HW1, EMA 521

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Problem 1

part a

The dimensions of A is $\frac{1}{L^3}$ (where L is the length dimension) and similarly for B. This makes the over expression $\frac{T}{T_0}$ dimensinless

part b

Let $q = \frac{T}{T_0}$ then for isolines it is required that $\nabla q.\text{ds}_i = 0$ which implies

$$\left(\frac{\partial q}{\partial x} i + \frac{\partial q}{\partial y} j\right).(\cos(\alpha)\, i + \sin(\alpha)\, j) = \left(\frac{\partial q}{\partial x} \cos(\alpha) + \frac{\partial q}{\partial y} \sin(\alpha)\right) = 0$$

The above simplifies to $\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \tan(\alpha) = 0$ or $\tan(\alpha) = -\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}}$, but $\tan(\alpha) = \frac{\text{dy}_i}{\text{dx}}$, hence $\frac{\text{dy}_i}{\text{dx}} = -\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}}$. Now y_i is found by integration.

First the scalar field is evaluated for the constants given

```
q = (A x^2 y + B y^2 x);
q = q /. {A → 1, B → 1}
```

```
x^2 y + x y^2
```

Now $\frac{\text{dy}_i}{\text{dx}}$ is found using the above definition

```
dyidx = - D[q, x] / D[q, y]
```

```
- (2 x y + y^2) / (x^2 + 2 x y)
```

y_i is found by integration

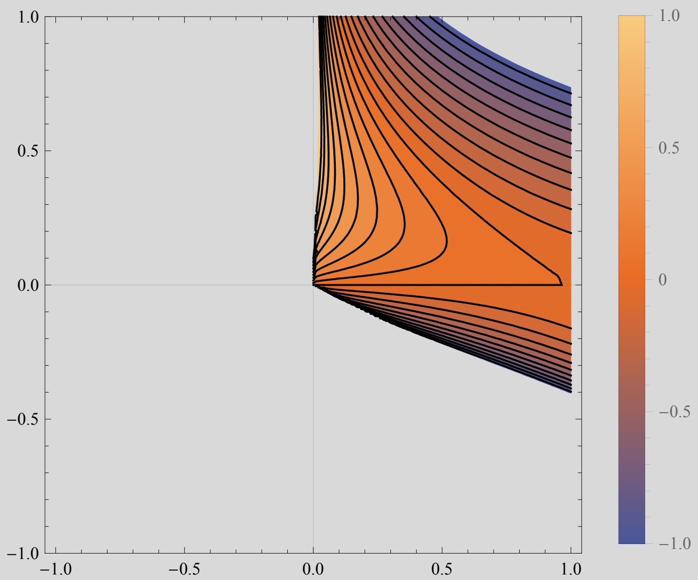
```
yi = Integrate[dyidx, x]
```

```
-y (Log[x]/2 + 3/2 Log[x + 2 y])
```

The lines where $\frac{T}{T_0}$ is constant is now plotted. The range used is $-1 < \frac{T}{T_0} < 1$ and similarly for x, y. The result is below.

20 Lines are used.

```
ContourPlot[yi, {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic,
 Contours → 20, PlotTheme → "Scientific", PlotRange → {-1, 1}, ImageSize → 300]
```



part d

The scalar field $\frac{T}{T_0} = x^2\, y + y^2\, x$ hence

```
(x^2 y + x y^2) /. {x → √2, y → √3} // N
```

```
7.70674
```

The above is dimensonless value that represents $\frac{T}{T_0}$and without knowing T_0 the temprature T can't be found.

part e

To find the slope along an arbitrary direction, the vector ds is found first. This comes from $\text{ds} = v_2 - v_1$ where $v_1 = -2\, i + 3\, j$ and $v_2 = 2\, i - 3\, j$. Hence

$$\begin{aligned} \text{ds} &= (2\, i - 3\, j) - (-2\, i + 3\, j) \\ &= 4\, i - 6\, j \end{aligned}$$

Therefore $\text{ds} = \text{dx}\, i + \text{dy}\, j$ where $\text{dx} = 4$ and $\text{dy} = -6$.

The length of $\text{ds} = \sqrt{\text{dx}^2 + \text{dy}^2}$. Now we can find $\cos(\alpha) = \frac{\text{dx}}{\text{ds}}$ and $\sin(\alpha) = \frac{\text{dy}}{\text{ds}}$. Therefore $e_s = \cos(\alpha)\, i + \sin(\alpha)\, j$.

$$e_s = \cos(\alpha)\, i + \sin(\alpha)\, j$$

The slope can be found using

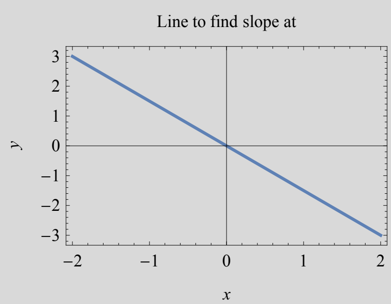
$$\frac{\partial q}{\partial s} = \nabla q.e_s$$
$$= \left(\frac{\partial q}{\partial x}\, i + \frac{\partial q}{\partial y}\, j\right).(\cos(\alpha)\, i + \sin(\alpha)\, j)$$

The following shows the calculations

```
v1 = {-2, 3};
v2 = {2, -3};
ds = v2 - v1
```

```
{4, -6}
```

```
ListLinePlot[{v1, v2}, Frame → True, ImageSize → 200,
 FrameLabel → {{y, None}, {x, "Line to find slope at"}}]
```



Unit vector along the line

```
es = ds / Norm[ds]
```

$$\left\{\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right\}$$

The temprature scalar field is

```
q = (A x^2 y + B y^2 x);
q = q /. {A → 1, B → 1}
```

$$x^2\, y + x\, y^2$$

The slope is

```
slope = {D[q, x], D[q, y]}.es
```

$$-\frac{3\left(x^2 + 2\, x\, y\right)}{\sqrt{13}} + \frac{2\left(2\, x\, y + y^2\right)}{\sqrt{13}}$$

Now this slope is evaluated at $x = 1$ on the line ds. We first need to find the y coordinate at this location. Since we know 2 points on the line, we can find the equation of the line and solve for y

```
Clear[y, x, m, x1, y1];
m = (y2 - y1) / (x2 - x1); (*slope of line*)
eq = y - y1 == m (x - x1)
```

$$y - y1 == \frac{(x - x1)\ (-y1 + y2)}{-x1 + x2}$$

Now replace x1, y1, x2, y2 from the points we are given to find the equation of the line

```
eq = eq /. {x1 → v1[[1]], y1 → v1[[2]], x2 → v2[[1]], y2 → v2[[2]]}
```

$$-3 + y == -\frac{3}{2}\ (2 + x)$$

Now that we have equation of the line, find y where $x = 1$

```
ylocation = y /. First@Solve[eq /. x → 1, y]
```

$$-\frac{3}{2}$$

Hence $x = 1,\ y = -3/2$ and now the slope can be found

```
slope /. {x → 1, y → ylocation}
```

$$\frac{9}{2\sqrt{13}}$$

Hence the slope of the scalar field temprature along this line at the point $x = 1,\ y = -1.5$ is

```
N[%]
```

1.24808

part f

The differential equation is found by solving $\frac{\text{dy}_i}{\text{dx}} = -\left(\frac{\frac{\partial q}{\partial y}}{\frac{\partial q}{\partial x}}\right)$

```
Clear[x, y, A, B];
q = (A x^2 y + B y^2 x);
q = q /. {A → 1, B → 1}
```

$$x^2\ y + x\ y^2$$

```
eq = ym'[x] == - D[q, y] / D[q, x]
```

$$ym'[x] == -\frac{x^2 + 2\,x\,y}{2\,x\,y + y^2}$$

The above represents the differential equation for the largest temprature change.

part g

y_m is replaced by $v(x)\,x$

```
eq = eq /. {ym'[x] → D[v[x] x, x], y → (v[x] x)}
```

$$v[x] + x\,v'[x] == -\frac{x^2 + 2\,x^2\,v[x]}{2\,x^2\,v[x] + x^2\,v[x]^2}$$

```
v[x] /. First@DSolve[eq, v[x], x];
eq = v[x] == %
```

$$v[x] == \frac{e^{C[1]} - x - \sqrt{e^{2\,C[1]} + 2\,e^{C[1]}\,x - 3\,x^2}}{2\,x}$$

Now replace $v(x)$ by y/x again in the above

```
eq = eq /. v[x] → (y / x)
```

$$\frac{y}{x} == \frac{e^{C[1]} - x - \sqrt{e^{2\,C[1]} + 2\,e^{C[1]}\,x - 3\,x^2}}{2\,x}$$

Solve for the constant, so that all non-constant terms are on one side. This way we can obtain $f(x,\ y) = C$ expression

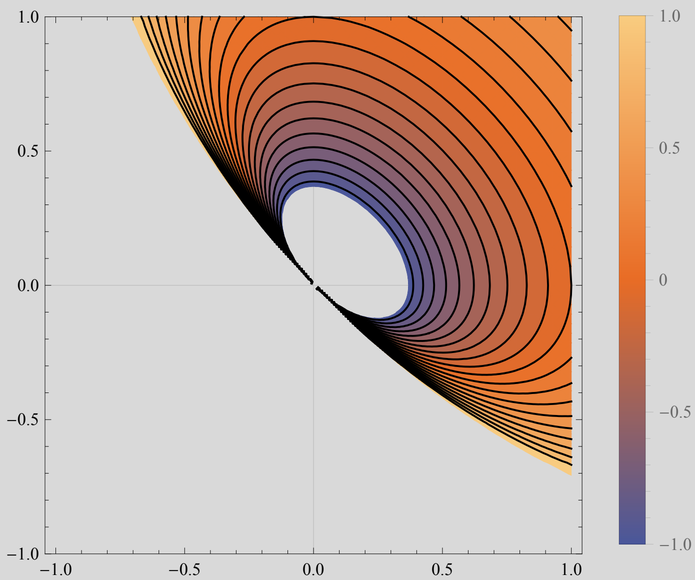
```
sol = C[1] /. First@Solve[eq, C[1]]
```

$$\text{Log}\left[\frac{x^2 + x\,y + y^2}{x + y}\right]$$

Therefore $f(x,\ y_m) = \log\left[\frac{x^2+x\,y+y^2}{x+y}\right] = \text{constant}$

part h

```
ContourPlot[sol, {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic,
 Contours → 20, PlotTheme → "Scientific", PlotRange → {-1, 1}, ImageSize → 300]
```



Problem 2