University Course

Math 703
Methods of Applied Mathematics I

University of Wisconsin, Madison
Fall 2014

My Class Notes

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Contents

Chapter 1

Introduction

Took this course in Fall 2014. Part of MSc. in Engineering Mechanics.

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Syllabus

The course introduces methods to solve mathematical problems that arise in areas of application such as physics, engineering, chemistry, biology, and statistics. Roughly speaking, we can divide these problems into two categories: (i) equilibrium (statics problems), and (ii) departures from equilibrium (dynamics problems).

The first part of the course will be devoted to the study of equilibrium: linear algebra provides a unifying framework for discrete equilibrium problems from several application areas. This algebraic structure is also the basis for numerical solution of both discrete and continuous equilibrium systems.

In the continuous case, equilibrium mechanics leads to boundary value problems for differential equations: in one dimension, one finds ordinary differential equations, e.g., Sturm-Liouville equations; for higher dimensional systems, one finds partial differential equations, e.g., Laplace's equation, Poisson's equation and the equations for Stokes flow. After review of some basic techniques for solving differential equations, asymptotic methods for the global analysis of ordinary differential equations will be introduced (boundary layer theory and WKB theory). The calculus of variations will also enable us to understand the different formulations of mechanics (by Newton, Lagrange and Hamilton).

Chapter 2

my class notes

Local contents

2.1 tuesday sept 2, 2014

covered parts of chapter 1. LU decomposition. We go from $Ax = b$ to $Ux = c$ where U contains the pivots on the diagonal and zeros in the lower triangle. To find c we use $Lc = b$ where L has ones on the diagonal, and has the multipliers used to obtain U just below the diagonal and has zero everywhere else. Putting these togother gives $LUx = b$. And this is the whole point of LU decomposition. For each new b' we find c' from $Lc' = b'$ and then find x from $Ux = c'$. We do not need to do the pivoting again to obtain U and L again. It is done once. It is. The cost now is n^2 each time to solve for x. i.e. the elimination is done only once.

The above is valid for any A. It does not have to be symmetric.

We go one step further. Let $U = D\tilde{U}$ where D is diagonal matrix and contains the pivotes on the diagonal. Again, A do not have to be symmteric for this.

Now, if A is symmetric, then $\tilde{U} = L^T$ and now we get special case of $\boxed{A = LDL^T}$.

If in addition, A has all positive pivots, then we can write $D = \sqrt{D}\sqrt{D}$, and $A = LDL^T$ becomes $\boxed{A = \tilde{L}\tilde{L}^T}$ where $\tilde{L} = L\sqrt{D}$

Did 1.2.1 to practice the pivoting and finding LU.

Chapter 3

HWs

Local contents

3.1 HW 1, Due sept 18, 2014

3.1.1 Problem 1.2.7

1.2.7 From the multiplication LS show that

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & 0 & 1 \end{bmatrix} \quad \text{is the inverse of} \quad S = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix}.$$

S subtracts multiples of row 1 and L adds them back.

Figure 3.1: the Problem statement

Solution

Multiplying LS gives

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $LS = I$ then $L = S^{-1}$ by definition.

3.1.2 Problem 1.2.8

1.2.8 Unlike the previous exercise, show that

$$L=\begin{bmatrix}1&&\\l_{21}&1&\\l_{31}&l_{32}&1\end{bmatrix}\quad\text{is } \textit{not} \text{ the inverse of}\quad S=\begin{bmatrix}1&&\\-l_{21}&1&\\-l_{31}&-l_{32}&1\end{bmatrix}.$$

If S is changed to

$$E=\begin{bmatrix}1&&\\0&1&\\0&-l_{32}&1\end{bmatrix}\begin{bmatrix}1&&\\-l_{21}&1&\\-l_{31}&0&1\end{bmatrix},$$

show that E is the correct inverse of L. E contains the elimination steps as they are actually done—subtractions of multiples of row 1 followed by subtraction of a multiple of row 2.

Figure 3.2: the Problem statement

Solution

Multiplying LS gives

$$\begin{pmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{pmatrix}\begin{pmatrix}1&0&0\\-l_{21}&1&0\\-l_{31}&-l_{32}&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\-l_{21}l_{32}&0&1\end{pmatrix}$$

Since $LS\neq I$ then L is not the inverse of S. Now let $S=E=\begin{pmatrix}1&0&0\\0&1&0\\0&-l_{32}&1\end{pmatrix}\begin{pmatrix}1&0&0\\-l_{21}&1&0\\-l_{31}&0&1\end{pmatrix}=$ $\begin{pmatrix}1&0&0\\-l_{21}&1&0\\l_{21}l_{32}-l_{31}&-l_{32}&1\end{pmatrix}$ and now evaluating LS gives

$$\begin{pmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{pmatrix}\begin{pmatrix}1&0&0\\-l_{21}&1&0\\l_{21}l_{32}-l_{31}&-l_{32}&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Therefore, with the new S matrix, now L is the inverse of S since $LS=I$

3.1.3 Problem 1.2.9

1.2.9 Find examples of 2 by 2 matrices such that

(a) $LU \neq UL$
(b) $A^2 = -I$, with real entries in A
(c) $B^2 = 0$, with no zeros in B
(d) $CD = -DC$, not allowing $CD = 0$.

Figure 3.3: the Problem statement

Solution

3.1.3.1 Part (a)

Take any random 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, By elimination $U = \begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}$. Now LU is found, giving back A as expected

$$\begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

UL is found

$$\begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{1}{a}bc & b \\ \frac{1}{a}c\left(d - \frac{1}{a}bc\right) & d - \frac{1}{a}bc \end{pmatrix}$$

Comparing LU and UL above, it can be seen that by setting $b = 0$ the $LU = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ while $UL = \begin{pmatrix} a & 0 \\ \frac{1}{a}cd & d \end{pmatrix}$, which means they will be different as long as $d \neq a$. So picking any A matrix which has $b = 0$ and which $d \neq a$ will work. An example is

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}}$$

To verify, $U=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ and $L=\begin{pmatrix}1&0\\5&1\end{pmatrix}$, hence $LU=\begin{pmatrix}1&0\\5&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}=\begin{pmatrix}1&0\\5&2\end{pmatrix}$ while $UL=\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}1&0\\5&1\end{pmatrix}=\begin{pmatrix}1&0\\10&2\end{pmatrix}$. They are different.

3.1.3.2 Part (b)

Take any random 2×2 matrix $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, then $A^2=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&d^2+bc\end{pmatrix}$ Now solving

$$\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&d^2+bc\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

gives 4 equations for a,b,c,d

$$\begin{aligned}a^2+bc&=1\\ab+bd&=0\\ac+cd&=0\\d^2+bc&=1\end{aligned}$$

Gives the following solutions

$$\begin{aligned}&a=-1,b=0,c=0,d=-1\\&a=1,b=0,c=0,d=-1\\&a=-1,b=0,c=0,d=1\\&a=1,b=0,c=0,d=1\end{aligned}$$

Any of the above solutions will satisfy $A^2=I$. For example, using the first one gives

$$\boxed{A=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}$$

3.1.3.3 Part (c)

As was done above, the following set of equations are solved.

$$\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&d^2+bc\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Hence

$$\begin{aligned}a^2+bc&=0\\ab+bd&=0\\ac+cd&=0\\d^2+bc&=0\end{aligned}$$

Solution is

```
eq1:=a^2+b*c=0;eq2:=a*b+b*d=0;eq3:=a*c+c*d=0;eq4:=a^2+b*c=0;
solve({eq1,eq2,eq3,eq4},{a,b,c,d});
 {a = a, b = b, c = -a^2/b, d = -a}, {a = 0, b = 0, c = 0, d = d}
```

Since we are looking for non-zero elements in B, then the first solution $\{a = a, b = b, c = -\frac{a^2}{b}, d = -a\}$ is used. For example, letting $a = 1, b = 2, c = -\frac{1}{2}, d = -1$ gives

$$\boxed{B = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix}}$$

To verify

$$\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3.1.3.4 Part (d)

Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, hence we want $CD = -DC$. To simplify this, let the diagonal be zero in both cases. This reduced the equations to 4 unknowns. Hence Let $C = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$ and

$$CD = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} = \begin{pmatrix} bg & 0 \\ 0 & cf \end{pmatrix}$$

$$DC = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} cf & 0 \\ 0 & bg \end{pmatrix}$$

Hence we want to solve $\begin{pmatrix} bg & 0 \\ 0 & cf \end{pmatrix} = -\begin{pmatrix} cf & 0 \\ 0 & bg \end{pmatrix}$ Hence this reduces to just solving

$$bg = -cf$$

Let $b = n, c = -n, g = n, f = n$ which satisfies the above. I.e. $n \times n = -(-n \times n) \Rightarrow n^2 = n^2$, therefore

$$C = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$$

To verify, $CD = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} n^2 & 0 \\ 0 & -n^2 \end{pmatrix}$ and $DC = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} = \begin{pmatrix} -n^2 & 0 \\ 0 & n^2 \end{pmatrix}$ hence $DC = -CD$. Let $n = 2$ for example, then

$$\boxed{C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

$$\boxed{D = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}}$$

3.1.4 Problem 1.3.2

1.3.2 Factor $A = \begin{bmatrix} 3 & 6 \\ 6 & 8 \end{bmatrix}$ into $A = LDL^T$. Is this matrix positive definite? Write x^TAx as a combination of two squares.

Figure 3.4: the Problem statement

Solution

$$A = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix}$$

Hence $U = \begin{pmatrix} 3 & 6 \\ 0 & -4 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, therefore $D = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$. D has the pivots on its diagonal. The pivots is the diagonal of U. Therefore

$$LDL^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix} = A$$

Since not all the pivots are positive and the matrix is symmetric, then this is not positive definite (P.D.). This can be confirmed by writing

$$\begin{aligned} x^TAx &= \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1 (3x_1 + 6x_2) + x_2 (6x_1 + 8x_2) \\ &= 3x_1^2 + 12x_1x_2 + 8x_2^2 \end{aligned}$$

We now need to complete the squares.

$$\begin{aligned} x^T A x &= 3\left(x_1 + a x_2\right)^2 + c x_2^2 \\ &= 3\left(x_1^2 + a^2 x_2^2 + 2 a x_1 x_2\right) + c x_2^2 \\ &= 3 x_1^2 + \left(3a^2 + c\right) x_2^2 + 6 a x_1 x_2 \end{aligned}$$

Comparing to $3x_1^2 + 12x_1x_2 + 8x_2^2$ we see that $a = 2$ and $c = 8 - 3a^2 = 8 - 12 = -4$, hence

$$x^T A x = 3\left(x_1 + 2x_2\right)^2 - 4x_2^2$$

This shows that $x^T A x$ is not positive for all x due to the -4 term. For example, if $x = \{1, -1\}$ then $x^T A x = -1$. Basically, we obtain the same result as before. For a symmetric matrix A, if not all the pivots are positive, then the matrix is not P.D. Using $x^T A x$ is another method to answer the same question. After completing the squares, we look to see if all the coefficients are positive or not.

3.1.5 Problem 1.3.6

1.3.6 In the 2 by 2 case, suppose the positive coefficients a and c dominate b in the sense that $a + c > 2b$. Is this enough to guarantee that $ac > b^2$ and the matrix is positive definite? Give a proof or a counterexample.

Figure 3.5: the Problem statement

Solution

A counter example is $a = 8, b = 2, c = 4$. We see that $a + b > 2c$ but $ac = 16$ and $c^2 = 16$, hence ac is not greater than b^2. So $a + c > 2b$ do not guarantee that $ac > c^2$. Therefore, we also can not guarantee that the matrix is P.D. this comes from the pivots. The pivots of $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are $\{a, c - \frac{b^2}{a}\}$. Since $a > 0$ as given, then we just need to check if $c - \frac{b^2}{a} > 0$. This means $ac - b^2 > 0$. But since we can't guarantee that $ac > b^2$ then this means the second pivot can be negative. Hence the matrix A with such property can not be guaranteed to be P.D.

3.1.6 Problem 1.3.7

1.3.7 Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Write A as $l_1 d_1 l_1^T$ and write A' as LDL^T.

Figure 3.6: the Problem statement

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

To show if A is P.D., we need to show that all the pivots are positive. This is the same as showing that $x^T A x > 0$ for all non-zero x.To obtain the pivots, we generate the U and look at the diagonal values. From the above, we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{U}$$

Hence using $l_1 = 1$ we see that the pivots are not all positive. There are zero pivot. Hence A is not P.D. For

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}^{U}$$

Hence all the pivots are positive. Therefore A' is P.D. We can write it as LDL^T

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

3.1.7 Problem 1.3.8

1.3.8 If each diagonal entry a_{ii} is larger than the sum of the absolute values $|a_{ij}|$ along the rest of its row, then the symmetric matrix A is positive definite. How large would c have to be in

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$$

for this statement to apply? How large does c actually have to be to assure that A is positive definite? Note that

$$x^T A x = (x_1 + x_2 + x_3)^2 + (c-1)(x_1^2 + x_2^2 + x_3^2);$$

when is this positive?

Figure 3.7: the Problem statement

Solution

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$$

$c > 2$ is enough to guarantee row dominant matrix. For P.D., looking at $x^T A x = (x_1 + x_2 + x_3)^2 + (c-1)\left(x_1^2 + x_2^2 + x_3^2\right)$ shows that $c - 1 > 0$ is the condition for P.D. which implies $c > 1$. Hence it is enough that $c > 1$.

3.1.8 Problem 1.3.11

1.3.11 A function $F(x, y)$ has a local minimum at any point where its first derivatives $\partial F/\partial x$ and $\partial F/\partial y$ are zero and the matrix of second derivatives

$$A = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x \partial y} \\ \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

is positive definite. Is this true for $F_1 = x^2 - x^2 y^2 + y^2 + y^3$ and $F_2 = \cos x \cos y$ at $x = y = 0$? Does F_1 have a global minimum or can it approach $-\infty$?

Figure 3.8: the Problem statement

Solution

For $F_1 = x^2 - x^2y^2 + y^2 + y^3$, we find $\frac{\partial F_1}{\partial y} = -2x^2y + 2y + 3y^2 = 0$ at $x = 0, y = 0$. And $\frac{\partial F_1}{\partial x} = 2x - 2xy^2 = 0$ at $x = 0, y = 0$. Now we need to look at the P.D. of

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 - 2y^2 & -4xy \\ -4xy & -2x^2 + 2 + 6y \end{pmatrix}$$

At $x = 0, y = 0$ the above becomes

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is already in U form. Since the diagonal is all positive, then this is P.D., which means it is true for $F_1(x, y)$. Now we check $F_2(x, y)$

$F_2(x, y) = \cos x \cos y$. Hence $\frac{\partial F_1}{\partial y} = -\sin y \cos x = 0$ at $x = 0, y = 0$. And $\frac{\partial F_1}{\partial x} = -\sin x \cos y = 0$ at $x = 0, y = 0$. Now we need to look at the P.D. of

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\cos x \cos y & \sin y \sin x \\ \sin y \sin x & -\cos y \cos x \end{pmatrix}$$

And at $x = 0, y = 0$ the above becomes

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence this is not P.D, since the pivots are negative. To answer the part about F_1 having global minimum. The point $x=0,y=0$ is local minimum for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ since $\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix}$ was found to be P.D. at $x=0,y=0$. But this is not global minimum. Only when the function can be written as quadratic form $x^T A x$ will the local minumum be global minumum. In this case, F_1 can approach $-\infty$, hence this is the global minimum.

Taking the limit $\lim_{x_1 \to -\infty} F_1 = \left(1 - y^2\right)\infty$. Taking the limit of this as $y \to \infty$ gives $-\infty$. Here is a plot of F_1 around $x=0,y=0$ showing it is a local minimum

```
F1 = x^2 - x^2 y^2 + y^2 + y^3
Plot3D[F1, {x, -3, 3}, {y, -3, 3},
PlotLabel -> "F1 function", AxesLabel -> {x, y}]
```

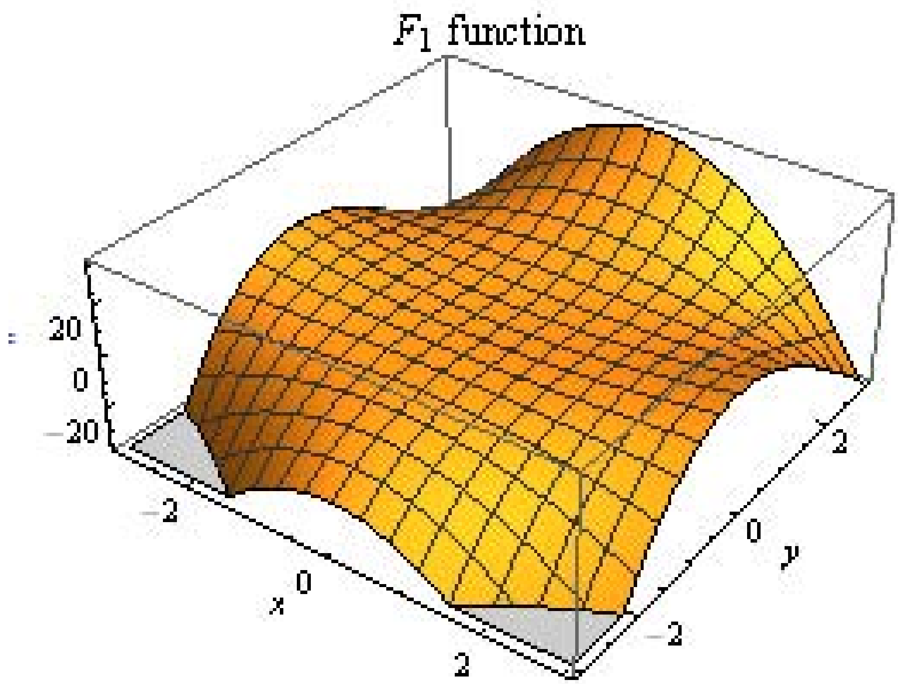


Figure 3.9: plot of the above

3.1.9 Problem 1.4.5

1.4.5 The best fit to b_1, b_2, b_3, b_4 by a horizontal line (a constant function $y = C$) is their average $C = (b_1 + b_2 + b_3 + b_4)/4$. Confirm this by least squares solution of

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [C] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

From calculus, which C minimizes the error $E = (b_1 - C)^2 + \cdots + (b_4 - c)^2$?

Figure 3.10: the Problem statement

Solution

The equation of the line is $y = C$, hence we obtain 4 equations.

$$\begin{aligned} b_1 &= C \\ b_2 &= C \\ b_3 &= C \\ b_4 &= C \end{aligned}$$

or

$$\overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}^{A} C = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Hence now we set $A^T A x = A^T b$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$4C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$4C = b_1 + b_2 + b_3 + b_4$$

Hence $C = \frac{b_1+b_2+b_3+b_4}{4}$, Which is the average. Using calculus, to minimize $E = (b_1 - C)^2 +$

$(b_2 - C)^2 + (b_3 - C)^2 + (b_4 - C)^2$

$$
\begin{aligned}
\frac{dE}{dC} &= -2(b_1 - C) - 2(b_2 - C) - 2(b_3 - C) - 2(b_4 - C) \\
0 &= 8C - 2b_1 - 2b_2 - 2b_3 - 2b_4 \\
8C &= 2b_1 + 2b_2 + 2b_3 + 2b_4 \\
C &= \frac{b_1 + b_2 + b_3 + b_4}{4}
\end{aligned}
$$

Which is the same found using $A^T Ax = A^T b$ solution.

3.1.10 Problem 1.4.7

1.4.7 For the three measurements $b = 0, 3, 12$ at times $t = 0, 1, 2$, find

(i) the best horizontal line $y = C$
(ii) the best straight line $y = C + Dt$
(iii) the best parabola $y = C + Dt + Et^2$.

Figure 3.11: the Problem statement

Solution

3.1.10.1 Part (a)

For $y = C$ we obtain the following equations

$$
\begin{aligned}
b_1 &= C \\
b_2 &= C \\
b_3 &= C
\end{aligned}
$$

Hence

$$
\overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{A} C = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
$$

Applying $A^TAx = A^Tb$ gives

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}C = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$
$$3C = b_1 + b_2 + b_3$$
$$C = \frac{b_1 + b_2 + b_3}{3}$$

Therefore $y = C = \frac{b_1+b_2+b_3}{3} = \frac{0+3+12}{3} = 5$, or

$$y = 5$$

3.1.10.2 Part (b)

For $y = C + Dt$ we obtain the following equations

$$b_1 = C + Dt$$
$$b_2 = C + Dt$$
$$b_3 = C + Dt$$

Applying the numerical values gives results in

$$0 = C$$
$$3 = C + D$$
$$12 = C + D\,(2)$$

Hence

$$\overbrace{\begin{pmatrix}1 & 0\\1 & 1\\1 & 2\end{pmatrix}}^{A}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}0\\3\\12\end{pmatrix}$$

Applying $A^TAx = A^Tb$ gives

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}0\\3\\12\end{pmatrix}$$
$$\begin{pmatrix}3 & 3\\3 & 5\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}15\\27\end{pmatrix}$$

Now we solve this using Gaussian elimination. First U is found

$$\begin{pmatrix}3 & 3\\3 & 5\end{pmatrix} \rightarrow \begin{pmatrix}3 & 3\\0 & 2\end{pmatrix}$$

Hence $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Lc = b$, then $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 27 \end{pmatrix}$ now c is found by forward substitution, giving $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Now we solve $Ux = c$ or $\begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$ by backward substitution, the result is

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

Hence the line is

$$\boxed{y = -1 + 6t}$$

Here is a plot of the fit found above

```
b = {0, 3, 12}; t = {0, 1, 2};
p1 = ListPlot[Transpose[{t, b}], PlotStyle -> Red];
p2 = Plot[-1 + 6 t, {t, -.5, 3}, PlotTheme -> "Detailed",
FrameLabel -> {{"y(t)", None}, {"t", "Fit by least squares"}}];
Show[p2, p1]
```

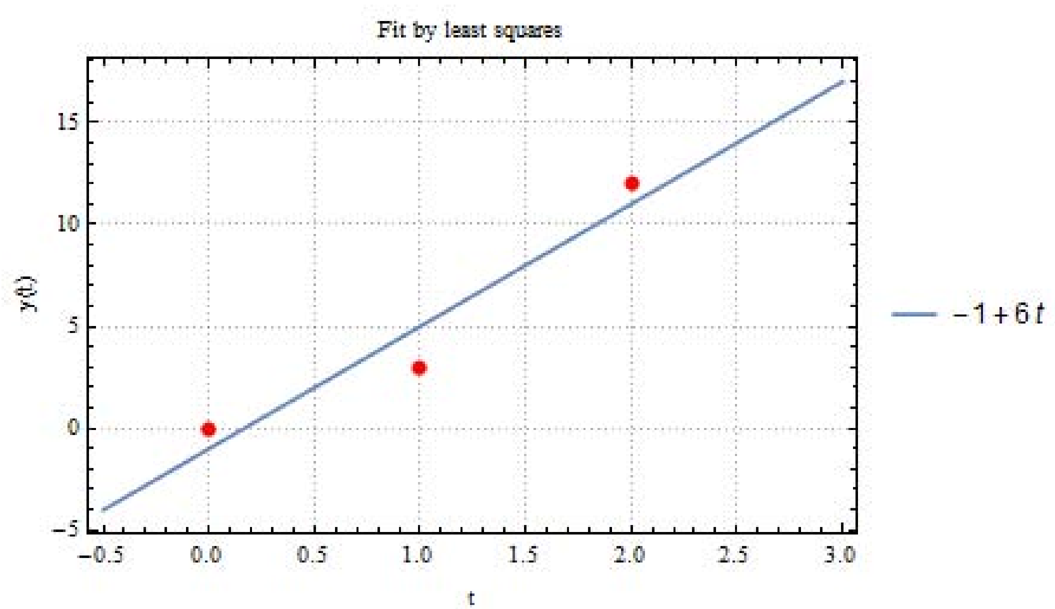


Figure 3.12: Plot of the above

3.1.10.3 Part c

For $y = C + Dt + Et^2$ we obtain the following equations

$$\begin{aligned} b_1 &= C + Dt + Et^2 \\ b_2 &= C + Dt + Et^2 \\ b_3 &= C + Dt + Et^2 \end{aligned}$$

Applying the numerical values gives results in

$$\begin{aligned} 0 &= C \\ 3 &= C + D + E \\ 12 &= C + 2D + 4E \end{aligned}$$

Hence

$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}^{A} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$$

Now we solve this using Gaussian elimination. We do not need to use $A^T A$ least squares since the number of rows is the same as number of columns. First U is found

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $Lc = b$, then $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$ now c is found by forward substitution, giving $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$

Now we solve $Ux = c$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ by backward substitution, giving

$$\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Hence the solution is

$$y = 3t^2$$

Here is a plot of the fit

```
b = {0, 3, 12}; t = {0, 1, 2};
p1 = ListPlot[Transpose[{t, b}], PlotStyle -> Red];
p2 = Plot[3 t^2, {t, -.5, 3}, PlotTheme -> "Detailed",
FrameLabel -> {{"y(t)", None}, {"t", "Fit by least squares"}}];
Show[p2, p1]
```

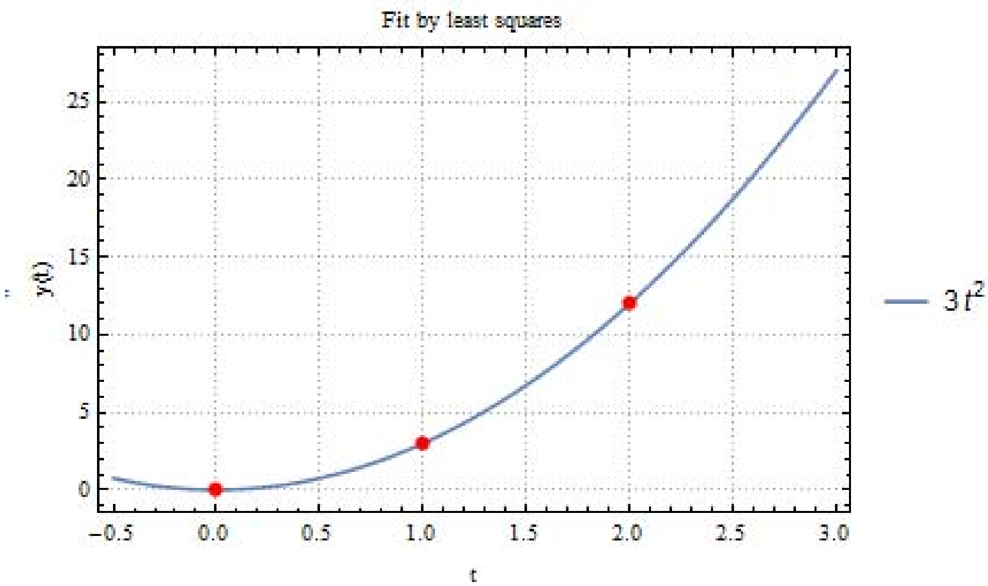


Figure 3.13: Plot of the above

We can see this is an exact fit since no least squares was used.

3.1.11 Problem 1.4.10

1.4.10 In a system with three springs and two forces and displacements write out the equations $e = Ax$, $y = Ce$, and $A^T y = f$. For unit forces and spring constants, what are the displacements?

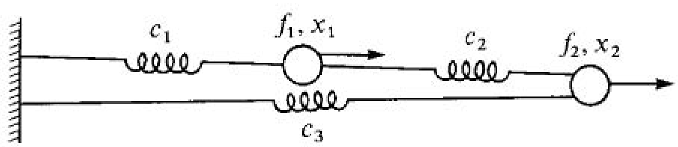


Figure 3.14: Problem description

Solution

From page 40 in textbook, y is force in spring, e is the elongation of spring from equilibrium and f external force at each mass. Hence for $Ax = e$, we see that $e_1 = x_1, e_2 = x_2 - x_1$ and $e_3 = x_2$. Therefore

$$\overbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

For $y = Ce$, here y is the internal force in spring. Hence $y_1 = c_1 e_1, y_2 = c_2 e_2, y_3 = c_3 e_3$, therefore

$$\overbrace{\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}}^{A} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

For $A^T y = f$, we need to find the external forces at each node first. From diagram we see that $f_1 = y_1 - y_2$ and $f_2 = y_2 + y_3$, therefore

$$\overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}^{A^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

3.1.12 Problem 1.4.11

1.4.11 Suppose the lowest spring in Fig. 1.7 is removed, leaving masses m_1, m_2, m_3 hanging from the three remaining springs. The equation $e = Ax$ becomes

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find the corresponding equations $y = Ce$ and $A^T y = f$, and solve the last equation for y. This is the ***determinate*** case, with square matrices, when the factors in $A^T CA$ can be inverted separately and y can be found before x.

Figure 3.15: Problem description

Solution

To find $y = Ce$. In this equation, e is the elongation of the spring and y is the internal force.

Hence from figure 1.7 we obtain

$$
\begin{aligned}
y_1 &= c_1 e_1 \\
y_2 &= c_2 e_2 \\
y_3 &= c_3 e_3
\end{aligned}
$$

Hence in matrix form

$$
\overbrace{\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}}^{C} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
$$

In the question $A^T y = f$, f is the external force. Hence by balance of force at each mass, we obtain

$$
\begin{aligned}
f_1 &= y_1 - y_2 \\
f_2 &= y_2 - y_3 \\
f_3 &= y_3
\end{aligned}
$$

or in matrix form

$$
\overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}^{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
$$

To solve, since already in U form, we will just need to do backward substitution. Hence

$$
\begin{aligned}
y_3 &= f_3 \\
y_2 &= f_2 + f_3 \\
y_1 &= f_1 + f_2 + f_3
\end{aligned}
$$

3.1.13 Problem 1.4.12

1.4.12 For the same 3 by 3 problem find $K = A^T CA$ and A^{-1} and K^{-1}. If the forces f_1, f_2, f_3 are all positive, acting in the same direction, how do you know that the displacements x_1, x_2, x_3 are also positive?

Figure 3.16: Problem description

Solution

$K = A^TCA$, but $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ given in problem 1.4.11, and $C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$. Hence

$$\begin{aligned} K = A^TCA &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^T \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix} \end{aligned}$$

And

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

And

$$\begin{aligned} K^{-1} &= \left(A^TCA\right)^{-1} \\ &= A^{-1}C^{-1}\left(A^T\right)^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & \frac{1}{c_3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{c_1} & \frac{1}{c_1} & \frac{1}{c_1} \\ \frac{1}{c_1} & \frac{1}{c_1} + \frac{1}{c_2} & \frac{1}{c_1} + \frac{1}{c_2} \\ \frac{1}{c_1} & \frac{1}{c_1} + \frac{1}{c_2} & \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \end{pmatrix} \end{aligned}$$

Since $f = Kx$ then $x = K^{-1}f$. Since we are told f_1, f_2, f_3 are all positive, and so the sign of x the displacement, is determined by the sign of K^{-1}. But K^{-1} has positive entries only, since c_i is positive by definition. Therefore all displacements x must be positive.

3.1.14 Problem 1.5.6

1.5.6 Solve the second-order system

$$\frac{d^2u}{dt^2}+\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}u=0 \quad \text{with} \quad u_0=\begin{bmatrix}2\\-1\\-1\end{bmatrix} \quad \text{and} \quad u_0'=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

These initial conditions do not activate the zero eigenvalue (see the following exercises).

Figure 3.17: Problem description

Solution

$$\begin{pmatrix}u_1''\\u_2''\\u_3''\end{pmatrix}+\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The solution is

$$\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix}=\left(a_1\cos\sqrt{\lambda_1}t+b_1\sin\sqrt{\lambda_1}t\right)\begin{pmatrix}v_{11}\\v_{21}\\v_{31}\end{pmatrix}+\left(a_2\cos\sqrt{\lambda_2}t+b_2\sin\sqrt{\lambda_2}t\right)\begin{pmatrix}v_{12}\\v_{22}\\v_{32}\end{pmatrix}+\left(a_3\cos\sqrt{\lambda_3}t+b_3\sin\sqrt{\lambda_3}t\right)\begin{pmatrix}v_{13}\\v_{23}\\v_{33}\end{pmatrix}$$

Where λ_i are the eigenvalues and v_i are the corresponding eigenvectors of A. The constants are found from initial conditions.

For the matrix A, the eigenvalues are found by solving

$$|A-\lambda I|=0$$

Solving for eigenvalues gives $\lambda_1=0,\lambda_2=1,\lambda_3=3$ and the corresponding eigenvectors are $\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}1\\-2\\1\end{pmatrix}$ hence the solution becomes

$$\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix}=a_1\begin{pmatrix}1\\1\\1\end{pmatrix}+(a_2\cos t+b_2\sin t)\begin{pmatrix}-1\\0\\1\end{pmatrix}+\left(a_3\cos\sqrt{3}t+b_3\sin\sqrt{3}t\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

At $t=0$

$$\begin{pmatrix}2\\-1\\-1\end{pmatrix}=a_1\begin{pmatrix}1\\1\\1\end{pmatrix}+a_2\begin{pmatrix}-1\\0\\1\end{pmatrix}+a_3\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1}$$

And taking derivative of the solution gives

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-a_2 \sin t + b_2 \cos t) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \left(-a_3\sqrt{3}\sin\sqrt{3}t + b_3\sqrt{3}\cos\sqrt{3}t\right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

At $t = 0$ the above becomes

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + b_3\sqrt{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2}$$

Now (1),(2) needs to be solved for the constants. From (1)

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

This is solved using Gaussian elimination. $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix}$

Hence $U = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and hence $Lc = b$ is solved first for c using forward substitution

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Which gives $c_1 = 2, c_2 = -3, c_3 = 3$, hence now we solved for x from $Ux = c$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

Giving $a_3 = \frac{1}{2}, a_2 = -\frac{3}{2}, a_1 = 0$.

Now we solve for the rest of the constant in same way. From (2)

$$\begin{pmatrix} 1 & -1 & \sqrt{3} \\ 1 & 0 & -2\sqrt{3} \\ 1 & 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} a_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is solved using Gaussian elimination. $\begin{pmatrix} 1 & -1 & \sqrt{3} \\ 1 & 0 & -2\sqrt{3} \\ 1 & 1 & \sqrt{3} \end{pmatrix} \to \begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 1 & -3\sqrt{3} \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 1 & -3\sqrt{3} \\ 0 & 0 & 6\sqrt{3} \end{pmatrix}$

Hence $U = \begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 1 & -3\sqrt{3} \\ 0 & 0 & 6\sqrt{3} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $Lc = b$ is solved first for c using forward substitution

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$c_1 = 0, c_2 = 0, c_3 = 0$, therefore now we solved for x from $Ux = c$

$$\begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 1 & -3\sqrt{3} \\ 0 & 0 & 6\sqrt{3} \end{pmatrix} \begin{pmatrix} a_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which gives $b_3 = 0, b_2 = 0, a_1 = 0$. Now that all constants are found the final solution is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = -\frac{1}{2} \cos t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \cos \sqrt{3} t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Hence

$$\begin{aligned} u_1(t) &= \frac{3}{2} \cos t + \frac{1}{2} \cos \sqrt{3} t \\ u_2(t) &= -\cos \sqrt{3} t \\ u_3(t) &= -\frac{3}{2} \cos t + \frac{1}{2} \cos \sqrt{3} t \end{aligned}$$

3.1.15 Problem 1.5.7

1.5.7 Suppose each column of A adds to zero, as in

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

(a) Prove that zero is an eigenvalue and A is singular, by showing that the vector of ones is an eigenvector of A^T. (A and A^T have the same eigenvalues, but not the same eigenvectors.)
(b) Find the other eigenvalues of this matrix A, and all three eigenvectors.

Figure 3.18: Problem description

Solution

3.1.15.1 Part (a)

$$A=\begin{pmatrix}3&-1&0\\-2&2&-1\\-1&-1&1\end{pmatrix}$$

$$A^T=\begin{pmatrix}3&-2&-1\\-1&2&-1\\0&-1&1\end{pmatrix}$$

For eigenvector v of ones, we write

$$A^T\boldsymbol{v}=\lambda\boldsymbol{v}$$

Hence

$$\begin{pmatrix}3&-2&-1\\-1&2&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\0\end{pmatrix}=\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Which implies $\lambda=0$. Since A^T has same eigenvalues of A then A has zero eigenvalue. But the determinant of A is the products of its eigenvalues. Since one eigenvalue is zero, then $|A|=0$, which means A is singular.

3.1.15.2 Part (b)

To find all three eigenvalues of A we solve $|\lambda I-A|=0$. Hence

$$\begin{vmatrix}\lambda-3&1&0\\2&\lambda-2&1\\1&1&\lambda-1\end{vmatrix}=0$$

$$\lambda^3-6\lambda^2+8\lambda=0$$

$$\lambda\left(\lambda^2-6\lambda+8\right)=0$$

Hence $\lambda=0,\lambda=2,\lambda=4$ .To find the eigenvectors, we solve $Av_i=\lambda_i v_i$ for each eigenvalue. This means solving $(\lambda_i I-A)\,v_i=0$ for each eigenvalue. For $\lambda=0$

$$-\begin{pmatrix}3&-1&0\\-2&2&-1\\-1&-1&1\end{pmatrix}\begin{pmatrix}v_{11}\\v_{21}\\v_{31}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

We always set $v_{1j} = 1$ and then go to find v_{2j}, v_{3j} in finding eigenvectors. Hence we solve

$$\begin{pmatrix} -3 & 1 & 0 \\ 2 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving gives $v_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ For the second eigenvalue $\lambda = 2$ we obtain

$$\begin{aligned} \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \right) \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ v_{22} \\ v_{32} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Solving gives $v_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$. For the last eigenvalue $\lambda = 4$ we obtain

$$\begin{aligned} \left(\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \right) \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ v_{23} \\ v_{33} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Solving gives $v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.Summary. The eigenvalues are $\{0, 2, 4\}$ and the eigenvectors are

$$\boxed{\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

3.1.16 Problem 1.5.11

1.5.11 Why is the sum of entries on the diagonal of AB equal to the sum along the diagonal of BA? In other words, what terms contribute to the trace of AB?

Figure 3.19: Problem description

Solution

The elements of the diagonal of AB come from multiplying row i in A with column i in B. Therefore, looking at the diagonal elements only, we can write, using a_{ij} as element in A and using b_{ij} as element in B

$$AB = \begin{pmatrix} \sum_i a_{1i}b_{i1} & & & \\ & \sum_i a_{2i}b_{i2} & & \\ & & \ddots & \\ & & & \sum_i a_{ni}b_{in} \end{pmatrix}$$

Hence the trace of AB is

$$tr\,(AB) = \sum_i a_{1i}b_{i1} + \sum_i a_{2i}b_{i2} + \cdots + \sum_i a_{ni}b_{in}$$

But the above can be combined as

$$tr\,(AB) = \sum_k \sum_i a_{ki}b_{ik} \tag{1}$$

Now if we consider BA, then the result comes from multiplying row i in B with column i in A

$$BA = \begin{pmatrix} \sum_i b_{1i}a_{i1} & & & \\ & \sum_i b_{2i}a_{i2} & & \\ & & \ddots & \\ & & & \sum_i b_{ni}a_{in} \end{pmatrix}$$

Hence the trace of BA is

$$tr\,(BA) = \sum_i b_{1i}a_{i1} + \sum_i b_{2i}a_{i2} + \cdots + \sum_i b_{ni}a_{in}$$

But the above can be combined as

$$tr\,(BA) = \sum_k \sum_i b_{ki}a_{ik} \tag{2}$$

Looking at (1) and (2) above we can see that both traces contain the same elements, but arranged differently. The indices can be changes in the sum without changing the value of the sum. This can be seen more directly by looking at specific example of 2×2 case. Let

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, hence the elements on the diagonal of AB are $\begin{pmatrix} ae+bg & \\ & cf+dh \end{pmatrix}$ while for BA the result is $\begin{pmatrix} ae+cf & \\ & bg+dh \end{pmatrix}$. We see that the trace is the same.

3.1.17 Problem 1.5.12

1.5.12 Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda), \qquad (*)$$

and making a clever choice of λ.

Figure 3.20: Problem description

Solution

$\det(A - \lambda I)$ is a polynomial in λ. Hence it can be factored in its roots as

$$\det(A - \lambda I) = P(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda)$$

Assuming there is n eigenvalues. When $\lambda = 0$ (which is the independent variable now, and not any specific eigenvalue, then (1) becomes

$$\det(A) = P(0) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$$

Hence

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$$

Which is what we are asked to show.

3.1.18 Problem 1.5.13

1.5.13 Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of (*). Next, look for all the terms in

$$\det(A-\lambda I)=\det\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

which involve $(-\lambda)^{n-1}$. Explain why they all come from the main diagonal, and find the coefficient of $(-\lambda)^{n-1}$ on the left side of (*). Compare.

Figure 3.21: Problem description

Solution

$$\det(A-\lambda I)=P(\lambda)=(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)\cdots(\lambda_n-\lambda) \tag{*}$$

Let look at the case of $n=2$

$$\begin{aligned} P(\lambda) &= (\lambda_1-\lambda)(\lambda_2-\lambda) \\ &= \lambda^2-\lambda(\lambda_1+\lambda_2)+\lambda_1\lambda_2 \end{aligned}$$

Hence the coefficient of $(-\lambda)^{n-1}$ which is $-\lambda$ is $(\lambda_1+\lambda_2)$ which is the sum of the eigenvalues. Lets look at $n=3$

$$\begin{aligned} P(\lambda) &= (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) \\ &= -\lambda^3+\lambda^2(\lambda_1+\lambda_2+\lambda_3)-\lambda(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+\lambda_1\lambda_2\lambda_3 \end{aligned}$$

So the pattern is now clear. The coefficient of $(-\lambda)^{n-1}$ is the sum of all the eigenvalues of A.

For $\det(A-\lambda I)$, looking at $n=2$ we write

$$\begin{aligned} \det(A-\lambda I) = \det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} &= (a_{11}-\lambda)(a_{22}-\lambda)-a_{21}a_{12} \\ &= \lambda^2-\lambda(a_{11}+a_{22})+(a_{11}a_{22}-a_{21}a_{12}) \end{aligned}$$

We see in this case that the coefficient of $(-\lambda)^{n-1} = -\lambda$ is the trace of A. Lets look at $n = 3$

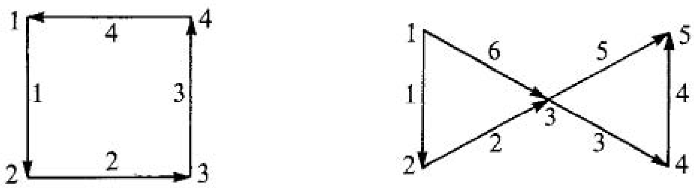
$$
\begin{aligned}
\det(A-\lambda I) &= \det\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix} \\
&= (a_{11}-\lambda)\det\begin{pmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22}-\lambda \\ a_{31} & a_{32} \end{pmatrix} \\
&= (a_{11}-\lambda)\left(\lambda^2 - \lambda(a_{22}+a_{33}) + (a_{22}a_{33} - a_{23}a_{32})\right) \\
&- a_{12}(a_{21}a_{33} - \lambda a_{21} - a_{31}a_{23}) + a_{13}(\lambda a_{31} + a_{21}a_{32} - a_{22}a_{31}) \\
&= -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda(a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}) \\
&+ (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{21}a_{13}a_{32} - a_{13}a_{22}a_{31})
\end{aligned}
$$

We see again that the coefficient of $(-\lambda)^{n-1} = \lambda^2$ is the trace of A. So by construction we can show that coefficient of $(-\lambda)^{n-1}$ is the trace of A. But we showed above that coefficient of $(-\lambda)^{n-1}$ is the sum of all the eigenvalues of A. Hence the sum of all the eigenvalues of $A = tr(A)$

3.2 HW 2, Due Oct 2, 2014

3.2.1 Problem 1.6.2

1.6.2 Write down the incidence matrices A_1 and A_2 for the following graphs:



For which right sides does $A_1 x = b$ have a solution? Which vectors are in the nullspace of A_1^T?

Figure 3.22: the Problem statement

In the incidence matrices, the rows indicate the edges, and the columns are the nodes. We put −1 for the node that the edge leaves and +1 for the node that the edges arrives at. Arrows are used to indicate direction.

$$A_1 = \begin{pmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & +1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & 0 & +1 \\ -1 & 0 & +1 & 0 & 0 \end{pmatrix}$$

We first note that matrix A_1 rank $r = 3, m = 4, n = 4$.

In $A_1 x = b$, the vectors b have to be in the column space of A_1. These are vectors in $R^m = R^4$, that span space of dimension $r = 3$. Since there is a cycle (starting from node 1 we end up at node 1 again by following the edges), this means that all the potentials at each node must be the same. But if the potential at each node is the same, then there can be no flow of current. Since flow of current represent the edge, it means each edge will have zero value. So b must be all vectors/edges that add up to $[0, 0, 0, 0]$ vector. For the case of A_1^T, we obtain the matrix

$$A_1^T = \begin{pmatrix} -1 & 0 & 0 & 1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{pmatrix}$$

$N\left(A_1^T\right)$ in the space of $R^m = R^4$ with vectors that span dimension space $m - r = 4 - 3 = 1$. So a line. So one basis vector is all what is needed.

And now we ask about the nodes of this graph. What values can they have? This is the graph associated with this matrix

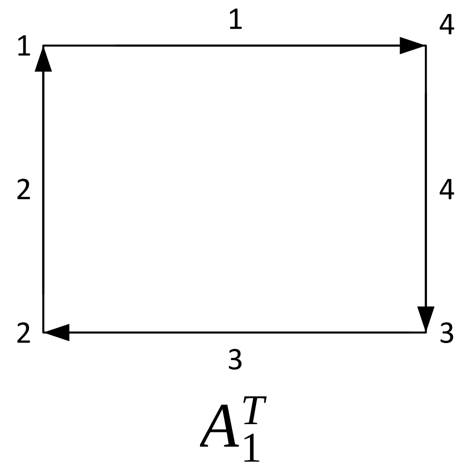


Figure 3.23: graph associated with this matrix

We now ask, what values should the nodes have in order for the edges to have zero flow in them? It is clear the nodes must all be equal [1,1,1,1] since if the potential is same at each node, then there will be no flow (i.e. zero potential difference) on the edges. Therefore

$$\boxed{N\left(A^T\right) = [1,1,1,1]}$$

We also know from fundamental theory of linear algebra, that $R(A)$ is orthogonal to $N\left(A^T\right)$.

3.2.2 Problem 1.6.3

1.6.3 The previous matrix A_2 should have $n-1$ independent rows; which are they? There should also be $m-n+1$ independent vectors in the nullspace of A_2^T, one from each loop; which are they?

Figure 3.24: Problem description

The matrix A_2 has rank $r=4, m=6, n=5$.The number of independent rows (edges) is $n-1$ or $5-1=4$ which is its rank. These can be read from the graph directly. Any 4 edges, as long as they do not complete a cycle, will qualify. Hence the edges that meet this condition are

$$
\begin{gathered}
6,5,4,2\\
6,5,4,1\\
6,5,3,2\\
6,5,3,1\\
6,4,3,2\\
6,4,3,1\\
5,4,2,1\\
5,3,2,1\\
4,3,2,1
\end{gathered}
$$

Notice that we could not have selected for example $6,5,4,3$ since $5,4,3$ are in one loop.

The $N\left(A_2^T\right)$ has $m-r=6-4=2$ dimensions. Now we take the edges on each loop. Since the loop is the null space. Since there are two loops, this give us the two independent rows. The left loop has

$$\boxed{edge\,(1)+edge\,(2)-edge\,(6)=[1,1,0,0,0,-1]} \tag{1}$$

Second loop has

$$\boxed{edge\,(3)+edge\,(4)-edge\,(5)=[0,0,+1,+1,-1,0]}$$

In other words, we put a 0 for the edge that is not there and put a +1 for the edge the goes one direction and −1 for the edge that goes in the opposite direction. For example, in (1) we put 1 for edge(1) since edge(1) is in the loop. We put 0 for edge (3) since edge (3) is not in the loop at all. We put −1 for edge(6) since it goes in the opposite direction from the others. It is arbitrary which direction is positive and which is negative, as long as one is consistent. Notice the above two basis vectors span $N\left(A_2^T\right)$ and live inside R^6 since $m=6$ in this case.

3.2.3 Problem 1.6.5

1.6.5 If A is the incidence matrix of a connected graph and $Ax=0$, show that $x_1=x_2=\cdots=x_n$. Each row of $Ax=0$ is an equation $x_j-x_k=0$; how do you prove that $x_j=x_k$ even when no edge goes from node j to node k?

Figure 3.25: Problem description

Since $Ax=0$ then we set up the equations from incidence matrix one for each edge as follows

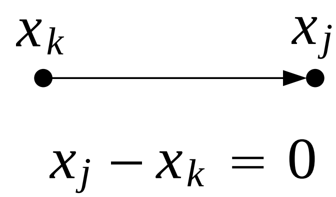


Figure 3.26: plot for prob 1.6.5

If we assign one node any arbitrary value, say $x_k = 1$, then $x_j = 1$ as well. But then any node on the other side of x_j, say x_i will now have value 1 as well. By transitivity, all other nodes will end up with the same value assigned to the first node. Hence all nodes have the same value.

For the case of a two nodes not connect. Assume the nodes are x_1 and x_4 and that there is no edge between them. Now assume there is an edge x_1x_2 and edge x_2x_3 and edge x_3x_4. Since $x_2 = x_1$ since $Ax = 0$ then this implies $x_3 = x_2 = x_1$ as well. This also implies $x_4 = x_3 = x_2 = x_1$ or $x_1 = x_4$ even though there is no direct edge.

3.2.4 Problem 1.6.6

1.6.6 In a graph with N nodes and N edges show that there must be a loop.

Figure 3.27: Problem description

Proof by contradiction: Assuming there is no loop. Hence the graph must be a spanning tree. But by definition, a spanning tree with N nodes have $N-1$ edges. But we are given that number of edges is the same as the number of nodes. Hence the assumption is not valid, and there must be a loop, called the fundamental loop or fundamental cycle.

3.2.5 Problem 1.6.7

1.6.7 For electrical networks x represents potentials, Ax represents potential differences, y represents currents, and $A^T y = 0$ is Kirchhoff's current law (Section 2.3). Tellegen's theorem says that Ax is perpendicular to y. How does this follow from the fundamental theorem of linear algebra?

Figure 3.28: Problem description

The fundamental theorem of linear algebra says that vectors in $R(A)$ are orthogonal to vectors in $N\left(A^T\right)$. Ax gives the vectors in $R(A)$ which is the potential difference. While currents y which results in $A^Ty = 0$ are in $N\left(A^T\right)$. The following diagram illustrates this

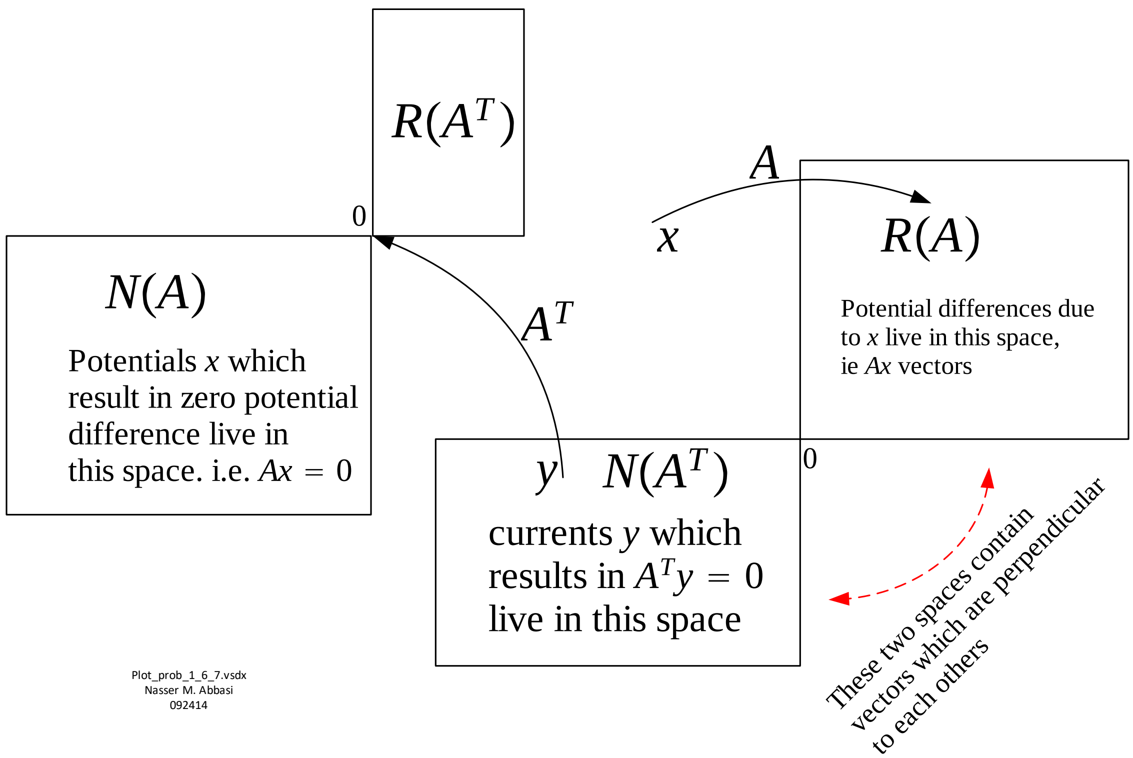


Figure 3.29: Plot for Problem 1.6.7

3.2.6 Problem 2.1.2

2.1.2 (a) Compute the 4 by 4 matrices $A_0^TA_0$ and $A_0^TCA_0$ for the network in Fig. 2.1. Notice that like the original A_0, its columns add up to the zero column.

(b) Verify that removing the last row and column of $A_0^TCA_0$ leaves A^TCA in equation (7). What is A^TA?

(c) Show that this A^TA is positive definite by applying one of the tests in Chapter 1 (for example, compute the determinants or the pivots).

Figure 3.30: Problem description

Figure 2.1 is the following

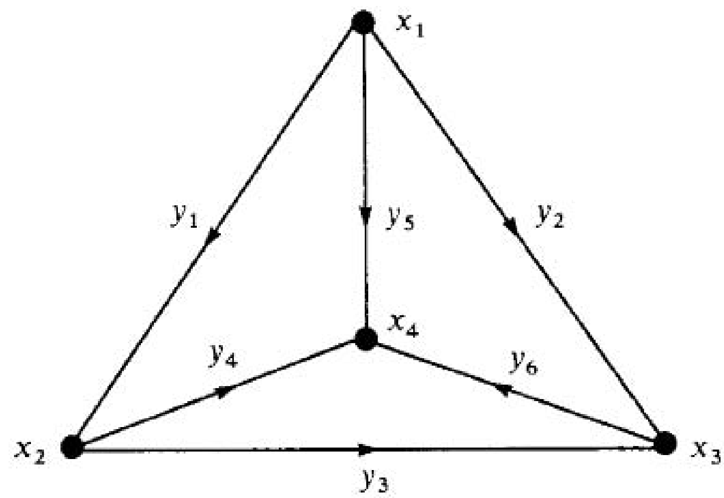


Fig. 2.1. Four nodal variables and six edge variables.

Figure 3.31: Figure 2.1 in book.

The A_o matrix, is the incidence matrix. Since we have 6 edges, the matrix will have 6 rows. Since we have 4 nodes, there will be 4 columns. The matrix is

$$A_o = \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

Hence $A_o^T A_o$ is

$$A_o^T A_o = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

And the C matrix is $m \times m$ where $m = 6$ since this is the number of rows in A_o. Hence

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{pmatrix}$$

Therefore

$$A_o^T C A_o = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{pmatrix} \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

Hence

$$\boxed{A_o^T C A_o = \begin{pmatrix} c_1 + c_2 + c_5 & -c_1 & -c_2 & -c_5 \\ -c_1 & c_1 + c_3 + c_4 & -c_3 & -c_4 \\ -c_2 & -c_3 & c_2 + c_3 + c_6 & -c_6 \\ -c_5 & -c_4 & -c_6 & c_4 + c_5 + c_6 \end{pmatrix}}$$

We notice that the diagonal entry on $A_o^T C A_o^T$ matches the sum on the rest of the row.

3.2.7 Problem 2.1.3

2.1.3 For the triangular network in Fig. 2.1, let $f_1 = f_2 = f_3 = 1$ and $f_4 = -3$. With $C = I$ and $b = 0$, solve the equilibrium equation $-A^T CAX = f$. (Note that f_4 and x_4 do not enter, because $x_4 = 0$ and the last column of A_0 was removed.) Solve also for y, and describe the flows through the network.

Figure 3.32: Problem description

From problem 2.1.2, we found

$$A_o = \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

We first start by removing the last column, hence $A = \begin{pmatrix} -1 & +1 & 0 \\ -1 & 0 & +1 \\ 0 & -1 & +1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ which means x is 3×1 vector now.

We are given that $C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, hence we need to solve the equilibrium equation $-A^TCAx = f - A^TCb$, but $b = 0$, hence this becomes

$$-A^TCAx = f$$

$$-\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We now solve the above by Gaussian elimination which gives

$$\boxed{x = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}}$$

To solve for y we use the first equation of the equilibrium equation after elimination, which is given on page 92 of the textbook as

$$\begin{pmatrix} C^{-1} & A \\ 0 & -A^TCA \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f - A^TCb \end{pmatrix}$$

The first equation gives

$$C^{-1}y + Ax = b$$

And for $b = 0$ this becomes

$$\begin{aligned} y &= -CAx \\ &= -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

or

$$\boxed{y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}}$$

y now is the edges, it is the flow. Hence the above says that in figure 2.1 network, shown again below

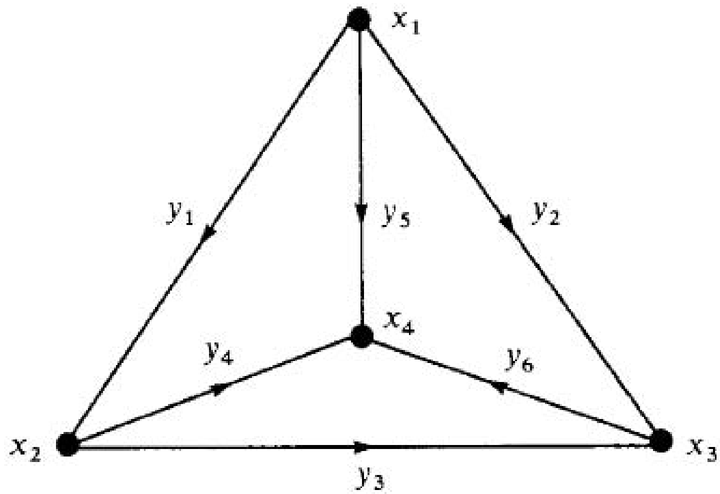


Fig. 2.1. Four nodal variables and six edge variables.

Figure 3.33: plot for 2.1.2

That there is now flow over edges $1,2,3$ (the outer cycle) and flow is only on the inner edges $4,5,6$ in opposite direction shown.

3.2.8 Problem 2.1.6

2.1.6 Suppose a network has N nodes and every pair is connected by an edge. Find m, the number of edges.

Figure 3.34: Problem description

The first node needs $N-1$ edges to connect to the other N. The second node needs $N-2$ edges to connect to the other nodes. We do not count the first one since it is already connected by now. The third node needs $N-3$ edges, and so on. The last node needs no edges, since by the time it is reach, it already has an edge from all the others to it. Hence

$$
\begin{aligned}
m &= (N-1) + (N-2) + \cdots + (N-N) \\
&= \sum_{i=1}^{N} (N-i) \\
&= \sum_{i=1}^{N} N - \sum_{i=1}^{N} i \\
&= N^2 - \frac{1}{2} N (N+1) \\
&= N^2 - \frac{1}{2} N^2 - \frac{1}{2} N \\
&= \frac{1}{2} N^2 - \frac{1}{2} N
\end{aligned}
$$

Hence

$$m = \frac{1}{2} N (N-1)$$

3.2.9 Problem 2.1.12

2.1.12 Draw a network with no loops (a *tree*). Check that with one node grounded the incidence matrix A is square, and find A^{-1}. All entries of the inverse are 1, -1, or 0.

Figure 3.35: Problem description

A tree is drawn with arbitrary directions

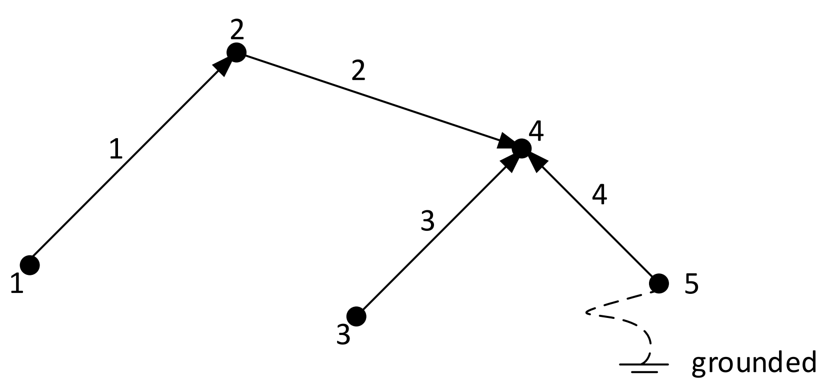


Figure 3.36: Tree for problem 2.1.12

Before grounded node 5 the A matrix is

$$A=\begin{pmatrix}-1 & +1 & 0 & 0 & 0\\ 0 & -1 & 0 & +1 & 0\\ 0 & 0 & -1 & +1 & 0\\ 0 & 0 & 0 & +1 & -1\end{pmatrix}$$

When node 5 is grounded, then column 5 is removed, now the matrix becomes

$$A=\begin{pmatrix}-1 & +1 & 0 & 0\\ 0 & -1 & 0 & +1\\ 0 & 0 & -1 & +1\\ 0 & 0 & 0 & +1\end{pmatrix}$$

And its inverse is

$$A^{-1}=\begin{pmatrix}-1 & -1 & 0 & +1\\ 0 & -1 & 0 & +1\\ 0 & 0 & -1 & +1\\ 0 & 0 & 0 & +1\end{pmatrix}$$

3.2.10 Problem 2.2.1

2.2.1 Minimize $Q=\frac{1}{2}(y_1^2+\frac{1}{3}y_2^2)$ subject to $y_1+y_2=8$ in two ways:

(a) Solve $\partial L/\partial y=0$, $\partial L/\partial x=0$ for the Lagrangian $L=Q+x_1(y_1+y_2-8)$.

(b) Solve the equilibrium equations (with $b=0$) for x and y.

What is the optimal y, and what is the minimum of Q? What is the dual quadratic $-P(x)$, and where is it maximized?

Figure 3.37: Problem description

3.2.10.1 part(a)

$Q=\frac{1}{2}\left(y_1^2+\frac{1}{3}y_2^2\right)$ and constraint $r=y_1+y_2-8=0$ hence $L=Q+xr$ where x here is the Lagrange multiplier. Hence

$$\boxed{L=\frac{1}{2}\left(y_1^2+\frac{1}{3}y_2^2\right)+x\left(y_1+y_2-8\right)}$$

Therefore

$$\begin{aligned}\frac{\partial L}{\partial y_1} &= y_1 + x\\ \frac{\partial L}{\partial y_2} &= \frac{1}{3}y_2 + x\\ \frac{\partial L}{\partial x} &= \left(y_1 + y_2 - 8\right)\end{aligned}$$

In matrix form it becomes

$$\nabla L = \begin{pmatrix} 1 & 0 & 1\\ 0 & \frac{1}{3} & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ x\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 8\end{pmatrix} \tag{1}$$

Solving by Gaussian elimination gives

$$\boxed{\begin{pmatrix} y_1\\ y_2\\ x\end{pmatrix} = \begin{pmatrix} 2\\ 6\\ -2\end{pmatrix}}$$

3.2.10.2 Part(b)

We now compare (1) above to the equilibrium matrix equation given by

$$\begin{pmatrix} C^{-1} & A\\ 0 & A^TCA\end{pmatrix}\begin{pmatrix} y\\ x\end{pmatrix} = \begin{pmatrix} b\\ f - A^TCb\end{pmatrix}$$

Which for $b = 0$ becomes

$$\begin{pmatrix} C^{-1} & A\\ 0 & A^TCA\end{pmatrix}\begin{pmatrix} y\\ x\end{pmatrix} = \begin{pmatrix} 0\\ f\end{pmatrix}$$

From the above, and comparing to (1) we see that $A = \begin{pmatrix} 1\\ 1\end{pmatrix}, C^{-1} = \begin{pmatrix} 1 & 0\\ 0 & \frac{1}{3}\end{pmatrix}, y = \begin{pmatrix} y_1\\ y_2\end{pmatrix}, f = 8$. Hence we first solve for x

$$\begin{aligned} A^TCAx &= f\\ \begin{pmatrix} 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix} 1\\ 1\end{pmatrix}x &= 8\\ 4x &= 8\\ x &= 2\end{aligned}$$

Now the first equation is used to solve for y

$$\begin{aligned}
C^{-1}y + Ax &= 0\\
y &= CAx\\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} x\\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} 2
\end{aligned}$$

Hence the optimal y is

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}}$$

Which is the same as in part(a). At this point, Q is now evaluated

$$\begin{aligned}
Q_{\min} &= \frac{1}{2}\left(y_1^2 + \frac{1}{3}y_2^2\right)\\
&= \frac{1}{2}\left(2^2 + \frac{1}{3}6^2\right)\\
&= 8
\end{aligned}$$

The dual quadratic is given on page 101 of the text

$$-P(x) = -\frac{1}{2}(Ax-b)^T C (Ax-b) - x^T f$$

And for $b=0$ it becomes

$$-P(x) = -\frac{1}{2}x^T A^T C A x - x^T f$$

But from above, $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, f = 8$ hence

$$\begin{aligned}
-P(x) &= -\frac{1}{2}x^T \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} x - 8x^T\\
&= -\frac{1}{2}x^T 4x - 8x^T\\
&= -2x^T x - 8x^T
\end{aligned}$$

But $x^T x = x^2$ so the above can be written as

$$\begin{aligned}
-P(x) &= -2x^2 - 8x\\
P(x) &= x(2x+8)
\end{aligned}$$

To find where it is maximum, since $\frac{dP}{dx} = 0 = 4x + 8$ hence $\boxed{x = -2}$. Therefore, $-P(x)$ is maximized at same x where $Q(x)$ is minimized.

3.2.11 Problem 2.2.2

2.2.2 Find the nearest point to the origin on the plane $y_1 + y_2 + \ldots + y_m = 1$ by solving for y_1, substituting into $Q = \frac{1}{2}(y_1^2 + \ldots + y_m^2)$, and minimizing with respect to the other y's. Then solve the same problem with Lagrange multipliers.

Figure 3.38: Problem description

$$Q = \frac{1}{2}\left(y_1^2 + y_2^2 + \cdots + y_m^2\right)$$

Constraints in $y_1 + y_2 + \cdots + y_m = 1$. Solving for y_1 from the constrains and substitute the result in Q. Hence

$$y_1 = 1 - \left(y_2 + y_3 + \cdots + y_m\right)$$

And Q becomes

$$\begin{aligned}
Q &= \frac{1}{2}\left(\left[1 - \left(y_2 + y_3 + \cdots + y_m\right)\right]^2 + \left(y_2^2 + \cdots + y_m^2\right)\right)\\
&= \frac{1}{2}\left(1 + \left(y_2 + y_3 + \cdots + y_m\right)^2 - 2\left(y_2 + y_3 + \cdots + y_m\right) + \left(y_2^2 + \cdots + y_m^2\right)\right)\\
&= \frac{1}{2} + \frac{1}{2}\left(y_2 + y_3 + \cdots + y_m\right)^2 - \left(y_2 + y_3 + \cdots + y_m\right) + \frac{1}{2}\left(y_2^2 + \cdots + y_m^2\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial Q}{\partial_{y_2}} &= \left(y_2 + y_3 + \cdots + y_m\right) - 1 + y_2 = 0\\
\frac{\partial Q}{\partial_{y_3}} &= \left(y_2 + y_3 + \cdots + y_m\right) - 1 + y_3 = 0\\
&\vdots\\
\frac{\partial Q}{\partial_{y_m}} &= \left(y_2 + y_3 + \cdots + y_m\right) - 1 + y_m = 0
\end{aligned}$$

The above can be written as

$$\begin{aligned}
2y_2 + y_3 + \cdots + y_m &= 1\\
y_2 + 2y_3 + \cdots + y_m &= 1\\
&\vdots\\
y_2 + y_3 + \cdots + 2y_m &= 1
\end{aligned}$$

In matrix form,

$$\boxed{\begin{pmatrix}2&1&1&\cdots&1\\1&2&1&\cdots&1\\1&1&2&\cdots&1\\1&1&1&\ddots&1\\1&1&1&\cdots&2\end{pmatrix}\begin{pmatrix}y_2\\y_3\\y_4\\\vdots\\y_m\end{pmatrix}=\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}}$$

Solving this gives

$$\boxed{y_2=y_3=\cdots=y_m=\frac{1}{m}}$$

This was done by solving for $m=3,4,5\cdots$ on the computer and seeing the result is always $\frac{1}{m}$. Now we solve for y_1. Since

$$y_1=1-\left(y_2+y_3+\cdots+y_m\right)$$

Then

$$\begin{aligned}y_1&=1-\left(\frac{1}{m}+\frac{1}{m}+\cdots+\frac{1}{m}\right)\\&=1-(m-1)\frac{1}{m}\\&=1-\left(1-\frac{1}{m}\right)\\&=\frac{1}{m}\end{aligned}$$

Therefore, all y_i have the value $\frac{1}{m}$

Now the last part is solved, which asks to solve the same problem using Lagrange multiplier. Since there is one constraint, then $n=1$ and since there are m number of y variables, there will be $n+m$ or $m+1$ equations.

$$L=Q+xR$$

Where R is the contraints. The above becomes

$$L=\frac{1}{2}\left(y_1^2+y_2^2+\cdots+y_m^2\right)+x\left(y_1+y_2+\cdots+y_m-1\right)$$

Now we take the derivatives and set up the system of equations

$$
\begin{aligned}
\frac{\partial L}{\partial y_1} &= y_1 + x = 0\\
\frac{\partial L}{\partial y_2} &= y_2 + x = 0\\
&\vdots\\
\frac{\partial L}{\partial y_m} &= y_m + x = 0\\
\frac{\partial L}{\partial x} &= \left(y_1 + y_2 + \cdots + y_m - 1\right) = 0
\end{aligned}
$$

In matrix form the above is

$$
\boxed{\begin{pmatrix} 1 & 0 & 0 & \cdots & 1\\ 0 & 1 & 0 & \cdots & 1\\ 0 & 0 & 1 & \cdots & 1\\ 0 & 0 & 0 & \ddots & 1\\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_3\\ \vdots\\ x \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ \vdots\\ 1 \end{pmatrix}}
$$

Solving this also gives the same answer as above, which is

$$
y_i = \frac{1}{m}
$$

and the Lagrange multipler is found, using any of the above equation, such as $y_1 + x = 0$ to be

$$
\boxed{x = -\frac{1}{m}}
$$

3.2.12 Problem 2.2.4

2.2.4 Find the rectangle with corners at points $(\pm y_1, \pm y_2)$ on the ellipse $y_1^2 + 4y_2^2 = 1$, such that the perimeter $4y_1 + 4y_2$ is as large as possible.

Figure 3.39: Problem description

We want to maximize $Q = 4y_1 + 4y_2$ subject to $y_1^2 + 4y_2^2 = 1$. Hence

$$
\begin{aligned}
L &= Q + x\left(y_1^2 + 4y_2^2 - 1\right)\\
&= 4y_1 + 4y_2 + x\left(y_1^2 + 4y_2^2 - 1\right)
\end{aligned}
$$

And

$$\frac{\partial L}{\partial y_1} = 4 + 2xy_1 = 0 \tag{1}$$

$$\frac{\partial L}{\partial y_2} = 4 + 8xy_2 = 0 \tag{2}$$

$$\frac{\partial L}{\partial x} = y_1^2 + 4y_2^2 - 1 = 0 \tag{3}$$

Or

$$y_1 = \frac{-2}{x} \tag{1}$$

$$y_2 = \frac{-1}{2x} \tag{2}$$

$$y_1^2 + 4y_2^2 = 1 \tag{3}$$

From (1),(2) we see that $y_1 = 4y_2$. Substituting in (3) gives

$$\begin{aligned}\left(4y_2\right)^2 + 4y_2^2 &= 1\\ 16y_2^2 + 4y_2^2 &= 1\\ y_2 &= \pm\sqrt{\frac{1}{20}}\end{aligned}$$

Hence

$$y_1 = \pm\sqrt{\frac{16}{20}} = \pm\sqrt{\frac{4}{5}}$$

So the corners are $\left(\pm\sqrt{\frac{4}{5}}, \pm\sqrt{\frac{1}{20}}\right)$. Here is a plot of the ellipse showing the 4 corners given by the above solution to verify

```
a = 1;
b = (1/2);
y1 = Sqrt[4/5]; y2 = Sqrt[1/20];
Graphics[
{
Circle[{0, 0}, {a, b}],
{EdgeForm[Thick], LightGray, Rectangle[{-y1, -y2}, {y1, y2}]}
},
Axes -> True]
```

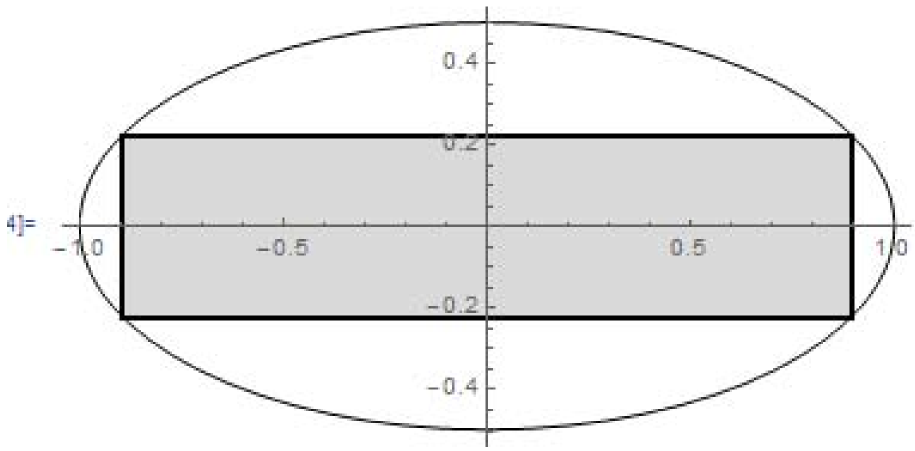


Figure 3.40: plot for prob 2.2.4

3.2.13 Problem 2.2.6

2.2.6 The minimum distance to the surface $A^T y = f$ equals the maximum distance to the hyperplanes which ____
Complete this statement of duality.

Figure 3.41: Problem description

From the duality statement on page 100 of the text, we can complete this sentence similarly by saying

The minimum distance to the surface $A^T y = f$ equals the maximum distance to the hyperplanes which go through those hyperplanes.

3.3 HW 3, Due Oct 16, 2014

3.3.1 Problem 2.2.7

2.2.7 How far is it from the origin $(0,0,0)$ to the plane $y_1+2y_2+2y_3=18$? Write this constraint as $A^Ty=18$, and solve for y in

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix}.$$

Figure 3.42: the Problem statement

The objective function is $\frac{1}{2}\|d\|^2$ where d is the distance from origin the plane. Hence $Q(y) = \frac{1}{2}\left(y_1^2+y_2^2+y_3^2\right)$. The constraint $R = y_1+2y_2+2y_3-18$. Therefore, the Lagrangian is

$$\begin{aligned} L(y,x) &= Q(y) + xR \\ &= \frac{1}{2}\left(y_1^2+y_2^2+y_3^2\right) + x\left(y_1+2y_2+2y_3-18\right) \end{aligned}$$

Now we set up the optimization problem

$$\begin{aligned} \frac{\partial L}{\partial y_1} &= y_1 + x = 0 \\ \frac{\partial L}{\partial y_2} &= y_2 + 2x = 0 \\ \frac{\partial L}{\partial y_3} &= y_3 + 2x = 0 \\ \frac{\partial L}{\partial x} &= y_1 + 2y_2 + 2y_3 - 18 \end{aligned}$$

In Matrix form

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}} \tag{1}$$

Comparing the above to the standard form given

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$$

We see that $A = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Now we solve (1) using Gaussian elimination

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

Hence $U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix}$. Therefore $Lc = x$ or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}$$

Hence $c_1 = 0, c_2 = 0, c_3 = 0$, $c_4 = 18$. Now solving $Ux = c$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}$$

Hence solution is Solution is:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ -2 \end{pmatrix}$$

So the Lagrangian multiplier is $\boxed{x = -2}$. Now we can calculate the distance

$$\begin{aligned} d &= \sqrt{y_1^2 + y_2^2 + y_3^2} \\ &= \sqrt{2^2 + 4^2 + 4^2} \\ &= 6 \end{aligned}$$

3.3.2 Problem 2.2.8

2.2.8 The previous question brings together several parts of mathematics if you answer it more than once:

(i) The vector y to the nearest point on the plane must be on the perpendicular ray. Therefore y must be a multiple of (1, 2, 2). What multiple lies on the plane $y_1 + 2y_2 + 2y_3 = 18$? What is the length of this y?

(ii) Since $A^T = [1\ 2\ 2]$ has length $(1+4+4)^{1/2} = 3$, the Schwarz inequality for inner products gives

$$A^T y \leq \| A \| \, \| y \| \quad \text{or} \quad 18 \leq 3 \| y \|.$$

What is the minimum possible length $\| y \|$? Conclusion: The distance to the plane $A^T y = f$ is $| f | / \| A \|$.

Figure 3.43: the Problem statement

3.3.2.1 Part (i)

Let us assume that

$$y = k \times [1, 2, 2]$$

where k is this multiple. This means $y_1 = k, y_2 = 2k, y_2 = 2k$. In other words, the vector is

$$y = [k, 2k, 2k]$$

But since the constraint is $y_1 + 2y_2 + 2y_3 = 18$ this substituting the values of each y_i in the constraint gives

$$\begin{aligned} k + 2(2k) + 2(2k) &= 18 \\ 9k &= 18 \end{aligned}$$

Hence

$$\boxed{k = 2}$$

Using this k, the vector is

$$\begin{aligned} y &= [k, 2k, 2k] \\ &= [2, 4, 4] \end{aligned}$$

Hence the norm of the vector is

$$\begin{aligned} \|y\| &= \sqrt{y_1^2 + y_2^2 + y_3^2} \\ &= \sqrt{2^2 + 4^2 + 4^2} \\ &= 6 \end{aligned}$$

3.3.2.2 Part(ii)

Using

$$\begin{aligned} 18 &\leq 3\left\Vert y\right\Vert \qquad (1)\\ \left\Vert y\right\Vert &\geq 6 \end{aligned}$$

Therefore minimum length of y must be 6.

In (1), $18 = f$ from the equation $A^T y = f$ and $3 = ||A||$. This means the

$$y_m in = \frac{f}{||A||}$$

3.3.3 Problem 2.2.9

2.2.9 In the first example of duality—"the minimum distance to points equals the maximum distance to planes"—how do you know immediately that maximum $\leq$ minimum? In other words explain *weak duality*: The distance to any plane through the line is not greater than the distance to any point on the line.

Figure 3.44: the Problem statement

The primal problem is minimization of $Q\left(y\right)$ over y (unconstrained optimization), and the dual problem is maximization of $-P(x)$ over x. The minimum of $Q\left(y\right)$ is the maximum of $-P(x)$. This is the weak duality. In this problem, the point on the line must also be on a point on the plane since the line is constrained to be on the plane.

So the distance to the plane can not be larger than the distance to the line. The distance to the plane is represented by $-P(x)$ and the distance to the the line is represented by $Q\left(y\right)$. So this leads to

$$-P(x) \leq Q\left(y\right)$$

3.3.4 Problem 2.2.10

2.2.10 If $b = (15, 10)$ in the geometry example of Fig. 2.4, what are the optimal Ax and y and what are the lengths in $\|Ax\|^2 + \|y\|^2 = \|b\|^2$?

Figure 3.45: the Problem statement

The figure mentioned in the problem is

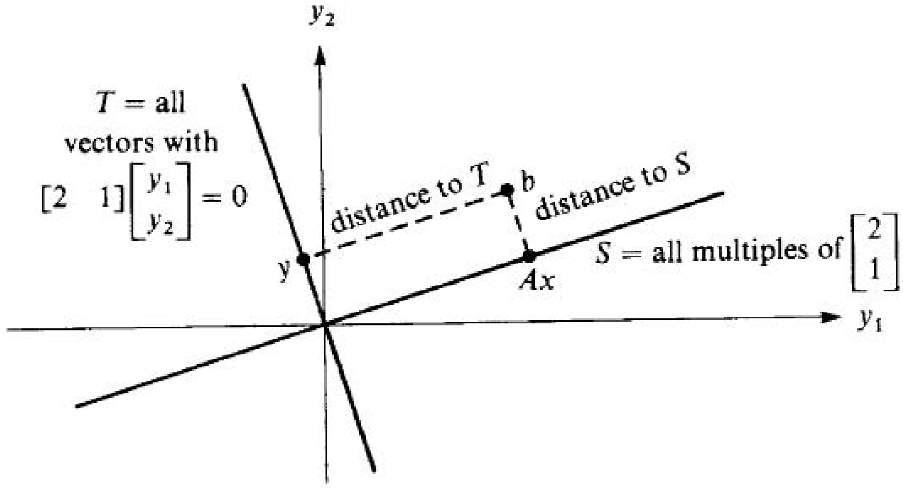


Fig. 2.4. Projection of b onto orthogonal subspaces S and T.

Figure 3.46: figure mentioned in problem 2.2.10

To find distance to S, we need to solve

$$
\begin{aligned}
(\text{distance to S})^2 &= \min_x (Ax - b)^T (Ax - b) \\
&= \min_x \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} x - \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right)^T \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} x - \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right) \\
&= \min_x (x - 10)^2 + (2x - 15)^2 \\
&= \min_x 5x^2 - 80x + 325
\end{aligned}
$$

Hence $\frac{d}{dx}\left(5x^2 - 80x + 325\right) = 10x - 80$ hence $x = \frac{80}{10} = 8$. Therefore

$$
Ax = \begin{pmatrix} 16 \\ 8 \end{pmatrix}
$$

To find y we need to solve

$$
\begin{aligned}
(\text{distance to T})^2 &= \min_{A^T y = 0} \left\| b - y \right\|^2 = \min_{A^T y = 0} y^T y - 2 b^T y + b^T b \\
&= \min_{A^T y = 0} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - 2 \begin{pmatrix} 15 \\ 10 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 15 \\ 10 \end{pmatrix}^T \begin{pmatrix} 15 \\ 10 \end{pmatrix} \\
&= y_1^2 - 30 y_1 + y_2^2 - 20 y_2 + 325
\end{aligned}
$$

Need to minimize the above subject to $A^T y = 0$ or $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, or $2y_1 + y_2 = 0$. Therefore, we setup an optimization problem

$$
\begin{aligned}
L &= Q + xR \\
&= y_1^2 - 30y_1 + y_2^2 - 20y_2 + 325 + x\left(2y_1 + y_2\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial L}{\partial y_1} &= 2y_1 - 30 + 2x = 0 \\
\frac{\partial L}{\partial y_2} &= 2y_2 - 20 + x = 0 \\
\frac{\partial L}{\partial x} &= 2y_1 + y_2 = 0
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 0 \end{pmatrix}
$$

Solving gives

$$
\begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 16 \end{pmatrix}
$$

Hence

$$
y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}
$$

Since now we know the optimal Ax and y, we can find the lengths.

$$
\boxed{||Ax|| = \left\| \begin{pmatrix} 16 \\ 8 \end{pmatrix} \right\| = 8\sqrt{5}}
$$

and

$$
\boxed{\|y\| = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = \sqrt{5}}
$$

and

$$
\boxed{||b|| = \left\| \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right\| = 5\sqrt{13}}
$$

Therefore

$$\left(8\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 = \left(5\sqrt{13}\right)^2$$
$$325 = 325$$

OK, verified.

3.3.5 Problem 2.2.16

2.2.16. In m dimensions, how far is it from the origin to the hyperplane $x_1 + x_2 + \cdots + x_m = 1$? Which point on the plane is nearest to the origin?

Figure 3.47: the Problem statement

The constraint is $x_1+x_2+\cdots+x_m = 1$ and the objective function is $\frac{1}{2}\|d\|^2 = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_m^2\right)$. Hence

$$L = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_m^2\right) + x\left(x_1+x_2+\cdots+x_m-1\right)$$

Setting up

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_1 + x = 0\\
\frac{\partial L}{\partial x_2} &= x_2 + x = 0\\
&\vdots\\
\frac{\partial L}{\partial x_n} &= x_n + x = 0\\
\frac{\partial L}{\partial x} &= x_1 + x_2 + \cdots + x_m - 1 = 0
\end{aligned}$$

Or in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 1\\ 0 & 1 & 0 & \cdots & 1\\ 0 & 0 & 1 & \cdots & 1\\ 0 & 0 & \cdots & 1 & 1\\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_m\\ x \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ \vdots\\ 1 \end{pmatrix}$$

Solving, for specific m to be able to see the pattern gives for $m = 3$

$$\begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 1 \end{pmatrix}$$

Solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

So $\boxed{x_i = \frac{1}{m}}$ so the distance is

$$\begin{aligned} \left\| x_1^2 + x_2^2 + \cdots + x_m^2 \right\| &= \sqrt{m \left(\frac{1}{m} \right)^2} \\ &= \sqrt{m} \end{aligned}$$

3.3.6 Problem 2.4.1

2.4.1 Write down *m*, *N*, *r*, and *n* for the three trusses in Fig. 2.10, and establish which is statically determinate, which is statically indeterminate, and which one has a mechanism. Describe the mechanism (the uncontrolled deformation).

Figure 3.48: the Problem statement

Figure 2.10 is

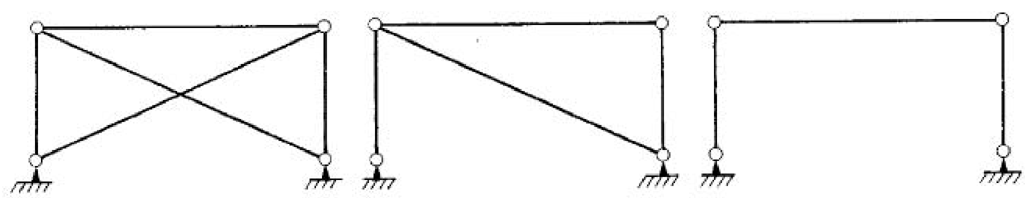


Fig. 2.10. Trusses with $m > n$ (indeterminate), $m = n$ (determinate), $m < n$ (unstable).

Figure 3.49: Figure 2.10 in book

m is number of bars, and N is number of nodes. Truss is stable if $m \geq 2N - r$ where r is the number of constraints. For determining rigid motion and mechanism, we need to solve $Ax = 0$ and look at the solutions.

	N (nodes)	m(bar)	r	$n = 2N - r$	determinate? $m = n$	indeterminate? $m > n$	stable?
1	4	5	4	4	No	yes	stable
2	4	4	4	4	Yes	No	stable
3	4	3	4	4	No	No	mechanism

For case (3), since it is neither determinate nor indeterminate, we need to look at $Ax = 0$. But it is clear that the truss in (3) will not move as a rigid body, but will deform. It is not stable. The table below summarizes the results.

3.3.7 Problem 2.4.4

2.4.4 For the truss in Fig. 2.10c, write down the equations $A^T y = f$ in three unknowns y_1, y_2, y_3 to balance the four external forces $f_H^1, f_H^2, f_V^1, f_V^2$. Under what condition on these forces will the equations have a solution (allowing the truss to avoid collapse)?

Figure 3.50: the Problem statement

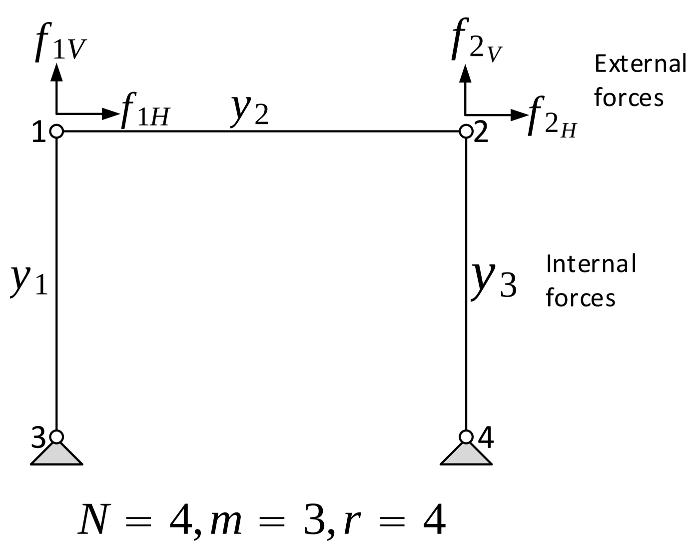


Figure 3.51: Figure for problem 2.4.4

The A matrix is found from $A^T y = f$. where f is a column vector of length 4 since there are 2 nodal forces, and each has 2 components. This represents a force at each node. So we first find A^T. To do this, we resolve internal forces y to balance the external nodal forces f. We assume there are nodal forces only on nodes 1, 2 in the above diagram and that $f_3 = f_4 = 0$.

Clearly $f_{1V} = y_1$ to make forces balance in the vertical direction at node 1 and that $f_{2V} = y_3$ for similar reason on node 2. On node 1, assuming y_2 is in positive, so in tension, then

$-f_{1H} = y_2$ and $+f_{2H} = y_2$. If we had assumed y_2 is in the negative direction then we will get same result but signs reversed.

Therefore

$$\begin{aligned} f_{1V} &= y_1 \\ f_{2V} &= y_3 \\ f_{1H} &= -y_2 \\ f_{2H} &= y_2 \end{aligned}$$

Hence $A^T y = f$ becomes

$$\boxed{\overbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{A^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_{1H} \\ f_{1V} \\ f_{2H} \\ f_{2V} \end{pmatrix}}$$

Hence

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix A has rank 3 and the same for A^T. For $A^T y = f$ to have solution, then f must be in the column space of A^T. For solution, (equilibrium) we need $\sum f_{iH} = 0$ and $\sum f_{iV} = 0$ and moments about a point zero.

3.3.8 Problem 2.4.10

2.4.10 If we create a new node in Fig. 2.10a where the diagonals cross, is the resulting truss statically determinate or indeterminate?

Figure 3.52: the Problem statement

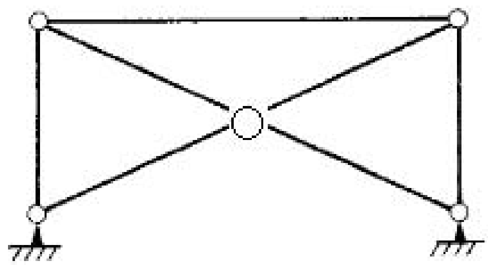


Figure 3.53: Figure for problem 2.4.10

With the new truss as above, the number of bars $m = 7$, and the number of nodes is $N = 5$. The number of constraints $r = 4$ (two from each support). Hence

$$\begin{aligned} n &= 2N - r \\ &= 10 - 4 \\ &= 6 \end{aligned}$$

Therefore $m > n$ and A is not square. Hence not statically determinate.

3.3.9 Problem 2.4.11

2.4.11 In continuum mechanics, work is the product of stress and strain integrated over the structure: $W = \int \sigma\varepsilon \, dV$. If a bar has uniform stress $\sigma = y/A$ and uniform strain $\varepsilon = e/L$, show by integrating over the volume of the bar that $W = ye$. Then the sum over all bars is $W_{\text{total}} = y^T e$; show that this equals $f^T x$.

Figure 3.54: the Problem statement

Work over the first bar, of say length L_1 is

$$\begin{aligned} W_i &= \int \sigma_1 \epsilon_1 dV \\ &= \int \frac{y_1}{A_1}\frac{e_1}{L_1} A_1 dL \\ &= y_1 \frac{e_1}{L_1} \int dL \\ &= y_1 \frac{e_1}{L_1} L_1 \\ &= y_1 e_1 \end{aligned}$$

Therefore, the sum all the truss is $y_1e_1 + y_2e_2 + \cdots + y_me_m$ or

$$
\begin{aligned}
W_{total} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} \\
&= y^T e \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
A^T y &= f \\
\left(A^T y\right)^T &= f^T \\
y^T A &= f^T \\
y^T &= f^T A^{-1} \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) gives

$$W_{total} = f^T A^{-1} e \qquad (3)$$

But

$$e = Ax$$

Hence (3) becomes

$$
\begin{aligned}
W_{total} &= f^T A^{-1} A x \\
&= f^T x
\end{aligned}
$$

This is an expression of the work done by external forces at nodes. So this says the internal work equals the external work.

3.3.10 Problem 2.4.12

2.4.12 At the equilibrium $x = K^{-1}f$, show that the strain energy U (the quadratic term in P) equals $-P_{\min}$, and therefore $U = Q_{\min}$.

Figure 3.55: the Problem statement

The potential energy is $P(x) = \frac{1}{2}x^T A^T C A x - f^T x$. This is minimum at $A^T C A x = f$. Hence

$$
\begin{aligned}
P_{\min}(x) &= \frac{1}{2}x^T A^T C A x - \left(A^T C A x\right)^T x \\
&= \frac{1}{2}x^T A^T C A x - x^T A^T C^T A x
\end{aligned}
$$

But $C = C^T$ since diagonal matrix, then

$$P_{\min}(x) = -\frac{1}{2}x^T A^T C A x$$
$$-P_{\min}(x) = \frac{1}{2}x^T A^T C A x$$

But strain energy is the quadratic term in $P(x)$, which is $\frac{1}{2}x^T A^T C A x$. Hence they are the same, which is what we are asked to show.

3.3.11 Problem 2.4.17

2.4.17 For *networks*, a typical row of $A_0^T C A_0$ (say row 1) is described on page 92: The diagonal entry is Σc_i, including all edges into node 1, and each $-c_i$ appears along the row. It is in column k if edge i connects nodes 1 and k. ($A^T C A$ is the same with the grounded row and column removed.) The problem is to describe $A_0^T C A_0$ for *trusses*, and the idea is to put together the special $A_0^T C A_0$ found in the previous exercise (*a 4 by 4 matrix for each bar*).

(a) Suppose bar i goes at angle θ_i from node 1 to node k. By assembling the $A_0^T C A_0$ for each bar, show how the 2 by 2 upper left corner of $A_0^T C A_0$ contains

$$\begin{bmatrix} \Sigma c_i \cos^2\theta_i & \Sigma c_i \cos\theta_i \sin\theta_i \\ \Sigma c_i \cos\theta_i \sin\theta_i & \Sigma c_i \sin^2\theta_i \end{bmatrix}$$

(b) Where do those terms appear (with minus signs) in the first two rows? All rows of $A_0^T C A_0$ add to zero.

Figure 3.56: the Problem statement

If we have a bar 1, then the elongation is due to total motion of bar two nodes due to motion of all bar attached as was shown on page 124 of the text, which is

$$e_1 = x_1 \cos\theta_1 - x_3 \cos\theta_1 + x_2 \sin\theta_1 - x_4 \sin\theta_1$$

The second bar 2 which could have one joint common with the bar 1, say (x_3, x_4) displacement, will then add to these when bar 2 itself deforms. Hence for bar 2 we have

$$e_2 = x_5 \cos\theta_2 - x_3 \cos\theta_2 + x_6 \sin\theta_2 - x_4 \sin\theta_2$$

Where in the above x_3, x_4 are kept the same as bar 1 since the joint is common. Now if bar 3 had joint (x_1, x_2) common with bar 1, it will have

$$e_3 = x_1 \cos\theta_3 - x_7 \cos\theta_3 + x_2 \sin\theta_3 - x_8 \sin\theta_3$$

When assembling the Ax matrix the pattern given should result using trigonometric relations.

3.4 HW 4, Due Oct 30, 2014

3.4.1 Problem 3.1.1

3.1.1 For a bar with constant c but with decreasing $f=1-x$, find $w(x)$ and $u(x)$ as in equations (8–10).

Figure 3.57: the Problem statement

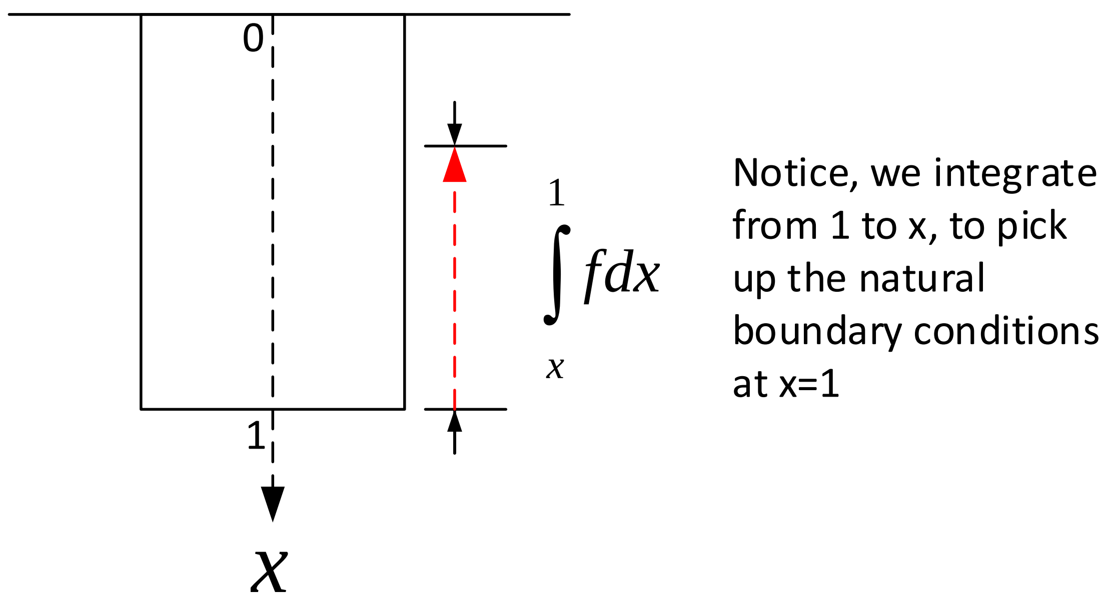


Figure 3.58: Figure for 3.1.1

Starting with the differential equation for u (which is the longitudinal deformation of the bar along the x axis)

$$-c\frac{d^2u}{dx^2} = f(x)$$

And using $f(x) = 1 - x$ and integrating both sides gives

$$-c\int_x^1 \frac{d^2u}{d\tau^2}d\tau = \int_x^1 (1-\tau)\,d\tau$$

$$-c\left[\frac{du}{d\tau}\right]_x^1 = \left[\tau - \frac{\tau^2}{2}\right]_x^1$$

But $\frac{du}{dx} = w$, and $w(1) = 0$, hence the above becomes

$$-c\left[e(1) - e(x)\right] = \left[\left(1 - \frac{1^2}{2}\right) - \left(x - \frac{x^2}{2}\right)\right]$$

But $ce = w$, hence the above can be written as

$$-\left[w(1) - w(x)\right] = \frac{1}{2} - x + \frac{x^2}{2}$$

But $w(1) = 0$, hence

$$w(x) = \frac{1}{2} - x + \frac{x^2}{2}$$

To find $u(x)$, we use the relation that

$$c\frac{du}{dx} = w(x)$$

This is the same as $ce = w(x)$, since strain $e = \frac{du}{dx}$. So we integrate one more time, but this time, we integrate from 0 to x instead from 1 to x. This is in order to pick up the essential boundary conditions on u at $x = 0$, since $u(1)$ is not known, it would be an error to use the first integration limits used earlier above. Hence

$$\begin{aligned}
\int_0^x c\frac{du}{d\tau}d\tau &= \int_0^x w(\tau)\,d\tau \\
c\int_0^x \frac{du}{d\tau}d\tau &= \int_0^x \frac{1}{2} - \tau + \frac{\tau^2}{2}d\tau \\
c\left[u\right]_0^x &= \left[\left(\frac{\tau}{2} - \frac{\tau^2}{2} + \frac{\tau^3}{6}\right)\right]_0^x \\
c\left(u(x) - u(0)\right) &= \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)
\end{aligned}$$

But $u(0) = 0$ since fixed there. This is the essential boundary conditions we are give. The above now simplifies to

$$u(x) = \frac{1}{c}\left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)$$

3.4.2 Problem 3.1.2

3.1.2 For a hanging bar with constant f but weakening elasticity $c(x) = 1 - x$, find the displacement $u(x)$. The first step $w = (1 - x)f$ is the same as in (9), but there will be stretching even at $x = 1$ where there is no force. (The condition is $w = c\, du/dx = 0$ at the free end, and $c = 0$ allows $du/dx \neq 0$.)

Figure 3.59: the Problem statement

Since $ce = w(x)$, then $w(x) = (1-x)e$ and since $e = \frac{du}{dx}$ then

$$w(x) = (1-x)\frac{du}{dx}$$

But $-\frac{dw}{dx} = f$, hence integrating both sides gives

$$-\int_x^1 \frac{dw}{d\tau} d\tau = \int_x^1 f d\tau$$

$$-[w]_x^1 = f\int_x^1 d\tau$$

$$-(w(1) - w(x)) = f(1-x)$$

But $w(1) = 0$, hence

$$w(x) = f(1-x)$$

We found from above that $w(x) = (1-x)\frac{du}{dx}$, therefore

$$(1-x)\frac{du}{dx} = f(1-x)$$

$$\frac{du}{dx} = f$$

Integrating one more time to find $u(x)$

$$\int_0^x \frac{du}{d\tau} d\tau = \int_0^x f d\tau$$

$$[u]_0^x = fx$$

$$u(x) - u(0) = fx$$

But $u(0) = 0$, hence

$$u(x) = fx$$

3.4.3 Problem 3.1.4

3.1.4 With the bar still free at both ends, what is the condition on the external force f in order that $-\dfrac{dw}{dx} = f(x)$, $w(0) = w(1) = 0$ has a solution? (Integrate both sides of the equation from 0 to 1.) This corresponds in the discrete case to solving $A_0^T y = f$; there is no solution for most f, because the left sides of the equations add to zero.

Figure 3.60: the Problem statement

Since $-\frac{dw}{dx}=f$, then integrating from 0 to 1, gives

$$-\int_0^1 \frac{dw}{d\tau}d\tau = \int_0^1 f d\tau$$

$$-\left[w(1)-w(0)\right] = \int_0^1 f d\tau$$

If $w(1)=0$ and $w(0)=0$, then this implies

$$\int_0^1 f d\tau = 0$$

Therefore the only possibility for solution is that $\int_0^1 f d\tau = 0$. For example, a constant none zero f will not work, since this will result in $f=0$ which is a contradiction.

3.4.4 Problem 3.1.5

3.1.5 Find the displacement for an exponential force, $-u''=e^x$ with $u(0)=u(1)=0$.

Note that $A+Bx$ is the general solution to $-u''=0$; it can be added to any particular solution for the given f, and A and B can be adjusted to fit the boundary conditions.

Figure 3.61: the Problem statement

The general solution is $u=u_h+u_p$. For the homogeneous solution $u_h=A+Bx$, now we find the particular solution. By inspection we see that $u_p=-e^x$ satisfies the differential equation. Hence

$$u = A + Bx - e^x$$

We now apply the boundary conditions to find A,B. At $x=0$,

$$\begin{aligned} 0 &= A - e^0 \\ 0 &= A - 1 \\ A &= 1 \end{aligned}$$

Therefore $u=1+Bx-e^x$. At $u=1$ we find

$$\begin{aligned} 0 &= 1 + B - e^1 \\ B &= e - 1 \end{aligned}$$

Hence the solution is

$$u = 1 + (e-1)x - e^x$$

3.4.5 Problem 3.1.6

3.1.6 Suppose the force f is constant but the elastic constant c jumps from $c=1$ for $x \le \frac{1}{2}$ to $c=2$ for $x > \frac{1}{2}$. Solve $-dw/dx = f$ with $w(1)=0$ as before, and then solve $c\, du/dx = w$ with $u(0)=0$. Even if c jumps, the combination $w = c\, du/dx$ remains smooth.

Figure 3.62: the Problem statement

Using $-\frac{dw}{dx} = f$, integrating both sides

$$\begin{aligned}
-\int_x^1 \frac{dw}{d\tau} d\tau &= \int_x^1 f d\tau \\
-\left[w(\tau)\right]_x^1 &= (1-x) f \\
-\left(w(1) - w(x)\right) &= (1-x) f \\
w(x) &= (1-x) f
\end{aligned}$$

Since $w(1) = 0$. Now we use $ce = w(x)$ to solve for u. Since $e = \frac{du}{dx}$. For $0 \le x \le \frac{1}{2}$ we solve, using $c = 1$

$$\begin{aligned}
c\frac{du}{dx} &= (1-x) f \\
\int_0^x \frac{du}{d\tau} d\tau &= \int_0^x (1-\tau) f d\tau \\
\left[u(\tau)\right]_0^x &= f\left[\tau - \frac{\tau^2}{2}\right]_0^x \\
u(x) - u(0) &= f\left(x - \frac{x^2}{2}\right)
\end{aligned}$$

But $u(0) = 0$, hence the solution is

$$u(x) = f\left(x - \frac{x^2}{2}\right) \qquad 0 \le x \le \frac{1}{2} \tag{1}$$

We now integrate over the second half, where $c = 2$

$$
\begin{aligned}
c\frac{du}{dx} &= (1-x)\,f \\
\int_{\frac{1}{2}}^{x} 2\frac{du}{d\tau}d\tau &= \int_{\frac{1}{2}}^{x}(1-\tau)\,f d\tau \\
2\left[u(\tau)\right]_{\frac{1}{2}}^{x} &= f\left[\tau - \frac{\tau^2}{2}\right]_{\frac{1}{2}}^{x} \\
2\left(u(x) - u\left(\frac{1}{2}\right)\right) &= f\left(\left(x - \frac{x^2}{2}\right) - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\right)\right) \\
2u(x) - 2u\left(\frac{1}{2}\right) &= f\left(-\frac{1}{2}x^2 + x - \frac{3}{8}\right) \qquad (2)
\end{aligned}
$$

To find $u\left(\frac{1}{2}\right)$ we use the earlier solution (1) above $u\left(\frac{1}{2}\right) = f\left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\right) = \frac{3}{8}f$, hence (2) becomes

$$
\begin{aligned}
2u(x) - \frac{3}{4}f &= \left(-\frac{1}{2}x^2 + x - \frac{3}{8}\right)f \\
2u(x) &= \left(-\frac{1}{2}x^2 + x - \frac{3}{8} + \frac{3}{4}\right)f \\
u(x) &= \left(-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{16}\right)f
\end{aligned}
$$

To verify, let us check that $u(x) = \frac{3}{8}f$ also using the second solution above. Let $x = \frac{1}{2}$ in the above, we find

$$
\begin{aligned}
u\left(\frac{1}{2}\right) &= \left(-\frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\frac{1}{2} + \frac{3}{16}\right)f \\
&= \frac{3}{8}
\end{aligned}
$$

Therefore the solution $u(x)$ is continuous and smooth at $x = \frac{1}{2}$ where the elasticity changes. This is a plot of the solution

```
In[63]:= u[x_] := Piecewise[{{(-1/4 x^2 + 1/2 x + 3/16), 1/2 ≤ x < 1}, {x - x^2/2, 0 < x < 1/2}}]
         Plot[u[x], {x, 0, 1}, PlotTheme → "Detailed", Frame → True,
          FrameLabel → {{"u(x)", None}, {"x", "Solution for problem 3.1.6"}}]
```

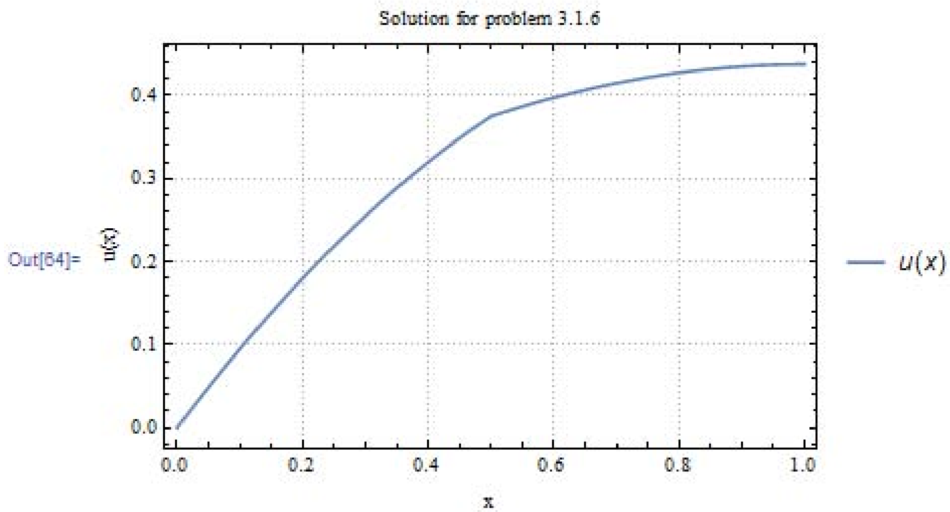


Figure 3.63: Figure for 3.1.6

3.4.6 Problem 3.2.2

3.2.2 What function $u(x)$ with $u(0) = 0$ and $u(1) = 0$ minimizes

$$P(u) = \int_0^1 \left[\tfrac{1}{2} \left(\frac{du}{dx} \right)^2 + x\, u(x) \right] dx?$$

Figure 3.64: the Problem statement

The general form of $P(u(x))$ is

$$P(u(x)) = \int_0^1 \left[\frac{1}{2} C \left(\frac{du(x)}{dx} \right)^2 - f(x)\, u(x) \right] dx \tag{1}$$

We will use theorem proved in class that function $\bar{u}(x)$ minimizes $p(\bar{u})$ iff

$$\int_0^1 C\frac{d\bar{u}}{dx}\frac{dv}{dx} - fvdx = 0$$

For any test function $v(x)$. However, this test function must satisfy the essential conditions on $u(x)$. Therefore, since we are told $u(1) = u(0) = 0$, then it follows that $v(1) = v(0) = 0$. Now we apply Integration by part to (1)

$$\left[C\frac{d\bar{u}}{dx}v\right]_0^1 - C\int_0^1 \frac{d^2\bar{u}}{dx^2}vdx - \int_0^1 fvdx = 0$$

$$C\left[\left.\frac{d\bar{u}}{dx}\right|_{x=1} v(1) - \left.\frac{d\bar{u}}{dx}\right|_{x=0} v(0)\right] - C\int_0^1 \frac{d^2\bar{u}}{dx^2}vdx - \int_0^1 fvdx = 0$$

Since $v(1) = v(0) = 0$ the above reduces to

$$-C\int_0^1 \frac{d^2\bar{u}}{dx^2}vdx = \int_0^1 fvdx$$

Since $v(x)$ is arbitrary function (other than having the same essential boundary conditions as $u(x)$) then the above implies

$$-C\frac{d^2\bar{u}}{dx^2} = f \tag{2}$$

Now we can apply this result to the problem at hand, which is to find $\bar{u}$ which minimizes

$$p(u) = \int_0^1 \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 + xu\right]dx \tag{3}$$

By comparing (3) and (1), we see that $C = 1$ and $f = -x$, hence from (2), we need to solve

$$-\frac{d^2\bar{u}}{dx^2} = -x$$

or

$$\frac{d^2\bar{u}}{dx^2} = x \tag{4}$$

With the boundary conditions $\bar{u}(0) = \bar{u}(1) = 0$. The homogeneous solution to (4) is $\bar{u}_h(x) = Ax + B$. Let the particular solution be $\bar{u}_p(x) = c_1x^3$, then applying this to (4) gives

$$6c_1x = x$$

Hence $c_1 = \frac{1}{6}$ and $\bar{u}_p(x) = \frac{1}{6}x^3$. Therefore the general solution is

$$\begin{aligned}\bar{u}(x) &= \bar{u}_h(x) + \bar{u}_p(x)\\ &= Ax + B + \frac{1}{6}x^3\end{aligned}$$

We now apply the essential conditions on the above. Which results in two equations to solve for A, B

$$\bar{u}(0) = 0 = B$$
$$\bar{u}(1) = 0 = A + \frac{1}{6}$$

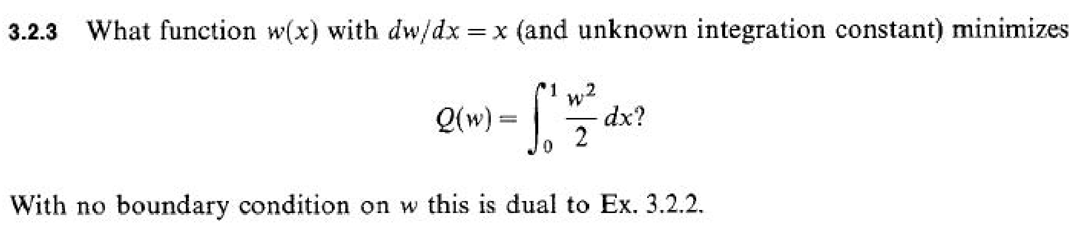
Hence $B = 0, A = -\frac{1}{6}$, and the solution is

$$\bar{u}(x) = -\frac{1}{6}x + \frac{1}{6}x^3$$

or

$$\bar{u}(x) = -\frac{x}{6}\left(1 - x^2\right)$$

3.4.7 Problem 3.2.3



3.2.3 What function $w(x)$ with $dw/dx = x$ (and unknown integration constant) minimizes

$$Q(w) = \int_0^1 \frac{w^2}{2}\,dx?$$

With no boundary condition on w this is dual to Ex. 3.2.2.

Figure 3.65: the Problem statement

We need to find $\bar{w}(x)$ which minimizes the functional $Q(w(x)) = \int_0^1 \frac{w^2}{2} dx$ with constraint $\frac{dw}{dx} = x$. Since we have a constraint, we need to set up a Lagrangian minimization. Hence we want to minimize

$$L(w, \lambda) = \int_0^1 \frac{w^2}{2} - \lambda\left(\frac{dw}{dx} + x\right) dx$$

Where λ is the Lagrangian. Now we follow the standard method, but work with L instead of Q.

$$L((w + v), \lambda) = L(w, \lambda) + \frac{\delta L(w, \lambda)}{\delta x} v + \cdots$$

Hence

$$
\begin{aligned}
\frac{\delta L(w,\lambda)}{\delta x}v &= L((w+v),\lambda) - L(w,\lambda)\\
&= \int_0^1 \frac{(w+v)^2}{2} - \lambda\left(\frac{d(w+v)}{dx} + x\right)dx - \int_0^1 \frac{w^2}{2} - \lambda\left(\frac{dw}{dx}+x\right)dx\\
&= \int_0^1 \frac{1}{2}\left(w^2+v^2+2vw\right) - \lambda\left(\frac{dw}{dx}+\frac{dv}{dx}+x\right) - \frac{w^2}{2} + \lambda\left(\frac{dw}{dx}+x\right)dx\\
&= \int_0^1 \frac{1}{2}\left(v^2+2vw\right) - \lambda\frac{dv}{dx}dx\\
&= \int_0^1 \frac{1}{2}v^2 dx + \int_0^1 \left(vw - \lambda\frac{dv}{dx}\right)dx
\end{aligned}
$$

But for small variation v the term $\int_0^1 \frac{1}{2}v^2 dx$ is always positive and can be made as small as needed. Hence we ignore it, and what is left is

$$
\frac{\delta L(w,\lambda)}{\delta x}v = \int_0^1 \left(vw - \lambda\frac{dv}{dx}\right)dx
$$

Since we want $\frac{\delta L(w,\lambda)}{\delta x} = 0$ for a minimum, and the above must be valid for any non trivial v then

$$
\int_0^1 \left(vw - \lambda\frac{dv}{dx}\right)dx = 0
$$

Applying integration by parts to $\int_0^1 \lambda\frac{dv}{dx}dx$ where $\int udv = [uv] - \int vdu$. Let $u=\lambda, dv = \frac{dv}{dx}$, hence the above becomes

$$
\begin{aligned}
0 &= \int_0^1 \left(vw - \lambda\frac{dv}{dx}\right)dx\\
&= \int_0^1 vw\, dx - \overbrace{\int_0^1 \lambda\frac{dv}{dx}dx}^{\text{by parts}}\\
&= \int_0^1 vw\, dx - \left[(\lambda v)_0^1 - \int_0^1 \frac{d\lambda}{dx}v dx\right]
\end{aligned}
$$

Assuming $v(0) = v(1) = 0$, then the above reduces to

$$\int_0^1 vw + \frac{d\lambda}{dx} v dx = 0$$

$$\int_0^1 \left(w + \frac{d\lambda}{dx}\right) v dx = 0$$

Since this is valid for any v, therefore

$$w + \frac{d\lambda}{dx} = 0$$

Hence the $w(x)$ which minimizes $\int_0^1 \frac{w^2}{2} dx$ with constraint $\frac{dw}{dx} = x$ is

$$w(x) = -\frac{d\lambda}{dx}$$

3.4.8 Problem 3.2.10

3.2.10 If the ends of a beam are fixed (zero boundary conditions) and the force is $f = 1$ with $c = 1$, solve $d^4u/dx^4 = 1$ and then find M. Why does it have to be done in that order?

Figure 3.66: the Problem statement

For a beam, the equation of deflection is $u^{(4)} = 1$. The solution is given by integrating 4 times resulting in

$$\begin{aligned}
u'''(x) &= x + c_1 \\
u'' &= \frac{x^2}{2} + c_1 x + c_2 \\
u' &= \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3 \\
u &= \frac{x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4
\end{aligned}$$

Since $u(0) = 0$ then $c_4 = 0$ and since $u'(0) = 0$ then $c_3 = 0$, hence

$$u(x) = \frac{x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2}$$

Now, assuming the beam has length 1. Then on the other end, we have also $u(1) = 0$, then

$$u(1) = 0 = \frac{1}{24} + c_1 \frac{1}{6} + c_2 \frac{1}{2} \tag{1}$$

And since also $u'(1) = 0$, then

$$u'(1) = 0 = \frac{1}{6} + c_1\frac{1}{2} + c_2 \tag{2}$$

From (1) and (2) we can solve for c_2, c_1, giving $c_2 = \frac{1}{12}, c_1 = -\frac{1}{2}$, hence

$$u(x) = \frac{x^4}{24} - \frac{1}{12}x^3 + \frac{1}{24}x^2$$

Now we can find $M(x)$ since $M(x) = c\frac{d^2u}{dx^2}$, hence

$$M(x) = \frac{x^2}{2} - \frac{1}{2}x + \frac{1}{12}$$

If we had used $M = u''$ directly (from page 173 on text, where $c = 1$ now), then the solution would be

$$\begin{aligned} Mx + c_1 &= u' \\ \frac{Mx^2}{2} + c_1x + c_2 &= u \end{aligned}$$

At $u(0) = 0$ then $c_2 = 0$, hence $\frac{Mx^2}{2} + c_1x = u$ and from $u(1) = 0$ we obtain $\frac{M}{2} + c_1 = 0$ or $M = -\frac{c_1}{2}$. But we are now stuck since we can't find c_1.

So to find M, we must first find $u(x)$ and then find $M = cu''$ after solving for u completely.

3.4.9 Problem 3.2.12

3.2.12 What is the shape of a uniform beam under zero force, $f = 0$ and $c = 1$, if $u(0) = u(1) = 0$ at the ends but $du/dx(0) = 1$ and $du/dx(1) = -1$? Sketch this shape.

Figure 3.67: the Problem statement

For a beam, the equation of deflection is $u^{(4)} = 0$. The solution is given by integrating 4 times resulting in

$$\begin{aligned} u'''(x) &= c_1 \\ u'' &= c_1x + c_2 \\ u' &= c_1\frac{x^2}{2} + c_2x + c_3 \\ u &= c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4 \end{aligned}$$

For $u(0) = 0$ gives $c_4 = 0$ and $u'(0) = 1$ gives $c_3 = 1$ and $u(1) = 0$ gives $0 = c_1\frac{1}{6} + c_2\frac{1}{2} + 1$ and $u'(1) = -1$ gives $-1 = c_1\frac{1}{2} + c_2 + 1$

Hence we need to solve these

$$-1 = c_1 \frac{1}{2} + c_2 + 1$$
$$0 = c_1 \frac{1}{6} + c_2 \frac{1}{2} + 1$$

For c_1, c_2. The solution is: $c_1 = 0, c_2 = -2$. Hence

$$u(x) = -x^2 + x$$

A plot is

```
Plot[x - x^2, {x, 0, 1}, Frame → True, AspectRatio → Automatic,
 FrameLabel → {{"u(x)", None}, {"x", "solution to u''''(x)=0"}}]
```

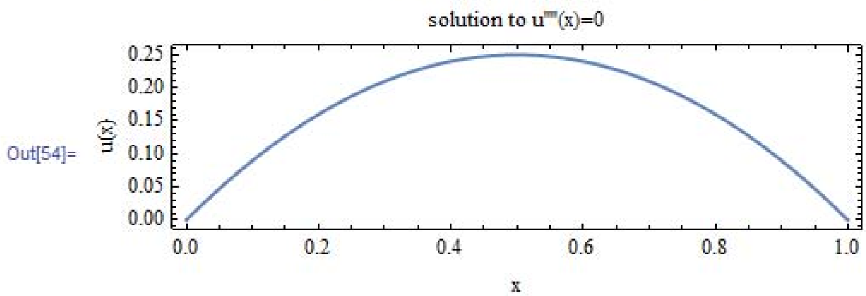


Figure 3.68: Plot for 3.2.12

3.4.10 Problem 3.3.3

3.3.3 *Discrete divergence theorem*: Why is the flow across the "cut" in the figure equal to the sum of the flows from the individual nodes *A,B,C,D*? *Note*: This is true even if flows like $d_1 - d_6$ from nodes like *A* are nonzero. If the current law holds and each node has zero net flow, then the exercise says that the flow across every cut is zero.

Figure 3.69: the Problem statement

3.4.11 Problem 3.3.4

3.3.4 *Discrete Stokes theorem*: Why is the voltage drop around the large triangle equal to the sum of the drops around the small triangles? *Note*: This is true even if voltage drops like $d_1 + d_7 + d_6$ around triangles like ABC are nonzero. If the voltage law holds and the drop around each small triangle is zero, then the exercise says that $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0$.

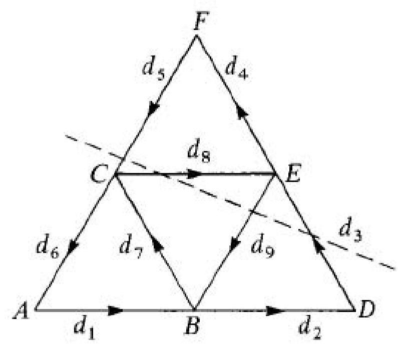


Figure 3.70: the Problem statement

3.4.12 Problem 3.3.5

3.3.5 On a graph the analogue of the gradient is the edge-node incidence matrix A_0. The analogue of the curl is the loop-edge matrix R with a row for each independent loop and a column for each edge. Draw a graph with four nodes and six directed edges, write down A_0 and R, and confirm that $RA_0 = 0$ in analogy with curl grad $= 0$.

Figure 3.71: the Problem statement

3.5 HW 5, Due Nov 20, 2014

3.5.1 Problem 4.1.1(d)

4.1.1 Find the Fourier series on $-\pi < x < \pi$ for
(a) $f(x) = \sin^3 x$, an odd function
(b) $f(x) = |\sin x|$, an even function
(c) $f(x) = x^2$, integrating either $x^2 \cos kx$ or the sine series for $f = x$
(d) $f(x) = e^x$, using the complex form of the series.
What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

Figure 3.72: the Problem statement

$$f(x) = e^x = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

Where

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x} dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(1-ik)x}}{1-ik}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(1-ik)}\left[\frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2}\right]
\end{aligned}$$

But $\frac{e^z}{2} - \frac{e^{-z}}{2} = \sinh(z)$, hence the above reduces to

$$c_k = \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) \tag{2}$$

Substituting (2) into (1) gives

$$e^x = \sum_{k=-\infty}^{\infty} \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) e^{ikx}$$

Here are few terms in the series generated using symbolic software:

```
ClearAll[x, k, n, f, ck]
ck[k_, x_] := 1/(2 Pi) Integrate[Exp[x] Exp[-I k x], {x, -Pi, Pi}]
f[k_, x_] := ck[k, x]*Exp[I k x];
term[n_] := If[n == 0, N@f[0, x], N@Simplify@ComplexExpand[f[-n, x] + f[n, x]]]
tbl = Table[{k, Simplify@TrigToExp@ck[k, x]}, {k, -5, 5, 1}];
Grid[Join[{{"k", "C_k"}}, tbl], Frame -> All]
```

k	C_k
-5	$\frac{(1-5i)e^{-\pi}-(1-5i)e^{\pi}}{52\pi}$
-4	$\frac{\left(\frac{1}{34}-\frac{2i}{17}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
-3	$\frac{(1-3i)e^{-\pi}-(1-3i)e^{\pi}}{20\pi}$
-2	$\frac{\left(\frac{1}{10}-\frac{i}{5}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
-1	$-\frac{\left(\frac{1}{4}-\frac{i}{4}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
0	$-\frac{e^{-\pi}-e^{\pi}}{2\pi}$
1	$\frac{(1+i)e^{-\pi}-(1+i)e^{\pi}}{4\pi}$
2	$\frac{\left(\frac{1}{10}+\frac{i}{5}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
3	$\frac{(1+3i)e^{-\pi}-(1+3i)e^{\pi}}{20\pi}$
4	$\frac{\left(\frac{1}{34}+\frac{2i}{17}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
5	$\frac{(1+5i)e^{-\pi}-(1+5i)e^{\pi}}{52\pi}$

Here is a plot of Fourier series of e^x for k increasing range to compare with e^x. To generate this plot the terms with $c_{-k}+c_k$ were added in order together to obtain a real valued function before plotting. Plotting was done from $x=-\pi\cdots\pi$. We see as more terms are added, the approximation improves. At 20 terms, the approximations became very good. Here is the plot

```
ck = 1/(2 Pi) Integrate[Exp[x] Exp[-I k1 x], {x, -Pi, Pi}]
f[k_] := (ck /. k1 -> k)*Exp[I k x];
fs[n_] := Sum[Simplify[f[-k] + f[k]], {k, 1, n}] + f[0];
tbl = Table[Plot[{fs[n], Exp[x]}, {x, -Pi, Pi}, Frame -> True, Axes -> False,
FrameLabel -> {{"f(x)", None},
{"x", Row[{"Using " <> ToString[n] <> " terms"}]}},
PlotStyle -> {Dashed, Red}], {n, 1, 20, 1}];
Grid[Partition[tbl, 4]]
```

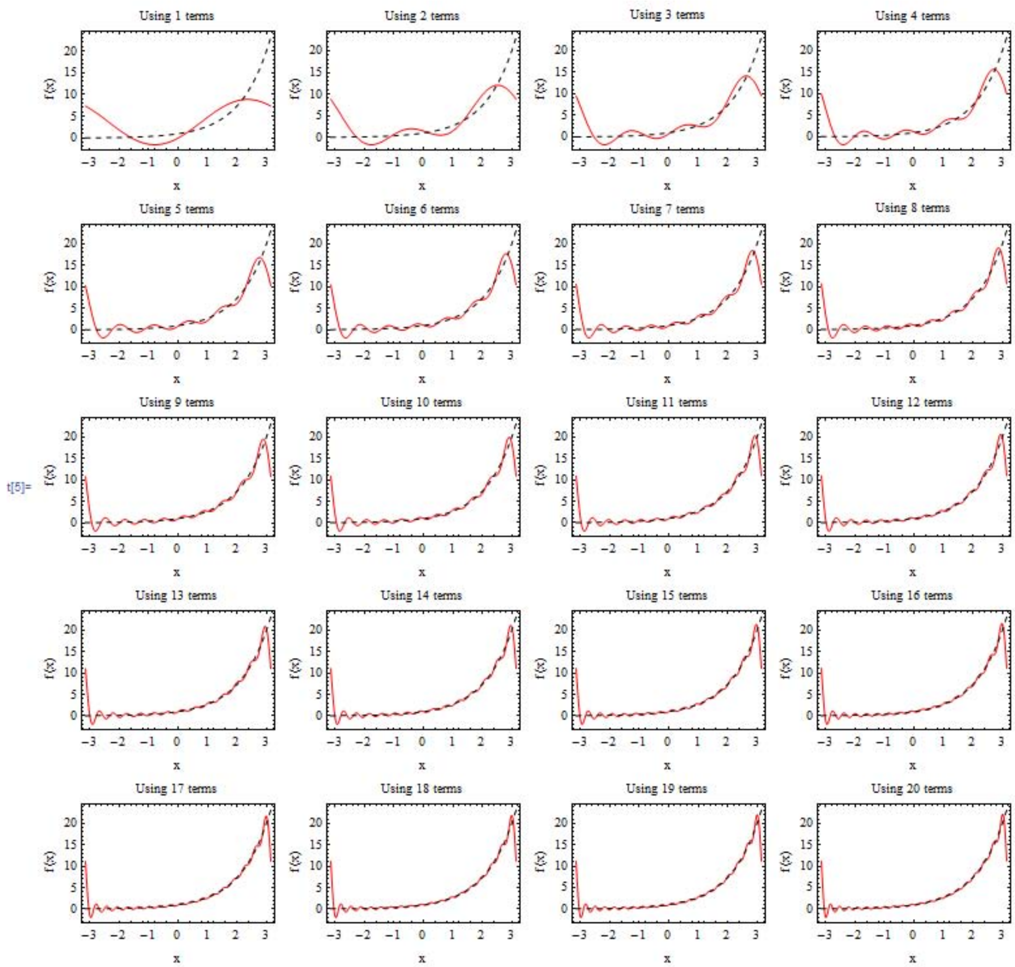


Figure 3.73: Plot for problem 4.1.1

The even part of e^x are given by $\frac{e^x+e^{-x}}{2} = \cosh x$ and the odd part is $\frac{e^x-e^{-x}}{2} = \sinh x$. For e^{ix}, the even part is $\frac{e^{ix}+e^{-ix}}{2} = \cos x$ and the odd part is $\frac{e^{ix}-e^{-ix}}{2} = i \sin x$

3.5.2 Problem 4.1.2

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = +1$ on the right side $0 < x < \pi$.

(1) Why are all the cosine coefficients $a_k = 0$?

(2) Find the sine series $\Sigma\, b_k \sin kx$ from equation (6).

Figure 3.74: the Problem statement

3.5.2.1 Part (a)

Since $f(-\pi) = -f(-\pi)$ then $f(x)$ is an odd function. For an odd function all the $a_k = 0$ since these go with the even part.

3.5.2.2 Part(b)

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(kx)\,dx + \int_{0}^{\pi} f(x)\sin(kx)\,dx\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(kx)\,dx + \int_{0}^{\pi}\sin(kx)\,dx\right)
\end{aligned}
$$

Changing the limits of integration changes the sign, hence the above can be written as

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\left(\int_{0}^{\pi}\sin(kx)\,dx + \int_{0}^{\pi}\sin(kx)\,dx\right) \\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin(kx)\,dx \\
&= \frac{2}{\pi}\left[\frac{-\cos kx}{k}\right]_0^{\pi} \\
&= \frac{-2}{\pi k}\left[\cos kx\right]_0^{\pi} \\
&= \frac{-2}{\pi k}\left[\cos k\pi - \cos 0\right] \\
&= \frac{2}{\pi k}\left(1-\cos k\pi\right) \qquad k = 1,2,3,\cdots
\end{aligned}
$$

Hence

$$
b_k = \begin{cases} \frac{4}{\pi k} & k = 1,3,5,\cdots \\ 0 & k = 2,4,6,\cdots \end{cases}
$$

Hence using $f(x) = \sum_{k=1}^{\infty} b_k \sin kx$, we can write the Fourier series of $f(x)$ as

$$
\begin{aligned}
f(x) &= \sum_{k=1,3,\cdots}^{\infty} \frac{4}{\pi k}\sin kx \\
&= \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots \\
&= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)
\end{aligned}
$$

Here is a plot showing the Fourier series approximation to the square wave from $x = -\pi \cdots \pi$ as more terms are added

```
Clear[f, k, x];
f[x_, k_] := Sum[2/(Pi n) (1 - Cos[n Pi]) Sin[n x], {n, 1, k}];
tbl = Partition[Table[
Plot[{Sign[x], f[x, k]}, {x, -Pi, Pi},
Exclusions -> None, PlotLabel -> Row[{"k=", k}],
PlotStyle -> {Thin, Red}], {k, 1, 20,2}], 3];
Grid[tbl, Frame -> All]
```

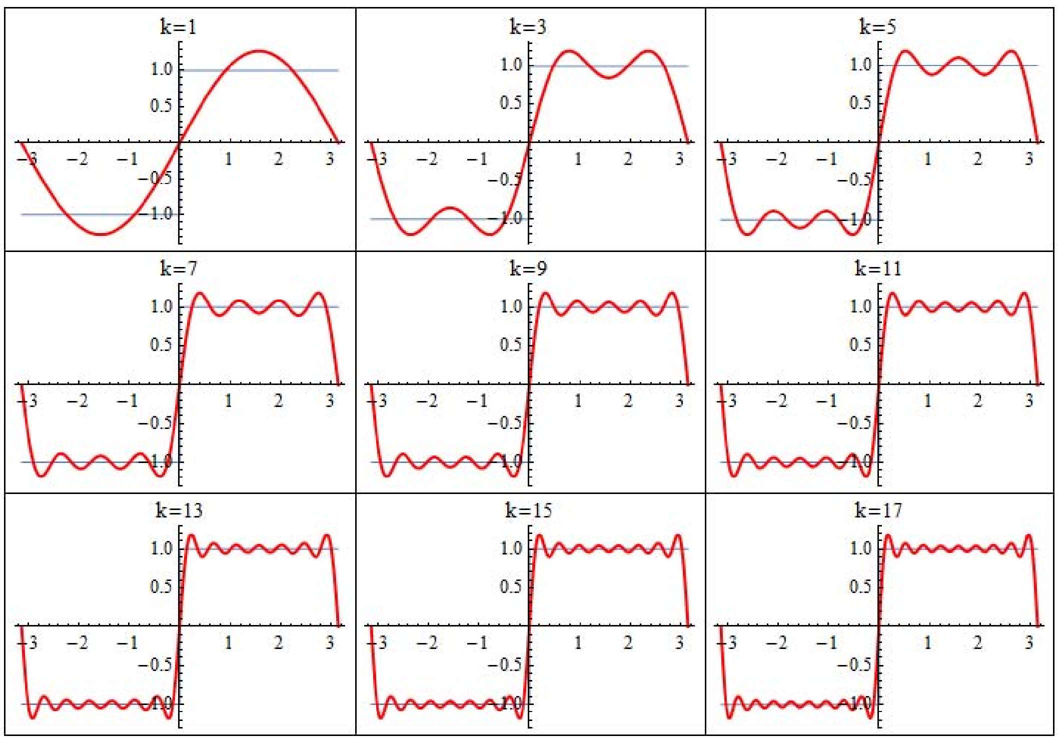


Figure 3.75: Plot for problem 4.1.2

3.5.3 Problem 4.1.3

4.1.3 Find this sine series for the square wave f in another way, by showing that

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically

(b) $2\delta(x) - 2\delta(x+\pi) = \dfrac{4}{\pi}(\cos x + \cos 3x + \cdots)$ from (10)

Integrate each term to find the square wave f.

Figure 3.76: the Problem statement

3.5.3.1 Part(a)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x+\pi)$. For $\delta(x)$ we find

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\cos kx dx = \frac{1}{\pi} \qquad (\text{since } \cos 0 = 1)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\sin kx dx = 0 \qquad (\text{since } \sin 0 = 0)$$

Hence

$$\begin{aligned}\delta(x) &= \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi}(\cos x + \cos 2x + \cos 3x + \cdots)\end{aligned}$$

Now to determine Fourier series for $\delta(x+\pi)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x+\pi)\,dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x+\pi)\cos kx dx = \frac{(-1)^k}{\pi} \qquad (\text{since } \cos(-k\pi) = \cos k\pi = (-1)^k)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\sin kx dx = 0 \qquad (\text{since } \sin(-k\pi) = 0)$$

Hence

$$\begin{aligned}\delta(x+\pi) &= \frac{1}{2\pi}+\sum_{k=1}^{\infty}a_k\cos kx\\ &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\cos kx\\ &= \frac{1}{2\pi}+\frac{1}{\pi}(-\cos x+\cos 2x-\cos 3x+\cdots)\end{aligned}$$

Therefore

$$\begin{aligned}2\delta(x)-2\delta(x+\pi) &= 2\left[\frac{1}{2\pi}+\frac{1}{\pi}(\cos x+\cos 2x+\cos 3x+\cdots)\right]-2\left[\frac{1}{2\pi}+\frac{1}{\pi}(-\cos x+\cos 2x-\cos 3x+\cdots)\right]\\ &= \frac{1}{\pi}+\frac{2}{\pi}(\cos x+\cos 2x+\cos 3x+\cdots)-\frac{1}{\pi}+\frac{2}{\pi}(\cos x-\cos 2x+\cos 3x-\cos 5x+\cdots)\\ &= \frac{2}{\pi}(2\cos x+2\cos 3x+2\cos 5x+\cdots)\\ &= \frac{4}{\pi}(\cos x+\cos 3x+\cos 5x+\cdots)\end{aligned}$$

Hence

$$\frac{df}{dx}=\frac{4}{\pi}(\cos x+\cos 3x+\cos 5x+\cdots)$$

Hence

$$f(x)=\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$

3.5.3.2 Part (b)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x+\pi)$. For $\delta(x)$ we find

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)e^{-ikx}dx=\frac{1}{2\pi}$$

Hence

$$\begin{aligned}\delta(x) &= \sum_{k=-\infty}^{\infty}c_ke^{ikx}\\ &= \sum_{k=-\infty}^{\infty}\frac{1}{2\pi}e^{ikx}\\ &= \frac{1}{2\pi}\left(1+e^{-ikx}+e^{ikx}+e^{-2ik}+e^{2ik}+\cdots\right)\\ &= \frac{1}{2\pi}(1+2\cos kx+2\cos 2kx+2\cos 3kx+\cdots)\end{aligned}$$

Now to determine Fourier series for $\delta(x+\pi)$

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x+\pi)e^{-ikx}dx=\frac{1}{2\pi}e^{ik\pi}=\frac{1}{2\pi}\cos k\pi=\frac{(-1)^k}{2\pi}$$

Hence

$$
\begin{aligned}
\delta(x+\pi) &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{2\pi} e^{ikx} \\
&= \frac{1}{2\pi}\left(1 - e^{-ix} - e^{ix} + e^{-2ix} + e^{2ix} - e^{-3ix} - e^{3ix} + \cdots\right) \\
&= \frac{1}{2\pi}\left(1 - \left(e^{-ix} + e^{ix}\right) + e^{-2ix} + e^{2ix} - \left(e^{-3ix} + e^{3ix}\right) + \cdots\right) \\
&= \frac{1}{2\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
2\delta(x) - 2\delta(x+\pi) &= 2\left[\frac{1}{2\pi}\left(1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right)\right] - 2\left[\frac{1}{2\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right)\right] \\
&= \frac{1}{\pi}\left(1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right) - \frac{1}{\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right) \\
&= \frac{1}{\pi}\left(4\cos x + 4\cos 3x + 4\cos 5x + \cdots\right) \\
&= \frac{4}{\pi}\left(\cos x + \cos 3x + \cos 5x + \cdots\right)
\end{aligned}
$$

Hence

$$
\frac{df}{dx} = \frac{4}{\pi}\left(\cos x + \cos 3x + \cos 5x + \cdots\right)
$$

Therefore

$$
f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)
$$

Which is the same as above using the a_k, b_k method.

3.5.4 Problem 4.1.4

4.1.4 At $x = \pi/2$ the square wave equals 1. From the Fourier series at this point find the alternating sum that equals π:

$$
\pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} \cdots\right).
$$

Figure 3.77: the Problem statement

From above we found that the Fourier series for square wave is

$$
f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)
$$

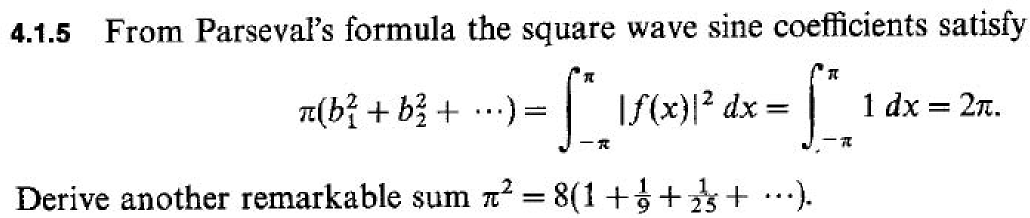
Therefore at $x = \frac{\pi}{2}$, the above becomes

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \cdots\right)$$

Hence

$$\begin{aligned}\pi &= 4\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \cdots\right)\\ &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\end{aligned}$$

3.5.5 Problem 4.1.5



4.1.5 From Parseval's formula the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |f(x)|^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Derive another remarkable sum $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

Figure 3.78: the Problem statement

We found that only the b_k survive for the Fourier series of the wave function. They are

$$b_k = \begin{cases} \frac{4}{\pi k} & k = 1,3,5,\cdots \\ 0 & k = 2,4,6,\cdots \end{cases}$$

Applying Parseval's formula leads to

$$\pi\left(b_1^2 + b_3^2 + b_5^2 + \cdots\right) = \int_{-\pi}^{\pi} \left|f(x)\right|^2 dx = 2\pi$$

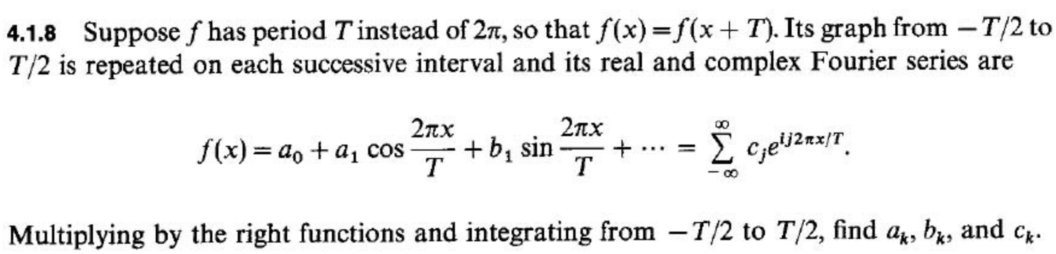
Where we used only the odd b_k terms since all others are zero. The above becomes

$$\begin{aligned}\pi\left(\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \cdots\right) &= 2\pi\\ \pi\left(\frac{1}{\pi^2}4^2 + \frac{1}{\pi^2}\left(\frac{4}{3}\right)^2 + \frac{1}{\pi^2}\left(\frac{4}{5}\right)^2 + \cdots\right) &= 2\pi\\ \left(4^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{5}\right)^2 + \cdots\right) &= 2\pi^2\\ \pi^2 = 8\left(1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \cdots\right)\end{aligned}$$

Hence

$$\pi^2 = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right)$$

3.5.6 Problem 4.1.8



4.1.8 Suppose f has period T instead of 2π, so that $f(x) = f(x+T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

$$f(x) = a_0 + a_1 \cos\frac{2\pi x}{T} + b_1 \sin\frac{2\pi x}{T} + \cdots = \sum_{-\infty}^{\infty} c_j e^{ij2\pi x/T}.$$

Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find a_k, b_k, and c_k.

Figure 3.79: the Problem statement

ps. In the solution below, I was using T when I should be using $\frac{T}{2}$ in all the limits. Need to correct later. Or just let period be $2T$ then the math works ok.

In this problem, the basic idea is to observe that when the period was 2π then

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Now when the period is a general value T we use $\left(\frac{2\pi}{T}k\right)$ in place of just k. So the above becomes

$$f(x) = \sum_{k=0}^{\infty} a_k \cos\left(k\frac{2\pi}{T}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{T}x\right) \tag{1}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(\frac{2\pi}{T}k\right)x} \tag{2}$$

We now need to determine a_k, b_k, c_k using (1) and (2) in similar way we did when the period was 2π.

To find a_k we multiply (1) by $\cos\left(m\frac{2\pi}{T}x\right)$ where m is some integer between $1\cdots\infty$, and integrating from $-T$ to T gives

$$\begin{aligned}
\int_{-T}^{T} f(x)\cos\left(m\frac{2\pi}{T}x\right)dx &= \int_{-T}^{T}\sum_{k=0}^{\infty} a_k \cos\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx + \int_{-T}^{T}\sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx \\
&= \sum_{k=0}^{\infty}\int_{-T}^{T} a_k \cos\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx + \sum_{k=1}^{\infty}\int_{-T}^{T} b_k \sin\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx
\end{aligned}$$

Due to orthogonality between the sin and cos, all the product of sin cos vanish, and only one term in the product of cos cos remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^{T} f(x)\cos\left(m\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} a_m\cos\left(m\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} a_k\cos^2\left(k\frac{2\pi}{T}x\right)dx \tag{3}$$

When $k = 0$ we find

$$\begin{aligned}\int_{-T}^{T} f(x)\,dx &= \int_{-T}^{T} a_0 dx\\ &= 2a_0T\end{aligned}$$

Hence

$$a_0 = \frac{1}{2T}\int_{-T}^{T} f(x)\,dx$$

Notice, when $T = \pi$, the above reduces to $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$. Now to find a_k for $k \geq 1$, then from (3)

$$\begin{aligned}\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx &= \int_{-T}^{T} a_k\cos^2\left(k\frac{2\pi}{T}x\right)dx\\ &= a_kT\end{aligned}$$

Hence

$$a_k = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx$$

Notice that when $T = \pi$ the above reduces to $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$ as before.

Now we find b_k similarly. We multiply (1) by $\sin\left(m\frac{2\pi}{T}x\right)$ where m is some integer between $1\cdots\infty$, and integrating from $-T$ to T gives

$$\int_{-T}^{T} f(x)\sin\left(m\frac{2\pi}{T}x\right)dx = \sum_{k=0}^{\infty}\int_{-T}^{T} a_k\cos\left(k\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx + \sum_{k=1}^{\infty}\int_{-T}^{T} b_k\sin\left(k\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx$$

Due to orthogonality between the sin and cos, all the products of sin cos vanish, and only one term in the product of sin sin remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^{T} f(x)\sin\left(m\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} b_m \sin\left(m\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\begin{aligned}\int_{-T}^{T} f(x)\sin\left(k\frac{2\pi}{T}x\right)dx &= \int_{-T}^{T} b_k \sin^2\left(k\frac{2\pi}{T}x\right)dx\\ &= b_k T\end{aligned}$$

Hence

$$b_k = \frac{1}{T}\int T^T f(x)\sin\left(k\frac{2\pi}{T}x\right)dx$$

Notice that when $T=\pi$ the above reduces to $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$ as before. We now find c_k.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(k\frac{2\pi}{T}\right)x}$$

Multiplying both side by $e^{-i\left(m\frac{2\pi}{T}\right)x}$ and integrating over the period

$$\int_{-T}^{T} f(x)\,e^{-i\left(m\frac{2\pi}{T}\right)x}dx = \sum_{k=-\infty}^{\infty}\int_{-T}^{T} c_k e^{i\left(k\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x}dx$$

All terms other than ones which $k=m$ remain. Hence the above becomes

$$\begin{aligned}\int_{-T}^{T} f(x)\,e^{-i\left(m\frac{2\pi}{T}\right)x}dx &= \int_{-T}^{T} c_m e^{i\left(m\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x}dx\\ &= \int_{-T}^{T} c_m dx\end{aligned}$$

Therefore, since m is now arbitrary, we rename it back to k and simplifying

$$\int_{-T}^{T} f(x)\,e^{-i\left(k\frac{2\pi}{T}\right)x}dx = 2Tc_k$$

$$c_k = \frac{1}{2T}\int_{-T}^{T} f(x)\,e^{-i\left(k\frac{2\pi}{T}\right)x}dx$$

3.5.7 Problem 4.1.10

4.1.10 What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

Figure 3.80: the Problem statement

The a_0 term in the Fourier series of $\cos^2 x$ is the constant term. Hence it is the constant that is closest to $\cos^2 x$ in the square sense. Therefore

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 x dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 x dx \\
&= \frac{1}{2}
\end{aligned}
$$

To find the multiple of $\cos x$ which is closest to $\cos^3 x$, we find a_1 term in the Fourier series of $\cos^3 x$ since that is the term which has $a_1 \cos x$ in it. Hence

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^3 x \cos x dx \\
&= \frac{1}{\pi}\left(\frac{3\pi}{4}\right) \\
&= \frac{3}{4}
\end{aligned}
$$

3.5.8 Problem 4.1.11

4.1.11 Sketch the graph and find the Fourier series of the even function $f = 1 - |x|/\pi$ (extended periodically) in either of two ways: integrate the square wave or compute (with $a_0 = \frac{1}{2}$)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{x}{\pi}\right)\cos kx\, dx.$$

Figure 3.81: the Problem statement

The function we are approximating using Fourier series is

```
f[x_] := Piecewise[{{1 + x/Pi, x < 0}, {1 - x/Pi, x >= 0}}];
Plot[f[x], {x, -Pi, Pi}]
```

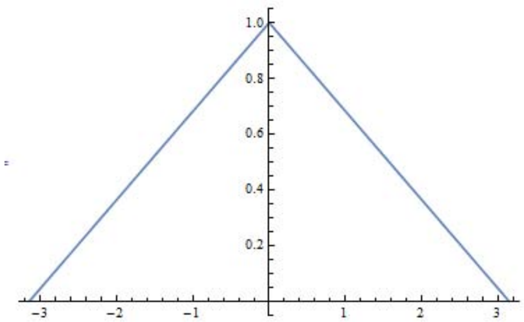


Figure 3.82: Plot for problem 4.1.11

Since it is even, we only need to determine a_k

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{x}{\pi}\right)\cos kx dx \\
&= \frac{2}{\pi}\left(\frac{1-\cos k\pi}{k^2\pi}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} a_k \cos kx \\
&= \frac{1}{2} + \frac{2}{\pi}\left(\frac{1-\cos\pi}{\pi}\right)\cos x + \frac{2}{\pi}\left(\frac{1-\cos 2\pi}{4\pi}\right)\cos 2x + \frac{2}{\pi}\left(\frac{1-\cos 3\pi}{9\pi}\right)\cos 3x + \cdots \\
&= \frac{1}{2} + \frac{2}{\pi}\left(\frac{2}{\pi}\right)\cos x + \frac{2}{\pi}\left(\frac{2}{9\pi}\right)\cos 3x + \frac{2}{\pi}\left(\frac{2}{25\pi}\right)\cos 5x + \cdots \\
&= \frac{1}{2} + \frac{4}{\pi^2}\cos x + \frac{4}{9\pi^2}\cos 3x + \frac{4}{25\pi}\cos 5x + \cdots
\end{aligned}
$$

Here is a plot showing the approximation as more terms are added. The label of each plot show the number of terms used. The more terms we use, the better the approximation

```
ck = (2/Pi) Integrate[(1 - x/Pi) Cos[k x], {x, 0, Pi}];
upTo[n_, x_] := (1/2) + Sum[(ck /. k -> m)* Cos[m x], {m, 1, n}];
tbl = Table[Plot[upTo[m, x], {x, -Pi, Pi},
PlotLabel -> Row[{"terms used =", m}]], {m, 0, 18, 2}];
Grid[Partition[tbl, 3], Frame -> All]
```

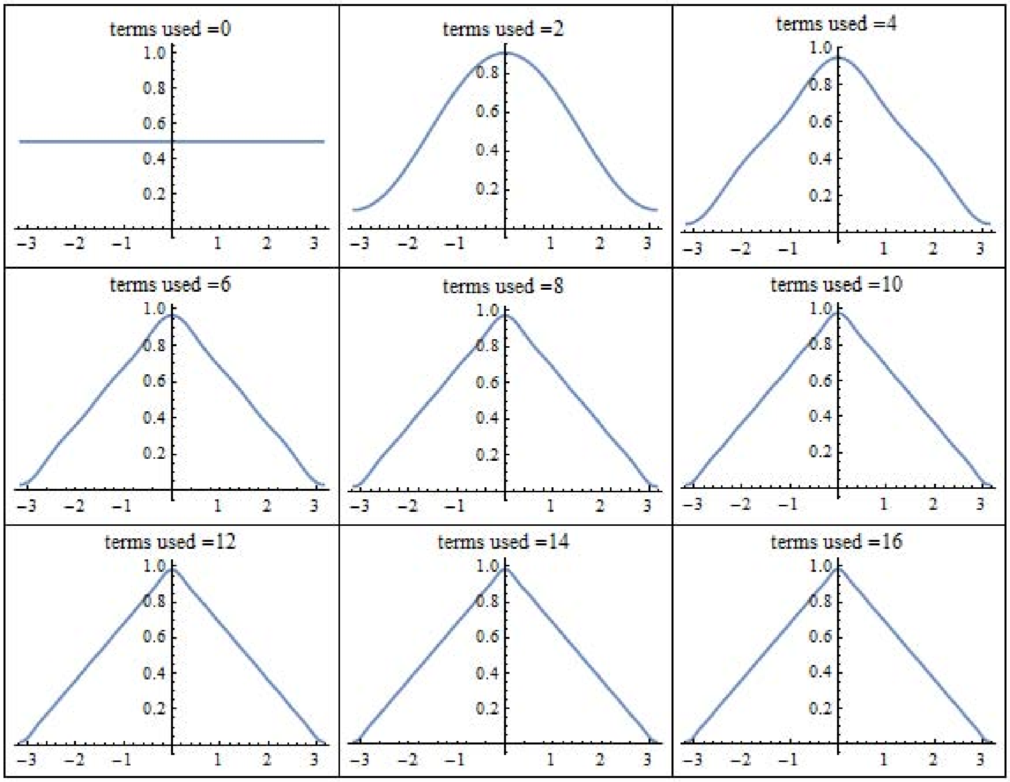


Figure 3.83: Plot for problem 4.1.11 part 2

3.5.9 Problem 4.1.16

4.1.16 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin?

Figure 3.84: the Problem statement

The first step is to obtain the a_k, b_k coefficients by expanding the boundary value of the solution using Fourier series. On the boundary

$$u_0 = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases}$$

Hence

$$a_0 = \frac{1}{2\pi}\int_0^\pi d\theta = \frac{1}{2}$$

And

$$a_k = \frac{1}{\pi}\int_0^{\pi} \cos k\theta d\theta = \frac{1}{k\pi}\left[\sin k\theta\right]_0^{\pi} = 0$$

And

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{\pi} \sin k\theta d\theta = \frac{1}{k\pi}\left[-\cos k\theta\right]_0^{\pi} = 0 = \frac{-1}{k\pi}\left[\cos k\pi - \cos 0\right] \\ &= \left\{\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \cdots\right\} \end{aligned}$$

Only odd values of k survive. Now that we found the Fourier coefficient, we use them in the solution given in equation (22), page 276 on the book

$$\begin{aligned} u(r,\theta) &= a_0 + b_1 r\sin\theta + b_3 r^3 \sin 3\theta + b_5 r^5 \sin^5\theta + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi}\left(r\sin\theta + \frac{1}{3}r^3\sin 3\theta + \frac{1}{5}r^5\sin^5\theta + \cdots\right) \end{aligned}$$

At the origin, let $r = 0$

$$u(0,\theta) = \frac{1}{2}$$

3.5.10 Problem 4.1.19

4.1.19 A plucked string goes linearly from $f(0) = 0$ to $f(p) = 1$ and back to $f(\pi) = 0$. The linear part $f = x/p$ reaches to $x = p$, followed by $f = (\pi - x)/(\pi - p)$ to $x = \pi$. Sketch f as an odd function and find a plucking point p for which the second harmonic $\sin 2x$ will not be heard ($b_2 = 0$).

Figure 3.85: the Problem statement

A sketch of the function (string) is below.

```
Clear[x, f, p];
f[x_, p_] := Piecewise[{{(-x - Pi)/(Pi - p), x < -p},
{(x + p)/p - 1, -p < x < 0}, {x/p, 0 < x < p},
{(x - Pi)/(p - Pi), p < x < Pi}}]
Plot[f[x, .8 Pi], {x, -Pi, Pi}, Frame -> True,
FrameLabel -> {{"f(x)", None}, {x, "problem 4.1.19"}}]
```

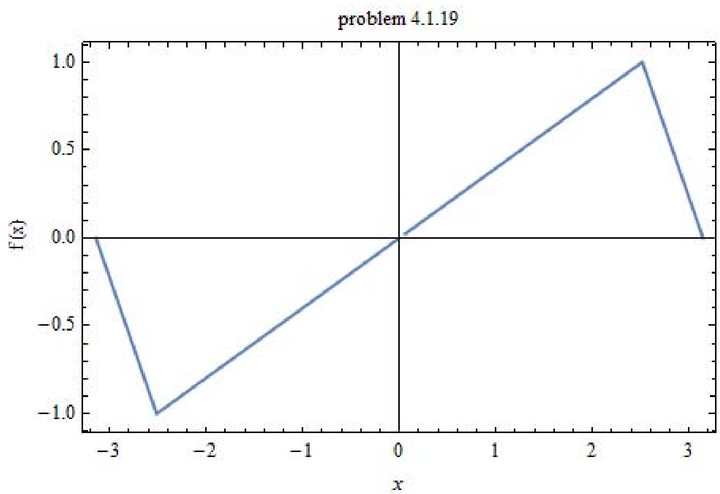


Figure 3.86: Plot for problem 4.1.19

Since $f(x)$ is odd, we only need to determine b_k

$$
\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^\pi f(x)\sin kx dx \\
&= \frac{2}{\pi}\left(\int_0^p \frac{x}{p}\sin kx dx + \int_p^\pi \frac{x-\pi}{p-\pi}\sin kx dx\right) \\
&= \frac{2}{\pi}\left(\frac{\sin kp - kp\cos kp}{k^2 p} + \frac{k(\pi-p)\cos kp + \sin kp - \sin k\pi}{k^2(\pi-p)}\right) \\
&= \frac{2(\pi\sin kp - p\sin k\pi)}{k^2 p\pi(\pi-p)}
\end{aligned}
$$

For $k=2$

$$
\begin{aligned}
b_2 &= \frac{(\pi\sin 2p - p\sin 2\pi)}{2p\pi(\pi-p)} \\
&= \frac{\pi\sin 2p}{2p\pi(\pi-p)}
\end{aligned}
$$

For zero, we need

$$
\begin{aligned}
0 &= \pi\sin 2p \\
\sin 2p &= 0
\end{aligned}
$$

Hence

$$
p = \frac{\pi}{2}
$$

3.5.11 Problem 4.1.20

4.1.20 Show that $P_2 = x^2 - \frac{1}{3}$ is orthogonal to $P_0 = 1$ and $P_1 = x$ over the interval $-1 \le x \le 1$. Can you find the next Legendre polynomial by choosing c to make $x^3 - cx$ orthogonal to P_0, P_1, and P_2?

Figure 3.87: the Problem statement

Two functions f, g are if the inner product is zero $\int_{-1}^{1} f(x) g(x)\, dx = 0$. Hence

$$\int_{-1}^{1} P_2 P_0 dx = \int_{-1}^{1} \left(x^2 - 1\right) dx = \left(\frac{x^3}{3} - x\right)_{-1}^{1} = 0$$

And

$$\int_{-1}^{1} P_2 P_1 dx = \int_{-1}^{1} \left(x^2 - 1\right) x dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{1} = 0$$

Now let $P_3 = x^3 - cx$, we want this to be orthogonal to P_0, P_1, P_2. Hence

$$\int_{-1}^{1} P_3 P_0 dx = \int_{-1}^{1} x^3 - cx dx = \left(\frac{x^4}{4} - c\frac{x^2}{2}\right)_{-1}^{1} = \left(\frac{1}{4} - c\frac{1}{2}\right) - \left(\frac{1}{4} - c\frac{1}{2}\right)$$
$$0 = 0$$

This equation did not help us find c. We try the next one

$$\int_{-1}^{1} P_3 P_1 dx = \int_{-1}^{1} \left(x^3 - cx\right) x dx = \left(\frac{x^5}{5} - c\frac{x^3}{3}\right)_{-1}^{1} = \left(\frac{1}{5} - c\frac{1}{3}\right) - \left(-\frac{1}{5} + c\frac{1}{3}\right) = \frac{2}{5} - \frac{2}{3}c$$

$$\begin{aligned} \frac{2}{5} - \frac{2}{3}c &= 0 \\ c &= \frac{2}{5}\frac{3}{2} \\ &= \frac{3}{5} \end{aligned}$$

Hence

$$P_3 = x^3 - \frac{3}{5}x$$

3.5.12 Problem 4.1.26

4.1.26 If f has the double sine series $\Sigma\Sigma\, b_{kl} \sin kx \sin ly$, show that Poisson's equation $-u_{xx} - u_{yy} = f$ is solved by the double sine series $u = \Sigma\Sigma\, b_{kl} \sin kx \sin ly/(k^2 + l^2)$. This is the solution with $u = 0$ on the boundary of the square $-\pi < x, y < \pi$.

Figure 3.88: the Problem statement

The proposed solution is

$$u\left(x,y\right) = \sum\sum \frac{b_{kl}\sin kx \sin ly}{\left(k^2+l^2\right)} \tag{1}$$

To see if this solves

$$-u_{xx} - u_{yy} = f = \sum\sum b_{kl}\sin kx \sin ly \tag{1A}$$

we will take (1) and substitute in the LHS of Poisson equation (1A) and see if we get the RHS of (1A) which is f.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \sum\sum \frac{b_{kl}k\cos kx \sin ly}{\left(k^2+l^2\right)}\\ \frac{\partial^2 u}{\partial x^2} &= \sum\sum \frac{-b_{kl}k^2\sin kx \sin ly}{\left(k^2+l^2\right)}\end{aligned} \tag{2}$$

And

$$\begin{aligned}\frac{\partial u}{\partial y} &= \sum\sum \frac{b_{kl}\sin\left(kx\right) l\cos ly}{\left(k^2+l^2\right)}\\ \frac{\partial^2 u}{\partial y^2} &= \sum\sum \frac{-b_{kl}\sin\left(kx\right) l^2\sin ly}{\left(k^2+l^2\right)}\end{aligned} \tag{3}$$

Substituting (2) and (3) in the LHS of (1A) gives

$$\begin{aligned}-u_{xx} - u_{yy} &= \sum\sum \frac{b_{kl}k^2\sin kx \sin ly}{\left(k^2+l^2\right)} + \sum\sum \frac{b_{kl}\sin\left(kx\right) l^2\sin ly}{\left(k^2+l^2\right)}\\ &= \sum\sum \frac{b_{kl}k^2\sin kx \sin ly + b_{kl}\sin\left(kx\right) l^2\sin ly}{\left(k^2+l^2\right)}\\ &= \sum\sum \frac{\left(b_{kl}\sin kx \sin ly\right)\left(k^2+l^2\right)}{\left(k^2+l^2\right)}\\ &= \sum\sum b_{kl}\sin kx \sin ly\end{aligned}$$

Which is f. Hence $u\left(x,y\right) = \sum\sum \frac{b_{kl}\sin kx \sin ly}{\left(k^2+l^2\right)}$ is the solution verified.

3.5.13 Problem 4.3.3

4.3.3 Find the inverse transforms of

(a) $\hat{f}(k) = \delta(k)$ (b) $\hat{f}(k) = e^{-|k|}$ (separate $k < 0$ from $k > 0$).

Figure 3.89: the Problem statement

3.5.13.1 Part (a)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{k=-\infty}^{\infty} \delta(k)\, e^{ikx} dk \\
&= \frac{1}{2\pi} \left[e^{ikx} \right]_{k=0} = \frac{1}{2\pi}
\end{aligned}
$$

3.5.13.2 Part (b)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{-|k|} e^{ikx} dk \\
&= \frac{1}{2\pi} \left(\int_{k=-\infty}^{0} e^{k} e^{ikx} dk + \int_{0}^{\infty} e^{-k} e^{ikx} dk \right) \\
&= \frac{1}{2\pi} \left(\int_{k=-\infty}^{0} e^{k(1+ix)} dk + \int_{0}^{\infty} e^{k(-1+ix)} dk \right) \\
&= \frac{1}{2\pi} \left(\left[\frac{e^{k(1+ix)}}{1+ix} \right]_{-\infty}^{0} + \left[\frac{e^{k(-1+ix)}}{-1+ix} \right]_{0}^{\infty} \right) \qquad (1)
\end{aligned}
$$

Looking at the first integral result

$$
\left[\frac{e^{k(1+ix)}}{1+ix} \right]_{-\infty}^{0} = \frac{1}{1+ix} - \frac{e^{-\infty(1+ix)}}{1+ix} = \frac{1}{1+ix}
$$

Where we looked at real part of $e^{-\infty(1+ix)} = 0$ so that we can make $e^{-\infty(1+ix)}$ to be zero.

Looking at the second integral result

$$
\left[\frac{e^{k(-1+ix)}}{-1+ix} \right]_{0}^{\infty} = \frac{e^{\infty(-1+ix)}}{-1+ix} - \frac{1}{-1+ix} = -\frac{1}{-1+ix}
$$

Where we looked at real part of $e^{\infty(-1+ix)} = 0$ so that we can make $e^{\infty(-1+ix)}$ to be zero. Hence, using the above two results in (1) gives

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\left(\frac{1}{1+ix}-\frac{1}{-1+ix}\right)\\
&= \frac{1}{2\pi}\left(\frac{1}{1+ix}+\frac{1}{1-ix}\right)\\
&= \frac{1}{2\pi}\left(\frac{(1-ix)+(1+ix)}{(1+ix)(1-ix)}\right)\\
&= \frac{1}{2\pi}\left(\frac{2}{1+x^2}\right)\\
&= \frac{1}{\pi}\frac{1}{1+x^2}
\end{aligned}
$$

3.5.14 Problem 4.3.5

4.3.5 Verify Plancherel's energy equation for $f=\delta$ and $f=e^{-x^2/2}$. Infinite energy is allowed.

Figure 3.90: the Problem statement

3.5.14.1 Part(a)

For $f(x)=\delta(x)$

$$2\pi\int_{-\infty}^{\infty}\delta^2(x)\,dx = 2\pi\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta(x)\,g_n(x)\,dx$$

Where $g_n(x)$ is sequence of Gaussian functions. The RHS above becomes

$$2\pi\int_{-\infty}^{\infty}\delta^2(x)\,dx = 2\pi\lim_{n\to\infty}g_n(0)$$

But $\lim_{n\to\infty}g_n(0)=\infty$ hence

$$2\pi\int_{-\infty}^{\infty}\delta^2(x)\,dx=\infty$$

Now $\hat{f}(k) = 1$ for the Dirac delta. Hence

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} (1)\, e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} e^{-ikx} dx \\
&= \left[\frac{e^{-ikx}}{-ik} \right]_{-\infty}^{\infty} = \frac{1}{-ik} \left(e^{-ik\infty} - e^{+ik\infty} \right) = \frac{1}{-ik} (0 - \infty) = \infty
\end{aligned}$$

Hence verified for δ OK.

3.5.14.2 Part (b)

For $f(x) = e^{-\frac{x^2}{2}}$ then

$$\begin{aligned}
2\pi \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx &= 2\pi \int_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} \right|^2 dx \\
&= 2\pi \int_{0}^{\infty} e^{-x^2} dx \\
&= 2\pi \left(\frac{\sqrt{\pi}}{2} \right) \\
&= \pi^{\frac{3}{2}}
\end{aligned}$$

Now $\hat{f}(k)$ for the above function is

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx \\
&= e^{-\frac{k^2}{2}} \sqrt{2\pi}
\end{aligned}$$

Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dk &= \int_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \sqrt{2\pi} \right|^2 dk \\
&= 2\pi \int_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \right|^2 dk \\
&= 2\pi \int_{0}^{\infty} e^{-k^2} dk \\
&= 2\pi \left(\sqrt{\frac{\pi}{2}} \right) \\
&= \pi^{\frac{3}{2}}
\end{aligned}
$$

Which is the same as before. Hence verified.

3.5.15 Problem 4.3.6

4.3.6 What are the half-widths W_x and W_k of the bell-shaped function $f = e^{-x^2/2}$ and its transform? Show that equality holds in the uncertainty principle.

Figure 3.91: the Problem statement

For $f(x) = e^{-\frac{x^2}{2}}$

$$
\begin{aligned}
W_x^2 &= \frac{\int_{-\infty}^{\infty} x^2 \left|f(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx} \\
&= \frac{\int_{-\infty}^{\infty} x^2 \left| e^{-\frac{x^2}{2}} \right|^2 dx}{\int_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} \right|^2 dx} \\
&= \frac{\int_{0}^{\infty} x^2 e^{-x^2} dx}{\int_{0}^{\infty} e^{-x^2} dx} \\
&= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2}
\end{aligned}
$$

Now $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx = e^{-\frac{k^2}{2}} \sqrt{2\pi}$, hence

$$\begin{aligned}
W_k^2 &= \frac{\int_{-\infty}^{\infty} k^2 \left|\hat{f}(k)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dx} \\
&= \frac{\int_{-\infty}^{\infty} k^2 \left|e^{-\frac{k^2}{2}} \sqrt{2\pi}\right|^2 dx}{\int_{-\infty}^{\infty} \left|e^{-\frac{k^2}{2}} \sqrt{2\pi}\right|^2 dx} \\
&= \frac{2\pi \int_0^{\infty} k^2 e^{-k^2} dx}{2\pi \int_0^{\infty} e^{-k^2} dx} \\
&= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2}
\end{aligned}$$

Hence

$$W_x W_k = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = \frac{1}{2}$$

But uncertainty principle says that $W_x W_k \geq \frac{1}{2}$. Hence verified OK.

3.5.16 Problem 4.3.7

4.3.7 What is the transform of $xe^{-x^2/2}$? What about $x^2 e^{-x^2/2}$, using **4L**?

Figure 3.92: the Problem statement

3.5.16.1 Part (a)

Using 4L(1), let $f(x) = e^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx = \sqrt{2\pi} e^{-\frac{k^2}{2}}$, hence $\frac{d}{dx} f(x)$ will have the transform $ik\hat{f}(k)$, therefore,

$$\mathscr{F}\left(\frac{d}{dx} f(x)\right) = \mathscr{F}\left(-xe^{-\frac{x^2}{2}}\right) = ik\sqrt{2\pi} e^{-\frac{k^2}{2}}$$

Therefore $xe^{-\frac{x^2}{2}}$ has the transform $-ik\sqrt{2\pi} e^{-\frac{k^2}{2}}$

3.5.16.2 Part(b)

Let $f(x) = xe^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = -ik\sqrt{2\pi}e^{-\frac{k^2}{2}}$ from part(a). But $\frac{d}{dx}f(x) = e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}$. Hence the transform of $\frac{d}{dx}f(x) = ik\hat{f}(k)$. Therefore

$$\mathscr{F}\left(e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}\right) = ik\left(-ik\sqrt{2\pi}e^{-\frac{k^2}{2}}\right)$$
$$\mathscr{F}\left(e^{-\frac{x^2}{2}}\right) - \mathscr{F}\left(x^2e^{-\frac{x^2}{2}}\right) = k^2\sqrt{2\pi}e^{-\frac{k^2}{2}}$$

But $\mathscr{F}\left(e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}}$, hence

$$\mathscr{F}\left(x^2e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}} - k^2\sqrt{2\pi}e^{-\frac{k^2}{2}}$$
$$\mathscr{F}\left(x^2e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}}\left(1-k^2\right)$$

Therefore

$$\mathscr{F}\left(x^2e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}}\left(1-k^2\right)$$

3.5.17 Problem 4.3.10

4.3.10 Solve the differential equation

$$\frac{du}{dx} + au = \delta(x)$$

by taking Fourier transforms to find $\hat{u}(k)$. What is the solution u (the Green's function for this equation)?

Figure 3.93: the Problem statement

Let $\hat{u}(k)$ be the Fourier transform of $u(x)$. Using $\mathscr{F}\left(\frac{du}{dx}\right) = ik\hat{u}(k)$ and $\mathscr{F}(\delta) = 1$, then applying Fourier transform on the ODE gives

$$ik\hat{u}(k) + a\hat{u}(k) = 1$$

Solving for $\hat{u}(k)$

$$\hat{u}(k)(a+ik) = 1$$
$$\hat{u}(k) = \frac{1}{a+ik}$$

Hence, from page 310 in text book, it gives the inverse Fourier transform for the above as

$$u(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$$

3.5.18 Problem 4.3.21

4.3.21 Apply Fourier transforms to $\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy - 2u(x) = f(x)$ to show that the solution is $u = -\frac{1}{2}f + \frac{1}{2}g$, where g comes from integrating f twice. (Its transform is $\hat{g} = \hat{f}/(i\omega)^2$.) If $f = e^{-|x|}$ find u and verify that it solves the integral equation.

Figure 3.94: the Problem statement

Comparing the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy - 2u(x) = f(x) \tag{1}$$

with the one in the textbook, page 322 in example one, where the Fourier transform of

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy = f(x)$$

Is given as

$$\frac{2}{1+\omega^2}\hat{u}(\omega) = \hat{f}(\omega)$$

The only difference is that in this problem we have an extra $-2u(x)$ term, whose Fourier transform is $-2\hat{u}(\omega)$. Hence the Fourier transform for (1) becomes

$$\frac{2}{1+\omega^2}\hat{u}(\omega) - 2\hat{u}(\omega) = \hat{f}(\omega)$$

Solving for $\hat{u}(\omega)$

$$\begin{aligned} \hat{u}(\omega)\left(\frac{2}{1+\omega^2} - 2\right) &= \hat{f}(\omega) \\ \hat{u}(\omega)\left(\frac{2 - 2\left(1+\omega^2\right)}{1+\omega^2}\right) &= \hat{f}(\omega) \\ \hat{u}(\omega) &= \frac{1+\omega^2}{-2\omega^2}\hat{f}(\omega) \end{aligned}$$

We need to write the above as $\hat{u}(\omega) = \frac{-1}{2}f + \frac{1}{2}g$. Hence

$$\hat{u}(\omega) = \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \tag{2}$$

Let $f(x) = e^{-|x|}$, then

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx \\
&= \int_{-\infty}^{0} e^{x} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx \\
&= \left[\frac{e^{x(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-x(1+i\omega)}}{1+i\omega}\right]_{0}^{\infty} \\
&= \frac{1}{1-i\omega} - \frac{1}{1+i\omega} \\
&= \frac{(1+i\omega)-(1-i\omega)}{(1-i\omega)(1+i\omega)} \\
&= \frac{2}{1+\omega^2}
\end{aligned}$$

Hence using (2)

$$\begin{aligned}
\hat{u}(\omega) &= \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \\
&= \frac{-1}{2}\frac{2}{1+\omega^2} + \frac{1}{-2\omega^2}\frac{2}{1+\omega^2} \\
&= -\frac{1}{\omega^2}
\end{aligned}$$

Hence

$$u(x) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2} e^{i\omega x} d\omega$$

Using tables $u(x) = \frac{-1}{2}|x|$.

3.5.19 Problem 4.3.27

4.3.27 Take Fourier transforms in the equation $d^4G/dx^4 - 2a^2 d^2G/dx^2 + a^4 G = \delta$ to find the transform $\hat{G}$ of the fundamental solution. How would it be possible to find G?

Figure 3.95: the Problem statement

The equation is

$$\frac{d^4 G(x)}{dx^4} - 2a^2 \frac{d^2 G(x)}{dx^2} + a^4 G(x) = \delta$$

Taking Fourier transform, and using $\frac{d^nG}{dx^n} \Longrightarrow (ik)^n \hat{g}(k)$,hence $G'(x) \Longrightarrow ik\hat{g}(k), G''(x) \Longrightarrow -k^2\hat{g}(k), G''''(x) \Longrightarrow (ik)^4 \hat{g}(k) = k^4\hat{g}(k)$. Therefore the Fourier transform of the above differential equation is

$$k^4\hat{g}(k) + 2a^2k^2\hat{g}(x) + a^4\hat{g}(k) = 1$$

Solving for $\hat{g}(k)$

$$\begin{aligned}
\hat{g}(k)\left(k^4 + 2a^2k^2 + a^4\right) = 1 \\
\hat{g}(k) &= \frac{1}{k^4 + 2a^2k^2 + a^4} \\
&= \frac{1}{\left(k^2 + a^2\right)^2}
\end{aligned}$$

To find $G(x)$ we need to find the inverse Fourier transform.

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(k^2 + a^2\right)^2} e^{ikx} dk$$

With the help of computer, I obtained the following result

$$G(x) = \frac{(1 + a|x|)}{4a^3} e^{-}a|x|$$

3.6 HW 6, Due Dec 10, 2014

3.6.1 Problem 1

Draw bifurcation diagrams for the normal form of the transcritical bifurcation: $\frac{dx}{dt} = rx - x^2$, and of the pitchfork bifurcation: $\frac{dx}{dt} = rx - x^3$

Solution:

3.6.1.1 Part(a) transcritical bifurcation

For transcritical bifurcation $\frac{dx}{dt} = f(r,x) = rx - x^2$. The critical points are $x^* = 0$ and $x^* = r$.

There are 3 cases to consider. $r = 0, r < 0$ and $r > 0$.The the vector field plot is first made, using x as the x-axis, and using x' as the y-axis.

Using Mathematica,a plot of the 3 above cases was generated

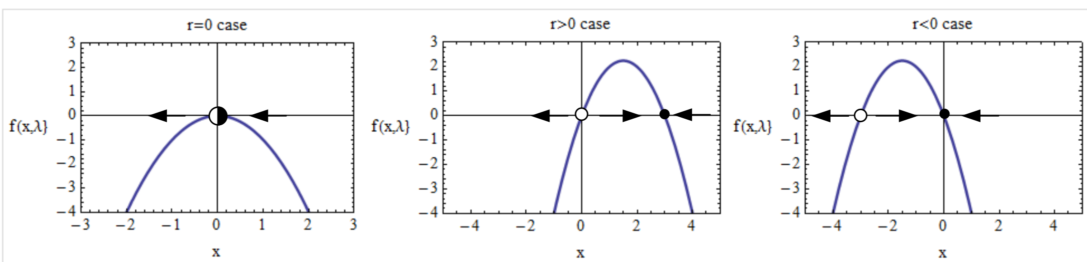


Figure 3.96: plot for problem 1

To plot the Bifurcation diagram, we have to now use r as the x-axis and use x for the y-axis. This was done by hand similar to what the textbook at page 50 shows.

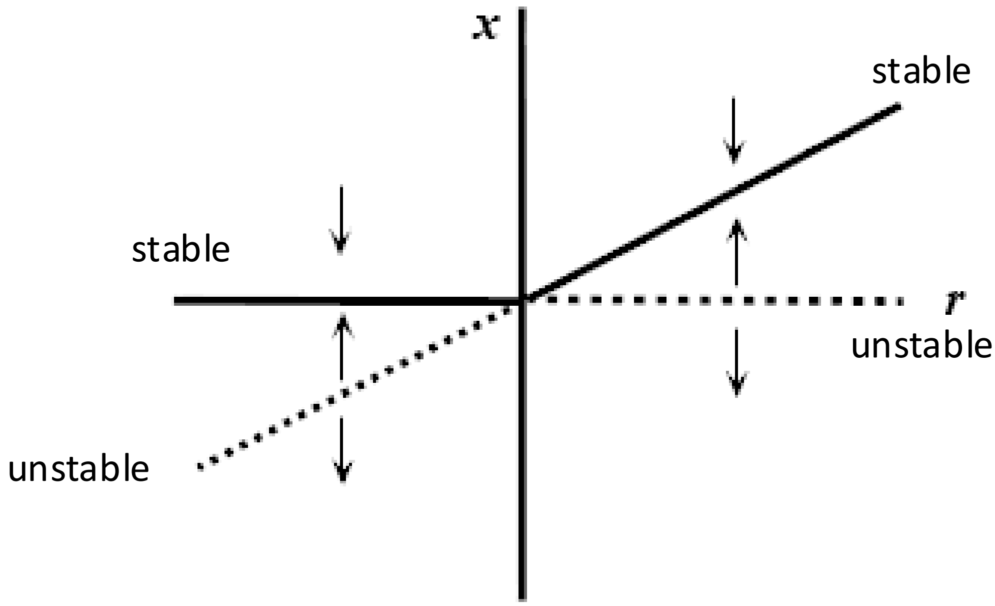


Figure 3.97: second plot for problem 1

3.6.1.2 Part (b) pitchfork bifurcation

$\frac{dx}{dt} = rx - x^3$. The critical points are $x\left(r - x^2\right) = 0$, hence $x^* = 0$ and $x^* = \pm\sqrt{r}$. When $r = 0$ then $x' = -x^3$. So it approaches $x = 0$ from the right and approaches $x = 0$ from the left. Hence $x^* = 0$ is stable in this case. When $r < 0$, then only $x^* = 0$ is fixed point (since we can't have complex values). So this is similar to $r = 0$ case. When $r > 0$ then there are 3 critical points now $x^* = 0, -\sqrt{r}, \sqrt{r}$. The following Bifurcation illustrates these cases (from textbook, Nonlinear Dynamics and Chaos, page 56)

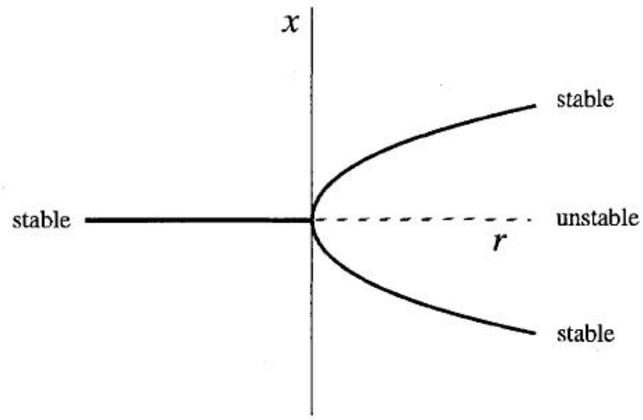


Figure 3.98: pitchfork bifurcation

3.6.2 Problem 2

Find a 2D dynamical system that undergoes Hopf bifurcation, and explain why the Hopf bifurcation occurs.

Solution:

Hopf bifurcation requires a minimum of 2D system to occur. Hopf bifurcation shows up when spiral changes from stable to unstable (or vice versa) with a new periodic solution showing up. So Hopf bifurcation considers when a 2D system with stable fixed point losses the stability at this point when a parameter changes. So changes in the parameters, causes one of the eigenvalues of the Jacobian to become positive, causing instability. An example from the textbook is given by

$$
\begin{aligned}
r' &= \mu r - r^3 \\
\theta' &= \omega + br^2
\end{aligned}
$$

The phase portrait is shown in figure below from the text book. This shows that when $\mu < 0$, the origin was stable. (spiral in). But when $\mu > 0$, a limit cycle show up with radius $r = \sqrt{\mu}$ and inside this radius, it is spiral out, hence the origin became unstable, moving to the limit cycle, and outside the limit cycle, it is stable and state trajectory moves towards the limit cycle. Here is the diagram from the text

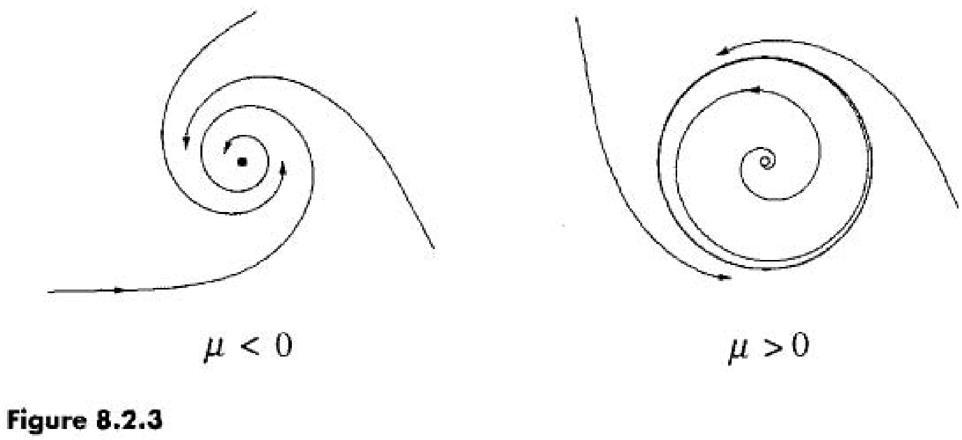


Figure 3.99: phase portrait

The eigenvalues of the Jacobian, evaluated at the origin (critical point) is shown to be $\lambda = \mu \pm i\omega$. So as μ changes from negative to positive, the system moves from being stable to unstable.

3.6.3 Problem 4.4.6

4.4.6 The derivative df/dz of an analytic function is also analytic; it still depends on the combination $z = x + iy$. Find df/dz if $f = 1 + z + z^2 + \cdots$ or $f = z^{1/2}$ (away from $z = 0$).

Figure 3.100: Problem description

Using the form $f(z) = z^{\frac{1}{2}}$, taking derivative w.r.t. gives $f'(z) = \frac{1}{2}\frac{1}{z^{\frac{1}{2}}}$. But $z = x + iy$, hence

$$f'(z) = \frac{1}{2}\frac{1}{\sqrt{(x+iy)}} = \frac{1}{2}\frac{\sqrt{(x-iy)}}{\sqrt{(x+iy)}\sqrt{(x-iy)}} = \frac{1}{2}\frac{\sqrt{(x-iy)}}{\sqrt{x^2+y^2}} = \frac{1}{2}\frac{1}{|z|}\sqrt{(x-iy)}$$

But $\sqrt{(x-iy)} = \bar{z}^{\frac{1}{2}}$ where $\bar{z}$ is complex conjugate of z. Hence

$$f'(z) = \frac{1}{2|z|}\bar{z}^{\frac{1}{2}}$$

3.6.4 Problem 4.4.7

4.4.7 Are the following functions analytic?

(a) $f = |z|^2 = x^2 + y^2$
(b) $f = \text{Re}\, z = x$
(c) $f = \sin z = \sin x \cosh y + i \cos x \sinh y$.

Can a function satisfy Laplace's equation without being analytic?

Figure 3.101: the Problem statement

A function $f(z)$ is analytic if it satisfies conditions as given in 4P, page 334

4P A function $f(z)$ is ***analytic*** at $z = a$ if in a neighborhood of that point

(1) it depends on the combination $z = x + iy$ and satisfies $i\partial f/\partial x = \partial f/\partial y$
(2) its real and imaginary parts are connected by the Cauchy-Riemann equations $u_x = s_y$ and $u_y = -s_x$
(3) it is the sum of a convergent power series $c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$

Figure 3.102: Problem description

3.6.4.1 Part(a)

$$f = |z|^2 = x^2 + y^2$$

Using 4P part(1), then $i\frac{\partial f}{\partial x} = i2x$ and $\frac{\partial f}{\partial y} = 2y$. Hence they are not the same. Therefore not analytic.

3.6.4.2 Part(b)

$$f = \text{Re}(z) = x$$

$i\frac{\partial f}{\partial x} = i$ and $\frac{\partial f}{\partial y} = 0$, hence not analytic.

3.6.4.3 Part(c)

$$
\begin{aligned}
f &= \sin x \cosh y + i \cos x \sinh y \\
&= u\left(x,y\right) + iv\left(x,y\right)
\end{aligned}
$$

Since

$$
i\frac{\partial f}{\partial x} = i\left(\cos x \cosh y - i \sin x \sinh y\right) = i \cos x \cosh y + \sin x \sinh y \tag{1}
$$

And

$$
\frac{\partial f}{\partial y} = \sin x \sinh y + i \cos x \cosh y \tag{2}
$$

We see that (1) and (2) are the same. Hence analytic.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}
$$

3.6.5 Problem 4.4.17

4.4.17 For the map $w = \frac{1}{2}(z + z^{-1})$ in Fig. 4.15, what happens to points $z = x > 1$ on the real axis? What happens to points $0 < x < 1$? What happens to the imaginary axis $z = iy$?

Figure 3.103: Problem description

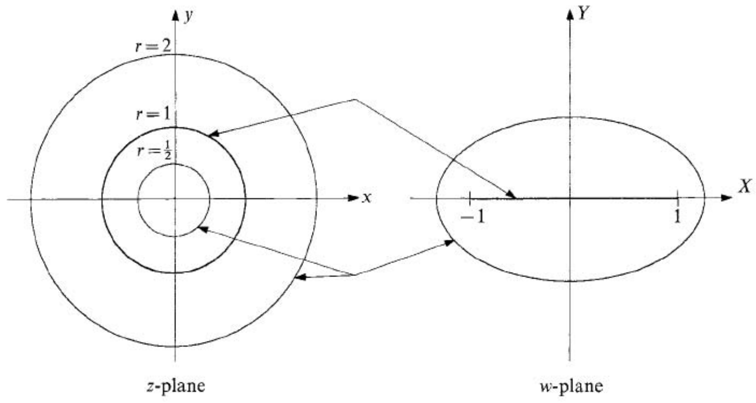


Fig. 4.15. The map from z to $w = \frac{1}{2}(z + z^{-1})$.

Figure 3.104: Problem description

3.6.5.1 Part (a)

The mapping $w = \frac{1}{2}\left(z + z^{-1}\right)$ is

$$
\begin{aligned}
w &= \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) \\
&= \frac{1}{2}\left(r\left(\cos\theta + i\sin\theta\right) + \frac{1}{r}\left(\cos\theta - i\sin\theta\right)\right) \\
&= \frac{1}{2}\left(\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right) \\
&= \frac{1}{2}\left(\frac{r^2+1}{r}\right)\cos\theta + i\frac{1}{2}\left(\frac{r^2-1}{r}\right)\sin\theta
\end{aligned}
$$

For example, for unit circle, $r = 1$ and $w = \cos\theta$. Hence all points on unit circle map to $X = \cos\theta$. i.e. the link between $X = -1\cdots 1$. To answer the question, it might be easier to write

$$
\begin{aligned}
w &= \frac{1}{2}\left(\left(x + iy\right) + \frac{1}{x + iy}\right) \\
&= \frac{1}{2}\left(\left(x + iy\right) + \frac{x - iy}{\left(x + iy\right)\left(x - iy\right)}\right) \\
&= \frac{1}{2}\left(\left(x + iy\right) + \frac{x - iy}{\left(x^2 + y^2\right)}\right) \\
&= \frac{1}{2}\left(x + iy + \frac{x}{\left(x^2 + y^2\right)} - i\frac{y}{\left(x^2 + y^2\right)}\right)
\end{aligned}
$$

Write as $w = X + iY$

$$w = \frac{1}{2}\left(x + \frac{x}{\left(x^2+y^2\right)}\right) + i\left(\frac{1}{2}\left(y - \frac{y}{\left(x^2+y^2\right)}\right)\right)$$

Hence for point $(x,0)$ it maps to $w = \frac{1}{2}\left(x + \frac{1}{x}\right) + i0 = \frac{1}{2}\left(\frac{x^2+1}{x}\right)$. Since $x > 1$ then $\frac{1}{2}\left(\frac{x^2+1}{x}\right)$ maps to all point on X that are larger than $X = 1$

3.6.5.2 Part (b)

For $0 < x < 1$, then from $w = \frac{1}{2}\left(x + \frac{1}{x}\right)$, we see that for example, of $x = 1/3$ then $X = \frac{1}{2}\left(\frac{1}{3} + 3\right) > 1$.

3.6.5.3 Part (c)

For $z = iy$, then $x = 0$, and the mapping becomes

$$w = i\left(\frac{1}{2}\left(y - \frac{1}{y}\right)\right)$$

Hence

$$Y = \frac{1}{2}\left(y - \frac{1}{y}\right)$$

So

$$\begin{aligned} y &= 0 \rightarrow Y = \infty \\ y &= 1 \rightarrow Y = 0 \\ y &= -1 \rightarrow Y = 0 \\ y &> 1 \rightarrow 0 < Y < 1 \end{aligned}$$

3.6.6 Problem 4.4.23

4.4.23 Solve Laplace's equation in the 45° wedge if the boundary condition is $u = 0$ on both sides $y = 0$ and $y = x$.

(a) Where does $F(z) = z^4$ map the wedge?
(b) Find a solution with zero boundary conditions other than $u \equiv 0$.

Figure 3.105: the Problem statement

3.6.6.1 Part(a)

We need to transform to XY plane using conformal mapping to be able to solve it in the standard Cartesian system instead of on the quarter circle. Since the angle is 45^0 we need to map it to the full 180^0. So this mapping will work $w^4 = e^{4i\theta}$. So a point on e^{i45^0} will map to e^{i180^0} and point at e^{i0^0} will map to e^{i0^0}, hence the top half plane is where the new XY coordinates is. So we need to solve

$$U_{XX} + U_{YY} = 0 \tag{1}$$

In the upper half plane, then transform the solution back to (x, y) space. Solution to (1) is $U = aX + bY$. Since $U_{XX} = 0$ and $U_{YY} = 0$, hence this solution satisfies (1). We now need to figure how to map this back to (x, y). Using

$$\begin{aligned} w &= (x + iy) \\ w^4 &= (x + iy)^4 = x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 \\ &= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) \end{aligned}$$

Hence $X = (x^4 - 6x^2y^2 + y^4)$ and $Y = 4x^3y - 4xy^3$. So the solution is

$$U = aX + bY = a(x^4 - 6x^2y^2 + y^4) + b(4x^3y - 4xy^3)$$

Where a, b are constant found from boundary conditions.

3.6.7 Problem 6.1.11

6.1.11 Find the solution with arbitrary constants C and D to

(a) $u'' - 9u = 0$ (b) $u'' - 5u' + 4u = 0$ (c) $u'' + 2u' + 5u = 0$

Figure 3.106: the Problem statement

3.6.7.1 Part(a)

$$u'' - 9u = 0$$

This is constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 - 9 = 0$, hence $\lambda = \pm 3$, therefore the solution is

$$u(t) = De^{3t} + Ce^{-3t}$$

We notice this is not stable ode.

3.6.7.2 Part(b)

$$u'' - 5u' + 4u = 0$$

This is also constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 - 5\lambda + 4 = 0$. Solution is $\lambda = \{4, 1\}$, therefore the solution is

$$u(t) = De^{4t} + Ce^{t}$$

This is also not stable ode.

3.6.7.3 Part(c)

$$u'' + 2u' + 5u = 0$$

This is also constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 + 2\lambda + 5 = 0$, Solution is: $\lambda = \{-1 + 2i, -1 - 2i\}$, therefore the solution is

$$\begin{aligned} u(t) &= De^{(-1+2i)t} + Ce^{(-1-2i)t} \\ &= e^{-t}\left(De^{2it} + C^{-2it}\right) \end{aligned}$$

Which can be written as

$$u(t) = e^{-t}(d\cos 2t + c\sin 2t)$$

3.6.8 Problem 6.1.12

6.1.12 Find an equation $u'' + pu' + qu = 0$ whose solutions are
(a) e^t, e^{-t} (b) $\sin 2t, \cos 2t$ (c) $1, t$ (d) $e^{-t}\sin t, e^{-t}\cos t$

Figure 3.107: Problem description

3.6.8.1 Part(a)

From the solutions, we see that roots of the characteristic equation are $\{1, -1\}$, which means the characteristic equation is

$$p(\lambda) = (\lambda - 1)(\lambda + 1) = \lambda^2 - 1$$

Which implies the ODE is $u'' - u = 0$

3.6.8.2 Part(b)

Since the solution contains no damping (no e^{-t} term), and only contain oscillation, then it means the ode much contain only friction term, hence the ode is of the form

$$u'' + qu = 0$$

Since oscillation frequency is 2, then $\lambda_1 = 2i, \lambda_2 = -2i$ so to be able to contain the sin/cos shown as the solutions. Hence

$$p(\lambda) = (\lambda - 2i)(\lambda + 2i) = \lambda^2 + 4$$

Therefore

$$u'' + 4u = 0$$

3.6.8.3 Part(c)

Let

$$u(t) = Au_1 + Bu_2$$

Where A, B are constants of integration. Then $u(t) = A + Bt$ or $u' = B$ or $u'' = 0$

3.6.8.4 Part(d)

Since the solution contains damping (has e^{-t} term), and since oscillation oscillation exist, then the solution must be of form

$$u'' + pu' + qu = 0$$

The roots of the characteristic equation are therefore $\lambda_1 = -1 + i, \lambda_2 = -1 - i$. Hence

$$p(\lambda) = (\lambda - (-1 + i))(\lambda - (-1 - i)) = \lambda^2 + 2\lambda + 2$$

Therefore the ODE is

$$u'' + 2u' + 2u = 0$$

3.6.9 Problem 6.2.2

6.2.2 What types of critical points can $u' = Au$ have if
(1) A is symmetric positive definite
(2) A is symmetric negative definite
(3) A is skew-symmetric
(4) A is negative definite plus skew-symmetric (choose example).

Figure 3.108: the Problem statement

3.6.9.1 Part(1)

Since eigenvalues of A are real and positive, then not stable

3.6.9.2 Part(2)

Since eigenvalues of A are real and negative, then stable

3.6.9.3 Part(3)

(real) skew symmetric matrix always have pure imaginary eigenvalues. Hence phase plane is circles. This is called marginally stable.

3.6.9.4 Part(4)

And example of negative definite is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and skew symmetric is $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, hence $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$

The eigenvalues are found from $\begin{vmatrix} \lambda + 1 & -2 \\ 2 & \lambda + 1 \end{vmatrix} = 0$ or $(\lambda + 1)^2 + 4 = 0$, hence $(\lambda + 1)^2 = -4$ or $\lambda + 1 = \pm 2i$, therefore $\lambda = -1 \pm 2i$

Hence the eigenvalues have negative real part and imaginary parts. This is stable, and spiral due to the sin/cos which will result in the solution. It will spiral in, since the real part is negative.

3.6.10 Problem 6.2.12

6.2.12 With internal competition the predator-prey system might be

$$u_1' = u_1 - u_1^2 - bu_1u_2, \qquad u_2' = u_2 - u_2^2 + cu_1u_2$$

Find all equilibrium points and their stability (for $c < 1$ and $c > 1$). Which points make sense biologically?

Figure 3.109: the Problem statement

$$\begin{aligned} u_1' &= u_1 - u_1^2 - bu_1u_2 = F_1(u_1, u_2) \\ u_2' &= u_2 - u_2^2 + cu_1u_2 = F_2(u_1, u_2) \end{aligned}$$

We first need to find critical points by solving $F_1(u_1, u_2) = 0$ and $F_2(u_1, u_2) = 0$

From $F_1(u_1, u_2) = 0$ we obtain

$$u_1(1 - u_1 - bu_2) = 0$$

Hence $u_1 = 0$ or $u_1 = 1 - bu_2$. looking at the second equation $F_2(u_1, u_2) = 0$ which gives

$$u_2(1 - u_2 + cu_1) = 0$$

Hence $u_2 = 0$ or $u_2 = 1 + cu_1$.

Considering the case of $u_1 = 0$, then $u_2 = 1$, and when $u_1 = 1 - bu_2$, then

$$\begin{aligned} u_2 &= 1 + c(1 - bu_2) \\ &= 1 + c - cbu_2 \\ u_2 + cbu_2 &= 1 + c \\ u_2 &= \frac{1 + c}{1 + cb} \end{aligned}$$

And when $u_2 = 0$ then $u_1 = 1$ and when $u_2 = \frac{1+c}{1+cb}$ then $u_1 = 1 - bu_2 = 1 - b\frac{1+c}{1+cb}$. Hence the critical points are

$$
\begin{aligned}
u_1 &= 0, u_2 = 0\\
u_1 &= 0, u_2 = 1\\
u_1 &= 1, u_2 = 0\\
u_1 &= -\frac{(b-1)}{bc+1}, u_2 = \frac{1+c}{1+cb}
\end{aligned}
$$

To find stability, we evaluate the Jacobian at each of the critical points. The Jacobian is

$$
J = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(u_1 - u_1^2 - bu_1u_2\right)}{\partial u_1} & \frac{\partial\left(u_1 - u_1^2 - bu_1u_2\right)}{\partial u_2} \\ \frac{\partial\left(u_2 - u_2^2 + cu_1u_2\right)}{\partial u_1} & \frac{\partial\left(u_2 - u_2^2 + cu_1u_2\right)}{\partial u_2} \end{bmatrix} = \begin{bmatrix} (1 - 2u_1 - bu_2) & -bu_1 \\ cu_2 & 1 - 2u_2 + cu_1 \end{bmatrix}
$$

At point $u_1 = 0, u_2 = 0$ we obtain $J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ this has eigenvalues $\lambda = 1$ (double). Hence not stable node.

At point $u_1 = 0, u_2 = 1$ we obtain $J = \begin{bmatrix} 1-b & 0 \\ c & -1 \end{bmatrix}$ which has eigenvalues: $\{1-b, -1\}$. Hence if $b > 1$ then both are stable. (negative), hence stable node. But if $b < 1$ then one is stable and the other is not. Which means unstable saddle point.

At point $u_1 = 1, u_2 = 0$ we obtain $J = \begin{bmatrix} -1 & -b \\ 0 & 1+c \end{bmatrix}$, eigenvalues: $c+1, -1$. Hence if $c < -1$ then both are stable, and we have stable node. If $c > -1$ then one is stable and the other is not, so we have unstable saddle.

3.6.11 Problem 6.2.13

6.2.13 According to Braun, reptiles, mammals, and plants on the island of Komodo have populations governed by

$$
\begin{aligned}
u_r' &= -au_r - bu_ru_m + cu_ru_p\\
u_m' &= -du_m + eu_ru_m\\
u_p' &= fu_p - gu_p^2 - hu_ru_p.
\end{aligned}
$$

Who is eating whom? Find all equilibrium solutions u^*.

Figure 3.110: the Problem statement

$$
\begin{aligned}
u_r' &= -au_r - bu_r u_m + cu_r u_p = F_1\left(u_r, u_m, u_p\right)\\
u_m' &= -du_m + eu_r u_m = F_2\left(u_r, u_m, u_p\right)\\
u_p' &= fu_p - gu_p^2 - hu_r u_p = F_3\left(u_r, u_m, u_p\right)
\end{aligned}
$$

We first need to find critical points by solving $F_1\left(u_r, u_m, u_p\right) = 0$ and $F_2\left(u_r, u_m, u_p\right) = 0$ and $F_3\left(u_r, u_m, u_p\right) = 0$. Solving using computer algebra gives

```
eq1:=-a*u[r]-b*u[r]*u[m]+c*u[r]*u[p]=0;
eq2:=-d*u[m]-e*u[r]*u[m]=0;
eq3:=f*u[p]-g*(u[p])^2-h*u[r]*u[p]=0;
solve({eq1,eq2,eq3},{u[r],u[p],u[m]});
```

$$
\begin{aligned}
&u_m = 0, u_p = 0, u_r = 0\\
&u_m = -\frac{a}{b}, u_p = 0, u_r = -\frac{d}{e}\\
&u_m = 0, u_p = \frac{f}{g}, u_r = 0\\
&u_m = -\frac{aeg - dch - cfe}{ebg}, u_p = \frac{hd + fe}{eg}, u_r = -\frac{d}{e}\\
&u_m = 0, u_p = \frac{a}{c}, u_r = -\frac{ag - cf}{ch}
\end{aligned}
$$

We now need to find the Jacobian and evaluate it at each of the above points to determine the type of stability.

```
jac:=Matrix([[diff(eq1,u[r]),diff(eq1,u[m]),diff(eq1,u[p])],
[diff(eq2,u[r]),diff(eq2,u[m]),diff(eq2,u[p])],
[diff(eq3,u[r]),diff(eq3,u[m]),diff(eq3,u[p])]]);
```

Which gives

$$
J = \begin{bmatrix} -bu_m + cu_p - a & -bu_r & cu_r \\ -eu_m & -eu_r - d & 0 \\ -hu_p & 0 & -2gu_p - hu_r + f \end{bmatrix}
$$

At point $u_m = 0, u_p = 0, u_r = 0$, $J = \begin{bmatrix} -a & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & f \end{bmatrix}$ so assuming all a, d, f are positive, this shows this point is not stable. It is $\boxed{\text{unstable spiral}}$ since one of the eigenvalues is positive.

At point $u_m = -\frac{a}{b}, u_p = 0, u_r = -\frac{d}{e}, J = \begin{bmatrix} -b\frac{a}{b} - a & b\frac{d}{e} & -c\frac{d}{e} \\ e\frac{a}{b} & e\frac{d}{e} - d & 0 \\ 0 & 0 & h\frac{d}{e} + f \end{bmatrix} = \begin{bmatrix} -2a & b\frac{d}{e} & -c\frac{d}{e} \\ e\frac{a}{b} & 0 & 0 \\ 0 & 0 & h\frac{d}{e} + f \end{bmatrix}$, eigenvalues are $\left\{-a - \sqrt{a(a+d)}, \sqrt{a(a+d)} - a, \frac{1}{e}\left(fe + dh\right)\right\}$. So for positive parameters $\sqrt{a(a+d)} - a > 0$, hence $\underline{\text{not stable}}$.

At $u_m = 0, u_p = \frac{f}{g}, u_r = 0, J = \begin{bmatrix} c\frac{f}{g} - a & 0 & 0 \\ 0 & -d & 0 \\ -h\frac{f}{g} & 0 & -2g\frac{f}{g} + f \end{bmatrix}$, eigenvalues: $-d, -f, -\frac{1}{g}\left(ag - cf\right)$. Therefore, for positive parameters, this is $\underline{\text{stable}}$ node.

3.6.12 Problem 6.2.19

6.2.19 (Epidemic theory). Suppose $u(t)$ people are healthy at time t and $v(t)$ are infected. If the latter become dead or otherwise immune at rate b and infection occurs at rate a, then $u' = -auv, \quad v' = auv - bv.$

(a) Show that $v' > 0$ if $u > b/a$, so the epidemic spreads.
(b) Show that $v' < 0$ if $u < b/a$, so the epidemic slows down. (It never starts if $u_0 < b/a$.)
(c) Show that $E = u + v - (b/a)\log u$ is constant during the epidemic.
(d) What is $v_{\max}$ (when $u = b/a$) in terms of u_0?

Figure 3.111: the Problem statement

$$\begin{aligned} u' &= -auv = F_1(u,v) \\ v' &= auv - bv = F_2(u,v) \end{aligned}$$

The critical points are $u = any, v = 0$.

3.6.12.1 Part(a)

If $u > \frac{b}{a}$, then we write $u = \frac{b+\varepsilon}{a}$ for $\varepsilon > 0$. Substituting in $v' = auv - bv$ results in

$$\begin{aligned} v' &= a\frac{b+\varepsilon}{a}v - bv \\ &= bv + \varepsilon v - bv \\ &= \varepsilon v \end{aligned}$$

Hence $v' > 0$ and the epidemic spreads.

3.6.12.2 Part(b)

If $u < \frac{b}{a}$, then we write $u = \frac{b-\varepsilon}{a}$ for $\varepsilon > 0$, Substituting in $v' = auv - bv$ results in

$$\begin{aligned} v' &= a\frac{b-\varepsilon}{a}v - bv \\ &= bv - \varepsilon v - bv \\ &= -\varepsilon v \end{aligned}$$

Hence $v' < 0$ and the epidemic slows down.

3.6.12.3 Part(d)

From second equation, $v(t) = Ae^{\int (u(t)a-b)dt}$, hence when $u(t) = \frac{b}{a}$, then $v(t) = k$. A constant u_0.

Chapter 4

cheat sheet

<u>Fourier Transform formulas</u>

$$\hat{u}(k) = \int_{-\infty}^{\infty} u(x)e^{-ikx}\,dx$$

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{ikx}\,dk$$

$$\hat{u}(k) = \hat{G}(k)\hat{h}(k)$$

$$u(x) = g(x)h(x) \Longleftrightarrow \hat{g}(k) * \hat{h}(k) = \hat{u}(k) = \int_{-\infty}^{\infty} \hat{g}(k-\tau)\hat{h}(\tau)\,d\tau$$

$$\hat{u}(k) = \hat{g}(k)\hat{h}(k) \Longleftrightarrow g(x) * h(x) = u(x) = \int_{-\infty}^{\infty} g(x-\tau)h(\tau)\,d\tau$$

$$e^{ixd}u(x) \Longleftrightarrow \hat{u}(k-d)$$

$$u(x-a) \Longleftrightarrow e^{-iak}\hat{u}(k)$$

$$\frac{du}{dx} \Longleftrightarrow ik\hat{u}(k)$$

$$\int_a^x u(x)\,dx \Longleftrightarrow \frac{\hat{u}(k)}{ik} + c\delta(k)$$

<u>Fourier Series formulas</u>

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos(kx) + b_k\sin(kx)\right)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)f(x)\,dx = c_k + c_{k}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)f(x)\,dx = i(c_k - c_{-k})$$

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$$

$$\text{square wave} \overset{\text{Fourier series}}{\longleftrightarrow} \sum_{k=1,3,5,\ldots}^{\infty} \frac{4}{\pi k}\sin(kx)$$

$$f(x) \overset{\text{Fourier series}}{\longleftrightarrow} \sum_{-\infty}^{\infty} c_k e^{ikx}$$

$$f(x) \overset{\text{Fourier transform}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x} \quad \text{where} \quad \omega = \frac{2\pi}{T}$$

$$\delta(x) \overset{\text{Fourier series}}{\longleftrightarrow} \frac{1}{2\pi} + \frac{1}{\pi}(\cos x + \cos 2x + \cos 3x + \ldots$$

$$x \overset{\text{Fourier series}}{\longleftrightarrow} 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

$$x \overset{\text{Fourier series}}{\longleftrightarrow} 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

$$|\sin(x)| \overset{\text{Fourier series}}{\longleftrightarrow} \begin{cases} a_0 & = \frac{2}{\pi} \\ a_k & = \begin{cases} 0, & \text{k odd.} \\ \frac{4}{\pi}\left(\frac{1}{1-k^2}\right), & \text{k even.} \end{cases} \end{cases}$$

$$\hat{H}(k) = \hat{f}(k)\hat{g}(k) \overset{F^{-1}}{\longleftrightarrow} H(x) = f(x) \circledast g(x) = \int_{-infty}^{\infty} f(\tau)g(x-\tau)\,d\tau$$

$$u(x) = f(x) \circledast g(x)$$

```
In[16]:= Integrate[Sin[x] Cos[k x], x]
Out[16]= (Cos[x] Cos[k x] + k Sin[x] Sin[k x]) / (-1 + k^2)

In[15]:= Integrate[Cos[x] Cos[k x], x]
Out[15]= (-Cos[k x] Sin[x] + k Cos[x] Sin[k x]) / (-1 + k^2)

In[17]:= Integrate[Sin[x] Sin[k x], x]
Out[17]= (-k Cos[k x] Sin[x] + Cos[x] Sin[k x]) / (-1 + k^2)

In[18]:= Integrate[Cos[x] Sin[k x], x]
Out[18]= (k Cos[x] Cos[k x] + Sin[x] Sin[k x]) / (1 - k^2)
```

1. When period T is not 2π replace k by $\frac{2\pi}{T}k$ in all formulas for Fourier series.

2. Plancherel formula $2\pi \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\,dk$

3. Parseval's formula $\int_{-\pi}^{\pi} |f(x)|^2\,dx = 2\pi \sum_{k=1}^{\infty} |c_k|^2$

4. Parseval's formula again $2\pi a_0^2 + \pi\left(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots\right) = \int_{-\infty}^{\infty} f^2(x)\,dx$

5. Inner products $2\pi \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(k)\bar{\hat{g}}(k)\,dk$

6. integration by parts $\int uv' = [uv] - \int u'v$ so pick the one that is easy to differentiate for u and the one that is easy to integrate for v.

7. properties of odd and even functions Let o, e be odd and even functions, then $e+e = e, o+o = o, e\times e = e, o\times o = e, o\times e = o, \frac{e}{e} = e, \frac{e}{o} = o$

8. trig identities

$$
\begin{aligned}
\sin^2(x) &= \frac{1}{2} - \frac{1}{2}\cos(2x)\\
\cos^2(x) &= \frac{1}{2} + \frac{1}{2}\cos(2x)\\
\sin^3(x) &= \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\\
\cos^3(x) &= \frac{3}{4}\cos(x) - \frac{1}{2}\cos(2x)\\
\sin(2x) &= 2\sin(x)\cos(x)\\
\cos(2x) &= \cos^2(x) - \sin^2(x)\\
&= 1 - 2\sin^2(x)\\
&= 2\cos^2(x) - 1\\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)}\\
\sin(A \pm B) &= \sin(A)\cos(B) \pm \cos(A)\cos(B)\\
\cos(A \pm B) &= \cos(A)\sin(B) \mp \sin(A)\sin(B)\\
\int \cos^n(x)\,dx &= \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}\,dx\\
&= \frac{1}{2}\cos x\sin x + \frac{x}{2} \quad \text{n even}\\
&= \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x \quad \text{n odd}\\
\int \sin^n(x)\,dx &= \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}\,dx\\
&= \frac{-1}{2}\sin x\cos x + \frac{x}{2} \quad \text{n even}\\
&= \frac{-1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x \quad \text{n odd}\\
\int x^n e^{ax}\,dx &= \frac{1}{a}\left(x^n e^{ax} - n\int x^{n-1}e^{ax}\,dx\right)
\end{aligned}
$$

9. exp/trig

$$\sin(x) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos(x) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$re^{i\theta} = r\left(\cos(\theta) + i\sin(\theta)\right)$$

$$\ln(re^{i\theta}) = \ln(r) + i\theta + 2k\pi i$$

$$F(e^{-\frac{x^2}{2}}) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx}\, dx = e^{-\frac{k^2}{2}}\sqrt{2\pi}$$

$$F(e^{-x^2}) = e^{-\frac{k^2}{4}}\sqrt{\pi}$$

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Laplace

1. To find solution to Laplace on disk, or radius r, use polar. The solution is

$$\begin{aligned} u(r,\theta) = a_0 &+ a_1 r\cos(\theta) + b_1 r\sin(\theta) \\ &+ a_2 r^2\cos(2\theta) + b_2 r^2 \sin(2\theta) \ldots \end{aligned}$$

Where the a_k and b_k found from finding Fourier series of $u(r,\theta)$ evaluated at boundary as normally done.

2. solution inside the circle is

$$u(r,\theta) = \frac{1}{2\pi} = \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \zeta)\, d\zeta}$$

If we are given $u_0 = \delta$ at point on circle as boundary conditions, use the above formula, much easier.

misc. items

1. The function that minimizes $\int_a^b \frac{1}{2}(u'(x))^2 - f u(x)\, dx$ is the solution of $u''(x) = f$

2. every function is made up of odd/even parts

$$f_{\text{odd part}} = \frac{f(x) - f(-x)}{2}$$

$$f_{\text{even part}} = \frac{f(x) + f(-x)}{2}$$

references:

1. schaum's mathematical handbook of formulas and tables by Spiegel
2. `http://www.integraltec.com/math`
3. `http://en.wikipedia.org/wiki/Integration_by_reduction_formulae`

$$Ax=z \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \xrightarrow{\text{Row elimination}} \begin{pmatrix} a & b \\ 0 & d-b\frac{c}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} z_1 \\ z_2-z_1\frac{c}{a} \end{pmatrix}$$

2 pivots

multiplier

$$A=LU,\text{ where } U=\begin{pmatrix} a & b \\ 0 & d-b\frac{c}{a} \end{pmatrix}, L=\begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \xrightarrow{\text{Only if A symmetric}} A=LDL^T \xrightarrow{\text{Only if all pivots >0}} A=\tilde{L}\tilde{L}^T$$

$\tilde{L}=\sqrt{D}L$

To solve $Ax=b$, Solve $Lz=b$ for z then solve $Ux=z$ for x

If $x^TAx>0$ for $x\neq 0$ then positive definite $\longrightarrow$ Pivots all>0 (this definition usually applied to symmetric matrices

To show quadratic form x^TAx is positive everywhere see if you can complete the squares. If so, then yes.

$p_1(m_1x_1+m_2x_2)^2+p_2x_2^2$

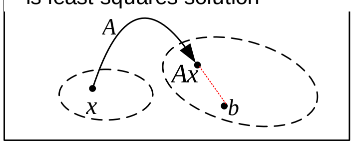
x^TAx minumum is also global min only if A is positive definite

This will be the pivots

These will be the multipliers

If $Ax=b$ has no solution, then solve $A^TAx=A^Tb$, hence $x=(A^TA)^{-1}A^Tb$, is least squares solution

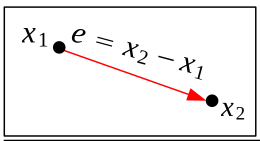
The x which minimized $\|Ax-b\|^2$ is least squares solution



$\|Ax-b\|^2=(Ax-b)^T(Ax-b)$

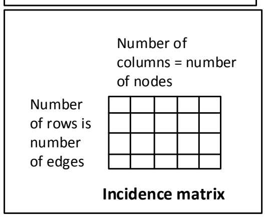
e =potential difference (elongation)
x =potential (node displacement)
y =flow (internal spring forces)
f =external nodal forces (applied node forces)

$$Ax=e,\quad Ce=y \longrightarrow y=CAx$$
$$A^Ty=f \longrightarrow A^TCAx=f$$



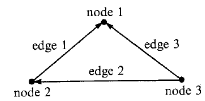
e = elongation of springs
x = mass displacement
c =elasticity
y =forces in spring
f =external forces

For fitting, First build $Ax=b$ then solve x from $A^TAx=A^Tb$



If A is an $n\times n$ square matrix with n linearly independent eigevectors then $A=S\Lambda S^{-1}$ where S is matrix with its columns the eigenvectors and Λ is diagonal matrix of the eigenvalues

$A=S\Lambda S^{-1}=Q\Lambda Q^T$.



$$A=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$K=A^TCA$$

spring potential

$P(x)=\frac{1}{2}kx^2=\frac{1}{2}x^TA^TCAx-x^Tf$

sum of squares:
if given $2x_1^2+8x_1x_2+11x_2^2$ write it as $2(x_1+ax_2)^2+bx_2^2$ and solve for a,b

A^TCA Is called the stiffness matrix K

$$A \overset{\text{square/invertible}}{\Longrightarrow} LU \overset{\text{symmetric}}{\Longrightarrow} LDL^T \overset{\text{positive pivots}}{\Longrightarrow} L\sqrt{D}\sqrt{D}L^T=\tilde{L}\tilde{L}^T$$

Solution to vectored second order ode. Supposed we have 2 second order ODE (coupled, say 2 masses), then the solution to

$$M\frac{d^2u}{dt^2}+ku=0 \text{ is } \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}=a_1\cos\omega_1t+b_1\sin\omega_1t\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}+a_2\cos\omega_2t+b_2\sin\omega_2t\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$$

Where $v's$ are the 2 eigenvectors and $\omega_1^2=\lambda_1$ and $\omega_2^2=\lambda_2$

1K The sum of the diagonal entries is the ***trace*** of A. It equals the sum of the eigenvalues. The product of the eigenvalues equals the ***determinant*** of A.

1M If A is real and symmetric then

(1) its eigenvalues are all real
(2) its eigenvectors can be chosen orthonormal (perpendicular with length one)
(3) $A=Q\Lambda Q^{-1}=Q\Lambda Q^T$.

If A is also positive definite then

(4) its eigenvalues are not only real but positive.

Content of this box is from Strang text book

Nasser M. Abbasi 092814

$P(x) = \frac{1}{2}x^TAx - x^Tb$ has min at $Ax = b$
$Q(x) = \frac{1}{2}y^TC^{-1}y - y^Tb$ has min at $C^{-1}y = b$
$A^Ty = f$ is the constraint

$$
\begin{aligned}
L &= Q(x) + x^T(A^Ty - f)\\
&= \frac{1}{2}y^TC^{-1}y - y^Tb + x^T(A^Ty - f)\\
&= \frac{1}{2}y^TC^{-1}y - y^Tb + (Ax)y - x^Tf
\end{aligned}
$$

$$\frac{\partial L}{\partial y} = C^{-1}y - b + (Ax)^T = 0$$

$$\frac{\partial L}{\partial x} = A^Ty - f = 0$$

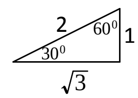
$$\begin{pmatrix} C^{-1} & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} y_i \\ x \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

Lagrangian $L(x,y)$ is minimized w.r.t y and at same time maximized w.r.t. x

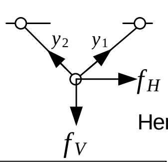
$P(x) = x^TAx - x^Tb$ has min at $Ax = b$
$Q(x) = y^TC^{-1}y - y^Tb$ has min at $C^{-1}y = b$

Use with optimization problems and constraint Q is called the objective function

If A is PD, then quadratic $P(x) = \frac{1}{2}\langle Ax,x\rangle - \langle x,b\rangle$ is min at $Ax = b$ or $x = A^{-1}b$ and its miniumum is $P(A^{-1}b) = -\frac{1}{2}\langle Ax,x\rangle = -\frac{1}{2}\langle b,A^{-1}b\rangle$



structures



$A^Ty = f$
$y = Ce$
$e = Ax$

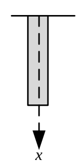
Use calculus to find min of the polynomial in x1,x2

Hence $A^TCe = f$ or $A^TCAx = f$

note: $\epsilon = \frac{e}{L}, \sigma = \frac{y}{A}, c = \frac{y}{e} = \frac{\sigma A}{\epsilon L} = \frac{EA}{L}$

$m = 2(bars)$
$N = 3(joints)$
$r = 4(restrictions)$
$n = 2N - r(d.o.f)$
If n /= m, then not statically determinate

continuous structure



$A = \frac{d}{dx}, A^T = -\frac{d}{dx}$

DISCRETE	CONTINUOUS
nodal unknown x	displacement or potential u
elongation $e = Ax$	strain $e = du/dx$
spring forces $y = Ce$	bar forces $w = ce$
equilibrium $A^Ty = f$	equilibrium $-dw/dx = f$
incidence matrix A	differential operator d/dx
diagonal matrix C	multiplication by $c(x)$
transposed matrix A^T	transposed operator $-d/dx$
matrix equation $A^TCAx = f$	differential equation $-\frac{d}{dx}\left(c\frac{du}{dx}\right) = f$
fixed (or grounded) node	displacement boundary condition $u(0) = 0$
unstretched spring	stress boundary condition $w(1) = 0$

Above table From Strang book

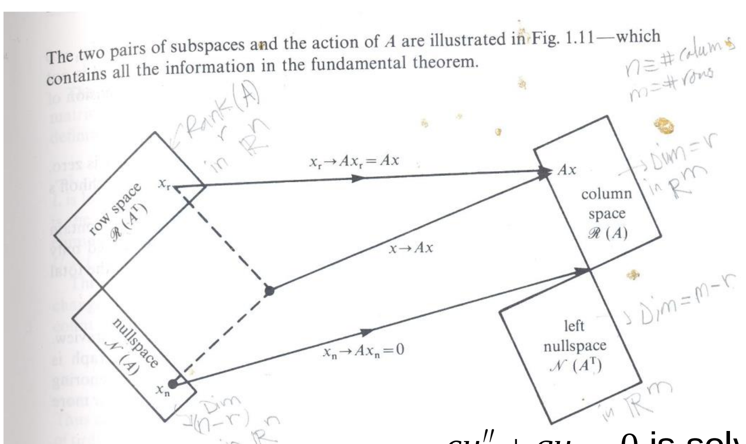


Fig. 1.11. The four subspaces and the action of A.

$-cu'' + qu = 0$ is solved by $u = Ae^{x\sqrt{q/c}} + Ae^{-x\sqrt{q/c}}$

Chapter 5

exams

Local contents

5.1 mid term

5.1.1 questions and key

Math 703 – MIDTERM EXAM – Fall 2014

YOUR NAME: ______________________________

1._(30 points)_ (a) Consider the matrix $M = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. What is the connection between a 3×3 matrix A and the matrices AM and MA? What is the inverse of the matrix M?

AM is obtained from A by adding the first column of A times α to the third column of A.

MA is obtained from A by adding the third line of A time α to the first line of A.

$$M^{-1} = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Is the matrix $M = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ positive definite? Explain your answer.

<u>Not</u> positive definite because $\det M = 0$.

(c) True or false: there exist 2×2 matrices A and B such that $AB = 0$, but $A \neq 0$ and $B \neq 0$. Explain your answer.

True : $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

1

2. *(30 points)* (a) Factor the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ into a product $A = LDL^T$, where L is lower triangular and D is a diagonal matrix. Use this factorization to write the quadratic form $x^T Ax$ as a sum of two squares.

$$A = \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 8/3 \end{pmatrix} \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$$

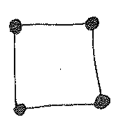
$$x^T A x = 3\left(x_1 + \frac{1}{3}x_2\right)^2 + \frac{8}{3}x_2^2$$

(b) Solve the 2nd order system $\frac{d^2u}{dt^2} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} u = 0$.

Eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with eigenvalues 1 & 3.

$$u(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (a_2 \cos\sqrt{3}\,t + b_2 \sin\sqrt{3}\,t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) What is the dimension of the nullspace of an incidence matrix A of a square-shaped graph? Explain your answer.

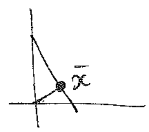


Graph is connected so $\dim(\text{nullspace}(A)) = 1$

3. *(30 points)* (a) In n dimensions, what is the distance from the origin to the hyperplane $x_1 + \sqrt{2}x_2 + ... + \sqrt{n}x_n = 1$? *(Hint: you may use the formula* $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$*.)*

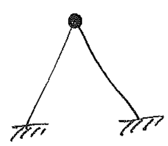
The distance is realized along the orthogonal line to the plane, so $\bar{x} = \alpha(1, \sqrt{2}, \dots, \sqrt{n})$.

Since $\bar{x}$ is on the plane $\Rightarrow 1\cdot\alpha + (\sqrt{2})^2\alpha + (\sqrt{3})^2\alpha + \dots + (\sqrt{n})^2\alpha = 1$



So $\alpha = \frac{2}{n(n+1)}$ Then the distance is $\|\bar{x}\| = \sqrt{\alpha^2 \cdot (1+2+\dots+n)} = \sqrt{\alpha} = \sqrt{\frac{2}{n(n+}}$

(b) Consider an A-shaped truss with fixed supports at the two bottom nodes. Is this truss statically determinate or indeterminate? Explain your answer.



# of degrees of freedom = # of constraints

$\Rightarrow$ Statically determinate

(c) Consider a solid bar with constant force $f = 1$, but such that the elastic constant jumps from $c = 2$ for $0 < x < 1/2$, to $c = 3$ for $1/2 < x < 1$. Calculate the displacement u and the bar forces w.

$$w = \int_x^1 f\,dx = 1 - x.$$

- If $x < \frac{1}{2}$:
  $u(x) = \frac{1}{2}\int_0^x (1-x)\,dx = \boxed{\frac{x}{2} - \frac{1}{4}x^2}$
  $u(\frac{1}{2}) = \frac{3}{16}$

- If $x > \frac{1}{2}$:
  $u(x) = \boxed{u(\frac{1}{2}) + \frac{1}{3}\int_{\frac{1}{2}}^x (1-x)\,dx}$

3

are distinct

4. *(10 points)* (a) True or false: if $a, b, c > 0$, then the matrix $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is non-singular (i.e., $\det A \neq 0$).

True: $\det A = a^3 + b^3 + c^3 - 3abc > 0$

because $\dfrac{a^3 + b^3 + c^3}{3} > \sqrt[3]{a^3 b^3 c^3} = abc$.

(b) How many 3×3 matrices A have the property that $A^3 = I$?

Infinitely many (all rotations by 120° around any line in $\mathbb{R}^3$ have this property)

(c) If a network has N edges, and every pair of nodes is connected by an edge, how many nodes does it have?

$n = \#$ of nodes $\Rightarrow N = \dfrac{n(n-1)}{2}$, so $n^2 - n - 2N = 0$

$\Rightarrow \boxed{n = \dfrac{1 + \sqrt{1 + 8N}}{2}}$

(d) If two solid bars are identical except one has $c = 1$ and the other has $c = 2$, which one is more flexible? Explain your answer.

$u = \dfrac{1}{c} \displaystyle\int_0^x w\, dx \Longrightarrow$ the bar with $c = 1$ has larger displacement, so it is more flexible.

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5.2 final

5.2.1 questions and key

Math 703 – FINAL EXAM – Fall 2014

YOUR NAME: ______________________________

1. *(30 points)* (a) Use the method of Lagrange multipliers to minimize the objective function $Q(y_1, y_2) = y_1^2 + 3y_2^2$ subject to the constraint $y_1 + y_2 = 4$.

(b) What is the dual quadratic $-P(x)$ for problem above, and where is it maximized? Explain your answer.

(c) True or false: there exist 2×2 positive definite matrices A and B such that $AB = 0$, but $BA \neq 0$. Explain your answer.

1

2. *(30 points)* (a) Consider the piecewise constant function $g(x)$ defined on $[-\pi, \pi]$ as follows: $g(x) = 5$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $g(x) = 1$ otherwise, and extended periodically with period 2π on $(-\infty, \infty)$.
Use the integral formulas for Fourier coefficients to calculate the trigonometric series decomposition of $g(x)$ on $(-\infty, \infty)$. Include all the details of your computation.

(b) Find the trigonometric series decomposition for the function $g(x)$ in a different way: express $\frac{dg}{dx}$ in terms of delta functions, and integrate each term in the trigonometric series decomposition for the delta function. Include all the details of your computation.

2

3. *(30 points)* (a) Calculate the Fourier transform of the function $h(x) = e^{-|x|}$.

(b) Verify Plancherel's energy equation for the solution you obtained above for $h(x)$.

(c) Apply the Fourier transform to the equation $a\frac{d^2H}{dx^2} + b\frac{dH}{dx} + cH = \delta$ and calculate the transform $\hat{H}$ of its fundamental solution. Choose some numbers a, b, c for which you can easily take the inverse Fourier transform of $\hat{H}$ to calculate H, and check that it satisfies the equation above.

3

4. *(10 points)* (a) Explain why the derivative of the Dirac delta function has the property $\int_{-\infty}^{\infty} f(x)\delta'(x)dx = -f'(0)$ for any smooth function $f(x)$.

(b) Give an example of a pitchfork bifurcation with bifurcation parameter λ such that the origin is unstable for $\lambda < 0$. Explain your answer.

(c) Is the origin a stable fixed point of the 2nd order system $\begin{cases} x' &= 2x - y \\ y' &= -x + 2y \end{cases}$? Explain your answer.

(d) Is the equation $\cos x + \frac{\sin 2x}{2} + \frac{\cos 3x}{3} + ... = \sin x + \frac{\cos 2x}{2} + \frac{\sin 3x}{3} + ...$ a trigonometric identity or not? Explain your answer.

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