n)\theta}$ . Therefore we need  $(m-n)\theta = \pi$  for a point to be on R.L. Hence

$$\theta = \frac{\pi}{m-n} + M(2\pi)$$

From some integer  $M$ . Lemma 4 will be discussed more next time.

## 2.20 Lecture 19, Nov. 3, 2015. No lecture

## 2.21 Lecture 20, Nov. 5, 2015. Roots Locus completed

Reminder: Exam 2 on Nov 12 Thursday. Learn relation between damping ratio  $\zeta$  and damping cone. Exam will cover up to root locus. Will finish root locus today. Quick reminder of lemmas covered so far

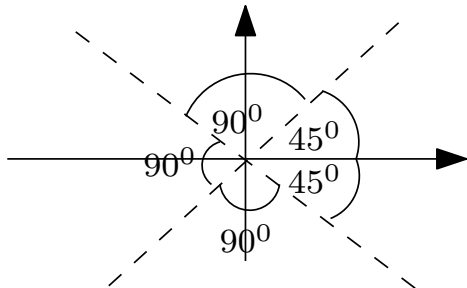
1. Mark open pole loops (where R.L. starts) and open loop zeros.
2. Tells us that R.L. starts at open loop poles and ends up with open loop zeros. Some branches left will end up at  $\pm\infty$
3. Tells us what happens on the real axis. The segments that R.L. will travel over on the real axis. Called the "odd" lemma
4. Tell us how to generate the asymptotes of R.L. using  $\theta = \frac{180^\circ \pm k360^\circ}{n-m}$ . Example, for  $n = 6, m = 3$ , (where  $n, m$  are number of open loop poles and zeros respectively) then  $\theta = 60^\circ \pm k120^\circ$  for  $k = 0, 1, \dots$

**Reader:** With  $n - m > 2$  argue that closed loop is unstable. (answer: with  $n - m > 2$ , an asymptote will be moving to the RHS. Hence one of the branches will eventually move to the RHS for large gain, which means unstable).

**Lemma 5:** This lemma for finding from which point on the real axis the asymptotes will start. The centroid is given by

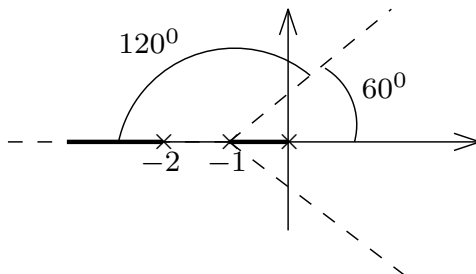
$$\sigma_c = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m}$$

The above is for poles and zeros of the open loop, not the closed loop!. For example, for  $GH = \frac{(s+2)^2}{(s+1)(s+6)(s+8)^4}$  then  $\sigma_c = -8.75$  and the angles at  $\theta = \frac{180^\circ \pm k360^\circ}{4} = 45^\circ \pm 90^\circ$ . So the asymptotes are



Example:  $GH = \frac{1}{s(s+1)(s+2)}$ . Lemma 1:  $n = 3, m = 0$ . Lemma 2: Start from open loop poles. Lemma 3: Real axis, use the odd criteria. Lemma 4: Find the asymptotes and the centroid.

$\sigma_c = \frac{0-1-2}{3} = -1$ .  $\theta = \frac{180 \pm k360}{3} = 60^\circ \pm k120$ . The result is this root locus



We still need to find where the break away points are and where root locus intersects the imaginary axis at.

**Reader:** Without using root locus, find the gain  $K$  what makes the closed loop unstable. (answer: use Routh table).

**Lemma 6:** This lemma tells us where the break points are on the real axis. Solve  $\frac{dK}{ds} = 0$ . Note not all solution points will be valid. Using the above example  $GH = \frac{1}{s(s+1)(s+2)}$ , then the characteristic polynomial is

$$\begin{aligned} K + GH &= 0 \\ (s+1)(s+2) + K &= 0 \\ K &= -s^3 - 3s^2 - 2s \\ \frac{dK}{ds} &= -3s^2 - 6s - 2 \end{aligned}$$

Hence we solve  $-3s^2 - 6s - 2 = 0$ , and find  $s = -1.577$  and  $s = -0.423$ . But  $s = -1.577$  is not on root locus (from above) so only  $s = -0.423$  is on root locus, and that is the breakaway point.

**Lemma 7** Departure angle. R.L. depart each pole. We want to find the departure angles. Use

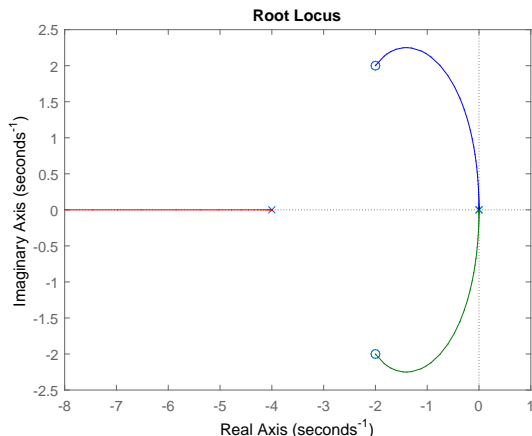
$$\sum \angle z_i - \sum \angle p_i = 180^\circ \pm k360^\circ \quad k = 0, 1, \dots$$

Where  $\sum \angle z_i$  is sum of all angles from all zeros to the pole in question (the one we want to find the departure angle from) and  $\sum \angle p_i$  is the sum of all the angles from each pole to the pole in question. In the above, the left side will contain only one unknown, which is  $\theta$ , the angle of departure of that one pole. Do the above for each pole at a time.

**Lemma 8** Apply the same method as lemma 7, but now do it for each zero at a time to find the arrival angles at each zero.

**Lemma 9** Find where root locus crosses the imaginary axis. Use Routh table for this.

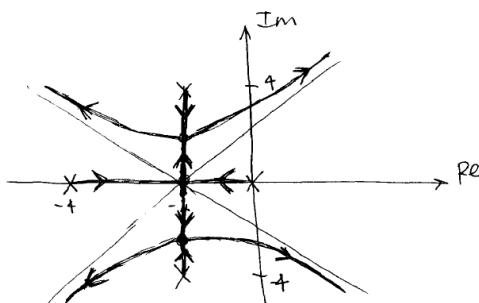
Example:  $GH = \frac{s^2+4s+8}{s^2(s+4)}$  **Reader:** reproduce the solution below



**Reader:** Do this problem sent to us by email also. Find root locus for  $G = \frac{K}{s(s+4)(s^2+4s+20)}$

EXAMPLE 2 Find RL for unity feedback system with  $G(s)$ :  

$$G(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$



See my HW6 for more detailed root locus steps.

### 2.21.1 Summary of root locus

1. Mark on plot all open loop poles and open zero locations.
2. R.L. starts from open loop poles and ends up at open loop zeros. Some will end up at  $\pm\infty$  when  $n - m > 0$  (which is almost always the case)
3. Mark on real line where R.L. exist. Use the odd criteria.

4. Find centroid of asymptotes  $\sigma = \frac{\sum p_i - \sum z_i}{n-m}$ . i.e. sum of all poles minus sum of all zeros.  $n$  is number of poles, and  $m$  is number of zeros (these are for the open loop, not the closed loop!)

5. Find asymptotes angles.  $\theta = \frac{180^\circ \pm 360^\circ}{n-m}$

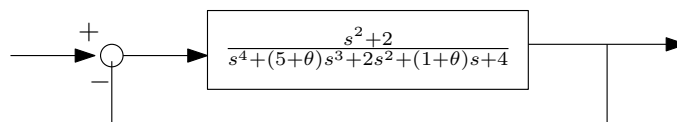
6. Find breakaway points. From  $1 + KG = 0$  find  $k = f(s)$  and solve  $\frac{dk}{ds} = 0$  for  $s$ . These are locations where the breakaway and break-in points on root locus will be.
7. Find departure angles for each pole (the complex ones, the ones on real line will have  $180^\circ$ ) and arrival angles for the zeros (also complex ones). This can be done using  $\sum \angle z_i - \sum \angle p_i = 180^\circ \pm 360^\circ k$ , where in the LHS, we have one unknown angle each time. This is the angle of either the departure or arrival. Then solve for it. The above is sum of angles that all other poles and zeros make with the point in question. See HW 6 for details.
8. Find where root locus crosses the imaginary line. Find  $K$  which makes the closed loop unstable. Then solve for the polynomial above that row in Routh table for this  $K$  and solve for  $s$ . This will be where it crosses the imaginary line. See HW6 for details.

## 2.22 Lecture 21, Nov 10 2015, Extension Root Locus

Review of what will be on midterm on Thursday

1. Not cumulative
2. Mason rule
3. Benefits of feedback. Nonlinear block. Can't use Mason. We solve graphically. Ask what is the new relation, given the old relation. Dead zone is lessened. Saturation moved further away. We can make the system closer to linear. Second benefit is sensitivity.  $S_\alpha^T = \frac{\alpha}{T} \frac{dT}{d\alpha}$ . Sensitivity is function of frequency, defined as % change in  $T$  (transfer function) per unit % change in parameter  $\alpha$ . Evaluate at nominal  $\alpha$ , then switch to  $s = j\omega$  and look at magnitude of  $S_\alpha^T$ .
4. Disturbance rejection. Avoid inversion, use low pass filter.
5. Second order system.  $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ . Benefits of second order system: Well understood. Can model many systems as second order. We talked about overshoot  $e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ . Rise time, resonance frequency, and location of resonance. Talked about damping cone.
6. Stability: Using Routh table. If polynomial has one sign change in its coefficient, then it is unstable right away, no need to even use Routh table. How to handle one zero in first column. How to handle a row of zeros. We can use Routh table to find  $K$  which makes the system unstable. Also we talked about shifted polynomial. One where we want all poles below some  $p$ .
7. Last topic was root locus. Ability to generate simple R.L. If  $n - m > 2$  then system is unstable at large gain even if all poles are in the LHS.

Today will start with extended root locus. If the open loop transfer function has a parameter, say  $\theta$  and we are not confident of its value, we want to know what happens as  $\theta$  increases. But we can not use standard root locus, as  $\theta$  is inside the  $GH$  itself and not a multiplier like  $K$  before. We need to convert it the transfer function to be in the form  $1 + \theta\tilde{G}$  where  $\tilde{G}$  is derived from original open loop  $G$  and is called the Fictitious system model. Here is an example. Given this original system



Warning, we can not use the original root locus 9 lemma's on the above as it stand. We have to convert it to Fictitious system model first as follows. The closed loop poles are obtained from

$$T = \frac{G}{1+G}$$

Hence set the denominator to zero to find the closed loop poles

$$1 + \frac{s^2 + 2}{s^4 + (5 + \theta)s^3 + 2s^2 + (1 + \theta)s + 4 + (s^2 + 2)} = 0$$

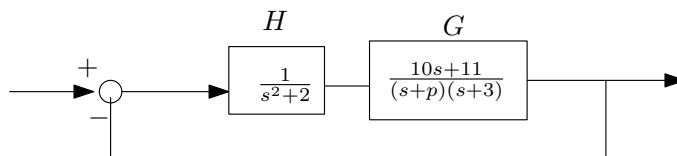
Which means

$$s^4 + (5 + \theta)s^3 + 2s^2 + (1 + \theta)s + 4 + (s^2 + 2) + s^2 + 2 = 0$$

Now factor out  $\theta$  which becomes

$$1 + \theta \frac{s^3 + s}{s^4 + 5s^3 + 3s^2 + s + 6} = 0$$

The above is now in the form of  $1 + K\tilde{G}$  as before, but now  $\tilde{G} = \frac{s^3 + s}{s^4 + 5s^3 + 3s^2 + s + 6}$ . Now we can apply root locus on the above. Example. Consider the following system with uncertain pole at  $-p$



We want root locus with respect to  $p$ . Covert to Fictitious system model as above.

$$1 + GH = 0$$

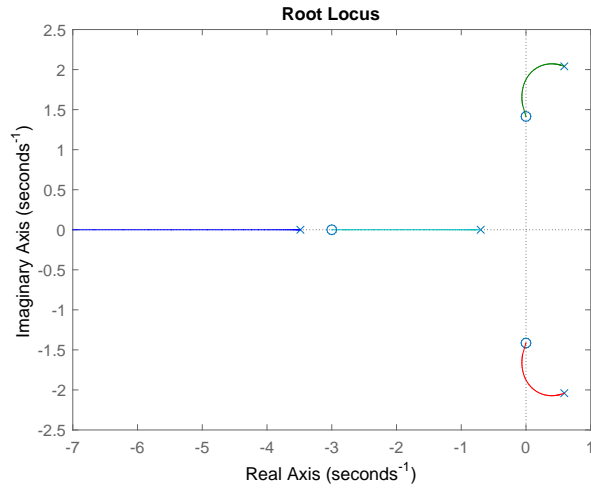
$$1 + \frac{1}{s^2 + 2} \frac{10s + 11}{(s + p)(s + 3)} = 0$$

$$1 + p \frac{s^3 + 3s^2 + 2s + 6}{s^4 + 3s^3 + 2s^2 + 16s + 11} = 0$$

Where  $\tilde{G} = \frac{s^3 + 3s^2 + 2s + 6}{s^4 + 3s^3 + 2s^2 + 16s + 11}$ . In this form, we can now apply root locus. Using Matlab we should get this

```
>> s=tf('s');
>> sys=(s^3+3*s^2+2*s+6)/(s^4+3*s^3+2*s^2+16*s+11);
>> rlocus(sys)
```





## **2.23 Lecture 22, Nov 12 2015, second exam**

Exam 2

## 2.24 Lecture 23, Nov 17 2015, Started Nyquist

There are two more HWs for the course. The final Exam will emphasis Nyquist and Bode. The rest of the course will cover Nyquist and Bode. Nyquist is considered the hardest part of the course. For motivation, we are moving to frequency domain now. So far we have not worked much in frequency domain. Now it will be the main emphasis. We will develop tools in the frequency domain. Some of the tools we did was Routh table for stability and second order system analysis. We did cover some frequency domain when we did noise attenuation and using low pass filter.

Nyquist method is a view of stability in the frequency domain. Why we care about this. We already have stability tools? Such as Routh table and Root locus also does tell us something about stability. Because Nyquist tells us more. Nyquist gives new information about stability and shows new ways a system can become unstable.

We have Routh table to check for stability. But Routh table assumes the model is perfect. But what if the model is not certain? What is there is amplifier that has frequency drift? This is called model imperfection.

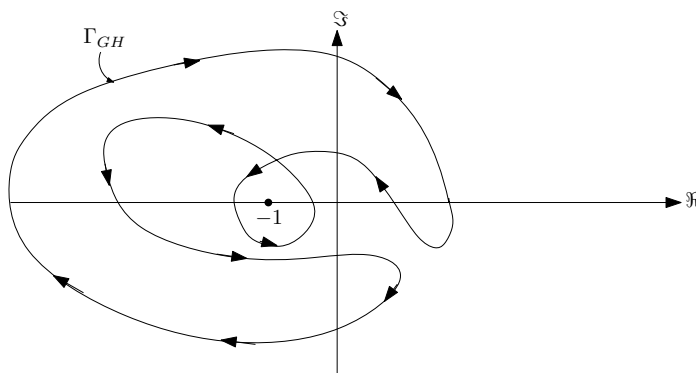
The main measures of stability are gain and phase margins. This tells us how tolerant the model to imperfections. Nyquist is a graphical description of stability, while Routh table is algebraic. Engineers use Nyquist since it is considered CAS based method.

### 2.24.1 What is the Nyquist criteria?

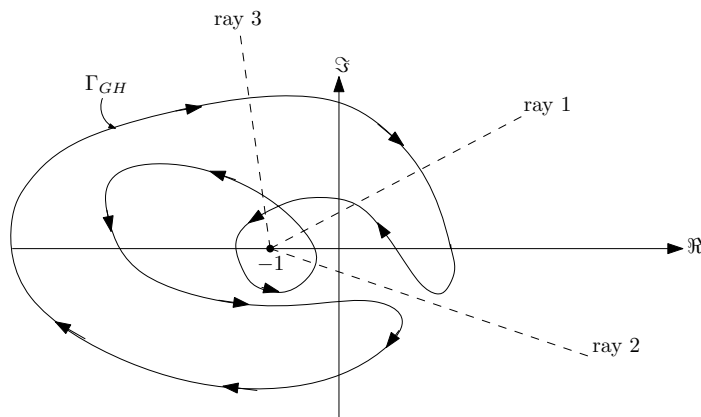
We will generate contour in complex plane, look at the contour and say right away if the system is stable or not. We will generate  $\Gamma_{GH}$  contour.  $\Gamma_{GH}$  is directed and closed curve. We use  $GH$  which is the open loop transfer function to generate  $\Gamma_{GH}$ . Then by looking at  $\Gamma_{GH}$  we will say if the closed loop is stable or not.

### 2.24.2 Preliminary example

Suppose we have generated  $\Gamma_{GH}$  that looks like the following (we will later learn how to generate  $\Gamma_{GH}$ ).



Now we apply the Nyquist criteria: The closed loop transfer function (we assume we have unity feedback), is stable, iff the number of net clockwise encirclements of the critical point  $-1$  on the real axis is equal to the number of unstable RHP poles of the open loop transfer function (strictly in the RHP). So we need to learn how to count encirclements. For this, we draw straight line from the point  $-1$  on the real line, outwards. It does not matter which direction we draw the line. Any one will end up giving the same result. This is due to how we will do the counting of the encirclements around  $-1$ . To show this, the above graph is redrawn below with 3 lines on it. (but we only need one, any one will do).



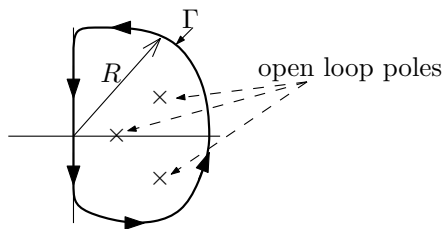
In the above we have randomly drawn 3 lines (or rays). Now we count the number of times the straight line cross the  $\Gamma_{GH}$  graph. Each time it crosses  $\Gamma_{GH}$  we count the direction of  $\Gamma_{GH}$  at that point, if it is clock wise, or anticlock wise. We add one when it is clock wise, and subtract one when it is counter clock wise. For example, for ray 1 in the above diagram, we see that there are 2 counter clock wise crossings, and one clock wise crossing. There for the final result is one counter clock wise crossing. This is the same as saying the result is  $-1$  clock wise crossing. So think of counter clock wise as negative numbers, and clock wise as positive numbers and simply add them. If the final number that results from the above, is the same as the number open open loop poles in the RHP, then the system is stable.

**Reader:** The other rays (lines 2 and 3) will give the same final result of one counter clock net encirclements. Show this.

Example, suppose we have  $GH$  which has two unstable poles in the RHP such as one with denominator  $(s - 1)(s^2 + s + 2)(s - 6)$ , then when we draw  $\Gamma_{GH}$  and count the encirclements, and find that the net number of clock wise encirclements is also 2, then we know the closed loop is stable.

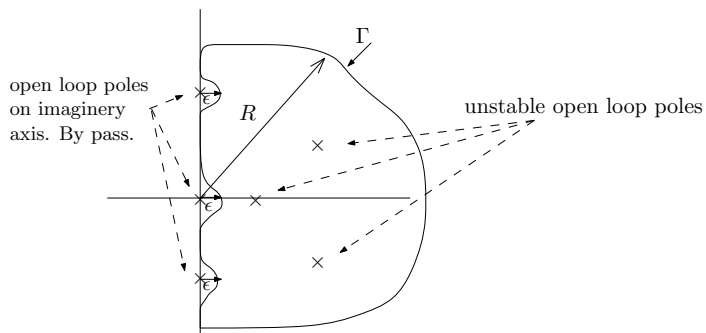
Now that we know how to interpret  $\Gamma_{GH}$  and how to use it to find if the closed loop is stable or not, we need to learn how to generate  $\Gamma_{GH}$ . The plan is to become an expert in generating  $\Gamma_{GH}$  and also learn how to use it to obtain other stability information from it. To generate  $\Gamma_{GH}$ , we first generate  $\Gamma$ , which is called the Nyquist contour. Then we map  $\Gamma$  to  $\Gamma_{GH}$ . But first we need to generate  $\Gamma$ . The first step is to mark all the open loop poles on the complex plane (we do not need to mark the open loop zeros, but we can do that as well if we want). Then we draw a counter clock wise curve around that incloses all these poles in the RHP. If

there are any open loop poles on the imaginary axis, we draw small circle around them to bypass them. Here are two examples



Example of  $\Gamma$  for open loop that has no poles on the imaginary axis. Let  $R \rightarrow \infty$

Now we show an example of  $\Gamma$  for the case when the open loop transfer function  $GH$  has poles on the imaginary axis.



Example of  $\Gamma$  for open loop that has additional three poles on the imaginary axis. Let  $R \rightarrow \infty$  and let  $\epsilon \rightarrow 0$

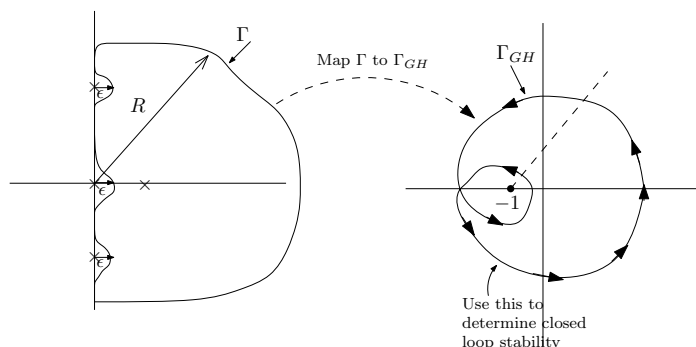
## 2.25 Lecture 24, Nov 19 2015, More Nyquist

The plane for today is more Nyquist, as well as next lecture. Then a HW set on Nyquist. Nyquist is most difficult subject in this course. When practising, use Matlab to verify the result.

Quick summary of Nyquist: We are interested in frequency based stability. This will take us to frequency analysis and Bode plots next.

Now that we learned how to draw  $\Gamma$ , the next step is to learn how to map  $\Gamma$  to  $\Gamma_{GH}$  since it is  $\Gamma_{GH}$  that we will use to count encirclements to determine if the closed loop is stable or not.

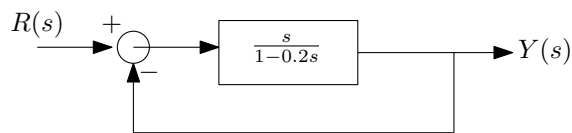
How to map  $\Gamma$  to  $\Gamma_{GH}$ ? We take each point on  $\Gamma$  and map it to new curve, which will be  $\Gamma_{GH}$ .



Nyquist says that the closed loop is stable iff  $\Gamma_{GH}$  net clock wise encirclement of  $-1$  is the same number as the open loop poles that are strictly in the RHS. In the above example, there is one open loop in the RHS, but we see that  $\Gamma_{GH}$  encircles  $-1$  two times in clockwise. Hence this shows that the closed loop is not stable.

### 2.25.1 Mechanism of $\Gamma_{GH}$ generation

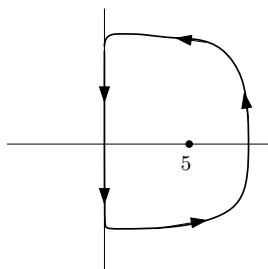
First we start with a simple example. Given this system  $G = \frac{s}{1-0.2s}$ , where the open loop is unstable, and has one pole in the RHS which is  $s = 5$ .



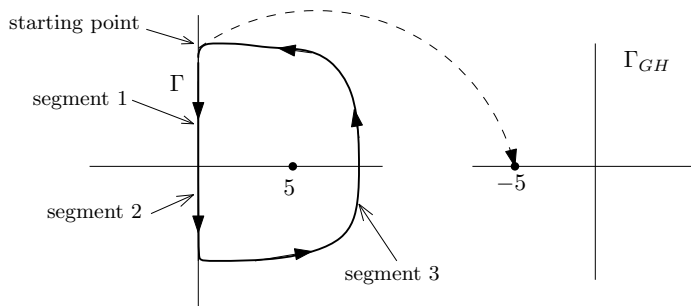
The closed loop TF is

$$\begin{aligned}\frac{Y}{R} &= \frac{G}{1+G} \\ &= \frac{s}{(1-0.2s)+s} \\ &= \frac{s}{1+0.8s}\end{aligned}$$

So the closed loop is stable, since it has no poles in the RHS. So we expect that  $\Gamma_{GH}$  to encircle  $-1$  one time only, in clockwise, since that is the number of open loop poles in RHS. So we start by drawing  $\Gamma$ . And we start to map  $\Gamma$  to  $\Gamma_{GH}$ .



We start always from  $s = j\infty$  And go down on  $j\omega$  axis of the  $\Gamma$  path. We call this branch 1. When  $s = j\infty$  then  $\frac{Y}{R} \approx \frac{s}{-0.2s} \approx -5$ , this means that on branch 1 on  $\Gamma$ , the starting point will map to  $s = -5$  on the  $\Gamma_{GH}$  path. So we have this diagram now

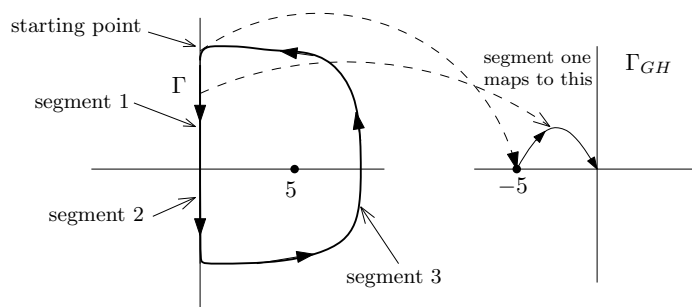


Now we move down the  $+j\omega$  axes and map segment 1 to  $\Gamma_{GH}$ . At segment 1, where  $s = j\omega$ , then  $G = \frac{s}{1-0.2s} = \frac{j\omega}{1-0.2(j\omega)}$ . Now we need to see the real part and imaginary part to be able to see the mapping to  $\Gamma_{GH}$ . Hence we write

$$\begin{aligned}
 G &= \frac{j\omega}{1 - 0.2(j\omega)} \\
 &= \frac{j\omega}{1 - 0.2(j\omega)} \frac{1 + 0.2(j\omega)}{1 + 0.2(j\omega)} \\
 &= \frac{j\omega - 0.2\omega^2}{1 + 0.04\omega}
 \end{aligned}$$

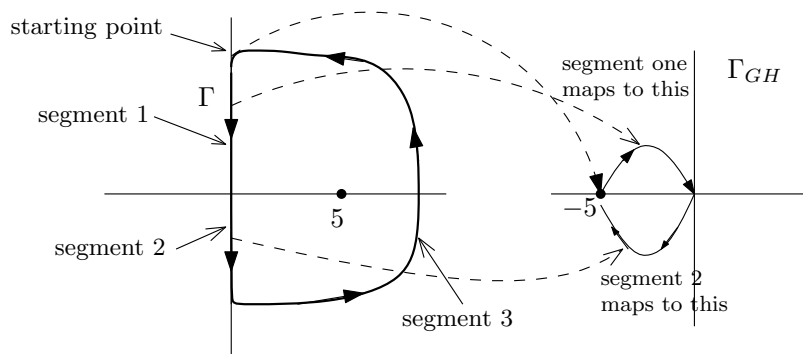
Therefore the real part is  $\frac{-0.2\omega^2}{1+0.04\omega}$  and the imaginary part is  $\frac{\omega}{1+0.04\omega}$ . So starting from large positive  $\omega$  going down the segment one, we see that the real part is negative and the imaginary part is negative. This means  $\Gamma_{GH}$  will be somewhere in the second quadrant. We do not care about the shape of  $\Gamma_{GH}$  that results from mapping segment one. We just know so far it starts from  $-5$  and remains in the second quadrant. What is important in Nyquist, is where  $\Gamma_{GH}$  crosses the imaginary and real axis, and now its shape in between. To get the crossing with the real axis, we set the imaginary part of  $GH$  to zero. And to get the imaginary axis crossings, we set the real part of  $GH$  to zero.

To find where  $\Gamma_{GH}$  crosses the real axis, then from  $\frac{-0.2\omega^2}{1+0.04\omega} = 0$  we find  $\omega = 0$ . And to find where  $\Gamma_{GH}$  crosses the imaginary axis, then from  $\frac{\omega}{1+0.04\omega} = 0$  we also get  $\omega = 0$ . So there is only one point where  $\Gamma_{GH}$  crosses the axes, which is the origin. At  $\omega = 0$ , we see that  $G = 0$ , therefore the full segment one now maps to  $\Gamma_{GH}$  as in the following diagram

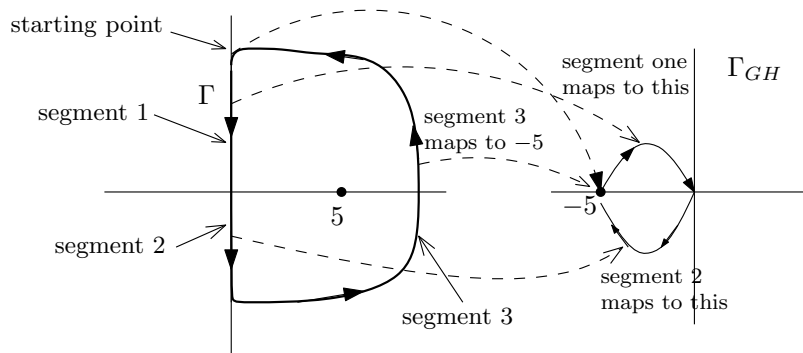


Now we go to segment 2. Along this segment (we are still moving down the imaginary axis), along this, the real part of  $GH$  is negative, and the imaginary part is also negative. We the mapping is now in the third quadrant of  $\Gamma_{GH}$ . We also know this will be the case due to symmetry about the real axis. So now  $\Gamma_{GH}$  will look as the following





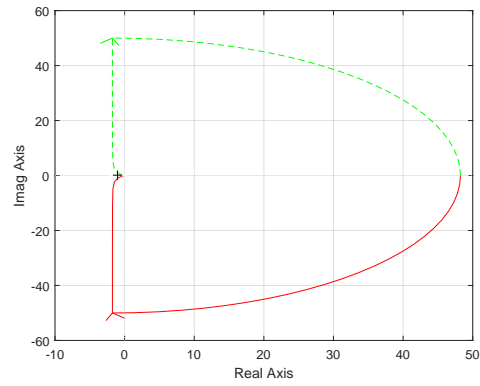
Now we just need to do segment three. For this segment  $s = Re^{j\theta}$  with  $R$  becoming very large and  $\theta$  going from  $-90^\circ$  to  $0^\circ$  and then to  $+90^\circ$  in the  $\Gamma$  path. But since  $G = \frac{s}{1-0.2s}$  then  $G = \frac{Re^{j\theta}}{1-0.2Re^{j\theta}} \approx \frac{Re^{j\theta}}{-0.2Re^{j\theta}}$  for large  $R$ . Hence  $G \approx -5$  as well for segment three. This means all of segment three in  $\Gamma$ , maps to the one point  $s = -5$  in  $\Gamma_{GH}$ . So the  $\Gamma_{GH}$  will not change. Here is the final result



Now that we have build  $\Gamma_{GH}$ , we can find the number of net encirclements around  $-1$ . We see that there is one clock wise encirclement. But this is the same as the number of unstable open loop poles. Therefore the closed loop is stable (as we knew before). But this shows how to Nyquist to find out. We now do a second example. Let  $GH = \frac{5(1-0.5s)}{s(1+0.1s)(1-0.25s)}$ .

**Reader:** Is the closed loop stable? Use Routh table to find out. The closed loop denominator is  $P(s) = 0.025s^3 + 0.15s^2 + 0.15s - 5 = 0$ . This is not stable closed loop. Use Nyquist to confirm. Here is the result using Matlab

```
s=tf('s');
sys=5*(1-0.5*s)/(s*(1+0.1*s)*(1-0.25*s));
close all;
nyquist1([2.5 -5],[0.025 0.15 -1 0]) %use http://ctms.engin.umich.edu/CTMS/Content/Intro
```



## 2.25.2 Nyquist Handouts

### 2.25.2.1 Nyquist Handout 1

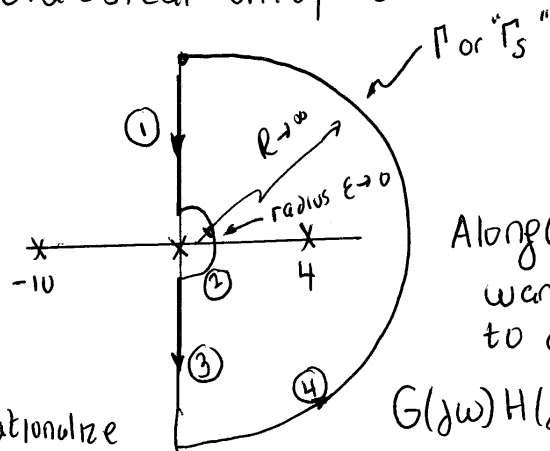
The following is Handout 1 Nyquist.

## Nyquist Example (From Class)

We consider

$$G(s)H(s) = \frac{5(1-0.5s)}{s(1+0.1s)(1-0.25s)}$$

in classical unity feedback configuration.



At  $s = j\omega$

$$GH \sim \frac{-2.5s}{-0.25s^3} \sim 0$$

Along ①,

want  $\text{Re } G(s)$  and  $\text{Im } G(s)$   
to determine quadrant

$$G(j\omega)H(j\omega) = \frac{5(1-0.5j\omega)}{j\omega(1+0.1j\omega)}$$

rationalize  
with  
complex conjugate

$$= \frac{-j\omega(1-0.1j\omega)(1+0.25j\omega) * 5(1-0.5j\omega)(1-0.25j\omega)}{\omega^2(1+0.01\omega^2)(1+0.1625\omega^2)}$$

$$\underbrace{\omega^2(1+0.01\omega^2)(1+0.1625\omega^2)}_{\text{denom } \neq 0 \text{ for } \omega \neq 0}$$

algebra

$$= \frac{-\omega^2 \left( \frac{\omega^2}{16} + \frac{7}{4} \right) + j\omega \left( \frac{\omega^2}{2} + 5 \right)}{\text{denom}}$$

Hence

$$\operatorname{Re} G(j\omega)H(j\omega) = \frac{-\omega^2(\frac{\omega^2}{16} + 1.75)}{\text{denom}}$$

$$\operatorname{Im} G(j\omega)H(j\omega) = \frac{-\omega(\frac{\omega^2}{2} + 5)}{\text{denom}}$$

Hence, along ①,  $\operatorname{Re} < 0$  and  $\operatorname{Im} < 0$  means  $\Gamma_{GH}$  is in ③ III (third quadrant)

Note that there are no axis crossings since  $\operatorname{Re} \neq 0$  and  $\operatorname{Im} \neq 0$ .

Along ②,  $s = \varepsilon e^{j\phi}$  with  $\phi$  going from  $90^\circ$  to  $-90^\circ$ . Now  $GH \sim \frac{5}{\varepsilon e^{j\phi}}$

So  $\Gamma_{GH}$  is an arc at infinity going from  $-90^\circ$  to  $+90^\circ$

Along ③,  $\Gamma_{GH}$  is the conjugate of ①. So we reflect w.r.t. real axis and see  $\Gamma_{GH}$  in ② II with no axial crossings.

②

Along ④,  $s = Re^{j\phi}$  with  $R \rightarrow \infty$  and  $\phi$  going from  $-90^\circ$  to  $90^\circ$ . Now

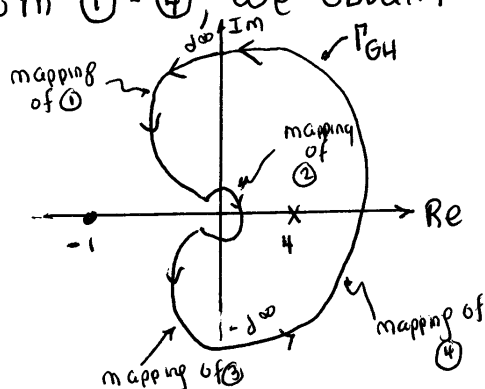
$$GH \sim \frac{5(-0.5Re^{j\phi})}{Re^{j\phi}(0.1Re^{j\phi})(-0.25Re^{j\phi})}$$

$$\sim \frac{+ \text{positive const.}}{R^2 e^{2j\phi}}$$

So  $\Gamma_{GH}$  is an infinitesimal arc "twirling" about the origin from  $+180^\circ$  to  $-180^\circ$ .

Note: Often, a computer generated Nyquist plot does not show such arcs. Also arcs at infinity are often not shown.

However, we show such arcs since they help us draw <sup>(the)</sup> plot. Now combining results from ① - ④, we obtain



Conclusions: We have

One net clockwise encirclements of  $-1+j0$ .

Nyquist demands 1 cw enc. since there is one Oh right half plane pole. Hence unstable ③

The following is my derivation of the above reader

$$\begin{aligned}
 G &= \frac{5(1 - 0.5s)}{s(1 + 0.1s)(1 - 0.25s)} \\
 &= \frac{5(1 - 0.5j\omega)}{j\omega(1 + 0.1j\omega)(1 - 0.25j\omega)}
 \end{aligned}$$

Multiply by complex conjugate of the denominator

$$\begin{aligned}
 G &= \frac{5(1 - 0.5j\omega)(-j\omega(1 - 0.1j\omega)(1 + 0.25j\omega))}{j\omega(1 + 0.1j\omega)(1 - 0.25j\omega)(-j\omega(1 - 0.1j\omega)(1 + 0.25j\omega))} \\
 &= \frac{-0.0625\omega^4 - 0.5j\omega^3 - 1.75\omega^2 - 5j\omega}{0.000625\omega^6 + 0.0725\omega^4 + \omega^2} \\
 &= \frac{-j(0.5j\omega^3 + 5\omega) - \omega^2(0.0625 + 1.75)}{\omega^2(1 + 0.01\omega^2)(1 + 0.0625\omega^2)} \\
 &= \frac{-j\omega\left(\frac{\omega^2}{2} + 5\right) - \omega^2\left(\frac{\omega^2}{16} + 1.75\right)}{\omega^2(1 + 0.01\omega^2)(1 + 0.0625\omega^2)}
 \end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{Re}(G) &= \frac{-\omega^2 \left( \frac{\omega^2}{16} + 1.75 \right)}{\omega^2 (1 + 0.01\omega^2) (1 + 0.0625\omega^2)} \\
&= \frac{-\left( \frac{\omega^2}{16} + 1.75 \right)}{(1 + 0.01\omega^2) (1 + 0.0625\omega^2)} \\
&= \frac{-(\omega^2 + 28)}{(1 + 0.01\omega^2) (16 + \omega^2)} \\
&= \frac{-100(\omega^2 + 28)}{(100 + \omega^2) (16 + \omega^2)} \\
\operatorname{Im}(G) &= \frac{-\omega \left( \frac{\omega^2}{2} + 5 \right)}{\omega^2 (1 + 0.01\omega^2) (1 + 0.0625\omega^2)} \\
&= \frac{-16 \left( \frac{\omega^2}{2} + 5 \right)}{\omega (1 + 0.01\omega^2) (16 + \omega^2)} \\
&= \frac{-1600 \left( \frac{\omega^2}{2} + 5 \right)}{\omega (100 + \omega^2) (16 + \omega^2)} \\
&= \frac{-800(\omega^2 + 10)}{\omega (100 + \omega^2) (16 + \omega^2)}
\end{aligned}$$

Therefore

$$\operatorname{Re}(G) = \frac{-100(\omega^2 + 28)}{(100 + \omega^2)(16 + \omega^2)}, \operatorname{Im}(G) = \frac{-800(\omega^2 + 10)}{\omega(100 + \omega^2)(16 + \omega^2)}$$

**Important note:** The net number of encirclements around  $-1$  must match the number of unstable open loop poles. But what sense depends on the initial  $\Gamma$  being clockwise or anti-clockwise. If  $\Gamma$  was anti-clockwise (like we use in class), then we want the net *clockwise* encirclements around  $-1$  to match the number of unstable open loop poles. If  $\Gamma$  was clockwise (like other books use), then we want the net *anti-clockwise* encirclements around  $-1$  to match the number of unstable open loop poles.

### 2.25.2.2 Nyquist Handout 2

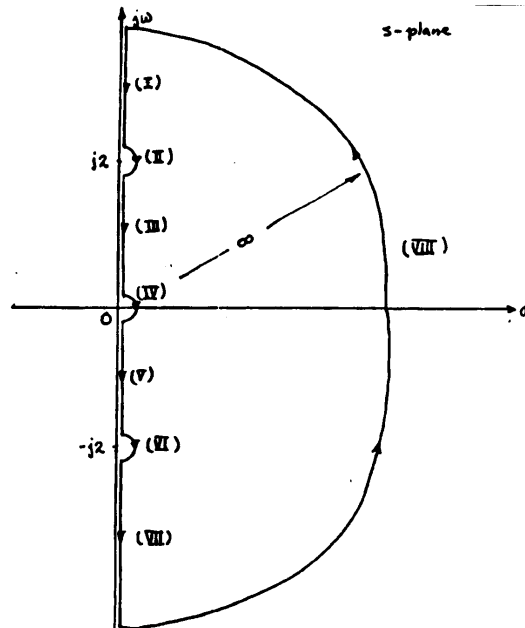
#### 2.25.2.1.3 Emailed on Monday Nov 23, 2015

# Nyquist Example

Consider

$$G(s)H(s) = \frac{50}{s(s+2)(s^2+4)}$$

Since  $G(s)H(s)$  has poles at  $+j2$  and  $-j2$ , it is necessary to choose the Nyquist path not passing these points. The Nyquist path is divided into eight sections as shown.



On Sec. (IV): We let  $s = ce^{j\theta}$  ( $c \rightarrow \infty$ ),  $\theta$  varies from  $+90^\circ$  to  $-90^\circ$  in clockwise direction. Then,

$$G(s)H(s) = \frac{50}{8ce^{j\theta}} = \frac{50}{8c}e^{-j\theta}$$

Thus, the GH plot of Sec. (IV) is a semicircle with infinite radius, and starts from  $-90^\circ$  to  $+90^\circ$  in the counterclockwise direction.



On Sec. (VIII): We let  $s = Re^{j\phi}$  ( $R \rightarrow \infty$ ),  $\phi$  varies from  $-90^\circ$  to  $+90^\circ$  in the counterclockwise direction. Then,

$$G(s)H(s) = \frac{50}{R^4 e^{j4\phi}} = 0e^{-4j\phi}$$

Thus, the GH-plot of Sec. (VIII) goes around the origin twice ( $720^\circ$ ) in the clockwise direction.

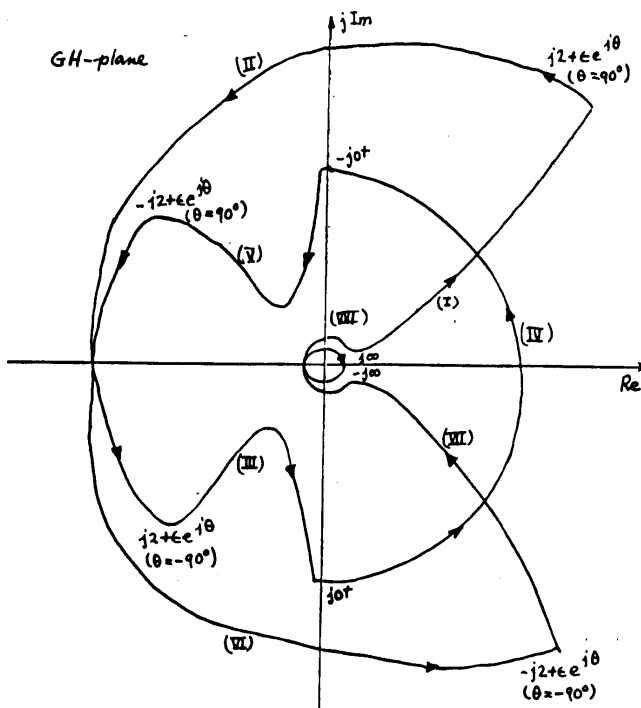
On Sec. (II): We let  $s = j2 + \epsilon e^{j\theta}$  ( $\epsilon \rightarrow 0$ ),  $\theta$  varies from  $+90^\circ$  to  $-90^\circ$  in the clockwise direction. Then

$$G(s)H(s) = \frac{50}{(j2 + \epsilon e^{j\theta})(2 + j2 + \epsilon e^{j\theta})(-4 + j4\epsilon e^{j\theta} + \epsilon^2 e^{j2\theta} + 4)}$$

or

$$G(s)H(s) = \frac{50}{j2(2 + j2)j4\epsilon e^{j\theta}} = \angle -\theta + 135^\circ$$

As  $\theta$  is varied from  $+90^\circ$  to  $-90^\circ$  along the Nyquist path [Sec. (II)], GH varies from  $45^\circ$  to  $225^\circ$  at infinity. Section (VI) is the mirror image of Sec. (II). The entire Nyquist plot of  $G(s)H(s)$  is shown in the following figure.



**The closed loop is unstable! 2 ENCIRCLEMENT WHEN ZERO REQUIRED**

2.25.2.2.1

2.25.2.2.2

## 2.26 Lecture 25, Tuesday Nov 24 2015, gain and phase margins

We will spend few minutes going over where we are. Today we will talk about: Discussion of Nyquist examples. Embellishment of Nyquist theory, Bridge into Bode analysis.

We have solution of last lecture example. We will talk about gain, phase margin and frequency response. Then we will go to Bode plot. We will take Bode analysis into design. This means we will design a controller based on Bode plot. Example from class (last lecture). Open loop is

$$GH = \frac{5(1 - 0.5s)}{s(1 + 0.1s)(1 - 0.25s)}$$

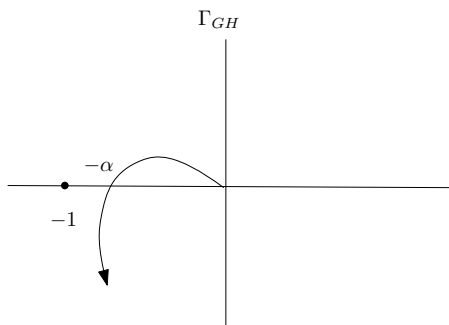
Review of the method in the handout that was send to class.

1. Mark the open loop poles on  $\Gamma$  plot
2. Make  $\Gamma$  to encircle all RHP open loop poles and small circle around all open loop poles on the imaginary axis.
3. Map  $\Gamma$  to  $\Gamma_{GH}$  segment by segment. For the segments on the imaginary axis, we do not have to do both, due to symmetry. (complex conjugate).
4. The important part is the real axis and the imaginary axis crossings. To do this, we need to find  $\text{Re}(GH)$  and  $\text{Im}(GH)$ . There might not be any crossings.

The second example was emailed. Which is  $GH = \frac{50}{s(s+2)(s^2+4)}$ . See last lecture for solution and go over it.

### 2.26.1 Gain and Phase margin

The most classical case is discussed. This is where the open loop is stable, and also the closed loop is stable. This means  $GH(s)$  has no poles in RHP. Imagine that we obtain  $\Gamma_{GH}$  that looks like this



Closed loop is stable (since it has zero net clockwise encirclement around  $-1$ ). Note that if closed loop is unstable, then we can not talk about gain and phase margins. This only applied to closed loop which is already stable. If  $\alpha$  point in the figure above is very close to  $-1$  then are close to being unstable. This are dangerous if  $\Gamma_{GH}$  is close to  $-1$ . (this is for this classical case). For example, say the true system is  $\gamma G(s)H(s)$ , but our  $\Gamma_{GH}$  is based on just  $G(s)H(s)$  and  $\gamma$  is the small variation of the true system.  $G(s)H(s)$  is our math model of the true system approximation. So we need to know the margin of safety because the system can be unstable if  $\gamma$  is large enough. Assume  $\gamma$  is an uncertain gain. Call it now  $K$ . example, an amplifier gain. The math model used to generate  $\Gamma_{GH}$  is based on  $K = 1$ .

**Reader:** What does  $K$  do to the Nyquist  $\Gamma_{GH}$  plot?

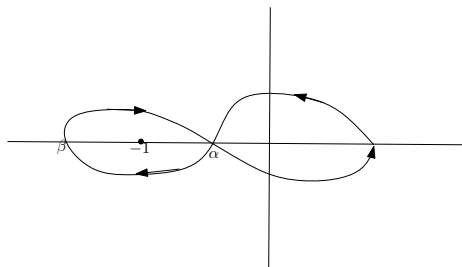
$K$  scales the  $\Gamma_{GH}$ . The scaling is centered at the origin. So large  $K$  magnifies  $\Gamma_{GH}$  and small  $K$  contracts  $\Gamma_{GH}$ . So we need to find the largest  $K_{\max}$  and still be stable. We need  $K_{\max}\alpha < 1$  or

$$K_{\max} = \frac{1}{\alpha}$$

This is called the gain margin. We express this in dB

$$(K_{\max})_{dB} = 20 \log_{10} \frac{1}{\alpha}$$

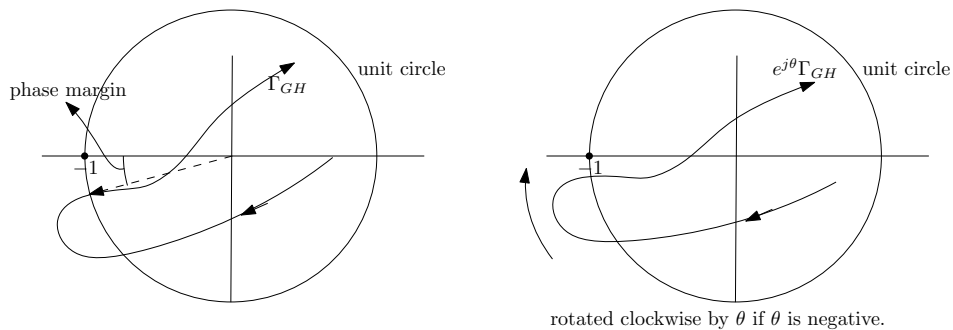
For example, if  $\alpha = 0.5$  then  $(K_{\max})_{dB} = 6dB$ . The gain margin is a measure of safety. We can get  $K_{\max}$  also using Routh table. We can also have two sided gain margin



Assume the  $GH$  now has one pole in RHP. So we want to have one net clockwise encirclement. So the  $\Gamma_{GH}$  plot shows that the closed loop is stable. But to maintain stability we need  $K\alpha < 1$  and  $K\beta > 1$ . This means for closed loop stability, we need

$$\frac{1}{\beta} < K < \frac{1}{\alpha}$$

This is two sided gain margin. Now we talk about phase margin. So far we have not talked about phase of  $GH(s)$ . Suppose the true system is  $G(s)H(s)e^{j\theta}$  where  $\theta$  is the phase error. How does small phase error affect stability? Could large phase error destabilize the closed loop? Consider now the classical case (again, this is where open loop is stable, and closed loop remain stable). Here is an example  $\Gamma_{GH}$  which we will use to analyze the effect of phase margin



How does multiplying  $G(s)H(s)$  by  $e^{j\theta}$  changes the above  $\Gamma_{GH}$ ? Each point  $z$  on  $\Gamma_{GH}$  now go to  $ze^{j\theta}$ . So  $\Gamma_{GH}$  rotates counter clock wise around the origin if  $\theta > 0$  and rotates clock wise if  $\theta < 0$ . How large the angle  $\theta$  become before the closed loop become unstable? We draw a unit circle around origin as shown and extend a straight line to where  $\Gamma_{GH}$  intersects the unit circle. The angle between this line and negative real axis is the phase margin. If  $\theta < 0$ , then it will rotate anti clock wise, and will become unstable after rotating  $-pm$ , where  $pm$ , is the phase margin.

## **2.27 Lecture 26, Thursday Nov. 26 2015, thanks giving, no class**

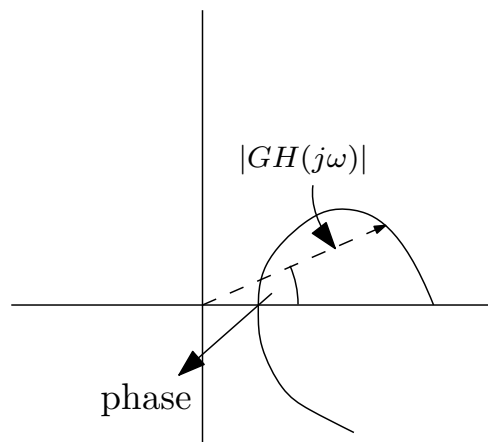
No class. Thanks giving day

## 2.28 Lecture 27, Tuesday December 1 2015, Start Bode frequency analysis

Reminder: Final exam on December 10. Final topic of course is Bode analysis. We touched on frequency response before but did not go heavily into frequency based design. When we look at the open loop transfer function, we would like to quickly find the frequency response.

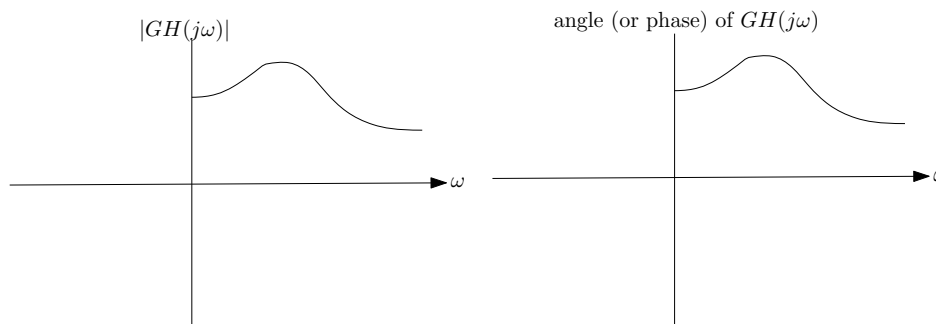
The frequency response is defined as the magnitude and phase of the transfer function, when viewed as complex function, which happens when we replace  $s$  by  $j\omega$  where  $\omega$  is the frequency. i.e  $|GH(j\omega)|$  and  $\angle GH(j\omega)$  are the frequency response.

Bode method is better for design in frequency domain than Nyquist as it is easier to determine the magnitude and phase. Nyquist plot already have frequency response information in it, but hard to read.



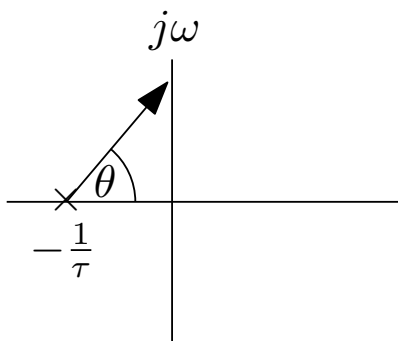
reading magnitude and phase from Nyquist plot

So if we plot the magnitude and phase, as  $\omega$  changes we get something like

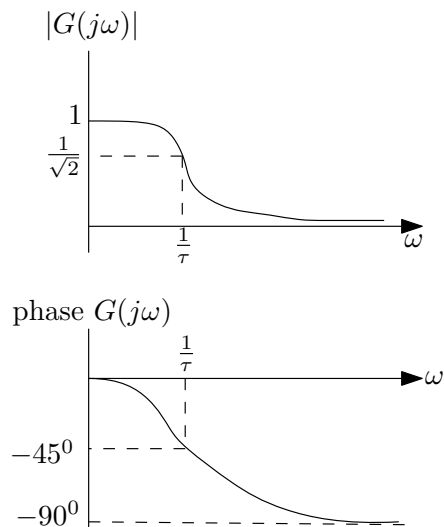


Simple motivating example: Say  $G(s) = \frac{1}{1+\tau s}$ . We will now use  $G(s)$  as the open loop. We could also use  $G(s)H(s)$ . We know that the  $|G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$  and  $\angle G(j\omega) = -\tan^{-1}(\omega\tau)$ . To easily see what happens to phase as  $\omega$  changes, we draw line from the pole, which we know

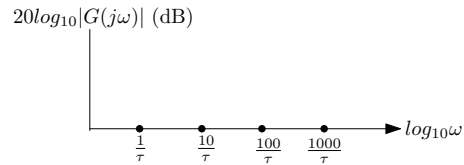
is at  $s = -\frac{1}{\tau}$  and the angle between this line and the real axis is the phase.



We now draw the  $|GH(j\omega)|$  and  $\angle GH(j\omega)$  on two separate plots



What bode analysis allows us to do, is to quickly make the magnitude and phase plots of more complicated transfer functions using approximations. We break the frequency into low region approximation and high frequency approximation and then join these curves together. The low frequency approximation is when  $\omega \ll \frac{1}{\tau}$ , where then we say  $|G(j\omega)| \approx 1$  and for large frequency approximation is when  $\omega \gg \frac{1}{\tau}$ , where then we say  $|G(j\omega)| \approx 0$ . For plotting  $|G(j\omega)|$ , for the  $y$  axis, which is  $|G(j\omega)|$ , we use  $\log \text{gain}(G)$ , which is dB. i.e.  $\log \text{gain}(G) = 20 \log_{10} |G(j\omega)|$ . So for  $\omega \ll \frac{1}{\tau}$  we find  $20 \log_{10} 1 = 0$ . The  $x$  axis, which is the frequency, is drawn using  $\log_{10} \omega$  scale and not linear  $\omega$  scale. So each step on the  $x$  axis jumps by 10 times its last value. Each step is called a decade, as follows



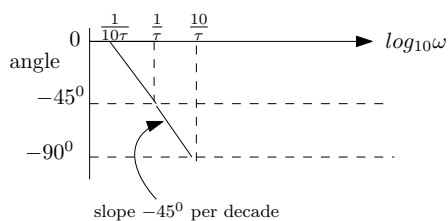
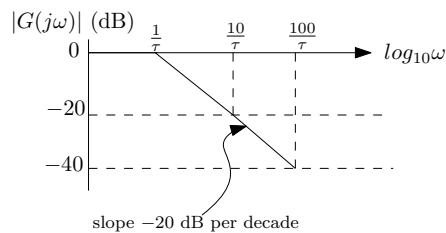
Now we go back to the first example  $G(s) = \frac{1}{1+\tau s}$ , then  $|G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$  hence

$$\begin{aligned} 20 \log_{10} |G(j\omega)| &= 20 \log_{10} 1 - 20 \log_{10} \sqrt{1 + \tau^2 \omega^2} \\ &= 0 - 10 \log_{10} (1 + \tau^2 \omega^2) \end{aligned}$$

Now we apply the Bode approximation. When  $\omega \gg \frac{1}{\tau}$  (large frequency approximation), we get

$$\begin{aligned} 20 \log_{10} |G(j\omega)| &= 0 - 10 \log_{10} (\tau^2 \omega^2) \\ &= -20 \log_{10} \tau \omega \end{aligned}$$

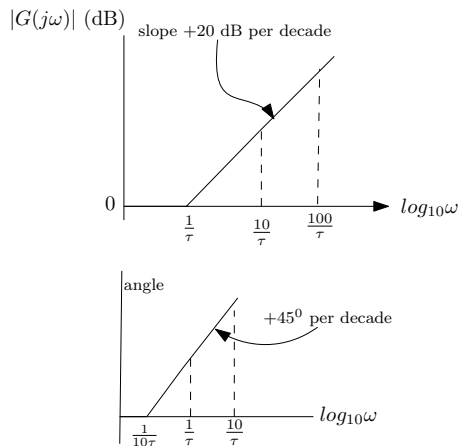
So for large  $\omega$  the  $|G(j\omega)|$  (in db) has a slope of  $-20$  per decade. So we have the following approximation of the magnitude



Notice: the phase plot break point starts the slope from  $\frac{1}{10\tau}$  value, and goes down  $-45^\circ$  per decade. The magnitude starts from the  $\frac{1}{\tau}$  break point and goes down  $-20$  db per decade.

**Reader:** Do Bode plot for  $G(s) = 1 + \tau s$ . Everything now is flipped from last example of  $G(s) = \frac{1}{1+\tau s}$ .





Now we build on this for more complicated transfer functions. Key remark: Using

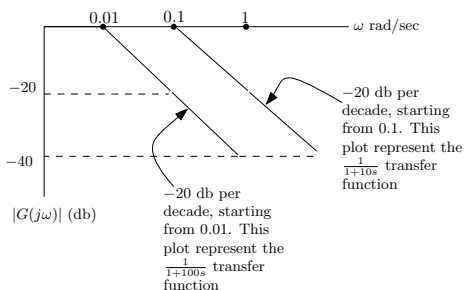
$$\log_{10}(ABC) = \log_{10} A + \log_{10} B = \log_{10} C$$

Then if  $ABC$  was a complicated transfer function, we break it into simple functions and make a Bode plot for each, and then just add them to make the overall Bode plot. For phase we use

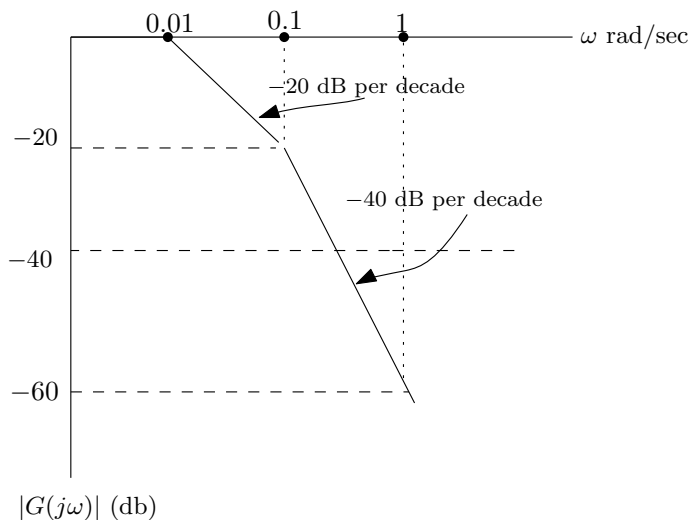
$$\angle(ABC) = \angle A + \angle B + \angle C$$

Here is an example. Let  $G(s) = \frac{1}{(1+10s)(1+100s)}$ . For  $\frac{1}{1+100s}$ ,  $\tau = 100$ , so the break point is  $\frac{1}{100} = 0.01$ .

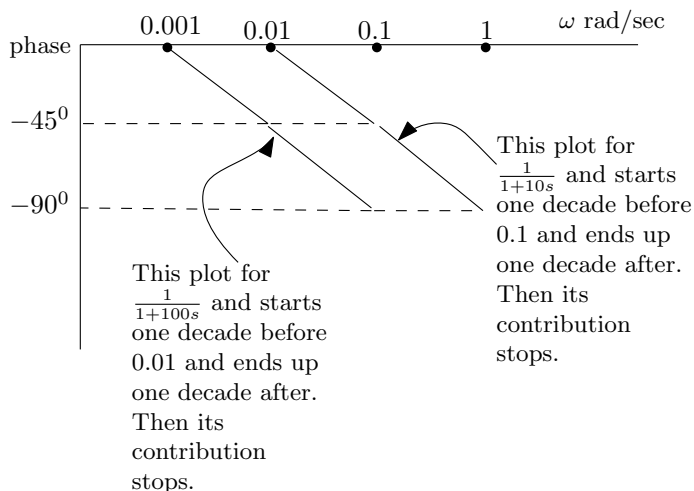
For  $\frac{1}{1+10s}$ , if we write it as  $\frac{1}{1+\tau s}$  then  $\tau = 10$ , and the break points for this term are  $\frac{1}{10} = 0.1$ . These values will go on the  $x$ -axis ( $\omega$ ). Looking at the magnitudes, we plot each transfer function on its own, like this



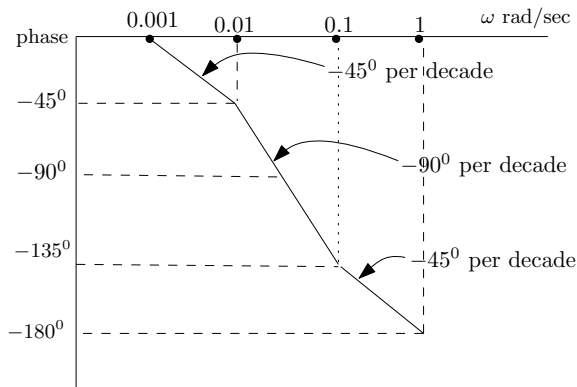
Next, we add them. We see that from 0.01 to 0.1, only one TF is active, so the slope is  $-20$  dB per decade. But from 0.1 to 1 and beyond, both transfer functions are active, and so the slope becomes  $-40$  dB per decade. So the result becomes



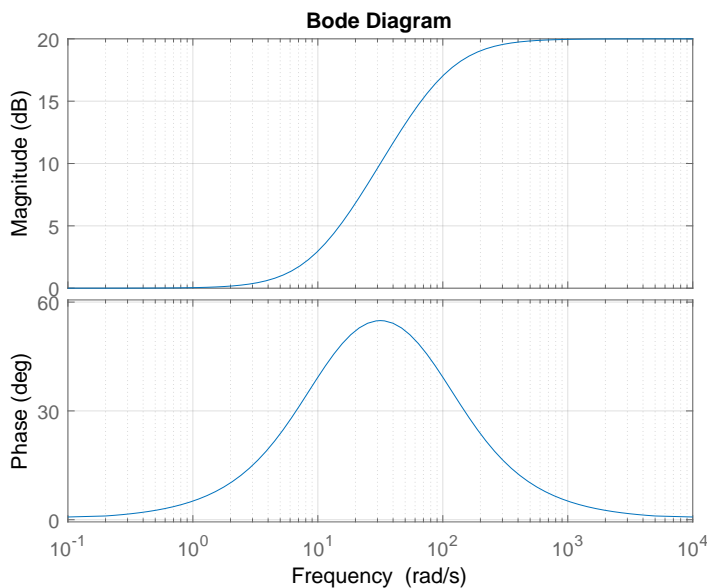
Now we will do the phase plot. We show each TF phase plot separately, then add them. For phase plot, each transfer function effect will extend up to 2 decades only. This is different from the magnitude plot. We go one decade before the break point, and one decade after the break point. This means each TF will contribute  $-45^\circ$  slope per decade, but only for the two decades around the break point.



Next, we add them. From 0.001 to 0.1 rad/sec  $\frac{1}{1+100s}$  is active, so the slope is  $-45^\circ$  per decade. 0.01 to 1 rad/sec, The second transfer function  $\frac{1}{1+10s}$  is active. After 1 rad/sec, the contribution stops. So the result becomes



**Reader:** Sketch  $\frac{1+0.1s}{1+0.01s}$ . The Denominator has break point at 100. The Numerator has break point at 10.



Notice that the  $1 + \tau s$  is very important here. We can't apply any of the approximation if the transfer function was not in this form. For example  $\frac{1}{s+10}$ . But we can easily convert everything to the  $1 + \tau s$  form. For example

$$G(s) = \frac{(s + 30)(s + 50)}{(s + 20)(s + 100)} \tag{1}$$

This is not the standard form. We convert each one at a time. Hence

$$\begin{aligned}(s + 30) &= 30 \left( \frac{s}{30} + 1 \right) \\(s + 50) &= 50 \left( \frac{s}{50} + 1 \right) \\(s + 20) &= 20 \left( \frac{s}{20} + 1 \right) \\(s + 100) &= 100 \left( \frac{s}{100} + 1 \right)\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}G(s) &= \frac{(30)(50)}{(20)(100)} \frac{\left(\frac{s}{30} + 1\right)\left(\frac{s}{50} + 1\right)}{\left(\frac{s}{20} + 1\right)\left(\frac{s}{100} + 1\right)} \\&= 0.75 \frac{\left(\frac{s}{30} + 1\right)\left(\frac{s}{50} + 1\right)}{\left(\frac{s}{20} + 1\right)\left(\frac{s}{100} + 1\right)}\end{aligned}$$

The effect of the constant 0.75 is to just shift the magnitude plot by  $20 \log_{10} 0.75$ . This constant will not affect the phase plot. (if it is positive, as in this example). For negative, it subtracts  $180^\circ$  from the phase.

## 2.29 Lecture 28, Thursday December 4 2015, More Bode analysis

Today we will finish bode approximation and look at the "sign" issue. Then we will move to margin analysis and design using Bode. Suppose we have  $G(s) = \frac{1}{s^k}$ . There is a pole of order  $k$  at zero. For this, we need separate Bode analysis. Since  $G(j\omega) = \frac{1}{(j\omega)^k}$ , hence  $|G(j\omega)| = \frac{1}{\omega^k}$ . So  $\log \text{gain}(G) = -20k \log_{10} \omega$ . This is exact. No straight line approximation as we did before. It falls off at  $-20k$  dB/decade at all frequencies, starting from  $\omega = 0.01$  rad/sec. So at  $\omega = 0.01$ , we have  $|G(j\omega)| = -20k \log(0.01)$ . So for  $k = 3$ , we get 120 dB. Then it will fall by  $-60$  dB/decade (not 20 dB/decade, since we have factor 3). What about the phase? Since  $\angle \frac{1}{(j\omega)^k} = -90k$  degree. So it is  $-90k$  at all frequencies. Not from one decade before to one decade after as we did for the approximation, as this is exact. Similarly for zero at  $s = 0$  of order  $k$ , as in  $G(s) = s^k$ . It will all be reverse. The phase will be  $+90k$  and the magnitude will have slope of  $+20k$  dB/decade.

Now we consider another special case. Which is second order system

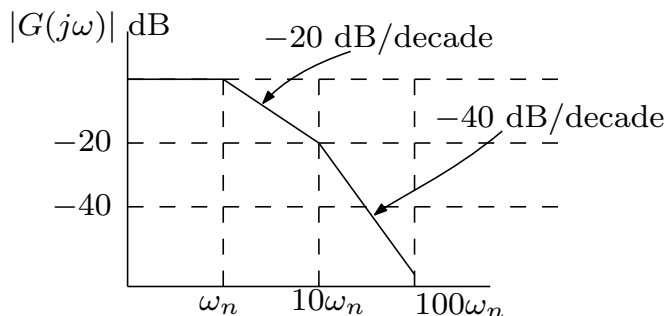
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where  $0 < \zeta < 1$ . If the poles are complex, we can't use the straight line approximation, since we can not put it in the form of  $\left(1 + \frac{s}{\tau}\right)$ . Only if the roots are real can we do this. For complex poles, we need special handling to make Bode plot. We start by rewriting  $G(s)$  as

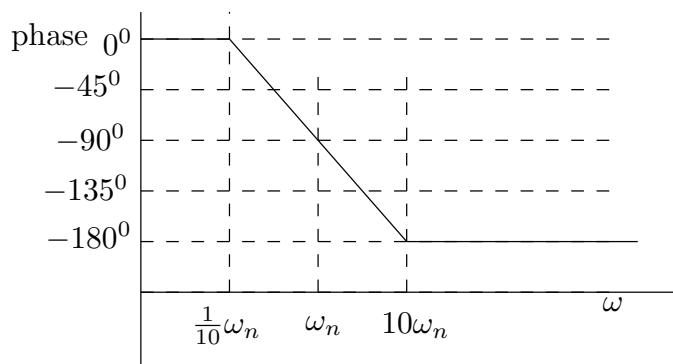
$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\frac{\zeta}{\omega_n}s + 1}$$

$$G(j\omega) = \frac{1}{\frac{\omega^2}{\omega_n^2} + 2\frac{\zeta}{\omega_n}\omega j + 1}$$

Now consider when  $\omega \ll \omega_n$ . Then  $G(j\omega) \approx 1$  or 0 dB. When  $\omega \gg \omega_n$  then will fall off by  $-40$  dB/decade since we have 2 poles.

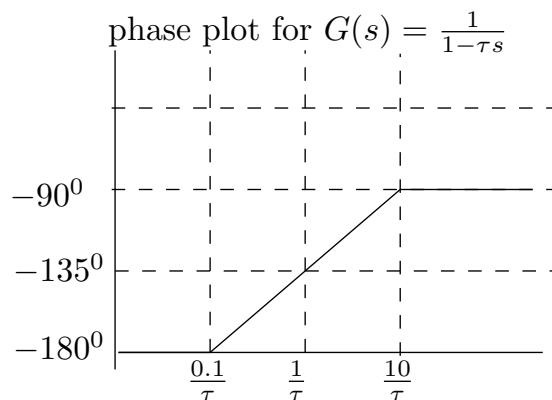


Notice that this has approximation errors when damping  $\zeta$  is very small. What about the phase? Since poles are complex, then at  $\omega = 0$  the phase cancel each others. So phase is zero at  $\omega = 0$ . For large  $\omega$ , one pole phase goes to  $90^\circ$  and another to  $90^\circ$ , so phase is  $+180^\circ$ . But the poles are in the denominator, so phase is  $-180^\circ$ .



**Reader:** Consider numerator with complex poles

Now will talk about the issue with "signs". Suppose  $G(s) = \frac{1}{1-\tau s}$  hence  $G(j\omega) = \frac{1}{1-\tau(j\omega)}$  therefore  $|G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$ , so log gain is the same as  $G(s) = \frac{1}{1+\tau s}$ , which is the standard form, but the phase is not the same. Now the pole is in the RHS. But since in the denominator, then phase is  $-180^\circ$  and not  $0^\circ$  as before. This means initial phase is at  $-180^\circ$  and not at  $0^\circ$ . Now, when  $\omega$  is very large, now the phase goes to  $+90^\circ$ . But since in the denominator, the phase goes to  $-90^\circ$ . Hence the phase plot only changes for  $\frac{1}{1-\tau s}$  and not the magnitude. The phase plot will be



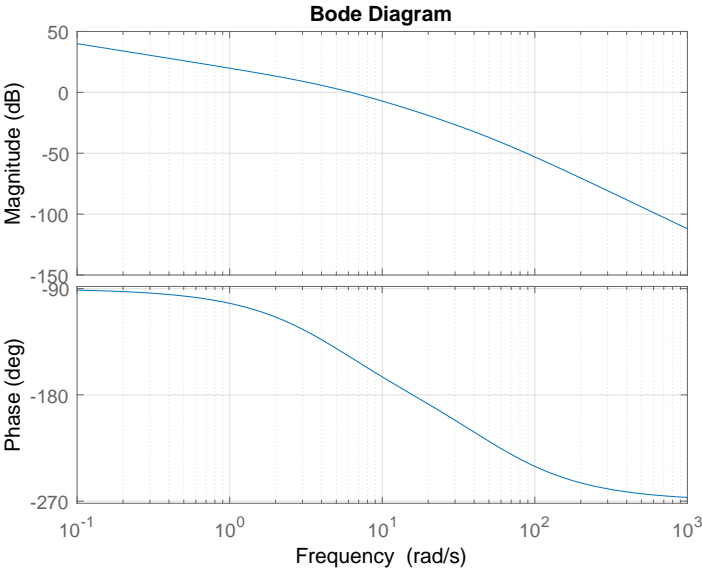
The magnitude plot is the same as for  $G(s) = \frac{1}{1+\tau s}$ . Remember to change this to standard form  $G(s) = \frac{1}{1+\left(\frac{s}{\frac{1}{\tau}}\right)}$  so that the corner frequency is more clear now that it is  $\left(\frac{1}{\tau}\right)$ .

We now go to design and margins. Example. Given open loop  $G(s) = \frac{2500}{s(s+5)(s+50)}$ . We use bode plot to find gain and phase margins. Always use the open loop transfer function, and not the closed loop. First we find corner frequencies. Need to write the above in standard for

$$G(s) = 2500 \frac{1}{s(5)\left(\frac{s}{5} + 1\right)(50)\left(\frac{s}{50} + 1\right)}$$

$$= 10 \frac{1}{s\left(\frac{s}{5} + 1\right)\left(\frac{s}{50} + 1\right)}$$

So corner frequencies are 5 and 50 rad/sec. The gain 10 causes a  $20 \log_{10} 10 = 20$  dB shift in the magnitude. We always start from  $\omega = 0.01$ . The pole at  $s = 0$  always starts at 40 dB, since  $20 \log_{10} \frac{1}{\omega} = -20 \log_{10} 0.01 = 40$  dB. Here is the bode plot using Matlab. See HW8 for more examples how to make Bode plot approximation by hand.





## 2.30 Lecture 29, Dec 8 2015, Bode gain and phase, effects of delay

Final exam on thursday. Three main topics on final: Root locus, Nyquist and Bode analysis.

Root locus: Know how to draw it and make inferences. What is important is the axis crossings. We can make some inferences without drawing also. For example, what happens at high frequency? Is the system stable or not at high gain? If the difference between number of poles and zeros is over two, then we know the system will be not stable at high gain since we know that some of the asymptotes will end up at infinity. Where do asymptotes begin? At centroid. We also covered the variant of root locus, where the system is in the form  $G(s) = \frac{s^2+ps+6}{(s+8)^3}$ . We can't apply Root locus on this form. We have to first convert it to standard form and only then apply root locus.

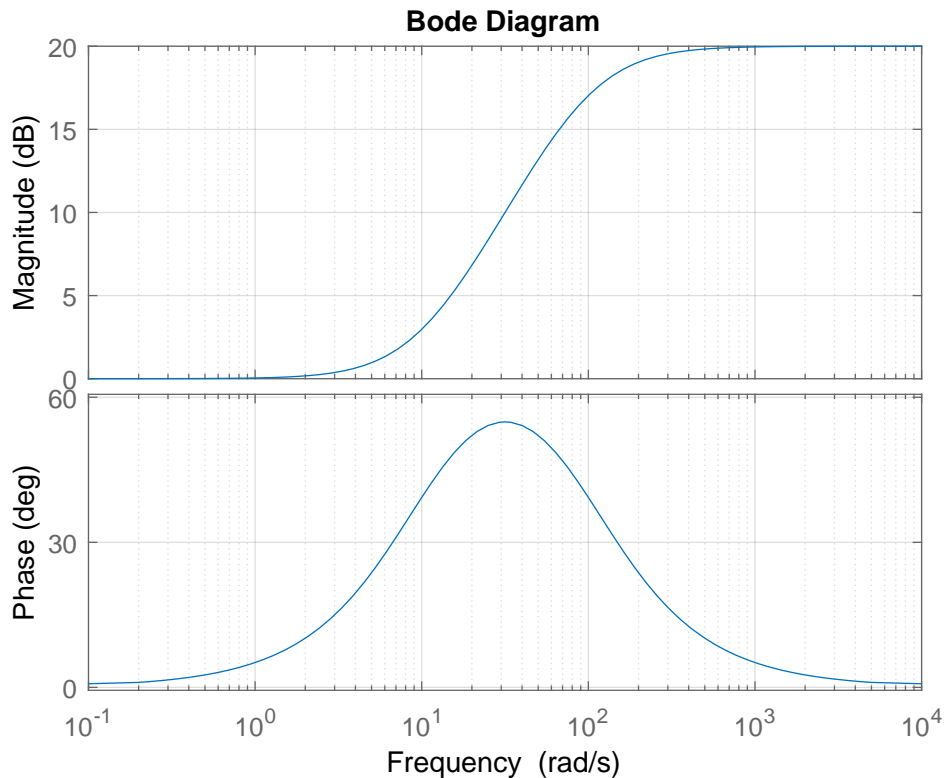
For Nyquist, need to know how to do the mapping from  $\Gamma$  to  $\Gamma_{GH}$ . Make sure to get the crossing correct. Count number of open loop poles. Then count number of net clockwise encirclements around  $-1$  and see if they match. Then the closed loop is stable, else it is not. We can also make the system stable by increasing the gain. We looked at effect of delay on Nyquist. It will cause the plot to rotate, which can cause it to become unstable.

For Bode, learn how to make quick sketch. Must first convert  $G(s)$  to standard form for approximation. We also looked at issue of signs. The difference between  $1 + \tau s$  and  $1 - \tau s$ . The magnitude remain the same, but the phase changes. Learn how to read gain and phase margins from Bode plots.

Now back to the lecture.

If there is pure gain  $H(s) = K$ , then its effect is only on the magnitude, not on the phase. It will cause the magnitude to shift by  $20 \log_{10} K$ .

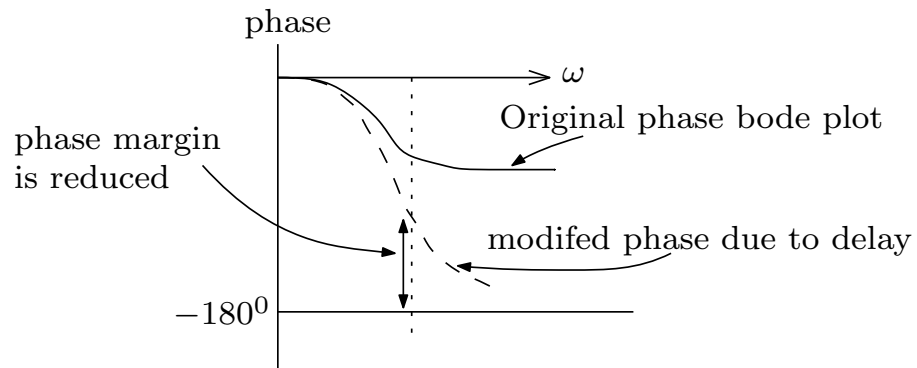
For the gain and phase margin. For example. Given  $G(s) = \frac{1}{2s(1+\frac{s}{2})}$ . The gain is  $\frac{1}{2}$  and there is corner frequency at 2 rad/sec. Here is the Bode plot



The gain margin is at frequency where phase is  $-180^\circ$ . Notice that in this example, the phase actually is never  $-180^\circ$  but can get as close to it as we want. So in theory, this has infinite gain margin. The phase has to actually dip below  $-180^\circ$  to get an actual crossing. Matlab reports infinite gain margin also. The frequency at which phase is  $-180^\circ$  is called  $\omega_{pc}$  and the frequency at which magnitude is 0 dB is called  $\omega_{gc}$ . From the plot the phase margin is about  $76^\circ$ .

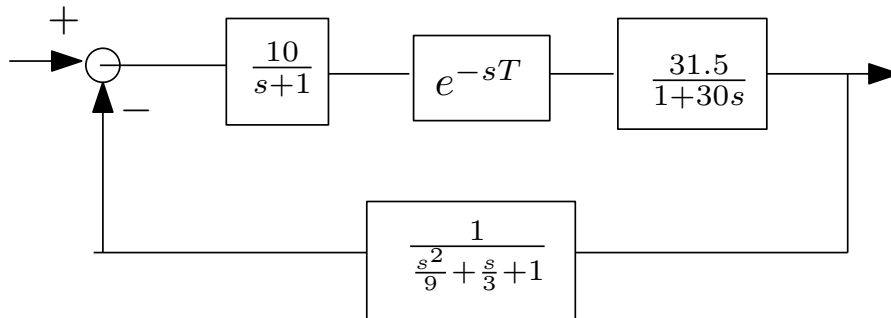
So a gain only controller  $H(s) = K$  only affects the magnitude but not the phase.

Now we will talk about the effect of delay on Bode plot. If we have  $e^{-j\omega T}$  in forward path, then since  $|e^{-j\omega T}| = 1$ , the delay only affects the phase and not the magnitude. Nyquist plot rotates clock wise by amount  $T$  at  $\omega = 1$ . For bode, as  $\omega$  increases, the phase will decrease more and more. So starting with a system that has positive phase margin, as  $\omega$  is increased for some  $T$ , we will see the phase margin decreasing, and the system can become unstable.



**Reader:** How large can  $T$  be in the example above before the system become unstable?

**Reader:** Make a bode plot of this system



## **2.31 Lecture 30, Dec 10 2015, Final exam**

Final exam.

## 2.32 Cheat sheet

### 2.32.1 Laplace Transforms for exam

	time	Laplace
impulse	$\delta(t)$	1
delayed impulse	$\delta(t-a)$	$e^{-as}$
unit step	$u(t)$	$\frac{1}{s}$
delayed unit impulse	$u(t-a)$	$\frac{1}{s}e^{-as}$
ramp	$t$	$\frac{1}{s^2}$
parabolic	$t^2$	$\frac{2}{s^3}$
	$tf(t)$	$-F'(s)$
scaled	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
	$e^{at}f(t)$	$F(s-a)$
	$f(t)$	$F(s)$
derivative	$f'(t)$	$sF(s) - f(0)$
second derivative	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
integrator	$\int_0^t f(\tau) d\tau$	$\frac{1}{s}F(s)$
delay in time	$f(t-a)u(t-a)$	$e^{-as}F(s)$
convolution	$f(t) \otimes g(t)$	$F(s)G(s)$
	$e^{-at}u(t)$	$\frac{1}{s+a}$
	$e^{-a t }$	$\frac{2a}{a^2-s^2}$
	$(1-e^{-at})u(t)$	$\frac{a}{s(s+a)}$
	$\sin(at)u(t)$	$\frac{a}{s^2+a^2}$
	$\cos(at)u(t)$	$\frac{s}{s^2+a^2}$
	$e^{-bt}\sin(at)u(t)$	$\frac{a}{(s+b)^2+a^2}$
	$e^{-bt}\cos(at)u(t)$	$\frac{s+b}{(s+b)^2+a^2}$

### 2.32.2 Partial fractions

1.

$$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

$$A = \lim_{s \rightarrow -a} \frac{1}{s+b}$$

$$B = \lim_{s \rightarrow -b} \frac{1}{s+a}$$

2.

$$\frac{1}{(s+a)(s+b)^2} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{(s+b)^2}$$

$$A = \lim_{s \rightarrow -a} \frac{1}{(s+b)^2}$$

$$B = \lim_{s \rightarrow -b} \frac{d}{ds} \left( \frac{1}{s+a} \right)$$

$$C = \lim_{s \rightarrow -b} \frac{1}{(s+a)^2}$$

3.

$$\frac{1}{(s+a)(s^2+b)} = \frac{A}{s+a} + \frac{Bs+C}{s^2+b}$$

$$A = \lim_{s \rightarrow -a} \frac{1}{s^2+b}$$

For  $B, C$ , expand now and compare coefficients (but there should be faster way)

### 2.32.3 Final value

Suppose  $F(s) = \frac{N(s)}{D(s)}$  is stable, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$F(s)$  is allowed to have only one pole at origin and still use FVT. But if  $F(s)$  has more than one pole at the origin, or unstable, we can't use FVT to determine  $\lim_{t \rightarrow \infty} f(t)$ .

### 2.32.4 Tracking

$\frac{E(s)}{R(s)} = \frac{1}{1+GH}$ . To have  $\lim_{t \rightarrow \infty} e(t) = 0$  when the input  $R(s)$  is step, we need to have integrator in  $G(s)H(s)$ . Since we want  $GH$  to be very large for  $s = 0$ . And integrator is  $\frac{1}{s}$ . If the input is ramp  $t$ , then we need  $\frac{1}{s^2}$  in  $GH$ . If the input is  $t^2$  then we need  $\frac{1}{s^3}$  in the controller. and so on.

### 2.32.5 Second order system

$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ ,  $y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (\sin \omega_n \sqrt{1-\zeta^2} t + \phi)$  where  $\phi = \cos^{-1} \zeta$ . Maximum overshoot is  $e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ . Resonance frequency  $\omega_r = \omega_n \sqrt{1-2\zeta^2}$ , and  $|G(\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

# Chapter 3

## HWs

### Local contents

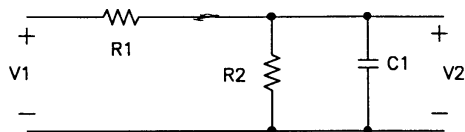
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## 3.1 HW 1

### 3.1.1 Problems

#### ECE 332 Homework 1

1. For the circuit below, find the transfer function  $\frac{V_2(s)}{V_1(s)}$ .



2. Find the inverse Laplace transforms of the following:

(a)  $F(s) = \frac{1}{(s+2)^4}$   
 (b)  $F(s) = \frac{1}{s^3+5s^2+8s+4}$   
 (c)  $F(s) = \frac{7s^2+48s+62}{s^3+7s^2+10s}$   
 (d)  $F(s) = \frac{10(s+2)}{s(s^2+4)(s+1)}e^{-s}$

3. A system is described by the differential equation

$$y^{(4)} + y^{(3)} + 5y^{(2)} + 7y^{(1)} + y = u^{(3)} + 2u^{(2)} + 3u^{(1)} + 7u$$

Find the transfer function  $\frac{Y(s)}{U(s)}$ .

4. For the system in Problem 3, find the step response using the *Matlab* toolbox *Simulink* (see handout).
- (a) Plot your output for 30 seconds and comment. Please include descriptive titles and axis labels.
- (b) In view of the result in (a), a feedback gain  $K$  is introduced into the system. This leads to a system with  $7y^{(1)}$  replaced by  $(7 + K)y^{(1)}$ . Using *Simulink*, find a suitable value of  $K$  so that the unstable oscillations in (a) are eliminated. Plot 30 seconds of the step response for the stabilized system. (Note: Try not to use an excessively large gain if avoidable.)

For this problem submit the two plots as well as a copy of the *Simulink* schematic file used to generate the plots (only one schematic is needed). Please enlarge the necessary blocks so that all of the numbers inside are visible, e.g. the transfer function block.

### 3.1.2 Problem 1 solution

Let the input impedance be  $Z_{in}$  and the output impedance be  $Z_{out}$ , then

$$Z_{out} = R_2 \parallel Z_{C_1}$$



Where  $Z_{c_1}$  is the capacitor impedance given by  $Z_{c_1} = \frac{1}{C_1s}$ . To find  $Z_{out}$ , we use the parallel formula

$$\begin{aligned}\frac{1}{Z_{out}} &= \frac{1}{R_2} + \frac{1}{Z_{c_1}} \\ &= \frac{Z_{c_1} + R_2}{R_2Z_{c_1}}\end{aligned}$$

Therefore

$$Z_{out} = \frac{R_2Z_{c_1}}{Z_{c_1} + R_2}$$

Now we find the input impedance, which is

$$\begin{aligned}Z_{in} &= R_1 + R_2 \parallel Z_{c_1} \\ &= R_1 + \frac{R_2Z_{c_1}}{Z_{c_1} + R_2}\end{aligned}$$

Simplifying gives

$$Z_{in} = \frac{R_1(Z_{c_1} + R_2) + R_2Z_{c_1}}{Z_{c_1} + R_2}$$

Now voltage divider is used, which gives

$$\begin{aligned}\frac{V_2(s)}{V_1(s)} &= \frac{Z_{out}}{Z_{in}} \\ &= \frac{\frac{R_2Z_{c_1}}{Z_{c_1} + R_2}}{\frac{R_1(Z_{c_1} + R_2) + R_2Z_{c_1}}{Z_{c_1} + R_2}} \\ &= \frac{R_2Z_{c_1}}{R_1(Z_{c_1} + R_2) + R_2Z_{c_1}} \\ &= \frac{R_2Z_{c_1}}{R_1Z_{c_1} + R_1R_2 + R_2Z_{c_1}} \\ &= \frac{1}{\frac{R_1}{R_2} + \frac{R_1}{Z_{c_1}} + 1} \\ &= \frac{1}{\frac{R_1}{R_2} + R_1Cs + 1} \\ &= \frac{R_2}{(R_1 + R_2) + R_2R_1Cs}\end{aligned}$$

Hence the transfer function is

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2}{(R_1 + R_2) + R_2R_1Cs}$$

The above is a low pass filter since making  $s = j\omega$  gives

$$G(\omega) = \frac{R_2}{(R_1 + R_2) + R_2R_1Cj\omega}$$

As the frequency  $\omega \rightarrow \infty$  then  $|G(\omega)| \rightarrow 0$  and as  $\omega \rightarrow 0$ ,  $|G(\omega)| \rightarrow \frac{R_2}{R_1+R_2}$ . So the above transfer function attenuate large frequencies.

### 3.1.3 Problem 2 solution

#### 3.1.3.1 Part (a)

$F(s) = \frac{1}{(s+2)^4}$ . Using the Laplace transform property

$$F(s - a) \iff e^{at} f(t) \quad (1)$$

Where  $f(t)$  is the inverse Laplace transform of  $F(s)$ . And also using the property

$$\frac{1}{s^n} \iff \frac{t^{n-1}}{(n-1)!} \quad (2)$$

Then, combining (1) and (2) gives

$$\frac{1}{(s-a)^n} \iff e^{at} \frac{t^{n-1}}{(n-1)!}$$

For  $a = -2$  and  $n = 4$  the above becomes

$$\boxed{\frac{1}{(s+2)^4} \iff e^{-2t} \frac{t^3}{3!} = e^{-2t} \frac{t^3}{6}}$$

#### 3.1.3.2 Part (b)

$F(s) = \frac{1}{s^3+5s^2+8s+4}$ . Using the computer, the factors are first found

$$s^3 + 5s^2 + 8s + 4 = (s+1)(s+2)^2$$

Applying partial fractions decomposition on  $\frac{1}{(s+1)(s+2)^2}$  gives

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \quad (1)$$

Multiplying both sides by  $(s+1)$  and evaluating at  $s = -1$  gives

$$A = \frac{1}{(s+2)^2}_{s=-1}$$

$$A = 1$$

Multiplying both sides by  $(s+2)^2$  and evaluating at  $s = -2$  gives

$$C = \frac{1}{(s+1)_{s=-2}}$$

$$C = -1$$

Therefore (1) now becomes

$$\begin{aligned} \frac{1}{(s+1)(s+2)^2} &= \frac{1}{(s+1)} + \frac{B}{(s+2)} - \frac{1}{(s+2)^2} \\ &= \frac{(s+2)^2 + B(s+2)(s+1) - (s+1)}{(s+1)(s+2)^2} \\ &= \frac{2B + 3s + Bs^2 + 3Bs + s^2 + 3}{(s+1)(s+2)^2} \end{aligned}$$

Comparing numerator of RHS and LHS gives

$$1 = (2B + 3) + (3 + 3B)s + (B + 1)s^2$$

Comparing coefficients (powers of  $s$ ) gives  $1 = 2B + 3$  or  $B = -1$ . Therefore the partial fraction decomposition of (1) becomes

$$\boxed{\frac{1}{(s+1)(s+2)^2} = \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2}}$$

The inverse Laplace transform is now applied to each term in the RHS and using  $F(s - a) \Leftrightarrow e^{at}f(t)$  which results in

$$\begin{aligned} \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2} &\Leftrightarrow e^{-t} - e^{-2t} - te^{-2t} \\ &= \boxed{e^{-t}(1 - e^{-t} - te^{-t})} \end{aligned}$$

### 3.1.3.3 Part (c)

$F(s) = \frac{7s^2 + 48s + 62}{s^3 + 7s^2 + 10s}$ . The denominator can be written as  $s(s^2 + 7s + 10)$  which is now factored to  $s(s+2)(s+5)$ . Applying partial fractions on this gives

$$\begin{aligned} F(s) &= \frac{7s^2 + 48s + 62}{s(s+2)(s+5)} \\ &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5} \end{aligned}$$

Hence

$$\begin{aligned} A &= \lim_{s \rightarrow 0} \frac{7s^2 + 48s + 62}{(s+2)(s+5)} = \frac{62}{10} \\ B &= \lim_{s \rightarrow (-2)} \frac{7s^2 + 48s + 62}{s(s+5)} = \frac{7(4) + 48(-2) + 62}{-2(-2+5)} = 1 \\ C &= \lim_{s \rightarrow (-5)} \frac{7s^2 + 48s + 62}{s(s+2)} = \frac{7(25) + 48(-5) + 62}{-5(-5+2)} = -\frac{1}{5} \end{aligned}$$

Therefore

$$F(s) = \frac{621}{10s} + \frac{1}{s+2} - \frac{1}{5s+5}$$

The inverse Laplace transform is

$$f(t) = \frac{62}{10} + e^{-2t} - \frac{1}{5}e^{-5t}$$

For  $t \geq 0$ .

### 3.1.3.4 Part (d)

$F(s) = \frac{10(s+2)}{s(s^2+4)(s+1)}e^{-s}$ . This is solved using time shifting property of Laplace transform, which says that

$$\mathcal{L}\{f(t-a)\} = e^{-as}F(s)$$

Where  $F(s)$  is the Laplace transform of  $f(t)$ . We see that  $a = 1$  in the above. So we just need to find  $\mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+4)(s+1)}\right)$  and then time shift (delay) the result by the amount  $a$  which is 1 in this case. Applying partial fractions on  $\frac{10(s+2)}{s(s^2+4)(s+1)}$  gives

$$\frac{10(s+2)}{s(s^2+4)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4} \quad (1)$$

Hence

$$A = \lim_{s \rightarrow 0} \frac{10(s+2)}{(s^2+4)(s+1)} = 5$$

$$B = \lim_{s \rightarrow -1} \frac{10(s+2)}{s(s^2+4)} = \frac{10}{-5} = -2$$

Therefore (1) becomes

$$\begin{aligned} \frac{10(s+2)}{s(s^2+4)(s+1)} &= \frac{5}{s} - \frac{2}{s+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{5(s+1)(s^2+4) - 2(s)((s^2+4)) + (Cs+D)(s)(s+1)}{s(s^2+4)(s+1)} \end{aligned}$$

Or

$$\begin{aligned} 10(s+2) &= 5(s+1)(s^2+4) - 2(s)((s^2+4)) + (Cs+D)(s)(s+1) \\ 10s+20 &= 12s+sD+Cs^2+Cs^3+s^2D+5s^2+3s^3+20 \\ &= s(12+D)+20+s^2(C+D+5)+(C+3)s^3 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 12+D &= 10 \\ D &= -2 \end{aligned}$$

And

$$\begin{aligned} 0 &= C + D + 5 \\ C &= -D - 5 \\ &= -3 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{10(s+2)}{s(s^2+4)(s+1)} &= \frac{A}{s} + \frac{B}{(s+1)} + \frac{Cs+D}{(s^2+4)} \\ &= \frac{5}{s} - \frac{2}{(s+1)} - \frac{3s+2}{(s^2+4)} \\ &= \frac{5}{s} - \frac{2}{(s+1)} - 3\frac{s}{(s^2+4)} - \frac{2}{(s^2+4)} \end{aligned} \quad (3)$$

Using tables,  $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$ , hence  $\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)}\right) = \cos 2t$ . Also, from tables  $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$ , hence  $\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \sin 2t$ . The complete the inverse Laplace transform of  $\frac{10(s+2)}{s(s^2+4)(s+1)}$  is now found to be

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+4)(s+1)}\right) \\ &= 5 - 2e^{-t} - 3\cos 2t - \sin 2t \end{aligned}$$

Therefore the final answer is just a time shifted version of the above, which is

$$\begin{aligned} f(t) &= g(t-1) \\ &= \boxed{5 - 2e^{-(t-1)} - 3\cos 2(t-1) - \sin 2(t-1)} \end{aligned}$$

For  $t \geq 1$  and zero otherwise.

### 3.1.4 Problem 3 solution

Taking Laplace transform of both sides, and assuming zero for all initial conditions gives

$$\begin{aligned} s^4Y(s) + s^3Y(s) + 5s^2Y(s) + 7sY(s) + Y(s) &= s^3U(s) + 2s^2U(s) + 3sU(s) + 7U(s) \\ Y(s)(s^4 + s^3 + 5s^2 + 7s + 1) &= U(s)(s^3 + 2s^2 + 3s + 7) \\ \frac{Y(s)}{U(s)} &= \frac{s^3 + 2s^2 + 3s + 7}{s^4 + s^3 + 5s^2 + 7s + 1} \end{aligned}$$

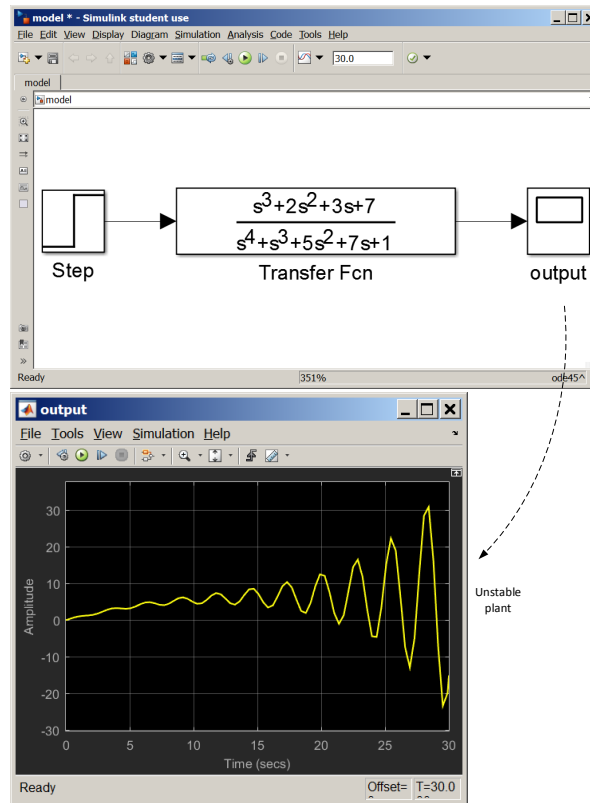
### 3.1.5 Problem 4 solution

#### 3.1.5.1 Part (a)

Simulink model was setup and run for 30 seconds. The step response shows that the plant is not stable. This is due to the numerator having roots with positive real parts

```
>> double(solve(s^4+s^3+5*s^2+7*s+1==0,s))
-1.1803 + 0.0000i
-0.1608 + 0.0000i
0.1706 - 2.2888i
0.1706 + 2.2888i
```

The following diagram show the simulink model and the output



### 3.1.5.2 Part (b)

Replacing  $7y^{(1)}$  with  $(7+k)y^{(1)}$  in the given differential equation leads to the following transfer function

$$s^4Y(s) + s^3Y(s) + 5s^2Y(s) + (7+k)sY(s) + Y(s) = s^3U(s) + 2s^2U(s) + 3sU(s) + 7U(s)$$

$$Y(s) [s^4 + s^3 + 5s^2 + (7+k)s + 1] = U(s) (s^3 + 2s^2 + 3s + 7)$$

Hence

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 2s^2 + 3s + 7}{s^4 + s^3 + 5s^2 + (7+k)s + 1}$$

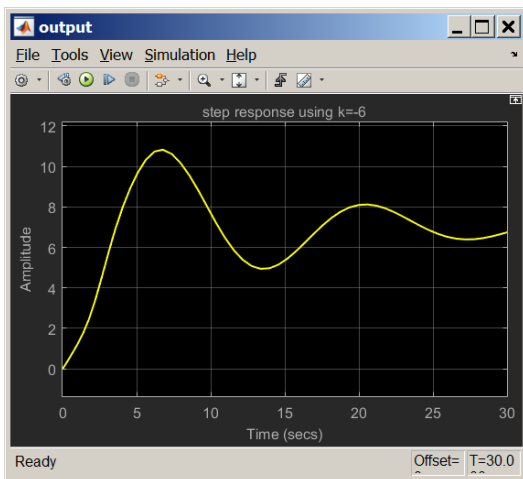
To determine what range of values for  $k$  to use, Routh stability table is first applied to the denominator polynomial

1	5	1
1	$7 + k$	0
$-(2 + k)$	1	0
$\frac{-(2+k)(7+k)-1}{-(2+k)}$	0	

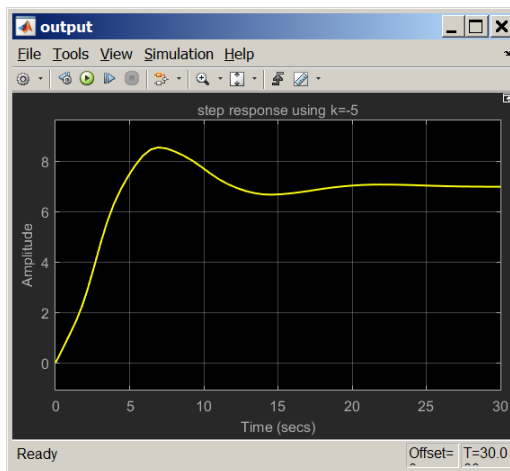
Therefore, for no sign change in the first column, we need  $-(2 + k) > 0$  or  $2 + k < 0$  or  $k < -2$  and we also need  $\frac{-(2+k)(7+k)-1}{-(2+k)} > 0$  or  $\frac{k^2+9k+15}{2+k} > 0$  which means  $k^2 + 9k + 15 > 0$ . The roots of this quadratic are  $k = -2.2087, k = -6.7913$ . Putting all these conditions together gives the range of value on  $k$  for a stable system as

$$-6.79 < k < -2.208$$

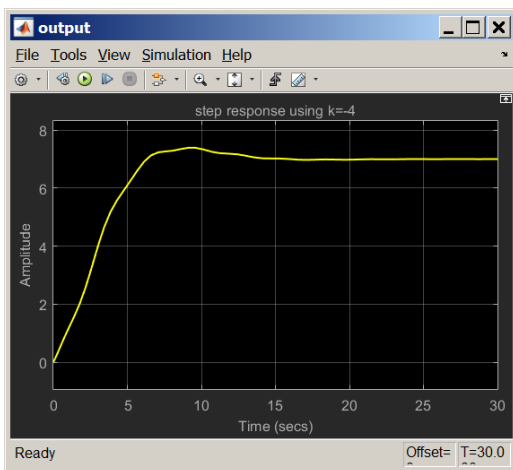
So any  $k$  in this range can be used. Here is the response using simulink using 4 different values of  $k$ , all within the above range to keep the system stable, showing that all the responses are stable, using the same 30 seconds duration.



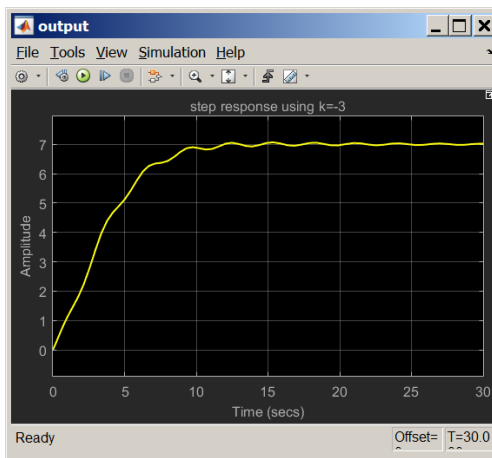
**K=-6**



**K=-5**



**K=-4**



**K=-3**

## 3.1.6 HW 1 key solution

## ECE 332 Homework 1 Solutions

1. By voltage division, we know that  $V_2 = \frac{Z_2}{Z_1 + Z_2} V_1$ , and in this case,  $Z_1 = R_1$  and

$$Z_2 = R_2 // \frac{1}{sC_1} = \frac{R_2 \frac{1}{sC_1}}{R_2 + \frac{1}{sC_1}}. \quad \text{Say } V_{-1} = 1 \text{ volt}$$

Therefore,

$$V_2 = \frac{\frac{\frac{R_2}{sC_1}}{R_2 + \frac{1}{sC_1}}}{R_1 + \frac{\frac{R_2}{sC_1}}{R_2 + \frac{1}{sC_1}}} = \frac{\frac{R_2}{sC_1}}{R_1 \left( R_2 + \frac{1}{sC_1} \right) + \frac{R_2}{sC_1}}$$

$$\boxed{V_2 = \frac{R_2}{R_1 R_2 C_1 s + (R_1 + R_2)}} \quad \text{Since } V_{-1} \text{ is not necessarily one volt, the quantity on the right hand side is the required transfer function } V_2(s)/V_{-1}(s)$$

2. (a)

$$F(s) = \frac{1}{(s+2)^4} \implies f(t) = \mathcal{L}^{-1} \left[ \frac{1}{(s+2)^4} \right]$$

$$f(t) = \frac{1}{3!} t^3 e^{-2t} u(t)$$

$$f(t) = \frac{t^3 e^{-2t}}{6} u(t)$$

(b)

$$F(s) = \frac{1}{s^3 + 5s^2 + 8s + 4} = \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$f(t) = \mathcal{L}^{-1}(F(s))$$

$$f(t) = (e^{-t} - e^{-2t} - te^{-2t}) u(t)$$

(c)

$$F(s) = \frac{7s^2 + 48s + 62}{s^3 + 7s^2 + 10s} = \frac{7s^2 + 48s + 62}{s(s+2)(s+5)}$$

$$F(s) = \frac{6.2}{s} + \frac{1}{s+2} - \frac{1}{5(s+5)}$$

$$f(t) = \left( 6.2 + e^{-2t} - \frac{1}{5} e^{-5t} \right) u(t)$$

;



(d)

$$\begin{aligned}
 F(s) &= \frac{10(s+2)}{s(s^2+4)(s+1)} e^{-s} = 10e^{-s} \left[ \frac{1}{s} - \frac{3}{10} \frac{s+\frac{1}{5}}{s^2+4} - \frac{1}{s+1} \right] \\
 &= 10e^{-s} \left[ \frac{5}{s} - \frac{3s+2}{s^2+4} - \frac{2}{s+1} \right] = 10e^{-s} \left[ \frac{5}{s} - 3 \frac{s}{s^2+4} - \frac{2}{s^2+4} - \frac{2}{s+1} \right] \\
 f(t) &= \left\{ 5 - 3 \cos[2(t-1)] - \sin[2(t-1)] - 2e^{-(t-1)} \right\} u(t-1)
 \end{aligned}$$

3. Given the differential equation

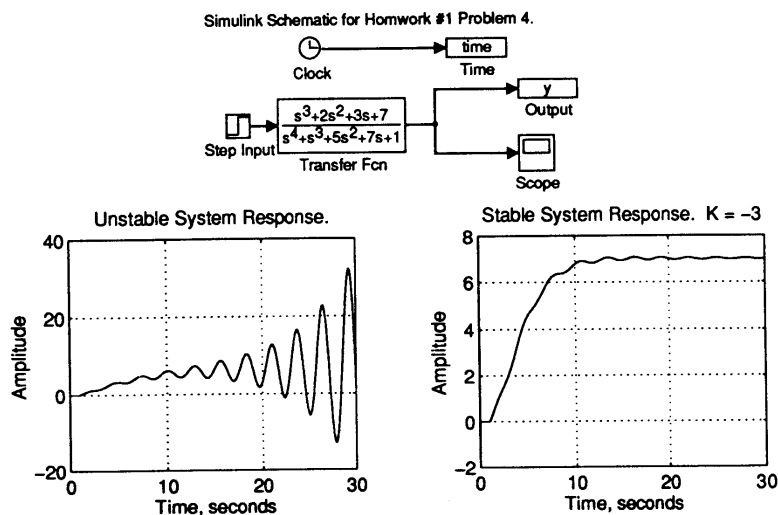
$$y^{(4)} + y^{(3)} + 5y^{(2)} + 7y^{(1)} + y = u^{(3)} + 2u^{(2)} + 3u^{(1)} + 7u$$

Taking the Laplace Transform of both sides yields:

$$s^4 Y(s) + s^3 Y(s) + 5s^2 Y(s) + 7s Y(s) + Y(s) = s^3 U(s) + 2s^2 U(s) + 3s U(s) + 7U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s^3 + 2s^2 + 3s + 7}{s^4 + s^3 + 5s^2 + 7s + 1}}$$

4. (a) The transfer function depicting a system which is unstable. The output oscillates about the point  $y = 7$ . (approximately)  
 (b)  $K = -3$  or  $K = -4$  seem to work well.



```

% Matlab script file used to generate the plots.
subplot(221), plot(time,yunst), grid, xlabel('Time, seconds')
ylabel('Amplitude'), title('Unstable System Response.')
subplot(222), plot(time,y), xlabel('Time, seconds'), ylabel('Amplitude')
grid, title('Stable System Response. K = -3')

```

You may wish to ignore the code above if you have been successful making Simulink run

## 3.2 HW 2

### 3.2.1 Problem 1

**Problem 1:** Determine the step, ramp and parabolic steady-state errors of the following unity-feedback control systems. The forward-path transfer functions are given

$$(a) G(s)H(s) = \frac{1000}{(1 + 0.1s)(1 + 10s)}$$

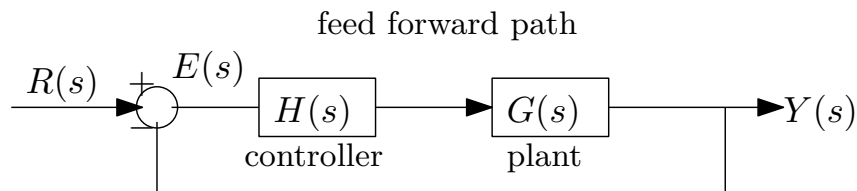
$$(b) G(s)H(s) = \frac{1000}{s(s + 10)(s + 100)}$$

$$(c) G(s)H(s) = \frac{K(1 + 2s)(1 + 4s)}{s^2(s^2 + s + 1)}$$

(d) What relationships can you find between the number of poles of  $G(s)$  at the origin and the type of input signal for which there is a constant steady-state error ( $\neq 0$ )? If there is a relation, state it; if there is no relation, give the evidence to support your claim.

SOLUTION:

In all of these systems, the feedback block diagram is configured as follows



$$\frac{E(s)}{R(s)} = \frac{1}{1 + H(s)G(s)}$$

Since we are looking at steady state, we need to obtain the transfer function between  $E(s)$  and  $R(s)$ . Given that  $E(s) = R(s) - Y(s)$  and  $Y(s) = E(s)G(s)H(s)$  then we solve these two equations for  $E(s)$  by eliminating  $Y(s)$  giving

$$E(s) = R(s) - E(s)G(s)H(s)$$

$$E(s)(1 + G(s)H(s)) = R(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

The above is the transfer function used for the different  $R(s)$  signals: unit step  $u(t)$ , ramp  $t$ , and parabolic  $t^2$ .

## 3.2.1.1 part (a)

The open loop transfer function is  $\frac{1000}{(1+0.1s)(1+10s)}$ . Since the number of poles at zero is zero, *the system type<sup>1</sup> is zero.*

When the input is a unit step  $u(t)$ , then  $R(s) = \frac{1}{s}$ . Using the steady state error transfer function found above gives

$$\begin{aligned} E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\ &= \frac{1}{s} \frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} = \frac{1}{s} \frac{\left(1 + \frac{1}{10}s\right)(1+10s)}{\left(1 + \frac{1}{10}s\right)(1+10s) + 1000} = \frac{1}{s} \frac{\left(1 + \frac{1}{10}s\right)(1+10s)}{s^2 + \frac{101}{10}s + 1001} \end{aligned}$$

We see that the poles are located at  $s = 0, s = -5.05 \pm 31.233i$ . Therefore this is stable  $E(s)$  as the real parts of the poles are negative. We are allowed one pole at the origin. Applying the final value theorem gives

$$\begin{aligned} e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} \\ &= \frac{1}{1 + \frac{1000}{\lim_{s \rightarrow 0}(1+0.1s)(1+10s)}} \\ &= \frac{1}{1 + 1000} \end{aligned}$$

Hence

$$e_{ss} = \frac{1}{1001}$$

When the input is a ramp, then  $R(s) = \frac{1}{s^2}$ , therefore

$$\begin{aligned} E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\ &= \frac{1}{s^2} \frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} \end{aligned}$$

There are two poles at the origin and the other two poles are the same as above at  $s = -5.05 \pm 31.233i$ . Since there are two poles at the origin, the final value is not defined (taken from now on as infinity in order to be compatible with the text book result and notation).

<sup>1</sup>The system type is the number of poles at zero of the open loop transfer function  $G(s)H(s)$ .

Finally, when the input is  $t^2$ , then  $R(s) = \frac{2}{s^3}$  and

$$\begin{aligned} E(s) &= \frac{2}{s^3} \frac{1}{1 + G(s)H(s)} \\ &= \frac{2}{s^3} \frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} \\ &= \frac{2(1+0.1s)(1+10s)}{s^3(1+0.1s)(1+10s) + 1000s^3} \\ &= \frac{2(1+0.1s)(1+10s)}{s^3(s^2 + 10.1s + 1001)} \end{aligned}$$

There are now three poles at the origin  $s = 0$ . As above, this means the final value is taken as infinity.

### 3.2.1.2 part (b)

The open loop transfer function is  $\frac{1000}{s(1+10)(s+100)}$ . There is one pole at the origin which means *the system type is one*.

When the input is a unit step, then  $R(s) = \frac{1}{s}$  and

$$\begin{aligned} E(s) &= \frac{1}{s} \frac{R(s)}{1 + G(s)H(s)} \\ &= \frac{1}{s} \frac{1}{1 + \frac{1000}{s(1+10)(s+100)}} = \frac{1}{s + \frac{1000}{(1+10)(s+100)}} = \frac{(1+10)(s+100)}{s(1+10)(s+100) + 1000} = \frac{(1+10)(s+100)}{11s^2 + 1100s + 1000} \end{aligned}$$

The poles are at  $s = -0.9175, s = -99.08$ . This is stable  $E(s)$  and we can now apply the final value theorem

$$\begin{aligned} e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \frac{\frac{1}{s}}{1 + \frac{1000}{s(1+10)(s+100)}} \\ &= \frac{1}{1 + \lim_{s \rightarrow 0} \frac{1000}{s(1+10)(s+100)}} \end{aligned}$$

Hence

$$e_{ss} = 0$$

When the input is a ramp, then  $R(s) = \frac{1}{s^2}$  and

$$\begin{aligned}
 E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\
 &= \frac{1}{s^2} \frac{1}{1 + \frac{1000}{s(1+10)(s+100)}} \\
 &= \frac{1}{s^2 + \frac{1000s}{(1+10)(s+100)}} \\
 &= \frac{(1+10)(s+100)}{s^2(1+10)(s+100) + 1000s} \\
 &= \frac{(1+10)(s+100)}{s(11s^2 + 1100s + 1000)}
 \end{aligned}$$

There is one pole at the origin  $s = 0$  and the other two poles are the same at  $s = -0.9175, s = -99.08$ . This is stable. Applying the final value theorem gives

$$\begin{aligned}
 e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\
 &= \lim_{s \rightarrow 0} \frac{1}{s} \frac{1}{1 + \frac{1000}{s(1+10)(s+100)}} \\
 &= \lim_{s \rightarrow 0} \frac{1}{s + \frac{1000}{(1+10)(s+100)}} \\
 &= \lim_{s \rightarrow 0} \frac{(1+10)(s+100)}{s(1+10)(s+100) + 1000} \\
 &= \frac{(1+10)(100)}{1000}
 \end{aligned}$$

Hence

$$e_{ss} = 1.1$$

When the input is a  $t^2$ , then  $R(s) = \frac{2}{s^3}$  and

$$\begin{aligned}
 E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\
 &= \frac{2}{s^3} \frac{1}{1 + \frac{1000}{s(1+10)(s+100)}} \\
 &= \frac{2}{s^2 + \frac{1000s^2}{(1+10)(s+100)}} \\
 &= \frac{(2)(11)(s+100)}{s^2(11s+2100)}
 \end{aligned}$$

There are now two poles at the origin. Therefore final value is taken as infinity.

**3.2.1.3 part (c)**

The open loop transfer function is  $\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}$ . There are two poles at the origin which means *the system type is 2*.

When the input is a unit step, then  $R(s) = \frac{1}{s}$  and

$$\begin{aligned} E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\ &= \frac{1}{s} \frac{1}{1 + \frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}} \\ &= \frac{1}{s + \frac{K(1+2s)(1+4s)}{s(s^2+s+1)}} \\ &= \frac{s(s^2+s+1)}{s^2(s^2+s+1) + K(1+2s)(1+4s)} \end{aligned}$$

We have to now assume that  $E(s)$  is stable to be able to apply the final value theorem as this depends on the value of  $k$  which is not given in the problem. Therefore

$$\begin{aligned} e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \left( \frac{1}{s} \frac{1}{1 + \frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}} \right) \\ &= \lim_{s \rightarrow 0} \frac{s^2(s^2+s+1)}{s^2(s^2+s+1) + K(1+2s)(1+4s)} \\ &= \frac{0}{K} \end{aligned}$$

Which means

$$e_{ss} = 0$$

When the input is a ramp, then  $R(s) = \frac{1}{s^2}$ . Applying the final value theorem gives

$$\begin{aligned}
 e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{1}{s^2} \frac{1}{1 + \frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}} \right) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{1}{s^2 + \frac{K(1+2s)(1+4s)}{(s^2+s+1)}} \right) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{(s^2 + s + 1)}{s^2(s^2 + s + 1) + K(1 + 2s)(1 + 4s)} \right) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{1}{K} \right)
 \end{aligned}$$

Hence

$$e_{ss} = 0$$

When the input is  $t^2$  then  $R(s) = \frac{2}{s^3}$  and

$$\begin{aligned}
 e_{ss} &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{2}{s^3} \frac{1}{1 + \frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}} \right) \\
 &= \lim_{s \rightarrow 0} s \left( \frac{2}{s} \frac{(s^2 + s + 1)}{s^2(s^2 + s + 1) + K(1 + 2s)(1 + 4s)} \right) \\
 &= \lim_{s \rightarrow 0} \left( 2 \frac{1}{K} \right)
 \end{aligned}$$

Hence

$$e_{ss} = \frac{2}{K}$$

### 3.2.1.4 Part (d)

Summary of results from the above parts is

	$G(s)H(s)$	system type (number of poles at origin)	$e_{ss}$ step	$e_{ss}$ ramp	$e_{ss} t^2$
part(a)	$\frac{1000}{(1+0.1s)(1+10s)}$	0	$\frac{1}{1001}$	$\infty$	$\infty$
part(b)	$\frac{1000}{s(1+10)(s+100)}$	1	0	$\frac{11}{10}$	$\infty$
part(c)	$\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}$	2	0	0	$\frac{2}{K}$

From the above table, we see that as the system type (number of poles at origin of the open

loop  $G(s)H(s)$  increases, then the system can handle more signal types while still producing zero steady state error (this is good). The input signal that gives constant (non zero) steady state error per system type is summarized below.

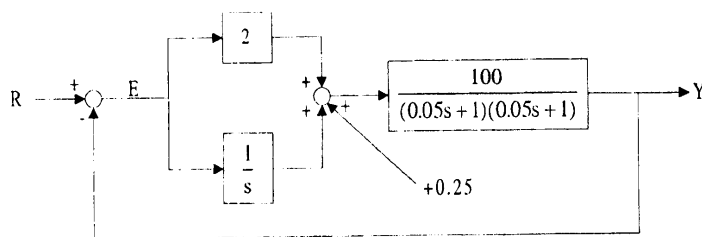
System type (open loop number of poles at origin)	Input that gives constant nonzero $e_{ss}$
0	step ( $t^0$ )
1	ramp ( $t^1$ )
2	parabolic ( $t^2$ )

So the relation between number of poles at origin of open loop and the type of signal that gives constant non zero steady state error can be written as

if the system type is  $m$  then nonzero constant  $e_{ss}$  is generated by signal  $t^m$ .

### 3.2.2 Problem 2

**Problem 2:** Consider the linear control system shown



and let  $R(t)=1.5t$ . What is the steady state error?

**SOLUTION:**

Let the first input  $R(s)$  be  $U_1(s)$  and the second input (the constant 0.25) be  $U_2(s)$ , then

$$Y(s) = \left[ \left( 2 + \frac{1}{s} \right) E(s) + U_2(s) \right] G(s)$$

And

$$E(s) = U_1(s) - Y(s)$$



Hence

$$\begin{aligned}
 E(s) &= U_1(s) - \left[ \left( 2 + \frac{1}{s} \right) E(s) + U_2(s) \right] G(s) \\
 &= U_1(s) - \left( 2 + \frac{1}{s} \right) E(s) G(s) - U_2(s) G(s) \\
 E(s) \left( 1 + \left( 2 + \frac{1}{s} \right) G(s) \right) &= U_1(s) - U_2(s) G(s) \\
 E(s) &= \frac{U_1(s) - U_2(s) G(s)}{1 + \left( 2 + \frac{1}{s} \right) G(s)} \tag{1}
 \end{aligned}$$

To obtain the error transfer function from  $E(s)$  to  $U_1(s)$ , the input  $U_2(s)$  is set to zero. To obtain the error transfer function from  $E(s)$  to  $U_2(s)$ , the input  $U_1(s)$  is set to zero. Applying these to (1) gives

$$\begin{aligned}
 \left. \frac{E(s)}{U_1(s)} \right|_{U_2=0} &= \frac{1}{1 + \left( 2 + \frac{1}{s} \right) G(s)} \\
 \left. \frac{E(s)}{U_2(s)} \right|_{U_1=0} &= \frac{-G(s)}{1 + \left( 2 + \frac{1}{s} \right) G(s)}
 \end{aligned}$$

In Matrix form,

$$E(s) = \begin{pmatrix} \frac{1}{1 + \left( 2 + \frac{1}{s} \right) G(s)} & \frac{-G(s)}{1 + \left( 2 + \frac{1}{s} \right) G(s)} \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

But  $U_1(s) = \frac{1.5}{s^2}$  and  $U_2(s) = \frac{0.25}{s}$ , and the above becomes

$$E(s) = \begin{pmatrix} \frac{1}{1 + \left( 2 + \frac{1}{s} \right) \frac{100}{(0.05s+1)^2}} & \frac{-\frac{100}{(0.05s+1)^2}}{1 + \left( 2 + \frac{1}{s} \right) \frac{100}{(0.05s+1)^2}} \end{pmatrix} \begin{pmatrix} \frac{1.5}{s^2} \\ \frac{0.25}{s} \end{pmatrix}$$

Hence

$$E(s) = \frac{1.5}{s} \frac{\overbrace{(0.05s+1)^2}^{E_1(s)}}{0.0025s^3 + 0.1s^2 + 201s + 100.0} - \frac{\overbrace{10000}^{E_2(s)}}{s^3 + 40s^2 + 80400s + 40000}$$

The poles of the first term are  $-19.751 \pm 282.83i$ ,  $s = -0.497$ ,  $s = 0$ , Hence this is stable and have at most one pole at origin. Then using F.V.T. gives

$$\begin{aligned}
 e_{ss1} &= \lim_{s \rightarrow 0} s E_1(s) \\
 &= \lim_{s \rightarrow 0} 1.5 \frac{(0.05s+1)^2}{0.0025s^3 + 0.1s^2 + 201s + 100} \\
 &= 0.015
 \end{aligned}$$

For  $E_2(s)$ , the poles are at ,  $s = -19.75 \pm 282.83i$ ,  $s = -0.498$ , Hence this is stable. Therefore

using F.V.T. gives

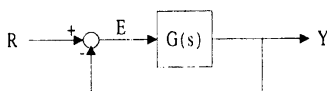
$$\begin{aligned} e_{ss2} &= \lim_{s \rightarrow 0} sE_2(s) \\ &= \lim_{s \rightarrow 0} \frac{10000s}{s^3 + 40s^2 + 80400s + 40000} \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} sE(s) \\ &= 0.015 - 0 \\ &= 0.015 \end{aligned}$$

### 3.2.3 Problem 3

**Problem 3:** Consider the closed loop system



and assume the following:

- (i) The steady state error for a step input is zero.
- (ii) The denominator of the closed loop transfer function  $\frac{Y(s)}{R(s)}$  (also called the characteristic polynomial of the closed loop system) is  $s^3 + 4s^2 + 6s + 4$ .

Find the transfer function  $G(s)$ . Also find the steady state error if the input is a unit ramp.

(Hint: Let  $n(s)$  and  $d(s)$  be the numerator and denominator of  $G(s)$ . Express the closed loop transfer function as a function of  $n(s)$  and  $d(s)$ )

**SOLUTION:**

Let  $G(s) = \frac{N(s)}{D(s)}$ . The closed loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{N(s)}{D(s)}}{1 + \frac{N(s)}{D(s)}} = \frac{N(s)}{D(s) + N(s)}$$

We are given that  $D(s) + N(s) = s^3 + 4s^2 + 6s + 4$ . The error transfer function is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + \frac{N(s)}{D(s)}} = \frac{D(s)}{D(s) + N(s)}$$

Substituting for  $D(s) + N(s)$  in the above with the given polynomial results in

$$\frac{E(s)}{R(s)} = \frac{D(s)}{s^3 + 4s^2 + 6s + 4} \quad (1)$$

We are told that  $\lim_{s \rightarrow 0} sE(s) = 0$  when  $R(s) = \frac{1}{s}$ . Applying this to (1) gives

$$E(s) = \frac{1}{s} \frac{D(s)}{s^3 + 4s^2 + 6s + 4}$$

$E(s)$  above is stable since the poles are at  $-2, -1 \pm i$  with another pole at zero. Hence F.V.T. can be applied to  $E(s)$

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \left( \frac{1}{s} \frac{D(s)}{s^3 + 4s^2 + 6s + 4} \right) \\ &= \lim_{s \rightarrow 0} \frac{D(s)}{4} \end{aligned}$$

We are also told that the above is zero. Hence

$$0 = \lim_{s \rightarrow 0} \frac{D(s)}{4}$$

The above implies that  $D(s)$  must contain only  $s$  terms and no constant terms, since we want  $D(s) = 0$  when  $s = 0$ .

Assuming proper transfer function  $G(s)$  where degree of  $N(s) \leq$  degree of  $D(s)$ , then  $D(s)$  can be  $s^3$  or  $s^3 + 4s^2$ , or  $s^3 + 4s^2 + 6s$ , since any of these will give  $\lim_{s \rightarrow 0} \frac{D(s)}{4} = 0$ . But  $D(s)$  can not be  $s^2$  for example, else  $G(s)$  will not proper  $G(s)$ .

There are actually an infinite number of  $D(s)$  polynomials which meets this condition (if we use fractions for the coefficients). Below is an example of two possible  $D(s)$  choices and the corresponding  $G(s)$

$$D_1(s) = s^3 + 4s^2 + 6s$$

Then

$$G(s) = \frac{4}{s^3 + 4s^2 + 6s}$$

For steady state when input is ramp, using the above  $G(s)$  gives

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s \left( \frac{1}{s^2} \frac{s^3 + 4s^2 + 6s}{s^3 + 4s^2 + 6s + 4} \right) \\ &= \lim_{s \rightarrow 0} \frac{s^2 + 4s + 6}{s^3 + 4s^2 + 6s + 4} \end{aligned}$$

Hence

$$e_{ss} = 1.5$$

Another choice is  $D_2(s) = s^3 + 4s^2$ . Using this,  $G(s) = \frac{6s+4}{s^3+4s^2}$ . Using this, and when the input

is ramp then

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s \left( \frac{1}{s^2} \frac{s^3 + 4s^2}{s^3 + 4s^2 + 6s + 4} \right) \\ &= \lim_{s \rightarrow 0} \frac{s^2 + 4s}{s^3 + 4s^2 + 6s + 4} \\ &= 0 \end{aligned}$$

So the steady state error for ramp depends on which  $G(s)$  is used.

### 3.2.4 Problem 4

**Problem 4:** Two feedback systems are shown in Figure 1 and Figure 2.

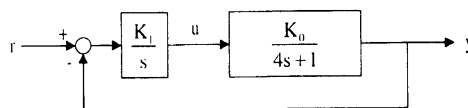


Figure 1: Feedback System 1

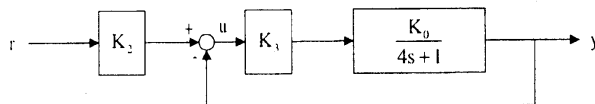


Figure 2: Feedback System 2

- Let  $K_0 = 1$ . Determine the values of  $K_1$  for system 1 and  $K_2$  and  $K_3$  for system 2 so that both of the systems exhibit zero steady error to step inputs and such the steady state error to a unit ramp is 1 in both cases.
- Suppose  $K_0$  changes from 1 to  $1 + \delta$ . Show that the steady state error with this perturbed  $K_0$  is still zero to a unit step input, for Figure 1. Also show that this is not the case for Figure 2.
- A control engineer would prefer the system in Figure 1 to the one in Figure 2. Do you agree with this statement? Justify.

SOLUTION:

#### 3.2.4.1 Part(a)

**For system 1.** Using  $K_0 = 0$  we first obtain expression for  $E(s)$  and  $Y(s)$

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ Y(s) &= E(s) \left( \frac{K_1}{s} \right) \left( \frac{1}{4s + 1} \right) \end{aligned}$$

Solving for  $E(s)$  from the above two equations gives

$$E(s) = R(s) - \left( E(s) \frac{K_1}{s(4s+1)} \right)$$

$$E(s) \left( 1 + \frac{K_1}{s(4s+1)} \right) = R(s)$$

$$E(s) = R(s) \frac{s(4s+1)}{s(4s+1) + K_1}$$

When  $R(s) = \frac{1}{s}$  we want  $e_{ss} = 0$ , therefore

$$e_{ss} = 0 = \lim_{s \rightarrow 0} sE(s)$$

$$0 = \lim_{s \rightarrow 0} \frac{s(4s+1)}{s(4s+1) + K_1}$$

$$= \lim_{s \rightarrow 0} \frac{s(4s+1)}{K_1}$$

$$= \frac{0}{K_1}$$

The above is true for any  $K_1$  since the numerator is already zero. Considering now the ramp input. When  $R(s) = \frac{1}{s^2}$  we want  $e_{ss} = 1$ , hence

$$e_{ss} = 1 = \lim_{s \rightarrow 0} sE(s)$$

$$1 = \lim_{s \rightarrow 0} s \frac{1}{s^2} \frac{s(4s+1)}{s(4s+1) + K_1}$$

$$= \lim_{s \rightarrow 0} \frac{4s+1}{s(4s+1) + K_1}$$

$$= \frac{1}{K_1}$$

Therefore

$$\boxed{K_1 = 1}$$

**For system 2**

$$E(s) = R(s) - Y(s) \tag{1}$$

But

$$Y(s) = U(s) K_3 \frac{1}{4s+1} \tag{2}$$

And

$$U(s) = R(s) K_2 - Y(s)$$

Hence (2) becomes

$$\begin{aligned}
 Y(s) &= (R(s)K_2 - Y(s))\frac{K_3}{4s+1} \\
 Y(s)\left(1 + \frac{K_3}{4s+1}\right) &= R(s)\frac{K_2K_3}{4s+1} \\
 Y(s) &= R(s)\frac{\frac{K_2K_3}{4s+1}}{1 + \frac{K_3}{4s+1}} \\
 &= R(s)\frac{K_2K_3}{4s+1+K_3}
 \end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$\begin{aligned}
 E(s) &= R(s) - R(s)\frac{K_2K_3}{4s+1+K_3} \\
 E(s) &= R(s)\left(1 - \frac{K_2K_3}{4s+1+K_3}\right)
 \end{aligned} \tag{4}$$

When  $R(s) = \frac{1}{s}$  we want  $e_{ss} = 0$ , hence

$$\begin{aligned}
 e_{ss} = 0 &= \lim_{s \rightarrow 0} sE(s) \\
 0 &= \lim_{s \rightarrow 0} s \frac{1}{s} \left(1 - \frac{K_2K_3}{4s+1+K_3}\right) \\
 &= 1 - \frac{K_2K_3}{1+K_3}
 \end{aligned}$$

For the above to be true, then

$$\boxed{\frac{K_2K_3}{1+K_3} = 1} \tag{5}$$

We now obtain a second equation from the ramp condition. When  $R(s) = \frac{1}{s^2}$  we want  $e_{ss} = 1$ , hence

$$\begin{aligned}
 e_{ss} = 1 &= \lim_{s \rightarrow 0} sE(s) \\
 1 &= \lim_{s \rightarrow 0} s \frac{1}{s^2} \left(1 - \frac{K_2K_3}{4s+1+K_3}\right) \\
 &= \lim_{s \rightarrow 0} \frac{1}{s} \left(1 - \frac{K_2K_3}{4s+1+K_3}\right) \\
 &= \lim_{s \rightarrow 0} \frac{1}{s} \left(\frac{4s+1+K_3 - (K_2K_3)}{4s+1+K_3}\right)
 \end{aligned}$$

Replacing  $K_2K_3$  in the above with  $1 + K_3$  found in (5) gives

$$\begin{aligned} 1 &= \lim_{s \rightarrow 0} \frac{1}{s} \left( \frac{4s + 1 + K_3 - (1 + k_3)}{4s + 1 + K_3} \right) \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \left( \frac{4s}{4s + 1 + K_3} \right) \\ &= \frac{4}{\lim_{s \rightarrow 0} (4s + 1 + K_3)} \\ 1 &= \frac{4}{1 + K_3} \end{aligned}$$

Hence  $1 + K_3 = 4$  or

$$\boxed{K_3 = 3}$$

Now that we found  $K_3$  we go back to (5) and solve for  $K_2$

$$\begin{aligned} \frac{K_2K_3}{1 + K_3} &= 1 \\ K_2 &= \frac{1 + K_3}{K_3} \\ &= \frac{1 + 3}{3} \end{aligned}$$

Hence

$$\boxed{K_2 = \frac{4}{3}}$$

Summary

	$K_1$	$K_2$	$K_3$
system 1	1	N/A	N/A
system 2	N/A	$\frac{4}{3}$	3

### 3.2.4.2 Part (b)

For system 1.

$$E(s) = R(s) - Y(s)$$

$$Y(s) = E(s) \frac{K_1(1 + \delta)}{s(4s + 1)}$$

Hence

$$\begin{aligned}
 E(s) &= R(s) - \left( E(s) \frac{K_1 (1 + \delta)}{s (4s + 1)} \right) \\
 E(s) \left( 1 + \frac{K_1 (1 + \delta)}{s (4s + 1)} \right) &= R(s) \\
 E(s) &= R(s) \frac{1}{1 + \frac{K_1 (1 + \delta)}{s (4s + 1)}} \\
 &= R(s) \frac{s (4s + 1)}{s (4s + 1) + K_1 (1 + \delta)}
 \end{aligned}$$

When  $R(s) = \frac{1}{s}$  then

$$\begin{aligned}
 e_{ss} &= \lim_{s \rightarrow 0} sE(s) \\
 &= \lim_{s \rightarrow 0} \frac{s (4s + 1)}{s (4s + 1) + K_1 (1 + \delta)} \\
 &= \frac{\lim_{s \rightarrow 0} s (4s + 1)}{K_1 (1 + \delta)} \\
 &= \frac{0}{K_1 (1 + \delta)}
 \end{aligned}$$

The above is zero for any  $K_1$  and any perturbation  $\delta$  since the numerator is already zero. This is the same condition we found in part(a). Perturbing  $K_0$  has no effect on the result of  $e_{ss}$  for step input.

**For system 2**

$$E(s) = R(s) - Y(s) \tag{1}$$

But

$$Y(s) = U(s) K_3 \frac{1 + \delta}{4s + 1} \tag{2}$$

And

$$U(s) = R(s) K_2 - Y(s)$$

Replacing  $U(s)$  into (2)

$$\begin{aligned}
 Y(s) &= (R(s) K_2 - Y(s)) K_3 \frac{1 + \delta}{4s + 1} \\
 Y(s) \left( 1 + \frac{K_3 (1 + \delta)}{4s + 1} \right) &= R(s) \frac{K_2 K_3 (1 + \delta)}{4s + 1} \\
 Y(s) &= R(s) \frac{\frac{K_2 K_3 (1 + \delta)}{4s + 1}}{1 + \frac{K_3 (1 + \delta)}{4s + 1}} \\
 &= R(s) \frac{K_2 K_3 (1 + \delta)}{4s + 1 + K_3 (1 + \delta)} \tag{3}
 \end{aligned}$$



Substituting  $Y(s)$  from (3) into (1) gives

$$\begin{aligned} E(s) &= R(s) - R(s) \frac{K_2 K_3 (1 + \delta)}{4s + 1 + K_3 (1 + \delta)} \\ &= R(s) \left( 1 - \frac{K_2 K_3 (1 + \delta)}{4s + 1 + K_3 (1 + \delta)} \right) \end{aligned} \quad (4)$$

When  $R(s) = \frac{1}{s}$  and using the F.V.T. gives

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \frac{1}{s} \left( 1 - \frac{K_2 K_3 (1 + \delta)}{4s + 1 + K_3 (1 + \delta)} \right) \\ &= \lim_{s \rightarrow 0} \left( 1 - \frac{K_2 K_3 (1 + \delta)}{4s + 1 + K_3 (1 + \delta)} \right) \\ &= 1 - \frac{K_2 K_3 (1 + \delta)}{1 + K_3 (1 + \delta)} \end{aligned}$$

For the above  $e_{ss}$  to be zero, then the condition is that

$$\frac{K_2 K_3 (1 + \delta)}{1 + K_3 (1 + \delta)} = 1$$

or

$$\begin{aligned} K_2 K_3 (1 + \delta) &= 1 + K_3 (1 + \delta) \\ K_2 K_3 - K_3 &= \frac{1}{1 + \delta} \end{aligned}$$

Using  $K_2 = \frac{4}{3}$  and  $K_3 = 3$  found in part (a) then the above becomes

$$\begin{aligned} \left( \frac{4}{3} \right) 3 - 3 &= \frac{1}{1 + \delta} \\ 1 &= \frac{1}{1 + \delta} \end{aligned}$$

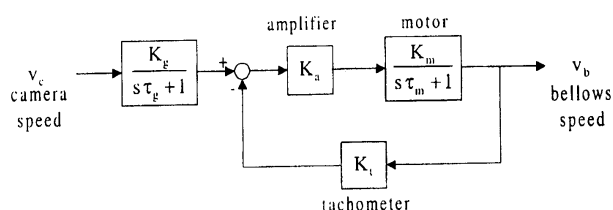
But this is **impossible** since the RHS must be either larger than one or smaller than one (depending on the sign of  $\delta$ ). This means if  $K_0$  is perturbed from unity, then it is no longer possible to obtain zero steady state error for a step input with the same  $k_2, k_3$ .

### 3.2.4.3 Part(c)

I agree. For first system, it gives  $e_{ss} = 0$  for a step input regardless of the value of  $K_1$  or  $K_0$  as was shown in part (b) above. But for system two,  $e_{ss} = 0$  for step input only when using specific values of  $K_i$ . Any small change in  $K_0$ , the steady state error is no longer zero. In other words, system one is more robust in this regard to changes in  $K_0$  and it is therefore the preferred system.

## 3.2.5 Problem 5

**Problem 5:** An important problem for television camera systems is the jumping and wobbling of the picture due to movement of the camera. This effect occurs when the camera is mounted on a moving truck or airplane. A system has been designed (shown below) which is intended to reduce the effect of rapid scanning motion. A maximum scanning motion of 25% is expected.



- (a) Determine the steady state error of the system for a step input  $V_c(s) = \frac{25}{s}$ . Assume that  $\tau_g$  is "negligible" and  $K_g = K_t = 1$ .
- (b) Determine the necessary loop gain  $K_a K_m$  when a  $1^\circ/\text{sec}$  steady state error is allowable. (Same assumptions as Part (a))
- (c) Show that the step response of the system is of the form

$$v_b(t) = \frac{k}{q} \{1 - e^{-qt}\}$$

under the assumptions in Part (a). Express  $k$  and  $q$  in terms of the system parameters.

- (d) The settling time is defined as the time it takes for the step response to be within 2% of the steady state value. Given the expression of the step response determined in Part (c), derive the expression for the settling time of  $v_b$ . Also, find the loop gain  $K_a K_m$  so that the settling time of  $v_b$  is less than or equal to 0.04 sec. Take  $\tau_m = 0.4$  sec as the motor time constant.

SOLUTION:

## 3.2.5.1 Part (a)

We first need to find  $\frac{E(s)}{V_c(s)}$ . From the block diagram<sup>2</sup>.

$$V_b(s) = E(s) K_a \left( \frac{K_m}{1 + s\tau_m} \right) \quad (1)$$

And

$$E(s) = V_c(s) \left( \frac{K_g}{1 + s\tau_g} \right) - K_t V_b(s) \quad (2)$$

<sup>2</sup>Notice that the problem is saying  $E(s)$  is the variable to the left of the amplifier  $K_a$  and this solution is based on this and not on using  $E(s) = V_b(s) - V_c(s)$

Replacing  $V_b(s)$  in (2) with (1) gives

$$E(s) = V_c(s) \left( \frac{K_g}{1 + s\tau_g} \right) - K_t E(s) K_a \left( \frac{K_m}{1 + s\tau_m} \right)$$

$$E(s) \left( 1 + K_t K_a \left( \frac{K_m}{1 + s\tau_m} \right) \right) = V_c(s) \left( \frac{K_g}{1 + s\tau_g} \right)$$

$$\frac{E(s)}{V_c(s)} = \frac{\left( \frac{K_g}{1 + s\tau_g} \right)}{1 + K_t K_a \left( \frac{K_m}{1 + s\tau_m} \right)}$$

Hence

$$\boxed{\frac{E(s)}{V_c(s)} = \left( \frac{1 + s\tau_m}{1 + s\tau_g} \right) \frac{K_g}{1 + s\tau_m + K_t K_a K_m}}$$

When  $V_c(s) = \frac{25}{s}$ ,  $K_g = K_t = 1$  then  $E(s)$  from above becomes

$$E(s) = \frac{25}{s} \left( \frac{1 + s\tau_m}{1 + s\tau_g} \right) \frac{1}{(1 + K_a K_m) + s\tau_m}$$

The above  $E(s)$  has one pole at the origin, and has a pole at  $s = \frac{-1}{\tau_g}$  and a pole at  $s = -\frac{1 + K_a K_m}{\tau_m}$ .

Hence this is stable (assuming  $K_a K_m > -1$ ). Applying F.V.T. gives

$$e_{ss} = \lim_{s \rightarrow 0} sE(s)$$

$$= \lim_{s \rightarrow 0} 25 \left( \frac{1 + s\tau_m}{1 + s\tau_g} \right) \frac{1}{(1 + K_a K_m) + s\tau_m}$$

Hence

$$e_{ss} = \frac{25}{1 + K_a K_m}$$

### 3.2.5.2 Part(b)

When  $e_{ss}$  is one degree per second, then from the above

$$1 = \frac{25}{1 + K_a K_m}$$

Or

$$\boxed{K_a K_m = 24}$$

## 3.2.5.3 Part(c)

To find the step response, we find the closed loop  $\frac{V_b(s)}{V_c(s)}$  transfer function first. Substituting (2) into (1) found in part (a) above to obtain an expression for  $V_b(s)$

$$V_b(s) = \left( V_c(s) \left( \frac{K_g}{1 + s\tau_g} \right) - K_t V_b(s) \right) K_a \left( \frac{K_m}{1 + s\tau_m} \right)$$

$$V_b(s) \left( 1 + \frac{K_m K_t K_a}{1 + s\tau_m} \right) = V_c(s) \frac{K_g K_a K_m}{(1 + s\tau_g)(1 + s\tau_m)}$$

Hence the closed loop transfer function is

$$\frac{V_b(s)}{V_c(s)} = \frac{\frac{K_g K_a K_m}{(1 + s\tau_g)(1 + s\tau_m)}}{1 + \frac{K_m K_t K_a}{1 + s\tau_m}}$$

$$= \frac{1}{(1 + s\tau_g)} \frac{K_g K_a K_m}{(1 + s\tau_m) + K_m K_t K_a}$$

Using same assumptions as part (a), and now using that  $\tau_g$  is negligible so that  $\frac{1}{(1 + s\tau_g)} \approx 1$  in the above, and using  $V_c(s) = \frac{1}{s}$  since we are told in this part it is a step input (should we have used  $\frac{25}{s}$  again here? It is not clear, but it says step input so I think  $\frac{1}{s}$  should be used in this part), then the above simplifies to

$$V_b(s) = \left( \frac{1}{s} \right) \frac{K_a K_m}{(1 + s\tau_m) + K_m K_a}$$

$$= \frac{1}{s} \frac{K_a K_m}{(1 + K_m K_a) + s\tau_m}$$

$$= \frac{K_a K_m}{\tau_m} \frac{1}{s} \frac{1}{s + \left( \frac{1 + K_m K_a}{\tau_m} \right)}$$

We now need to find the inverse Laplace transform. Using partial fractions

$$\left( \frac{K_a K_m}{\tau_m} \right) \frac{1}{s} \frac{1}{s + \left( \frac{1 + K_m K_a}{\tau_m} \right)} = \frac{A}{s} + \frac{B}{s + \left( \frac{1 + K_m K_a}{\tau_m} \right)} \quad (3)$$

Hence

$$A = \lim_{s \rightarrow 0} \left( \frac{K_a K_m}{\tau_m} \right) \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s} = \frac{K_a K_m}{1 + K_m K_a}$$

And

$$B = \lim_{s \rightarrow -\frac{1 + K_m K_a}{\tau_m}} \frac{K_a K_m}{\tau_m} \frac{1}{s} = \frac{K_a K_m}{\tau_m} \frac{1}{\frac{-(1 + K_m K_a)}{\tau_m}} = -\frac{K_a K_m}{1 + K_m K_a}$$

Now that we found  $A, B$  using partial fractions, we replace these values in (3) to obtain  $V_b(s)$

$$V_b(s) = \frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s} \quad (4)$$

Now we can apply inverse Laplace transform. Hence

$$\begin{aligned} v_b(t) &= \left( \frac{K_a K_m}{1 + K_m K_a} - \frac{K_a K_m}{1 + K_m K_a} e^{-\frac{1 + K_m K_a}{\tau_m} t} \right) u(t) \\ &= \frac{K_a K_m}{1 + K_m K_a} \left( 1 - e^{-\frac{1 + K_m K_a}{\tau_m} t} \right) u(t) \end{aligned}$$

Let

$$q = \frac{1 + K_m K_a}{\tau_m}$$

and

$$k = \frac{K_a K_m}{1 + K_m K_a}$$

Then  $v_b(t)$  can be written as required

$$v_b(t) = \frac{k}{q} \left( 1 - e^{-qt} \right) u(t)$$

### 3.2.5.4 Part(d)

We first need to find the steady state  $v_b(t)$ . From (4) found above in part (c)

$$V_b(s) = \frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s}$$

Then applying F.V.T. assuming stability

$$\begin{aligned} V_b(\infty) &= \lim_{s \rightarrow 0} s V_b(s) \\ &= \lim_{s \rightarrow 0} s \left( \frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s} \right) \\ &= \frac{K_a K_m}{1 + K_m K_a} \\ &= \frac{k}{q} \end{aligned}$$

Let the settling time be  $t_s$ , then we want to solve for  $t_s$  from

$$\begin{aligned} v_b(t_s) &= 0.98 V_b(\infty) \\ \frac{k}{q} \left( 1 - e^{-qt_s} \right) &= 0.98 \frac{k}{q} \\ 1 - e^{-qt_s} &= 0.98 \\ e^{-qt_s} &= 0.02 \end{aligned}$$

Taking natural logs on both sides gives

$$\begin{aligned} -qt_s &= \ln(0.02) \\ qt_s &= 3.912 \end{aligned}$$

Hence

$$t_s = \frac{3.912}{\left(\frac{1+K_m K_a}{t_m}\right)}$$

Using  $t_m = 0.4$  seconds in the above gives

$$t_s = \frac{1.5648}{1 + K_m K_a}$$

For  $t_s \leq 0.04$  then

$$\begin{aligned} \frac{1.5648}{(1 + K_m K_a)} &\leq 0.04 \\ 1 + K_m K_a &\geq \frac{1.5648}{0.04} \\ 1 + K_m K_a &\geq 39.12 \end{aligned}$$

Hence

$$K_m K_a \geq 38.12$$

## 3.2.6 HW 2 key solution

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## HW #2 Solutions

$$1\text{-}(a) \quad W_e(s) = \frac{s^2 + 10.1s + 1}{s^2 + 10.1s + 1001}$$

$$e_{ss}|_{\text{step}} = \frac{1}{1001} \quad e_{ss}|_{\text{ramp}} = \infty \quad e_{ss}|_{\text{parabola}} = \infty$$

$$1\text{-}(b) \quad W_e(s) = \frac{s(s+10)(s+100)}{s(s+10)(s+100)+1000}$$

$$e_{ss}|_{\text{step}} = 0 \quad e_{ss}|_{\text{ramp}} = 1 \quad e_{ss}|_{\text{parabola}} = \infty$$

$$1\text{-}(c) \quad W_e(s) = \frac{s^2(s^2 + s + 1)}{s^2(s^2 + s + 1) + K(1 + 2s)(1 + 4s)}$$

$$e_{ss}|_{\text{step}} = 0 \quad e_{ss}|_{\text{ramp}} = 0 \quad e_{ss}|_{\text{parabola}} = \frac{1}{K}$$

1-(d) Systems with  $e_{ss}|_{\text{step}} = k$  (being  $k$  finite and different from 0) have  $G(s)$  with no poles at the origin ( $s = 0$ ); Systems with  $e_{ss}|_{\text{ramp}} = k$  (being  $k$  finite and different from 0) have  $G(s)$  with one pole at the origin; Systems with  $e_{ss}|_{\text{parabola}} = k$  (being  $k$  finite and different from 0) have  $G(s)$  with two poles at the origin.

;

2

$$E(s) = \frac{1}{1 + \left(2 + \frac{1}{s}\right) \frac{100}{(0.05s+1)(0.05s+1)}} R(s) + \frac{-\frac{100}{(0.05s+1)(0.05s+1)}}{1 + \left(2 + \frac{1}{s}\right) \frac{100}{(0.05s+1)(0.05s+1)}} D(s)$$

Where  $D(s)$  is the constant input  $D(s) = \frac{0.25}{s}$ .

So  $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$

$$\begin{aligned} &= \lim_{s \rightarrow 0} \left( s \frac{s(0.05s+1)(0.05s+1)}{s(0.05s+1)(0.05s+1) + (2s+1)100} \frac{0.25}{s} - s \frac{-100s}{s(0.05s+1)(0.05s+1) + (2s+1)100} \frac{15}{s^2} \right) \\ &= \frac{15}{100} = 0.015 \end{aligned}$$

3 Let  $G(s) = \frac{n(s)}{d(s)}$ , where  $n(s)$  and  $d(s)$  are polynomials. Then

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + \frac{n(s)}{d(s)}} = \frac{d(s)}{d(s) + n(s)}$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{n(s)}{d(s)}}{1 + \frac{n(s)}{d(s)}} = \frac{n(s)}{d(s) + n(s)}$$

Since the denominator of the closed loop function is  $s^3 + 4s^2 + 6s + 4$  then we have

$$n(s) + d(s) = s^3 + 4s^2 + 6s + 4$$

Now, since the steady state error for a step input is zero, we have

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{d(s)}{n(s) + d(s)} R(s) = \lim_{s \rightarrow 0} s \frac{d(s)}{s^3 + 4s^2 + 6s + 4} \frac{1}{s} = 0$$

;



The equation above implies that  $\lim_{s \rightarrow 0} d(s) = 0$ , or in other words  $d(s) = s p(s)$ , where  $p(s)$  is a polynomial. So just take

$$d(s) = s^3 + 4s^2 + 6s$$

and as consequence we have

$$n(s) = 4$$

$$G(s) = \frac{4}{s^3 + 4s^2 + 6s}$$

Note that this is not the only possible answer.  
The steady state to a unit ramp is

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{s^3 + 4s^2 + 6s}{s^3 + 4s^2 + 6s + 4} \frac{1}{s^2} = \frac{6}{4} = 1.5$$

4-(a) First find the transfer functions for each system

$$\text{Figure 1: } \frac{Y(s)}{R(s)} = \frac{K_1 K_0}{s(4s+1) + K_1 K_0}$$

$$\text{Figure 2: } \frac{Y(s)}{R(s)} = \frac{K_3 K_2 K_0}{(4s+1) + K_3 K_0}$$

Next calculate the steady state error due to a step.

$$\text{Figure 1: } W_e(s) = R(s) - Y(s) = R(s) \left[ 1 - \frac{Y(s)}{R(s)} \right]$$

$$E(s) = R(s) \left[ 1 - \frac{K_1 K_0}{s(4s+1) + K_1 K_0} \right]$$

$$E(s) = R(s) \left[ \frac{4s^2 + s}{4s^2 + s + K_1 K_0} \right]$$

$$e_{\text{step}} = \lim_{s \rightarrow 0} \frac{4s^2 + s}{4s^2 + s + K_1 K_0} = 0$$

Figure 2: 
$$W_e(s) = R(s) - Y(s) = R(s) \left[ 1 - \frac{Y(s)}{R(s)} \right]$$

$$E(s) = R(s) \left[ 1 - \frac{K_3 K_2 K_0}{(4s+1) + K_3 K_0} \right]$$

$$E(s) = R(s) \left[ \frac{(4s+1) + K_3 K_0 - K_3 K_2 K_0}{(4s+1) + K_3 K_0} \right]$$

$$e_{ss}|_{\text{step}} = \lim_{s \rightarrow 0} \frac{(4s+1) + K_3 K_0 - K_3 K_2 K_0}{(4s+1) + K_3 K_0}$$

$$e_{ss}|_{\text{step}} = \frac{1 + K_3 K_0 - K_3 K_2 K_0}{1 + K_3 K_0} = 0 \quad \#$$

When  $1 + K_3 K_0 - K_3 K_2 K_0 = 0$  then the step response will have zero steady-state error. The constants  $K_2$  and  $K_3$  will be chosen so that this is true.

Now that we know that both of the systems have zero steady-state error to step inputs, we can say that for each transfer function,  $TF(s)$ , the DC gain (gain at frequency zero) is 1 which means  $TF(0) = 1$  for both systems.

Next, we can find the steady state error due to unit ramp. From the equations above, we see the pattern is  $W_e(s) = R(s)[1 - TF(s)]$  which means that

$$e_{ss}|_{\text{ramp}} = \lim_{s \rightarrow 0} s[1 - TF(s)] \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1 - TF(s)}{s} = \left. \frac{d(TF(s))}{ds} \right|_{s=0}$$

Given that we want  $e_{ss}|_{\text{ramp}} = 1$ , we have the following equations for systems that we're studying

Figure 1: 
$$-\left. \frac{d(TF(s))}{ds} \right|_{s=0} = \frac{(8s+1)K_1 K_0}{(4s^2 + s + K_1 K_0)^2} \Big|_{s=0} = \frac{1}{K_1 K_0} = \frac{1}{K_1} = 1$$

$$\Rightarrow K_1 = 1$$

Figure 2: 
$$-\left. \frac{d(TF(s))}{ds} \right|_{s=0} = \frac{4K_3 K_2}{(1 + K_3)^2} = 1$$

$$\Rightarrow 4K_3 K_2 = (1 + K_3)^2 \quad *$$

From equation # we get  $1 + K_3 = K_3 K_2$ . Substituting this into equation \* we get

$$4(1 + K_3) = (1 + K_3)^2 \Rightarrow K_3 = -1 \quad \text{or} \quad K_3 = 3$$

Solving for  $K_2$  in equation \* we find that

$$K_2 = \frac{1 + K_3}{K_3}$$

So, we get  $K_2 = 0$  or  $K_2 = \frac{4}{3}$  but  $K_2 = 0$  is not feasible.

The final result is then

$$K_1 = 1 \quad K_2 = \frac{4}{3} \quad K_3 = 3$$

4-(b) Let  $K_0 = 1 + \delta$

$$\text{Figure 1: } e_{ss}|_{step} = \lim_{s \rightarrow 0} \frac{4s^2 + s}{4s^2 + s + K_1 K_0} = \lim_{s \rightarrow 0} \frac{4s^2 + s}{4s^2 + s + K_1(1 + \delta)} = 0$$

which is independent of  $\delta$ .

$$\text{Figure 2: } e_{ss}|_{step} = \lim_{s \rightarrow 0} \frac{(4s + 1) + K_3 K_0 - K_3 K_2 K_0}{(4s + 1) + K_3 K_0}$$

$$e_{ss}|_{step} = \lim_{s \rightarrow 0} \frac{(4s + 1) + K_3(1 + \delta) - K_3 K_2(1 + \delta)}{(4s + 1) + K_3(1 + \delta)}$$

Substituting with the results obtained in Part (a) we get

$$e_{ss}|_{step} = \frac{\delta(3 - 3\frac{4}{3})}{1 + 3 + 3\delta} \neq 0$$

Hence, the steady state error is not zero in this case

4-(c) I agree because the first system is clearly robust to variations in  $K_0$ , while the second is not.

;

5. Under stated assumption, the error TF is

$$\frac{E(s)}{V_c(s)} = \frac{1}{1 + \frac{K_a K_m}{s\tau_m + 1}}$$

A straightforward calculation with  $V_c(s) = 25/s$  leads to

$$e_{s\mathfrak{z}} = \lim_{s \rightarrow 0} sE(s) = \frac{25}{1 + K_a K_m}$$

5-(b) The condition

$$\frac{25}{1 + K_a K_m} = 1$$

leads to  $K_a K_m = 24$ .

5-(c) With  $R(s) = 1/s$ , a Mason calculation gives bellows output

$$V_b(s) = \frac{\frac{K_a K_m}{\tau_m}}{s(s + \frac{1 + K_a K_m}{\tau_m})} = \frac{K}{s(s + q)}$$

where

$$K = \frac{K_a K_m}{\tau_m}; \quad q = \frac{1 + K_a K_m}{\tau_m}$$

Now, the inverse Laplace transform gives the desired result.

5-(d) Enforcing the error requirement

$$e(t) = \frac{K}{q} e^{-qt} \leq \frac{0.02K}{q} \quad \text{at } t = 0.04$$

leads to

$$.04q \geq 4; \quad \text{i.e., } 0.04\left(\frac{1 + K_a K_m}{0.4}\right) \geq 4$$

with resulting loop gain requirement  $K_a K_m \geq 39$ .

### 3.3 HW 3

#### 3.3.1 Problem 1

1. Consider the system with transfer function

$$G(s) = \frac{Ks + 1}{s^3 + s^2 + (K + k)s + 1}.$$

(a) For  $K = 1$ , find the sensitivity  $S_k^G$  of this transfer function with respect to  $k$  assuming nominal value  $k = 2$ . Then plot its magnitude as a function of frequency.

(b) Repeat (a) with  $K = 100$  and compare the effect of a large loop gain on the sensitivity.

SOLUTION:

$$G(s) = \frac{Ks + 1}{s^3 + s^2 + (K + k)s + 1}$$

#### 3.3.1.1 Part (a)

For  $K = 1$  the above becomes

$$G(s) = \frac{s + 1}{s^3 + s^2 + (1 + k)s + 1}$$

Hence

$$\begin{aligned} S_k^G &= \frac{dG}{dk} \frac{k}{G} \\ &= \frac{d}{dk} \frac{s + 1}{s^3 + s^2 + (1 + k)s + 1} \left( \frac{k}{\frac{s + 1}{s^3 + s^2 + (1 + k)s + 1}} \right) \\ &= \frac{-s(s + 1)}{(s + ks + s^2 + s^3 + 1)^2} \frac{k(s^3 + s^2 + (1 + k)s + 1)}{s + 1} \\ &= \frac{-ks}{1 + (1 + k)s + s^2 + s^3} \end{aligned}$$

At nominal  $k = 2$  the above becomes

$$S_k^G|_{k=2} = \frac{-2s}{3s + s^2 + s^3 + 1}$$

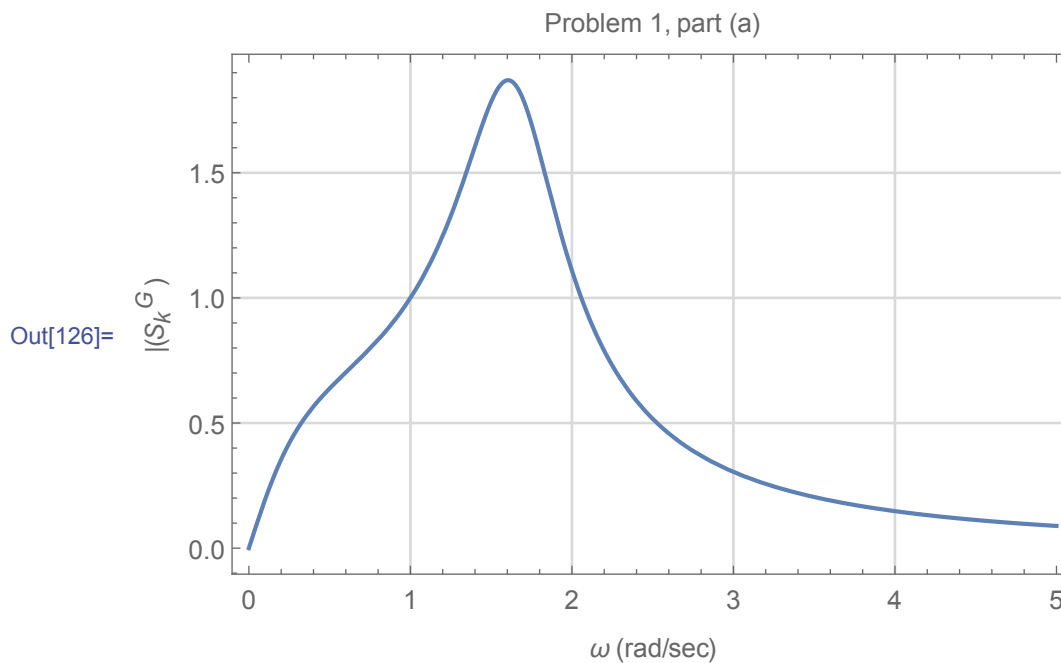
Let  $s = j\omega$  then

$$S_k^G = \frac{-2(j\omega)}{3j\omega + (j\omega)^2 + (j\omega)^3 + 1} = \frac{-2j\omega}{j(3\omega - \omega^3) + (1 - \omega^2)}$$

Taking the magnitude

$$|S_k^G| = \frac{2\omega}{\sqrt{(1 - \omega^2)^2 + (3\omega - \omega^3)^2}}$$

Here is a plot of  $|S_k^G|$  as function of  $\omega$



### 3.3.1.2 Part (b)

For  $K = 100$  the transfer function becomes

$$\begin{aligned} G(s) &= \frac{Ks + 1}{s^3 + s^2 + (K + k)s + 1} \\ &= \frac{100s + 1}{s^3 + s^2 + (100 + k)s + 1} \end{aligned}$$

Hence

$$\begin{aligned}
 S_k^G &= \frac{dG}{dk} \frac{k}{G} \\
 &= \frac{d}{dk} \frac{100s+1}{s^3+s^2+(100+k)s+1} \left( \frac{k}{\frac{100s+1}{s^3+s^2+(100+k)s+1}} \right) \\
 &= \frac{-s(100s+1)}{(100s+ks+s^2+s^3+1)^2} \frac{k(s^3+s^2+(100+k)s+1)}{100s+1} \\
 &= \frac{-ks}{100s+ks+s^2+s^3+1}
 \end{aligned}$$

At nominal  $k = 2$  the above becomes

$$\begin{aligned}
 S_k^G|_{k=2} &= \frac{-2s}{100s+2s+s^2+s^3+1} \\
 &= \frac{-2s}{102s+s^2+s^3+1}
 \end{aligned}$$

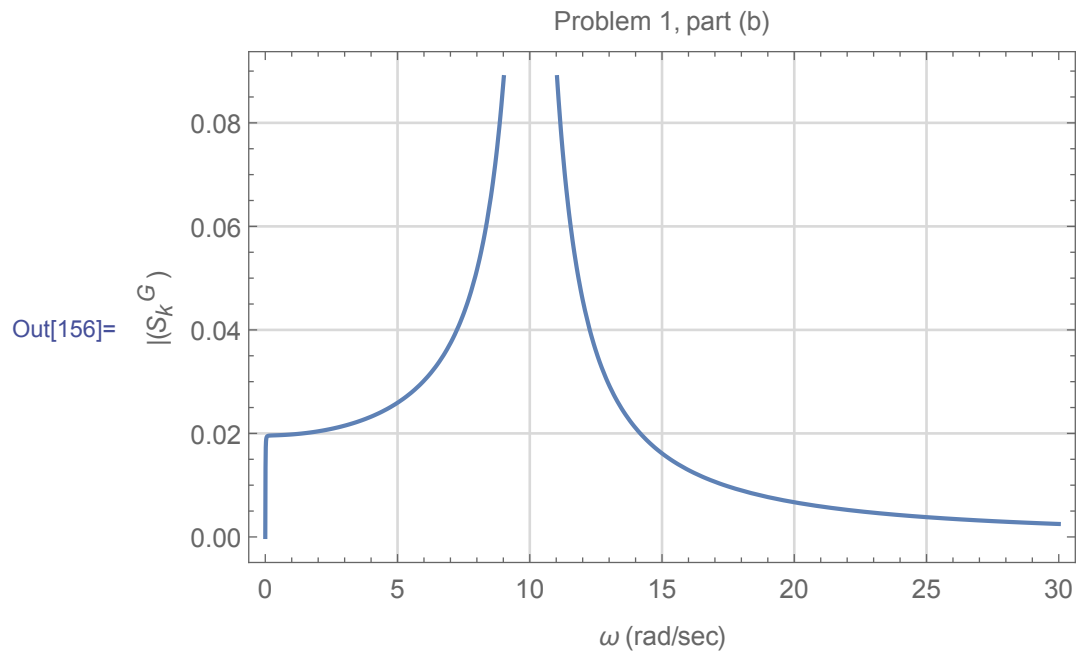
Let  $s = j\omega$  then

$$\begin{aligned}
 S_k^G &= \frac{-2(j\omega)}{102j\omega + (j\omega)^2 + (j\omega)^3 + 1} \\
 &= \frac{-2j\omega}{102j\omega - \omega^2 - j\omega^3 + 1} \\
 &= \frac{-2j\omega}{j(102\omega - \omega^3) + (1 - \omega^2)}
 \end{aligned}$$

Taking the magnitude

$$|S_k^G| = \frac{2\omega}{\sqrt{(1 - \omega^2)^2 + (102\omega - \omega^3)^2}}$$

Here is a plot of  $|S_k^G|$  as function of  $\omega$



We clearly see that as  $K$  became much larger, resonance occurs near  $\omega = 10$ . This shows that sensitivity of transfer function to changes in  $k$ , depends on the value of  $K$ .



## 3.3.2 Problem 2

2. In many instances, steady state errors in control system are due to some non-linearities such as dead zones. A 'Class B' amplifier is a typical example of a device having a dead zone where it takes  $\sim 1$  Volt of input signal to turn on the transistor. Once the device is on, however, it can be assumed to function linearly. The Class B amplifier in Figure 1 can be characterized mathematically (assuming a 1V threshold voltage and  $V_{cc} = \infty$ ) by  $N(\cdot)$  which is given by:

$$Y = \begin{cases} 0 & \text{if } -1 \leq U \leq 1, \\ U - 1 & \text{if } U > 1, \\ U + 1 & \text{if } U < -1. \end{cases}$$

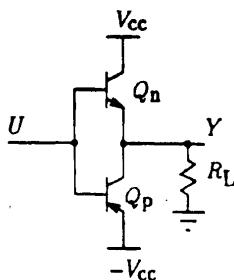


Figure 1: Class B Amplifier Circuit

- (a) Develop a plot of  $Y$  versus  $R$  with  $K = 1$  in Figure 2.

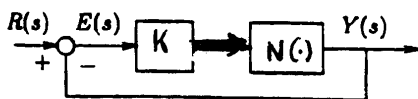
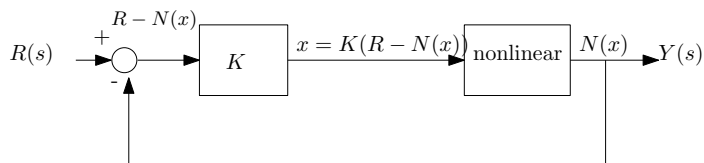


Figure 2: Feedback System

- (b) If  $r(t) = 5 \sin(t)$ , sketch  $y(t)$ .
- (c) Now let  $K = 10$  and develop a plot of  $Y(s)$  versus  $R(s)$ . Comment on the difference in the output due to the sinusoid - consider amplitude, distortion, etc.
- (d) Comment on the change to  $Y(s)$  versus  $R(s)$  if the gain block  $K$  were placed in the feedback loop instead of the forward path.

SOLUTION:

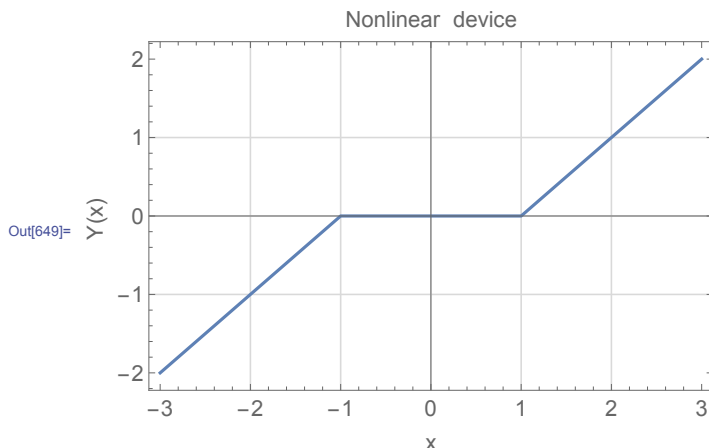
3.3.2.1 Part (a)



From the above we see that

$$Y = N(x) \tag{1}$$

The plot of  $Y(x)$  is given below based on the definition given in the problem



At the output of the controller we have

$$\begin{aligned} x &= k(R - N(x)) \\ x &= k(R - Y) \\ x &= kR - kY \\ Y &= R - \frac{x}{k} \end{aligned} \tag{2}$$

Equations (1) and (2) must both hold. We now setup a table of  $R$  and corresponding  $Y$  values, and using  $k = 1$  for this part, we obtain

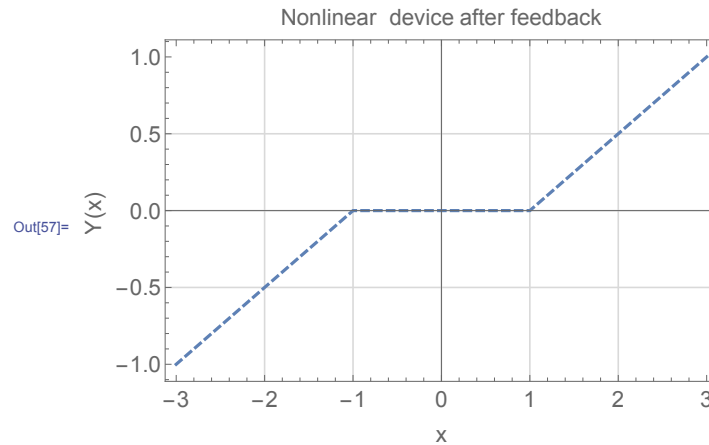
$R$	$Y = R - x$	solution of $Y = N(x)$	$Y$ at solution
0	$-x$	$x = 0$	0
0.1	$0.1 - x$	$x = 0$	0
0.2	$0.2 - x$	$\vdots$	$\vdots$
see program	$\vdots$	$\vdots$	$\vdots$

Small code was written to finish the above table, using  $R = -2 \dots 2$  range with increments of 0.1. Here is the generated table, followed by the plot

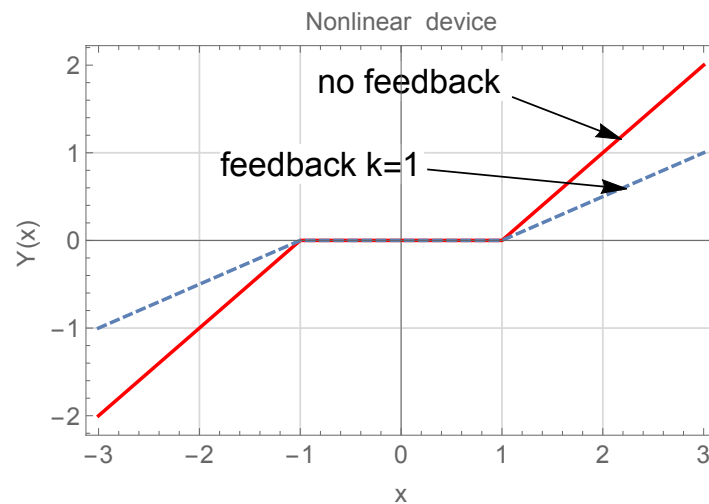
Out[181]=

r	r-x/k	solution of $N(x) = r-x/k$	Y at solution
-2.	-2.-x	-1.5	-0.5
-1.9	-1.9-x	-1.45	-0.45
-1.8	-1.8-x	-1.4	-0.4
-1.7	-1.7-x	-1.35	-0.35
-1.6	-1.6-x	-1.3	-0.3
-1.5	-1.5-x	-1.25	-0.25
-1.4	-1.4-x	-1.2	-0.2
-1.3	-1.3-x	-1.15	-0.15
-1.2	-1.2-x	-1.1	-0.1
-1.1	-1.1-x	-1.05	-0.05
-1.	-1.-x	-1.	0
-0.9	-0.9-x	-0.9	0
-0.8	-0.8-x	-0.8	0
-0.7	-0.7-x	-0.7	0
-0.6	-0.6-x	-0.6	0
-0.5	-0.5-x	-0.5	0
-0.4	-0.4-x	-0.4	0
-0.3	-0.3-x	-0.3	0
-0.2	-0.2-x	-0.2	0
-0.1	-0.1-x	-0.1	0
0.	0.-x	0	0
0.1	0.1-x	0.1	0
0.2	0.2-x	0.2	0
0.3	0.3-x	0.3	0
0.4	0.4-x	0.4	0
0.5	0.5-x	0.5	0
0.6	0.6-x	0.6	0
0.7	0.7-x	0.7	0
0.8	0.8-x	0.8	0
0.9	0.9-x	0.9	0
1.	1.-x	1.	0
1.1	1.1-x	1.05	0.05
1.2	1.2-x	1.1	0.1
1.3	1.3-x	1.15	0.15
1.4	1.4-x	1.2	0.2
1.5	1.5-x	1.25	0.25
1.6	1.6-x	1.3	0.3
1.7	1.7-x	1.35	0.35
1.8	1.8-x	1.4	0.4
1.9	1.9-x	1.45	0.45
2.	2.-x	1.5	0.5

And plot of  $Y$  vs.  $R$  is below



This below is the above plot, but with the original device output without feedback, in order to better see the effect of feedback with  $k = 1$ .



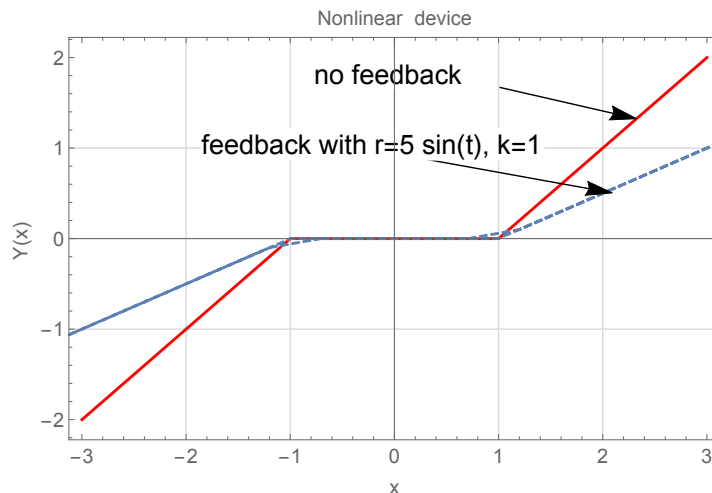
For  $k = 1$ , the dead zone did not change. But the slope became small after  $x = \pm 1$

### 3.3.2.2 Part (b)

When  $r(t) = 5 \sin(t)$ , the following table shows result for  $t = -2 \dots 2$ .

t	$r = 5 \sin(t)$	$r-x/k$	solution of $N(x)=r-x/k$	Y at solution
-2.	-4.54649	-4.54649-x	-2.77324	-1.77324
-1.9	-4.7315	-4.7315-x	-2.86575	-1.86575
-1.8	-4.86924	-4.86924-x	-2.93462	-1.93462
-1.7	-4.95832	-4.95832-x	-2.97916	-1.97916
-1.6	-4.99787	-4.99787-x	-2.99893	-1.99893
-1.5	-4.98747	-4.98747-x	-2.99374	-1.99374
-1.4	-4.92725	-4.92725-x	-2.96362	-1.96362
-1.3	-4.81779	-4.81779-x	-2.9089	-1.9089
-1.2	-4.6602	-4.6602-x	-2.8301	-1.8301
-1.1	-4.45604	-4.45604-x	-2.72802	-1.72802
-1.	-4.20735	-4.20735-x	-2.60368	-1.60368
-0.9	-3.91663	-3.91663-x	-2.45832	-1.45832
-0.8	-3.58678	-3.58678-x	-2.29339	-1.29339
-0.7	-3.22109	-3.22109-x	-2.11054	-1.11054
-0.6	-2.82321	-2.82321-x	-1.91161	-0.911606
-0.5	-2.39713	-2.39713-x	-1.69856	-0.698564
-0.4	-1.94709	-1.94709-x	-1.47355	-0.473546
-0.3	-1.4776	-1.4776-x	-1.2388	-0.238801
-0.2	-0.993347	-0.993347-x	-0.993347	0
-0.1	-0.499167	-0.499167-x	-0.499167	0
0.	0.	0.-x	0	0
0.1	0.499167	0.499167-x	0.499167	0
0.2	0.993347	0.993347-x	0.993347	0
0.3	1.4776	1.4776-x	1.2388	0.238801
0.4	1.94709	1.94709-x	1.47355	0.473546
0.5	2.39713	2.39713-x	1.69856	0.698564
0.6	2.82321	2.82321-x	1.91161	0.911606
0.7	3.22109	3.22109-x	2.11054	1.11054
0.8	3.58678	3.58678-x	2.29339	1.29339
0.9	3.91663	3.91663-x	2.45832	1.45832
1.	4.20735	4.20735-x	2.60368	1.60368
1.1	4.45604	4.45604-x	2.72802	1.72802
1.2	4.6602	4.6602-x	2.8301	1.8301
1.3	4.81779	4.81779-x	2.9089	1.9089
1.4	4.92725	4.92725-x	2.96362	1.96362
1.5	4.98747	4.98747-x	2.99374	1.99374
1.6	4.99787	4.99787-x	2.99893	1.99893
1.7	4.95832	4.95832-x	2.97916	1.97916
1.8	4.86924	4.86924-x	2.93462	1.93462
1.9	4.7315	4.7315-x	2.86575	1.86575
2.	4.54649	4.54649-x	2.77324	1.77324

the following is the plot of the output with the feedback for  $k = 1$



Matlab code to plot the solution

```
%matlab code to generate plot for part(b), HW3, problem 2
%ECE 332
close all; clear all;
figure
t=0:.1:10;
f=@(x) (x+1).*(x<-1)+(x-1).*(x>1)+0; %non-linear device
r = 5 *sin(t); %input
k =1; %change to 10 for second part
x = fsolve(@(x) f(x)-(r-x/k),r);

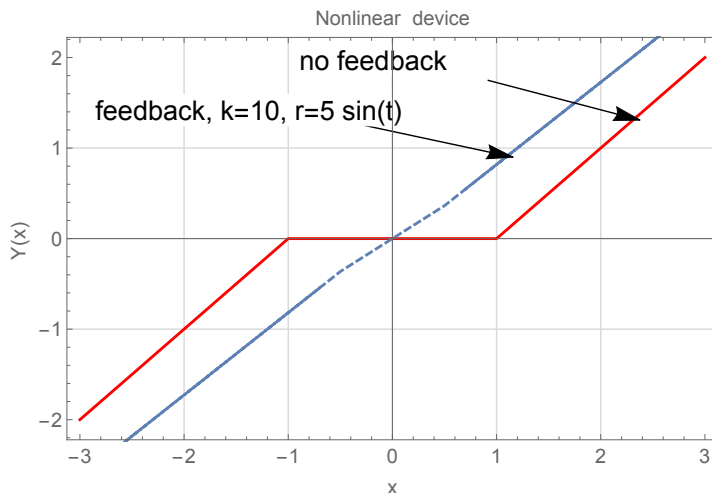
plot(t,f(x));
grid;
title('output of 5*sin(t), k=1');
```

### 3.3.2.3 Part (c)

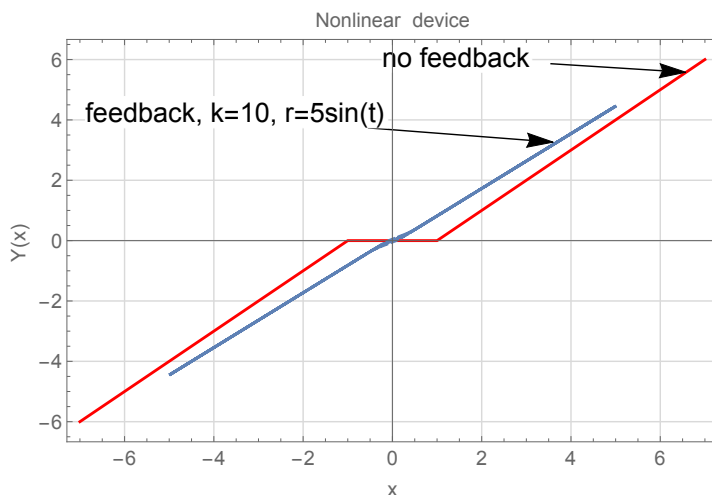
Now  $k = 10$ , and part(b) was repeated. the following table shows result for  $t = -2 \dots 2$ .

t	$r = 5 \sin(t)$	$r-x/10$	solution of $N(x)=r-x/10$	Y at solution
-2.	-4.54649	-4.54649-x	-5.04226	-4.04226
-1.9	-4.7315	-4.7315-x	-5.21045	-4.21045
-1.8	-4.86924	-4.86924-x	-5.33567	-4.33567
-1.7	-4.95832	-4.95832-x	-5.41666	-4.41666
-1.6	-4.99787	-4.99787-x	-5.45261	-4.45261
-1.5	-4.98747	-4.98747-x	-5.44316	-4.44316
-1.4	-4.92725	-4.92725-x	-5.38841	-4.38841
-1.3	-4.81779	-4.81779-x	-5.2889	-4.2889
-1.2	-4.6602	-4.6602-x	-5.14563	-4.14563
-1.1	-4.45604	-4.45604-x	-4.96003	-3.96003
-1.	-4.20735	-4.20735-x	-4.73396	-3.73396
-0.9	-3.91663	-3.91663-x	-4.46967	-3.46967
-0.8	-3.58678	-3.58678-x	-4.1698	-3.1698
-0.7	-3.22109	-3.22109-x	-3.83735	-2.83735
-0.6	-2.82321	-2.82321-x	-3.47565	-2.47565
-0.5	-2.39713	-2.39713-x	-3.0883	-2.0883
-0.4	-1.94709	-1.94709-x	-2.67917	-1.67917
-0.3	-1.4776	-1.4776-x	-2.25236	-1.25236
-0.2	-0.993347	-0.993347-x	-1.81213	-0.812133
-0.1	-0.499167	-0.499167-x	-1.36288	-0.362879
0.	0.	0.-x	0	0
0.1	0.499167	0.499167-x	1.36288	0.362879
0.2	0.993347	0.993347-x	1.81213	0.812133
0.3	1.4776	1.4776-x	2.25236	1.25236
0.4	1.94709	1.94709-x	2.67917	1.67917
0.5	2.39713	2.39713-x	3.0883	2.0883
0.6	2.82321	2.82321-x	3.47565	2.47565
0.7	3.22109	3.22109-x	3.83735	2.83735
0.8	3.58678	3.58678-x	4.1698	3.1698
0.9	3.91663	3.91663-x	4.46967	3.46967
1.	4.20735	4.20735-x	4.73396	3.73396
1.1	4.45604	4.45604-x	4.96003	3.96003
1.2	4.6602	4.6602-x	5.14563	4.14563
1.3	4.81779	4.81779-x	5.2889	4.2889
1.4	4.92725	4.92725-x	5.38841	4.38841
1.5	4.98747	4.98747-x	5.44316	4.44316
1.6	4.99787	4.99787-x	5.45261	4.45261
1.7	4.95832	4.95832-x	5.41666	4.41666
1.8	4.86924	4.86924-x	5.33567	4.33567
1.9	4.7315	4.7315-x	5.21045	4.21045
2.	4.54649	4.54649-x	5.04226	4.04226

And the following is the plot of the result



The above plot shows that with  $k = 10$ , the dead zone has shrunk to almost zero, and the output of the nonlinear device is now linear. This is good. This is another plot, for larger range of input values.

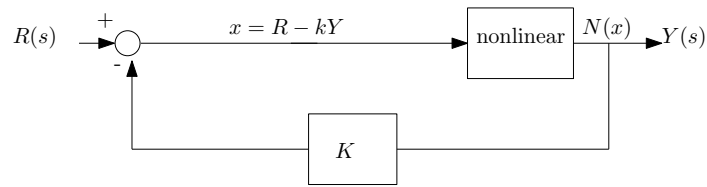


As range of input values become large, the output of the feedback linear device approaches the open loop device output. This is outside the dead zone region as can be seen from the above. Very close to the origin, there is very small non-linearity remains, but it is hard to see.

**3.3.2.4 part (d)**

When the gain  $k$  is in the feedback loop, as shown in the following diagram





Therefore

$$x = R - kY$$

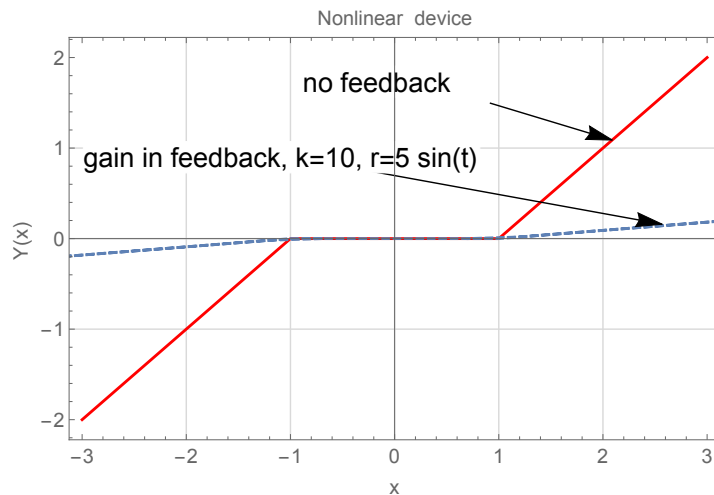
$$Y = \frac{(R - x)}{k}$$

Before, when the gain was in the feedforward,  $Y = R - \frac{x}{k}$ , so now  $k$  affects  $R$  as well. Using this new  $x$ , the above plot was reproduced for the case of  $r(t) = 5 \sin(t)$ .

the following table shows result for  $t = -2 \dots 2$ .

t	$r = 5 \sin(t)$	$(r-x)/10$	solution of $N(x)=(r-x)/10$	Y at solution
-2.	-4.54649	$(-4.54649-x)/10$	-1.32241	-0.322408
-1.9	-4.7315	$(-4.7315-x)/10$	-1.33923	-0.339227
-1.8	-4.86924	$(-4.86924-x)/10$	-1.35175	-0.351749
-1.7	-4.95832	$(-4.95832-x)/10$	-1.35985	-0.359848
-1.6	-4.99787	$(-4.99787-x)/10$	-1.36344	-0.363443
-1.5	-4.98747	$(-4.98747-x)/10$	-1.3625	-0.362498
-1.4	-4.92725	$(-4.92725-x)/10$	-1.35702	-0.357023
-1.3	-4.81779	$(-4.81779-x)/10$	-1.34707	-0.347072
-1.2	-4.6602	$(-4.6602-x)/10$	-1.33275	-0.332745
-1.1	-4.45604	$(-4.45604-x)/10$	-1.31419	-0.314185
-1.	-4.20735	$(-4.20735-x)/10$	-1.29158	-0.291578
-0.9	-3.91663	$(-3.91663-x)/10$	-1.26515	-0.265149
-0.8	-3.58678	$(-3.58678-x)/10$	-1.23516	-0.235162
-0.7	-3.22109	$(-3.22109-x)/10$	-1.20192	-0.201917
-0.6	-2.82321	$(-2.82321-x)/10$	-1.16575	-0.165747
-0.5	-2.39713	$(-2.39713-x)/10$	-1.12701	-0.127012
-0.4	-1.94709	$(-1.94709-x)/10$	-1.0861	-0.0860992
-0.3	-1.4776	$(-1.4776-x)/10$	-1.04342	-0.0434183
-0.2	-0.993347	$(-0.993347-x)/10$	-0.993347	0
-0.1	-0.499167	$(-0.499167-x)/10$	-0.499167	0
0.	0.	$(0.-x)/10$	0	0
0.1	0.499167	$(0.499167-x)/10$	0.499167	0
0.2	0.993347	$(0.993347-x)/10$	0.993347	0
0.3	1.4776	$(1.4776-x)/10$	1.04342	0.0434183
0.4	1.94709	$(1.94709-x)/10$	1.0861	0.0860992
0.5	2.39713	$(2.39713-x)/10$	1.12701	0.127012
0.6	2.82321	$(2.82321-x)/10$	1.16575	0.165747
0.7	3.22109	$(3.22109-x)/10$	1.20192	0.201917
0.8	3.58678	$(3.58678-x)/10$	1.23516	0.235162
0.9	3.91663	$(3.91663-x)/10$	1.26515	0.265149
1.	4.20735	$(4.20735-x)/10$	1.29158	0.291578
1.1	4.45604	$(4.45604-x)/10$	1.31419	0.314185
1.2	4.6602	$(4.6602-x)/10$	1.33275	0.332745
1.3	4.81779	$(4.81779-x)/10$	1.34707	0.347072
1.4	4.92725	$(4.92725-x)/10$	1.35702	0.357023
1.5	4.98747	$(4.98747-x)/10$	1.3625	0.362498
1.6	4.99787	$(4.99787-x)/10$	1.36344	0.363443
1.7	4.95832	$(4.95832-x)/10$	1.35985	0.359848
1.8	4.86924	$(4.86924-x)/10$	1.35175	0.351749
1.9	4.7315	$(4.7315-x)/10$	1.33923	0.339227
2.	4.54649	$(4.54649-x)/10$	1.32241	0.322408

And the following is the plot



We see the effect of reducing  $R$  (since it is now divided by  $k > 1$ ) and the output from the non-linear device is not as good as when the gain was in the feedforward. The dead zone has returned back and the output after the dead zone is much smaller in amplitude than the original open loop output. Putting the gain in the feedback loop does not appear to be a good choice in this case.

## 3.3.3 Problem 3

3. The block diagram of a feedback control system is shown in Figure 3

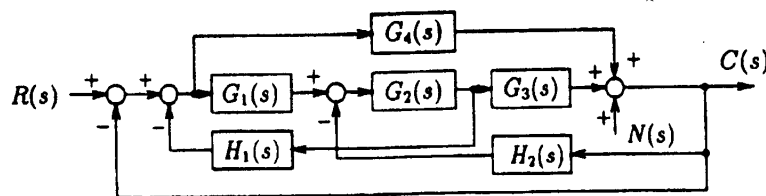


Figure 3: Block Diagram

- (a) Apply Mason's gain formula to the block diagram to find the transfer functions

$$\left. \frac{C(s)}{R(s)} \right|_{N=0} \quad \left. \frac{C(s)}{N(s)} \right|_{R=0}$$

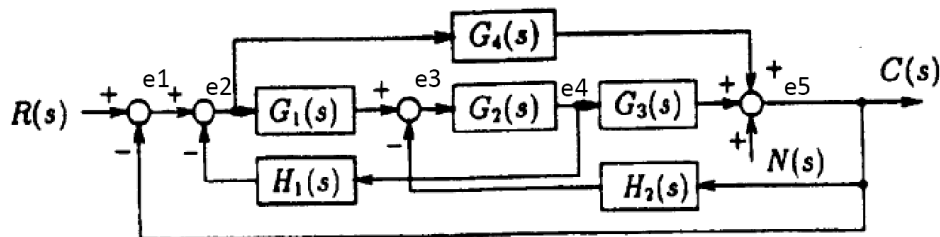
Express  $C(s)$  in terms of  $R(s)$  and  $N(s)$  when both inputs are applied simultaneously.

- (b) Find the desired relation among the transfer functions  $G_1(s)$ ,  $G_2(s)$ ,  $G_3(s)$ ,  $G_4(s)$ ,  $H_1(s)$  and  $H_2(s)$  so that the output  $C(s)$  is not affected by the disturbance signal  $N(s)$  at all.

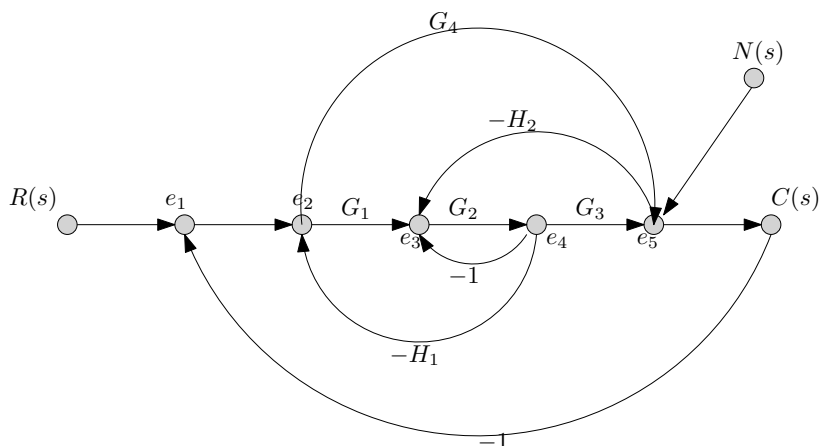
SOLUTION:

## 3.3.3.1 Part(a)

The first step is to convert the block diagram to signal flow diagram. By assigning variables as shown below, the following signal diagram we drawn



Converted to signal flow as



For finding  $\frac{C(s)}{R(s)}$  then  $N(s)$  is set to zero. There are two forward paths from  $R(s)$  to  $C(s)$  they are

$$M_1 = \{1, 1, G_1, G_2, G_3, 1\} = G_1 G_2 G_3$$

$$M_2 = \{1, 1, G_4, 1\} = G_4$$

The corresponding Mason deltas are

$$\Delta_1 = 1$$

$$\Delta_2 = 1 - (G_2)(-1) = 1 + G_2$$

The loops, one at a time are

$$\begin{aligned} L_1 &= (G_2)(-1), (G_1)(G_2)(-H_1), (G_2)(G_3)(-H_2), (1)(G_1)(G_2)(G_3)(1)(-1), (G_4)(1)(-1) \\ &= -G_2, -G_1 G_2 H_1, -H_2 G_2 G_3, -G_1 G_2 G_3, -G_4 \end{aligned}$$

Two at a time are

$$\begin{aligned} L_2 &= \{(G_2)(-1) \times (G_4)(1)(-1)(1)\} \\ &= G_2 G_4 \end{aligned}$$

Hence

$$\begin{aligned} \Delta &= 1 - \sum (-G_2 - G_1 G_2 H_1 - H_2 G_2 G_3 - G_1 G_2 G_3 - G_4) + \sum G_2 G_4 \\ &= 1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4 \end{aligned}$$

Hence

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{\sum M_i \Delta_i}{\Delta} \\ &= \frac{G_1 G_2 G_3 (1) + G_4 (1 + G_2)}{1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4} \end{aligned}$$

Hence

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_4 + G_4 G_2}{1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4}}$$

For finding  $\frac{C(s)}{N(s)}$  then  $R(s)$  is set to zero. There is now one forward path from  $N(s)$  to  $C(s)$

$$M_1 = \{1, 1\} = 1$$

The corresponding Mason deltas are

$$\Delta_1 = 1 - \{(G_2)(-1) + (G_1)(G_2)(-H_1)\} = 1 + G_2 + G_1G_2H_1$$

The loops remain the same as above. Hence the mason delta do not change. Therefore

$$\begin{aligned} \frac{C(s)}{N(s)} &= \frac{\sum M_i \Delta_i}{\Delta} \\ &= \frac{(1)(1 + G_2 + G_1G_2H_1)}{1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4} \end{aligned}$$

Hence

$$\boxed{\frac{C(s)}{N(s)} = \frac{1 + G_2 + G_1G_2H_1}{1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4}}$$

### 3.3.3.2 Part (b)

Since

$$C(s) = \frac{1 + G_2 + G_1G_2H_1}{1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4} N(s)$$

Then we want  $1 + G_2 + G_1G_2H_1 = 0$  or for the denominator

$$1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4$$

to be very large. Both of these will cause  $C(s)$  to remain zero for any value of  $N(s)$ . But since the denominator is the same as for  $\frac{C(s)}{R(s)}$  then making this very large will also affect  $\frac{C(s)}{R(s)}$  which we do not want to. Hence the choice left is

$$1 + G_2 + G_1G_2H_1 = 0$$

### 3.3.4 Problem 4

Consider the signal flow graph shown in Figure 3.

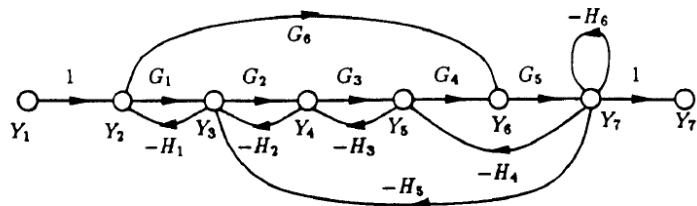


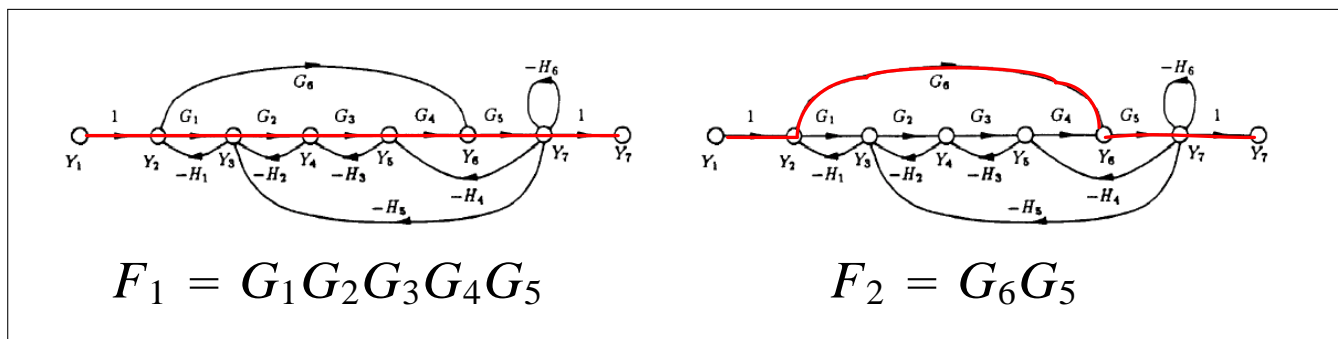
Figure 3: Signal Flow Graph

- (a) Identify all the forward paths and their loop gains.
- (b) Identify all the loops.
- (c) Find the transfer function from  $Y_1$  to  $Y_7$  and from  $Y_1$  to  $Y_2$  using Mason's rule.

SOLUTION:

#### 3.3.4.1 Part (a)

For the  $\frac{Y_7}{Y_1}$ , There are two forward paths. The following diagrams shows them with the gain on each.



$$F_1 = G_1 G_2 G_3 G_4 G_5$$

$$F_2 = G_6 G_5$$

Now  $\Delta_k$  is found for each forward loop.  $\Delta_k$  is the Mason  $\Delta$  but with  $F_k$  removed from the graph. Removing  $F_1$  removes all the loops, hence

$$\Delta_1 = 1$$

When removing  $F_2$  what remains is  $L_2$  and  $L_3$ , hence

$$\begin{aligned}\Delta_2 &= 1 - (L_2 + L_3) \\ &= 1 - (-H_2G_2 - H_3G_3) \\ &= 1 + (H_2G_2 + H_3G_3)\end{aligned}$$

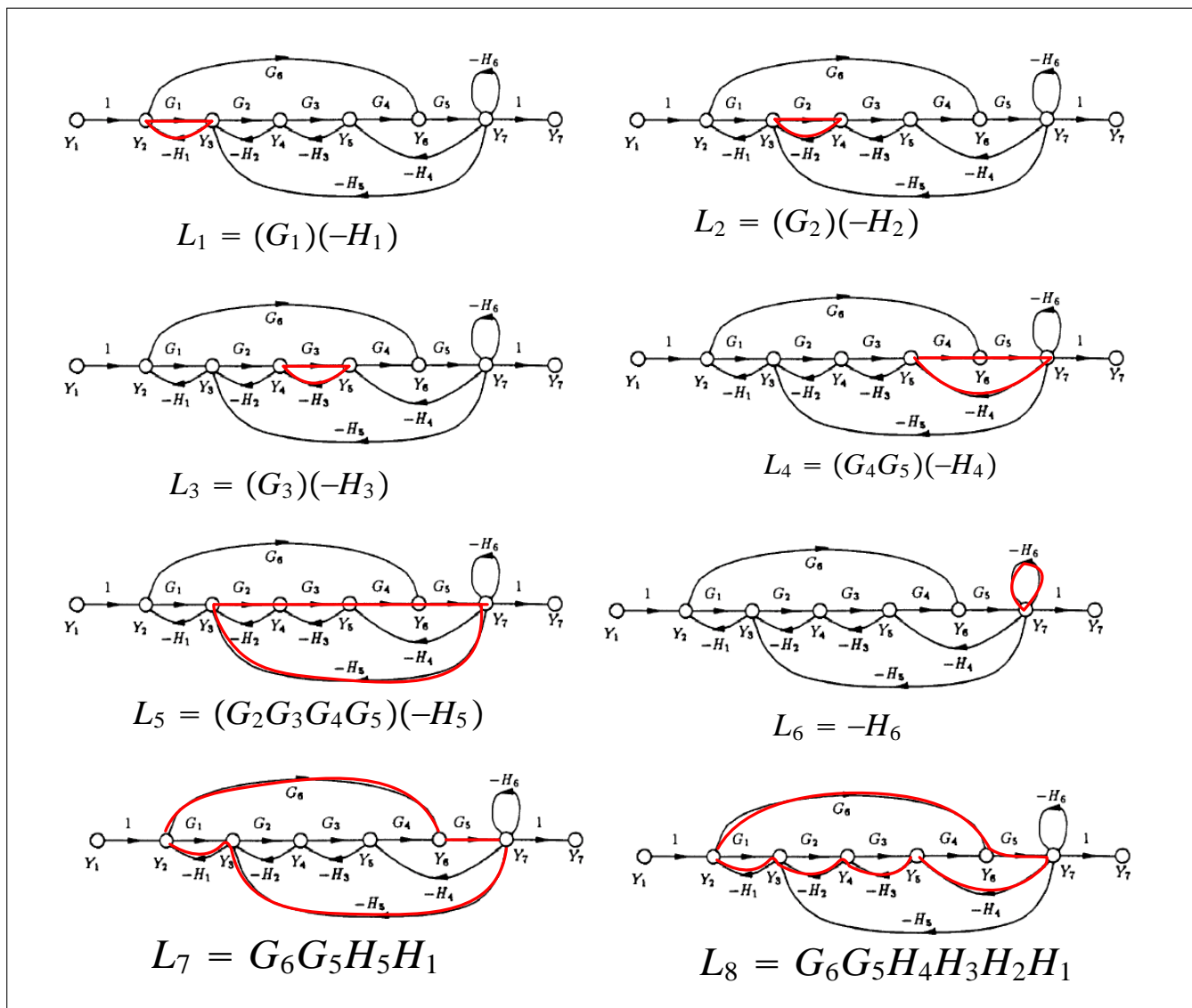
For the  $\frac{Y_2}{Y_1}$ , there is one forward path  $F_1 = 1$ , the associated  $\Delta_1$  is

$$\begin{aligned}\Delta_1 &= 1 - \sum -G_2H_2 - G_3H_3 - G_4G_5H_4 - H_6 - G_2G_3G_4G_5H_5 \\ &\quad + \sum (-G_2H_2)(-G_4G_5H_4) + (-G_2H_2)(-H_6) + (-G_3H_3)(-H_6) \\ &= 1 + \underbrace{G_2H_2 + G_3H_3 + G_4G_5H_4 + H_6 + G_2G_3G_4G_5H_5}_{\text{one at a time}} + \underbrace{G_2H_2G_4G_5H_4 + G_2H_2H_6 + G_3H_3H_6}_{\text{two at a time}}\end{aligned}$$

### 3.3.4.2 Part(b)

There are 8 loops. The following diagrams shows the loops with the gains





$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_1L_3 + L_1L_4 + L_1L_6 + L_2L_4 + L_2L_6 + L_3L_6 + L_3L_7) - L_1L_3L_6$$

Therefore

$$\Delta = 1 + \overbrace{H_1G_1 + H_2G_2 + H_3G_3 + H_4G_4G_5 + H_5G_2G_3G_4G_5 + H_6 - G_5G_6H_1H_5 - G_6G_5H_4H_3H_2H_1}^{\text{one at a time}} \tag{1}$$

$$+ \overbrace{(H_1G_1H_3G_3 + H_1G_1H_4G_4G_5 + H_1H_6G_1 + H_2G_2H_4G_4G_5 + H_2G_2H_6 + H_3G_3H_6 - G_3H_3G_6G_5H_5H_1)}^{\text{two at a time}}$$

$$+ \overbrace{H_1G_1H_3G_3H_6}^{\text{three at a time}}$$

**3.3.4.3 Part (c)**

For  $G(s) = \frac{Y_7}{Y_1}$ , and using result found above in part (a) and part (b)

$$\begin{aligned} G(s) &= \frac{Y_7}{Y_1} \\ &= \frac{\Delta_1 F_1 + \Delta_2 F_2}{\Delta} \\ &= \frac{(G_1 G_2 G_3 G_4 G_5) + G_6 G_5 (1 + H_2 G_2 + H_3 G_3)}{\Delta} \end{aligned}$$

Where  $\Delta$  is given in (1) found in part(b). To obtain  $\frac{Y_2}{Y_1}$

$$\begin{aligned} \frac{Y_2}{Y_1} &= \frac{\Delta_1 F_1}{\Delta} \\ &= \frac{1 + \overbrace{G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6}^{\text{one at a time}} + \overbrace{G_2 G_3 G_4 G_5 H_5 + G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6}^{\text{two at a time}}}{\Delta} \\ &= \frac{1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 + G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6}{\Delta} \end{aligned}$$

## 3.3.5 HW 3 key solution

$$1. G(s) = \frac{Ks+1}{s^3+s^2+(K+k)s+1}$$

$$(a) S_k^G = \frac{\partial G}{\partial k} \cdot \frac{k}{G} = \frac{-(Ks+1)}{(s^3+s^2+(K+k)s+1)^2} \cdot \frac{k}{\frac{Ks+1}{s^3+s^2+(K+k)s+1}}$$

$$S_k^G = \frac{-ks}{s^3+s^2+(K+k)s+1}$$

$$|S_k^G(j\omega)| = \left| \frac{-kj\omega}{(j\omega)^3 + (j\omega)^2 + (K+k)j\omega + 1} \right| = \left| \frac{-kj\omega}{(1-\omega^2) + j((K+k)\omega - \omega^3)} \right|$$

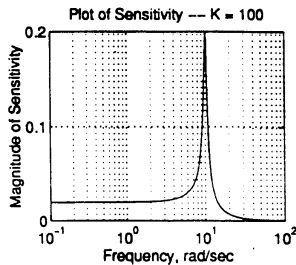
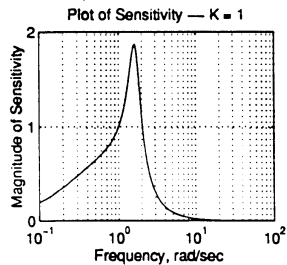
$$|S_k^G(j\omega)| = \frac{k\omega}{\sqrt{(1-\omega^2)^2 + ((K+k)\omega - \omega^3)^2}}$$

$$\text{When } k=2 \text{ and } K=1, |S_k^G(j\omega)| = \frac{2\omega}{\sqrt{(1-\omega^2)^2 + (3\omega - \omega^3)^2}}$$

See printout below.

$$(b) \text{ When } k=2 \text{ and } K=100, |S_k^G(j\omega)| = \frac{2\omega}{\sqrt{(1-\omega^2)^2 + (102\omega - \omega^3)^2}}$$

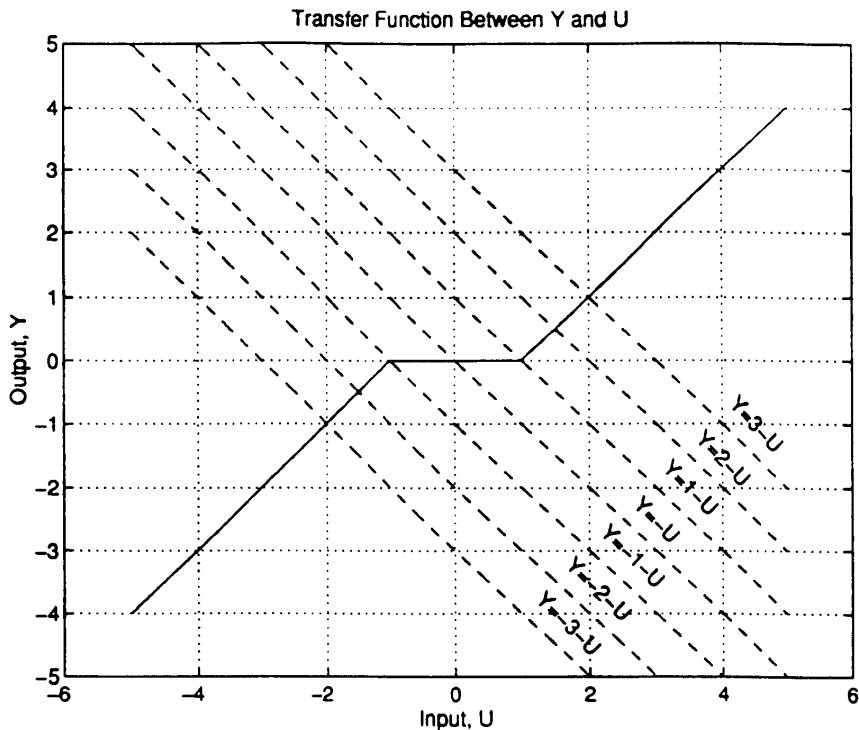
See printout below.



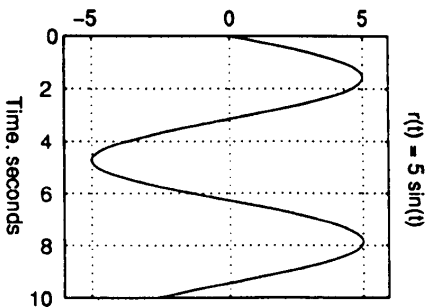
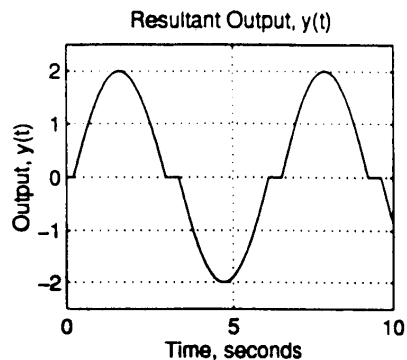
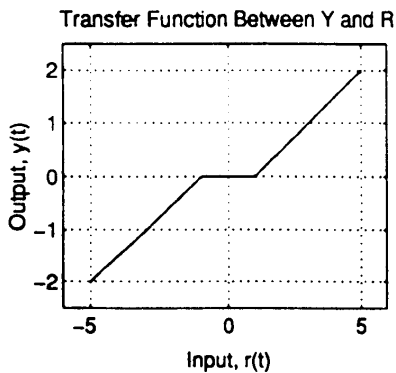
```
w=logspace(-1,2,1000);
SGk = 2*w./sqrt((1-w.^2).^2 + (3*w-w.^3).^2);
SGk1 = 2*w./sqrt((1-w.^2).^2 + (102*w-w.^3).^2);
subplot(221),semilogx(w,SGk),grid,title('Plot of Sensitivity -- K = 1')
xlabel('Frequency, rad/sec'),ylabel('Magnitude of Sensitivity')
axis([1e-1 1e2 0 2])
subplot(222),semilogx(w,SGk1),grid,title('Plot of Sensitivity -- K = 100')
xlabel('Frequency, rad/sec'),ylabel('Magnitude of Sensitivity')
axis([1e-1 1e2 0 .2])
```

;

2. (a) If  $K = 1$ , then  $U = R - Y \implies Y = R - U$   
 For  $R = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$  draw the line  $Y = R - U$ . The intersection points give the corresponding values of  $Y$ .



Resulting  $Y$  versus  $R$ :



- (b)  $r(t) = 5 \sin(t)$ . Using the transfer function between  $Y$  and  $R$ , we can generate  $y(t)$  graphically by “bouncing”  $r(t)$  (left) off of the transfer function graph (upper left) onto  $y(t)$  (top). The time at which the output becomes greater than 1 occurs at  $t = \sin^{-1}(\frac{1}{5})$ .

;

(c) When  $K = 10$ , then

$$U = 10(R - Y) \implies Y = R - \frac{U}{10} \quad \star$$

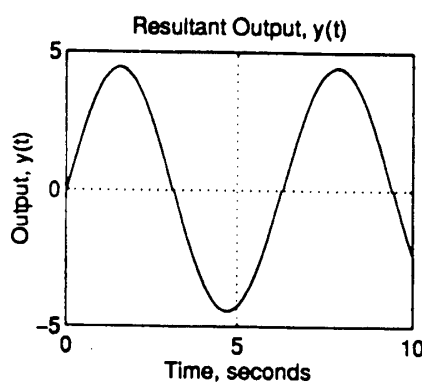
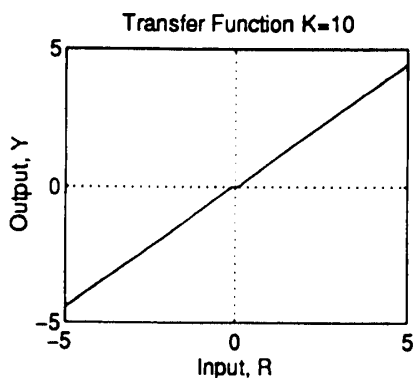
For  $|R| \leq \frac{1}{10}$   $Y = 0$

For  $R > \frac{1}{10}$   $Y = U - 1$  (From  $N(\cdot)$ )

$$U = Y + 1 \quad \#$$

Plugging # into  $\star$  we get  $Y = R - \left(\frac{Y+1}{10}\right) \implies 10Y = 10R - Y - 1$

$$11Y = 10R - 1 \implies \boxed{Y = \frac{10}{11}R - \frac{1}{11}}$$

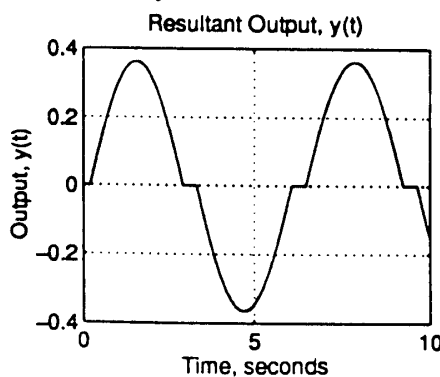
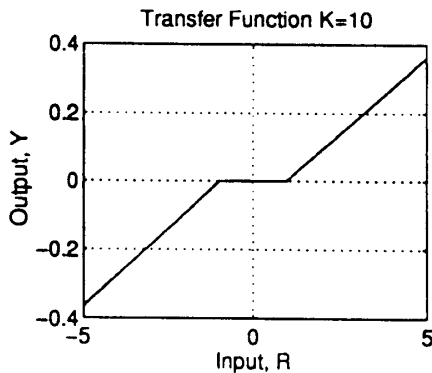


In general,

$$Y = \begin{cases} 0 & \text{if } -\frac{1}{K} \leq R \leq \frac{1}{K} \\ \frac{KR - 1}{K + 1} & \text{if } R > \frac{1}{K} \\ \frac{KR + 1}{K + 1} & \text{if } R < -\frac{1}{K} \end{cases}$$

As for  $y(t)$ , the amplitude approaches 1 as  $K \rightarrow \infty$  and the missing middle section of the output diminishes to zero.

(d) When we place the gain in the feedback loop, there is no effect on the missing middle section of the output. Instead, the slope decreases so that in the limit we have a flat output signal which would theoretically have no distortion.



In general,

$$Y = \begin{cases} 0 & \text{if } -1 \leq R \leq 1 \\ \frac{1}{K+1}(R-1) & \text{if } R > 1 \\ \frac{1}{K+1}(R+1) & \text{if } R < -1 \end{cases}$$

The key here is that the “circuit” connected to node  $Y_2$  can affect (and most likely will) the transfer function between  $Y_1$  and  $Y_2$ . For instance, if  $Y_1$  were 10 and  $Y_2$  were defined to be zero, then  $Y_2$  would not equal  $Y_1$ .

$$3. \quad (a) \quad \left. \frac{C(s)}{R(s)} \right|_{N=0} = \frac{G_1(s)G_2(s)G_3(s) + G_4(s)}{\Delta}$$

$$\left. \frac{C(s)}{N(s)} \right|_{R=0} = \frac{1 + G_1(s)G_2(s)H_1(s)}{\Delta}$$

where

$$\Delta = 1 + G_1(s)G_2(s)H_1(s) + G_2(s)G_3(s)H_2(s) + G_2(s)G_4(s)H_1(s)H_2(s) + G_4 + G_1G_2G_3$$

$$Y(s) = \left. \frac{C(s)}{R(s)} \right|_{N=0} R(s) + \left. \frac{C(s)}{N(s)} \right|_{R=0} N(s)$$

(b) When  $1 + G_1(s)G_2(s)H_1(s) = 0$ ,  $Y(s)$  is not affected by  $N(s)$ .

;

- (a) Forward paths and their respective loop gains for the
- $\frac{Y_2}{Y_1}$
- transfer function:

$$M_1 = G_1 G_2 G_3 G_4 G_5 \quad \Delta_1 = 1$$

$$M_2 = G_6 G_5 \quad \Delta_2 = 1 + G_2 H_2 + G_3 H_3$$

Forward paths and their respective loop gains for the  $\frac{Y_1}{Y_2}$  transfer function:

$$M_1 = 1$$

$$\Delta_1 = 1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 + G_2 G_4 G_5 H_5 H_4 + G_2 H_2 H_6 + G_2 H_3 H_6$$

- (b) All loops:

$$\Delta = 1 + G_1 H_1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 - G_5 G_6 H_1 H_5 - G_5 G_6 H_1 H_2 H_3 H_4 + G_1 G_3 H_1 H_3 + G_1 G_4 G_5 H_1 H_4 + G_1 H_1 H_6 + G_2 G_4 G_5 H_2 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6 - G_3 G_5 G_6 H_1 H_3 H_5 + G_1 G_3 H_1 H_3 H_6$$

$$(c) \frac{Y_2}{Y_1} = \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 (1 + G_2 H_2 + G_3 H_3)}{\Delta}$$

$$\frac{Y_1}{Y_2} = \frac{1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5}{\Delta + \frac{G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_2 H_3 H_6}{\Delta}}$$

The key here is that the “circuit” connected to node  $Y_2$  can affect (and most likely will) the transfer function between  $Y_1$  and  $Y_2$ . For instance, if  $Y_1$  were 10 and  $Y_2$  were defined to be zero, then  $Y_2$  would not equal  $Y_1$ .

;

## 3.4 HW 4

### 3.4.1 Problem 1

**Problem 2:** Consider the block diagram set-up for the disturbance attenuation problem formulated in lecture with

$$G_1(s) = \frac{s + 1}{s^2 + 10s + 100}$$

and

$$G_2(s) = \frac{1}{s + 2}.$$

(a) Design control blocks  $H_1(s)$  and  $H_2(s)$  such that the following two specifications are satisfied: First,

$$\left| \frac{Y(j\omega)}{N(j\omega)} \right| \leq 0.01$$

for all frequencies  $\omega \geq 0$ . Second, the output  $y(t)$  should respond to command to  $r(t)$  in approximately the same manner in the closed loop as in the open loop; i.e. for the closed loop, we desire

$$Y(s) \approx G_1(s)G_2(s)R(s).$$

Note: In class, we did not fully solve for  $H_1$ ; i.e., we never found the constant  $\alpha$ . In this homework, a specific solution is sought.

(b) To make your solution “proper,” introduce a second order lowpass filter as appropriate and solve for the filter parameter  $\epsilon$ .

(c) For the compensated system resulting from (b), generate a frequency response plot for the closed loop transfer function  $|Y(j\omega)/R(j\omega)|$  and compare it to the target transfer function  $|G_1(j\omega)G_2(j\omega)|$ . Plot the error between these two frequency responses as a function of the frequency  $\omega \geq 0$ .

SOLUTION:

#### 3.4.1.1 Part (a)

The second condition which says that the closed loops should approximate the open loop response, implies that we should use  $H_2(s) = \frac{1}{G_1G_2}$ , i.e. to apply the inversion. This is because

$\frac{Y(s)}{R(s)} = \frac{H_1G_1G_2}{1+H_1G_1G_2H_2} \Big|_{N=0}$  and this becomes  $\frac{Y(s)}{R(s)} \approx G_1G_2$  when we set  $H_2 = \frac{1}{G_1G_2}$  and also by making  $H_1 = \alpha$  where  $\alpha$  is a large gain. So now we just need to worry about finding  $\alpha$  s.t.



$$\left| \frac{Y(j\omega)}{N(j\omega)} \right| \leq 0.01 \text{ for all } \omega > 0.$$

We know that  $\frac{Y(s)}{N(s)} = \frac{G_2}{1+G_1G_2H_2H_1}$ , but since we are using the inversion, this reduces to

$$\frac{Y(s)}{N(s)} = \frac{G_2}{1+H_1}$$

By setting  $H_1(s) = \alpha$  and using  $G_2 = \frac{1}{s+2}$  and moving to the frequency domain, the above becomes

$$\frac{Y(j\omega)}{N(j\omega)} = \frac{\frac{1}{j\omega+2}}{1+\alpha} = \frac{1}{(j\omega+2)(1+\alpha)} = \frac{1}{(1+\alpha)j\omega+2(1+\alpha)}$$

Taking the magnitude

$$\left| \frac{Y(j\omega)}{N(j\omega)} \right| = \frac{1}{\sqrt{(1+\alpha)^2\omega^2+4(1+\alpha)^2}}$$

We want the above to be smaller than 0.01 for all  $\omega$ , which implies

$$\begin{aligned} \frac{1}{\sqrt{(1+\alpha)^2\omega^2+4(1+\alpha)^2}} &\leq 0.01 \\ \frac{1}{(1+\alpha)^2\omega^2+4(1+\alpha)^2} &\leq 0.01^2 \\ (1+\alpha)^2\omega^2+4(1+\alpha)^2 &\geq 10000 \\ \omega^2 &\geq \frac{10000-4(1+\alpha)^2}{(1+\alpha)^2} \\ \omega^2 &\geq \frac{10000}{(1+\alpha)^2} - 4 \\ \omega &\geq \sqrt{\frac{10000}{(1+\alpha)^2} - 4} \end{aligned}$$

The smallest  $\alpha$  to allow the above is when  $\omega = 0$ , hence we need to solve for  $\alpha$  from

$$\begin{aligned} \sqrt{\frac{10000}{(1+\alpha)^2} - 4} &= 0 \\ \frac{10000}{(1+\alpha)^2} - 4 &= 0 \\ \frac{1}{(1+\alpha)^2} &= \frac{4}{10000} \\ (1+\alpha)^2 &= \frac{10000}{4} = 2500 \\ 1+\alpha &= 50 \end{aligned}$$

Hence

$$\alpha \geq 49$$

Therefore  $H_1(s) = \alpha$  where  $\alpha \geq 49$  and  $H_1(s) = \frac{1}{G_1(s)G_2(s)}$ . This complete this part.

### 3.4.1.2 Part(b)

One problem with the above inversion method for finding  $H_2(s) = \frac{1}{G_1G_2}$  is that  $H_2(s)$  becomes improper:

$$\begin{aligned} H_2(s) &= \frac{1}{G_1G_2} = \frac{1}{\frac{s+1}{s^2+10s+100} \frac{1}{s+2}} \\ &= \frac{(s^2 + 10s + 100)(s + 2)}{s + 1} \\ &= \frac{s^3 + 12s^2 + 120s + 200}{s + 1} \end{aligned}$$

$H_2(s)$  is improper, since the numerator has a degree larger than the denominator. This introduces differentiator in the feedback loop which is something we do not like to have.

We will now replace  $H_2 = \frac{1}{G_1G_2}$  by  $\left(\frac{1}{G_1G_2}\right)H_{LP}(s)$  where  $H_{LP}(s) = \frac{1}{(\varepsilon s+1)^k}$  is a low pass filter where  $k$  is an integer and  $\varepsilon$  is some parameter, both are positive. The goal is to block high frequency noise content and also make  $\left(\frac{1}{G_1G_2}\right)H_{LP}(s)$  become a proper transfer function. We also want to make sure  $\left|\frac{Y(j\omega)}{N(j\omega)}\right|$  remain less than 0.01.

Let

$$\begin{aligned} H_2(s) &= \frac{1}{G_1G_2} \frac{1}{(\varepsilon s + 1)^k} \\ &= \frac{s^3 + 12s^2 + 120s + 200}{(s + 1)} \frac{1}{(\varepsilon s + 1)^k} \end{aligned}$$

The degree of the numerator is 3. So we want  $k$  to be at least 2 (it can be more), so that the denominator has at least degree 3 as well. If we want strict proper, then we make  $k = 3$ . Using  $k = 2$  we now have

$$H_2(s) = \frac{1}{G_1G_2} \frac{1}{(\varepsilon s + 1)^2}$$

Therefore,  $\frac{Y(j\omega)}{N(j\omega)}$  now becomes

$$\begin{aligned} \frac{Y(s)}{N(s)} &= \frac{G_2}{1 + G_1G_2H_1\left(\frac{1}{G_1G_2} \frac{1}{(\varepsilon s+1)^2}\right)} \\ &= \frac{\frac{1}{s+2}}{1 + \frac{\alpha}{(\varepsilon s+1)^2}} \end{aligned} \tag{1}$$

Where we used  $\alpha$  for  $H_1$ . We now move to the frequency domain and take the magnitude in order to solve for  $\varepsilon$ . We will use the same  $\alpha$  found in part (1), otherwise, there will be two free parameters to adjust at the same time, which would make this a hard problem, and the problem seems to indicate we are to use same  $\alpha$  value found in part (1) although it did not say that explicitly. Therefore (1) becomes (using  $\alpha = 49$ )

$$\frac{Y(s)}{N(s)} = \frac{\frac{(\varepsilon s+1)^2}{s+2}}{(\varepsilon s+1)^2 + 49}$$

Hence

$$\begin{aligned} \left| \frac{Y(j\omega)}{N(j\omega)} \right| &= \frac{\left| \frac{(\varepsilon j\omega+1)^2}{j\omega+2} \right|}{\left| (\varepsilon j\omega+1)^2 + 49 \right|} \\ &= \frac{\frac{\varepsilon^2\omega^2+1}{\sqrt{\omega^2+4}}}{\left| -\varepsilon^2\omega^2 + 1 + 2\varepsilon j\omega + 49 \right|} \\ &= \frac{\frac{\varepsilon^2\omega^2+1}{\sqrt{\omega^2+4}}}{\sqrt{4\varepsilon^2\omega^2 + (50 - \varepsilon^2\omega^2)^2}} \end{aligned}$$

Hence

$$\left| \frac{Y(j\omega)}{N(j\omega)} \right|^2 = \frac{(\varepsilon^2\omega^2 + 1)^2}{(\omega^2 + 4)(4\varepsilon^2\omega^2 + (50 - \varepsilon^2\omega^2)^2)}$$

We now find  $\omega$  where  $\left| \frac{Y(j\omega)}{N(j\omega)} \right|$  is maximum, which is the same as where the above is maximum.

The above is maximum when the denominator is minimum. Hence

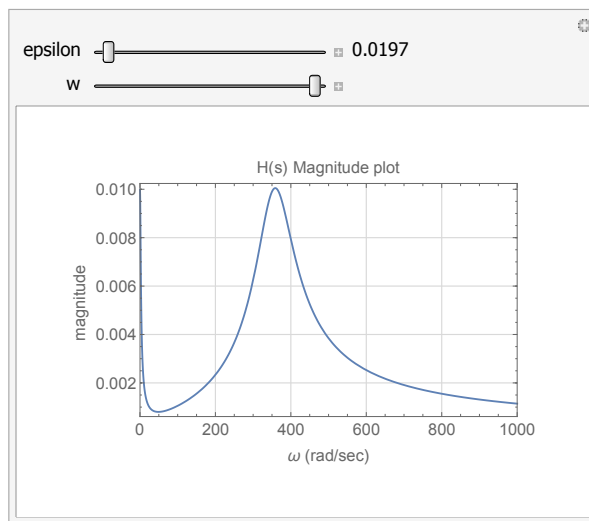
$$\frac{d}{d\omega} (\omega^2 + 4)(4\varepsilon^2\omega^2 + (50 - \varepsilon^2\omega^2)^2) = 0$$

Solving for  $\omega$  from the above using computer algebra (the algebra is too complicated to do by hand. May be there is a short cut) in terms of  $\varepsilon$ , and plugging the solution  $\omega_{\max}$  back to  $\left| \frac{Y(j\omega)}{N(j\omega)} \right|$  and setting the result to 0.01 and solving numerically for  $\varepsilon$  that satisfy the equation gives

$$\varepsilon = 0.0197$$

To verify this, a small demo was made to plot  $\left| \frac{Y(j\omega)}{N(j\omega)} \right|$  for different  $\varepsilon$  values. The following

plot shows  $\left| \frac{Y(j\omega)}{N(j\omega)} \right|$  using  $k = 2, \varepsilon = 0.0197$  and the maximum magnitude was checked to be just less than 0.01



### 3.4.1.3 Part (c)

We will now use

$$H_1 = 49$$

$$H_2 = \frac{1}{G_1 G_2} \frac{1}{(0.0197s + 1)^2}$$

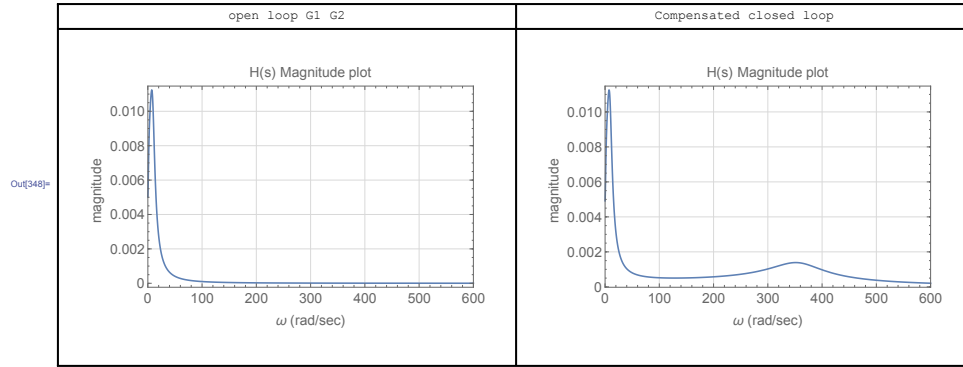
And plot  $\left| \frac{Y(s)}{R(s)} \right| = \left| \frac{H_1 G_1 G_2}{1 + H_1 G_1 G_2 H_2} \right|$  against the  $|G_1 G_2|$  to see how good the choice of  $H_1$  and  $H_2$  are.

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{H_1 G_1 G_2}{1 + H_1 G_1 G_2 H_2} \\ &= \frac{49 G_1 G_2}{1 + 49 G_1 G_2 \frac{1}{G_1 G_2} \frac{1}{(0.0197s + 1)^2}} \\ &= \frac{49 \frac{s+1}{s^2+10s+100} \frac{1}{s+2}}{1 + 49 \frac{1}{(0.0197s+1)^2}} \\ &= \frac{49 \frac{s+1}{s^2+10s+100} \frac{1}{s+2} (0.0197s + 1)^2}{(0.0197s + 1)^2 + 49} \end{aligned}$$

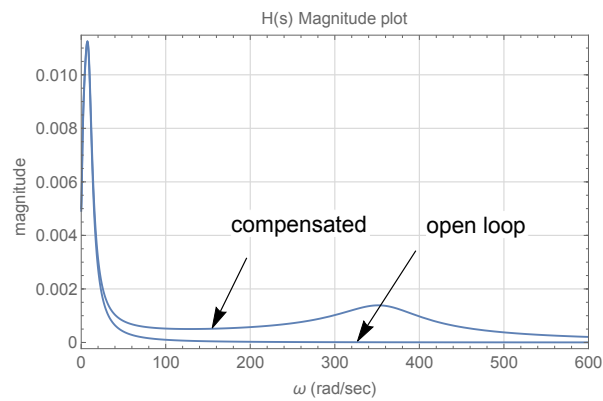
While

$$G_1 G_2 = \frac{s+1}{s^2+10s+100} \frac{1}{s+2}$$

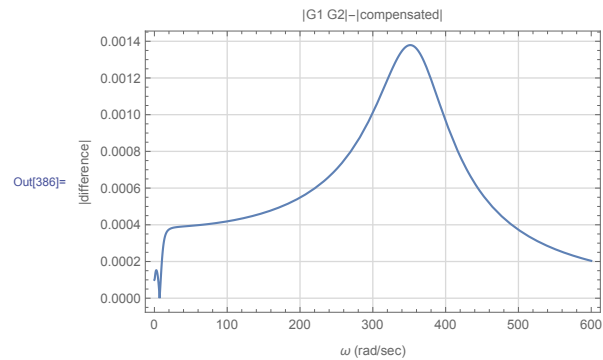
The following plot shows  $\left| \frac{Y(s)}{R(s)} \right|$  vs.  $|G_1 G_2|$  side by side



The following plot shows both on the same plot



The following plot show difference between the magnitudes



**Observations:**

From the above difference plot, we see that the maximum difference between  $|G_1G_2|$  and the compensated  $\left| \frac{H_1G_1G_2}{1+H_1G_1G_2H_2} \right|$  occurred at around  $\omega = 350$  and had value of about 0.0014. This value seems relatively small, and seems to indicate that  $H_1$  and  $H_2$  used for compensation were a good choice.

## 3.4.2 HW 4 key solution

Disturbance Attenuation Problem

By Mason's Rule:

$$Y(s) = \frac{G_2(s)}{1 + H_1(s)H_2(s)G_1(s)G_2(s)} N(s) + \frac{H_1(s)G_1(s)G_2(s)}{1 + H_1(s)H_2(s)G_1(s)G_2(s)} R(s)$$

Let  $H_1 = \alpha$      $H_2(s) = \frac{1}{G_1G_2}$  (later modify with lowpass)

So

$$Y(s) = \frac{G_2(s)}{1+\alpha} N(s) + \alpha \frac{G_1(s)G_2(s)}{1+\alpha} R(s)$$

Want

$$\left| \frac{G_2(j\omega)}{1+\alpha} \right| \leq 0.01 \text{ at all } \omega \gg 0$$

$$G_2(j\omega) = \frac{1}{j\omega+2}$$

$$\Rightarrow \left| \frac{1}{(j\omega+2)(1+\alpha)} \right| \leq 0.01$$

;

$$\Rightarrow \frac{1}{(1+d)\sqrt{4+\omega^2}} \leq 0.01$$

Suffices to satisfy

$$\frac{1}{2(1+d)} \leq 0.01$$

$$\Rightarrow d \geq 49.$$

Let's take  $d = 50$

(b) Now we bring in lowpass:  $H_{LP}(s) = \frac{1}{(s+1)^2}$

Now demand

$$\left| \frac{G_2(j\omega)}{1+d H_{LP}(j\omega)} \right| \leq 0.01$$

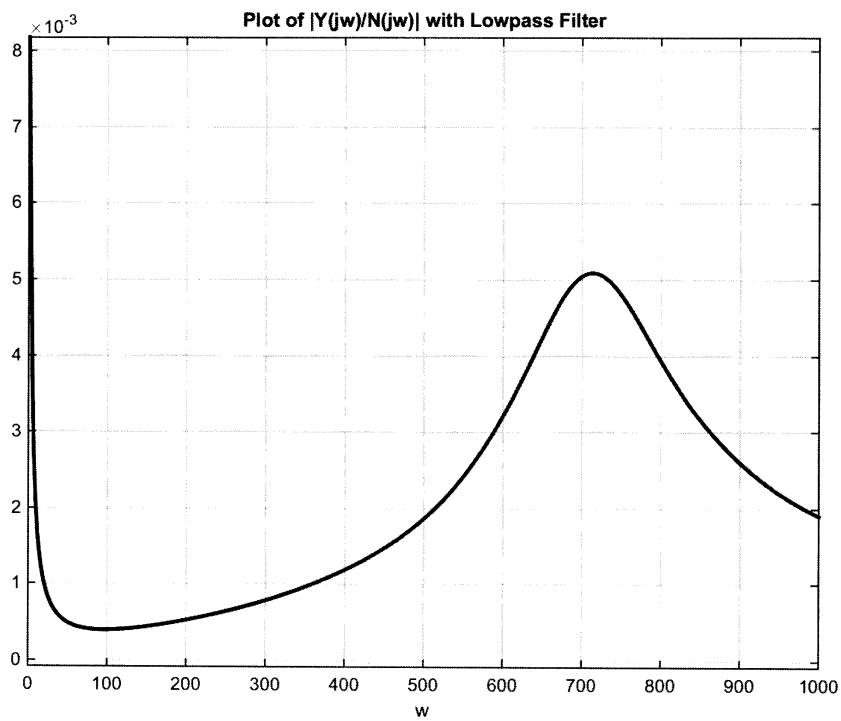
↑  
50

$$\Rightarrow \frac{\left| \frac{1}{j\omega+2} \right|}{\left| 1 + \frac{50}{(j\omega+1)^2} \right|} \leq 0.01$$

Experimenting with various values of epsilon, we arrive at epsilon = 0.01 as reasonable for satisfaction of the performance specification above. For this value of epsilon, we plot the quantity above

;

$$\left| \frac{Y(j\omega)}{N(j\omega)} \right| = \left| \frac{G_2(j\omega)}{1 + Q' H_{LP}(j\omega)} \right|$$



;



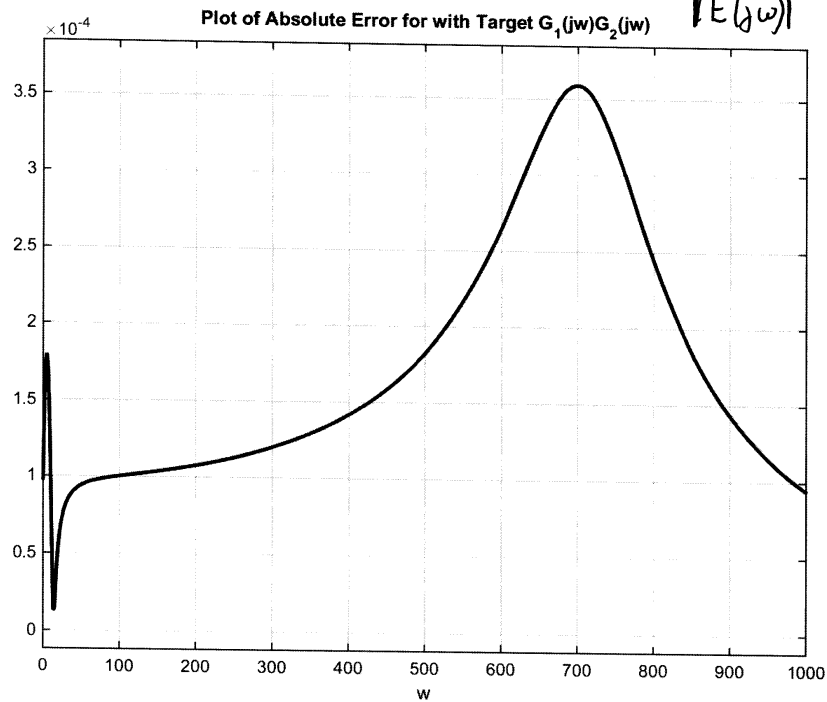
(c) Target:  $T(j\omega) = G_1(j\omega) G_2(j\omega)$

Actual:  $A(j\omega) = \frac{\alpha G_1(j\omega) G_2(j\omega)}{1 + \alpha H_{LP}(j\omega)}$

Error  $E(j\omega) = T(j\omega) - A(j\omega)$

Plot

$|E(j\omega)|$



;

## 3.5 HW 5

### 3.5.1 Problem 1

**Problem 1:** Performances specifications for a second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with  $0 < \zeta < 1$  require that the maximum overshoot to a step input not exceed 30% and that the settling time be less than 0.25 seconds. Using the settling time approximation

$$t_s \approx \frac{3.2}{\zeta\omega_n},$$

find and sketch the region in the complex plane where the two poles need to be located.

**SOLUTION:**

There are two inequalities to satisfy. The first is given by the settling time requirement

$$t_s = \frac{3.2}{\zeta\omega_n} < 0.25 \quad (1)$$

The second is given by the overshoot requirement

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.3 \quad (2)$$

From (2), taking the log of both sides gives

$$-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} < \ln(0.3)$$

Multiplying both sides by  $-1$  changes the inequality from  $<$  to  $>$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} > -\ln(0.3)$$

Simplifying gives

$$\begin{aligned} \frac{\pi\zeta}{\sqrt{1-\zeta^2}} &> \ln\left(\frac{1}{0.3}\right) \\ &> 1.204 \end{aligned}$$

Squaring both sides and solving for  $\zeta$

$$\begin{aligned}\frac{\zeta^2}{1-\zeta^2} &> \left(\frac{1.204}{\pi}\right)^2 \\ \zeta^2 &> 0.14688(1-\zeta^2) \\ 1.14688\zeta^2 &> 0.14688 \\ \zeta^2 &> 0.12807\end{aligned}$$

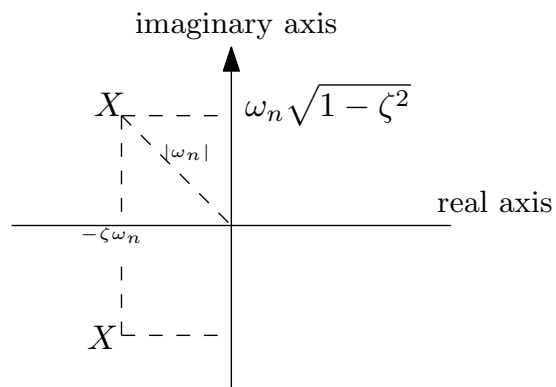
Since  $\zeta$  has to be positive then the positive root is used giving

$$\zeta > 0.35787$$

Back to the (1) specifications, which says

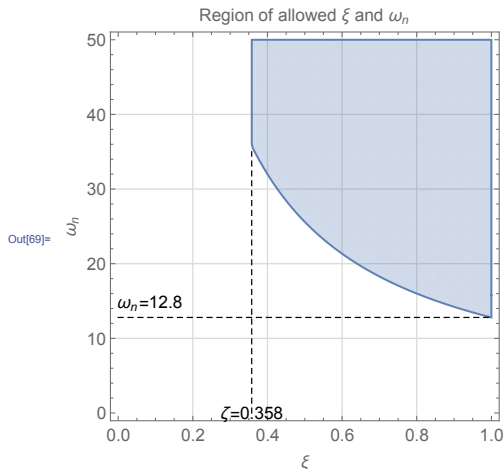
$$\begin{aligned}\frac{3.2}{\zeta\omega_n} &< 0.25 \\ \zeta\omega_n &> 12.8\end{aligned}$$

For each  $\zeta_i > 0.3578$ , we solve for  $\omega_n$  from  $\zeta\omega_n > 12.8$ . This will give full description of where the poles are located.



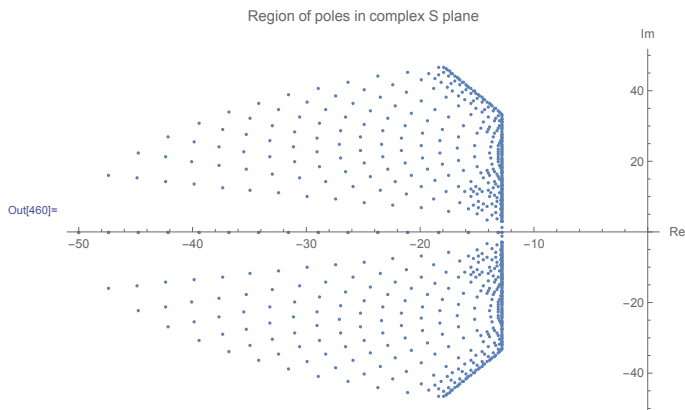
Here is a plot of the  $(\zeta, \omega_n)$  space showing allowed values of  $\zeta, \omega_n$ .

```
In[69]:= RegionPlot[1 > z > 0.35787 && z w > 12.8, {z, 0, 1}, {w, 0, 50}, GridLines -> Automatic, GridLinesStyle -> LightGray, Frame -> True,
FrameLabel -> {{ "ωn", None}, {"ξ", "Region of allowed ξ and ωn"}, BaseStyle -> 14, Epilog -> {
{Dashed, Line[{{0.3578, 0}, {0.3578, 35}]}, Line[{{0, 12.8}, {1, 12.8}}]},
Text["ξ=0.358", {0.358, 0}],
Text["ωn=12.8", {0.084, 14.9}]
}]
```



By taking each point  $(\zeta, \omega_n)$  from the above plot, then the pole location with coordinates  $-\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}$  is generated. The following shows the final result, showing the region where the poles have to be located in order to meet the performance requirements.

```
In[456]:= p = Normal@RegionPlot[1 > z > 0.35787 && z w > 12.8, {z, 0, 1}, {w, 0, 50}];
pts = DeleteDuplicates@Flatten[Cases[p, Polygon[x_] -> x, Infinity], 1];
data = {-First@#Last@#, Last@#Sqrt[1 - (First@#)^2]} & /@pts;
data2 = {-First@#Last@#, -Last@#Sqrt[1 - (First@#)^2]} & /@pts;
ListPlot[Union[data, data2], AxesOrigin -> {0, 0}, AxesLabel -> {"Re", "Im"} (*AxesLabel -> {"-ξ ωn", "ωn √(1 - ξ2)"}*), PlotLabel -> "Region of poles in co
ImageSize -> 500]
```

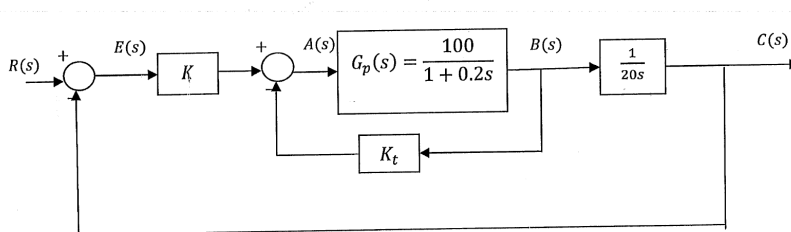


The above diagram shows the location of each pair of poles as a small dot. Complex poles come in pair of conjugates. One pole will be above the real axis and its pair below the real axis.

### 3.5.2 Problem 2

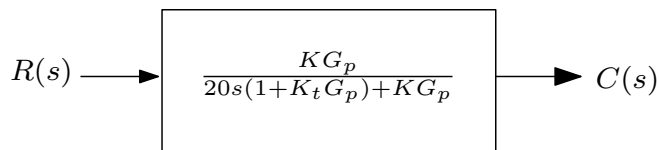
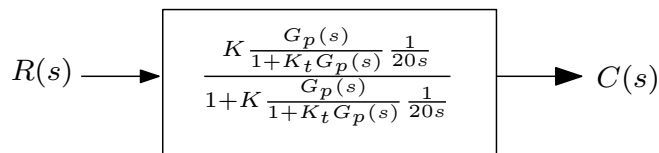
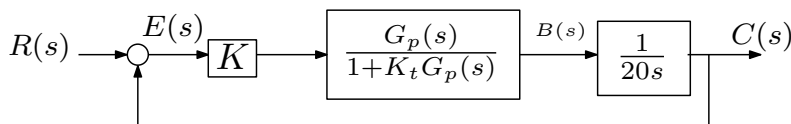
**Problem 2:** For the system shown in the figure below, find gains  $K$  and  $K_t$  so that the maximum overshoot of the output  $c(t)$  to a unit step input  $r(t)$  is about 20% and that the rise time is approximately 0.05 seconds. Find the resulting closed loop transfer function  $C(s)/R(s)$  and simulate the step response in Matlab. In your solution, use the rise time approximation

$$t_r \approx \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n}$$



SOLUTION:

The closed loop transfer function, in terms of  $K$  and  $K_t$  can be found using either Mason rule or simple block reduction. For this problem block reduction seems easier.



Using  $G_p = \frac{100}{1+0.2s}$  the above becomes

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{K\left(\frac{100}{1+0.2s}\right)}{20s\left(1 + K_t\frac{100}{1+0.2s}\right) + K\frac{100}{1+0.2s}} \\ &= \frac{100K}{20s(1 + 0.2s + 100K_t) + 100K} \\ &= \frac{100K}{4s^2 + (20 + 2000K_t)s + 100K} \\ &= \frac{25K}{s^2 + (5 + 500K_t)s + 25K}\end{aligned}$$

The standard form is  $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$  therefore by comparing to the above we find

$$\begin{aligned}\omega_n^2 &= 25K \\ \omega_n &= 5\sqrt{K}\end{aligned}\tag{1}$$

And

$$\begin{aligned}5 + 500K_t &= 2\zeta\omega_n \\ &= 2\zeta(5\sqrt{K}) \\ \zeta &= \frac{5 + 500K_t}{10\sqrt{K}}\end{aligned}\tag{2}$$

Hence the transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where  $\omega_n = 5\sqrt{K}$  and  $\zeta = \frac{5+500K_t}{10\sqrt{K}}$ . We now apply the user specifications in order to determine  $K$  and  $K_t$ . From the overshoot requirement, we write

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \leq 0.2\tag{3}$$

And from the rise time requirements we have

$$\frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.05\tag{4}$$

From (3) and (4) we can now solve for  $\omega_n$  and  $\zeta$  and this allow us to find  $K$  and  $K_t$  by using

(1,2). From (3), taking logs and solving for  $\zeta$  gives

$$\begin{aligned} -\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\leq \ln(0.2) \\ \frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\geq \ln(5) \\ \frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\geq 1.6094 \\ \pi^2\zeta^2 &\geq (1.6094)^2(1-\zeta^2) \\ \pi^2\zeta^2 &\geq 2.5902 - 2.5902\zeta^2 \\ (\pi^2 + 2.5902)\zeta^2 &\geq 2.5902 \\ \zeta &\geq \sqrt{\frac{2.5902}{(\pi^2 + 2.5902)}} \end{aligned}$$

Hence

$$\zeta \geq 0.456$$

Any  $0.456 \leq \zeta < 1$  can be used. In order to find  $\omega_n$ , let us choose

$$\zeta = 0.46$$

For the rest of the calculations . From (4) we find

$$\begin{aligned} \frac{1 - 0.4167(0.46) + 2.917(0.46)^2}{\omega_n} &= 0.05 \\ \omega_n &= \frac{1.4256}{0.05} \end{aligned}$$

Therefore

$$\omega_n = 28.512 \text{ rad/sec}$$

Now that we found  $\zeta$  and  $\omega_n$ , we use (1,2) to find the gains. From (1)

$$\begin{aligned} \omega_n &= 5\sqrt{K} \\ K &= \frac{\omega_n^2}{25} = \frac{28.512^2}{25} \end{aligned}$$

Therefore

$$K = 32.517$$

And from (2)

$$\begin{aligned} \zeta &= \frac{5 + 500K_t}{10\sqrt{K}} \\ 0.46 &= \frac{5 + 500K_t}{10\sqrt{32.517}} \end{aligned}$$

Hence

$$K_t = 0.04246$$

The final transfer function is

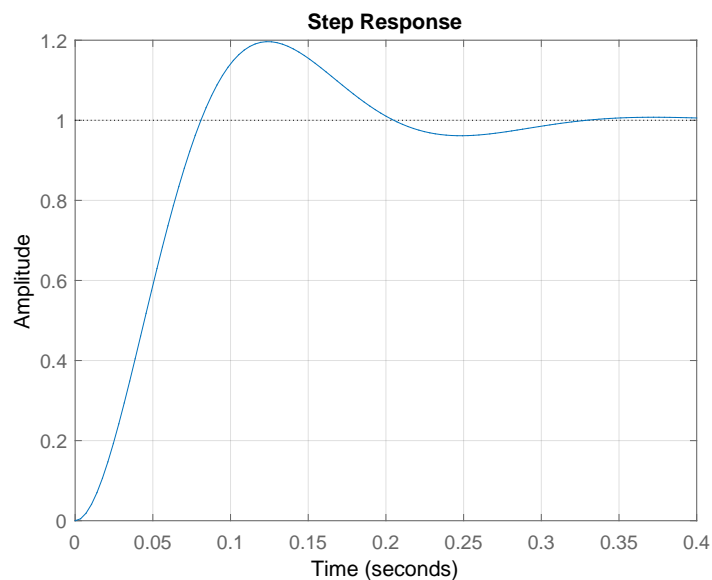
$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \\ &= \frac{25(32.517)}{s^2 + (5 + 500(0.04246))s + 25(32.517)} \end{aligned}$$

Or

$$\frac{C(s)}{R(s)} = \frac{812.93}{s^2 + 26.23s + 812.93}$$

Matlab is used to simulate the step response, and to also verify the user requirements are met.

```
close all;
clear all;
s = tf('s');
sys = 812.93/(s^2+26.23*s+812.93);
step(sys)
grid
```



The step information was also verified using the command `stepinfo()` which showed the specifications was indeed met.

```
stepinfo(sys)
```

```
RiseTime: 0.0549
SettlingTime: 0.2916
```



SettlingMin: 0.9225  
 SettlingMax: 1.1963  
 Overshoot: 19.6310  
 Undershoot: 0  
 Peak: 1.1963  
 PeakTime: 0.1229

### 3.5.3 Problem 3

**Problem 3:** A system with adjustable gain  $K$  and open loop transfer function

$$G(s) = \frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}$$

connected in a classical unity feedback configuration. Use the Routh-Hurwitz criterion to find the range of  $K$  for which closed loop stability is assured.

SOLUTION:

$$G(s) = \frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}$$

In classical unity feedback, the closed loop transfer function  $T(s)$  is

$$\begin{aligned}
 T(s) &= \frac{G(s)}{1 + G(s)} \\
 &= \frac{Ks(20s^2 + 1)}{(s^4 + 5s^2 + 10s + 15) + Ks(20s^2 + 1)} \\
 &= \frac{Ks(20s^2 + 1)}{s^4 + 20Ks^3 + 5s^2 + (10 + K)s + 15}
 \end{aligned}$$

Applying Routh-Hurwitz to the denominator  $D(s) = s^4 + 20Ks^3 + 5s^2 + (10 + K)s + 15$  gives

$s^4$	1	5	15
$s^3$	$20K$	$(10 + K)$	0
$s^2$	$\frac{20K(5) - (10+K)}{20K}$	15	0
$s^1$	$\frac{\frac{(20K(5) - (10+K))}{20K} (10+K) - 20K(15)}{\frac{20K(5) - (10+K)}{20K}}$	0	0
$s^0$	15		

Simplifying gives

$s^4$	1	5	15
$s^3$	$20K$	$(10 + K)$	0
$s^2$	$\frac{1}{20K}(99K - 10)$	15	0
$s^1$	$\frac{1}{10-99K}(5901K^2 - 980K + 100)$	0	0
$s^0$	15		

For stability, we need the first column to be positive. Hence the conditions are

$$20K > 0$$

$$\frac{1}{20K}(99K - 10) > 0$$

$$\frac{1}{10 - 99K}(5901K^2 - 980K + 100) > 0$$

The first just says that  $K > 0$ . The second says  $99K - 10 > 0$  or  $K > \frac{10}{99}$ . Now for the third condition

$$\frac{1}{10 - 99K}(5901K^2 - 980K + 100) > 0$$

Since  $K > \frac{10}{99}$  is required, then  $\frac{1}{10-99K}$  is negative quantity since  $10 - 99K$  is negative for  $K > \frac{10}{99}$ . This means the above becomes

$$5901K^2 - 980K + 100 < 0$$

Notice the change of inequality from  $>$  to  $<$  since we multiplied both sides by a negative quantity ( $10 - 99K$ ) to cancel it out. But  $5901K^2 - 980K + 100 < 0$  can not be satisfied with a positive  $K > \frac{10}{99}$ . For example, using the minimum allowed  $K$  which is  $\frac{10}{99}$ , then the value of  $5901K^2 - 980K + 100$  becomes

$$5901 \left(\frac{10}{99}\right)^2 - 980 \left(\frac{10}{99}\right) + 100 = 61.218$$

But it needs to be negative. So *there does not exist K which makes the closed loop stable.*

### 3.5.4 Problem 4

**Problem 4:** Consider the system with open loop transfer function

$$G(s) = \frac{K}{s(1 + Ts)}$$

is connected in a unity feedback configuration. Given  $a > 0$ , the specification is that all closed loop poles have real part less than  $-a$ . Show how the Routh-Hurwitz criterion can be modified to address this problem. Subsequently, for  $T = 1$ , find and sketch the region in the  $(a, K)$  plane associated with closed loop stability.

SOLUTION:

Given  $G(s) = \frac{K}{s(1+Ts)}$  then the closed loop transfer function is

$$\begin{aligned} G_{closed}(s) &= \frac{G}{1+G} \\ &= \frac{K}{s(1+Ts) + K} \\ &= \frac{K}{Ts^2 + s + K} \end{aligned}$$

Therefore  $D(s) = Ts^2 + s + K$ . For the closed loop poles with real part to be less than  $-a$ , let  $s_1 = s + a$ . Then  $s = s_1 - a$ . We apply Routh-Hurwitz to  $D(s)$  but with  $s = s_1 - a$ . The new denominator polynomial becomes

$$D(s_1) = T(s_1 - a)^2 + (s_1 - a) + K$$

Expanding gives

$$\begin{aligned} D(s_1) &= T(s_1^2 + a^2 - 2s_1a) + s_1 - a + K \\ &= Ts_1^2 + s_1(1 - 2Ta) + (Ta^2 - a + K) \end{aligned}$$

Routh table applied to the above polynomial is

$s_1^2$	$T$	$Ta^2 - a + K$
$s_1^1$	$1 - 2Ta$	$0$
$s_1^0$	$Ta^2 - a + K$	

We need all entries in the first column to be same sign (positive in this case, since  $T = 1$ ) for stability to hold (This is in addition to having the poles be with real part less than  $-a$ ). For  $T = 1$  the above becomes

$s^2$	$1$	$a^2 + a + K$
$s^1$	$1 - 2a$	$0$
$s^0$	$a^2 - a + K$	

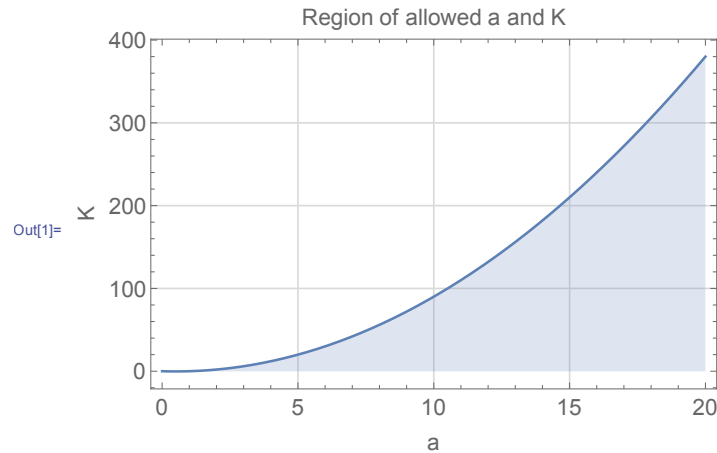
The conditions for stability are

$$\begin{aligned} 1 - 2a &> 0 \\ a(1 - a) + K &> 0 \end{aligned}$$

The first condition gives  $a > \frac{1}{2}$ . The second condition gives

$$K > a^2 - a$$

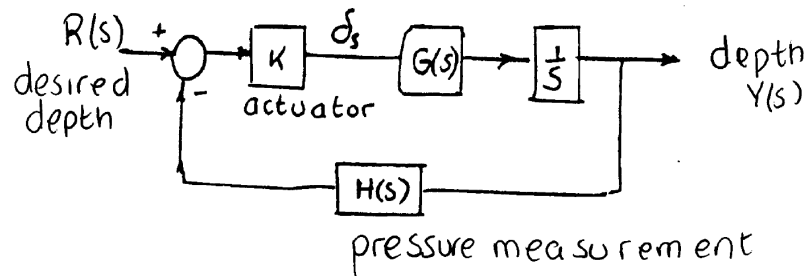
Here is plot of the region in the  $(a, K)$  plane associated with closed loop stability.



### 3.5.5 Problem 5

**Problem 5:** An automatic depth control system for a submarine is depicted in the figure below. The depth is measured by a pressure transducer. For what values of  $K$  will the system be stable? Take  $H(s) = 1$  and submarine transfer function

$$G(s) = \frac{(s+2)^2}{s^2+0.01}$$



SOLUTION:

The closed loop transfer function is

$$T(s) = \frac{KG(s)\frac{1}{s}}{1 + HKG(s)\frac{1}{s}}$$

Replacing  $H(s) = 1$  and  $G = \frac{(s+2)^2}{s^2+0.01}$  the above becomes

$$\begin{aligned} T(s) &= \frac{\frac{K(s+2)^2}{s^2+0.01} \frac{1}{s}}{1 + \frac{K(s+2)^2}{s^2+0.01} \frac{1}{s}} \\ &= \frac{K(s+2)^2}{s(s^2+0.01) + K(s+2)^2} \\ &= \frac{K(s+2)^2}{s^3 + Ks^2 + (0.01 + 4K)s + 4K} \end{aligned}$$

The Routh table for  $D(s) = s^3 + Ks^2 + (0.01 + 4K)s + 4K$  is

$s^3$	1	$0.01 + 4K$
$s^2$	$K$	$4K$
$s^1$	$4K - 3.99$	0
$s^0$	$4K$	

Therefore for stability we need

$$\begin{aligned} K &> 0 \\ 4K &> 3.99 \\ 4K &> 0 \end{aligned}$$

The first and the third conditions give  $K > 0$ . From the second condition,  $K > \frac{3.99}{4} = 0.9975$ . Therefore

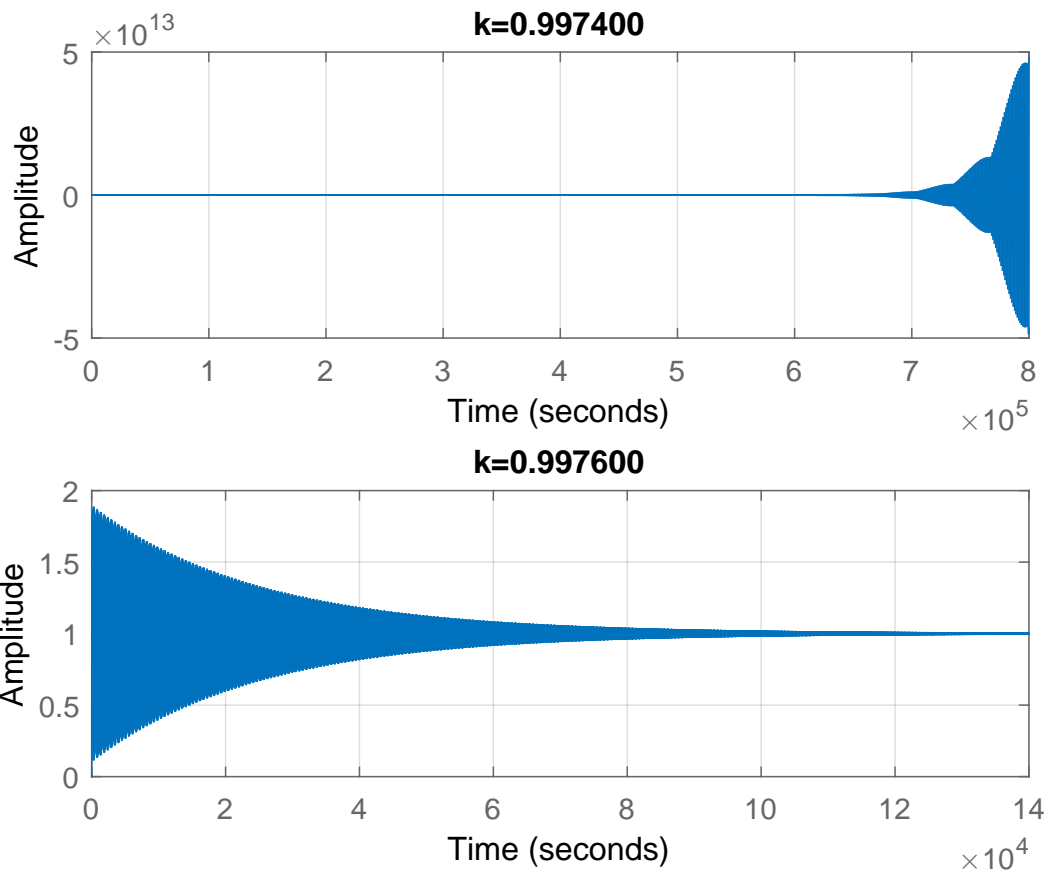
$$K > 0.9975$$

To verify, here is the step response for  $k = 0.9974$  and  $k = 0.9976$ , showing one is unstable and the second is stable.

```
close all; clear all;
s = tf('s');
G = (s+2)^2/(s^2+0.01);

k = .9974;
sys = feedback(k*G*1/s,1);
subplot(2,1,1);
step(sys);
title(sprintf('k=%f',k)); grid

subplot(2,1,2);
k = .9976;
sys = feedback(k*G*1/s,1);
step(sys);
title(sprintf('k=%f',k)); grid
```



## 3.5.6 HW 5 key solution

## 1) HOMEWORK OVERTHOOT SETTLING:

Performance specifications for a second-order system require that the maximum overshoot to a step input must not exceed 1.3 (30% overshoot) and that the settling time should be less than 0.25 seconds. Interpret these requirements in terms of desired  $\zeta$  and  $\omega_n$ . Plot the region in the  $s$ -plane where the two poles of the system would need to be located in order to meet these design specifications.

For maximum overshoot, use  $y_{\max} - 1 = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

For settling time, use  $t_s = \frac{3.2}{\omega_n}$

$$y_{\max} \leq 1.3 \quad t_s \leq 0.25$$

$$1.3 - 1 \geq e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$-1.204 \geq -\pi\zeta/\sqrt{1-\zeta^2}$$

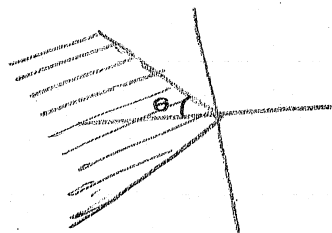
$$1.204 \leq \pi\zeta/\sqrt{1-\zeta^2}$$

$$1.449(1-\zeta^2) \leq \pi^2\zeta^2$$

$$1.449 \leq 11.319\zeta^2$$

$$\boxed{\zeta \geq 0.358}$$

Recall that  $\zeta = \cos \theta$   
 $\therefore \cos \theta \geq 0.358$



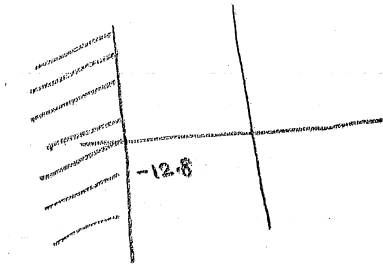
The above shaded region represents  $\cos \theta \geq 0.358$ . The  $\theta$  in the above figure is  $\cos^{-1}(0.358) = 69^\circ$

For the settling time constraint, we have,

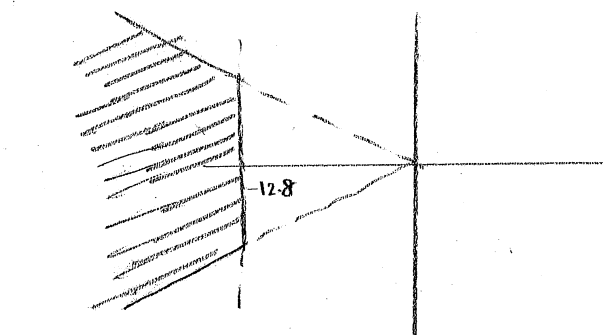
$$0.25 \geq \frac{3.2}{\xi \omega_n}$$

$$\xi \omega_n \geq 12.8$$

This corresponds to



Since we want both conditions to be satisfied, the allowed region for the poles is the common shaded region from the above two figures:



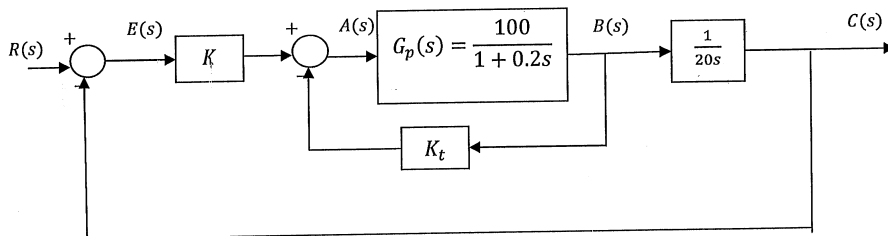
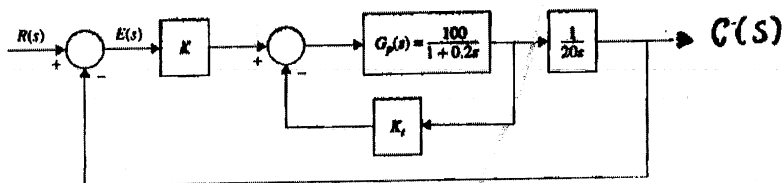
;



## 2) HOMEWORK OVERSHOOT

## 1. Overshoot

**Problem 3:** For the system shown in the figure below, find gains  $K$  and  $K_t$  so that the maximum overshoot of the output  $c(t)$  to a unit step input  $r(t)$  is about 20% and the rise time is approximately 0.05 seconds. Find the resulting closed loop transfer function  $C(s)/R(s)$  and simulate the step response in Matlab.



$$E(s) = R(s) - C(s)$$

$$A(s) = E(s) * K - B(s) * K_t$$

$$B(s) = A(s) * G_p(s) = A(s) * \frac{100}{1 + 0.2s}$$

$$C(s) = B(s) * \frac{1}{20s}$$

$$B(s) = 20s * C(s)$$

;

$$\begin{aligned}
 A(s) &= B(s) * \frac{1 + 0.2s}{100} \\
 &= 20s * C(s) * \frac{1 + 0.2s}{100} \\
 &= C(s) * \frac{20s + 4s^2}{100}
 \end{aligned}$$

$$E(s) = \frac{A(s) + K_t B(s)}{K}$$

$$R(s) - C(s) = \frac{A(s) + K_t B(s)}{K}$$

$$R(s) - C(s) = \frac{\left( C(s) * \frac{20s + 4s^2}{100} \right) + K_t (20s * C(s))}{K}$$

$$R(s) - C(s) = C(s) * \frac{20s + 4s^2}{100K} + C(s) \frac{K_t (20s)}{K}$$

$$C(s) \left[ \frac{20s + 4s^2}{100K} + \frac{K_t (20s)}{K} + 1 \right] = R(s)$$

$$\begin{aligned}
 \frac{C(s)}{R(s)} &= \frac{1}{\frac{20s + 4s^2}{100K} + \frac{K_t (20s)}{K} + 1} \\
 &= \frac{1}{\frac{20s + 4s^2}{100K} + \frac{2000K_t s}{100K} + \frac{100K}{100K}} \\
 &= \frac{1}{\frac{20s + 4s^2 + 2000K_t s + 100K}{100K}} \\
 &= \frac{100K}{4s^2 + (2000K_t + 20)s + 100K} \\
 &= \frac{25K}{s^2 + (500K_t + 5)s + 25K}
 \end{aligned}$$

;

$$\text{Percent maximum overshoot} = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\text{Maximum overshoot} = 20\%$$

$$100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 20$$

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.2$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.2$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = 1.61$$

$$\pi\zeta = 1.61\sqrt{1-\zeta^2}$$

$$\frac{\pi\zeta}{1.61} = \sqrt{1-\zeta^2}$$

$$\frac{\pi^2\zeta^2}{2.59} = 1 - \zeta^2$$

$$\pi^2\zeta^2 = 2.59 - 2.59\zeta^2$$

$$(\pi^2 + 2.59)\zeta^2 = 2.59$$

$$\zeta^2 = \frac{2.59}{(\pi^2 + 2.59)}$$

$$\zeta = 0.456$$

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} \quad 0 < \zeta < 1$$

$$t_r = \text{rise time} = 0.05 \text{ s}$$

$$0.05 = \frac{1 - 0.4167(0.456) + 2.917(0.456)^2}{\omega_n}$$

$$0.05\omega_n = 1.416$$

$$\omega_n = 28.33$$

;

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{25K}{s^2 + (500K_t + 5)s + 25K}$$

$$2\zeta\omega_n = 500K_t + 5$$

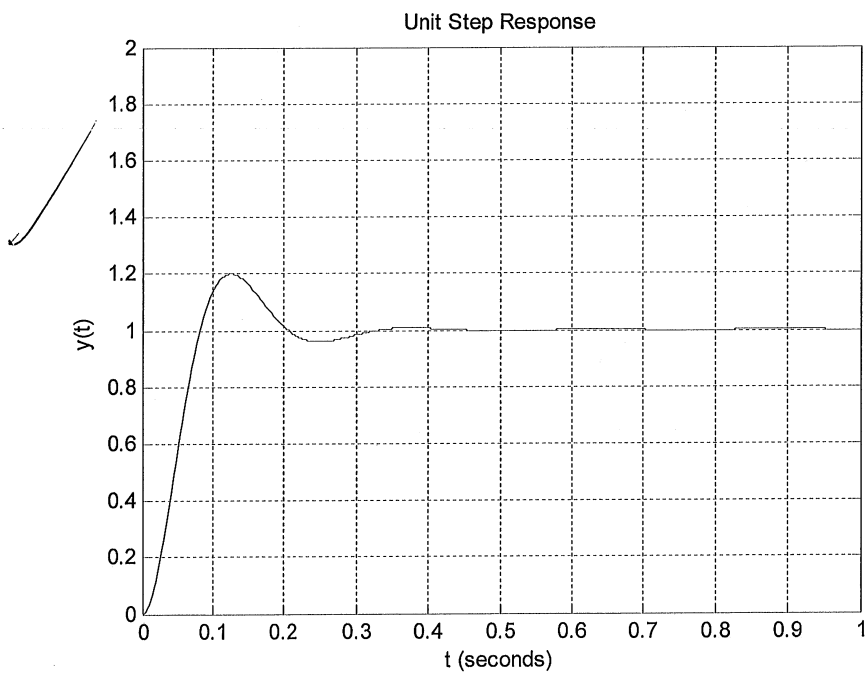
$$2 * (0.456) * (28.33) = 500K_t + 5$$

$$K_t = 0.0417$$

$$\omega_n^2 = 25K$$

$$(28.33)^2 = 25K$$

$$K = 32.1$$



;

$$G(s) = \frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}$$

is connected in a classical unity feedback configuration. Find the range of gains  $K$  for which closed loop stability is assured.

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + G(s)} \\ &= \frac{\frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}}{1 + \frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}} \\ &= \frac{\frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}}{\frac{s^4 + 5s^2 + 10s + 15 + Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15}} \\ &= \frac{Ks(20s^2 + 1)}{s^4 + 5s^2 + 10s + 15 + Ks(20s^2 + 1)} \\ &= \frac{K(20s^3 + s)}{s^4 + 20Ks^3 + 5s^2 + s(10 + K) + 15} \end{aligned}$$

;

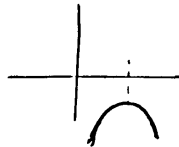
$$\begin{array}{rcccc}
 \textcircled{1} & S^4 & 1 & 5 & 15 \\
 & S^3 & 20K & (10+K) & 0 \\
 & S^2 & \frac{99K-10}{20K} & 15 & 0 \\
 & S^1 & \left\{ \frac{(10+K)(99K-10) - 6000K^2}{(99K-10)} \right\} & 0 & 0 \\
 & S^0 & 15 & 0 & 0
 \end{array}$$

To avoid sign changes  $\overset{\textcircled{1}}{20K} \neq 0 \Rightarrow K \neq 0$

$$\textcircled{2} \quad 99K - 10 \neq 0 \Rightarrow K \neq \frac{10}{99}$$

$$\begin{aligned}
 \textcircled{3} \quad & (10+K)(99K-10) - 6000K^2 \neq 0 \\
 \Rightarrow & -5901K^2 + 980K - 100 \neq 0 \quad \star
 \end{aligned}$$

\* No real value of  $K$  satisfies  $\star$  above



$\Rightarrow$  System is unstable for all real  $K$ .

$$G(s) = \frac{K}{s(1+Ts)}$$

connected in classical unity feedback configuration. Given  $a > 0$ , the specification for this system is that all closed loop poles have real part less than  $-a$ . That is, all closed loop poles should lie to the left of the line  $z = -a$  in the complex plane. Show how the Routh-Hurwitz criterion can be modified to address this problem. Subsequently, with  $T = 1$ , find the region in the  $(a, K)$  plane associated with closed loop stability.

② Let  $s_1 = s + a$ . Now  $s_1$  being left of zero is like  $s$  being left of  $-a$ .

$$\Rightarrow G(s) = \frac{K}{(s_1 - a)(1 + s_1 T - aT)}$$

Obtain characteristic equation in  $s_1$ .

$$1(s_1) = Ts_1^2 + (1 - 2aT)s_1 - a + a^2T + K$$

(Look at  $1 + GH, H=1$ )

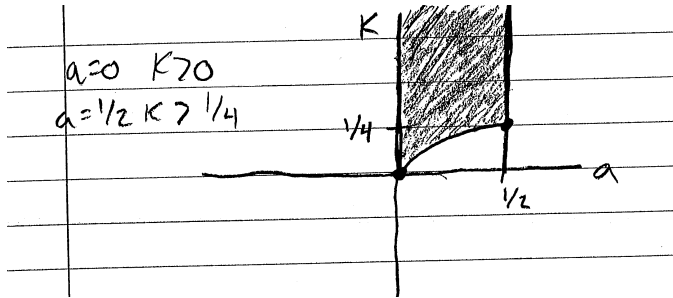
$$s_1^2 \quad T \quad -a + a^2T + K$$

$$s_1 \quad 1 - 2aT \quad 0$$

$$s^0 \quad -a + a^2T + K$$

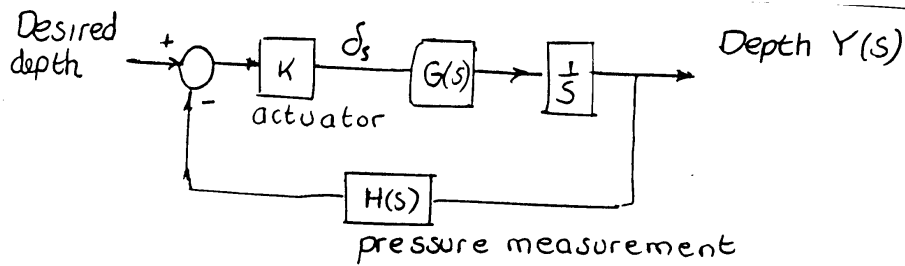
For example, we can obtain a stable

stem if  $T > 0, 1 - 2aT > 0, K + a^2T - a > 0$   
 (or reverse inequalities)  $T=1 \Rightarrow a < \frac{1}{2}$   
 $K > a^2 - a$



5) **PROBLEM 3:**

**Problem 3:** An automatic depth control system for a submarine is depicted in the figure below.



The depth is measured by a pressure transducer. For what values of the actuator  $K$  will the system be stable? The submarine has an approximate transfer function

$$G(s) = \frac{(s + 0.2)^2}{s^2 + 0.01}$$

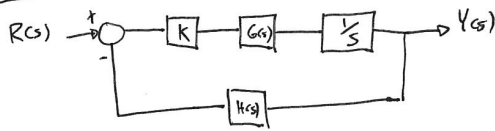
and the transducer may be taken to be

$$H(s) = 1.$$

;



Problem #5



$$G(s) = \frac{(s+2)^2}{s^2 + 0.1}$$

$$H(s) = 1$$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K(s+2)^2}{s(s^2 + 0.1)} = \frac{K(s+2)^2}{K(s+2)^2 + s(s^2 + 0.1)}$$

Characteristic equation:

$$s^3 + ks^2 + (.01 + 4k)s + 4k$$

Routh table:

$s^3$	1	.01 + 4k	0
$s^2$	k	4k	0
$s^1$	a	0	
$s^0$	4k		

$$a = \frac{k(.01 + 4k) - 4k}{k}$$

$$= (.01 - 4) + 4k \geq 0$$

$$\Rightarrow k > \frac{3.99}{4} \quad \textcircled{I}$$

$$k > 0 \quad \textcircled{II}$$

$k > .9975$  ensures stability

;

## 3.6 HW 6

### 3.6.1 Problem 1

Problem 1

The open-loop transfer functions of a unity negative feedback control system are given as follows:

$$(a) G(s) = \frac{K}{s(s+2)(s+5)(s+10)}$$

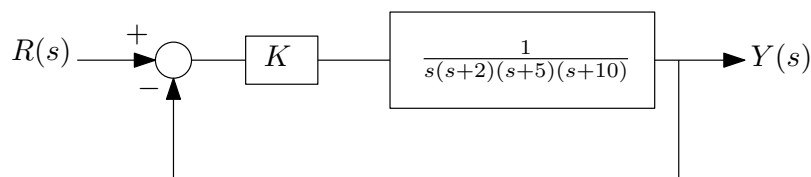
$$(b) G(s) = \frac{K}{(s^2+s+2)(s+1)}$$

$$(c) G(s) = \frac{K(s^2+2s+10)}{s(s+5)(s+10)}$$

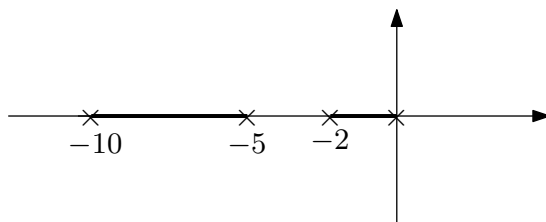
For each of these, plot the root locus for  $0 \leq K \leq +\infty$  without using any computer

SOLUTION:

#### 3.6.1.1 Part(a)

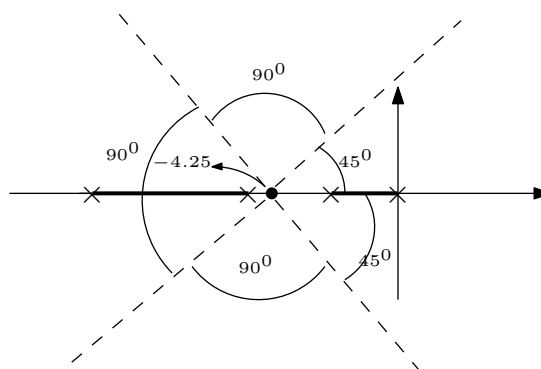


- $n = 4$  (The number of open loop poles),  $m = 0$  (The number of open loop zeros).
- Root Locus (R.L.) starts at the open loop poles  $s = 0, s = -2, s = -5, s = -10$  when  $k = 0$ . Since there are no open loop zeros, the branches will end up at  $\pm\infty$
- On the real axis, R.L. exists on segments based on number of poles and zeros to the right of the segment. If this sum (including multiplicity) is odd, then the segment is on the R.L., else it is not. The plot below shows the segments found.



- Center of asymptotes is now found using  $\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0-2-5-10)}{4} = -4.25$

5. The asymptotes angles are  $\theta = \frac{180^\circ \pm k360^\circ}{n-m}$  for  $k = 0, 1, \dots$ . Hence  $\theta = \frac{180^\circ \pm k360^\circ}{4} = 45^\circ \pm k90^\circ$ . The R.L. is updated below



6. Finding the break away points. We need to find where on the two segments shown above the root locus will break away from the real line. Since

$$1 + G = 0$$

$$1 + \frac{K}{s(s+2)(s+5)(s+10)} = 0$$

$$K = -s(s+2)(s+5)(s+10)$$

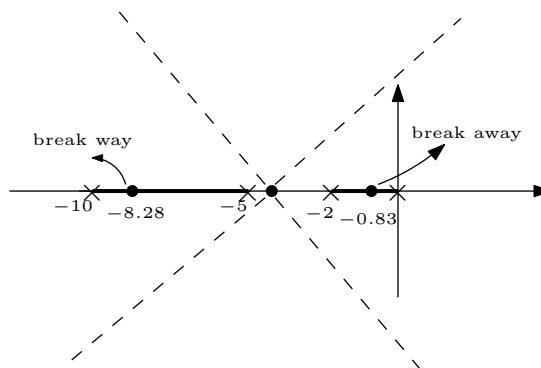
$$K = -s^4 - 17s^3 - 80s^2 - 100s$$

Then we now solve for  $s$

$$\frac{dK}{ds} = -4s^3 - 51s^2 - 160s - 100 = 0$$

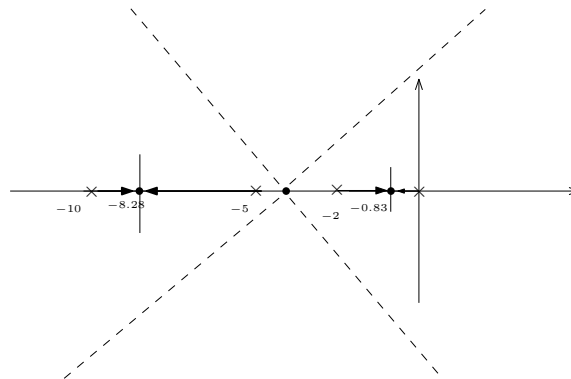
$$0 = 4s^3 + 51s^2 + 160s + 100$$

The roots are  $s = -8.287, s = -0.835, s = -3.632$ . Not all these points will be break away points. Looking at the segments on the real line, we see that  $s = -0.835$  and  $s = -8.287$  are on the R.L. but  $s = -3.632$  is not. We mark these points now on the current plot and update the plot again

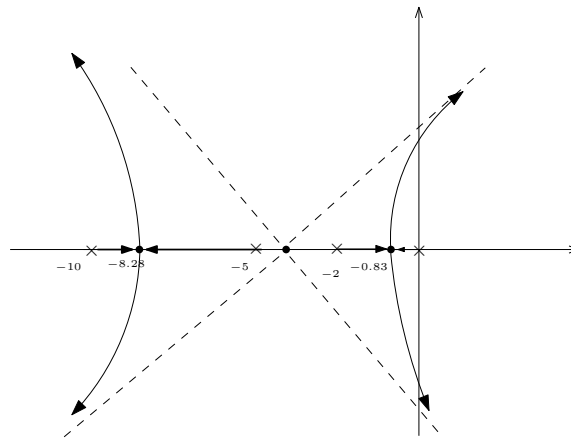


7. Departure angles from poles. It is clear that R.L. depart from open loop poles on the

real axis as shown, moving towards the break away points. Hence the plot now looks like the following



8. We now extend the branch from the break away points towards the asymptotes, since there are no zeros, no need to calculate arrival angles. The branch will move to the asymptotes. So we end up with the following



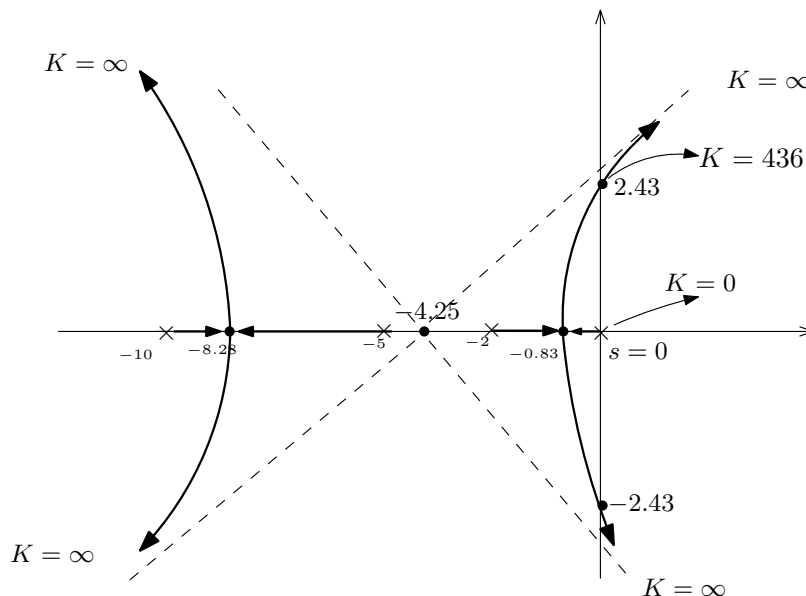
It is important to remember that root locus will always be symmetric with respect to the real axis.

9. Now the only lemma left is to find where R.L. crosses the imaginary axis. For this we use Routh stability stable as follows. Since the characteristic polynomial is  $K + s(s + 2)(s + 5)(s + 10) = 0$  then

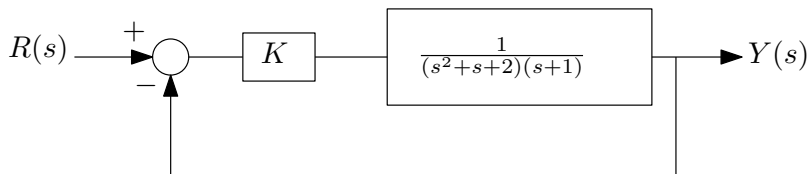
$$P(s) = s^4 + 17s^3 + 80s^2 + 100s + K$$

$s^4$	1	80	$K$
$s^3$	17	100	
$s^2$	74.118	$K$	
$s^1$	$100 - 0.229K$		
$s^0$	$K$		

We just need to find where it cross the imaginary axis. Setting the  $s^1$  row to zero gives  $K = 436.68$ . This means the closed loop will become unstable for  $k > 436.68$ . To find where R.L. crosses the imaginary axes, we go back to the even polynomial  $s^2$  (the row above the line  $s^1$ ) in above Routh table, which yields  $74.118s^2 + K = 0$  or  $74.118s^2 + 436.68 = 0$ , therefore  $s = \pm 2.43i$ . This complete R.L. Here is the final plot

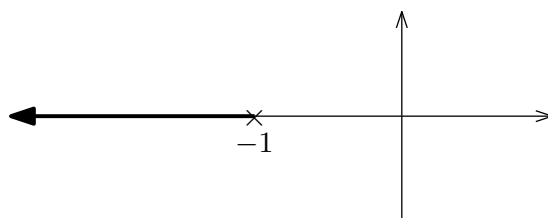


3.6.1.2 Part(b)



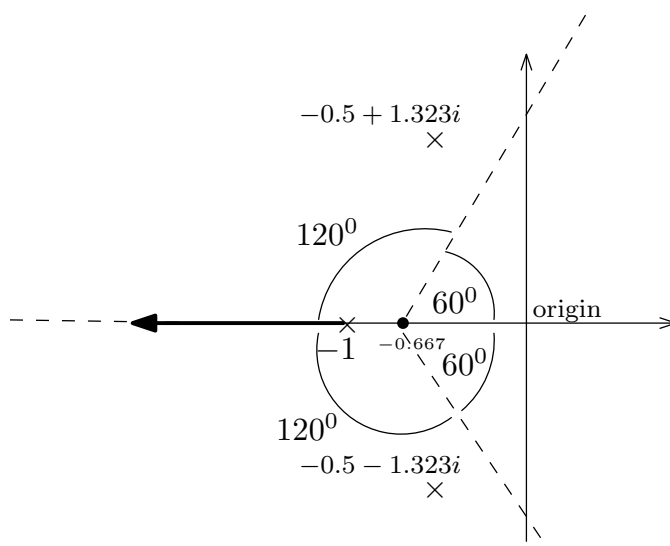
$$G(s) = \frac{K}{(s^2 + s + 2)(s + 1)}$$

1.  $n = 3$  (The number of open loop poles),  $m = 0$  (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles  $s = -1, s = -0.5 \pm 1.323i$  when  $k = 0$ . Since there are no open loop zeros, the branches end up at  $\pm\infty$
3. On the real axis, R.L., since one pole on the real axis, then the R.L. segment is to the left of the only pole, which is  $s = -1$



4. Center of asymptotes is now found.  $\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(-1-0.5-0.5)}{3} = -0.667$

5. The asymptotes angles are  $\theta = \frac{180^\circ \pm k360^\circ}{n-m}$  for  $k = 0, 1, \dots$ . Hence  $\theta = \frac{180^\circ \pm k360^\circ}{3} = 60^\circ \pm k120^\circ$ . Therefore the R.L. now looks like the following



6. Break away points. There is no break away points, since there are no segments between two poles on the real line. We can see this also if we try to solve  $\frac{dK}{ds} = 0$  as follows

$$1 + G = 0$$

$$1 + \frac{K}{(s^2 + s + 2)(s + 1)} = 0$$

$$K = (s^2 + s + 2)(s + 1)$$

$$K = s^3 + 2s^2 + 3s + 2$$

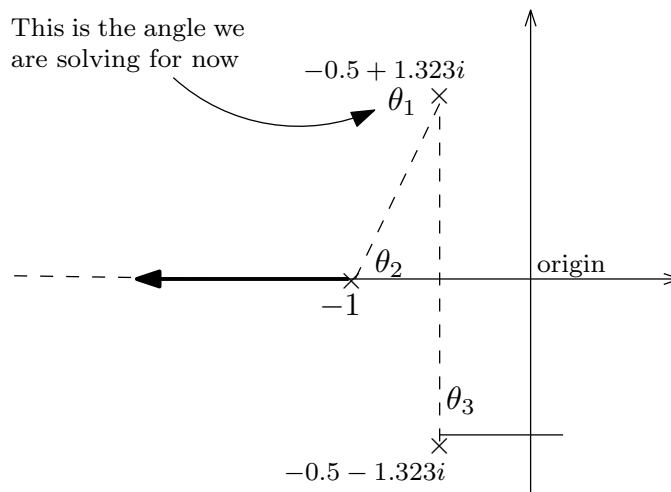
Then we now solve for

$$\frac{dK}{ds} = 3s^2 + 4s + 3 = 0$$

The solution gives only complex roots. So we go to the next lemma.

7. Departure angles from poles. It is clear that R.L. depart from open loop pole on the

real axis  $s = -1$  as shown moving to  $-\infty$  along one of the asymptotes. But we need to find departure angles from the two complex poles  $s = -0.5 \pm 1.323i$ . For  $s = -0.5 + 1.323i$ , let this unknown angle be  $\theta_1$  and we write



$$\sum_{\text{zeros}} \angle z_i - \sum_{\text{poles}} \angle p_i = 180^\circ + k360^\circ \quad k = 0, 1, \dots$$

$$-(\theta_1 + \theta_2 + \theta_3) = 180^\circ + k360^\circ$$

$$\theta_1 = -180^\circ - \theta_2 - \theta_3 - k360^\circ$$

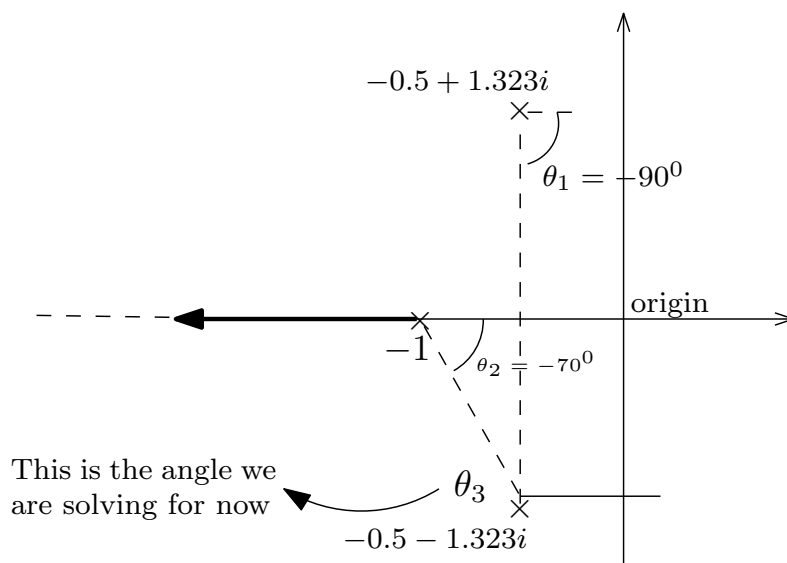
We see from the diagram that  $\theta_3 = 90^\circ$  and  $\theta_2 = \tan^{-1}\left(\frac{1.323}{0.5}\right) = 70^\circ$ , hence from above

$$\theta_1 = -180^\circ - 70^\circ - 90^\circ - k360^\circ$$

$$= -340^\circ$$

$$= 20^\circ$$

By symmetry (R.L. is symmetric with respect to the real axis), the departure angle for the other complex pole  $s = -0.5 - 1.323i$  must be  $-20^\circ$ . We can calculate it to make sure it is indeed  $-20^\circ$  as follows.



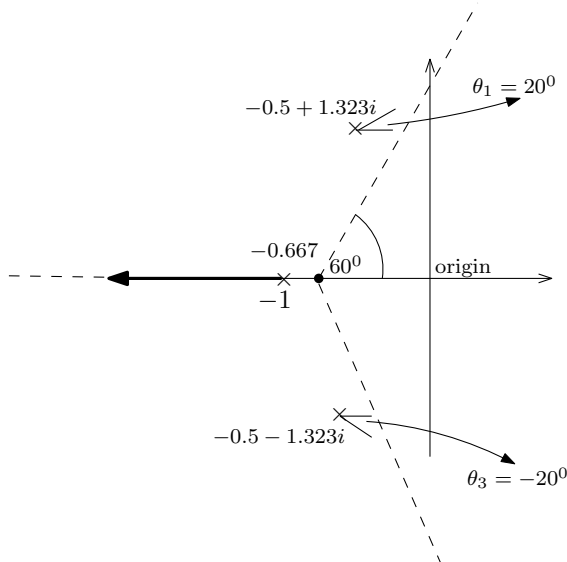
Let  $\theta_3$  be the departure angle for the lower complex pole  $s = -0.5 - 1.323i$ , then

$$\sum_{\text{zeros}} \angle z_i - \sum_{\text{poles}} \angle p_i = 180^\circ + k360^\circ \quad k = 0, 1, \dots$$

$$-(\theta_1 + \theta_2 + \theta_3) = 180^\circ + k360^\circ$$

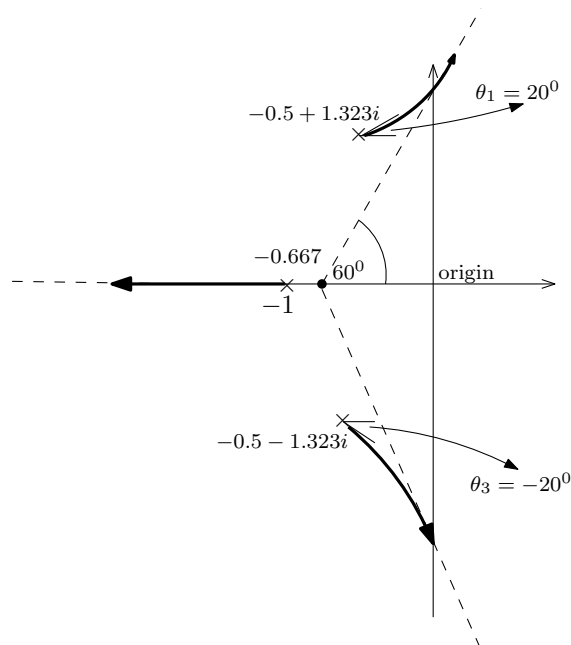
$$\theta_3 = -180^\circ - \theta_2 - \theta_1 - k360^\circ$$

But  $\theta_2 = -70^\circ$  and  $\theta_1 = -90^\circ$  then  $\theta_3 = -180^\circ + 70 + 90 = -20^\circ$  as expected. The root locus plot is now updated





8. We now extend R.L. to the asymptotes, since there are no zeros, there is no need to calculate arrival angles and we end up with the following

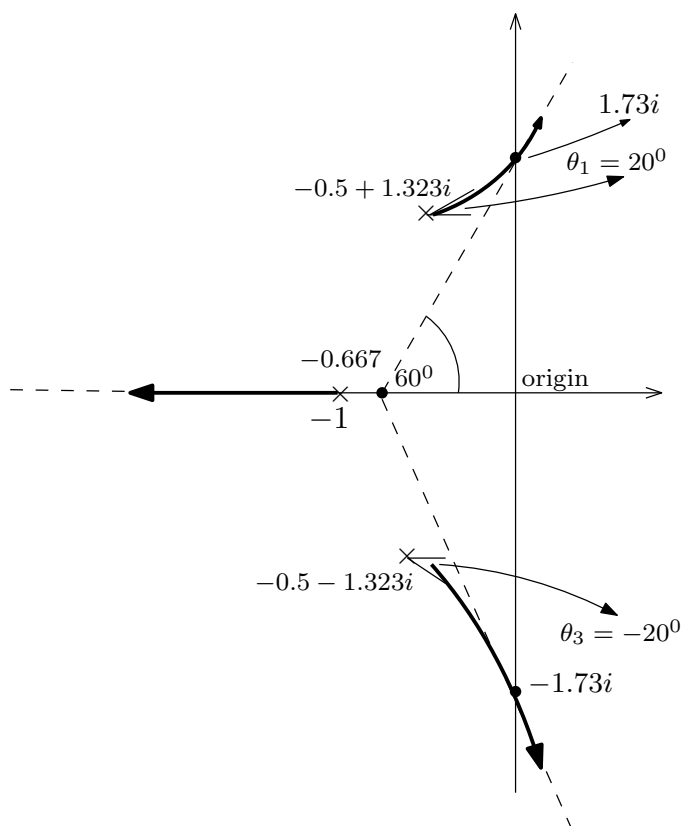


9. Now the only lemma left is to find where R.L. crosses the imaginary axis. For this we use Routh stability stable as follows. Since the characteristic polynomial is  $K + (s^2 + s + 2)(s + 1) = 0$  then

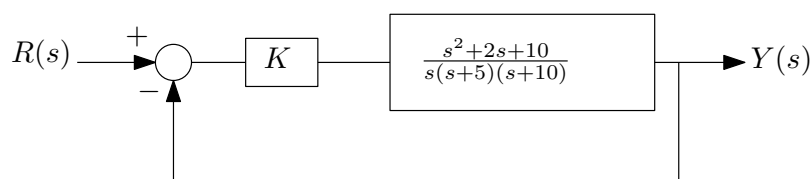
$$P(s) = s^3 + 2s^2 + 3s + K + 2$$

$s^3$	1	3
$s^2$	2	$K + 2$
$s^1$	$\frac{4-k}{2}$	
$s^0$	$K + 2$	

We need  $K = 4$  (by setting the  $s^1$  row to zero). This means system will be unstable for  $k > 4$ . To find where R.L. crosses the imaginary axes, we go back to the  $s^2$  polynomial above the line  $s^1$  in Routh table, which yields  $2s^2 + K + 2 = 0$  or  $2s^2 + 6 = 0$ , therefore  $s = \pm 1.73i$ . This complete R.L. Here is the final plot

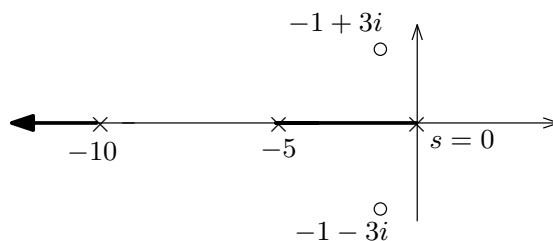


### 3.6.1.3 Part(c)



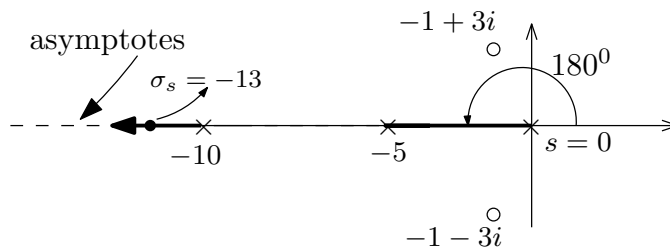
$$G(s) = \frac{K(s^2 + 2s + 10)}{s(s+5)(s+10)}$$

1.  $n = 3$  (The number of open loop poles),  $m = 2$  (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles  $s = 0, s = -5, s = -10$  when  $k = 0$  and two of the branches end up at zeros  $s = -1 \pm 3i$ . One branch will end up at  $\pm\infty$ .
3. On the real axis, there is R.L. segment between  $s = 0, s = -5$  and another segment to the left of  $s = -10$ . No R.L. segment exist between  $s = -5$  and  $s = -10$ . Current diagram now looks like the following



4. Center of asymptotes is now found.  $\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{-15 - (-1-1)}{1} = -13$

5. The asymptotes angles are  $\theta = \frac{180^\circ \pm k360^\circ}{n-m}$  for  $k = 0, 1, \dots$ . Hence  $\theta = \frac{180^\circ \pm k360^\circ}{1} = 180^\circ \pm k360^\circ$ . Therefore the R.L. now looks like the following



6. Finding the break away points

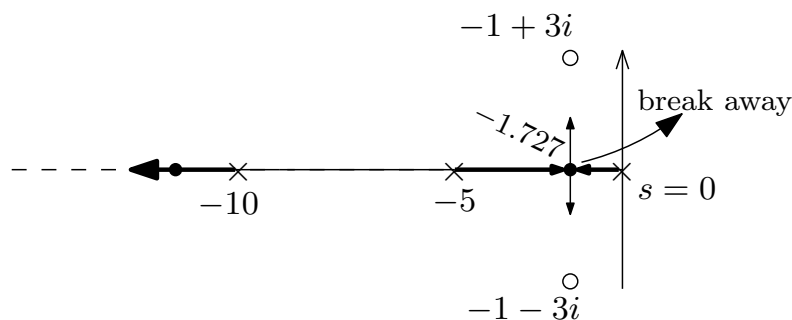
$$\begin{aligned}
 1 + G &= 0 \\
 1 + \frac{K(s^2 + 2s + 10)}{s(s+5)(s+10)} &= 0 \\
 \frac{s(s+5)(s+10) + K(s^2 + 2s + 10)}{s(s+5)(s+10)} &= 0 \\
 s(s+5)(s+10) + K(s^2 + 2s + 10) &= 0 \\
 K &= \frac{-s(s+5)(s+10)}{(s^2 + 2s + 10)} \\
 &= \frac{-s^3 - 15s^2 - 50s}{s^2 + 2s + 10}
 \end{aligned}$$

We now solve for  $\frac{dK}{ds} = 0$

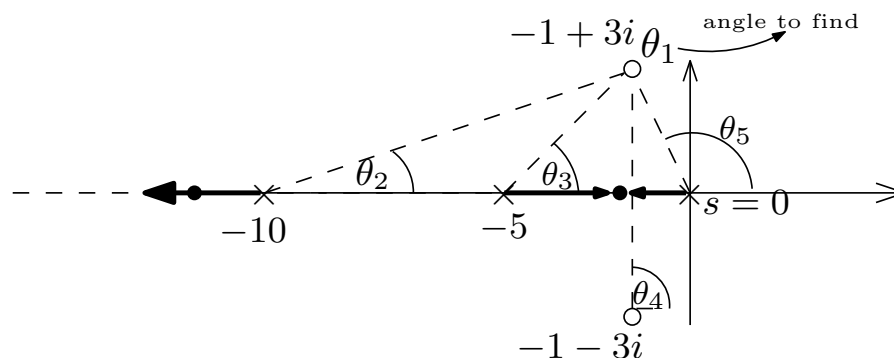
$$\begin{aligned} \frac{dK}{ds} &= 0 \\ \frac{d}{ds} \left( \frac{-s^3 - 15s^2 - 50s}{s^2 + 2s + 10} \right) &= 0 \\ \frac{s^4 + 4s^3 + 10s^2 + 300s + 500}{(s^2 + 2s + 10)^2} &= 0 \\ s^4 + 4s^3 + 10s^2 + 300s + 500 &= 0 \end{aligned}$$

The roots are  $\{s = 2.43 \pm 5.895i, s = -1.727, s = -7.12\}$  We want breakaway between  $s = 0$  and  $s = -5$ , hence only valid value is  $-1.727$ . We now go to the next lemma.

7. Departure angles from poles. Since poles are all on the real axis, the angles of departure are as shown below. The root locus plot now appears as follows



8. Since there are zeros, we now need to calculate arrival angles. There are two angles to calculate since there are two zeros. But we really need to calculate one, since the other will be symmetrical. Let us pick the top zero.  $s = -1 + 3i$ . Let its unknown arrival angle be  $\theta_1$ . This is the angle we want to solve for. So we set up the following diagram to use for the calculation



$$\sum_{\text{zeros}} \angle z_i - \sum_{\text{poles}} \angle p_i = 180^\circ + k360^\circ \quad k = 0, 1, \dots$$

$$(\theta_4 + \theta_1) - (\theta_2 + \theta_3 + \theta_5) = 180^\circ + k360^\circ$$

Now we do a little geometry to calculate the angles.  $\theta_4 = 90^\circ$ ,  $\theta_2 = \tan^{-1}\left(\frac{3}{10-1}\right) = 18.435^\circ$ ,  $\theta_3 = \tan^{-1}\left(\frac{3}{4}\right) = 36.8^\circ$  and  $\theta_5 = 180 - \tan^{-1}\left(\frac{3}{1}\right) = 108.43^\circ$ . The above now becomes

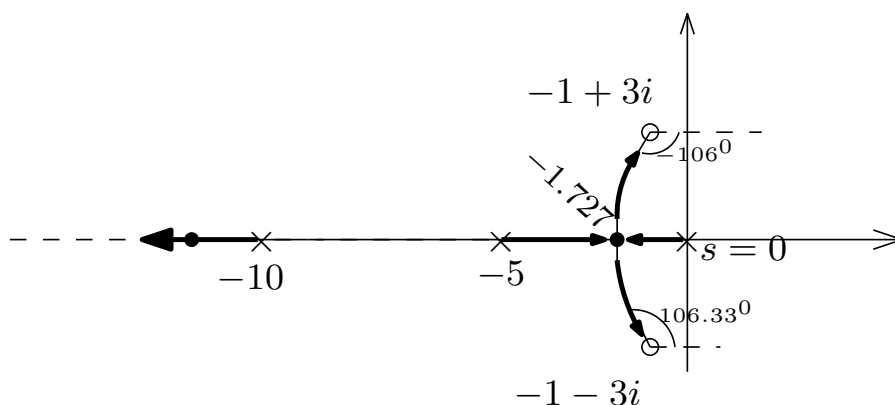
$$(90^\circ + \theta_1) - (18.435^\circ + 36.8^\circ + 108.43^\circ) = 180^\circ + k360^\circ$$

$$\theta_1 = 180^\circ - 90^\circ + 18.435^\circ + 36.8^\circ + 108.43^\circ + k360^\circ$$

$$= 253.67^\circ$$

$$= -106.33^\circ$$

Therefore, the arrival angle at the lower zero will be  $-253.67^\circ$  or  $106.33^\circ$ . The root locus now is as follows



9. The only lemma left is to find where R.L. crosses the imaginary axis. But we do not have to do this, since the zeros are to the left of the imaginary axis and the third branch goes to  $-\infty$ . So we are done. The above is the final root locus.

## 3.6.2 Problem 2

## Problem 2

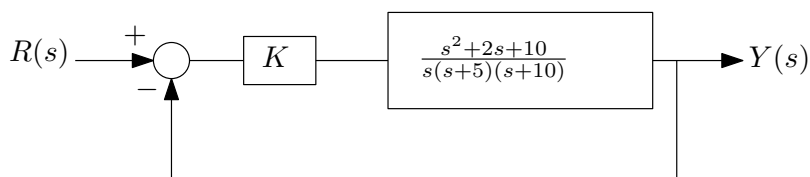
The steering control system for a mobile robot for security purpose has a unity negative feedback with

$$G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)}$$

- Sketch the root locus for  $0 \leq K \leq +\infty$  without using any computer tool.
- Verify your plot in (a) using *Matlab*.
- Find  $K$  for all breakaway points on the real axis.

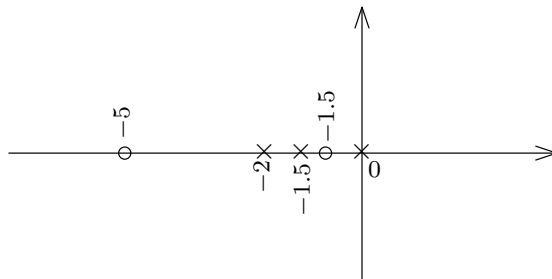
SOLUTION:

## 3.6.2.1 part a

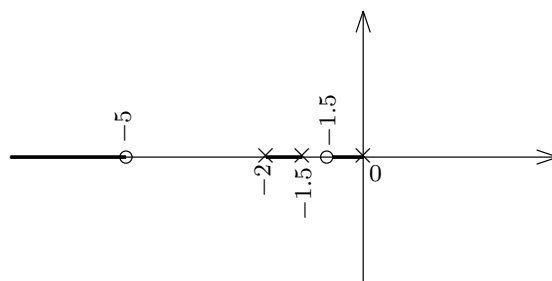


$$G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)}$$

- $n = 3$  (The number of open loop poles),  $m = 2$  (The number of open loop zeros).
- Root Locus (R.L.) starts at the open loop poles  $s = 0, s = -1.5, s = -2$  when  $k = 0$  and two of the branches end up at zeros  $s = -1$  and  $s = -5$ . One branch will end up at  $\pm\infty$ .



- On the real axis, the following diagram shows the segments.



4. Center of asymptotes is now found.  $\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{-3.5 - (-6)}{1} = 2.5$

5. The asymptotes angles are  $\theta = \frac{180^\circ \pm k360^\circ}{n-m}$  for  $k = 0, 1, \dots$ . Hence  $\theta = \frac{180^\circ \pm k360^\circ}{1} = 180^\circ \pm k360^\circ$ . So the asymptote is the real axis itself. This is clear since all zeros and poles are on the real axis.

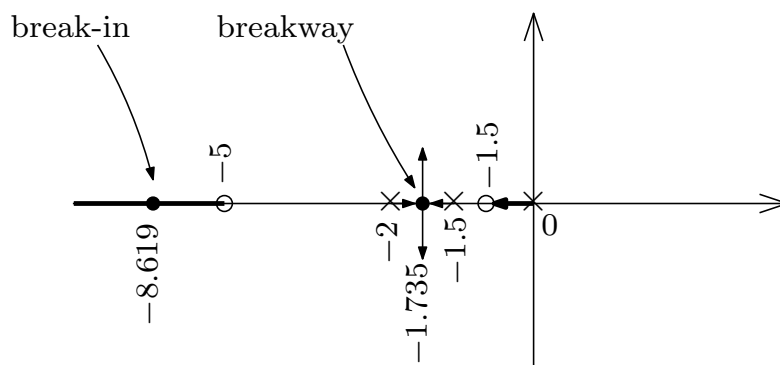
6. Break away points.

$$\begin{aligned}
 1 + G &= 0 \\
 1 + \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)} &= 0 \\
 \frac{s(s+1.5)(s+2) + K(s+1)(s+5)}{s(s+1.5)(s+2)} &= 0 \\
 s(s+1.5)(s+2) + K(s+1)(s+5) &= 0 \\
 K &= \frac{-s(s+1.5)(s+2)}{(s+1)(s+5)} \\
 &= \frac{-s^3 - 3.5s^2 - 3s}{s^2 + 6s + 5}
 \end{aligned}$$

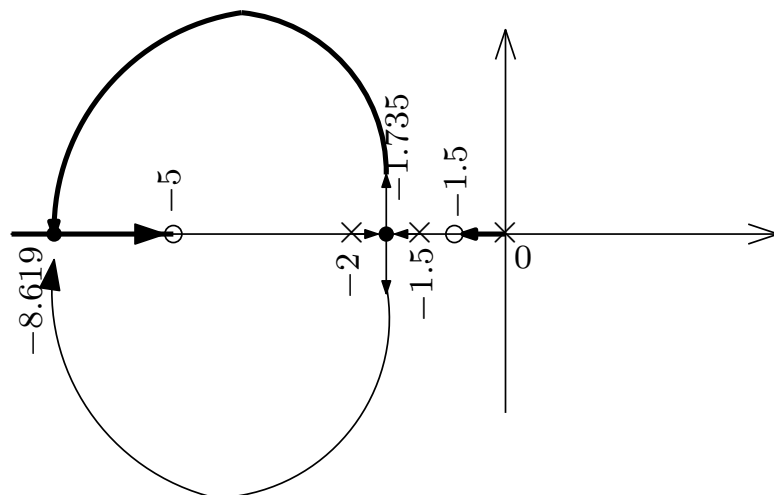
Then we now solve for  $\frac{dK}{ds} = 0$

$$\begin{aligned}
 \frac{dK}{ds} &= 0 \\
 \frac{d}{ds} \left( \frac{-s^3 - 3.5s^2 - 3s}{s^2 + 6s + 5} \right) &= 0 \\
 \frac{s^4 + 12s^3 + 33s^2 + 35s + 15}{s^4 + 12s^3 + 46s^2 + 60s + 25} &= 0 \\
 s^4 + 12s^3 + 33s^2 + 35s + 15 &= 0
 \end{aligned}$$

The roots are  $s = -0.823 \pm 0.571i$ ,  $s = -8.619$ ,  $s = -1.735$ . We want breakaway between  $s = -1.5$  and  $s = -2$ , since these are two poles facing each others, one of the breakaway points is  $s = -1.735$  on that segment. The complex root is discarded since it is not on the real line. The point  $s = -8.619$  is valid since it is on a segment on the real line. It will be a break-in point since it is on a segment with only a zero on it. So the current plot is now as follows



7. Departure angles from poles. Since poles are all on the real axis and no complex zeros exist, then the angles of departure are as shown above. There is nothing to do in this step.
8. Since all the zeros are on the real axis, there is nothing to do for this step. We just need to connect the break away branches to the break-in point on the real axis as follows



9. Now the only lemma left is to find where R.L. cross the imaginary axis. Again, we do not have to do this since the zeros are all to the left of the imaginary axis and the third branch goes to  $-\infty$ . So we are done. The following is the final plot. Since we need to have 3 branches (since  $n = 3$ ) and  $m = 2$ , then one of the branches going into the break in point at  $s = -8.619$  will go towards the zero at  $s = -5$ , but the other branch go to  $-\infty$ . We do not know which of the two branches will go to the zero and which will go to  $-\infty$ . It does not really matter. We pick the bottom branch going to the zero and the top branch going to  $-\infty$ . So the final plot is





**3.6.2.3 Part(c)**

The gain  $K$  any point on the roots locus is given by multiplying the distances from each pole to the that point, divided by the product of the distances from all the zeros to the same point. This comes from

$$1 + KG_{openloop} = 0$$

$$K = \frac{1}{|G_{openloop}|}$$

In other words, if we want to find gain at some point  $r$ , then

$$K = \frac{\prod |p_i r|}{\prod |z_i r|}$$

Since the first breakaway point in this case is  $r = -1.735$  (the break away point), then the above becomes

$$K = \frac{(1.735)(1.735 - 1.5)(2 - 1.735)}{(1.735 - 1)(5 - 1.735)}$$

The above was done by just looking the diagram of the root locus and measuring the distance from each pole to the breakaway point, and similarly for the zeros. The above reduces to

$$K = 0.045$$

For the second break-in point in this case is  $r = -8.617$ , therefore

$$K = \frac{(8.617)(8.617 - 1.5)(8.617 - 2)}{(8.617 - 1)(8.617 - 5)}$$

The above was done by just looking the diagram of the root locus and measuring the distance from each pole to the break-in point, and similarly for the zeros. The above reduces to

$$K = 14.73$$

3.6.3 HW 6 key solution

1.(a)  $G(s) = \frac{K}{s(s+2)(s+5)(s+10)} = \frac{K}{s^4 + 17s^3 + 80s^2 + 100s}$

Poles: 0, -2, -5, -10

Zero: None

# of branches = 4

Symmetric with real axis

Angle of asymptotes =  $\frac{(2k+1)\pi}{4}$   $k=0,1,2,3$  for  $K \geq 0$

=  $45^\circ, 135^\circ, 225^\circ, 315^\circ$

Angle of asymptotes =  $\frac{2k\pi}{4}$   $k=0,1,2,3$  for  $K \leq 0$

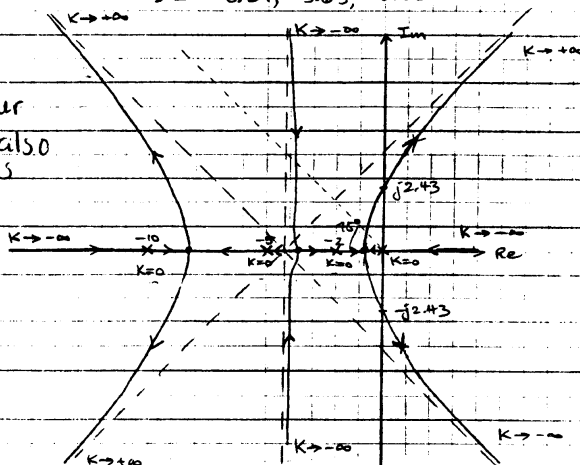
=  $0^\circ, 90^\circ, 180^\circ, 270^\circ$

Center of asymptotes =  $\frac{0-2-5-10}{4} = -4.25$

Breakaway point:  $0 = 1(4s^3 + 51s^2 + 160s + 100)$

$s = -8.29, -3.63, -0.83$

Though not required, for your interest, we also show root locus for  $K < 0$  too!



Intersection on Im axis

Characteristic eq:  $s^4 + 17s^3 + 80s^2 + 100s + K$

Routh-Hurwitz

$s^4$	1	80	K	limiting value of K =
$s^3$	17	100		
$s^2$	$\frac{7412}{74.12}$	K	$\frac{7412-17K}{74.12} = 0$	
$s^1$	$\frac{242-17K}{74.12}$	0		
$s^0$	K		K = 436	

$A(s) = 74.12s^2 + K = 0$

$s = \pm j2.43$

When  $s = 0.707$ ,  $K \approx 5.2 \times 4.2 \times 4 \times 7.4 = 646 *$

(6)  $G(s) = \frac{K}{(s^2 + s + 2)(s + 1)} = \frac{K}{s^3 + 2s^2 + 3s + 2}$

Poles:  $-1, -0.5 + j1.32, -0.5 - j1.32$

Zero: None

# of branches = 3

Symmetric about real axis

Angle of asymptotes =  $\frac{(2k+1)180^\circ}{3}$   $k=0,1,2$  for  $K \geq 0$

=  $60^\circ, 180^\circ, 300^\circ$

Angle of asymptotes =  $\frac{(2k)180^\circ}{3}$   $k=0,1,2$  for  $K \leq 0$

=  $0^\circ, 120^\circ, 240^\circ$

Center of asymptotes =  $\frac{-1 - 0.5 - 0.5}{3} = -0.67$

Breakaway point:  $0 = 1 \cdot (3s^2 + 4s + 3)$

$s = -0.67 \pm j0.75$  Not Valid

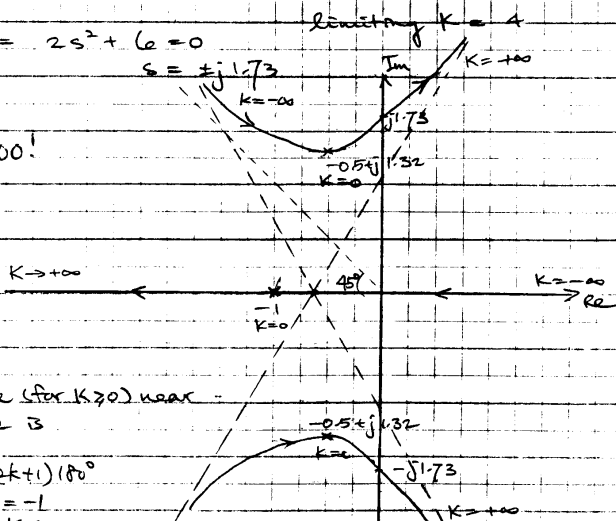
Intersection on Im axis:

Char. eqn.  $s^3 + 2s^2 + 3s + (2+K)$

$s^3$	1	3		$a = \frac{b - (2+K)}{2} = \frac{4-K}{2}$
$s^2$	2	$2+K$		
$s^1$	$a$			$b = 0$
$s^0$	$b$			

$A(s) = 2s^2 + (2+K) = 0$

$K \leq 0$  shown too!



↳ Departure (for  $K \geq 0$ ) naar  $-0.5 + j1.32$  is

$-70^\circ - 90^\circ - \theta = (2k+1)180^\circ$

$\theta \approx 20^\circ$  if  $k = -1$   
Similarly for  $K \leq 0$

When  $s = -0.101$  no possible  $K$  for stable system.

1(c) Root locus for  $\frac{K(s^2 + 2s + 10)}{s^3 + 15s^2 + 50s}$

Poles 0, -5, -10

Zeros -1 + j3, -1 - j3

# branches = 3

Asymptotes  $\frac{(2k+1)180}{3-2} \rightarrow 180$

$$\sigma = \frac{-5 - 10 - (-1 + j3) - (-1 - j3)}{3-2} = -13$$

Breakaway  $\frac{dK}{ds} = 0$

$$(s^2 + 2s + 10)(3s^2 + 30s + 50)$$

$$s^4 + 4s^3 + 10s^2 + 300s + 50 = 0$$

$$s = 2.43 \pm j5.89 \text{ (discard)}$$

$$s = -1.72, s = -7.12 \text{ (discard)}$$

Imag

Axis (Routh Hurwitz)

Crossing

$$s^3 \quad 1 \quad 50 + 2K$$

$$s^2 \quad 15 + K \quad 10K$$

$$s^1 \quad a$$

$$s^0 \quad b$$

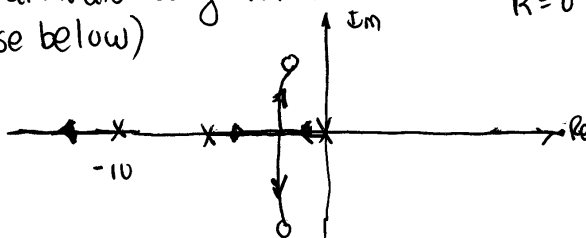
$$s^3 + (15 + K)s^2 + (50 + 2K)s + 10K = 0$$

$$a = \frac{(15 + K)(50 + 2K) - 10K}{15 + K}$$

$$b = 10K$$

vanishes only for K=0

Estimate arrivals using formula (departures) (± improvise below)



$$2. \quad G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)} = \frac{K(s^2+6s+5)}{s^3+3.5s^2+3s}$$

(a) poles: 0, -1.5, -2  
 zeros: -1, -5

Root locus obtained following same procedure as above

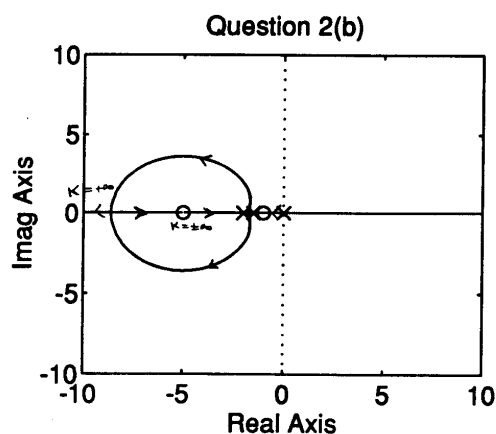
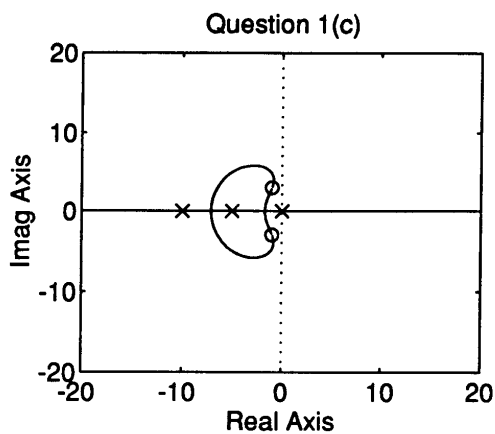
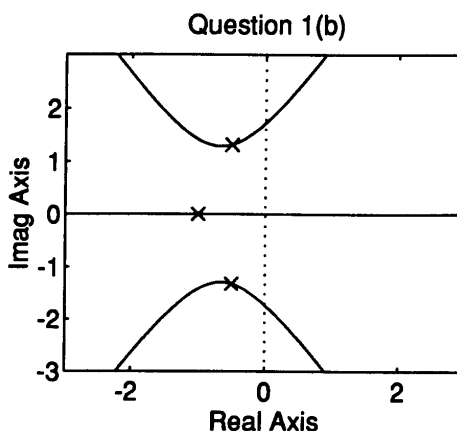
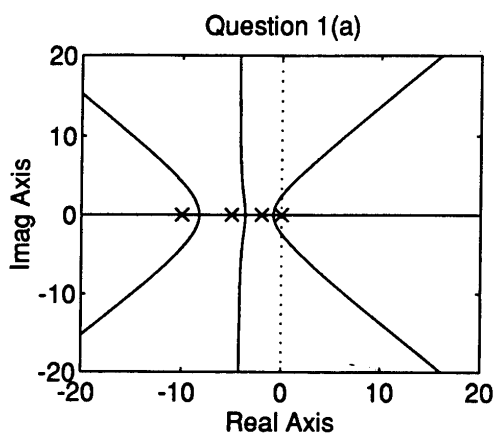
(b) See plot. AT END OF SOLUTIONS

(c) Breakaway point:

$$(s^2 + 6s + 5)(3s^2 + 7s + 3) = (s^3 + 3.5s^2 + 3s)(2s + 6)$$

$$s^4 + 12s^3 + 33s^2 + 35s + 15 = 0$$

$s = -8.62, -1.74$  on real axis.



## 3.7 HW 7

### 3.7.1 Problem 1

1. For each of the open loop transfer functions below, assume a classical unity feedback connection and sketch the Nyquist plot. Find the values of the gain  $K > 0$  for which closed loop stability is guaranteed.

$$(a) G(s)H(s) = \frac{K}{s(s^2+s+4)}$$

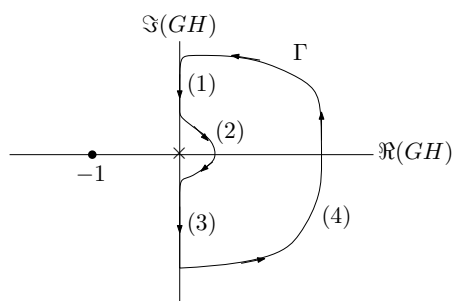
$$(b) G(s)H(s) = \frac{K(s+1)}{s^2(s+2)}$$

SOLUTION:

#### 3.7.1.1 Part(a)

$$GH = \frac{K}{s(s^2 + s + 4)} \quad (1)$$

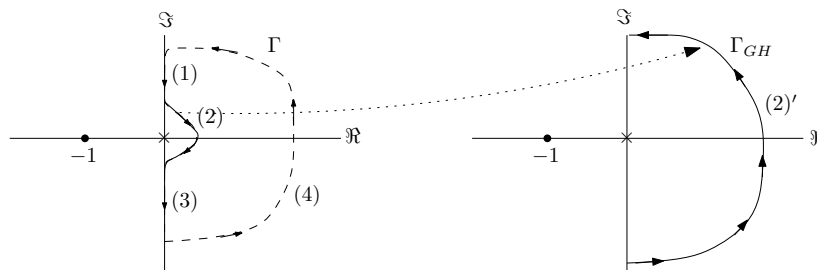
The poles of the open loop are at  $s = 0$  and  $s = -0.5 \pm j1.94$ . So we draw  $\Gamma$  which encloses all the poles in the RHS making sure we avoid the pole at  $s = 0$  by making small circle. Here is the result. (we do not care about open loop poles in the LHS).



We now start by mapping each segment from  $\Gamma$  to  $\Gamma_{GH}$ . Starting with *segment 2*. We see that

$$\lim_{s \rightarrow 0} \frac{K}{s(s^2 + s + 4)} = \lim_{s \rightarrow 0} \frac{K}{s} = \infty e^{-j\theta}$$

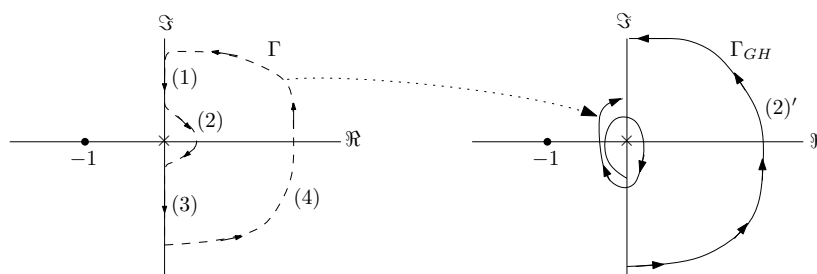
Where  $\theta$  goes from  $+90^\circ$  to  $-90^\circ$  in  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (2) will map to a very large circle which goes from  $-90^\circ$  to  $+90^\circ$  (anti-clockwise). We update the plot after making each segment so we see the progress.



We now go to *segment 4* in  $\Gamma$ .

$$\lim_{s \rightarrow \infty} \frac{K}{s(s^2 + s + 4)} = \lim_{s \rightarrow \infty} \frac{K}{s^3} = 0e^{-j3\theta}$$

Where  $\theta$  goes from  $-90^\circ$  to  $+90^\circ$  on  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (4) will map to a very small circle which goes from  $+270^\circ$  to  $-270^\circ$  on  $\Gamma_{GH}$ . This is 1.5 circle rotation in clockwise that goes from around  $540^\circ$ . updating the plot gives



We now do *segment 1*. For this we need to find the real and imaginary part of  $GH$  since we need the axis crossings. From (1)

$$GH = \frac{K}{s^3 + s^2 + 4s} = \frac{K}{(j\omega)^3 + (j\omega)^2 + 4(j\omega)} = \frac{K}{-j\omega^3 - \omega^2 + 4j\omega} = \frac{K}{j(4\omega - \omega^3) - \omega^2}$$

Multiply numerator and denominator by complex conjugate denominator gives

$$\begin{aligned} GH &= \frac{K}{j(4\omega - \omega^3) - \omega^2} \frac{(-j(4\omega - \omega^3) - \omega^2)}{(-j(4\omega - \omega^3) - \omega^2)} \\ &= \frac{K(-j(4\omega - \omega^3) - \omega^2)}{(j(4\omega - \omega^3) - \omega^2)(-j(4\omega - \omega^3) - \omega^2)} \\ &= \frac{-jK(4\omega - \omega^3) - K\omega^2}{\omega^6 - 7\omega^4 + 16\omega^2} \end{aligned}$$



Hence

$$\begin{aligned}\operatorname{Re}(GH) &= \frac{-K}{\omega^4 - 7\omega^2 + 16} \\ \operatorname{Im}(GH) &= \frac{-K(4\omega - \omega^3)}{\omega^6 - 7\omega^4 + 16\omega^2} = \frac{K(\omega^2 - 4)}{\omega(\omega^4 - 7\omega^2 + 16)}\end{aligned}$$

To find the crossing.  $\Gamma_{GH}$  will cross the real axis when  $\operatorname{Im}(GH) = 0$ , hence

$$\begin{aligned}\frac{K(\omega^2 - 4)}{\omega(\omega^4 - 7\omega^2 + 16)} &= 0 \\ K(\omega^2 - 4) &= 0 \\ \omega^2 - 4 &= 0 \\ \omega^2 &= 4 \\ \omega &= \pm 2 \text{ rad/sec}\end{aligned}$$

Then

$$\begin{aligned}GH(j2) &= \frac{K}{(j2)^3 + (j2)^2 + 4(j2)} \\ &= \frac{K}{(-j8) + (-4) + (j8)} \\ &= \frac{-1}{4}K\end{aligned}$$

So  $\Gamma_{GH}$  will cross the real axis at  $-0.25K$ .

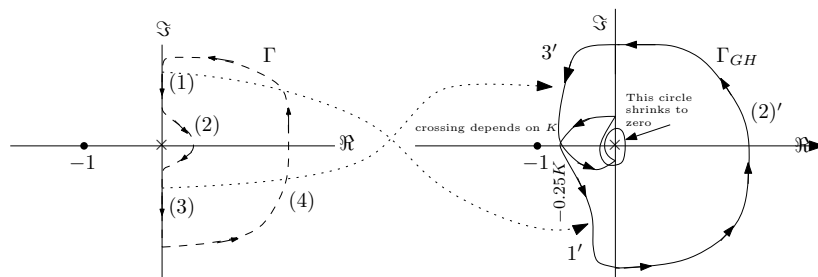
Since  $K > 0$ , this will be somewhere on the negative real axis. To find where  $\Gamma_{GH}$  crosses the imaginary axis we set  $\operatorname{Re}(GH) = 0$ , hence

$$\frac{-K}{\omega^4 - 7\omega^2 + 16} = 0$$

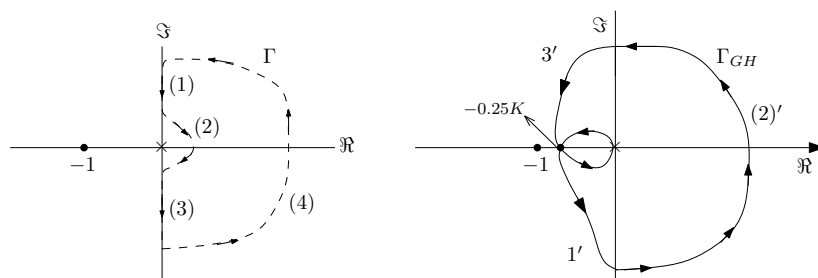
Which gives  $\omega = \pm\infty$ . When  $\omega = \pm\infty$  then  $GH(\pm\infty) = 0^\pm$ . This means the crossing at origin. This makes sense, as the small circle around the origin shrinks to zero size.

To find where segment (1) maps to, we see that for  $\omega = +\infty$ ,  $\operatorname{Re}(GH) < 0$  and  $\operatorname{Im}(GH) > 0$ , so it starts in first quadrant.  $\omega = 0^+$ ,  $\operatorname{Re}(GH) < 0$  and  $\operatorname{Im}(GH) < 0$ , which means segment (1) starts in the first quadrant but ends in the third quadrant.

Now for segment (3). we see that  $\omega = 0^-$ ,  $\operatorname{Re}(GH) < 0$  and  $\operatorname{Im}(GH) > 0$ , which means segment (3) starts in the first quadrant.  $\omega = -\infty$  then  $\operatorname{Re}(GH) < 0$  and  $\operatorname{Im}(GH) < 0$  which means segment (3) in  $\Gamma_{GH}$  starts in quadrant 1 and ends up in quadrant 3. There will be crossing at the real axis at  $-0.25K$ . Therefore the plot now looks like this



As the circle around  $s = 0$  shrinks to zero,  $\Gamma_{GH}$  becomes as follows



We are now ready to answer the final question about stability and  $K$ . Since open loop has zero poles in RHS, then we need to have zero net clockwise encirclements around  $-1$  for the closed loop to be stable. Only condition that will meet that, is to keep the crossing point  $-0.25K$  to the right of  $-1$ . This means we need  $0.25K < 1$ . This insures zero clockwise encirclements. This means

$$K < 4$$

To verify, Routh table is used to determined  $K$  using the closed loop transfer function. The closed loop is given by

$$\begin{aligned} T &= \frac{GH}{1 + GH} \\ &= \frac{K}{s(s^2 + s + 4) + K} \\ &= \frac{K}{s^3 + s^2 + 4s + K} \end{aligned}$$

Hence the Routh table is

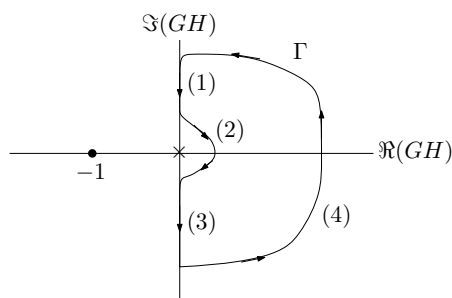
$s^3$	1	4
$s^2$	1	$K$
$s^1$	$4 - K$	
$s^0$	$K$	

For no sign change in first column, we need  $K > 0$  and  $4 - K > 0$ . Which means  $K < 4$  as was found above. Verified OK.

## 3.7.1.2 Part(b)

$$GH = \frac{K(s+1)}{s^2(s+2)} = \frac{K(s+1)}{s^3+2s^2} \quad (1)$$

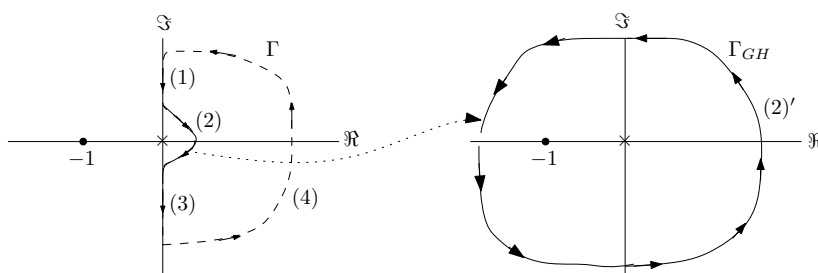
The poles of the open loop are at  $s = 0$  (two poles) and  $s = -2$ . So we draw  $\Gamma$  which encloses all the poles in the RHS making sure we avoid the poles at  $s = 0$  by making small circle. Here is the result. (we do not care about open loop pole in the LHS and about the zeros of the open loop).



We now start by mapping each segment from  $\Gamma$  to  $\Gamma_{GH}$ . Starting with *segment 2*. We see that

$$\lim_{s \rightarrow 0} \frac{K(s+1)}{s^3+2s^2} = \lim_{s \rightarrow 0} \frac{K}{s^2} = \infty e^{-2j\theta}$$

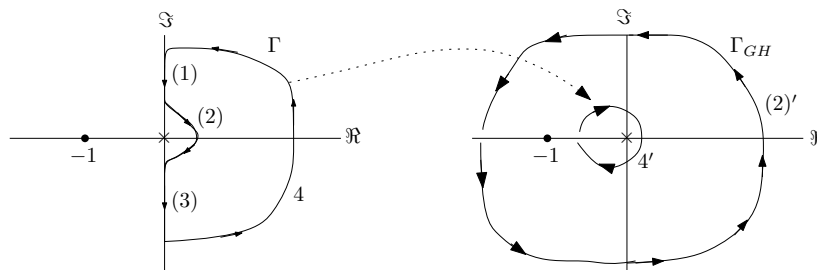
Where  $\theta$  goes from  $+90^\circ$  to  $-90^\circ$  in  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (2) will map to a very large circle which goes from  $-180^\circ$  to  $+180^\circ$ . This is a full circle in the anti-clockwise. We update the plot after making each segment so we see the progress.



We now go to *segment 4* in  $\Gamma$ .

$$\lim_{s \rightarrow \infty} \frac{K(s+1)}{s^3+2s^2} = \lim_{s \rightarrow \infty} \frac{Ks}{s^3} = \lim_{s \rightarrow \infty} \frac{K}{s^2} = 0 e^{-j2\theta}$$

Where  $\theta$  goes from  $-90^\circ$  to  $+90^\circ$  on  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (4) will map to a very small circle which goes from  $+180^\circ$  to  $-180^\circ$  on  $\Gamma_{GH}$ . This is basically a full circle in clockwise around zero. updating the plot gives



We now do *segment 1*. For this we need to find the real and imaginary part of  $GH$  since we need the axis crossings. From (1)

$$GH = \frac{K(s+1)}{s^3+2s^2} = \frac{K(j\omega+1)}{(j\omega)^3+2(j\omega)^2} = \frac{K(j\omega+1)}{-j\omega^3-2\omega^2}$$

Multiply numerator and denominator by complex conjugate denominator gives

$$\begin{aligned} GH &= \frac{K(j\omega+1)(j\omega^3-2\omega^2)}{-j\omega^3-2\omega^2(j\omega^3-2\omega^2)} \\ &= \frac{j(-K\omega^3) - (K\omega^4 + 2K\omega^2)}{\omega^4(\omega^2+4)} \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Re}(GH) &= \frac{-(K\omega^4 + 2K\omega^2)}{\omega^4(\omega^2+4)} \\ \operatorname{Im}(GH) &= \frac{-K\omega^3}{\omega^4(\omega^2+4)} = \frac{-K}{\omega(\omega^2+4)} \end{aligned}$$

To find the crossing,  $\Gamma_{GH}$  will cross the real axis when  $\operatorname{Im}(GH) = 0$ , hence

$$\begin{aligned} \frac{-K}{\omega(\omega^2+4)} &= 0 \\ \omega &= \pm\infty \text{ rad/sec} \end{aligned}$$

Therefore

$$\begin{aligned} GH(j\infty) &= \lim_{\omega \rightarrow \infty} \frac{K(j\omega+1)}{-j\omega^3-2\omega^2} \\ &= \lim_{\omega \rightarrow \infty} \frac{j\omega}{-j\omega^3} \\ &= \lim_{\omega \rightarrow \infty} \frac{j}{-j\omega^2} \\ &= 0^- \end{aligned}$$

So  $\Gamma_{GH}$  will cross the real axis at 0. Which is where the small circle shrinks to zero. To find

where  $\Gamma_{GH}$  crosses the imaginary axis we set  $\text{Re}(GH) = 0$ , hence

$$\begin{aligned} \frac{-(K\omega^4 + 2K\omega^2)}{\omega^4(\omega^2 + 4)} &= 0 \\ K\omega^4 + 2K\omega^2 &= 0 \\ \omega^2(\omega^2 + 2) &= 0 \end{aligned}$$

Hence

$$\omega = 0$$

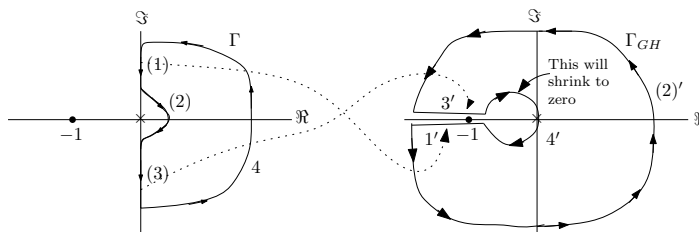
And

$$\omega = \pm i\sqrt{2}$$

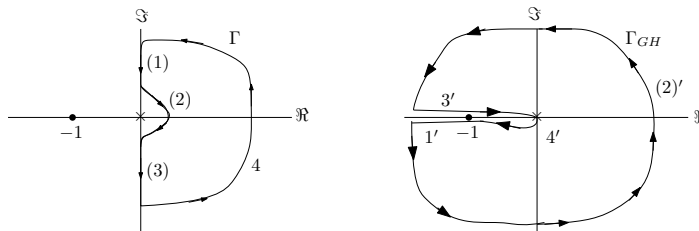
Since  $\omega$  has to be real. Then  $\omega = 0$  is only used. When  $\omega = 0$  then  $GH(j0) = \lim_{\omega \rightarrow 0} \frac{K(j\omega+1)}{-j\omega^3-2\omega^2} = \lim_{\omega \rightarrow 0} \frac{K}{-2\omega^2} = \infty$ . Hence  $\Gamma_{GH}$  will cross the imaginary axis at  $\infty$ .

To find which quadrants segment (1) maps to, we see that for positive  $\omega$  then  $\text{Re}(GH)$  is negative and  $\text{Im}(GH)$  is negative (since  $K > 0$ ). Therefore segment (1) maps to third quadrant. And for segment (3), we see that negative  $\omega$  then  $\text{Re}(GH)$  is negative and  $\text{Im}(GH)$  is positive (since  $K > 0$ ). Therefore segment (3) maps to first quadrant

Therefore the plot now looks like this



After the small circle shrinks to zero, and since the crossing on the real axis was found at 0 then the final plot looks like



We are now ready to answer the final question about stability and  $K$ . Since open loop has zero poles in RHS, then we need to have zero net clockwise encirclements around  $-1$  for the closed loop to be stable. We see that there is no encirclements around  $-1$ . No matter what  $K > 0$  value is. So the closed loop is stable for all positive  $K$ . To verify, Routh table is

used to determine  $K$  using the closed loop transfer function. The closed loop is given by

$$T = \frac{GH}{1 + GH}$$

$$= \frac{K(s+1)}{s^3 + 2s^2 + K(s+1)}$$

Hence the Routh table is

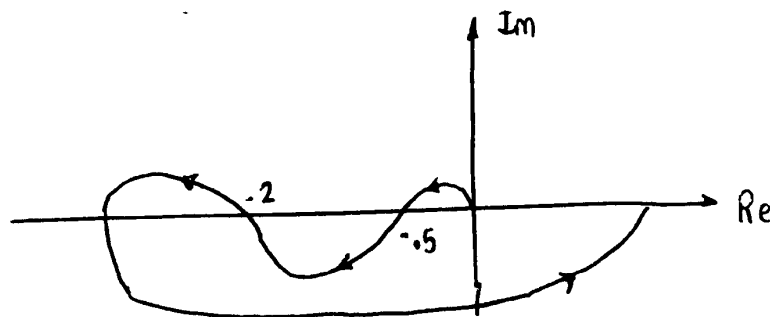
$s^3$	1	$K$
$s^2$	2	$K$
$s^1$	$\frac{2K-K}{2} = \frac{K}{2}$	
$s^0$	$K$	

For no sign change in first column, we need  $K > 0$  and  $\frac{K}{2} > 0$ . Which is always true since  $K$  is positive. Verified OK.

### 3.7.2 Problem 2

2. The Nyquist plot for a transfer function  $KG(s)$  is shown below for  $K = 1$  and  $\omega \geq 0$ .

Assuming a unity feedback configuration with  $G(s)$  having no poles in the closed right half plane, find the ranges for  $K > 0$  under which closed loop stability is assured.



SOLUTION:

Since there is zero open loop poles in RHP, we need zero net clockwise encirclements around  $-1$ . This means we need to keep point  $-0.5$  to the right of  $-1$  and keep the point  $-2$  to the left of  $-1$ . In other words, we need to satisfy

$$0.5K < 1$$

$$2K > 1$$

Or

$$K < 2$$

And

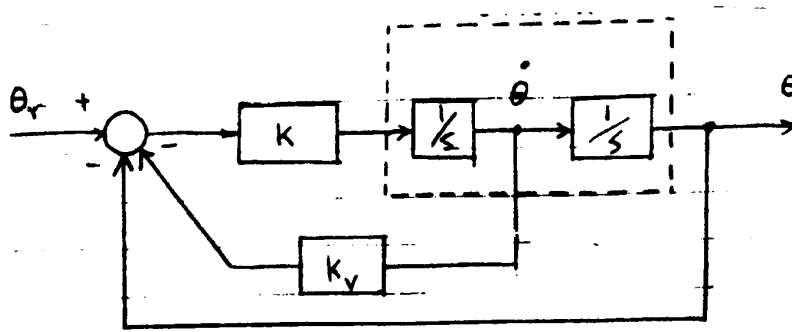
$$K > \frac{1}{2}$$

Hence the range of stable closed loop is for  $K$  to meet the following requirement

$$0.5 < K < 2$$

### 3.7.3 Problem 3

3. (a) For the attitude control system shown below, assume that the velocity feedback gain  $K_v$  is zero and sketch the Nyquist for open loop system  $G(s)H(s)$ .



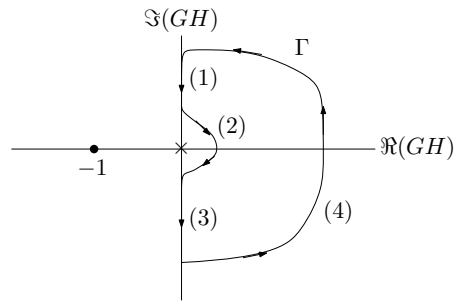
- (b) For the closed loop system above, for what range of gains  $K > 0$  is stability assured?

- (c) Now consider the case with both gains  $K$  and  $K_v$  non-zero. Using an appropriate Nyquist plot, find the range for these gains under which closed loop stability is assured.

SOLUTION:

#### 3.7.3.1 Part(a)

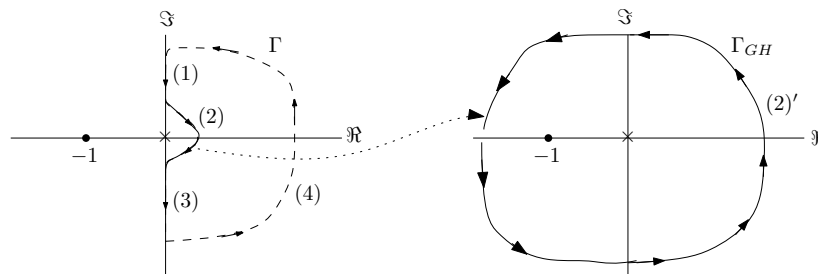
When  $K_v = 0$  the open loop is  $GH = K \frac{1}{s^2}$ . This has no poles in RHP and two poles at zero. Hence  $\Gamma$  is



We start at segment 2.

$$\lim_{s \rightarrow 0} \frac{K}{s^2} = \lim_{\epsilon \rightarrow 0} \frac{K}{(\epsilon e^{j\theta})^2} = \infty e^{-2j\theta}$$

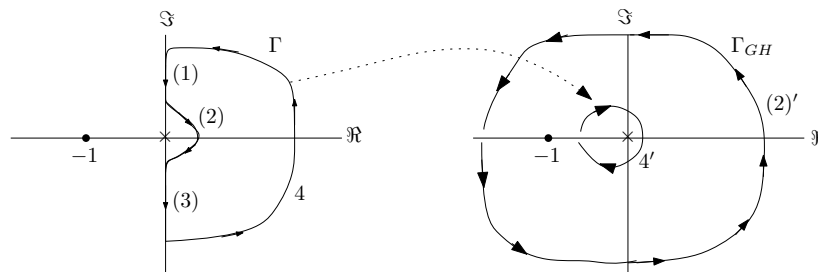
As  $\theta$  goes from  $+90^\circ$  to  $-90^\circ$  on  $\Gamma$  segment 2 goes from  $-180^\circ$  to  $+180$  on  $\Gamma_{GH}$  with  $\infty$  radius in anti-clockwise. Hence the plot now looks as follows



For segment 4 in  $\Gamma$ . We have

$$\lim_{s \rightarrow \infty} \frac{K}{s^2} = \lim_{R \rightarrow \infty} \frac{K}{(R e^{j\theta})^2} = 0 e^{-2j\theta}$$

$\theta$  goes from  $-90^\circ$  to  $+90^\circ$  on  $\Gamma$ , segment 4 goes from  $+180^\circ$  to  $-180$  on  $\Gamma_{GH}$  with 0 radius in clockwise. Hence the plot now looks as follows

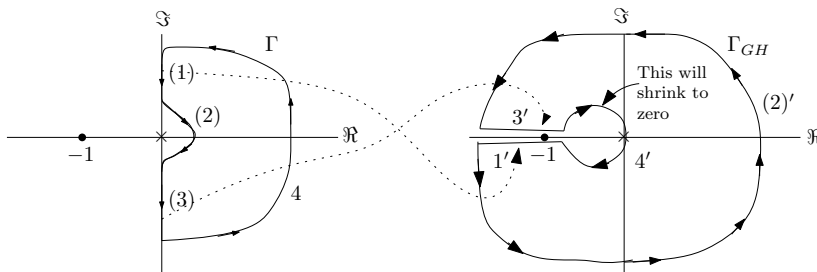


Now

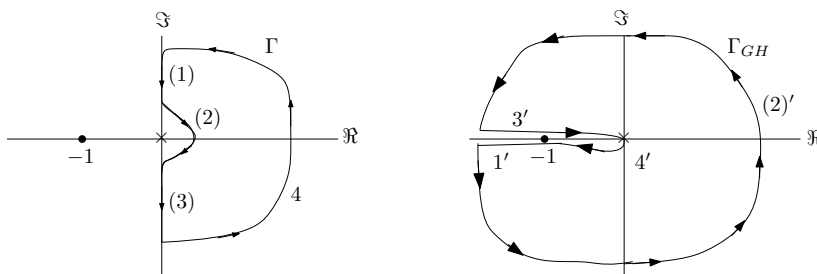
$$\frac{K}{s^2} = \frac{K}{(j\omega)^2} = \frac{K}{-\omega^2}$$



Hence  $\text{Re}(GH) = \frac{K}{-\omega^2}$  and there is no imaginary part. Therefore, there is no crossing on the real axis. There is crossing on the imaginary axis at  $\omega = \pm\infty$  which occurs at  $GH = 0$ . This is when the small circle shrinks to zero size. It is clear that segment 1 will map to third quadrant and segment 3 will map to second quadrant since there is no crossing. The final plot is



After the small circle shrinks to zero, and since the crossing on the real axis was found at 0 then the final plot looks like



**3.7.3.2 Part(b)**

Since open loop has zero poles in RHP, we need  $\Gamma_{GH}$  to have zero net clockwise encirclements. Since  $\Gamma_{GH}$  has no crossing on the real axis that depends on  $K$  then  $\Gamma_{GH}$  will remain as shown for any  $K$ . So closed loop is stable for all  $K > 0$ . To verify, we set the Routh table for the closed loop polynomial

The closed loop is given by

$$T = \frac{GH}{1 + GH} = \frac{K}{s^2 + K}$$

Hence the Routh table is

$s^2$	1	$K$
$s^1$	0	0
$s^0$	0	

We see that there is no sign change in first column, no matter what  $K$  is.

**3.7.3.3 Part (c)**

We first find the closed loop transfer function. Let  $E(s)$  be the error (just after the summing junction) then

$$E = \theta_r - \theta - K_v s \theta$$

$$\theta = EK \frac{1}{s^2}$$

From the second equation  $E = \frac{\theta s^2}{K}$ , hence the first equation becomes

$$\frac{\theta s^2}{K} = \theta_r - \theta - K_v s \theta$$

$$\frac{\theta s^2}{K} + \theta + K_v s \theta = \theta_r$$

$$\theta \left( \frac{s^2}{K} + 1 + K_v s \right) = \theta_r$$

Therefore the closed loop transfer function  $T(s) = \frac{\theta}{\theta_r}$  is

$$\begin{aligned} \frac{\theta}{\theta_r} &= \frac{1}{\frac{s^2}{K} + 1 + K_v s} \\ &= \frac{K}{s^2 + KK_v s + K} \end{aligned}$$

We now find the open loop transfer function with unity feedback using the closed loop transfer function. Since  $T(s) = \frac{G}{1+G}$  where  $G(s)$  is the closed loop transfer function, then letting  $G(s) = \frac{N}{D}$  we have

$$\begin{aligned} \frac{K}{s^2 + KK_v s + K} &= \frac{G}{1+G} \\ &= \frac{\frac{N}{D}}{1 + \frac{N}{D}} \\ &= \frac{N}{N+D} \end{aligned}$$

Therefore  $N = K$  and  $N + D = s^2 + KK_v s + K$  which means  $D(s) = s^2 + KK_v s$ . Therefore the open loop transfer function is

$$G(s) = \frac{N}{D} = \frac{K}{s^2 + KK_v s}$$

Hence

$$GH = \frac{K}{s(s+KK_v)}$$

Therefore, the open loop has a pole at  $s = 0$  and at  $s = -KK_v$ .

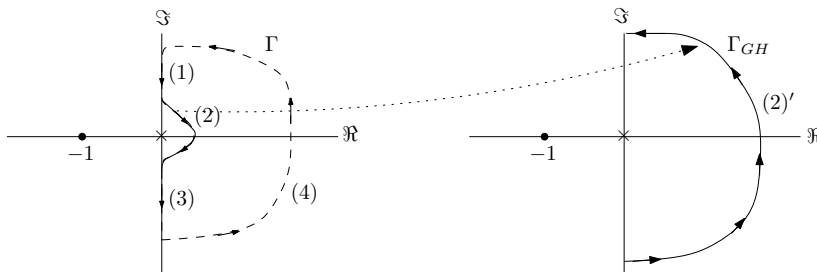
Assuming positive gains,  $-KK_v$  is in the LHP. Hence the open loop is stable. Which means

the Nyquist plot should have zero net clockwise encirclement around  $-1$  for the closed loop to be stable.

We now start by mapping each segment from  $\Gamma$  to  $\Gamma_{GH}$ . Starting with *segment 2*. We see that

$$\lim_{s \rightarrow 0} \frac{K}{s^2 + sKK_v} = \lim_{s \rightarrow 0} \frac{1}{sK_v} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon e^{j\theta} K_v} = \infty e^{-j\theta}$$

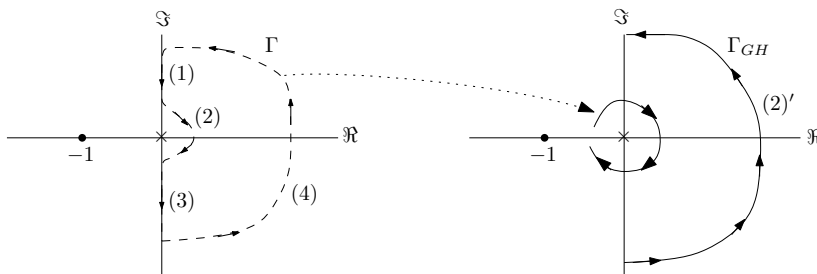
Where  $\theta$  goes from  $+90^\circ$  to  $-90^\circ$  in  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (2) will map to a half circle which goes from  $-90^\circ$  to  $+90^\circ$  in anti-clockwise.



We now go to *segment 4* in  $\Gamma$ .

$$\lim_{s \rightarrow \infty} \frac{K}{s^2 + sKK_v} = \lim_{s \rightarrow \infty} \frac{K}{s^2} = \lim_{R \rightarrow \infty} \frac{1}{Re^{2j\theta}} = 0e^{-j2\theta}$$

Where  $\theta$  goes from  $-90^\circ$  to  $+90^\circ$  on  $\Gamma$ . This means on  $\Gamma_{GH}$  segment (4) will map to a very small circle which goes from  $+180^\circ$  to  $-180^\circ$  on  $\Gamma_{GH}$ . This is basically a full circle in clockwise around zero. updating the plot gives

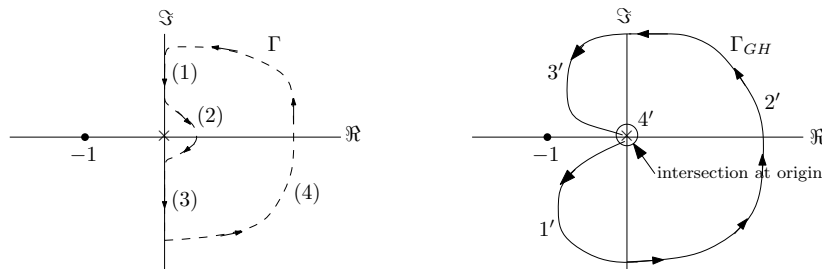


To find the intersections,

$$\begin{aligned} GH &= \frac{K}{s^2 + sKK_v} = \frac{K}{-\omega^2 + j\omega KK_v} = \frac{K}{(-\omega^2 + j\omega KK_v)} \frac{(-\omega^2 - j\omega KK_v)}{(-\omega^2 - j\omega KK_v)} \\ &= \frac{-K\omega^2 - j\omega K^2 K_v}{K^2\omega^2 K_v^2 + \omega^4} = \frac{-K}{K^2 K_v^2 + \omega^2} - j \frac{K^2 K_v}{K^2\omega K_v^2 + \omega^3} \end{aligned}$$

Hence  $\text{Re}(GH) = \frac{-K}{K^2 K_v^2 + \omega^2}$  and  $\text{Im}(GH) = -\frac{K^2 K_v}{K^2\omega K_v^2 + \omega^3}$ . When  $\text{Re}(GH) = 0$ , we get  $\omega = \pm\infty$ . When means  $GH = 0$  at this frequency. So  $\Gamma_{GH}$  crosses the imaginary axis at the origin.

When  $\text{Im}(GH) = 0$ , we also get  $\omega = \pm\infty$  which also means  $\Gamma_{GH}$  crosses the real axis at zero. By continuation, and since segment 1 must follow segment 4, then segment 1 maps to third quadrant in  $\Gamma_{GH}$  and segment 3 must map to quadrant 2 in  $\Gamma_{GH}$ . As  $\varepsilon \rightarrow 0$  the small circle become a point at origin and we get the final plot



Since the intersection is always at the origin,  $\Gamma_{GH}$  will never move to the left passed  $-1$  to make any encirclement around  $-1$ . We need at least one net encirclement for the closed loop to be unstable. Hence the closed loop is stable for all positive  $K, K_v$ .

To verify, we show Routh table for the closed loop found above, which is  $\frac{K}{s^2 + KK_v s + K}$ . The Routh table is

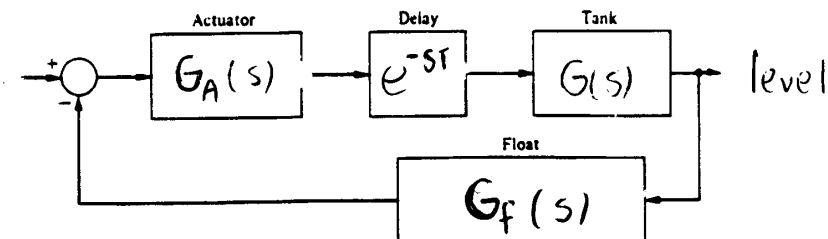
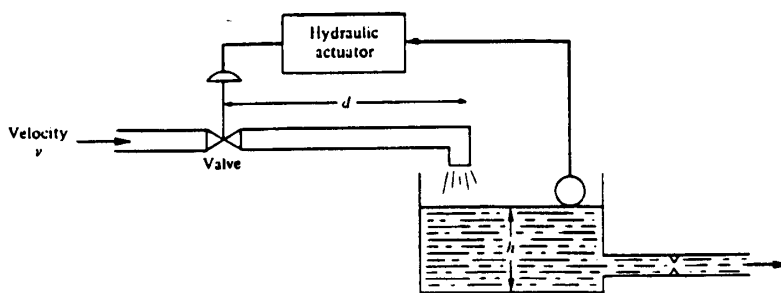
$s^2$	1	$K$
$s^1$	$KK_v$	0
$s^0$	$K$	

For positive  $K, K_v$ , we see that there can not be a sign change. Hence closed loop is stable for all  $K, K_v$ . (Note, I assume  $K, K_v > 0$ . Verified this with instructor via email).

**3.7.4 Problem 4**

4. A level control system and its transfer function model is depicted below. Assuming zero delay ( $T = 0$ ), use Matlab to generate the appropriate Nyquist plot which can be used to analyze closed loop stability. Take

$$G_A(s) = \frac{10}{s+1}; \quad G(s) = \frac{3.15}{30s+1}; \quad G_f(s) = \frac{1}{(s^2/9) + (s/3) + 1}$$



(b) Using the Nyquist plot from part (a), estimate the gain and phase margins.

(c) With time delay  $T > 0$ , explain in detail what is meant by the following: “The appropriate Nyquist plot is obtained from the plot in Part (a) by a frequency dependent angular rotation.” For the case when  $T = 1$ , is the closed loop stable? Explain.

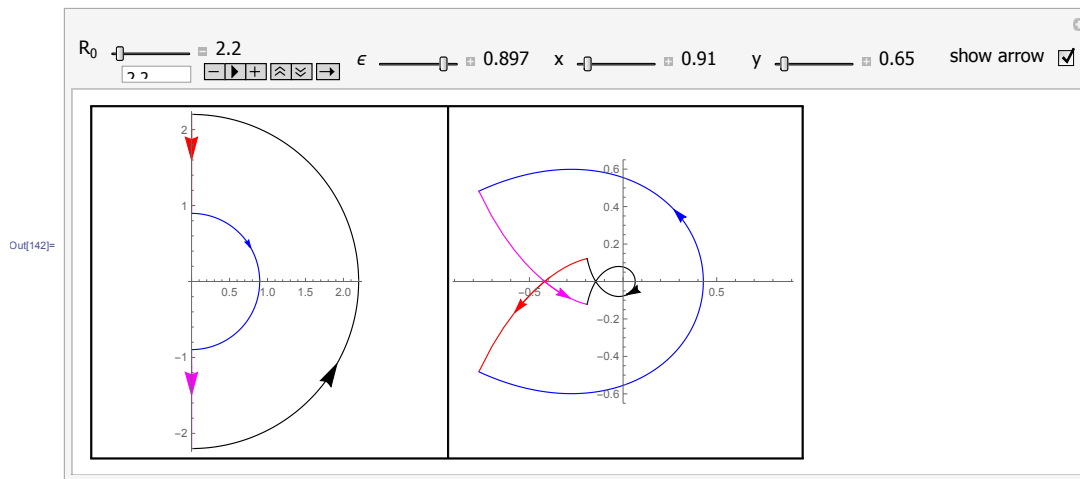
SOLUTION:

## 3.7.4.1 Part(a)

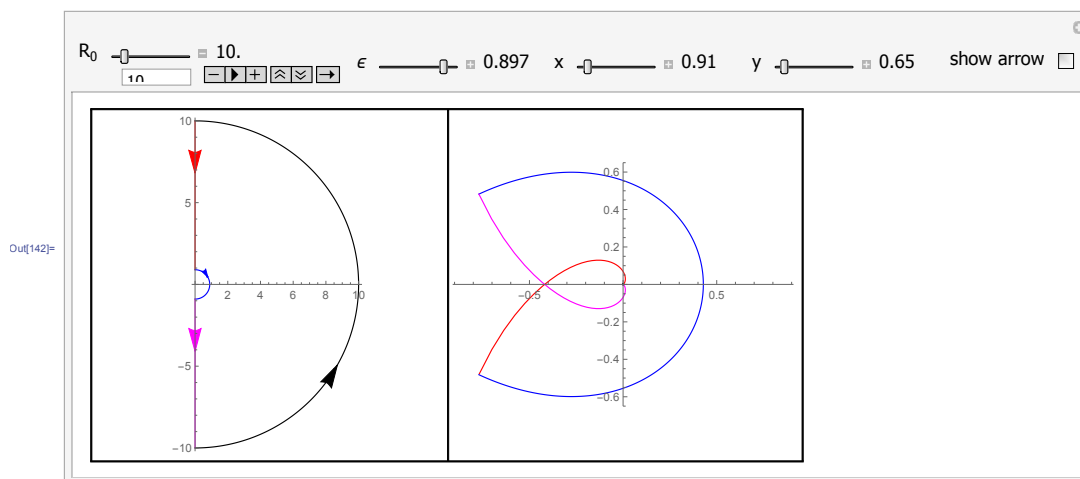
The open loop transfer function  $T(s)$  is  $G_A(s)e^{-sT}G(s)G_f(s)$ . For  $T = 0$ , we have

$$\begin{aligned} T(s) &= G_A(s)G(s)G_f(s) \\ &= \frac{10}{s+1} \frac{3.15}{30s+1} \frac{1}{\frac{s^2}{9} + \frac{s}{3} + 1} \end{aligned}$$

The Nyquist plot is (using the program I wrote which shows  $\Gamma$  and  $\Gamma_{GH}$  side by side)



In the limit, as  $R$  becomes very large we obtain

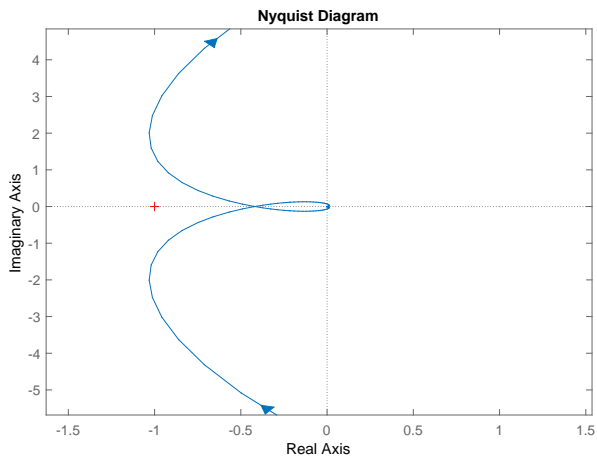


Here is also Matlab nyquist output (zoomed in version)

```

s=tf('s');
sys=10/(s+1)*(3.15)/(30*s+1)*1/(s^2/9+s/3+1);
sys =
850.5
-----
90 s^4 + 363 s^3 + 1092 s^2 + 846 s + 27
nyquist([850.5],[90 363 1092 846 27])

```



### 3.7.4.2 Part(b)

For the gain margin, the  $\Gamma_{GH}$  curve crosses the real axis at about  $-0.41$ . Therefore we need

$$0.41K_{\max} < 1$$

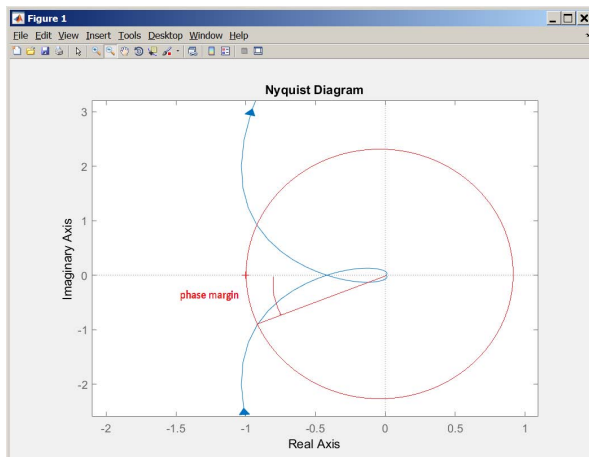
For stability. Hence

$$K_{\max} = \frac{1}{0.41} = 2.439$$

In dB, the above becomes

$$\begin{aligned}
 gm &= 20 \log_{10} 2.439 \\
 &= 7.74 \text{ db}
 \end{aligned}$$

For the phase margin, we draw a unit circle and find the intersection with  $\Gamma_{GH}$  and estimate the angle between the line from origin to the intersection and the  $-180^\circ$  line. As follows



The angle seems to be approximately between  $30^0$  and  $35^0$ . This is the phase margin. To get exact values, Matlab margin command can be used as follows

```
s=tf('s');
sys=10/(s+1)*(3.15)/(30*s+1)*1/(s^2/9+s/3+1);
sys =
850.5
-----
90 s^4 + 363 s^3 + 1092 s^2 + 846 s + 27
>> [Gm,Pm,~,~] = margin(sys)
Gm =
2.3854
Pm =
35.6025
```

Converting the Gm value given in Matlab to dB, gives the result shown above. Matlab gives  $35.6^0$  as the exact phase margin.

### 3.7.4.3 Part(c)

Let the open loop  $GH$  when  $T = 0$  (which is what we analyzed in part (b)) be called  $GH(s)$  which can be written, in frequency domain as

$$GH(s)|_{s=j\omega} = |GH| e^{j\theta}$$

Where both the magnitude  $|GH|$  and the phase  $\theta$  in the above, are functions of the frequency  $\omega$ . The above is polar representation of the complex quantity  $GH(j\omega)$ . When  $T > 0$ , then the open loop is now  $e^{-sT}GH(s)$ , which can be written in frequency domain as

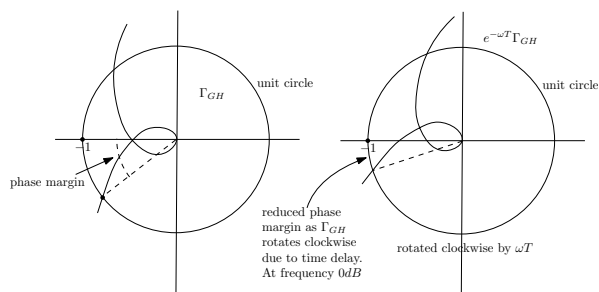
$$e^{-j\omega T}GH(j\omega) = |e^{-j\omega T}| |GH| e^{j(\theta-\omega T)}$$



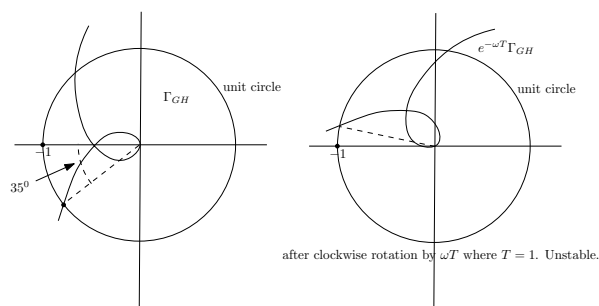
In other words, magnitudes multiply and phases are added. But  $|e^{-j\omega T}| = 1$ , so the above is

$$e^{-j\omega T}GH(j\omega) = |GH|e^{j(\theta-\omega T)}$$

We see that the resulting open loop has the same magnitude as before, but its phase has change. We subtract angle  $\omega T$  from the original phase  $\theta$ . subtract angle  $\omega T$  is the same as rotating the complex vector representation clockwise  $\omega T$ . So this causes the whole Nyquist plot, which is a frequency plot  $GH(j\omega)$ , to just rotate by  $\omega T$  clockwise (since negative angle) to what it was before. This makes  $\Gamma_{GH}$  become closer to  $-1$ . This is illustrated in the following diagram



When  $T = 1$ , the angle is  $\omega$  radians. Since we found the phase margin to be  $35^\circ$  or about 0.61 radians, then the closed loop, which corresponds to the open  $e^{-j\omega T}GH(j\omega)$ , will have new phase reduced by  $\omega$  radians. Since phase margin is measured at  $0dB$  angle (or  $\omega = 1$  radian, or  $57.3^\circ$ ). This is larger than the phase margin  $35^\circ$ . Therefore the new system is unstable.  $\Gamma_{GH}$  will rotate and will cross over  $-1$ .



What the above shows, is that adding delays  $e^{-sT}$  makes the system less stable (closer to becoming unstable). Delays causes the phase margin to reduce. We can find the amount of delay  $T$  before the system becomes unstable. We need  $\omega T < 35^\circ$  or  $T < \frac{35^\circ}{57.3^\circ} < 0.611$  seconds. This is the maximum delay  $T$  we can have before the closed loop phase margin is all used and the system become unstable.

## 3.7.5 HW 7 key solution

1. For each of the open loop transfer functions below, assume a classical unity feedback connection and sketch the Nyquist plot. Find the values of the gain  $K > 0$  for which closed loop stability is guaranteed.

$$(a) G(s)H(s) = \frac{K}{s(s^2+s+4)}$$

$$(b) G(s)H(s) = \frac{K(s+1)}{s^2(s+2)}$$

a)

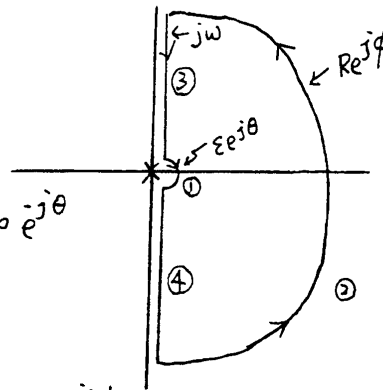
$$G(s)H(s) = \frac{K}{s(s^2+s+4)}$$

• Small circle :  $s = \epsilon e^{j\theta}$

$$G(s)H(s) \Big|_{s=\epsilon e^{j\theta}} = \frac{K}{4\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

• Big circle :  $s = R e^{j\phi}$

$$G(s)H(s) \Big|_{s=R e^{j\phi}} = \frac{K}{R^3 e^{j3\phi}} = 0 e^{-j3\phi}$$



• when  $s = j\omega$

$$G(s)H(s) \Big|_{s=j\omega} = \frac{K}{j\omega(-\omega^2+j\omega+4)}$$

$$= \frac{K[-\omega^2 - j\omega(4-\omega^2)]}{[(4-\omega^2)^2\omega^2 + \omega^4]}$$

$$\text{Im} \{ G(j\omega)H(j\omega) \} = 0, \quad -K\omega(4-\omega^2) = 0$$

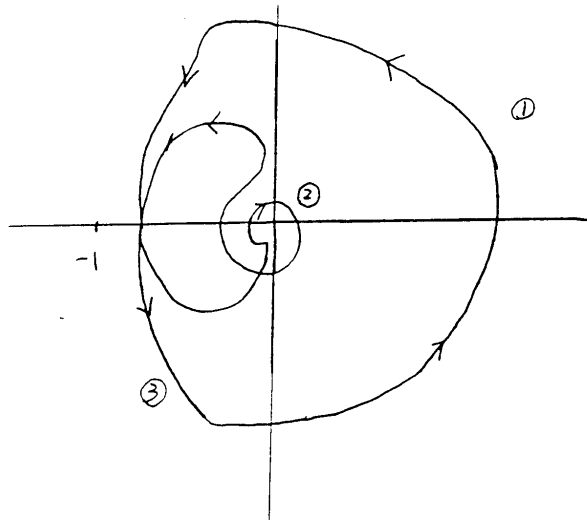
$$\Rightarrow \text{Cross Real axis at } \omega = 2$$

$$\operatorname{Re} \{ G(j\omega)H(j\omega) \} = 0 \Rightarrow \frac{-k\omega^{-1}}{\omega^4} = 0$$

$\Rightarrow$  does not cross imaginary axis for

$$0 < \omega < \infty$$

when  $s = -j\omega$ , apply similar arguments



Increasing  $k$  "expands" the diagram.

Since the real axis crossing is at  $\omega=2$ ,

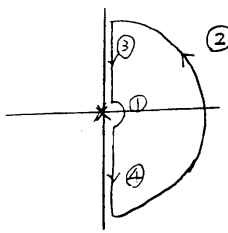
$$\operatorname{Re} \{ G(j\omega)H(j\omega) \}_{\omega=2} = -\frac{k}{4}$$

For stability, want  $(-k/4) > -1$ , or  $k < 4$ .

(Since for  $k > 4$ , there are 2 encirclements)

;

b)  $G(s)H(s) = \frac{k(s+1)}{s^2(s+2)}$



①  $s = \varepsilon e^{j\theta}$

$$G(s)H(s) \Big|_{s=\varepsilon e^{j\theta}} = \frac{k}{2\varepsilon^2 e^{j2\theta}} = \infty e^{-j2\theta}$$

②  $s = R e^{j\phi}$

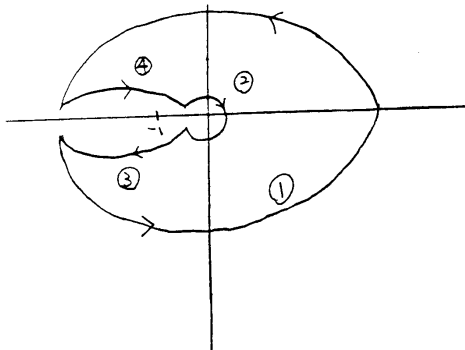
$$G(s)H(s) \Big|_{s=R e^{j\phi}} = \frac{k R e^{j\phi}}{R^3 e^{j3\phi}} = \frac{k}{R^2 e^{j2\phi}}$$

which equals  $0 \cdot e^{-j2\phi}$

③  $s = j\omega$

$$G(s)H(s) \Big|_{s=j\omega} = \frac{k(j\omega+1)}{(j\omega)^2(j\omega+2)} = \frac{k(-\omega^2-2-j\omega)}{-\omega^2(-\omega^2-4)}$$

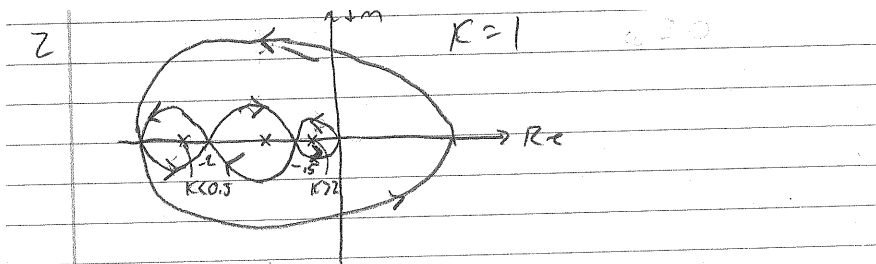
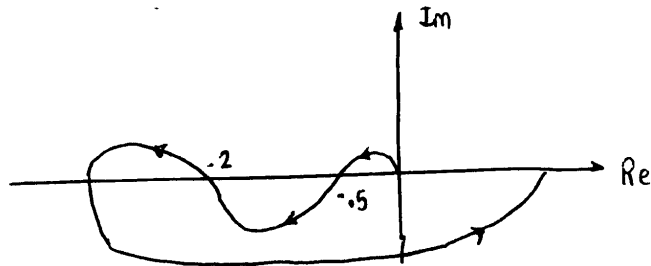
$\Rightarrow$  no axis crossings



Never encircles the point  $(-1, 0)$   
stable for all  $k > 0$

;

2. The Nyquist plot for a transfer function  $KG(s)$  is shown below for  $K = 1$  and  $\omega \geq 0$ . Assuming a unity feedback configuration with  $G(s)$  having no poles in the closed right half plane, find the ranges for  $K > 0$  under which closed loop stability is assured.



$P=0$

$Z = P - N$  For stability  $Z=0$

$0 = 0 - N$

$N=0$  need 0 net cw encirclements for CL stability

at  $K=1$

$N=0$  Stable

at  $K < 0.5$

$N=-2$

at  $K > 2$

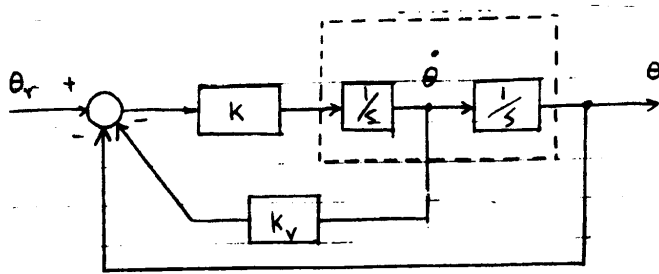
Not stable

$N=-2$  Not stable

CL is stable when

$0.5 < K < 2$

3. (a) For the attitude control system shown below, assume that the velocity feedback gain  $K_v$  is zero and sketch the Nyquist for open loop system  $G(s)H(s)$ .



(b) For the closed loop system above, for what range of gains  $K > 0$  is stability assured?

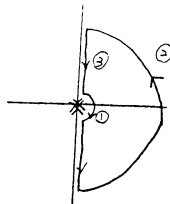
(c) Now consider the case with both gains  $K$  and  $K_v$  non-zero. Using an appropriate Nyquist plot, find the range for these gains under which closed loop stability is assured.

a)  $K G(s) = \frac{K}{s^2}$

①:  $s = \epsilon e^{j\theta} \Rightarrow$

$$G(s) \Big|_{s=\epsilon e^{j\theta}} \approx \frac{K}{\epsilon^2 e^{j2\theta}}$$

$$= \infty e^{-j2\theta}$$



②:  $s = R e^{j\phi} \Rightarrow$

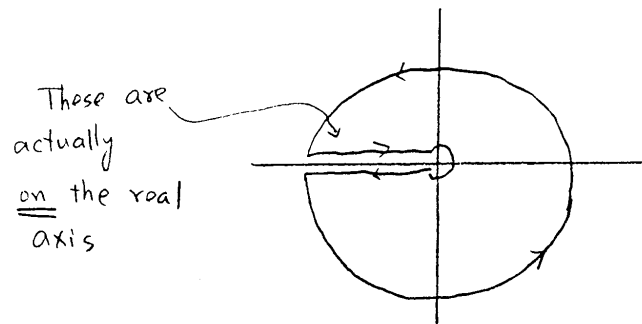
$$G(s) \Big|_{s=R e^{j\phi}} \approx \frac{K}{R^2 e^{j2\phi}} = 0 e^{-j2\phi}$$

③:  $s = j\omega \Rightarrow$

$$G(s) \Big|_{s=j\omega} = \frac{K}{-\omega^2}$$

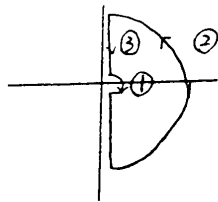
;

Real axis crossing for all  $0 < \omega < \infty$



(b) The system is marginally stable for all  $k > 0$

with  $k_v > 0$ ,  $k > 0$ ,  $G(s) = \frac{k}{s(s+k k_v)}$



①:  $s = \epsilon e^{j\theta} \Rightarrow$

$$G(s) \Big|_{s=\epsilon e^{j\theta}} \approx \frac{k}{k k_v \epsilon e^{j\theta}} = \infty e^{-j\theta}$$

②:  $s = R e^{j\phi} \Rightarrow$

$$G(s) \Big|_{s=R e^{j\phi}} \approx \frac{k}{R^2 e^{j2\phi}} = 0 e^{-j2\phi}$$

③  $s = j\omega \Rightarrow$

$$G(s) \Big|_{s=j\omega} = \frac{k}{-\omega^2 + j\omega k k_v}$$

;

$$= \frac{-k\omega^2 - j\omega k^2 k_v}{\omega^4 + \omega^2 k k_v}$$

To find the real axis crossing

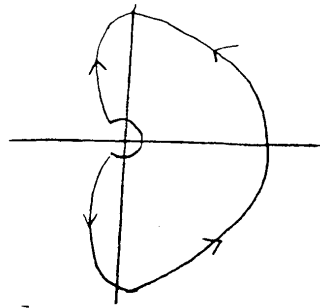
$$\text{let } -\omega k^2 k_v = 0$$

$\Rightarrow$  no crossing for  $0 < \omega < \infty$

To find imaginary axis crossings,

$$\text{let } -k\omega^2 = 0$$

$\Rightarrow$  no crossing for  $0 < \omega < \infty$



No encirclements of  $(-1, 0)$  for any  $k, k_v > 0$

So, the system is stable for all  $k > 0$

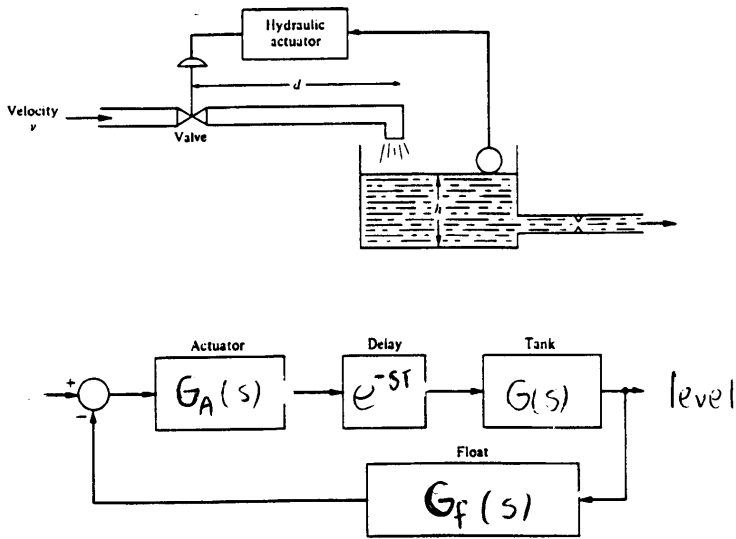
$$k_v > 0$$

;



4. A level control system and its transfer function model is depicted below. Assuming zero delay ( $T = 0$ ), use Matlab to generate the appropriate Nyquist plot which can be used to analyze closed loop stability. Take

$$G_A(s) = \frac{10}{s+1}; \quad G(s) = \frac{3.15}{30s+1}; \quad G_f(s) = \frac{1}{(s^2/9) + (s/3) + 1}.$$



(b) Using the Nyquist plot from part (a), estimate the gain and phase margins.

(c) With time delay  $T > 0$ , explain in detail what is meant by the following: “The appropriate Nyquist plot is obtained from the plot in Part (a) by a frequency dependent angular rotation.” For the case when  $T = 1$ , is the closed loop stable? Explain.

;

```

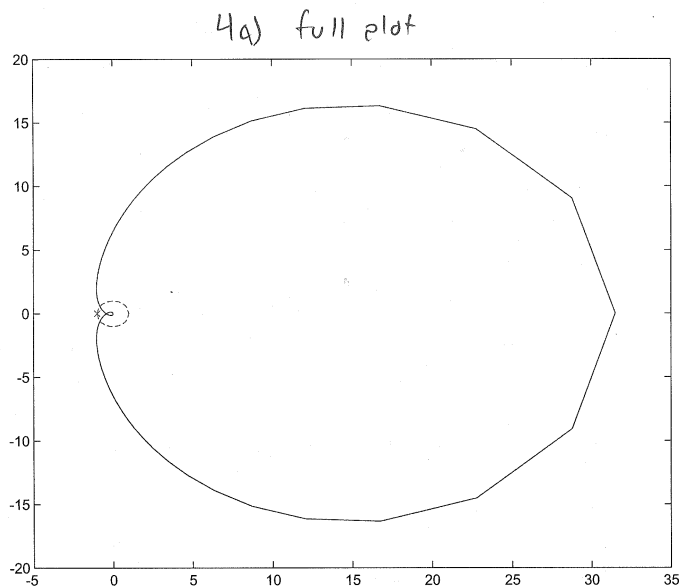
clear all; clc; clf;
R=1000;
svec = -R:R/100000:R;
s=svec*i;
thvec = -pi/2:pi/360:pi/2;
for j = length(svec)+1:length(svec)+length(thvec)
    s(j) = R.*cos(thvec(j-length(svec))) + R.*sin(thvec(j-length(svec))).*i;
end

T=0;
GA = 10./(s+1);
G = 3.15./(30.*s+1);
GF = 1./((s.^2./9)+(s./3)+1);
DEL = exp(-T.*s);
GH = GA.*DEL.*G.*GF;
Re = real(GH);
Im = imag(GH);
plot(Re,Im)

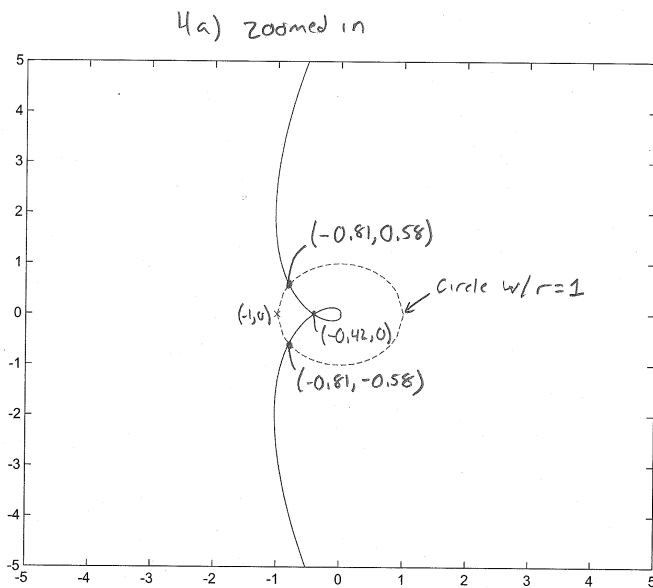
hold on
x=-1:.1:1;
y=sqrt(1-x.^2);
y2=-sqrt(1-x.^2);
plot(x,y,'r--',x,y2,'r--')
plot(-1,0,'rx')

%axis([-5 5 -5 5])

```



;



b) Poles:  $-0.033, -1, -1.5 \pm 2.598i$

$$P=0$$

from plot real axis crossings  
at  $-0.42$

$$-0.42K = -1$$

$$K = 2.381$$

Need  $N=0$  for stability

$$0 = 0 - N \Rightarrow N=0$$

$$K < 2.381$$

$$N=0$$

Stable

$$K > 2.381$$

$$N=-2$$

Unstable

Gain Margin:  $K=2.381$   
stable for  $K < 2.381$

Circle of radius 1 intersects nyquist at  
 $(-0.81, \pm 0.58)$

$$\theta = \pm 35.6^\circ$$

$$\theta = \tan^{-1}\left(\frac{\pm 0.58}{0.81}\right)$$

$$\theta=0 \quad N=0$$

all

Phase Margin:  $\theta = \pm 35.6^\circ$   
stable for  $-35.6^\circ < \theta < 35.6^\circ$

c) You need to rotate the plot by the phase corresponding to the appropriate delay and reexamine the -1 point to check for stability. Due to  $e^{-sT} = e^{-j\omega T}$  factor, some points will be rotated more than others, depending upon the frequency.

```

> s = tf('s')

Transfer function:
s

>> G_a = 10/(s+1)

Transfer function:
 10
----
s + 1

>> D = exp(-s)

Transfer function:
exp(-1*s) * (1)

>> G = 3.15/(30*s+1)

Transfer function:
 3.15
-----
30 s + 1

>> G_f = 1/(s^2/9+s/3+1)

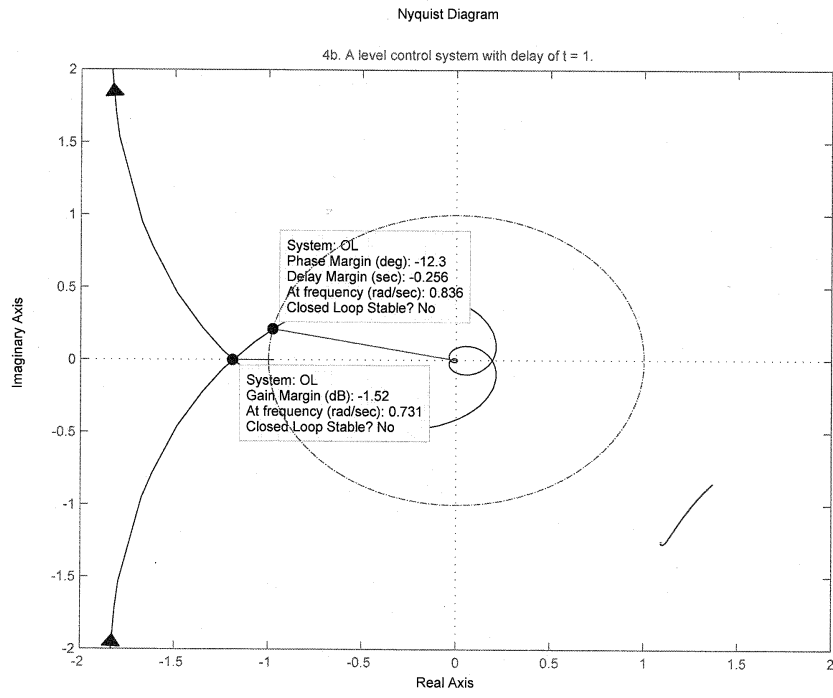
Transfer function:
      1
-----
0.1111 s^2 + 0.3333 s + 1

>> OL = G_a*D*G*G_f

Transfer function:
                                31.5
exp(-1*s) * -----
      3.333 s^4 + 13.44 s^3 + 40.44 s^2 + 31.33 s + 1

>> nyquist(OL)
>> axis([-2 2 -2 2])
>>
;

```



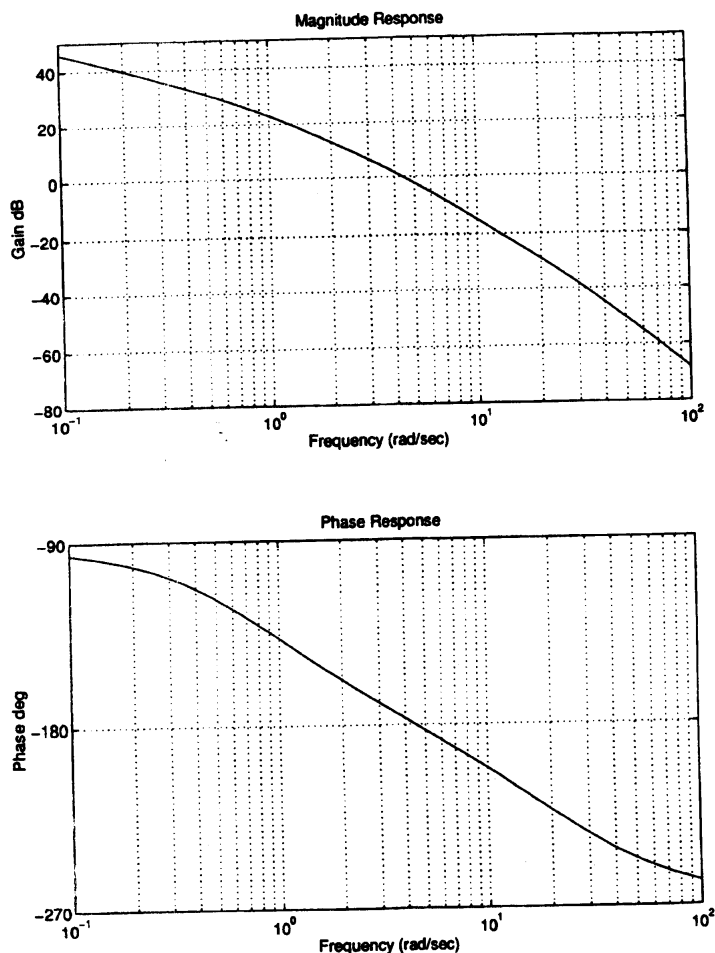
;

## 3.8 HW 8

## 3.8.1 Problem 1

1. Derive a model for the open-loop transfer function for the system whose frequency response plots are given on the last page. Turn in the plot with the straight-line Bode approximations of phase and log gain drawn on top of the true Bode plot. Will the closed-loop system be stable with negative unity-feedback? What are the gain and phase margins?

Bode plot for Problem 1.



## SOLUTION:

The first step is to find number of poles and number of zeros. Looking at the phase plot, we see that at high frequency the phase is  $-270^\circ$ . Therefore, there are 3 more poles than zeros.

(A pole adds  $-90^\circ$  at high frequency and zero adds  $+90^\circ$ ). Now we look at the magnitude plot and look for any slope change in the positive direction. There is none. The slope starts at  $-20$  dB/decade and remains negative going to  $-40$  dB/decade, then  $-60$  dB/decade.

This implies there are no zeros since a zero makes the slope positive. We now know that there are 3 poles and no zeros.

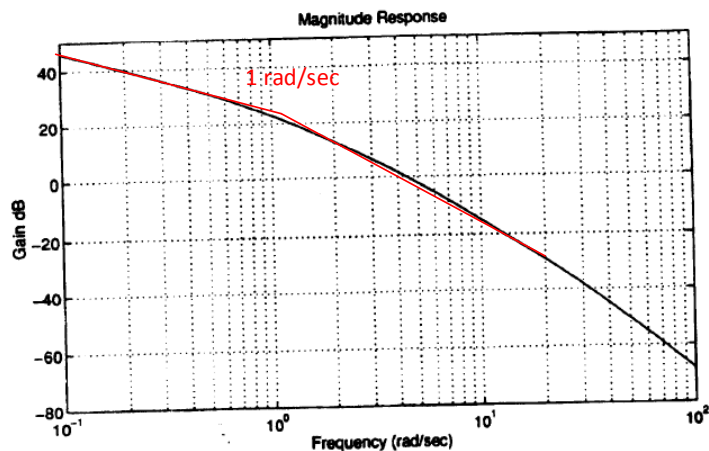
Next we look at where the phase starts. We see it starting at  $-90^\circ$ . This means this is type 1 system (i.e. one pole at the origin.). So now we can say that our system has this general form

$$G(s) = \frac{K}{s \left(1 + \frac{s}{\tau_1}\right) \left(1 + \frac{s}{\tau_2}\right)}$$

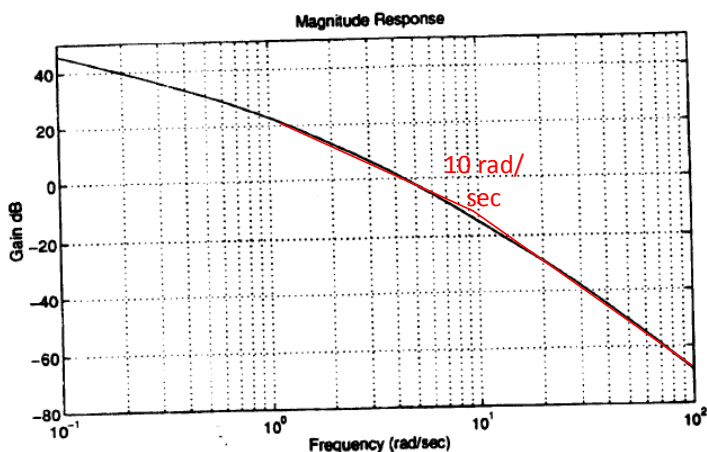
A pole at zero always starts at 40 dB at low frequency since  $20 \log \frac{1}{\omega} = -20 \log \omega$ , and using  $\omega = 0.01$  as the small frequency value (by convention), we obtain  $-20 \log 0.01 = 40$  dB. It drops by  $-20$  dB/decade. Now we need to find the locations of the break points  $\tau_1$  and  $\tau_2$  (also called corner frequencies).

To find  $\tau_1$  we draw an asymptotic lines between the first segment which has slop of  $-20$  dB/decade and the next segment which has slope of  $-40$  dB per decade and look for the intersection point. We find it is  $\tau_1 \approx 1$  rad/sec as illustrated below. Similarly between the second segment which has slope of  $-40$  dB/decade and the third segment which has slope of  $-60$  dB/decade we find the intersection to be around  $\tau_1 \approx 10$  rad/sec

Bode plot for Problem 1.



Bode plot for Problem 1.



Now that we found the corner frequencies, our system has this form

$$G(s) = \frac{K}{s(1+s)\left(1+\frac{s}{10}\right)}$$

The only thing left is to determine gain  $K$ . This is done by looking at low frequency. By convention this is  $\omega = 0.01$  rad/sec. At  $\omega = 0.1$  we see the gain is about 45 dB, and since the slope is  $-20$  dB/decade we go back one decade, and conclude that magnitude at  $\omega = 0.01$  rad/sec must be 65 dB.

Since pole at zero at 40 dB, then the difference, which is 25 dB, must be due to the gain  $K$ . Hence we solve for  $K$  from

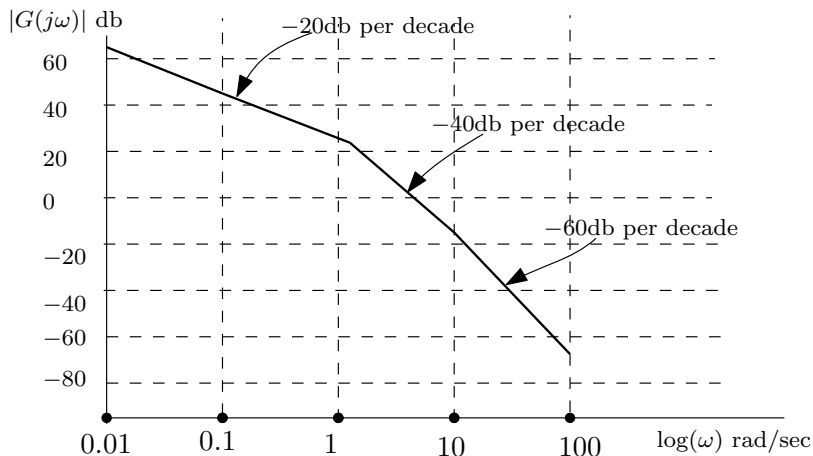
$$\begin{aligned} 20 \log K &= 25 \\ K &= 10^{\frac{25}{20}} = 17.78 \end{aligned}$$



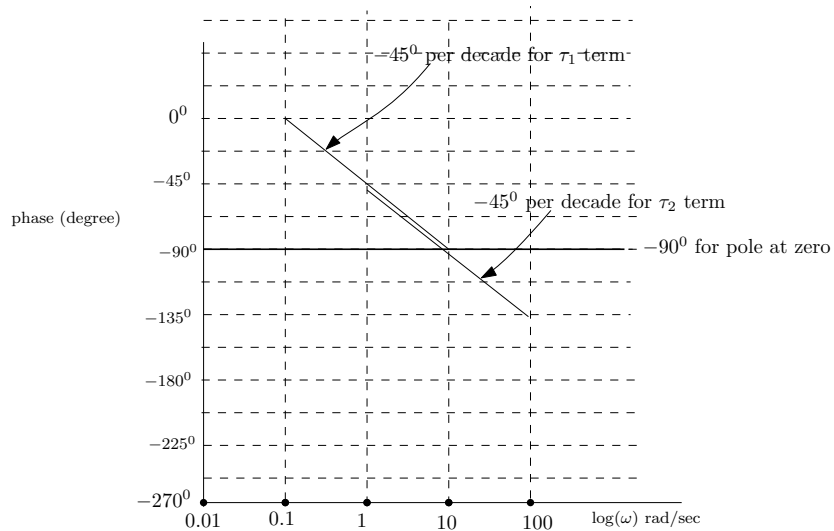
Therefore our system is now complete. The open loop transfer function is

$$G(s) = \frac{17.78}{s(1+s)(1+\frac{s}{10})}$$

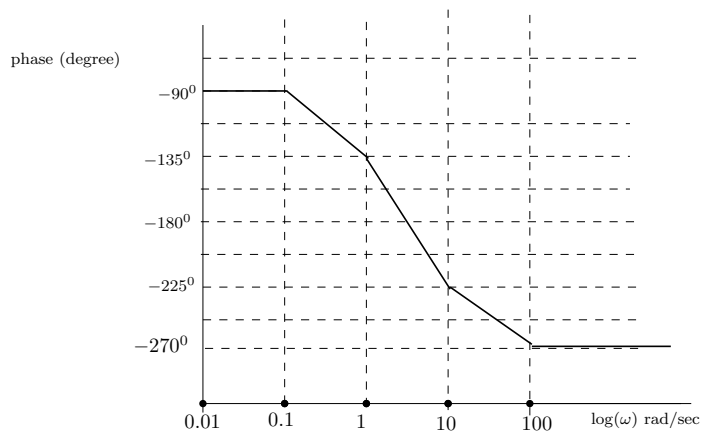
Now we Draw the magnitude straight line approximation (this below was drawn by hand using drawing program. This is not computer generated)



Now we draw the phase straight line approximation. Phase goes down by  $-45^\circ$  for each pole, starting one decade before the corner frequency, and ending one decade after the corner frequency. This is only for the approximation factors in the form  $\frac{1}{1+\frac{s}{\tau}}$ . For the exact pole  $\frac{1}{s}$ , it starts at  $-90^\circ$  and remains over the whole frequency range. For  $\tau_1 = 1$  rad/sec, we start from 0.1 up to 10 rad/sec. For  $\tau_1 = 10$  rad/sec, we start from 1 rad/sec up to 100 rad/sec. Using this information, below is sketch of the phase straight line approximation.

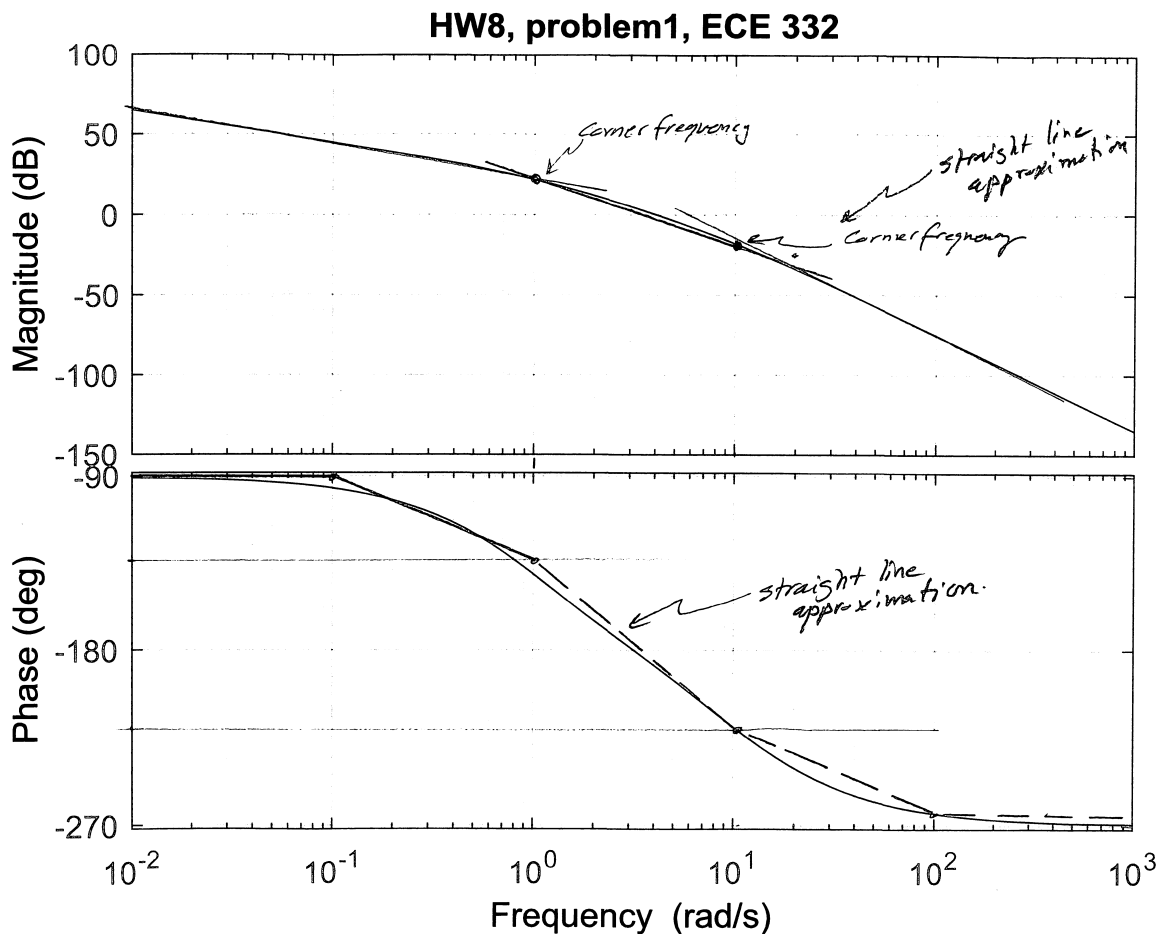


Sum each segment, we obtain the plot below



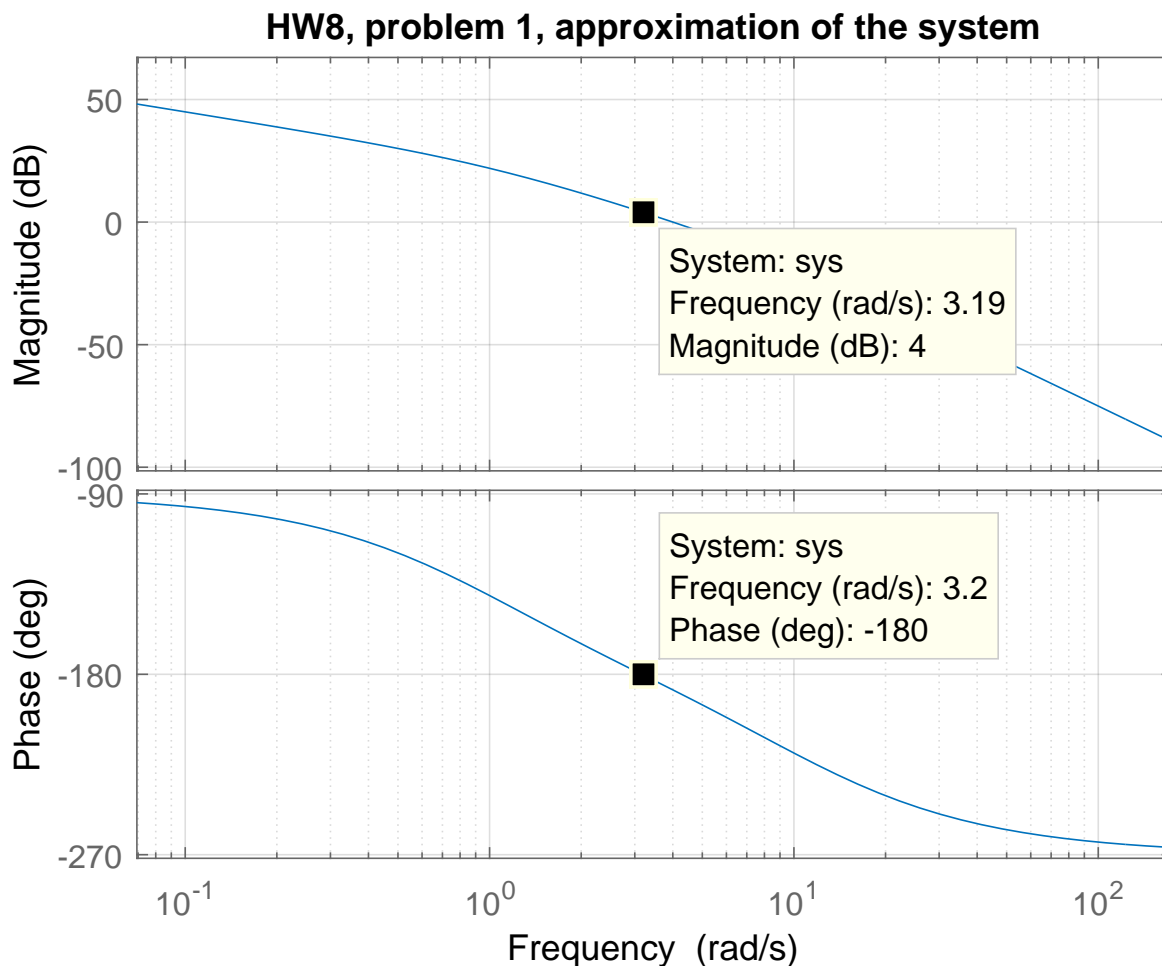
Below is the straight line approximation on top of the true bode plot as required to show.

```
s=tf('s'); sys=17.78/(s*(1+s)*(1+s/10))
bode(sys)
```



Since the open loop is stable as it has poles at  $\{0, -1, -0.1\}$ , then the closed loop will be stable if there is zero net clockwise encirclement around  $-1$ . This translates to having positive phase margin when magnitude of  $|G(j\omega_{gc})|$  is unity and positive gain margin with phase is  $-180^\circ$ .

To find the gain margin and the phase margins, we first plot our approximation of the system found above:  $G(s) = \frac{17.78}{s(1+s)(1+\frac{s}{10})}$  using the bode command. Here it is, showing on it the  $\omega_{pc}$  frequency (where phase is  $-180^\circ$ ) and the corresponding  $|G(j\omega_{pc})|$  in dB found, which we will use to find the gain margin from



We see that at  $-180^\circ$ , the frequency is 3.2 rad/sec. This is called  $\omega_{pc}$ . Going back to the magnitude plot, we see that at  $\omega_{pc}$  then  $|G(j\omega_{pc})|_{dB} = 4$  dB. This means the gain margin  $GM_{dB}$  is

$$GM_{dB} = -4 \text{ dB}$$

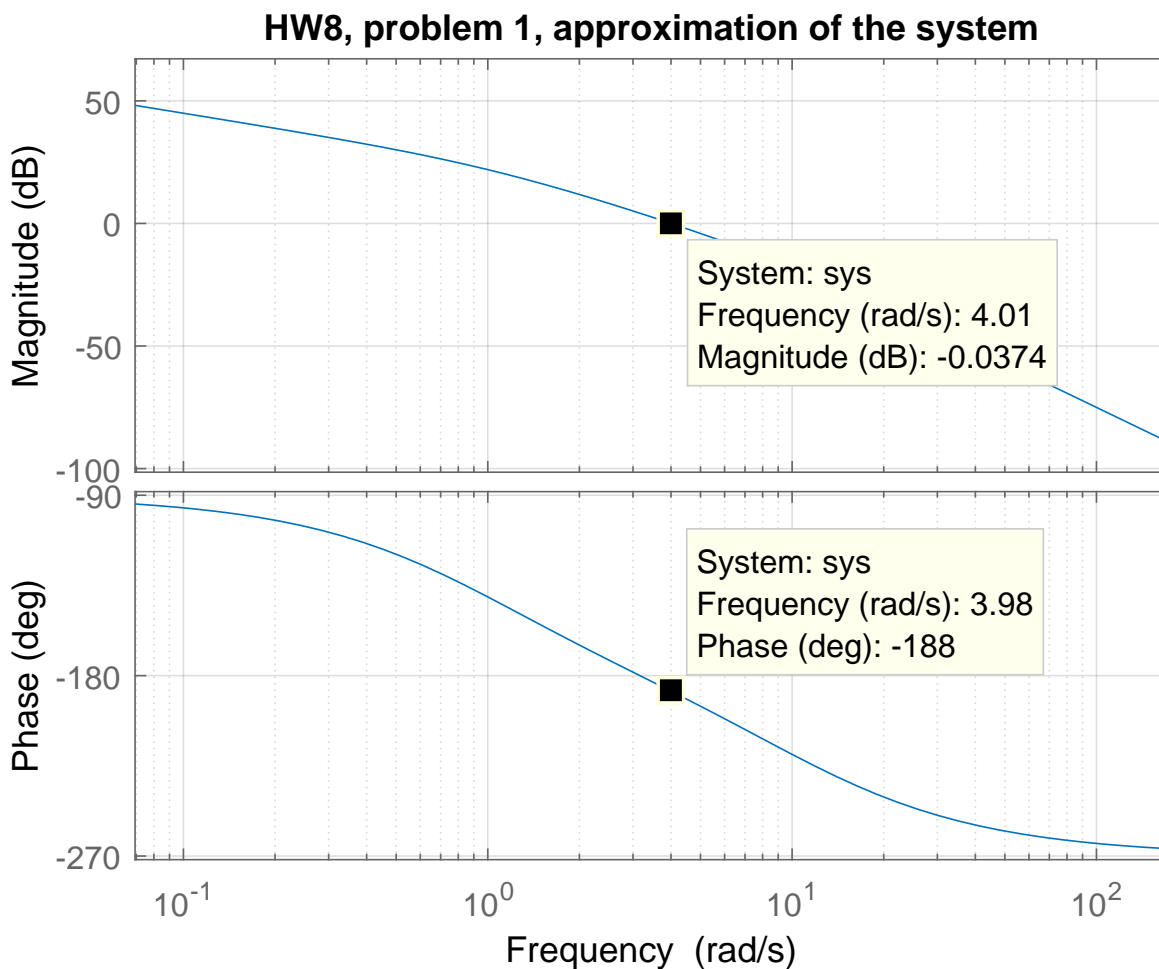
Notice that  $GM_{dB}$  is negative of  $|G(j\omega_{pc})|_{dB}$ . The reason is due to the definition of  $GM_{dB}$ , which is

$$GM_{dB} = 20 \log_{10} \frac{1}{|G(j\omega_{pc})|}$$

But  $|G(j\omega_{pc})| = 10^{\frac{|G(j\omega_{pc})|_{dB}}{20}}$ . Substituting this in the above gives

$$\begin{aligned}
 GM_{dB} &= 20 \log_{10} \frac{1}{10^{\frac{|G(j\omega_{pc})|_{dB}}{20}}} \\
 &= -20 \log_{10} 10^{\frac{|G(j\omega_{pc})|_{dB}}{20}} \\
 &= -|G(j\omega_{pc})|_{dB} \log_{10} 10 \\
 &= -|G(j\omega_{pc})|_{dB}
 \end{aligned}$$

Therefore,  $GM_{db}$  is the negative of  $|G(j\omega_{pc})|_{db}$  as read from bode plot. Since  $GM_{dB} < 0$  then closed loop is not stable. To find the phase margin. we find the frequency  $\omega_{gm}$  which is where magnitude plot is at 0 dB. We see that the frequency is  $\omega_{gm} = 4$  rad/sec as shown in the plot below.



At  $\omega_{gm}$ , the phase is  $-188^{\circ}$ . Subtracting  $180^{\circ}$  from this phase gives  $-7^{\circ}$ . Hence

$$\text{phase margin} = -7^\circ$$

Phase margin must be positive for stable closed loop stable. The closed loop is not stable.

Note that both the phase margin and the gain margin must be positive for stable closed loop system.

### 3.8.2 Problem 2

2. Given the forward loop transfer function for a negative unity-feedback system

$$G(s) = \frac{(s+5)(s+3)}{s(s+1)(s^2+s+4)}$$

- (a) investigate the stability of the system using both Nyquist and Bode in Matlab,
- (b) find the gain and phase margins from each method and their corresponding crossover frequencies. Display the gain and phase margins on the plots where they occur.

SOLUTION:

The open loop

$$G(s) = \frac{(s+5)(s+3)}{s(s+1)(s^2+s+4)}$$

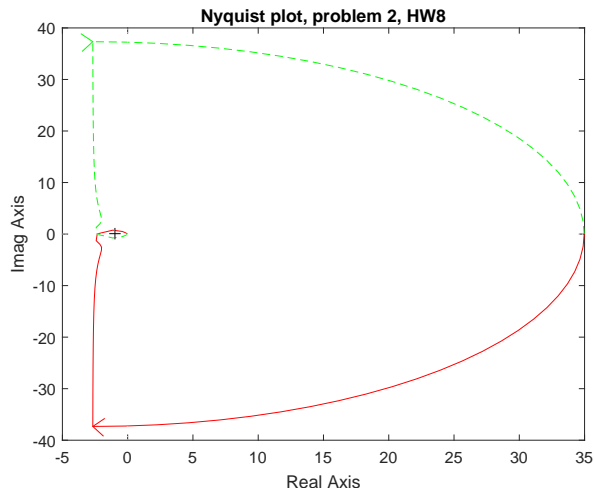
has poles at  $s = 0, s = -1$  and  $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$ , therefore it is stable. So we expect the closed loop to be stable only if the Nyquist plot has a net of zero clockwise encirclement around  $-1$ . When looking at the bode plot, the rules are these. The closed loop is stable, if both these conditions are met:

1. The gain margin  $GM_{db}$  is positive. Or in other words, if  $\left|G(j\omega_{gc})\right|_{db}$  is negative as read from the bode plot.
2. The phase margin is positive.

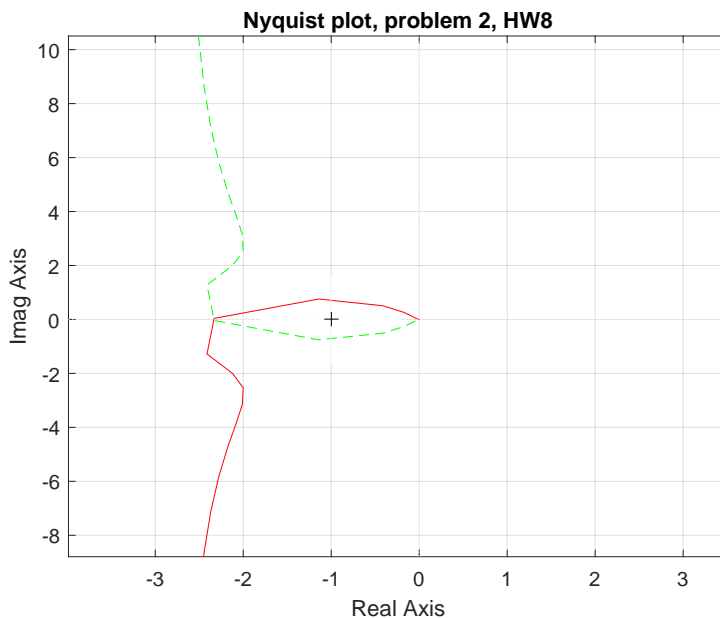
#### 3.8.2.1 Part(a)

Here is Nyquist plot

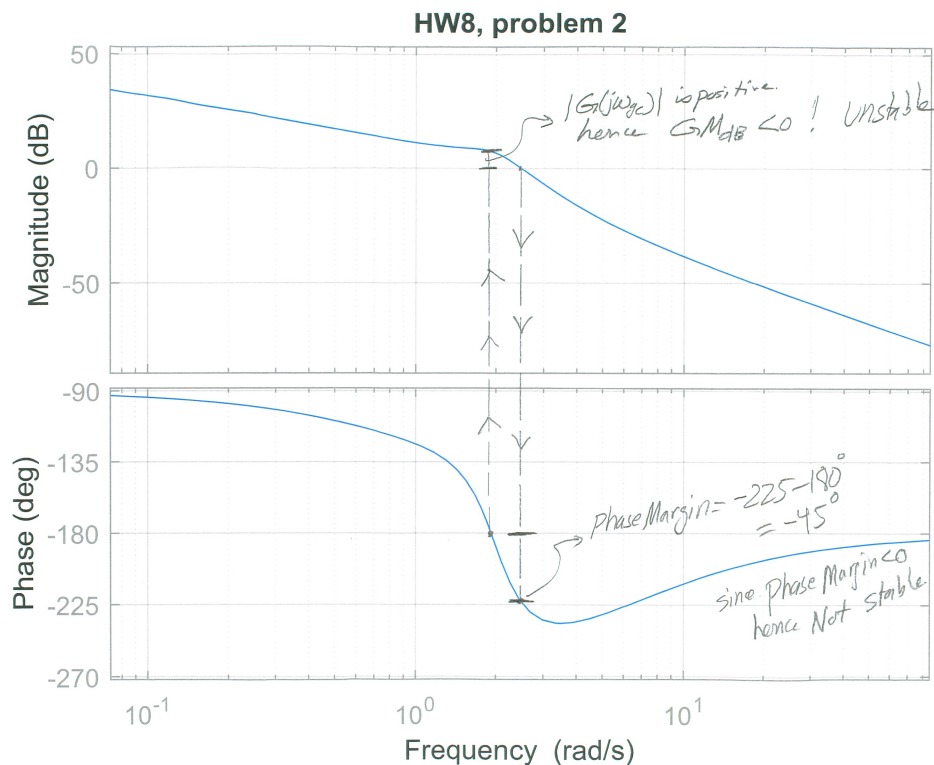
```
s=tf('s');
sys=(s+5)*(s+3)/(s*(s+1)*(s^2+s+4));
nyquist1(sys)
```



It shows there is one encirclement around  $-1$ . We can zoom in to make sure:



The above shows that the closed loop is not stable. We will now look at Bode plot. Here is the result, where I showed the gain and phase margins on the generated Matlab plot. This shows that the gain margin is negative, hence not stable, and it also shows that the phase margin is negative, also indicating it is not stable.



We can also ask Matlab to give us the margins and the corresponding break frequencies. Matlab was correct and found that the closed system is also not stable:

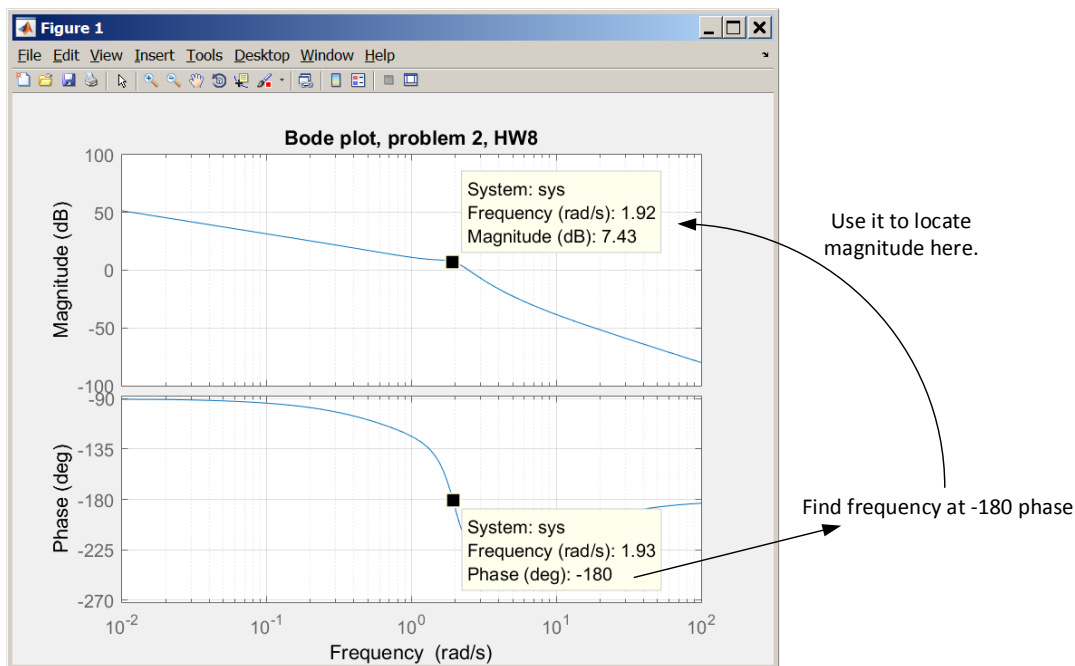
```
>> [gm,pm,gwc,pwc]=margin(sys)
Warning: The closed-loop system is unstable.
> In ctrlMsgUtils.warning (line 25)
In DynamicSystem/margin (line 65)
gm =
0.4254
pm =
-42.9450
gwc =
1.9239
pwc =
2.4785
```

Notice that Matlab gives the gain margin  $GM$  in linear value. We see it says  $GM = 0.4254$  above, which is  $-7.42$  dB. Since it is negative, then closed loop is not stable.

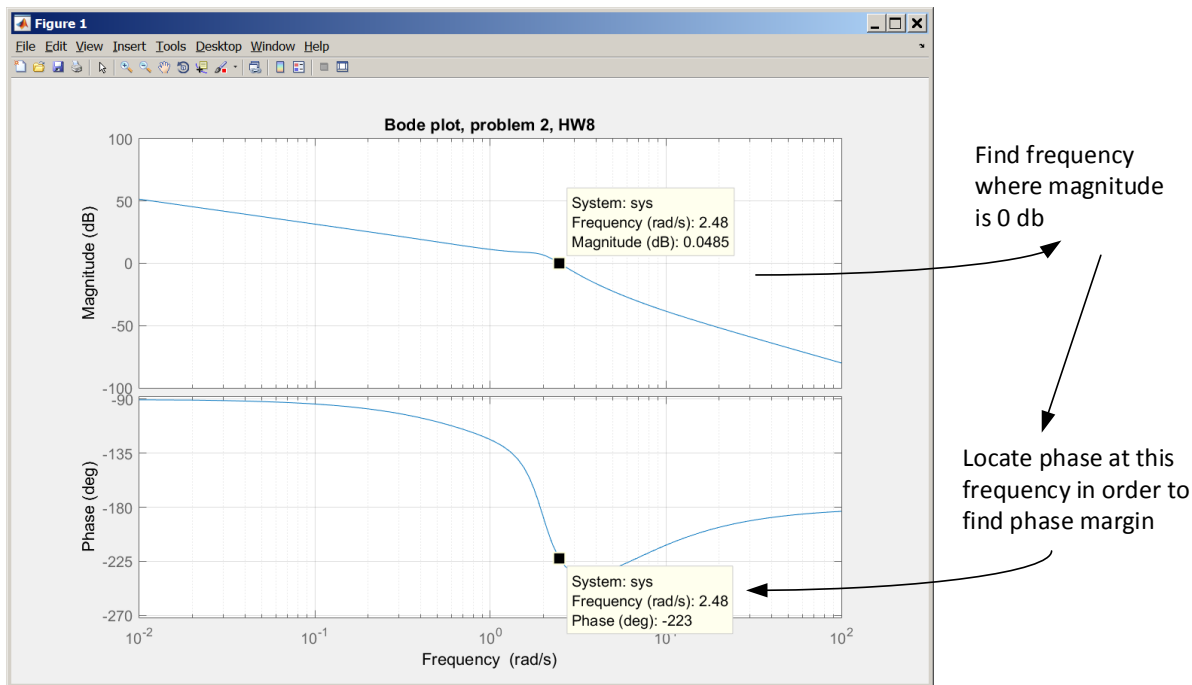


3.8.2.2 Part(b)

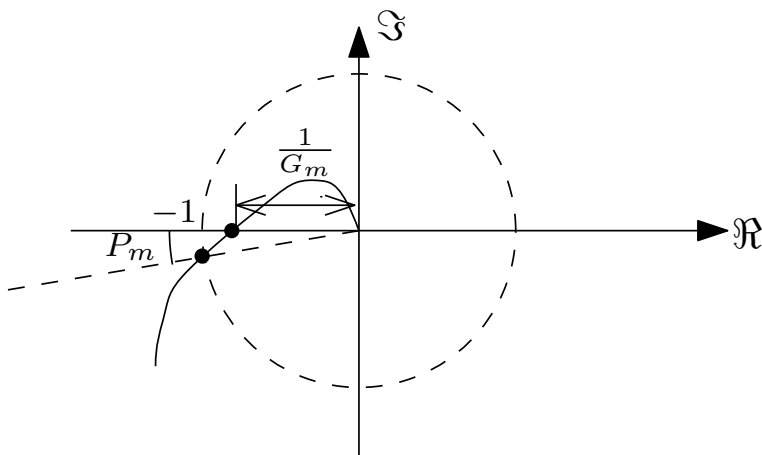
We can display the cross over frequencies using Matlab either by using the margin command or using the GUI by using the mouse as shown below. First we find the frequency where the phase is  $-180^0$ , this is called  $\omega_{pc}$ . We see it is 1.93 rad/sec. Then using the mouse, we locate this frequency on the magnitude plot and read  $|G(j\omega)|$  in dB. We see that  $|G(j\omega)|$  in dB is positive, hence GM is negative, and closed loop is not stable.



To determine the stability using phase margin, we do the reverse. We locate the frequency where the magnitude is zero dB using the mouse. This is called  $\omega_{gm}$ . We see it is at 2.48 rad/sec. Then on the phase plot, we locate the phase at this frequency using the mouse. We see it is  $-228^0$ . Adding  $-180^0$  gives  $-48^0$ . Since this is negative, then the close loop is not stable.

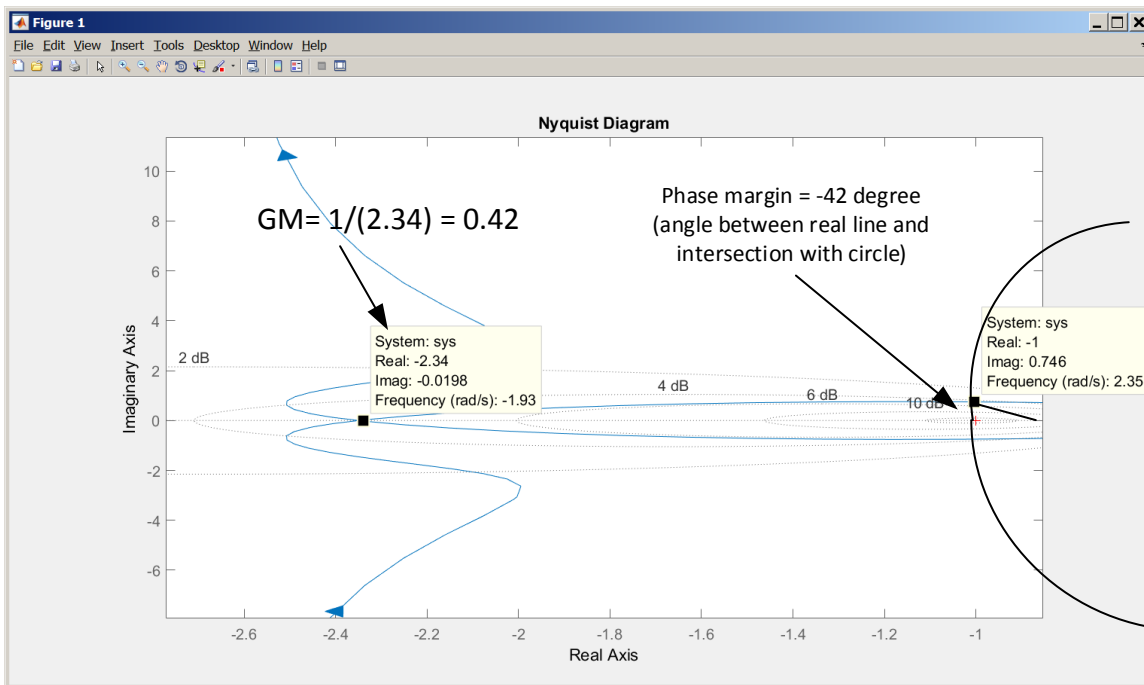


Using Nyquist to determine stability, we plot Nyquist. Then make a unit circle around the origin to locate the gain and phase margin, as illustrated below



Locating gain and phase margin on Nyquist plot

Plotting the Nyquist plot using Matlab, and zooming in, and making a unit circle (Circle was added by hand on top of the Matlab Nyquist plot), one can measure the gain and phase margin as below



### 3.8.3 Problem 3

3. The block diagram shown in Figure 1 represents a model of a hydroelectric alternator, turbine, and penstock with transfer function  $G_1(s)$  being given by

$$G_1(s) = \frac{25(1 + 5s)}{(s^3 + 49.277s^2 + 25.772s + 2.526)}$$

The parameter values are  $T = 1$ ,  $K = 0.05$ ,  $M = 10$ , and  $D = 1$  with  $L(s) = 0$ .

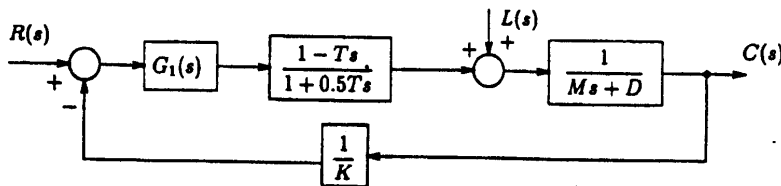


Figure 1: Hydroelectric System Block Diagram

- (a) Using the Bode plots, generated using *Matlab* if you wish, find the maximum value of  $K$  to retain stability.
- (b) Find the gain margin, phase margin and the corresponding crossover frequencies and label them on the Bode plots.

SOLUTION:

## 3.8.3.1 Part (a)

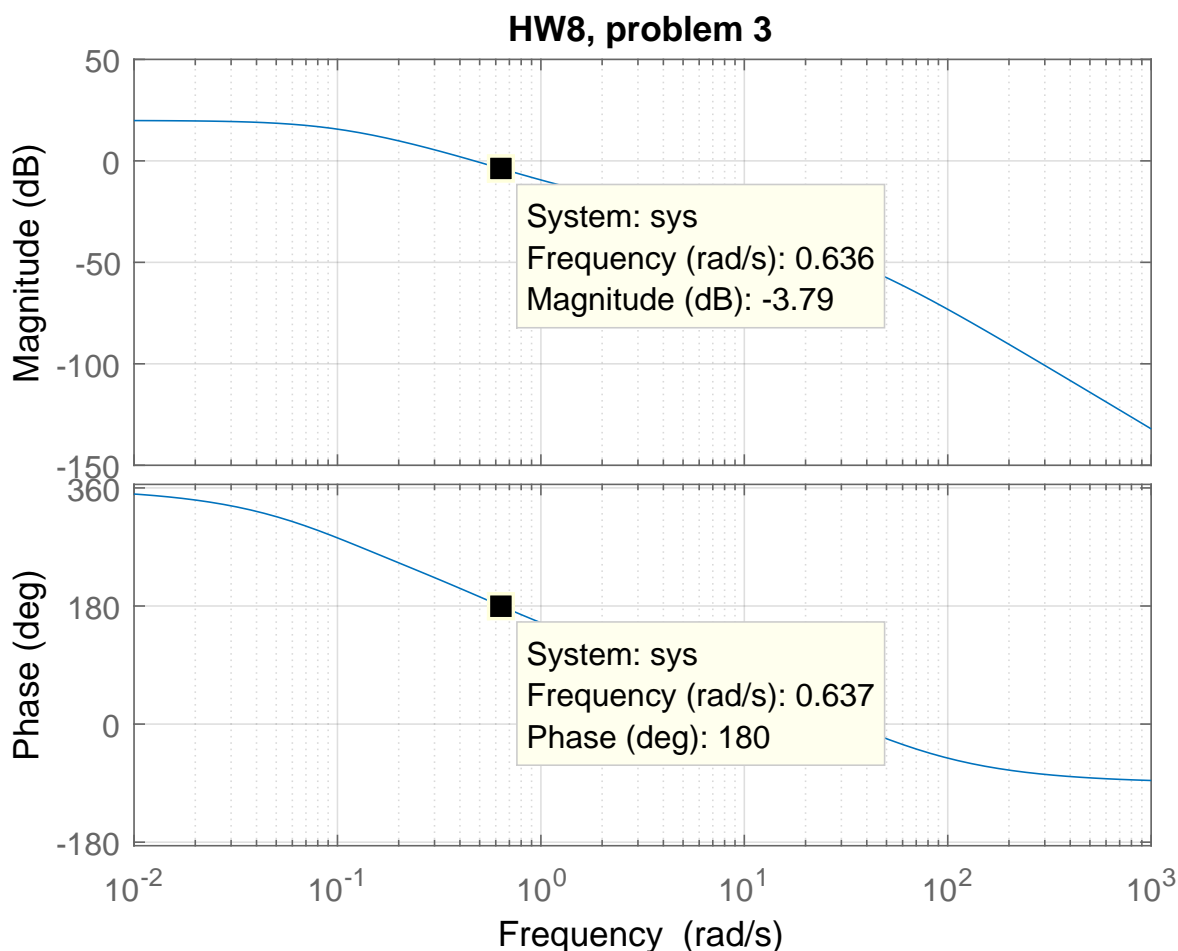
The open loop transfer function is

$$G(s) = \frac{25(1+5s)}{s^3 + 42.277s^2 + 25.772s + 2.526} \left( \frac{1-Ts}{1+0.5Ts} \right) \left( \frac{1}{Ms+D} \right) \left( \frac{1}{K} \right)$$

We start with  $T = 1, M = 10, D = 1, K = 1$ .

$$G(s) = \frac{25(1+5s)}{s^3 + 42.277s^2 + 25.772s + 2.526} \left( \frac{1-s}{1+0.5s} \right) \left( \frac{1}{10s+1} \right)$$

And make a bode plot, then find the  $|G(j\omega_{pc})|$  at corresponding  $-180^\circ$ .



But the phase do not cross  $-180^\circ$ . This indicates an infinite gain margin. Similarly we find that the phase margin is infinite. Hence we conclude that we can make  $K$  as close to zero (since it is in denominator) as we want, while keeping the system stable and can make it as large as we want. Note that we are assuming that gain itself can only be positive here.

## 3.8.3.2 Part (b)

Both the gain and the phase margin are infinite. There is no corresponding crossover frequencies.

## 3.8.4 Problem 4

Problem 4: The open loop transfer function of a position control system is

$$G(s) = \frac{K(1 + 0.1s)}{s(1 + 0.2s)(1 + 0.4s)}$$

Assume a negative unity feedback configuration.

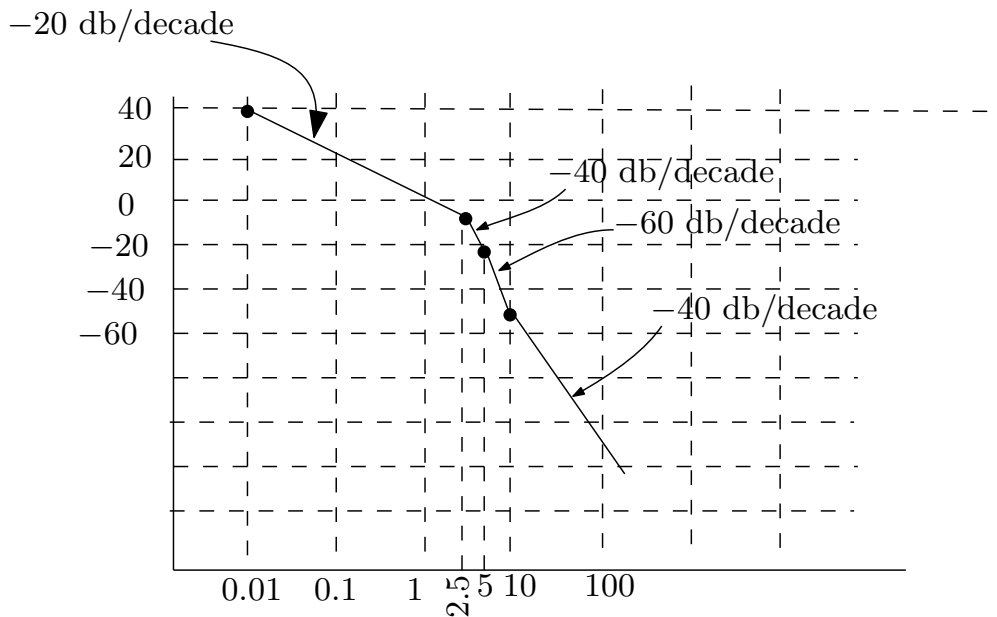
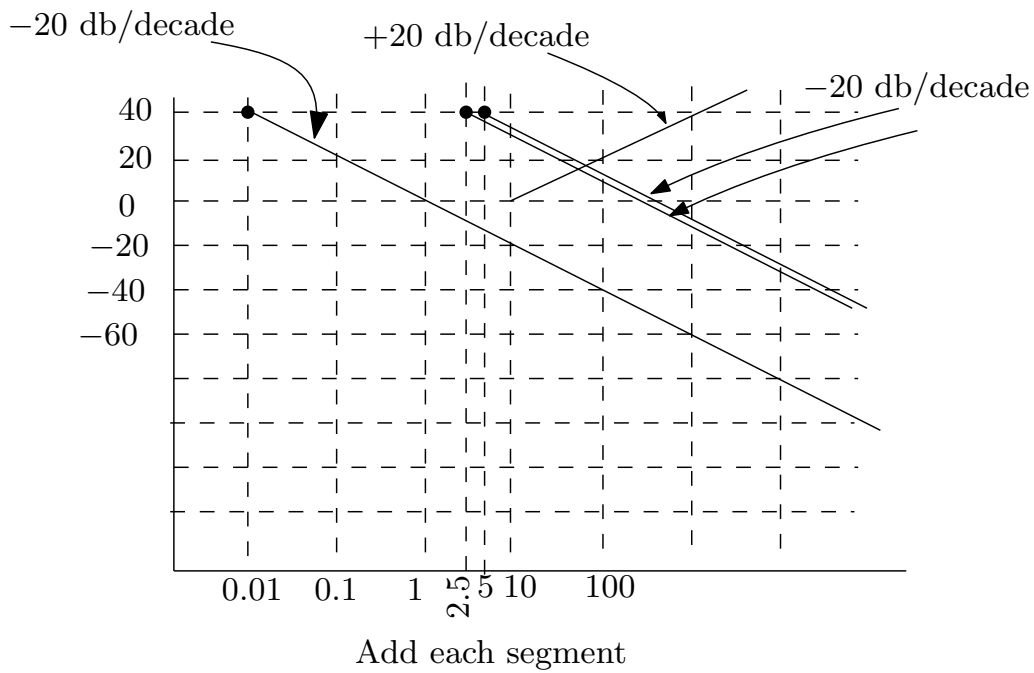
- (a) Sketch the Bode diagram for  $K = 1$  (Do Not Use MATLAB). Find the gain margin, gain-crossover frequency, phase margin and the phase-crossover frequency for the system.
- (b) Use MATLAB to check your plot
- (c) Determine the value of  $K$  which will satisfy the following design criteria
  1. A phase margin of  $55^\circ$ .
  2. A gain margin of 26 dB.

SOLUTION:

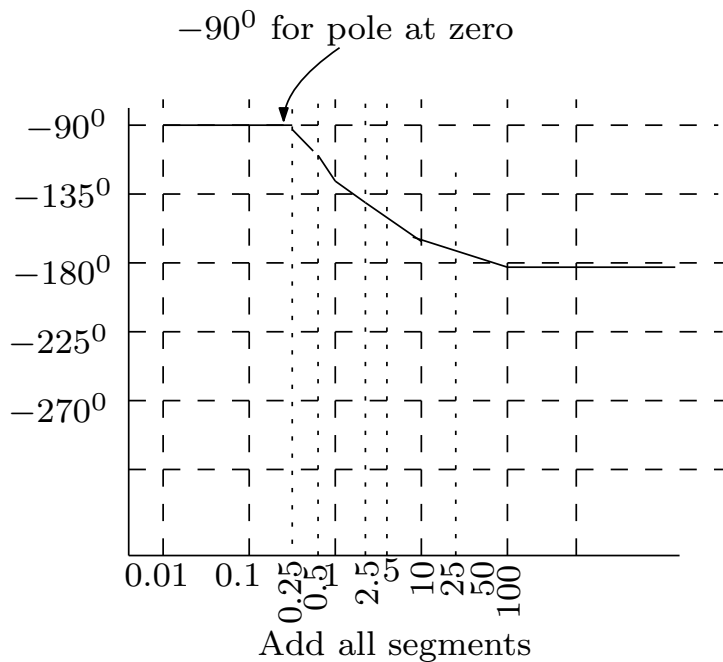
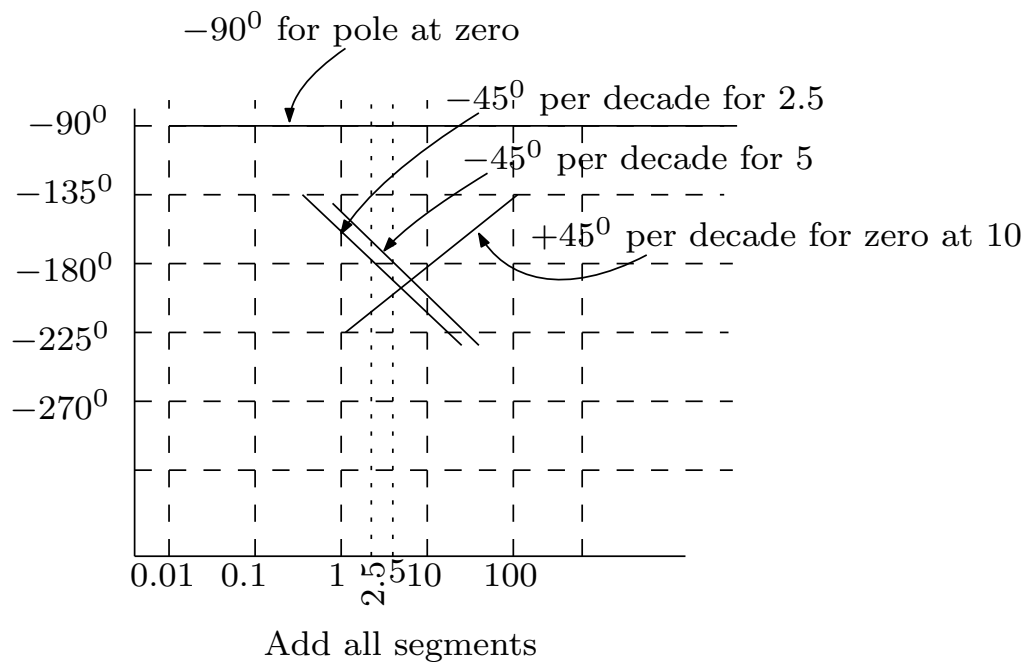
## 3.8.4.1 part(a)

$$G(s) = \frac{\left(1 + \frac{s}{10}\right)}{s\left(1 + \frac{s}{5}\right)\left(1 + \frac{s}{2.5}\right)}$$

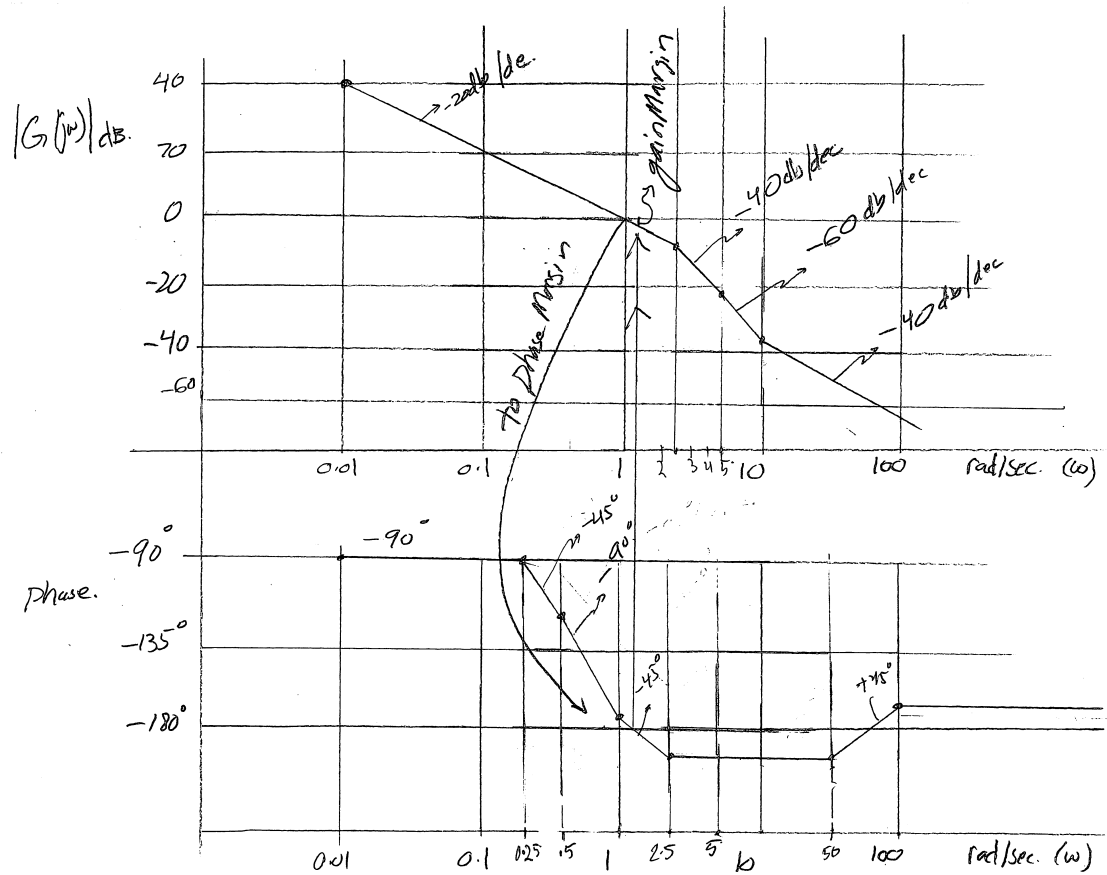
For the magnitude, the corner frequencies are at  $s = 0, s = 5, s = 2.5$  for the poles and  $s = 10$  for the zeros. The pole at zero starts at 40 dB (we use  $\omega = 0.01$  as starting point by convention) with slope of  $-20$  dB/decade. Each pole in the denominator adds a  $-20$  dB/decade slope, while each zero adds  $+20$  dB/decade. Here is the magnitude plot approximation



For the phase, since this is type 1 system, it starts at  $-90^\circ$  at the  $\omega = 0.01$ . This stays for all the range of frequencies. At  $\omega = 5$  and  $\omega = 2.5$ , A  $-45^\circ$  slope is added. This slope extends one decade before the corner frequency up to one decade above it. At  $\omega = 10$  rad/sec, we add  $+45^\circ$  for the zero. Here is the result



Now to answer the part about the gain and phase margins. For this, we show both  $|G(j\omega)|$  and phase plot that we sketched above in one diagram and mark on them the gain and phase related quantities. This is the result.



In next part, we use Matlab to get accurate margin values, which shows that gain margin is 30 dB and phase margin is  $64^\circ$ . The gain cross over frequency is 7 rad/sec and the phase cross over frequency is 0.92 rad/sec. The closed loop is stable.

### 3.8.4.2 Part(b)

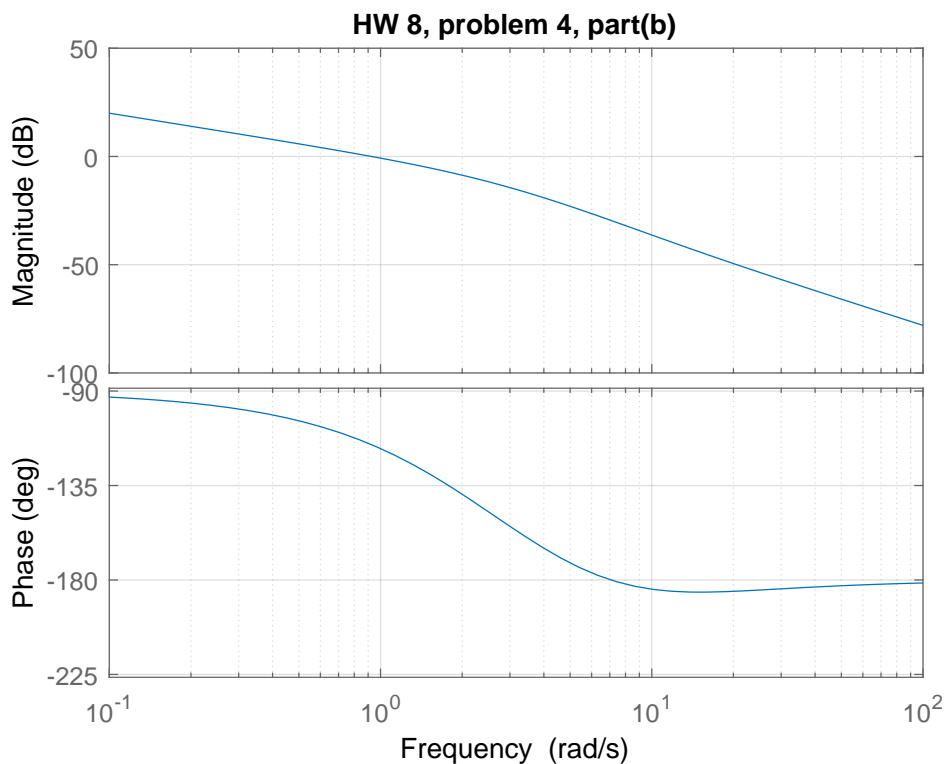
Using Matlab



```

clear
s=tf('s');
sys=tf((1+s/10)/(s*(1+s/5)*(1+s/2.5)));
bode(sys);
grid
[gm,pm,gcw,pcw]=margin(sys)
gm =
30.0007
pm =
64.4735
gcw =
7.0711
pcw =
0.9260

```



### 3.8.4.3 Part(c)

**3.8.4.3.1 Part(1)** For phase margin of  $55^\circ$  we want the phase at  $-125^\circ$  to correspond to 0 dB in the magnitude plot. At phase  $-125^\circ$  the frequency is 1.3 rad/sec from the plot. At this frequency the magnitude is  $-3.55$  dB and not zero dB. Hence positive gain margin of 3.55 dB. We want to shift this up to 0 dB. So we need to solve for additional gain from

$$20 \log_{10} K = 3.55$$

$$K = 10^{\left(\frac{3.55}{20}\right)}$$

$$= 1.5$$

Hence

$$K = 1.5$$

To verify, here is the Matlab margin command, which shows the phase margin is indeed now  $55^\circ$  when using  $K = 1.5$

```
clear
s=tf('s');
K=1.5;
sys=tf( K* (1+s/10)/(s*(1+s/5)*(1+s/2.5)));
bode(sys);
grid
[gm,pm,gcw,pcw]=margin(sys)
gm =
20.0005
pm =
55.3798
gcw =
7.0711
pcw =
1.2991
```

**3.8.4.3.2 Part(2)** At  $-180^\circ$  in the phase plot, we want the corresponding gain margin to be 26 dB which means we want  $|G(j\omega_{gc})| = -26$  dB. Currently, we see that at  $-180^\circ$ , the frequency is 7.1. The magnitude is  $-30$  dB at this frequency. We want magnitude to be  $-26$  dB instead. Hence we want to shift up by 4 dB the magnitude plot, or  $K = 10^{\left(\frac{4}{20}\right)} = 1.585$

$$K = 1.585$$

To verify, here is the Matlab margin command, which shows the gain margin is close to 26 dB now using  $K = 1.585$ . (Matlab gives  $gm = 18.92$  which is 25.54 dB)

```
clear
s=tf('s');
K=1.585;
sys=tf( K* (1+s/10)/(s*(1+s/5)*(1+s/2.5)));
[gm,pm,gcw,pcw]=margin(sys)
gm =
18.9279
pm =
54.0559
gcw =
7.0711
pcw =
1.3568
>> 20*log10(gm)
25.5420
```

## 3.8.5 HW 8 key solution

## Problem 1:

1. Derive a model for the open-loop transfer function for the system whose frequency response plots are given on the last page. Turn in the plot with the straight line Bode approximations of phase and log gain drawn on top of the true Bode plot. Will the closed loop system be stable with negative unity-feedback? What are the gain and phase margin?

ends at  $-270^\circ$ ; 3 more poles than zeroes  
 phase starts at  $-90^\circ$ ; 1 pole is 0  
 phase is steepest between  $1+10$ ; 2 poles "on"

log gain changes at  $1+10$ ; poles are  $1+10$

at  $\omega = 1 \text{ rad/s}$   $\log \text{ gain} \approx 22$ ;

$$20 \log K = 22$$

$$K = 12.6$$

$$G(s)H(s) = \frac{12.6}{s(s+1)(0.1s+1)}$$

$$\omega_{pc} = 4.5 \text{ rad/s}$$

$$\log \text{ gain } |G(j\omega_{pc})| = 0 \quad \phi M = 0$$

The system is marginally stable

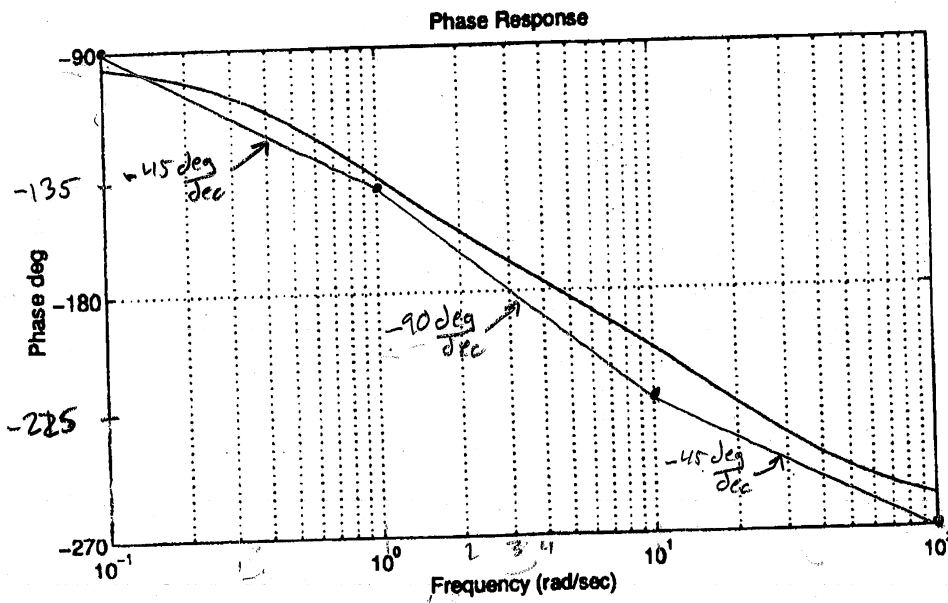
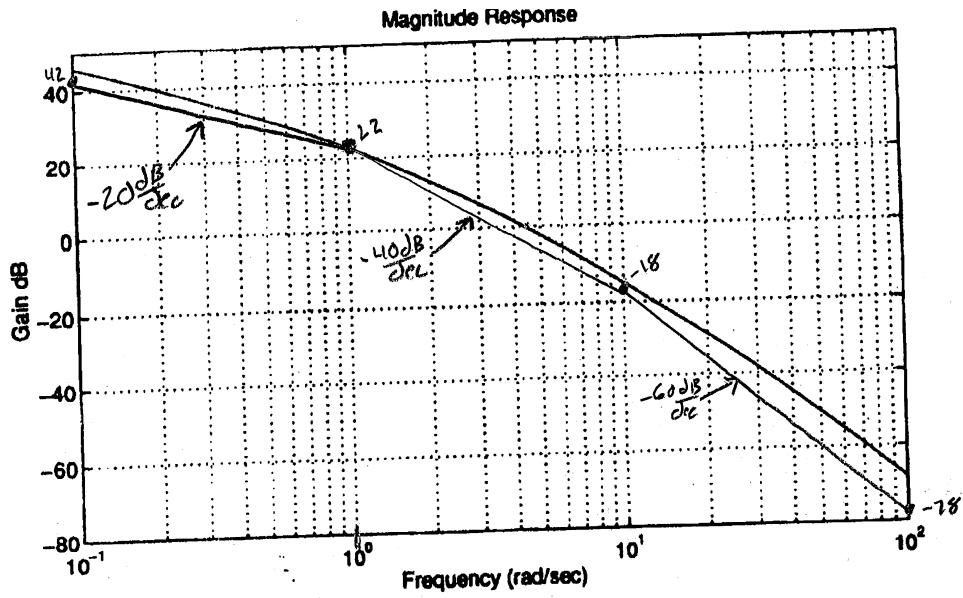
$$\omega_{gc} = 4.5 \text{ rad/s}$$

$$\angle G(j\omega_{gc}) = -180^\circ$$

$$\phi M = 180 - 180 = 0^\circ$$

$$\phi M = \phi M = 0$$

marginally stable



**Remark:**

This problem has no single correct solution. Any solution, backed up with a valid reasoning is acceptable.

## Problem 2:

2. Given the forward loop transfer function for a negative unity-feedback system

$$G(s) = \frac{(s+5)(s+3)}{s(s+1)(s^2+s+4)}$$

- (a) investigate the stability of the system using both Nyquist and Bode in Matlab,  
 (b) find the gain and phase margins from each method and their corresponding crossover frequencies. Display the gain and phase margins on the plots where they occur.

First a SUMMARY. Then supporting plots to follow

2) Nyquist It is easy to verify (for example use Routh) that closed loop is unstable. Hence there are no gain and phase margins.

$$\omega_{pc} = 1.92 \text{ rad/s}$$

$$\zeta = 2.36$$

$$\omega_{gc} = 2.48 \text{ rad/s}$$

$$x = -1.73 \quad y = 1.681$$

$$\sigma M = 20 \log\left(\frac{1}{2.36}\right)$$

$$\sigma M = -7.46 \text{ db}$$

This is not really a gain margin. But it reflects the decibel deficiency in the closed loop

$$\phi M = \tan^{-1}\left(\frac{1.73}{1.681}\right)$$

$$\phi M = -47.0^\circ$$

negative because unstable

Same comment here. THIS is not really a phase margin. It reflects the phase deficiency.

Bode

$$\omega_{pc} = 1.92 \text{ rad/s}$$

$$\omega_{gc} = 2.48 \text{ rad/s}$$

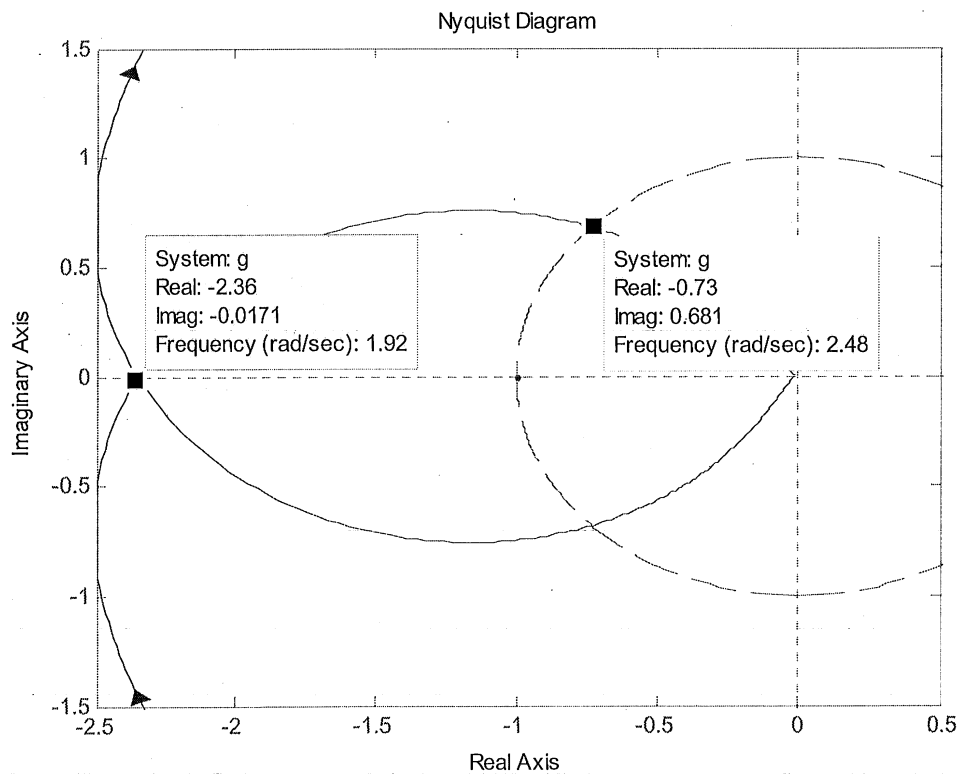
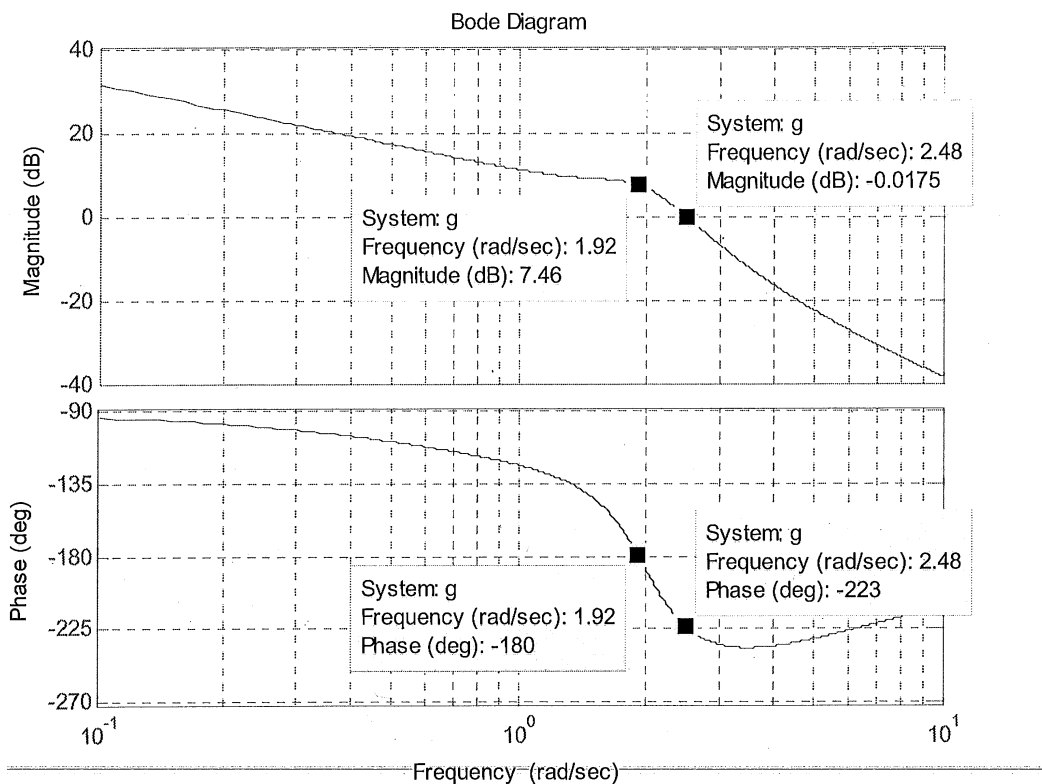
$$\sigma M = 0 - 7.46$$

$$\sigma M = -7.46 \text{ db}$$

$$\phi M = -223^\circ + 180^\circ$$

$$\phi M = -43^\circ$$

Note that Bode and Nyquist lead to the same "phony" gain and phase margins.



### Problem 3:

3. The block diagram shown in Figure 1 represents a model of a hydroelectric alternator, turbine, and penstock with transfer function  $G_1(s)$  being given by:

$$G_1(s) = \frac{25(1 + 5s)}{(s^3 + 49.277s^2 + 25.772s + 2.526)}$$

The parameter values are  $T = 1$ ,  $K = 0.05$ ,  $M = 10$ , and  $D = 1$  with  $L(s) = 0$ .

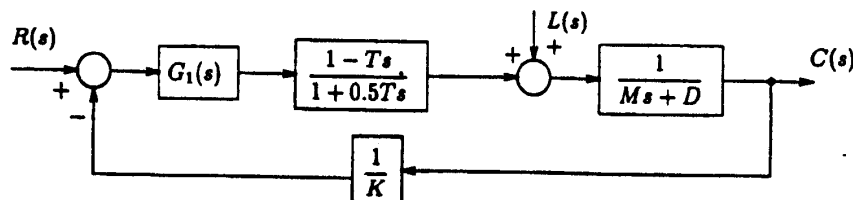


Figure 1: Hydroelectric System Block Diagram

- Using the Bode plots, generated using *Matlab* if you wish, find the maximum value of  $K$  to retain stability.
- Find the gain margin, phase margin and the corresponding crossover frequencies and label them on the Bode plots.

Multiplying all the factors and simplifying, we get,

$G(s)H(s) =$

$$= \frac{25(1+5s)(1-s)}{K(s^3 + 49.277s^2 + 25.772s + 2.526)(1+0.5s)(10s+1)}$$

$$= \frac{-125s^2 + 100s + 25}{K(5s^5 + 256.9s^4 + 647.3s^3 + 332.5s^2 + 52.3s + 2.5)}$$

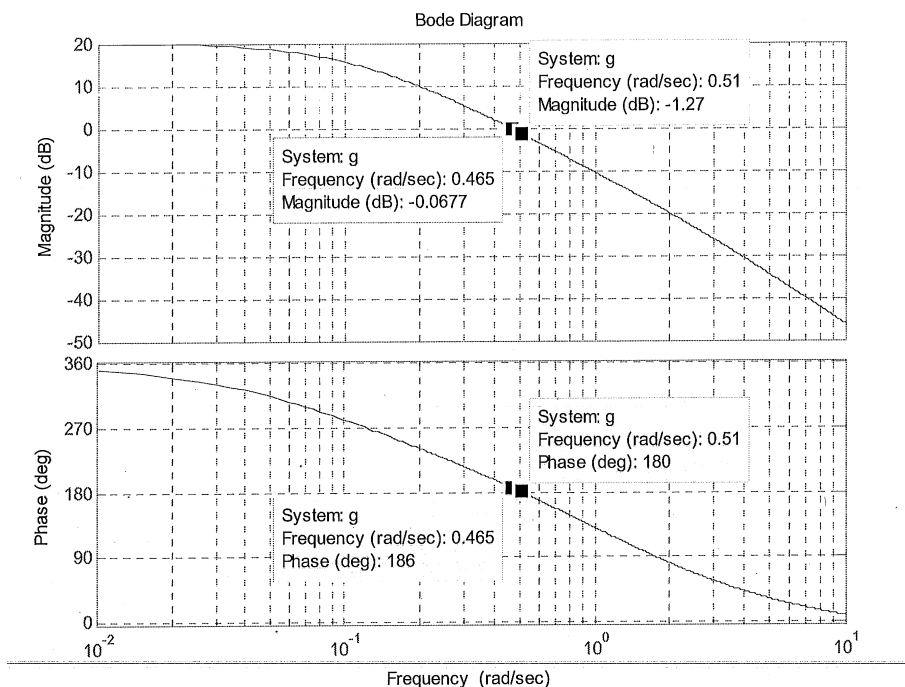
$\omega_{pc} = 0.51 \text{ rad/s}$	$\log g_{a.n} = -1.27 \text{ db}$
$\omega_{gc} = 0.465 \text{ rad/s}$	$\phi = 186^\circ$
$GM = 1.27 \text{ db}$	$\gamma = \frac{1}{K}$
$\phi M = 186 - 180$	
$\phi M = 6^\circ$	



$$20 \log \delta < 1,27$$

$$\delta < 1,15$$

$$K > 0,864$$



Note: The inequality above for K describes the minimal value for stability. Note that the problem statement asks for the maximum value. In fact, arbitrarily large  $K > 0$  leads to stability.

;

**Problem 3: Part (b):**

```
>> G_1 = 25*(1+5*s)/(s^3+49.277*s^2+25.772*s+2.526)
```

```
Transfer function:
```

```
125 s + 25
```

```
-----  
s^3 + 49.28 s^2 + 25.77 s + 2.526
```

```
>> H_1 = (1-s)/(1+.5*s)
```

```
Transfer function:
```

```
-s + 1
```

```
-----  
0.5 s + 1
```

```
>> H_2 = 1/(10*s+1)
```

```
Transfer function:
```

```
1
```

```
-----  
10 s + 1
```

```
>> K_FB = 1/.05
```

```
K_FB =
```

```
-----  
20
```

```
>> trans_HE_ALT = G_1*H_1*H_2*K_FB
```

```
Transfer function:
```

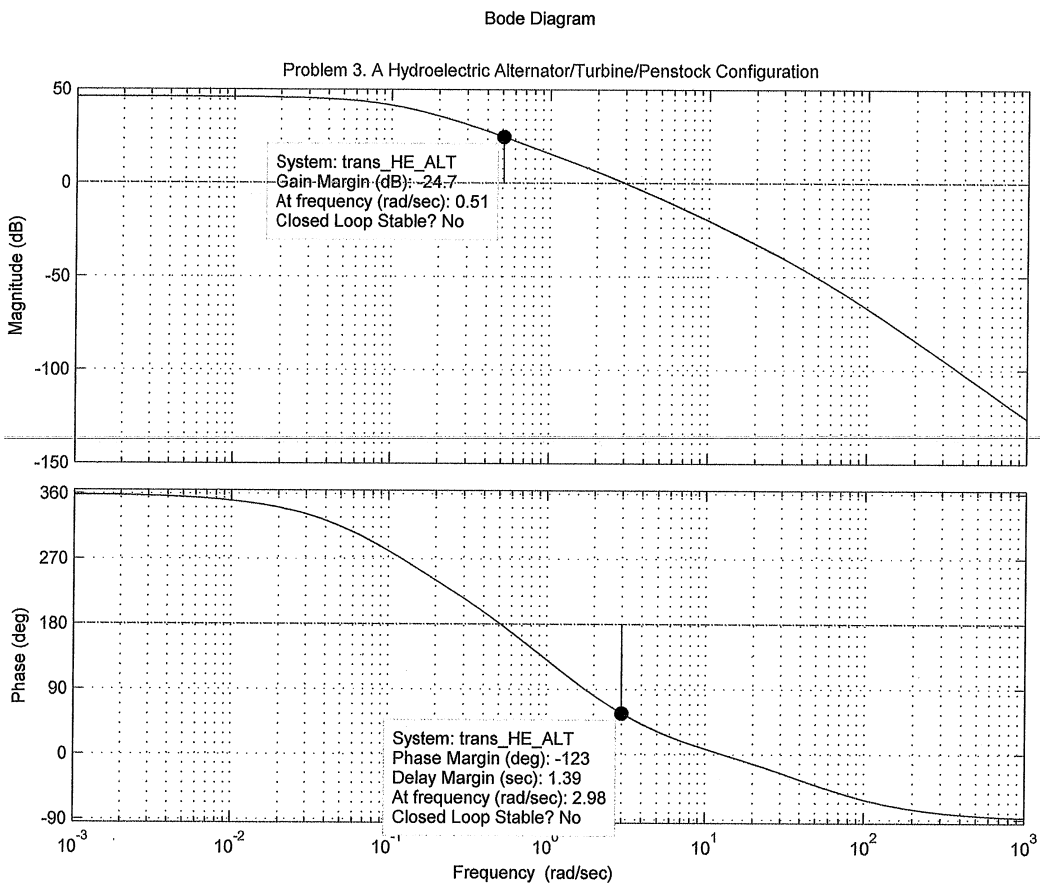
```
-2500 s^2 + 2000 s + 500
```

```
-----  
5 s^5 + 256.9 s^4 + 647.3 s^3 + 332.5 s^2 + 52.29 s + 2.526
```

```
>> bode(trans_HE_ALT)
```

```
>>
```

```
;
```



;

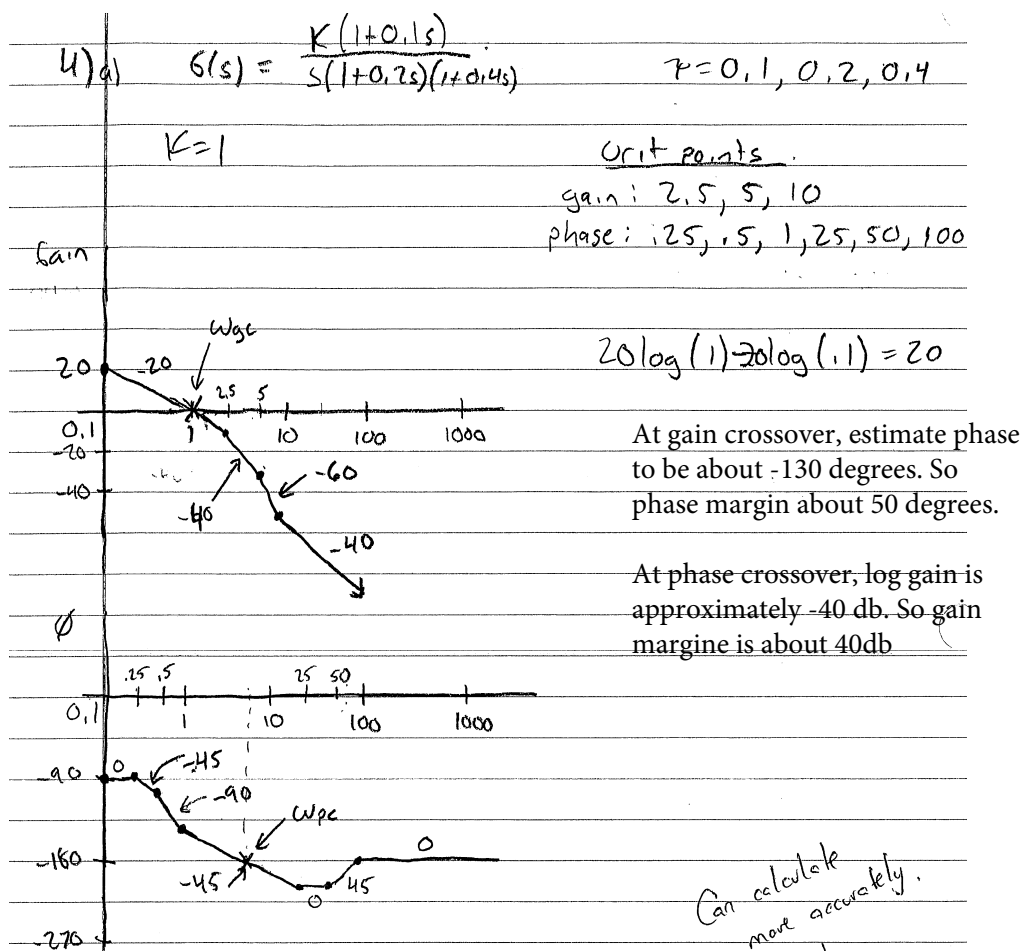
**Problem 4:**

Problem 4: The open loop transfer function of a position control system is

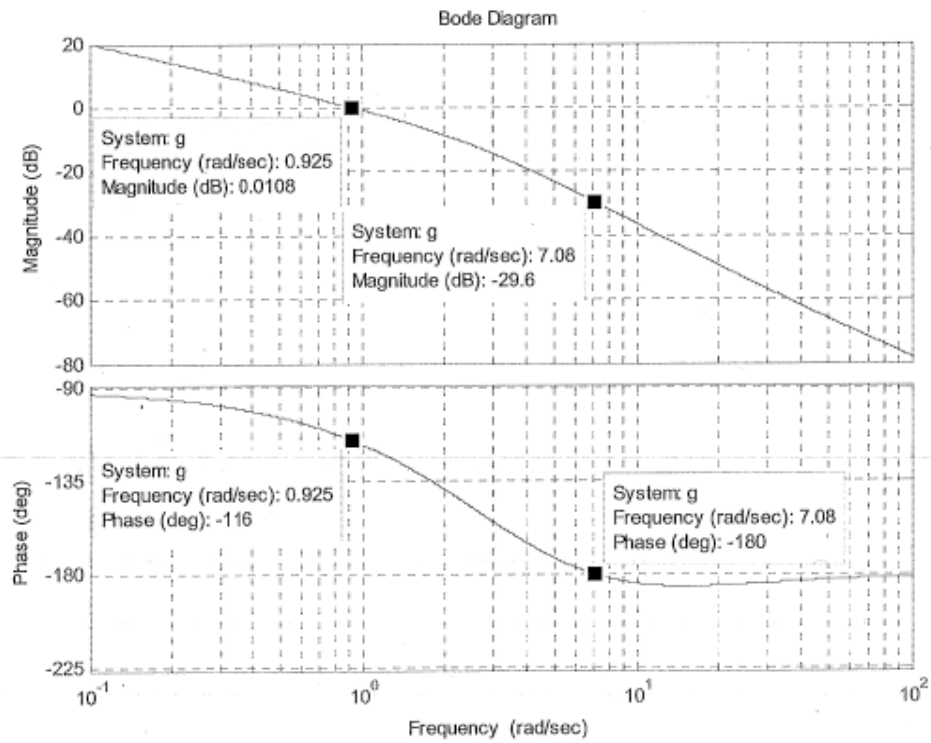
$$G(s) = \frac{K(1 + 0.1s)}{s(1 + 0.2s)(1 + 0.4s)}$$

Assume a negative unity feedback configuration.

- (a) Sketch the Bode diagram for  $K = 1$  (Do Not Use MATLAB). Find the gain margin, gain-crossover frequency, phase margin and the phase-crossover frequency for the system.
- (b) Use MATLAB to check your plot
- (c) Determine the value of  $K$  which will satisfy the following design criteria
  1. A phase margin of  $55^\circ$ .
  2. A gain margin of 26 dB.



**Problem 4: Part (b):**



;

$\omega_{gc} = 0.925 \text{ rad/s}$	$\phi = -116^\circ$
$\omega_{pc} = 7.09 \text{ rad/s}$	mag = -29.6 dB
$GM = 29.6 \text{ dB} = 30.2$	/
$\phi_M = 64^\circ$	

**Problem 4: Part (c):**

From the above plot, we see that  $K=1$  satisfies the required criterion.

To find value of  $K$  that satisfies the criterion exactly, we can proceed as follows – More precisely, we obtain a  $K$  which approximately meets both specs.

$GM = 26 \text{ dB}$	$\phi_M = 55^\circ \therefore$
$29.6 - 26 = 3.6 \text{ dB}$	$\phi = -125^\circ$ at $\omega_{gc}$
$20 \log K = 3.6 \text{ dB}$	from plot
$K = 1.511$	$\omega_{gc} = 1.32 \text{ rad/s}$
	for this to occur plot
	must be shifted up
	3.66 dB
at a gain $K = 1.52$	$20 \log K = 3.66$
	$K = 1.52$
$GM = 26 \text{ dB} + \phi_M = 55^\circ$	$K = 1.52$

;

# Chapter 4

## Exams

### Local contents

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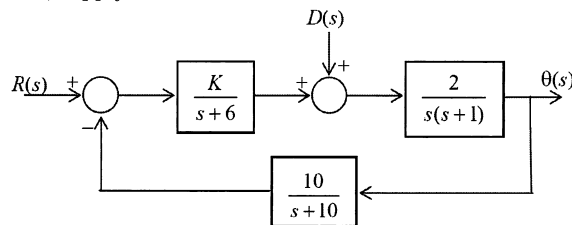
## 4.1 First exam

### 4.1.1 questions

ECE 332 - Test #1 September 29, 2015

**INSTRUCTIONS:** Closed notes and closed book. Be sure your name is on the exam booklet(s). Answer all questions showing your work.

**QUESTION 1 (25%):** (a) For the system with block diagram below, with  $R(s)$  and  $D(s)$  corresponding to unit steps, find  $\theta(s)$  as a quotient of 2 polynomials. (b) With  $K = 1$ , supply a few lines of Matlab code to find the closed loop poles.



**QUESTION 2 (25%):** A tank with its transfer function  $G(s) = \frac{5}{s^2+5s+6}$  is connected with a PID controller  $H(s) = K_1 + \frac{K_2}{s} + K_3s$  in a classical unity feedback configuration. (a) Find gains  $K_1, K_2$  and  $K_3$  so that the resulting closed loop transfer function is  $T(s) = \frac{5}{s+5}$ . (b) For this  $T(s)$ , find the step response  $y(t)$ .

**QUESTION 3 (25%):** The open loop system with input  $u$ , output  $y$  and parameter  $\zeta$  described by differential equation  $(50 + 10\zeta)\frac{dy}{dt} + (100 + 10\zeta)y = 3u(t)$ . (a) Find the open loop transfer function. (b) For this open loop system with a unit step input, is there a choice of parameter  $\zeta$  leading to  $\lim_{t \rightarrow \infty} y(t) = 1$ ? Explain.

**QUESTION 4 (25%):** A single-input two-output system with transfer function matrix  $G(s)$  is connected in a classical unity feedback configuration with controller having  $1 \times 2$  transfer function matrix  $H(s)$ . (a) Given the four matrix entries  $G_{11}(s) = 1/s, G_{21}(s) = 1, H_{11}(s) = 0, H_{12}(s) = 2$ , find  $y_1(t)$  with inputs  $r_1(t) = 0$  and  $r_2(t)$  being a unit step. (b) Write a Matlab code using symbolic computation to find the closed loop transfer function matrix.



4.1.2 key solution

ECE 332 - Test # 1 - Solutions

Barmish 2015

Q1) By signal addition  $\Theta(s) = \frac{2K}{s(s+1)(s+6)} \cdot \frac{1}{s} + \frac{2}{s(s+1)} \cdot \frac{1}{s}$

algebra  $\frac{2K(s+10) + 2(s+6)(s+10)}{s[s(s+1)(s+6)(s+10) + 20K]} = \frac{2K(s+10) + 2(s+6)(s+10)}{s^5 + 17s^4 + 76s^3 + 60s^2 + 20Ks}$

(b)  $p = [1 \ 17 \ 76 \ 60 \ 20 \ 0]$   
roots(p)

Q2) Calculate  $T(s) = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{\left(\frac{5}{s^2+5s+6}\right) / (K_1+K_2+K_3s)}{1 + \left(\frac{5}{s^2+5s+6}\right) (K_1+K_2+K_3s)}$

$= \frac{5}{(s+3)(s+2)} \frac{K_3s^2 + K_1s + K_2}{s}$  Choose  $K_3s^2 + K_1s + K_2 = (s+3)(s+2) = s^2 + 5s + 6$

$K_3=1, K_2=5, K_1=6 \Rightarrow T(s) = \frac{5}{s+5}$  (b)  $Y(s) = \left(\frac{5}{s+5}\right) \frac{1}{s} = \frac{1}{s} - \frac{1}{s+5}$  Partial frac

$\mathcal{L}^{-1}$  gives  $y(t) = 1 - e^{-5t}$

Q3)  $\mathcal{L} \Rightarrow (50+10s)Y(s) + (100+10s)Y(s) = 3U(s)$

$\frac{Y(s)}{U(s)} = \frac{3}{(50+10s)s + 100 + 10s}$  By FVT, assuming temporarily, it applies  $\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$

$= \frac{3}{100+10s}$ . Hence iff  $\xi$  exists, it must satisfy  $\frac{3}{100+10s} = 1 \Rightarrow s = -9.7$

However, when  $s = -9.7$ , denom =  $-47s + 3 \Rightarrow s = 3/47$  UNSTABLE!

So NO  $\xi$  exists! FVT does not apply for unstable case!

Q4)  $T(s) = [I + G(s)H(s)]^{-1} G(s)H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1/5 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1/5 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2/5 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2/5 \\ 0 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & -2/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2/5 \\ 0 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 & 2/5 \\ 0 & 2 \end{bmatrix}$  (b) sym s  $G = [1/5 \ 1]$   
 $H = [0 \ 2]$   
 $T = \text{inv}(eye(2) + GH)$

$Y(s) = \frac{1}{3} \begin{bmatrix} 0 & 2/5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/15 \\ 2/3 \end{bmatrix}$  So  $y_1(t) = \mathcal{L}^{-1} \frac{2}{s^2} = 2t$

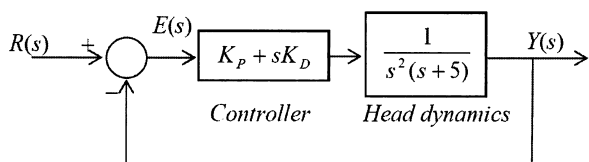
## 4.2 Second exam

### 4.2.1 questions

ECE 332 - Test #2 November 12, 2015

**INSTRUCTIONS:** Closed notes and closed book. Be sure your name is on the exam booklet(s). Answer all questions showing your work.

**QUESTION 1** (25%): For the disk storage data-head positioning system depicted in the block diagram below, find the conditions on  $K_P$  and  $K_D$  for which all the closed loop poles have real part less than  $\sigma = -1.0$ . Depict the stability region in the  $(K_P, K_D)$  plane.



**QUESTION 2** (25%): A system with transfer function  $G(s) = 1/(s + 0.5)$  is connected in a standard unity feedback configuration with controller  $H(s) = K/s$ . Find the sensitivity of the closed loop transfer function with respect to variations in the controller gain about  $K = 1$ . Plot the magnitude of this function and determine the frequency at which the sensitivity is a maximum.

**QUESTION 3** (25%): Consider the second order system with transfer function

$$G(s) = \frac{\alpha^2}{s^2 + 2\alpha^{5/2}s}$$

and pure gain controller controller  $H(s) = \beta$  connected in unity feedback configuration. Select the tunable parameter  $\beta$  such that the undamped natural frequency of the closed loop is  $\omega_n = \sqrt{\alpha}$ . Then, given a unit step input, determine the value of the tunable parameter  $\alpha \geq 0$  leading to maximum percentage overshoot which is approximately 10%. Formula:  $OS_{max} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$

**QUESTION 4** (25%): A nonlinear device with input-output relationship given by  $Y = 2|X|$  if  $|X| \leq 2$  and  $Y = 4$  for  $|X| > 2$  is connected in a classical unity feedback configuration with pure gain controller  $H(s) = 10$ . (a) Generate a sketch for the input-output relationship  $Y$  versus  $R$ . (b) For the resulting closed loop system and find and sketch the output  $y(t)$  when the command input is given by  $r(t) = 8 \sin t$ .

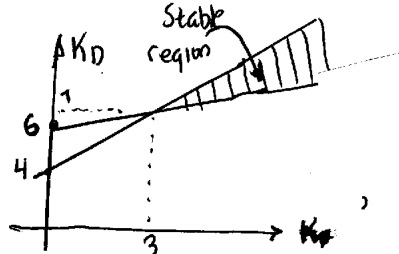
4.2.2 key solution

ECE 332 - Test # 2 - Solutions

Barmish  
2015

Q1)  $1 + \frac{K_p + 5K_D}{s^2(s+5)} = 0 \Rightarrow$  denominator of closed loop  $D(s) = s^3 + 5s^2 + K_D s + K_p$   
 To assure roots  $\leq -\sigma$  substitute  $s \rightarrow s - \sigma; \sigma = 1 \Rightarrow (s-1)^3 + 5(s-1)^2 + K_D(s-1) + K_p$   
 Consider POLY =  $s^3 + 2s^2 + (K_D - 7)s + 4 - K_D + K_p$   
 Routh Array

$s^3$	1	$K_D - 7$	} For stability $3K_D - K_p - 18 > 0$ $4 - K_D + K_p > 0 \Rightarrow K_D > \frac{1}{3}K_p + 6$ $K_D < 4 + K_p$
$s^2$	2	$4 - K_D + K_p$	
$s$	$\frac{3K_D - K_p - 18}{2}$		
$s^0$	$4 - K_D + K_p$		



Q2

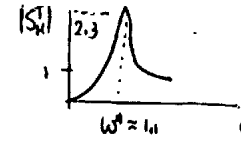
$$T(s) = \frac{1}{s+5} \frac{K}{s} = \frac{2K}{2s^2 + s + 2K}; \quad \frac{dT}{dK} = \frac{2(2s^2 + s)}{(2s^2 + s + 2K)^2}$$

$$S_K^T = \frac{K}{T} \frac{dT}{dK} = \frac{2s^2 + 2s + K}{2K} \cdot \frac{K - 2(2s^2 + s)}{(2s^2 + s + 2K)^2} = \frac{s(2s+1)}{2s^2 + s + 2K}$$

So  $S_K^T|_{K=1} = \frac{s(2s+1)}{2s^2 + s + 2}$

$$|S_K^T(j\omega)| = \frac{\omega \sqrt{1+4\omega^2}}{\sqrt{(2-2\omega^2)^2 + \omega^2}}$$

To plot look @  $\omega=0, \omega \rightarrow \infty$  and a few points



Q3

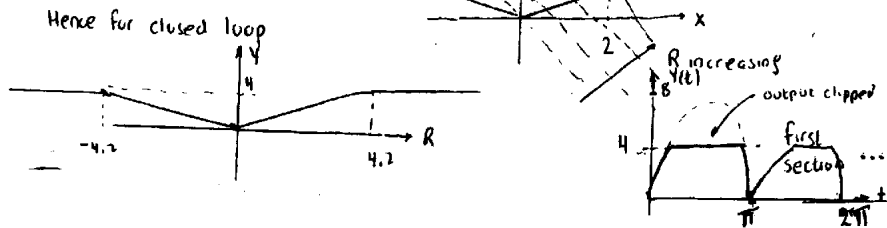
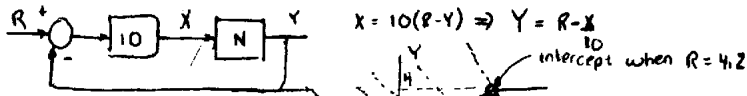
$$\frac{Y(s)}{R(s)} = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{\alpha^2 \beta}{s^2 + 2\alpha^{5/2}s + \alpha^2 \beta}$$

$\omega_n^2 = \alpha^2 \beta = \alpha \left[ \beta = \frac{1}{\alpha} \right]$

Now  $T(s) = \frac{\alpha}{s^2 + 2\alpha^{5/2}s + \alpha}$   $2\zeta\omega_n = 2\alpha^{5/2} = 2\alpha^{5/2} \Rightarrow \boxed{\zeta = \alpha^2}$

Now equate  $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$  Take logs  $-\frac{\pi\alpha^2}{\sqrt{1-\alpha^4}} = -2.3 \Rightarrow \alpha = \left[ \frac{5.29}{\pi^2 + 5.29} \right]^{1/4} = \underline{0.77}$

Q4



## 4.3 Final exam

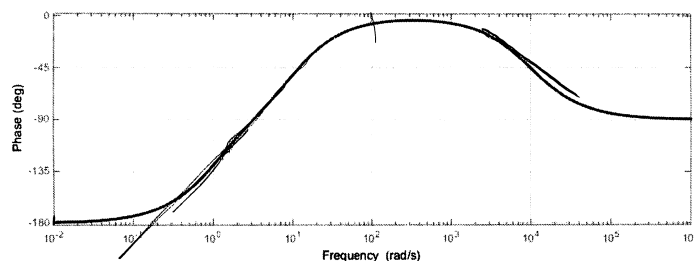
### 4.3.1 questions

ECE 332 - Test #3 December 10, 2015

**INSTRUCTIONS:** Closed notes and closed book. Be sure your name is on the exam booklet(s). Answer all questions showing your work.

**Question 1** (25%): For the open loop transfer function  $G(s) = (s+p)/(s^2+0.25ps)$  connected in a classical unity feedback configuration, find and sketch the Root Locus corresponding to variations  $p \geq 0$ .

**Question 2** (25%): A transfer function  $G(s) = K(s+a)/((s+b)(s+c))$  with one real zero and two real poles has log gain of -20db at  $\omega = 0$  and phase plot shown below. Estimate the transfer function parameters  $a, b, c$  and  $K$ .



**Question 3** (25%): Answer TRUE or FALSE and provide a brief explanation.

(a) When loop transfer function  $G(s) = K(s^5+18s^4+13s^3+s^2+s+1)^2/(s^2+8s+2)^7$  is connected in a classical unity feedback configuration, the resulting closed loop system will be stable when  $K > 0$  is suitably large.

(b) For open loop transfer function  $G(s)H(s) = K(s+5)^5/(s-10)^5$ , for  $K > 0$  suitably large, the Nyquist plot will have five net clockwise encirclements of the critical point  $s = -1$ .

(c) When the transfer function  $G(s) = 1/(s^3 + 3s^2 + 5s + 5)$  is connected in a classical unity feedback configuration, the resulting closed loop system will have a gain margin of 20 decibels.

**Question 4** (25%): For the system  $G(s) = (1-s)/(s^2+3s+1)$  connected in a unity feedback configuration, generate a Nyquist plot and estimate the phase margin.

4.3.2 key solution

Permish  
2015

ECE 332 - Test #3 - Solutions

Q1) Need fictitious system:  $D(s) = s^2 + 0.25ps + s + p$

So  $D(s) = 0$  gives  $1 + p \frac{(0.25s+1)}{s(s+1)}$  RL for this. Apply lemmas

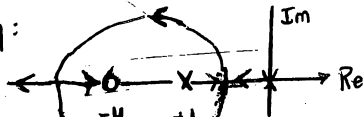
2 branches,  
 $K=0$ , RL @  $0-1$ ,  $K=\infty$  RL @  $-4$

Only  $180^\circ$  asymptotes  
Centroid, arrivals departure  
no issue. Real axis from  
 $0$  to  $-1$  and less than  $-4$

Breakaway:

$\frac{dK}{ds} = 0 \Rightarrow$

$(2.5s+1)(2s+1) = (s^2+s)(.25) \Rightarrow s = -7.46, -0.53$



By Routh, no imag axis crossing. Stable for  $K$

Q2) Using decades below and above for phase, estimate the breakpoints  $\frac{1}{\tau_1} = 1$ ,  $\frac{1}{\tau_2} = 100$ ,  $\frac{1}{\tau_3} = 10,000$

must be a zero LHP

must be a RHP pole

must be a LHP pole

These are only way phase variations are consistent

Get estimate  $G(s) = K \frac{(1+s)}{(s+100)(s+10000)}$

$20 \log K = -20$   
 $K = 0.1$

$(1 + \frac{1}{100}s) (1 + \frac{1}{10,000}s) \rightarrow = 10^5 (s+1)$

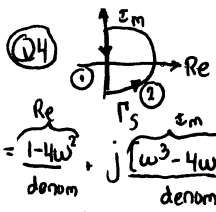
Q3) (a) FALSE,  $n-m = 4$

implies RHP asymptote. Unstable for  $K$  large

$K = -10^5, a=1, b=-100, c=10,000$

(b) TRUE. For  $K$  large RL goes to  $-5$ . No asymptotes. So stable system must be reflected in Nyquist encirclement count. (c) TRUE

$\left. \begin{matrix} s^3 + 5s^2 + 5s + K \\ s^2(10-K)s + 5s + K \\ s^0 5 + K \end{matrix} \right\} \begin{matrix} K < 10 \\ \text{for stability} \\ GM = 20 \log 10 \\ = 20 \text{ db} \end{matrix}$

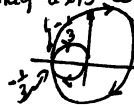


Along  $\odot$ ,  $s = j\omega$   $G(j\omega) = \frac{(1-j\omega)}{(1-\omega^2) + 3j\omega}$

Along  $\odot$  begin with  $\frac{1}{\omega} > 0$ ,  $\text{Im} > 0$  when  $\omega$  large  $\Rightarrow$  in  $\odot 3$

with  $|G| = \text{small}$ . Crossings Real axis  $\omega = 2$  Imag axis  $\omega = \frac{1}{2}$  When  $\omega = \frac{1}{2}$ ,

Calculate real crossing  $G(j2) = \frac{1-j^2}{-3+j6} = -\frac{1}{3}$



$GM = 20 \log (\frac{1}{3})$   
 $\phi M = \infty \approx 9.5 \text{ db}$