My ECE 717 Linear systems web page
Fall 2014, University of Wisconsin, Madison

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Fall 2014 Compiled on December 31, 2019 at 12:42am [public]

Contents

Chapter 1

Lecture notes

This book is compilation of my own write up based on note taking during this course given by Professor B. Ross Barmish. This book includes all the material for this course archived in one place.

Most of the time I followed what was being said, but sometimes I added additional text. Therefore, If there are any errors in these notes, all blame goes to me (the student) and not to the instructor of the course.

I added copy of all my HWs in the appendix at the end. The key solutions are also included.

1.1 Summary of lecture topics covered

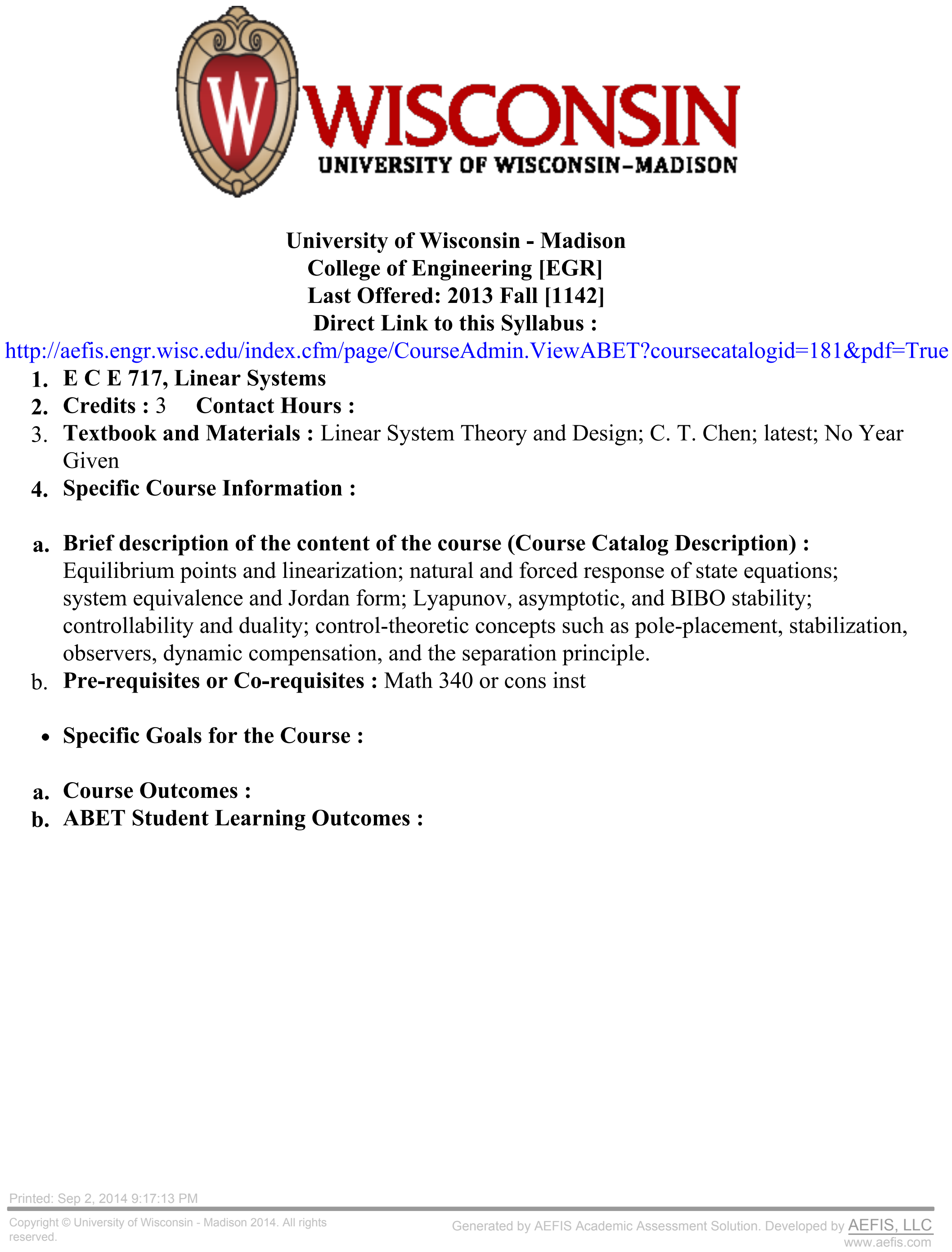
#	date	topics
1	Tuesday Sept. 3	Introduction. Mechanical system to ODE to state space
2	Thursday Sept. 5	discrete time state equation, into to nonlinear state space
3	Tuesday Sept. 9	more non-linear state space, linearization, electric circuit, Laplace transform
4	Thursday Sept. 11	State space realization
5	Tuesday Sept. 16	State space realization, Mason rules and examples using it.
6	Thursday Sept. 18	Realization theorem, MIMO, state space feedback
7	Tuesday Sept. 23	controllability, observability, Mapping using T
8	Thursday Sept. 25	Pole assignment, design using state space feedback
9	Tuesday Sept. 30	Separation theorem, Observer design
10	Thursday Oct. 2	No lecture
11	Tuesday Oct. 7 2:30	Vector spaces preliminaries, norms, piecewise and uniform convergence
12	Tuesday Oct. 7 6:00	first midterm
13	Thursday Oct. 9	Norms, convergence
14	Tuesday Oct. 14	More on convergence
15	Thursday Oct. 16	More on convergences, 4 lemmas
16	Tuesday Oct. 21	Solution of state space using fundamental matrix, its properties
17	Thursday Oct. 23	How to determine e^{At}, LTI vs. LTV
18	Tuesday Oct. 28	Solving state equation
19	Thursday Oct. 30	Start of physical controllability, linear independence of time vectors
20	Tuesday Nov 4	More on controllability LTV
21	Thursday Nov 6	Analytic functions, M test for controllability, LTI

22	Thursday Nov 6, 6pm	second exam
23	Tuesday Nov 11	No class
24	Thursday Nov 13	Controllability of LTV, Cayley Hamilton, differential Controllability
25	Tuesday Nov 18	Observability of LTVm duel system, transition matrix, Canonical decomposition
26	Thursday Nov 20	More on Canonical decomposition, starting stability
27	Tuesday Nov 25	No class
28	Thursday Nov 27	Holiday
29	Tuesday Dec 1	Stability, Hurwitz
30	Thursday Dec 4	More robust stability, q's and intervals. Start of Lyapunov stability
31	Tuesday Dec 9	Review of topics for finals, Routh table examples, future courses
32	Thursday Dec 11	Final exam

1.2 Lecture 0. Tuesday September 3, 2014

Instructor: B. Ross Barmish

1.2.1 Official Syllabus from school website



University of Wisconsin - Madison
College of Engineering [EGR]
Last Offered: 2013 Fall [1142]
Direct Link to this Syllabus :
http://aefis.engr.wisc.edu/index.cfm/page/CourseAdmin.ViewABET?coursecatalogid=181&pdf=True

1. **E C E 717, Linear Systems**
2. **Credits :** 3 **Contact Hours :**
3. **Textbook and Materials :** Linear System Theory and Design; C. T. Chen; latest; No Year Given
4. **Specific Course Information :**

a. **Brief description of the content of the course (Course Catalog Description) :** Equilibrium points and linearization; natural and forced response of state equations; system equivalence and Jordan form; Lyapunov, asymptotic, and BIBO stability; controllability and duality; control-theoretic concepts such as pole-placement, stabilization, observers, dynamic compensation, and the separation principle.
b. **Pre-requisites or Co-requisites :** Math 340 or cons inst

- **Specific Goals for the Course :**

a. **Course Outcomes :**
b. **ABET Student Learning Outcomes :**

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1.2.2 Handout, Syllabus

Barmish 2014

ECE 717 – Handout Description

Audience: This course is intended for graduate students interested in the fundamentals of linear systems. The contents of the course are particularly relevant to areas such as control, communications, signal processing, power systems and circuits. The coverage of material will be suitable for students outside ECE.

Prerequisites: Math 340 or consent of instructor.

Topics: Various models in time and frequency domain, linearization, transformations and realizations, canonical forms and equivalent systems, minimal realizations, pole assignment and stability and robustness, Lyapunov functions, vector space concepts for time-varying systems, fundamental matrix solutions, mathematics of controllability, observability and duality, Jordan forms, spectral theory, functions of matrices, decoupling and compensator design, state estimators and Luenberger observers, separation of estimation and control, linear quadratic regulators.

Lectures: Professor B. R. Barmish

1.2.3 Handout, Organization

Barmish 2014

ECE 717 – Handout Organization

- *Lectures*

B. R. Barmish
Office: 3613 Engineering Hall
E-mail: barmish@engr.wisc.edu
Office Hours: Wednesday 1:30-3:00 PM

- *Recommended Textbook*

W. J. Rugh, Linear System Theory, Prentice Hall, New York.

- *Additional References*

C. T. Chen, Linear System Theory and Design, Oxford University Press, New York.
T. Kailath, Linear Systems, Prentice-Hall, New York.
R. W. Brockett, Finite Dimensional Linear Systems, Wiley, New York.
P. J. Antsaklis and A. N. Michel, Linear Systems, McGraw-Hill, New York.

- *Homework*

Approximately weekly

- *Computer Use*

Matlab and Simulink

- *Grading*

Test 1: 25%
Test 2: 25%
Test 3: 25%
Homework & Special Problem: 25%
The instructor may exercise up to 5% discretion in grading categories above.

- *Scheduling Information*

No lectures on October 2, November 11 and November 25.
Makeup: Reserve 6 PM on October 7, November 6, December 11.
Test 1: Tuesday October 7 (in class OR @ 6 PM)
Test 2: Thursday November 6 (in class OR @ 6 PM)
Test 3: Thursday December 11 (in class OR @ 6 PM)
No Office Hours: October 1 and November 12

- *Discussion of Prerequisites*

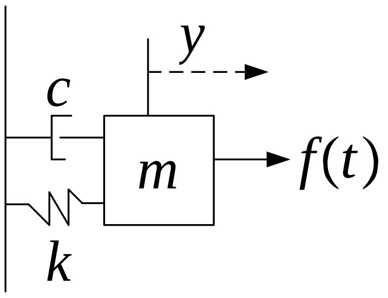
Differential equations, Laplace Transforms and transfer functions, matrix algebra, classical feedback control or state space systems. Use of Matlab and Simulink.

1.3 Lecture 1: Introduction, mechanical system to ODE to state space

Handout organization: Approximately one HW per week. About 9 in total.

Will give back detailed key solution. Requires using Matlab and simulink. See handouts.

What is state space model? Given m inputs to system and r outputs. The input are the controls (since we can manipulate them). This is the vector $u(t)$. The states of the system are $x(t)$ and the output is $y(t)$. A simple example of spring mass damper is now given.



The differential equation is

$$my'' + cy' + ky = f(t)$$

Let $x_1 = y, x_2 = y'$ be the two states, hence $x_1' = x_2$ and $x_2' = \frac{f(t)}{m} - \frac{k}{m}x_1 - \frac{c}{m}x_2$. Therefore the state space representation is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} f(t)$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [0] f(t)$$

More generally,

$$\begin{aligned} x' &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Reader: Find LTI for $y''' + 6y' - 2y = 2u(t)$

Answer:

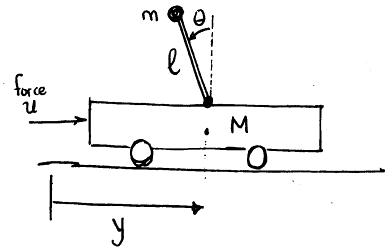
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + [0] u(t)$$

The matrix A has one on the superdiagonal. What if the input had additional term? Such as $\frac{du}{dt}$ or $\frac{d^2u}{dt^2}$ in it? We need a state space realization to handle this.

1.4 Lecture 2. Thursday September 5 2014

1.4.1 Handout, Pendulum on moving cart

Handout Cart- Pendulum



$$(M+m)\frac{d^2y}{dt^2} + m\ell\cos\theta\frac{d^2\theta}{dt^2} + \overset{\text{viscous friction}}{c}\frac{dy}{dt} - m\ell\sin\theta\left(\frac{d\theta}{dt}\right)^2 = u$$

$$m\ell\cos\theta\frac{d^2y}{dt^2} + (\overset{\text{moment of inertia}}{J} + m\ell^2)\frac{d^2\theta}{dt^2} + \overset{\text{viscous friction}}{\gamma}\frac{d\theta}{dt} + mg\ell\sin\theta = 0$$

Reader: With state $x = \begin{bmatrix} y \\ \frac{dy}{dt} \\ \theta \\ d\theta/dt \end{bmatrix}$ find the linearized system $\Sigma = (A,B,C,D)$ for equilibrium $(\bar{x},\bar{u}) = (0,0)$

1.4.2 Lecture: Discrete time state space, introduction to nonlinear state space

Reader: Recall the mass, spring, damper with k, c, m and we generated the second order ODE for it. Let $u = 0$ be the input. Initial conditions are $y(0) = 1, y'(0) = 0$.

Experiment with various values of the parameters k, m, c and observe variety of responses. Also consider $u(t)$ as unit step.

Reader: Consider discrete time state equation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

with zero initial conditions. Solve for $x(N)$ and $y(N)$.

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A(Ax(0) + Bu(0)) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1) \\ x(3) &= A(A^2x(0) + ABu(0) + Bu(1)) + Bu(2) \\ &= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\ &\vdots \end{aligned}$$

Hence

$$x(N)=A^{N}x(0)+\sum_{k=0}^{N-1}A^{N-1-k}Bu(k)$$

In solving continuous time state equations, the solution will contain terms such as $\sin/\cos$ and exponential and t multipliers. These are the only things that come up in linear system theory.

Sometimes an LTI system is stable and sometimes it is not stable.

Stable vs. not stable: if A has all its eigenvalues with real part negative, then it is stable, else not stable. But even if one one eigenvalue had real part which is positive, it might still be stable. This depends if the initial conditions activate the mode with this eigenvalue. For Example

$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, B=0$, then $x_1'=x_1$ and $x_2'=-x_2$. Then $x_1=e^{t}x_1(0)$ and $x_2=e^{-t}x_2(0)$. Now if the initial conditions are $x_1(0)=0$ and $x_2(0)=1$, then $x_1(t)=0$ even though the eigenvalue was positive.

Now we will talk about non-linear systems. Consider

$$\begin{aligned}x_1'&=x_1x_2+ux_3\\ x_2'&=x_2+2x_2x_3\\ x_3'&=x_2+x_3\end{aligned}$$

More generally, $x'=f(x,u)$. There are no A,B,C,D in nonlinear system. there is only $f(x,u)$ and $g(x,u)$. To linearize this, we need to talk about equilibrium. There is stable and there is unstable equilibrium. Always linearize around the stable equilibrium point. To find $(\bar{x},\bar{u})$ solve $x'=0$. i.e. it is when $f(\bar{x},\bar{u})=0$.

How does linearization work? Since we assume $f(x,u)$ is smooth function, we expand in Taylor series around equilibrium $(\bar{x},\bar{u})$

$$\begin{aligned}f_1(\bar{x}+\Delta x)=\ &\overbrace{f\left(x_{eq}\right)}^{\textit{zero}\text{ at equilibrium}}+\left.\frac{\partial f_1}{\partial x_1}\right|_{(\bar{x},\bar{u})}\Delta x_1+\frac{1}{2}\left.\frac{\partial^2 f_1}{\partial x_1^2}\right|_{(\bar{x},\bar{u})}(\Delta x_1)^2+\cdots\\ &+\left.\frac{\partial f_1}{\partial x_2}\right|_{(\bar{x},\bar{u})}\Delta x_2+\frac{1}{2}\left.\frac{\partial^2 f_1}{\partial x_2^2}\right|_{(\bar{x},\bar{u})}(\Delta x_2)^2+\cdots\\ &+\left.\frac{\partial f_1}{\partial x_3}\right|_{(\bar{x},\bar{u})}\Delta x_3+\frac{1}{2}\left.\frac{\partial^2 f_1}{\partial x_3^2}\right|_{(\bar{x},\bar{u})}(\Delta x_3)^2+\cdots\end{aligned}$$

Similarly for each $f_i(x,u)$. For small Δx we obtain, after dropping all higher order terms

$$f(\bar{x}+\Delta x)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix}$$

Two roles for the small Δx: approximates linear behavior, and remain around domain of influence so system returns to x_{eq}.

Reader Argue that incremental dynamics are now $\Delta x'=\overbrace{\left.\frac{\partial f}{\partial x}\right|_{x_{eq}}}^{A}\Delta x$

Reader: Generalize to $f(x,u)$ instead of just $f(x)$: Define $(\bar{x},\bar{u})$ s.t. $f(\bar{x},\bar{u})=0$. So that above become $f(\bar{x}+\Delta x,\bar{u}+\Delta u)$ then

$$B\left(\bar{u}+\Delta u\right)=\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix}$$

Can also introduce the output equation $y=g\left(x,u\right)$ and now obtain linearized C,D as we did above for A,B, but now using $g\left(x,u\right)$ in place of $f\left(x,u\right)$

1.5 Lecture 3. Tuesday September 9 2014 (non-linear state space, linearization, Laplace)

Linear systems are described by A,B,C,D. as in

$$\begin{aligned} x' &= Ax+Bu \\ y &= Cx+Du \end{aligned}$$

For non-linear systems we have

$$\begin{aligned} x' &= f\left(x,u\right) \\ y &= g\left(x,u\right) \end{aligned}$$

We assume f,g are smooth and solution exists (may be using numerical). We talked about equilibrium point $\left(x,u\right)$. This can be stable or unstable equilibrium.

We want to linearize the equations above. Linearization must be done about a stable point. Do not linearize around an unstable equilibrium. The linearized system

$$\Delta x' = \overbrace{\left.\frac{\partial f}{\partial x}\right|_{\left(x,u\right)_{eq}}}^{A}\Delta x+\overbrace{\left.\frac{\partial f}{\partial u}\right|_{\left(x,u\right)_{eq}}}^{B}\Delta u$$

So solution, near x_{eq} is

$$x\left(t\right)=x_{eq}+\Delta x$$

Output equation is

$$\Delta y = \overbrace{\left.\frac{\partial g}{\partial x}\right|_{\left(x,u\right)_{eq}}}^{C}\Delta x+\overbrace{\left.\frac{\partial g}{\partial u}\right|_{\left(x,u\right)_{eq}}}^{D}\Delta u$$

And

$$y\left(t\right)=g\left(x,y\right)+\Delta y$$

Reader: Distinction between domain of attraction and linear approximation.

Region of attraction: Domain of initial states that converges to x_{eq}.

Transfer functions: Motivating example. Given $H\left(s\right)=\frac{Y(s)}{U(s)}=\frac{5}{s^2+6s+7}$. Recall, $\mathscr{L}\frac{d^k f(t)}{dt^k}=s^kF\left(s\right)$ with initial added. Example, $\mathscr{L}\left\{f'\left(t\right)\right\}=sF\left(s\right)-f\left(0\right)$ and $\mathscr{L}\left\{f''\left(t\right)\right\}=s^2F\left(s\right)-sf\left(0\right)-f'\left(0\right)$.

Note: $H\left(s\right)$ is derived assuming all initial conditions are zero.

Reader: Find $H\left(s\right)$ for $y''+6y'+7y=5u\left(t\right)$ and find the state space realization.

Suppose we are given A,B,C,D. There is SISO (single input, single output) and MIMO (multiple input, multiple output).

Consider

$$\begin{aligned} x' &= Ax+Bu \\ y &= Cx+Du \end{aligned}$$

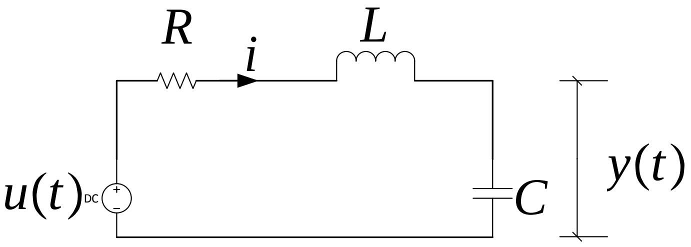
Take Laplace transform results in $X(s) = (sI - A)^{-1} BU(s)$ and the output becomes

$$\begin{aligned} Y(s) &= C(sI - A)^{-1} BU + DU \\ &= \left(C(sI - A)^{-1} B + D\right) U \end{aligned}$$

Hence

$$H(s) = \frac{Y}{U} = C(sI - A)^{-1} B + D$$

Reader:



Find SISO $H(s)$ for the above. Writing loop equations, using $v = Ri$ for voltage across resistor, results in

$$Ri + L\frac{di}{dt} + \frac{1}{c}\int_0^t i\,d\tau = u(t)$$

Taking derivative gives

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{c}i = u'(t)$$

The output equation is $y = \frac{1}{c}\int_0^t i\,d\tau$

Reader: Given $A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Find MIMO $H(s)$. Can solve using syms.

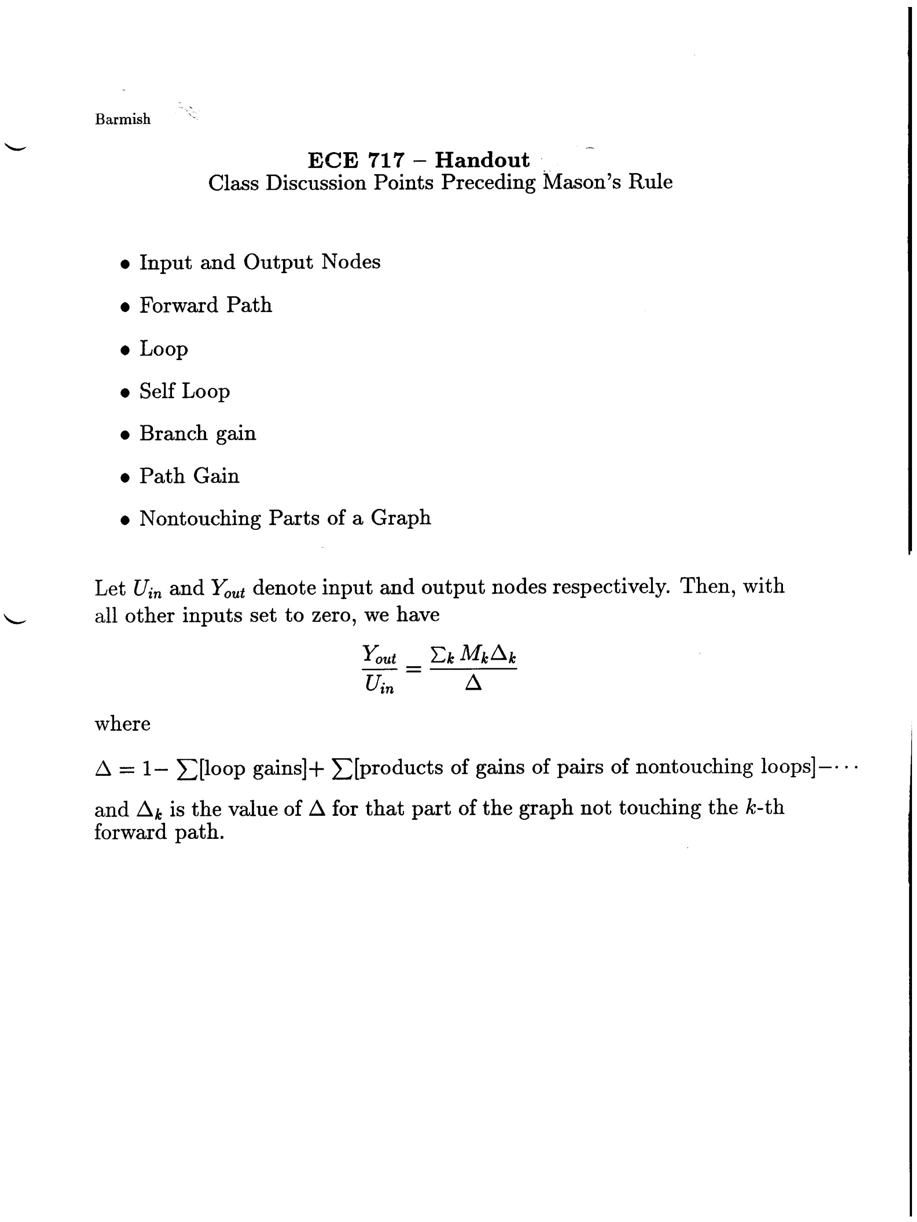
Given transfer function matrix, can we find state space realization? Remark on MIMO: Consider $Y_{r\times 1}(s) = H_{r\times m}(s)U_{m\times 1}(s)$, so the i^{th} output is $\sum_{j=1}^{m} H_{ij}(s)U_j(s)$. So entry $H_{ij}(s)$ in the matrix transfer function is the transfer function between the j^{th} input to the i^{th} output.

$$H_{ij}(s) = \frac{Y_i}{U_j}$$

This suggests experimental method to find $H_{ij}(s)$. Zero out all inputs except for one. Measure the output at the port of interest. This finds $H(s)$ between the input which is not zero and the output port being measured.

1.6 Lecture 4. Thursday September 11 2014

1.6.1 Handout, Mason



Barmish

ECE 717 – Handout
Class Discussion Points Preceding Mason's Rule

- Input and Output Nodes
- Forward Path
- Loop
- Self Loop
- Branch gain
- Path Gain
- Nontouching Parts of a Graph

Let U_{in} and Y_{out} denote input and output nodes respectively. Then, with all other inputs set to zero, we have

$$\frac{Y_{out}}{U_{in}} = \frac{\sum_k M_k \Delta_k}{\Delta}$$

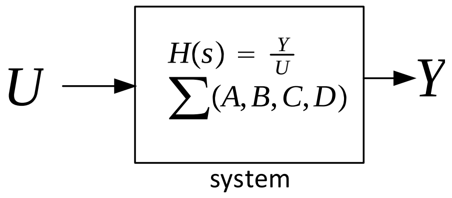
where

$\Delta = 1 - \sum$[loop gains]$+ \sum$[products of gains of pairs of nontouching loops]$- \cdots$

and Δ_k is the value of Δ for that part of the graph not touching the k-th forward path.

1.6.2 Lecture: state space realization

State space realization. Fundamental to state space.



Assume we do not have to look inside the system and we want to model the system? When we model the system, we have idea of what is relevant. The input and the output. From the input/output point of view, it does not matter what the internal of the system are. The constraint is only that the internal states are bounded.

Given system (A,B,C,D), we write $H_*(s)$, which is the realization of this system, as

$$H_*(s) = C(sI - A)^{-1}B + D$$

This is called the transfer function of the system. Let $H(s)$ be some given transfer function matrix, of dimensions $r \times m$, where r is the number of outputs and m is the number of inputs. Each entry in this transfer function matrix is a ratio of two polynomials in s. We say that $H(s)$ is a realization of $\sum(A,B,C,D)$ if $H_*(s) = H(s)$.

In other words, we say $H(s)$ is a realization, if we can find $\sum(A,B,C,D)$ or construct $\sum(A,B,C,D)$ such that $H_*(s) = H(s)$.

When is $H(s)$ realizable? i.e. given some $H(s)$, can we find $\sum(A,B,C,D)$ whose transfer function is this $H(s)$?

Is the set of realizable transfer functions common or rare? If we can realize $H(s)$, then let $\sum$ be a realization. Now do the "Gedankan" experiment. Pick any non singular $n \times n$ matrix T, and form

$$\begin{aligned}
\tilde{A} &= TAT^{-1}\\
\tilde{B} &= TB\\
\tilde{C} &= CT^{-1}\\
\tilde{D} &= D
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{H}_*(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\\
&= CT^{-1}(sI - TAT^{-1})^{-1}TB + D\\
&= CT^{-1}(sITT^{-1} - TAT^{-1})^{-1}TB + D\\
&= CT^{-1}(T(sI - A)T^{-1})^{-1}TB + D\\
&= CT^{-1}T(sI - A)^{-1}T^{-1}TB + D\\
&= C(sI - A)^{-1}B + D
\end{aligned}$$

So we see that $\tilde{C}, \tilde{A}, \tilde{B}, \tilde{D}$ has the same realization as A, B, C, D. So if one realization exist, then there are infinite number of realization that can be found using the T transformation as above.

Reader: How does state x relates to state $\tilde{x}$ under T?.

Let $\tilde{x} = Tx$, then

$$\begin{aligned}
\tilde{x}' &= Tx'\\
&= T\,(Ax + Bu)\\
&= T\left(AT^{-1}\tilde{x} + T^{-1}\tilde{B}u\right)\\
&= TAT^{-1}\tilde{x} + TT^{-1}\tilde{B}u\\
&= \tilde{A}\tilde{x} + \tilde{B}u
\end{aligned}$$

So new system is the same as original before transformation. The big question is: When is $H(s)$ realizable. We start with SISO, after that we will talk about MIMO. Let $H(s) = \frac{N(s)}{D(s)}$ be some given $H(s)$ that we want to realize. Define a proper T.F. as one which has $\deg(N(s)) \leq \deg(D(s))$. Define a strict proper T.F. as one which has $\deg(N(s)) < \deg(D(s))$.

Every proper T.F. is realizable. In this, the word proper is important. Is the improper case important? Example was given for a system where the input is step function, showing the output is Dirac delta $\delta(t)$.

Theorem 1: If $H(s)$ is proper, then it is realizable. Example: $H(s) = \frac{s^3+3s^2+2s+4}{s^3+6s^2-2s-7}$, the associated ODE is $y''' + 6y'' - 2y' - 7y = 4u''' + 3u'' + 2u' + 4u$.

A recipe to realize $H(s)$: If $H(s)$ is proper, make it strict proper by long division and write it as $H_{proper}(s) = \gamma + H_{strict}(s)$. Doing long division gives

$$H = 4 + \frac{-21s^2 + 10s + 32}{s^3 + 6s^2 - 2s - 7}$$

Reader: Verify the following is a realization of the above $H(s)$:

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 7 & 2 & - \end{pmatrix}, B = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}, C = \begin{pmatrix} 32 & 10 & -21 \end{pmatrix}, D = [4]$$

We generalize the above to $H(s) = \gamma + \frac{\beta_2 s^2+\beta_1 s+\beta_0}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}$. Notice we always keep the leading term in the denominator as unity. Realization of this is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}, D = \begin{bmatrix} \gamma \end{bmatrix}$

Reader: Propose a realization for general case. Mason rule is used to generalize to $n \times n$ case.

HW 1 assigned.

1.7 Lecture 5. Tuesday September 16 2014

1.7.1 Lecture: Mason rule and examples

We were talking about realization. Every proper transfer function $H(s)$ is a realization of some $\sum(A,B,C,D)$. We saw the recipe before, but not the justification.

Reader: Consider $H(s) = \frac{s^5+7s^4+19s^3+25s^2+16s+4}{s^5+12s^4+56s^3+12s^2+125s+54}$, is there a realization with $n = 5$ states? Yes. Now, do the realization. What about with $n = 6$ states? How about with $n < 5$ states? If there is zero/pole cancellation. So do factorization first.

If we have a transfer function, then *minimal realization* is one with no cancellations. If the system is uncontrollable or unobservable, there will always be some cancellation. Any system that is controllable and observable is minimal.

Mason rule:

For multi-input, start by zeroing out all input except for one that is of interest. Example given of using Mason rule now. See handout of Mason that was given during the class. Example now given for electrical network. The first step is to find the equations. Once the equations are found, then Mason rule is used to find the transfer function between one input and the output.

Reader: Solve $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ using Mason rule. I did this, see note on my pages.

HW 2 assigned.

1.8 Lecture 6. Thursday September 18 2014 (Realization theorem, MIMO, state feedback)

If the transfer function $H(s)$ is proper, then is it realizable. (SISO for now). Reminder: Need to show this for the general case.

Proof: We must find $\sum(A,B,C,D)$ such that $H_*(s) = H(s)$ where $H_*(s)$ is the transfer function obtained from $\sum(A,B,C,D)$ and $H(s)$ is the transfer function we are given. We propose $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{pmatrix}$. Let $H(s) = \gamma + \frac{\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+\cdots+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_0}$ and propose $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and propose $C = \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{pmatrix}$ and $D = \begin{bmatrix} \gamma \end{bmatrix}$, now we need to show that $H_*(s) = H(s)$ using Mason rule.

Reader: Use Mason rule to show that this realization works. Now what about MIMO? Assume we are given $H(s) = \begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix}$. We can do each on its own, then need to "patched" to show that the matrix of them all work. Example using 2 × 2.

If each $H_{ij}(s)$ is proper, let $\sum_{ij} = A_{ij}, B_{ij}, C_{ij}, D_{ij}$ be realization for $H_{ij}(s)$. Note each A_{ij} can be different size. Propose $A = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & 0 \\ B_{12} & 0 \\ 0 & B_{21} \\ 0 & B_{22} \end{pmatrix}$ and $C = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{pmatrix}$ and $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$. Now we claim (A, B, C, D) is realization of $\begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix}$. We need to calculate

$$H_*(s) = C(sI - A)^{-1} B + D$$

$$= \begin{pmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{pmatrix} \left(sI - \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{pmatrix} \right)^{-1} \begin{pmatrix} B_{11} & 0 \\ B_{12} & 0 \\ 0 & B_{21} \\ 0 & B_{22} \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Reader: The above reduces to

$$H_*(s) = \begin{pmatrix} C_{11}(sI - A_{11})^{-1} B_{11} + D_{11} & C_{12}(sI - A_{12})^{-1} B_{12} + D_{12} \\ C_{21}(sI - A_{21})^{-1} B_{21} + D_{21} & C_{22}(sI - A_{22})^{-1} B_{22} + D_{22} \end{pmatrix}$$

What about other dimensions?

Reader: Propose realization with $H(s)$ that is 3 × 2. i.e. $\begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \\ H_{31}(s) & H_{32}(s) \end{pmatrix}$. Try it. What should A, B, C, D look like? Note: Even though $H_{ij}(s)$ might each be minimal, when we obtain the realization, it might no longer be minimal. Some realization are "nicer" than others for analysis and design.

Motivation example: $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$. When we add feedback, we ask what is the effect of feedback? This system is nice to study feedback. We often add feedback to improve time performance. $u(t) = k_1 x_1 + k_2 x_2 + k_3 x_3 + v$ where v is new input. We can pick k_i. The closed loop becomes

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (k_1 x_1 + k_2 x_2 + k_3 x_3 + v)$$

Reader Determine the new A matrix from the old.

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 - \alpha_0 & k_2 - \alpha_1 & k_3 - \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

Notice: State feedback preservers the companion form of A and b. Find closed form transfer function.

$$H_{closed} = \frac{\beta_0 + \beta_1 s + \beta_2 s^2}{s^3 + (\alpha_2 - k_3) s^2 + (\alpha_1 - k_2) s + (\alpha_0 - k_1)}$$

Note: k_i affects one coefficient each. So poles of closed loop can be arbitrarily assigned anywhere we want.

1.9 Lecture 7. Tuesday September 23 2014

1.9.1 Handout, linearization

ECE 717 – Handout Linearization

We now slightly generalize on the definition of equilibrium given in class. Indeed, we consider the nonlinear state space system

$$\dot{x} = f(x,u); \quad y = g(x,u)$$

with f and g assumed continuously differentiable. Then a pair $(\overline{x}, \overline{u})$ is said to define an *equilibrium* if $f(\overline{x}, \overline{u}) = 0$. Hence, if at some time $t^* \geq 0$, we see state $x(t) = \overline{x}$, then, with constant input $u(t) = \overline{u}$ for $t \geq t^*$, the state will remain at $x(t^*)$ and the output will remain at $\overline{y} = g(\overline{x}, \overline{u})$.

Now, motivated by series expansion, for an equilibrium pair $(\overline{x}, \overline{u})$, we define *linearization matrices* (A,B,C,D) whose entries are given as follows:

$$A \text{ has } (i,j)\text{-th entry } \frac{\partial f_i}{\partial x_j}|_{(\overline{x},\overline{u})}$$

$$B \text{ has } (i,j)\text{-th entry } \frac{\partial f_i}{\partial u_j}|_{(\overline{x},\overline{u})}$$

$$C \text{ has } (i,j)\text{-th entry } \frac{\partial g_i}{\partial x_j}|_{(\overline{x},\overline{u})}$$

$$D \text{ has } (i,j)\text{-th entry } \frac{\partial g_i}{\partial u_j}|_{(\overline{x},\overline{u})}$$

for i and j in their appropriate ranges. Now, the key idea underlying the application of these ideas is as follows: If A is strictly stable (all its eigenvalues have negative real part), it is often possible to use the *linearized (incremental) system*

$$\Delta\dot{x} = A\Delta x + B\Delta u; \quad \Delta y = C\Delta x + D\Delta u$$

to approximate suitably small deviations then for $x(t)$ and $u(t)$ about the equilibrium pair $(\overline{x}, \overline{u})$. That is, the actual state is recovered from the incremental system as $x(t) \approx \overline{x} + \Delta x(t)$ and $y(t) \approx \overline{y} + \Delta y(t)$.

As seen in Homework Satellite, the ideas above can be extended even further to address linearization about a trajectory pair $(\overline{x}^*(t), \overline{u}^*(t))$. That is, in the definitions of (A, B, C, D) above, we use these time-varying quantities in lieu of $(\overline{x}, \overline{u})$, when evaluating the partial derivatives; i.e., the linearization is time-varying.

1.9.2 Handout, transformation

ECE 717 – Handout Transformation

We consider state space systems $\Sigma = (A, B, C, D)$ and $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ each with m inputs, r outputs and n states. We further assume that these systems are input-output equivalent; i.e.,

$$H_\Sigma(s) = H_{\tilde{\Sigma}}(s).$$

Then assuming compatible matrix dimensions above, the question arises whether there there exists a nonsingular transformation matrix T such that

$$\tilde{A} = TAT^{-1};\ \ \tilde{B} = TB;\ \ \tilde{C} = CT^{-1};\ \ \tilde{D} = D.$$

To address this issue, we develop some necessary conditions which such a matrix T must satisfy. Reader: If T exists, verify that it must true that

$$\tilde{B} = TB;\ \ \tilde{A}\tilde{B} = TAB;\ \ \tilde{A}^2\tilde{B} = TA^2B; \cdots \tilde{A}^{n-1}\tilde{B} = TA^{n-1}B.$$

Next, we define the pair of $n \times nm$ block matrices

$$\mathcal{C}_\Sigma = [B\ \ AB\ \ A^2B \cdots A^{n-1}B];$$

$$\mathcal{C}_{\tilde{\Sigma}} = [\tilde{B}\ \ \tilde{A}\tilde{B}\ \ \tilde{A}^2\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}]$$

which are called *controllability matrices*. Based on the preceding discussion, the conditions are expressed compactly as

$$\mathcal{C}_{\tilde{\Sigma}} = T\mathcal{C}_\Sigma.$$

We now consider 2 cases. For single-input systems, the controllability matrices above are square. Hence, if $\mathcal{C}_\Sigma$ is nonsingular, we solve above and obtain

$$T = \mathcal{C}_{\tilde{\Sigma}}\mathcal{C}_\Sigma^{-1}.$$

More generally, for a system with multiple inputs, since the controllability matrices are non-square, we cannot invert them. To solve for T, we consider the case when

$$\operatorname{rank} \mathcal{C}_\Sigma = n. \quad (*)$$

Then, it follows that the $n \times n$ matrix $\mathcal{C}_\Sigma\mathcal{C}_\Sigma^T$ is invertible (Reader) and we obtain solution

$$T = \mathcal{C}_{\tilde{\Sigma}}\mathcal{C}_\Sigma^T\left[\mathcal{C}_\Sigma\mathcal{C}_\Sigma^T\right]^{-1}.$$

To summarize, if Condition $(*)$ is satisfied above, we say that Σ *satisfies the controllability rank condition* and we have a straightforward way to find the transformation matrix T.

1.9.3 Lecture: controllability, observability

Reader: Consider $H(s) = \gamma + \frac{\beta_3 s^3+\beta_2 s^2+\beta s+\beta_0}{s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha s+\alpha_0}$ and show that $A = \begin{pmatrix}0 & 0 & 0 & -\alpha_0\\ 1 & 0 & 0 & -\alpha_1\\ 0 & 1 & 0 & -\alpha_2\\ 0 & 0 & 1 & -\alpha_3\end{pmatrix}, B = \begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, C = \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}, D = \left[\gamma\right]$ is a realization.

We established realization before, when we obtained the controllable form. One way is to use Mason rule. Another way is using syms and find $C(sI-A)^{-1}B+D$.

This realization is called the observable canonical form. Some realizations are better than other for different things. For state estimation, observable form is better, for state control, the controllable form is better.

Reader: Generalized reader above to proper $H(s)$, can use Mason.

Reader: Even more general. Suppose $\sum(A,B,C,D)$ is a realization of SISO transfer function $H(s)$, show that $\sum_*\left(A^T,C^T,B^T,D^T\right)$ is also realization. So $H_*(s) = C_*(sI-A_*)^{-1}B_*+D_*$ where $C_* = B^T, A_* = A^T, B_* = C^T$. So $H_*(s) = B^T\left(SI-A^T\right)^{-1}C^T+D$. Since this is SISO, it is scalar, so its transpose do not change. Take the transpose of the above gives

$$\begin{aligned} H_*(s) &= C\left(\left(sI-A^T\right)^{-1}\right)^T B+D \\ &= C(sI-A)^{-1}B+D \end{aligned}$$

Reader: For MIMO get $H_*(s) = H^T(s)$

Transformation: $\sum \rightarrow T \rightarrow \sum_*$ where $\sum_*$ is equivalent. Design in $\sum_*$, then when design is completed, transform back to $\sum$.

Handout: Given $H(s)_{\sum} = H(s)_{\sum_*}$ when can $\sum$ be transformed to controllable or observable forms using T? Necessary conditions for existence of T is that

$$\mathbb{C}_* = T\mathbb{C}$$

Where $\mathbb{C}$ is the controllability matrix for the original $\sum$ and $\mathbb{C}_*$ is controllability matrix for the $\sum_*$. For SISO, if $\mathbb{C}$ is invertible, then $T = \mathbb{C}_*\mathbb{C}^{-1}$. What is MIMO? system is good if rank $\mathbb{C}$ is n. i.e. controllable system. When this is satisfied, we say (A,B) is controllable pair.

Reader: (A,B) is controllable pair implies $\mathbb{C}\mathbb{C}^T$ is invertible. Proof: Assume $\rho(\mathbb{C}) = n$, show that $\mathbb{C}\mathbb{C}^T$ is invertible. Use proof by contradiction. Assume no inverse,hence this means there is non-zero vector $\vec{x}$ s.t. $\mathbb{C}\mathbb{C}^T\vec{x} = 0$, so $x^T\mathbb{C}\mathbb{C}^Tx = 0$ or $y^Ty = 0$, so $y = 0$ or $\mathbb{C}^Tx = 0$, but $x \neq 0$ so contradiction.

There is also observability rank condition. **Reader**: Mimic the controllability analysis above for pair (A,C). Sketch steps: Develop $\mathbb{Q} = \begin{pmatrix} C\\ CA\\ \vdots\\ CA^{n-1}\end{pmatrix}$. We need to relate $\mathbb{Q}$ to $\mathbb{Q}_*$ using T. Since $C_* = CT^{-1}$, then $C_*A_* = \left(CT^{-1}\right)\left(T^{-1}AT\right) = CA$. So we get condition that $\rho(\mathbb{Q}) = n$ is necessary condition for existence of T.

If the controllability failed, try the observability.

Reader: Show the controllability canonical form always satisfies $\rho(\mathbb{C}) = n$. i.e. if we can put the system in the controllable form, then it is controllable. To show, start with 3×3 matrix and find its $\mathbb{C}$ to show it is 3.

Reader: Can controllable form fail to be observable? (yes).

Reader: Consider $H(s) = \frac{s+1}{s^2+2s+1} = \frac{1}{s+1}$ and study the controllability and observability rank.

More generally, if $H(s)$ has no pole/zero cancellation, then it is minimal and $\mathbb{C}$ and $\mathbb{Q}$ both have rank n (full rank). So if there is no pole/zero cancellation, then it is both controllable and observable. Consider $H(s) = \frac{(s^2+s+1)(s^3+2s^2+4)}{(s^2+s+1)^2(s+1)^2}$, try and see.

1.10 Lecture 8. Thursday September 25 2014 (Pole assignment, state feedback)

Review: We talked about transformation while preserving transfer functions. Can we find transformation to a specific target? Role of $\mathbb{C}$ (controllability matrix) and $\mathbb{Q}$ (observability matrix). **Reader**: Suppose A is selected randomly and so is B, example within normal distribution, find probability that rank $\mathbb{C}$ is n. The probability is 1. So almost all (A, B) are controllable. But if some entries of A, B are hardwired to some specific values due to design, they the chance of getting uncontrollable (A, B) starts to increase. For example, the controllable canonical form of A has hardwired entries in A. Even when we get close to be uncontrollable numerically we will get into more problems.

Now we talk about nice properties of companion forms. Let us use $n = 4$ for illustration. Pole assignment: Select k s.t. $A + Bk$ has pre-specified eigenvalues. Here A has bad eigenvalues, but $A + Bk$ will have good eigenvalues. Note, we are using controllable canonical form, also assume we have access to all states. This is simple

$$A_{closed} = A + Bk = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_0 & k_1 & k_2 & k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_0 - \alpha_0 & k_1 - \alpha_1 & k_2 - \alpha_2 & k_3 - \alpha_3 \end{pmatrix}$$

Hence

$$\det(\lambda I - A_{closed}) = \lambda^4 + (\alpha_3 - k_3)^3 \lambda^3 + (\alpha_2 - k_2)^2 \lambda^2 + (\alpha_1 - k_1)\lambda + (\alpha_0 - k_0) \tag{1}$$

Now pick target eigenvalues, say $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ (if complex, use complex conjugates). The desired

$$\begin{aligned} P(\lambda) &= (\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ &= \lambda^4 + \alpha_3^* \lambda^3 + \alpha_2^* \lambda^2 + \alpha_1^* \lambda + \alpha_0^* \end{aligned} \tag{2}$$

Equate (1) and (2)

$$\lambda^4 + \alpha_3^* \lambda^3 + \alpha_2^* \lambda^2 + \alpha_1^* \lambda + \alpha_0^* = \lambda^4 + (\alpha_3 - k_3)^3 \lambda^3 + (\alpha_2 - k_2)^2 \lambda^2 + (\alpha_1 - k_1)\lambda + (\alpha_0 - k_0)$$

Equate like coefficients, we obtain

$$\begin{aligned} k_0 &= \alpha_0 - \alpha_0^* \\ k_1 &= \alpha_1 - \alpha_1^* \\ k_2 &= \alpha_2 - \alpha_2^* \\ k_3 &= \alpha_3 - \alpha_3^* \end{aligned}$$

Example: See handout. Given system A is 3×3 and B is 3×1, (A, B) is not in controllable form. The open loop eigenvalues are $-0.222, 1.11 \pm j1.8$. Then transform to controllable form, then design in the controllable form, then transform back to original system to set the k in the original system. To set controllable form, $\det(\lambda I - A)$ and this gives the last row of

$A_{companion}$. This is all what we need.

$$A_{comp} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & 2 \end{pmatrix}$$

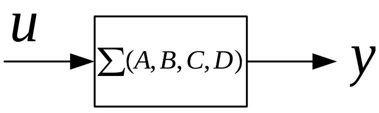
$$B_{comp} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assuming original A is controllable, we find $T = \mathbb{C}_{comp}\mathbb{C}_{original}^{-1}$

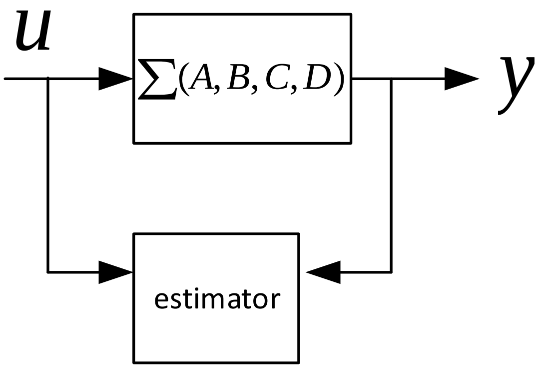
Reader: Design k such that 3 eigenvalues are $\lambda = -2$, so $p^* = (\lambda + 2)^3$. compare to $\det\left(\lambda I - A_{comp}\right)$. This gives k_{comp}, now transform back to original A.

closed loop $\tilde{A} + \tilde{B}\tilde{k}$,but $\tilde{A} = TAT^{-1}, \tilde{B} = TB$, $\tilde{k} = kT$

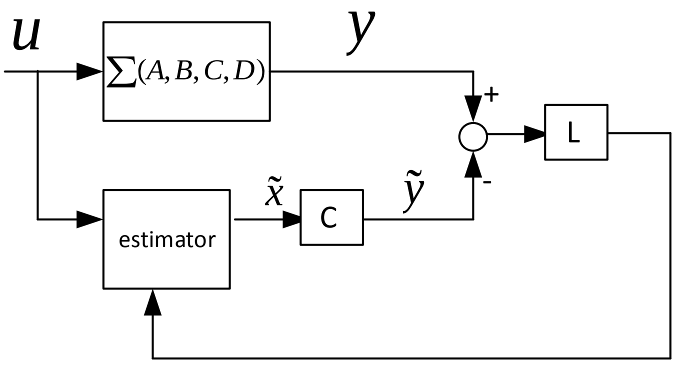
We will now do observer design. State estimation and observer design.



We see y, u and need to estimate state x. i.e. supposed we are given few states and we need from these to estimate all other states.



Use Lvenberger observer to build estimator.



Observer equations

$$\tilde{x}' = A\tilde{x} + Bu + L\left(y - \tilde{y}\right)$$

L is called the observer gain matrix, which we design.

$$\tilde{x}' = (A - LC)\,\tilde{x} + Bu + LCx$$

Reader: Adding Du to y do not add affect design.

How does this observer perform? The error is $e = x - \tilde{x}$, so

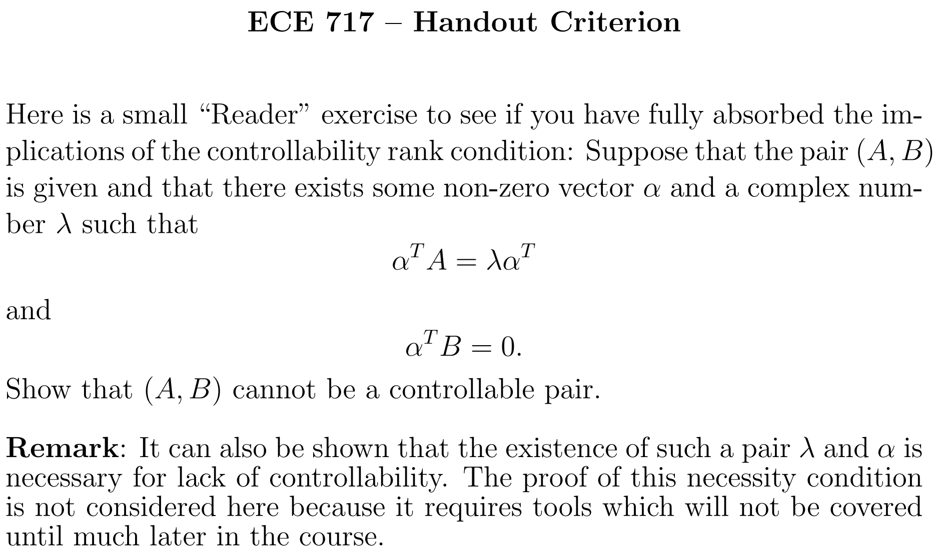
$$\begin{aligned} e' &= (Ax + Bu) - \left(A\tilde{x} + Bu + L\left(y - \tilde{y}\right)\right) \\ &= (A - LC)(x - \tilde{x}) \\ &= (A - LC)\, e \end{aligned}$$

We want $e \rightarrow 0$ fast. So we need to design L to make $(A - LC)$ do so.

If (A, C) is observable pair, then eigenvalues for $(A - LC)$ can put anywhere by choice of L. If (A, C) is observable, then $\left(A^T, C^T\right)$ is controllable pair using duality.

1.11 Lecture 9. Tuesday September 30 2014

1.11.1 Handout, controllability criterion



ECE 717 – Handout Criterion

Here is a small "Reader" exercise to see if you have fully absorbed the implications of the controllability rank condition: Suppose that the pair (A, B) is given and that there exists some non-zero vector α and a complex number λ such that

$$\alpha^T A = \lambda \alpha^T$$

and

$$\alpha^T B = 0.$$

Show that (A, B) cannot be a controllable pair.

Remark: It can also be shown that the existence of such a pair λ and α is necessary for lack of controllability. The proof of this necessity condition is not considered here because it requires tools which will not be covered until much later in the course.

1.11.2 Lecture: Separation theorem, observer design

Oct. 2, no class. Things covered today not on exam. Exam covers up to HW3. Solution to HW3 will be send oct6. Test on Thursday Oct. 7.2014. Closed books, closed notes, open minds. Remember Mason rules and realization.

We studied controllers (state space feedback) and studied observers. What happens when we combine them? Recall

$$\tilde{x}' = A\tilde{x} + Bu + L\,(Cx - C\tilde{x})$$

Where $\tilde{x}$ is the full estimated state from the observer. The idea is to look a the error and use L to reduce the error.

$$\tilde{x}' = (A - LC)\,\tilde{x} + LCx + Bu$$

How good is this observer? Study error $e = x - \tilde{x}$. In perfect world, $e \rightarrow 0$ quickly with no overshoot. Find

$$e' = x' - \tilde{x}'$$

Do some algebra

$$e' = (A - LC)\, e$$

note on initial states: $\tilde{x}(0) \neq x(0)$. We want to pick L which is $n \times r$ dimensions so that $A - LC$ has desired eigenvalues. We want L to be stabilizing. At bare minimum we want $A - LC$ stable. What can we do to generate the eigenvalues of $(A - LC)$?. If (A, C) is observable, then we can make $eig\,(A - LC)$ any value we want. Consider SISO system where (A, C) is observable Then rank of the observability matrix Θ is n. Then since (A, C) is observable, by duality, $\left(A^T, C^T\right)$ is controllable. We will use now the controllability results fro pole assignment that tells us we can select gain K s.t. $\left(A^T + C^T K\right)$ with desired eigenvalues.

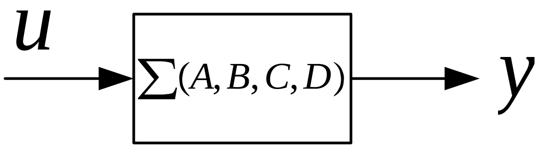
I wrote this below for a HW assignment, I copy it here. For example of pole assignment for the observer $(A - LC)$.

We need to determine L such that the eigenvalues of $(A - LC)$ are $\lambda_1 = -1$ and $\lambda_2 = -2$. Before showing the design steps using the actual data given in the problem, the design steps are given below for the general case.

1.11.3 Design steps for finding L

1. Input is A, C and set of desired eigenvalues λ_i
2. Verify that (A, C) is observable. If so then let $A_o = A^T, B_o = C^T$, hence (A_o, B_o) is controllable.
3. Find controllability matrix $\mathbb{C}(A_o, B_o)$
4. Write down the controllability companion form for A_o, B_o. Let them be called $\tilde{A}_o, \tilde{B}_o$. To do this, we only need to find the characteristic polynomial for A_o and read the coefficients in reverse and change the signs. $\tilde{B}_o$ will always have zeros other than the last row.
5. Find controllability matrix $\tilde{\mathbb{C}}\left(\tilde{A}_o, \tilde{B}_o\right)$
6. Find $T = \tilde{\mathbb{C}}\mathbb{C}^{-1}$
7. Find the closed loop matrix $\left[\tilde{A}_o + \tilde{B}_o\tilde{K}\right]$ where $\tilde{K} = [k_0, k_1, \cdots, k_{n-1}]$ is the gain matrix we looking to determine.
8. Find the characteristic polynomial of $\left[\tilde{A}_o + \tilde{B}_o\tilde{K}\right]$, it will be a function of k_i
9. Set up the desired polynomial $p(\lambda) = (\lambda - \lambda_o)(\lambda - \lambda_o)\cdots(\lambda - \lambda_{n-1})$ where λ_i are the desired eigenvalues given.
10. Compare coefficients of polynomial from step (9) with the polynomial of step (7) and solve for k_i
11. Now we have found $\tilde{K} = [k_0, k_1, \cdots, k_{n-1}]$. Convert it to K using T as follows: $K = \tilde{K}T$
12. Find $L = -K^T$. This completes the design.
13. The observer A matrix now becomes $[A - LC]$

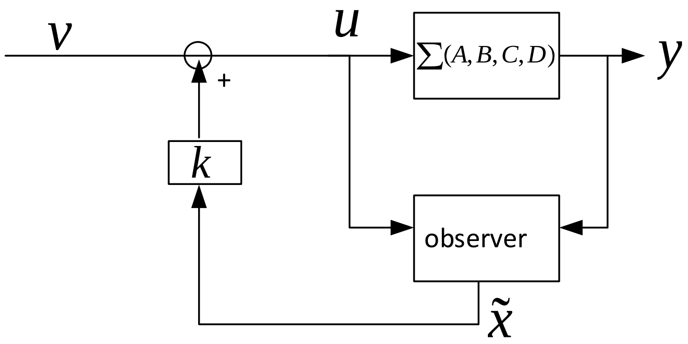
Now we will start talking about combining controller/observer systems. The key result is separation theorem. This system



This can have states that are hidden, which can blow up

can have states we want to control, that can not be observed/measured. We need an observer

$$u = k\tilde{x} + v$$



We now have $2n$ equations, n for the observer and n for the original $\sum$ system. They are cross coupled. Assuming (A,B) and (A,C) are controllable and observable, and without loss of generality, let $v=0$ then we have

$$\begin{aligned} x' &= Ax + B\overbrace{k\tilde{x}}^{u} \\ \tilde{x}' &= A\tilde{x} + Bk\tilde{x} + LC\left(x-\tilde{x}\right) \end{aligned}$$

So now we have

$$\begin{pmatrix} x' \\ \tilde{x}' \end{pmatrix} = \overbrace{\begin{pmatrix} A & Bk \\ LC & A+Bk-LC \end{pmatrix}}^{2n\times 2n \text{ called Augmented } A_{+}} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

We have $A+Bk$ stable and we know that $A-LC$ is stable by design. But we do not know if A_{+} is stable. (the augmented A above). We know eigenvalues of A_{+} is the eigenvalues of $\left(TA_{+}T^{-1}\right)$ if T is not singular. So let us make special

$$T = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}$$

Hence $T^{-1} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$, now calculate

$$\begin{aligned} \left(TA_{+}T^{-1}\right) &= \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} A & Bk \\ LC & A+Bk-LC \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \\ &= \begin{pmatrix} A & Bk \\ A-LC & LC-A \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \\ &= \begin{pmatrix} A+Bk & -Bk \\ 0 & A-LC \end{pmatrix} \end{aligned}$$

Since diagonal matrix, then the eigenvalues on the diagonal. So the eigenvalues are the union of the eigenvalues of $A+Bk$ and eigenvalues of $A-LC$. So A_{+} is stable if $A+Bk$ and $A-LC$ are stable.

HW3 assigned.

1.12 Lecture 10. Thursday October 2 2014 (no lecture)

No lecture.

1.13 Lecture 11. Tuesday October 7, 2014, 2:30 PM (Vector spaces preliminaries, norms)

Test at 6 pm, 75 min, closed notes, closed books. no cheat sheet. Talked little about what can be on the exam. Does there exist T that takes $\sum_1 \rightarrow \sum_2$? depends on controllability and observability. There are canonical forms: controllable (good for feedback) and observable (good for state estimation). We talked about duality and observer design. State feedback control.

Vector spaces preliminaries:

Most common vector space is $\Re^n$. $x=\{x_1,x_2,\cdots,x_n\}$. Need vector spaces where vectors are functions. For this, the function space must have these three operators defined on it: $+,\times,0$. The first $+$ is addition, as in $\vec{x}+\vec{y}$ or $f_1(t)+f_2(t)$. Second is scalar multiplication, as in $5\vec{x}=5\{x_1,x_2,\cdots,x_n\}=\{5x_1,5x_2,\cdots,5x_n\}$ and the third is the zero vector $\{0,0,0,\cdots\}$.

Examples: We can have spaces of vectors that are infinite dimension sequences.

Reader: Consider the continuous functions on $[0,1]$ as vector spaces.

Reader: Generalize to n-dimensional continuous functions on time interval $[0,T]$

Now we will talk about another important function space. This is the space of bounded functions. A function $f(t)$ is said to be bounded on $[0,T]$ if $\exists$ some $\beta > 0$ s.t. $|f(t)| \leq \beta$ for all $t \in [0,T]$. Bounded functions need not be continuous.

Reader: Consider associated vector spaces $B([0,T])$ where B means bounded (I do not understand this)

Reader: Generalize to $B([0,T], \Re^n)$

A critical point in solving $x' = Ax$ is to know where solution sequence converges to actual solution. Convergence in function spaces. Need $\|\|$ defined so we can say $\|x^{(k)} - x^*\| \to 0$ as $k \to \infty$. So need notion of norm. Vector spaces with norm are called normed vector spaces. A norm is mapping from X to the reals, where X is the vector space. Norm must satisfy the following

1. $\|\vec{0}\| = 0$
2. For any $\vec{x} \in X$ and λ real, then $\|\lambda\vec{x}\| = \lambda\|\vec{x}\|$
3. Triangle inequality: for any $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$

On $\Re^n$ we typically use the Euclidean norm defined as $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. We should write this as $\|x\|_2$. Other norms are possible, such as $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$ and also $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ which is used in control theory. **Reader**: Verify these are norms. Need to check the triangle inequality.

Reader: Verify that max is norm. i.e given two vectors, say $a = \{3, 6, 8\}$,$b = \{4, 8, 18\}$ then show that $\max\{a+b\} \leq \max\{a\} + \max\{b\}$

Now consider vector spaces of $m \times n$ matrices. A norm $\|M\| = \sqrt{\lambda_{\max}(M^T M)}$. **Reader**: Verify for $n = 1$ this reduces to L^2 norm above. i.e. this become normal vector $\|\|_2$ norm. Proof: For example, let $M = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $\|M\| = \sqrt{\lambda_{\max}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}$, but $\lambda_{\max}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 5$, hence $\|M\| = \sqrt{5}$. Now $\|M\|_2 = \sqrt{1^2 + 2^2} = \sqrt{5}$. The same.

1.14 Lecture 12. Tuesday October 7 2014, 6:00 PM. First exam

First exam.

1.15 Lecture 13. Thursday October 9 2014. Default norms, convergence, Picard

Will finish material on vector spaces today. Next we will solve state space equation.

Default norms:

There are many norms in $\Re^n$, we will use $\|\|$ to indicate default Euclidean norm vs. $\|\|_\infty$ for maximum norm (the norm of the vector is its maximum component) Same idea will be used for other vector spaces. For matrix norm, we will talk about induced norm. Say $M \in \Re^{m \times n}$ is a matrix. (i.e. matrix of dimensions m, n with elements in the real. Then define

$$\|M\| = \max \|Mx\| \quad \text{for all } \|\vec{x}\| = 1 \tag{1}$$

What this means, is that we apply Mx to all vectors $\vec{x}$ which has norm $\|\vec{x}\|_2 = 1$, and look at the generated vector $v = Mx$, then apply standard vector norm to v (Euclidean) as in $\|v\|_2$. We pick the largest norm $\|v\|_2$ that results. We call this norm as the norm of M. This value is the induced norm of $\|M\|$.

Reader: Does the above definition define a norm? Recall a norm $\|\|$ must satisfy three properties from last lecture. For the zero matrix, easy to show. Scaling is also easy. Now for the triangle inequality, which says $\|M_1 + M_2\| \leq \|M_1\| + \|M_2\|$. Show this.

Reader: Show that (1) is equivalent to $\|M\| = \max \frac{\|Mx\|}{\|x\|}$ for all $\left\|\vec{x}\right\| \neq 0$

Reader: Show that (1) is equivalent to $\|M\| = \sqrt{\lambda_{\max}\left(M^T M\right)}$ where $\lambda_{\max}\left(M^T M\right)$ means the the largest eigenvalue of $M^T M$. Sketch of proof was given, but needs more time to understand it.

Reader: Find the matrix norm induced by $\|x\|_\infty$

We will use the space of bounded functions and the subset of this space we will use most are the bounded continuous functions over some interval. note: Any continuous function is bounded function. We will call it $B\left([t_0, t_1], \Re^n\right)$.

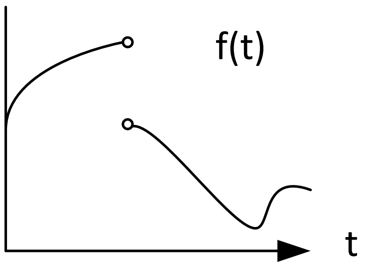
A function $f(t)$ is bounded if $\left\|f(t)\right\| \leq \beta$ for some $\beta < \infty$. The continuous functions $C\left([t_0, t], \Re^n\right) \in \not{B}$, We need a norm for $C\left([t_0, t], \Re^n\right)$. We will use $\left\|f\right\|$ as the norm, which is the largest value of the function over $[t_0, t_1]$. Make sure not to confuse $\left\|f\right\|$ and $\left\|f(t)\right\|$. The first one is called the induced norm. i.e.

$$\left\|f\right\| = \max_{t_0 \leq t \leq t_1} \left\|f(t)\right\| \tag{2}$$

While $\left\|f(t)\right\|$ is just normal Euclidean norm, and is defined only for specific t. i.e. we fix $t = t_0$ then calculate $\left\|f(t_0)\right\|$, but $\left\|f\right\|$ has no t in it. So this is the norm over the whole range and defined as in (2) above.

Reader: Show that (2) defines a norm. (I have a side note here about for non-negative functions, check what this is for??)

Now we will talk about norms on B where $f(t)$ is not necessarily continuous function.



When $f(t)$ is not continuous, we will use $\sup$ instead of $\max$ in the definition, i.e. we write (2) as

$$\left\|f\right\| = \sup_{t_0 \leq t \leq t_1} \left\|f(t)\right\| \tag{2A}$$

We need one more thing before going to solve the state equation, which is

Reader: Show that $\left\|\int_0^t f(t)\,dt\right\| \leq \int_0^t \left\|f(t)\right\| dt$ question: ask about what norm this is $\|\|$ here. Use Riemann sum to proof this?

Reader: Similarly, using matrix norms, show that $\left\|\int_0^t A(t)\,dt\right\| \leq \int_0^t \|A(t)\|\,dt$ where A is now matrix in $\Re^{m\times n}$

Convergence:

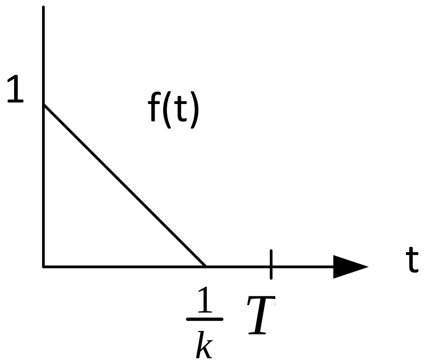
A sequence $\{x_k\}_{k=1}^{\theta}$ in a normal vector space X is said to converge to $x^* \in X$ if

$$\lim_{k\to\infty} \left\|x^* - x^k\right\| \to 0$$

Example: $\lim_{k\to\infty} \begin{pmatrix} 1+\frac{1}{k} \\ -1^k e^{-k} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or we can just write $\begin{pmatrix} 1+\frac{1}{k} \\ -1^k e^{-k} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

But vector $\begin{pmatrix} -1^k \\ \frac{1}{k} \end{pmatrix}$ does not converge.

Reader:what about convergence of this function, defined over $0 \leq t \leq T$



For pointwise convergence, at $t = 0, f(0) = 1$, and for $0 < t \leq \frac{1}{k} < T$, $f(t) = 1 - kt$ hence the limit goes to 1 also. So this converges pointwise. Since $\|f_k - f\| = \max_{0 \leq t \leq T} \|f(t)\| = 1$, it does not converge uniformly. For uniform convergence, we need to have $\|f_k - f\| \to 0$ as $k \to \infty$. In space of bounded functions, we always mean uniform convergence.

Summary: $f_k \to f$ in B mean uniform convergence. But $f_k(t) \to f^*(t)$ mean pointwise.

Reader: Show that uniform convergence implies pointwise convergence. Proof:

$$\begin{aligned} \|f_k - f\|_I &= \max_{t \in I} \|f_k(t) - f(t)\| \\ &\leq \|f_k(t) - f(t)\| \end{aligned}$$

But if f_k converges uniformly to f then $\|f_k - f\|_I \to 0$ as $k \to \infty$, hence

$$\lim_{k \to \infty} \|f_k(t) - f(t)\| = 0$$
$$\lim_{k \to \infty} f_k(t) = f(t)$$

Therefore, $f_k(t)$ converges to $f(t)$ pointwise. QED.

HW4 assigned.

1.16 Lecture 14. Tuesday October 14 2014 (More on convergence, the 4 lemmas)

Notes on first exam:

If one can put state space in controllable canonical form, then this implies it is controllable.

Lack of cancellation of poles/zero in a transfer function implies minimal system. Hence it is observable and controllable.

If $H(s)$ is proper, then it is realizable (can obtain A, B, C, D). However, if it is not proper $H(s)$ then we can't decide. It might still be possible to obtain A, B, C, D. Think of an example.

But if we are given (A, B, C, D) then $H(s) = C(sI - A)^{-1}B + D$ must come out to be proper by construction.

Next goal is to solve the state space equation. All our solutions live in the space of bounded functions $B([t_0, t_1], \Re^n)$

In these notes, I will use f^* for the uniform convergence limit and use $f^*(t)$ for pointwise limit.

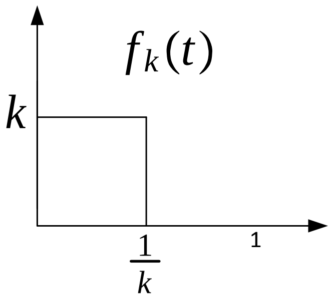
Uniform convergence: Lemma 1

Suppose $f_k \rightarrow f^*$ convergence uniformly in B, i.e. $\|f_k - f^*\| \rightarrow 0$ (notice, when we write f^* and not $f^*(t)$, then this means uniform convergence, then

$$\lim_{k\rightarrow\infty} \int_{t_0}^{t} f_k(\tau)\,d\tau = \int_{t_0}^{t} f^*(\tau)\,d\tau$$

In words, if a sequence of functions converges to some limit, then the limit of the integral is the integral of the limit. This only applies for uniform converges. This does not hold (most of the time) for pointwise convergence.

Reader: proof this. Here is a case where the above fails for pointwise convergence. Note: A sequence of functions $f_k(t)$, converges pointwise, if when we fix t to some specific t_0, then the sequence $f_k(t_0)$ converges to some limit. I.e. we have to fix t and only after that, generate the sequence and see if $|f_k(t_0)|$ converges to some $f^*(t_0)$ as $k \rightarrow \infty$. This can be written as $|f_k(t_0) - f^*(t_0)| < \epsilon$ wherever $n > N$ where N is some integer. Given the function shown below



where in the above, $f(0) = 0$ and $f(t) = k$ for $0 < t \leq \frac{1}{k}$. To find pointwise limit $f^*(t)$: At $t = 0, f_k(0) = 0$, and at $0 < t \leq \frac{1}{k}$, $f_k(t) = k$ and for $t > \frac{1}{k}$ it is already zero. Hence as $k \rightarrow \infty$ we see that $f(t) \rightarrow 0$ everywhere. So $f^* = 0$ is the pointwise limit.

Back to the proof of the lemma above for uniform convergence.

proof of lemma:

$$\begin{aligned} error(k) &= \left\| \int_{t_0}^{t} f_k(\tau)\,d\tau - \int_{t_0}^{t} f^*(\tau)\,d\tau \right\| \\ &= \left\| \int_{t_0}^{t} f_k(\tau) - f^*(\tau)\,d\tau \right\| \\ &\leq \int_{t_0}^{t} \|f_k(\tau) - f^*(\tau)\|\,d\tau \end{aligned}$$

Since uniform convergence. But the above is

$$\int_{t_0}^{t} \|f_k(\tau) - f^*(\tau)\|\,d\tau \leq \int_{t_0}^{t} \max \|f_k(t) - f^*(t)\|\,d\tau$$

But now $\max \|f_k(t) - f^*(t)\|$ is fixed, so we can take it out of the integral

$$\begin{aligned} \int_{t_0}^{t} \|f_k(\tau) - f^*(\tau)\|\,d\tau &\leq \max \|f_k - f^*\| \int_{t_0}^{t} d\tau \\ \int_{t_0}^{t} \|f_k(\tau) - f^*(\tau)\|\,d\tau &\leq \max \|f_k - f^*\| (t - t_0) \end{aligned}$$

But since we assumed $f_k(t)$ convergence uniformly then $\lim_{k\rightarrow\infty} \|f_k - f^*\| = \max \|f_k - f^*\| = 0$, therefore RHS above is zero. Hence $\int_{t_0}^{t} \|f_k(\tau) - f^*(\tau)\|\,d\tau = 0$ or $\left\| \int_{t_0}^{t} f_k(\tau) - f^*(\tau)\,d\tau \right\| = 0$

or $\int_{t_0}^{t} f_k(\tau) - f^*(\tau)\, d\tau = 0$ or $\lim_{k\to\infty} \int_{t_0}^{t} f_k(\tau)\, d\tau - \int_{t_0}^{t} f^*(\tau)\, d\tau = 0$ or

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(\tau)\, d\tau = \int_{t_0}^{t} f^*(\tau)\, d\tau$$

Which is the lemma we wanted to proof.

Series in bounded spaces:

We will look at Series in $B([t_0, t_1], \Re^n)$

We now look at series of functions $f_k(t)$ in B. And use Weierstrass M-test to see if the series converges or not. Looking at partial sum

$$S(k) = \sum_{i=1}^{k} f_i(t)$$

where $f_i(t) \in B$. Does this converge? If we can find constants M_i s.t. $\|f_i\| \leq M_i$, (M_i can for example be the $\sup$ norm of $f_i(t)$), and if then we can determine that $\sum_{i=1}^{\infty} M_i < \infty$ then we say that $S(k) \underset{\text{uniform}}{\longrightarrow} S^* \in B$

(need an example, see references)

Solving state space:

Now we start talking about solving the state space equation $x' = A(t)x(t)$. We start with the zero input case. Only initial conditions will drive this system. We look at using Picard iterations to solve it. By integration both sides of the above, we obtain

$$x(t) - x(0) = \int_{0}^{t} A(\tau)x(\tau)\, d\tau$$

Define $x^0 = x(0)$ and define this iteration scheme for $k = 0, 1, 2, \cdots$

$$x^{k+1} = x^0 + \int_{0}^{t} A(\tau)x^k(\tau)\, d\tau$$

Reader:

For $A(t) = \begin{pmatrix} 1 & t & -t^2 \\ 0 & t+1 & t^3 \\ 0 & 1 & -2 \end{pmatrix}$ and $x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ find $x^3(t)$

Done in class. Direct integration.

Reader:

For scalar $x' = ax$ show that Picard iteration gives $x = e^{at}x(0)$.

Proof:

$$\begin{aligned}
x^1 &= x^0 + \int_0^t a x^0 d\tau = x^0 + a x^0 t\\
x^2 &= x^0 + \int_0^t a x^1 d\tau\\
&= x^0 + \int_0^t a\left(x^0 + a x^0 t\right) d\tau\\
&= x^0 + a x^0 t + a^2 x^0 \frac{t^2}{2}\\
x^3 &= x^0 + \int_0^t a x^2 d\tau\\
&= x^0 + \int_0^t a\left(x^0 + a x^0 t + a^2 x^0 \frac{t^2}{2}\right) d\tau\\
&= x^0 + a x^0 t + a^2 x^0 \frac{t^2}{2} + a^3 x^0 \frac{t^3}{2\times 3}
\end{aligned}$$

and so on. Hence the result is

$$\begin{aligned}
x^\infty = x(t) &= x^0\left(1 + at + a^2\frac{t^2}{2} + a^3\frac{t^3}{3!} + \cdots\right)\\
&= x(0)\sum_{k=0}^{\infty}\frac{(at)^k}{k!}\\
&= x(0)\, e^{at}
\end{aligned}$$

Reader

Show the solution for scalar time varying $x' = a(t)x$ using Picard. This should become $x(t) = x(0)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^t a(td\tau)\right)^k$

(need to work it out).

Convergence of Picard iterate $x^k(t)$:

Helpful function is $\Pi(t) = \int_0^t \|A(\tau)\| d\tau$. It has the following properties

1. $\Pi(0) = 0$
2. $\Pi(t)$ is not decreasing
3. $\frac{d}{dt}\Pi(t) = \|A(t)\|$

The above is reader, need to show.

Lemma 1:

The Picard iterate $x^k(t) \in B([t_0, t_1], \mathbb{R}^n)$ for all k. We want to show that for each k, $x^{(k)}(t)$ is bounded. This is done using induction.

proof:

For $k = 0$, it is clear that $x^0 = x(0)$ is bounded, since initial conditions. Now assume that for some k it is true that $x^k(t)$ is bounded, then we need to show that for $k + 1$ it is also bounded. Form

$$
\begin{aligned}
x^{k+1}(t) &= x^0 + \int_0^t A(\tau)\, x^k(\tau)\, d\tau \\
\left\|x^{k+1}(t)\right\| &= \left\| x^0 + \int_0^t A(\tau)\, x^k(\tau)\, d\tau \right\| \\
&\leq \left\|x^0\right\| + \left\| \int_0^t A(\tau)\, x^k(\tau)\, d\tau \right\| \\
&\leq \left\|x^0\right\| + \int_0^t \left\|A(\tau)\, x^k(\tau)\right\| d\tau \\
&\leq \left\|x^0\right\| + \int_0^t \|A(\tau)\| \left\|x^k(\tau)\right\| d\tau \\
&\leq \left\|x^0\right\| + \int_0^t \|A(\tau)\| \left(\sup \left\|x^k(\tau)\right\|\right) d\tau
\end{aligned}
$$

Since $\sup \left\|x^k(t)\right\|$ is fixed, then we can remove it out of the integral

$$
\left\|x^{k+1}(t)\right\| \leq \left\|x^0\right\| + \sup \left\|x^k(t)\right\| \int_0^t \|A(\tau)\|\, d\tau
$$

But $\int_0^t \|A(\tau)\|\, d\tau = \pi(t)$ which is non-decreasing. So it was take its maximum value $\pi(t_1)$ we can limit the above from below and write

$$
\left\|x^{k+1}(t)\right\| \leq \left\|x^0\right\| + \sup \left\|x^k(t)\right\| \Pi(t_1)
$$

Therefore we just showed that $x^{k+1}(t)$ is bounded. Since $\sup \left\|x^k(t)\right\|$ is bounded by assumption.

1.17 Lecture 15. Thursday October 16 2014 (More on converges, Lemmas)

We have $x'(t) = A(t)\, x(t), x(0) = x^0$ with Picard iterate $x(0) \equiv x^0(t)$. Define $x^{k+1}(t) = x^0 + \int_0^t A(\tau)\, x^k(\tau)\, d\tau$ for $k = 0, 1, 2, \cdots$

This sequence lives in the bounded space $B([t_0, t_1]\, \mathbb{R}^n)$. Lemma 1 from last lecture shows that $x^k(t)$ is bounded. i.e. $x^k(t) \in B$. Now we go to lemma 2

lemma 2:

Convergence: The Picard iteration satisfy

$$
\overbrace{\left\|x^{k+1}(t) - x^k(t)\right\|} \leq \frac{\left\|x^0\right\| \Pi^{k+1}(t)}{(k+1)!}
$$

for $k = 0, 1, 2, \cdots$. Notice the LHS is norm in $\mathbb{R}^n$ (pointwise convergence) since we used $x(t)$ inside.

proof: By induction. For $k = 0$,

$$\begin{aligned}
\left\|x^1(t) - x^0(t)\right\| &= \left\|x^0 + \int_0^t A(\tau)\, x^0(\tau)\, d\tau - x^0\right\| \\
&= \left\|\int_0^t A(\tau)\, x^0(\tau)\, d\tau\right\| \\
&\leq \int_0^t \left\|A(\tau)\, x^0(\tau)\right\| d\tau \\
&\leq \int_0^t \|A(\tau)\| \left\|x^0(\tau)\right\| d\tau \\
&= \left\|x^0(\tau)\right\| \int_0^t \|A(\tau)\|\, d\tau
\end{aligned}$$

But $\Pi(t) = \int_0^t \|A(\tau)\|\, d\tau$ and $\Pi(t)$ is non decreasing. It maximum is $\Pi(t_1)$ Hence

$$\left\|x^1(t) - x^0(t)\right\| \leq \left\|x^0(\tau)\right\| \Pi(t_1)$$

So true for $k = 0$. Now assume lemma is true for k and we need to show it is true for $k+1$. We form

$$\begin{aligned}
\left\|x^{k+1}(t) - x^k(t)\right\| &= \left\|\left(x^0 + \int_0^t A(\tau)\, x^k(\tau)\, d\tau\right) - \left(x^0 + \int_0^t A(\tau)\, x^{k-1}(\tau)\, d\tau\right)\right\| \\
&= \left\|\int_0^t A(\tau)\, x^k(\tau)\, d\tau - \int_0^t A(\tau)\, x^{k-1}(\tau)\, d\tau\right\| \\
&= \left\|\int_0^t A(\tau)\left(x^k(\tau) - x^{k-1}(\tau)\right) d\tau\right\| \\
&\leq \int_0^t \left\|A(\tau)\left(x^k(\tau) - x^{k-1}(\tau)\right)\right\| d\tau \\
&\leq \int_0^t \|A(\tau)\| \left\|\left(x^k(\tau) - x^{k-1}(\tau)\right)\right\| d\tau
\end{aligned}$$

Since we assumed it is true for k, i.e. $\left\|\left(x^k(\tau) - x^{k-1}(\tau)\right)\right\| \leq \frac{\|x^0\|\Pi^k(t)}{k!}$ is true by assumption. Then the above becomes

$$\begin{aligned}
\left\|x^{k+1}(t) - x^k(t)\right\| &\leq \int_0^t \|A(\tau)\| \frac{\left\|x^0\right\| \Pi^k(\tau)}{k!} d\tau \\
&= \frac{\left\|x^0\right\|}{k!} \int_0^t \|A(\tau)\|\, \Pi^k(\tau)\, d\tau
\end{aligned}$$

But $\frac{d}{d\tau}\Pi(\tau) = \|A(\tau)\|$ then $d\Pi = \|A(\tau)\|\, d\tau$ and the above can be written as

$$\begin{aligned}
\left\|x^{k+1}(t) - x^k(t)\right\| &\leq \frac{\left\|x^0\right\|}{k!} \int_0^t \Pi^k(\tau)\, d\Pi \\
&= \frac{\left\|x^0\right\|}{k!} \left(\frac{\Pi^{k+1}(\tau)}{k+1}\right)_0^t \\
&= \frac{\left\|x^0\right\|}{k!} \left(\frac{\Pi^{k+1}(t)}{k+1} - \frac{\Pi^{k+1}(0)}{k+1}\right)
\end{aligned}$$

But $\Pi(0) = 0$ from properties of Π, then the above reduces to

$$\left\|x^{k+1}(t) - x^k(t)\right\| \leq \left\|x^0\right\| \frac{\Pi^{k+1}(t)}{(k+1)!}$$

and this proofs the lemma.

Lemma 3:

$x^k(t)$ converges to the some limit. We need to show that $x^k(t)$ converges uniformly to some $x^*(t) \in B([t_0, t_1], \mathbb{R}^n)$. When we say a function converges in bounded space B, we always mean uniform convergence.

proof:

We need to generate a telescoping sequence, as in $x^{(4)} = \left(x^{(4)} - x^{(3)}\right) + \left(x^{(3)} - x^{(2)}\right) + \left(x^{(2)} - x^{(1)}\right) + \left(x^{(1)} - x^{(0)}\right) + x^{(0)}$, which mean

$$x^n(t) = x^0(t) + \sum_{k=0}^{n-1} \left(x^{k+1}(t) - x^k(t)\right)$$

We now need to use the M-test to bound $\left\|x^{k+1}(t) - x^k(t)\right\|$. From lemma 2

$$\left\|x^{k+1} - x^k\right\|_I \leq \sup_{0 \leq t \leq t_1} \left\|x^0\right\| \frac{\Pi^{k+1}(t)}{(k+1)!} = \left\|x^0\right\| \frac{\Pi^{k+1}(t_1)}{(k+1)!}$$

Since Π is non-decreasing, then we can bound the above from below by some $M_k = \left\|x^0\right\| \frac{\Pi^{k+1}(t_1)}{(k+1)!}$, so now can use M-test

$$\begin{aligned}\sum_{k=0}^{\infty} M_k &= \left\|x^0\right\| \sum_{k=0}^{\infty} \frac{\Pi^{k+1}(t_1)}{(k+1)!} \\ &= \left\|x^0\right\| \left(e^{\Pi(t_1)} - 1\right)\end{aligned}$$

Since $\sum_{k=0}^{\infty} M_k$ is finite, then by the M-test we conclude that $\sum_{k=0}^{n-1} \left(x^{k+1} - x^k\right)$ will converge to some limiting value, which implies $x^{(n)}$ in the limit will also converge (uniformly) to some limit x^*

Lemma 4:

The x^* obtained from lemma 3 solves the state equation $x' = A(t)x(t)$

proof: We know that $x^{k+1} = x^0 + \int_0^t A(\tau) x^k(\tau) d\tau$ and we also know that $x^{k+1}(t)$ will converge uniformly to some limit $x^*(t)$ by lemma 3. Taking the limit of both sides of the Picard iteration formula above gives

$$\begin{aligned}\lim_{k \to \infty} x^{k+1}(t) &= \lim_{k \to \infty} \left(x^0(t) + \int_0^t A(\tau) x^k(\tau) d\tau\right) \\ x^*(t) &= x^0(t) + \lim_{k \to \infty} \int_0^t A(\tau) x^k(\tau) d\tau\end{aligned}$$

To take the limit inside the integral, we need to first show that $A(\tau)x^k(\tau)$ converges uniformly to $A(\tau)x^*(\tau)$

$$\begin{aligned}\left\|Ax^k - Ax^*\right\|_I &\leq \sup \|A(t)\| \overbrace{\left\|x^k(t) - x^*(t)\right\|}^{\text{converges uniformly to say z}} \\ &\leq \|A(t)\| z\end{aligned}$$

But $\|A(t)\|$ is bounded, hence $A(\tau)x^k(\tau)$ converges uniformly and now we can take the limit inside the integral.

$$x^*(t) = x^0 + \int_0^t \lim_{k \to \infty} A(\tau) x^k(\tau) d\tau$$

But $\lim_{k \to \infty} x^k(t) = x^*(t)$ by lemma 3, hence the above becomes

$$x^*(t) = x^0 + \int_0^t A(\tau) x^*(\tau) d\tau$$

This proofs the lemma.

We now need to establish uniqueness. Which means we need to show that x^* is the only solution to $x' = A(t)x(t)$

We will use what is called Granwall's inequality.

If $u(t)$ and $\theta(t)$ are non-negative continuous functions on $[0,t]$ satisfying $\theta(t) \leq \int_0^t u(\tau)\theta(\tau)\,d\tau$ then $\theta(t) = 0$ everywhere. Now we assume there are two solutions to state equation. $x_1' = Ax_1$ and $x_2' = Ax_2$. Therefore

$$x_1(t) - x_1(0) = \int_0^t A(\tau)x_1(\tau)\,d\tau$$

$$x_2(t) - x_2(0) = \int_0^t A(\tau)x_2(\tau)\,d\tau$$

Hence

$$\begin{aligned}\|x_1(t) - x_2(t)\| &= \left\|\int_0^t A(\tau)(x_1(\tau) - x_2(\tau))\,d\tau\right\| \\ &\leq \int_0^t \|A(\tau)(x_1(\tau) - x_2(\tau))\|\,d\tau \\ &\leq \int_0^t \|A(\tau)\|\,\|(x_1(\tau) - x_2(\tau))\|\,d\tau\end{aligned}$$

Let $x_1(t) - x_2(t) \equiv \theta(t)$ and $A(t) \equiv u(t)$ then by Granwall inequality $x_1(t) - x_2(t) = 0$ or $x_1 = x_2$. Therefore the solution to state space is unique.

Can we get unique solution to the state space problem? For large family of $A(t)$ we can. We need to formulate the fundamental matrix.

1.18 Lecture 16. Tuesday October 21 2014

Properties of $\Phi(t,\tau)$.

We developed Picard for solution of linear time varying $x' = A(t)x$ in the last two lectures. Established: That solution exist and the solution is unique. Some disadvantages of Picard method are

1. Each time the initial conditions x^0 changes, we have to run the method again to find the solution.
2. No closed form solution, so we lose insight by not being able to do some qualitative analysis on the solution if it were analytical solution.
3. LTI system always have closed for solution, and for many LTV, there is also closed form solution, so we should try to find closed form solution.
4. If input $u(t)$ changes, we have to run Picard method again

To find closed form solution, we need to obtain what is called the fundamental matrix. Let $X^{01}, X^{02}, \cdots, X^{0n}$ be n linearly independent initial conditions for a system with n states. For example, for $n = 3$, always take these as $X^{01} = \begin{pmatrix}1\\0\\0\end{pmatrix}, X^{02} = \begin{pmatrix}0\\1\\0\end{pmatrix}, X^{03} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ and so on for more states. For each one of these X^{0i}, let Ψ^i be the corresponding solution of $x' = A(t)x$. i.e. $\Psi^1(0) = X^{01}, \Psi^2(0) = X^{02}, \Psi^3(0) = X^{03}$. i.e.

$$\dot{\Psi}^i(t) = A(t)\Psi^i(t)$$

Now form the fundamental matrix solution

$$\Psi(t) = \begin{pmatrix}\Psi^1(t) & \Psi^2(t) & \cdots & \Psi^n(t)\end{pmatrix}$$

Each Ψ^i is $n \times 1$, and there are n such columns, hence $\Psi(t)$ is $n \times n$ matrix. Any solution can now be found with the help of this $\Psi(t)$, for any initial conditions. Remark: Matrix $\Psi(t)$ satisfies the state equation.

$$\Psi'(t) = A(t)\Psi(t)$$

At $t = 0, \Psi(0)$ has n linearly independent columns by construction. What about for $t > 0$?

Reader: Show that $\Psi(t)$ has n linearly independent columns for $t > 0$. Proof: By contradiction. Assume at t^* , $\Psi(t^*)$ no longer has linearly independent columns. Then there exist vector $\vec{x}(t^*)$ not zero s.t. $\Psi(t^*)\vec{x}(t^*) = \vec{0}$. This implies that $x'(t^*) = 0$, which means that $x(t) = 0$, hence contradiction.

Example: Let $x_1' = x_1 + tx_2, x_2' = x_2$. Hence $x' = \begin{pmatrix}1 & t\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$. Now let $X^{01} = \begin{pmatrix}1\\0\end{pmatrix}$ to be one linearly independent initial conditions. We use this to solve the state equation. Next to use the second $X^{02} = \begin{pmatrix}0\\1\end{pmatrix}$ and repeat the process. So we end up with two solutions. These make up Ψ matrix. Using X^{01}, we see that $x_1(0) = 1, x_2(0) = 0$. Now we solve the state equation. $x_1' = x_1 + tx_2, x_2' = x_2(t)$. This results in $\Psi^1(t) = \begin{pmatrix}e^t\\0\end{pmatrix}$. Now using initial conditions $x_1(0) = 0, x_2(0) = 1$ we solve the same state equation again, this results in $\Psi^2(t) = \begin{pmatrix}\frac{1}{2}t^2e^t\\e^t\end{pmatrix}$, hence

$$\Psi(t) = \begin{pmatrix}e^t & \frac{1}{2}t^2e^t\\0 & e^t\end{pmatrix}$$

```
DSolve[{x1'[t] == x1[t] + t x2[t], x2'[t] == x2[t], x1[0] == 1, x2[0] == 0},
{x1[t], x2[t]}, t]
{{x1[t] -> E^t, x2[t] -> 0}}
DSolve[{x1'[t] == x1[t] + t x2[t], x2'[t] == x2[t], x1[0] == 0, x2[0] == 1},
{x1[t], x2[t]}, t]
{{x1[t] -> (E^t t^2)/2, x2[t] -> E^t}}
```

Now that we have found $\Psi(t)$ we need to find the general solution to $x' = A(t)x(t) + B(t)u$ with given any $x(0)$ (this initial condition has nothing to do with X^{0i} used to find $\Psi(t)$, this is the actual initial condition for the problem itself.

Assume the general solution is

$$x(t) = \Psi(t)\theta(t)$$

where $\theta(t)$ is some function to be found. Plugging this solution into the state space equation, we obtain

$$\Psi'(t)\theta(t) + \Psi(t)\theta'(t) = A(t)\Psi(t)\theta(t) + B(t)u$$

But $\Psi'(t) = A(t)\Psi(t)$, so the above simplifies to

$$\begin{aligned}\Psi(t)\theta'(t) &= B(t)u\\ \theta'(t) &= \Psi^{-1}(t)B(t)u\end{aligned}$$

Integrating

$$\theta(t) - \theta(0) = \int_0^t \Psi^{-1}(\tau)B(\tau)u(\tau)d\tau$$

But $\theta(0) = \Psi^{-1}(0)X(0)$ where $X(0)$ is the initial conditions. (why??). Hence the above becomes

$$\theta(t) = \Psi^{-1}(0)X(0) + \int_0^t \Psi^{-1}(\tau)B(\tau)u(\tau)d\tau$$

Therefore, since $x(t) = \Psi(t)\,\theta(t)$. Then

$$\begin{aligned} x(t) &= \Psi(t)\left(\Psi^{-1}(0)\,X(0) + \int_0^t \Psi^{-1}(\tau)\,B(\tau)\,u(\tau)\,d\tau\right) \\ &= \Psi(t)\,\Psi^{-1}(0)\,X(0) + \Psi(t)\int_0^t \Psi^{-1}(\tau)\,B(\tau)\,u(\tau)\,d\tau \\ &= \Psi(t)\,\Psi^{-1}(0)\,X(0) + \int_0^t \Psi(t)\,\Psi^{-1}(\tau)\,B(\tau)\,u(\tau)\,d\tau \end{aligned}$$

Let

$$\Psi(t)\,\Psi^{-1}(\tau) = \Phi(t,\tau)$$

called the transition matrix, then the above becomes

$$\boxed{x(t) = \Phi(t,0)\,X(0) + \int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau}$$

Reader: Find $\Phi(t,\tau)$ for the last example, and then find $x(t)$ for unit step $u(t)$.

Properties of $\Phi(t,\tau)$:

1. $\Phi(0,0) = I$. Note that $\Psi(t)$ does not depend on the actual initial conditions for the problem. (these are eigenfunctions of the system).
2. $\Phi(t_3,t_1) = \Phi(t_3,t_2)\,\Phi(t_2,t_1)$. Proof: $\Psi(3)\,\Psi^{-1}(1) = \Psi(3)\,\Psi^{-1}(2)\,\Psi(2)\,\Psi^{-1}(1) = \Psi(3)\,I\Psi^{-1}(1) = \Psi(3)\,\Psi^{-1}(1)$
3. $\Phi(t_m,t_1) = \Phi(t_m,t_{m-1})\,\Phi(t_{m-1},t_{m-2})\cdots\Phi(t_2,t_1)$
4. $\Phi(t_m,t_m) = I$
5. $\Phi(t,t_0) = \Phi^{-1}(t_0,t)$
6. $\Phi(t,\tau)$ satisfies the state equation under appropriate conditions. proof: $\frac{\partial\Phi(t,\tau)}{\partial t} = \frac{\partial\Psi(t)\Psi^{-1}(\tau)}{\partial t} = \overbrace{\frac{\partial\Psi(t)}{\partial t}}^{A(t)\Psi(t)}\Psi^{-1}(\tau) = A(t)\,\Psi(t)\,\Psi^{-1}(\tau) = A(t)\,\Phi(t,\tau)$. But we can't differentiate w.r.t. τ in the above. **Reader**: Think about $\frac{\partial\Phi(t,\tau)}{\partial\tau}$

side note: Remember this $\frac{\partial}{\partial t}\int_{t_0}^t f(t,\tau)\,d\tau = \int_{t_0}^t \frac{\partial}{\partial t}f(t,\tau)\,d\tau + f(t,\tau)\big|_{\tau=t}$

1.19 Lecture 17. Thursday October 23 2014

How to determine e^{At}

Summary: We solve $x' = A(t)\,x(t) + B(t)\,u(t)$ with $X(0) = x^0$. We assumed continuity on A,B (piecewise continuous is OK). First step was to find $\Psi(t)$, where $\Psi(t) = \begin{pmatrix}\Psi^1(t) & \Psi^2(t) & \cdots & \Psi^n(t)\end{pmatrix}$. This matrix is $n\times n$ and is not unique. Now we formed $\Phi(t,\tau) = \Psi(t)\,\Psi^{-1}(\tau)$ called the transition matrix, which is unique (Q: How can $\Phi(t,\tau)$ be unique if Ψ is not?). Then we found

$$x(t) = \Phi(t,0)\,X(0) + \int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

We can also easily get the output equation as well. Which is

$$y(t) = C(t)\,\Phi(t,0)\,X(0) + \int_0^t C(t)\,\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau + D(t)\,u(t)$$

Today we will talk about LTI (linear time invariant), where A,B,C,D matrices are now constants and do not depend on time

$$
\begin{aligned}
x' &= Ax(t) + Bu(t) \\
y &= Cx(t) + Du(t)
\end{aligned}
$$

Where now A, B, C, D are constant matrices. We want to find solution to this as special case of LTV. Is $\Phi(t,\tau)$ easy to get now? Would we still need Picard iterations? Yes, it is easier to get and we do not need to use Picard iterations to solve the LTI. We will introduce matrix exponential e^{At} where A is matrix. Define as

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2t^2}{2!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}
\end{aligned}
$$

Is this even well defined? We ask, is it convergent sum? Let us view the partial sum S_k as sequence in space of bounded functions and show that this converges uniformly. $S_k = \sum_{i=0}^{k} \frac{A^i t^i}{i!}$. This is an $n \times n$ matrix, it is continuous since we only get polynomials in t as entries in this matrix. View as vector in space of bounded functions $B([0,T], M^{m\times n})$. The norm of this space is the $\sup$ norm since this is a bounded space. Now let us look at $\|S_k(t)\|$

$$
\begin{aligned}
\|S_k\|_I &= \sup \|S_k(t)\| \\
&= \sup \left\| \sum_{i=0}^{k} \frac{A^i t^i}{i!} \right\| \\
&\leq \sum_{i=0}^{k} \frac{\left\|A^i\right\| t^i}{i!}
\end{aligned}
$$

But $\left\|A^i\right\| = \|AA\cdots A\| \leq \|A\|\|A\|\cdots\|A\| = \|A\|^i$ so the above becomes

$$
\|S_k\|_I \leq \sum_{i=0}^{k} \frac{\|A\|^i t^i}{i!}
$$

Now we use Weierstrass M test. Let $M^i = (\|A\| t)^i$ then we need to see if $\sum_{i=0}^{\infty} \frac{M^i}{i!}$converges. But

$$
\sum_{i=0}^{\infty} \frac{M^i}{i!} = e^M
$$

Since it converges, then this implies that e^{At} converges uniformly. So we found out that e^{At} is continuous and converges uniformly. So it is well defined definition we have above. OK, now we have introduced e^{At}, but now we need to see how to use it to solve the LTI.

reader: e^{At} is fundamental matrix $\Psi(t)$ for LTI system $x' = Ax$. One thing to check is that at $t = 0$ the matrix $\Psi(0)$ has n linearly independent columns. We also need each column to be a solution of the state equation $\Psi' = A\Psi$. Since

$$
\begin{aligned}
\frac{d}{dt} e^{At} &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \cdots\right) \\
&= 0 + A + A^2 t + \frac{A^3t^2}{2!} + \cdots \\
&= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right) \\
&= Ae^{At}
\end{aligned}
$$

Therefore, e^{At} satisfies the state equation. What about transition matrix? Let

$$
\begin{aligned}
\Phi(t,\tau) &= \Psi(t)\Psi^{-1}(\tau) \\
&= e^{At}\left(e^{A\tau}\right)^{-1}
\end{aligned}
$$

Reader: Show that $\left(e^{A\tau}\right)^{-1} = e^{-A\tau}$.

Proof: (For case of distinct eigenvalues only): Using $e^{At} = V\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}V^{-1}$ which is $e^{At} = V\Lambda V^{-1}$, then $\left(e^{At}\right)^{-1} = \left(V\Lambda V^{-1}\right)^{-1}$, but for matrices, $(AB)^{-1} = B^{-1}A^{-1}$, hence $\left(e^{At}\right)^{-1} = V^{-1}(V\Lambda)^{-1} = V\Lambda^{-1}V^{-1}$, but $\Lambda^{-1} = \begin{pmatrix} e^{-\lambda_1 t} & & \\ & \ddots & \\ & & e^{-\lambda_n t}\end{pmatrix}$, hence $\left(e^{At}\right)^{-1} = e^{-At}$. QED. Therefore the above becomes $\Phi(t,\tau) = e^{At}e^{-A\tau}$. Question: I assumed distinct eigenvalues for A in the above proof for the reader. What about if A has repeated eigenvalues?

Reader: Show that $e^{At}e^{-A\tau} = e^{A(t-\tau)}$. To show this, use the series definition above, multiply things out and simplify. To do

So now that we showed e^{At} is fundamental matrix for $x' = Ax$ we can write the state solution using it as

$$x(t) = e^{A(t-0)}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Note: in LTV, $\Phi(t,\tau)$ was a function of 2 parameters t and τ. Here $e^{A(t-\tau)}$ is function of only one parameter, which is the difference $t-\tau$.

Some properties of e^{At}:

1. **Reader**: show that e^{At} commute with A. i.e. $Ae^{At} = e^{At}A$.
2. **Reader**: is $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$?
3. **Reader**: Is $e^{A_1}e^{A_2} = e^{A_1+A_2}$?
4. **Reader**: Is $e^{A_1}e^{A_2} = e^{A_2}e^{A_2}$? (no, in general).

How to determine e^{At}:

There are many ways to determine e^{At} (18 or more). We will cover two ways. One uses the eigenvector/eigenvalues approach and one is good for hand calculations

First method: This method assume there are n distinct eigenvalues and n distinct eigenvector. This method will not work as is if there are no n distinct eigenvalue. Most of A matrices have distinct eigenvalues, unless we hardcoded some values in them in practice. Now, let $v^1, v^2, \cdots v^n$ be the n eigenvectors and let $\lambda_1, \lambda_2, \cdots \lambda_n$ be the eigenvalues. Where $Av^i = \lambda_i v^i$. Form the modal matrix $V = \begin{pmatrix} v^1 & v^2 & \cdots & v^n\end{pmatrix}$. This matrix diagonalizes A. Hence we write

$$V^{-1}AV = \Lambda$$

Where $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n\end{pmatrix}$, hence we have

$$A = V\Lambda V^{-1}$$

$$e^{At} = \sum_{k=0}^{\infty}\frac{\left(V\Lambda V^{-1}\right)^k t^k}{k!}$$

VV^{-1} cancel leaving

$$
\begin{aligned}
e^{At} &= V\left(\sum_{k=0}^{\infty}\frac{\Lambda^k t^k}{k!}\right)V^{-1}\\
&= V\left(\sum_{k=0}^{\infty}\frac{\begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \lambda_n\end{pmatrix}^k t^k}{k!}\right)V^{-1}\\
&= V\begin{pmatrix}\sum_{k=0}^{\infty}\frac{\lambda_1^k t^k}{k!} & 0 & 0 & 0\\ 0 & \sum_{k=0}^{\infty}\frac{\lambda_2^k t^k}{k!} & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \sum_{k=0}^{\infty}\frac{\lambda_n^k t^k}{k!}\end{pmatrix}V^{-1}\\
&= V\begin{pmatrix}e^{\lambda_1 t} & 0 & 0 & 0\\ 0 & e^{\lambda_2 t} & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & e^{\lambda_2 t}\end{pmatrix}V^{-1}
\end{aligned}
$$

Next time we will look at the other method to find e^{At}

HW5 assigned.

1.20 Lecture 18. Tuesday October 28, 2014 (solving the state equation)

Summary of where we are: In middle of solving the state equation. We did LTV. In the case of LTI, we end up with e^{At}. We found it using using the first method. When has has distinct eigenvalues then we write

$$e^{At} = V\begin{pmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & e^{\lambda_n t}\end{pmatrix}V^{-1}$$

Where V is called the modal matrix. (it has as its columns the eigenvectors of A). If we scale the eigenvectors, they still remain eigenvectors.

Reader: Show e^{At} is invariant under scaling of V.

Example: $x_1' = 2x_1$ and $x_2' = -3x_1 - 3x_2$. We want to find e^{At}. Hence $A = \begin{pmatrix}2 & 0\\ -3 & -3\end{pmatrix}$. The eigenvalues are $\lambda_1 = 2, \lambda_2 = -3$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix}1\\ -\frac{3}{5}\end{pmatrix}, v_1 = \begin{pmatrix}0\\ 1\end{pmatrix}$, hence $V = \begin{pmatrix}1 & 0\\ -\frac{3}{5} & 1\end{pmatrix}$,therefore

$$
\begin{aligned}
e^{At} &= \begin{pmatrix}1 & 0\\ -\frac{3}{5} & 1\end{pmatrix}\begin{pmatrix}e^{2t} & 0\\ 0 & e^{-3t}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{3}{5} & 1\end{pmatrix}^{-1}\\
&= \begin{pmatrix}e^{2t} & 0\\ -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-3t} & e^{-3t}\end{pmatrix}
\end{aligned}
$$

Notice at $t = 0$ then $e^{At} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Reader: Show that $\frac{d}{dt}e^{At} = A$

What if we take the k^{th} derivative? then $\frac{d^k}{dt^k}e^{At} = A^k$

We will now do another example with complex eigenvalues. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, eigenvalues are $\lambda_1 = \frac{1}{2}i\sqrt{3} - \frac{1}{2}, \lambda_2 = -\frac{1}{2}i\sqrt{3} - \frac{1}{2}$

Reader: Find e^{At} for the above.

We will find the eigenvectors to make the modal matrix. The eigenvectors are $\begin{pmatrix} \frac{1}{2}i\sqrt{3} - \frac{1}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{2}i\sqrt{3} - \frac{1}{2} \\ 1 \end{pmatrix}$, hence $V = \begin{pmatrix} \frac{1}{2}i\sqrt{3} - \frac{1}{2} & -\frac{1}{2}i\sqrt{3} - \frac{1}{2} \\ 1 & 1 \end{pmatrix}$, therefore

$$e^{At} = \begin{pmatrix} \frac{1}{2}i\sqrt{3} - \frac{1}{2} & -\frac{1}{2}i\sqrt{3} - \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\left(-\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)t} & 0 \\ 0 & e^{\left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}i\sqrt{3} - \frac{1}{2} & -\frac{1}{2}i\sqrt{3} - \frac{1}{2} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-\frac{1}{2}t-\frac{1}{2}i\sqrt{3}t} + \frac{1}{2}e^{\frac{1}{2}i\sqrt{3}t-\frac{1}{2}t} + \frac{1}{6}i\sqrt{3}e^{-\frac{1}{2}t-\frac{1}{2}i\sqrt{3}t} - \frac{1}{6}i\sqrt{3}e^{\frac{1}{2}i\sqrt{3}t-\frac{1}{2}t} & \frac{1}{3}i\sqrt{3}\exp\left(-t\left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)\right) - \frac{1}{3}i\sqrt{3}e^{t\left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)} \\ \frac{1}{3}i\sqrt{3}e^{t\left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)} - \frac{1}{3}i\sqrt{3}\exp\left(-t\left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)\right) & \frac{1}{2}e^{-\frac{1}{2}t-\frac{1}{2}i\sqrt{3}t} + \frac{1}{2}e^{\frac{1}{2}i\sqrt{3}t-\frac{1}{2}t} - \frac{1}{6}i\sqrt{3}e^{-\frac{1}{2}t-\frac{1}{2}i\sqrt{3}t} + \frac{1}{6}i\sqrt{3}e^{\frac{1}{2}i\sqrt{3}t-\frac{1}{2}t} \end{pmatrix}$$

Now we will show another method to find e^{At}. This is using Laplace transform.

Reader: Show that $e^{At} = \mathscr{L}^{-1}(sI - A)^{-1}$. Why is this true?

$$\begin{aligned} x' &= Ax \\ sX(s) - x(0) &= AX(s) \\ X(s)(sI - A) &= x(0) \\ X(s) &= (sI - A)^{-1}x(0) \\ x(t) &= \mathscr{L}^{-1}(sI - A)^{-1}x(0) \end{aligned}$$

Compare to $x(t) = e^{At}x(0)$ we see that $e^{At} = \mathscr{L}^{-1}(sI - A)^{-1}$ for any $x(0)$

Now we will given the third method to find e^{At}. This is called expansion of natural frequencies method. Here we allow repeated eigenvalues. In the first method (using modal matrix) the eigenvalues has to be distinct. Let the eigenvalues be $\lambda_1, \lambda_1, \cdots, \lambda_m$ with corresponding multiplies $n_1, n_1, \cdots, n_m$. We will propose a form for e^{At} with some unknowns, then solve for these unknowns. Since all solution must have exp and t multipliers (for repeated eigenvalues), let

$$e^{At} = \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} Y_{k,i}t^k e^{\lambda_i t} \tag{1}$$

Where $Y(k,i)$ are the unknowns. To find $Y_{k,i}$, we use $\frac{d^k}{dt^k}e^{At} = A^k$. Let us implement this on the first example we did above

$$A = \begin{pmatrix} 2 & 0 \\ -3 & -3 \end{pmatrix}$$

$\lambda_1 = 2, \lambda_2 = -3$, hence $m = 2$, and the multiplies are $n_1 = 1, n_2 = 1$, hence using (1) gives

$$e^{At} = Y_{0,1}e^{2t} + Y_{0,2}e^{-3t} \tag{2}$$

$$\begin{aligned} e^{At}\Big|_{t=0} &= I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Y_{0,1} + Y_{0,2} \\ \frac{d}{dt}e^{At}\Big|_{t=0} &= A = \begin{pmatrix} 2 & 0 \\ -3 & -3 \end{pmatrix} = 2Y_{0,1} - 3Y_{0,2} \end{aligned}$$

We have 2 equations above in 2 unknowns $Y_{0,1}, Y_{0,2}$. We solve for these, then using (2) gives e^{At}. Solving gives $Y_{0,1} = \begin{pmatrix} 1 & 0 \\ \frac{-3}{5} & 0 \end{pmatrix}, Y_{0,2} = \begin{pmatrix} 0 & 0 \\ \frac{3}{5} & 1 \end{pmatrix}$, hence (2) becomes

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 0 \\ \frac{-3}{5} & 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 & 0 \\ \frac{3}{5} & 1 \end{pmatrix} e^{-3t} \\ &= \begin{pmatrix} e^{2t} & 0 \\ \frac{3}{5}e^{-3t} - \frac{3}{5}e^{2t} & e^{-3t} \end{pmatrix} \end{aligned}$$

Lets now do repeated eigenvalues. See my expansion of natural frequencies method notes for this larger example and more examples using this method using a symbolic function written to process this method.

1.21 Lecture 19. Thursday October 30 2014 (Controllability)

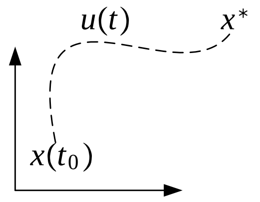
Today will be on controllability of $\sum = (A, B)$. We talked before about controllability for LTI. We said that when rank of the controllability matrix is n then (A, B) is controllable. This is an algebraic view. When (A, B) is controllable, it means we can do some useful transformations. We talked about minimal realization. These are all algebraic properties. Today will talk about what physically it means for system to be controllable. This is the physical meaning to saying $\rho(\mathbb{C}) = n$. Also, if we want to manipulate the input we need physically controllability.

Physically controllability has to do with only A, B, from $x' = Ax + Bu$. It is about the ability to steer the system with an input. What this means, for given state $X(0)$ we want to be able to transfer the system to new state $X(t)$. This is called the target state.

So system is controllable at t_0 if the following is true: (Note, we the "at t_0" is important, since this is now LTV and system can change from time to time, so we always talk about controllability at some specific time with LTV).

Formal definition of physical controllability: Given any initial conditions $x(t_0) = X_0$ any target state x^* then there exist a future time $t_1 > t_0$ and input $u(t)$ over $[t_0, t_1]$ leading to $x(t_1) = x^*$.

Notice there is not constraint on $u(t)$, it can be anything and as large as needed. (but the time interval to arrive at target state must be finite).



Phase space diagram showing u(t) moving state from x(t0) to x*

There can be many $u(t)$ which will do the above, but we only need to find one. Why "at t_0" is important? Looking at 2 extreme cases

1. $B(t) = 0$ then clearly the system is not controllable. No input.
2. $B(t) = I$ the identity matrix. **Reader**: Show the system is always controllable with such $B(t)$.

There are cases in between the above extreme cases where it is not clear. For example, given

$$\begin{aligned} x_1' &= x_1 + u \\ x_2' &= x_2 + u \end{aligned}$$

Not controllable. This is not coupled. We can control x_1 state on its own, and x_2 on its own, but not both at same time which would be necessary for complete controllability.

Reader: If system is controllable at t_0, is it controllable at $t_0 < t$? Yes. (but need to understand the argument given).

Suppose we have 2 vectors and they are time dependent. So we need to define what linear independent means in this case (when the vectors depend on time as well).

On linear independence of time vectors:

Let

$$
\begin{aligned}
f_1(t) &= \left[f_{11}(t), f_{12}(t), \cdots, f_{1p}(t)\right]\\
f_2(t) &= \left[f_{21}(t), f_{22}(t), \cdots, f_{2p}(t)\right]\\
&\vdots\\
f_n(t) &= \left[f_{n1}(t), f_{n2}(t), \cdots, f_{np}(t)\right]
\end{aligned}
$$

We say that f_i vectors are L.D. (Linear dependent) on time interval $[t_0, t_1]$ if the following occurs: There exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ not all zero, such that $\sum_{i=1}^{n} \alpha_i f_i(t) = 0$ for all $t = [t_0, t_1]$. Otherwise they are L.I.

Examples: $f_1(t) = -t, f_2(t) = t^2$ on $[0,2]$. Can we find α_1, α_2 such that $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$. No. So L.I., notice that the same α has to be used for all t.

Reader: Show that $f_1(t) = [1, t], f_2(t) = \left[t^2, t^3\right]$ is L.I. on $[-2, 2]$

Reader: Show $\left[1, t, t^2, \cdots, t^n\right]$ are L.I. on any $[t_0, t_1]$ with $t_1 > t_0$

Reader: Show $\left[e^t, e^{-t}, e^{3t}, \cdots\right]$ are L.I. on any $[t_0, t_1]$ with $t_1 > t_0$

Reader: Does being L.I. on $[t_0, t_1]$ implies L.I. on $\left[t_0', t_1'\right]$ where $\left[t_0', t_1'\right] \supseteq [t_0, t_1]$? Yes. What about if $\left[t_0', t_1'\right] \sqsubseteq [t_0, t_1]$? NO. Not necessarily.

Theorem: Given f_i and $[t_0, t_1]$, define matrix $F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$, so $F(t)$ is $n \times p$ size matrix.

Now define the Gramian

$$
W_{[t_0, t_1]} = \int_{t_0}^{t_1} F(t) F^T(t)\, dt
$$

Then $f_i(t)$ are L.I. on $[t_0, t_1]$ iff $W[t_0, t_1]$ is not singular. Proof:

Necessity: Assume f_i are L.I. Show $\Im$ not singular. Proof by contradiction. Assume W singular then $\Im\vec{\alpha} = 0$ for non-zero $\vec{\alpha}$ vector. Also $\vec{\alpha}^T \Im \vec{\alpha} = 0$. Hence $\int \vec{\alpha}^T W \vec{\alpha} dt = 0$ or $\int \vec{\alpha}^T F(t) F^T \vec{\alpha} dt$. Let $F^T \vec{\alpha} = \xi(t)$ and $\vec{\alpha}^T F(t) = \xi^T(t)$. Then we have $\int_{t_0}^{t_1} \sum_{i=1}^{n} \xi_i^2(t)\, dt = 0$ which implies $\xi(t) = 0$ identically. But this means $F^T \vec{\alpha} = 0$, which means f_i are L.I. But this is contradiction to assumption. Hence f_i are L.I. implies W not singular.

Now to proof the sufficiency: Assume $\Im$ not singular, show f_i are L.I. Proof by contradiction. Assume f_i are L.D., then $\vec{\alpha}$ exist such that $\vec{\alpha}^T F(t) = 0$ which implies $\vec{\alpha}^T W = 0$. But this means $\Im$ is singular. Which is contradiction. This complete the proof that $f_i(t)$ are L.I. on $[t_0, t_1]$ iff $W[t_0, t_1]$ is not singular.

1.22 Lecture 20. Tuesday November 4 2014. (Controllability of LTV)

No lecture next Tuesday. Second midterm next Thursday at 6 pm.

Keywords for test 2: Not cumulative. Covers material from first exam. Test 2, starts with vector spaces, definition of vector space, norms, sequences, convergence. We used space of bounded function $B\left([t_0,t_1],\Re^n\right)$. This is a function space. Can be made of vector functions. The default convergence in this space is uniform convergence. But there is a weaker convergence called pointwise. This is important for integrals (we can move the limit inside if the function converges uniformly).

Picard iterations. M-test to test for uniform convergence. Solution of state equation using Picard iterations. We proofed many things about Picard, such as convergence and uniqueness. We used Granwall's inequality.

We looked at negative aspects of Picard iterations. We want closed form. Using fundamental matrix $\Psi(t)$. Once we have $\Psi(t)$ we have solution for any input. $\Psi(t)$ is not unique, but $\Phi(t,\tau)$ is. $\Phi(t,\tau)=\Psi(t)\Psi^{-1}(\tau)$. We also talked at LTI. It simplifies. We used $\Psi(t)=e^{At}$. We looked at three methods to find e^{At}.

Back to lecture. We are talking about controllability of $x'(t)=A(t)x(t)+B(t)u(t)$. We started talking about physical controllability. This is the ability to take the system from $x(t_0)$ to $x(t_1)$ for $t_1>t_0$ by using some $u(t)$.

How to test controllability of LTV system? Define $W=\int FF^T dt$ and check of W is not singular. In our case $F=\Phi(t_0,\tau)B(\tau)$.

Theorem: LTV system is controllable at t_0 iff the rows of $F=\Phi(t_0,\tau)B(\tau)$ are linearly independent time functions on $[t_0,t_1]$ for some $t_1\geq t_0$. From last lecture, we defined

$$W(t_0,t_1)=\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\,d\tau$$

$W(t_0,t_1)$ must be not singular for the system to be controllable at t_0.

Proof: Sufficiency $\Longleftarrow$. We need to proof this: If W not singular, then LTV is controllable at t_0. For necessity $\Longrightarrow$ we need to proof this: If LTV is controllable at t_0 then W is not singular.

We start with Sufficiency. Let W be not singular. Let $x(t_0),x(t_1)$ be arbitrarily given states, where $x(t_1)$ is the target state at time t_1. We must be able to construct $u(t)$ that steers the system from $x(t_0)$ to $x(t_1)$. i.e. we want $x(t_1)=\Phi(t_1,t_0)x(t_0)+\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$. Pre-multiplying both sides by $\Phi(t_0,t_1)$ gives

$$\begin{aligned}
\Phi(t_0,t_1)x(t_1) &= \overbrace{\Phi(t_0,t_1)\Phi(t_1,t_0)}^{I}x(t_0)+\int_{t_0}^{t}\overbrace{\Phi(t_0,t_1)\Phi(t,\tau)}^{\Phi(t_0,\tau)}B(\tau)u(\tau)d\tau\\
\Phi(t_0,t_1)x(t_1) &= x(t_0)+\int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)u(\tau)d\tau\\
\Phi(t_0,t_1)x(t_1)-x(t_0) &= \int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)u(\tau)d\tau
\end{aligned}$$

Let

$$u(\tau)=-B^T(\tau)\Phi^T(t_1,\tau)W^{-1}(t_0,t_1)\left[\Phi(t_1,t_0)x(t_0)-x(t_1)\right]$$

Reader Show that this $u(t)$ leads to $x(t_0)\longrightarrow x(t_1)$

We now do proof of necessity $\Longrightarrow$ If LTV is controllable at t_0 then W is not singular. Equivalently, show that if LTV is controllable at t_0 then rows of $\Phi(t_0,\tau)B(\tau)$ are linearly independent. Proof by contradiction: Assume W is singular but LTV is controllable at t_0 and show a contradiction. Since W is singular, then there exist a vector $\vec{\alpha}\neq 0$ s.t. $\vec{\alpha}^T\Phi(t_0,\tau)B(\tau)=0$ for all $\tau\in[t_0,t_1]$. Now construct $x(t_0),x(t_1)$. Let $x(t_0)=\vec{\alpha}$ and let $x(t_1)=0$ (i.e. the origin vector in the state space). Since LTV is controllable, then there exist $u(t)$ such that

$$
\begin{aligned}
x(t_1) &= \Phi(t_1,t_0)\,x(t_0) + \int\limits_{t_0}^{t_1} \Phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau \\
&= \Phi(t_1,t_0)\,\vec{\alpha} + \int\limits_{t_0}^{t_1} \Phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau
\end{aligned}
$$

Pre-multiply by $\Phi(t_0,t_1)$ both sides gives

$$
\Phi(t_0,t_1)\,x(t_1) = \overbrace{\Phi(t_0,t_1)\,\Phi(t_1,t_0)}^{I}\,\vec{\alpha} + \int\limits_{t_0}^{t_1} \overbrace{\Phi(t_0,t_1)\,\Phi(t_1,\tau)}^{\Phi(t_0,\tau)}\,B(\tau)\,u(\tau)\,d\tau
$$

So the above becomes

$$
\Phi(t_0,t_1)\,x(t_1) = \vec{\alpha} + \int\limits_{t_0}^{t} \Phi(t_0,\tau)\,B(\tau)\,u(\tau)\,d\tau
$$

Now pre-multiplying both sides by $\vec{\alpha}^T$

$$
\vec{\alpha}^T\Phi(t_0,t_1)\,x(t_1) = \vec{\alpha}^T\vec{\alpha} + \int\limits_{t_0}^{t} \vec{\alpha}^T\Phi(t_0,\tau)\,B(\tau)\,u(\tau)\,d\tau
$$

But $\vec{\alpha}^T\vec{\alpha} = \|\alpha\|^2$ and $\vec{\alpha}^T\Phi(t_0,t_1)\,x(t_1) = 0$ since $x(t_1) = 0$, hence the above becomes

$$
0 = \|\alpha\|^2 + \int\limits_{t_0}^{t} \vec{\alpha}^T\Phi(t_0,\tau)\,B(\tau)\,u(\tau)\,d\tau
$$

But we assumed $\vec{\alpha}^T\Phi(t_0,\tau)\,B(\tau) = 0$ above, hence the above reduces to

$$
0 = \|\alpha\|^2
$$

Which means $\vec{\alpha} = 0$. But this contradicts our assumption that $\vec{\alpha} \neq 0$. Hence our assumption that W is singular but LTV is controllable at t_0 has been found to produce a contradiction. Hence it must be that out assumption of W being singular is not valid. QED.

Advanced reader: The proof of necessity above contains an error. Try to find it.

Remark: There are infinite many controls $u(t)$ that can take $x(t_0)$ to $x(t_1)$. We considered one of them in the above proof $u(t) = B^T(\tau)\,\Phi^T(t_0,\tau)\,W^{-1}(t_0,t_1)\,[\Phi(t_0,t_1)\,x(t_1) - x(t_0)]$. In this u special in some sense?

Minimum energy theorem: Suppose $\sum$ is controllable at t_0, i.e, controllability $W(t_0,t_1)$ is not singular. So we can steer $x(t_0)$ to $x(t_1)$. So given pair $x(t_0), x(t_1)$ and as associated control law $u(t)$, define the energy $E(u) = \int\limits_{t_0}^{t_1} u^T u\,dt = \int\limits_{t_0}^{t_1} \|u\|^2\,dt$. This gives energy needed. $u(t)$ used above in the proof minimizes $E(u)$. Next lecture we will show that $u(t) = B^T(\tau)\,\Phi^T(t_0,\tau)\,W^{-1}(t_0,t_1)\,[\Phi(t_0,t_1)\,x(t_1) - x(t_0)]$ is energy minimizer.

1.23 Lecture 21. Thursday Nov. 6, 2014, 2:30 PM (controllability, Gramian, proofs)

$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)\,B(\tau)\,B^T(\tau)\,\Phi^T(t_0,\tau)\,d\tau$ is called the controllability Gramian. Is there a shorter method to check for controllability other having to build $W(t_0,t_1)$ and check it is not singular? The short cut will be sufficient for LTV. For LTI, this short cut with minor change. will become sufficient and necessary for controllability (it will lead to the rank condition on the pair A, B).

Lemma: Consider $n \times m$ matrix, we called it $F(t)$ of continuous and smooth functions on $[t_0, t_1]$. This means each column is a vector functions. Suppose further that the matrix $\Im(t) = \begin{pmatrix} F(t) & F'(t) & \cdots & F^{(n-1)}(t) \end{pmatrix}$. This is of size $n \times nm$. If $\Im(t)$ has rank n for some $t^* \in [t_0, t_1]$, then it follows that $F^{(i)}(t)$ are all linearly independent time functions on $[t_0, t_1]$.

Proof by contradiction. Suppose we have such $t^* \in [t_0, t_1]$ where rank $\Im(t) = n$, and assume $F^{(i)}(t)$ are all linearly dependent time functions. Hence we can find $\vec{\alpha} \neq 0$, s.t. $\vec{\alpha}^T F(t) = 0$. This mean $F(t) = 0$, so that all columns of $\Im(t)$ are zero. Hence rank of $\Im(t)$ is not n. Hence $F^{(i)}(t)$ are all linearly independent time functions. QED.

Is this useful to study controllability? Use

$$F(\tau) = \Phi(t_0, \tau) B(\tau)$$

Hence

$$\begin{aligned}
F'(\tau) &= \frac{d\Phi(t_0,\tau)}{d\tau} B(\tau) + \Phi(t_0,\tau)\frac{dB(\tau)}{d\tau} \\
F''(\tau) &= \frac{d^2\Phi(t_0,\tau)}{d\tau} B(\tau) + \frac{d\Phi(t_0,\tau)}{d\tau}\frac{dB(\tau)}{d\tau} + \frac{d\Phi(t_0,\tau)}{d\tau}\frac{dB(\tau)}{d\tau} + \Phi(t_0,\tau)\frac{d^2B(\tau)}{d\tau^2}
\end{aligned}$$

Reader: Generalize to many $F^{(n)}$ using recursive formula.

Let $M_0(\tau) = B(\tau)$, then

$$M_{k+1}(\tau) = -A(\tau) M_k(\tau) + \frac{dM_k(\tau)}{d\tau}$$

For $k = 0 \cdots n-2$.

Reader: Show the above is true.

Now we will use the above lemma. System is controllable at t_0 if there exist $t^* > t_0$ such that rank$(M(t^*)) = n$

Example: Let $x' = \begin{pmatrix} \cos t & t \\ e^t & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix} u(t)$. Is this controllable at $t_0 = 0$?

$$\begin{aligned}
M_0(t) &= \begin{pmatrix} 1 \\ t \end{pmatrix} \\
M_1(t) &= -\begin{pmatrix} \cos t & t \\ e^t & 2 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\cos t + t^2 \\ -\left(e^t + 2t\right) + 1 \end{pmatrix}
\end{aligned}$$

Hence

$$M(t) = \begin{pmatrix} 1 & -\cos t + t^2 \\ t & -\left(e^t + 2t\right) + 1 \end{pmatrix}$$

Check the determinant. If it is not zero, then rank is 2 and hence controllable. Need to use $t^* > 0$ to find numerical solution for determinant to check if zero or not. Basically we need to check if there exist $t^* > t$ where the above matrix is not singular.

Few words about analytic functions: Of we have smooth function $f(t)$ and $f(t) = 0$ on some region $[t_0, t_1]$ with $t_0 \neq t$, then this means $f(t) = 0$ on all time t and not just in this region. This is because an analytical function can not have any place with its derivative does not exist (no sharp corners). Also, for analytic functions, we can expand them locally around a point using Taylor series. Now we make a bridge to LTI. We need more result about linear independence.

Lemma: Suppose $F(t)$ is analytic on $[t_0, t_1]$, (this is the new addition for LTI, which we did not use for LTV), define

$$\Im(t) = \begin{pmatrix} F(t) & F'(t) & \cdots & F^{(n-1)}(t) & \cdots \end{pmatrix}$$

Notice, there are infinite many columns now. Unlike with LTV. Now the lemma says: $F_i(t)$ are L.I. on $[t_0, t_1]$ iff rank $\Im(t) = n$ for some $t^* \in [t_0, t_1]$.

Proof: sufficiency: $\Longleftarrow$. Assume rank $\Im(t) = n$ for some $t^* \in [t_0, t_1]$ we need to show that $F_i(t)$ are L.I. on $[t_0, t_1]$. By contradiction: Assume $\Im(t) = n$ for some $t^* \in [t_0, t_1]$ but $F_i(t)$ are

L.D. on $[t_0, t_1]$. Hence there exist vector $\vec{\alpha} \neq 0$ such that $\vec{\alpha} F_i(t) = 0$. Hence $F(t) = 0$ and it follows that $F'(t) = 0$ etc.. for all columns of $\Im(t)$. Hence rank $\Im(t) \neq n$. QED.

necessity: $\Longrightarrow$. Assume $F_i(t)$ are L.I. on $[t_0, t_1]$ we need to show that there exist $t^* \in [t_0, t_1]$ such that rank $\Im(t^*) = n$. Proof by contradiction. Assume $F_i(t)$ are L.I. on $[t_0, t_1]$ but no such t^* exist, so rank $\Im(t^*) < n$ for all $[t_0, t_1]$. Pick any t in the range and expand $F(t)$ around t^* using Taylor (since analytic)

$$F(t) = \sum_{k=0}^{\infty} \frac{F^{(k)}(t - t^*)^k}{k!} \tag{1}$$

Since rank rank $\Im(t^*) < n$, then there exist vector $\vec{\alpha} \neq 0$ such that $\vec{\alpha}\Im(t^*) = 0$. Now multiply (1) by $\vec{\alpha}^T$ we get $\vec{\alpha}^T F(t) = 0$ (**Reader**) on $[t^* - \varepsilon, t^* + \varepsilon]$. But since we assumed $F(t)$ analytic, then $F(t) = 0$ everywhere. Which contradicts that $F_i(t)$ are L.I. on $[t_0, t_1]$. QED.

1.24 Lecture 22. Thursday Nov. 6, 2014, 6:00 PM. Second Exam

1.25 Lecture 23. Tuesday November 11 2014 (no lecture)

No lecture

HW6 assigned.

1.26 Lecture 24. Thursday November 13 2014 (physical controllability)

Went over second exam: Meaning of uniform convergence. Many had trouble with part (c) of first problem. For problem three use the commute property. Much easier that calculus.

Now back to lecture. We said before that

$$\Im(t) = \begin{pmatrix} F(t) & F'(t) & \cdots & F^{(n-1)}(t) & \cdots \end{pmatrix} \tag{1}$$

We were talking about Linear independence and we had these functions $F(t)$. We had the extra condition that they are analytic and wanted to check of there linearly independent on $[t_0, t_1]$. Yes they are iff rank $\Im(t) = n$ for all $t \in [t_0, t_1]$. This was the stepping stone to physical controllability of LTI system. For LTI the Gramian matrix $W(t_0, t)$ simplifies to the controllability matrix $\mathbb{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$.

Reader: If LTI system is controllable at t_0 then it is controllable for every t. So in LTI, we do not need to keep saying at t_0 and we drop it, and just saying that LTI is controllable, period. This implies for any t. Proof: Suppose (A, B) is controllable at $t_0 = 0$,. we need to show it is controllable at $t_0' > t_0$. Argument: Since it is controllable at t_0 we can find $u(t)$ to steer the system to $x(t_0')$. Now shift $u(t)$ by t_0', hence $u(t - t_0')$ is applied again to show it will take the system from $x(t_0')$ to $x(t_1)$. Note: I am not sure I follow this argument. Need to check with the prof. on this. I do not understand how applying $u(t - t_0')$ makes it controllable at t_0'.

Now we build $\Im(t)$ for LTI. Instead of using $\Phi(t_0, t) B(t)$ for the $F(t)$ functions, we now use $e^{-At}B$, since LTI. Hence (1) becomes

$$\Im(t) = \begin{pmatrix} e^{-At}B & -e^{-At}AB & e^{-At}A^2B & \cdots & (-1)^n e^{-At}A^nB & \cdots \end{pmatrix} \tag{2}$$

The system is controllable iff rank $\Im(t) = n$ Since e^{At} is non-singular, we factor it out

$$\Im(t) = e^{-At} \begin{pmatrix} B & -AB & A^2B & \cdots & (-1)^n A^nB & \cdots \end{pmatrix}$$

The sign do not affect the rank, so we make all the signs the same

$$\Im(t) = e^{-At}\begin{pmatrix} B & AB & A^2B & \cdots & A^nB & \cdots \end{pmatrix}$$

And since e^{-At} is always non-singular, it does not affect the rank, so we write

$$\Im(t) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B & A^nB & \cdots \end{pmatrix}$$

Now, we need to find a way to remove all terms beyond $A^{n-1}B$. To do this we use Cayley Hamilton. This says that for matrix A of size $n \times n$ with characteristic polynomial $\Delta(\lambda) = \det(\lambda I - A)$ then $\Delta(A) = 0$.

Example: Given $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & -1 & 3 \end{bmatrix}$, then $\Delta(\lambda) = \det(\lambda I - A) = (\lambda - 2)(\lambda + 4)(\lambda - 4) = \lambda^3 - \lambda^2 - 14\lambda + 24$, hence by Cayley Hamilton we have

$$A^3 - A^2 - 14A + 24I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Reader: Using Cayley Hamilton show that

$$\rho\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B & A^nB & \cdots \end{pmatrix} = \rho\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

Hence $\Im(t) = \mathbb{C}$ is controllable iff $\rho(\mathbb{C}) = n$

We need to show that columns from n to ∞ do not contribute to rank of $\Im(t)$. This means A^n is linear combination of $\{I, A, A^2, \cdots, A^{n-1}\}$. Using Cayley Hamilton applied for A^k we obtain that

$$0 = A^n + \sum_{i=0}^{n-1} a_i A^i$$
$$A^n = \sum_{i=0}^{n-1} a_i A^i$$

Hence A^n is linear combination of $\{I, A, A^2, \cdots, A^{n-1}\}$. We now do the same for A^{n+1} to show it is linear combination of A^n and all the other matrices, and so on. Hence all matrices A after $n-1$ do not contribute to rank of $\Im(t)$. This complete the proof that $\Im(t)$ reduces to $\mathbb{C}$ for LTI systems.

More on controllability: Differential controllability. We will start with LTV. We say $\sum$ is differentially controllable at t_0 if we can get to new state in as small time as we want. If given $\varepsilon > 0$, arbitrary small, and any $x(t_0)$ state, then there exist $u(t)$ steering $x(t_0)$ to $x(t_0 + \varepsilon)$.

Reader: Give criteria and short cut for differential controllability at t_0. Use $W(t_0, t_0 + \varepsilon)$. For LTI, use short cut M.

For LTI: **Reader**: If $\sum$ is controllable, then it is always differentially controllable. But this is not necessarily true for LTV. To show, let

$$x^1 = e^{At_1}x^0 + \int_{t_0}^{t_1} e^{(t_1-\tau)}Bu(\tau)\,d\tau$$
$$x^\varepsilon = e^{A\varepsilon}x^0 + \int_{t_0}^{\varepsilon} e^{(\varepsilon-\tau)}B\tilde{u}(\tau)\,d\tau$$

Relate $u(t)$ to $\tilde{u}(t)$ so that we can get to x^1 in as short time as we want. Note: The above is not clear to me, need to clean up.

Controllability with bounded control: i.e. we now have a bound on the magnitude of $u(t)$. There is whole theory on controllability with bounded input. Next we will do the same with observability, using duality to speed all the derivations by using results obtained from the controllability. We will later study decomposition, then stability.

1.27 Lecture 25. Tuesday November 18 2014

1.27.1 Handout, Observability summary

ECE 717 – Handout Observability Summary

For the continuous LTV system Σ

$$\dot{x} = A(t)x + B(t)u; \quad y(t) = Cx(t) + D(t)u$$

the definition of "observability at" t_0 will first be given.

Reader: Does observability at t_0 imply observability at $t_0' > t_0$? $t_0' < t_0$?

Gramian Condition for Observability: We begin with $x(t_0) = x^0$ and

$$y(t) = C(t)\Phi(t, t_0)x^0 + C(t)\int_0^t \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t).$$

We want to determine x^0.

Reader: The integral above and $Du(t)$ do not matter in the development of a criterion. Hence, without loss of generality, we consider output $y(t) = C(t)\Phi(t, t_0)x^0$ and obtain the condition

$$\int_0^{t_1} \Phi^T(\tau, t_0)C^T(\tau)C(\tau)\Phi(\tau, t_0)d\tau x^0 = \int_0^{t_1} \Phi^T(\tau, t_0)C^T(\tau)y(\tau)d\tau.$$

This motivates defining Gramian

$$W_o(t_0, t_1) \doteq \int_0^{t_1} \Phi^T(\tau, t_0)C^T(\tau)C(\tau)\Phi(\tau, t_0)d\tau$$

whose nonsingularity for some $t_1 > t_0$ is both necessary and sufficient for observability at t_0. This being the case, we recover the initial state

$$x^0 = W_o^{-1}(t_0, t_1)\int_0^{t_1} \Phi^T(\tau, t_0)C^T(\tau)y(\tau)d\tau.$$

Reader: Notice the the ideas above can equally well be stated in terms of linear independence of functions. Namely, Σ is observable at t_0 if and only if the rows of the matrix $\Phi^T(\tau, t_0)C^T(\tau)$ are linearly independent on some time interval $[t_0, t_1]$.

Reader: With dual system $\Sigma^* = (A^T(t), C^T(t), B^(T), D^T(t))$, establish: (i) Σ is observable at t_0 if and only if Σ^* and controllable at t_0. (i) Σ is controllable t_0 if and only if Σ^* and observable at t_0.

Reader: For the case of a "smooth" system, consider shortcut matrices

$$L_0(t) = C(t);\ \ L_{k+1}(t) = L_k(t)A(t) + \frac{d}{dt}L_k(t);\ k = 0, 1, ..., n - 2$$

and

$$\mathcal{L}(t) \doteq \begin{bmatrix} L_0(t) \\ L_1(t) \\ \cdot \\ \cdot \\ \cdot \\ L_{n-1}(t) \end{bmatrix}.$$

Now provide a sufficient condition for observability at t_0. This summary is completed by considering the LTI case where we drop "at t_0" and bring analyticity into play to arrive at

$$\text{rank}\mathcal{O}_\Sigma = n$$

as the necessary and sufficient condition for observability.

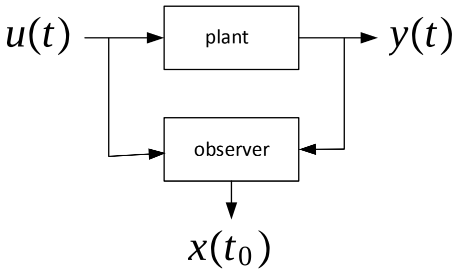
1.27.2 Lecture: Canonical decomposition theorem

We already talked about algebraic observability (observability matrix). We also talked about the observer design. Now we want to talk about physical observability. Starting with LTV system

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Formal definition of physical observability: System is observable at t_0 if the following condition holds: With $x(t_0) = x^0$ unknown, suppose $u(t)$ and $y(t)$ are known, then there exist time $t_1 \geq t_0$ such that $x(t_0)$ can be determined from knowing $u(t)$ and $y(t)$ over $[t_0, t_1]$.

This is true for any $x(t_0)$.



Definition from Chen book: pair $(A(t), C(t))$ is observable at t_0 iff there exists time $t_1 > t_0$ such that the $n \times n$ matrix $W_o(t_0,t_1) = \int_{t_0}^{t} \Phi^T(\tau,t_0) C^T(\tau) C(\tau) \Phi(\tau,t_0) d\tau$ where $\Phi(t,\tau)$ is the state transition matrix for $x'(t) = A(t)x(t)$ is nonsingular.

Controllability of the primal system equals the observability of the duel. If system is observable at t_0 what about at $t > t_0$? The answer is not necessarily. What about for $t < t_0$? The answer is yes. We can always solve for $x(0)$

$$y(t) = C(t)\Phi(t,t_0)x^0 + \overbrace{C(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)}^{\text{not function of } x^0 \text{ can be ignored}}$$

$$\tilde{y}(t) = C(t)\Phi(t,t_0)x^0$$

Pre-multiply by $\Phi^T(\tau,t_0)C^T(\tau)$ and integrate

$$\int_{t_0}^{t}\Phi^T(\tau,t_0)C^T(\tau)\tilde{y}(t)d\tau = \overbrace{\left(\int_{t_0}^{t}\Phi^T(\tau,t_0)C^T(\tau)C(\tau)\Phi(\tau,t_0)d\tau\right)}^{W(t_0,t)}x^0$$

Since observable we can invert W, hence

$$x^0 = W^{-1}(t_0,t)\int_{t_0}^{t}\Phi^T(\tau,t_0)C^T(\tau)\tilde{y}(t)d\tau$$

So system is observable iff W is non-singular for $t > t_0$. This means rows of $\Phi^T(\tau,t_0)C^T(\tau)$ are linearly independent.

Reader express state transition matrix for the duel system $\left(\tilde{A},\tilde{B},\tilde{C},\tilde{D}\right)$ where

$$\begin{aligned}\tilde{A} &= -A^T\\ \tilde{B} &= -C^T\\ \tilde{C} &= B^T\\ \tilde{D} &= D\end{aligned}$$

Hence, the duel system

$$\begin{aligned}\tilde{x}'(t) &= -A^T(t)\tilde{x}(t) - C^T(t)u(t)\\ \tilde{y}(t) &= B^T(t)\tilde{x}(t) + D^T(t)u(t)\end{aligned}$$

Now we will establish the relation between the system transfer matrix Φ for the primal and the duel. For the primal, we have

$$\frac{d\Psi^{-1}(t)}{dt} = -\Psi^{-1}(t)A(t)$$

Take the transpose of both sides gives

$$\frac{d\left[\Psi^{-1}(t)\right]^T}{dt} = -A^T(t)\left[\Psi^{-1}(t)\right]^T \tag{1}$$

Now for the duel, we have

$$\frac{d\tilde{\Psi}(t)}{dt} = \tilde{A}(t)\tilde{\Psi}(t)$$

However $\tilde{A}(t) = -A^T(t)$ therefore the above becomes

$$\frac{d\tilde{\Psi}(t)}{dt} = -A^T(t)\,\tilde{\Psi}(t) \tag{2}$$

Comparing LHS and RHS of (1) and (2) we see that

$$\left[\Psi^{-1}(t)\right]^T = \tilde{\Psi}(t) \tag{3}$$

The above also mean that

$$\left[\Psi^T(t)\right]^{-1} = \tilde{\Psi}(t)$$

Now taking the inverse of both sides

$$\Psi^T(t) = \tilde{\Psi}^{-1}(t) \tag{4}$$

(3) and (4) is all what we need to establish that

$$\begin{aligned}
\tilde{\Phi}(t,\tau) &= \tilde{\Psi}(t)\,\tilde{\Psi}^{-1}(\tau)\\
&= \left[\Psi^T(t)\right]^{-1}\Psi^T(\tau)\\
&= \left[\Psi^{-1}(t)\right]^T\Psi^T(\tau)\\
&= \left[\Psi(\tau)\,\Psi(t)^{-1}\right]^T\\
&= \Phi^T(\tau,t)
\end{aligned}$$

Hence

$$\boxed{\tilde{\Phi}(t,\tau) = \Phi^T(\tau,t)} \tag{5}$$

Now we describe the short cut method to determine observability. For $n-1$ differential system

$$L_0(t) = C(t)$$

$$L_{k+1}(t) = L_k(t)\,A(t) + \frac{d}{dt}L_k(t)$$

For $k = 0\cdots n-2$. Then the system is observable at t_0 iff there exist $t_1 > t_0$ such that $\rho\left[L(t)\right] = \rho\begin{pmatrix} L_0(t_1)\\ L_1(t_1)\\ \vdots\\ L_{n-1}(t_1)\end{pmatrix} = n$

Example: $A(t) = \begin{pmatrix} 0 & 1 & t\\ t^2 & -t & e^t\\ 1 & -2 & 1\end{pmatrix}, C(t) = \begin{pmatrix} 1 & t & e^{-t}\end{pmatrix}$

$L_0(t) = \begin{pmatrix} 1 & t & e^{-t}\end{pmatrix}$

$$L_1(t) = \begin{pmatrix} 1 & t & e^{-t}\end{pmatrix}\begin{pmatrix} 0 & 1 & t\\ t^2 & -t & e^t\\ 1 & -2 & 1\end{pmatrix} + \begin{pmatrix} 0 & 1 & -e^{-t}\end{pmatrix} = \begin{pmatrix} e^{-t}+t^3 & 2-t^2-2e^{-t} & t+te^t\end{pmatrix}$$

$$L_2(t) = \begin{pmatrix} e^{-t}+t^3 & 2-t^2-2e^{-t} & t+te^t\end{pmatrix}\begin{pmatrix} 0 & 1 & t\\ t^2 & -t & e^t\\ 1 & -2 & 1\end{pmatrix} + \begin{pmatrix} -e^{-t}+3t^2 & -2t+2e^{-t} & 1+e^t+te^t\end{pmatrix} =$$

$$\bigl(t-e^{-t}-t^2\left(2e^{-t}+t^2-2\right)+te^t+3t^2 \quad 3e^{-t}-4t+t\left(2e^{-t}+t^2-2\right)-2te^t+t^3 \quad t+e^t+t\left(e^{-t}+t^3\right)-e^t\bigl(2e^{-t}+$$

Hence

$$L = \begin{pmatrix} 1 & t & e^{-t}\\ e^{-t}+t^3 & 2-t^2-2e^{-t} & t+te^t\\ t-e^{-t}-t^2\left(2e^{-t}+t^2-2\right)+te^t+3t^2 & 3e^{-t}-4t+t\left(2e^{-t}+t^2-2\right)-2te^t+t^3 & t+e^t+t\left(e^{-t}+t^3\right)-e^t\left(2e^{-t}+t^2-2\right)+2te^t+1\end{pmatrix}$$

The rank of the above must be 3 for observable system.

Canonical decomposition theorem:

We want minimal realization. Canonical transformation: System $\sum = (A,B,C,D)$ can be transformed to equivalent system via non-singular matrix T.

$$\begin{aligned}A^{*} &= TAT^{-1}\\ B^{*} &= TB\\ C^{*} &= CT^{-1}\\ D^{*} &= D\end{aligned}$$

Which has the following structure. A^{*} is block structure of the form

$$A^{*}=\begin{pmatrix}A^{*}_{c\bar{o}} & A^{*}_{12} & A^{*}_{13}\\ 0 & A^{*}_{co} & A^{*}_{23}\\ 0 & 0 & A^{*}_{\bar{c}}\end{pmatrix},B^{*}=\begin{pmatrix}B^{*}_{c\bar{o}}\\ B^{*}_{co}\\ 0\end{pmatrix},C^{*}=\begin{pmatrix}0 & C^{*}_{co} & C^{*}_{\bar{c}}\end{pmatrix}$$

$A^{*}_{c\bar{o}}$ means controllable but not observable, A^{*}_{co} means controllable and observable and $A^{*}_{\bar{c}}$ means not controllable.

Hence, since $x^{*}=Tx$, then the $\overset{*}{\sum}$ is

$$\begin{aligned}x^{*\prime} &= A^{*}x^{*}+B^{*}u\left(t\right)\\ y &= C^{*}x^{*}+D^{*}u\left(t\right)\end{aligned}$$

So the controllable and observable part is $\sum_{co}=\left(A^{*}_{co},B^{*}_{co},C^{*}_{co}\right)$. And The not controllable and not observable part is $\sum_{\bar{c}\bar{o}}=\left(A^{*}_{\bar{c}\bar{o}},0,C^{*}_{\bar{c}}\right)$. Notice that

$$H^{*}_{co}\left(s\right)=H\left(s\right)$$

Minimal realization is both controllable and observable.

Original A,B,C,D can be anything, and we can always find T to do the above transformation.

1.28 Lecture 26. Thursday November 20 2014 (Handout, Kharitonov's Theorem)

CHAPTER 5

The Spark: Kharitonov's Theorem

Synopsis

This chapter is devoted to the seminal theorem of Kharitonov. The technical ideas underlying the proof serve as a pedagogical stepping stone for development of the more general value set concept which unifies many results in later chapters. In fact, the Kharitonov rectangle which we introduce is actually a value set corresponding to a rather specialized uncertainty structure.

5.1. Introduction

The main result in this chapter, Kharitonov's Theorem, addresses a rather specialized problem—robust stability of an interval polynomial family. The elegance of the solution immediately sets one's thought processes in motion; i.e., seeing such a dramatic breakthrough for the robust stability problem for interval polynomials, one cannot help but wonder what powerful results are possible for more general robustness problems. In a sense, most of the chapters to follow are testimonials to the new way of thinking which comes from the proof of Kharitonov's Theorem.

5.2. Independent Uncertainty Structures

In this section, we introduce the independent uncertainty structure. Results for this highly specialized structure should not be viewed as an end in itself. With this simpler theory under our belts, however, we are prepared to deal with more general polytopic and multilinear uncertainty structures in the chapters to follow.

Perhaps the most compelling motivation for the study of independent uncertainty structures is derived from the following scenario: An engineer generates a fixed model for a control system and obtains the associated characteristic polynomial $p(s)$. Although the presence of parametric uncertainty is acknowledged, the dependence on q is complicated and highly nonlinear. Despite the fact that the uncertainty structure is too complicated to analyze mathematically, it is still important to know something about the degree of robustness. In such cases, a sound argument can be made for imposition of an independent uncertainty structure. For example, using an independent uncertainty structure, we can use the theory in this chapter to determine what percentage variations in the coefficients of polynomial $p(s)$ can be tolerated.

It is also worth noting that in many cases, a more complicated uncertainty structure admits a certain type of overbounding by an independent uncertainty structure. Hence, once we have results for the independent case, we often obtain sufficient conditions for the more complicated case at hand. To illustrate, after overbounding a complicated uncertainty structure by an independent uncertainty structure, one might compute a robustness margin of 13% when the true robustness margin is 16%. It can be argued that the conservatism resulting from overbounding is not critical when the performance specification is still met.

DEFINITION 5.2.1 (Independent Uncertainty Structure): An uncertain polynomial

$$p(s,q) = \sum_{i=0}^{n} a_i(q)s^i$$

is said to have an *independent uncertainty structure* if each component q_i of q enters into only one coefficient.

EXERCISE 5.2.2 (Independent Uncertainty Structure): Does the uncertain polynomial

$$p(s,q) = s^3 + (q_1 + 4q_2 + 6)s^2 + (q_1 - 3q_4)s + (q_0 + 5)$$

have an independent uncertainty structure? Explain.

5.3. Interval Polynomial Family

In this section, we define interval polynomial families and the concept of lumping. By lumping, we mean combining uncertainties so as to obtain a description of the same family of polynomials involving a smaller number of uncertain parameters.

DEFINITION 5.3.1 (Interval Polynomial Family): A family of polynomials $\mathscr{P} = \{p(\cdot,q) : q \in Q\}$ is said to be an *interval polynomial family* if $p(s,q)$ has an independent uncertainty structure, each coefficient depends continuously on q and Q is a box. For brevity, we often drop the word "family" and simply refer to $\mathscr{P}$ as an *interval polynomial*.

EXAMPLE 5.3.2 (Simple Interval Polynomial): An interval polynomial family $\mathscr{P}$ arises from the uncertain polynomial described by $p(s,q) = (5+q_4)s^4 + (3+q_3)s^3 + (2+q_2)s^2 + (4+q_1)s + (6+q_0)$ with uncertainty bounds $|q_i| \leq 1$ for $i = 0,1,2,3,4$.

EXAMPLE 5.3.3 (Some Coefficients Fixed): Notice that the definition of interval polynomial does not rule out the possibility that some coefficients of $p(s,q)$ are fixed rather than uncertain; e.g., consider $p(s,q) = (5+q_4)s^4 + 3s^3 + (2+q_2)s^2 + (4+q_1)s + 6$ with a given box Q for the uncertainty bounding set.

EXAMPLE 5.3.4 (Lumping Interval Polynomials): The uncertainty representation often involves a certain type of redundancy. For example, if $p(s,q) = s^3 + (5+q_2+2q_3)s^2 + (6+2q_1+5q_4)s + (3+q_0)$ and bounds $|q_i| \leq 0.5$ for $i = 0,1,2,3,4$, one can "lump" the uncertainty as follows: Define new uncertain parameters $\tilde{q}_2 = 5+q_2+2q_3$, $\tilde{q}_1 = 6+2q_1+5q_4$ and $\tilde{q}_0 = 3+q_0$, a new uncertainty bounding set $\tilde{Q}$ by $2.5 \leq \tilde{q}_0 \leq 3.5$, $2.5 \leq \tilde{q}_1 \leq 9.5$ and $3.5 \leq \tilde{q}_2 \leq 6.5$ and a new uncertain polynomial $\tilde{p}(s,\tilde{q}) = s^3 + \tilde{q}_2 s^2 +$

;

$\tilde{q}_1 s+\tilde{q}_0$. We call $\tilde{\mathscr{P}} = \{\tilde{p}(\cdot,\tilde{q}) : \tilde{q} \in Q\}$ a *lumped version* of the original family $\mathscr{P}$ and leave it to the reader to verify that $\tilde{\mathscr{P}} = \mathscr{P}$.

EXERCISE 5.3.5 (Lumping with More Complicated Dependence): The objective of this exercise is to demonstrate that lumping is possible with more complicated dependence on q. To this end, consider an interval polynomial family $\mathscr{P}$ described by

$$p(s,q) = (5+e^{q_1}\cos q_2)s^2 + (\sin(q_3+q_4)+4)s + (q_5 q_6^2 + e^{q_7})$$

and $|q_i| \leq 1$ for $i = 1,2,\ldots,7$. Provide a characterization of a lumped version $\tilde{\mathscr{P}}$ of $\mathscr{P}$.

EXERCISE 5.3.6 (A Lumping Theorem): This exercise generalizes on the one above. Indeed, consider an interval polynomial family $\mathscr{P} = \{p(\cdot,q) : q \in Q\}$ with $p(s,q)$ having coefficients depending continuously on q. Prove that there exists a second interval polynomial family $\tilde{\mathscr{P}} = \{\tilde{p}(\cdot,\tilde{q}) : \tilde{q} \in \tilde{Q}\}$ with $\tilde{p}(s,\tilde{q})$ of the form $\tilde{p}(s,\tilde{q}) = \sum_{i=0}^{n} \tilde{q}_i s^i$ and, moreover, $\tilde{\mathscr{P}} = \mathscr{P}$.

5.4. Shorthand Notation

In view of the discussion of lumping above, we henceforth work with an uncertain polynomial of the form

$$p(s,q) = \sum_{i=0}^{n} q_i s^i$$

when dealing with an interval family. Such a family is completely described by the shorthand notation

$$p(s,q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i$$

with $[q_i^-, q_i^+]$ denoting the bounding interval for the i-th component of uncertainty q_i. In the context of this convenient abuse of notation, we can refer to $p(s,q)$ as an *interval polynomial*.

5.5. The Kharitonov Polynomials

In order to describe Kharitonov's Theorem for robust stability, we first define four fixed polynomials associated with an interval polynomial family $\mathscr{P}$. In the definition below, note that the polynomials are fixed in the sense that only the bounds q_i^- and q_i^+ enter into the description but not the q_i themselves. We also emphasize that the number of polynomials is four—independent of the degree of $p(s,q)$. That is, four is a magic number.

DEFINITION 5.5.1 (The Kharitonov Polynomials): Associated with the interval polynomial $p(s,q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i$ are the four fixed *Kharitonov polynomials*

$$K_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + q_6^+ s^6 + \cdots;$$

;

$$
\begin{aligned}
K_2(s) &= q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + q_6^- s^6 + \cdots;\\
K_3(s) &= q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + q_6^- s^6 + \cdots;\\
K_4(s) &= q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + q_6^+ s^6 + \cdots.
\end{aligned}
$$

EXAMPLE 5.5.2 (Construction of Kharitonov Polynomials): The Kharitonov polynomials are easily constructed by inspection. To illustrate, the four Kharitonov polynomials corresponding to the interval polynomial

$$p(s,q) = [1,2]s^5 + [3,4]s^4 + [5,6]s^3 + [7,8]s^2 + [9,10]s + [11,12]$$

are

$$
\begin{aligned}
K_1(s) &= 11 + 9s + 8s^2 + 6s^3 + 3s^4 + s^5;\\
K_2(s) &= 12 + 10s + 7s^2 + 5s^3 + 4s^4 + 2s^5;\\
K_3(s) &= 12 + 9s + 7s^2 + 6s^3 + 4s^4 + s^5;\\
K_4(s) &= 11 + 10s + 8s^2 + 5s^3 + 3s^4 + 2s^5.
\end{aligned}
$$

5.6. Kharitonov's Theorem

We now present the celebrated theorem of Kharitonov (1978a) and also illustrate its application. The proof of the theorem is relegated to the next two sections.

THEOREM 5.6.1 (Kharitonov (1978a)): *An interval polynomial family $\mathscr{P}$ with invariant degree is robustly stable if and only if its four Kharitonov polynomials are stable.*

EXAMPLE 5.6.2 (Application of Kharitonov's Theorem): For the interval polynomial

$$p(s,q) = [0.25, 1.25]s^3 + [2.75, 3.25]s^2 + [0.75, 1.25]s + [0.25, 1.25],$$

the four Kharitonov polynomials are

$$
\begin{aligned}
K_1(s) &= 0.25 + 0.75s + 3.25s^2 + 1.25s^3;\\
K_2(s) &= 1.25 + 1.25s + 2.75s^2 + 0.25s^3;\\
K_3(s) &= 1.25 + 0.75s + 2.75s^2 + 1.25s^3;\\
K_4(s) &= 0.25 + 1.25s + 3.25s^2 + 0.25s^3.
\end{aligned}
$$

Using the classical Hurwitz criterion, it is easy to verify that all four Kharitonov polynomials above are stable. Hence, we conclude that the interval polynomial family is robustly stable.

EXERCISE 5.6.3 (Application of Kharitonov's Theorem): Consider the interval polynomial family which is given in Example 5.5.2. Is it robustly stable?

5.7. Machinery for the Proof

For some readers, there is a temptation to skip sections containing technical proofs. For the case of Kharitonov's Theorem, however, the author's advice is to continue reading. The ideas introduced in this section and the next are at the heart of many generalizations presented in later chapters. In addition, the geometrical ideas in the proof suggest ideas for computer-aided analysis. Most notably, the proof makes use of the so-called Kharitonov rectangle. This rectangle is in fact a special type of "value set" which plays a major role in later chapters.

5.7.1. The Kharitonov Rectangle

In this subsection, we consider an elementary geometry problem: Given an interval polynomial $p(s,q) = \sum_{i=0}^{n}[q_i^-,q_i^+]s^i$ and a fixed frequency $\omega = \omega_0$, describe the set of possible values that $p(j\omega_0,q)$ can assume as q ranges over the box Q. More formally, we want to describe the subset of the complex plane given by

$$p(j\omega_0,Q) = \{p(j\omega_0,q) : q \in Q\}.$$

We call $p(j\omega_0,Q)$ the *Kharitonov rectangle* at frequency $\omega = \omega_0$. To justify this name, we now prove that $p(j\omega_0,Q)$ is a rectangle with vertices which are obtained by evaluating the four *fixed* Kharitonov polynomials $K_1(s)$, $K_2(s)$, $K_3(s)$ and $K_4(s)$ at $s = j\omega_0$; i.e., the vertices of $p(j\omega_0,Q)$ are precisely the $K_i(j\omega_0)$.

To establish rectangularity, we examine the real and imaginary parts of $p(j\omega_0,q)$. Indeed, we first observe that

$$Re\; p(j\omega_0,q) = \sum_{i\;even} q_i(j\omega_0)^i = q_0 - q_2\omega_0^2 + q_4\omega_0^4 - q_6\omega_0^6 + q_8\omega_0^8 - \cdots$$

and

$$Im\; p(j\omega_0,q) = \frac{1}{j}\sum_{i\;odd} q_i(j\omega_0)^i = q_1\omega_0 - q_3\omega_0^3 + q_5\omega_0^5 - q_7\omega_0^7 + q_9\omega_0^9 - \cdots.$$

Notice that no q_i which enters $Re\; p(j\omega_0,q)$ enters $Im\; p(j\omega_0,q)$ and vice versa. In view of this decoupling between real and imaginary parts, the set $p(j\omega_0,Q)$ consists of all complex numbers z such that

$$Re\, z = q_0 - q_2\omega_0^2 + q_4\omega_0^4 - q_6\omega_0^6 + q_8\omega_0^8 - \cdots$$

for some admissible $q \in Q$ and

$$Im\, z = q_1\omega_0 - q_3\omega_0^3 + q_5\omega_0^5 - q_7\omega_0^7 + q_9\omega_0^9 - \cdots$$

for some admissible $q \in Q$.

We now argue that the set of all generatable pairs $(Re\, z, Im\, z)$ above is a rectangle which is obtained by finding the minimum and maximum values of $Re\; p(j\omega_0,q)$ and $Im\; p(j\omega_0,q)$ with respect to $q \in Q$. Indeed, since each q_i enters only one coefficient of $p(s,q)$, for $Re\, p(j\omega_0,q)$, we can minimize or maximize each term individually to obtain

$$\begin{aligned}\min_{q\in Q} Re\, p(j\omega_0,q) &= q_0^- - q_2^+\omega_0^2 + q_4^-\omega_0^4 - q_6^+\omega_0^6 + q_8^-\omega_0^8 - \cdots \\ &= Re\, K_1(j\omega_0)\end{aligned}$$

and

$$\begin{aligned}\max_{q\in Q} Re\, p(j\omega_0,q) &= q_0^+ - q_2^-\omega_0^2 + q_4^+\omega_0^4 - q_6^-\omega_0^6 + q_8^+\omega_0^8 - \cdots \\ &= Re\, K_2(j\omega_0).\end{aligned}$$

As far as $Im\, p(j\omega_0,q)$ is concerned, one must pay attention to the sign of ω_0 in deciding whether to use q_i^- or q_i^+ when minimizing or maximizing. Keeping this issue in mind, for $\omega_0 \geq 0$, we obtain

$$\min_{q\in Q} Im\, p(j\omega_0,q) = q_1^-\omega_0 - q_3^+\omega_0^3 + q_5^-\omega_0^5 - q_7^+\omega_0^7 + \cdots$$

and similarly, for $\omega_0 < 0$,

$$\min_{q\in Q} Im\, p(j\omega_0,q) = q_1^+\omega_0 - q_3^-\omega_0^3 + q_5^+\omega_0^5 - q_7^-\omega_0^7 + \cdots.$$

Combining these two cases, we arrive at

$$\min_{q\in Q} Im\, p(j\omega_0,q) = \begin{cases} Im\, K_3(j\omega_0) & \text{if } \omega_0 \geq 0; \\ Im\, K_4(j\omega_0) & \text{if } \omega_0 < 0. \end{cases}$$

For the maximization problem, the same type of reasoning leads to

$$\max_{q\in Q} Im\, p(j\omega_0,q) = \begin{cases} Im\, K_4(j\omega_0) & \text{if } \omega_0 \geq 0; \\ Im\, K_3(j\omega_0) & \text{if } \omega_0 < 0. \end{cases}$$

Thus far, our arguments indicate that $p(j\omega_0,Q)$ is bounded by the rectangle given in Figure 5.7.1; i.e., if $z \in p(j\omega_0,Q)$ and $\omega_0 \geq 0$, then

$$Re\, K_1(j\omega_0) \leq Re\, z \leq Re\, K_2(j\omega_0);$$

$$Im\, K_3(j\omega_0) \leq Im\, z \leq Im\, K_4(j\omega_0).$$

To complete the argument, we now claim that this bounding rectangle is precisely equal to $p(j\omega_0,Q)$. That is, every value in this rectangle is realizable by some $q \in Q$. Indeed, by viewing $Re\, p(j\omega_0,q)$ as a mapping of $(q_0,q_2,q_4,\ldots)$ to $\mathbf{R}$ and $Im\, p(j\omega_0,q)$ as a mapping from $(q_1,q_3,q_5,\ldots)$ to $\mathbf{R}$, a simple intermediate value argument guarantees that for each z satisfying the two inequalities above, there exists some uncertainty $q_z \in Q$ such that $p(j\omega_0,q_z) = z$. In summary, the set $p(j\omega_0,Q)$ is precisely the rectangle depicted in Figure 5.7.1.

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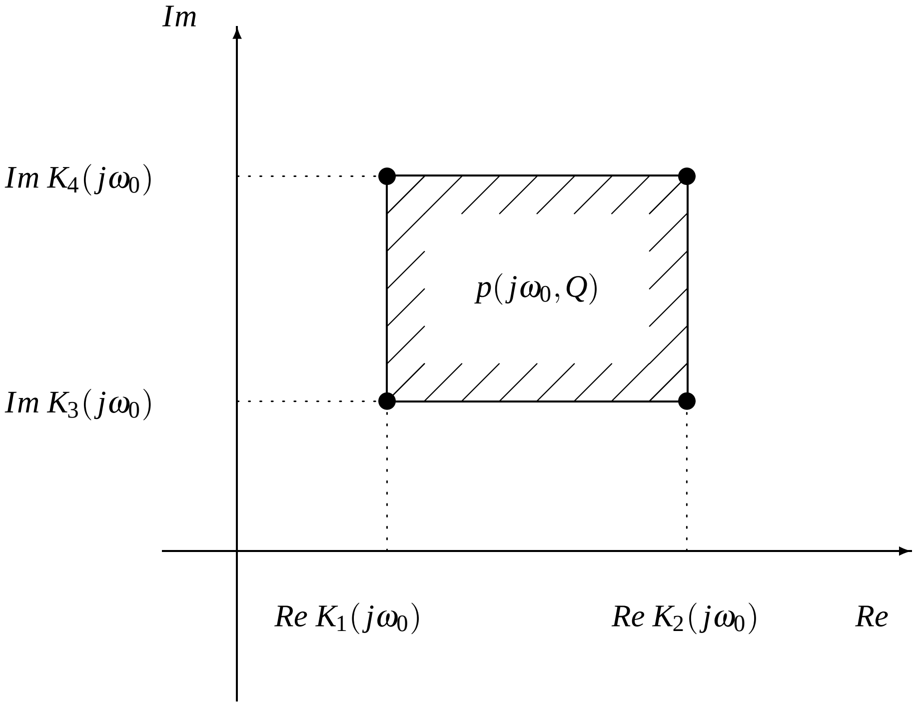

Figure 5.7.1. The Kharitonov Rectangle for $\omega_0 \geq 0$

We now relate the vertices of the rectangle $p(j\omega_0,Q)$ to the Kharitonov polynomials:

$$\begin{aligned}\text{Southwest Vertex} &= Re\,K_1(j\omega_0) + jIm\,K_3(j\omega_0)\\ &= Re\,K_1(j\omega_0) + jIm\,K_1(j\omega_0)\\ &= K_1(j\omega_0);\end{aligned}$$

$$\begin{aligned}\text{Northeast Vertex} &= Re\,K_2(j\omega_0) + jIm\,K_4(j\omega_0)\\ &= Re\,K_2(j\omega_0) + jIm\,K_2(j\omega_0)\\ &= K_2(j\omega_0);\end{aligned}$$

$$\begin{aligned}\text{Southeast Vertex} &= Re\,K_2(j\omega_0) + jIm\,K_3(j\omega_0)\\ &= Re\,K_3(j\omega_0) + jIm\,K_3(j\omega_0)\\ &= K_3(j\omega_0);\end{aligned}$$

$$\begin{aligned}\text{Northwest Vertex} &= Re\,K_1(j\omega_0) + jIm\,K_4(j\omega_0)\\ &= Re\,K_4(j\omega_0) + jIm\,K_4(j\omega_0)\\ &= K_4(j\omega_0).\end{aligned}$$

This leads to our final depiction of the *Kharitonov rectangle* given in Figure 5.7.2. The key point to note is that each vertex is associated with a unique Kharitonov polynomial.

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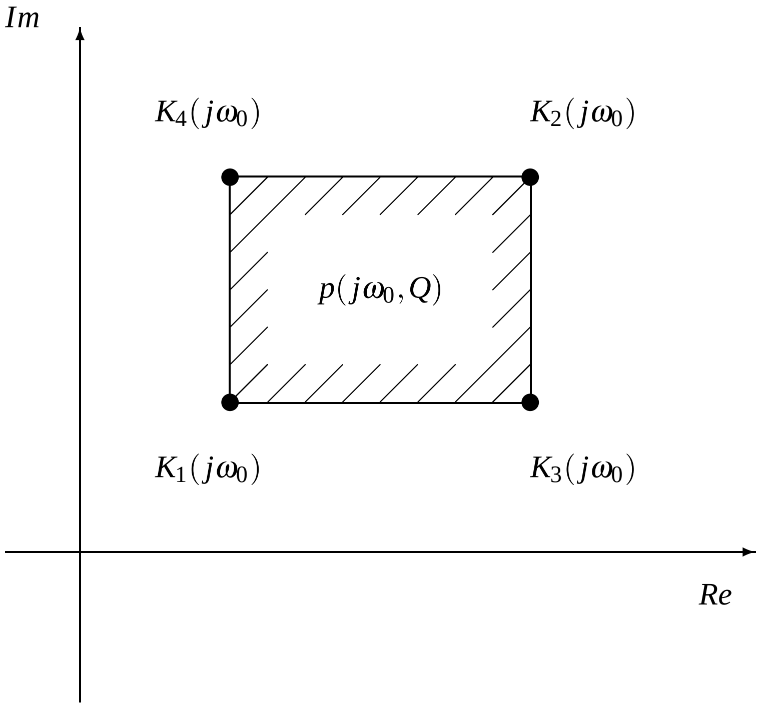

Figure 5.7.2. Simplified Kharitonov Rectangle for $\omega_0 \geq 0$

EXERCISE 5.7.2 (Kharitonov Rectangle for $\omega_0 < 0$ and $\omega_0 = 0$): Sketch the Kharitonov rectangle $p(j\omega_0,Q)$ for $\omega_0 < 0$ with vertices carefully labeled. For $\omega_0 = 0$, notice that $p(j\omega_0,Q) = [q_0^-,q_0^+]$.

REMARKS 5.7.3 (Motion of Kharitonov Rectangle): Thus far, the discussion of the Kharitonov rectangle has been in the context of a frozen frequency $\omega = \omega_0$. We now entertain the notion of sweeping the frequency. Indeed, we begin at $\omega = 0$ and imagine ω increasing. This results in motion of the Kharitonov rectangle. That is, we have a rectangle moving around the complex plane with vertices $K_i(j\omega)$ obtained by evaluation of the Kharitonov polynomials. Generally, the dimensions of this rectangle vary with the frequency ω.

EXAMPLE 5.7.4 (Illustration of Motion): For the interval polynomial

$$p(s,q) = [0.25,1.25]s^3 + [2.75,3.25]s^2 + [0.75,1.25]s + [0.25,1.25]$$

which we analyzed in Example 5.6.2, we illustrate the motion of the Kharitonov rectangle $p(j\omega,Q)$ in Figure 5.7.3 for twenty frequencies evenly spaced between $\omega = 0$ and $\omega = 1$. Notice that this rectangle begins at $\omega = 0$ as an interval on the positive real axis and then moves from the first to the second quadrant as ω is increased.

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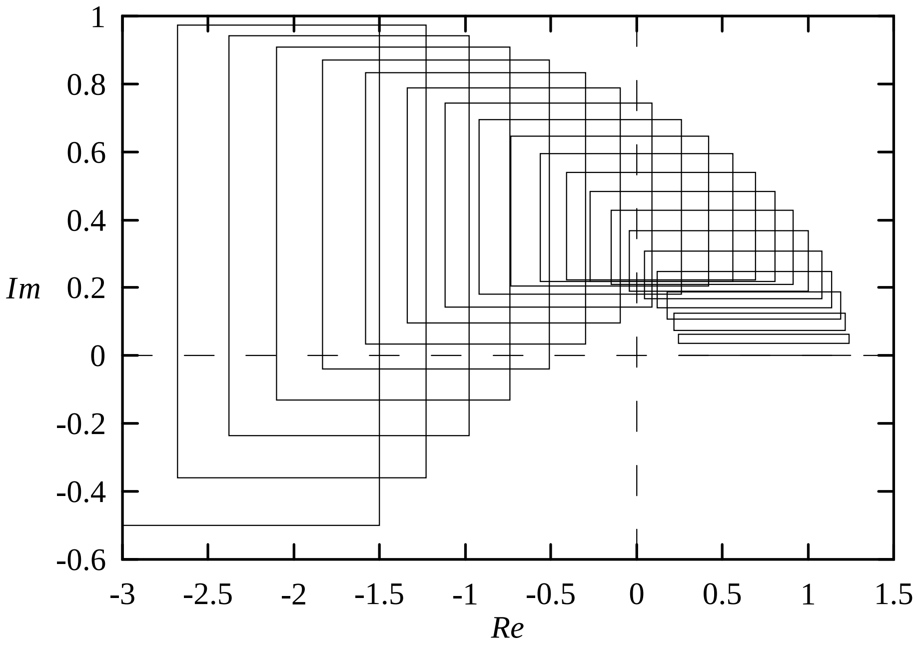

Figure 5.7.3. Motion of Kharitonov Rectangle for Example 5.7.4

5.7.5. Angle Considerations

In this subsection, we review some basic facts about the angle of a polynomial as a function of frequency. We include the proof of the well-known lemma below because the underlying ideas are useful in later chapters. In a control setting, the lemma below is often credited to Mikhailov (1938).

LEMMA 5.7.6 (Monotonic Angle Property): *Suppose that $p(s)$ is a stable polynomial. Then the angle of $p(j\omega)$ is a strictly increasing function of $\omega \in \mathbf{R}$. Furthermore, as ω varies from 0 to $+\infty$, $\measuredangle p(j\omega)$ experiences an increment of $n\pi/2$.*

PROOF: First, we write $p(s) = K\prod_{i=1}^{n}(s-z_i)$, where $K \in \mathbf{R}$ and $Re\, z_i < 0$ for $i = 1,2,\ldots,n$. The angle of $p(j\omega)$ is given by

$$\measuredangle p(j\omega) = \measuredangle K + \sum_{i=1}^{n} \measuredangle (j\omega - z_i).$$

With $\theta_i(\omega) = \measuredangle (j\omega - z_i)$ and the aid of Figure 5.7.4, we make the following observations, noting that z_i lies in the strict left half plane: If z_i is purely real, then as ω varies from 0 to $+\infty$, $\theta_i(\omega)$ is strictly increasing and experiences a net increment of $\pi/2$. If z_i is complex, we work with z_i in combination with its conjugate z_i^*. Now, as ω increases from 0 to $+\infty$, the corresponding angles $\theta_i(\omega)$ are strictly increasing and contribute a net increment total of π. The proof of the lemma is completed by summing over the

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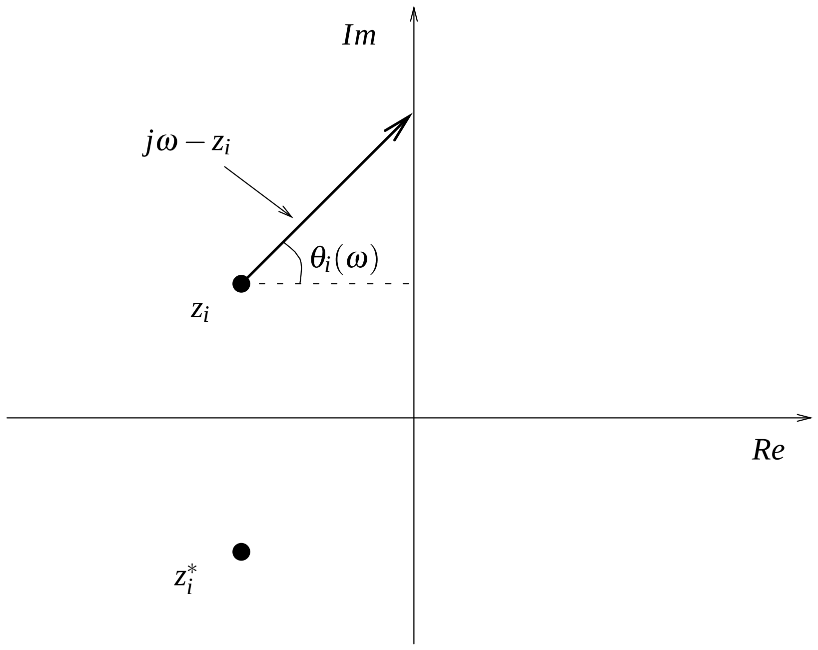

Figure 5.7.4. $\theta_i(\omega)$ is a Strictly Increasing Function of ω

$\theta_i(\omega)$. ∎

EXERCISE 5.7.7 (More General Angle Considerations): Suppose $p(s)$ is an n-th order polynomial with n_1 roots in the strict left half plane and n_2 roots in the strict right half plane. Assume that $n_1 + n_2 = n$ and show that as ω varies from 0 to $+\infty$, $\measuredangle p(j\omega)$ experiences a total change in angle of $(n_1 - n_2)\pi/2$. Also modify the result to allow for the case when $p(s)$ has some roots on the imaginary axis.

5.7.8. The Zero Exclusion Condition

In this subsection, we introduce the Zero Exclusion Condition. The technical ideas associated with this condition arise time and time again throughout the remainder of this text. Since we are currently working within the framework of interval polynomials, the lemma below is not stated in full generality; the most general version which we provide is given in Theorem 7.4.2. In addition to facilitating the proof of Kharitonov's Theorem, the lemma below is also of practical use because it suggests a simple test for robust stability which is easy to implement in graphics.

LEMMA 5.7.9 (Zero Exclusion Condition): *Suppose that an interval polynomial family $\mathscr{P} = \{p(\cdot, q) : q \in Q\}$ has invariant degree and at least one stable member $p(s, q^0)$. Then $\mathscr{P}$ is robustly stable if and only if $z = 0$ is excluded from the Kharitonov rectangle at all nonnegative frequencies; i.e.,*

$$0 \notin p(j\omega, Q)$$

for all frequencies $\omega \geq 0$.

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PROOF: We first justify the restriction to nonnegative frequencies. To this end, note that $z \in p(j\omega, Q)$ if and only if $z^* \in p(-j\omega, Q)$. Hence, without loss of generality, we restrict our attention to $\omega \geq 0$.

To establish necessity, we assume that $\mathscr{P}$ is robustly stable and must prove that $0 \notin p(j\omega, Q)$ for all $\omega \in \mathbf{R}$. Proceeding by contradiction, suppose that $0 \in p(j\omega^*, Q)$ for some frequency $\omega^* \in \mathbf{R}$. Then $p(j\omega^*, q^*) = 0$ for some $q^* \in Q$; i.e., the polynomial $p(s, q^*)$ has a root at $s = j\omega^*$ which contradicts robust stability of $\mathscr{P}$.

To establish sufficiency, we assume that $0 \notin p(j\omega, Q)$ for all $\omega \in \mathbf{R}$ and must show that $\mathscr{P}$ is robustly stable. Proceeding by contradiction, if $\mathscr{P}$ is not robustly stable, then $p(s, q^1)$ is unstable for some $q^1 \in Q$. Now, for $\lambda \in [0, 1]$, let

$$\tilde{p}(s, \lambda) = p(s, \lambda q^1 + (1 - \lambda) q^0)$$

and notice that $\tilde{p}(s, \lambda) \in \mathscr{P}$ because $\lambda q^1 + (1 - \lambda) q^0 \in Q$. Moreover, for $\lambda = 0$, $\tilde{p}(s, 0) = p(s, q^0)$ has all roots in the strict left half plane and for $\lambda = 1$, $\tilde{p}(s, 1) = p(s, q^1)$ has at least one root in the closed right half plane. Since the roots of $\tilde{p}(s, \lambda)$ depend continuously on λ (Lemma 4.8.2), there exists a $\lambda^* \in [0, 1]$ such that $\tilde{p}(s, \lambda^*)$ has a root on the imaginary axis. Equivalently, $p(j\omega^*, \lambda^* q^1 + (1 - \lambda^*) q^0) = 0$ for some $\omega^* \in \mathbf{R}$. This implies that $0 \in p(j\omega^*, Q)$, which is the contradiction we seek. ∎

REMARKS 5.7.10 (Real Versus Complex Coefficients): When working with the Zero Exclusion Condition for the complex coefficient case, we can no longer restrict attention to $\omega \geq 0$; i.e., we cannot exploit the fact that $z \in p(j\omega, Q)$ if and only if $z^* \in p(-j\omega, Q)$. In this case, the lemma above requires a minor modification: Under the standing hypotheses, $\mathscr{P}$ is robustly stable if and only if $0 \notin p(j\omega, Q)$ for all $\omega \in \mathbf{R}$. This arises in Chapter 6 when we consider the complex coefficient version of Kharitonov's Theorem.

5.8. Proof of Kharitonov's Theorem

The proof of necessity is trivial; i.e., if $\mathscr{P}$ is robustly stable, it follows that the four Kharitonov polynomials are stable because $K_i(s) \in \mathscr{P}$ for $i = 1, 2, 3, 4$. To establish sufficiency, we assume that the four Kharitonov polynomials are stable and must prove that $\mathscr{P}$ is robustly stable. Proceeding by contradiction, suppose that $\mathscr{P}$ is not robustly stable. Using the standard notation $p(s, q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i$, we consider two cases.

Case 1: $0 \in [q_0^-, q_0^+]$. Recalling the invariant degree assumption, it must be true that q_n^- and q_n^+ have the same sign. Without loss of generality, say that the signs of q_n^- and q_n^+ are positive. Then it follows that at least one of the four Kharitonov polynomials, call it $K_{i^*}(s)$, has coefficient of s^n, which is positive, and coefficient of s^0, which is nonpositive. This contradicts the assumed stability of $K_{i^*}(s)$ because a stable polynomial must have nonzero coefficients which all have the same sign.

Case 2: $0 \notin [q_0^-, q_0^+]$. Since $p(j0, Q) = [q_0^-, q_0^+]$ is the Kharitonov rectangle at $\omega = 0$, we have $0 \notin p(j0, Q)$. On the other hand, since $\mathscr{P}$ is not robustly stable, we know by the Zero Exclusion Condition (Lemma 5.7.9) that $0 \in p(j\omega^*, Q)$ for some $\omega^* \in \mathbf{R}$. Now, using the fact that $0 \notin p(j0, Q)$, the continuous motion of the vertices $K_i(j\omega)$ of $p(j\omega, Q)$ guarantees that there must be some frequency $\hat{\omega} > 0$ for which $z = 0$ pierces

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the boundary of the rectangle $p(j\hat{\omega},Q)$. Without loss of generality, assume that this piercing occurs on the southern boundary of $p(j\hat{\omega},Q)$ as shown in Figure 5.8.1. Also,

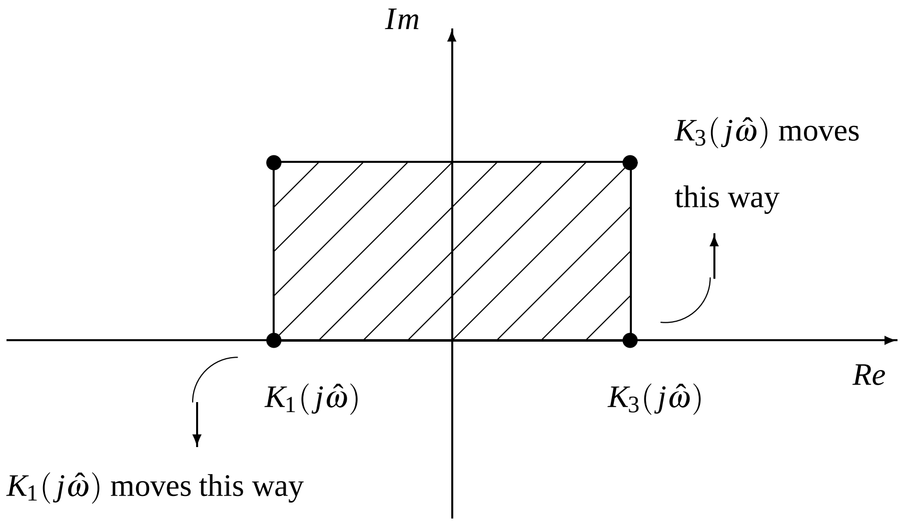

Figure 5.8.1. Piercing the Boundary of the Kharitonov Rectangle

note that $z=0$ cannot be coincident with $K_1(j\hat{\omega})$ or $K_3(j\hat{\omega})$ because $K_1(s)$ and $K_3(s)$ are assumed stable. To complete the proof, we exploit continuity of the $K_i(j\omega)$ and the Monotonic Angle Property (Lemma 5.7.6). Namely, for $\delta\hat{\omega}>0$ suitably small, it follows that

$$0^0 < \measuredangle K_3(j(\hat{\omega}+\delta\hat{\omega})) < 90^0$$

and

$$180^0 < \measuredangle K_1(j(\hat{\omega}+\delta\hat{\omega})) < 270^0.$$

We now have the contradiction which we seek because simultaneous satisfaction of the two angle inequalities above makes it impossible for the southern boundary of the rectangle $p(j(\hat{\omega}+\delta\hat{\omega}),Q)$ to remain parallel to the real axis. ∎

5.9. Formula for the Robustness Margin

For an interval polynomial family, by combining the results of this chapter with those of Chapter 4, we obtain the robustness margin formulas of Fu and Barmish (1988). To this end, we describe an n-th order interval polynomial family with stable nominal $p_0(s)$ and variable uncertainty bound $r\geq 0$ by writing

$$p_r(s,q) = p_0(s) + r\sum_{i=0}^{n-1}[-\varepsilon_i,\varepsilon_i]s^i.$$

We view the $\varepsilon_i \geq 0$ above as scale factors which determine the aspect ratios of the uncertainty bounding set Q_r. Letting $\mathscr{P}_r$ denote the resulting family of polynomials, our objective is to provide a formula for the robustness margin

$$r_{max} = \sup\{r : \mathscr{P}_r \text{ is robustly stable}\}.$$

To obtain the desired formula, we first argue that Kharitonov's Theorem enables us to reduce the robustness margin problem to four separate problems for the uncertain polynomials $\{p_0(s)+qp_{1,i}(s)\}_{i=1}^{4}$, where

$$p_{1,1}(s) = -\varepsilon_0 - \varepsilon_1 s + \varepsilon_2 s^2 + \varepsilon_3 s^3 - \varepsilon_4 s^4 - \varepsilon_5 s^5 + \varepsilon_6 s^6 + \cdots;$$

$$p_{1,2}(s) = \varepsilon_0 + \varepsilon_1 s - \varepsilon_2 s^2 - \varepsilon_3 s^3 + \varepsilon_4 s^4 + \varepsilon_5 s^5 - \varepsilon_6 s^6 - \cdots;$$

$$p_{1,3}(s) = \varepsilon_0 - \varepsilon_1 s - \varepsilon_2 s^2 + \varepsilon_3 s^3 + \varepsilon_4 s^4 - \varepsilon_5 s^5 - \varepsilon_6 s^6 + \cdots;$$

$$p_{1,4}(s) = -\varepsilon_0 + \varepsilon_1 s + \varepsilon_2 s^2 - \varepsilon_3 s^3 - \varepsilon_4 s^4 + \varepsilon_5 s^5 + \varepsilon_6 s^6 - \cdots.$$

Now, applying Theorem 4.7.6 and taking the worst case with respect to $i = 1,2,3,4$, we arrive at the formula

$$r_{max} = \min_{i\leq 4} \frac{1}{\lambda_{max}^{+}(-H^{-1}(p_0)H(p_{1,i}))}.$$

5.10. Robust Stability Testing via Graphics

The Zero Exclusion Condition (see Lemma 5.7.9) suggests a simple graphical procedure for checking robust stability—watch the motion of the Kharitonov rectangle $p(j\omega,Q)$ as ω varies from 0 to $+\infty$ and determine by inspection if the condition $0 \notin p(j\omega,Q)$ is satisfied. This raises the following question: Can we find some finite precomputable *cutoff frequency* $\omega_c > 0$ such that $0 \notin p(j\omega,Q)$ for all $\omega \geq \omega_c$? That is, can we terminate the frequency sweep at the frequency $\omega = \omega_c$?

The existence of ω_c is easily established using the invariant degree condition. Indeed, suppose that $p(s,q) = \sum_{i=0}^{n}[q_i^-,q_i^+]s^i$ and, without loss of generality, assume that $q_i^- > 0$ for $i = 0,1,\ldots,n$. Then given any $q \in Q$, it is easy to see that for $\omega \geq 0$,

$$|p(j\omega,q)| \geq q_n^-\omega^n - \sum_{i=0}^{n-1} q_i^+\omega^i.$$

Since the right-hand side tends to $+\infty$ as $\omega \to +\infty$, it follows that for any prescribed $\beta > 0$ there exists an $\omega_c > 0$ such that $|p(j\omega,q)| \geq \beta$ for all $\omega > \omega_c$. Hence, $0 \notin p(j\omega,Q)$ for all $\omega > \omega_c$.

In fact, we can easily compute an appropriate ω_c. For example, one can take ω_c to be the largest real root of the polynomial

$$f(\omega) = q_n^-\omega^n - \sum_{i=1}^{n-1} q_i^+\omega^i.$$

Other possibilities for estimating ω_c (often less conservatively) are suggested from classical bounds on the roots of a polynomial. For example, in Marden (1966), it is seen that the roots of a fixed positive coefficient polynomial $p(s) = \sum_{i=0}^{n} a_i s^i$ lie in a disc of radius

$$R = 1 + \frac{\max\{a_0,a_1,\ldots,a_{n-1}\}}{a_n}.$$

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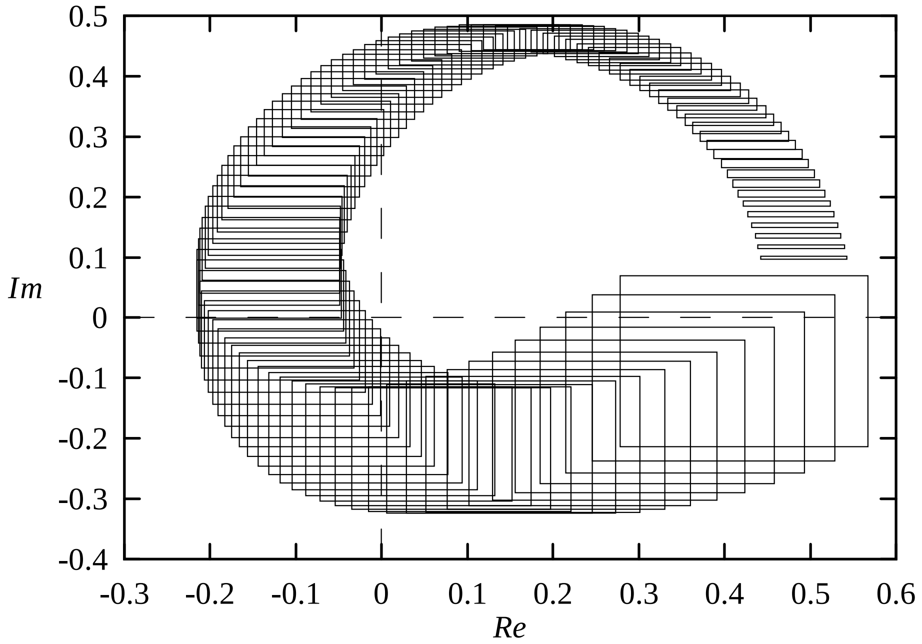

Figure 5.10.1. Graphical Robust Stability Test for Example 5.10.1

Hence, for the interval polynomial $p(s,q)$, with $q_n^- > 0$, it follows that an appropriate cutoff frequency is given by

$$\omega_c = 1 + \frac{\max\{q_0^+, q_1^+, \ldots, q_{n-1}^+\}}{q_n^-}.$$

EXAMPLE 5.10.1 (Illustration of Graphics Method): We consider the interval polynomial family $\mathscr{P} = \{p(\cdot,q) : q \in Q\}$ described by

$$\begin{aligned} p(s,q) \;=\; & s^6 + [3.95, 4.05]s^5 + [3.95, 4.05]s^4 + [5.95, 6.05]s^3 \\ & + [2.95, 3.05]s^2 + [1.95, 2.05]s + [0.45, 0.55]. \end{aligned}$$

In accordance with Lemma 5.7.9, the first step in the graphical test for robust stability requires that we guarantee that at least one polynomial in $\mathscr{P}$ is stable. Using the midpoint of each interval above, we obtain $p(s,q^0) = s^6 + 4s^5 + 4s^4 + 6s^3 + 3s^2 + 2s + 0.5$, whose roots are $s_1 \approx -3.2681$, $s_{2,3} \approx -0.1328 \pm 0.9473j$, $s_{4,5} \approx -0.0731 \pm 0.7190j$ and $s_6 \approx -0.3201$.

Next, in accordance with the discussion of cutoff frequencies above, we compute the largest real root of the test polynomial $f(\omega)$; that is, with

$$f(\omega) = \omega^6 - 4.05\omega^5 - 4.05\omega^4 - 6.05\omega^3 - 3.05\omega^2 - 2.05\omega - 0.55,$$

we obtain $\omega_c \approx 5.1023$ as an acceptable cutoff frequency for the required Kharitonov rectangle plot. In Figure 5.10.1, we provide a “zoom” of the required plot using 100 evenly spaced frequencies in the critical range $0 \leq \omega \leq 1$. For ω in this range, the Kharitonov rectangle makes its closest approach to $z = 0$. Since $0 \notin p(j\omega, Q)$, we conclude that the family of polynomials $\mathscr{P}$ is robustly stable.

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5.11. Overbounding via Interval Polynomials

As mentioned in the introduction to this chapter, the independent uncertainty structure is restrictive because uncertain parameters typically enter into more than one coefficient. For such "dependent" uncertainty structures, we consider two alternatives: The first alternative is to develop more general results; this is the topic of later chapters. The second alternative is the so-called *overbounding method*, which is described below. One warning, however, is in order: Although the overbounding method is easy to use, it may lead to unduly conservative results; i.e., we only obtain sufficient conditions for robustness. In short, associated with overbounding is a trade-off between ease of use and degree of conservatism.

In the remainder of this section, we no longer require the polynomials $p(s,q)$ to have an independent uncertainty structure, and, in addition, Q is not necessarily a box. We begin with the uncertain polynomial $p(s,q) = \sum_{i=0}^{n} a_i(q)s^i$ and an uncertainty bounding set Q which is closed and bounded. Assuming the coefficient functions $a_i(q)$ depend continuously on q, we define the bounds

$$\overline{q}_i^- = \min_{q \in Q} a_i(q)$$

and

$$\overline{q}_i^+ = \max_{q \in Q} a_i(q)$$

and simply observe that the family of polynomials $\overline{\mathscr{P}}$ described by

$$\overline{p}(s,\overline{q}) = \sum_{i=0}^{n} [\overline{q}_i^-, \overline{q}_i^+] s^i$$

is a superset of $\mathscr{P}$. Therefore, any robustness property which holds for the interval polynomial family $\overline{\mathscr{P}}$ must hold for $\mathscr{P}$. In particular, robust stability of $\overline{\mathscr{P}}$ implies robust stability of $\mathscr{P}$. Note, however, that the converse is not true. These points are illustrated via the examples below.

EXAMPLE 5.11.1 (Success of Overbounding): Consider the family of polynomials $\mathscr{P}$ described by

$$\begin{aligned} p(s,q) = s^4 &+ (5 + 0.2q_1q_2 + 0.1q_1 - 0.1q_2)s^3 + (6 + 3q_1q_2 - 4q_2)s^2 \\ &+ (6 + 6q_1 - 8q_2)s + (0.5 - 3q_1q_2) \end{aligned}$$

and uncertainty bound $|q_i| \leq 0.25$ for $i = 1,2$. The objective is to determine whether $\mathscr{P}$ is robustly stable. To this end, we compute bounds

$$\begin{aligned} \overline{q}_0^- &= \min_{q \in Q} a_0(q) = \min_{-0.25 \leq q_i \leq 0.25} (0.5 - 3q_1q_2) = 0.3125; \\ \overline{q}_0^+ &= \max_{q \in Q} a_0(q) = \max_{-0.25 \leq q_i \leq 0.25} (0.5 - 3q_1q_2) = 0.6875; \\ \overline{q}_1^- &= \min_{q \in Q} a_1(q) = \min_{-0.25 \leq q_i \leq 0.25} (6 + 6q_1 - 8q_2) = 2.5; \\ \overline{q}_1^+ &= \max_{q \in Q} a_1(q) = \max_{-0.25 \leq q_i \leq 0.25} (6 + 6q_1 - 8q_2) = 9.5. \end{aligned}$$

;

Similar computations yield $\overline{q}_2^- = 4.8125, \overline{q}_2^+ = 7.1875, \overline{q}_3^- = 4.9475$ and $\overline{q}_3^+ = 5.0375$. Hence, an interval polynomial family $\overline{\mathscr{P}}$ used for overbounding is described by

$$\begin{aligned}\overline{p}(s,\overline{q}) \;=\; & s^4 + [4.9475, 5.0375]s^3 + [4.8125, 7.1875]s^2 \\ & + [2.5, 9.5]s + [0.3125, 0.6875].\end{aligned}$$

By applying Kharitonov's Theorem to the overbounding family $\overline{\mathscr{P}}$ above, it is straightforward to verify that the four Kharitonov polynomials are stable. Hence, from the robust stability of $\overline{\mathscr{P}}$, we conclude that the original family $\mathscr{P}$ must also be robustly stable.

EXERCISE 5.11.2 (Failure of Overbounding): In this exercise, the objective is to illustrate how overbounding can fail. To this end, consider the family of polynomials $\mathscr{P}$ given in Wei and Yedavalli (1989); i.e., the family $\mathscr{P}$ is described by

$$p(s,q) = s^4 + s^3 + 2qs^2 + s + q$$

with uncertainty bounding set $Q = [1.5, 4]$. Argue that $\mathscr{P}$ is robustly stable but the overbounding family

$$\overline{p}(s,\overline{q}) = s^4 + s^3 + [3,8]s^2 + s + [1.5, 4]$$

has an unstable Kharitonov polynomial.

5.12. Conclusion

In a sense, Kharitonov's Theorem raises more questions than it answers. To illustrate the type of questions suggested by Kharitonov's Theorem, we consider the robust Schur stability problem for an interval polynomial family $\mathscr{P}$: Indeed, if the associated four Kharitonov polynomials have all their roots in the interior of the unit disc, does it follow that $\mathscr{P}$ is robustly Schur stable? If not, does it suffice to test polynomials associated with all the vertices of Q? More generally, for what type of root location regions does a Kharitonov-like extreme point result hold? The list of possible questions seems endless. In Chapter 13, we characterize classes of $\mathscr{D}$ regions for which $\mathscr{D}$-stability of the polynomials associated with the extreme points of the Q box implies robust $\mathscr{D}$-stability of the associated interval polynomial family.

Notes and Related Literature

NRL 5.1 The paper by Faedo (1953) appears to have provided important motivation for Kharitonov's work.

NRL 5.2 Kharitonov's original proof is based on the Hermite–Biehler Theorem; e.g., see Gantmacher (1959). Indeed, consider a polynomial $p(s)$ decomposed into even and odd parts $p(s) = p_{even}(s^2) + sp_{odd}(s^2)$. Then, according to the Hermite–Biehler Theorem, $p(s)$ is stable if and only if $p_{even}(x)$ and $p_{odd}(x)$ have highest order coefficients of the same sign and negative real

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distinct interlacing roots; e.g., if polynomial $p(s)$ has odd degree and $x_{e,1} < x_{e,2} < \cdots < x_{e,m}$ and $x_{o,1} < x_{o,2} < \cdots < x_{o,m}$ are the roots of $p_{even}(x)$ and $p_{odd}(x)$, respectively, then $x_{o,1} < x_{e,1} < x_{o,2} < x_{e,2} < \cdots < x_{o,m} < x_{e,m}$. The key idea behind the original proof of Kharitonov's Theorem is as follows: Given an interval polynomial family $\mathscr{P}$, one creates "root intervals" for the even and odd parts. The endpoints of these intervals are associated with the Kharitonov polynomials. Subsequently, it is argued that satisfaction of the root interlacing condition for each Kharitonov polynomial implies satisfaction of the root interlacing condition for the entire family. The Hermite–Biehler line of attack is not pursued in this text because we want to explain as many results as possible within the unifying framework of value sets. The Kharitonov rectangle is in fact an example illustrating the more general value set concept of Chapter 7.

NRL 5.3 The key ideas underlying our proof of Kharitonov's Theorem come from Dasgupta (1988) and Minnichelli, Anagnost and Desoer (1989). More specifically, we note that Dasgupta (1988) exposes the rectangular geometry of $p(j\omega, Q)$ and Minnichelli, Anagnost and Desoer (1989) exploits rectangularity and the Zero Exclusion Condition to obtain a simple proof of the theorem.

NRL 5.4 The paper by Frazer and Duncan (1929) appears to be the first to use the Zero Exclusion Condition in a robust stability context.

NRL 5.5 For more complicated uncertainty structures, Wei and Yedavalli (1989) propose a transformation technique in lieu of overbounding. Their approach involves applying a q-dependent linear transformation to the even or odd parts of $p(s,q)$. As a simple illustration, take $p(s) = p_{even}(s^2) + sp_{odd}(s^2)$ and suppose, $R(q)$ and $I(q)$ are positive functions of q. Defining the transformed polynomial $\tilde{p}(s,q) = R(q)p_{even}(s^2) + sI(q)p_{odd}(s^2)$, it is easy to show that robust stability remains invariant and in some cases, a reduction of conservatism may result. The potential for further research involving such methods is illustrated by the family $\mathscr{P}$ in Exercise 5.11.2. A robust stability test based on overbounding by an interval polynomial is inconclusive but multiplication of the even part by $1/q$ and the odd part by unity leads to an interval polynomial whose robust stability is easily verified by Kharitonov's Theorem.

NRL 5.6 There are a number of papers in the literature involving transformations aimed at facilitating robust stability analysis. For example, using the shifted circles in Petersen (1989), one can deal with the so-called Delta transform for a discrete-time system; for similar extreme point results involving Delta transformation, see also Soh (1991). The paper by Vaidyanathan (1990) provides another example of a transformation used for discrete-time problems.

NRL 5.7 Some alternatives to the technique described in Section 5.11 are given in papers by Djaferis (1991) and Pujara (1990). These papers describe different overbounding families which are sometimes useful.

NRL 5.8 Rather than working with the original coefficients, one can consider a bounding box B in the space of Markov parameters. By breaking an n-th order $p(s)$ into its even and odd parts as $p(s) = p_{even}(s^2) + sp_{odd}(s^2)$, a continued fraction expansion for $p_{odd}(x)/p_{even}(x)$ leads to the set of Markov parameters; see Gantmacher (1959). If $n = 2m$, we obtain parameters $(b_0, b_1, \ldots, b_{2m-1})$, and if $n = 2m-1$, we obtain $(b_{-1}, b_0, \ldots, b_{2m-1})$. With this representation, robust stability is guaranteed if and only if two distinguished polynomials are stable. For example, if $n = 2m$ and the box B is described by $b_i^- \le b_i \le b_i^+$ for $i = 0, 1, 2, \ldots, 2m-1$, the first distinguished polynomial has Markov parameters $(b_0^-, b_1^+, b_2^-, \ldots, b_{2m-1}^+)$ and the second distinguished polynomial has Markov parameters $(b_0^+, b_1^-, b_2^+, \ldots, b_{2m-1}^-)$; see Hollot (1989) for

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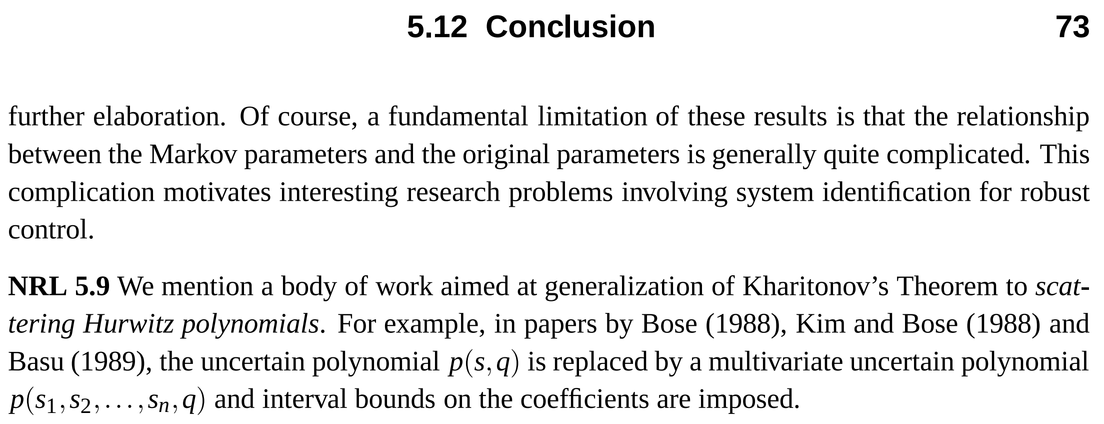

5.12 Conclusion 73

further elaboration. Of course, a fundamental limitation of these results is that the relationship between the Markov parameters and the original parameters is generally quite complicated. This complication motivates interesting research problems involving system identification for robust control.

NRL 5.9 We mention a body of work aimed at generalization of Kharitonov's Theorem to *scattering Hurwitz polynomials*. For example, in papers by Bose (1988), Kim and Bose (1988) and Basu (1989), the uncertain polynomial $p(s,q)$ is replaced by a multivariate uncertain polynomial $p(s_1,s_2,\ldots,s_n,q)$ and interval bounds on the coefficients are imposed.

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1.28.1 Lecture: Stability, duel systems

No class next Tuesday.

Note, for LTV, the duel system

$$\begin{aligned}\tilde{A}&=-A^T(t)\\ \tilde{B}&=C^T(t)\\ \tilde{C}&=B^T(t)\\ \tilde{D}&=D(t)\end{aligned}$$

But for LTI, the duel system is

$$\begin{aligned}\tilde{A}&=A^T\\ \tilde{B}&=C^T\\ \tilde{C}&=B^T\\ \tilde{D}&=D\end{aligned}$$

Reader: show that for the duel $\tilde{\Phi}(t,\tau)=\Phi^T(\tau,t)$ and that $\left[\Psi^T(t)\right]^{-1}=\tilde{\Psi}(t)$

Summary of LT and LTV:

Duality (LTI)	$A,B,C,D \Longleftrightarrow A^T,C^T,B^T,D$ $\begin{array}{c}x'(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)+Du(t)\end{array} \Longleftrightarrow \begin{array}{c}x'(t)=A^Tx(t)+C^Tu(t)\\ y(t)=B^Tx(t)+D^Tu(t)\end{array}$
Duality (LTV)	primal $\Longleftrightarrow$ duel $A(t),B(t),C(t),D(t) \Longleftrightarrow -A^T(t),C^T(t),B^T(t),D^T(t)$ $\Phi(t_0,\tau) \Longleftrightarrow \Phi^T(\tau,t_0)$ $\begin{array}{c}x'(t)=A(t)x(t)+B(t)u(t)\\ y(t)=C(t)x(t)+D(t)u(t)\end{array} \Longleftrightarrow \begin{array}{c}z'(t)=-A^T(t)z(t)+C^T(t)v(t)\\ w(t)=B^T(t)z(t)+D^T(t)v(t)\end{array}$
controllability Gramian (LTV)	$W(t_0,t_1)=\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\,dt$
observability Gramian (LTV)	$W_o(t_0,t_1)=\int_{t_0}^{t_1}\Phi^T(\tau,t_0)C^T(\tau)C(\tau)\Phi(\tau,t_0)\,dt$
Controllability Matrix (LTI)	$\mathbb{C}=\left[\begin{array}{c\|c\|c\|c\|c} B & AB & A^2B & \cdots & A^{n-1}B\end{array}\right]$
Controllability Matrix (LTV)	$M=\left[\begin{array}{c\|c\|c\|c} M_0 & M_1 & \cdots & M_{n-1}\end{array}\right]$ $M_0=B(t),M_{k+1}=-A(t)M_k+\frac{d}{dt}M_k$ for $k=0\cdots n-2$
Observability Matrix (LTI)	$\mathbb{Q}=\begin{pmatrix}C\\ CA\\ CA^2\\ \vdots\\ CA^{n-1}\end{pmatrix}$

Observability Matrix (LTV)	$L(t) = \begin{pmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \\ L_{n-1} \end{pmatrix}$ $L_0(t) = C(t), L_{k+1}(t) = L_k(t)A(t) + \frac{d}{dt}L_k(t)$ for $k = 0 \cdots n-2$
definition of physical controllability (LTI)	System is controllable if for any initial state x_0 and any final state x_1 $\exists$ input $u(t)$ that transfer x_0 to x_1 in finite time.
definition of physical controllability (LTV)	System is controllable at t_0 if $\exists$ input u over $[t_0, t_1]$ that transfers $x(t_0)$ to any $x(t_1)$ where $t_1 > t_0$
definition of physical observability (LTI)	System is observable if $\exists$ time $t_1 > t_0$ such that knowing input u and output y over $[t_0, t_1]$ suffices to determine state $x(t_0)$
definition of physical observability (LTV)	System is observable at t_0 if the following condition holds: With $x(t_0) = x^0$ unknown, suppose $u(t)$ and $y(t)$ are known, then there exist time $t_1 \geq t_0$ such that $x(t_0)$ can be determined from knowing $u(t)$ and $y(t)$ over $[t_0, t_1]$. This is true for any $x(t_0)$
State solution (LTV). $A(t)$ commutes with itself, i.e. $A(t)A(\tau) = A(\tau)A(t)$ then $\Psi(t) = e^{\int_{t_0}^{t} A(\zeta)\,d\zeta}$ $\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$ $\Phi(t,\tau) = e^{\int_{t}^{\tau} A(\zeta)\,d\zeta}$	$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\,d\tau$
State solution (LTV) $A(t)$ does not commutes with itself, but $A(t)$ commutes with its integral. i.e. $A(t)e^{\int_0^t A(\tau)\,d\tau} = e^{\int_0^t A(\tau)\,d\tau}A(t)$ then the same applied as above.	$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\,d\tau$
State solution (LTV) None of the above conditions apply. This is the hard case. Need to actually solve for $\Phi(t,\tau)$ by solving the state equations.	$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\,d\tau$
State solution (LTI)	$x(t_1) = e^{A(t_1 - t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1 - \tau)}Bu(\tau)\,d\tau$
State solution (LTI) with $t_0 = 0$	$x(t_1) = e^{A(t_1)}x(0) + \int_{0}^{t_1} e^{A(t_1 - \tau)}Bu(\tau)\,d\tau$

Back to canonical decomposition. Notice that decomposition is for LTI only, not for LTV.

Stability:

system characteristic equation $P(s) = \sum_{i=0}^{n} a_i s^i$. Let us assume all signs of $P(s)$ is the same to start with (if they are the same, then the polynomial is not stable. Also, assume all are positive. (we can always multiply by -1 to force this if needed.)

To find roots of $P(s)$ we can solve and check if $\operatorname{Re}(.)$ of each root is negative. If so, we say the system is stable. But we can check for stability without finding the roots using

Routh-Hurwitz. The proof is complicated. To use, here is an example for $n = 5$

$$H_{Hurwitz} = \begin{pmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & a_0 & a_2 & a_4 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 \end{pmatrix}$$

Now find Δ_i for each leading principle minor. Hence

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$\vdots$$

The system is stable if all $\Delta_i > 0$. A necessary condition for stability is that all a_i must be same sign. But this is not sufficient. Therefore, always start by checking for this. If there is sign change, no need to do Hurwitz, since not stable. Otherwise, have to do the above to determine stability.

Example: $P(s) = s^3 + 3s^2 + 3s + 1$.

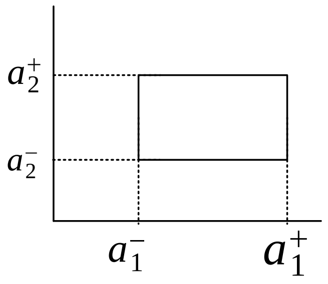
$$H_{Hurwitz} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$\Delta_1 = 1, \Delta_2 = 8, \Delta_3 = 8$, hence stable.

Reader: Suppose we want to check for stable where $\operatorname{Re}(.)$ of all roots are such that $\operatorname{Re}(.) < -\alpha$. Modify $P(s)$ to become $P(s+\alpha)$. Now we want to generalize to robust control. When we created $\sum = (A, B, C, D)$ we have an approximation to the system. So we have actually $A_{true} = A + \Delta A$, i.e. some perturbation of A on both sides. So the true A can become unstable. So we want $P(s)+$ some perturbation. Consider

$P_{true}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ where now we way that $a_i^- \leq a_i \leq a_i^+$ and the limits are known. This is called interval polynomial.

This is robustly stable no matter what values of a_i can have between the limits. The robust analysis problem was solved only in the last 30 years. Motivation for solution. Assume we have only 2 a_i which are a_1, a_2. Each with known limits. Hence we have the following diagram.



One approach is to make grid and solve for each combination, but this will become very large as more $a's$ are added. But Kharitonov's theorem reduces this to only 4 parameters. See hand out on Kharitonov's theorem. So we only have to check stability for 4 different polynomials instead of thousands and millions of them as the case would be with the grid method.

1.29 Lecture 27. Tuesday, November 25 2014 (no lecture)

No lecture.

1.30 Lecture 28. Thursday November 27 2014 (holiday)

Holiday

1.31 Lecture 29. Tuesday December 1 2014

1.31.1 Lecture: Stability, Hurwitz

Next 3 lectures will be on stability. One HW on stability due next Tuesday As well the special problem.

When dealing with linear systems, we form the characteristic polynomial $P(s)$. If we know the location of the roots, we can also find other properties.

reader: if $P(S)$ is stable and we reverse the order of the coefficients to obtain $\hat{P}(s)$, is $\hat{P}(s)$ stable?

Next we look at interval polynomial. $P(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$. We are interested in robust stability where each coefficients a_i has some range of values it can take

$$a_i^- \leq a_i \leq a_i^+$$

Robust stability: This polynomial is stable no matter what values a_i takes in this range. There are also robust tracking, robust damping, robust dynamic systems and other area where robustness is applied.

We can ask: How sensitive is system due to change in parameters? For n, a_i coefficients there is 2^n vertices If the extreme points define stable polynomial (i.e. max and min values of each interval), we would expect it to be stable for values in between.

Kharitonov theorem gave 4 fixed polynomials to check for stability of robust polynomial. We define the four polynomials as

$$\begin{aligned}
K_1(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + \dots \\
K_2(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \dots \\
K_3(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 \\
K_4(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4
\end{aligned}$$

Kharitonov theorem The uncertain polynomial $P(s)$ is robustly stable iff the above four polynomials are stable.

The proof of necessity is easy. The sufficiency proof is hard.

Examples: Let $P(s) = s^3 + 4s^2 + 5s + 2$. This is stable by construction $(s+1)^2(s+2)$. Roots are negative. Suppose we have interval polynomial $P(s) = s^3 + [3.5, 4.5]s^2 + [4.5, 5.5]s + [1.5, 2.5]$ then the four polynomials are

$$\begin{aligned}
K_1(s) &= s^3 + 4.5s^2 + 5.5s + 1.5 \\
K_2(s) &= s^3 + 3.5s^2 + 4.5s + 2.5 \\
K_3(s) &= s^3 + 4.5s^2 + 4.5s + 1.5 \\
K_4(s) &= s^3 + 3.5s^2 + 5.5s + 2.5
\end{aligned}$$

For low order polynomials, sometimes have to check less than 4 polynomials Sometimes only need to check 3 or just 2 as some will come up duplicate.

reader: check stability of the above.

If we start with stable polynomial $P(s)$, we can ask, by what percentage can we perturb the coefficients while preserving stability? Similar to asking for radius of stability.

How to set it up? How to use percentage?

$$\begin{aligned} P(s) &= s^3 + 4(1+\epsilon)s^2 + 5(1+\epsilon)s + 2(1+\epsilon) \\ &= s^3 + 4[1-\epsilon, 1+\epsilon]s^2 + 5(1-\epsilon, 1+\epsilon)s + 2[1-\epsilon, 1+\epsilon] \end{aligned}$$

For $\epsilon \ll 1$ we know it is stable. If ϵ denotes the percentage perturbation, we want

$$\epsilon_{max} = \sup \epsilon : \text{s.t. robust stability is guaranteed}$$

HW7 assigned.

1.32 Lecture 30. Thursday December 4 2014

1.32.1 Lecture: Stability, Lyapunov

On the special problem: up to 10 points of the course. Due same time as last HW. Today topic is related to special problem. can be done with computer simulation or theory.

Back to robust stability. Important to distinguish between dependent and independent uncertainty When we have intervals $[a_i^-, a_i^+]$ then each coefficient is independent of any other coefficient. But we can also have coefficients that are dependents on each others, or correlated. When we do, we call them $q's$ and write $P(s,q) = s^n + \sum_{k=0}^{n-1} a_k(q)s^k$. For example

$$P(s,q) = s^3 + (6 + q_1 + 2q_2)s^2 + (q_1 + 4)s + (q_3 + 6 + q_2)$$

This generalized the interval polynomial because we can now write

$$P(s,q) = s^n + \sum_{k=0}^{n-1} q_k s^k$$

This framework handles non-linear dependence on q. For example, assume we have problem with interval matrix. i.e. study the stability of interval matrix

$$A(q) = \begin{pmatrix} -1 + q(1) & 2 + q(2) \\ q(3) & -2 + q(4) \end{pmatrix}$$

Where $|q(i)| \leq r$. We now find the determinant of $A(q)$ as function of q_i which comes to

$$P(s) = s^2 + s(3 - q_4 - q_1) + (2 - q_4 - 2q_1 + q_1q_2 - 2q_3 - q_3q_2)$$

reader: how large r can be with robust stability still guaranteed

We can over-bound by applying Kharitonov polynomials. Let $r = \frac{1}{2}$ then

$$P(s) = s^2 + [2,4]s + \ldots$$

And now check stability of the four polynomials. Note that, if the result is negative, i.e. Kharitonov polynomials say that $P(s)$ is not stable, it might still be stable. So then we need to try other tests to check. An example is now given using $P(s) = s^4 + s^3 + 2qs^2 + s + q$ where $1.5 \leq qleq4$ and by over-bounding, we obtain the interval polynomial

$$P(s) = s^4 + s^3 + 2[-1.5, 4]s^2 + s + [1.5, 4]$$

reader: Show the above leads to 4 polynomials which one or more will be unstable, hence by over-bounding, we conclude it is not stable. But the it is robustly stable, which can be verified using Routh table.

Lyapunov Lemma:

This is important for non-linear systems. For linear systems it is not as important. It is also useful with dependent certainty. To start with, let us forget about uncertainty for now.

Initial $q = 0$ and consider $\dot{x} = Ax$. Lyapunov method will generalize. But using eigenvalues to check for stability will only work in linear systems.

What does Lyapunov says about stability of $\dot{x} = Ax$? Need energy like function $V(x) = x^T x = \sum_{i=1}^{n} x_i^2$.

We will study behavior of $V(x(t))$ along trajectory $x(t)$. If $V(x)$ goes to zero, then $x(t) = 0$. It is easier to study scalar function $V(x)$ for stability.

To determine if $V(x)$ goes to zero, we look at $\frac{dV(x)}{dt} < 0$

$$\begin{aligned}\frac{dv}{dt} &= \frac{d}{dx}x^T x\\ &= x^T Ax + \dot{x}^T x\\ &= x^T Ax + (Ax)^T x\\ &= x^T Ax + x^T A^T x\\ &= x^T (A + A^T) x\end{aligned}$$

iff $(A + A^T)$ is negative definite, then stable. But this definition of $V(x)$ is not satisfactory. Here is a counter example. Consider stable system $A = \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix}$ but $A + A^T = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$ so not stable. (ps. I must have copied something wrong. Since is stable, need to check notes with someone else).

So this energy function $V(x) = x^T x$ is not good and need to try a better one.

lemma the $n \times n$ matrix A is stable the following conditions is satisfied: Given any $n \times n$ positive definite symmetric matrix Q, the equation $A^T P + PA = -Q$ has positive definite symmetric solution P

Back to the above example. Solving it, using $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, gives

```
with(LinearAlgebra):
A:=<<-1,3;0,-2>>;
                              [-1   3]
                         A := [      ]
                              [ 0  -2]

P,s:=LyapunovSolve(A,<<-1,0;0,-1>>);

               [ 1.25000000000000  0.250000000000000]
       P, s := [                                    ], 1.
               [0.250000000000000  0.250000000000000]

Eigenvalues(P);
                   [1.30901699437495 + 0. I ]
                   [                        ]
                   [0.190983005625053 + 0. I]
```

Back to energy function. Use $V(x) = x^T Px$ instead of $V(x) = x^T x$. If stable, this energy function will satisfy $\frac{dv}{dt} = -x^T Qx < 0$.

So for robust stability, use Lyapunov method above and not the eigenvalues method.

perturbation on A:

Consider system $\dot{x} = \left[A_o + \Delta A(q)\right] x$ and q is bounded as before, and we are interested in robust stability. q above can even be time changing and the method will still work. But eigenvalues method will not work here. Assuming A_o is stable without loss of generality, the system is stable if

$$||\Delta A(t)||_2 \leq \frac{\lambda_{min} Q}{2\lambda_{max} P}$$

1.33 Lecture 31. Tuesday September 9 2014 (Review finals, Routh table)

Review of material for finals:

1. not cumulative
2. state transition
3. controlability, definitions, short cut M method. Gramian matrix, LTI and LTV cases. Cayley Hamilton
4. Duality for finding conditions for stability.
5. canonical decomposition. Minimal realization.
6. Hurwitz matrix, Routh table.
7. Robust stability, Kharitonov polynomials
8. uncertainty with correlated and uncorrelated parameters in the polynomial.
9. For robust stability, use lyapunov method.

Rest of lecture, examples on using Routh table. Showing it can shed more light on the system stability more than Hurwitz matrix. For example, it can tell one how many eigenvalues are unstable as well.

Review of future courses offered in the department. ECE 817,730,719. Taking math 521 is important for graduate work also. Optimal control vs. analysis direction of study.

1.34 Lecture 32. Thursday December 11 2014. Final Exam

Chapter 2

HW's

These are all my HW's. See the key solutions above for the official solution.

2.1 HW 1

2.1.1 Questions

Barmish

ECE 717 – Homework Newton

For the mass-spring system depicted on the next page, the input $u(t)$ is taken to be the displacement of the supporting platform.

(a) Apply Newton's Laws to obtain the two governing differential equations of motion in s_1, s_2 and u.

(b) With states taken to be $x_1 = s_1, x_2 = s_2, x_3 = \frac{ds_1}{dt}, x_4 = \frac{ds_2}{dt}$ and outputs $y_1 = s_1, y_2 = s_2$, obtain linear time-invariant state equations in the matrix form $\dot{x} = Ax + Bu,\ y = Cx + Du$.

(c) Use Simulink to obtain the unit step response for y_1 and y_2 using normalized parameter values $k_1 = k_2 = 0.5, m_1 = 1, m_2 = 2$. Assume the system is initially at rest; i.e., $x(0) = 0$.

(d) Experiment in your simulation with other values of the parameters to see the variety of possibilities for $y_1(t)$ and $y_2(t)$.

(e) For the state space system $\dot{x} = Ax + Bu,\ y = Cx + Du$ above with parameters indicated in part (c), use syms in Matlab to obtain the transfer functions $H_1(s)$ from u to y_1 and $H_2(s)$ from u to y_2.

(f) Using transfer function $H_1(s)$, find the differential equation relating output y_1 to input u.

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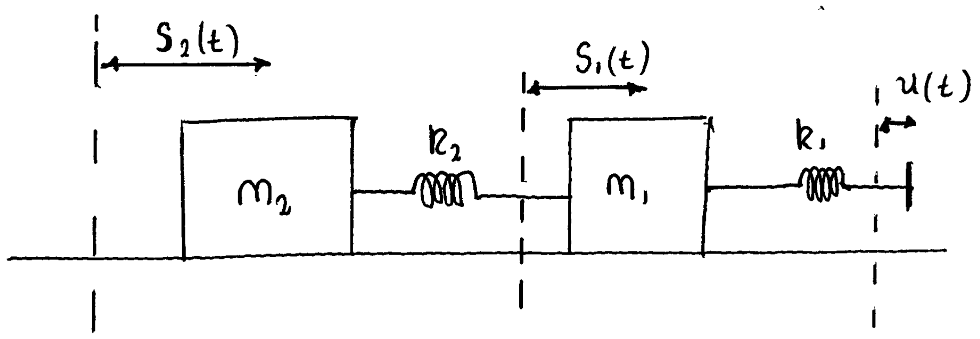

$S_1(t)$, $S_2(t)$ and $u(t)$ are displacements from equilibrium

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Barmish

ECE 717 – Homework Satellite

The motion of a satellite in earth orbit is modelled using polar coordinates (r, θ, ϕ) with corresponding thrust components (u_r, u_θ, u_ϕ). Now, with m denoting the mass, the kinetic energy is given by

$$\frac{m}{2}[\dot{r}^2 + (r\dot{\phi})^2 + (r\dot{\theta}\cos\phi)^2]$$

and the potential energy is

$$P = -\frac{km}{r}$$

where k is a fixed constant. For this system with Lagrangian defined by

$$\mathcal{L} \doteq K - P,$$

the dynamic behavior of the system is modelled as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = u_r(t);$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = u_\theta(t);$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = u_\phi(t).$$

(a) Define states $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, $x_5 = \phi$, $x_6 = \dot{\phi}$, inputs $u_1 = u_r$, $u_2 = u_\theta$, $u_3 = u_\phi$ and outputs $y_1 = r$, $y_2 = \theta$, $y_3 = \phi$. Now generate state equations and ouput equations in the standard form $\dot{x} = f(x, u)$; $y = g(x, u)$.

(b) Of particular interest (for example, for communication satellites), we consider a circular equatorial orbit obtained from (a) with $r(t) \equiv$ constant $= r_0$; $\dot{\theta} \equiv$ constant $= \omega$; $\phi(t) \equiv$ constant $= 0$; $u(t) \equiv$ constant $= 0$ and $r_0^3\omega^2 = k$. Argue that the satellite will remain in this orbit in the absence of disturbances to the system. Hence, describe a steady state solution to the state equation.

(c) Assuming the satellite strays slightly from the trajectory in (b), describe the state space pair (A, B) associated with linearization of the system. Take $r_0 = m_0 = 1$ and again use $r_0^3\omega^2 = k$.

(d) An alert engineer makes the observation: "The incremental dynamics for the azimuthal angle is decoupled from the rest of the system." Explain more fully what is meant by this statement.

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Barmish

ECE 717 – Homework Wave Theory

The objective in this problem is to show that some situations which do not appear to be amenable to our state equation paradigm can actually be "massaged" into our required $\dot{x} = f(x)$ format. Indeed, we begin with the so partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi$$

from the theory of nonlinear waves. This is the so-called Sine-Gordon equation.

(a) If we are seeking a travelling wave solution with velocity v, it is obtained by assuming the special form

$$\phi(x, t) = \tilde{\phi}(x - vt).$$

Now, taking

$$\zeta \doteq x - vt,$$

find the ordinary differential equation which $\tilde{\zeta}$ must satisfy.

(b) Next, with state assignment

$$x_1 = \tilde{\phi}; \;\; x_2 = \frac{d\tilde{\phi}}{d\zeta},$$

obtain state equations of the form

$$\dot{x} = f(x).$$

(c) A special case of the above is the so-called "kink" solution. Assuming velocity satisfying

$$0 < v < 1,$$

find special initial conditions $x_1(0), x_2(0)$ resulting in closed form solution

$$x_1(\zeta) = 4 \tan^{-1} \left(\exp(-\frac{\zeta}{\sqrt{1 - v^2}}) \right).$$

This is the famous "kink" solution. When it was first discovered, it was surprising to wave theorists because it is rather unusual for a highly nonlinear partial differential equation to admit a closed form solution.

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Barmish

ECE 717 – Homework Robotic Manipulator

A single-link robotic manipulator with a flexible joint is modelled via the pair of second order differential equations

$$I\frac{d^2\theta_1}{dt^2} + mgl\sin\theta_1 + k(\theta_1 - \theta_2) = 0;$$

$$J\frac{d^2\theta_2}{dt^2} - k(\theta_1 - \theta_2) = F(t)$$

where θ_1 and θ_2 are angular positions, I and J are moments of inertia, m,l and g are respectively, link mass, length and the gravitational constant, k is the link spring constant and f is the applied force.

(a) Obtain a nonlinear state space model $\dot{x} = f(x,u)$ for this system. Note: Four states are needed.

(b) Using equilibrium point $\overline{x}, \overline{u} = (0,0)$, obtain a linearized state space model (A,B).

(c) Using normalized unit-free constants $I = J = mgl = k = 1$, determine if the linearized model which you obtained is stable.

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Barmish

ECE 717 – Homework Time-Varying

Typically, the linear time varying system $\dot{x} = A(t)x$ cannot be solved in closed form; numerical methods are needed. However, for some special cases, a closed form solution can be found in an ad hoc manner. Indeed, consider the state equation

$$\dot{x} = \begin{bmatrix} -\frac{t}{1+t^2} & 1 \\ 0 & -\frac{4t}{1+t^2} \end{bmatrix} x$$

with initial condition $x(0) = x_0$ with components x_{10}, x_{20}. Using basic calculus manipulations, find the solutions $x_1(t)$ and $x_2(t)$. Then with $x_{10} = 1$, find $\lim_{t\rightarrow\infty} x_2(t)$. Reader: Even though the time-varying eigenvalues of $A(t)$ are negative, notice that $x_2(t)$ does not tend to zero.

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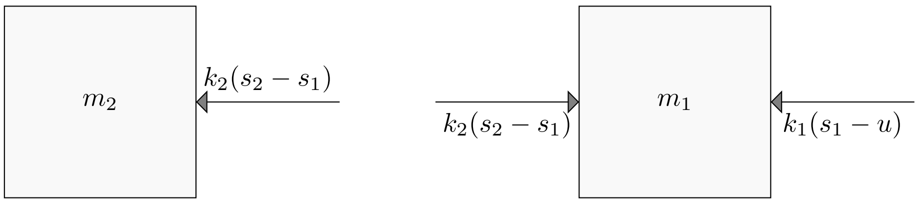
2.1.2 Problem 1

part (a)

Starting with the assumption that the ground surface is smooth and there is no friction. Assuming that all parts are moving in the positive direction to the right. Taking a snap shot when $s_2 > s_1$ so that the spring k_2 is in compression. Spring k_1 is in compression by also assuming that $s_1 > u$ at this instance.

Any other assumptions will also lead to the same set of equations as long as they are used in consistent way when finding the forces in the springs.

Starting with drawing a free body diagram of each body showing all forces acting on them based on the above assumption, and then using $F = ma$ to find the equation of motion of each body m_1, m_2. The free body diagrams is shown below



Now $F=ma$ is applied to each body to obtain the equation of motions. For mass m_2

$$\begin{aligned} m_2 s_2'' &= -k_2\left(s_2-s_1\right)\\ s_2'' &= -\frac{k_2}{m_2}\left(s_2-s_1\right)\end{aligned}$$

And for mass m_1

$$\begin{aligned} m_1 s_1'' &= k_2\left(s_2-s_1\right)-k_1\left(s_1-u\right)\\ s_1'' &= \frac{k_2}{m_1}\left(s_2-s_1\right)-\frac{k_1}{m_1}\left(s_1-u\right)\end{aligned}$$

Part (b)

Now the state space equations are found.

$$\begin{aligned} \begin{pmatrix} x_1=s_1\\ x_2=s_2\\ x_3=s_1'\\ x_4=s_2'\end{pmatrix} &\overset{\frac{d}{dt}}{\Longrightarrow} \begin{pmatrix} x_1'=s_1'=x_3\\ x_2'=s_2'=x_4\\ x_3'=s_1''=\frac{k_2}{m_1}\left(s_2-s_1\right)-\frac{k_1}{m_1}\left(s_1-u\right)=\frac{k_2}{m_1}\left(x_2-x_1\right)-\frac{k_1}{m_1}\left(x_1-u\right)\\ x_4'=s_2''=-\frac{k_2}{m_2}\left(s_2-s_1\right)=-\frac{k_2}{m_2}\left(x_2-x_1\right)\end{pmatrix}\\ &= \begin{pmatrix} x_3\\ x_4\\ x_1\left(-\frac{k_2}{m_1}-\frac{k_1}{m_1}\right)+\frac{k_2}{m_1}x_2+\frac{k_1}{m_1}u\\ \frac{k_2}{m_2}x_1-\frac{k_2}{m_2}x_2\end{pmatrix}\end{aligned}$$

Hence

$$\begin{pmatrix} x_1'\\ x_2'\\ x_3'\\ x_4'\end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\left(\frac{k_2}{m_1}+\frac{k_1}{m_1}\right) & \frac{k_2}{m_1} & 0 & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0\end{pmatrix}}^{A(n\times n)} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} + \overbrace{\begin{pmatrix} 0\\ 0\\ \frac{k_1}{m_1}\\ 0\end{pmatrix}}^{B(n\times m)} u\left(t\right)$$

$$\begin{pmatrix} y_1\\ y_2\end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}}^{C(r\times n)} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} + \overbrace{\begin{pmatrix} 0\\ 0\end{pmatrix}}^{D(r\times m)} u\left(t\right)$$

The above is in the form of $x'=Ax+Bu$ and $y=Cx+Du$ where $r=2$ is number of outputs, $m=1$ is the number of input and $n=4$ is the number of states.

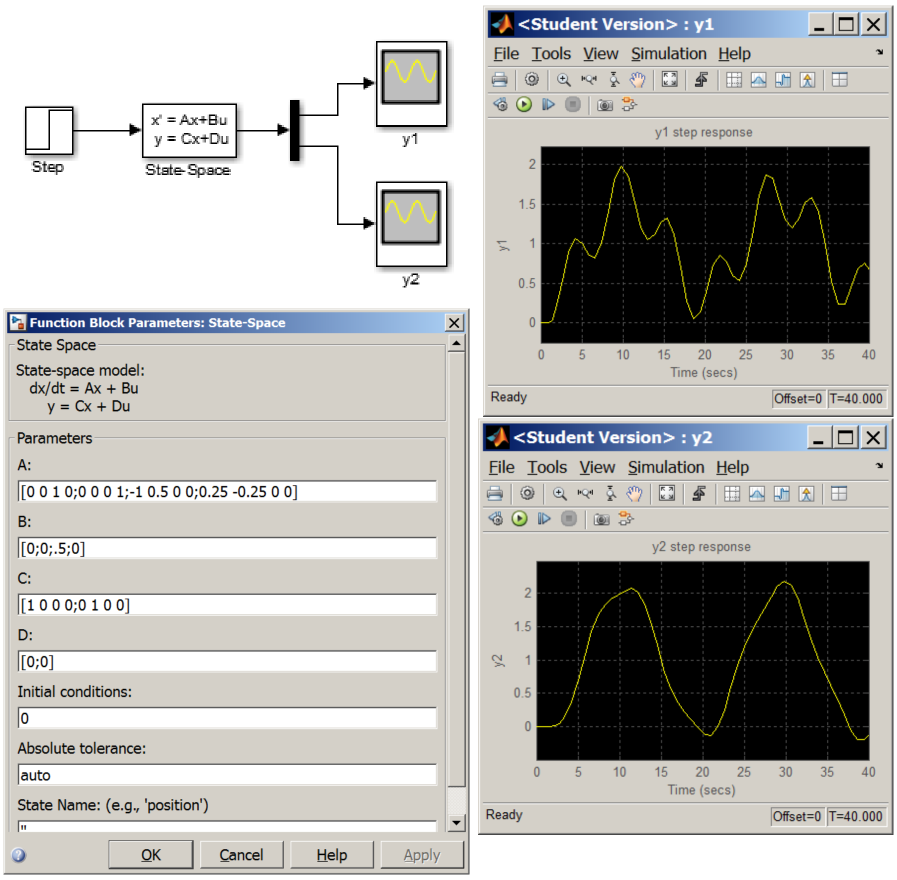
Part (c)

Using $k_1 = k_2 = 0.5, m_1 = 1, m_2 = 2$ and $x(0) = 0$ now the unit step response for y_1, y_2 is found using Simulink. With the above values the system becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0.5 & 0 & 0 \\ 0.25 & -0.25 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} u(t)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u(t)$$

Using simulink, state space block was used to implement the above. A step input source was used. Demux was used to send the y_1 and y_2 responses to two different time scopes. Simulation was set for 40 seconds to obtain long enough view of the response. The following figure shows the step response and the model used.



Function Block Parameters: State-Space

State Space

State-space model:
dx/dt = Ax + Bu
y = Cx + Du

Parameters

A: [0 0 1 0;0 0 0 1;-1 0.5 0 0;0.25 -0.25 0 0]

B: [0;0;.5;0]

C: [1 0 0 0;0 1 0 0]

D: [0;0]

Initial conditions: 0

Absolute tolerance: auto

State Name: (e.g., 'position') ''

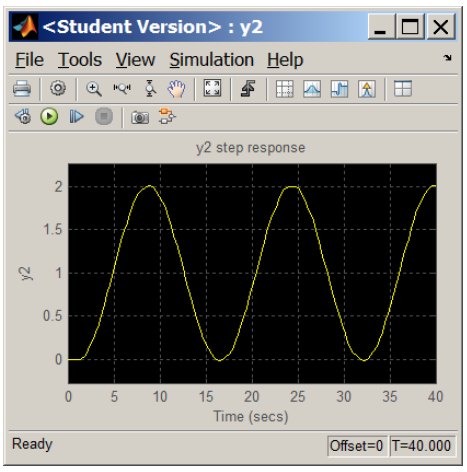
Part (d)

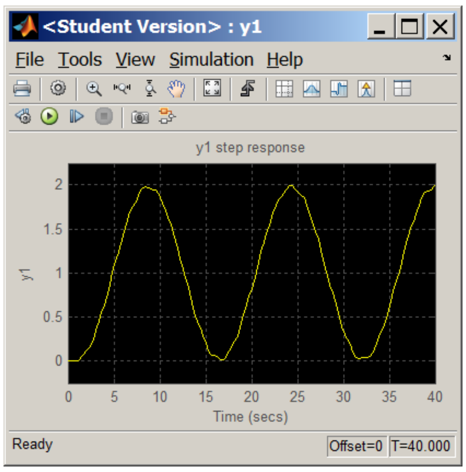
Different values of k_1, k_2 are used to see the effect on the responses. When the spring stiffness increased, the frequency of oscillation increased. For example, this is a simulation using $k_1 = 0.5, k_2 = 10, m_1 = 1, m_2 = 2$ and $x(0) = 0$. With the above values the system becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.5 & 10 & 0 & 0 \\ 5 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} u(t)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u(t)$$

The step response for y_1, y_2 is





Part (e)

Given

$$x' = Ax + Bu$$
$$y = Cx + Du$$

Applying the Laplace transform to the above and using zero initial conditions gives the following result. In the following, $X(s)$ is the Laplace transform of $x(t)$, $Y(s)$ is the Laplace transform of $y(t)$, and $U(s)$ is the Laplace transform of $u(t)$

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

The first equation above gives $X(s) = (sI - A)^{-1} BU(s)$. Substituting this value of $X(s)$ in the second equation above results in

$$Y(s) = C(sI - A)^{-1} BU(s)$$

Where $DU(s)$ was not used since D is zero matrix in this example. Therefore the system transfer function matrix is

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B$$

Using numerical values of A, B, C from part(c) gives

$$G(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \left(s \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0.5 & 0 & 0 \\ 0.25 & -0.25 & 0 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}$$

Hence $G(s)$ is a 2×1 vector. The first entry is the transfer function between u and y_1 and the second entry is the transfer function between u and y_2. The above is evaluated using Matlab syms as follows

```
A=[0 0 1 0;0 0 0 1;-1 0.5 0 0;0.25 -0.25 0 0];
B=[0 0 .5 0]';
C=[1 0 0 0;0 1 0 0];
syms s
G=C*inv(s*eye(4)-A)*B
G =
(4*s^2 + 1)/(8*s^4 + 10*s^2 + 1)
1/(8*s^4 + 10*s^2 + 1)
```

From above, the transfer functions are

$$\boxed{H_1(s) = \frac{Y_1(s)}{U(s)} = \frac{4s^2+1}{8s^4+10s^2+1}}$$

$$\boxed{H_2(s) = \frac{Y_2(s)}{U(s)} = \frac{1}{8s^4+10s^2+1}}$$

Part (f)

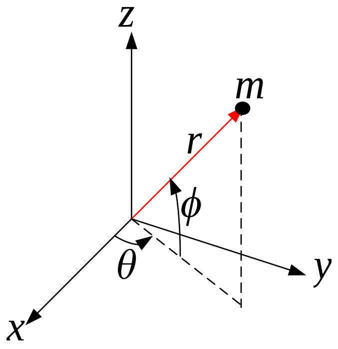
Using $H_1(s)$ above

$$Y_1(s)\left(8s^4+10s^2+1\right) = \left(4s^2+1\right)U(s)$$

And taking inverse Laplace transform gives (each s adds one derivative in time)

$$\boxed{8\frac{d^4y_1}{dt^4} + 10\frac{d^2y_1}{dt^2} + y_1(t) = 4\frac{d^2u}{dt^2} + u(t)}$$

2.1.3 Problem 2



The kinetic energy is given by

$$K = \frac{m}{2}\left(\dot{r}^2 + \left(r\dot{\phi}\right)^2 + \left(r\dot{\theta}\cos\phi\right)^2\right)$$

And the potential energy is

$$P = -\frac{km}{r}$$

Where k is constant and m is mass of satellite. The Lagrangian is

$$\begin{aligned} L &= K - P \\ &= \frac{m}{2}\left(\dot{r}^2 + \left(r\dot{\phi}\right)^2 + \left(r\dot{\theta}\cos\phi\right)^2\right) + \frac{km}{r} \end{aligned}$$

The equations of motions of the mass m in each degree of freedom r,θ,ϕ are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = u_r(t) \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = u_\theta(t) \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = u_\phi(t) \tag{3}$$

Part (a)

Starting with (1) gives find $\frac{\partial L}{\partial r} = mr\dot{\phi}^2 + mr\left(\dot{\theta}\cos\phi\right)^2 - \frac{km}{r^2}$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, hence (1) becomes

$$m\ddot{r} - mr\left(\dot{\phi}^2 + \left(\dot{\theta}\cos\phi\right)^2\right) + \frac{km}{r^2} = u_r(t)$$

Hence

$$\boxed{\ddot{r} = r\left(\dot{\phi}^2 + \left(\dot{\theta}\cos\phi\right)^2\right) - \frac{k}{r^2} + \frac{1}{m}u_r(t)}$$

Similarly for (2), $\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}r^2\cos^2\phi$, and

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= m\left(\ddot{\theta}r^2\cos^2\phi + 2\dot{\theta}r\dot{r}\cos^2\phi - 2\dot{\theta}\dot{\phi}r^2\cos\phi\sin\phi\right)\\ &= mr\cos\phi\left(\ddot{\theta}r\cos\phi + 2\dot{\theta}\dot{r}\cos\phi - 2\dot{\theta}\dot{\phi}r\sin\phi\right)\end{aligned}$$

And (2) becomes

$$\begin{aligned}mr\cos\phi\left(\ddot{\theta}r\cos\phi + 2\dot{\theta}\dot{r}\cos\phi - 2\dot{\theta}\dot{\phi}r\sin\phi\right) &= u_\theta(t)\\ \ddot{\theta}r\cos\phi + 2\dot{\theta}\left(\dot{r}\cos\phi - \dot{\phi}r\sin\phi\right) &= \frac{1}{mr\cos\phi}u_\theta(t)\\ \ddot{\theta}r\cos\phi &= -2\dot{\theta}\left(\dot{r}\cos\phi - \dot{\phi}r\sin\phi\right) + \frac{1}{mr\cos\phi}u_\theta(t)\end{aligned}$$

Hence

$$\boxed{\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + 2\dot{\theta}\dot{\phi}\tan\phi + \frac{1}{mr^2\cos^2\phi}u_\theta(t)}$$

Similarly for (3), $\frac{\partial L}{\partial \phi} = -mr^2\dot{\theta}^2\cos\phi\sin\phi$ and $\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$, and $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = m\left(2r\dot{r}\dot{\phi} + r^2\ddot{\phi}\right)$ hence (3) becomes

$$\begin{aligned}mr\left(2\dot{r}\dot{\phi} + r\left(\ddot{\phi} + \dot{\theta}^2\cos\phi\sin\phi\right)\right) &= u_\phi(t)\\ 2\dot{r}\dot{\phi} + r\ddot{\phi} + r\dot{\theta}^2\cos\phi\sin\phi &= \frac{1}{mr}u_\phi(t)\end{aligned}$$

Hence

$$\boxed{\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} - \dot{\theta}^2\cos\phi\sin\phi + \frac{1}{mr^2}u_\phi(t)}$$

The state space becomes

$$\begin{pmatrix} x_1 = r \\ x_2 = \dot r \\ x_3 = \theta \\ x_4 = \dot\theta \\ x_5 = \phi \\ x_6 = \dot\phi \end{pmatrix} \overset{\frac{d}{dt}}{\Longrightarrow} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} \dot r \\ \ddot r \\ \dot\theta \\ \ddot\theta \\ \dot\phi \\ \ddot\phi \end{pmatrix} = \begin{pmatrix} \dot r \\ r\dot\phi^2 + r\dot\theta^2\cos^2\phi - \frac{k}{r^2} + \frac{1}{m}u_r(t) \\ \dot\theta \\ -\frac{2}{r}\dot\theta\dot r + 2\dot\theta\dot\phi\tan\phi + \frac{1}{mr^2\cos^2\phi}u_\theta(t) \\ \dot\phi \\ -\frac{2}{r}\dot r\dot\phi - \dot\theta^2\cos\phi\sin\phi + \frac{1}{mr^2}u_\phi(t) \end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ x_1\left(x_6^2 + (x_4\cos x_5)^2\right) - \frac{k}{x_1^2} + \frac{1}{m}u_r(t) \\ x_4 \\ -\frac{2}{x_1}x_4x_2 + 2x_4x_6\tan x_5 + \frac{1}{mx_1^2\cos^2 x_5}u_\theta(t) \\ x_6 \\ -\frac{2}{x_1}x_2x_6 - x_4^2\cos x_5\sin x_5 + \frac{1}{mx_1^2}u_\phi(t) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \\ f_5(x,u) \\ f_6(x,u) \end{pmatrix}$$

The output equation is now found. $y_1 = r = x_1, y_2 = \theta = x_3, y_3 = \phi = x_5$, hence

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Part (b)

Applying the values given to the state vector x results in

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} r \\ \dot r \\ \theta \\ \dot\theta \\ \phi \\ \dot\phi \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ \omega t \\ \omega \\ 0 \\ 0 \end{pmatrix}$$

And

$$\bar u = \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is seen that the state vector x now has zero in all the components that can change the orbit from being in the equatorial orbit. Since the input $u = 0$ then this state will not change. The satellite will remain in this orbit.

Part (c)

To obtain x' at $(\bar{x}+\Delta x, \bar{u}+\Delta u)$ then $f(\bar{x}+\Delta x, \bar{u}+\Delta u)$ is evaluated using Taylor expansion and higher order terms are ignored. This results in the A, B matrices as follows

$$f(\bar{x}+\Delta x, \bar{u}+\Delta u) = \overbrace{f(\bar{x}, \bar{u})}^{\text{0 since equilibrium}} + \left.\frac{\partial f}{\partial x}\Delta x\right|_{x=\bar{x}} + \left.\frac{\partial f}{\partial u}\Delta u\right|_{u=\bar{u}} + H.O.T$$

In matrix form, the above is

$$f(\bar{x}+\Delta x, \bar{u}+\Delta u) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Delta x_1 & \frac{\partial f_1}{\partial x_2}\Delta x_2 & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial x_6}\Delta x_6 \\ \frac{\partial f_2}{\partial x_1}\Delta x_1 & \frac{\partial f_2}{\partial x_2}\Delta x_2 & \cdots & \cdots & \cdots & \frac{\partial f_2}{\partial x_6}\Delta x_6 \\ \vdots & & & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ \frac{\partial f_6}{\partial x_1}\Delta x_1 & \frac{\partial f_6}{\partial x_2}\Delta x_2 & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial x_6}\Delta x_6 \end{pmatrix}_{(\bar{x},\bar{u})} + \begin{pmatrix} \frac{\partial f_1}{\partial u_1}\Delta u_1 & \frac{\partial f_1}{\partial u_2}\Delta u_2 & \frac{\partial f_1}{\partial u_3}\Delta u_3 \\ \frac{\partial f_2}{\partial u_1}\Delta u_1 & \frac{\partial f_2}{\partial u_2}\Delta u_2 & \frac{\partial f_2}{\partial u_3}\Delta u_3 \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_6}{\partial u_1}\Delta u_1 & \frac{\partial f_6}{\partial u_2}\Delta u_2 & \frac{\partial f_6}{\partial u_3}\Delta u_3 \end{pmatrix}_{(\bar{x},\bar{u})}$$

Therefore

$$f(\bar{x}+\Delta x, \bar{u}+\Delta u) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & & & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial x_6} \end{pmatrix}_{(\bar{x},\bar{u})} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta x_6 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_6}{\partial u_1} & \frac{\partial f_6}{\partial u_2} & \frac{\partial f_6}{\partial u_3} \end{pmatrix}_{(\bar{x},\bar{u})} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix}$$

Each component in the above is now evaluated and A, B are evaluated at $\bar{x} = \begin{pmatrix} r_0 & 0 & \omega t & \omega & 0 & 0 \end{pmatrix}^T, \bar{u} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ in order to obtain A, B. Since $f_1(x,u) = x_2$, then $\frac{\partial f_1}{\partial x_2} = 1$ and all other values are zero. Since $f_2(x,u) = x_1\left(x_6^2 + (x_4 \cos x_5)^2\right) - \frac{k}{x_1^2} + \frac{1}{m}u_r(t)$ then

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= \left(x_6^2 + (x_4 \cos x_5)^2\right) + 2\frac{k}{x_1^3} \\ \frac{\partial f_2}{\partial x_2} &= 0 \\ \frac{\partial f_2}{\partial x_3} &= 0 \\ \frac{\partial f_2}{\partial x_4} &= 2x_1 x_4 \cos^2 x_5 \\ \frac{\partial f_2}{\partial x_5} &= -2x_1 x_4^2 \cos x_5 \sin x_5 \\ \frac{\partial f_2}{\partial x_6} &= 0 \end{aligned}$$

Since $f_3(x,u) = x_4$ then $\frac{\partial f_3}{\partial x_4} = 1$ and all other components are zero. And since $f_4(x,u) = -\frac{2}{x_1}x_4x_2 + 2x_4x_6\frac{\sin x_5}{\cos x_5} + \frac{1}{mx_1^2\cos^2 x_5}u_\theta(t)$ then

$$\begin{aligned}
\frac{\partial f_4}{\partial x_1} &= 0\\
\frac{\partial f_4}{\partial x_2} &= -\frac{2}{x_1}x_4\\
\frac{\partial f_4}{\partial x_3} &= 0\\
\frac{\partial f_4}{\partial x_4} &= -\frac{2}{x_1}x_2 + 2x_6\frac{\sin x_5}{\cos x_5}\\
\frac{\partial f_4}{\partial x_5} &= 2x_4x_6\sec^2(x_5) + \frac{2u_\theta\sec^2 x_5\tan x_5}{mx_1^2}\\
\frac{\partial f_2}{\partial x_6} &= 2x_4\tan x_5
\end{aligned}$$

Since $f_5(x,u) = x_6$ then $\frac{\partial f_5}{\partial x_6} = 1$ and all other components are zero. Finally, since $f_6(x,u) = -\frac{2}{x_1}x_2x_6 - x_4^2\cos x_5\sin x_5 + \frac{1}{mx_1^2}u_\phi(t)$ then

$$\begin{aligned}
\frac{\partial f_6}{\partial x_1} &= -\frac{2}{mx_1^3}u_\phi\\
\frac{\partial f_6}{\partial x_2} &= -\frac{2}{x_1}x_2\\
\frac{\partial f_6}{\partial x_3} &= 0\\
\frac{\partial f_6}{\partial x_4} &= -2x_4\cos x_5\sin x_5\\
\frac{\partial f_6}{\partial x_5} &= -x_4^2\cos^2 x_5 + x_4^2\sin^2 x_5\\
\frac{\partial f_6}{\partial x_6} &= -\frac{2}{x_1}x_2
\end{aligned}$$

Therefore, the linearized A matrix is

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0\\
\left(x_6^2 + (x_4\cos x_5)^2\right) + 2\frac{k}{x_1^3} & 0 & 0 & 2x_1x_4\cos^2 x_5 & -2x_1x_4^2\cos x_5\sin x_5 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & -\frac{2}{x_1}x_4 & 0 & -\frac{2}{x_1}x_2 + 2x_6\frac{\sin x_5}{\cos x_5} & 2x_4x_6\sec^2(x_5) + \frac{2u_\theta\sec^2 x_5\tan x_5}{mx_1^2} & 2x_4\tan x_5\\
0 & 0 & 0 & 0 & 0 & 1\\
-\frac{2}{mx_1^3}u_\phi & -\frac{2}{x_1}x_2 & 0 & -2x_4\cos x_5\sin x_5 & -x_4^2\cos^2 x_5 + x_4^2\sin^2 x_5 & -\frac{2}{x_1}x_2
\end{pmatrix}$$

The above is evaluated at $\bar{x} = \begin{pmatrix} r_0 & 0 & \omega t & \omega & 0 & 0 \end{pmatrix}^T, \bar{u} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ which results in

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \left(0 + (\omega \cos 0)^2\right) + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega\cos^2 0 & -2r_0\omega^2 \cos 0 \sin 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{r_0}\omega & 0 & -\frac{2}{r_0}(0) + 2(0)\frac{\sin 0}{\cos 0} & 2\omega(0)\sec^2(0) + \frac{2u_\theta \sec^2(0)\tan(0)}{mr_0^2} & 2\omega\tan(0) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{2}{mr_0^3}(0) & -\frac{2}{r_0}(0) & 0 & -2\omega\cos 0 \sin 0 & -\omega^2\cos^2 0 + \omega^2 \sin^2 0 & -\frac{2}{r_0}(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \omega^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{r_0}\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{pmatrix}$$

Using $r_0 = 1, m = 1$ and $k = r_0^3\omega^2$ then the above becomes

$$\boxed{A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{pmatrix}}$$

The B matrix is now found. Since $f_1(x,u) = x_2$, then $\frac{\partial f_1}{\partial u_i} = 0$ for $i = 1 \cdots 3$. And since $f_2(x,u) = x_1\left(x_6^2 + (x_4 \cos x_5)^2\right) - \frac{k}{x_1^2} + \frac{1}{m}u_r(t)$ then $\frac{\partial f_2}{\partial u_r} = \frac{1}{m}$ and the other two components are zero. Since $f_3(x,u) = x_4$ then $\frac{\partial f_3}{\partial u_i} = 0$ for $i = 1 \cdots 3$. And since $f_4(x,u) = -\frac{2}{x_1}x_4x_2 + 2x_4x_6\frac{\sin x_5}{\cos x_5} + \frac{1}{mx_1^2\cos^2 x_5}u_\theta(t)$ then $\frac{\partial f_4}{\partial u_\theta} = \frac{1}{mx_1^2\cos^2 x_5}$and the other two components are zero. Since $f_5(x,u) = x_6$ then $\frac{\partial f_5}{\partial u_i} = 0$ for $i = 1 \cdots 3$. Finally, since $f_6(x,u) = -\frac{2}{x_1}x_2x_6 - x_4^2\cos x_5 \sin x_5 + \frac{1}{mx_1^2}u_\phi(t)$ then $\frac{\partial f_6}{\partial u_\phi} = \frac{1}{mx_1^2}$ and the other two components are zero. Hence the B matrix becomes

$$B = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{mx_1^2\cos^2 x_5} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{mx_1^2} \end{pmatrix}$$

The above is evaluated at $\bar{x} = \begin{pmatrix} r_0 & 0 & \omega t & \omega & 0 & 0 \end{pmatrix}^T$ which results in

$$B = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{mr_0^2\cos^2 0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{mr_0^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{mr_0^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{mr_0^2} \end{pmatrix}$$

And using $r_0 = 1, m = 1$ it reduces to

$$\boxed{B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Part d

The linearized A matrix found above is

$$A = \left(\begin{array}{cccc:cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{array}\right)$$

This shows that due to zeros everywhere in the linkage between the states r, r', θ, θ' and the states ϕ, ϕ', these states are decoupled. What this means is that motion can be analyzed in the ϕ, ϕ' states as its own system without having to carry along other terms from the other states. This simplifies both the analysis and design for these parts of the system since they are decoupled from each others. The above decoupling is also present in the B and C matrices.

2.1.4 Problem 3

part (a)

Using chain rule, and using $\frac{\partial \zeta}{\partial x} = 1$ and $\frac{\partial \zeta}{\partial t} = -v$ then[1]

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\phi}{d\zeta}\frac{\partial \zeta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\phi}{d\zeta}\right) = \frac{\partial}{\partial \zeta}\left(\frac{d\phi}{d\zeta}\right)\frac{\partial \zeta}{\partial x} = \frac{\partial}{\partial \zeta}\left(\frac{d\phi}{d\zeta}\right) = \frac{d^2\phi}{d\zeta^2}$$

And

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{d\phi}{d\zeta}\frac{\partial \zeta}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{d\phi}{d\zeta}(-v)\right) = \frac{\partial}{\partial \zeta}\left(\frac{d\phi}{d\zeta}(-v)\right)\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial \zeta}\left(\frac{d\phi}{d\zeta}(-v)\right)(-v) \\ &= v^2\frac{d^2\phi}{d\zeta^2} \end{aligned}$$

Hence the PDE becomes the ODE

$$\frac{d^2\phi}{d\zeta^2} - v^2\frac{d^2\phi}{d\zeta^2} = \sin\phi(\zeta)$$

Hence the differential equation is

$$\boxed{\left(1 - v^2\right)\frac{d^2\phi}{d\zeta^2} = \sin\phi(\zeta)} \tag{1}$$

Part (b)

Let $x_1 = \phi, x_2 = \frac{d\phi}{d\zeta}$, hence

$$\begin{pmatrix} x_1 = \phi \\ x_2 = \frac{d\phi}{d\zeta} \end{pmatrix} \overset{\frac{d}{dt}}{\Longrightarrow} \begin{pmatrix} x_1' = \frac{d\phi}{d\zeta} \\ x_2' = \frac{d^2\phi}{d\zeta^2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{\sin x_1}{1 - v^2} \end{pmatrix}$$

---

[1] In the following, $\tilde{\phi}$ is written as ϕ to make the notation more clear, but it is meant to be the special form $\tilde{\phi}$ that is being used in all these calculations

Hence $x' = f(x)$ becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{\sin x_1}{1-v^2} \end{pmatrix}$$
$$\equiv f(x)$$

Part (c)

Equation (1) is solved, and then the result is compared to the so called "kink" solution provided in order to determine what the constant of integration are. The constant of integration will be $x_1(0)$ and $x_2(0)$. Starting by multiplying both sides of (1) by $\frac{d\phi}{d\zeta}$ results in

$$\left(1-v^2\right)\frac{d^2\phi}{d\zeta^2}\frac{d\phi}{d\zeta} = \sin\phi\frac{d\phi}{d\zeta}$$

Or

$$\left(1-v^2\right)\frac{d^2\phi}{d\zeta^2}\frac{d\phi}{d\zeta}d\zeta = \sin\phi d\phi$$

Now both sides are integrated. The RHS gives $\int \sin\phi d\phi = -\cos\left(\phi\right) + c_1$ and the LHS gives

$$\int \left(1-v^2\right)\frac{d^2\phi}{d\zeta^2}\frac{d\phi}{d\zeta}d\zeta = \left(1-v^2\right)\frac{1}{2}\left(\frac{d\phi}{d\zeta}\right)^2$$

This is becuase if $\frac{1}{2}\left(\frac{d\phi}{d\zeta}\right)^2$ is differentiated, using chain rule, the result will be the integrand. Since differentiating $\frac{1}{2}\left(\frac{d\phi}{d\zeta}\right)^2$ w.r.t. ζ gives $\frac{d\phi}{d\zeta}\frac{d^2\phi}{d\zeta^2}$ which is the integrand in the LHS. No need to introduced a new integration of constant again here as it can be absorbed with c_1. Therefore, the result after integration once is

$$\left(1-v^2\right)\frac{1}{2}\left(\frac{d\phi}{d\zeta}\right)^2 = -\cos\left(\phi\right) + c_1 \tag{2}$$

To make some progress now, assuming the following initial conditions $x_1(0) = 0 = \phi$ and $x_2(0) = 0 = \frac{d\phi}{d\zeta}$. Using these initial conditions results in

$$\boxed{c_1 = 1}$$

Equation (2) now becomes

$$\frac{\left(\frac{d\phi}{d\zeta}\right)^2}{1-\cos\left(\phi\right)} = \frac{2}{\left(1-v^2\right)}$$
$$\frac{1}{\sqrt{1-\cos\left(\phi\right)}}\frac{d\phi}{d\zeta} = \frac{\sqrt{2}}{\sqrt{1-v^2}} \tag{3}$$

From trigonometric tables the relation $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ is used, therefore $\sqrt{1-\cos\left(\phi\right)} = \pm\sqrt{2}\sin\frac{\phi}{2}$ and (3) becomes

$$\frac{1}{\pm\sqrt{2}\sin\frac{\phi}{2}}\frac{d\phi}{d\zeta} = \frac{\sqrt{2}}{\sqrt{1-v^2}}$$
$$\pm\frac{d\phi}{\sin\frac{\phi}{2}} = \frac{2}{\sqrt{1-v^2}}d\zeta$$

Doing integration again

$$\pm\int\frac{d\phi}{\sin\frac{\phi}{2}} = \frac{2}{\sqrt{1-v^2}}\int d\zeta \tag{4}$$

From tables (or using substitutions) $\int \frac{d\phi}{\sin\frac{\phi}{2}} = 2\ln\left(\tan\left(\frac{\phi}{4}\right)\right)$ hence (4) becomes

$$\pm 2\ln\left(\tan\left(\frac{\phi}{4}\right)\right) = \frac{2}{\sqrt{1-v^2}}\left(\zeta - \zeta_0\right)$$

$$\tan\left(\frac{\phi}{4}\right) = \exp\left(\pm\frac{\zeta-\zeta_0}{\sqrt{1-v^2}}\right)$$

Therefore

$$\frac{\phi}{4} = \arctan\left(\pm\exp\left(\frac{\zeta-\zeta_0}{\sqrt{1-v^2}}\right)\right)$$

$$\phi\left(\zeta\right) = x_1 = 4\arctan\left(\pm\exp\left(\frac{\zeta-\zeta_0}{\sqrt{1-v^2}}\right)\right)$$

If $\zeta_0 = 0$ then

$$\boxed{x_1 = 4\arctan\left(\pm\exp\left(\frac{\zeta}{\sqrt{1-v^2}}\right)\right)}$$

This is the answer we are asked to show. Hence the initial conditions needed to obtain this answer are $x_1(0) = 0$ and $x_2(0) = 0$

2.1.5 Problem 4

The robotic arm coupled differential equations are

$$I\theta_1'' + mgl\sin\theta_1 + k\left(\theta_1 - \theta_2\right) = 0$$

$$J\theta_2'' - k\left(\theta_1 - \theta_2\right) = F(t)$$

Part a

Let

$$\begin{pmatrix} x_1 = \theta_1 \\ x_2 = \theta_2 \\ x_3 = \theta_1' \\ x_4 = \theta_2' \end{pmatrix} \overset{\frac{d}{dt}}{\Longrightarrow} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \theta_1' \\ \theta_2' \\ \theta_1'' \\ \theta_2'' \end{pmatrix} = \begin{pmatrix} \theta_1' \\ \theta_2' \\ \frac{mgl}{I}\sin\theta_1 + \frac{k}{I}\left(\theta_1-\theta_2\right) \\ \frac{k}{J}\left(\theta_1-\theta_2\right) + \frac{1}{J}F(t) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ -\frac{mgl}{I}\sin x_1 - \frac{k}{I}\left(x_1 - x_2\right) \\ \frac{k}{J}\left(x_1-x_2\right) + \frac{1}{J}F(t) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \end{pmatrix}$$

Where u, the input, is $F(t)$ in this example, but the letter u is used since it is the common notation.

Part b

Let equilibrium point be $(\bar{x},\bar{u}) = (0,0)$. The system is now linearized around this point.

$$f\left(\bar{x}+\Delta x, \bar{u}+\Delta u\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{pmatrix}_{(\bar{x},\bar{u})} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}_{(\bar{x},\bar{u})} \Delta u$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgl}{I}\cos x_1 - \frac{k}{I} & \frac{k}{I} & 0 & 0 \\ \frac{k}{J} & -\frac{k}{J} & 0 & 0 \end{pmatrix}_{(\bar{x},\bar{u})} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{pmatrix}_{(\bar{x},\bar{u})} \Delta u$$

The above is evaluated at $(\bar{x},\bar{u}) = (0,0)$ giving

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgl}{I} - \frac{k}{I} & \frac{k}{I} & 0 & 0 \\ \frac{k}{J} & -\frac{k}{J} & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{pmatrix}$$

Part c

Using values $I = J = mgl = k = 1$, the A matrix becomes

$$\boxed{A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}}$$

The real part of the eigenvalues of this matrix are

```
A=[0 0 1 0;0 0 0 1;-2 1 0 0;1 -1 0 0];
EDU>> real(eig(A))
ans =
2.9302e-18
2.9302e-18
-3.1554e-30
-3.1554e-30
```

There is no positive real part. The real part of the eigenvalues are effectively zero. They are pure complex conjugate values. Hence the system is stable (sometimes also called marginally stable in this case).

2.1.6 Problem 5

solution

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{t}{1+t^2} & 1 \\ 0 & \frac{-4t}{1+t^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

With initial conditions $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$

$x_2(t)$ is first solved since it does not depend on $x_1(t)$ and then the solution is used to solve for $x_1(t)$. The differential equation for $x_2(t)$ is

$$\frac{dx_2}{dt} = \frac{-4t}{1+t^2}x_2$$
$$\frac{dx_2}{x_2} = \frac{-4t}{1+t^2}dt$$

Integrating

$$\ln x_2 = \ln\left(\frac{1}{\left(1+t^2\right)^2}\right) + c$$
$$x_2 = c\frac{1}{\left(1+t^2\right)^2}$$

When $t=0$, $x_2(0)=x_{20}$, hence $c=x_{20}$ and

$$\boxed{x_2(t)=\frac{x_{20}}{\left(1+t^2\right)^2}}$$

Now since $x_2(t)$ was found, it is used to obtain $x_1(t)$. The differential equation for $x_1(t)$ is

$$\begin{aligned}\frac{dx_1}{dt}&=-\frac{t}{1+t^2}x_1+x_2\\ \frac{dx_1}{dt}+\frac{t}{1+t^2}x_1&=x_{20}\frac{1}{\left(1+t^2\right)^2}\end{aligned}$$

The integrating factor is $I=e^{\int\frac{t}{1+t^2}dt}=\sqrt{\left(1+t^2\right)}$, hence the solution to the above is

$$d\left(Ix_1\right)=x_{20}\frac{I}{\left(1+t^2\right)^2}$$

Integrating

$$\begin{aligned}Ix_1&=\int x_{20}\frac{I}{\left(1+t^2\right)^2}dt\\ &=x_{20}\int\frac{\sqrt{\left(1+t^2\right)}}{\left(1+t^2\right)^2}dt\\ &=x_{20}\int\frac{1}{\left(1+t^2\right)^{3/2}}dt\\ &=x_{20}\frac{t}{\sqrt{1+t^2}}+c_2\end{aligned}$$

Hence, dividing by I gives the final solution

$$x_1(t)=x_{20}\frac{t}{1+t^2}+\frac{c_2}{\sqrt{\left(1+t^2\right)}}$$

When $t=0,x_1(0)=x_{10}$, hence $c_2=x_{10}$ and the solution becomes

$$x_1(t)=x_{20}\frac{t}{1+t^2}+\frac{x_{10}}{\sqrt{\left(1+t^2\right)}}$$

Now we are asked to let $x_{10}=1$, hence $x_1(t)$ becomes

$$\boxed{x_1(t)=x_{20}\frac{t}{1+t^2}+\frac{1}{\sqrt{\left(1+t^2\right)}}}$$

Now taking the limit of $x_1(t)_{t\to\infty}$ gives

$$x_1(t)\to 0$$

Using Matlab syms, these can be solved as follows

```
clear all
syms x1(t) x2(t) t x10 x20
eq1=diff(x1,t)== -t/(1+t^2)*x1+x2;
eq2=diff(x2,t)== -(4*t)/(1+t^2)*x2;
[x1Sol,x2Sol]=dsolve(eq1,eq2,x1(0)==1)

x1Sol =

1/(t^2 + 1)^(1/2) + (C2*t)/(t^2 + 1)

x2Sol =

C2/(t^2 + 1)^2
```

2.1.7 key solution

(a) SOLUTION NEWTON

Applying Newton's law $m_1 \frac{d^2 s_1}{dt^2} = k_1(u(t) - s_1(t)) - k_2(s_1(t) - s_2(t))$

$$m_2 \frac{d^2 s_2}{dt^2} = k_2(s_1(t) - s_2(t))$$

(b) With state variables $x_1(t) \triangleq s_1(t)$, $x_2(t) \triangleq s_2(t)$ $x_3(t) = \frac{ds_1}{dt}$, $x_4(t) = \frac{ds_2}{dt}$ and obtain state equations in matrix form.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -k_2/m_2 & 0 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ k_1/m_1 \\ 0 \end{bmatrix}}_{B} u(t)$$

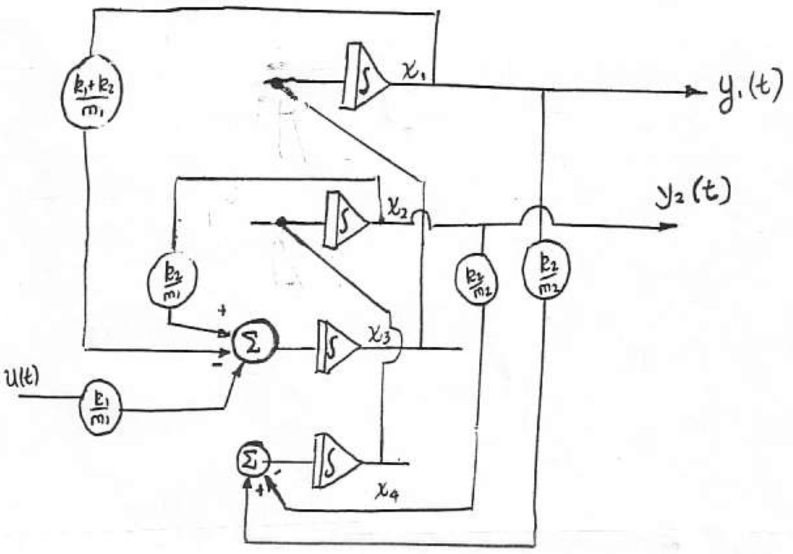
measured outputs : displacements of masses

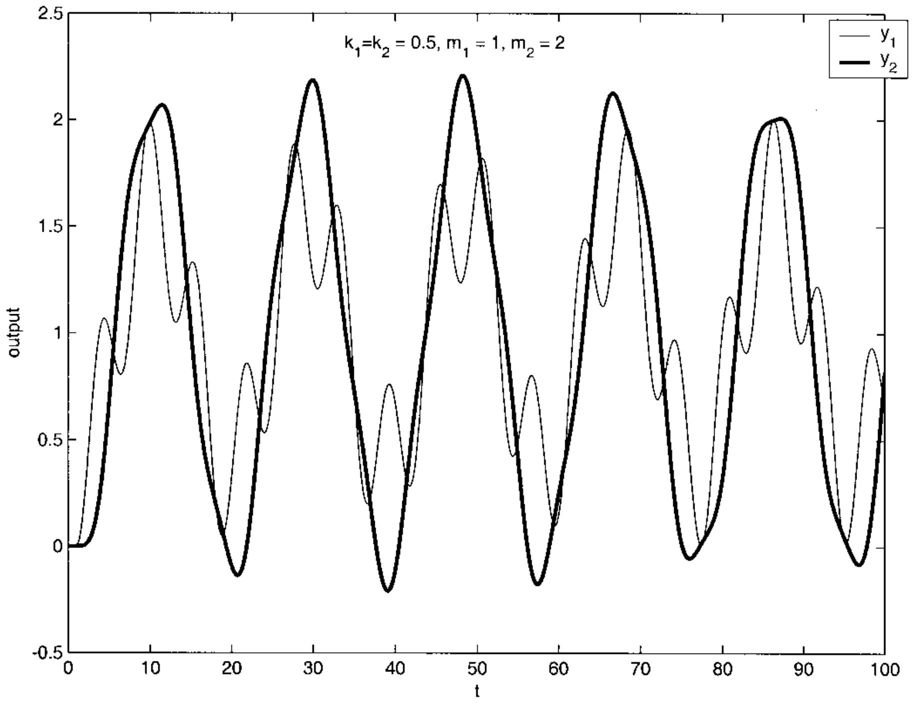
So $y_1(t) = x_1(t)$

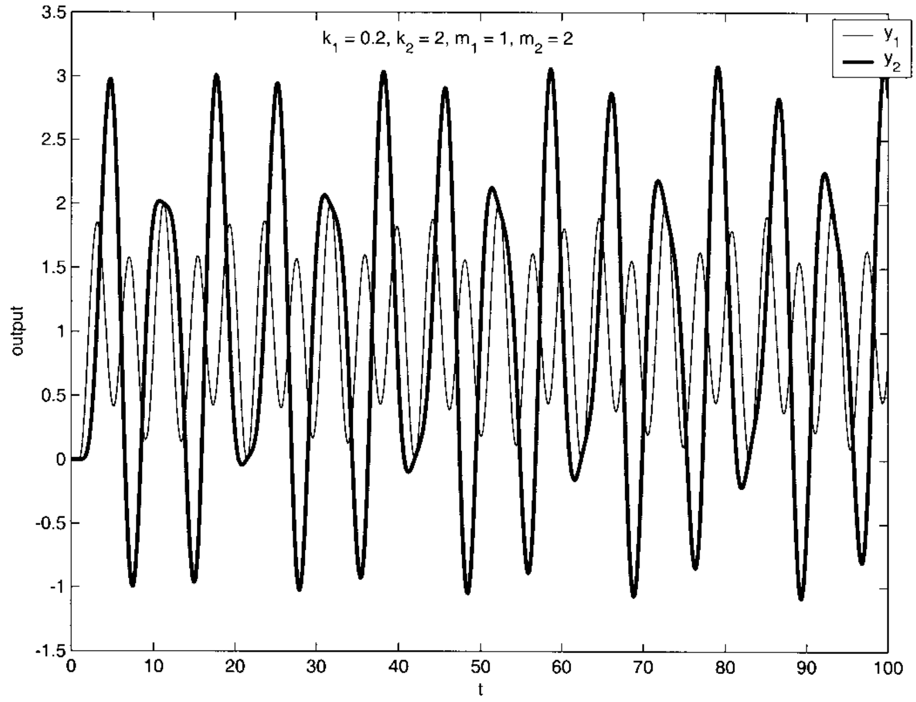
$y_2(t) = x_2(t)$

In matrix form $y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)$

Simulation




k_1=k_2 = 0.5, m_1 = 1, m_2 = 2
y_1
y_2
output
t
2.5
2
1.5
1
0.5
0
-0.5
0
10
20
30
40
50
60
70
80
90
100


k$_1$ = 0.2, k$_2$ = 2, m$_1$ = 1, m$_2$ = 2
y$_1$
y$_2$
output
t
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
10
20
30
40
50
60
70
80
90
100

Part (e): Simple Matlab code

```
syms s
k_1 = 0.5;
k_2 = 0.5;
m_1 = 1;
m_2 = 2;
A = [0 0 1 0;0 0 0 1;-(k_1 + k_2)/m_1 k_2/m_1 0 0;...
 k_2/m_2 -k_2/m_2 0 0];
B = [0; 0; k_1/m_1; 0];
C_1 = [1 0 0 0];
C_2 = [0 1 0 0];
D = 0;
H_1 = C_1*inv(s*eye(4) - A)*B + D
H_2 = C_2*inv(s*eye(4) - A)*B + D
```

with output

```
>> H_1 =
(4*s^2+1)/(8*s^4+10*s^2+1)
>> H_2 =
1/(8*s^4+10*s^2+1)
```

Solution Satellite

Calculating derivatives in Lagrange's equations, we obtain

$$m\left(\ddot{r}(t) - r(t)\dot{\theta}^2(t)\cos^2\phi(t) - r\dot{\phi}^2(t) + \frac{k}{r^2(t)}\right) = u_r(t) \qquad ①$$

to simply notation, drop "t" ⇒

$$m\left(\ddot{\theta} r^2 \cos^2\phi + 2r\dot{r}\dot{\theta}\cos^2\phi - 2r^2\dot{\theta}\dot{\phi}\cos\phi\sin\phi\right) = u_\theta \qquad ②$$

$$m\left(\ddot{\phi} r^2 + r^2\dot{\theta}^2\cos\phi\sin\phi + 2r\dot{r}\dot{\phi}\right) = u_\phi \qquad ③$$

Substitute state variables and re-write vectorially

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} x_2 \\ x_1 x_4^2 \cos^2 x_5 + x_1 x_6^2 - \frac{k}{x_1^2} + \frac{u_1}{m} \\ x_4 \\ -\frac{2x_2 x_4}{x_1} + \frac{2x_4 x_6 \sin x_5}{\cos x_5} + \frac{u_2}{m x_1^2 \cos^2 x_5} \\ x_6 \\ -x_4^2 \cos x_5 \sin x_5 - \frac{2x_2 x_6}{x_1} + \frac{u_3}{m x_1^2} \end{bmatrix}}_{f(x(t),\,u(t))} \quad \bigstar$$

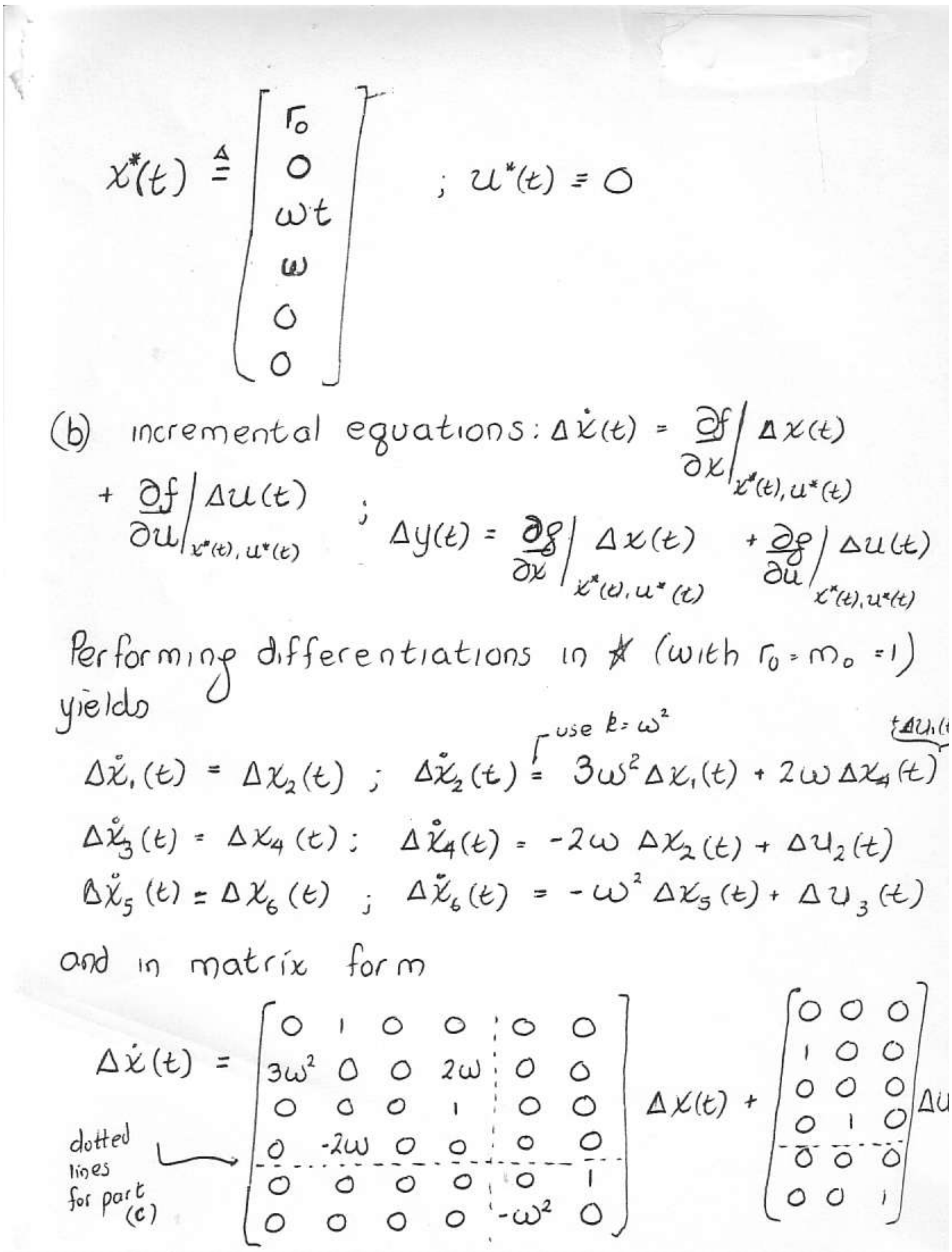
$y_1 = x_1$, $y_2 = x_3$ $y_3 = x_5$ and in vector/matrix form:

$$y(t) = \underbrace{\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}}_{g(x(t),u(t)) \,=\, Cx(t)} x(t) \qquad \text{" linear"}$$

Let $x_1 = r_0$, $x_2 = 0$, $x_3 = \omega t$, $x_4 = \omega$, $x_5 = x_6 = 0$ and $u(t) \equiv 0$ in ★ and obtain

$$\dot{x}(t) = \begin{bmatrix}0\\ r_0\omega^2 - \frac{k}{r_0^2}\\ \omega\\ 0\\ 0\\ 0\end{bmatrix} \underset{\substack{\uparrow\\ \text{using}\\ r_0^3\omega^2 = k}}{=} \begin{bmatrix}0\\ 0\\ \omega\\ 0\\ 0\\ 0\end{bmatrix}$$

i.e., all states remain at their constant value except for Θ which changes at a constant rate (as it should) Hence, the trajectory remains fixed. Therefore an equilibrium solution is given by



$$x^*(t) \triangleq \begin{bmatrix} r_0 \\ 0 \\ \omega t \\ \omega \\ 0 \\ 0 \end{bmatrix} \quad ; \; u^*(t) \equiv 0$$

(b) incremental equations: $\Delta\dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{x^*(t),u^*(t)} \Delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{x^*(t),u^*(t)} \Delta u(t)$; $\Delta y(t) = \left.\frac{\partial g}{\partial x}\right|_{x^*(t),u^*(t)} \Delta x(t) + \left.\frac{\partial g}{\partial u}\right|_{x^*(t),u^*(t)} \Delta u(t)$

Performing differentiations in ✯ (with $r_0 = m_0 = 1$) yields

use $k = \omega^2$ $\qquad$ $+\Delta u_1($

$$\Delta\dot{x}_1(t) = \Delta x_2(t) \; ; \; \Delta\dot{x}_2(t) \overset{!}{=} 3\omega^2 \Delta x_1(t) + 2\omega \Delta x_4(t)$$

$$\Delta\dot{x}_3(t) = \Delta x_4(t) ; \; \Delta\dot{x}_4(t) = -2\omega \, \Delta x_2(t) + \Delta u_2(t)$$

$$\Delta\dot{x}_5(t) = \Delta x_6(t) \; ; \; \Delta\dot{x}_6(t) = -\omega^2 \Delta x_5(t) + \Delta u_3(t)$$

and in matrix form

$$\Delta\dot{x}(t) = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{array}\right] \Delta x(t) + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \Delta u$$

dotted lines for part (c)

Solution of the "$\Delta \dot{x}_5$, $\Delta \dot{x}_6$" equations are independent of $\Delta x_{1,2,3,4}$ and Δu_3 only enters into "$\Delta \dot{x}_5$, $\Delta \dot{x}_6$" equations ; i.e, we can decouple the 6-dimensional system into 2 smaller systems

$$\left.\begin{aligned}
&\Delta \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \Delta u(t) \\
&\text{and} \\
&\Delta \dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \Delta \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta \hat{u}(t)
\end{aligned}\right\} \begin{array}{c}\text{Non} \\ \text{interacting} \\ \text{Subsystems} \\ \text{of} \\ \text{the} \\ \text{original} \\ \text{system}\end{array}$$

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin\phi$$

$$\phi(x,t) = \tilde{\phi}(x - \nu t)$$

a) Let's define $\xi = x - \nu t$, then $d\xi = dx$

$d\xi = -\nu\, dt$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \tilde{\phi}}{\partial x} = \frac{\partial \tilde{\phi}}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial \tilde{\phi}}{\partial \xi}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \tilde{\phi}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{\phi}}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{\phi}}{\partial x}\right)\frac{\partial \xi}{\partial x}$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{\phi}}{\partial \xi}\right)\frac{\partial \xi}{\partial x}$$

$$= \frac{\partial^2 \tilde{\phi}}{\partial \xi^2}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \tilde{\phi}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial \xi}\frac{\partial \xi}{\partial t} = -\nu \frac{\partial \tilde{\phi}}{\partial \xi}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \tilde{\phi}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \tilde{\phi}}{\partial t}\right) = \frac{\partial}{\partial \xi}\left(-\nu \frac{\partial \tilde{\phi}}{\partial \xi}\right)\frac{\partial \xi}{\partial t}$$

$$= \nu^2 \frac{\partial \tilde{\phi}}{\partial \xi^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{\phi}}{\partial \xi^2}(1 - \nu^2) = \sin\tilde{\phi}}$$

b) states : $x_1 = \tilde{\phi}$

$x_2 = \frac{\partial \tilde{\phi}}{\partial \xi}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{\sin x_1}{(1 - \nu^2)}$$

c) See simulation attached

d) $x_1(\xi) = 4 \tan^{-1}\left[\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\right]$

$x_1(0) = 4 \tan^{-1}(0) = \pi$

To find $x_2(0)$, we get $x_2(\xi)$ by

$$x_2(\xi) = \dot{x}_1(\xi) = \frac{4 \exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\left(-\frac{1}{\sqrt{1-\nu^2}}\right)}{1 + \exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)^2}$$

$$\Rightarrow \quad x_2(\xi) = \frac{-4}{\sqrt{1-\nu^2}} \cdot \frac{\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)}{1 + \exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)}$$

$$x_2(0) = \frac{-4}{\sqrt{1-\nu^2}} \cdot \frac{1}{2} = \frac{-2}{\sqrt{1-\nu^2}}$$

<u>note that</u> $0 < \nu < 1$

Let's check the solution

$$x_1(\xi) = 4 \tan^{-1}\left[\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\right]$$

$$\dot{x}_1(\xi) = \frac{-4}{\sqrt{1-\nu^2}} \; \frac{\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)}{1+\exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)}$$

we have to verify $\dot{x}_2(\xi) = \dfrac{\sin(x_1)}{1-\nu^2}$

(1) Let's compute $\dot{x}_2(\xi)$ first, since $\dot{x}_1 = x_2$, then $\dot{x}_2 = \ddot{x}_1$

$$\dot{x}_2(\xi) = \ddot{x}_1(\xi) = -\frac{4}{\sqrt{1-\nu^2}}\left[\frac{\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\left(\frac{-1}{\sqrt{1-\nu^2}}\right)\left[1+\exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)\right] - \star}{\left[1+\exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)\right]^2}\right.$$

$$\left.\star \; \exp\left(-\frac{\xi}{\sqrt{1-\nu^2}}\right)\exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)\left(\frac{-2}{\sqrt{1-\nu^2}}\right)\right]$$

$$\dot{x}_2(\xi) = \frac{4}{1-\nu^2}\left[\frac{\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\left[1-\exp\left(-\frac{2\xi}{\sqrt{1-\nu^2}}\right)\right]}{\left[1+\exp\left(\frac{-2\xi}{\sqrt{1-\nu^2}}\right)\right]^2}\right]$$

(2) Now let's compute $\dfrac{\sin(x_1)}{1-\nu^2}$

$$\frac{\sin(x_1)}{1-\nu^2} = \frac{1}{1-\nu^2}\sin\left[4\tan^{-1}\left(\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\right)\right]$$

$$= \frac{1}{1-\nu^2}\sin\left[4\alpha\right]$$

where $\alpha = \tan^{-1}\left[\exp\left(\frac{-\xi}{\sqrt{1-\nu^2}}\right)\right]$

$$\sin(4\alpha) = 2\sin(2\alpha)\cos(2\alpha)$$

$$= 2 \cdot \frac{2\,tg\,\alpha}{1+tg^2\alpha} \cdot \frac{1-tg^2\alpha}{1+tg^2\alpha}$$

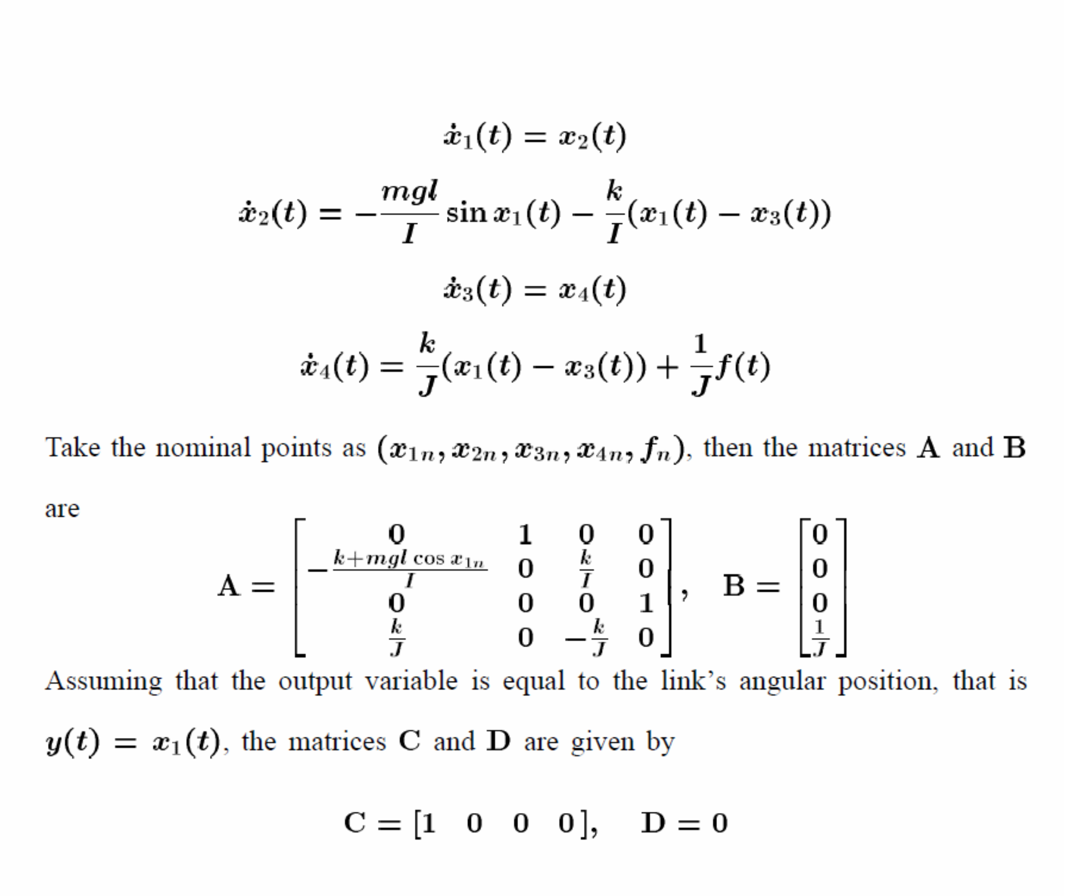
$$= 4\,\frac{tg\,\alpha\,(1-tg^2\alpha)}{(1+tg^2\alpha)^2}$$

$$\alpha = \tan^{-1}\left[\exp\left(\frac{-\tau}{\sqrt{1-\nu^2}}\right)\right]$$

then

$$\sin(4\alpha) = \frac{4\left[\exp\left(\frac{-\tau}{\sqrt{1-\nu^2}}\right)\left(1-\exp\left(\frac{-2\tau}{\sqrt{1-\nu^2}}\right)\right)\right]}{\left[1+\exp\left(\frac{-2\tau}{\sqrt{1-\nu^2}}\right)\right]^2}$$

then. $\frac{\sin(x_1)}{1-\nu^2} = \frac{1}{1-\nu^2}\sin(4\alpha) = \dot{x}_2(\tau)$ verified.



$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{mgl}{I}\sin x_1(t) - \frac{k}{I}(x_1(t) - x_3(t))$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{k}{J}(x_1(t) - x_3(t)) + \frac{1}{J}f(t)$$

Take the nominal points as $(x_{1n}, x_{2n}, x_{3n}, x_{4n}, f_n)$, then the matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k+mgl\cos x_{1n}}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

Assuming that the output variable is equal to the link's angular position, that is $y(t) = x_1(t)$, the matrices $\mathbf{C}$ and $\mathbf{D}$ are given by

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0], \qquad \mathbf{D} = 0$$

Solution Time-Varying

Begin with

$$\mathring{x}_2 = \frac{-4t}{1+t^2} x$$

[Note: Scalar eqⁿ $\dot{x} = a(t)x$ has solution $x(t) = e^{\int_0^t a(\sigma)d\tau} x(0)$

with solution

$$x_2(t) = e^{-\int_0^t \frac{4\tau}{1+\tau^2} d\tau} x_2(0)$$

$$= e^{-2 \log(1+\tau^2)\big|_0^t}$$

$$= \frac{x_{20}}{(1+t^2)^2}$$

So we have $\lim_{t \to \infty} x_2(t) = 0$

Now substitute into $\mathring{x}_1 \Rightarrow$

$$\mathring{x}_1 = -\frac{t}{1+t^2} x_1 + \frac{x_{20}}{(1+t^2)^2}$$

Note: The scalar equation $\mathring{x} = a(t)x + b(t)u$ has solution

$$x(t) = e^{\int_0^t a(\tau)} x(0) + \int_0^t e^{\int_\tau^t a(\xi)d\xi} b(\tau)u(\tau)d\tau$$

Hence:

$$x_1(t) = e^{-\int_0^t \frac{\tau}{1+\tau^2} d\tau} x_{10} + \int_0^t e^{-\int_\tau^t \frac{t}{1+\xi^2} d\xi} \frac{x_{20}}{(1+\tau^2)^2} d\tau$$

$$\Rightarrow \quad x_1(t) = e^{-\frac{1}{2}\log(1+\tau^2)\big|_0^t} x_{10} + \int_0^t e^{-\frac{1}{2}\log(1+\xi^2)\big|_\tau^t} \frac{x_{20}}{(1+\tau^2)} d\tau$$

$$\overset{\text{algebra}}{=} \frac{x_{10}}{\sqrt{1+t^2}} + \int_0^t \frac{1}{\sqrt{1+t^2}} \frac{x_{20}}{(1+\tau^2)^{3/2}} d\tau$$

$$\overset{x_{10}=1}{=} \frac{1}{\sqrt{1+t^2}} + \frac{x_{20}}{1+t^2}$$

So $x_1(t) \to 0$ too

Reader: Using the transpose of $A(t)$ instead, it is easy to obtain

$$x_1(t) = \frac{x_{10}}{\sqrt{1+t^2}}$$

and a challenging exercise is to show

$$x_2(t) = \frac{\sqrt{1+t^2}\left(t^3/4 + 5t/8\right) + \frac{3}{8}\sinh^{-1} t}{(1+t^2)^2}$$

and then

$$x_2(t) \to 1/4$$

2.2 HW2

2.2.1 Questions

ECE 717 – Homework Set 2

Due Thursday, September 25, 2014

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Barmish

ECE 334 – Homework Glider

(a) The two state equations below were derived from a model for normalized glider dynamics:

$$\dot{x}_1 = -\sin x_2 - ax_1^2;$$

$$\dot{x}_2 = \frac{-\cos x_2 + x_1^2}{x_1}$$

For $a = 1$, characterize the set of equilibria.

(b) Verify that one of the equilibria is given by

$$x^* = \begin{bmatrix} 0.8409 \\ -0.7854 \end{bmatrix}.$$

(c) Develop a Simulink Model for this nonlinear system and carry out some simulations for initial conditions which are in some region around the equilibrium x^* in (b) above. Show your simulations in the so-called phase plane. That is, begin at the point $x^*(0)$ in the (x_1, x_2) plane and obtain a plot of $(x_1(t), x_2(t))$.

(d) Do your simulations indicate that x^* is a stable equilibrium. If so, estimate a circular domain of attraction via simulation.

(e) Verify your result in (d) by finding the appropriate linearization matrix and then obtaining its eigenvalues.

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Consider the signal flow graph shown in Figure 3.

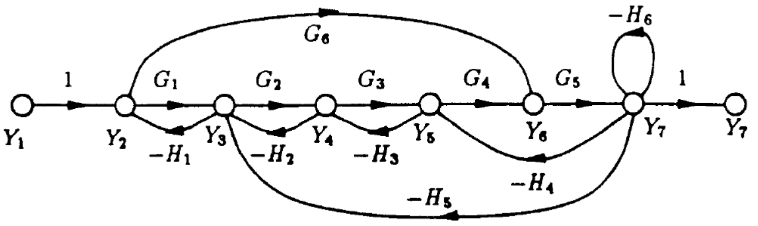

Figure 3: Signal Flow Graph

(a) Identify all the forward paths and their loop gains.

(b) Identify all the loops.

(c) Find the transfer function from Y_1 to Y_7 and from Y_1 to Y_2 using Mason's rule.

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Barmish

ECE 717 – Homework Dimension

(a) For the transfer function

$$H(s) = \frac{1}{(s+1)(s+2)},$$

show that a realization is given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \quad D = 0.$$

(b) Is the realization above minimal? Explain.

(c) When a unit step is applied to the state space system above which is initially at rest, an engineer is surprised to see that the some of the integrators saturate; i.e., one expects output

$$y(t) = \mathcal{L}^{-1}H(s)U(s) = \mathcal{L}^{-1}\frac{1}{s(s+1)(s+2)} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Explain.

(d) With unit step input, plot all three state responses $x_i(t)$ and explain the stable response you see in view of the saturation in (a).

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Barmish

ECE 717 – Homework Realization

Given the 3×2 transfer function matrix

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \\ H_{31}(s) & H_{32}(s) \end{bmatrix},$$

whose entries are proper, describe a state space realization

$$\Sigma = (A, B, C, D)$$

in terms of the individual realizations

$$\Sigma_{ij} = (A_{ij}, B_{ij}, C_{ij}, D_{ij}).$$

Then prove that your realization works by showing that

$$H_{\Sigma} = H(s).$$

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Barmish

ECE 717 – Homework Singular Value

(a) Consider a MIMO compensator $H(s)$ of appropriate dimension which is connected in series with a MIMO system $G(s)$ in a unity feedback configuration. Argue that the closed loop transfer matrix is

$$T(s) = (I + G(s)H(s))^{-1}G(s)H(s).$$

(b) For the MIMO system with transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{s}{2s^2+3s+1} \\ \\ \frac{2}{s-1} & \frac{s}{s^2+1} \end{bmatrix}$$

with compensator

$$H(s) = \begin{bmatrix} \frac{1}{s} & 2 \\ \\ -3 & -\frac{1}{s+1} \end{bmatrix},$$

find $T(s)$ and generate its associated closed-loop singular value plot.

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Barmish

ECE 717 – Homework Block

(a) Find the transfer function matrix $H_\Sigma(s)$ for the LTI system Σ described by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(b) Find a realization for the transfer function matrix $H(s)$ having entries

$$H_{11}(s) = \frac{-(s^2 - 4s - 5)}{s^3 - s^2 - 9s + 9}; \quad H_{12}(s) = \frac{s}{s-1}.$$

(c) Is your realization in part (b) minimal? Explain.

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2.2.2 Problem 1

Part (a)

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{Bmatrix} = \begin{Bmatrix} -\sin x_2 - ax_1^2 \\ \frac{-\cos x_2 + x_1^2}{x_1} \end{Bmatrix} \equiv \begin{Bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{Bmatrix} = f(x)$$

Letting $a = 1$, equilibrium is found by setting $\dot{x} = 0$ giving

$$f(x_1, x_2) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\sin x_2 - x_1^2 \\ \frac{-\cos x_2 + x_1^2}{x_1} \end{Bmatrix}$$

The following two equations are solved for x_1, x_2

$$-\sin x_2 - x_1^2 = 0 \tag{1}$$

$$\frac{-\cos x_2 + x_1^2}{x_1} = 0 \tag{2}$$

Equation (1) gives $x_1^2 = -\sin x_2$. Substituting this in (2) gives

$$\frac{-\cos x_2 - \sin x_2}{x_1} = 0$$

Assuming the state x_1 is finite, the above implies $-\cos x_2 - \sin x_2 = 0$ or $\tan x_2 = -1$ giving

$$\boxed{x_2 = \arctan(-1) = \frac{-\pi}{4} \pm 2n\pi}$$

For $n = 0, 1, 2, \cdots$ integer values . Substituting this value for x_2 back in (1) gives

$$\begin{aligned} x_1^2 &= -\sin x_2 \\ &= -\sin\left(\frac{-\pi}{4} \pm 2n\pi\right) \\ &= \sin\frac{\pi}{4} \\ &= \sqrt{\frac{1}{2}} \end{aligned}$$

Therefore

$$x_1 = \pm\left(\frac{1}{2}\right)^{\frac{1}{4}}$$

The equilibrium points are $\left\{\left(\frac{1}{2}\right)^{\frac{1}{4}}, \frac{-\pi}{4}\right\}$ and $\left\{-\left(\frac{1}{2}\right)^{\frac{1}{4}}, \frac{-\pi}{4}\right\}$. There are infinite number of equilibrium points for different n values but using $n = 0$ the above are the two equilibrium points considered. Approximate numerical values of the points are

$$\begin{aligned} &\{+0.8409, -0.7854\} \\ &\{-0.8409, -0.7854\} \end{aligned}$$

Part (b)

The point x^* was found in part(a). To verify that a point is an equilibrium point, $\dot{x}$ is evaluated at this point to see if $\dot{x} = 0$. Replacing x_1 by 0.8409 and x_2 by −0.7854 in

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{Bmatrix} = \begin{Bmatrix} -\sin x_2 - x_1^2 \\ \frac{-\cos x_2 + x_1^2}{x_1} \end{Bmatrix}$$

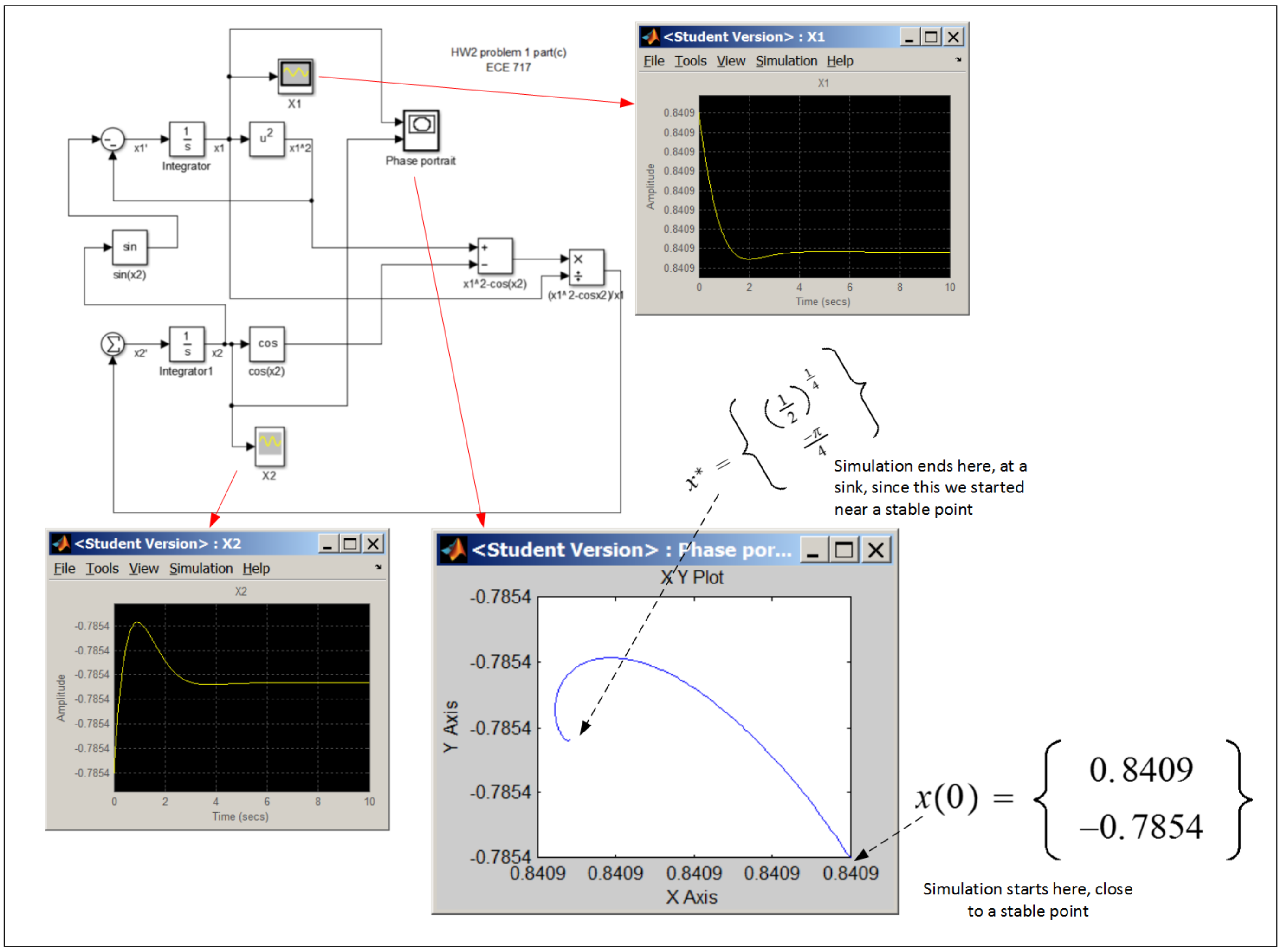
Gives

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Therefore x^* is an equilibrium point.

Part(c)

Simulink model was developed that implements part(a). Simulation was run for 10 seconds. The block XYgraph was used to generate phase portrait by having $x_1(t)$ being the X input to the block and $x_2(t)$ being the Y input to the block. Initial values of $x_1(0), x_2(0)$ used are near x^* found above, which is $x^* = \begin{Bmatrix} \left(\frac{1}{2}\right)^{\frac{1}{4}} \\ -\frac{\pi}{4} \end{Bmatrix}$. We see that the trajectory stays near the starting point used and is a sink stable point. Increasing the simulation time has no effect, since the trajectory will move to the sink and not leave it since it is a stable sink. The following shows the model used, a plot of $x_1(t), x_2(t)$ done separately, and the phase portrait plot.



$$x^* = \left\{ \begin{array}{c} \left(\frac{1}{2}\right)^{\frac{1}{4}} \\ \frac{-\pi}{4} \end{array} \right\}$$

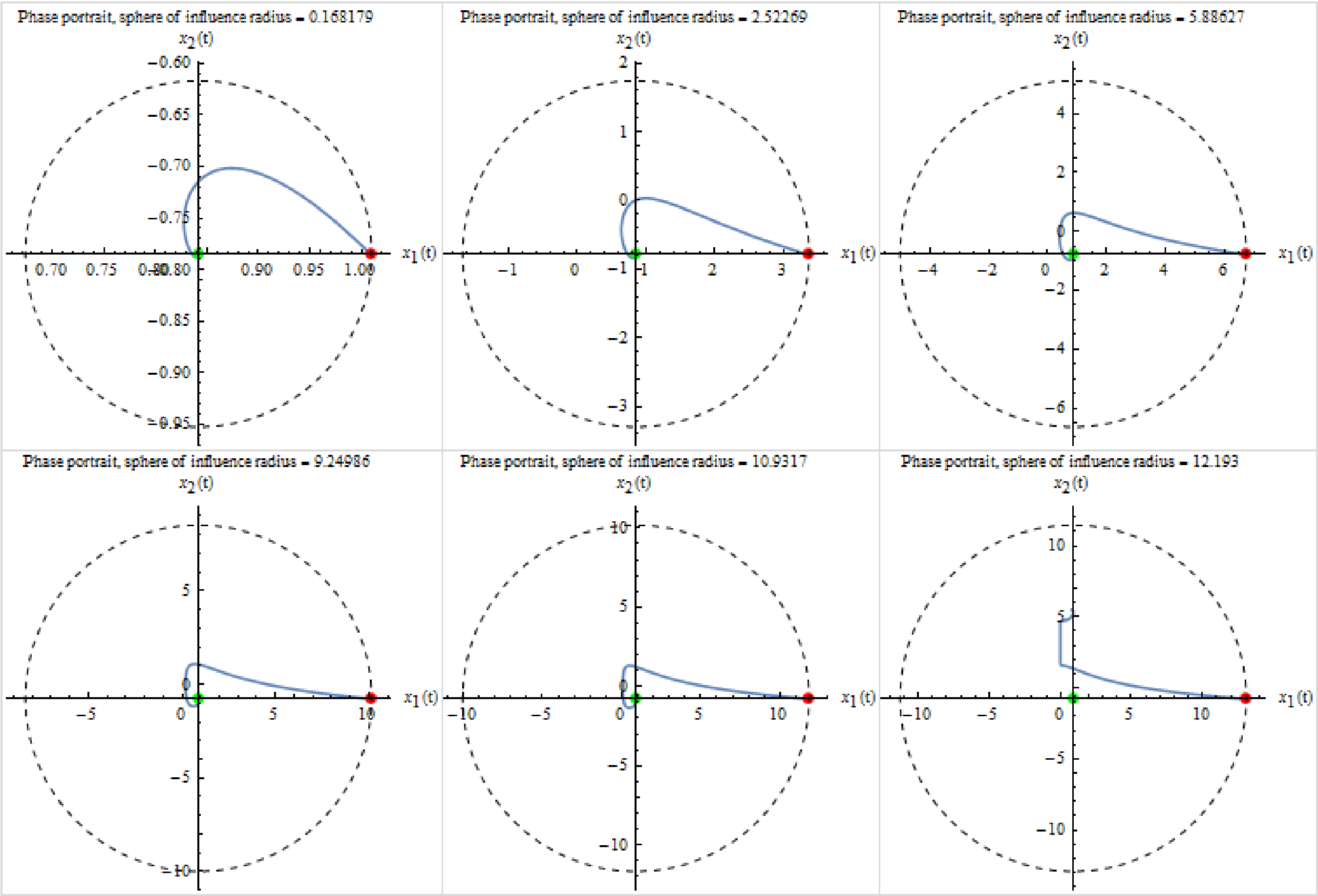
Simulation ends here, at a sink, since this we started near a stable point

$$x(0) = \left\{ \begin{array}{c} 0.8409 \\ -0.7854 \end{array} \right\}$$

Simulation starts here, close to a stable point

Part (d)

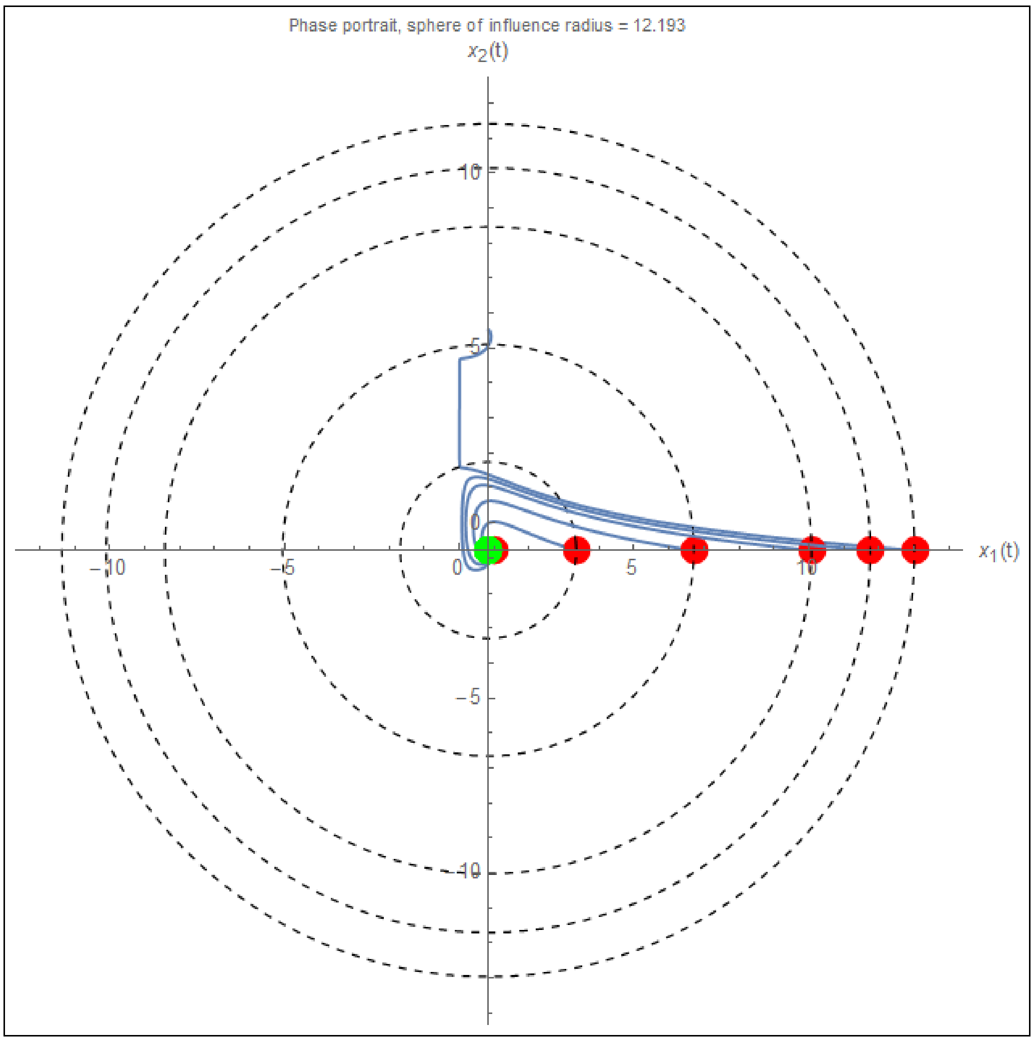
Yes, simulation indicates x^* is stable equilibrium. To estimate the circular domain of attraction, a circle centered at x^* with a radius r was used. The radius was increased in small increments. A starting initial point at the end of the radius was used to start the simulation. If the trajectory remained inside the circle and went to the sink at x^* then the radius was increased and the simulation is run again until the trajectory no longer remained inside the circle. The following are plots show this process for different values of r. The result shows that the sphere of influence around x^* has radius about 12.



Small code written to generate the above plot

```
makePlot[f_] := Module[{r, f1, f2, eq1, eq2, x10, sol, x20, t, x1, x2},
f1 = -Sin[x2[t]] - x1[t]^2;
f2 = (-Cos[x2[t]] + x1[t]^2)/x1[t];
eq1 = x1'[t] == f1;  eq2 = x2'[t] == f2;
x10 = Sqrt[Sqrt[1/2]];   x20 = -Pi/4;
r = f x10;
sol = {x1[t], x2[t]} /. First@NDSolve[{eq1, eq2, x1[0] == r, x2[0] == x20},
{x1[t], x2[t]}, {t, 0, 100}];
p1 = ParametricPlot[{sol[[1]], sol[[2]]}, {t, 0, 40},
                     AxesLabel -> {"x1(t)", "x2(t}"
```

The following plot is another view of the above but displayed on the same plot



Part (e)

The linearized A matrix for the system, which is the Jacobian of f, is now found.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(-\sin x_2 - ax_1^2\right) & \frac{\partial}{\partial x_2}\left(-\sin x_2 - ax_1^2\right) \\ \frac{\partial}{\partial x_1}\left(\frac{-\cos x_2 + x_1^2}{x_1}\right) & \frac{\partial}{\partial x_2}\left(\frac{-\cos x_2 + x_1^2}{x_1}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -2ax_1 & -\cos x_2 \\ \frac{1}{x_1^2}\left(x_1^2 + \cos x_2\right) & \frac{1}{x_1}\sin x_2 \end{pmatrix}$$

For $a = 1$

$$A = \begin{pmatrix} -2x_1 & -\cos x_2 \\ 1 + \frac{\cos x_2}{x_1^2} & \frac{1}{x_1}\sin x_2 \end{pmatrix}$$

The eigenvalues are found from

$$\det\left(\lambda I - A\right) = 0$$

$$p\left(\lambda\right) = \begin{vmatrix} \lambda + 2x_1 & \cos x_2 \\ -\left(1 + \frac{\cos x_2}{x_1^2}\right) & \lambda - \frac{1}{x_1}\sin x_2 \end{vmatrix}$$

$$= \lambda^2 + \lambda\left(2x_1 - \frac{\sin x_2}{x_1}\right) + \left(\cos x_2 - 2\sin x_2 + \cos^2 x_2\right)$$

If the real part of each λ_i is negative, then the system is stable. The numerical values of the equilibrium points found above are substituted in $p\left(\lambda\right)$ and the roots of the characteristic equation are found to determine the type of stability. For $x_1 = 0.8409$, $x_2 = -0.7854$ the roots are

$$\boxed{\lambda = -1.26135 \pm j1.1124}$$

The system is stable since the real part of the eigenvalues is negative. The type of stability is a sink. For the second equilibrium point $x_1 = -0.8409$, $x_2 = -0.7854$ the roots are

$$\boxed{\lambda = 1.26135 \pm j1.1124}$$

At this point the system is not stable since the real part is positive. The type of instability is a focus. The following is a summary of the result

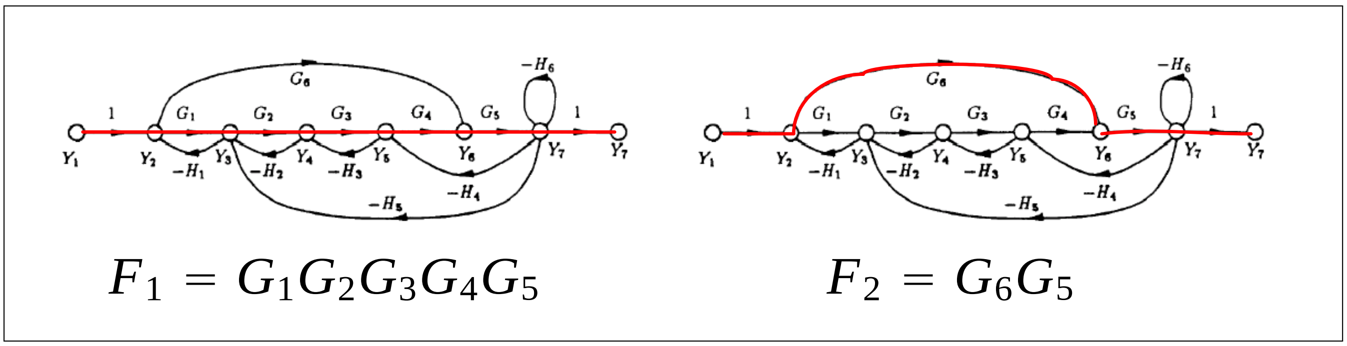
point	eigenvalues of A	stable/unstable
$\{x_1 = 0.8409, x_2 = -0.7854\}$	$-1.26135 \pm j1.1124$	Stable, sink
$\{x_1 = -0.8409, x_2 = -0.7854\}$	$1.26135 \pm j1.1124$	Not stable, focus

2.2.3 Problem 2

SOLUTION:

Part (a)

For the $\frac{Y_7}{Y_1}$, There are two forward paths. The following diagrams shows them with the gain on each.



$F_1 = G_1G_2G_3G_4G_5$ $F_2 = G_6G_5$

$$
\begin{aligned}
F_1 &= G_1G_2G_3G_4G_5\\
F_2 &= G_6G_5
\end{aligned}
$$

Now Δ_k is found for each forward loop. Δ_k is the Mason Δ but with F_k removed from the graph. Removing F_1 removes all the loops, hence

$$\Delta_1 = 1$$

When removing F_2 what remains is L_2 and L_3, hence

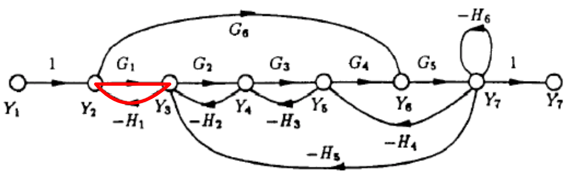
$$
\begin{aligned}
\Delta_2 &= 1 - (L_2 + L_3)\\
&= 1 - (-H_2G_2 - H_3G_3)\\
&= 1 + (H_2G_2 + H_3G_3)
\end{aligned}
$$

For the $\frac{Y2}{Y_1}$, there is one forward path $F_1 = 1$, the associated Δ_1 is

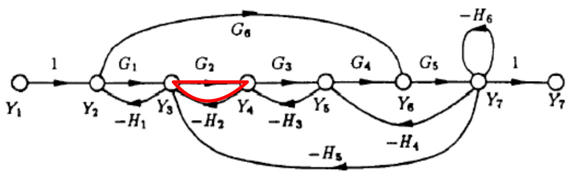
$$
\begin{aligned}
\Delta_1 &= 1 - \sum -G_2H_2 - G_3H_3 - G_4G_5H_4 - H_6 - G_2G_3G_4G_5H_5\\
&\quad + \sum (-G_2H_2)(-G_4G_5H_4) + (-G_2H_2)(-H_6) + (-G_3H_3)(-H_6)\\
&= 1 + \overbrace{G_2H_2 + G_3H_3 + G_4G_5H_4 + H_6 + G_2G_3G_4G_5H_5}^{\text{one at a time}} + \overbrace{G_2H_2G_4G_5H_4 + G_2H_2H_6 + G_3H_3H_6}^{\text{two at a time}}
\end{aligned}
$$

Part(b)

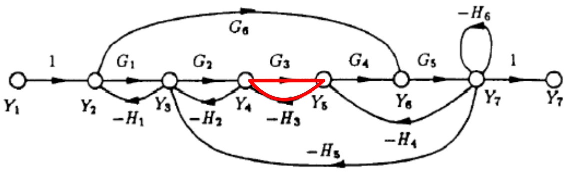
There are 8 loops. The following diagrams shows the loops with the gains



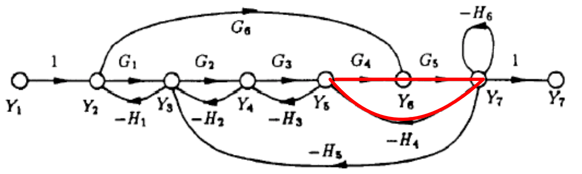
$L_1 = (G_1)(-H_1)$



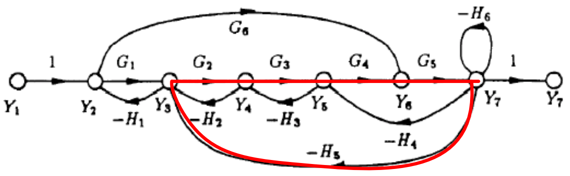
$L_2 = (G_2)(-H_2)$



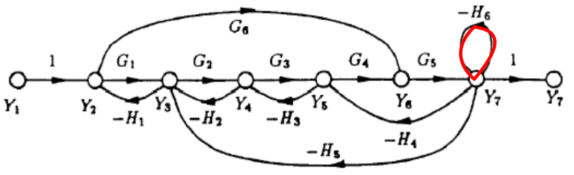
$L_3 = (G_3)(-H_3)$



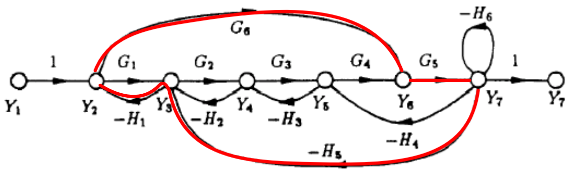
$L_4 = (G_4G_5)(-H_4)$



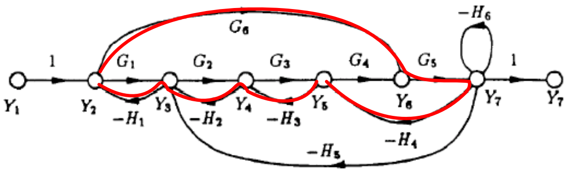
$L_5 = (G_2G_3G_4G_5)(-H_5)$



$L_6 = -H_6$



$L_7 = G_6G_5H_5H_1$



$L_8 = G_6G_5H_4H_3H_2H_1$

$$\begin{aligned}\Delta = 1 &- (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8)\\ &+ (L_1L_3 + L_1L_4 + L_1L_6 + L_2L_4 + L_2L_6 + L_3L_6 + L_3L_7) - L_1L_3L_6\end{aligned}$$

Therefore

$$\Delta = 1 + \overbrace{H_1G_1 + H_2G_2 + H_3G_3 + H_4G_4G_5 + H_5G_2G_3G_4G_5 + H_6 - G_5G_6H_1H_5 - G_6G_5H_4H_3H_2H_1}^{\text{one at a time}} \tag{1}$$

$$+ \overbrace{(H_1G_1H_3G_3 + H_1G_1H_4G_4G_5 + H_1H_6G_1 + H_2G_2H_4G_4G_5 + H_2G_2H_6 + H_3G_3H_6 - G_3H_3G_6G_5H_5H_1)}^{\text{two at time}}$$

$$+ \overbrace{H_1G_1H_3G_3H_6}^{\text{three at time}}$$

Part (c)

For $G(s) = \frac{Y_7}{Y_1}$, and using result found above in part (a) and part (b)

$$\begin{aligned}G(s) &= \frac{Y_7}{Y_1}\\ &= \frac{\Delta_1F_1 + \Delta_2F_2}{\Delta}\\ &= \frac{(G_1G_2G_3G_4G_5) + G_6G_5(1 + H_2G_2 + H_3G_3)}{\Delta}\end{aligned}$$

Where Δ is given in (1) found in part(b). To obtain $\frac{Y_2}{Y_1}$

$$
\begin{aligned}
\frac{Y_2}{Y_1} &= \frac{\Delta_1 F_1}{\Delta}\\
&= \frac{1+\overbrace{G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5}^{\text{one at a time}}+\overbrace{G_2H_2G_4G_5H_4+G_2H_2H_6+G_3H_3H_6}^{\text{two ata time}}}{\Delta}\\
&= \frac{1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5+G_2H_2G_4G_5H_4+G_2H_2H_6+G_3H_3H_6}{\Delta}
\end{aligned}
$$

2.2.4 Problem 3

Part (a)

Writing $H(s)$ as

$$H(s) = \frac{1}{s^2+3s+2}$$

The transfer function from A,B,C,D is

$$
\begin{aligned}
H_*(s) &= C\left(sI-A\right)^{-1}B+D\\
&= \begin{pmatrix}1 & 0 & 0\end{pmatrix}\left(\begin{pmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{pmatrix}-\begin{pmatrix}1 & 1 & 0\\ 0 & -2 & 1\\ 0 & 0 & -1\end{pmatrix}\right)^{-1}\begin{pmatrix}0\\ 1\\ -2\end{pmatrix}\\
&= \frac{1}{(s-1)(s+2)}-\frac{2}{(s-1)(s+1)(s+2)}\\
&= \frac{(s+1)-2}{(s-1)(s+1)(s+2)}\\
&= \frac{(s-1)}{(s-1)(s+1)(s+2)}
\end{aligned}
$$

There is a zero/pole cancellation due to common factor, which results in

$$H_*(s) = \frac{1}{s^2+3s+2}$$

```
A=[1 1 0;0 -2 1;0 0 -1];
B=[0;1;-2];
C=[1 0 0];
syms s;
C*inv((s*eye(3)-A))*B
ans =
1/((s - 1)*(s + 2)) - 2/((s - 1)*(s + 1)*(s + 2))
simplify(ans) %this causes pole/zero cancelation
1/(s^2 + 3*s + 2)
```

Hence it is a realization of $H(s)$

Part (b)

(A,B,C,D) is $\boxed{\text{not a minimal realization}}$ of $H(s)$. The actual plant given by $H(s)$ is a second order. The corresponding differential equation is second order

$$y''(t)+3y'(t)+2y(t)=u(t)$$

Therefore only two states are needed. These are normally taken to be the position and the velocity (for dynamic system) $\left(y,y'\right)$. These variables become x_1,x_2 in the state space formulation. However, the state space realization contains three states x_1,x_2,x_3. Therefore it is not minimal.

One way to check if (A,B,C,D) is minimal, is to compare the eigenvalues of A to the poles of the transfer function to see if they are the same. In this case the eigenvalues of A are found from

$$\det\left(\lambda I - A\right) = \left|\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix} - \begin{pmatrix}1 & 1 & 0\\ 0 & -2 & 1\\ 0 & 0 & -1\end{pmatrix}\right| = 0$$

Hence solving $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$, gives $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = -2$. However the poles of $H(s)$ are $\{-1, -2\}$, therefore, since the eigenvalues of A are not the same as the poles of $H(s)$ then the realization is not minimal.

Another way to verify if the system is minimal or not, is to check if the system is both controllable and observable. If one of these tests fail, then it is not a minimal realization.

```
A=[1 1 0;0 -2 1;0 0 -1];
B=[0;1;-2];
z=ctrb(A,B)
z =
0     1    -3
1    -4    10
-2     2    -2
rank(z)
2
```

Since the rank of the controllability matrix is less than the dimension of the matrix, then the realization shown is not controllable, which implies it is not minimal. No need to check for observability.

Part (c)

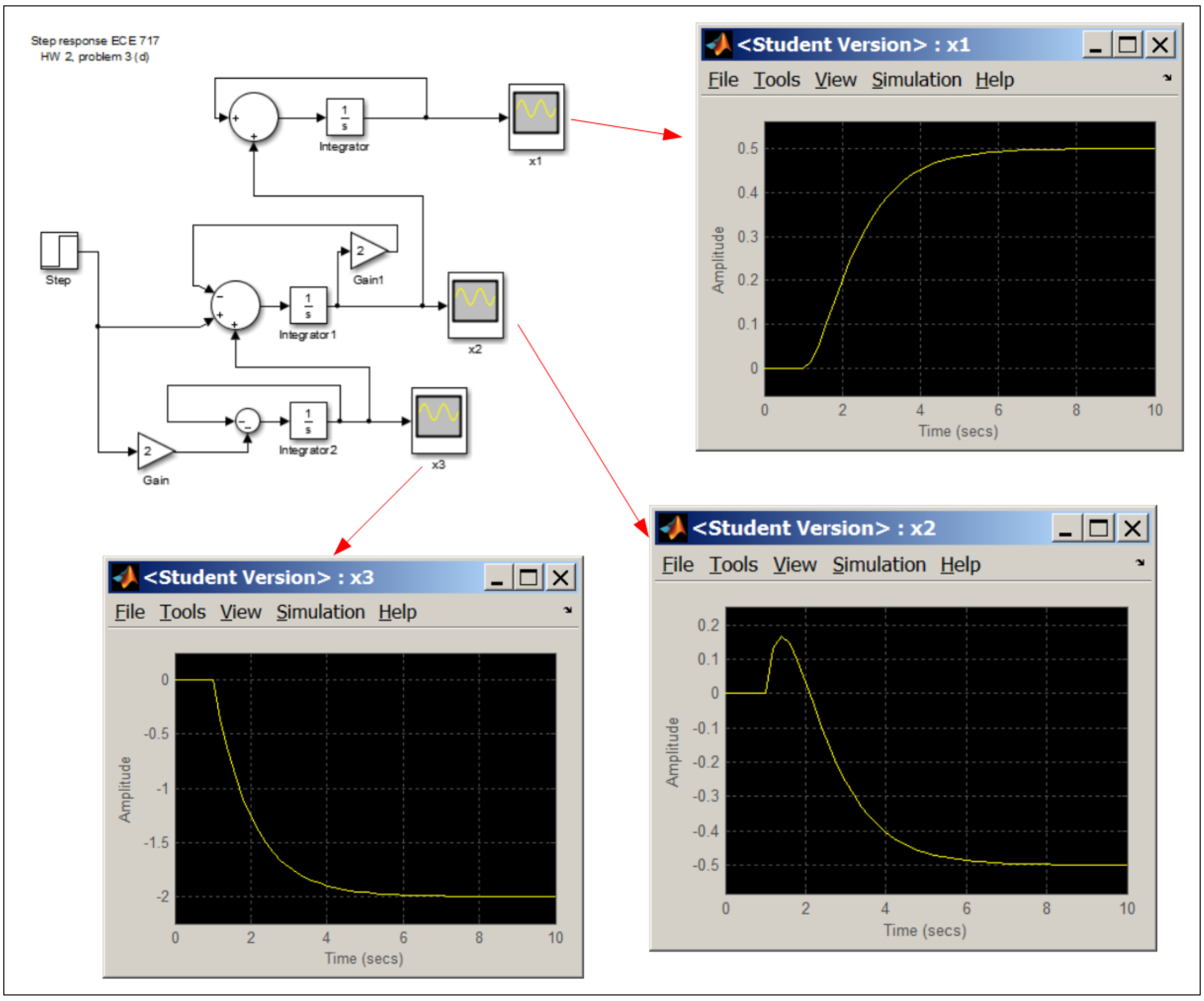
The differential equation of the system given by realization, not using the pole/zero cancellation is found from the transfer function $\frac{(s-1)}{(s-1)(s+1)(s+2)}$ giving

$$\boxed{y''' + 2y'' - y - 2 = u' - u}$$

When the input $u(t)$ is a unit step, its derivative becomes a Dirac delta $\delta(t)$ which causes a short time spike at $t = 0$ causing the integrator saturation. When any input contains a derivative of unit step and higher order derivatives (doublets and triplets function), they will cause Dirac delta to show up at $t = 0$. (the time the input is applied). Therefore, the system trajectory in state space is no longer unique and hence the given state vector x can not be used as state vector.

Part (d)

The following simulink model shows plot of the three states



The stable response shown above can be explained as follows. Even though the derivative of the unit step causes a Dirac delta spike, its duration is very short and instantaneous and occurs at $t = 0$. Hence it did not affect the overall response shown in the plot above at steady state since the transient response have died away by then.

2.2.5 Problem 4

The system transfer function is $H(s)$ of order $r \times m$ where r is the number of the output and m is the number of the input. Hence there are 2 inputs and 3 outputs in this example, i.e. D has size $r \times m = 3 \times 2$.

$$H(s) = \overbrace{\begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \\ H_{31}(s) & H_{32}(s) \end{pmatrix}}^{\text{number of input } (m)}$$

Let

$$A = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{32} \end{pmatrix}$$

And

$$B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \\ B_{31} & 0 \\ 0 & B_{32} \end{pmatrix}$$

And

$$C = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{31} & C_{32} \end{pmatrix}$$

And

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{pmatrix}$$

Now $H_*(s) = C(sI - A)^{-1} B + D$ is evaluated to show it is the same as the given system transfer function.

$$(sI - A)^{-1} = \begin{pmatrix} sI - A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & sI - A_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & sI - A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & sI - A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & sI - A_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI - A_{32} \end{pmatrix}^{-1}$$

This $sI - A$ is a diagonal matrix, then its inverse is the matrix with each elements on the diagonal inverted. Hence the above becomes

$$(sI - A)^{-1} = \begin{pmatrix} (sI - A_{11})^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (sI - A_{12})^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & (sI - A_{21})^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (sI - A_{22})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & (sI - A_{31})^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (sI - A_{32})^{-1} \end{pmatrix}$$

Now $C(sI - A)^{-1}$ is evaluated

$$C(sI - A)^{-1} = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{31} & C_{32} \end{pmatrix} \begin{pmatrix} (sI-A_{11})^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (sI-A_{12})^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & (sI-A_{21})^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (sI-A_{22})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & (sI-A_{31})^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (sI-A_{32})^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} C_{11}(sI-A_{11})^{-1} & C_{12}(sI-A_{12})^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{21}(sI-A_{21})^{-1} & C_{22}(sI-A_{22})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{31}(sI-A_{31})^{-1} & C_{32}(sI-A_{32})^{-1} \end{pmatrix}$$

Now $C(sI - A)^{-1} B$ is evaluated

$$C(sI - A)^{-1} B = \begin{pmatrix} C_{11}(sI - A_{11})^{-1} & C_{12}(sI - A_{12})^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{21}(sI - A_{21})^{-1} & C_{22}(sI - A_{22})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{31}(sI - A_{31})^{-1} & C_{32}(sI - A_{32})^{-1} \end{pmatrix}$$

Which reduces to

$$C(sI - A)^{-1} B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \\ B_{31} & 0 \\ 0 & B_{32} \end{pmatrix}$$

$$= \begin{pmatrix} C_{11}(sI - A_{11})^{-1} B_{11} & C_{12}(sI - A_{12})^{-1} B_{12} \\ C_{21}(sI - A_{21})^{-1} B_{21} & C_{22}(sI - A_{22})^{-1} B_{22} \\ C_{31}(sI - A_{31})^{-1} B_{31} & C_{32}(sI - A_{32})^{-1} B_{32} \end{pmatrix}$$

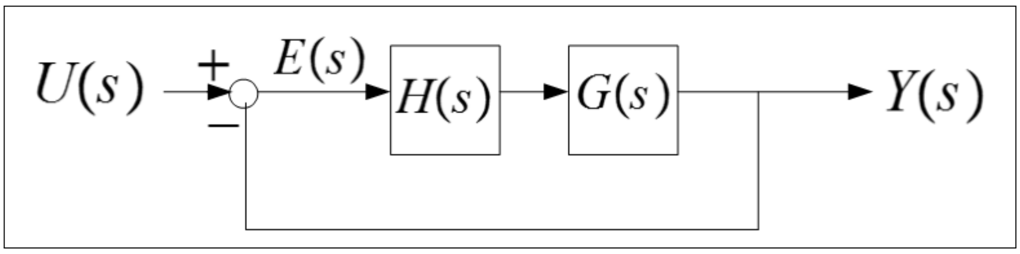
Finally, $C(sI-A)^{-1}B+D$ is evaluated giving

$$C(sI-A)^{-1}B+D=\begin{pmatrix}C_{11}(sI-A_{11})^{-1}B_{11} & C_{12}(sI-A_{12})^{-1}B_{12}\\ C_{21}(sI-A_{21})^{-1}B_{21} & C_{22}(sI-A_{22})^{-1}B_{22}\\ C_{31}(sI-A_{31})^{-1}B_{31} & C_{32}(sI-A_{32})^{-1}B_{32}\end{pmatrix}+\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\\ D_{31} & D_{32}\end{pmatrix}$$
$$=\begin{pmatrix}C_{11}(sI-A_{11})^{-1}B_{11}+D_{11} & C_{12}(sI-A_{12})^{-1}B_{12}+D_{12}\\ C_{21}(sI-A_{21})^{-1}B_{21}+D_{21} & C_{22}(sI-A_{22})^{-1}B_{22}+D_{22}\\ C_{31}(sI-A_{31})^{-1}B_{31}+D_{31} & C_{32}(sI-A_{32})^{-1}B_{32}+D_{32}\end{pmatrix}$$

But the above is $\begin{pmatrix}H_{11}(s) & H_{12}(s)\\ H_{21}(s) & H_{22}(s)\\ H_{31}(s) & H_{32}(s)\end{pmatrix}$ which is what we are asked to show.

2.2.6 Problem 5

Part(a)



Using standard method used in SISO with attention to dimensions, one can write

$$E(s)=U(s)-Y(s)$$
$$Y(s)=E(s)H(s)G(s)$$

Substituting the first equation above in the second equation to eliminate $E(s)$ gives (the letter s is dropped below to make the notation it more clear)

$$\begin{aligned}Y&=(U-Y)HG\\&=UHG-YHG\end{aligned}$$

Hence

$$Y+YHG=UHG$$

Factoring out $H(s)$ but since these are matrices, this operation generates an identity matrix with ones on the diagonal now and not scalar one as the case with SISO

$$Y(I+HG)=UHG$$

Hence

$$\frac{Y}{U}\equiv T(s)=(I+H(s)G(s))^{-1}H(s)G(s) \tag{1}$$

Which is what we asked to show. Another method is to use Mason rule. There is one forward path given by $H(s)G(s)$ and one loop given by $-H(s)G(s)$ where the negative sign is due to negative feedback, which is assumed throughout. Hence

$$\begin{aligned}\Delta&=I-(-H(s)G(s))\\&=I+H(s)G(s)\end{aligned}$$

and $\Delta_1=1$ since removing the forward path removes the loop. Hence

$$T(s)=\frac{\Delta_1(H(s)G(s))}{\Delta}$$

Since these are matrices, one uses matrix inversion in place of division , and the above becomes

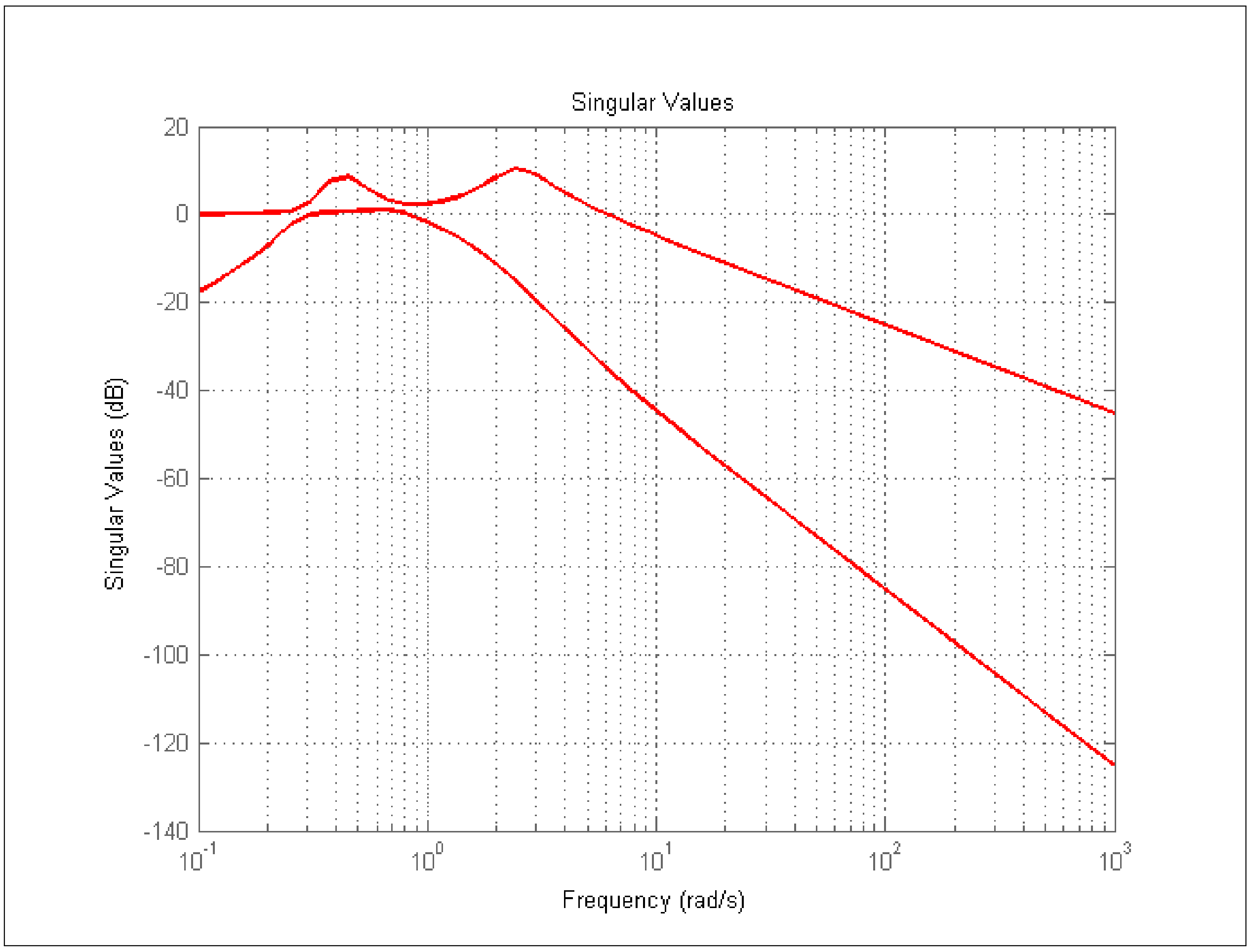
$$\boxed{T(s)=(I+H(s)G(s))^{-1}H(s)G(s)} \tag{2}$$

Which is the same as (1).

Part (b)

Matlab was used to plot the singular value of $T(s)$. The plots shows 2 lines, one for each eigenvalue, plotted from low frequency of 0.1 to 10^3 rad/sec. The following shows the plot and the code used

```
close all
s = tf('s');
G = [1/s s/(2*s^2+3*s+1); 2/(s-1)  s/(s^2+1)];
H = [1/s 2; -3  -1/(s+1)];
T = inv(eye(2)+G*H)*G*H
sigma(T,logspace(-1,3));
h = findobj(gcf,'type','line');
set(h,'linewidth',1.5);
set(h,'color','r');
grid
```



2.2.7 Problem 6

Part (a)

The transfer function matrix is given by

$$
\begin{aligned}
H(s) &= C(sI-A)^{-1}B+D \\
&= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\left(s\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s-1 & -2 & 0 \\ -4 & s+1 & 0 \\ 0 & 0 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (1)
\end{aligned}
$$

But

$$\begin{pmatrix} s-1 & -2 & 0 \\ -4 & s+1 & 0 \\ 0 & 0 & s-1 \end{pmatrix}^{-1} = \frac{adjugate\,(A)}{\det\,(A)}$$

Where

$$\begin{aligned} |(sI-A)| &= (s-1)\,(s+1)\,(s-1) + 2\,(-4\,(s-1)) \\ &= s^3 - s^2 - 9s + 9 \end{aligned}$$

And adjugate of $(sI-A)$ is $cofactor\,(sI-A)^T$ where

$$cofactor\,(sI-A) = \begin{pmatrix} (s+1)\,(s-1) & 4\,(s-1) & 0 \\ 2\,(s-1) & (s+1)\,(s-1) & 0 \\ 0 & 0 & (s-1)\,(s+1)-8 \end{pmatrix}$$

Hence

$$\begin{aligned} adjugate\,(sI-A) &= cofactor\,(sI-A)^T \\ &= \begin{pmatrix} (s-1)\,(s+1) & 2s-2 & 0 \\ 4s-4 & (s-1)\,(s+1) & 0 \\ 0 & 0 & (s-1)\,(s+1)-8 \end{pmatrix} \end{aligned}$$

Therefore

$$(sI-A)^{-1} = \frac{1}{s^3-s^2-9s+9}\begin{pmatrix} s^2-1 & 2s-2 & 0 \\ 4s-4 & s^2-1 & 0 \\ 0 & 0 & s^2-9 \end{pmatrix}$$

And (1) now becomes

$$\begin{aligned} H\,(s) &= \frac{1}{s^3-s^2-9s+9}\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s^2-1 & 2s-2 & 0 \\ 4s-4 & s^2-1 & 0 \\ 0 & 0 & s^2-9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{s^3-s^2-9s+9}\begin{pmatrix} 4s-4 & s^2-1 & 9-s^2 \\ 0 & 0 & s^2-9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{s^3-s^2-9s+9}\begin{pmatrix} -s^2+4s+5 \\ s^2-9 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{-s^2+4s+5}{-s^3+s^2+9s-9} \\ 1-\frac{s^2-9}{-s^3+s^2+9s-9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\left(s^2-2s-8\right)}{s^3-s^2-9s+9} \\ \frac{s}{s-1} \end{pmatrix} \end{aligned}$$

Therefore

$$\boxed{H_{11}\,(s) = \frac{-\left(s^2-4s-5\right)}{s^3-s^2-9s+9}}$$

and

$$\boxed{H_{21} = \frac{s}{s-1}}$$

Part (b)

Given the system transfer function

$$H\,(s) = \begin{pmatrix} \frac{-\left(s^2-4s-5\right)}{s^3-s^2-9s+9} & \frac{s}{s-1} \end{pmatrix}$$

This new system has one output and two inputs. The system in part(a) had two outputs and one input. Let

$$A=\begin{pmatrix} A_{11} & 0\\ 0 & A_{12}\end{pmatrix}$$

And

$$B=\begin{pmatrix} B_{11} & 0\\ 0 & B_{12}\end{pmatrix}$$

And

$$C=\begin{pmatrix} C_{11} & C_{12}\end{pmatrix}$$

And

$$D=\begin{pmatrix} D_{11} & D_{12}\end{pmatrix}$$

Now $H_{*}(s)=C(sI-A)^{-1}B+D$ is evaluated giving

$$\begin{aligned}H_{*}(s)&=\begin{pmatrix} C_{11} & C_{12}\end{pmatrix}\begin{pmatrix}(sI-A_{11})^{-1} & 0\\ 0 & (sI-A_{12})^{-1}\end{pmatrix}\begin{pmatrix} B_{11} & 0\\ 0 & B_{12}\end{pmatrix}+\begin{pmatrix} D_{11} & D_{12}\end{pmatrix}\\ &=\begin{pmatrix} C_{11}(sI-A_{11})^{-1} & C_{12}(sI-A_{12})^{-1}\end{pmatrix}\begin{pmatrix} B_{11} & 0\\ 0 & B_{12}\end{pmatrix}+\begin{pmatrix} D_{11} & D_{12}\end{pmatrix}\\ &=\begin{pmatrix} C_{11}(sI-A_{11})^{-1}B_{11}+D_{11} & C_{12}(sI-A_{12})^{-1}B_{12}+D_{12}\end{pmatrix}\end{aligned}$$

Now the following two equations are solved

$$\begin{aligned}C_{11}(sI-A_{11})^{-1}B_{11}+D_{11}&=\frac{-\left(s^{2}-4s-5\right)}{s^{3}-s^{2}-9s+9}\\ C_{12}(sI-A_{12})^{-1}B_{12}+D_{12}&=\frac{s}{s-1}\end{aligned}$$

Using the companion form, the first equation above results in

$$A_{11}=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -9 & 9 & 1\end{pmatrix}$$

$$B_{11}=\begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}$$

$$C_{11}=\begin{pmatrix} 5 & 4 & -1\end{pmatrix}$$

$$D_{11}=\begin{pmatrix} 0\end{pmatrix}$$

And for the second equation $\frac{s}{s-1}$ it is first converted to strict proper transfer function by long division, given $1+\frac{1}{s-1}$ and now the conversion is carried out for the companion form giving

$$A_{12}=\begin{pmatrix} 1\end{pmatrix}$$

$$B_{12}=\begin{pmatrix} 1\end{pmatrix}$$

$$C_{12}=\begin{pmatrix} 1\end{pmatrix}$$

$$D_{12}=\begin{pmatrix} 1\end{pmatrix}$$

Therefore the realization is now found by patching the above into larger matrices as follows

$$A=\begin{pmatrix} A_{11} & 0\\ 0 & A_{12}\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -9 & 9 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

And

$$B=\begin{pmatrix}B_{11} & 0\\ 0 & B_{12}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$$

And

$$C=\begin{pmatrix}C_{11} & C_{12}\end{pmatrix}=\begin{pmatrix}5 & 4 & -1 & 1\end{pmatrix}$$

And

$$D=\begin{pmatrix}D_{11} & D_{12}\end{pmatrix}=\begin{pmatrix}0 & 1\end{pmatrix}$$

Hence in $x' = Ax + Bu$ and $y = Cx + Du$ it becomes

$$\boxed{\begin{pmatrix}x_1'\\ x_2'\\ x_3'\\ x_4'\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -9 & 9 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}}$$

And

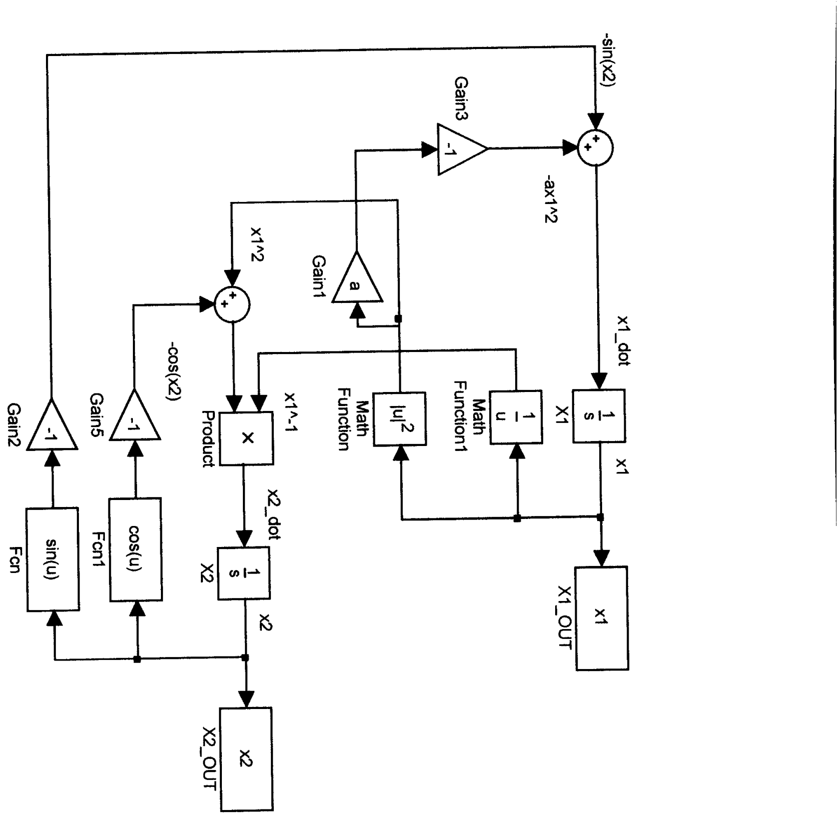
$$\boxed{y=\begin{pmatrix}5 & 4 & -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}}$$

is the realization.

Part (c)

The above is $\boxed{\text{not a minimal realization}}$. There are 4 states in the realization while the maximum number of poles in $H(s)$ is 3 which is located in $H_{11}(s)$. In other words, the largest part of the system is a third order differential equation, which needs only 3 states to fully describe. The system can be found to be not observable but it is controllable. Hence it fails one of the tests needed to qualify as a minimal realization.

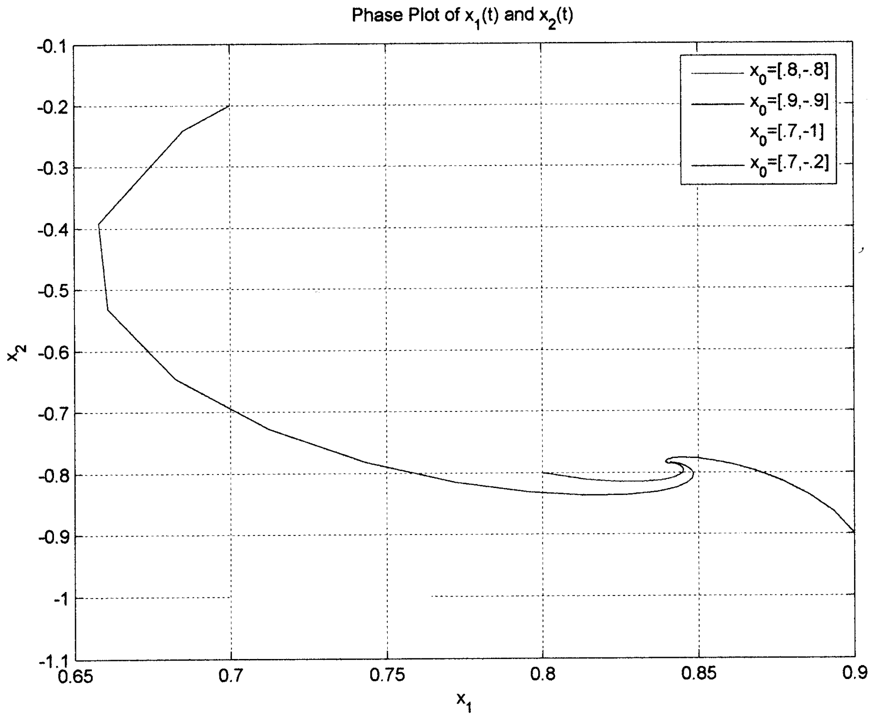
2.2.8 key solution



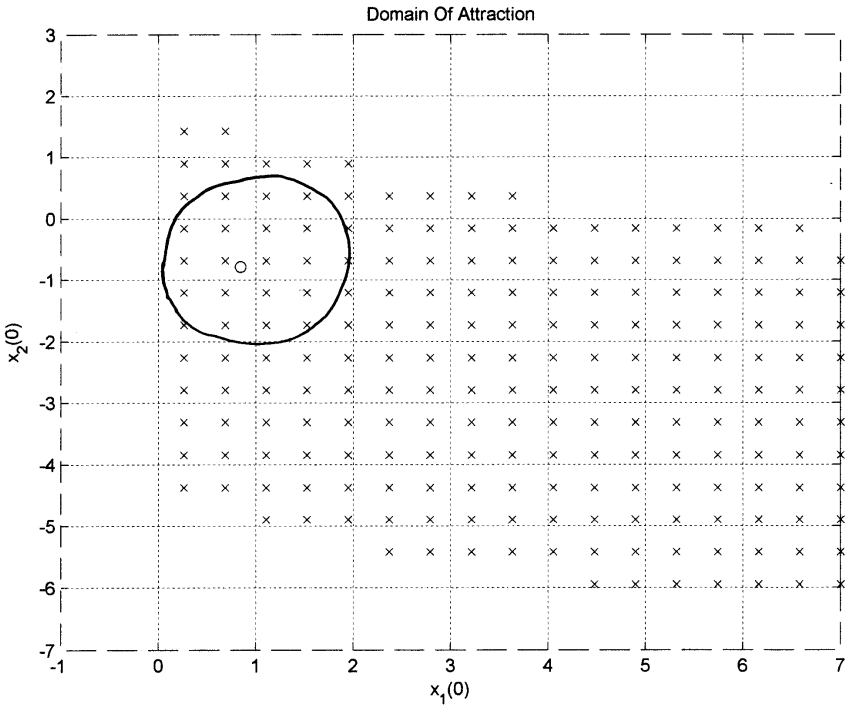
With $a=1$,

$X_{o,1} = \begin{bmatrix} .8409 \\ -.7854 \end{bmatrix}$

$X_{o,d} = \begin{bmatrix} .8409 \\ 5.4978 \end{bmatrix}$



Student Version of MATLAB



X: Converge to $\bar{X} = \begin{bmatrix} .8409 \\ -.7854 \end{bmatrix}$

* : does not converge

$\Omega_{\bar{X}} \approx$ Circle of radius .84 For $\bar{X}$ as above

Student Version of MATLAB

(d) First find partials

$$\frac{\partial f_1}{dx_1} = -2x_1 \quad \frac{\partial f_1}{dx_2} = -\cos x_2;$$

$$\frac{\partial f_2}{dx_1} = 1 + \frac{1}{x_1^2}\cos x_2; \quad \frac{\partial f_2}{dx_2} = \frac{\sin x_2}{x_1}.$$

Now evaluating these at x^* leads to linearization matrix

$$A = \begin{bmatrix} -1.6818 & -0.7071 \\ 2 & -.8409 \end{bmatrix}.$$

Now, using $eig(A)$ in Matlab gives $\lambda = -1.26 \pm j1.11$. Hence these eigenvalues have negative real part; i.e., the linearization is stable which is consistent with simulations.

(a) Forward paths and their respective loop gains for the $\frac{Y_7}{Y_1}$ transfer function:
$M_1 = G_1G_2G_3G_4G_5 \quad \Delta_1 = 1$
$M_2 = G_6G_5 \qquad\qquad \Delta_2 = 1 + G_2H_2 + G_3H_3$

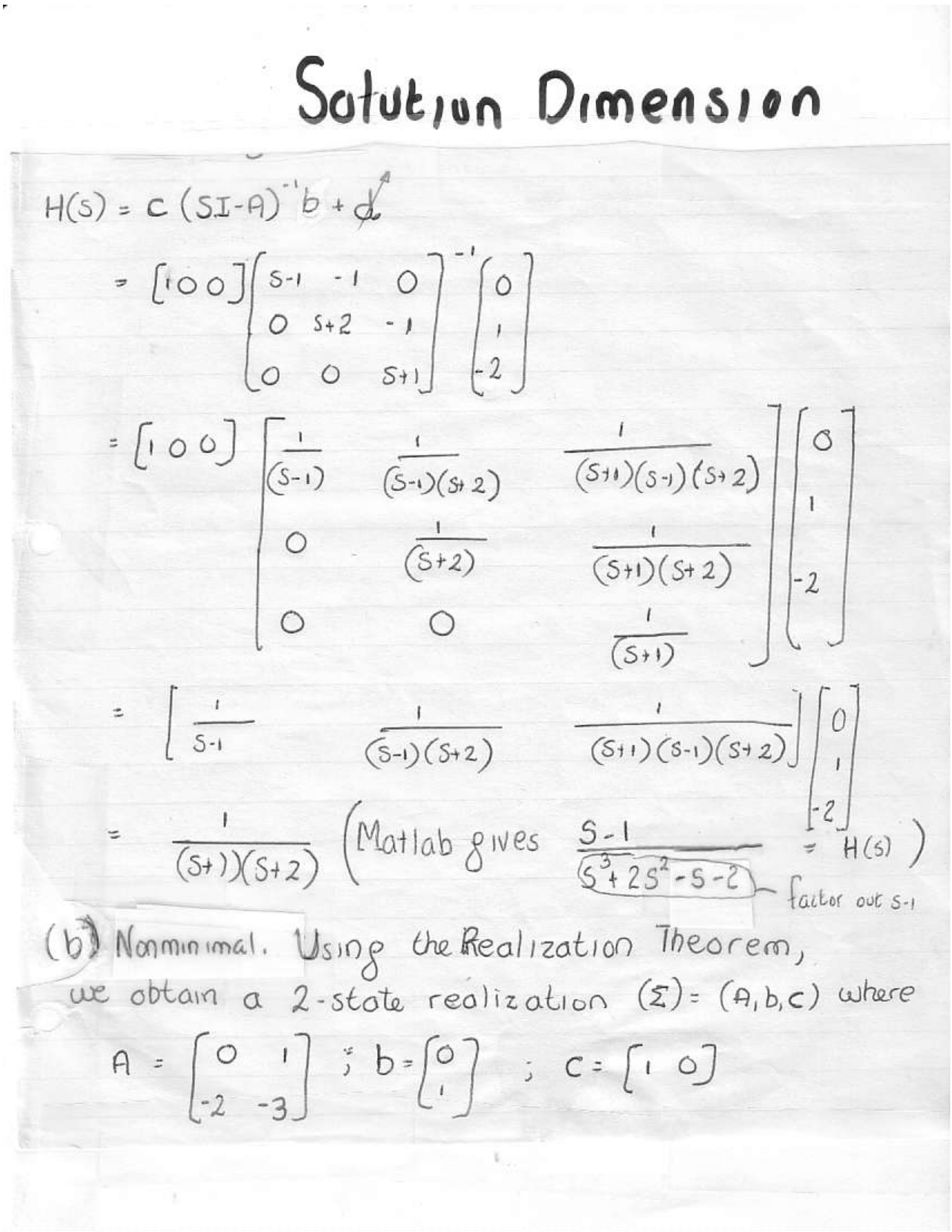
Forward paths and their respective loop gains for the $\frac{Y_2}{Y_1}$ transfer function:
$M_1 = 1$
$\Delta_1 = 1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5+G_2G_4G_5H_5H_4+G_2H_2H_6+G_2H_3H_6$

(b) All loops:
$\Delta = 1 + G_1H_1 + G_2H_2 + G_3H_3 + G_4G_5H_4 + H_6 + G_2G_3G_4G_5H_5 - G_5G_6H_1H_5 - G_5G_6H_1H_2H_3H_4+G_1G_3H_1H_3+G_1G_4G_5H_1H_4+G_1H_1H_6+G_2G_4G_5H_2H_4+G_2H_2H_6+G_3H_3H_6 - G_3G_5G_6H_1H_3H_5 + G_1G_3H_1H_3H_6$

(c)
$$\frac{Y_7}{Y_1} = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1 + G_2H_2 + G_3H_3)}{\Delta}$$

$$\frac{Y_2}{Y_1} = \frac{1 + G_2H_2 + G_3H_3 + G_4G_5H_4 + H_6 + G_2G_3G_4G_5H_5}{\Delta} + \frac{G_2H_2G_4G_5H_4 + G_2H_2H_6 + G_2H_3H_6}{\Delta}$$

The key here is that the "circuit" connected to node Y_2 can affect (and most likely will) the transfer function between Y_1 and Y_2. For instance, if Y_1 were 10 and Y_2 were defined to be zero, then Y_2 would not equal Y_1.



Solution Dimension

$$H(s) = c\,(sI-A)^{-1}b + \cancel{d}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s-1 & -1 & 0 \\ 0 & s+2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-1)} & \frac{1}{(s-1)(s+2)} & \frac{1}{(s+1)(s-1)(s+2)} \\ 0 & \frac{1}{(s+2)} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{(s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s+2)} & \frac{1}{(s+1)(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \quad \left(\text{Matlab gives } \frac{s-1}{\boxed{s^3+2s^2-s-2}} = H(s)\right)$$

factor out s-1

(b) Nonminimal. Using the Realization Theorem, we obtain a 2-state realization $(\Sigma) = (A, b, c)$ where

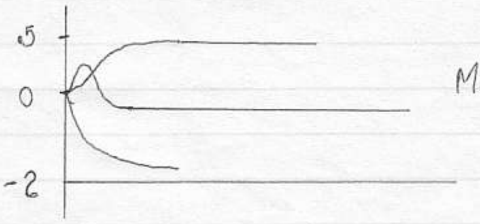
$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} ; \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(c) Obtain eig(A) = 1, -2, -1 with Matlab

However the poles of $H(s)$ are -1, -2. They are different because eig(A) describes all internal modes (e^{t}, e^{-2t}, e^{-t}) whereas $H(s)$ poles describe modes at the output (e^{-2t}, e^{-t}).

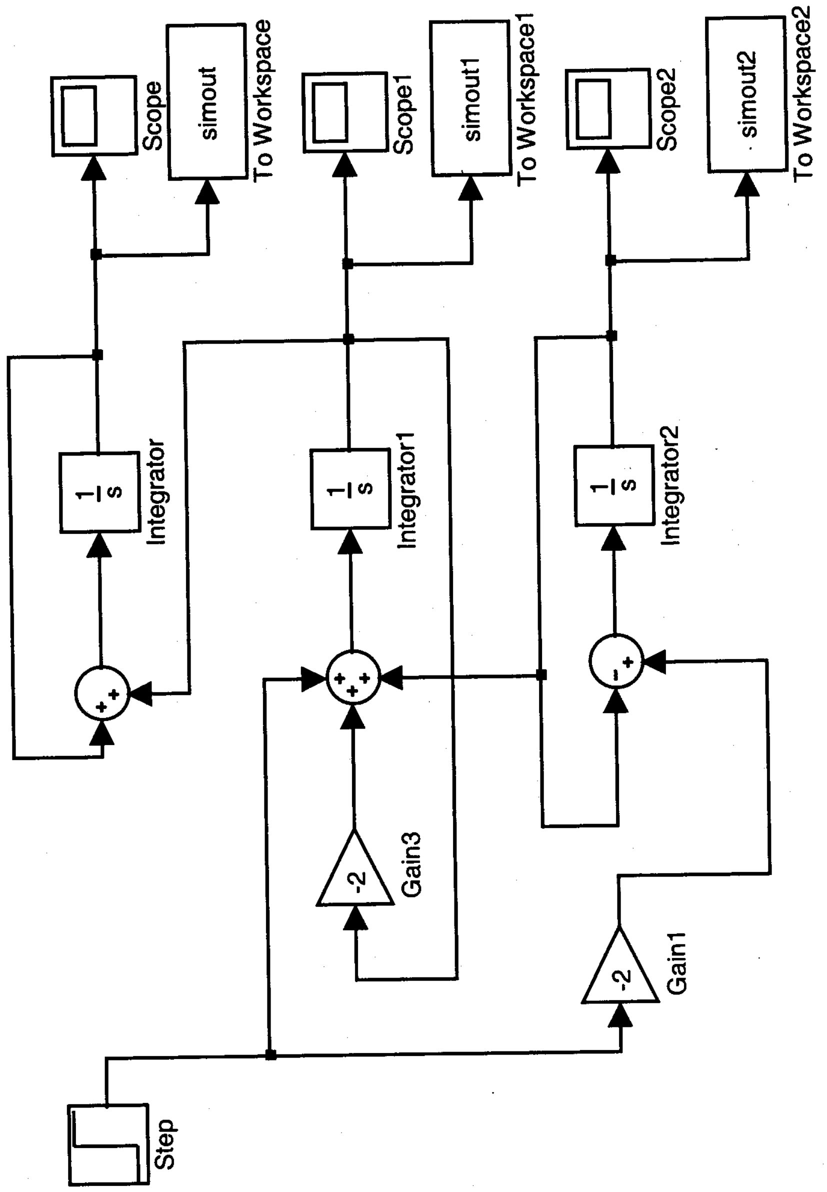
In view of above, there is a "hidden" internal mode which is unstable! The output, however, looks fine. The unstable internal mode leads to large signals and saturation of operational amplifiers

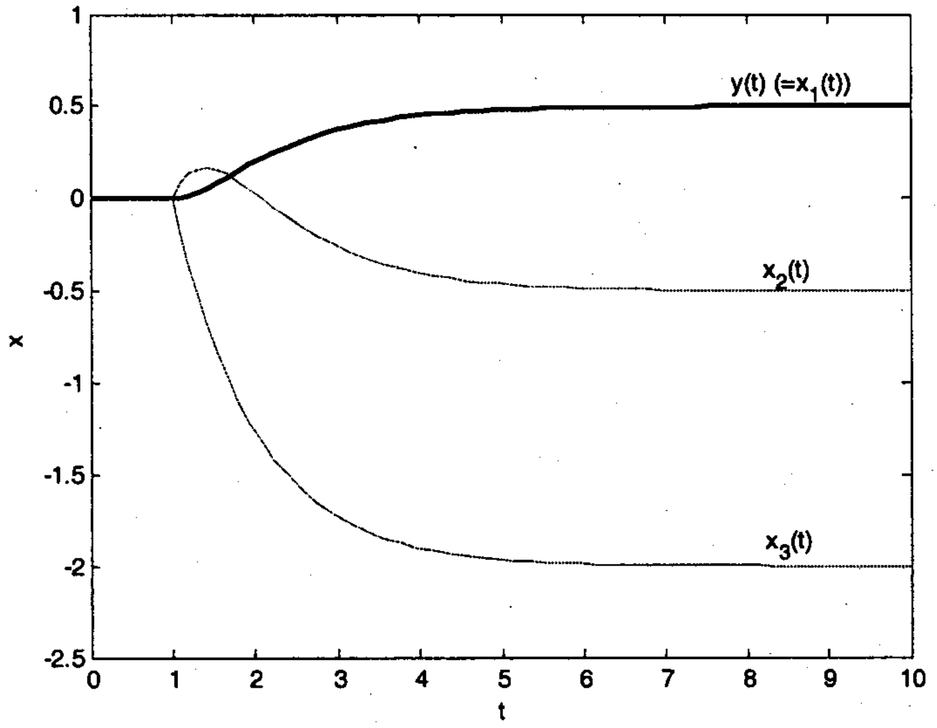
(d)

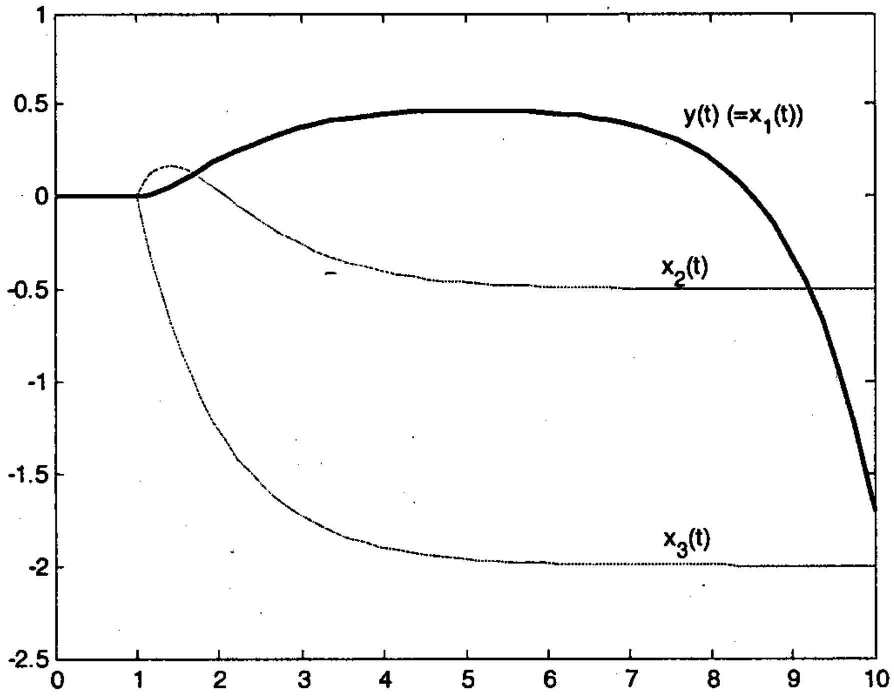


Matlab Plots look like this

Not all initial conditions lead to an unbounded response. The Step response uses $x(0) = 0$ which is not a "bad" initial condition. Other initial conditions lead to "blow-up" of state


Step
-2
Gain3
-2
Gain1
Integrator
1/s
Integrator1
1/s
Integrator2
1/s
Scope
simout
To Workspace
Scope1
simout1
To Workspace1
Scope2
simout2
To Workspace2


y(t) (=x_1(t))
x_2(t)
x_3(t)
x
t
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
9
10


y(t) (=x_1(t))
x_2(t)
x_3(t)
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
9
10

Homework Realization ·

$H(s)$: 3×2 .

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \\ H_{31}(s) & H_{32}(s) \end{bmatrix}$$

with $H_{ij}(s)$ assumed proper $i = 1, 2, 3.$ $j = 1, 2.$

According to the realization theorem, any proper TF $H(s)$ admits a realization Σ.

Let's $\Sigma_{ij} = (A_{ij}, B_{ij}, C_{ij}, D_{ij})$ be individual. realization for H_{ij} with $i = 1, 2, 3$ $j = 1, 2.$

We propose:

$$A = \begin{bmatrix} A_{11} & & & & & \\ & A_{12} & & & & \\ & & A_{21} & & & \\ & & & A_{22} & & \\ & & & & A_{31} & \\ & & & & & A_{32} \end{bmatrix}$$

Block diagonal.

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \\ B_{31} & 0 \\ 0 & B_{32} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{31} & C_{32} \end{bmatrix}$$

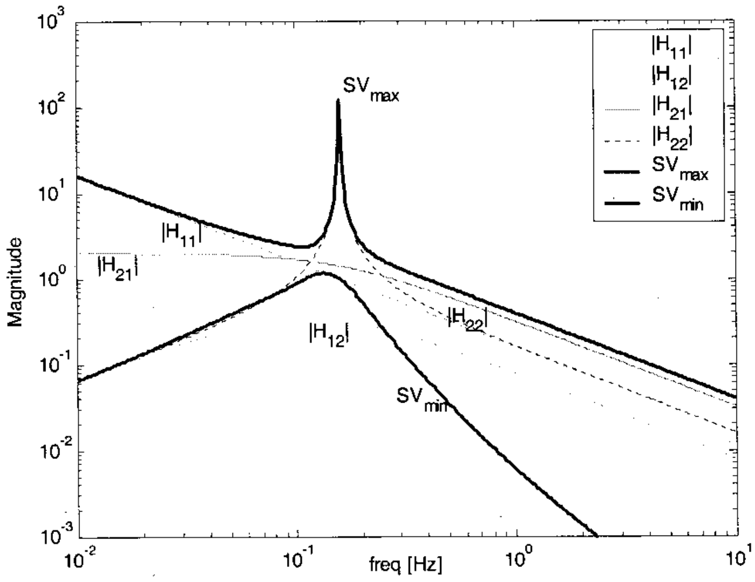
$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix}$$

Now, let's verify it.

$$H = C(SI-A)^{-1}B + D.$$

$$(s) = \begin{bmatrix} C_{11} & C_{12} & & & & \\ & & C_{21} & C_{22} & & \\ & & & & C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} SI-A_{11} & & & & & \\ & SI-A_{12} & & & & \\ & & SI-A_{21} & & & \\ & & & SI-A_{22} & & \\ & & & & SI-A_{31} & \\ & & & & & SI-A_{3} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \\ B_{31} & 0 \\ 0 & B_{32} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11}(SI-A_{11})^{-1}B_{11} + D_{11} & C_{12}(SI-A_{12})^{-1}B_{12} + D_{12} \\ C_{21}(SI-A_{21})^{-1}B_{21} + D_{21} & C_{22}(SI-A_{22})^{-1}B_{22} + D_{22} \\ C_{31}(SI-A_{31})^{-1}B_{31} + D_{31} & C_{32}(SI-A_{32})^{-1}B_{32} + D_{32} \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \\ H_{31}(s) & H_{32}(s) \end{bmatrix}$$


$|H_{11}|$
$|H_{12}|$
$|H_{21}|$
$|H_{22}|$
SV_{max}
SV_{min}
SV_{max}
$|H_{11}|$
$|H_{21}|$
$|H_{12}|$
$|H_{22}|$
SV_{min}
Magnitude
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
freq [Hz]
10^{0}
10^{1}

Solution Block

Compute

$$H_{\Sigma}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} s-1 & -2 & 0 \\ -4 & s+1 & 0 \\ 0 & 0 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\overset{\text{algebra}}{=} \begin{bmatrix} -\dfrac{s^2+4s-5}{s^3-s^2-9s+9} \\ \dfrac{s}{s-1} \end{bmatrix}$$

(b) For $H_{11}(s)$: $A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 9 & 1 \end{bmatrix}$; $b_{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$C_{11} = \begin{bmatrix} 5 & 4 & -1 \end{bmatrix}$; $d_{11} = 0$

Rewrite $H_{12}(s)$ as $\dfrac{s}{s-1} = 1 + \dfrac{1}{s-1}$

$\Rightarrow A_{12} = 1$; $b_{12} = 1$; $C_{12} = 1$; $d_{12} = 1$

Now realize H(s)

$$A=\begin{bmatrix} A_{11} & \\ & A_{12}\end{bmatrix};\ b=\begin{bmatrix} b_{11} & 0\\ 0 & b_{12}\end{bmatrix};\ C=\begin{bmatrix} C_{11} & \overset{C_{12}}{0}\\ \text{[illegible]} & \end{bmatrix};\ d=\begin{bmatrix} d_{11} & \overset{d_{12}}{\downarrow}\\ \text{[illegible]} & \end{bmatrix}$$

$$\Rightarrow\quad A=\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -9 & +9 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ b=\begin{pmatrix} 0 & \overset{\downarrow}{0}\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$$

$$C=\begin{bmatrix} 5 & 4 & -1 & 1.0\\ \text{[illegible]} & & & \end{bmatrix};\quad d=\begin{bmatrix} 0 & 1\\ \text{[illegible]} & \end{bmatrix}$$

(c) This realization is non-minimal. To prove this, take the realization in part (a), $\Sigma=(A,B,C,D)$ which has $H_\Sigma = H^T(s)$. We convert it into a new realization $\Sigma^* = (\underset{A_*}{A^T}, \underset{B_*}{C^T}, \underset{C_*}{B^T}, \underset{D_*}{D^T})$

Now calculate

$$H_{\Sigma_*}(s) = C_*(sI - A_*)^{-1}B_* + D^*$$

$$= B^T(sI - A^T)^{-1}C^T + D^T$$

$$= \left[C(sI - A)^{-1}B + D\right]^T$$

$$= H_{\Sigma}^T(s) = \left[H_{11}(s) \quad H_{12}(s)\right]$$

Hence, we have a realization of dimension 3 whereas the realization in part (b) is of dimension 2.

2.3 HW3

2.3.1 Questions

ECE 717 – Homework Set 3

Due Monday, October 6, 2014 @ 9 AM
Please slide under my door

;

Barmish

ECE 717 – Homework Invariance

For the state space system $\Sigma = (A, B, C)$, a linear feedback control

$$u(t) = Kx(t) + v(t)$$

is applied. For simplicity, assume that the system has a single input. Hence use column vector $B = b$ and row vector $K = k$.

(a) Find the transfer function for the closed loop system Σ_{cl}.

(b) Again assuming a single input (B = b), prove that the controllability matrices for the open loop and closed loop systems have the same rank; i.e., we say "controllability is invariant under linear state feedback." HINT: Show that any vector x which can be written as a linear combination of the columns of $\mathcal{C}_\Sigma$ can also be written as a linear combination of the columns of $\mathcal{C}_{\Sigma_{cl}}$ and vice versa.

;

Barmish

ECE 717 – Homework Design

Consider the LTI system $\Sigma = (A, b)$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}; \; b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Determine if this system is controllable.

(b) Find the open loop eigenvalues of the system. Is the system stable?

(c) Find a transformation matrix T taking this system to its companion canonical form $\tilde{\Sigma}$.

(d) Design a feedback gain matrix $\tilde{K}$ such that $\tilde{\Sigma}$ has two closed loop eigenvalues at $-1 + j$ and two at $-1 - j$.

(e) For the original system Σ find the gain matrix K leading to the same closed loop eigenvalues as in (d).

;

Barmish

ECE 717 – Homework Observability

Suppose $\Sigma_1 = (A_1, B_1, C_1, D_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2)$ are two equivalent systems with the same number of states n, inputs m and outputs r and that there exits a nonsingular transformation matrix T relating their states. Assuming the following *observability rank condition* is satisfied for each system:

$$\text{rank } \mathcal{O}_\Sigma = n$$

where

$$\mathcal{O}_\Sigma = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}.$$

Now find a formula for the nonsingular transformation T taking Σ_1 to Σ_2.

;

Barmish

ECE 717 – Homework Observer

(a) For the harmonic oscillator described by the state equations $\dot{x_1} = \omega_0 x_2$ and $\dot{x_2} = -\omega_0 x_1$ and measured output $y = x_1$, design an observer gain matrix L so that the error dynamics have eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Note that the gains in L will be functions of ω_0.

(b) Develop a Simulink program to study the performance of your observer. To this end, for $\omega_0 = 1, 10, 100, 1000$, perform the following experiment: With system initialized to $x_1(0) = x_2(0) = 1$ and observer initialized to $\hat{x}_0 = 0$, generate plots of $x_1(t)$ and $\hat{x}_1(t)$ on the same graph and comment on the observer's ability to track the state. Similarly, study the tracking of $x_2(t)$.

(c) With initial conditions as given in (b) above, study the performance of the observer in the $x_1 - x_2$ phase plain. That is, compare the phase plot of $x(t)$ with that of $\hat{x}(t)$.

(d) Now tune the observer gains with the goal of improving the tracking performance. Once you have decided on your final set of gains, repeat the experiment in part (c). Also explain the rationale for your choice of gains.

(e) Notice that the initial conditions for the system and the observer are different. Explain why the problem was formulated in this way.

;

2.3.2 Problem 1

Part(a)

Given

$$x' = Ax + bu$$
$$y = Cx$$

Replacing u with $kx+v$ results in

$$\begin{aligned} x' &= Ax + b\left(kx+v\right) \\ &= Ax + bkx + bv \\ &= \left(A+bk\right)x + bv \end{aligned}$$

In the above the dimensions are $A_{n\times n}, b_{n\times 1}, k_{1\times n}, v_{1\times 1}, x_{n\times 1}$. The transfer function is

$$H_{cl}\left(s\right) = C\left(sI - \left(A+bk\right)\right)^{-1} b \tag{1}$$

2.3.3 Part (b)

Let the controllability matrix for the open loop system (A, b) be $\mathbb{C}$ with some rank m, not necessarily full rank.

$$\mathbb{C} = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix}$$

We need to show that the rank of closed loop controllability matrix $\mathbb{C}_{cl}$ will also have the same rank m.

$$\mathbb{C}_{cl} = \begin{bmatrix} b & \left(A+bk\right)b & \left(A+bk\right)^2 b & \cdots & \left(A+bk\right)^{n-1} b \end{bmatrix}$$

Given any matrix, we know that we can perform elementary column or row operations on it without changing its rank. In other words, column operations are rank-preserving. And this is the main tool used to proof this.

For example, we can add the first column to the second, and this will not change the rank of the matrix. So the idea of the is this: We will perform column operations on each column $\mathbb{C}_{cl}$ to convert it back to the same corresponding column of $\mathbb{C}$.

The first step is to expand $\mathbb{C}_{cl}$ columns in order to see more clearly what operations are needed. Only the first 3 columns are expanded due to space limitation and this is sufficient to show the point

$$\begin{aligned} \mathbb{C}_{cl} &= \begin{bmatrix} b & Ab + bkb & \left(A^2 + \left(bk\right)^2 + Abk + bkA\right)b & \cdots \end{bmatrix} \\ &= \begin{bmatrix} b & Ab + bkb & A^2b + bkbkb + Abkb + bkAb & \cdots \end{bmatrix} \end{aligned} \tag{2}$$

The first column of $\mathbb{C}_{cl}$ is the same as the first column of $\mathbb{C}$, so we go to the next column which is $Ab + bkb$ which we want to make it Ab. post-multiplying the first column by kb and subtracting the result from the second column makes the second column become Ab.

Now we will work on the third column which is $A^2b + \left(bk\right)^2 b + Abkb + bkAb$ and search for column operations that converts this to A^2b. If we post-multiply the second column of $\mathbb{C}_{cl}$ by kb and subtract the result from the third column, now the third column becomes

$$\begin{aligned} \mathbb{C}_{cl}\left(3\right) &= \begin{bmatrix} A^2b + bkbkb + Abkb + bkAb \end{bmatrix} - \left[Ab + bkb\right]kb \\ &= \begin{bmatrix} A^2b + bkbkb + Abkb + bkAb \end{bmatrix} - \left[Abkb + bkbkb\right] \\ &= A^2b + bkAb \end{aligned}$$

We still have $bkAb$ left to remove. So we need to do more column operations. If we now post-multiply the first column by kAb and subtract the result from $\mathbb{C}_{cl}\left(3\right)$, then we finally obtain $\mathbb{C}_{cl}\left(3\right) = A^2b$. We continue doing this for each column in $\mathbb{C}_{cl}$ converting each column to the same columns as $\mathbb{C}$.

This shows that whatever rank $\mathbb{C}$ had, then $\mathbb{C}_{cl}$ will have the same rank. This is what we are asked to show.

2.3.4 Problem 2

Part(a)

The controllability matrix $\mathbb{C}$ is

$$\begin{aligned}\mathbb{C} &= \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix}\\ &= \boxed{\begin{bmatrix} 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1\\ 1 & 0 & -2 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix}}\end{aligned}$$

We find the rank of this matrix we can exchange the rows to convert it to row echelon form

$$\begin{bmatrix} 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1\\ 1 & 0 & -2 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We now see that it is full rank, since there are no zero pivots. Hence the rank 4. Since the rank is the same as n the size of A therefore

$$\boxed{\text{the open loop system is controllable}}$$

part(b)

$$\begin{aligned}p(\lambda) &= |\lambda I - A|\\ &= \begin{vmatrix} \lambda & 0 & -1 & 0\\ 0 & \lambda & 0 & -1\\ 2 & 1 & \lambda & 0\\ -1 & 1 & 0 & \lambda \end{vmatrix}\\ &= \lambda^4 + 3\lambda^2 + 3\end{aligned}$$

Now we solve $p(\lambda) = 0$. The roots of this characteristic equation (the same as eigenvalues of A) are found to be

$$\begin{aligned}\lambda_1 &= -0.34 + 1.27j\\ \lambda_2 &= -0.34 - 1.27j\\ \lambda_3 &= +0.34 + 1.27j\\ \lambda_4 &= +0.34 + 1.27j\end{aligned}$$

We see that there are two eigenvalues whose real part is positive, hence the

$$\boxed{\text{open loop system is not stable}}$$

part(c)

The target system is the companion form, which is

$$x' = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}}^{\tilde{A}} x + \overbrace{\begin{bmatrix} 0\\ 0\\ 0\\ 1 \end{bmatrix}}^{\tilde{b}} u$$

Where the last row of $\tilde{A}$ is taken from the characteristic polynomial terms in original A but in reverse order and by changing the sign. The characteristic polynomial of the original A was found above, here it is again

$$\begin{aligned}P(s) &= a_4s^4 + a_3s^2 + a_2s + a_0\\ &= s^4 + 3s^2 + 3\end{aligned}$$

Hence $a_0 = 3, a_1 = 0, a_2 = 3, a_3 = 0$, therefore the target system is

$$x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Now we find $\mathbb{C}, \tilde{\mathbb{C}}$ and then find

$$T = \tilde{\mathbb{C}}\mathbb{C}^{-1} \tag{3}$$

The controllability matrix $\mathbb{C}$ of the original system was found in part (a) as

$$\mathbb{C} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence

$$\mathbb{C}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The controllability matrix $\tilde{\mathbb{C}}$ is given by the following

$$\begin{aligned} \tilde{\mathbb{C}} &= \begin{bmatrix} \tilde{b} & \tilde{A}\tilde{b} & \tilde{A}^2\tilde{b} & \tilde{A}^3\tilde{b} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \end{bmatrix} \end{aligned}$$

Now we can find T using (3)

$$\begin{aligned} T &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}} \end{aligned}$$

To check T we apply it to A and see if we obtain $\tilde{A}$

$$\begin{aligned} \tilde{A} &= TAT^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & 0 \end{bmatrix} \end{aligned}$$

So T has been verified OK.

Part(d)

Let the control input be $u = \tilde{K}x + v$, where $\tilde{K} = \begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix}$. Therefore the closed loop system become

$$
\begin{aligned}
x' &= \tilde{A}x + \tilde{b}\left(\tilde{K}x + v\right) \\
&= \overbrace{\left(\tilde{A} + \tilde{b}\tilde{K}\right)}^{A_{\text{closed}}} x + \tilde{b}v
\end{aligned}
$$

Hence

$$
\begin{aligned}
A_{\text{closed}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_0 - 3 & k_1 & k_2 - 3 & k_3 \end{bmatrix} \qquad (4)
\end{aligned}
$$

The characteristic polynomial of the closed loop A_{closed} is found from

$$
\begin{aligned}
p(\lambda) &= \left|\lambda I - A_{\text{closed}}\right| \\
&= \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 3 - k_0 & -k_1 & 3 - k_2 & \lambda - k_3 \end{vmatrix} \\
&= \lambda^4 - \lambda^3 k_3 + \lambda^2 (3 - k_2) - \lambda k_1 + (3 - k_0) \qquad (5)
\end{aligned}
$$

We want the above polynomial to be equal to the polynomial with the desired roots given below, where the two unstable roots of the open loop have now been replaced with the given two stable roots. The stable roots of the original system are not modified since they are already stable.

$$
\begin{aligned}
\lambda_1 &= -0.34 + 1.27j \\
\lambda_2 &= -0.34 - 1.27j \\
\lambda_3 &= -1 + 1j \\
\lambda_4 &= -1 - j
\end{aligned}
$$

In other words, we want to force (5) to be the same as the following desired characteristic polynomial

$$
\begin{aligned}
p_{\text{design}}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \\
&= \left(\lambda - \left(-0.34 + 1.27j\right)\right)\left(\lambda - \left(-0.34 - 1.27j\right)\right)\left(\lambda - \left(-1 + 1j\right)\right)\left(\lambda - \left(-1 - j\right)\right) \\
&= \lambda^4 + 2.68\lambda^3 + 5.088\,5\lambda^2 + 4.817\lambda + 3.457 \qquad (6)
\end{aligned}
$$

Comparing coefficients of (5) with (6) and solving for k_i gives

$$
\begin{aligned}
k_3 &= -2.68 \\
3 - k_2 &= 5.088\,5 \\
k_1 &= -4.817 \\
(3 - k_0) &= 3.457
\end{aligned}
$$

Hence

$$
\begin{aligned}
k_3 &= -2.68 \\
k_2 &= -2.088\,5 \\
k_1 &= -4.817 \\
k_0 &= -0.457
\end{aligned}
$$

And the required gain vector is

$$
\boxed{\tilde{K} = \begin{bmatrix} -0.457 & -4.817 & -2.0885 & -2.68 \end{bmatrix}}
$$

Part(e)

In the above the gain $\tilde{K}$ vector was found for $\tilde{A}$ based system (the controllable form), however our original system is A. The gain vector is transformed using T found earlier

$$\begin{aligned}\tilde{K} &= KT^{-1}\\ K &= \tilde{K}T\\ &= \begin{bmatrix}-0.457 & -4.817 & -2.0885 & -2.68\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}\end{aligned}$$

Hence

$$\boxed{K = \begin{bmatrix}-2.0885 & 1.631\,5 & -2.68 & -2.137\end{bmatrix}}$$

To verify the above, we now find the eigenvalues of $[A - bK]$ and see if it gives the same eigenvalues we have designed for under $\tilde{A}$.

$$\begin{aligned}[A + bK] &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}-2.0885 & 1.6315 & -2.68 & -2.137\end{bmatrix}\\ &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -4.0885 & 0.6315 & -2.68 & -2.137\\ 1 & -1 & 0 & 0\end{bmatrix}\end{aligned}$$

The eigenvalues of the above matrix is $\left\{-0.34 - 1.27j, -0.34 + 1.27j, -1 - 1j, -1 + 1j\right\}$ and these are the same eigenvalues used in the design under $\tilde{A}$.

2.3.5 Problem 3

Let the first system be

$$\begin{aligned}x_1' &= A_1x_1 + B_1u\\ y &= C_1x_1 + D_1u\end{aligned} \tag{1}$$

And the second system be

$$\begin{aligned}x_2' &= A_2x_2 + B_2u\\ y &= C_2x_1 + D_2u\end{aligned}$$

And assume there exists a non-singular constant matrix T such that $x_2 = Tx_1$. We need to T. By applying this transformation to (1) we obtain the transformations

$$\begin{aligned}A_2 &= TA_1T^{-1}\\ B_2 &= TB_1\\ C_2 &= C_1T^{-1}\\ D_2 &= D_1\end{aligned}$$

Now, let Θ_2 be the observability matrix for first system given by

$$\Theta_2 = \begin{pmatrix}C_2\\ C_2A_2\\ C_2A_2^2\\ \vdots\\ C_2A_2^{n-1}\end{pmatrix}$$

Applying the above transformations to Θ_2 results in

$$\Theta_2 = \begin{pmatrix} C_2 \\ C_2A_2 \\ C_2A_2^2 \\ \vdots \\ C_2A_2^{n-1} \end{pmatrix} = \begin{pmatrix} C_1T^{-1} \\ \left(C_1T^{-1}\right)\left(TA_1T^{-1}\right) \\ \left(C_1T^{-1}\right)\left(TA_1^2T^{-1}\right) \\ \vdots \\ \left(\left(C_1T^{-1}\right)\left(TA_1^{n-1}T^{-1}\right)\right) \end{pmatrix} = \begin{pmatrix} C_1T^{-1} \\ C_1A_1T^{-1} \\ C_1A_1^2T^{-1} \\ \vdots \\ C_1A_1^{n-1}T^{-1} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1A_1 \\ C_1A_1^2 \\ \vdots \\ C_1A_1^{n-1} \end{pmatrix} T^{-1} = \Theta_1 T^{-1}$$

Therefore

$$\Theta_2 = \Theta_1 T^{-1}$$
$$\Theta_2 T = \Theta_1$$

Hence

$$\boxed{T = \Theta_2^{-1}\Theta_1}$$

2.3.6 Problem 4

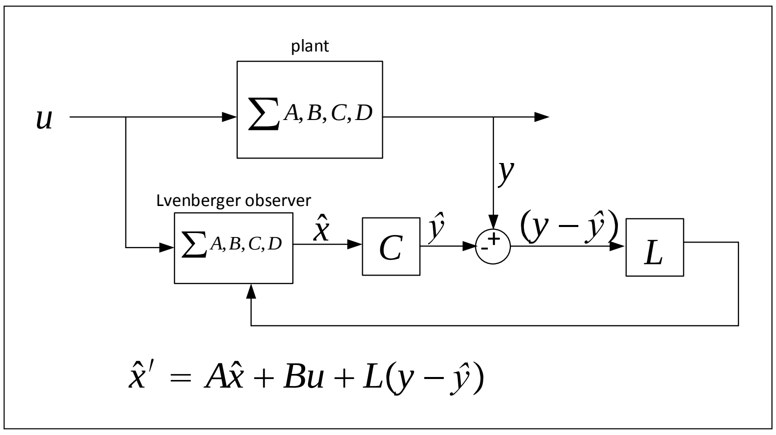
Part(a)

The system is given by $x' = Ax; y = Cx$

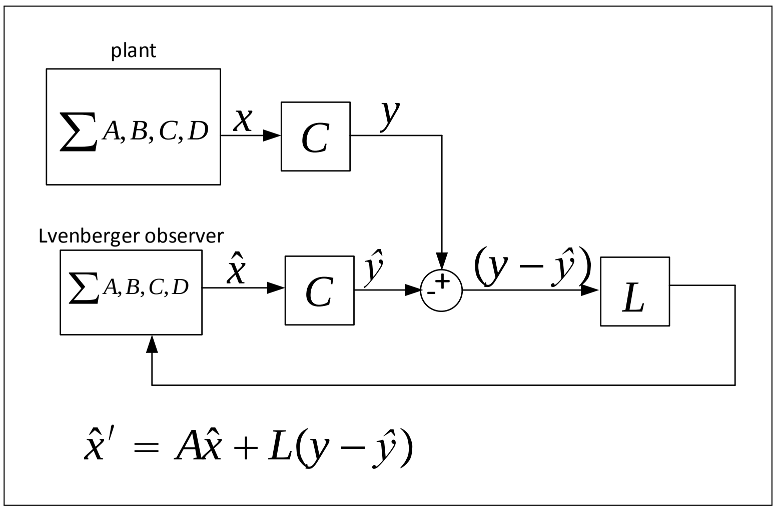
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$y = \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The observer state estimator is given by $\hat{x}' = A\hat{x} + L\left(y - \hat{y}\right)$

This diagram shows the flow for the observer



In our case, there is no input $u(t)$ since it is a free system, and it simplifies to



And the goal is to determine L based on eigenvalue requirements. In the above diagram, $y = Cx$ and $\hat{y} = C\hat{x}$.

Now, Let the error in state estimation be $e = (\hat{x} - x)$, therefore the rate of change of the error becomes

$$e' = (A - LC)\, e$$

We need to determine L such that the eigenvalues of $(A - LC)$ are $\lambda_1 = -1$ and $\lambda_2 = -2$. Before showing the design steps using the actual data given in the problem, the design steps are given below for the general case.

Design steps for finding L

1. Input is A, C and set of desired eigenvalues λ_i
2. Verify that (A, C) is observable. If so then let $A_o = A^T, B_o = C^T$, hence (A_o, B_o) is controllable.
3. Find controllability matrix $\mathbb{C}(A_o, B_o)$
4. Write down the controllability companion form for A_o, B_o. Let them be called $\tilde{A}_o, \tilde{B}_o$. To do this, we only need to find the characteristic polynomial for A_o and read the coefficients in reverse and change the signs. $\tilde{B}_o$ will always have zeros other than the last row.
5. Find controllability matrix $\tilde{\mathbb{C}}\left(\tilde{A}_o, \tilde{B}_o\right)$
6. Find $T = \tilde{\mathbb{C}}\mathbb{C}^{-1}$
7. Find the closed loop matrix $\left[\tilde{A}_o + \tilde{B}_o\tilde{K}\right]$ where $\tilde{K} = [k_0, k_1, \cdots, k_{n-1}]$ is the gain matrix we looking to determine.
8. Find the characteristic polynomial of $\left[\tilde{A}_o + \tilde{B}_o\tilde{K}\right]$, it will be a function of k_i
9. Set up the desired polynomial $p(\lambda) = (\lambda - \lambda_o)(\lambda - \lambda_o)\cdots(\lambda - \lambda_{n-1})$ where λ_i are the desired eigenvalues given.
10. Compare coefficients of polynomial from step (9) with the polynomial of step (7) and solve for k_i
11. Now we have found $\tilde{K} = [k_0, k_1, \cdots, k_{n-1}]$. Convert it to K using T as follows: $K = \tilde{K}T$
12. Find $L = -K^T$. This completes the design.
13. The observer A matrix now becomes $[A - LC]$

Applying the design of L to the problem

The first step is to check if (A, C) is observable. The observability matrix is

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \omega_0 \end{pmatrix}$$

Since the determinant is ω_0, hence not zero. Then this is invertible and full rank. Hence (A, C) is observable. Therefore $\left(A^T, C^T\right)$ is controllable pair. Lets call them (A_o, B_o) so that we do not have to use transpose in all the notation. Hence $A_o = A^T = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$ and $B_o = C^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore we can design $A_o + B_oK$ as we did for state feedback to find K, then use K to determine L using $L = -K^T$. The controllability matrix for (A_o, B_o) is

$$\begin{aligned} \mathbb{C} &= \begin{pmatrix} B_o & A_oB_o \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \omega_0 \end{pmatrix} \end{aligned}$$

And the characteristic equation is

$$\begin{aligned}\left|sI-A^T\right| &= 0\\ \begin{vmatrix} s & \omega_0\\ -\omega_0 & s\end{vmatrix} &= 0\\ s^2+\omega_0^2 &= 0\end{aligned}$$

Hence, the controllability companion form is

$$\begin{aligned}\tilde{A}_o &= \begin{pmatrix}0 & 1\\ -\alpha_0 & -\alpha_1\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\omega_0^2 & 0\end{pmatrix}\\ \tilde{B}_o &= \begin{pmatrix}0\\ 1\end{pmatrix}\end{aligned}$$

Hence the controllability matrix of the companion form is

$$\begin{aligned}\tilde{\mathbb{C}} &= \begin{pmatrix}\tilde{B}_o & \tilde{A}_o\tilde{B}_o\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\end{aligned}$$

Therefore the transformation operator T is

$$\begin{aligned}T &= \tilde{\mathbb{C}}\mathbb{C}^{-1}\\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \omega_0\end{pmatrix}^{-1} = \begin{pmatrix}0 & \frac{1}{\omega_0}\\ 1 & 0\end{pmatrix}\end{aligned}$$

Now we want

$$\begin{aligned}A_{\text{closed}} &= \tilde{A}_o + \tilde{B}_o\tilde{K}\\ &= \begin{pmatrix}0 & 1\\ -\omega_0^2 & 0\end{pmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k_0 & k_1\end{bmatrix}\\ &= \begin{bmatrix}0 & 1\\ k_0-\omega_0^2 & k_1\end{bmatrix}\end{aligned}$$

It has the following characteristic polynomial

$$p(\lambda) = \lambda^2 - \lambda k_1 + \left(\omega_0^2 - k_0\right)$$

The desired $p^*(\lambda) = (\lambda+1)(\lambda+2) = \lambda^2+3\lambda+2$. Comparing coefficients of this polynomial to the above gives

$$\begin{aligned}k_1 &= -3\\ \omega_0^2 - k_0 &= 2\\ k_0 &= \omega_0^2 - 2\end{aligned}$$

Hence, the gain vector is found to be

$$\tilde{K} = \begin{bmatrix}k_0 & k_1\end{bmatrix} = \begin{bmatrix}\omega_0^2-2 & -3\end{bmatrix}$$

The above $\tilde{K}$ was designed for the controllable companion form. We need to transform it back to the original $\left(A^T, C^T\right)$ system using T found earlier

$$\begin{aligned}K &= \tilde{K}T\\ &= \begin{bmatrix}\omega_0^2-2 & -3\end{bmatrix}\begin{pmatrix}0 & \frac{1}{\omega_0}\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}-3 & \frac{1}{\omega_0}\left(\omega_0^2-2\right)\end{pmatrix}\end{aligned}$$

Therefore, the observability gain vector is found as

$$\begin{aligned} L &= -K^T \\ &= -\begin{pmatrix} -3 & \frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix}^T \\ &= \begin{pmatrix} 3 \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix} \end{aligned}$$

Before continuing, let us verify the eigenvalues of $(A-LC)$ are where they should be now.

$$\begin{aligned} (A-LC) &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 & \omega_0 \\ \frac{1}{\omega_0}\left(\omega_0^2-2\right)-\omega_0 & 0 \end{pmatrix} \end{aligned}$$

The eigenvalues are $-1,-2$. Verified.

Now we continue the observer design. The observer is the following system

$$\begin{aligned} \hat{x}' &= A\hat{x} + L\left(y-\hat{y}\right) \\ &= A\hat{x} + L\left(y-C\hat{x}\right) \end{aligned}$$

Or

$$\begin{aligned} \begin{pmatrix} \hat{x}_1' \\ \hat{x}_2' \end{pmatrix} &= \overbrace{\begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}}^{A} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix} \left(\overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{y} - \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}}^{\hat{y}} \right) \\ &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix} (x_1-\hat{x}_1) \\ &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 3(x_1-\hat{x}_1) \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right)(x_1-\hat{x}_1) \end{pmatrix} \end{aligned}$$

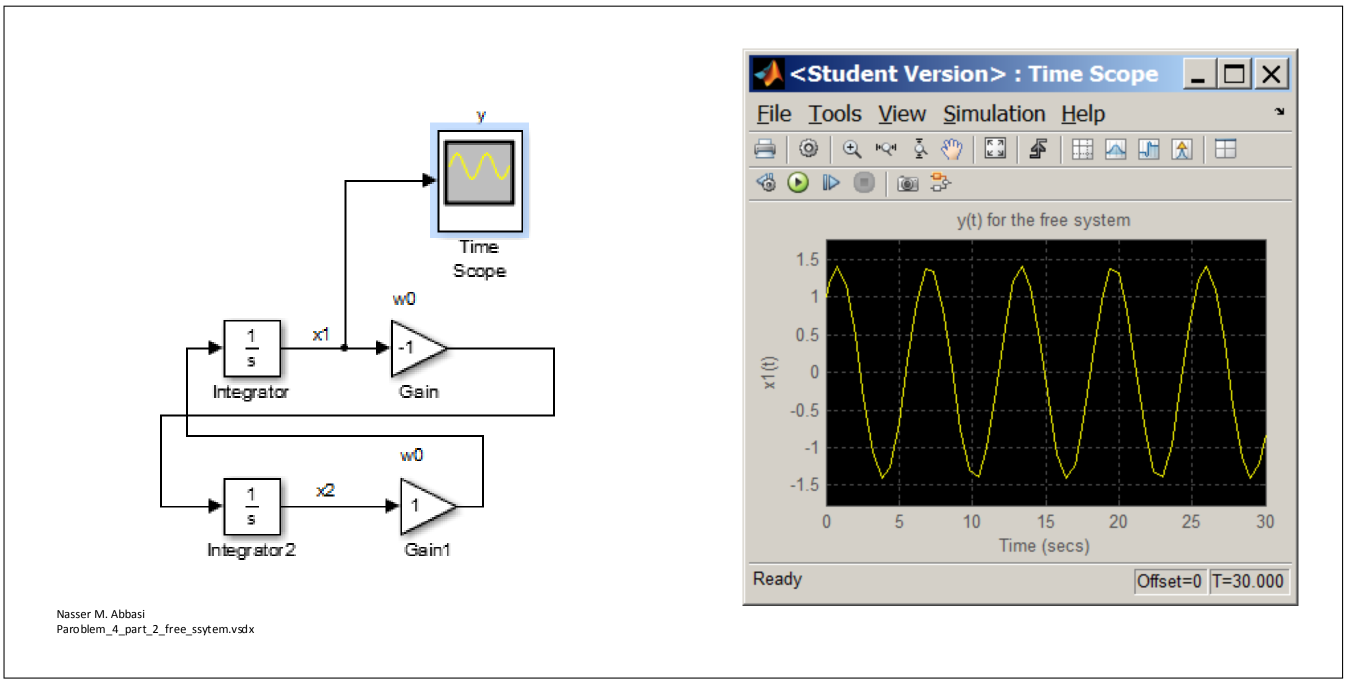
Therefore

$$\begin{aligned} \hat{x}_1' &= \omega_0\hat{x}_2 + L(1)(x_1-\hat{x}_1) \\ \hat{x}_2' &= -\omega_0\hat{x}_1 + L(2)(x_1-\hat{x}_1) \end{aligned}$$

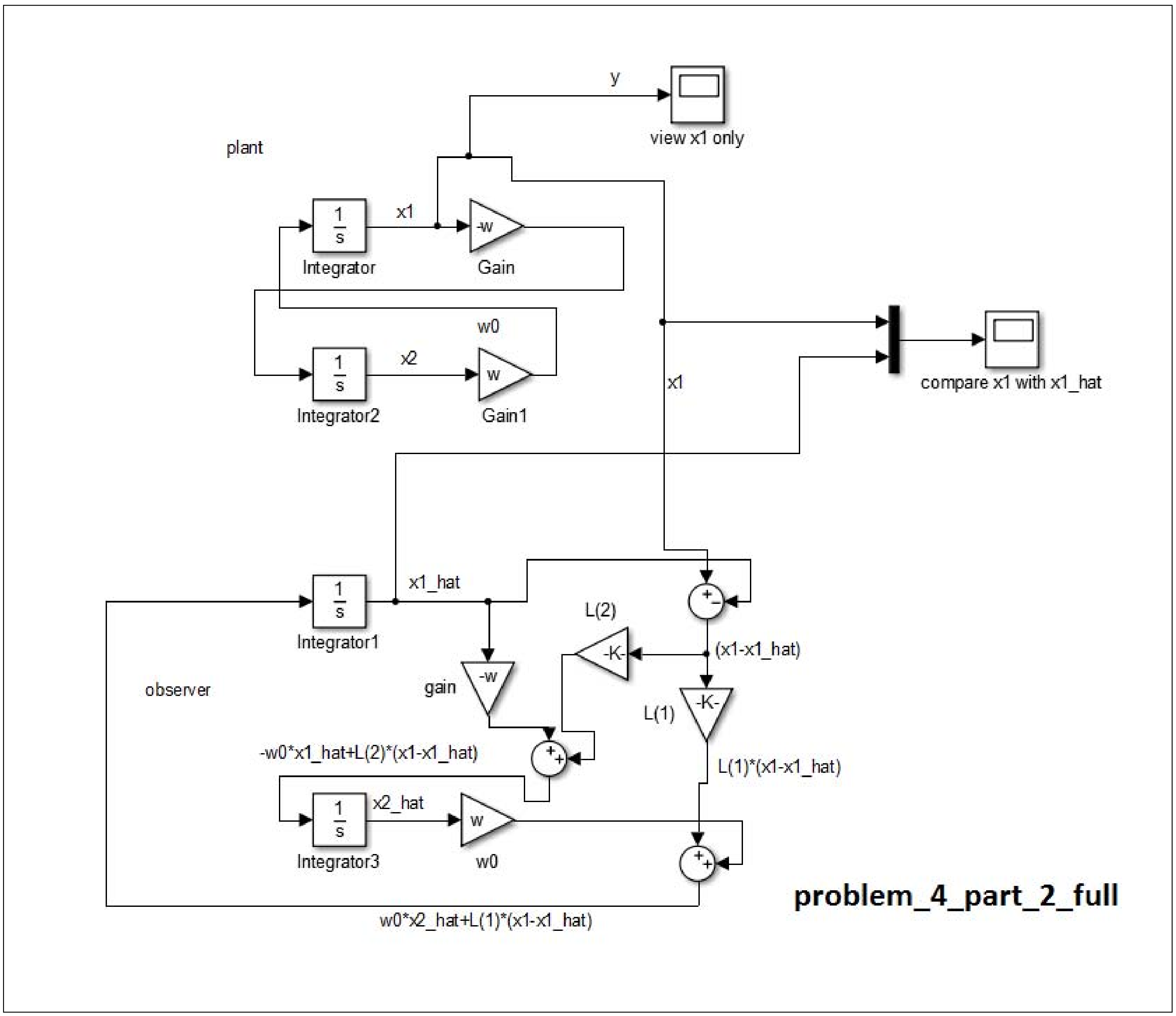
Where $L(1)=3$ and $L(2) = -\frac{1}{\omega_0}\left(\omega_0^2-2\right)$. In part(d), we will change these values to tune the observer. A Matlab script is written to generate L from different design eigenvalue locations.

Part(b)

The system we are given is free system, which means it is driven only by initial conditions. Therefore the model for the plant itself is the following, where $\omega_0=1$ was used to test the free system before adding the observer. The states x_1, x_2 were initialized to 1 in this example



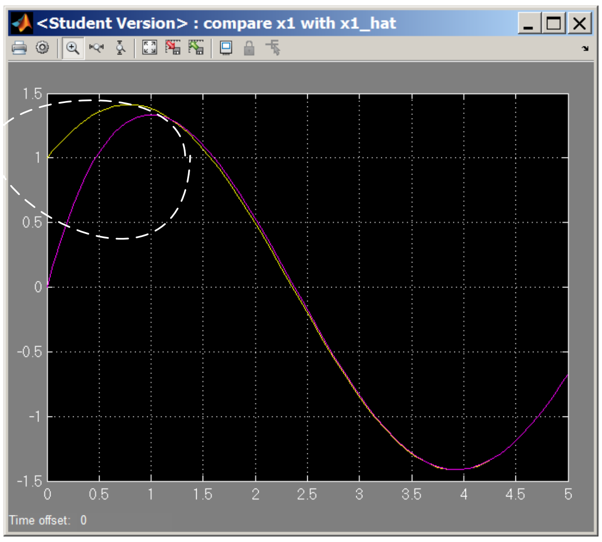
Now we will add the observer designed in part(a) and compare the observer state estimation to the actual x_1 of the plant. The model is the following



Tracking of x1 ω_0 is given the values $\{1,10,100,1000\}$ rad/sec and result showing $x_1(t), \hat{x}_1(t)$ on the same plot is displayed to see how well the observer will estimate the true $x_1(t)$ as the frequency changes. The result is the following

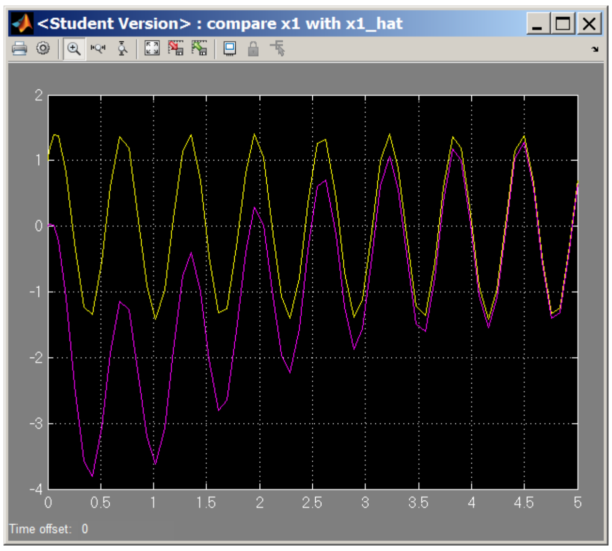
Yellow line is original x1 and red line is estimated x1

$\omega_0 = 1$



It took about 1 second for the observer to lock into x1 with good estimation after that. This difference is also due to using different initial conditions

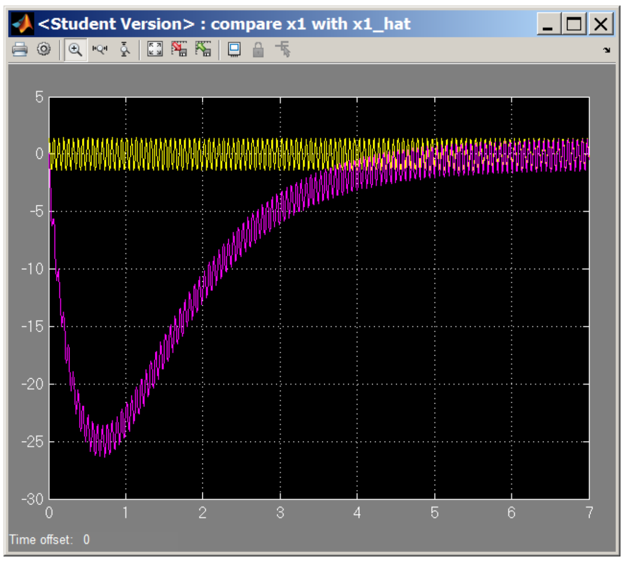
$\omega_0 = 10$



It took about 5 seconds now for the observer to lock into x1(t). The initial error was also larger.

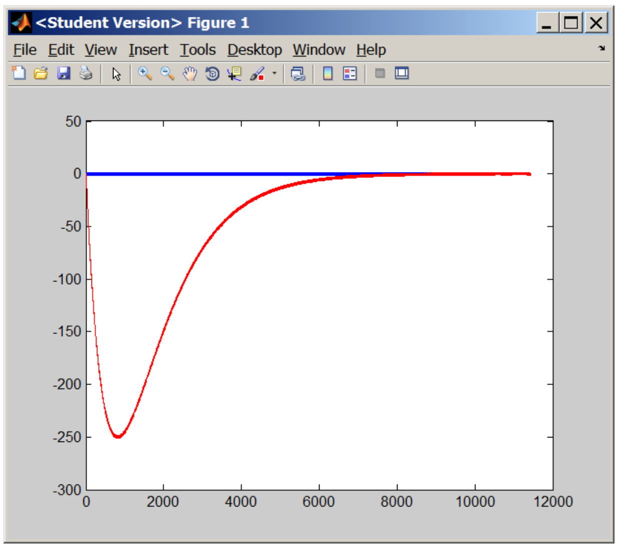
Yellow line is original x1 and red line is estimated x1

$\omega_0 = 100$



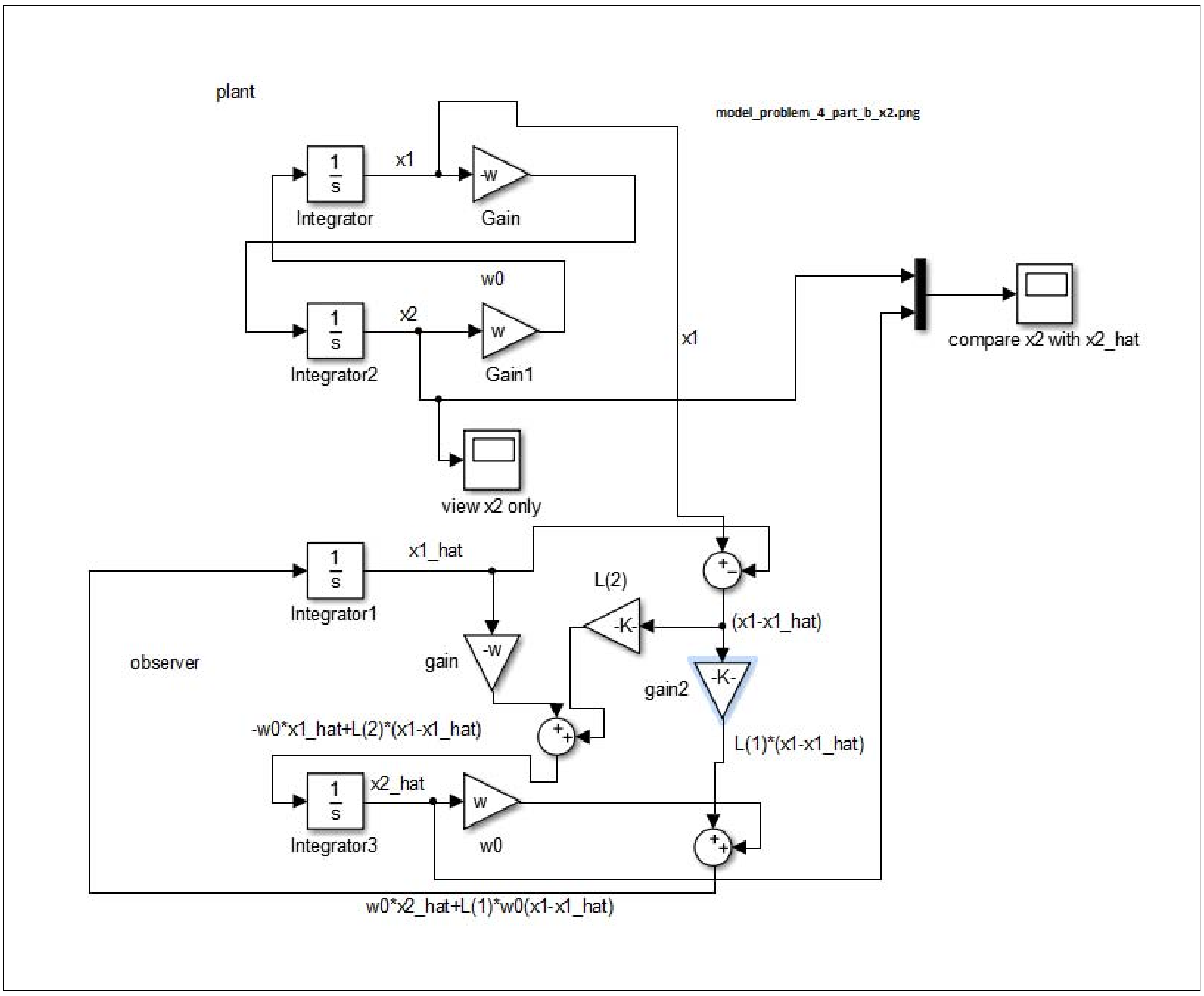
Initial overshoot is now larger even larger. At about 7 seconds the estimate converged to the actual.

$\omega_0 = 1000$



Now it took about 10 seconds for the estimate to converge, but initial error is now much larger. Almost 250 times as large as x1(t) for the first one second.

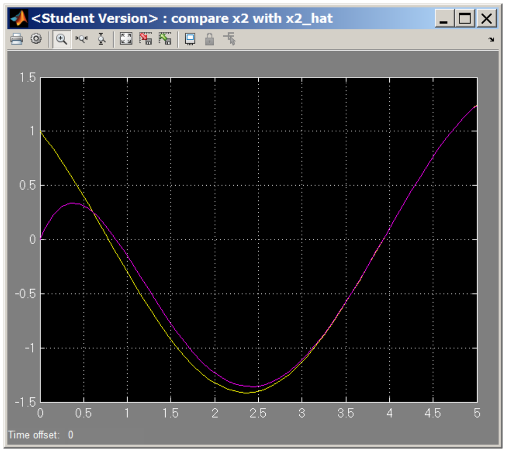
Tracking of $x_2(t)$ A plot showing the true x_2 and $\hat{x}_2$ is now given, similar to the above. The model was changed slightly to add a sink to plot $x_2, \hat{x}_2$ as follows



Now the frequency was set to $\omega_0 = 1, 10, 100, 1000$ rad/sec and the simulation was run. The following is the result and the observations

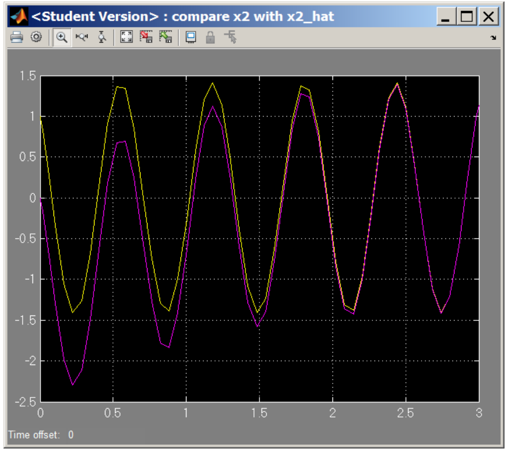
Yellow line is original x2 and red line is estimated x2

$\omega_0 = 1$ rad/sec



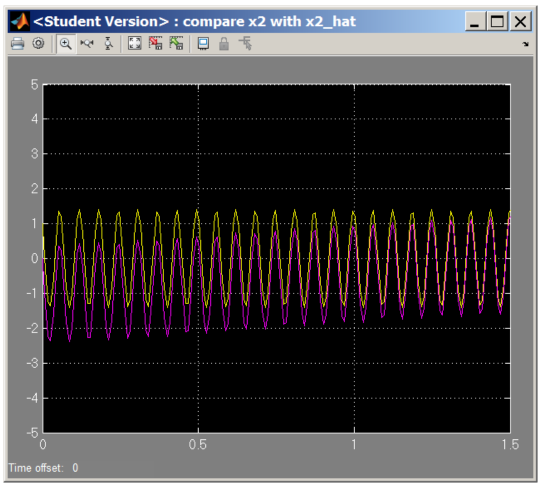
Convergence to x2 took about the same time as with x1. About 3.5 seconds. The overshoot is very small compared to when the frequency is higher

$\omega_0 = 10$ rad/sec



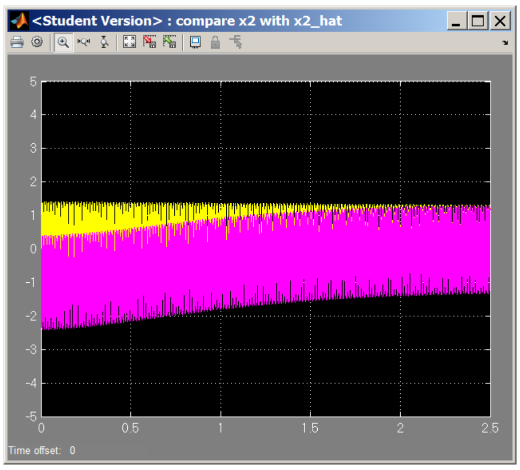
Convergence to x2 took about 2.5 seconds, but we notice the overshoot on the negative sign is larger now

$\omega_0 = 100$ rad/sec



Convergence to x2 took 2 seconds, but now we notice that the overshoot on the negative sign did not get worst than w=10. I was expecting the overshoot to get worst to follow from the last result

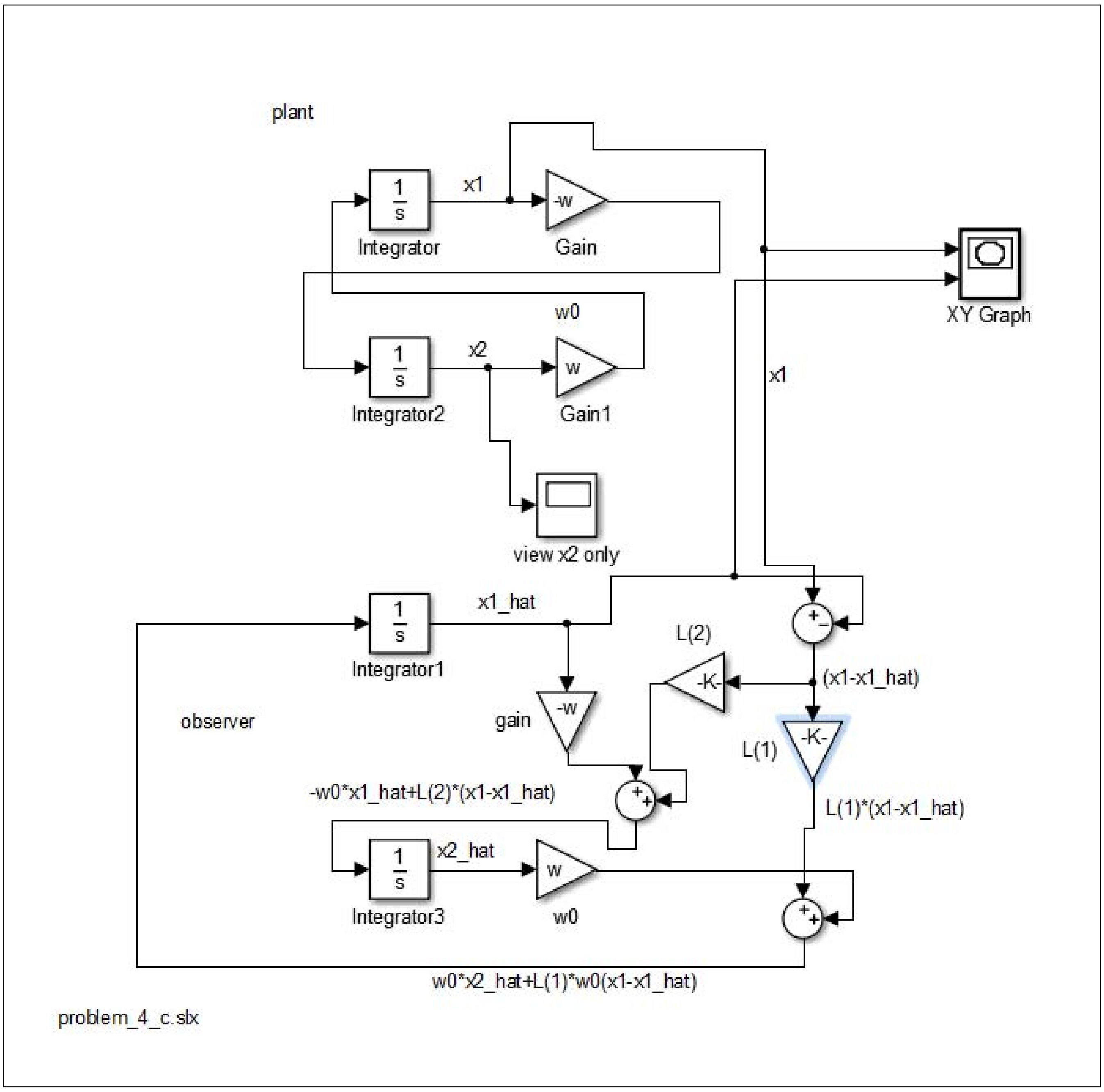
$\omega_0 = 1000$ rad/sec



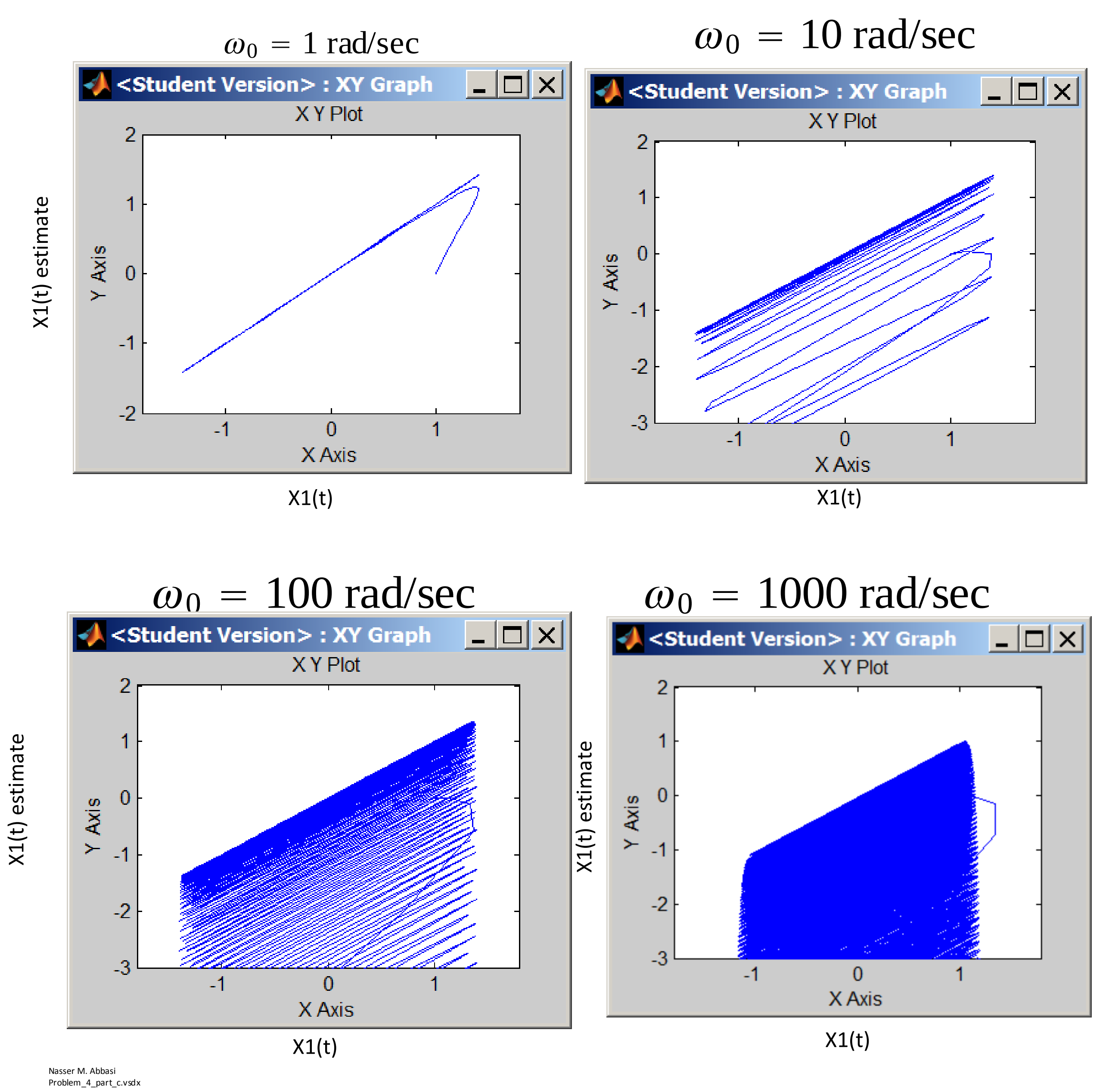
The convergence do not see to change with x2(t) as the frequency is made higher. It takes about 2.5 second to convergence and the overshoot remains at the same value it was for w=100

Part (c)

The model was changed slightly to add an XY graph as follows



The result of the simulation to generate the phase plots is shown below

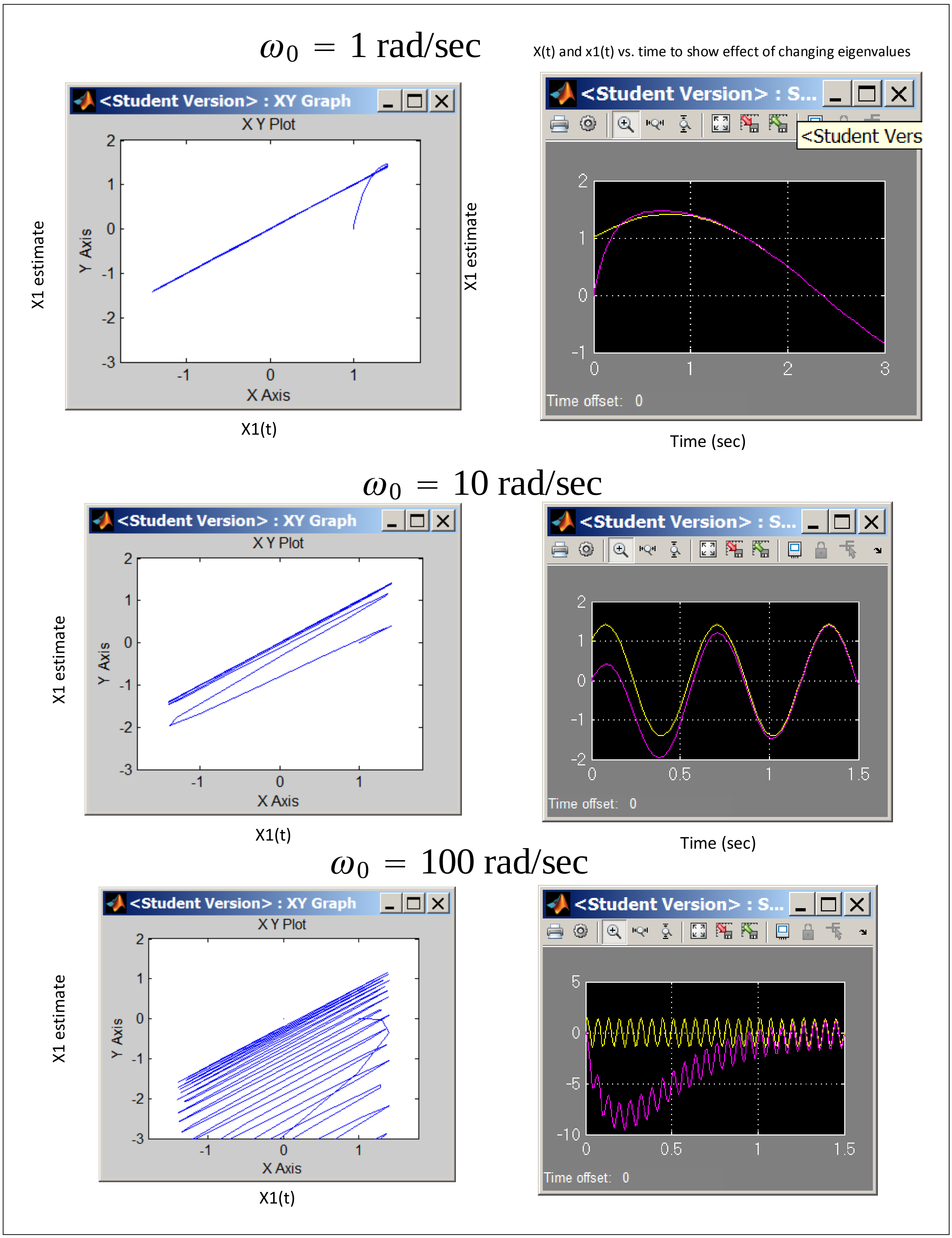


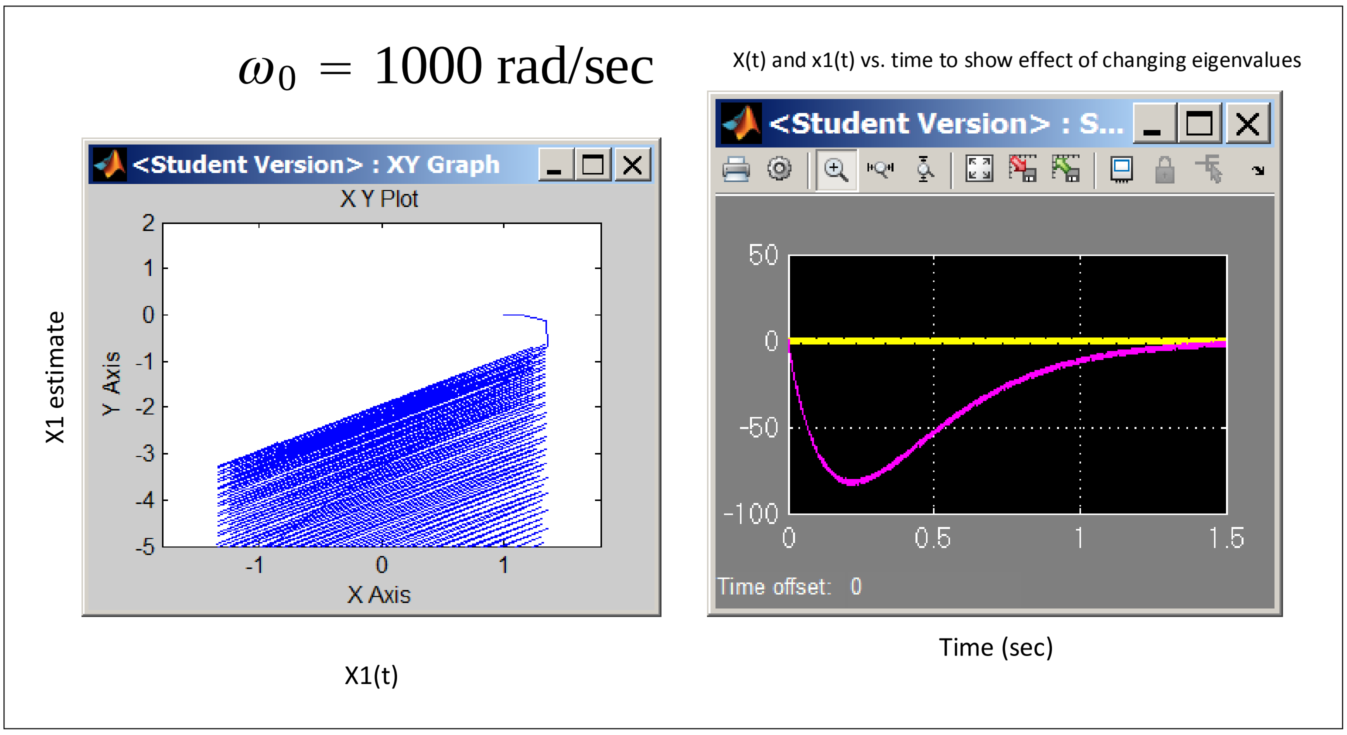
Part (d)

A small Matlab program was written to tune the observer. This was done by changing the location of the design eigenvalues and generating new L observer gain vector for each new set of eigenvalues, then using the new L in the simulink model in part(c) to see the effect on the phase plot. The goal is to obtain a straight line in the phase plane, since a straight line indicates that $\hat{x}_1$ is tracking x_1 well. Few eigenvalues are tried. This table shows summary of each pair of eigenvalues and the corresponding L vector generated. We show one final result which was found to be the best one from the ones tried

eigenvalues	L generated by the design script
$[-1,-2]$ (original eigenvalues)	$\begin{pmatrix} 3 \\ -\frac{1}{\omega_0}\left(\omega_0^2-2\right) \end{pmatrix}$
$[-1.5,-2]$	$\begin{pmatrix} 3.5 \\ -\frac{1}{\omega_0}\left(\omega_0^2-3\right) \end{pmatrix}$
$[-2,-3]$	$\begin{pmatrix} 5 \\ -\frac{1}{\omega_0}\left(\omega_0^2-6\right) \end{pmatrix}$
$[-2.5,-3.5]$	$\begin{pmatrix} 6 \\ -\frac{1}{\omega_0}\left(\omega_0^2-8.75\right) \end{pmatrix}$
$[-3,-4]$	$\begin{pmatrix} 7 \\ -\frac{1}{\omega_0}\left(\omega_0^2-12\right) \end{pmatrix}$
$[-4,-5]$	$\begin{pmatrix} 9 \\ -\frac{1}{\omega_0}\left(\omega_0^2-20\right) \end{pmatrix}$

Using the eigenvalues at $[-4,-5]$ the initial overshoot was found to be become small. This was noticed most for large frequencies. Here is the phase plot of $x_1 - \hat{x}_1$ using the last entry in the above table. To make it easier to compare with the original eigenvalues design, a plot of $x_1, \hat{x}_1$ vs. time was also added. This plot shows more clearly that by making the eigenvalues more negative, the convergence became faster.


$\omega_0 = 1$ rad/sec
X(t) and x1(t) vs. time to show effect of changing eigenvalues
<Student Version> : XY Graph
X Y Plot
X1 estimate
Y Axis
X Axis
X1(t)
<Student Version> : S...
<Student Vers
X1 estimate
Time offset: 0
Time (sec)
$\omega_0 = 10$ rad/sec
<Student Version> : XY Graph
X Y Plot
X1 estimate
Y Axis
X Axis
X1(t)
<Student Version> : S...
Time offset: 0
Time (sec)
$\omega_0 = 100$ rad/sec
<Student Version> : XY Graph
X Y Plot
X1 estimate
Y Axis
X Axis
X1(t)
<Student Version> : S...
Time offset: 0



```
%nma_design_observer
%Script by Nasser M. Abbasi
%HW3, ECE 717, problem 4 part(d)

%This scripts design the oberver gain vector L to
%allow one to tune the obsever. The input is A,C
%This is only meant for 2x2 case for the HW assignment
%This is not a general purpose script.
%One can modify the location of the desired eigenvalues
%for the matrix (A−LC) to improve the error behavior.
%modify the lambda line below to change the locations of
%the desired eigvalues and run this script, it will print
%the final L vector to use in simulink

clear all;
syms w k0 k1 s;
A=[0 w;−w 0];
C=[1 0];
%lambda=[−1,−2]; %HW ones
%lambda=[−1.5,−2.5]; %design eigevalues, change as needed
lambda=[−1,−2]; %design eigevalues, change as needed

%go to the controllability framework so to be able to generate T
%and get the gain vector
Ao = A.';
Bo = C.';
controllabilityMatrix = [Bo Ao∗Bo];
alpha                 = charpoly(Ao);

%obtain the controllable companion form
AoCompanion           = [0 1;−alpha(end) −alpha(end−1)];
BoCompanion           = [0;1];

%find the transformation T matrix
controllabilityMatrixCompanion = [BoCompanion AoCompanion∗BoCompanion];
T = controllabilityMatrixCompanion∗inv(controllabilityMatrix);

%now design the controllability gain vector
Aclosed       =  AoCompanion + BoCompanion∗[k0 k1];
Aclosed_poly = charpoly(Aclosed);
design_poly  = (s−lambda(1))∗(s−lambda(2));
coeff        = sym2poly(design_poly);
```

```
%now solve for k0,k1
k0  = solve( coeff(end)==Aclosed_poly(end),k0);
k1  = solve( coeff(end-1)==Aclosed_poly(end-1),k1);
gainVector  = [k0 k1];
gainVector  = gainVector*T;
L           = -gainVector.'
```

Example use is

```
EDU>> nma_design_observer
L =
3
-(w^2 - 2)/w
```

Part(e)

source code listing There are two main reasons, as was explained in class. One is that we do not know what the initial conditions that the plant starts at, and this could change each time. But most importantly, the observer could be started at any time during the operation of the overall system and it does not have to be started at the same instance as the plant. Since the observer could start at later time, the initial conditions that the plant was in have been lost and no longer available to the observer. So there will always be some initial settling time. So having different initial conditions for the plant and the observer is the more common case.

2.3.7 key solution

ECE 717 – Solution Set 3

Solution Invariance

(a) For the closed loop,

$$\mathring{x} = \underbrace{(A+BK)}_{A_{cl}}x + bv; \quad y = Cx + Du, \ \Sigma_{cl}$$

the transfer function is

$$H_{\Sigma_{cl}}(s) = C(sI - A_{cl})^{-1}B + D$$

$$= C(sI - A - BK)^{-1}B + D$$

(b) To show rank $\mathcal{C}_{\Sigma}$ = rank $\mathcal{C}_{\Sigma_{cl}}$, we show that any linear combination of the columns of $\mathcal{C}_{\Sigma}$ is also a linear combination of the columns of $\mathcal{C}_{\Sigma_{cl}}$ (and vice versa). Proceeding by induction, for $n=1$, since $\mathcal{C}_{\Sigma} = \mathcal{C}_{\Sigma_{cl}} = b$, the result holds.

Now consider $n=k$ and assume the spanning condition involving linear combinations above holds for $n=k$. We must show that it holds for $n=k+1$. Indeed, for $n=k+1$, say $x = \sum_{i=0}^{k}\alpha_i A^i b$. We must show this vector can be written as a linear combination of the $(A+BK)^i b$. Indeed, we write

$$x = \alpha_k A^k B + \sum_{i=0}^{k-1}\alpha_i A^i B = \alpha_k A(A^{k-1}B) + \sum_{i=0}^{k-1}\alpha_i A^i B$$

Now using the inductive hypothesis for $n = k$ on $A^{k-1}B$ and $A^{i}B$ above, write

$$X = \alpha_k A \sum_{i=0}^{k-1} \tilde{\beta}_i (A+BK)^i B + \sum_{i=0}^{k-1} \beta_i (A+BK)^i B \quad \star$$

with appropriate scalars $\tilde{\beta}_i$, β_i above.

Re-writing $\star$ above,

$$X = \alpha_k (A+BK) \sum_{i=0}^{k-1} \tilde{\beta}_i (A+BK)^i B + \sum_{i=0}^{k-1} \beta_i (A+BK)^i B - \alpha_k \underbrace{BK \sum_{i=0}^{k-1} \tilde{\beta}_i (A+BK)^i B}_{\text{Scalar } \gamma}$$

By inspection, the vector X is a linear combination of $B, (A+BK)B, \dots, (A+BK)^n B$

So columns of $\mathcal{C}_\Sigma$ are spanned by columns of $\mathcal{C}_{\Sigma_{c1}}$

Completion of Proof: Remains to show columns of $\mathcal{C}_{\Sigma_{c1}}$ are spanned by columns of $\mathcal{C}_\Sigma$. Again, assuming this condition holds for $n = k$, consider $n = k+1$. Now suppose

$$X = \sum_{i=0}^{k} \alpha_i (A+BK)^i B = \alpha_k (A+BK)^k B + \sum_{i=0}^{k-1} \alpha_i (A+BK)^i B$$

Now using inductive hypothesis, rewrite above as

$$X = \alpha_k (A+BK) \sum_{i=0}^{k-1} \tilde{\beta}_i A^i B + \sum_{i=0}^{k-1} \beta_i A^i B$$

$$= \sum_{i=0}^{k-1} \alpha_k \tilde{\beta}_i A^{i+1} B + \sum_{i=0}^{k-1} \beta_i A^i B + B \underbrace{K \sum_{i=0}^{k-1} \alpha_k \tilde{\beta}_i A^i B}_{\text{Scalar}}$$

Hence, we see that x above is a linear combination of $B, AB, \ldots, A^{n-1}B$. The proof is now complete.

Homework Design

$\Sigma = (A, b)$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a) $\mathcal{C}_\Sigma = [b \quad Ab \quad A^2 b \quad A^3 b]$

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad Ab = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad A^2 b = A(Ab) = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$A^3 b = A(A^2 b) = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathcal{C}_\Sigma = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\text{rank}(\mathcal{C}_\Sigma) = 4$ then the system is controllable

b) $$sI - A = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 2 & -1 & s & 0 \\ -1 & 1 & 1 & s \end{bmatrix}$$

$\det(sI - A) = s^4 + 3s^2 + 1$

Roots are:
-.34 +/- j1.27
.34 +/-j 1.27
So open loop unstable

The eigenvalues are on the imaginary axis, then the system is marginally stable, but not asymptotically stable because the real parts are not negative.

Step 1: Recalling the characteristic poly $\Delta(\lambda) = \det(\lambda I - A) = \lambda^4 + 3\lambda^2 + 1$, the target canonical form is

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix} ; \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with controllability matrix

$$\mathcal{C}_{\tilde{\Sigma}} = \left[\tilde{b} \ \tilde{A}\tilde{b} \ \tilde{A}^2\tilde{b} \ \tilde{A}^3\tilde{b} \right] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \end{pmatrix}$$

Now calculate

$$T = \mathcal{C}_{\tilde{\Sigma}} \mathcal{C}_{\Sigma}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Then $\tilde{A} = TAT^{-1}$, $\tilde{b} = Tb$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix} , \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2. Find the companion feedback

For the transformed system

$\dot{z} = \tilde{A}z + \tilde{b}u$ with $u = \tilde{k}z = [\tilde{k}_1 \ \tilde{k}_2 \ \tilde{k}_3 \ \tilde{k}_4]$

$\dot{z} = (\tilde{A} + \tilde{b}\tilde{k})z = \tilde{A}_{cl}$

$$\tilde{A}_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1+\tilde{k}_1 & \tilde{k}_2 & -3+\tilde{k}_3 & \tilde{k}_4 \end{bmatrix}$$

and $\Delta(\lambda) = \lambda^4 - \tilde{k}_4\lambda^3 - (\tilde{k}_3 - 3)\lambda^2 - \tilde{k}_2\lambda - (\tilde{k}_1 - 1)$

we want

$$\Delta(\lambda) = [(\lambda - (-1+j))^2(\lambda - (-1-j))^2]$$

$$= \lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4$$

then $\tilde{k} = [-3 \quad -8 \quad -5 \quad -4]$

Step 3. Transform back to the original system $k = \tilde{k}T$

therefore $k = [-5 \quad 2 \quad -4 \quad -4]$

Solution Observability

Via a "mimic" of the controllability analysis from class, we have necessary conditions

$$C_2 = C_1 T^{-1}$$

$$C_2 A_2 = C_1 A_1 T^{-1}$$

$$C_2 A_2^2 = C_1 A_1^2 T^{-1}$$

$$\vdots$$

$$C_2 A_2^{n-1} = C_1 A_1^{n-1} T^{-1}$$

In matrix form

$$\Theta_{\Sigma_2} = \Theta_{\Sigma_1} T^{-1}$$

$$\Rightarrow \quad \Theta_{\Sigma_1} = \Theta_{\Sigma_2} T$$

$$\Rightarrow \quad \Theta_{\Sigma_2}^T \Theta_{\Sigma_1} = \Theta_{\Sigma_2}^T \Theta_{\Sigma_2} T$$

Assuming observability $\Theta_{\Sigma_2}^T \Theta_{\Sigma_2}$ is invertible.

Hence,

$$T = \left(\Theta_{\Sigma_2}^T \Theta_{\Sigma_2}\right)^{-1} \Theta_{\Sigma_2}^T \Theta_{\Sigma_1}$$

Homework observer

$\dot{x}_1 = w_o x_2$

$\dot{x}_2 = -w_o x_1$

$y = x_1$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & w_o \\ -w_o & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & w_o \\ -w_o & 0 \end{bmatrix} \qquad C = [1 \quad 0]$$

$\dot{x} = Ax$

Observer $\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x})$

then $\dot{e} = Ae - LCe = (A - LC)e$ $\qquad$ e : error dynamics

Design $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ such that with eigenvalue.

$\lambda_1 = -1 \qquad \lambda_2 = -2$

$$A - LC = \begin{bmatrix} -L_1 & w_o \\ -w_o - L_2 & 0 \end{bmatrix} = A_{obs}$$

$$\Delta(\lambda) = \det(\lambda I - A_{obs}) = \begin{vmatrix} \lambda + L_1 & -w_o \\ w_o + L_2 & \lambda \end{vmatrix}$$

$$= \lambda^2 + L_1\lambda + w_o(w_o + L_2)$$

The desired error dynamic system is.

$$\Delta(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$$

therefore $L_1 = 3$

$L_2 = \frac{2}{w_o} - w_o$

$$\Rightarrow \quad L = \begin{bmatrix} 3 \\ \frac{2-w_o^2}{w_o} \end{bmatrix}$$

b) $$\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x})$$

$$= \begin{bmatrix} 0 & w_o \\ -w_o & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 3 \\ \frac{2-w_o^2}{w_o} \end{bmatrix} (x_1 - \hat{x}_1)$$

$$\dot{\hat{x}}_1 = w_o \hat{x}_2 + 3(x_1 - \hat{x}_1)$$

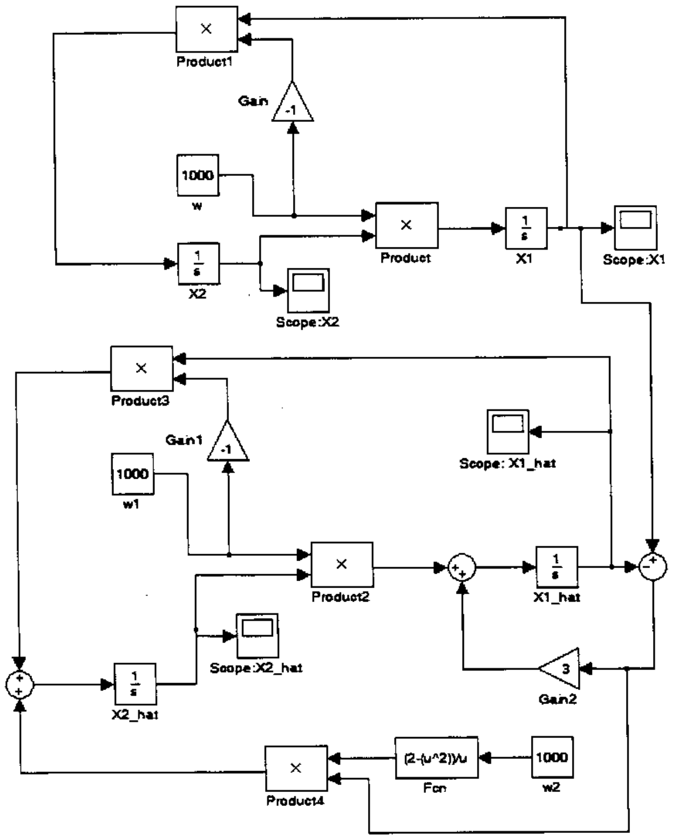
$$\dot{\hat{x}}_2 = -w_o \hat{x}_1 + \left(\frac{2-w_o^2}{w_o}\right)(x_1 - \hat{x}_1)$$

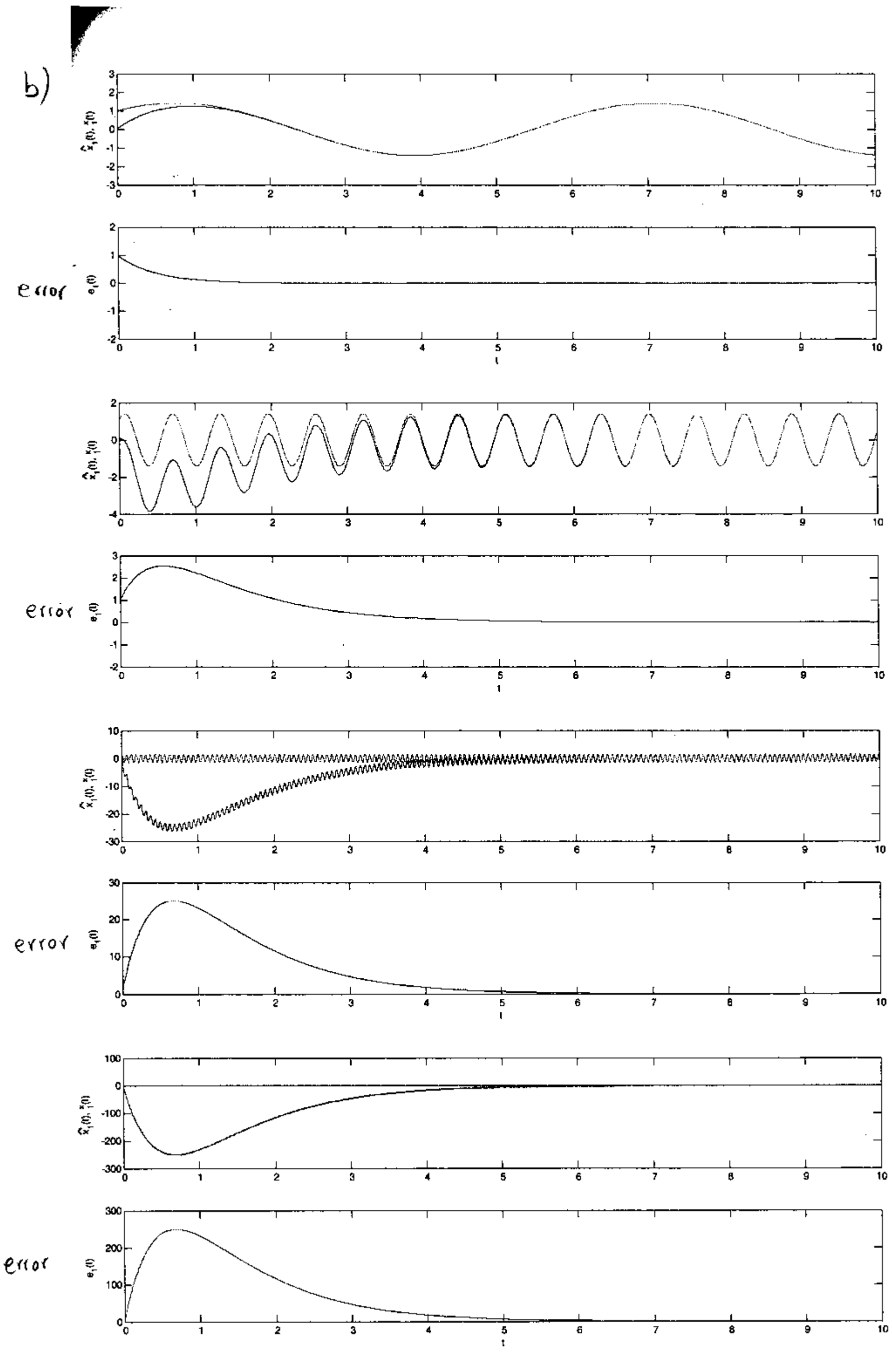
From the graphs, we can conclude that at low frequencies, it can track very quickly, whereas at high frequencies, it takes more time to track Note that not only frequencies affect strongly the tracking performances, but also the eigenvalues. The estimate of $x_1(t)$ has large overshoot as the frequecy increases, where the estimate of $x_2(t)$ has less effect.

c) The origind system has its phase portrait pretly "regulor" with simple oscillation in certain region as the graphs show
On the other hand, the obsever behave much less regular.

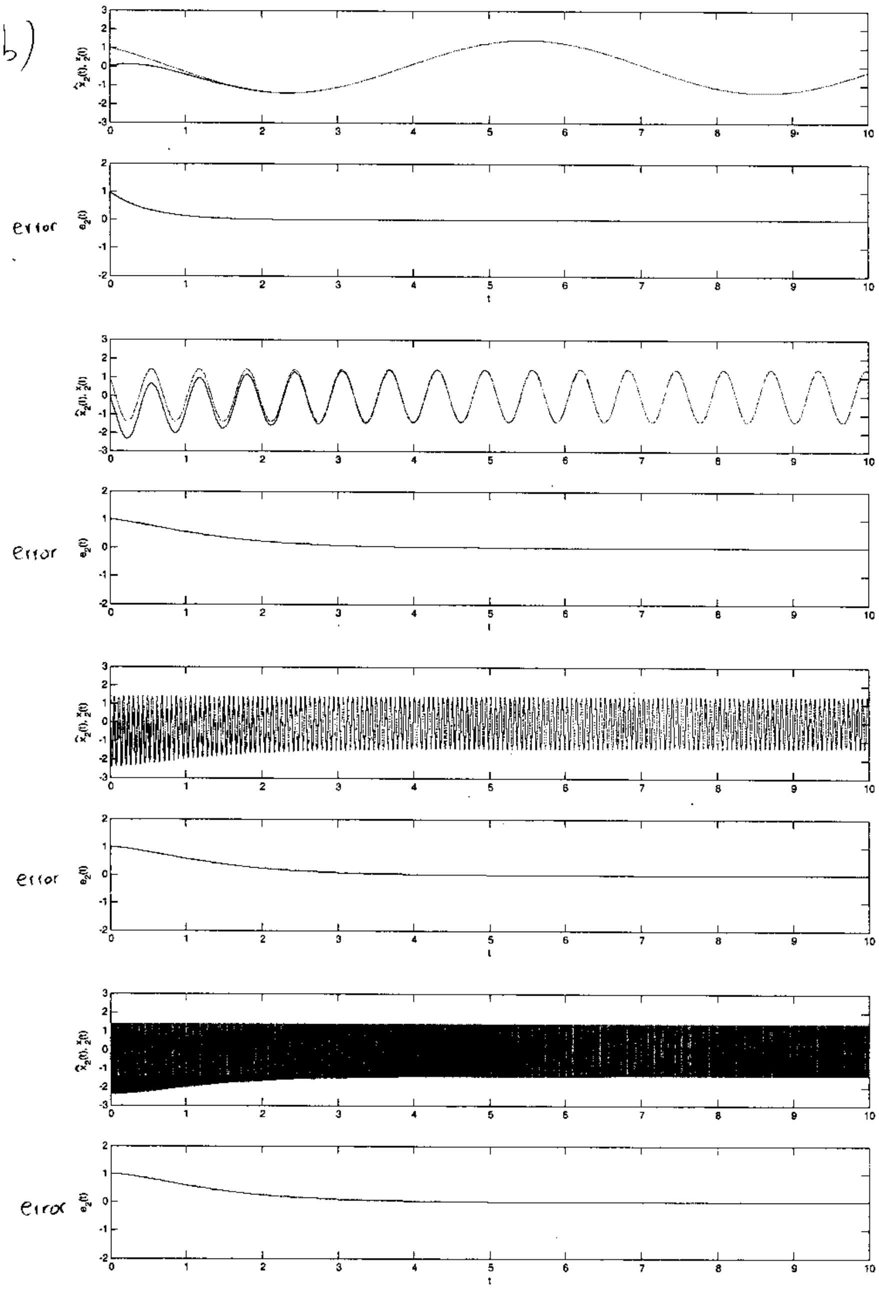
d) We can improve the performace of the tracking by making the eigenvalue of the error dynamic further into the left half plane. However, it may track well for higher frequencies, but it may degrade the tracking performance for lower frequeies.
For example, the observer was modified to be 100 time faster., it experiences overshoot for lower frequencies, and very good ticking performance for $\omega_0 = 100$, but still less efficient for higher frequeues

e) In real application, the initial condition of the system is unknown. We need to estimate the state at the current time. and a good observer should track the state as quick as possible with any initial conditions.

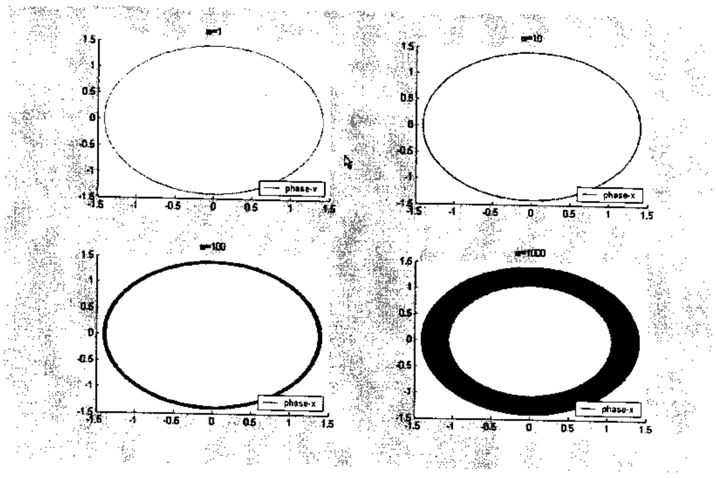

Product1
Gain
-1
1000
w
Product
1/s
X1
Scope:X1
1/s
X2
Scope:X2
Product3
Gain1
-1
Scope: X1_hat
1000
w1
Product2
1/s
X1_hat
Scope:X2_hat
1/s
X2_hat
3
Gain2
(2-(u^2))/u
Fcn
1000
w2
Product4

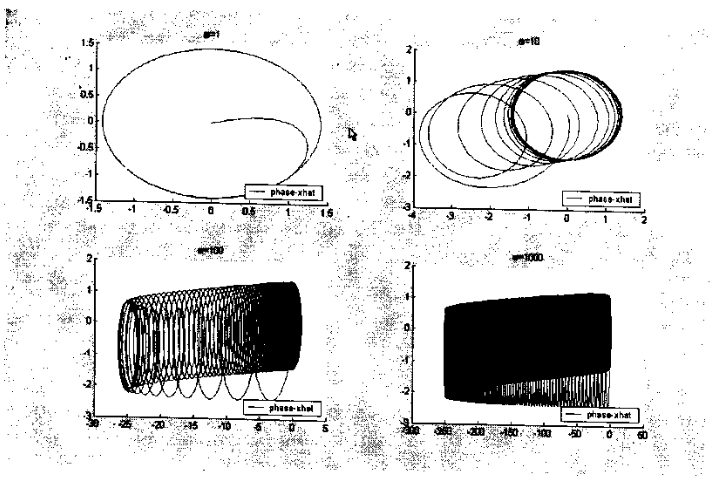

b)
$\hat{x}_1(t), x_1(t)$
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
7
8
9
10
error
$e_1(t)$
2
1
0
-1
-2
t
$\hat{x}_1(t), x_1(t)$
2
0
-2
-4
error
$e_1(t)$
3
2
1
0
-1
-2
t
$\hat{x}_1(t), x_1(t)$
10
0
-10
-20
-30
error
$e_1(t)$
30
20
10
0
t
$\hat{x}_1(t), x_1(t)$
100
0
-100
-200
-300
error
$e_1(t)$
300
200
100
0
t

b)


$\hat{x}_2(t), x_2(t)$
error
$e_2(t)$
$\hat{x}_2(t), x_2(t)$
error
$e_2(t)$
$\hat{x}_2(t), x_2(t)$
error
$e_2(t)$
$\hat{x}_2(t), x_2(t)$
error
$e_2(t)$
t
0 1 2 3 4 5 6 7 8 9 10

c)


w=1
w=10
w=100
w=1000
phase-x
-1.5
-1
-0.5
0
0.5
1
1.5


w=1
w=10
w=100
w=1000
phase-xhat
-1.5
-1
-0.5
0
0.5
1
1.5
-4
-3
-2
2
-30
-25
-20
-15
-10
-5
5
-300
-250
-200
-150
-100
-50
50

2.4 HW4

2.4.1 Questions

ECE 717 – Homework Set 4

Due Friday, October 17, 2013

Please slide submissions under my door.

;

Barmish

ECE 717 – Homework Criterion

For the state-space system $\Sigma = (A, B)$, suppose that there exists some non-zero vector α and a complex number λ such that

$$\alpha^T A = \lambda \alpha^T$$

and

$$\alpha^T B = 0.$$

Now show that (A, B) is NOT a controllable pair. Remark: This condition is also necessary for lack of controllability but its proof involves results to be developed later in the course.

;

Barmish

ECE 717 – Homework Convergence

Consider the sequence of functions defined on $[0, \infty)$ by

$$f_n(t) = \frac{t^2}{1 + nt^2}$$

for $n = 1, 2, 3.....$ Find the pointwise limit f. Does this sequence converge uniformly to f? Explain.

;

Barmish

ECE 717 – Homework Integrals

Consider the sequence of functions defined on $[0, 1]$ by

$$f_k(t) = k^2 t(1 - t^2)^k$$

for $k = 1, 2, 3....$ Compute

$$\lim_{k \to \infty} \int_0^1 f_k(t) dt.$$

Now let f be the pointwise limit of the f_k and compute the integral of f (also from 0 to 1). Are the two computed quantities equal? Discuss.

;

Barmish

ECE 717 – Homework Nonlinear Picard

For the nonlinear state equation

$$\dot{x} = f(x,t),$$

consider the Picard iteration scheme beginning with $x^0(t) \equiv x^0$ with iterative step

$$x^{k+1}(t) = x^0 + \int_0^t f(x^k(\eta), \eta) d\eta.$$

Then, for the two state nonlinear system described by

$$\dot{x}_1 = \cos x_1$$

and

$$\dot{x}_2 = tx_1 + e^{-t} x_2,$$

find the first three Picard iterates $x^1(t), x^2(t)$ and $x^3(t)$ corresponding to initial conditions

$$x_1(0) = 2; \quad x_2(0) = -1.$$

Also provide plots of $x_1(t)$ and $x_2(t)$ for each Picard iterate. Are your solutions converging? Discuss.

Note: To maximize learning, I suggest you do this problem by hand with the integral for $x^3(t)$ facilitated with Matlab syms.

;

Barmish

ECE 717 – Homework Saturation

A scalar nonlinear system is described by

$$\dot{x} = f(x)$$

with $f(x)$ being a saturation nonlinearity given by

$$f(x) = \frac{1}{2}x$$

for $|x| \leq 2$,

$$f(x) = 1$$

for $x > 2$ and

$$f(x) = -1$$

for $x < -2$. Now for $x(0) = 1$, first plot $f(x)$ and then, by hand, find Picard iterates $x^1(t)$ and $x^2(t)$.

;

2.4.2 Problem 1

A Matrix $\mathbb{C}$ is rank deficient if there exist a non-zero vector $\boldsymbol{\alpha}$ such that $\mathbb{C}\boldsymbol{\alpha} = 0$. This means that the Null space of $\mathbb{C}$ is not empty. The idea of the proof is to apply this to the controllability matrix itself to check if $\mathbb{C}$ is full rank or not. The left null space is used instead. If the left null space of $\mathbb{C}$ is not empty, this implies Null space of $\mathbb{C}$ is also not empty, hence $\mathbb{C}$ is rank deficient, which gives the proof.

The first step is to find $\mathbb{C}$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$\boldsymbol{\alpha}^T\mathbb{C}$ is now found to see if it produces zero row vector. If it does, then the left null space of $\mathbb{C}$ is not empty.

$$\boldsymbol{\alpha}^T\mathbb{C} = \boldsymbol{\alpha}^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

But α^T pre-multiplied by a matrix is α^T pre-multiplied by each one of the columns of the matrix. The above becomes

$$\begin{aligned}\alpha^T\mathbb{C} &= \begin{bmatrix}\alpha^T B & \alpha^T AB & \alpha^T A^2B & \cdots & \alpha^T A^{n-1}B\end{bmatrix}\\ &= \begin{bmatrix}\alpha^T B & \left(\alpha^T A\right)B & \left(\alpha^T A\right)AB & \cdots & \left(\alpha^T A\right)A^{n-2}B\end{bmatrix}\end{aligned}$$

$\alpha^T A$ is replaced by $\lambda\alpha^T$ in the above giving

$$\begin{aligned}\alpha^T\mathbb{C} &= \begin{bmatrix}\alpha^T B & \left(\lambda\alpha^T\right)B & \left(\lambda\alpha^T\right)AB & \cdots & \left(\lambda\alpha^T\right)A^{n-2}B\end{bmatrix}\\ &= \begin{bmatrix}\alpha^T B & \lambda\left(\alpha^T B\right) & \lambda\left(\alpha^T A\right)B & \cdots & \left(\lambda\alpha^T A\right)A^{n-3}B\end{bmatrix}\end{aligned}$$

Applying $\alpha^T A = \lambda\alpha^T$ again on the above

$$\alpha^T\mathbb{C} = \begin{bmatrix}\alpha^T B & \lambda\left(\alpha^T B\right) & \lambda\lambda\alpha^T B & \cdots & \left(\lambda\lambda\alpha^T A\right)A^{n-4}B\end{bmatrix}$$

This process is continued until the final result is

$$\alpha^T\mathbb{C} = \begin{bmatrix}\left(\alpha^T B\right) & \lambda\left(\alpha^T B\right) & \lambda^2\left(\alpha^T B\right) & \cdots & \lambda^{n-1}\left(\alpha^T B\right)\end{bmatrix}$$

Letting $\alpha^T B = 0$ in the above results in

$$\alpha^T\mathbb{C} = \begin{bmatrix}0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

Hence

$$\alpha^T\mathbb{C} = 0^T$$

Since α is not zero then the above implies the left null space of $\mathbb{C}$ is not empty. Taking the transpose of both sides gives

$$\begin{aligned}\left(\alpha^T\mathbb{C}\right)^T &= 0\\ \mathbb{C}^T\alpha &= 0\end{aligned}$$

The null space of the transpose of $\mathbb{C}$ is not empty. Hence $\mathbb{C}^T$ is not full rank which means $\mathbb{C}$ is is not full rank (Transposing a matrix does not change its rank). This implies (A, B) is not controllable by definition.

2.4.3 Problem 2

A function sequence f_n on D is said to converge pointwise to f if $\lim_{n\to\infty} f_n(t)$ exist for each t in D. This means, not only the limit needs to be found, if it exist, but there should be a limit for each t in the interval the function is defined over. If for even a single t_0 there is no limit, then the function does not converge pointwise.

$$\lim_{n\to\infty}\frac{t^2}{1+nt} = t^2\lim_{n\to\infty}\frac{1}{1+nt}$$

When $t = 0$ the limit is zero. For all other values, $0 < t < \infty$ the limit is $t^2\frac{1}{\infty} = 0$. So the limit exists for each t. $\boxed{\text{The pointwise limit is the function } f^*(t) = 0}$.

Now to find if the sequence converges uniformly. A function sequence $f_n(t)$ is uniformly convergent on D if for each $\epsilon > 0$ one can find an integer N such that $\|f_n - f\| < \epsilon$ and all $n \geq N$ and each t in D.The integer N here depends only on ϵ and does not depend on t. For pointwise convergence, N depends on both t and ϵ.

Since the sequence convergence pointwise to $f^*(t) = 0$ then one needs to show that

$$\|f_n - f\|_I = \sup_{0\leq t<\infty}\left|f_n(t) - f^*(t)\right|$$

goes to zero as $n \to \infty$. But $f^*(t) = 0$ from above, hence

$$\begin{aligned}\|f_n - f\|_I &= \sup_{0\leq t<\infty}\left|f_n(t)\right|\\ &= \sup_{0\leq t<\infty}\left|\frac{t^2}{1+nt}\right|\end{aligned}$$

To find the maximum of $f_n(t) = \frac{t^2}{1+nt}$, the equation $f_n'(t) = 0$ is first solved for t

$$\frac{d}{dt}\left(\frac{t^2}{1+nt}\right) = 0$$
$$\frac{t(2+nt)}{(1+nt)^2} = 0$$

Hence $t(2+nt) = 0$ gives the solutions $t = 0$ and $t = -\frac{2}{n}$. When $t = 0$ then $f_n(0) = 0$ and when $t = -\frac{2}{n}$ then $f_n\left(-\frac{2}{n}\right) = \frac{\left(-\frac{2}{n}\right)^2}{1+n\left(-\frac{2}{n}\right)} = -\frac{4}{n^2}$

The maximum of these is $\frac{4}{n^2}$ in absolute terms. Hence

$$\sup_{0\leq t<\infty}\left|\frac{t^2}{1+nt}\right| = \frac{4}{n^2}$$

Therefore

$$\left\|f_n - f\right\|_I = \frac{4}{n^2}$$

Taking the limit $n \to \infty$ gives

$$\lim_{n\to\infty}\left\|f_n - f\right\|_I = 0$$

Since the limit is zero, then $\boxed{\text{the sequence does convergences uniformly}}$.

2.4.4 Problem 3

$I = \int_0^1 t\left(1-t^2\right)^k dt$ is first evaluated. This is done using substitution Let $u = 1-t^2$, hence $du = -2tdt$. When $t = 0, u = 1$ and when $t = 1, u = 0$. Therefore the integral becomes

$$\begin{aligned} I &= \int_1^0 tu^k \frac{du}{-2t} \\ &= \frac{-1}{2}\int_1^0 u^k du \\ &= \frac{-1}{2}\left[\frac{u^{k+1}}{k+1}\right]_1^0 \\ &= \frac{-1}{2(k+1)}[0-1] \\ &= \frac{1}{2(k+1)} \end{aligned}$$

Hence

$$\begin{aligned} \lim_{k\to\infty}\int_0^1 f_k(t)\,dt = \lim_{k\to\infty}\int_0^1 k^2t\left(1-t^2\right)^k dt &= \lim_{k\to\infty} k^2\frac{1}{2(k+1)} \\ &= \frac{1}{2}\left(\lim_{k\to\infty} k + \lim_{k\to\infty} k^2\right) \\ &= \infty \end{aligned}$$

Let $f^*(t)$ be the pointwise limit of $f_k(t)$. At $t = 0$, $\lim_{k\to\infty} f_k(0) = 0$ and at $t = 1$, $\lim_{k\to\infty} f_k(1) = 0$. For $0 < t < 1$

$$\begin{aligned} \lim_{k\to\infty} f_k(t) &= \lim_{k\to\infty} k^2t\left(1-t^2\right)^k \\ &= \lim_{k\to\infty} k^2te^{k\ln\left(1-t^2\right)} \end{aligned}$$

Since $0 < 1 - t^2 < 1$ for $0 < t < 1$ then $\ln\left(1 - t^2\right)$ is negative, hence $e^{k\ln\left(1-t^2\right)}$ will go to zero in the limit much faster than k^2 going towards infinity. Hence

$$\lim_{k\to\infty} f_k(t) = 0$$

This shows that the pointwise convergence is $\boxed{f^*(t) = 0}$.Therefore $\int_0^1 f^*(t)\,dt = 0$

From the first part it was found that $\boxed{\lim_{k\to\infty} \int_0^1 f_k(t)\,dt = \infty}$ and from the second part $\boxed{\int_0^1 \lim_{k\to\infty} f_k(t) = 0}$.It is clear the quantities are the not the same. To be able to move the limit inside the integral, the sequence must be uniformly convergent.

The above indirectly indicates that $\boxed{f_k(t) \text{ is not uniform convergent}}$. This can be confirmed by trying to find the uniform convergence limit to show that it does not exist:

$$\left\|f_k - f\right\|_I = \max_{0\le t\le 1}\left|f_k(t) - f^*(t)\right| = \max_{0\le t\le 1}\left|f_k(t)\right|$$

Since $f^*(t) = 0$ identically. Hence

$$\left\|f_k - f\right\|_I = \max_{0\le t\le 1}\left|k^2 t\left(1 - t^2\right)^k\right|$$

At $t = 0$, $f_k = 0$ and at $t = 1, f_k(0) = 0$. The maximum value between zero and one is found from calculus:

$$\begin{aligned} f_k'(t) &= 0 \\ k^2\left(1 - t^2\right)^k - 2k^3 t^2\left(1 - t^2\right)^{k-1} &= 0 \\ k\left(1 - t^2\right)^{k-1}\left((1 + 2k)t^2 - 1\right) &= 0 \end{aligned}$$

The solution is $t = \pm\frac{1}{\sqrt{1+2k}}$. Substituting this back into $f_k(t)$ gives the value (using the positive root)

$$\begin{aligned} f_{k\max} &= k^2 t\left(1 - t^2\right)^k \\ &= k^2\frac{1}{\sqrt{1 + 2k}}\left(1 - \frac{1}{1 + 2k}\right)^k \end{aligned}$$

Therefore

$$\left\|f_n - f\right\|_I = \left|k^2\frac{1}{\sqrt{1 + 2k}}\left(1 - \frac{1}{1 + 2k}\right)^k\right|$$

Taking the limit of the above as $k \to \infty$ gives

$$\begin{aligned}
\left\| f_k - f \right\|_I &= \lim_{k\to\infty} k^2 \frac{1}{\sqrt{1+2k}} \left(1 - \frac{1}{1+2k}\right)^k \\
&= \left(\lim_{k\to\infty} k^2 \frac{1}{\sqrt{1+2k}}\right) \lim_{k\to\infty} \left(1 - \frac{1}{1+2k}\right)^k \\
&= \left(\lim_{k\to\infty} k^2 \frac{1}{\sqrt{1+2k}}\right) \lim_{k\to\infty} \left(1 - \frac{\frac{1}{k}}{\frac{1}{k}+2}\right)^k \\
&= \left(\lim_{k\to\infty} k^2 \frac{1}{\sqrt{1+2k}}\right) \frac{1}{\sqrt{e}} \\
&= \left(\lim_{k\to\infty} \frac{1}{\sqrt{\frac{1}{k^4} + \frac{2}{k^3}}}\right) \frac{1}{\sqrt{e}} \\
&= \left(\frac{1}{0}\right) \frac{1}{\sqrt{e}} \\
&= \infty \frac{1}{\sqrt{e}} \\
&= \infty
\end{aligned}$$

Therefore since the limit does not go to zero, then $f_k(t)$ does not converge uniformly. This explains why $\lim_{k\to\infty} \int_0^1 f_k(t)\,dt \neq \int_0^1 \lim_{k\to\infty} f_k(t)\,dt$

2.4.5 Problem 4

The nonlinear state space system is given by

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = f(x,t) = \begin{pmatrix} \cos x_1(t) \\ t x_1(t) + e^{-t} x_2(t) \end{pmatrix}$$

With the initial conditions

$$x^0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Let the initial guess of the solution x^0 be the same as initial conditions [2].

The first iteration gives

$$\begin{aligned}
x^1 &= x^0 + \int_0^t \begin{pmatrix} \cos x_1^0 \\ \eta x_1^0 + e^{-\eta} x_2^0 \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos 2 \\ 2\eta - e^{-\eta} \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left[\begin{pmatrix} \eta \cos 2 \\ \eta^2 + e^{-\eta} \end{pmatrix} \right]_0^t \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} t \cos 2 \\ t^2 + e^{-t} - 1 \end{pmatrix}
\end{aligned}$$

Therefore

$$\boxed{x^1 = \begin{pmatrix} 2 + t\cos 2 \\ t^2 + e^{-t} - 2 \end{pmatrix}}$$

---

[2] Initial guess does not have to be the same as initial conditions $x(0)$ and can be any other value. In this problem the initial guess is taken the same as initial conditions.

The second iteration is

$$
\begin{aligned}
x^2 &= x^0 + \int\limits_0^t \begin{pmatrix} \cos x_1^1 \\ \eta x_1^1 + e^{-\eta} x_2^1 \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int\limits_0^t \begin{pmatrix} \cos\left(2 + \eta \cos 2\right) \\ \eta\left(2 + \eta\cos 2\right) + e^{-\eta}\left(\eta^2 + e^{-\eta} - 2\right) \end{pmatrix} d\eta \qquad (1)
\end{aligned}
$$

The top integral $\int\limits_0^t \cos\left(2 + \eta\cos 2\right) d\eta$ is evaluated using substitution. Let $u = 2 + \eta\cos 2$ hence $du = \cos 2 d\eta$. When $\eta = 0, u = 2$ and when $\eta = t, u = 2 + t\cos 2$. Therefore the top integral becomes

$$
\begin{aligned}
\int_0^t \cos\left(2 + \eta\cos 2\right) d\eta &= \int_2^{2+t\cos 2} \cos\left(u\right) \frac{du}{\cos 2} \\
&= \frac{1}{\cos 2}\int_2^{2+t\cos 2} \cos\left(u\right) du \\
&= \frac{1}{\cos 2}\left[\sin\left(u\right)\right]_2^{2+t\cos 2} \\
&= \frac{1}{\cos 2}\left(\sin\left(2 + t\cos 2\right) - \sin 2\right) \\
&= \sec\left(2\right)\sin\left(2 + t\cos 2\right) - \tan 2 \qquad (2)
\end{aligned}
$$

The lower integral in (1) is now evaluate. The first part is of this integral is

$$
\begin{aligned}
\int\limits_0^t \eta\left(2 + \eta\cos 2\right) d\eta = \int\limits_0^t \left(2\eta + \eta^2\cos 2\right) d\eta &= \left[\eta^2 + \frac{\eta^3}{3}\cos 2\right]_0^t \\
= t^2 + \frac{t^3}{3}\cos 2 \qquad (3)
\end{aligned}
$$

The second part is

$$
\int\limits_0^t \eta^2 e^{-\eta} + e^{-2\eta} - 2e^{-\eta} d\eta \qquad (3A)
$$

The first part of the above is solved using integration by parts. $udv = uv - \int vdu$. Let $u = \eta^2, dv = e^{-\eta}, du = 2\eta, v = -e^{-\eta}$, therefore

$$
\begin{aligned}
\int_0^t \eta^2 e^{-\eta} d\eta &= \left[-\eta^2 e^{-\eta}\right]_0^t + \int_0^t 2\eta e^{-\eta} du \\
&= -t^2 e^{-t} + 2\int_0^t \eta e^{-\eta} du
\end{aligned}
$$

The integral $\int_0^t \eta e^{-\eta} du$ is solved also by integration by parts. $udv = uv - \int vdu$. Let $u = \eta, dv = e^{-\eta}, du = 1, v = -e^{-\eta}$, therefore

$$
\begin{aligned}
\int_0^t \eta^2 e^{-\eta} d\eta &= -t^2 e^{-t} + 2\left(\left[-\eta e^{-\eta}\right]_0^t + \int_0^t e^{-\eta} du\right) \\
&= -t^2 e^{-t} + 2\left(-te^{-t} + \left[-e^{-\eta}\right]_0^t\right) \\
&= -t^2 e^{-t} + 2\left(-te^{-t} - e^{-t} + 1\right) \\
&= -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2
\end{aligned}
$$

The remaining parts of (3A) are direct integrations that requires no special treatment, hence (3A) becomes

$$
\begin{aligned}
\int\limits_0^t \eta^2 e^{-\eta} + e^{-2\eta} - 2e^{-\eta} d\eta &= \left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2\right) + \left[\frac{e^{-2\eta}}{-2}\right]_0^t + 2\left[e^{-\eta}\right]_0^t \\
&= \left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2\right) - \frac{1}{2}\left(e^{-2t} - 1\right) + 2\left(e^{-t} - 1\right) \\
&= \frac{1}{2} - 2te^{-t} - t^2 e^{-t} - \frac{1}{2}e^{-2t} \qquad (4)
\end{aligned}
$$

Putting (4),(3) and (2) into (1) gives

$$x^2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} \sec(2)\sin(2+t\cos 2) - \tan 2 \\ \frac{2}{3}t^2 + \frac{t^3}{3}\cos 2 + \frac{1}{2} - 2te^{-t} - t^2e^{-t} - \frac{1}{2}e^{-2t} \end{pmatrix}$$

Hence the second iteration results in

$$\boxed{x^2 = \begin{pmatrix} 2 + \sec(2)\sin(2+t\cos 2) - \tan 2 \\ -1 + t^2 + \frac{t^3}{3}\cos 2 + \frac{1}{2} - 2te^{-t} - t^2e^{-t} - \frac{1}{2}e^{-2t} \end{pmatrix}}$$

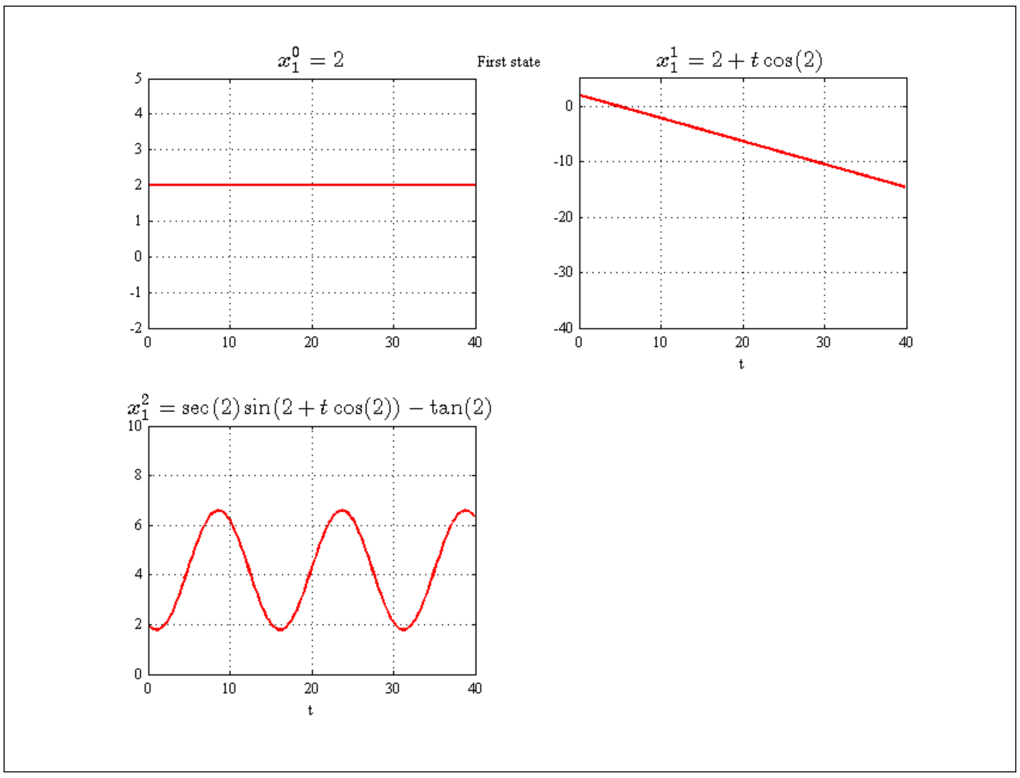
The third iteration x^3 is now found using

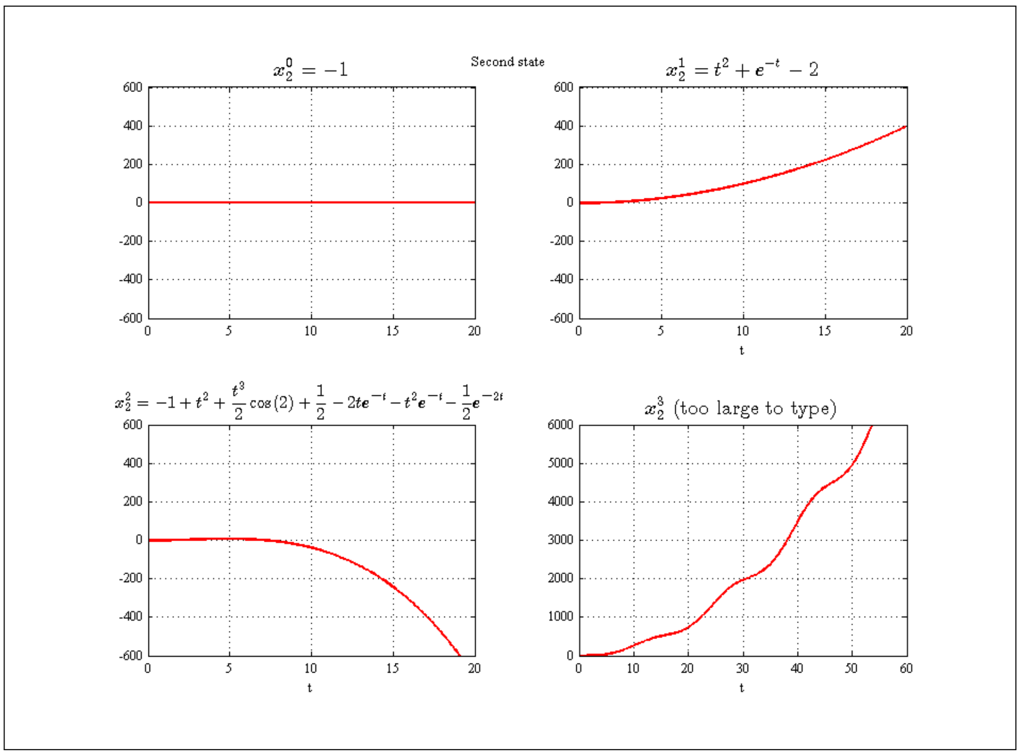
$$\begin{aligned} x^3 &= x^0 + \int_0^t f\left(x^2\right) d\eta \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos x_1^2 \\ \eta x_1^2 + e^{-\eta} x_2^2 \end{pmatrix} d\eta \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} 2 + \sec(2)\sin\left(2 + \eta\cos 2\right) - \tan 2 \\ -1 + \eta^2 + \frac{\eta^3}{3}\cos 2 + \frac{1}{2} - 2\eta e^{-\eta} - \eta^2 e^{-\eta} - \frac{1}{2}e^{-2\eta} \end{pmatrix} d\eta \end{aligned}$$

The top integral $\left(x_1^3\right)$ could not be evaluated using syms. A numerical solution is needed. The lower integral which gives the second state can be evaluated directly and requires no special treatment, giving

```
1 + (1/6)*(-1 + E^(-3*t)) - ((1/2)*(-1 + E^(2*t) - 2*t))/E^(2*t)
+ t^2 - (2 + t*(2 + t))/E^t +(1/4)*(-1 + (1 + 2*t + 2*t^2)/E^(2*t)) +
(1/3)*(6 + (-6 - t*(6 + t*(3 + t)))/E^t)*Cos[2] +
(1/2)*(-1 + Cosh[t] - Sinh[t]) + Sec[2]^2*((-t)*Cos[2 + t*Cos[2]] +
Sec[2]*Sin[2 + t*Cos[2]] - Tan[2]) - (1/2)*t^2*Tan[2]
```

A small function was written using syms to evaluate the Picard iterations and plot the solution. For the third iteration x^3 the first state was not solved due to complexity of the integral. Numerical solution would be needed. The following plots show the first state and the second state.





Source code

```
function res = nma_x(k)
%function to evaluate Picard iterations
%by Nasser M. Abbasi, ECE 717, Fall 2014, HW4, problem 4
if k==0
   res = [2;-1];
else
   syms z t;
   last  = nma_x(k-1);
   x1    = last(1); x2=last(2);
   x1    = subs(x1,t,z);
   x2    = subs(x2,t,z);
   res   = [2;-1] + int( [cos(sym(x1));z*x1+exp(-z)*x2],z,0,t);
end
res;
end



%script to plot the Picard iterations
%Nasser M. Abbasi, HW4, ECE 717

x0=nma_x(0);
x1=nma_x(1);
x2=nma_x(2);
x3=nma_x(3);
max_t=40; max_y=10;

close all; set(0,'DefaultAxesFontName', 'Times New Roman');
set(0,'DefaultAxesFontSize',8);
set(0,'DefaultTextFontname', 'Times New Roman'); set(0,'DefaultTextFontSize', 12);

subplot(2,2,1);
h(1)=plot([0,max_t],[x0(1),x0(1)]);
grid on; set(h(1),'linewidth',1.5); set(h(1),'color','r');
xlim([0,max_t]); ylim([-2,5]);
title('$$x^0_1 = 2$$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,2);
h(2)=ezplot(x1(1),[0,max_t]);
grid on; set(h(2),'linewidth',1.5); set(h(2),'color','r'); ylim([-40,5]);
title('$$x^1_1 = 2+t \cos(2) $$', 'FontSize', 12,'interpreter','latex');
```

```
subplot(2,2,3);
h(3)=ezplot(x2(1),[0,max_t]);
grid on; set(h(3),'linewidth',1.5); set(h(3),'color','r'); ylim([0,10]);
title('$$x^2_1 = \sec(2) \sin(2+t\cos(2))-\tan(2)$$', 'FontSize', 12,'interpreter','latex');

ax=axes('Units','Normal','Position',[.075 .075 .85 .85],'Visible','off');
set(get(ax,'Title'),'Visible','on')
title('First state');

%now do x_2
figure;
max_t=20; max_y=600;
subplot(2,2,1);
h(1)=plot([0,max_t],[x0(2),x0(2)]);
grid on; set(h(1),'linewidth',1.5); set(h(1),'color','r');
xlim([0,max_t]); ylim([-max_y,max_y]);
title('$$x^0_2 = -1$$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,2);
h(2)=ezplot(x1(2),[0,max_t]);
grid on; set(h(2),'linewidth',1.5); set(h(2),'color','r'); ylim([-max_y,max_y]);
title('$$x^1_2=t^2+e^{-t}-2 $$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,3);
h(3)=ezplot(x2(2),[0,max_t]);
grid on; set(h(3),'linewidth',1.5); set(h(3),'color','r'); ylim([-max_y,max_y]);
title('$$x^2_2 = -1+t^2+\frac{t^3}{2}\cos(2)+\frac{1}{2}-2 t e^{-t}-t^2 e^{-t}-\frac{1}{2}e^{-

subplot(2,2,4);
h(4)=ezplot(x3(2),[0,60]);
grid on; set(h(4),'linewidth',1.5); set(h(4),'color','r'); ylim([0,6000]);
title('$$x^3_2$$ (too large to type)', 'FontSize', 12,'interpreter','latex');

ax=axes('Units','Normal','Position',[.075 .075 .85 .85],'Visible','off');
set(get(ax,'Title'),'Visible','on')
title('Second state');
```

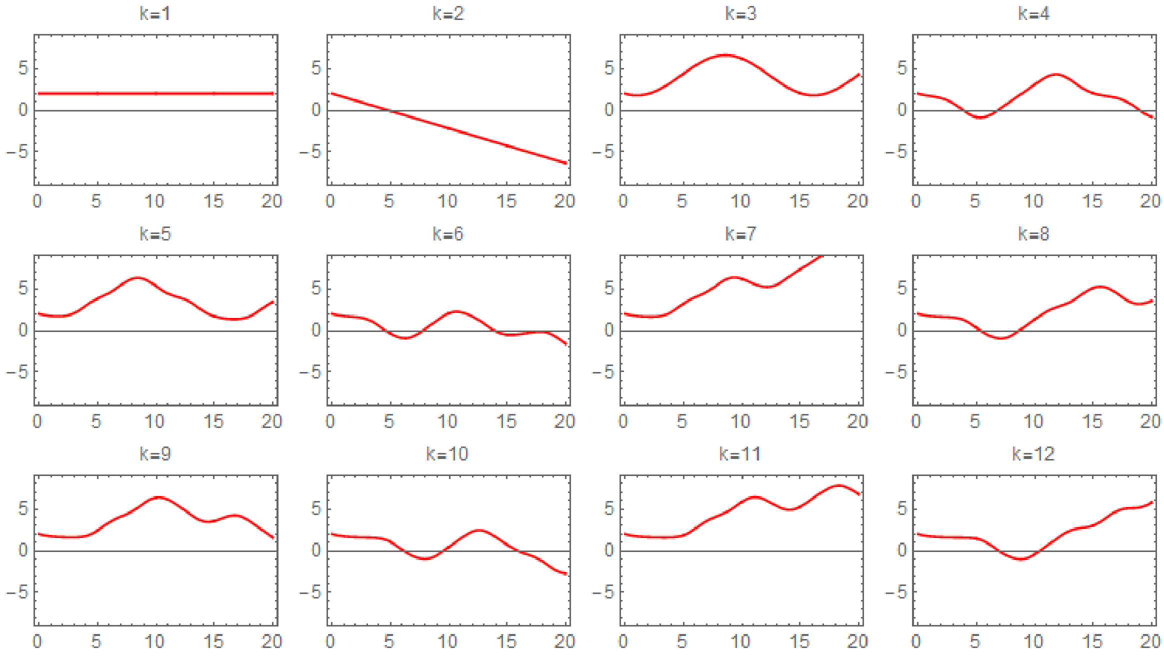
Example using Picard iteration function Example use is

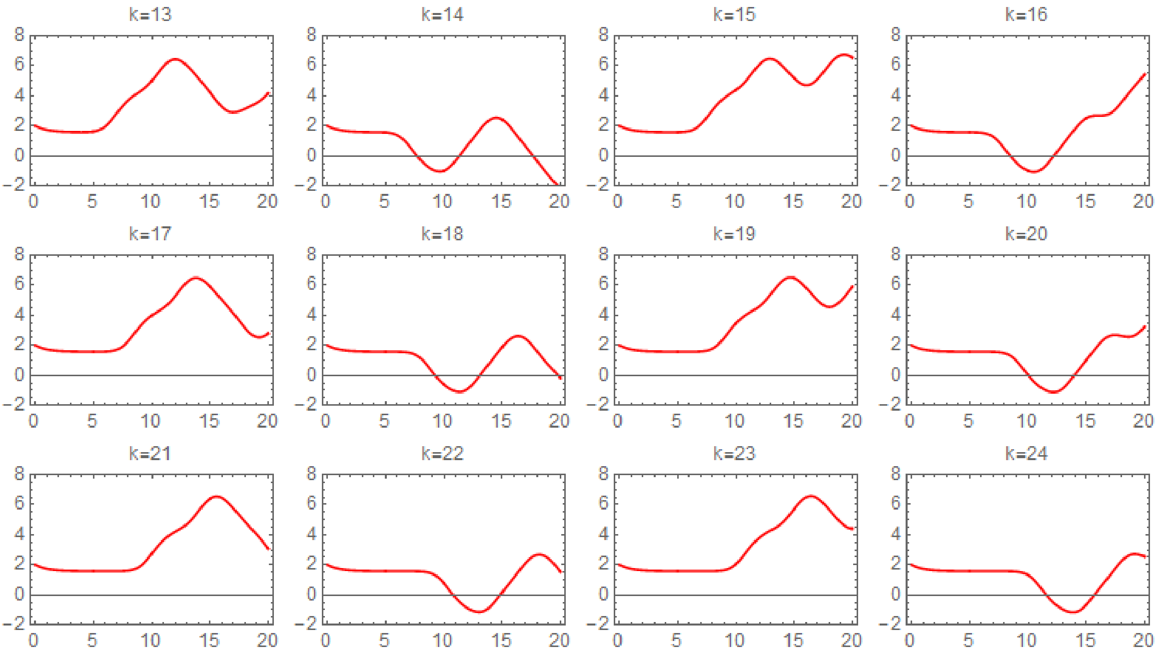
```
EDU>> nma_x(0)
2
-1
EDU>> nma_x(1)
t*cos(2) + 2
exp(-t) + t^2 - 2
EDU>> nma_x(2)
(sin(t*cos(2) + 2) - sin(2))/cos(2) + 2
(t^3*cos(2))/3 - 2*t*exp(-t) - t^2*exp(-t) - exp(-2*t)/2 + t^2 - 1/2
EDU>> nma_x(3)
Warning: Explicit integral could not be found.
int(cos((sin(z*cos(2) + 2) - sin(2))/cos(2) + 2), z == 0..t) + 2
exp(-t)/2 + exp(-3*t)/6 - (sin(2) - sin(t*cos(2) + 2) +
t*cos(2)*cos(t*cos(2) + 2))/cos(2)^3
- exp(-t)*(t^2 + 2*t + 2) + (exp(-2*t)*(2*t + 1))/2 +
t^2 - (cos(2)*(exp(-t)*(t^3 + 3*t^2 + 6*t + 6) - 6))/3 +
(exp(-2*t)*(4*t^2 + 4*t + 2))/8 - (t^2*sin(2))/(2*cos(2)) - 5/12
```

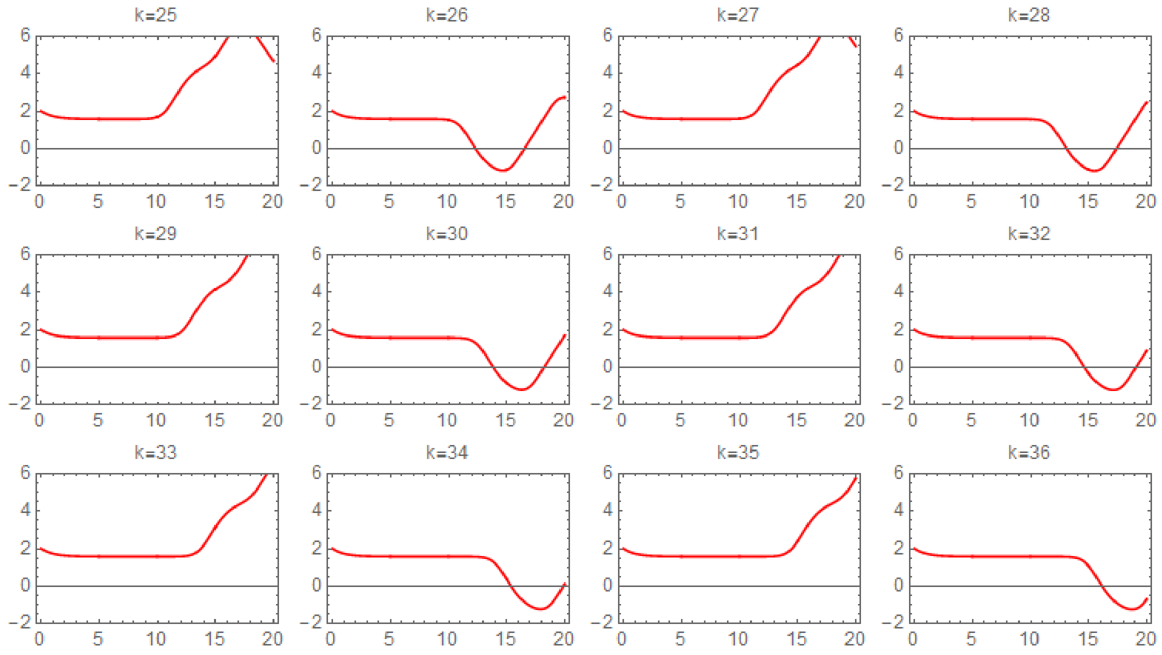
Convergence of solution using numerical integration

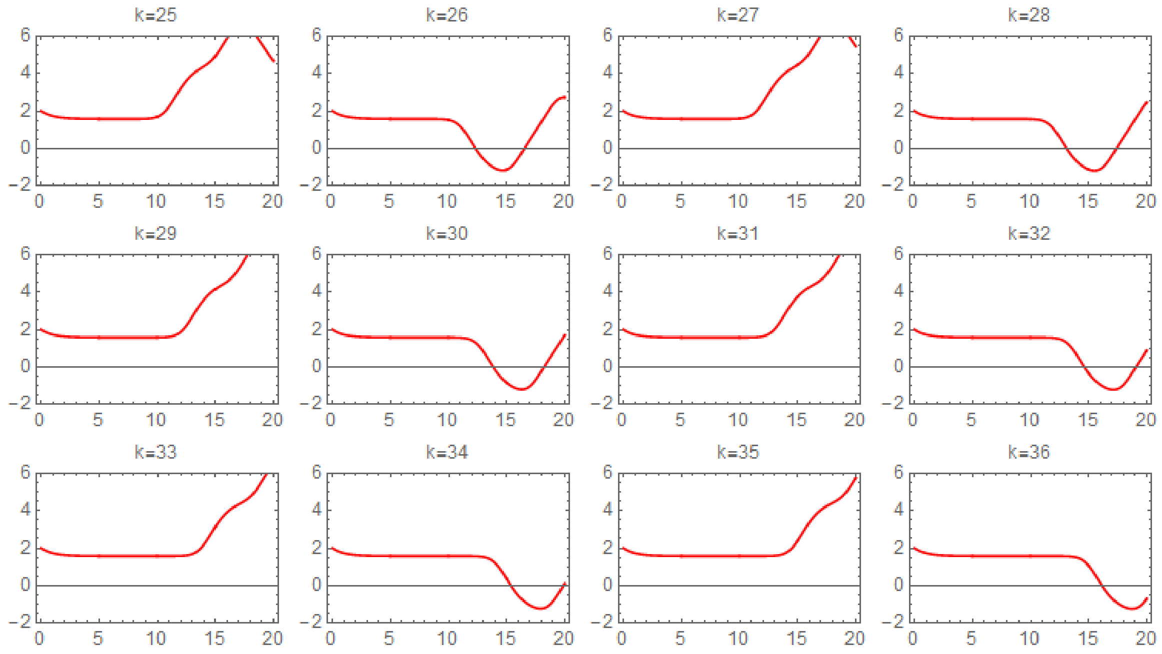
Picard iteration was integrated numerically due to difficulty of obtaining symbolic solution for each step. The following sequence of plots shows the convergence of each iteration. The first state required about 60 iterations to converge to the numerical ODE solver solution. The following shows the sequence of the iterations for the first state. Each one of these plots is 20 seconds long, and the title shows the iteration number.

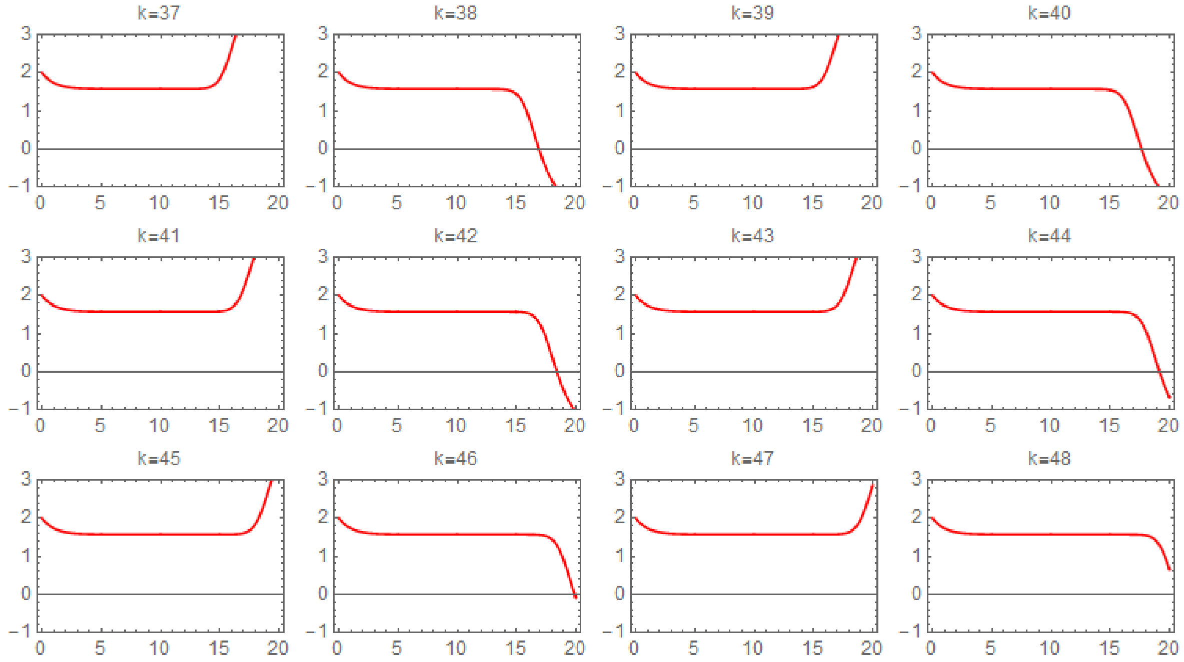
First state iterations

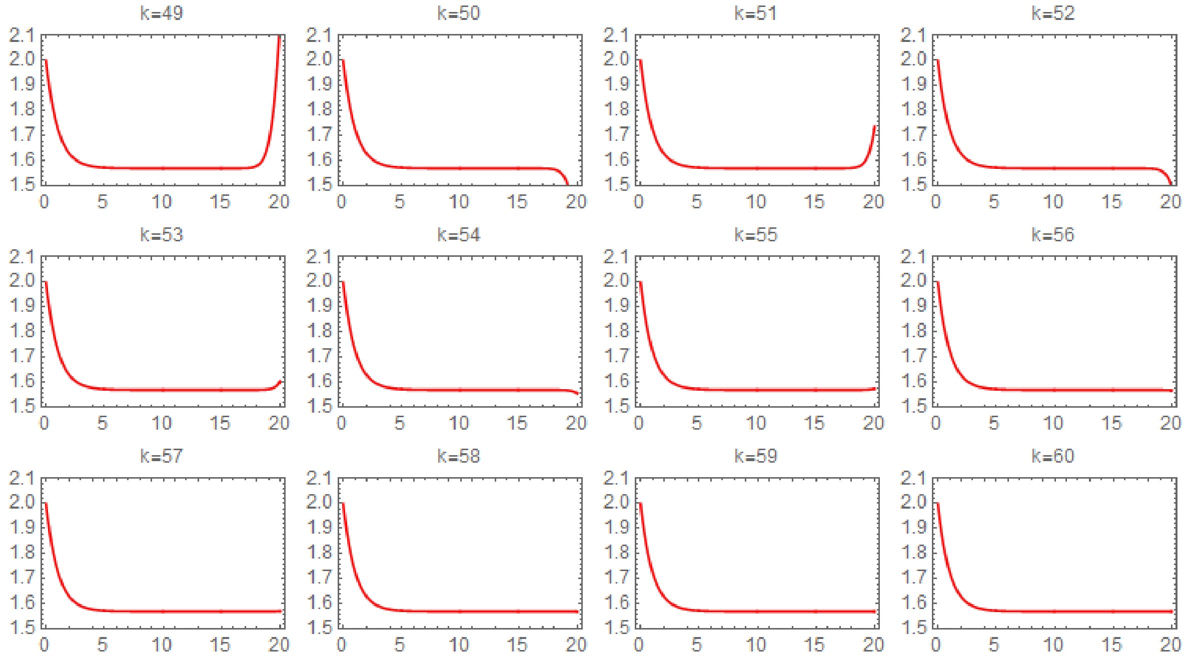




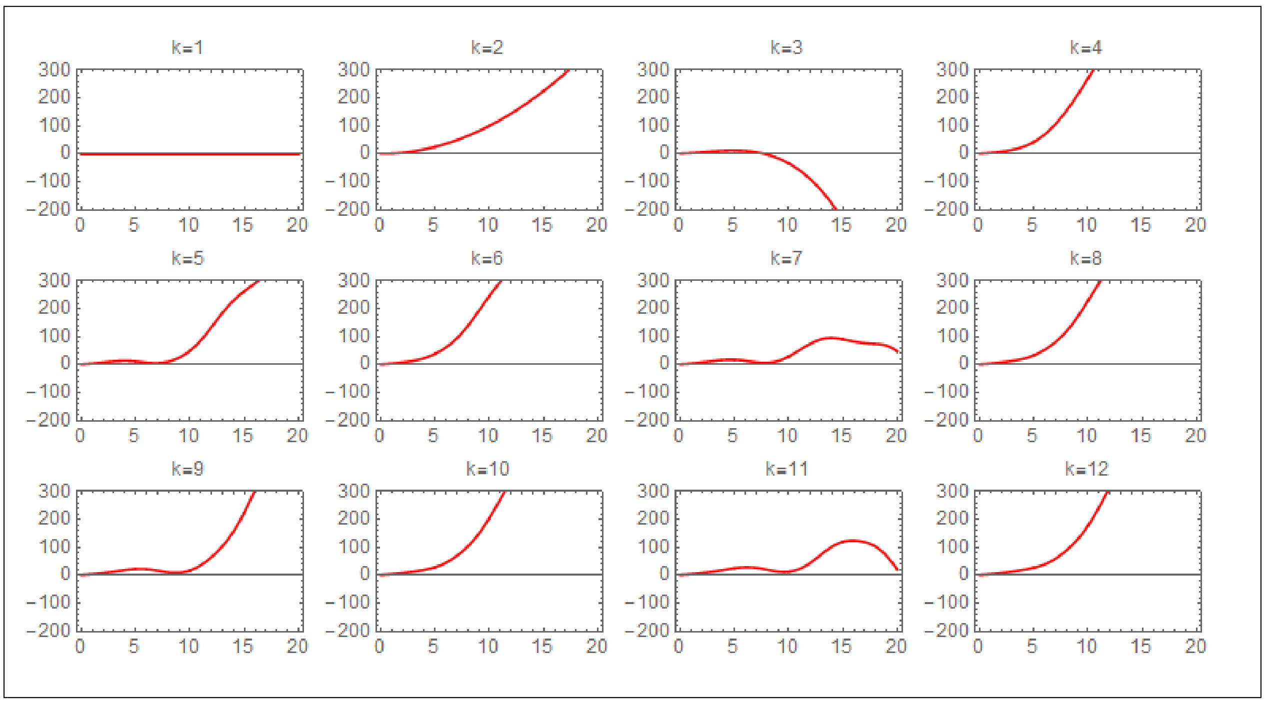

k=25
k=26
k=27
k=28
k=29
k=30
k=31
k=32
k=33
k=34
k=35
k=36
6
4
2
0
−2
0
5
10
15
20

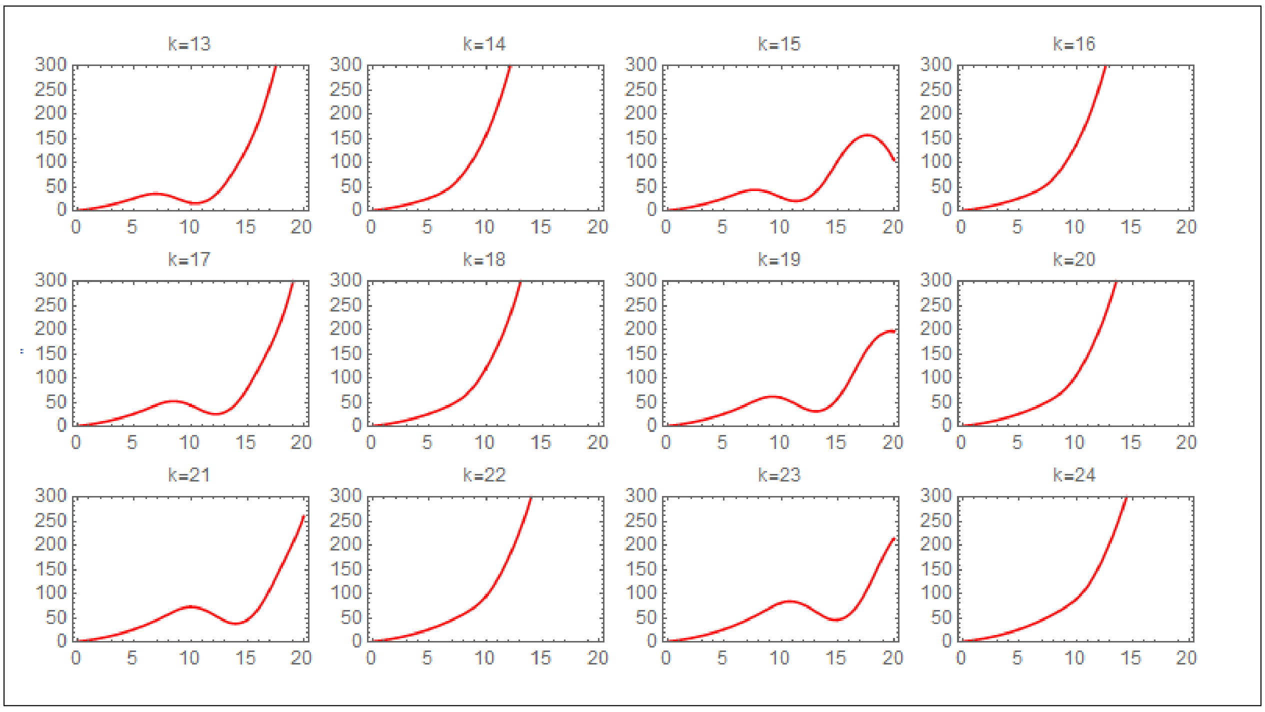

k=25
k=26
k=27
k=28
k=29
k=30
k=31
k=32
k=33
k=34
k=35
k=36
6
4
2
0
−2
0
5
10
15
20

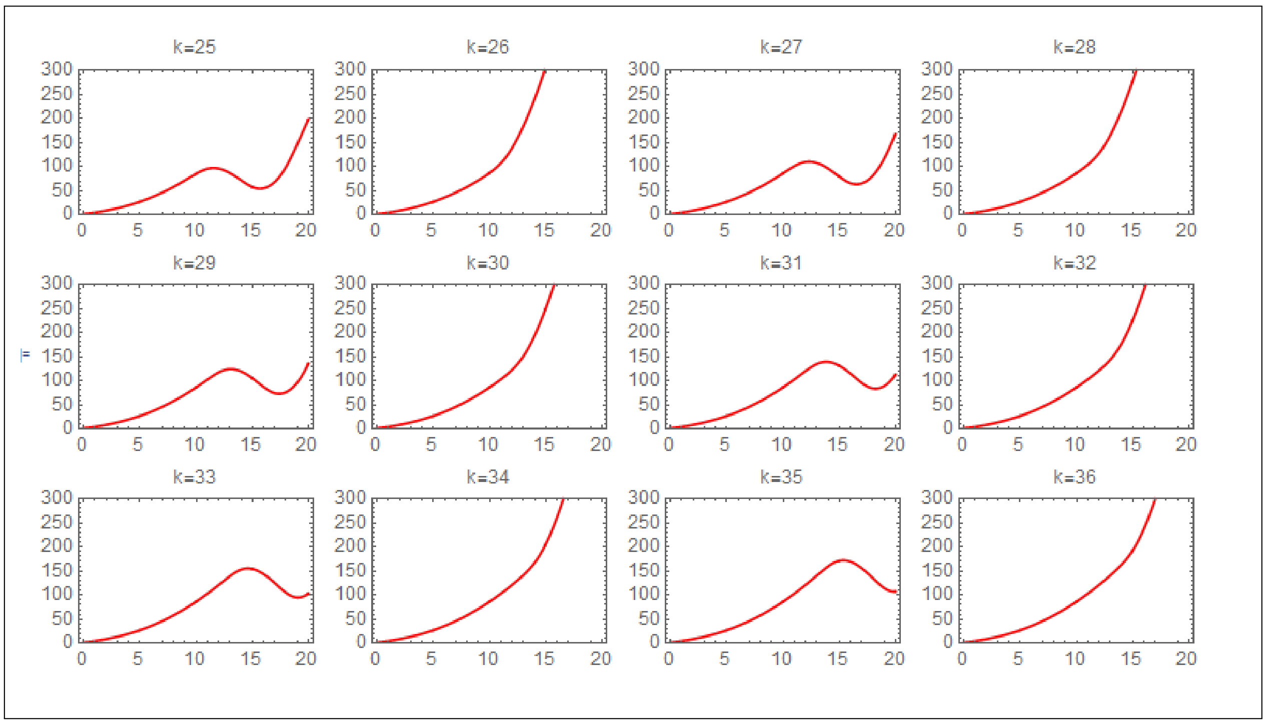

k=37
k=38
k=39
k=40
k=41
k=42
k=43
k=44
k=45
k=46
k=47
k=48
3
2
1
0
−1
0
5
10
15
20

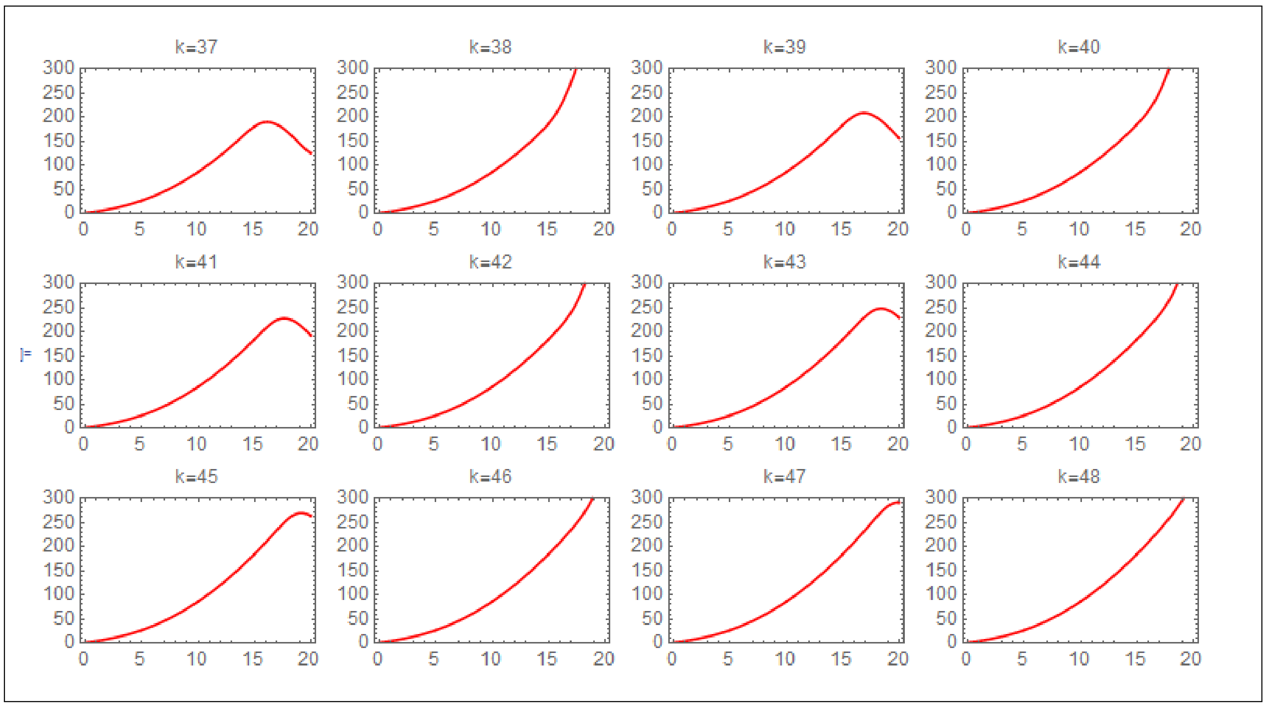


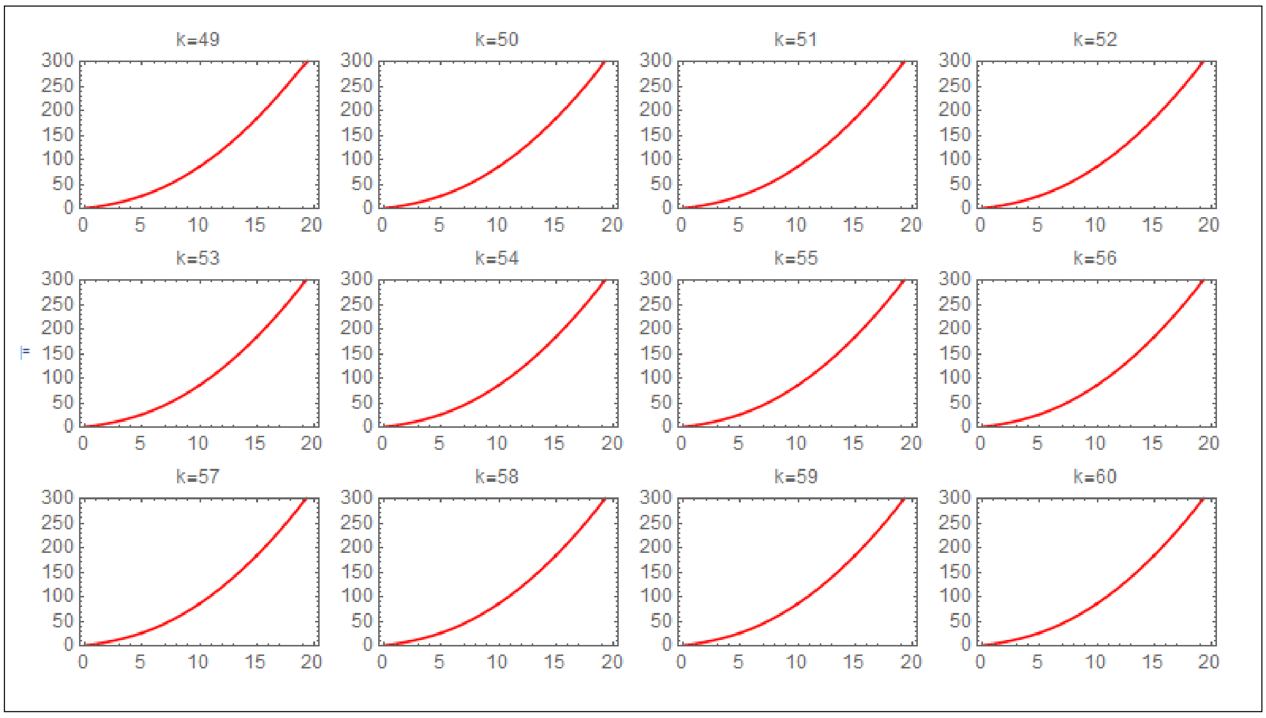
second state iterations It also took about 60 Picard iterations for the second state to converge. The following is the sequence of the iterations





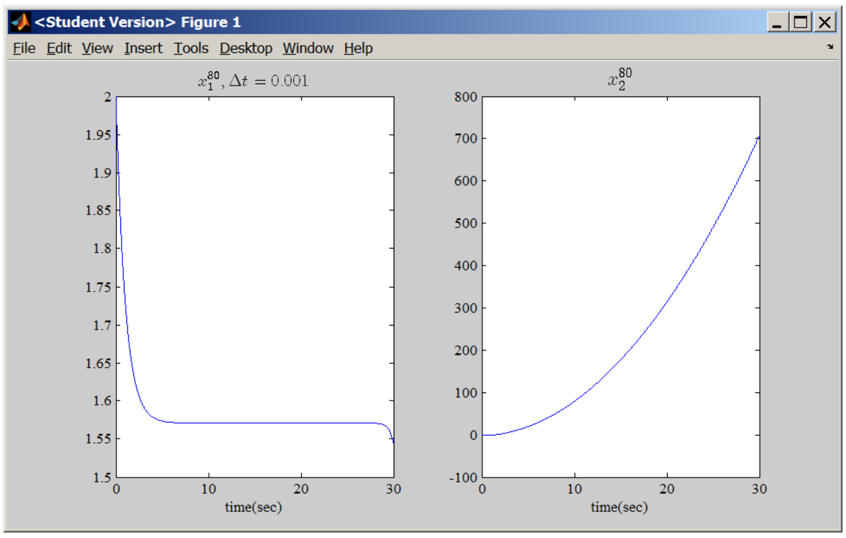

k=25
k=26
k=27
k=28
k=29
k=30
k=31
k=32
k=33
k=34
k=35
k=36
0 50 100 150 200 250 300
0 5 10 15 20


k=37
k=38
k=39
k=40
k=41
k=42
k=43
k=44
k=45
k=46
k=47
k=48
0 50 100 150 200 250 300
0 5 10 15 20

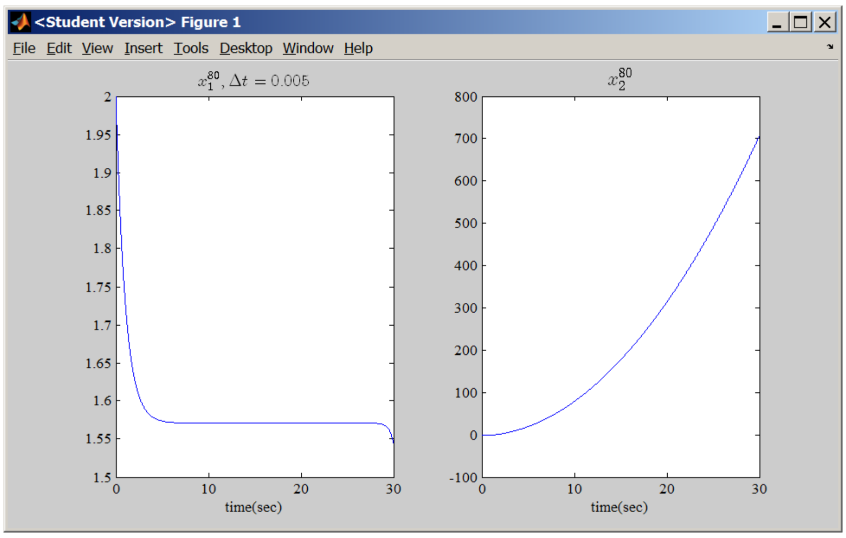

k=49
k=50
k=51
k=52
k=53
k=54
k=55
k=56
k=57
k=58
k=59
k=60
0 50 100 150 200 250 300
0 5 10 15 20

effect of changing Δt on convegence

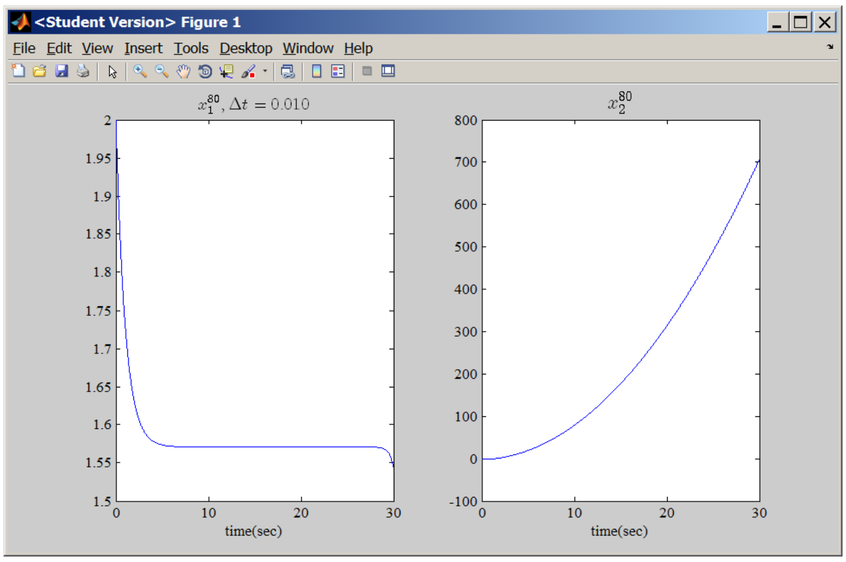
Since numerical integral was used, it would be useful to see what effect changing the sampling period on convergence. Three different values of Δt were tried. They are $(0.01, 0.005, 0.001)$, with units in seconds. There was no visible effect on the result. Running the program for 80 iterations for each case, they all converged to the same solution at the end. The following plots shows the result



Sampling 0.001 sec
80 Picard iterations



Sampling 0.05 sec
80 Picard iterations



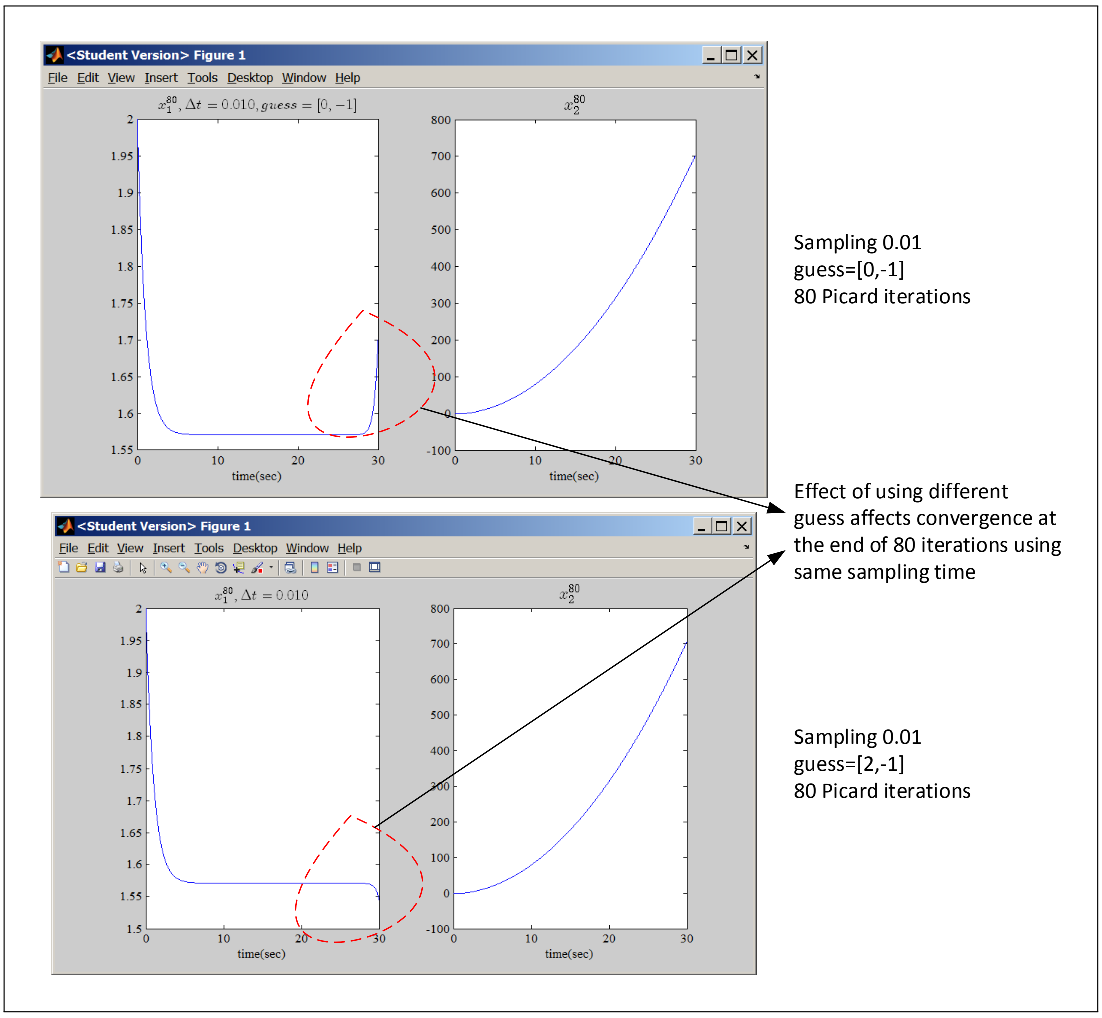
Sampling 0.01 sec
80 Picard iterations

effect of changing initial guess on convegence

Since initial guess can be any value, other than the initial conditions, it would be useful to see what effect, if any, changing the guess would have on convergence.

It was found that changing the guess to be different from initial conditions, resulted in different shape at the end of the 80 iterations. This indicates the guess used have an effect on speed of convergence. More analysis is required to inverstigate this more.

For example, this plot shows the difference at the end of 80 iterations, all using the same sampling time, with the only difference is that one used the initial conditions $[2,-1]$ as the guess, and the second used $[0,1]$ as the guess. One can see the final solution is different.



Sampling 0.01
guess=[0,-1]
80 Picard iterations

Effect of using different guess affects convergence at the end of 80 iterations using same sampling time

Sampling 0.01
guess=[2,-1]
80 Picard iterations

Conclusion

Picard iteration does converge for this non-linear system. Numerical integration was required to allow higher number of iterations to be performed, as it was not possible to do more than 3 iterations using symbolic computation.

It was found that changing the guess value from initial conditions does have an effect on convergence. But more analysis is needed to study this effect.

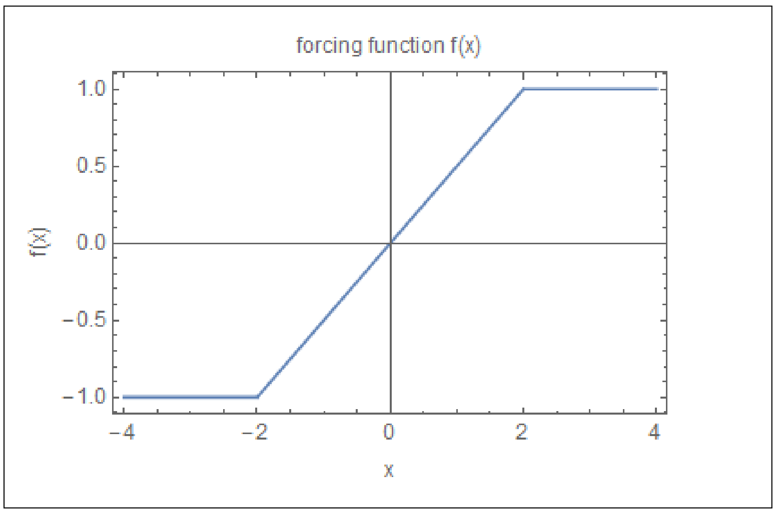
Function to do numerical Picard iterations

```
function   nma_picard()
%version oct 17, 2014
%study of picard iteration method for non-linear state space system
%Matlab 2013a
%The following parameters can be changed: max simulation time,
%time spacing between each sample, initial conditions, and
%initial guess.
%by Nasser M. Abbasi

close all

number_of_iterations = 100;    %How many Picard iterations to do?
initial_conditions   = [2 -1]; %initial conditions for x_1 and x_2
initial_guess        = [2 -1]; %initial guess
max_time             = 50;     %simulation time in seconds
delT                 = 0.02;   %time spacing for sampling, numerical integration

nSamples             = round(max_time/delT);
first_K              = bsxfun(@times, initial_guess,ones(nSamples,2));
```

```
next_K               = zeros(nSamples,2);
t                    = delT*[0;cumsum(ones(nSamples-1,1))]; %used for numerical integration

%obtain numerical ODE45 solution to use to compare with
odeTime                     = 0:delT:max_time-delT;
[odeTime,ODE45_solution]    = ode45(@rhs,odeTime,initial_conditions);

for n = 1:number_of_iterations
    for i = 1:nSamples %numerical integration as time is increased
                       %t vector above hold incremental time values.
        next_K(i,1) = initial_conditions(1) + delT*trapz(cos(first_K(1:i,1)));
        z = t(1:i).*first_K(1:i,1)+exp(-t(1:i)).*first_K(1:i,2);
        next_K(i,2) = initial_conditions(2) + delT*trapz(z);
    end

    makePlot(first_K,t,n,delT,initial_guess,max_time,ODE45_solution);
    first_K  = next_K;
end
    function dxdt=rhs(t,x)
        dxdt = [cos(x(1));t*x(1)+exp(-t)*x(2)];
    end
end

%-------------------------------
function makePlot(x,t,n,delT,guess,max_time,ODE45_solution)
if n==1
    scrsz = get(groot,'ScreenSize');
    figure('Position',[.25*scrsz(3) .35*scrsz(4) .5*scrsz(3) .5*scrsz(4)]);
    set(0,'DefaultAxesFontName', 'Times New Roman');
    set(0,'DefaultAxesFontSize',10);
    set(0,'DefaultTextFontname', 'Times New Roman');
    set(0,'DefaultTextFontSize', 12);
end

minY1   = -1;
maxY1   = 2.5;
minY2   = -10;
maxY2   = 1000;


subplot(1,2,1);
hold off;
plot(t,x(:,1));
hold on;
plot(t,ODE45_solution(:,1),'r:');
title(sprintf('$$x_1^{%d}, \\Delta t=%3.3f, guess=[%d,%d]$$',n,delT,guess(1),guess(2)),'FontSi
xlabel('time(sec)');
ylim([minY1,maxY1]);
xlim([0,max_time]);
subplot(1,2,2);
hold off;
plot(t,x(:,2));
hold on;
plot(t,ODE45_solution(:,2),'r:');
title(sprintf('$$x_2^{%d}$$',n),'FontSize', 14,'interpreter','latex');
xlabel('time(sec)');
ylim([minY2,maxY2]);
xlim([0,max_time]);
drawnow
end
```

2.4.6 Problem 5 (corrected after)

The following plot shows $f(x)$



$$\begin{aligned}
x^1(t) &= x^0 + \int_0^t f\left(x^0(\tau),\tau\right) d\tau\\
&= 1 + \int_0^t f(1)\, d\tau\\
&= 1 + \int_0^t \frac{1}{2} d\tau\\
&= 1 + \frac{1}{2}t
\end{aligned}$$

Therefore

$$\boxed{x^1(t) = 1 + \tfrac{1}{2}t}$$

Now for the second iteration

$$\begin{aligned}
x^{(2)}(t) &= x^0 + \int_0^t f\left(x^1(\tau),\tau\right) d\tau\\
&= x^0 + \int_0^t f\left(1 + \frac{1}{2}\tau\right) d\tau
\end{aligned}$$

For $0 \leq t \leq 2$ then $f\left(1 + \frac{1}{2}\tau\right) = \frac{1}{2}\left(1 + \frac{1}{2}\tau\right)$, Therefore

$$\begin{aligned}
x^{(2)}(t) &= x^0 + \int_0^t \frac{1}{2}\left(1 + \frac{1}{2}\tau\right) d\tau\\
&= 1 + \frac{1}{2}\left[\left(\tau + \frac{1}{4}\tau^2\right)\right]_0^t\\
&= 1 + \frac{1}{2}\left(t + \frac{1}{4}t^2\right)\\
&= \frac{t^2}{8} + \frac{t}{2} + 1
\end{aligned}$$

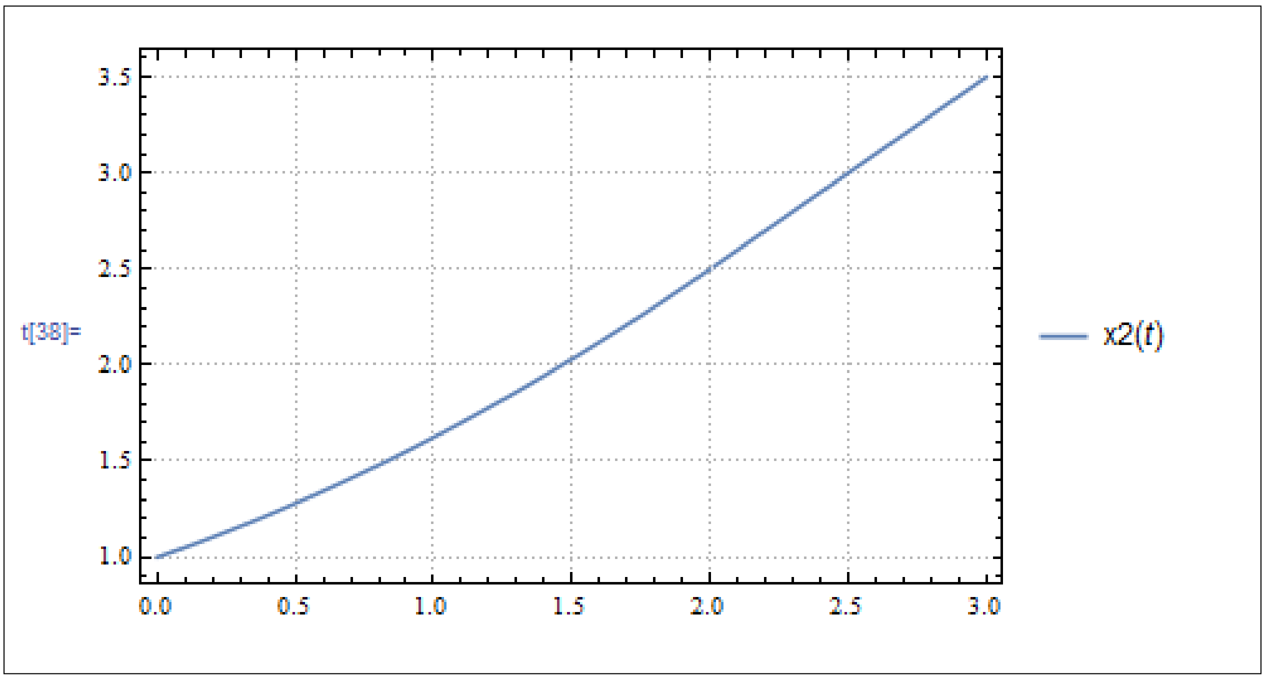
For $t > 2$, $f(t) = 1$, then

$$\begin{aligned}
x^{(2)}(t) &= x^0 + \int_0^2 \frac{1}{2}\left(1 + \frac{1}{2}t\right) d\tau + \int_2^t d\tau\\
&= t + \frac{1}{2}
\end{aligned}$$

Therefore

$$x^2(t) = \begin{cases} 1 + \frac{t}{2} + \frac{t^2}{8} & t \leq 2 \\ t + \frac{1}{2} & t > 2 \end{cases}$$

A plot of $x^2(t)$ is shown below



2.4.7 key solution

ECE 717 – Solution Set 4

Solution Criterion

(a) Sufficiency: Assume there exist $\alpha \neq 0$ and λ complex such that $\alpha^T A = \alpha^T \lambda$ and $\alpha^T B = 0$. Then $\alpha^T A^k B = \lambda^k \alpha^T B$ from which it follows that

$$\alpha^T \mathcal{C}_\Sigma = \alpha^T [B \;\; AB \;\; A^2B \cdots A^{n-1}B]$$
$$= [\alpha^T B \;\; \lambda \alpha^T B \;\; \lambda^2 \alpha^T B \cdots \lambda^{n-1} \alpha^T B]$$
$$= 0$$

Hence, the rows of $\mathcal{C}_\Sigma$ are dependent; i.e., rank $\mathcal{C}_\Sigma < n$ and (A, B) is $\underline{\text{not}}$ a controllable pair.

Solution Convergence

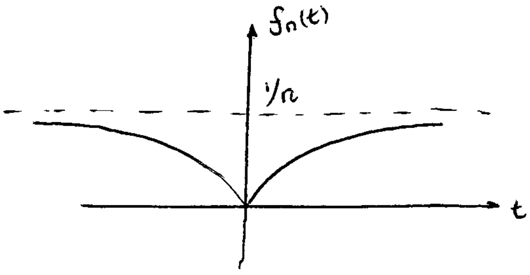
For fixed t, have

$$\lim_{n\to\infty} \frac{t^2}{1+nt^2} = 0$$

Hence $f(t) \equiv 0$ is the pontwise limit

Yes, the limit is uniform ... To see this,

Look at $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{B}}$



Clearly $\|f_n - f\| = 1/n$

So $\lim_{n\to\infty} \|f_n - f\| = 0$ done.

Solution Integrals

$$\int_0^1 f_k(t)\,dt = k^2 \int_0^1 t(1-t^2)^k$$

$$= \frac{k^2}{2k+1}$$

Hence,

$$\lim_{k\to\infty} \int_0^1 f_k(t)\,dt = \infty$$

Now describe pointwise limit:

$$f(t) = \lim_{k\to\infty} k^2 t(1-t^2)^k$$

$$= 0$$

Hence $\int_0^1 f(t)\,dt = 0$

So $\lim_{k\to\infty} \int_0^\infty f_k(t)\,dt \neq \int_0^1 f(t)\,dt$

Note: This is consistent with result in class; i.e., f_k above does not converge uniformly to f.

4 Problem 4

For the nonlinear state equation

$$\dot{x} = f(x,t),$$

consider the Picard iteration scheme beginning with $x^0(t) \equiv x^0$ with iterative step

$$x^{k+1}(t) = x^0 + \int_0^t f(x^k(\eta), \eta) d\eta.$$

Then, for the two state nonlinear system described by

$$\dot{x}_1 = \cos x_1$$

and

$$\dot{x}_2 = t x_1 + e^{-t} x_2,$$

find the first three Picard iterates $x^1(t), x^2(t)$ and $x^3(t)$ corresponding to initial conditions

$$x_1(0) = 2; \quad x_2(0) = -1.$$

Also provide plots of $x_1(t)$ and $x_2(t)$ for each Picard iterate. Are your solutions converging? Discuss.

Note: To maximize learning, I suggest you do this problem by hand with the integral for $x^3(t)$ facilitated with Matlab syms.

The nonlinear state space system is given by

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = f(x,t) = \begin{pmatrix} \cos x_1(t) \\ t x_1(t) + e^{-t} x_2(t) \end{pmatrix}$$

With the initial conditions

$$x^0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Let the initial guess of the solution x^0 be the same as initial conditions [1].

---

[1] Initial guess does not have to be the same as initial conditions $x(0)$ and can be any other value. In this problem the initial guess is taken the same as initial conditions.

The first iteration gives

$$\begin{aligned}
x^1 &= x^0 + \int_0^t \begin{pmatrix} \cos x_1^0 \\ \eta x_1^0 + e^{-\eta} x_2^0 \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos 2 \\ 2\eta - e^{-\eta} \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left[\begin{pmatrix} \eta \cos 2 \\ \eta^2 + e^{-\eta} \end{pmatrix} \right]_0^t \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} t \cos 2 \\ t^2 + e^{-t} - 1 \end{pmatrix}
\end{aligned}$$

Therefore

$$\boxed{x^1 = \begin{pmatrix} 2 + t\cos 2 \\ t^2 + e^{-t} - 2 \end{pmatrix}}$$

The second iteration is

$$\begin{aligned}
x^2 &= x^0 + \int_0^t \begin{pmatrix} \cos x_1^1 \\ \eta x_1^1 + e^{-\eta} x_2^1 \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos(2 + \eta\cos 2) \\ \eta(2 + \eta\cos 2) + e^{-\eta}\left(\eta^2 + e^{-\eta} - 2\right) \end{pmatrix} d\eta \qquad (1)
\end{aligned}$$

The top integral $\int_0^t \cos(2 + \eta\cos 2)\, d\eta$ is evaluated using substitution. Let $u = 2 + \eta\cos 2$ hence $du = \cos 2 d\eta$. When $\eta = 0, u = 2$ and when $\eta = t, u = 2 + t\cos 2$. Therefore the top integral becomes

$$\begin{aligned}
\int_0^t \cos(2 + \eta\cos 2)\, d\eta &= \int_2^{2+t\cos 2} \cos(u) \frac{du}{\cos 2} \\
&= \frac{1}{\cos 2} \int_2^{2+t\cos 2} \cos(u)\, du \\
&= \frac{1}{\cos 2} [\sin(u)]_2^{2+t\cos 2} \\
&= \frac{1}{\cos 2} (\sin(2 + t\cos 2) - \sin 2) \\
&= \sec(2) \sin(2 + t\cos 2) - \tan 2 \qquad (2)
\end{aligned}$$

The lower integral in (1) is now evaluate. The first part is of this integral is

$$\begin{aligned}
\int_0^t \eta(2 + \eta\cos 2)\, d\eta = \int_0^t \left(2\eta + \eta^2 \cos 2\right) d\eta &= \left[\eta^2 + \frac{\eta^3}{3}\cos 2\right]_0^t \\
&= t^2 + \frac{t^3}{3}\cos 2 \qquad (3)
\end{aligned}$$

The second part is

$$\int_0^t \eta^2 e^{-\eta} + e^{-2\eta} - 2e^{-\eta} d\eta \tag{3A}$$

The first part of the above is solved using integration by parts. $udv = uv - \int vdu$. Let $u = \eta^2, dv = e^{-\eta}, du = 2\eta, v = -e^{-\eta}$, therefore

$$\begin{aligned}\int_0^t \eta^2 e^{-\eta} d\eta &= \left[-\eta^2 e^{-\eta}\right]_0^t + \int_0^t 2\eta e^{-\eta} du \\ &= -t^2 e^{-t} + 2\int_0^t \eta e^{-\eta} du\end{aligned}$$

The integral $\int_0^t \eta e^{-\eta} du$ is solved also by integration by parts. $udv = uv - \int vdu$. Let $u = \eta, dv = e^{-\eta}, du = 1, v = -e^{-\eta}$, therefore

$$\begin{aligned}\int_0^t \eta^2 e^{-\eta} d\eta &= -t^2 e^{-t} + 2\left(\left[-\eta e^{-\eta}\right]_0^t + \int_0^t e^{-\eta} du\right) \\ &= -t^2 e^{-t} + 2\left(-te^{-t} + \left[-e^{-\eta}\right]_0^t\right) \\ &= -t^2 e^{-t} + 2\left(-te^{-t} - e^{-t} + 1\right) \\ &= -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2\end{aligned}$$

The remaining parts of (3A) are direct integrations that requires no special treatment, hence (3A) becomes

$$\begin{aligned}\int_0^t \eta^2 e^{-\eta} + e^{-2\eta} - 2e^{-\eta} d\eta &= \left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2\right) + \left[\frac{e^{-2\eta}}{-2}\right]_0^t + 2\left[e^{-\eta}\right]_0^t \\ &= \left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2\right) - \frac{1}{2}\left(e^{-2t} - 1\right) + 2\left(e^{-t} - 1\right) \\ &= \frac{1}{2} - 2te^{-t} - t^2 e^{-t} - \frac{1}{2}e^{-2t} \qquad (4)\end{aligned}$$

Putting (4),(3) and (2) into (1) gives

$$x^2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} \sec(2)\sin(2 + t\cos 2) - \tan 2 \\ \frac{2}{3}t^2 + \frac{t^3}{3}\cos 2 + \frac{1}{2} - 2te^{-t} - t^2 e^{-t} - \frac{1}{2}e^{-2t} \end{pmatrix}$$

Hence the second iteration results in

$$\boxed{x^2 = \begin{pmatrix} 2 + \sec(2)\sin(2 + t\cos 2) - \tan 2 \\ -1 + t^2 + \dfrac{t^3}{3}\cos 2 + \dfrac{1}{2} - 2te^{-t} - t^2 e^{-t} - \dfrac{1}{2}e^{-2t} \end{pmatrix}}$$

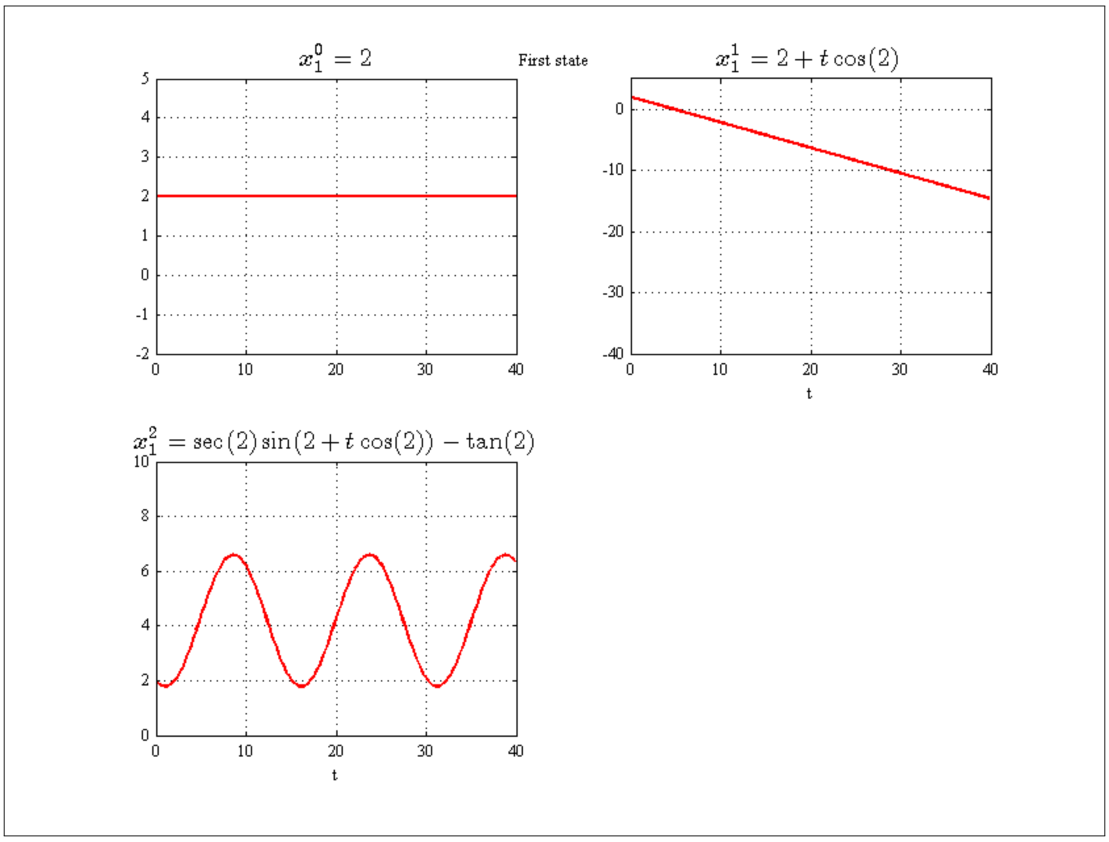
The third iteration x^3 is now found using

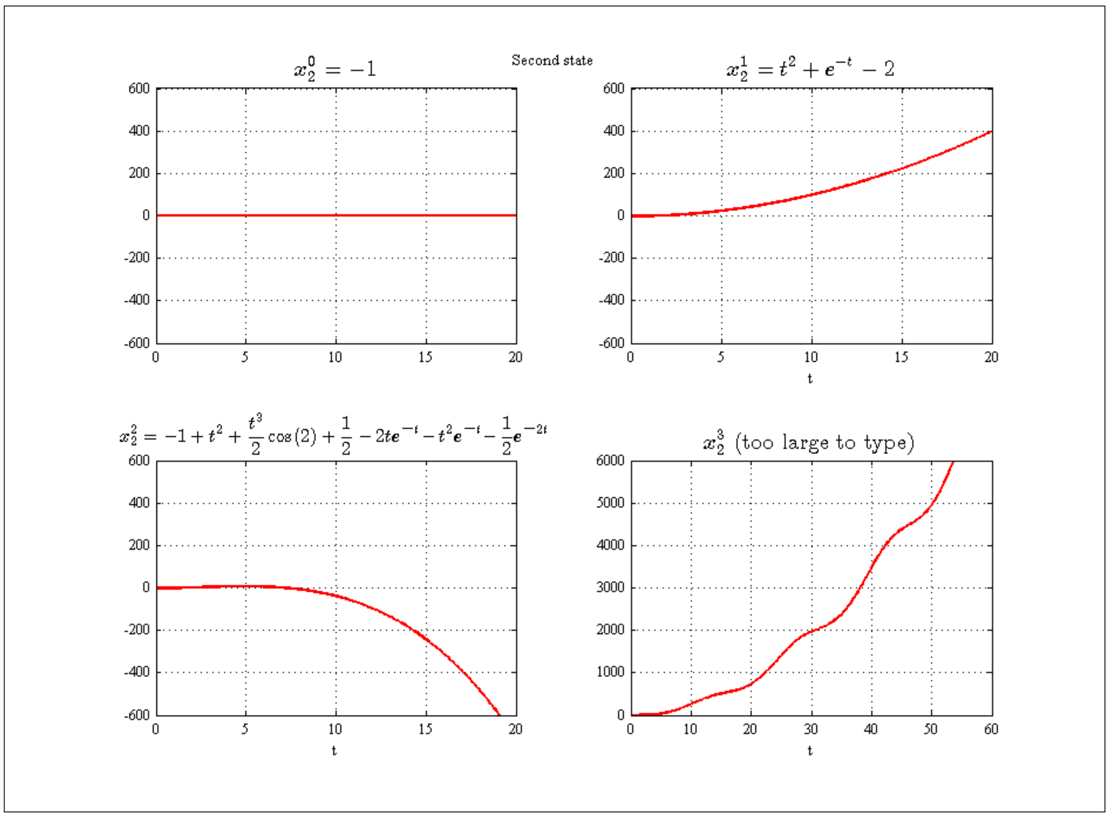
$$
\begin{aligned}
x^3 &= x^0 + \int_0^t f\left(x^2\right) d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos x_1^2 \\ \eta x_1^2 + e^{-\eta} x_2^2 \end{pmatrix} d\eta \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} 2 + \sec\left(2\right) \sin\left(2 + \eta \cos 2\right) - \tan 2 \\ -1 + \eta^2 + \frac{\eta^3}{3} \cos 2 + \frac{1}{2} - 2\eta e^{-\eta} - \eta^2 e^{-\eta} - \frac{1}{2} e^{-2\eta} \end{pmatrix} d\eta
\end{aligned}
$$

The top integral $\left(x_1^3\right)$ could not be evaluated using syms. A numerical solution is needed. The lower integral which gives the second state can be evaluated directly and requires no special treatment, giving

```
1 + (1/6)*(-1 + E^(-3*t)) - ((1/2)*(-1 + E^(2*t) - 2*t))/E^(2*t)
+ t^2 - (2 + t*(2 + t))/E^t +(1/4)*(-1 + (1 + 2*t + 2*t^2)/E^(2*t)) +
(1/3)*(6 + (-6 - t*(6 + t*(3 + t)))/E^t)*Cos[2] +
(1/2)*(-1 + Cosh[t] - Sinh[t]) + Sec[2]^2*((-t)*Cos[2 + t*Cos[2]] +
Sec[2]*Sin[2 + t*Cos[2]] - Tan[2]) - (1/2)*t^2*Tan[2]
```

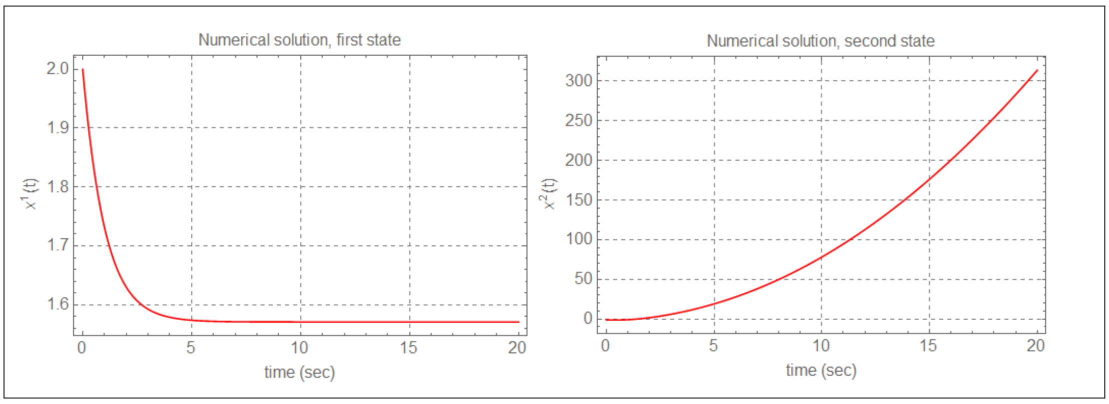
A small function was written using syms to evaluate the Picard iterations and plot the solution. For the third iteration x^3 the first state was not solved due to complexity of the integral. Numerical solution would be needed. The following plots show the first state and the second state.


$x_1^0 = 2$
First state
$x_1^1 = 2 + t\cos(2)$
$x_1^2 = \sec(2)\sin(2 + t\cos(2)) - \tan(2)$
5
4
3
2
1
0
-1
-2
0
-10
-20
-30
-40
10
8
6
4
2
0
0
10
20
30
40
t

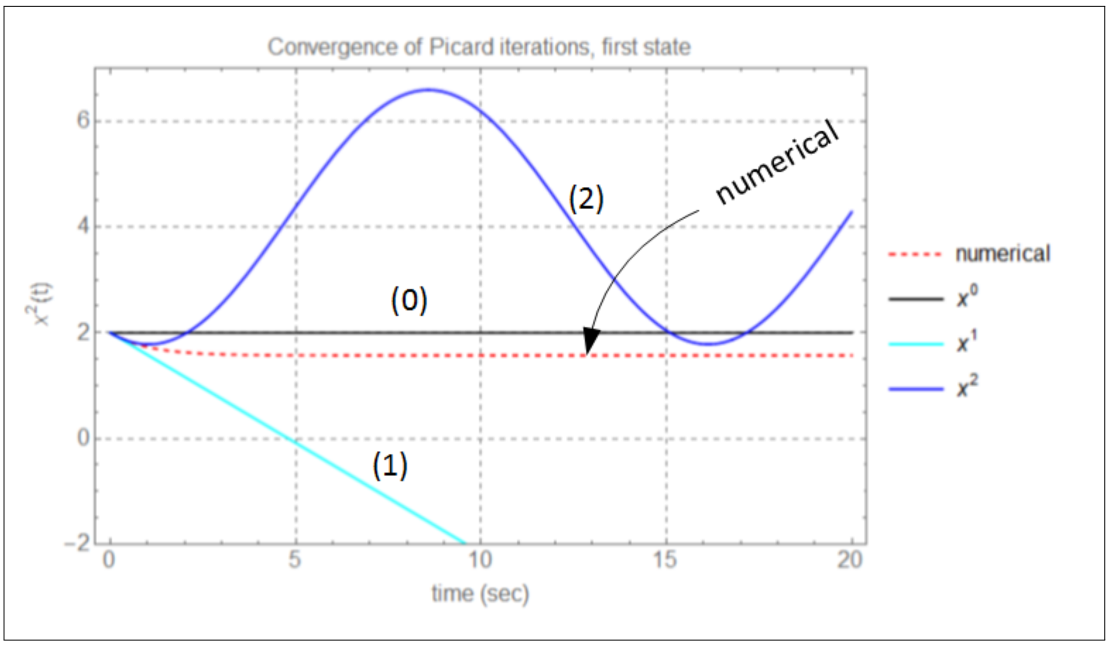


4.1 Convergence of solution for the symbolic computation

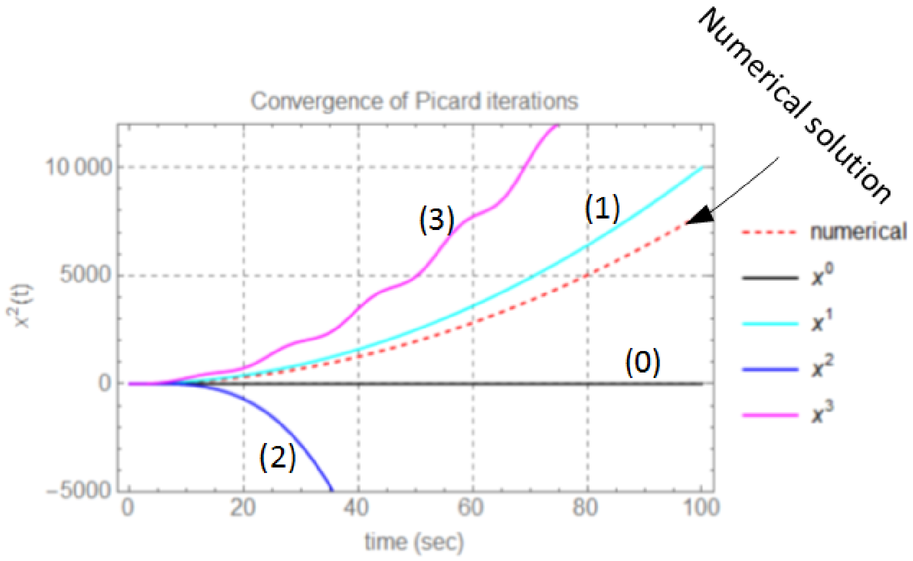
For linear system, Picard iterations will converge to the unique solution. For non-linear system it is not so clear. One way to check convergence is by numerically solving the system using numerical ODE solver and plotting both the Picard iterations against the numerical solution in order to see that the solution is getting closer the numerical solution. This was done below for $t = 40\,\text{sec}$. The system was solved numerically and both the numerically generated $x_1(t)$ and $x_2(t)$ solutions were plotted with the Picard generated x_1, x_2 for each iteration, on the same figure. If the Picard solution convergence, the more iterations are made, the closer the Picard solution should approach the numerical solution from the ODE solver. This result is given below. The numerical solution is first plotted for the first and second state.



Now the convergence of the first state on the same plot as the numerical is shown in order to compare convergence



Similarly for the second state



Comparing each Picard iteration with numerical solution for second state

Due to the small number of iterations, it is hard to decide on convergence. To fully decide on the issue of convergence of Picard iterations for this non-linear system, each Picard iteration will have to be numerically integrated in order to be able to generate more iterations. In the next section below, the numerical integration results are given, which shows the that Picard iterations for this system does indeed converge, but the convergence is slow.

4.2 Source code

4.2.1 Function to generate Picard iterations

```
function res = nma_x(k)
%function to evaluate Picard iterations
%by Nasser M. Abbasi, ECE 717, Fall 2014, HW4, problem 4
if k==0
   res = [2;−1];
else
   syms z t;
   last  = nma_x(k−1);
   x1    = last(1); x2=last(2);
   x1    = subs(x1,t,z);
   x2    = subs(x2,t,z);
   res   = [2;−1] + int( [cos(sym(x1));z*x1+exp(−z)*x2],z,0,t);
end
res;
end
```

4.2.2 Script to plot Picard iterations

```
%script to plot the Picard iterations
%Nasser M. Abbasi, HW4, ECE 717

x0=nma_x(0);
x1=nma_x(1);
x2=nma_x(2);
x3=nma_x(3);
max_t=40; max_y=10;

close all; set(0,'DefaultAxesFontName', 'Times New Roman');
set(0,'DefaultAxesFontSize',8);
set(0,'DefaultTextFontname', 'Times New Roman'); set(0,'DefaultTextFontSize', 12);

subplot(2,2,1);
h(1)=plot([0,max_t],[x0(1),x0(1)]);
grid on; set(h(1),'linewidth',1.5); set(h(1),'color','r');
xlim([0,max_t]); ylim([-2,5]);
title('$$x^0_1 = 2$$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,2);
h(2)=ezplot(x1(1),[0,max_t]);
grid on; set(h(2),'linewidth',1.5); set(h(2),'color','r'); ylim([-40,5]);
title('$$x^1_1 = 2+t \cos(2) $$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,3);
h(3)=ezplot(x2(1),[0,max_t]);
grid on; set(h(3),'linewidth',1.5); set(h(3),'color','r'); ylim([0,10]);
title('$$x^2_1 = \sec(2) \sin(2+t\cos(2))-\tan(2)$$', 'FontSize', ...
    12,'interpreter','latex');

ax=axes('Units','Normal','Position',[.075 .075 .85 .85],'Visible','off');
set(get(ax,'Title'),'Visible','on')
title('First state');

%now do x_2
figure;
max_t=20; max_y=600;
subplot(2,2,1);
h(1)=plot([0,max_t],[x0(2),x0(2)]);
grid on; set(h(1),'linewidth',1.5); set(h(1),'color','r');
xlim([0,max_t]); ylim([-max_y,max_y]);
title('$$x^0_2 = -1$$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,2);
h(2)=ezplot(x1(2),[0,max_t]);
grid on; set(h(2),'linewidth',1.5); set(h(2),'color','r'); ylim([-max_y,max_y]);
title('$$x^1_2=t^2+e^{-t}-2 $$', 'FontSize', 12,'interpreter','latex');

subplot(2,2,3);
h(3)=ezplot(x2(2),[0,max_t]);
grid on; set(h(3),'linewidth',1.5); set(h(3),'color','r'); ylim([-max_y,max_y]);
title('$$x^2_2 = -1+t^2+\frac{t^3}{2}\cos(2)+\frac{1}{2}-2 t e^{-t}-t^2 ...
    e^{-t}-\frac{1}{2}e^{-2t}$$', 'FontSize',10,'interpreter','latex');

subplot(2,2,4);
```

```
h(4)=ezplot(x3(2),[0,60]);
grid on; set(h(4),'linewidth',1.5); set(h(4),'color','r'); ylim([0,6000]);
title('$$x^3_2$$ (too large to type)', 'FontSize', 12,'interpreter','latex');

ax=axes('Units','Normal','Position',[.075 .075 .85 .85],'Visible','off');
set(get(ax,'Title'),'Visible','on')
title('Second state');
```

4.2.3 Example using Picard iteration function

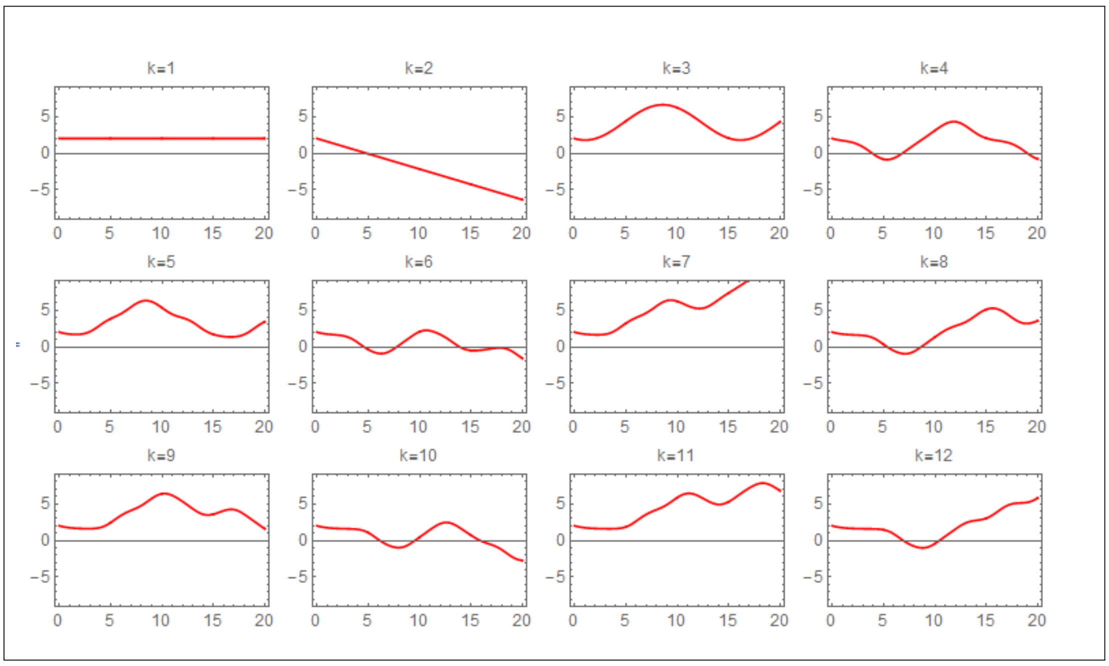
Example use is

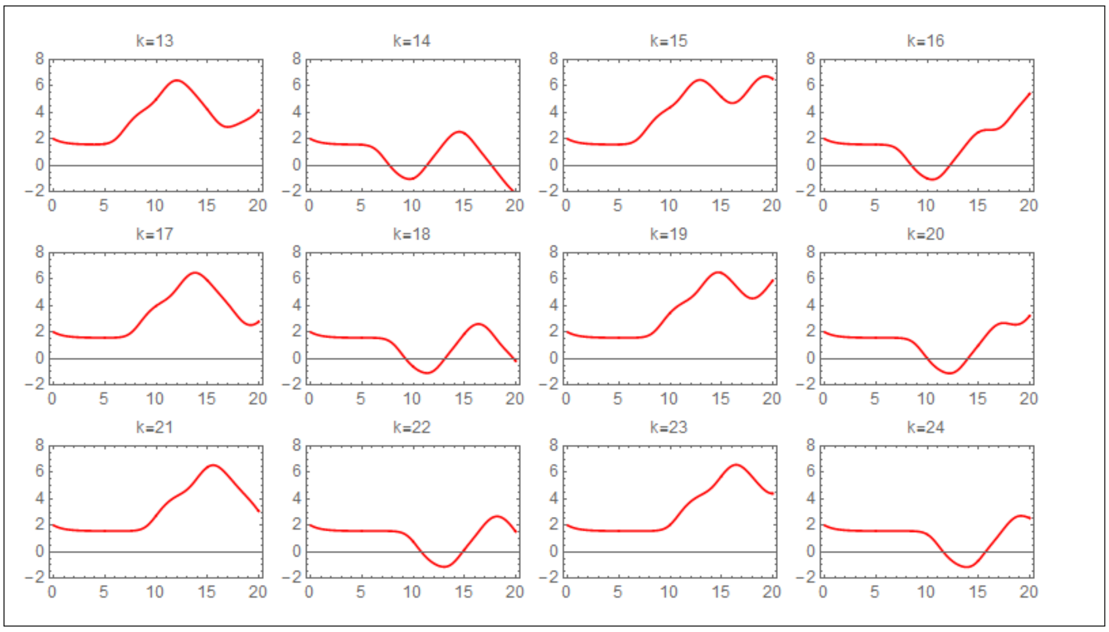
```
EDU>> nma_x(0)
2
-1

EDU>> nma_x(1)

t*cos(2) + 2
exp(-t) + t^2 - 2

EDU>> nma_x(2)
(sin(t*cos(2) + 2) - sin(2))/cos(2) + 2
(t^3*cos(2))/3 - 2*t*exp(-t) - t^2*exp(-t) - exp(-2*t)/2 + t^2 - 1/2

EDU>> nma_x(3)
Warning: Explicit integral could not be found.
int(cos((sin(z*cos(2) + 2) - sin(2))/cos(2) + 2), z == 0..t) + 2
exp(-t)/2 + exp(-3*t)/6 - (sin(2) - sin(t*cos(2) + 2) +
t*cos(2)*cos(t*cos(2) + 2))/cos(2)^3
- exp(-t)*(t^2 + 2*t + 2) + (exp(-2*t)*(2*t + 1))/2 +
t^2 - (cos(2)*(exp(-t)*(t^3 + 3*t^2 + 6*t + 6) - 6))/3 +
(exp(-2*t)*(4*t^2 + 4*t + 2))/8 - (t^2*sin(2))/(2*cos(2)) - 5/12
```

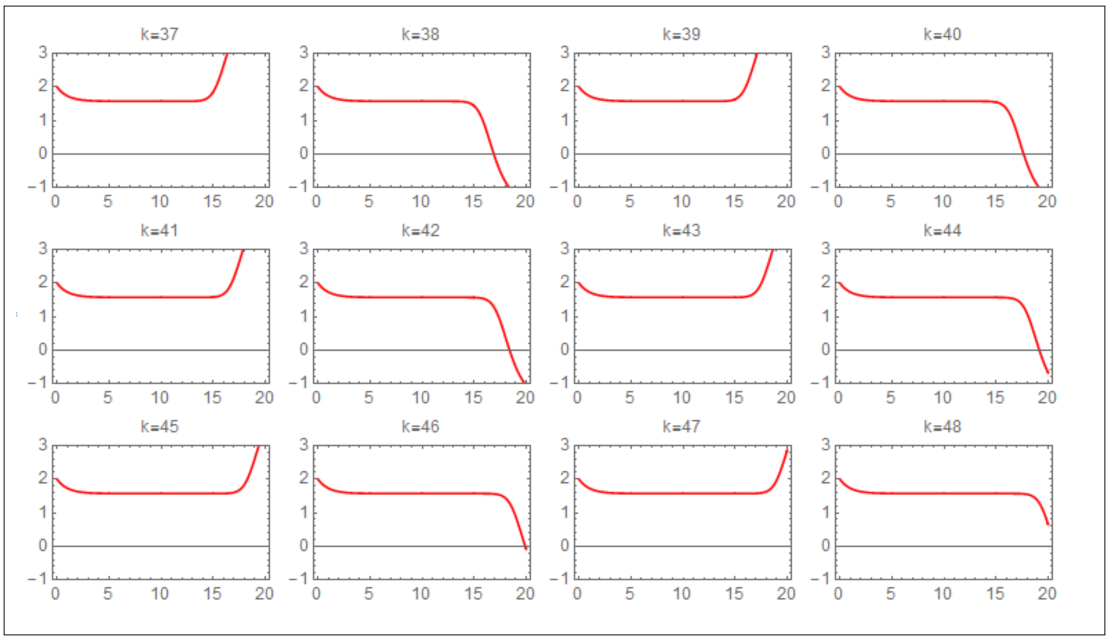
4.3 Convergence of solution using numerical integration

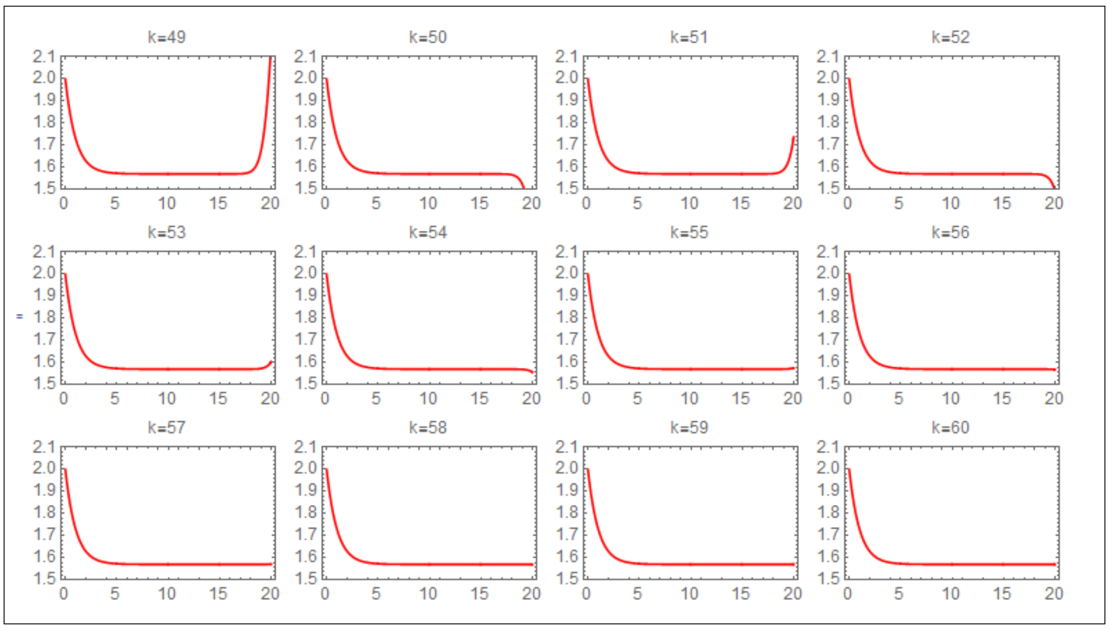
Picard iteration was integrated numerically due to difficulty of obtaining symbolic solution for each step. The following sequence of plots shows the convergence of each iteration. The first state required about 60 iterations to converge to the numerical ODE solver solution. The following shows the sequence of the iterations for the first state. Each one of these plots is 20 seconds long, and the title shows the iteration number.

4.3.1 First state iterations



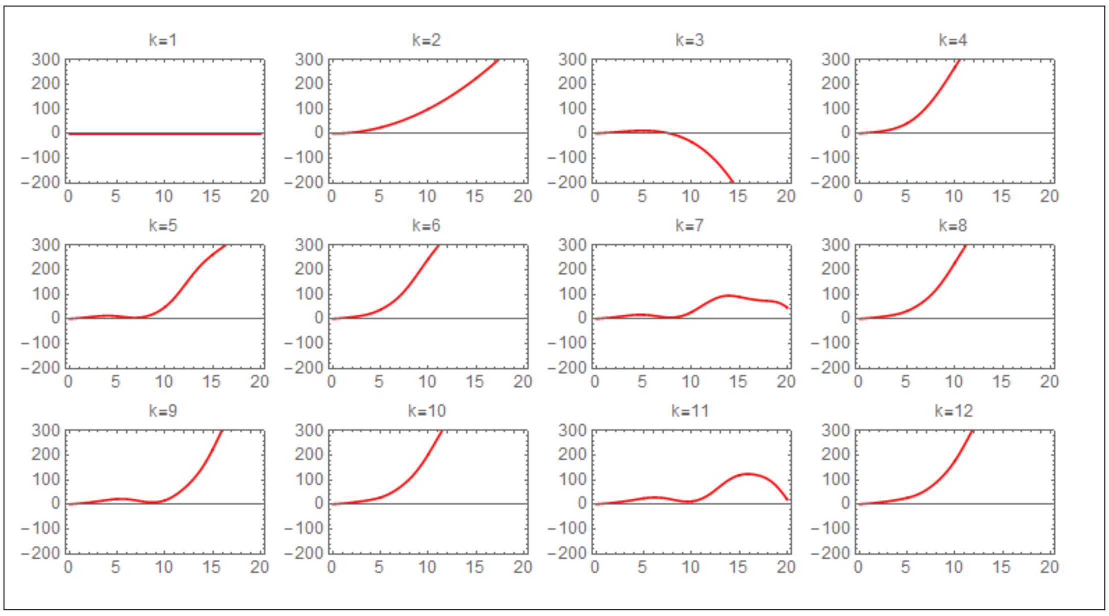


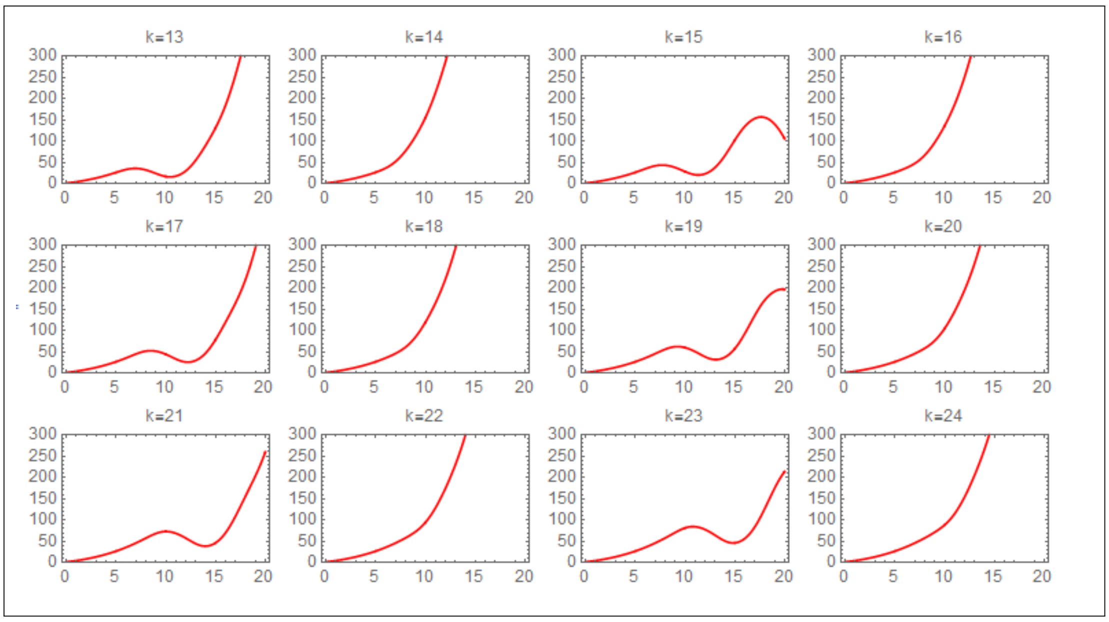


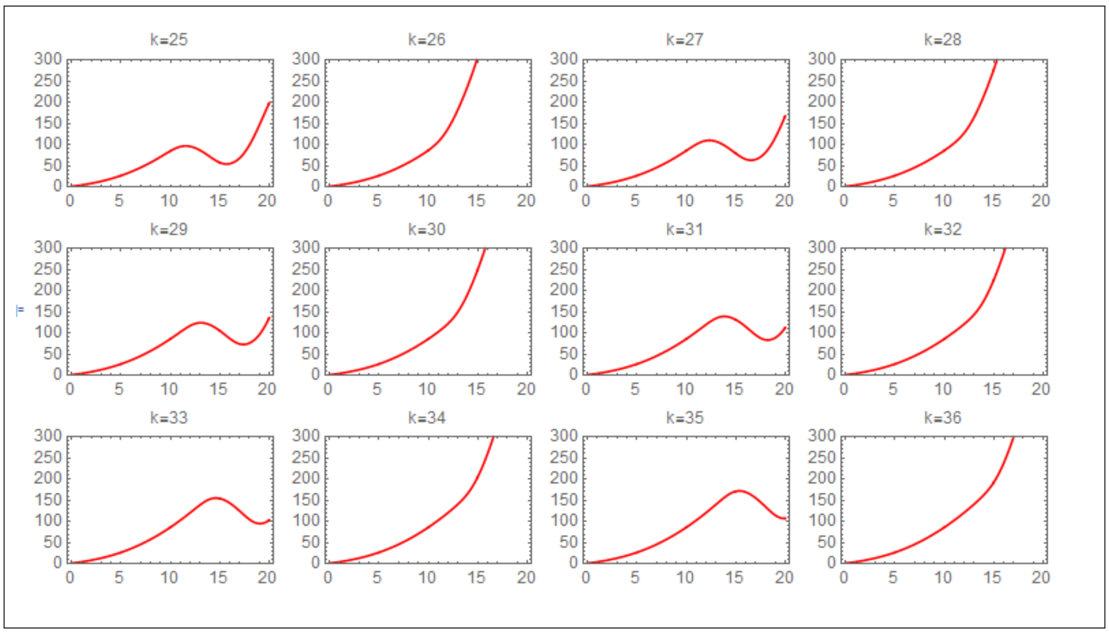


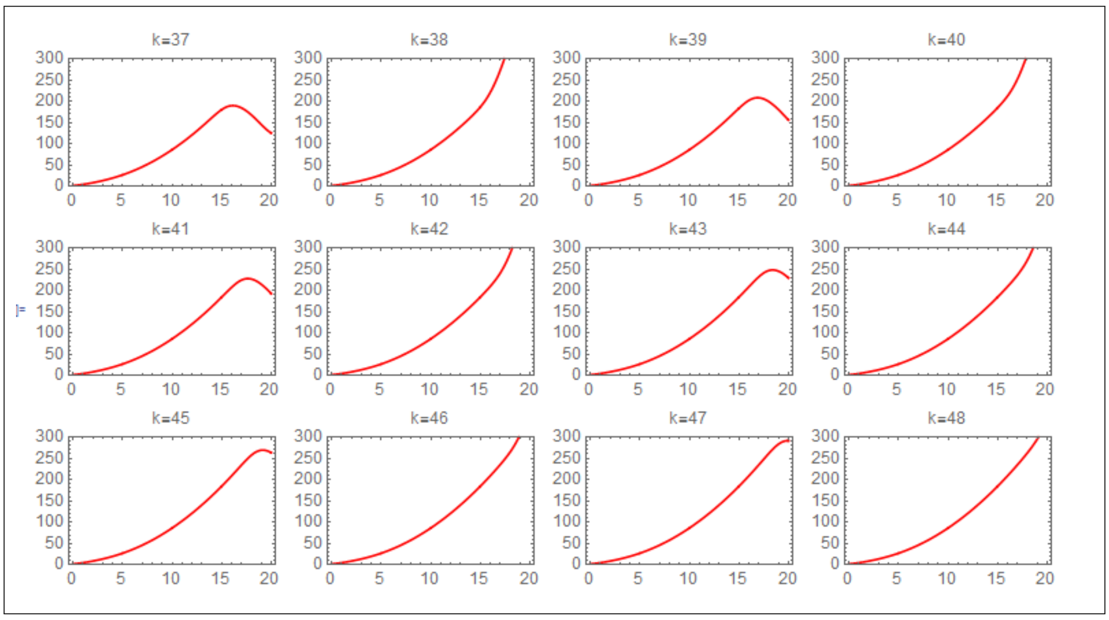
4.3.2 second state iterations

It also took about 60 Picard iterations for the second state to converge. The following is the sequence of the iterations


k=1
k=2
k=3
k=4
k=5
k=6
k=7
k=8
k=9
k=10
k=11
k=12
300
200
100
0
−100
−200
0
5
10
15
20


k=13
k=14
k=15
k=16
k=17
k=18
k=19
k=20
k=21
k=22
k=23
k=24
300
250
200
150
100
50
0
0
5
10
15
20


k=25
k=26
k=27
k=28
k=29
k=30
k=31
k=32
k=33
k=34
k=35
k=36
300
250
200
150
100
50
0
0
5
10
15
20

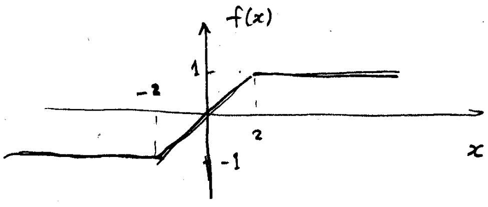

k=37
k=38
k=39
k=40
k=41
k=42
k=43
k=44
k=45
k=46
k=47
k=48
300
250
200
150
100
50
0
0
5
10
15
20

a) We have $\dot{x} = f(x)$

$\dot{x} = f(x) \Rightarrow x(t) - x(0) = \int_0^t f(x(\eta))d\eta$

Therefore $x^{K+1}(t) = x^0 + \int_0^t f(x^K(\eta))d\eta$

b)



$x^0 = 1.$

$x^1(t) = x^0 + \int_0^t f(x^0(\eta))d\eta = 1 + \int_0^t \frac{1}{2}d\eta = 1 + \frac{t}{2}$

$x^2(t) = x^0 + \int_0^t f(x^1(\eta))d\eta = x^0 + \int_0^t f(1+\frac{\eta}{2})d\eta.$

If $0 \le t \le 2$, $0 \le 1+\frac{\eta}{2} \le 2$, $0 \le \eta \le t \le 2.$

then $x^2(t) = 1 + \int_0^t \frac{1}{2}(1+\frac{\eta}{2})d\eta = 1 + \frac{t^2}{2} + \frac{t^2}{8}$

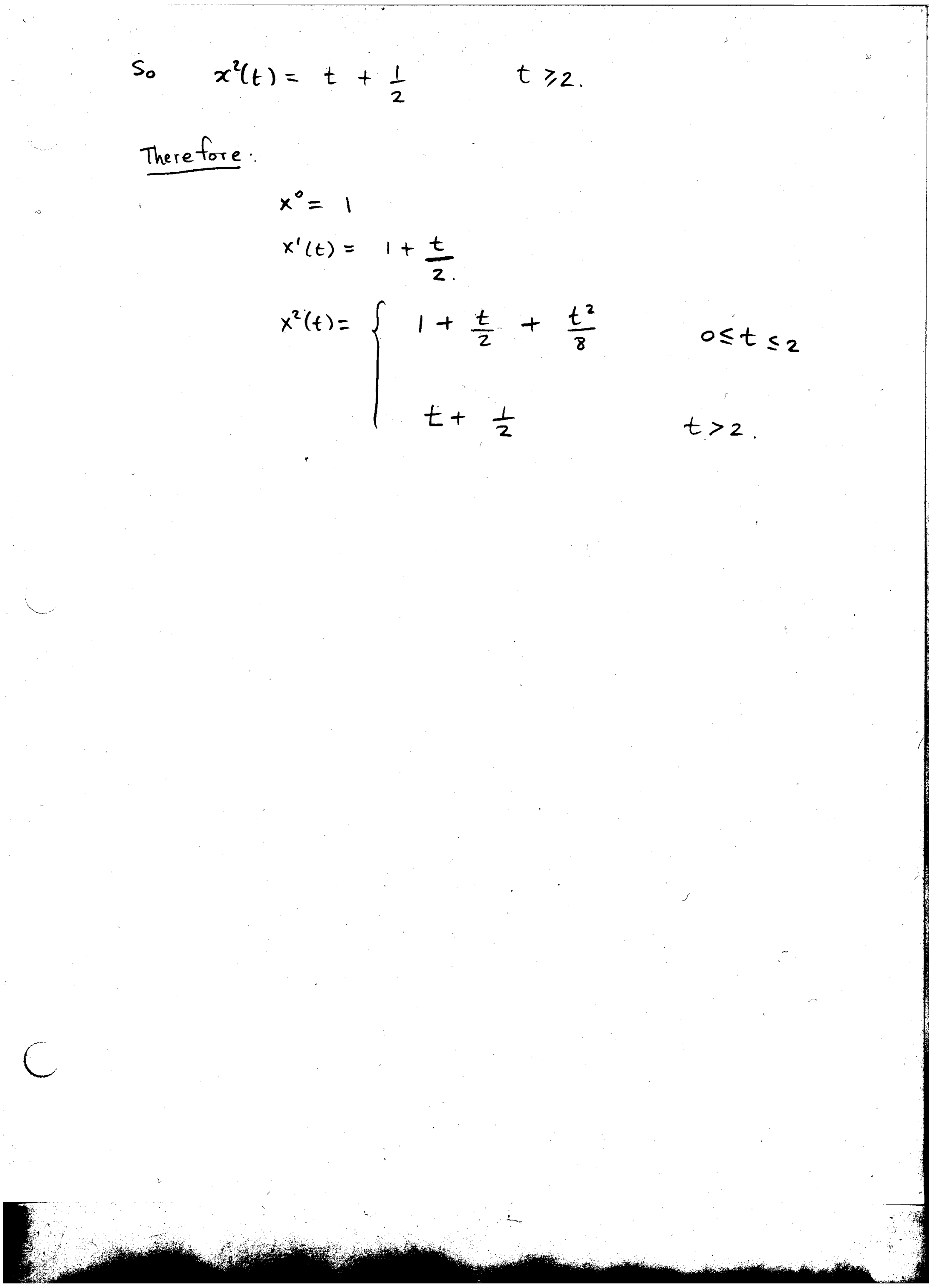
for $0 \le t \le 2.$

If $t \geq 2$, $1 + \frac{\eta}{2} \geq 2.$

then $x^2(t) = 1 + \int_0^2 \frac{1}{2}(1+\frac{\eta}{2})d\eta + \int_2^t d\eta$

$= 1 + \frac{1}{2}\eta\Big|_0^2 + \frac{\eta^2}{8}\Big|_0^2 + t - 2.$

$= \cancel{1} + \cancel{1} + \frac{1}{2} + t - \cancel{2}$



So $x^2(t) = t + \frac{1}{2}$ $\qquad t \geqslant 2.$

Therefore:

$$x^0 = 1$$

$$x^1(t) = 1 + \frac{t}{2}.$$

$$x^2(t) = \begin{cases} 1 + \frac{t}{2} + \frac{t^2}{8} & 0 \leq t \leq 2 \\ t + \frac{1}{2} & t > 2. \end{cases}$$

2.5 HW5

2.5.1 Questions

ECE 717 – Homework Set 5

Due Thursday, October 30, 2014

;

Barmish

ECE 717 – Homework Diagonalize

Suppose M_1 and M_2 are two $n \times n$ matrices with distinct eigenvalues which commute; i.e.,

$$M_1 M_2 = M_2 M_1.$$

Prove that there is an nonsingular matrix T which simultaneously diagonalizes M_1 and M_2. That is, both TM_1T^{-1} and TM_2T^{-1} are diagonal.

;

Barmish

ECE 717 – Homework Exponentials

Consider the time-varying system $\dot{x} = A(t)x$ and assume that $A(t)$ commutes with its integral; i.e.,

$$A(t) \int_0^t A(\eta) d\eta = \int_0^t A(\eta) d\eta \, A(t).$$

Now prove that the matrix

$$\Psi(t) = e^{\int_0^t A(\eta) d\eta}$$

satisfies the state equation with initial condition $\Psi(0) = I$ and the state solution is given by

$$x(t) = e^{\int_0^t A(\eta) d\eta} x(0).$$

(b) Again considering the LTV system $\dot{x} = A(t)x$, instead of beginning with the assumption that $A(t)$ commutes with its integral, assume that the commutation condition

$$A(t_1)A(t_2) = A(t_2)A(t_1)$$

is satisfied for . Now describe the solution to the state equation.

;

Barmish

ECE 717 – Homework Revisit

In this homework problem, we consider the LTV system $\dot{x} = A(t)x$ and revisit a result obtained in Homework Exponentials — under the strengthened hypothesis that $A(t)$ has distinct eigenvalues for all t. To this end, we again assume that the commutation condition

$$A(t_1)A(t_2) = A(t_2)A(t_1)$$

is satisfied for all pairs (t_1, t_2). Letting $\Lambda(t)$ be a diagonal matrix whose entries are the eigenvalues of $A(t)$, prove that there is a constant (time-invariant) matrix T such that the matrix

$$\Psi(t) = T^{-1} e^{\int_0^t \Lambda(\eta) d\eta} T$$

satisfies the state equation with initial condition $\Psi(0) = I$. Note: If your T matrix depends on time, you have not solved the problem.

;

Barmish

ECE 717 – Homework Transition

Find the state transition matrix $\Phi(t,\tau)$, in closed form, associated with each of the $A(t)$ matrices below. Note: Picard iteration should not be used. A closed form is requested. For the first $A(t)$ matrix, to guarantee well-posedness, assume times $t \geq t_0 = 1$.

$$A(t) = \begin{bmatrix} -\frac{4}{t} & -\frac{2}{t^2} \\ 1 & 0 \end{bmatrix};$$

$$A(t) = \begin{bmatrix} 2 & -e^t \\ e^{-t} & 1 \end{bmatrix}.$$

;

2.5.2 Problem 1

Let α_i be the i^{th} eigenvalue of M_1 and let v_i be an eigenvector associated with α_i. This implies

$$M_1 v_i = \alpha_i v_i$$

Similarly, let β_i be the i^{th} eigenvalue of M_2 and let u_i be an eigenvector associated with β_i. This implies

$$M_2 u_i = \beta_i u_i$$

We start by post multiplying $M_1 M_2$ with an eigenvector of M_1 associated with eigenvalue α_i, this results in

$$M_1 M_2 v_i = M_2 M_1 v_i$$

Where we just took advantage of commuting $M_1 M_2$ by changing the order in the RHS above. But $M_1 v_i = \alpha_i v_i$, hence the above becomes

$$M_1 M_2 v_i = M_2 \alpha_i v_i$$

Since α_i is scalar, we can move it to the left and obtain

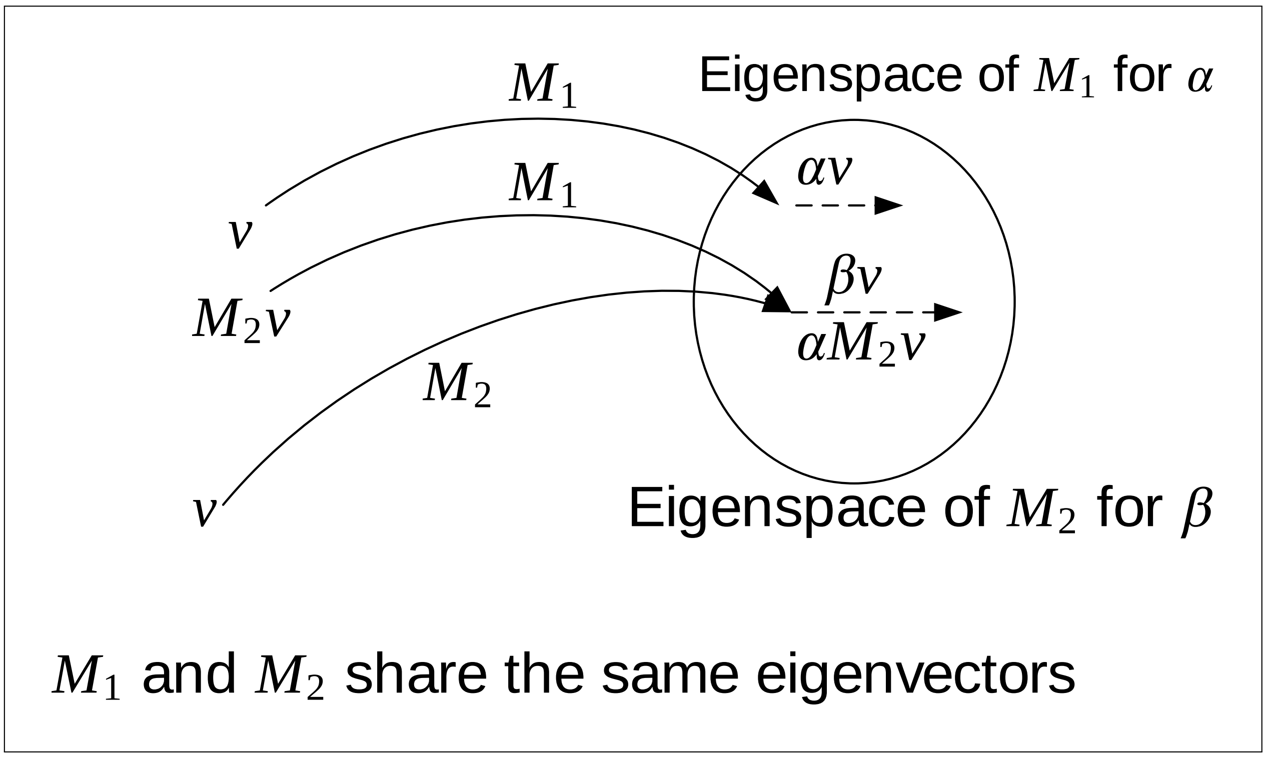
$$M_1 (M_2 v_i) = \alpha_i (M_2 v_i)$$

We see now that $\boxed{M_2 v_i \text{ itself is an eigenvector of } M_1}$.

What the above means is that if v_i is an eigenvector of M_1 associated with an eigenvalue α_i, then so will be $M_2 v_i$. Now an important point follows: Since the eigenvalues are distinct, then all the eigenvectors that belong to each eigenvalues are scalar multiple of each others. What this means, is that $M_2 v_i$ is some scaled version of v_i since both are in the same eigenspace associated with α_i. The eigenspace associated with an eigenvalue is just the space spanned by all the eigenvectors of this eigenvalue. This means this space is one dimensional in this case.

This is critical, since it then tells us that $M_2 v_i = \beta_i v_i$ where β_i is the above scalar, which is the eigenvalue of M_2. Without this restriction, we could not say that $M_2 v_i = \beta_i v_i$.

Therefore, the above means that $\boxed{\text{each eigenvector of } M_1 \text{ is also an eigenvector of } M_2}$. Or said in other way, the matrix M_1 and M_2 share the same eigenspaces.



But this complete the proof. Since the nonsingular matrix T which diagonalizes a matrix is made of up of the eigenvectors of the matrix. The columns of T are the eigenvectors of the matrix. And since M_1, M_2 share the same eigenvectors, hence the same T will diagonalize both of them at the same time.

QED

2.5.3 Problem 2

Part (a)

We want to show that $\frac{d}{dt}\Psi(t) = A(t)\Psi(t)$. Where $\Psi(t) = e^{\int_0^t A(\tau)d\tau}$. To expand $e^{\int_0^t A(\tau)d\tau}$we will use the definition of matrix exponential

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots$$

Therefore

$$e^{\int_0^t A(\tau)d\tau} = I+\int_0^t A(\tau)\,d\tau+\frac{1}{2}\left(\int_0^t A(\tau)\,d\tau\right)\left(\int_0^t A(\tau)\,d\tau\right)+\frac{1}{3!}\left(\int_0^t A(\tau)\,d\tau\right)\left(\int_0^t A(\tau)\,d\tau\right)\left(\int_0^t A(\tau)\,d\tau\right)+\cdots$$

To make it easier to see, we will expand only the first 2 terms in expansion:

$$e^{\int_0^t A(\tau)d\tau} = I + \int_0^t A(\tau)\,d\tau + \frac{1}{2}\left[\int_0^t A(\tau)\,d\tau \int_0^t A(\tau)\,d\tau\right] + \cdots \tag{1}$$

Taking the time derivative of the above and using the product rule $\frac{d}{dt}(XY) = X\frac{d}{dt}Y + Y\frac{d}{dt}X$ gives

$$\begin{aligned}
\frac{d}{dt}\Psi(t) &= \frac{d}{dt}\left(e^{\int_0^t A(\tau)d\tau}\right)\\
&= \frac{d}{dt}\left(I + \int_0^t A(\tau)\,d\tau + \frac{1}{2}\left[\int_0^t A(\tau)\,d\tau \int_0^t A(\tau)\,d\tau\right] + \cdots\right)\\
&= \overbrace{\frac{d}{dt}I}^{0} + \frac{d}{dt}\int_0^t A(\tau)\,d\tau + \frac{1}{2}\frac{d}{dt}\left[\int_0^t A(\tau)\,d\tau \int_0^t A(\tau)\,d\tau\right] + \cdots\\
&= A(t) + \frac{1}{2}\left[\left(\int_0^t A(\tau)\,d\tau\right)A(t) + A(t)\left(\int_0^t A(\tau)\,d\tau\right)\right] + \cdots
\end{aligned}$$

Taking advantage of the commute property we write the second term above as

$$\frac{d}{dt}\Psi(t) = A(t) + \frac{1}{2}\left[A(t)\left(\int_0^t A(\tau)\,d\tau\right) + A(t)\left(\int_0^t A(\tau)\,d\tau\right)\right] + \cdots$$

Therefore

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= A(t)+\frac{1}{2}\left[2A(t)\left(\int_0^t A(\tau)\,d\tau\right)\right]+\cdots\\ &= A(t)+A(t)\int_0^t A(\tau)\,d\tau+\cdots\end{aligned}$$

Since all the $A(t)$ are on the left side, we can now factor $A(t)$ out and obtain

$$\frac{d}{dt}\Psi(t)=A(t)\overbrace{\left(I+\int_0^t A(\tau)\,d\tau+\cdots\right)}^{\Psi(t)}$$

Comparing the term inside $(\cdot)$ in the above expression above with equation (1) we see it is $\Psi(t)$. (If we have expanded more terms, it would be more clear, but the idea is the same as shown above). Therefore we conclude that

$$\frac{d}{dt}e^{\int_0^t A(\tau)d\tau}=A(t)\,e^{\int_0^t A(\tau)d\tau}$$

Or

$$\boxed{\frac{d}{dt}\Psi(t)=A(t)\,\Psi(t)}$$

Hence $\Psi(t)$ satisfies the state equation. Now we need to show that $x(t)=e^{\int_0^t A(\tau)d\tau}x(0)$ is the state solution. Since $\Psi(t)$ is the fundamental matrix, then each of its columns is an independent solution to $x'=A(t)\,x$ by definition. Hence a linear combinations of the columns of $\Psi(t)$ gives the solu

As shown in class, we now obtain the general solution by assuming $x(t)=\Psi(t)\,\theta(t)$ and then from this end up with the fundamental solution $x(t)$ as

$$\vec{x}(t)=\Psi(t)\,\Psi^{-1}(0)\,x(0)+\int_0^t \Psi(t)\,\Psi^{-1}(\tau)\,B(\tau)\,u(\tau)\,d\tau$$

But since this is free system, so there is no input $u(t)$ and since $\Psi(0)=I$ then $\Psi^{-1}(0)=I$ and the above reduces to

$$\vec{x}(t)=\Psi(t)\,x(0)$$

But $\Psi(t)=e^{\int_0^t A(\tau)d\tau}$ hence

$$\vec{x}(t)=e^{\int_0^t A(\tau)d\tau}x(0)$$

Part (b)

We are told that

$$A(t)\,A(\tau)=A(\tau)\,A(t)$$

Lets integrate both sides from 0 to t w.r.t to τ. The equality will remain since we are integrating over the same interval of equal quantities, hence

$$\int_0^t A(t)\,A(\tau)\,d\tau=\int_0^t A(\tau)\,A(t)\,d\tau$$

Now the integral on the LHS has $A(t)$ which can be taken out of the integral, keeping the order to the left, and the integral on RHS has $A(t)$ which can now be taken out of the integral, keeping the order to the right, which results in

$$A(t)\left(\int_0^t A(\tau)\,d\tau\right)=\left(\int_0^t A(\tau)\,d\tau\right)A(t)$$

But the above is the assumptions we used in part (a). Therefore,

$$A(t)\,A(\tau)=A(\tau)\,A(t)\overset{\text{implies}}{\Rightarrow}A(t)\left(\int_0^t A(\tau)\,d\tau\right)=\left(\int_0^t A(\tau)\,d\tau\right)A(t)$$

Therefore we can use the same solution found in (a)

$$\vec{x}(t)=e^{\int_0^t A(\tau)d\tau}x(0)$$

2.5.4 Problem 3

Since $A(t)A(\tau) = A(\tau)A(t)$, then from problem (2) we know that

$$\Psi(t) = e^{\int_0^t A(\tau)d\tau}$$

is the fundamental matrix for $x'(t) = A(t)x(t)$. We now need to show that, given that $A(t)$ has distinct eigenvalues for each t, the fundamental matrix can be written as

$$\Psi(t) = T^{-1}e^{\int_0^t \Lambda(\tau)d\tau}T$$

For some constant matrix T. The important point is that T must be constant in the above. In addition, we need to show that the above $\Psi(t)$ satisfies $\frac{d}{dt}\Psi(t) = A(t)\Psi(t)$.

The first step is to find the constant T matrix. Since $A(t)A(\tau) = A(\tau)A(t)$, then by selecting $\tau = 0$, which is the initial time, then $A(t)A(0) = A(0)A(t)$. Therefore, each $A(t)$ commutes with the same matrix $A(0)$. i.e. $A(t_1)$ will commute with $A(0)$ and $A(t_2)$ will commute with $A(0)$ and so on. But by problem 1, we showed that when two matrices commute, then they have the same eigenvectors. Therefore, we can select the eigenvectors of $A(0)$ to use to construct the T matrix from, by using the n linearly independent eigenvectors of $A(0)$ as the columns of T. Lets call it T_0. Therefore, T_0 is now constant and do not change. Now that we found a constant T_0 matrix to use for diagonalization of each $A(t)$ matrix, we will show the rest of the solution using T_0. Since

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots = \sum_{i=0}^{\infty}\frac{M^i}{i!}$$

Therefore, applying the above to

$$\begin{aligned}\Psi(t) &= e^{\int_0^t A(\tau)d\tau}\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\int_0^t A(\tau)\,d\tau\right)^i\end{aligned}$$

Since A has distinct eigenvalues at all time, we can diagonalize it using the constant T_0, hence

$$\begin{aligned}\Psi(t) &= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\int_0^t T_0^{-1}\Lambda(\tau)T_0 d\tau\right)^i\\ &= I + \int_0^t T_0^{-1}\Lambda(\tau)T_0 d\tau + \frac{1}{2}\int_0^t T_0^{-1}\Lambda(\tau)T_0 d\tau \int_0^t T_0^{-1}\Lambda(\tau)T_0 d\tau + \cdots\\ &= I + T_0^{-1}\left(\int_0^t \Lambda(\tau)\,d\tau\right)T_0 + \frac{1}{2}T_0^{-1}\left(\int_0^t \Lambda(\tau)\,d\tau\right)\left(T_0T_0^{-1}\right)\left(\int_0^t \Lambda(\tau)\,d\tau\right)T_0 + \cdots\end{aligned}$$

All the inner $T_0T_0^{-1}$ result in I since T_0 is invertible, therefore the above become

$$\Psi(t) = I + T_0^{-1}\left(\int_0^t \Lambda(\tau)\,d\tau\right)T_0 + \frac{1}{2!}T_0^{-1}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^2 T_0 + \frac{1}{3!}T_0^{-1}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^3 T_0 + \cdots$$

Pre-multiply both sides by T_0

$$T_0\Psi(t) = T_0 + \left(\int_0^t \Lambda(\tau)\,d\tau\right)T_0 + \frac{1}{2!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^2 T_0 + \frac{1}{3!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^3 T_0 + \cdots$$

Post multiply both sides by T_0^{-1}, and again replacing all of the $T_0T_0^{-1}$ products with I gives

$$\begin{aligned}T_0\Psi(t)T_0^{-1} &= I + \left(\int_0^t \Lambda(\tau)\,d\tau\right)T_0T_0^{-1} + \frac{1}{2!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^2 T_0T_0^{-1} + \frac{1}{3!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^3 T_0T_0^{-1} + \cdots\\ &= I + \left(\int_0^t \Lambda(\tau)\,d\tau\right) + \frac{1}{2!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^2 + \frac{1}{3!}\left(\int_0^t \Lambda(\tau)\,d\tau\right)^3 + \cdots\\ &= e^{\int_0^t \Lambda(\tau)d\tau}\end{aligned}$$

Therefore

$$\Psi(t) = T_0^{-1}e^{\int_0^t \Lambda(\tau)d\tau}T_0 \tag{1}$$

Given equation (1), we need now to show that it leads to $\frac{d}{dt}\Psi(t) = A(t)\Psi(t)$.

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= \frac{d}{dt}\left(T_0^{-1}e^{\int_0^t \Lambda(\tau)d\tau}T_0\right)\\ &= T_0^{-1}\left(\frac{d}{dt}e^{\int_0^t \Lambda(\tau)d\tau}\right)T_0 \end{aligned} \tag{2}$$

Since $\Lambda(\tau)$ is a diagonal matrix (by definition, it has the eigenvalues on the diagonal), therefore it commutes with another $\Lambda(t)$ (any diagonal matrix commutes with another diagonal matrix). Hence

$$\boxed{\Lambda(\tau)\Lambda(t) = \Lambda(t)\Lambda(\tau)} \tag{3}$$

What this means is that we can expand $e^{\int_0^t \Lambda(\tau)d\tau}$ in power series and simplified as follows

$$e^{\int_0^t \Lambda(\tau)d\tau} = I + \int_0^t \Lambda(\tau)\,d\tau + \frac{1}{2}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \frac{1}{3!}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \cdots$$

Substituting this into (2)

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= T_0^{-1}\left(\frac{d}{dt}\left[I + \int_0^t \Lambda(\tau)\,d\tau + \frac{1}{2}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \frac{1}{3!}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \cdots\right]\right)T_0\\ &= T_0^{-1}\left(\left[\Lambda(t) + \frac{1}{2}\left(\Lambda(t)\int_0^t \Lambda(\tau)\,d\tau + \int_0^t \Lambda(\tau)\,d\tau\Lambda(t)\right) + \cdots\right]\right)T_0\end{aligned}$$

Since $\Lambda(\tau)$ commute, then using (3)

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= T_0^{-1}\left(\left[\Lambda(t) + \frac{1}{2}\left(\Lambda(t)\int_0^t \Lambda(\tau)\,d\tau + \Lambda(t)\int_0^t \Lambda(\tau)\,d\tau\right) + \cdots\right]\right)T_0\\ &= T_0^{-1}\left(\left[\Lambda(t) + \Lambda(t)\int_0^t \Lambda(\tau)\,d\tau + \cdots\right]\right)T_0\\ &= T_0^{-1}\left(\Lambda(t)\left[I + \int_0^t \Lambda(\tau)\,d\tau + \frac{1}{2}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \cdots\right]\right)T_0\\ &= \overbrace{\left[T_0^{-1}\Lambda(t)\,T_0\right]}^{A(t)}\overbrace{T^{-1}\left(I + \int_0^t \Lambda(\tau)\,d\tau + \frac{1}{2}\int_0^t \Lambda(\tau)\,d\tau\int_0^t \Lambda(\tau)\,d\tau + \cdots\right)T_0}^{\Psi(t)\text{ from (1)}}\end{aligned}$$

Hence

$$\frac{d}{dt}\Psi(t) = A(t)\Psi(t)$$

2.5.5 Problem 4

Part (a)

For $A(t) = \begin{pmatrix} -\frac{4}{t} & -\frac{2}{t^2} \\ 1 & 0 \end{pmatrix}$, we first need to find the fundamental matrix $\Psi(t)$ and then $\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$. Let the 2 linearly independent initial conditions be

$$X^{01} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, X^{02} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We know solve $x' = A(t)x$ using both of these initial conditions and obtain two linearly independent solutions to use to construct $\Psi(t)$ with. Using the first initial conditions $\boxed{x_1(1) = 1, x_2(1) = 0}$. The two equations to solve are

$$x_1' = -\frac{4}{t}x_1 - \frac{2}{t^2}x_2 \tag{1}$$

$$x_2' = x_1 \tag{2}$$

From the second equation

$$\frac{d}{dt}x_2 = x_1$$

Integrate both sides

$$\int_1^t dx_2 = \int_1^t x_1(\tau)\,d\tau$$
$$x_2(t) - x_2(1) = \int_1^t x_1(\tau)\,d\tau$$

But $x_2(1) = 0$, hence $x_2(t) = \int_1^t x_1(\tau)\,d\tau$. Substituting this in (1) gives

$$x_1' = -\frac{4}{t}x_1 - \frac{2}{t^2}\int_1^t x_1(\tau)\,d\tau$$

Multiply both sides by $\frac{t^2}{2}$

$$\frac{t^2}{2}x' = -2tx_1 - \int_1^t x_1(\tau)\,d\tau$$

Taking derivative of both sides with respect to t gives

$$tx_1' + \frac{t^2}{2}x_1'' = -2x_1 - 2tx_1' - x_1(t)$$
$$\frac{t^2}{2}x_1'' + 3tx_1' + 3x_1 = 0$$
$$t^2x_1'' + 6tx_1' + 3x_1 = 0 \tag{3}$$

This second order differential is now solved for $x_1(t)$. The initial conditions is $x_1(1) = 1$ and $x_1'(1)$. However, we do not know $x_1'(1)$, as not given, but we can obtain it from the first equation (1) by noting that at $t = 1$ we find $x'(1) = -\frac{4}{1}x_1(1) - \frac{2}{1^2}x_2(1) = -4$. Therefore (3) can now be solved for x_1 since we have two initial conditions. Hence the problem to solve is

$$t^2x_1'' + 6tx_1' + 6x_1 = 0$$
$$x_1(1) = 1$$
$$x_1'(1) = -4$$

Equation (3) is in the form of Euler equation. Euler ODE has solution of the form $x_1(t) = t^\alpha$. Substituting this trial solution in (3) gives

$$t^2\left(\alpha(\alpha-1)t^{\alpha-2}\right) + 6t\alpha t^{\alpha-1} + 6t^\alpha = 0$$
$$\alpha(\alpha-1)t^\alpha + 6\alpha t^\alpha + 6t^\alpha = 0$$

For non-trivial solution, and assuming $t > 0$ which is the case here, dividing the above by t^α gives

$$\alpha(\alpha-1) + 6\alpha + 6 = 0$$
$$\alpha^2 + 5\alpha + 6 = 0$$

Hence

$$\alpha = \{-2, -3\}$$

Therefore the solution is a combination of solutions using these, which is

$$x_1(t) = \frac{c_1}{t^2} + \frac{c_2}{t^3} \tag{4}$$

Now we apply the initial conditions. At $t = 1, x_1(1) = 1$, hence

$$1 = c_1 + c_2 \tag{5}$$

And

$$x_1'(t) = -2\frac{c_1}{t^3} - 3\frac{c_2}{t^4}$$

And we have $x_1'(1) = -4$ hence

$$-4 = -2c_1 - 3c_2 \tag{6}$$

We now have (5),(6), which is two equations in two unknowns. The solution is

$$1 = c_1 + c_2$$
$$-4 = -2c_1 - 3c_2$$

The solution is: $c_1 = -1, c_2 = 2$. Hence the the solution is now found, using (4), it is

$$\boxed{x_1(t) = \frac{-1}{t^2} + \frac{2}{t^3}}$$

Now that we know $x_1(t)$, we can find $x_2(t)$ from $x_2(t) = \int_1^t x_1(\tau)\,d\tau$, therefore

$$x_2(t) = \int_1^t \frac{-1}{\tau^2} + \frac{2}{\tau^3} d\tau$$

Hence

$$\boxed{x_2(t) = \frac{t-1}{t^2}}$$

This gives us the first column of

$$\Psi^1 = \begin{pmatrix} \frac{-1}{t^2} + \frac{2}{t^3} \\ \frac{t-1}{t^2} \end{pmatrix}$$

Now we need to do the same the X^{02}.

Using the second initial conditions $\boxed{x_1(1) = 0, x_2(1) = 1}$. The two equations to solve are

$$x_1' = -\frac{4}{t}x_1 - \frac{2}{t^2}x_2 \tag{1A}$$

$$x_2' = x_1 \tag{2A}$$

From the second equation

$$\frac{d}{dt}x_2 = x_1$$

Integrate both sides

$$\int_1^t dx_2 = \int_1^t x_1(\tau)\,d\tau$$

$$x_2(t) - x_2(1) = \int_1^t x_1(\tau)\,d\tau$$

But $x_2(1) = 1$, hence $x_2(t) = 1 + \int_1^t x_1(\tau)\,d\tau$. Substituting this in (1A) gives

$$x_1' = -\frac{4}{t}x_1 - \frac{2}{t^2}\left(1 + \int_1^t x_1(\tau)\,d\tau\right)$$

Multiply both sides by $\frac{t^2}{2}$

$$\frac{t^2}{2}x' = -2tx_1 - 1 - \int_1^t x_1(\tau)\,d\tau$$

Taking derivative of both sides with respect to t gives

$$tx_1' + \frac{t^2}{2}x_1'' = -2x_1 - 2tx_1' - x_1(t)$$

$$\frac{t^2}{2}x_1'' + 3tx_1' + 3x_1 = 0$$

$$t^2x_1'' + 6tx_1' + 3x_1 = 0$$

This is the same second order differential as was found for X^{01} but the initial conditions are now different. The initial conditions are $x_1(1) = 0$ and $x_1'(1)$. However, we do not know $x_1'(1)$, as not given, but we can obtain it from the first equation (1) by noting that at $t = 1$ we find $x'(1) = -\frac{4}{1}x_1(1) - \frac{2}{1^2}x_2(1) = -2$. Therefore (3A) can now be solved for x_1 since we have two initial conditions. Hence the problem to solve is

$$t^2x_1'' + 6tx_1' + 6x_1 = 0 \tag{3A}$$

$$x_1(1) = 0$$

$$x_1'(1) = -2$$

Equation (3A) is in the form of Euler equation. Euler ODE has solution of the form

$x_1(t) = t^{\alpha}$. Substituting this trial solution in (3A) gives

$$\begin{aligned} t^2\left(\alpha(\alpha-1)t^{\alpha-2}\right) + 6t\alpha t^{\alpha-1} + 6t^{\alpha} &= 0 \\ \alpha(\alpha-1)t^{\alpha} + 6\alpha t^{\alpha} + 6t^{\alpha} &= 0 \end{aligned}$$

For non-trivial solution, and assuming $t > 0$ which is the case here, diving the above by t^{α} gives

$$\begin{aligned} \alpha(\alpha-1) + 6\alpha + 6 &= 0 \\ \alpha^2 + 5\alpha + 6 &= 0 \end{aligned}$$

Hence

$$\alpha = \{-2, -3\}$$

Therefore the solution is a combination of solutions using these, which is

$$x_1(t) = \frac{c_1}{t^2} + \frac{c_2}{t^3} \tag{4A}$$

Now we apply the initial conditions. At $t = 1, x_1(1) = 0$, hence

$$0 = c_1 + c_2 \tag{5A}$$

And

$$x_1'(t) = -2\frac{c_1}{t^3} - 3\frac{c_2}{t^4}$$

And we have $x_1'(1) = -2$ hence

$$-2 = -2c_1 - 3c_2 \tag{6A}$$

We now have (5A),(6A), which is two equations in two unknowns. The solution is

$$\begin{aligned} 0 &= c_1 + c_2 \\ -2 &= -2c_1 - 3c_2 \end{aligned}$$

The solution is: $c_1 = -2, c_2 = 2$. Hence the the solution is now found, using (4A), it is

$$x_1(t) = \frac{-2}{t^2} + \frac{2}{t^3}$$

Now that we know $x_1(t)$, we can find $x_2(t)$ from $x_2(t) = 1 + \int_1^t x_1(\tau)\,d\tau$, therefore

$$x_2(t) = 1 + \int_1^t \frac{-2}{\tau^2} + \frac{2}{\tau^3} d\tau$$

Hence

$$\boxed{x_2(t) = \frac{2t-1}{t^2}}$$

This gives us the second column of

$$\Psi^2 = \begin{pmatrix} \frac{-2}{t^2} + \frac{2}{t^3} \\ \frac{2t-1}{t^2} \end{pmatrix}$$

Hence the fundamental matrix is

$$\Psi = \begin{pmatrix} \frac{-1}{t^2} + \frac{2}{t^3} & \frac{-2}{t^2} + \frac{2}{t^3} \\ \frac{t-1}{t^2} & \frac{2t-1}{t^2} \end{pmatrix}$$

The inverse is now found.

$$\Psi^{-1} = \frac{\begin{pmatrix} \frac{2t-1}{t^2} & \frac{2}{t^2} - \frac{2}{t^3} \\ -\frac{t-1}{t^2} & \frac{-1}{t^2} + \frac{2}{t^3} \end{pmatrix}}{1/t^4} = \begin{pmatrix} 2t^3 - t^2 & 2t^2 - 2t \\ t^2 - t^3 & -\frac{t^2-t^3}{t-t^2}(t-2) \end{pmatrix}$$

Therefore the state transition function is

$$\begin{aligned}
\Phi(t,\tau) &= \Psi(t)\,\Psi^{-1}(\tau)\\
&= \begin{pmatrix} \frac{-1}{t^2}+\frac{2}{t^3} & \frac{-2}{t^2}+\frac{2}{t^3}\\ \frac{t-1}{t^2} & \frac{2t-1}{t^2}\end{pmatrix}\begin{pmatrix} 2\tau^3-\tau^2 & 2\tau^2-2\tau\\ \tau^2-\tau^3 & -\frac{\tau^2-\tau^3}{\tau-\tau^2}(\tau-2)\end{pmatrix}\\
&= \begin{pmatrix} -\frac{1}{t^3}\tau^2(t-2\tau) & -\frac{2}{t^3}\tau(t-\tau)\\ \frac{1}{t^2}\tau^2(t-\tau) & -\frac{1}{t^2}\tau(\tau-2t)\end{pmatrix}\\
&= \frac{\tau}{t^2}\begin{pmatrix} -\frac{\tau^2}{t}(t-2\tau) & -\frac{2}{t}(t-\tau)\\ \tau(t-\tau) & -(\tau-2t)\end{pmatrix}
\end{aligned}$$

Part (b)

For $A(t)=\begin{pmatrix}2 & -e^t\\ e^t & 1\end{pmatrix}$ we first need to find the fundamental matrix $\Psi(t)$ and then $\Phi(t,\tau)=\Psi(t)\,\Psi^{-1}(\tau)$. Let the two linearly independent initial conditions be

$$X^{01}=\begin{pmatrix}1\\0\end{pmatrix}, X^{02}=\begin{pmatrix}1\\0\end{pmatrix}$$

We know solve $x'=A(t)\,x$ using both of these initial conditions and obtain two linearly independent solutions to use to construct $\Psi(t)$ with. Using the first initial conditions $\boxed{x_1(1)=1, x_2(1)=0}$. The two equations to solve are

$$x_1' = 2x_1 - e^t x_2 \tag{1}$$
$$x_2' = e^{-t}x_1 + x_2 \tag{2}$$

Starting with (2), $x_2'-x_2=e^{-t}x_1$, this is in the form $x'+p(t)\,x=f(t)$, hence the integrating factor is $e^{\int p(t)dt}=e^{-\int dt}=e^{-t}$ and the solution is

$$\frac{d}{dt}\left(e^{-t}x_2\right)=e^{-t}\left(e^{-t}x_1\right)$$

Integrating both sides

$$\begin{aligned}
\left[e^{-\tau}x_2(\tau)\right]_1^t &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
e^{-t}x_2(t)-\overbrace{e^{-1}x_2(1)}^{\text{zero}} &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
e^{-t}x_2(t) &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau
\end{aligned}$$

Hence

$$x_2 = e^t\int_1^t e^{-2\tau}x_1(\tau)\,d\tau \tag{3}$$

Substituting this solution in (1) gives

$$\begin{aligned}
x_1' &= 2x_1 - e^{2t}\int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
e^{-2t}x_1' - 2x_1e^{-2t} &= -\int_1^t e^{-2\tau}x_1(\tau)\,d\tau
\end{aligned}$$

Differentiating

$$\begin{aligned}
-2e^{-2t}x_1' + e^{-2t}x_1'' - 2x_1'e^{-2t} + 4x_1e^{-2t} &= -e^{-2t}x_1(t)\\
e^{-2t}x_1'' - 4e^{-2t}x_1' + 5e^{-2t}x_1 &= 0\\
x_1'' - 4x_1' + 5x_1 &= 0
\end{aligned}$$

This is a constant coefficient ODE. Its solution can be found from the characteristic polynomial. $\lambda^2-4\lambda+5=0$, the solution is $\{2+i, 2-i=0\}$, hence

$$x_1 = c_1e^{(2+i)t}+c_2e^{(2-i)t}$$

Since the roots are complex, this can be written as $\sin / \cos$,

$$\begin{aligned}x_1 &= c_1 e^{2t} e^{it} + c_2 e^{2t} e^{-it}\\ &= e^{2t}\left(c_1 e^{it} + c_2 e^{-it}\right)\\ &= e^{2t}\left(c_1 (\cos t + i \sin t) + c_2 (\cos t - i \sin t)\right)\\ &= e^{2t}\left(\cos t\,(c_1 + c_2) + \sin t\,(ic_1 - ic_2)\right)\end{aligned}$$

Let $c_1 + c_2 = A$ and $i(c_1 - c_2) = B$, some new constants. Hence the above becomes

$$x_1(t) = e^{2t}(A \cos t + B \sin t) \tag{4}$$

From initial conditions, $x_1(1) = 1$. But we are not given $x_1'(1)$. We can find this from (1) $x_1' = 2x_1 - e^t x_2$ by noting that at $t = 1$,

$$\begin{aligned}x'(1) &= 2x_1(1) - e^1 x_2(1)\\ &= 2\end{aligned}$$

Hence now we have the two initial conditions to find A, B from (4). At $t = 1$, (4) becomes

$$1 = e^2(A \cos 1 + B \sin 1) \tag{5}$$

Taking derivative of (4)

$$x_1'(t) = 2e^{2t}(A \cos t + B \sin t) + e^{2t}(-A \sin t + B \cos t)$$

And at $t = 1$ this becomes

$$2 = 2e^2(A \cos 1 + B \sin 1) + e^2(-A \sin 1 + B \cos 1) \tag{6}$$

From (5),(6) we can solve for A, B,

$$\begin{aligned}1 &= e^2(A \cos 1 + B \sin 1)\\ 2 &= 2e^2(A \cos 1 + B \sin 1) + e^2(-A \sin 1 + B \cos 1)\end{aligned}$$

The solution is $A = \frac{\cos 1}{e^2}, B = \frac{\sin 1}{e^2}$. Therefore from (4) we obtain

$$\begin{aligned}x_1(t) &= e^{2t}\left(\frac{\cos 1 \cos t}{e^2} + \frac{\sin 1 \sin t}{e^2}\right)\\ &= e^{2t}\left(\frac{\cos 1 \cos t + \sin 1 \sin t}{e^2}\right)\end{aligned}$$

But $\cos 1 \cos t + \sin 1 \sin t = \cos(1 - t)$, hence

$$x_1(t) = e^{2(t-1)} \cos(1 - t)$$

Now that we found $x_1(t)$ we go to (3) and find $x_2(t)$

$$\begin{aligned}x_2 &= e^t \int_1^t e^{-2\tau} x_1(\tau)\, d\tau\\ &= e^t \int_1^t e^{-2\tau} e^{2(\tau-1)} \cos(1 - \tau)\, d\tau\\ &= e^t \int_1^t e^{-2} \cos(1 - \tau)\, d\tau\end{aligned}$$

Hence

$$x_2 = -e^{t-2} \sin(1 - t)$$

Therefore, the first columns of the fundamental matrix is found

$$\Psi^1 = \begin{pmatrix} e^{2t-2} \cos(1 - t)\\ -e^{t-2} \sin(1 - t)\end{pmatrix}$$

We now find the second column Ψ^2. Using the second initial conditions $\boxed{x_1(1) = 0, x_2(1) = 1}$. The two equations to solve are

$$\begin{aligned}x_1' &= 2x_1 - e^t x_2 &\text{(1A)}\\ x_2' &= e^{-t} x_1 + x_2 &\text{(2A)}\end{aligned}$$

Starting with (2), $x_2' - x_2 = e^{-t} x_1$, this is in the form $x' + p(t)x = f(t)$, hence the integrating factor is $e^{\int p(t)dt} = e^{-\int dt} = e^{-t}$ and the solution is

$$\frac{d}{dt}\left(e^{-t} x_2\right) = e^{-t}\left(e^{-t} x_1\right)$$

Integrating both sides

$$\begin{aligned}
\left[e^{-\tau}x_2(\tau)\right]_1^t &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
e^{-t}x_2(t)-e^{-1}x_2(1) &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
e^{-t}x_2(t)-e^{-1} &= \int_1^t e^{-2\tau}x_1(\tau)\,d\tau\\
x_2(t) &= e^t\int_1^t e^{-2\tau}x_1(\tau)\,d\tau + e^{t-1} \qquad (3A)
\end{aligned}$$

Substituting this solution in (1A) gives

$$\begin{aligned}
x_1' &= 2x_1 - e^t\left(e^t\int_1^t e^{-2\tau}x_1(\tau)\,d\tau + e^{t-1}\right)\\
x_1' &= 2x_1 - e^{2t}\int_1^t e^{-2\tau}x_1(\tau)\,d\tau - e^{2t-1}\\
e^{-2t}x_1' - 2x_1e^{-2t} &= -\int_1^t e^{-2\tau}x_1(\tau)\,d\tau - e^{-1}
\end{aligned}$$

Differentiating

$$\begin{aligned}
-2e^{-2t}x_1' + e^{-2t}x_1'' - 2x_1'e^{-2t} + 4x_1e^{-2t} &= -e^{-2t}x_1(t)\\
e^{-2t}x_1'' - 4e^{-2t}x_1' + 5e^{-2t}x_1 &= 0\\
x_1'' - 4x_1' + 5x_1 &= 0
\end{aligned}$$

This is a constant coefficient ODE. Its solution can be found from the characteristic polynomial. $\lambda^2-4\lambda+5=0$, the solution is $\{2+i,2-i=0\}$, hence

$$x_1 = c_1e^{(2+i)t} + c_2e^{(2-i)t}$$

Since the roots are complex, this can be written as $\sin/\cos$, giving, as above

$$x_1(t) = e^{2t}(A\cos t + B\sin t) \qquad (4A)$$

From initial conditions, $x_1(1)=0$. But we are not given $x_1'(1)$. We can find this from (1A) $x_1' = 2x_1 - e^tx_2$ by noting that at $t=1$,

$$\begin{aligned}
x'(1) &= 2x_1(1) - e^1x_2(1)\\
&= -e^1
\end{aligned}$$

Hence now we have the two initial conditions to find A,B from (4). At $t=1$, (4A) becomes

$$\begin{aligned}
0 &= e^2(A\cos 1 + B\sin 1)\\
0 &= A\cos 1 + B\sin 1 \qquad (5A)
\end{aligned}$$

Taking derivative of (4A)

$$x_1'(t) = 2e^{2t}(A\cos t + B\sin t) + e^{2t}(-A\sin t + B\cos t)$$

And at $t=1$ this becomes

$$-e^1 = 2e^2(A\cos 1 + B\sin 1) + e^2(-A\sin 1 + B\cos 1) \qquad (6A)$$

From (5A),(6A) we can solve for A,B,

$$\begin{aligned}
0 &= e^2(A\cos 1 + B\sin 1)\\
-e^1 &= 2e^2(A\cos(1) + B\sin(1)) + e^2(-A\sin(1) + B\cos(1))
\end{aligned}$$

The solution is $A=\frac{\sin 1}{e}, B=\frac{-\cos 1}{e}$. Therefore (4A) becomes

$$\begin{aligned}
x_1(t) &= e^{2t}(A\cos t + B\sin t)\\
&= e^{2t}\left(\frac{\sin 1\cos t}{e} - \frac{\cos 1\sin t}{e}\right)\\
&= e^{2t}\left(\frac{\sin 1\cos t - \cos 1\sin t}{e}\right)
\end{aligned}$$

But $\sin 1\cos t - \cos 1\sin t = \sin(1-t)$, hence

$$x_1(t) = e^{2t-1}\sin(1-t)$$

Now that we found $x_1(t)$ we go to (3A) and find $x_2(t)$

$$\begin{aligned}
x_2(t) &= e^t \int_1^t e^{-2\tau} x_1(\tau)\, d\tau + e^{t-1}\\
&= e^{t-1} + e^t \int_1^t e^{-2\tau} e^{2\tau-1} \sin(1-\tau)\, d\tau\\
&= e^{t-1} + e^t \int_1^t e^{-1} \sin(1-\tau)\, d\tau\\
&= e^{t-1} + e^{t-1} \int_1^t \sin(1-\tau)\, d\tau\\
&= e^{t-1} + e^{t-1}(-1+\cos(1-t))\\
&= e^{t-1}\cos(1-t)
\end{aligned}$$

Therefore, the second column of the fundamental matrix is found

$$\Psi^2 = \begin{pmatrix} e^{2t-1}\sin(1-t)\\ e^{t-1}\cos(1-t)\end{pmatrix}$$

Hence the fundamental matrix is

$$\Psi = \begin{pmatrix} e^{2t-2}\cos(1-t) & e^{2t-1}\sin(1-t)\\ -e^{t-2}\sin(1-t) & e^{t-1}\cos(1-t)\end{pmatrix}$$

The inverse is now found.

$$\Psi^{-1} = \begin{pmatrix} e^{2-2t}\cos(1-t) & -e^{2-t}\sin(1-t)\\ e^{1-2t}\sin(1-t) & e^{1-t}\cos(1-t)\end{pmatrix}$$

Therefore the state transition function, after some simplification, is

$$\begin{aligned}
\Phi(t,\tau) &= \Psi(t)\Psi^{-1}(\tau)\\
&= \begin{pmatrix} e^{2t-2}\cos(1-t) & e^{2t-1}\sin(1-t)\\ -e^{t-2}\sin(1-t) & e^{t-1}\cos(1-t)\end{pmatrix}\begin{pmatrix} e^{2-2\tau}\cos(1-\tau) & -e^{2-\tau}\sin(1-\tau)\\ e^{1-2\tau}\sin(1-\tau) & e^{1-\tau}\cos(1-\tau)\end{pmatrix}\\
&= \begin{pmatrix} e^{2(t-\tau)}\cos(t-\tau) & -e^{2t-\tau}\sin(t-\tau)\\ e^{t-2\tau}\sin(t-\tau) & e^{t-\tau}\cos(t-\tau)\end{pmatrix}
\end{aligned}$$

2.5.6 key solution

ECE 717 – Solution Set 5

ECE 717 – Solution Diagonalize

Let T be a matrix whose columns are eigenvectors of M_1. We claim that its columns are also eigenvectors of M_2.If we establish this, it follows that M_1 and M_2 are simultaneously diagonalizable.

Indeed, suppose v is an eigenvector of M_1 corresponding to eigenvalue λ. Then we know that

$$M_1 v = \lambda v.$$

Hence, to show that v is also an eigenvector of M_2, we observe, using the commutating property, that

$$M_1[M_2 v] = M_2 M_1 v = \lambda[M_2 v].$$

This says that $M_2 v$ is also an eigenvector of M_1 corresponding to eigenvalue λ. Since all eigenvectors for λ are scalar multiples of each other, it follows that

$$M_2 v = \rho v$$

for some scalar ρ. The equality above implies that v is an eigenvector of M_2 corresponding to eigenvalue ρ .

Solution Exponentials

(a) Propose FMS : $\Psi(t) = e^{\int_0^t A(\eta)d\eta}$. Suffices to show that ① $\Psi(0) = I$ ② $\dot{\Psi}(t) = A(t)\Psi(t)$

①) follows since $e^0 = I$. ② Look at $\frac{d}{dt}\Psi(t) = \frac{d}{dt} e^{\int_0^t A(\eta)d\eta}$

$$= \frac{d}{dt}\left[I + \left(\int_0^t A(\eta)\,d\eta\right) + \frac{\left(\int_0^t A(\eta)\,d\eta\right)^2}{2!} + \cdots\cdots \frac{\left(\int_0^t A(\eta)\,d\eta\right)^k}{k!} + \cdots \right]$$

$$= A(t) + \frac{2\left(\int_0^t A(\eta)\,d\eta\right)A(t)}{2!} + \cdots\cdots \frac{k\left(\int_0^t A(\eta)\,d\eta\right)^{k-1}A(t)}{k!}$$

$\overset{=}{\uparrow}$ using commutation

$$A(t) + \frac{2A(t)\left(\int_0^t A(\eta)\,d\eta\right)}{2!} + \cdots \frac{kA(t)\left(\int_0^t A(\eta)\,d\eta\right)^{k-1}}{k!} + \cdots\cdots$$

$$= \sum_{k=1}^{\infty} A(t)\frac{\left(\int_0^t A(\eta)\,d\eta\right)^{k-1}}{(k-1)!} = A(t)\sum_{k=0}^{\infty}\frac{\left(\int_0^t A(\eta)\,d\eta\right)^{k}}{k!}$$

$= A(t)\, e^{\int_0^t A(\eta)d\eta} = A(t)\Psi(t)$ Hence $\Psi(t)$ is an FMS

③ Suppose $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all t_1, t_2 ✡

Then $A(t)\int_0^t A(\eta)d\eta = \int_0^t A(t)A(\eta)d\eta \overset{\text{by ✡}}{=} \int_0^t A(\eta)A(t)d\eta$

$= \int_0^t A(\eta)d\eta \; A(t)$ so commutation holds.

Hence solution is again

$$X(t) = e^{\int_0^t A(\eta)d\eta} X(0)$$

Homework Revisit:

From homework Exponentials, we have

$$\Psi(t) = e^{\int_0^t A(\eta)\,d\eta}$$

$A(t)$ has distinct eigenvalues for all t, then

$$A(t) = T(t)^{-1} \Lambda(t)\, T(t) \quad \text{for all } t.$$

Since $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all pairs (t_1, t_2), from homework Diagonalize, $\exists\, P$ (time invariant) that diagonalize all $A(t)$ to $\Lambda(t)$.

Therefore, $A(\eta) = P\Lambda(\eta)P^{-1}$

Then

$$\begin{aligned}
\Psi(t) &= e^{\int_0^t A(\eta)\,d\eta} \\
&= e^{P\int_0^t \Lambda(\eta)\,d\eta\, P^{-1}} \\
&= \sum_{k=0}^{\infty} \frac{\left(P\int_0^t \Lambda(\eta)\,d\eta\, P^{-1}\right)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{P\left(\int_0^t \Lambda(\eta)\,d\eta\right)^k P^{-1}}{k!} \\
&= P \sum_{k=0}^{\infty} \frac{\left(\int_0^t \Lambda(\eta)\,d\eta\right)^k}{k!}\, P \\
&= P\, e^{\int_0^t \Lambda(\eta)\,d\eta}\, P^{-1}
\end{aligned}$$

Let $T = P^{-1}$, then.

$$\Psi(t) = T^{-1} e^{\int_0^t \Lambda(\eta)\, d\eta} T.$$

and it satisfies the state equation. and

$$\Psi(0) = I.$$

Solution Transition

(a)

We consider solution of the form

$x_1(t) = \pm\frac{1}{t^i}$; $x_2(t) = \pm\frac{1}{t^j}$. Now by trial and error substitution for $\Psi(t)$ obtain

$$\psi^1(t) = \begin{bmatrix} -\frac{1}{t^2} \\ \frac{1}{t} \end{bmatrix} ; \quad \psi^2(t) = \begin{bmatrix} \frac{2}{t^3} \\ -1/t^2 \end{bmatrix}$$

and, with $t_0 = 1$

have $\psi^1(t_0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $\psi^2(t_0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ which are independent as required.

Next compute

$$\Phi(t,\tau) = \Psi(t)\,\Psi^{-1}(\tau) = \begin{bmatrix} -\frac{1}{t^2} & \frac{2}{t^3} \\ \frac{1}{t} & -\frac{1}{t^2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\tau^2} & \frac{2}{\tau^3} \\ \frac{1}{\tau} & -\frac{1}{\tau^2} \end{bmatrix}^{-1}$$

$$\overset{\text{algebra}}{=} \begin{bmatrix} -t^{-2}\tau^2 + 2t^{-3}\tau^3 & -2t^{-2}\tau + 2t^{-3}\tau^2 \\ t^{-1}\tau^2 - t^{-2}\tau^3 & 2t^{-1}\tau - t^{-2}\tau^2 \end{bmatrix}$$

done.

(b) Similar to part (a). We first consider solutions of the form $x_1(t) = e^{\alpha t}\cos\beta t$ and similar forms for $x_2(t)$. After trial and error for $x^0 = \begin{bmatrix}1\\0\end{bmatrix}$, obtain $\psi^1(t) = \begin{bmatrix}e^{2t}\cos t\\ e^{t}\sin t\end{bmatrix}$ and for $x^0 = \begin{bmatrix}0\\1\end{bmatrix}$; $\psi^2(t) = \begin{bmatrix}-e^{2t}\sin t\\ e^{t}\cos t\end{bmatrix}$

(The above requires algebra... substitution into $\dot{x}_i$ equations, etc.)

Now

$$\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau) = \begin{bmatrix} e^{2t}\cos t & -e^{2t}\sin t \\ e^{t}\sin t & e^{t}\cos t \end{bmatrix}$$

$$\begin{bmatrix} e^{2\tau}\cos\tau & -e^{-2\tau}\sin\tau \\ e^{\tau}\sin\tau & e^{\tau}\cos\tau \end{bmatrix}^{-1}$$

lots of algebra ... can use syms in Matlab

$$= \begin{bmatrix} e^{2(t-\tau)}\cos(t-\tau) & -e^{(2t-\tau)}\sin(t-\tau) \\ e^{t-2\tau}\sin(t-\tau) & e^{t-\tau}\cos(t-\tau) \end{bmatrix}$$

2.6 HW6

2.6.1 Questions

ECE 717 – Homework Set 6

Due Tuesday, November 18, 2014

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Barmish

ECE 717 – Homework Controllability

Show that the linear time-varying system described by

$$A(t) = \begin{bmatrix} 2 & -e^{t} \\ e^{-t} & 1 \end{bmatrix}$$

and

$$b(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is controllable at $t_0 = 0$.

;

Barmish

ECE 717 – Homework P Transformation

Given a continuous LTV system $\Sigma = (A(t), B(t))$, let

$$z(t) = P(t)x(t)$$

where $P(t)$ is a continuously differentiable square nonsingular matrix. Denoting the resulting the z-system by $\tilde{\Sigma}$, establish the following result: Σ is controllable at t_0 if and only if $\tilde{\Sigma}$ is controllable at t_0. HINT: Relate $\Psi_{\tilde{\Sigma}}(t)$ to $\Psi_{\Sigma}(t)$ $\Phi_{\tilde{\Sigma}}(t,\tau)$ to $\Psi_{\Sigma}(t,\tau)$ and finally $W_{\tilde{\Sigma}}(t_0,t_1)$ to $W_{\Sigma}(t_0,t_1)$.

;

Barmish

ECE 717 – Homework Solve

For the LTI state equation

$$\dot{x} = Ax$$

with A having distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, recall that the fundamental matrix solution

$$\Psi(t) = e^{At}$$

can be expanded as

$$e^{At} = \sum_{i=1}^{n} Y_{0i} e^{\lambda_i t}$$

(a) For $n = 3$, show that the following formula provides a closed form solution for the Y_{0i}:

$$Y_{0i} = \prod_{j=1, j \neq i}^{n} \frac{A - \lambda_j I}{\lambda_i - \lambda_j}.$$

(b) Use the result above to obtain e^{At} with

$$A = -\frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

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Barmish

ECE 717 – Homework Circuit

A two-loop RLC circuit with continuously differentiable time-varying resistance $R(t)$ and input voltage $u(t)$ is described by the pair of state equations

$$\dot{x}_1 = -\frac{x_1}{R(t)} - x_2 + \frac{u}{R(t)};$$

$$\dot{x_2} = x_1$$

(a) Determine if this system is controllable at $t_0 > 0$.

(b) Determine if this system is differentially controllable at $t_0 > 0$.

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Barmish

ECE 717 – Homework Control Effort

For the controllable LTI system $\Sigma = (A, B)$ with all eigenvalues of A in the strict left half plane, establish the following *control effort* property: Given any initial condition $x(0) = x^0$ and any $\beta > 0$, there exists a control $u(t)$ and a future time $t_1 > 0$ such that

$$x(t_1) = 0$$

and

$$||u(t)|| \leq \beta$$

for all $t \in [0, t_1]$.

;

Barmish

ECE 717 – Homework Range

For the continuous LTV system

$$\dot{x} = A(t)x + B(t)u$$

with $t_0 \leq t \leq t_1$, prove that an initial state $x(t_0) = x^0$ can be steered to zero at time t_1 if and only if x^0 is in the range of $W(t_0, t_1)$. That is

$$x^0 \in \mathcal{R}(W(t_0, t_1)) = \{W(t_0, t_1)\overline{x} : \overline{x} \in \mathbf{R}^n\}.$$

HINT: If a vector x^0 is not in the range of $W(t_0, t_1)$, from matrix algebra, there exists a non-zero vector η such that

$$\eta^T x^0 \neq 0$$

and

$$\eta^T W(t_0, t_1) = 0.$$

;

2.6.2 Problem 1 Controllability

$n = 2$. Since $A(t), b(t)$ are $n-1$ or 1 time differentiable, we can obtain $M(t) = \begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix}$ and check that its rank is n using the theorem that $\sum$ is controllable at t_0 if there exist

$t > t_0$ such that $\rho(M(t)) = n$.

$$\begin{aligned} M_0(t) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ M_1(t) &= -A(t) M_0(t) + \frac{d}{dt} M_0(t) \\ &= -\begin{bmatrix} 2 & -e^t \\ e^{-t} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t \\ -1 \end{bmatrix} \end{aligned}$$

Hence

$$M(t) = \begin{bmatrix} 0 & e^t \\ 1 & -1 \end{bmatrix}$$

The determinant is $\Delta = -e^t$ which is not zero for any $t > 0$. Hence $M(t)$ is not singular and so has rank 2. Hence $\sum$ is controllable at $t = 0$. Note: This system is not stable.

2.6.3 Problem 2 P Transformation

$$\begin{aligned} z &= Px \\ z' &= P'x + Px' \end{aligned}$$

Hence

$$\begin{aligned} x' &= P^{-1}(z' - P'x) \\ &= P^{-1}\left(z' - P'P^{-1}z\right) \end{aligned}$$

Therefore, the state space $x' = Ax + Bu$ becomes

$$\begin{aligned} P^{-1}\left(z' - P'P^{-1}z\right) &= AP^{-1}z + Bu \\ z' - P'P^{-1}z &= PAP^{-1}z + PBu \\ z' &= P'P^{-1}z + PAP^{-1}z + PBu \\ &= \left(P'P^{-1} + PAP^{-1}\right) z + PBu \end{aligned}$$

Therefore

$$\tilde{A} = \left(P'P^{-1} + PAP^{-1}\right)$$

And

$$\tilde{B}(t) = P(t) B(t)$$

Now the state equation solution for $\sum$ is given by

$$x(t) = \Phi(t, 0) x(0) + \int_0^t \Phi(t, \tau) B(\tau) u(\tau) d\tau$$

Applying the transformation to the above results in

$$\begin{aligned} P^{-1}(t) z(t) &= \Phi(t, 0) P^{-1}(t) z(0) + \int_0^t \Phi(t, \tau) P^{-1}(\tau) \tilde{B} u(\tau) d\tau \\ z(t) &= \overbrace{P(t) \Phi(t, 0) P^{-1}(t)}^{\tilde{\Phi}(t,0)} z(0) + \int_0^t \overbrace{P(t) \Phi(t, \tau) P^{-1}(\tau)}^{\tilde{\Phi}(t,\tau)} \tilde{B}(\tau) u(\tau) d\tau \\ z(t) &= \tilde{\Phi}(t, 0) z(0) + \int_0^t \tilde{\Phi}(t, \tau) \tilde{B}(\tau) u(\tau) d\tau \end{aligned}$$

Hence

$$\tilde{\Phi}(t, \tau) = P(t) \Phi(t, \tau) P^{-1}(\tau)$$

Now that we found $\tilde{\Phi}(t, \tau)$ and $\tilde{B}(t)$, we are now ready to do the proof.

Theorem: (A, B) is controllable at t_0 iff $\left(\tilde{A}, \tilde{B}\right)$ is controllable at t_0.

Necessity $\Longrightarrow$. We need to show: If (A,B) is controllable at t_0 then $\left(\tilde{A},\tilde{B}\right)$ is controllable at t_0

Sufficiency $\Longleftarrow$. We need to show: If $\left(\tilde{A},\tilde{B}\right)$ is controllable at t_0 then (A,B) is controllable at t_0

Proof of Necessity: Given that (A,B) is controllable at t_0, show that $\left(\tilde{A},\tilde{B}\right)$ is controllable at t_0.

Since (A,B) is controllable at t_0, then the following controllability Gramian $W(t_0,t)$ is not singular

$$W(t_0,t)=\int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau \tag{1}$$

We want to show the above implies that

$$\tilde{W}(t_0,t)=\int_{t_0}^{t}\tilde{\Phi}(t_0,\tau)\tilde{B}(\tau)\tilde{B}^T(\tau)\tilde{\Phi}^T(t_0,\tau)d\tau \tag{2}$$

is also not singular.

Applying the transformations found to (2) gives

$$\begin{aligned}\tilde{W}(t_0,t)&=\int_{t_0}^{t}\left[P(t)\Phi(t_0,\tau)P^{-1}(\tau)\right][P(\tau)B(\tau)][P(\tau)B(\tau)]^T\left[P(t)\Phi(t_0,\tau)P^{-1}(\tau)\right]^T d\tau\\&=\int_{t_0}^{t}P(t)\Phi(t_0,\tau)B(\tau)B^T(\tau)P^T(\tau)\left(P^T(\tau)\right)^{-1}\Phi^T(t_0,\tau)P^T(t)\,d\tau\end{aligned}$$

Notice in the above we used $\left(P^{-1}(\tau)\right)^T=\left(P^T(\tau)\right)^{-1}$. Therefore the above simplifies to

$$\begin{aligned}\tilde{W}(t_0,t)&=\int_{t_0}^{t}P(t)\Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)P^T(t)\,d\tau\\&=P(t)\left(\int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\,d\tau\right)P^T(t)\\&=P(t)\,W(t_0,t)\,P^T(t)\end{aligned}$$

Since $W(t_0,t)$ is not singular, and $P(t)$ is given as not singular, then $P(t)\,W(t_0,t)\,P^T(t)$ is not singular also and this implies $\tilde{W}(t_0,t)$ is not singular.

Proof of sufficiency: $\Longleftarrow$. We need to show: If $\left(\tilde{A},\tilde{B}\right)$ is controllable at t_0 then (A,B) is controllable at t_0. Since $\left(\tilde{A},\tilde{B}\right)$ is controllable at t_0, then the controllability Gramian $\tilde{W}(t_0,t)$ is not singular

$$\tilde{W}(t_0,t)=\int_{t_0}^{t}\tilde{\Phi}(t_0,\tau)\,\tilde{B}(\tau)\,\tilde{B}^T(\tau)\,\tilde{\Phi}^T(t_0,\tau)\,d\tau \tag{3}$$

We want to show the above implies that

$$W(t_0,t)=\int_{t_0}^{t}\Phi(t_0,\tau)\,B(\tau)\,B^T(\tau)\,\Phi^T(t_0,\tau)\,d\tau \tag{4}$$

Applying the transformations to (4) gives

$$
\begin{aligned}
W(t_0,t) &= \int_{t_0}^{t} \left[P^{-1}(t)\,\tilde{\Phi}(t_0,\tau)\,P(\tau)\right]\left[P^{-1}(\tau)\,\tilde{B}(\tau)\right]\left[P^{-1}(\tau)\,\tilde{B}(\tau)\right]^T\left[P^{-1}(t)\,\tilde{\Phi}(t_0,\tau)\,P(\tau)\right]^T d\tau \\
&= \int_{t_0}^{t} P^{-1}(t)\,\tilde{\Phi}(t_0,\tau)\,\tilde{B}(\tau)\,\tilde{B}^T(\tau)\,P^T(\tau)^{-1}\left[P^T(\tau)\left[P^{-1}(t)\,\tilde{\Phi}(t_0,\tau)\right]^T\right]d\tau \\
&= \int_{t_0}^{t} P^{-1}(t)\,\tilde{\Phi}(t_0,\tau)\,\tilde{B}(\tau)\,\tilde{B}^T(\tau)\left(P^{-T}(\tau)\right)^{-1}P^T(\tau)\,\tilde{\Phi}^T(t_0,\tau)\,P^{-1}(t)^T\,d\tau \\
&= \int_{t_0}^{t} P^{-1}(t)\,\tilde{\Phi}(t,\tau)\,\tilde{B}(\tau)\,\tilde{B}^T(\tau)\,\tilde{\Phi}^T(t,\tau)\,P^{-1}(t)^T\,d\tau \\
&= P^{-1}(t)\left(\int_{t_0}^{t} \tilde{\Phi}(t_0,\tau)\,\tilde{B}(\tau)\,\tilde{B}^T(\tau)\,\tilde{\Phi}^T(t_0,\tau)\,d\tau\right)P^{-1}(t)^T \\
&= P^{-1}(t)\,\tilde{W}(t_0,t)\,P^{-1}(t)^T
\end{aligned}
$$

Similar to the same argument used for the Necessity, since $\tilde{W}(t_0,t)$ is not singular, and $P(t)$ is given as not singular, then $P(t)\,\tilde{W}(t_0,t)\,P^T(t)$ is not singular and this implies $W(t_0,t)$ is not singular.

2.6.4 Problem 3 Solve

Part (a)

$$e^{At} = Y_{01}e^{\lambda_1 t} + Y_{02}e^{\lambda_2 t} + Y_{03}e^{\lambda_3 t}$$

Where

$$
\begin{aligned}
Y_{01} &= \frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\
Y_{02} &= \frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\
Y_{03} &= \frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}
\end{aligned}
$$

We know that $e^{At}\big|_{t=0} = I$ and $\frac{d}{dt}e^{At}\Big|_{t=0} = A$ and $\frac{d^2}{dt^2}e^{At}\Big|_{t=0} = A^2$. So now need to verify that using the above expressions these remain satisfied.

$$
\begin{aligned}
e^{At}\big|_{t=0} &= I \\
Y_{01}+Y_{02}+Y_{03} &= \frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}
\end{aligned}
$$

Using common denominator $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)$ results in

$$
\begin{aligned}
Y_{01}+Y_{02}+Y_{03} &= \frac{(A-\lambda_2 I)(A-\lambda_3 I)(\lambda_3-\lambda_2) + (A-\lambda_1 I)(A-\lambda_3 I)(\lambda_1-\lambda_3) - (A-\lambda_1 I)(A-\lambda_2 I)(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)} \\
&= \frac{\left(A^2-\lambda_3 A-\lambda_2 A+\lambda_2\lambda_3 I\right)(\lambda_3-\lambda_2) + \left(A^2-\lambda_3 A-\lambda_1 A+\lambda_1\lambda_3 I\right)(\lambda_1-\lambda_3) - \left(A^2-\lambda_2 A-\lambda_1 A+\lambda_1\lambda_2\right)(\lambda_1-\lambda_2 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)}
\end{aligned}
$$

Expanding the numerator and simplifying results in $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)\,I$, hence

$$
\begin{aligned}
Y_{01}+Y_{02}+Y_{03} &= \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)\,I}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)} \\
&= I
\end{aligned}
$$

Now we need to verify the second equation

$$
\begin{aligned}
\frac{d}{dt}e^{At}\Big|_{t=0} &= A \\
\frac{d}{dt}\left(Y_{01}e^{\lambda_1 t} + Y_{02}e^{\lambda_2 t} + Y_{03}e^{\lambda_3 t}\right)\Big|_{t=0} &= \lambda_1 Y_{01} + \lambda_2 Y_{02} + \lambda_3 Y_{03}
\end{aligned}
$$

But

$$\begin{aligned}
\lambda_1 Y_{01}+\lambda_2 Y_{02}+\lambda_3 Y_{03} &= \lambda_1\frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\lambda_2\frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\lambda_3\frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\\
&= \frac{-(A-\lambda_1)(A\lambda_1-\lambda_2\lambda_3)+(A-\lambda_2)(A-\lambda_3)\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\\
&= \frac{A\lambda_1^2-A\lambda_1\lambda_2-A\lambda_1\lambda_3+A\lambda_2\lambda_3}{\lambda_1^2-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_2\lambda_3}\\
&= \frac{A\left(\lambda_1^2-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_2\lambda_3\right)}{\lambda_1^2-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_2\lambda_3}\\
&= A
\end{aligned}$$

Now we need to verify the third equation

$$\frac{d^2}{dt^2}e^{At}\bigg|_{t=0} = A^2$$

$$\frac{d^2}{dt^2}\left(Y_{01}e^{\lambda_1 t}+Y_{02}e^{\lambda_2 t}+Y_{03}e^{\lambda_3 t}\right)\bigg|_{t=0} = \lambda_1^2 Y_{01}+\lambda_2^2 Y_{02}+\lambda_3^2 Y_{03}$$

But

$$\begin{aligned}
\lambda_1^2 Y_{01}+\lambda_2^2 Y_{02}+\lambda_3^2 Y_{03} &= \lambda_1^2\frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\lambda_2^2\frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\lambda_3^2\frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\\
&= \frac{-(A-\lambda_1)(A\lambda_1\lambda_2+(A\lambda_1-(A+\lambda_1)\lambda_2)\lambda_3)+(A-\lambda_2)(A-\lambda_3)\lambda_1^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\\
&= \frac{A^2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\\
&= A^2
\end{aligned}$$

Verified for $n=3$ OK.

Part (b)

$$A=-\frac{1}{2}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$$

The eigenvalues are $\lambda_1=-1,\lambda_2=-2$. Hence

$$e^{At}=Y_{01}e^{\lambda_1 t}+Y_{02}e^{\lambda_2 t}$$

Where

$$\begin{aligned}
Y_{01} &= \frac{(A-\lambda_2 I)}{(\lambda_1-\lambda_2)} = \frac{\left(\begin{bmatrix}\frac{-3}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-3}{2}\end{bmatrix}-\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}\right)}{(-1+2)} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\\
Y_{02} &= \frac{(A-\lambda_1 I)}{(\lambda_2-\lambda_1)} = \frac{\left(\begin{bmatrix}\frac{-3}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-3}{2}\end{bmatrix}-\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)}{(-2+1)} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
e^{At} &= \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}e^{-t}+\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}e^{-2t}\\
&= \begin{bmatrix}\frac{1}{2}e^{-t}\left(e^{-t}+1\right) & -\frac{1}{2}e^{-t}\left(e^{-t}-1\right)\\ -\frac{1}{2}e^{-t}\left(e^{-t}-1\right) & \frac{1}{2}e^{-t}\left(e^{-t}+1\right)\end{bmatrix}
\end{aligned}$$

2.6.5 Problem 4 Circuit

Part (a)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{R(t)} & -1 \\ 1 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{R(t)} \\ 0 \end{bmatrix}}^{B} u(t)$$

Since A, B are continuously differentiable, we can use the short cut M based method to determine if (A, B) is controllable at some instance of time and we do not need to compute the controllability Gramian W. First we will find M

$$M_0 = B(t) = \begin{bmatrix} \frac{1}{R(t)} \\ 0 \end{bmatrix}$$

$$\begin{aligned} M_1(t) &= -A(t) M_0(t) + \frac{d}{dt} M_0(t) \\ &= -\begin{bmatrix} -\frac{1}{R(t)} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{R(t)} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-1}{R^2(t)} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{R^2(t)} \\ -\frac{1}{R(t)} \end{bmatrix} - \begin{bmatrix} \frac{1}{R^2(t)} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{R(t)} \end{bmatrix} \end{aligned}$$

Hence

$$M = \begin{bmatrix} \frac{1}{R(t)} & 0 \\ 0 & -\frac{1}{R(t)} \end{bmatrix}$$

The determinant of M is

$$\Delta = \frac{-1}{R^2(t)}$$

The system is not controllable at t_0 if the determinant is zero at that instance of time. But for the determinant to become zero means that $R(t)$ has to become ∞. Therefore, assuming $R(t)$ remain finite for all $t > 0$ which is expected in a working physical system, then we conclude the system is indeed controllable for any $t_0 > 0$.

Part (b)

A system is differentially controllable at some time t_0 if there exist $u(t)$ which will steer $x(t_0)$ to $x(t_1)$ no matter how small $t_1 - t_0$ is. Clearly if the system is differentially controllable at t_0, then it is also controllable at t_0 by making $t_1 - t_0$ as large as we want. The question is asking to show the system is differentially controllable for $t_0 > 0$.

This actually follows from the fact that A, B are analytic functions. By definition, analytic functions over $[0, \infty]$ are linearly independent iff they are linearly independent over any sub interval no matter how small the interval is. But I think we need to proof this using calculus. Therefore an attempt to do so is given below:

Let $t_1 = t_0 + \varepsilon$ where ε is the time increment we will make as small as we want. Since the system is controllable at t_0 then $W(t_0, t_0 + \varepsilon)$ is nonsingular.

Now I will use the same result used in proofing controllability itself, which is to claim the following $u(t)$ will steer the system from $x(t_0)$ to $x(t_0 + \varepsilon)$

$$u(t) = -B^T(t) \Phi^T(t_0 + \varepsilon, t) W^{-1}(t_0, t_0 + \varepsilon) [\Phi(t_0 + \varepsilon, t_0) x(t_0) - x(t_0 + \varepsilon)]$$

To show that the above u results in system moving to $x(t_0 + \varepsilon)$ from $x(t_0)$, we substitute the above u into the state solution

$$\Delta = \Phi(t_0 + \varepsilon, t_0) x(t_0) + \int_{t_0}^{t_0+\varepsilon} \Phi(t_0 + \varepsilon, \tau) B(\tau) u(\tau) d\tau$$

And this will result in $\Delta = x(t_0 + \varepsilon)$.

$$
\begin{aligned}
\Delta &= \Phi(t_0+\varepsilon,t_0)\,x(t_0) - \int_{t_0}^{t_0+\varepsilon} \Phi(t_0+\varepsilon,\tau)\,B(\tau)\,\overbrace{\left(-B^T(t)\,\Phi^T(t_0+\varepsilon,\tau)\,W^{-1}(t_0,t_0+\varepsilon)\,[\Phi(t_0+\varepsilon,t_0)\,x(t_0)-x(t_0+\varepsilon)]\right)}^{u(\tau)}\,d\tau \\
&= \Phi(t_0+\varepsilon,t_0)\,x(t_0) - \overbrace{\left(\int_{t_0}^{t_0+\varepsilon} \Phi(t_0+\varepsilon,\tau)\,B(\tau)\,B^T(t)\,\Phi^T(t_0+\varepsilon,\tau)\,d\tau\right)}^{W(t_0,t_0+\varepsilon)} W^{-1}(t_0,t_0+\varepsilon)\,[\Phi(t_0+\varepsilon,t_0)\,x(t_0)-x(t_0+\varepsilon)] \\
&= \Phi(t_0+\varepsilon,t_0)\,x(t_0) - \overbrace{W(t_0,t_0+\varepsilon)\,W^{-1}(t_0,t_0+\varepsilon)}^{I}\,[\Phi(t_0+\varepsilon,t_0)\,x(t_0)-x(t_0+\varepsilon)] \\
&= \overbrace{\Phi(t_0+\varepsilon,t_0)\,x(t_0) - \Phi(t_0+\varepsilon,t_0)\,x(t_0)} + x(t_0+\varepsilon) \\
&= x(t_0+\varepsilon)
\end{aligned}
$$

The only requirement for the above proof was the condition that $W(t_0,t_0+\varepsilon)$ is nonsingular at t_0 which was established in part(a).

I give another proof just in case the above is not acceptable. Consider

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)\,B(\tau)\,B^T(\tau)\,\Phi^T(t_0,\tau)\,d\tau$$

We know the above is nonsingular since the system is controllable at $t_0 > 0$ from part (a). Using $\Phi(t_0,\tau) = \Phi(t_0,t_0+\varepsilon)\,\Phi(t_0+\varepsilon,\tau)$ we can rewrite the above as

$$
\begin{aligned}
W(t_0,t_1) &= \int_{t_0}^{t_1} \Phi(t_0,t_0+\varepsilon)\,\Phi(t_0+\varepsilon,\tau)\,B(\tau)\,B^T(\tau)\,(\Phi(t_0,t_0+\varepsilon)\,\Phi(t_0+\varepsilon,\tau))^T\,d\tau \\
&= \int_{t_0}^{t_1} \Phi(t_0,t_0+\varepsilon)\,\Phi(t_0+\varepsilon,\tau)\,B(\tau)\,B^T(\tau)\,\Phi^T(t_0+\varepsilon,\tau)\,\Phi^T(t_0,t_0+\varepsilon)\,d\tau
\end{aligned}
$$

Now $\Phi(t_0,t_0+\varepsilon)$ and $\Phi^T(t_0,t_0+\varepsilon)$ do not depend on τ and can be removed outside the integral

$$W(t_0,t) = \Phi(t_0,t_0+\varepsilon)\left(\int_{t_0}^{t_1} \Phi(t_0+\varepsilon,\tau)\,B(\tau)\,B^T(\tau)\,\Phi^T(t_0+\varepsilon,\tau)\,d\tau\right)\Phi^T(t_0,t_0+\varepsilon)$$

The integral inside the controllability Gramian $W(t_0+\varepsilon,t_1)$, hence

$$W(t_0,t) = \Phi(t_0,t_0+\varepsilon)\,W(t_0+\varepsilon,t)\,\Phi^T(t_0,t_0+\varepsilon)$$

Therefore

$$W(t_0+\varepsilon,t) = \Phi^{-1}(t_0,t_0+\varepsilon)\,W(t_0,t)\,\Phi^{-T}(t_0,t_0+\varepsilon)$$

Since $W(t_0,t)$ is nonsingular, and since $\Phi(t_0,t_0+\varepsilon)$ is also nonsingular, then $W(t_0+\varepsilon,t)$ is also nonsingular for any ε. Therefore the system is controllable at any time after t_0 no matter how small ε is.

2.6.6 Problem 5 Control effort

Future state is given by

$$x(t_1) = e^{A(t_1)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau \tag{1}$$

Let M be the controllability matrix, which we know is nonsingular since the system is controllable. The following $u(t)$ will bring the system from $x(t_0)$ to $x(t_1)$

$$u(t) = -B^T\left(e^{A(t_1-t)}\right)^T M^{-1}\left(e^{A(t_1)}x(t_0) - x(t_1)\right)$$

Substituting this in (1) shows that this is the case

$$
\begin{aligned}
e^{A(t_1)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau &= e^{A(t_1)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}B\left[-B^T\left(e^{A(t_1-t)}\right)^T M^{-1}\left(e^{A(t_1)}x(t_0)-x(t_1)\right)\right]d\tau \\
&= e^{A(t_1)}x(t_0) - \overbrace{\left(\int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^T\left(e^{A(t_1-t)}\right)^T d\tau\right)}^{\text{for LTI = } M \text{ matrix}} M^{-1}\left(e^{A(t_1)}x(t_0)-x(t_1)\right) \\
&= e^{A(t_1)}x(t_0) - MM^{-1}\left(e^{A(t_1)}x(t_0)-x(t_1)\right) \\
&= e^{A(t_1)}x(t_0) - e^{A(t_1)}x(t_0) + x(t_1) \\
&= x(t_1)
\end{aligned}
$$

Hence we know that $u(t) = -B^T\left(e^{A(t_1-t)}\right)^T M^{-1}\left(e^{A(t_1)}x(t_0)-x(t_1)\right)$ will steer the system from $x(t_0)$ to $x(t_1)$. Now if we set $x(t_1)=0$ as the goal state, then $u(t)$ simplifies to

$$u(t) = -B^T\left(e^{A(t_1-t)}\right)^T M^{-1}e^{A(t_1)}x(t_0)$$

This control will steer the system from state $x(t_0)$ to state 0. Now we need to show that $\|u(t)\| \leq \beta$ for any given $\beta > 0$. In the above B and $x(t_0)$ are fixed and given and do not change with time. The same for M. This is because this is an LTI system. The only effect on the norm of $u(t)$ comes from $e^{A(t_1-t)}$ matrix, since this is the only quantity in the above that changes with time. Therefore, to reduce the norm of $u(t)$ is means we can change t where $u(t)$ is applied such that the resulting $e^{A(t_1-t)}$ is such that $\|u(t)\| \leq \beta$. We might have to make (t_1-t) very small, but we can always do that in order to cause $\|u(t)\| \leq \beta$.

2.6.7 Problem 6 Range

Need to proof the following: $x(t_0)$ can be steered to $x(t_1)=0$ iff $x(t_0)$ is in range of $W(t_0,t_1)$.

Proof: The above is equivalent to proofing this: $x(t_0)$ can be steered to $x(t_1) = 0$ iff $W(t_0,t_1)v = x(t_0)$ for $\vec{v} \neq 0$. But the ability to steer from $x(t_0)$ to $x(t_1) = 0$ is the same as saying the system is controllable at t_0. Therefore, what we want to proof is the following

$$\boxed{\text{The system is controllable at } t_0 \text{ iff } W(t_0,t_1)v = x(t_0) \text{ for } \vec{v} \neq 0}$$

Since if the system is controllable, then by definition, we can find control $u(t)$ to steer $x(t_0)$ to $x(t_1) = 0$. Now we will start by proofing the above.

Necessity: $\Longrightarrow$ If The system is controllable at t_0 then $W(t_0,t_1)v = x(t_0)$ for $\vec{v} \neq 0$

sufficient: $\Longleftarrow$ If $W(t_0,t_1)v = x(t_0)$ for $\vec{v} \neq 0$ then the system is controllable at t_0.

Proof of Necessity: Since the system is controllable at t_0 then we can find $u(t)$ such that

$$x(t_1) = 0 = \Phi(t_1,t_0)x(t_0) + \int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau$$

Premultiply both sides by $\Phi(t_0,t_1)$ then

$$
\begin{aligned}
0 &= \overbrace{\Phi(t_0,t_1)\Phi(t_1,t_0)}^{I}x(t_0) + \int_{t_0}^{t_1}\overbrace{\Phi(t_0,t_1)\Phi(t_1,\tau)}^{\Phi(t_0,\tau)}B(\tau)u(\tau)\,d\tau \\
0 &= x(t_0) + \int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau \\
-x(t_0) &= \int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau \qquad (1)
\end{aligned}
$$

Let the control be $u(\tau) = -B^T(\tau)\Phi^T(t_0,\tau)\vec{v}(t)$ for some none zero $\vec{v}(t)$. Since $B^T(t)$ has size $m\times n$ and $\Phi^T(t_0,t)$ has size $n\times n$ then $\vec{v}(t)$ will have size $m\times 1$. Substituting this control law into (1) gives

$$-x(t_0) = \left(-\int_{t_0}^{t_1} \Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau)\, d\tau \right) \vec{v}(t)$$

where we moved v outside the integral since it does not depend on t. But

$$\int_{t_0}^{t_1} \Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau)\, d\tau = W(t_0,t_1)$$

Hence the above becomes

$$W(t_0,t_1)\vec{v}(t) = \vec{x}(t_0)$$

Therefore $x(t_0)$ is in the range of $W(t_0,t_1)$.

Proof of sufficient: $\Longleftarrow$ If $W(t_0,t_1)v = x(t_0)$ for $\vec{v} \neq 0$ then the system is controllable at t_0.

Since $W(t_0,t_1)\vec{v}(t) = x(t_0)$ then

$$\begin{aligned} x(t_0) &= W(t_0,t_1)\vec{v}(t) \\ &= \left(\int_{t_0}^{t_1} \Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau)\, d\tau \right) \\ &= \int_{t_0}^{t_1} \Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau) \vec{v}(t)\, d\tau \end{aligned}$$

Premultiply both sides by $\Phi(t_1,t)$

$$\Phi(t_1,t)x(t_0) = \int_{t_0}^{t_1} \Phi(t_1,t)\Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau) \vec{v}(t)\, d\tau$$

$$0 = -\Phi(t_1,t)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau) \vec{v}(t)\, d\tau$$

Let $B^T(\tau)\Phi^T(t_0,\tau)\vec{v}(t) = -u(t)$, then the above can be written as

$$0 = -\Phi(t_1,t)x(t_0) - \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) u(t)\, d\tau$$

Since $x(t_1) = 0$ then the LHS above is $x(t_1)$ then

$$x(t_1) = \Phi(t_1,t)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) u(t)\, d\tau$$

But the above means $x(t_0)$ is steered to $x(t_1) = 0$. This completes the proof.

2.6.8 key solution

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Solution Controllability

Two possible solutions: With

$$A(t) = \begin{bmatrix} 2 & -e^t \\ e^{-t} & 1 \end{bmatrix} ; \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Sigma'$$

One can use $\phi(t,\tau)$ from Homework Transition and use the W-test. Alternatively, we form

$$M_0(t) = B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad M_1(t) = -A(t)M_0(t) + \frac{d}{dt} M_0(t)$$

$$= \begin{bmatrix} e^t \\ -1 \end{bmatrix}$$

Hence $\mathcal{M}(t) = \begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix} = \begin{pmatrix} 0 & e^t \\ 1 & -1 \end{pmatrix}$

Since $\det \mathcal{M}(t) = -e^t \neq 0$, we have rank $\mathcal{M}(t) = 2$. for all $t \geq 0$. Hence Σ is controllable at t_0.

Solution P-Transformation

Find state equations for $z(t)$:

$$\dot{z}(t) = \frac{d}{dt} P(t)\,x(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t)$$

$$= \dot{P}(t)x(t) + P(t)\left[A(t)x(t) + B(t)u(t)\right]$$

$$= \left[\dot{P}(t) + P(t)A(t)\right]x(t) + \underbrace{P(t)B(t)}_{\tilde{B}(t)}u(t)$$

$$\underbrace{\left[\dot{P}(t) + P(t)A(t)\right]P^{-1}(t)}_{\tilde{A}(t)}z(t) + \underbrace{P(t)B(t)}_{\tilde{B}(t)}u(t)$$

Next, we propose $\Psi_{\tilde{\Sigma}}(t) = P(t)\,\Psi_{\Sigma}(t)$: Must show

① $\Psi_{\tilde{\Sigma}}(0)$ has lin. ind columns

② $\dot{\Psi}_{\tilde{\Sigma}}(t) = \tilde{A}(t)\Psi_{\tilde{\Sigma}}(t)$; $\hat{A}(t)$ as above

① : To prove ①, notice $\Psi_{\tilde{\Sigma}}(0) = P(0)\Psi_{\Sigma}(0)$. Since $P(t)$ is non-singular and columns of $\Psi_{\Sigma}(0)$ are lin. indep.

$\Rightarrow$ $P(0)\Psi_{\Sigma}(0)$ is non-singular; i.e, $\Psi_{\tilde{\Sigma}}(0)$ has lin. indep. columns

Next, we look at

$$\dot{\Psi}_{\tilde{\Sigma}}(t) = \frac{d}{dt} P(t)\Psi_{\Sigma}(t) = \dot{P}(t)\Psi_{\Sigma}(t) + P(t)\dot{\Psi}_{\Sigma}(t)$$

$$\underset{\overset{\uparrow}{\dot{\Psi}_\Sigma = A\Psi_\Sigma}}{=} \dot{P}(t)\Psi_\Sigma(t) + P(t)A(t)\Psi_\Sigma(t)$$

$$= [\dot{P}(t) + P(t)A(t)]\Psi_\Sigma(t)$$

$$= [\dot{P}(t) + P(t)A(t)]P^{-1}(t)\Psi_{\tilde\Sigma}(t)$$

$$= \tilde{A}(t)\Psi_{\tilde\Sigma}(t)$$

$\Rightarrow \Psi_{\tilde\Sigma}(t)$ is an FMS for $\tilde\Sigma$

$$\Rightarrow \Phi_{\tilde\Sigma}(t,\tau) = \Psi_{\tilde\Sigma}(t)\Psi_{\tilde\Sigma}^{-1}(\tau) = [P(t)\Psi_\Sigma(t)][P(\tau)\Psi_\Sigma(\tau)]^{-1}$$

$$= P(t)\Psi_\Sigma(t)\Psi_\Sigma^{-1}(\tau)P^{-1}(\tau)$$

$$\Rightarrow \Phi_{\tilde\Sigma}(t,\tau) = P(t)\Phi_\Sigma(t,\tau)P^{-1}(\tau)$$

Let $W_\Sigma(t_0,t_1)$ and $W_{\tilde\Sigma}(t_0,t_1)$ denote controllability matrices for Σ and $\tilde\Sigma$. From above, we know that $\Phi_{\tilde\Sigma}(t,\tau) = P(t)\Phi_\Sigma(t,\tau)P^{-1}(\tau)$.

$$\Rightarrow W_{\tilde\Sigma}(t_0,t_1) = \int_{t_0}^{t_1} \Phi_{\tilde\Sigma}(t_0,\tau)\tilde{B}(\tau)\tilde{B}'(\tau)\Phi'_{\tilde\Sigma}(t_0,\tau)\,d\tau$$

$$= \int_{t_0}^{t_1} P(t_0)\Phi_\Sigma(t_0,\tau)P^{-1}(\tau)P(\tau)B(\tau)B'(\tau)P^{-1}(\tau)[P'(\tau)]^{-1}\Phi'_\Sigma(t_0,\tau)P'(t_0)\,d\tau$$

$= \int_{t_0}^{t} P(t_0)\,\Phi_{\Sigma}(t_0,\tau)\,B(\tau)\,B'(\tau)\,P'(t_0)\,d\tau$

$= P(t_0)\,W_{\Sigma}(t_0,t_1)\,P'(t_0)$. Since $P(t_0)$ is nonsingular it follows that $W_{\Sigma}(t_0,t_1)$ is nonsingular if and only if $W_{\tilde{\Sigma}}(t_0,t_1)$ is nonsingular. Hence Σ is controllable at t_0 if and only if $\tilde{\Sigma}$ is controllable at t_0.

Solution Solve

(a) Differentiate $\Psi(t)$; we require $Y_{01} + Y_{02} + Y_{03} = I$
$\lambda_1 Y_{01} + \lambda_2 Y_{02} + \lambda_3 Y_{03} = A$; $\lambda_1^2 Y_{01} + \lambda_2^2 Y_{02} + \lambda_3^2 Y_{03} = A^2$ ✭

(Now verify ✭) $Y_{01} = \dfrac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$; $Y_{02} = \dfrac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$

$Y_{03} = \dfrac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$. Next compute

$+(\lambda_1-\lambda_2)(A-\lambda_1 I)(A-\lambda_2 I)$

$$Y_{01} + Y_{02} + Y_{03} = \frac{(\lambda_2-\lambda_3)(A-\lambda_2 I)(A-\lambda_3 I) - (\lambda_1-\lambda_3)(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}$$

$$= \frac{(\lambda_2-\lambda_3)\left[A^2-(\lambda_2+\lambda_3)A+\lambda_2\lambda_3 I\right] - (\lambda_1-\lambda_3)\left[A^2-(\lambda_1+\lambda_3)A+\lambda_1\lambda_3 I\right] + (\lambda_1-\lambda_2)\left[A^2-(\lambda_1+\lambda_2)A+\lambda_1\lambda_2 I\right]}{(\text{denom.})}$$

$$= \frac{(\cancel{\lambda_2}-\lambda_3-\lambda_1+\lambda_3+\lambda_1-\cancel{\lambda_2})A^2 + \left[\lambda_3^2-\lambda_2^2+\lambda_1^2-\lambda_3^2+\lambda_2^2-\lambda_1^2\right]A + (\lambda_2-\lambda_3)\lambda_2\lambda_3 I - (\lambda_1-\lambda_3)\lambda_1\lambda_3 I + (\lambda_1-\lambda_2)\lambda_1\lambda_2 I}{\text{denom.}}$$

$$= \frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)I}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} ✓ = I$$

Next

$+\lambda_3(A-\lambda_1 I)(A-\lambda_3 I)$

$$\lambda_1 Y_{01} + \lambda_2 Y_{02} + \lambda_3 Y_{03} = \frac{\lambda_1(\lambda_2-\lambda_3)(A-\lambda_2 I)(A-\lambda_3 I) - \lambda_2(\lambda_1-\lambda_3)(A-\lambda_1 I)(A-\lambda}{\text{denom}} \overset{\text{arith}}{=} A \quad (\text{long arith.})$$

Similarly, a lengthy calculation leads to

$$\lambda_1^2 Y_{01} + \lambda_2^2 Y_{02} + \lambda_3^2 Y_{03} = A^2$$

(b) Now with $A = -\frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$; we have eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ and the formula gives $Y_{01} = \frac{A - \lambda_2 I}{\lambda_2 - \lambda_1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$Y_{02} = \frac{A - \lambda_1 I}{\lambda_1 - \lambda_2} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence we obtain

$$e^{At} = Y_{01} e^{\lambda_1 t} + Y_{02} e^{\lambda_2 t}$$

$$= \begin{bmatrix} \frac{1}{2} e^{-2t} + \frac{1}{2} e^{t} & -\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-t} \\ -\frac{1}{2} e^{-2t} + \frac{1}{2} e^{t} & \frac{1}{2} e^{-2t} + \frac{1}{2} e^{-t} \end{bmatrix}$$

Solution Circuit

$$A(t) = \begin{bmatrix} -\frac{1}{R(t)} & -1 \\ 1 & 0 \end{bmatrix} \; ; \quad b(t) = \begin{bmatrix} \frac{1}{R(t)} \\ 0 \end{bmatrix}$$

(a)

$$M_0(t) = b(t) = \begin{bmatrix} \frac{1}{R(t)} \\ 0 \end{bmatrix} \; ; \; M_1(t) = -A(t)\,M_0(t) + \frac{d}{dt} M_0(t)$$

$$= \begin{bmatrix} +\frac{1}{R^2(t)} \\ -\frac{1}{R(t)} \end{bmatrix} + \begin{bmatrix} -\frac{1}{R^2(t)}\frac{dR}{dt} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R^2(t)}\left[1 - \frac{dR}{dt}\right] \\ -\frac{1}{R(t)} \end{bmatrix}$$

$$\Rightarrow \mathcal{M}(t) = \begin{bmatrix} \frac{1}{R(t)} & \frac{1}{R^2}\left[1 - \frac{dR}{dt}\right] \\ 0 & -\frac{1}{R(t)} \end{bmatrix}$$

So with $R(t) \neq 0$, $\det \mathcal{M}(t^*) \neq 0$ for any $t^* > t_0$

$\Rightarrow$ Controllable

(b) On any $[t_0, t_0+\varepsilon]$, we have $\det \mathcal{M}(t^*) \neq 0$ for any $t^* \in [t_0, t_0+\varepsilon]$. Hence, we have differential controllability

Solution Control Effort

Controller has form

$$u(\tau) = \begin{cases} 0 & \text{for } t \leq T \\ -B'\Phi'(T,\tau)W^{-1}(T,T+1)\Phi(T,t_0)x_0 & \text{for } T \leq t \leq T+1 \end{cases}$$

where W is the controllability Grammian and $T > 0$ needs to be specified. Note that $u(\tau)$ takes $x(t)$ to zero at time $T+1$; i.e, with $x(T) = \Phi(T, t_0)x_0$, the control above coincides with the one in the proof of the Controllability Theorem. Hence, it remains to show that T can be selected such that $\|u(\tau)\| \leq \beta$ for $\tau \in [T, T+1]$.

To obtain T: Let

$$\xi = \max_{\tau \in [0,1]} \| B'\Phi'(0,\tau)W^{-1}(0,1)\|$$

and pick T such that $\|x(T)\| < \frac{\beta}{\|\xi\|}$.
(We know T exists because $x(t) \to 0$ with $u(\tau) \equiv 0$; i.e., have A stable). Now, to see that our construction works, begin with state $x_0 = x(T) = \phi(T,t_0)x_0$ and consider the norm bound

$$\|u(\tau)\| = \|-B'\phi'(T,\tau)W^{-1}(T,T+1)\phi(T,t_0)x_0\|$$
$$\leq \|B'\phi'(T,\tau)W^{-1}(T,T+1)\|\ \|x(T)\|$$

Now using time invariance study $u(\rho)$ for $\rho \in [0,1]$ instead of $u(\tau)$ for $\tau \in [T,T+1]$. We get

$$\|u(\rho)\| \leq \underbrace{\|B'\phi'(0,\tau)W^{-1}(0,1)\|}_{\leq \xi} \beta/\xi$$
$$< \beta$$

Solution Range

(Sufficieny) Suppose $x_0 \in \mathcal{R}$ and let $x_1 = 0$. Must show we can steer to the origin. To prove this, first pick $\bar{x} \in \mathbb{R}^n$ such that $x_0 = W(t_0,t_1)\bar{x}$. Now, let

$$\tilde{u}(\tau) = -B^T(\tau)\phi^T(t_0,\tau)\bar{x}$$

. With this control,

$$x(t_1) = \phi(t_1,t_0)x_0 + \int_{t_0}^{t_1}\phi(t_1,\tau)B(\tau)\tilde{u}(\tau)\,d\tau$$

$$= \phi(t_1,t_0)W(t_0,t_1)\bar{x} - \int_{t_0}^{t_1}\phi(t_1,\tau)B(\tau)B^T(\tau)\phi^T(t_0,\tau)\bar{x}\,d\tau$$

$$= \phi(t_1,t_0)\left[W(t_0,t_1)\bar{x} - \underbrace{\int_{t_0}^{t_1}\phi(t_0,\tau)B(\tau)B^T(\tau)\phi^T(t_0,\tau)\,d\tau}_{W(t_0,t_1)}\,\bar{x}\right]$$

$$= 0$$

(Necessity) Suppose x_0 can be steered to zero (with control $\tilde{u}(\cdot)$) at time t_1. We claim $x_0 \in \mathcal{R}[W(t_0,t_1)]$ Proceeding by contradiction, suppose $x_0 \notin \mathcal{R}[W(t_0,t_1)]$ Then there is a vector $\eta \neq 0$ such that $\eta^T x_0 \neq 0$ and $W^T(t_1,t_0)\eta = 0$

2.7 HW7

2.7.1 Questions

Problem 1: Consider a unity feedback control system with interval plant

$$P(s,\rho)=\frac{(3+\rho[0,1])s+(1+\rho[-0.5,0.25])}{s^4+(3+\rho[-1,1])s^3+(6+\rho[-1,1])s^2+(3+\rho[-1,0])s+(4+\rho[-0.5,0.75])}$$

(a) Verify that the system is stable when $\rho = 0$.

(b) Using Kharitonov's Theorem, find the largest $\rho > 0$ such that the closed loop system is still stable; call this value $\rho = \rho_{max}$, the robustness margin.

(c) Now instead of unity feedback, a compensator $H(s)=\frac{1}{s}$ is used. Determine if the resulting family of polynomials is stable for $\rho = 0.5\rho_{max}$ with ρ_{max} from Part (b).

Problem 2: A plant with transfer function

$$G(s) = K\frac{3s+1}{s^4+s^3+as+b}$$

and uncertain parameters $12 \leq a \leq 36$ and $1 \leq b \leq 2$ is connected in a classical unity feedback configuration. Find the largest value of the gain $K \geq 0$, call it K_{max}, under which robust stability of the closed loop is guaranteed. Note: Since there are only two uncertain parameters, consider solving this problem via direct argument using the Routh-Hurwitz criterion instead of Kharitonov's Theorem.

Problem 3: The interval plant

$$G(s) = \frac{s+1+\rho[-1,1]}{s^2+(2+\rho[-1,1])s+3}$$

with variable radius of uncertainty $\rho \geq 0$ is compensated with controller

$$H(s) = \frac{s+1}{s-1}$$

in a classical unity feedback configuration. Find the largest value of ρ, call it ρ_{max} under which robust stability of the closed loop is guaranteed. We call ρ_{max} the robustness margin.

Problem 4: Suppose $p_0(s)$ and $p_1(s)$ are stable polynomials with positive coefficients. For $0 \leq \lambda \leq 1$ define

$$p_\lambda = (1-\lambda)p_0(s) + \lambda p_1(s).$$

Given that $p_\lambda(s)$ is stable at the endpoints $\lambda = 0$ and $\lambda = 1$, an engineer claims that $p_\lambda(s)$ will be stable for all intermediate values of λ as well. Do you agree with the engineer? Explain by providing a technical argument supporting the engineer's point of view or generate a counterexample showing that the claim can be false.

ECE 717 – Homework P Matrix

(a) For the linear time-invariant system with

$$A = \begin{bmatrix} -2 & 4 & -3 & 1 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

using $Q = I$, find the solution of the associated Lyapunov equation by hand and give the associated V function in expanded form.

(b) Verify the solution obtained in (a) via Matlab.

;

ECE 717 – Homework Matrix Norm

(a) Find the matrix norm of

$$A = \begin{bmatrix} 0.95 & 0.48 & 0.45 \\ 0.23 & 0.89 & 0.01 \\ 0.60 & 0.76 & 0.82 \end{bmatrix},$$

by hand.

(b) Verify the solution obtained in (a) via Matlab.

;

ECE 717 – Homework Robustness Bound

For the linear time-invariant system described by

$$\dot{x} = (A + \Delta A)x$$

with

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 7 \\ 0 & -2 & -1 & 5 & -3 \\ 0 & 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix},$$

determine a robustness bound on $||\Delta A||$ under which this system is stable.

;

2.7.2 problem 1

Solution:

The plant is

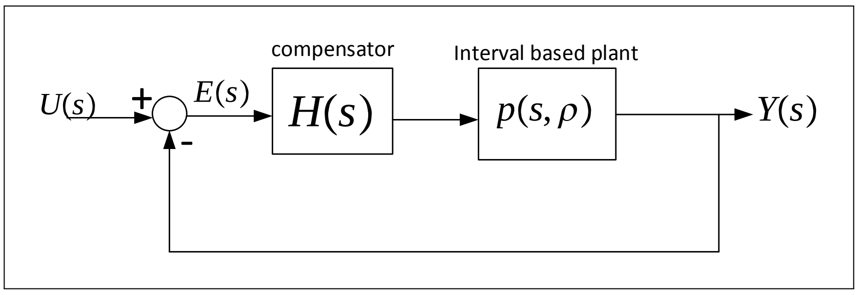
$$P(s,\rho) = \frac{\left(3+\rho\,[0,1]\right)s + \left(1+\rho\,[-0.5,0.25]\right)}{s^4 + \left(3+\rho\,[-1,1]\right)s^3 + \left(6+\rho\,[-1,1]\right)s^2 + \left(3+\rho\,[-1,0]\right)s + \left(4+\rho\,[-0.5,0.75]\right)}$$

Part (a)

When $\rho = 0$

$$\boxed{P(s) = \frac{3s+1}{s^4+3s^3+6s^2+3s+4}}$$

First we find the closed loop transfer function. Using the following diagram



$$E = U - Y$$
$$Y = EHP$$

Replacing E in second equation with E from the first equation

$$\begin{aligned} Y &= (U-Y)HP \\ &= UHP - YHP \end{aligned}$$

Hence

$$Y(1+HP) = UHP$$

The closed loop transfer function is $\frac{Y}{U}$. From the above we obtain

$$\frac{Y}{U} = G_{cl}(s) = \frac{HP}{1+HP}$$

For unity feedback, $H = 1$, the above reduces to

$$\boxed{G_{cl}(s) = \frac{P(s)}{1+P(s)}}$$

This is stable if poles of $G_{cl}(s)$ are stable. This is the same as saying the zeros of denominator of $G_{cl}(s)$ all have negative real parts. Writing $P(s) = \frac{N(s)}{D(s)}$ then

$$G_{cl}(s) = \frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} = \frac{N(s)}{D(s)+N(s)} = \frac{3s+1}{s^4+3s^3+6s^2+3s+4+3s+1} = \frac{3s+1}{s^4+3s^3+6s^2+6s+5}$$

We now need to check stability of the denominator of $G_{cl}(s)$ given by $s^4+3s^3+6s^2+6s+5$. Using Hurwitz matrix where $s^4+3s^3+6s^2+6s+5 \equiv a_4s^4+a_3s^3+a_2s^2+a_1s+a_0$ gives

$$H = \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 5 & 6 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 5 & 6 & 1 \end{bmatrix}$$

Hence $\Delta_1 = 6, \Delta_2 = 21, \Delta_3 = 27, \Delta_4 = 27$. Since all $\Delta_i > 0$ then the

$$\boxed{\text{denominator polynomial of } G_{cl}(s) \text{ is stable}}$$

Hence closed loop system is stable. To verify, using the computer, the roots of $s^4+3s^3+6s^2+6s+5$ are $\{-0.296974, -0.296974, -1.20303, -1.20303\}$. Since they are all negative, this verifies system is stable as well.

Part(b)

Still using the unity compensator but now using the interval plant gives

$$G_{cl}(s)=\frac{P(s)}{1+P(s)}=\frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}}=\frac{N(s)}{D(s)+N(s)}$$

$$=\frac{\left(3+\rho[0,1]\right)s+\left(1+\rho[-0.5,0.25]\right)}{s^4+\left(3+\rho[-1,1]\right)s^3+\left(6+\rho[-1,1]\right)s^2+\left(3+\rho[-1,0]\right)s+\left(4+\rho[-0.5,0.75]\right)+\left(3+\rho[0,1]\right)s+\left(1+\rho[-0.5,0.25]\right)}$$

The denominator polynomial from above is

$$\Delta(s)=s^4+\left(3+\rho[-1,1]\right)s^3+\left(6+\rho[-1,1]\right)s^2+\left(6+\rho[-1,0]+\rho[0,1]\right)s+\left(5+\rho[-0.5,0.75]+\rho[-0.5,0.25]\right)\tag{1}$$

But[3]

$$\left(6+\rho[-1,0]+\rho[0,1]\right)s=\left(6+\rho[-1,1]\right)s$$

And

$$\left(5+\rho[-0.5,0.75]+\rho[-0.5,0.25]\right)=\left(5+\rho[-1,1]\right)$$

Therefore (1) becomes

$$\boxed{\Delta(s)=s^4+\left(3+\rho[-1,1]\right)s^3+\left(6+\rho[-1,1]\right)s^2+\left(6+\rho[-1,1]\right)s+\left(5+\rho[-1,1]\right)}$$

The above is the polynomial to examine for finding the maximum ρ. Notice when $\rho=0$ we obtain $s^4+3s^3+6s^2+6s+5$ as in part(a) which is stable. Note that if the nominal polynomial is not stable, then there will be no point in checking for robust stability. The four Kharitonov polynomials are from the above are

$$\begin{aligned}K_1&=\left(5-\rho\right)+\left(6-\rho\right)s+\left(6+\rho\right)s^2+\left(3+\rho\right)s^3+s^4\\K_2&=\left(5+\rho\right)+\left(6+\rho\right)s+\left(6-\rho\right)s^2+\left(3-\rho\right)s^3+s^4\\K_3&=\left(5+\rho\right)+\left(6-\rho\right)s+\left(6-\rho\right)s^2+\left(3+\rho\right)s^3+s^4\\K_4&=\left(5-\rho\right)+\left(6+\rho\right)s+\left(6+\rho\right)s^2+\left(3-\rho\right)s^3+s^4\end{aligned}$$

We want to find the maximum ρ such that the four polynomials above are still stable. We setup the Hurwitz matrix for each and determine the condition on ρ needed. For K_1

$$\begin{bmatrix}a_1&a_3&0&0\\a_0&a_2&a_4&0\\0&a_1&a_3&0\\0&a_0&a_2&a_4\end{bmatrix}=\begin{bmatrix}\left(6-\rho\right)&\left(3+\rho\right)&0&0\\\left(5-\rho\right)&\left(6+\rho\right)&1&0\\0&\left(6-\rho\right)&\left(3+\rho\right)&0\\0&\left(5-\rho\right)&\left(6+\rho\right)&1\end{bmatrix}$$

Hence $\Delta_1=\left(6-\rho\right)>0$ which means $\rho<6$. And $\Delta_2=21-2\rho>0$ hence $\rho<10.5$ and $\Delta_3=27+27\rho-3\rho^2>0$. Hence $-0.9083<\rho<9.908$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_1 we find the following

$$\left\{\rho<6,\rho<10.5,-0.9083<\rho<9.908\right\}$$

For K_2

$$\begin{bmatrix}\left(6+\rho\right)&\left(3-\rho\right)&0&0\\\left(5+\rho\right)&\left(6-\rho\right)&1&0\\0&\left(6+\rho\right)&\left(3-\rho\right)&0\\0&\left(5+\rho\right)&\left(6-\rho\right)&1\end{bmatrix}$$

Hence $\Delta_1=\left(6+\rho\right)>0$ which means $\rho>-6$. And $\Delta_2=21+2\rho>0$ hence $\rho>-10.5$ and $\Delta_3=27-27\rho-3\rho^2>0$. Hence $-9.908<\rho<0.908$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_2 we find the following

$$\left\{\rho>-6,\rho>-10.5,-9.908<\rho<0.908\right\}$$

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[3] Using properties of interval arithmetic $[a,b]+[c,d]=[a+c,b+d]$ and $[a,b]-[c,d]=[a-d,b-c]$

For K_3

$$\begin{bmatrix} \left(6-\rho\right) & \left(3+\rho\right) & 0 & 0 \\ \left(5+\rho\right) & \left(6-\rho\right) & 1 & 0 \\ 0 & \left(6-\rho\right) & \left(3+\rho\right) & 0 \\ 0 & \left(5+\rho\right) & \left(6-\rho\right) & 1 \end{bmatrix}$$

Hence $\Delta_1 = \left(6-\rho\right) > 0$ which means $\rho < 6$. And $\Delta_2 = 21 - 20\rho > 0$ hence $\rho < 1.05$ and $\Delta_3 = 27 - 27\rho - 21\rho^2 > 0$. Hence $-1.936 < \rho < 0.66059$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_3 we find the following

$$\left\{\rho < 6, \rho < 1.05, -1.936 < \rho < 0.66059\right\}$$

For K_4

$$\begin{bmatrix} \left(6+\rho\right) & \left(3-\rho\right) & 0 & 0 \\ \left(5-\rho\right) & \left(6+\rho\right) & 1 & 0 \\ 0 & \left(6+\rho\right) & \left(3-\rho\right) & 0 \\ 0 & \left(5-\rho\right) & \left(6+\rho\right) & 1 \end{bmatrix}$$

Hence $\Delta_1 = \left(6+\rho\right) > 0$ which means $\rho > -6$. And $\Delta_2 = 21 + 20\rho > 0$ hence $\rho > -1.05$ and $\Delta_3 = 27 + 27\rho - 21\rho^2 > 0$. Hence $-0.66059 < \rho < 1.956$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_4 we find the following

$$\left\{\rho > -6., \rho > -1.05, -0.66059 < \rho < 1.956\right\}$$

We now have found all the range for ρ from each polynomial. We put them together in order to determine the largest ρ allowed

$$\begin{aligned} K_1 &\Rightarrow \left\{\rho < 6, \rho < 10.5, -0.9083 < \rho < 9.908\right\} \\ K_2 &\Rightarrow \left\{\rho > -6, \rho > -10.5, -9.908 < \rho < 0.908\right\} \\ K_3 &\Rightarrow \left\{\rho < 6, \rho < 1.05, -1.936 < \rho < 0.66059\right\} \\ K_4 &\Rightarrow \left\{\rho > -6., \rho > -1.05, -0.66059 < \rho < 1.956\right\} \end{aligned}$$

We see that the largest allowed positive ρ is

$$\boxed{\rho_{\max} = 0.66}$$

Part(c)

Using

$$\rho = \frac{1}{2}\rho_{\max} = \frac{1}{2}\left(0.908\right) = 0.454$$

The plant becomes

$$\begin{aligned} p\left(s\right) &= \frac{\left(3 + 0.454\left[0,1\right]\right)s + \left(1 + 0.454\left[-0.5,0.25\right]\right)}{s^4 + \left(3 + 0.454\left[-1,1\right]\right)s^3 + \left(6 + 0.454\left[-1,1\right]\right)s^2 + \left(3 + 0.454\left[-1,0\right]\right)s + \left(4 + 0.454\left[-0.5,0.75\right]\right)} \\ &= \frac{\left(3 + \left[0,0.454\right]\right)s + \left(1 + \left[-0.5\left(0.454\right), 0.25\left(0.454\right)\right]\right)}{s^4 + \left(3 + \left[-0.454,0.454\right]\right)s^3 + \left(6 + \left[-0.454,0.454\right]\right)s^2 + \left(3 + \left[-0.454,0\right]\right)s + \left(4 + \left[-0.5\left(0.454\right), 0.75\left(0.454\right)\right]\right)} \\ &= \frac{\left(3 + \left[0,0.454\right]\right)s + \left(1 + \left[-0.227,0.1135\right]\right)}{s^4 + \left(3 + \left[-0.454,0.454\right]\right)s^3 + \left(6 + \left[-0.454,0.454\right]\right)s^2 + \left(3 + \left[-0.454,0\right]\right)s + \left(4 + \left[-0.227,0.3405\right]\right)} \end{aligned}$$

But $a + \left[b,c\right] = \left[a+b, a+c\right]$, hence we can simplify the above to

$$p\left(s\right) = \frac{\left[3,3.454\right]s + \left[0.773,1.1135\right]}{s^4 + \left[2.546,3.454\right]s^3 + \left[5.546,6.454\right]s^2 + \left[2.546,3\right]s + \left[3.773,4.3405\right]}$$

Now the compensator is no longer unity but $H(s)=\frac{1}{s}$. Hence the closed loop transfer function is

$$
\begin{aligned}
G_{cl}(s) &= \frac{H(s)P(s)}{1+H(s)P(s)}\\
&= \frac{p(s)H(s)}{1+\left(\frac{1}{s}\right)\left(\frac{[3,3.454]s+[0.773,1.1135]}{s^4+[2.546,3.454]s^3+[5.546,6.454]s^2+[2.546,3]s+[3.773,4.3405]}\right)}\\
&= \frac{p(s)H(s)}{1+\left(\frac{[3,3.454]s+[0.773,1.1135]}{s^5+[2.546,3.454]s^4+[5.546,6.454]s^3+[2.546,3]s^2+[3.773,4.3405]s}\right)}\\
&= \frac{[3,3.454]s+[0.773,1.1135]}{s^5+[2.546,3.454]s^4+[5.546,6.454]s^3+[2.546,3]s^2+[3.773,4.3405]s+[3,3.454]s+[0.773,1.1135]}
\end{aligned}
$$

But $[3.773,4.3405]s+[3,3.454]s=[6.773,7.7945]s$, and the above becomes

$$
G_{cl}(s)=\frac{[3,3.454]s+[0.773,1.1135]}{s^5+[2.546,3.454]s^4+[5.546,6.454]s^3+[2.546,3]s^2+[6.773,7.7945]s+[0.773,1.1135]}
$$

The system is stable if the zeros of the denominator of $G_{cl}(s)$ are stable. The interval polynomial to check for robust stability is

$$
\boxed{s^5+[2.546,3.454]s^4+[5.546,6.454]s^3+[2.546,3]s^2+[6.773,7.7945]s+[0.773,1.1135]}
$$

The four Kharitonov polynomials from the above is

$$
\begin{aligned}
K_1 &= 0.773+6.773s+3s^2+6.454s^3+2.546s^4+s^5\\
K_2 &= 1.1135+7.7945s+2.546s^2+5.546s^3+3.454s^4+s^5\\
K_3 &= 1.1135+6.773s+2.546s^2+6.454s^3+3.454s^4+s^5\\
K_4 &= 0.773+7.7945s+3s^2+5.546s^3+2.546s^4+s^5
\end{aligned}
$$

Finding the real part of the roots of each polynomial gives

$$
\begin{aligned}
K_1 &= \{-1.29,-1.29,-0.119,0.0806,0.0806\}\\
K_2 &= \{-1.92,-1.92,-0.148,0.270,0.270\}\\
K_3 &= \{-1.83,-1.83,-0.171,0.186,0.186\}\\
K_4 &= \{-1.42,-1.42,-0.102,0.199,0.199\}
\end{aligned}
$$

Since some roots have positive real parts, the polynomials are $\boxed{\text{not stable}}$.

2.7.3 problem 2

$$
G(s)=k\frac{3s+1}{s^4+s^3+as^2+s+b}
$$

Where $12\leq a\leq 36, 1\leq b\leq 2$. The closed loop transfer function for unity feedback is

$$
\begin{aligned}
G_{cl}(s) &= \frac{G(s)}{1+G(s)}\\
&= \frac{k\frac{3s+1}{s^4+s^3+as^2+s+b}}{1+k\frac{3s+1}{s^4+s^3+as^2+s+b}}\\
&= \frac{k(3s+1)}{\left(s^4+s^3+as^2+s+b\right)+k(3s+1)}\\
&= \frac{k(3s+1)}{s^4+s^3+as^2+s(1+3k)+(b+k)}
\end{aligned}
$$

We need to find the largest k such that the zeros of $s^4+s^3+as^2+s(1+3k)+(b+k)$ remain stable. Writing this using uncertainties

$$
\Delta = s^4+s^3+[12,36]s^2+s(1+3k)+([1,2]+k)
$$

The four Kharitonov polynomials are

$$K_1 = (k+1) + (1+3k)s + 36s^2 + s^3 + s^4$$
$$K_2 = (k+2) + (1+3k)s + 12s^2 + s^3 + s^4$$
$$K_3 = (k+2) + (1+3k)s + 12s^2 + s^3 + s^4$$
$$K_4 = (k+1) + (1+3k)s + 36s^2 + s^3 + s^4$$

We want to find the maximum k such that the four polynomials above are stable. We setup the Hurwitz matrix for each and determine the condition on k needed. For K_1

$$\begin{bmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix} = \begin{bmatrix} (1+3k) & 1 & 0 & 0 \\ 1+k & 36 & 1 & 0 \\ 0 & (1+3k) & 1 & 0 \\ 0 & 1+k & 36 & 1 \end{bmatrix}$$

Hence $\Delta_1 = 1+3k > 0$ which means $k > \frac{-1}{3}$. And $\Delta_2 = 107k + 35 > 0$ hence $k > \frac{-35}{107} = -0.3271$ and $\Delta_3 = -9k^2 + 101k + 34 > 0$. Hence $-0.3271 < k < 11.5493$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_1 we find the following

$$\{k > -0.333, k > -0.3271, -0.3271 < k < 11.5493\}$$

Looking at K_2

$$\begin{bmatrix} (1+3k) & 1 & 0 & 0 \\ 2+k & 12 & 1 & 0 \\ 0 & (1+3k) & 1 & 0 \\ 0 & 2+k & 12 & 1 \end{bmatrix}$$

Hence $\Delta_1 = 1+3k > 0$ which means $k > \frac{-1}{3}$. And $\Delta_2 = 35k + 10 > 0$ hence $k > -0.285714$ and $\Delta_3 = -9k^2 + 29k + 9 > 0$. Hence $-0.285 < k < 3.50734$. Δ_4 is the same as Δ_3 hence no new information is obtained from it. Therefore, from K_2 we find the following

$$\{k > -0.333, k > -0.285714, -0.285 < k < 3.50734\}$$

Looking at K_3

$$\begin{bmatrix} (1+3k) & 1 & 0 & 0 \\ 2+k & 12 & 1 & 0 \\ 0 & (1+3k) & 1 & 0 \\ 0 & 2+k & 12 & 1 \end{bmatrix}$$

This is the same as K_2.Finally, looking at K_4

$$\begin{bmatrix} (1+3k) & 1 & 0 & 0 \\ 2+k & 12 & 1 & 0 \\ 0 & (1+3k) & 1 & 0 \\ 0 & 2+k & 12 & 1 \end{bmatrix}$$

This is the same as K_1.We now have found all the range for k from each polynomial. We put them together in order to determine the largest k allowed

$$K_1 \Rightarrow \{k > -0.333, k > -0.3271, -0.3271 < k < 11.5493\}$$
$$K_2 \Rightarrow \{k > -0.333, k > -0.285714, -0.285 < k < 3.50734\}$$

We see the range of positive k values for robust stability is $0 < k < 3.50734$. Therefore

$$k_{\max} = 3.50734$$

2.7.4 problem 3

$$G(s) = \frac{s + 1 + \rho[-1,1]}{s^2 + \left(2 + \rho[-1,1]\right)s + 3}$$
$$H(s) = \frac{s+1}{s-1}$$

The closed loop transfer function is

$$\begin{aligned} G_{cl}(s) &= \frac{H(s)\,G(s)}{1+H(s)\,G(s)} \\ &= \frac{H(s)\,G(s)}{1+\left(\frac{s+1}{s-1}\right)\left(\frac{s+1+\rho[-1,1]}{s^2+(2+\rho[-1,1])s+3}\right)} \end{aligned}$$

The denominator of the $G_{cl}(s)$ becomes

$$\begin{aligned} \Delta &= (s-1)\left(s^2+\left(2+\rho[-1,1]\right)s+3\right)+(s+1)\left(s+1+\rho[-1,1]\right) \\ &= \left(s^3+\left(2+\rho[-1,1]\right)s^2+3s\right)-\left(s^2+\left(2+\rho[-1,1]\right)s+3\right)+\left(s^2+s+\rho[-1,1]s\right)+\left(s+1+\rho[-1,1]\right) \\ &= s^3+\left(2+\rho[-1,1]\right)s^2+3s-s^2-\left(2+\rho[-1,1]\right)s-3+s^2+s+\rho[-1,1]s+s+1+\rho[-1,1] \\ &= s^3+\left(2+\rho[-1,1]\right)s^2+3s-\rho[-1,1]s+\rho[-1,1]s-2+\rho[-1,1] \end{aligned}$$

But $\rho[-1,1]s-\rho[-1,1]s=\rho[-2,2]s$, hence

$$\Delta = s^3+s^2\left(2+\rho[-1,1]\right)+s\left(3+\rho[-2,2]s\right)-2+\rho[-1,1]$$

We need to first check that the nominal polynomial is stable before checking for robust stability. When $\rho=0$ the denominator becomes

$$\Delta = s^3+2s^2+3s-2$$

Since there is a sign change, then the nominal polynomial is not stable. This means the closed loop is not stable. Therefore no need to do robust stability. There is no $\rho_{\max}$.

2.7.5 problem 4

$$p_\lambda = (1-\lambda)\,p_0(s)+\lambda p_1(s)$$

For $0\leq\lambda\leq 1$. We are given that

$$p_{\lambda=0}=p_0(s)$$

is stable and

$$p_{\lambda=1}=p_1(s)$$

is stable. In other-words, $p_0(s),p_1(s)$ are both stable polynomials. For any value of $0<\lambda<1$ we then have a sum of two stable polynomials, each being multiplied by a constant.

$$p_\lambda(s) = (1-\lambda)\,p_0(s)+\lambda p_1(s) \tag{1}$$

Let the zeros of p_0 be $r_i, i=1\cdots n$ where n is the order of $p_0(s)$ and let the zeros of $p_1(s)$ be $z_j, j=1\cdots m$ where m is the order of $p_1(s)$. Using the fundamental theorem of algebra, we can write

$$\begin{aligned} p_0(s) &= (s-r_1)(s-r_2)\cdots(s-r_n) \\ p_1(s) &= (s-z_1)(s-z_2)\cdots(s-z_m) \end{aligned}$$

Equation (1) becomes

$$p_\lambda(s) = (1-\lambda)(s-r_1)(s-r_2)\cdots(s-r_n)+\lambda(s-z_1)(s-z_2)\cdots(s-z_m) \tag{2}$$

Now we will proof that $p_\lambda(s)$ can only have negative zeros. Proof is by contradiction. Assume that $p_\lambda(s)$ have a positive root, say $\xi>0$, then this root when substituted in (2) will result in zero by definition

$$p_\lambda(\xi) = \overbrace{(1-\lambda)(\xi-r_1)(\xi-r_2)\cdots(\xi-r_n)}^{\Delta_1}+\overbrace{\lambda(\xi-z_1)(\xi-z_2)\cdots(\xi-z_m)}^{\Delta_2} \tag{3}$$

Each term $(\xi-r_i)$ is therefore positive quantity, since each r_i is negative since $p_0(s)$ is stable. We also have $(1-\lambda)>0$. Therefore the product shown as Δ_1 in (3) it a positive quantity.

Similarly, each term $\left(\xi-z_j\right)$ is positive quantity, since each z_j is negative since $p_1(s)$ is stable. We also have $\lambda>0$. Therefore the product shown as Δ_2 in (3) it a positive quantity.

This shows that (3) is the sum of two positive quantities. Hence the sum can not be zero. This contradicts our assumptions $p_\lambda(\xi)=0$ due to assuming $\xi>0$. Therefore zeros of $p_\lambda(s)$ can not be positive.

Similarly, we show that $\xi=0$ is also not possible root of $p_\lambda(s)$. Assume $\xi=0$ is a root, then this leads to contradiction as above, since we will have positive quantities added to zero, which is not possible.

Therefore, the only possible choice left is for all zeros of $p_{\lambda}(s)$ to be negative.

Hence $p_{\lambda}(s)$ is be stable for any $0 < \lambda < 1$. Therefore the engineer was correct. QED.

2.7.6 problem 5

The Lyapunov equation is

$$A^T P + PA = -Q$$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & -5 & 0 & 0 \\ -3 & 2 & -1 & 0 \\ 1 & -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix} \begin{bmatrix} -2 & 4 & -3 & 1 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} -4p_{11} & 4p_{11} - 7p_{12} & 2p_{12} - 3p_{11} - 3p_{13} & p_{11} - p_{12} + 2p_{13} - 6p_{14} \\ 4p_{11} - 7p_{12} & 8p_{12} - 10p_{22} & 4p_{13} - 3p_{12} + 2p_{22} - 6p_{23} & p_{12} - p_{22} + 4p_{14} + 2p_{23} - 9p_{24} \\ 2p_{12} - 3p_{11} - 3p_{13} & 4p_{13} - 3p_{12} + 2p_{22} - 6p_{23} & 4p_{23} - 6p_{13} - 2p_{33} & p_{13} - 3p_{14} - p_{23} + 2p_{24} + 2p_{33} - 5p_{34} \\ p_{11} - p_{12} + 2p_{13} - 6p_{14} & p_{12} - p_{22} + 4p_{14} + 2p_{23} - 9p_{24} & p_{13} - 3p_{14} - p_{23} + 2p_{24} + 2p_{33} - 5p_{34} & 2p_{14} - 2p_{24} + 4p_{34} - 8p_{44} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.1}$$

There are 10 unknowns. They are $p_{11}, p_{12}, p_{13}, p_{14}, p_{22}, p_{23}, p_{24}, p_{33}, p_{34}$. The 10 equations to solve are from the upper triangle above

$$-4p_{11} = -1 \tag{1}$$

$$4p_{11} - 7p_{12} = 0 \tag{2}$$

$$2p_{12} - 3p_{11} - 3p_{13} = 0 \tag{3}$$

$$p_{11} - p_{12} + 2p_{13} - 6p_{14} = 0 \tag{4}$$

$$8p_{12} - 10p_{22} = -1 \tag{5}$$

$$4p_{13} - 3p_{12} + 2p_{22} - 6p_{23} = 0 \tag{6}$$

$$p_{12} - p_{22} + 4p_{14} + 2p_{23} - 9p_{24} = 0 \tag{7}$$

$$4p_{23} - 6p_{13} - 2p_{33} = -1 \tag{8}$$

$$p_{13} - 3p_{14} - p_{23} + 2p_{24} + 2p_{33} - 5p_{34} = 0 \tag{9}$$

$$2p_{14} - 2p_{24} + 4p_{34} - 8p_{44} = -1 \tag{10}$$

From (1) $p_{11} = \frac{1}{4}$, substituting in (2) gives $p_{12} = \frac{4}{7}\left(\frac{1}{4}\right) = \frac{1}{7}$, substituting these in (3) gives $3p_{13} = 2p_{12} - 3p_{11}$ or $p_{13} = \frac{2p_{12}-3p_{11}}{3} = \frac{2\left(\frac{1}{7}\right)-3\left(\frac{1}{4}\right)}{3} = -\frac{13}{84}$. Substituting these in (4) gives $6p_{14} = p_{11} - p_{12} + 2p_{13}$ or $p_{14} = \frac{p_{11}-p_{12}+2p_{13}}{6} = \frac{\frac{1}{4}-\frac{1}{7}-2\left(\frac{13}{84}\right)}{6} = -\frac{17}{504}$.

From (5) we find $8p_{12} = 10p_{22} - 1$ hence $p_{22} = \frac{8p_{12}+1}{10} = \frac{8\left(\frac{1}{7}\right)+1}{10} = \frac{3}{14}$. From (6) $4p_{13} - 3p_{12} + 2p_{22} = 6p_{23}$. hence $p_{23} = \frac{4p_{13}-3p_{12}+2p_{22}}{6} = \frac{4\left(-\frac{13}{84}\right)-3\left(\frac{1}{7}\right)+2\left(\frac{3}{14}\right)}{6} = -\frac{13}{126}$. From (7) $p_{12} - p_{22} + 4p_{14} + 2p_{23} = 9p_{24}$, hence $p_{24} = \frac{p_{12}-p_{22}+4p_{14}+2p_{23}}{9} = \frac{\frac{1}{7}-\frac{3}{14}+4\left(-\frac{17}{504}\right)+2\left(-\frac{13}{126}\right)}{9} = -\frac{26}{567}$. And from (8), $4p_{23} - 6p_{13} - 2p_{33} = -1$, hence $p_{33} = \frac{4p_{23}-6p_{13}+1}{2} = \frac{4\left(-\frac{13}{126}\right)-6\left(-\frac{13}{84}\right)+1}{2} = \frac{191}{252}$. From (9), $p_{13} - 3p_{14} - p_{23} + 2p_{24} + 2p_{33} = 5p_{34}$, hence $p_{34} = \frac{p_{13}-3p_{14}-p_{23}+2p_{24}+2p_{33}}{5} = \frac{-\frac{13}{84}-3\left(-\frac{17}{504}\right)-\left(-\frac{13}{126}\right)+2\left(-\frac{26}{567}\right)+2\left(\frac{191}{252}\right)}{5} = \frac{191}{648}$, and finally from (10) $2p_{14} - 2p_{24} + 4p_{34} = 8p_{44} - 1$, hence $p_{44} = \frac{2p_{14}-2p_{24}+4p_{34}+1}{8} = \frac{2\left(-\frac{17}{504}\right)-2\left(-\frac{26}{567}\right)+4\left(\frac{191}{648}\right)+1}{8} = \frac{4997}{18144}$. Therefore the solution is

$$P=\begin{bmatrix}p_{11} & p_{12} & p_{13} & p_{14}\\ p_{12} & p_{22} & p_{23} & p_{24}\\ p_{13} & p_{23} & p_{33} & p_{34}\\ p_{14} & p_{24} & p_{34} & p_{44}\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & \frac{1}{7} & -\frac{13}{84} & -\frac{17}{504}\\ \frac{1}{7} & \frac{3}{14} & -\frac{13}{126} & -\frac{26}{567}\\ -\frac{13}{84} & -\frac{13}{126} & \frac{191}{252} & \frac{191}{648}\\ -\frac{17}{504} & -\frac{26}{567} & \frac{191}{648} & \frac{4997}{18144}\end{bmatrix}$$

$$=\begin{bmatrix}0.25 & 0.14286 & -0.15476 & -0.03373\\ 0.14286 & 0.21429 & -0.10317 & -4.5855\times 10^{-2}\\ -0.15476 & -0.10317 & 0.75794 & 0.29475\\ -0.03373 & -4.5855\times 10^{-2} & 0.29475 & 0.27541\end{bmatrix}$$

The associated $V(x(t))$ function is

$V(x(t))=X^TPX$

$$=\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}\begin{bmatrix}\frac{1}{4} & \frac{1}{7} & -\frac{13}{84} & -\frac{17}{504}\\ \frac{1}{7} & \frac{3}{14} & -\frac{13}{126} & -\frac{26}{567}\\ -\frac{13}{84} & -\frac{13}{126} & \frac{191}{252} & \frac{191}{648}\\ -\frac{17}{504} & -\frac{26}{567} & \frac{191}{648} & \frac{4997}{18144}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

$$=\frac{1}{4}x_1^2+\frac{2}{7}x_1x_2-\frac{13}{42}x_1x_3-\frac{17}{252}x_1x_4+\frac{3}{14}x_2^2-\frac{13}{63}x_2x_3-\frac{52}{567}x_2x_4+\frac{191}{252}x_3^2+\frac{191}{324}x_3x_4+\frac{4997}{18144}x_4^2$$

or

$$V(x(t))=0.25x_1^2+0.286x_1x_2-0.309x_1x_3-0.067x_1x_4+0.214x_2^2-0.206x_2x_3-0.091x_2x_4+0.758x_3^2+0.589x_3x_4+0.275x_4^2$$

Part(b)

Verification using Matlab

```
EDU>> A=[-2 4 -3 1;0 -5 2 -1;0 0 -1 2;0 0 0 -4]

    -2     4    -3     1
     0    -5     2    -1
     0     0    -1     2
     0     0     0    -4

EDU>> syms x1 x2 x3 x4;
EDU>> P=lyap(A',eye(4))

         0.25       0.14286      -0.15476      -0.03373
      0.14286       0.21429      -0.10317     -0.045855
     -0.15476      -0.10317       0.75794       0.29475
     -0.03373     -0.045855       0.29475       0.27541

EDU>> x=[x1;x2;x3;x4];
EDU>> V=x.'*P*x;
EDU>> vpa(expand(V),3)

0.25*x1^2 + 0.286*x1*x2 - 0.31*x1*x3 - 0.0675*x1*x4 + 0.214*x2^2
    - 0.206*x2*x3 - 0.0917*x2*x4 + 0.758*x3^2 + 0.59*x3*x4 + 0.275*x4^2
```

2.7.7 problem 6

Part (a)

Assuming the problem is asking for the 2-norm. This is defined as positive square root of the largest eigenvalue of AA^T. Therefore, we first find AA^T, then find the eigenvalues, then pick the largest one in absolute terms, then take the square root.

$$AA^T = \begin{bmatrix} 0.95 & 0.48 & 0.45 \\ 0.23 & 0.89 & 0.01 \\ 0.6 & 0.76 & 0.82 \end{bmatrix} \begin{bmatrix} 0.95 & 0.23 & 0.6 \\ 0.48 & 0.89 & 0.76 \\ 0.45 & 0.01 & 0.82 \end{bmatrix} = \begin{bmatrix} 1.3354 & 0.6502 & 1.3038 \\ 0.6502 & 0.8451 & 0.8226 \\ 1.3038 & 0.8226 & 1.61 \end{bmatrix}$$

Now we find the eigenvalues.

$$p(\lambda) = \left|\lambda I - AA^T\right| = \begin{vmatrix} \lambda - 1.3354 & -0.6502 & -1.3038 \\ -0.6502 & \lambda - 0.8451 & -0.8226 \\ -1.3038 & -0.8226 & \lambda - 1.61 \end{vmatrix}$$
$$= \lambda^3 - 3.7905\lambda^2 + 1.8398\lambda - 0.1908$$

The roots of this polynomials are $\lambda = 0.14584, \lambda = 0.40366, \lambda = 3.241$. Hence the largest eigenvalue is $\lambda = 3.241$. Therefore the 2-norm is

$$\boxed{\sqrt{3.241} = 1.8003}$$

Part (b)

```
EDU>> A=[0.95 0.48 0.45;0.23 0.89 0.01;0.6 0.76 0.82];
EDU>> norm(A,2)
ans =
       1.8003
```

2.7.8 problem 7

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 7 \\ 0 & -2 & -1 & 5 & -3 \\ 0 & 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

The robustness bound is given by $||\Delta A||_2$ defined as

$$\boxed{||\Delta A||_2 = \frac{\lambda_{\min}[Q]}{2\lambda_{\max}[P]}}$$

We first need to solve the Lyapunov equation to find P.

$$A^T P + PA = -Q$$

Using Matlab, and use $Q = I_5$ gives the solution P

```
EDU>> A=[-1 2 1 3 7;0 -2 -1 5 -3;0 0 -3 4 0;0 0 0 -4 1;0 0 0 0 -5]

A =

    -1     2     1     3     7
     0    -2    -1     5    -3
     0     0    -3     4     0
     0     0     0    -4     1
     0     0     0     0    -5

EDU>> P=lyap(A',eye(5))

P =

          0.5      0.33333     0.041667      0.66667      0.52778
      0.33333      0.58333    -0.033333      0.85278      0.35595
     0.041667    -0.033333      0.19167     0.076984      0.08006
      0.66667      0.85278     0.076984        1.768      0.83996
      0.52778      0.35595      0.08006      0.83996      0.79331
```

Now we find the largest eigenvalue of P

```
EDU>> eig(P)

     0.073178
      0.10973
      0.21435
      0.45009
       2.9889

EDU>> max(ans)
       2.9889
```

Therefore

$$\boxed{\|\Delta A\|_2 = \frac{1}{2\,(2.9889)} = 0.16729}$$

2.7.9 key solution

1-(a) With $\rho = 0$, we get

$$\text{TF} = \frac{3s+1}{s^4+3s^3+6s^2+6s+5}$$

which is stable. The first column of the Routh-Hurwitz table is [1 3 4 2.25 5]'.

1-(b) First generate the four Kharitonov polynomials

$$p_1(s) = s^4 + (3+\rho)s^3 + (6+\rho)s^2 + (6-\rho)s + (5-\rho)$$

$$p_2(s) = s^4 + (3+\rho)s^3 + (6-\rho)s^2 + (6-\rho)s + (5+\rho)$$

$$p_3(s) = s^4 + (3-\rho)s^3 + (6+\rho)s^2 + (6+\rho)s + (5-\rho)$$

$$p_4(s) = s^4 + (3-\rho)s^3 + (6-\rho)s^2 + (6+\rho)s + (5+\rho)$$

Now using Routh-Hurwitz, for each of the above polynomials, compute the maximum value of ρ for the polynomial is stable.

Routh-Hurwitz table for $p_1(s)$

s^4	1	$6+\rho$	$5-\rho$
s^3	$3+\rho$	$6-\rho$	
s^2	$\dfrac{12+10\rho+\rho^2}{3+\rho}$	$5-\rho$	
s^1	$\dfrac{3(9+9\rho-\rho^2)}{12+10\rho+\rho^2}$		
s^0	$5-\rho$		

From the table above we get $\rho < 9.908$.

Routh-Hurwitz table for $p_2(s)$

$$\begin{array}{llll}
s^4 & 1 & 6-\rho & 5+\rho \\
s^3 & 3+\rho & 6-\rho & \\
s^2 & \dfrac{12+4\rho-\rho^2}{3+\rho} & 5+\rho & \\
s^1 & \dfrac{3(-9+9\rho+7\rho^2)}{-12-4\rho+\rho^2} & & \\
s^0 & 5+\rho & &
\end{array}$$

From the table above we get $\rho < 0.66$.

Routh-Hurwitz table for $p_3(s)$

$$\begin{array}{llll}
s^4 & 1 & 6+\rho & 5-\rho \\
s^3 & 3-\rho & 6+\rho & \\
s^2 & \dfrac{-12+4\rho+\rho^2}{\rho-3} & 5-\rho & \\
s^1 & \dfrac{3(-9-9\rho+7\rho^2)}{-12+4\rho+\rho^2} & & \\
s^0 & 5-\rho & &
\end{array}$$

From the table above we get $\rho < 1.946$.

Routh-Hurwitz table for $p_4(s)$

$$\begin{array}{llll}
s^4 & 1 & 6-\rho & 5+\rho \\
s^3 & 3-\rho & 6+\rho & \\
s^2 & \dfrac{-12+10\rho-\rho^2}{\rho-3} & 5+\rho & \\
s^1 & \dfrac{3(9-9\rho-\rho^2)}{12-10\rho+\rho^2} & & \\
s^0 & 5+\rho & &
\end{array}$$

From the table above we get $\rho < 0.908$.

Therefore, we get $\rho_{\max} = 0.66$.

1-(c) Now, with the compensator $H(s) = \frac{1}{s}$, we get the following closed-loop transfer function:

$$TF = \frac{(3+\rho[0,1])s + (1+\rho[-0.5,0.25])}{s^5 + (3+\rho[-1,1])s^4 + (6+\rho[-1,1])s^3 + (3+\rho[-1,0])s^2 + (7+\rho[-0.5,1.75])s + (1+\rho[-0.5,0.25])}$$

And now, try $\rho = 0$ and test for stability and we find that the system is unstable for $\rho = 0$. The first column of the Routh-Hurwitz table is [1 3 5 -1 11.67 1]'. Therefore the system will not be stable for any value larger than $\rho = 0$ and we need not go any further.

Problem 2

$$G(s) = \frac{K(3s+1)}{s^4 + s^3 + as^2 + s + b}$$

where a $\in [12, 36]$ and $b \in [1, 2]$.

The closed-loop transfer function (with unity feedback) can be written as:

$$\frac{Y(s)}{R(s)} = \frac{K(3s+1)}{s^4 + s^3 + as^2 + (1+3K)s + (K+b)}.$$

The first column of the Routh array is

$$\begin{bmatrix}1 & 1 & (a-1-3K) & [(a-1-b)+(3a-7)K-9K^2] & (K+b)\end{bmatrix}'.$$

For stability we require:

$$K < \frac{a-1}{3}, \quad K > -b,$$

and

$$\frac{1}{18}\left[(3a-7) - \sqrt{9a^2 - 6a - 36b + 13}\right] < K < \frac{1}{18}\left[(3a-7) + \sqrt{9a^2 - 6a - 36b + 13}\right]$$

Plugging in our extremum values for a and b and the above conditions, we find

$$K_{max} = 3.5073$$

3. Consider

$$G(s) = \frac{s+1+q_1}{s^2+(2+q_2)s+3}$$

with uncertainty $|q_1| \leq \rho$, $|q_2| \leq \rho$ and controller $H(s) = (s+1)/(s-1)$. This leads, by Mason, to closed loop polynomial given by

$$\begin{aligned} p(s,q) &= (s+1)(s+1+q_1)+(s-1)((s^2+(2+q_2)s+3) \\ &= s^3+(2+q_2)s^2+(3+q_1-q_2)s+q_1-2 \end{aligned}$$

Notice above that even an infinitesimally small value of $\rho > 0$, this polynomial has a a negative coefficient. Noting that the Routh criterion always fails if the polynomial coefficients are not of the same sign, there is no value of $\rho > 0$ for which robust stability is assured. Hence, there is no ρ_{max} or one might equally well say $\rho_{max} = 0$.

4. The engineer is wrong. It is possible to have a polynomial which is stable for $\lambda = 0$ and $\lambda = 1$ but unstable for some intermediate values $0 < \lambda < 1$. To illustrate how this can occur, let

$$p_0(s) = 10s^3 + s^2 + 6s + 0.57$$

and

$$p_1(s) = 10s^3 + 2s^2 + 8s + 1.57$$

Then, with $p_\lambda(s) = (1-\lambda)p_0(s) + \lambda p_1(s)$ it is a simply matter, using Routh Hurwitz, to verify that $p_0(s)$ is stable, $p_1(s)$ is stable but with $\lambda = 0.5$ the polynomial

$$p_\lambda(s) = (1-\lambda)p_0(s) + \lambda p_1(s) = 10s^3 + 1.5s^2 + 7s + 1.07$$

is unstable.

Homework: P matrix

a) $A = \begin{bmatrix} -2 & 4 & -3 & 1 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ → A is stable because it is upper tria and the diagonal entries are ne (they are eigenvalues of A)

Solve $A^TP + PA = -Q$, $Q = I$

$P = P^T$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & -5 & 0 & 0 \\ -3 & 2 & -1 & 0 \\ 1 & -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12} & P_{22} & P_{23} & P_{24} \\ P_{13} & P_{23} & P_{33} & P_{34} \\ P_{14} & P_{24} & P_{34} & P_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12} & P_{22} & P_{23} & P_{24} \\ P_{13} & P_{23} & P_{33} & P_{34} \\ P_{14} & P_{24} & P_{34} & P_{44} \end{bmatrix} \begin{bmatrix} -2 & 4 & -3 & 1 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ Q & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$-2P_{11} - 2P_{11} = -1 \Rightarrow \boxed{P_{11} = 1/4}$$

$$4P_{11} - 7P_{12} = 0 \Rightarrow \boxed{P_{12} = 1/7}$$

$$-3P_{11} + 2P_{12} - 3P_{13} = 0 \Rightarrow \boxed{P_{13} = \frac{-13}{84}}$$

$$P_{11} - P_{12} + 2P_{13} - 6P_{14} = 0 \Rightarrow \boxed{P_{14} = -17/504}$$

$$4P_{12} - 10P_{22} + 4P_{12} = -1 \Rightarrow \boxed{P_{22} = 3/14}$$

$$4P_{13} - 6P_{23} - 3P_{12} + 2P_{22} = 0 \Rightarrow \boxed{P_{23} = -13/126}$$

$$P_{12} - P_{22} + 2P_{23} - 9P_{24} + 4P_{14} = 0 \Rightarrow \boxed{P_{24} = -26/567}$$

$$-3p_{13} + 2p_{23} - 2p_{33} - 3p_{13} + 2p_{23} = -1 \Rightarrow \boxed{p_{33} = 191/252}$$

$$p_{13} - p_{23} + 2p_{33} - 5p_{34} - 3p_{14} + 2p_{24} = 0 \Rightarrow \boxed{p_{34} = 191/648}$$

$$p_{14} - p_{24} + 2p_{34} - 8p_{44} + p_{14} - p_{24} + 2p_{34} = -1 \Rightarrow \boxed{p_{44} = 0.2754}$$

$$P = \begin{bmatrix} 0.25 & 0.1429 & -0.1548 & -0.0337 \\ 0.1429 & 0.2143 & -0.1032 & -0.0459 \\ -0.1548 & -0.1032 & 0.7579 & 0.2948 \\ -0.0337 & -0.0459 & 0.2948 & 0.2754 \end{bmatrix} > 0$$

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, Lyapunov function is

$$V(x) = \underline{x}^T P \underline{x}$$

$$= 0.25x_1^2 + 0.2857x_1x_2 - 0.3095x_1x_3 - 0.0675x_1x_4$$
$$+0.2143x_2^2 - 0.2063x_2x_3 - 0.0917x_2x_4$$
$$+0.758x_3^2 + 0.5895x_3x_4 + 0.2754x_4^2 \quad \#$$

Homework: Matrix Norm

a) Find a norm of matrix $A = \begin{bmatrix} 0.95 & 0.48 & 0.45 \\ 0.23 & 0.89 & 0.01 \\ 0.60 & 0.76 & 0.82 \end{bmatrix}$ without use of Matlab.

$$\|A\| = \sqrt{\lambda_{max}(A^TA)}$$

$$A^TA = \begin{bmatrix} 0.95 & 0.23 & 0.60 \\ 0.48 & 0.89 & 0.76 \\ 0.45 & 0.01 & 0.82 \end{bmatrix} \begin{bmatrix} 0.95 & 0.48 & 0.45 \\ 0.23 & 0.89 & 0.01 \\ 0.60 & 0.76 & 0.82 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3154 & 1.1167 & 0.9218 \\ 1.1167 & 1.6001 & 0.8481 \\ 0.9218 & 0.8481 & 0.8750 \end{bmatrix}$$

$$\det(sI - A^TA) = \det \begin{bmatrix} s-1.3154 & -1.1167 & -0.9218 \\ -1.1167 & s-1.6001 & -0.8481 \\ -0.9218 & -0.8481 & s-0.8750 \end{bmatrix}$$

$$= s^3 - 3.7905s^2 + 1.8398s - 0.1908 = 0.$$

$\Rightarrow$ eigenvalues of $A^TA \Rightarrow s = 3.2410,\ 0.4037,\ 0.1458$

$$\lambda_{max}(A^TA) = 3.2410$$

$$\|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{3.2410} = 1.8003 \quad \#$$

Home work : Robustness Bound

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 7 \\ 0 & -2 & -1 & 5 & -3 \\ 0 & 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad Q = \text{Identity matrix}$$

$$= I_{5\times 5}$$

In Matlab : with A above

```
Q = eye(5);
P = lyap( A', Q )
Bound = min(real(eig(Q)))/(2* max(real(eig(P)
```

$$\Rightarrow \quad 0.1673 = \frac{\lambda_{min}[Q]}{2\lambda_{max}[P]} \qquad \#$$

2.8 special problem

2.8.1 special problem

ECE 717 – Special Problem

Consider the uncertain LTI state-space system

$$\dot{x} = A(q)x$$

where $\varepsilon > 0$ is a VERY small parameter,

$$A(q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(.5+\varepsilon+3q_1+3q_2+2q_1q_2) & -(1+q_1+q_2) & -(1+q_1+q_2) \end{bmatrix}$$

and $q \in Q$ is described by the known bounds

$$0 \leq q_i \leq 1; \quad i = 1, 2.$$

(a) Carry out a robust stability analysis with respect to $0 < \varepsilon < 0.1$. For each such ε which you consider in this range, declare whether the resulting family of polynomials, call it $\mathcal{P}_\varepsilon$ is robustly stable. For those ε when robust stability fails, provide a characterization of both the stable and unstable subsets of Q. Describe how these two sets evolve with respect to ε. Given the low order of this system, you should include a theoretical analysis which supports any numerical computations which you perform.

(b) Instead of considering robust stability, suppose we view q as a random variable which is uniformly distributed over Q. Generate a plot of the probability of stability, call it p_ε, versus ε over the range of interest. Can obtain a formula for p_ε be given? Explain.

Part (a)

The first step is to obtain the matrix $A(q)$ characteristic polynomial

$$A(q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\left(0.5+\varepsilon+3q_1+3q_2+2q_1q_2\right) & -\left(1+q_1+q_2\right) & -\left(1+q_1+q_2\right) \end{bmatrix}$$

Therefore

$$\begin{aligned} p(s,q) &= \left|sI - A(q)\right| \\ &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \left(0.5+\varepsilon+3q_1+3q_2+2q_1q_2\right) & \left(1+q_1+q_2\right) & s+\left(1+q_1+q_2\right) \end{vmatrix} \\ &= \left(\varepsilon+3q_1+3q_2+2q_1q_2+\frac{1}{2}\right) + \left(1+q_1+q_2\right)s + \left(1+q_1+q_2\right)s^2 + s^3 \\ &= a_0(q) + a_1(q)s + a_2(q)s^2 + a_3(q)s^3 \end{aligned}$$

Checking for robust stability

The method of polynomial over-bounding was tried first, but it was inconclusive. The attempt is included in the appendix. A graphical method was then tried based on the zero exclusion principle using set value of polytope of polynomials, but that also was inconclusive as the polygon seen crossing the zero as the frequency increased. The result of this attempt is described in the appendix.

Analysis using Hurwitz matrix Using $P(s) = \left(\varepsilon + 3q_1 + 3q_2 + 2q_1q_2 + \frac{1}{2}\right) + s\left(1 + q_1 + q_2\right) + s^2\left(1 + q_1 + q_2\right) + s^3$ with $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1$. We setup the Hurwitz matrix for the above polynomial to find the conditions under which it is stable.

$$H = \begin{bmatrix} 1 + q_1 + q_2 & 1 & 0 \\ 0.5 + \varepsilon + 3q_1 + 3q_2 + 2q_1q_2 & 1 + q_1 + q_2 & 0 \\ 0 & 1 + q_1 + q_2 & 1 \end{bmatrix}$$

The leading minors are

$$\begin{aligned} \Delta_1 &= 1 + q_1 + q_2 \\ \Delta_2 &= 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2 \\ \Delta_3 &= \Delta_2 \end{aligned}$$

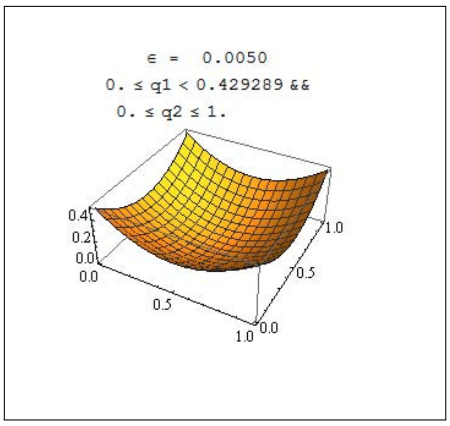
Hence we only need to examine two cases. For $\Delta_1 = 1 + q_1 + q_2$, we see this is positive for all q, since $0 \leq q_i \leq 1$.

For $\Delta_2 = 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2$ we need to determine the conditions which makes this positive.

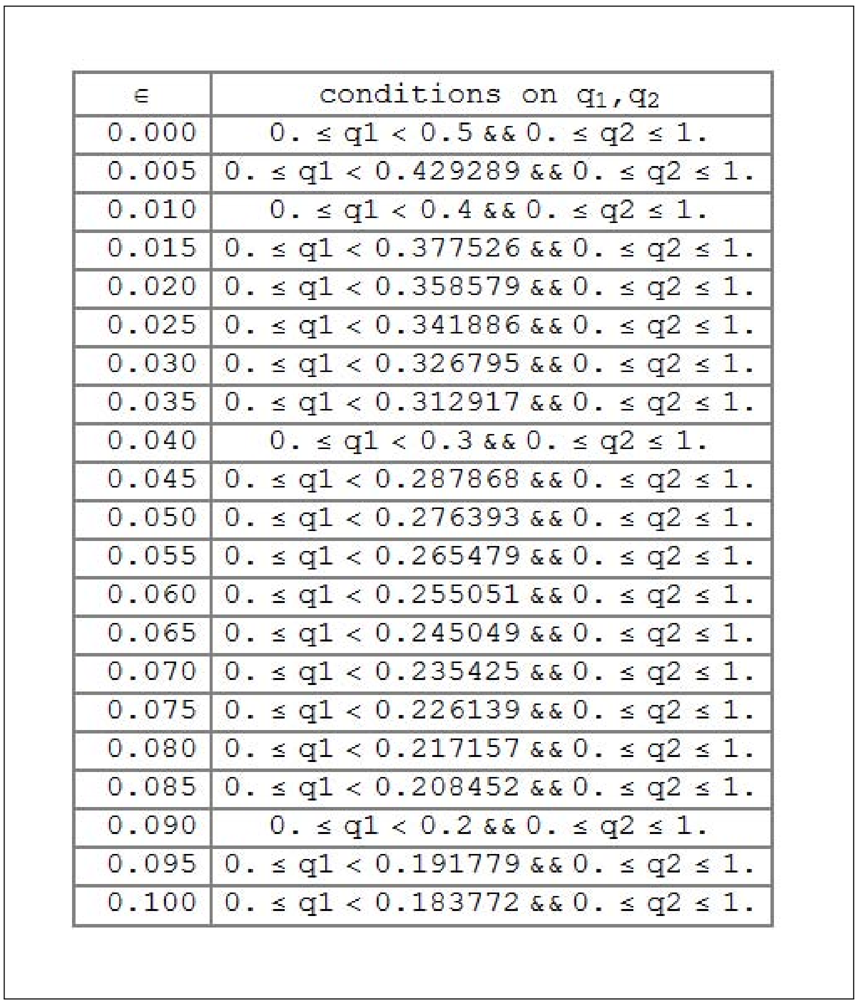
$$0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2 > 0$$

In other words, the minimum of Δ_2 should be positive to insure stability. This is global minimization with constrain problem. However, an algebraic reduce method was used instead to obtain the limits on q_1, q_2 for each different ε_i where ε_i was incremented by 0.005 from 0 to 0.1

There are 20 increments, and for each ε_i an algebraic conditions was found on using the computer on q_1, q_2 which insures that $\Delta_2 > 0$. The result is tabulated below. In addition, 3D plot of Δ_2 is given, using q_1, q_2 as x, y and using the value of Δ_2 as the z-axis. This gives a visual view of the Δ_2 showing that it is indeed positive all the time using the constraints found on q_i for some specific ε used. A typical 3D for some ε is shown below for illustration



The above shows that for $\varepsilon = 0.005$, $\Delta_2 > 0$, and hence the system is stable under the conditions $0 \leq q_1 \leq 0.4294, 0 \leq q_2 \leq 1$.



ε	conditions on q_1, q_2
0.000	0. ≤ q1 < 0.5 && 0. ≤ q2 ≤ 1.
0.005	0. ≤ q1 < 0.429289 && 0. ≤ q2 ≤ 1.
0.010	0. ≤ q1 < 0.4 && 0. ≤ q2 ≤ 1.
0.015	0. ≤ q1 < 0.377526 && 0. ≤ q2 ≤ 1.
0.020	0. ≤ q1 < 0.358579 && 0. ≤ q2 ≤ 1.
0.025	0. ≤ q1 < 0.341886 && 0. ≤ q2 ≤ 1.
0.030	0. ≤ q1 < 0.326795 && 0. ≤ q2 ≤ 1.
0.035	0. ≤ q1 < 0.312917 && 0. ≤ q2 ≤ 1.
0.040	0. ≤ q1 < 0.3 && 0. ≤ q2 ≤ 1.
0.045	0. ≤ q1 < 0.287868 && 0. ≤ q2 ≤ 1.
0.050	0. ≤ q1 < 0.276393 && 0. ≤ q2 ≤ 1.
0.055	0. ≤ q1 < 0.265479 && 0. ≤ q2 ≤ 1.
0.060	0. ≤ q1 < 0.255051 && 0. ≤ q2 ≤ 1.
0.065	0. ≤ q1 < 0.245049 && 0. ≤ q2 ≤ 1.
0.070	0. ≤ q1 < 0.235425 && 0. ≤ q2 ≤ 1.
0.075	0. ≤ q1 < 0.226139 && 0. ≤ q2 ≤ 1.
0.080	0. ≤ q1 < 0.217157 && 0. ≤ q2 ≤ 1.
0.085	0. ≤ q1 < 0.208452 && 0. ≤ q2 ≤ 1.
0.090	0. ≤ q1 < 0.2 && 0. ≤ q2 ≤ 1.
0.095	0. ≤ q1 < 0.191779 && 0. ≤ q2 ≤ 1.
0.100	0. ≤ q1 < 0.183772 && 0. ≤ q2 ≤ 1.

Figure 2.1: conditions on q_1 and q_2 for each ε used for the visualization that follows

Visualization of the solution space for positive Δ_2

Robust stability depends on positive Δ_2. In the above, we obtained conditions on ε, q_1, q_2 which when met, will insure $\Delta_2 > 0$ and hence a stable polynomial. To visualize the solution in 3D, where we will use q_1, q_2, ε as the three axis, and use two colors: green to indicate the region where $\Delta_2 > 0$ and red color for the remaining region where $\Delta_2 \leq 0$. Therefore, the space is a 3D cube enclosed in $1, 1, \varepsilon$. Using the above inequalities, the 3D plots are made.

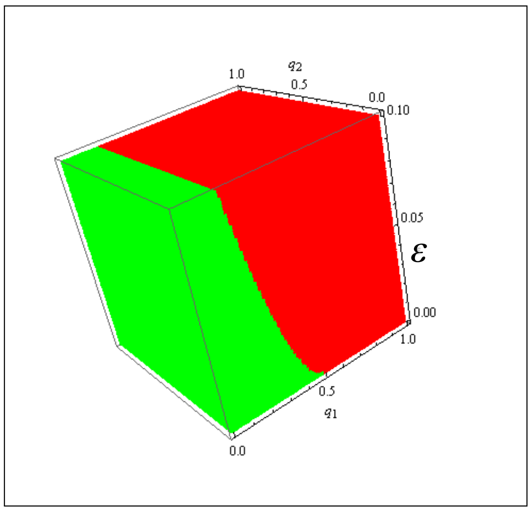


Figure 2.2: points which results in stable polynomial are green, otherwise they are red

Another 3D plot was made, using different 3D plot, called the surface plot, to help visualize the regions in different way. (some parts of the cube are not fully shown due to limited sampling used in producing the data).

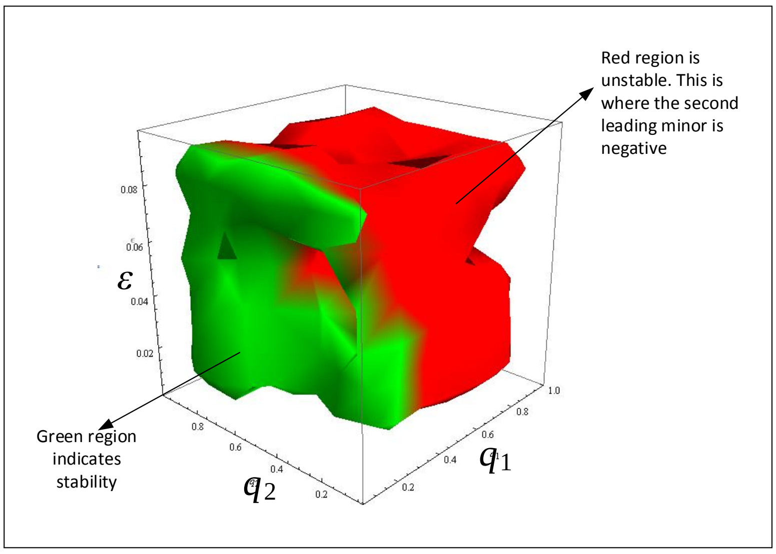


Figure 2.3: Surface around region of stable points is colored green, otherwise red

We see from the above, that stability emerges for the region enclosed by $0 \le q_1 \le 0.5$ and $0 \le q_2 \le 1$ for low values of ε and for the region $0 \le q_1 \le 0.2$ and $0 \le q_2 \le 1$ for the large values of ε. The small ε is, and the smaller q_1 is, the larger the stability region becomes.

Theoretical support for the numerical computation

Attempts were first made to obtain answer using zero exclusion condition. But both attempts resulted in inclusive result, as the polygon crosses the zero point. Algebraic reduction was used to obtain the constraints on solving for conditions on q_1, q_2 for making $\Delta_2 = 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2$ positive as ε was incremented by small amount as shown above. Writing Δ_2 as

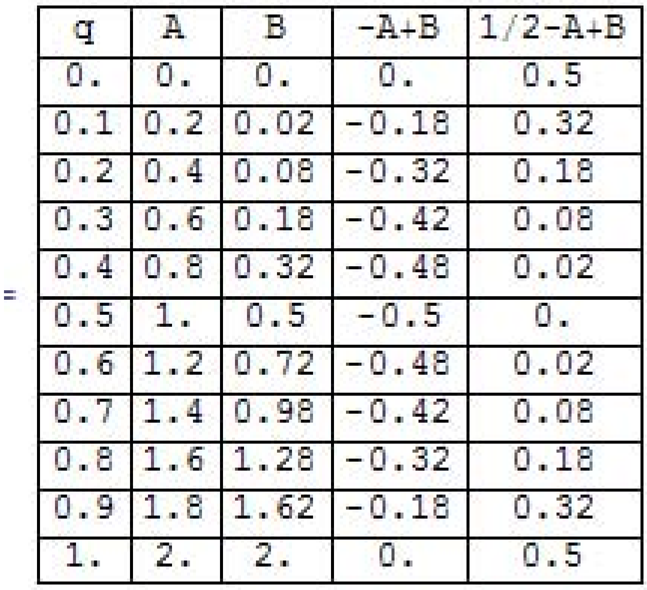
$$\Delta_2 = 0.5 - \varepsilon - \overbrace{\left(q_1 + q_2\right)}^{A} + \overbrace{\left(q_1^2 + q_2^2\right)}^{B}$$

Since each $0 \le q_i \le 1$ then $q_i^2 < q_i$. This means that the smaller the q_i becomes then A will dominate over B more in size but they are both small, and hence Δ_2 is positive .

When both q are around mid point of their range, for example $q_i = \frac{1}{2}$, then $-A + B = -\frac{1}{2}$ and the result is $\Delta_2 = -\varepsilon$, hence not stable.

As the q_i becomes larger than $\frac{1}{2}$ then A does not dominate over B as much, but they are both larger now and the difference between A, B becomes smaller again as when they were both below $\frac{1}{2}$, which means now Δ_2 becomes larger, hence stable.

Here is a small table showing this variation. The above implies that the critical condition is where both q_i are close to each other in value. As they get closer to 0.5 then ε has to become smaller in order to keep Δ_2 positive.



q	A	B	-A+B	1/2-A+B
0.	0.	0.	0.	0.5
0.1	0.2	0.02	-0.18	0.32
0.2	0.4	0.08	-0.32	0.18
0.3	0.6	0.18	-0.42	0.08
0.4	0.8	0.32	-0.48	0.02
0.5	1.	0.5	-0.5	0.
0.6	1.2	0.72	-0.48	0.02
0.7	1.4	0.98	-0.42	0.08
0.8	1.6	1.28	-0.32	0.18
0.9	1.8	1.62	-0.18	0.32
1.	2.	2.	0.	0.5

Solving as constraint minimization problem Another attempt at theoretical support for the numerical computation, is to view this as constraint minimization problem, where we want to find the minimum of Δ_2 subject to constrain $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1$ and $0 < \varepsilon < 0.1$, then the method of Lagrangian multipliers can be used.

Let the objective function be $Q = \Delta_2 = 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2$ subject to $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1, 0 < \varepsilon < 0.1$. Hence the Lagrangian L is

$$\begin{aligned} L &= Q + \lambda_1 \left(1 - q_1\right) + \lambda_2 \left(1 - q_2\right) + \lambda_3 (0.1 - \varepsilon) + \lambda_4 q_1 + \lambda_5 q_2 + \lambda_6 \varepsilon \\ &= 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2 + \lambda_1 \left(1 - q_1\right) + \lambda_2 \left(1 - q_2\right) + \lambda_3 (0.1 - \varepsilon) + \lambda_4 q_1 + \lambda_5 q_2 + \lambda_6 \varepsilon \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial q_1} &= -1 + 2q_1 - \lambda_1 + \lambda_4 = 0 \\ \frac{\partial L}{\partial q_2} &= -1 + 2q_2 - \lambda_2 + \lambda_5 = 0 \\ \frac{\partial L}{\partial \varepsilon} &= -1 - \lambda_3 + \lambda_6 = 0 \\ \frac{\partial L}{\partial \lambda_1} &= 1 - q_1 = 0 \\ \frac{\partial L}{\partial \lambda_2} &= 1 - q_2 = 0 \\ \frac{\partial L}{\partial \lambda_3} &= 0.1 - \varepsilon = 0 \\ \frac{\partial L}{\partial \lambda_4} &= q_1 = 0 \\ \frac{\partial L}{\partial \lambda_5} &= q_2 = 0 \\ \frac{\partial L}{\partial \lambda_6} &= \varepsilon = 0 \end{aligned}$$

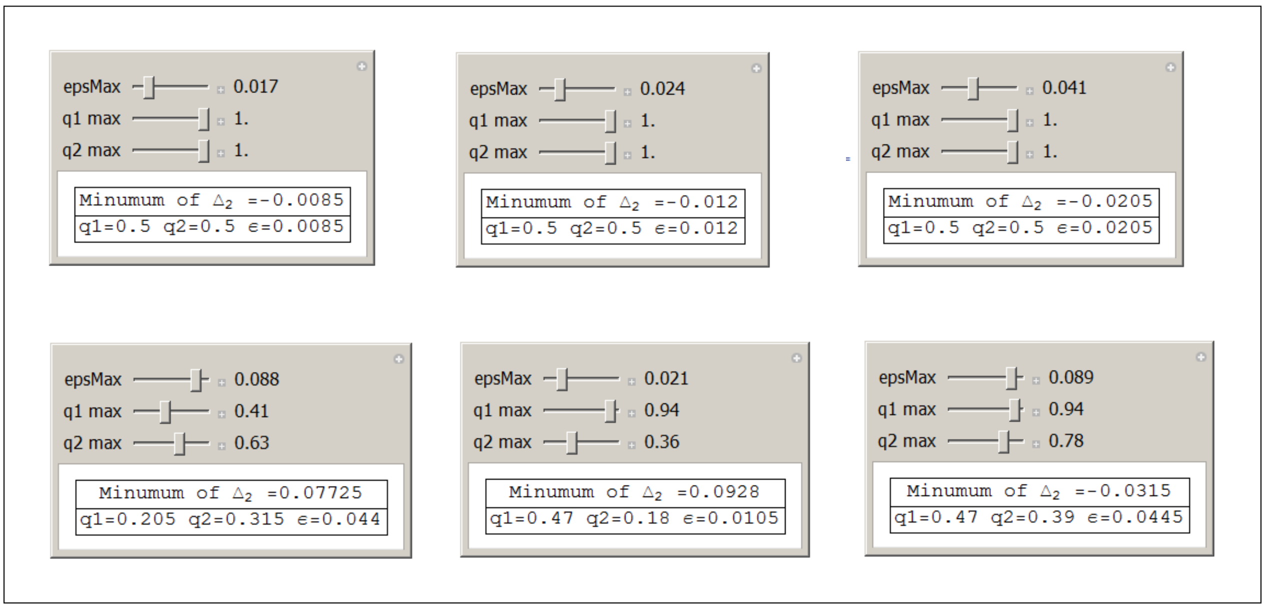
And

$$\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \varepsilon \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -0.1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above $Ax = b$ system using least squares gave the solution as $q_1 = 0.5, q_2 = 0.5, \varepsilon = 0.05$, which is where the minimum is. Using the solution we can find Δ_2 at these values

$$\Delta_2 = -0.05$$

The above Lagrangian multiplier method to find the minimum was implemented in separate program allowing one to change the limits of ε and q_1, q_2 to see where the minimum shows up for each different combination. Here are few screen shots showing different results. This method was used to verify the numerical 3D based plots shown above by verifying the stable points are where they are shown in the 3D plots.



epsMax 0.017

q1 max 1.

q2 max 1.

```
Minumum of Δ2 =-0.0085
q1=0.5 q2=0.5 ϵ=0.0085
```

epsMax 0.024

q1 max 1.

q2 max 1.

```
Minumum of Δ2 =-0.012
q1=0.5 q2=0.5 ϵ=0.012
```

epsMax 0.041

q1 max 1.

q2 max 1.

```
Minumum of Δ2 =-0.0205
q1=0.5 q2=0.5 ϵ=0.0205
```

epsMax 0.088

q1 max 0.41

q2 max 0.63

```
Minumum of Δ2 =0.07725
q1=0.205 q2=0.315 ϵ=0.044
```

epsMax 0.021

q1 max 0.94

q2 max 0.36

```
Minumum of Δ2 =0.0928
q1=0.47 q2=0.18 ϵ=0.0105
```

epsMax 0.089

q1 max 0.94

q2 max 0.78

```
Minumum of Δ2 =-0.0315
q1=0.47 q2=0.39 ϵ=0.0445
```

Part (b)

Plotting P_ε We now treat q_1, q_2 as random variables. Stability is still decided by

$$\Delta_2 = 0.5 - \varepsilon - q_1 + q_1^2 - q_2 + q_2^2$$

If we call Δ_2 as the random variable Z which is now function of random variables q_1, q_2 renamed to be X, Y and are drawn from uniform distribution, then we can write

$$Z = 0.5 - \varepsilon - X + X^2 - Y + Y^2$$

Then

$$\Pr(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_1} f_Z(z)\, dz$$

or

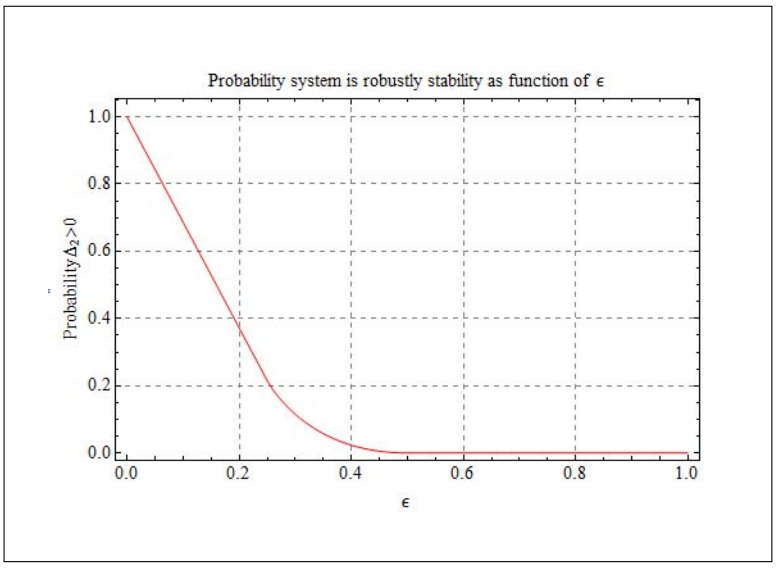
$$\Pr(Z \leq z) = F_Z(z) = \int_{-\infty}^{z} f_Z(z)\, dz$$

Where $F_Z(z)$ is the cumulative distribution function. To find $\Pr(Z > 0)$ then

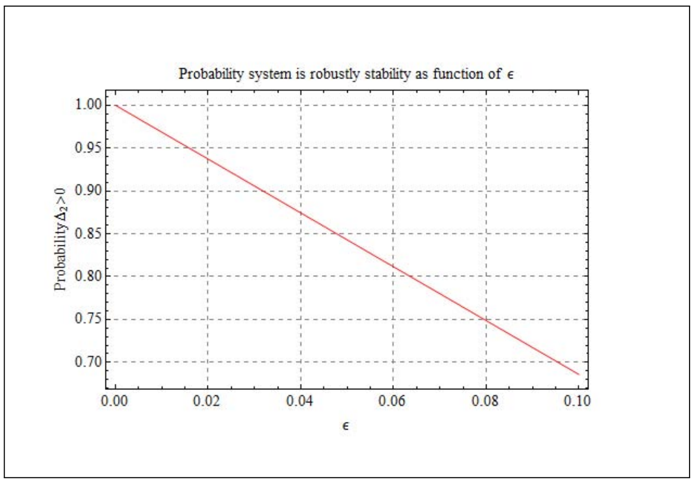
$$\Pr(Z > 0) = \int_{0}^{\infty} f_Z(z)\, dz$$

ε is fixed each time before finding $f_z(z)$. Hence for each ε there will be a different $f_Z(z)$ which we then use to find $\Pr(Z>0)$ for robust stability, since $\Pr(Z>0)$ is the same as asking what is the probability that $\Delta_2>0$ for a given ε.

P_ε was drawn for $\varepsilon = 0\cdots 1$ to see how it shows up before zooming in.



The above shows that for $0\le\varepsilon\le 0.1$ the probability of robust stability is high. We can zoom in to that region



We see that at $\varepsilon=0.1$ there is about 68% chance that the system will be stable and for $\varepsilon=0$ the probability the system is stable is 100%.

Finding the formula for the probability P_ε $f_Z(z)$ is the probability density function of the random variable Z. To obtain $f_Z(z)$, we use the following two definitions. For random variable $Z=X+Y$ where X is random variable drawn from f_X distribution and Y is random variable drawn from f_Y distribution, then the random variable Z will be drawn from distribution made from the convolution of f_X with f_Y

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(z-x)\,f_Y(x)\,dx$$

Where f_X is the pdf of the uniform distribution for Q which is defined for $q=[0,1]$ as

$$f_X(x)=\begin{cases}1 & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

For $Z=XY$ evaluation, we need a product of two random variables. This is given by

$$f_Z(z)=\int_{-\infty}^{\infty} f(x)\,f(z/x)\,\frac{1}{|x|}dx$$

Using the above, we can find that $f_Z(z)$ for $Z=0.5-\varepsilon-X+X^2-Y+Y^2$.

Let $Z_1=-X-Y=-(X+Y)$, hence

$$f_{Z_1}(z)=-\int_{-\infty}^{\infty} f_U(z-x)\,f_U(x)\,dx$$

Now let $Z_2 = X^2$. Using the product formula, we write

$$f_{Z_2}(z) = \int_{-\infty}^{\infty} f_U(x) f_U(z/x) \frac{1}{|x|} dx$$

Therefore for $Z_3 = -X - Y + X^2$ we now have $Z_3 = Z_1 + Z_2$ and now we use the addition formula

$$f_{Z_3}(z) = \int_{-\infty}^{\infty} f_{Z_1}(z - x) f_{Z_2}(x)\, dx$$

For $-Y^2$, let $Z_4 = -Y^2$ and using the product formula gives

$$f_{Z_4}(z) = -\int_{-\infty}^{\infty} f_U(x) f_U(z/x) \frac{1}{|x|} dx$$

Hence we now have the following $Z_5 = Z_3 + Z_4$ and the pdf is

$$f_{Z_5}(z) = \int_{-\infty}^{\infty} f_{Z_3}(z - x) f_{Z_4}(x)\, dx$$

Finally, we have $Z = 0.5 - \varepsilon + Z_5$ which have the pdf

$$f_Z(z) = \frac{1}{|0.5 - \varepsilon|} f_{Z_5}(z - (0.5 - \varepsilon))$$

With the help of the computer, formula for the pdf of $f_Z(z)$ was obtained which was used to generate the above plots of P_ε

$$f_Z(z) = \begin{cases} \pi & 0 < q + \epsilon \leq 0.25 \\ 2\left(\operatorname{arcsec}\left(2\sqrt{q+\varepsilon}\right) - \arctan\left(\sqrt{-1 + 4q + 4\varepsilon}\right)\right) & 0.25 \leq q + \varepsilon < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

2.8.2 Appendix

polynomial over-bounding method

The first method to try is the method of polynomial over-bounding If this method says the polynomial is robustly stable, then we are done. However, if this method says the polynomial is not robustly stable, then it can still be stable. Hence the polynomial over-bounding method is called inconclusive, and we need to try other methods.

But we start with this method since it is simple to use to check. Using the method of over-bounding, we first need to determine the bounds on $\bar{q}_i$. In other words, we convert the polynomial in q to an interval polynomial in $\bar{q}$

$$\bar{q}_0^- = \min_{q\in Q} a_0\left(q\right) = \min_{0\leq q\leq 1}\left(\varepsilon + 3q_1 + 3q_2 + 2q_1q_2 + \frac{1}{2}\right)$$

Setting up the following table

q_1	q_2	$\varepsilon + 3q_1 + 3q_2 + 2q_1q_2 + \frac{1}{2}$
0	0	$\varepsilon + 0.5$
0	1	$\varepsilon + 3.5$
1	0	$\varepsilon + 3.5$
1	1	$\varepsilon + 3 + 3 + 2 + \frac{1}{2} = \varepsilon + 8.5$

Hence

$$\bar{q}_0^- = \varepsilon + 0.5$$

And

$$\bar{q}_0^+ = \max_{q\in Q} a_0\left(q\right) = \varepsilon + 8.5$$

And

$$\bar{q}_1^- = \min_{q\in Q} a_1\left(q\right) = \min_{0\leq q\leq 1}\left(1 + q_1 + q_2\right)$$

Setting up a table

q_1	q_2	$1 + q_1 + q_2$
0	0	1
0	1	2
1	0	2
1	1	3

Hence

$$\bar{q}_1^- = 1$$

And

$$\bar{q}_1^+ = \max_{q\in Q} a_1\left(q\right) = 3$$

And similarly, $\bar{q}_2^- = 1, \bar{q}_2^+ = 3$. And for $\bar{q}_3$, since it has no uncertainties, then $\bar{q}_3^- = \bar{q}_3^+ = 1$, Hence the over-bounding interval polynomial is

$$\begin{aligned}\bar{p}\left(s,\bar{q}\right) &= \left[\bar{q}_0^-, \bar{q}_0^+\right] + \left[\bar{q}_1^-, \bar{q}_1^+\right]s + \left[\bar{q}_2^-, \bar{q}_2^+\right]s^2 + \left[\bar{q}_3^-, \bar{q}_3^+\right]s^3 \\ &= \left[\varepsilon + 0.5, \varepsilon + 8.5\right] + \left[1,3\right]s + \left[1,3\right]s^2 + \left[1,1\right]s^3\end{aligned}$$

We now construct the four Kharitonov polynomials to check for stability using Hurwitz matrix method

$$\begin{aligned}K_1(s) &= (\varepsilon + 0.5) + s + 3s^2 + s^3 \\ K_2(s) &= (\varepsilon + 8.5) + 3s + s^2 + s^3 \\ K_3(s) &= (\varepsilon + 8.5) + s + s^2 + s^3 \\ K_4(s) &= (\varepsilon + 0.5) + 3s + 3s^2 + s^3\end{aligned}$$

Hence $H_1 = \begin{bmatrix} 1 & 1 & 0 \\ (\varepsilon + 0.5) & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and for stability we require that $\Delta_1 = 1 > 0$, $\Delta_2 = 3 - (\varepsilon + 0.5) = 2.5 - \varepsilon > 0$ and $\Delta_3 = 2.5 - \varepsilon > 0$. Since $0 < \varepsilon < 0.1$, then we see that $\Delta_i, i = 1 \cdots 3$ are all positive. Now we consider K_2.

$H_2 = \begin{bmatrix} 3 & 1 & 0 \\ (\varepsilon + 8.5) & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ and for stability we require that $\Delta_1 = 3 > 0$, $\Delta_2 = -5.5 - \varepsilon$. We see that $\Delta_2 < 0$ since $0 < \varepsilon < 0.1$. Hence

$$\boxed{\text{the over-bounding polynomial method was not conclusive}}.$$

Therefore we need to try a different method.

Graphical method based on zero exclusion using set value of polytope of polynomials The $P(s) = \left(\varepsilon + 3q_1 + 3q_2 + 2q_1q_2 + \frac{1}{2}\right) + s\left(1 + q_1 + q_2\right) + s^2\left(1 + q_1 + q_2\right) + s^3$ with $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1$ is multilinear in q. To use the value set for polytope of polynomials and apply graphical zero exclusion method in the hope to confirm robust stability, we have to convert the polynomial to an over-bounding affine linear polynomial in q by introducing new $q_3 = q_1q_2$[4]

The polynomial $P(s) = \left(\varepsilon + 3q_1 + 3q_2 + 2q_3 + \frac{1}{2}\right) + s\left(1 + q_1 + q_2\right) + s^2\left(1 + q_1 + q_2\right) + s^3$ with $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1, 0 \leq q_3 \leq 1$ and $0 < \varepsilon < \varepsilon^+$. Hence the uncertainty bounding set Q has 8 extremes $q^1 = (0,0,0), q^2 = (0,0,1), q^3 = (0,1,0), q^4 = (0,1,1), q^5 = (1,0,0), q^6 = (1,0,1), q^7 = (1,1,0), q^8 = (1,1,1)$. The eight associated polynomials generated are

$$\begin{aligned}
P\left(s,q^1\right) &= (\varepsilon + 0.5) + s + s^2 + s^3 \\
P\left(s,q^2\right) &= \varepsilon + 2.5 + s + s^2 + s^3 \\
P\left(s,q^3\right) &= (\varepsilon + 3.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^4\right) &= (\varepsilon + 5.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^5\right) &= (\varepsilon + 3.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^6\right) &= (\varepsilon + 5.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^7\right) &= (\varepsilon + 6.5) + 3s + 3s^2 + s^3 \\
P\left(s,q^8\right) &= (\varepsilon + 8.5) + 3s + 3s^2 + s^3
\end{aligned}$$

From the above the nodes will be determined only by the unique polynomials. We will use only six out of the eight above, they are

$$\begin{aligned}
P\left(s,q^1\right) &= (\varepsilon + 0.5) + s + s^2 + s^3 \\
P\left(s,q^2\right) &= \varepsilon + 2.5 + s + s^2 + s^3 \\
P\left(s,q^3\right) &= (\varepsilon + 3.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^4\right) &= (\varepsilon + 5.5) + 2s + 2s^2 + s^3 \\
P\left(s,q^7\right) &= (\varepsilon + 6.5) + 3s + 3s^2 + s^3 \\
P\left(s,q^8\right) &= (\varepsilon + 8.5) + 3s + 3s^2 + s^3
\end{aligned}$$

We need to check that we have at least one nominal stable polynomial in the family of polynomials. Using $P\left(s,q^1\right) = 0.5 + s + s^2 + s^3$ as the stable member, we check it for stability first:

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

For $\Delta_1 = 1 > 0$, and $\Delta_2 = 0.5 > 0$ and $\Delta_3 = \Delta_2 > 0$ as well. Hence we verified the stable member exist. Now we need to generate the polygonal value set for each of the polynomials

---

[4] Similar to method in example 8.2.8, page 128, reference [1]

above.

$$\begin{aligned} P\left(j\omega, q^1\right) &= 0.5 + j\omega + \left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= 0.5 + j\omega - \omega^2 - j\omega^3 \\ &= 0.5 - \omega^2 + j\left(\omega - \omega^3\right) \end{aligned}$$

And

$$\begin{aligned} P\left(j\omega, q^2\right) &= 3.5 + 2j\omega + 2\left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= 3.5 + 2j\omega - 2\omega^2 - j\omega^3 \\ &= 3.5 - 2\omega^2 + j\left(2\omega - \omega^3\right) \end{aligned}$$

Polynomial $P\left(s, q^3\right)$ is the same as $P\left(s, q^2\right)$, so corner q^2 and q^3 map to same point in complex plane.

$$\begin{aligned} P\left(j\omega, q^4\right) &= 8.5 + 3j\omega + 3\left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= 8.5 + 3j\omega - 3\omega^2 - j\omega^3 \\ &= 8.5 - 3\omega^2 + j\left(3\omega - \omega^3\right) \end{aligned}$$

And

$$\begin{aligned} P\left(j\omega, q^5\right) &= (\varepsilon^+ + 0.5) + j\omega + \left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= (\varepsilon^+ + 0.5) + j\omega - \omega^2 - j\omega^3 \\ &= (\varepsilon^+ + 0.5) - \omega^2 + j\left(\omega - \omega^3\right) \end{aligned}$$

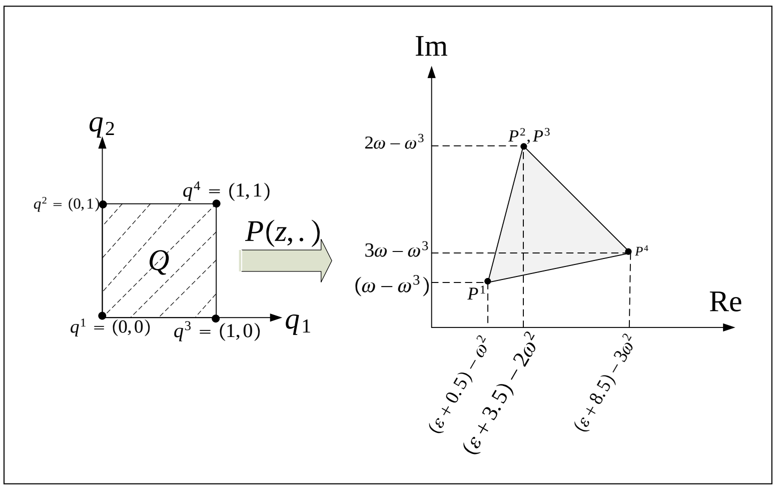
And

$$\begin{aligned} P\left(j\omega, q^6\right) &= (\varepsilon^+ + 3.5) + 2j\omega + 2\left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= (\varepsilon^+ + 3.5) + 2j\omega - 2\omega^2 - j\omega^3 \\ &= (\varepsilon^+ + 3.5) - 2\omega^2 + j\left(2\omega - \omega^3\right) \end{aligned}$$

Polynomial $P\left(s, q^7\right)$ is the same as $P\left(s, q^6\right)$, so corner q^6 and q^7 map to same point in complex plane.

$$\begin{aligned} P\left(j\omega, q^8\right) &= (\varepsilon^+ + 8.5) + 3j\omega + 3\left(j\omega\right)^2 + \left(j\omega\right)^3 \\ &= (\varepsilon^+ + 8.5) + 3j\omega - 3\omega^2 - j\omega^3 \\ &= (\varepsilon^+ + 8.5) - 3\omega^2 + j\left(3\omega - \omega^3\right) \end{aligned}$$

This diagram illustrates the mapping[5]



To find the cut off frequency ω_c, using $\omega_c = 1 + \frac{\max\left\{q_0^+, q_1^+, \cdots, q_{n-1}^+\right\}}{q_n^{-1}}$, where here we use the interval polynomial $\bar{p}\left(s, \bar{q}\right) = [\varepsilon + 0.5, \varepsilon + 8.5] + [1,3]\,s + [1,3]\,s^2 + [1,1]\,s^3$ found above. This results in

$$\omega_c = 1 + \frac{\max\left\{\varepsilon + 8.5, 3, 3\right\}}{1} = 1 + \varepsilon + 8.5 = 9.5 + \varepsilon = 9.51$$

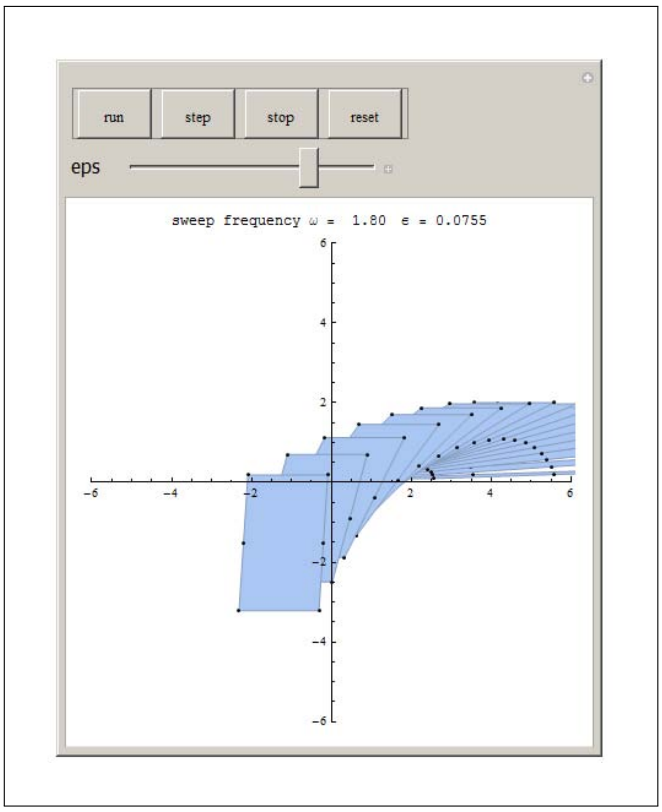
---

[5] This diagram shows the mapping done before converting the polynomial to linear in q. So the original multilinear form was used, that is why the square was mapped to triangle as shown. Zero exclusion principle also was inconclusive in this case, and that is why an attempt was made using the linear form as described in example 8.2.8 in [1]

Therefore the sweep frequency from be $0 < \omega < 9.51$ for the simulation[6]. A program was written to simulate the value sets for this problem. For each ε value, the value set is displayed over the sweep frequency to see if the polygon will include the origin or not. ε was changed by small increments and the simulation was run again.

Unfortunately, the result of this method was inconclusive as well

The polygons generated did include the origin as the frequency ω was increased and different ε tried. Below is an example running a program written to implement this method



2.8.3 References

1. Barmish, B. Ross. New tools for robustness of linear systems. Macmillian publishing company, 1994.

2. Class notes, ECE 717, University of Wisconsin, Madison. Fall 2014.

3. Strang, Gilbert. Introduction to applied mathematics. Wellesley-Cambridge press, 1986.

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[6] In the simulation, I did not have to sweep to such high frequency to see the polygon cross the zero

2.8.4 key solution

ECE 717 – Special Problem Notes

(a) The problem can be solved analytically without recourse to numerical computation. In is arguable that graphical computer-generated plots provide no significant insight over and above the solution obtained by hand. We see below that the $(q_,q_2)$ rectangle includes a circular "island of instability" with radius $r = \sqrt{\varepsilon}$. Indeed, noting that $A(q)$ is a companion form, its characteristic polynomial is

$$p(s,q) = \det(sI - A(q)) = s^3 + a(q)s^2 + a(q)s + b(q,\varepsilon)$$

where

$$a(q) \doteq 1 + q_1 + q_2$$

and

$$b(q,\varepsilon) \doteq 3q_1 + 3q_2 + 2q_1q_2 + 0.5 + \varepsilon.$$

To establish the claimed result, for the positive-coefficient polynomial above, we first generate the Hurwitz matrix

$$\mathcal{H} = \begin{bmatrix} a(q) & b(q,\varepsilon) & 0 \\ 1 & a(q) & 0 \\ 0 & a(q) & b(q,\varepsilon) \end{bmatrix}.$$

Now enforcing the positive minor condition for stability, we characterize the stable set by

$$a(q) > 0; \;\; a^2(q) - b(q,\varepsilon) > 0; \;\; b(q,\varepsilon) > 0.$$

Over the range of variation $0 \leq q_i \leq 1$, we have $a(q) > 0$ and $b(q,\varepsilon) > 0$ trivially satisfied. Hence, stability is determined by the condition

$$\begin{aligned} 0 &< a^2(q) - b(q,\varepsilon) \\ &= (1 + q_1 + q_2)^2 - (3q_1 + 3q_2 + 2q_1q_2 + 0.5 + \varepsilon) \\ &= q_1^2 + q_2^2 - q_1 - q_2 + 0.5 - \varepsilon \\ &= (q_1 - 0.5)^2 + (q_2 - 0.5)^2 - \varepsilon. \end{aligned}$$

From the above, the stable set is the complement of a circular region centered at $(q_1, q_2) = (0.5, 0.5)$ with radius $r = \sqrt{\varepsilon}$. Notice that as ε tends to zero, the instability domain gets smaller and smaller shrinking down to a single point when $\varepsilon = 0$.

(b) When q is viewed as a random vector which is uniformly distributed over the unit square, the probability of stability is given by

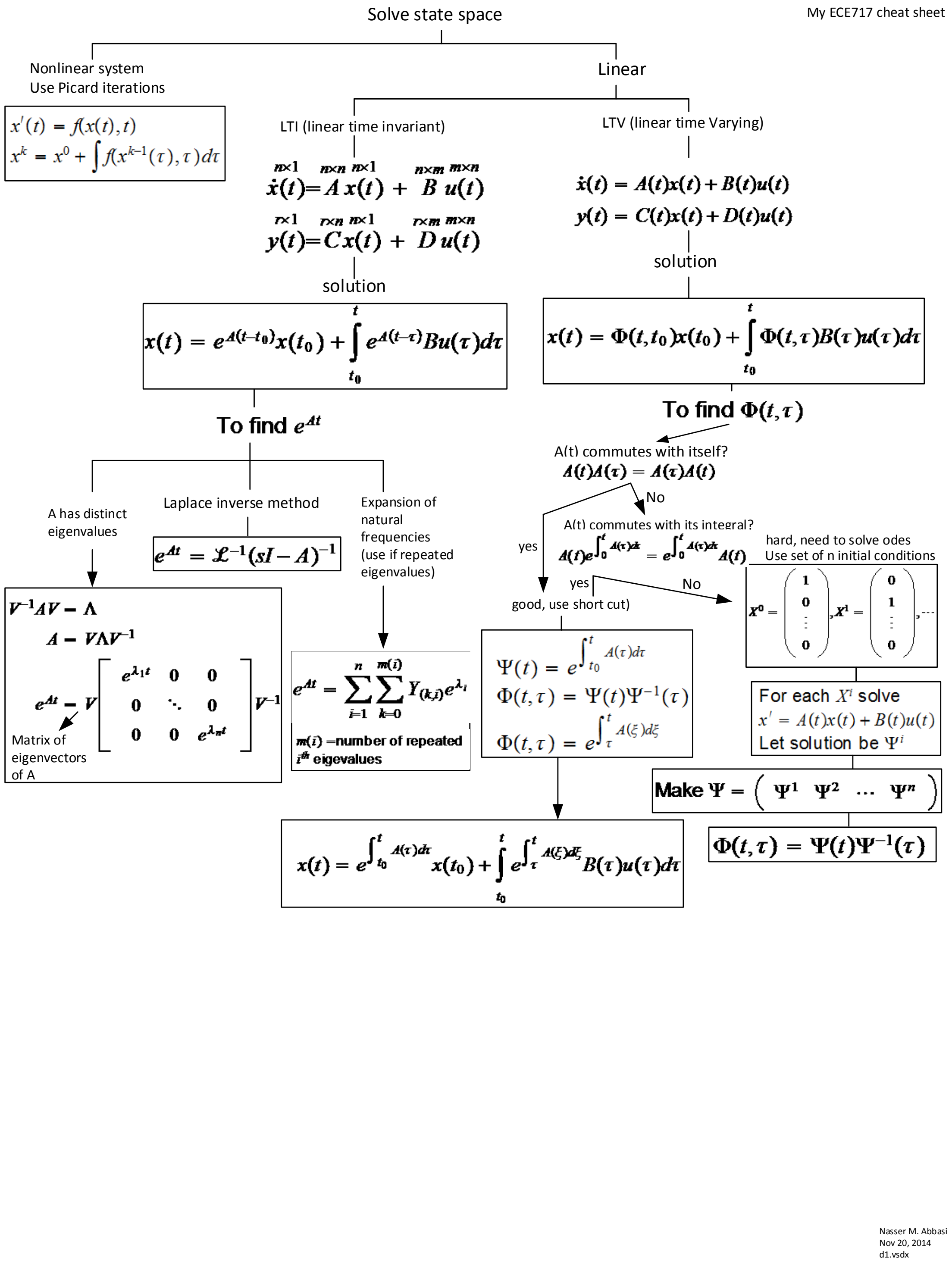
$$p_\varepsilon = 1 - \text{area of the unstable set} = 1 - \pi\epsilon.$$

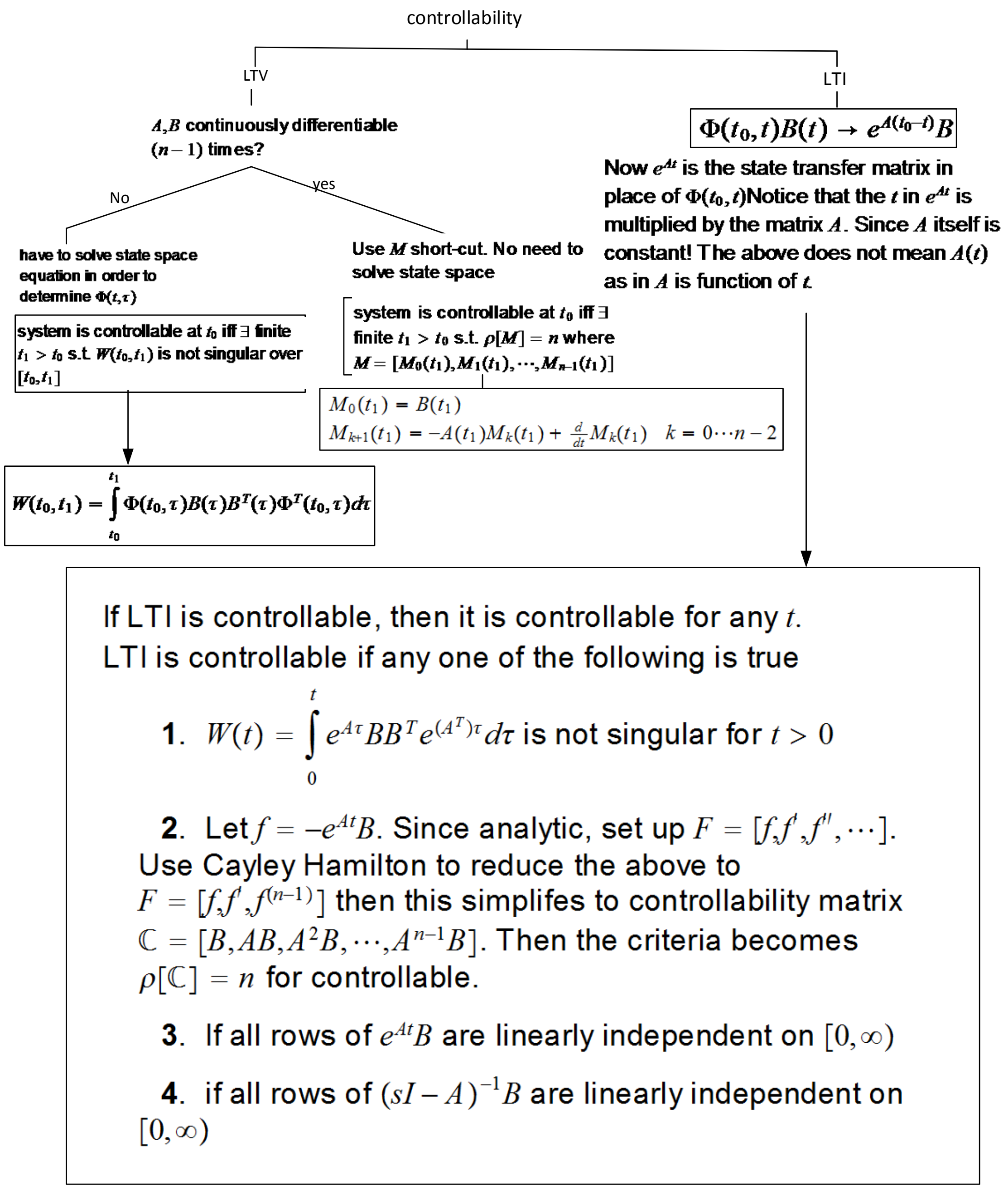
This can readily plotted as a straight line for $0 \leq \varepsilon \leq 0.25$ which is the range guaranteeing inclusion of the unstable set within the unit square.

Chapter 3

Appendix

3.1 my class cheat sheets





controllability

LTV

LTI

A,B continuously differentiable $(n-1)$ times?

No

yes

have to solve state space equation in order to determine $\Phi(t,\tau)$

system is controllable at t_0 iff $\exists$ finite $t_1 > t_0$ s.t. $W(t_0,t_1)$ is not singular over $[t_0,t_1]$

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau$$

Use M short-cut. No need to solve state space

system is controllable at t_0 iff $\exists$ finite $t_1 > t_0$ s.t. $\rho[M] = n$ where $M = [M_0(t_1), M_1(t_1), \cdots, M_{n-1}(t_1)]$

$$M_0(t_1) = B(t_1)$$
$$M_{k+1}(t_1) = -A(t_1)M_k(t_1) + \tfrac{d}{dt}M_k(t_1) \quad k = 0 \cdots n-2$$

$$\boxed{\Phi(t_0,t)B(t) \rightarrow e^{A(t_0-t)}B}$$

Now e^{At} is the state transfer matrix in place of $\Phi(t_0,t)$Notice that the t in e^{At} is multiplied by the matrix A. Since A itself is constant! The above does not mean $A(t)$ as in A is function of t.

If LTI is controllable, then it is controllable for any t.
LTI is controllable if any one of the following is true

1. $W(t) = \int_0^t e^{A\tau}BB^Te^{(A^T)\tau}d\tau$ is not singular for $t > 0$

2. Let $f = -e^{At}B$. Since analytic, set up $F = [f, f', f'', \cdots]$. Use Cayley Hamilton to reduce the above to $F = [f, f', f^{(n-1)}]$ then this simplifes to controllability matrix $\mathbb{C} = [B, AB, A^2B, \cdots, A^{n-1}B]$. Then the criteria becomes $\rho[\mathbb{C}] = n$ for controllable.

3. If all rows of $e^{At}B$ are linearly independent on $[0,\infty)$

4. if all rows of $(sI-A)^{-1}B$ are linearly independent on $[0,\infty)$

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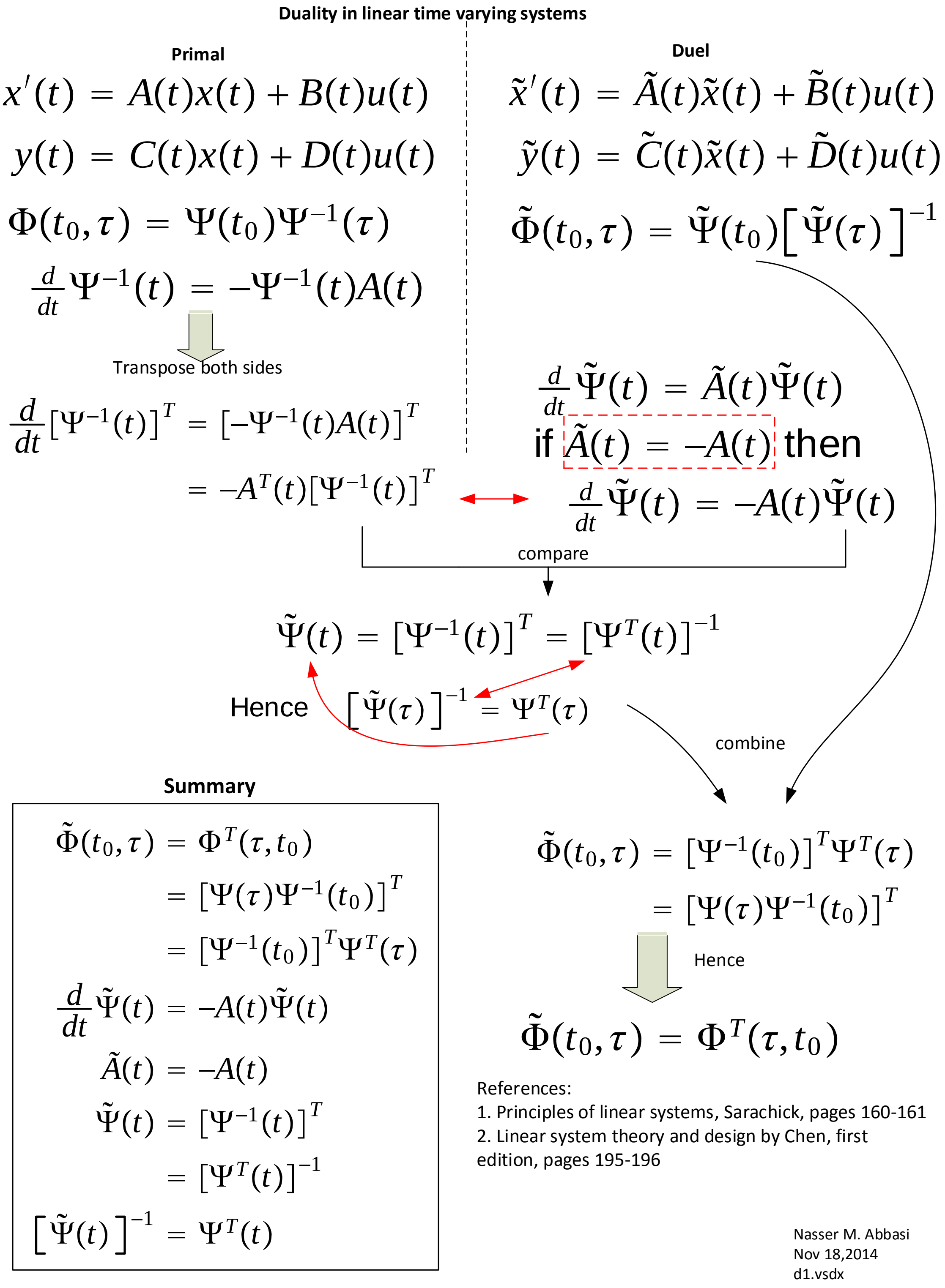
Nov 21, 2014
d2.vsdx



Duality in linear time varying systems

Primal

$$x'(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

$$\Phi(t_0,\tau) = \Psi(t_0)\Psi^{-1}(\tau)$$

$$\frac{d}{dt}\Psi^{-1}(t) = -\Psi^{-1}(t)A(t)$$

Transpose both sides

$$\frac{d}{dt}\left[\Psi^{-1}(t)\right]^T = \left[-\Psi^{-1}(t)A(t)\right]^T$$

$$= -A^T(t)\left[\Psi^{-1}(t)\right]^T$$

Duel

$$\tilde{x}'(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t)$$

$$\tilde{y}(t) = \tilde{C}(t)\tilde{x}(t) + \tilde{D}(t)u(t)$$

$$\tilde{\Phi}(t_0,\tau) = \tilde{\Psi}(t_0)\left[\tilde{\Psi}(\tau)\right]^{-1}$$

$$\frac{d}{dt}\tilde{\Psi}(t) = \tilde{A}(t)\tilde{\Psi}(t)$$

if $\tilde{A}(t) = -A(t)$ then

$$\frac{d}{dt}\tilde{\Psi}(t) = -A(t)\tilde{\Psi}(t)$$

compare

$$\tilde{\Psi}(t) = \left[\Psi^{-1}(t)\right]^T = \left[\Psi^T(t)\right]^{-1}$$

Hence $\left[\tilde{\Psi}(\tau)\right]^{-1} = \Psi^T(\tau)$

combine

$$\tilde{\Phi}(t_0,\tau) = \left[\Psi^{-1}(t_0)\right]^T\Psi^T(\tau)$$

$$= \left[\Psi(\tau)\Psi^{-1}(t_0)\right]^T$$

Hence

$$\tilde{\Phi}(t_0,\tau) = \Phi^T(\tau,t_0)$$

Summary

$$\tilde{\Phi}(t_0,\tau) = \Phi^T(\tau,t_0)$$

$$= \left[\Psi(\tau)\Psi^{-1}(t_0)\right]^T$$

$$= \left[\Psi^{-1}(t_0)\right]^T\Psi^T(\tau)$$

$$\frac{d}{dt}\tilde{\Psi}(t) = -A(t)\tilde{\Psi}(t)$$

$$\tilde{A}(t) = -A(t)$$

$$\tilde{\Psi}(t) = \left[\Psi^{-1}(t)\right]^T$$

$$= \left[\Psi^T(t)\right]^{-1}$$

$$\left[\tilde{\Psi}(t)\right]^{-1} = \Psi^T(t)$$

References:
1. Principles of linear systems, Sarachick, pages 160-161
2. Linear system theory and design by Chen, first edition, pages 195-196

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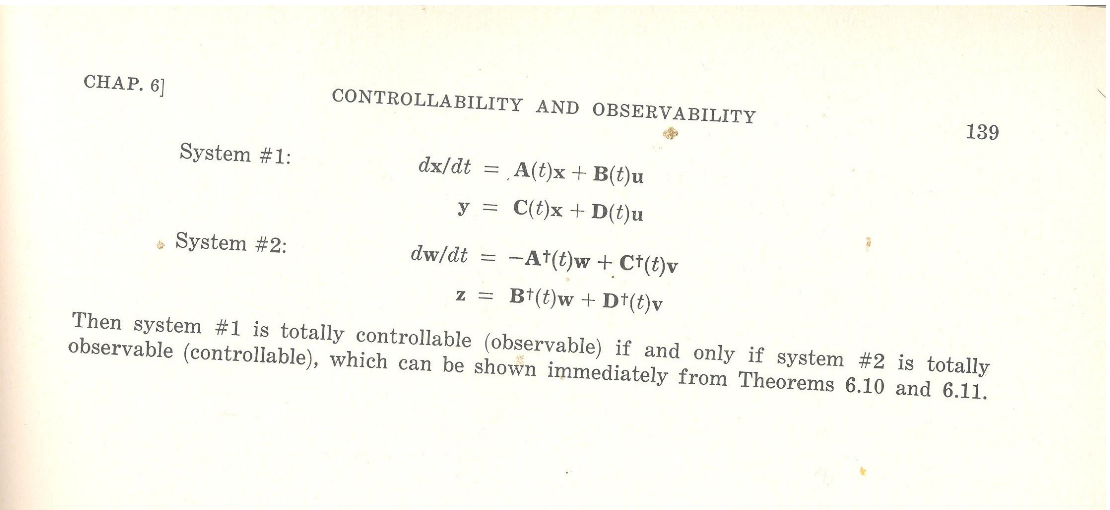
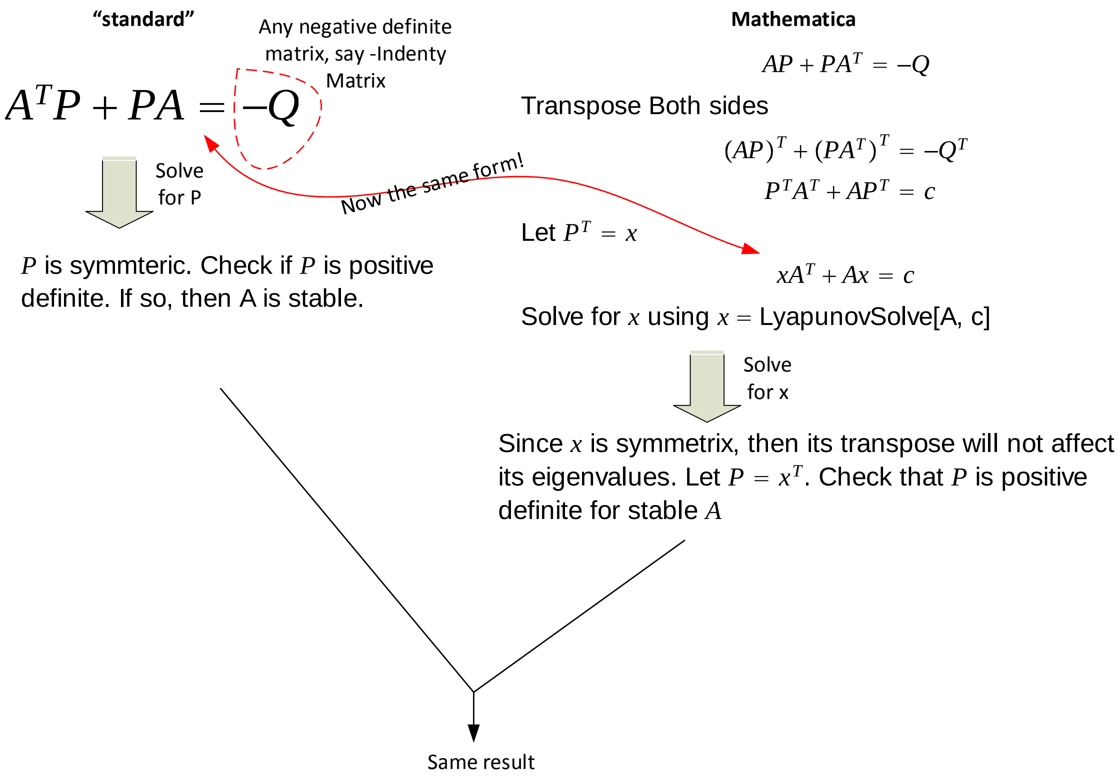


CHAP. 6] CONTROLLABILITY AND OBSERVABILITY 139

System #1: $$d\mathbf{x}/dt = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u}$$

System #2: $$d\mathbf{w}/dt = -\mathbf{A}^\dagger(t)\mathbf{w} + \mathbf{C}^\dagger(t)\mathbf{v}$$
$$\mathbf{z} = \mathbf{B}^\dagger(t)\mathbf{w} + \mathbf{D}^\dagger(t)\mathbf{v}$$

Then system #1 is totally controllable (observable) if and only if system #2 is totally observable (controllable), which can be shown immediately from Theorems 6.10 and 6.11.



"standard"

$$A^TP + PA = -Q$$

P is symmteric. Check if *P* is positive definite. If so, then A is stable.

Mathematica

$$AP + PA^T = -Q$$

Transpose Both sides

$$(AP)^T + (PA^T)^T = -Q^T$$

$$P^TA^T + AP^T = c$$

Let $P^T = x$

$$xA^T + Ax = c$$

Solve for x using $x =$ LyapunovSolve[A, c]

Since x is symmetrix, then its transpose will not affect its eigenvalues. Let $P = x^T$. Check that P is positive definite for stable A