HW6 ECE 332 Feedback Control

Fall 2015
Electrical engineering department
University of Wisconsin, Madison

Instructor: Professor B Ross Barmish

By

Nasser M. Abbasi

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0.1 Problem 1

Problem 1

The open-loop transfer functions of a unity negative feedback control system are given as follows:

(a) $G(s) = \frac{K}{s(s+2)(s+5)(s+10)}$

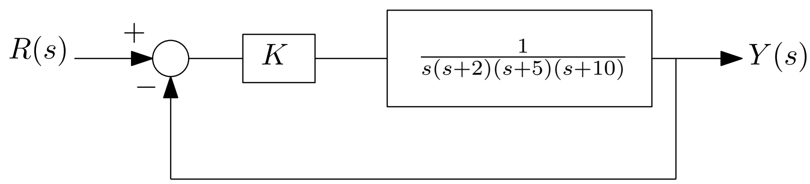
(b) $G(s) = \frac{K}{(s^2+s+2)(s+1)}$

(c) $G(s) = \frac{K(s^2+2s+10)}{s(s+5)(s+10)}$

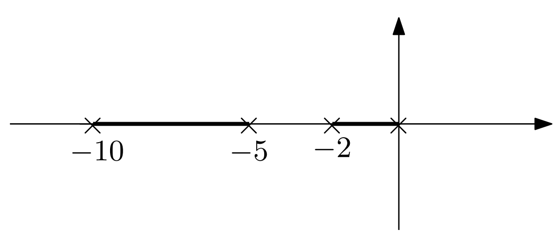
For each of these, plot the root locus for $0 \leq K \leq +\infty$ **without** using any computer

SOLUTION:

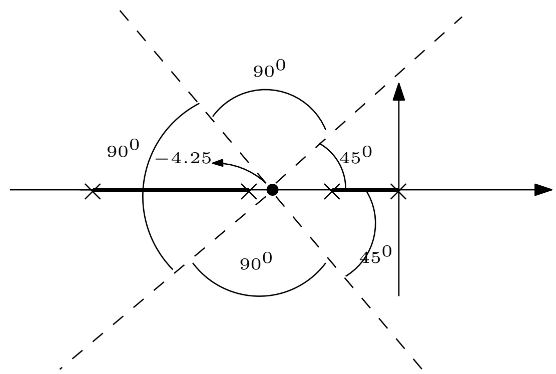
0.1.1 Part(a)



1. $n = 4$ (The number of open loop poles), $m = 0$ (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles $s = 0, s = -2, x = -5, s - 10$ when $k = 0$. Since there are no open loop zeros, the branches will end up at $\pm\infty$
3. On the real axis, R.L. exists on segments based on number of poles and zeros to the right of the segment. If this sum (including multiplicity) is odd, then the segment is on the R.L., else it is not. The plot below shows the segments found.



4. Center of asymptotes is now found using $\sigma = \frac{\sum poles - \sum zeros}{n-m} = \frac{(0-2-5-10)}{4} = -4.25$
5. The asymptotes angles are $\theta = \frac{180^0 \pm k360^0}{n-m}$ for $k = 0, 1, \cdots$. Hence $\theta = \frac{180^0 \pm k360^0}{4} = 45^0 \pm k90^0$. The R.L. is updated below



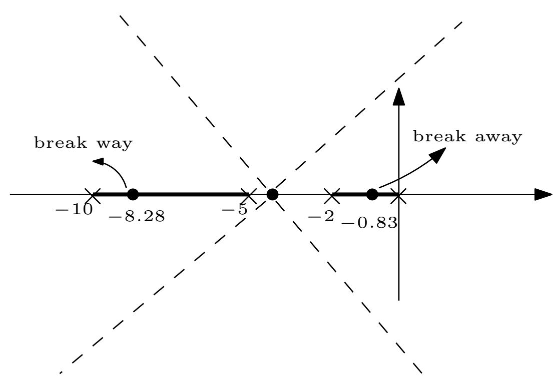
6. Finding the break away points. We need to find where on the two segments shown above the root locus will break away from the real line. Since

$$
\begin{aligned}
1+G &= 0\\
1+\frac{K}{s(s+2)(s+5)(s+10)} &= 0\\
K &= -s(s+2)(s+5)(s+10)\\
K &= -s^4-17s^3-80s^2-100s
\end{aligned}
$$

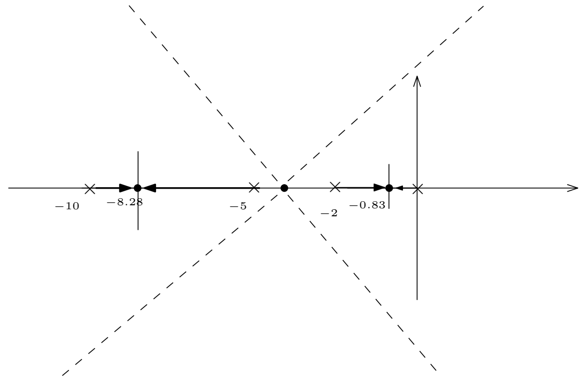
Then we now solve for s

$$
\begin{aligned}
\frac{dK}{ds} &= -4s^3-51s^2-160s-100=0\\
0 &= 4s^3+51s^2+160s+100
\end{aligned}
$$

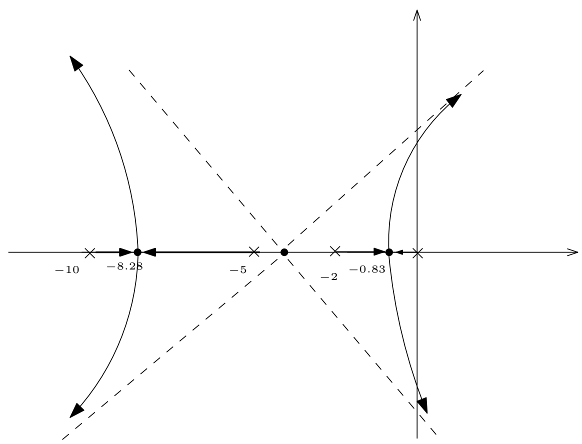
The roots are $s=-8.287, s=-0.835, s=-3.632$. Not all these points will be break away points. Looking at the segments on the real line, we see that $s=-0.835$ and $s=-8.287$ are on the R.L. but $s=-3.632$ is not. We mark these points now on the current plot and update the plot again



7. Departure angles from poles. It is clear that R.L. depart from open loop poles on the real axis as shown, moving towards the break away points. Hence the plot now looks like the following



8. We now extend the branch from the break away points towards the asymptotes, since there are no zeros, no need to calculate arrival angles. The branch will move to the asymptotes. So we end up with the following



   It is important to remember that root locus will always be symmetric with respect to the real axis.

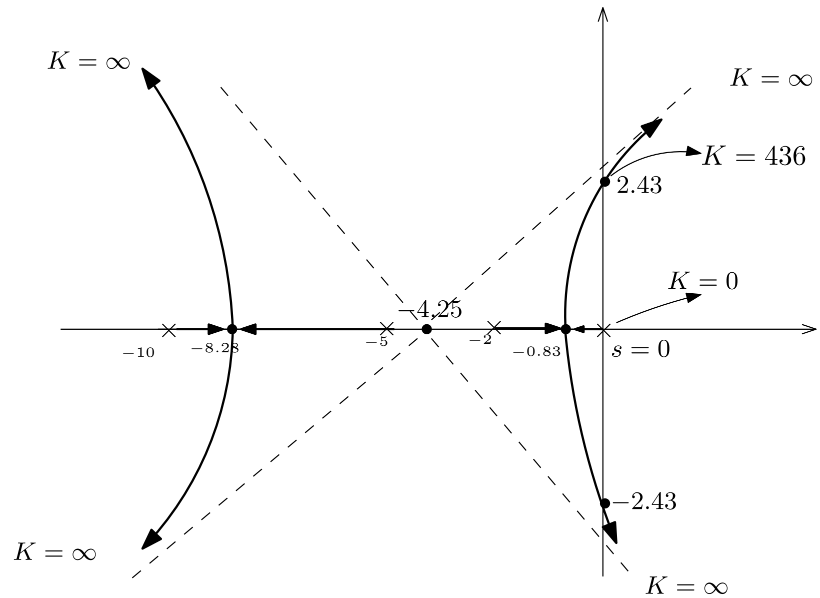
9. Now the only lemma left is to find where R.L. crosses the imaginary axis. For this we use Routh stability stable as follows. Since the characteristic polynomial is $K + s(s+2)(s+5)(s+10) = 0$ then

$$P(s) = s^4 + 17s^3 + 80s^2 + 100s + K$$

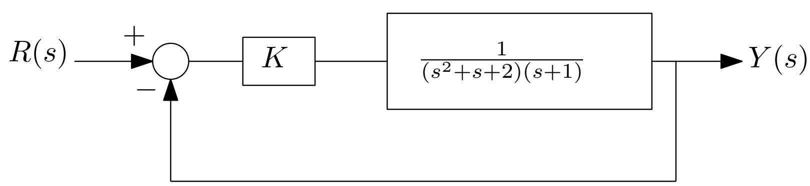
s^4	1	80	K
s^3	17	100	
s^2	74.118	K	
s^1	$100 - 0.229K$		
s^0	K		

   We just need to find where it cross the imaginary axis. Setting the s^1 row to zero gives $K = 436.68$. This means the closed loop will become unstable for $k > 436.68$.

To find where R.L. crosses the imaginary axes, we go back to the even polynomial s^2 (the row above the line s^1) in above Routh table, which yields $74.118s^2 + K = 0$ or $74.118s^2 + 436.68 = 0$, therefore $s = \pm 2.43i$. This complete R.L. Here is the final plot

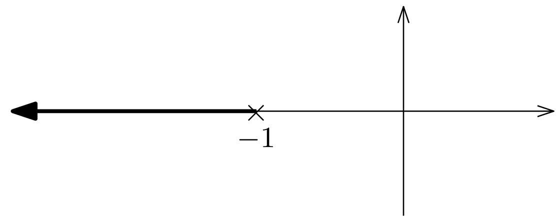


0.1.2 Part(b)



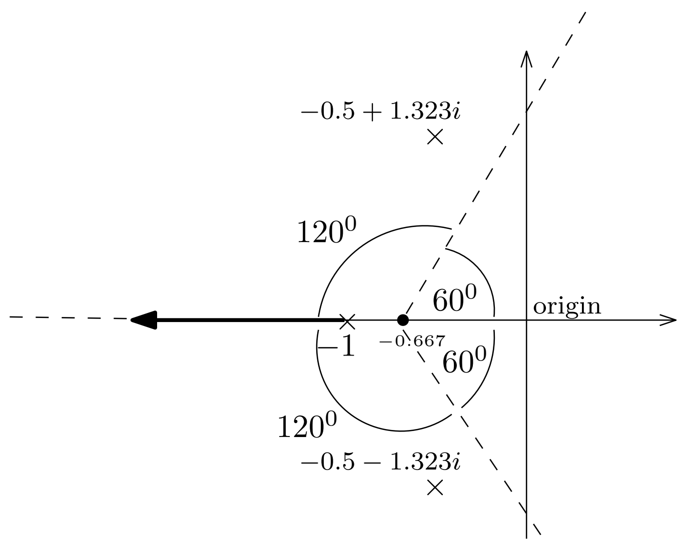
$$G(s) = \frac{K}{\left(s^2 + s + 2\right)(s + 1)}$$

1. $n = 3$ (The number of open loop poles), $m = 0$ (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles $s = -1, s = -0.5 \pm 1.323i$ when $k = 0$. Since there are no open loop zeros, the branches end up at $\pm\infty$
3. On the real axis, R.L., since one pole on the real axis, then the R.L. segment is to the left of the only pole, which is $s = -1$



4. Center of asymptotes is now found. $\sigma = \frac{\sum poles - \sum zeros}{n-m} = \frac{(-1-0.5-0.5)}{3} = -0.667$

5. The asymptotes angles are $\theta = \frac{180^0 \pm k360^0}{n-m}$ for $k = 0, 1, \cdots$. Hence $\theta = \frac{180^0 \pm k360^0}{3} = 60^0 \pm k120^0$. Therefore the R.L. now looks like the following



6. Break away points. There is no break away points, since there are no segments between two poles on the real line. We can see this also if we try to solve $\frac{dK}{ds} = 0$ as follows

$$1 + G = 0$$

$$1 + \frac{K}{\left(s^2 + s + 2\right)(s + 1)} = 0$$

$$K = \left(s^2 + s + 2\right)(s + 1)$$

$$K = s^3 + 2s^2 + 3s + 2$$

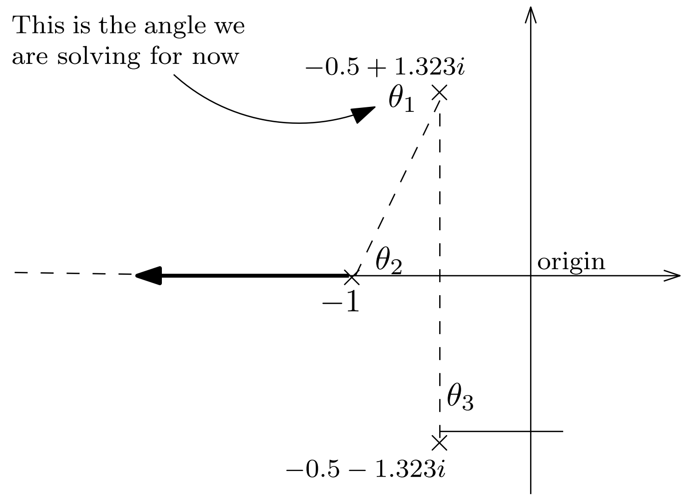
Then we now solve for

$$\frac{dK}{ds} = 3s^2 + 4s + 3 = 0$$

The solution gives only complex roots. So we go to the next lemma.

7. Departure angles from poles. It is clear that R.L. depart from open loop pole on the

real axis $s = -1$ as shown moving to $-\infty$ along one of the asymptotes. But we need to find departure angles from the two complex poles $s = -0.5 \pm 1.323i$. For $s = -0.5 + 1.323i$, let this unknown angle be θ_1 and we write

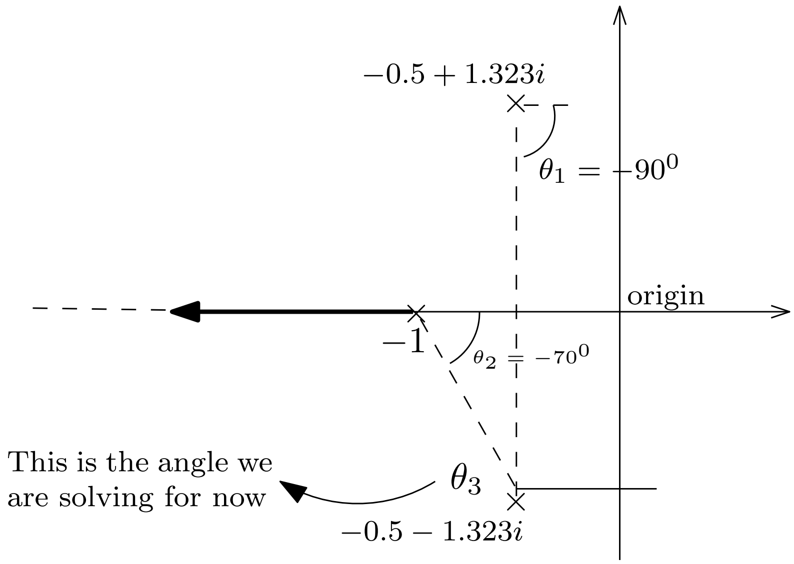


$$
\begin{aligned}
\sum_{zeros} \sphericalangle z_i - \sum_{poles} \sphericalangle p_i &= 180^0 + k360^0 \qquad k = 0, 1, \cdots \\
-(\theta_1 + \theta_2 + \theta_3) &= 180^0 + k360^0 \\
\theta_1 &= -180^0 - \theta_2 - \theta_3 - k360^0
\end{aligned}
$$

We see from the diagram that $\theta_3 = 90^0$ and $\theta_2 = \tan^{-1}\left(\frac{1.323}{0.5}\right) = 70^0$, hence from above

$$
\begin{aligned}
\theta_1 &= -180^0 - 70^0 - 90^0 - k360^0 \\
&= -340^0 \\
&= 20^0
\end{aligned}
$$

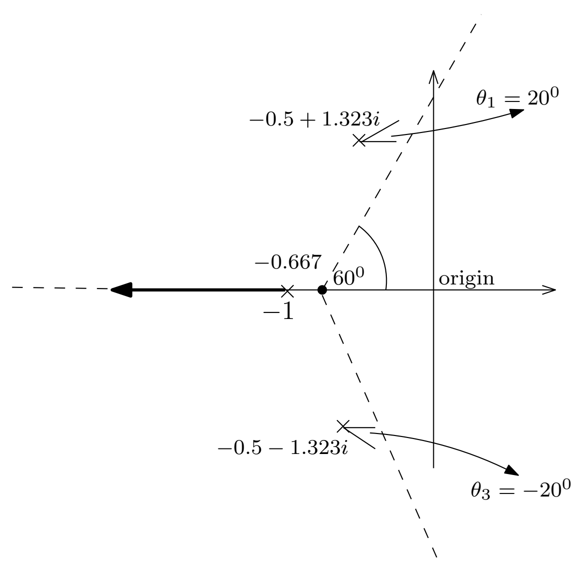
By symmetry (R.L. is symmetric with respect to the real axis), the departure angle for the other complex pole $s = -0.5 - 1.323i$ must be -20^0. We can calculate it to make sure it is indeed -20^0 as follows.



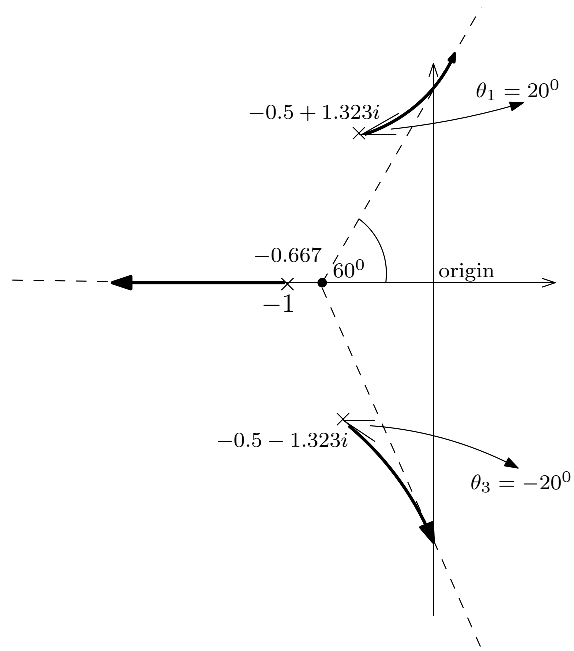
Let θ_3 be the departure angle for the lower complex pole $s = -0.5 - 1.323i$, then

$$\begin{aligned}\sum_{zeros} \sphericalangle z_i - \sum_{poles} \sphericalangle p_i &= 180^0 + k360^0 \qquad k = 0, 1, \cdots \\ -(\theta_1 + \theta_2 + \theta_3) &= 180^0 + k360^0 \\ \theta_3 &= -180^0 - \theta_2 - \theta_1 - k360^0\end{aligned}$$

But $\theta_2 = -70^0$ and $\theta_1 = -90^0$ then $\theta_3 = -180^0 + 70 + 90 = -20^0$ as expected. The root locus plot is now updated



8. We now extend R.L. to the asymptotes, since there are no zeros, there is no need to calculate arrival angles and we end up with the following

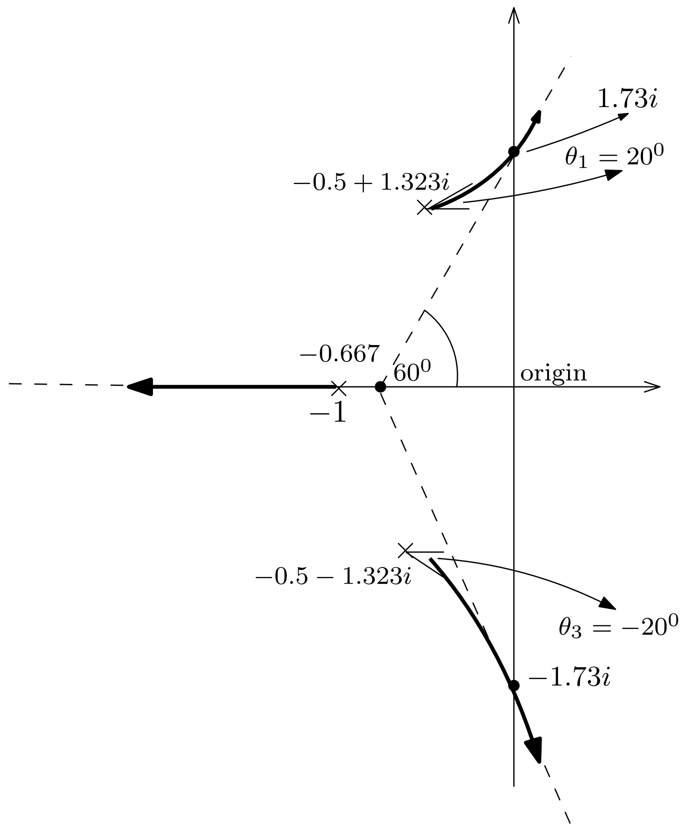


9. Now the only lemma left is to find where R.L. crosses the imaginary axis. For this we use Routh stability stable as follows. Since the characteristic polynomial is $K + \left(s^2 + s + 2\right)(s + 1) = 0$ then

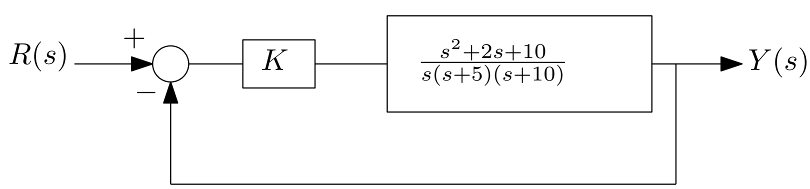
$$P(s) = s^3 + 2s^2 + 3s + K + 2$$

s^3	1	3
s^2	2	$K + 2$
s^1	$\frac{4-k}{2}$	
s^0	$K + 2$	

We need $K = 4$ (by setting the s^1 row to zero). This means system will be unstable for $k > 4$. To find where R.L. crosses the imaginary axes, we go back to the s^2 polynomial above the line s^1 in Routh table, which yields $2s^2 + K + 2 = 0$ or $2s^2 + 6 = 0$, therefore $s = \pm 1.73i$. This complete R.L. Here is the final plot

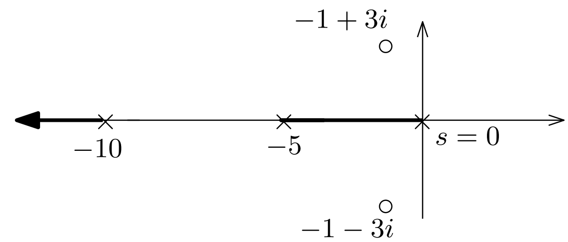


0.1.3 Part(c)



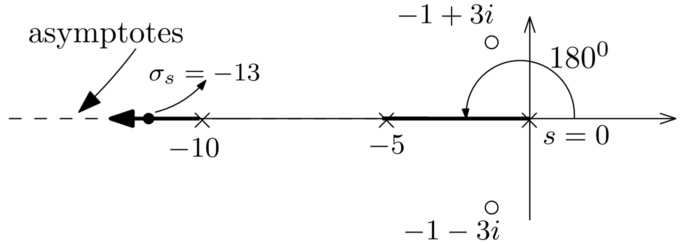
$$G(s) = \frac{K\left(s^2 + 2s + 10\right)}{s\,(s+5)\,(s+10)}$$

1. $n = 3$ (The number of open loop poles), $m = 2$ (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles $s = 0, s = -5, s = -10$ when $k = 0$ and two of the branches end up at zeros $s = -1 \pm 3i$. One branch will end up at $\pm\infty$.
3. On the real axis, there is R.L. segment between $s = 0, s = -5$ and another segment to the left of $s = -10$. No R.L. segment exist between $s = -5$ and $s = -10$. Current diagram now looks like the following



4. Center of asymptotes is now found. $\sigma = \frac{\sum poles - \sum zeros}{n-m} = \frac{-15-(-1-1)}{1} = -13$

5. The asymptotes angles are $\theta = \frac{180^0 \pm k360^0}{n-m}$ for $k = 0, 1, \cdots$. Hence $\theta = \frac{180^0 \pm k360^0}{1} = 180^0 \pm k360^0$. Therefore the R.L. now looks like the following



6. Finding the break away points

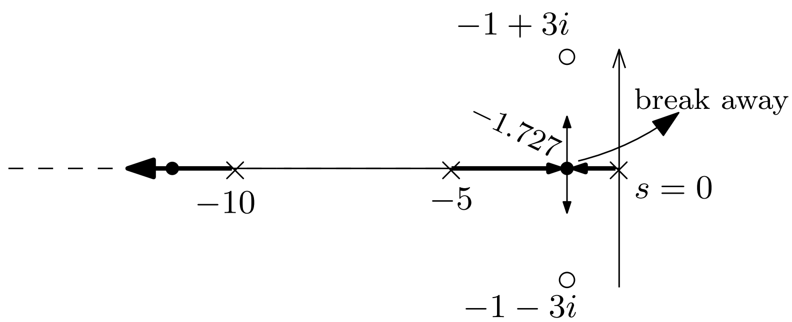
$$
\begin{aligned}
1 + G &= 0 \\
1 + \frac{K\left(s^2+2s+10\right)}{s\,(s+5)\,(s+10)} &= 0 \\
\frac{s\,(s+5)\,(s+10) + K\left(s^2+2s+10\right)}{s\,(s+5)\,(s+10)} &= 0 \\
s\,(s+5)\,(s+10) + K\left(s^2+2s+10\right) &= 0 \\
K &= \frac{-s\,(s+5)\,(s+10)}{\left(s^2+2s+10\right)} \\
&= \frac{-s^3-15s^2-50s}{s^2+2s+10}
\end{aligned}
$$

We now solve for $\frac{dK}{ds} = 0$

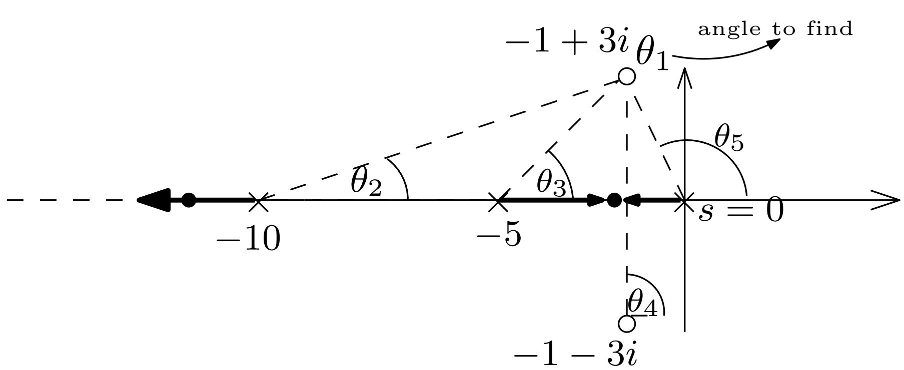
$$
\begin{aligned}
\frac{dK}{ds} &= 0 \\
\frac{d}{ds}\left(\frac{-s^3 - 15s^2 - 50s}{s^2 + 2s + 10}\right) &= 0 \\
\frac{s^4 + 4s^3 + 10s^2 + 300s + 500}{\left(s^2 + 2s + 10\right)^2} &= 0 \\
s^4 + 4s^3 + 10s^2 + 300s + 500 &= 0
\end{aligned}
$$

The roots are $\{s = 2.43 \pm 5.895i, s = -1.727, s = -7.12\}$ We want breakaway between $s = 0$ and $s = -5$, hence only valid value is -1.727. We now go to the next lemma.

7. Departure angles from poles. Since poles are all on the real axis, the angles of departure are as shown below. The root locus plot now appears as follows



8. Since there are zeros, we now need to calculate arrival angles. There are two angles to calculate since there are two zeros. But we really need to calculate one, since the other will be symmetrical. Let us pick the top zero. $s = -1 + 3i$. Let its unknown arrival angle be θ_1. This is the angle we want to solve for. So we set up the following diagram to use for the calculation

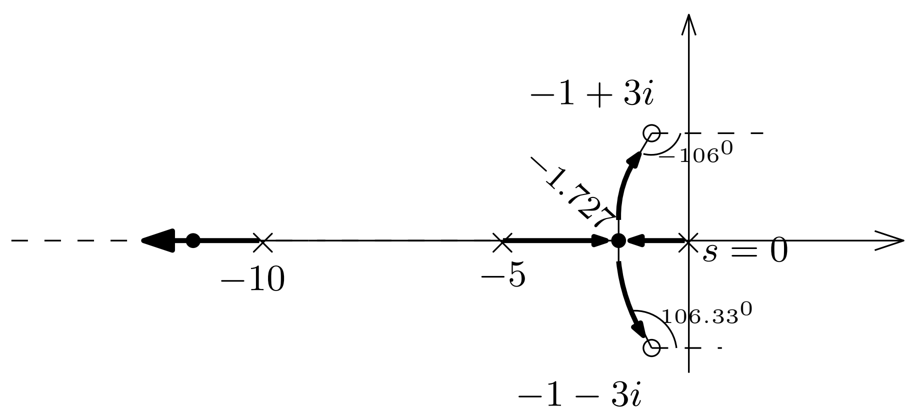


$$\sum_{zeros} \sphericalangle z_i - \sum_{poles} \sphericalangle p_i = 180^0 + k360^0 \qquad k = 0, 1, \cdots$$
$$(\theta_4 + \theta_1) - (\theta_2 + \theta_3 + \theta_5) = 180^0 + k360^0$$

Now we do a little geometry to calculate the angles. $\theta_4 = 90^0, \theta_2 = \tan^{-1}\left(\frac{3}{10-1}\right) = 18.435^0, \theta_3 = \tan^{-1}\left(\frac{3}{4}\right) = 36.8^0$ and $\theta_5 = 180 - \tan^{-1}\left(\frac{3}{1}\right) = 108.43^0$. The above now becomes

$$\begin{aligned}
\left(90^0 + \theta_1\right) - \left(18.435^0 + 36.8^0 + 108.43^0\right) &= 180^0 + k360^0 \\
\theta_1 &= 180^0 - 90^0 + 18.435^0 + 36.8^0 + 108.43^0 + k360^0 \\
&= 253.67^0 \\
&= -106.33^0
\end{aligned}$$

Therefore, the arrival angle at the lower zero will be -253.67^0 or 106.33^0. The root locus now is as follows



9. The only lemma left is to find where R.L. crosses the imaginary axis. But we do not have to do this, since the zeros are to the left of the imaginary axis and the third branch goes to $-\infty$. So we are done. The above is the final root locus.

0.2 Problem 2

Problem 2

The steering control system for a mobile robot for security purpose has a unity negative feedback with

$$G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)}$$

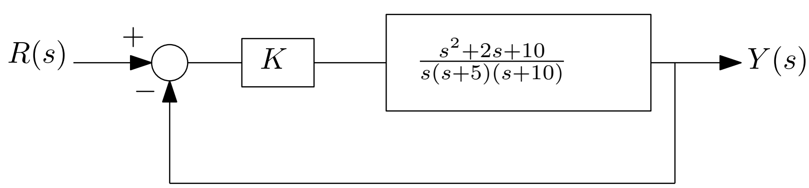
(a) Sketch the root locus for $0 \le K \le +\infty$ **without** using any computer tool.

(b) Verify your plot in (a) using ***Matlab***.

(c) Find K for all breakaway points on the real axis.

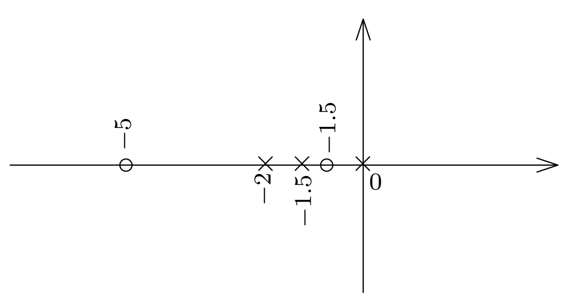
SOLUTION:

0.2.1 part a

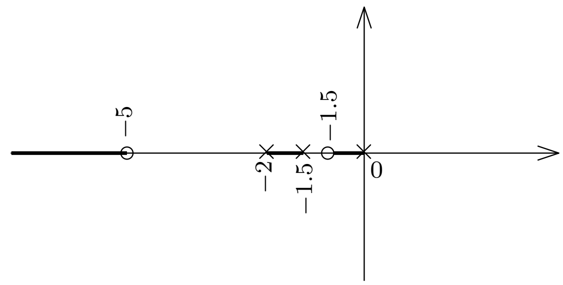


$$G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)}$$

1. $n = 3$ (The number of open loop poles), $m = 2$ (The number of open loop zeros).
2. Root Locus (R.L.) starts at the open loop poles $s = 0, s = -1.5, s = -2$ when $k = 0$ and two of the branches end up at zeros $s = -1$ and $s = -5$. One branch will end up at $\pm\infty$.



3. On the real axis, the following diagram shows the segments.



4. Center of asymptotes is now found. $\sigma = \frac{\sum poles - \sum zeros}{n-m} = \frac{-3.5-(-6)}{1} = 2.5$

5. The asymptotes angles are $\theta = \frac{180^0 \pm k360^0}{n-m}$ for $k = 0,1,\cdots$. Hence $\theta = \frac{180^0 \pm k360^0}{1} = 180^0 \pm k360^0$. So the asymptote is the real axis itself. This is clear since all zeros and poles are on the real axis.

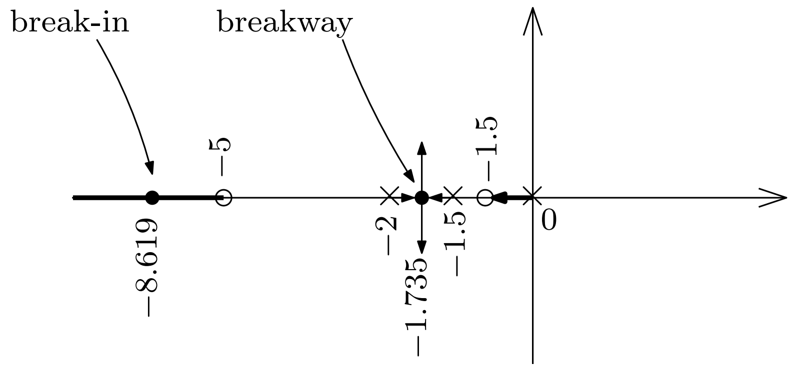
6. Break away points.

$$
\begin{aligned}
1+G &= 0\\
1+\frac{K(s+1)(s+5)}{s(s+1.5)(s+2)} &= 0\\
\frac{s(s+1.5)(s+2)+K(s+1)(s+5)}{s(s+1.5)(s+2)} &= 0\\
s(s+1.5)(s+2)+K(s+1)(s+5) &= 0\\
K &= \frac{-s(s+1.5)(s+2)}{(s+1)(s+5)}\\
&= \frac{-s^3-3.5s^2-3s}{s^2+6s+5}
\end{aligned}
$$

Then we now solve for $\frac{dK}{ds} = 0$

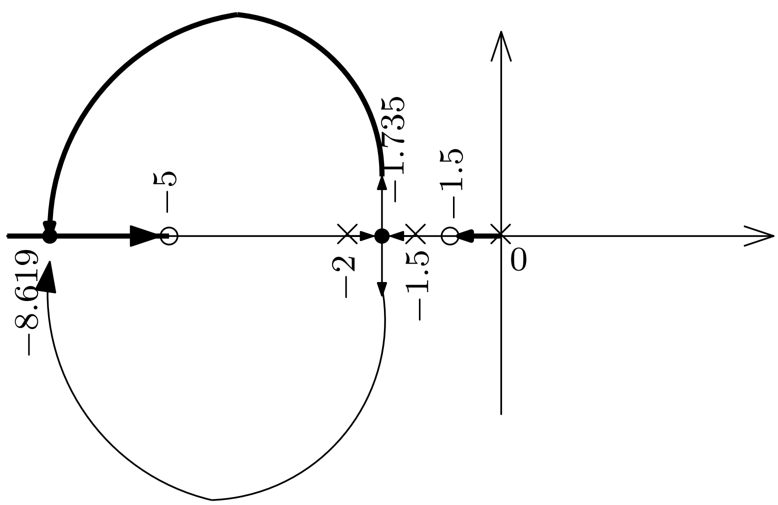
$$
\begin{aligned}
\frac{dK}{ds} &= 0\\
\frac{d}{ds}\left(\frac{-s^3-3.5s^2-3s}{s^2+6s+5}\right) &= 0\\
-\frac{s^4+12s^3+33s^2+35s+15}{s^4+12s^3+46s^2+60s+25} &= 0\\
s^4+12s^3+33s^2+35s+15.0 &= 0
\end{aligned}
$$

The roots are $s = -0.823 \pm 0.57i, s = -8.619, s = -1.735$ We want breakaway between $s = -1.5$ and $s = -2$, since these are two poles facing each others, one of the breakaway points is $s = -1.735$ on that segment. The complex root is discarded since it is not on the real line. The point $s = -8.619$ is valid since it is on a segment on the real line. It will be a break-in point since it is on a segment with only a zero on it. So the current plot is now as follows

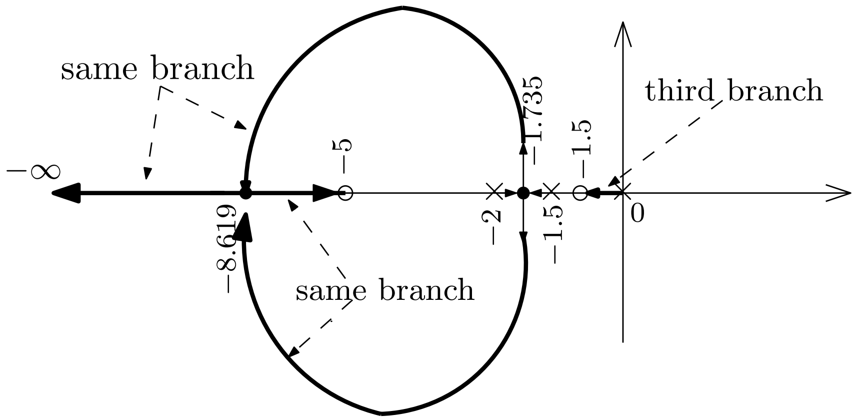


7. Departure angles from poles. Since poles are all on the real axis and no complex zeros exist, then the angles of departure are as shown above. There is nothing to do in this step.

8. Since all the zeros are on the real axis, there is nothing to do for this step. We just need to connect the break away branches to the break-in point on the real axis as follows

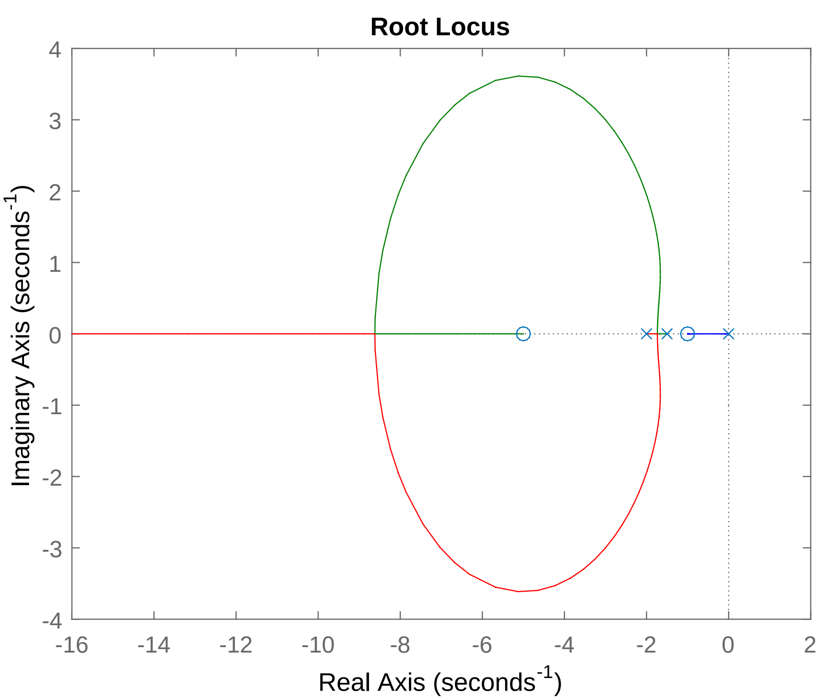


9. Now the only lemma left is to find where R.L. cross the imaginary axis. Again, we do not have to do this since the zeros are all to the left of the imaginary axis and the third branch goes to $-\infty$. So we are done. The following is the final plot. Since we need to have 3 branches (since $n = 3$) and $m = 2$, then one of the branches going into the break in point at $s = -8.619$ will go towards the zero at $s = -5$, but the other branch go to $-\infty$. We do not know which of the two branches will go to the zero and which will to go $-\infty$. It does not really matter. We pick the bottom branch going to the zero and the top branch going to $-\infty$. So the final plot is



0.2.2 Part (b)

```
>> s=tf('s');
>> sys=((s+1)*(s+5)/(s*(s+1.5)*(s+2)))
sys =
s^2 + 6 s + 5
--------------------
s^3 + 3.5 s^2 + 3 s
Continuous-time transfer function.
>> rlocus(sys)
```



0.2.3 Part (c)

The gain K any point on the roots locus is given by multiplying the distances from each pole to the that point, divided by the product of the distances from all the zeros to the same point. This comes from

$$1 + KG_{openloop} = 0$$
$$K = \frac{1}{\left|G_{openloop}\right|}$$

In other words, if we want to find gain at some point r, then

$$K = \frac{\prod \left|p_i r\right|}{\prod \left|z_i r\right|}$$

Since the first breakaway point in this case is $r = -1.735$ (the break away point), then the above becomes

$$K = \frac{(1.735)(1.735-1.5)(2-1.735)}{(1.735-1)(5-1.735)}$$

The above was done by just looking the diagram of the root locus and measuring the distance from each pole to the breakaway point, and similarly for the zeros. The above reduces to

$$\boxed{K = 0.045}$$

For the second break-in point in this case is $r = -8.617$, therefore

$$K = \frac{(8.617)(8.617-1.5)(8.617-2)}{(8.617-1)(8.617-5)}$$

The above was done by just looking the diagram of the root locus and measuring the distance from each pole to the break-in point, and similarly for the zeros. The above reduces to

$$\boxed{K = 14.73}$$