University Course

# EMA 547
# Engineering analysis I

## University of Wisconsin, Madison
## Fall 2013

My Class Notes

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Fall 2013

# Contents

# Chapter 1

# Introduction

Took this course in Fall 2013. Part of MSc. in Engineering Mechanics.

Instructor: Professor Douglass L. Henderson

Hard course, lots of Math, and lots of HWs. but very useful. Teacher was very good in the subject, and very helpful, spending lots of time after class answering all students questions. Text book was good also.

Textbook: Peter V. OŇiel, Advanced Engineering Mathematics, 6th edition 2007.

Here is the `syllubus`

Link to school course description

# Chapter 2

# HW

## Local contents

## 2.1 Summary of HWs

### Local contents

| HW | about | grade |
|---|---|---|
| HW1 | first chapter. Solving first order ODE. Different types, Exact, integrating factor, generlaized integrating factors, Bernulli and Riccati. | 54.5/55 |
| HW2 | reduction of order, linear operators, linear dependent and independent, inhomogeneous ODE with constant coefficients, finding particular solutions | 50/55 |
| HW3 | variation of parameters, reduction of order, Euler Cauchy ODE, Wronskian | 39/40 |
| HW4 | Using Laplace to solve ODE's, system of equations, Electric circuit spring mass system, partial fractions | 41/41 |
| HW5 | More Laplace and inverse Laplace. Using Laplace on periodic functions convolution, using Laplace for solving PDE (the long problem) | 38/38 |
| extra credit | lots of differential equations | 25/25 |
| HW6 | solving integral equations | 46/48 |
| HW7 | system of equations, reduction/diagonalization, variation of parameters | 40/42 |
| HW8 | Fourier series | 26/26 |
| HW9 | Fourier transform | 23/23 |
| HW10 | complex variables | 33/36 |
| HW11 | complex variables | 56/70 |
| second extra credit | complex variables | 25/25 |

## 2.2 HW 1

### 2.2.1 Problems to solve

Homework Set No. 1
Due September 13, 2013

NEEP 547
DLH

Separable Eqs.; Solve the initial valued problems:

1. (4pts) O'Neal, page 20, prob. 14: $2yy' = e^{x-y^2}$; $y(4) = -2$
2. (4pts) O'Neal, page 20, prob. 15: $y\,y' = 2\,x\sec(3y)$; $y(2/3) = \pi/3$

Exact Differential Eqs.; Solve the initial valued problems:

3. (5pts) $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; $y(1) = \ln(2)$
4. (5pts) O'Neal, page 32, prob. 14: $e^y + (xe^y - 1)y' = 0$; $y(5) = 0$

General Integrating Factor:

5. (6pts) $(3x - y)\,dx + (3y + x)\,dy = 0$
6. (6pts) O'Neal, page 38, prob. 17; Solve the initial valued problem: $2xy + 3y' = 0$; y(0) = 4 (Hint; try $\mu(x,y) = y^a\,e^{bx^2}$, where $a$ and $b$ are constants)
7. (6pts) O'Neal, page 38, prob. 20; Solve the initial valued problem: $3x^2y + y^3 + 2xy^2y' = 0$; $y(2) = 1$

Homogenous, Bernoulli and Ricatti Eqs.:

8. (5pts) O'Neal, page 45 prob. 12; find the general solution: $x^3y' = x^2y - y^3$
9. (7pts) O'Neal, page 45, prob. 17; find the general solution: $y' = \frac{3x-y-9}{x+y+1}$
10. (5pts) Find the general solution: $(2x^2 - y^2)\,dx + 3xy\,dy = 0$
11. (4pts) Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new dependent variable $z$.

### 2.2.2 Problem 1, O'Neal page 20 (section 1.2) , problem 14

$2yy' = e^{x-y^2}; y(4) = -2$

Writing the above as

$$2y\frac{dy}{dx} = \frac{e^x}{e^{y^2}}$$
$$2ye^{y^2}dy = e^x dx$$

Hence the ode is separable. Integrating both sides gives

$$2\int ye^{y^2}\,dy = \int e^x\,dx \tag{1}$$

Using $\int ye^{y^2}\,dy = \frac{1}{2}e^{y^2}$ the above reduces to

$$e^{y^2} = e^x + C$$

Initial conditions are now used to find the constant of integration $C$. Letting $y = -2$ and $x = 4$ in the above gives

$$e^4 = e^4 + C$$
$$C = 0$$

Hence the solution is

$$e^{y^2} = e^x$$
$$y^2 = x$$

Or

$$y = \pm\sqrt{x}$$

To verify the solution, it is substituted back into the ode. When $y = \sqrt{x}$ the differential equation becomes $2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = e^{x-x}$ or $1 = 1$.

### 2.2.3 Problem 2, O'Neal page 20 (section 1.2), problem 15

Solve $yy' = 2x\sec(3y)$ with initial conditions $y\left(\frac{2}{3}\right) = \frac{\pi}{3}$

Writing the ODE as

$$\begin{aligned} y\frac{dy}{dx} &= 2x\sec\left(3y\right) \\ \frac{y}{\sec\left(3y\right)}dy &= 2xdx \end{aligned}$$

Hence it is separable. Integrating both side

$$\begin{aligned} \int\frac{y}{\sec\left(3y\right)}dy &= 2\int xdx \\ \int y\cos\left(3y\right)dy &= 2\int xdx \qquad (1) \end{aligned}$$

Using integration by parts for $\int y\cos\left(3y\right)dy$. $\int udv = [uv] - \int vdu$. Let $u = y; dv = \cos\left(3y\right)$ then

$$\begin{aligned} \int y\cos\left(3y\right)dy &= \left[y\frac{\sin\left(3y\right)}{3}\right]_{y_0}^{y} - \int\frac{\sin\left(3y\right)}{3}dy \\ &= \left[\frac{1}{3}y\sin\left(3y\right) - y_0\frac{\sin\left(3y_0\right)}{3}\right] - \left[\frac{1}{3}\left(-\frac{1}{3}\cos\left(3y\right)\right) + C\right] \\ &= \frac{1}{3}y\sin\left(3y\right) + \frac{1}{9}\cos\left(3y\right) + C_1 \end{aligned}$$

Where $C_1$ is new constant that includes $C$ and $y_0\frac{\sin\left(3y_0\right)}{3}$. Eq. (1) becomes

$$\frac{1}{3}y\sin\left(3y\right) + \frac{1}{9}\cos\left(3y\right) = x^2 + C_2$$

Where $C_2$ is new constant. Initial conditions are now used to find $C_2$. Letting $x = \frac{2}{3}$ and $y = \frac{\pi}{3}$ the above becomes

$$\begin{aligned} \frac{1}{3}\frac{\pi}{3}\sin\left(3\frac{\pi}{3}\right) + \frac{1}{9}\cos\left(3\frac{\pi}{3}\right) &= \left(\frac{2}{3}\right)^2 + C_2 \\ C_2 &= -\frac{1}{9} - \frac{4}{9} = \frac{-5}{9} \end{aligned}$$

Hence solution is

$$\boxed{3y\sin\left(3y\right) + \cos\left(3y\right) = 9x^2 - 5}$$

### 2.2.4 Problem 3

Solve $\left(2xy + e^y\right)dx + \left(x^2 + xe^y\right)dy = 0; y(1) = \ln(2)$

This ODE is not separable. To check if it is exact, it is written as

$$M\left(x,y\right)dx + N\left(x,y\right)dy = 0$$

Then $\frac{\partial M}{\partial y} = 2x + e^y$ and $\frac{\partial N}{\partial x} = 2x + e^y$. Since they are the same, the differential equation is exact.

Now let $\frac{\partial\varphi(x,y)}{\partial x} = M\left(x,y\right)$ and $\frac{\partial\varphi(x,y)}{\partial y} = N\left(x,y\right)$ and the ode becomes

$$\frac{\partial\varphi\left(x,y\right)}{\partial x} + \frac{\partial\varphi\left(x,y\right)}{\partial y}\frac{dy}{dx} = 0$$

or

$$\frac{d}{dx}\varphi\left(x,y\left(x\right)\right) = 0$$

Which means that $\varphi\left(x,y\left(x\right)\right) = C$. To find $\varphi\left(x,y\left(x\right)\right)$ the equation $\frac{\partial\varphi(x,y)}{\partial x} = M\left(x,y\right)$ is used and integrated as follows

$$\begin{aligned}\frac{\partial\varphi\left(x,y\right)}{\partial x} &= M\left(x,y\right) = 2xy + e^{y}\\ \varphi\left(x,y\right) &= \int\left(2xy + e^{y}\right)dx\\ &= x^{2}y + xe^{y} + g\left(y\right) \qquad (1)\end{aligned}$$

Where $g\left(y\right)$ is a function of $y$ that needs to be found. Since $\frac{\partial\varphi(x,y)}{\partial y} = N\left(x,y\right)$ then

$$\frac{\partial\varphi\left(x,y\right)}{\partial y} = \left(x^{2} + xe^{y}\right) \qquad (2)$$

But from Eq. (1) $\frac{\partial\varphi(x,y)}{\partial y} = x^{2} + xe^{y} + g'\left(y\right)$, hence Eq. (2) becomes

$$\begin{aligned}x^{2} + xe^{y} + g'\left(y\right) &= \left(x^{2} + xe^{y}\right)\\ g'\left(y\right) &= x^{2} + xe^{y} - x^{2} - xe^{y}\\ &= 0\end{aligned}$$

Hence, since $g'\left(y\right) = 0$ then $g\left(y\right) = 0$ can be chosen as solution. Therefore Eq. (1) becomes $\varphi\left(x,y\right) = x^{2}y + xe^{y}$ but $\varphi\left(x,y\left(x\right)\right) = C$ hence

$$x^{2}y + xe^{y} = C$$

Initial conditions are used to find $C$. Since $y = \ln(2)$ when $x = 1$, the above becomes $\ln(2) + e^{\ln(2)} = C$ or $C = \ln(2) + 2$.Therefore, the implicit solution is

$$\boxed{x^{2}y + xe^{y} = \ln(2) + 2}$$

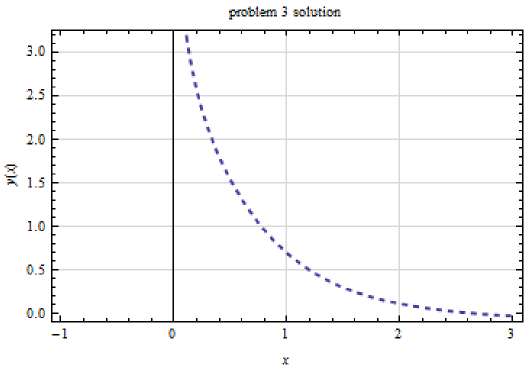


Figure 2.1: problem 3 solution

To verify: Taking derivative w.r.t. $x$ gives

$$\begin{aligned}2xy + x^{2}\frac{dy}{dx} + e^{y} + x\frac{dy}{dx}e^{y} &= 0\\ \left(2xy + e^{y}\right) + \frac{dy}{dx}\left(x^{2} + xe^{y}\right) &= 0\end{aligned}$$

Which is the same as the original ODE.

### 2.2.5 Problem 4 O'Neal page 32 (section 1.4), problem 14

Solve $e^{y} + \left(xe^{y} - 1\right)y' = 0; y(5) = 0$

Writing the above as $e^{y}dx+(xe^{y}-1)\,dy=0$.Letting $M\left(x,y\right)=e^{y}$ and $N\left(x,y\right)=(xe^{y}-1)$. To verify first this is exact: $\frac{\partial M}{\partial y}=e^{y}$ and $\frac{\partial N}{\partial x}=e^{y}$ hence they are the same and the differential equation is exact.

Letting $\frac{\partial\varphi(x,y)}{\partial x}=M\left(x,y\right)$ and $\frac{\partial\varphi(x,y)}{\partial y}=N\left(x,y\right)$ the ode becomes $\frac{\partial\varphi(x,y)}{\partial x}+\frac{\partial\varphi(x,y)}{\partial y}\frac{dy}{dx}=0$ or $\frac{d}{dx}\varphi\left(x,y\left(x\right)\right)=0$ hence $\varphi\left(x,y\left(x\right)\right)=C$. To find $\varphi\left(x,y\left(x\right)\right)$ the first equation $\frac{\partial\varphi(x,y)}{\partial x}=M\left(x,y\right)$ is used

$$\begin{aligned}\frac{\partial\varphi\left(x,y\right)}{\partial x}&=M\left(x,y\right)=e^{y}\\ \varphi\left(x,y\right)&=\int e^{y}dx\\ &=e^{y}+g\left(y\right) \qquad (1)\end{aligned}$$

Where $g\left(y\right)$ is a function of $y$ that needs to be found. Since $\frac{\partial\varphi(x,y)}{\partial y}=N\left(x,y\right)$ then

$$\frac{\partial\varphi\left(x,y\right)}{\partial y}=(xe^{y}-1) \qquad (2)$$

From Eq. (1) $\frac{\partial\varphi(x,y)}{\partial y}=e^{y}+g'\left(y\right)$, hence Eq. (2) becomes

$$\begin{aligned}e^{y}+g'\left(y\right)&=xe^{y}-1\\ g'\left(y\right)&=e^{y}\left(x-1\right)-1\end{aligned}$$

Hence $g\left(y\right)=\int\left(e^{y}\left(x-1\right)-1\right)dy=e^{y}\left(x-1\right)-y+C_{1}$ Therefore

$$\begin{aligned}\varphi\left(x,y\right)&=e^{y}+e^{y}\left(x-1\right)-y+C_{1}\\ &=xe^{y}-y+C_{1}\end{aligned}$$

but $\varphi\left(x,y\left(x\right)\right)=C$ hence

$$xe^{y}-y=C_{2}$$

Initial conditions are used to find $C_{2}$. Letting $y=0$ when $x=5$ the above becomes $5\left(1\right)-0=C_{2}$ hence $C_{2}=5$. Therefore, the implicit solution is

$$\begin{aligned}xe^{y}-y&=5\\ y\left(x\right)&=xe^{y(x)}-5\end{aligned}$$

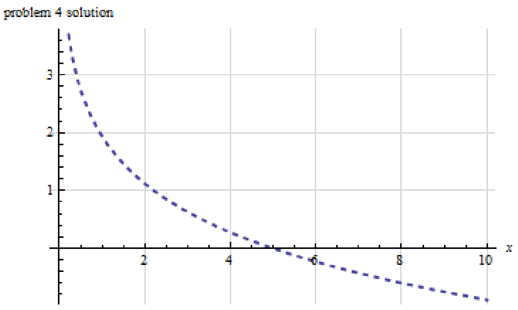


Figure 2.2: Problem 4 plot of solution

### 2.2.6 Problem 5

Solve $\left(3x-y\right)dx+\left(3y+x\right)dy=0$ using general integrating factor

The ODE is Checked if it is exact: $M\left(x,y\right)=3x-y$ hence $\frac{\partial M}{\partial y}=-1$ and $N\left(x,y\right)=3y+x$, then $\frac{\partial N}{\partial x}=1$. Therefore it is not exact. The ODE can be made exact by using a general integration factor.

Trying $I = \left(x^2+y^2\right)^p$ the ODE becomes

$$\left(x^2+y^2\right)^p\left(3x-y\right)dx + \left(x^2+y^2\right)^p\left(3y+x\right)dy = 0$$

Hence

$$\begin{aligned} M &= \left(x^2+y^2\right)^p\left(3x-y\right) \\ \frac{\partial M}{\partial y} &= p\left(x^2+y^2\right)^{p-1}\left(2y\right)\left(3x-y\right) - \left(x^2+y^2\right)^p \end{aligned}$$

and

$$\begin{aligned} N &= \left(x^2+y^2\right)^p\left(3y+x\right) \\ \frac{\partial N}{\partial x} &= p\left(x^2+y^2\right)^{p-1}(2x)\left(3y+x\right) + \left(x^2+y^2\right)^p \end{aligned}$$

For an exact ode the following is required

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \\ 2py\left(x^2+y^2\right)^{p-1}\left(3x-y\right) - \left(x^2+y^2\right)^p &= 2px\left(x^2+y^2\right)^{p-1}\left(3y+x\right) + \left(x^2+y^2\right)^p \end{aligned}$$

Dividing by $\left(x^2+y^2\right)^{p-1}$

$$\begin{aligned} p\left(2y\right)\left(3x-y\right) - \left(x^2+y^2\right) &= p(2x)\left(3y+x\right) + \left(x^2+y^2\right) \\ 2py\left(3x-y\right) - \left(x^2+y^2\right) - 2px\left(3y+x\right) - \left(x^2+y^2\right) &= 0 \end{aligned}$$

and Letting $p=-1$

$$\begin{aligned} -2y\left(3x-y\right) - \left(x^2+y^2\right) + 2x\left(3y+x\right) - \left(x^2+y^2\right) &= 0 \\ -6yx + 2y^2 - x^2 - y^2 + 6xy + 2x^2 - x^2 - y^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence the integrating factor is

$$I_f = \left(x^2+y^2\right)^{-1}$$

The ode is now multiplied by this integrating factor

$$\begin{aligned} I_f\left(3x-y\right)dx + I_f\left(3y+x\right)dy &= 0 \\ \left(x^2+y^2\right)^{-1}\left(3x-y\right)dx + \left(x^2+y^2\right)^{-1}\left(3y+x\right)dy &= 0 \end{aligned}$$

Now $M\left(x,y\right) = \left(x^2+y^2\right)^{-1}\left(3x-y\right)$ and $N\left(x,y\right) = \left(x^2+y^2\right)^{-1}\left(3y+x\right)$.

Let $\frac{\partial\varphi(x,y)}{\partial x} = M\left(x,y\right)$ and $\frac{\partial\varphi(x,y)}{\partial y} = N\left(x,y\right)$ the ode becomes $\frac{\partial\varphi(x,y)}{\partial x} + \frac{\partial\varphi(x,y)}{\partial y}\frac{dy}{dx} = 0$ or $\frac{d}{dx}\varphi\left(x,y(x)\right) = 0$ hence $\varphi\left(x,y(x)\right) = C$. To find $\varphi\left(x,y(x)\right)$,the first equation $\frac{\partial\varphi(x,y)}{\partial x} = M\left(x,y\right)$ is used

$$\begin{aligned} \frac{\partial\varphi\left(x,y\right)}{\partial x} &= M\left(x,y\right) = \left(x^2+y^2\right)^{-1}\left(3x-y\right) \\ \varphi\left(x,y\right) &= \int\frac{\left(3x-y\right)}{\left(x^2+y^2\right)}dx \\ &= \frac{3}{2}\ln\left(x^2+y^2\right) - \arctan\left(\frac{x}{y}\right) + g\left(y\right) \qquad (1) \end{aligned}$$

Where $g\left(y\right)$ is a function of $y$ that needs to be found. Since $\frac{\partial\varphi(x,y)}{\partial y} = N\left(x,y\right)$ then

$$\frac{\partial\varphi\left(x,y\right)}{\partial y} = \frac{\left(3y+x\right)}{x^2+y^2} \qquad (2)$$

But from Eq. (1) $\frac{\partial\varphi(x,y)}{\partial y} = \frac{x+3y}{y^2+x^2} + g'\left(y\right)$, hence Eq. (2) becomes

$$\begin{aligned} \frac{x+3y}{y^2+x^2} + g'\left(y\right) &= \frac{\left(3y+x\right)}{x^2+y^2} \\ g'\left(y\right) &= 0 \end{aligned}$$

Since $g'\left(y\right)=0$ then $g\left(y\right)=0$ is assumed. Therefore $\varphi\left(x,y\right)=\frac{3}{2}\ln\left(x^2+y^2\right)-\arctan\left(\frac{x}{y}\right)$ but $\varphi\left(x,y\left(x\right)\right)=C$ hence

$$\boxed{\frac{3}{2}\ln\left(x^2+y^2\right)-\arctan\left(\frac{x}{y}\right)=C}$$

To find $C$ initial conditions can be used but in this problem these are not given.

### 2.2.7 Problem 6 O'Neal page 38 (section 1.5), problem 17

Solve $2xy+3y'=0; y\left(0\right)=4$, using $I_f=y^ae^{bx^2}$ where $a,b$ are constants

Here $M\left(x,y\right)=2xy$ and $N\left(x,y\right)=3$, hence the ODE is not exact. Multiplying the ODE by $I_f$ gives

$$y^ae^{bx^2}\left(2xy\right)dx+y^ae^{bx^2}\left(3\right)dy=0$$

Hence

$$\begin{aligned} M&=2xy^{a+1}e^{bx^2}\\ \frac{\partial M}{\partial y}&=2\left(a+1\right)xy^ae^{bx^2}\end{aligned}$$

and

$$\begin{aligned} N&=3y^ae^{bx^2}\\ \frac{\partial N}{\partial x}&=6bxy^ae^{bx^2}\end{aligned}$$

So for exact the following is required

$$\begin{aligned} \frac{\partial M}{\partial y}&=\frac{\partial N}{\partial x}\\ 2\left(a+1\right)xy^ae^{bx^2}&=6bxy^ae^{bx^2}\\ \left(a+1\right)&=3b\end{aligned}$$

Hence $b=1,a=2$. Therefore

$$I_f=y^2e^{x^2}$$

Multiplying the ODE by the above integrating factor gives

$$\left(y^2e^{x^2}\right)2xy+\left(y^2e^{x^2}\right)3y'=0=0$$

Where now $M\left(x,y\right)=\left(y^2e^{x^2}\right)2xy$ and $N\left(x,y\right)=3\left(y^2e^{x^2}\right)$

Letting $\frac{\partial\varphi(x,y)}{\partial x}=M\left(x,y\right)$ and $\frac{\partial\varphi(x,y)}{\partial y}=N\left(x,y\right)$ the ode becomes $\frac{\partial\varphi(x,y)}{\partial x}+\frac{\partial\varphi(x,y)}{\partial y}\frac{dy}{dx}=0$ or $\frac{d}{dx}\varphi\left(x,y\left(x\right)\right)=0$ hence $\varphi\left(x,y\left(x\right)\right)=C$. To find $\varphi\left(x,y\left(x\right)\right)$ the first equation $\frac{\partial\varphi(x,y)}{\partial x}=M\left(x,y\right)$ is used

$$\begin{aligned} \frac{\partial\varphi\left(x,y\right)}{\partial x}&=M\left(x,y\right)=\left(y^2e^{x^2}\right)2xy\\ \varphi\left(x,y\right)&=\int 2y^3xe^{x^2}dx\\ &=y^3e^{x^2}+g\left(y\right) \qquad (1)\end{aligned}$$

Where $g\left(y\right)$ is a function of $y$ that needs to be found. Since $\frac{\partial\varphi(x,y)}{\partial y}=N\left(x,y\right)$ then

$$\frac{\partial\varphi\left(x,y\right)}{\partial y}=3y^2e^{x^2} \qquad (2)$$

From Eq. (1) $\frac{\partial\varphi(x,y)}{\partial y}=3y^2e^{x^2}+g'\left(y\right)$, hence Eq. (2) becomes

$$\begin{aligned} 3y^2e^{x^2}+g'\left(y\right)&=3y^2e^{x^2}\\ g'\left(y\right)&=0\end{aligned}$$

Since $g'\left(y\right)=0$ then $g\left(y\right)=0$ is assumed. Therefore $\varphi\left(x,y\right)=y^3e^{x^2}$ but $\varphi\left(x,y\left(x\right)\right)=C$

hence

$$y^3 e^{x^2} = C$$

Initial conditions are used to find $C$

$$\begin{aligned}(4)^3 e^0 &= C \\ 4^3 &= C\end{aligned}$$

Hence the solution is

$$y(x) = \left(4^3 e^{-x^2}\right)^{\frac{1}{3}}$$

or

$$y(x) = 4e^{-\frac{x^2}{3}}$$

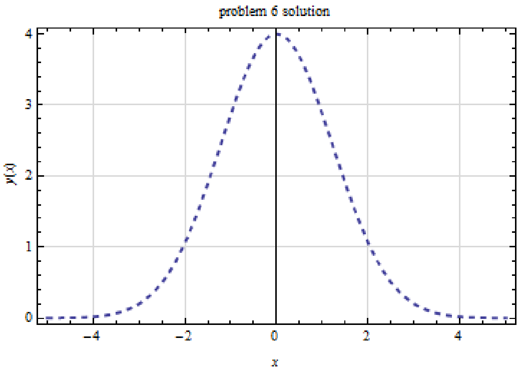


Figure 2.3: problem 6 plot of solution

#### 2.2.7.1 small question on the above problem

Why can't one start by dividing by $y$ in order to obtain $2x + \frac{3}{y}y' = 0$. Hence now $M = 2x$ and $N = \frac{3}{y}$ and so it is exact. Therefore

$\frac{\partial \phi}{\partial x} = M = 2x$, hence $\phi = \int 2x dx = x^2 + g\left(y\right)$. This means $\frac{\partial \phi}{\partial y} = g'\left(y\right)$ but we also know that $\frac{\partial \phi}{\partial y} = N = \frac{3}{y}$, therefore

$$g'\left(y\right) = \frac{3}{y}$$

Hence $g\left(y\right) = 3\ln y + c_1$ where we can set $c_1$ to any values and we choose zero. Therefore since the potential $\phi\left(x,y\right) = C$ some constant, this means

$$\begin{aligned}C &= x^2 + g\left(y\right) \\ C &= x^2 + 3\ln y\end{aligned}$$

From initial conditions $C = 0 + 3\ln 4$, hence

$$\begin{aligned}3\ln 4 &= x^2 + 3\ln y \\ \ln y &= \ln 4 - \frac{1}{3}x^2\end{aligned}$$

Or

$$\begin{aligned}y &= e^{\ln 4 - \frac{1}{3}x^2} \\ &= 4e^{\frac{x^2}{3}}\end{aligned}$$

Which is the same solution found inthe key solution.

### 2.2.8 Problem 7 O'Neal page 38 (section 1.5), problem 20

Solve $3x^2y + y^3 + 2xy^2y' = 0; y(2) = 1$

Here $M\left(x,y\right) = 3x^2y + y^3$ and $N\left(x,y\right) = 2xy^2$, hence $\frac{\partial M}{\partial y} = 3x^2 + 3y^2$ and $\frac{\partial N}{\partial x} = 2y^2$, therefore the ODE is not exact.

Multiplying the ODE by $I_f = \frac{1}{y}$ gives

$$3x^2 + y^2 + 2xyy' = 0$$

To verify that the ODE is not exact

$$\begin{aligned} M &= 3x^2 + y^2 \\ \frac{\partial M}{\partial y} &= 2y \end{aligned}$$

and

$$\begin{aligned} N &= 2xy \\ \frac{\partial N}{\partial x} &= 2y \end{aligned}$$

They are the same, hence the ODE is exact. Picking the first equation $\frac{\partial \varphi(x,y)}{\partial x} = M\left(x,y\right)$ and integrating

$$\begin{aligned} \frac{\partial \varphi\left(x,y\right)}{\partial x} &= 3x^2 + y^2 \\ \varphi\left(x,y\right) &= \int 3x^2 + y^2 dx \\ &= x^3 + y^2 x + g\left(y\right) \end{aligned} \tag{1}$$

Where $g\left(y\right)$ is a function of $y$ that needs to be found. Since $\frac{\partial \varphi(x,y)}{\partial y} = N\left(x,y\right)$ then

$$\frac{\partial \varphi\left(x,y\right)}{\partial y} = 2xy \tag{2}$$

But from Eq. (1) $\frac{\partial \varphi(x,y)}{\partial y} = 2yx + g'\left(y\right)$, hence Eq. (2) becomes

$$\begin{aligned} 2yx + g'\left(y\right) &= 2xy \\ g'\left(y\right) &= 0 \end{aligned}$$

Since $g'\left(y\right) = 0$ then $g\left(y\right) = 0$ is assumed. Therefore $\varphi\left(x,y\right) = x^3 + y^2 x$ but $\varphi\left(x, y\left(x\right)\right) = C$ hence

$$x^3 + y^2 x = C$$

Initial conditions are used to find the constant of integration. Using $y\left(2\right) = 1$

$$\begin{aligned} 2^3 + 1^2\left(2\right) &= C \\ C &= 10 \end{aligned}$$

Hence the solution

$$\begin{aligned} x^3 + y^2 x &= 10 \\ y &= \pm\sqrt{\frac{10 - x^3}{x}} \end{aligned}$$

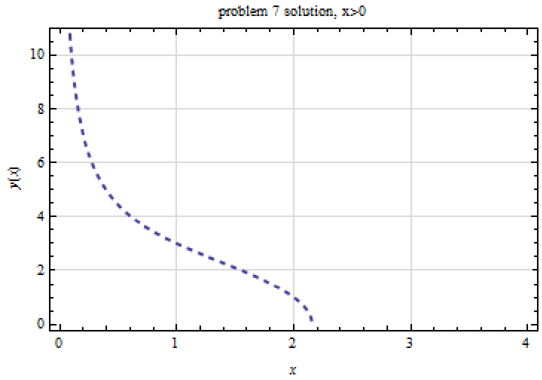


Figure 2.4: problem 7 plot of solution

### 2.2.9 Problem 8 O'Neal page 45 (section 1.6), problem 12

Find the general solution to $x^3 y' = x^2 y - y^3$

The ODE is

$$\frac{dy}{dx} - \frac{1}{x} y = -\frac{1}{x^3} y^3 \tag{1}$$

hence it is in the form

$$y' + p(x) y = f(x) y^n$$

where $p(x) = \frac{-1}{x}$ and $f(x) = \frac{-1}{x^3}$ therefore a Bernoulli equation of third order since $n = 3$. To solve, it is first linearized using $u = y^{1-n}$

$$\begin{aligned} u &= y^{-2} \qquad (2) \\ \frac{du}{dy} &= -2y^{-3} \end{aligned}$$

Hence $\frac{dy}{du} = \frac{-1}{2} y^3$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{-1}{2} y^3 \frac{du}{dx} \end{aligned}$$

Also from Eq. (2)

$$y = u\, y^3$$

Substituting the above 2 equations into Eq. (1) gives

$$\frac{-1}{2} y^3 \frac{du}{dx} - \frac{1}{x} u\, y^3 = -\frac{1}{x^3} y^3$$

Dividing by $y^3$ gives

$$\begin{aligned} \frac{-1}{2}\frac{du}{dx} - \frac{1}{x} u &= -\frac{1}{x^3} \\ \frac{du}{dx} + \frac{2}{x} u &= \frac{2}{x^3} \end{aligned}$$

This is a linear ODE in $u(x)$. Multiplying by integrating factor $I$

$$I\frac{du}{dx} + I\frac{2}{x} u = I\frac{2}{x^3}$$

But $\frac{d}{dx}(Iu) = I'u + Iu'$, hence comparing to the above shows that $I' = I\frac{2}{x}$ or $I = e^{\int^x \frac{2}{t} dt} = e^{2\ln x} = x^2$. The ode can now be written as

$$\begin{aligned} \frac{d}{dx}(Iu) &= I\frac{2}{x^3} \\ ux^2 &= \int \frac{2}{x^3} x^2 dx + C \\ &= 2\ln x + C \\ u(x) &= \frac{1}{x^2}(2\ln x + C) \end{aligned}$$

But $u = y^{-2}$ or $y = \frac{1}{\sqrt{u}}$, hence

$$\begin{aligned} y(x) &= \pm\frac{1}{\sqrt{\frac{1}{x^2}(2\ln x + C)}} \\ &= \pm\frac{1}{x\sqrt{2\ln x + C}} \end{aligned}$$

### 2.2.10 Problem 9 O'Neal page 45 (section 1.6), problem 17

Solve $y' = \frac{3x-y-9}{x+y+1}$

This is of the form $y' = F\left(\frac{ax+by+c}{dx+ey+r}\right)$, since $ae = 3$ and $bd = -1$ hence $ae - bd = 4 \neq 0$, therefore this can be transformed to nearly homogeneous using $X = x + h$ and $Y = y + k$ as follows.

$x = X - h$, hence $dx = dX$ and $y = Y - k$, hence $dy = dY$. The ODE becomes

$$\begin{aligned}\frac{dY}{dX} &= \frac{3(X-h)-(Y-k)-9}{(X-h)+(Y-k)+1}\\ &= \frac{3X-3h-Y+k-9}{X-h+Y-k+1}\\ &= \frac{3X-Y+(-3h+k-9)}{X+Y+(-h-k+1)}\end{aligned}$$

Now $h,k$ are found to make $(-3h+k-9)=0$ and $(-h-k+1)=0$. Solving for these gives $h=-2,k=3$, hence

$$\begin{aligned}x &= X+2\\ y &= Y-3\end{aligned}$$

And the ODE becomes

$$\begin{aligned}\frac{dY}{dX} &= \frac{3X-Y}{X+Y}\\ &= \frac{3-Y/X}{1+Y/X} \qquad (1)\end{aligned}$$

For $\boxed{X \neq 0}$ or $x \neq 2$. Letting $Y = UX$, then $\frac{dY}{dX} = \frac{dU}{dX}X + U$ and Eq. (1) becomes

$$\begin{aligned}\frac{dU}{dX}X + U &= \frac{3-U}{1+U}\\ \frac{dU}{dX}X &= \frac{3-U}{1+U} - U\\ &= \frac{3-U-U(1+U)}{1+U}\\ &= \frac{3-2U-U^2}{1+U}\end{aligned}$$

Hence it is now separable

$$\begin{aligned}\frac{1+U}{3-2U-U^2}dU &= \frac{1}{X}dX\\ \int \frac{1+U}{3-2U-U^2}dU &= \ln(X) + C\\ -\frac{1}{2}\ln\left(3-2U-U^2\right) &= \ln(X) + C\\ 3-2U-U^2 &= e^{-2\ln(X)+C}\\ 3-2U-U^2 &= CX^{-2}\end{aligned}$$

Since $Y = UX$ then

$$\begin{aligned}3-2\left(\frac{Y}{X}\right)-\left(\frac{Y}{X}\right)^2 &= CX^{-2}\\ 3X^2-2XY-Y^2 &= C\end{aligned}$$

Transforming back to $x,y$. From $X = x-2$ and $Y = y+3$, the above becomes

$$3(x-2)^2 - 2(x-2)\left(y+3\right) - \left(y+3\right)^2 = C$$

The above is the general solution.

### 2.2.11 Problem 10

Find general solution to $\left(2x^2 - y^2\right)dx + 3xydy = 0$

This is of the form

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2-2x^2}{3xy}\\ \frac{dy}{dx} &= \frac{y^2}{3xy} - \frac{2x^2}{3xy}\\ \frac{dy}{dx} - \frac{1}{3x}y &= -\frac{2x}{3}y^{-1} \qquad (1)\\ y' + p(x)y &= f(x)y^n\end{aligned}$$

Where $p(x) = \frac{-1}{3x}$ and $f(x) = -\frac{2x}{3}$ and $n = -1$, hence it is a Bernoulli equation of order $-1$. Using the substitution $u = y^{1-n}$

$$\begin{aligned} u &= y^2 \qquad (2)\\ \frac{du}{dy} &= 2y \end{aligned}$$

Hence $\frac{dy}{du} = \frac{1}{2y}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{1}{2y}\frac{du}{dx} \end{aligned}$$

And from Eq. (2)

$$y = \frac{u}{y}$$

Substituting the above 2 equations into Eq (1) gives

$$\frac{1}{2y}\frac{du}{dx} - \frac{1}{3x}\frac{u}{y} = -\frac{2x}{3y}$$

The term $\frac{1}{y}$ cancels resulting in

$$\begin{aligned} \frac{1}{2}\frac{du}{dx} - \frac{1}{3x}u &= -\frac{2x}{3}\\ \frac{du}{dx} - \frac{2}{3x}u &= -\frac{4x}{3} \end{aligned}$$

This is a linear ODE in $u(x)$. Multiplying by an integrating factor $I$

$$I\frac{du}{dx} - I\frac{2}{3x}u = I\left(\frac{-4x}{3}\right)$$

But $\frac{d}{dx}(Iu) = I'u + Iu'$, hence $I' = -I\frac{2}{3x}$ or $I = e^{\int^x \frac{-2}{3t}dt} = e^{-\frac{2}{3}\ln x} = x^{-\frac{2}{3}}$, and the above ode becomes

$$\begin{aligned} \frac{d}{dx}(Iu) &= I\left(\frac{-4x}{3}\right)\\ Iu &= \int\left(\frac{-4x}{3}\right)Idx + C\\ u &= \frac{1}{I}\int\left(\frac{-4x}{3}\right)Idx + \frac{C}{I} \end{aligned}$$

But $I = x^{-\frac{2}{3}}$, therefore

$$\begin{aligned} u &= \frac{1}{x^{-\frac{2}{3}}}\int\left(\frac{-4x}{3}\right)x^{-\frac{2}{3}}dx + \frac{C}{x^{-\frac{2}{3}}}\\ &= -\frac{4}{3}x^{\frac{2}{3}}\int x^{\frac{1}{3}}dx + Cx^{\frac{2}{3}}\\ &= -\frac{4}{3}x^{\frac{2}{3}}\frac{3}{4}x^{\frac{4}{3}} + Cx^{\frac{2}{3}}\\ &= -x^2 + Cx^{\frac{2}{3}} \end{aligned}$$

Since $u = y^2$, then $y = \pm\sqrt{u}$, and the solution becomes

$$y(x) = \pm\sqrt{-x^2 + Cx^{\frac{2}{3}}}$$

### 2.2.12 Problem 11

Problem: Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new dependent variable $z$.

Using $y = u + \frac{1}{z}$ then $y' = u' - \frac{1}{z^2}z'$. Substituting this into the original ODE

$$\begin{aligned}
u' - \frac{1}{z^2}z' &= P\left(u + \frac{1}{z}\right)^2 + Q\left(u + \frac{1}{z}\right) + R \\
z^2u' - z' &= z^2P\left(u^2 + \frac{1}{z^2} + 2\frac{u}{z}\right) + z^2Qu + z^2\frac{Q}{z} + z^2R \\
z^2u' - z' &= P\left(z^2u^2 + 1 + 2zu\right) + z^2Qu + zQ + z^2R \\
z' &= z^2u' - Pz^2u^2 - P - 2Pzu - z^2Qu - zQ - z^2R \\
&= z^2\left(u' - Pu^2 - Qu - R\right) + z\left(-2Pu - Q\right) - P \qquad (1)
\end{aligned}$$

For this to be linear, the term multiplying $z^2$ must vanish. Hence

$$\Delta = u' - Pu^2 - Qu - R$$

must be shown to be zero. Now, since $u' = y'$ and $u = y$ and $u^2 = y^2$ this term can be written as

$$\Delta = y' - \left(Py^2 + Qy + R\right)$$

But from the original ODE itself, it is seen that $y' = Py^2 + Qy + R$, therefore

$$\Delta = 0$$

And Eq(1) becomes

$$z' = z\left(-2Pu - Q\right) - P$$

Or

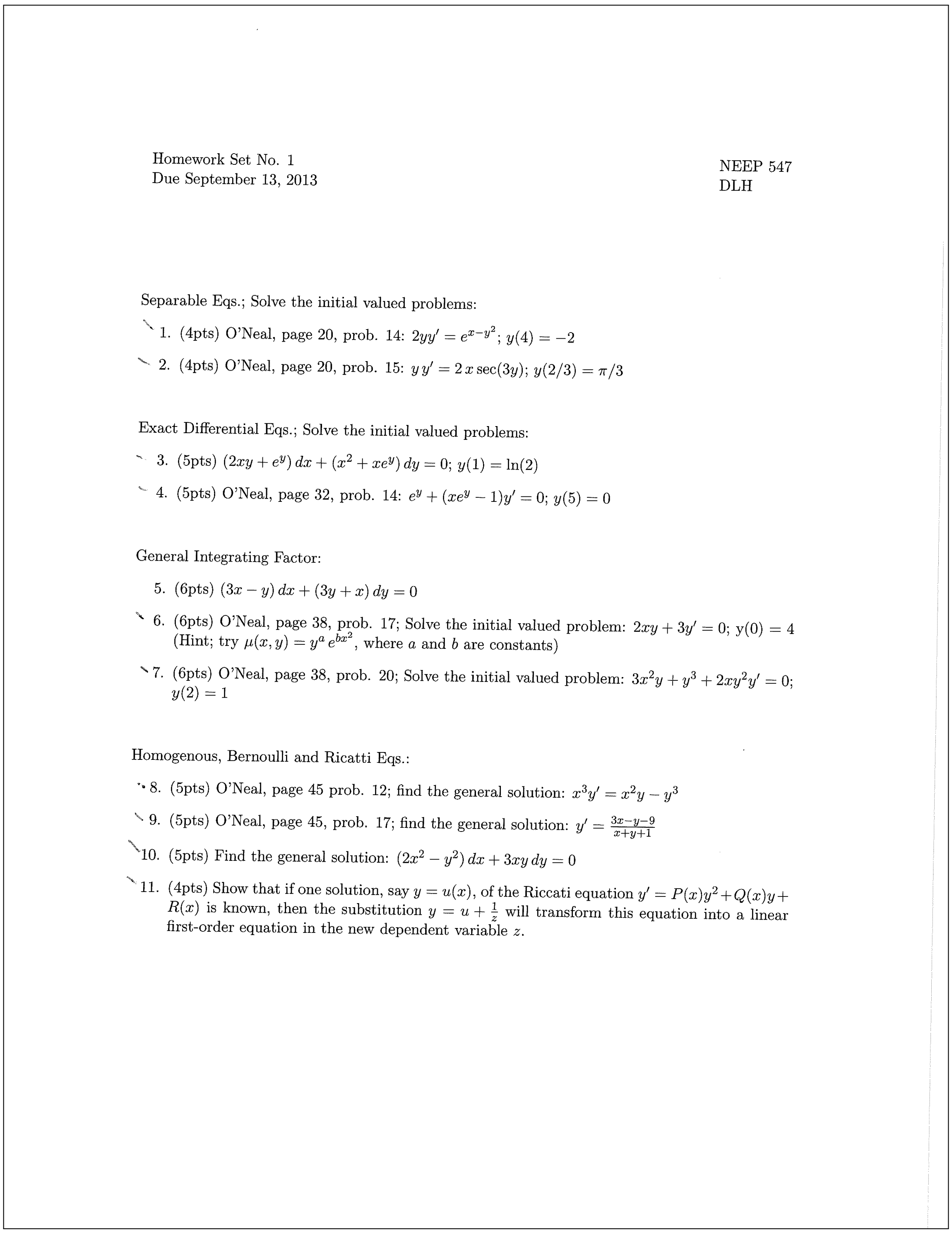
$$z'(x) + z(x)\left(2P(x)u(x) + Q(x)\right) = -P(x)$$

This is now in the form

$$z'(x) + A(x)z(x) = B(x)$$

Where $A(x) = 2P(x)u(x) + Q(x)$ and $B(x) = -P(x)$. Since this derivation was carried out using general expressions, then it is valid for any solution $u(x)$. Hence if one solution is known, the ODE can be transformed to linear first order in the new variable.

### 2.2.13 key solution



Homework Set No. 1
Due September 13, 2013

NEEP 547
DLH

Separable Eqs.; Solve the initial valued problems:

1. (4pts) O'Neal, page 20, prob. 14: $2yy' = e^{x-y^2}$; $y(4) = -2$
2. (4pts) O'Neal, page 20, prob. 15: $y\,y' = 2\,x\sec(3y)$; $y(2/3) = \pi/3$

Exact Differential Eqs.; Solve the initial valued problems:

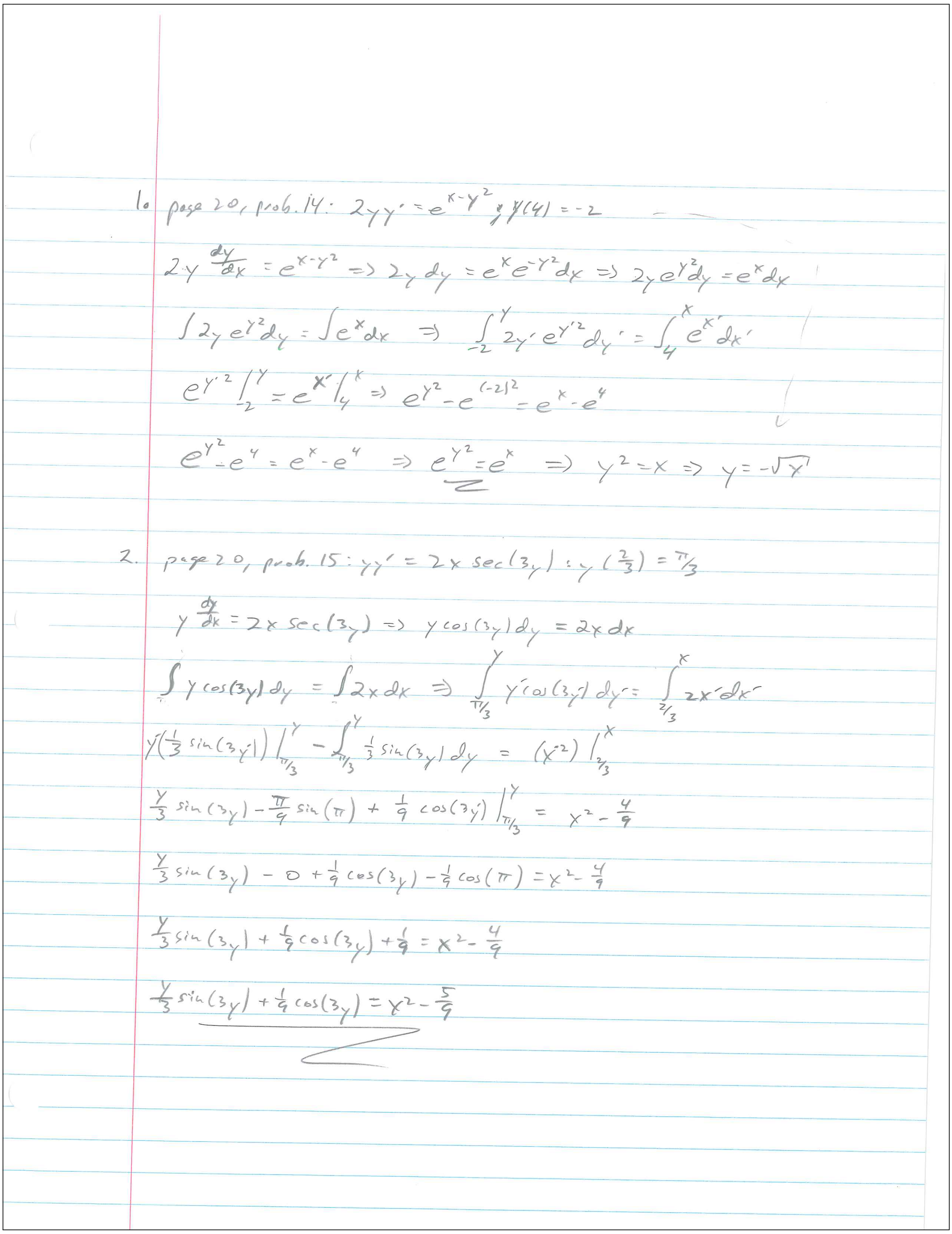
3. (5pts) $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; $y(1) = \ln(2)$
4. (5pts) O'Neal, page 32, prob. 14: $e^y + (xe^y - 1)y' = 0$; $y(5) = 0$

General Integrating Factor:

5. (6pts) $(3x - y)\,dx + (3y + x)\,dy = 0$
6. (6pts) O'Neal, page 38, prob. 17; Solve the initial valued problem: $2xy + 3y' = 0$; y(0) = 4 (Hint; try $\mu(x, y) = y^a\,e^{bx^2}$, where $a$ and $b$ are constants)
7. (6pts) O'Neal, page 38, prob. 20; Solve the initial valued problem: $3x^2y + y^3 + 2xy^2y' = 0$; $y(2) = 1$

Homogenous, Bernoulli and Ricatti Eqs.:

8. (5pts) O'Neal, page 45 prob. 12; find the general solution: $x^3y' = x^2y - y^3$
9. (5pts) O'Neal, page 45, prob. 17; find the general solution: $y' = \frac{3x-y-9}{x+y+1}$
10. (5pts) Find the general solution: $(2x^2 - y^2)\,dx + 3xy\,dy = 0$
11. (4pts) Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new dependent variable $z$.



1. page 20, prob. 14: $2yy' = e^{x-y^2}$; $y(4) = -2$

$$2y\frac{dy}{dx} = e^{x-y^2} \Rightarrow 2y\,dy = e^{x}e^{-y^2}dx \Rightarrow 2ye^{y^2}dy = e^{x}dx$$

$$\int 2ye^{y^2}dy = \int e^{x}dx \Rightarrow \int_{-2}^{y} 2y'e^{y'^2}dy' = \int_{4}^{x} e^{x'}dx'$$

$$e^{y'^2}\Big|_{-2}^{y} = e^{x'}\Big|_{4}^{x} \Rightarrow e^{y^2} - e^{(-2)^2} = e^{x} - e^{4}$$

$$e^{y^2} - e^{4} = e^{x} - e^{4} \Rightarrow e^{y^2} = e^{x} \Rightarrow y^2 = x \Rightarrow y = -\sqrt{x}$$

2. page 20, prob. 15: $yy' = 2x\sec(3y)$ : $y\left(\frac{2}{3}\right) = \pi/3$

$$y\frac{dy}{dx} = 2x\sec(3y) \Rightarrow y\cos(3y)dy = 2x\,dx$$

$$\int y\cos(3y)dy = \int 2x\,dx \Rightarrow \int_{\pi/3}^{y} y'\cos(3y')dy' = \int_{2/3}^{x} 2x'dx'$$

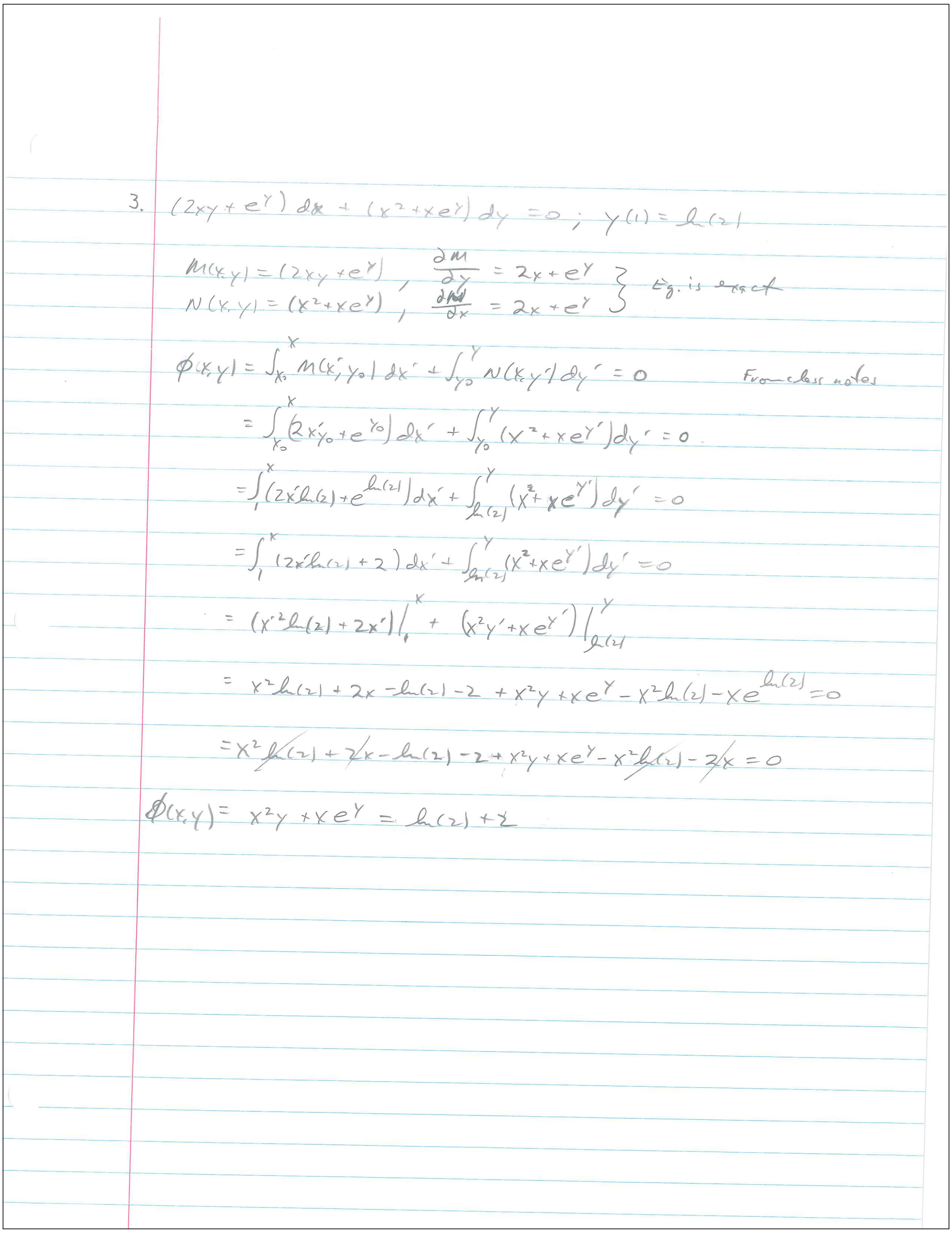
$$y'\left(\frac{1}{3}\sin(3y')\right)\Big|_{\pi/3}^{y} - \int_{\pi/3}^{y}\frac{1}{3}\sin(3y)dy = (x'^2)\Big|_{2/3}^{x}$$

$$\frac{y}{3}\sin(3y) - \frac{\pi}{9}\sin(\pi) + \frac{1}{9}\cos(3y')\Big|_{\pi/3}^{y} = x^2 - \frac{4}{9}$$

$$\frac{y}{3}\sin(3y) - 0 + \frac{1}{9}\cos(3y) - \frac{1}{9}\cos(\pi) = x^2 - \frac{4}{9}$$

$$\frac{y}{3}\sin(3y) + \frac{1}{9}\cos(3y) + \frac{1}{9} = x^2 - \frac{4}{9}$$

$$\underline{\frac{y}{3}\sin(3y) + \frac{1}{9}\cos(3y) = x^2 - \frac{5}{9}}$$



3. $(2xy+e^y)\,dx + (x^2+xe^y)\,dy = 0 \; ; \quad y(1) = \ln(2)$

$M(x,y) = (2xy+e^y)\,, \quad \dfrac{\partial M}{\partial y} = 2x+e^y$

$N(x,y) = (x^2+xe^y)\,, \quad \dfrac{\partial N}{\partial x} = 2x+e^y$

Eq. is exact

$$\phi(x,y) = \int_{x_0}^{x} M(x',y_0)\,dx' + \int_{y_0}^{y} N(x,y')\,dy' = 0$$

From class notes

$$= \int_{x_0}^{x} (2x'y_0+e^{y_0})\,dx' + \int_{y_0}^{y} (x^2+xe^{y'})\,dy' = 0$$

$$= \int_{1}^{x} (2x'\ln(2)+e^{\ln(2)})\,dx' + \int_{\ln(2)}^{y} (x^2+xe^{y'})\,dy' = 0$$

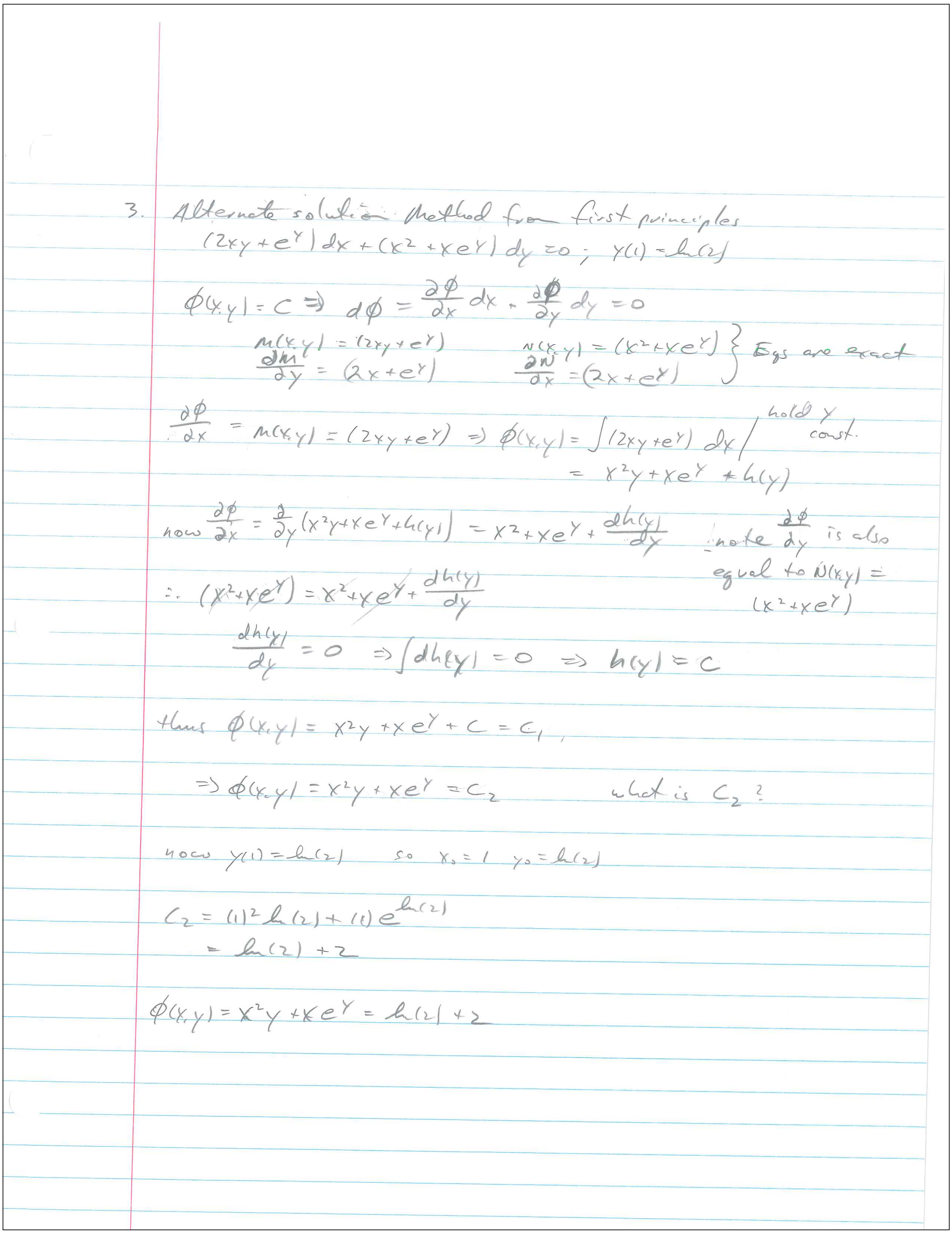
$$= \int_{1}^{x} (2x'\ln(2)+2)\,dx' + \int_{\ln(2)}^{y} (x^2+xe^{y'})\,dy' = 0$$

$$= (x'^2\ln(2)+2x')\Big|_{1}^{x} + (x^2y'+xe^{y'})\Big|_{\ln(2)}^{y}$$

$$= x^2\ln(2)+2x-\ln(2)-2+x^2y+xe^y-x^2\ln(2)-xe^{\ln(2)} = 0$$

$$= x^2\cancel{\ln(2)}+\cancel{2}x-\ln(2)-2+x^2y+xe^y-\cancel{x^2\ln(2)}-\cancel{2x} = 0$$

$$\phi(x,y) = x^2y+xe^y = \ln(2)+2$$



3. Alternate solution method from first principles

$(2xy+e^y)dx+(x^2+xe^y)dy=0\ ;\quad y(1)=\ln(2)$

$\phi(x,y)=C \Rightarrow d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=0$

$M(x,y)=(2xy+e^y)$ $\quad N(x,y)=(x^2+xe^y)$

$\frac{\partial M}{\partial y}=(2x+e^y)$ $\quad \frac{\partial N}{\partial x}=(2x+e^y)$ $\Big\}$ Eqs are exact

$\frac{\partial\phi}{\partial x}=M(x,y)=(2xy+e^y) \Rightarrow \phi(x,y)=\int(2xy+e^y)\,dx\Big/_{\text{hold } y \text{ const.}}$

$=x^2y+xe^y+h(y)$

now $\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial y}(x^2y+xe^y+h(y))=x^2+xe^y+\frac{dh(y)}{dy}$ $\quad$ note $\frac{\partial\phi}{\partial y}$ is also equal to $N(x,y)=(x^2+xe^y)$

$\therefore\ (\cancel{x^2+xe^y})=\cancel{x^2+xe^y}+\frac{dh(y)}{dy}$

$\frac{dh(y)}{dy}=0 \Rightarrow \int dh(y)=0 \Rightarrow h(y)=C$

thus $\phi(x,y)=x^2y+xe^y+C=C_1$

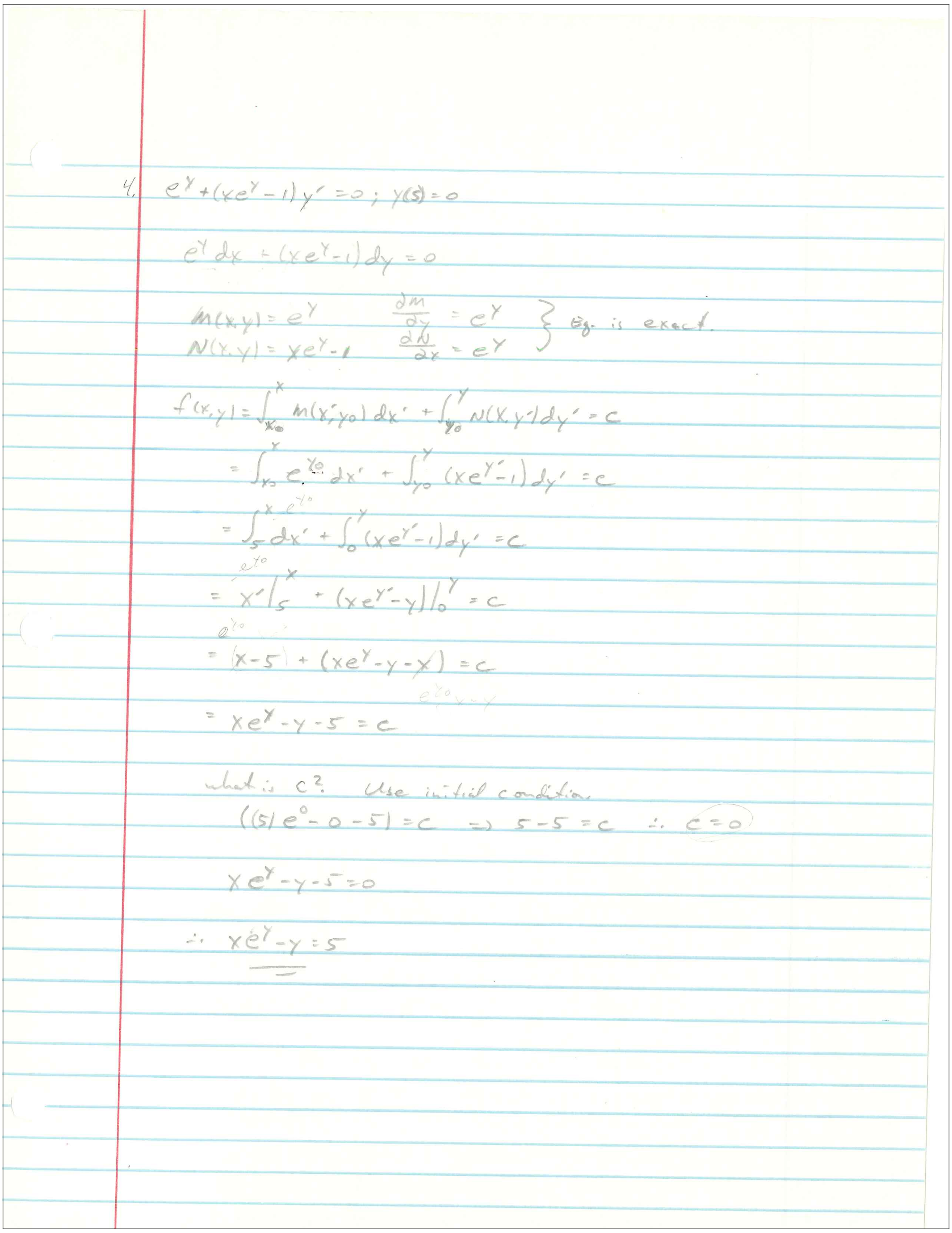
$\Rightarrow \phi(x,y)=x^2y+xe^y=C_2$ $\qquad$ what is $C_2$?

now $y(1)=\ln(2)$ $\quad$ so $x_0=1$ $\quad y_0=\ln(2)$

$C_2=(1)^2\ln(2)+(1)e^{\ln(2)}$

$=\ln(2)+2$

$\phi(x,y)=x^2y+xe^y=\ln(2)+2$



4. $e^y + (xe^y - 1)y' = 0 \; ; \; y(5) = 0$

$e^y\,dx + (xe^y - 1)\,dy = 0$

$\left.\begin{array}{ll} M(x,y) = e^y & \dfrac{\partial M}{\partial y} = e^y \\ N(x,y) = xe^y - 1 & \dfrac{\partial N}{\partial x} = e^y \end{array}\right\}$ Eq. is exact.

$f(x,y) = \int_{x_0}^{x} M(x', y_0)\,dx' + \int_{y_0}^{y} N(x, y')\,dy' = c$

$= \int_{x_0}^{x} e^{y_0}\,dx' + \int_{y_0}^{y} (xe^{y'} - 1)\,dy' = c$

$= \int_{5}^{x} e^{y_0}\,dx' + \int_{0}^{y} (xe^{y'} - 1)\,dy' = c$

$= \overset{e^{y_0}}{x'}\Big|_5^x + (xe^{y'} - y')\Big|_0^y = c$

$= \overset{e^{y_0}}{(x-5)} + (xe^y - y - \cancel{x}) = c$

$e^{y_0}x - y$

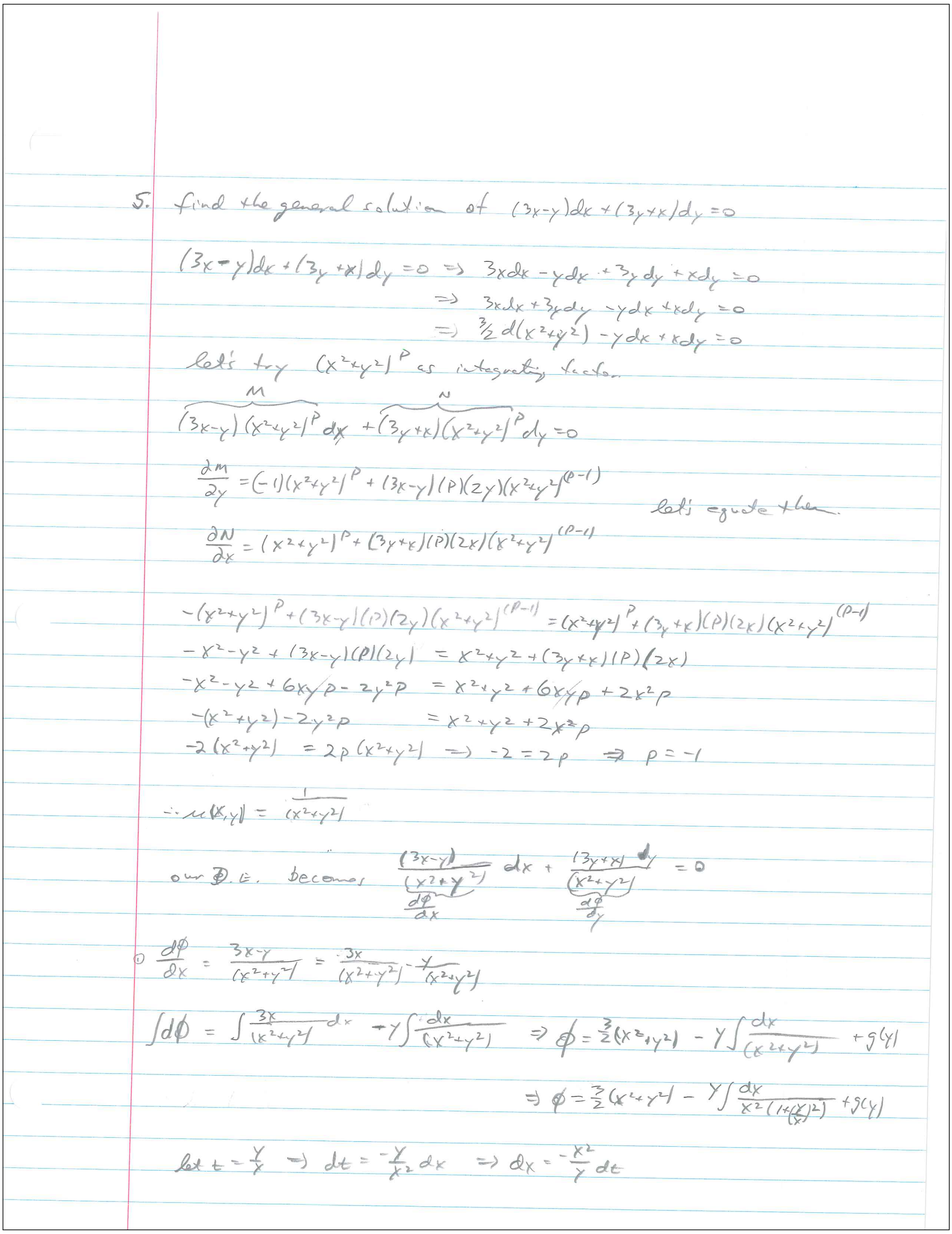
$= xe^y - y - 5 = c$

what is c? Use initial condition

$((5)e^0 - 0 - 5) = c \Rightarrow 5 - 5 = c \quad \therefore \boxed{c = 0}$

$xe^y - y - 5 = 0$

$\therefore \underline{\underline{xe^y - y = 5}}$



5. find the general solution of $(3x-y)dx+(3y+x)dy=0$

$$(3x-y)dx+(3y+x)dy=0 \Rightarrow 3xdx-ydx+3ydy+xdy=0$$
$$\Rightarrow 3xdx+3ydy-ydx+xdy=0$$
$$\Rightarrow \tfrac{3}{2}d(x^2+y^2)-ydx+xdy=0$$

let's try $(x^2+y^2)^P$ as integrating factor

$$\overbrace{(3x-y)(x^2+y^2)^P}^{M}dx+\overbrace{(3y+x)(x^2+y^2)^P}^{N}dy=0$$

$$\frac{\partial m}{\partial y}=(-1)(x^2+y^2)^P+(3x-y)(P)(2y)(x^2+y^2)^{(P-1)}$$

let's equate them.

$$\frac{\partial N}{\partial x}=(x^2+y^2)^P+(3y+x)(P)(2x)(x^2+y^2)^{(P-1)}$$

$$-(x^2+y^2)^P+(3x-y)(P)(2y)(x^2+y^2)^{(P-1)}=(x^2+y^2)^P+(3y+x)(P)(2x)(x^2+y^2)^{(P-1)}$$
$$-x^2-y^2+(3x-y)(P)(2y)=x^2+y^2+(3y+x)(P)(2x)$$
$$-x^2-y^2+\cancel{6xyp}-2y^2p=x^2+y^2+\cancel{6xyp}+2x^2p$$
$$-(x^2+y^2)-2y^2p=x^2+y^2+2x^2p$$
$$-2(x^2+y^2)=2p(x^2+y^2) \Rightarrow -2=2p \Rightarrow p=-1$$

$$\therefore \mu(x,y)=\frac{1}{(x^2+y^2)}$$

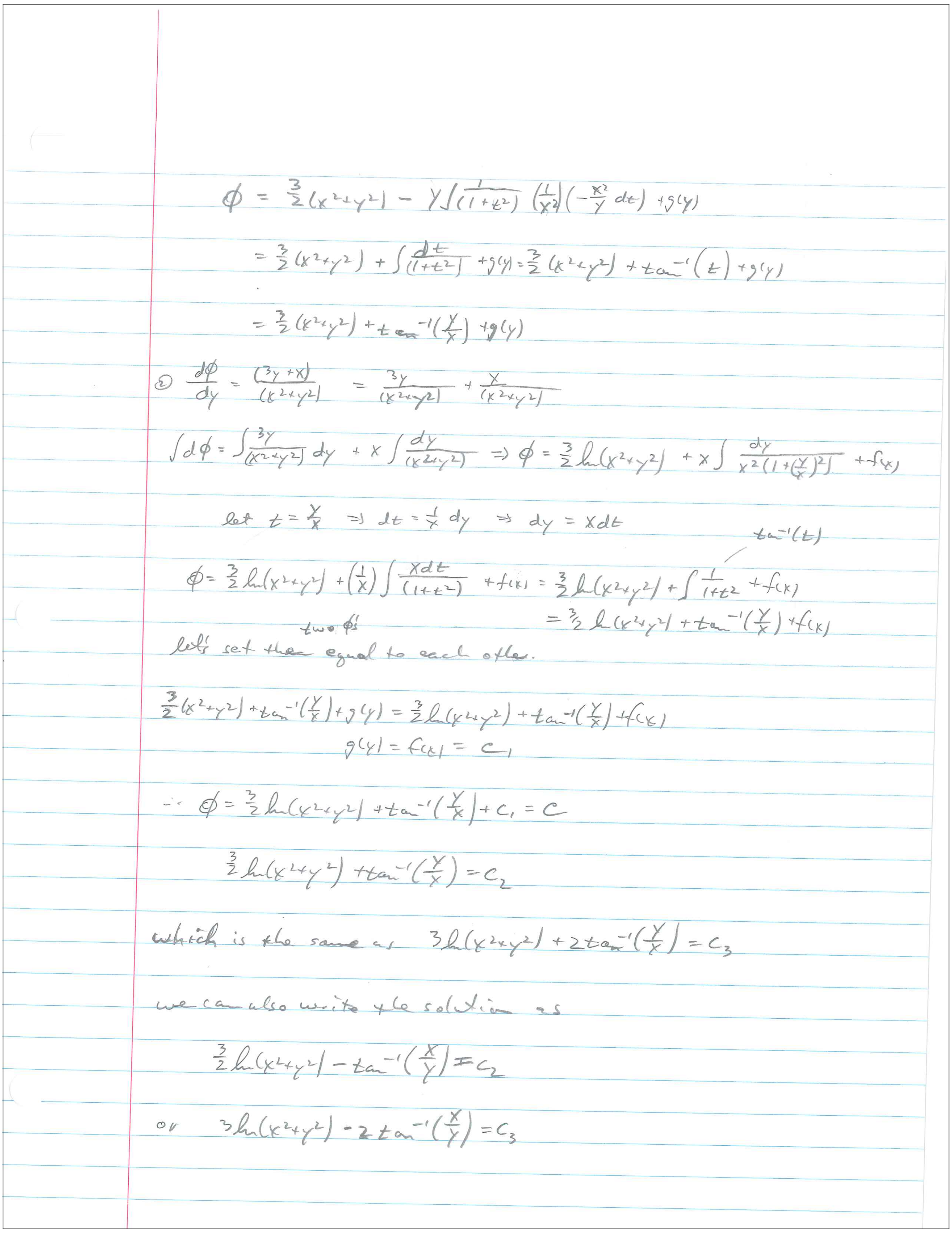
our D.E. becomes $\underbrace{\frac{(3x-y)}{(x^2+y^2)}}_{\frac{d\phi}{dx}}dx+\underbrace{\frac{(3y+x)}{(x^2+y^2)}}_{\frac{d\phi}{dy}}dy=0$

① $$\frac{d\phi}{dx}=\frac{3x-y}{(x^2+y^2)}=\frac{3x}{(x^2+y^2)}-\frac{y}{(x^2+y^2)}$$

$$\int d\phi=\int\frac{3x}{(x^2+y^2)}dx-y\int\frac{dx}{(x^2+y^2)} \Rightarrow \phi=\frac{3}{2}(x^2+y^2)-y\int\frac{dx}{(x^2+y^2)}+g(y)$$

$$\Rightarrow \phi=\frac{3}{2}(x^2+y^2)-y\int\frac{dx}{x^2(1+(\frac{y}{x})^2)}+g(y)$$

let $t=\frac{y}{x} \Rightarrow dt=\frac{-y}{x^2}dx \Rightarrow dx=\frac{-x^2}{y}dt$



$$\phi = \frac{3}{2}(x^2+y^2) - y\int \frac{1}{(1+t^2)}\left(\frac{1}{x^2}\right)\left(-\frac{x^2}{y}\,dt\right) + g(y)$$

$$= \frac{3}{2}(x^2+y^2) + \int \frac{dt}{(1+t^2)} + g(y) = \frac{3}{2}(x^2+y^2) + \tan^{-1}(t) + g(y)$$

$$= \frac{3}{2}(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + g(y)$$

② $\frac{d\phi}{dy} = \frac{(3y+x)}{(x^2+y^2)} = \frac{3y}{(x^2+y^2)} + \frac{x}{(x^2+y^2)}$

$$\int d\phi = \int \frac{3y}{(x^2+y^2)}\,dy + x\int \frac{dy}{(x^2+y^2)} \Rightarrow \phi = \frac{3}{2}\ln(x^2+y^2) + x\int \frac{dy}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)} + f(x)$$

let $t = \frac{y}{x} \Rightarrow dt = \frac{1}{x}\,dy \Rightarrow dy = x\,dt$

$$\phi = \frac{3}{2}\ln(x^2+y^2) + \left(\frac{1}{x}\right)\int \frac{x\,dt}{(1+t^2)} + f(x) = \frac{3}{2}\ln(x^2+y^2) + \overset{\tan^{-1}(t)}{\int \frac{1}{1+t^2}} + f(x)$$

$$= \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

two $\phi$'s

let's set them equal to each other.

$$\frac{3}{2}(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + g(y) = \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

$$g(y) = f(x) = c_1$$

$$\therefore \phi = \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + c_1 = c$$

$$\frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = c_2$$

which is the same as $3\ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = c_3$

we can also write the solution as

$$\frac{3}{2}\ln(x^2+y^2) - \tan^{-1}\left(\frac{x}{y}\right) = c_2$$

or $3\ln(x^2+y^2) - 2\tan^{-1}\left(\frac{x}{y}\right) = c_3$

5. Alternative way to find the solution of $(3x-y)dx + (3y+x)dy = 0$

$$(3x-y)dx + (3y+x)dy = 0 \Rightarrow (3x-y)dx = -(3y+x)dy$$

$$\frac{-(3x-y)}{3y+x} = \frac{dy}{dx} \Rightarrow \frac{-(3-\frac{y}{x})}{3\frac{y}{x}+1} = \frac{dy}{dx}$$ note ~~eq.~~ is a homogenous eq.

let $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$ now to substitute

$$u + x\frac{du}{dx} = \frac{-(3-u)}{3u+1} \Rightarrow x\frac{du}{dx} = \frac{u-3}{3u+1} - u = \frac{u-3-u(3u+1)}{3u+1}$$

$$x\frac{du}{dx} = \frac{u-3-3u^2-u}{3u+1} = \frac{-3(u^2+1)}{3u+1}$$

$$du\frac{(3u+1)}{3(u^2+1)} = -\frac{dx}{x} \Rightarrow \frac{1}{3}\int\frac{3u}{1+u^2}du + \frac{1}{3}\int\frac{du}{1+u^2} = -\int\frac{dx}{x}$$

$$\frac{1}{2}\ln(1+u^2) + \frac{1}{3}\tan^{-1}(u) = -\ln(x) + c$$

$$3\ln(1+u^2) + 2\tan^{-1}(u) = -6\ln(x) + 6c$$

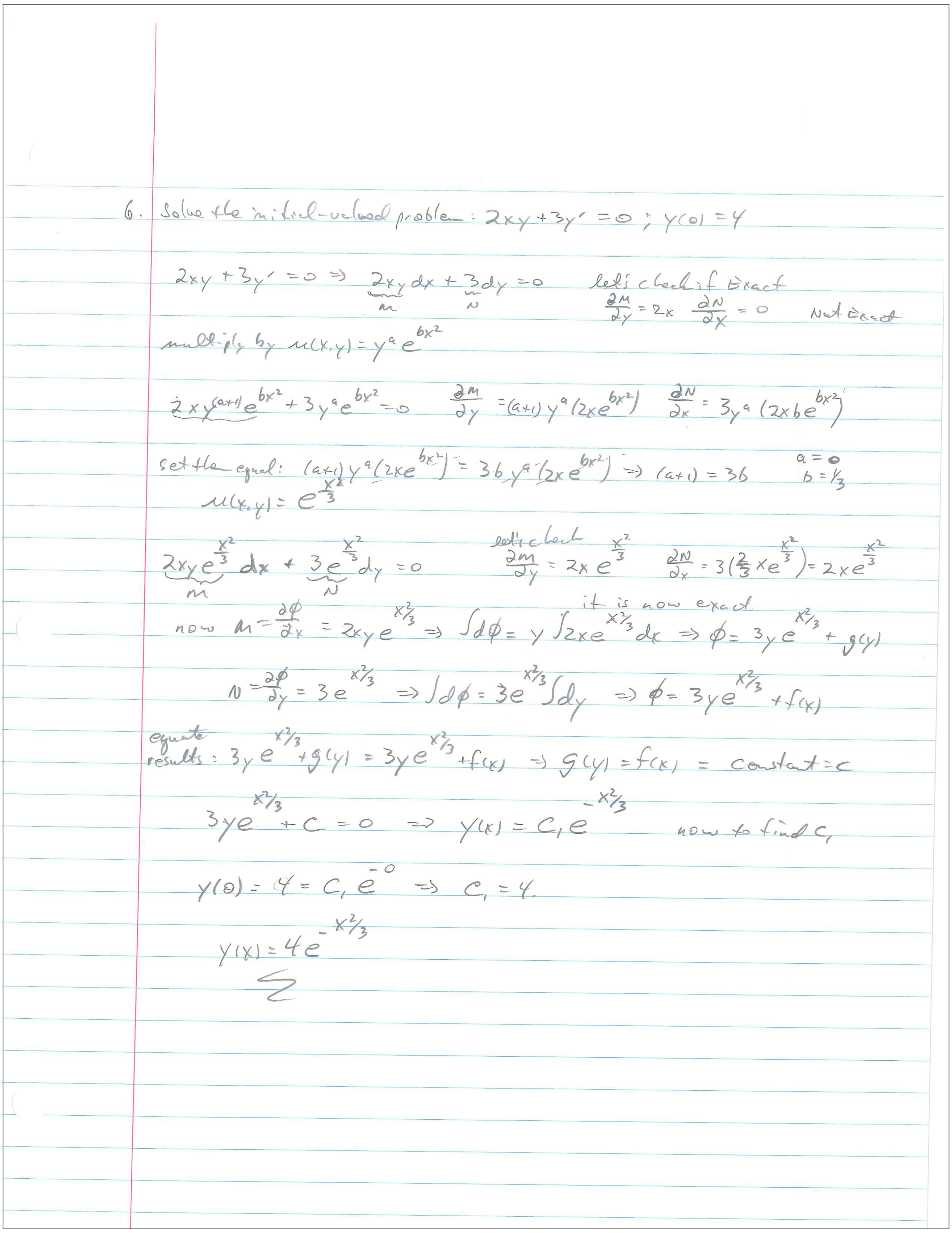
$$3(\ln(1+u^2) + 2\ln(x)) + 2\tan^{-1}(u) = c_1$$

$$3(\ln(1+u^2) + \ln(x^2)) + 2\tan^{-1}(u) = c_1$$ recall $u = \frac{y}{x}$

$$3(\ln(1+(\frac{y}{x})^2) + \ln(x^2)) + 2\tan^{-1}(\frac{y}{x}) = c_1$$

$$3\ln((1+(\frac{y}{x})^2)x^2) + 2\tan^{-1}(\frac{y}{x}) = c_1$$

$$3\ln(y^2+x^2) + 2\tan^{-1}(\frac{y}{x}) = c_1$$



6. Solve the initial-valued problem: $2xy+3y'=0$ ; $y(0)=4$

$2xy+3y'=0 \Rightarrow \underbrace{2xy}_{M}dx+\underbrace{3}_{N}dy=0$ let's check if Exact

$\frac{\partial M}{\partial y}=2x \quad \frac{\partial N}{\partial x}=0$ Not Exact

multiply by $\mu(x,y)=y^a e^{bx^2}$

$\underbrace{2xy^{(a+1)}e^{bx^2}}+3y^a e^{bx^2}=0 \quad \frac{\partial M}{\partial y}=(a+1)y^a(2xe^{bx^2}) \quad \frac{\partial N}{\partial x}=3y^a(2xbe^{bx^2})$

set them equal: $(a+1)y^a(2xe^{bx^2})=3b\,y^a(2xe^{bx^2}) \Rightarrow (a+1)=3b$ $a=0$ $b=1/3$

$\mu(x,y)=e^{\frac{x^2}{3}}$

$\underbrace{2xye^{\frac{x^2}{3}}}_{M}dx+\underbrace{3e^{\frac{x^2}{3}}}_{N}dy=0$ let's check $\frac{\partial M}{\partial y}=2xe^{\frac{x^2}{3}} \quad \frac{\partial N}{\partial x}=3(\frac{2}{3}xe^{\frac{x^2}{3}})=2xe^{\frac{x^2}{3}}$

it is now exact

now $M=\frac{\partial \phi}{\partial x}=2xye^{x^2/3} \Rightarrow \int d\phi=y\int 2xe^{x^2/3}dx \Rightarrow \phi=3ye^{x^2/3}+g(y)$

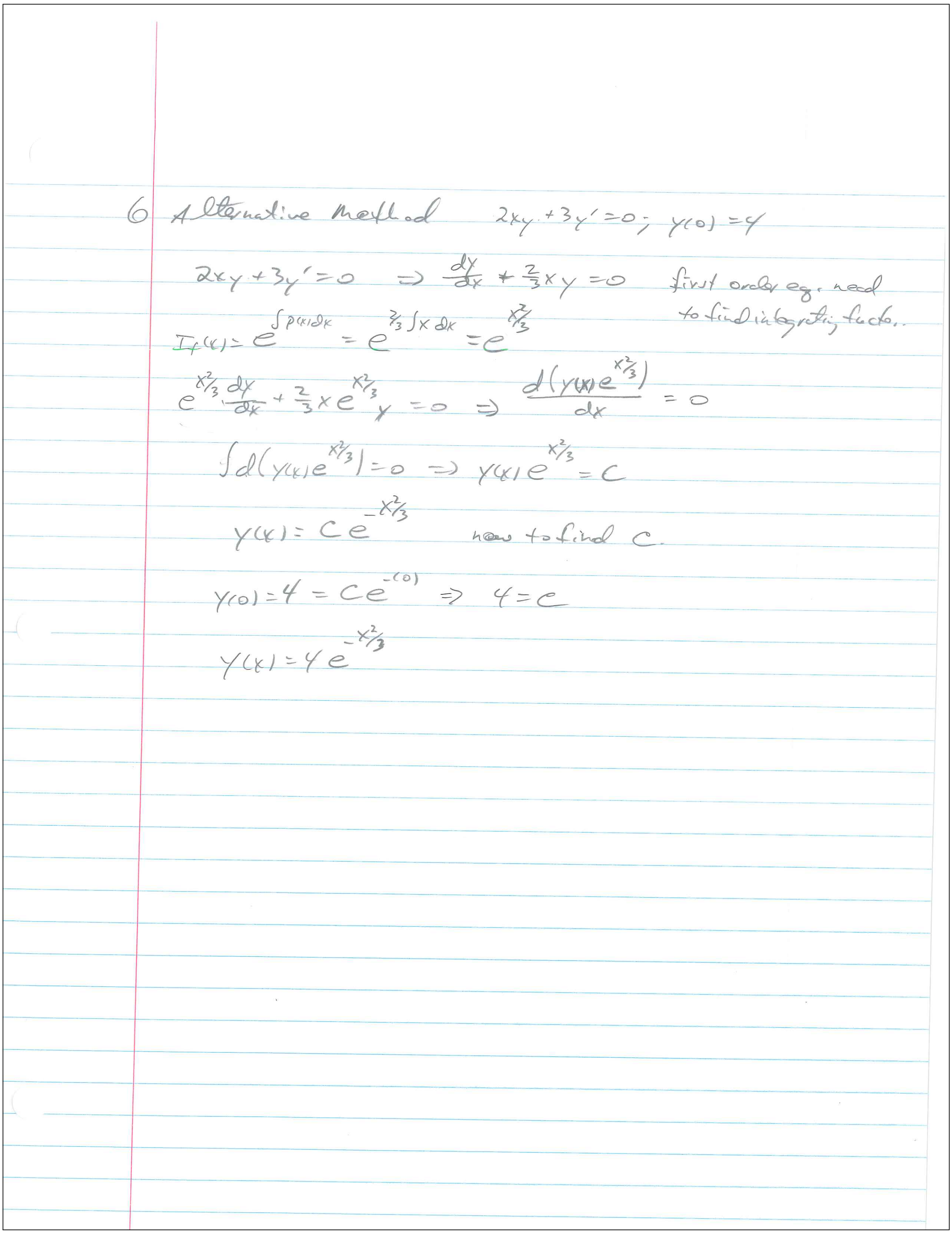
$N=\frac{\partial \phi}{\partial y}=3e^{x^2/3} \Rightarrow \int d\phi=3e^{x^2/3}\int dy \Rightarrow \phi=3ye^{x^2/3}+f(x)$

equate results: $3ye^{x^2/3}+g(y)=3ye^{x^2/3}+f(x) \Rightarrow g(y)=f(x)=\text{constant}=c$

$3ye^{x^2/3}+c=0 \Rightarrow y(x)=c_1e^{-x^2/3}$ now to find $c_1$

$y(0)=4=c_1e^{-0} \Rightarrow c_1=4.$

$y(x)=4e^{-x^2/3}$



6 Alternative method $2xy+3y'=0$; $y(0)=4$

$2xy+3y'=0 \Rightarrow \frac{dy}{dx}+\frac{2}{3}xy=0$ first order eq. need to find integrating factor.

$I_f(x)=e^{\int p(x)dx}=e^{\frac{2}{3}\int x\,dx}=e^{x^2/3}$

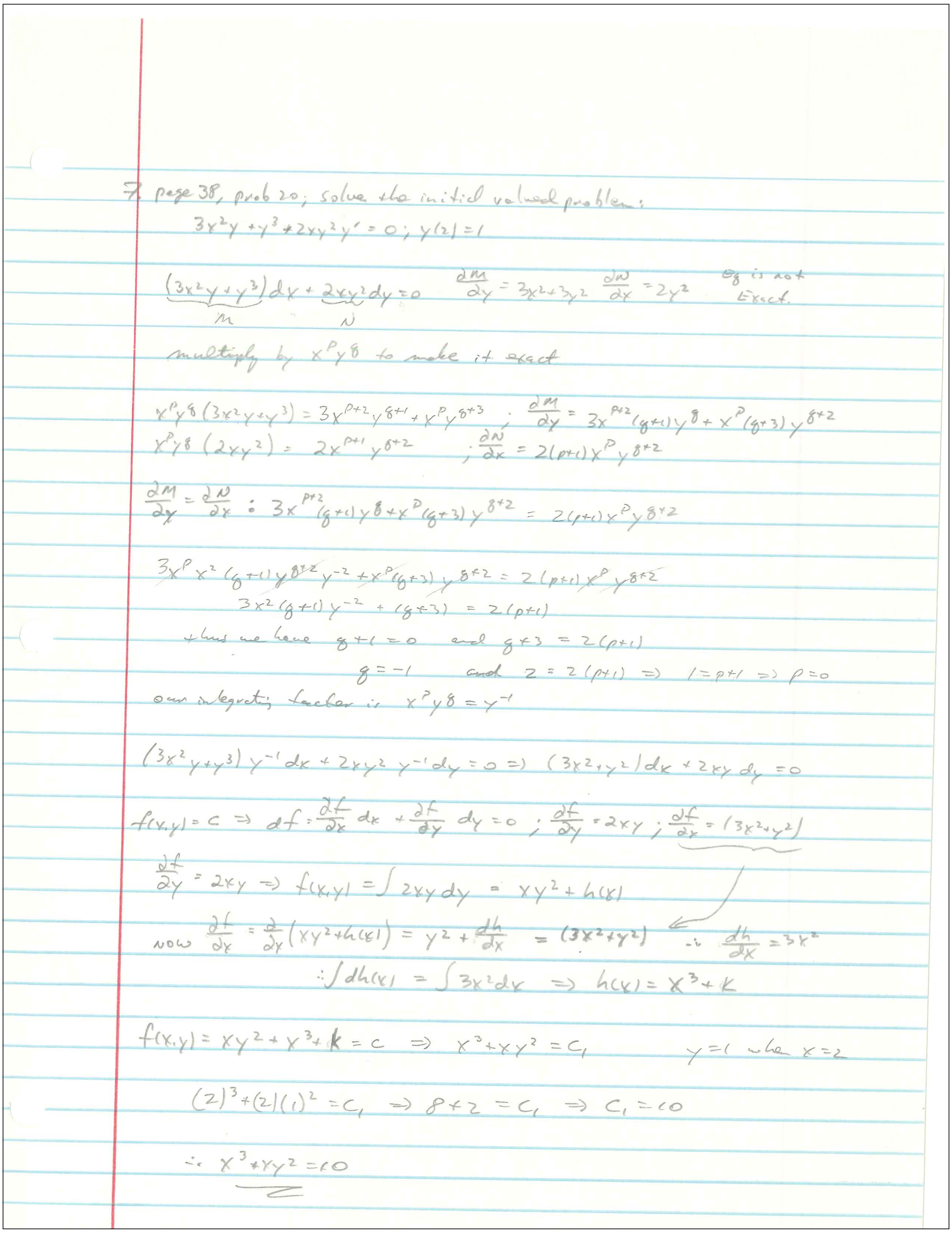
$e^{x^2/3}\frac{dy}{dx}+\frac{2}{3}xe^{x^2/3}y=0 \Rightarrow \frac{d(y(x)e^{x^2/3})}{dx}=0$

$\int d(y(x)e^{x^2/3})=0 \Rightarrow y(x)e^{x^2/3}=C$

$y(x)=Ce^{-x^2/3}$ now to find C.

$y(0)=4=Ce^{-(0)} \Rightarrow 4=c$

$y(x)=4e^{-x^2/3}$



7. page 38, prob 20; solve the initial valued problem:

$3x^2y + y^3 + 2xy^2y' = 0; \ y(2) = 1$

$\underbrace{(3x^2y+y^3)}_{M}dx + \underbrace{2xy^2}_{N}dy = 0 \qquad \frac{\partial M}{\partial y} = 3x^2+3y^2 \quad \frac{\partial N}{\partial x} = 2y^2$

Eq is not Exact.

multiply by $x^py^q$ to make it exact

$x^py^q(3x^2y+y^3) = 3x^{p+2}y^{q+1} + x^py^{q+3} \ ; \ \frac{\partial M}{\partial y} = 3x^{p+2}(q+1)y^q + x^p(q+3)y^{q+2}$

$x^py^q(2xy^2) = 2x^{p+1}y^{q+2} \ ; \ \frac{\partial N}{\partial x} = 2(p+1)x^py^{q+2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} : 3x^{p+2}(q+1)y^q + x^p(q+3)y^{q+2} = 2(p+1)x^py^{q+2}$

$3\cancel{x^p}x^2(q+1)\cancel{y^{q+2}}y^{-2} + \cancel{x^p}(q+3)\cancel{y^{q+2}} = 2(p+1)\cancel{x^p}\cancel{y^{q+2}}$

$3x^2(q+1)y^{-2} + (q+3) = 2(p+1)$

thus we have $q+1=0$ and $q+3 = 2(p+1)$

$q = -1$ and $2 = 2(p+1) \Rightarrow 1 = p+1 \Rightarrow p = 0$

our integrating factor is $x^py^q = y^{-1}$

$(3x^2y+y^3)y^{-1}dx + 2xy^2y^{-1}dy = 0 \Rightarrow (3x^2+y^2)dx + 2xy\,dy = 0$

$f(x,y) = c \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \ ; \ \frac{\partial f}{\partial y} = 2xy \ ; \ \underbrace{\frac{\partial f}{\partial x} = (3x^2+y^2)}$

$\frac{\partial f}{\partial y} = 2xy \Rightarrow f(x,y) = \int 2xy\,dy = xy^2 + h(x)$

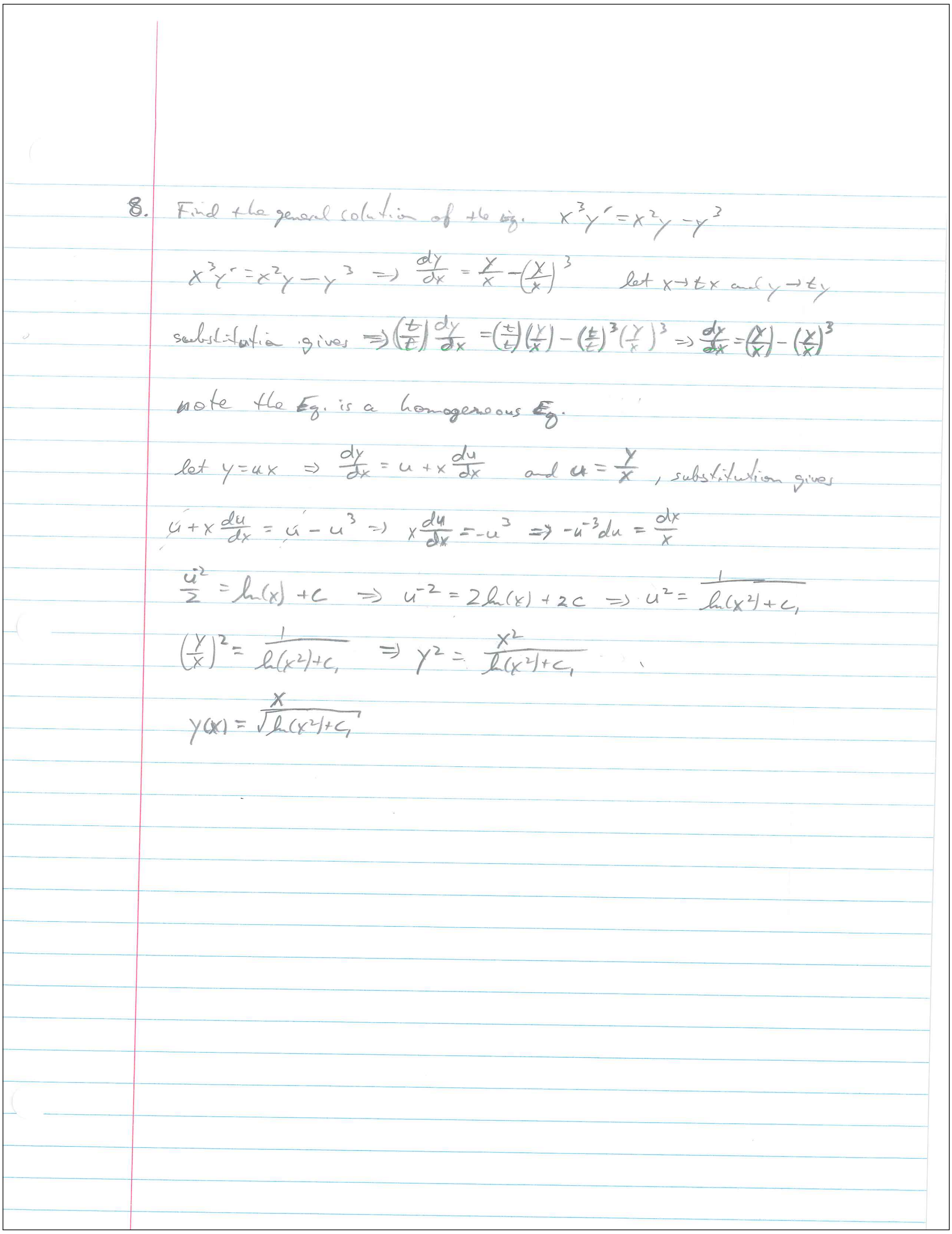
now $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2+h(x)) = y^2 + \frac{dh}{dx} = (3x^2+y^2) \quad \therefore \frac{dh}{dx} = 3x^2$

$\therefore \int dh(x) = \int 3x^2dx \Rightarrow h(x) = x^3 + k$

$f(x,y) = xy^2 + x^3 + k = c \Rightarrow x^3 + xy^2 = c_1 \qquad y = 1$ when $x = 2$

$(2)^3 + (2)(1)^2 = c_1 \Rightarrow 8 + 2 = c_1 \Rightarrow c_1 = 10$

$\therefore \underline{\underline{x^3 + xy^2 = 10}}$



8. Find the general solution of the Eq. $x^3 y' = x^2 y - y^3$

$x^3 y' = x^2 y - y^3 \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^3$ let $x \to tx$ and $y \to ty$

substitution gives $\Rightarrow \cancel{\left(\frac{t}{t}\right)} \frac{dy}{dx} = \cancel{\left(\frac{t}{t}\right)}\left(\frac{y}{x}\right) - \cancel{\left(\frac{t}{t}\right)^3}\left(\frac{y}{x}\right)^3 \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^3$

note the Eq. is a homogeneous Eq.

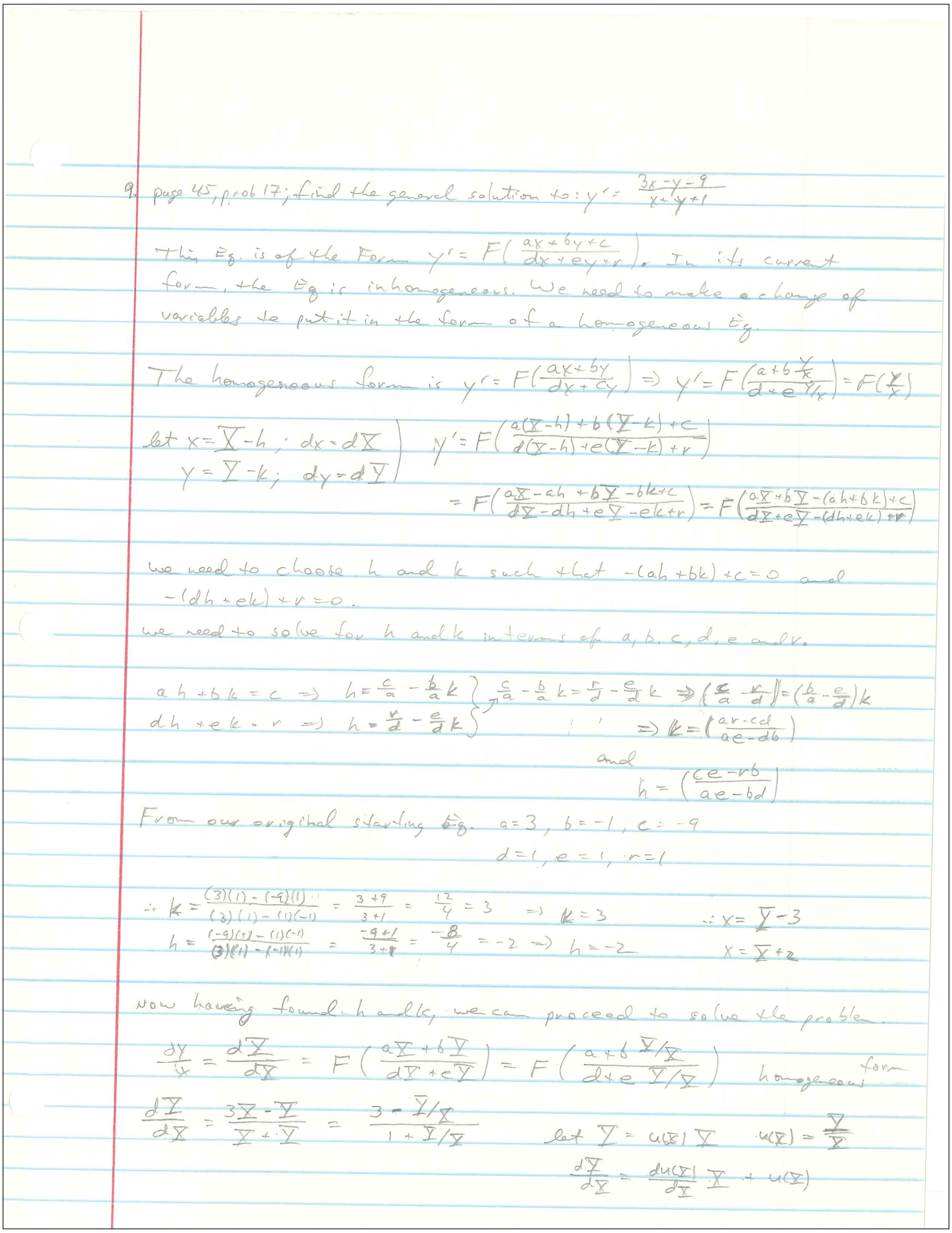
let $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$ and $u = \frac{y}{x}$, substitution gives

$\cancel{u} + x\frac{du}{dx} = \cancel{u} - u^3 \Rightarrow x\frac{du}{dx} = -u^3 \Rightarrow -u^{-3}du = \frac{dx}{x}$

$\frac{u^{-2}}{2} = \ln(x) + c \Rightarrow u^{-2} = 2\ln(x) + 2c \Rightarrow u^2 = \frac{1}{\ln(x^2) + c_1}$

$\left(\frac{y}{x}\right)^2 = \frac{1}{\ln(x^2)+c_1} \Rightarrow y^2 = \frac{x^2}{\ln(x^2)+c_1}$

$y(x) = \frac{x}{\sqrt{\ln(x^2)+c_1}}$



9. page 45, prob 17; find the general solution to: $y' = \frac{3x-y-9}{x+y+1}$

This Eq. is of the Form $y' = F\left(\frac{ax+by+c}{dx+ey+r}\right)$. In its current form, the Eq is inhomogeneous. We need to make a change of variables to put it in the form of a homogeneous Eq.

The homogeneous form is $y' = F\left(\frac{ax+by}{dx+cy}\right) \Rightarrow y' = F\left(\frac{a+b\frac{y}{x}}{d+e\,y/x}\right) = F\left(\frac{y}{x}\right)$

let $x = X - h$; $dx = dX$
$y = Y - k$; $dy = dY$

$$y' = F\left(\frac{a(X-h)+b(Y-k)+c}{d(X-h)+e(Y-k)+r}\right)$$

$$= F\left(\frac{aX - ah + bY - bk + c}{dX - dh + eY - ek + r}\right) = F\left(\frac{aX + bY - (ah+bk) + c}{dX + eY - (dh+ek) + r}\right)$$

we need to choose $h$ and $k$ such that $-(ah+bk)+c=0$ and $-(dh+ek)+r=0$.

we need to solve for $h$ and $k$ in terms of $a, b, c, d, e$ and $r$.

$ah + bk = c \Rightarrow h = \frac{c}{a} - \frac{b}{a}k$
$dh + ek = r \Rightarrow h = \frac{r}{d} - \frac{e}{d}k$

$\frac{c}{a} - \frac{b}{a}k = \frac{r}{d} - \frac{e}{d}k \Rightarrow \left(\frac{c}{a} - \frac{r}{d}\right) = \left(\frac{b}{a} - \frac{e}{d}\right)k$

$\Rightarrow k = \left(\frac{ar-cd}{ae-db}\right)$

and

$h = \left(\frac{ce-rb}{ae-bd}\right)$

From our original starting Eq. $a=3$, $b=-1$, $c=-9$
$d=1$, $e=1$, $r=1$

$\therefore k = \frac{(3)(1)-(-9)(1)}{(3)(1)-(1)(-1)} = \frac{3+9}{3+1} = \frac{12}{4} = 3 \Rightarrow k = 3$

$h = \frac{(-9)(1)-(1)(-1)}{(3)(1)-(-1)(1)} = \frac{-9+1}{3+1} = \frac{-8}{4} = -2 \Rightarrow h = -2$

$\therefore x = Y - 3$
$x = X + 2$

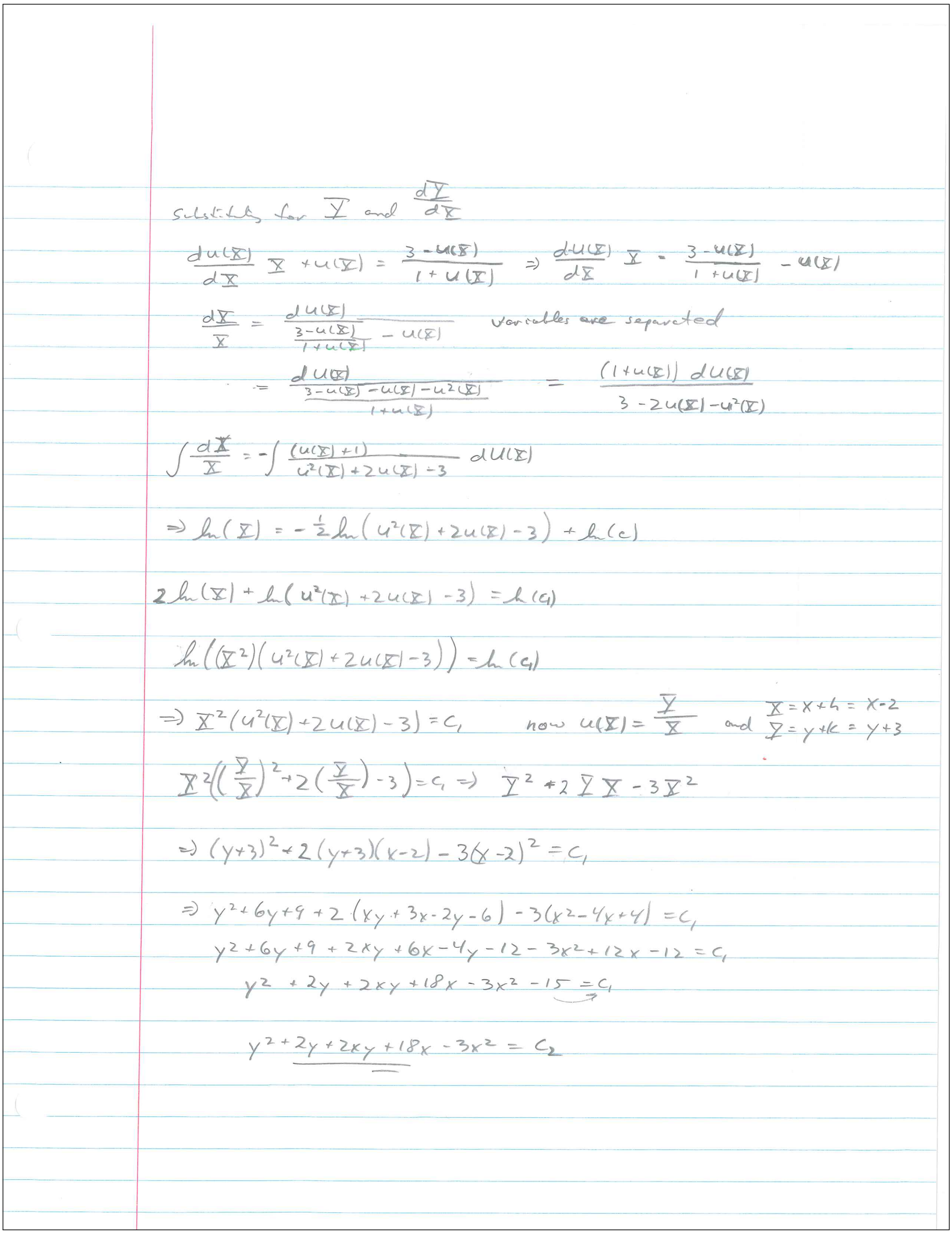
Now having found $h$ and $k$, we can proceed to solve the problem.

$\frac{dy}{x} = \frac{dY}{dX} = F\left(\frac{aX+bY}{dX+eY}\right) = F\left(\frac{a+b\,Y/X}{d+e\,Y/X}\right)$ homogeneous form

$\frac{dY}{dX} = \frac{3X-Y}{X+Y} = \frac{3-Y/X}{1+Y/X}$

let $Y = u(X)\,X$ $\quad u(X) = \frac{Y}{X}$

$\frac{dY}{dX} = \frac{du(X)}{dX}X + u(X)$



Substituting for $\underline{Y}$ and $\frac{d\underline{Y}}{d\underline{X}}$

$$\frac{du(\underline{X})}{d\underline{X}}\underline{X}+u(\underline{X})=\frac{3-u(\underline{X})}{1+u(\underline{X})} \Rightarrow \frac{du(\underline{X})}{d\underline{X}}\underline{X}=\frac{3-u(\underline{X})}{1+u(\underline{X})}-u(\underline{X})$$

$$\frac{d\underline{X}}{\underline{X}}=\frac{du(\underline{X})}{\frac{3-u(\underline{X})}{1+u(\underline{X})}-u(\underline{X})}$$ variables are separated

$$=\frac{du(\underline{X})}{\frac{3-u(\underline{X})-u(\underline{X})-u^2(\underline{X})}{1+u(\underline{X})}}=\frac{(1+u(\underline{X}))\,du(\underline{X})}{3-2u(\underline{X})-u^2(\underline{X})}$$

$$\int\frac{d\underline{X}}{\underline{X}}=-\int\frac{(u(\underline{X})+1)}{u^2(\underline{X})+2u(\underline{X})-3}\,du(\underline{X})$$

$$\Rightarrow \ln(\underline{X})=-\tfrac{1}{2}\ln\left(u^2(\underline{X})+2u(\underline{X})-3\right)+\ln(c)$$

$$2\ln(\underline{X})+\ln(u^2(\underline{X})+2u(\underline{X})-3)=\ln(c_1)$$

$$\ln\left((\underline{X}^2)(u^2(\underline{X})+2u(\underline{X})-3)\right)=\ln(c_1)$$

$$\Rightarrow \underline{X}^2(u^2(\underline{X})+2u(\underline{X})-3)=c_1$$ now $u(\underline{X})=\frac{\underline{Y}}{\underline{X}}$ and $\underline{X}=x+h=x-2$, $\underline{Y}=y+k=y+3$

$$\underline{X}^2\left(\left(\frac{\underline{Y}}{\underline{X}}\right)^2+2\left(\frac{\underline{Y}}{\underline{X}}\right)-3\right)=c_1 \Rightarrow \underline{Y}^2+2\underline{Y}\,\underline{X}-3\underline{X}^2$$

$$\Rightarrow (y+3)^2+2(y+3)(x-2)-3(x-2)^2=c_1$$

$$\Rightarrow y^2+6y+9+2(xy+3x-2y-6)-3(x^2-4x+4)=c_1$$

$$y^2+6y+9+2xy+6x-4y-12-3x^2+12x-12=c_1$$

$$y^2+2y+2xy+18x-3x^2-15=c_1$$

$$y^2+\underline{2y+2xy+18}x-3x^2=c_2$$

This problem can also be solved as an exact Eq.

$$\frac{dy}{dx} = \frac{3x-y-9}{x+y+1} \Rightarrow \underbrace{(x+y+1)}_{N}dy - \underbrace{(3x-y-9)}_{M}dx = 0$$

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 1 \qquad \text{Eq. is exact.}$$

$$f(x,y) = C \quad : \quad df = \underbrace{\frac{\partial f}{\partial x}}_{M(x,y)}dx + \underbrace{\frac{\partial f}{\partial y}}_{N(x,y)}dy = 0$$

$$\frac{\partial f}{\partial x} = -3x+y+9 \Rightarrow df = (-3x+y+9)dx \Rightarrow \int df = \int(-3x+y+9)\,dx$$

$$f(x,y) = -\frac{3}{2}x^2 + xy + 9x + h(y)$$ take $\frac{\partial f}{\partial y}$ of this Eq. and set equal to $N(x,y)$

$$\frac{\partial f}{\partial y} = 0 + x + 0 + \frac{\partial h(y)}{\partial y} = N(x,y) = x+y+1$$

$$\Rightarrow x + \frac{\partial h(y)}{\partial y} = x+y+1 \Rightarrow \frac{\partial h(y)}{\partial y} = y+1$$

$$\Rightarrow \int d(h(y)) = \int (y+1)\,dy \Rightarrow h(y) = \frac{y^2}{2} + y + k$$

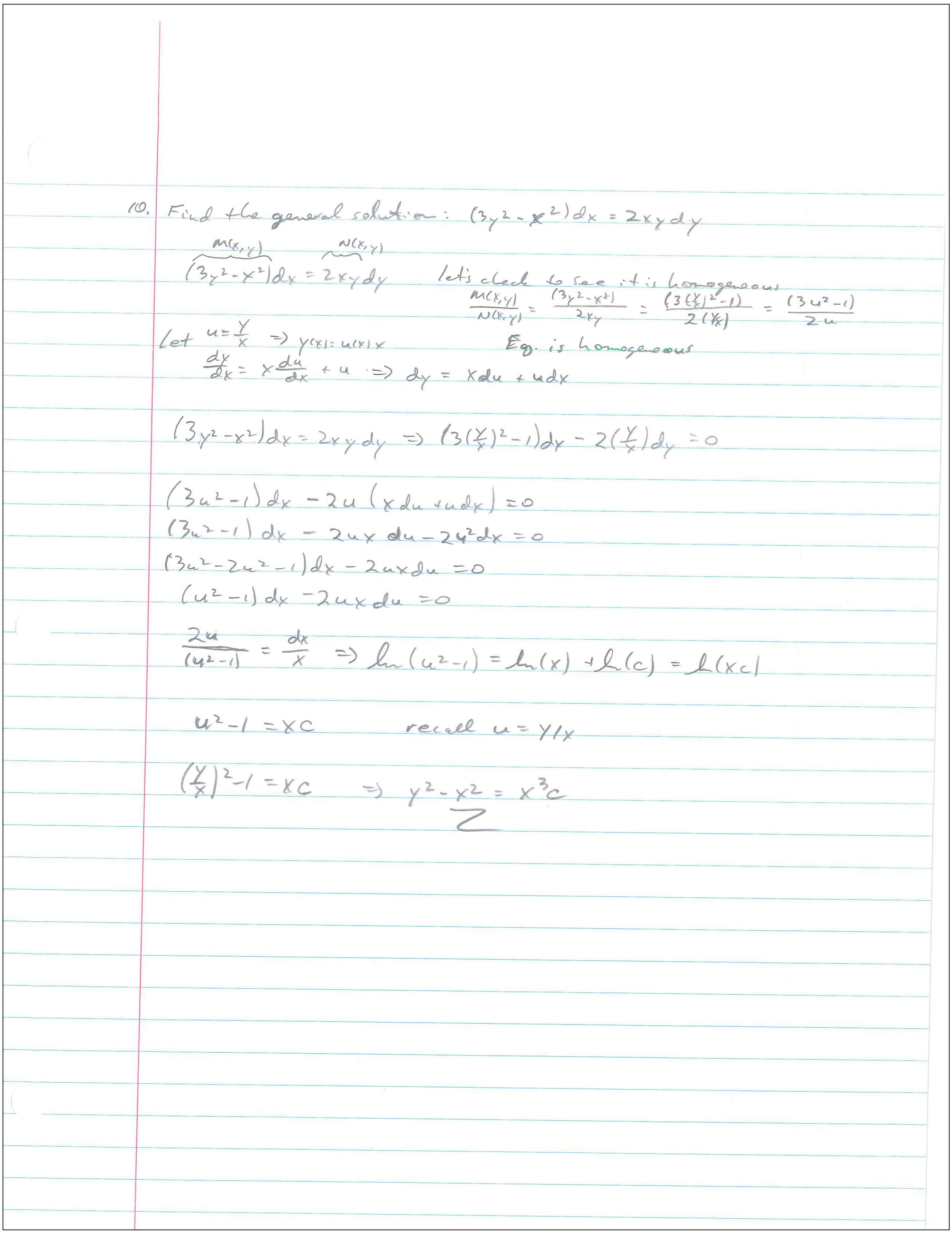
$$\therefore f(x,y) = -\frac{3}{2}x^2 + xy + 9x + \frac{y^2}{2} + y + k = c$$

$$-\frac{3}{2}x^2 + xy + 9x + \frac{y^2}{2} + y = c - k = c_1$$

now let's multiply through by 2

$$-3x^2 + 2xy + 18x + y^2 + 2y = c_2 \qquad \text{where } c_2 = 2c_1$$

same as solution on previous page



10. Find the general solution: $(3y^2 - x^2)dx = 2xy\,dy$

$$\underbrace{(3y^2-x^2)}_{M(x,y)}dx = \underbrace{2xy}_{N(x,y)}dy$$

let's check to see it is homogeneous

$$\frac{M(x,y)}{N(x,y)} = \frac{(3y^2-x^2)}{2xy} = \frac{(3(\frac{y}{x})^2-1)}{2(y/x)} = \frac{(3u^2-1)}{2u}$$

Eq. is homogeneous

Let $u = \frac{y}{x} \Rightarrow y(x) = u(x)x$

$$\frac{dy}{dx} = x\frac{du}{dx} + u \Rightarrow dy = x\,du + u\,dx$$

$$(3y^2-x^2)dx = 2xy\,dy \Rightarrow (3(\tfrac{y}{x})^2-1)dx - 2(\tfrac{y}{x})dy = 0$$

$$(3u^2-1)dx - 2u(x\,du + u\,dx) = 0$$

$$(3u^2-1)dx - 2ux\,du - 2u^2dx = 0$$

$$(3u^2-2u^2-1)dx - 2ux\,du = 0$$

$$(u^2-1)dx - 2ux\,du = 0$$

$$\frac{2u}{(u^2-1)} = \frac{dx}{x} \Rightarrow \ln(u^2-1) = \ln(x) + \ln(c) = \ln(xc)$$

$u^2 - 1 = xc$ recall $u = y/x$

$$\left(\frac{y}{x}\right)^2 - 1 = xc \Rightarrow \underline{\underline{y^2 - x^2 = x^3c}}$$

10. Find the general solution: $(2x^2 - y^2)\,dx + 3xy\,dy = 0$

let $x \to xt$ $\quad M(2x^2 - y^2) \to M(2t^2x^2 - t^2y^2) = t^2 M(2x^2 - y^2)$

$y \to yt$ $\quad N(3xy) \to N(3txyt) = t^2 N(3xy)$

Eq. is Homogeneous

let $y = ux \Rightarrow dy = u\,dx + x\,du$ substitution in the above Eq. gives

$(2x^2 + u^2x^2)\,dx + 3x^2u\,(u\,dx + x\,du) = 0$

$(2x^2 - u^2x^2 + 3x^2u^2)\,dx + 3x^3u\,du = 0$

$(2x^2 + 2x^2u^2)\,dx = -3x^3u\,du$

$2x^2(1 + u^2)\,dx = -3x^3u\,du$

$$\frac{2x^2}{x^3}\,dx = -3\left(\frac{u}{1+u^2}\right)du$$

$\int \frac{2}{x}\,dx = \int -3\left(\frac{u}{1+u^2}\right)du \Rightarrow 2\ln(x) = -\frac{3}{2}\ln(1+u^2) + c$

$-4\ln(x) = \cdot 3\ln(1+u^2) + c_1 \Rightarrow \ln(x^{-4}) = \ln((1+u^2)^3) + c_1$

$x^{-4}C_2 = (1+u^2)^3$ recall $u = \frac{y}{x}$

$x^{-4}C_2 = \left(1 + \frac{y^2}{x^2}\right)^3 \Rightarrow x^{-4}C_2 = (x^2+y^2)^3\left(\frac{1}{x^6}\right)$

$\Rightarrow x^2C_2 = (x^2+y^2)^3$

11. Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new variable $z$.

$$y' = P(x)y^2 + Q(x)y + R(x)$$

let $y(x) = u + \frac{1}{z}$

$$\frac{dy}{d(x)} = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx}$$
$$= u' - \frac{1}{z^2} z'$$

substitute new variables into the above equation.

$$u' - \frac{1}{z^2} z' = P(x)\left(u^2 + 2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\left(u + \frac{1}{z}\right) + R(x)$$

$$u' - \frac{1}{z^2} z' = P(x)u^2 + Q(x)u + R(x) + P(x)\left(2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\frac{1}{z}$$

Note: since $u(x)$ is a solution to the differential equation, it satisfies the original D.E. $u' = P(x)u^2 + Q(x)u + R(x)$.
Hence we are left with

$$-\frac{1}{z^2} z' = P(x)\left(2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\frac{1}{z}$$
$$= P(x)\left(\frac{1}{z^2}\right) + \left(P(x)2u + Q(x)\right)\left(\frac{1}{z}\right)$$
$$= -P(x) - \left(2P(x)u(x) + Q(x)\right)z$$

$$\therefore\ z' + \left(2P(x)u(x) + Q(x)\right)z = -P(x)$$

where $P(x)$ and $Q(x)$ are functions from the original equation and $u(x)$ is the known solution.

This can also be written as

$$z' + H(x)z = -P(x) \qquad \text{where } H(x) = 2P(x)u(x) + Q(x)$$

## 2.3 HW 2

### 2.3.1 Problems to solve

Homework Set No. 2 NEEP 547
Due September 20, 2013 DLH

Nonlinear Eqs. reducible to first order:

1. (5pts) Find the general solution to the differential equation: $y'' = \left[1+(y')^2\right]^{3/2}$
2. (5pts) page 72, prob. 13c; Find the general solution to the differential equation: $y\,y'' = y^2\,y' + (y')^2$

Linear Operators

3. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} - y = 0$$
4. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
$$x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3\,x^2 - x$$

Linear dependent or independent solutions.

5. (4pts) page 69, prob. 8. : Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of $x^2\,y'' - 2\,x\,y' + 2\,y = 0$ on [-1,1], but that $W(0) = 0$. Why does this not contradict Theorem 2.3.1 in this interval?
*Theorem 2.3: Wronskian Test : Let $y_1$ and $y_2$ be solutions of $y'' + p(x)\,y' + q(x)\,y = 0$ on the open interval I. Then,*
*2.3.1. Either $W(x) = 0$ for all $x$ in $I$, or $W(x) \neq 0$ for all $x$ in $I$.*
*2.3.2. $y_1$ and $y_2$ are linearly independent on $I$ if and only if $W(x) \neq 0$ on $I$.*
6. (4pts) page 69, prob. 10: Show that $y_1(x) = 3\,e^{2x} - 1$ and $y_2(x) = e^{-x} + 2$ are solutions of $y\,y'' + 2\,y' - (y')^2 = 0$, but neither $2\,y_1$ nor $y_1 + y_2$ is a solution. Why does this not contradict Theorem 2.2?
*Theorem 2.2: Let $y_1$ and $y_2$ be solutions of $y'' + p(x)\,y' + q(x)\,y = 0$ on an interval I. Then any linear combination of these solutions is also a solution.*

Homogeneous Linear Differential Equations with Constant Coefficients:

7. (6pts) Solve the initial-value problem: $(D^3 - 6\,D^2 + 11\,D - 6)\,y = 0$ where $D^n = \frac{d^n}{dx^n}$; with conditions: $y = y' = 0$ and $y'' = 2$ when $x = 0$.
8. (6pts) Solve the initial-value problem: $8y''' - 4y'' + 6y' + 5y = 0$ with conditions: $y = 0, y'' = y' = 1$ when $x = 0$.

Nonhomogeneous Equations with Constant Coefficients

9. (6pts) O'Neil, page 93 prob. 16; find the general solution: $y'' - 2\,y' + y = 3\,x + 25\,\sin(3x)$
10. (7pts) find the general solution: $y^{iv} + 3\,y'' - 4\,y = \sinh(x) - \sin^2(x)$

### 2.3.2 Problem 1 reduction of order (book 2.3 section)

Nonlinear Eq, reducible to first order.

Find the general solution to $y'' = \left(1 + \left(y'\right)^2\right)^{\frac{3}{2}}$

This is non-linear, second order differential equation. Since $x$ does not appear explicitly, let $u = y'$, then $u' = y''$ and the above differential equation becomes

$$u' = \left(1 + u^2\right)^{\frac{3}{2}}$$

The above is now separable and solved for $u$

$$\begin{aligned}\frac{du}{\left(1+u^2\right)^{\frac{3}{2}}} &= dx\\ \frac{u}{\sqrt{1+u^2}} &= x+C\end{aligned}$$

The above is solved explicitly for $u$

$$\begin{aligned}\frac{u^2}{1+u^2} &= (x+C)^2\\ u^2 &= \left(1+u^2\right)(x+C)^2\\ u^2-u^2(x+C)^2 &= (x+C)^2\\ u^2 &= \frac{(x+C)^2}{1-(x+C)^2}\\ u &= \pm\sqrt{\frac{(x+C)^2}{1-(x+C)^2}} = \pm\frac{(x+C)}{\sqrt{1-(x+C)^2}}\end{aligned}$$

Since $u=y'$ therefore

$$y' = \pm\frac{(x+C)}{\sqrt{1-(x+C)^2}}$$

This is separable, hence the solution is

$$\begin{aligned}y(x) &= \pm\int\frac{(x+C)}{\sqrt{1-(x+C)^2}}+C_2\\ &= \pm\sqrt{1-(x+C)^2}+C_2\\ &= \pm\sqrt{1-x^2-2xC+C^2}+C_2\end{aligned}$$

### 2.3.3 Problem 2 O'Neil page 72, problem 13c

Find general solution to $yy''=y^2y'+\left(y'\right)^2$

Solution: This is non-linear, second order differential equation.

$$\begin{aligned}y\frac{d^2y}{dx^2} &= y^2\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\\ \frac{d^2y}{dx^2} &= y\frac{dy}{dx}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2\end{aligned}$$

Multiply by $\frac{dx}{dy}$

$$\frac{d^2y}{dx^2}\frac{dx}{dy} = y+\frac{1}{y}\frac{dy}{dx}$$

Let

$$u\left(y\right)=\frac{dy}{dx}$$

$u$ here is function of $y$.The differential equation becomes

$$\begin{aligned}\frac{du}{dx}\frac{1}{u} &= y+\frac{1}{y}u\\ \frac{du}{dx} &= yu+\frac{1}{y}u^2\end{aligned}$$

Multiply by $\frac{dx}{dy}$ and using that $u=\frac{dy}{dx}$ the above reduces to

$$\begin{aligned}\frac{du}{dx}\frac{dx}{dy} &= yu\frac{dx}{dy}+\frac{u^2}{y}\frac{dx}{dy}\\ \frac{du}{dy} &= y+\frac{1}{y}\left(\frac{dy}{dx}\right)^2\frac{dx}{dy}\\ &= y+\frac{1}{y}\left(\frac{dy}{dx}\right)\\ &= y+\frac{1}{y}u\end{aligned}$$

Hence

$$\frac{du}{dy}-\frac{u}{y}=y$$

This is solved for $u\left(y\right)$.The integrating factor is $I_f=y^{-1}$ hence

$$\begin{aligned}d\left(y^{-1}u\right) &= y^{-1}y=1\\ y^{-1}u &= y+C_1\\ u &= y^2+C_1y\end{aligned}$$

But $u=\frac{dy}{dx}$ hence

$$\begin{aligned}\frac{dy}{dx} &= y^2+C_1y\\ \frac{dy}{dx}-y^2-C_1y &= 0\end{aligned}$$

This is first order non-linear ODE. It is separable, hence

$$\frac{dy}{\left(y^2+C_1y\right)}=dx$$

Applying partial fractions to the LHS gives

$$\frac{dy}{C_1y}-\frac{1}{C_1}\frac{dy}{C_1+y}=dx$$

Integrating both sides now gives

$$\begin{aligned}\frac{1}{C_1}\ln y-\frac{1}{C_1}\ln\left(y+C_1\right) &= x+C_2\\ \ln y-\ln\left(y+C_1\right) &= C_1x+C_3\end{aligned}$$

Where $C_3=C_1C_2$

$$\begin{aligned}\ln\frac{y}{y+C_1} &= C_1x+C_3\\ \frac{y}{y+C_1} &= C_4e^{C_1x}\\ y &= yC_4e^{C_1x}+C_1C_4e^{C_1x}\\ y-yC_4e^{C_1x} &= C_1C_4e^{C_1x}\\ y\left(1-C_4e^{C_1x}\right) &= C_1C_4e^{C_1x}\\ y &= \frac{C_1C_4e^{C_1x}}{1-C_4e^{C_1x}}\end{aligned}$$

### 2.3.4 problem 3, linear operators

First factor the equation using operator notation and then find the general solution

$$x^2y''+xy'-y=0$$

Let $D\equiv\frac{d}{dx}$.The ODE can be written as

$$\left(x^2D^2+xD-1\right)y=0$$

The roots of the characteristic equation $x^2\lambda^2 + x\lambda - 1$ are $m = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-x\pm\sqrt{x^2+4x^2}}{2x^2} = \frac{-x\pm x\sqrt{5}}{2x^2} = \frac{-1\pm\sqrt{5}}{2x} = \frac{-1}{2x} \pm \frac{\sqrt{5}}{2x}$. Hence the roots are

$$
\begin{aligned}
m_1 &= \frac{-1+\sqrt{5}}{2x} \\
m_2 &= \frac{-1-\sqrt{5}}{2x}
\end{aligned}
$$

The ODE becomes

$$(D - m_1)(D - m_2)\, y = 0$$

Let

$$(D - m_2)\, y = v \tag{1}$$

hence

$$(D - m_1)\, v = 0$$

Solution of $(D - m)\, v = 0$ is solution of $v' - mv = 0$ which is $v(x) = Ae^{mx}$ hence the solution of the above becomes

$$
\begin{aligned}
v(x) &= Ae^{m_1 x} \\
&= Ae^{\left(\frac{-1+\sqrt{5}}{2x}\right)x} \\
&= Ae^{\left(\frac{-1+\sqrt{5}}{2}\right)}
\end{aligned}
$$

Hence $v(x)$ is constant and does not depend on $x$. Let $Ae^{\left(\frac{-1+\sqrt{5}}{2}\right)} = C_1$.Now from Eq. (1)

$$
\begin{aligned}
(D - m_2)\, y = v &= C_1 \\
\frac{dy}{dx} - m_2 y &= C_1 \\
\frac{dy}{dx} + \frac{1+\sqrt{5}}{2x} y &= C_1
\end{aligned}
$$

The solution to the homogenous equation is

$$
\begin{aligned}
\frac{dy_h}{dx} + \frac{1+\sqrt{5}}{2x} y &= 0 \\
\frac{dy_h}{y} &= -\frac{1+\sqrt{5}}{2x} dx \\
\ln y_h &= \frac{\left(-1-\sqrt{5}\right)}{2} \ln x + C \\
y_h &= C_2 x e^{\frac{\left(-1-\sqrt{5}\right)}{2}} \\
y_h &= C_3 x
\end{aligned}
$$

For the particular solution, using the trial $y_p = C$, hence $\frac{dy_p}{dx} - m_2 y_p = C_1$ or $0 - \frac{1+\sqrt{5}}{2x} C = C_1$, hence $C_1 = \frac{C_4}{x}$, so

$$y_p = \frac{C_4}{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= C_3 x + \frac{C_4}{x}
\end{aligned}
$$

Where $C_4, C_3$ are constants that can be determined from initial or boundary conditions

### 2.3.5 problem 4

$$xy'' + y' = 3x^2 - x$$

First the homogenous equation is solved. Let $D \equiv \frac{d}{dx}$ hence

$$\left(xD^2 + D\right) y_h = 0$$

The roots of the characteristic equation $\left(x\lambda^2 + \lambda\right)$ are $m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1}}{2x} = \frac{-1\pm 1}{2x}$ hence the roots are

$$\begin{aligned} m_1 &= \frac{-1+1}{2x} = 0 \\ m_2 &= \frac{-1-1}{2x} = -\frac{1}{x} \end{aligned}$$

Therefore

$$\begin{aligned} (D - m_2)(D - m_1) y_h &= 0 \\ (D - m_2)(D) y_h &= 0 \qquad (1) \end{aligned}$$

Let

$$(D) y_h = v \qquad (2)$$

Hence

$$\begin{aligned} (D - m_2) v(x) &= 0 \\ \frac{dv}{dx} - m_2 v(x) &= 0 \\ \frac{dv}{v} &= m_2 dx \\ \ln v &= \int \frac{-1}{x} dx + C \\ \ln v &= -\ln x + C \end{aligned}$$

Hence

$$v(x) = \frac{C_1}{x}$$

Where $C_1$ is new constant. Eq. (2) becomes

$$\begin{aligned} (D) y_h = v &= \frac{C_1}{x} \\ y_h' &= \frac{C_1}{x} \\ dy_h &= \frac{C_1}{x} dx \\ y_h &= C_1 \ln x + C_2 \end{aligned}$$

To find particular solution, let

$$y_p = ax^3 + bx^2 + cx + d$$

and $y_p' = 3ax^2 + 2bx + c$ and $y_p'' = 6ax + 2b$ hence the original ODE becomes

$$\begin{aligned} x(6ax + 2b) + \left(3ax^2 + 2bx + c\right) &= 3x^2 - x \\ 9ax^2 + 4bx + c &= 3x^2 - x \end{aligned}$$

Hence $c = 0$ and $a = \frac{1}{3}$ and $4b = -1$ or $b = -\frac{1}{4}$, therefore

$$y_p = \frac{1}{3}x^2 - \frac{1}{4}x$$

And the full solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 \ln x + C_2 + \frac{1}{3}x^2 - \frac{1}{4}x \end{aligned}$$

### 2.3.6 Problem 5 (linear dependent and independent solution)

Problem page 69, problem 8

Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions to $x^2y'' - 2xy' + 2y = 0$ on $[-1, 1]$ but that $W(0) = 0$. Why does this not contradict theorem 2.3.1 in this interval?

Theorem 2.3: Wronskian test: Let $y_1, y_2$ be solutions of $y'' + p(x) y' + q(x) y = 0$ on the open interval $I$, then the following is true

1. Either $W(x) = 0$ for all $x$ in $I$, or $W(x) \neq 0$ for all $x$ in $I$
2. $y_1$ and $y_2$ are linearly independent on $I$ iff $W(x) \neq 0$ on $I$

**Answer**:

First we show that $y_1, y_2$ are solutions to the ODE. Looking at $y_1$, then $y_1' = 1, y'' = 0$. Substituting into the ODE gives

$$-2x + 2x = 0$$

Hence $y_1$ is a solution. Looking now at $y_2$, then $y_2' = 2x, y'' = 2$. Substituting into the ODE gives

$$2x^2 - 4x^2 + 2x^2 = 0$$

Hence $y_2$ is also a solution. Now we will show they are linearly independent. Let

$$ay_1 + by_2 = 0$$

Where $a, b$ are constants. If there are non-zero constants $a, b$ that will make the above true, then $y_1, y_2$ are linearly dependent. Another way to say this, is that if and only if when $a = b = 0$ then the above is true, then $y_1, y_2$ are linearly independent.

Let us assume that for all $x$ the following is true

$$ax + bx^2 = 0$$

Let $x = 1$, then $a + b = 0$. Let $x = -1$ then $b - a = 0$. Solving for $a, b$ from these two equations shows that $2b = 0$ or $b = 0$, hence $a = 0$. Therefore, for $ay_1 + by_2$ to be zero then $a = b = 0$. This shows that $y_1, y_2$ are linearly independent.

The above showed that $y_1, y_2$ are solutions to the ODE and that they are linearly independent functions. Now the Wronskian test is applied

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

At point 0 we see that $W(0) = 0$. This seems like a conflict. But the Abel's stronger statement applies only for solutions of an ODE, which says that for second order ODE, if $y_1, y_2$ are linearly independent solutions of the ODE, then $W$ can not be zero at any point in the interval. However, there is no conflict in this case, since at $x = 0$ this statement does not even apply, as we see that when $x = 0$ the first and second terms of the ODE itself vanish and we no longer have an ODE in first place. At any other point $x$, where the ODE remain in effect as stated, then $W(x) \neq 0$, and hence there is no conflict.

Summary: To show that two functions are linearly independent on an interval, it is enough to show that the $W$ is not zero on any one point in the interval. We do not need to check at each point. It is only when these two functions are also solutions of the ODE, then we need to check that $W$ is not zero on each point in the interval, where the ODE is defined. In this problem, it happened that at $x = 0$ the ODE itself is not defined since $a_0 = 0$ there.

### 2.3.7 Problem 6 page 69 problem 10

Show that $y_1(x) = 3e^{2x} - 1$ and $y_2(x) = e^{-x} + 2$ are solutions of $yy'' + 2y' - \left(y'\right)^2 = 0$ but neither $2y_1$ nor $y_1 + y_2$ is a solution. Why does this not contradict theorem 2.2?

Theorem 2.2: Let $y_1, y_2$ be solutions of $y'' + p(x) y' + q(x) y = 0$ on interval $I$, then any linear combination of these solutions is also a solution.

**Solution**

First we show that the $y_1$ and $y_2$ are solutions. This is done by substitution into the ODE and checking for identity. Starting with $y_1$

$y_1' = 6e^{2x}, y_1'' = 12e^{2x}$, hence the ODE become

$$\begin{aligned} y_1y_1'' + 2y_1' - \left(y_1'\right)^2 &= \left(3e^{2x}-1\right)\left(12e^{2x}\right) + 2\left(6e^{2x}\right) - \left(6e^{2x}\right)^2 \\ &= 36e^{4x} - 12e^{2x} + 12e^{2x} - 36e^{4x} \\ &= 0 \end{aligned}$$

This shows that $y_1$ is a solution. Now for $y_2$ we have $y_2' = -e^{-x}, y_2'' = e^{-x}$, hence the ODE become

$$\begin{aligned} y_1y_1'' + 2y_1' - \left(y_1'\right)^2 &= \left(e^{-x}+2\right)\left(e^{-x}\right) + 2\left(-e^{-x}\right) - \left(-e^{-x}\right)^2 \\ &= e^{-2x} + 2e^{-x} - 2e^{-x} - e^{-2x} \\ &= 0 \end{aligned}$$

Therefore $y_2$ is also a solution. now to Check if $2y_1$ is a solution. Let $y_3 = 2y_1 = 6e^{2x} - 2$ hence $y_3' = 12e^{2x}$ and $y_3'' = 24e^{2x}$. Substitution into the ODE gives

$$\begin{aligned} y_3y_3'' + 2y_3' - \left(y_3'\right)^2 &= \left(6e^{2x}-2\right)\left(24e^{2x}\right) + 2\left(12e^{2x}\right) - \left(12e^{2x}\right)^2 \\ &= 144e^{4x} - 48e^{2x} + 24e^{2x} - 144e^{4x} \\ &= -24e^{2x} \\ &\neq 0 \end{aligned}$$

Hence $y_3 = 2y_1$ is not a solution.

Now to check that $y_1+y_2$ is a solution or not. Let $y_4 = y_1+y_2 = 3e^{2x}-1+e^{-x}+2 = 3e^{2x}+e^{-x}+1$ , hence $y_4' = 6e^{2x} - e^{-x}$ and $y_4'' = 12e^{2x} + e^{-x}$, and substitution into the ODE gives

$$\begin{aligned} y_4y_4'' + 2y_4' - \left(y_4'\right)^2 &= \left(3e^{2x}+e^{-x}+1\right)\left(12e^{2x}+e^{-x}\right) + 2\left(6e^{2x}-e^{-x}\right) - \left(6e^{2x}-e^{-x}\right)^2 \\ &= 36e^{4x} + 3e^{x} + 12e^{x} + e^{-2x} + 12e^{2x} + e^{-x} + 12e^{2x} - 2e^{-x} - 36e^{4x} - e^{-2x} + 12e^{x} \\ &= 27e^{x} - e^{-x} + 24e^{2x} \\ &\neq 0 \end{aligned}$$

Hence $y_4 = y_1 + y_2$ is not a solution.

Now to answer the question. Since the ODE given is not linear, and not of the form $y'' + p(x)y' + q(x)y = 0$, then we need to check first that when using the solution $2y_1$ or $y_1 + y_2$, the ODE remains of the same form shown above for these to be also solutions.

Let us try $2y_1$ and substituting this into the ODE. This results in

$$\begin{aligned} yy'' + 2y' - \left(y'\right)^2 &= 0 \\ \left(2y_1\right)\left(2y_1\right)'' + 2\left(2y_1\right)' - \left[\left(2y_1\right)'\right]^2 &= 0 \\ \left(2y_1\right)2y_1'' + 2\left(2y_1'\right) - \left(2y_1'\right)^2 &= 0 \\ 4y_1y_1'' + 4y_1' - 4\left(y_1'\right)^2 &= 0 \end{aligned}$$

Dividing by 4

$$y_1y_1'' + y_1' - \left(y_1'\right)^2 = 0$$

Comparing this with the original ODE, we see it is not the same ODE. The second term was $2y'$ and now it is $y_1'$. Hence $2y_1$ is not a solution. The reason is due to the ***nonlinearity*** of the ODE, the theorem did not apply to it.

Checking now for the second trial solution $y_1 + y_2$ and substituting this into the ODE

$$\begin{aligned}
yy'' + 2y' - \left(y'\right)^2 &= 0\\
\left(y_1 + y_2\right)\left(y_1 + y_2\right)'' + 2\left(y_1 + y_2\right)' - \left[\left(y_1 + y_2\right)'\right]^2 &= 0\\
\left(y_1 + y_2\right)\left(y_1'' + y_2''\right) + 2\left(y_1' + y_2'\right) - \left(y_1' + y_2'\right)^2 &= 0\\
\left(y_1 + y_2\right)\left(y_1'' + y_2''\right) + 2\left(y_1' + y_2'\right) - \left(y_1'\right)^2 - \left(y_2'\right)^2 - 2y_1'y_2' &= 0\\
\left(y_1y_1'' + y_1y_2''\right) + \left(y_2y_1'' + y_2y_2''\right) + 2y_1' + 2y_2' - \left(y_1'\right)^2 - \left(y_2'\right)^2 - 2y_1'y_2' &= 0\\
\left[y_1y_1'' + 2y_1' - \left(y_1'\right)^2\right] + \left[y_2y_2'' + 2y_2' - \left(y_2'\right)^2\right] + y_1y_2'' + y_2y_1'' - 2y_1'y_2' &= 0
\end{aligned}$$

The terms in square brackets are zero, since they are solutions of the ODE and hence vanish, hence the above reduces to

$$y_1y_2'' + y_2y_1'' - 2y_1'y_2' = 0$$

This is not the same ODE we started with. For $y_3 = y_1 + y_2$ to be a solution, the ODE obtain $y_3y_3'' + 2y_3' - \left(y_3'\right)^2 = 0$. The reason is due to the ***nonlinearity*** of the ODE.

### 2.3.8 Problem 7

Solve the IC problem $\left(D^3 - 6D^2 + 11D - 6\right)y = 0$ with IC $y = y' = 0$ and $y'' = 2$ when $x = 0$

We need to factor the characteristic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$. Guessing a root, we see that $\lambda = 2$ is a root. Long division gives $\frac{\lambda^3 - 6\lambda^2 + 11\lambda - 6}{\lambda - 2} = \lambda^2 - 4\lambda + 3$, hence the characteristic equation is $\left(\lambda^2 - 4\lambda + 3\right)(\lambda - 2)$. Now we factor the quadratic giving the final answer of $(\lambda - 1)(\lambda - 3)(\lambda - 2)$. The ODE is now written as

$$(D-1)(D-3)(D-2)y = 0$$

Let $(D-2)y = v$ then

$$(D-1)(D-3)v = 0$$

Let $(D-3)v = u$ then

$$\begin{aligned}
(D-1)u &= 0\\
u' - u &= 0\\
\frac{du}{dx} &= u(x)\\
\ln u &= x + c_1\\
u &= c_1e^x
\end{aligned}$$

Backtracking to the previous ODE

$$\begin{aligned}
(D-3)v &= u\\
\frac{dv}{dx} - 3v &= c_1e^x
\end{aligned}$$

Integrating factor is $I_f = e^{-3x}$ hence

$$\begin{aligned}
\frac{d}{dx}\left(I_fv\right) &= I_fc_1e^x\\
I_fv &= \int I_fc_1e^xdx + c_2\\
&= c_1\int e^{-2x}dx + c_2\\
&= c_1\left(\frac{-1}{2}e^{-2x}\right) + c_2\\
v &= \frac{-c_1}{2e^{-3x}}e^{-2x} + \frac{c_2}{e^{-3x}}\\
&= \frac{-c_1}{2}e^x + c_2e^{3x}
\end{aligned}$$

Now backtracking to the first ODE

$$(D-2)y = v$$
$$\frac{dy}{dx} - 2y = \frac{-c_1}{2}e^x + c_2e^{3x}$$

Integrating factor is $I_f = e^{-2x}$ hence

$$\frac{d}{dx}\left(I_f y\right) = I_f\left(\frac{-c_1}{2}e^x + c_2e^{3x}\right)$$
$$e^{-2x}y = \int e^{-2x}\left(\frac{-c_1}{2}e^x + c_2e^{3x}\right)dx + c_3$$
$$= \int\left(\frac{-c_1}{2}e^{-x} + c_2e^x\right)dx + c_3$$
$$= \frac{c_1}{2}e^{-x} + c_2e^x + c_3$$
$$y = \frac{c_1}{2}e^x + c_2e^{3x} + c_3e^{2x}$$

or letting $\frac{c_1}{2} = c_1$ (new constant) then

$$y(x) = c_1e^x + c_2e^{3x} + c_3e^{2x}$$

Now the constants are found from IC. $y = y' = 0$ and $y'' = 2$

When $x = 0$ then $y = 0$, hence

$$0 = c_1 + c_2 + c_3 \tag{1}$$

Taking derivative, then

$$y'(x) = c_1e^x + 3c_2e^{3x} + 2c_3e^{2x}$$

Hence

$$0 = c_1 + 3c_2 + 2c_3 \tag{2}$$

Taking derivative again

$$y''(x) = c_1e^x + 9c_2e^{3x} + 4c_3e^{2x}$$

At $x = 0$

$$2 = c_1 + 9c_2 + 4c_3 \tag{3}$$

Solving Eqs. (1),(2),(3) for the constants gives $c_1 = 1, c_2 = 1, c_3 = -2$. The final solution is

$$y(x) = e^x + e^{3x} - 2e^{2x}$$

### 2.3.9 Problem 8

Solve the IC problem $8y''' - 4y'' + 6y' + 5y = 0$ with IC $y = 0, y'' = y' = 1$ when $x = 0$

**Solution**:

Writing the ODE as $\left(8D^3 - 4D^2 + 6D + 5\right)y = 0$. The first step is to factor the characteristic equation $8\lambda^3 - 4\lambda^2 + 6\lambda + 5 = 0$.

By guessing an initial root as $\lambda = -\frac{1}{2}$ with some trials, now performing long Division to reduce it to a quadratic and then applying the quadratic equation to obtain the remaining two roots. Hence$\frac{8\lambda^3-4\lambda^2+6\lambda+5}{\lambda+\frac{1}{2}} = 8\lambda^2 - 8\lambda + 10$.

The characteristic equation now becomes $\left(\lambda + \frac{1}{2}\right)\left(8\lambda^2 - 8\lambda + 10\right)$. Factoring the quadratic gives $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{8\pm\sqrt{64-4(8)(10)}}{16} = \frac{8\pm\sqrt{64-320}}{16} = \frac{8\pm16i}{16} = \frac{1\pm2i}{2}$. This means the roots are $\frac{1}{2} \pm i$. Hence the ODE becomes

$$\left(D - \left(\frac{1}{2} + i\right)\right)\left(D - \left(\frac{1}{2} - i\right)\right)\left(D + \frac{1}{2}\right)y = 0$$

Let $\left(D + \frac{1}{2}\right)y = v$ The ODE becomes

$$\left(D - \left(\frac{1}{2} + i\right)\right)\left(D - \left(\frac{1}{2} - i\right)\right)v = 0$$

Let $\left(D-\left(\frac{1}{2}-i\right)\right)v=u$. The ODE becomes

$$\begin{aligned}\left(D-\left(\frac{1}{2}+i\right)\right)u&=0\\ \frac{du}{dx}-\left(\frac{1}{2}+i\right)u&=0\end{aligned}$$

This is separable with solution $u=c_1e^{\left(\frac{1}{2}+i\right)x}$ backtracking to the previous ODE and solving

$$\begin{aligned}\left(D-\left(\frac{1}{2}-i\right)\right)v&=c_1e^{\left(\frac{1}{2}+i\right)x}\\ \frac{dv}{dx}-\left(\frac{1}{2}-i\right)v&=c_1e^{\left(\frac{1}{2}+i\right)x}\end{aligned}$$

Integrating factor is $I_f=e^{-\left(\frac{1}{2}-i\right)x}$ hence

$$\begin{aligned}\frac{d}{dx}\left(I_fv\right)&=I_fc_1e^{\left(\frac{1}{2}+i\right)x}\\ ve^{-\left(\frac{1}{2}-i\right)x}&=\int c_1e^{-\left(\frac{1}{2}-i\right)x}e^{\left(\frac{1}{2}+i\right)x}dx+c_2\\ &=c_1\int e^{\left[-\left(\frac{1}{2}-i\right)+\left(\frac{1}{2}+i\right)\right]x}dx+c_2\\ &=c_1\int e^{2ix}dx+c_2\\ &=c_1\frac{e^{2ix}}{2}+c_2\end{aligned}$$

Therefore

$$\begin{aligned}v\left(x\right)&=c_1\frac{e^{2ix}}{2}e^{\left(\frac{1}{2}-i\right)x}+c_2e^{\left(\frac{1}{2}-i\right)x}\\ &=c_1e^{\left(i+\frac{1}{2}\right)x}+c_2e^{\left(-i+\frac{1}{2}\right)x}\end{aligned}$$

Where $c_1=\frac{c_1}{2}$. Backtracking to the first ODE, we now solve

$$\begin{aligned}\left(D+\frac{1}{2}\right)y&=v\\ \frac{dy}{dx}+\frac{1}{2}y&=c_1e^{\left(i+\frac{1}{2}\right)x}+c_2e^{\left(-i+\frac{1}{2}\right)x}\end{aligned}$$

The integrating factor is $e^{\frac{1}{2}x}$ hence

$$\begin{aligned}\frac{d}{dx}\left(I_fy\right)&=I_f\left(c_1e^{\left(i+\frac{1}{2}\right)x}+c_2e^{\left(-i+\frac{1}{2}\right)x}\right)\\ I_fy&=\int e^{\frac{1}{2}x}\left(c_1e^{\left(i+\frac{1}{2}\right)x}+c_2e^{\left(-i+\frac{1}{2}\right)x}\right)dx+c_3\\ &=\int c_1e^{(i+1)x}+c_2e^{(-i+1)x}dx+c_3\\ &=c_1\frac{e^{(1+i)x}}{1+i}+c_2\frac{e^{(1-i)x}}{1-i}+c_3\end{aligned}$$

Therefore

$$\begin{aligned}y&=c_1\frac{e^{(1+i)x}}{1+i}e^{\frac{-1}{2}x}+c_2\frac{e^{(1-i)x}}{1-i}e^{\frac{-1}{2}x}+c_3e^{\frac{-1}{2}x}\\ &=e^{\frac{1}{2}x}\left(c_1\frac{e^{ix}}{1+i}+c_2\frac{e^{-ix}}{1-i}\right)+c_3e^{\frac{-1}{2}x}\end{aligned}$$

But $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$, hence combining the above gives

$$\begin{aligned}
y &= e^{\frac{1}{2}x}\left(c_1\frac{\cos x + i\sin x}{1+i} + c_2\frac{\cos x - i\sin x}{1-i}\right) + c_3 e^{\frac{-1}{2}x} \\
&= e^{\frac{1}{2}x}\left(\frac{c_1(1-i)(\cos x + i\sin x) + c_2(1+i)(\cos x - i\sin x)}{(1+i)(1-i)}\right) + c_3 e^{\frac{-1}{2}x} \\
&= e^{\frac{1}{2}x}\left(\frac{c_1(\cos x + i\sin x - i(\cos x + i\sin x)) + c_2(\cos x - i\sin x + i(\cos x - i\sin x))}{2}\right) + c_3 e^{\frac{-1}{2}x} \\
&= e^{\frac{1}{2}x}\left(\frac{c_1(\cos x + i\sin x - i\cos x + \sin x) + c_2(\cos x - i\sin x + i\cos x + \sin x)}{2}\right) + c_3 e^{\frac{-1}{2}x} \\
&= e^{\frac{1}{2}x}\left(\frac{\cos x(c_1 - ic_1 + c_2 + ic_2) + \sin x(c_1 + ic_1 - ic_2 + c_2)}{2}\right) + c_3 e^{\frac{-1}{2}x}
\end{aligned}$$

Let $\frac{(c_1 - ic_1 + c_2 + ic_2)}{2} = c_4$ and let $\frac{(c_1 + ic_1 - ic_2 + c_2)}{2} = c_5$, then the above reduces to

$$y = e^{\frac{1}{2}x}(c_4\cos x + c_5\sin x) + c_3 e^{\frac{-1}{2}x}$$

This is the general solution. $c_3, c_4, c_5$ are found from IC. $y = 0, y'' = y' = 1$

When $x = 0$ and $y = 0$

$$0 = c_4 + c_3 \tag{1}$$

Now

$$y' = \frac{1}{2}e^{\frac{1}{2}x}(c_4\cos x + c_5\sin x) + e^{\frac{1}{2}x}(-c_4\sin x + c_5\cos x) - \frac{1}{2}c_3 e^{\frac{-1}{2}x}$$

Hence at $x = 0$

$$1 = \frac{1}{2}c_4 + c_5 - \frac{1}{2}c_3 \tag{2}$$

and

$$\begin{aligned}
y'' = &\frac{1}{4}e^{\frac{1}{2}x}(c_4\cos x + c_5\sin x) + \frac{1}{2}e^{\frac{1}{2}x}(-c_4\sin x + c_5\cos x) \\
&+ \frac{1}{2}e^{\frac{1}{2}x}(-c_4\sin x + c_5\cos x) + \frac{1}{2}e^{\frac{1}{2}x}(-c_4\cos x - c_5\sin x) + \frac{1}{4}c_3 e^{\frac{-1}{2}x}
\end{aligned}$$

Hence at $x = 0$

$$\begin{aligned}
1 &= \frac{1}{4}c_4 + \frac{1}{2}c_5 + \frac{1}{2}c_5 - \frac{1}{2}c_4 + \frac{1}{4}c_3 \\
&= \frac{1}{4}c_3 - \frac{1}{4}c_4 + c_5 \qquad (3)
\end{aligned}$$

Solving Eqs. (1),(2),(3) for the constants gives $c_3 = 0, c_4 = 0, c_5 = 1$, hence the solution is

$$y = e^{\frac{1}{2}x}\sin x$$

A plot of the solution is

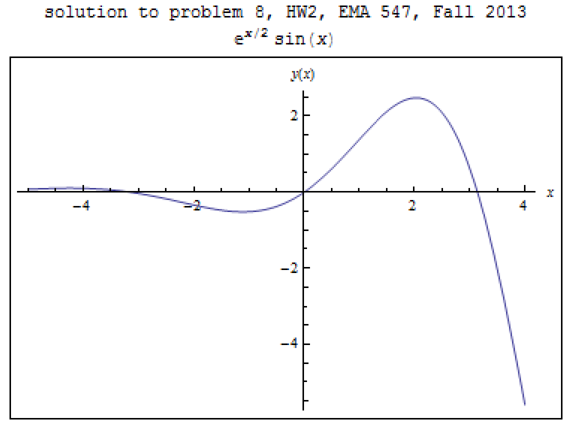


Figure 2.5: plot of solution to problem 8 HW2

### 2.3.10 Problem 9, Nonhomogeneous equations with constant coefficients

O'Neil. page 93, problem 16. Find general solution to $y'' - 2y' + y = 3x + 25\sin(3x)$

Write as $\left(D^2 - 2D + 1\right) y = 3x + 25 \sin(3x)$, where $L \equiv D^2 - 2D + 1 = (D-1)(D-1)$. This will be solved two ways. The first using variation of parameters to obtain the particular solution, and the second by finding particular solution to each separate forcing function and adding.

#### 2.3.10.1 First method (variation of parameters)

$$(D-1)(D-1) y_h = 0$$

Let $(D-1) y_h = v$, then

$$\begin{aligned}(D-1) v &= 0 \\ \frac{dv}{dx} - v &= 0\end{aligned}$$

Solution is $v = c_1 e^x$. We now backtrack and solve

$$\begin{aligned}(D-1) y_h &= v \\ \frac{dy_{1,h}}{dx} - y_h &= c_1 e^x\end{aligned}$$

Integrating factor is $e^{-x}$ hence

$$\begin{aligned}\frac{d}{dx}\left(I_f y_h\right) &= e^{-x}\left(c_1 e^x\right) \\ e^{-x} y_h &= c_1 x + c_2 \\ y_h &= c_1 x e^x + c_2 e^x\end{aligned}$$

Hence $y_1 = xe^x$ and $y_2 = e^x$ are the two linearly independent solutions of the homogenous ODE. Let the particular solution be

$$y_p = u_1 y_1 + u_2 y_2$$

where $u_1(x), u_2(x)$ are functions of $x$ to be found. Hence

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

and

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2''$$

Therefore, the ODE $y_p'' - 2y_p' + y_p = 3x + 25\sin(3x)$ becomes

$$\begin{aligned}u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2'' \\ -2\left(u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right) + u_1 y_1 + u_2 y_2 \\ = 3x + 25\sin(3x)\end{aligned}$$

Collecting terms

$$u_1\left[y_1'' - 2y_1' + y_1\right] + u_2\left[y_2'' - 2y_2' + y_2\right] + u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_2'' y_2 + u_2' y_2' + u_2' y_2' - 2\left(u_1' y_1 + u_2' y_2\right) = 3x + 25\sin(3x)$$

But terms in brackets vanish since this is the ODE with the homogeneous solutions, hence the above reduces to

$$\begin{aligned}u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_2'' y_2 + u_2' y_2' + u_2' y_2' - 2\left(u_1' y_1 + u_2' y_2\right) &= 3x + 25\sin(3x) \\ \overbrace{u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2'} + u_1' y_1' + u_2' y_2' - 2\left(u_1' y_1 + u_2' y_2\right) &= 3x + 25\sin(3x) \\ \frac{d}{dx}\left(u_1' y_1 + u_2' y_2\right) + \left(u_1' y_1' + u_2' y_2'\right) - 2\left(u_1' y_1 + u_2' y_2\right) &= 3x + 25\sin(3x)\end{aligned}$$

If

$$u_1' y_1 + u_2' y_2 = 0 \tag{1}$$

then the above becomes

$$\left(u_1' y_1' + u_2' y_2'\right) = f(x) = 3x + 25\sin(3x) \tag{2}$$

Hence we have two equations Eqs. (1),(2) to solve for $u_1, u_2$

$$u_1 = \int \frac{-y_2}{y_1 y_2' - y_2 y_1'} f(x)\, dx = \int \frac{-y_2}{W(x)} f(x)\, dx$$

But

$$W(x)=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}xe^x & e^x\\ e^x+xe^x & e^x\end{vmatrix}=xe^{2x}-\left(e^{2x}+xe^{2x}\right)=-e^{2x}$$

Hence

$$\begin{aligned}u_1&=\int\frac{-e^x}{-e^{2x}}\left(3x+25\sin\left(3x\right)\right)dx\\&=\int e^{-x}\left(3x+25\sin\left(3x\right)\right)dx\\&=3\int xe^{-x}+25\int e^{-x}\sin\left(3x\right)dx\\&=e^{-x}\left(-3-3x-\frac{15}{2}\cos\left(3x\right)-\frac{5}{2}\sin\left(3x\right)\right)\end{aligned}$$

and

$$\begin{aligned}u_2&=\int\frac{y_1}{W\left(x\right)}f\left(x\right)dx\\&=\int\frac{xe^x}{-e^{2x}}\left(3x+25\sin\left(3x\right)\right)dx\\&=-\int xe^{-x}\left(3x+25\sin\left(3x\right)\right)dx\\&=-3\int x^2e^{-x}dx-25\int e^{-x}x\sin\left(3x\right)dx\\&=e^{-x}\left(6+6x+3x^2+\frac{3}{2}\cos 3x+\frac{15}{2}x\cos 3x-2\sin 3x+\frac{5}{2}x\sin 3x\right)\end{aligned}$$

Therefore

$$\begin{aligned}y_p&=u_1y_1+u_2y_2\\&=e^{-x}\left[-3-3x-\frac{15}{2}\cos\left(3x\right)-\frac{5}{2}\sin\left(3x\right)\right]xe^x\\&\quad+\left(e^{-x}\left(6+6x+3x^2+\frac{3}{2}\cos 3x+\frac{15}{2}x\cos 3x-2\sin 3x+\frac{5}{2}x\sin 3x\right)\right)e^x\\&=-3x-3x^2-\frac{15}{2}x\cos\left(3x\right)-\frac{5}{2}x\sin\left(3x\right)+6+6x+3x^2+\frac{3}{2}\cos 3x+\frac{15}{2}x\cos 3x-2\sin 3x+\frac{5}{2}x\sin 3x\\&=3x+\frac{3}{2}\cos 3x-2\sin 3x+6\end{aligned}$$

Hence the total solution is

$$\begin{aligned}y&=y_h+y_p\\&=c_1xe^x+c_2e^x+3x+\frac{3}{2}\cos 3x-2\sin 3x+6\end{aligned}$$

#### 2.3.10.2 Second method (using linearity to add the two separate particular solutions)

This will be solved by breaking the forcing functions and solving for each separately and then adding the solutions at the end since the ODE is linear. Hence we will solve the following two ODE's

$$\begin{aligned}y_1''-2y_1'+y_1&=3x\\y_2''-2y_2'+y_2&=25\sin\left(3x\right)\end{aligned}$$

and the solution will be $y=y_1+y_2$. Starting with the first one, we solve for the homogeneous and then for the particular.

$$\left(D-1\right)\left(D-1\right)y_{1,h}=0$$

Now we processed as before. Let $\left(D-1\right)y_{1,h}=v$, then

$$\begin{aligned}\left(D-1\right)v&=0\\\frac{dv}{dx}-v&=0\end{aligned}$$

Solution is $v = c_1 e^x$. We now backtrack and solve

$$\begin{aligned}(D-1)\,y_{1,h} &= v\\ \frac{dy_{1,h}}{dx} - y_{1,h} &= c_1 e^x\end{aligned}$$

Integrating factor is $e^{-x}$ hence

$$\begin{aligned}\frac{d}{dx}\left(I_f y_{1,h}\right) &= e^{-x}\,(c_1 e^x)\\ e^{-x} y_{1,h} &= c_1 x + c_2\\ y_{1,h} &= c_1 x e^x + c_2 e^x\end{aligned}$$

Now we find the particular solution $y_{1,p}$.

Let $y_{1,p} = ax^2 + bx + c$, hence $y'_{1,p} = 2ax + b$ and $y''_{1,p} = 2a$, therefore the ODE becomes

$$\begin{aligned}2a - 2\,(2ax + b) + ax^2 + bx + c &= 3x\\ x^2\,(a) + x\,(-4a + b) + 2a - 2b + c &= 3x\end{aligned}$$

Hence $a = 0$ and $-2b + c = 0$ and $b = 3$. Therefore $c = 6$ and the forcing function is $y_{1,p} = 3x + 6$, hence

$$y_1 = c_1 x e^x + c_2 e^x + 3x + 6$$

We now solve the second ode

$$y_2'' - 2y_2' + y_2 = 25\sin(3x)$$

The homogenous solution is the same as above, $y_{2,h} = c_1 x e^x + c_2 e^x$. Only the particular solution needs to be found again. Let $y_{2,p} = A\sin 3x + B\cos 3x$, hence $y'_{2,p} = 3A\cos 3x - 3B\sin 3x$ and $y''_{2,p} = -9A\sin 3x - 9B\cos x$. The ODE becomes

$$\begin{aligned}-9A\sin 3x - 9B\cos x - 2\,(3A\cos 3x - 3B\sin 3x) + A\sin 3x + B\cos 3x &= 25\sin(3x)\\ \sin 3x\,(-9A + 6B + A) + \cos 3x\,(-9B - 6A + B) &= 25\sin(3x)\end{aligned}$$

Therefore, $(-8A + 6B) = 25$ and $(-8B - 6A) = 0$, from the first equation $A = \frac{6B-25}{8}$, and from the second $-8B - 6\frac{6B-25}{8} = 0$ or $-64B - 36B + 150 = 0$ or $B = 1.5$, hence $A = \frac{9-25}{8} = -2$, therefore

$$y_{2,p} = -2\sin 3x + \frac{3}{2}\cos 3x$$

And the general solution is

$$y = c_1 x e^x + c_2 e^x + 3x + 6 - 2\sin 3x + \frac{3}{2}\cos 3x$$

This answer matches the answer obtained above using variation of parameters.

### 2.3.11 Problem 10

Find general solution $y^{(4)} + 3y'' - 4y = \sinh(x) - \sin^2 x$

First the homogenous solution is find using the operator method. Let

$$\left(D^4 + 3D^2 - 4\right)y = \sinh(x) - \sin^2 x$$

The characteristic equation is $\lambda^4 + 3\lambda^2 - 4 = 0$. Let $\lambda^2 = u$, hence $u^2 - u - 4 = 0$, and the roots are $u = \{1, -4\}$. Hence when $u = 1, \lambda = \pm 1$ and when $u = -4, \lambda = \pm 2i$, therefore we obtain the 4 roots as $\{1, -1, 2i, -2i\}$ and the factorization is

$$(D-1)\,(D+1)\,(D-2i)\,(D+2i)\,y = \sinh(x) - \sin^2 x$$

Solving the homogenous part first.

$$(D-1)\,(D+1)\,(D-2i)\,(D+2i)\,y = 0$$

Let $(D+2i)\,y = v$, hence

$$(D-1)\,(D+1)\,(D-2i)\,v = 0$$

Let $(D-2i)\,v = u$ hence

$$(D-1)\,(D+1)\,u = 0$$

Let $(D+1)u=r$ hence

$$\begin{aligned}(D-1)r&=0\\ \frac{dr}{dx}-r&=0\end{aligned}$$

And the solution is $r(x)=c_1e^x$, backtracking now we solve

$$\begin{aligned}(D+1)u&=c_1e^x\\ \frac{du}{dx}+u&=c_1e^x\end{aligned}$$

Integration factor is $e^x$, hence

$$\begin{aligned}\frac{d}{dx}(e^xu)&=e^x(c_1e^x)\\ e^xu&=c_1\int e^{2x}dx+c_2\\ &=c_1\frac{e^{2x}}{2}+c_2\end{aligned}$$

Therefore

$$u=c_1\frac{e^x}{2}+c_2e^{-x}$$

Let $c_1=\frac{1}{2}c_1$, hence

$$u=c_1e^x+c_2e^{-x}$$

Backtracking, we now solve

$$\begin{aligned}(D-2i)v&=u\\ \frac{dv}{dx}-2iv&=c_1e^x+c_2e^{-x}\end{aligned}$$

Integration factor is $e^{-2ix}$ hence

$$\begin{aligned}\frac{d}{dx}\left(e^{-2ix}v\right)&=e^{-2ix}(c_1e^x+c_2e^{-x})\\ e^{-2ix}v&=\int e^{-2ix}(c_1e^x+c_2e^{-x})\,dx+c_3\\ &=c_1\int e^{-2ix+x}dx+c_2\int e^{-2ix-x}dx+c_3\\ &=c_1\frac{e^{-2ix+x}}{-2i+1}+c_2\frac{e^{-2ix-x}}{-2i-1}+c_3\end{aligned}$$

Hence

$$\begin{aligned}v&=e^{2ix}c_1\frac{e^{-2ix+x}}{-2i+1}+e^{2ix}c_2\frac{e^{-2ix-x}}{-2i-1}+e^{2ix}c_3\\ &=c_1\frac{e^x}{-2i+1}+c_2\frac{e^{-x}}{-2i-1}+e^{2ix}c_3\end{aligned}$$

Now we backtrack one last time and solve for $y_h$

$$\begin{aligned}(D+2i)y_h&=v\\ \frac{dy_h}{dx}+2iy_h&=c_1\frac{e^x}{-2i+1}+c_2\frac{e^{-x}}{-2i-1}+e^{2ix}c_3\end{aligned}$$

Integration factor is $e^{2ix}$ hence

$$\begin{aligned}\frac{d}{dx}\left(e^{2ix}y_h\right)&=e^{2ix}\left(c_1\frac{e^x}{-2i+1}+c_2\frac{e^{-x}}{-2i-1}+e^{2ix}c_3\right)\\ e^{2ix}y_h&=\int e^{2ix}\left(c_1\frac{e^x}{-2i+1}+c_2\frac{e^{-x}}{-2i-1}+e^{2ix}c_3\right)dx+c_4\\ &=\frac{c_1}{-2i+1}\int e^{x+2ix}dx+\frac{c_2}{-2i-1}\int e^{-x+2ix}dx+\int e^{4ix}c_3dx+c_4\\ &=\frac{c_1}{-2i+1}\frac{e^{x+2ix}}{1+2i}-\frac{c_2}{-2i-1}\frac{e^{-x+2ix}}{-1+2i}+\frac{c_3}{4i}e^{4ix}+c_4\\ &=\frac{c_1}{5}e^{x+2ix}-\frac{c_2}{5}e^{-x+2ix}+\frac{c_3}{4i}e^{4ix}+c_4\end{aligned}$$

Hence

$$
\begin{aligned}
y_h &= e^{-2ix}\left(\frac{c_1}{5}e^{x+2ix} - \frac{c_2}{5}e^{-x+2ix} + \frac{c_3}{4i}e^{4ix} + c_4\right)\\
&= \frac{c_1}{5}e^{x} - \frac{c_2}{5}e^{-x} + \frac{c_3}{4i}e^{2ix} + c_4 e^{-2ix}
\end{aligned}
$$

Let $\frac{c_1}{5} = c_1$ and $\frac{-c_2}{5} = c_2$ and $\frac{-c_3}{4} = c_3$ the above simplifies to

$$
\begin{aligned}
y_h &= c_1 e^{x} + c_2 e^{-x} - c_3\frac{e^{2ix}}{i} + c_4 e^{-2ix}\\
&= c_1 e^{x} + c_2 e^{-x} + c_3 i e^{2ix} + c_4 e^{-2ix}
\end{aligned}
$$

Convert to trig using Euler's we obtain

$$
\begin{aligned}
y_h &= c_1 e^{x} + c_2 e^{-x} + c_3 i\left(\cos 2x + i\sin 2x\right) + c_4\left(\cos 2x - i\sin 2x\right)\\
&= c_1 e^{x} + c_2 e^{-x} + \cos(2x)\left(ic_3 + c_4\right) + \sin(2x)\left(-c_3 - ic_4\right)
\end{aligned}
$$

Let $(ic_3 + c_4) = c_5$ and $(-c_3 - ic_4) = c_6$, new constants, hence

$$
y_h = c_1 e^{x} + c_2 e^{-x} + c_5\cos(2x) + c_6\sin(2x)
$$

#### 2.3.11.1 Finding the particular solutions

To find the particular solution, using superposition. Since $\left(D^4 + 3D^2 - 4\right)y = \sinh(x) - \sin^2 x$, we solve first for the first forcing function

$$
\left(D^4 + 3D^2 - 4\right)y = \sinh(x)
$$

$\sinh(x)$ can not be used for trail solution, as the homogeneous solution include $e^x$ in it and $\sinh(x) = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$. Therefore we will use $Axe^x + Cxe^{-x}$ as trial solution. Hence

$$
\begin{aligned}
y_{p1} &= Ae^{x} + Bxe^{x} + Ce^{-x} + Dxe^{-x}\\
y'_{p1} &= Ae^{x} + Be^{x} + Bxe^{x} - Ce^{-x} + De^{-x} - Dxe^{-x}\\
y''_{p1} &= Ae^{x} + Be^{x} + Be^{x} + Bxe^{x} + Ce^{-x} - De^{-x} - De^{-x} + Dxe^{-x}\\
&= Ae^{x} + 2Be^{x} + Bxe^{x} + Ce^{-x} - 2De^{-x} + Dxe^{-x}\\
y'''_{p1} &= Ae^{x} + 2Be^{x} + Be^{x} + Bxe^{x} - Ce^{-x} + 2De^{-x} + De^{-x} - Dxe^{-x}\\
&= Ae^{x} + 3Be^{x} + Bxe^{x} - Ce^{-x} + 3De^{-x} - Dxe^{-x}\\
y''''_{p1} &= Ae^{x} + 3Be^{x} + Be^{x} + Bxe^{x} + Ce^{-x} - 3De^{-x} - De^{-x} + Dxe^{-x}\\
&= Ae^{x} + 4Be^{x} + Bxe^{x} + Ce^{-x} - 4De^{-x} + Dxe^{-x}
\end{aligned}
$$

Therefore the ODE becomes, and using $\frac{e^x}{2} - \frac{e^{-x}}{2}$ for $\sinh(x)$

$$
\begin{aligned}
\left(D^4 + 3D^2 - 4\right)y_{p1} &= \left(Ae^{x} + 4Be^{x} + Bxe^{x} + Ce^{-x} - 4De^{-x} + Dxe^{-x}\right)\\
&\quad + 3\left(Ae^{x} + 2Be^{x} + Bxe^{x} + Ce^{-x} - 2De^{-x} + Dxe^{-x}\right)\\
&\quad - 4\left(Ae^{x} + Bxe^{x} + Ce^{-x} + Dxe^{-x}\right) = \frac{e^{x}}{2} - \frac{e^{-x}}{2}
\end{aligned}
$$

Hence, comparing coefficients

$$
e^{x}\left(A + 4B + 3A + 6B - 4A\right) + e^{-x}\left(C - 4D + 3C - 6D - 4C\right) + xe^{x}\left(B + 3B - 4B\right) + xe^{-x}\left(D + 3D - 4D\right) = \frac{e^{x}}{2} - \frac{e^{-x}}{2}
$$

Hence

$$
\begin{aligned}
A + 4B + 3A + 6B - 4A &= \frac{1}{2}\\
C - 4D + 3C - 6D - 4C &= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
10B &= \frac{1}{2}\\
-10D &= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
B &= \frac{1}{20}\\
D &= \frac{1}{20}
\end{aligned}
$$

Therefore

$$
y_{1p} = \frac{1}{20} x e^{x} + \frac{1}{20} x e^{-x}
$$

To find second particular solution,

$$
\left(D^4 + 3D^2 - 4\right) y = -\sin^2 x
$$

Since $\sin^2 x = \frac{1}{2} - \frac{1}{4}\left(e^{-2ix} + e^{2ix}\right)$ and the functions $e^{\pm 2ix}$ are in the homogeneous solution, let the trial function be $y_{p2} = a + bxe^{-2ix} + cxe^{2ix}$. Plug-in this into the ODE and expanding gives

$$
e^{-2ix}\left(32ib + 16xb - 12ib - 4bx\right) + e^{2ix}\left(-32ic + 16xc + 12ic - 12cx - 4bx\right) - 4a = \frac{1}{2} - \frac{1}{4}\left(e^{-2ix} + e^{2ix}\right)
$$

This can be used to find $y_{p2}$ (need to more time to work this out). The final solution will then be

$$
\begin{aligned}
y &= y_h + y_{p1} + y_{p2}\\
y_h &= c_1 e^{x} + c_2 e^{-x} + c_5 \cos\left(2x\right) + c_6 \sin\left(2x\right) + \frac{1}{20} x e^{x} + \frac{1}{20} x e^{-x} + y_{p2}
\end{aligned}
$$

**note**:

I verified the solution using Mathematica. The homogeneous solution appears to be correct, but need to work more on the particular solution. Here is the result

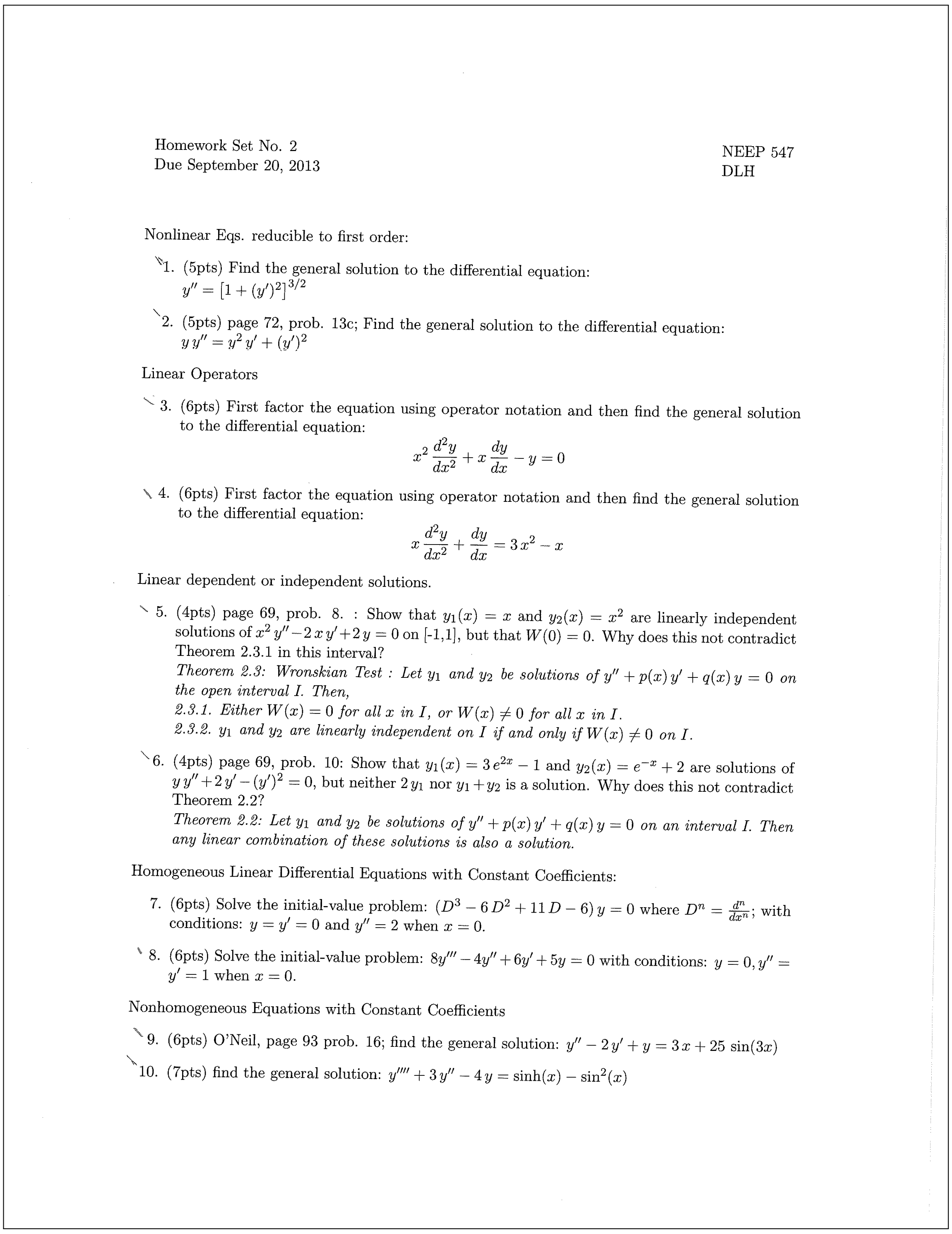
$$
y\left(x\right) = c_1 e^{x} + c_2 e^{-x} + c_5 \cos\left(2x\right) + c_6 \sin\left(2x\right) + y_p
$$

Where $y_p$ was given as $\frac{1}{800} e^{-x} \Delta$ where

$$
\begin{aligned}
\Delta = {} & -80e^{x} - 20e^{2x} + 40e^{2x}x + 40x - 20e^{x}\sin^2(2x) + 20e^{x}x\sin(2x)\\
& + 5e^{x}\sin(2x)\sin(4x) + 10e^{x}\cos^3(2x) - 20e^{x}\cos^2(2x) +\\
& 16e^{x}\cos(2x) - 32e^{x}\sin^2(2x)\sinh(x) - 32e^{x}\cos^2(2x)\sinh(x) + 20
\end{aligned}
$$

I tried using the variational method, but needed more time to complete finding the particular solution.

### 2.3.12 key solution



Homework Set No. 2
Due September 20, 2013

NEEP 547
DLH

Nonlinear Eqs. reducible to first order:

1. (5pts) Find the general solution to the differential equation:
   $y'' = \left[1 + (y')^2\right]^{3/2}$
2. (5pts) page 72, prob. 13c; Find the general solution to the differential equation:
   $y\,y'' = y^2\,y' + (y')^2$

Linear Operators

3. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$
4. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
   $$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3\,x^2 - x$$

Linear dependent or independent solutions.

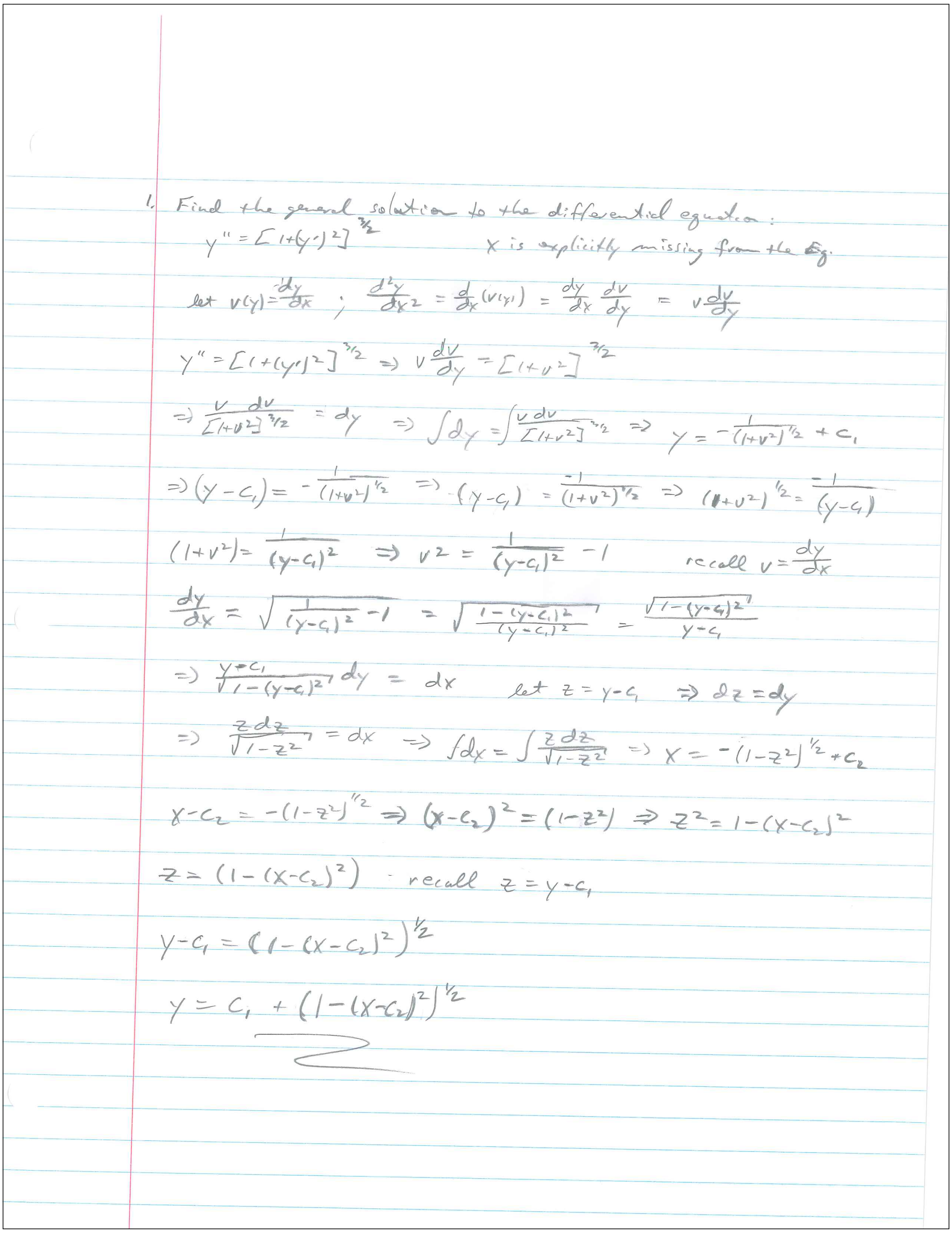
5. (4pts) page 69, prob. 8. : Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of $x^2\,y'' - 2\,x\,y' + 2\,y = 0$ on [-1,1], but that $W(0) = 0$. Why does this not contradict Theorem 2.3.1 in this interval?
   *Theorem 2.3: Wronskian Test : Let $y_1$ and $y_2$ be solutions of $y'' + p(x)\,y' + q(x)\,y = 0$ on the open interval I. Then,*
   *2.3.1. Either $W(x) = 0$ for all $x$ in $I$, or $W(x) \neq 0$ for all $x$ in $I$.*
   *2.3.2. $y_1$ and $y_2$ are linearly independent on I if and only if $W(x) \neq 0$ on $I$.*
6. (4pts) page 69, prob. 10: Show that $y_1(x) = 3\,e^{2x} - 1$ and $y_2(x) = e^{-x} + 2$ are solutions of $y\,y'' + 2\,y' - (y')^2 = 0$, but neither $2\,y_1$ nor $y_1 + y_2$ is a solution. Why does this not contradict Theorem 2.2?
   *Theorem 2.2: Let $y_1$ and $y_2$ be solutions of $y'' + p(x)\,y' + q(x)\,y = 0$ on an interval I. Then any linear combination of these solutions is also a solution.*

Homogeneous Linear Differential Equations with Constant Coefficients:

7. (6pts) Solve the initial-value problem: $(D^3 - 6\,D^2 + 11\,D - 6)\,y = 0$ where $D^n = \frac{d^n}{dx^n}$; with conditions: $y = y' = 0$ and $y'' = 2$ when $x = 0$.
8. (6pts) Solve the initial-value problem: $8y''' - 4y'' + 6y' + 5y = 0$ with conditions: $y = 0, y'' = y' = 1$ when $x = 0$.

Nonhomogeneous Equations with Constant Coefficients

9. (6pts) O'Neil, page 93 prob. 16; find the general solution: $y'' - 2\,y' + y = 3\,x + 25\,\sin(3x)$
10. (7pts) find the general solution: $y'''' + 3\,y'' - 4\,y = \sinh(x) - \sin^2(x)$



1. Find the general solution to the differential equation:

$y'' = [1+(y')^2]^{3/2}$ x is explicitly missing from the Eq.

let $v(y) = \frac{dy}{dx}$ ; $\frac{d^2y}{dx^2} = \frac{d}{dx}(v(y)) = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$

$y'' = [1+(y')^2]^{3/2} \Rightarrow v\frac{dv}{dy} = [1+v^2]^{3/2}$

$\Rightarrow \frac{v\,dv}{[1+v^2]^{3/2}} = dy \Rightarrow \int dy = \int \frac{v\,dv}{[1+v^2]^{3/2}} \Rightarrow y = -\frac{1}{(1+v^2)^{1/2}} + c_1$

$\Rightarrow (y-c_1) = -\frac{1}{(1+v^2)^{1/2}} \Rightarrow (y-c_1) = \frac{-1}{(1+v^2)^{1/2}} \Rightarrow (1+v^2)^{1/2} = \frac{-1}{(y-c_1)}$

$(1+v^2) = \frac{1}{(y-c_1)^2} \Rightarrow v^2 = \frac{1}{(y-c_1)^2} - 1$ recall $v = \frac{dy}{dx}$

$\frac{dy}{dx} = \sqrt{\frac{1}{(y-c_1)^2} - 1} = \sqrt{\frac{1-(y-c_1)^2}{(y-c_1)^2}} = \frac{\sqrt{1-(y-c_1)^2}}{y-c_1}$

$\Rightarrow \frac{y-c_1}{\sqrt{1-(y-c_1)^2}}dy = dx$ let $z = y-c_1 \Rightarrow dz = dy$

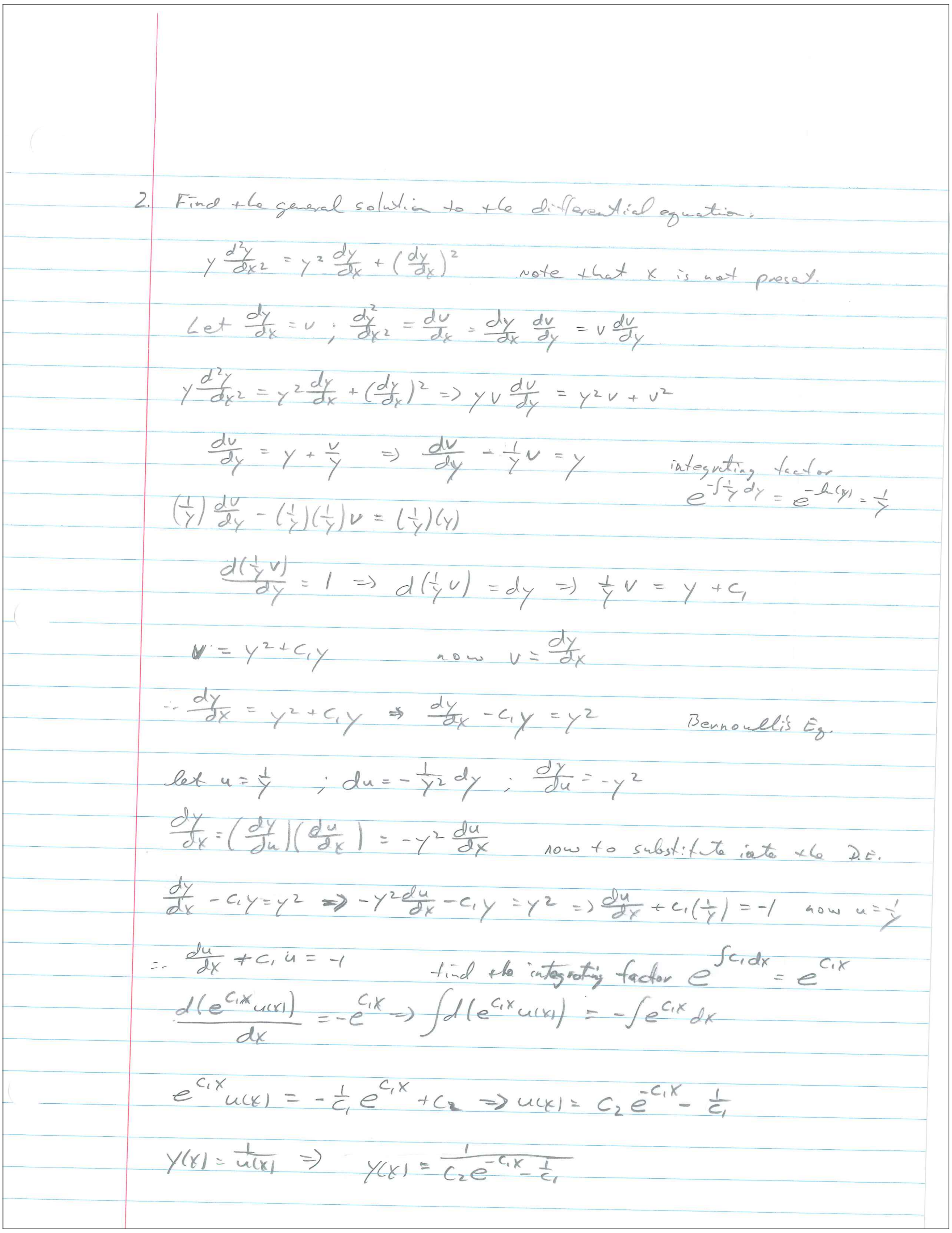
$\Rightarrow \frac{z\,dz}{\sqrt{1-z^2}} = dx \Rightarrow \int dx = \int \frac{z\,dz}{\sqrt{1-z^2}} \Rightarrow x = -(1-z^2)^{1/2} + c_2$

$x - c_2 = -(1-z^2)^{1/2} \Rightarrow (x-c_2)^2 = (1-z^2) \Rightarrow z^2 = 1-(x-c_2)^2$

$z = (1-(x-c_2)^2)$ recall $z = y-c_1$

$y - c_1 = (1-(x-c_2)^2)^{1/2}$

$y = c_1 + (1-(x-c_2)^2)^{1/2}$



2. Find the general solution to the differential equation:

$$y\frac{d^2y}{dx^2} = y^2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$ note that x is not present.

Let $\frac{dy}{dx} = v$ ; $\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$

$$y\frac{d^2y}{dx^2} = y^2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 \Rightarrow yv\frac{dv}{dy} = y^2v + v^2$$

$$\frac{dv}{dy} = y + \frac{v}{y} \Rightarrow \frac{dv}{dy} - \frac{1}{y}v = y$$

integrating factor $e^{-\int\frac{1}{y}dy} = e^{-\ln(y)} = \frac{1}{y}$

$$\left(\frac{1}{y}\right)\frac{dv}{dy} - \left(\frac{1}{y}\right)\left(\frac{1}{y}\right)v = \left(\frac{1}{y}\right)(y)$$

$$\frac{d(\frac{1}{y}v)}{dy} = 1 \Rightarrow d\left(\frac{1}{y}v\right) = dy \Rightarrow \frac{1}{y}v = y + c_1$$

$v = y^2 + c_1y$ now $v = \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = y^2 + c_1y \Rightarrow \frac{dy}{dx} - c_1y = y^2$ Bernoulli's Eq.

let $u = \frac{1}{y}$ ; $du = -\frac{1}{y^2}dy$ ; $\frac{dy}{du} = -y^2$

$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = -y^2\frac{du}{dx}$ now to substitute into the D.E.

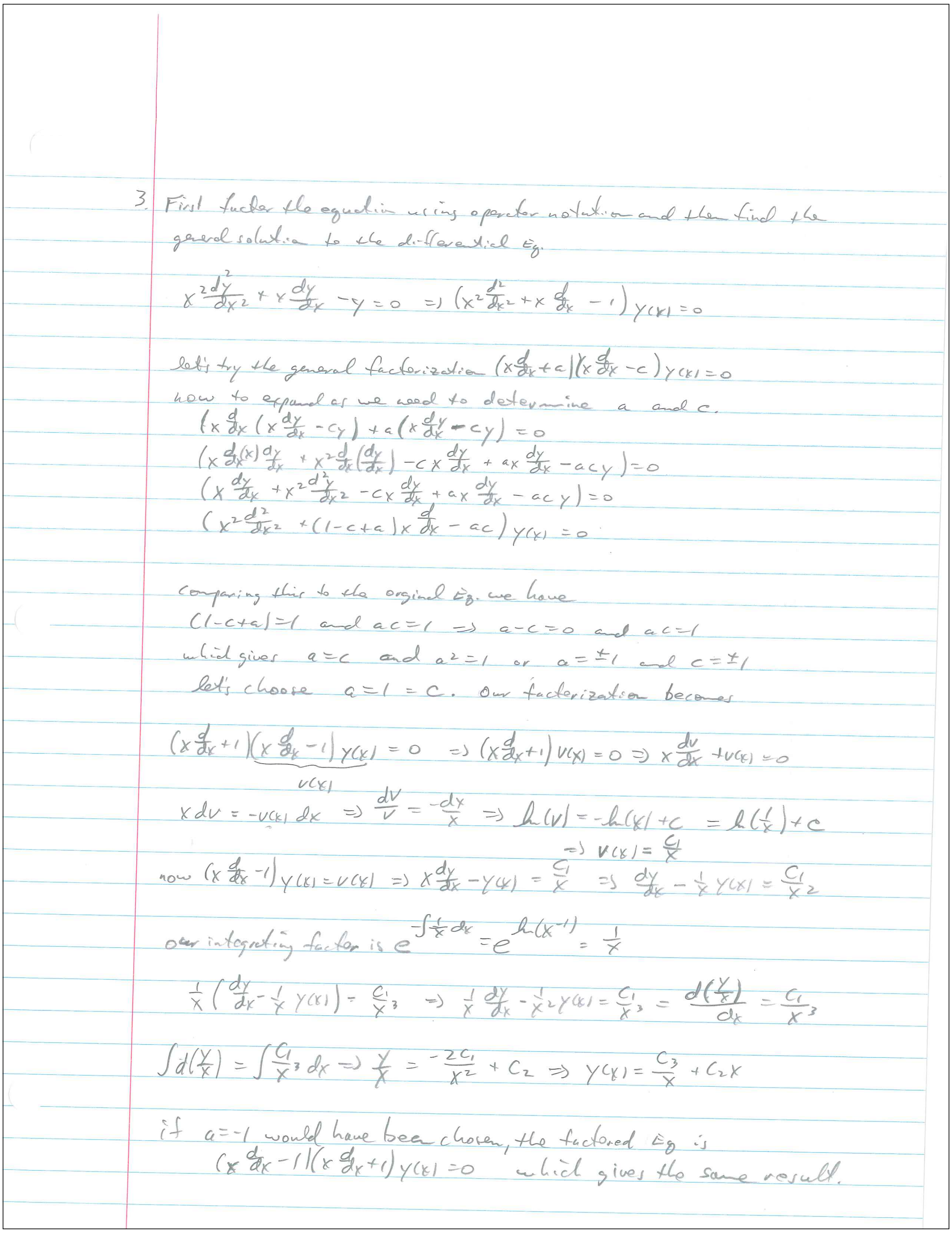
$\frac{dy}{dx} - c_1y = y^2 \Rightarrow -y^2\frac{du}{dx} - c_1y = y^2 \Rightarrow \frac{du}{dx} + c_1\left(\frac{1}{y}\right) = -1$ now $u = \frac{1}{y}$

$\therefore \frac{du}{dx} + c_1u = -1$ find the integrating factor $e^{\int c_1dx} = e^{c_1x}$

$$\frac{d(e^{c_1x}u(x))}{dx} = -e^{c_1x} \Rightarrow \int d(e^{c_1x}u(x)) = -\int e^{c_1x}dx$$

$$e^{c_1x}u(x) = -\frac{1}{c_1}e^{c_1x} + c_2 \Rightarrow u(x) = c_2e^{-c_1x} - \frac{1}{c_1}$$

$$y(x) = \frac{1}{u(x)} \Rightarrow y(x) = \frac{1}{c_2e^{-c_1x} - \frac{1}{c_1}}$$



3. First factor the equation using operator notation and then find the general solution to the differential Eq.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \Rightarrow \left(x^2\frac{d^2}{dx^2}+x\frac{d}{dx}-1\right)y(x)=0$$

let's try the general factorization $(x\frac{d}{dx}+a)(x\frac{d}{dx}-c)y(x)=0$

now to expand as we need to determine a and c.

$$(x\frac{d}{dx}(x\frac{dy}{dx}-cy)+a(x\frac{dy}{dx}-cy)=0$$

$$(x\frac{d}{dx}(x)\frac{dy}{dx}+x^2\frac{d}{dx}\left(\frac{dy}{dx}\right)-cx\frac{dy}{dx}+ax\frac{dy}{dx}-acy)=0$$

$$(x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}-cx\frac{dy}{dx}+ax\frac{dy}{dx}-acy)=0$$

$$\left(x^2\frac{d^2}{dx^2}+(1-c+a)x\frac{d}{dx}-ac\right)y(x)=0$$

comparing this to the orginal Eq. we have

$(1-c+a)=1$ and $ac=1$ $\Rightarrow$ $a-c=0$ and $ac=1$

which gives $a=c$ and $a^2=1$ or $a=\pm 1$ and $c=\pm 1$

let's choose $a=1=c$. Our factorization becomes

$$(x\frac{d}{dx}+1)\underbrace{(x\frac{d}{dx}-1)y(x)}_{v(x)}=0 \Rightarrow (x\frac{d}{dx}+1)v(x)=0 \Rightarrow x\frac{dv}{dx}+v(x)=0$$

$$x\,dv=-v(x)\,dx \Rightarrow \frac{dv}{v}=\frac{-dx}{x} \Rightarrow \ln(v)=-\ln(x)+c=\ln\left(\frac{1}{x}\right)+c$$

$$\Rightarrow v(x)=\frac{c_1}{x}$$

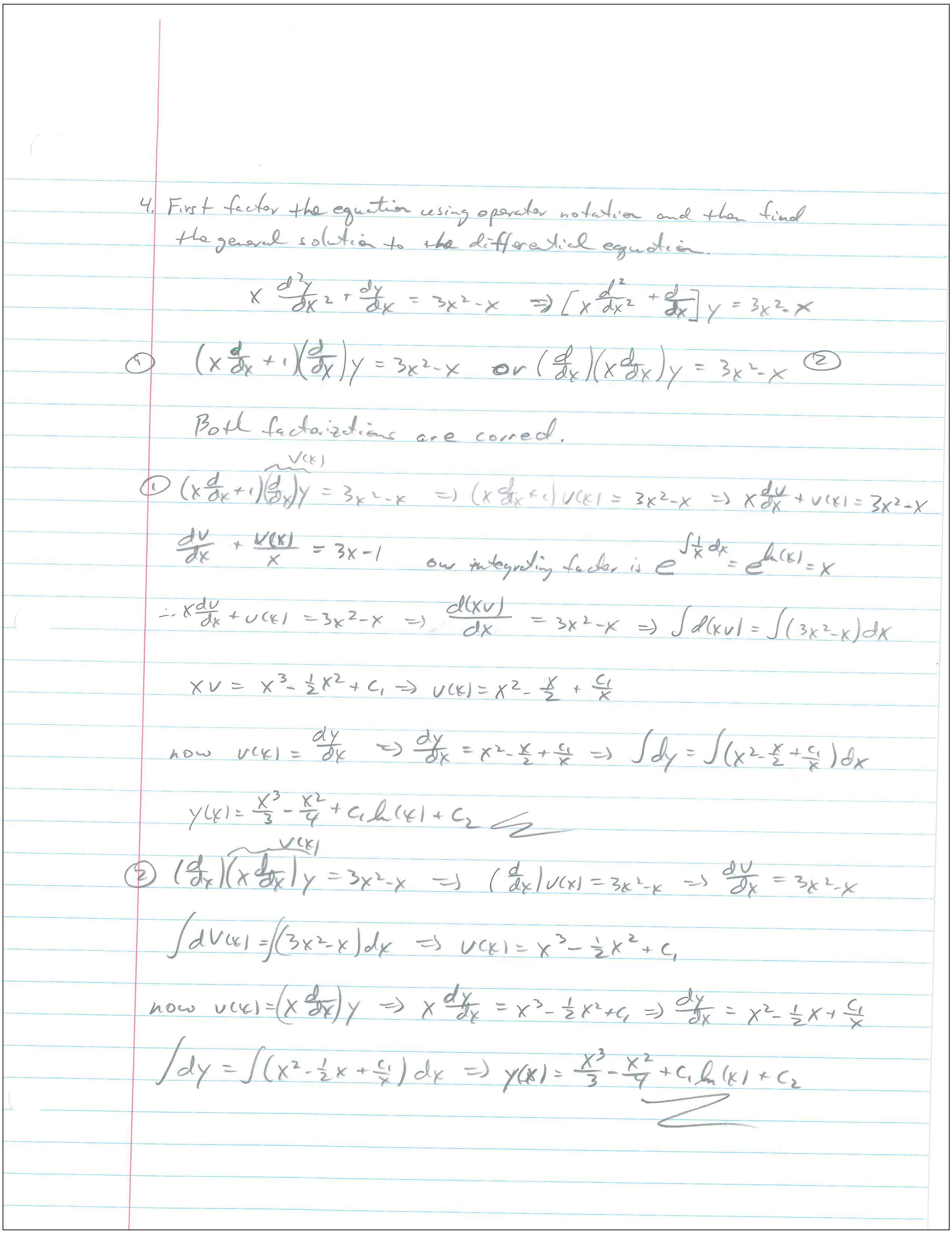
now $(x\frac{d}{dx}-1)y(x)=v(x) \Rightarrow x\frac{dy}{dx}-y(x)=\frac{c_1}{x} \Rightarrow \frac{dy}{dx}-\frac{1}{x}y(x)=\frac{c_1}{x^2}$

our integrating factor is $e^{-\int\frac{1}{x}dx}=e^{\ln(x^{-1})}=\frac{1}{x}$

$$\frac{1}{x}\left(\frac{dy}{dx}-\frac{1}{x}y(x)\right)=\frac{c_1}{x^3} \Rightarrow \frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y(x)=\frac{c_1}{x^3}=\frac{d\left(\frac{y}{x}\right)}{dx}=\frac{c_1}{x^3}$$

$$\int d\left(\frac{y}{x}\right)=\int\frac{c_1}{x^3}dx \Rightarrow \frac{y}{x}=\frac{-2c_1}{x^2}+C_2 \Rightarrow y(x)=\frac{C_3}{x}+C_2x$$

if $a=-1$ would have been chosen, the factored Eq is
$(x\frac{d}{dx}-1)(x\frac{d}{dx}+1)y(x)=0$ which gives the same result.



4. First factor the equation using operator notation and then find the general solution to the differential equation.

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=3x^2-x \Rightarrow \left[x\frac{d^2}{dx^2}+\frac{d}{dx}\right]y=3x^2-x$$

① $\left(x\frac{d}{dx}+1\right)\left(\frac{d}{dx}\right)y=3x^2-x$ or $\left(\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)y=3x^2-x$ ②

Both factorizations are correct.

① $\left(x\frac{d}{dx}+1\right)\underbrace{\left(\frac{d}{dx}\right)y}^{v(x)}=3x^2-x \Rightarrow \left(x\frac{d}{dx}+1\right)v(x)=3x^2-x \Rightarrow x\frac{dv}{dx}+v(x)=3x^2-x$

$\frac{dv}{dx}+\frac{v(x)}{x}=3x-1$ ow integrating factor is $e^{\int\frac{1}{x}dx}=e^{\ln(x)}=x$

$\therefore x\frac{dv}{dx}+v(x)=3x^2-x \Rightarrow \frac{d(xv)}{dx}=3x^2-x \Rightarrow \int d(xv)=\int(3x^2-x)dx$

$xv=x^3-\frac{1}{2}x^2+c_1 \Rightarrow v(x)=x^2-\frac{x}{2}+\frac{c_1}{x}$

now $v(x)=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=x^2-\frac{x}{2}+\frac{c_1}{x} \Rightarrow \int dy=\int\left(x^2-\frac{x}{2}+\frac{c_1}{x}\right)dx$

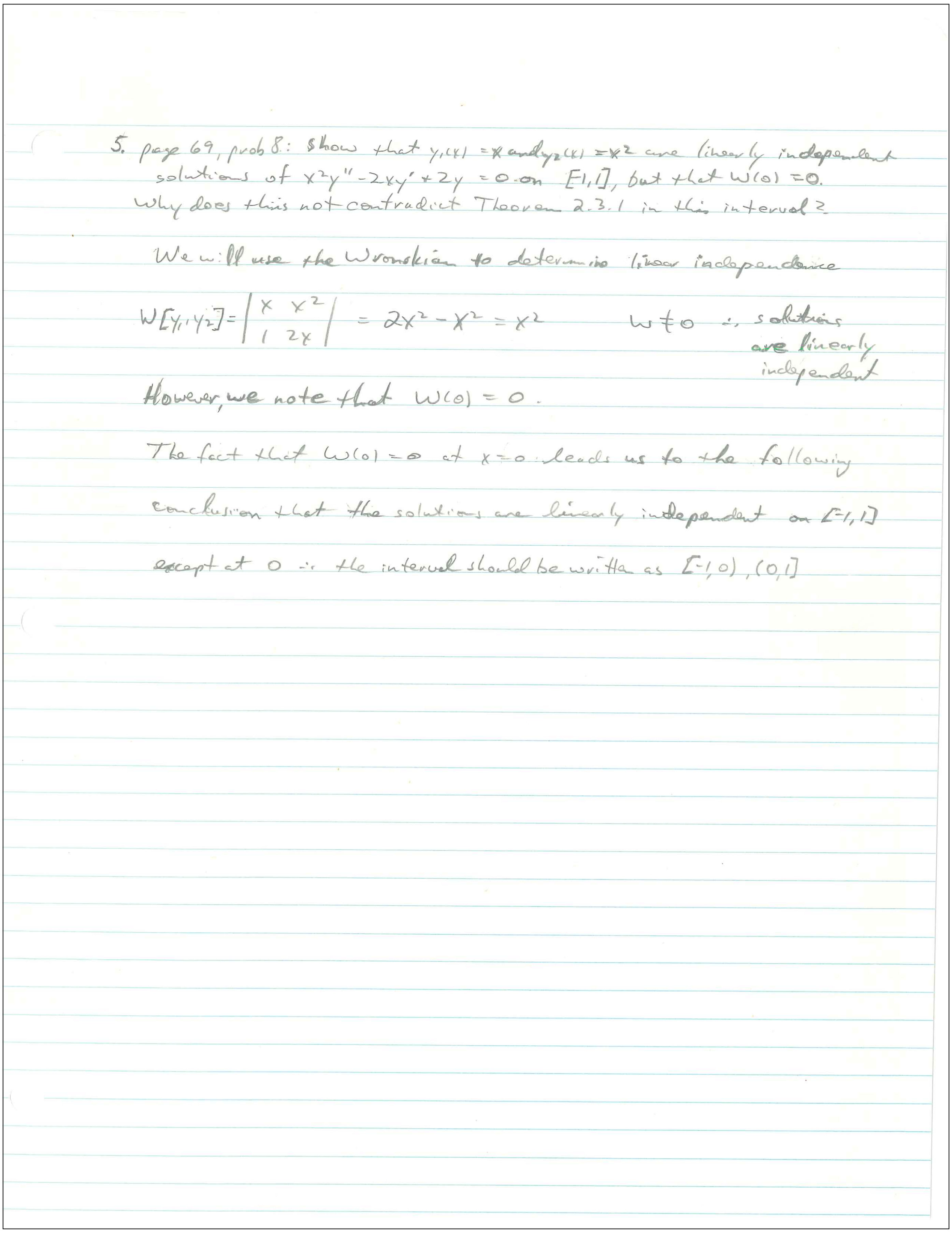
$y(x)=\frac{x^3}{3}-\frac{x^2}{4}+c_1\ln(x)+c_2$

② $\left(\frac{d}{dx}\right)\overbrace{\left(x\frac{d}{dx}\right)y}^{v(x)}=3x^2-x \Rightarrow \left(\frac{d}{dx}\right)v(x)=3x^2-x \Rightarrow \frac{dv}{dx}=3x^2-x$

$\int dv(x)=\int(3x^2-x)dx \Rightarrow v(x)=x^3-\frac{1}{2}x^2+c_1$

now $v(x)=\left(x\frac{d}{dx}\right)y \Rightarrow x\frac{dy}{dx}=x^3-\frac{1}{2}x^2+c_1 \Rightarrow \frac{dy}{dx}=x^2-\frac{1}{2}x+\frac{c_1}{x}$

$\int dy=\int\left(x^2-\frac{1}{2}x+\frac{c_1}{x}\right)dx \Rightarrow y(x)=\frac{x^3}{3}-\frac{x^2}{4}+c_1\ln(x)+c_2$



5. page 69, prob 8: Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of $x^2y'' - 2xy' + 2y = 0$ on $[-1,1]$, but that $W(0) = 0$. Why does this not contradict Theorem 2.3.1 in this interval?

We will use the Wronskian to determine linear independence

$$W[y_1, y_2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \qquad W \neq 0 \therefore \text{solutions are linearly independent}$$

However, we note that $W(0) = 0$.

The fact that $W(0) = 0$ at $x = 0$ leads us to the following conclusion that the solutions are linearly independent on $[-1,1]$ except at 0 $\therefore$ the interval should be written as $[-1,0), (0,1]$

6. page 69, prob 10: show that $y_1(x) = 3e^{2x}-1$ and $y_2(x) = e^{-x}+2$ are solutions of $yy'' + 2y' - (y')^2 = 0$, but neither $2y_1$ nor $y_1+y_2$ is a solution. Why does this not contradict Theorem 2.2?

$y_1(x) = 3e^{2x}-1,\ y_1' = 6e^{2x},\ y_1'' = 12e^{2x}$

$(3e^{2x}-1)(12e^{2x}) + 2(6e^{2x}) - (6e^{2x})^2 \overset{?}{=} 0$

$36e^{4x} - 12\cancel{e^{2x}} + 12\cancel{e^{2x}} - 36e^{4x} = 0$ ✓ solution satisfies the equation

$y_2(x) = e^{-x}+2,\ y_2' = -e^{-x},\ y_2'' = e^{-x}$

$(e^{-x}+2)e^{-x} + 2(-e^{-x}) - (-e^{-x})^2 \overset{?}{=} 0$

$e^{-2x} + 2\cancel{e^{-x}} - 2\cancel{e^{-x}} - e^{-2x} = 0$ ✓ solution satisfies the equation

let's check the Wronskian

$$W[y_1,y_2] \quad \begin{bmatrix} 3e^{2x}-1 & e^{-x}+2 \\ 6e^{2x} & -e^{-x} \end{bmatrix} = (3e^{2x}-1)(-e^{-x}) - (6e^{2x})(e^{-x}+2)$$

$$= -3e^{x}+e^{-x} - 6e^{x} - 12e^{2x} -$$

$$= -9e^{x}+e^{-x}-12e^{2x}$$

$W[y_1,y_2] \neq 0$ linearly independent solution

$y_3 = 2y_1 = 6e^{2x}-2$ = new solution. Will it work?

$y_3(x) = 6e^{2x}-2,\ y_3' = 12e^{2x},\ y_3'' = 24e^{2x}$

$(6e^{2x}-2)(24e^{2x}) + 2(12e^{2x}) - (12e^{2x})^2 \overset{?}{=} 0$

$\cancel{144}e^{4x} - 48e^{2x} + 24e^{2x} - \cancel{144}e^{4x} = -24e^{2x}$ does not satisfy the equation

Thus $2y_1$ is not a solution

$y_3 = y_1+y_2 = 3e^{2x}-1+e^{-x}+2,\ y_3' = 6e^{2x} - e^{-x},\ y_3'' = 12e^{2x}+e^{-x}$

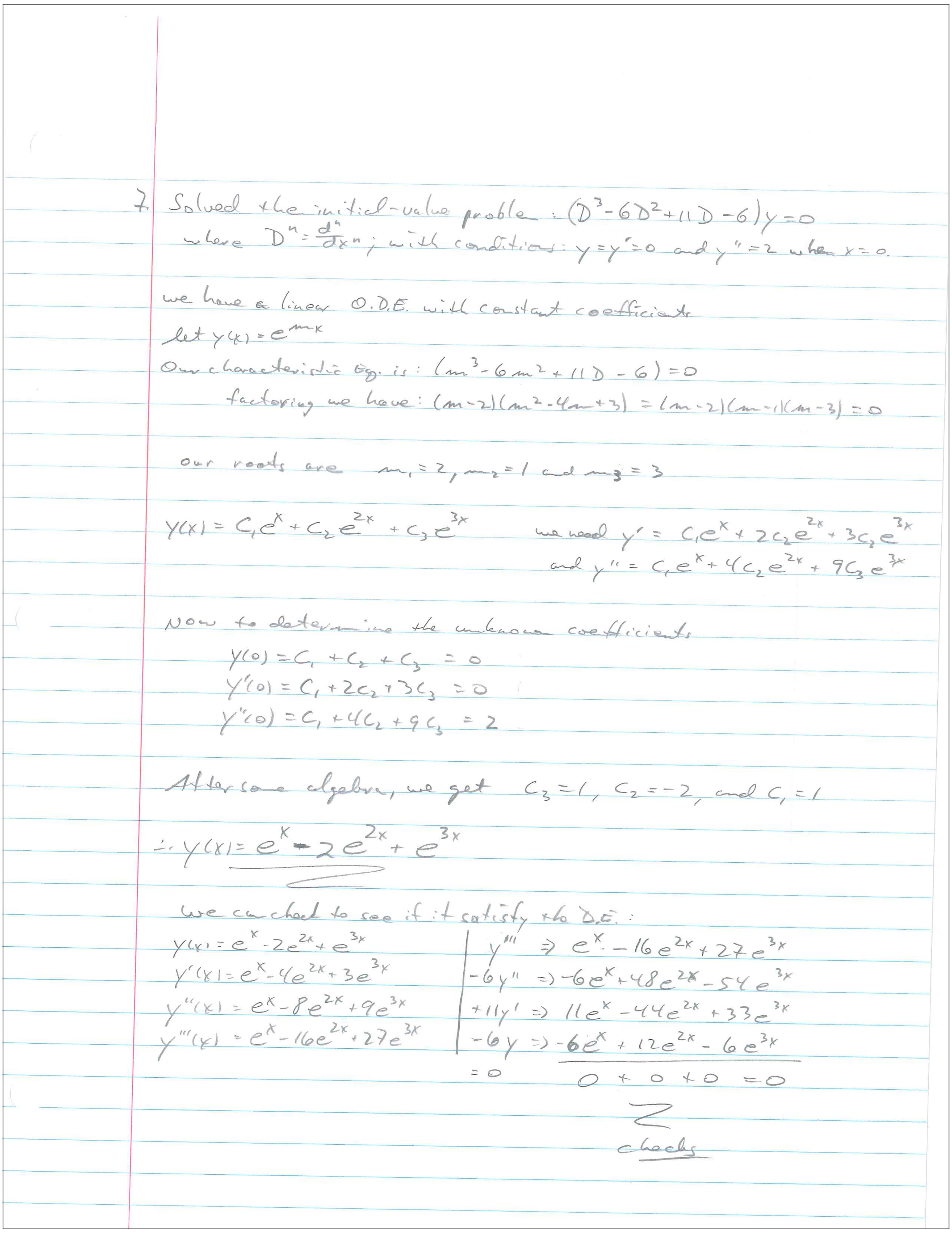
$(3e^{2x}-1+e^{-x}+2)(12e^{2x}+e^{-x}) + 2(6e^{2x}-e^{-x}) - (6e^{2x}-e^{-x})^2 \overset{?}{=} 0$

$36e^{4x} - 12e^{2x} + 12e^{x} + 24e^{2x} + 3e^{x} - e^{-x} + e^{-2x} + 2e^{-x} + 12e^{2x} - 2e^{-x} - 36e^{4x} - e^{-2x} + 12e^{x} \overset{?}{=} 0$

$12e^{x} + 24e^{2x} + 3e^{x} - e^{-x} + 12e^{x} = 24e^{2x} + 27e^{x} - e^{-x} \neq 0$

does not satisfy the equation

The D.E. is a nonlinear Eq. The theorem is for linear Eqs.



7. Solved the initial-value problem: $(D^3 - 6D^2 + 11D - 6)y = 0$ where $D^n = \frac{d^n}{dx^n}$; with conditions: $y = y' = 0$ and $y'' = 2$ when $x = 0$.

we have a linear O.D.E. with constant coefficients

let $y(x) = e^{mx}$

Our characteristic Eq. is: $(m^3 - 6m^2 + 11D - 6) = 0$

factoring we have: $(m-2)(m^2-4m+3) = (m-2)(m-1)(m-3) = 0$

our roots are $m_1 = 2$, $m_2 = 1$ and $m_3 = 3$

$y(x) = C_1e^x + C_2e^{2x} + C_3e^{3x}$

we need $y' = C_1e^x + 2C_2e^{2x} + 3C_3e^{3x}$

and $y'' = C_1e^x + 4C_2e^{2x} + 9C_3e^{3x}$

Now to determine the unknown coefficients

$$y(0) = C_1 + C_2 + C_3 = 0$$
$$y'(0) = C_1 + 2C_2 + 3C_3 = 0$$
$$y''(0) = C_1 + 4C_2 + 9C_3 = 2$$

After some algebra, we get $C_3 = 1$, $C_2 = -2$, and $C_1 = 1$

$$\therefore \underline{y(x) = e^x - 2e^{2x} + e^{3x}}$$

we can check to see if it satisfy the D.E.:

$y(x) = e^x - 2e^{2x} + e^{3x}$

$y'(x) = e^x - 4e^{2x} + 3e^{3x}$

$y''(x) = e^x - 8e^{2x} + 9e^{3x}$

$y'''(x) = e^x - 16e^{2x} + 27e^{3x}$

$y''' \Rightarrow e^x - 16e^{2x} + 27e^{3x}$

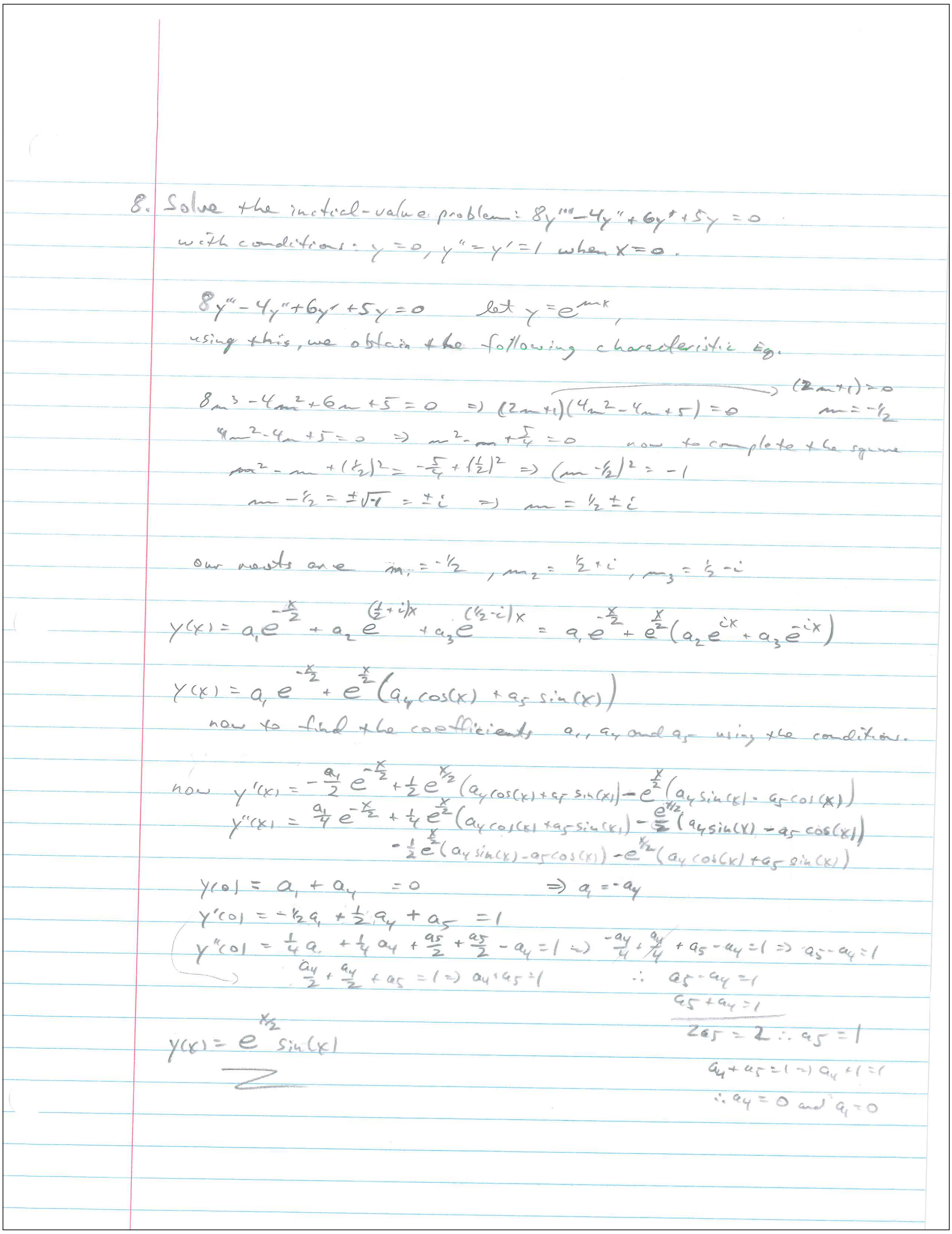
$-6y'' \Rightarrow -6e^x + 48e^{2x} - 54e^{3x}$

$+11y' \Rightarrow 11e^x - 44e^{2x} + 33e^{3x}$

$-6y \Rightarrow -6e^x + 12e^{2x} - 6e^{3x}$

$= 0 \qquad 0 + 0 + 0 = 0$

checks



8. Solve the initial-value problem: $8y''' - 4y'' + 6y' + 5y = 0$ with conditions: $y = 0$, $y'' = y' = 1$ when $x = 0$.

$8y''' - 4y'' + 6y' + 5y = 0$ let $y = e^{mx}$, using this, we obtain the following characteristic Eq.

$8m^3 - 4m^2 + 6m + 5 = 0 \Rightarrow (2m+1)(4m^2 - 4m + 5) = 0$ $\longrightarrow$ $(2m+1) = 0$, $m = -\frac{1}{2}$

$4m^2 - 4m + 5 = 0 \Rightarrow m^2 - m + \frac{5}{4} = 0$ now to complete the square

$m^2 - m + (\frac{1}{2})^2 = -\frac{5}{4} + (\frac{1}{2})^2 \Rightarrow (m - \frac{1}{2})^2 = -1$

$m - \frac{1}{2} = \pm\sqrt{-1} = \pm i \Rightarrow m = \frac{1}{2} \pm i$

our roots are $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{2} + i$, $m_3 = \frac{1}{2} - i$

$$y(x) = a_1 e^{-\frac{x}{2}} + a_2 e^{(\frac{1}{2}+i)x} + a_3 e^{(\frac{1}{2}-i)x} = a_1 e^{-\frac{x}{2}} + e^{\frac{x}{2}}(a_2 e^{ix} + a_3 e^{-ix})$$

$$y(x) = a_1 e^{-\frac{x}{2}} + e^{\frac{x}{2}}(a_4 \cos(x) + a_5 \sin(x))$$

now to find the coefficients $a_1$, $a_4$ and $a_5$ using the conditions.

now $y'(x) = -\frac{a_1}{2} e^{-\frac{x}{2}} + \frac{1}{2} e^{x/2}(a_4 \cos(x) + a_5 \sin(x)) - e^{\frac{x}{2}}(a_4 \sin(x) - a_5 \cos(x))$

$y''(x) = \frac{a_1}{4} e^{-\frac{x}{2}} + \frac{1}{4} e^{\frac{x}{2}}(a_4 \cos(x) + a_5 \sin(x)) - \frac{e^{x/2}}{2}(a_4 \sin(x) - a_5 \cos(x))$
$- \frac{1}{2} e^{\frac{x}{2}}(a_4 \sin(x) - a_5 \cos(x)) - e^{x/2}(a_4 \cos(x) + a_5 \sin(x))$

$y(0) = a_1 + a_4 = 0 \Rightarrow a_1 = -a_4$

$y'(0) = -\frac{1}{2}a_1 + \frac{1}{2}a_4 + a_5 = 1$

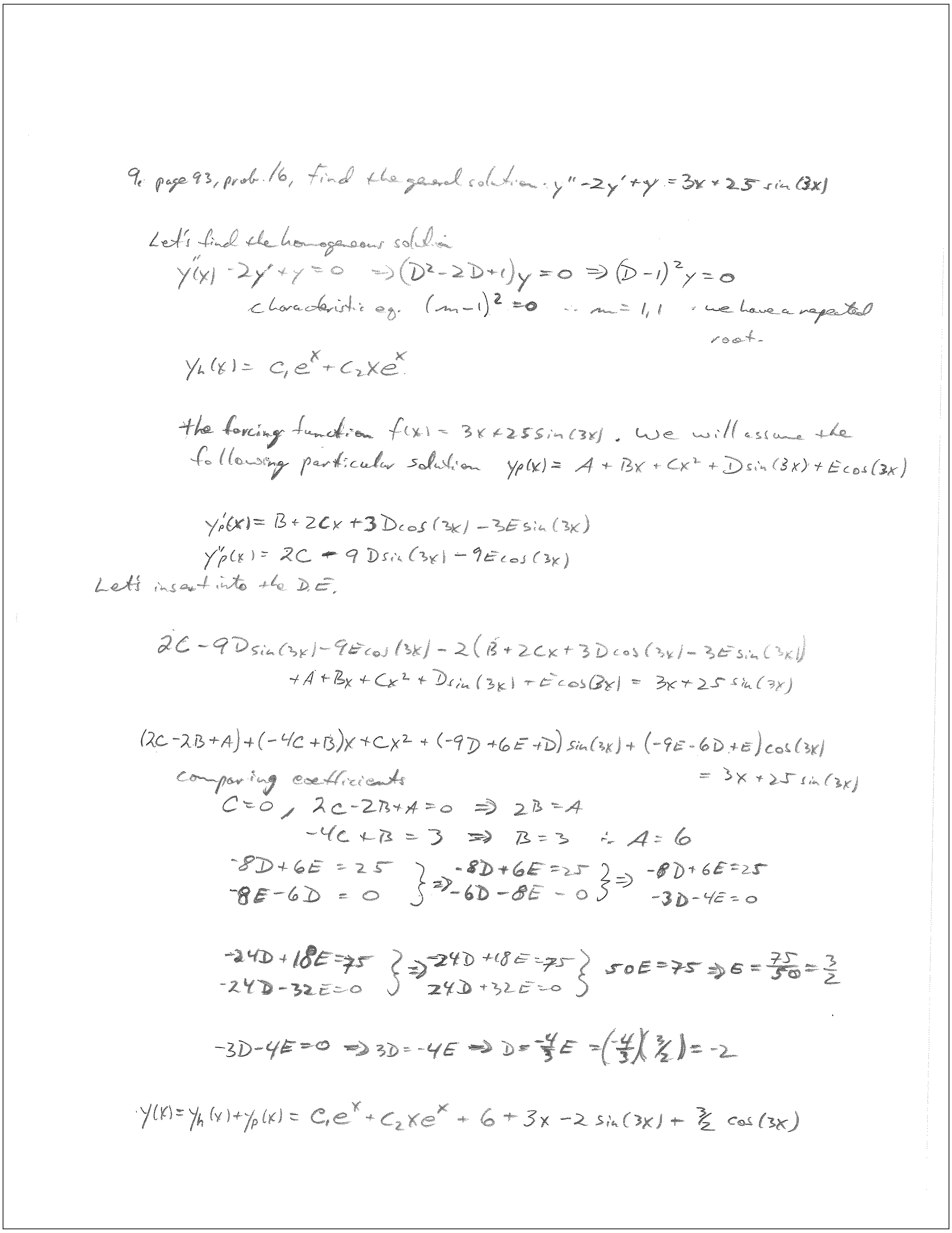
$y''(0) = \frac{1}{4}a_1 + \frac{1}{4}a_4 + \frac{a_5}{2} + \frac{a_5}{2} - a_4 = 1 \Rightarrow -\frac{a_4}{4} + \frac{a_4}{4} + a_5 - a_4 = 1 \Rightarrow a_5 - a_4 = 1$

$\frac{a_4}{2} + \frac{a_4}{2} + a_5 = 1 \Rightarrow a_4 + a_5 = 1$

$\therefore a_5 - a_4 = 1$
$a_5 + a_4 = 1$
$2a_5 = 2 \therefore a_5 = 1$

$a_4 + a_5 = 1 \Rightarrow a_4 + 1 = 1$
$\therefore a_4 = 0$ and $a_1 = 0$

$$y(x) = e^{x/2} \sin(x)$$



9: page 93, prob. 16, find the general solution: $y'' - 2y' + y = 3x + 25\sin(3x)$

Let's find the homogeneous solution

$y''(x) - 2y' + y = 0 \Rightarrow (D^2 - 2D + 1)y = 0 \Rightarrow (D-1)^2 y = 0$

characteristic eq. $(m-1)^2 = 0$ $\therefore m = 1, 1$ $\therefore$ we have a repeated root.

$y_h(x) = c_1 e^x + c_2 x e^x$

the forcing function $f(x) = 3x + 25\sin(3x)$. We will assume the following particular solution $y_p(x) = A + Bx + Cx^2 + D\sin(3x) + E\cos(3x)$

$y_p'(x) = B + 2Cx + 3D\cos(3x) - 3E\sin(3x)$

$y_p''(x) = 2C + 9D\sin(3x) - 9E\cos(3x)$

Let's insert into the D.E.

$2C - 9D\sin(3x) - 9E\cos(3x) - 2(B + 2Cx + 3D\cos(3x) - 3E\sin(3x))$
$+ A + Bx + Cx^2 + D\sin(3x) + E\cos(3x) = 3x + 25\sin(3x)$

$(2C - 2B + A) + (-4C + B)x + Cx^2 + (-9D + 6E + D)\sin(3x) + (-9E - 6D + E)\cos(3x)$
$= 3x + 25\sin(3x)$

comparing coefficients

$C = 0$, $2C - 2B + A = 0 \Rightarrow 2B = A$

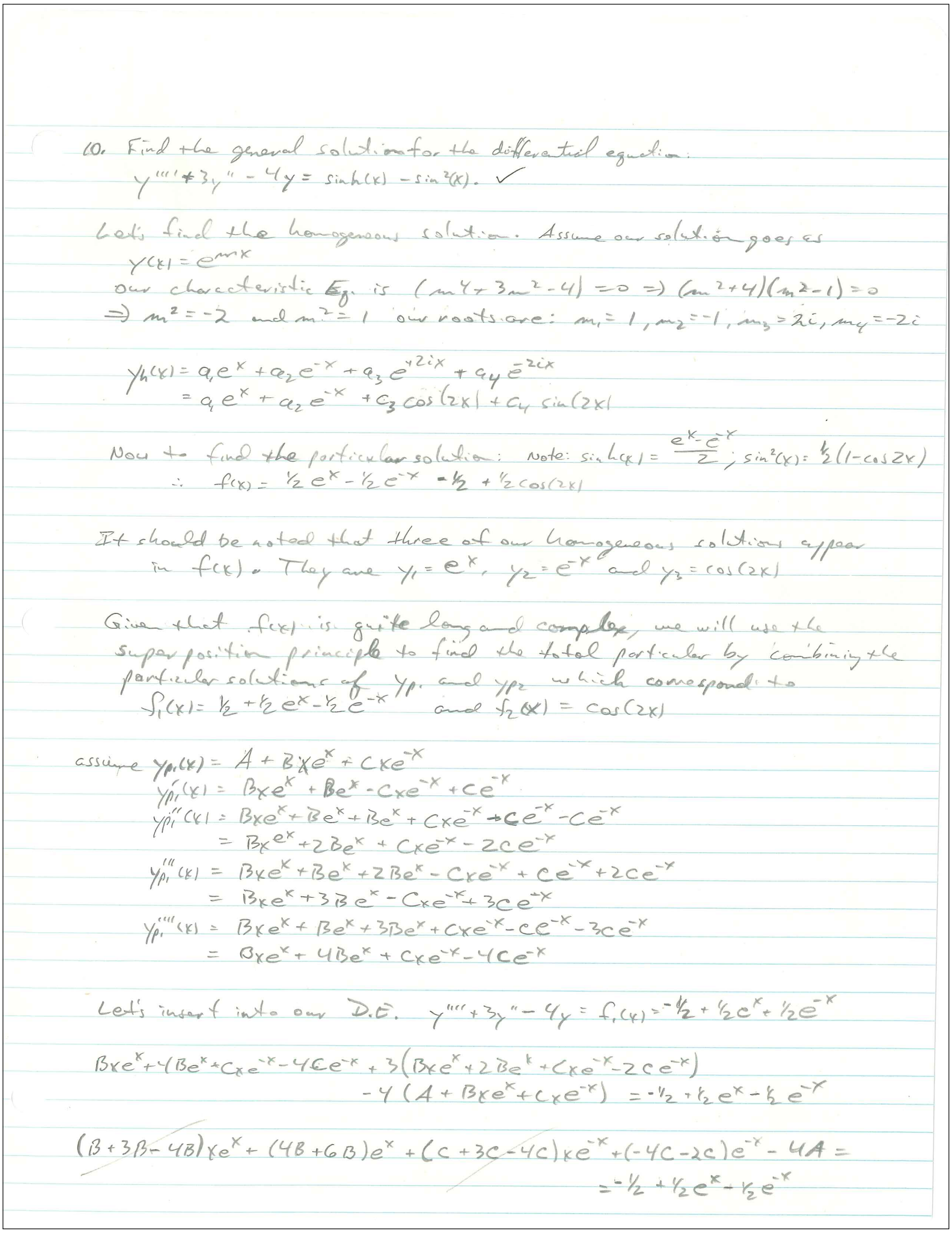
$-4C + B = 3 \Rightarrow B = 3$ $\therefore A = 6$

$$\left.\begin{array}{l} -8D + 6E = 25 \\ -8E - 6D = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -8D + 6E = 25 \\ -6D - 8E = 0 \end{array}\right\} \Rightarrow \begin{array}{l} -8D + 6E = 25 \\ -3D - 4E = 0 \end{array}$$

$$\left.\begin{array}{l} -24D + 18E = 75 \\ -24D - 32E = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -24D + 18E = 75 \\ 24D + 32E = 0 \end{array}\right\} \quad 50E = 75 \Rightarrow E = \frac{75}{50} = \frac{3}{2}$$

$-3D - 4E = 0 \Rightarrow 3D = -4E \Rightarrow D = -\frac{4}{3}E = \left(-\frac{4}{3}\right)\left(\frac{3}{2}\right) = -2$

$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 x e^x + 6 + 3x - 2\sin(3x) + \frac{3}{2}\cos(3x)$



10. Find the general solution for the differential equation:

$y'''' + 3y'' - 4y = \sinh(x) - \sin^2(x)$. ✓

Let's find the homogeneous solution. Assume our solution goes as

$y(x) = e^{mx}$

our characteristic Eq. is $(m^4 + 3m^2 - 4) = 0 \Rightarrow (m^2+4)(m^2-1) = 0$

$\Rightarrow m^2 = -2$ and $m^2 = 1$ our roots are: $m_1 = 1, m_2 = -1, m_3 = 2i, m_4 = -2i$

$$y_h(x) = a_1 e^x + a_2 e^{-x} + a_3 e^{+2ix} + a_4 e^{-2ix}$$
$$= a_1 e^x + a_2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$$

Now to find the particular solution: Note: $\sinh(x) = \frac{e^x - e^{-x}}{2}$; $\sin^2(x) = \frac{1}{2}(1-\cos 2x)$

$\therefore \quad f(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} - \frac{1}{2} + \frac{1}{2}\cos(2x)$

It should be noted that three of our homogeneous solutions appear in $f(x)$. They are $y_1 = e^x$, $y_2 = e^{-x}$ and $y_3 = \cos(2x)$

Given that $f(x)$ is quite long and complex, we will use the superposition principle to find the total particular by combining the particular solutions of $y_{p_1}$ and $y_{p_2}$ which correspond to $f_1(x) = \frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ and $f_2(x) = \cos(2x)$

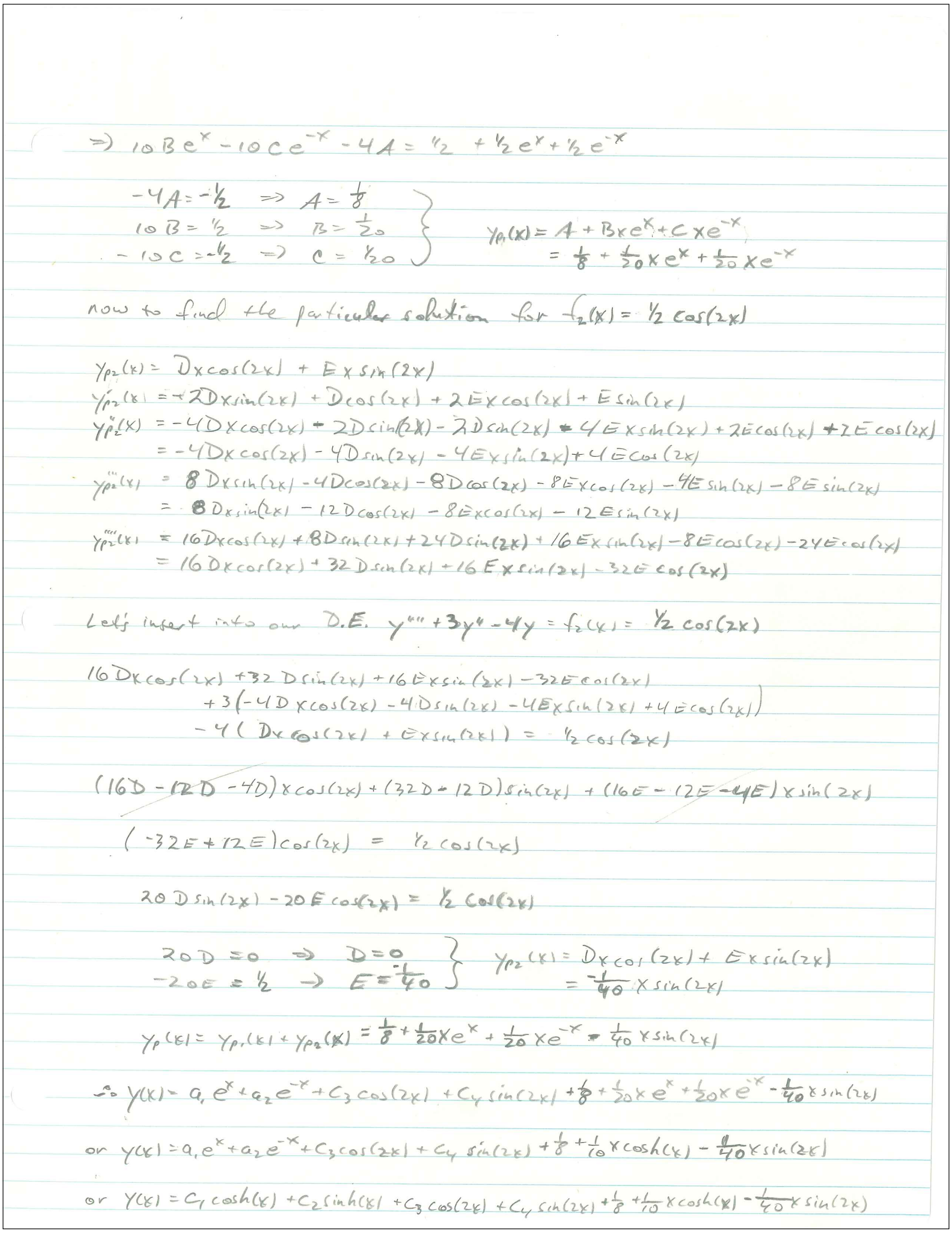
assume $y_{p_1}(x) = A + Bxe^x + Cxe^{-x}$

$$y_{p_1}'(x) = Bxe^x + Be^x - Cxe^{-x} + Ce^{-x}$$
$$y_{p_1}''(x) = Bxe^x + Be^x + Be^x + Cxe^{-x} - Ce^{-x} - Ce^{-x}$$
$$= Bxe^x + 2Be^x + Cxe^{-x} - 2Ce^{-x}$$
$$y_{p_1}'''(x) = Bxe^x + Be^x + 2Be^x - Cxe^{-x} + Ce^{-x} + 2Ce^{-x}$$
$$= Bxe^x + 3Be^x - Cxe^{-x} + 3Ce^{-x}$$
$$y_{p_1}''''(x) = Bxe^x + Be^x + 3Be^x + Cxe^{-x} - Ce^{-x} - 3Ce^{-x}$$
$$= Bxe^x + 4Be^x + Cxe^{-x} - 4Ce^{-x}$$

Let's insert into our D.E. $y'''' + 3y'' - 4y = f_1(x) = -\frac{1}{2} + \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$$Bxe^x + 4Be^x + Cxe^{-x} - 4Ce^{-x} + 3(Bxe^x + 2Be^x + Cxe^{-x} - 2Ce^{-x})$$
$$-4(A + Bxe^x + Cxe^{-x}) = -\frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$\cancel{(B + 3B - 4B)xe^x} + (4B + 6B)e^x + \cancel{(C + 3C - 4C)xe^{-x}} + (-4C - 2C)e^{-x} - 4A =$$
$$= -\frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$



$\Rightarrow 10Be^{x} - 10Ce^{-x} - 4A = \frac{1}{2} + \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$$\left.\begin{array}{lll} -4A = -\frac{1}{2} & \Rightarrow & A = \frac{1}{8} \\ 10B = \frac{1}{2} & \Rightarrow & B = \frac{1}{20} \\ -10C = -\frac{1}{2} & \Rightarrow & C = \frac{1}{20} \end{array}\right\} \quad \begin{aligned} y_{p_1}(x) &= A + Bxe^{x} + Cxe^{-x} \\ &= \frac{1}{8} + \frac{1}{20}xe^{x} + \frac{1}{20}xe^{-x} \end{aligned}$$

now to find the particular solution for $f_2(x) = \frac{1}{2}\cos(2x)$

$$\begin{aligned} y_{p_2}(x) &= Dx\cos(2x) + Ex\sin(2x) \\ y_{p_2}'(x) &= -2Dx\sin(2x) + D\cos(2x) + 2Ex\cos(2x) + E\sin(2x) \\ y_{p_2}''(x) &= -4Dx\cos(2x) - 2D\sin(2x) - 2D\sin(2x) - 4Ex\sin(2x) + 2E\cos(2x) + 2E\cos(2x) \\ &= -4Dx\cos(2x) - 4D\sin(2x) - 4Ex\sin(2x) + 4E\cos(2x) \\ y_{p_2}'''(x) &= 8Dx\sin(2x) - 4D\cos(2x) - 8D\cos(2x) - 8Ex\cos(2x) - 4E\sin(2x) - 8E\sin(2x) \\ &= 8Dx\sin(2x) - 12D\cos(2x) - 8Ex\cos(2x) - 12E\sin(2x) \\ y_{p_2}''''(x) &= 16Dx\cos(2x) + 8D\sin(2x) + 24D\sin(2x) + 16Ex\sin(2x) - 8E\cos(2x) - 24E\cos(2x) \\ &= 16Dx\cos(2x) + 32D\sin(2x) + 16Ex\sin(2x) - 32E\cos(2x) \end{aligned}$$

Let's insert into our D.E. $y'''' + 3y'' - 4y = f_2(x) = \frac{1}{2}\cos(2x)$

$$\begin{aligned} &16Dx\cos(2x) + 32D\sin(2x) + 16Ex\sin(2x) - 32E\cos(2x) \\ &\quad + 3\left(-4Dx\cos(2x) - 4D\sin(2x) - 4Ex\sin(2x) + 4E\cos(2x)\right) \\ &\quad - 4\left(Dx\cos(2x) + Ex\sin(2x)\right) = \frac{1}{2}\cos(2x) \end{aligned}$$

$$(16D - 12D - 4D)x\cos(2x) + (32D - 12D)\sin(2x) + (16E - 12E - 4E)x\sin(2x)$$

$$(-32E + 12E)\cos(2x) = \frac{1}{2}\cos(2x)$$

$$20D\sin(2x) - 20E\cos(2x) = \frac{1}{2}\cos(2x)$$

$$\left.\begin{array}{lll} 20D = 0 & \Rightarrow & D = 0 \\ -20E = \frac{1}{2} & \rightarrow & E = \frac{-1}{40} \end{array}\right\} \quad \begin{aligned} y_{p_2}(x) &= Dx\cos(2x) + Ex\sin(2x) \\ &= \frac{-1}{40}x\sin(2x) \end{aligned}$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{1}{8} + \frac{1}{20}xe^{x} + \frac{1}{20}xe^{-x} - \frac{1}{40}x\sin(2x)$$

$\therefore y(x) = a_1e^{x} + a_2e^{-x} + C_3\cos(2x) + C_4\sin(2x) + \frac{1}{8} + \frac{1}{20}xe^{x} + \frac{1}{20}xe^{-x} - \frac{1}{40}x\sin(2x)$

or $y(x) = a_1e^{x} + a_2e^{-x} + C_3\cos(2x) + C_4\sin(2x) + \frac{1}{8} + \frac{1}{10}x\cosh(x) - \frac{1}{40}x\sin(2x)$

or $y(x) = C_1\cosh(x) + C_2\sinh(x) + C_3\cos(2x) + C_4\sin(2x) + \frac{1}{8} + \frac{1}{10}x\cosh(x) - \frac{1}{40}x\sin(2x)$

## 2.4 HW 3

### 2.4.1 Problems to solve

Homework Set No. 3 NEEP 547
Due September 27, 2013 DLH

Nonhomogeneous Equation (Variation of Parameters)

1. (6pts) page 93, prob. 22: Find the general solution to the differential equation: $x^2\,y'' + 3x\,y' + \,y = 4/x$

2. (6pts) Using the variation of parameters show that

$$y = c_1\cosh(kx) + c_2\sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))\,f(s)\,ds$$

is a complete solution of the equation $y'' - k^2\,y = f(x)$, where $k \neq 0$ and $f$ is everywhere continuous. *Hint:* Introduce the dummy variable $s$ in the integrals which define $u_1$ and $u_2$. Then move $y_1(x)$ and $y_2(x)$ into the integrands of the respective integrals and combine the two integrals.

Reduction of Order

3. (6pts) page 72, prob. 8.: Verify that the given function is a solution of the differential equation. Derive the equation satisfied by $u(x)$, give its solution and give the general solution of the second order equation: $y'' - (2x/(1+x^2))\,y' + (2/(1+x^2))\,y = 0$; $y_1(x) = x$.

4. (6pts) Use the one solution indicated to find the complete solution: $(2x - x^2)\,y'' + 2\,(x-1)\,y' - 2\,y = 0$; $y_1(x) = x - 1$

Euler Equation

5. (6pts) page 81, prob. 20: $x^2\,y'' - 9x\,y' + 24\,y = 0$; $y(1) = 1$, $y'(1) = 10$.

6. (6pts) To reduce the Euler equation to a linear equation, we use the substitution, $z = \log(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation $D = \mathrm{d/dx}$ and $\mathcal{D} = \mathrm{d/dz}$, show that

$$\text{i).}\frac{dy}{dx} = Dy = \frac{1}{x}\mathcal{D}y \qquad \text{or} \qquad x\,Dy = \mathcal{D}y$$

$$\text{ii).}\frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}\left(\mathcal{D}^2y - \mathcal{D}y\right) \qquad \text{or} \qquad x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$$

$$\text{iii). and hence,} \quad \text{that} \quad x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

7. (6pts) Find the complete solution of the equation: $x^3\,y''' + 4x^2y'' - 5xy' - 15y = x^4$

First Order Equation

8. (6pts) The differential equation below has the boundary condition $y(1) = b$. Find the only value of $b$ for which $y(0)$ is finite.

$$\frac{dy}{dx} + (\frac{1}{x} - 1)y = \frac{e^{2x}}{x}.$$

### 2.4.2 Problem 1 Page 93, problem 22 (Variation of parameters)

Find the general solution to $x^2y'' + 3xy' + y = \frac{4}{x}$

**Solution**: This is Euler differential equation. The homogeneous solution $y_h$ is first found, then variation of parameters method is used to find the particular solution $y_p$. The general solution can then be written as $y = y_h + y_p$

Comparing the homogeneous part to the standard form of Euler differential equation, which is given by

$$x^2y'' + Axy' + By = 0$$

Where in the above $y(x)$ is a function of $x$, shows that $A = 3$ and $B = 1$.

Applying the transformation[1] $t=\ln(x)$ to the original ODE converts it to

$$
\begin{aligned}
y''+(A-1)y'+By&=0\\
y''+2y'+y&=0 \qquad (1)
\end{aligned}
$$

Where $y(t)$ is now a function of $t$ and not $x$. This new ODE is solved for $y(t)$. The solution is then converted back to be a function of $x$.

Since Eq. (1) now is a constant coefficients ODE, direct application of characteristic roots method can be used. The roots of $\lambda^2+2\lambda+1=0$ are $\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-4}}{2}=-1$ (repeated). Therefore the solution to Eq. (1) is

$$y(t)=c_1e^{-t}+c_2te^{-t}$$

The above solution is converted back to be a function of $x$ using $t=\ln(x)$.This results in

$$
\begin{aligned}
y_h(x)&=c_1e^{-\ln(x)}+c_2\ln(x)\,e^{-\ln(x)}\\
&=\frac{c_1}{x}+c_2\frac{\ln(x)}{x} \qquad (2)
\end{aligned}
$$

This is valid for $x>0$ and $x<0$ but not for $x=0$. The solution can also be written as

$$y_h(x)=\frac{c_1}{x}+c_2\frac{\ln(|x|)}{x}$$

Now that the homogeneous solution is found, the particular solution is obtained using variation of parameters. Let the two linearly independent solutions of the homogeneous part of the solution to the ODE as shown in Eq. (2) be

$$
\begin{aligned}
y_1&=\frac{1}{x}\\
y_2&=\frac{\ln(x)}{x}
\end{aligned}
$$

The particular solution $y_p$ is

$$y_p=u_1y_1+u_2y_2$$

Where $u_1,u_2$ are two functions to be determined. Using the standard formula derived in class, these functions are

$$
\begin{aligned}
u_1&=\int\frac{-y_2}{W(x)}\frac{f(x)}{a_0}dx \qquad (1)\\
u_2&=\int\frac{y_1}{W(x)}\frac{f(x)}{a_0}dx \qquad (2)
\end{aligned}
$$

Where $f(x)=\frac{4}{x}$ and $a_0=x^2$. The Wronskian is

$$
\begin{aligned}
W(x)&=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}\frac{1}{x} & \frac{\ln(x)}{x}\\ -\frac{1}{x^2} & -\frac{1}{x^2}\ln(x)+\frac{1}{x^2}\end{vmatrix}=\frac{1}{x}\left(-\frac{1}{x^2}\ln(x)+\frac{1}{x^2}\right)+\left(\frac{1}{x^2}\right)\frac{\ln(x)}{x}\\
&=-\frac{\ln(x)}{x^3}+\frac{1}{x^3}+\frac{\ln(x)}{x^3}\\
&=\frac{1}{x^3}
\end{aligned}
$$

$u_1$ is found from Eq (1)

$$
\begin{aligned}
u_1&=\int\frac{-\frac{\ln(x)}{x}}{\frac{1}{x^3}}\frac{\frac{4}{x}}{x^2}dx=-4\int\frac{\ln(x)}{x}dx=-4\left(\frac{\ln(x)^2}{2}\right)\\
&=-2\ln(x)^2
\end{aligned}
$$

$u_2$ is found from Eq. (2)

$$
\begin{aligned}
u_2&=\int\frac{\frac{1}{x}}{\frac{1}{x^3}}\frac{\frac{4}{x}}{x^2}dx=4\int\frac{1}{x}dx\\
&=4\ln(x)
\end{aligned}
$$

---

[1] See page 79, textbook *Advanced engineering mathematics*, 6th ed. by Peter O'Neil.

Therefore the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= -2\ln(x)^2 \frac{1}{x} + 4\ln(x)\frac{\ln(x)}{x} \\ &= -2\frac{\ln(x)^2}{x} + 4\frac{\ln(x)^2}{x} \\ &= 2\frac{\ln(x)^2}{x} \end{aligned}$$

And finally the general solution is obtained

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{c_1}{x} + c_2\frac{\ln(x)}{x} + 2\frac{\ln(x)^2}{x} \end{aligned}$$

Hence

$$y = \frac{1}{x}\left(c_1 + c_2\ln(x) + 2\ln(x)^2\right)$$

### 2.4.3 Problem 2 (Variation of parameters)

Using variation of parameters, show that

$$y = c_1\cosh(kx) + c_2\sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))\, f(s)\, ds$$

Is a complete solution of the equation $y'' - k^2 y = f(x)$, where $k \neq 0$ and $f$ is everywhere continuous. Hint: Introduce the dummy variable $s$ in the integrals which define $u_1$ and $u_2$. Then move $y_1(x)$ and $y_2(x)$ into the integrands of the respective integrals and combine the two integrals.

**solution**: Since the ODE is constant coefficients, direct application of the roots of the characteristic equation is used to obtain the homogeneous solution $y_h$. The roots of the characteristic equation are $\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\pm\sqrt{4k^2}}{2} = \pm k$, hence

$$\begin{aligned} y_h &= c_1 e^{kx} + c_2 e^{-kx} \\ &= c_1\cosh(kx) + c_2\sin(kx) \end{aligned}$$

Let

$$\begin{aligned} y_1 &= \cosh(kx) \\ y_2 &= \sinh(kx) \end{aligned}$$

The Wronskian is

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh(kx) & \sinh(kx) \\ k\sinh(kx) & k\cosh(kx) \end{vmatrix} = k\cosh(x)^2 + k\sinh(kx)^2 \\ &= k \end{aligned}$$

Let the particular solution be

$$y_p = u_1 y_1 + u_2 y_2$$

hence

$$u_1 = \int \frac{-y_2}{W(x)}\frac{f(x)}{a_0}dx \tag{1}$$

$$u_2 = \int \frac{y_1}{W(x)}\frac{f(x)}{a_0}dx \tag{2}$$

Therefore

$$\begin{aligned} u_1 &= \int \frac{-\sinh(kx)}{k} f(x)\, dx \\ u_2 &= \int \frac{\cosh(kx)}{k} f(x)\, dx \end{aligned}$$

Applying Eqs. (1) and (2) gives the particular solution

$$y_p = \frac{1}{k}\cosh(kx)\left(\int -\sinh(kx) f(x)\, dx\right) + \frac{1}{k}\sinh(kx)\left(\int \cosh(kx) f(x)\, dx\right)$$

Let $s = x$, hence $ds = dx$. The integration remains non-definite and can now be written as

$$y_p = \frac{1}{k}\cosh(kx)\left(\int -\sinh(ks) f(s)\,ds\right) + \frac{1}{k}\sinh(kx)\left(\int \cosh(ks) f(s)\,ds\right)$$

Now $\cosh(kx)$ and $\sinh(kx)$ can be moved inside the integrals since they do not depend on the new dummy variable $s$ and are hence treated as constants inside the integration. This results in

$$\begin{aligned} y_p &= \frac{1}{k}\left(\int -\cosh(kx)\sinh(ks) f(s)\,ds\right) + \frac{1}{k}\left(\int \sinh(kx)\cosh(ks) f(s)\,ds\right) \\ &= \frac{1}{k}\int (\sinh(kx)\cosh(ks) - \cosh(kx)\sinh(ks))\ f(s)\ ds \qquad (3) \end{aligned}$$

Using the trigonometric relation

$$\sinh A \cosh B - \cosh A \sinh B = \sinh(A - B)$$

Eq. (3) becomes

$$y_p = \frac{1}{k}\int \sinh(k(x-s))\ f(s)\ ds$$

Therefore, the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\cosh(kx) + c_2\sin(kx) + \frac{1}{k}\int \sinh(k(x-s))\ f(s)\ ds \end{aligned}$$

which is what was asked to show. *Note*: The question asks to show the final integral as definite with limits $\int_0^x$. However in the solution obtained above, the integral is non-definite $\int$. Need more clarification on this point.

### 2.4.4 Problem 3, reduction of order

Problem 8, page 72: Verify that the given function is a solution of the differential equation. Derive the equation

satisfied by $u(x)$, give its solution and give the general solution of the second order equation: $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0; y_1(x) = x$

**Solution:**

Let the second solution of the ODE be $y_2 = uy_1$ where $u(x)$ is a function of $x$ to be determined. The derivatives of $y_2$ are now found and substituted back into the ODE to solve for $u$.

$$\begin{aligned} y_2' &= u'y_1 + uy_1' \qquad (1) \\ y_2'' &= u''y_1 + u'y_1' + u'y_1' + uy_1'' \qquad (2) \end{aligned}$$

Since $y_2$ is assumed to be a solution of the original ODE, then it satisfies it. Hence

$$y_2'' - \frac{2x}{1+x^2}y_2' + \frac{2}{1+x^2}y_2 = 0 \qquad (3)$$

Using Eqs. (1) and (2) into (3) gives

$$\begin{aligned} \left(u''y_1 + u'y_1' + u'y_1' + uy_1''\right) - \frac{2x}{1+x^2}\left(u'y_1 + uy_1'\right) + \frac{2}{1+x^2}\left(uy_1\right) = 0 \\ u''\left(y_1\right) + u'\left(2y_1' - \frac{2x}{1+x^2}y_1\right) + u\left(y_1'' - \frac{2x}{1+x^2}y_1' + \frac{2}{1+x^2}y_1\right) = 0 \end{aligned}$$

But $y_1$ is a solution of the ODE, hence the last term in the above vanish resulting in

$$u''y_1 + u'\left(2y_1' - \frac{2x}{1+x^2}y_1\right) = 0$$

But $y_1 = x$ and $y_1' = 1$ hence the above becomes

$$\begin{aligned} u'' + u'\left(2 - \frac{2x^2}{1+x^2}\right)\frac{1}{x} = 0 \\ u'' + u'\left(\frac{2}{x+x^3}\right) = 0 \end{aligned}$$

Let $u' = v$, the above becomes

$$v' + v\left(\frac{2}{x+x^3}\right) = 0$$

This is now separable

$$\frac{v'}{v} = -\left(\frac{2}{x+x^3}\right)$$

Integrating both sides

$$\begin{aligned}
\ln v &= -2\int\left(\frac{1}{x+x^3}\right)dx + c_1\\
&= -2\int\frac{1}{x} - \frac{x}{1+x^2}dx + c_1\\
&= -2\int\frac{1}{x}dx + 2\int\frac{x}{1+x^2}dx + c_1\\
&= -2\ln x + 2\left(\frac{1}{2}\ln\left(1+x^2\right)\right) + c_1\\
&= -2\ln x + \ln\left(1+x^2\right) + c_1
\end{aligned}$$

Hence

$$\begin{aligned}
v &= c_1 e^{-2\ln x + \ln\left(1+x^2\right)}\\
&= c_1\left(e^{-2\ln x}e^{\ln\left(1+x^2\right)}\right)\\
&= c_1\frac{1}{x^2}\left(1+x^2\right)\\
&= c_1\left(1+\frac{1}{x^2}\right)
\end{aligned}$$

Since only one second solution $y_2$ is needed, let $c_1 = 1$.

Now that $v(x)$ is found, then $u$ is found by solving

$$\begin{aligned}
u' &= v\\
\frac{du}{dx} &= 1 + \frac{1}{x^2}
\end{aligned}$$

Hence

$$\begin{aligned}
u &= \int 1 + \frac{1}{x^2}dx + c_2\\
&= \left(x - \frac{1}{x}\right) + c_2
\end{aligned}$$

Since only one second solution $y_2$ is needed, let $c_2 = 0$ hence

$$u = \left(x - \frac{1}{x}\right)$$

Therefore, since $y_2 = uy_1$, and $y_1 = x$ then

$$\begin{aligned}
y_2 &= u\,y_1\\
&= \left(x - \frac{1}{x}\right)x
\end{aligned}$$

Hence

$$y_2 = \left(x^2 - 1\right)$$

And finally, the general solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2\\
&= c_1x + c_2\left(x^2 - 1\right)
\end{aligned}$$

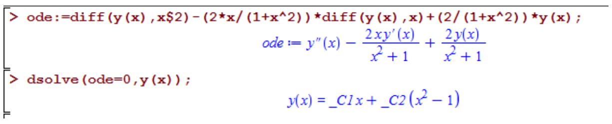
To verify the above solution, it is substituted back into the ODE $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y$ to check if the result is zero.

$$\begin{aligned}
y' &= c_1 + c_2(2x)\\
y'' &= 2c_2
\end{aligned}$$

Hence the ODE becomes

$$
\begin{aligned}
0 &= y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y \\
&= (2c_2) - \frac{2x}{1+x^2}\left(c_1 + c_2\left(2x\right)\right) + \frac{2}{1+x^2}\left(c_1x + c_2\left(x^2-1\right)\right) \\
&= 2c_2 - c_1\frac{2x}{1+x^2} - 2xc_2\frac{2x}{1+x^2} + c_1x\frac{2}{1+x^2} + c_2\left(x^2-1\right)\frac{2}{1+x^2} \\
&= 2c_2\left(1+x^2\right) - 2c_1x - 4x^2c_2 + 2c_1x + 2c_2\left(x^2-1\right) \\
&= 2c_2 + 2c_2x^2 - 2c_1x - 4x^2c_2 + 2c_1x + 2c_2x^2 - 2c_2 \\
&= 4c_2x^2 - 4x^2c_2 \\
&= 0
\end{aligned}
$$

Verified OK.

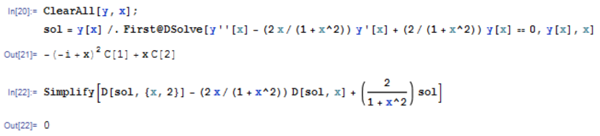


```
> ode:=diff(y(x),x$2)-(2*x/(1+x^2))*diff(y(x),x)+(2/(1+x^2))*y(x);
```

$$ode := y''(x) - \frac{2xy'(x)}{x^2+1} + \frac{2y(x)}{x^2+1}$$

```
> dsolve(ode=0,y(x));
```

$$y(x) = \_C1\,x + \_C2\left(x^2-1\right)$$



```
In[20]:= ClearAll[y, x];
         sol = y[x] /. First@DSolve[y''[x] - (2 x / (1 + x^2)) y'[x] + (2 / (1 + x^2)) y[x] == 0, y[x], x]
Out[21]= - (-i + x)^2 C[1] + x C[2]

In[22]:= Simplify[D[sol, {x, 2}] - (2 x / (1 + x^2)) D[sol, x] + (2 / (1 + x^2)) sol]

Out[22]= 0
```

### 2.4.5 Problem 4 reduction of order

Use the one solution indicated to find the complete solution. $\left(2x-x^2\right)y'' + 2\left(x-1\right)y' - 2y = 0; y_1\left(x\right) = x - 1$

**Solution**:

The ODE can be written as $y'' + \frac{2(x-1)}{\left(2x-x^2\right)}y' - \frac{2}{\left(2x-x^2\right)}y = 0$. (assuming $x \neq 0$) Let the second solution of the ODE be

$$y_2 = uy_1$$

where $u\left(x\right)$ is a function of $x$ to be determined. The derivatives of $y_2$ are now found and substituted back into the ODE to solve for $u$.

$$y_2' = u'y_1 + uy_1' \tag{1}$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' \tag{2}$$

Since $y_2$ is assumed to be a solution of the original ODE, then it satisfies it. Hence

$$y_2'' + \frac{2\left(x-1\right)}{\left(2x-x^2\right)}y_2' - \frac{2}{\left(2x-x^2\right)}y_2 = 0 \tag{3}$$

Using Eqs. (1) and (2) into (3) gives

$$\left(u''y_1 + u'y_1' + u'y_1' + uy_1''\right) + \frac{2\left(x-1\right)}{\left(2x-x^2\right)}\left(u'y_1 + uy_1'\right) - \frac{2}{\left(2x-x^2\right)}uy_1 = 0$$

$$u''y_1 + u'\left(2y_1' + \frac{2\left(x-1\right)}{\left(2x-x^2\right)}y_1\right) + u\left(y_1'' + \frac{2\left(x-1\right)}{\left(2x-x^2\right)}y_1' - \frac{2}{\left(2x-x^2\right)}y_1\right) = 0$$

But $y_1$ is a solution of the ODE, hence the last term in the above vanishes resulting in

$$u''y_1 + u'\left(2y_1' + \frac{2\left(x-1\right)}{\left(2x-x^2\right)}y_1\right) = 0$$

But $y_1 = x - 1$ and $y_1' = 1$ hence the above becomes (assuming $x \neq 1$)

$$\begin{aligned}
u''(x-1) + u'\left(2 + \frac{2(x-1)}{\left(2x - x^2\right)}(x-1)\right) &= 0\\
u'' + u'\left(\frac{2}{(x-1)} + \frac{2(x-1)}{\left(2x-x^2\right)}\right) &= 0\\
u'' + u'\left(\frac{2\left(2x-x^2\right) + 2(x-1)(x-1)}{\left(2x-x^2\right)(x-1)}\right) &= 0\\
u'' + u'\left(\frac{4x - 2x^2 + 2x^2 + 2 - 4x}{\left(2x-x^2\right)(x-1)}\right) &= 0\\
u'' + u'\left(\frac{2}{3x^2 - 2x - x^3}\right) &= 0
\end{aligned}$$

Let $u' = v$, the above becomes

$$v' + v\left(\frac{2}{3x^2 - 2x - x^3}\right) = 0$$

This is now separable

$$\frac{v'}{v} = -\left(\frac{2}{3x^2 - 2x - x^3}\right)$$

Integrating both sides

$$\ln v = -2\int \frac{1}{3x^2 - 2x - x^3}dx + c_1$$

Partial fraction decomposition on the integrand gives

$$\begin{aligned}
\ln v &= -2\int \frac{1}{-2(x-2)} + \frac{1}{x-1} - \frac{1}{2x}dx + c_1\\
&= \int \frac{1}{(x-2)} - 2\int \frac{1}{x-1} + \int \frac{1}{x}dx + c_1\\
&= \ln(x-2) - 2\ln(x-1) + \ln x + c_1
\end{aligned}$$

Hence

$$\begin{aligned}
v &= c_1 e^{\ln(x-2) - 2\ln(x-1) + \ln x}\\
&= c_1 e^{\ln(x-2)} e^{-2\ln(x-1)} e^{\ln x}\\
&= c_1 \frac{(x-2)x}{(x-1)^2}
\end{aligned}$$

Since only one second solution $y_2$ is needed, let $c_1 = 1$.

Now that $v(x)$ is found, then $u$ is found by solving

$$\begin{aligned}
u' &= v\\
\frac{du}{dx} &= \frac{(x-2)x}{(x-1)^2}
\end{aligned}$$

Hence

$$\begin{aligned}
u &= \int \frac{x^2 - 2x}{(x-1)^2}dx + c_2\\
&= x + \frac{1}{x-1} + c_2
\end{aligned}$$

Since only one second solution $y_2$ is needed, let $c_2 = 0$ hence

$$u = x + \frac{1}{x-1}$$

Therefore, since $y_2 = uy_1$, and $y_1 = x - 1$ then

$$\begin{aligned}
y_2 &= u\, y_1\\
&= \left(x + \frac{1}{x-1}\right)(x-1)\\
&= x(x-1) + 1
\end{aligned}$$

And finally, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x-1) + c_2 (x(x-1)+1) \\ &= c_1 (x-1) + c_2 \left(x^2 - x + 1\right) \end{aligned}$$

By letting $c_3 = (c_1 - c_2)$ the above can be simplified to

$$y(x) = c_3(x-1) + c_2 x^2$$

Or by constants renaming

$$\boxed{y(x) = c_1(x-1) + c_2 x^2}$$

To verify the above solution, it is substituted back into the ODE $y'' + \frac{2(x-1)}{\left(2x-x^2\right)} y' - \frac{2}{\left(2x-x^2\right)} y = 0$ to check if the result is zero.

$$\begin{aligned} y' &= c_1 + 2c_2 x \\ y'' &= 2c_2 \end{aligned}$$

Hence the ODE becomes

$$\begin{aligned} 0 &= \left(2x - x^2\right) y'' + 2(x-1) y' - 2y \\ &= \left(2x - x^2\right) 2c_2 + (2x-2)(c_1 + 2c_2 x) - 2\left(c_1(x-1) + c_2 x^2\right) \\ &= 4c_2 x - 2c_2 x^2 + 2c_1 x + 4c_2 x^2 - 2c_1 - 4c_2 x - 2c_1 x + 2c_1 - 2c_2 x^2 \\ &= 4c_2 x + 2c_1 x - 2c_1 - 4c_2 x - 2c_1 x + 2c_1 \\ &= 4c_2 x - 4c_2 x \\ &= 0 \end{aligned}$$

Verified OK.

### 2.4.6 Problem 5 Euler equation, page 81

Problem 20, page 81. Solve $x^2 y'' - 9xy' + 24y = 0; y(1) = 1; y'(1) = 10$

**Solution**:

This is Euler differential equation. Comparing to the standard form of Euler differential equation, which is given by

$$x^2 y'' + Axy' + By = 0$$

Where in the above $y(x)$ is a function of $x$, shows that $A = -9$ and $B = 24$.

Applying the transformation[2] $t = \ln(x)$ to the original ODE converts it to

$$\begin{aligned} y'' + (A-1)y' + By &= 0 \\ y'' - 10y' + 24y &= 0 \qquad (1) \end{aligned}$$

Where $y(t)$ is now a function of $t$ and not $x$. This new ODE is solved for $y(t)$. The solution is then converted back to be a function of $x$.

Since Eq. (1) is now a constant coefficients ODE, direct application of characteristic roots method can be used. The roots of $\lambda^2 - 10\lambda + 24 = 0$ are $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 4(24)}}{2} = \frac{10 \pm \sqrt{4}}{2} = \frac{10 \pm 2}{2} = \{6, 4\}$. Therefore the solution to Eq. (1) is

$$y(t) = c_1 e^{6t} + c_2 e^{4t}$$

The above solution is converted back to be a function of $x$ using $t = \ln(x)$.This results in

$$\begin{aligned} y(x) &= c_1 e^{6\ln(x)} + c_2 e^{4\ln(x)} \\ &= c_1 x^6 + c_2 x^4 \qquad (2) \end{aligned}$$

This is valid for $x > 0$ and $x < 0$ but not for $x = 0$.

$$y' = 6c_1 x^5 + 4c_2 x^3$$

---

[2] See page 79, textbook *Advanced engineering mathematics*, 6th ed. by Peter O'Neil.

At $x=1$, $y=1$, hence

$$1=c_1+c_2$$

At $x=1,y'(1)=10$, hence

$$10=6c_1+4c_2$$

Hence $c_1=1-c_2$, then $10=6(1-c_2)+4c_2$ or $10=6-2c_2$ or $c_2=-2$, hence $c_1=1+2=3$, therefore the final solution is

$$\boxed{y=3x^6-2x^4}$$

### 2.4.7 Problem 6

**Question**:

To reduce the Euler equation to a linear equation, we use the substitution, $z=\ln(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation[3] $D_x\equiv\frac{d}{dx}$ and $D_z\equiv\frac{d}{dz}$ show that

1. $\frac{dy}{dx}=D_xy=\frac{1}{x}D_zy$ or $xD_xy=D_zy$
2. $\frac{d^2y}{dx^2}=D_x^2y=\frac{1}{x^2}\left(D_z^2y-D_zy\right)$ or $x^2D_x^2y=D_z\left(D_z-1\right)y$
3. $x^3D_x^3y=D_z\left(D_z-1\right)\left(D_z-2\right)y$

**Answer**

#### 2.4.7.1 Part 1

$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}$ but $\frac{dz}{dx}=\frac{1}{x}$ hence

$$\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dz}$$

Using operator notation

$$D_xy=\frac{1}{x}D_zy$$

or

$$xD_xy=D_zy$$

Part 2

$\frac{d^2y}{dx^2}=D_x^2y$ by definitions. This can be written as

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

but from part(1) it was found that $\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dz}$, hence the above becomes

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

---

[3] I used the notation $D_x$ and $D_z$ instead of those given in the problem statment as easier to read on the screen

Applying chain rule

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\frac{dy}{dz}\\ &= \frac{-1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)\\ &= \frac{-1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)\\ &= \frac{-1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{1}{x}\right)\\ &= \frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)\end{aligned}$$

Using operator notation

$$D_x^2y=\frac{1}{x^2}\left(D_z^2y-D_zy\right)$$

or

$$x^2D_x^2y=D_z\left(D_z-1\right)y$$

#### 2.4.7.2 Part 3

$\frac{d^3y}{dx^3}=D_x^3y$ by definitions. This can be written as

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

but from part(2) it was found that $\frac{d^2y}{dx^2}=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$, hence the above becomes

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)\right)$$

Applying chain rule

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{-2}{x^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{x^2}\left(\frac{d}{dx}\frac{d^2y}{dz^2}-\frac{d}{dx}\frac{dy}{dz}\right)\\ &= \frac{-2}{x^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{x^2}\left(\frac{d}{dz}\frac{d^2y}{dz^2}\frac{dz}{dx}-\frac{d}{dz}\frac{dy}{dz}\frac{dz}{dx}\right)\\ &= \frac{-2}{x^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{x^2}\left(\frac{d^3y}{dz^3}\frac{1}{x}-\frac{d^2y}{dz^2}\frac{1}{x}\right)\\ &= \frac{-2}{x^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{x^3}\left(\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\right)\\ &= \frac{1}{x^3}\left[\left(-2\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right)+\left(\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\right)\right]\\ &= \frac{1}{x^3}\left[\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right]\end{aligned}$$

Using operator notation

$$\begin{aligned}D_x^3y &= \frac{1}{x^3}\left(D_z^3y-3D_z^2y+2D_zy\right)\\ x^3D_x^3y &= \left(D_z^3-3D_z^2+2D_z\right)y\end{aligned}$$

Writing the RHS as $\left(\lambda^3-3\lambda^2+2\lambda\right)$, then it is seen it can be factored as $\lambda\left(\lambda^2-3\lambda+2\right)=\lambda\left(\lambda-1\right)\left(\lambda-2\right)$, hence the above can be written as

$$x^3D_x^3y=D_z\left(D_z-1\right)\left(D_z-2\right)y$$

### 2.4.8 Problem 7

Find the complete solution of $x^3y'''+4x^2y''-5xy'-15y=x^4$

**solution**:

This is a Euler differential equation since it is of the form $a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\cdots+$

$a_1\frac{d}{dx}y + a_0 y = f(x)$. Let $z = \ln(x)$, or $x = e^z$ to convert the equation from $y(x)$ to an equation for $Y(z)$. hence $\frac{dz}{dx} = \frac{1}{x}$ and using results from problem 6 above summarized below

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}\frac{dY}{dz}\\ \frac{d^2y}{dx^2} &= \frac{1}{x^2}\left(\frac{d^2Y}{dz^2} - \frac{dY}{dz}\right)\\ \frac{d^3y}{dx^3} &= \frac{1}{x^3}\left[\frac{d^3Y}{dz^3} - 3\frac{d^2Y}{dz^2} + 2\frac{dY}{dz}\right]\end{aligned}$$

The homogeneous part of the ODE is first solved. Substituting the above three relations into the ODE gives

$$x^3\frac{1}{x^3}\left[\frac{d^3Y}{dz^3} - 3\frac{d^2Y}{dz^2} + 2\frac{dY}{dz}\right] + 4x^2\frac{1}{x^2}\left(\frac{d^2Y}{dz^2} - \frac{dY}{dz}\right) - 5x\frac{1}{x}\frac{dY}{dz} - 15Y = 0$$

Where $Y$ is function of $z$ and $y$ is the original function of $x$.The above becomes

$$\begin{aligned}\frac{d^3Y}{dz^3} - 3\frac{d^2Y}{dz^2} + 2\frac{dY}{dz} + 4\left(\frac{d^2Y}{dz^2} - \frac{dY}{dz}\right) - 5\frac{dY}{dz} - 15Y &= 0\\ \frac{d^3Y}{dz^3} + \frac{d^2Y}{dz^2} - 7\frac{dY}{dz} - 15Y &= 0\end{aligned}$$

This is now a constant coefficient ODE, which can be solved directly using the characteristic roots method.

$$\lambda^3 + \lambda^2 - 7\lambda - 15 = 0$$

The roots are $\{3, -2-i, -2+i\}$, hence the solution is

$$\begin{aligned}Y(z) &= c_1e^{3z} + c_2e^{(-2-i)z} + c_3e^{(-2+i)z}\\ &= c_1e^{3z} + c_2e^{(-2-i)z} + c_3e^{(-2+i)z}\\ &= c_1e^{3z} + c_2e^{-2z}e^{-iz} + c_3e^{-2z}e^{iz}\\ &= c_1e^{3z} + e^{-2z}\left(c_2e^{-iz} + c_3e^{iz}\right)\\ &= c_1e^{3z} + e^{-2z}\left(c_2\left(\cos z - i\sin z\right) + c_3\left(\cos z + i\sin z\right)\right)\\ &= c_1e^{3z} + e^{-2z}\left(c_2\cos z - c_2 i\sin z + c_3\cos z + c_3 i\sin z\right)\\ &= c_1e^{3z} + e^{-2z}\left(\left(c_2 + c_3\right)\cos z + \left(c_3 - c_2\right)i\sin z\right)\end{aligned}$$

Let $(c_2 + c_3)$ be new constant $c_4$ and $(c_3 - c_2)\,i$ new constant $c_5$, hence

$$Y(z) = c_1e^{3z} + e^{-2z}\left(c_4\cos z + c_5\sin z\right)$$

Converting back to $x$ using $z = \ln(x)$

$$\begin{aligned}y(x) &= c_1e^{3\ln x} + e^{-2\ln x}\left(c_4\cos(\ln x) + c_5\sin(\ln x)\right)\\ &= c_1x^3 + \frac{1}{x^2}\left(c_4\cos(\ln x) + c_5\sin(\ln x)\right)\end{aligned}$$

The above is the homogeneous part of the solution. The particular solution is now found. Since the homogeneous solution has the following forms of solutions in it $x^3, \frac{1}{x^2}\cos(\ln x), \frac{1}{x^2}\sin(\ln x)$, then using variation of parameters, assume

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

where

$$\begin{aligned}y_1 &= x^3\\ y_2 &= \frac{1}{x^2}\cos(\ln x)\\ y_3 &= \frac{1}{x^2}\sin(\ln x)\end{aligned}$$

Therefore

$$\begin{aligned}u_1 &= \int\frac{W_1(x)f(x)}{W(x)}dx\\ u_2 &= \int\frac{W_2(x)f(x)}{W(x)}dx\\ u_3 &= \int\frac{W_3(x)f(x)}{W(x)}dx\end{aligned}$$

Where $f(x) = x^4$ and

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} x^3 & \frac{1}{x^2}\cos(\ln x) & \frac{1}{x^2}\sin(\ln x) \\ 3x^2 & \frac{-2}{x^3}\cos(\ln x) - \frac{1}{x^3}\sin(\ln x) & \frac{-2}{x^3}\sin(\ln x) + \frac{1}{x^3}\cos(\ln x) \\ 6x & \frac{6}{x^4}\cos(\ln x) + \frac{2}{x^4}\sin(\ln x) + \frac{3}{x^4}\sin(\ln x) - \frac{1}{x^4}\cos(\ln x) & \frac{-6}{x^4}\sin(\ln x) - \frac{2}{x^4}\cos(\ln x) - \frac{3}{x^4}\cos(\ln x) + \frac{1}{x^4}\sin(\ln x) \end{vmatrix} \\
&= \frac{1}{x^4}\left(26\cos^2(\ln x) + 50\cos(\ln x)\sin(\ln x) + 36\sin^2(\ln x)\right)
\end{aligned}$$

And

$$\begin{aligned}
W_1(x) &= (-1)^{3-1} W\left(y_2, y_3\right) \\
W_2(x) &= (-1)^{3-2} W\left(y_1, y_3\right) \\
W_3(x) &= (-1)^{3-3} W\left(y_1, y_2\right)
\end{aligned}$$

Hence

$$\begin{aligned}
W_1(x) &= (-1)^2 \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} = \begin{vmatrix} \frac{1}{x^2}\cos(\ln x) & \frac{1}{x^2}\sin(\ln x) \\ \frac{-2}{x^3}\cos(\ln x) - \frac{1}{x^3}\sin(\ln x) & \frac{-2}{x^3}\sin(\ln x) + \frac{1}{x^3}\cos(\ln x) \end{vmatrix} \\
&= \frac{1}{x^5}\left(\cos^2(\ln x) + \sin^2(\ln x)\right)
\end{aligned}$$

$$W_2(x) = (-1)^{3-2} \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} = -\begin{vmatrix} x^3 & \frac{1}{x^2}\sin(\ln x) \\ 3x^2 & \frac{-2}{x^3}\sin(\ln x) + \frac{1}{x^3}\cos(\ln x) \end{vmatrix} = 5\sin(\ln x) - \cos(\ln x)$$

$$W_3(x) = (-1)^{3-3} \begin{vmatrix} x^3 & \frac{1}{x^2}\cos(\ln x) \\ 3x^2 & \frac{-2}{x^3}\cos(\ln x) - \frac{1}{x^3}\sin(\ln x) \end{vmatrix} = -5\cos(\ln x) - \sin(\ln x)$$

Hence[4]

$$\begin{aligned}
u_1 &= \int \frac{W_1(x)}{W(x)} \frac{f(x)}{a_0} dx \\
&= \int \frac{\frac{1}{x^5}\left(\cos^2(\ln x) + \sin^2(\ln x)\right)}{\frac{1}{x^4}\left(26\cos^2(\ln x) + 50\cos(\ln x)\sin(\ln x) + 36\sin^2(\ln x)\right)} x dx \\
&= \frac{x}{26}
\end{aligned}$$

And

$$\begin{aligned}
u_2 &= \int \frac{W_2(x)}{W(x)} \frac{f(x)}{a_0} dx \\
&= \int \frac{5\sin(\ln x) - \cos(\ln x)}{\frac{1}{x^4}\left(26\cos^2(\ln x) + 50\cos(\ln x)\sin(\ln x) + 36\sin^2(\ln x)\right)} x dx \\
&= \frac{-11}{962} x^6 \cos(\ln(x)) + \frac{29}{962} x^6 \sin(\ln x)
\end{aligned}$$

and

$$\begin{aligned}
u_3 &= \int \frac{W_3(x)}{W(x)} \frac{f(x)}{a_0} dx \\
&= \int \frac{-5\cos(\ln x) - \sin(\ln x)}{\frac{1}{x^4}\left(26\cos^2(\ln x) + 50\cos(\ln x)\sin(\ln x) + 36\sin^2(\ln x)\right)} x dx \\
&= \frac{-29}{962} x^6 \cos(\ln(x)) - \frac{11}{962} x^6 \sin(\ln x)
\end{aligned}$$

Hence

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\
&= \frac{x}{26} x^3 + \left(\frac{-11}{962} x^6 \cos(\ln(x)) + \frac{29}{962} x^6 \sin(\ln x)\right) \frac{1}{x^2}\cos(\ln x) + \left(\frac{-29}{962} x^6 \cos(\ln(x)) - \frac{11}{962} x^6 \sin(\ln x)\right) \frac{1}{x^2}\sin(\ln x)
\end{aligned}$$

---

[4] CAS was used to evaluate these integrals

The above reduces to

$$y_p = \frac{x^4}{37}$$

Hence the final solution is

$$y = c_1 x^3 + c_4 \frac{1}{x^2} \cos(\ln x) + c_5 \frac{1}{x^2} \sin(\ln x) + \frac{x^4}{37}$$

### 2.4.9 Problem 8

The differential equation $\frac{dy}{dx} + \left(\frac{1}{x} - 1\right) y = \frac{e^{2x}}{x}$ has boundary condition $y(1) = b$. Find the only value of $b$ for which $y(0)$ is finite.

**Solution**:

The complete solution is first found. The $y_h$ is found first

$$\frac{dy}{dx} + \left(\frac{1}{x} - 1\right) y = 0$$

This is separable

$$\begin{aligned} \frac{dy}{y} &= \left(1 - \frac{1}{x}\right) dx \\ \ln y &= x - \ln x + c \\ y &= ce^{x - \ln x} \\ &= \frac{ce^x}{x} \end{aligned}$$

For the particular solution, try $y_p = A\frac{e^{2x}}{x}$, hence $y_p' = A\left(\frac{-1}{x^2}e^{2x} + \frac{2}{x}e^{2x}\right)$, and the ODE becomes

$$\begin{aligned} A\left(\frac{-1}{x^2}e^{2x} + \frac{2}{x}e^{2x}\right) + \left(\frac{1}{x} - 1\right) A\frac{e^{2x}}{x} &= \frac{e^{2x}}{x} \\ \frac{-A}{x^2}e^{2x} + \frac{2A}{x}e^{2x} + \frac{A}{x}\frac{e^{2x}}{x} - A\frac{e^{2x}}{x} &= \frac{e^{2x}}{x} \\ \frac{2A}{x}e^{2x} - A\frac{e^{2x}}{x} &= \frac{e^{2x}}{x} \\ A\frac{e^{2x}}{x} &= \frac{e^{2x}}{x} \end{aligned}$$

Hence $A = 1$ and the complete solution is

$$y = \frac{ce^x}{x} + \frac{e^{2x}}{x}$$

At $y(1)$ the above becomes

$$\begin{aligned} b &= ce + e^2 \\ &= ce(1 + e) \end{aligned}$$

Hence

$$c = \frac{b}{e(1 + e)}$$

Therefore the solution is

$$\begin{aligned} y &= \frac{b}{e(1 + e)} \frac{e^x}{x} + \frac{e^{2x}}{x} \\ &= \frac{1}{x}\left(\frac{b}{1 + e}e^{x-1} + e^{2x}\right) \end{aligned}$$

Now at $x = 0$ the solution is required to be finite.

$$\begin{aligned}\lim_{x\to 0} y(x) &= \lim_{x\to 0} \frac{1}{x} \lim_{x\to 0}\left(\frac{b}{1+e} e^{x-1} + e^{2x}\right)\\ &= \lim_{x\to 0}\frac{1}{x}\left(\frac{b}{1+e}e^{-1} + 1\right)\\ &= \lim_{x\to 0}\frac{1}{x}\left(\frac{b}{e+e^2} + 1\right)\\ &= \lim_{x\to 0}\frac{1}{x}\left(\frac{b+e+e^2}{e+e^2}\right)\end{aligned}$$

But $e + e^2 = k = 10.107$, a known constant, hence the above

$$\lim_{x\to 0} y(x) = \frac{1}{k} \lim_{x\to 0} \frac{1}{x}(b+k)$$

If $b + k = x_0 = 0$, then $\lim_{x\to 0} \frac{b+k}{x} \to 0$, since $\lim_{x\to 0} \frac{0}{x} = 0$, therefore

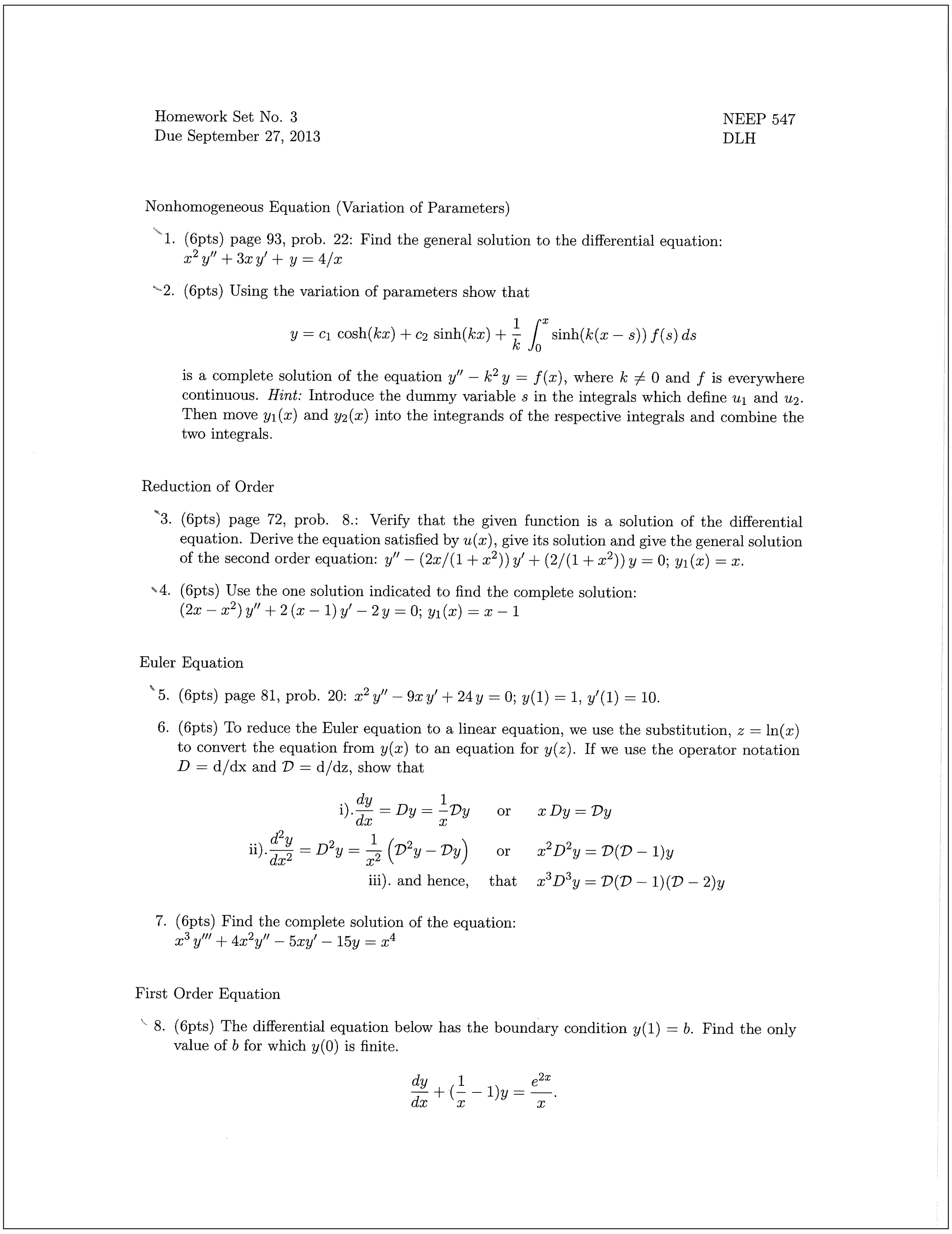
$$b = -k$$

Hence

$$b = -10.107$$

Is the only value.

## 2.4.10 key solution



Homework Set No. 3
Due September 27, 2013

NEEP 547
DLH

Nonhomogeneous Equation (Variation of Parameters)

1. (6pts) page 93, prob. 22: Find the general solution to the differential equation: $x^2 y'' + 3x\,y' + y = 4/x$
2. (6pts) Using the variation of parameters show that

$$y = c_1 \cosh(kx) + c_2 \sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))\, f(s)\, ds$$

is a complete solution of the equation $y'' - k^2 y = f(x)$, where $k \neq 0$ and $f$ is everywhere continuous. *Hint:* Introduce the dummy variable $s$ in the integrals which define $u_1$ and $u_2$. Then move $y_1(x)$ and $y_2(x)$ into the integrands of the respective integrals and combine the two integrals.

Reduction of Order

3. (6pts) page 72, prob. 8.: Verify that the given function is a solution of the differential equation. Derive the equation satisfied by $u(x)$, give its solution and give the general solution of the second order equation: $y'' - (2x/(1+x^2))\,y' + (2/(1+x^2))\,y = 0$; $y_1(x) = x$.
4. (6pts) Use the one solution indicated to find the complete solution: $(2x - x^2)\,y'' + 2\,(x-1)\,y' - 2\,y = 0$; $y_1(x) = x - 1$

Euler Equation

5. (6pts) page 81, prob. 20: $x^2 y'' - 9x\,y' + 24\,y = 0$; $y(1) = 1$, $y'(1) = 10$.
6. (6pts) To reduce the Euler equation to a linear equation, we use the substitution, $z = \ln(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation $D = \mathrm{d/dx}$ and $\mathcal{D} = \mathrm{d/dz}$, show that

$$\text{i).}\frac{dy}{dx} = Dy = \frac{1}{x}\mathcal{D}y \quad \text{or} \quad x\,Dy = \mathcal{D}y$$

$$\text{ii).}\frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}\left(\mathcal{D}^2y - \mathcal{D}y\right) \quad \text{or} \quad x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$$

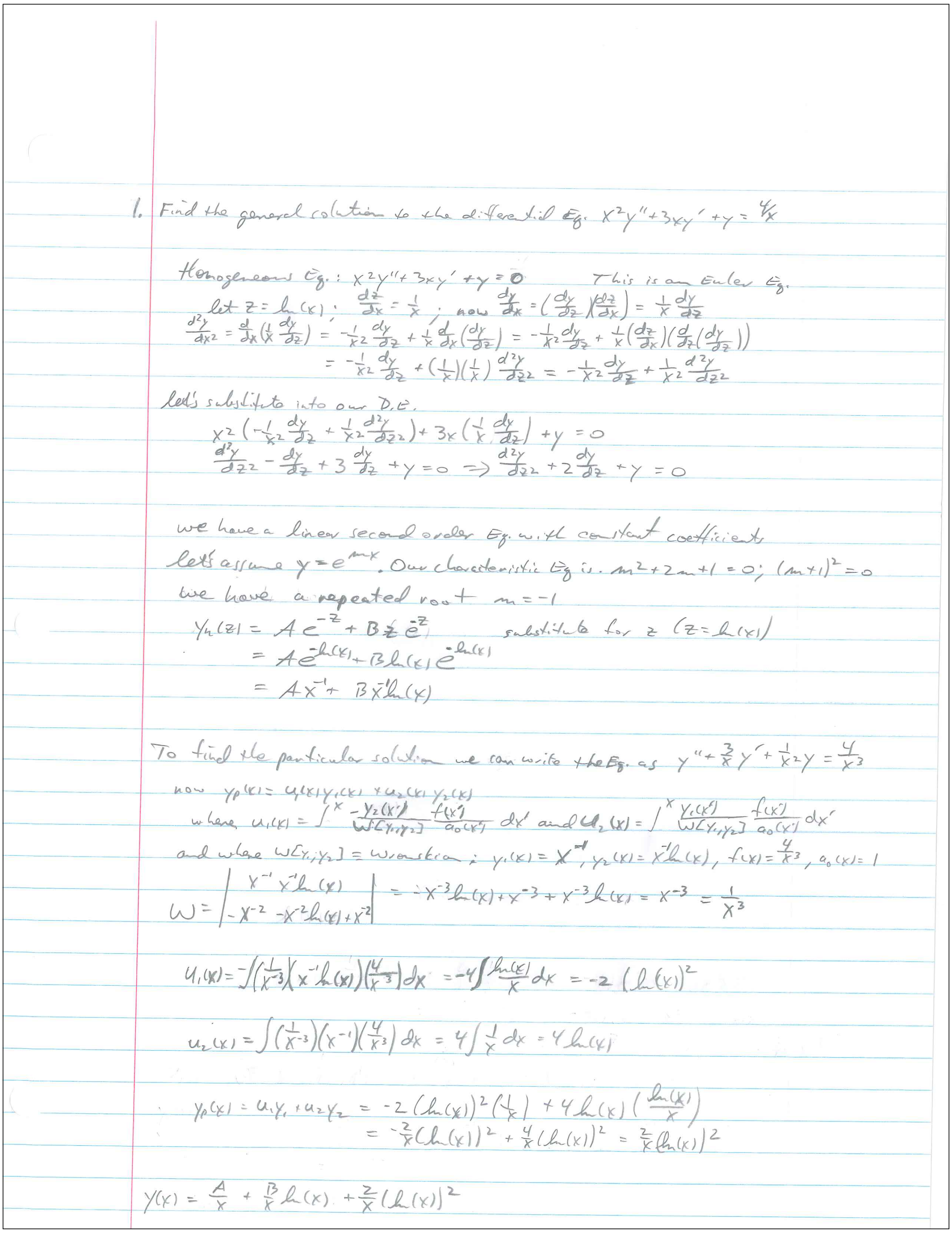
$$\text{iii). and hence,} \quad \text{that} \quad x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

7. (6pts) Find the complete solution of the equation: $x^3 y''' + 4x^2y'' - 5xy' - 15y = x^4$

First Order Equation

8. (6pts) The differential equation below has the boundary condition $y(1) = b$. Find the only value of $b$ for which $y(0)$ is finite.

$$\frac{dy}{dx} + (\frac{1}{x} - 1)y = \frac{e^{2x}}{x}.$$



1. Find the general solution to the differential Eq. $x^2y''+3xy'+y=4/x$

Homogeneous Eq.: $x^2y''+3xy'+y=0$ This is an Euler Eq.

let $z=\ln(x)$; $\frac{dz}{dx}=\frac{1}{x}$; now $\frac{dy}{dx}=\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)=\frac{1}{x}\frac{dy}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\left(\frac{dz}{dx}\right)\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\right)$$
$$=-\frac{1}{x^2}\frac{dy}{dz}+\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\frac{d^2y}{dz^2}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

let's substitute into our D.E.

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}\right)+3x\left(\frac{1}{x}\frac{dy}{dz}\right)+y=0$$
$$\frac{d^2y}{dz^2}-\frac{dy}{dz}+3\frac{dy}{dz}+y=0 \Rightarrow \frac{d^2y}{dz^2}+2\frac{dy}{dz}+y=0$$

we have a linear second order Eq. with constant coefficients

let's assume $y=e^{mx}$. Our characteristic Eq is. $m^2+2m+1=0$; $(m+1)^2=0$

we have a repeated root $m=-1$

$$y_h(z)=Ae^{-z}+Bze^{-z}$$ substitute for $z$ ($z=\ln(x)$)
$$=Ae^{-\ln(x)}+B\ln(x)e^{-\ln(x)}$$
$$=Ax^{-1}+Bx^{-1}\ln(x)$$

To find the particular solution we can write the Eq. as $y''+\frac{3}{x}y'+\frac{1}{x^2}y=\frac{4}{x^3}$

now $y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$

where $u_1(x)=\int^x\frac{-y_2(x')}{W[y_1,y_2]}\frac{f(x')}{a_0(x')}dx'$ and $u_2(x)=\int^x\frac{y_1(x')}{W[y_1,y_2]}\frac{f(x')}{a_0(x')}dx'$

and where $W[y_1,y_2]\equiv$ Wronskian; $y_1(x)=x^{-1}$, $y_2(x)=x^{-1}\ln(x)$, $f(x)=\frac{4}{x^3}$, $a_0(x)=1$

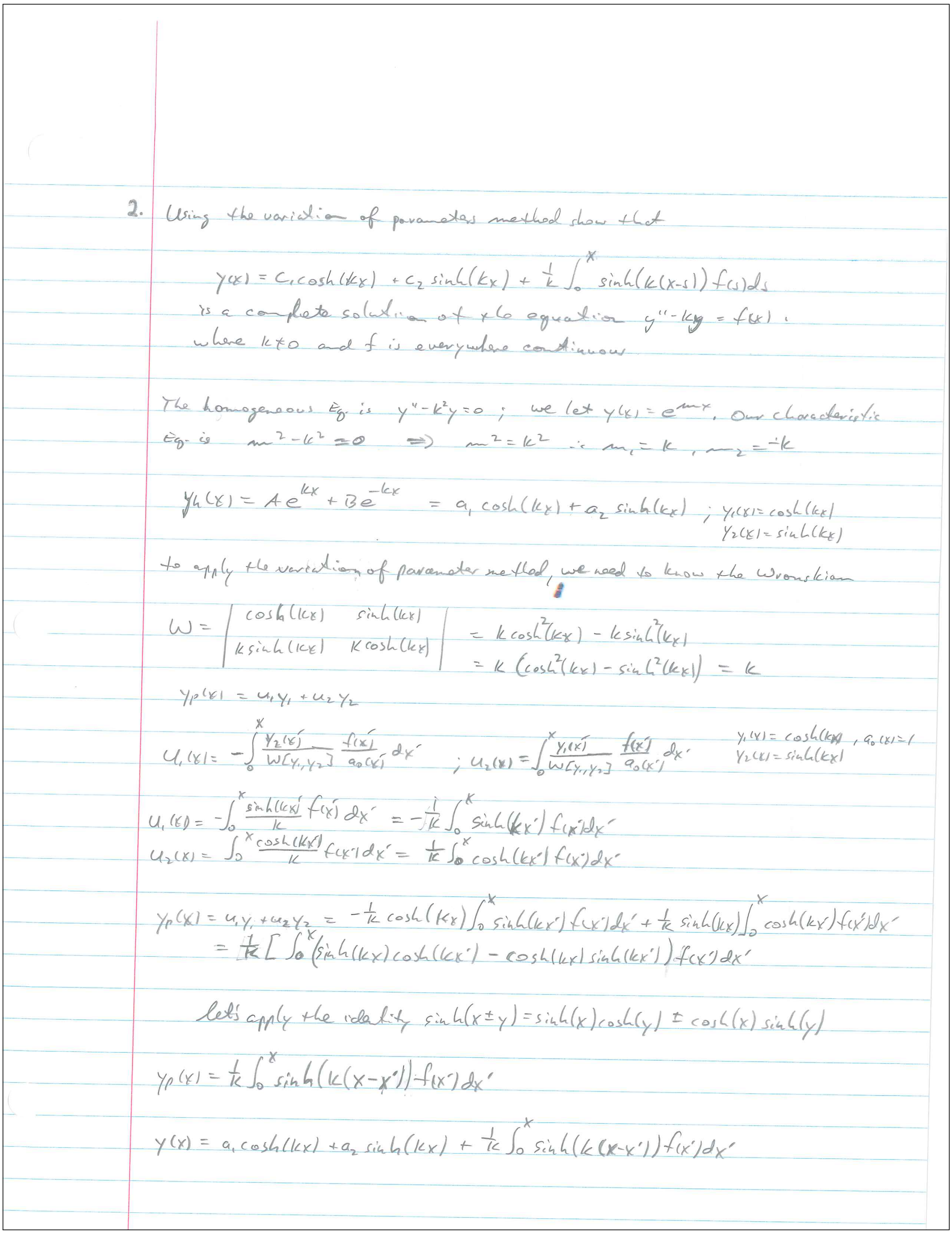
$$W=\begin{vmatrix}x^{-1} & x^{-1}\ln(x)\\ -x^{-2} & -x^{-2}\ln(x)+x^{-2}\end{vmatrix}=-x^{-3}\ln(x)+x^{-3}+x^{-3}\ln(x)=x^{-3}=\frac{1}{x^3}$$

$$u_1(x)=-\int\left(\frac{1}{x^{-3}}\right)\left(x^{-1}\ln(x)\right)\left(\frac{4}{x^3}\right)dx=-4\int\frac{\ln(x)}{x}dx=-2\left(\ln(x)\right)^2$$

$$u_2(x)=\int\left(\frac{1}{x^{-3}}\right)\left(x^{-1}\right)\left(\frac{4}{x^3}\right)dx=4\int\frac{1}{x}dx=4\ln(x)$$

$$y_p(x)=u_1y_1+u_2y_2=-2(\ln(x))^2\left(\frac{1}{x}\right)+4\ln(x)\left(\frac{\ln(x)}{x}\right)$$
$$=-\frac{2}{x}(\ln(x))^2+\frac{4}{x}(\ln(x))^2=\frac{2}{x}(\ln(x))^2$$

$$y(x)=\frac{A}{x}+\frac{B}{x}\ln(x)+\frac{2}{x}(\ln(x))^2$$



2\. Using the variation of parameters method show that

$$y(x) = C_1 \cosh(kx) + C_2 \sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))f(s)ds$$

is a complete solution of the equation $y'' - ky = f(x)$.
where $k \neq 0$ and $f$ is everywhere continuous

The homogeneous Eq. is $y'' - k^2 y = 0$ ; we let $y(x) = e^{mx}$, Our characteristic Eq. is $m^2 - k^2 = 0 \Rightarrow m^2 = k^2 \therefore m_1 = k, \; m_2 = -k$

$$y_h(x) = Ae^{kx} + Be^{-kx} = a_1 \cosh(kx) + a_2 \sinh(kx) \; ; \; \begin{array}{l} y_1(x) = \cosh(kx) \\ y_2(x) = \sinh(kx) \end{array}$$

to apply the variation of parameter method, we need to know the Wronskian

$$W = \begin{vmatrix} \cosh(kx) & \sinh(kx) \\ k\sinh(kx) & k\cosh(kx) \end{vmatrix} = k\cosh^2(kx) - k\sinh^2(kx)$$

$$= k\left(\cosh^2(kx) - \sinh^2(kx)\right) = k$$

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$u_1(x) = -\int_0^x \frac{y_2(x')}{W[y_1,y_2]} \frac{f(x')}{a_0(x')} dx' \; ; \; u_2(x) = \int_0^x \frac{y_1(x')}{W[y_1,y_2]} \frac{f(x')}{a_0(x')} dx' \qquad \begin{array}{l} y_1(x) = \cosh(kx), \; a_0(x) = 1 \\ y_2(x) = \sinh(kx) \end{array}$$

$$u_1(x) = -\int_0^x \frac{\sinh(kx')}{k} f(x')dx' = -\frac{1}{k}\int_0^x \sinh(kx')f(x')dx'$$

$$u_2(x) = \int_0^x \frac{\cosh(kx')}{k} f(x')dx' = \frac{1}{k}\int_0^x \cosh(kx')f(x')dx'$$

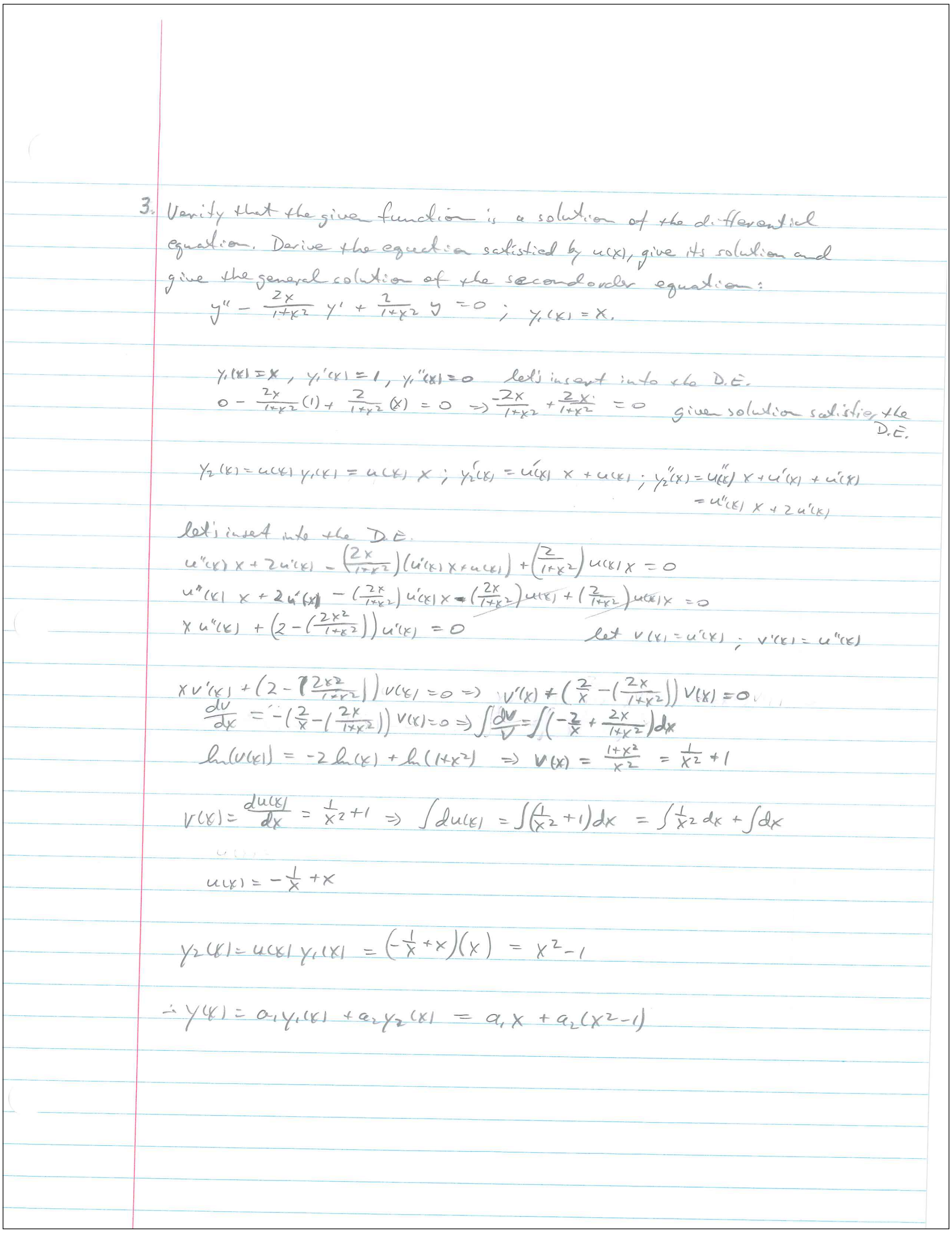
$$y_p(x) = u_1 y_1 + u_2 y_2 = -\frac{1}{k}\cosh(kx)\int_0^x \sinh(kx')f(x')dx' + \frac{1}{k}\sinh(kx)\int_0^x \cosh(kx')f(x')dx'$$

$$= \frac{1}{k}\left[\int_0^x \left(\sinh(kx)\cosh(kx') - \cosh(kx)\sinh(kx')\right)f(x')dx'\right.$$

let's apply the identity $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

$$y_p(x) = \frac{1}{k}\int_0^x \sinh(k(x-x'))f(x')dx'$$

$$y(x) = a_1\cosh(kx) + a_2\sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-x'))f(x')dx'$$



3. Verify that the given function is a solution of the differential equation. Derive the equation satisfied by u(x), give its solution and give the general solution of the second order equation:

$$y'' - \frac{2x}{1+x^2}\,y' + \frac{2}{1+x^2}\,y = 0 \; ; \quad y_1(x) = x.$$

$y_1(x) = x$, $y_1'(x) = 1$, $y_1''(x) = 0$ let's insert into the D.E.

$$0 - \frac{2x}{1+x^2}(1) + \frac{2}{1+x^2}(x) = 0 \Rightarrow \frac{-2x}{1+x^2} + \frac{2x}{1+x^2} = 0$$

given solution satisfies the D.E.

$$y_2(x) = u(x)\,y_1(x) = u(x)\,x \; ; \; y_2'(x) = u'(x)\,x + u(x) \; ; \; y_2''(x) = u''(x)\,x + u'(x) + u'(x)$$
$$= u''(x)\,x + 2u'(x)$$

let's insert into the D.E.

$$u''(x)\,x + 2u'(x) - \left(\frac{2x}{1+x^2}\right)\left(u'(x)\,x + u(x)\right) + \left(\frac{2}{1+x^2}\right)u(x)\,x = 0$$

$$u''(x)\,x + 2u'(x) - \left(\frac{2x}{1+x^2}\right)u'(x)\,x - \cancel{\left(\frac{2x}{1+x^2}\right)u(x)} + \cancel{\left(\frac{2}{1+x^2}\right)u(x)\,x} = 0$$

$$x\,u''(x) + \left(2 - \left(\frac{2x^2}{1+x^2}\right)\right)u'(x) = 0$$

let $v(x) = u'(x)$ ; $v'(x) = u''(x)$

$$x\,v'(x) + \left(2 - \left(\frac{2x^2}{1+x^2}\right)\right)v(x) = 0 \Rightarrow v'(x) + \left(\frac{2}{x} - \left(\frac{2x}{1+x^2}\right)\right)V(x) = 0$$

$$\frac{dv}{dx} = -\left(\frac{2}{x} - \left(\frac{2x}{1+x^2}\right)\right)v(x) = 0 \Rightarrow \int \frac{dv}{v} = \int\left(-\frac{2}{x} + \frac{2x}{1+x^2}\right)dx$$

$$\ln(v(x)) = -2\ln(x) + \ln(1+x^2) \Rightarrow V(x) = \frac{1+x^2}{x^2} = \frac{1}{x^2} + 1$$

$$v(x) = \frac{du(x)}{dx} = \frac{1}{x^2} + 1 \Rightarrow \int du(x) = \int\left(\frac{1}{x^2} + 1\right)dx = \int \frac{1}{x^2}\,dx + \int dx$$

$$u(x) = -\frac{1}{x} + x$$

$$y_2(x) = u(x)\,y_1(x) = \left(-\frac{1}{x} + x\right)(x) = x^2 - 1$$

$$\therefore y(x) = a_1 y_1(x) + a_2 y_2(x) = a_1 x + a_2(x^2 - 1)$$

4). Use the one solution indicated and find the complete solution.

$$(2x-x^2)y'' + 2(x-1)y' - 2y = 0 \; ; \; y_1(x) = x-1$$

$$y_2(x) = u(x)y_1(x) = u(x)(x-1), \quad y_2'(x) = u'(x)(x-1) + u(x)$$

$$y_2''(x) = u''(x)(x-1) + u'(x) + u'(x)$$

$$= u''(x)(x-1) + 2u'(x)$$

$$(2x-x^2)(u''(x)(x-1) + 2u'(x)) + 2(x-1)(u'(x-1) + u(x)) - 2(u(x)(x-1)) = 0$$

$$(2x-x^2)(x-1)u''(x) + 2(2x-x^2)u'(x) + 2(x-1)^2u'(x) + \cancel{2(x-1)u(x)} - \cancel{2u(x)(x-1)} = 0$$

$$(2x-x^2)(x-1)u''(x) + (2(2x-x^2) + 2(x-1)^2)u'(x) = 0$$

$$(2x-x^2)(x-1)u''(x) + 2[(2x-x^2) + (x-1)^2]u'(x) = 0 \qquad \text{let } v(x) = u'(x)$$

$$v'(x) = u''(x)$$

$$(2x-x^2)(x-1)v'(x) + 2[(2x-x^2) + (x-1)^2]v(x) = 0$$

$$v'(x) = -\frac{2[(2x-x^2) + (x-1)^2]}{(2x-x^2)(x-1)}v(x) \Rightarrow \frac{dv(x)}{dx} = -\frac{2[(2x-x^2) + (x-1)^2]}{(2x^2-x^2)(x-1)}v(x)$$

$$\frac{dv(x)}{v(x)} = -\frac{2[(2x-x^2) + (x-1)^2]}{(2x-x^2)(x-1)}dx = \left[-2\left(\frac{1}{x-1}\right) - 2\left(\frac{x-1}{2x-x^2}\right)\right]dx$$

$$\frac{dv(x)}{v(x)} = -2\int \frac{1}{x-1}dx + \int \frac{2x-2}{x^2-2x}dx$$

$$\ln(v(x)) = -2\ln(x-1) + \ln(x^2-2x) \Rightarrow v(x) = (x-1)^{-2}(x^2-2x)$$

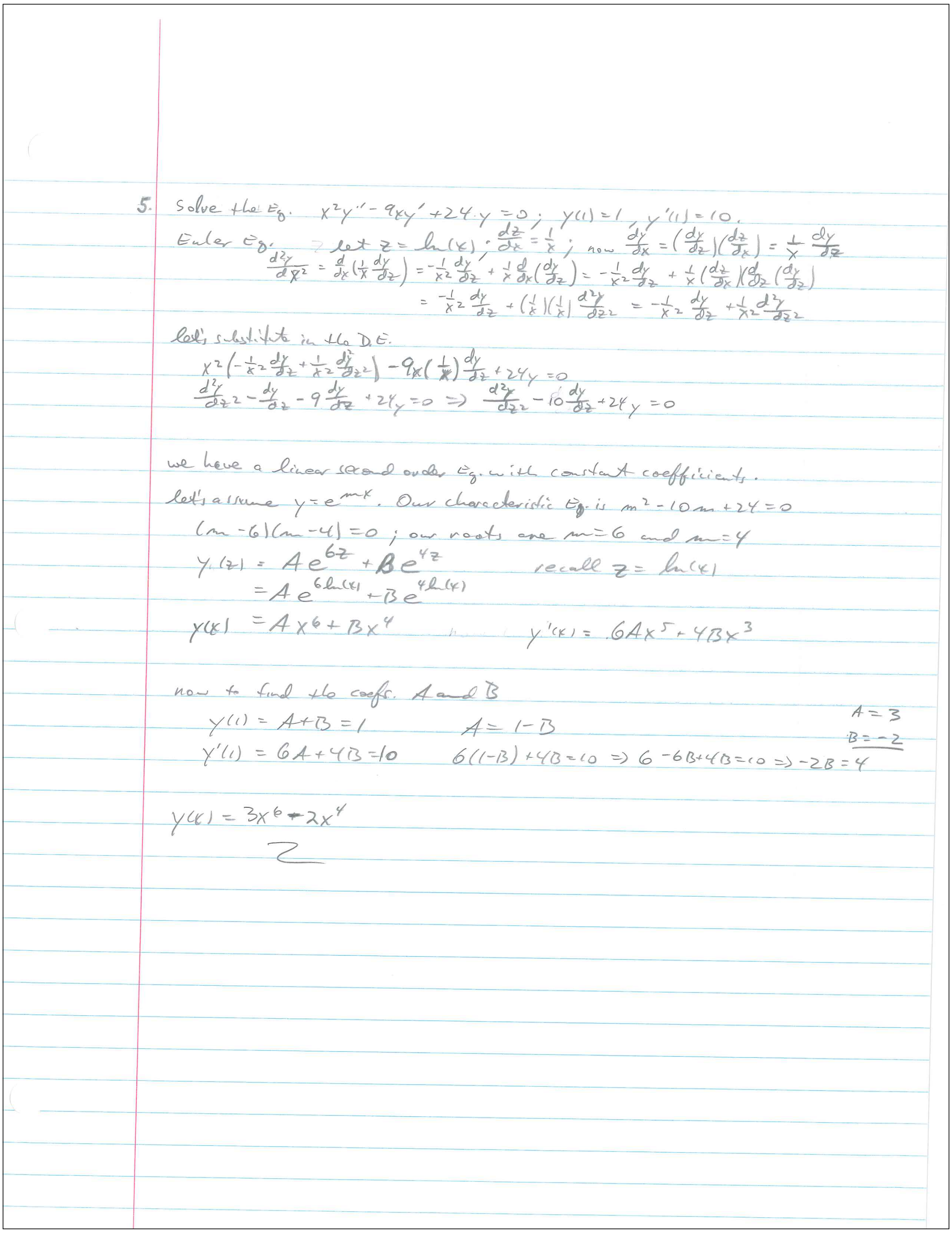
$$v(x) = \frac{du(x)}{dx} = \frac{x^2-2x}{(x-1)^2} \Rightarrow \int du(x) = \int \frac{x^2-2x}{(x-1)^2}dx$$

$$u(x) = \int \frac{x^2-2x+1}{(x-1)^2}dx - \int \frac{1}{(x-1)^2}dx = \int \frac{(x-1)^2}{(x-1)^2}dx - \int \frac{1}{(x-1)^2}dx$$

$$= \int dx - \int \frac{1}{(x-1)^2}dx = x + \frac{1}{(x-1)}$$

$$y_2(x) = u(x)y_1(x) = \left(x + \frac{1}{(x-1)}\right)(x-1) = x^2 - x + 1$$

$$\therefore y(x) = a_1y_1(x) + a_2y_2(x) = \underline{a_1(x-1) + a_2(x^2-x+1)}$$



5. Solve the Eq. $x^2y'' - 9xy' + 24 y = 0$; $y(1) = 1$, $y'(1) = 10$.

Euler Eq. $\Rightarrow$ let $z = \ln(x)$; $\frac{dz}{dx} = \frac{1}{x}$; now $\frac{dy}{dx} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right) = \frac{1}{x}\frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{dz}{dx}\right)\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\frac{d^2y}{dz^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

let's substitute in the D.E.

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) - 9x\left(\frac{1}{x}\right)\frac{dy}{dz} + 24y = 0$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 9\frac{dy}{dz} + 24y = 0 \Rightarrow \frac{d^2y}{dz^2} - 10\frac{dy}{dz} + 24y = 0$$

we have a linear second order Eq. with constant coefficients.

let's assume $y = e^{mx}$. Our characteristic Eq. is $m^2 - 10m + 24 = 0$

$(m-6)(m-4) = 0$ ; our roots are $m = 6$ and $m = 4$

$y(z) = Ae^{6z} + Be^{4z}$ recall $z = \ln(x)$

$= Ae^{6\ln(x)} + Be^{4\ln(x)}$

$y(x) = Ax^6 + Bx^4$ $y'(x) = 6Ax^5 + 4Bx^3$

now to find the coefs. A and B

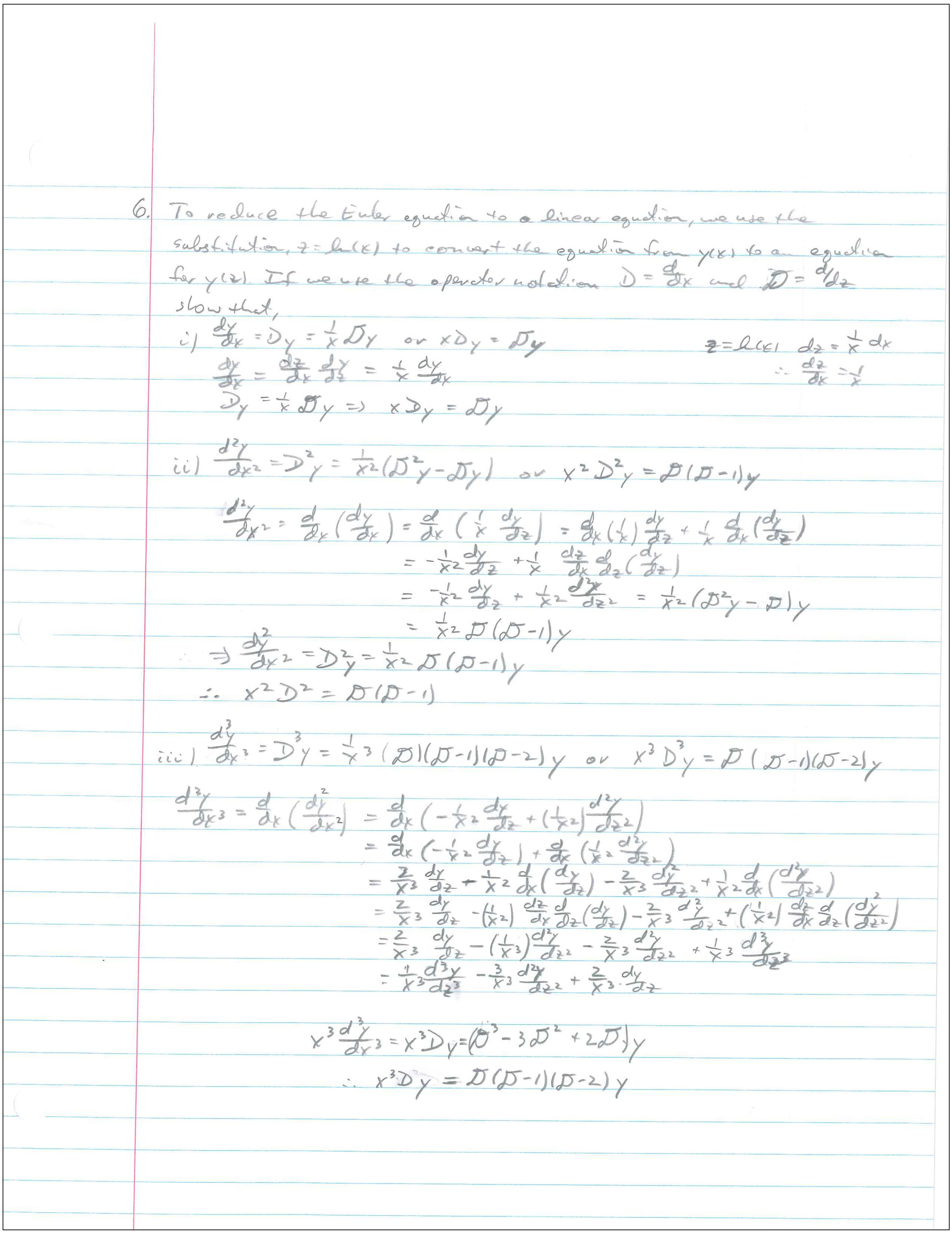
$y(1) = A + B = 1$ $A = 1 - B$

$y'(1) = 6A + 4B = 10$ $6(1-B) + 4B = 10 \Rightarrow 6 - 6B + 4B = 10 \Rightarrow -2B = 4$

$A = 3$

$B = -2$

$y(x) = 3x^6 + 2x^4$



6. To reduce the Euler equation to a linear equation, we use the substitution, $z = \ln(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation $D = \frac{d}{dx}$ and $\mathcal{D} = d/dz$ show that,

i) $\frac{dy}{dx} = Dy = \frac{1}{x}\mathcal{D}y$ or $xDy = \mathcal{D}y$

$z = \ln(x)$ $dz = \frac{1}{x}dx$

$\therefore \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dx}$

$Dy = \frac{1}{x}\mathcal{D}y \Rightarrow xDy = \mathcal{D}y$

ii) $\frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}(\mathcal{D}^2y - \mathcal{D}y)$ or $x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{dz}{dx}\frac{d}{dz}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}(\mathcal{D}^2y - \mathcal{D})y$$

$$= \frac{1}{x^2}\mathcal{D}(\mathcal{D}-1)y$$

$\Rightarrow \frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}\mathcal{D}(\mathcal{D}-1)y$

$\therefore x^2D^2 = \mathcal{D}(\mathcal{D}-1)$

iii) $\frac{d^3y}{dx^3} = D^3y = \frac{1}{x^3}(\mathcal{D})(\mathcal{D}-1)(\mathcal{D}-2)y$ or $x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz} + \left(\frac{1}{x^2}\right)\frac{d^2y}{dz^2}\right)$$

$$= \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz}\right) + \frac{d}{dx}\left(\frac{1}{x^2}\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} + \frac{1}{x^2}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \left(\frac{1}{x^2}\right)\frac{dz}{dx}\frac{d}{dz}\left(\frac{dy}{dz}\right) - \frac{2}{x^3}\frac{d^2y}{dz^2} + \left(\frac{1}{x^2}\right)\frac{dz}{dx}\frac{d}{dz}\left(\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \left(\frac{1}{x^3}\right)\frac{d^2y}{dz^2} - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^3}\frac{d^3y}{dz^3}$$

$$= \frac{1}{x^3}\frac{d^3y}{dz^3} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{2}{x^3}\frac{dy}{dz}$$

$$x^3\frac{d^3y}{dx^3} = x^3Dy = (\mathcal{D}^3 - 3\mathcal{D}^2 + 2\mathcal{D})y$$

$$\therefore x^3Dy = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

7. Find the complete solution of the equation

$$x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$$

let $z = \ln(x)$

$e^z = x$

using the results from problem 6

$$[D(D-1)(D-2) + 4D(D-1) - 5D - 15]y = e^{4z}$$
$$[D(D^2-3D+2) + 4D^2 - 4D - 5D - 15]y = e^{4z}$$
$$[D^3 - 3D^2 + 2D + 4D^2 - 9D - 15]y = e^{4z}$$
$$[D^3 + D^2 - 7D - 15]y = e^{4z}$$

homogeneous Eq.: $[D^3 + D^2 - 7D - 15]y = 0$ let $y(z) = e^{mz}$

characteristic Eq: $m^3 + m^2 - 7m - 15 = 0$

$$(m-3)(m^2+4m+5) = 0 \Rightarrow (m-3)(m+2-i)(m+2+i) = 0$$

$m = 3$, $m = -2-i$, $m = -2+i$

$$y_h(z) = c_1e^{3z} + c_2e^{(-2-i)z} + c_3e^{(-2+i)z}$$
$$= C_1e^{3z} + e^{-2z}(C_2e^{-iz} + C_3e^{iz})$$
$$= C_1e^{3z} + e^{-2z}(C_4\cos(z) + C_5\sin(z))$$

$y_p(z) = Ae^{4z}$, $y_p'(z) = 4Ae^{4z}$, $y_p''(z) = 16Ae^{4z}$, $y_p'''(x) = 64Ae^{4z}$

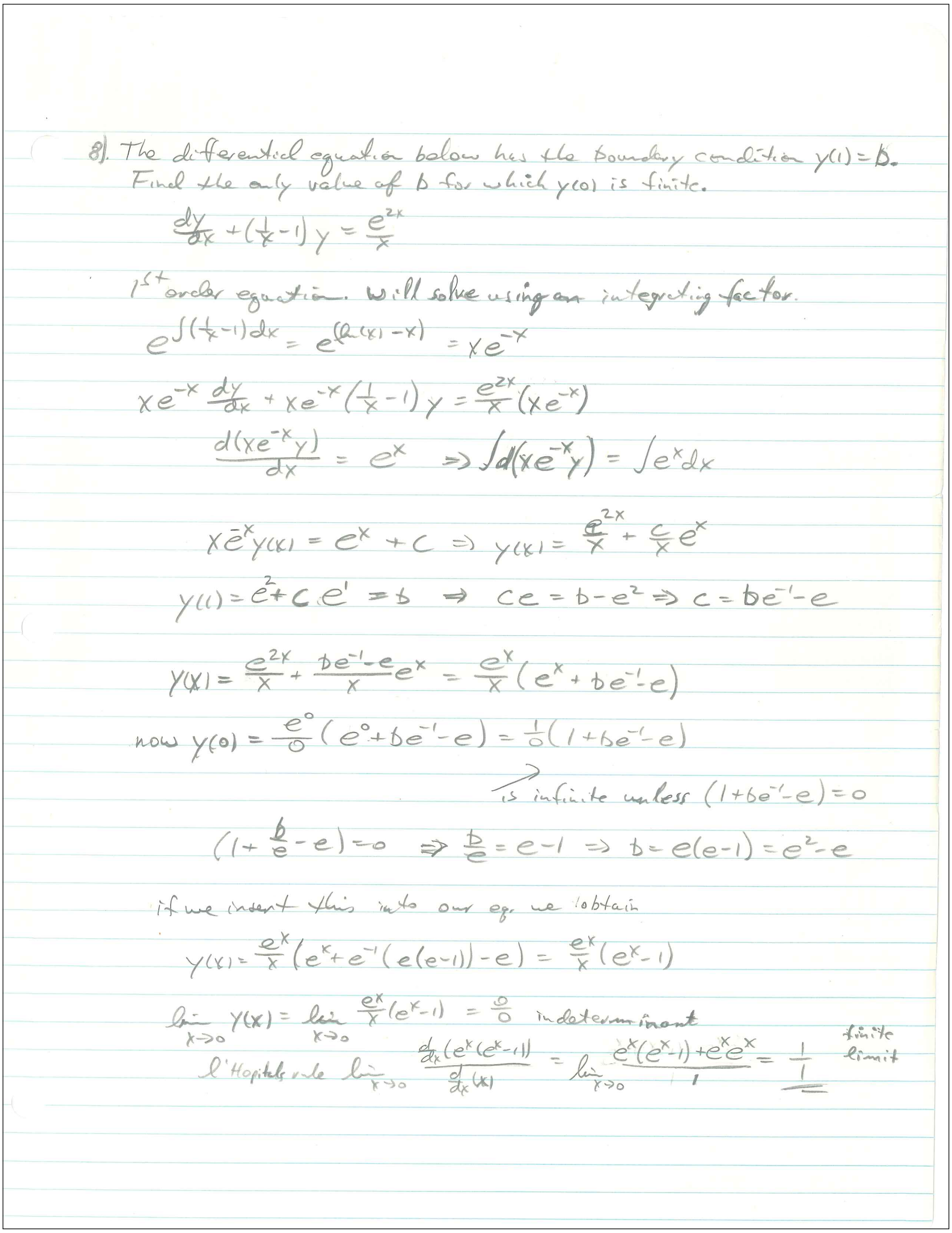
$$64Ae^{4z} + 16Ae^{4z} - 7(4Ae^{4z}) - 15Ae^{4z} = e^{4z}$$
$$64A + 16A - 28A - 15A = 1 \Rightarrow 80A - 43A = 1 \Rightarrow A = \frac{1}{37}$$

$$y(z) = y_h(z) + y_p(z) = C_1e^{3z} + e^{-2z}(C_4\cos(z) + C_5\sin(z)) + \frac{1}{37}e^{4z}$$

recall $z = \ln(x)$

$$y(x) = C_1e^{3\ln(x)} + e^{-2\ln(x)}(C_4\cos(\ln(x)) + C_5\sin(\ln(x))) + \frac{1}{37}e^{4\ln(x)}$$

$$\Rightarrow y(x) = C_1x^3 + x^{-2}(C_4\cos(\ln(x)) + C_5\sin(\ln(x)) + \frac{1}{37}x^4$$



8). The differential equation below has the boundary condition $y(1)=b$. Find the only value of $b$ for which $y(0)$ is finite.

$$\frac{dy}{dx} + \left(\frac{1}{x}-1\right)y = \frac{e^{2x}}{x}$$

1st order equation. will solve using an integrating factor.

$$e^{\int(\frac{1}{x}-1)dx} = e^{(\ln(x)-x)} = xe^{-x}$$

$$xe^{-x}\frac{dy}{dx} + xe^{-x}\left(\frac{1}{x}-1\right)y = \frac{e^{2x}}{x}(xe^{-x})$$

$$\frac{d(xe^{-x}y)}{dx} = e^x \Rightarrow \int d(xe^{-x}y) = \int e^x dx$$

$$xe^{-x}y(x) = e^x + c \Rightarrow y(x) = \frac{e^{2x}}{x} + \frac{c}{x}e^x$$

$$y(1) = e^2 + ce^1 = b \Rightarrow ce = b - e^2 \Rightarrow c = be^{-1} - e$$

$$y(x) = \frac{e^{2x}}{x} + \frac{be^{-1}-e}{x}e^x = \frac{e^x}{x}(e^x + be^{-1} - e)$$

now $y(0) = \frac{e^0}{0}(e^0 + be^{-1} - e) = \frac{1}{0}(1 + be^{-1} - e)$

$\nearrow$ is infinite unless $(1+be^{-1}-e)=0$

$$\left(1+\frac{b}{e}-e\right)=0 \Rightarrow \frac{b}{e} = e-1 \Rightarrow b = e(e-1) = e^2-e$$

if we insert this into our eq. we obtain

$$y(x) = \frac{e^x}{x}(e^x + e^{-1}(e(e-1)) - e) = \frac{e^x}{x}(e^x-1)$$

$$\lim_{x\to 0} y(x) = \lim_{x\to 0}\frac{e^x}{x}(e^x-1) = \frac{0}{0} \text{ indeterminant}$$

l'Hopitals rule $\lim_{x\to 0}\frac{\frac{d}{dx}(e^x(e^x-1))}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{e^x(e^x-1)+e^xe^x}{1} = \underline{\underline{\frac{1}{1}}}$ finite limit

## 2.5 HW 4

### 2.5.1 Problems to solve

Homework Set No. 4 NEEP 547
Due October 4, 2013 DLH

1. (5pts) Use the Laplace transform to solve the following problem:
   $y'' - 4\,y' + 4\,y = t^3 e^{2t}$ with initial conditions; $y(0) = 0$ and $y'(0) = 0$.
2. (6pts) Use the Laplace transform to solve the following problem:
   $y'' - 4\,y' + 3\,y = 1 - H(t-2) - H(t-4) + H(t-6)$, with initial conditions; $y(0) = 0$ and $y'(0) = 0$.
3. (6pts) Use the Laplace transform to solve the following problem:
   $y'' - 4\,y' + 13\,y = \delta(t-\pi) + \delta(t-3\pi)$ with initial conditions; $y(0) = 1$ and $y'(0) = 0$.
4. (6pts) Solve the boundary valued problem using the Laplace Transform:
   $$t\,\frac{d^2y}{dt^2} - (t+3)\,\frac{dy}{dt} + 4y = t - 1 \qquad \text{where: } y(0) = y(1) = 0.$$
5. (7pts) Solve the following system of equations for the unknown functions, $y(t)$ and $z(t)$:
   $$3\,y' + 8\,y + 2\,z' + 5\,z = e^{-t}$$
   $$y' + z' + z = 0$$
   where the initial conditions are $y(0) = 2$ and $z(0) = -2$.
6. (8pts) page 148, prob. 12: Solve for the currents in the circuit of Figure 3.37, assuming that the currents and charges are initially zero and that $E(t) = 2\,H(t-4) - H(t-5)$.
7. (8pts) page 149, prob. 15: Solve for the displacement functions in the system of Figure 3.38 if $f_1(t) = 1 - H(t-2)$ and $f_2(t) = 0$. Assume zero initial displacements and velocities.

### 2.5.2 Problem 1

Use the Laplace transform to solve $y'' - 4y' + 4y = t^3 e^{2t}$ with IC $y(0) = 0$ and $y'(0) = 0$

**Solution:**

Let $Y(s) = \left(y(t)\right)$. Taking Laplace transform of the above ODE gives

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) - 4\left(sY(s) - y(0)\right) + 4Y(s) = \mathscr{L}\left(t^3 e^{2t}\right) \tag{1}$$

But $\mathscr{L}\left(t^n e^{at}\right) = (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}\left(e^{at}\right)\right)$, where $\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a}$, therefore

$$\begin{aligned}\mathscr{L}\left(t^3 e^{2t}\right) &= (-1)^3 \frac{d^3}{ds^3}\left(\mathscr{L}\left(e^{2t}\right)\right)\\ &= (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s-2}\right)\\ &= \frac{6}{(s-2)^4}\end{aligned}$$

Eq. (1) becomes

$$\begin{aligned}
s^2Y(s) - 4sY(s) + 4Y(s) &= \frac{6}{(s-2)^4}\\
Y(s)\left(s^2 - 4s + 4\right) &= \frac{6}{(s-2)^4}\\
Y(s) &= \frac{6}{(s-2)^4\left(s^2-4s+4\right)}\\
&= \frac{6}{(s-2)^4(s-2)^2}\\
&= \frac{6}{(s-2)^6}
\end{aligned}$$

Using the property $\mathscr{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = e^{at}\mathscr{L}^{-1}\left(\frac{1}{s^n}\right)$, the above reduces to

$$y(t) = 6e^{2t}\mathscr{L}^{-1}\left(\frac{1}{s^6}\right)$$

Since $\mathscr{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$, hence $\left(\frac{1}{s^6}\right) = \frac{t^5}{120}$, and the above becomes

$$y(t) = \frac{1}{20}t^5e^{2t}$$

### 2.5.3 Problem 2

Use Laplace transform to solve

$y'' - 4y' + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$ with IC $y(0) = 0$ and $y'(0) = 0$

**solution**:

Let $Y(s) = \mathscr{L}\big(y(t)\big)$. Taking Laplace transform of the above ODE gives

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - 4\left(sY(s) - y(0)\right) + 3Y(s) = \mathscr{L}(1 - H(t-2) - H(t-4) + H(t-6)) \quad (1)$$

But $\mathscr{L}(1) = \frac{1}{s}$ and $\mathscr{L}(H(t-a)) = \frac{e^{-as}}{s}$, hence the above becomes

$$\begin{aligned}
s^2Y(s) - 4sY(s) + 3Y(s) &= \frac{1}{s} - \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-6s}}{s}\\
Y(s) &= \frac{1 - e^{-2s} - e^{-4s} + e^{-6s}}{s\left(s^2 - 4s + 3\right)}\\
&= \frac{1 - e^{-2s} - e^{-4s} + e^{-6s}}{s(s-3)(s-1)}
\end{aligned}$$

The inverse Laplace transform of $\frac{1}{s(s-3)(s-1)}$ is found first. Let the result be $f(t)$. Then the relation $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$ is used to obtain the final answer. Using partial fractions. Let

$$\frac{1}{s(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

Then $A = \lim_{s\to 0}\frac{1}{(s-3)(s-1)} = \frac{1}{3}$ and $B = \lim_{s\to 3}\frac{1}{s(s-1)} = \frac{1}{6}$ and $C = \lim_{s\to 1}\frac{1}{s(s-3)} = -\frac{1}{2}$, hence

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)(s-1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{3}\frac{1}{s} + \frac{1}{6}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-1}\right\}\\
&= \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t}
\end{aligned}$$

The above is $f(t)$. Therefore, using $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(1 - e^{-2s} - e^{-4s} + e^{-6s}\right)F(s)\\
&= f(t) - f(t-2)H(t-2) - f(t-4)H(t-4) + f(t-6)H(t-6)
\end{aligned}$$

Hence the answer is

$$\begin{aligned} y(t) =& \left(\frac{1}{3}+\frac{1}{6}e^{3t}-\frac{1}{2}e^{t}\right)-\left(\frac{1}{3}+\frac{1}{6}e^{3(t-2)}-\frac{1}{2}e^{(t-2)}\right)H(t-2)\\ &-\left(\frac{1}{3}+\frac{1}{6}e^{3(t-4)}-\frac{1}{2}e^{(t-4)}\right)H(t-4)+\left(\frac{1}{3}+\frac{1}{6}e^{3(t-6)}-\frac{1}{2}e^{(t-6)}\right)H(t-6)\end{aligned}$$

### 2.5.4 Problem 3

Use Laplace transform to solve $y''-4y'+13y=\delta(t-\pi)+\delta(t-3\pi)$ with IC $y(0)=1, y'(0)=0$

**Solution:**

The property $\mathscr{L}(\delta(t-a))=e^{-as}$ will be used. Taking the Laplace transform of the ODE gives

$$\left(s^2Y(s)-sy(0)-y'(0)\right)-4\left(sY(s)-y(0)\right)+13Y(s)=e^{-\pi s}+e^{-3\pi s} \tag{1}$$

Hence

$$\begin{aligned}\left(s^2Y(s)-s\right)-4(sY(s)-1)+13Y(s)&=e^{-\pi s}+e^{-3\pi s}\\ \left[s^2Y(s)-4sY(s)+13Y(s)\right]+(-s+4)&=e^{-\pi s}+e^{-3\pi s}\\ Y(s)\left(s^2-4s+13\right)&=e^{-\pi s}+e^{-3\pi s}+(s-4)\\ Y(s)&=\frac{e^{-\pi s}+e^{-3\pi s}+(s-4)}{s^2-4s+13}\\ &=\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}+\frac{(s-4)}{s^2-4s+13}\\ &=\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}+\frac{(s-2)-2}{(s-2)^2+9}\end{aligned}$$

To find the inverse Laplace of $\frac{(s-2)-2}{(s-2)^2+9}$ the following property is used $\mathscr{L}^{-1}\left(\frac{(s-a)}{(s-a)^n+b}\right)=e^{at}\mathscr{L}^{-1}\left(\frac{s}{s^n+b}\right)$, Hence

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{(s-2)-2}{(s-2)^2+9}\right)&=e^{2t}\mathscr{L}^{-1}\left(\frac{s-2}{s^2+9}\right)\\ &=e^{2t}\left[\mathscr{L}^{-1}\left(\frac{s}{s^2+9}\right)-2\mathscr{L}^{-1}\left(\frac{1}{s^2+9}\right)\right]\\ &=e^{2t}\left[\mathscr{L}^{-1}\left(\frac{s}{s^2+3^2}\right)-\frac{2}{3}\mathscr{L}^{-1}\left(\frac{3}{s^2+3^2}\right)\right]\\ &=e^{2t}\left(\cos 3t-\frac{2}{3}\sin 3t\right)\\ &=\frac{1}{3}e^{2t}(3\cos 3t-2\sin 3t)\end{aligned} \tag{2}$$

Now the inverse Laplace transform of $\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}$ is found. Writing this as

$$\begin{aligned}\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}&=\frac{e^{-\pi s}}{s^2-4s+13}+\frac{e^{-3\pi s}}{s^2-4s+13}\\ &=\frac{e^{-\pi s}}{(s-2)^2+9}+\frac{e^{-3\pi s}}{(s-2)^2+9}\end{aligned}$$

To be able to use the property $\mathscr{L}^{-1}\left(\frac{F(s-a)}{(s-a)^n+b}\right)=e^{at}\mathscr{L}^{-1}\left(\frac{F(s)}{s^n+b}\right)$ the terms in the numerator are converted as follows

$$\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}=e^{-2\pi}\frac{e^{-\pi(s-2)}}{(s-2)^2+9}+e^{-6\pi}\frac{e^{-3\pi(s-2)}}{(s-2)^2+9}$$

Now the property can be used, hence

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}\right)&=e^{-2\pi}e^{2t}\mathscr{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+9}\right)+e^{-6\pi}e^{2t}\mathscr{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right)\\ &=e^{2t}\left(e^{-2\pi}\mathscr{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+9}\right)+e^{-6\pi}\mathscr{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right)\right)\end{aligned}$$

Now another property is used to find $\mathscr{L}^{-1}$ of the remaining terms. This is $\mathscr{L}^{-1}\left(e^{-as}F(s)\right)=$

$f(t-a)H(t-a)$. The above becomes

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{e^{-\pi s}+e^{-3\pi s}}{s^2-4s+13}\right) &= e^{2t}\left(e^{-2\pi}\mathscr{L}^{-1}\left(\frac{1}{s^2+3^2}\right)H(t-\pi)+e^{-6\pi}\mathscr{L}^{-1}\left(\frac{1}{s^2+3^2}\right)H(t-3\pi)\right)\\ &= e^{2t}\left(\frac{1}{3}e^{-2\pi}\mathscr{L}^{-1}\left(\frac{3}{s^2+3^2}\right)H(t-\pi)+\frac{1}{3}e^{-6\pi}\mathscr{L}^{-1}\left(\frac{3}{s^2+3^2}\right)H(t-3\pi)\right)\\ &= e^{2t}\left(\frac{1}{3}e^{-2\pi}\sin(3(t-\pi))H(t-\pi)+\frac{1}{3}e^{-6\pi}\sin(3(t-3\pi))H(t-3\pi)\right) \qquad (3)\end{aligned}$$

The full solution is now found, combining Eq. (2) and Eq. (3) gives

$$y(t) = -\frac{1}{3}e^{2t}(3\cos 3t-2\sin 3t)+e^{2t}\left(\frac{1}{3}e^{-2\pi}\sin(3(t-\pi))H(t-\pi)+\frac{1}{3}e^{-6\pi}\sin(3(t-3\pi))H(t-3\pi)\right)$$

Taking common factors out gives

$$y(t) = -\frac{1}{3}e^{2t}\left(2\sin(3t)-3\cos(3t)+e^{-2\pi}\sin(3(t-\pi))H(t-\pi)+e^{-6\pi}\sin(3(t-3\pi))H(t-3\pi)\right)$$

The above can be reduced more. Since $\sin(n(t-\pi)) = -1^n \sin nt$ for integer $n$, hence the above simplifies to

$$y(t) = -\frac{1}{3}e^{2t}\left(2\sin(3t)-3\cos(3t)-e^{-2\pi}\sin(3t)H(t-\pi)-e^{-6\pi}\sin(3t)H(t-3\pi)\right)$$

For $t \geq 0$.

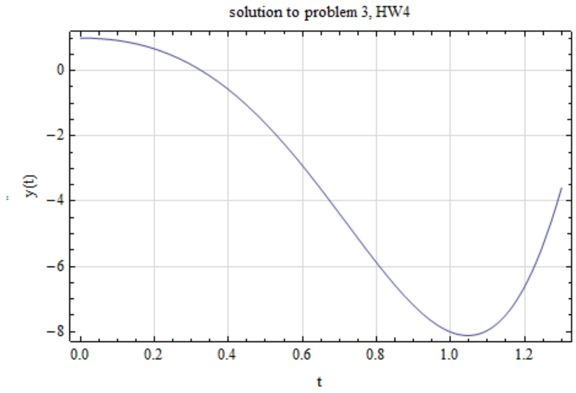


Figure 2.6: plot of solution problem 3 HW 4

### 2.5.5 Problem 4

Solve the boundary valued problem using Laplace transform

$ty'' - (t+3)y' + 4y = t - 1$ where $y(0) = y(1) = 0$

**Solution**:

The following property will be used $\mathscr{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}\left(f(t)\right)\right)$. Taking Laplace transform of the above gives

$$\begin{aligned}
-\frac{d}{ds}\left(s^2Y(s)-sy(0)-y'(0)\right)+\frac{d}{ds}\left(sY(s)-y(0)\right)-3\left(sY(s)-y(0)\right)+4Y(s)&=\frac{1}{s^2}-\frac{1}{s}\\
-\frac{d}{ds}\left(s^2Y(s)-y'(0)\right)+\frac{d}{ds}sY(s)-3sY(s)+4Y(s)&=\frac{1}{s^2}-\frac{1}{s}\\
-\left(2sY(s)+s^2Y'(s)\right)+\left(Y(s)+sY'(s)\right)-3sY(s)+4Y(s)&=\frac{1}{s^2}-\frac{1}{s}\\
-2sY(s)-s^2Y'(s)+Y(s)+sY'(s)-3sY(s)+4Y(s)&=\frac{1}{s^2}-\frac{1}{s}\\
Y'(s)\left(-s^2+s\right)-5sY(s)+5Y(s)&=\frac{1}{s^2}-\frac{1}{s}\\
Y'(s)-\frac{5s}{\left(-s^2+s\right)}Y(s)+\frac{5}{\left(-s^2+s\right)}Y(s)&=\frac{\frac{1}{s^2}-\frac{1}{s}}{\left(-s^2+s\right)}\\
Y'(s)+\frac{5s}{\left(s^2-s\right)}Y(s)-\frac{5}{\left(s^2-s\right)}Y(s)&=\frac{1-s}{s^2\left(-s^2+s\right)}\\
Y'(s)+\frac{5}{s}Y(s)&=\frac{1}{s^3}
\end{aligned}$$

Integrating factor is $\ln\left(I_f\right)=\int\frac{5}{s}ds$, hence $I_f=e^{5\int\frac{1}{s}ds}=e^{5\ln s}=s^5$, therefore

$$\begin{aligned}
d\left(I_fY\right)&=s^5\frac{1}{s^3}\\
I_fY&=\int s^2ds+c\\
&=\frac{s^3}{3}+c
\end{aligned}$$

Then

$$Y(s)=\frac{1}{3s^2}+\frac{c}{s^5}$$

Using the property $\mathscr{L}^{-1}\left(\frac{n!}{s^{n+1}}\right)=t^n$ for $n=1,2,3,\cdots$ then $\mathscr{L}^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!}$, therefore taking the inverse Laplace transform of the above gives

$$\begin{aligned}
y(t)&=\frac{1}{3}t+c\frac{t^4}{4!}\\
&=\frac{1}{3}t+c\frac{t^4}{24}
\end{aligned}$$

To find the constant $c$, from the second boundary condition

$$\begin{aligned}
y(1)&=0\\
\frac{1}{3}+c\frac{1}{24}&=0\\
c&=-\frac{24}{3}=-8
\end{aligned}$$

Hence

$$y(t)=\frac{1}{3}t\left(1-t^3\right)$$

### 2.5.6 Problem 5

Solve the following system of equations for $y(t)$ and $z(t)$

$$\begin{aligned}
3y'+8y+2z'+5z&=e^{-t}\\
y'+z'+z&=0
\end{aligned}$$

Where the IC are $y(0)=2$ and $z(0)=-2$

**Solution**:

Taking Laplace transform of the system of equation gives

$$\begin{aligned} 3\left(sY(s)-y(0)\right)+8Y(s)+2\left(sZ(s)-z(0)\right)+5Z(s) &= \frac{1}{s+1} \\ \left(sY(s)-y(0)\right)+\left(sZ(s)-z(0)\right)+Z(s) &= 0 \end{aligned}$$

Applying IC gives

$$\begin{aligned} 3\left(sY(s)-2\right)+8Y(s)+2\left(sZ(s)+2\right)+5Z(s) &= \frac{1}{s+1} \\ \left(sY(s)-2\right)+\left(sZ(s)+2\right)+Z(s) &= 0 \end{aligned}$$

Simplifying in order to solve for $Y(s)$ and $Z(s)$

$$\begin{aligned} Y(s)(3s+8)+Z(s)(2s+5) &= \frac{1}{s+1}+2 \qquad (1) \\ sY(s)+Z(s)(1+s) &= 0 \end{aligned}$$

From the second Eq.

$$Y(s) = -\frac{Z(s)(1+s)}{s}$$

Substituting this in the first equation above gives

$$-\frac{Z(s)(1+s)}{s}(3s+8)+Z(s)(2s+5) = \frac{1}{s+1}+2$$

Simplifying

$$\begin{aligned} -Z(s)(1+s)(3s+8)+Z(s)s(2s+5) &= \frac{s}{s+1}+2s \\ Z(s)\left[-(1+s)(3s+8)+s(2s+5)\right] &= \frac{s+2s(s+1)}{s+1} \\ Z(s)\left[-s^2-6s-8\right] &= \frac{s+2s(s+1)}{s+1} \\ Z(s) &= -\frac{s(2s+3)}{(s+1)\left(s^2+6s+8\right)} \\ &= -\frac{s(2s+3)}{(s+1)(s+4)(s+2)} \end{aligned}$$

Using Partial fractions, let $\frac{s(2s+3)}{(s+1)(s+4)(s+2)} = \frac{A}{(s+1)}+\frac{B}{(s+4)}+\frac{C}{s+2}$, hence

$$\begin{aligned} A &= \lim_{s\to-1}\frac{s(2s+3)}{(s+4)(s+2)} = \frac{-1(-2+3)}{(-1+4)(-1+2)} = -\frac{1}{3} \\ B &= \lim_{s\to-4}\frac{s(2s+3)}{(s+1)(s+2)} = \frac{-4(-8+3)}{(-4+1)(-4+2)} = \frac{10}{3} \\ C &= \lim_{s\to-2}\frac{s(2s+3)}{(s+1)(s+4)} = \frac{-2(-4+3)}{(-2+1)(-2+4)} = -1 \end{aligned}$$

Or

$$\begin{aligned} Z(s) &= -\left(\frac{A}{(s+1)}+\frac{B}{(s+4)}+\frac{C}{s+2}\right) \\ &= \frac{1}{3}\frac{1}{(s+1)}-\frac{10}{3}\frac{1}{(s+4)}+\frac{1}{s+2} \qquad (2) \end{aligned}$$

Hence

$$z(t) = \frac{1}{3}e^{-t}-\frac{10}{3}e^{-4t}+e^{-2t}$$

Now $Y(s)$ will be found and solved for. Using the value for $Z(s)$ from Eq. (2) and substituting this in Eq. (1) gives

$$\begin{aligned} Y(s)(3s+8)+\left(\frac{1}{3}\frac{1}{(s+1)}-\frac{10}{3}\frac{1}{(s+4)}+\frac{1}{s+2}\right)(2s+5) &= \frac{1}{s+1}+2 \\ Y(s)(3s+8) &= \frac{1}{s+1}+2-\frac{1}{3}\frac{(2s+5)}{(s+1)}+\frac{10}{3}\frac{(2s+5)}{(s+4)}-\frac{(2s+5)}{s+2} \end{aligned}$$

Hence

$$\begin{aligned}
Y(s) &= \frac{1}{(s+1)(3s+8)} + \frac{2}{(3s+8)} - \frac{1}{3}\frac{(2s+5)}{(3s+8)(s+1)} + \frac{10}{3}\frac{(2s+5)}{(3s+8)(s+4)} - \frac{(2s+5)}{(s+2)(3s+8)} \\
&= \frac{2s+3}{s^2+6s+8} \\
&= \frac{2s+3}{(s+4)(s+2)}
\end{aligned}$$

Using Partial fractions, let $\frac{2s+3}{(s+4)(s+2)} = \frac{A}{(s+4)} + \frac{B}{s+2}$, hence

$$\begin{aligned}
A &= \lim_{s\to -4}\frac{(2s+3)}{(s+2)} = \frac{(-8+3)}{(-4+2)} = \frac{5}{2} \\
B &= \lim_{s\to -2}\frac{(2s+3)}{(s+4)} = \frac{(-4+3)}{(-2+4)} = \frac{-1}{2}
\end{aligned}$$

Or

$$\begin{aligned}
Y(s) &= \frac{A}{(s+4)} + \frac{B}{s+2} \\
&= \frac{5}{2(s+4)} - \frac{1}{2(s+2)}
\end{aligned}$$

Then

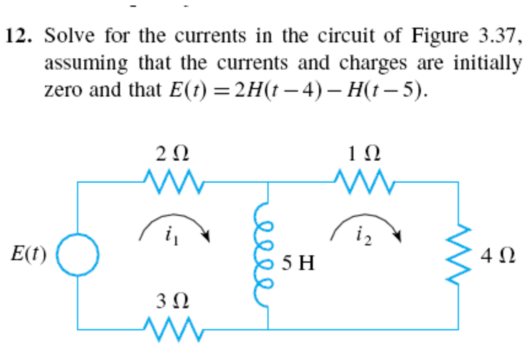
$$y(t) = \frac{5}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

Summary

$$\begin{aligned}
z(t) &= \frac{1}{3}e^{-t} - \frac{10}{3}e^{-4t} + e^{-2t} \\
y(t) &= \frac{5}{2}e^{-4t} - \frac{1}{2}e^{-2t}
\end{aligned}$$

### 2.5.7 Problem 6 page 148, problem 12

Solve for the currents in the circuit assuming currents and charges are initially zero and act that $E(t) = 2H(t-4) - H(t-5)$



**12.** Solve for the currents in the circuit of Figure 3.37, assuming that the currents and charges are initially zero and that $E(t) = 2H(t-4) - H(t-5)$.

**Answer:**

For reference, these are the laws to remember: Ohm's law $V = Ri$ for voltage across resistor, $V = Li'$ for voltage across inductor, $V = Q/C$ for voltage across capacitor, $i = q'$ for current capacitor relation, hence $V = \frac{1}{c}\int_0^t i(\tau)\,d\tau$ for voltage across capacitor.

Applying Kirchoff's voltage law to each loop gives

$$\begin{aligned}
5i_1 + 5\frac{d}{dt}i_1 - 5\frac{d}{dt}i_2 &= E(t) \\
5i_2 + 5\frac{d}{dt}i_2 - 5\frac{d}{dt}i_1 &= 0
\end{aligned}$$

Taking Laplace transform, and using property $\mathscr{L}(H(t-a)) = \frac{1}{s}e^{-as}$ gives (writing $I_1$ to mean $I_1(s)$ and $I_2$ to mean $I_2(s)$)

$$\begin{aligned}
5I_1 + 5(sI_1 - i_1(0)) - 5(sI_2 - i_2(0)) &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s} \\
5I_2 + 5(sI_2 - i_2(0)) - 5(sI_1 - i_1(0)) &= 0
\end{aligned}$$

Setting initial conditions, the above becomes

$$\begin{aligned} I_1(5+5s) - 5sI_2 &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s} \qquad (1)\\ I_2(5+5s) - 5sI_1 &= 0 \end{aligned}$$

Solving for $I_1(s)$ from the second equation gives

$$I_1 = \frac{I_2(5+5s)}{5s}$$

Substituting this into Eq. (1) gives

$$\begin{aligned} \frac{I_2(5+5s)}{5s}(5+5s) - 5sI_2 &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s}\\ I_2\left(25s^2+50s+25\right) - 25s^2I_2 &= 10e^{-4s} - 5e^{-5s}\\ I_2(50s+25) &= 10e^{-4s} - 5e^{-5s}\\ I_2 &= \frac{10e^{-4s} - 5e^{-5s}}{50s+25}\\ &= \frac{10}{25}\left(\frac{e^{-4s}}{2s+1}\right) - \frac{5}{25}\left(\frac{e^{-5s}}{2s+1}\right)\\ &= \frac{10}{25}\frac{1}{2}\left(\frac{e^{-4s}}{s+\frac{1}{2}}\right) - \frac{5}{25}\frac{1}{2}\left(\frac{e^{-5s}}{s+\frac{1}{2}}\right)\\ &= \frac{1}{5}\left(\frac{e^{-4s}}{s+\frac{1}{2}}\right) - \frac{1}{10}\left(\frac{e^{-5s}}{s+\frac{1}{2}}\right) \qquad (2) \end{aligned}$$

To be able to use the property $\mathscr{L}^{-1}\left(\frac{e^{-as}}{(s+b)}\right) = H(t-a)\mathscr{L}^{-1}\left(\frac{1}{s+b}\right)$ and the property that $\mathscr{L}^{-1}\left(\frac{1}{s+b}\right) = e^{-bt}$ Hence the inverse Laplace transform is

$$\begin{aligned} i_2(t) &= \frac{1}{5}H(t-4)e^{\left(-\frac{1}{2}(t-4)\right)} - \frac{1}{10}H(t-5)e^{\left(-\frac{1}{2}(t-5)\right)}\\ &= \frac{1}{5}H(t-4)e^{\left(-\frac{1}{2}t+2\right)} - \frac{1}{10}H(t-5)e^{\left(-\frac{1}{2}t+\frac{5}{2}\right)} \end{aligned}$$

Hence

$$i_2(t) = \frac{1}{5}e^{-\frac{t}{2}}\left(e^2H(t-4) - \frac{1}{2}e^{\frac{5}{2}}H(t-5)\right)$$

Now $I_2$ from Eq. (2) is substituted back into Eq. (1) to solve for $I_1$. Hence Eq. (1) becomes

$$\begin{aligned} I_1(5+5s) - 5s\left(\frac{1}{5}\left(\frac{e^{-4s}}{s+\frac{1}{2}}\right) - \frac{1}{10}\left(\frac{e^{-5s}}{s+\frac{1}{2}}\right)\right) &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s}\\ I_1(5+5s) &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s} + 5s\left(\frac{1}{5}\left(\frac{e^{-4s}}{s+\frac{1}{2}}\right) - \frac{1}{10}\left(\frac{e^{-5s}}{s+\frac{1}{2}}\right)\right)\\ &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s} + s\left(\frac{e^{-4s}}{s+\frac{1}{2}}\right) - \frac{1}{2}s\left(\frac{e^{-5s}}{s+\frac{1}{2}}\right)\\ &= \frac{\left(s+\frac{1}{2}\right)\frac{2}{s}e^{-4s} - \left(s+\frac{1}{2}\right)\frac{1}{s}e^{-5s} + se^{-4s} - \frac{1}{2}se^{-5s}}{s+\frac{1}{2}} \end{aligned}$$

Hence

$$\begin{aligned}
I_1 &= \frac{\left(s+\frac{1}{2}\right)\frac{2}{s}e^{-4s}-\left(s+\frac{1}{2}\right)\frac{1}{s}e^{-5s}+se^{-4s}-\frac{1}{2}se^{-5s}}{5\left(1+s\right)\left(s+\frac{1}{2}\right)}\\
&= \frac{\left(s+\frac{1}{2}\right)\frac{2}{s}e^{-4s}}{5\left(1+s\right)\left(s+\frac{1}{2}\right)}-\frac{\left(s+\frac{1}{2}\right)\frac{1}{s}e^{-5s}}{5\left(1+s\right)\left(s+\frac{1}{2}\right)}+\frac{1}{5}\frac{se^{-4s}}{\left(1+s\right)\left(s+\frac{1}{2}\right)}-\frac{1}{10}\frac{se^{-5s}}{\left(1+s\right)\left(s+\frac{1}{2}\right)}\\
&= \frac{2}{5}\frac{e^{-4s}}{s\left(1+s\right)}-\frac{1}{5}\frac{e^{-5s}}{s\left(1+s\right)}+\frac{1}{5}\frac{se^{-4s}}{\left(1+s\right)\left(s+\frac{1}{2}\right)}-\frac{1}{10}\frac{se^{-5s}}{\left(1+s\right)\left(s+\frac{1}{2}\right)}
\end{aligned}$$

The inverse Laplace transform of each term is now found. The following properties will be used

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{e^{-as}}{\left(s+b\right)}\right) &= H\left(t-a\right)\mathscr{L}^{-1}\left(\frac{1}{s+b}\right)=H\left(t-a\right)f\left(t-a\right)\\
\mathscr{L}^{-1}\left(\frac{1}{s+b}\right) &= e^{-bt}
\end{aligned}$$

Finding partial fractions of $\frac{1}{s(1+s)}=\frac{A}{s}+\frac{B}{s+1}$, hence

$$\begin{aligned}
A &= \lim_{s\to 0}\frac{1}{\left(1+s\right)}=1\\
B &= \lim_{s\to -1}\frac{1}{s}=-1
\end{aligned}$$

Hence

$$\frac{1}{s\left(1+s\right)}=\frac{1}{s}-\frac{1}{s+1}$$

Therefore, inverse Laplace of first term in Eq. (3) gives

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{2}{5}\frac{e^{-4s}}{s\left(1+s\right)}\right) &= \frac{2}{5}H\left(t-4\right)\mathscr{L}^{-1}\left(\frac{1}{s}-\frac{1}{s+1}\right)\\
&= \frac{2}{5}H\left(t-4\right)\left(H\left(t-4\right)-e^{-\left(t-4\right)}\right)\\
&= \frac{2}{5}\left(H\left(t-4\right)-H\left(t-4\right)e^{-\left(t-4\right)}\right)
\end{aligned}$$

And for the second term in Eq. (3)

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{5}\frac{e^{-5s}}{s\left(1+s\right)}\right) &= \frac{1}{5}H\left(t-5\right)\mathscr{L}^{-1}\left(\frac{1}{s}-\frac{1}{s+1}\right)\\
&= \frac{1}{5}H\left(t-5\right)\left(H\left(t-4\right)-e^{-\left(t-5\right)}\right)\\
&= \frac{1}{5}\left(H\left(t-5\right)-H\left(t-5\right)e^{-\left(t-5\right)}\right)
\end{aligned}$$

For the third term, partial fractions of $\frac{s}{(1+s)\left(s+\frac{1}{2}\right)}$ is needed. Let

$$\frac{s}{\left(1+s\right)\left(s+\frac{1}{2}\right)}=\frac{A}{\left(1+s\right)}+\frac{B}{\left(s+\frac{1}{2}\right)}$$

$$\begin{aligned}
A &= \lim_{s\to -1}\frac{s}{\left(s+\frac{1}{2}\right)}=\frac{-1}{-\frac{1}{2}}=2\\
B &= \lim_{s\to -\frac{1}{2}}\frac{s}{\left(1+s\right)}=\frac{-\frac{1}{2}}{\left(1-\frac{1}{2}\right)}=-1
\end{aligned}$$

Hence

$$\mathscr{L}^{-1}\left(\frac{1}{5}\frac{se^{-4s}}{(1+s)\left(s+\frac{1}{2}\right)}\right) = \frac{1}{5}H(t-4)\,\mathscr{L}^{-1}\left(\frac{2}{(1+s)} - \frac{1}{\left(s+\frac{1}{2}\right)}\right)$$
$$= \frac{1}{5}H(t-4)\left(2e^{-(t-4)} - e^{-\frac{1}{2}(t-4)}\right)$$

For the last term in Eq. (3)

$$\mathscr{L}^{-1}\left(\frac{1}{10}\frac{se^{-5s}}{(1+s)\left(s+\frac{1}{2}\right)}\right) = \frac{1}{10}H(t-5)\,\mathscr{L}^{-1}\left(\frac{2}{(1+s)} - \frac{1}{\left(s+\frac{1}{2}\right)}\right)$$
$$= \frac{1}{10}H(t-5)\left(2e^{-(t-5)} - e^{-\frac{1}{2}(t-5)}\right)$$

Putting all this together gives

$$\begin{aligned} i_1(t) = {}& \frac{2}{5}\left(H(t-4) - H(t-4)\,e^{-(t-4)}\right) \\ & - \frac{1}{5}\left(H(t-5) - H(t-5)\,e^{-(t-5)}\right) \\ & + \frac{1}{5}H(t-4)\left(2e^{-(t-4)} - e^{-\frac{1}{2}(t-4)}\right) \\ & - \frac{1}{10}H(t-5)\left(2e^{-(t-5)} - e^{-\frac{1}{2}(t-5)}\right) \end{aligned}$$

This can be simplified to

$$\begin{aligned} i_1(t) = {}& \frac{2}{5}H(t-4) + H(t-4)\left(-\frac{2}{5}e^{-(t-4)} + \frac{2}{5}e^{-(t-4)} - \frac{1}{5}e^{-\frac{1}{2}(t-4)}\right) \\ & - \frac{1}{5}H(t-5) + H(t-5)\left(\frac{1}{5}e^{-(t-5)} - \frac{1}{5}e^{-(t-5)} + \frac{1}{10}e^{-\frac{1}{2}(t-5)}\right) \end{aligned}$$

Hence, summary of final results:

$$\begin{aligned} i_1(t) &= \frac{2}{5}H(t-4) - \frac{1}{5}H(t-5) + \frac{1}{5}H(t-4)\,e^{-\frac{1}{2}(t-4)} + \frac{1}{10}H(t-5)\,e^{-\frac{1}{2}(t-5)} \\ i_2(t) &= \frac{1}{5}e^{-\frac{t}{2}}\left(e^2 H(t-4) - \frac{1}{2}e^{\frac{5}{2}}H(t-5)\right) \end{aligned}$$

Here is a plot of the solutions for $t = 0 \cdots 8\,\text{sec}$ showing the input $E(t)$ and the currents $i_1(t)$ and $i_2(t)$.

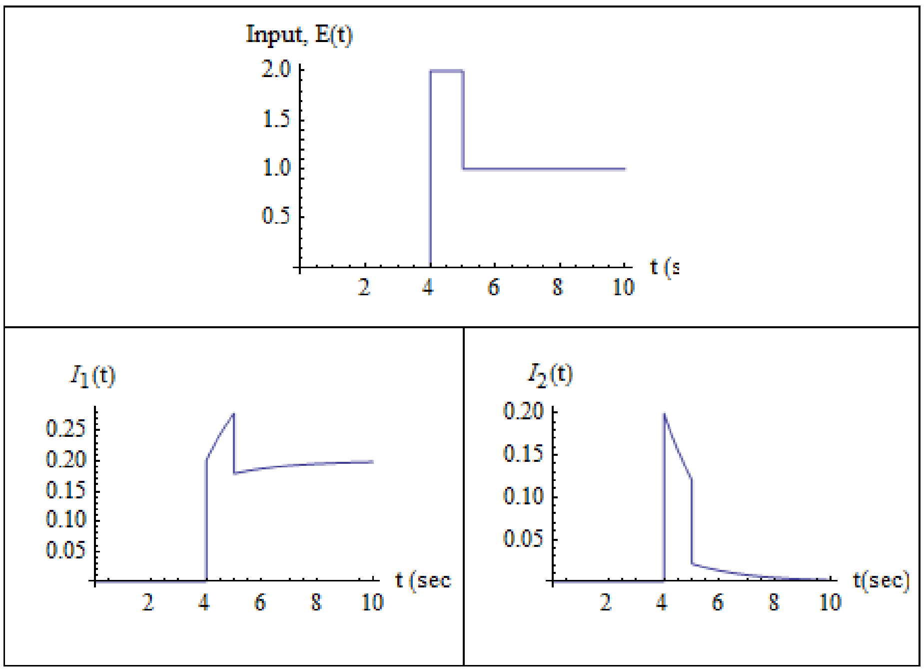
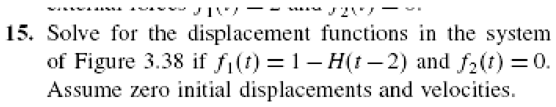


Figure 2.7: plot of solution problem 6

### 2.5.8 Problem 7, page 149, problem 15



**15.** Solve for the displacement functions in the system of Figure 3.38 if $f_1(t) = 1 - H(t-2)$ and $f_2(t) = 0$. Assume zero initial displacements and velocities.

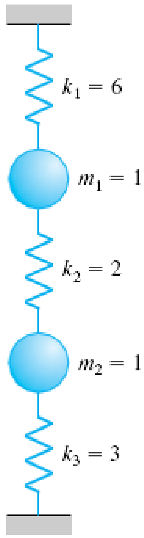


**FIGURE 3.38**

**Solution**

Assuming the generalized coordinate for $m_1$ is $y_1$ and for $m_2$ is $y_2$ and these are measured when the system is relaxed, hence force due to weight is already accounted for. Assume positive is upwards. For mass $m_1$ applying $F = ma$ gives

$$-k_1 y_1 - k_2 \left(y_1 - y_2\right) + f_1(t) = m_1 y_1''(t)$$

and for mass $m_2$

$$-k_3 y_1 - k_2 \left(y_1 - y_2\right) = m_2 y_2''(t)$$

Or

$$\begin{aligned} m_1 y_1''(t) + y_1(k_1 + k_2) - k_2 y_2 &= f_1(t) \\ m_2 y_2''(t) - y_2(k_2) + y_1(k_3 + k_2) &= 0 \end{aligned} \tag{1}$$

Or, in matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_3 + k_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 - H(t-2) \\ 0 \end{pmatrix}$$

The stiffness matrix is symmetric and positive definite. OK. Now Laplace transform is applied to the above, using $Y$ to mean $Y(s)$, the Laplace transform of $y(t)$, hence

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} s^2 Y_1 - s y_1(0) - y_1'(0) \\ s^2 Y_2 - s y_2(0) - y_2'(0) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_3 + k_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} - \frac{1}{s} e^{-2s} \\ 0 \end{pmatrix}$$

Applying IC, gives

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} s^2 Y_1 \\ s^2 Y_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_3 + k_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} - \frac{1}{s} e^{-2s} \\ 0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} m_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2 s^2 + (k_3 + k_2) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} - \frac{1}{s} e^{-2s} \\ 0 \end{pmatrix}$$

To speed the typing, numerical values are now substituted for the symbolic values above (but it is best to delay this to the end otherwise so that same solution can be applied to different numerical values)

$$\begin{pmatrix} s^2 + 8 & -2 \\ -2 & s^2 + 5 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} - \frac{1}{s} e^{-2s} \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned}\begin{pmatrix}Y_1\\Y_2\end{pmatrix} &= \begin{pmatrix}s^2+8 & -2\\ -2 & s^2+5\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{s}-\frac{1}{s}e^{-2s}\\ 0\end{pmatrix}\\ &= \frac{1}{\left(s^2+8\right)\left(s^2+5\right)-4}\begin{pmatrix}s^2+5 & 2\\ 2 & s^2+8\end{pmatrix}\begin{pmatrix}\frac{1}{s}-\frac{1}{s}e^{-2s}\\ 0\end{pmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}Y_1 &= \frac{s^2+5}{\left(s^2+8\right)\left(s^2+5\right)-4}\left(\frac{1}{s}-\frac{1}{s}e^{-2s}\right)\\ Y_2 &= \frac{2}{\left(s^2+8\right)\left(s^2+5\right)-4}\left(\frac{1}{s}-\frac{1}{s}e^{-2s}\right)\end{aligned}$$

Or

$$\begin{aligned}Y_1 &= \frac{1}{s}\frac{s^2+5}{\left(s^2+8\right)\left(s^2+5\right)-4}-\frac{1}{s}e^{-2s}\frac{s^2+5}{\left(s^2+8\right)\left(s^2+5\right)-4}\\ Y_2 &= \frac{1}{s}\frac{2}{\left(s^2+8\right)\left(s^2+5\right)-4}-\frac{1}{s}e^{-2s}\frac{2}{\left(s^2+8\right)\left(s^2+5\right)-4}\end{aligned}$$

But $\left(s^2+8\right)\left(s^2+5\right)-4=\left(s^2+4\right)\left(s^2+9\right)$, hence the above becomes (after writing $s^2+5$ as $\left(s^2+4\right)+1$)

$$\begin{aligned}Y_1 &= \frac{1}{s}\frac{\left(s^2+4\right)+1}{\left(s^2+4\right)\left(s^2+9\right)}-\frac{1}{s}e^{-2s}\frac{\left(s^2+4\right)+1}{\left(s^2+4\right)\left(s^2+9\right)}\\ Y_2 &= \frac{1}{s}\frac{2}{\left(s^2+4\right)\left(s^2+9\right)}-\frac{1}{s}e^{-2s}\frac{2}{\left(s^2+4\right)\left(s^2+9\right)}\end{aligned} \tag{2}$$

This can be written as

$$\begin{aligned}Y_1 &= \left(\frac{1}{s\left(s^2+9\right)}\right)+\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)-e^{-2s}\frac{1}{s\left(s^2+9\right)}-e^{-2s}\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\\ Y_2 &= 2\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)-2e^{-2s}\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)\end{aligned} \tag{2}$$

So there are only 3 common terms that needs to be inverse Laplace. These are $\frac{1}{s\left(s^2+9\right)}$ and $\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}$

### 2.5.9 First term

$$\frac{1}{s\left(s^2+9\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+9}$$

Then

$$1=A\left(s^2+9\right)+Bs^2+Cs$$

Let $s=0$ hence $A=\frac{1}{9}$. Now, comparing $s^2$ coefficients gives $\frac{1}{9}+B=0$ or $B=-\frac{1}{9}$. And finally comparing $s$ coefficients shows that $C=0$, therefore

$$\frac{1}{s\left(s^2+9\right)}=\frac{1}{9}\frac{1}{s}-\frac{1}{9}\frac{s}{s^2+9}$$

Hence

$$\mathscr{L}^{-1}\left(\frac{1}{9}\frac{1}{s}-\frac{1}{9}\frac{s}{s^2+9}\right)=\frac{1}{9}-\frac{1}{9}\cos 3t$$

and

$$\mathscr{L}^{-1}\left(e^{-2s}\frac{1}{s\left(s^2+9\right)}\right)=H(t-2)\left(\frac{1}{9}H(t-2)-\frac{1}{9}\cos\left(3(t-2)\right)\right)$$

#### 2.5.9.1 Second term

Now looking at the term

$$\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}+\frac{Ds+G}{s^2+9}$$

Comparing coefficients gives

$$\begin{aligned}1&=A\left(s^4+13s^2+36\right)+(Bs+C)\left(s^3+9s\right)+(Ds+G)\left(s^3+4s\right)\\1&=A\left(s^4+13s^2+36\right)+Bs^4+9Bs^2+Cs^3+9Cs+Ds^4+4Ds^2+Gs^3+4Gs\end{aligned}$$

Hence

$$1=s^4\left(A+B+D\right)+s^3\left(C+G\right)+s^2\left(13A+9B+4D\right)+s\left(9C+4G\right)+36A$$

Therefore, $A=\frac{1}{36}$ and

$$\begin{aligned}A+B+D&=0\\C+G&=0\\13A+9B+4D&=0\\9C+4G&=0\end{aligned}$$

$C=0$ and $G=0$ since the second and the fourth equation do not have solution other than zero. Hence the above becomes

$$\begin{aligned}A+B+D&=0\\13A+9B+4D&=0\end{aligned}$$

From the first equation, $B=-\frac{1}{36}-D$, hence from the second equation $\frac{13}{36}+9\left(-\frac{1}{36}-D\right)+4D=0$ or $\frac{4}{36}-5D=0$, hence $D=\frac{4}{36\times5}=\frac{1}{45}$. Therefore $B=-\frac{1}{36}-D=-\frac{1}{36}-\frac{1}{45}=-\frac{1}{20}$

Therefore

$$\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}=\frac{1}{36}\frac{1}{s}-\frac{1}{20}\frac{s}{s^2+4}+\frac{1}{45}\frac{s}{\left(s^2+9\right)}$$

Now the inverse Laplace transform can be taken

$$\mathscr{L}^{-1}\left(\frac{1}{36}\frac{1}{s}-\frac{1}{20}\frac{s}{s^2+4}+\frac{1}{45}\frac{s}{\left(s^2+9\right)}\right)=\frac{1}{36}-\frac{1}{20}\cos 2t+\frac{1}{45}\cos 3t$$

Hence

$$\mathscr{L}^{-1}\left(e^{-2s}\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)=H\left(t-2\right)\left(\frac{1}{36}H\left(t-2\right)-\frac{1}{20}\cos 2\left(t-2\right)+\frac{1}{45}\cos 3\left(t-2\right)\right)$$

### 2.5.10 Final solution

Now that all terms have been inverse Laplace, the solution can be written down. From above,

$$\begin{aligned}Y_1&=\left(\frac{1}{s\left(s^2+9\right)}\right)+\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)-e^{-2s}\frac{1}{s\left(s^2+9\right)}-e^{-2s}\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\\Y_2&=2\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)-2e^{-2s}\left(\frac{1}{s\left(s^2+4\right)\left(s^2+9\right)}\right)\end{aligned}\tag{2}$$

Hence

$$\begin{aligned}y_1\left(t\right)=&\frac{1}{9}-\frac{1}{9}\cos 3t+\frac{1}{36}-\frac{1}{20}\cos 2t+\frac{1}{45}\cos 3t\\&-H\left(t-2\right)\left(\frac{1}{9}H\left(t-2\right)-\frac{1}{9}\cos\left(3\left(t-2\right)\right)\right)\\&-H\left(t-2\right)\left(\frac{1}{36}H\left(t-2\right)-\frac{1}{20}\cos 2\left(t-2\right)+\frac{1}{45}\cos 3\left(t-2\right)\right)\end{aligned}$$

Simplifying

$$y_1\left(t\right)=\frac{5}{36}-\frac{1}{20}\cos 2t-\frac{4}{45}\cos 3t-H\left(t-2\right)\left(\frac{5}{36}-\frac{4}{45}\cos\left(3\left(t-2\right)\right)-\frac{1}{20}\cos\left(2\left(t-2\right)\right)\right)$$

and

$$\begin{aligned} y_2(t) &= \frac{2}{36} - \frac{2}{20}\cos 2t + \frac{2}{45}\cos 3t - 2H(t-2)\left(\frac{1}{36}H(t-2) - \frac{1}{20}\cos 2(t-2) + \frac{1}{45}\cos 3(t-2)\right) \\ &= \frac{1}{18} - \frac{1}{10}\cos 2t + \frac{2}{45}\cos 3t - H(t-2)\left(\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos 3(t-2)\right) \end{aligned}$$

This is a plot of the solutions, with the input force

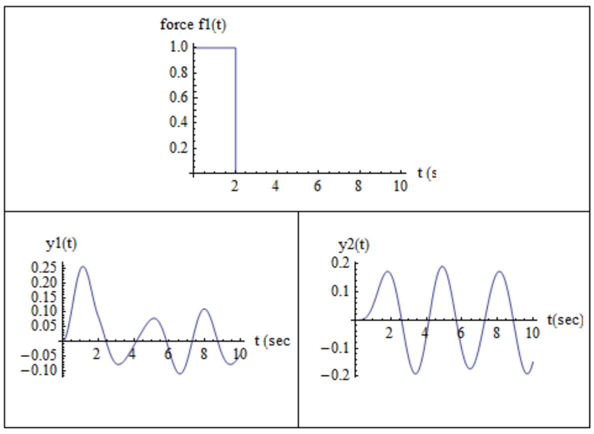
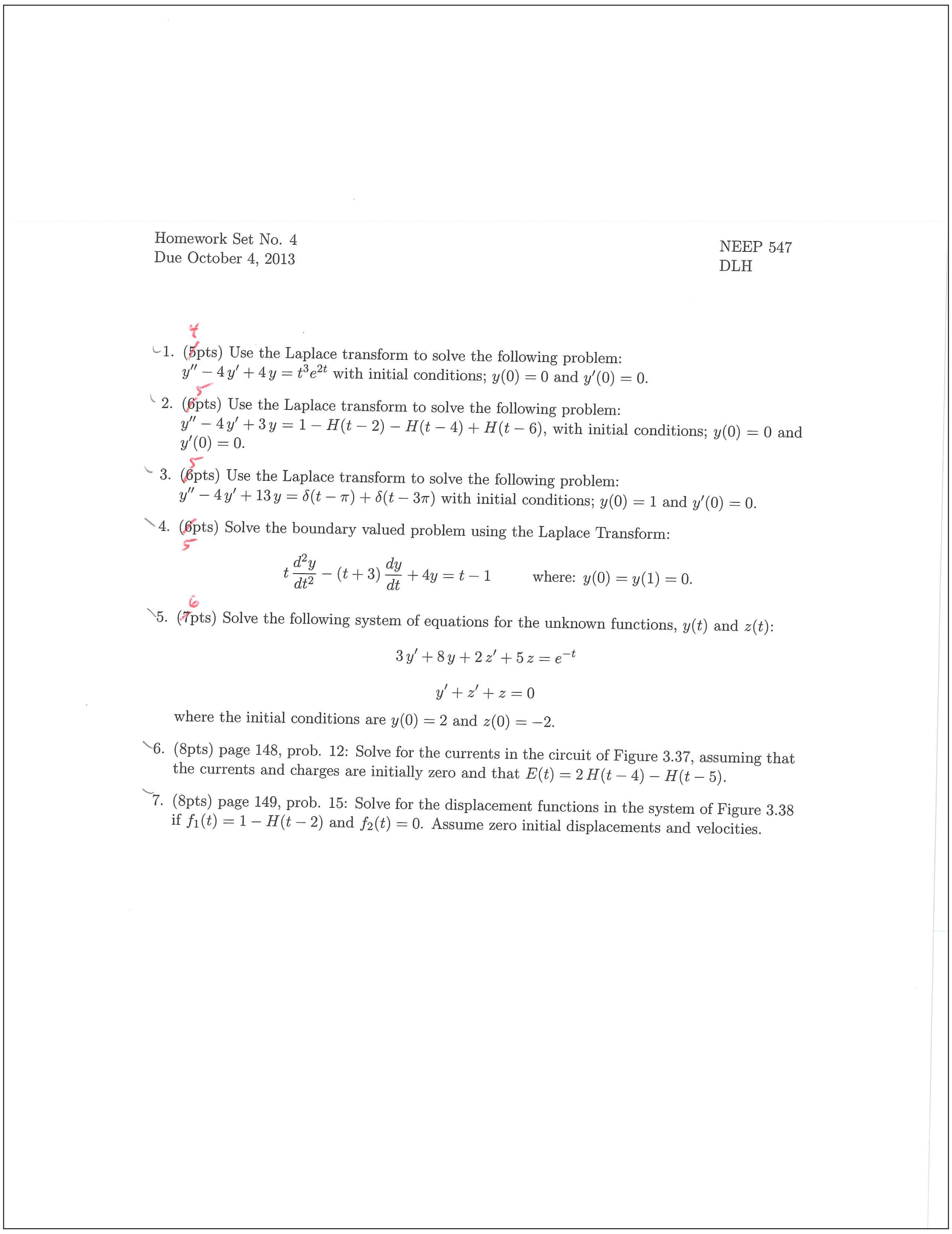


Figure 2.8: plot of solution problem 7 HW 4

### 2.5.11 key solution



Homework Set No. 4
Due October 4, 2013

NEEP 547
DLH

1. (~~5~~ 4pts) Use the Laplace transform to solve the following problem:
$y'' - 4y' + 4y = t^3 e^{2t}$ with initial conditions; $y(0) = 0$ and $y'(0) = 0$.

2. (~~6~~ 5pts) Use the Laplace transform to solve the following problem:
$y'' - 4y' + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$, with initial conditions; $y(0) = 0$ and $y'(0) = 0$.

3. (~~6~~ 5pts) Use the Laplace transform to solve the following problem:
$y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$ with initial conditions; $y(0) = 1$ and $y'(0) = 0$.

4. (~~6~~ 5pts) Solve the boundary valued problem using the Laplace Transform:

$$t\,\frac{d^2y}{dt^2} - (t+3)\,\frac{dy}{dt} + 4y = t - 1 \qquad \text{where: } y(0) = y(1) = 0.$$

5. (~~7~~ 6pts) Solve the following system of equations for the unknown functions, $y(t)$ and $z(t)$:

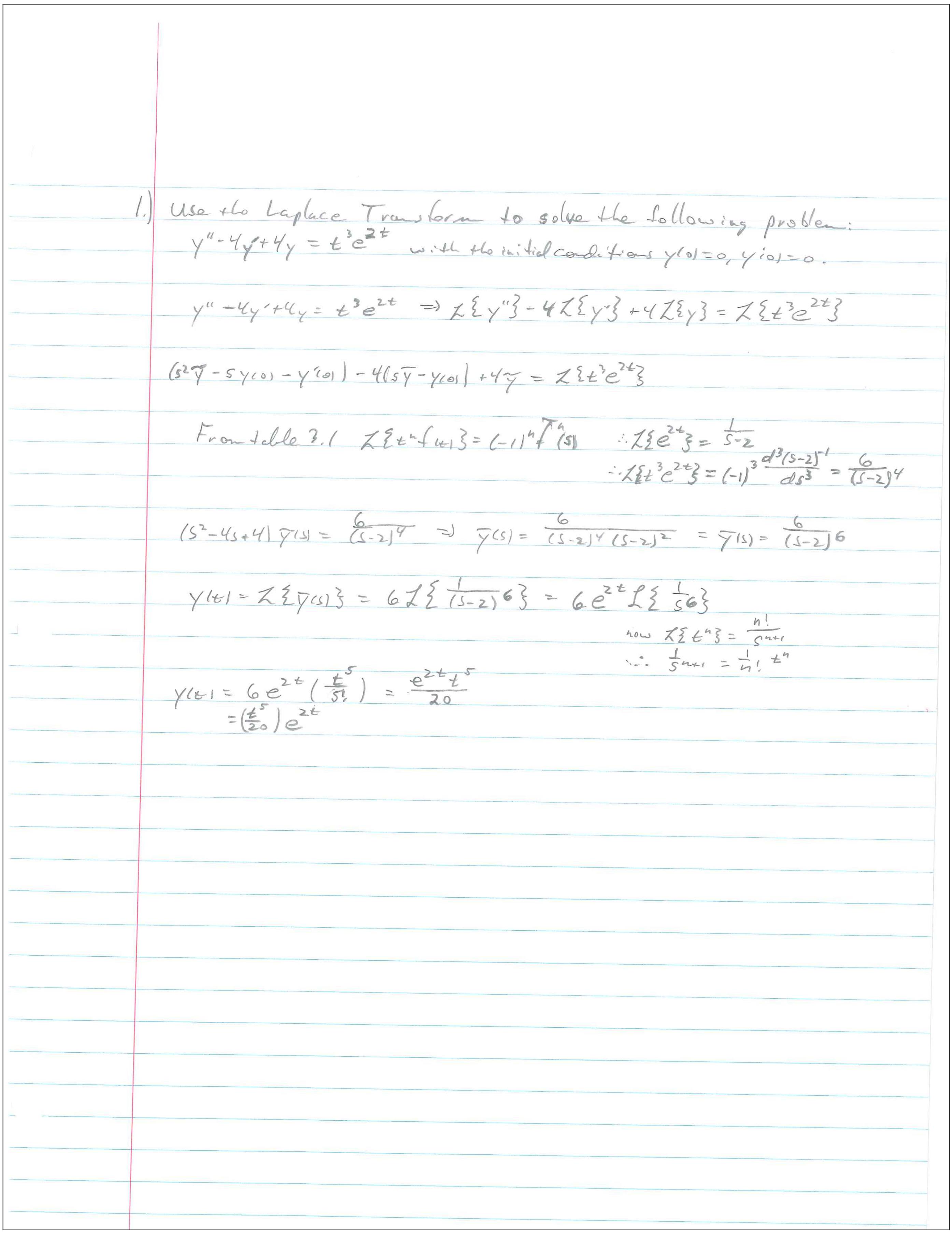
$$3y' + 8y + 2z' + 5z = e^{-t}$$

$$y' + z' + z = 0$$

where the initial conditions are $y(0) = 2$ and $z(0) = -2$.

6. (8pts) page 148, prob. 12: Solve for the currents in the circuit of Figure 3.37, assuming that the currents and charges are initially zero and that $E(t) = 2H(t-4) - H(t-5)$.

7. (8pts) page 149, prob. 15: Solve for the displacement functions in the system of Figure 3.38 if $f_1(t) = 1 - H(t-2)$ and $f_2(t) = 0$. Assume zero initial displacements and velocities.



1.) Use the Laplace Transform to solve the following problem:

$y''-4y'+4y = t^3e^{2t}$ with the initial conditions $y(0)=0$, $y'(0)=0$.

$$y''-4y'+4y = t^3e^{2t} \Rightarrow \mathcal{L}\{y''\}-4\mathcal{L}\{y'\}+4\mathcal{L}\{y\}=\mathcal{L}\{t^3e^{2t}\}$$

$$(s^2\bar{y}-sy(0)-y'(0))-4(s\bar{y}-y(0))+4\bar{y}=\mathcal{L}\{t^3e^{2t}\}$$

From table 3.1 $\mathcal{L}\{t^nf(t)\}=(-1)^n\bar{f}^n(s)$ $\therefore \mathcal{L}\{e^{2t}\}=\frac{1}{s-2}$

$$\therefore \mathcal{L}\{t^3e^{2t}\}=(-1)^3\frac{d^3(s-2)^{-1}}{ds^3}=\frac{6}{(s-2)^4}$$

$$(s^2-4s+4)\bar{y}(s)=\frac{6}{(s-2)^4} \Rightarrow \bar{y}(s)=\frac{6}{(s-2)^4(s-2)^2}=\bar{y}(s)=\frac{6}{(s-2)^6}$$

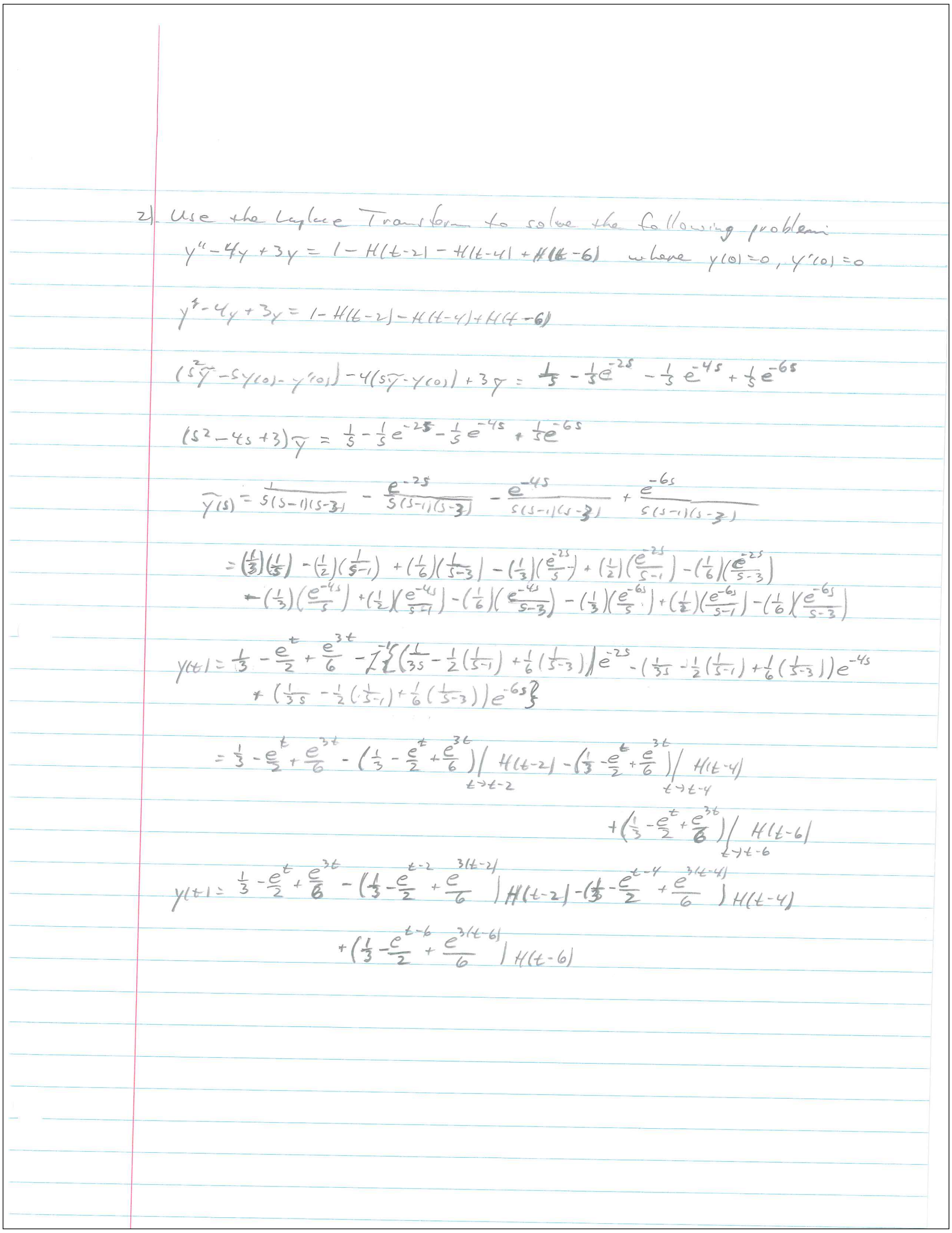
$$y(t)=\mathcal{L}\{\bar{y}(s)\}=6\mathcal{L}\{\frac{1}{(s-2)^6}\}=6e^{2t}\mathcal{L}\{\frac{1}{s^6}\}$$

now $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$

$\therefore \frac{1}{s^{n+1}}=\frac{1}{n!}t^n$

$$y(t)=6e^{2t}\left(\frac{t^5}{5!}\right)=\frac{e^{2t}t^5}{20}$$

$$=\left(\frac{t^5}{20}\right)e^{2t}$$



2) Use the Laplace Transform to solve the following problem

$y'' - 4y + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$ where $y(0)=0,\ y'(0)=0$

$$y'' - 4y + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$$

$$(s^2\bar{y} - sy(0) - y'(0)) - 4(s\bar{y} - y(0)) + 3\bar{y} = \frac{1}{s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s} + \frac{1}{s}e^{-6s}$$

$$(s^2 - 4s + 3)\bar{y} = \frac{1}{s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s} + \frac{1}{s}e^{-6s}$$

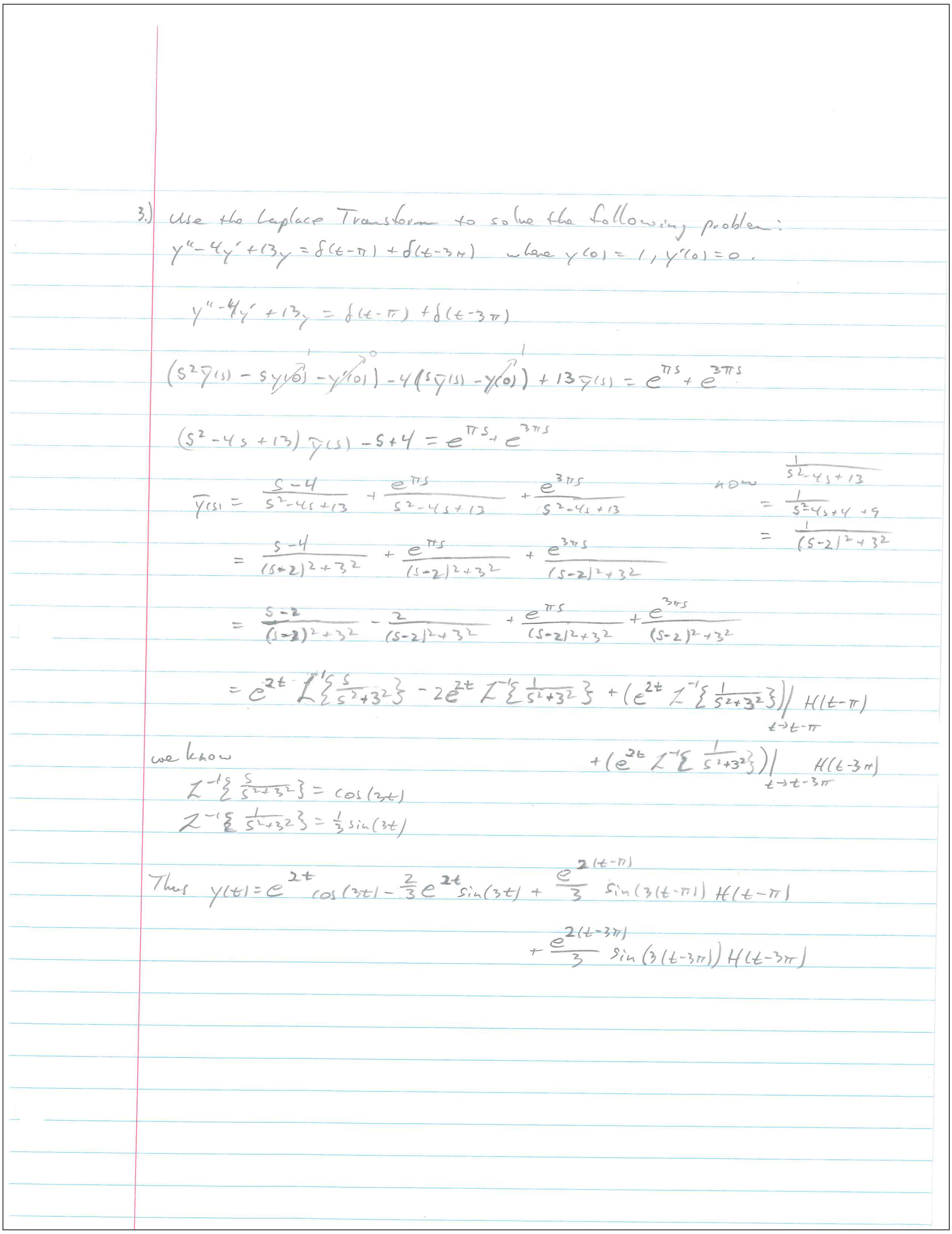
$$\bar{y}(s) = \frac{1}{s(s-1)(s-3)} - \frac{e^{-2s}}{s(s-1)(s-3)} - \frac{e^{-4s}}{s(s-1)(s-3)} + \frac{e^{-6s}}{s(s-1)(s-3)}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{s-1}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{s-3}\right) - \left(\frac{1}{3}\right)\left(\frac{e^{-2s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-2s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-2s}}{s-3}\right)$$
$$- \left(\frac{1}{3}\right)\left(\frac{e^{-4s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-4s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-4s}}{s-3}\right) - \left(\frac{1}{3}\right)\left(\frac{e^{-6s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-6s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-6s}}{s-3}\right)$$

$$y(t) = \frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6} - \mathcal{L}^{-1}\left\{\left(\frac{1}{3s} - \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s-3}\right)\right)e^{-2s} - \left(\frac{1}{3s} - \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s-3}\right)\right)e^{-4s}\right.$$
$$\left. + \left(\frac{1}{3s} - \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s-3}\right)\right)e^{-6s}\right\}$$

$$= \frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6} - \left(\frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6}\right)\Big|_{t \to t-2} H(t-2) - \left(\frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6}\right)\Big|_{t \to t-4} H(t-4)$$
$$+ \left(\frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6}\right)\Big|_{t \to t-6} H(t-6)$$

$$y(t) = \frac{1}{3} - \frac{e^{t}}{2} + \frac{e^{3t}}{6} - \left(\frac{1}{3} - \frac{e^{t-2}}{2} + \frac{e^{3(t-2)}}{6}\right)H(t-2) - \left(\frac{1}{3} - \frac{e^{t-4}}{2} + \frac{e^{3(t-4)}}{6}\right)H(t-4)$$
$$+ \left(\frac{1}{3} - \frac{e^{t-6}}{2} + \frac{e^{3(t-6)}}{6}\right)H(t-6)$$



3.) Use the Laplace Transform to solve the following problem:
$y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$ where $y(0) = 1$, $y'(0) = 0$.

$$y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$$

$$(s^2\overline{y}(s) - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}}) - 4(s\overline{y}(s) - \overset{1}{\cancel{y(0)}}) + 13\overline{y}(s) = e^{\pi s} + e^{3\pi s}$$

$$(s^2 - 4s + 13)\overline{y}(s) - s + 4 = e^{\pi s} + e^{3\pi s}$$

$$\overline{y}(s) = \frac{s-4}{s^2-4s+13} + \frac{e^{\pi s}}{s^2-4s+13} + \frac{e^{3\pi s}}{s^2-4s+13}$$

now $\frac{1}{s^2-4s+13}$

$= \frac{1}{s^2-4s+4+9}$

$= \frac{1}{(s-2)^2+3^2}$

$$= \frac{s-4}{(s+2)^2+3^2} + \frac{e^{\pi s}}{(s-2)^2+3^2} + \frac{e^{3\pi s}}{(s-2)^2+3^2}$$

$$= \frac{s-2}{(s-2)^2+3^2} - \frac{2}{(s-2)^2+3^2} + \frac{e^{\pi s}}{(s-2)^2+3^2} + \frac{e^{3\pi s}}{(s-2)^2+3^2}$$

$$= e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} + \left(e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right)\Big|_{t\to t-\pi} H(t-\pi)$$

$$+ \left(e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right)\Big|_{t\to t-3\pi} H(t-3\pi)$$

we know

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{1}{3}\sin(3t)$$

Thus $y(t) = e^{2t}\cos(3t) - \frac{2}{3}e^{2t}\sin(3t) + \frac{e^{2(t-\pi)}}{3}\sin(3(t-\pi))H(t-\pi)$

$$+ \frac{e^{2(t-3\pi)}}{3}\sin(3(t-3\pi))H(t-3\pi)$$

4). solve the boundary valued problem

$$t\frac{d^2y}{dt^2}-(t+3)\frac{dy}{dt}+4y=t-1 \qquad \text{where: } y(0)=y(1)=0$$

$$\mathcal{L}\{tf(t)\}=-\frac{d}{ds}(\mathcal{L}\{f(t)\})=-\frac{d}{ds}(\bar{f}(s))$$

$$\mathcal{L}\{t\frac{d^2y}{dt^2}\}=-\frac{d}{ds}(s^2\bar{y}(s)-sy(0)-y'(0))$$

$$=-2s\bar{y}(s)-s^2\frac{d\bar{y}(s)}{ds}$$

$$\mathcal{L}\{t\frac{dy}{dt}\}=-\frac{d}{ds}(s\bar{y}(s)-y(0))=-\bar{y}(s)-s\frac{d\bar{y}(s)}{ds}$$

$$-2s\bar{y}(s)-s^2\frac{d\bar{y}}{ds}+\bar{y}(s)+s\frac{d\bar{y}}{ds}-3(s\bar{y}(s)-y(0))+4(\bar{y}(s))=\frac{1}{s^2}-\frac{1}{s}$$

$$(-s^2+s)\frac{d\bar{y}(s)}{ds}-(2s+3s)\bar{y}(s)+5\bar{y}(s)=\frac{1}{s^2}-\frac{1}{s}$$

$$(-s^2+s)\frac{d\bar{y}(s)}{ds}-(5s-5)\bar{y}(s)=\frac{1}{s^2}-\frac{1}{s}$$

$$-s(s-1)\frac{d\bar{y}(s)}{ds}-5(s-1)\bar{y}(s)=-\frac{(s-1)}{s^2}\Rightarrow\frac{d\bar{y}(s)}{ds}-\frac{5(s-1)}{(-s)(s-1)}\bar{y}(s)=\frac{-(s-1)}{(s^2)(-s)(s-1)}$$

$$\frac{d\bar{y}(s)}{ds}+\frac{5}{s}\bar{y}(s)=\frac{1}{s^3}$$ First order Eq. Integrating factor $e^{\int\frac{5}{s}ds}=e^{5\ln(s)}=s^5$

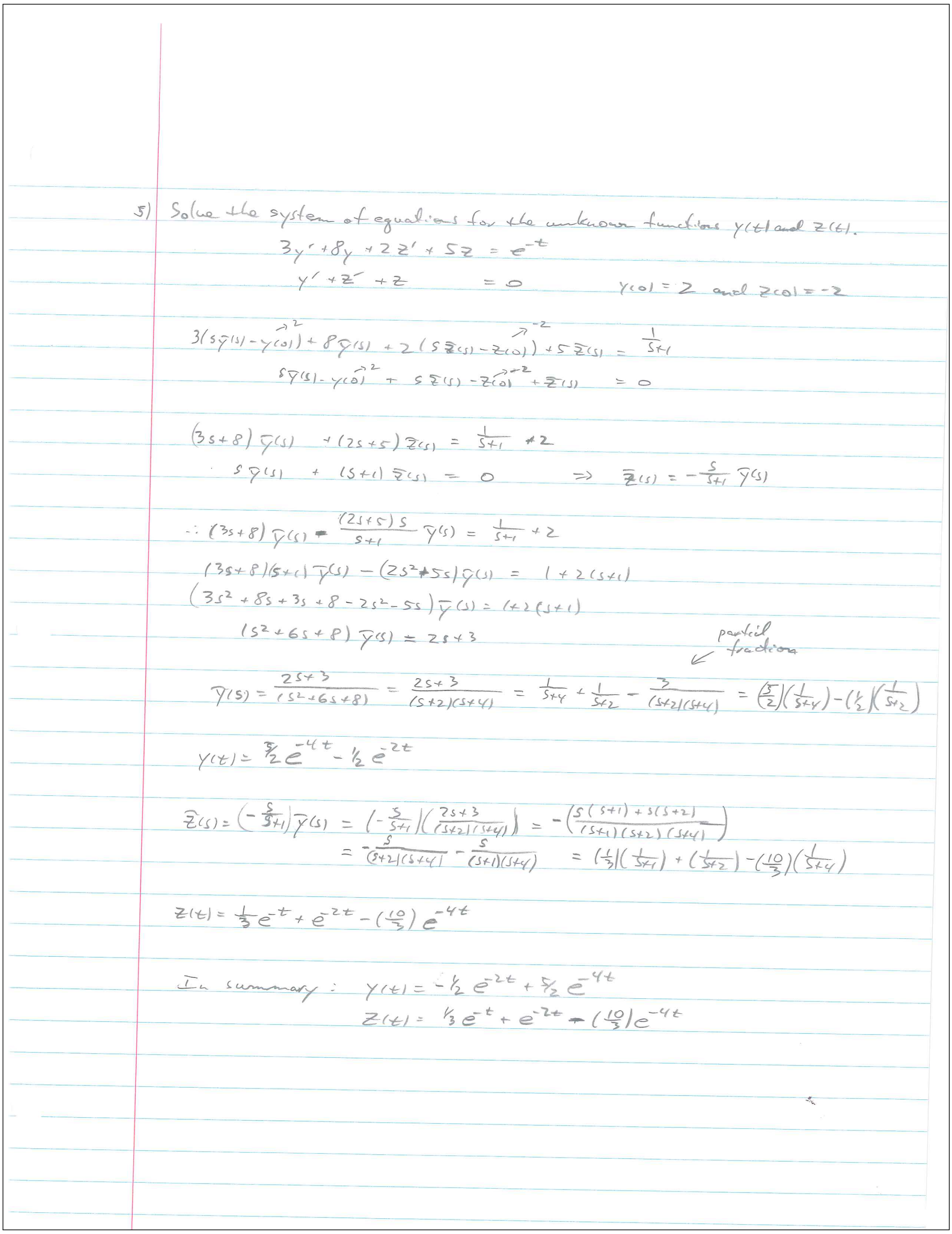
$$\int d(s^5\bar{y}(s))=\int(s^5/s^3)ds\Rightarrow s^5\bar{y}(s)=\frac{1}{3}s^3+c$$

$$\bar{y}(s)=\frac{1}{3s^2}+\frac{c}{s^5}$$

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=y(t)=\frac{t}{3}+\frac{t^4}{24}c$$ what is c ?

$$y(1)=0=\frac{1}{3}+\frac{1}{24}c\Rightarrow c=-\frac{24}{3}=-8$$

$$y(t)=\frac{t}{3}-\frac{t^4}{3}$$



5) Solve the system of equations for the unknown functions $y(t)$ and $z(t)$.

$$3y' + 8y + 2z' + 5z = e^{-t}$$
$$y' + z' + z = 0 \qquad y(0) = 2 \text{ and } z(0) = -2$$

$$3(s\bar{y}(s) - \overset{\nearrow 2}{y(0)}) + 8\bar{y}(s) + 2(s\bar{z}(s) - \overset{\nearrow -2}{z(0)}) + 5\bar{z}(s) = \frac{1}{s+1}$$
$$s\bar{y}(s) - \overset{\nearrow 2}{y(0)} + s\bar{z}(s) - \overset{\nearrow -2}{z(0)} + \bar{z}(s) = 0$$

$$(3s+8)\bar{y}(s) + (2s+5)\bar{z}(s) = \frac{1}{s+1} + 2$$
$$s\bar{y}(s) + (s+1)\bar{z}(s) = 0 \qquad \Rightarrow \qquad \bar{z}(s) = -\frac{s}{s+1}\bar{y}(s)$$

$$\therefore (3s+8)\bar{y}(s) - \frac{(2s+5)s}{s+1}\bar{y}(s) = \frac{1}{s+1} + 2$$
$$(3s+8)(s+1)\bar{y}(s) - (2s^2+5s)\bar{y}(s) = 1 + 2(s+1)$$
$$(3s^2 + 8s + 3s + 8 - 2s^2 - 5s)\bar{y}(s) = 1 + 2(s+1)$$
$$(s^2+6s+8)\bar{y}(s) = 2s+3$$

partial fractions ↙

$$\bar{y}(s) = \frac{2s+3}{(s^2+6s+8)} = \frac{2s+3}{(s+2)(s+4)} = \frac{1}{s+4} + \frac{1}{s+2} - \frac{3}{(s+2)(s+4)} = \left(\frac{5}{2}\right)\left(\frac{1}{s+4}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{s+2}\right)$$

$$y(t) = \frac{5}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

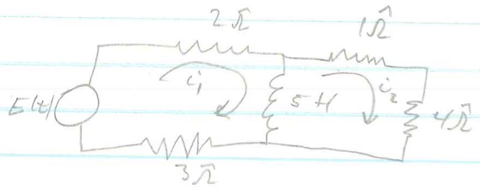
$$\bar{z}(s) = \left(-\frac{s}{s+1}\right)\bar{y}(s) = \left(-\frac{s}{s+1}\right)\left(\frac{2s+3}{(s+2)(s+4)}\right) = -\left(\frac{s(s+1) + s(s+2)}{(s+1)(s+2)(s+4)}\right)$$
$$= -\frac{s}{(s+2)(s+4)} - \frac{s}{(s+1)(s+4)} = \left(\frac{1}{3}\right)\left(\frac{1}{s+1}\right) + \left(\frac{1}{s+2}\right) - \left(\frac{10}{3}\right)\left(\frac{1}{s+4}\right)$$

$$z(t) = \frac{1}{3}e^{-t} + e^{-2t} - \left(\frac{10}{3}\right)e^{-4t}$$

In summary: $y(t) = -\frac{1}{2}e^{-2t} + \frac{5}{2}e^{-4t}$

$z(t) = \frac{1}{3}e^{-t} + e^{-2t} - \left(\frac{10}{3}\right)e^{-4t}$

6). page 148, prob 12



$i_1(0) = i_2(0) = 0$

$E(t) = 2H(t-4) - H(t-5)$

$1^{st}$ loop $\quad 2i_1(t) + 5\frac{d}{dt}\left(i_1(t) - i_2(t)\right) + 3i_1(t) = E(t) = 2H(t-4) - H(t-5)$

$2^{nd}$ loop $\quad i_2(t) + 4i_2(t) + 5\frac{d}{dt}\left(i_2(t) - i_1(t)\right) = 0$

Take the Laplace Transform of both loop Equations

$1^{st}$ $\quad 2\tilde{i}_1(s) + 5\left(s\tilde{i}_1(s) - \cancel{i_1(0)}\right) - 5\left(s\tilde{i}_2(s) - i_2(0)\right) + 3\tilde{i}_1(s) = \left(\frac{2}{s}\right)e^{-4s} - \left(\frac{1}{s}\right)e^{-5s}$

$2^{nd}$ $\quad 5\tilde{i}_2(s) + 5\left(s\tilde{i}_2(s) - \cancel{i_2(0)}\right) - 5\left(s\tilde{i}_1(s) - \cancel{i_1(0)}\right) = 0$

$1^{st}$ loop: $\quad 5(s+1)\tilde{i}_1(s) - 5s\tilde{i}_2(s) = 2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)e^{-5s}$

$2^{nd}$ loop: $\quad 5(s+1)\tilde{i}_2(s) - 5s\tilde{i}_1(s) = 0 \quad \Longrightarrow \quad \tilde{i}_1(s) = \frac{(s+1)}{s}\tilde{i}_2(s)$

$$5(s+1)\tilde{i}_1(s) - 5s\tilde{i}_2(s) = 2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)e^{-5s}$$

$$5(s+1)\left(\frac{s+1}{s}\right)\tilde{i}_2(s) - 5s\tilde{i}_2(s) =$$

$$5\left(\frac{(s+1)^2}{s} - \frac{s^2}{s}\right)\tilde{i}_2(s) =$$

$$5\left(\frac{2s+1}{s}\right)\tilde{i}_2(s) =$$

$$\tilde{i}_2(s) = \left(\frac{1}{5}\right)\left(\frac{s}{2s+1}\right)\left(2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)e^{-5s}\right)$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{2s+1}\right)\left(2e^{-4s} - e^{-5s}\right)$$

and

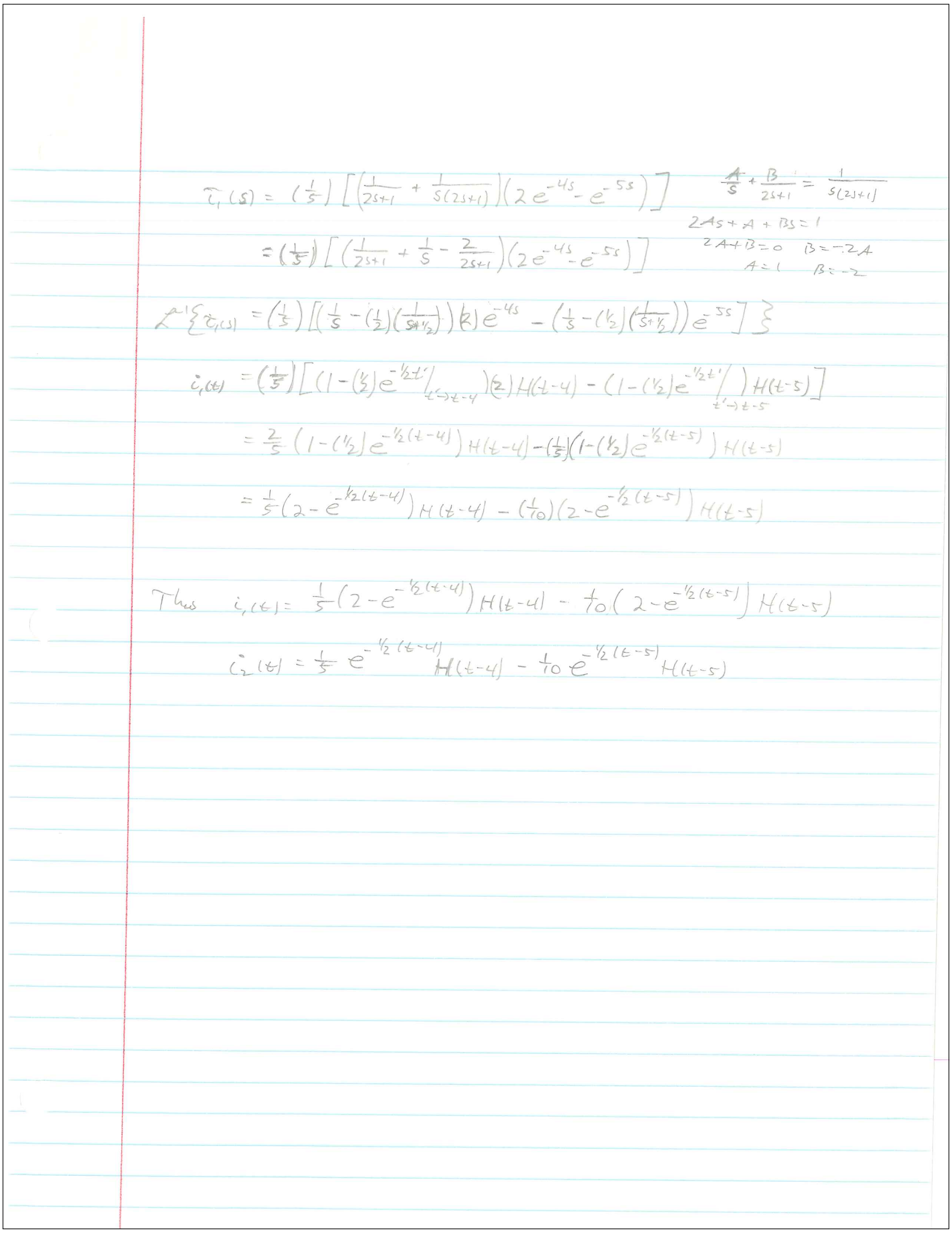
$$\tilde{i}_1(s) = \left(\frac{s+1}{s}\right)\tilde{i}_2(s) = \left(\frac{1}{5}\right)\left(\frac{s+1}{s(2s+1)}\right)\left(2e^{-4s} - e^{-5s}\right)$$

now to invert

$$\mathcal{L}^{-1}\left\{\tilde{i}_2(s) = \left(\frac{1}{5}\right)\left[\frac{1}{(s+\frac{1}{2})}e^{-4s} - \frac{1}{2}\left(\frac{1}{s+\frac{1}{2}}\right)e^{-5s}\right]\right\}$$

$$i_2(t) = \left(\frac{1}{5}\right)\left[e^{-\frac{1}{2}t'}\Big|_{t'\to t-4} H(t-4) - \frac{1}{2}e^{-\frac{1}{2}t'}\Big|_{t'\to t-5} H(t-5)\right]$$

$$i_2(t) = \left(\frac{1}{5}\right)\left[e^{-\frac{1}{2}(t-4)}H(t-4) - \frac{1}{2}e^{-\frac{1}{2}(t-5)}H(t-5)\right]$$



$$\tilde{\imath}_1(s) = \left(\frac{1}{5}\right)\left[\left(\frac{1}{2s+1} + \frac{1}{s(2s+1)}\right)\left(2e^{-4s} - e^{-5s}\right)\right]$$

$$\frac{A}{s} + \frac{B}{2s+1} = \frac{1}{s(2s+1)}$$

$$2As + A + Bs = 1$$

$$2A + B = 0 \quad B = -2A$$

$$A = 1 \quad B = -2$$

$$= \left(\frac{1}{5}\right)\left[\left(\frac{1}{2s+1} + \frac{1}{s} - \frac{2}{2s+1}\right)\left(2e^{-4s} - e^{-5s}\right)\right]$$

$$\mathcal{L}^{-1}\left\{\tilde{\imath}_1(s) = \left(\frac{1}{5}\right)\left[\left(\frac{1}{s} - \left(\frac{1}{2}\right)\left(\frac{1}{s+1/2}\right)\right)(2)e^{-4s} - \left(\frac{1}{s} - (1/2)\left(\frac{1}{s+1/2}\right)\right)e^{-5s}\right]\right\}$$

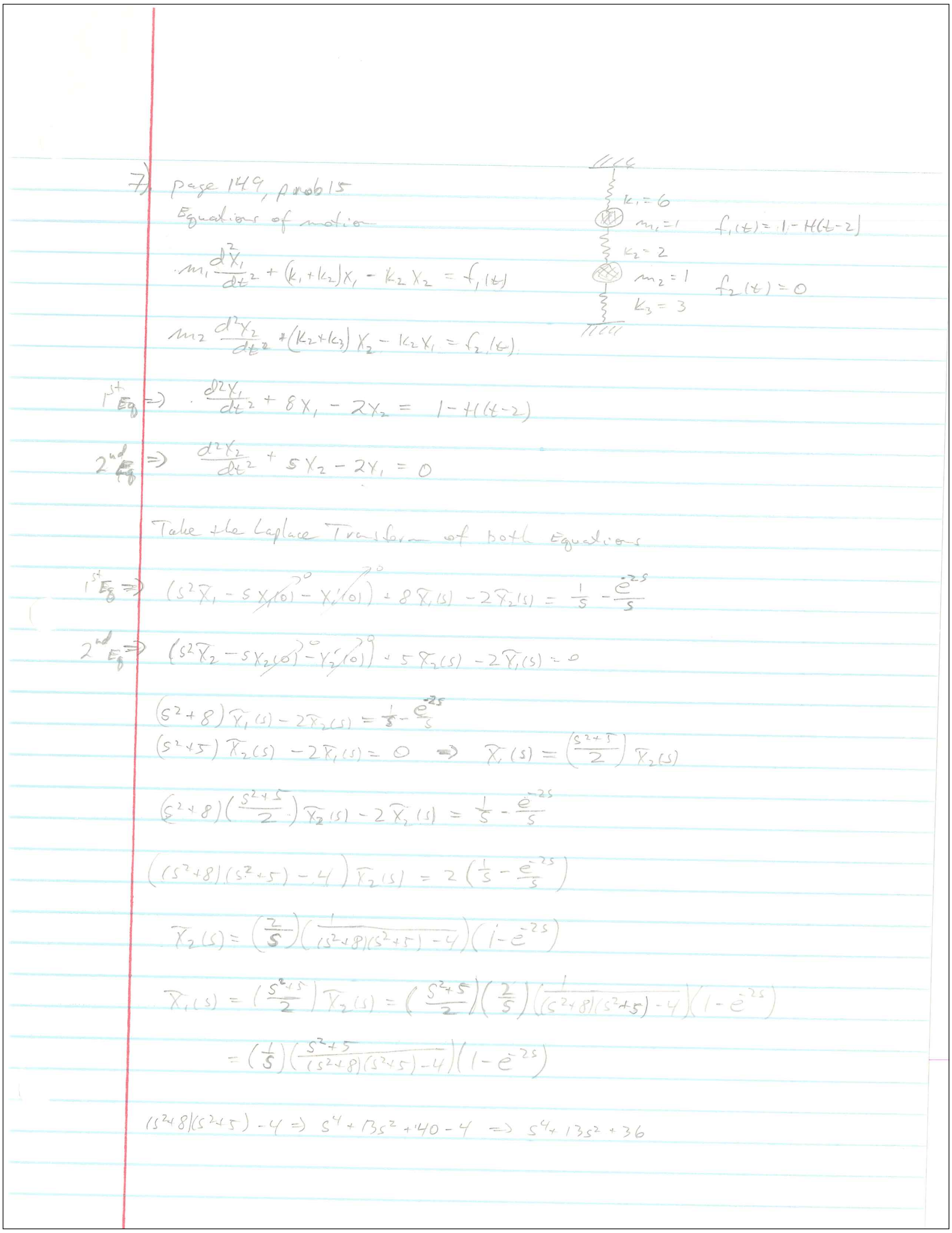
$$i_1(t) = \left(\frac{1}{5}\right)\left[\left(1 - (1/2)e^{-1/2t'}\Big|_{t' \to t-4}\right)(2)H(t-4) - \left(1 - (1/2)e^{-1/2t'}\Big|_{t' \to t-5}\right)H(t-5)\right]$$

$$= \frac{2}{5}\left(1 - (1/2)e^{-1/2(t-4)}\right)H(t-4) - \left(\frac{1}{5}\right)\left(1 - (1/2)e^{-1/2(t-5)}\right)H(t-5)$$

$$= \frac{1}{5}\left(2 - e^{-1/2(t-4)}\right)H(t-4) - \left(\frac{1}{10}\right)\left(2 - e^{-1/2(t-5)}\right)H(t-5)$$

Thus $i_1(t) = \frac{1}{5}\left(2 - e^{-1/2(t-4)}\right)H(t-4) - \frac{1}{10}\left(2 - e^{-1/2(t-5)}\right)H(t-5)$

$$i_2(t) = \frac{1}{5}e^{-1/2(t-4)}H(t-4) - \frac{1}{10}e^{-1/2(t-5)}H(t-5)$$



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Equations of motion

$k_1 = 6$

$m_1 = 1$ $f_1(t) = 1 - H(t-2)$

$k_2 = 2$

$m_2 = 1$ $f_2(t) = 0$

$k_3 = 3$

$$m_1 \frac{d^2X_1}{dt^2} + (k_1+k_2)X_1 - k_2X_2 = f_1(t)$$

$$m_2 \frac{d^2X_2}{dt^2} + (k_2+k_3)X_2 - k_2X_1 = f_2(t)$$

1st Eq ⇒ $$\frac{d^2X_1}{dt^2} + 8X_1 - 2X_2 = 1 - H(t-2)$$

2nd Eq ⇒ $$\frac{d^2X_2}{dt^2} + 5X_2 - 2X_1 = 0$$

Take the Laplace Transform of both Equations

1st Eq ⇒ $$\left(s^2\bar{X}_1 - s\cancelto{0}{X_1(0)} - \cancelto{0}{X_1'(0)}\right) + 8\bar{X}_1(s) - 2\bar{X}_2(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

2nd Eq ⇒ $$\left(s^2\bar{X}_2 - s\cancelto{0}{X_2(0)} - \cancelto{0}{X_2'(0)}\right) + 5\bar{X}_2(s) - 2\bar{X}_1(s) = 0$$

$$(s^2+8)\bar{X}_1(s) - 2\bar{X}_2(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$(s^2+5)\bar{X}_2(s) - 2\bar{X}_1(s) = 0 \Rightarrow \bar{X}_1(s) = \left(\frac{s^2+5}{2}\right)\bar{X}_2(s)$$

$$(s^2+8)\left(\frac{s^2+5}{2}\right)\bar{X}_2(s) - 2\bar{X}_1(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

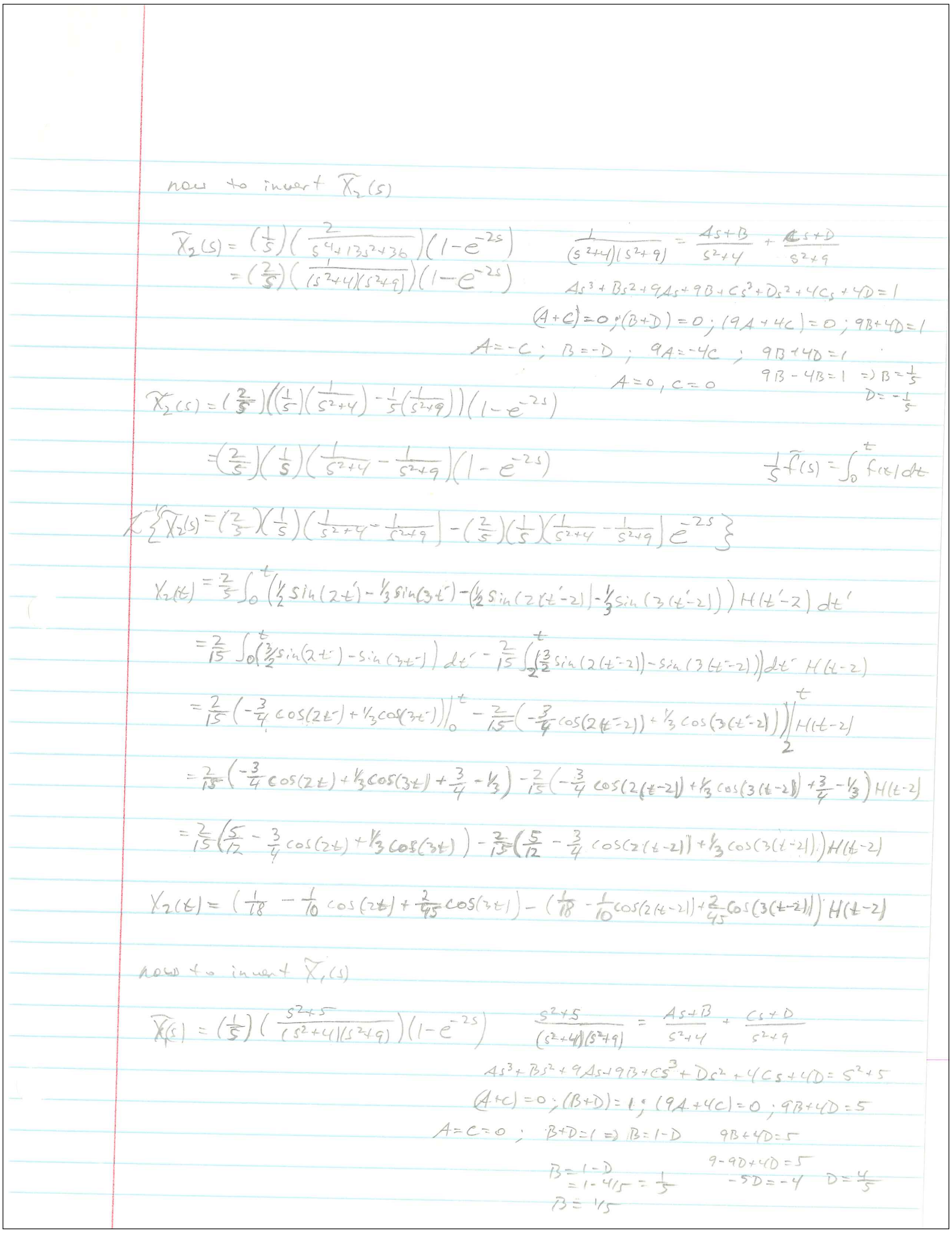
$$\left((s^2+8)(s^2+5) - 4\right)\bar{X}_2(s) = 2\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)$$

$$\bar{X}_2(s) = \left(\frac{2}{s}\right)\left(\frac{1}{(s^2+8)(s^2+5)-4}\right)\left(1 - e^{-2s}\right)$$

$$\bar{X}_1(s) = \left(\frac{s^2+5}{2}\right)\bar{X}_2(s) = \left(\frac{s^2+5}{2}\right)\left(\frac{2}{s}\right)\left(\frac{1}{(s^2+8)(s^2+5)-4}\right)\left(1 - e^{-2s}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{s^2+5}{(s^2+8)(s^2+5)-4}\right)\left(1 - e^{-2s}\right)$$

$$(s^2+8)(s^2+5) - 4 \Rightarrow s^4 + 13s^2 + 40 - 4 \Rightarrow s^4 + 13s^2 + 36$$



now to invert $\bar{X}_2(s)$

$$\bar{X}_2(s) = \left(\frac{1}{s}\right)\left(\frac{2}{s^4+13s^2+36}\right)\left(1-e^{-2s}\right)$$
$$= \left(\frac{2}{s}\right)\left(\frac{1}{(s^2+4)(s^2+9)}\right)\left(1-e^{-2s}\right)$$

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$As^3+Bs^2+9As+9B+Cs^3+Ds^2+4Cs+4D=1$$

$$(A+C)=0;\ (B+D)=0;\ (9A+4C)=0;\ 9B+4D=1$$

$$A=-C;\ B=-D;\ 9A=-4C;\ 9B+4D=1$$

$$A=0,\ C=0 \qquad 9B-4B=1 \Rightarrow B=\frac{1}{5}$$
$$D=-\frac{1}{5}$$

$$\bar{X}_2(s) = \left(\frac{2}{s}\right)\left(\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4}\right) - \frac{1}{5}\left(\frac{1}{s^2+9}\right)\right)\left(1-e^{-2s}\right)$$

$$= \left(\frac{2}{s}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)\left(1-e^{-2s}\right)$$

$$\frac{1}{s}\bar{f}(s) = \int_0^t f(\tau)\,dt$$

$$\mathcal{L}^{-1}\left\{\bar{X}_2(s) = \left(\frac{2}{s}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right) - \left(\frac{2}{s}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)e^{-2s}\right\}$$

$$X_2(t) = \frac{2}{5}\int_0^t \left(\tfrac{1}{2}\sin(2t') - \tfrac{1}{3}\sin(3t') - \left(\tfrac{1}{2}\sin(2(t'-2)) - \tfrac{1}{3}\sin(3(t'-2))\right)H(t'-2)\right)dt'$$

$$= \frac{2}{15}\int_0^t \left(\tfrac{3}{2}\sin(2t') - \sin(3t')\right)dt' - \frac{2}{15}\int_2^t \left(\tfrac{3}{2}\sin(2(t'-2)) - \sin(3(t'-2))\right)dt'\ H(t-2)$$

$$= \frac{2}{15}\left(-\frac{3}{4}\cos(2t') + \tfrac{1}{3}\cos(3t')\right)\Big|_0^t - \frac{2}{15}\left(-\frac{3}{4}\cos(2(t'-2)) + \tfrac{1}{3}\cos(3(t'-2))\right)\Big|_2^t H(t-2)$$

$$= \frac{2}{15}\left(-\frac{3}{4}\cos(2t) + \tfrac{1}{3}\cos(3t) + \frac{3}{4} - \tfrac{1}{3}\right) - \frac{2}{15}\left(-\frac{3}{4}\cos(2(t-2)) + \tfrac{1}{3}\cos(3(t-2)) + \frac{3}{4} - \tfrac{1}{3}\right)H(t-2)$$

$$= \frac{2}{15}\left(\frac{5}{12} - \frac{3}{4}\cos(2t) + \tfrac{1}{3}\cos(3t)\right) - \frac{2}{15}\left(\frac{5}{12} - \frac{3}{4}\cos(2(t-2)) + \tfrac{1}{3}\cos(3(t-2))\right)H(t-2)$$

$$X_2(t) = \left(\frac{1}{18} - \frac{1}{10}\cos(2t) + \frac{2}{45}\cos(3t)\right) - \left(\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos(3(t-2))\right)H(t-2)$$

now to invert $\bar{X}_1(s)$

$$\bar{X}_1(s) = \left(\frac{1}{s}\right)\left(\frac{s^2+5}{(s^2+4)(s^2+9)}\right)\left(1-e^{-2s}\right)$$

$$\frac{s^2+5}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

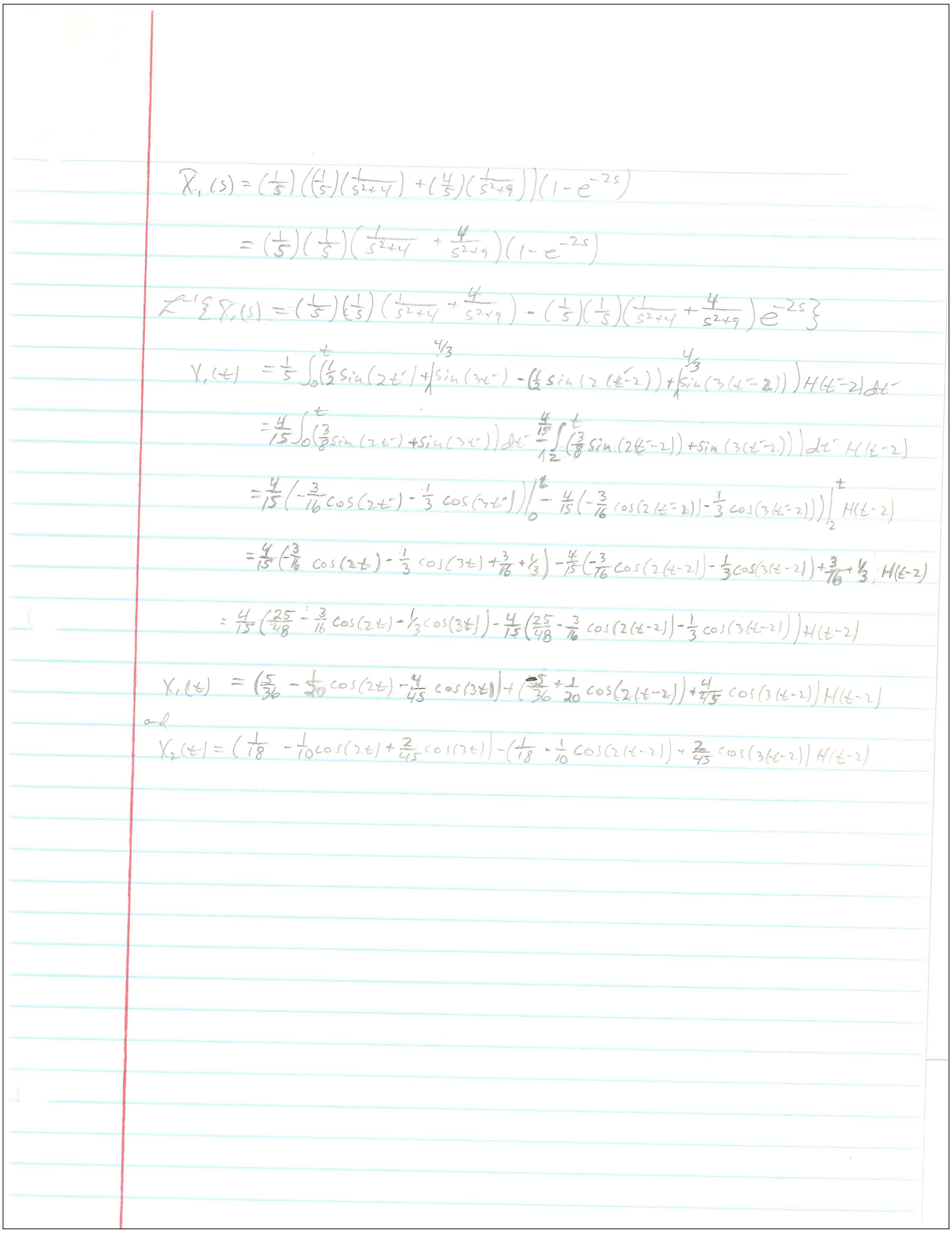
$$As^3+Bs^2+9As+9B+Cs^3+Ds^2+4Cs+4D=s^2+5$$

$$(A+C)=0;\ (B+D)=1;\ (9A+4C)=0;\ 9B+4D=5$$

$$A=C=0;\ B+D=1 \Rightarrow B=1-D \qquad 9B+4D=5$$

$$9-9D+4D=5$$
$$-5D=-4 \qquad D=\frac{4}{5}$$

$$B=1-D$$
$$=1-4/5=\frac{1}{5}$$
$$B=1/5$$



$$X_1(s) = \left(\frac{1}{5}\right)\left(\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{s^2+9}\right)\right)\left(1-e^{-2s}\right)$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right)\left(1-e^{-2s}\right)$$

$$\mathcal{L}^{-1}\left\{X_1(s) = \left(\frac{1}{5}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right) - \left(\frac{1}{5}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right)e^{-2s}\right\}$$

$$x_1(t) = \frac{1}{5}\int_0^t \left(\frac{1}{2}\sin(2t') + \overset{4/3}{1}\sin(3t') - \left(\frac{1}{2}\sin(2(t'-2)) + \overset{4/3}{1}\sin(3(t'-2))\right)H(t'-2)\right)dt'$$

$$= \frac{4}{15}\int_0^t \left(\frac{3}{8}\sin(2t') + \sin(3t')\right)dt' - \frac{\frac{4}{15}}{12}\int^t \left(\frac{3}{8}\sin(2(t'-2)) + \sin(3(t'-2))\right)dt' \, H(t-2)$$

$$= \frac{4}{15}\left(-\frac{3}{16}\cos(2t') - \frac{1}{3}\cos(3t')\right)\Big|_0^t - \frac{4}{15}\left(-\frac{3}{16}\cos(2(t'-2)) - \frac{1}{3}\cos(3(t'-2))\right)\Big|_2^t H(t-2)$$

$$= \frac{4}{15}\left(-\frac{3}{16}\cos(2t) - \frac{1}{3}\cos(3t) + \frac{3}{16} + \frac{1}{3}\right) - \frac{4}{15}\left(-\frac{3}{16}\cos(2(t-2)) - \frac{1}{3}\cos(3(t-2)) + \frac{3}{16} + \frac{1}{3}, H(t-2)\right)$$

$$= \frac{4}{15}\left(\frac{25}{48} - \frac{3}{16}\cos(2t) - \frac{1}{3}\cos(3t)\right) - \frac{4}{15}\left(\frac{25}{48} - \frac{3}{16}\cos(2(t-2)) - \frac{1}{3}\cos(3(t-2))\right)H(t-2)$$

$$x_1(t) = \left(\frac{5}{36} - \frac{1}{20}\cos(2t) - \frac{4}{45}\cos(3t)\right) + \left(-\frac{5}{36} + \frac{1}{20}\cos(2(t-2)) + \frac{4}{45}\cos(3(t-2))\right)H(t-2)$$

and

$$x_2(t) = \left(\frac{1}{18} - \frac{1}{10}\cos(2t) + \frac{2}{45}\cos(3t)\right) - \left(\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos(3(t-2))\right)H(t-2)$$

## 2.6 HW 5

### 2.6.1 Problems to solve

| | |
|---|---|
| Homework Set No. 5 | NEEP 547 |
| Due October 11, 2013 | DLH |

1. (4pts) Find the inverse Laplace transform of the following function:

$$g(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$$

2. (6pts) Use the Laplace transform to solve the following problem:
$(1-t)\,y'' + t\,y' - y = 0$ with initial conditions; $y(0) = 3$ and $y'(0) = -1$.

Periodic function

3. (8pts) Solve the initial value problem where is $f(t)$ is a periodic function:
$y' + 4\,y + 3\,\int_0^t y(t')\,dt' = f(t)$ ; $y(0) = 1$, with $f(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ -1 & \text{for } 2 < t < 4. \end{cases}$

Convolution Theorem

4. (10pts) Consider the equation of motion for a damped harmonic oscillator:

$$m\,\frac{d^2x}{dt^2} + 2\lambda\,\frac{dx}{dt} + \omega_o^2 x = f(t)$$

with a general forcing function $f(t)$ and initial conditions $x(0) = 0$ and $x'(0) = v_0$. Express the general solution of the equation in terms of the convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0\,\sin(\omega\,t)$. Note: the oscillator is under damped, i.e. $\omega_o > \lambda$.

5. (10pts) Heat Conduction Equation ($0 < x < \ell$ and $t > 0$):

$$\frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} = a\,e^{-\alpha\,t} \quad \text{with I.C.: } U(x,0) = 0 \text{ and B.C.: } U(0,t) = U(\ell,t) = 0$$

where $k$, $a$ and $\alpha$ are positive constants.

### 2.6.2 Problem 1

Find the inverse Laplace transform of

$$G(s) = \ln\left(\frac{s^2+1}{s(s+1)}\right)$$

**Answer**:

Using the property $\mathscr{L}\left(tg\left(t\right)\right)=-\frac{d}{ds}G\left(s\right)$, then $tg\left(t\right)=-\mathscr{L}^{-1}\left(\frac{d}{ds}G\left(s\right)\right)$. Let $f\left(s\right)=\frac{s^2+1}{s(s+1)}$ then

$$\begin{aligned}
\frac{d}{ds}G\left(s\right) &= \frac{d}{ds}\ln\left(f\left(s\right)\right)\\
&= \frac{f'\left(s\right)}{f\left(s\right)}\\
&= \frac{s\left(s+1\right)}{s^2+1}\frac{d}{ds}\left(\frac{s^2+1}{s\left(s+1\right)}\right)\\
&= \frac{s\left(s+1\right)}{s^2+1}\frac{d}{ds}\left(\frac{s^2}{s\left(s+1\right)}+\frac{1}{s\left(s+1\right)}\right)\\
&= \frac{s\left(s+1\right)}{s^2+1}\left[\frac{d}{ds}\left(\frac{s}{\left(s+1\right)}\right)+\frac{d}{ds}\left(\frac{1}{s\left(s+1\right)}\right)\right]\\
&= \frac{s\left(s+1\right)}{s^2+1}\left[\frac{1}{\left(s+1\right)^2}-\frac{1}{s^2}\frac{2s+1}{\left(s+1\right)^2}\right]\\
&= \frac{s}{s^2+1}\left[\frac{1}{s+1}-\frac{1}{s^2}\frac{2s+1}{\left(s+1\right)}\right]\\
&= \frac{s}{\left(s^2+1\right)\left(s+1\right)}-\frac{1}{s}\frac{2s+1}{\left(s^2+1\right)\left(s+1\right)}\\
&= \frac{s^2-2s-1}{s\left(s^2+1\right)\left(s+1\right)}
\end{aligned}$$

Using partial fractions $\frac{s^2-2s-1}{s\left(s^2+1\right)\left(s+1\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}+\frac{D}{s+1}$, hence

$$\begin{aligned}
s^2-2s-1 &= A\left(s^2+1\right)\left(s+1\right)+\left(Bs+C\right)s\left(s+1\right)+Ds\left(s^2+1\right)\\
&= A\left(s^3+s^2+s+1\right)+\left(Bs^3+Bs^2+Cs^2+Cs\right)+Ds^3+Ds\\
&= s^3\left(A+B+D\right)+s^2\left(A+B+C\right)+s\left(A+C+D\right)+A
\end{aligned}$$

Comparing coefficients gives,

$$A=-1$$

And

$$\begin{aligned}
B+D&=1\\
B+C&=2\\
C+D&=0
\end{aligned}$$

Hence $B=1-D$, and from second equation $1-D+C=2$ or $C=D+1$, hence from third equation $D+1+D=-1$ or $D=-1$ hence $C=0$ and $B=2$, therefore

$$\frac{s^2-2s-1}{s\left(s^2+1\right)\left(s+1\right)}=-\frac{1}{s}+\frac{2s}{s^2+1}-\frac{1}{s+1}$$

Now the inverse Laplace transform can be found. Using $\mathscr{L}^{-1}\left(\frac{s}{s^2+a}\right)=\cos\left(at\right)$ then $\mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right)=\cos t$ and

$$\begin{aligned}
tg\left(t\right) &= -\mathscr{L}^{-1}\left(-\frac{1}{s}+\frac{2s}{s^2+1}-\frac{1}{s+1}\right)\\
&= -\left(-1+2\cos t-e^{-t}\right)
\end{aligned}$$

Therefore

$$g\left(t\right)=\frac{1}{t}\left(1-2\cos t+e^{-t}\right)$$

### 2.6.3 Problem 2

Use Laplace transform to solve $\left(1-t\right)y''+ty'-y=0$ with IC $y\left(0\right)=3$ and $y'\left(0\right)=-1$

**Answer**:

$$\begin{aligned}
\mathscr{L}\left(\left(1-t\right)y''+ty'-y\right) &= \mathscr{L}\left(y''-ty''+ty'-y\right)\\
&= \left(s^2Y-sy\left(0\right)-y'\left(0\right)\right)+\frac{d}{ds}\left(s^2Y-sy\left(0\right)-y'\left(0\right)\right)-\frac{d}{ds}\left(sY-y\left(0\right)\right)-Y
\end{aligned}$$

Substituting IC gives

$$\begin{aligned}\mathscr{L}\left((1-t)\,y''+ty'-y\right)&=\left(s^2Y-3s+1\right)+\frac{d}{ds}\left(s^2Y-3s+1\right)-\frac{d}{ds}\left(sY-3\right)-Y\\&=s^2Y-3s+1+2sY+s^2Y'-3-Y-sY'-Y\\&=Y'\left(s^2-s\right)+Y\left(s^2+2s-2\right)-3s-2\end{aligned}$$

Therefore

$$\begin{aligned}Y'\left(s^2-s\right)+Y\left(s^2+2s-2\right)&=3s+2\\Y'+Y\frac{\left(s^2+2s-2\right)}{\left(s^2-s\right)}&=\frac{3s+2}{\left(s^2-s\right)}\end{aligned}$$

Integrating factor is $I_f=e^{\int\frac{\left(s^2+2s-2\right)}{\left(s^2-s\right)}ds}$ where

$$\begin{aligned}\int\frac{\left(s^2+2s-2\right)}{\left(s^2-s\right)}ds&=\int 1+\frac{1}{s-1}+\frac{2}{s}ds\\&=s+\ln\left(s-1\right)+2\ln s\end{aligned}$$

Hence

$$\begin{aligned}I_f&=e^{(s+2\ln s+\ln(s-1))}\\&=e^s s^2\left(s-1\right)\end{aligned}$$

Hence

$$\begin{aligned}d\left(I_fY\right)&=I_f\frac{3s+2}{\left(s^2-s\right)}\\I_fY&=\int I_f\frac{3s+2}{\left(s^2-s\right)}ds+c\\&=\int e^s s^2\left(s-1\right)\frac{3s+2}{\left(s^2-s\right)}ds+c\\&=\int e^s s\left(3s+2\right)ds+c\\&=\int 3s^2e^sds+\int 2se^sds+c\\&=3e^s\left(s^2-2s+2\right)+2e^s\left(s-1\right)+c\end{aligned}$$

Hence

$$\begin{aligned}Y&=\frac{3e^s\left(s^2-2s+2\right)+2e^s\left(s-1\right)}{e^s s^2\left(s-1\right)}+\frac{c}{e^s s^2\left(s-1\right)}\\&=3\frac{\left(s^2-2s+2\right)}{s^2\left(s-1\right)}+2\frac{1}{s^2}+c\frac{e^{-s}}{s^2\left(s-1\right)}\\&=\left(3\frac{1}{s-1}-6\frac{1}{s^2}\right)+2\frac{1}{s^2}+c\frac{e^{-s}}{s^2\left(s-1\right)}\\&=\frac{3}{s-1}-4\frac{1}{s^2}+ce^{-s}\left(\frac{1}{s-1}-\frac{1}{s}-\frac{1}{s^2}\right)\end{aligned}$$

Therefore, inverse Laplace transform is

$$\begin{aligned}y\left(t\right)&=3e^t-4t+cH\left(t-1\right)\left(e^{(t-1)}-1-\left(t-1\right)\right)\\&=3e^t-4t+cH\left(t-1\right)\left(e^{(t-1)}-t\right)\end{aligned}$$

And the solution is

$$y\left(t\right)=\begin{cases}3e^t-4t & 0\leq t<1\\3e^t-4t+c\left(e^{(t-1)}-t\right) & 1\leq t\end{cases}$$

Notice that there is not enough information given in the problem to find the constant $c$ for the solution $1\leq t$.

### 2.6.4 Problem 3, periodic function

Solve the initial value problem where $f(t)$ is periodic

$$y' + 4y + 3\int_0^t y(\tau)\,d\tau = f(t)$$

And

$$f(t) = \begin{cases} 1 & 0 < t < 2 \\ -1 & 2 < t < 4 \end{cases}$$

And $y(0) = 1$.

**Solution**:

The Laplace transform of periodic function $f(t)$ with period $k$ is

$$F(s) = \frac{1}{1 - e^{-ks}} \int_0^k f(t)\,e^{-st}dt$$

For the above function $k = 4$ and

$$\begin{aligned}
\int_0^4 f(t)\,e^{-st}dt &= \int_0^2 e^{-st}dt - \int_2^4 e^{-st}dt \\
&= \left.\frac{e^{-st}}{-s}\right|_0^2 - \left.\frac{e^{-st}}{-s}\right|_2^4 \\
&= \frac{-1}{s}\left(e^{-2s} - 1\right) + \frac{1}{s}\left(e^{-4s} - e^{-2s}\right) \\
&= \frac{-e^{-2s}}{s} + \frac{1}{s} + \frac{e^{-4s}}{s} - \frac{e^{-2s}}{s} \\
&= \frac{1}{s}\left(1 - 2e^{-2s} + e^{-4s}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
F(s) &= \left(\frac{1}{1 - e^{-4s}}\right)\frac{1 - 2e^{-2s} + e^{-4s}}{s} \\
&= \frac{1}{s}\left(\frac{1}{1 - e^{-4s}}\right)\left(1 - e^{-2s}\right)^2 \\
&= \frac{1}{s}\left(\frac{1}{\left(1 - e^{-2s}\right)\left(1 + e^{-2s}\right)}\right)\left(1 - e^{-2s}\right)^2 \\
&= \frac{1}{s}\frac{\left(1 - e^{-2s}\right)}{\left(1 + e^{-2s}\right)}
\end{aligned}$$

Using $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ the above can be written as (using $e^{-2s}$ as $x$)

$$\begin{aligned}
F(s) &= \frac{1}{s}\left(1 - e^{-2s}\right)\left(1 - e^{-2s} + e^{-4s} - e^{-6s} + e^{-8s} - e^{-10s}\cdots\right) \\
&= \frac{1}{s}\left[\left(1 - e^{-2s} + e^{-4s} - e^{-6s} + e^{-8s} - e^{-10s} + \cdots\right) + \left(-e^{-2s} + e^{-4s} - e^{-6s} + e^{-8s} - e^{-10s}\cdots\right)\right] \\
&= \frac{1}{s}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)
\end{aligned}$$

Now taking the Laplace transform of the ODE gives

$$\left(sY - y(0)\right) + 4Y + 3\mathscr{L}\left(\int_0^t y(\tau)\,d\tau\right) = \frac{1}{s}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)$$

Using the property that $\mathscr{L}^{-1}\left(\frac{1}{s}Y(s)\right) = \int_0^t y(\tau)\,d\tau$, therefore $\frac{1}{s}Y(s) = \mathscr{L}\left(\int_0^t y(\tau)\,d\tau\right)$ where

$Y(s) = \mathscr{L}\left(y(t)\right)$, hence the above becomes

$$\begin{aligned}
(sY-1) + 4Y + 3\frac{Y}{s} &= \frac{1}{s}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)\\
Y\left(s + 4 + \frac{3}{s}\right) &= 1 + \frac{1}{s}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)\\
Y &= \frac{s}{s^2+4s+3} + \frac{1}{s\left(s+4+\frac{3}{s}\right)}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)\\
&= \frac{s}{s^2+4s+3} + \frac{1}{s^2+4s+3}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)\\
&= \frac{s}{(s+3)(s+1)} + \frac{1}{(s+3)(s+1)}\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} + \cdots\right)\\
&= \frac{s}{(s+3)(s+1)} + \frac{1}{(s+3)(s+1)} - \frac{2e^{-2s}}{(s+3)(s+1)}\\
&+ \frac{2e^{-4s}}{(s+3)(s+1)} - \frac{2e^{-6s}}{(s+3)(s+1)} + \frac{2e^{-8s}}{(s+3)(s+1)} - \cdots
\end{aligned}$$

Partial fractions of $\frac{2}{(s+3)(s+1)} = \frac{1}{s+1} - \frac{1}{s+3}$ and Partial fractions of $\frac{1}{(s+3)(s+1)} = \frac{1}{2(s+1)} - \frac{1}{2(s+3)}$ and partial fractions of $\frac{s}{(s+3)(s+1)} = \frac{3}{2(s+3)} - \frac{1}{2(s+1)}$ and using the property $\mathscr{L}^{-1}\left(e^{-as}f(s)\right) = H(t-a)f(t-a)$ and since $\mathscr{L}^{-1}\frac{1}{s+1} = e^{-t}$ and $\mathscr{L}^{-1}\frac{1}{s+3} = e^{-3t}$, hence the inverse Transform of the above can be written as

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(\frac{3}{2(s+3)} - \frac{1}{2(s+1)}\right)\\
&+ \mathscr{L}^{-1}\left(\frac{1}{2(s+1)} - \frac{1}{2(s+3)}\right)\\
&- \mathscr{L}^{-1}\left(\frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+3}\right)\\
&+ \mathscr{L}^{-1}\left(\frac{e^{-4s}}{s+1} - \frac{e^{-4s}}{s+3}\right)\\
&- \mathscr{L}^{-1}\left(\frac{e^{-6s}}{s+1} - \frac{e^{-6s}}{s+3}\right) + \cdots
\end{aligned}$$

Hence

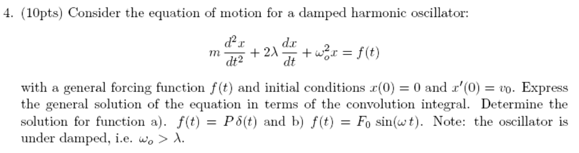
$$\begin{aligned}
y(t) &= \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t}\\
&+ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\\
&- H(t-2)\left(e^{-(t-2)} - e^{-3(t-2)}\right)\\
&+ H(t-4)\left(e^{-(t-4)} - e^{-3(t-4)}\right)\\
&- H(t-6)\left(e^{-(t-6)} - e^{-3(t-6)}\right) + \cdots
\end{aligned}$$

Or

$$y(t) = e^{-3t} - H(t-2)\left(e^{-(t-2)} - e^{-3(t-2)}\right) + H(t-4)\left(e^{-(t-4)} - e^{-3(t-4)}\right) - H(t-6)\left(e^{-(t-6)} - e^{-3(t-6)}\right) + \cdots$$

Here is a plot of the solution. It shows the response to be periodic, of same period as the forcing function.

### 2.6.5 Problem 4 convolution



4. (10pts) Consider the equation of motion for a damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_o^2 x = f(t)$$

with a general forcing function $f(t)$ and initial conditions $x(0) = 0$ and $x'(0) = v_0$. Express the general solution of the equation in terms of the convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0\sin(\omega t)$. Note: the oscillator is under damped, i.e. $\omega_o > \lambda$.

**Solution**:

The problem as stated had a typo in it. It was updated to read

$$x'' + 2\lambda x' + \omega_0^2 x = \frac{f(t)}{m}$$

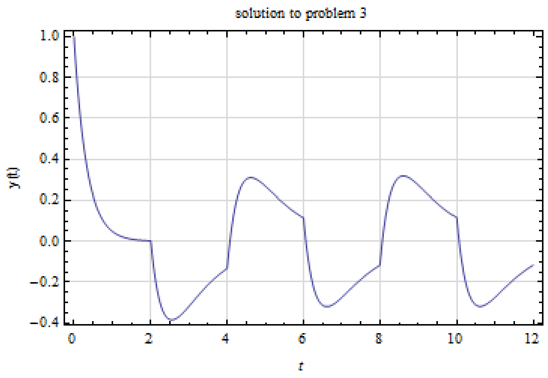


Figure 2.9: plot of solution problem 3 HW 5

Where $\lambda = \frac{\rho}{2m}$ where $\rho$ is the damping coefficient and $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency and $k$ is the stiffness. Using the above form, and letting $F(s)$ be the Laplace transform of $f(t)$ and $X(s)$ the Laplace transform of the solution $x(t)$ and applying the Laplace transform results in

$$\left(s^2X - sx(0) - x'(0)\right) + 2\lambda\left(sX - x(0)\right) + \omega_0^2 X = \frac{1}{m}F(s)$$

Applying initial conditions

$$\begin{aligned}\left(s^2X - v_0\right) + 2\lambda sX + \omega_0^2 X &= \frac{1}{m}F\\ X\left(s^2 + 2\lambda s + \omega_0^2\right) &= \frac{1}{m}F + v_0\\ X &= \frac{1}{m}\frac{F}{s^2 + 2\lambda s + \omega_0^2} + \frac{v_0}{s^2 + 2\lambda s + \omega_0^2} \qquad (1)\end{aligned}$$

Finding the roots of the characteristic equation $s^2 + 2\lambda s + \omega^2$ gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$$

Since underdamped, therefore $\omega > \lambda$ then the quantity under the radical is negative and the roots are complex conjugate

$$r = -\lambda \pm i\sqrt{\omega_0^2 - \lambda^2} = -\lambda \pm i\omega_0\sqrt{1 - \frac{\lambda^2}{\omega_0^2}} = -\frac{\rho}{2m} \pm i\omega_0\sqrt{1 - \frac{\rho^2}{4m^2\omega_0^2}}$$

But $\frac{\rho^2}{4m^2\omega_0^2} = \xi^2$ where $\xi = \frac{\rho}{2m\omega_0}$ is the damping ratio. Hence the above can be written in the more common form as

$$r = -\frac{\rho}{2m} \pm i\omega_0\sqrt{1 - \frac{\rho^2}{4m^2\omega^2}} = -\xi\omega_0 \pm i\omega_0\sqrt{1 - \xi^2}$$

In addition, defining

$$\omega_d = \omega_0\sqrt{1 - \xi^2}$$

as the natural damped frequency, hence

$$r = -\frac{\rho}{2m} \pm i\omega_0\sqrt{1 - \xi^2} = -\xi\omega_0 \pm i\omega_d$$

Let $r_1 = -\xi\omega_0 + i\omega_d, r_2 = -\xi\omega_0 - i\omega_d$. Eq. (1) can now be written as

$$X = \frac{1}{m}\frac{F}{(s - r_1)(s - r_2)} + \frac{v_0}{(s - r_1)(s - r_2)}$$

Let

$$\begin{aligned}G(s) &= \frac{1}{(s - r_1)(s - r_2)}\\ &= \frac{A}{(s - r_1)} + \frac{B}{(s - r_2)}\end{aligned}$$

Where

$$A = \lim_{s \to r_1}\frac{1}{(s - r_2)} = \frac{1}{(-\xi\omega_0 + i\omega_d - (-\xi\omega_0 - i\omega_d))} = \frac{1}{2i\omega_d}$$

And

$$B = \lim_{s\to r_2} \frac{1}{(s-r_1)} = \frac{1}{(-\xi\omega_0 - i\omega_d - (-\xi\omega_0 + i\omega_d))} = \frac{1}{-2i\omega_d}$$

Therefore

$$\begin{aligned}
\mathscr{L}^{-1}(G(s)) &= \mathscr{L}^{-1}\left(\frac{1}{2i\omega_0\sqrt{1-\xi}}\frac{1}{(s-r_1)} - \frac{1}{2i\omega_0\sqrt{1-\xi}}\frac{1}{(s-r_2)}\right)\\
&= \frac{1}{2i\omega_0\sqrt{1-\xi}}\mathscr{L}^{-1}\left(\frac{1}{(s-r_1)} - \frac{1}{(s-r_2)}\right)\\
&= \frac{1}{2i\omega_d}\left(e^{r_1 t} - e^{r_2 t}\right)\\
&= \frac{1}{2i\omega_d}\left(e^{(-\xi\omega_0 + i\omega_d)t} - e^{(-\xi\omega_0 - i\omega_d)t}\right)\\
&= \frac{1}{2i\omega_d}\left(e^{-\xi\omega_0 t}e^{i\omega_d t} - e^{-\xi\omega_0 t}e^{-i\omega_d t}\right)\\
&= \frac{e^{-\xi\omega_0 t}}{2i\omega_d}\left(e^{i\omega_d t} - e^{-i\omega_d t}\right)\\
&= \frac{e^{-\xi\omega_0 t}}{2i\omega_d}\left(2i\sin(\omega_d t)\right)\\
&= \frac{e^{-\xi\omega_0 t}}{\omega_d}\sin(\omega_d t)
\end{aligned}$$

Therefore the general solution is

$$x(t) = \frac{1}{m}\mathscr{L}^{-1}(F(s)\,G(s)) + v_0\frac{e^{-\xi\omega_0 t}}{\omega_d}\sin(\omega_d t)$$

But $\mathscr{L}^{-1}(F(s)\,G(s)) = \int_0^t f(t-\tau)\,g(\tau)\,dt$, hence the above becomes

$$\begin{aligned}
x(t) &= \frac{1}{m}\int_0^t f(t-\tau)\left[\overbrace{\frac{e^{-\xi\omega_0\tau}}{\omega_d}\sin(\omega_d\tau)}^{g(\tau)}\right]d\tau + v_0\frac{e^{-\xi\omega_0 t}}{\omega_d}\sin(\omega_d t)\\
&= \frac{1}{m\omega_d}\left[\int_0^t f(t-\tau)\,e^{-\xi\omega_0\tau}\sin(\omega_d\tau)\,d\tau + mv_0 e^{-\xi\omega_0 t}\sin(\omega_d t)\right]
\end{aligned}$$

Now the problem can be solved for different $f(t)$.

#### 2.6.5.1 Part (a)

When $f(t) = P\delta(t)$ then the solution is

$$x(t) = \frac{1}{m\omega_d}\left[\int_0^t P\delta(t-\tau)\,e^{-\xi\omega_0\tau}\sin(\omega_d\tau)\,d\tau + mv_0\frac{e^{-\xi\omega_0 t}}{\omega_d}\sin(\omega_d t)\right]$$

But

$$\int_0^t \delta(t-\tau)\,r(\tau)\,d\tau = r(t)$$

Hence

$$\begin{aligned}
x(t) &= \frac{1}{m\omega_d}\left[Pe^{-\xi\omega_0 t}\sin(\omega_d t) + mv_0 e^{-\xi\omega_0 t}\sin(\omega_d t)\right]\\
&= (P + mv_0)\frac{e^{-\xi\omega_0 t}}{m\omega_d}\sin(\omega_d t)
\end{aligned}$$

We see that the effect of having an initial velocity is to contribute an additional impulse of magnitude $mv_0$. Therefore, this response is the same to one where an impulse of amplitude $(P + mv_0)$ instead of $P$ was applied to the same system when all its initial conditions are zero.

To express the above solution using the terms in the original ODE, then using $\xi = \frac{\rho}{2m\omega_0}$

and $\omega_d = \omega_0\sqrt{1-\left(\frac{\rho}{2m\omega_0}\right)^2}$ then the solution is

$$x(t) = (P+mv_0)\frac{e^{-\frac{\rho}{2m}t}}{m\omega_0\sqrt{1-\left(\frac{\rho}{2m\omega_0}\right)^2}}\sin\left(\omega_0\sqrt{1-\left(\frac{\rho}{2m\omega_0}\right)^2}\,t\right)$$

#### 2.6.5.2 Part(b)

When $f(t) = F_0\sin(\omega t)$ then the solution is

$$x(t) = \frac{1}{m\omega_d}\left[\int_0^t F_0\sin(\omega(t-\tau))\,e^{-\xi\omega_0\tau}\sin(\omega_d\tau)\,d\tau + mv_0e^{-\xi\omega_0 t}\sin(\omega_d t)\right]$$

The $\int_0^t \sin(\omega(t-\tau))\,e^{-\xi\omega_0\tau}\sin(\omega_d\tau)\,d\tau$ can be done be repeated use of integration by parts by starting with writing

$$\sin(\omega(t-\tau)) = \cos\omega\tau\sin\omega t - \cos\omega t\sin\omega\tau$$

The final result can be found to be[5]

$$x(t) = e^{-\xi\omega_0\tau}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{m\omega_0^2}\frac{\left(\left(1-r^2\right)\sin\omega t - 2\xi r\cos\omega t\right)}{\left(1-r^2\right)^2+(2\xi r)^2}$$

Where

$$r = \frac{\omega}{\omega_0}$$

and

$$A = \frac{2rF\xi}{m\omega_0^2}\frac{1}{\left(1-r^2\right)^2+(2\xi r)^2}$$

and

$$B = \frac{v_0}{\omega_d} - \frac{F\left(1-r^2\right)}{m\omega_0\omega_d}\frac{r}{\left(1-r^2\right)^2+(2\xi r)^2} + \frac{2rF\xi}{m\omega_0\omega_d}\frac{1}{\left(1-r^2\right)^2+(2\xi r)^2}$$

### 2.6.6 Problem 5, heat conduction

For $0<x<L$ and $t>0$

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = ae^{-\alpha t}$$

IC $u(x,0)=0$ and BC: $u(0,t) = u(L,t) = 0$

Solve using convolution.

**Solution**:

Lower case $u$ is used for $u(x,t)$ in the time domain so to be able to use $U(x,s)$ as the transform so to keep the notation simple. Now, Taking Laplace transform, using $t$ as the corresponding independent variable for $s$, hence

$$\begin{aligned}\mathscr{L}\left(\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2}\right) &= a\mathscr{L}\left(e^{-\alpha t}\right)\\ (sU(x,s) - u(x,0)) - k\frac{d^2U(x,s)}{dx^2} &= \frac{a}{s+\alpha}\\ sU(x,s) - k\frac{d^2U(x,s)}{dx^2} &= \frac{a}{s+\alpha}\end{aligned}$$

Now this is a second order ODE in $U(x,s)$

$$\frac{d^2U(x,s)}{dx^2} - \frac{s}{k}U(x,s) = \frac{-1}{k}\frac{a}{s+\alpha}$$

To solve, it requires two boundary conditions. The boundary condition are transformed

---

[5] CAS was used for the integration

to Laplace, hence $U(0,s)=0$ and $U(L,s)=0$. The homogenous solution is

$$U_h(x,s)=Ae^{-\sqrt{\frac{s}{k}}x}+Be^{\sqrt{\frac{s}{k}}x}$$

and the particular solution is (since RHS is constant)

$$U_p(x,s)=\frac{1}{s}\frac{a}{s+\alpha}$$

Hence the solution is

$$\begin{aligned}U(x,s)&=U_h+U_p\\&=\left(Ae^{-\sqrt{\frac{s}{k}}x}+Be^{\sqrt{\frac{s}{k}}x}\right)+\frac{1}{s}\frac{a}{s+\alpha}\end{aligned}$$

When $x=0$

$$\frac{-1}{s}\frac{a}{s+\alpha}=(A+B)\tag{1}$$

and when $x=L$

$$\frac{-1}{s}\frac{a}{s+\alpha}=\left(Ae^{-\sqrt{\frac{s}{k}}L}+Be^{\sqrt{\frac{s}{k}}L}\right)\tag{2}$$

From (1), $A=-\left(\frac{1}{s}\frac{a}{s+\alpha}+B\right)$, hence from Eq. (2)

$$\begin{aligned}\frac{-1}{s}\frac{a}{s+\alpha}&=-\left(\frac{1}{s}\frac{a}{s+\alpha}+B\right)e^{-\sqrt{\frac{s}{k}}L}+Be^{\sqrt{\frac{s}{k}}L}\\\frac{-1}{s}\frac{a}{s+\alpha}&=-\frac{1}{s}\frac{a}{s+\alpha}e^{-\sqrt{\frac{s}{k}}L}-Be^{-\sqrt{\frac{s}{k}}L}+Be^{\sqrt{\frac{s}{k}}L}\\\frac{-1}{s}\frac{a}{s+\alpha}+\frac{1}{s}\frac{a}{s+\alpha}e^{-\sqrt{\frac{s}{k}}L}&=B\left(e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}\right)\\\frac{1}{s}\frac{a}{s+\alpha}\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)&=2B\left(\sinh\left(\sqrt{\frac{s}{k}}L\right)\right)\\B&=\frac{\frac{1}{s}\frac{a}{s+\alpha}\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\\&=\frac{1}{s}\frac{a}{s+\alpha}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\end{aligned}$$

Therefore

$$\begin{aligned}A&=\frac{-1}{s}\frac{a}{s+\alpha}-\frac{1}{s}\frac{a}{s+\alpha}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\\&=\frac{1}{s}\frac{a}{s+\alpha}\left(-1+\frac{1}{2}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}\right)\end{aligned}$$

And the solution is

$$
\begin{aligned}
U(x,s) &= \left(Ae^{-\sqrt{\frac{s}{k}}x}+Be^{\sqrt{\frac{s}{k}}x}\right)+\frac{1}{s}\frac{a}{s+\alpha}\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left(-1+\frac{1}{2}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}\right)e^{-\sqrt{\frac{s}{k}}x}+\frac{1}{s}\frac{a}{s+\alpha}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}e^{\sqrt{\frac{s}{k}}x}+\frac{1}{s}\frac{a}{s+\alpha}\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[\left(-1+\frac{1}{2}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}\right)e^{-\sqrt{\frac{s}{k}}x}+\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}e^{\sqrt{\frac{s}{k}}x}+1\right]\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[-e^{-\sqrt{\frac{s}{k}}x}+1+\frac{1}{2}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}e^{-\sqrt{\frac{s}{k}}x}+\frac{\left(e^{-\sqrt{\frac{s}{k}}L}-1\right)}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}e^{\sqrt{\frac{s}{k}}x}\right]\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[-e^{-\sqrt{\frac{s}{k}}x}+1+\frac{1}{2}\frac{\left(e^{-\sqrt{\frac{s}{k}}L}e^{-\sqrt{\frac{s}{k}}x}-e^{-\sqrt{\frac{s}{k}}x}\right)+\left(e^{-\sqrt{\frac{s}{k}}L}e^{\sqrt{\frac{s}{k}}x}-e^{\sqrt{\frac{s}{k}}x}\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}\right]\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[-e^{-\sqrt{\frac{s}{k}}x}+1+\frac{1}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\left(e^{-\sqrt{\frac{s}{k}}L}e^{-\sqrt{\frac{s}{k}}x}-e^{-\sqrt{\frac{s}{k}}x}+e^{-\sqrt{\frac{s}{k}}L}e^{\sqrt{\frac{s}{k}}x}-e^{\sqrt{\frac{s}{k}}x}\right)\right]\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[-e^{-\sqrt{\frac{s}{k}}x}+1+\frac{1}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\left(e^{-\sqrt{\frac{s}{k}}(x+L)}+e^{\sqrt{\frac{s}{k}}(x-L)}+-\left(e^{-\sqrt{\frac{s}{k}}x}+e^{\sqrt{\frac{s}{k}}x}\right)\right)\right]\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left[-e^{-\sqrt{\frac{s}{k}}x}+1+\frac{1}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}\left(e^{\sqrt{\frac{s}{k}}(x-L)}+e^{-\sqrt{\frac{s}{k}}(x+L)}-\left(2\cosh\left(\sqrt{\frac{s}{k}}L\right)\right)\right)\right]
\end{aligned}
$$

Hence

$$
\begin{aligned}
Y(s) &= \frac{1}{s}\frac{a}{s+\alpha}\left(1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}-\frac{\cosh\left(\sqrt{\frac{s}{k}}L\right)}{\sinh\left(\sqrt{\frac{s}{k}}L\right)}\right)\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left(1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{\left(e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}\right)}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{\left(e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}\right)}-\coth\left(\sqrt{\frac{s}{k}}L\right)\right)\\
&= \frac{1}{s}\frac{a}{s+\alpha}\left(1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{2\sinh\left(\sqrt{\frac{s}{k}}L\right)}-\coth\left(\sqrt{\frac{s}{k}}L\right)\right)
\end{aligned}
$$

Now the inverse Laplace transform is found. Let

$$G(s)=\frac{1}{s}\frac{a}{s+\alpha}$$

and

$$
\begin{aligned}
F(s) &= 1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{\sqrt{\frac{s}{k}}L}+e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\\
&= 1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}
\end{aligned}
$$

hence

$$\begin{aligned}u(x,t) &= \mathscr{L}^{-1}\left(F(s)\,G(s)\right)\\ &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \qquad (3)\end{aligned}$$

Where

$$\begin{aligned}g(t) &= \mathscr{L}^{-1}\left(\frac{1}{s}\frac{a}{s+\alpha}\right)\\ &= \mathscr{L}^{-1}\left(\frac{a}{s\alpha}-\frac{a}{\alpha(s+\alpha)}\right)\\ &= \frac{a}{\alpha}-\frac{a}{\alpha}e^{-\alpha t}\end{aligned}$$

And

$$f(t) = \mathscr{L}^{-1}\left(1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right) \qquad (4)$$

Note:

$$\mathscr{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}x}\right) = \frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} \qquad \text{for}\sqrt{\frac{1}{k}}x>0$$

To find the inverse Laplace of $\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}$ , dividing by $e^{\sqrt{\frac{s}{k}}L}$the numerator and denominator gives

$$\frac{e^{\sqrt{\frac{s}{k}}(x-2L)}}{\left(1-e^{-2\sqrt{\frac{s}{k}}L}\right)}$$

Now $\frac{1}{1-z}=1+z+z^2+\cdots$ by series expansion, hence the above becomes

$$\begin{aligned}\frac{e^{\sqrt{\frac{s}{k}}(x-2L)}}{\left(1-e^{-2\sqrt{\frac{s}{k}}L}\right)} &= e^{\sqrt{\frac{s}{k}}(x-2L)}\left(1+e^{-\sqrt{\frac{s}{k}}2L}+e^{-\sqrt{\frac{s}{k}}4L}+e^{-\sqrt{\frac{s}{k}}6L}+\cdots\right)\\ &= \left(e^{\sqrt{\frac{s}{k}}(x-2L)}+e^{\sqrt{\frac{s}{k}}(x-4L)}+e^{\sqrt{\frac{s}{k}}(x-6L)}+\cdots\right)\end{aligned}$$

Similarly, to find Laplace of $\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}$ ,dividing by $e^{\sqrt{\frac{s}{k}}L}$ the numerator and denominator gives

$$\frac{e^{-\sqrt{\frac{s}{k}}(x+2L)}}{\left(1-e^{-2\sqrt{\frac{s}{k}}L}\right)}$$

Now $\frac{1}{1-z}=1+z+z^2+\cdots$ by series expansion, hence the above becomes

$$\begin{aligned}\frac{e^{-\sqrt{\frac{s}{k}}(x+2L)}}{\left(1-e^{-2\sqrt{\frac{s}{k}}L}\right)} &= e^{-\sqrt{\frac{s}{k}}(x+2L)}\left(1+e^{-\sqrt{\frac{s}{k}}2L}+e^{-\sqrt{\frac{s}{k}}4L}+e^{-\sqrt{\frac{s}{k}}6L}+\cdots\right)\\ &= e^{-\sqrt{\frac{s}{k}}(x+2L)}+e^{-\sqrt{\frac{s}{k}}(x+4L)}+e^{-\sqrt{\frac{s}{k}}(x+6L)}+e^{-\sqrt{\frac{s}{k}}(x+8L)}+\cdots\end{aligned}$$

Similarly, to find Laplace of $\frac{e^{\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}$ ,dividing by $e^{\sqrt{\frac{s}{k}}L}$ the numerator and denominator gives

$$\frac{1}{1-e^{-2\sqrt{\frac{s}{k}}L}}$$

Now $\frac{1}{1-z}=1+z+z^2+\cdots$ by series expansion, hence the above becomes

$$\frac{1}{1-e^{-2\sqrt{\frac{s}{k}}L}}=\left(1+e^{-2\sqrt{\frac{s}{k}}2L}+e^{-4\sqrt{\frac{s}{k}}L}+e^{-6\sqrt{\frac{s}{k}}L}+\cdots\right)$$

And to find Laplace of $\frac{e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}$ ,dividing by $e^{\sqrt{\frac{s}{k}}L}$ the numerator and denominator gives

$$\frac{e^{-2\sqrt{\frac{s}{k}}L}}{1-e^{-2\sqrt{\frac{s}{k}}L}}$$

Now $\frac{1}{1-z}=1+z+z^2+\cdots$ by series expansion, hence the above becomes

$$\begin{aligned}\frac{e^{-2\sqrt{\frac{s}{k}}L}}{1-e^{-2\sqrt{\frac{s}{k}}L}}&=e^{-2\sqrt{\frac{s}{k}}L}\left(1+e^{-2\sqrt{\frac{s}{k}}L}+e^{-4\sqrt{\frac{s}{k}}L}+e^{-6\sqrt{\frac{s}{k}}L}+\cdots\right)\\&=e^{-2\sqrt{\frac{s}{k}}L}+e^{-4\sqrt{\frac{s}{k}}2L}+e^{-6\sqrt{\frac{s}{k}}L}+e^{-8\sqrt{\frac{s}{k}}L}+\cdots\end{aligned}$$

Applying all these results to Eq. (4) above

$$\begin{aligned}f(t)&=\mathscr{L}^{-1}\left(1-e^{-\sqrt{\frac{s}{k}}x}+\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}+\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}-\frac{e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right)\\&=\mathscr{L}^{-1}1-\mathscr{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}x}\right)\\&+\mathscr{L}^{-1}\left(\frac{e^{\sqrt{\frac{s}{k}}(x-L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right)\\&+\mathscr{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s}{k}}(x+L)}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right)\\&-\mathscr{L}^{-1}\left(\frac{e^{\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right)\\&-\mathscr{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s}{k}}L}}{e^{\sqrt{\frac{s}{k}}L}-e^{-\sqrt{\frac{s}{k}}L}}\right)\end{aligned}$$

Hence

$$\begin{aligned}f(t)&=\delta(t)-\frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}}\\&+\mathscr{L}^{-1}\left(e^{\sqrt{\frac{s}{k}}(x-2L)}+e^{\sqrt{\frac{s}{k}}(x-4L)}+e^{\sqrt{\frac{s}{k}}(x-6L)}+\cdots\right)\\&+\mathscr{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}(x+2L)}+e^{-\sqrt{\frac{s}{k}}(x+4L)}+e^{-\sqrt{\frac{s}{k}}(x+6L)}+e^{-\sqrt{\frac{s}{k}}(x+8L)}+\cdots\right)\\&-\mathscr{L}^{-1}\left(1+e^{-2\sqrt{\frac{s}{k}}L}+e^{-4\sqrt{\frac{s}{k}}L}+e^{-6\sqrt{\frac{s}{k}}L}+\cdots\right)\\&-\mathscr{L}^{-1}\left(e^{-2\sqrt{\frac{s}{k}}L}+e^{-4\sqrt{\frac{s}{k}}2L}+e^{-6\sqrt{\frac{s}{k}}L}+e^{-8\sqrt{\frac{s}{k}}L}+\cdots\right)\end{aligned}$$

This can be written as

$$\begin{aligned}
f(t) &= \delta(t) - \frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} \\
&\quad + \mathscr{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}(2L-x)} + e^{-\sqrt{\frac{s}{k}}(4L-x)} + e^{-\sqrt{\frac{s}{k}}(6L-x)} + \cdots\right) \\
&\quad + \mathscr{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}(x+2L)} + e^{-\sqrt{\frac{s}{k}}(x+4L)} + e^{-\sqrt{\frac{s}{k}}(x+6L)} + e^{-\sqrt{\frac{s}{k}}(x+8L)} + \cdots\right) \\
&\quad - \mathscr{L}^{-1}\left(1 + 2e^{-2\sqrt{\frac{s}{k}}L} + 2e^{-4\sqrt{\frac{s}{k}}L} + 2e^{-6\sqrt{\frac{s}{k}}L} + 2e^{-8\sqrt{\frac{s}{k}}L} + \cdots\right)
\end{aligned}$$

Hence the result becomes

$$\begin{aligned}
f(t) &= \delta(t) - \frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} \\
&\quad + \frac{(2L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L-x)^2}{4kt}} + \frac{(4L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L-x)^2}{4kt}} + \frac{(6L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L-x)^2}{4kt}} + \cdots \\
&\quad + \frac{(x+2L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+2L)^2}{4kt}} + \frac{(x+4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+4L)^2}{4kt}} + \frac{(x+6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+6L)^2}{4kt}} + \cdots \\
&\quad - \delta(t) - \frac{2L}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L)^2}{4kt}} - \frac{(4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L)^2}{4kt}} - \frac{(6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L)^2}{4kt}} - \cdots \\
&\quad \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
f(t) &= -\frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} \\
&\quad + \frac{(2L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L-x)^2}{4kt}} + \frac{(4L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L-x)^2}{4kt}} + \frac{(6L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L-x)^2}{4kt}} + \cdots \\
&\quad + \frac{(x+2L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+2L)^2}{4kt}} + \frac{(x+4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+4L)^2}{4kt}} + \frac{(x+6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+6L)^2}{4kt}} + \cdots \\
&\quad - \frac{2L}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L)^2}{4kt}} - \frac{(4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L)^2}{4kt}} - \frac{(6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L)^2}{4kt}} - \cdots
\end{aligned}$$

Now convolution can be carried out. From Eq. (3)

$$\begin{aligned}
u(x,t) &= \int_0^t f(\tau)\, g(t-\tau)\, d\tau \\
&= \int_0^t f(\tau)\left(\frac{a}{\alpha} - \frac{a}{\alpha}e^{-\alpha(t-\tau)}\right) d\tau \\
&= \frac{a}{\alpha}\left(\int_0^t f(\tau)\, d\tau - \int_0^t f(\tau)\, e^{-\alpha(t-\tau)} d\tau\right)
\end{aligned}$$

Now

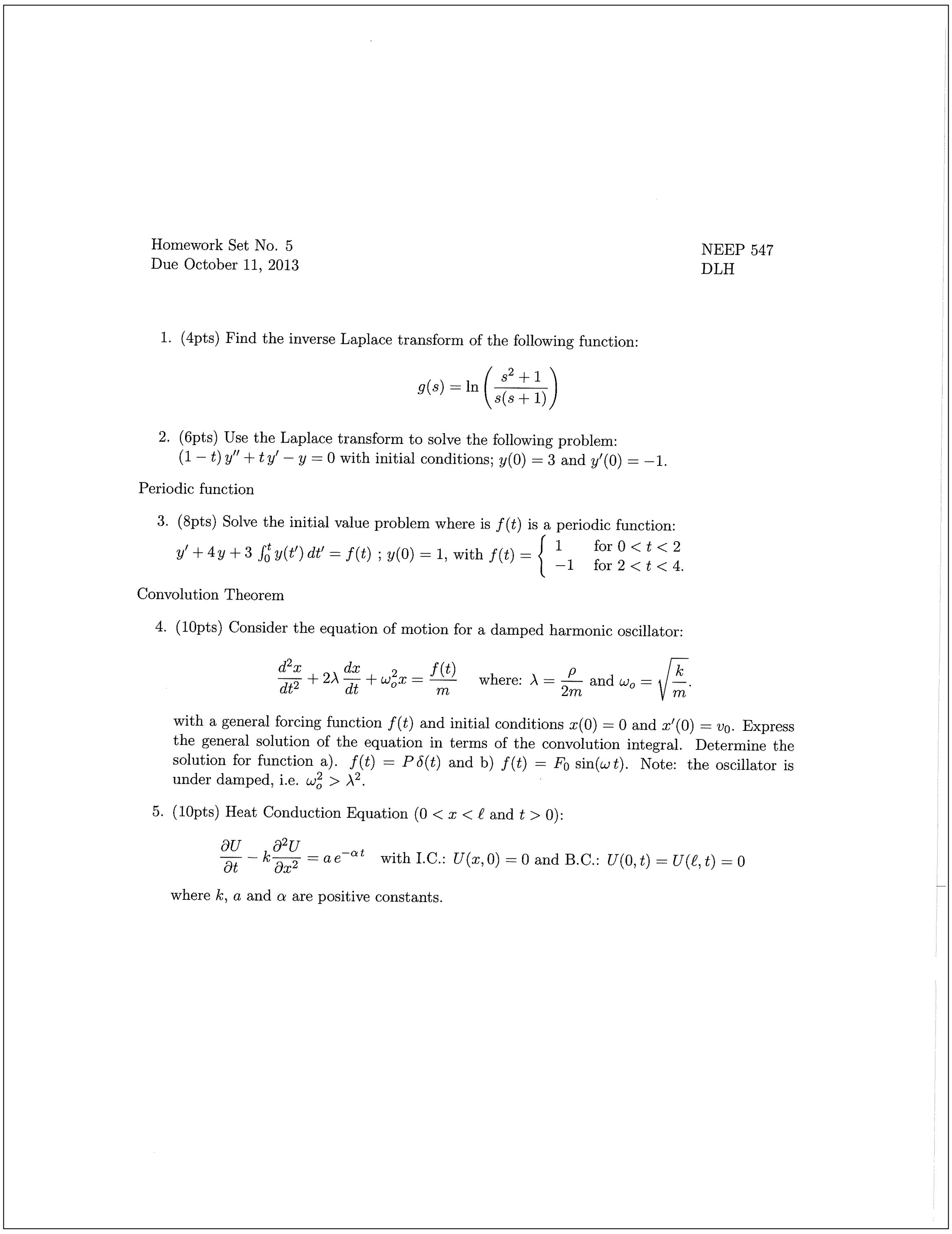
$$\begin{aligned}
\int_0^t f(\tau)\, d\tau = \int_0^t & -\frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} + + \frac{(2L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L-x)^2}{4kt}} + \frac{(4L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L-x)^2}{4kt}} + \frac{(6L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L-x)^2}{4kt}} + \cdots \\
& + \frac{(x+2L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+2L)^2}{4kt}} + \frac{(x+4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+4L)^2}{4kt}} + \frac{(x+6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(x+6L)^2}{4kt}} + \cdots \\
& - \frac{2L}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L)^2}{4kt}} - \frac{(4L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(4L)^2}{4kt}} - \frac{(6L)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(6L)^2}{4kt}} - \cdots
\end{aligned}$$

and

$$\int_0^t f(\tau)\, e^{-\alpha(t-\tau)} d\tau = \int_0^t \left(-\frac{x}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-x^2}{4kt}} + \frac{(2L-x)}{2\sqrt{\pi k}}\frac{1}{t^{\frac{3}{2}}}e^{\frac{-(2L-x)^2}{4kt}} + \cdots\right) e^{-\alpha(t-\tau)} d\tau$$

The last 2 integral, when evaluated give the solution $u(x,t)$

### 2.6.7 key solution



Homework Set No. 5
Due October 11, 2013

NEEP 547
DLH

1. (4pts) Find the inverse Laplace transform of the following function:

$$g(s) = \ln\left(\frac{s^2+1}{s(s+1)}\right)$$

2. (6pts) Use the Laplace transform to solve the following problem:
$(1-t)\,y'' + t\,y' - y = 0$ with initial conditions; $y(0) = 3$ and $y'(0) = -1$.

Periodic function

3. (8pts) Solve the initial value problem where is $f(t)$ is a periodic function:
$y' + 4y + 3\int_0^t y(t')\,dt' = f(t)$ ; $y(0) = 1$, with $f(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ -1 & \text{for } 2 < t < 4. \end{cases}$

Convolution Theorem

4. (10pts) Consider the equation of motion for a damped harmonic oscillator:

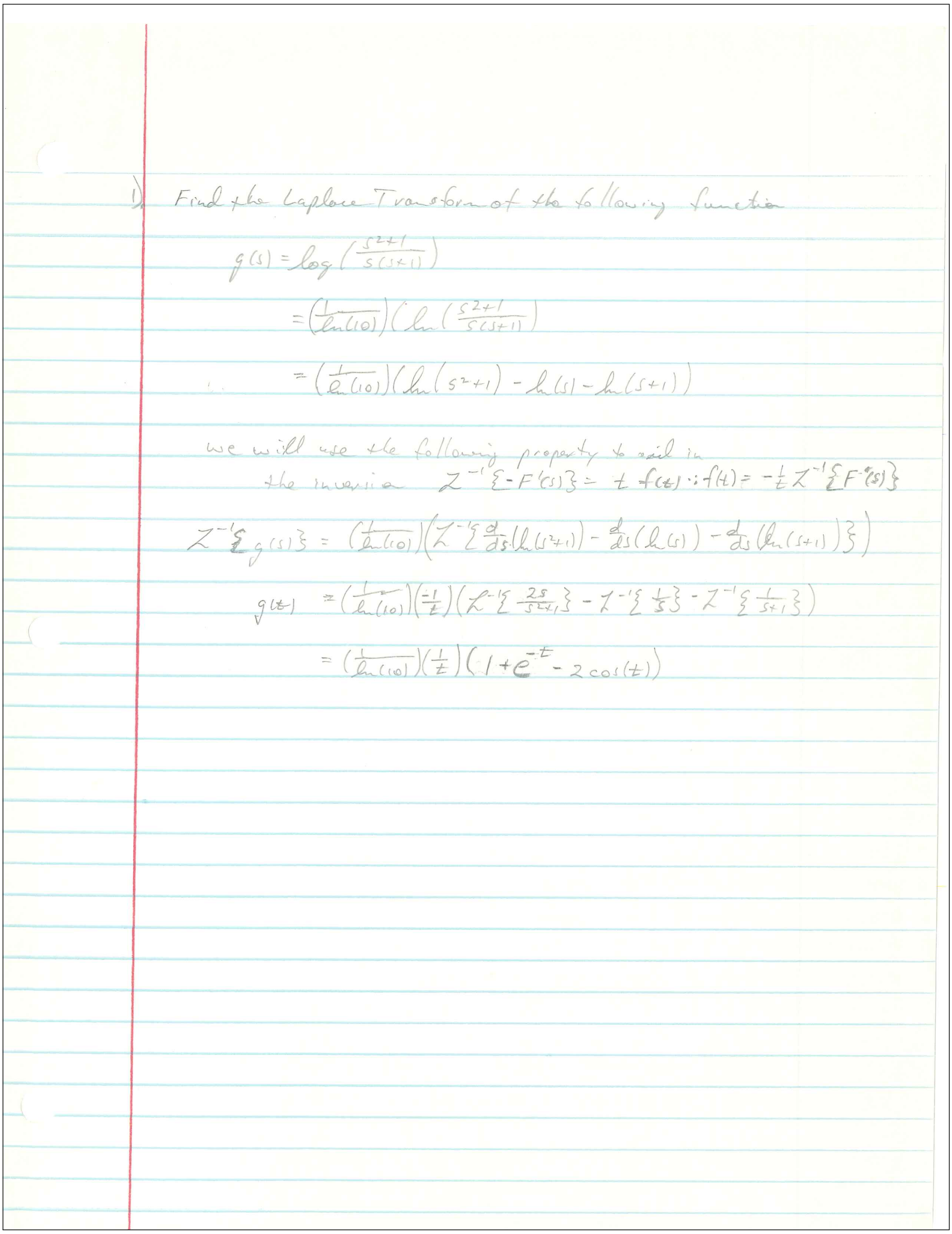
$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_o^2 x = \frac{f(t)}{m} \quad \text{where: } \lambda = \frac{\rho}{2m} \text{ and } \omega_o = \sqrt{\frac{k}{m}}.$$

with a general forcing function $f(t)$ and initial conditions $x(0) = 0$ and $x'(0) = v_0$. Express the general solution of the equation in terms of the convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0\,\sin(\omega\,t)$. Note: the oscillator is under damped, i.e. $\omega_o^2 > \lambda^2$.

5. (10pts) Heat Conduction Equation ($0 < x < \ell$ and $t > 0$):

$$\frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} = a\,e^{-\alpha t} \quad \text{with I.C.: } U(x,0) = 0 \text{ and B.C.: } U(0,t) = U(\ell,t) = 0$$

where $k$, $a$ and $\alpha$ are positive constants.



1) Find the Laplace Transform of the following function

$$g(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$$

$$= \left(\frac{1}{\ln(10)}\right)\left(\ln\left(\frac{s^2+1}{s(s+1)}\right)\right)$$

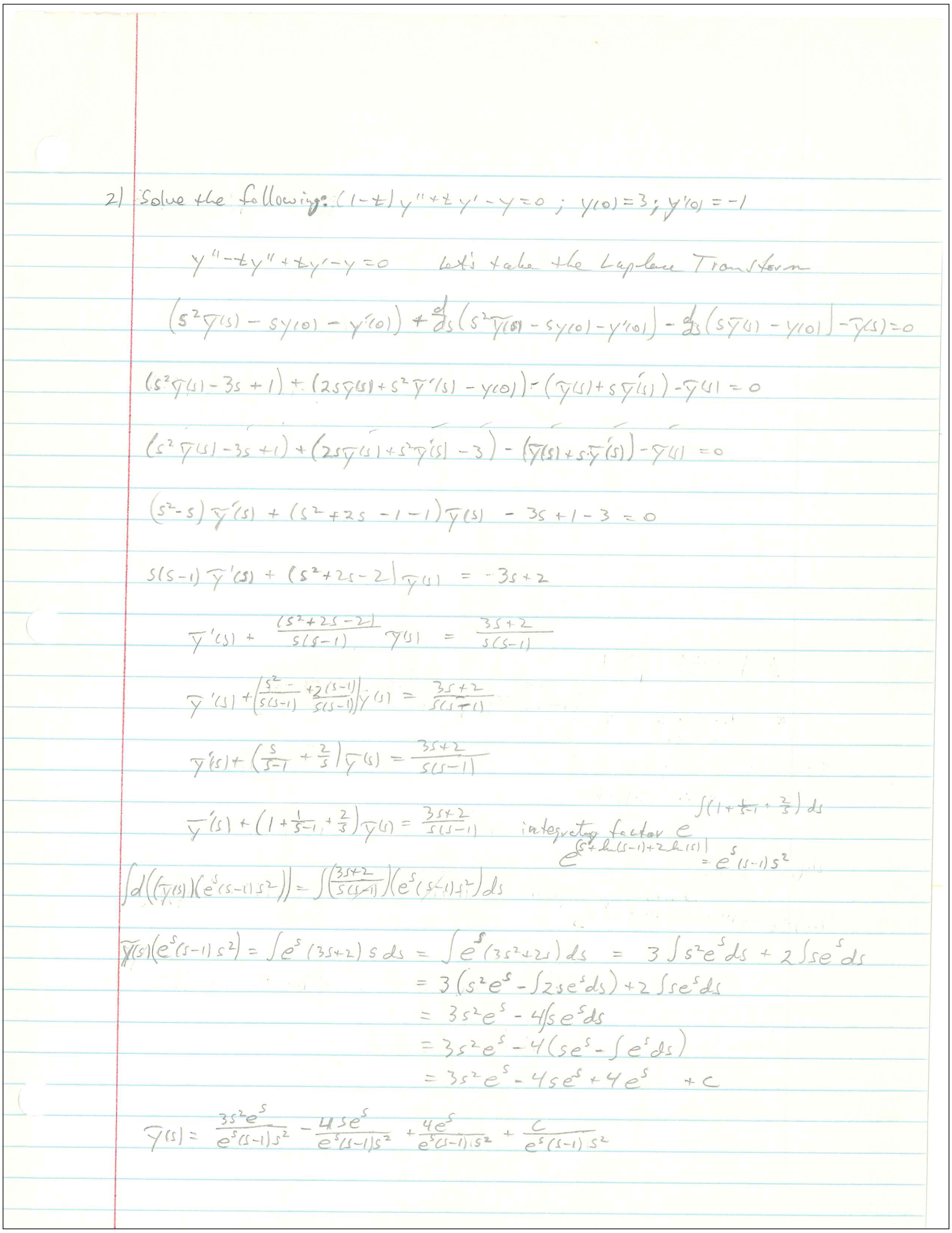
$$= \left(\frac{1}{\ln(10)}\right)\left(\ln(s^2+1) - \ln(s) - \ln(s+1)\right)$$

we will use the following property to aid in the inversion $\mathcal{L}^{-1}\{-F'(s)\} = t\, f(t) \therefore f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

$$\mathcal{L}^{-1}\{g(s)\} = \left(\frac{1}{\ln(10)}\right)\left(\mathcal{L}^{-1}\left\{\frac{d}{ds}\ln(s^2+1)\right) - \frac{d}{ds}(\ln(s)) - \frac{d}{ds}(\ln(s+1))\right\}\right)$$

$$g(t) = \left(\frac{1}{\ln(10)}\right)\left(\frac{-1}{t}\right)\left(\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right)$$

$$= \left(\frac{1}{\ln(10)}\right)\left(\frac{1}{t}\right)\left(1 + e^{-t} - 2\cos(t)\right)$$



2) Solve the following: $(1-t)y''+ty'-y=0$ ; $y(0)=3$; $y'(0)=-1$

$y''-ty''+ty'-y=0$ Let's take the Laplace Transform

$$\left(s^2\bar{y}(s)-sy(0)-y'(0)\right)+\frac{d}{ds}\left(s^2\bar{y}(s)-sy(0)-y'(0)\right)-\frac{d}{ds}\left(s\bar{y}(s)-y(0)\right)-\bar{y}(s)=0$$

$$(s^2\bar{y}(s)-3s+1)+(2s\bar{y}(s)+s^2\bar{y}'(s)-y(0))-(\bar{y}(s)+s\bar{y}'(s))-\bar{y}(s)=0$$

$$(s^2\bar{y}(s)-3s+1)+(2s\bar{y}(s)+s^2\bar{y}'(s)-3)-(\bar{y}(s)+s\cdot\bar{y}'(s))-\bar{y}(s)=0$$

$$(s^2-s)\bar{y}'(s)+(s^2+2s-1-1)\bar{y}(s)-3s+1-3=0$$

$$s(s-1)\bar{y}'(s)+(s^2+2s-2)\bar{y}(s)=-3s+2$$

$$\bar{y}'(s)+\frac{(s^2+2s-2)}{s(s-1)}\bar{y}(s)=\frac{3s+2}{s(s-1)}$$

$$\bar{y}'(s)+\left(\frac{s^2}{s(s-1)}+\frac{2(s-1)}{s(s-1)}\right)\bar{y}(s)=\frac{3s+2}{s(s-1)}$$

$$\bar{y}'(s)+\left(\frac{s}{s-1}+\frac{2}{s}\right)\bar{y}(s)=\frac{3s+2}{s(s-1)}$$

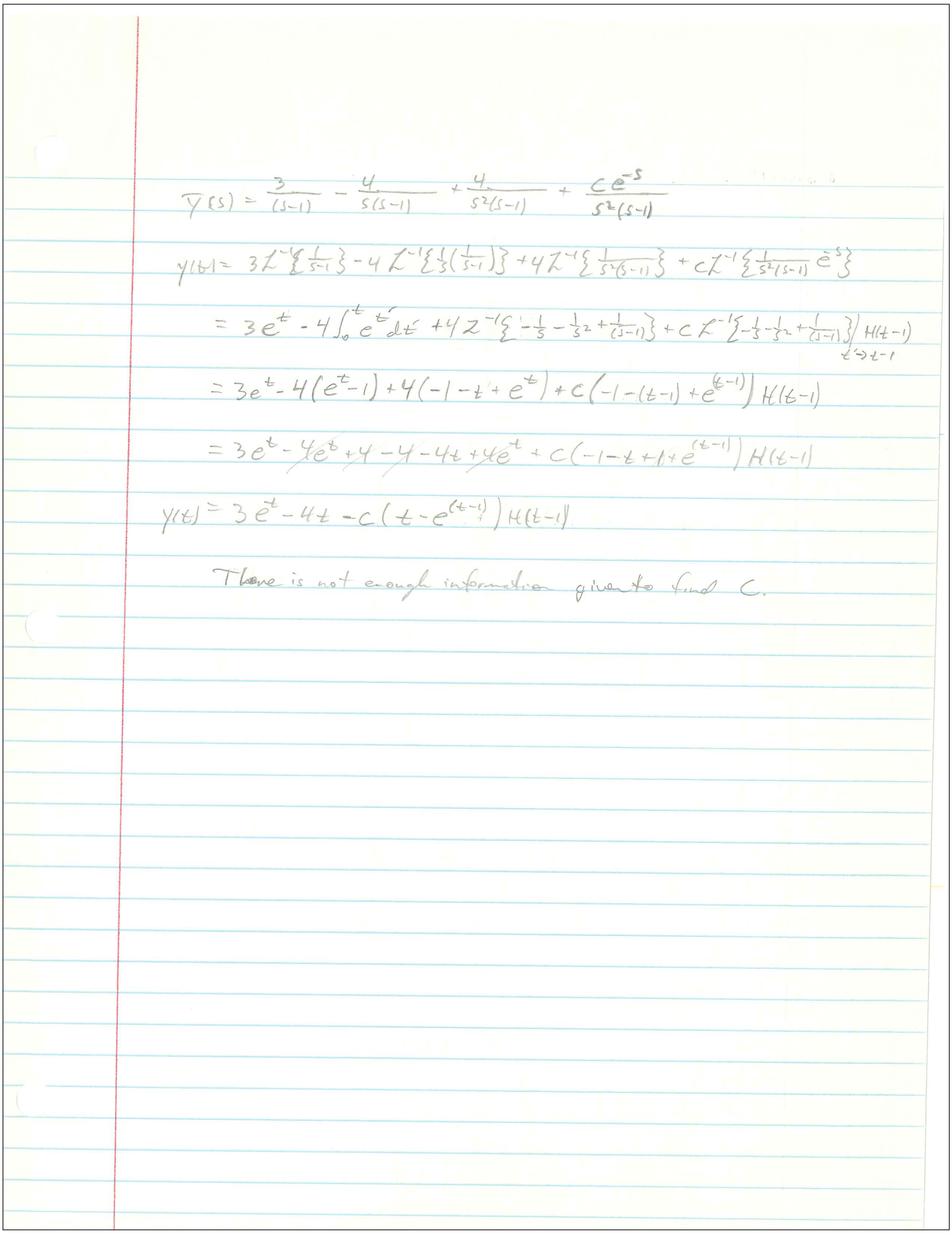
$$\bar{y}'(s)+\left(1+\frac{1}{s-1}+\frac{2}{s}\right)\bar{y}(s)=\frac{3s+2}{s(s-1)}$$ integrating factor $e^{\int(1+\frac{1}{s-1}+\frac{2}{s})ds}$

$e^{(s+\ln(s-1)+2\ln(s))}=e^s(s-1)s^2$

$$\int d\left((\bar{y}(s))(e^s(s-1)s^2)\right)=\int\left(\frac{3s+2}{s(s-1)}\right)(e^s(s-1)s^2)ds$$

$$\begin{aligned}\bar{y}(s)(e^s(s-1)s^2)=\int e^s(3s+2)s\,ds&=\int e^s(3s^2+2s)ds=3\int s^2e^sds+2\int se^sds\\&=3\left(s^2e^s-\int 2se^sds\right)+2\int se^sds\\&=3s^2e^s-4\int se^sds\\&=3s^2e^s-4\left(se^s-\int e^sds\right)\\&=3s^2e^s-4se^s+4e^s+c\end{aligned}$$

$$\bar{y}(s)=\frac{3s^2e^s}{e^s(s-1)s^2}-\frac{4se^s}{e^s(s-1)s^2}+\frac{4e^s}{e^s(s-1)s^2}+\frac{c}{e^s(s-1)s^2}$$



$$\bar{Y}(s) = \frac{3}{(s-1)} - \frac{4}{s(s-1)} + \frac{4}{s^2(s-1)} + \frac{ce^{-s}}{s^2(s-1)}$$

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s-1}\right)\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} + c\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}e^{-s}\right\}$$

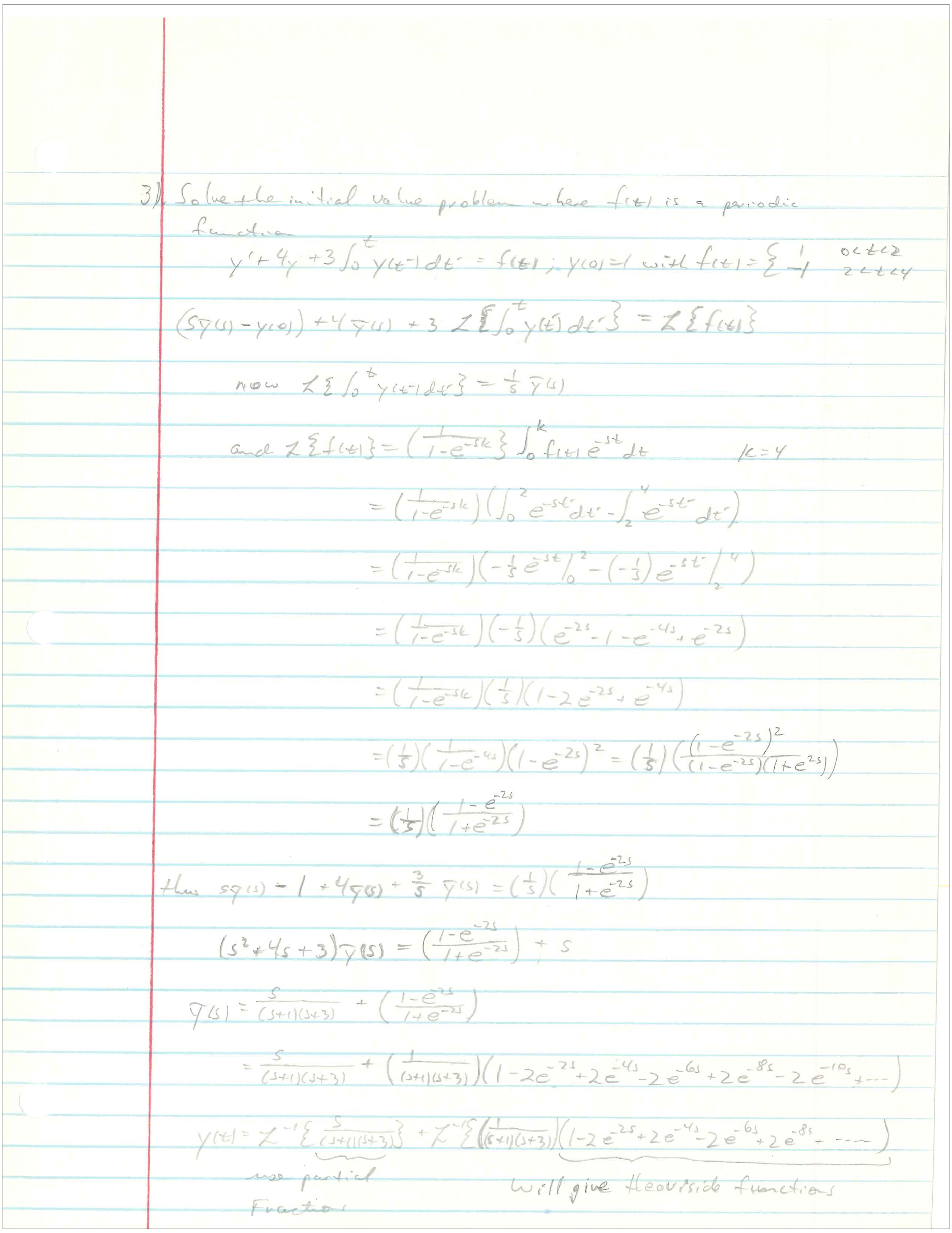
$$= 3e^t - 4\int_0^t e^{t'}dt' + 4\mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)}\right\} + c\mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)}\right\}\Big|_{t'\to t-1} H(t-1)$$

$$= 3e^t - 4(e^t - 1) + 4(-1 - t + e^t) + c\left(-1 - (t-1) + e^{(t-1)}\right) H(t-1)$$

$$= 3e^t \cancel{-4e^t} \cancel{+4} \cancel{-4} - 4t \cancel{+4e^t} + c(-1 - t + \cancel{1} + e^{(t-1)}) H(t-1)$$

$$y(t) = 3e^t - 4t - c(t - e^{(t-1)}) H(t-1)$$

There is not enough information given to find C.



3) Solve the initial value problem where $f(t)$ is a periodic function

$$y' + 4y + 3\int_0^t y(t')\,dt' = f(t) \, ; \; y(0) = 1 \text{ with } f(t) = \begin{cases} 1 & 0 < t < 2 \\ -1 & 2 < t < 4 \end{cases}$$

$$\left(s\bar{Y}(s) - y(0)\right) + 4\bar{Y}(s) + 3\,\mathcal{L}\left\{\int_0^t y(t')\,dt'\right\} = \mathcal{L}\{f(t)\}$$

now $\mathcal{L}\left\{\int_0^t y(t')\,dt'\right\} = \frac{1}{s}\bar{Y}(s)$

and $\mathcal{L}\{f(t)\} = \left(\frac{1}{1-e^{-sk}}\right\} \int_0^k f(t)\,e^{-st}\,dt$ $\quad k=4$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(\int_0^2 e^{-st'}\,dt' - \int_2^4 e^{-st'}\,dt'\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(-\frac{1}{s}e^{-st}\Big|_0^2 - \left(-\frac{1}{s}\right)e^{-st'}\Big|_2^4\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(-\frac{1}{s}\right)\left(e^{-2s} - 1 - e^{-4s} + e^{-2s}\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(\frac{1}{s}\right)\left(1 - 2e^{-2s} + e^{-4s}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{1-e^{-4s}}\right)\left(1-e^{-2s}\right)^2 = \left(\frac{1}{s}\right)\left(\frac{\left(1-e^{-2s}\right)^2}{\left(1-e^{-2s}\right)\left(1+e^{2s}\right)}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$$

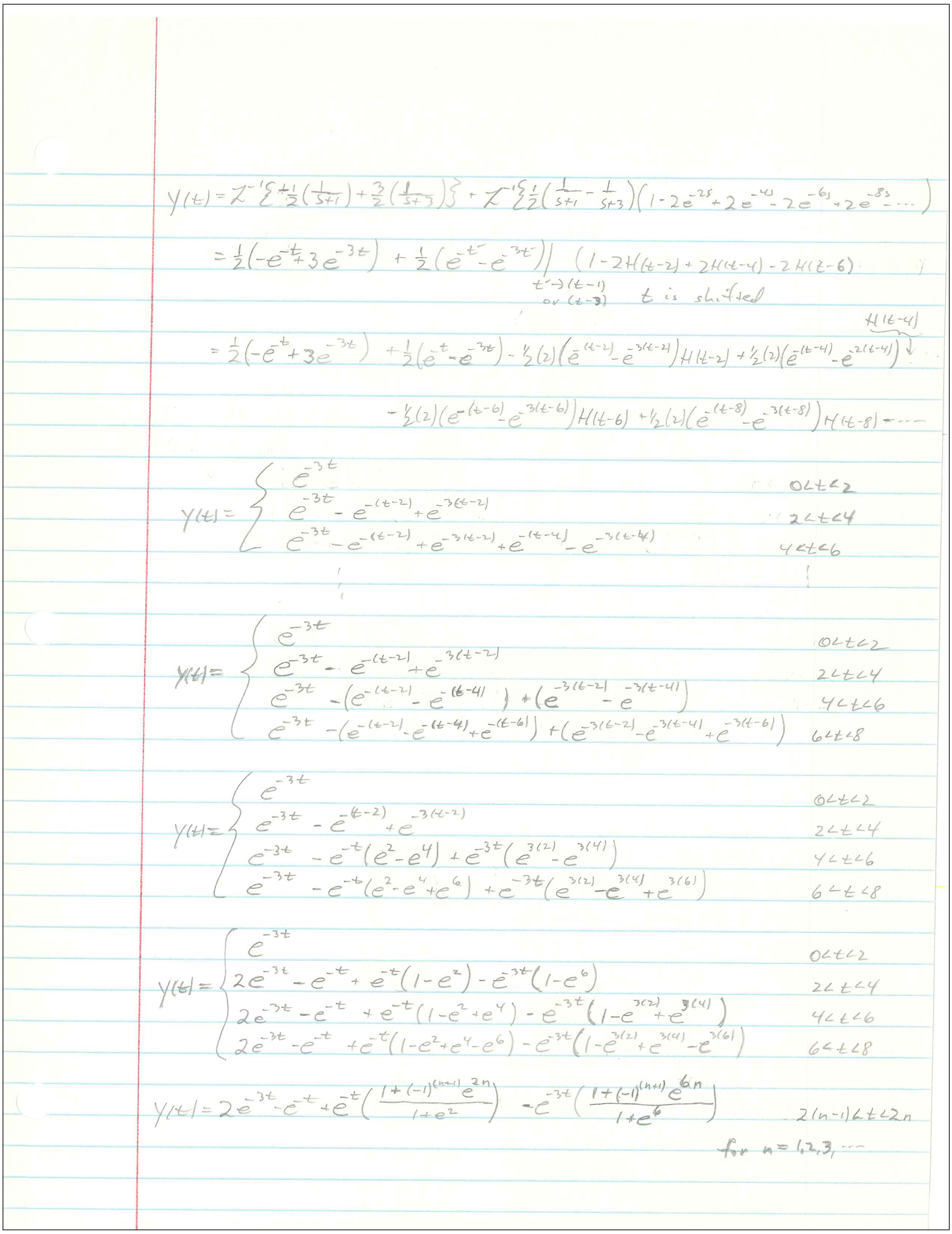
thus $s\bar{Y}(s) - 1 + 4\bar{Y}(s) + \frac{3}{s}\bar{Y}(s) = \left(\frac{1}{s}\right)\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$

$$\left(s^2 + 4s + 3\right)\bar{Y}(s) = \left(\frac{1-e^{-2s}}{1+e^{-2s}}\right) + s$$

$$\bar{Y}(s) = \frac{s}{(s+1)(s+3)} + \left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$$

$$= \frac{s}{(s+1)(s+3)} + \left(\frac{1}{(s+1)(s+3)}\right)\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} - 2e^{-10s} + \cdots\right)$$

$$y(t) = \mathcal{L}^{-1}\Big\{\underbrace{\frac{s}{(s+1)(s+3)}}_{\text{use partial Fraction}}\Big\} + \mathcal{L}^{-1}\Big\{\left(\frac{1}{(s+1)(s+3)}\right)\underbrace{\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} - \cdots\right)}_{\text{Will give Heaviside functions}}$$



$$y(t)=\mathcal{L}^{-1}\left\{-\frac{1}{2}\left(\frac{1}{s+1}\right)+\frac{3}{2}\left(\frac{1}{s+3}\right)\right\}+\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{s+1}-\frac{1}{s+3}\right)\left(1-2e^{-2s}+2e^{-4s}-2e^{-6s}+2e^{-8s}-\cdots\right)\right.$$

$$=\frac{1}{2}\left(-e^{-t}+3e^{-3t}\right)+\frac{1}{2}\left(e^{-t}-e^{-3t}\right)\Big|\ \left(1-2H(t-2)+2H(t-4)-2H(t-6)\right.$$

$t \to (t-1)$ or $(t-3)$ t is shifted

$$=\frac{1}{2}\left(-e^{-t}+3e^{-3t}\right)+\frac{1}{2}\left(e^{-t}-e^{-3t}\right)-\frac{1}{2}(2)\left(e^{-(t-2)}-e^{-3(t-2)}\right)H(t-2)+\frac{1}{2}(2)\left(e^{-(t-4)}-e^{-2(t-4)}\right)\overset{H(t-4)}{\downarrow}$$

$$-\frac{1}{2}(2)\left(e^{-(t-6)}-e^{-3(t-6)}\right)H(t-6)+\frac{1}{2}(2)\left(e^{-(t-8)}-e^{-3(t-8)}\right)H(t-8)+\cdots$$

$$y(t)=\begin{cases} e^{-3t} & 0<t<2\\ e^{-3t}-e^{-(t-2)}+e^{-3(t-2)} & 2<t<4\\ e^{-3t}-e^{-(t-2)}+e^{-3(t-2)}+e^{-(t-4)}-e^{-3(t-4)} & 4<t<6\\ \vdots & \vdots \end{cases}$$

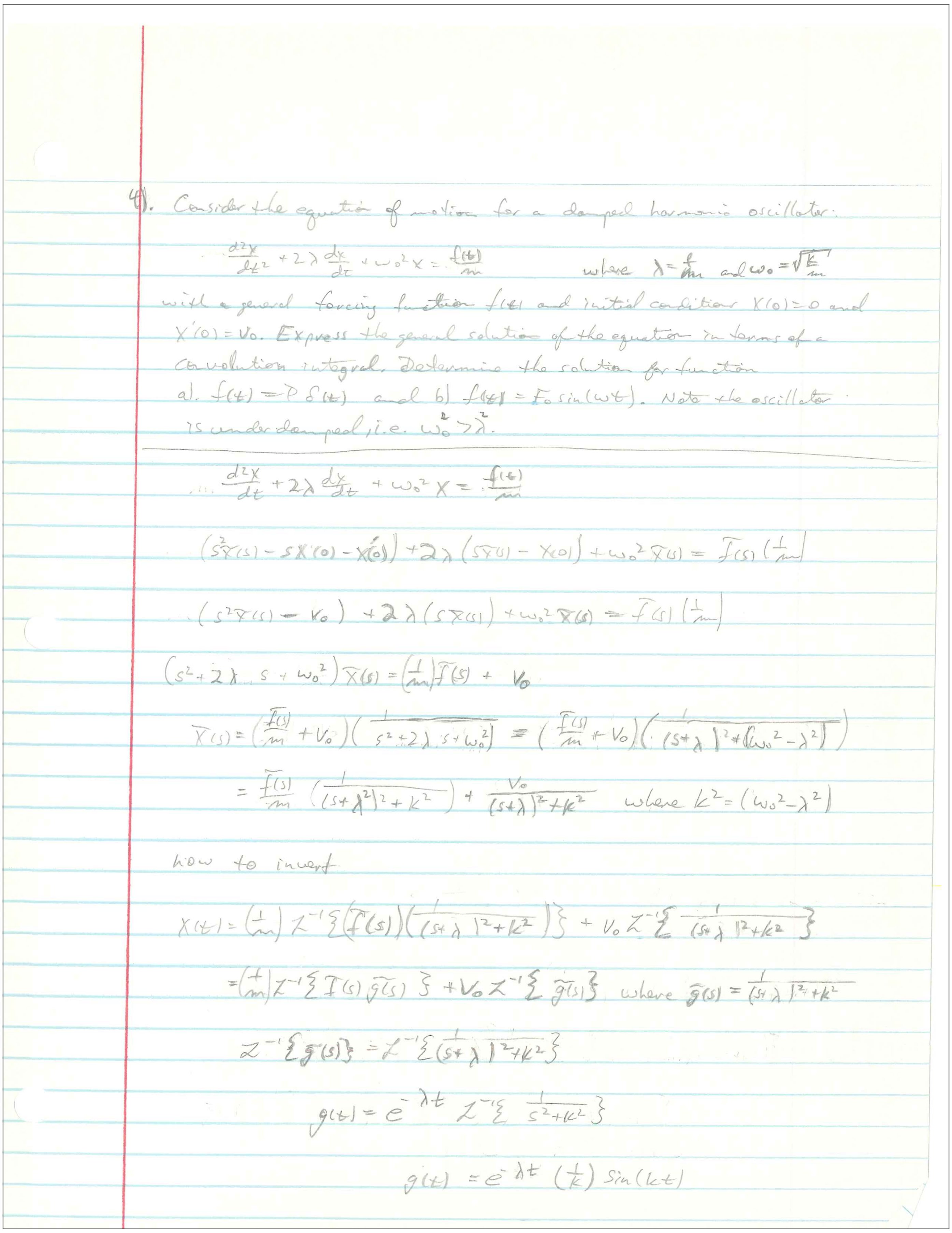
$$y(t)=\begin{cases} e^{-3t} & 0<t<2\\ e^{-3t}-e^{-(t-2)}+e^{-3(t-2)} & 2<t<4\\ e^{-3t}-\left(e^{-(t-2)}-e^{-(t-4)}\right)+\left(e^{-3(t-2)}-e^{-3(t-4)}\right) & 4<t<6\\ e^{-3t}-\left(e^{-(t-2)}-e^{-(t-4)}+e^{-(t-6)}\right)+\left(e^{-3(t-2)}-e^{-3(t-4)}+e^{-3(t-6)}\right) & 6<t<8 \end{cases}$$

$$y(t)=\begin{cases} e^{-3t} & 0<t<2\\ e^{-3t}-e^{-(t-2)}+e^{-3(t-2)} & 2<t<4\\ e^{-3t}-e^{-t}\left(e^{2}-e^{4}\right)+e^{-3t}\left(e^{3(2)}-e^{3(4)}\right) & 4<t<6\\ e^{-3t}-e^{-t}\left(e^{2}-e^{4}+e^{6}\right)+e^{-3t}\left(e^{3(2)}-e^{3(4)}+e^{3(6)}\right) & 6<t<8 \end{cases}$$

$$y(t)=\begin{cases} e^{-3t} & 0<t<2\\ 2e^{-3t}-e^{-t}+e^{-t}\left(1-e^{2}\right)-e^{-3t}\left(1-e^{6}\right) & 2<t<4\\ 2e^{-3t}-e^{-t}+e^{-t}\left(1-e^{2}+e^{4}\right)-e^{-3t}\left(1-e^{3(2)}+e^{3(4)}\right) & 4<t<6\\ 2e^{-3t}-e^{-t}+e^{-t}\left(1-e^{2}+e^{4}-e^{6}\right)-e^{-3t}\left(1-e^{3(2)}+e^{3(4)}-e^{3(6)}\right) & 6<t<8 \end{cases}$$

$$y(t)=2e^{-3t}-e^{-t}+e^{-t}\left(\frac{1+(-1)^{(n+1)}e^{2n}}{1+e^{2}}\right)-e^{-3t}\left(\frac{1+(-1)^{(n+1)}e^{6n}}{1+e^{6}}\right)\qquad 2(n-1)<t<2n$$

for $n=1,2,3,\cdots$



4). Consider the equation of motion for a damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega_0^2 x = \frac{f(t)}{m} \qquad \text{where } \lambda = \frac{f}{m} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}$$

with a general forcing function $f(t)$ and initial condition $X(0) = 0$ and $X'(0) = V_0$. Express the general solution of the equation in terms of a convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0 \sin(\omega t)$. Note the oscillator is underdamped, i.e. $\omega_0^2 > \lambda^2$.

---

$$\frac{d^2X}{dt} + 2\lambda \frac{dx}{dt} + \omega_0^2 X = \frac{f(t)}{m}$$

$$\left(s^2\overline{X}(s) - sX(0) - X'(0)\right) + 2\lambda\left(s\overline{X}(s) - X(0)\right) + \omega_0^2\overline{X}(s) = \overline{f}(s)\left(\frac{1}{m}\right)$$

$$\left(s^2\overline{X}(s) - V_0\right) + 2\lambda\left(s\overline{X}(s)\right) + \omega_0^2\overline{X}(s) = \overline{f}(s)\left(\frac{1}{m}\right)$$

$$\left(s^2 + 2\lambda s + \omega_0^2\right)\overline{X}(s) = \left(\frac{1}{m}\right)\overline{f}(s) + V_0$$

$$\overline{X}(s) = \left(\frac{\overline{f}(s)}{m} + V_0\right)\left(\frac{1}{s^2 + 2\lambda s + \omega_0^2}\right) = \left(\frac{\overline{f}(s)}{m} + V_0\right)\left(\frac{1}{(s+\lambda)^2 + (\omega_0^2 - \lambda^2)}\right)$$

$$= \frac{\overline{f}(s)}{m}\left(\frac{1}{(s+\lambda^2)^2 + k^2}\right) + \frac{V_0}{(s+\lambda)^2 + k^2} \qquad \text{where } k^2 = (\omega_0^2 - \lambda^2)$$

how to invert

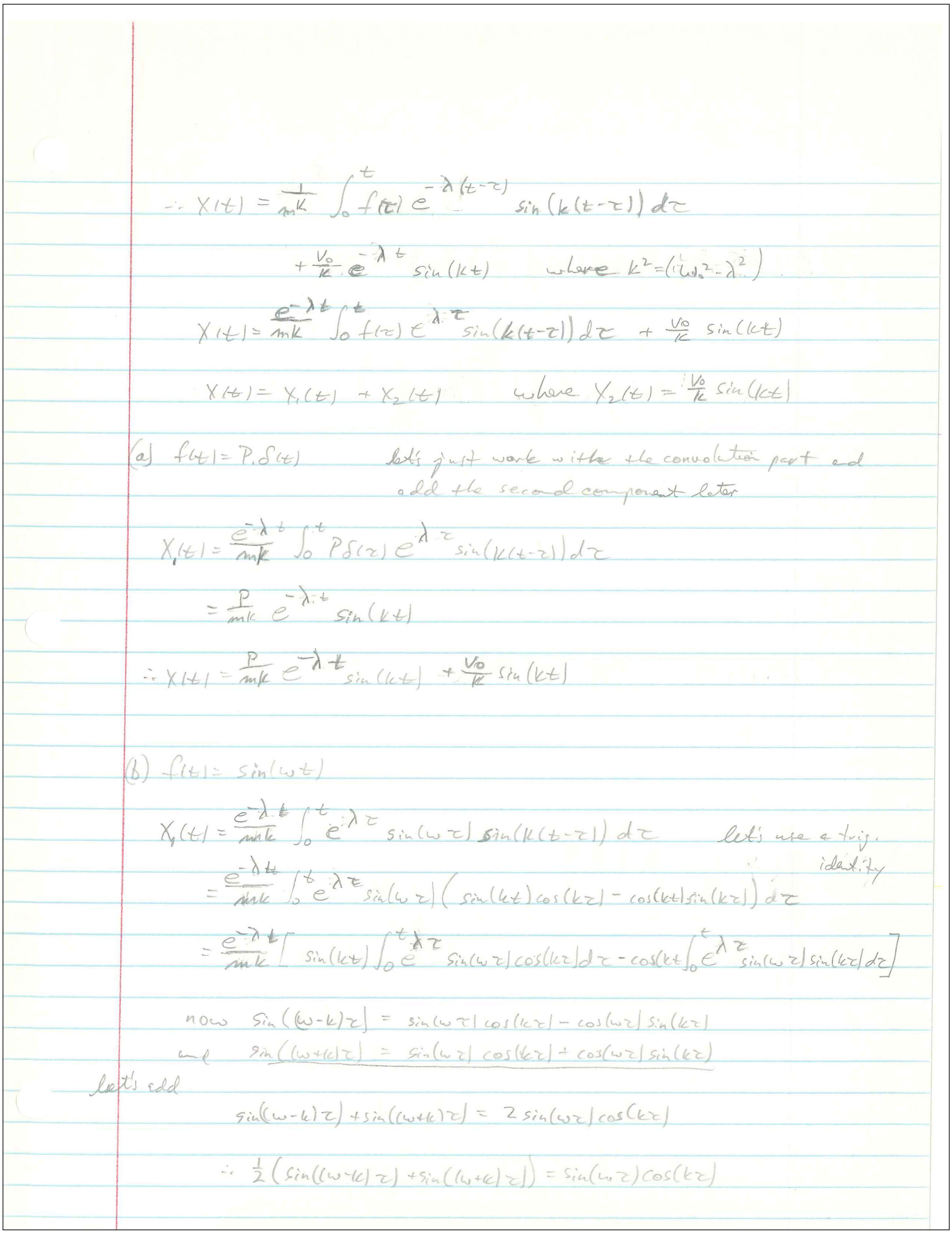
$$X(t) = \left(\frac{1}{m}\right)\mathcal{L}^{-1}\left\{\left(\overline{f}(s)\right)\left(\frac{1}{(s+\lambda)^2 + k^2}\right)\right\} + V_0\,\mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)^2 + k^2}\right\}$$

$$= \left(\frac{1}{m}\right)\mathcal{L}^{-1}\left\{\overline{f}(s)\,\overline{g}(s)\right\} + V_0\,\mathcal{L}^{-1}\left\{\overline{g}(s)\right\} \qquad \text{where } \overline{g}(s) = \frac{1}{(s+\lambda)^2 + k^2}$$

$$\mathcal{L}^{-1}\left\{\overline{g}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)^2 + k^2}\right\}$$

$$g(t) = e^{-\lambda t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\}$$

$$g(t) = e^{-\lambda t}\left(\frac{1}{k}\right)\sin(kt)$$



$$\therefore X(t) = \frac{1}{mk}\int_0^t f(\tau)\, e^{-\lambda(t-\tau)} \sin(k(t-\tau))\, d\tau$$

$$+ \frac{V_0}{k} e^{-\lambda t} \sin(kt) \qquad \text{where } k^2 = (\omega_0^2 - \lambda^2)$$

$$X(t) = \frac{e^{-\lambda t}}{mk}\int_0^t f(\tau)\, e^{\lambda\tau} \sin(k(t-\tau))\, d\tau + \frac{V_0}{k}\sin(kt)$$

$$X(t) = X_1(t) + X_2(t) \qquad \text{where } X_2(t) = \frac{V_0}{k}\sin(kt)$$

(a) $f(t) = P.\delta(t)$ let's just work with the convolution part and add the second component later

$$X_1(t) = \frac{e^{-\lambda t}}{mk}\int_0^t P\delta(\tau)\, e^{\lambda\tau} \sin(k(t-\tau))\, d\tau$$

$$= \frac{P}{mk} e^{-\lambda t} \sin(kt)$$

$$\therefore X(t) = \frac{P}{mk} e^{-\lambda t} \sin(kt) + \frac{V_0}{k}\sin(kt)$$

(b) $f(t) = \sin(\omega t)$

$$X_1(t) = \frac{e^{-\lambda t}}{mk}\int_0^t e^{\lambda\tau} \sin(\omega\tau) \sin(k(t-\tau))\, d\tau$$ let's use a trig. identity

$$= \frac{e^{-\lambda t}}{mk}\int_0^t e^{\lambda\tau} \sin(\omega\tau)\left(\sin(kt)\cos(k\tau) - \cos(kt)\sin(k\tau)\right) d\tau$$

$$= \frac{e^{-\lambda t}}{mk}\left[\sin(kt)\int_0^t e^{\lambda\tau} \sin(\omega\tau)\cos(k\tau)\, d\tau - \cos(kt)\int_0^t e^{\lambda\tau} \sin(\omega\tau)\sin(k\tau)\, d\tau\right]$$

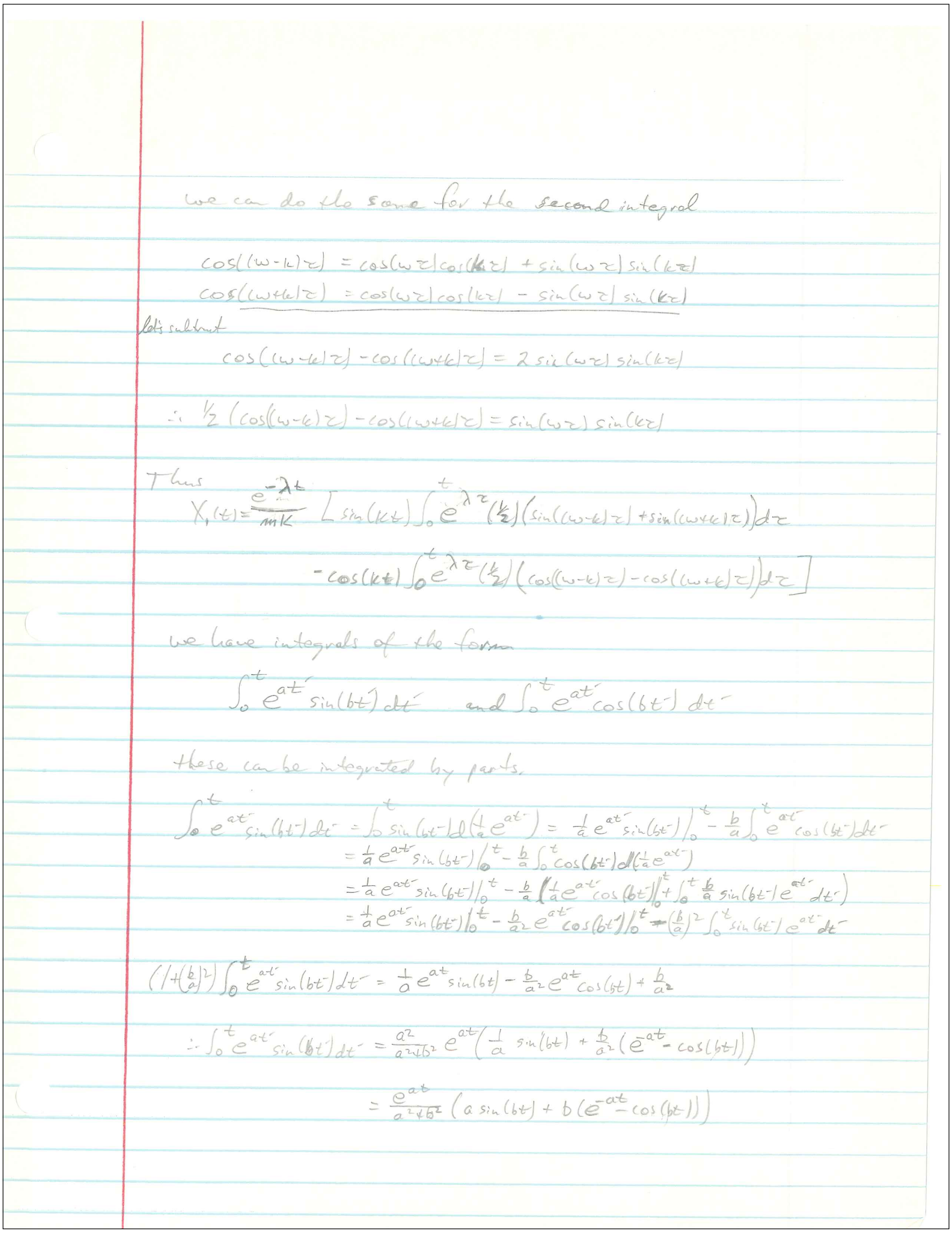
now $\sin((\omega-k)\tau) = \sin(\omega\tau)\cos(k\tau) - \cos(\omega\tau)\sin(k\tau)$

and $\underline{\sin((\omega+k)\tau) = \sin(\omega\tau)\cos(k\tau) + \cos(\omega\tau)\sin(k\tau)}$

let's add

$$\sin((\omega-k)\tau) + \sin((\omega+k)\tau) = 2\sin(\omega\tau)\cos(k\tau)$$

$$\therefore \frac{1}{2}\left(\sin((\omega-k)\tau) + \sin((\omega+k)\tau)\right) = \sin(\omega\tau)\cos(k\tau)$$



we can do the same for the second integral

$$\cos((\omega-k)\tau) = \cos(\omega\tau)\cos(k\tau) + \sin(\omega\tau)\sin(k\tau)$$

$$\cos((\omega+k)\tau) = \cos(\omega\tau)\cos(k\tau) - \sin(\omega\tau)\sin(k\tau)$$

let's subtract

$$\cos((\omega-k)\tau) - \cos((\omega+k)\tau) = 2\sin(\omega\tau)\sin(k\tau)$$

$$\therefore \tfrac{1}{2}\left(\cos((\omega-k)\tau) - \cos((\omega+k)\tau)\right) = \sin(\omega\tau)\sin(k\tau)$$

Thus

$$X_1(t) = \frac{e^{-\lambda t}}{mK}\Big[\sin(Kt)\int_0^t e^{\lambda\tau}\left(\tfrac{1}{2}\right)\left(\sin((\omega-k)\tau) + \sin((\omega+k)\tau)\right)d\tau$$

$$-\cos(kt)\int_0^t e^{\lambda\tau}\left(\tfrac{1}{2}\right)\left(\cos((\omega-k)\tau) - \cos((\omega+k)\tau)\right)d\tau\Big]$$

we have integrals of the form

$$\int_0^t e^{at'}\sin(bt')\,dt' \quad \text{and} \int_0^t e^{at'}\cos(bt')\,dt'$$

these can be integrated by parts.

$$\int_0^t e^{at'}\sin(bt')\,dt' = \int_0^t \sin(bt')\,d\left(\tfrac{1}{a}e^{at'}\right) = \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \tfrac{b}{a}\int_0^t e^{at'}\cos(bt')\,dt'$$

$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \tfrac{b}{a}\int_0^t \cos(bt')\,d\left(\tfrac{1}{a}e^{at'}\right)$$

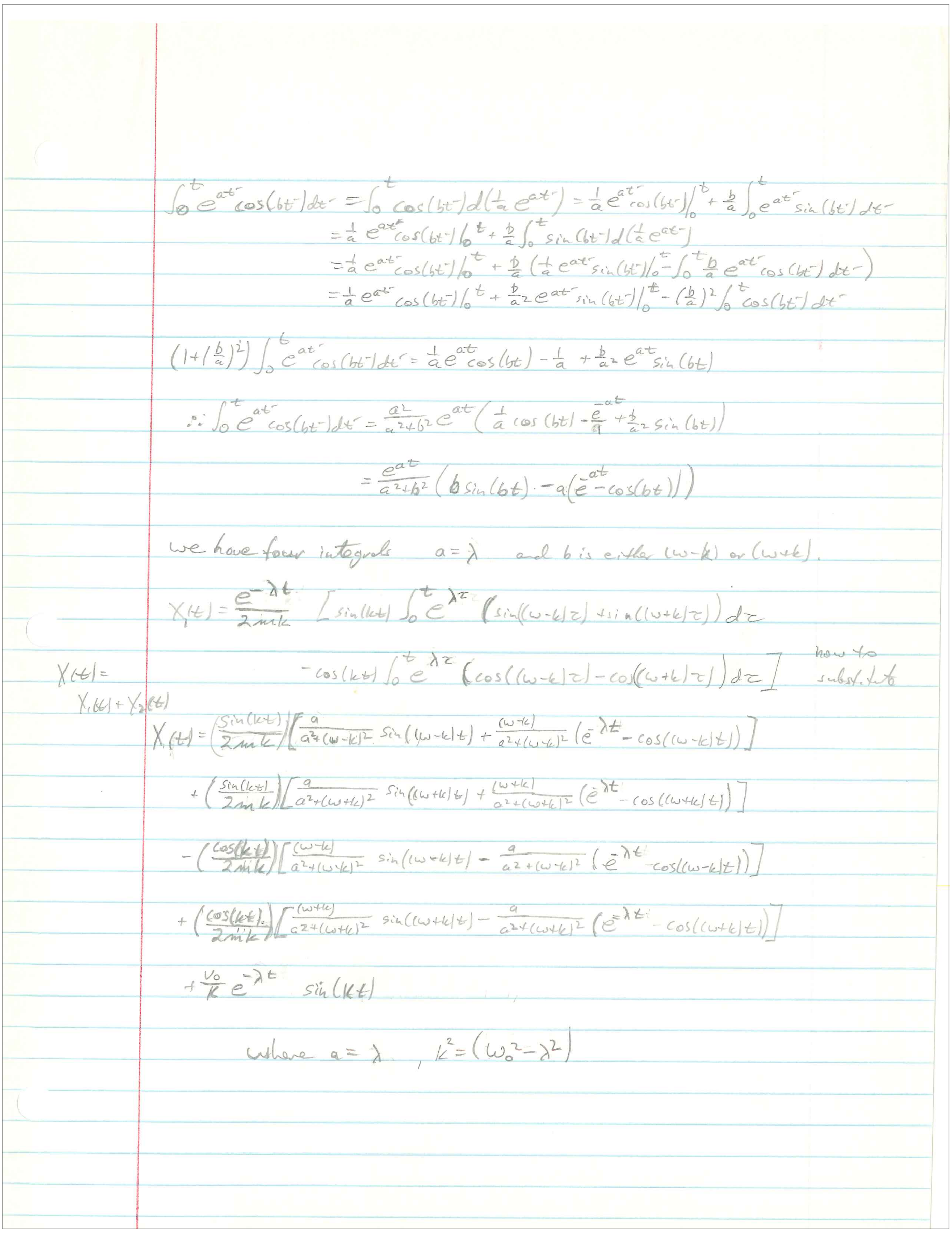
$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \tfrac{b}{a}\left(\tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \int_0^t \tfrac{b}{a}\sin(bt')e^{at'}\,dt'\right)$$

$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \tfrac{b}{a^2}e^{at'}\cos(bt')\Big|_0^t - \left(\tfrac{b}{a}\right)^2\int_0^t \sin(bt')e^{at'}\,dt'$$

$$\left(1+\left(\tfrac{b}{a}\right)^2\right)\int_0^t e^{at'}\sin(bt')\,dt' = \tfrac{1}{a}e^{at}\sin(bt) - \tfrac{b}{a^2}e^{at}\cos(bt) + \tfrac{b}{a^2}$$

$$\therefore \int_0^t e^{at'}\sin(bt')\,dt' = \frac{a^2}{a^2+b^2}e^{at}\left(\frac{1}{a}\sin(bt) + \frac{b}{a^2}\left(e^{-at} - \cos(bt)\right)\right)$$

$$= \frac{e^{at}}{a^2+b^2}\left(a\sin(bt) + b\left(e^{-at} - \cos(bt)\right)\right)$$



$$\int_0^t e^{at'}\cos(bt')\,dt' = \int_0^t \cos(bt')\,d\left(\tfrac{1}{a}e^{at'}\right) = \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\int_0^t e^{at'}\sin(bt')\,dt'$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\int_0^t \sin(bt')\,d\left(\tfrac{1}{a}e^{at'}\right)$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\left(\tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \int_0^t \tfrac{b}{a}e^{at'}\cos(bt')\,dt'\right)$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a^2}e^{at'}\sin(bt')\Big|_0^t - \left(\tfrac{b}{a}\right)^2\int_0^t \cos(bt')\,dt'$$

$$\left(1+\left(\tfrac{b}{a}\right)^2\right)\int_0^t e^{at'}\cos(bt')\,dt' = \tfrac{1}{a}e^{at}\cos(bt) - \tfrac{1}{a} + \tfrac{b}{a^2}e^{at}\sin(bt)$$

$$\therefore \int_0^t e^{at'}\cos(bt')\,dt' = \frac{a^2}{a^2+b^2}e^{at}\left(\tfrac{1}{a}\cos(bt) - \tfrac{e^{-at}}{a} + \tfrac{b}{a^2}\sin(bt)\right)$$

$$= \frac{e^{at}}{a^2+b^2}\left(b\sin(bt) - a\left(e^{-at} - \cos(bt)\right)\right)$$

we have four integrals $a=\lambda$ and $b$ is either $(\omega-k)$ or $(\omega+k)$.

$$X_1(t) = \frac{e^{-\lambda t}}{2mk}\Big[\sin(kt)\int_0^t e^{\lambda\tau}\left(\sin((\omega-k)\tau) + \sin((\omega+k)\tau)\right)d\tau$$

$$-\cos(kt)\int_0^t e^{\lambda\tau}\left(\cos((\omega-k)\tau) - \cos((\omega+k)\tau)\right)d\tau\Big]$$

how to substitute

$X(t) =$
$X_1(t) + X_2(t)$

$$X_1(t) = \left(\frac{\sin(kt)}{2mk}\right)\left[\frac{a}{a^2+(\omega-k)^2}\sin((\omega-k)t) + \frac{(\omega-k)}{a^2+(\omega-k)^2}\left(e^{-\lambda t} - \cos((\omega-k)t)\right)\right]$$

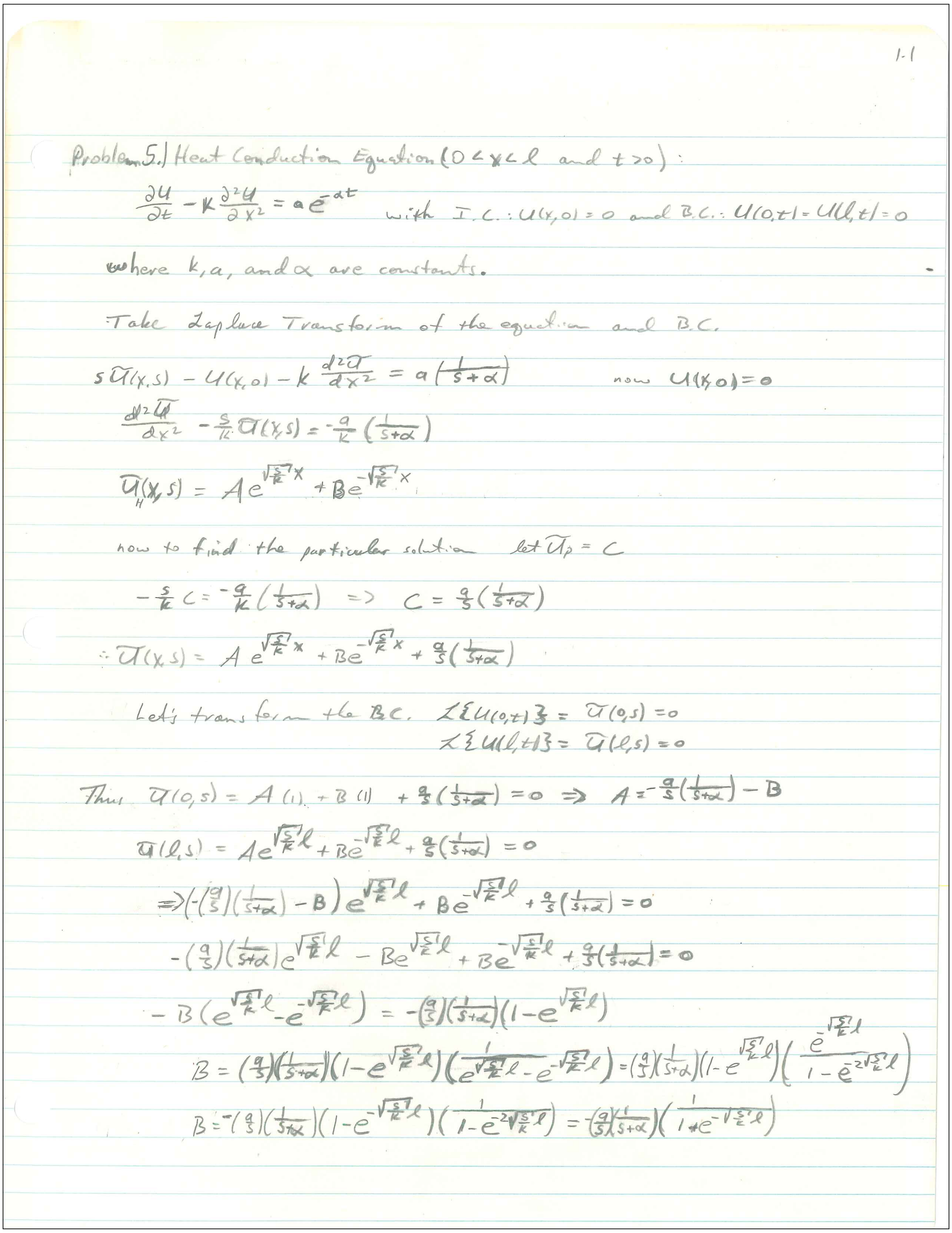
$$+\left(\frac{\sin(kt)}{2mk}\right)\left[\frac{a}{a^2+(\omega+k)^2}\sin((\omega+k)t) + \frac{(\omega+k)}{a^2+(\omega+k)^2}\left(e^{-\lambda t} - \cos((\omega+k)t)\right)\right]$$

$$-\left(\frac{\cos(kt)}{2mk}\right)\left[\frac{(\omega-k)}{a^2+(\omega-k)^2}\sin((\omega+k)t) - \frac{a}{a^2+(\omega-k)^2}\left(e^{-\lambda t} - \cos((\omega-k)t)\right)\right]$$

$$+\left(\frac{\cos(kt)}{2mk}\right)\left[\frac{(\omega+k)}{a^2+(\omega+k)^2}\sin((\omega+k)t) - \frac{a}{a^2+(\omega+k)^2}\left(e^{-\lambda t} - \cos((\omega+k)t)\right)\right]$$

$$+\frac{V_0}{k}e^{-\lambda t}\sin(kt)$$

where $a=\lambda$, $k^2 = (\omega_0^2 - \lambda^2)$



1.1

Problem 5.) Heat Conduction Equation ($0 < x < l$ and $t > 0$):

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = a e^{-\alpha t} \quad \text{with I.C.: } u(x,0) = 0 \text{ and B.C.: } u(0,t) = u(l,t) = 0$$

where $k$, $a$, and $\alpha$ are constants.

Take Laplace Transform of the equation and B.C.

$$s\bar{u}(x,s) - u(x,0) - k\frac{d^2\bar{u}}{dx^2} = a\left(\frac{1}{s+\alpha}\right) \qquad \text{now } u(x,0) = 0$$

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u}(x,s) = -\frac{a}{k}\left(\frac{1}{s+\alpha}\right)$$

$$\bar{u}_H(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x}$$

now to find the particular solution let $\bar{u}_p = C$

$$-\frac{s}{k}C = -\frac{a}{k}\left(\frac{1}{s+\alpha}\right) \Rightarrow C = \frac{a}{s}\left(\frac{1}{s+\alpha}\right)$$

$$\therefore \bar{u}(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right)$$

Let's transform the B.C. $\mathcal{L}\{u(0,t)\} = \bar{u}(0,s) = 0$

$\mathcal{L}\{u(l,t)\} = \bar{u}(l,s) = 0$

Thus $\bar{u}(0,s) = A(1) + B(1) + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0 \Rightarrow A = -\frac{a}{s}\left(\frac{1}{s+\alpha}\right) - B$

$$\bar{u}(l,s) = Ae^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

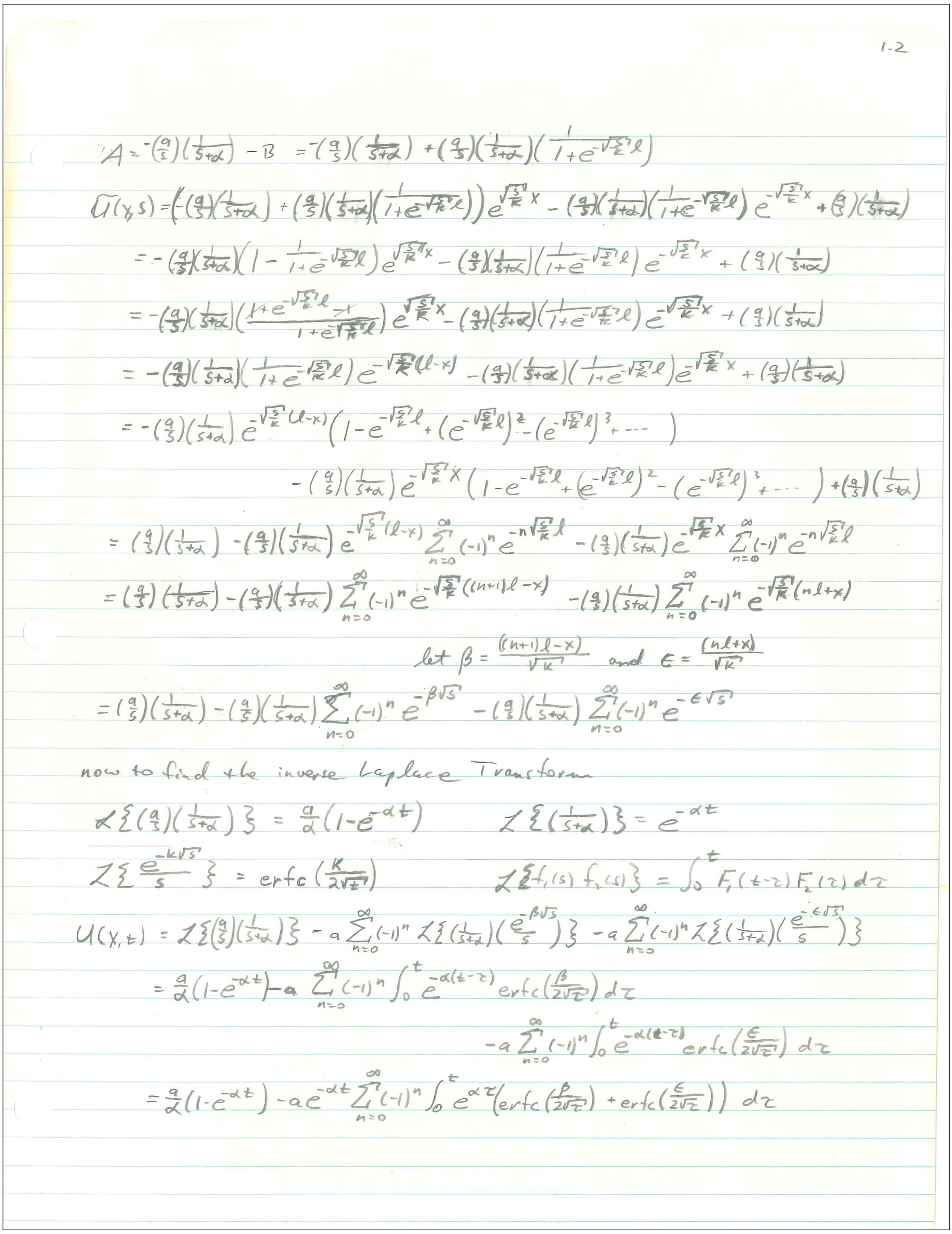
$$\Rightarrow \left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - B\right)e^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

$$-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{\sqrt{\frac{s}{k}}l} - Be^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

$$-B\left(e^{\sqrt{\frac{s}{k}}l} - e^{-\sqrt{\frac{s}{k}}l}\right) = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)$$

$$B = \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)\left(\frac{1}{e^{\sqrt{\frac{s}{k}}l} - e^{-\sqrt{\frac{s}{k}}l}}\right) = \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)\left(\frac{e^{-\sqrt{\frac{s}{k}}l}}{1 - e^{-2\sqrt{\frac{s}{k}}l}}\right)$$

$$B = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{-\sqrt{\frac{s}{k}}l}\right)\left(\frac{1}{1 - e^{-2\sqrt{\frac{s}{k}}l}}\right) = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1 + e^{-\sqrt{\frac{s}{k}}l}}\right)$$



1-2

$$A = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - B = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)$$

$$\bar{U}(x,s) = \left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)\right)e^{\sqrt{\frac{s}{k}}x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{-\sqrt{\frac{s}{k}}x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - \frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{\sqrt{\frac{s}{k}}x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{-\sqrt{\frac{s}{k}}x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{\cancel{1}+e^{-\sqrt{\frac{s}{k}}l}\cancel{-1}}{1+e^{\sqrt{\frac{s}{k}}l}}\right)e^{\sqrt{\frac{s}{k}}x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{-\sqrt{\frac{s}{k}}x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{-\sqrt{\frac{s}{k}}(l-x)} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}l}}\right)e^{-\sqrt{\frac{s}{k}}x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}(l-x)}\left(1 - e^{-\sqrt{\frac{s}{k}}l} + \left(e^{-\sqrt{\frac{s}{k}}l}\right)^2 - \left(e^{-\sqrt{\frac{s}{k}}l}\right)^3 + --- \right)$$

$$- \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}x}\left(1 - e^{-\sqrt{\frac{s}{k}}l} + \left(e^{-\sqrt{\frac{s}{k}}l}\right)^2 - \left(e^{-\sqrt{\frac{s}{k}}l}\right)^3 + --- \right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}(l-x)}\sum_{n=0}^{\infty}(-1)^n e^{-n\sqrt{\frac{s}{k}}l} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}x}\sum_{n=0}^{\infty}(-1)^n e^{-n\sqrt{\frac{s}{k}}l}$$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{s}{k}}((n+1)l-x)} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{s}{k}}(nl+x)}$$

let $\beta = \frac{((n+1)l-x)}{\sqrt{k}}$ and $\epsilon = \frac{(nl+x)}{\sqrt{k}}$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\beta\sqrt{s}} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\epsilon\sqrt{s}}$$

now to find the inverse Laplace Transform

$$\mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\right\} = \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) \qquad \mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\right\} = e^{-\alpha t}$$

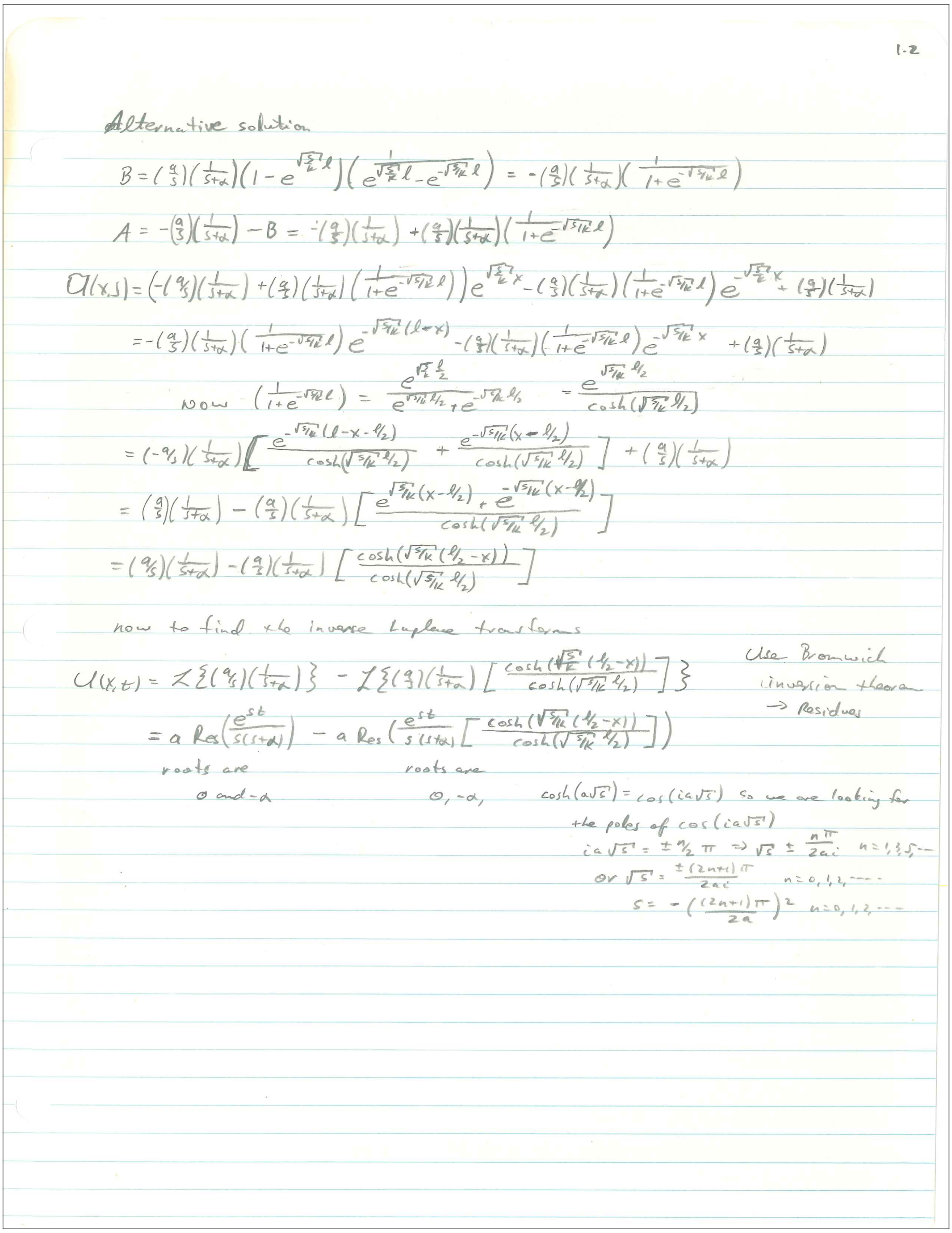
$$\mathcal{L}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{K}{2\sqrt{t}}\right) \qquad \mathcal{L}\left\{f_1(s)\,f_2(s)\right\} = \int_0^t F_1(t-\tau)F_2(\tau)\,d\tau$$

$$U(x,t) = \mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\right\} - a\sum_{n=0}^{\infty}(-1)^n\mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\left(\frac{e^{-\beta\sqrt{s}}}{s}\right)\right\} - a\sum_{n=0}^{\infty}(-1)^n\mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\left(\frac{e^{-\epsilon\sqrt{s}}}{s}\right)\right\}$$

$$= \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) - a\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{-\alpha(t-\tau)}\operatorname{erfc}\left(\frac{\beta}{2\sqrt{\tau}}\right)d\tau$$

$$- a\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{-\alpha(t-\tau)}\operatorname{erfc}\left(\frac{\epsilon}{2\sqrt{\tau}}\right)d\tau$$

$$= \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) - a e^{-\alpha t}\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{\alpha\tau}\left(\operatorname{erfc}\left(\frac{\beta}{2\sqrt{\tau}}\right) + \operatorname{erfc}\left(\frac{\epsilon}{2\sqrt{\tau}}\right)\right)d\tau$$



1.2

Alternative solution

$$B=\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1-e^{\sqrt{\frac{s}{k}}l}\right)\left(\frac{1}{e^{\sqrt{\frac{s}{k}}l}-e^{-\sqrt{s/k}\,l}}\right)=-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)$$

$$A=-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)-B=-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)+\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)$$

$$U(x,s)=\left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)+\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)\right)e^{\sqrt{\frac{s}{k}}x}-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{\frac{s}{k}}x}+\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$=-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{s/k}(l-x)}-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{s/k}\,x}+\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

now $\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)=\frac{e^{\sqrt{\frac{s}{k}}\frac{l}{2}}}{e^{\sqrt{s/k}\,l/2}+e^{-\sqrt{s/k}\,l/2}}=\frac{e^{\sqrt{s/k}\,l/2}}{\cosh(\sqrt{s/k}\,l/2)}$

$$=(-a/s)\left(\frac{1}{s+\alpha}\right)\left[\frac{e^{-\sqrt{s/k}(l-x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)}+\frac{e^{-\sqrt{s/k}(x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)}\right]+\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$=\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{e^{\sqrt{s/k}(x-l/2)}+e^{-\sqrt{s/k}(x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)}\right]$$

$$=(a/s)\left(\frac{1}{s+\alpha}\right)-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{\cosh(\sqrt{s/k}(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]$$

now to find the inverse Laplace transforms

$$U(x,t)=\mathcal{L}\left\{(a/s)\left(\frac{1}{s+\alpha}\right)\right\}-\mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{\cosh(\sqrt{\frac{s}{k}}(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]\right\}$$

Use Bromwich inversion theorem → Residues

$$=a\,\text{Res}\left(\frac{e^{st}}{s(s+\alpha)}\right)-a\,\text{Res}\left(\frac{e^{st}}{s(s+\alpha)}\left[\frac{\cosh(\sqrt{s/k}(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]\right)$$

roots are $0$ and $-\alpha$ roots are $0,-\alpha,$

$\cosh(a\sqrt{s})=\cos(ia\sqrt{s})$ so we are looking for the poles of $\cos(ia\sqrt{s})$

$ia\sqrt{s}=\pm\frac{n}{2}\pi \Rightarrow \sqrt{s}\pm\frac{n\pi}{2ai}$ $n=1,3,5,\ldots$

or $\sqrt{s}=\frac{\pm(2n+1)\pi}{2ai}$ $n=0,1,2,\ldots$

$s=-\left(\frac{(2n+1)\pi}{2a}\right)^2$ $n=0,1,2,\ldots$

## 2.7 HW 6

### 2.7.1 Problems to solve

Homework Set No. 6 NEEP 547
Due October 25, 2013 DLH

Integral Equation (Homogeneous)

1. (6pts) Show that the only values of $\lambda$ for which

$$f(x)=\lambda\int_0^1 xy\,(x+y)\,f(y)\,dy$$

has a non-trivial solution are the roots of the equation

$$\lambda^2+120\lambda-240=0.$$

(Hint: This is an eigenvalue problem for the case of a homogeneous Fredholm Eq. It can be solved by the differential equation method.)

Integral Equation (Neumann series solution method)

2. (6pts) Solve the following integral equation using the Neumann series method:

$$\psi(x)=1+\frac{x^2}{2!}-\int_0^x (x-t)\,\psi(t)\,dt$$

3. (6pts) Determine $\psi(x)$ using the Neumann series method:

$$\psi(x)=x\cos x+\int_0^x t\,\psi(t)\,dt$$

Integral Equation (Differential equation solution method)

$$\begin{aligned} 4.(6\text{pts})\ \ 2\cosh(x)-\sinh(x)-(2-x) &= 1+\int_0^x (2-x+t)\,\psi(t)\,dt \\ 5.(6\text{pts})\ \ u(x) &= \cos(x)-x-2+\int_0^x (t-x)\,u(t)\,dt \\ 6.(6\text{pts})\ \ u(x) &= x+\lambda\int_0^1 (1+x+t)\,u(t)\,dt \\ 7.(6pts)\ \frac{dy(t)}{dt} &= 2-\frac{t^2}{2}-\frac{1}{4}\int_0^t y(\tau)\,d\tau,\ \text{with I.C.: } y(0)=0. \end{aligned}$$

Integral Equation

8. (6pts) Solve the integral equation using any technique:

$$y(x)=f(x)-A\int_a^x xt\,e^{\lambda(x-t)}\,y(t)dt$$

### 2.7.2 Problem 1 (integral equation (homogeneous)

Show that the only value of $\lambda$ for which

$$f(x)=\lambda\int_0^1 xy(x+y)f(y)\,\mathrm{d}y$$

has a non-trivial solution the roots of the equation

$$\lambda^2+120\lambda-240=0$$

**solution**

The integral equation is a homogeneous Fredholm of the second kind. Normally it is written as

$$\varphi(x)=\lambda\int_0^1 K\left(x,y\right)\varphi(y)\,\mathrm{d}y \tag{1}$$

Where in our case the kernel $K\left(x,y\right)=xy(x+y)$. Hence the kernel is separable and can be written as

$$\begin{aligned}K\left(x,y\right) &= x^2y + xy^2\\ &= \sum_j^2 M_j\left(x\right)N_j\left(y\right)\end{aligned}$$

Where $M_1 = x^2, N_1 = y, M_2 = x, N_2 = y^2$, therefore Eq. (1) can be written as

$$\varphi(x) = \lambda \int_0^1 \left(\sum_j^2 M_j\left(x\right)N_j\left(y\right)\right)\varphi(y)\,\mathrm{d}y$$

Interchanging summation with integration

$$\varphi(x) = \lambda \sum_j^2 M_j\left(\int_0^1 N_j\left(y\right)\varphi(y)\,\mathrm{d}y\right) \tag{2}$$

Now $\int_0^1 N_j\left(y\right)\varphi(y)\,\mathrm{d}y$ is a constant. Lets call it $c_j$,

$$c_j = \int_0^1 N_j\left(y\right)\varphi(y)\,\mathrm{d}y$$

Therefore Eq. (2) becomes

$$\varphi(x) = \lambda \sum_j^2 M_j c_j$$

To find $c_i$ we multiply both side by $N_i$ and integrate w.r.t. $x$ which gives

$$\int_0^1 N_i\left(x\right)\varphi(x)\,\mathrm{d}x = \lambda \sum_j^2 a_{ij}c_j \tag{3}$$

Where

$$a_{ij} = \int_0^1 N_i\left(x\right)M_j\left(x\right)\,\mathrm{d}x$$

Which can be evaluated to a numerical values. Going back to Eq. (3) we see that the LHS is just $c_i$, hence Eq. (3) can be written as

$$c_i = \lambda \sum_j^2 a_{ij}c_j \tag{4}$$

Writing this in matrix for

$$\begin{aligned}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \lambda\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}\\ \begin{pmatrix}c_1\\c_2\end{pmatrix}\left(I - \lambda\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\right) &= 0\\ \begin{pmatrix}c_1\\c_2\end{pmatrix}\begin{pmatrix}1-\lambda a_{11} & -\lambda a_{12}\\-\lambda a_{21} & 1-\lambda a_{22}\end{pmatrix} &= 0\end{aligned}$$

For a non-trivial solution to exist, $\begin{pmatrix}c_1\\c_2\end{pmatrix}$ can't be zero. Hence the only possibility is that

$$\begin{vmatrix}1-\lambda a_{11} & -\lambda a_{12}\\-\lambda a_{21} & 1-\lambda a_{22}\end{vmatrix} = 0$$

Now we evaluate the $a_{ij}$ coefficients. Recall from eariler that $M_1\left(x\right) = x^2, N_1\left(y\right) = y, M_2\left(x\right) =$

$x, N_2\left(y\right) = y^2$, hence

$$\begin{aligned}
a_{11} &= \int_0^1 N_1\left(x\right) M_1\left(x\right) \, \mathrm{d}x = \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4} \\
a_{12} &= \int_0^1 N_1\left(x\right) M_2\left(x\right) \, \mathrm{d}x = \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3} \\
a_{21} &= \int_0^1 N_2\left(x\right) M_1\left(x\right) \, \mathrm{d}x = \int_0^1 x^4 \, \mathrm{d}x = \frac{1}{5} \\
a_{22} &= \int_0^1 N_2\left(x\right) M_2\left(x\right) \, \mathrm{d}x = \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4}
\end{aligned}$$

Therefore

$$\begin{vmatrix} 1 - \lambda\frac{1}{4} & -\lambda\frac{1}{3} \\ -\lambda\frac{1}{5} & 1 - \lambda\frac{1}{4} \end{vmatrix} = 0$$

or

$$\begin{aligned}
\left(\frac{4-\lambda}{4}\right)\left(\frac{4-\lambda}{4}\right) - \left(\frac{-\lambda}{3}\right)\left(\frac{-\lambda}{5}\right) &= 0 \\
-\frac{1}{240}\lambda^2 - \frac{1}{2}\lambda + 1 &= 0 \\
\lambda^2 + 120\lambda - 240 &= 0
\end{aligned}$$

Therefore, the values of $\lambda$ which satisfy this polynomial will allow a non-trivial solution. In otherwords, the roots of this polynomial. This is what was required to show.

### 2.7.3 Problem 2 (integral equation, Neumann series method)

Solve the following integral equation using Neumann series method

$$\varphi(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)\,\varphi(t)\,\mathrm{d}t$$

**Solution**:

The above is in the form

$$\varphi(x) = f\left(x\right) - \int_0^x K\left(x,t\right)\varphi(t)\,\mathrm{d}t \tag{1}$$

Lets makes it in standard form, by changing the sign in front of the integral to + instead of −

$$\varphi(x) = f\left(x\right) + \int_x^0 K\left(x,t\right)\varphi(t)\,\mathrm{d}t$$

Hence it is a Volterra of the second kind. We start by approximating $\varphi(x)$ to $f\left(x\right)$

$$\varphi_0(x) \simeq f\left(x\right)$$

Then

$$\begin{aligned}
\varphi_1(x) &\simeq \int_x^0 K\left(x,t\right)\varphi_0(t)\,\mathrm{d}t \\
\varphi_2(x) &\simeq \int_x^0 K\left(x,t\right)\varphi_1(t)\,\mathrm{d}t \\
&\vdots \\
\varphi_n(x) &\simeq \int_x^0 K\left(x,t\right)\varphi_{n-1}(t)\,\mathrm{d}t
\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi_0(x) &= 1 + \frac{x^2}{2!} \\
\varphi_1(x) &= \int_x^0 (x-t)\left(1 + \frac{t^2}{2!}\right)\mathrm{d}t = -\left(\frac{x^2}{2} + \frac{x^4}{24}\right) \\
\varphi_2(x) &= -\int_x^0 (x-t)\left(\frac{t^2}{2} + \frac{t^4}{24}\right)\mathrm{d}t = \left(\frac{x^4}{24} + \frac{x^6}{720}\right) \\
\varphi_3(x) &= \int_x^0 (x-t)\left(\frac{t^4}{24} + \frac{t^6}{720}\right)\mathrm{d}t = -\left(\frac{x^6}{720} + \frac{x^8}{40320}\right)
\end{aligned}$$

Therefore, we now see the sequence as

$$\begin{aligned}
\varphi(x) &\simeq \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \cdots \\
&= \left(1 + \frac{x^2}{2!}\right) - \left(\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(\frac{x^4}{24} + \frac{x^6}{720}\right) - \left(\frac{x^6}{720} + \frac{x^8}{40320}\right) + \cdots \\
&= 1 + \frac{x^2}{2!} - \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots \\
&= 1
\end{aligned}$$

Hence the solution is

$$\varphi(x) = 1$$

### 2.7.4 Problem 3

Solve the following integral equation using Neumann series method

$$\varphi(x) = x\cos x + \int_0^x t\varphi(t)\,\mathrm{d}t$$

**Solution**:

The above is in the form

$$\varphi(x) = f(x) + \int_0^x K(x,t)\,\varphi(t)\,\mathrm{d}t \tag{1}$$

Hence it is a Volterra of the second kind. We start by approximating $\varphi(x)$ to $f(x)$

$$\varphi_0(x) \simeq f(x) = x\cos x$$

Then

$$\begin{aligned}
\varphi_1(x) &\simeq \int_x^0 K(x,t)\,\varphi_0(t)\,\mathrm{d}t \\
\varphi_2(x) &\simeq \int_x^0 K(x,t)\,\varphi_1(t)\,\mathrm{d}t \\
&\vdots \\
\varphi_n(x) &\simeq \int_x^0 K(x,t)\,\varphi_{n-1}(t)\,\mathrm{d}t
\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi_0(x) &= x\cos x \\
\varphi_1(x) &= \int_0^x t\,(t\cos t)\,\mathrm{d}t \\
&= 2x\cos x + \left(-2 + x^2\right)\sin x = x^2\sin x + 2x\cos x - 2\sin x
\end{aligned}$$

$$\begin{aligned}
\varphi_2(x) &= \int_0^x t\left(2\,(t\cos t - \sin t) + t^2\sin t\right)\,\mathrm{d}t \\
&= -x^3\cos x + 5x^2\sin x - 12\sin x + 12x\cos x
\end{aligned}$$

$$\begin{aligned}
\varphi_3(x) &= \int_0^x t\left(-t^3\cos t + 5t^2\sin t - 12\sin t + 12t\cos t\right)\,\mathrm{d}t \\
&= -x^4\sin x - 9x^3\cos x + 39x^2\sin x - 90\sin x + 90x\cos x
\end{aligned}$$

$$\begin{aligned}
\varphi_4(x) &= \int_0^x t\left(-t^4\sin t - 9t^3\cos t + 39t^2\sin t - 90\sin t + 90t\cos t\right)\,\mathrm{d}t \\
&= x^5\cos x - 14x^4\sin x - 95x^3\cos x + 375x^2\sin x - 840\sin x + 840x\cos x
\end{aligned}$$

$$\begin{aligned}
\varphi_5(x) &= \int_0^x t\left(t^5\cos t - 14t^4\sin t - 95t^3\cos t + 375t^2\sin t - 840\sin t + 840t\cos t\right)\,\mathrm{d}t \\
&= x^6\sin x + 20x^5\cos x - 195x^4\sin x - 1155x^3\cos x + 4305x^2\sin x - 9450\sin x + 9450x\cos x
\end{aligned}$$

Therefore, we now see the sequence as

$$\begin{aligned}\varphi(x) &\simeq \varphi_0(x)+\varphi_1(x)+\varphi_2(x)+\varphi_3(x)+\cdots\\ &= (x\cos x)\\ &+ x^2\sin x+2x\cos x-2\sin x\\ &- x^3\cos x+5x^2\sin x-12\sin x+12x\cos x\\ &- x^4\sin x-9x^3\cos x+39x^2\sin x-90\sin x+90x\cos x\\ &+ x^5\cos x-14x^4\sin x-95x^3\cos x+375x^2\sin x-840\sin x+840x\cos x\\ &+ x^6\sin x+20x^5\cos x-195x^4\sin x-1155x^3\cos x+4305x^2\sin x-9450\sin x+9450x\cos x\\ &+\cdots\end{aligned}$$

Collecting

$$\begin{aligned}\varphi(x) = &\cos x\,(x+2x+12x+90x+840x+9450x+\cdots)\\ &+\cos x\left(-x^3-9x^3-95x^3-1155x^3\cdots\right)\\ &+\cos x\left(x^5+20x^5+\cdots\right)\\ &\vdots\\ &+\sin x\,(-2-12-90-840-9450\cdots)\\ &+\sin x\left(x^2+5x^2+39x^2+375x^2+4305x^2+\cdots\right)\\ &+\sin x\left(-14x^4-195x^4-\cdots\right)\\ &+\sin x\left(x^6+\cdots\right)\end{aligned}$$

or

$$\begin{aligned}\varphi(x) &= \cos x\begin{pmatrix} x+2x+12x+90x+840x+9450x+\cdots\\ -x^3-9x^3-95x^3-1155x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}\\ &+\sin x\begin{pmatrix}-2-12-90-840-9450\cdots\\ +x^2+5x^2+39x^2+375x^2+4305x^2+\cdots\\ -14x^4-195x^4-\cdots\\ x^6+\cdots\end{pmatrix}\end{aligned}$$

But $\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$ and $\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots$ then the above can be written as

$$\begin{aligned}\varphi(x) &= \left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)\begin{pmatrix} x+2x+12x+90x+840x+9450x+\cdots\\ -x^3-9x^3-95x^3-1155x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}\\ &+\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)\begin{pmatrix}-2-12-90-840-9450\cdots\\ +x^2+5x^2+39x^2+375x^2+4305x^2+\cdots\\ -14x^4-195x^4-\cdots\\ x^6+\cdots\end{pmatrix}\end{aligned}$$

Looking at few terms

$$\begin{aligned}\varphi(x) = &\begin{pmatrix} x+2x+12x+90x+\\ -x^3-9x^3-95x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}-\frac{x^2}{2}\begin{pmatrix} x+2x+12x+90x+\\ -x^3-9x^3-95x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}+\frac{x^4}{24}\begin{pmatrix} x+2x+12x+90x+\\ -x^3-9x^3-95x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}\cdots\\ &+x\begin{pmatrix}-2-12-90-\cdots\\ +x^2+5x^2+39x^2\cdots\\ -14x^4-195x^4-\cdots\\ x^6+\cdots\end{pmatrix}-\frac{x^3}{6}\begin{pmatrix}-2-12-90-\cdots\\ +x^2+5x^2+39x^2\cdots\\ -14x^4-195x^4-\cdots\\ x^6+\cdots\end{pmatrix}+\frac{x^5}{120}\begin{pmatrix}-2-12-90-\cdots\\ +x^2+5x^2+39x^2\cdots\\ -14x^4-195x^4-\cdots\\ x^6+\cdots\end{pmatrix}\cdots\end{aligned}$$

Hence

$$\varphi(x)=\begin{pmatrix} x+2x+12x+90x+\\ -x^3-9x^3-95x^3\cdots\\ +x^5+20x^5+\cdots\end{pmatrix}+\begin{pmatrix} -\frac{x^3}{2}-x^3-6x^3-45x^3+\\ +\frac{x^5}{2}+\frac{9}{2}x^5+\frac{95}{2}x^5\cdots\\ -\frac{x^7}{2}-10x^7+\cdots\end{pmatrix}+\begin{pmatrix} \frac{x^5}{24}+\frac{x^5}{12}+\frac{x^5}{2}+\frac{45}{12}x^5+\\ -\frac{x^7}{24}-\frac{9}{24}x^7-\frac{95}{24}x^7\cdots\\ +\frac{x^9}{24}+\frac{20}{24}x^9+\cdots\end{pmatrix}\cdots$$

$$+\begin{pmatrix} -2x-12x-90x-\cdots\\ +x^3+5x^3+39x^3\cdots\\ -14x^5-195x^5-\cdots\\ x^7+\cdots\end{pmatrix}+\begin{pmatrix} +\frac{1}{3}x^3+2x^3+15x^3-\cdots\\ -\frac{x^5}{6}-\frac{5}{6}x^5-\frac{39}{6}x^5\cdots\\ +\frac{14}{6}x^7+\frac{195}{6}x^7-\cdots\\ -\frac{x^9}{6}+\cdots\end{pmatrix}+\begin{pmatrix} -\frac{x^5}{60}-\frac{1}{10}x^5-\frac{3}{4}x^5-\cdots\\ +\frac{1}{120}x^7+\frac{5}{120}x^7+\frac{39}{120}x^7\cdots\\ -\frac{14}{120}x^9-\frac{195}{120}x^9-\cdots\\ \frac{x^{10}}{120}+\cdots\end{pmatrix}\cdots$$

Hence, collecting terms. In this we start with the smallest coefficients of each power of $x$ found in all the above, and add them

$$\begin{aligned}\varphi(x)&=x\,(1+2+12+90+\cdots-2-12-90-\cdots)\\&\quad+x^3\left(\frac{1}{3}-\frac{1}{2}-1-1+2+5-6+15+39-45-\cdots\right)\\&\quad+x^5\left(-\frac{1}{60}+\frac{1}{24}+\frac{1}{12}+\frac{45}{12}-\frac{1}{10}\right)\end{aligned}$$

or

$$\varphi(x)=x+x^3\left(-\frac{1}{6}+O\left(h\right)\right)+x^5\left(\frac{451}{120}+O\left(h\right)\right)$$

May be a

$$\boxed{\sin\left(x\right)}$$

since $\sin\left(x\right)=x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots$? This problem is too hard but the above is my final answer.

### 2.7.5 Problem 4, integral equations, differential equation method

Solve

$$2\cosh\left(x\right)-\sinh\left(x\right)-(2-x)=1+\int_0^x(2-x+t)\,\varphi\left(t\right)dt$$

**Solution**:

Taking derivative w.r.t. $x$ gives

$$\begin{aligned}2\sin\left(x\right)-\cosh\left(x\right)+1&=\int_0^x\frac{d}{dx}(2-x+t)\,\varphi\left(t\right)dt+\frac{dx}{dx}\Big((2-x+x)\,\varphi\left(x\right)\Big)-0\\ 2\sin\left(x\right)-\cosh\left(x\right)+1&=\int_0^x-\varphi\left(t\right)dt+2\varphi\left(x\right)\\ 2\sinh\left(x\right)-\cosh\left(x\right)+1-2\varphi\left(x\right)&=-\int_0^x\varphi\left(t\right)dt\\ \cosh\left(x\right)-2\sinh\left(x\right)-1+2\varphi\left(x\right)&=\int_0^x\varphi\left(t\right)dt\qquad(1)\end{aligned}$$

Differneting again w.r.t. $x$

$$\begin{aligned}\sinh\left(x\right)-2\cosh\left(x\right)+2\varphi'\left(x\right)&=\int_0^x\frac{d}{dx}\varphi\left(t\right)dt+\frac{dx}{dx}\Big(\varphi\left(x\right)\Big)-0\\&=\varphi\left(x\right)\end{aligned}$$

Hence the ODE is

$$\varphi'\left(x\right)-\frac{1}{2}\varphi\left(x\right)=\cosh\left(x\right)-\frac{1}{2}\sinh\left(x\right)$$

Integrating factor is $e^{-\frac{x}{2}}$hence the solution is

$$\begin{aligned}d\left(e^{-\frac{x}{2}}\varphi\left(x\right)\right) &= \int e^{-\frac{x}{2}}\cosh\left(x\right)-\frac{1}{2}\int e^{-\frac{x}{2}}\sinh\left(x\right)\\ e^{-\frac{x}{2}}\varphi\left(x\right) &= \cosh\left(\frac{x}{2}\right)-\frac{1}{3}\cosh\left(\frac{3x}{2}\right)+\sinh\left(\frac{x}{2}\right)+\frac{1}{3}\sinh\left(\frac{3x}{2}\right)\\ &\quad -\frac{1}{2}\left[\sinh\left(\frac{x}{2}\right)-\frac{1}{3}\sinh\left(\frac{3x}{2}\right)+\cosh\left(\frac{x}{2}\right)+\frac{1}{3}\cosh\left(\frac{3x}{2}\right)\right]+C\end{aligned}$$

Simplfying RHS

$$\begin{aligned}e^{-\frac{x}{2}}\varphi\left(x\right) &= \cosh\left(\frac{x}{2}\right)-\frac{1}{3}\cosh\left(\frac{3x}{2}\right)+\sinh\left(\frac{x}{2}\right)+\frac{1}{3}\sinh\left(\frac{3x}{2}\right)-\frac{1}{2}\sinh\left(\frac{x}{2}\right)\\ &\quad+\frac{1}{6}\sinh\left(\frac{3x}{2}\right)-\frac{1}{2}\cosh\left(\frac{x}{2}\right)-\frac{1}{6}\cosh\left(\frac{3x}{2}\right)+C\\ \\ &= \frac{1}{2}\cosh\left(\frac{1}{2}x\right)-\frac{1}{2}\cosh\left(\frac{3}{2}x\right)+\frac{1}{2}\sinh\left(\frac{1}{2}x\right)+\frac{1}{2}\sinh\left(\frac{3}{2}x\right)+C\end{aligned}$$

Therefore

$$\varphi\left(x\right)=\frac{e^{\frac{x}{2}}}{2}\left(\cosh\left(\frac{1}{2}x\right)-\cosh\left(\frac{3}{2}x\right)+\sinh\left(\frac{1}{2}x\right)+\sinh\left(\frac{3}{2}x\right)\right)+Ce^{\frac{x}{2}}$$

Where the constant of integration $C$. To find this constant, looking at Eq. (1) above, repeated below

$$\cosh\left(x\right)-2\sinh\left(x\right)-1+2\varphi\left(x\right)=\int_0^x\varphi\left(t\right)dt$$

We see that at $x=0$ the above gives

$$\begin{aligned}1-1+2\varphi\left(0\right)&=0\\ \varphi\left(0\right)&=0\end{aligned}$$

Therefore, we let $x=0$ in the solution itself and find $C$ which gives

$$\begin{aligned}\varphi\left(0\right)=0&=\frac{1}{2}\left(1-1+0+0\right)+C\\ C&=0\end{aligned}$$

Hence the solution is

$$\varphi\left(x\right)=\frac{e^{\frac{x}{2}}}{2}\left(\cosh\left(\frac{1}{2}x\right)-\cosh\left(\frac{3}{2}x\right)+\sinh\left(\frac{1}{2}x\right)+\sinh\left(\frac{3}{2}x\right)\right)$$

### 2.7.6 Problem 5

Solve

$$u\left(x\right)=\cos\left(x\right)-x-2+\int_0^x\left(t-x\right)u\left(t\right)dt$$

**solution**

Taking derivative w.r.t. $x$

$$\begin{aligned}u'\left(x\right)&=-\sin\left(x\right)-1+\int_0^x\frac{d}{dx}\left(t-x\right)u\left(t\right)dt+\left[\frac{dx}{dx}\left(x-x\right)u\left(x\right)-0\right]\\ &=-\sin\left(x\right)-1-\int_0^x u\left(t\right)dt\end{aligned}$$

Taking another derivative w.r.t. $x$

$$\begin{aligned}u''\left(x\right)&=-\cos\left(x\right)-\left(\int_0^x\frac{d}{dx}u\left(t\right)dt+\frac{dx}{dx}u\left(x\right)-0\right)\\ &=-\cos\left(x\right)-u\left(x\right)\end{aligned}$$

The differential equation is

$$u''\left(x\right)+u\left(x\right)=-\cos x \tag{1}$$

The homogeneous solution is

$$u_h = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Where $\lambda_{1,2} = \frac{\pm\sqrt{-4}}{2} = \pm i$, hence

$$u_h = A\cos x + B\sin x$$

To find the particular solution, let

$$u_p = c_1 x\cos x + c_2 x\sin x$$

hence

$$\begin{aligned} u_p' &= c_1\cos x - c_1 x\sin x + c_2\sin x + c_2 x\cos x \\ u_p'' &= -c_1\sin x - c_1\sin x - c_1 x\cos x + c_2\cos x + c_2\cos x - c_2 x\sin x \end{aligned}$$

Substituting all the above in the original ODE Eq. (1)

$$\begin{aligned} -c_1\sin x - c_1\sin x - c_1 x\cos x + c_2\cos x + c_2\cos x - c_2 x\sin x + (c_1 x\cos x + c_2 x\sin x) &= -\cos x \\ 2c_2\cos x - 2c_1\sin x &= -\cos x \end{aligned}$$

Hence $c_1 = 0$ and $c_2 = -\frac{1}{2}$, therefore, the particular solution is

$$u_p = -\frac{1}{2}x\sin x$$

This gives the complete solution

$$u = A\cos x + B\sin x - \frac{1}{2}x\sin x \tag{1}$$

At $x = 0$, the original integral equation becomes

$$\begin{aligned} u(0) &= \cos(0) - 0 - 2 + \int_0^0 (t-x)\,u(t)\,dt \\ &= 1 - 2 = -1 \end{aligned}$$

Hence Eq.(1) when $x = 0$ gives

$$-1 = A$$

Hence the solution now is

$$u = -\cos x + B\sin x - \frac{1}{2}x\sin x \tag{2}$$

Now to find $B$ we take derivative of the integral equation and evaluate it at $x = 0$ which gives

$$\begin{aligned} u'(0) &= -\sin(0) - 1 - \int_0^0 u(t)\,dt \\ &= -1 \end{aligned}$$

But the derivative of Eq (2) is

$$u' = \sin x + B\cos x - \frac{1}{2}(\sin x + x\cos x)$$

And at $x = 0$ it gives

$$u'(0) = B$$

Hence $B = -1$ and the solution is

$$u = -\cos x - \sin x - \frac{1}{2}x\sin x$$

### 2.7.7 Problem 6

Solve $u(x) = x + \lambda\int_0^1 (1 + x + t)\,u(t)\,dt$

**solution**

Taking derivative w.r.t. $x$

$$\begin{aligned} u'(x) &= 1+\lambda\int_0^1 \frac{d}{dx}(1+x+t)\,u(t)\,dt \\ &= 1+\lambda\int_0^1 u(t)\,dt \end{aligned}$$

Taking derivative w.r.t. $x$ again

$$\begin{aligned} u''(x) &= \lambda\int_0^1 \frac{d}{dx}u(t)\,dt \\ &= 0 \end{aligned}$$

Hence the solution is

$$u(x) = Ax+B$$

We can rewrite the integral equation as

$$u(x) = x+\lambda\left(\int_0^x (1+x+t)\,u(t)\,dt+\int_x^1 (1+x+t)\,u(t)\,dt\right)$$

And it will give the same answer ofcourse. $x$ is some point between 0 and 1. Now we processed as before. Taking derivative w.r.t. $x$

$$\begin{aligned} u'(x) &= 1+\lambda\left(\int_0^x \frac{d}{dx}(1+x+t)\,u(t)\,dt+\frac{dx}{dx}(1+x+x)\,u(x)-0\right) \\ &\quad +\lambda\left(\int_x^1 \frac{d}{dx}(1+x+t)\,u(t)\,dt+0-\frac{dx}{dx}(1+x+x)\,u(x)\right) \\ &= 1+\lambda\left(\int_0^x u(t)\,dt+(1+2x)\,u(x)\right)+\lambda\left(\int_x^1 u(t)\,dt-(1+2x)\,u(x)\right) \end{aligned}$$

Taking derivative w.r.t. $x$ one more time

$$\begin{aligned} u''(x) &= \lambda\left(\int_0^x \frac{d}{dx}u(t)\,dt+\left[\frac{dx}{dx}u(x)-0\right]+\frac{d}{dx}(1+2x)\,u(x)\right) \\ &\quad +\lambda\left(\int_x^1 \frac{d}{dx}u(t)\,dt+\left[0-\frac{dx}{dx}u(x)\right]-\frac{d}{dx}(1+2x)\,u(x)\right) \\ &= \lambda\left(u(x)+2u(x)+(1+2x)\,u'(x)\right)+\lambda\left(-u(x)-2u(x)-(1+2x)\,u'(x)\right) \\ &= \lambda\left((1+2x)\,u'(x)\right)+\lambda\left(-(1+2x)\,u'(x)\right) \\ &= 0 \end{aligned}$$

Hence the solution is

$$u(x) = Ax+B \tag{1}$$

To find the constants of integrations, we see that at $x=0$ the integral equation gives

$$u(0) = 0+\lambda\int_0^0 (1+x+t)\,u(t)\,dt = 0$$

Hence from Eq. (1) at $x=0$ we find $B=0$, hence the solution is

$$u(x) = Ax \tag{2}$$

Taking derivative $u'(x)=A$, but from original integral equation we found that

$$u'(x) = 1+\lambda\int_0^1 u(t)\,dt$$

Hence at $x=0$

$$u'(0) = 1$$

Therefore $A=1$ and the solution is

$$u(x) = x$$

### 2.7.8 Problem 7

Solve $\frac{dy(t)}{dt} = 2 - \frac{t^2}{2} - \frac{1}{4}\int_0^t y(\tau)\,d\tau$

**Solution**

If we integrate from 0 to $t$, then

$$\begin{aligned}\int_0^t \frac{dy(\tau)}{d\tau}d\tau &= \int_0^t 2d\tau - \int_0^t \frac{\tau^2}{2}d\tau - \frac{1}{4}\int_0^t \left(\int_0^{t_1} y(\tau_2)\,d\tau_2\right)d\tau\\ y(t) - y(0) &= 2t - \frac{t^3}{6} - \frac{1}{4}\int_0^t (t-\tau_2)\,y(\tau_2)\,d\tau_2\\ y(t) &= 2t - \frac{t^3}{6} - \frac{1}{4}\int_0^t (t-\tau)\,y(\tau)\,d\tau\end{aligned}$$

Check the above. Can I do this below? Taking derivative w.r.t. $t$

$$\begin{aligned}\frac{d^2y(t)}{dt^2} &= -t - \frac{1}{4}\left(\int_0^t \frac{d}{dt}y(\tau)\,d\tau + \frac{dt}{dt}y(t) - 0\right)\\ &= -t - \frac{1}{4}y(t)\end{aligned}$$

Hence

$$y'' + \frac{1}{4}y(t) = -t$$

The roots are $\frac{i}{2}, -\frac{i}{2}$, hence

$$y_h = B\cos\frac{t}{2} + C\sin\frac{t}{2}$$

For particular solution, let

$$y_p = c_1 t + c_2$$

and substituting in the ODE gives $\frac{1}{4}c_1 = -1$ or $c_1 = -4$, so

$$y_p = -4t$$

Hence the solution is

$$y = B\cos\frac{t}{2} + C\sin\frac{t}{2} - 4t$$

When $x = 0$

$$0 = B$$

Hence the solution becomes

$$y(t) = C\sin\frac{t}{2} - 4t \tag{1}$$

To find the constant $C$, from the integral equation, at $t = 0$

$$\begin{aligned}\frac{dy(0)}{dt} &= 2 - \frac{0}{2} - \frac{1}{4}\int_0^0 y(\tau)\,d\tau\\ &= 2\end{aligned}$$

And from Eq. (1) $y'(t) = C\frac{1}{2}\cos\frac{t}{2} - 4$, hence at $t = 0$

$$\begin{aligned}2 &= C\frac{1}{2} - 4\\ C &= 12\end{aligned}$$

Hence the solution in Eq. (1) becomes

$$y(t) = 12\sin\frac{t}{2} - 4t$$

### 2.7.9 Problem 8

Solve the integral equation using any method

$$y(x) = f(x) - A\int_0^x xte^{\lambda(x-t)}y(t)\,dt$$

**Solution**:

The above is a Volterra integral equation, inhomogeneous since $f(x)$ exist, and second kind since the function we solving for is under the integral as well. We start by removing $x$ dependencies from inside the integral to the outside

$$y(x) = f(x) - Axe^{\lambda x}\int_0^x te^{-\lambda t} y(t)\,dt$$

Now divide by $xe^{\lambda x}$ (notice that $x$ can not be zero, hence initial conditions must start at some other value).

$$\frac{y(x)}{xe^{\lambda x}} = \frac{f(x)}{xe^{\lambda x}} - A\int_0^x te^{-\lambda t} y(t)\,dt$$

Let $\phi(x) = \frac{y(x)}{xe^{\lambda x}}$ and $F(x) = \frac{f(x)}{xe^{\lambda x}}$ the above becomes

$$\phi(x) = F(x) - A\int_0^x te^{-\lambda t} y(t)\,dt$$

Now inside the integral, multiply by $\frac{te^{\lambda t}}{te^{\lambda t}}$ in order to obtain the same form as on LHS

$$\begin{aligned}\phi(x) &= F(x) - A\int_0^x te^{-\lambda t}te^{\lambda t}\left(\frac{y(t)}{te^{\lambda t}}\right)dt\\ &= F(x) - A\int_0^x t^2\phi(t)\,dt\end{aligned}$$

Taking derivative

$$\begin{aligned}\phi'(x) &= F'(x) - A\left(\int_0^x \frac{d}{dx}t^2\phi(t)\,dt + \frac{dx}{dx}x^2\phi(x) - 0\right)\\ &= F'(x) - Ax^2\phi(x)\\ \phi'(x) + Ax^2\phi(x) &= F'(x)\end{aligned}$$

Integrating factor is $e^{\int Ax^2 dx} = e^{A\frac{x^3}{3}}$, hence

$$d\left(e^{A\frac{x^3}{3}}\phi(x)\right) = e^{A\frac{x^3}{3}}F'(x)$$

Integrate both sides from 0 to $x$

$$\begin{aligned}\int_0^x d\left(e^{A\frac{z^3}{3}}\phi(z)\right) &= \int_0^x e^{A\frac{z^3}{3}}\frac{dF}{dz}dz\\ e^{A\frac{x^3}{3}}\phi(x) - \phi(0) &= \left[e^{A\frac{z^3}{3}}F\right]_0^x - \int_0^x Fd\left(e^{A\frac{z^3}{3}}\right)\\ e^{A\frac{x^3}{3}}\phi(x) - \phi(0) &= \left(e^{A\frac{x^3}{3}}F(x) - F(0)\right) - \int_0^x FAz^2e^{A\frac{z^3}{3}}dz\\ e^{A\frac{x^3}{3}}\phi(x) &= \phi(0) + e^{A\frac{x^3}{3}}F(x) - F(0) - \int_0^x FAz^2e^{A\frac{z^3}{3}}dz\\ \phi(x) &= e^{-A\frac{x^3}{3}}\phi(0) + F(x) - e^{-A\frac{x^3}{3}}F(0) - Ae^{-A\frac{x^3}{3}}\int_0^x Fz^2e^{A\frac{z^3}{3}}dz\end{aligned}$$

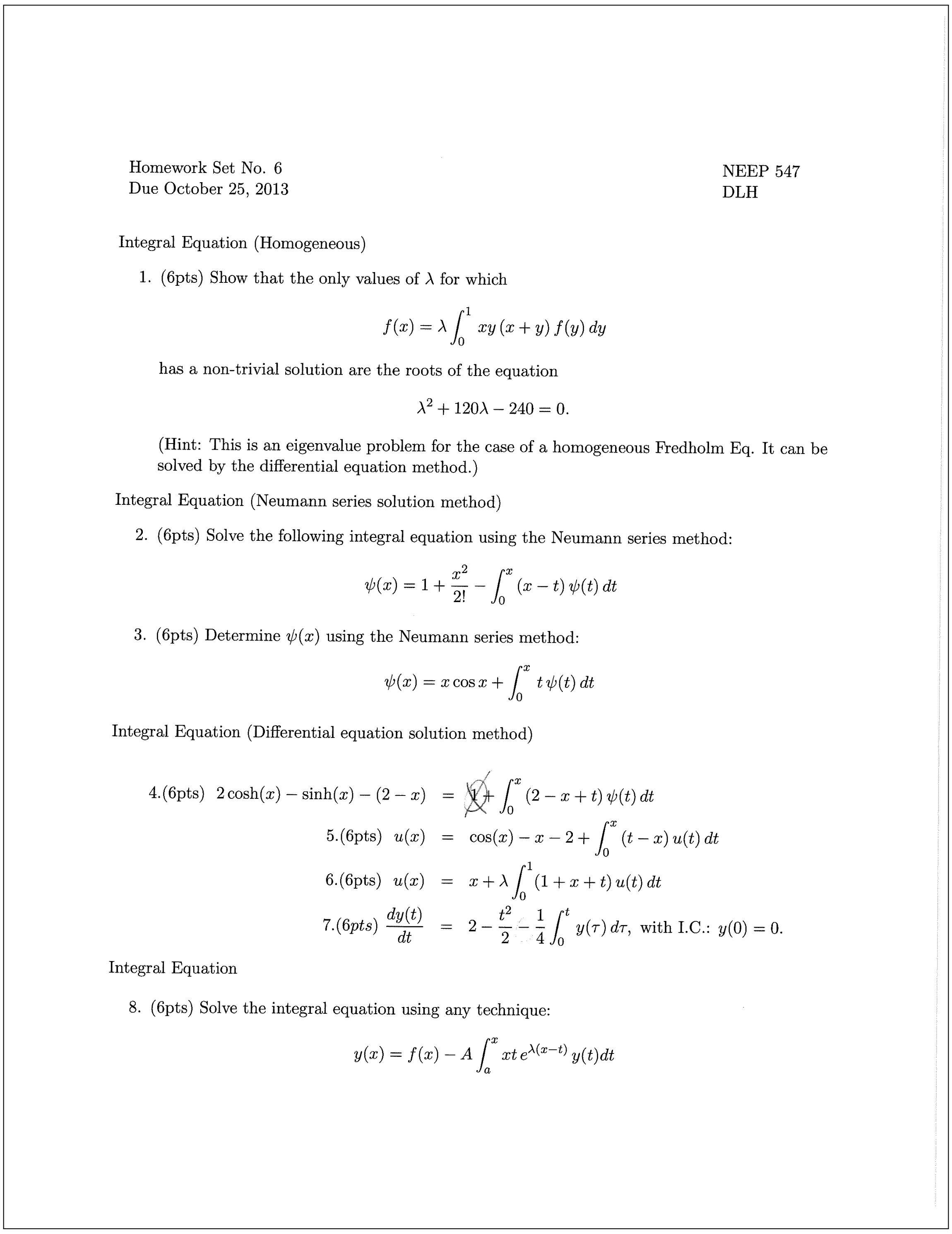
But $\phi(x) = \frac{y(x)}{xe^{\lambda x}}$ and $F(x) = \frac{f(x)}{xe^{\lambda x}}$, therefore the above becomes (what to do with the division by zero?)

$$\begin{aligned}\frac{y(x)}{xe^{\lambda x}} &= e^{-A\frac{x^3}{3}}\frac{y(0)}{\lim_{x\to 0}xe^{\lambda x}} + \frac{f(x)}{xe^{\lambda x}} - e^{-A\frac{x^3}{3}}\frac{f(0)}{\lim_{x\to 0}xe^{\lambda x}} - Ae^{-A\frac{x^3}{3}}\int_a^x \frac{f(z)}{ze^{\lambda z}}z^2e^{A\frac{z^3}{3}}dz\\ y(x) &= xe^{\left(-A\frac{x^3}{3}+\lambda x\right)}\frac{y(0)}{\lim_{x\to 0}xe^{\lambda x}} + f(x) - xe^{\left(-A\frac{x^3}{3}+\lambda x\right)}\frac{f(0)}{\lim_{x\to 0}xe^{\lambda x}} - xAe^{\left(-A\frac{x^3}{3}+\lambda x\right)}\int_a^x \frac{f(z)}{e^{\lambda z}}ze^{A\frac{z^3}{3}}dz\end{aligned}$$

Assuming zero initial conditions for now, which means $f(0) = 0$ and $y(0) = 0$, then in the limit the above reduces to

$$y(x) = f(x) - \frac{xAe^{\lambda x}}{e^{\frac{Ax^3}{3}}}\int_a^x zf(z)\frac{e^{\frac{Az^3}{3}}}{e^{\lambda z}}dz$$

### 2.7.10 key solution



Homework Set No. 6
Due October 25, 2013

NEEP 547
DLH

Integral Equation (Homogeneous)

1. (6pts) Show that the only values of $\lambda$ for which

$$f(x) = \lambda \int_0^1 xy\,(x+y)\,f(y)\,dy$$

has a non-trivial solution are the roots of the equation

$$\lambda^2 + 120\lambda - 240 = 0.$$

(Hint: This is an eigenvalue problem for the case of a homogeneous Fredholm Eq. It can be solved by the differential equation method.)

Integral Equation (Neumann series solution method)

2. (6pts) Solve the following integral equation using the Neumann series method:

$$\psi(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)\,\psi(t)\,dt$$

3. (6pts) Determine $\psi(x)$ using the Neumann series method:

$$\psi(x) = x\cos x + \int_0^x t\,\psi(t)\,dt$$

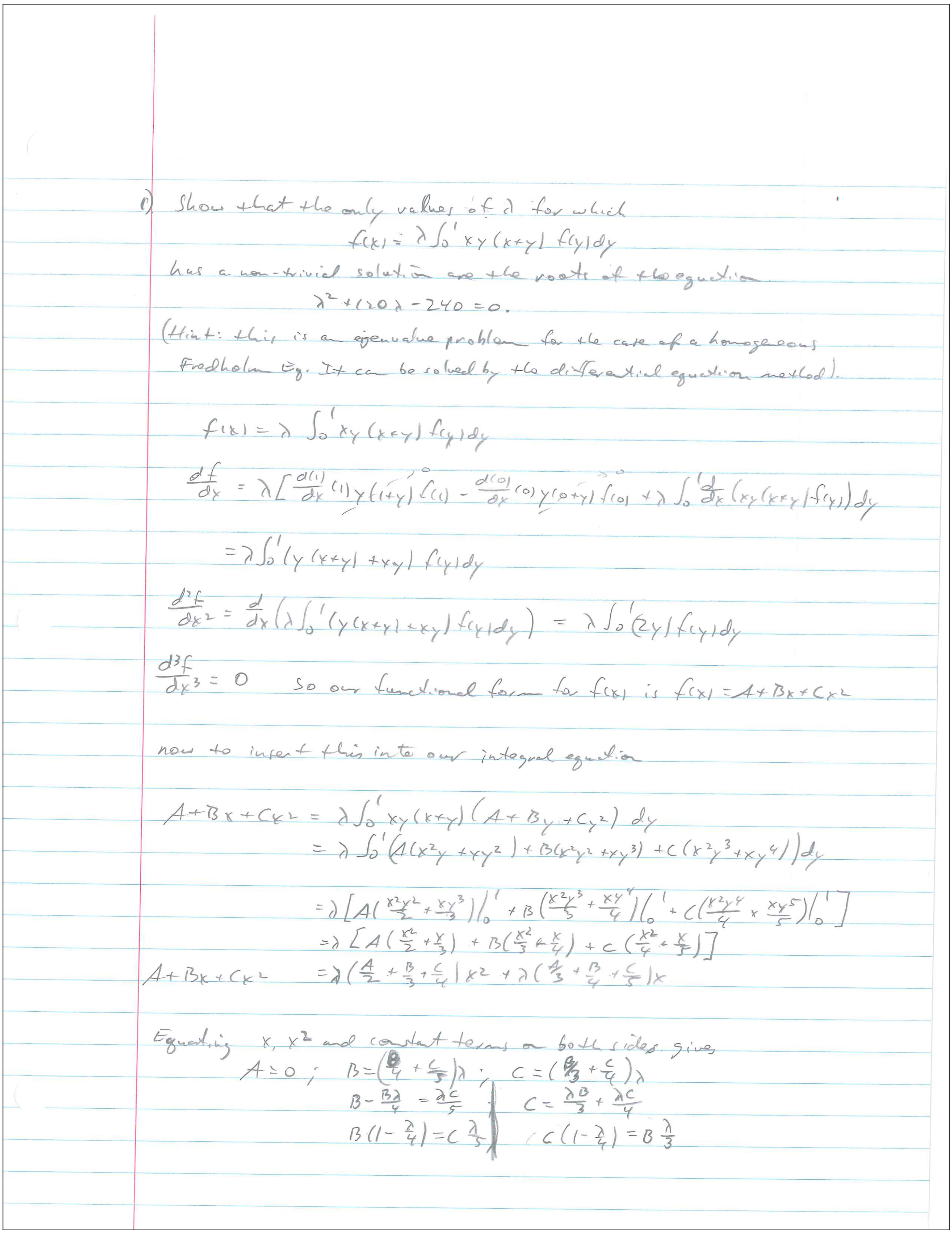
Integral Equation (Differential equation solution method)

$$\begin{aligned}
4.(6\text{pts})\ \ 2\cosh(x) - \sinh(x) - (2-x) &= \cancel{1+} \int_0^x (2-x+t)\,\psi(t)\,dt\\
5.(6\text{pts})\ \ u(x) &= \cos(x) - x - 2 + \int_0^x (t-x)\,u(t)\,dt\\
6.(6\text{pts})\ \ u(x) &= x + \lambda\int_0^1 (1+x+t)\,u(t)\,dt\\
7.(6pts)\ \frac{dy(t)}{dt} &= 2 - \frac{t^2}{2} - \frac{1}{4}\int_0^t y(\tau)\,d\tau,\ \text{with I.C.: } y(0) = 0.
\end{aligned}$$

Integral Equation

8. (6pts) Solve the integral equation using any technique:

$$y(x) = f(x) - A\int_a^x xt\,e^{\lambda(x-t)}\,y(t)dt$$



1) Show that the only values of $\lambda$ for which

$$f(x) = \lambda \int_0^1 xy(x+y)\, f(y)\,dy$$

has a non-trivial solution are the roots of the equation

$$\lambda^2 + 120\lambda - 240 = 0.$$

(Hint: this is an eigenvalue problem for the case of a homogeneous Fredholm Eq. It can be solved by the differential equation method).

$$f(x) = \lambda \int_0^1 xy(x+y)\, f(y)\,dy$$

$$\frac{df}{dx} = \lambda\Big[\frac{d(1)}{dx}(1)\cancelto{0}{y(1+y)}f(1) - \frac{d(0)}{dx}(0)\cancelto{0}{y(0+y)}f(0) + \lambda \int_0^1 \frac{d}{dx}\big(xy(x+y)f(y)\big)dy$$

$$= \lambda \int_0^1 \big(y(x+y) + xy\big) f(y)\,dy$$

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\Big(\lambda \int_0^1 \big(y(x+y)+xy\big) f(y)\,dy\Big) = \lambda \int_0^1 (2y) f(y)\,dy$$

$\frac{d^3f}{dx^3} = 0$ so our functional form for $f(x)$ is $f(x) = A + Bx + Cx^2$

now to insert this into our integral equation

$$A + Bx + Cx^2 = \lambda \int_0^1 xy(x+y)(A + By + Cy^2)\,dy$$

$$= \lambda \int_0^1 \Big(A(x^2y + xy^2) + B(x^2y^2 + xy^3) + C(x^2y^3 + xy^4)\Big)dy$$

$$= \lambda\Big[A\Big(\frac{x^2y^2}{2} + \frac{xy^3}{3}\Big)\Big|_0^1 + B\Big(\frac{x^2y^3}{5} + \frac{xy^4}{4}\Big)\Big|_0^1 + C\Big(\frac{x^2y^4}{4} + \frac{xy^5}{5}\Big)\Big|_0^1\Big]$$

$$= \lambda\Big[A\Big(\frac{x^2}{2} + \frac{x}{3}\Big) + B\Big(\frac{x^2}{3} + \frac{x}{4}\Big) + C\Big(\frac{x^2}{4} + \frac{x}{5}\Big)\Big]$$

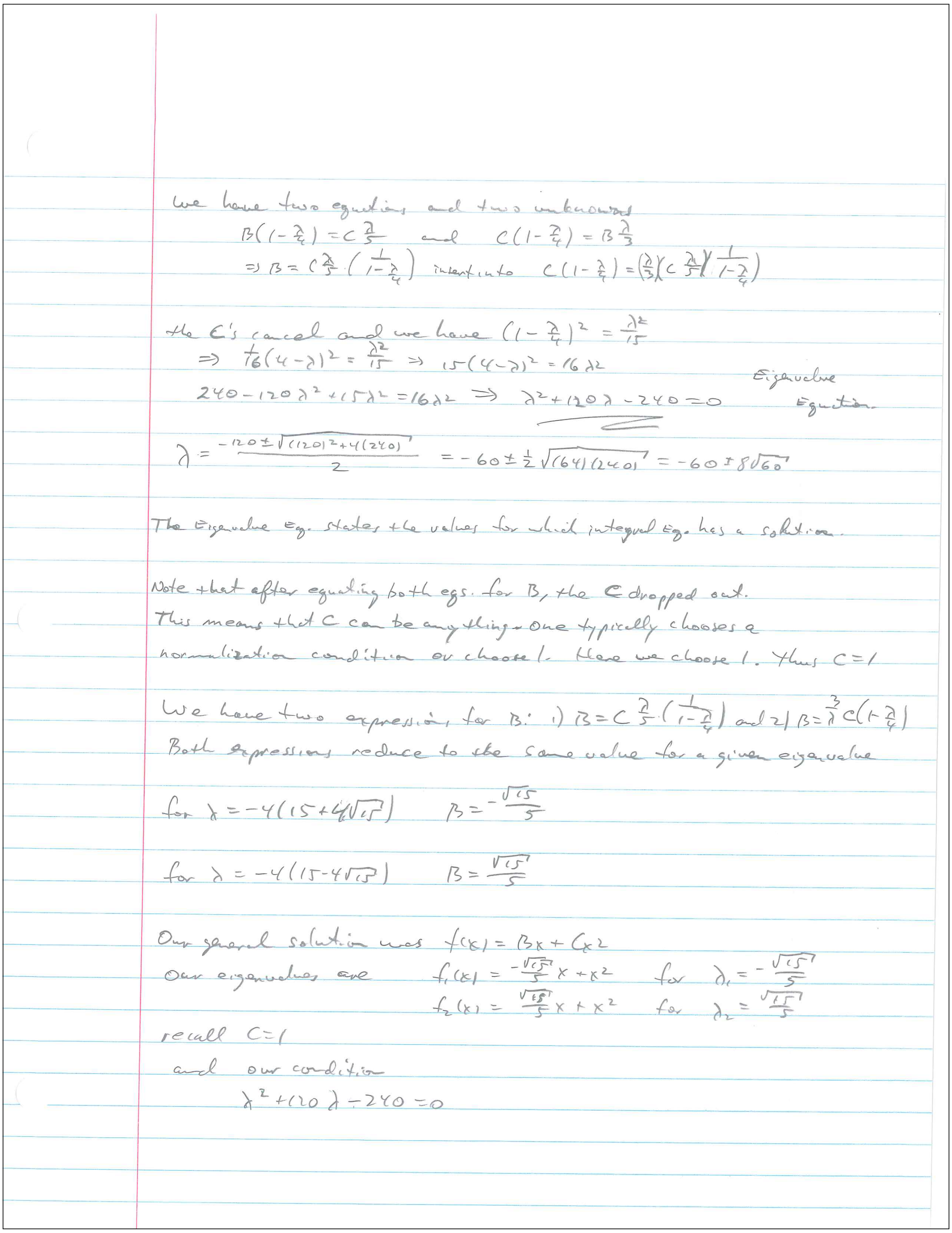
$$A + Bx + Cx^2 = \lambda\Big(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\Big)x^2 + \lambda\Big(\frac{A}{3} + \frac{B}{4} + \frac{C}{5}\Big)x$$

Equating $x$, $x^2$ and constant terms on both sides gives

$$A = 0; \quad B = \Big(\frac{B}{4} + \frac{C}{5}\Big)\lambda; \quad C = \Big(\frac{B}{3} + \frac{C}{4}\Big)\lambda$$

$$B - \frac{B\lambda}{4} = \frac{\lambda C}{5} \qquad C = \frac{\lambda B}{3} + \frac{\lambda C}{4}$$

$$B\Big(1 - \frac{\lambda}{4}\Big) = C\frac{\lambda}{5} \qquad C\Big(1 - \frac{\lambda}{4}\Big) = B\frac{\lambda}{3}$$



we have two equations and two unknowns

$$B\left(1-\frac{\lambda}{4}\right)=C\frac{\lambda}{5} \quad \text{and} \quad C\left(1-\frac{\lambda}{4}\right)=B\frac{\lambda}{3}$$

$$\Rightarrow B = C\frac{\lambda}{5}\cdot\left(\frac{1}{1-\frac{\lambda}{4}}\right) \text{ insert into } C\left(1-\frac{\lambda}{4}\right)=\left(\frac{\lambda}{3}\right)\left(C\frac{\lambda}{5}\right)\left(\frac{1}{1-\frac{\lambda}{4}}\right)$$

the C's cancel and we have $\left(1-\frac{\lambda}{4}\right)^2=\frac{\lambda^2}{15}$

$$\Rightarrow \frac{1}{16}(4-\lambda)^2=\frac{\lambda^2}{15} \Rightarrow 15(4-\lambda)^2=16\lambda^2$$

$$240-120\lambda^2+15\lambda^2=16\lambda^2 \Rightarrow \underline{\underline{\lambda^2+120\lambda-240=0}}$$

Eigenvalue Equation.

$$\lambda=\frac{-120\pm\sqrt{(120)^2+4(240)}}{2}=-60\pm\frac{1}{2}\sqrt{(64)(240)}=-60\pm8\sqrt{60}$$

The Eigenvalue Eq. states the values for which integral Eq. has a solution.

Note that after equating both eqs. for B, the C dropped out.
This means that C can be anything. One typically chooses a normalization condition or choose 1. Here we choose 1. Thus C=1

We have two expressions for B: 1) $B=C\frac{\lambda}{5}\cdot\left(\frac{1}{1-\frac{\lambda}{4}}\right)$ and 2) $B=\frac{3}{\lambda}C\left(1-\frac{\lambda}{4}\right)$
Both expressions reduce to the same value for a given eigenvalue

for $\lambda=-4(15+4\sqrt{15})$ $\quad B=-\frac{\sqrt{15}}{5}$

for $\lambda=-4(15-4\sqrt{15})$ $\quad B=\frac{\sqrt{15}}{5}$

Our general solution was $f(x)=Bx+Cx^2$

Our eigenvalues are $f_1(x)=-\frac{\sqrt{15}}{5}x+x^2$ for $\lambda_1=-\frac{\sqrt{15}}{5}$

$f_2(x)=\frac{\sqrt{15}}{5}x+x^2$ for $\lambda_2=\frac{\sqrt{15}}{5}$

recall C=1

and our condition

$$\lambda^2+120\lambda-240=0$$

2) Solve the following integral equation using the Neuman series method:

$$\varphi(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)\,\varphi(t)\,dt \qquad \varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$$

$$\varphi_0(x) = 1 + \frac{x^2}{2!}$$

$$\varphi_1(x) = -\int_0^x (x-t)\left(1+\frac{t^2}{2!}\right)dt = -\int_0^x (x)\left(1+\frac{t^2}{2!}\right)dt + \int_0^x (t)\left(1+\frac{t^2}{2!}\right)dt$$

$$= -x\int_0^x \left(1+\frac{t^2}{2!}\right)dt + \int_0^x \left(t+\frac{t^3}{2!}\right)dt$$

$$= -x\left(t+\frac{t^3}{3!}\right)\Big|_0^x + \left(\frac{t^2}{2}+\frac{t^4}{2\cdot 4}\right)\Big|_0^x$$

$$= -x\left(x+\frac{x^3}{2\cdot 3}\right) + \left(\frac{x^2}{2}+\frac{x^4}{2\cdot 4}\right) = -x^2 - \frac{x^4}{2\cdot 3} + \frac{x^2}{2} + \frac{x^4}{2\cdot 4}$$

$$= -\frac{x^2}{2} - \frac{x^4}{2\cdot 3\cdot 4} = -\frac{x^2}{2!} - \frac{x^4}{4!} = -\left(\frac{x^2}{2!}+\frac{x^4}{4!}\right)$$

$$\varphi_2(x) = -\int_0^x (x-t)\left(-\frac{t^2}{2!}-\frac{t^4}{4!}\right)dt = \int_0^x (x)\left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt - \int_0^x (t)\left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt$$

$$= x\int_0^x \left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt - \int_0^x \left(\frac{t^3}{2!}+\frac{t^5}{4!}\right)dt$$

$$= x\left(\frac{t^3}{3!}+\frac{t^5}{5!}\right)\Big|_0^x - \left(\frac{t^4}{2\cdot 4}+\frac{t^6}{6\cdot 4!}\right)\Big|_0^x$$

$$= x\left(\frac{x^3}{3!}+\frac{x^5}{5!}\right) - \left(\frac{x^4}{2\cdot 4}+\frac{x^6}{6\cdot 4!}\right) = \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^4}{2\cdot 4} - \frac{x^6}{6\cdot 4!}$$

$$= \left(\frac{4}{2\cdot 3\cdot 4} - \frac{3}{2\cdot 3\cdot 4}\right)x^4 + \left(\frac{6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\right)x^6 = \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$\varphi_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{4!}+\frac{x^6}{6!}\right)dt = -\int_0^x (x)\left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt + \int_0^x (t)\left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt$$

$$= -x\int_0^x \left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt + \int_0^x \left(\frac{t^5}{4!}+\frac{t^7}{6!}\right)dt$$

$$= -x\left(\frac{t^5}{5!}+\frac{t^7}{7!}\right)\Big|_0^x + \left(\frac{t^6}{6\cdot 4!}+\frac{t^8}{8\cdot 6!}\right)\Big|_0^x = -\frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^6}{6\cdot 4!} + \frac{x^8}{8\cdot 6!}$$

$$= -\left(\frac{6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\right)x^6 - \left(\frac{8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} - \frac{7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\right)x^8$$

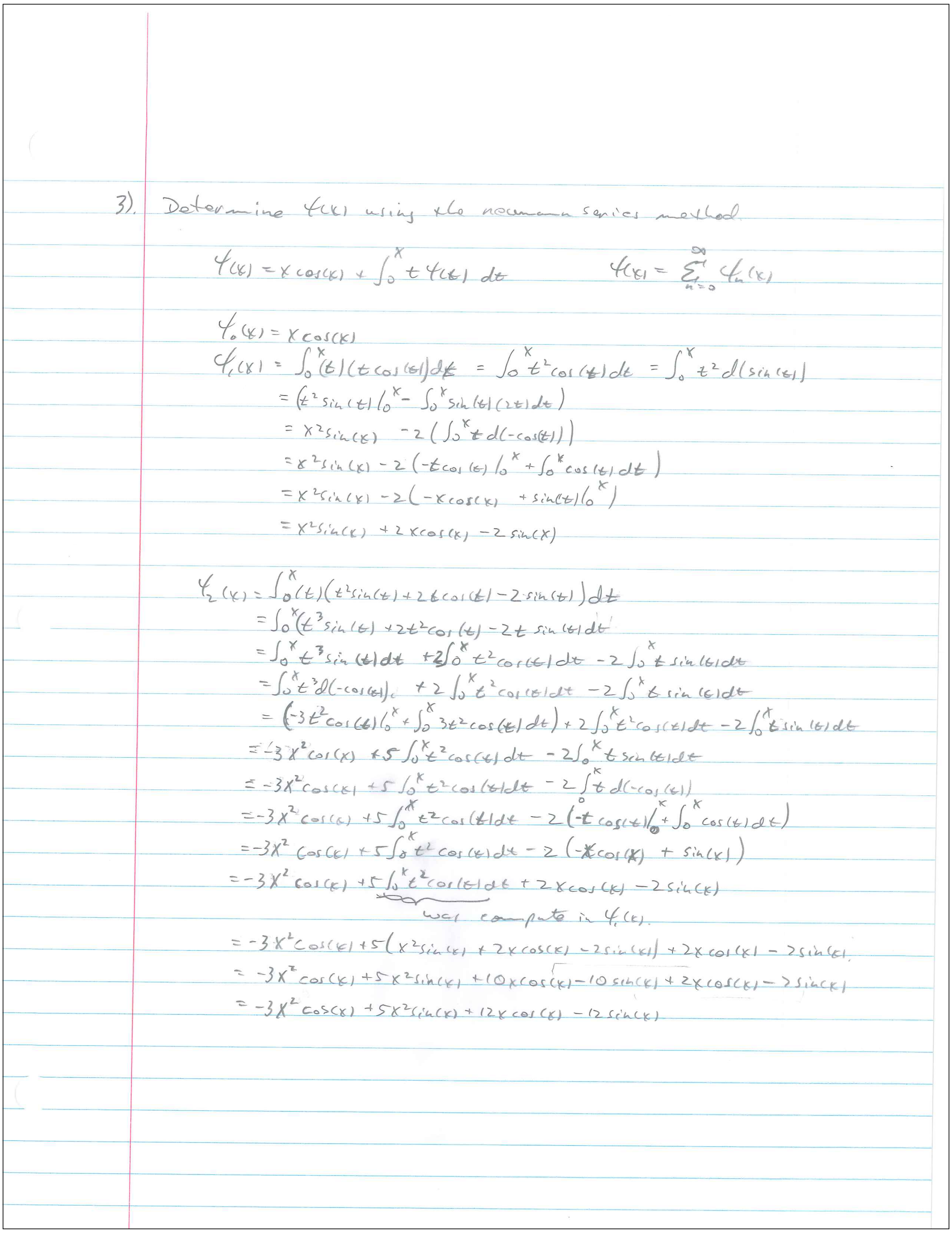
$$= -\frac{x^6}{6!} - \frac{x^8}{8!}$$

$$\therefore \varphi(x) = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots$$

$$= 1 + \cancel{\frac{x^2}{2!}} - \cancel{\frac{x^2}{2!}} - \cancel{\frac{x^4}{4!}} + \cancel{\frac{x^4}{4!}} + \cancel{\frac{x^6}{6!}} - \cancel{\frac{x^6}{6!}} - \cancel{\frac{x^8}{8!}} + \cancel{\frac{x^8}{8!}} + \cancel{\frac{x^{10}}{10!}} - - - -$$

$$= 1$$

2



3). Determine $\phi(x)$ using the neumann series method

$$\phi(x) = x\cos(x) + \int_0^x t\,\phi(t)\,dt \qquad \phi(x) = \sum_{n=0}^{\infty} \phi_n(x)$$

$$\phi_0(x) = x\cos(x)$$

$$\phi_1(x) = \int_0^x (t)(t\cos(t))dt = \int_0^x t^2\cos(t)dt = \int_0^x t^2 d(\sin(t))$$

$$= \left(t^2\sin(t)\Big|_0^x - \int_0^x \sin(t)(2t)dt\right)$$

$$= x^2\sin(x) \quad -2\left(\int_0^x t\,d(-\cos(t))\right)$$

$$= x^2\sin(x) - 2\left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)dt\right)$$

$$= x^2\sin(x) - 2\left(-x\cos(x) \quad + \sin(t)\Big|_0^x\right)$$

$$= x^2\sin(x) \; + 2x\cos(x) \; -2\sin(x)$$

$$\phi_2(x) = \int_0^x (t)(t^2\sin(t) + 2t\cos(t) - 2\sin(t))dt$$

$$= \int_0^x (t^3\sin(t) + 2t^2\cos(t) - 2t\sin(t)dt$$

$$= \int_0^x t^3\sin(t)dt \; + 2\int_0^x t^2\cos(t)dt \; - 2\int_0^x t\sin(t)dt$$

$$= \int_0^x t^3 d(-\cos(t)) \; + 2\int_0^x t^2\cos(t)dt \; - 2\int_0^x t\sin(t)dt$$

$$= \left(-3t^2\cos(t)\Big|_0^x + \int_0^x 3t^2\cos(t)dt\right) + 2\int_0^x t^2\cos(t)dt - 2\int_0^x t\sin(t)dt$$

$$= -3x^2\cos(x) \; + 5\int_0^x t^2\cos(t)dt \; - 2\int_0^x t\sin(t)dt$$

$$= -3x^2\cos(x) \; + 5\int_0^x t^2\cos(t)dt \; - 2\int_0^x t\,d(-\cos(t))$$

$$= -3x^2\cos(x) \; + 5\int_0^x t^2\cos(t)dt \; - 2\left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)dt\right)$$

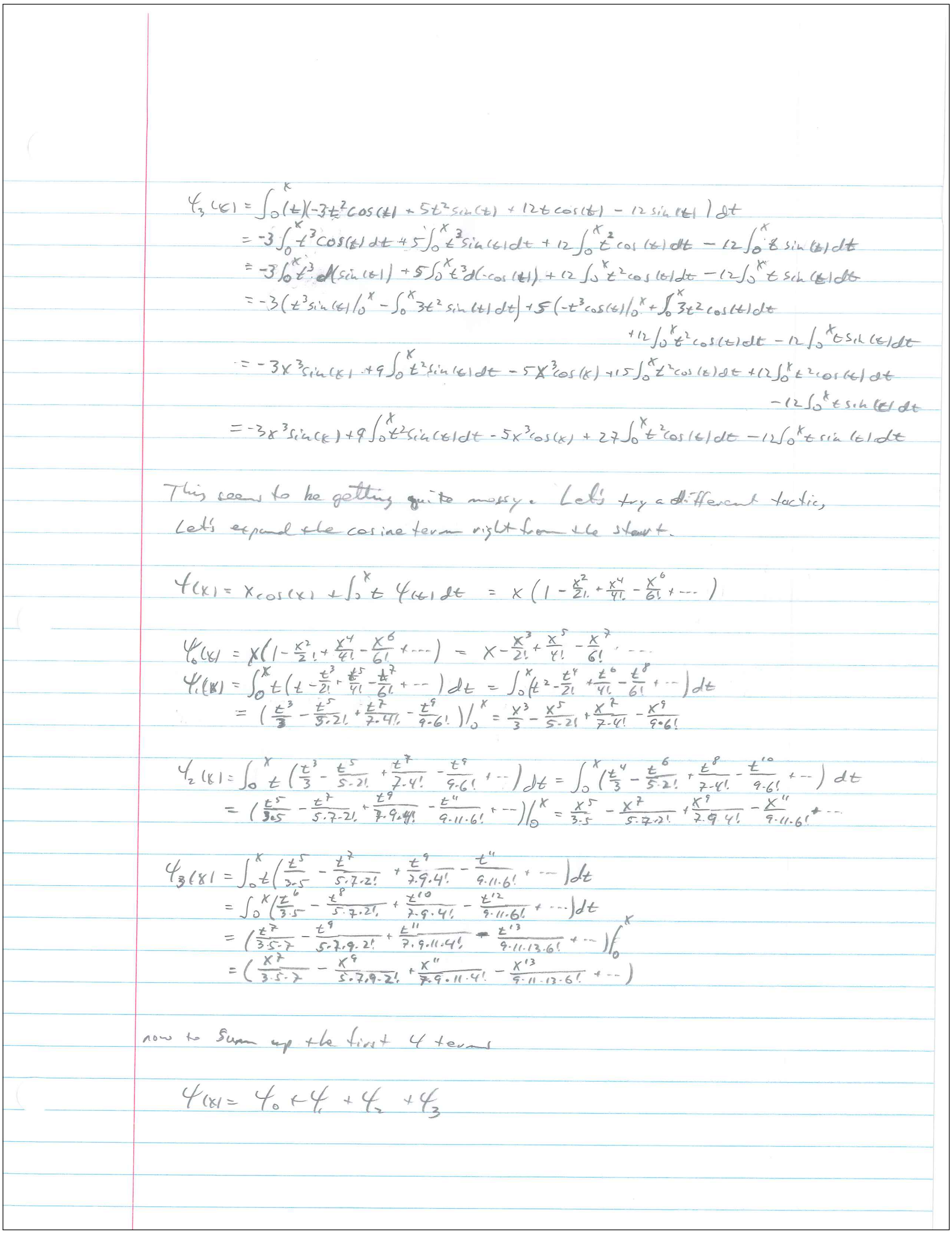
$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt - 2\left(-x\cos(x) \; + \sin(x)\right)$$

$$= -3x^2\cos(x) + 5\underbrace{\int_0^x t^2\cos(t)dt}_{\text{was compute in } \phi_1(x).} + 2x\cos(x) - 2\sin(x)$$

$$= -3x^2\cos(x) + 5\left(x^2\sin(x) + 2x\cos(x) - 2\sin(x)\right) + 2x\cos(x) - 2\sin(x)$$

$$= -3x^2\cos(x) + 5x^2\sin(x) + 10x\cos(x) - 10\sin(x) + 2x\cos(x) - 2\sin(x)$$

$$= -3x^2\cos(x) + 5x^2\sin(x) + 12x\cos(x) - 12\sin(x)$$



$$\phi_3(x) = \int_0^x (t)(-3t^2\cos(t) + 5t^2\sin(t) + 12t\cos(t) - 12\sin(t))\,dt$$
$$= -3\int_0^x t^3\cos(t)\,dt + 5\int_0^x t^3\sin(t)\,dt + 12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$
$$= -3\int_0^x t^3\,d(\sin(t)) + 5\int_0^x t^3\,d(-\cos(t)) + 12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$
$$= -3\left(t^3\sin(t)\Big|_0^x - \int_0^x 3t^2\sin(t)\,dt\right) + 5\left(-t^3\cos(t)\Big|_0^x + \int_0^x 3t^2\cos(t)\,dt\right.$$
$$+ 12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$
$$= -3x^3\sin(x) + 9\int_0^x t^2\sin(t)\,dt - 5x^3\cos(x) + 15\int_0^x t^2\cos(t)\,dt + 12\int_0^x t^2\cos(t)\,dt$$
$$- 12\int_0^x t\sin(t)\,dt$$
$$= -3x^3\sin(x) + 9\int_0^x t^2\sin(t)\,dt - 5x^3\cos(x) + 27\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$

This seems to be getting quite messy. Let's try a different tactic, Let's expand the cosine term right from the start.

$$\phi(x) = x\cos(x) + \int_0^x t\,\phi(t)\,dt \quad = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

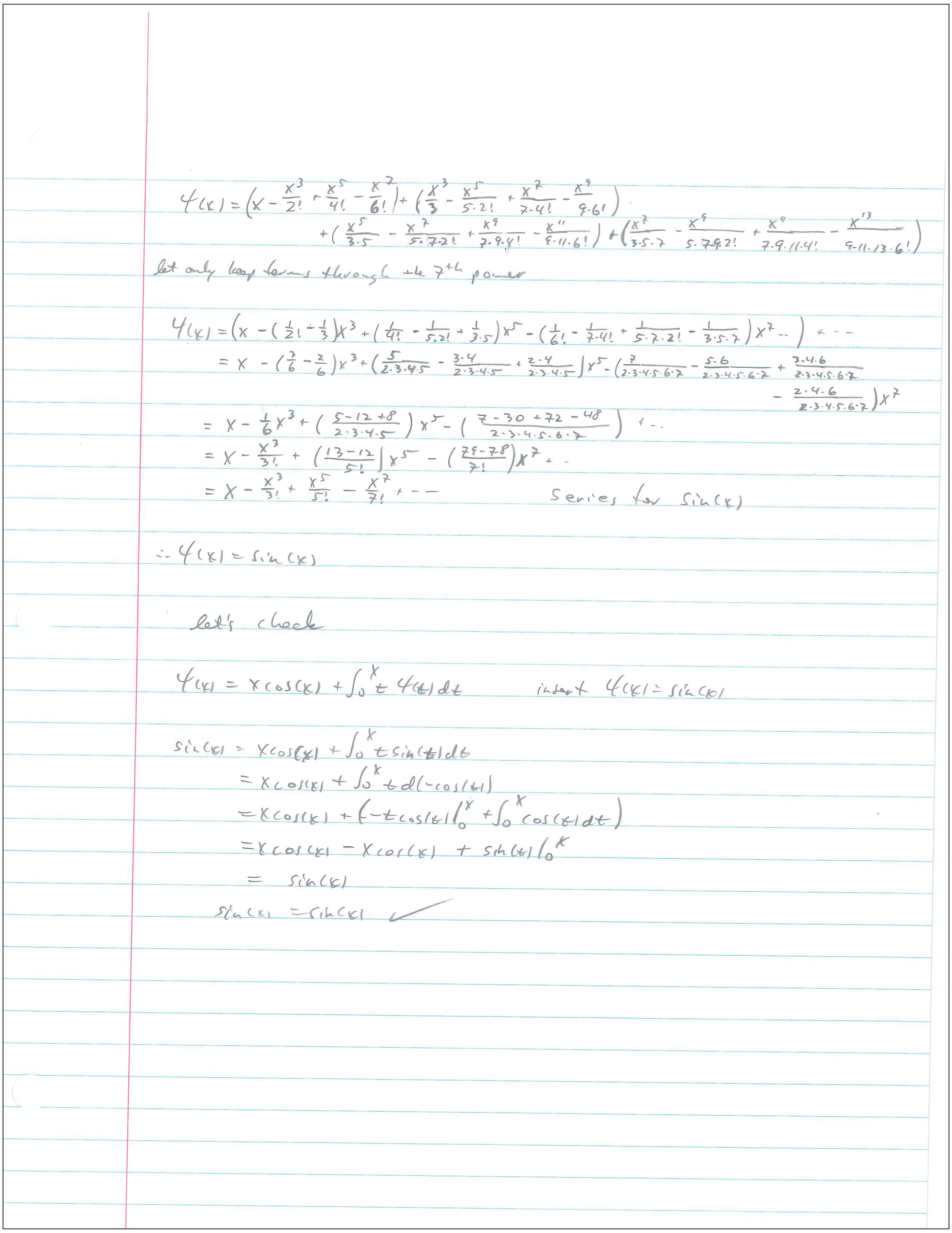
$$\phi_0(x) = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} \cdots$$
$$\phi_1(x) = \int_0^x t\left(t - \frac{t^3}{2!} + \frac{t^5}{4!} - \frac{t^7}{6!} + \cdots\right)dt = \int_0^x \left(t^2 - \frac{t^4}{2!} + \frac{t^6}{4!} - \frac{t^8}{6!} + \cdots\right)dt$$
$$= \left(\frac{t^3}{3} - \frac{t^5}{5\cdot 2!} + \frac{t^7}{7\cdot 4!} - \frac{t^9}{9\cdot 6!}\right)\Big|_0^x = \frac{x^3}{3} - \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 4!} - \frac{x^9}{9\cdot 6!}$$

$$\phi_2(x) = \int_0^x t\left(\frac{t^3}{3} - \frac{t^5}{5\cdot 2!} + \frac{t^7}{7\cdot 4!} - \frac{t^9}{9\cdot 6!} + \cdots\right)dt = \int_0^x \left(\frac{t^4}{3} - \frac{t^6}{5\cdot 2!} + \frac{t^8}{7\cdot 4!} - \frac{t^{10}}{9\cdot 6!} + \cdots\right)dt$$
$$= \left(\frac{t^5}{3\cdot 5} - \frac{t^7}{5\cdot 7\cdot 2!} + \frac{t^9}{7\cdot 9\cdot 4!} - \frac{t^{11}}{9\cdot 11\cdot 6!} + \cdots\right)\Big|_0^x = \frac{x^5}{3\cdot 5} - \frac{x^7}{5\cdot 7\cdot 2!} + \frac{x^9}{7\cdot 9\cdot 4!} - \frac{x^{11}}{9\cdot 11\cdot 6!} + \cdots$$

$$\phi_3(x) = \int_0^x t\left(\frac{t^5}{3\cdot 5} - \frac{t^7}{5\cdot 7\cdot 2!} + \frac{t^9}{7\cdot 9\cdot 4!} - \frac{t^{11}}{9\cdot 11\cdot 6!} + \cdots\right)dt$$
$$= \int_0^x \left(\frac{t^6}{3\cdot 5} - \frac{t^8}{5\cdot 7\cdot 2!} + \frac{t^{10}}{7\cdot 9\cdot 4!} - \frac{t^{12}}{9\cdot 11\cdot 6!} + \cdots\right)dt$$
$$= \left(\frac{t^7}{3\cdot 5\cdot 7} - \frac{t^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{t^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{t^{13}}{9\cdot 11\cdot 13\cdot 6!} + \cdots\right)\Big|_0^x$$
$$= \left(\frac{x^7}{3\cdot 5\cdot 7} - \frac{x^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{x^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{x^{13}}{9\cdot 11\cdot 13\cdot 6!} + \cdots\right)$$

now to sum up the first 4 terms

$$\phi(x) = \phi_0 + \phi_1 + \phi_2 + \phi_3$$



$$\psi(x) = \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}\right) + \left(\frac{x^3}{3} - \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 4!} - \frac{x^9}{9\cdot 6!}\right)$$

$$+ \left(\frac{x^5}{3\cdot 5} - \frac{x^7}{5\cdot 7\cdot 2!} + \frac{x^9}{7\cdot 9\cdot 4!} - \frac{x^{11}}{9\cdot 11\cdot 6!}\right) + \left(\frac{x^7}{3\cdot 5\cdot 7} - \frac{x^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{x^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{x^{13}}{9\cdot 11\cdot 13\cdot 6!}\right)$$

let only keep terms through the 7th power

$$\psi(x) = \left(x - \left(\frac{1}{2!} - \frac{1}{3}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{5\cdot 2!} + \frac{1}{3\cdot 5}\right)x^5 - \left(\frac{1}{6!} - \frac{1}{7\cdot 4!} + \frac{1}{5\cdot 7\cdot 2!} - \frac{1}{3\cdot 5\cdot 7}\right)x^7 \cdots\right) + \cdots$$

$$= x - \left(\frac{3}{6} - \frac{2}{6}\right)x^3 + \left(\frac{5}{2\cdot 3\cdot 4\cdot 5} - \frac{3\cdot 4}{2\cdot 3\cdot 4\cdot 5} + \frac{2\cdot 4}{2\cdot 3\cdot 4\cdot 5}\right)x^5 - \left(\frac{7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} - \frac{5\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{3\cdot 4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} - \frac{2\cdot 4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\right)x^7$$

$$= x - \frac{1}{6}x^3 + \left(\frac{5-12+8}{2\cdot 3\cdot 4\cdot 5}\right)x^5 - \left(\frac{7-30+72-48}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\right) + \cdots$$

$$= x - \frac{x^3}{3!} + \left(\frac{13-12}{5!}\right)x^5 - \left(\frac{79-78}{7!}\right)x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$ Series for $\sin(x)$

$\therefore \psi(x) = \sin(x)$

let's check

$\psi(x) = x\cos(x) + \int_0^x t\,\psi(t)\,dt$ insert $\psi(x) = \sin(x)$

$$\sin(x) = x\cos(x) + \int_0^x t\sin(t)\,dt$$

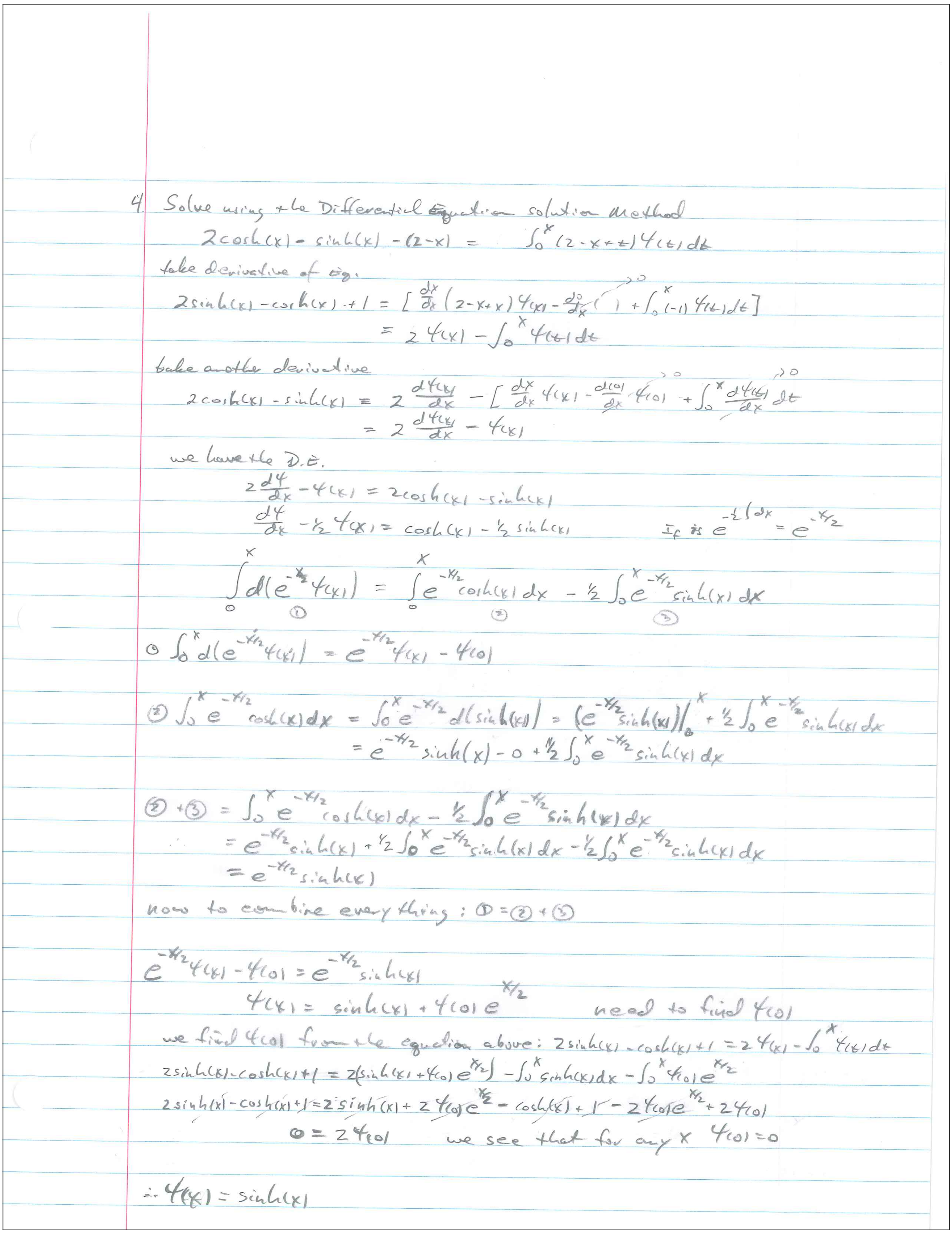
$$= x\cos(x) + \int_0^x t\,d(-\cos(t))$$

$$= x\cos(x) + \left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)\,dt\right)$$

$$= x\cos(x) - x\cos(x) + \sin(t)\Big|_0^x$$

$$= \sin(x)$$

$\sin(x) = \sin(x)$ ✓



4. Solve using the Differential equation solution method

$$2\cosh(x) - \sinh(x) - (2-x) = \int_0^x (2-x+t)\phi(t)\,dt$$

take derivative of eq.

$$2\sinh(x) - \cosh(x) + 1 = \left[\frac{dx}{dx}(2-x+x)\phi(x) - \cancelto{0}{\frac{d0}{dx}(\ )} + \int_0^x (-1)\phi(t)\,dt\right]$$

$$= 2\phi(x) - \int_0^x \phi(t)\,dt$$

take another derivative

$$2\cosh(x) - \sinh(x) = 2\frac{d\phi(x)}{dx} - \left[\frac{dx}{dx}\phi(x) - \cancelto{0}{\frac{d(0)}{dx}\phi(0)} + \cancelto{0}{\int_0^x \frac{d\phi(t)}{dx}\,dt}\right.$$

$$= 2\frac{d\phi(x)}{dx} - \phi(x)$$

we have the D.E.

$$2\frac{d\phi}{dx} - \phi(x) = 2\cosh(x) - \sinh(x)$$

$$\frac{d\phi}{dx} - \tfrac{1}{2}\phi(x) = \cosh(x) - \tfrac{1}{2}\sinh(x) \qquad I_f \text{ is } e^{-\frac{1}{2}\int dx} = e^{-x/2}$$

$$\underset{①}{\int_0^x d(e^{-x/2}\phi(x))} = \underset{②}{\int_0^x e^{-x/2}\cosh(x)\,dx} - \underset{③}{\tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx}$$

$$① \int_0^x d(e^{-x/2}\phi(x)) = e^{-x/2}\phi(x) - \phi(0)$$

$$② \int_0^x e^{-x/2}\cosh(x)\,dx = \int_0^x e^{-x/2}\,d(\sinh(x)) = \left(e^{-x/2}\sinh(x)\right)\Big|_0^x + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

$$= e^{-x/2}\sinh(x) - 0 + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

$$② + ③ = \int_0^x e^{-x/2}\cosh(x)\,dx - \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

$$= e^{-x/2}\sinh(x) + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx - \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

$$= e^{-x/2}\sinh(x)$$

now to combine everything : ① = ② + ③

$$e^{-x/2}\phi(x) - \phi(0) = e^{-x/2}\sinh(x)$$

$$\phi(x) = \sinh(x) + \phi(0)e^{x/2} \qquad \text{need to find } \phi(0)$$

we find $\phi(0)$ from the equation above: $2\sinh(x) - \cosh(x) + 1 = 2\phi(x) - \int_0^x \phi(t)\,dt$

$$2\sinh(x) - \cosh(x) + 1 = 2(\sinh(x) + \phi(0)e^{x/2}) - \int_0^x \sinh(x)\,dx - \int_0^x \phi(0)e^{x/2}$$

$$\cancel{2\sinh(x)} - \cancel{\cosh(x)} + \cancel{1} = \cancel{2\sinh(x)} + \cancel{2\phi(0)e^{x/2}} - \cancel{\cosh(x)} + \cancel{1} - \cancel{2\phi(0)e^{x/2}} + 2\phi(0)$$

$$0 = 2\phi(0) \qquad \text{we see that for any } x \ \phi(0) = 0$$

$$\therefore \phi(x) = \sinh(x)$$

5) Solve using the differential Eq. method

$$u(x) = \cos(x) - x - 2 + \int_0^x (t-x)u(t)\,dt$$

$$\frac{du}{dx} = -\sin(x) - 1 + \left[\frac{dx}{dx}\cancelto{0}{(x-x)u(x)} - \cancelto{0}{\frac{d(0)}{dx}(\ )} + \int_0^x (-1)u(t)\,dt\right]$$

$$= -\sin(x) - 1 - \int_0^x u(t)\,dt$$

$$\frac{d^2u}{dx^2} = -\cos(x) - \left[\frac{dx}{dx}u(x) - \cancelto{0}{\frac{d(0)}{dx}u(0)} + \cancelto{0}{\int_0^x \frac{du(t)}{dx}\,dt}\right]$$

$$\frac{d^2u}{dx^2} = -\cos(x) - u(x)$$

$$\frac{d^2u}{dx^2} + u(x) = -\cos(x)$$ 2nd order Eq. with conditions: $u(0) = -1$

$u'(0) = -1$

$$u(x) = u_h(x) + u_p(x)$$

$$u_h(x) = A\cos(x) + B\sin(x)$$

$$u_p = c\,x\cos(x) + d\,x\sin(x)$$

$$u_p' = c(\cos(x) - x\sin(x)) + d(\sin(x) + x\cos(x))$$

$$u_p'' = c(-\sin(x) - \sin(x) - x\cos(x)) + d(\cos(x) + \cos(x) - x\sin(x))$$

substitute in D.E.

$$c(-2\sin(x) - x\cos(x)) + d(2\cos(x) - x\sin(x)) + c\,x\cos(x) + d\,x\sin(x) = -\cos(x)$$

$$c(-2\sin(x) \cancel{- x\cos(x) + x\cos(x)}) + d(2\cos(x) \cancel{- x\sin(x) + x\sin(x)}) = -\cos(x)$$

$$-2c\sin(x) + 2d\cos(x) = -\cos(x)$$

$$\therefore c = 0 \qquad 2d = -1 \implies d = -\tfrac{1}{2}$$

$$u_p = -\tfrac{1}{2}x\sin(x)$$

$$u(x) = A\cos(x) + B\sin(x) - \tfrac{1}{2}x\sin(x)$$

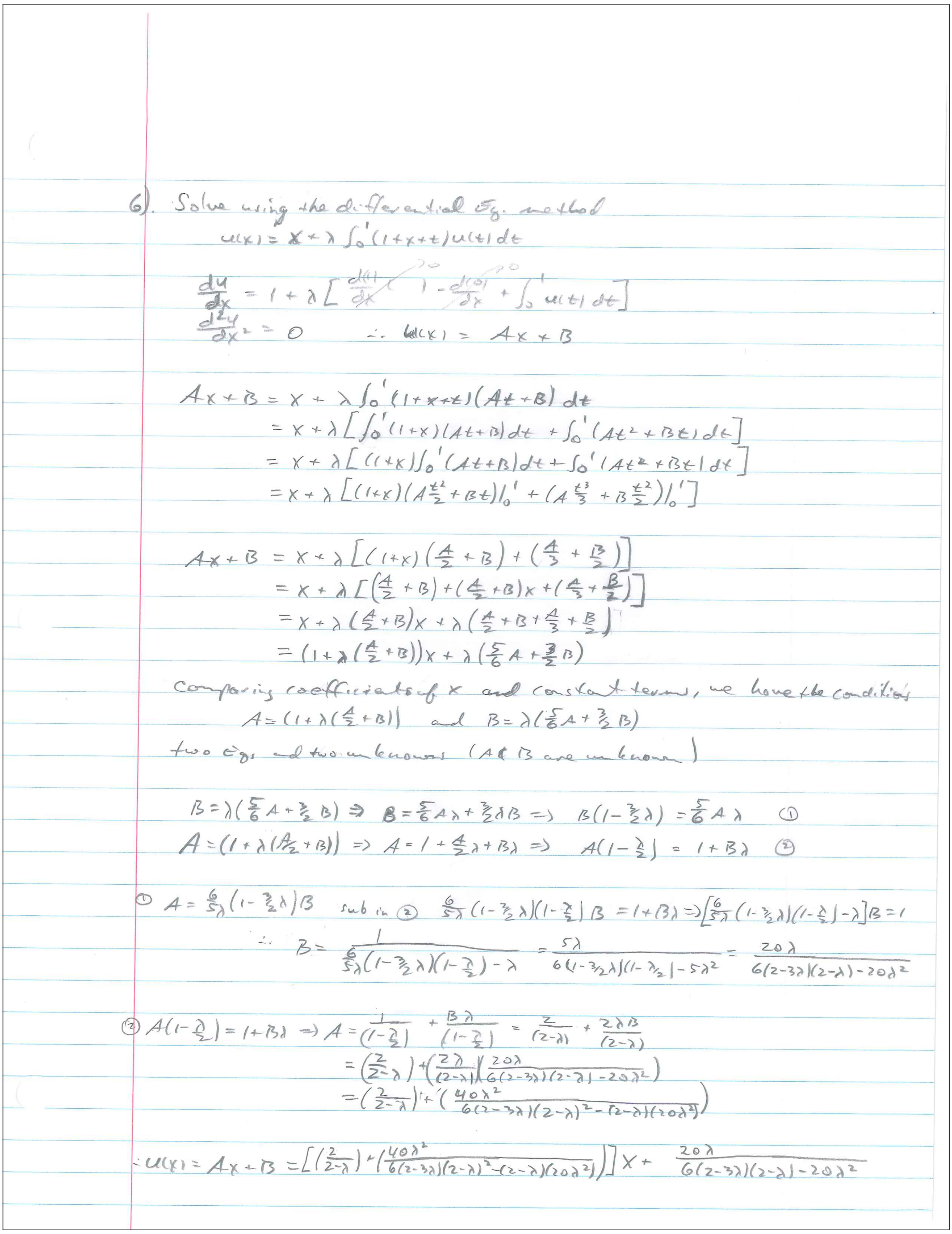
$$u'(x) = -A\sin(x) + B\cos(x) - \tfrac{1}{2}\sin(x) - \tfrac{1}{2}x\cos(x)$$

now to find A & B from the I.C. $u(0) = -1$, $u'(0) = -1$

$$u(0) = A + 0 - 0 = -1 \quad \therefore A = -1$$

$$u'(0) = -0 + B - 0 - 0 = -1 \quad \therefore B = -1$$

$$u(x) = -\cos(x) - \sin(x) - \tfrac{1}{2}x\sin(x)$$



6). Solve using the differential Eq. method

$$u(x) = x + \lambda \int_0^1 (1+x+t)\,u(t)\,dt$$

$$\frac{du}{dx} = 1 + \lambda\left[\cancelto{0}{\frac{d(1)}{dx}}(\quad) - \cancelto{0}{\frac{d(0)}{dx}} + \int_0^1 u(t)\,dt\right]$$

$$\frac{d^2y}{dx^2} = 0 \qquad \therefore\ u(x) = Ax + B$$

$$\begin{aligned}
Ax + B &= x + \lambda \int_0^1 (1+x+t)(At+B)\,dt \\
&= x + \lambda\left[\int_0^1 (1+x)(At+B)\,dt + \int_0^1 (At^2 + Bt)\,dt\right] \\
&= x + \lambda\left[(1+x)\int_0^1 (At+B)\,dt + \int_0^1 (At^2 + Bt)\,dx\right] \\
&= x + \lambda\left[(1+x)\left(A\frac{t^2}{2} + Bt\right)\Big|_0^1 + \left(A\frac{t^3}{3} + B\frac{t^2}{2}\right)\Big|_0^1\right]
\end{aligned}$$

$$\begin{aligned}
Ax + B &= x + \lambda\left[(1+x)\left(\frac{A}{2} + B\right) + \left(\frac{A}{3} + \frac{B}{2}\right)\right] \\
&= x + \lambda\left[\left(\frac{A}{2} + B\right) + \left(\frac{A}{2} + B\right)x + \left(\frac{A}{3} + \frac{B}{2}\right)\right] \\
&= x + \lambda\left(\frac{A}{2} + B\right)x + \lambda\left(\frac{A}{2} + B + \frac{A}{3} + \frac{B}{2}\right) \\
&= \left(1 + \lambda\left(\frac{A}{2} + B\right)\right)x + \lambda\left(\frac{5}{6}A + \frac{3}{2}B\right)
\end{aligned}$$

comparing coefficients of $x$ and constant terms, we have the conditions

$$A = \left(1 + \lambda\left(\frac{A}{2} + B\right)\right) \quad \text{and} \quad B = \lambda\left(\frac{5}{6}A + \frac{3}{2}B\right)$$

two Eqs and two unknowns ($A$ & $B$ are unknown)

$$B = \lambda\left(\frac{5}{6}A + \frac{3}{2}B\right) \Rightarrow B = \frac{5}{6}A\lambda + \frac{3}{2}\lambda B \Rightarrow B\left(1 - \frac{3}{2}\lambda\right) = \frac{5}{6}A\lambda \quad ①$$

$$A = \left(1 + \lambda\left(\frac{A}{2} + B\right)\right) \Rightarrow A = 1 + \frac{A}{2}\lambda + B\lambda \Rightarrow A\left(1 - \frac{\lambda}{2}\right) = 1 + B\lambda \quad ②$$

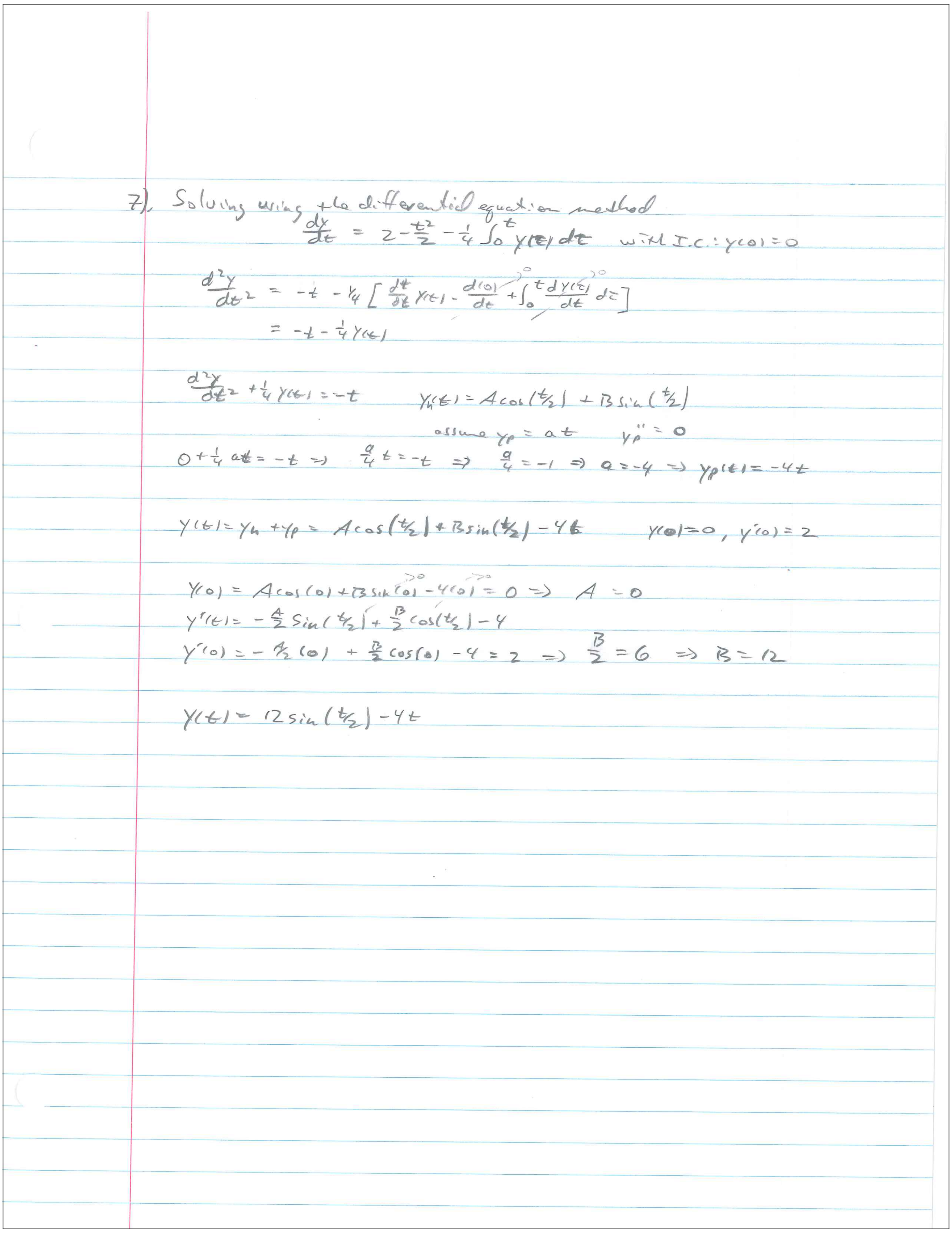
① $A = \frac{6}{5\lambda}\left(1 - \frac{3}{2}\lambda\right)B$ sub in ② $\frac{6}{5\lambda}\left(1 - \frac{3}{2}\lambda\right)\left(1 - \frac{\lambda}{2}\right)B = 1 + B\lambda \Rightarrow \left[\frac{6}{5\lambda}\left(1 - \frac{3}{2}\lambda\right)\left(1 - \frac{\lambda}{2}\right) - \lambda\right]B = 1$

$$\therefore\ B = \frac{1}{\frac{6}{5\lambda}\left(1 - \frac{3}{2}\lambda\right)\left(1 - \frac{\lambda}{2}\right) - \lambda} = \frac{5\lambda}{6\left(1 - 3/2\,\lambda\right)\left(1 - \lambda/2\right) - 5\lambda^2} = \frac{20\lambda}{6(2-3\lambda)(2-\lambda) - 20\lambda^2}$$

② $A\left(1 - \frac{\lambda}{2}\right) = 1 + B\lambda \Rightarrow A = \frac{1}{\left(1 - \frac{\lambda}{2}\right)} + \frac{B\lambda}{\left(1 - \frac{\lambda}{2}\right)} = \frac{2}{(2-\lambda)} + \frac{2\lambda B}{(2-\lambda)}$

$$\begin{aligned}
&= \left(\frac{2}{2-\lambda}\right) + \left(\frac{2\lambda}{(2-\lambda)}\right)\left(\frac{20\lambda}{6(2-3\lambda)(2-\lambda) - 20\lambda^2}\right) \\
&= \left(\frac{2}{2-\lambda}\right) + \left(\frac{40\lambda^2}{6(2-3\lambda)(2-\lambda)^2 - (2-\lambda)(20\lambda^2)}\right)
\end{aligned}$$

$$\therefore u(x) = Ax + B = \left[\left(\frac{2}{2-\lambda}\right) + \left(\frac{40\lambda^2}{6(2-3\lambda)(2-\lambda)^2 - (2-\lambda)(20\lambda^2)}\right)\right]x + \frac{20\lambda}{6(2-3\lambda)(2-\lambda) - 20\lambda^2}$$



7). Solving using the differential equation method

$$\frac{dx}{dt} = 2-\frac{t^2}{2}-\frac{1}{4}\int_0^t y(\tau)\,d\tau \quad \text{with I.C.: } y(0)=0$$

$$\frac{d^2y}{dt^2} = -t-\frac{1}{4}\left[\frac{dt}{dt}y(t)-\cancelto{0}{\frac{d(0)}{dt}}+\cancelto{0}{\int_0^t \frac{dy(\tau)}{dt}\,d\tau}\right]$$

$$= -t-\frac{1}{4}y(t)$$

$$\frac{d^2y}{dt^2}+\frac{1}{4}y(t) = -t \qquad y_h(t) = A\cos(t/2)+B\sin(t/2)$$

assume $y_p = at$ $\quad y_p'' = 0$

$$0+\frac{1}{4}at = -t \Rightarrow \frac{a}{4}t = -t \Rightarrow \frac{a}{4} = -1 \Rightarrow a = -4 \Rightarrow y_p(t) = -4t$$

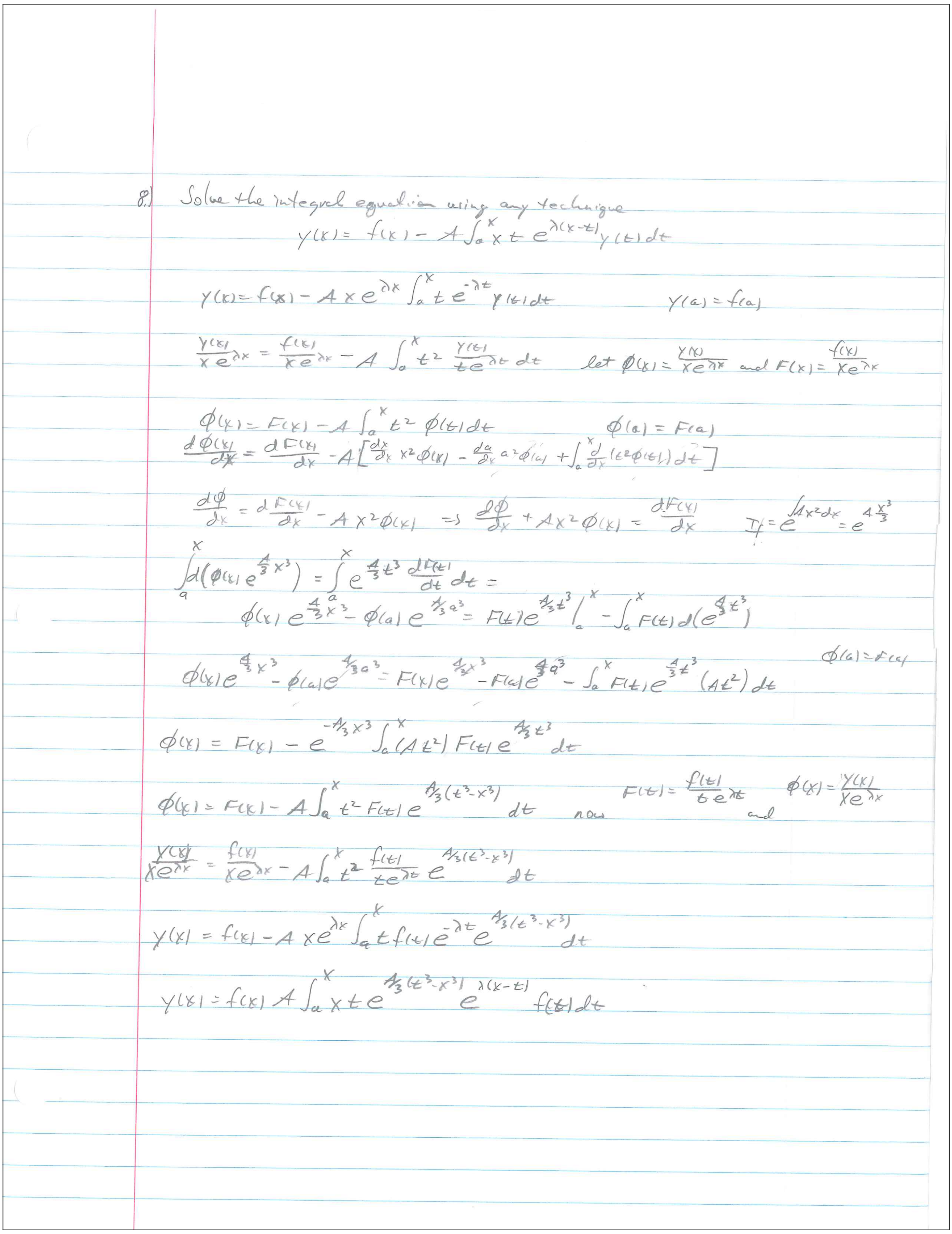
$$y(t) = y_h+y_p = A\cos(t/2)+B\sin(t/2)-4t \qquad y(0)=0,\ y'(0)=2$$

$$y(0) = A\cos(0)+B\cancelto{0}{\sin(0)}-\cancelto{0}{4(0)} = 0 \Rightarrow A = 0$$

$$y'(t) = -\frac{A}{2}\sin(t/2)+\frac{B}{2}\cos(t/2)-4$$

$$y'(0) = -A/2\,(0)+\frac{B}{2}\cos(0)-4 = 2 \Rightarrow \frac{B}{2} = 6 \Rightarrow B = 12$$

$$y(t) = 12\sin(t/2)-4t$$



8.) Solve the integral equation using any technique

$$y(x) = f(x) - A\int_a^x x\,t\,e^{\lambda(x-t)}y(t)\,dt$$

$$y(x) = f(x) - A\,x\,e^{\lambda x}\int_a^x t\,e^{-\lambda t}y(t)\,dt \qquad y(a) = f(a)$$

$$\frac{y(x)}{xe^{\lambda x}} = \frac{f(x)}{xe^{\lambda x}} - A\int_0^x t^2\,\frac{y(t)}{te^{\lambda t}}\,dt \qquad \text{let } \phi(x) = \frac{y(x)}{xe^{\lambda x}} \text{ and } F(x) = \frac{f(x)}{xe^{\lambda x}}$$

$$\phi(x) = F(x) - A\int_a^x t^2\,\phi(t)\,dt \qquad \phi(a) = F(a)$$

$$\frac{d\phi(x)}{dx} = \frac{dF(x)}{dx} - A\left[\frac{dx}{dx}x^2\phi(x) - \frac{da}{dx}a^2\phi(a) + \int_a^x\frac{d}{dx}(t^2\phi(t))\,dt\right]$$

$$\frac{d\phi}{dx} = \frac{dF(x)}{dx} - A\,x^2\phi(x) \Rightarrow \frac{d\phi}{dx} + Ax^2\phi(x) = \frac{dF(x)}{dx} \qquad I_f = e^{\int Ax^2dx} = e^{A\frac{x^3}{3}}$$

$$\int_a^x d\left(\phi(x)e^{\frac{A}{3}x^3}\right) = \int_a^x e^{\frac{A}{3}t^3}\frac{dF(t)}{dt}\,dt =$$

$$\phi(x)e^{\frac{A}{3}x^3} - \phi(a)e^{A/3\,a^3} = F(t)e^{A/3\,t^3}\Big|_a^x - \int_a^x F(t)\,d\left(e^{\frac{A}{3}t^3}\right)$$

$$\phi(x)e^{\frac{A}{3}x^3} - \phi(a)e^{A/3\,a^3} = F(x)e^{A/3\,x^3} - F(a)e^{\frac{A}{3}a^3} - \int_a^x F(t)e^{\frac{A}{3}t^3}(At^2)\,dt \qquad \phi(a) = F(a)$$

$$\phi(x) = F(x) - e^{-A/3\,x^3}\int_a^x (At^2)\,F(t)\,e^{A/3\,t^3}\,dt$$

$$\phi(x) = F(x) - A\int_a^x t^2\,F(t)\,e^{A/3(t^3-x^3)}\,dt \qquad \text{now} \quad F(t) = \frac{f(t)}{te^{\lambda t}} \quad \text{and} \quad \phi(x) = \frac{y(x)}{xe^{\lambda x}}$$

$$\frac{y(x)}{xe^{\lambda x}} = \frac{f(x)}{xe^{\lambda x}} - A\int_a^x t^2\,\frac{f(t)}{te^{\lambda t}}\,e^{A/3(t^3-x^3)}\,dt$$

$$y(x) = f(x) - A\,xe^{\lambda x}\int_a^x t\,f(t)\,e^{-\lambda t}\,e^{A/3(t^3-x^3)}\,dt$$

$$y(x) = f(x)\,A\int_a^x x\,t\,e^{A/3(t^3-x^3)}\,e^{\lambda(x-t)}\,f(t)\,dt$$

## 2.8 HW 7

### 2.8.1 Problems to solve

Homework Set No. 7 NEEP 547
Due November 1, 2013 DLH

Solve the system with use of the Fundamental Matrix

1. (6pts) Solve the following system of equations with the initial conditions, $x(1) = 3$ and $y(1) = 1$:

$$\begin{aligned} x' &= 3x + y - 2\ \sin(t) \\ y' &= 4x + 3y + 6\ \cos(t). \end{aligned}$$

Solve the system with use of the variation of parameters

2. (6pts) Find the complete solution of the system with the initial conditions, $x(0) = -1$, $y(0) = 2$ and $z(0) = 8$.

$$\begin{aligned} x' &= 3x - z \\ y' &= -2x + 2y + z \\ z' &= 8x - 3z. \end{aligned}$$

Diagonalization

3. (6pts) page 339, prob. 6

Solve system with Diagonalization

4. (6pts) Find the general solution of the system:

$$\begin{aligned} x' &= -x + 3y \\ y' &= 3x - y \\ z' &= -2x - 2y + 6z \end{aligned}$$

5. (6pts) Find the general solution of the following system:

$$\begin{aligned} 2x' + x + y' + 2y &= e^t \\ 3x' - 7x + 3y' + y &= 0. \end{aligned}$$

Matrix Exponential

6. (6pts) Using the relation $\int\ e^{\mathbf{A}t}\ dt = e^{\mathbf{A}t} \times \mathbf{A}^{-1}$, determine the general solution of the following matrix equation;

$$\frac{d\overline{N}}{dt} = \mathbf{A}\,\overline{N}(t) + \overline{F}(t)$$

where $\overline{F}$(t) $= \overline{B}t^2$ and $\overline{B}$ is a constant vector.

7. (6pts) Solve problem 5 using the Matrix Exponential method outlined in class.

### 2.8.2 problem 1 (fundamental matrix, variation of parameters)

Solve the following system of equations using initial conditions $x(0) = 3; y(0) = 1$

$$\begin{aligned} x' &= 3x + y - 2\sin(t) \\ y' &= 4x + 3y + 6\cos(t) \end{aligned}$$

**Solution:**

Writing the above in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2\sin(t) \\ 6\cos(t) \end{pmatrix}$$

In vector/matrix notations it becomes

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}$$

The eigenvalues of $A$ are $\{5,1\}$ and the matrix whose columns are the corresponding eigenvectors (in same order as the eigenvalues) is

$$P=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 1\end{pmatrix}$$

The fundamental matrix $\Omega$ is given by

$$\Omega=\begin{pmatrix}\frac{1}{2}e^{5t} & -\frac{1}{2}e^{t}\\ e^{5t} & e^{t}\end{pmatrix}$$

Since $|P|=1\neq 0$, hence these solutions are linearly independent eigenvectors, (This check was not needed in this case, since the eigenvalues are distinct).

$$\begin{aligned}\boldsymbol{x}_h &= \Omega\boldsymbol{c}\\ &=\begin{pmatrix}\frac{1}{2}e^{5t} & -\frac{1}{2}e^{t}\\ e^{5t} & e^{t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\end{aligned}$$

Now the particular solution $\boldsymbol{x}_p$ is found using the variation of parameters method for systems of equations.

$$\boldsymbol{x}=\boldsymbol{x}_h+\boldsymbol{x}_p \tag{4}$$

To find the particular solution, assume

$$\boldsymbol{x}_p=\Omega\boldsymbol{u}$$

and $\boldsymbol{u}$ is found

$$\boldsymbol{x}_p'=\Omega'\boldsymbol{u}+\Omega\boldsymbol{u}'$$

Substituting the above in the ode $\boldsymbol{x}'=A\boldsymbol{x}+\boldsymbol{f}$ gives

$$\Omega'\boldsymbol{u}+\Omega\boldsymbol{u}'=(A\Omega)\,\boldsymbol{u}+\boldsymbol{f}$$

But $\Omega'=A\Omega$ (since $\Omega$ is a fundamental solution) then the above becomes

$$\begin{aligned}\Omega'\boldsymbol{u}+\Omega\boldsymbol{u}' &= \Omega'\boldsymbol{u}+\boldsymbol{f}\\ \Omega\boldsymbol{u}' &= \boldsymbol{f}\\ \boldsymbol{u}' &= \Omega^{-1}\boldsymbol{f}\\ \boldsymbol{u} &= \int\Omega^{-1}\boldsymbol{f}\,\mathrm{d}t\end{aligned}$$

But $\Omega=\begin{pmatrix}\frac{1}{2}e^{5t} & -\frac{1}{2}e^{t}\\ e^{5t} & e^{t}\end{pmatrix}$, hence $\Omega^{-1}=\begin{pmatrix}\frac{1}{e^{5t}} & \frac{1}{2e^{5t}}\\ -\frac{1}{e^{t}} & \frac{1}{2e^{t}}\end{pmatrix}$, and $\boldsymbol{f}=\begin{pmatrix}-2\sin(t)\\ 6\cos(t)\end{pmatrix}$ hence the above becomes

$$\begin{aligned}\boldsymbol{u} &= \int\begin{pmatrix}\frac{1}{e^{5t}} & \frac{1}{2e^{5t}}\\ -\frac{1}{e^{t}} & \frac{1}{2e^{t}}\end{pmatrix}\begin{pmatrix}-2\sin(t)\\ 6\cos(t)\end{pmatrix}\mathrm{d}t\\ &=\int\begin{pmatrix}3\frac{\cos t}{e^{5t}}-2\frac{\sin t}{e^{5t}}\\ 3\frac{\cos t}{e^{t}}+\frac{2}{e^{t}}\sin t\end{pmatrix}\mathrm{d}t\\ &=\begin{pmatrix}\frac{1}{2}e^{-5t}(\sin t-\cos t)\\ \frac{1}{2}e^{-t}(\sin t-5\cos t)\end{pmatrix}\end{aligned}$$

Then [6], since $\boldsymbol{x}_p = \Omega \boldsymbol{u}$ then

$$\begin{aligned}
\boldsymbol{x}_p &= \begin{pmatrix} \frac{1}{2}e^{5t} & -\frac{1}{2}e^{t} \\ e^{5t} & e^{t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{-5t}\left(\sin t - \cos t\right) \\ \frac{1}{2}e^{-t}\left(\sin t - 5\cos t\right) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{4}e^{t}e^{-t}\left(\sin t - 5\cos t\right) - \frac{1}{4}e^{-5t}e^{5t}\left(\cos t - \sin t\right) \\ \frac{1}{2}e^{t}e^{-t}\left(\sin t - 5\cos t\right) - \frac{1}{2}e^{-5t}e^{5t}\left(\cos t - \sin t\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos t \\ \sin t - 3\cos t \end{pmatrix}
\end{aligned}$$

Substituting the above into Eq. (4) gives

$$\begin{aligned}
\boldsymbol{x} &= \Omega \boldsymbol{c} + \boldsymbol{x}_p \\
&= \begin{pmatrix} \frac{1}{2}e^{5t} & -\frac{1}{2}e^{t} \\ e^{5t} & e^{t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t - 3\cos t \end{pmatrix}
\end{aligned}$$

Applying initial conditions $x(0) = 3; y(0) = 1$ the constants are found by solving the

---

[6] Another way to find particular solution is by gussing. But problem asks to use the variation of parameters method. But this is how the guessing method would work: Let

$$x_p = A\sin t + B\cos t$$

And

$$y_p = C\sin t + D\cos t$$

Then $x_p' = A\cos t - B\sin t$ and $y_p' = C\cos t - D\sin t$ and the ODE system becomes

$$\begin{aligned}
A\cos t - B\sin t &= 3\left(A\sin t + B\cos t\right) + \left(C\sin t + D\cos t\right) - 2\sin(t) \\
C\cos t - D\sin t &= 4\left(A\sin t + B\cos t\right) + 3\left(C\sin t + D\cos t\right) + 6\cos(t)
\end{aligned}$$

comparing coefficients

$$\begin{aligned}
\sin t\left(-B - 3A - C\right) + \cos(t)\left(A - 3B - D\right) &= -2\sin(t) \\
\sin t\left(-D - 4A - 3C\right) + \cos t\left(C - 4B - 3D\right) &= 6\cos(t)
\end{aligned}$$

Hence

$$\begin{aligned}
-B - 3A - C &= -2 \\
A - 3B - D &= 0 \\
-D - 4A - 3C &= 0 \\
C - 4B - 3D &= 6
\end{aligned}$$

4 equations in 4 unknowns

$$\begin{pmatrix} -3 & -1 & -1 & 0 \\ 1 & -3 & 0 & -1 \\ -4 & 0 & -3 & -1 \\ 0 & -4 & 1 & -3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 6 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

Therefore

$$\begin{aligned}
x_p &= A\sin t + B\cos t \\
&= \cos t
\end{aligned}$$

and

$$\begin{aligned}
y_p &= C\sin t + D\cos t \\
&= \sin t - 3\cos t
\end{aligned}$$

following

$$\begin{aligned}\begin{pmatrix}3\\1\end{pmatrix}&=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}1\\-3\end{pmatrix}\\\begin{pmatrix}2\\4\end{pmatrix}&=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}\end{aligned}$$

Hence

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\1&1\end{pmatrix}^{-1}\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}$$

And the final solution is

$$\begin{aligned}\boldsymbol{x}&=\Omega\boldsymbol{c}+\boldsymbol{x}_p\\\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}\frac{1}{2}e^{5t}&-\frac{1}{2}e^{t}\\e^{5t}&e^{t}\end{pmatrix}\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}\cos t\\\sin t-3\cos t\end{pmatrix}\end{aligned}$$

Or

$$\begin{aligned}x&=\cos t+2e^{5t}\\y&=\sin t-3\cos t+4e^{5t}\end{aligned}$$

### 2.8.3 Problem 2 (fundamental matrix, diagonalization)

Find the complete solution of the system with initial conditions $x(0)=-1, y(0)=2, z(0)=8$

$$\begin{aligned}x'&=3x-z\\y'&=-2x+2y+z\\z'&=8x-3z\end{aligned}$$

**solution**:

$$\begin{aligned}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}&=\begin{pmatrix}3&0&-1\\-2&2&1\\8&0&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\\boldsymbol{x}'&=A\boldsymbol{x}\end{aligned}$$

To diagonalize $A$, its eigenvectors matrix $P$ is found. Then the matrix $D$ is

$$P^{-1}AP=D$$

$D$ is a matrix with the eigenvalues of $A$ on its diagonal. $P$ has the eigenvectors as its columns. In this problem it is found that

$$P=\begin{pmatrix}\frac{1}{4}&\frac{1}{2}&0\\-\frac{1}{6}&0&1\\1&1&0\end{pmatrix}$$

And

$$D=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}$$

Now to decouple $A$, let

$$\boldsymbol{x}=P\boldsymbol{z}$$

Hence the problem becomes

$$\begin{aligned}\boldsymbol{x}'&=P\boldsymbol{z}'\\&=A\boldsymbol{x}\end{aligned}$$

Therefore

$$\begin{aligned} Pz' &= Ax \\ &= APz \end{aligned}$$

or

$$\begin{aligned} z' &= P^{-1}APz \\ &= Dz \end{aligned}$$

The new system is which is diagonalizable (decoupled) hence easy to solve. Solving for $z$ gives

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Hence

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ \boldsymbol{z} &= \Omega_z \boldsymbol{c} \end{aligned}$$

Where $c_i$ are the constants of integration and $\Omega_z$ is the fundamental matrix in the $z$ space. Now, since $x = Pz$ then the solution is converted back

$$\begin{aligned} \boldsymbol{x} &= P\Omega_z \boldsymbol{c} \\ &= \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}e^{-t} & \frac{1}{2}e^{t} & 0 \\ -\frac{1}{6}e^{-t} & 0 & e^{2t} \\ e^{-t} & e^{t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \end{aligned}$$

Applying initial conditions $x(0) = -1, y(0) = 2, z(0) = 8$

$$\begin{pmatrix} -1 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Hence

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 20 \\ -12 \\ \frac{16}{3} \end{pmatrix}$$

Therefore, the final solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} \frac{1}{4}e^{-t} & \frac{1}{2}e^{t} & 0 \\ -\frac{1}{6}e^{-t} & 0 & e^{2t} \\ e^{-t} & e^{t} & 0 \end{pmatrix} \begin{pmatrix} 20 \\ -12 \\ \frac{16}{3} \end{pmatrix} \\ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} 5e^{-t} - 6e^{t} \\ \frac{16}{3}e^{2t} - \frac{10}{3}e^{-t} \\ 20e^{-t} - 12e^{t} \end{pmatrix} \end{aligned}$$

### 2.8.4 problem 3 (Show matrix is diagonalizable)

Produce a matrix that diagonalizes $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$, or show the matrix is not diagonalizable.

**Solution**:

The eigenvalues of $A$ are

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda 3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\sqrt{17}+\frac{3}{2}\\ \frac{3}{2}-\frac{1}{2}\sqrt{17}\\ 0\end{pmatrix}$$

Since the eigenvalues are distinct then the matrix is ***diagonalizable***. The corresponding eigenvectors are

$$P=\begin{pmatrix}0 & 0 & -2\\ \frac{1}{2}\sqrt{17}-\frac{3}{2} & -\frac{1}{2}\sqrt{17}-\frac{3}{2} & -3\\ 1 & 1 & 1\end{pmatrix}$$

Where the first column is the first eigenvector associated with first eigenvalue $\lambda_1$ and the second column is second first eigenvector associated with the second eigenvalue $\lambda_2$. Therefore the diagonalized matrix of $A$ is the matrix $D$ given by

$$D=P^{-1}AP$$

$D$ does not need to be computed from the above, since it is given also by

$$\begin{aligned}D&=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda 3\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{2}\sqrt{17}+\frac{3}{2} & 0 & 0\\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{17} & 0\\ 0 & 0 & 0\end{pmatrix}\\ &=\begin{pmatrix}3.5616 & 0 & 0\\ 0 & -0.5616 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Note that Jordan form of the matrix is its diagonalizable matrix (if it is diagonalizable).

### 2.8.5 problem 4 (solve system of equations with diagonalization)

Find the general solution to

$$\begin{aligned}x'&=-x+3y\\ y'&=3x-y\\ z'&=-2x-2y+6z\end{aligned}$$

**solution**:

$$\begin{aligned}\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}&=\begin{pmatrix}-1 & 3 & 0\\ 3 & -1 & 0\\ -2 & -2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\\ \mathbf{x}'&=A\mathbf{x}\end{aligned}$$

The eigenvalues of $A$ are $\{-4,2,6\}$, hence the matrix is diagonalizable (since all its eigenvalues are distinct). Therefore

$$\begin{aligned}P^{-1}AP&=D\\ &=\begin{pmatrix}-4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 6\end{pmatrix}\end{aligned}$$

The eigenvector matrix $P$ is

$$P=\begin{pmatrix}-1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$$

Where the first column if the eigenvector associated with the first eigenvalues −4 and so on. Now, let

$$\mathbf{x} = P\mathbf{z}$$

Which leads to (as was done in problem 2) in order to decouple the system of equations in the $z$ space

$$\mathbf{z}' = D\mathbf{z}$$

Now the new system is decoupled. Solving for $\mathbf{z}$

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Hence

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\mathbf{z} = \Omega_z \mathbf{c}$$

Where $c$ are the constants of integration and $\Omega_z$ is the fundamental matrix in the $z$ space. Since $\mathbf{x} = P\mathbf{z}$ then

$$\begin{aligned} \mathbf{x} &= P\Omega_z\mathbf{c} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} -e^{-4t} & e^{2t} & 0 \\ e^{-4t} & e^{2t} & 0 \\ 0 & e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} c_2e^{2t} - c_1e^{-4t} \\ c_2e^{2t} + c_1e^{-4t} \\ c_2e^{2t} + c_3e^{6t} \end{pmatrix} \end{aligned}$$

### 2.8.6 Problem 5 (solve system of equations with diagonalization)

Find the general solution for

$$\begin{aligned} 2x' + x + y' + 2y &= e^t \\ 3x' - 7x + 3y' + y &= 0 \end{aligned}$$

**solution**:

First the equations are transformed such that each equation contains only $x'$ or $y'$ on its own. This is to allow the system to be written as $\mathbf{x}' = A\mathbf{x}$. Solving for $x', y'$ gives

$$\begin{aligned} x' &= -\frac{10}{3}x - \frac{5}{3}y + e^t \\ y' &= \frac{17}{3}x + \frac{4}{3}y - e^t \end{aligned}$$

Hence

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

The eigenvalues of $A$ are $\{-1 + 2i, -1 - 2i\}$. Since they are distinct, the matrix is diagonalizable. The $P$ matrix of eigenvectors is

$$P = \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 + 2i & 0 \\ 0 & -1 - 2i \end{pmatrix}$$

Let

$$\boldsymbol{x} = P\boldsymbol{z}$$

Hence $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}$ becomes decoupled as follows

$$\begin{aligned} \boldsymbol{z}' &= D\boldsymbol{z} + P^{-1}\boldsymbol{f} \\ &= D\boldsymbol{z} + \boldsymbol{G} \end{aligned}$$

Where $\boldsymbol{G} = P^{-1}\boldsymbol{f} = \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2} + \frac{5}{6}i\right) e^t \\ -\left(\frac{1}{2} - \frac{5}{6}i\right) e^t \end{pmatrix}$

Solving for $z$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -1 + 2i & 0 \\ 0 & -1 - 2i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -\left(\frac{1}{2} + \frac{5}{6}i\right) e^t \\ -\left(\frac{1}{2} - \frac{5}{6}i\right) e^t \end{pmatrix}$$

Since the system is decoupled it can be solved as each equation on its own as follows

$$\begin{aligned} z_1' &= (-1 + 2i)\, z_1 - \left(\frac{1}{2} + \frac{5}{6}i\right) e^t \\ z_2' &= (-1 - 2i)\, z_2 - \left(\frac{1}{2} - \frac{5}{6}i\right) e^t \end{aligned}$$

The solution is

$$\begin{aligned} z_1' + (1 - 2i)\, z_1 &= \left(-\frac{1}{2} - \frac{5}{6}i\right) e^t \\ z_2' + (1 + 2i)\, z_2 &= \left(-\frac{1}{2} + \frac{5}{6}i\right) e^t \end{aligned}$$

Hence, using integrating factor

$$\begin{aligned} d\left(e^{\int (1-2i)\, \mathrm{d}t} z_1\right) &= e^{\int (1-2i)\, d\tau} \left(-\frac{1}{2} - \frac{5}{6}i\right) e^t \\ d\left(e^{\int (1+2i)\, \mathrm{d}t} z_2\right) &= e^{\int (1+2i)\, d\tau} \left(-\frac{1}{2} + \frac{5}{6}i\right) e^t \end{aligned}$$

Therefore

$$\begin{aligned} d\left(e^{(1-2i)t} z_1\right) &= e^{(1-2i)t} \left(-\frac{1}{2} - \frac{5}{6}i\right) e^t \\ d\left(e^{(1+2i)t} z_2\right) &= e^{(1+2i)t} \left(-\frac{1}{2} + \frac{5}{6}i\right) e^t \end{aligned}$$

Hence

$$\begin{aligned} e^{(1-2i)t} z_1 &= \left(-\frac{1}{2} - \frac{5}{6}i\right) \int e^{(1-2i)t+t}\, \mathrm{d}t + c_1 \\ e^{(1+2i)t} z_2 &= \left(-\frac{1}{2} + \frac{5}{6}i\right) \int e^{(1+2i)t+t}\, \mathrm{d}t + c_2 \end{aligned}$$

or

$$\begin{aligned} e^{(1-2i)t} z_1 &= \left(-\frac{1}{2} - \frac{5}{6}i\right) \left(\frac{1}{4} + \frac{i}{4}\right) e^{-2it+2t} + c_1 \\ e^{(1+2i)t} z_2 &= \left(-\frac{1}{2} + \frac{5}{6}i\right) \left(\frac{1}{4} - \frac{i}{4}\right) e^{2it+2t} + c_2 \end{aligned}$$

or

$$
\begin{aligned}
z_1 &= \left(-\frac{1}{2}-\frac{5}{6}i\right)\left(\frac{1}{4}+\frac{i}{4}\right)e^{-2it+2t-(1-2i)t}+c_1e^{-(1-2i)t}\\
z_2 &= \left(-\frac{1}{2}+\frac{5}{6}i\right)\left(\frac{1}{4}-\frac{i}{4}\right)e^{2it+2t-(1+2i)t}+c_2e^{-(1+2i)t}
\end{aligned}
$$

Hence

$$
\begin{aligned}
z_1 &= \left(\frac{1}{12}-\frac{1}{3}i\right)e^{t}+c_1e^{-(1-2i)t}\\
z_2 &= \left(\frac{1}{12}+\frac{1}{3}i\right)e^{t}+c_2e^{-(1+2i)t}
\end{aligned}
$$

But $x = Pz$ therefore

$$
\begin{aligned}
\boldsymbol{x} &= \begin{pmatrix}-\frac{7}{17}+\frac{6}{17}i & -\frac{7}{17}-\frac{6}{17}i\\ 1 & 1\end{pmatrix}\begin{pmatrix}\left(\frac{1}{12}-\frac{1}{3}i\right)e^{t}+c_1e^{-(1-2i)t}\\ \left(\frac{1}{12}+\frac{1}{3}i\right)e^{t}+c_2e^{-(1+2i)t}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{6}e^{t}-\left(\frac{7}{17}-\frac{6}{17}i\right)c_1e^{-(1-2i)t}-\left(\frac{7}{17}+\frac{6}{17}i\right)c_2e^{-(1+2i)t}\\ \frac{1}{6}e^{t}+c_1e^{-(1-2i)t}+c_2e^{-(1+2i)t}\end{pmatrix}
\end{aligned}
$$

The final solution is

$$
\begin{aligned}
x(t) &= \frac{1}{6}e^{t}-\left(\frac{7}{17}-\frac{6}{17}i\right)c_1e^{-(1-2i)t}-\left(\frac{7}{17}+\frac{6}{17}i\right)c_2e^{-(1+2i)t}\\
y(t) &= \frac{1}{6}e^{t}+c_1e^{-(1-2i)t}+c_2e^{-(1+2i)t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= -\left(\frac{7}{17}c_1e^{-(1-2i)t}-\frac{6}{17}ic_1e^{-(1-2i)t}\right)-\left(\frac{7}{17}c_2e^{-(1+2i)t}+\frac{6}{17}ic_2e^{-(1+2i)t}\right)+\frac{1}{6}e^{t}\\
y(t) &= c_1e^{-t}e^{2it}+c_2e^{-t}e^{-2it}+\frac{1}{6}e^{t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= \frac{1}{6}e^{t}-e^{-t}\left(\frac{7}{17}c_1e^{2it}-\frac{6}{17}ic_1e^{2it}\right)-e^{-t}\left(\frac{7}{17}c_2e^{-2it}+\frac{6}{17}ic_2e^{-2it}\right)\\
y(t) &= e^{-t}\left(c_1\left(\cos 2t+i\sin 2t\right)+c_2\left(\cos 2t-i\sin 2t\right)\right)+\frac{1}{6}e^{t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= -e^{-t}\left(\frac{7}{17}c_1\left(\cos 2t+i\sin 2t\right)-\frac{6}{17}ic_1\left(\cos 2t+i\sin 2t\right)\right)\\
&\quad -e^{-t}\left(\frac{7}{17}c_2\left(\cos 2t-i\sin 2t\right)+\frac{6}{17}ic_2\left(\cos 2t-i\sin 2t\right)\right)+\frac{1}{6}e^{t}\\
y(t) &= e^{-t}\left(c_1\cos 2t+c_1i\sin 2t+c_2\cos 2t-ic_2\sin 2t\right)+\frac{1}{6}e^{t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= -e^{-t}\left(\frac{7}{17}c_1\cos 2t+\frac{7}{17}ic_1\sin 2t-ic_1\frac{6}{17}\cos 2t+c_1\frac{6}{17}\sin 2t\right)\\
&\quad -e^{-t}\left(\frac{7}{17}c_2\cos 2t-i\frac{7}{17}c_2\sin 2t+\frac{6}{17}ic_2\cos 2t+\frac{6}{17}c_2\sin 2t\right)+\frac{1}{6}e^{t}\\
y(t) &= e^{-t}\left(\left(c_1+c_2\right)\cos 2t+i\left(c_1-c_2\right)\sin 2t\right)+\frac{1}{6}e^{t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= \frac{1}{6}e^{t}-e^{-t}\begin{pmatrix}\frac{7}{17}c_1\cos 2t+\frac{7}{17}ic_1\sin 2t-ic_1\frac{6}{17}\cos 2t+c_1\frac{6}{17}\sin 2t\\ -\frac{7}{17}c_2\cos 2t+i\frac{7}{17}c_2\sin 2t-\frac{6}{17}ic_2\cos 2t-\frac{6}{17}c_2\sin 2t\end{pmatrix}\\
y(t) &= e^{-t}\left(\left(c_1+c_2\right)\cos 2t+i\left(c_1-c_2\right)\sin 2t\right)+\frac{1}{6}e^{t}
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= \frac{1}{6}e^{t}-e^{-t}\left(\frac{7}{17}\left(c_1-c_2\right)\cos 2t+i\frac{7}{17}\left(c_1+c_2\right)\sin 2t-i\frac{6}{17}\left(c_1+c_2\right)\cos 2t+\frac{6}{17}\left(c_1-c_2\right)\sin 2t\right)\\
y(t) &= e^{-t}\left(\left(c_1+c_2\right)\cos 2t-i\left(c_1-c_2\right)\sin 2t\right)+\frac{1}{6}e^{t}
\end{aligned}
$$

Let $(c_1 + c_2)\, i = A$ and let $(c_1 - c_2) = B$ then above becomes

$$x(t) = \frac{1}{6}e^t - e^{-t}\left(\frac{7}{17}B\cos 2t - \frac{7}{17}A\sin 2t + \frac{6}{17}A\cos 2t + \frac{6}{17}B\sin 2t\right)$$

$$y(t) = \frac{1}{6}e^t - ie^{-t}\left(B\sin 2t + A\cos 2t\right)$$

I am not sure how to move the remaining complex number into the constants over both solutions. According to CAS, the answer should be real

$$y(t) = \frac{1}{6}e^t - \frac{1}{5}e^{-t}\left(7B\cos 2t + 7A\sin 2t + 6A\cos 2t - 6B\sin 2t\right)$$

$$x(t) = \frac{1}{6}e^t + e^{-t}\left(B\sin 2t + A\cos 2t\right)$$

May be I am close, but do not see it now. So will stop here.

### 2.8.7 problem 6 Matrix exponential

Using $\int e^{At}dt = e^{At}A^{-1}$ determine general solution of the following matrix equation $\frac{dN(t)}{dt} = AN(t) + \boldsymbol{F}(t)$ where $\boldsymbol{F}(t) = \boldsymbol{B}t^2$ and $\boldsymbol{B}$ is constant vector.

**solution**:

$$\begin{aligned}
\frac{d\boldsymbol{N}(t)}{dt} &= A\boldsymbol{N}(t) + \boldsymbol{F}(t)\\
\frac{d\boldsymbol{N}(t)}{dt} - A\boldsymbol{N}(t) &= \boldsymbol{F}(t)\\
\int_0^t d\left(e^{-A\tau}\boldsymbol{N}(\tau)\right) &= e^{-At}\boldsymbol{F}(t)\\
e^{-At}\boldsymbol{N}(t) - \boldsymbol{N}(0) &= \int e^{-At}\boldsymbol{F}(t)\, \mathrm{d}\tau\\
e^{-At}\boldsymbol{N}(t) &= \boldsymbol{N}(0) + \int e^{-At}\boldsymbol{F}(t)\, \mathrm{d}\tau\\
\boldsymbol{N}(t) &= e^{At}\boldsymbol{N}(0) + e^{At}\int e^{-At}\boldsymbol{F}(t)\, \mathrm{d}\tau\\
\boldsymbol{N}(t) &= e^{At}\boldsymbol{N}(0) + e^{At}\int e^{-At}\boldsymbol{B}t^2\, \mathrm{d}\tau
\end{aligned}$$

But $e^{At} = Pe^{Dt}P^{-1}$,where $P$ be the matrix of eigenvectors of $A$ and $D$ be the matrix of eigenvalues of $A$ on its diagonal. then

$$\begin{aligned}
\boldsymbol{N}(t) &= Pe^{Dt}P^{-1}\boldsymbol{N}(0) + Pe^{Dt}P^{-1}\int Pe^{-Dt}P^{-1}\boldsymbol{B}t^2\, \mathrm{d}\tau\\
&= Pe^{Dt}P^{-1}\boldsymbol{N}(0) + Pe^{Dt}\int e^{-Dt}P^{-1}\boldsymbol{B}t^2\, \mathrm{d}\tau
\end{aligned}$$

The solution due to the forcing function is contained in $Pe^{Dt}\int e^{-Dt}P^{-1}\boldsymbol{B}t^2\, \mathrm{d}\tau$

### 2.8.8 Problem 7 (Matrix exponential)

Solve problem 5 above using the matrix exponential method.

**solution**:

$$\begin{aligned}
2x' + x + y' + 2y &= e^t\\
3x' - 7x + 3y' + y &= 0
\end{aligned}$$

**solution**:

First we solve the above such that each equation contains only $x'$ or $y'$ on its own. This is to allow us to write the system as $x' = Ax$. By solving for $x', y'$ we find

$$\begin{aligned} x' &= -\frac{10}{3}x - \frac{5}{3}y + e^t \\ y' &= \frac{17}{3}x + \frac{4}{3}y - e^t \end{aligned}$$

Hence

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} -\frac{10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix} \\ \boldsymbol{x}' &= A\boldsymbol{x} + \boldsymbol{f} \end{aligned}$$

The general solution is

$$\boldsymbol{x} = Pe^{Dt}P^{-1}\boldsymbol{x}(0) + Pe^{Dt}\int_0^t e^{-D\tau}P^{-1}\boldsymbol{F}\,\mathrm{d}t \tag{1}$$

Where the eigenvalues of $A$ are $\{-1+2i, -1-2i\}$, hence, as was found in problem 5

$$\begin{aligned} P &= \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix} \\ D &= \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix} \\ \Phi &= \begin{pmatrix} \left(-\frac{7}{17} + \frac{6}{17}i\right)e^{(-1+2i)t} & \left(-\frac{7}{17} - \frac{6}{17}i\right)e^{(-1-2i)t} \\ e^{(-1+2i)t} & e^{(-1-2i)t} \end{pmatrix} \end{aligned}$$

Now

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{(-1-2i)t} \end{pmatrix}$$

and

$$e^{-D\tau} = \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} = \begin{pmatrix} e^{(1-2i)t} & 0 \\ 0 & e^{(1+2i)t} \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{17}{12}i & \frac{1}{2} - \frac{7}{12}i \\ \frac{17}{12}i & \frac{1}{2} + \frac{7}{12}i \end{pmatrix}$$

Hence Eq. (1) becomes

$$\begin{aligned} \boldsymbol{x} &= Pe^{Dt}P^{-1}\boldsymbol{x}(0) + Pe^{Dt}\int_0^t e^{-D\tau}P^{-1}\boldsymbol{F}\,d\tau \\ &= \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{(-1-2i)t} \end{pmatrix} \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix}^{-1} \boldsymbol{x}(0) \\ &+ \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{(-1-2i)t} \end{pmatrix} \\ &\int_0^t \begin{pmatrix} e^{(1-2i)t} & 0 \\ 0 & e^{(1+2i)t} \end{pmatrix} \begin{pmatrix} -\frac{7}{17} + \frac{6}{17}i & -\frac{7}{17} - \frac{6}{17}i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} \mathrm{d}t \end{aligned}$$

Simplifying gives

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} \left(\frac{1}{2} - \frac{7}{12}i\right)e^{-(1+2i)t} + \left(\frac{1}{2} + \frac{7}{12}i\right)e^{-(1-2i)t} & \frac{5}{12}ie^{-(1-2i)t} - \frac{5}{12}ie^{-(1+2i)t} \\ \frac{17}{12}ie^{-(1+2i)t} - \frac{17}{12}ie^{-(1-2i)t} & \left(\frac{1}{2} + \frac{7}{12}i\right)e^{-(1+2i)t} + \left(\frac{1}{2} - \frac{7}{12}i\right)e^{-(1-2i)t} \end{pmatrix} \boldsymbol{x}(0) \\ &+ \begin{pmatrix} -\left(\frac{7}{17} - \frac{6}{17}i\right)e^{-(1-2i)t} & -\left(\frac{7}{17} + \frac{6}{17}i\right)e^{-(1+2i)t} \\ e^{-(1-2i)t} & e^{-(1+2i)t} \end{pmatrix} \int_0^t \begin{pmatrix} -\left(\frac{1}{2} + \frac{5}{6}i\right)e^{(2-2i)t} \\ -\left(\frac{1}{2} - \frac{5}{6}i\right)e^{(2+2i)t} \end{pmatrix} \mathrm{d}t \end{aligned}$$

The integration yields

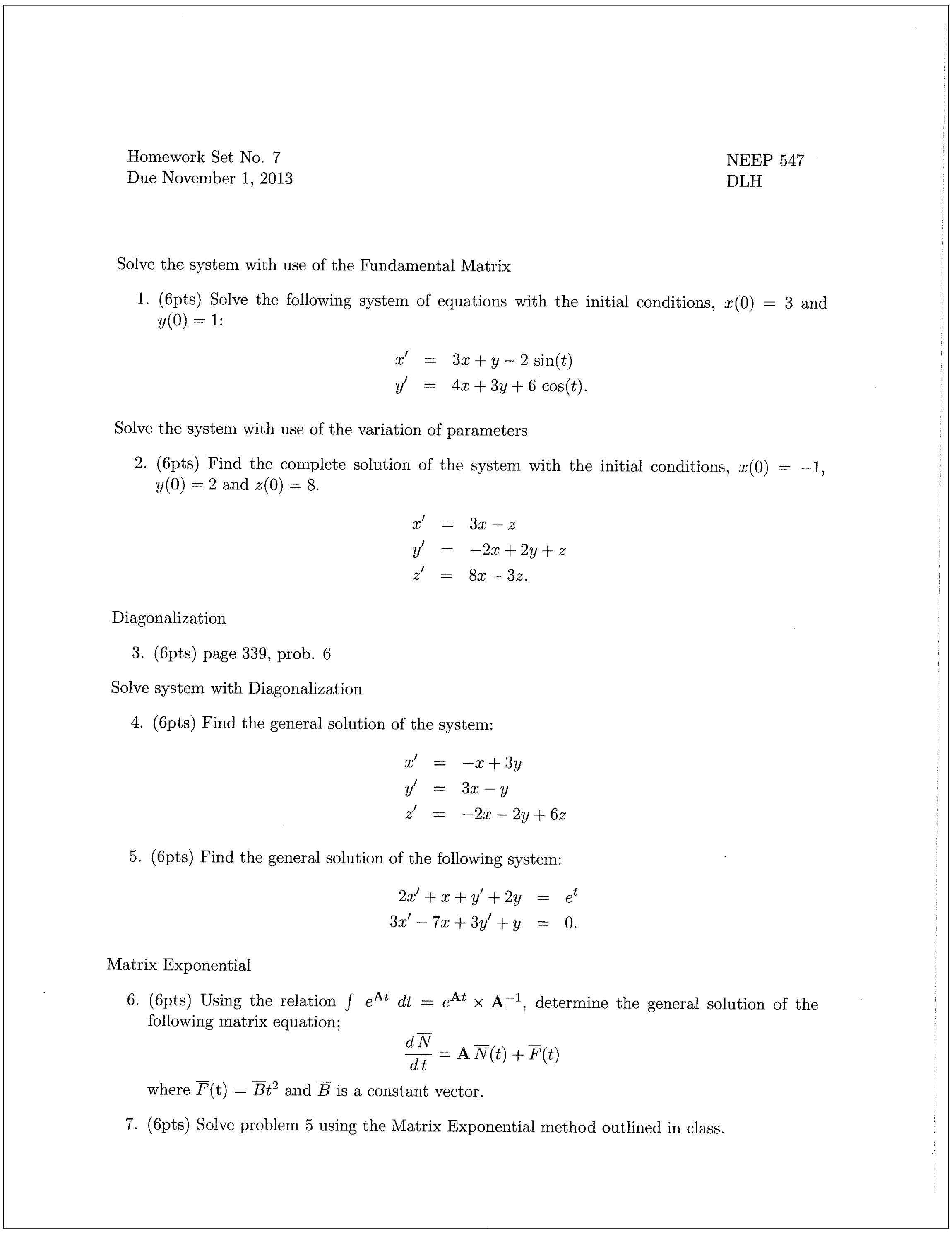
$$\begin{pmatrix} -\frac{1}{12} + \frac{i}{3} + \frac{e^{(2-2i)t}}{12} - \frac{ie^{(2-2i)t}}{3} \\ -\frac{1}{12} + \frac{i}{3} + \frac{e^{(2+2i)t}}{12} + \frac{ie^{(2+2i)t}}{3} \end{pmatrix}$$

Hence the above simplifies to

$$x = \begin{pmatrix} \left(\frac{1}{2} - \frac{7}{12}i\right) e^{-(1+2i)t} + \left(\frac{1}{2} + \frac{7}{12}i\right) e^{-(1-2i)t} & \frac{5}{12} i e^{-(1-2i)t} - \frac{5}{12} i e^{-(1+2i)t} \\ \frac{17}{12} i e^{-(1+2i)t} - \frac{17}{12} i e^{-(1-2i)t} & \left(\frac{1}{2} + \frac{7}{12}i\right) e^{-(1+2i)t} + \left(\frac{1}{2} - \frac{7}{12}i\right) e^{-(1-2i)t} \end{pmatrix} x(0)$$

$$+ \begin{pmatrix} \left(\frac{31}{204} - \frac{11}{102}i\right) e^{-(1+2i)t} - \left(\frac{1}{12} + \frac{1}{6}i\right) e^{-(1-2i)t} + \left(\frac{1}{12} - \frac{1}{6}i\right) e^{-(1+2i)t} e^{(2+2i)t} + \left(\frac{1}{12} + \frac{1}{6}i\right) e^{-(1-2i)t} e^{(2-2i)t} \\ \left(\frac{1}{12} + \frac{1}{3}i\right) e^{-(1+2i)t} e^{(2+2i)t} - \left(\frac{1}{12} - \frac{1}{3}i\right) e^{-(1-2i)t} - \left(\frac{1}{12} - \frac{1}{3}i\right) e^{-(1+2i)t} + \left(\frac{1}{12} - \frac{1}{3}i\right) e^{-(1-2i)t} e^{(2-2i)t} \end{pmatrix}$$

### 2.8.9 key solution



Homework Set No. 7
Due November 1, 2013

NEEP 547
DLH

Solve the system with use of the Fundamental Matrix

1. (6pts) Solve the following system of equations with the initial conditions, $x(0) = 3$ and $y(0) = 1$:

$$\begin{aligned} x' &= 3x + y - 2\sin(t) \\ y' &= 4x + 3y + 6\cos(t). \end{aligned}$$

Solve the system with use of the variation of parameters

2. (6pts) Find the complete solution of the system with the initial conditions, $x(0) = -1$, $y(0) = 2$ and $z(0) = 8$.

$$\begin{aligned} x' &= 3x - z \\ y' &= -2x + 2y + z \\ z' &= 8x - 3z. \end{aligned}$$

Diagonalization

3. (6pts) page 339, prob. 6

Solve system with Diagonalization

4. (6pts) Find the general solution of the system:

$$\begin{aligned} x' &= -x + 3y \\ y' &= 3x - y \\ z' &= -2x - 2y + 6z \end{aligned}$$

5. (6pts) Find the general solution of the following system:

$$\begin{aligned} 2x' + x + y' + 2y &= e^t \\ 3x' - 7x + 3y' + y &= 0. \end{aligned}$$

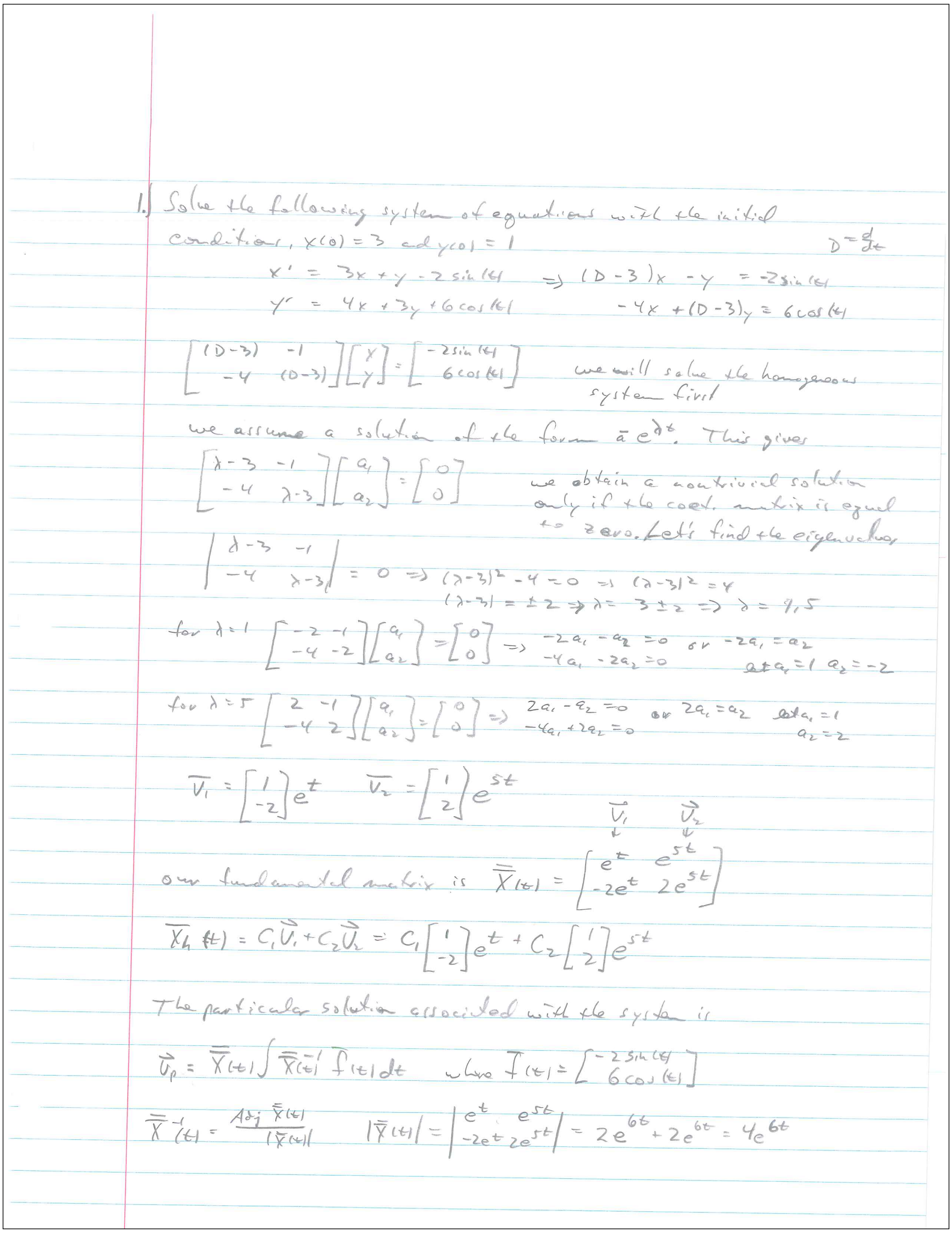
Matrix Exponential

6. (6pts) Using the relation $\int e^{\mathbf{A}t}\,dt = e^{\mathbf{A}t} \times \mathbf{A}^{-1}$, determine the general solution of the following matrix equation;

$$\frac{d\overline{N}}{dt} = \mathbf{A}\,\overline{N}(t) + \overline{F}(t)$$

where $\overline{F}$(t) $= \overline{B}t^2$ and $\overline{B}$ is a constant vector.

7. (6pts) Solve problem 5 using the Matrix Exponential method outlined in class.



1.) Solve the following system of equations with the initial conditions, $x(0)=3$ and $y(0)=1$

$D=\frac{d}{dt}$

$$x' = 3x+y-2\sin(t) \quad \Rightarrow \quad (D-3)x - y = -2\sin(t)$$

$$y' = 4x+3y+6\cos(t) \qquad -4x+(D-3)y = 6\cos(t)$$

$$\begin{bmatrix} (D-3) & -1 \\ -4 & (D-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix}$$

we will solve the homogenous system first

we assume a solution of the form $\bar{a}e^{\lambda t}$. This gives

$$\begin{bmatrix} \lambda-3 & -1 \\ -4 & \lambda-3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we obtain a nontrivial solution only if the coef. matrix is equal to zero. Let's find the eigenvalues

$$\begin{vmatrix} \lambda-3 & -1 \\ -4 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-3)^2-4=0 \Rightarrow (\lambda-3)^2=4$$

$$(\lambda-3)=\pm 2 \Rightarrow \lambda = 3\pm 2 \Rightarrow \lambda = 1,5$$

for $\lambda=1$ $\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2a_1-a_2=0 \\ -4a_1-2a_2=0 \end{matrix}$ or $-2a_1=a_2$

let $a_1=1$ $a_2=-2$

for $\lambda=5$ $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2a_1-a_2=0 \\ -4a_1+2a_2=0 \end{matrix}$ or $2a_1=a_2$ let $a_1=1$

$a_2=2$

$$\bar{V}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} \qquad \bar{V}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t}$$

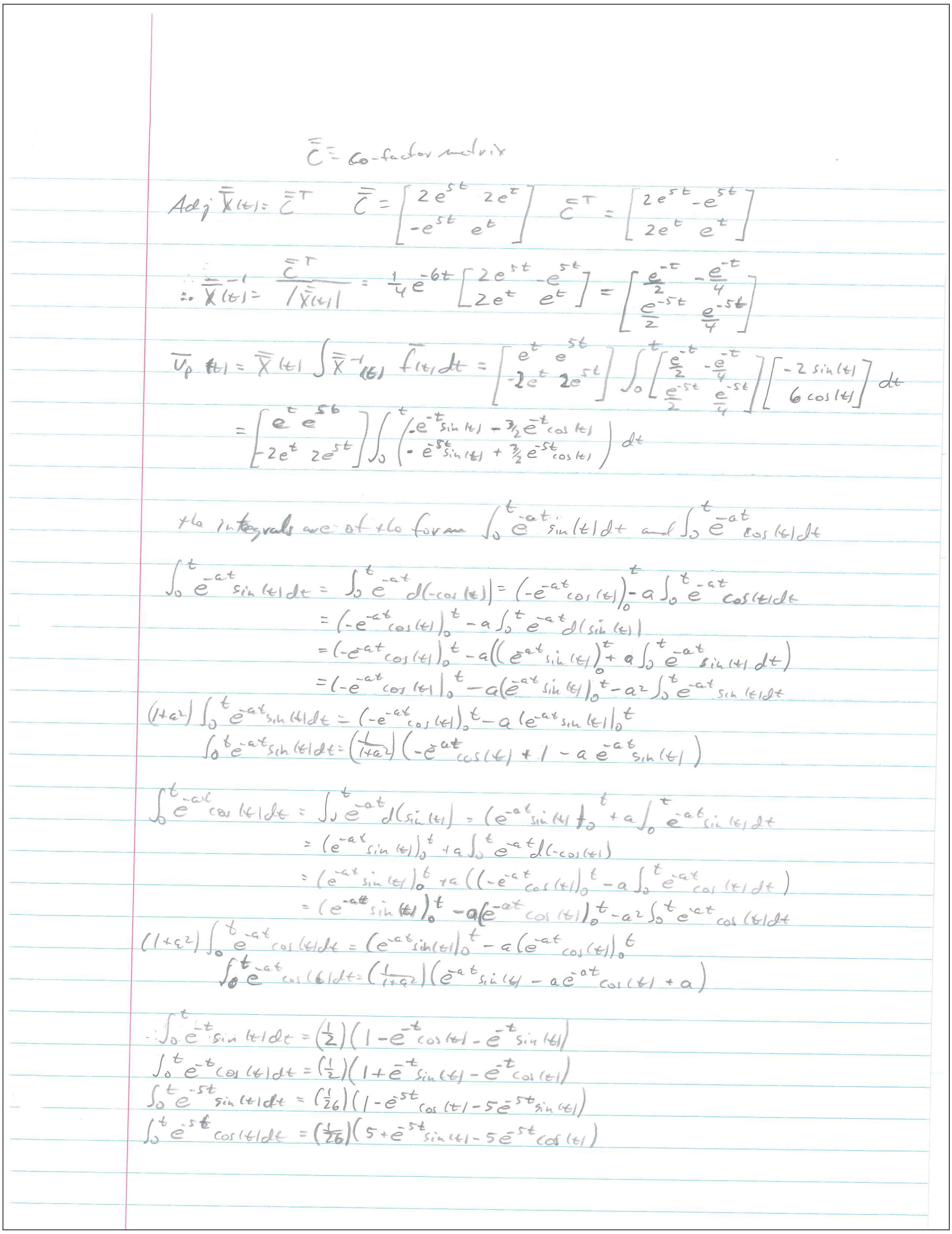
our fundamental matrix is $\bar{\bar{X}}(t) = \begin{bmatrix} \overset{\vec{V}_1}{\downarrow} & \overset{\vec{V}_2}{\downarrow} \\ e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}$

$$\bar{X}_h(t) = C_1\vec{V}_1 + C_2\vec{V}_2 = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t}$$

The particular solution associated with the system is

$$\vec{U}_p = \bar{\bar{X}}(t)\int \bar{\bar{X}}(t)^{-1}\,\bar{f}(t)\,dt \quad \text{where } \bar{f}(t) = \begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix}$$

$$\bar{\bar{X}}^{-1}(t) = \frac{\mathrm{Adj}\,\bar{\bar{X}}(t)}{|\bar{\bar{X}}(t)|} \qquad |\bar{\bar{X}}(t)| = \begin{vmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{vmatrix} = 2e^{6t}+2e^{6t} = 4e^{6t}$$



$\bar{\bar{C}}$ = co-factor matrix

$$Adj\ \bar{\bar{X}}(t) = \bar{\bar{C}}^T \qquad \bar{\bar{C}} = \begin{bmatrix} 2e^{5t} & 2e^{t} \\ -e^{5t} & e^{t} \end{bmatrix} \qquad \bar{\bar{C}}^T = \begin{bmatrix} 2e^{5t} & -e^{5t} \\ 2e^{t} & e^{t} \end{bmatrix}$$

$$\therefore \bar{\bar{X}}^{-1}(t) = \frac{\bar{\bar{C}}^T}{|\bar{\bar{X}}(t)|} = \frac{1}{4}e^{-6t}\begin{bmatrix} 2e^{5t} & -e^{5t} \\ 2e^{t} & e^{t} \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}}{2} & -\frac{e^{-t}}{4} \\ \frac{e^{-5t}}{2} & \frac{e^{-5t}}{4} \end{bmatrix}$$

$$\bar{U}_p(t) = \bar{\bar{X}}(t)\int \bar{\bar{X}}^{-1}(t)\,\bar{f}(t)\,dt = \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}\int_0^t \begin{bmatrix} \frac{e^{-t}}{2} & -\frac{e^{-t}}{4} \\ \frac{e^{-5t}}{2} & \frac{e^{-5t}}{4} \end{bmatrix}\begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}\int_0^t \begin{pmatrix} -e^{-t}\sin(t) - \frac{3}{2}e^{-t}\cos(t) \\ -e^{-5t}\sin(t) + \frac{3}{2}e^{-5t}\cos(t) \end{pmatrix} dt$$

the integrals are of the form $\int_0^t e^{-at}\sin(t)\,dt$ and $\int_0^t e^{-at}\cos(t)\,dt$

$$\int_0^t e^{-at}\sin(t)\,dt = \int_0^t e^{-at}\,d(-\cos(t)) = \left(-e^{-at}\cos(t)\right)_0^t - a\int_0^t e^{-at}\cos(t)\,dt$$

$$= \left(-e^{-at}\cos(t)\right)_0^t - a\int_0^t e^{-at}\,d(\sin(t))$$

$$= \left(-e^{-at}\cos(t)\right)_0^t - a\left(\left(e^{-at}\sin(t)\right)_0^t + a\int_0^t e^{-at}\sin(t)\,dt\right)$$

$$= \left(-e^{-at}\cos(t)\right)_0^t - a\left(e^{-at}\sin(t)\right)_0^t - a^2\int_0^t e^{-at}\sin(t)\,dt$$

$$(1+a^2)\int_0^t e^{-at}\sin(t)\,dt = \left(-e^{-at}\cos(t)\right)_0^t - a\left(e^{-at}\sin(t)\right)_0^t$$

$$\int_0^t e^{-at}\sin(t)\,dt = \left(\frac{1}{1+a^2}\right)\left(-e^{-at}\cos(t) + 1 - a e^{-at}\sin(t)\right)$$

$$\int_0^t e^{-at}\cos(t)\,dt = \int_0^t e^{-at}\,d(\sin(t)) = \left(e^{-at}\sin(t)\right)_0^t + a\int_0^t e^{-at}\sin(t)\,dt$$

$$= \left(e^{-at}\sin(t)\right)_0^t + a\int_0^t e^{-at}\,d(-\cos(t))$$

$$= \left(e^{-at}\sin(t)\right)_0^t + a\left(\left(-e^{-at}\cos(t)\right)_0^t - a\int_0^t e^{-at}\cos(t)\,dt\right)$$

$$= \left(e^{-at}\sin(t)\right)_0^t - a\left(e^{-at}\cos(t)\right)_0^t - a^2\int_0^t e^{-at}\cos(t)\,dt$$

$$(1+a^2)\int_0^t e^{-at}\cos(t)\,dt = \left(e^{-at}\sin(t)\right)_0^t - a\left(e^{-at}\cos(t)\right)_0^t$$

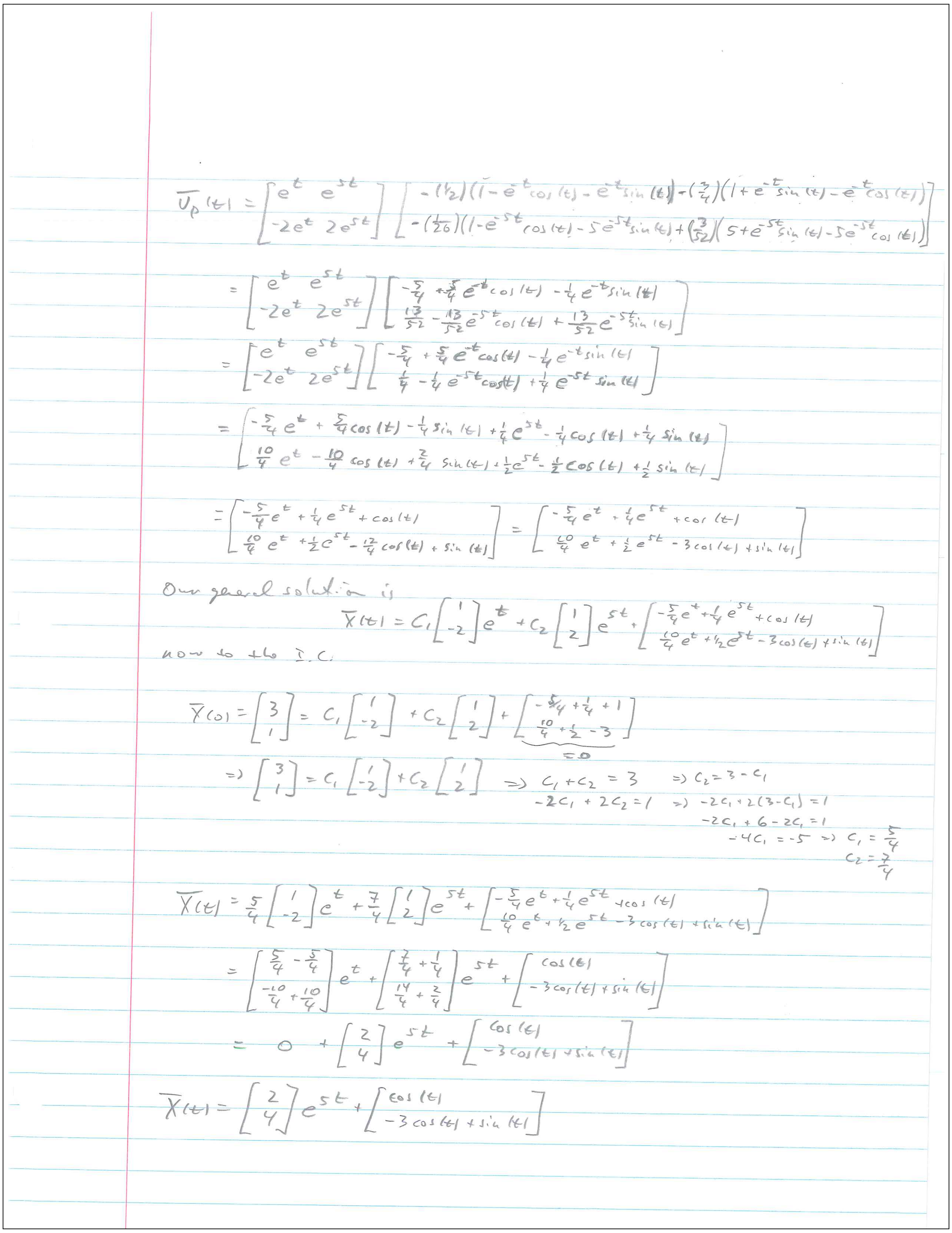
$$\int_0^t e^{-at}\cos(t)\,dt = \left(\frac{1}{1+a^2}\right)\left(e^{-at}\sin(t) - a e^{-at}\cos(t) + a\right)$$

$$\int_0^t e^{-t}\sin(t)\,dt = \left(\frac{1}{2}\right)\left(1 - e^{-t}\cos(t) - e^{-t}\sin(t)\right)$$

$$\int_0^t e^{-t}\cos(t)\,dt = \left(\frac{1}{2}\right)\left(1 + e^{-t}\sin(t) - e^{-t}\cos(t)\right)$$

$$\int_0^t e^{-5t}\sin(t)\,dt = \left(\frac{1}{26}\right)\left(1 - e^{-5t}\cos(t) - 5e^{-5t}\sin(t)\right)$$

$$\int_0^t e^{-5t}\cos(t)\,dt = \left(\frac{1}{26}\right)\left(5 + e^{-5t}\sin(t) - 5e^{-5t}\cos(t)\right)$$



$$\overline{U}_p(t) = \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -(\tfrac{1}{2})(1-e^{-t}\cos(t) - e^{-t}\sin(t)) - (\tfrac{3}{4})(1+e^{-t}\sin(t) - e^{-t}\cos(t)) \\ -(\tfrac{1}{26})(1-e^{-5t}\cos(t) - 5e^{-5t}\sin(t)) + (\tfrac{3}{52})(5+e^{-5t}\sin(t) - 5e^{-5t}\cos(t)) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -\tfrac{5}{4} + \tfrac{5}{4}e^{-t}\cos(t) - \tfrac{1}{4}e^{-t}\sin(t) \\ \tfrac{13}{52} - \tfrac{13}{52}e^{-5t}\cos(t) + \tfrac{13}{52}e^{-5t}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -\tfrac{5}{4} + \tfrac{5}{4}e^{-t}\cos(t) - \tfrac{1}{4}e^{-t}\sin(t) \\ \tfrac{1}{4} - \tfrac{1}{4}e^{-5t}\cos(t) + \tfrac{1}{4}e^{-5t}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\tfrac{5}{4}e^{t} + \tfrac{5}{4}\cos(t) - \tfrac{1}{4}\sin(t) + \tfrac{1}{4}e^{5t} - \tfrac{1}{4}\cos(t) + \tfrac{1}{4}\sin(t) \\ \tfrac{10}{4}e^{t} - \tfrac{10}{4}\cos(t) + \tfrac{2}{4}\sin(t) + \tfrac{1}{2}e^{5t} - \tfrac{1}{2}\cos(t) + \tfrac{1}{2}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\tfrac{5}{4}e^{t} + \tfrac{1}{4}e^{5t} + \cos(t) \\ \tfrac{10}{4}e^{t} + \tfrac{1}{2}e^{5t} - \tfrac{12}{4}\cos(t) + \sin(t) \end{bmatrix} = \begin{bmatrix} -\tfrac{5}{4}e^{t} + \tfrac{1}{4}e^{5t} + \cos(t) \\ \tfrac{10}{4}e^{t} + \tfrac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

Our general solution is

$$\overline{X}(t) = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + \begin{bmatrix} -\tfrac{5}{4}e^{t} + \tfrac{1}{4}e^{5t} + \cos(t) \\ \tfrac{10}{4}e^{t} + \tfrac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

now to the I.C.

$$\overline{X}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{\begin{bmatrix} -\tfrac{5}{4} + \tfrac{1}{4} + 1 \\ \tfrac{10}{4} + \tfrac{1}{2} - 3 \end{bmatrix}}_{=0}$$

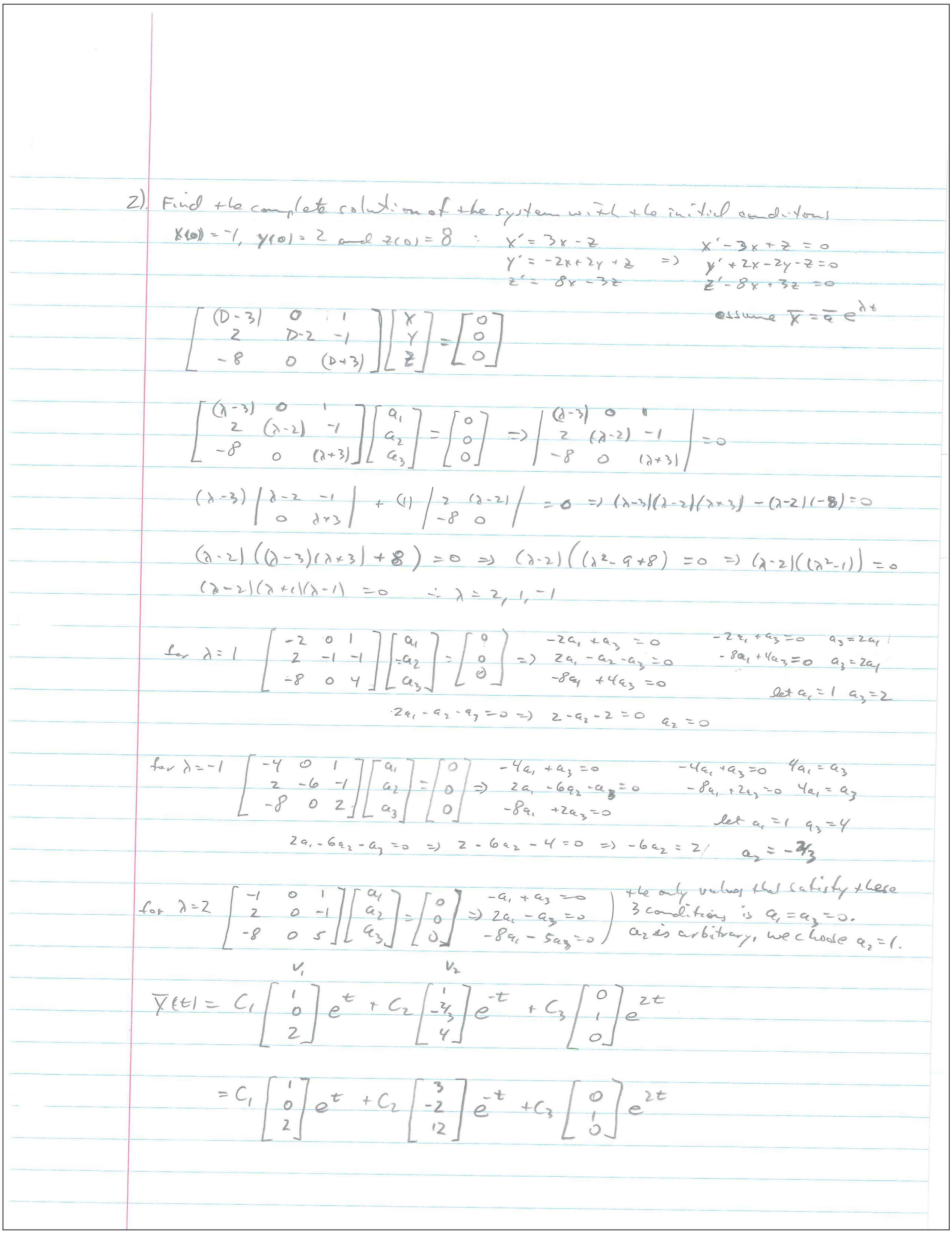
$$\Rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} C_1 + C_2 = 3 \\ -2C_1 + 2C_2 = 1 \end{matrix} \quad \begin{matrix} \Rightarrow C_2 = 3 - C_1 \\ \Rightarrow -2C_1 + 2(3-C_1) = 1 \\ -2C_1 + 6 - 2C_1 = 1 \\ -4C_1 = -5 \Rightarrow C_1 = \tfrac{5}{4} \\ C_2 = \tfrac{7}{4} \end{matrix}$$

$$\overline{X}(t) = \frac{5}{4}\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + \frac{7}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + \begin{bmatrix} -\tfrac{5}{4}e^{t} + \tfrac{1}{4}e^{5t} + \cos(t) \\ \tfrac{10}{4}e^{t} + \tfrac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{5}{4} - \tfrac{5}{4} \\ \tfrac{-10}{4} + \tfrac{10}{4} \end{bmatrix}e^{t} + \begin{bmatrix} \tfrac{7}{4} + \tfrac{1}{4} \\ \tfrac{14}{4} + \tfrac{2}{4} \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$

$$= 0 + \begin{bmatrix} 2 \\ 4 \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$

$$\overline{X}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$



2) Find the complete solution of the system with the initial conditions

$x(0) = -1$, $y(0) = 2$ and $z(0) = 8$ :

$$x' = 3x - z \qquad x' - 3x + z = 0$$
$$y' = -2x + 2y + z \quad \Rightarrow \quad y' + 2x - 2y - z = 0$$
$$z' = 8x - 3z \qquad z' - 8x + 3z = 0$$

assume $\bar{x} = \bar{a}e^{\lambda t}$

$$\begin{bmatrix} (D-3) & 0 & 1 \\ 2 & D-2 & -1 \\ -8 & 0 & (D+3) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (\lambda-3) & 0 & 1 \\ 2 & (\lambda-2) & -1 \\ -8 & 0 & (\lambda+3) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} (\lambda-3) & 0 & 1 \\ 2 & (\lambda-2) & -1 \\ -8 & 0 & (\lambda+3) \end{vmatrix} = 0$$

$$(\lambda-3)\begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda+3 \end{vmatrix} + (1)\begin{vmatrix} 2 & (\lambda-2) \\ -8 & 0 \end{vmatrix} = 0 \Rightarrow (\lambda-3)(\lambda-2)(\lambda+3) - (\lambda-2)(-8) = 0$$

$$(\lambda-2)\left((\lambda-3)(\lambda+3) + 8\right) = 0 \Rightarrow (\lambda-2)\left((\lambda^2-9+8\right) = 0 \Rightarrow (\lambda-2)\left((\lambda^2-1)\right) = 0$$

$$(\lambda-2)(\lambda+1)(\lambda-1) = 0 \qquad \therefore \lambda = 2, 1, -1$$

for $\lambda = 1$

$$\begin{bmatrix} -2 & 0 & 1 \\ 2 & -1 & -1 \\ -8 & 0 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2a_1 + a_3 = 0 \\ 2a_1 - a_2 - a_3 = 0 \\ -8a_1 + 4a_3 = 0 \end{matrix}$$

$-2a_1 + a_3 = 0 \quad a_3 = 2a_1$

$-8a_1 + 4a_3 = 0 \quad a_3 = 2a_1$

let $a_1 = 1$ $a_3 = 2$

$2a_1 - a_2 - a_3 = 0 \Rightarrow 2 - a_2 - 2 = 0 \quad a_2 = 0$

for $\lambda = -1$

$$\begin{bmatrix} -4 & 0 & 1 \\ 2 & -6 & -1 \\ -8 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -4a_1 + a_3 = 0 \\ 2a_1 - 6a_2 - a_3 = 0 \\ -8a_1 + 2a_3 = 0 \end{matrix}$$

$-4a_1 + a_3 = 0 \quad 4a_1 = a_3$

$-8a_1 + 2a_3 = 0 \quad 4a_1 = a_3$

let $a_1 = 1$ $a_3 = 4$

$2a_1 - 6a_2 - a_3 = 0 \Rightarrow 2 - 6a_2 - 4 = 0 \Rightarrow -6a_2 = 2 \quad a_2 = -\frac{2}{3}$

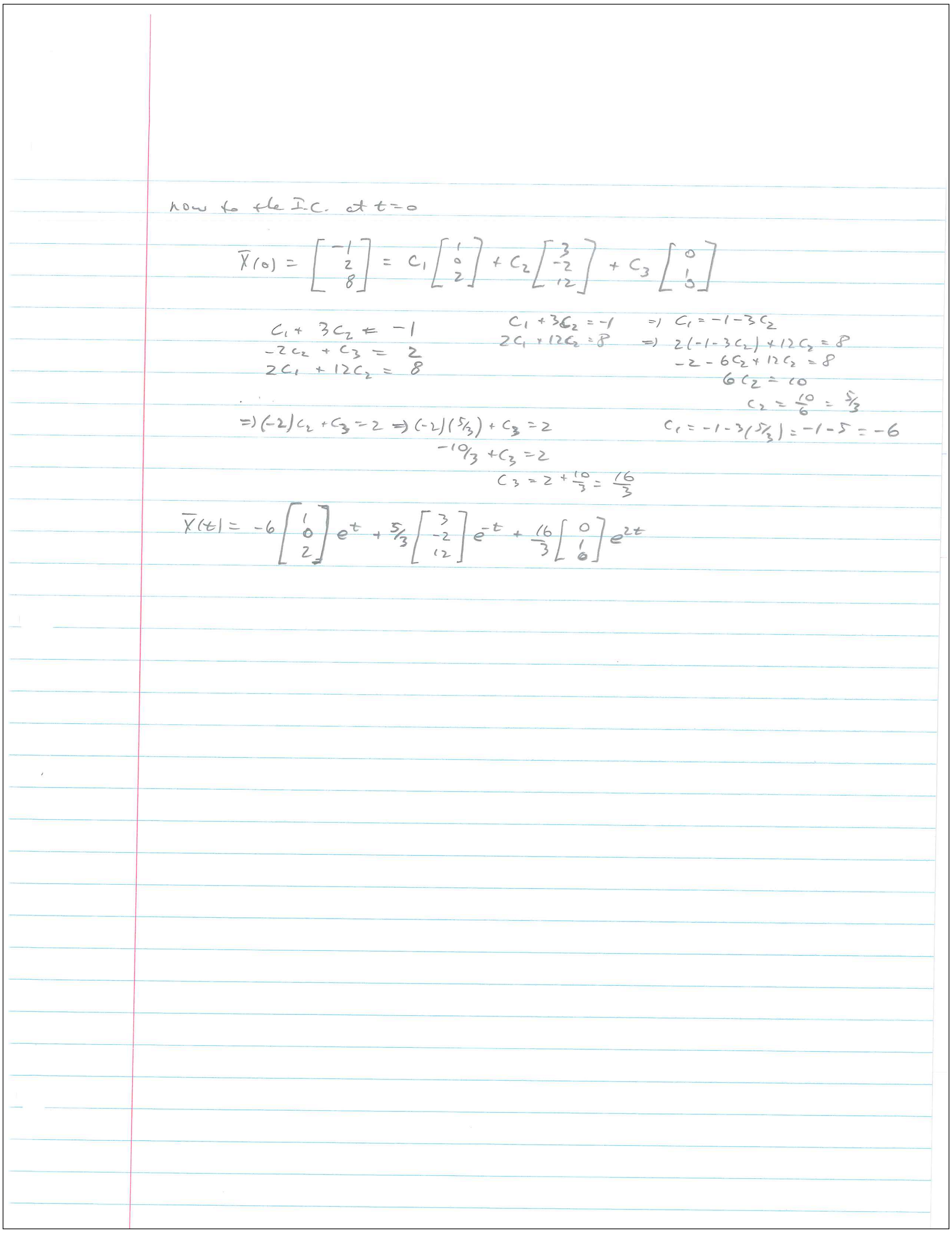
for $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ -8 & 0 & 5 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -a_1 + a_3 = 0 \\ 2a_1 - a_3 = 0 \\ -8a_1 - 5a_3 = 0 \end{matrix}\right)$$

the only values that satisfy these 3 conditions is $a_1 = a_3 = 0$. $a_2$ is arbitrary, we choose $a_2 = 1$.

$$\bar{x}(t) = C_1 \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} e^{t} + C_2 \overset{v_2}{\begin{bmatrix} 1 \\ -\frac{2}{3} \\ 4 \end{bmatrix}} e^{-t} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

$$= C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} e^{-t} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$



now to the I.C. at $t=0$

$$\bar{X}(0) = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} c_1 + 3c_2 &= -1 \\ -2c_2 + c_3 &= 2 \\ 2c_1 + 12c_2 &= 8 \end{aligned}$$

$c_1 + 3c_2 = -1$

$2c_1 + 12c_2 = 8$

$\Rightarrow c_1 = -1 - 3c_2$

$\Rightarrow 2(-1-3c_2) + 12c_2 = 8$

$-2 - 6c_2 + 12c_2 = 8$

$6c_2 = 10$

$c_2 = \frac{10}{6} = \frac{5}{3}$

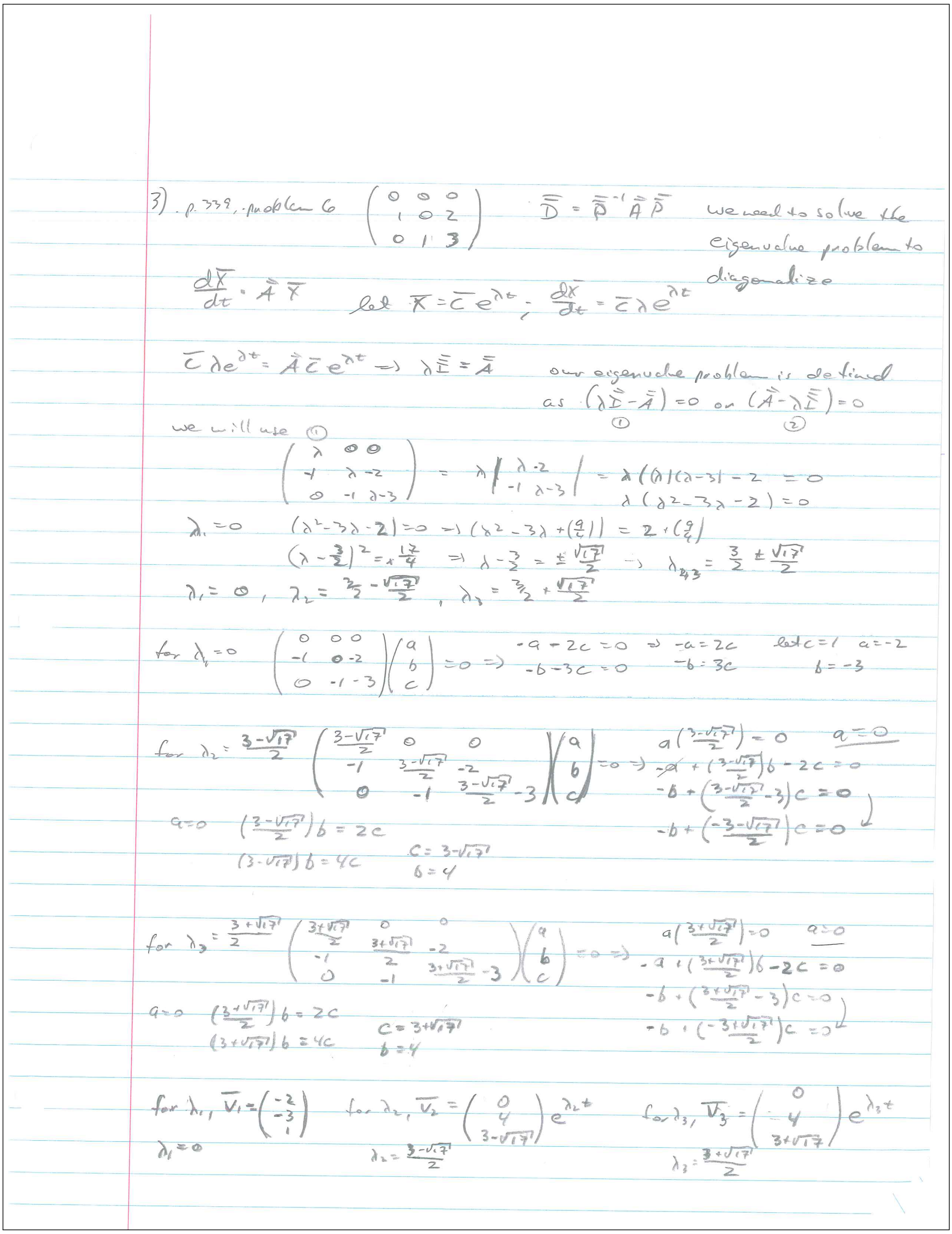
$c_1 = -1 - 3(\frac{5}{3}) = -1 - 5 = -6$

$\Rightarrow (-2)c_2 + c_3 = 2 \Rightarrow (-2)(\frac{5}{3}) + c_3 = 2$

$-\frac{10}{3} + c_3 = 2$

$c_3 = 2 + \frac{10}{3} = \frac{16}{3}$

$$\bar{X}(t) = -6 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + \frac{5}{3} \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} e^{-t} + \frac{16}{3} \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix} e^{2t}$$



3) p. 339, problem 6 $\begin{pmatrix}0&0&0\\1&0&2\\0&1&3\end{pmatrix}$ $\bar{\bar{D}} = \bar{\bar{P}}^{-1}\bar{\bar{A}}\bar{\bar{P}}$ we need to solve the eigenvalue problem to diagonalize

$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X}$ let $\bar{X} = \bar{C}e^{\lambda t}$; $\frac{d\bar{X}}{dt} = \bar{C}\lambda e^{\lambda t}$

$\bar{C}\lambda e^{\lambda t} = \bar{\bar{A}}\bar{C}e^{\lambda t} \Rightarrow \lambda\bar{\bar{I}} = \bar{\bar{A}}$ our eigenvalue problem is defined as $\underset{\textcircled{1}}{(\lambda\bar{\bar{I}}-\bar{\bar{A}})} = 0$ or $\underset{\textcircled{2}}{(\bar{\bar{A}}-\lambda\bar{\bar{I}})} = 0$

we will use ①

$$\begin{pmatrix}\lambda&0&0\\-1&\lambda&-2\\0&-1&\lambda-3\end{pmatrix} = \lambda\begin{vmatrix}\lambda&-2\\-1&\lambda-3\end{vmatrix} = \lambda((\lambda)(\lambda-3)-2 = 0$$

$$\lambda(\lambda^2-3\lambda-2) = 0$$

$\lambda_1 = 0$ $(\lambda^2-3\lambda-2) = 0 \Rightarrow (\lambda^2-3\lambda+(\frac{9}{4})) = 2+(\frac{9}{4})$

$(\lambda-\frac{3}{2})^2 = \frac{17}{4} \Rightarrow \lambda-\frac{3}{2} = \pm\frac{\sqrt{17}}{2} \Rightarrow \lambda_{2,3} = \frac{3}{2} \pm \frac{\sqrt{17}}{2}$

$\lambda_1 = 0$, $\lambda_2 = \frac{3}{2} - \frac{\sqrt{17}}{2}$, $\lambda_3 = \frac{3}{2} + \frac{\sqrt{17}}{2}$

for $\lambda_1 = 0$ $\begin{pmatrix}0&0&0\\-1&0&-2\\0&-1&-3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \Rightarrow$ $-a-2c = 0 \Rightarrow -a = 2c$ let $c = 1$ $a = -2$

$-b-3c = 0$ $-b = 3c$ $b = -3$

for $\lambda_2 = \frac{3-\sqrt{17}}{2}$ $\begin{pmatrix}\frac{3-\sqrt{17}}{2}&0&0\\-1&\frac{3-\sqrt{17}}{2}&-2\\0&-1&\frac{3-\sqrt{17}}{2}-3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \Rightarrow$ $a(\frac{3-\sqrt{17}}{2}) = 0$ $\underline{a = 0}$

$-\cancel{a} + (\frac{3-\sqrt{17}}{2})b - 2c = 0$

$-b + (\frac{3-\sqrt{17}}{2}-3)c = 0$

$-b + (\frac{-3-\sqrt{17}}{2})c = 0$

$a = 0$ $(\frac{3-\sqrt{17}}{2})b = 2c$

$(3-\sqrt{17})b = 4c$ $c = 3-\sqrt{17}$

$b = 4$

for $\lambda_3 = \frac{3+\sqrt{17}}{2}$ $\begin{pmatrix}\frac{3+\sqrt{17}}{2}&0&0\\-1&\frac{3+\sqrt{17}}{2}&-2\\0&-1&\frac{3+\sqrt{17}}{2}-3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \Rightarrow$ $a(\frac{3+\sqrt{17}}{2}) = 0$ $\underline{a = 0}$

$-a + (\frac{3+\sqrt{17}}{2})b - 2c = 0$

$-b + (\frac{3+\sqrt{17}}{2}-3)c = 0$

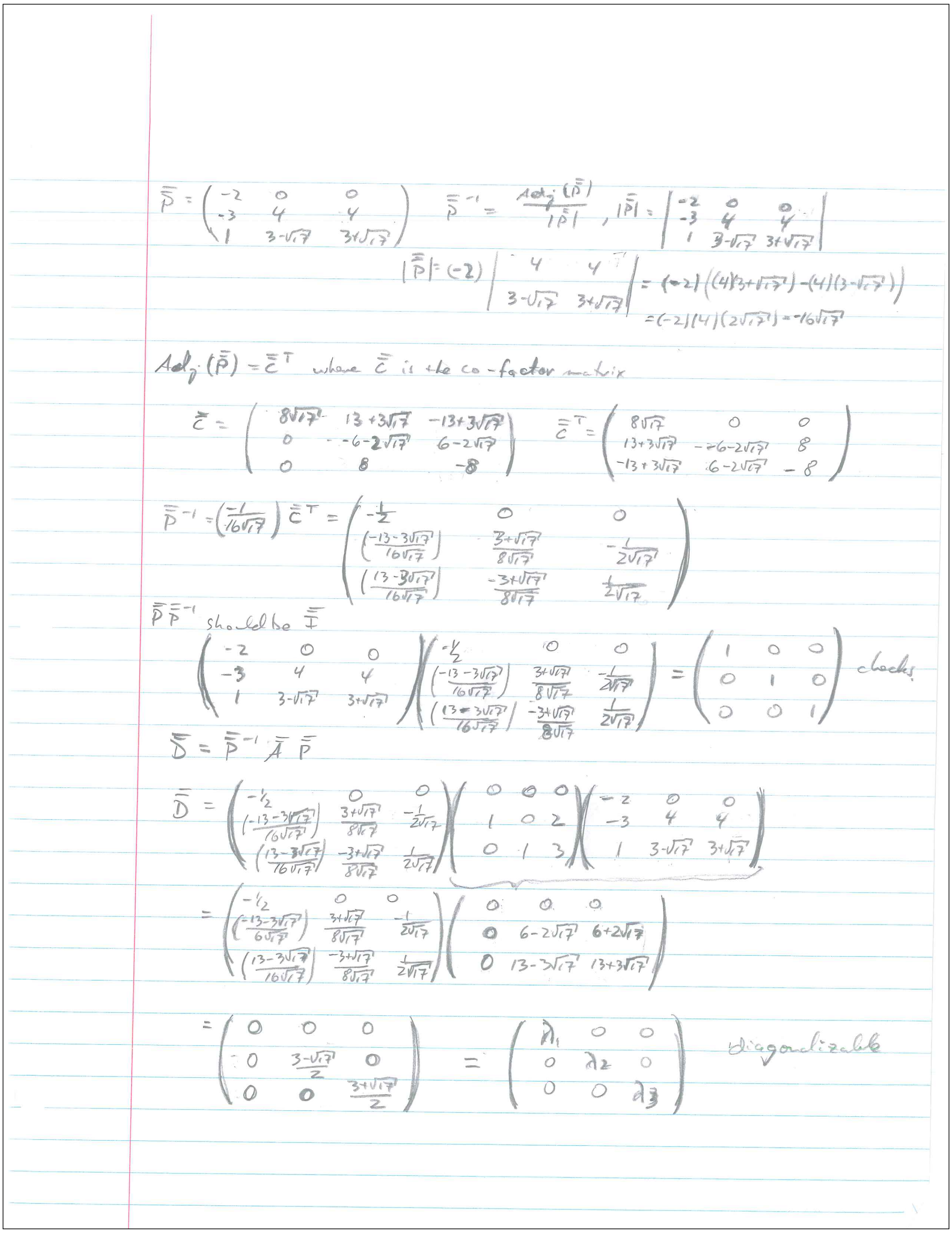
$-b + (\frac{-3+\sqrt{17}}{2})c = 0$

$a = 0$ $(\frac{3+\sqrt{17}}{2})b = 2c$

$(3+\sqrt{17})b = 4c$ $c = 3+\sqrt{17}$

$b = 4$

for $\lambda_1$, $\bar{V}_1 = \begin{pmatrix}-2\\-3\\1\end{pmatrix}$ $\lambda_1 = 0$ for $\lambda_2$, $\bar{V}_2 = \begin{pmatrix}0\\4\\3-\sqrt{17}\end{pmatrix}e^{\lambda_2 t}$ $\lambda_2 = \frac{3-\sqrt{17}}{2}$ for $\lambda_3$, $\bar{V}_3 = \begin{pmatrix}0\\4\\3+\sqrt{17}\end{pmatrix}e^{\lambda_3 t}$ $\lambda_3 = \frac{3+\sqrt{17}}{2}$



$$\bar{\bar{P}} = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \frac{\text{Adj}(\bar{\bar{P}})}{|\bar{\bar{P}}|}, \quad |\bar{\bar{P}}| = \begin{vmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{vmatrix}$$

$$|\bar{\bar{P}}| = (-2) \begin{vmatrix} 4 & 4 \\ 3-\sqrt{17} & 3+\sqrt{17} \end{vmatrix} = (-2)\left((4)(3+\sqrt{17}) - (4)(3-\sqrt{17})\right)$$

$$= (-2)(4)(2\sqrt{17}) = -16\sqrt{17}$$

$\text{Adj}(\bar{\bar{P}}) = \bar{\bar{C}}^T$ where $\bar{\bar{C}}$ is the co-factor matrix

$$\bar{\bar{C}} = \begin{pmatrix} 8\sqrt{17} & 13+3\sqrt{17} & -13+3\sqrt{17} \\ 0 & -6-2\sqrt{17} & 6-2\sqrt{17} \\ 0 & 8 & -8 \end{pmatrix} \qquad \bar{\bar{C}}^T = \begin{pmatrix} 8\sqrt{17} & 0 & 0 \\ 13+3\sqrt{17} & -6-2\sqrt{17} & 8 \\ -13+3\sqrt{17} & 6-2\sqrt{17} & -8 \end{pmatrix}$$

$$\bar{\bar{P}}^{-1} = \left(\frac{-1}{16\sqrt{17}}\right) \bar{\bar{C}}^T = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix}$$

$\bar{\bar{P}}\,\bar{\bar{P}}^{-1}$ should be $\bar{\bar{I}}$

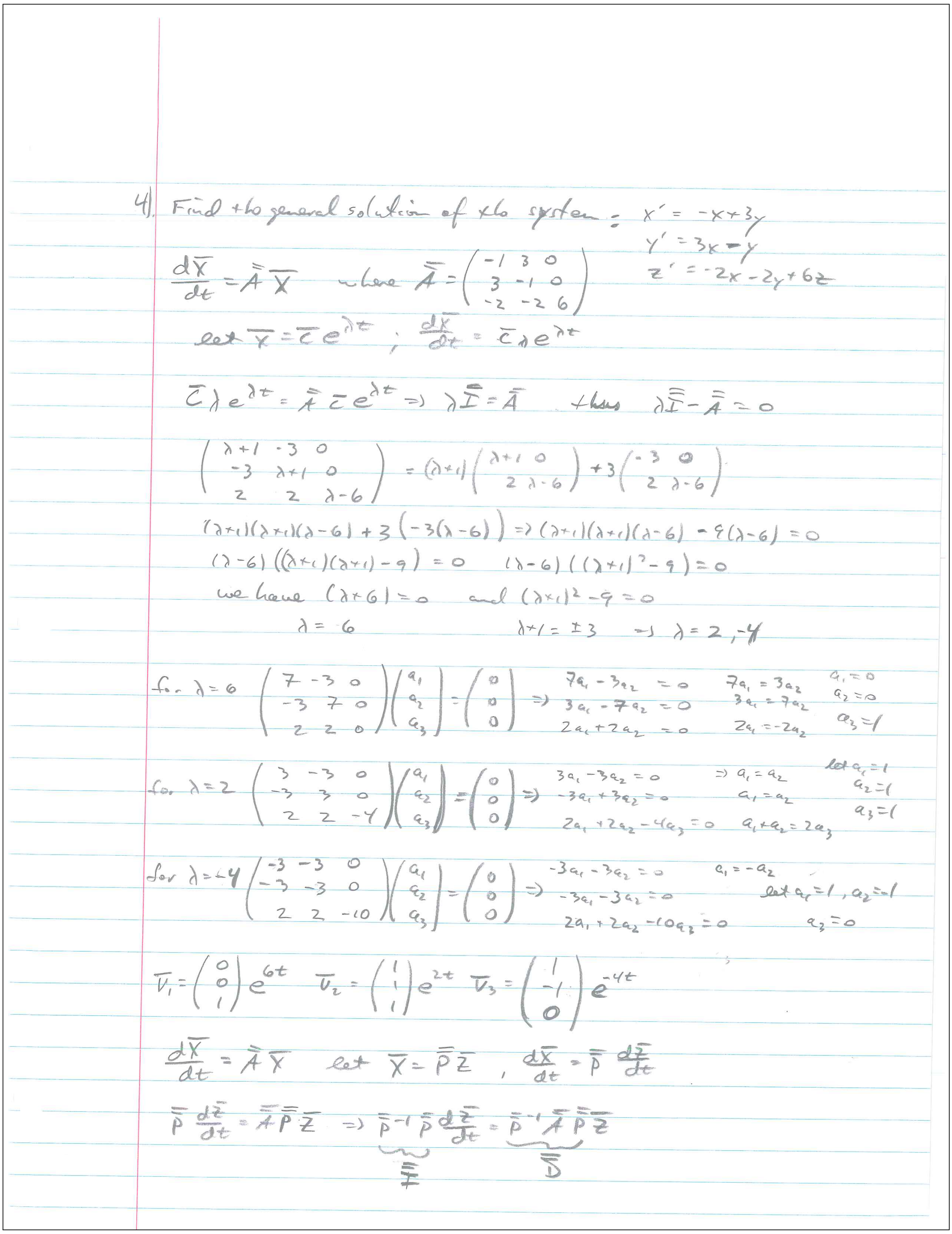
$$\begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ checks}$$

$$\bar{\bar{D}} = \bar{\bar{P}}^{-1} \bar{A} \bar{\bar{P}}$$

$$\bar{\bar{D}} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix}}$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{6\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6-2\sqrt{17} & 6+2\sqrt{17} \\ 0 & 13-3\sqrt{17} & 13+3\sqrt{17} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3-\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{3+\sqrt{17}}{2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{diagonalizable}$$



4) Find the general solution of the system: $x' = -x+3y$

$y' = 3x - y$

$z' = -2x-2y+6z$

$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\,\bar{X}$ where $\bar{\bar{A}} = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{pmatrix}$

let $\bar{X} = \bar{C} e^{\lambda t}$ ; $\frac{d\bar{X}}{dt} = \bar{C}\lambda e^{\lambda t}$

$\bar{C}\lambda e^{\lambda t} = \bar{\bar{A}}\,\bar{C} e^{\lambda t} \Rightarrow \lambda\bar{\bar{I}} = \bar{\bar{A}}$ thus $\lambda\bar{\bar{I}} - \bar{\bar{A}} = 0$

$$\begin{pmatrix} \lambda+1 & -3 & 0 \\ -3 & \lambda+1 & 0 \\ 2 & 2 & \lambda-6 \end{pmatrix} = (\lambda+1)\begin{pmatrix} \lambda+1 & 0 \\ 2 & \lambda-6 \end{pmatrix} + 3\begin{pmatrix} -3 & 0 \\ 2 & \lambda-6 \end{pmatrix}$$

$(\lambda+1)(\lambda+1)(\lambda-6) + 3\big(-3(\lambda-6)\big) \Rightarrow (\lambda+1)(\lambda+1)(\lambda-6) - 9(\lambda-6) = 0$

$(\lambda-6)\big((\lambda+1)(\lambda+1)-9\big) = 0 \qquad (\lambda-6)\big((\lambda+1)^2-9\big) = 0$

we have $(\lambda+6) = 0$ and $(\lambda+1)^2 - 9 = 0$

$\lambda = 6 \qquad \lambda+1 = \pm 3 \Rightarrow \lambda = 2, -4$

for $\lambda = 6$ $\begin{pmatrix} 7 & -3 & 0 \\ -3 & 7 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$7a_1 - 3a_2 = 0 \qquad 7a_1 = 3a_2 \qquad a_1 = 0$

$3a_1 - 7a_2 = 0 \qquad 3a_1 = 7a_2 \qquad a_2 = 0$

$2a_1 + 2a_2 = 0 \qquad 2a_1 = -2a_2 \qquad a_3 = 1$

for $\lambda = 2$ $\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$3a_1 - 3a_2 = 0 \quad \Rightarrow a_1 = a_2 \qquad$ let $a_1 = 1$

$-3a_1 + 3a_2 = 0 \qquad a_1 = a_2 \qquad a_2 = 1$

$2a_1 + 2a_2 - 4a_3 = 0 \qquad a_1 + a_2 = 2a_3 \qquad a_3 = 1$

for $\lambda = -4$ $\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 2 & 2 & -10 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$-3a_1 - 3a_2 = 0 \qquad a_1 = -a_2$

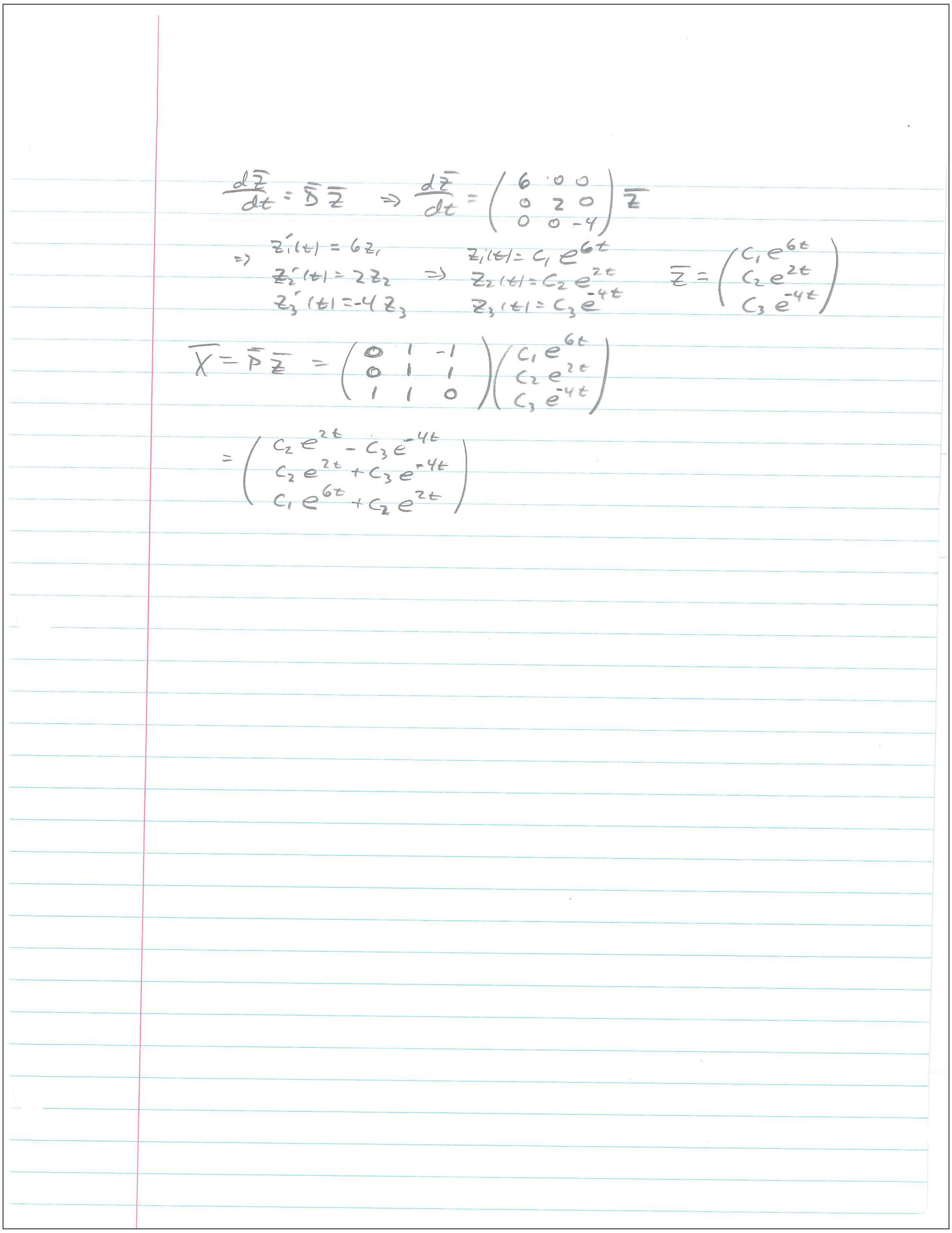
$-3a_1 - 3a_2 = 0 \qquad$ let $a_1 = 1, a_2 = -1$

$2a_1 + 2a_2 - 10a_3 = 0 \qquad a_3 = 0$

$\bar{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{6t} \quad \bar{V}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} \quad \bar{V}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-4t}$

$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\,\bar{X}$ let $\bar{X} = \bar{\bar{P}}\bar{Z}$ , $\frac{d\bar{X}}{dt} = \bar{\bar{P}}\frac{d\bar{Z}}{dt}$

$$\bar{\bar{P}}\frac{d\bar{Z}}{dt} = \bar{\bar{A}}\bar{\bar{P}}\bar{Z} \Rightarrow \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{P}}}_{\bar{\bar{I}}}\frac{d\bar{Z}}{dt} = \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{A}}\bar{\bar{P}}}_{\bar{\bar{D}}}\bar{Z}$$

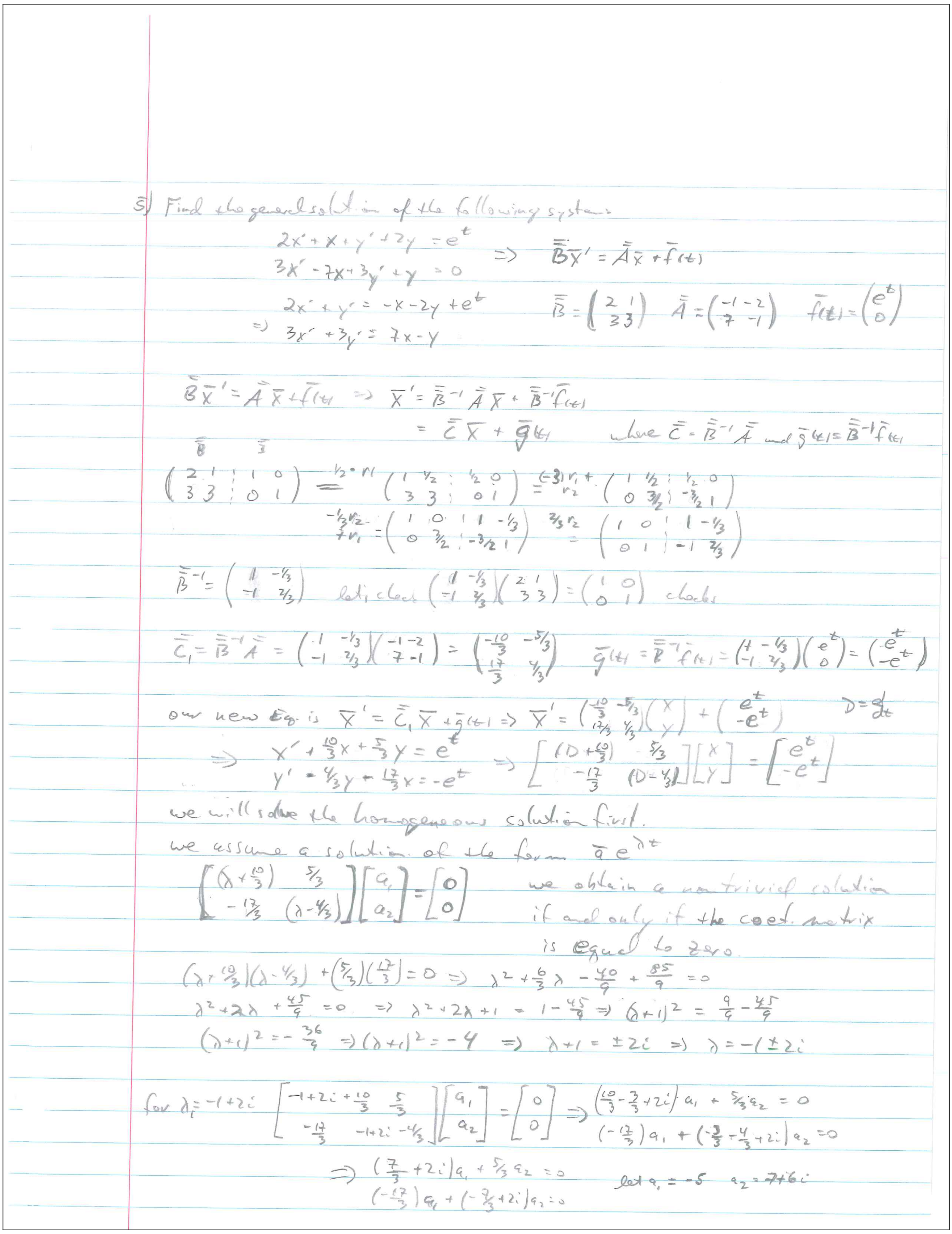


$$\frac{d\bar{Z}}{dt} = \bar{D}\bar{Z} \Rightarrow \frac{d\bar{Z}}{dt} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \bar{Z}$$

$$\Rightarrow \begin{matrix} Z_1'(t) = 6Z_1 \\ Z_2'(t) = 2Z_2 \\ Z_3'(t) = -4Z_3 \end{matrix} \Rightarrow \begin{matrix} Z_1(t) = C_1 e^{6t} \\ Z_2(t) = C_2 e^{2t} \\ Z_3(t) = C_3 e^{-4t} \end{matrix} \qquad \bar{Z} = \begin{pmatrix} C_1 e^{6t} \\ C_2 e^{2t} \\ C_3 e^{-4t} \end{pmatrix}$$

$$\bar{X} = \bar{P}\bar{Z} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 e^{6t} \\ C_2 e^{2t} \\ C_3 e^{-4t} \end{pmatrix}$$

$$= \begin{pmatrix} C_2 e^{2t} - C_3 e^{-4t} \\ C_2 e^{2t} + C_3 e^{-4t} \\ C_1 e^{6t} + C_2 e^{2t} \end{pmatrix}$$



5) Find the general solution of the following system

$$\begin{aligned}2x'+x+y'+2y&=e^t\\3x'-7x+3y'+y&=0\end{aligned}\Rightarrow \bar{\bar{B}}\bar{X}'=\bar{\bar{A}}\bar{x}+\bar{f}(t)$$

$$\begin{aligned}2x'+y'&=-x-2y+e^t\\ \Rightarrow 3x'+3y'&=7x-y\end{aligned}\qquad \bar{\bar{B}}=\begin{pmatrix}2&1\\3&3\end{pmatrix}\quad \bar{\bar{A}}=\begin{pmatrix}-1&-2\\7&-1\end{pmatrix}\quad \bar{f}(t)=\begin{pmatrix}e^t\\0\end{pmatrix}$$

$$\bar{\bar{B}}\bar{X}'=\bar{\bar{A}}\bar{X}+\bar{f}(t)\Rightarrow \bar{X}'=\bar{\bar{B}}^{-1}\bar{\bar{A}}\bar{X}+\bar{\bar{B}}^{-1}\bar{f}(t)$$

$$=\bar{\bar{C}}\bar{X}+\bar{g}(t)\qquad \text{where } \bar{\bar{C}}=\bar{\bar{B}}^{-1}\bar{\bar{A}} \text{ and } \bar{g}(t)=\bar{\bar{B}}^{-1}\bar{f}(t)$$

$$\overset{\bar{\bar{B}}\qquad\quad \bar{\bar{I}}}{\left(\begin{array}{cc|cc}2&1&1&0\\3&3&0&1\end{array}\right)}\overset{\frac{1}{2}\cdot r_1}{=}\left(\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{2}&0\\3&3&0&1\end{array}\right)\overset{(-3)r_1+r_2}{=}\left(\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{2}&0\\0&\frac{3}{2}&-\frac{3}{2}&1\end{array}\right)$$

$$\overset{-\frac{1}{3}r_2+r_1}{=}\left(\begin{array}{cc|cc}1&0&1&-\frac{1}{3}\\0&\frac{3}{2}&-\frac{3}{2}&1\end{array}\right)\overset{\frac{2}{3}r_2}{=}\left(\begin{array}{cc|cc}1&0&1&-\frac{1}{3}\\0&1&-1&\frac{2}{3}\end{array}\right)$$

$\bar{\bar{B}}^{-1}=\begin{pmatrix}1&-\frac{1}{3}\\-1&\frac{2}{3}\end{pmatrix}$ let's check $\begin{pmatrix}1&-\frac{1}{3}\\-1&\frac{2}{3}\end{pmatrix}\begin{pmatrix}2&1\\3&3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ checks

$$\bar{\bar{C}}_1=\bar{\bar{B}}^{-1}\bar{\bar{A}}=\begin{pmatrix}1&-\frac{1}{3}\\-1&\frac{2}{3}\end{pmatrix}\begin{pmatrix}-1&-2\\7&-1\end{pmatrix}=\begin{pmatrix}-\frac{10}{3}&-\frac{5}{3}\\\frac{17}{3}&\frac{4}{3}\end{pmatrix}\qquad \bar{g}(t)=\bar{\bar{B}}^{-1}\bar{f}(t)=\begin{pmatrix}1&-\frac{1}{3}\\-1&\frac{2}{3}\end{pmatrix}\begin{pmatrix}e^t\\0\end{pmatrix}=\begin{pmatrix}e^t\\-e^t\end{pmatrix}$$

our new Eq. is $\bar{X}'=\bar{\bar{C}}_1\bar{X}+\bar{g}(t)\Rightarrow \bar{X}'=\begin{pmatrix}-\frac{10}{3}&-\frac{5}{3}\\\frac{17}{3}&\frac{4}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}e^t\\-e^t\end{pmatrix}$ $\qquad D=\frac{d}{dt}$

$$\Rightarrow\begin{aligned}x'+\tfrac{10}{3}x+\tfrac{5}{3}y&=e^t\\y'-\tfrac{4}{3}y+\tfrac{17}{3}x&=-e^t\end{aligned}\Rightarrow\begin{bmatrix}(D+\frac{10}{3})&\frac{5}{3}\\-\frac{17}{3}&(D-\frac{4}{3})\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}e^t\\-e^t\end{bmatrix}$$

we will solve the homogeneous solution first.

we assume a solution of the form $\bar{a}e^{\lambda t}$

$$\begin{bmatrix}(\lambda+\frac{10}{3})&\frac{5}{3}\\-\frac{17}{3}&(\lambda-\frac{4}{3})\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

we obtain a non trivial solution if and only if the coef. matrix is equal to zero.

$$(\lambda+\tfrac{10}{3})(\lambda-\tfrac{4}{3})+(\tfrac{5}{3})(\tfrac{17}{3})=0\Rightarrow \lambda^2+\tfrac{6}{3}\lambda-\tfrac{40}{9}+\tfrac{85}{9}=0$$

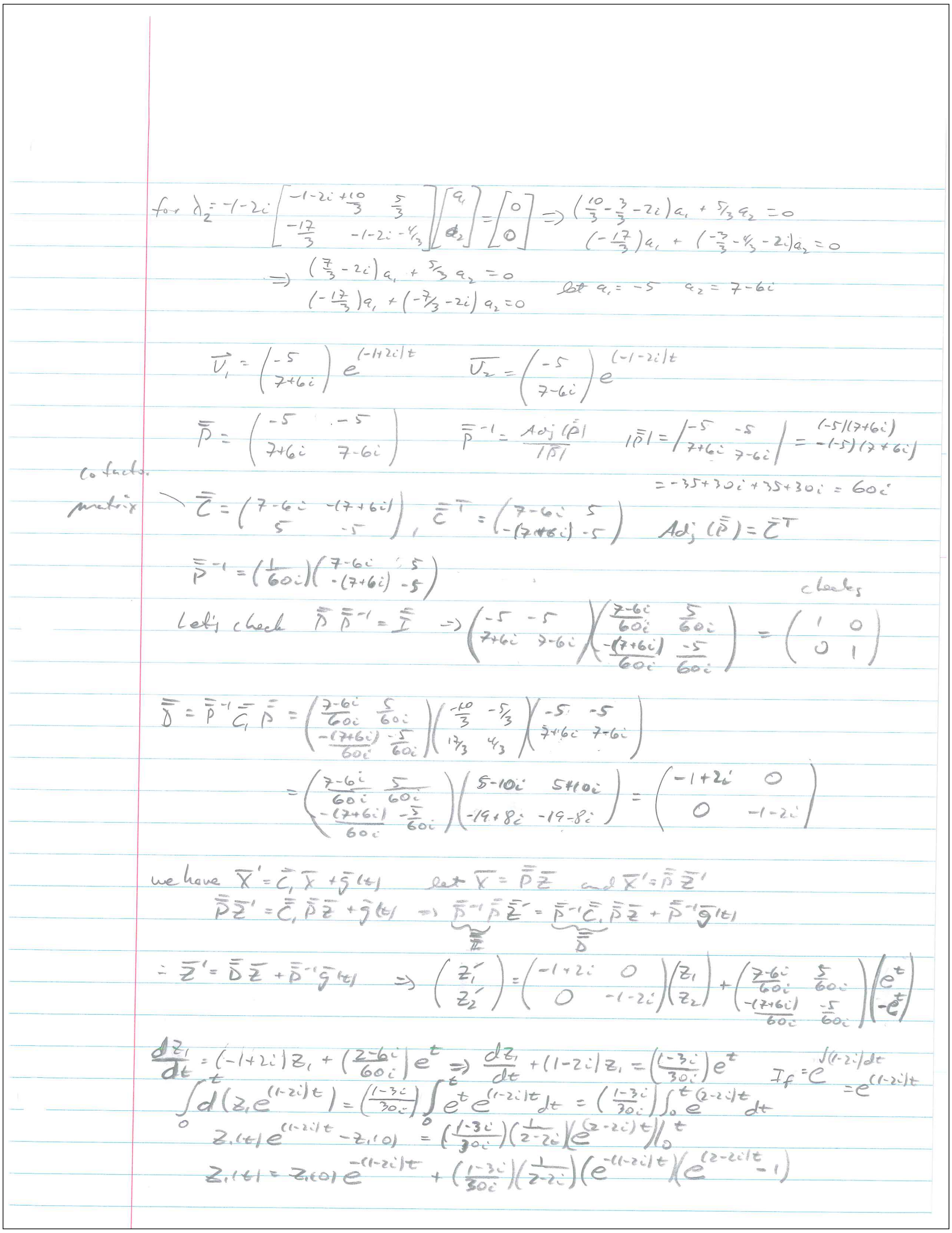
$$\lambda^2+2\lambda+\tfrac{45}{9}=0\Rightarrow \lambda^2+2\lambda+1=1-\tfrac{45}{9}\Rightarrow(\lambda+1)^2=\tfrac{9}{9}-\tfrac{45}{9}$$

$$(\lambda+1)^2=-\tfrac{36}{9}\Rightarrow(\lambda+1)^2=-4\Rightarrow\lambda+1=\pm2i\Rightarrow\lambda=-1\pm2i$$

for $\lambda_1=-1+2i$ $\begin{bmatrix}-1+2i+\frac{10}{3}&\frac{5}{3}\\-\frac{17}{3}&-1+2i-\frac{4}{3}\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\begin{aligned}(\tfrac{10}{3}-\tfrac{3}{3}+2i)a_1+\tfrac{5}{3}a_2&=0\\(-\tfrac{17}{3})a_1+(-\tfrac{3}{3}-\tfrac{4}{3}+2i)a_2&=0\end{aligned}$

$$\Rightarrow\begin{aligned}(\tfrac{7}{3}+2i)a_1+\tfrac{5}{3}a_2&=0\\(-\tfrac{17}{3})a_1+(-\tfrac{7}{3}+2i)a_2&=0\end{aligned}$$

let $a_1=-5$ $\quad a_2=7+6i$



for $\lambda_2 = -1-2i$ $\begin{bmatrix} -1-2i+\frac{10}{3} & \frac{5}{3} \\ -\frac{17}{3} & -1-2i-\frac{4}{3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \left(\frac{10}{3}-\frac{3}{3}-2i\right)a_1 + \frac{5}{3}a_2 = 0 \\ \left(-\frac{17}{3}\right)a_1 + \left(-\frac{3}{3}-\frac{4}{3}-2i\right)a_2 = 0 \end{matrix}$

$$\Rightarrow \begin{matrix} \left(\frac{7}{3}-2i\right)a_1 + \frac{5}{3}a_2 = 0 \\ \left(-\frac{17}{3}\right)a_1 + \left(-\frac{7}{3}-2i\right)a_2 = 0 \end{matrix}$$

let $a_1 = -5$ $a_2 = 7-6i$

$$\vec{V}_1 = \begin{pmatrix} -5 \\ 7+6i \end{pmatrix} e^{(-1+2i)t} \qquad \bar{V}_2 = \begin{pmatrix} -5 \\ 7-6i \end{pmatrix} e^{(-1-2i)t}$$

$$\bar{\bar{P}} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \frac{Adj(\bar{\bar{P}})}{|\bar{\bar{P}}|} \qquad |\bar{\bar{P}}| = \begin{vmatrix} -5 & -5 \\ 7+6i & 7-6i \end{vmatrix} = \begin{matrix} (-5)(7+6i) \\ -(-5)(7+6i) \end{matrix}$$

$$= -35+30i+35+30i = 60i$$

co factor matrix

$$\bar{\bar{C}} = \begin{pmatrix} 7-6i & -(7+6i) \\ 5 & -5 \end{pmatrix}, \quad \bar{\bar{C}}^T = \begin{pmatrix} 7-6i & 5 \\ -(7+6i) & -5 \end{pmatrix} \qquad Adj(\bar{\bar{P}}) = \bar{\bar{C}}^T$$

$$\bar{\bar{P}}^{-1} = \left(\frac{1}{60i}\right)\begin{pmatrix} 7-6i & 5 \\ -(7+6i) & -5 \end{pmatrix}$$

checks

Let's check $\bar{\bar{P}}\,\bar{\bar{P}}^{-1} = \bar{\bar{I}} \Rightarrow \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\bar{\bar{D}} = \bar{\bar{P}}^{-1}\bar{\bar{C}}_1\bar{\bar{P}} = \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} \frac{-10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} 5-10i & 5+10i \\ -19+8i & -19-8i \end{pmatrix} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}$$

we have $\bar{X}' = \bar{\bar{C}}_1\bar{X} + \bar{g}(t)$ let $\bar{X} = \bar{\bar{P}}\bar{Z}$ and $\bar{X}' = \bar{\bar{P}}\bar{Z}'$

$$\bar{\bar{P}}\bar{Z}' = \bar{\bar{C}}_1\bar{\bar{P}}\bar{Z} + \bar{g}(t) \Rightarrow \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{P}}}_{\bar{\bar{I}}}\bar{Z}' = \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{C}}_1\bar{\bar{P}}}_{\bar{\bar{D}}}\bar{Z} + \bar{\bar{P}}^{-1}\bar{g}(t)$$

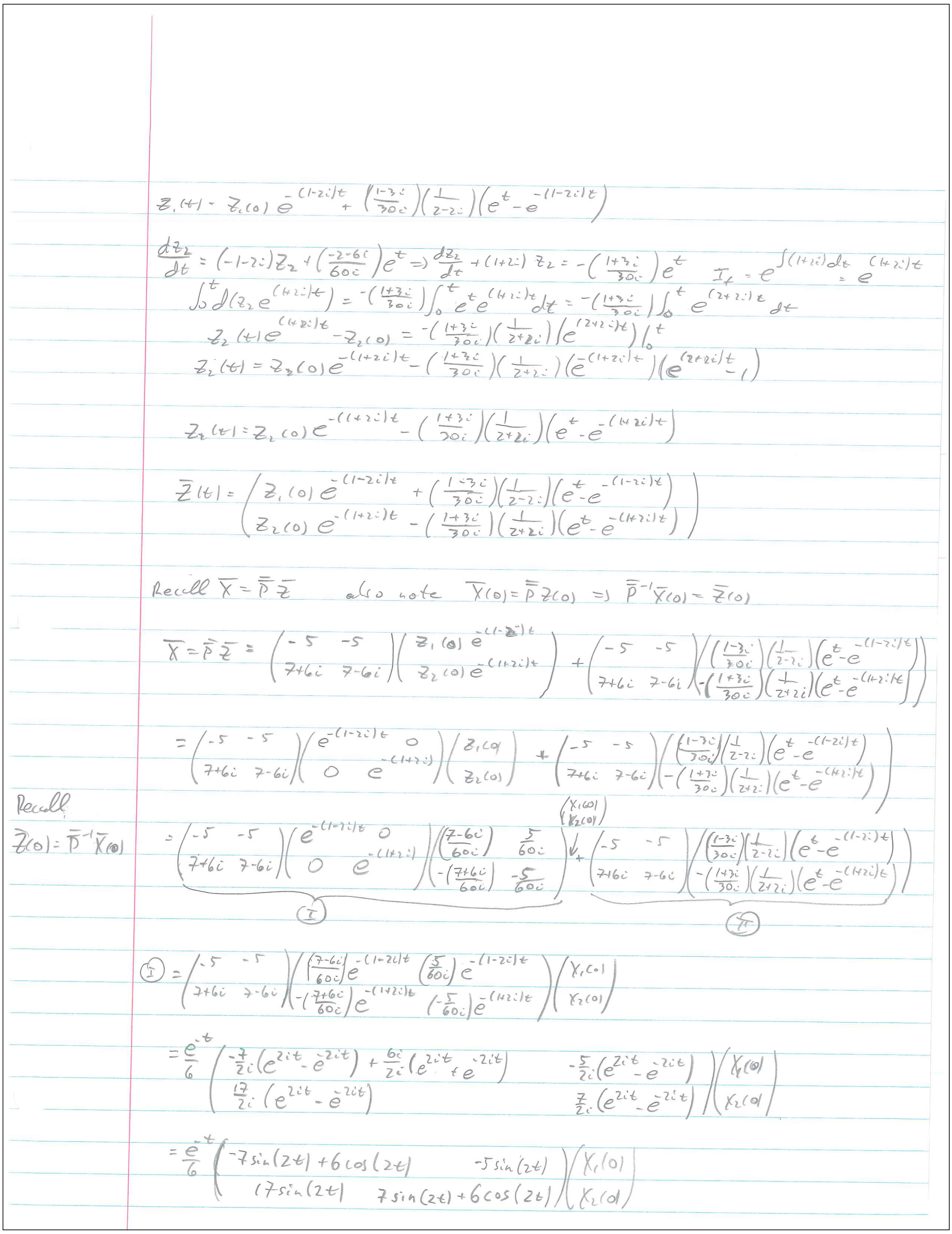
$$\therefore \bar{Z}' = \bar{\bar{D}}\bar{Z} + \bar{\bar{P}}^{-1}\bar{g}(t) \Rightarrow \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\frac{dz_1}{dt} = (-1+2i)z_1 + \left(\frac{2-6i}{60i}\right)e^t \Rightarrow \frac{dz_1}{dt} + (1-2i)z_1 = \left(\frac{1-3i}{30i}\right)e^t \qquad I_f = e^{\int(1-2i)dt} = e^{(1-2i)t}$$

$$\int_0^t d\left(z_1 e^{(1-2i)t}\right) = \left(\frac{1-3i}{30i}\right)\int_0^t e^t e^{(1-2i)t}dt = \left(\frac{1-3i}{30i}\right)\int_0^t e^{(2-2i)t}dt$$

$$z_1(t)e^{(1-2i)t} - z_1(0) = \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{(2-2i)t}\right)\Big|_0^t$$

$$z_1(t) = z_1(0)e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{-(1-2i)t}\right)\left(e^{(2-2i)t} - 1\right)$$



$$Z_1(t) = Z_1(0)\, e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right)$$

$$\frac{dZ_2}{dt} = (-1-2i)Z_2 + \left(\frac{-2-6i}{60i}\right)e^{t} \Rightarrow \frac{dZ_2}{dt} + (1+2i)\,Z_2 = -\left(\frac{1+3i}{30i}\right)e^{t} \qquad I_f = e^{\int (1+2i)\,dt} = e^{(1+2i)t}$$

$$\int_0^t d\left(Z_2 e^{(1+2i)t}\right) = -\left(\frac{1+3i}{30i}\right)\int_0^t e^{t} e^{(1+2i)t}\,dt = -\left(\frac{1+3i}{30i}\right)\int_0^t e^{(2+2i)t}\,dt$$

$$Z_2(t)\,e^{(1+2i)t} - Z_2(0) = -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{(2+2i)t}\right)\Big|_0^t$$

$$Z_2(t) = Z_2(0)\,e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{-(1+2i)t}\right)\left(e^{(2+2i)t} - 1\right)$$

$$Z_2(t) = Z_2(0)\,e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right)$$

$$\bar{Z}(t) = \begin{pmatrix} Z_1(0)\,e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ Z_2(0)\,e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

Recall $\bar{X} = \bar{\bar{P}}\,\bar{Z}$ also note $\bar{X}(0) = \bar{\bar{P}}\,\bar{Z}(0) \Rightarrow \bar{\bar{P}}^{-1}\bar{X}(0) = \bar{Z}(0)$

$$\bar{X} = \bar{\bar{P}}\,\bar{Z} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} Z_1(0)\,e^{-(1-2i)t} \\ Z_2(0)\,e^{-(1+2i)t} \end{pmatrix} + \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{-(1-2i)t} & 0 \\ 0 & e^{-(1+2i)} \end{pmatrix}\begin{pmatrix} Z_1(0) \\ Z_2(0) \end{pmatrix} + \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

Recall

$\bar{Z}(0) = \bar{\bar{P}}^{-1}\bar{X}(0)$

$$= \underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{-(1-2i)t} & 0 \\ 0 & e^{-(1+2i)} \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right) & \frac{5}{60i} \\ -\left(\frac{7+6i}{60i}\right) & \frac{-5}{60i} \end{pmatrix}}_{①}\overset{\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}}{\downarrow} + \underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}}_{Ⅱ}$$

$$① = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{-(1-2i)t} & \left(\frac{5}{60i}\right)e^{-(1-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{-(1+2i)t} & \left(-\frac{5}{60i}\right)e^{-(1+2i)t} \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -\frac{7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) & -\frac{5}{2i}\left(e^{2it} - e^{-2it}\right) \\ \frac{17}{2i}\left(e^{2it} - e^{-2it}\right) & \frac{7}{2i}\left(e^{2it} - e^{-2it}\right) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t) + 6\cos(2t) & -5\sin(2t) \\ (7\sin(2t) & 7\sin(2t) + 6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$\textcircled{\scriptsize II} = \begin{pmatrix} (-5)\left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t}-e^{-(1-2i)t}\right)+(5)\left(\frac{1+3i}{60i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t}-e^{-(1+2i)t}\right) \\ (7+6i)\left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t}-e^{-(1-2i)t}\right)-(7-6i)\left(\frac{1+3i}{60i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t}-e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\left(\frac{1-3i}{6i}\right)\left(\frac{2+2i}{8}\right)\left(e^{t}-e^{-(1-2i)t}\right)+\left(\frac{1+3i}{6i}\right)\left(\frac{2-2i}{8}\right)\left(e^{t}-e^{-(1+2i)t}\right) \\ \left(\frac{25-15i}{30i}\right)\left(\frac{2+2i}{8}\right)\left(e^{t}-e^{-(1-2i)t}\right)+\left(\frac{25+15i}{30i}\right)\left(\frac{2-2i}{8}\right)\left(e^{t}-e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -\left(\frac{2-i}{12i}\right)\left(e^{2t}-e^{2it}\right)+\left(\frac{2+i}{12i}\right)\left(e^{2t}-e^{-2it}\right) \\ \left(\frac{4+i}{12i}\right)\left(e^{2t}-e^{2it}\right)-\left(\frac{4-i}{12i}\right)\left(e^{2t}-e^{-2it}\right) \end{pmatrix}$$

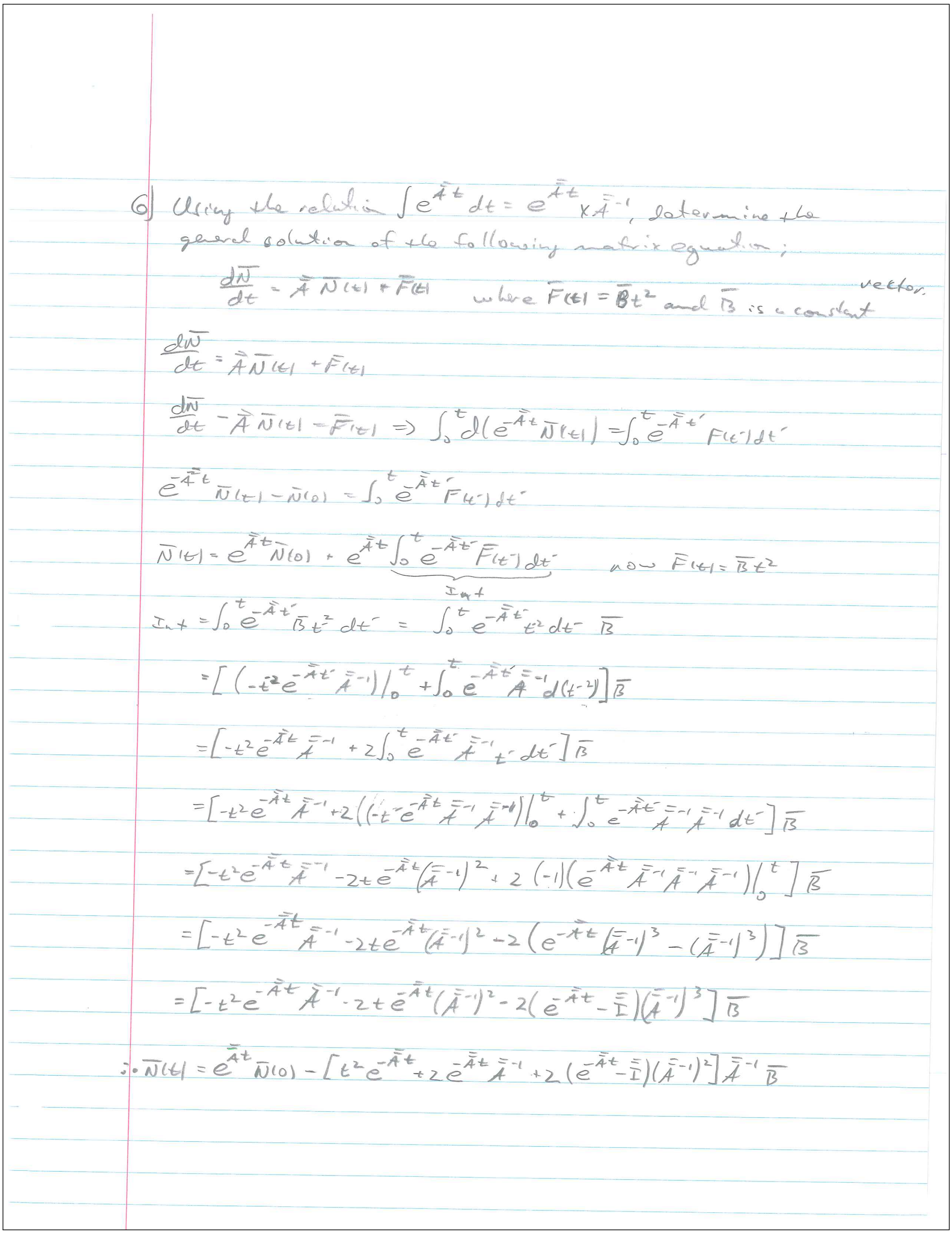
$$= e^{-t}\begin{pmatrix} -\left(\frac{1}{12i}\right)\left(\cancel{2e^{2t}}-2e^{2it}-ie^{2t}+ie^{2it}\cancel{-2e^{2t}}+2e^{-2it}-ie^{2t}+ie^{-2it}\right) \\ \left(\frac{1}{12i}\right)\left(\cancel{4e^{2t}}-4e^{2it}+ie^{2t}-ie^{2it}\cancel{-4e^{2t}}+4e^{-2it}+ie^{2t}-ie^{-2it}\right) \end{pmatrix}$$

$$= \frac{e^{-t}}{12i}\begin{pmatrix} 2\left(e^{2it}-e^{-2it}\right)+2ie^{2t}-i\left(e^{2it}+e^{-2it}\right) \\ -4\left(e^{2it}-e^{-2it}\right)+2ie^{2t}-i\left(e^{2it}+e^{-2it}\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \frac{1}{3}\sin(2t)+\frac{1}{6}e^{2t}-\frac{1}{6}\cos(2t) \\ \frac{-2}{3}\sin(2t)+\frac{1}{6}e^{2t}-\frac{1}{6}\cos(2t) \end{pmatrix} = \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t)+e^{2t}-\cos(2t) \\ -4\sin(2t)+e^{2t}-\cos(2t) \end{pmatrix}$$

$\overline{X}(t) = \textcircled{\scriptsize I} + \textcircled{\scriptsize II}$ :

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t)+6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t)+6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} + \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t)+e^{2t}-\cos(2t) \\ -4\sin(2t)+e^{2t}-\cos(2t) \end{pmatrix}$$



6) Using the relation $\int e^{\bar{\bar{A}}t}dt = e^{\bar{\bar{A}}t} \times \bar{\bar{A}}^{-1}$, determine the general solution of the following matrix equation;

$$\frac{d\bar{N}}{dt} = \bar{\bar{A}}\bar{N}(t) + \bar{F}(t)$$ where $\bar{F}(t) = \bar{B}t^2$ and $\bar{B}$ is a constant vector.

$$\frac{d\bar{N}}{dt} = \bar{\bar{A}}\bar{N}(t) + \bar{F}(t)$$

$$\frac{d\bar{N}}{dt} - \bar{\bar{A}}\bar{N}(t) = \bar{F}(t) \Rightarrow \int_0^t d\left(e^{-\bar{\bar{A}}t}\bar{N}(t)\right) = \int_0^t e^{-\bar{\bar{A}}t'} F(t')dt'$$

$$e^{-\bar{\bar{A}}t}\bar{N}(t) - \bar{N}(0) = \int_0^t e^{-\bar{\bar{A}}t'} F(t')dt'$$

$$\bar{N}(t) = e^{\bar{\bar{A}}t}\bar{N}(0) + e^{\bar{\bar{A}}t}\underbrace{\int_0^t e^{-\bar{\bar{A}}t'}\bar{F}(t')dt'}_{Int} \quad \text{now } \bar{F}(t) = \bar{B}t^2$$

$$Int = \int_0^t e^{-\bar{\bar{A}}t'}\bar{B}t'^2 dt' = \int_0^t e^{-\bar{\bar{A}}t'}t'^2 dt' \, \bar{B}$$

$$= \left[\left(-t'^2 e^{-\bar{\bar{A}}t'}\bar{\bar{A}}^{-1}\right)\Big|_0^t + \int_0^t e^{-\bar{\bar{A}}t'}\bar{\bar{A}}^{-1} d(t'^2)\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} + 2\int_0^t e^{-\bar{\bar{A}}t'}\bar{\bar{A}}^{-1} t' dt'\right]\bar{B}$$

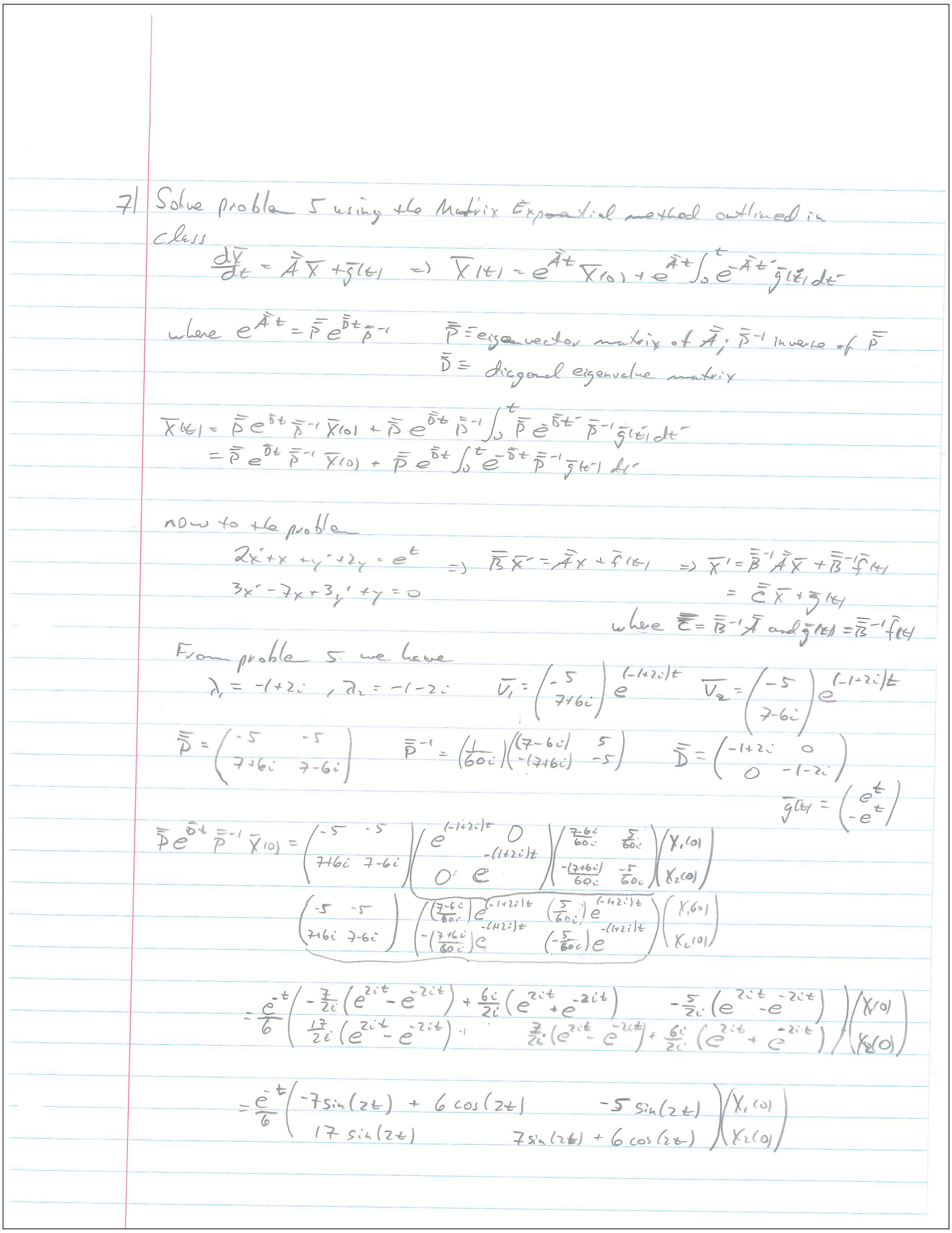
$$= \left[-t^2 e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} + 2\left(\left(-t' e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1}\bar{\bar{A}}^{-1}\right)\Big|_0^t + \int_0^t e^{-\bar{\bar{A}}t'}\bar{\bar{A}}^{-1}\bar{\bar{A}}^{-1}dt'\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} - 2t e^{-\bar{\bar{A}}t}(\bar{\bar{A}}^{-1})^2 + 2(-1)\left(e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1}\bar{\bar{A}}^{-1}\bar{\bar{A}}^{-1}\right)\Big|_0^t\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} - 2t e^{-\bar{\bar{A}}t}(\bar{\bar{A}}^{-1})^2 - 2\left(e^{-\bar{\bar{A}}t}(\bar{\bar{A}}^{-1})^3 - (\bar{\bar{A}}^{-1})^3\right)\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} - 2t e^{-\bar{\bar{A}}t}(\bar{\bar{A}}^{-1})^2 - 2\left(e^{-\bar{\bar{A}}t} - \bar{\bar{I}}\right)(\bar{\bar{A}}^{-1})^3\right]\bar{B}$$

$$\therefore \bar{N}(t) = e^{\bar{\bar{A}}t}\bar{N}(0) - \left[t^2 e^{-\bar{\bar{A}}t} + 2e^{-\bar{\bar{A}}t}\bar{\bar{A}}^{-1} + 2\left(e^{-\bar{\bar{A}}t} - \bar{\bar{I}}\right)(\bar{\bar{A}}^{-1})^2\right]\bar{\bar{A}}^{-1}\bar{B}$$



7| Solve problem 5 using the Matrix Exponential method outlined in class

$$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X} + \bar{g}(t) \Rightarrow \bar{X}(t) = e^{\bar{\bar{A}}t}\bar{X}(0) + e^{\bar{\bar{A}}t}\int_0^t e^{-\bar{\bar{A}}t'}\bar{g}(t')\,dt'$$

where $e^{\bar{\bar{A}}t} = \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}$ $\bar{\bar{P}}$ = eigenvector matrix of $\bar{\bar{A}}$; $\bar{\bar{P}}^{-1}$ inverse of $\bar{\bar{P}}$

$\bar{\bar{D}}$ = diagonal eigenvalue matrix

$$\bar{X}(t) = \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) + \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\int_0^t \bar{\bar{P}}e^{\bar{\bar{D}}t'}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt'$$

$$= \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) + \bar{\bar{P}}e^{\bar{\bar{D}}t}\int_0^t e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt'$$

now to the problem

$$2x' + x + y' + 2y = e^t \qquad \Rightarrow \bar{\bar{B}}\bar{X}' = \bar{\bar{A}}x + \bar{f}(t) \Rightarrow \bar{X}' = \bar{\bar{B}}^{-1}\bar{\bar{A}}\bar{X} + \bar{\bar{B}}^{-1}\bar{f}(t)$$

$$3x' - 7x + 3y' + y = 0 \qquad = \bar{\bar{C}}\bar{X} + \bar{g}(t)$$

where $\bar{\bar{C}} = \bar{\bar{B}}^{-1}\bar{\bar{A}}$ and $\bar{g}(t) = \bar{\bar{B}}^{-1}\bar{f}(t)$

From problem 5 we have

$$\lambda_1 = -1+2i, \quad \lambda_2 = -1-2i \qquad \bar{V}_1 = \begin{pmatrix} -5 \\ 7+6i \end{pmatrix} e^{(-1+2i)t} \qquad \bar{V}_2 = \begin{pmatrix} -5 \\ 7-6i \end{pmatrix} e^{(-1-2i)t}$$

$$\bar{\bar{P}} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \left(\frac{1}{60i}\right)\begin{pmatrix} (7-6i) & 5 \\ -(7+6i) & -5 \end{pmatrix} \qquad \bar{\bar{D}} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}$$

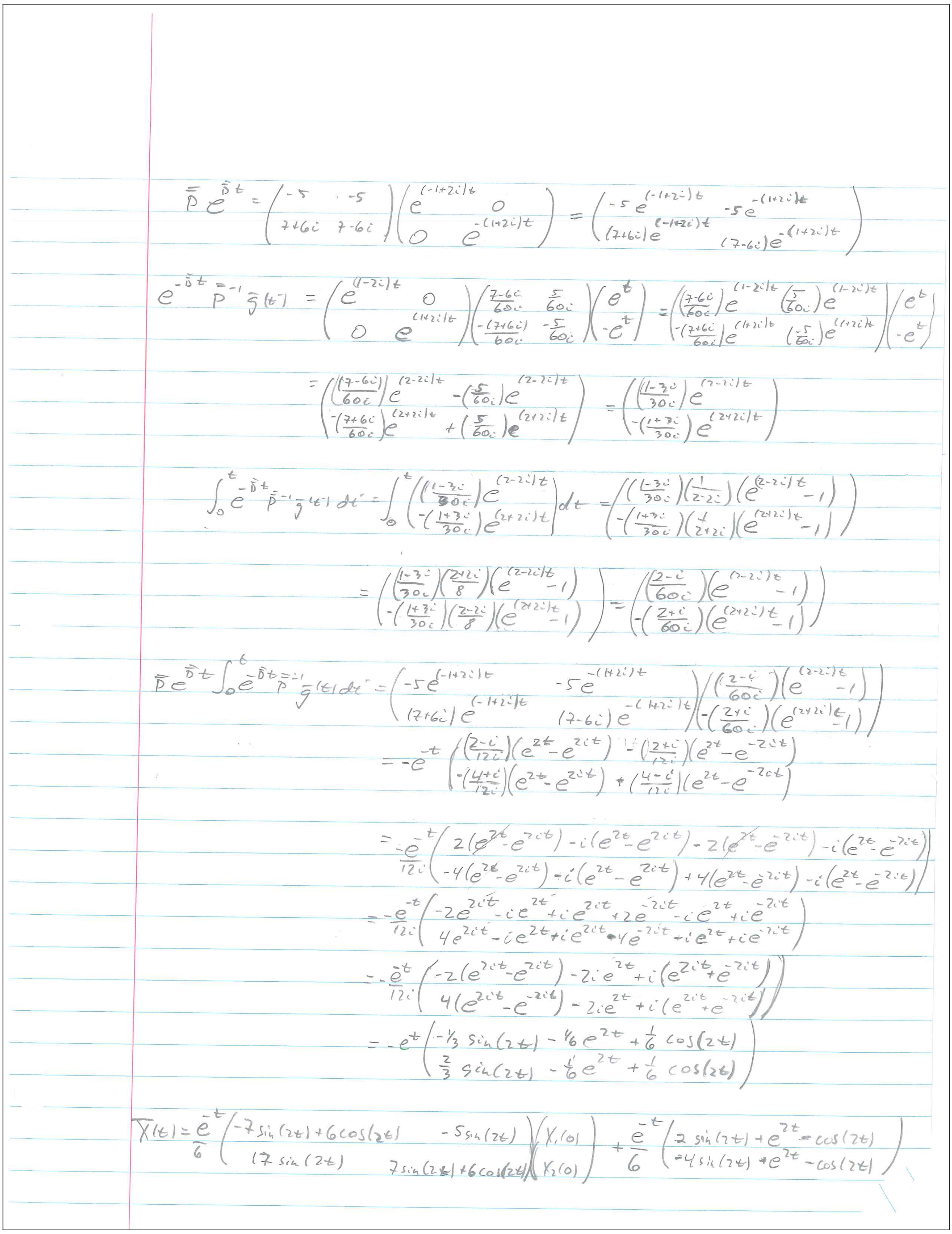
$$\bar{g}(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\underbrace{\begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{-(1+2i)t} \end{pmatrix}\begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ -\frac{(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{(-1+2i)t} & \left(\frac{5}{60i}\right)e^{(-1+2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{-(1+2i)t} & \left(-\frac{5}{60i}\right)e^{-(1+2i)t} \end{pmatrix}}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -\frac{7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) & -\frac{5}{2i}\left(e^{2it} - e^{-2it}\right) \\ \frac{17}{2i}\left(e^{2it} - e^{-2it}\right) & \frac{7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) \end{pmatrix}\begin{pmatrix} X(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t) + 6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t) + 6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$



$$\bar{\bar{P}} e^{\bar{\bar{D}} t} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{-(1+2i)t}\end{pmatrix} = \begin{pmatrix} -5e^{(-1+2i)t} & -5e^{-(1+2i)t} \\ (7+6i)e^{(-1+2i)t} & (7-6i)e^{-(1+2i)t}\end{pmatrix}$$

$$e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t') = \begin{pmatrix} e^{(1-2i)t} & 0 \\ 0 & e^{(1+2i)t}\end{pmatrix}\begin{pmatrix}\frac{7-6i}{60i} & \frac{5}{60i} \\ -\frac{(7+6i)}{60i} & -\frac{5}{60i}\end{pmatrix}\begin{pmatrix} e^t \\ -e^t\end{pmatrix} = \begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{(1-2i)t} & \left(\frac{5}{60i}\right)e^{(1-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{(1+2i)t} & \left(\frac{-5}{60i}\right)e^{(1+2i)t}\end{pmatrix}\begin{pmatrix} e^t \\ -e^t\end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{(7-6i)}{60i}\right)e^{(2-2i)t} - \left(\frac{5}{60i}\right)e^{(2-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{(2+2i)t} + \left(\frac{5}{60i}\right)e^{(2+2i)t}\end{pmatrix} = \begin{pmatrix} \left(\frac{1-3i}{30i}\right)e^{(2-2i)t} \\ -\left(\frac{1+3i}{30i}\right)e^{(2+2i)t}\end{pmatrix}$$

$$\int_0^t e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt' = \int_0^t \begin{pmatrix} \left(\frac{1-3i}{30i}\right)e^{(2-2i)t} \\ -\left(\frac{1+3i}{30i}\right)e^{(2+2i)t}\end{pmatrix} dt = \begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{2+2i}{8}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{2-2i}{8}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix} = \begin{pmatrix} \left(\frac{2-i}{60i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{2+i}{60i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$\bar{\bar{P}}e^{\bar{\bar{D}}t}\int_0^t e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt' = \begin{pmatrix} -5e^{(-1+2i)t} & -5e^{-(1+2i)t} \\ (7+6i)e^{(-1+2i)t} & (7-6i)e^{-(1+2i)t}\end{pmatrix}\begin{pmatrix} \left(\frac{2-i}{60i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{2+i}{60i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$= -e^{-t}\begin{pmatrix} \left(\frac{2-i}{12i}\right)\left(e^{2t}-e^{2it}\right) - \left(\frac{2+i}{12i}\right)\left(e^{2t}-e^{-2it}\right) \\ -\left(\frac{4+i}{12i}\right)\left(e^{2t}-e^{2it}\right) + \left(\frac{4-i}{12i}\right)\left(e^{2t}-e^{-2it}\right)\end{pmatrix}$$

$$= -\frac{e^{-t}}{12i}\begin{pmatrix} 2(\cancel{e^{2t}}-e^{2it}) - i(e^{2t}-e^{2it}) - 2(\cancel{e^{2t}}-e^{-2it}) - i(e^{2t}-e^{-2it}) \\ -4(e^{2t}-e^{2it}) - i(e^{2t}-e^{2it}) + 4(e^{2t}-e^{-2it}) - i(e^{2t}-e^{-2it})\end{pmatrix}$$

$$= -\frac{e^{-t}}{12i}\begin{pmatrix} -2e^{2it} - ie^{2t} + ie^{2it} + 2e^{-2it} - ie^{2t} + ie^{-2it} \\ 4e^{2it} - ie^{2t} + ie^{2it} + 4e^{-2it} + ie^{2t} + ie^{-2it}\end{pmatrix}$$

$$= -\frac{e^{-t}}{12i}\begin{pmatrix} -2(e^{2it}-e^{2it}) - 2ie^{2t} + i(e^{2it}+e^{-2it}) \\ 4(e^{2it}-e^{-2it}) - 2ie^{2t} + i(e^{2it}+e^{-2it})\end{pmatrix}$$

$$= -e^{t}\begin{pmatrix} -\frac{1}{3}\sin(2t) - \frac{1}{6}e^{2t} + \frac{1}{6}\cos(2t) \\ \frac{2}{3}\sin(2t) - \frac{1}{6}e^{2t} + \frac{1}{6}\cos(2t)\end{pmatrix}$$

$$\bar{X}(t) = \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t)+6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t)+6\cos(2t)\end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0)\end{pmatrix} + \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t)+e^{2t}-\cos(2t) \\ -4\sin(2t)+e^{2t}-\cos(2t)\end{pmatrix}$$

# 2.9 HW 8

## 2.9.1 Problems to solve

Homework Set No. 8 NEEP 547
Due November 8, 2013 DLH

Fourier expansions

1. (4pts) Find the Fourier expansions of the periodic function whose definition on one period is
$$f(t) = \begin{cases} t & \text{for } 0 < t < 2 \\ 4 - t & \text{for } 2 < t < 4. \end{cases}$$

2. (6pts) Find the complex exponential Fourier series of the periodic function whose definition on one period is $f(t) = \cosh(t) \quad -1 < t < 1$.

3. (8pts) Find the solution of the following differential equation which satisfies the given initial conditions:
$$y'' - 3y' + 2y = f(t) \quad ; y(0) = y'(0) = 0 \text{ and } f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi. \end{cases}$$
(Hint: solve for the homogeneous Eqn. using O.D.E techniques and expand $f(t)$ in a Fourier series).

4. (8pts) A vibrating string, clamped at $x = 0$ and at $x = \ell$, is in a resisting medium which damps it motion. Its motion is described by the damped wave equation
$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial t}$$
with I.C.: $u(x,0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t) = u(\ell,t) = 0$.
where $v$ and $k$ are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a Fourier expansion of the form
$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{\ell}x\right).$$
Why did we use the sine series and not the cosine series?

## 2.9.2 problem 1 (Fourier expansion, periodic function)

Find Fourier expansion of the periodic function whose definition on one period is
$$f(t) = \begin{cases} t & 0 < t < 2 \\ 4 - t & 2 < t < 4 \end{cases}$$

**Solution:**

The function $f(t)$ is (for 3 periods)

```
f[t_] := Piecewise[{{t, 0 < t < 2}, {4 - t, 2 < t < 4}, {0, True}}]
Plot[UnitStep[#] f[t - #] & /@ {0, 4, 8}, {t, 0, 12},
 PlotRange -> All, Frame -> False, AxesLabel -> {"t", "f(t)"}]
```

The period is $T = 4$. Let Fourier series approximation of $f(t)$ be $\tilde{f}(t)$, hence from the definition
$$\tilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right) + b_n \sin\left(\frac{2\pi}{T}nt\right)$$
This is an even function, hence $b_n = 0$ and only $a_n$ needs to be found. Hence
$$\tilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right)$$

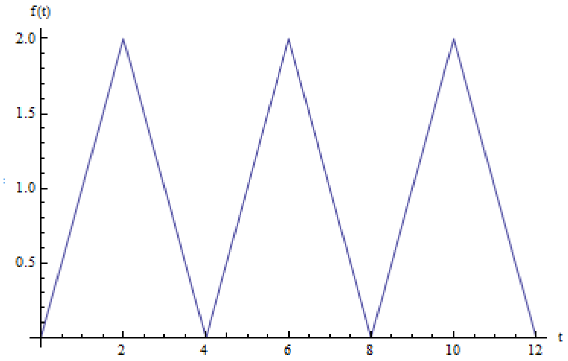


Figure 2.10: $f(t)$ drawn for 3 periods for illustration

$$\begin{aligned} a_0 &= \frac{1}{T/2}\int_0^T f(t)\,dt = \frac{1}{2}\left(\int_0^2 t dt + \int_2^4 (4-t)\,dt\right) \\ &= \frac{1}{2}\left(\left[\frac{t^2}{2}\right]_0^2 + \left[4t - \frac{t^2}{2}\right]_2^4\right) = \frac{1}{2}\left(2 + \left(16 - \frac{16}{2}\right) - \left(8 - \frac{4}{2}\right)\right) \\ &= \frac{1}{2}(2+8-6) \\ &= 2 \end{aligned}$$

And

$$a_n = \frac{1}{T/2}\int_0^T \cos\left(\frac{2\pi}{T}nt\right) f(t)\,dt = \frac{1}{2}\left(\int_0^2 t\cos\left(\frac{2\pi}{T}nt\right)dt + \int_2^4 (4-t)\cos\left(\frac{2\pi}{T}nt\right)dt\right)$$

But $T=4$, hence the above simplifies to

$$a_n = \frac{8\sin^2\left(\frac{\pi n}{2}\right)\cos(\pi n)}{\pi^2 n^2}$$

Looking at few terms

$$\begin{aligned} a_n &= 8\left\{\frac{\sin^2\left(\frac{\pi}{2}\right)\cos(\pi)}{\pi^2}, \frac{\sin^2(\pi)\cos(2\pi)}{\pi^2 2^2}, \frac{\sin^2\left(\frac{3\pi}{2}\right)\cos(3\pi)}{\pi^2 3^2}, \frac{\sin^2(2\pi)\cos(4\pi)}{\pi^2 4^2}, \cdots\right\} \\ &= 8\left\{\frac{\cos(\pi)}{\pi^2}, 0, \frac{\cos(3\pi)}{\pi^2 3^2}, 0, \frac{\cos(5\pi)}{\pi^2 5^2}, \cdots\right\} \\ &= 8\left\{\frac{-1}{\pi^2}, 0, \frac{-1}{\pi^2 3^2}, 0, \frac{-1}{\pi^2 5^2}, \cdots\right\} \end{aligned}$$

Hence

$$a_n = \frac{-8}{\pi^2 n^2} \qquad n = 1,3,5,\cdots$$

Therefore

$$\begin{aligned} \tilde{f}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2}\cos\left(\frac{\pi}{2}nt\right) \end{aligned}$$

This is a plot of $\tilde{f}(t)$ showing the approximation as $n$ is increased for few terms. Sum needed to go to only $n=7$ to obtain a very good approximation.

```
tb = Table[
   Plot[{UnitStep[#] f[t - #] & /@ {0, 4, 8}, g[t, n]}, {t, 0, 8},
    PlotRange -> All, AxesLabel -> {"t", Style[Row[{"N=", n}], 16]},
    ImageSize -> 300], {n, 1, 7, 2}];
Grid[Partition[tb, 2]]
```

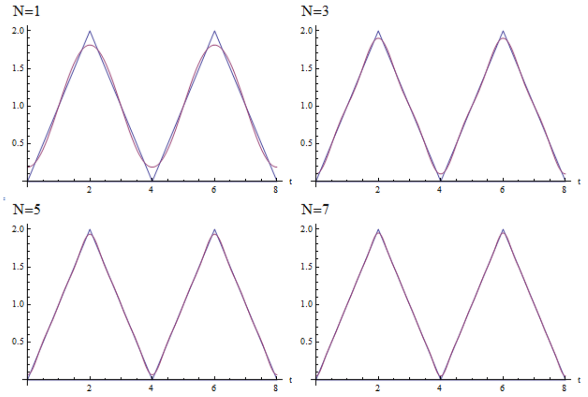


Figure 2.11: Showing Fourier series approximation

### 2.9.3 Problem 2

Find the complex exponential Fourier series of the periodic function whose definition on one period is $f(t)=\cosh(t);-1<t<1$

**Solution**:

The period is $T=2$. Let Fourier series approximation of $f(t)$ be $\tilde{f}(t)$, hence from the definition

$$\tilde{f}(t)=\sum_{n=-\infty}^{\infty}c_n e^{i\frac{2\pi}{T}nt}$$

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{-i\frac{2\pi}{T}nt}dt \qquad n=0,\pm1,\pm2,\cdots$$

Hence

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1}\cosh(t)e^{-i\frac{2\pi}{T}nt}dt\\
&= \frac{1}{2}\int_{-1}^{1}\cosh(t)e^{-i\pi nt}dt\\
&= \frac{1}{2}\int_{-1}^{1}\left(\frac{e^{-t}}{2}+\frac{e^{t}}{2}\right)e^{-i\pi nt}dt\\
&= \frac{1}{4}\left(\int_{-1}^{1}e^{(-i\pi n-1)t}dt+\int_{-1}^{1}e^{(-i\pi n+1)t}dt\right)\\
&= \frac{1}{4}\left(\left[\frac{e^{(-i\pi n-1)t}}{-i\pi n-1}\right]_{-1}^{1}+\left[\frac{e^{(-i\pi n+1)t}}{-i\pi n+1}\right]_{-1}^{1}\right)\\
&= \frac{1}{4}\left(\frac{e^{(-i\pi n-1)}-e^{(i\pi n+1)}}{-i\pi n-1}+\frac{e^{(-i\pi n+1)}-e^{(i\pi n-1)}}{-i\pi n+1}\right)\\
&= \frac{1}{4}\left(\frac{(-i\pi n+1)e^{(-i\pi n-1)}-(-i\pi n+1)e^{(i\pi n+1)}+(-i\pi n-1)e^{(-i\pi n+1)}-(-i\pi n-1)e^{(i\pi n-1)}}{(-i\pi n-1)(-i\pi n+1)}\right)\\
&= \frac{1}{4}\left(\frac{-i\pi ne^{(-i\pi n-1)}+e^{(-i\pi n-1)}+i\pi ne^{(i\pi n+1)}-e^{(i\pi n+1)}-i\pi ne^{(-i\pi n+1)}-e^{(-i\pi n+1)}+i\pi ne^{(i\pi n-1)}+e^{(i\pi n-1)}}{-\pi^2n^2-1}\right)\\
&= \frac{1}{4}\left(\frac{-i\pi ne^{(-i\pi n-1)}+e^{(-i\pi n-1)}+i\pi ne^{(i\pi n+1)}-e^{(i\pi n+1)}-i\pi ne^{(-i\pi n+1)}-e^{(-i\pi n+1)}+i\pi ne^{(i\pi n-1)}+e^{(i\pi n-1)}}{-\pi^2n^2-1}\right)
\end{aligned}$$

Collect terms

$$\begin{aligned}
c_n &= \frac{1}{4}\left(\frac{-i\pi n e e^{-i\pi n} + e^{-1}e^{-i\pi n} + i\pi n e e^{i\pi n} - e e^{i\pi n} - i\pi n e^{-i\pi n} - e e^{-i\pi n} + i\pi n e^{-1} e^{i\pi n} + e^{-1}e^{-i\pi n}}{-\pi^2 n^2 - 1}\right)\\
&= \frac{1}{4}\left(\frac{e^{-1}e^{-i\pi n}(-i\pi n + 1 + i\pi n + 1) + e e^{i\pi n}\left(in\pi - 1 - i\pi n - 1\right)}{-\pi^2 n^2 - 1}\right)\\
&= \frac{1}{4}\left(\frac{2e^{-1}e^{-i\pi n} - 2e e^{i\pi n}}{-\pi^2 n^2 - 1}\right)\\
&= \frac{1}{2}\left(\frac{e^{-1}\left(\cos n\pi - i\sin n\pi\right) - e\left(\cos n\pi + i\sin n\pi\right)}{-\pi^2 n^2 - 1}\right)
\end{aligned}$$

But $n$ is an integer therefore $\sin n\pi = 0$

$$\begin{aligned}
c_n &= \frac{1}{2}\left(\frac{e^{-1}\cos n\pi - e\cos n\pi}{-\pi^2 n^2 - 1}\right)\\
&= \frac{1}{2}\left(\frac{\cos n\pi\left(1 - e^2\right)}{e\left(-\pi^2 n^2 - 1\right)}\right)\\
&= \frac{1}{2e}\left(\frac{\cos n\pi\left(e^2 - 1\right)}{1 + \pi^2 n^2}\right)
\end{aligned}$$

Looking at few terms

$$\begin{aligned}
c_n &= \left\{\overbrace{\frac{1}{2e}\left(\frac{\cos(-\pi)\left(e^2-1\right)}{1+\pi^2}\right)}^{n=-1}, \overbrace{\frac{1}{2e}\left(\frac{\cos(0)\left(e^2-1\right)}{1+\pi^2}\right)}^{n=0}, \overbrace{\frac{1}{2e}\left(\frac{\cos\pi\left(e^2-1\right)}{1+\pi^2}\right)}^{n=1}, \overbrace{\frac{1}{2e}\left(\frac{\cos 2\pi\left(e^2-1\right)}{1+\pi^2 2^2}\right)}^{n=2}, \cdots\right\}\\
&= \left\{\frac{1}{2e}\left(\frac{-1\left(e^2-1\right)}{1+\pi^2}\right), \frac{1}{2e}\left(\frac{\left(e^2-1\right)}{1+\pi^2}\right), \frac{1}{2e}\left(\frac{-1\left(e^2-1\right)}{1+\pi^2}\right), \frac{1}{2e}\left(\frac{\left(e^2-1\right)}{1+\pi^2 2^2}\right), \frac{1}{2e}\left(\frac{-1\left(e^2-1\right)}{1+\pi^2 3^2}\right), \cdots\right\}
\end{aligned}$$

Hence

$$\begin{aligned}
c_n &= (-1)^{|n|}\frac{\left(e^2-1\right)}{2e\left(1+\pi^2 n^2\right)} \qquad n = \pm 1, \pm 2, \cdots\\
&= \frac{\left(e^2-1\right)}{2e} \qquad n = 0
\end{aligned}$$

The first few terms are

$$-0.108118, 0.02903, -0.013083, 0.0073952, \cdots\}$$

Hence

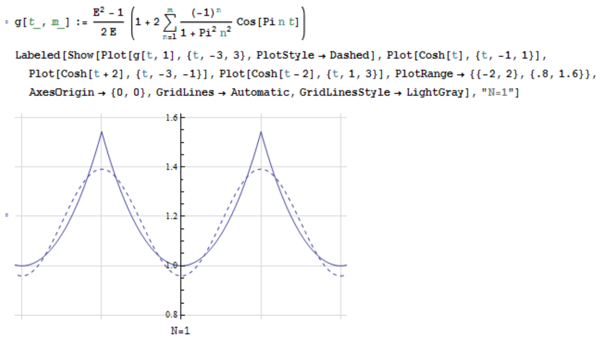
$$\begin{aligned}
\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T}nt}\\
&= \frac{\left(e^2-1\right)}{2e} + \frac{\left(e^2-1\right)}{2e}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{|n|}}{\left(1+\pi^2 n^2\right)}e^{i\pi n t}\\
&= \frac{\left(e^2-1\right)}{2e}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+\pi^2 n^2\right)}\left(e^{i\pi n t} + e^{-i\pi n t}\right)\right)\\
&= \frac{\left(e^2-1\right)}{2e}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+\pi^2 n^2\right)}\left(2\cos\left(\pi n t\right)\right)\right)\\
&= \frac{\left(e^2-1\right)}{2e}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+\pi^2 n^2\right)}\cos\left(\pi n t\right)\right)
\end{aligned}$$

Here is a plot of the function, and its approximation $\tilde{f}(t)$ for few terms. This shows the convergence is rapid and at $N = 4$ it was very close to the original periodic function

### 2.9.4 Problem 3

**Solution**

Since the D.E. is constant coefficients, then the homogeneous solution is found from the



$$g[t\_,\ m\_] := \frac{E^2 - 1}{2\,E}\left(1 + 2\sum_{n=1}^{m}\frac{(-1)^n}{1 + Pi^2\, n^2}\,Cos[Pi\ n\ t]\right)$$

```
Labeled[Show[Plot[g[t, 1], {t, -3, 3}, PlotStyle → Dashed], Plot[Cosh[t], {t, -1, 1}],
  Plot[Cosh[t + 2], {t, -3, -1}], Plot[Cosh[t - 2], {t, 1, 3}], PlotRange → {{-2, 2}, {.8, 1.6}},
  AxesOrigin → {0, 0}, GridLines → Automatic, GridLinesStyle → LightGray], "N=1"]
```

Figure 2.12: Complex Fourier series approximation for $N = 1$

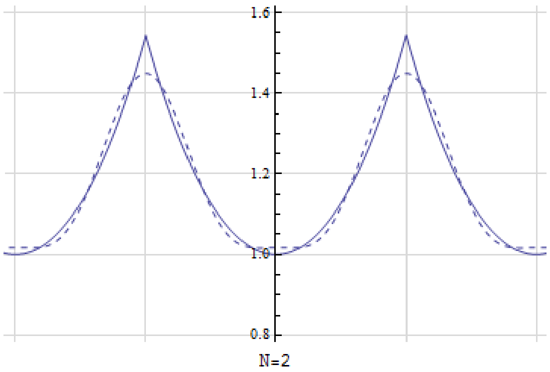


Figure 2.13: Approximation at $N = 2$

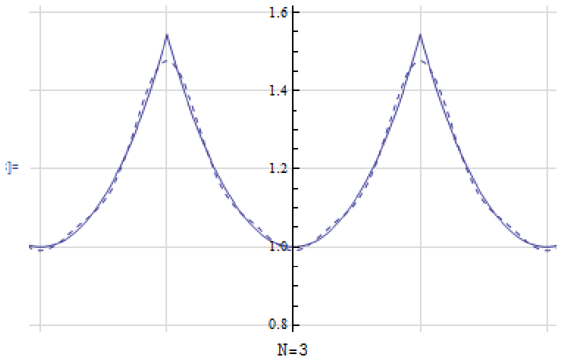


Figure 2.14: Approximation at $N = 3$

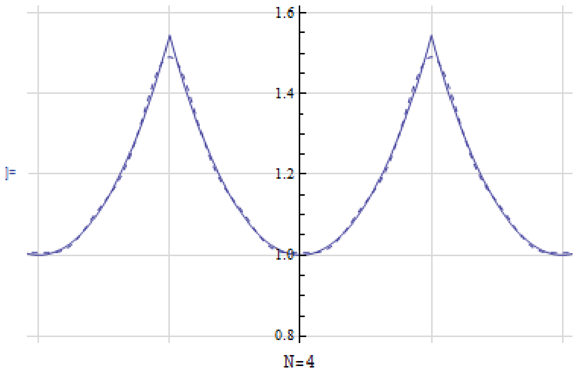
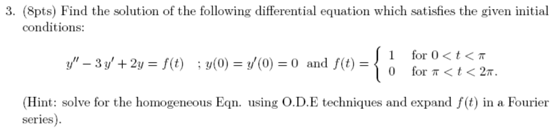


Figure 2.15: Approximation at $N = 4$



3. (8pts) Find the solution of the following differential equation which satisfies the given initial conditions:

$$y'' - 3\,y' + 2y = f(t) \quad ;\ y(0) = y'(0) = 0 \ \text{ and } f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi. \end{cases}$$

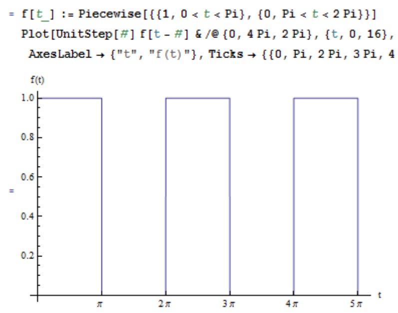
(Hint: solve for the homogeneous Eqn. using O.D.E techniques and expand $f(t)$ in a Fourier series).

Figure 2.16: Problem 3 description

roots of the characteristic equations. The roots are $\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{3\pm\sqrt{9-(4)(2)}}{2} = \frac{3\pm1}{2}$, hence $\lambda_1 = 2, \lambda_2 = 1$ and the homogeneous solution is

$$y_h = Ae^{2t} + Be^t$$

To find the particular solution, $f(t)$ is first expressed in its Fourier series approximation form. A plot of $f(t)$ is (assuming it is periodic) shows the period is $T = 2\pi$ Let Fourier series approximation of $f(t)$ be $\tilde{f}(t)$, hence from the definition



```
f[t_] := Piecewise[{{1, 0 < t < Pi}, {0, Pi < t < 2 Pi}}]
Plot[UnitStep[#] f[t - #] & /@ {0, 4 Pi, 2 Pi}, {t, 0, 16}, PlotRange → All, Frame → False,
 AxesLabel → {"t", "f(t)"}, Ticks → {{0, Pi, 2 Pi, 3 Pi, 4 Pi, 5 Pi, 6 Pi}, Automatic}]
```

Figure 2.17: Showing the periodic forcing function

$$\begin{aligned}
\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T}nt} \\
c_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\frac{2\pi}{T}nt} dt \qquad n = 0, \pm1, \pm2, \cdots
\end{aligned}$$

For $n = 0$

$$\begin{aligned}
c_0 &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, dt \\
&= \frac{1}{2\pi}\int_0^{\pi} dt \\
&= \frac{1}{2}
\end{aligned}$$

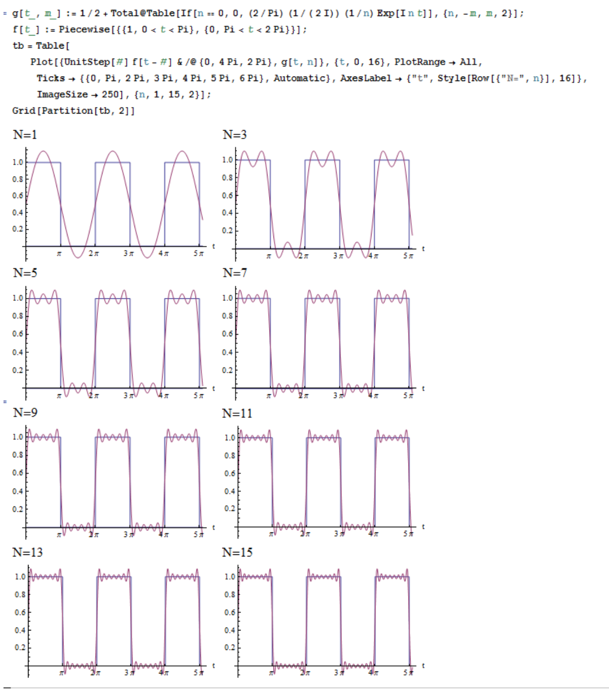
For all other values of $n$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\left(\int_0^{\pi} 1\times e^{-i\frac{2\pi}{T}nt} dt + \int_{\pi}^{2\pi} 0\times e^{-i\frac{2\pi}{T}nt} dt\right) \\
&= \frac{1}{2\pi}\int_0^{\pi} e^{-int} dt \\
&= \frac{1}{2\pi}\left[\frac{e^{-int}}{-in}\right]_0^{\pi} = \frac{i}{2\pi n}\left[e^{-int}\right]_0^{\pi} \\
&= \frac{i}{2\pi n}\left(e^{-in\pi} - 1\right) \\
&= \left(\frac{1}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(1 - e^{-in\pi}\right)
\end{aligned}$$

But $1 - e^{-in\pi} = 1 - (-1)^n$ which is zero for $n$ even and 2 for $n$ odd, Hence

$$\begin{aligned}
\tilde{f}(t) &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq0}}^{\infty} c_n e^{int} \\
&= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int}
\end{aligned}$$

This is a plot of $\tilde{f}(t)$ showing the approximation as $n$ is increased for few terms.



```
g[t_, m_] := 1/2 + Total@Table[If[n == 0, 0, (2/Pi) (1/(2 I)) (1/n) Exp[I n t]], {n, -m, m, 2}];
f[t_] := Piecewise[{{1, 0 < t < Pi}, {0, Pi < t < 2 Pi}}];
tb = Table[
   Plot[{UnitStep[#] f[t - #] & /@ {0, 4 Pi, 2 Pi}, g[t, n]}, {t, 0, 16}, PlotRange → All,
    Ticks → {{0, Pi, 2 Pi, 3 Pi, 4 Pi, 5 Pi, 6 Pi}, Automatic}, AxesLabel → {"t", Style[Row[{"N=", n}], 16]},
    ImageSize → 250], {n, 1, 15, 2}];
Grid[Partition[tb, 2]]
```

Figure 2.18: Showing Fourier series approximation of the periodic forcing function

Now that the forcing function form is found, a particular solution can be guessed. Let

$$\begin{aligned} y_p &= \sum_{n=-\infty}^{\infty} c_n e^{int} \\ y_p' &= \sum_{n=-\infty}^{\infty} i n c_n e^{int} \\ y_p'' &= \sum_{n=-\infty}^{\infty} -n^2 c_n e^{int} \end{aligned}$$

Now $c_n$ needs to be found. The D.E. becomes

$$\begin{aligned} y_p'' - 3y_p' + 2y_p &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq 0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int} \\ \sum_{n=-\infty}^{\infty} -n^2 c_n e^{int} - 3 \sum_{n=-\infty}^{\infty} i n c_n e^{int} + 2 \sum_{n=-\infty}^{\infty} c_n e^{int} &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq 0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int} \\ \sum_{n=-\infty}^{\infty} \left(-n^2 - 3in + 2\right) c_n e^{int} &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq 0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int} \\ 2c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-n^2 - 3in + 2\right) c_n e^{int} &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq 0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int} \end{aligned}$$

Therefore, by comparing

$$c_0 = \frac{1}{4}$$

And then rest of the terms are given by

$$\begin{aligned} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-n^2 - 3in + 2\right) c_n &= \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n\neq 0}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) \\ c_n &= \frac{\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)}{-n^2 - 3in + 2} \qquad n \text{ odd}, n \neq 0 \\ &= \frac{2}{\pi}\left(\frac{1}{2i}\right)\frac{1}{-n^3 - 3in^2 + 2n} \qquad n \text{ odd}, n \neq 0 \end{aligned}$$

Therefore,

$$y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=-\infty \\ n \text{ odd} \\ n \neq 0}}^{\infty} \frac{1}{-n^3 - 3in^2 + 2n} \left( \frac{1}{2i} \right) e^{int}$$

The above is converted to a sum from 1 to infinity. The first step is to break it into 2 sums

$$y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=-\infty \\ odd}}^{-1} \frac{1}{-n^3 - 3in^2 + 2n} \left( \frac{1}{2i} \right) e^{int} + \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{1}{-n^3 - 3in^2 + 2n} \left( \frac{1}{2i} \right) e^{int}$$

To take care of the *odd* part of the sum, for the $\sum_{n=-\infty}^{-1}$ part, let $n = 2m+1$ and the $\sum_{n=1}^{\infty}$ part of the sum let $n = 2m - 1$. The above becomes

$$\begin{aligned} y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{m=-\infty}^{-1} \frac{1}{-(2m+1)^3 - 3i(2m+1)^2 + 2(2m+1)} \left( \frac{1}{2i} \right) e^{i(2m+1)t} \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{-(2m-1)^3 - 3i(2m-1)^2 + 2(2m-1)} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

$\sum_{m=-\infty}^{-1}$ is changed to $\sum_{m=\infty}^{1}$ by replacing $m$ with $-m$ in the first sum above, giving

$$\begin{aligned} y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{m=\infty}^{1} \frac{1}{-(2(-m)+1)^3 - 3i(2(-m)+1)^2 + 2(2(-m)+1)} \left( \frac{1}{2i} \right) e^{i(2(-m)+1)t} \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{-(2m-1)^3 - 3i(2m-1)^2 + 2(2m-1)} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

Simplifying by extracting a minus from the first sum to make all terms the same as the second sum

$$\begin{aligned} y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{-(-(2m-1))^3 - 3i(-(2m-1))^2 + 2(-(2m-1))} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{-(2m-1)^3 - 3i(2m-1)^2 + 2(2m-1)} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

Or

$$\begin{aligned} y_p = \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3 - 3i(2m-1)^2 - 2(2m-1)} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{-(2m-1)^3 - 3i(2m-1)^2 + 2(2m-1)} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

To simplify this more, let $(2m-1)^3 = A, 3(2m-1)^2 = B, 2(2m-1) = C$, hence the above becomes

$$\begin{aligned} y_p &= \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{A - Bi - C} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} + \frac{1}{-A - Bi + C} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \\ &= \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{(A-C) - Bi} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} + \frac{1}{-(A-C) - Bi} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \\ &= \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{(A-C) - Bi} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} - \frac{1}{(A-C) + Bi} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

Let $A - C = D$ then

$$\begin{aligned} y_p &= \frac{1}{4} + \frac{2}{\pi} \left( \sum_{m=1}^{\infty} \frac{1}{D - Bi} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} - \frac{1}{D + Bi} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \right) \\ &= \frac{1}{4} + \frac{2}{\pi} \left( \sum_{m=1}^{\infty} \frac{1}{D - Bi} \frac{D + Bi}{D + Bi} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} - \frac{1}{D + Bi} \frac{D - Bi}{D - Bi} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \right) \\ &= \frac{1}{4} + \frac{2}{\pi} \left( \sum_{m=1}^{\infty} \frac{D + Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} - \frac{D - Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \right) \\ &= \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{D + Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{D - Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \\ &= \frac{1}{4} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{D}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{-i(2m-1)t} \\ &- \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{D}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{i(2m-1)t} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{Bi}{B^2 + D^2} \left( \frac{1}{2i} \right) e^{i(2m-1)t} \end{aligned}$$

Now common terms can be combined

$$y_p = \frac{1}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{D}{B^2+D^2}\left(\frac{1}{2i}\right)\left(e^{-i(2m-1)t} - e^{i(2m-1)t}\right)$$
$$+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{Bi}{B^2+D^2}\left(\frac{1}{2i}\right)\left(e^{-i(2m-1)t} + e^{i(2m-1)t}\right)$$

Hence

$$\begin{aligned}y_p &= \frac{1}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{-D}{B^2+D^2}\sin(nt) + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{B}{B^2+D^2}\cos(nt)\\ &= \frac{1}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{B}{B^2+D^2}\cos(nt) - \frac{D}{B^2+D^2}\sin(nt)\end{aligned}$$

Replacing back values for $B$ and $D$. Since $D = A-C = (2n-1)^3 - 2(2n-1) = 8n^3 - 12n^2 + 2n + 1$ and $B = 3(2n-1)^2$, therefore

$$y_p = \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3(2n-1)^2}{9(2n-1)^4 + \left((2n-1)^3 - 2(2n-1)\right)^2}\cos(nt)$$
$$-\frac{(2n-1)^3 - 2(2n-1)}{9(2n-1)^4 + \left((2n-1)^3 - 2(2n-1)\right)^2}\sin(nt)$$

or

$$y_p = \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}\cos(nt)$$
$$+\frac{-4n^2 + 4n + 1}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}\sin(nt)$$

And the solution is

$$y = Ae^{2t} + Be^{t} + y_p$$

When $t = 0; y = 0$ hence

$$0 = A + B + \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10} \tag{1}$$

Taking derivative

$$y' = 2Ae^{2t} + Be^{t} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-3n\sin(nt)}{16n^4 - 32n^3 + 44n^2 - 28n + 10}$$
$$+\frac{\left(-4n^2 + 4n + 1\right)n\cos(nt)}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}$$

When $t = 0, y' = 0$ hence

$$0 = 2A + B + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(-4n^2 + 4n + 1\right)n}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10} \tag{2}$$

Subtract Eq. (1) from Eq. (2) gives

$$0 = A + -\frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(-4n^2 + 4n + 1\right)n}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}$$
$$-\frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}$$
$$\begin{aligned}A &= \frac{1}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(-4n^2 + 4n + 1\right)n}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10} - \frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}\\ &= \frac{1}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-4n^3 + 4n^2 - 5n + 3}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}\end{aligned}$$

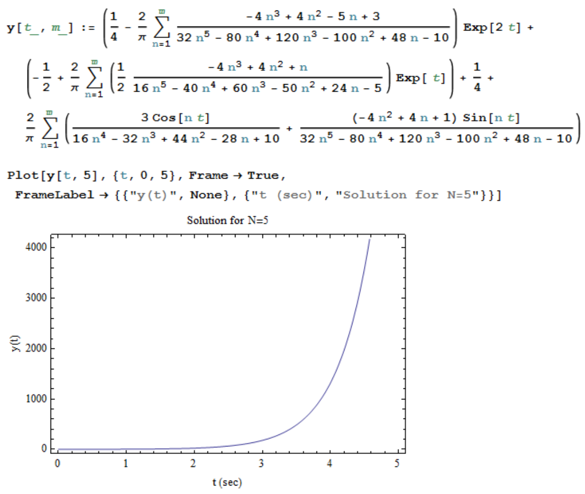
And from Eq. (1), $B$ can now be found

$$
\begin{aligned}
B &= -A - \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}\\
&= -\frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-4n^3 + 4n^2 - 5n + 3}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}\\
&- \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}\\
B &= -\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-4n^3 + 4n^2 - 5n + 3}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10} + \frac{3}{16n^4 - 32n^3 + 44n^2 - 28n + 10}\\
&= -\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2}n\frac{-4n^2 + 4n + 1}{16n^5 - 40n^4 + 60n^3 - 50n^2 + 24n - 5}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= Ae^{2t} + Be^{t} + y_p t\\
&= \left(\frac{1}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-4n^3 + 4n^2 - 5n + 3}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}\right)e^{2t}\\
&+ \left(-\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2}\frac{-4n^3 + 4n^2 + n}{16n^5 - 40n^4 + 60n^3 - 50n^2 + 24n - 5}\right)e^{t}\\
&+ \frac{1}{4}\\
&+ \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{3\cos(nt)}{16n^4 - 32n^3 + 44n^2 - 28n + 10} + \frac{\left(-4n^2 + 4n + 1\right)\sin(nt)}{32n^5 - 80n^4 + 120n^3 - 100n^2 + 48n - 10}\right)
\end{aligned}
$$

This is a plot of the solution for $n = 4$. Notice that the system as given is unstable. This is because the damping is negative. Hence the solution below blows up. I think the ODE should have been $y'' + 3y + 2y = f(t)$ and not $y'' - 3y + 2y = f(t)$. May be a typo. But here is the solution plot



$$
y[t\_, m\_] := \left(\frac{1}{4} - \frac{2}{\pi}\sum_{n=1}^{m}\frac{-4 n^3 + 4 n^2 - 5 n + 3}{32 n^5 - 80 n^4 + 120 n^3 - 100 n^2 + 48 n - 10}\right)\text{Exp}[2 t] + \left(-\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{m}\left(\frac{1}{2}\frac{-4 n^3 + 4 n^2 + n}{16 n^5 - 40 n^4 + 60 n^3 - 50 n^2 + 24 n - 5}\right)\text{Exp}[t]\right) + \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{m}\left(\frac{3\,\text{Cos}[n t]}{16 n^4 - 32 n^3 + 44 n^2 - 28 n + 10} + \frac{(-4 n^2 + 4 n + 1)\,\text{Sin}[n t]}{32 n^5 - 80 n^4 + 120 n^3 - 100 n^2 + 48 n - 10}\right)
$$

```
Plot[y[t, 5], {t, 0, 5}, Frame → True,
 FrameLabel → {{"y(t)", None}, {"t (sec)", "Solution for N=5"}}]
```

Figure 2.19: Solution showing the system is unstable, for $N = 5$ terms.

### 2.9.4.1 Appendix for problem 3

The Fourier series approximation using the classical definition can be obtained as follows

$$
\tilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right) + b_n \sin\left(\frac{2\pi}{T}nt\right)
$$

Where

$$
\begin{aligned}
a_0 &= \frac{1}{T/2}\int_0^T f(t)\,dt = \frac{1}{\pi}\left(\int_0^{\pi} dt + \int_{\pi}^{2\pi} 0dt\right)\\
&= 1
\end{aligned}
$$

And

$$
a_n = \frac{1}{\pi}\int_0^{\pi}\cos\left(\frac{2\pi}{T}nt\right)dt = \frac{1}{\pi}\int_0^{\pi}\cos(nt)\,dt = \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_0^{\pi} = \frac{1}{n\pi}\sin n\pi = 0
$$

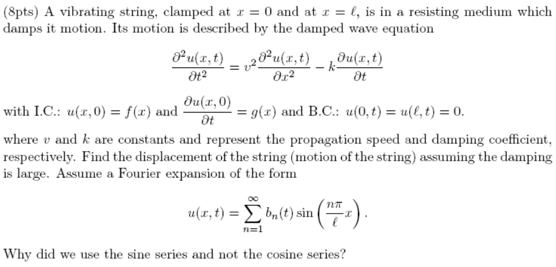
And

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin\left(\frac{2\pi}{T}nt\right)dt = \frac{1}{\pi}\int_0^{\pi} \sin(nt)\,dt = \frac{1}{\pi}\left[\frac{-\cos nt}{n}\right]_0^{\pi} = \frac{-1}{n\pi}(\cos n\pi - 1)$$
$$= \frac{-1}{n\pi}\left((-1)^n - 1\right)$$
$$= \frac{1+(-1)^{n+1}}{n\pi}$$

Therefore

$$\tilde{f}(t) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T}nt\right)$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n\pi}\sin(nt)$$

### 2.9.5 Problem 4



4. (8pts) A vibrating string, clamped at $x = 0$ and at $x = \ell$, is in a resisting medium which damps it motion. Its motion is described by the damped wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial t}$$

with I.C.: $u(x,0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t) = u(\ell,t) = 0$.

where $v$ and $k$ are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a Fourier expansion of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{\ell}x\right).$$

Why did we use the sine series and not the cosine series?

Figure 2.20: problem 4 description

**Solution:**

The reason to select sin series for the spatial solution is to satisfy boundary conditions. We are told the solution at $t > 0$ must be zero at $x = 0$ and at $x = L$. This is satisfied by $\sin\left(\frac{n\pi}{L}x\right)$. If we had a $\cos\left(\frac{n\pi}{L}x\right)$ term in the spatial solution then that will not vanish at $x = 0$ nor at $x = L$.

In this problem, we need to find $b_n(t)$, which is the temporal part of the general solution since the spatial solution is provided and it already satisfies the boundary conditions. Hence we just need to determine the form of $b_n(t)$ which will satisfy the two initial conditions given. $b_n(t)$ will contain two constants to be determined from these initial conditions, and this will complete the solution.

To find the general temporal solution, we can start using the standard method of separation of variables. This will lead to second order differential equation to solve for in order to find $b_n(t)$. But with the eigenvalues determined from the spatial solution. This is standard separation of method approach.

Let

$$u(x,t) = T(t)X(x)$$

where $T(t)$ is a function that depends on time only and $X(x)$ is a function that depends on $x$ only. We know what $X(x)$ in this problem. We are given this as $\sin\left(\frac{n\pi}{L}x\right)$. However, we will continue the separation of variables approach in order to find $T(t)$ and assign this to $b_n(t)$ and in order to find the eigenvalues.

Since we assumed $u(x,t) = T(t)X(x)$, we will now take derivatives and substitute all these back into the PDE.

$$\frac{\partial u}{\partial t} = T'X$$
$$\frac{\partial^2 u}{\partial t^2} = T''X$$
$$\frac{\partial u}{\partial x} = TX'$$
$$\frac{\partial^2 u}{\partial x^2} = TX''$$

Therefore the PDE becomes

$$T''X = v^2 TX'' - kT'X$$

Dividing by $TX$

$$\frac{T''X}{TX} = v^2\frac{TX''}{TX} - k\frac{T'X}{TX}$$
$$\frac{T''}{T} + k\frac{T'}{T} = v^2\frac{X''}{X}$$

We now follow the standard argument of separation of variables and say that the LHS is a function of time only and this is equal to the RHS which is a function of $x$ only. Therefore, for this to hold, both sides must be equal to some constant, and we choose this constant to be negative, say $-\omega^2$. (A positive or zero eigenvalues will not lead to real solutions). Therefore, we end up with two differential equations

$$\frac{T''}{T} + k\frac{T'}{T} = -\omega^2$$
$$v^2\frac{X''}{X} = -\omega^2$$

We always start by the spatial differential equation, which leads to $X'' + \frac{\omega^2}{v^2}X = 0$. Since the stiffness term $\frac{\omega^2}{v^2}$is positive, then the solution is sinusoidal and stable, hence

$$X = A\cos\left(\frac{\omega}{v}x\right) + B\sin\left(\frac{\omega}{v}x\right)$$

This will lead to the solution given in the problem. But to verify, let us find $A, B$. When $x = 0$, $X(0) = 0$, hence $0 = A$. Hence $X(x) = B\sin\left(\frac{\omega}{v}x\right)$. When $x = L$, $X(L) = 0$, therefore $0 = B\sin\left(\frac{\omega}{v}L\right)$. Since we can't have $B = 0$ else there will be no solution left to use, we force $\sin\left(\frac{\omega}{v}L\right) = 0$ which means $\frac{\omega}{v}L = n\pi$ for integer $n$ and for any $B \neq 0$. Therefore,

$$\omega_n = v\frac{n\pi}{L} \qquad n = 1, 2, \cdots$$

(We do not need the constant $B$ any more at this stage, as we can choose $B = 1$ here. The constants will come from the temporal part of the solution). The spatial solution therefore

$$X(x) = \sin\left(\frac{n\pi}{L}x\right)$$

Which is the solution given.

Now we are ready to find the general solution for $T(t)$ since we now know the eigenvalues $\omega_n$. We go back to the ODE for $T(t)$ and solve it. Here it is again

$$\frac{T''}{T} + k\frac{T'}{T} = -\omega_n^2$$
$$T'' + kT' + \omega_n^2 T = 0$$

This is a standard second order ODE that represents a damped system with stiffness term. The solution depends on the damping. For underdamped system, the solution would look like

$$T_n(t) = e^{\frac{-k}{2}t}\left(A_n\cos(\omega_d t) + B_n\sin(\omega_d t)\right)$$

Where $\omega_d$ is the damped natural frequency (this is due to the roots of the characteristic equation being complex conjugates). But here we are told that the damping is large. This must mean it is overdamped, and hence the roots of the characteristic equation are both real and distinct (The solution would have been easier if it is underdamped). Therefore, we can write down the solution as [7]

$$\begin{aligned}T_n(t) &= A_n e^{\lambda_1 t} + B_n e^{\lambda_2 t}\\ &= A_n e^{\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} + B_n e^{\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t}\end{aligned}$$

Where $\omega_n = \frac{vn\pi}{L}$. These fundamental solutions can now be added to obtain the general solution

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} T_n X_n \qquad (1)\\ &= \sum_{n=1}^{\infty}\left\{A_n e^{\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} + B_n e^{\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t}\right\}\sin\left(\frac{n\pi}{L}x\right)\end{aligned}$$

We see now that

$$b_n(t) = A_n e^{\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} + B_n e^{\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t}$$

We are now ready to find $A_n, B_n$ from initial conditions. When $t=0$, $u(x,0)=f(x)$, hence from Eq. (1) we obtain

$$f(x) = \sum_{n=1}^{\infty}(A_n+B_n)\sin\left(\frac{n\pi}{L}x\right) \qquad 0\leq x\leq L$$

To find $A_n$ we need to apply the orthogonality relation by multiplying each side by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating and using the fact that

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases}0 & m\neq n\\ \frac{1}{2}L & m=n\end{cases}$$

Therefore

$$\begin{aligned}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \sum_{n=1}^{\infty}(A_n+B_n)\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\\ \int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \frac{L}{2}(A_m+B_m)\end{aligned}$$

Hence

$$A_n + B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad (2)$$

Now using the second initial conditions. First, we need to take time derivative of the general solution, which results in

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left[A_n\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)e^{\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} + B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)e^{\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t}\right]\sin\left(\frac{n\pi}{L}x\right)$$

At $t=0$ the above becomes

$$g(x) = \sum_{n=1}^{\infty}\left[A_n\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right) + B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

---

[7] This can be seen by finding the roots of the characteristic equation which are

$$\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-k\pm\sqrt{k^2-4\omega_n^2}}{2}$$

But $\omega_n^2 = \left(\frac{vn\pi}{L}\right)^2$. Hence we can write the above as

$$\lambda = -\frac{k}{2}\pm\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}$$

Hence the roots are both real.

To find $B_n$ we need to apply the orthogonality relation by multiplying each side by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating. Hence

$$\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty}\left[A_n\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)+B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\right]$$
$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)$$
$$\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\left[A_n\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)+B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\right]$$

Hence

$$A_n\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)+B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \tag{2}$$

We now have two equations to solve for $A_n, B_n$. From Eq (1),

$$A_n = -B_m + \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

Substituting this in Eq. (2) gives

$$\left(-B_n+\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)$$
$$+B_n\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Solving for $B_n$ gives

$$B_n\left(\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)$$
$$+\frac{2}{L}\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Hence

$$B_n=\frac{\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx-\frac{2}{L}\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{2\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}}$$

Now that $B_n$ is found, we can find $A_n$ from

$$A_n=-B_n+\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

This complete the solution, all terms are now found. The solution is

$$u(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

Where

$$
\begin{aligned}
b_n(t) &= A_n e^{\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} + B_n e^{\left(-\frac{k}{2}+\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)t} \\
A_n &= -B_n + \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
B_n &= \frac{\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx - \frac{2}{L}\left(-\frac{k}{2}-\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}\right)\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{2\sqrt{\left(\frac{k}{2}\right)^2-\omega_n^2}} \\
\omega_n &= \left(\frac{vn\pi}{L}\right)^2
\end{aligned}
$$

### 2.9.5.1 Appendix for problem 4

In the above solution, system was assumed to be overdamped. This was done because the problem said the damping is large. It was not clear if this meant overdamped or not. Here is a solution assuming underdamped, which means the solution $T_n(t)$ will have complex roots. Assuming underdamped, the time solution is

$$T_n(t) = e^{\frac{-k}{2}t}\left(A_n\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right) + B_n\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)$$

Where $\omega_n = \frac{vn\pi}{L}$. These fundamental solutions can now be added to obtain the general solution

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} T_n X_n \qquad (1)\\
&= \sum_{n=1}^{\infty} e^{\frac{-k}{2}t}\left(A_n\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right) + B_n\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)\sin\left(\frac{n\pi}{L}x\right)
\end{aligned}
$$

We see now that

$$b_n(t) = e^{\frac{-k}{2}t}\left(A_n\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right) + B_n\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)$$

We are now ready to find $A_n, B_n$ from initial conditions. When $t=0$, $u(x,0)=f(x)$, hence from Eq. (1) we obtain

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \qquad 0 \le x \le L$$

To find $A_n$ we need to apply the orthogonality relation by multiplying each side by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating and using the fact that

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2}L & m = n \end{cases}$$

Therefore

$$
\begin{aligned}
\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) \\
\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \frac{L}{2}A_m
\end{aligned}
$$

Hence

$$A_n = \int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

We are not given the form of $f(x)$ so we determine the exact value of $A_n$. Now we will try to find $B_n$ using the second initial conditions. First, we need to take time derivative of the general solution, which results in

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{n=1}^{\infty} -\frac{k}{2}e^{\frac{-k}{2}t}\left(A_n\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)+B_n\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)\sin\left(\frac{n\pi}{L}x\right)\\
&\quad+\sum_{n=1}^{\infty} e^{\frac{-k}{2}t}\left(-A_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)+B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)\\
&\qquad\sin\left(\frac{n\pi}{L}x\right)
\end{aligned}
$$

At $t=0$ the above becomes

$$g(x)=\sum_{n=1}^{\infty}\left(-\frac{k}{2}A_n+B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\right)\sin\left(\frac{n\pi}{L}x\right)$$

To find $B_n$ we need to apply the orthogonality relation by multiplying each side by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating. Hence

$$
\begin{aligned}
\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \sum_{n=1}^{\infty}\left(-\frac{k}{2}A_n+B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\right)\int_0^L\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\\
\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \frac{L}{2}\left(-\frac{k}{2}A_n+B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\right)\\
-\frac{k}{2}A_n+B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2} &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx\\
B_n\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2} &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{k}{2}A_n\\
B_n &= \frac{\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{k}{2}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}}
\end{aligned}
$$

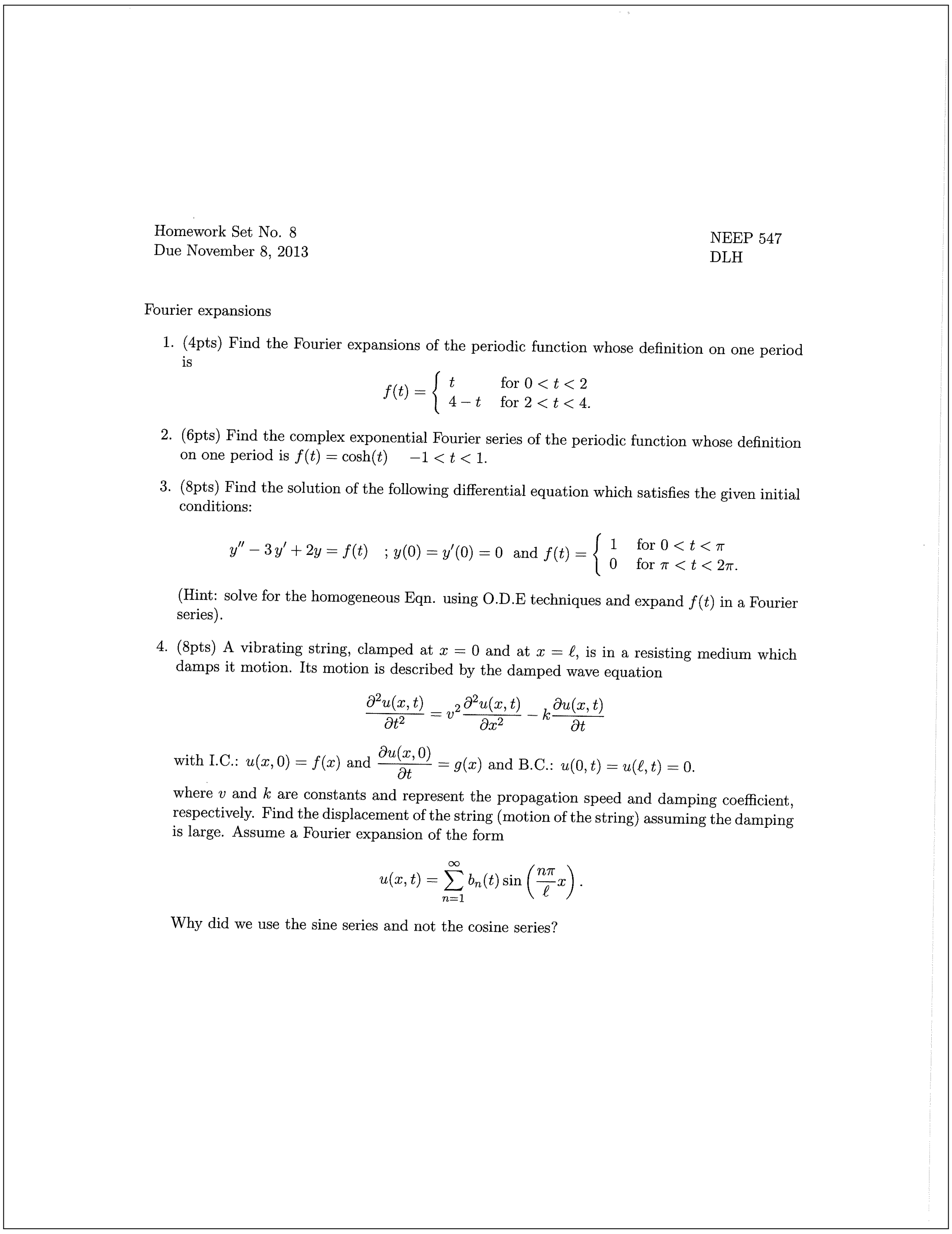
This complete the solution. The solution is

$$u(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

Where

$$
\begin{aligned}
b_n(t) &= e^{\frac{-k}{2}t}\left(A_n\cos\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)+B_n\sin\left(\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}\,t\right)\right)\\
A_n &= \int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\\
B_n &= \frac{\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{k}{2}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\sqrt{\omega_n^2-\left(\frac{k}{2}\right)^2}}\\
\omega_n &= \left(\frac{vn\pi}{L}\right)^2
\end{aligned}
$$

### 2.9.6 key solution



Homework Set No. 8
Due November 8, 2013

NEEP 547
DLH

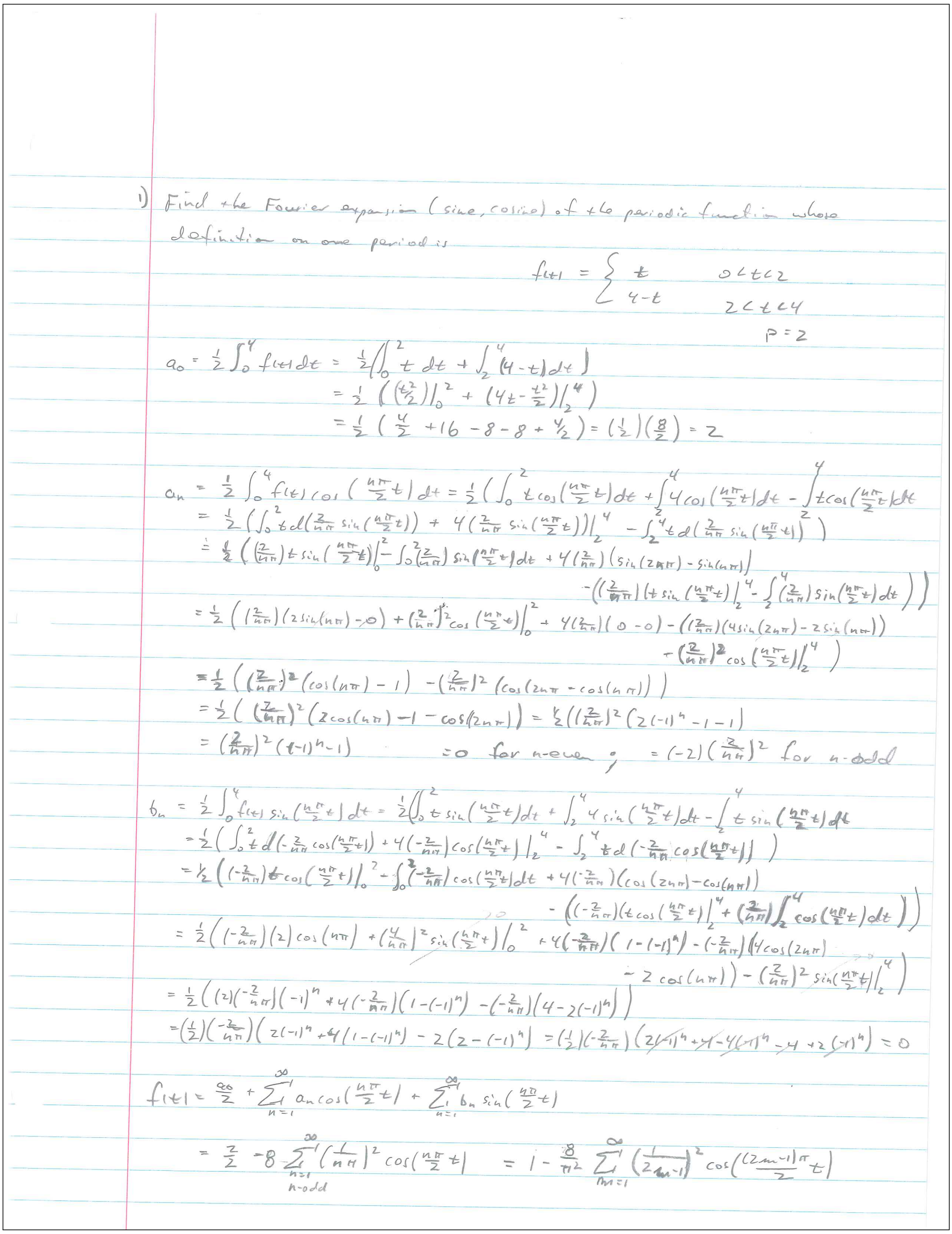
Fourier expansions

1. (4pts) Find the Fourier expansions of the periodic function whose definition on one period is
$$f(t) = \begin{cases} t & \text{for } 0 < t < 2 \\ 4 - t & \text{for } 2 < t < 4. \end{cases}$$

2. (6pts) Find the complex exponential Fourier series of the periodic function whose definition on one period is $f(t) = \cosh(t) \quad -1 < t < 1$.

3. (8pts) Find the solution of the following differential equation which satisfies the given initial conditions:
$$y'' - 3y' + 2y = f(t) \quad ; y(0) = y'(0) = 0 \text{ and } f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi. \end{cases}$$
(Hint: solve for the homogeneous Eqn. using O.D.E techniques and expand $f(t)$ in a Fourier series).

4. (8pts) A vibrating string, clamped at $x = 0$ and at $x = \ell$, is in a resisting medium which damps it motion. Its motion is described by the damped wave equation
$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial t}$$
with I.C.: $u(x,0) = f(x)$ and $\dfrac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t) = u(\ell,t) = 0$.
where $v$ and $k$ are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a Fourier expansion of the form
$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{\ell} x\right).$$
Why did we use the sine series and not the cosine series?



1) Find the Fourier expansion (sine, cosine) of the periodic function whose definition on one period is

$$f(t) = \begin{cases} t & 0<t<2 \\ 4-t & 2<t<4 \end{cases}$$

$p=2$

$$a_0 = \frac{1}{2}\int_0^4 f(t)\,dt = \frac{1}{2}\left(\int_0^2 t\,dt + \int_2^4 (4-t)\,dt\right)$$
$$= \frac{1}{2}\left(\left(\frac{t^2}{2}\right)\Big|_0^2 + \left(4t-\frac{t^2}{2}\right)\Big|_2^4\right)$$
$$= \frac{1}{2}\left(\frac{4}{2}+16-8-8+\frac{4}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{8}{2}\right) = 2$$

$$a_n = \frac{1}{2}\int_0^4 f(t)\cos\left(\frac{n\pi}{2}t\right)dt = \frac{1}{2}\left(\int_0^2 t\cos\left(\frac{n\pi}{2}t\right)dt + \int_2^4 4\cos\left(\frac{n\pi}{2}t\right)dt - \int_2^4 t\cos\left(\frac{n\pi}{2}t\right)dt\right.$$
$$= \frac{1}{2}\left(\int_0^2 t\,d\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right) + 4\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 - \int_2^4 t\,d\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right)\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)t\sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 - \int_0^2\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}t\right)dt + 4\left(\frac{2}{n\pi}\right)(\sin(2n\pi)-\sin(n\pi))\right.$$
$$\left. - \left(\left(\frac{2}{n\pi}\right)\left(t\sin\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 - \int_2^4\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}t\right)dt\right)\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)(2\sin(n\pi)-0) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right)\Big|_0^2 + 4\left(\frac{2}{n\pi}\right)(0-0) - \left(\left(\frac{2}{n\pi}\right)(4\sin(2n\pi)-2\sin(n\pi)\right)\right.$$
$$\left. - \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right)\Big|_2^4\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(\cos(n\pi)-1) - \left(\frac{2}{n\pi}\right)^2(\cos(2n\pi-\cos(n\pi))\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(2\cos(n\pi)-1-\cos(2n\pi))\right) = \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(2(-1)^n-1-1)\right.$$
$$= \left(\frac{2}{n\pi}\right)^2((-1)^n-1) \qquad = 0 \text{ for } n\text{-even}; \qquad = (-2)\left(\frac{2}{n\pi}\right)^2 \text{ for } n\text{-odd}$$

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\left(\frac{n\pi}{2}t\right)dt = \frac{1}{2}\left(\int_0^2 t\sin\left(\frac{n\pi}{2}t\right)dt + \int_2^4 4\sin\left(\frac{n\pi}{2}t\right)dt - \int_2^4 t\sin\left(\frac{n\pi}{2}t\right)dt\right.$$
$$= \frac{1}{2}\left(\int_0^2 t\,d\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right) + 4\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}t\right)\Big|_2^4 - \int_2^4 t\,d\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{2}{n\pi}\right)t\cos\left(\frac{n\pi}{2}t\right)\Big|_0^2 - \int_0^2\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}t\right)dt + 4\left(-\frac{2}{n\pi}\right)(\cos(2n\pi)-\cos(n\pi))\right.$$
$$\left. - \left(\left(-\frac{2}{n\pi}\right)\left(t\cos\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 + \left(\frac{2}{n\pi}\right)\int_2^4\cos\left(\frac{n\pi}{2}t\right)dt\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{2}{n\pi}\right)(2)\cos(n\pi) + \cancelto{0}{\left(\frac{4}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}t\right)\Big|_0^2} + 4\left(-\frac{2}{n\pi}\right)(1-(-1)^n) - \left(-\frac{2}{n\pi}\right)\left(4\cos(2n\pi)\right.\right.$$
$$\left.\left. - 2\cos(n\pi)\right) - \cancelto{0}{\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}t\right)\Big|_2^4}\right)$$
$$= \frac{1}{2}\left((2)\left(-\frac{2}{n\pi}\right)(-1)^n + 4\left(-\frac{2}{n\pi}\right)(1-(-1)^n) - \left(-\frac{2}{n\pi}\right)(4-2(-1)^n)\right)$$
$$= \left(\frac{1}{2}\right)\left(-\frac{2}{n\pi}\right)\left(2(-1)^n + 4(1-(-1)^n) - 2(2-(-1)^n)\right) = \left(\frac{1}{2}\right)\left(-\frac{2}{n\pi}\right)\left(\cancel{2(-1)^n} + \cancel{4} - \cancel{4(-1)^n} - \cancel{4} + \cancel{2(-1)^n}\right) = 0$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{2}t\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{2}t\right)$$

$$= \frac{2}{2} - 8\sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}\left(\frac{1}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right) \qquad = 1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\left(\frac{1}{2m-1}\right)^2\cos\left(\frac{(2m-1)\pi}{2}t\right)$$

2\. Find the complex exponential Fourier series of the periodic function whose definition over one period is

$$f(t) = \cosh(t) \qquad -1 < t < 1 \qquad p = 1$$

$$f(t) = \sum_{n=-\infty}^{\infty}{}' C_n e^{\frac{n i \pi t}{p}} \qquad \text{where } C_n = \tfrac{1}{2}\int_{-1}^{1} f(t) e^{-n i \pi t/p}$$

$$C_n = \tfrac{1}{2}\int_{-1}^{1} \cosh(t) e^{-in\pi t}\, dt = \tfrac{1}{2}\left(-\tfrac{1}{in\pi}\right)\cosh(t) e^{-in\pi t}\Big|_{-1}^{1} + \tfrac{1}{2}\left(\tfrac{1}{in\pi}\right)\int_{-1}^{1} \sinh(t) e^{-in\pi t} dt$$

$$= \left(\tfrac{1}{2}\right)\left(\frac{-\cosh(1)}{in\pi}\right)\left(e^{-in\pi} - e^{in\pi}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)\left(\left(\tfrac{-1}{in\pi}\right)\sinh(t) e^{-in\pi t}\Big|_{-1}^{1}\right) + \left(\tfrac{1}{in\pi}\right)\int_{-1}^{1} \cosh(t) e^{-in\pi t} dt\Big)\Big)$$

$$= \left(\tfrac{1}{2}\right)\left(\frac{\cosh(1)}{in\pi}\right)\left(e^{in\pi} - e^{-in\pi}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2\left(\sinh(1)\left(e^{in\pi} + e^{-in\pi}\right)\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2 \int_{-1}^{1} \cosh(t) e^{-in\pi t} dt\Big)$$

$$\therefore \left(1 - \left(\tfrac{1}{in\pi}\right)^2\right)\left(\tfrac{1}{2}\right)\int_{-1}^{1} \cosh(t) e^{-in\pi t} dt = \left(\tfrac{1}{2}\right)\left(\frac{\cosh(1)}{in\pi}\right)\left(e^{in\pi} - e^{-in\pi}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2\left(\sinh(1)\left(e^{in\pi} + e^{-in\pi}\right)\right)$$

$$\text{thus } C_n = \tfrac{1}{2}\int_{-1}^{1} \cosh(t) e^{-in\pi t} dt = \left(\frac{(in\pi)^2}{(in\pi)^2 - 1}\right)\left(\frac{\cosh(1)}{n\pi}\underbrace{\left(\frac{e^{in\pi} - e^{-in\pi}}{2i}\right)}_{\sin(n\pi)} + \left(\tfrac{1}{n\pi}\right)^2(\sinh(1))\underbrace{\left(\frac{e^{in\pi} + e^{-in\pi}}{2}\right)}_{\cos(n\pi)}\right)$$

$$C_n = \left(\frac{(in\pi)^2}{(in\pi)^2 - 1}\right)\left(\frac{\cosh(1)}{n\pi}\cancelto{0}{\sin(n\pi)} + \left(\tfrac{1}{n\pi}\right)^2(\sinh(1))\underbrace{\cos(n\pi)}_{(-1)^n}\right)$$

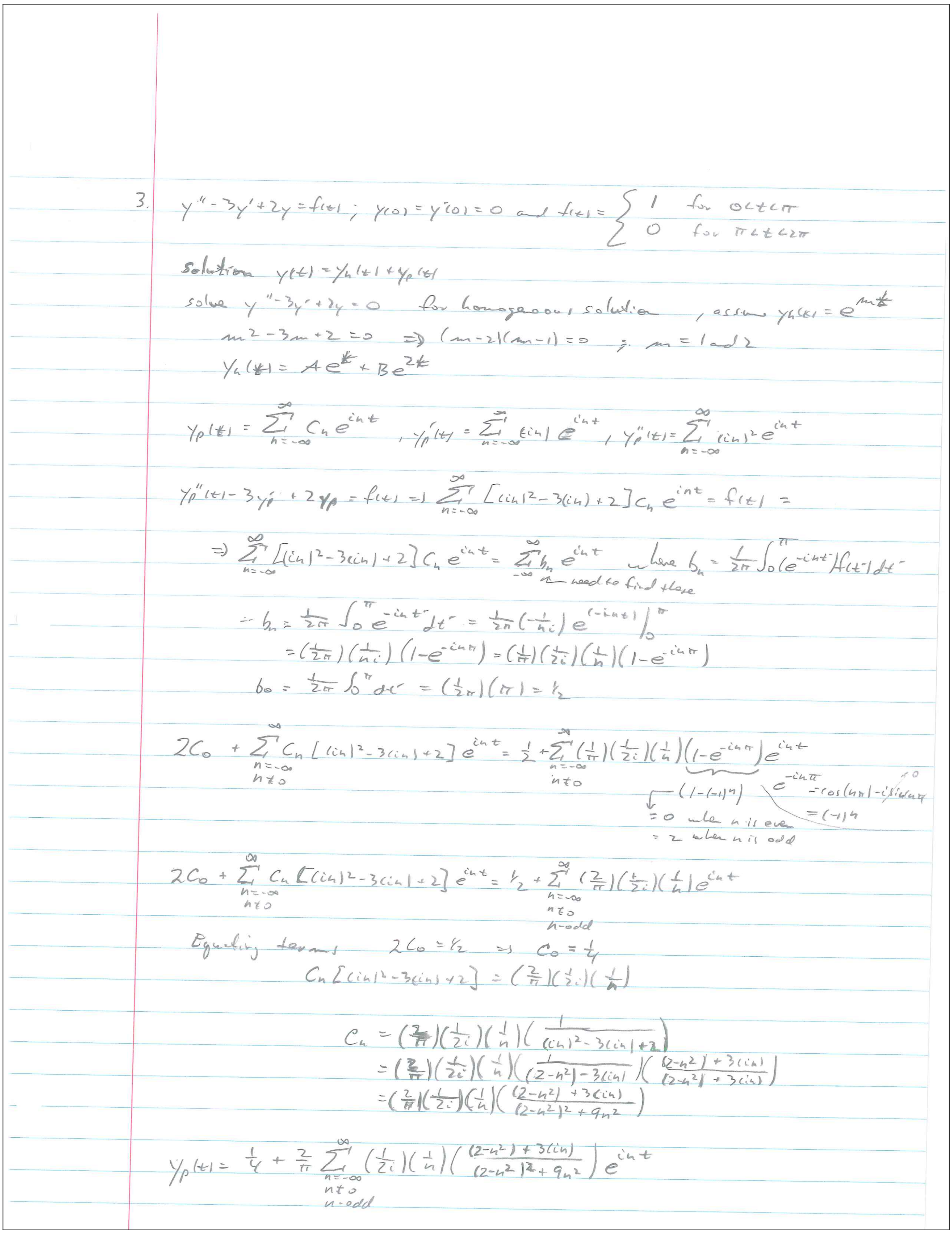
$$= \left(\frac{(in\pi)^2}{-(n\pi)^2 - 1}\right)\left(\left(\tfrac{1}{n\pi}\right)^2 \sinh(1)\,(-1)^n\right) = \left(\frac{(n\pi)^2}{(n\pi)^2 + 1}\right)\left(\tfrac{1}{n\pi}\right)^2 \sinh(1)\,(-1)^n$$

$$= \left(\frac{(-1)^n}{(n\pi)^2 + 1}\right)\sinh(1)$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty}{}' (-1)^n \left(\frac{1}{(n\pi)^2 + 1}\right)\sinh(1)\, e^{in\pi t}$$

$$= \sinh(1) + \sum_{n=1}^{\infty}{}' (-1)^n \left(\frac{1}{(n\pi)^2 + 1}\right)\sinh(1)\left(\frac{e^{in\pi t} + e^{-in\pi t}}{2}\right)(2)$$

$$= \sinh(1) + \sum_{n=1}^{\infty}{}' (-1)^n \left(\frac{2}{(n\pi)^2 + 1}\right)\sinh(1)\cos(n\pi t)$$



3. $y''-3y'+2y=f(t)$ ; $y(0)=y'(0)=0$ and $f(t)=\begin{cases} 1 & \text{for } 0<t<\pi \\ 0 & \text{for } \pi<t<2\pi \end{cases}$

Solution $y(t)=y_h(t)+y_p(t)$

solve $y''-3y'+2y=0$ for homogeneous solution, assume $y_h(t)=e^{mt}$

$m^2-3m+2=0 \Rightarrow (m-2)(m-1)=0$ ∴ $m=1$ and $2$

$y_h(t)=Ae^{t}+Be^{2t}$

$y_p(t)=\sum_{n=-\infty}^{\infty} C_n e^{int}$ , $y_p'(t)=\sum_{n=-\infty}^{\infty}(in)e^{int}$ , $y_p''(t)=\sum_{n=-\infty}^{\infty}(in)^2e^{int}$

$y_p''(t)-3y_p'+2y_p=f(t) \Rightarrow \sum_{n=-\infty}^{\infty}\left[(in)^2-3(in)+2\right]C_n e^{int}=f(t)=$

$\Rightarrow \sum_{n=-\infty}^{\infty}\left[(in)^2-3(in)+2\right]C_n e^{int}=\sum_{-\infty}^{\infty}b_n e^{int}$ where $b_n=\frac{1}{2\pi}\int_0^{\pi}(e^{-int'})f(t')dt'$

← need to find these

∴ $b_n=\frac{1}{2\pi}\int_0^{\pi}e^{-int'}dt'=\frac{1}{2\pi}\left(-\frac{1}{ni}\right)e^{(-int)}\Big|_0^{\pi}$

$=\left(\frac{1}{2\pi}\right)\left(\frac{1}{ni}\right)\left(1-e^{-in\pi}\right)=\left(\frac{1}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(1-e^{-in\pi}\right)$

$b_0=\frac{1}{2\pi}\int_0^{\pi}dt'=\left(\frac{1}{2\pi}\right)(\pi)=\frac{1}{2}$

$$2C_0+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}C_n\left[(in)^2-3(in)+2\right]e^{int}=\frac{1}{2}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\frac{1}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\underbrace{\left(1-e^{-in\pi}\right)}_{(1-(-1)^n)}e^{int}$$

$(1-(-1)^n)$

$=0$ when $n$ is even

$=2$ when $n$ is odd

$e^{-in\pi}=\cos(n\pi)-i\cancel{\sin(n\pi)}^{\,0}$

$=(-1)^n$

$$2C_0+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}C_n\left[(in)^2-3(in)+2\right]e^{int}=\frac{1}{2}+\sum_{\substack{n=-\infty\\n\neq 0\\n\text{-odd}}}^{\infty}\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)e^{int}$$

Equating terms $2C_0=\frac{1}{2} \Rightarrow C_0=\frac{1}{4}$

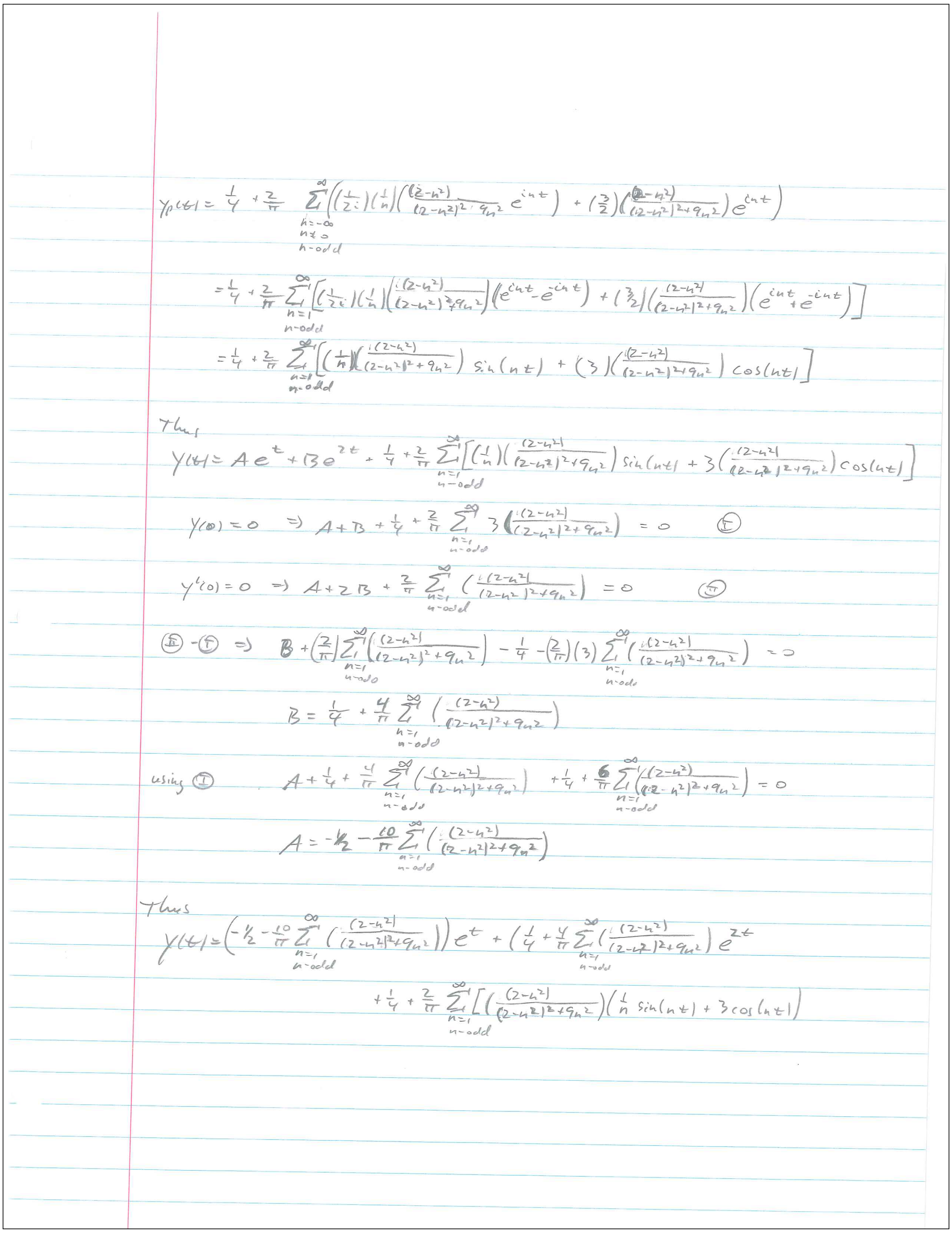
$C_n\left[(in)^2-3(in)+2\right]=\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)$

$$C_n=\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{1}{(in)^2-3(in)+2}\right)$$

$$=\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{1}{(2-n^2)-3(in)}\right)\left(\frac{(2-n^2)+3(in)}{(2-n^2)+3(in)}\right)$$

$$=\left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2)+3(in)}{(2-n^2)^2+9n^2}\right)$$

$$y_p(t)=\frac{1}{4}+\frac{2}{\pi}\sum_{\substack{n=-\infty\\n\neq 0\\n\text{-odd}}}^{\infty}\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2)+3(in)}{(2-n^2)^2+9n^2}\right)e^{int}$$



$$y_p(t) = \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0 \\ n\text{-odd}}}^{\infty} \left( \left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2} e^{int}\right) + \left(\frac{3}{2}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) e^{int} \right)$$

$$= \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left[ \left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(e^{int}-e^{-int}\right) + \left(\frac{3}{2}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(e^{int}+e^{-int}\right) \right]$$

$$= \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left[ \left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) \sin(nt) + (3)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\cos(nt) \right]$$

Thus

$$y(t) = Ae^{t} + Be^{2t} + \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left[ \left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\sin(nt) + 3\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\cos(nt) \right]$$

$$y(0) = 0 \Rightarrow A + B + \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} 3\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0 \qquad \textcircled{I}$$

$$y'(0) = 0 \Rightarrow A + 2B + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0 \qquad \textcircled{II}$$

$$\textcircled{II} - \textcircled{I} \Rightarrow B + \left(\frac{2}{\pi}\right) \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) - \frac{1}{4} - \left(\frac{2}{\pi}\right)(3) \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0$$

$$B = \frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)$$

using $\textcircled{I}$

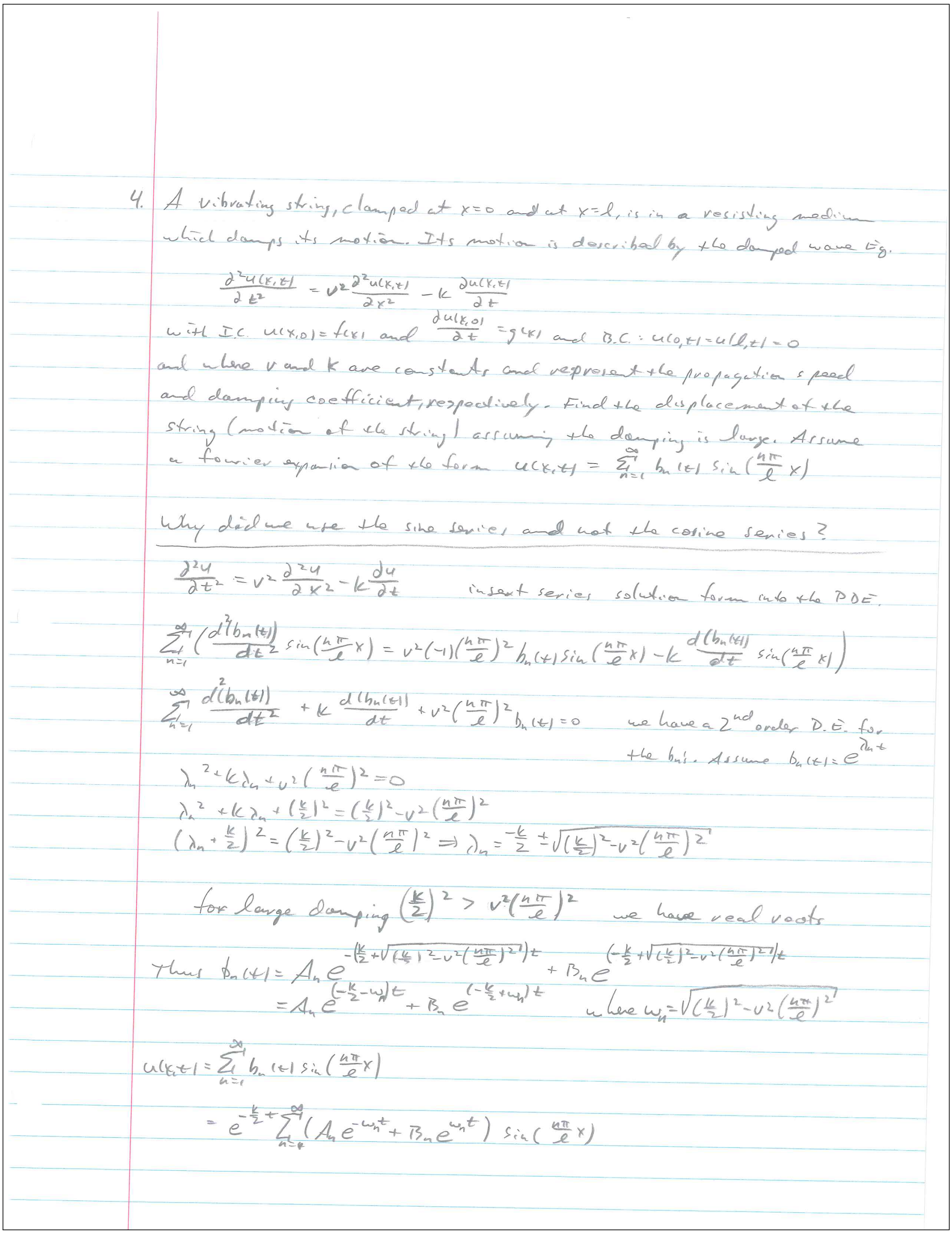
$$A + \frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) + \frac{1}{4} + \frac{6}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0$$

$$A = -\frac{1}{2} - \frac{10}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)$$

Thus

$$y(t) = \left(-\frac{1}{2} - \frac{10}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\right) e^{t} + \left(\frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\right) e^{2t}$$

$$+ \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty} \left[\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(\frac{1}{n}\sin(nt) + 3\cos(nt)\right)\right]$$



4. A vibrating string, clamped at $x=0$ and at $x=l$, is in a resisting medium which damps its motion. Its motion is described by the damped wave Eq.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - k \frac{\partial u(x,t)}{\partial t}$$

with I.C. $u(x,0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t) = u(l,t) = 0$ and where $v$ and $k$ are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a fourier expansion of the form $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{l} x\right)$

Why did we use the sine series and not the cosine series?

---

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial t}$$ insert series solution form into the PDE.

$$\sum_{n=1}^{\infty} \left( \frac{d^2 b_n(t)}{dt^2} \sin\left(\frac{n\pi}{l} x\right) = v^2(-1)\left(\frac{n\pi}{l}\right)^2 b_n(t) \sin\left(\frac{n\pi}{l} x\right) - k \frac{d(b_n(t))}{dt} \sin\left(\frac{n\pi}{l} x\right) \right)$$

$$\sum_{n=1}^{\infty} \frac{d^2(b_n(t))}{dt^2} + k \frac{d(b_n(t))}{dt} + v^2\left(\frac{n\pi}{l}\right)^2 b_n(t) = 0$$

we have a $2^{nd}$ order D.E. for the $b_n$'s. Assume $b_n(t) = e^{\lambda_n t}$

$$\lambda_n^2 + k\lambda_n + v^2\left(\frac{n\pi}{l}\right)^2 = 0$$

$$\lambda_n^2 + k\lambda_n + \left(\frac{k}{2}\right)^2 = \left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2$$

$$\left(\lambda_n + \frac{k}{2}\right)^2 = \left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2 \Rightarrow \lambda_n = \frac{-k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2}$$

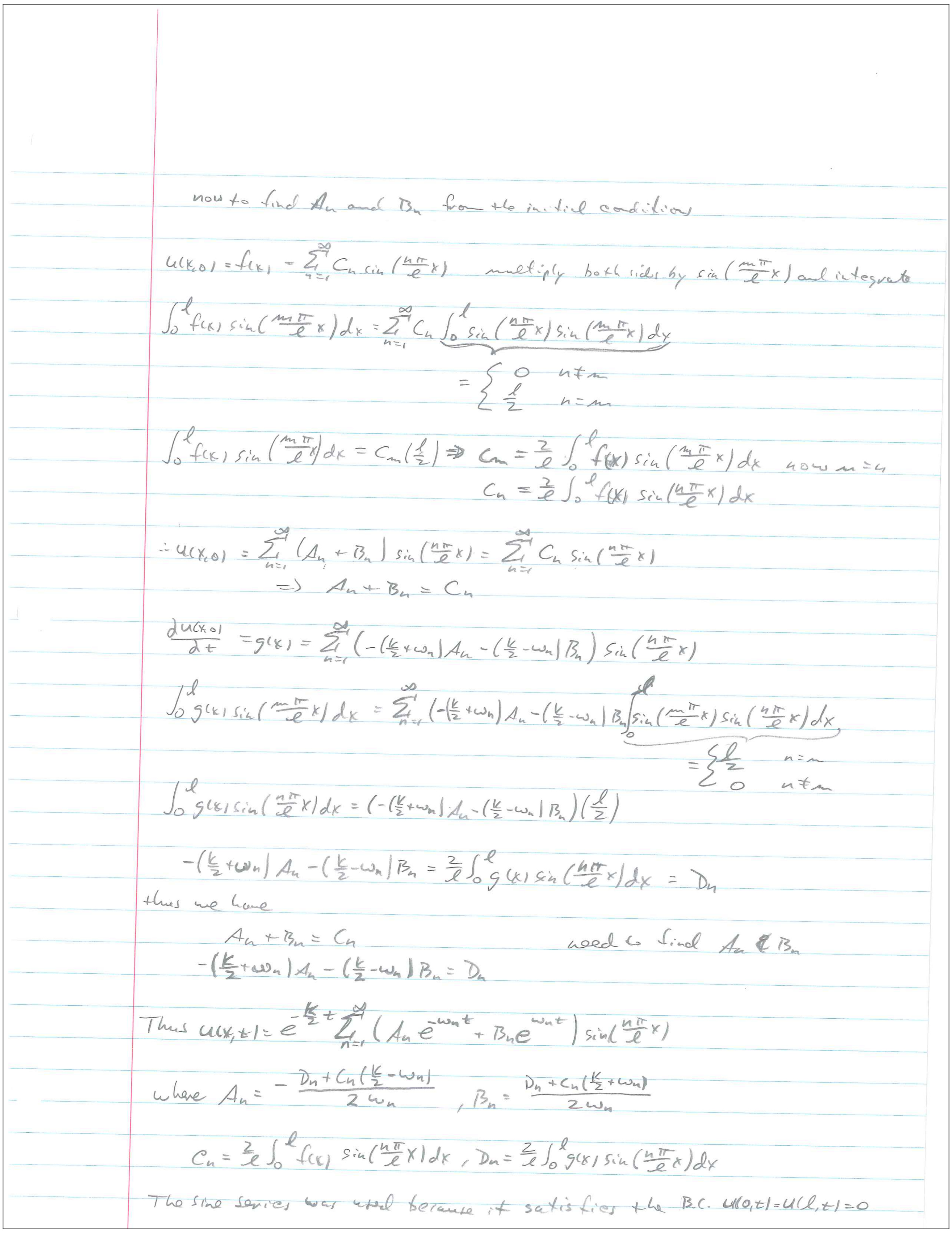
for large damping $\left(\frac{k}{2}\right)^2 > v^2\left(\frac{n\pi}{l}\right)^2$ we have real roots

$$\text{Thus } b_n(t) = A_n e^{-\left(\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2}\right)t} + B_n e^{\left(-\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2}\right)t}$$

$$= A_n e^{\left(-\frac{k}{2} - w_n\right)t} + B_n e^{\left(-\frac{k}{2} + w_n\right)t}$$ where $w_n = \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{l}\right)^2}$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{l} x\right)$$

$$= e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \left(A_n e^{-w_n t} + B_n e^{w_n t}\right) \sin\left(\frac{n\pi}{l} x\right)$$



now to find $A_n$ and $B_n$ from the initial conditions

$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{\ell}x)$ multiply both sides by $\sin(\frac{m\pi}{\ell}x)$ and integrate

$$\int_0^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)\,dx = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^{\ell} \sin(\frac{n\pi}{\ell}x)\sin(\frac{m\pi}{\ell}x)\,dx}_{= \begin{cases} 0 & n\neq m \\ \frac{\ell}{2} & n=m \end{cases}}$$

$\int_0^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)\,dx = C_m(\frac{\ell}{2}) \Rightarrow C_m = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)\,dx$ now $m=n$

$$C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{n\pi}{\ell}x)\,dx$$

$$\therefore u(x,0) = \sum_{n=1}^{\infty}(A_n + B_n)\sin(\frac{n\pi}{\ell}x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{\ell}x)$$

$$\Rightarrow A_n + B_n = C_n$$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = \sum_{n=1}^{\infty}\left(-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n\right)\sin(\frac{n\pi}{\ell}x)$$

$$\int_0^{\ell} g(x)\sin(\frac{m\pi}{\ell}x)\,dx = \sum_{n=1}^{\infty}(-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n)\underbrace{\int_0^{\ell}\sin(\frac{m\pi}{\ell}x)\sin(\frac{n\pi}{\ell}x)\,dx}_{= \begin{cases} \frac{\ell}{2} & n=m \\ 0 & n\neq m \end{cases}}$$

$$\int_0^{\ell} g(x)\sin(\frac{n\pi}{\ell}x)\,dx = (-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n)(\frac{\ell}{2})$$

$$-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n = \frac{2}{\ell}\int_0^{\ell} g(x)\sin(\frac{n\pi}{\ell}x)\,dx = D_n$$

thus we have

$A_n + B_n = C_n$ need to find $A_n$ & $B_n$

$-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n = D_n$

Thus $u(x,t) = e^{-\frac{k}{2}t}\sum_{n=1}^{\infty}(A_n e^{-\omega_n t} + B_n e^{\omega_n t})\sin(\frac{n\pi}{\ell}x)$

where $A_n = -\frac{D_n + C_n(\frac{k}{2}-\omega_n)}{2\omega_n}$, $B_n = \frac{D_n + C_n(\frac{k}{2}+\omega_n)}{2\omega_n}$

$C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{n\pi}{\ell}x)\,dx$, $D_n = \frac{2}{\ell}\int_0^{\ell} g(x)\sin(\frac{n\pi}{\ell}x)\,dx$

The sine series was used because it satisfies the B.C. $u(0,t) = u(\ell,t) = 0$

# 2.10 HW 9

## 2.10.1 Problems to solve

Homework Set No. 9 NEEP 547
Due November 15, 2013 DLH

Fourier Cosine, Sine and Integral Transforms

1. (5pts) page 672, prob. 2: Find the fourier cosine and sine transform of $f(t) = t\,e^{-a\,t}$.
2. (5pts) Find the inverse transform of
$$\frac{1-e^{-2i\,\omega}}{-\omega^2+4i\,\omega+3}.$$
3. (10pts) Find the deflection in the beam
$$E\,I\,\frac{d^4\,y}{d\,x^4}+k\;y(x)=-p(x)\text{ where }p(x)=\begin{cases}0 & \text{for } -\infty<x<-\ell\\ P_0(\ell+x)/\ell^2 & \text{for } -\ell<x<0\\ P_0(\ell-x)/\ell^2 & \text{for } 0<x<\ell\\ 0 & \text{for } \ell<x<\infty.\end{cases}$$
4. (10pts) Find the solution to the one Dimensional Wave Equation: $(-\infty<x<\infty$ and $t>0)$:
$$\frac{\partial^2 y}{\partial t^2}=k^2\frac{\partial^2 y}{\partial x^2}\qquad\text{with I.C.: } y(x,0)=\sin\left(\frac{\pi x}{a}\right)(H(x)-H(x-a))$$
$$\text{and }\left.\frac{\partial y}{\partial t}\right|_{t=0}=F_o\,(H(x)-H(x-a))$$
and where $k$ is a positive constant and $H(x)$ is the Heaviside function

## 2.10.2 problem 1

Find the Fourier cosine and Fourier sin transforms of $f(t)=te^{-at}$ for positive $a$

**solution**

$f(t)$ is plotted in time domain to see what its shape using $a=0.5$ for illustration

```
a = .5;
Plot[t Exp[-a t], {t, -1.5, 1.5}, AxesLabel -> {t, Defer[f[t]]},
 AxesOrigin -> {0, 0}]
```

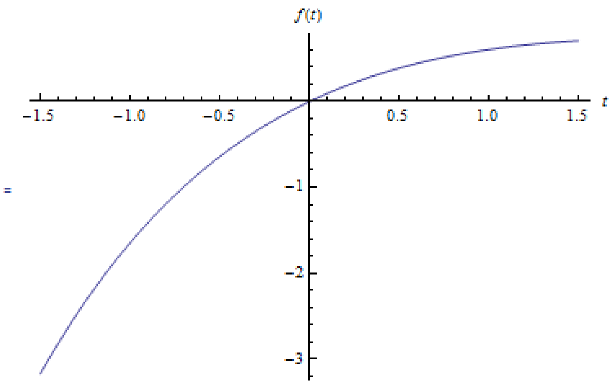


Figure 2.21: showing $f(t)$ for some positive $a$

#### 2.10.2.1 part 1 (cosine Fourier transform)

The Fourier cosine transform of $f(t)$ is

$$F_c(\omega)=\int_0^\infty f(t)\cos(\omega t)\,dt$$

$f(t)$ is first verified that it is smooth and $\int_0^\infty \left|f(t)\right| dt < \infty$ (i.e absolutely integrable function). $f(t)$ is clearly smooth function since it is the product of two smooth functions. $\int_0^\infty \left|f(t)\right| dt = \int_0^\infty \left|te^{-at}\right| dt = \int_0^\infty te^{-at}dt = \frac{1}{a^2}$, hence the integral converges. This means the Fourier cosine transform can be used. Evaluating $F_c(\omega)$.

$$F_c(\omega) = \int_0^\infty te^{-at}\cos(\omega t)\,dt$$

Integration by parts 2 times in order to obtain the same original integral, Let

$$I = \int_0^\infty te^{-at}\cos(\omega t)\,dt$$

Using $\int udv = [uv] - \int vdu$, let $u = te^{-at}$ and $dv = \cos(\omega t)$, then $du = e^{-at} - ate^{-at}$ and $v = \int \cos(\omega t)\,dt = \frac{\sin(\omega t)}{\omega}$, hence

$$\begin{aligned}
I &= \overbrace{\left[te^{-at}\frac{\sin(\omega t)}{\omega}\right]_0^\infty}^{0} - \int \frac{\sin(\omega t)}{\omega}\left(e^{-at} - ate^{-at}\right)dt\\
&= -\int \frac{\sin(\omega t)}{\omega}e^{-at}dt + \int \frac{\sin(\omega t)}{\omega}ate^{-at}dt\\
&= -\frac{1}{\omega}\int e^{-at}\sin(\omega t)\,dt + \frac{a}{\omega}\int te^{-at}\sin(\omega t)\,dt
\end{aligned}$$

$\int e^{-at}\sin(\omega t)\,dt$ can be integrated by parts giving $\frac{\omega}{a^2+\omega^2}$. For the second term $\int te^{-at}\sin(\omega t)\,dt$, integrating by parts again, but now let $u = te^{-at}$ and $dv = \sin(\omega t)$, then $du = e^{-at} - ate^{-at}$ and $v = \int \sin(\omega t)\,dt = \frac{-\cos(\omega t)}{\omega}$, hence the above becomes

$$\begin{aligned}
I &= -\frac{1}{\omega}\left(\frac{\omega}{a^2+\omega^2}\right) + \frac{a}{\omega}\left(\overbrace{\left[-te^{-at}\frac{\cos(\omega t)}{\omega}\right]_0^\infty}^{0} - \int_0^\infty \frac{-\cos(\omega t)}{\omega}\left(e^{-at} - ate^{-at}\right)dt\right)\\
&= -\frac{1}{\omega}\left(\frac{\omega}{a^2+\omega^2}\right) + \frac{a}{\omega}\left(\frac{1}{\omega}\int_0^\infty \cos(\omega t)\left(e^{-at} - ate^{-at}\right)dt\right)\\
&= -\frac{1}{\omega}\left(\frac{\omega}{a^2+\omega^2}\right) + \frac{a}{\omega^2}\left(\overbrace{\int_0^\infty \cos(\omega t)\,e^{-at}}^{\frac{a}{a^2+\omega^2}} - a\overbrace{\int_0^\infty te^{-at}\cos(\omega t)\,dt}^{I}\right)
\end{aligned}$$

The original integral $I$ is obtained in the RHS. Solving for $I$ and simplifying gives

$$\begin{aligned}
I &= -\frac{1}{\omega}\left(\frac{\omega}{a^2+\omega^2}\right) + \frac{a}{\omega^2}\left(\frac{a}{a^2+\omega^2} - aI\right)\\
&= -\left(\frac{1}{a^2+\omega^2}\right) + \frac{a^2}{\omega^2\left(a^2+\omega^2\right)} - \frac{a^2 I}{\omega^2}\\
I + \frac{a^2 I}{\omega^2} &= \frac{-1}{a^2+\omega^2} + \frac{a^2}{\omega^2\left(a^2+\omega^2\right)}\\
I\left(\frac{\omega^2+a^2}{\omega^2}\right) &= \frac{-\omega^2+a^2}{\omega^2\left(a^2+\omega^2\right)}\\
I &= \frac{\frac{\left(a^2-\omega^2\right)}{\omega^2\left(a^2+\omega^2\right)}}{\left(\frac{\omega^2+a^2}{\omega^2}\right)}\\
I &= \frac{\left(a^2-\omega^2\right)\omega^2}{\omega^2\left(a^2+\omega^2\right)\left(\omega^2+a^2\right)}
\end{aligned}$$

Therefore

$$I = F_c(\omega) = \frac{\left(a^2-\omega^2\right)}{\left(a^2+\omega^2\right)^2}$$

#### 2.10.2.2 Part 2, sin Fourier transform

From the definition, the Fourier sin transform of $f(t)$ is

$$F_s(\omega) = \int_0^\infty te^{-at}\sin(\omega t)\,dt$$

This can be integrated by parts 2 times in order to obtain the same integral we started with. Let

$$I = \int_0^\infty te^{-at}\sin(\omega t)\,dt$$

Using $\int udv = [uv] - \int vdu$, let $u = te^{-at}$ and $dv = \sin(\omega t)$, then $du = e^{-at} - ate^{-at}$ and $v = \int \sin(\omega t)\,dt = -\frac{\cos(\omega t)}{\omega}$, hence

$$\begin{aligned}
I &= \overbrace{\left[-te^{-at}\frac{\cos(\omega t)}{\omega}\right]_0^\infty}^{0} + \int \frac{\cos(\omega t)}{\omega}\left(e^{-at} - ate^{-at}\right)dt \\
&= \int \frac{\cos(\omega t)}{\omega}e^{-at}dt - \int \frac{\cos(\omega t)}{\omega}ate^{-at}dt \\
&= \frac{1}{\omega}\int e^{-at}\cos(\omega t)\,dt - \frac{a}{\omega}\int te^{-at}\cos(\omega t)\,dt
\end{aligned}$$

$\int e^{-at}\cos(\omega t)\,dt$ can be integrated by parts giving $\frac{a}{a^2+\omega^2}$. For the second term $\int te^{-at}\cos(\omega t)\,dt$, integrating by parts again, but now let $u = te^{-at}$ and $dv = \cos(\omega t)$, then $du = e^{-at} - ate^{-at}$ and $v = \int \cos(\omega t)\,dt = \frac{\sin(\omega t)}{\omega}$, hence the above becomes

$$\begin{aligned}
I &= \frac{1}{\omega}\left(\frac{a}{a^2+\omega^2}\right) - \frac{a}{\omega}\left(\overbrace{\left[te^{-at}\frac{\sin(\omega t)}{\omega}\right]_0^\infty}^{0} - \int_0^\infty \frac{\sin(\omega t)}{\omega}\left(e^{-at} - ate^{-at}\right)dt\right) \\
&= \frac{1}{\omega}\left(\frac{a}{a^2+\omega^2}\right) + \frac{a}{\omega}\left(\frac{1}{\omega}\int_0^\infty \sin(\omega t)\left(e^{-at} - ate^{-at}\right)dt\right) \\
&= \frac{1}{\omega}\left(\frac{a}{a^2+\omega^2}\right) + \frac{a}{\omega^2}\left(\overbrace{\int_0^\infty \sin(\omega t)e^{-at}}^{\frac{\omega}{a^2+\omega^2}} - a\overbrace{\int_0^\infty te^{-at}\sin(\omega t)\,dt}^{I}\right)
\end{aligned}$$

The original integral $I$ appeared again in the RHS. Solving for $I$ and simplifying gives

$$\begin{aligned}
I &= \frac{1}{\omega}\left(\frac{a}{a^2+\omega^2}\right) + \frac{a}{\omega^2}\left(\frac{\omega}{a^2+\omega^2} - aI\right) \\
&= \frac{a}{\omega\left(a^2+\omega^2\right)} + \frac{a\omega}{\omega^2\left(a^2+\omega^2\right)} - \frac{a^2 I}{\omega^2} \\
I + \frac{a^2 I}{\omega^2} &= \frac{a}{\omega\left(a^2+\omega^2\right)} + \frac{a}{\omega\left(a^2+\omega^2\right)} \\
I\left(\frac{\omega^2+a^2}{\omega^2}\right) &= \frac{2a}{\omega\left(a^2+\omega^2\right)} \\
I &= \frac{\frac{2a}{\omega\left(a^2+\omega^2\right)}}{\left(\frac{\omega^2+a^2}{\omega^2}\right)} \\
I &= \frac{2a\omega^2}{\omega\left(a^2+\omega^2\right)\left(\omega^2+a^2\right)}
\end{aligned}$$

Therefore

$$I = F_s(\omega) = \frac{2a\omega}{\left(a^2+\omega^2\right)^2}$$

### 2.10.3 Problem 2 Inverse Fourier Transform

Find the inverse Fourier transform of $F(\omega) = \frac{1-e^{-2i\omega}}{-\omega^2+4i\omega+3}$

**solution**:

Using the following definitions

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \\
F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\end{aligned}
$$

Now

$$
\begin{aligned}
F(\omega) &= \frac{1-e^{-2i\omega}}{-\omega^2+4i\omega+3} = \frac{1-e^{-2i\omega}}{-\left(\omega^2-4i\omega-3\right)} = \frac{1-e^{-2i\omega}}{-(\omega-3i)(\omega-i)} \\
&= \frac{e^{-2i\omega}-1}{(\omega-3i)(\omega-i)}
\end{aligned}
$$

Let

$$
\frac{1}{(\omega-3i)(\omega-i)} = \frac{A}{(\omega-3i)} + \frac{B}{(\omega-i)}
$$

Hence

$$
A = \lim_{\omega\to 3i} \frac{1}{(\omega-i)} = \frac{1}{2i} = \frac{-i}{2}
$$

And

$$
B = \lim_{\omega\to i} \frac{1}{(\omega-3i)} = \frac{1}{-2i} = \frac{i}{2}
$$

Hence

$$
\begin{aligned}
\frac{1}{(\omega-3i)(\omega-i)} &= \frac{1}{2}\frac{-i}{(\omega-3i)} + \frac{1}{2}\frac{i}{(\omega-i)} \\
\frac{e^{-2i\omega}-1}{(\omega-3i)(\omega-i)} &= \left(\frac{-ie^{-2i\omega}}{2(\omega-3i)} + \frac{1}{2}\frac{ie^{-2i\omega}}{(\omega-i)}\right) - \left(\frac{1}{2}\frac{-i}{(\omega-3i)} + \frac{1}{2}\frac{i}{(\omega-i)}\right) \\
&= \frac{-i}{2}\frac{e^{-2i\omega}}{(\omega-3i)} + \frac{i}{2}\frac{e^{-2i\omega}}{(\omega-i)} + \frac{i}{2}\frac{1}{(\omega-3i)} - \frac{i}{2}\frac{1}{(\omega-i)} \\
&= \frac{1}{2}\frac{e^{-2i\omega}}{(i\omega+3)} - \frac{1}{2}\frac{e^{-2i\omega}}{(i\omega+1)} - \frac{1}{2}\frac{1}{(i\omega+3)} + \frac{1}{2}\frac{1}{(i\omega+1)}
\end{aligned}
$$

From table,

$$
H(t)e^{-\alpha t} \iff \frac{1}{i\omega+\alpha}
$$

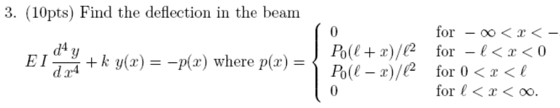
And

$$
H(t-t_0) f(t-t_0) \iff e^{-i\omega t_0} F(\omega)
$$

Hence, using the above two relations we now find the inverse Fourier transform

$$
\begin{aligned}
F^{-1}\left(\frac{e^{-2i\omega}-1}{(\omega-3i)(\omega-i)}\right) &= H(t-2)\frac{1}{2}e^{-3(t-2)} - H(t-2)\frac{1}{2}e^{-(t-2)} - H(t)\frac{1}{2}e^{-3t} + H(t)\frac{1}{2}e^{-t} \\
&= \frac{1}{2}H(t-2)\left(e^{-3t+6} - e^{-t+2}\right) + \frac{1}{2}H(t)\left(e^{-t} - e^{-3t}\right)
\end{aligned}
$$

### 2.10.4 Problem 3, deflection in beam



3. (10pts) Find the deflection in the beam

$$
E\,I\,\frac{d^4\,y}{d\,x^4} + k\;y(x) = -p(x) \text{ where } p(x) = \begin{cases} 0 & \text{for } -\infty < x < -\ell \\ P_0(\ell+x)/\ell^2 & \text{for } -\ell < x < 0 \\ P_0(\ell-x)/\ell^2 & \text{for } 0 < x < \ell \\ 0 & \text{for } \ell < x < \infty. \end{cases}
$$

Figure 2.22: Problem description

**Solution**:

Method overview: Load $p(x)$ is expressed in the Fourier transform space $\omega$. The LHS is also converted to Fourier transform space. $Y(\omega)$ is then solved for algebraically. The inverse Fourier transform is used to obtain $y(t)$, the solution.

Plotting the load $p(x)$ for some values, in order to see its shape (let $L = 1$, $P_0 = 1$), gives the following

```
p0 = 1; L = 1;
p[x_] := Piecewise[{{p0 (L + x)/L^2, -L < x < 0}, {p0 (L - x)/L^2,
    0 < x < L}, {0, True}}]
Plot[p[x], {x, -2 L, 2 L}]
```

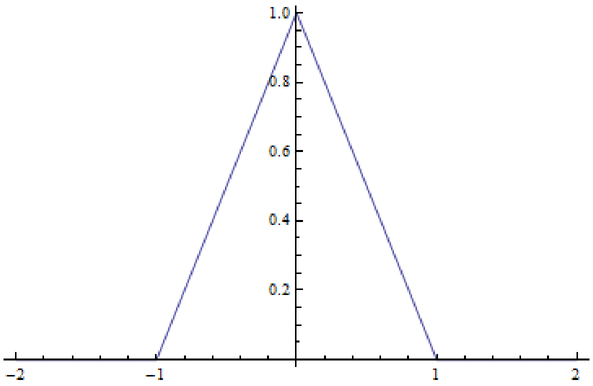


Figure 2.23: showing that $p(x)$ is an even function

$p(x)$ is an even function. Assuming simply supported beam, symmetry of load implies that the deflection will also be even. Hence an even function for a solution is assumed and the cos Fourier transform is therefore used.

$$y(x) = \frac{2}{\pi}\int_0^\infty Y\cos(\omega x)\,d\omega$$

where $Y \equiv Y(\omega)$ is the Fourier transform on $y(x)$ to be found. Taking $4^{th}$ derivative w.r.t. $x$ gives

$$y^{(4)}(x) = \frac{2}{\pi}\int_0^\infty \omega^4 Y\cos(\omega x)\,d\omega$$

The load $p(x)$ is

$$p(x) = \frac{2}{\pi}\int_0^\infty P(\omega)\cos(\omega x)\,d\omega \tag{1}$$

Where $P(\omega)$ is the Fourier transform of $p(x)$. $P(\omega)$ can be found since $p(x)$ is given

$$\begin{aligned}P(\omega) &= \int_0^\infty p(x)\cos(\omega x)\,dx\\ &= \int_0^L \frac{P_0(L-x)}{L^2}\cos(\omega x)\,dx\\ &= \int_0^L \frac{P_0}{L}\cos(\omega x)\,dx - \frac{P_0}{L^2}x\cos(\omega x)\,dx\\ &= \frac{P_0}{L}\left[\frac{\sin(\omega x)}{\omega}\right]_0^L - \frac{P_0}{L^2}\int_0^L x\cos(\omega x)\,dx \qquad (2)\end{aligned}$$

Solving $\int_0^L x\cos(\omega x)\,dx$, by integration by parts. Using $\int udv = [uv] - \int vdu$, let $u = x$ and $dv = \cos(\omega t)$, then $du = 1$ and $v = \int \cos(\omega x)\,dx = \frac{\sin(\omega x)}{\omega}$, hence

$$\begin{aligned}\int_0^L x\cos(\omega x)\,dx &= \left[x\frac{\sin(\omega x)}{\omega}\right]_0^L - \int_0^L \frac{\sin(\omega x)}{\omega}dx\\ &= \frac{L}{\omega}\sin(\omega L) - \frac{1}{\omega}\left[\frac{-\cos(\omega x)}{\omega}\right]_0^L\\ &= \frac{L}{\omega}\sin(\omega L) + \frac{1}{\omega^2}(\cos(\omega L) - 1)\end{aligned}$$

Substituting the above in Eq. (2) results in

$$\begin{aligned} P(\omega) &= \frac{P_0}{L}\frac{\sin(\omega L)}{\omega} - \frac{P_0}{L^2}\left(\frac{L}{\omega}\sin(\omega L) + \frac{1}{\omega^2}(\cos(\omega L) - 1)\right) \\ &= \frac{P_0}{\omega L}\sin(\omega L) - \frac{P_0}{\omega L}\sin(\omega L) - \frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1) \\ &= -\frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1) \end{aligned}$$

Substituting the above in Eq. (1) gives

$$p(x) = \frac{-2}{\pi}\int_0^\infty \frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1)\cos(\omega x)\,d\omega$$

Therefore, the original ODE now becomes

$$EIy^{(4)}(x) + ky(x) = p(x)$$

$$EI\frac{2}{\pi}\int_0^\infty \omega^4 Y\cos(\omega x)\,d\omega + k\frac{2}{\pi}\int_0^\infty Y\cos(\omega x)\,d\omega = \frac{-2}{\pi}\int_0^\infty \frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1)\cos(\omega x)\,d\omega$$

Hence

$$\frac{2}{\pi}\int_0^\infty \left[EI\omega^4 Y + kY + \frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1)\right]\cos(\omega x)\,d\omega = 0$$

Since the integral is zero, and $\cos(\omega x) \neq 0$, then it must be that

$$EI\omega^4 Y + kY + \frac{P_0}{\omega^2 L^2}(\cos(\omega L) - 1) = 0$$

Solving for $Y$ from the above gives

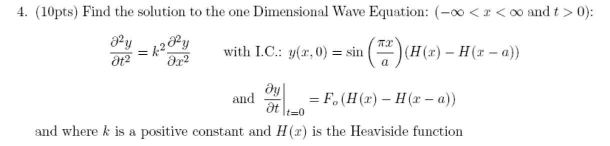
$$\begin{aligned} Y\left(EI\omega^4 + k\right) &= -\frac{P_0}{\omega^2 L^2}\cos(\omega L) + \frac{P_0}{\omega^2 L^2} \\ Y &= \frac{-P_0}{L^2}\frac{\cos(\omega L)}{EI\omega^6 + \omega^2 k} + \frac{P_0}{L^2}\frac{1}{EI\omega^6 + \omega^2 k} \end{aligned}$$

Taking the inverse Fourier cosine transform in order to find $y(x)$

$$\begin{aligned} y(x) &= \frac{1}{2\pi}\int_0^\infty \left(\frac{-P_0}{L^2}\frac{\cos(\omega L)}{EI\omega^6 + \omega^2 k} + \frac{P_0}{L^2}\frac{1}{EI\omega^6 + \omega^2 k}\right)\cos(\omega x)\,d\omega \\ &= \frac{-1}{2\pi}\frac{P_0}{L^2}\int_0^\infty \frac{\cos(\omega L)\cos(\omega x)}{EI\omega^6 + \omega^2 k}d\omega + \frac{1}{2\pi}\frac{P_0}{L^2}\int_0^\infty \frac{\cos(\omega x)}{EI\omega^6 + \omega^2 k}d\omega \end{aligned}$$

The final step to compute the above was not required to do per class announcement.

### 2.10.5 Problem 4, wave equation



4. (10pts) Find the solution to the one Dimensional Wave Equation: $(-\infty < x < \infty$ and $t > 0)$:

$$\frac{\partial^2 y}{\partial t^2} = k^2\frac{\partial^2 y}{\partial x^2} \qquad \text{with I.C.: } y(x,0) = \sin\left(\frac{\pi x}{a}\right)(H(x) - H(x-a))$$

$$\text{and } \left.\frac{\partial y}{\partial t}\right|_{t=0} = F_o(H(x) - H(x-a))$$

and where $k$ is a positive constant and $H(x)$ is the Heaviside function

Figure 2.24: problem 4 description

**Solution**

Method review: Laplace transform is used to obtain a differential equation in $Y(x,s)$. Then Fourier transform is used to obtain an algebraic equation in $Y(\omega,s)$. The inverse Laplace transform is used to obtain $Y(\omega,t)$ and finally the inverse Fourier transform is used to obtain $y(x,t)$ .Let

$$f(x) = y(x,0) = \sin\left(\frac{\pi x}{a}\right)(H(x) - H(x-a))$$

And let its Fourier transform be

$$F(\omega) = \mathscr{F}\big(f(x)\big) = \int_0^a \sin\left(\frac{\pi x}{a}\right)e^{-i\omega x}dx = \frac{a\left(1 + e^{-ia\omega}\right)\pi}{\pi^2 - a^2\omega^2}$$

And let

$$g(x) = y'(x,0) = F_0(H(x) - H(x-a))$$

And let its Fourier transform be

$$G(\omega)=\mathscr{F}\big(y'(x,0)\big)=\int_0^a F_0e^{-i\omega x}dx=F_0\left[\frac{e^{-i\omega x}}{-i\omega}\right]_0^a=\frac{F_0}{-i\omega}\left(e^{-i\omega a}-1\right)=\frac{iF_0}{\omega}\left(e^{-i\omega a}-1\right)$$

Now, taking the Laplace transform of the PDE gives

$$\begin{aligned}s^2Y(x,s)-sy(x,0)-y'(x,0)&=k^2\frac{d^2Y(x,s)}{dx^2}\\ s^2Y(x,s)-sf(x)-g(x)&=k^2\frac{d^2Y(x,s)}{dx^2}\end{aligned}\tag{1}$$

Taking the Fourier transform of the above equation. The following relation is used $\frac{d^ny(x)}{dx^n}\iff(i\omega)^nY(\omega)$, hence $\frac{d^2y(x)}{dx^2}\iff-\omega^2Y(\omega)$ hence

$$s^2Y(\omega,s)-sF(\omega)-G(\omega)=-k^2\omega^2Y(\omega)$$

Solving for $Y(\omega,s)$ gives

$$\begin{aligned}Y(\omega,s)&=\frac{sF(\omega)+G(\omega)}{\left(s^2+k^2\omega^2\right)}\\ &=F(\omega)\frac{s}{s^2+k^2\omega^2}+G(\omega)\frac{1}{s^2+k^2\omega^2}\end{aligned}\tag{2}$$

Taking inverse Laplace transform of $\frac{s}{s^2+k^2\omega^2}$.Using $\frac{s}{\left(s^2+a^2\right)}\iff\cos(at)$, hence $\frac{s}{s^2+k^2\omega^2}\iff\cos(k\omega t)$. Taking inverse Laplace transform of $\frac{1}{s^2+k^2\omega^2}$, from $\frac{a}{\left(s^2+a^2\right)}\iff\sin(at)$, hence $\frac{1}{k\omega}\frac{1}{s^2+k^2\omega^2}\iff\frac{\sin(k\omega t)}{k\omega}$

Therefore inverse Laplace of Eq. (2) is

$$Y(\omega,t)=F(\omega)\cos(k\omega t)+G(\omega)\frac{\sin(k\omega t)}{k\omega}\tag{3}$$

Taking the inverse Fourier transform of the above. Writing $\cos(k\omega t)=\frac{e^{ik\omega t}+e^{-ik\omega t}}{2}$ and $\sin(k\omega t)=\frac{e^{ik\omega t}-e^{-ik\omega t}}{2i}$ and applying the definition of inverse Fourier transform, results in

$$\begin{aligned}y(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}\left(F(\omega)\cos(k\omega t)+G(\omega)\frac{\sin(k\omega t)}{k\omega}\right)d\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}\left(F(\omega)\frac{e^{ik\omega t}+e^{-ik\omega t}}{2}+\frac{G(\omega)}{k\omega}\frac{e^{ik\omega t}-e^{-ik\omega t}}{2i}\right)d\omega\\ &=\frac{1}{4\pi}\int_{-\infty}^{\infty}F(\omega)\left(e^{i(x+kt)\omega}+e^{i(x-kt)\omega}\right)d\omega+\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{G(\omega)}{ik\omega}\left(e^{i(x+kt)\omega}-e^{i(x-kt)\omega}\right)d\omega\end{aligned}$$

Hence, breaking the above into 4 integrals gives

$$\begin{aligned}y(x,t)=&\frac{1}{2}\left(\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i(x+kt)\omega}d\omega}^{f(x+kt)}\right)+\frac{1}{2}\left(\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i(x-kt)\omega}d\omega}^{f(x-kt)}\right)\\ &+\frac{1}{2k}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega)\frac{e^{i(x+kt)\omega}}{i\omega}d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega)\frac{e^{i(x-kt)\omega}}{i\omega}d\omega\right)\end{aligned}$$

Or

$$\begin{aligned}y(x,t)&=\frac{1}{2}f(x+kt)+\frac{1}{2}f(x-kt)+\frac{1}{2k}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega)\frac{e^{i(x+kt)\omega}-e^{i(x-kt)\omega}}{i\omega}d\omega\right)\\ &=\frac{1}{2}f(x+kt)+\frac{1}{2}f(x-kt)+\frac{1}{2k}\left(\int_{x-kt}^{x+kt}\overbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega)e^{i\xi\omega}d\omega\right)}^{g(\xi)}d\xi\right)\\ &=\frac{1}{2}f(x+kt)+\frac{1}{2}f(x-kt)+\frac{1}{2k}\int_{x-kt}^{x+kt}g(\xi)d\xi\end{aligned}$$

Where

$$g(\xi) = F_0 (H(\xi) - H(\xi - a))$$

Hence the final solution is

$$\begin{aligned} y(x,t) &= \frac{1}{2} f(x+kt) + \frac{1}{2} f(x-kt) + \frac{1}{2k} \int_{x-kt}^{x+kt} F_0 (H(\xi) - H(\xi - a))\, d\xi \\ &= \frac{1}{2} \sin\left(\frac{\pi (x+kt)}{a}\right) (H(x+kt) - H(x+kt-a)) \\ &+ \frac{1}{2} \sin\left(\frac{\pi (x-kt)}{a}\right) (H(x-kt) - H(x-kt-a)) \\ &+ \frac{1}{2k} \int_{x-kt}^{x+kt} F_0 (H(\xi) - H(\xi - a))\, d\xi \end{aligned}$$

But

$$\int_{x-kt}^{x+kt} F_0 (H(\xi) - H(\xi - a))\, d\xi = F_0 \int_{x-kt}^{x+kt} H(\xi)\, d\xi - F_0 \int_{x-kt}^{x+kt} H(\xi - a)\, d\xi$$

Hence

$$F_0 \int_{x-kt}^{x+kt} H(\xi)\, d\xi - F_0 \int_{x-kt}^{x+kt} H(\xi - a)\, d\xi = F_0 ((x+kt) - (x-kt)) = \begin{matrix} 2F_0 kt & 0 < x - kt \text{ and } x + kt < a \\ 0 & \textit{otherwise} \end{matrix}$$

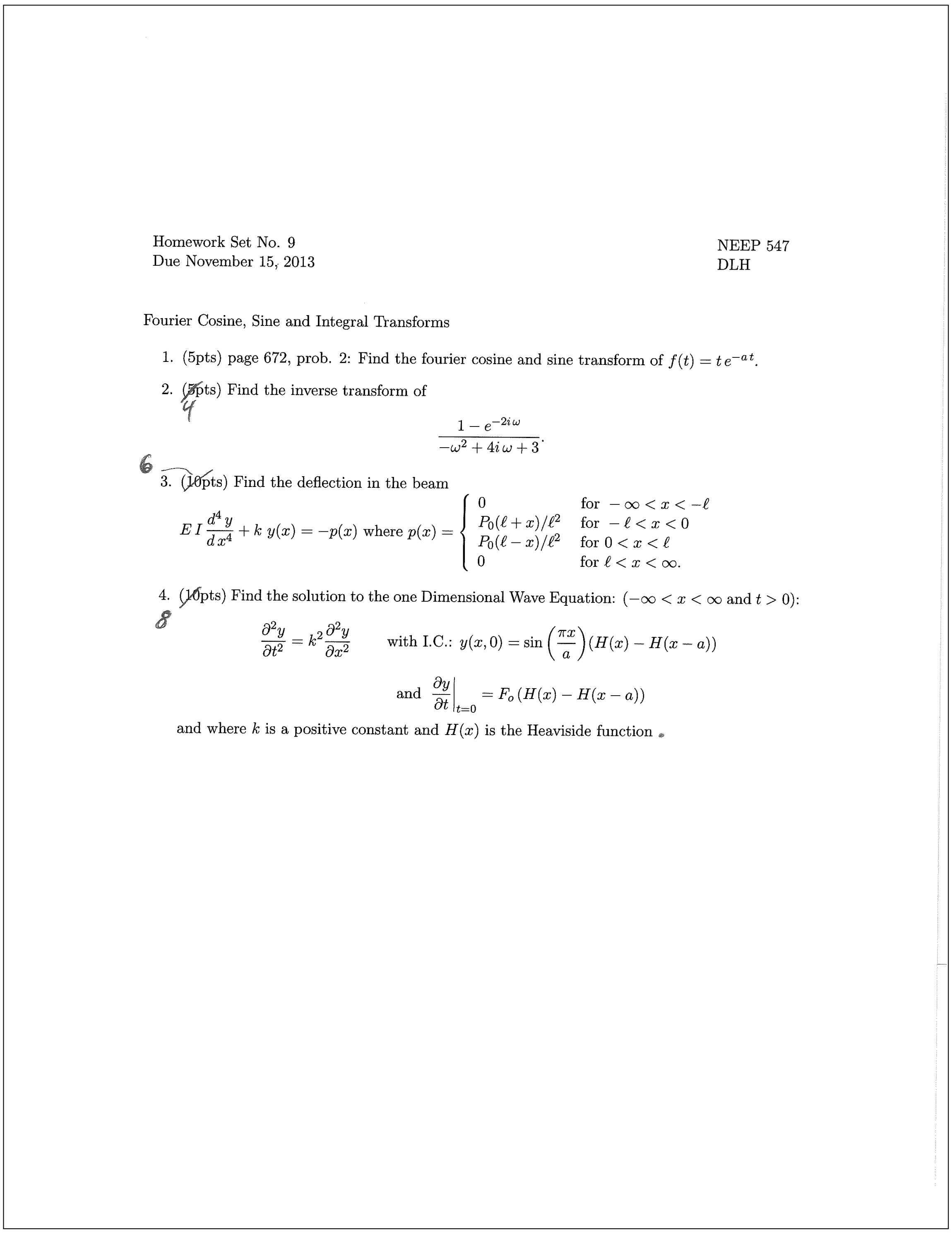
Hence, for $0 < x - kt$ and $x + kt < a$, the solution is

$$\begin{aligned} y(x,t) = \frac{1}{2} &\left( \sin\left(\frac{\pi (x+kt)}{a}\right) (H(x+kt) - H(x+kt-a)) + \sin\left(\frac{\pi (x-kt)}{a}\right) (H(x-kt) - H(x-kt-a)) \right) \\ &+ 2F_0 kt \end{aligned}$$

Otherwise, the solution is

$$y(x,t) = \frac{1}{2} \left( \sin\left(\frac{\pi (x+kt)}{a}\right) (H(x+kt) - H(x+kt-a)) + \sin\left(\frac{\pi (x-kt)}{a}\right) (H(x-kt) - H(x-kt-a)) \right)$$

### 2.10.6 key solution

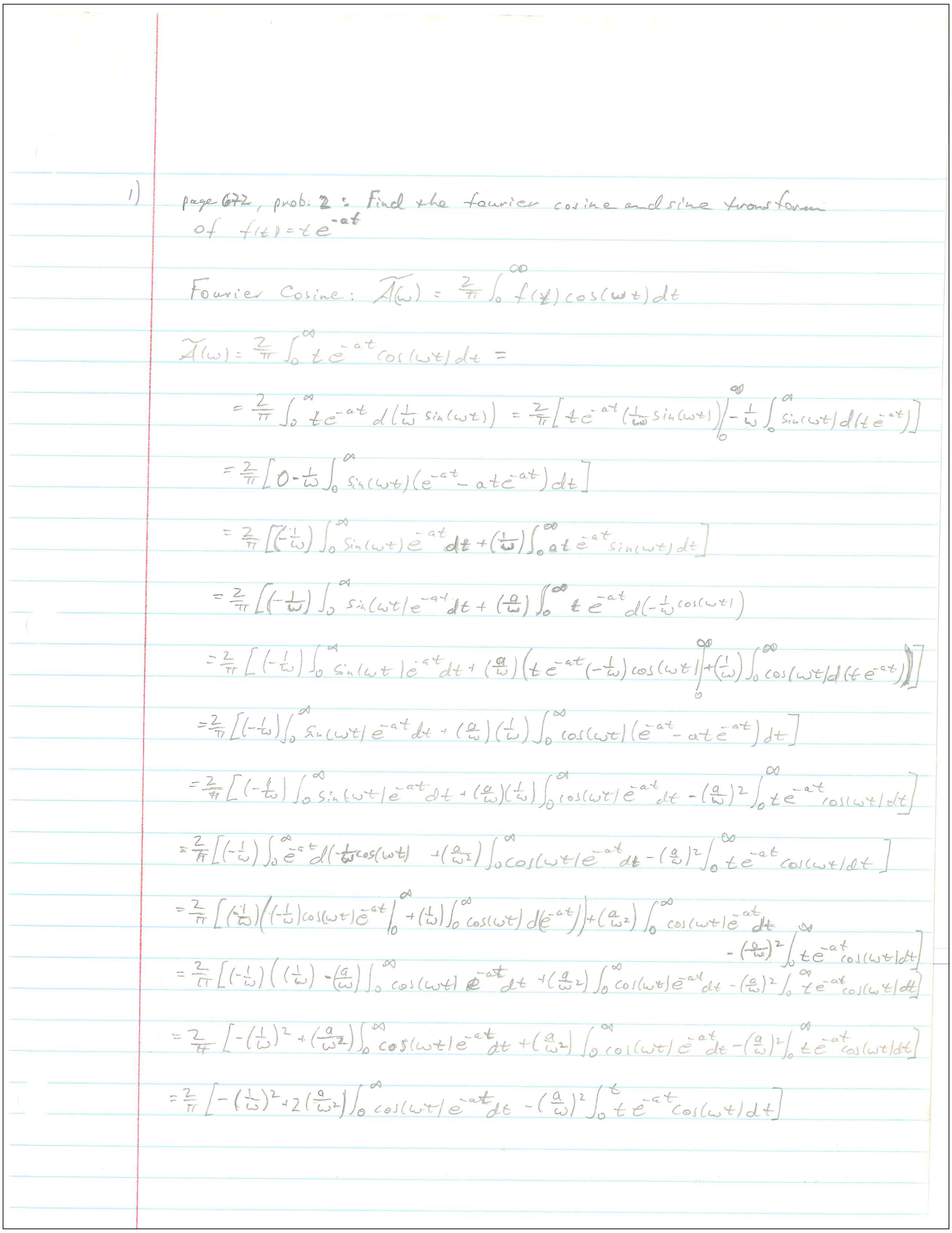


Homework Set No. 9
Due November 15, 2013

NEEP 547
DLH

Fourier Cosine, Sine and Integral Transforms

1. (5pts) page 672, prob. 2: Find the fourier cosine and sine transform of $f(t) = t\,e^{-at}$.
2. (~~5~~pts) Find the inverse transform of
   4
$$\frac{1-e^{-2i\omega}}{-\omega^2+4i\,\omega+3}.$$
6
3. (~~10~~pts) Find the deflection in the beam
$$E\,I\,\frac{d^4\,y}{d\,x^4}+k\;y(x)=-p(x)\text{ where }p(x)=\begin{cases}0 & \text{for } -\infty<x<-\ell\\ P_0(\ell+x)/\ell^2 & \text{for } -\ell<x<0\\ P_0(\ell-x)/\ell^2 & \text{for } 0<x<\ell\\ 0 & \text{for } \ell<x<\infty.\end{cases}$$
4. (~~10~~pts) Find the solution to the one Dimensional Wave Equation: $(-\infty<x<\infty$ and $t>0)$:
   8
$$\frac{\partial^2 y}{\partial t^2}=k^2\frac{\partial^2 y}{\partial x^2}\qquad\text{with I.C.: } y(x,0)=\sin\left(\frac{\pi x}{a}\right)(H(x)-H(x-a))$$
$$\text{and }\left.\frac{\partial y}{\partial t}\right|_{t=0}=F_o\,(H(x)-H(x-a))$$
   and where $k$ is a positive constant and $H(x)$ is the Heaviside function.



1) page 672, prob. 2: Find the fourier cosine and sine transform of $f(t)=te^{-at}$

Fourier Cosine: $\tilde{A}(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)dt$

$$\tilde{A}(\omega) = \frac{2}{\pi}\int_0^\infty te^{-at}\cos(\omega t)dt =$$

$$= \frac{2}{\pi}\int_0^\infty te^{-at} d\left(\frac{1}{\omega}\sin(\omega t)\right) = \frac{2}{\pi}\left[te^{-at}\left(\frac{1}{\omega}\sin(\omega t)\right)\Big|_0^\infty - \frac{1}{\omega}\int_0^\infty \sin(\omega t)d(te^{-at})\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{\omega}\int_0^\infty \sin(\omega t)(e^{-at} - ate^{-at})dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty \sin(\omega t)e^{-at}dt + \left(\frac{1}{\omega}\right)\int_0^\infty ate^{-at}\sin(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty \sin(\omega t)e^{-at}dt + \left(\frac{a}{\omega}\right)\int_0^\infty te^{-at}d\left(-\frac{1}{\omega}\cos(\omega t)\right)\right.$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty \sin(\omega t)e^{-at}dt + \left(\frac{a}{\omega}\right)\left(te^{-at}\left(-\frac{1}{\omega}\right)\cos(\omega t)\Big|_0^\infty + \left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)d(te^{-at})\right)\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty \sin(\omega t)e^{-at}dt + \left(\frac{a}{\omega}\right)\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)(e^{-at} - ate^{-at})dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty \sin(\omega t)e^{-at}dt + \left(\frac{a}{\omega}\right)\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty te^{-at}\cos(\omega t)dt\right]$$

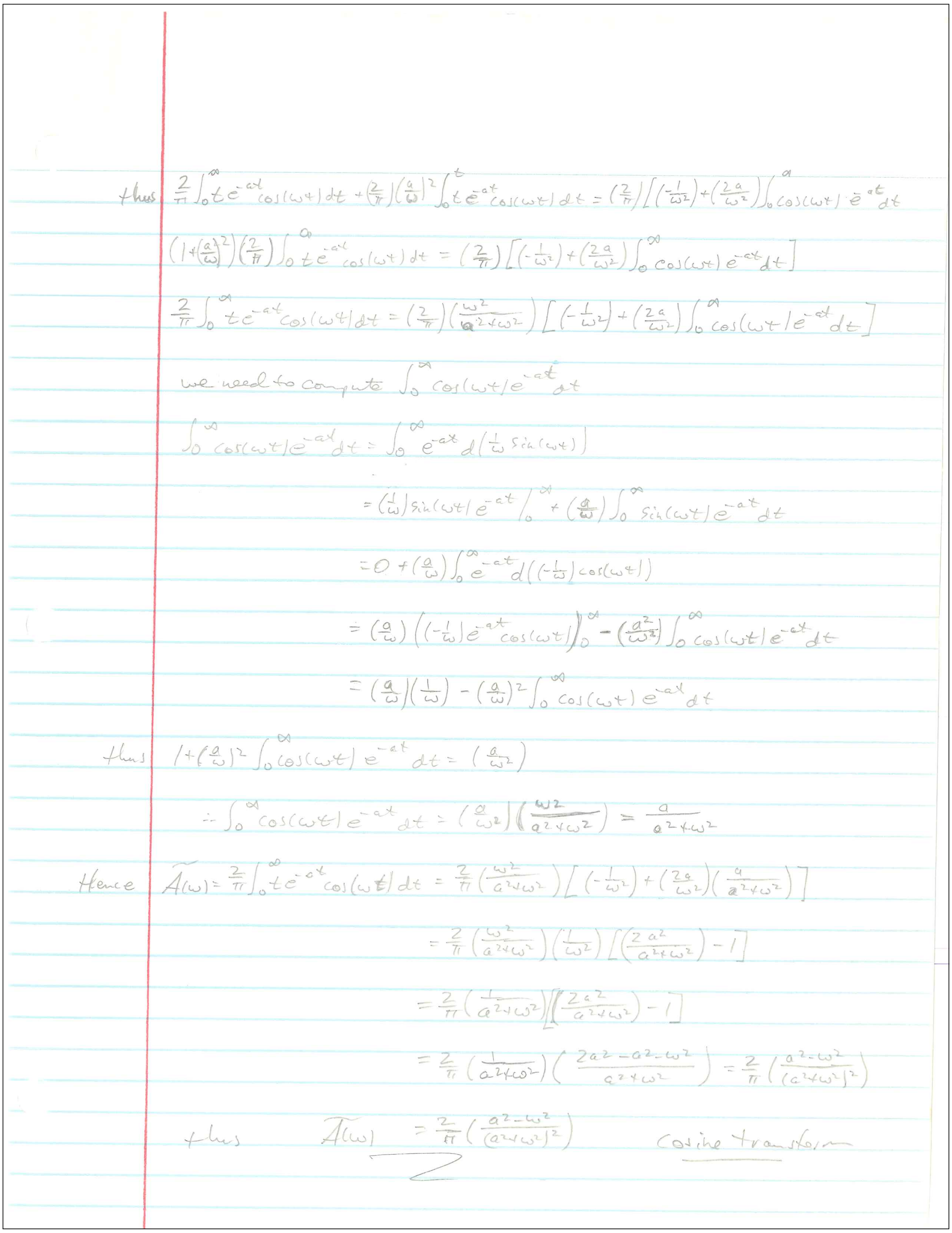
$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\int_0^\infty e^{-at}d\left(-\frac{1}{\omega}\cos(\omega t)\right) + \left(\frac{a}{\omega^2}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty te^{-at}\cos(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\left(\left(-\frac{1}{\omega}\right)\cos(\omega t)e^{-at}\Big|_0^\infty + \left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)d(e^{-at})\right) + \left(\frac{a}{\omega^2}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty te^{-at}\cos(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\left(\left(\frac{1}{\omega}\right) - \left(\frac{a}{\omega}\right)\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty te^{-at}\cos(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[-\left(\frac{1}{\omega}\right)^2 + \left(\frac{a}{\omega^2}\right)\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty te^{-at}\cos(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[-\left(\frac{1}{\omega}\right)^2 + 2\left(\frac{a}{\omega^2}\right)\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^t te^{-at}\cos(\omega t)dt\right]$$



thus $\frac{2}{\pi}\int_0^\infty te^{-at}\cos(\omega t)\,dt+\left(\frac{2}{\pi}\right)\left(\frac{a}{\omega}\right)^2\int_0^t t\bar{e}^{at}\cos(\omega t)\,dt=\left(\frac{2}{\pi}\right)\left[\left(-\frac{1}{\omega^2}\right)+\left(\frac{2a}{\omega^2}\right)\int_0^\infty\cos(\omega t)\,\bar{e}^{at}\,dt\right.$

$$\left(1+\left(\frac{a}{\omega}\right)^2\right)\left(\frac{2}{\pi}\right)\int_0^\infty te^{-at}\cos(\omega t)\,dt=\left(\frac{2}{\pi}\right)\left[\left(-\frac{1}{\omega^2}\right)+\left(\frac{2a}{\omega^2}\right)\int_0^\infty\cos(\omega t)e^{-at}\,dt\right]$$

$$\frac{2}{\pi}\int_0^\infty te^{-at}\cos(\omega t)\,dt=\left(\frac{2}{\pi}\right)\left(\frac{\omega^2}{a^2+\omega^2}\right)\left[\left(-\frac{1}{\omega^2}\right)+\left(\frac{2a}{\omega^2}\right)\int_0^\infty\cos(\omega t)e^{-at}\,dt\right]$$

we need to compute $\int_0^\infty\cos(\omega t)e^{-at}\,dt$

$$\int_0^\infty\cos(\omega t)e^{-at}\,dt=\int_0^\infty e^{-at}\,d\left(\frac{1}{\omega}\sin(\omega t)\right)$$

$$=\left(\frac{1}{\omega}\right)\sin(\omega t)e^{-at}\Big|_0^\infty+\left(\frac{a}{\omega}\right)\int_0^\infty\sin(\omega t)e^{-at}\,dt$$

$$=0+\left(\frac{a}{\omega}\right)\int_0^\infty e^{-at}\,d\left(\left(-\frac{1}{\omega}\right)\cos(\omega t)\right)$$

$$=\left(\frac{a}{\omega}\right)\left(\left(-\frac{1}{\omega}\right)e^{-at}\cos(\omega t)\right)\Big|_0^\infty-\left(\frac{a^2}{\omega^2}\right)\int_0^\infty\cos(\omega t)e^{-at}\,dt$$

$$=\left(\frac{a}{\omega}\right)\left(\frac{1}{\omega}\right)-\left(\frac{a}{\omega}\right)^2\int_0^\infty\cos(\omega t)e^{-at}\,dt$$

thus $1+\left(\frac{a}{\omega}\right)^2\int_0^\infty\cos(\omega t)e^{-at}\,dt=\left(\frac{a}{\omega^2}\right)$

$$\therefore\int_0^\infty\cos(\omega t)e^{-at}\,dt=\left(\frac{a}{\omega^2}\right)\left(\frac{\omega^2}{a^2+\omega^2}\right)=\frac{a}{a^2+\omega^2}$$

Hence $\tilde{A}(\omega)=\frac{2}{\pi}\int_0^\infty te^{-at}\cos(\omega t)\,dt=\frac{2}{\pi}\left(\frac{\omega^2}{a^2+\omega^2}\right)\left[\left(-\frac{1}{\omega^2}\right)+\left(\frac{2a}{\omega^2}\right)\left(\frac{a}{a^2+\omega^2}\right)\right]$

$$=\frac{2}{\pi}\left(\frac{\omega^2}{a^2+\omega^2}\right)\left(\frac{1}{\omega^2}\right)\left[\left(\frac{2a^2}{a^2+\omega^2}\right)-1\right]$$

$$=\frac{2}{\pi}\left(\frac{1}{a^2+\omega^2}\right)\left[\left(\frac{2a^2}{a^2+\omega^2}\right)-1\right]$$

$$=\frac{2}{\pi}\left(\frac{1}{a^2+\omega^2}\right)\left(\frac{2a^2-a^2-\omega^2}{a^2+\omega^2}\right)=\frac{2}{\pi}\left(\frac{a^2-\omega^2}{(a^2+\omega^2)^2}\right)$$

thus $\underline{\tilde{A}(\omega)=\frac{2}{\pi}\left(\frac{a^2-\omega^2}{(a^2+\omega^2)^2}\right)}$ Cosine transform

Sine Transform

$$B(\omega) = \frac{2}{\pi}\int_0^\infty t e^{-at}\sin(\omega t)dt$$

$$= \frac{2}{\pi}\int_0^\infty t e^{-at} d\left(-\frac{1}{\omega}\cos(\omega t)\right) = \frac{2}{\pi}\left[\left(-\frac{1}{\omega}\right)\cos(\omega t)\, t e^{-at}\Big|_0^\infty + \left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)\, d(t e^{-at})\right]$$

$$= \frac{2}{\pi}\left[0 + \left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)\left(e^{-at} - a t e^{-at}\right)dt\right]$$

$$= \frac{2}{\pi}\left[0 + \left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)\, e^{-at}dt - \left(\frac{a}{\omega}\right)\int_0^\infty \cos(\omega t)\, t e^{-at}dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)\int_0^\infty t e^{-at} d\left(\frac{1}{\omega}\sin(\omega t)\right)dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)\left(t e^{-at}\left(\frac{1}{\omega}\right)\sin(\omega t)\right)\Big|_0^\infty + \left(\frac{a}{\omega^2}\right)\int_0^\infty \sin(\omega t)\, d(t e^{-at})\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\int_0^\infty \sin(\omega t)\left(e^{-at} - a t e^{-at}\right)dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\int_0^\infty \sin(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$

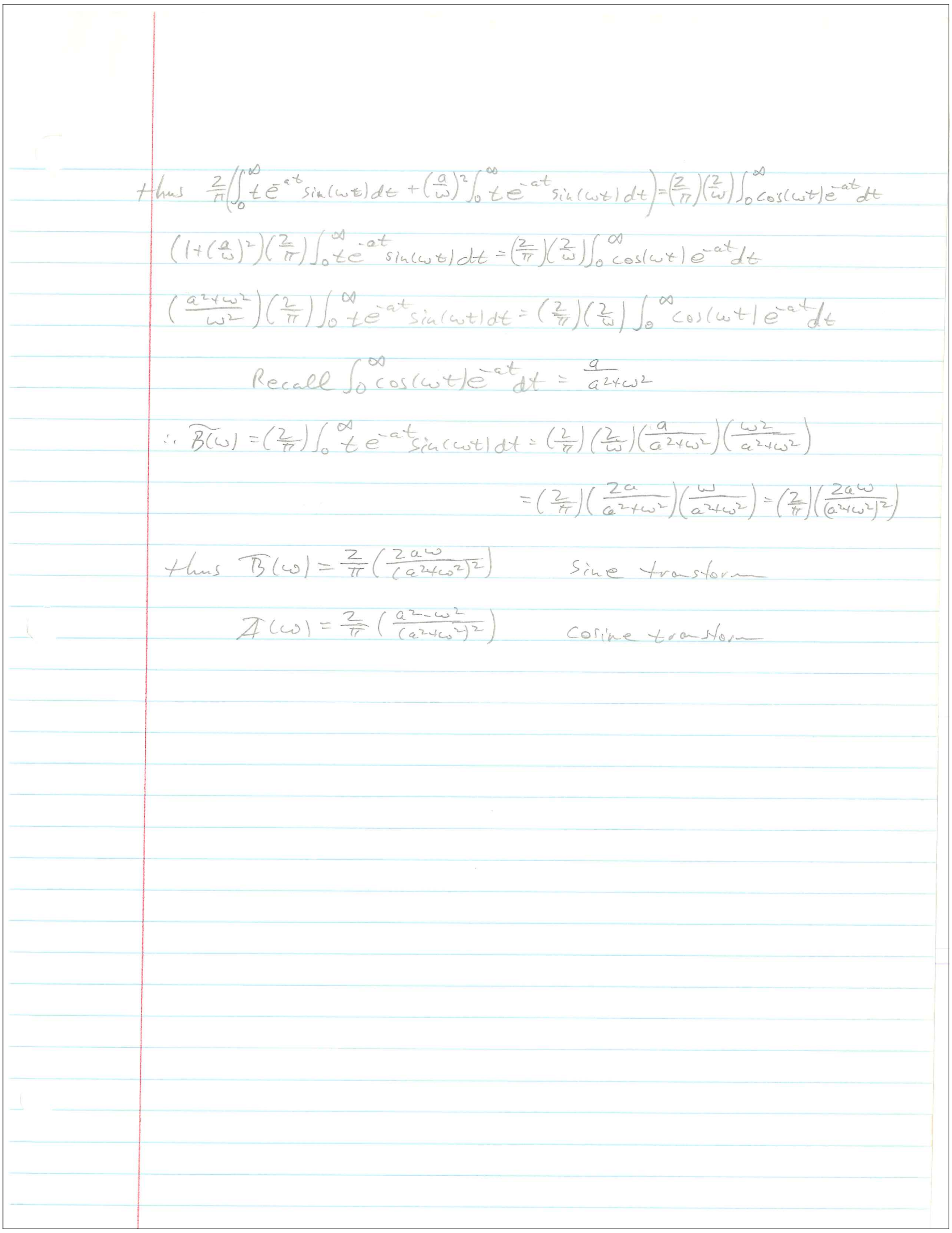
$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\int_0^\infty \sin(\omega t)\, d\left(-\frac{1}{a}e^{-at}\right) - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\left(\left(-\frac{1}{a}\right)e^{-at}\sin(\omega t)\Big|_0^\infty + \frac{1}{a}\int_0^\infty e^{-at}d(\sin(\omega t))\right)\right.$$
$$\left. - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{a}{\omega^2}\right)\left(0 + \frac{\omega}{a}\int_0^\infty e^{-at}\cos(\omega t)dt\right)\right.$$
$$\left. - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\omega}\right)\int_0^\infty \cos(\omega t)e^{-at}dt + \left(\frac{1}{\omega}\right)\int_0^\infty e^{-at}\cos(\omega t)dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{\omega}\int_0^\infty \cos(\omega t)e^{-at}dt - \left(\frac{a}{\omega}\right)^2\int_0^\infty t e^{-at}\sin(\omega t)dt\right]$$



thus $\frac{2}{\pi}\left(\int_0^\infty t e^{-at} \sin(\omega t)\,dt + \left(\frac{a}{\omega}\right)^2 \int_0^\infty t e^{-at} \sin(\omega t)\,dt\right) = \left(\frac{2}{\pi}\right)\left(\frac{2}{\omega}\right)\int_0^\infty \cos(\omega t) e^{-at}\,dt$

$$\left(1+\left(\frac{a}{\omega}\right)^2\right)\left(\frac{2}{\pi}\right)\int_0^\infty t e^{-at} \sin(\omega t)\,dt = \left(\frac{2}{\pi}\right)\left(\frac{2}{\omega}\right)\int_0^\infty \cos(\omega t) e^{-at}\,dt$$

$$\left(\frac{a^2+\omega^2}{\omega^2}\right)\left(\frac{2}{\pi}\right)\int_0^\infty t e^{-at} \sin(\omega t)\,dt = \left(\frac{2}{\pi}\right)\left(\frac{2}{\omega}\right)\int_0^\infty \cos(\omega t) e^{-at}\,dt$$

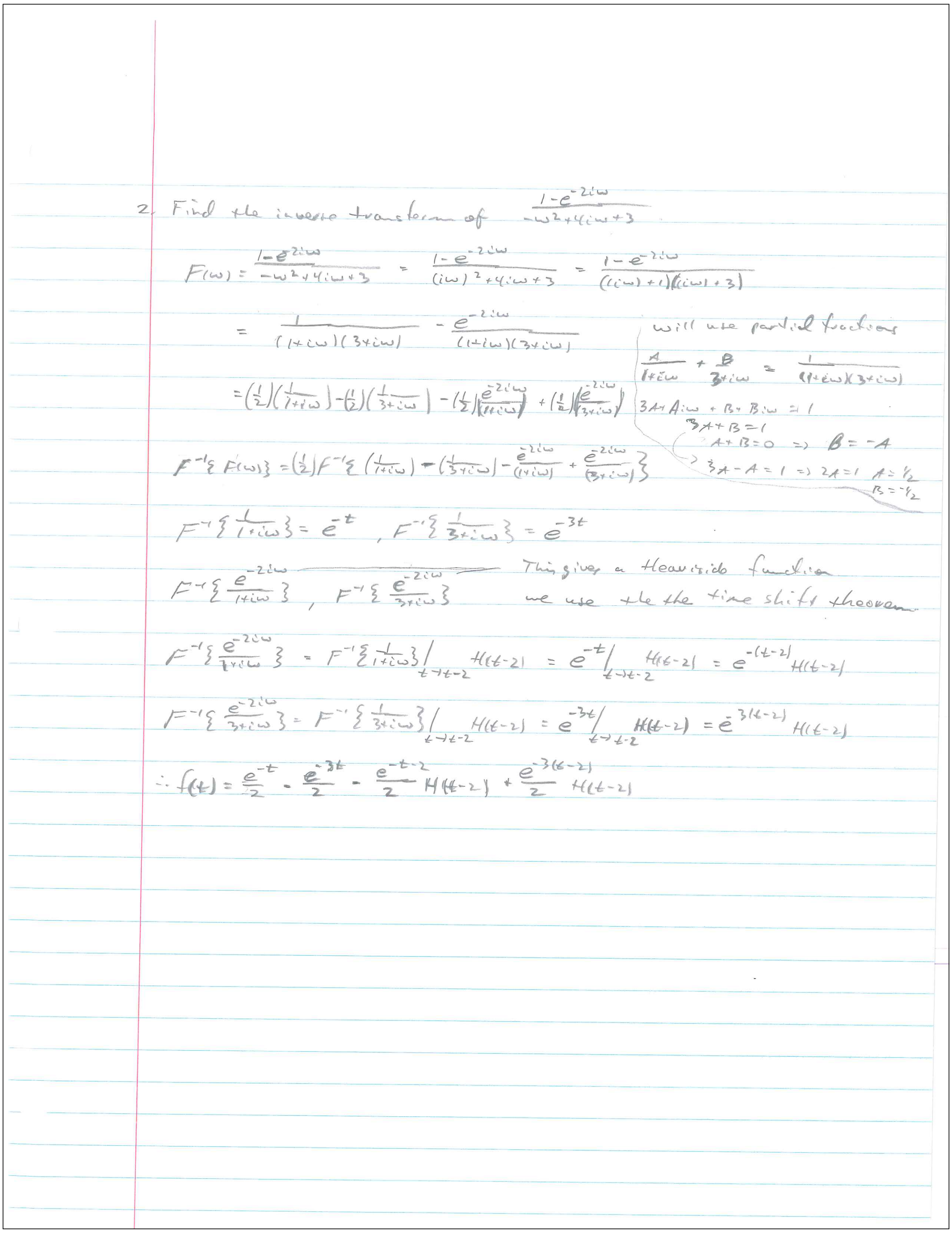
Recall $\int_0^\infty \cos(\omega t) e^{-at}\,dt = \frac{a}{a^2+\omega^2}$

$$\therefore \overline{B(\omega)} = \left(\frac{2}{\pi}\right)\int_0^\infty t e^{-at} \sin(\omega t)\,dt = \left(\frac{2}{\pi}\right)\left(\frac{2}{\omega}\right)\left(\frac{a}{a^2+\omega^2}\right)\left(\frac{\omega^2}{a^2+\omega^2}\right)$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{2a}{a^2+\omega^2}\right)\left(\frac{\omega}{a^2+\omega^2}\right) = \left(\frac{2}{\pi}\right)\left(\frac{2a\omega}{(a^2+\omega^2)^2}\right)$$

thus $B(\omega) = \frac{2}{\pi}\left(\frac{2a\omega}{(a^2+\omega^2)^2}\right)$ Sine transform

$A(\omega) = \frac{2}{\pi}\left(\frac{a^2-\omega^2}{(a^2+\omega^2)^2}\right)$ Cosine transform



2. Find the inverse transform of $\dfrac{1-e^{-2i\omega}}{-\omega^2+4i\omega+3}$

$$F(\omega) = \frac{1-e^{2i\omega}}{-\omega^2+4i\omega+3} = \frac{1-e^{-2i\omega}}{(i\omega)^2+4i\omega+3} = \frac{1-e^{-2i\omega}}{((i\omega)+1)((i\omega)+3)}$$

$$= \frac{1}{(1+i\omega)(3+i\omega)} - \frac{e^{-2i\omega}}{(1+i\omega)(3+i\omega)}$$

will use partial fractions

$$\frac{A}{1+i\omega} + \frac{B}{3+i\omega} = \frac{1}{(1+i\omega)(3+i\omega)}$$

$3A + Ai\omega + B + Bi\omega = 1$

$3A+B=1$

$A+B=0 \Rightarrow B=-A$

$3A-A=1 \Rightarrow 2A=1 \quad A=1/2$

$B=-1/2$

$$= \left(\tfrac{1}{2}\right)\left(\frac{1}{1+i\omega}\right) - \left(\tfrac{1}{2}\right)\left(\frac{1}{3+i\omega}\right) - \left(\tfrac{1}{2}\right)\left(\frac{e^{-2i\omega}}{1+i\omega}\right) + \left(\tfrac{1}{2}\right)\left(\frac{e^{-2i\omega}}{3+i\omega}\right)$$

$$F^{-1}\{F(\omega)\} = \left(\tfrac{1}{2}\right)F^{-1}\left\{\left(\frac{1}{1+i\omega}\right) - \left(\frac{1}{3+i\omega}\right) - \frac{e^{-2i\omega}}{(1+i\omega)} + \frac{e^{-2i\omega}}{(3+i\omega)}\right\}$$

$$F^{-1}\left\{\frac{1}{1+i\omega}\right\} = e^{-t} \quad , \quad F^{-1}\left\{\frac{1}{3+i\omega}\right\} = e^{-3t}$$

$$F^{-1}\left\{\frac{e^{-2i\omega}}{1+i\omega}\right\} \quad , \quad F^{-1}\left\{\frac{e^{-2i\omega}}{3+i\omega}\right\}$$

This gives a Heaviside function
we use the the time shift theorem

$$F^{-1}\left\{\frac{e^{-2i\omega}}{1+i\omega}\right\} = F^{-1}\left\{\frac{1}{1+i\omega}\right\}\Big|_{t \to t-2} H(t-2) = e^{-t}\Big|_{t \to t-2} H(t-2) = e^{-(t-2)}H(t-2)$$

$$F^{-1}\left\{\frac{e^{-2i\omega}}{3+i\omega}\right\} = F^{-1}\left\{\frac{1}{3+i\omega}\right\}\Big|_{t \to t-2} H(t-2) = e^{-3t}\Big|_{t \to t-2} H(t-2) = e^{-3(t-2)} H(t-2)$$

$$\therefore f(t) = \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} - \frac{e^{-t-2}}{2}H(t-2) + \frac{e^{-3(t-2)}}{2}H(t-2)$$

3) Find the deflection in the beam using either the fourier cosine, sine or complex transform.

$$EI\frac{d^4y}{dx^4} + k\,y(x) = -p(x) \quad \text{where } p(x) = \begin{cases} 0 & \text{for } -\infty < x < \ell \\ P_0(\ell + x) & \text{for } -\ell < x < 0 \\ P_0(\ell - x) & \text{for } 0 < x < \ell \\ 0 & \text{for } \ell < x < \infty \end{cases}$$

$p(x)$ is an even function so we will assume that $y(x)$ will also be an even function. We will use the cosine transform.

$$p(x) = \frac{2}{\pi}\int_0^\infty \bar{P}_c(\omega)\cos(\omega x)\,d\omega \quad \text{and} \quad y(x) = \frac{2}{\pi}\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega$$

now let's expand the load in a fourier cosine integral

$$p(x) = \frac{2}{\pi}\int_0^\infty p_c(\omega)\cos(\omega x)\,d\omega \quad \text{where } p_c(\omega) = \int_0^\infty p(x)\cos(\omega x)\,dx$$

$$p_c(\omega) = \int_0^\infty p(x)\cos(\omega x)\,dx = \int_0^\ell \left(\frac{P_0}{\ell^2}\right)(\ell - x)\cos(\omega x)\,dx$$

$$= \left(\frac{P_0}{\ell^2}\right)\int_0^\ell (\ell x \omega)\cos(\omega x)\,dx = \left(\frac{P_0}{\ell^2}\right)\int_0^\ell \ell\cos(\omega x)\,dx - \left(\frac{P_0}{\ell^2}\right)\int_0^\ell x\cos(\omega x)\,dx$$

$$= \left(\frac{P_0}{\ell}\right)\left(\frac{1}{\omega}\right)(\sin(\omega x))\Big|_0^\ell - \left(\frac{P_0}{\ell^2}\right)\left(\left(\frac{x}{\omega}\sin(\omega x)\right)\Big|_0^\ell - \left(\frac{1}{\omega}\right)\int_0^\ell \sin(\omega x)\,dx\right)$$

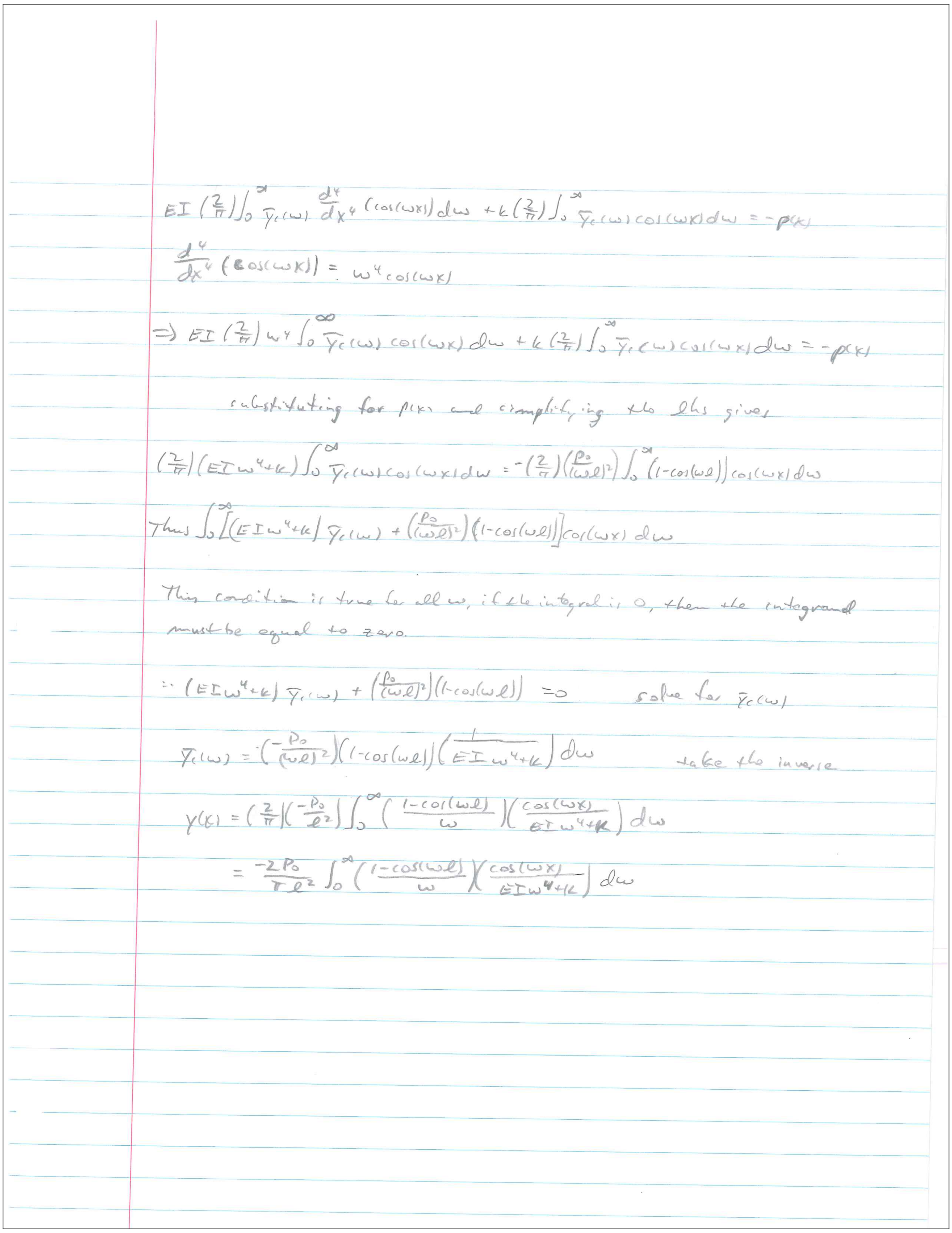
$$= \left(\frac{P_0}{\omega\ell}\right)(\sin(\omega\ell)) - \left(\frac{P_0}{\ell^2}\right)\left(\left(\frac{\ell}{\omega}\right)\sin(\omega\ell) + \left(\frac{1}{\omega^2}\right)(\cos(\omega x))\Big|_0^\ell\right)$$

$$= \left(\frac{P_0}{\omega\ell}\right)\sin(\omega\ell) - \left(\frac{P_0}{\omega\ell}\right)\sin(\omega\ell) - \left(\frac{P_0}{(\omega\ell)^2}\right)(\cos(\omega\ell) - 1)$$

$$\therefore\ p(x) = \frac{2}{\pi}\int_0^\infty p_c(\omega)\cos(\omega x)\,d\omega = \frac{2}{\pi}\left(\frac{P_0}{(\omega\ell)^2}\right)\int_0^\infty (1 - \cos(\omega\ell))\cos(\omega x)\,d\omega$$

$y(x) = \frac{2}{\pi}\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega$ let's insert this into the D.E.

$$EI\frac{d^4}{dx^4}\left(\frac{2}{\pi}\right)\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega + k\left(\frac{2}{\pi}\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega = -p(x)$$



$$EI\left(\frac{2}{\pi}\right)\int_0^\infty \bar{y}_c(\omega)\frac{d^4}{dx^4}(\cos(\omega x))\,d\omega + k\left(\frac{2}{\pi}\right)\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega = -p(x)$$

$$\frac{d^4}{dx^4}(\cos(\omega x)) = \omega^4\cos(\omega x)$$

$$\Rightarrow EI\left(\frac{2}{\pi}\right)\omega^4\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega + k\left(\frac{2}{\pi}\right)\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega = -p(x)$$

substituting for $p(x)$ and simplifying the lhs gives

$$\left(\frac{2}{\pi}\right)(EI\omega^4+k)\int_0^\infty \bar{y}_c(\omega)\cos(\omega x)\,d\omega = -\left(\frac{2}{\pi}\right)\left(\frac{P_0}{(\omega\ell)^2}\right)\int_0^\infty (1-\cos(\omega\ell))\cos(\omega x)\,d\omega$$

$$\text{Thus }\int_0^\infty \left[(EI\omega^4+k)\bar{y}_c(\omega) + \left(\frac{P_0}{(\omega\ell)^2}\right)(1-\cos(\omega\ell))\right]\cos(\omega x)\,d\omega$$

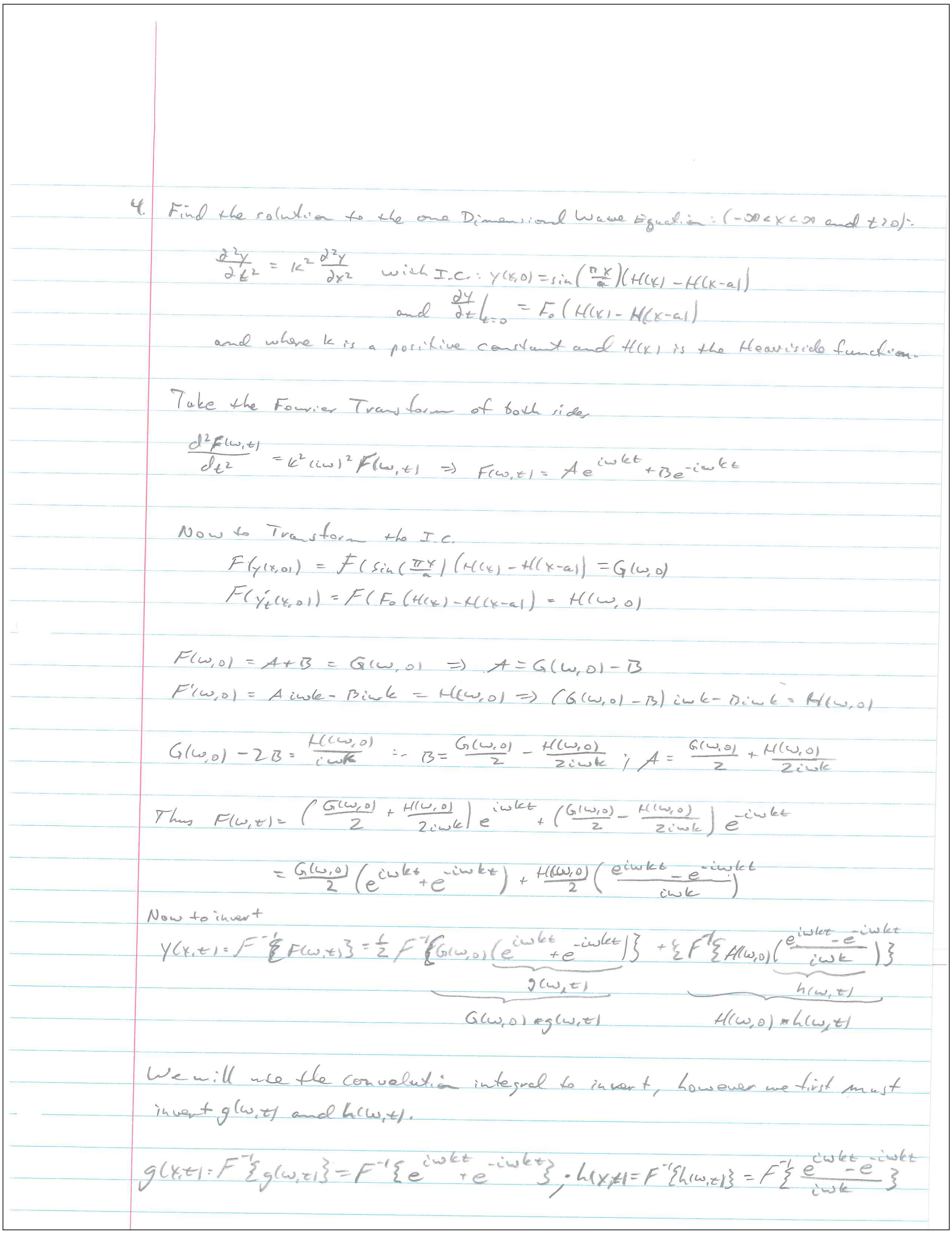
This condition is true for all $\omega$, if the integral is 0, then the integrand must be equal to zero.

$$\therefore (EI\omega^4+k)\bar{y}_c(\omega) + \left(\frac{P_0}{(\omega\ell)^2}\right)(1-\cos(\omega\ell)) = 0$$ solve for $\bar{y}_c(\omega)$

$$\bar{y}_c(\omega) = \left(\frac{-P_0}{(\omega\ell)^2}\right)(1-\cos(\omega\ell))\left(\frac{1}{EI\omega^4+k}\right)d\omega$$ take the inverse

$$y(x) = \left(\frac{2}{\pi}\right)\left(\frac{-P_0}{\ell^2}\right)\int_0^\infty \left(\frac{1-\cos(\omega\ell)}{\omega}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega$$

$$= \frac{-2P_0}{\pi\ell^2}\int_0^\infty \left(\frac{1-\cos(\omega\ell)}{\omega}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega$$



4. Find the solution to the one Dimensional Wave Equation: $(-\infty < x < \infty$ and $t > 0)$:

$$\frac{\partial^2 y}{\partial t^2} = k^2 \frac{\partial^2 y}{\partial x^2} \quad \text{with I.C.: } y(x,0) = \sin\left(\frac{\pi x}{a}\right)\left(H(x) - H(x-a)\right)$$

$$\text{and } \left.\frac{\partial y}{\partial t}\right|_{t=0} = F_0\left(H(x) - H(x-a)\right)$$

and where $k$ is a positive constant and $H(x)$ is the Heaviside function.

Take the Fourier Transform of both sides

$$\frac{d^2 F(\omega,t)}{dt^2} = k^2 (i\omega)^2 F(\omega,t) \Rightarrow F(\omega,t) = Ae^{i\omega kt} + Be^{-i\omega kt}$$

Now to Transform the I.C.

$$F(y(x,0)) = F\left(\sin\left(\frac{\pi x}{a}\right)\left(H(x) - H(x-a)\right)\right) = G(\omega,0)$$

$$F(y_t'(x,0)) = F\left(F_0\left(H(x) - H(x-a)\right)\right) = H(\omega,0)$$

$$F(\omega,0) = A + B = G(\omega,0) \Rightarrow A = G(\omega,0) - B$$

$$F'(\omega,0) = Ai\omega k - Bi\omega k = H(\omega,0) \Rightarrow (G(\omega,0) - B)i\omega k - Bi\omega k = H(\omega,0)$$

$$G(\omega,0) - 2B = \frac{H(\omega,0)}{i\omega k} \therefore B = \frac{G(\omega,0)}{2} - \frac{H(\omega,0)}{2i\omega k} \; ; \; A = \frac{G(\omega,0)}{2} + \frac{H(\omega,0)}{2i\omega k}$$

$$\text{Thus } F(\omega,t) = \left(\frac{G(\omega,0)}{2} + \frac{H(\omega,0)}{2i\omega k}\right)e^{i\omega kt} + \left(\frac{G(\omega,0)}{2} - \frac{H(\omega,0)}{2i\omega k}\right)e^{-i\omega kt}$$

$$= \frac{G(\omega,0)}{2}\left(e^{i\omega kt} + e^{-i\omega kt}\right) + \frac{H(\omega,0)}{2}\left(\frac{e^{i\omega kt} - e^{-i\omega kt}}{i\omega k}\right)$$

Now to invert

$$y(x,t) = F^{-1}\{F(\omega,t)\} = \frac{1}{2}F^{-1}\{\underbrace{G(\omega,0)\underbrace{\left(e^{i\omega kt} + e^{-i\omega kt}\right)}_{g(\omega,t)}}_{G(\omega,0)*g(\omega,t)}\} + \frac{1}{2}F^{-1}\{\underbrace{H(\omega,0)\underbrace{\left(\frac{e^{i\omega kt} - e^{-i\omega kt}}{i\omega k}\right)}_{h(\omega,t)}}_{H(\omega,0)*h(\omega,t)}\}$$

We will use the convolution integral to invert, however we first must invert $g(\omega,t)$ and $h(\omega,t)$.

$$g(x,t) = F^{-1}\{g(\omega,t)\} = F^{-1}\{e^{i\omega kt} + e^{-i\omega kt}\} \; ; \; h(x,t) = F^{-1}\{h(\omega,t)\} = F^{-1}\left\{\frac{e^{i\omega kt} - e^{-i\omega kt}}{i\omega k}\right\}$$

$$g(x,t) = F^{-1}\{e^{i\omega kt} + e^{-i\omega kt}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{i\omega kt} + e^{-i\omega kt})e^{i\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{i\omega(kt+x)} + e^{-i\omega(kt-x)})\,d\omega$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x+kt)}\,d\omega}_{\delta(x+kt)} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-kt)}\,d\omega}_{\delta(x-kt)}$$

$$\therefore g(x,t) = \delta(x+kt) + \delta(x-kt)$$

$$h(x,t) = F^{-1}\left\{\frac{e^{i\omega kt} - e^{-i\omega kt}}{i\omega k}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{i\omega kt} - e^{-i\omega kt}}{i\omega k}\right)e^{i\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(kt+x)}}{i\omega k}\,d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(kt-x)}}{i\omega k}\,d\omega$$

$$= \frac{1}{k}\Big(\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(x+kt)}}{i\omega}\,d\omega}_{H(x+kt)} - \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(x-kt)}}{i\omega}\,d\omega}_{H(x-kt)}$$

$\therefore h(x,t) = H(x+b) - H(x-c)$ where $b = kt+x$ and $c = kt-x$

recall $y(x,t) = \frac{1}{2}F^{-1}\{G(\omega,0)\,g(\omega,t)\} + \frac{1}{2}F^{-1}\{H(\omega,0)\,h(\omega,t)\}$

$$= \frac{1}{2}\int_{-\infty}^{\infty}(\delta(\lambda+kt) + \delta(\lambda-kt))(H(x-\lambda) - H(x-\lambda-a))\sin\left(\frac{\pi(x-\lambda)}{a}\right)d\lambda$$
$$+ \frac{F_0}{2k}\int_{-\infty}^{\infty}(H(x-\lambda+kt) - H(x-\lambda-kt))(H(\lambda) - H(\lambda-a))\,d\lambda$$

$$= \frac{1}{2}(H(x+kt) - H(x+kt-a))\sin\left(\frac{\pi(x+kt)}{a}\right) + \frac{1}{2}(H(x-kt) - H(x-kt-a))\sin\left(\frac{\pi(x-kt)}{a}\right)$$
$$+ \frac{F_0}{2k}\underbrace{\int_0^a H(x-\lambda+kt) - H(x-\lambda-kt)\,d\lambda}_{①}$$

$$① = \int_0^a H(x-\lambda+kt) - H(\lambda-\lambda-kt)\,d\lambda$$

$$= \int_{x+kt}^{a} d\lambda\, H(x-a+kt) - \int_{x+kt}^{0} d\lambda\, H(x+kt) - \int_{x-kt}^{a} d\lambda\, H(x-a-kt) + \int_{x-kt}^{0} d\lambda\, H(x-kt)$$

$$= (a-x-kt)H(x-a+kt) + (x+kt)H(x+kt) - (a-x+kt)H(x-a-kt) - (x-kt)H(x-kt)$$

$$= (x+kt)(H(x+kt) - H(x-a+kt)) - (x-kt)(H(x-kt) - H(x-a-kt))$$
$$+ a(H(x-a+kt - H(x-a-kt))$$

Thus $y(x,t) = \frac{1}{2}(H(x+kt) - H(x+kt-a))\sin\left(\frac{\pi(x+kt)}{a}\right) + \frac{1}{2}(H(x-kt) - H(x-kt-a))\sin\left(\frac{\pi(x-kt)}{a}\right)$

$$+ \frac{F_0}{2k}\Big((x+kt)(H(x+kt - H(x-a+kt)) - (x-kt)(H(x-kt) - H(x-a-kt))$$
$$+ a(H(x-a+kt) - H(x-a-kt))\Big)$$

## 2.11 HW 10

### 2.11.1 Problems to solve

Homework Set No. 10 NEEP 547
Due November 22, 2013 DLH

1. Evaulate the following integrals:

   (a) (3pts) $\int_0^{1+i} e^{2z}\,dz$ and (b) (3pts) $\int_C \dfrac{z+4}{(z^2+1)(z-1)}\,dz$, where $C$ is the circle $|z|=2$.

2. What is the value of

   $$\int_C \frac{\sin(2z)}{z^2-4\,z+5}\,dz, \quad \text{for:}$$

   (a) (3pts) if $C$ is the circle $|z|=1$ ?
   (b) (3pts) if $C$ is the circle $|z-2i|=3$ ?
   (c) (3pts) if $C$ is the circle $|z-1+2i|=2$ ?

3. page 1005, problems 4, 6 and 10: Evaluate the $\int f(z)\,dz$ for the given function and closed (positively) oriented path ($\Gamma$).

   4). (3pts) $f(z)=\dfrac{2\,z^3}{(z-2)^2}$; where $\Gamma$ is the rectangle having vertices $4\pm i$ and $-4\pm i$.
   6). (3pts) $f(z)=\dfrac{\cos(z-i)}{(z+2i)^3}$; where $\Gamma$ is any path enclosing $-2i$.
   10). (3pts) $f(z)=(z-i)^2$ where $\Gamma$ is the semicircle of radius 1 about 0 from $i$ to $-i$.

4. Expand $f(z)=1/(z^2+3\,z+2)$ in a Taylor series (a) (3pts) about the point $z=0$ and (b) (3pts) about the point $z=2$. Determine the radius of convergence for each case.

5. page 1018, problems 6 and 8: Find the Taylor series of the function about the indicated point and determine the radius of convergence: (6) (3pts) $f(z)=1/(2+z)$, point: $1-8i$ and (8) (3pts) $f(z)=1+1/(2+z^2)$, point: $i$.

6. Obtain two distinct Laurent expansions for $f(z)=(3z+1)/(z^2-1)$ around $z=1$ and tell where each converges.

### 2.11.2 problem 1

Evaluate $(a)\int_0^{1+i} e^{2z}dz$ and $(b)\int_C \frac{z+4}{(z^2+1)(z-1)}dz$ where $C$ is the circle $|z|=2$

**Solution**:

**part a**.

Checking if the function $f(z)$ is analytic ( This is not actually needed for this part, since integration is not over a closed path).

Let $z=x+iy$, hence $f(z)=e^{2z}=e^{2(x+iy)}$, hence

$$\begin{aligned} f(z) &= e^{2x}\left(\cos 2y + i\sin 2y\right)\\ &= e^{2x}\cos 2y + ie^{2x}\sin 2y\\ &= u+iv \end{aligned}$$

Hence $u=e^{2x}\cos 2y, v=e^{2x}\sin 2y$ and

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2e^{2x}\cos 2y\\ \frac{\partial v}{\partial y} &= 2e^{2x}\cos 2y \end{aligned}$$

Hence $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and

$$\begin{aligned} \frac{\partial u}{\partial y} &= -2e^{2x}\sin 2y\\ \frac{\partial v}{\partial x} &= 2e^{2x}\sin 2y \end{aligned}$$

Hence $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$, therefore it is analytic function.

Complex integration can be done along any path. Using the direct path gives

$$\begin{aligned}\int_0^{1+i} e^{2z}dz &= \left[\frac{e^{2z}}{2}\right]_0^{1+i}\\ &= \frac{1}{2}\left(e^{2+2i}-1\right)\end{aligned}$$

**part b**:

$$f(z)=\frac{z+4}{\left(z^2+1\right)(z-1)}$$

This function can be verified to be analytic. Verified using CAS

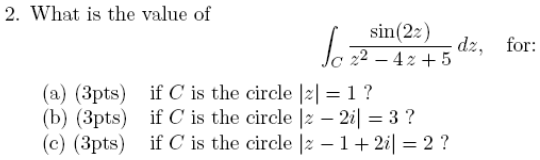
```
f = (z + 4)/((z^2 + 1) (z - 1))
u = ComplexExpand@Re[f /. z -> (x + I y)];
v = ComplexExpand@Im[f /. z -> (x + I y)];
Simplify[D[u, x] == D[v, y]]

    True

Simplify[D[u, y] == - D[v, x]]

  True
```

The function $f(z)$ is analytic but it does have singularity at $z=1,-i,+i$, therefore it is not an entire function (An entire function is analytic everywhere and has no singularity).

$f(z)=\frac{z+4}{(z-i)(z+i)(z-1)}$ has all its poles inside $C$, therefore, using residual theory $\int_C f(z)\,dz = 2\pi i\sum_k residual\left(f(z)\right)_k$ gives

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\left(\frac{z+4}{(z-i)(z+i)}\right)_{z=1}+\left(\frac{z+4}{(z-i)(z-1)}\right)_{z=-i}+\left(\frac{z+4}{(z+i)(z-1)}\right)_{z=i}\right]\\ &= 2\pi i\left(\frac{1+4}{(1-i)(1+i)}+\frac{-i+4}{(-i-i)(-i-1)}+\frac{i+4}{(i+i)(i-1)}\right)\\ &= 2\pi i\left(\frac{5}{2}+\frac{-i+4}{-2+2i}+\frac{i+4}{-2-2i}\right)\\ &= 2\pi i\left(\frac{5}{2}+\frac{-i+4}{-2+2i}+\frac{i+4}{-2-2i}\right)\\ &= 2\pi i\left(\frac{5}{2}-\frac{5}{2}\right)\\ &= 0\end{aligned}$$

### 2.11.3 Problem 2



2. What is the value of

$$\int_C \frac{\sin(2z)}{z^2-4z+5}\,dz,\quad \text{for:}$$

(a) (3pts) if $C$ is the circle $|z|=1$ ?
(b) (3pts) if $C$ is the circle $|z-2i|=3$ ?
(c) (3pts) if $C$ is the circle $|z-1+2i|=2$ ?

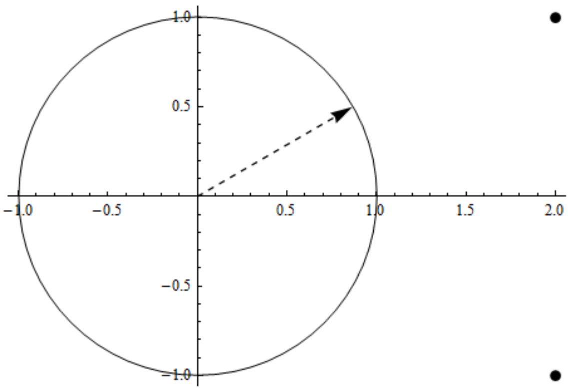
Figure 2.25: Problem description

**Solution**

The integrand $f(z)$ is $\frac{\sin(2z)}{(z-(2-i))(z-(2+i))}$, hence it is poles at $z_1=2-i$ and $z_2=2+i$

**part a**

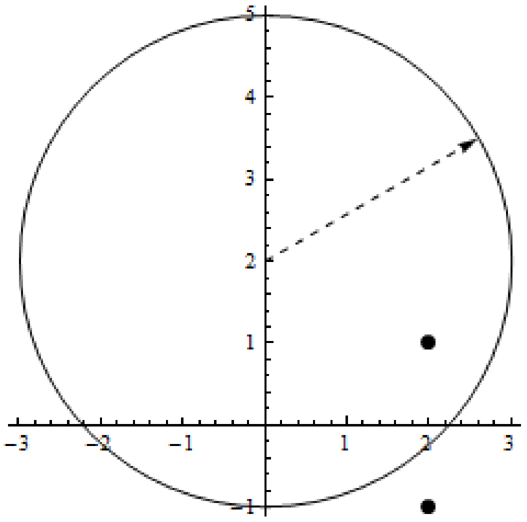
In this case, both poles are outside $C$, which is unit circle centered at 0, hence $\int_C \frac{\sin(2z)}{(z-(2-i))(z-(2+i))}dz = 0$



**part b**

In this case, the circle is centered at $2i$ and its radius is 3. A plot shows the location so of poles

```
Labeled[Graphics[{
   Circle[{0, 2}, 3],
   {PointSize[Large], Point[{{2, -1}, {2, 1}}]},
   {Dashed, Arrow[{{0, 2}, {3 Cos[30 Degree], 2 + 3 Sin[30 Degree]}}]}
   }, Axes -> True], "Poles locations reference to C loop"]
```



Poles locations reference to C loop

We see there is one pole inside $C$ from the plot. But to determine if each pole is inside $C$ or not, we check for $|z_i - center|$ and see if this is less than the circle radius or not. We do this for each pole. For pole $z_1 = 2 - i$ we obtain $|(2-i)-(2i)| = |2+3i| = \sqrt{4+9} = \sqrt{13} = 3.605$ hence this pole is outside $C$ as shown in the diagram. For pole $z_2 = 2 + i$ we obtain $|(2+i)-(2i)| = |2-i| = \sqrt{4+1} = \sqrt{5} = 2.236$ which is smaller that the radius of the circle $C$ hence this pole is inside. Therefore, using the residual theorem we obtain

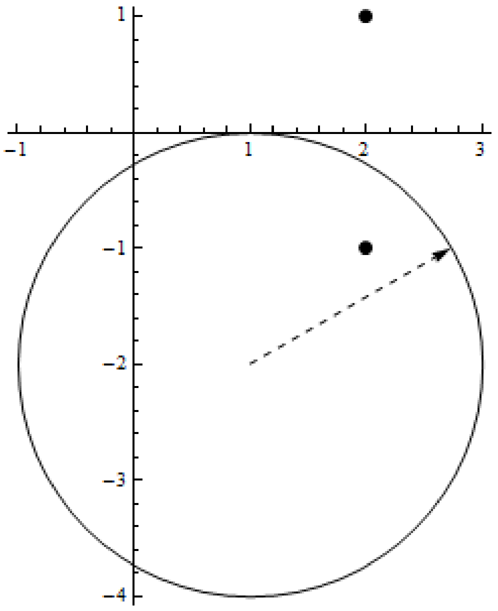
$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\left(\frac{\sin(2z)}{(z-(2-i))}\right)_{z=(2+i)}\right]\\ &= 2\pi i\left(\frac{\sin(2(2+i))}{((2+i)-(2-i))}\right)\\ &= 2\pi i\frac{\sin(4+2i)}{2i}\\ &= \pi\sin(4+2i)\end{aligned}$$

**Part c**

In this case, the circle is $|z-(1-2i)| = 2$ , hence it is centered at point $(1,-2i)$ and its radius is 2.

For pole $z_1 = 2 - i$ we obtain $|(2-i)-(1-2i)| = |1+i| = \sqrt{2}$ hence this pole is inside $C$ as shown in the diagram below. For pole $z_2 = 2+i$ we obtain $|(2+i)-(1-2i)| = |1+3i| = \sqrt{1+9} = \sqrt{10} = 3.1623$ which is larger than the radius of the circle $C$ hence this pole is outside.

```
Labeled[Graphics[{
   Circle[{1, -2}, 2],
   {PointSize[Large], Point[{{2, -1}, {2, 1}}]},
   {Dashed,
    Arrow[{{1, -2}, {1 + 2 Cos[30 Degree], -2 + 2 Sin[30 Degree]}}]}
   }, Axes -> True], "Poles locations reference to C loop"]
```

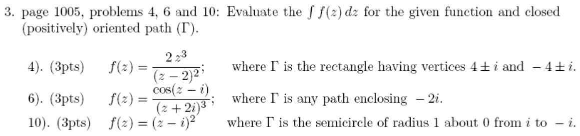


Poles locations reference to C loop

Therefore, using the residual theorem

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\left(\frac{\sin(2z)}{(z-(2+i))}\right)_{z=(2-i)}\right]\\ &= 2\pi i\left(\frac{\sin(2(2-i))}{((2-i)-(2+i))}\right)\\ &= 2\pi i\frac{\sin(4-2i)}{-2i}\\ &= -\pi\sin(4+2i)\end{aligned}$$

### 2.11.4 problem 3



3. page 1005, problems 4, 6 and 10: Evaluate the $\int f(z)\,dz$ for the given function and closed (positively) oriented path ($\Gamma$).

4). (3pts) $f(z) = \frac{2z^3}{(z-2)^2}$; where $\Gamma$ is the rectangle having vertices $4 \pm i$ and $-4 \pm i$.
6). (3pts) $f(z) = \frac{\cos(z-i)}{(z+2i)^3}$; where $\Gamma$ is any path enclosing $-2i$.
10). (3pts) $f(z) = (z-i)^2$ where $\Gamma$ is the semicircle of radius 1 about 0 from $i$ to $-i$.

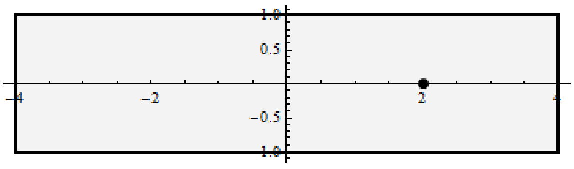
Figure 2.26: Problem description

#### Solution

#### Part (a)

The poles of $f(z) = \frac{2z^3}{(z-2)^2}$ are at $z = 2$ (order 2). The following plot shows the location of the pole relative to $\Gamma$

```
Labeled[Graphics[{
   {EdgeForm[Thick], Opacity[.05], Rectangle[{-4, -1}, {4, 1}]},
   {PointSize[Large], Point[{{2, 0}}]}
   }, Axes -> True], "Pole locations reference to C loop"]
```



Pole locations reference to C loop

The pole is inside Γ hence using residual we obtain

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}2z^3\right)_{z=2}\right]$$

Where $n = 2$ in the above, since that is the order of the pole. Therefore

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\left(\frac{d}{dz}2z^3\right)_{z=2}\right]\\ &= (2\pi i)\left(6z^2\right)_{z=2}\\ &= (2\pi i)\left(6(2)^2\right)\\ &= 48\pi i\end{aligned}$$

**Part b**

The poles of $f(z) = \frac{\cos(z-i)}{(z+2i)^3}$. The poles are at $z = -2i$ of order 3. Therefore Γ includes the pole inside it, and we can use the residual, hence

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\cos(z-i)\right)_{z=-2i}\right]$$

Where $n = 3$ in the above, since that is the order of the pole. Therefore

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\frac{1}{2}\left(\frac{d^2}{dz^2}\cos(z-i)\right)_{z=-2i}\right]\\ &= 2\pi i\left[\frac{1}{2}\left(\frac{d}{dz}(-\sin(z-i))\right)_{z=-2i}\right]\\ &= 2\pi i\left[\frac{1}{2}(-\cos(z-i))_{z=-2i}\right]\\ &= 2\pi i\frac{1}{2}(-\cos(-2i-i))\\ &= -\pi i\cos(-3i)\end{aligned}$$

Or

$$\int_C f(z)\,dz = -i\pi\cosh(3)$$

**Part c**

$f(z) = (z-i)^2$ has no poles. Hence if we can show it is analytic, then the closed path integral will be zero by Cauchy theorem. We can verify also it is analytic. Let $z = x + iy$, hence

$$\begin{aligned}f(z) &= \left(\left(x+iy\right)-i\right)^2\\ &= x^2+2ixy-2ix-y^2+2y-1\\ &= \left(x^2-y^2-1\right)+i\left(2xy-2x\right)\end{aligned}$$

Therefore $u = x^2 - y^2 - 1$ and $v = 2xy - 2x$ and

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2x\\ \frac{\partial v}{\partial y} &= 2x\end{aligned}$$

Hence $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and

$$\begin{aligned}\frac{\partial u}{\partial y}&=-2y\\ \frac{\partial v}{\partial x}&=2y-2\end{aligned}$$

Hence $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$, therefore it is analytic function and the integral is zero.

### 2.11.5 problem 4

Expand $f(z)=\frac{1}{z^2+3z+2}$ in Taylor series. (a) About point $z=0$ (b) about point $z=2$. Determine the radius of convergence for each case.

**Solution**:

#### 2.11.5.1 part (a)

Taylor series is defined as

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots+\frac{(z-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

However, we do not calculate the series directly from the above definition. Instead, using power series method, we can formulate this to finding the series

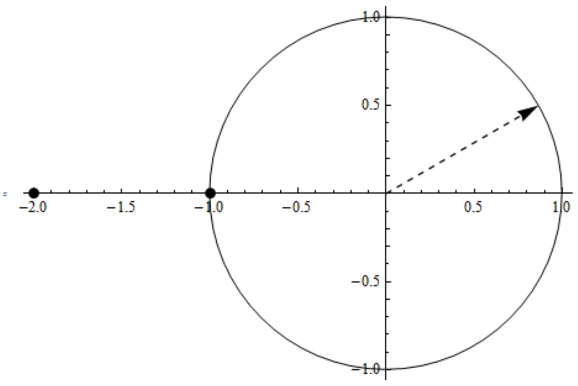
$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n$$

Where $z_0$ is the point to expand around. In this case it is zero. We need to find $c_n$.

To find radius of convergence $\rho$, we draw a circle, centered at the point of expansion, and extend it all the way to the first pole. $f(z)=\frac{1}{(z+2)(z+1)}$ hence it has a pole at $z=-2$ and $z=-1$ .therefore,

$$\rho=1$$

Here is a diagram



Poles locations reference unit circle

Now the Taylor series is found. We need to write $f(z)$ in the form $f(z)=\frac{1}{1-u}$ so we can use the geometric series which will be valid for only when $|u|<1$

$$\begin{aligned}f(z)&=\frac{1}{z^2+3z+2}=\frac{1}{(z+2)(z+1)}\\ &=\frac{1}{z+1}-\frac{1}{z+2}\\ &=\frac{1}{1-(-z)}-\frac{1}{2}\frac{1}{1-\left(\frac{-z}{2}\right)}\\ &=\sum_{n=0}^{\infty}(-1)^n z^n-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n\end{aligned}$$

Where the first geometric series converges for $|z|<1$ and the second converges for $\left|\frac{z}{2}\right|<1$

or $|z| < 2$. The series is

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n \\
&= \left(1 - z + z^2 - z^3 + z^4 + \cdots\right) - \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \frac{z^4}{2^4} + \cdots\right) \\
&= \left(1 - z + z^2 - z^3 + z^4 + \cdots\right) + \left(-\frac{1}{2} + \frac{z}{2^2} - \frac{z^2}{2^3} + \frac{z^3}{2^4} - \frac{z^4}{2^5} + \cdots\right) \\
&= \frac{1}{2} - z\left(1 - \frac{1}{2^2}\right) + z^2\left(1 - \frac{1}{2^3}\right) - z^3\left(1 - \frac{1}{2^4}\right) + \cdots \\
&= \frac{1}{2} - \frac{3}{4}z + \frac{7}{8}z^3 - \frac{15}{16}z^3 + \cdots
\end{aligned}$$

#### 2.11.5.2 Part b

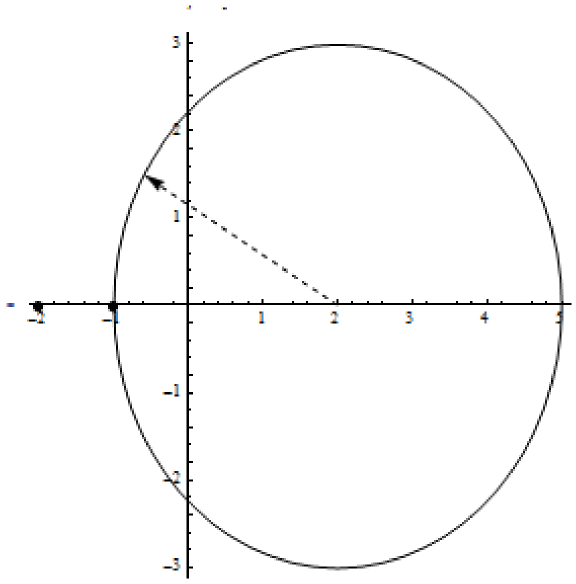
$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Where $z_0$ is the point to expand around. In this case it is 2.

To find radius of convergence $\rho$, we make a circle, centered at the point of expansion, and extend it all the way to the first pole. Therefore radius of convergence is

$$\rho = 3$$

As shown in this diagram



Poles locations reference unit circle

We need to find $c_n$. Writing

$$\begin{aligned}
f(z) &= \frac{1}{z^2 + 3z + 2} = \frac{1}{(z+2)(z+1)} \\
&= \frac{1}{z+1} - \frac{1}{z+2} \\
&= \frac{1}{(z-2)+3} - \frac{1}{(z-2)+4} \\
&= \frac{1}{3}\frac{1}{1+\left(\frac{z-2}{3}\right)} - \frac{1}{4}\frac{1}{1+\left(\frac{z-2}{4}\right)} \\
&= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{4}\right)^n
\end{aligned}$$

Where the first geometric series converges for $\left|\frac{z-2}{3}\right| < 1$ or $|z-2| < 3$ or $|z| < 5$ and the second converges for $\left|\frac{z-2}{4}\right| < 1$ or $|z-2| < 4$ or $|z| < 6$. And the series is

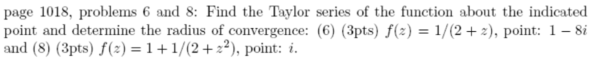
$$f(z) = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{4}\right)^n$$

$$= \frac{1}{3}\left(1 - \frac{1}{3}(z-2) + \left(\frac{1}{3}\right)^2(z-2)^2 - \left(\frac{1}{3}\right)^2(z-2)^3 + \cdots\right)$$
$$- \frac{1}{4}\left(1 - \frac{1}{4}(z-2) + \left(\frac{1}{4}\right)^2(z-2)^2 - \left(\frac{1}{4}\right)^3(z-2)^3 + \cdots\right)$$

$$= \left(\frac{1}{3} - \frac{1}{9}(z-2) + \frac{1}{27}(z-2)^2 - \frac{1}{27}(z-2)^3 + \cdots\right)$$
$$- \left(\frac{1}{4} - \frac{1}{16}(z-2) + \frac{1}{64}(z-2)^2 - \frac{1}{256}(z-2)^3 + \cdots\right)$$

$$= \frac{1}{12} - \frac{7}{144}(z-2) + \frac{37}{1728}(z-2)^2 - \frac{229}{6912}(z-2)^3 + \cdots$$

### 2.11.6 problem 5



page 1018, problems 6 and 8: Find the Taylor series of the function about the indicated point and determine the radius of convergence: (6) (3pts) $f(z) = 1/(2+z)$, point: $1-8i$ and (8) (3pts) $f(z) = 1 + 1/(2+z^2)$, point: $i$.

**solution**

**part (a)**

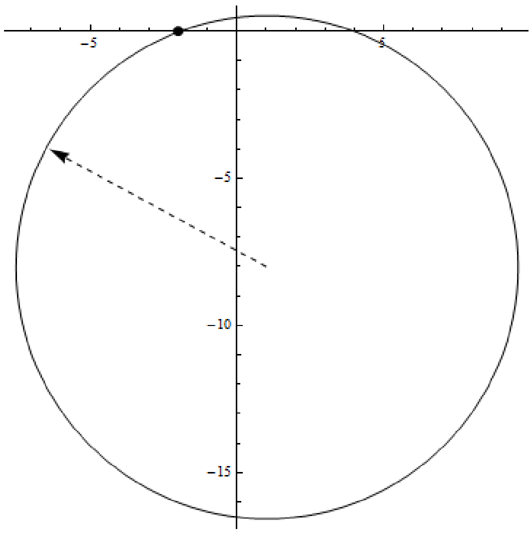
$$f(z) = \frac{1}{2+z}$$

around point $1-8i$.

To find radius of convergence $\rho$, we make a circle, centered at the point of expansion, and extend it all the way to the first pole. $f(z) = \frac{1}{2+z}$ hence it has a pole at $z = -2$, Therefore,

$$\rho = \sqrt{3^2 + 8^2} = \sqrt{73}$$

As seen by this diagram.



Poles locations reference unit circle

Now the Taylor series is found

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Where $z_0$ is the point to expand around. In this case it is $1-8i$. We need to find $c_n$. Writing

$$\begin{aligned}
f(z) &= \frac{1}{2+z} = \frac{1}{2+z-(1-8i)+(1-8i)} \\
&= \frac{1}{(3-8i)+(z-(1-8i))} \\
&= \frac{1}{(3-8i)}\frac{1}{1+\frac{1}{3-8i}(z-(1-8i))} \\
&= \frac{1}{(3-8i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3-8i}\right)^n(z-(1-8i))^n
\end{aligned}$$

Hence

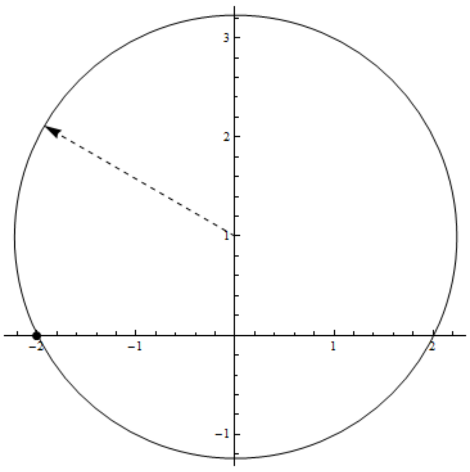
$$\begin{aligned}
f(z) &= \frac{1}{(3-8i)}\left(1-\left(\frac{1}{3-8i}\right)(z-(1-8i))+\left(\frac{1}{3-8i}\right)^2(z-(1-8i))^2+\cdots\right) \\
&= \frac{1}{(3-8i)}-\frac{1}{(3-8i)^2}(z-(1-8i))+\frac{1}{(3-8i)^3}(z-(1-8i))^2+\cdots \\
&= \left(\frac{3}{73}+\frac{8i}{73}\right)+\left(\frac{55}{5329}-\frac{48i}{5329}\right)(z-(1-8i))-\left(\frac{549}{389017}+\frac{296i}{389017}\right)(z-(1-8i))^2+\cdots
\end{aligned}$$

**part (b)**

$f(z)=1+\frac{1}{2+z^2}$ around point $i$. In this case, the center of circle is at $i$ and hence

$$\rho=\sqrt{1^2+2^2}=\sqrt{5}$$

As shown in this diagram



Poles locations reference unit circle

And Taylor series is

$$\begin{aligned}
f(z) &= 1+\frac{1}{2+z^2} \\
&= 1+\frac{1}{\left(z-i\sqrt{2}\right)\left(z+i\sqrt{2}\right)}
\end{aligned}$$

Doing partial fractions gives

$$f(z)=1+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}-iz}+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}+iz}$$

Convert so that $z-i$ appears. $iz=i(z-i)-1$, hence

$$\begin{aligned}
f(z) &= 1+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}-(i(z-i)-1)}+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}+(i(z-i)-1)} \\
&= 1+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}-i(z-i)+1}+\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}+i(z-i)-1} \\
&= 1+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)-i(z-i)}+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)+i(z-i)} \\
&= 1+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}\frac{1}{1-\frac{i}{\left(\sqrt{2}+1\right)}(z-i)}+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\frac{1}{1+\frac{i}{\left(\sqrt{2}-1\right)}(z-i)} \qquad (1)
\end{aligned}$$

Now geometric series can be used.

$$\begin{aligned}\frac{1}{1-\frac{i}{\left(\sqrt{2}+1\right)}\left(z-i\right)} &= \sum_{n}^{\infty}\left(\frac{i}{\left(\sqrt{2}+1\right)}\right)^{n}\left(z-i\right)^{n}\\ &= 1+\frac{i}{\sqrt{2}+1}\left(z-i\right)-\frac{1}{\left(\sqrt{2}+1\right)^{2}}\left(z-i\right)^{2}-\frac{i}{\left(\sqrt{2}+1\right)^{3}}\left(z-i\right)^{3}+\frac{1}{\left(\sqrt{2}+1\right)^{4}}\left(z-i\right)^{4}\cdots\end{aligned}$$

And

$$\begin{aligned}\frac{1}{1+\frac{i}{\left(\sqrt{2}-1\right)}\left(z-i\right)} &= \sum_{n}^{\infty}\left(-1\right)^{n}\left(\frac{i}{\left(\sqrt{2}-1\right)}\right)^{n}\left(z-i\right)^{n}\\ &= 1-\frac{i}{\sqrt{2}-1}\left(z-i\right)+\frac{1}{\left(\sqrt{2}-1\right)^{2}}\left(z-i\right)^{2}-\frac{i}{\left(\sqrt{2}-1\right)^{3}}\left(z-i\right)^{3}+\frac{1}{\left(\sqrt{2}-1\right)^{4}}\left(z-i\right)^{4}\cdots\end{aligned}$$

Substituing the above into Eq. (1) gives

$$\begin{aligned}f\left(z\right) = 1 &+ \frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}\left(1+\frac{i}{\sqrt{2}+1}\left(z-i\right)-\frac{1}{\left(\sqrt{2}+1\right)^{2}}\left(z-i\right)^{2}-\frac{i}{\left(\sqrt{2}+1\right)^{3}}\left(z-i\right)^{3}+\frac{1}{\left(\sqrt{2}+1\right)^{4}}\left(z-i\right)^{4}\cdots\right)\\ &+ \frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\left(1-\frac{i}{\sqrt{2}-1}\left(z-i\right)+\frac{1}{\left(\sqrt{2}-1\right)^{2}}\left(z-i\right)^{2}+\frac{i}{\left(\sqrt{2}-1\right)^{3}}\left(z-i\right)^{3}-\frac{1}{\left(\sqrt{2}-1\right)^{4}}\left(z-i\right)^{4}\cdots\right)\end{aligned}$$

Looking at coefficient of term $(z-i)^n$ power. For $n=0$ power the term is

$$\begin{aligned}c_0 &= 1+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\\ &= 2\end{aligned}$$

And for $c_1$ it is

$$\begin{aligned}c_1 &= \frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}\frac{i}{\sqrt{2}+1}-\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\frac{i}{\sqrt{2}-1}\\ &= -2i\end{aligned}$$

And for $c_2$

$$\begin{aligned}c_2 &= -\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}\frac{1}{\left(\sqrt{2}+1\right)^{2}}-\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\frac{1}{\left(\sqrt{2}-1\right)^{2}}\\ &= -5\end{aligned}$$

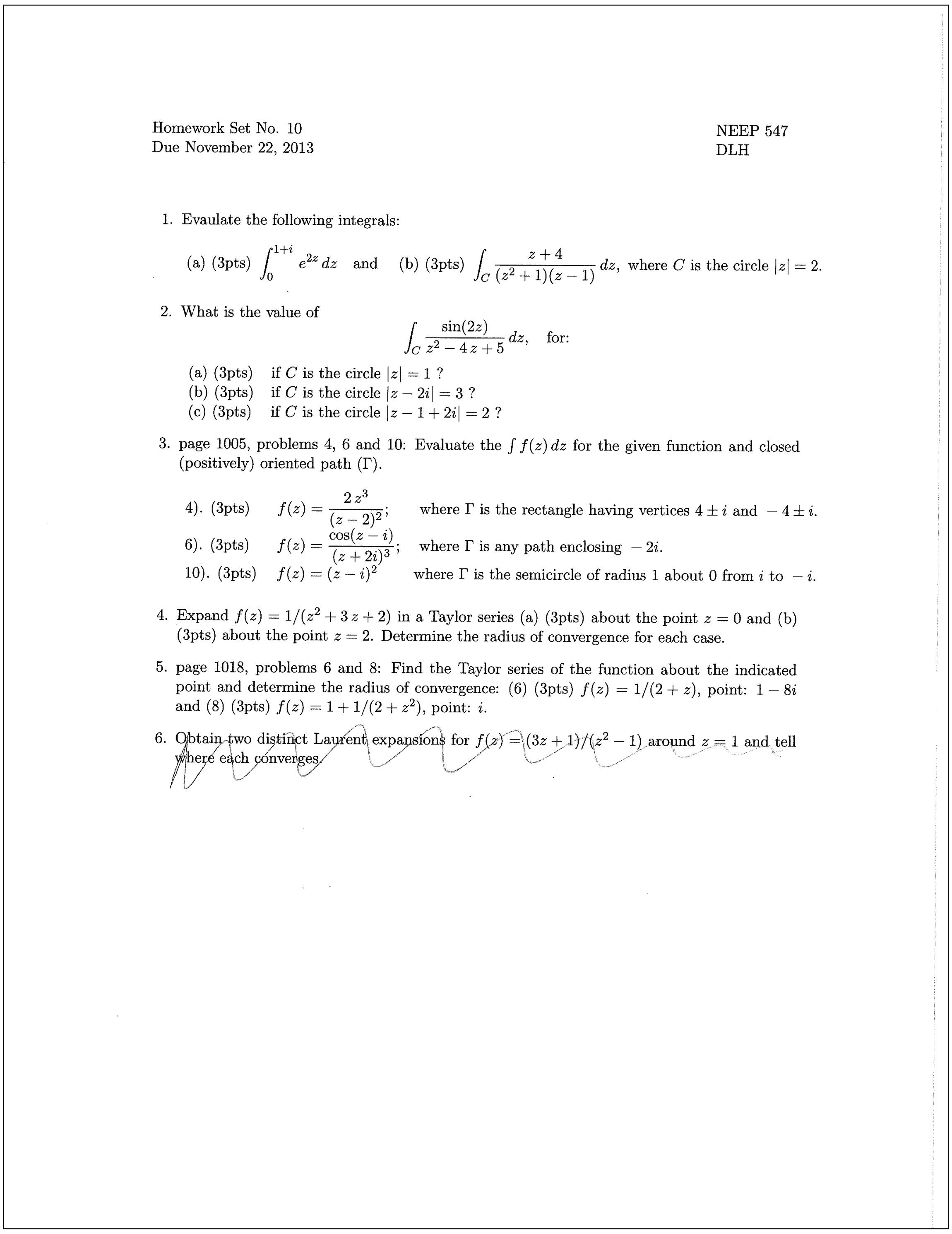
And for $c_3$

$$\begin{aligned}c_2 &= -\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}+1\right)}\frac{i}{\left(\sqrt{2}+1\right)^{3}}+\frac{1}{2\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)}\frac{i}{\left(\sqrt{2}-1\right)^{3}}\\ &= 12i\end{aligned}$$

and so on. Hence the series is

$$\begin{aligned}f\left(z\right) &= c_0+c_1\left(z-i\right)+c_2\left(z-i\right)^{2}+c_3\left(z-i\right)^{3}+\cdots\\ &= 2-2i\left(z-i\right)-5\left(z-i\right)^{2}+12i\left(z-i\right)^{3}+\cdots\end{aligned}$$

### 2.11.7 key solution



Homework Set No. 10
Due November 22, 2013

NEEP 547
DLH

1. Evaulate the following integrals:

   (a) (3pts) $\int_0^{1+i} e^{2z}\,dz$ and (b) (3pts) $\int_C \frac{z+4}{(z^2+1)(z-1)}\,dz$, where $C$ is the circle $|z|=2$.

2. What is the value of

   $$\int_C \frac{\sin(2z)}{z^2-4z+5}\,dz, \quad \text{for:}$$

   (a) (3pts) if $C$ is the circle $|z|=1$ ?
   (b) (3pts) if $C$ is the circle $|z-2i|=3$ ?
   (c) (3pts) if $C$ is the circle $|z-1+2i|=2$ ?

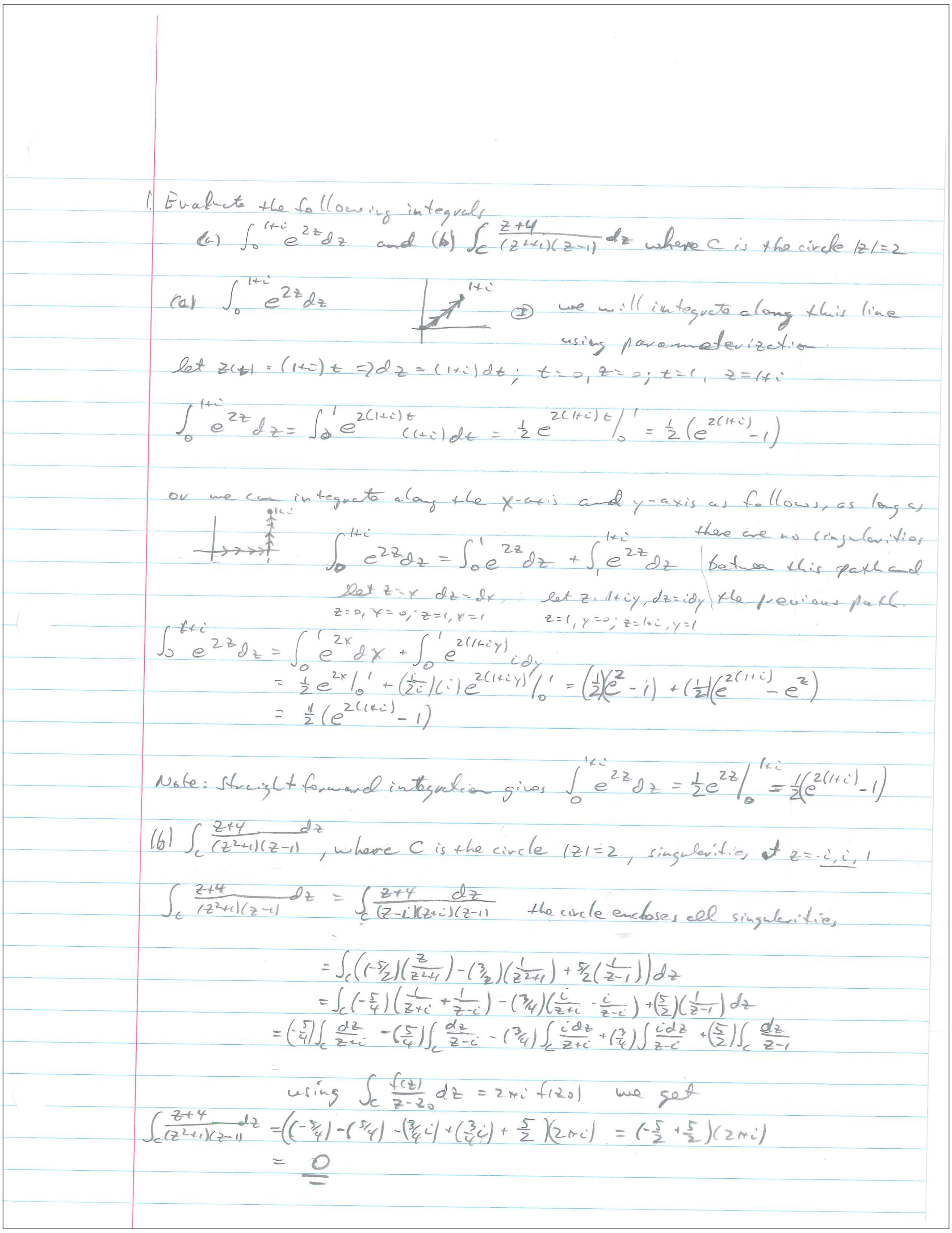
3. page 1005, problems 4, 6 and 10: Evaluate the $\int f(z)\,dz$ for the given function and closed (positively) oriented path ($\Gamma$).

   4). (3pts) $f(z)=\frac{2z^3}{(z-2)^2}$; where $\Gamma$ is the rectangle having vertices $4\pm i$ and $-4\pm i$.
   6). (3pts) $f(z)=\frac{\cos(z-i)}{(z+2i)^3}$; where $\Gamma$ is any path enclosing $-2i$.
   10). (3pts) $f(z)=(z-i)^2$ where $\Gamma$ is the semicircle of radius 1 about 0 from $i$ to $-i$.

4. Expand $f(z)=1/(z^2+3z+2)$ in a Taylor series (a) (3pts) about the point $z=0$ and (b) (3pts) about the point $z=2$. Determine the radius of convergence for each case.

5. page 1018, problems 6 and 8: Find the Taylor series of the function about the indicated point and determine the radius of convergence: (6) (3pts) $f(z)=1/(2+z)$, point: $1-8i$ and (8) (3pts) $f(z)=1+1/(2+z^2)$, point: $i$.

6. Obtain two distinct Laurent expansions for $f(z)=(3z+1)/(z^2-1)$ around $z=1$ and tell where each converges.



1. Evaluate the following integrals
(a) $\int_0^{1+i} e^{2z}\,dz$ and (b) $\int_C \frac{z+4}{(z^2+1)(z-1)}dz$ where C is the circle $|z|=2$

(a) $\int_0^{1+i} e^{2z}\,dz$

① we will integrate along this line using parameterization.

let $z(t) = (1+i)t \Rightarrow dz = (1+i)\,dt$; $t=0, z=0$; $t=1$, $z=1+i$

$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2(1+i)t}(1+i)\,dt = \frac{1}{2}e^{2(1+i)t}\Big|_0^1 = \frac{1}{2}\left(e^{2(1+i)}-1\right)$$

or we can integrate along the x-axis and y-axis as follows, as long as there are no singularities between this path and the previous path.

$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2z}\,dz + \int_1^{1+i} e^{2z}\,dz$$

let $z=x$ $dz=dx$, $z=0, x=0$; $z=1, x=1$ let $z=1+iy$, $dz=i\,dy$ $z=1, y=0$; $z=1+i, y=1$

$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2x}\,dx + \int_0^1 e^{2(1+iy)}\,i\,dy$$
$$= \frac{1}{2}e^{2x}\Big|_0^1 + \left(\frac{1}{2i}\right)(i)e^{2(1+iy)}\Big|_0^1 = \left(\frac{1}{2}\right)\left(e^2-1\right) + \left(\frac{1}{2}\right)\left(e^{2(1+i)}-e^2\right)$$
$$= \frac{1}{2}\left(e^{2(1+i)}-1\right)$$

Note: Straight forward integration gives $\int_0^{1+i} e^{2z}\,dz = \frac{1}{2}e^{2z}\Big|_0^{1+i} = \frac{1}{2}\left(e^{2(1+i)}-1\right)$

(b) $\int_C \frac{z+4}{(z^2+1)(z-1)}dz$, where C is the circle $|z|=2$, singularities at $z=\underline{-i,i},1$

$\int_C \frac{z+4}{(z^2+1)(z-1)}dz = \int_C \frac{z+4}{(z-i)(z+i)(z-1)}dz$ the circle encloses all singularities

$$= \int_C \left(\left(-\tfrac{5}{2}\right)\left(\frac{z}{z^2+1}\right) - \left(\tfrac{3}{2}\right)\left(\frac{1}{z^2+1}\right) + \tfrac{5}{2}\left(\frac{1}{z-1}\right)\right)dz$$
$$= \int_C \left(-\tfrac{5}{4}\right)\left(\frac{1}{z+i}+\frac{1}{z-i}\right) - \left(\tfrac{3}{4}\right)\left(\frac{i}{z+i}-\frac{i}{z-i}\right) + \left(\tfrac{5}{2}\right)\left(\frac{1}{z-1}\right)dz$$
$$= \left(-\tfrac{5}{4}\right)\int_C \frac{dz}{z+i} - \left(\tfrac{5}{4}\right)\int_C \frac{dz}{z-i} - \left(\tfrac{3}{4}\right)\int_C \frac{i\,dz}{z+i} + \left(\tfrac{3}{4}\right)\int \frac{i\,dz}{z-i} + \left(\tfrac{5}{2}\right)\int_C \frac{dz}{z-1}$$

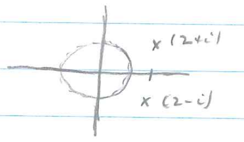
using $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$ we get

$$\int_C \frac{z+4}{(z^2+1)(z-1)}dz = \left(\left(-\tfrac{5}{4}\right) - \left(\tfrac{5}{4}\right) - \left(\tfrac{3}{4}i\right) + \left(\tfrac{3}{4}i\right) + \tfrac{5}{2}\right)(2\pi i) = \left(-\tfrac{5}{2}+\tfrac{5}{2}\right)(2\pi i)$$
$$= \underline{\underline{0}}$$

2 what is the value of $\int_C \frac{\sin(2z)}{z^2-4z+5}dz = \int_C \frac{\sin(2z)}{(z-2)^2+1}dz = \int_C \frac{\sin(2z)}{(z-(2+i))(z-(2-i))}dz$

poles at $z=2+i$, $z=2-i$

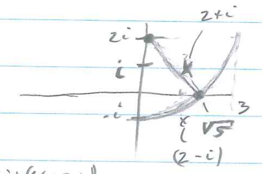
(a) if $C$ is the circle $|z|=1$?



neither singularity is in the circle, hence

$$\int_C \frac{\sin(2z)}{(z-(2+i))(z-(2-i))}dz = \underline{0}$$

(b) if $C$ is the circle $|z-2i|=3$?



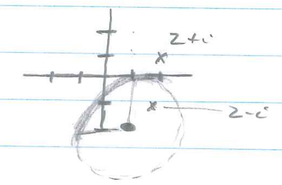
only the $z=2+i$ root is enclosed

$$\int_C \frac{f(z)}{(z-(2+i))}dz = 2\pi i f(z=2+i)$$

$$=(2\pi i)\frac{\sin(2(2+i))}{(2+i)-(2-i)} = (2\pi i)\frac{\sin(4+2i)}{2i} = \underline{\pi \sin(4+2i)}$$

$|z-(1-2i)|$

(c) if $C$ is the circle $|z-1+2i|=2$?

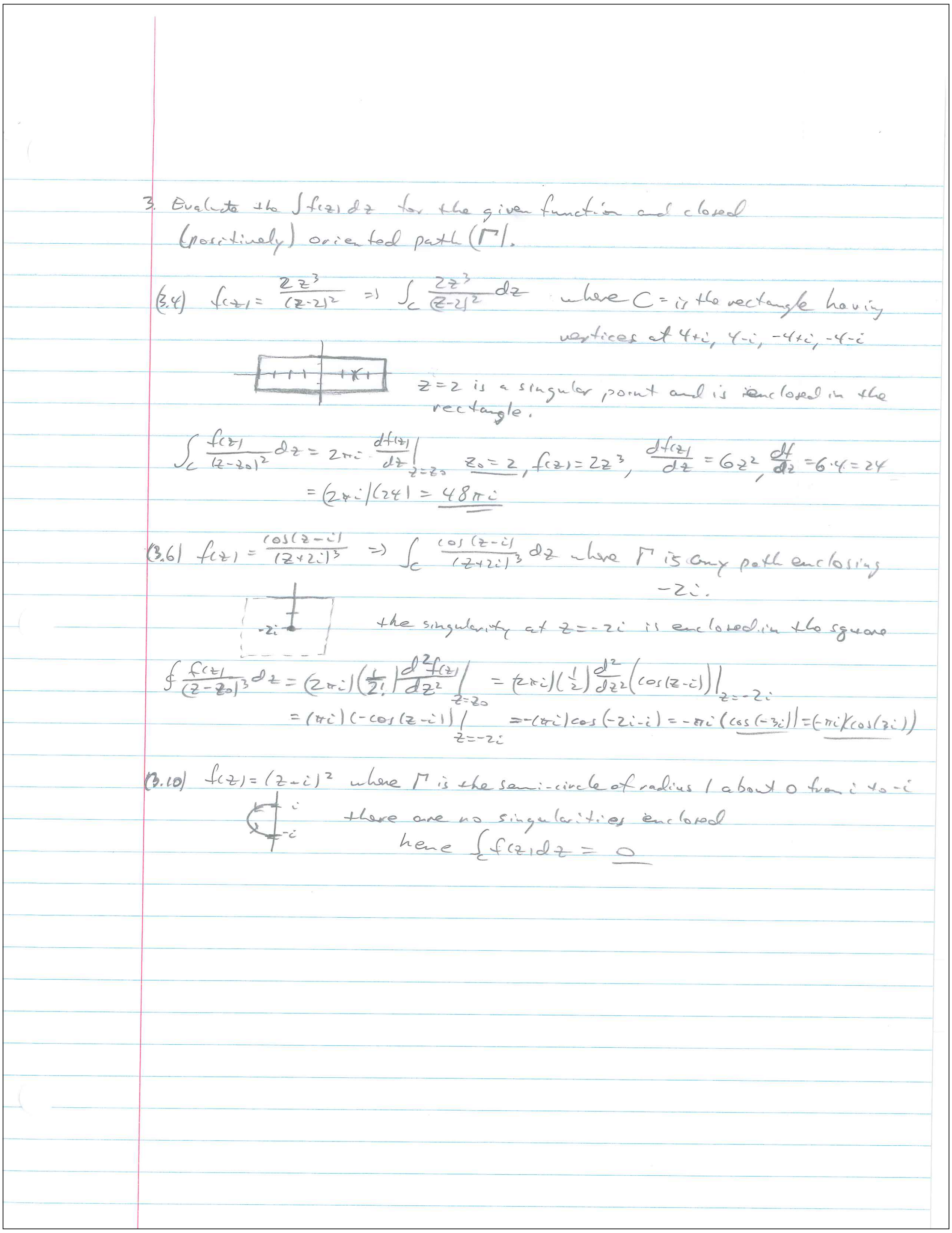


only the $z=2-i$ root is enclosed

$$\int_C \frac{f(z)}{(z-(2-i))}dz = 2\pi i\, f(z=2-i)$$

$$=(2\pi i)\frac{\sin(2(2-i))}{(2-i)-(2+i)}$$

$$=(2\pi i)\frac{\sin(4-2i)}{-2i} = \underline{-\pi \sin(4-2i)}$$



3. Evalute the $\int f(z)\,dz$ for the given function and closed (positively) oriented path $(\Gamma)$.

(3.4) $f(z) = \frac{2z^3}{(z-2)^2} \Rightarrow \int_C \frac{2z^3}{(z-2)^2}dz$ where C = is the rectangle having vertices at 4+i, 4-i, -4+i, -4-i

z=2 is a singular point and is enclosed in the rectangle.

$$\int_C \frac{f(z)}{(z-z_0)^2}dz = 2\pi i \cdot \frac{df(z)}{dz}\Big|_{z=z_0} \quad \underline{z_0 = 2}, f(z) = 2z^3, \frac{df(z)}{dz} = 6z^2, \frac{df}{dz} = 6\cdot 4 = 24$$

$$= (2\pi i)(24) = \underline{\underline{48\pi i}}$$

(3.6) $f(z) = \frac{\cos(z-i)}{(z+2i)^3} \Rightarrow \int_C \frac{\cos(z-i)}{(z+2i)^3}dz$ where $\Gamma$ is any path enclosing $-2i$.

the singularity at $z=-2i$ is enclosed in the square

$$\oint \frac{f(z)}{(z-z_0)^3}dz = (2\pi i)\left(\frac{1}{2!}\right)\frac{d^2f(z)}{dz^2}\Big|_{z=z_0} = (2\pi i)\left(\frac{1}{2}\right)\frac{d^2}{dz^2}\left(\cos(z-i)\right)\Big|_{z=-2i}$$

$$= (\pi i)(-\cos(z-i))\Big|_{z=-2i} = -(\pi i)\cos(-2i-i) = -\pi i(\underline{\cos(-3i)}) = (-\pi i)(\underline{\cos(3i)})$$

(3.10) $f(z) = (z-i)^2$ where $\Gamma$ is the semi-circle of radius 1 about 0 from $i$ to $-i$

there are no singularities enclosed

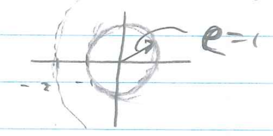
hence $\oint_C f(z)dz = \underline{0}$

4. Expand $f(z) = \frac{1}{z^2+3z+2}$ in a Taylor series and determine the radius of convergence

(a) about the point $z=0$.

$f(z) = \frac{1}{(z+2)(z+1)}$ singularities at $z=-1,-2$

Radius of convergence is $\underline{\rho=1}$



$$f(z) = \frac{1}{(z+2)(z+1)} = \frac{1}{z+1} - \frac{1}{z+2}$$

will use $\frac{1}{1-u} = 1+u+u^2+\dots$ for $|u|<1$

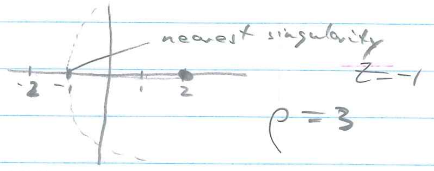
$$= \frac{1}{1-(-z)} - \frac{1}{2}\left(\frac{1}{1-(-z/2)}\right)$$

$$= (1-z+z^2-z^3+\dots) - \left(\frac{1}{2}\right)\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3\dots\right)$$

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^n - \left(\frac{1}{2}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)(z^n)\left(\frac{1}{2}\right)\left(1-\frac{1}{2^{n+1}}\right)$$

(b) about the point $z=2$.

Radius of convergence is $\rho=\underline{3}$

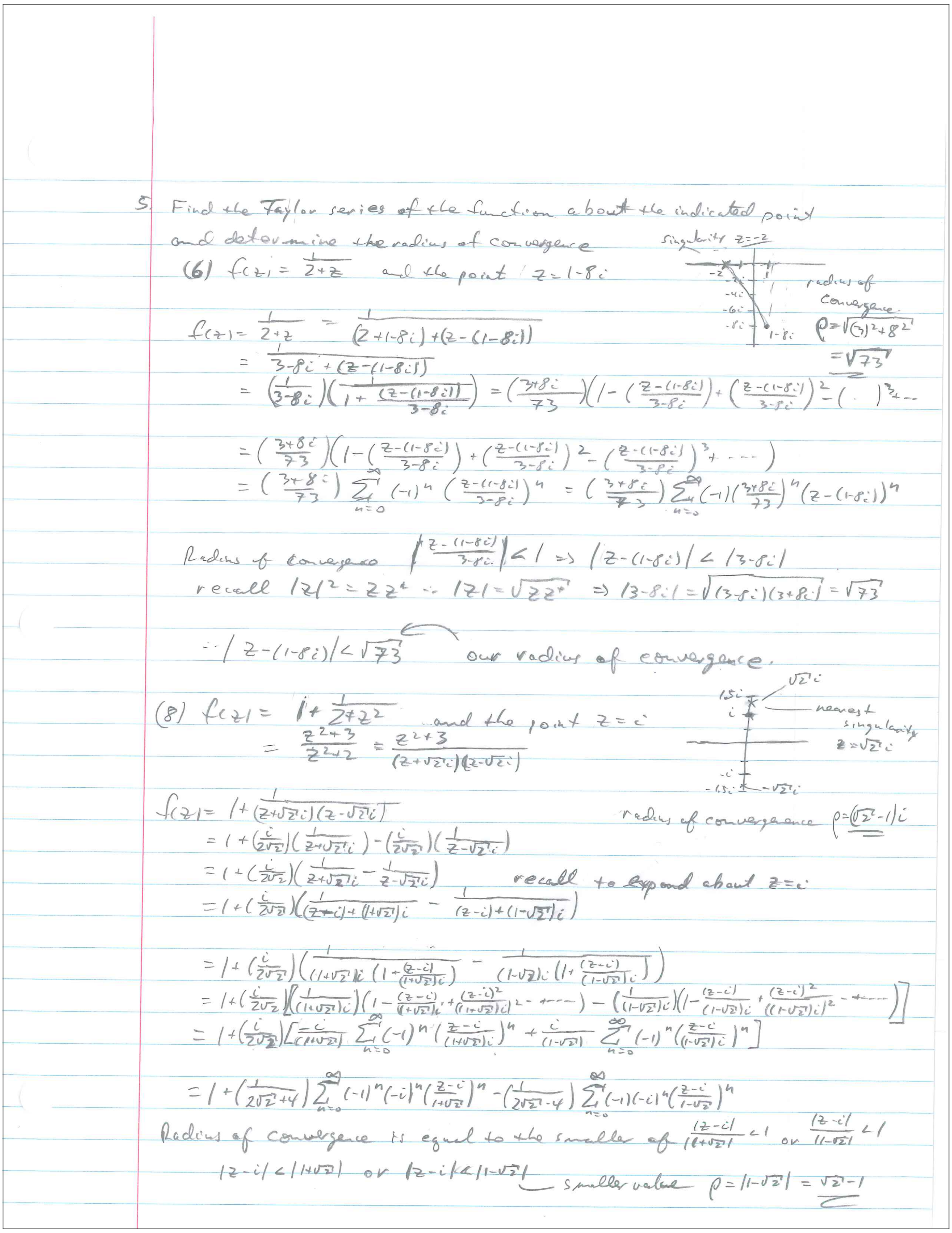


$$f(z) = \frac{1}{z+1} - \frac{1}{z+2} = \frac{1}{(z-2)+3} - \frac{1}{(z-2)+4}$$

$$= \frac{1}{3+(z-2)} - \frac{1}{4+(z-2)} = \left(\frac{1}{3}\right)\left(\frac{1}{1+\left(\frac{z-2}{3}\right)}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{1+\left(\frac{z-2}{4}\right)}\right)$$

$$= \left(\frac{1}{3}\right)\left(1-\left(\frac{z-2}{3}\right)+\left(\frac{z-2}{3}\right)^2-\left(\frac{z-2}{3}\right)^3+\dots\right) - \left(\frac{1}{4}\right)\left(1-\left(\frac{z-2}{4}\right)+\left(\frac{z-2}{4}\right)^2-\left(\frac{z-2}{4}\right)^3+\dots\right)$$

$$= \left(\frac{1}{3}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n - \left(\frac{1}{4}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{4}\right)^2 = \sum_{n=0}^{\infty}(-1)^n(z-2)^n\left(\frac{1}{3^{n+1}}-\frac{1}{4^{n+1}}\right)$$



5. Find the Taylor series of the function about the indicated point and determine the radius of convergence

(6) $f(z) = \frac{1}{2+z}$ and the point $z = 1-8i$

singularity $z=-2$

$-2$, $-2i$, $-4i$, $-6i$, $-8i$, $1-8i$

radius of convergence $\rho = \sqrt{(3)^2+8^2} = \underline{\sqrt{73}}$

$$f(z) = \frac{1}{2+z} = \frac{1}{(2+1-8i)+(z-(1-8i))}$$

$$= \frac{1}{3-8i+(z-(1-8i))}$$

$$= \left(\frac{1}{3-8i}\right)\left(\frac{1}{1+\frac{(z-(1-8i))}{3-8i}}\right) = \left(\frac{3+8i}{73}\right)\left(1-\left(\frac{z-(1-8i)}{3-8i}\right)+\left(\frac{z-(1-8i)}{3-8i}\right)^2-(\quad)^3+\cdots\right.$$

$$= \left(\frac{3+8i}{73}\right)\left(1-\left(\frac{z-(1-8i)}{3-8i}\right)+\left(\frac{z-(1-8i)}{3-8i}\right)^2-\left(\frac{z-(1-8i)}{3-8i}\right)^3+\cdots\right)$$

$$= \left(\frac{3+8i}{73}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-(1-8i)}{3-8i}\right)^n = \left(\frac{3+8i}{73}\right)\sum_{n=0}^{\infty}(-1)\left(\frac{3+8i}{73}\right)^n(z-(1-8i))^n$$

Radius of convergence $\left|\frac{z-(1-8i)}{3-8i}\right| < 1 \Rightarrow |z-(1-8i)| < |3-8i|$

recall $|z|^2 = zz^* \therefore |z| = \sqrt{zz^*} \Rightarrow |3-8i| = \sqrt{(3-8i)(3+8i)} = \sqrt{73}$

$\therefore |z-(1-8i)| < \sqrt{73}$ our radius of convergence.

(8) $f(z) = 1+\frac{1}{2+z^2}$ and the point $z = i$

$$= \frac{z^2+3}{z^2+2} = \frac{z^2+3}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$$

$\sqrt{2}i$, $1.5i$, $i$, nearest singularity $z=\sqrt{2}i$, $-i$, $-1.5i$, $-\sqrt{2}i$

$f(z) = 1+\frac{1}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$ radius of convergence $\rho = \underline{(\sqrt{2}-1)}i$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z+\sqrt{2}i}\right)-\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z-\sqrt{2}i}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z+\sqrt{2}i}-\frac{1}{z-\sqrt{2}i}\right)$$ recall to expand about $z=i$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{(z-i)+(1+\sqrt{2})i}-\frac{1}{(z-i)+(1-\sqrt{2})i}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{(1+\sqrt{2})i\left(1+\frac{(z-i)}{(1+\sqrt{2})i}\right)}-\frac{1}{(1-\sqrt{2})i\left(1+\frac{(z-i)}{(1-\sqrt{2})i}\right)}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left[\left(\frac{1}{(1+\sqrt{2})i}\right)\left(1-\frac{(z-i)}{(1+\sqrt{2})i}+\frac{(z-i)^2}{((1+\sqrt{2})i)^2}-+\cdots\right)-\left(\frac{1}{(1-\sqrt{2})i}\right)\left(1-\frac{(z-i)}{(1-\sqrt{2})i}+\frac{(z-i)^2}{((1-\sqrt{2})i)^2}-+\cdots\right)\right]$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left[\frac{-i}{(1+\sqrt{2})}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{(1+\sqrt{2})i}\right)^n+\frac{i}{(1-\sqrt{2})}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{(1-\sqrt{2})i}\right)^n\right]$$

$$= 1+\left(\frac{1}{2\sqrt{2}+4}\right)\sum_{n=0}^{\infty}(-1)^n(-i)^n\left(\frac{z-i}{1+\sqrt{2}}\right)^n-\left(\frac{1}{2\sqrt{2}-4}\right)\sum_{n=0}^{\infty}(-1)(-i)^n\left(\frac{z-i}{1-\sqrt{2}}\right)^n$$

Radius of convergence is equal to the smaller of $\frac{|z-i|}{|1+\sqrt{2}|} < 1$ or $\frac{|z-i|}{|1-\sqrt{2}|} < 1$

$|z-i| < |1+\sqrt{2}|$ or $|z-i| < |1-\sqrt{2}|$ smaller value $\rho = |1-\sqrt{2}| = \underline{\sqrt{2}-1}$

## 2.12 HW 11

### 2.12.1 Problems to solve

Homework Set No. 11 NEEP 547
Due December 13, 2013 DLH

1. (5pts) Obtain two distinct Laurent expansions for $f(z) = (3z+1)/(z^2-1)$ around $z=1$ and tell where each converges.

2. If $C$ is the circle $|z-1| = \frac{3}{2}$, evaluate $\int_C f(z)dz$ using the Residue Theorem for each of the following:

   a). (3pts) $f(z) = \dfrac{z+1}{z^2\,(z+2)}$ b). (3pts) $f(z) = \dfrac{z^2}{(z^2+3z+2)^2}$ c). (3pts) $f(z) = \dfrac{1}{z(z^2+6z+4)}$

3. (4pts) Show that the following function has a simple pole at the origin and find its residue there:
$$f(z) = \frac{cosh(z)-1}{sinh(z)-z}.$$

4. Evaluate the following integrals by the method of residues:

   $a)$. (5pts) $\displaystyle\int_0^{\pi} \frac{\cos(2\theta)\,d\theta}{4\cos(\theta)+5}$ $b)$. (5pts) $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)\,d\theta}{a+b\cos(\theta)}$ where $0<b<a$ $c)$.(5pts) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^6}$

5. (6pts) Evaluate the following integral by integration around suitably indented contours in the complex plane:
$$\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx \qquad \text{where a} > 0 \text{ and b} > 0.$$

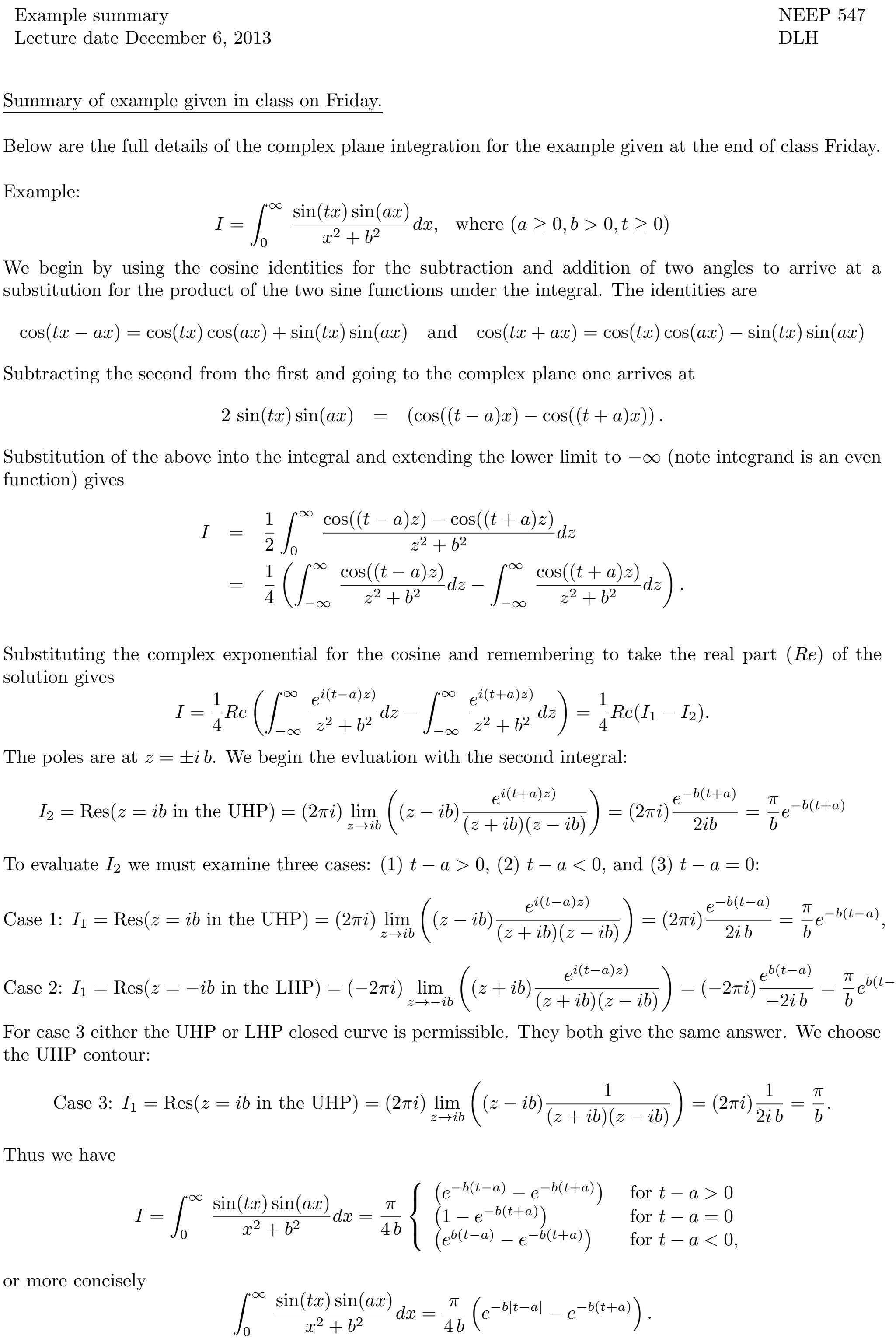
6. Evaluate the integrals:

   a). (6pts) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^x}\,dx$ where $0<p<1$ and $0<q<1$ b). (5pts) $\displaystyle\int_0^{\infty} \frac{\ln(x^2+1)}{1+x^2}dx$

7. Determine the Laplace inversion of the following functions:

   a). (6pts) $F(s) = \dfrac{s+1}{s^2(s^2+s+1)}$ b). (6pts) $F(s) = \dfrac{1}{(s+b)\,\cosh(a\sqrt{s})}$

8. (8pts) In homework 9, problem 3, we solved for the deflection of the beam, y(x), in Fourier transform space using the following equation:
$$E\,I\,\frac{d^4\,y}{d\,x^4} + k\;y(x) = -p(x) \text{ where } p(x) = \begin{cases} 0 & \text{for } -\infty < x < -\ell \\ P_0(\ell+x)/\ell^2 & \text{for } -\ell < x < 0 \\ P_0(\ell-x)/\ell^2 & \text{for } 0 < x < \ell \\ 0 & \text{for } \ell < x < \infty. \end{cases}$$
and obtained the following integral:
$$y(x) = \frac{-2\,P_o}{\pi\,\ell^2}\int_0^{\infty}\left(\frac{1-\cos(\omega\ell)}{\omega^2}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega.$$
Evaluate the integral in the complex plane using the Residue theorem to obtain the complete solution for y(x).



Example summary  
Lecture date December 6, 2013

NEEP 547  
DLH

Summary of example given in class on Friday.

Below are the full details of the complex plane integration for the example given at the end of class Friday.

Example:

$$I=\int_0^\infty \frac{\sin(tx)\sin(ax)}{x^2+b^2}dx,\quad \text{where } (a\geq 0, b>0, t\geq 0)$$

We begin by using the cosine identities for the subtraction and addition of two angles to arrive at a substitution for the product of the two sine functions under the integral. The identities are

$$\cos(tx-ax)=\cos(tx)\cos(ax)+\sin(tx)\sin(ax)\quad \text{and}\quad \cos(tx+ax)=\cos(tx)\cos(ax)-\sin(tx)\sin(ax)$$

Subtracting the second from the first and going to the complex plane one arrives at

$$2\,\sin(tx)\sin(ax) \;=\; \left(\cos((t-a)x)-\cos((t+a)x)\right).$$

Substitution of the above into the integral and extending the lower limit to $-\infty$ (note integrand is an even function) gives

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \frac{\cos((t-a)z)-\cos((t+a)z)}{z^2+b^2}dz \\ &= \frac{1}{4}\left(\int_{-\infty}^\infty \frac{\cos((t-a)z)}{z^2+b^2}dz-\int_{-\infty}^\infty \frac{\cos((t+a)z)}{z^2+b^2}dz\right).\end{aligned}$$

Substituting the complex exponential for the cosine and remembering to take the real part ($Re$) of the solution gives

$$I=\frac{1}{4}Re\left(\int_{-\infty}^\infty \frac{e^{i(t-a)z)}}{z^2+b^2}dz-\int_{-\infty}^\infty \frac{e^{i(t+a)z)}}{z^2+b^2}dz\right)=\frac{1}{4}Re(I_1-I_2).$$

The poles are at $z=\pm i\,b$. We begin the evluation with the second integral:

$$I_2=\text{Res}(z=ib \text{ in the UHP})=(2\pi i)\lim_{z\to ib}\left((z-ib)\frac{e^{i(t+a)z)}}{(z+ib)(z-ib)}\right)=(2\pi i)\frac{e^{-b(t+a)}}{2ib}=\frac{\pi}{b}e^{-b(t+a)}$$

To evaluate $I_2$ we must examine three cases: (1) $t-a>0$, (2) $t-a<0$, and (3) $t-a=0$:

$$\text{Case 1: } I_1=\text{Res}(z=ib \text{ in the UHP})=(2\pi i)\lim_{z\to ib}\left((z-ib)\frac{e^{i(t-a)z)}}{(z+ib)(z-ib)}\right)=(2\pi i)\frac{e^{-b(t-a)}}{2i\,b}=\frac{\pi}{b}e^{-b(t-a)},$$

$$\text{Case 2: } I_1=\text{Res}(z=-ib \text{ in the LHP})=(-2\pi i)\lim_{z\to -ib}\left((z+ib)\frac{e^{i(t-a)z)}}{(z+ib)(z-ib)}\right)=(-2\pi i)\frac{e^{b(t-a)}}{-2i\,b}=\frac{\pi}{b}e^{b(t-a)}.$$

For case 3 either the UHP or LHP closed curve is permissible. They both give the same answer. We choose the UHP contour:

$$\text{Case 3: } I_1=\text{Res}(z=ib \text{ in the UHP})=(2\pi i)\lim_{z\to ib}\left((z-ib)\frac{1}{(z+ib)(z-ib)}\right)=(2\pi i)\frac{1}{2i\,b}=\frac{\pi}{b}.$$

Thus we have

$$I=\int_0^\infty \frac{\sin(tx)\sin(ax)}{x^2+b^2}dx=\frac{\pi}{4\,b}\begin{cases}\left(e^{-b(t-a)}-e^{-b(t+a)}\right) & \text{for } t-a>0\\ \left(1-e^{-b(t+a)}\right) & \text{for } t-a=0\\ \left(e^{b(t-a)}-e^{-b(t+a)}\right) & \text{for } t-a<0,\end{cases}$$

or more concisely

$$\int_0^\infty \frac{\sin(tx)\sin(ax)}{x^2+b^2}dx=\frac{\pi}{4\,b}\left(e^{-b|t-a|}-e^{-b(t+a)}\right).$$

### 2.12.2 problem 1

Obtain two distinct Laurent expansions for $f(z)=\frac{3z+1}{z^2-1}$ around $z=1$ and tell where each converges.

#### Solution

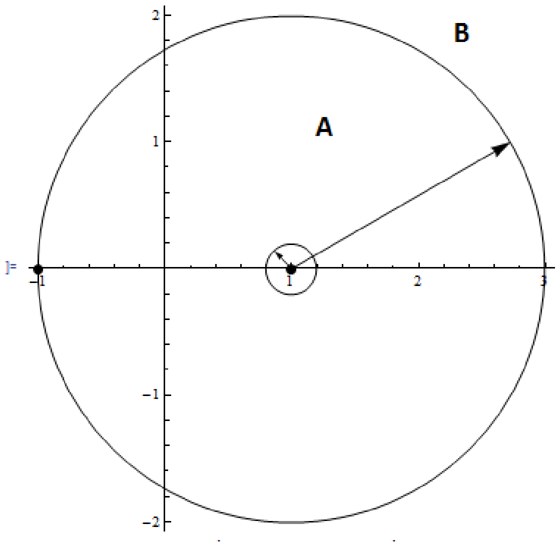
$f(z)$ has singularities at $z=\pm 1$ and the expansion is around one of these singularities. Looking at the diagram

Region $A$ is annulus between $z=1$ and $z=-1$ but does not include $z=1$ where the small circle is shown since that is a singularity. Region $B$ is all the region outside the large circle shown.

$$f(z)=\frac{3z+1}{(z-1)(z+1)}=\frac{2}{z-1}+\frac{1}{z+1}$$

#### Region A

For $\frac{2}{z-1}$, since its pole is at $z=1$ and so we expand outwards, and hence it is already in form of Laurent series around $z=1$, and for $\frac{1}{z+1}$ since its pole is at $z=-1$, hence we expand



inwards, and so it is expanded in Taylor series

$$f(z)=\overbrace{\frac{2}{z-1}}^{Laurent}+\overbrace{\frac{1}{z+1}}^{Taylor}$$

Looking at the second term above, expand in Taylor series

$$\begin{aligned}\frac{1}{z+1}&=\frac{1}{(z-1)+1+1}=\frac{1}{(z-1)+2}=\frac{1}{2}\frac{1}{1+\frac{1}{2}(z-1)}\\&=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n(z-1)^n\qquad |z-1|<2\end{aligned}$$

Therefore, for region A

$$\begin{aligned}f(z)&=\frac{2}{z-1}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n(z-1)^n\\&=\frac{2}{z-1}+\frac{1}{2}\left(1-\frac{1}{2}(z-1)+\frac{1}{4}(z-1)^2-\frac{1}{16}(z-1)^3+\cdots\right)\\&=\frac{2}{z-1}+\frac{1}{2}-\frac{1}{4}(z-1)+\frac{1}{8}(z-1)^2-\frac{1}{16}(z-1)^3+\cdots\end{aligned}$$

This is valid for $0<|z-1|<2, z\neq 1$

**Region B**

This is the region outside the large circle to infinity. Since expanding outwards, both terms will use Laurent series now.

$$f(z)=\overbrace{\frac{2}{z-1}}^{Laurent}+\overbrace{\frac{1}{z+1}}^{Laurent}$$

$\frac{2}{z-1}$ is already in Laurent series, for the second term

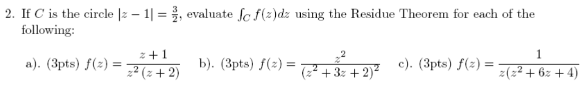
$$\begin{aligned}\frac{1}{z+1}&=\frac{1}{(z-1)+1+1}=\frac{1}{(z-1)+2}=\frac{1}{(z-1)}\frac{1}{1+\frac{2}{z-1}}\\&=\frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}\qquad |z-1|>2\end{aligned}$$

Hence

$$\begin{aligned}f(z)&=\frac{2}{z-1}+\frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}\\&=\frac{2}{z-1}+\frac{1}{(z-1)}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-\frac{8}{(z-1)^3}+\cdots\right)\\&=\frac{2}{z-1}+\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\cdots\\&=\frac{3}{z-1}-\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\cdots\end{aligned}$$

This is valid for $|z-1|>2$

### 2.12.3 problem 2



2. If $C$ is the circle $|z-1|=\frac{3}{2}$, evaluate $\int_C f(z)dz$ using the Residue Theorem for each of the following:

a). (3pts) $f(z)=\dfrac{z+1}{z^2(z+2)}$ b). (3pts) $f(z)=\dfrac{z^2}{(z^2+3z+2)^2}$ c). (3pts) $f(z)=\dfrac{1}{z(z^2+6z+4)}$

#### 2.12.3.1 Part (a)

$f(z)=\frac{z+1}{z^2(z+2)}$, Poles are $z=0$ order 2 and $z=-2$ order 1. The pole $z=-2$ is outside $C$ hence it will not have an effect. To find the residue due to pole $z=0$

$$\begin{aligned}
residue\,(z=0) &= \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left(z^2\frac{z+1}{z^2(z+2)}\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z+1}{z+2}\right)\\
&= \lim_{z\to 0}\left(\frac{1}{z+2}-\frac{(z+1)}{(z+2)^2}\right)\\
&= \frac{1}{2}-\frac{1}{4}\\
&= \frac{1}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\oint f(z)\,dz &= 2\pi i\left(\frac{1}{4}\right)\\
&= \frac{1}{2}\pi i
\end{aligned}$$

#### 2.12.3.2 Part (b)

$f(z)=\frac{z^2}{\left(z^2+3z+2\right)^2}$ the poles are the roots of $\left(z^2+3z+2\right)^2=0$ which is $((z+2)(z+1))^2=0$ or $(z+2)^2(z+1)^2=0$, hence poles at $(z+2)^2=0$ or $z=-2$ and $(z+1)^2=0$ or $z=-1$ Only pole at $z=-1$ is inside $C$. This is order 2 since the denominator is order 2. Hence

$$\begin{aligned}
residue\,(z=-1) &= \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}\left((z+1)^2 f(z)\right)\\
&= \lim_{z\to -1}\frac{d}{dz}\left((z+1)^2\frac{z^2}{(z+2)^2(z+1)^2}\right)\\
&= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z^2}{(z+1)^2}\right)\\
&= \lim_{z\to -1}\left(\frac{2z}{(z+2)^2}-2z^2(z+2)^{-3}\right)\\
&= \lim_{z\to -1}\left(\frac{2z}{(z+2)^2}-\frac{2z^2}{(z+2)^3}\right)\\
&= \left(\frac{-2}{(-1+2)^2}-\frac{2(-1)^2}{(-1+2)^3}\right)\\
&= (-2-2)\\
&= -4
\end{aligned}$$

Hence

$$\begin{aligned}
\oint f(z)\,dz &= 2\pi i(-4)\\
&= -8\pi i
\end{aligned}$$

#### 2.12.3.3 Part (c)

$f(z) = \frac{1}{z\left(z^2+6z+4\right)} = \frac{1}{z\left(z-\left(\sqrt{5}-3\right)\right)\left(z-\left(-\sqrt{5}-3\right)\right)} = \frac{1}{z\left(z-\left(\sqrt{5}-3\right)\right)\left(z-\left(-\sqrt{5}-3\right)\right)}$. Since $\sqrt{5}-3 = -5.2361$ and $-\sqrt{5}-3 = -0.76393$ then only $z = \left(-\sqrt{5}-3\right)$ is inside $C$.

Hence poles are $z=0$ and $z = \left(-\sqrt{5}-3\right)$ and both is order 1.

$$\begin{aligned}
\mathit{residue}\,(z=0) &= \lim_{z\to 0}\left(zf(z)\right)\\
&= \lim_{z\to 0}\left(z\frac{1}{z\left(z^2+6z+4\right)}\right)\\
&= \lim_{z\to 0}\left(\frac{1}{\left(z^2+6z+4\right)}\right)\\
&= \frac{1}{4}
\end{aligned}$$

And

$$\begin{aligned}
\mathit{residue}\left(z=\left(-\sqrt{5}-3\right)\right) &= \lim_{z\to\left(-\sqrt{5}-3\right)}\left(\left(z-\left(-\sqrt{5}-3\right)\right)f(z)\right)\\
&= \lim_{z\to\left(-\sqrt{5}-3\right)}\left(\left(z-\left(-\sqrt{5}-3\right)\right)\frac{1}{z\left(z-\left(\sqrt{5}-3\right)\right)\left(z-\left(-\sqrt{5}-3\right)\right)}\right)\\
&= \lim_{z\to\left(-\sqrt{5}-3\right)}\left(\frac{1}{z\left(z-\left(\sqrt{5}-3\right)\right)}\right)\\
&= \frac{1}{\left(-\sqrt{5}-3\right)\left(\left(-\sqrt{5}-3\right)-\left(\sqrt{5}-3\right)\right)}\\
&= \frac{1}{\left(-\sqrt{5}-3\right)\left(-\sqrt{5}-3-\sqrt{5}+3\right)}\\
&= \frac{1}{\left(-\sqrt{5}-3\right)\left(-2\sqrt{5}\right)}\\
&= \frac{1}{10+6\sqrt{5}}
\end{aligned}$$

Hence

$$\begin{aligned}
\oint f(z)\,dz &= 2\pi i\left(\frac{1}{4}+\frac{1}{10+6\sqrt{5}}\right)\\
&= \left(\frac{1}{2}+\frac{1}{5+3\sqrt{5}}\right)\pi i\\
&= 0.5854\pi i
\end{aligned}$$

### 2.12.4 Problem 3

Show that $f(z) = \frac{\cosh(z)-1}{\sinh(z)-z}$ has simple pole at $z=0$ and find its residue there.

**Solution**

Expanding $\cosh(z)$ and $\sinh(z)$ in series gives

$$f(z) = \frac{\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}\right)-1}{\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}\right)-z} = \frac{\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}}{\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}}$$

Divide by $z^2$

$$\begin{aligned}
f(z) &= \frac{\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}}{\frac{z}{3!}+\frac{z^3}{5!}+\frac{z^5}{7!}}\\
&= \frac{\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}}{z\left(\frac{1}{3!}+\frac{z^2}{5!}+\frac{z^4}{7!}\right)}
\end{aligned}$$

Therefore, $f(z)$ has simple pole at $z=0$. The residue is

$$\begin{aligned}
residue\,(z=0) &= \lim_{z\to 0}\left(zf(z)\right)\\
&= \lim_{z\to 0}\left(z\frac{\cosh(z)-1}{\sinh(z)-z}\right)\\
&= \lim_{z\to 0}\left(z\frac{\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}}{z\left(\frac{1}{3!}+\frac{z^2}{5!}+\frac{z^4}{7!}\right)}\right)\\
&= \lim_{z\to 0}\left(\frac{\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}}{\left(\frac{1}{3!}+\frac{z^2}{5!}+\frac{z^4}{7!}\right)}\right)\\
&= \frac{\frac{1}{2!}}{\frac{1}{3!}}\\
&= \frac{6}{2}\\
&= 3
\end{aligned}$$

### 2.12.5 Problem 4

Evaluate the following using the method of residues (a) $\int_0^\pi \frac{\cos(2\theta)}{5+4\cos\theta}d\theta$ (b) $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta$ where $0<b<a$ and (c) $\int_{-\infty}^{\infty}\frac{x^2}{1+x^6}dx$

#### 2.12.5.1 part(a)

Let

$$I=\int_0^\pi \frac{\cos(2\theta)}{5+4\cos\theta}d\theta$$

Since the integrand is even then

$$I=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos(2\theta)}{5+4\cos\theta}d\theta$$

Let $I_1=\int_{-\pi}^{\pi}\frac{\cos(2\theta)}{5+4\cos\theta}d\theta$, hence

$$I=\frac{1}{2}I_1$$

Now we evaluate $I_1$. Let $z=re^{i\theta}$, and for a unit circle, $r=1$, hence $dz=ie^{i\theta}d\theta$ or $d\theta=-i\frac{dz}{e^{i\theta}}=-i\frac{dz}{z}$. Now we need to convert the integrand from function of $\theta$ to function of $z$.

$\cos(2\theta)=\frac{e^{i2\theta}+e^{-i2\theta}}{2}=\frac{1}{2}\left(z^2+z^{-2}\right)$ and $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{1}{2}\left(z+z^{-1}\right)$, therefore the integral becomes

$$I_1=\operatorname{Re}\oint\frac{\frac{1}{2}\left(z^2+z^{-2}\right)}{5+4\left(\frac{1}{2}\left(z+z^{-1}\right)\right)}\left(-i\frac{dz}{z}\right)$$

The contour is over the unit circle. Let $I_2=\oint\frac{\frac{1}{2}\left(z^2+z^{-2}\right)}{5+4\left(\frac{1}{2}\left(z+z^{-1}\right)\right)}\left(-i\frac{dz}{z}\right)$, hence $I_1=\operatorname{Re}(I_2)$ and now we evaluate $I_2$.

$$\begin{aligned}
I_2 &= \oint \frac{\frac{1}{2}\left(z^2+z^{-2}\right)}{5+4\left(\frac{1}{2}\left(z+z^{-1}\right)\right)}\left(-i\frac{dz}{z}\right)\\
&= -\frac{i}{2}\oint \frac{z^2+z^{-2}}{5+2\left(z+z^{-1}\right)}\frac{dz}{z}\\
&= -\frac{i}{2}\oint \frac{\frac{z^4+1}{z^2}}{5+2\left(\frac{z^2+1}{z}\right)}\frac{dz}{z}\\
&= -\frac{i}{2}\oint \frac{\frac{z^4+1}{z^2}}{\frac{5z+2\left(z^2+1\right)}{z}}\frac{dz}{z}\\
&= -\frac{i}{2}\oint \frac{z^4+1}{5z^2+2z\left(z^2+1\right)}\frac{dz}{z}\\
&= -\frac{i}{2}\oint \frac{z^4+1}{5z^3+2z^2\left(z^2+1\right)}dz\\
&= -\frac{i}{2}\oint \frac{z^4+1}{5z^3+2z^4+2z^2}dz\\
&= -\frac{i}{2}\oint \frac{z^4+1}{z^2\left(2z^2+5z+2\right)}dz\\
&= -\frac{i}{4}\oint \frac{z^4+1}{z^2\left(z^2+\frac{5}{2}z+1\right)}dz\\
&= -\frac{i}{4}\oint \frac{z^4+1}{z^2\left(z+2\right)\left(z+\frac{1}{2}\right)}dz
\end{aligned}$$

The poles are $z=0$ of order 2 and $z=-2$ and $z=-\frac{1}{2}$, hence only $z=-\frac{1}{2}$ is inside the unit circle. Lets find the residues of each now.

$$\begin{aligned}
\text{residue}\left(z=0\right) &= \lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}\left(z^2 f\left(z\right)\right) = \frac{d}{dz}\left(\frac{z^4+1}{\left(z+2\right)\left(z+\frac{1}{2}\right)}\right)\\
&= \lim_{z\to 0}\left(\frac{2}{\left(2z^2+5z+2\right)^2}\left(4z^5+15z^4+8z^3-4z-5\right)\right)\\
&= \left(\frac{2}{\left(2\right)^2}\left(-5\right)\right) = -\frac{5}{2}
\end{aligned}$$

And

$$\begin{aligned}
\text{residue}\left(z=-\frac{1}{2}\right) &= \lim_{z\to-\frac{1}{2}}\left(\left(z+\frac{1}{2}\right)f\left(z\right)\right) = \lim_{z\to-\frac{1}{2}}\left(\frac{z^4+1}{z^2\left(z+2\right)}\right)\\
&= \frac{\left(-\frac{1}{2}\right)^4+1}{\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+2\right)}\\
&= \frac{17}{6}
\end{aligned}$$

Hence

$$\begin{aligned}
\oint \frac{z^4+1}{z^2\left(z+2\right)\left(z+\frac{1}{2}\right)}dz &= 2\pi i\left(-\frac{5}{2}+\frac{17}{6}\right)\\
&= \frac{2}{3}i\pi
\end{aligned}$$

Hence

$$\begin{aligned}-\frac{i}{4}\oint\frac{z^4+1}{z^2(z+2)\left(z+\frac{1}{2}\right)}dz &= \left(-\frac{i}{4}\right)\frac{2}{3}i\pi\\ &= \left(\frac{1}{4}\right)\frac{2}{3}\pi\\ &= \frac{1}{6}\pi\end{aligned}$$

Hence

$$I_2=\frac{1}{6}\pi$$

But $I_1=\operatorname{Re}(I_2)$, hence $I_1=\frac{1}{3}\pi$. And $I=\frac{1}{2}I_1$, hence

$$I=\frac{1}{12}\pi$$

#### 2.12.5.2 Part(b)

$I=\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta$ where $0<b<a$.

We need to convert this to contour integral over the unit circle. Let $z=re^{i\theta}$, and for a unit circle, $r=1$, hence $dz=ie^{i\theta}d\theta$ or $d\theta=-i\frac{dz}{e^{i\theta}}=-i\frac{dz}{z}$. Now we need to convert the integrand from function of $\theta$ to function of $z$.

$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{1}{2}\left(z+z^{-1}\right)$ and $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{1}{2i}\left(z-z^{-1}\right)$, Hence

$$\begin{aligned}\sin^2\theta &= \left(\frac{1}{2i}\left(z-z^{-1}\right)\right)^2\\ &= \frac{-1}{4}\left(z-z^{-1}\right)^2=\frac{-1}{4}\left(z^2+z^{-2}-2\right)\end{aligned}$$

Therefore the integral becomes

$$\begin{aligned}I &= \oint\frac{\frac{-1}{4}\left(z^2+z^{-2}-2\right)}{a+b\frac{1}{2}\left(z+z^{-1}\right)}\left(-i\frac{dz}{z}\right)\\ &= \frac{i}{4}\oint\frac{z^2+z^{-2}-2}{a+b\frac{1}{2}\left(\frac{z^2+1}{z}\right)}\frac{dz}{z}\\ &= \frac{i}{4}\oint\frac{\frac{z^4+1-2z^2}{z^2}}{az+b\frac{1}{2}\left(z^2+1\right)}dz\\ &= \frac{i}{2}\oint\frac{1}{z^2}\frac{z^4-2z^2+1}{2az+b\left(z^2+1\right)}dz\\ &= \frac{i}{2}\oint\frac{1}{z^2}\frac{z^4-2z^2+1}{bz^2+2az+b}dz\\ &= \frac{i}{2b}\oint\frac{1}{z^2}\frac{z^4-2z^2+1}{z^2+\frac{2a}{b}z+1}dz\end{aligned}$$

Roots of $z^2+\frac{2a}{b}z+1$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\frac{2a}{b}\pm\sqrt{\left(\frac{2a}{b}\right)^2-4}}{2}=-\frac{a}{b}\pm\frac{1}{2}\sqrt{\frac{4a^2}{b^2}-4}=-\frac{a}{b}\pm\sqrt{\left(\frac{a}{b}\right)^2-1}$ Hence

$$r_1=-\frac{a}{b}-\sqrt{\left(\frac{a}{b}\right)^2-1}$$

and

$$r_2=-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}$$

Since $a>b$ then $r_1$ is outside the unit circle. but $r_2$ is inside. The integral is now

$$I=\frac{i}{2b}\oint\frac{1}{z^2}\frac{z^4-2z^2+1}{(z-r_1)(z-r_2)}dz$$

pole at $z=0$ or order 2. To find the residue for this pole

$$\begin{aligned} res\left(z=0\right) &= \lim_{z\to 0}\frac{1}{\left(2-1\right)!}\left(z^2 f\left(z\right)\right) \\ &= \lim_{z\to 0}\left(\frac{z^4-2z^2+1}{\left(z-r_1\right)\left(z-r_2\right)}\right) \\ &= \frac{1}{\left(-r_1\right)\left(-r_2\right)} \\ &= \frac{1}{r_1 r_2} \end{aligned}$$

To find the residue for this $r_2$

$$\begin{aligned} res\left(z=r_2\right) &= \lim_{z\to r_2}\left(\left(z-r_2\right)f\left(z\right)\right) \\ &= \lim_{z\to r_2}\left(\frac{z^4-2z^2+1}{z^2\left(z-r_1\right)}\right) \\ &= \frac{r_2^4-2r_2^2+1}{r_2^2\left(r_2-r_1\right)} \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \frac{i}{2b}\oint\frac{1}{z^2}\frac{z^4-2z^2+1}{z^2+\frac{2a}{b}z+1}dz \\ &= \frac{i}{2b}\left(2\pi i\left(\frac{1}{r_1 r_2}+\frac{r_2^4-2r_2^2+1}{r_2^2\left(r_2-r_1\right)}\right)\right) \\ &= \frac{-\pi}{b}\left(\frac{1}{r_1 r_2}+\frac{r_2^4-2r_2^2+1}{r_2^2\left(r_2-r_1\right)}\right) \end{aligned}$$

This is the value of the integral, where $r_1, r_2$ are given as above. To simplify this final result, CAS was used for this step

$$\begin{aligned} I &= \frac{-\pi}{b}\left(\frac{1}{\left(-\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}\right)\left(-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right)}+\frac{\left(-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right)^4-2\left(-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right)^2+1}{\left(-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right)\left(\left(-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right)-\left(-\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}\right)\right)}\right) \\ &= -\frac{\pi}{b}\left(1+\frac{2\sqrt{\left(a-b\right)\left(a+b\right)}}{b}\right) \\ &= -\frac{\pi}{b}\left(1+\frac{2\sqrt{a^2-b^2}}{b}\right) \end{aligned}$$

Therefore

$$I=\left(\frac{-b-2\sqrt{a^2-b^2}}{b^2}\right)\pi$$

#### 2.12.5.3 Part(c)

$$I=\int_{-\infty}^{\infty}\frac{x^2}{1+x^6}dx$$

We first need to determine if the integral over the upper half plan vanish when $R\to\infty$, and for this we just need to show that $\lim_{z\to\infty} zf\left(z\right)=0$ where $f\left(z\right)=\frac{z^2}{1+z^6}$ Hence

$$\begin{aligned} \lim_{z\to\infty} zf\left(z\right) &= \lim_{z\to\infty}\left(z\frac{z^2}{1+z^6}\right) \\ &= \lim_{z\to\infty}\left(\frac{z^3}{1+z^6}\right)=\lim_{z\to\infty}\left(\frac{1}{\frac{1}{z^3}+z^3}\right)=\frac{1}{0+\infty}=0 \end{aligned}$$

Hence the integral becomes

$$I=\int_{-\infty}^{\infty}\frac{z^2}{1+z^6}dz=2\pi i\sum^{UHP} res\left(f\left(z\right)\right)$$

Now need to find the poles in UHP. From $f(z)=\frac{z^2}{1+z^6}$ the poles are the roots of $1+z^6=0$ or $z^6=-1$, therefore $z=-1^{\left(\frac{1}{6}\right)}$, but $-1=e^{i\pi}$ Hence need to spread 6 poles around $2\pi$, which means the phase changes by $\frac{2\pi}{6}$ between each, hence

$$\begin{aligned} z &= \left\{e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{6}+\frac{2\pi}{6}}, e^{i\frac{\pi}{6}+\frac{4\pi}{6}}, e^{i\frac{\pi}{6}+\frac{6\pi}{6}}, e^{i\frac{\pi}{6}+\frac{8\pi}{6}}, e^{i\frac{\pi}{6}+\frac{10\pi}{6}}\right\} \\ &= \left\{e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{7\pi}{6}}, e^{i\frac{9\pi}{6}}, e^{i\frac{11\pi}{6}}\right\} \end{aligned}$$

Now need to find which of these roots is in UHP. Looking at the phase, we see that $e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}$ are in UHP since phase is less than $\pi$. So now we need to find residue at each pole.

$$I = 2\pi i\left(res\left(e^{i\frac{\pi}{6}}\right) + res\left(e^{i\frac{\pi}{2}}\right) + res\left(e^{i\frac{5\pi}{6}}\right)\right)$$

Using $\left[\frac{N(z)}{D'(z)}\right]_{z->z_o}$ to find residue at $z=z_0$ (since each is a simple pole). But $D'(z)=\frac{d}{dz}\left(1+z^6\right)=6z^5$, hence

$$\begin{aligned} res\left(e^{i\frac{\pi}{6}}\right) &= \left(\frac{z^2}{6z^5}\right)_{z=e^{i\frac{\pi}{6}}} \\ &= \frac{e^{i\frac{2\pi}{6}}}{6e^{i\frac{5\pi}{6}}} = \frac{e^{i\left(\frac{2\pi}{6}-\frac{5\pi}{6}\right)}}{6} = \frac{1}{6}e^{i\frac{-\pi}{2}} = \frac{-i}{6} \end{aligned}$$

And

$$\begin{aligned} res\left(e^{i\frac{3\pi}{6}}\right) &= \left(\frac{z^2}{6z^5}\right)_{z=e^{i\frac{3\pi}{6}}} \\ &= \frac{e^{i\frac{6\pi}{6}}}{6e^{i\frac{15\pi}{6}}} = \frac{e^{i\left(\frac{6\pi}{6}-\frac{15\pi}{6}\right)}}{6} = \frac{1}{6}e^{i\frac{-9\pi}{6}} = \frac{1}{6}e^{-i\frac{3}{2}\pi} = \frac{i}{6} \end{aligned}$$

And

$$\begin{aligned} res\left(e^{i\frac{5\pi}{6}}\right) &= \left(\frac{z^2}{6z^5}\right)_{z=e^{i\frac{5\pi}{6}}} \\ &= \frac{e^{i\frac{10\pi}{6}}}{6e^{i\frac{25\pi}{6}}} = \frac{e^{i\left(\frac{10\pi}{6}-\frac{25\pi}{6}\right)}}{6} = \frac{1}{6}e^{i\frac{-15\pi}{6}} = \frac{1}{6}e^{-i\frac{\pi}{2}} = \frac{-i}{6} \end{aligned}$$

Therefore

$$\begin{aligned} I &= 2\pi i\left(res\left(e^{i\frac{\pi}{6}}\right) + res\left(e^{i\frac{\pi}{2}}\right) + res\left(e^{i\frac{5\pi}{6}}\right)\right) \\ &= 2\pi i\left(\frac{-i}{6}+\frac{i}{6}-\frac{i}{6}\right) \end{aligned}$$

Hence

$$I = \frac{\pi}{3}$$

### 2.12.6 Problem 5

Evaluate the following integration by integrating around a suitably indented contour in the complex plane $I=\int_0^\infty \frac{\sin(ax)}{x\left(x^2+b^2\right)}dx$ where $a>0$, $b>0$

**Solution**

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(ax)}{x\left(x^2+b^2\right)}dx \\ &= \frac{1}{2}\operatorname{Im}\left(\int_{-\infty}^{\infty}\frac{e^{iaz}}{z\left(z^2+b^2\right)}dz\right) \\ &= \frac{1}{2}\operatorname{Im}\left(\int_{-\infty}^{\infty}\frac{e^{iaz}}{z\left(z-ib\right)\left(z+ib\right)}dz\right) \end{aligned}$$

To show that the UHP integral vanish we need to show that $\lim_{|z|\to\infty} zf(z) = 0$ then we can just do the integration over the real line using residues of poles in UHP.

$$\lim_{|z\to\infty|}\left(z\frac{e^{iaz}}{z\left(z^2+b^2\right)}\right)=\lim_{|z\to\infty|}\left(\frac{e^{iaz}}{z^2+b^2}\right)\to 0$$

Since $\left|e^{iaz}\right| = 1$. The poles at the real line are $z = 0$ and poles in complex plane are $z = -ib$ and $z = ib$. Since $b > 0$ then $z = ib$ is only in UHP.

$$I=\frac{1}{2}\operatorname{Im}\left(\pi i\left[res\left(0\right)\right]+2\pi i\left[res\left(ib\right)\right]\right)$$

To find residue at $z = 0$, using

$$res\left(0\right)=\lim_{z\to 0}\left(z\frac{e^{iaz}}{z\left(z^2+b^2\right)}\right)=\lim_{z\to 0}\left(\frac{e^{iaz}}{\left(z^2+b^2\right)}\right)=\frac{1}{b^2}$$

And

$$res\left(ib\right)=\lim_{z\to ib}\left(\left(z-ib\right)\frac{e^{iaz}}{z\left(z-ib\right)\left(z+ib\right)}\right)=\lim_{z\to ib}\left(\frac{e^{iaz}}{z\left(z+ib\right)}\right)=\frac{e^{-ab}}{-2b^2}$$

Therefore

$$\begin{aligned}I&=\frac{1}{2}\operatorname{Im}\left(\pi i\left[\frac{1}{b^2}\right]+2\pi i\left[\frac{e^{-ab}}{-2b^2}\right]\right)\\&=\frac{1}{2}\left(\pi\left[\frac{1}{b^2}\right]+2\pi\left[\frac{e^{-ab}}{-2b^2}\right]\right)\\&=\frac{1}{2}\left(\frac{\pi}{b^2}-\frac{\pi e^{-ab}}{b^2}\right)\end{aligned}$$

Hence

$$I=\frac{\pi-\pi e^{-ab}}{2b^2}$$

### 2.12.7 Problem 6

Evaluate the integrals (a) $\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^x}dx$ where $0 < p < 1, 0 < q < 1$ (b) $\int_0^{\infty}\frac{\ln\left(x^2+1\right)}{1+x^2}dx$

**Solution**

#### 2.12.7.1 Part(a)

$$I=\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^x}dx$$

Break the integral into 2 and use the rectangle grid method to show that each leg of the integral vanish

$$I=I_1-I_2$$

Where

$$\begin{aligned}I_1&=\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}dz\\I_2&=\int_{-\infty}^{\infty}\frac{e^{qz}}{1-e^z}dz\end{aligned}$$

Looking at $I_1$ for now.

$$\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}dz=\int_{-R}^{R}\frac{e^{pz}}{1-e^z}dz+\overbrace{\int_{R}^{R+2\pi i}\frac{e^{pz}}{1-e^z}dz}^{leg1}+\int_{R+2\pi i}^{-R+2\pi i}\frac{e^{pz}}{1-e^z}dz+\overbrace{\int_{-R+2\pi i}^{-R}\frac{e^{pz}}{1-e^z}dz}^{leg2}$$

looking at leg1 integral, $\int_R^{R+2\pi i}\frac{e^{pz}}{1-e^z}dz$. Let $z = 2\pi it + R$ or $t = \frac{z-R}{2\pi i}$, hence $dt = \frac{dz}{2\pi i}$. When $z = R, t = 0$ and when $z = R + 2\pi i, t = 1$, hence the integral for leg1 becomes

$$\int_0^1\frac{e^{p(2\pi it+R)}}{1-e^{2\pi it+R}}2\pi idt=2\pi i\int_0^1\frac{e^{p(2\pi it+R)}}{1-e^{2\pi it+R}}dt$$

Now we need to show that the above goes to zero as $R$ goes to infinity. Writing the above as

$$2\pi i\int_0^1 \frac{e^{p2\pi it}e^{pR}}{1-e^{2\pi it}e^R}dt = 2\pi i\int_0^1 \frac{e^{p2\pi it}}{e^{-pR}-e^{2\pi it}e^{R(1-p)}}dt$$

Now $\lim_{R\to\infty} e^{-pR} = 0$ since $p > 0$ and $\lim_{R\to\infty} e^{R(1-p)} = \infty$ since $1-p$ is positive, since $p < 1$. Therefore the above becomes

$$2\pi i\int_0^1 \frac{e^{p2\pi it}}{0-e^{2\pi it}\infty}dt$$

And since $\left|e^{ix}\right| = 1$ then the integrand becomes $\frac{1}{-\infty} \to 0$, hence the integral vanishes. We need to do the same for the second leg

$$\int_{-R+2\pi i}^{-R} \frac{e^{pz}}{1-e^z}dz$$

Let $z = 2\pi it + R$ or $t = \frac{z+R}{2\pi i}$, hence $dt = \frac{dz}{2\pi i}$. When $z = -R, t = 0$ and when $z = -R + 2\pi i, t = 1$, hence the integral for leg2 becomes

$$\int_1^0 \frac{e^{p(2\pi it+R)}}{1-e^{2\pi it+R}}2\pi idt = 2\pi i\int_1^0 \frac{e^{p(2\pi it+R)}}{1-e^{2\pi it+R}}dt$$

Now we need to show that the above goes to zero as $R$ goes to infinity. Writing the above as

$$2\pi i\int_0^1 \frac{e^{p2\pi it}e^{qR}}{1-e^{2\pi it}e^R}dt = 2\pi i\int_0^1 \frac{e^{p2\pi it}}{e^{-qR}-e^{2\pi it}e^{R(1-p)}}dt$$

Now $\lim_{R\to\infty} e^{-pR} = 0$ since $p > 0$ and $\lim_{R\to\infty} e^{R(1-p)} = \infty$ since $1-p$ is positive, since $p < 1$. Therefore the above becomes

$$2\pi i\int_0^1 \frac{e^{p2\pi it}}{0-e^{2\pi it}\infty}dt$$

And since $\left|e^{ix}\right| = 1$ then the integrand becomes $\frac{1}{-\infty} \to 0$, hence the integral vanishes. So we are left with these now

$$\int_{-\infty}^{\infty} \frac{e^{pz}}{1-e^z}dz = \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz + \int_{R+2\pi i}^{-R+2\pi i} \frac{e^{pz}}{1-e^z}dz$$

Now we use $t = z - 2\pi i$ for the top edge only. $dt = dz$, when $z = R + 2\pi i, t = R$ and when $z = -R + 2\pi i, t = -R$, hence the above becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{pz}}{1-e^z}dz &= \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz + \int_{R}^{-R} \frac{e^{p(t+2\pi i)}}{1-e^t}dt \\
&= \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz - \int_{-R}^{R} \frac{e^{p(t+2\pi i)}}{1-e^t}dt \\
&= \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz - \int_{-R}^{R} \frac{e^{pt}e^{p2\pi i}}{1-e^t}dt \\
&= \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz - e^{p2\pi i}\int_{-R}^{R} \frac{e^{pt}}{1-e^t}dt \\
&= \int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz - e^{p2\pi i}\int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz \\
&= \left(1-e^{p2\pi i}\right)\int_{-R}^{R} \frac{e^{pz}}{1-e^z}dz
\end{aligned}$$

But in the limit $\lim_{R\to\infty}\left(1-e^{p2\pi i}\right)\int_{-R}^{R}\frac{e^{pz}}{1-e^z}dz \to \left(1-e^{p2\pi i}\right)\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}dz = 2\pi i\sum res$ Hence

$$\int_{-\infty}^{\infty} \frac{e^{pz}}{1-e^z}dz = \frac{1}{\left(1-e^{p2\pi i}\right)}\left(2\pi i\sum^{UHP} res + \pi i\sum^{line} res\right)$$

Now we need to find poles in the rectangle. Poles of $f(z) = \frac{e^{pz}}{1-e^z}$ are $e^z = 1$, hence $z = 0, i2\pi, i4\pi, \cdots$ so only $z = 0$ and $z = i2\pi$ is of interest to use. The pole at $z = 0$ is on the real line, so this get $i\pi$ and the pole at $i2\pi$ get $i2\pi$ contribution. But

$$res(2\pi i) = \left[\frac{N(z)}{D'(z)}\right]_{z=2\pi i} = \left[\frac{e^{pz}}{-e^z}\right]_{z=2\pi i} = \frac{e^{p2\pi i}}{-e^{2\pi i}} = -e^{p2\pi i}$$

And

$$res\,(0) = \left[\frac{N\,(z)}{D'\,(z)}\right]_{z=0} = \left[\frac{e^{pz}}{-e^{z}}\right]_{z=0} = -1$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^{z}}dz &= \frac{1}{\left(1-e^{p2\pi i}\right)}\left(2\pi i\left(-e^{p2\pi i}\right)+\pi i\,(-1)\right)\\ &= \frac{-2\pi i e^{2\pi i p}-\pi i}{\left(1-e^{p2\pi i}\right)}\\ &= \pi i\left(\frac{-2e^{2\pi i p}-1}{1-e^{p2\pi i}}\right)\end{aligned}$$

For $I_2$, the result will be similar, hence

$$I_2 = \pi i\left(\frac{-2e^{2\pi i q}-1}{1-e^{q2\pi i}}\right)$$

Hence

$$\begin{aligned}I &= I_1 - I_2\\ &= \pi i\left(\frac{-2e^{2\pi i p}-1}{1-e^{p2\pi i}}\right)-\pi i\left(\frac{-2e^{2\pi i q}-1}{1-e^{q2\pi i}}\right)\\ &= \pi i\left(\frac{-2e^{2\pi i p}-1}{1-e^{p2\pi i}}-\frac{-2e^{2\pi i q}-1}{1-e^{q2\pi i}}\right)\end{aligned}$$

### 2.12.7.2 Part (b)

$$I = \int_0^{\infty}\frac{\ln\left(x^2+1\right)}{1+x^2}dx$$

parameterize it as

$$I\,(a) = \int_0^{\infty}\frac{\ln\left(ax^2+1\right)}{1+x^2}dx$$

Differentia w.r.t. $a$

$$\begin{aligned}I'\,(a) &= \int_0^{\infty}\frac{d}{da}\frac{\ln\left(ax^2+1\right)}{1+x^2}dx\\ &= \int_0^{\infty}\frac{x^2}{\left(1+x^2\right)\left(1+ax^2\right)}dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{z^2}{\left(1+z^2\right)\left(1+az^2\right)}dz\\ &= \frac{1}{2a}\int_{-\infty}^{\infty}\frac{z^2}{\left(1+z^2\right)\left(\frac{1}{a}+z^2\right)}dz\\ &= \frac{1}{2a}\int_{-\infty}^{\infty}\frac{z^2}{(z-i)\,(z+i)\left(z-\frac{i}{\sqrt{a}}\right)\left(z+\frac{i}{\sqrt{a}}\right)}dz\end{aligned}$$

Poles are at $z=\pm i$ and $z=\pm\frac{i}{\sqrt{a}}$. Pole $z=\{i,\frac{i}{\sqrt{a}}\}$ are in UHP assuming $a>0$. Residue for $z=i$ is

$$\begin{aligned}res\,(i) &= \lim_{z\to i}\frac{z^2}{(z+i)\left(z-\frac{i}{\sqrt{a}}\right)\left(z+\frac{i}{\sqrt{a}}\right)}\\ &= \frac{-1}{(2i)\left(i-\frac{i}{\sqrt{a}}\right)\left(i+\frac{i}{\sqrt{a}}\right)}\\ &= -i\frac{a}{2a-2}\end{aligned}$$

Residue for $z = \frac{i}{\sqrt{a}}$

$$
\begin{aligned}
res\left(i\sqrt{a}\right) &= \lim_{z\to\frac{i}{\sqrt{a}}} \frac{z^2}{(z-i)(z+i)\left(z+\frac{i}{\sqrt{a}}\right)} \\
&= \frac{\frac{-1}{a}}{\left(\frac{i}{\sqrt{a}}-i\right)\left(\frac{i}{\sqrt{a}}+i\right)\left(\frac{i}{\sqrt{a}}+\frac{i}{\sqrt{a}}\right)} \\
&= i\frac{\sqrt{a}}{2a-2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{1}{2a}\int_{-\infty}^{\infty} \frac{z^2}{(z-i)(z+i)\left(z-\frac{i}{\sqrt{a}}\right)\left(z+\frac{i}{\sqrt{a}}\right)}dz &= \frac{1}{2a}2\pi i\left(-i\frac{a}{2a-2}+i\frac{\sqrt{a}}{2a-2}\right) \\
&= \frac{1}{2}\frac{\pi}{a+\sqrt{a}}
\end{aligned}
$$

Therefore

$$
I'(a) = \frac{1}{2}\frac{\pi}{a+\sqrt{a}}
$$

Integrate

$$
\begin{aligned}
I(a) &= \frac{\pi}{2}\int \frac{1}{a+\sqrt{a}}da \\
&= \frac{\pi}{2}2\log\left(\sqrt{a}+1\right)+C \\
&= \pi\log\left(\sqrt{a}+1\right)+C
\end{aligned}
$$

To find $C$, from

$$
I(a) = \int_0^{\infty} \frac{\ln\left(ax^2+1\right)}{1+x^2}dx
$$

we see that at $a=0$ then $I(0)=0$ hence

$$
\begin{aligned}
0 &= \pi\log(0+1)+C \\
C &= 0
\end{aligned}
$$

Hence

$$
I(a) = \pi\log\left(\sqrt{a}+1\right)
$$

To obtain the integral we started with, let $a=1$, hence

$$
\begin{aligned}
I(1) &= \int_0^{\infty} \frac{\ln\left(x^2+1\right)}{1+x^2}dx \\
&= \left[\pi\log\left(\sqrt{a}+1\right)\right]_{a=1} \\
&= \pi\log(1+1)
\end{aligned}
$$

Hence

$$
\int_0^{\infty} \frac{\ln\left(x^2+1\right)}{1+x^2}dx = \pi\log(2)
$$

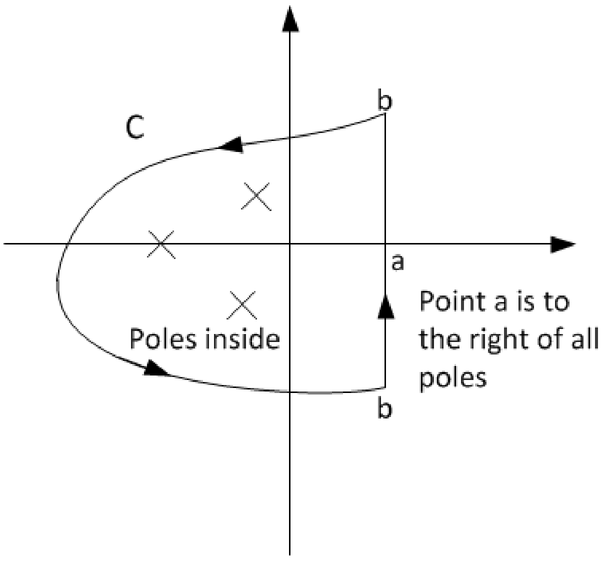
### 2.12.8 Problem 7

Determine the Laplace inversion for (a) $F(s) = \frac{s+1}{s^2\left(s^2+s+1\right)}$ (b)$F(s) = \frac{1}{(a+b)\cosh\left(a\sqrt{s}\right)}$

### 2.12.8.1 Part(a)

Using Bromwich formula

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib} F(s)\,e^{st}ds \\ &= \frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib} \frac{s+1}{s^2\left(s^2+s+1\right)}e^{st}ds \end{aligned}$$



The integral is broken into 2 parts, the vertical leg and the integration of the curve shown above labeled $C$.

$$f(t) = \frac{1}{2\pi i}\lim_{b\to\infty}\left(\int_{a-ib}^{a+ib} \frac{s+1}{s^2\left(s^2+s+1\right)}e^{st}ds + \oint_C \frac{s+1}{s^2\left(s^2+s+1\right)}e^{st}ds\right)$$

We can use the residue theorem on the above integral, but provided the function $F(s)\,e^{st}$ is analytic inside the curve shown above (except of course at the poles, if any, inside the curve). Proof that $F(s)\,e^{st}$ satisfies this condition is hence assumed. The proof was not given in class. So we need to calculate the residues of $F(s)\,e^{st}$ for all the poles.

$$f(t) = \frac{1}{2\pi i}\left(2\pi i\sum_{i=1}^{poles} res\left(F\left(s_i e^{s_i t}\right)\right)\right)$$

So let us find the poles and the residue of each. Given $\frac{s+1}{s^2\left(s^2+s+1\right)}e^{st} = \frac{s+1}{s^2\left(s-\left(-\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)\right)\left(s-\left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)\right)}e^{st}$ hence the poles are $s=0$ or order 2 and $s_1 = \left(-\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)$ and $s_2 = \left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)$.

Residue for $s=0$ is

$$\begin{aligned} res(0) &= \lim_{s\to 0}\frac{d}{ds}\left(\frac{s^2(s+1)}{s^2(s-s_1)(s-s_2)}e^{st}\right) \\ &= \lim_{s\to 0}\frac{d}{ds}\left(\frac{(s+1)}{(s-s_1)(s-s_2)}e^{(st)}\right) \\ &= t \end{aligned}$$

And

$$\begin{aligned} res(s_1) &= \lim_{s\to s_1}\frac{(s+1)}{s^2(s-s_2)}e^{st} \\ &= -\frac{ie^{-\frac{1}{2}\left(1+i\sqrt{3}\right)t}}{\sqrt{3}} \end{aligned}$$

And

$$\begin{aligned} res(s_2) &= \lim_{s\to s_2}\frac{(s+1)}{s^2(s-s_1)}e^{st} \\ &= -\frac{(-i+\sqrt{3})e^{\frac{1}{2}\left(1+i\sqrt{3}\right)t}}{3i+\sqrt{3}} \end{aligned}$$

Therefore,

$$
\begin{aligned}
f(t) &= t - \frac{ie^{-\frac{1}{2}\left(1+i\sqrt{3}\right)t}}{\sqrt{3}} - \frac{(-i+\sqrt{3})e^{\frac{1}{2}\left(1+i\sqrt{3}\right)t}}{3i+\sqrt{3}} \\
&= t - \frac{2e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}}
\end{aligned}
$$

#### 2.12.8.2 Part (b)

The setup is similar to part(a) and not will be repeated, we will go the step of finding the residues. For this, we need to first find the poles of

$$
F(s)\,e^{st} = \frac{e^{st}}{(a+b)\cosh\left(a\sqrt{s}\right)}
$$

The function $\cosh(x)$ is zero at $\pm\frac{i\pi}{2} + 2i\pi n$ where $n$ is the set of integers. For $\cosh\left(a\sqrt{s}\right)$ this becomes $-\frac{1}{4}\frac{\pi^2}{a^2}(1+2n)^2$ for $n$ integer. Hence, for $n=0$

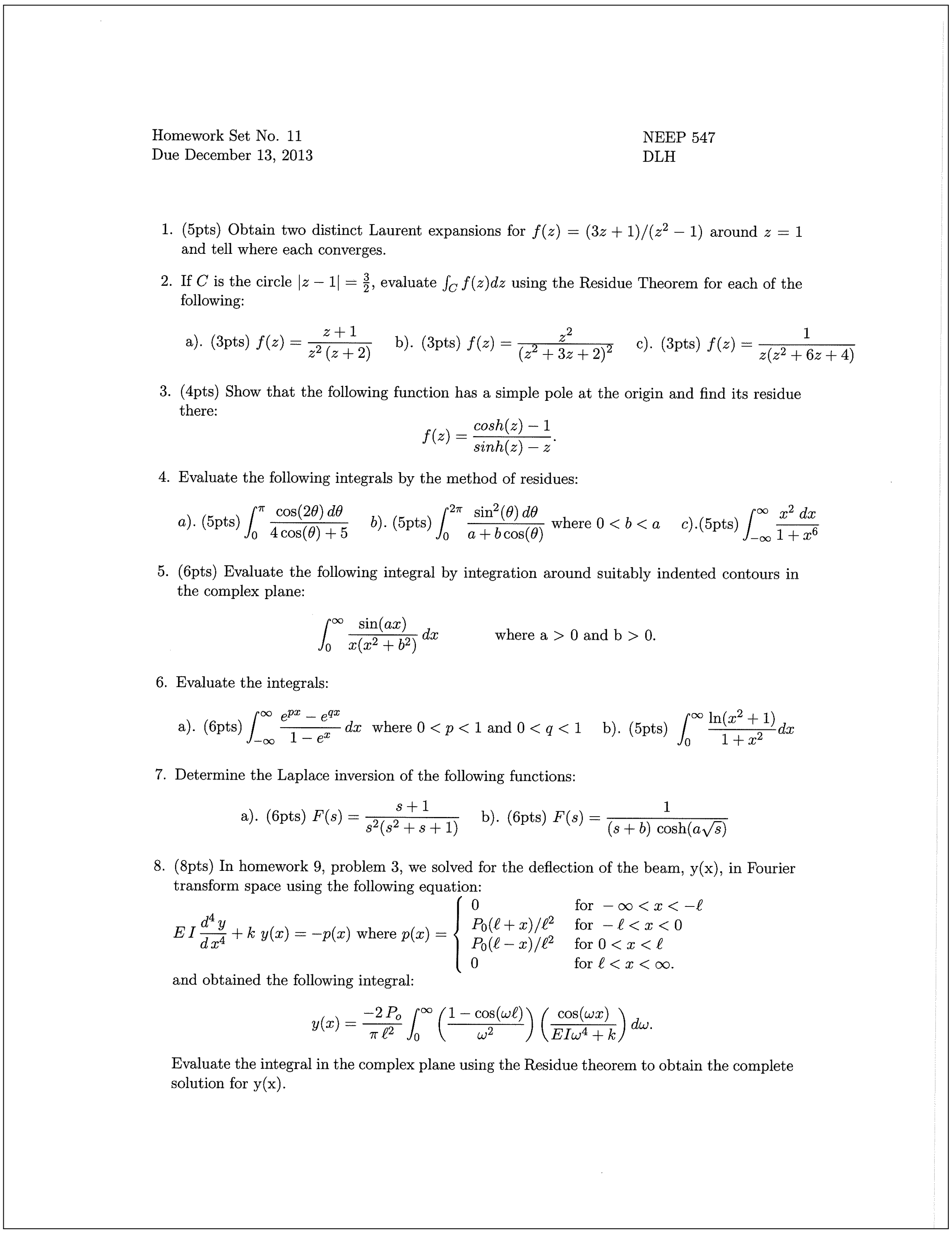
$$
\begin{aligned}
res\left(s \rightarrow \frac{-\pi^2}{4a^2}\right) &= \frac{1}{(a+b)} \lim_{s\rightarrow\frac{-\pi^2}{4a^2}} \left(s + \frac{\pi^2}{4a^2}\right)\frac{N(s)}{D'(s)} \\
&= \frac{1}{(a+b)} \lim_{s\rightarrow\frac{-\pi^2}{4a^2}} \frac{\left(s+\frac{\pi^2}{4a^2}\right)2\sqrt{s}e^{st}}{a\sinh\left(a\sqrt{s}\right)} \\
&= \frac{1}{(a+b)} \frac{(0)\,2\sqrt{\frac{-\pi^2}{4a^2}}e^{\frac{-\pi^2}{4a^2}t}}{a\sinh\left(a\sqrt{\frac{-\pi^2}{4a^2}}\right)} \\
&= \frac{1}{(a+b)} \frac{(0)\,2\sqrt{\frac{-\pi^2}{4a^2}}e^{\frac{-\pi^2}{4a^2}t}}{a\sinh\left(\frac{\pi i}{2}\right)} \\
&= \frac{1}{(a+b)} \frac{(0)\,2\sqrt{\frac{-\pi^2}{4a^2}}e^{\frac{-\pi^2}{4a^2}t}}{ai} \\
&= 0
\end{aligned}
$$

Hence the residue is zero. If we try for $n = 1, 2, \cdots$ we'll find all residues are zero. Hence

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi i}\left(2\pi i \sum_{i=1}^{poles} res\left(s_i\right)\right) \\
&= 0
\end{aligned}
$$

I have something wrong. Need more time to work on this.

### 2.12.9 key solution



Homework Set No. 11 NEEP 547
Due December 13, 2013 DLH

1. (5pts) Obtain two distinct Laurent expansions for $f(z) = (3z+1)/(z^2-1)$ around $z = 1$ and tell where each converges.

2. If $C$ is the circle $|z-1| = \frac{3}{2}$, evaluate $\int_C f(z)dz$ using the Residue Theorem for each of the following:

   a). (3pts) $f(z) = \dfrac{z+1}{z^2(z+2)}$ b). (3pts) $f(z) = \dfrac{z^2}{(z^2+3z+2)^2}$ c). (3pts) $f(z) = \dfrac{1}{z(z^2+6z+4)}$

3. (4pts) Show that the following function has a simple pole at the origin and find its residue there:
$$f(z) = \frac{cosh(z) - 1}{sinh(z) - z}.$$

4. Evaluate the following integrals by the method of residues:

   $a$). (5pts) $\displaystyle\int_0^{\pi} \frac{\cos(2\theta)\, d\theta}{4\cos(\theta)+5}$ $b$). (5pts) $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)\, d\theta}{a+b\cos(\theta)}$ where $0 < b < a$ $c$).(5pts) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\, dx}{1+x^6}$

5. (6pts) Evaluate the following integral by integration around suitably indented contours in the complex plane:
$$\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\, dx \qquad \text{where a > 0 and b > 0.}$$

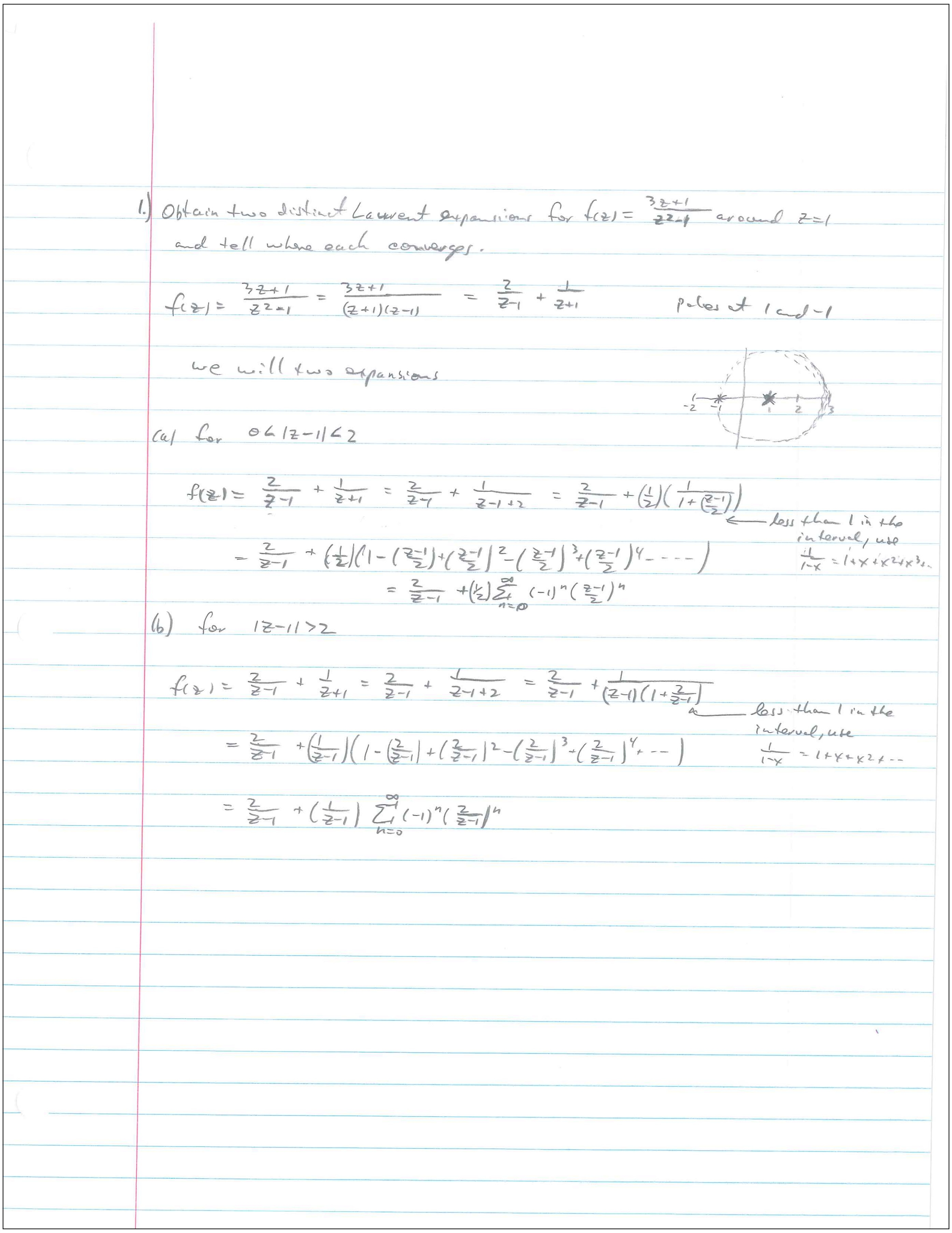
6. Evaluate the integrals:

   a). (6pts) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1-e^{x}}\, dx$ where $0 < p < 1$ and $0 < q < 1$ b). (5pts) $\displaystyle\int_0^{\infty} \frac{\ln(x^2+1)}{1+x^2} dx$

7. Determine the Laplace inversion of the following functions:

   a). (6pts) $F(s) = \dfrac{s+1}{s^2(s^2+s+1)}$ b). (6pts) $F(s) = \dfrac{1}{(s+b)\cosh(a\sqrt{s})}$

8. (8pts) In homework 9, problem 3, we solved for the deflection of the beam, y(x), in Fourier transform space using the following equation:
$$EI\frac{d^4y}{dx^4} + k\, y(x) = -p(x) \text{ where } p(x) = \begin{cases} 0 & \text{for } -\infty < x < -\ell \\ P_0(\ell+x)/\ell^2 & \text{for } -\ell < x < 0 \\ P_0(\ell-x)/\ell^2 & \text{for } 0 < x < \ell \\ 0 & \text{for } \ell < x < \infty. \end{cases}$$
and obtained the following integral:
$$y(x) = \frac{-2\,P_o}{\pi\,\ell^2}\int_0^{\infty}\left(\frac{1-\cos(\omega\ell)}{\omega^2}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right) d\omega.$$
Evaluate the integral in the complex plane using the Residue theorem to obtain the complete solution for y(x).



1.) Obtain two distinct Laurent expansions for $f(z)=\frac{3z+1}{z^2-1}$ around $z=1$ and tell where each converges.

$$f(z)=\frac{3z+1}{z^2-1}=\frac{3z+1}{(z+1)(z-1)}=\frac{2}{z-1}+\frac{1}{z+1}$$

poles at 1 and -1

we will two expansions

(a) for $0<|z-1|<2$

$$f(z)=\frac{2}{z-1}+\frac{1}{z+1}=\frac{2}{z-1}+\frac{1}{z-1+2}=\frac{2}{z-1}+\left(\frac{1}{2}\right)\left(\frac{1}{1+\left(\frac{z-1}{2}\right)}\right)$$

← less than 1 in the interval, use $\frac{1}{1-x}=1+x+x^2+x^3+\dots$

$$=\frac{2}{z-1}+\left(\frac{1}{2}\right)\left(1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\left(\frac{z-1}{2}\right)^4-\cdots\right)$$

$$=\frac{2}{z-1}+\left(\frac{1}{2}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$$

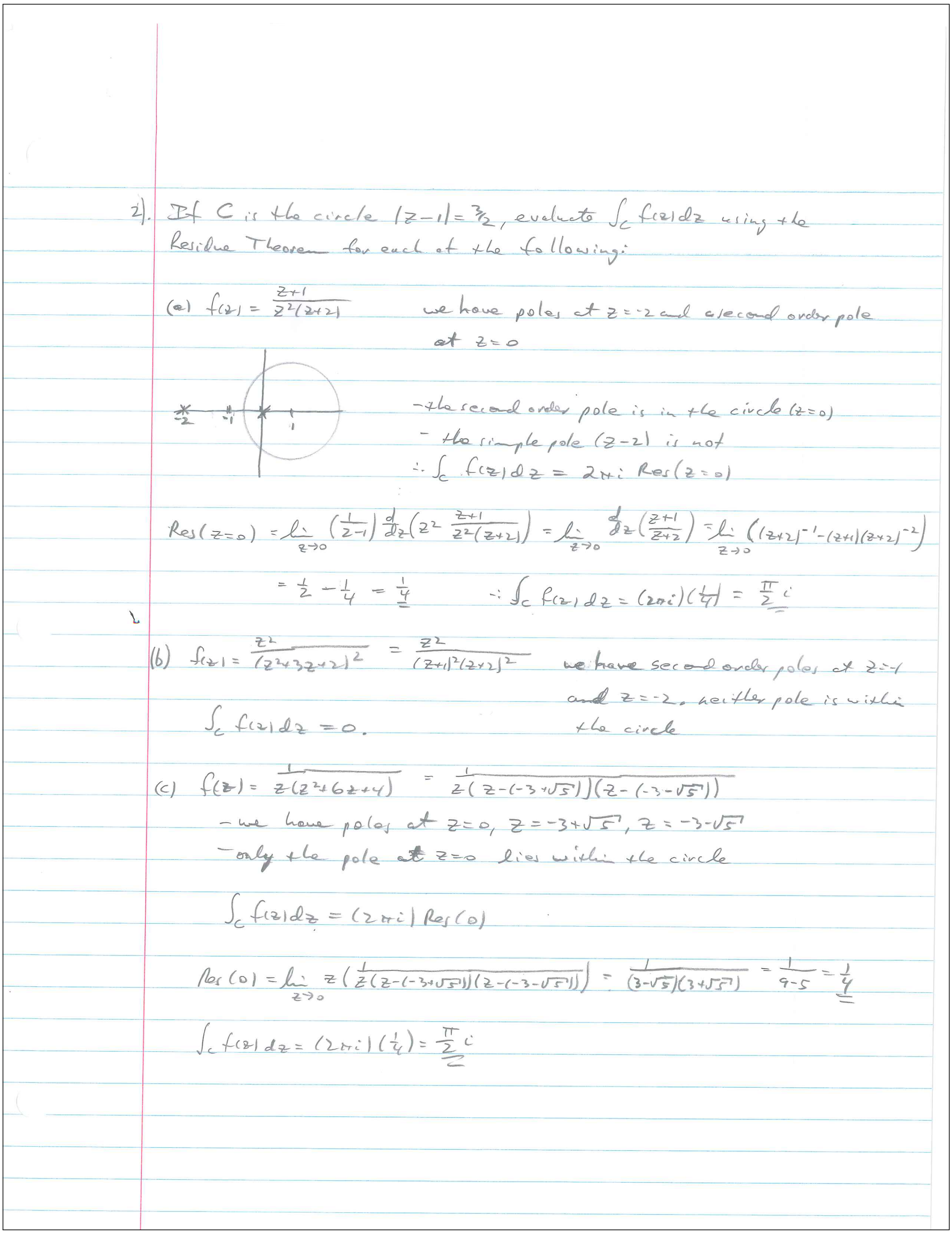
(b) for $|z-1|>2$

$$f(z)=\frac{2}{z-1}+\frac{1}{z+1}=\frac{2}{z-1}+\frac{1}{z-1+2}=\frac{2}{z-1}+\frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)}$$

less than 1 in the interval, use $\frac{1}{1-x}=1+x+x^2+\dots$

$$=\frac{2}{z-1}+\left(\frac{1}{z-1}\right)\left(1-\left(\frac{2}{z-1}\right)+\left(\frac{2}{z-1}\right)^2-\left(\frac{2}{z-1}\right)^3+\left(\frac{2}{z-1}\right)^4+\cdots\right)$$

$$=\frac{2}{z-1}+\left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z-1}\right)^n$$



2). If C is the circle $|z-1| = 3/2$, evaluate $\int_C f(z)dz$ using the Residue Theorem for each of the following:

(a) $f(z) = \frac{z+1}{z^2(z+2)}$ we have poles at $z=-2$ and a second order pole at $z=0$

- the second order pole is in the circle ($z=0$)
- the simple pole ($z-2$) is not

$\therefore \int_C f(z)dz = 2\pi i \text{ Res}(z=0)$

$$\text{Res}(z=0) = \lim_{z\to 0}\left(\frac{1}{2-1}\right)\frac{d}{dz}\left(z^2\frac{z+1}{z^2(z+2)}\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z+1}{z+2}\right) = \lim_{z\to 0}\left((z+2)^{-1}-(z+1)(z+2)^{-2}\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \underline{\underline{\frac{1}{4}}} \qquad \therefore \int_C f(z)dz = (2\pi i)\left(\frac{1}{4}\right) = \underline{\underline{\frac{\pi}{2}i}}$$

(b) $f(z) = \frac{z^2}{(z^2+3z+2)^2} = \frac{z^2}{(z+1)^2(z+2)^2}$ we have second order poles at $z=-1$ and $z=-2$. neither pole is within the circle

$\int_C f(z)dz = 0.$

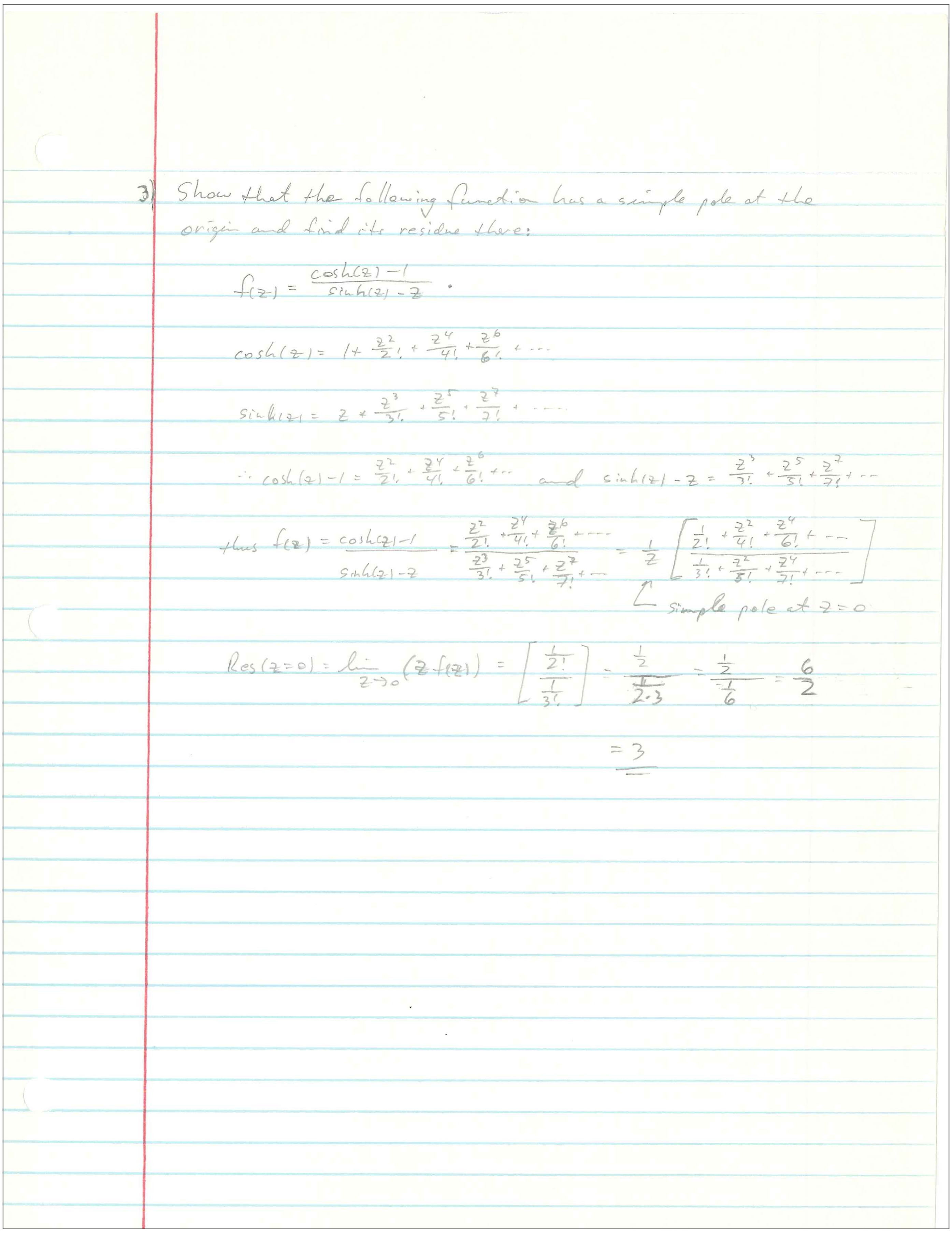
(c) $f(z) = \frac{1}{z(z^2+6z+4)} = \frac{1}{z(z-(-3+\sqrt{5}))(z-(-3-\sqrt{5}))}$

- we have poles at $z=0$, $z=-3+\sqrt{5}$, $z=-3-\sqrt{5}$
- only the pole at $z=0$ lies within the circle

$\int_C f(z)dz = (2\pi i)\text{ Res}(0)$

$$\text{Res}(0) = \lim_{z\to 0} z\left(\frac{1}{z(z-(-3+\sqrt{5}))(z-(-3-\sqrt{5}))}\right) = \frac{1}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{1}{9-5} = \underline{\underline{\frac{1}{4}}}$$

$\int_C f(z)dz = (2\pi i)\left(\frac{1}{4}\right) = \underline{\underline{\frac{\pi}{2}i}}$



3) Show that the following function has a simple pole at the origin and find its residue there:

$$f(z) = \frac{\cosh(z) - 1}{\sinh(z) - z} .$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\therefore \cosh(z) - 1 = \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \quad \text{and} \quad \sinh(z) - z = \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\text{thus } f(z) = \frac{\cosh(z) - 1}{\sinh(z) - z} = \frac{\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots}{\frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots} = \frac{1}{z}\left[\frac{\frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots}{\frac{1}{3!} + \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots}\right]$$

simple pole at $z = 0$

$$\text{Res}(z=0) = \lim_{z \to 0} (z f(z)) = \left[\frac{\frac{1}{2!}}{\frac{1}{3!}}\right] = \frac{\frac{1}{2}}{\frac{1}{2 \cdot 3}} = \frac{\frac{1}{2}}{\frac{1}{6}} = \frac{6}{2}$$

$$= \underline{\underline{3}}$$

4a) Evaluate the following integral by the method of Residues:

$$\int_0^{\pi} \frac{\cos(2\theta)}{4\cos(\theta)+5}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos(2\theta)}{4\cos(\theta)+5}\, d\theta$$

$z = re^{i\theta}$ ; $r=1$, $z = e^{i\theta}$ ; $dz = ie^{i\theta}\, d\theta$ ; $d\theta = \frac{-i\,dz}{z}$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}(z+z^{-1})$$

$$\cos(2\theta) = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{1}{2}(z^2+z^{-2})$$

$$\frac{1}{2}\int_c \frac{\frac{1}{2}(z^2+z^{-2})(-i\frac{dz}{z})}{4(\frac{1}{2})(z+z^{-1})+5} = \frac{-i}{4}\int_c \frac{(z^2+z^{-2})}{(2(z+z^{-1})+5)}\left(\frac{dz}{z}\right) = \frac{-i}{4}\int_c \left(\frac{1}{z^2}\right)\left(\frac{z^4+1}{2(z+z^{-1})+5}\right)\left(\frac{dz}{z}\right)$$

$$= \left(\frac{-i}{4}\right)\int_c \left(\frac{1}{z^2}\right)\left(\frac{(z^4+1)\,z}{2(z^2+1)+5z}\right)\left(\frac{dz}{z}\right) = \left(\frac{-i}{8}\right)\int_c \left(\frac{1}{z^2}\right)\left(\frac{z^4+1}{z^2+1+\frac{5}{2}z}\right)dz$$

$$= \left(\frac{-i}{8}\right)\int_c \frac{(z^4+1)}{(z^2)(z+2)(z+\frac{1}{2})}\,dz$$

the 2nd order pole $z=0$ and the simple pole $z=-\frac{1}{2}$ lie within the unit circle.

$$\therefore I = \left(-\tfrac{i}{8}\right)(2\pi i)\left(\text{Res}(0) + \text{Res}(z=-\tfrac{1}{2})\right)$$

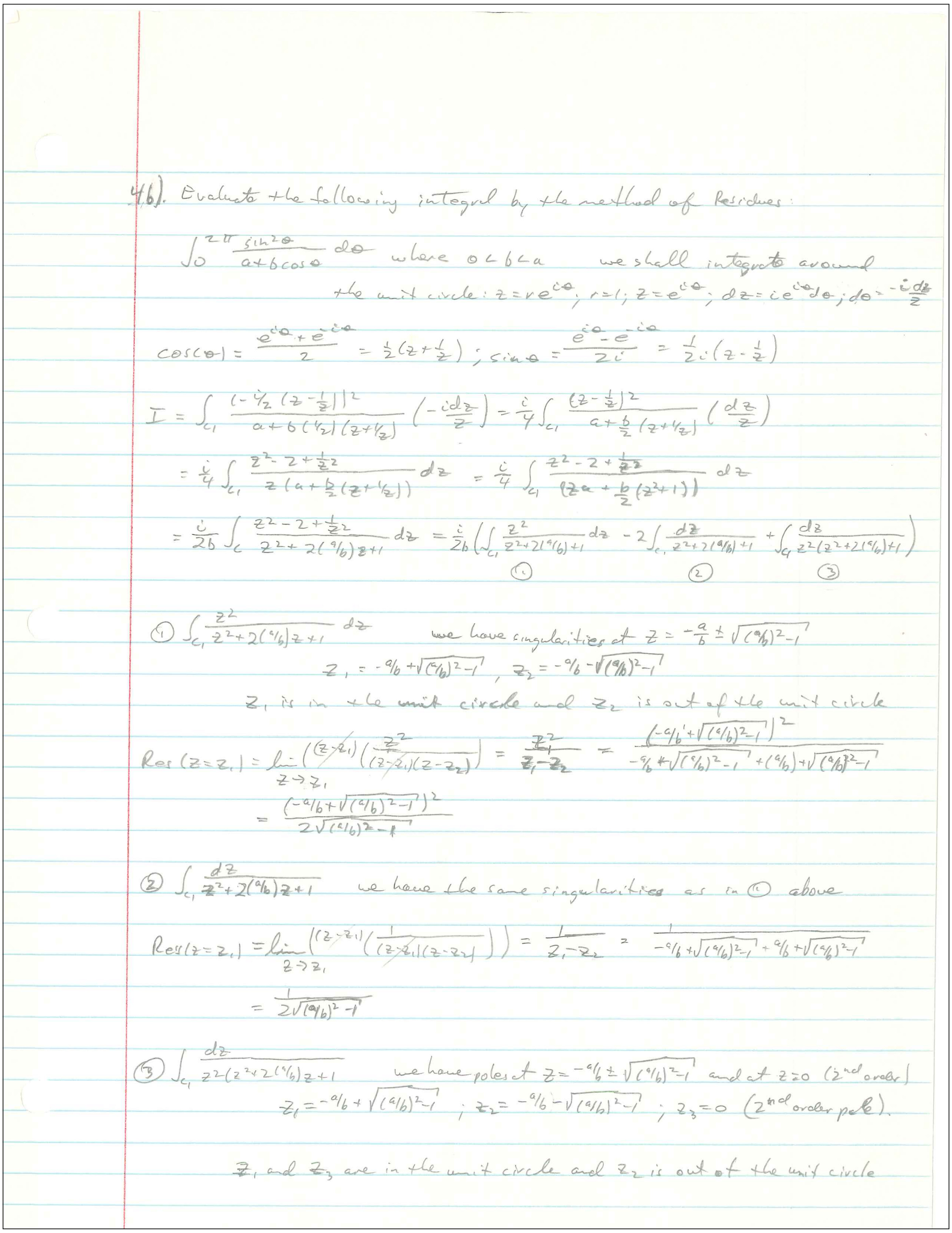
$$\text{Res}(-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \left(\cancel{(z+\tfrac{1}{2})}\right)\left(\frac{1+z^4}{(z^2)(z+2)\cancel{(z+\frac{1}{2})}}\right) = \frac{1+(-\frac{1}{2})^4}{(-\frac{1}{2})^2(2-\frac{1}{2})} = \frac{1+\frac{1}{16}}{(\frac{1}{4})(\frac{3}{2})} = \frac{\frac{17}{16}}{\frac{3}{8}} = \frac{17}{6}$$

$$\text{Res}(0) = \lim_{z\to 0} \frac{1}{(2-1)!}\frac{d}{dz}\left((z^2)\left(\frac{1+z^4}{(z)^2(z+2)(z+\frac{1}{2})}\right)\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{1+z^4}{(z+2)(z+\frac{1}{2})}\right)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{1+z^4}{z^2+\frac{5}{2}z+1}\right) = \lim_{z\to 0}\frac{(4z^3)(z^2+\frac{5}{2}z+1)-(1+z^4)(2z+\frac{5}{2})}{(z^2+\frac{5}{2}z+1)^2}$$

$$= -\frac{(\frac{5}{2})}{(1)^2} = -\frac{5}{2}$$

$$I = \left(-\tfrac{i}{8}\right)(2\pi i)\left(\left(-\tfrac{5}{2}\right)+\left(\tfrac{17}{6}\right)\right) = \left(\frac{\pi}{4}\right)\left(\frac{17}{6}-\frac{15}{6}\right) = \left(\frac{\pi}{4}\right)\left(\frac{1}{3}\right) = \underline{\underline{\frac{\pi}{12}}}$$



4b). Evaluate the following integral by the method of Residues:

$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$ where $0<b<a$ we shall integrate around the unit circle: $z=re^{i\theta}$, $r=1$; $z=e^{i\theta}$; $dz=ie^{i\theta}d\theta$; $d\theta=\frac{-i\,dz}{z}$

$$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{1}{2}\left(z+\frac{1}{z}\right);\ \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$I=\int_{C_1}\frac{\left(-\frac{i}{2}\left(z-\frac{1}{z}\right)\right)^2}{a+b\left(\frac{1}{2}\right)\left(z+\frac{1}{z}\right)}\left(-\frac{i\,dz}{z}\right)=\frac{i}{4}\int_{C_1}\frac{\left(z-\frac{1}{z}\right)^2}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)}\left(\frac{dz}{z}\right)$$

$$=\frac{i}{4}\int_{C_1}\frac{z^2-2+\frac{1}{z^2}}{z\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)}dz=\frac{i}{4}\int_{C_1}\frac{z^2-2+\frac{1}{z^2}}{\left(za+\frac{b}{2}(z^2+1)\right)}dz$$

$$=\frac{i}{2b}\int_C\frac{z^2-2+\frac{1}{z^2}}{z^2+2\left(\frac{a}{b}\right)z+1}dz=\frac{i}{2b}\Big(\underset{①}{\int_{C_1}\frac{z^2}{z^2+2\left(\frac{a}{b}\right)+1}dz}-2\underset{②}{\int_{C_1}\frac{dz}{z^2+2\left(\frac{a}{b}\right)+1}}+\underset{③}{\int_{C_1}\frac{dz}{z^2(z^2+2\left(\frac{a}{b}\right)+1}}\Big)$$

① $\int_{C_1}\frac{z^2}{z^2+2\left(\frac{a}{b}\right)z+1}dz$ we have singularities at $z=-\frac{a}{b}\pm\sqrt{\left(\frac{a}{b}\right)^2-1}$

$z_1=-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}$, $z_2=-\frac{a}{b}-\sqrt{\left(\frac{a}{b}\right)^2-1}$

$z_1$ is in the unit circle and $z_2$ is out of the unit circle

$$\operatorname{Res}(z=z_1)=\lim_{z\to z_1}\left(\cancel{(z-z_1)}\left(\frac{z^2}{\cancel{(z-z_1)}(z-z_2)}\right)\right)=\frac{z_1^2}{z_1-z_2}=\frac{\left(-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}\right)^2}{-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}+\left(\frac{a}{b}\right)+\sqrt{\left(\frac{a}{b}\right)^2-1}}$$

$$=\frac{\left(-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}\right)^2}{2\sqrt{\left(\frac{a}{b}\right)^2-1}}$$

② $\int_{C_1}\frac{dz}{z^2+2\left(\frac{a}{b}\right)z+1}$ we have the same singularities as in ① above

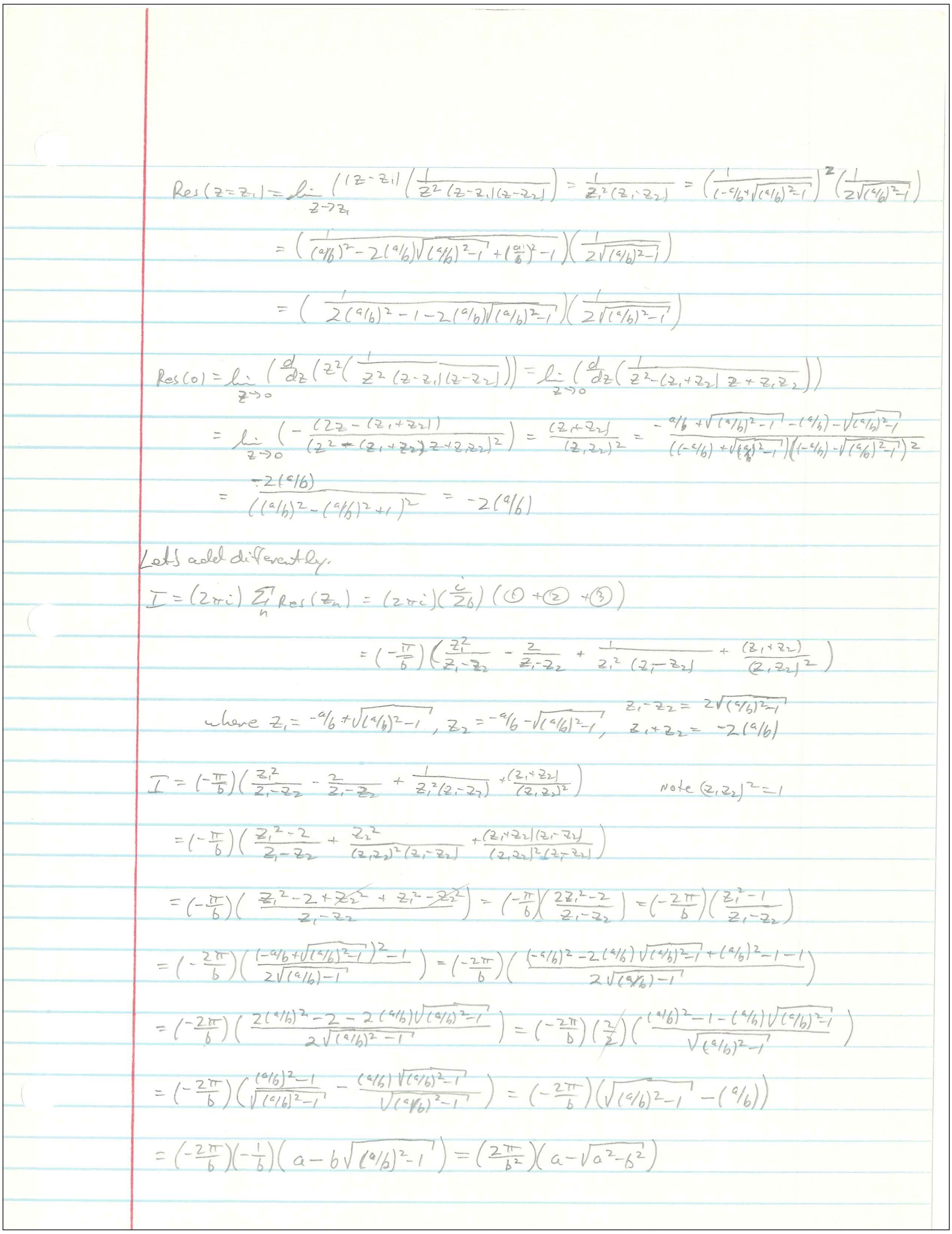
$$\operatorname{Res}(z=z_1)=\lim_{z\to z_1}\left(\cancel{(z-z_1)}\left(\frac{1}{\cancel{(z-z_1)}(z-z_2)}\right)\right)=\frac{1}{z_1-z_2}=\frac{1}{-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}+\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}}$$

$$=\frac{1}{2\sqrt{\left(\frac{a}{b}\right)^2-1}}$$

③ $\int_{C_1}\frac{dz}{z^2(z^2+2\left(\frac{a}{b}\right)z+1}$ we have poles at $z=-\frac{a}{b}\pm\sqrt{\left(\frac{a}{b}\right)^2-1}$ and at $z=0$ ($2^{nd}$ order)

$z_1=-\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1}$ ; $z_2=-\frac{a}{b}-\sqrt{\left(\frac{a}{b}\right)^2-1}$ ; $z_3=0$ ($2^{nd}$ order pole).

$z_1$ and $z_3$ are in the unit circle and $z_2$ is out of the unit circle



$$\text{Res}(z=z_1)=\lim_{z\to z_1}\left((z-z_1)\left(\frac{1}{z^2(z-z_1)(z-z_2)}\right)=\frac{1}{z_1^2(z_1-z_2)}=\left(\frac{1}{(-a/b+\sqrt{(a/b)^2-1})}\right)^2\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$=\left(\frac{1}{(a/b)^2-2(a/b)\sqrt{(a/b)^2-1}+(\frac{a}{b})^2-1}\right)\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$=\left(\frac{1}{2(a/b)^2-1-2(a/b)\sqrt{(a/b)^2-1}}\right)\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$\text{Res}(0)=\lim_{z\to 0}\left(\frac{d}{dz}\left(z^2\left(\frac{1}{z^2(z-z_1)(z-z_2)}\right)\right)\right)=\lim_{z\to 0}\left(\frac{d}{dz}\left(\frac{1}{z^2-(z_1+z_2)z+z_1z_2}\right)\right)$$

$$=\lim_{z\to 0}\left(-\frac{(2z-(z_1+z_2))}{(z^2-(z_1+z_2)z+z_1z_2)^2}\right)=\frac{(z_1+z_2)}{(z_1z_2)^2}=\frac{-a/b+\sqrt{(a/b)^2-1}-(a/b)-\sqrt{(a/b)^2-1}}{((-a/b)+\sqrt{(a/b)^2-1})((-a/b)-\sqrt{(a/b)^2-1})^2}$$

$$=\frac{-2(a/b)}{((a/b)^2-(a/b)^2+1)^2}=-2(a/b)$$

Lets add differently.

$$I=(2\pi i)\sum_n \text{Res}(z_n)=(2\pi i)\left(\frac{i}{2b}\right)(① + ② + ③)$$

$$=\left(-\frac{\pi}{b}\right)\left(\frac{z_1^2}{z_1-z_2}-\frac{2}{z_1-z_2}+\frac{1}{z_1^2(z_1-z_2)}+\frac{(z_1+z_2)}{(z_1z_2)^2}\right)$$

where $z_1=-a/b+\sqrt{(a/b)^2-1}$, $z_2=-a/b-\sqrt{(a/b)^2-1}$, $z_1-z_2=2\sqrt{(a/b)^2-1}$, $z_1+z_2=-2(a/b)$

$$I=\left(-\frac{\pi}{b}\right)\left(\frac{z_1^2}{z_1-z_2}-\frac{2}{z_1-z_2}+\frac{1}{z_1^2(z_1-z_2)}+\frac{(z_1+z_2)}{(z_1z_2)^2}\right)$$

note $(z_1z_2)^2=1$

$$=\left(-\frac{\pi}{b}\right)\left(\frac{z_1^2-2}{z_1-z_2}+\frac{z_2^2}{(z_1z_2)^2(z_1-z_2)}+\frac{(z_1+z_2)(z_1-z_2)}{(z_1z_2)^2(z_1-z_2)}\right)$$

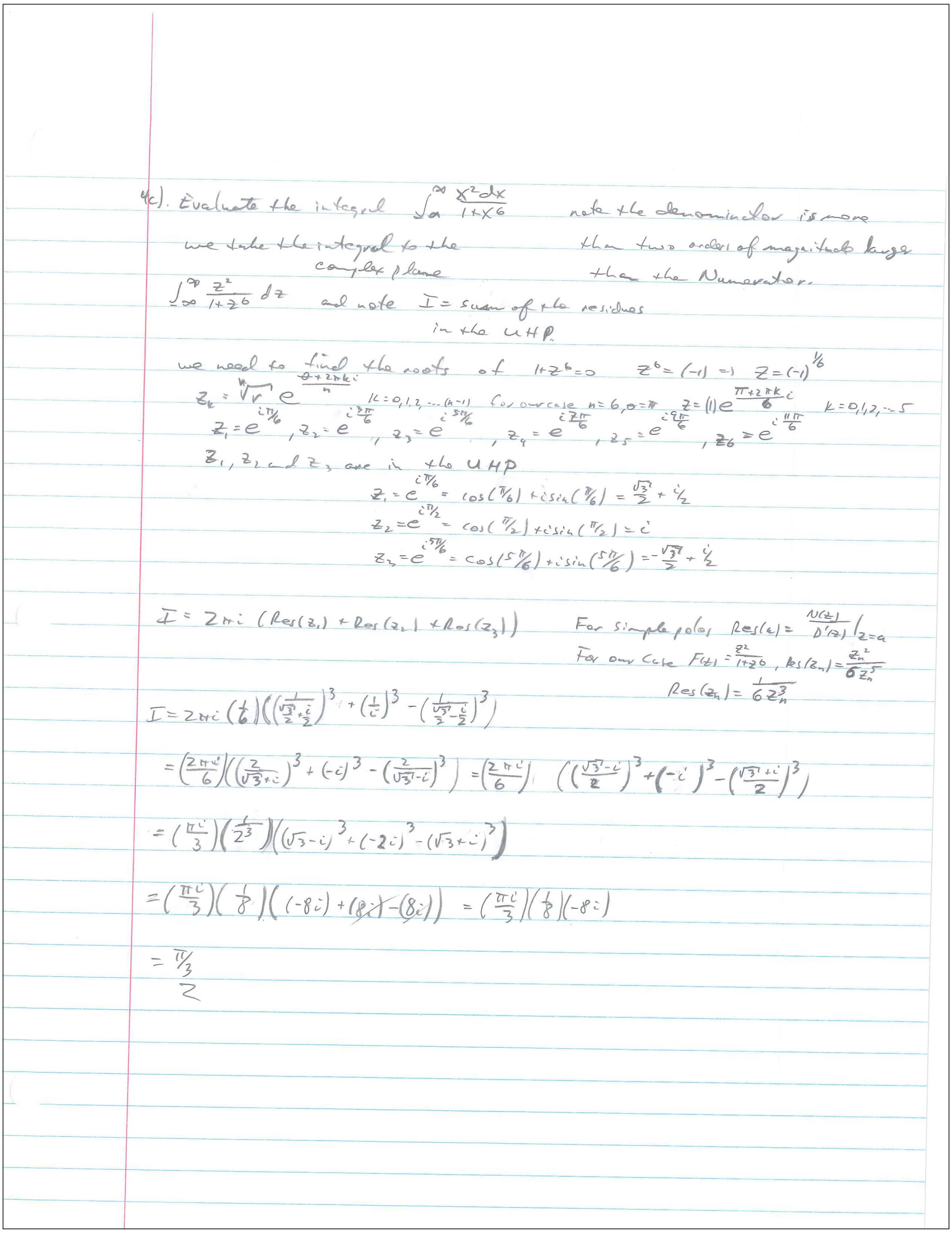
$$=\left(-\frac{\pi}{b}\right)\left(\frac{z_1^2-2+\cancel{z_2^2}+z_1^2-\cancel{z_2^2}}{z_1-z_2}\right)=\left(-\frac{\pi}{b}\right)\left(\frac{2z_1^2-2}{z_1-z_2}\right)=\left(-\frac{2\pi}{b}\right)\left(\frac{z_1^2-1}{z_1-z_2}\right)$$

$$=\left(-\frac{2\pi}{b}\right)\left(\frac{(-a/b+\sqrt{(a/b)^2-1})^2-1}{2\sqrt{(a/b)-1}}\right)=\left(-\frac{2\pi}{b}\right)\left(\frac{(-a/b)^2-2(a/b)\sqrt{(a/b)^2-1}+(a/b)^2-1-1}{2\sqrt{(a/b)-1}}\right)$$

$$=\left(-\frac{2\pi}{b}\right)\left(\frac{2(a/b)^2-2-2(a/b)\sqrt{(a/b)^2-1}}{2\sqrt{(a/b)^2-1}}\right)=\left(-\frac{2\pi}{b}\right)\left(\cancel{\frac{2}{2}}\right)\left(\frac{(a/b)^2-1-(a/b)\sqrt{(a/b)^2-1}}{\sqrt{(a/b)^2-1}}\right)$$

$$=\left(-\frac{2\pi}{b}\right)\left(\frac{(a/b)^2-1}{\sqrt{(a/b)^2-1}}-\frac{(a/b)\sqrt{(a/b)^2-1}}{\sqrt{(a/b)^2-1}}\right)=\left(-\frac{2\pi}{b}\right)\left(\sqrt{(a/b)^2-1}-(a/b)\right)$$

$$=\left(-\frac{2\pi}{b}\right)\left(-\frac{1}{b}\right)\left(a-b\sqrt{(a/b)^2-1}\right)=\left(\frac{2\pi}{b^2}\right)\left(a-\sqrt{a^2-b^2}\right)$$



4c). Evaluate the integral $\int_{\infty}^{\infty} \frac{x^2 dx}{1+x^6}$ note the denominator is more than two orders of magnitude large than the Numerator.

we take the integral to the complex plane

$\int_{-\infty}^{\infty} \frac{z^2}{1+z^6} dz$ and note $I$ = sum of the residues in the UHP.

we need to find the roots of $1+z^6=0$ $z^6=(-1) \Rightarrow z=(-1)^{1/6}$

$z_k = \sqrt[n]{r}\, e^{\frac{\theta+2\pi k i}{n}}$ $k=0,1,2,\dots(n-1)$ for our case $n=6, \theta=\pi$ $z=(1)e^{\frac{\pi+2\pi k}{6}i}$ $k=0,1,2,\dots 5$

$z_1=e^{i\pi/6}$, $z_2=e^{i\frac{3\pi}{6}}$, $z_3=e^{i\,5\pi/6}$, $z_4=e^{i\frac{7\pi}{6}}$, $z_5=e^{i\frac{9\pi}{6}}$, $z_6=e^{i\frac{11\pi}{6}}$

$z_1, z_2$ and $z_3$ are in the UHP

$$z_1=e^{i\pi/6} = \cos(\pi/6)+i\sin(\pi/6) = \frac{\sqrt{3}}{2}+i/2$$

$$z_2=e^{i\pi/2} = \cos(\pi/2)+i\sin(\pi/2) = i$$

$$z_3=e^{i5\pi/6} = \cos(5\pi/6)+i\sin(5\pi/6) = -\frac{\sqrt{3}}{2}+i/2$$

$I = 2\pi i\,(Res(z_1)+Res(z_2)+Res(z_3))$

For simple poles $Res(a) = \frac{N(z)}{D'(z)}\Big|_{z=a}$

For our Case $F(z)=\frac{z^2}{1+z^6}$, $Res(z_n)=\frac{z_n^2}{6z_n^5}$

$Res(z_n) = \frac{1}{6z_n^3}$

$$I = 2\pi i\left(\frac{1}{6}\right)\left(\left(\frac{1}{\frac{\sqrt{3}}{2}+\frac{i}{2}}\right)^3+\left(\frac{1}{i}\right)^3-\left(\frac{1}{\frac{\sqrt{3}}{2}-\frac{i}{2}}\right)^3\right)$$

$$=\left(\frac{2\pi i}{6}\right)\left(\left(\frac{2}{\sqrt{3}+i}\right)^3+(-i)^3-\left(\frac{2}{\sqrt{3}-i}\right)^3\right) = \left(\frac{2\pi i}{6}\right)\left(\left(\frac{\sqrt{3}-i}{2}\right)^3+(-i)^3-\left(\frac{\sqrt{3}+i}{2}\right)^3\right)$$

$$=\left(\frac{\pi i}{3}\right)\left(\frac{1}{2^3}\right)\left((\sqrt{3}-i)^3+(-2i)^3-(\sqrt{3}+i)^3\right)$$

$$=\left(\frac{\pi i}{3}\right)\left(\frac{1}{8}\right)\left((-8i)\cancel{+(8i)-(8i)}\right) = \left(\frac{\pi i}{3}\right)\left(\frac{1}{8}\right)(-8i)$$

$= \pi/3$

$2$

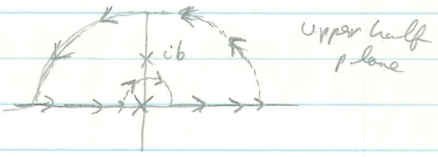
5. Evaluate the following integral by integration around a suitable indented contour in the complex plane:

$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx \qquad \text{where } a>0 \text{ and } b>0$$

$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{1}{2} I_m \int_{-\infty}^{\infty} \frac{e^{iaz}}{z(z^2+b^2)}\,dz$$

we have singularities at $z=0$ and $z=\pm ib$

Our contour of integration is



we have $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = \pi i \operatorname{Res}(z=0) + 2\pi i \operatorname{Res}(z=ib)$

let's use $\displaystyle \operatorname{Res}(z_n) = \frac{N(z)}{D'(z)} = \frac{e^{iaz}}{(z^2+b^2)+2z^2}$

$$\pi i \operatorname{Res}(0) = (\pi i)\left(\frac{1}{(0+b^2)+0}\right) = \frac{\pi i}{b^2}$$

$$2\pi i \operatorname{Res}(ib) = (2\pi i)\left(\frac{e^{-ab}}{(ib)^2+b^2+2(ib)^2}\right) = \frac{2\pi i e^{-ab}}{b^2-b^2-2b^2} = -\frac{\pi i e^{-ab}}{b^2}$$

$$I = \frac{1}{2} I_m\left(\frac{\pi i}{b^2} - \frac{\pi i e^{-ab}}{b^2}\right) = \underline{\underline{\frac{\pi}{2b^2}\left(1-e^{-ab}\right)}}$$

If we use our usual way of finding the residue, we have

$$\operatorname{Res}(0) = \lim_{z\to 0} \frac{\cancel{z} e^{iaz}}{\cancel{z}(z-ib)(z+ib)} = \frac{1}{(-ib)(ib)} = \frac{1}{b^2}$$

$$\operatorname{Res}(ib) = \lim_{z\to ib}\left(\cancel{(z-ib)}\frac{e^{iaz}}{z\cancel{(z-ib)}(z+ib)}\right) = \frac{e^{ia(ib)}}{(ib)(ib+ib)} = \frac{e^{-ab}}{(ib)(2ib)} = -\frac{e^{-ab}}{2b^2}$$

$$I = \frac{1}{2} I_m\left(\pi i\left(\frac{1}{b^2}\right) + 2\pi i\left(-\frac{e^{ab}}{2b^2}\right)\right) = \frac{1}{2} I_m\left(\frac{\pi i}{b^2} - \frac{\pi i e^{-ab}}{b^2}\right)$$

$$= \frac{\pi}{2b^2}\left(1-e^{-ab}\right)$$

6a) Evaluate $\int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^{x}}\,dx$ where $0<p<1$ and $0<q<1$

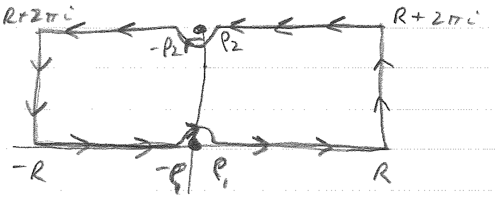
$$\int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^{x}}\,dx = \underset{①}{\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx} - \underset{②}{\int_{-\infty}^{\infty} \frac{e^{qx}}{1-e^{x}}\,dx}$$

we will solve the first integral and deduce the result of the second

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \int_{-\infty}^{\infty} \frac{e^{pz}}{1-e^{z}}\,dz$$

the poles are at $z=0, 2\pi i, 4\pi i, \dots$

we will use the following contour



$$\int_C \frac{e^{pz}}{1-e^{z}}\,dz = \int_{-R}^{-\rho} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{C_{\rho_1}} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{\rho_1}^{R} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{R}^{R+2\pi i} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{R+2\pi i}^{\rho_2} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{C_{\rho_2}} \frac{e^{pz}}{1-e^{z}}\,dz$$

$$+ \int_{-\rho_2}^{-R+2\pi i} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{-R+2\pi i}^{-R} \frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n \operatorname{Res}(z_n)$$

$$= \int_{-R}^{R} \frac{e^{px}}{1-e^{x}}\,dx - (\pi i)\operatorname{Res}(0) + \int_{R}^{R+2\pi i} \frac{e^{pz}}{1-e^{z}}\,dz + \int_{R+2\pi i}^{R-2\pi i} \frac{e^{pz}}{1-e^{z}}\,dz - (\pi i)\operatorname{Res}(2\pi i)$$

$$+ \int_{-R+2\pi i}^{-R} \frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n \operatorname{Res}(z_n)$$

let $t = z - 2\pi i$

$dt = dz$ $\quad z = -R+2\pi i \quad t = -R$

$\quad z = R+2\pi i \quad t = R$

$$\int_C \frac{e^{pz}}{1-e^{z}}\,dz = \int_{-R}^{R} \frac{e^{px}}{1-e^{x}}\,dx - (\pi i)\operatorname{Res}(0) + \cancel{\int_{R}^{R+2\pi i} \frac{e^{pz}}{1-e^{z}}\,dz} + \int_{R}^{-R} \frac{e^{p(t+2\pi i)}}{1-e^{(t+2\pi i)}}\,dt - (\pi i)\operatorname{Res}(2\pi i)$$

$$+ \int_{-R+2\pi i}^{-R} \frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n \operatorname{Res}(z_n)$$

vanish
as $R \to \infty$
see class notes

$$= \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx - e^{2p\pi i}\, p\int_{-\infty}^{\infty} \frac{e^{pt}}{1-e^{t}\underbrace{e^{2\pi i}}_{\to 1}}\,dt - (\pi i)\operatorname{Res}(0) - (\pi i)\operatorname{Res}(2\pi i) = 2\pi i \sum_n \operatorname{Res}(z_n)$$

$$\Rightarrow \; = \left(1-e^{p2\pi i}\right) \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = 2\pi i \sum_n \operatorname{Res}(z_n) + (\pi i)\operatorname{Res}(0) + (\pi i)\operatorname{Res}(2\pi i)$$

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \left(\frac{1}{1-e^{p2\pi i}}\right)\left[2\pi i \sum_n \operatorname{Res}(z_n) + (\pi i)\operatorname{Res}(0) + (\pi i)\operatorname{Res}(2\pi i)\right]$$

There are no poles inside the contour $\therefore$ $\text{Res}(z_n) = 0$

$$\text{Res}(0) = \left.\frac{N(z)}{D'(z)}\right|_{z=0} = \left.\frac{e^{pz}}{-e^{z}}\right|_{z=0} = \frac{e^{0}}{-e^{0}} = -1$$

$$\text{Res}(2\pi i) = \left.\frac{N(z)}{D'(z)}\right|_{z=2\pi i} = \left.\frac{e^{pz}}{-e^{z}}\right|_{z=2\pi i} = \frac{e^{p2\pi i}}{-e^{2\pi i}} = -e^{p2\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \left(\frac{1}{1-e^{p2\pi i}}\right)\left[\pi i(-1) + \pi i(-e^{p2\pi i})\right]$$

$$= \left(\frac{-\pi i}{1-e^{p2\pi i}}\right)\left(1+e^{p2\pi i}\right) = -\pi i\left(\frac{1+e^{p2\pi i}}{1-e^{p2\pi i}}\right) = \left(\frac{e^{p\pi i}+e^{-p\pi i}}{e^{p\pi i}-e^{-p\pi i}}\right)$$

$$= (-\pi i)\left(\frac{e^{p\pi i}+e^{-p\pi i}}{2}\right)\left(\frac{2}{e^{-p\pi i}-e^{p\pi i}}\right)$$

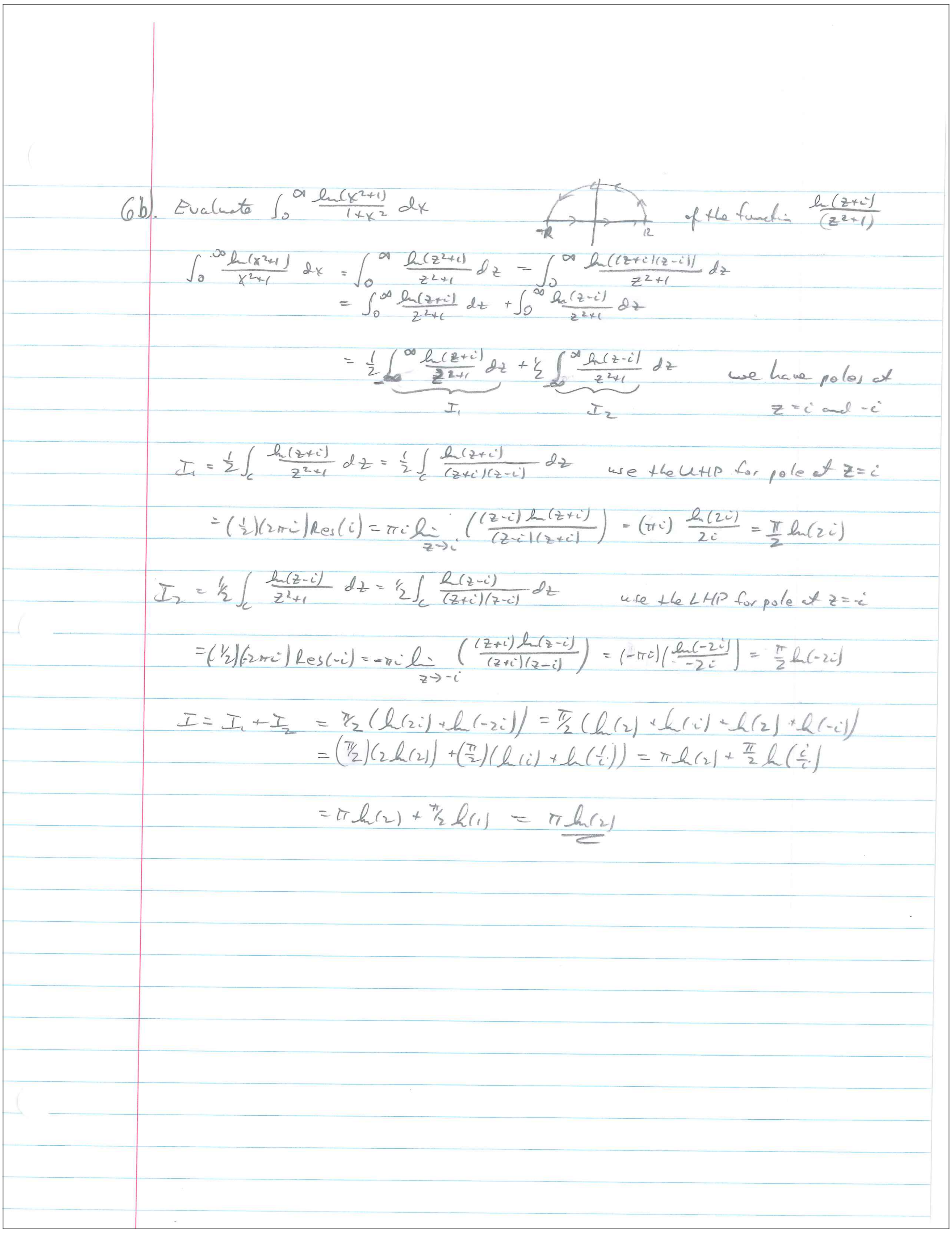
$$= \pi \underbrace{\left(\frac{e^{p\pi i}+e^{-p\pi i}}{2}\right)}_{\cos p\pi} \underbrace{\left(\frac{2i}{e^{p\pi i}-e^{-p\pi i}}\right)}_{1/\sin p\pi} = \pi \frac{\cos \pi p}{\sin \pi p}$$

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \pi \cot(\pi p)$$

we infer from the above analysis $\int_{-\infty}^{\infty} \frac{e^{qx}}{1-e^{x}}\,dx = \pi \cot(\pi q)$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^{x}}\,dx = \pi \cot(\pi p) - \pi \cot(\pi q)$$

$$= \pi\left(\cot(\pi p) - \cot(\pi q)\right)$$



6b). Evaluate $\int_0^\infty \frac{\ln(x^2+1)}{1+x^2}\,dx$ of the function $\frac{\ln(z+i)}{(z^2+1)}$

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^\infty \frac{\ln(z^2+1)}{z^2+1}\,dz = \int_0^\infty \frac{\ln((z+i)(z-i))}{z^2+1}\,dz$$

$$= \int_0^\infty \frac{\ln(z+i)}{z^2+1}\,dz + \int_0^\infty \frac{\ln(z-i)}{z^2+1}\,dz$$

$$= \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\ln(z+i)}{z^2+1}\,dz}_{I_1} + \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\ln(z-i)}{z^2+1}\,dz}_{I_2}$$

we have poles at $z = i$ and $-i$

$$I_1 = \frac{1}{2}\int_C \frac{\ln(z+i)}{z^2+1}\,dz = \frac{1}{2}\int_C \frac{\ln(z+i)}{(z+i)(z-i)}\,dz$$

use the UHP for pole at $z=i$

$$= \left(\frac{1}{2}\right)(2\pi i)\,\mathrm{Res}(i) = \pi i \lim_{z\to i}\left(\frac{(z-i)\ln(z+i)}{(z-i)(z+i)}\right) = (\pi i)\,\frac{\ln(2i)}{2i} = \frac{\pi}{2}\ln(2i)$$

$$I_2 = \frac{1}{2}\int_C \frac{\ln(z-i)}{z^2+1}\,dz = \frac{1}{2}\int_C \frac{\ln(z-i)}{(z+i)(z-i)}\,dz$$

use the LHP for pole at $z=-i$

$$= \left(\frac{1}{2}\right)(-2\pi i)\,\mathrm{Res}(-i) = -\pi i \lim_{z\to -i}\left(\frac{(z+i)\ln(z-i)}{(z+i)(z-i)}\right) = (-\pi i)\left(\frac{\ln(-2i)}{-2i}\right) = \frac{\pi}{2}\ln(-2i)$$

$$I = I_1 + I_2 = \frac{\pi}{2}\left(\ln(2i) + \ln(-2i)\right) = \frac{\pi}{2}\left(\ln(2) + \ln(i) + \ln(2) + \ln(-i)\right)$$

$$= \left(\frac{\pi}{2}\right)(2\ln(2)) + \left(\frac{\pi}{2}\right)\left(\ln(i) + \ln\left(\frac{1}{i}\right)\right) = \pi\ln(2) + \frac{\pi}{2}\ln\left(\frac{i}{i}\right)$$

$$= \pi\ln(2) + \frac{\pi}{2}\ln(1) = \underline{\underline{\pi\ln(2)}}$$

7a) $F(s) = \frac{s+1}{s^2(s^2+s+1)}$ we have a 2nd order pole at $s=0$ and simple poles at $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\mathcal{L}^{-1}(F(s)) = \operatorname{Res}(F(s)e^{st})$ we will use partial Fractions to help

$$\frac{s+1}{s^2(s^2+s+1)} = \frac{(A+Bs)}{s^2} + \frac{C+Ds}{s^2+s+1} \qquad A=1,\ B=0,\ C=-1,\ D=0$$

$$= \frac{1}{s^2} - \frac{1}{s^2+s+1}$$

$$\mathcal{L}^{-1}(F(s)) = \operatorname{Res}(s=0) - \operatorname{Res}\left(s=-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) - \operatorname{Res}\left(s=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)$$

$$\operatorname{Res}(s=0) = \lim_{s\to 0} \frac{1}{2-1} \frac{d}{ds}\left((s^2)\left(\frac{e^{st}}{s^2}\right)\right) = \lim_{s\to 0} \frac{d}{ds}(e^{st})$$

$$= \lim_{s\to 0} t e^{st} = t$$

$$\operatorname{Res}\left(s=-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) = \lim_{s\to -\frac{1}{2}-\frac{\sqrt{3}}{2}i}\left(\cancel{\left(s-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)} \frac{e^{st}}{\cancel{\left(s-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)}\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right)$$

$$= \lim_{s\to -\frac{1}{2}-\frac{\sqrt{3}}{2}i}\left(\frac{e^{st}}{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right) = \frac{e^{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}}{\cancel{-\frac{1}{2}}-\frac{\sqrt{3}}{2}i\cancel{+\frac{1}{2}}-\frac{\sqrt{3}}{2}i}$$

$$= e^{-\frac{1}{2}t} e^{-\frac{\sqrt{3}}{2}it} / (-\sqrt{3}i)$$

$$\operatorname{Res}\left(s=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = \lim_{s\to -\frac{1}{2}+\frac{\sqrt{3}}{2}i}\left(\cancel{\left(s-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)} \frac{e^{st}}{\cancel{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\left(s-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)}\right)$$

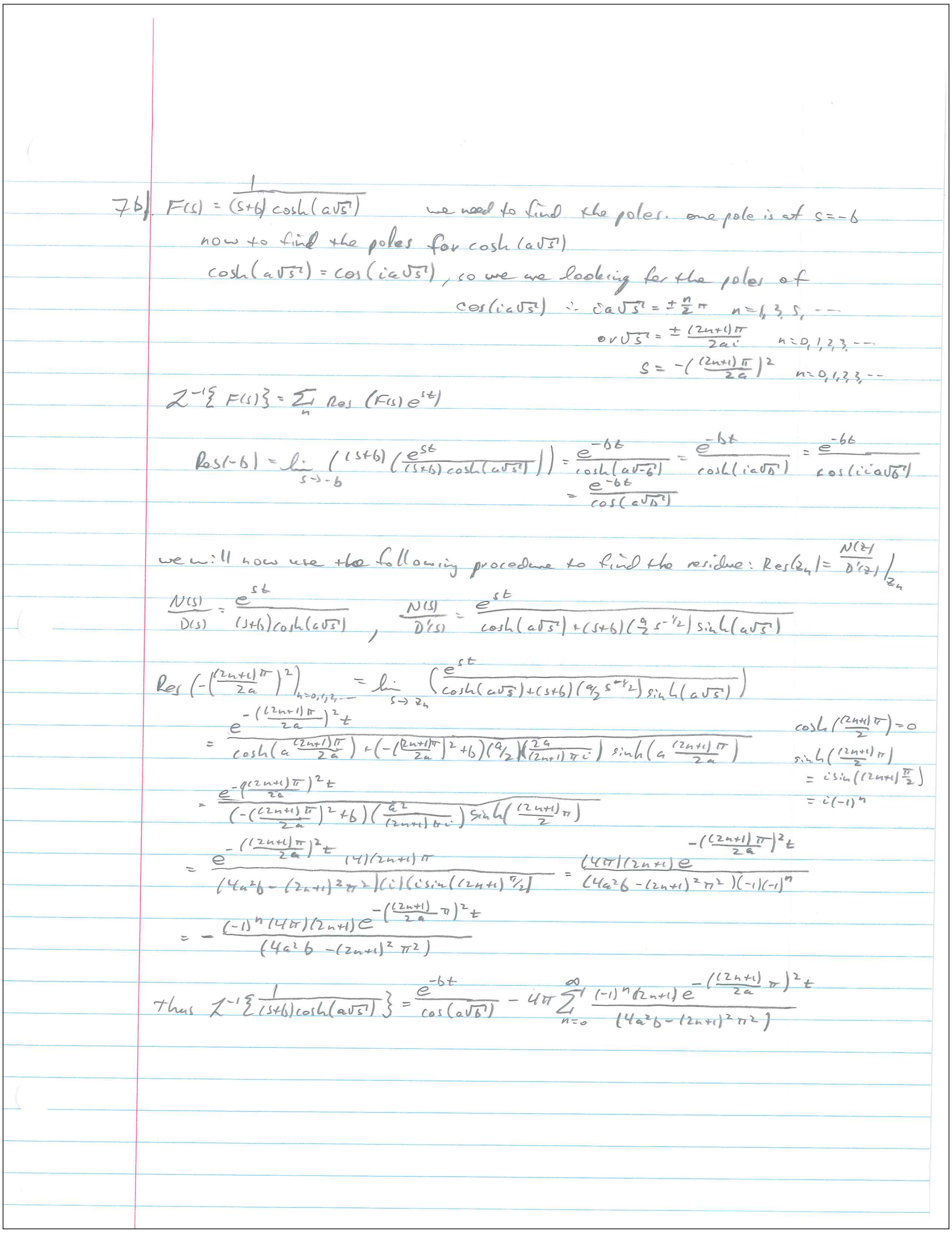
$$= \lim_{s\to -\frac{1}{2}+\frac{\sqrt{3}}{2}i}\left(\frac{e^{st}}{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right) = \frac{e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)t}}{\cancel{-\frac{1}{2}}+\frac{\sqrt{3}}{2}i\cancel{+\frac{1}{2}}+\frac{\sqrt{3}}{2}i}$$

$$= e^{-\frac{1}{2}t} e^{\frac{\sqrt{3}}{2}it} / \sqrt{3}i$$

$$\mathcal{L}^{-1}(F(s)) = t - \left(e^{-t/2} e^{-\frac{\sqrt{3}}{2}it} / -\sqrt{3}i + e^{-t/2} e^{\frac{\sqrt{3}}{2}it} / \sqrt{3}i\right)$$

$$= t - \frac{2}{\sqrt{3}} e^{-t/2}\left(\frac{e^{\frac{\sqrt{3}}{2}it} - e^{-\frac{\sqrt{3}}{2}it}}{2i}\right)$$

$$= t - \underline{\underline{\frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t}}$$



7b) $F(s) = \frac{1}{(s+b)\cosh(a\sqrt{s})}$ we need to find the poles. one pole is at $s=-b$

now to find the poles for $\cosh(a\sqrt{s})$

$\cosh(a\sqrt{s}) = \cos(ia\sqrt{s})$, so we we looking for the poles of

$$\cos(ia\sqrt{s}) \therefore ia\sqrt{s} = \pm\frac{n}{2}\pi \quad n=1,3,5,\dots$$

$$\text{or } \sqrt{s} = \pm\frac{(2n+1)\pi}{2ai} \quad n=0,1,2,3,\dots$$

$$s = -\left(\frac{(2n+1)\pi}{2a}\right)^2 \quad n=0,1,2,3,\dots$$

$$\mathcal{L}^{-1}\{F(s)\} = \sum_n \text{Res}\,(F(s)e^{st})$$

$$\text{Res}(-b) = \lim_{s\to -b}\left((s+b)\left(\frac{e^{st}}{(s+b)\cosh(a\sqrt{s})}\right)\right) = \frac{e^{-bt}}{\cosh(a\sqrt{-b})} = \frac{e^{-bt}}{\cosh(ia\sqrt{b})} = \frac{e^{-bt}}{\cos(iia\sqrt{b})}$$
$$= \frac{e^{-bt}}{\cos(a\sqrt{b})}$$

we will now use the following procedure to find the residue: $\text{Res}(z_n) = \left.\frac{N(z)}{D'(z)}\right|_{z_n}$

$$\frac{N(s)}{D(s)} = \frac{e^{st}}{(s+b)\cosh(a\sqrt{s})}, \quad \frac{N(s)}{D'(s)} = \frac{e^{st}}{\cosh(a\sqrt{s}) + (s+b)(\frac{a}{2}s^{-1/2})\sinh(a\sqrt{s})}$$

$$\text{Res}\left(-\left(\frac{(2n+1)\pi}{2a}\right)^2\right)_{n=0,1,2,\dots} = \lim_{s\to z_n}\left(\frac{e^{st}}{\cosh(a\sqrt{s}) + (s+b)(a/2\, s^{-1/2})\sinh(a\sqrt{s})}\right)$$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{\cosh\left(a\frac{(2n+1)\pi}{2a}\right) + \left(-\left(\frac{(2n+1)\pi}{2a}\right)^2 + b\right)(a/2)\left(\frac{2a}{(2n+1)\pi i}\right)\sinh\left(a\frac{(2n+1)\pi}{2a}\right)}$$

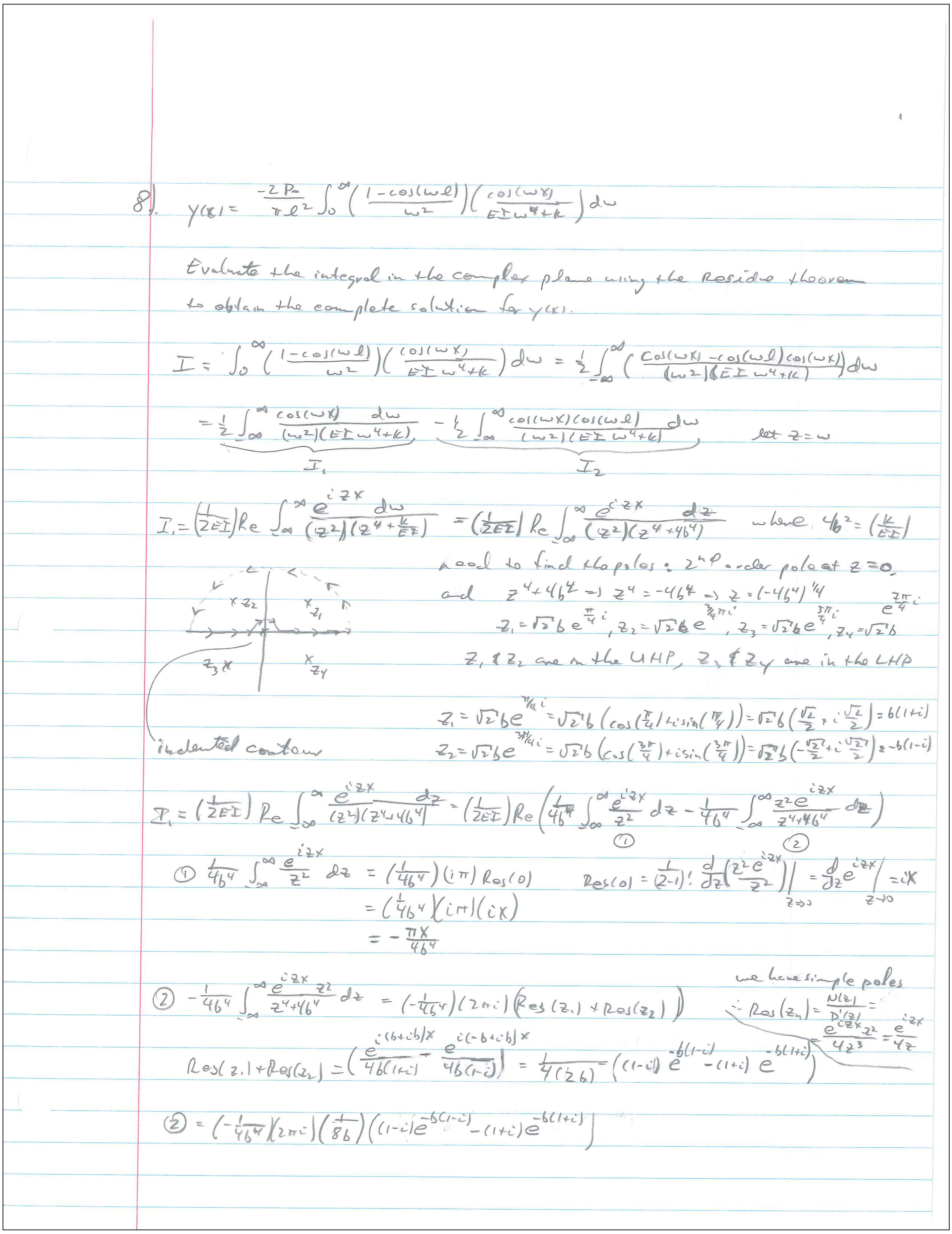
$$\cosh\left(\frac{(2n+1)\pi}{2}\right) = 0$$
$$\sinh\left(\frac{(2n+1)}{2}\pi\right) = i\sin\left((2n+1)\frac{\pi}{2}\right) = i(-1)^n$$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{\left(-\left(\frac{(2n+1)\pi}{2a}\right)^2 + b\right)\left(\frac{a^2}{(2n+1)\pi i}\right)\sinh\left(\frac{(2n+1)}{2}\pi\right)}$$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}(4)(2n+1)\pi}{(4a^2b - (2n+1)^2\pi^2)(i)(i\sin((2n+1)\pi/2))} = \frac{(4\pi)(2n+1)e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{(4a^2b - (2n+1)^2\pi^2)(-1)(-1)^n}$$

$$= -\frac{(-1)^n(4\pi)(2n+1)e^{-\left(\frac{(2n+1)}{2a}\pi\right)^2 t}}{(4a^2b - (2n+1)^2\pi^2)}$$

thus $\mathcal{L}^{-1}\left\{\frac{1}{(s+b)\cosh(a\sqrt{s})}\right\} = \frac{e^{-bt}}{\cos(a\sqrt{b})} - 4\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)e^{-\left(\frac{(2n+1)}{2a}\pi\right)^2 t}}{(4a^2b - (2n+1)^2\pi^2)}$



8). $y(x) = \frac{-2P_o}{\pi l^2}\int_0^\infty \left(\frac{1-\cos(\omega l)}{\omega^2}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega$

Evaluate the integral in the complex plane using the Residue theorem to obtain the complete solution for y(x).

$I = \int_0^\infty \left(\frac{1-\cos(\omega l)}{\omega^2}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(\omega x)-\cos(\omega l)\cos(\omega x)}{(\omega^2)(EI\omega^4+k)}\right)d\omega$

$= \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(\omega x)\,d\omega}{(\omega^2)(EI\omega^4+k)}}_{I_1} \underbrace{-\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(\omega x)\cos(\omega l)}{(\omega^2)(EI\omega^4+k)}d\omega}_{I_2}$ let $z=\omega$

$I_1 = \left(\frac{1}{2EI}\right)\mathrm{Re}\int_{-\infty}^\infty \frac{e^{izx}\,d\omega}{(z^2)(z^4+\frac{k}{EI})} = \left(\frac{1}{2EI}\right)\mathrm{Re}\int_{-\infty}^\infty \frac{e^{izx}\,dz}{(z^2)(z^4+4b^4)}$ where $4b^2 = \left(\frac{k}{EI}\right)$

need to find the poles: 2nd order pole at $z=0$,

and $z^4+4b^4 \to z^4 = -4b^4 \Rightarrow z = (-4b^4)^{1/4}$

$z_1 = \sqrt{2}be^{\frac{\pi}{4}i}$, $z_2 = \sqrt{2}be^{\frac{3}{4}\pi i}$, $z_3 = \sqrt{2}be^{\frac{5\pi}{4}i}$, $z_4 = \sqrt{2}b\,e^{\frac{7\pi}{4}i}$

$z_1$ & $z_2$ are in the UHP, $z_3$ & $z_4$ are in the LHP

$z_2$ $z_1$ $z_3$ $z_4$

indented contour

$z_1 = \sqrt{2}be^{\pi/4\, i} = \sqrt{2}b\left(\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4})\right) = \sqrt{2}b\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = b(1+i)$

$z_2 = \sqrt{2}be^{3\pi/4\, i} = \sqrt{2}b\left(\cos(\frac{3\pi}{4})+i\sin(\frac{3\pi}{4})\right) = \sqrt{2}b\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = -b(1-i)$

$I_1 = \left(\frac{1}{2EI}\right)\mathrm{Re}\int_{-\infty}^\infty \frac{e^{izx}\,dz}{(z^2)(z^4+4b^4)} = \left(\frac{1}{2EI}\right)\mathrm{Re}\left(\underbrace{\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}}{z^2}dz}_{①} - \underbrace{\frac{1}{4b^4}\int_{-\infty}^\infty \frac{z^2e^{izx}}{z^4+4b^4}dz}_{②}\right)$

① $\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}}{z^2}dz = \left(\frac{1}{4b^4}\right)(i\pi)\,\mathrm{Res}(0)$ $\mathrm{Res}(0) = \frac{1}{(2-1)!}\frac{d}{dz}\left(z^2\frac{e^{izx}}{z^2}\right)\Big|_{z\to 0} = \frac{d}{dz}e^{izx}\Big|_{z\to 0} = ix$

$= \left(\frac{1}{4b^4}\right)(i\pi)(ix)$

$= -\frac{\pi x}{4b^4}$

② $-\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}z^2}{z^4+4b^4}dz = \left(-\frac{1}{4b^4}\right)(2\pi i)\left(\mathrm{Res}(z_1)+\mathrm{Res}(z_2)\right)$

we have simple poles $\therefore \mathrm{Res}(z_n) = \frac{N(z)}{D'(z)} = \frac{e^{izx}z^2}{4z^3} = \frac{e^{izx}}{4z}$

$\mathrm{Res}(z_1)+\mathrm{Res}(z_2) = \left(\frac{e^{i(b+ib)x}}{4b(1+i)} - \frac{e^{i(-b+ib)x}}{4b(1-i)}\right) = \frac{1}{4(2b)}\left((1-i)e^{-b(1-i)} - (1+i)e^{-b(1+i)}\right)$

② $= \left(-\frac{1}{4b^4}\right)(2\pi i)\left(\frac{1}{8b}\right)\left((1-i)e^{-b(1-i)} - (1+i)e^{-b(1+i)}\right)$

$$② = \left(\frac{-\pi i}{16 b^5}\right)\left((1-i)\, e^{-b(1-i)x} - (1+i)\, e^{-b(1+i)x}\right)$$

$$= \left(\frac{-\pi i}{16 b^5}\right)(-i)\left(e^{-b(1-i)x} + e^{-b(1+i)x}\right) - \left(\frac{\pi i}{16 b^5}\right)\left(e^{-b(1-i)x} - e^{-b(1+i)x}\right)$$

$$= \left(-\frac{\pi e^{-bx}}{8 b^5}\right)\left(\frac{e^{ibx}+e^{-ibx}}{2}\right) + \left(\frac{\pi e^{-bx}}{8 b^5}\right)\left(\frac{e^{ibx}-e^{-ibx}}{2i}\right)$$

$$= \left(-\frac{\pi e^{-bx}}{8 b^5}\right)\cos(bx) + \left(\frac{\pi e^{-bx}}{8 b^5}\right)(\sin(bx))$$

$$= \left(\frac{\pi e^{-bx}}{8 b^5}\right)(-\cos(bx) + \sin(bx))$$

$$I_1 = \left(\frac{1}{2EI}\right)\mathrm{Re}\,(① + ②) = \left(\frac{1}{2EI}\right)\left[\frac{\pi}{8 b^5}\left(e^{-bx}(-\cos(bx) + \sin(bx)) - 2bx\right)\right]$$

$$I_2 = \left(-\frac{1}{2EI}\right)\int_{-\infty}^{\infty}\frac{\cos(zx)\cos(z\ell)}{(z^2)(z^4+4b^4)}\,dz$$

$$\cos((x+\ell)z) = \cos(zx)\cos(z\ell) - \sin(zx)\sin(z\ell)$$

$$+\ \cos((x-\ell)z) = \cos(zx)\cos(z\ell) + \sin(zx)\sin(z\ell)$$

$$\overline{\cos((x+\ell)z) + \cos((x-\ell)z) = 2\cos(zx)\cos(z\ell)}$$

$$= \left(-\frac{1}{2EI}\right)\left(\frac{1}{2}\right)\int_{-\infty}^{\infty}\frac{\cos((x+\ell)z) + \cos((x-\ell)z)}{(z^2)(z^4+4b^4)}\,dz$$

$$= \left(-\frac{1}{4EI}\right)\int_{-\infty}^{\infty}\frac{\cos((x+\ell)z)}{(z^2)(z^4+4b^4)}\,dz + \left(-\frac{1}{4EI}\right)\int_{-\infty}^{\infty}\frac{\cos((x-\ell)z)}{(z^2)(z^4+4b^4)}\,dz$$

$$= \left(-\frac{1}{4EI}\right)\mathrm{Re}\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(x+\ell)z}}{(z^2)(z^4+4b^4)}\,dz}_{①} + \left(-\frac{1}{4EI}\right)\mathrm{Re}\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}\,dz}_{②}$$

① since $(x+\ell>0)$, the evaluation of ① is similar to the evaluation of $I_1$, where the $x$ in $I_1$ becomes $(x+\ell)$ in ①.

$$① = \left(\frac{\pi}{8 b^5}\right)\left[e^{-b(x+\ell)}\left(-\cos(b(x+\ell)) + \sin(b(x+\ell))\right) - 2b(x+\ell)\right]$$

② There are three cases to consider (1) $(x-\ell)>0$, (2) $(x-\ell)=0$, and (3) $(x-\ell)<0$.

Case (1) $(x-\ell)>0$, the evaluation of ② is similar to the evaluation of $I_1$, where the $x$ in $I_1$ becomes $(x-\ell)$ in ② for case 1.

$$\text{Case (1)} = \left(\frac{\pi}{8 b^5}\right)\left[e^{-b(x-\ell)}\left(\cos(b(x-\ell)) + \sin(b(x-\ell))\right) - 2b(x-\ell)\right] \quad \text{for } (x-\ell)>0.$$

$x>\ell$

Case (2) $(x-\ell)=0$ $\int_{-\infty}^{\infty}\frac{1}{(z^2)(z^4+4b^4)}dz = \frac{1}{4b^4}\underbrace{\int_{-\infty}^{\infty}\frac{1}{z^2}dz}_{①} - \frac{1}{4b^2}\underbrace{\int_{-\infty}^{\infty}\frac{z^2}{z^4+4b^4}dz}_{②}$

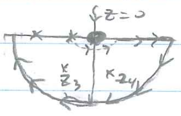
① $\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{1}{z^2}dz = \left(\frac{1}{4b^4}\right)(i\pi)\,\text{Res}(0)$  $\text{Res}(0)=\frac{1}{(2-1)!}\frac{d}{dz}\left(\frac{z^2}{z^2}\right)\Big|_{z\to0} = \frac{d}{dz}(1) = 0$

② $-\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{z^2}{z^4+4b^4}dz = \left(-\frac{1}{4b^4}\right)(2\pi i)\left(\text{Res}(z_1)+\text{Res}(z_2)\right)$  $z_1=b(1+i)$, $z_2=-b(1-i)$  $\text{Res}(z_n)=\frac{N(z)}{D'(z)}=\frac{z^2}{4z^3}=\frac{1}{4z}$

$$\left(\text{Res}(z_1)+\text{Res}(z_2)\right)=\left(\frac{1}{4b(1+i)}-\frac{1}{4b(1-i)}\right)=\left(\frac{1}{8b}\right)\left((1-i)-(1+i)\right)=\left(\frac{-2i}{8b}\right)=\frac{-i}{4b}$$

$$\therefore\left(-\frac{1}{4b^4}\right)(2\pi i)\left(\frac{-i}{4b}\right)=\left(-\frac{1}{4b^4}\right)\left(\frac{\pi}{2b}\right)=\underline{\underline{-\frac{\pi}{8b^5}}}$$

Case (3) $(x-\ell)<0$ $\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}dz$ need to integrate in the LHP. need $z_3$ & $z_4$



$$z_3=\sqrt{2}\,b\,e^{\frac{5\pi}{4}i}=\sqrt{2}\,b\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right)=\sqrt{2}\,b\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-b(1+i)$$

$$z_4=\sqrt{2}\,b\,e^{\frac{7\pi}{4}i}=\sqrt{2}\,b\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right)=\sqrt{2}\,b\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=b(1-i)$$

$$\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}dz=\frac{1}{4b^4}\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^2}dz}_{①}-\underbrace{\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{z^2e^{i(x-\ell)z}}{z^4+4b^4}dz}_{②}$$

① $\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^2}dz=\left(\frac{1}{4b^4}\right)(-i\pi)\,\text{Res}(0)$  $\text{Res}(0)=\frac{1}{(2-1)!}\frac{d}{dz}\left(\frac{z^2e^{i(x-\ell)z}}{z^2}\right)\Big|_{z\to0}=\frac{d}{dz}e^{i(x-\ell)z}\Big|_{z\to0}$

$=\left(\frac{1}{4b^4}\right)(-i\pi)(i(x-\ell))$  $=i(x-\ell)$

$=\frac{\pi}{4b^4}(x-\ell) =$

② $-\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^4+4b^4}dz=\left(-\frac{1}{4b^4}\right)(-2\pi i)\left(\text{Res}(z_3)+\text{Res}(z_4)\right)$  $\therefore\text{Res}(z_n)=\frac{N(z)}{D'(z)}=\frac{z^2e^{i(x-\ell)z}}{4z^3}=\frac{e^{i(x-\ell)z}}{4z}$

$$\text{Res}(z_3)+\text{Res}(z_4)=\left(-\frac{e^{i(x-\ell)(-b)(1+i)}}{4b(1+i)}+\frac{e^{i(x-\ell)(b)(1-i)}}{4b(1-i)}\right)$$

$$=\left(-\frac{e^{b(x-\ell)(1-i)}}{4b(1+i)}+\frac{e^{b(x-\ell)(1+i)}}{4b(1-i)}\right)=\left(\frac{1}{8b}\right)\left((1+i)e^{b(x-\ell)(1+i)}-(1-i)e^{b(x-\ell)(1-i)}\right)$$

$$②=\left(-\frac{1}{4b^4}\right)(-2\pi i)\left(\frac{1}{8b}\right)\left((1+i)e^{b(x-\ell)(1+i)}-(1-i)e^{b(x-\ell)(1-i)}\right)$$

$$=\left(\frac{\pi i}{16b^5}\right)\left((1+i)e^{b(x-\ell)(1+i)}-(1-i)e^{b(x-\ell)(1-i)}\right)$$

$$=\left(\frac{\pi i}{16b^5}\right)\left(e^{b(x-\ell)(1+i)}-e^{b(x-\ell)(1-i)}\right)+\left(\frac{\pi i}{16b^5}\right)(i)\left(e^{b(x-\ell)(1+i)}+e^{b(x-\ell)(1-i)}\right)$$

$$=\left(\frac{\pi i}{16b^5}\right)e^{-b(l-x)}\left(e^{-ib(l-x)}-e^{ib(l-x)}\right)-\left(\frac{\pi}{16b^5}\right)e^{-b(l-x)}\left(e^{-ib(l-x)}+e^{ib(l-x)}\right)$$

for $(x-l)<0$
or $(l-x)>0$

$$=\left(\frac{\pi}{8b^5}\right)e^{-b(l-x)}\left(\frac{e^{ib(l-x)}-e^{-ib(l-x)}}{2i}\right)-\left(\frac{\pi}{8b^5}\right)e^{-b(l-x)}\left(\frac{e^{ib(l-x)}+e^{-ib(l-x)}}{2}\right)$$

$$=\left(\frac{\pi}{8b^5}\right)e^{-b(l-x)}\left(\sin(b(l-x))-\cos(b(l-x))\right)$$

Case(3): $\int_{\infty}^{\infty}\frac{e^{i(x-l)z}}{(z^2)(z^4+4(b^4)}dz = ① + ②$

$$=\frac{\pi}{4b^4}(x-l)+\left(\frac{\pi}{8b^5}\right)e^{-b(l-x)}\left(\sin(b(l-x)-\cos(b(l-x))\right)$$

$$=\left(\frac{\pi}{8b^5}\right)\left[e^{-b(l-x)}\left(\sin(b(l-x))-\cos(b(l-x))\right)-2b(l-x)\right]$$

for $(l-x)>0$

$$I_2=\left(\frac{1}{-4EI}\right)\left(\frac{\pi}{8b^5}\right)\Bigg[\left[e^{-b(x+l)}(\sin(b(x+l))-\cos(b(x+l))-2b(x+l)\right]$$

$$+\left\{\begin{array}{ll}\left[e^{-b(x-l)}(-\sin(b(x-l))+\cos(b(x-l))-2b(x-l))\right] & \text{for } x>l\\ (-1) & \text{for } x=l\\ \left[e^{-b(l-x)}(\sin(b(l-x))-\cos(b(l-x))-2b(l-x)\right] & \text{for } l>x\end{array}\right.\Bigg]$$

recall

$$I_1=\left(\frac{1}{2EI}\right)\left(\frac{\pi}{8b^5}\right)\left[e^{-bx}\left(\sin(bx)-\cos(bx)\right)-2bx\right]$$

$$y(x)=\frac{-2P_0}{\pi l^2}\left[I_1+I_2\right] \qquad \text{where } 4b^4=\left(\frac{k}{EI}\right)$$

## 2.13 HW Extra credit problems

### 2.13.1 Problems to solve

Extra credit NEEP 547
October 18, 2013 DLH
Turn in the day of the exam

I am counting on everyone treating this as an individual effort not a group effort. Not all differential equation methods we examined are included below.

1. (3pts) Find the solution to the differential equation that satisfies the initial condition $(2\,xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; where $y(1) = \ln(2)$.
2. (3pts) Solve the following differential equation $2\,xy^3(y\,dx + x\,dy) = (y\,dx - x\,dy)\sin(x/y)$
3. (3pts) Find the solution which satisfies the given condition $(x^4 + y^4)\,dx = 2\,x^3\,y\,dy$; where $y(1) = 0$
4. (3pts) Find the general solution for the equation, using the indicated solution to the homogeneous equation to reduce the order of the equation $(2x+1)y'' - 4(x+1)y' + 4y = (2x+1)^2/x+1); y_1 = e^{2x}$.
5. (3pts) Find the complete solution for the follow equation: $(D^3 + D^2 + 3D - 5)\,y = e^x$.
6. (3pts) Find a solution for the following equation which satisfies the given conditions: $3\,xy' + y + x^2y^4 = 0$
7. (3pts) Solve the following equation: $(y')^2\,y'' = 1 + (y')^2$.
8. (4pts) Obtain the solution of the simultaneous equations

$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$

which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.

### 2.13.2 Problem 1

Find solution of the differential equation that satisfies the initial conditions $\left(2xy + e^y\right)dx + \left(x^2 + xe^y\right)dy = 0$ where $y(1) = \ln 2$

**Answer**:

This is not separable, so we will try to see if it is exact. We write the ODE as

$$M\left(x,y\right)dx + N\left(x,y\right)dy = 0$$

The condition for the ODE to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Now $\frac{\partial M}{\partial y} = 2x + e^y$ and $\frac{\partial N}{\partial x} = 2x + e^y$, therefore it is exact. Let

$$M = \frac{\partial \varphi\left(x,y\right)}{\partial x} \tag{1}$$

$$N = \frac{\partial \varphi\left(x,y\right)}{\partial y} \tag{2}$$

Hence the ODE can be written as

$$\frac{\partial \varphi\left(x,y\right)}{\partial x}dx + \frac{\partial \varphi\left(x,y\right)}{\partial y}dy = 0$$

$$\frac{\partial \varphi\left(x,y\right)}{\partial x} + \frac{\partial \varphi\left(x,y\right)}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(\varphi\left(x,y\left(x\right)\right)\right) = 0$$

This means that $\varphi\left(x,y\left(x\right)\right) = C$, a constant. We need now to find $\varphi\left(x,y\right)$. At this point we can pick either Eq. (1) or Eq. (2). Using Eq. (1) gives

$$2xy+e^{y}=\frac{\partial\varphi\left(x,y\right)}{\partial x}$$

Integrating

$$\begin{aligned}\varphi\left(x,y\right)&=\int 2xy+e^{y}dx\\&=x^{2}y+xe^{y}+g\left(y\right)\end{aligned}$$

Where $g\left(y\right)$ acts here as the constant of integration, since $y\left(x\right)$ is a function of $x$. Taking derivative of the above w.r.t. $y$ and equating the result to Eq.(2) (or to $N\left(x,y\right)$) gives

$$\begin{aligned}x^{2}+xe^{y}+g'\left(y\right)&=x^{2}+xe^{y}\\g'\left(y\right)&=0\end{aligned}$$

Hence $g\left(y\right)$ is constant. We can choose any value for this constant, so we pick zero. Therefore

$$\varphi\left(x,y\right)=x^{2}y+xe^{y}$$

But $\varphi\left(x,y\right)=C$, hence

$$x^{2}y+xe^{y}=C$$

What is left is to find $C$. For this we use the boundary conditions $y\left(1\right)=\ln 2$, which gives

$$\begin{aligned}1^{2}\left(\ln\left(2\right)\right)+1\times e^{\ln(2)}&=C\\C&=\ln\left(2\right)+2\end{aligned}$$

Hence the solution is

$$x^{2}y+xe^{y}=2+\ln\left(2\right)$$

We can try to find explicit form for $y$

$$y+\frac{e^{y}}{x}=\frac{2+\ln\left(2\right)}{x^{2}}$$

Will leave it at this for now.

### 2.13.3 Problem 2

Solve the following differential equation $2xy^{3}\left(ydx+xdy\right)=\left(ydx-xdy\right)\sin\left(\frac{x}{y}\right)$

**Answer**:

Let us see if it is exact or not. Simplifying

$$\begin{aligned}2xy^{4}dx+2x^{2}y^{3}dy&=\sin\left(\frac{x}{y}\right)ydx-x\sin\left(\frac{x}{y}\right)dy\\\left(2xy^{4}-\sin\left(\frac{x}{y}\right)y\right)dx+\left(2x^{2}y^{3}+x\sin\left(\frac{x}{y}\right)\right)dy&=0\end{aligned}$$

Now we write the ODE as

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0$$

Where

$$\begin{aligned}M\left(x,y\right)&=2xy^{4}-\sin\left(\frac{x}{y}\right)y\\N\left(x,y\right)&=2x^{2}y^{3}+x\sin\left(\frac{x}{y}\right)\end{aligned}$$

The condition for the ODE to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$\begin{aligned}\frac{\partial M}{\partial y}&=8xy^{3}-\left(\sin\left(\frac{x}{y}\right)+y\left(\frac{-x}{y^{2}}\cos\left(\frac{x}{y}\right)\right)\right)\\&=8xy^{3}-\sin\left(\frac{x}{y}\right)+\frac{x}{y}\cos\left(\frac{x}{y}\right)\end{aligned}$$

and

$$\frac{\partial N}{\partial x} = 4xy^3 + \sin\left(\frac{x}{y}\right) + \frac{x}{y}\cos\left(\frac{x}{y}\right)$$

Therefore it is not exact. To make it exact we need to use a general integrating factor. Trying $I = \frac{1}{y^2}$. We multiply the ODE by this factor, which gives

$$\begin{aligned}
M\left(x,y\right) &= \frac{1}{y^2}2xy^4 - \frac{1}{y^2}\sin\left(\frac{x}{y}\right)y \\
&= 2xy^2 - \frac{1}{y}\sin\left(\frac{x}{y}\right) \\
N\left(x,y\right) &= \frac{1}{y^2}2x^2y^3 + \frac{1}{y^2}x\sin\left(\frac{x}{y}\right) \\
&= 2x^2y + \frac{x}{y^2}\sin\left(\frac{x}{y}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial M}{\partial y} &= 4xy - \left(-\frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{1}{y}\left(-\frac{x}{y^2}\right)\cos\left(\frac{x}{y}\right)\right) \\
&= 4xy + \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{x}{y^3}\cos\left(\frac{x}{y}\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= 4xy + \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{x}{y^2}\frac{1}{y}\cos\left(\frac{x}{y}\right) \\
&= 4xy + \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{x}{y^3}\cos\left(\frac{x}{y}\right)
\end{aligned}$$

Therefore it is exact now. Hence

$$\begin{aligned}
M\left(x,y\right) &= 2xy^2 - \frac{1}{y}\sin\left(\frac{x}{y}\right) \\
N\left(x,y\right) &= 2x^2y + \frac{x}{y^2}\sin\left(\frac{x}{y}\right)
\end{aligned}$$

Let

$$M = \frac{\partial \varphi\left(x,y\right)}{\partial x} \tag{1}$$

$$N = \frac{\partial \varphi\left(x,y\right)}{\partial y} \tag{2}$$

Hence the ODE can be written as

$$\begin{aligned}
\frac{\partial \varphi\left(x,y\right)}{\partial x}dx + \frac{\partial \varphi\left(x,y\right)}{\partial y}dy &= 0 \\
\frac{\partial \varphi\left(x,y\right)}{\partial x} + \frac{\partial \varphi\left(x,y\right)}{\partial y}\frac{dy}{dx} &= 0 \\
\frac{d}{dx}\left(\varphi\left(x,y\left(x\right)\right)\right) &= 0
\end{aligned}$$

This means that $\varphi\left(x,y\left(x\right)\right) = C$, a constant. We need now to find $\varphi\left(x,y\right)$. At this point we can pick either Eq. (1) or Eq. (2). Using Eq. (1) gives

$$2xy^2 - \frac{1}{y}\sin\left(\frac{x}{y}\right) = \frac{\partial \varphi\left(x,y\right)}{\partial x}$$

Integrating

$$\begin{aligned}
\varphi\left(x,y\right) &= \int 2xy^2 - \frac{1}{y}\sin\left(\frac{x}{y}\right)dx \\
&= x^2y^2 + \frac{1}{y}y\cos\left(\frac{x}{y}\right) + g\left(y\right) \\
&= x^2y^2 + \cos\left(\frac{x}{y}\right) + g\left(y\right)
\end{aligned}$$

Where $g\left(y\right)$ acts here as the constant of integration, since $y\left(x\right)$ is a function of $x$. Taking derivative of the above w.r.t. $y$ and equating the result to Eq.(2) (or to $N\left(x,y\right)$) gives

$$\begin{aligned}2yx^2-\left(-\frac{x}{y^2}\right)\sin\left(\frac{x}{y}\right)+g'\left(y\right)&=2x^2y+\frac{x}{y^2}\sin\left(\frac{x}{y}\right)\\2yx^2+\frac{x}{y^2}\sin\left(\frac{x}{y}\right)+g'\left(y\right)&=2x^2y+\frac{x}{y^2}\sin\left(\frac{x}{y}\right)\\g'\left(y\right)&=0\end{aligned}$$

Hence $g\left(y\right)$ is constant. We can choose any value for this constant, so we pick zero. Therefore

$$\varphi\left(x,y\right)=x^2y^2+\cos\left(\frac{x}{y}\right)$$

But $\varphi\left(x,y\right)=C$, hence

$$x^2y^2+\cos\left(\frac{x}{y}\right)=C$$

We are not given more information to find $C$ so we stop here.

### 2.13.4 Problem 3

Find the solution which satisfies the given condition $\left(x^4+y^4\right)dx=2x^3ydy$ where $y\left(1\right)=0$

**Solution**:

This ODE is not separable, so we first check if it is exact. Writing the above as

$$\begin{aligned}\left(x^4+y^4\right)dx-2x^3ydy&=0\\M\left(x,y\right)+N\left(x,y\right)\frac{dy}{dx}&=0\end{aligned}$$

Hence $M\left(x,y\right)=\left(x^4+y^4\right)$ and $N\left(x,y\right)=-2x^3y$. Lets check if it is exact first.

$$\begin{aligned}\frac{\partial M}{\partial y}&=4y^3\\\frac{\partial N}{\partial x}&=-6yx^2\end{aligned}$$

Therefore it is not exact. Finding the generalized integrating factor for the above was not easy. Using a small program and with the help of a CAS, the following integrating factor was found

$$I_f=\frac{1}{x\left(y^2-x^2\right)^2}$$

Hence the new $M\left(x,y\right)$ is $\frac{\left(x^4+y^4\right)}{x\left(y^2-x^2\right)^2}$ and the new $N\left(x,y\right)$ is $\frac{-2x^3y}{x\left(y^2-x^2\right)^2}$. Therefore, now the following ODE is exact

$$\frac{\left(x^4+y^4\right)}{x\left(y^2-x^2\right)^2}dx-\frac{2x^3y}{x\left(y^2-x^2\right)^2}y=0$$

Let

$$M=\frac{\left(x^4+y^4\right)}{x\left(y^2-x^2\right)^2}=\frac{\partial\varphi\left(x,y\right)}{\partial x}\tag{1}$$

$$N=-\frac{2x^3y}{x\left(y^2-x^2\right)^2}=\frac{\partial\varphi\left(x,y\right)}{\partial y}\tag{2}$$

This means that $\varphi\left(x,y\left(x\right)\right)=C$, a constant. We need now to find $\varphi\left(x,y\right)$. At this point we can pick either Eq. (1) or Eq. (2). Using Eq. (1) gives

$$\frac{\left(x^4+y^4\right)}{x\left(y^2-x^2\right)^2}=\frac{\partial\varphi\left(x,y\right)}{\partial x}$$

Integrating

$$\varphi\left(x,y\right)=\int\frac{\left(x^4+y^4\right)}{x\left(y^2-x^2\right)^2}dx$$
$$=\ln\left(x\right)-\frac{y^2}{x^2-y^2}+g\left(y\right)$$

Taking derivative of the above w.r.t. $y$, and comparing the result to $N\left(x,y\right)$ gives

$$\frac{-2y^3}{\left(x^2-y^2\right)^2}-\frac{2y}{x^2-y^2}+g'\left(y\right)=\frac{-2x^3y}{x\left(y-x\right)^2\left(y+x\right)^2}$$

Simplifying

$$g'\left(y\right)=\frac{-2x^3y}{x\left(y-x\right)^2\left(y+x\right)^2}+\frac{2y^3}{\left(x^2-y^2\right)^2}+\frac{2y}{x^2-y^2}$$
$$g'\left(y\right)=0$$

Hence $g\left(y\right)$ is constant. We can choose any value for this constant, so we pick zero. Therefore

$$\ln\left(x\right)-\frac{y^2}{x^2-y^2}=C$$

is the solution. Using the condition $y\left(1\right)=0$ hence

$$\ln\left(1\right)-\frac{0}{1-0}=C$$
$$C=0$$

Hence the solution is

$$\ln\left(x\right)-\frac{y^2}{x^2-y^2}=0$$
$$\left(x^2-y^2\right)\ln\left(x\right)-y^2=0$$
$$x^2\ln\left(x\right)-y^2\ln\left(x\right)-y^2=0$$
$$y^2\left(1+\ln\left(x\right)\right)=x^2\ln\left(x\right)$$
$$y^2=\frac{x^2\ln\left(x\right)}{1+\ln\left(x\right)}$$

Therefore

$$y=\pm\frac{x\sqrt{\ln\left(x\right)}}{\sqrt{1+\ln\left(x\right)}}$$

### 2.13.5 Problem 4

Find the general solution for the equation, using the indicated solution for the homogeneous equation to reduce the order of the equation. $\left(2x+1\right)y''-4\left(x+1\right)y'+4y=\frac{\left(2x+1\right)^2}{x+1}$where $y_1=e^{2x}$

**Solution**:

Summary of method of solution: We let the second homogeneous solution be $y_2=u\left(x\right)y_1$ and the substitute this back into the ODE. This gives a new ODE in $u$ which we solve for $u$. Once $u$ is found, then homogeneous solution for the original ODE are found. Next we find the particular solution.

Let the second independent solution of the original ODE be

$$y_2\left(x\right)=u\left(x\right)y_1\left(x\right)$$

Hence

$$y_2'=u'y_1+uy_1'$$
$$y_2''=u''y_1+u'y_1'+u'y_1'+uy_1''$$
$$=u''y_1+2u'y_1'+uy_1''$$

Substituting these back in the original ODE gives

$$(2x+1)\left(u''y_1 + 2u'y_1' + uy_1''\right) - 4(x+1)\left(u'y_1 + uy_1'\right) + 4\left(uy_1\right) = 0$$
$$(2x+1)u''y_1 + u'\left((2x+1)2y_1' - 4(x+1)y_1\right) + u\left((2x+1)y_1'' - 4(x+1)y_1' + 4y_1\right) = 0$$

But $(2x+1)y_1'' - 4(x+1)y_1' + 4y_1$ in the last term above is the ODE itself, hence it is zero, therefore

$$(2x+1)u''y_1 + u'\left((2x+1)2y_1' - 4(x+1)y_1\right) = 0$$

Now we substitute the actual value of $y_1(x)$ and $y_1'(x)$ into the above

$$(2x+1)u''e^{2x} + u'\left((2x+1)4e^{2x} - 4(x+1)e^{2x}\right) = 0$$

Dividing through by $e^{2x}$ gives

$$\begin{aligned}(2x+1)u'' + 4u'((2x+1)-(x+1)) &= 0\\ (2x+1)u'' + 4xu' &= 0\end{aligned}$$

Let $v = u'$, hence

$$\begin{aligned}(2x+1)v' + 4xv &= 0\\ \frac{v'}{v} + \frac{4x}{(2x+1)} &= 0\\ \ln v &= -\int \frac{4x}{(2x+1)}dx\\ &= -\int 2 - \frac{2}{2x+1}dx\\ &= -\int 2dx + 2\int \frac{1}{2x+1}dx\\ &= -2x + \ln(2x+1) + A\end{aligned}$$

Hence

$$v = Ae^{-2x}(2x+1)$$

Where $A$ is constant. Since only one second solution is needed, let $A = 1$. Therefore we have

$$\begin{aligned}\frac{du}{dx} &= e^{-2x}(2x+1)\\ u &= \int e^{-2x}(2x+1)dx\\ &= \int 2xe^{-2x}dx + \int e^{-2x}dx\end{aligned}$$

Integration by parts is used for the first integral. The above simplifies to

$$u = -2e^{-2x}\left(\frac{1}{4} + \frac{x}{2}\right) + \frac{e^{-2x}}{-2} + B$$

Where $B$ is constant which can be set to zero. Hence

$$\begin{aligned}y_2(x) &= u(x)y_1(x)\\ &= e^{2x}\left(-2e^{-2x}\left(\frac{1}{4} + \frac{x}{2}\right) + \frac{e^{-2x}}{-2}\right)\\ &= -1 - x\end{aligned}$$

Therefore, the homogeneous is

$$\begin{aligned}y_h &= c_1y_1 + c_2y_2\\ &= c_1e^{2x} - c_2(1+x)\\ &= c_1e^{2x} + c_3(1+x)\end{aligned}$$

Now we work on finding the particular solution. The forcing function is $\frac{(2x+1)^2}{x+1}$. Using variation of parameters, since $y_1 = e^{2x}, y_2 = (1+x)$ then

$$W(x) = \begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix} = \begin{vmatrix} e^{2x} & (1+x)\\ 2e^{2x} & 1\end{vmatrix} = e^{2x} - 2e^{2x}(1+x) = -e^{2x}(1+2x)$$

Now we assume the particular solution is

$$y_p = u_1y_1 + u_2y_2$$

And recalling that the ODE is

$$(2x+1)\,y'' - 4\,(x+1)\,y' + 4y = \frac{(2x+1)^2}{x+1}$$

Hence

$$\begin{aligned}
u_1 &= \int \frac{-y_2 f\,(x)}{W\,(x)\,a}dx = \int \frac{-\,(1+x)\,\frac{(2x+1)^2}{x+1}}{-e^{2x}\,(1+2x)\,[2x+1]}dx = \int \frac{(2x+1)}{e^{2x}\,(2x+1)}dx = \int e^{-2x}dx \\
&= \frac{e^{-2x}}{-2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
u_2 &= \int \frac{y_1 f\,(x)}{W\,(x)\,a}dx = \int \frac{e^{2x}\frac{(2x+1)^2}{x+1}}{-e^{2x}\,(1+2x)\,[2x+1]}dx = -\int \frac{1}{x+1}dx \\
&= -\ln\,(1+x)
\end{aligned}$$

Therefore, the particular solution is

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \frac{e^{-2x}}{-2}e^{2x} - \ln\,(1+x)\,(1+x) \\
&= -\frac{1}{2} - \ln\,(1+x)\,(1+x)
\end{aligned}$$

Hence the total solution is $y = y_h + y_p$ or

$$y = c_1 e^{2x} + c_3\,(1+x) - \frac{1}{2} - \ln\,(1+x)\,(1+x)$$

In expanded form

$$y = c_1 e^{2x} + c_3 + c_3 x - \frac{1}{2} - \ln\,(1+x) - x\ln\,(1+x)$$

The solution contains two constants of integration which can be found when given initial conditions.

### 2.13.6 Problem 5

Find the complete solution of $\left(D^3 + D^2 + 3D - 5\right)y = e^x$

**Solution**:

To find the roots of the characteristic equation, since it is a cubic polynomial, we guess the first root. we see that 1 is a root for $\lambda^3 + \lambda^2 + 3\lambda - 5 = 0$, now we do long division $\left(\lambda^3 + \lambda^2 + 3\lambda - 5\right)/(\lambda - 1)$ which gives $\lambda^2 + 2\lambda + 5$, the roots of this are $\frac{-b\pm\sqrt{b^2-4(c)}}{2}$ or $\frac{-2\pm\sqrt{-16}}{2}$ or $-1 \pm 2i$, therefore the homogeneous solution is

$$y_h = c_1 e^x + c_2 e^{-1+2i} + c_3 e^{-1-2i}$$

Or in terms of sin and cos the above is

$$y_h = c_1 e^x + e^{-x}\,(A\cos\,(2x) + B\sin\,(2x))$$

Now we need to find $y_p$. Since $e^x$ is part of $y_h$ we have to guess $y_p = c_4 x e^x$. Substituting this into the original ODE gives

$$y_p''' + y_p'' + 3y_p' - 5y_p = e^x$$

Now plugging the results of all the derivatives and dividing through by $e^x$ gives

$$\begin{aligned}
c_4\,(1+1+1+x) + c_4\,(1+1+x) + 3c_4\,(1+x) - 5c_4 x &= 1 \\
c_4\,(8) + x\,(c_4 + c_4 + 3c_4 - 5c_4) &= 1 \\
8c_4 &= 1 \\
c_4 &= \frac{1}{8}
\end{aligned}$$

Therefore

$$y_p = \frac{1}{8}xe^x$$

hence the full solution is $y = y_h + y_p$ or

$$y = c_1 e^x + e^{-x}(A\cos(2x) + B\sin(2x)) + \frac{1}{8}xe^x$$

### 2.13.7 Problem 6

Find the solution for the following equation $3xy' + y + x^2y^4 = 0$

**Solution:**

Dividing by $3x$ gives

$$y' + \frac{1}{3x}y + \frac{x}{3}y^4 = 0$$

This is of the form

$$y' + f(x)y + g(x)y^n = 0$$

which is a Bernoulli ODE. Hence we start by dividing by $y^n$ or $y^4$ in this case, this gives

$$y^{-4}y' + \frac{1}{3x}y^{-3} + \frac{x}{3} = 0$$

Now let $v(x) = y^{-3}$, therefore $v'(x) = -3y^{-4}y'(x)$. Now we substitute these into the above ODE which turns it to an ODE in $v$ that we can solve

$$\begin{aligned}\frac{-1}{3}v' + \frac{1}{3x}v + \frac{x}{3} &= 0\\ v' - \frac{1}{x}v &= x\end{aligned}$$

Hence

$$v(x) = (x+c)x$$

or

$$\begin{aligned}\frac{1}{y^3} &= (x+c)x\\ y^3 &= \frac{1}{(x+c)x}\\ y &= \left(\frac{1}{(x+c)x}\right)^{\frac{1}{3}}\end{aligned}$$

### 2.13.8 Problem 7

Solve $(y')^2 y'' = 1 + (y')^2$

**Solution**

We start by writing the ODE as

$$y'' = \overbrace{(y')^{-2} + 1}^{f(x,y,y')}$$

Therefore, $f(x,y,y') = 1 + (y')^2$ So we see that $f$ is missing both $x$ and $y$. i.e. $y'' = f(y')$. So we will try both cases covered in class that handle missing $x$ and missing $y$ and see which produces a solution.

#### 2.13.8.1 case 1 (missing $x$)

Let $u(x) = y'(x)$, hence $u' = y''$ and the ODE becomes

$$\begin{aligned}u^2u' &= 1 + u^2\\ \frac{du}{dx} &= \frac{1}{u^2} + 1\end{aligned}$$

This is now separable

$$\begin{aligned}\frac{du}{\frac{1}{u^2}+1} &= dx\\ \int \frac{u^2}{1+u^2}du &= \int dx\\ \int 1-\frac{1}{u^2+1}du &= x+c\\ u-\arctan(u) &= x+c\end{aligned}$$

Since $u(x) = y'(x)$, the above becomes

$$y' - \arctan\left(y'\right) = x + c$$

This is an implicit solution for $y(x)$.

#### 2.13.8.2 case 2 (missing $y$)

Let $v\left(y\right) = y'(x)$, but we notice now that $v\left(y\right)$ is function of $y$. Hence $y'' = v\frac{dv}{dy}$ and the ODE becomes

$$v\frac{dv}{dy} = \frac{1}{v^2}+1$$

This is now separable

$$\begin{aligned}\frac{v}{\frac{1}{v^2}+1}dv &= dy\\ \int \frac{v^3}{1+v^2}dv &= \int dy\\ \int \frac{v^3}{1+v^2}dv &= \int dy\\ \int v-\frac{v}{v^2+1}dv &= y+c\\ \frac{v^2}{2}-\frac{1}{2}\ln\left(1+v^2\right) &= y+c\end{aligned}$$

Since $v = y'(x)$, the above becomes

$$\left(y'\right)^2 - \ln\left(1+\left(y'\right)^2\right) = 2y + c_1$$

This is an implicit solution for $y(x)$.

### 2.13.9 Problem 8

Obtain the solutions of the simultaneous equations

$$\begin{aligned}x'+y'+x &= -e^{-t}\\ x'+2y'+2x+2y &= 0\end{aligned}$$

with initial conditions $x(0) = -1; y(0) = 1$

**Solution**:

Taking the Laplace transform of the above system of equations, and using $X \equiv \mathscr{L}(x(t))$ and $Y = \mathscr{L}\left(y(t)\right)$ gives

$$\begin{aligned}sX - x(0) + sY - y(0) + X &= -\frac{1}{s+1}\\ sX - x(0) + 2\left(sY - y(0)\right) + 2X + 2Y &= 0\end{aligned}$$

Substituting initial conditions gives

$$\begin{aligned}sX + 1 + sY - 1 + X &= -\frac{1}{s+1}\\ sX + 1 + 2(sY-1) + 2X + 2Y &= 0\end{aligned}$$

Simplifying

$$
\begin{aligned}
X(1+s)+sY &= -\frac{1}{s+1}\\
X(s+2)+Y(2s+2) &= 1
\end{aligned}
$$

Hence

$$
\begin{pmatrix} 1+s & s\\ s+2 & 2(s+1) \end{pmatrix}\begin{pmatrix} X\\ Y \end{pmatrix}=\begin{pmatrix} -\frac{1}{s+1}\\ 1 \end{pmatrix}
$$

Therefore

$$
\begin{aligned}
\begin{pmatrix} X\\ Y \end{pmatrix} &= \begin{pmatrix} 1+s & s\\ s+2 & 2(s+1) \end{pmatrix}^{-1}\begin{pmatrix} -\frac{1}{s+1}\\ 1 \end{pmatrix}\\
&= \frac{1}{s^2+2s+2}\begin{pmatrix} 2s+2 & -s\\ -s-2 & s+1 \end{pmatrix}\begin{pmatrix} -\frac{1}{s+1}\\ 1 \end{pmatrix}\\
&= \frac{1}{s^2+2s+2}\begin{pmatrix} (2s+2)\left(-\frac{1}{s+1}\right)-s\\ (-s-2)\left(-\frac{1}{s+1}\right)+s+1 \end{pmatrix}\\
&= \frac{1}{s^2+2s+2}\begin{pmatrix} -s-\frac{2s+2}{s+1}\\ s+\frac{1}{s+1}(s+2)+1 \end{pmatrix}\\
&= \begin{pmatrix} -\frac{s+\frac{2s+2}{s+1}}{s^2+2s+2}\\ \frac{s+\frac{1}{s+1}(s+2)+1}{s^2+2s+2} \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
X &= -\frac{s+\frac{2s+2}{s+1}}{s^2+2s+2}=-\frac{s+2}{s^2+2s+2}\\
Y &= \frac{s+\frac{1}{s+1}(s+2)+1}{s^2+2s+2}=\frac{\left(s^2+3s+3\right)}{(s+1)\left(s^2+2s+2\right)}
\end{aligned}
$$

Now the inverse Laplace transform of each is found. Looking at $X$

$$
\begin{aligned}
X &= -\frac{s}{s^2+2s+2}-\frac{2}{s^2+2s+2}\\
&= -\frac{s}{(s-a)(s-b)}-2\frac{1}{(s-a)(s-b)}
\end{aligned}
$$

Where $a=-1-i, b=-1+i$. Now from tables, $\mathscr{L}^{-1}\left(\frac{s}{(s-a)(s-b)}\right)=\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$, and $\mathscr{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right)=\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$, hence

$$
\begin{aligned}
x(t) &= \frac{-1}{a-b}\left(ae^{at}-be^{bt}\right)-2\frac{1}{a-b}\left(e^{at}-e^{bt}\right)\\
&= \frac{-1}{(-1-i)-(-1+i)}\left((-1-i)e^{at}-(-1+i)e^{bt}\right)\\
&-2\frac{1}{(-1-i)-(-1+i)}\left(e^{at}-e^{bt}\right)\\
\\
&= \frac{1}{2i}\left(-e^{at}-ie^{at}+e^{bt}-ie^{bt}\right)-2\frac{1}{-2i}\left(e^{at}-e^{bt}\right)\\
&= -\frac{1}{2i}e^{at}-\frac{1}{2i}ie^{at}+\frac{1}{2i}e^{bt}-\frac{1}{2i}ie^{bt}+\frac{1}{i}e^{at}-\frac{1}{i}e^{bt}\\
&= \frac{1}{2i}e^{at}-\frac{1}{2i}e^{bt}-\frac{1}{2i}ie^{at}-\frac{1}{2i}ie^{bt}
\end{aligned}
$$

Now substitute the values for $a$ and $b$ in the exponents

$$\begin{aligned}
x(t) &= \frac{1}{2i}e^{(-1-i)t} - \frac{1}{2i}e^{(-1+i)t} - \frac{1}{2i}ie^{(-1-i)t} - \frac{1}{2i}ie^{(-1+i)t}\\
&= \frac{1}{2i}\left(e^{-t}e^{-it}\right) - \frac{1}{2i}\left(e^{-t}e^{it}\right) - \frac{1}{2}\left(e^{-t}e^{-it}\right) - \frac{1}{2}\left(e^{-t}e^{it}\right)\\
&= e^{-t}\left(\frac{e^{-it}-e^{it}}{2i}\right) - e^{-t}\left(\frac{e^{-it}+e^{it}}{2}\right)\\
&= e^{-t}\left(\frac{-ie^{-it}+ie^{it}}{2}\right) - e^{-t}\cos t\\
&= -e^{-t}(\sin t) - e^{-t}\cos t\\
&= -e^{-t}(\sin t + \cos t)
\end{aligned}$$

Now to find the inverse Laplace of $Y$

$$\begin{aligned}
Y &= \frac{\left(s^2+3s+3\right)}{(s+1)\left(s^2+2s+2\right)}\\
&= \frac{1}{s^2+2s+2} + \frac{1}{s+1}\\
&= \frac{1}{(s-a)(s-b)} + \frac{1}{s+1}
\end{aligned}$$

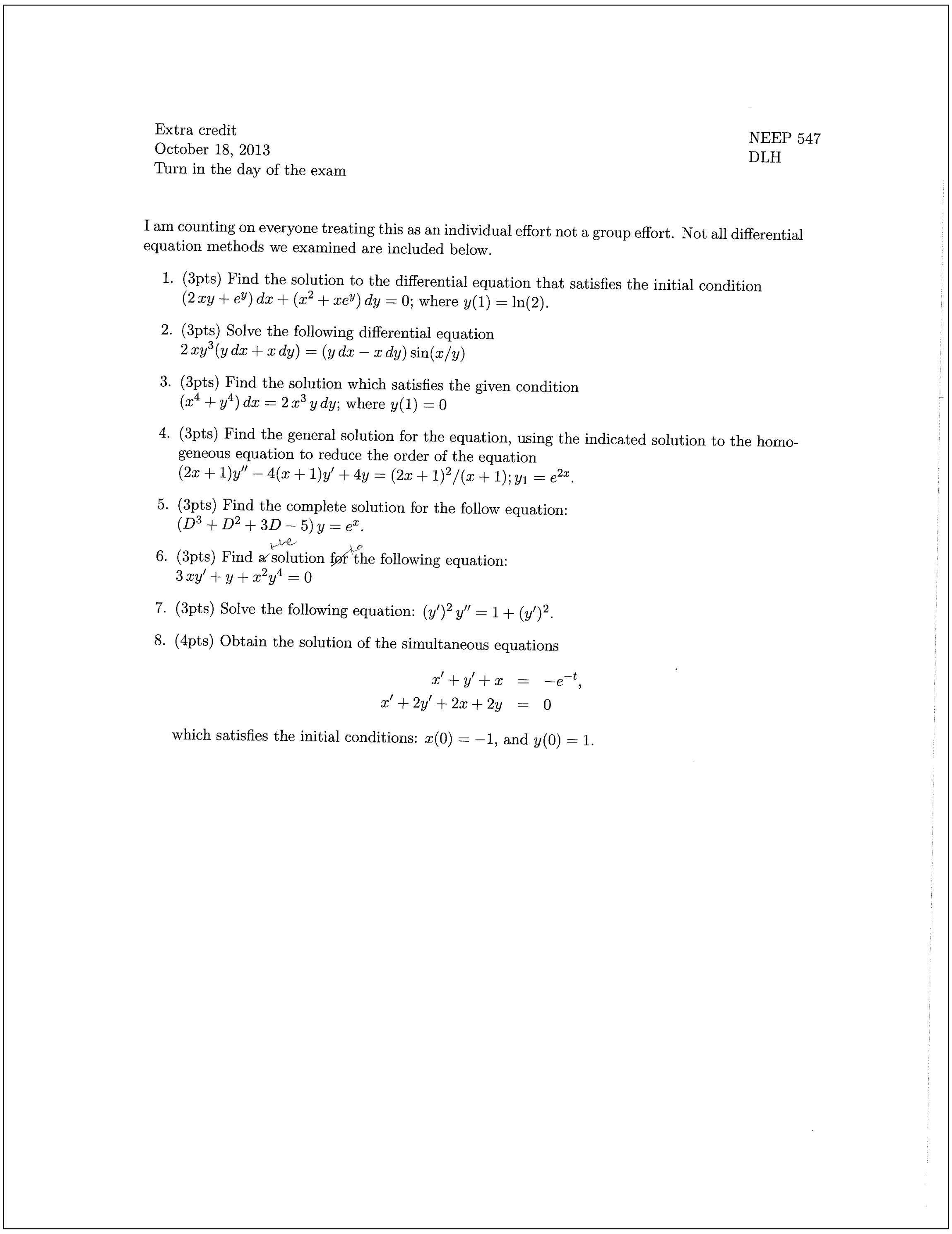
Where $a = -1 - i, b = -1 + i$. But $\mathscr{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right) = \frac{1}{-2i}\left(e^{(-1-i)t} - e^{(-1-i)t}\right)$, Hence

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) &= \frac{1}{-2i}\left(e^{-t}e^{-it} - e^{-t}e^{-it}\right)\\
&= -e^{-t}\left(\frac{-ie^{-it}+ie^{-it}}{2}\right)\\
&= e^{-t}\left(\frac{ie^{-it}-ie^{-it}}{2}\right)\\
&= e^{-t}\sin t
\end{aligned}$$

And $\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$, hence

$$y(t) = e^{-t}\sin t + e^{-t}$$

### 2.13.10 key solution



Extra credit
October 18, 2013
Turn in the day of the exam

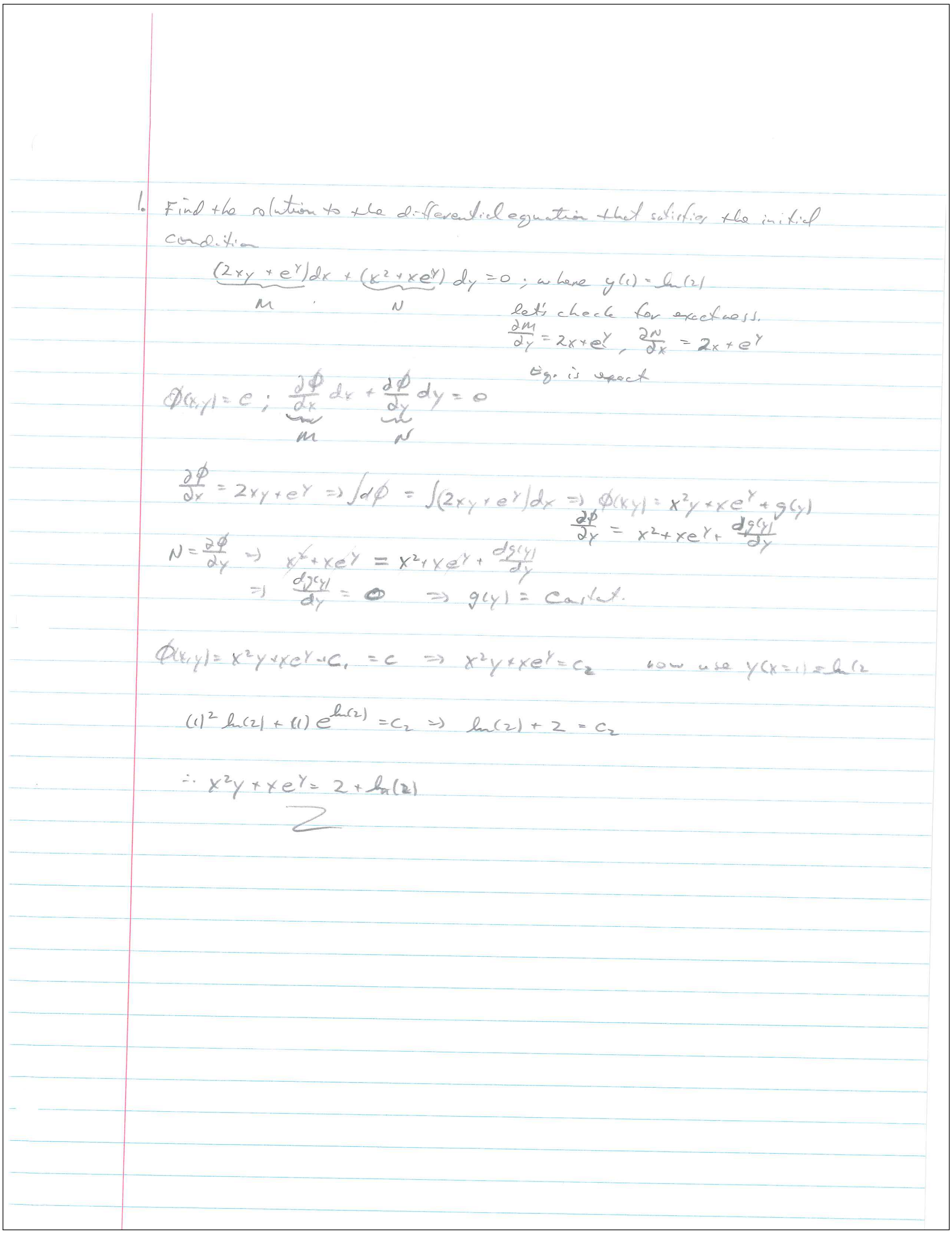
NEEP 547
DLH

I am counting on everyone treating this as an individual effort not a group effort. Not all differential equation methods we examined are included below.

1. (3pts) Find the solution to the differential equation that satisfies the initial condition $(2\,xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; where $y(1) = \ln(2)$.
2. (3pts) Solve the following differential equation $2\,xy^3(y\,dx + x\,dy) = (y\,dx - x\,dy)\sin(x/y)$
3. (3pts) Find the solution which satisfies the given condition $(x^4 + y^4)\,dx = 2\,x^3\,y\,dy$; where $y(1) = 0$
4. (3pts) Find the general solution for the equation, using the indicated solution to the homogeneous equation to reduce the order of the equation $(2x+1)y'' - 4(x+1)y' + 4y = (2x+1)^2/(x+1); y_1 = e^{2x}$.
5. (3pts) Find the complete solution for the follow equation: $(D^3 + D^2 + 3D - 5)\,y = e^x$.
6. (3pts) Find ~~a~~ the solution ~~for~~ to the following equation: $3\,xy' + y + x^2y^4 = 0$
7. (3pts) Solve the following equation: $(y')^2\,y'' = 1 + (y')^2$.
8. (4pts) Obtain the solution of the simultaneous equations

$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$

which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.



1. Find the solution to the differential equation that satisfies the initial condition

$$\underbrace{(2xy+e^y)}_{M}dx+\underbrace{(x^2+xe^y)}_{N}dy=0 \; ; \text{ where } y(1)=\ln(2)$$

let's check for exactness.

$\frac{\partial M}{\partial y}=2x+e^y$ , $\frac{\partial N}{\partial x}=2x+e^y$

Eq. is exact

$$\phi(x,y)=c \; ; \; \underbrace{\frac{\partial\phi}{\partial x}}_{M}dx+\underbrace{\frac{\partial\phi}{\partial y}}_{N}dy=0$$

$$\frac{\partial\phi}{\partial x}=2xy+e^y \Rightarrow \int d\phi=\int(2xy+e^y)dx \Rightarrow \phi(x,y)=x^2y+xe^y+g(y)$$

$$\frac{\partial\phi}{\partial y}=x^2+xe^y+\frac{dg(y)}{dy}$$

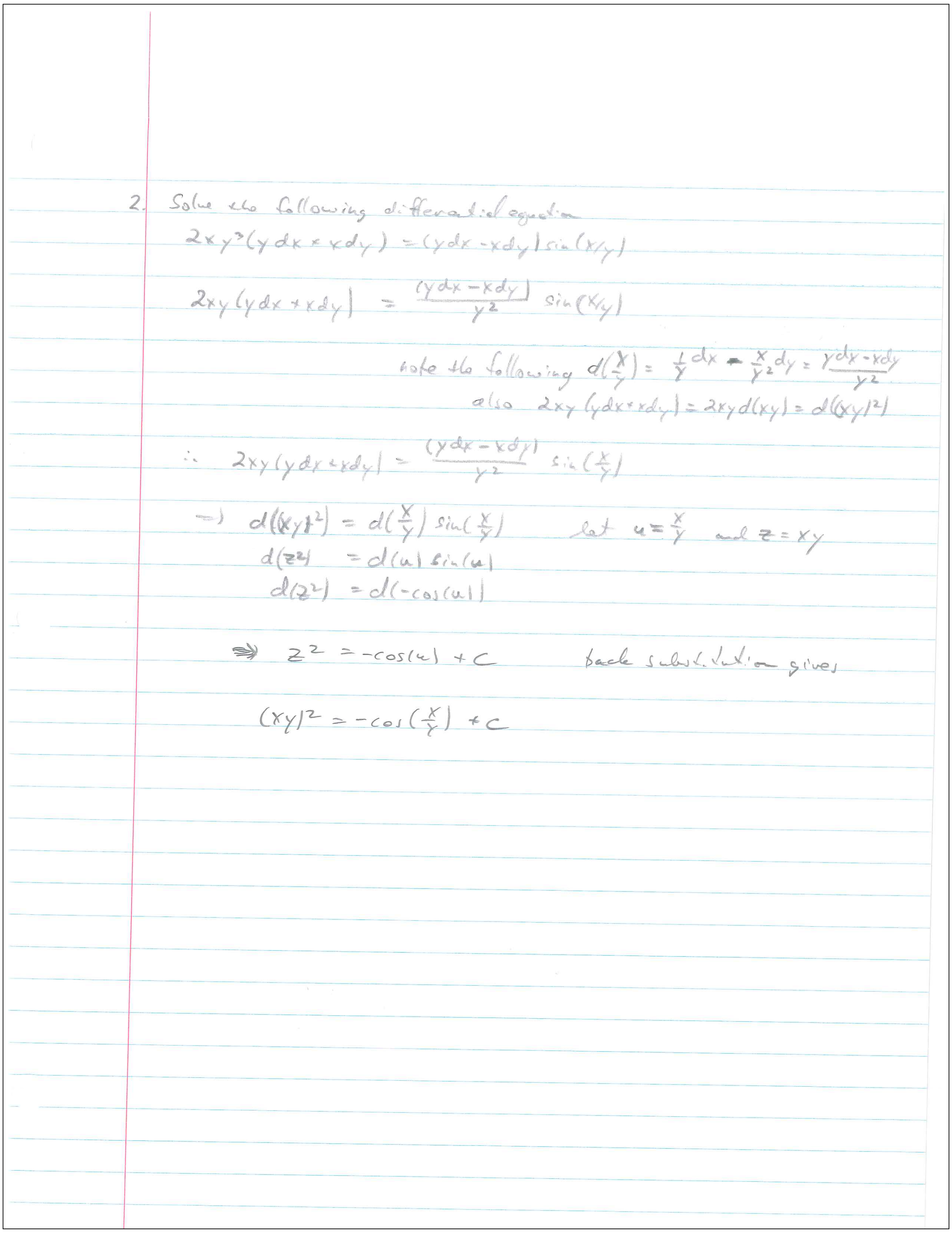
$$N=\frac{\partial\phi}{\partial y} \Rightarrow \cancel{x^2}+xe^y=x^2+xe^y+\frac{dg(y)}{dy}$$

$$\Rightarrow \frac{dg(y)}{dy}=0 \Rightarrow g(y)=\text{Constant.}$$

$\phi(x,y)=x^2y+xe^y+C_1=c \Rightarrow x^2y+xe^y=C_2$ now use $y(x=1)=\ln(2$

$$(1)^2\ln(2)+(1)e^{\ln(2)}=C_2 \Rightarrow \ln(2)+2=C_2$$

$$\therefore x^2y+xe^y=2+\ln(2)$$



2. Solve the following differential equation

$2xy^3(y\,dx+x\,dy)=(y\,dx-x\,dy)\sin(x/y)$

$2xy(y\,dx+x\,dy)=\frac{(y\,dx-x\,dy)}{y^2}\sin(x/y)$

note the following $d\left(\frac{x}{y}\right)=\frac{1}{y}dx-\frac{x}{y^2}dy=\frac{y\,dx-x\,dy}{y^2}$

also $2xy(y\,dx+x\,dy)=2xy\,d(xy)=d((xy)^2)$

$\therefore\ 2xy(y\,dx+x\,dy)=\frac{(y\,dx-x\,dy)}{y^2}\sin\left(\frac{x}{y}\right)$

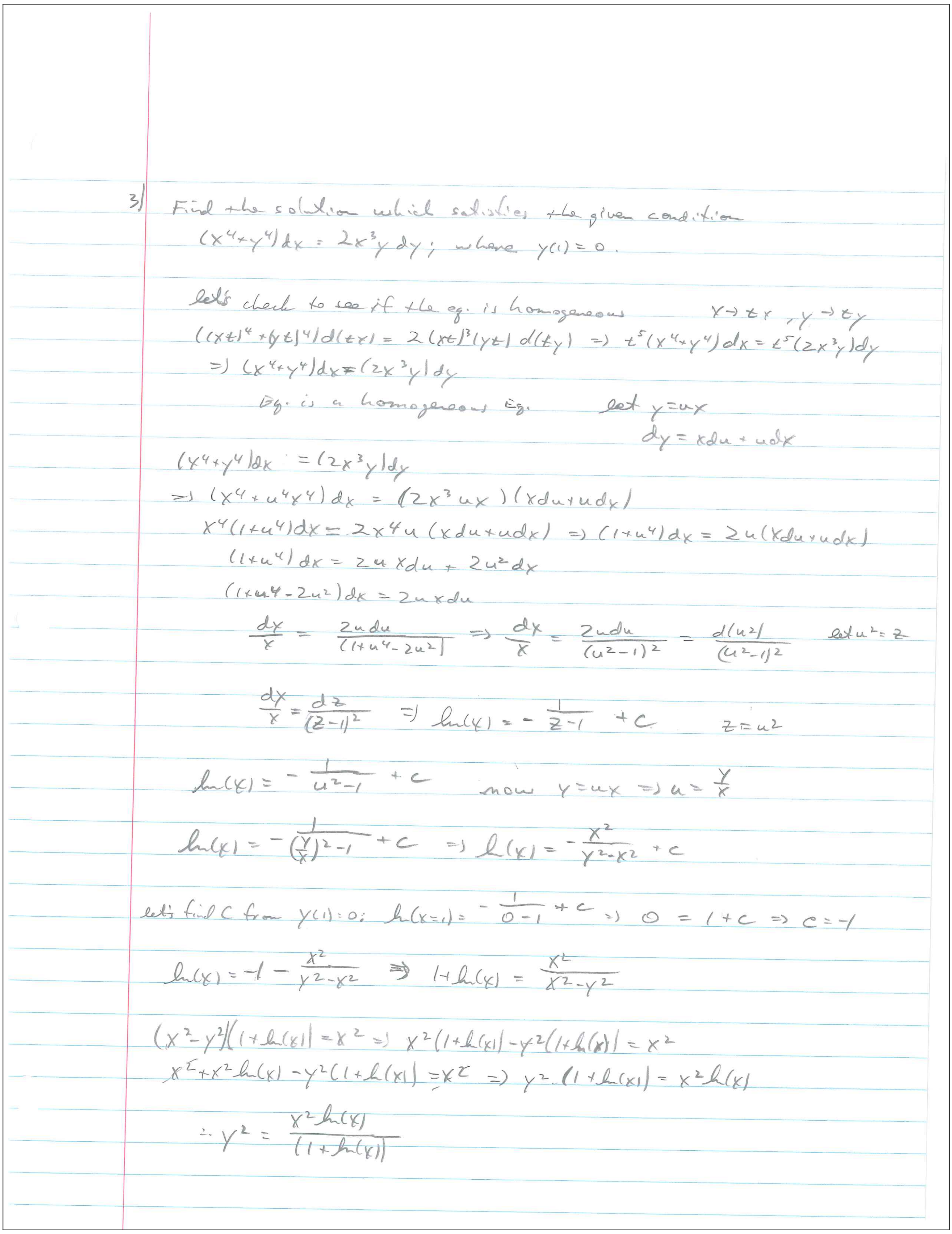
$\Rightarrow d((xy)^2)=d\left(\frac{x}{y}\right)\sin\left(\frac{x}{y}\right)$ let $u=\frac{x}{y}$ and $z=xy$

$d(z^2)=d(u)\sin(u)$

$d(z^2)=d(-\cos(u))$

$\Rightarrow z^2=-\cos(u)+C$ back substitution gives

$(xy)^2=-\cos\left(\frac{x}{y}\right)+C$



3) Find the solution which satisfies the given condition

$(x^4+y^4)dx = 2x^3y\,dy$; where $y(1)=0$.

let's check to see if the eq. is homogeneous $x \to tx$, $y \to ty$

$((xt)^4+(yt)^4)d(tx) = 2(xt)^3(yt)\,d(ty) \Rightarrow t^5(x^4+y^4)dx = t^5(2x^3y)dy$

$\Rightarrow (x^4+y^4)dx = (2x^3y)dy$

Eq. is a homogeneous Eq. let $y=ux$

$dy = xdu + udx$

$(x^4+y^4)dx = (2x^3y)dy$

$\Rightarrow (x^4+u^4x^4)dx = (2x^3ux)(xdu+udx)$

$x^4(1+u^4)dx = 2x^4u(xdu+udx) \Rightarrow (1+u^4)dx = 2u(xdu+udx)$

$(1+u^4)dx = 2u\,xdu + 2u^2dx$

$(1+u^4-2u^2)dx = 2u\,xdu$

$\frac{dx}{x} = \frac{2udu}{(1+u^4-2u^2)} \Rightarrow \frac{dx}{x} = \frac{2udu}{(u^2-1)^2} = \frac{d(u^2)}{(u^2-1)^2}$ let $u^2=z$

$\frac{dx}{x} = \frac{dz}{(z-1)^2} \Rightarrow \ln(x) = -\frac{1}{z-1} + c$ $z=u^2$

$\ln(x) = -\frac{1}{u^2-1} + c$ now $y=ux \Rightarrow u=\frac{y}{x}$

$\ln(x) = -\frac{1}{(\frac{y}{x})^2-1} + c \Rightarrow \ln(x) = -\frac{x^2}{y^2-x^2} + c$

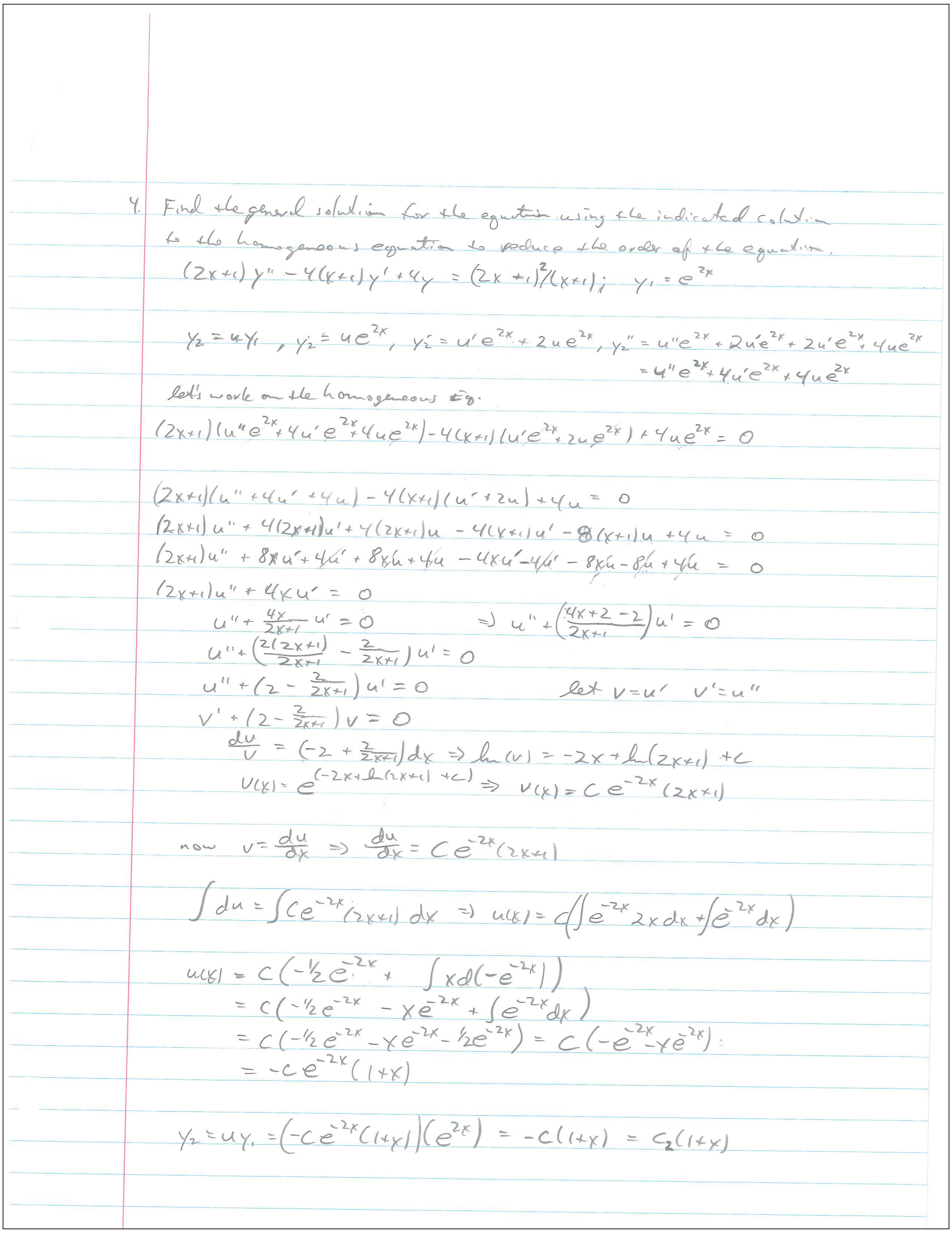
let's find C from $y(1)=0$: $\ln(x=1) = -\frac{1}{0-1} + c \Rightarrow 0 = 1+c \Rightarrow c=-1$

$\ln(x) = -1 - \frac{x^2}{y^2-x^2} \Rightarrow 1+\ln(x) = \frac{x^2}{x^2-y^2}$

$(x^2-y^2)(1+\ln(x)) = x^2 \Rightarrow x^2(1+\ln(x)) - y^2(1+\ln(x)) = x^2$

$\cancel{x^2} + x^2\ln(x) - y^2(1+\ln(x)) = \cancel{x^2} \Rightarrow y^2(1+\ln(x)) = x^2\ln(x)$

$\therefore y^2 = \frac{x^2\ln(x)}{(1+\ln(x))}$



4. Find the general solution for the equation using the indicated solution to the homogeneous equation to reduce the order of the equation.

$(2x+1)y'' - 4(x+1)y' + 4y = (2x+1)^2/(x+1); \quad y_1 = e^{2x}$

$y_2 = uy_1, \quad y_2 = ue^{2x}, \quad y_2' = u'e^{2x} + 2ue^{2x}, \quad y_2'' = u''e^{2x} + 2u'e^{2x} + 2u'e^{2x} + 4ue^{2x}$

$= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$

let's work on the homogeneous eq.

$(2x+1)(u''e^{2x} + 4u'e^{2x} + 4ue^{2x}) - 4(x+1)(u'e^{2x} + 2ue^{2x}) + 4ue^{2x} = 0$

$(2x+1)(u'' + 4u' + 4u) - 4(x+1)(u' + 2u) + 4u = 0$

$(2x+1)u'' + 4(2x+1)u' + 4(2x+1)u - 4(x+1)u' - 8(x+1)u + 4u = 0$

$(2x+1)u'' + 8xu' + \cancel{4u'} + \cancel{8xu} + \cancel{4u} - 4xu' - \cancel{4u'} - \cancel{8xu} - \cancel{8u} + \cancel{4u} = 0$

$(2x+1)u'' + 4xu' = 0$

$u'' + \frac{4x}{2x+1}u' = 0 \quad \Rightarrow u'' + \left(\frac{4x+2-2}{2x+1}\right)u' = 0$

$u'' + \left(\frac{2(2x+1)}{2x+1} - \frac{2}{2x+1}\right)u' = 0$

$u'' + \left(2 - \frac{2}{2x+1}\right)u' = 0 \qquad$ let $v = u' \quad v' = u''$

$v' + \left(2 - \frac{2}{2x+1}\right)v = 0$

$\frac{du}{v} = \left(-2 + \frac{2}{2x+1}\right)dx \Rightarrow \ln(v) = -2x + \ln(2x+1) + c$

$v(x) = e^{(-2x + \ln(2x+1) + c)} \Rightarrow v(x) = Ce^{-2x}(2x+1)$

now $v = \frac{du}{dx} \Rightarrow \frac{du}{dx} = Ce^{-2x}(2x+1)$

$\int du = \int Ce^{-2x}(2x+1)\,dx \Rightarrow u(x) = c\left(\int e^{-2x}2x\,dx + \int e^{-2x}dx\right)$

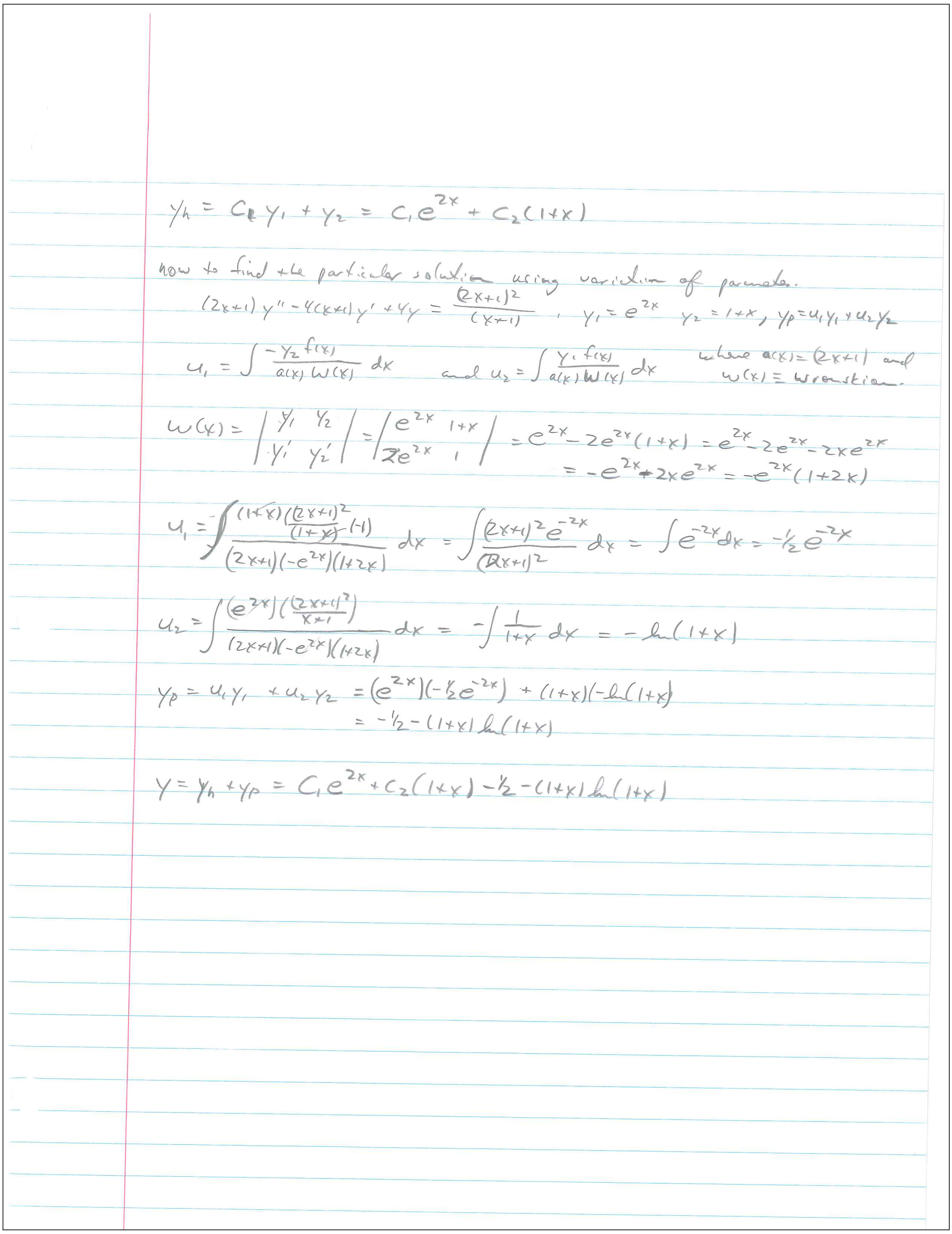
$u(x) = c\left(-\tfrac{1}{2}e^{-2x} + \int x\,d(-e^{-2x})\right)$

$= c\left(-\tfrac{1}{2}e^{-2x} - xe^{-2x} + \int e^{-2x}dx\right)$

$= c\left(-\tfrac{1}{2}e^{-2x} - xe^{-2x} - \tfrac{1}{2}e^{-2x}\right) = c\left(-e^{-2x} - xe^{-2x}\right)$

$= -ce^{-2x}(1+x)$

$y_2 = uy_1 = \left(-ce^{-2x}(1+x)\right)\left(e^{2x}\right) = -c(1+x) = c_2(1+x)$



$$y_h = C_1 y_1 + y_2 = c_1 e^{2x} + c_2(1+x)$$

now to find the particular solution using variation of parameter.

$$(2x+1)y'' - 4(x+1)y' + 4y = \frac{(2x+1)^2}{(x+1)}, \quad y_1 = e^{2x} \quad y_2 = 1+x, \; y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-y_2 f(x)}{a(x)\,W(x)}\,dx \quad \text{and } u_2 = \int \frac{y_1 f(x)}{a(x)\,W(x)}\,dx$$

where $a(x) = (2x+1)$ and $W(x) \equiv$ Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & 1+x \\ 2e^{2x} & 1 \end{vmatrix} = e^{2x} - 2e^{2x}(1+x) = e^{2x} - 2e^{2x} - 2xe^{2x}$$

$$= -e^{2x} + 2xe^{2x} = -e^{2x}(1+2x)$$

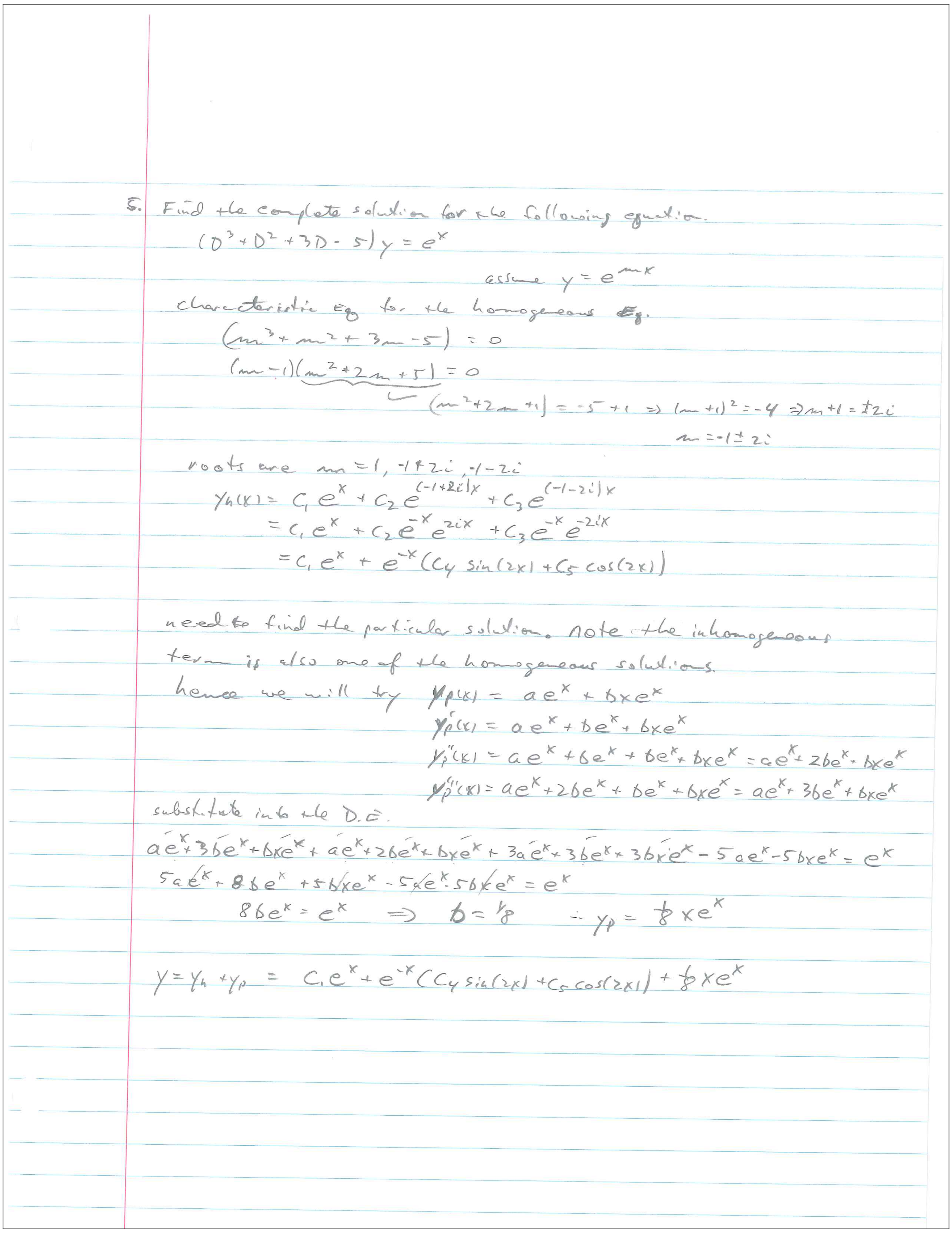
$$u_1 = \int \frac{\cancel{(1+x)}\left(\frac{(2x+1)^2}{\cancel{(1+x)}}\right)(-1)}{(2x+1)(-e^{2x})(1+2x)}\,dx = \int \frac{(2x+1)^2 e^{-2x}}{(2x+1)^2}\,dx = \int e^{-2x}dx = -\tfrac{1}{2}e^{-2x}$$

$$u_2 = \int \frac{(e^{2x})\left(\frac{(2x+1)^2}{x+1}\right)}{(2x+1)(-e^{2x})(1+2x)}\,dx = -\int \frac{1}{1+x}\,dx = -\ln(1+x)$$

$$y_p = u_1 y_1 + u_2 y_2 = (e^{2x})(-\tfrac{1}{2}e^{-2x}) + (1+x)(-\ln(1+x))$$

$$= -\tfrac{1}{2} - (1+x)\ln(1+x)$$

$$y = y_h + y_p = C_1 e^{2x} + c_2(1+x) - \tfrac{1}{2} - (1+x)\ln(1+x)$$



5. Find the complete solution for the following equation.

$$(D^3+D^2+3D-5)y=e^x$$

assume $y=e^{mx}$

characteristic Eq for the homogeneous Eq.

$$(m^3+m^2+3m-5)=0$$

$$(m-1)\underbrace{(m^2+2m+5)}=0$$

$$(m^2+2m+1)=-5+1 \Rightarrow (m+1)^2=-4 \Rightarrow m+1=\pm 2i$$

$$m=-1\pm 2i$$

roots are $m=1,\ -1+2i,\ -1-2i$

$$y_h(x)=c_1e^x+c_2e^{(-1+2i)x}+c_3e^{(-1-2i)x}$$

$$=c_1e^x+c_2e^{-x}e^{2ix}+c_3e^{-x}e^{-2ix}$$

$$=c_1e^x+e^{-x}(c_4\sin(2x)+c_5\cos(2x))$$

need to find the particular solution. note the inhomogeneous term is also one of the homogeneous solutions.

hence we will try $y_p(x)=ae^x+bxe^x$

$$y_p'(x)=ae^x+be^x+bxe^x$$

$$y_p''(x)=ae^x+be^x+be^x+bxe^x=ae^x+2be^x+bxe^x$$

$$y_p'''(x)=ae^x+2be^x+be^x+bxe^x=ae^x+3be^x+bxe^x$$

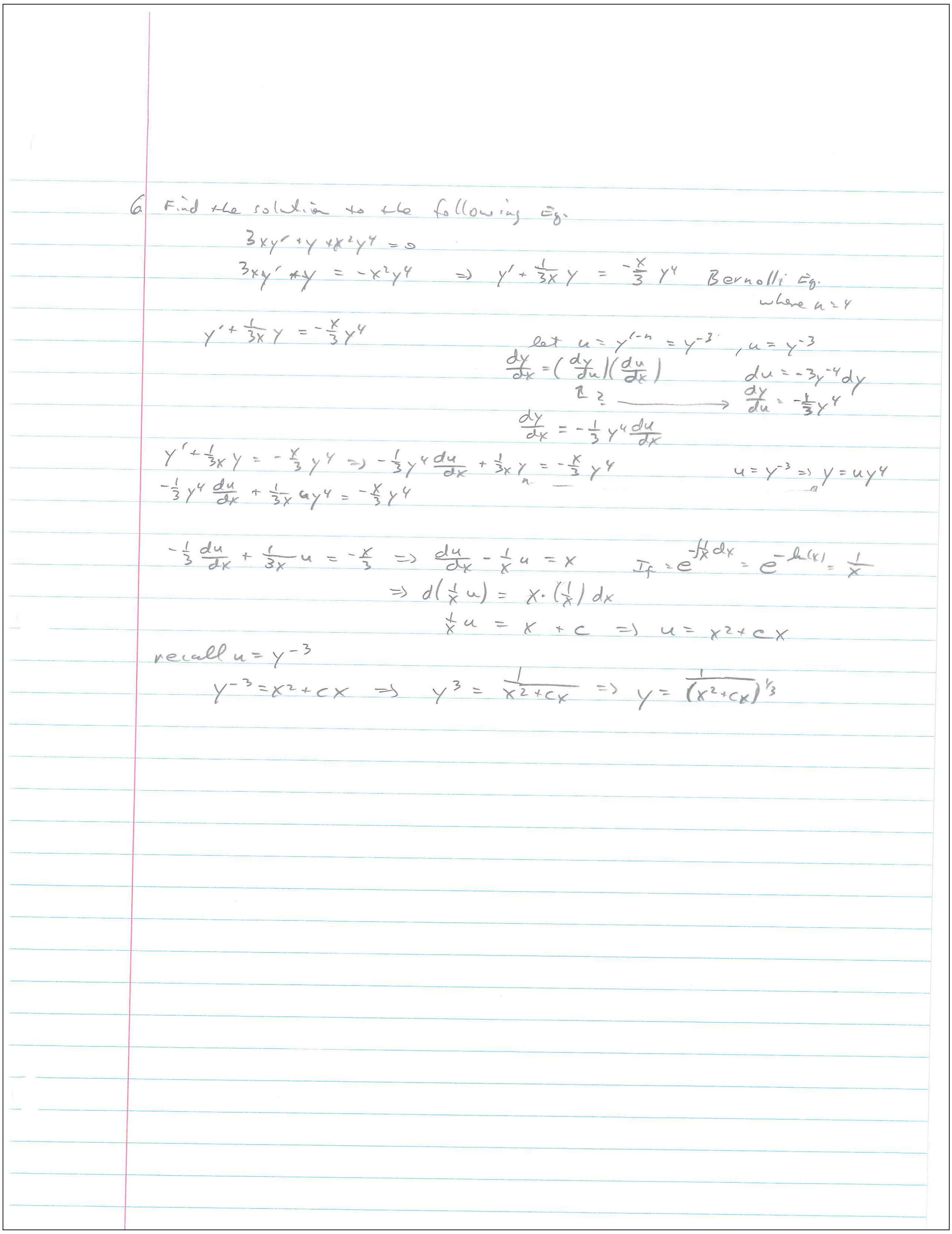
substitute into the D.E.

$$ae^x+3be^x+bxe^x+ae^x+2be^x+bxe^x+3ae^x+3be^x+3bxe^x-5ae^x-5bxe^x=e^x$$

$$5ae^x+8be^x+5\cancel{b}xe^x-5\cancel{a}e^x-5b\cancel{x}e^x=e^x$$

$$8be^x=e^x \Rightarrow b=1/8 \qquad \therefore y_p=\frac{1}{8}xe^x$$

$$y=y_h+y_p=c_1e^x+e^{-x}(c_4\sin(2x)+c_5\cos(2x))+\frac{1}{8}xe^x$$



6. Find the solution to the following Eq.

$$3xy' + y + x^2y^4 = 0$$

$$3xy' + y = -x^2y^4 \Rightarrow y' + \frac{1}{3x}y = -\frac{x}{3}y^4$$ Bernolli Eq. where $n = 4$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4$$

let $u = y^{1-n} = y^{-3}$, $u = y^{-3}$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$du = -3y^{-4}dy$

$$\overset{?}{\uparrow} \longrightarrow \frac{dy}{du} = -\frac{1}{3}y^4$$

$$\frac{dy}{dx} = -\frac{1}{3}y^4\frac{du}{dx}$$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4 \Rightarrow -\frac{1}{3}y^4\frac{du}{dx} + \frac{1}{3x}y = -\frac{x}{3}y^4$$

$u = y^{-3} \Rightarrow y = uy^4$

$$-\frac{1}{3}y^4\frac{du}{dx} + \frac{1}{3x}uy^4 = -\frac{x}{3}y^4$$

$$-\frac{1}{3}\frac{du}{dx} + \frac{1}{3x}u = -\frac{x}{3} \Rightarrow \frac{du}{dx} - \frac{1}{x}u = x$$

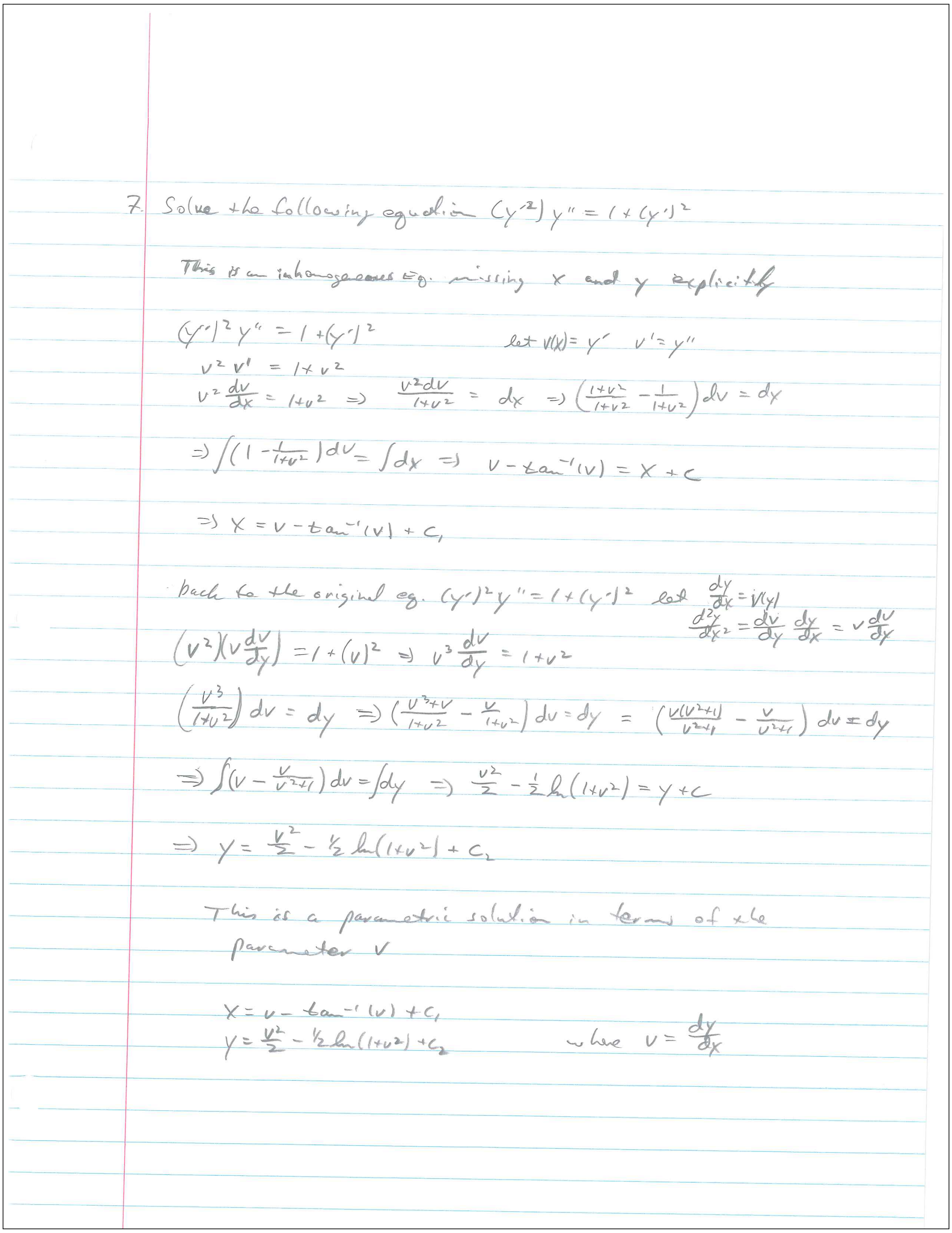
$$I_f = e^{-\int\frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\Rightarrow d\left(\frac{1}{x}u\right) = x\cdot\left(\frac{1}{x}\right)dx$$

$$\frac{1}{x}u = x + c \Rightarrow u = x^2 + cx$$

recall $u = y^{-3}$

$$y^{-3} = x^2 + cx \Rightarrow y^3 = \frac{1}{x^2+cx} \Rightarrow y = \frac{1}{(x^2+cx)^{1/3}}$$



7. Solve the following equation $(y'^2)\, y'' = 1 + (y')^2$

This is an inhomogeneous Eq. missing $x$ and $y$ explicitly

$(y')^2 y'' = 1 + (y')^2$ let $v(x) = y'$ $v' = y''$

$v^2 v' = 1 + v^2$

$v^2 \frac{dv}{dx} = 1 + v^2 \Rightarrow \frac{v^2 dv}{1+v^2} = dx \Rightarrow \left(\frac{1+v^2}{1+v^2} - \frac{1}{1+v^2}\right) dv = dx$

$\Rightarrow \int \left(1 - \frac{1}{1+v^2}\right) dv = \int dx \Rightarrow v - \tan^{-1}(v) = x + c$

$\Rightarrow x = v - \tan^{-1}(v) + c_1$

back to the original eq. $(y')^2 y'' = 1 + (y')^2$ let $\frac{dy}{dx} = v(y)$

$\frac{d^2y}{dx^2} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$

$(v^2)\left(v\frac{dv}{dy}\right) = 1 + (v)^2 \Rightarrow v^3 \frac{dv}{dy} = 1 + v^2$

$\left(\frac{v^3}{1+v^2}\right) dv = dy \Rightarrow \left(\frac{v^3+v}{1+v^2} - \frac{v}{1+v^2}\right) dv = dy = \left(\frac{v(v^2+1)}{v^2+1} - \frac{v}{v^2+1}\right) dv = dy$

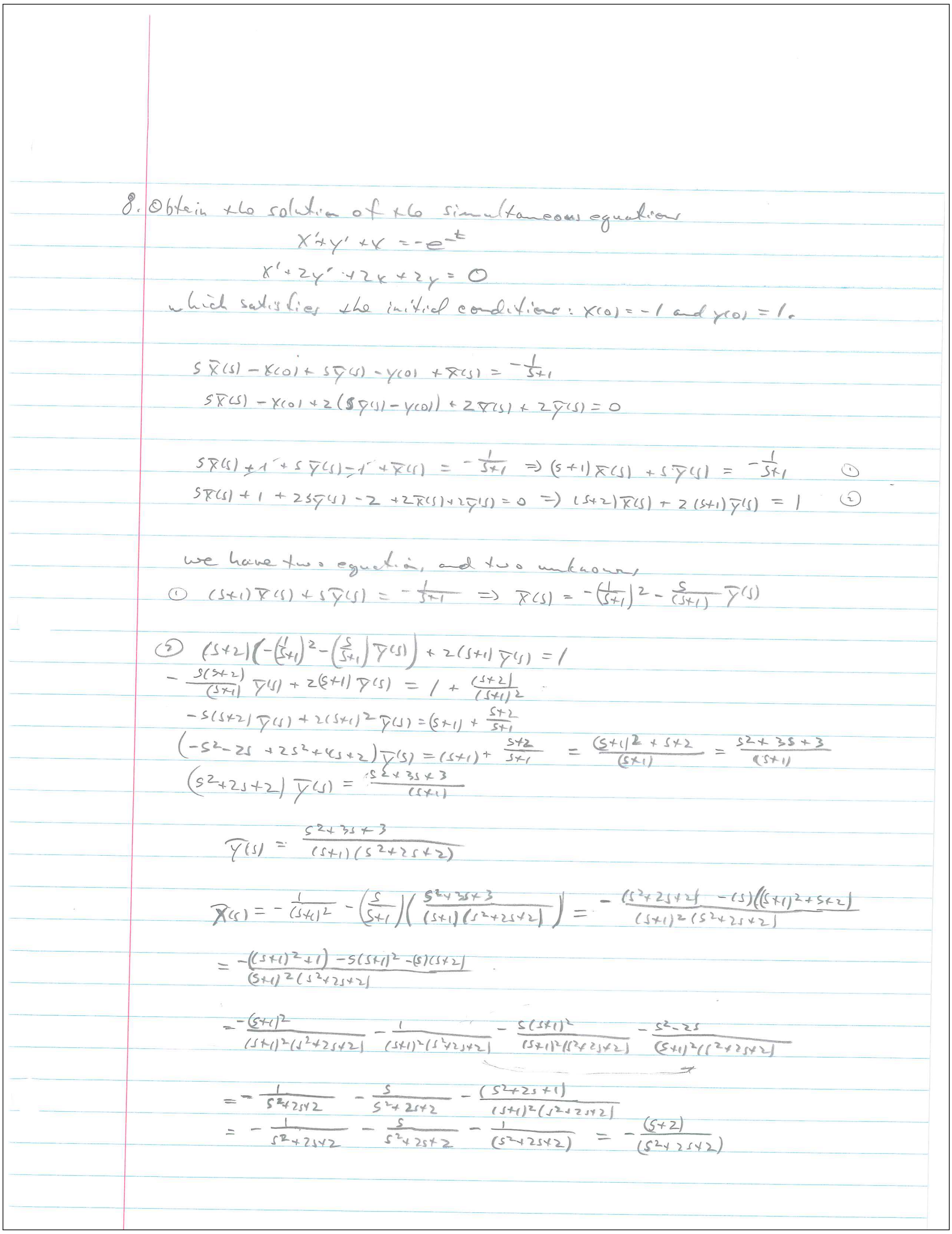
$\Rightarrow \int \left(v - \frac{v}{v^2+1}\right) dv = \int dy \Rightarrow \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) = y + c$

$\Rightarrow y = \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) + c_2$

This is a parametric solution in terms of the parameter $v$

$x = v - \tan^{-1}(v) + c_1$

$y = \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) + c_2$ where $v = \frac{dy}{dx}$



8. Obtain the solution of the simultaneous equations

$$x' + y' + x = -e^{-t}$$

$$x' + 2y' + 2x + 2y = 0$$

which satisfies the initial conditions: $x(0) = -1$ and $y(0) = 1$.

$$s\bar{X}(s) - x(0) + s\bar{Y}(s) - y(0) + \bar{X}(s) = -\frac{1}{s+1}$$

$$s\bar{X}(s) - x(0) + 2(s\bar{Y}(s) - y(0)) + 2\bar{X}(s) + 2\bar{Y}(s) = 0$$

$$s\bar{X}(s) \cancel{+1} + s\bar{Y}(s) \cancel{-1} + \bar{X}(s) = -\frac{1}{s+1} \Rightarrow (s+1)\bar{X}(s) + s\bar{Y}(s) = -\frac{1}{s+1} \quad ①$$

$$s\bar{X}(s) + 1 + 2s\bar{Y}(s) - 2 + 2\bar{X}(s) + 2\bar{Y}(s) = 0 \Rightarrow (s+2)\bar{X}(s) + 2(s+1)\bar{Y}(s) = 1 \quad ②$$

we have two equation, and two unknowns

$$① \quad (s+1)\bar{X}(s) + s\bar{Y}(s) = -\frac{1}{s+1} \Rightarrow \bar{X}(s) = -\left(\frac{1}{s+1}\right)^2 - \frac{s}{(s+1)}\bar{Y}(s)$$

$$② \quad (s+2)\left(-\left(\frac{1}{s+1}\right)^2 - \left(\frac{s}{s+1}\right)\bar{Y}(s)\right) + 2(s+1)\bar{Y}(s) = 1$$

$$-\frac{s(s+2)}{(s+1)}\bar{Y}(s) + 2(s+1)\bar{Y}(s) = 1 + \frac{(s+2)}{(s+1)^2}$$

$$-s(s+2)\bar{Y}(s) + 2(s+1)^2\bar{Y}(s) = (s+1) + \frac{s+2}{s+1}$$

$$(-s^2 - 2s + 2s^2 + 4s + 2)\bar{Y}(s) = (s+1) + \frac{s+2}{s+1} = \frac{(s+1)^2 + s+2}{(s+1)} = \frac{s^2+3s+3}{(s+1)}$$

$$(s^2+2s+2)\bar{Y}(s) = \frac{s^2+3s+3}{(s+1)}$$

$$\bar{Y}(s) = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)}$$

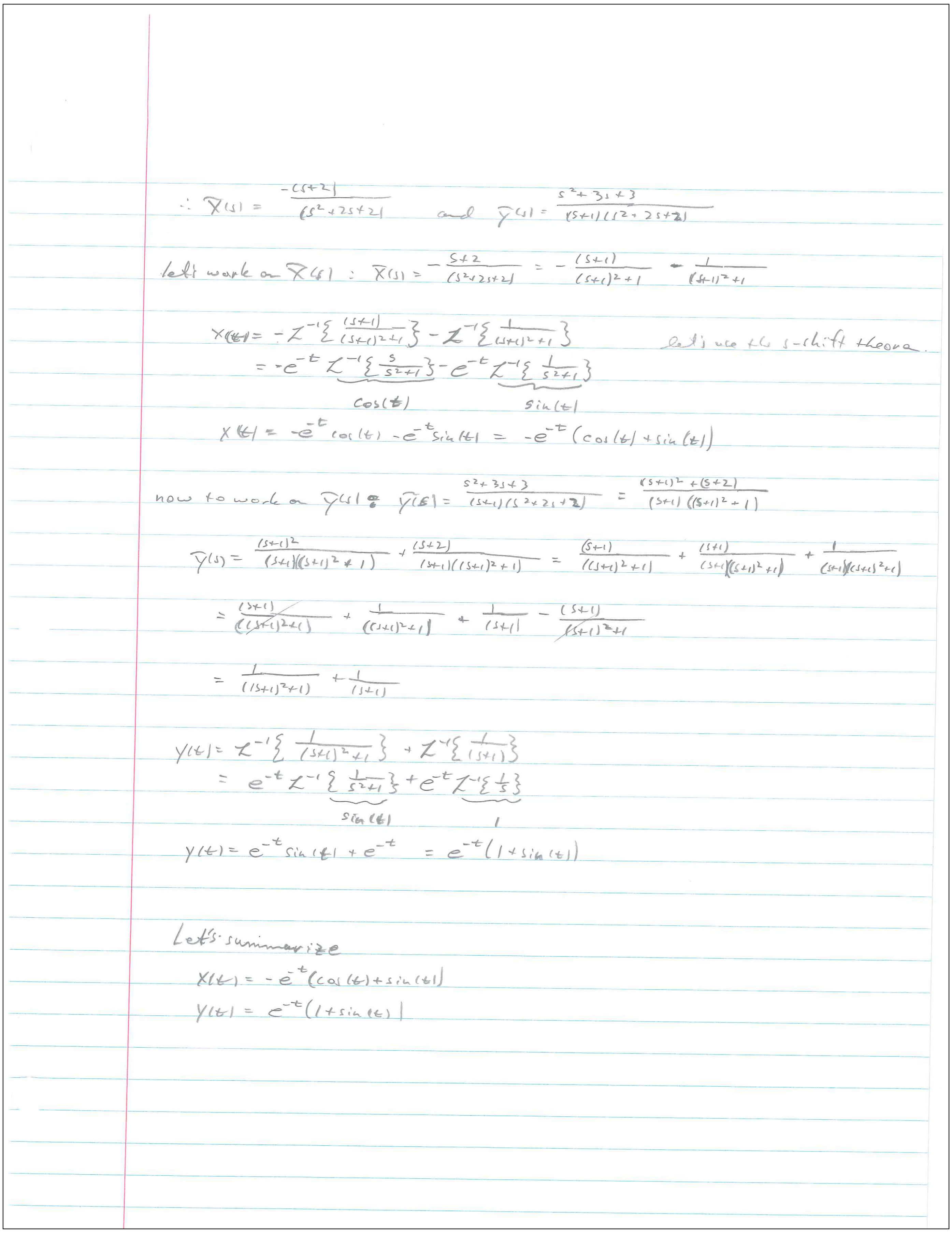
$$\bar{X}(s) = -\frac{1}{(s+1)^2} - \left(\frac{s}{s+1}\right)\left(\frac{s^2+3s+3}{(s+1)(s^2+2s+2)}\right) = -\frac{(s^2+2s+2) - (s)((s+1)^2+s+2)}{(s+1)^2(s^2+2s+2)}$$

$$= \frac{-((s+1)^2+1) - s(s+1)^2 - (s)(s+2)}{(s+1)^2(s^2+2s+2)}$$

$$= \frac{-(s+1)^2}{(s+1)^2(s^2+2s+2)} - \frac{1}{(s+1)^2(s^2+2s+2)} - \frac{s(s+1)^2}{(s+1)^2(s^2+2s+2)} - \frac{s^2-2s}{(s+1)^2(s^2+2s+2)}$$

$$= -\frac{1}{s^2+2s+2} - \frac{s}{s^2+2s+2} - \frac{(s^2+2s+1)}{(s+1)^2(s^2+2s+2)}$$

$$= -\frac{1}{s^2+2s+2} - \frac{s}{s^2+2s+2} - \frac{1}{(s^2+2s+2)} = -\frac{(s+2)}{(s^2+2s+2)}$$



$$\therefore \bar{X}(s) = \frac{-(s+2)}{(s^2+2s+2)} \quad \text{and} \quad \bar{Y}(s) = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)}$$

lets work on $\bar{X}(s)$ : $\bar{X}(s) = -\frac{s+2}{(s^2+2s+2)} = -\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$

$$x(t) = -\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

let's use the s-shift theorem.

$$= -e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos(t)} - e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin(t)}$$

$$x(t) = -e^{-t}\cos(t) - e^{-t}\sin(t) = -e^{-t}(\cos(t)+\sin(t))$$

now to work on $\bar{Y}(s)$: $\bar{Y}(s) = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)} = \frac{(s+1)^2+(s+2)}{(s+1)((s+1)^2+1)}$

$$\bar{Y}(s) = \frac{(s+1)^2}{(s+1)((s+1)^2+1)} + \frac{(s+2)}{(s+1)((s+1)^2+1)} = \frac{(s+1)}{((s+1)^2+1)} + \frac{(s+1)}{(s+1)((s+1)^2+1)} + \frac{1}{(s+1)((s+1)^2+1)}$$

$$= \cancel{\frac{(s+1)}{((s+1)^2+1)}} + \frac{1}{((s+1)^2+1)} + \frac{1}{(s+1)} - \cancel{\frac{(s+1)}{(s+1)^2+1}}$$

$$= \frac{1}{((s+1)^2+1)} + \frac{1}{(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\}$$

$$= e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin(t)} + e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1}$$

$$y(t) = e^{-t}\sin(t) + e^{-t} = e^{-t}(1+\sin(t))$$

Let's summarize

$$x(t) = -e^{-t}(\cos(t)+\sin(t))$$

$$y(t) = e^{-t}(1+\sin(t))$$

## 2.14 problem 6 from HW 10 and second extra credit HW

### 2.14.1 Problems to solve

| | |
|---|---|
| Extra credit | NEEP 547 |
| November 27, 2013 | DLH |

Turn in the day of the exam.
( I am counting on everyone treating this as an individual effort not a group effort.)

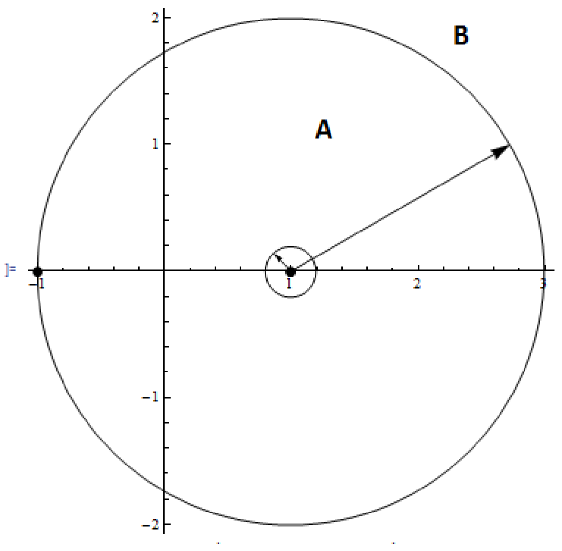
1. (3pts) Find the Fourier expansions of the periodic function whose definition on one period is
$$f(t) = \begin{cases} 0 & \text{for } -\pi \le t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi. \end{cases}$$
2. (3pts) Find the solution of the following differential equation which satisfies the given initial conditions and where $f(t)$ is a periodic function:
$$y'' + 9y = f(t) \quad ; y(0) = y'(0) = 0 \text{ and } f(t) = |t| \quad \text{for } -\pi \le t \le \pi.$$
3. (3pts) Find the Fourier transform of the function $f(t) = \begin{cases} 0 & \text{for } -\infty < t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi \le t < \infty \end{cases}$
using the basic definition of Fourier transform.
4. (3pts) Find the inverse transform of
$$f(\omega) = \frac{\sin(\omega - 2)}{\omega - 2}.$$
(Use theorems discussed in class such as the shifting theorem to solve for the inverse. Do not just copy the inverse from a table of Fourier Transforms).
5. (3pts) Solve the integral equation:
$$\psi(x) = 1 + \lambda^2 \int_0^x (t - x)\,\psi(t)\,dt.$$
6. (3pts) Solve the integral equation:
$$\frac{dy(t)}{dt} = 1 - \sin(t) - \int_0^t y(\tau)\,d\tau, \text{ with I.C.: } y(0) = 0.$$
7. (2pts) Given the matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, show that the particular solution integral is given by $\mathbf{v(t)} = \mathbf{X(t)} \int \mathbf{X^{-1}(t) f(t)\,dt}$ where $\mathbf{X(t)}$ is the fundamental matrix of the homogeneous equation.
8. (3pts) Obtain the solution of the simultaneous equations
$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$
which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.
9. (4pts) Solve the following differential equation using Fourier Transforms: $3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions $y(0) = -1$ and $y'(0) = 0$. Invert the resulting transform expression in the complex plane to obtain the final result for $y(t)$. (Recall that one can use Fourier Transforms on the half-space $(0 < t < \infty)$. The definition of the transforms for $y'(t)$ and $y''(t)$ are modified and incorporate the initial conditions similar to Laplace transforms).

### 2.14.2 problem 6, Laurent series, left from HW 10

Obtain two distinct Laurent expansions for $f(z) = \frac{3z+1}{z^2-1}$ around $z = 1$ and tell where each converges.

#### Solution

$f(z)$ has singularities at $z = \pm 1$ and the expansion is around one of these singularities. Looking at the diagram Region $A$ is annulus between $z = 1$ and $z = -1$ but does not include $z = 1$ where the small circle is shown since that is a singularity. Region $B$ is all the region



outside the large circle shown.

$$f(z)=\frac{3z+1}{(z-1)(z+1)}=\frac{2}{z-1}+\frac{1}{z+1}$$

**Region A**

For $\frac{2}{z-1}$, since its pole is at $z=1$ and so we expand outwards, and hence it is already in form of Laurent series around $z=1$, and for $\frac{1}{z+1}$ since its pole is at $z=-1$, hence we expand inwards, and so it is expanded in Taylor series

$$f(z)=\overbrace{\frac{2}{z-1}}^{Laurent}+\overbrace{\frac{1}{z+1}}^{Taylor}$$

Looking at the second term above, expand in Taylor series

$$\begin{aligned}\frac{1}{z+1}&=\frac{1}{(z-1)+1+1}=\frac{1}{(z-1)+2}=\frac{1}{2}\frac{1}{1+\frac{1}{2}(z-1)}\\&=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n(z-1)^n\qquad |z-1|<2\end{aligned}$$

Therefore, for region A

$$\begin{aligned}f(z)&=\frac{2}{z-1}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n(z-1)^n\\&=\frac{2}{z-1}+\frac{1}{2}\left(1-\frac{1}{2}(z-1)+\frac{1}{4}(z-1)^2-\frac{1}{16}(z-1)^3+\cdots\right)\\&=\frac{2}{z-1}+\frac{1}{2}-\frac{1}{4}(z-1)+\frac{1}{8}(z-1)^2-\frac{1}{16}(z-1)^3+\cdots\end{aligned}$$

This is valid for $0<|z-1|<2, z\neq 1$

**Region B**

This is the region outside the large circle to infinity. Since expanding outwards, both terms will use Laurent series now.

$$f(z)=\overbrace{\frac{2}{z-1}}^{Laurent}+\overbrace{\frac{1}{z+1}}^{Laurent}$$

$\frac{2}{z-1}$ is already in Laurent series, for the second term

$$\begin{aligned}\frac{1}{z+1}&=\frac{1}{(z-1)+1+1}=\frac{1}{(z-1)+2}=\frac{1}{(z-1)}\frac{1}{1+\frac{2}{z-1}}\\&=\frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}\qquad |z-1|>2\end{aligned}$$

Hence

$$
\begin{aligned}
f(z) &= \frac{2}{z-1}+\frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}\\
&= \frac{2}{z-1}+\frac{1}{(z-1)}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-\frac{8}{(z-1)^3}+\cdots\right)\\
&= \frac{2}{z-1}+\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\cdots\\
&= \frac{3}{z-1}-\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\cdots
\end{aligned}
$$

This is valid for $|z-1|>2$

### 2.14.3 problem 1

Find the Fourier expansion of the periodic function whose definition on one period is

$$
f(t)=\begin{cases} 0 & -\pi\le t\le 0\\ \sin t & 0\le t\le \pi\end{cases}
$$

**Solution**

The function is The period $T=2\pi$. This function is neither even nor odd. Using the complex expansion form

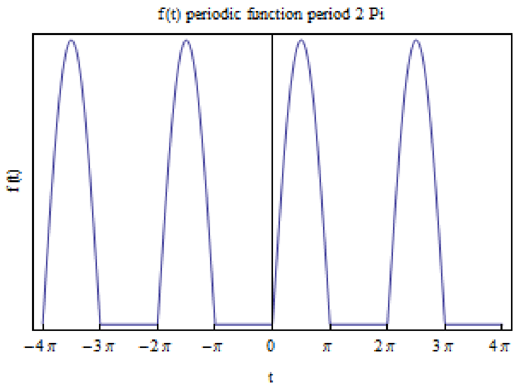


Figure 2.27: The periodic function $f(t)$

$$
\tilde{f}(t)=\sum_{n=-\infty}^{\infty}c_n e^{i\frac{2\pi}{T}nt}=\sum_{n=-\infty}^{\infty}c_n e^{int}
$$

$$
c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,e^{-i\frac{2\pi}{T}nt}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}
$$

Hence

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(t)\,e^{-int}dt+\int_{0}^{\pi} f(t)\,e^{-int}dt\right)\\
&= \frac{1}{2\pi}\int_{0}^{\pi}\sin(t)\,e^{-int}dt
\end{aligned}
$$

Integration by parts, $\int udv=uv-\int vdu$, let $u=\sin(t)\rightarrow du=\cos(t)$ and $dv=e^{-int}\rightarrow v=\frac{e^{-int}}{-in}$, therefore

$$\begin{aligned}
I &= [uv]_0^\pi - \int_0^\pi v du\\
&= \left[\sin(t)\frac{e^{-int}}{-in}\right]_0^\pi - \int_0^\pi \cos(t)\frac{e^{-int}}{-in}dt\\
&= \overbrace{\left[\sin(\pi)\frac{e^{-in\pi}}{-in} - \sin(0)\frac{e^{-in0}}{-in}\right]_0^\pi}^{0} + \frac{1}{in}\int_0^\pi \cos(t)e^{-int}dt\\
&= \frac{1}{in}\int_0^\pi \cos(t)e^{-int}dt
\end{aligned}$$

Integration by parts again, $\int udv = uv - \int vdu$, let $u = \cos(t) \rightarrow du = -\sin(t)$ and $dv = e^{-int} \rightarrow v = \frac{e^{-int}}{-in}$, therefore

$$\begin{aligned}
I &= \frac{1}{in}\left(\left[\cos(t)\frac{e^{-int}}{-in}\right]_0^\pi - \int_0^\pi -\sin(t)\frac{e^{-int}}{-in}dt\right)\\
&= \frac{1}{in}\left(\left[\cos(\pi)\frac{e^{-in\pi}}{-in} - \cos(0)\frac{e^{-in0}}{-in}\right] - \frac{1}{in}\int_0^\pi \sin(t)e^{-int}dt\right)\\
&= \frac{1}{in}\left(\left[\frac{e^{-in\pi}}{in} + \frac{1}{in}\right] - \frac{1}{in}\int_0^\pi \sin(t)e^{-int}dt\right)
\end{aligned}$$

But $e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n$ for integer $n$, hence

$$\begin{aligned}
I &= \frac{1}{in}\left(\frac{(-1)^n + 1}{in} - \frac{1}{in}\int_0^\pi \sin(t)e^{-int}dt\right)\\
&= \frac{(-1)^n + 1}{-n^2} + \frac{1}{n^2}\int_0^\pi \sin(t)e^{-int}dt
\end{aligned}$$

But $\int_0^\pi \sin(t)e^{-int}dt$ in the RHS above is $I$ itself. Hence solving for $I$ gives

$$\begin{aligned}
I &= \frac{(-1)^n + 1}{-n^2} + \frac{1}{n^2}I\\
I\left(1 - \frac{1}{n^2}\right) &= \frac{(-1)^n + 1}{-n^2}\\
I\left(\frac{n^2 - 1}{n^2}\right) &= \frac{(-1)^n + 1}{-n^2}\\
I &= \frac{(-1)^n + 1}{-n^2}\frac{n^2}{n^2 - 1}\\
&= \frac{(-1)^n + 1}{1 - n^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
c_n &= \frac{1}{2\pi}I\\
&= \frac{(-1)^n + 1}{2\pi\left(1 - n^2\right)}
\end{aligned}$$

And hence

$$\begin{aligned}
\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n + 1}{2\pi\left(1 - n^2\right)}e^{int}
\end{aligned}$$

For $c_0$

$$c_0 = \frac{1}{\pi}$$

Looking at some terms, for $n = 1$ and $n = -1$, there can not be substituted in the denominator since that will produce a division by 0, hence L'hospital rule is used. Since $-1 = e^{i\pi}$ then write $-1^n = e^{i\pi n}$ to simplify taking derivatives.

$$\lim_{n->1} \frac{f(n)}{g(n)} = \lim_{n->1} \frac{f'(n)}{g'(n)} = \lim_{n->1} \frac{\frac{d}{dn}\left((-1)^n + 1\right)}{\frac{d}{dn}\left(2\pi\left(1-n^2\right)\right)} = \lim_{n->1} \frac{i\pi e^{i\pi n}}{-4\pi n} = \frac{i\pi e^{i\pi}}{-4\pi} = \frac{-i\pi}{-4\pi} = \frac{i}{4}$$

and

$$\lim_{n->-1} \frac{f(n)}{g(n)} = \lim_{n->-1} \frac{f'(n)}{g'(n)} = \lim_{n->-1} \frac{\frac{d}{dn}\left((-1)^n + 1\right)}{\frac{d}{dn}\left(2\pi\left(1-n^2\right)\right)} = \lim_{n->-1} \frac{i\pi e^{i\pi n}}{-4\pi n} = \frac{i\pi e^{-i\pi}}{4\pi} = \frac{-i\pi}{4\pi} = \frac{-i}{4}$$

For all other odd $n$, all terms cancel. This means that for $n = \pm 3, \pm 5, \cdots$ all $c_n$ terms are zero. For example, for $n = 3$ the term is $c_3 = \frac{(-1)^3+1}{\pi(1-9)} = 0$ since all the numerators are zero for *odd n*.

For all even $n$, terms with $n = \pm 2, \pm 4, \cdots$ are combined into one term as follows

$$n = 2 \rightarrow c_2 e^{int} = \frac{(-1)^n + 1}{2\pi\left(1-n^2\right)} e^{int} = \frac{(-1)^2 + 1}{2\pi(1-4)} e^{i2t} = \frac{1}{-3\pi} e^{i2t}$$

$$n = -2 \rightarrow c_{-2} e^{int} = \frac{(-1)^n + 1}{2\pi\left(1-n^2\right)} e^{int} = \frac{(-1)^{-2} + 1}{2\pi(1-4)} e^{-i2t} = \frac{1}{-3\pi} e^{-i2t}$$

Hence

$$c_2 e^{i2t} + c_{-2} e^{-2it} = \frac{1}{-3\pi}\left(e^{i2t} + e^{-2it}\right) = \frac{2}{-3\pi}\cos(2t)$$

The same for all other even $n$. Therefore, all the *odd n* terms vanish other than for $n = \pm 1$ and all the even values produce a cosine terms. Hence

$$\begin{aligned}
\tilde{f}(t) &= \overbrace{\frac{1}{\pi}}^{c_0} + \overbrace{\left(\frac{i}{4}e^{it} + \frac{-i}{4}e^{-it}\right)}^{c_1 e^{it} + c_{-1}e^{-it}} + \sum_{n=2,4,6,\cdots}^{\infty} \frac{2}{2\pi\left(1-n^2\right)}\left(e^{int} + e^{-int}\right) \\
&= \frac{1}{\pi} + \frac{1}{2}\left(\frac{-1}{2i}e^{-it} + \frac{1}{2i}e^{it}\right) + \frac{2}{\pi}\sum_{n=2,4,6,\cdots}^{\infty} \frac{\cos nt}{\left(1-n^2\right)} \\
&= \frac{1}{\pi} + \frac{1}{2}\left(\frac{e^{it} - e^{-it}}{2i}\right) + \frac{2}{\pi}\sum_{n=2,4,6,\cdots}^{\infty} \frac{\cos nt}{\left(1-n^2\right)}
\end{aligned}$$

But $\left(\frac{e^{it}-e^{-it}}{2i}\right) = \sin t$ hence

$$\tilde{f}(t) = \frac{1}{2}\sin(t) + \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=2,4,6,\cdots}^{\infty} \frac{\cos nt}{\left(1-n^2\right)}$$

Therefore

$$\tilde{f}(t) = \frac{1}{2}\sin(t) + \frac{1}{\pi} + \frac{2}{\pi}\frac{\cos 2t}{-3} + \frac{2}{\pi}\frac{\cos 4t}{-15} + \frac{2}{\pi}\frac{\cos 6t}{-35} + \cdots$$

To verify this is correct, here is a plot of the Fourier series approximation of $f(t)$ for increasing number of terms.

```
f[t_, m_] :=  Sin[t]/2 + 1/Pi + 2/Pi  Sum[ Cos[n t]/(1 - n^2), {n, 2, m, 2}]
p[n_] := Plot[f[t, n], {t, -3 Pi, 3 Pi},
   Ticks -> {{-6 Pi, -4 Pi, -3 Pi, -2 Pi, -Pi, 0, Pi, 2 Pi, 3 Pi,
      4 Pi, 6 Pi}, Automatic}, AxesLabel -> Row[{"N=", n}],
   ImageSize -> 300, ImagePadding -> 40];
Grid[Partition[Table[p[n], {n, 2, 8, 2}], 2], Frame -> All]
```

### 2.14.4 problem 2

Find the solution of $y'' + 9y = f(t); y(0) = y'(0) = 0$ and $f(t) = |t|$ for $-\pi \leq t \leq \pi$

**solution**

The function $f(t)$ is

```
f[t_] := Abs[t];
```

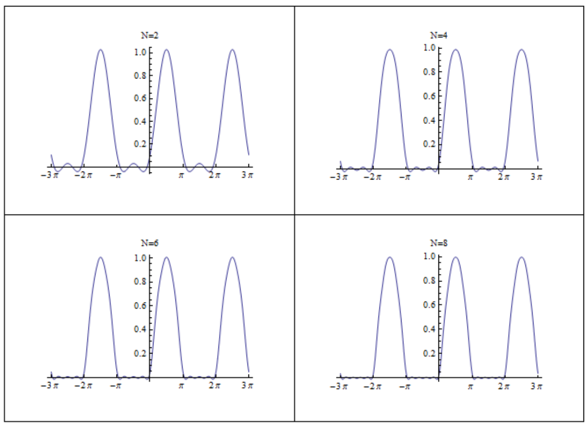
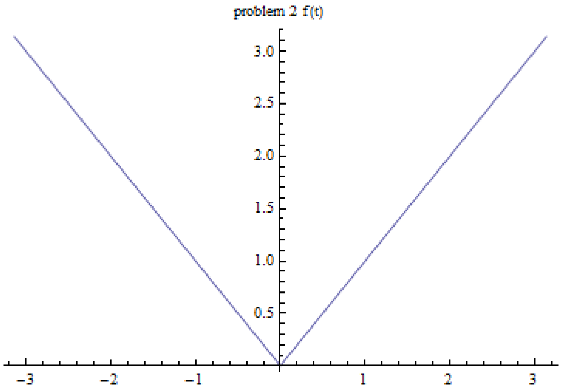


Figure 2.28: Fourier series approximation. $N = 8$ is needed to obtain good approximation.

```
Plot[f[t], {t, -Pi, Pi}, AxesLabel -> "problem 2 f(t)"]
```



The problem did not say if this is one period of a periodic function of if the function is zero beyond the given range. I assumed it is periodic and hence using Fourier series to solve this.

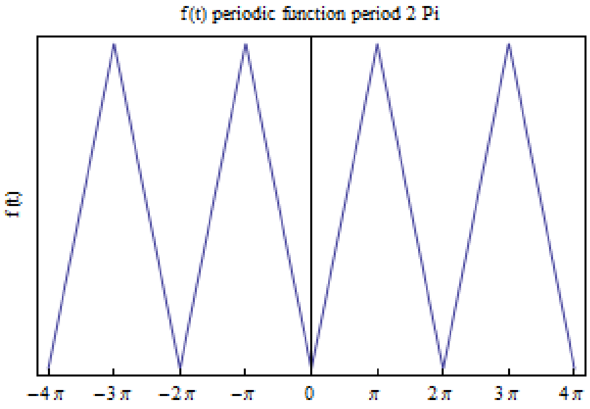


Figure 2.29: The input assumed to be periodic of period $2\pi$

The period is therefore $T = 2\pi$ and using Fourier series gives

$$\begin{aligned}\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int}\\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -te^{-int}dt + \frac{1}{2\pi}\int_{0}^{\pi} te^{-int}dt\end{aligned}$$

For $n = 0, c_0 = \frac{1}{2\pi}\int_{-\pi}^{0} -tdt + \frac{1}{2\pi}\int_{0}^{\pi} tdt = \frac{-1}{2\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_{0}^{\pi} = \frac{-1}{4\pi}\left[0-(-\pi)^2\right] + \frac{1}{4\pi}\left[\pi^2\right]$, hence

$$\begin{aligned}c_0 &= \frac{1}{4\pi}\pi^2 + \frac{1}{4\pi}\pi^2\\ &= \frac{\pi}{2}\end{aligned}$$

For other $n$ values, Integrate by parts. For the first integral,

$$I_1 = \frac{-1}{2\pi}\int_{-\pi}^{0} te^{-int}dt$$

it becomes $\int te^{-int}dt \Rightarrow [uv] - \int vdu$, let $u = t, dv = e^{-int}$, hence $du = 1; v = \frac{e^{-int}}{-in}$ and $\int te^{-int}dt = \left[\frac{te^{-int}}{-in}\right] + \frac{1}{in}\int e^{-int}dt$

Therefore

$$\begin{aligned} I_1 &= \frac{-1}{2\pi}\left(\left[\frac{te^{-int}}{-in}\right]_{-\pi}^{0} + \frac{1}{in}\int_{-\pi}^{0} e^{-int}dt\right) \\ &= \frac{-1}{2\pi}\left(-\frac{1}{in}\left[0 - \left(-\pi e^{in\pi}\right)\right] + \frac{1}{in}\left[\frac{e^{-int}}{-in}\right]_{-\pi}^{0}\right) \\ &= \frac{-1}{2\pi}\left(\frac{-1}{in}\pi e^{in\pi} + \frac{1}{n^2}\left[e^{-int}\right]_{-\pi}^{0}\right) \\ &= \frac{-1}{2\pi}\left(\frac{-1}{in}\pi e^{in\pi} + \frac{1}{n^2}\left[1 - e^{in\pi}\right]\right) \\ &= \frac{-1}{2\pi}\left(\frac{-1}{in}\pi e^{in\pi} + \frac{1}{n^2} - \frac{1}{n^2}e^{in\pi}\right) \end{aligned}$$

For the second integral,

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_{0}^{\pi} te^{-int}dt \\ &= \frac{1}{2\pi}\left(\left[\frac{te^{-int}}{-in}\right]_{0}^{\pi} + \frac{1}{in}\int_{0}^{\pi} e^{-int}dt\right) \\ &= \frac{1}{2\pi}\left(\frac{-1}{in}\left[\pi e^{-in\pi} - 0\right] + \frac{1}{in}\left[\frac{e^{-int}}{-in}\right]_{0}^{\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{-1}{in}\pi e^{-in\pi} + \frac{1}{n^2}\left[e^{-int}\right]_{0}^{\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi e^{-in\pi}}{in} + \frac{1}{n^2}\left[e^{-in\pi} - 1\right]\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi e^{-in\pi}}{in} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}\right) \end{aligned}$$

Hence

$$\begin{aligned} c_n &= I_1 + I_2 \\ &= \frac{-1}{2\pi}\left(\frac{-\pi e^{in\pi}}{in} + \frac{1}{n^2} - \frac{1}{n^2}e^{in\pi}\right) + \frac{1}{2\pi}\left(\frac{-\pi e^{-in\pi}}{in} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi e^{in\pi}}{in} - \frac{1}{n^2} + \frac{1}{n^2}e^{in\pi} + \frac{-\pi e^{-in\pi}}{in} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi}{n}\left(\frac{e^{in\pi}}{i} - \frac{e^{-in\pi}}{i}\right) + \frac{1}{n^2}\left(e^{in\pi} + e^{-in\pi}\right) - \frac{2}{n^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi}{n}\overbrace{(2\sin n\pi)}^{0} + \frac{1}{n^2}(2\cos n\pi) - \frac{2}{n^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{n^2}2(-1)^n - \frac{2}{n^2}\right) \\ &= \frac{1}{\pi n^2}\left(-1^n - 1\right) \end{aligned}$$

The above is for all $n$ other than $n = 0$ where $c_0 = \frac{\pi}{2}$. The above shows that for $c_n = 0$ for even $n$ and $c_n = \frac{-2}{\pi n^2}$ for odd $n$. Hence, by combining negative and positive $n$

$$
\begin{aligned}
\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int} \\
&= \frac{\pi}{2} + \sum_{\substack{n=-\infty \\ odd\ n, n\neq 0}}^{\infty} \frac{-2}{\pi n^2} e^{int} \\
&= \frac{\pi}{2} + \sum_{\substack{n=-\infty \\ odd\ n, n\neq 0}}^{\infty} \frac{-2}{\pi n^2} e^{int}
\end{aligned}
$$

To verify, here is a plot showing the approximation

```
f[t_, m_] := Pi/2 + Sum[If[n == 0, 0, -2/(Pi n^2) Exp[I n t]], {n, -m, m, 2}]
p[n_] := Plot[f[t, n], {t, -3 Pi, 3 Pi},
   Ticks -> {{-6 Pi, -4 Pi, -3 Pi, -2 Pi, -Pi, 0, Pi, 2 Pi, 3 Pi,
      4 Pi, 6 Pi}, Automatic}, AxesLabel -> Row[{"N=", n}],
   ImageSize -> 300, ImagePadding -> 40, PlotRange -> All];

Grid[Partition[Table[p[n], {n, 1, 7, 2}], 2], Frame -> All]
```

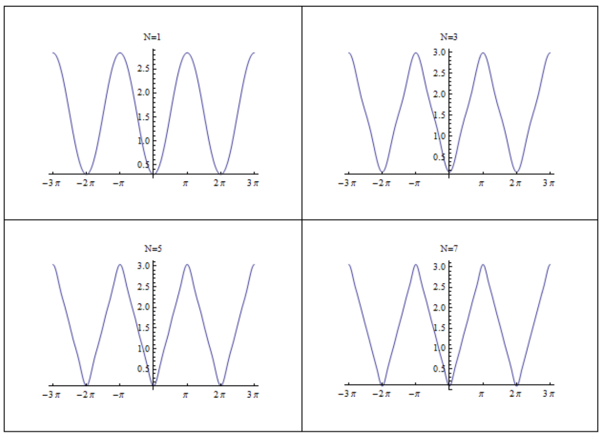


Figure 2.30: Approximation of the forcing function using Fourier series

Therefore, the differential equation is

$$
y'' + 9y = \frac{\pi}{2} + \sum_{\substack{n=-\infty \\ odd\ n, n\neq 0}}^{\infty} \frac{-2}{\pi n^2} e^{int}
$$

The homogeneous solution is now found

$$
y_h'' + 9y_h = 0
$$

The characteristic polynomial is $\lambda^2 + 9 = 0$, hence $\lambda = \pm i3$ and the solution is

$$
y_h = A\cos 3t + B\sin 3t
$$

For $y_p$, assume it has a Fourier series (using same frequency as forcing function, since linear system)

$$
\begin{aligned}
y_p &= C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{int} \\
y_p' &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} inC_n e^{int} \\
y_p'' &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -n^2 C_n e^{int}
\end{aligned}
$$

Hence the DE becomes

$$
\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -n^2 C_n e^{int} + 9C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} 9C_n e^{int} = \frac{\pi}{2} + \sum_{\substack{n=-\infty \\ odd\ n, n\neq 0}}^{\infty} \frac{-2}{\pi n^2} e^{int}
$$

For $n = 0$,

$$\begin{aligned} 9C_0 &= \frac{\pi}{2} \\ C_0 &= \frac{\pi}{18} \end{aligned}$$

For all other $n$ the terms can be combined as

$$\begin{aligned} \left(-n^2 C_n + 9C_n\right) &= \frac{-2}{\pi n^2} \\ C_n &= \frac{-2}{\left(9-n^2\right)\pi n^2} \qquad n \neq 0, n \neq \pm 3, n \textit{ odd} \end{aligned}$$

Notice that for $n = \pm 3$ the above is not defined since the denominator becomes zero. So this term is not used (what else to do?)

Hence,

$$\begin{aligned} y_p &= \frac{\pi}{18} + \sum_{n=1,5,7,\cdots}^{\infty} \frac{-2}{\left(9-n^2\right)\pi n^2}\left(e^{int} + e^{-int}\right) \\ &= \frac{\pi}{18} + \sum_{n=1,5,7,\cdots}^{\infty} \frac{-2}{\left(9-n^2\right)\pi n^2} 2\cos nt \\ &= \overbrace{\frac{\pi}{18}}^{n=0} \overbrace{- \frac{1}{2\pi}(\cos t)}^{n=1} - \frac{4}{\pi} \sum_{n=5,7,9,\cdots}^{\infty} \frac{1}{\left(9-n^2\right)n^2}\cos nt \\ &= \frac{\pi}{18} - \frac{1}{2\pi}\cos t - \frac{4}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{\cos nt}{\left(9-n^2\right)n^2} \end{aligned}$$

The final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= A\cos 3t + B\sin 3t + \frac{\pi}{18} - \frac{1}{2\pi}\cos t - \frac{4}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{\cos nt}{\left(9-n^2\right)n^2} \end{aligned}$$

Constants from initial conditions are now found

$$\begin{aligned} y(0) = 0 &= A + \frac{\pi}{18} - \frac{1}{2\pi} - \frac{4}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{1}{\left(9-n^2\right)n^2} \\ A &= -\frac{\pi}{18} + \frac{2}{4\pi} - \frac{1}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{-4}{\left(9-n^2\right)n^2} \end{aligned}$$

But, using CAS, found that

$$\sum_{n=5,7,9\cdots}^{\infty} \frac{-4}{\left(9-n^2\right)n^2} = \frac{1}{162}\left(91 - 9\pi^2\right)$$

Hence

$$\begin{aligned} A &= -\frac{\pi}{18} + \frac{2}{4\pi} - \frac{1}{\pi}\frac{1}{162}\left(91 - 9\pi^2\right) \\ &= -\frac{5}{81\pi} \end{aligned}$$

Taking derivative of the general solution

$$y'(t) = -3A\sin 3t + 3B\cos 3t + \frac{\pi}{18} + \frac{2}{4\pi}\sin t + \frac{1}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{4n\sin(nt)}{\left(9-n^2\right)n^2}$$

Using initial conditions $y'(0) = 0$

$$0 = 3B$$

Hence $B = 0$, and solution is

$$y(t) = -\frac{5}{81\pi}\cos 3t + \frac{\pi}{18} - \frac{2}{4\pi}\cos t + \frac{1}{\pi}\sum_{n=5,7,9,\cdots}^{\infty} \frac{-4\cos(nt)}{\left(9-n^2\right)n^2}$$

### 2.14.5 problem 3

Find the Fourier transform of $f(t)=\begin{cases} 0 & -\infty<t\leq 0\\ \sin(t) & 0\leq t\leq\pi \\ 0 & \pi\leq t<\infty\end{cases}$

**Solution**

By definition

$$\begin{aligned}\mathscr{F}\big(f(t)\big) &= \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}dt\\ &= \int_0^{\pi}\sin(t)\,e^{-i\omega t}dt\end{aligned}$$

Let

$$I=\int_0^{\pi}\sin(t)\,e^{-i\omega t}dt$$

Integration by part $\int udv\Rightarrow[uv]-\int vdu$, let $u=\sin t, dv=e^{-i\omega t}$, hence $du=\cos t; v=\frac{e^{-i\omega t}}{-i\omega}$ hence

$$\begin{aligned}I&=\left[\sin t\frac{e^{-i\omega t}}{-i\omega}\right]_0^{\pi}-\int\cos t\frac{e^{-i\omega t}}{-i\omega}dt\\ &=0+\frac{1}{i\omega}\int\cos te^{-i\omega t}dt\\ &=\frac{1}{i\omega}\int\cos te^{-i\omega t}dt\end{aligned}$$

Integration by part $\int udv\Rightarrow[uv]-\int vdu$, let $u=\cos t, dv=e^{-i\omega t}$, hence $du=-\sin t; v=\frac{e^{-i\omega t}}{-i\omega}$ hence

$$\begin{aligned}I&=\frac{1}{i\omega}\left(\left[\cos t\frac{e^{-i\omega t}}{-i\omega}\right]_0^{\pi}-\frac{1}{i\omega}\overbrace{\int\sin te^{-i\omega t}dt}^{I}\right)\\ &=\frac{1}{i\omega}\left(\frac{-1}{i\omega}\left[\cos te^{-i\omega t}\right]_0^{\pi}-\frac{1}{i\omega}I\right)\\ &=\frac{1}{i\omega}\left(\frac{-1}{i\omega}\left[\cos\pi e^{-i\omega\pi}-1\right]-\frac{1}{i\omega}I\right)\\ &=\frac{1}{\omega^2}\left[\cos\pi e^{-i\omega\pi}-1\right]+\frac{1}{\omega^2}I\\ I\left(1-\frac{1}{\omega^2}\right)&=\frac{1}{\omega^2}\left[\cos\pi e^{-i\omega\pi}-1\right]\\ I\left(\frac{\omega^2-1}{\omega^2}\right)&=\frac{\cos\pi e^{-i\omega\pi}-1}{\omega^2}\\ I&=\frac{\cos\pi e^{-i\omega\pi}-1}{\omega^2-1}=\frac{-e^{-i\omega\pi}-1}{\omega^2-1}=\frac{e^{-i\omega\pi}+1}{1-\omega^2}\end{aligned}$$

Hence

$$\mathscr{F}\big(f(t)\big)=\frac{1+e^{-i\omega\pi}}{1-\omega^2}$$

### 2.14.6 problem 4

Find the inverse transform of $F(\omega)=\frac{\sin(\omega-2)}{\omega-2}$

**Solution**

The inverse Fourier transform of $F(\omega)$ is

$$\begin{aligned}f(x)=\mathscr{F}^{-1}(F(\omega))&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\,e^{i\omega x}d\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega-2)}{\omega-2}e^{i\omega x}d\omega\end{aligned}$$

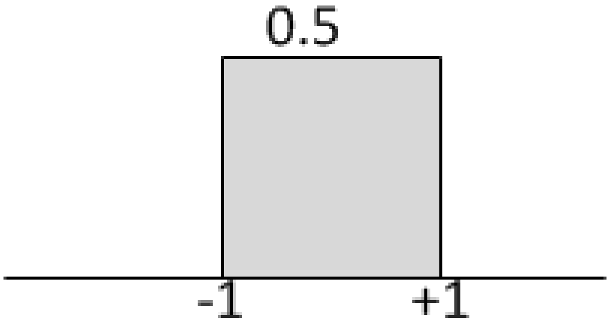
The first step is to use the frequency shift property

$$F(\omega-\omega_0) \iff e^{-i\omega_0 t} f(t) \tag{1}$$

In this case, $F(\omega-\omega_0) \equiv \frac{\sin(\omega-2)}{\omega-2}$ where $\omega_0 = 2$, hence

$$\frac{\sin(\omega-2)}{\omega-2} \iff e^{i2x}\mathscr{F}^{-1}\left(\frac{\sin\omega}{\omega}\right) \tag{2}$$

Now to find $\mathscr{F}^{-1}\left(\frac{\sin\omega}{\omega}\right)$. If we take the Fourier transform of a rectangular pulse centered at 0 with height $\frac{1}{2}$ we see it will give $\frac{\sin\omega}{\omega}$



This can be done as follows

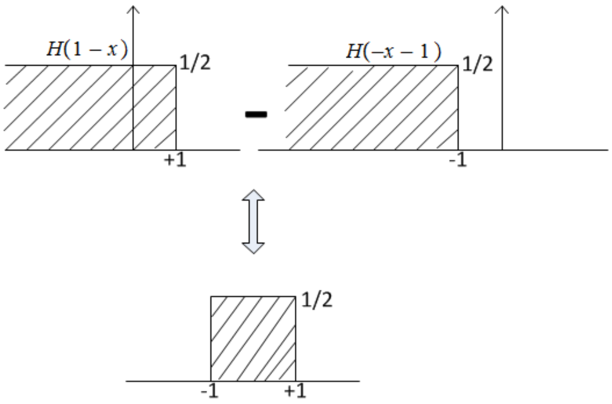
$$\begin{aligned}
F\left(\frac{1}{2}rect(x)\right) &= \int_{-1}^{1}\frac{1}{2}e^{-i\omega x}dx = \frac{1}{2}\left(\frac{e^{-i\omega x}}{-i\omega}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{e^{-i\omega}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right) = \frac{1}{2}\frac{-1}{\omega}\left(\frac{e^{-i\omega}-e^{i\omega}}{i}\right)\\
&= \frac{1}{2}\frac{1}{\omega}\left(\frac{e^{i\omega}-e^{-i\omega}}{i}\right)\\
&= \frac{1}{2}\frac{1}{\omega}(2\sin\omega)\\
&= \left(\frac{\sin\omega}{\omega}\right)
\end{aligned}$$

Therefore

$$\frac{\sin\omega}{\omega} \iff \frac{1}{2}rect(x)$$

But $rect(x)$ with height of $\frac{1}{2}$can be constructed from the difference of two unit step functions each of height $\frac{1}{2}$as shown below. Hence

$$\frac{1}{2}rect(x) = \frac{1}{2}H(1-x) - \frac{1}{2}H(-x-1)$$



Therefore

$$\begin{aligned}
\frac{\sin\omega}{\omega} &\iff \frac{1}{2}rect(x)\\
&\iff \frac{1}{2}(H(1-x)-H(-x-1))
\end{aligned}$$

Adding the frequency shifting from Eq. (2) gives the final result

$$\begin{aligned}\frac{\sin(\omega-2)}{\omega-2} &\Longleftrightarrow e^{i2x}\mathscr{F}^{-1}\left(\frac{\sin\omega}{\omega}\right)\\ &\Longleftrightarrow \frac{e^{i2x}}{2}\left(\frac{1}{2}\left(H(1-x)-H(-x-1)\right)\right)\\ &= \frac{e^{i2x}}{4}\left(H(1-x)-H(-x-1)\right)\end{aligned}$$

Or

$$\frac{\sin(\omega-2)}{\omega-2} \Longleftrightarrow \frac{1}{4}\left(\cos 2x + i\sin(2x)\right)\left(H(1-x)-H(-x-1)\right)$$

Here is a plot of the real part

```
f = 1/4 (Cos[2 x] + I Sin[2 x]) (UnitStep[1 - x] - UnitStep[-x - 1])
Plot[Re[f], {x, -3, 3}, Exclusions -> None]
```

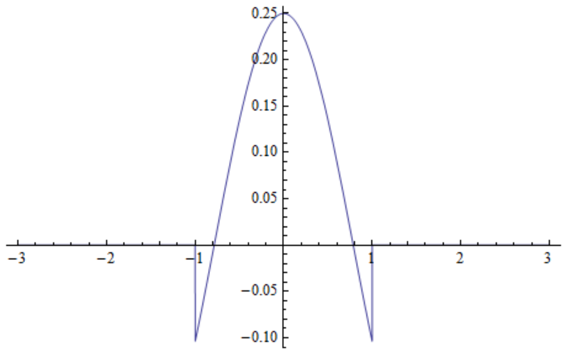


Figure 2.31: Real part of the inverse Fourier transform

Here is a plot of the imaginary part $\sin(2x)\frac{\pi}{2}(\operatorname{sign}(1-2\pi x)+\operatorname{sign}(1-2\pi x))$

```
Plot[Im[f], {x, -3, 3}, Exclusions -> None]
```

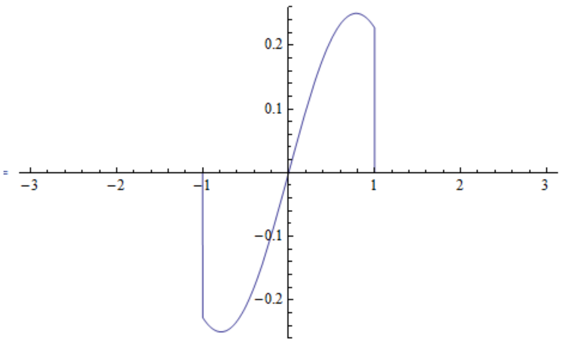


Figure 2.32: Complex part of solution

### 2.14.7 problem 5

Solve the integral equation $\psi(x) = 1+\lambda^2\int_0^x (t-x)\,\psi(t)\,dt$

**solution**

Taking derivative w.r.t. $x$

$$\begin{aligned}\psi'(x) &= \lambda^2\left[\int_0^x \frac{d}{dx}(t-x)\,\psi(t)\,dt + \frac{dx}{dx}\left((t-x)\,\psi(t)\right)_{t=x} - 0\right]\\ &= \lambda^2\left[\int_0^x \frac{d}{dx}\left(t\psi(t)-x\psi(t)\right)dt + \overbrace{\left((x-x)\,\psi(x)\right)}^{0}\right]\\ &= -\lambda^2\int_0^x \psi(t)\,dt\end{aligned}$$

Taking another derivative

$$\begin{aligned}\psi''(x) &= -\lambda^2\left[\int_0^x \frac{d}{dx}\psi(t)\,dt + \frac{dx}{dx}\left(\psi(t)\right)_{t=x} - 0\right]\\ &= -\lambda^2\psi(x)\end{aligned}$$

Hence

$$\psi''(x) + \lambda^2\psi(x) = 0$$

The solution is

$$\psi(x) = A\cos\lambda x + B\sin\lambda x$$

When $x = 0$ then

$$\begin{aligned}\psi(0) &= 1 + \lambda^2\int_0^0 (t-x)\,\psi(t)\,dt\\ &= 1\end{aligned}$$

Hence

$$\begin{aligned}A\cos(\lambda 0) + B\sin(\lambda 0) &= 1\\ A &= 1\end{aligned}$$

When $x = 0$ we also have

$$\begin{aligned}\psi'(0) &= -\lambda^2\int_0^0 \psi(t)\,dt\\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}0 &= -A\lambda\sin\lambda x + B\lambda\cos\lambda x\\ &= B\lambda\end{aligned}$$

Hence $B = 0$ and therefore the final solution is

$$\psi(x) = \cos(\lambda x)$$

### 2.14.8 problem 6

Solve the integral equation $\frac{dy(t)}{dt} = 1 - \sin(t) - \int_0^t y(\tau)\,d\tau$ with IC $y(0) = 0$

**Solution**:

Taking derivative w.r.t. $t$ gives

$$\begin{aligned}y''(t) &= -\cos(t) - \left[\int_0^t \frac{d}{dt}y(\tau)\,d\tau + \frac{dt}{dt}y(t) - 0\right]\\ &= -\cos(t) - y(t)\end{aligned}$$

Hence

$$y''(t) + y(t) = -\cos(t)$$

Hence $y_h(t) = A\cos t + B\sin t$ and for particular solution, since the homogeneous solution contains $\cos$ then add $t$ to each trial solution, hence let

$$y_p = c_1 t\cos t + c_2 t\sin t$$

Therefore

$$\begin{aligned}y_p' &= c_1(\cos t - t\sin t) + c_2(\sin t + t\cos t)\\ &= c_1\cos t - c_1 t\sin t + c_2\sin t + c_2 t\cos t\\ y_p'' &= c_1(-\sin t - (\sin t + t\cos t)) + c_2(\cos t + (\cos t - t\sin t))\\ &= -c_1\sin t - c_1\sin t - c_1 t\cos t + c_2\cos t + c_2\cos t - c_2 t\sin t\end{aligned}$$

DE becomes

$$\begin{aligned}-c_1\sin t - c_1\sin t - c_1 t\cos t + c_2\cos t + c_2\cos t - c_2 t\sin t + (c_1 t\cos t + c_2 t\sin t) &= -\cos(t)\\ \cos t(-c_1 t + 2c_2 + c_1 t) + \sin t(-2c_1 - c_2 t + c_2 t) &= -\cos(t)\\ \cos t(2c_2) + \sin t(-2c_1) &= -\cos(t)\end{aligned}$$

Hence

$$\begin{aligned} 2c_2 &= -1 \\ c_2 &= \frac{-1}{2} \end{aligned}$$

And

$$\begin{aligned} -2c_1 &= 0 \\ c_1 &= 0 \end{aligned}$$

Hence

$$y_p = \frac{-1}{2} t \sin t$$

Hence the general solution is

$$y(t) = A\cos t + B\sin t - \frac{1}{2} t \sin t$$

Since $y(0) = 0$ then

$$0 = A$$

And the solution becomes

$$y(t) = B\sin t - \frac{1}{2} t \sin t$$

But

$$\begin{aligned} y'(t) &= B\cos t - \frac{1}{2}(\sin t + t\cos t) \\ y'(0) &= B \end{aligned}$$

But from the integral equation $y'(0) = 1 - \sin(0) - \int_0^0 y(\tau)\, d\tau = 1$Hence

$$B = 1$$

And the solution is

$$y(t) = \sin t - \frac{1}{2} t \sin t$$

### 2.14.9 problem 7

Given the matrix equation $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}$ show that the particular solution integral is given by $\boldsymbol{v}(t) = X(t)\int X^{-1}(t)\boldsymbol{f}(t)\,dt$ where $X(t)$ is the fundamental matrix for the homogeneous equation

**Solution**

Will use $\Omega(t)$ for the fundamental matrix and not $X(t)$ since that is what we used before and in the text book, to make it less confusing. Hence the problem is to show that particular solution integral is given by $\boldsymbol{v}(t) = \Omega(t)\int \Omega^{-1}(t)\boldsymbol{f}(t)\,dt$ where $\Omega(t)$ is the fundamental matrix for the homogeneous equation

$$\begin{aligned} \boldsymbol{x}' &= A\boldsymbol{x} + \boldsymbol{f} \\ \boldsymbol{x}' - A\boldsymbol{x} &= \boldsymbol{f} \end{aligned}$$

The homogeneous solution is the solution to

$$\boldsymbol{x}_h' - A\boldsymbol{x}_h = 0$$

The homogeneous solution is given by

$$\boldsymbol{x}_h = \Omega(t)\boldsymbol{c}$$

Where $\Omega(t)$ is the fundamental matrix and $\boldsymbol{c}$ is vector of the constants of integration. The

Now the particular solution $\boldsymbol{v}_p$ is found using the variation of parameters method for systems of equations.

$$\boldsymbol{x} = \boldsymbol{x}_h + \boldsymbol{v}_p$$

To find the particular solution, assume

$$\boldsymbol{v}_p = \Omega\boldsymbol{u} \tag{1}$$

And $u$ is now found.

$$v_p' = \Omega' u + \Omega u'$$

Substituting the above in the original system ode $x' = Ax + f$ gives

$$\begin{aligned} v_p' &= Av_p + f \\ \Omega' u + \Omega u' &= (A\Omega)\, u + f \end{aligned}$$

But $\Omega' = A\Omega$ (since $\Omega$ is a fundamental solution, hence it satisfies the homogeneous DE) then the above simplifies to

$$\begin{aligned} \Omega' u + \Omega u' &= \Omega' u + f \\ \Omega u' &= f \\ u' &= \Omega^{-1} f \\ u &= \int \Omega^{-1} f \mathrm{d}t \end{aligned}$$

Hence, Eq (1) becomes

$$v(t) = \Omega(t) \int \Omega^{-1}(t) f(t)\, \mathrm{d}t$$

Which is what we are asked to show.

### 2.14.10 problem 8

Obtain solution to

$$\begin{aligned} x' + y' + x &= -e^{-t} \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$

With IC $x(0) = -1, y(0) = 1$

**Solution**:

The first step is to convert the system to $x' = Ax + f$ form. From first equation $x' = -y' - x - e^{-t}$, substituting this in the second equation gives

$$-y' - x - e^{-t} + 2y' + 2x + 2y = 0$$

Solving for $y'$ gives

$$y' = -x - 2y + e^{-t}$$

Substituting this in the first equation gives

$$\begin{aligned} x' + e^{-t} - x - 2y + x &= -e^{-t} \\ x' &= -e^{-t} - e^{-t} + x + 2y - x \\ &= 2y - 2e^{-t} \end{aligned}$$

Therefore the system becomes

$$\begin{aligned} x' &= 2y - 2e^{-t} \\ y' &= -x - 2y + e^{-t} \end{aligned}$$

Or

$$\overbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}^{x'} = \overbrace{\begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x \\ y \end{pmatrix}}^{x} + \overbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}}^{f}$$

To find the eigenvalues of $A$, the roots of the characteristic polynomial are found.

$$\begin{aligned}|A-\lambda I| &= 0\\ \left|\begin{pmatrix}0 & 2\\ -1 & -2\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| &= 0\\ \left|\begin{pmatrix}-\lambda & 2\\ -1 & -2-\lambda\end{pmatrix}\right| &= 0\\ (-\lambda)(-2-\lambda)+2 &= 0\\ \lambda^2+2\lambda+2 &= 0\end{aligned}$$

The roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$, hence eigenvalues are

$$\{-1-i,-1+i\}$$

To find the eigenvectors

$$\begin{aligned}(A-I\lambda)v_1 &= 0\\ \begin{pmatrix}-\lambda & 2\\ -1 & -2-\lambda\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{21}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

For $\lambda_1=-1-i$, hence

$$\begin{aligned}\begin{pmatrix}-(-1-i) & 2\\ -1 & -2-(-1-i)\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{21}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}1+i & 2\\ -1 & -1+i\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{21}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}(1+i)v_{11}+2v_{21}\\ -v_{11}+(-1+i)v_{21}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

Let $v_{21}=1$, and find $v_{11}$, hence from first equation $(1+i)v_{11}+2=0$ then $v_{11}=\frac{-2}{1+i}=\frac{-2(1-i)}{(1+i)(1-i)}=\frac{-2+2i}{1+1}=-1+i$

Hence the first eigenvector is

$$v_1=\begin{pmatrix}-1+i\\ 1\end{pmatrix}$$

For $\lambda_1=-1+i$,

$$\begin{aligned}\begin{pmatrix}-(-1+i) & 2\\ -1 & -2-(-1+i)\end{pmatrix}\begin{pmatrix}v_{12}\\ v_{22}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}1-i & 2\\ -1 & -1-i\end{pmatrix}\begin{pmatrix}v_{12}\\ v_{22}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}(1-i)v_{12}+2v_{22}\\ -v_{12}+(-1-i)v_{21}\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

Let $v_{22}=1$, and find $v_{12}$, hence from first equation $(1-i)v_{12}+2=0$ then $v_{12}=\frac{-2}{1-i}=\frac{-2(1+i)}{(1-i)(1+i)}=\frac{-2-2i}{1+1}=-1-i$

Hence the second eigenvector is

$$v_2=\begin{pmatrix}-1-i\\ 1\end{pmatrix}$$

Therefore, the matrix of eigenvectors is

$$\begin{aligned}P &= \begin{pmatrix}v_1 & v_2\end{pmatrix}\\ &= \begin{pmatrix}-1+i & -1-i\\ 1 & 1\end{pmatrix}\end{aligned}$$

And the matrix of eigenvalues is

$$D = \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}$$

Hence the fundamental matrix is

$$\Omega = \begin{pmatrix} (-1+i)\,e^{(-1-i)t} & (-1-i)\,e^{(-1+i)t} \\ e^{(-1-i)t} & e^{(-1+i)t} \end{pmatrix}$$

Now the system is decoupled in order to solve it. Let $\boldsymbol{x} = P\boldsymbol{z}$ hence $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}$ becomes

$$P\boldsymbol{z}' = AP\boldsymbol{z} + f$$

Premultiply by $P^{-1}$ simplifies to (since $P^{-1}P = I$)

$$\begin{aligned} \boldsymbol{z}' &= P^{-1}AP\boldsymbol{z} + P^{-1}\boldsymbol{f} \\ &= D\boldsymbol{z} + \boldsymbol{G} \end{aligned}$$

Where

$$\boldsymbol{G} = P^{-1}\boldsymbol{f} = \begin{pmatrix} -1+i & -1-i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}$$

But $\begin{pmatrix} -1+i & -1-i \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1-i \\ i & 1+i \end{pmatrix}$, so

$$\begin{aligned} \boldsymbol{G} &= \frac{1}{2}\begin{pmatrix} -i & 1-i \\ i & 1+i \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} \\ &= \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} e^{-t} \end{aligned}$$

The above system is decoupled since $D$ is diagonal

$$\begin{aligned} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} &= \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \boldsymbol{G} \\ &= \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} (-1-i)\,z_1 + \frac{1}{2}(1+i)\,e^{-t} \\ (-1+i)\,z_2 + \frac{1}{2}(1-i)\,e^{-t} \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} z_1'(t) &= (-1-i)\,z_1(t) + \frac{1}{2}(1+i)\,e^{-t} \\ z_2'(t) &= (-1+i)\,z_2(t) + \frac{1}{2}(1-i)\,e^{-t} \end{aligned}$$

The first equation gives

$$z_1'(t) + (1+i)\,z_1(t) = \frac{1}{2}(1+i)\,e^{-t}$$

The integrating factor is $e^{\int (1+i)dt} = e^{(1+i)t}$, hence the solution is

$$\begin{aligned}
d\left(e^{(1+i)t}z_1(t)\right) &= e^{(1+i)t}\frac{1}{2}(1+i)e^{-t}\\
e^{(1+i)t}z_1(t) &= \int e^{(1+i)t}\frac{1}{2}(1+i)e^{-t}dt + C_1\\
z_1(t) &= e^{-(1+i)t}\int e^{(1+i)t}\frac{1}{2}(1+i)e^{-t}dt + e^{-(1+i)t}C_1\\
&= \frac{1}{2}e^{-(1+i)t}\int e^{it}(1+i)\,dt + e^{-(1+i)t}C_1\\
&= \frac{1}{2}e^{-(1+i)t}\int \left(e^{it} + ie^{it}\right)dt + e^{-(1+i)t}C_1\\
&= \frac{1}{2}e^{-t}e^{-it}\left(\frac{e^{it}}{i} + i\frac{e^{it}}{i}\right) + e^{-(1+i)t}C_1\\
&= \frac{1}{2}e^{-t}e^{-it}\left(-ie^{it} + e^{it}\right) + e^{-(1+i)t}C_1\\
&= \frac{1}{2}e^{-t}(-i+1) + e^{-(1+i)t}C_1\\
&= -\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1+i)t}C_1
\end{aligned}$$

And

$$\begin{aligned}
z_2'(t) &= (-1+i)z_2(t) + \frac{1}{2}(1-i)e^{-t}\\
z_2'(t) + (1-i)z_2(t) &= \frac{1}{2}(1-i)e^{-t}
\end{aligned}$$

The integrating factor is $e^{\int (1-i)dt} = e^{(1-i)t}$, hence the solution is

$$\begin{aligned}
d\left(e^{(1-i)t}z_2(t)\right) &= e^{(1-i)t}\frac{1}{2}(1-i)e^{-t}\\
e^{(1-i)t}z_2(t) &= \int e^{(1-i)t}\frac{1}{2}(1-i)e^{-t}dt + C_2\\
z_2(t) &= e^{-(1-i)t}\int e^{(1-i)t}\frac{1}{2}(1-i)e^{-t}dt + e^{-(1-i)t}C_2\\
&= \frac{1}{2}e^{-(1-i)t}\int e^{-it}(1-i)\,dt + e^{-(1-i)t}C_2\\
&= \frac{1}{2}e^{-(1-i)t}\int \left(e^{-it} - ie^{-it}\right)dt + e^{-(1-i)t}C_2\\
&= \frac{1}{2}e^{-t}e^{it}\left(\frac{e^{-it}}{-i} - i\frac{e^{-it}}{-i}\right) + e^{-(1-i)t}C_2\\
&= \frac{1}{2}e^{-t}e^{it}\left(ie^{-it} + e^{-it}\right) + e^{-(1-i)t}C_2\\
&= \frac{1}{2}e^{-t}(i+1) + e^{-(1-i)t}C_2\\
&= \frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1-i)t}C_2
\end{aligned}$$

Hence

$$\begin{pmatrix} z_1(t)\\ z_2(t)\end{pmatrix} = \begin{pmatrix} -\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1+i)t}C_1\\ \frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1-i)t}C_2\end{pmatrix}$$

And since $x = Pz$ then

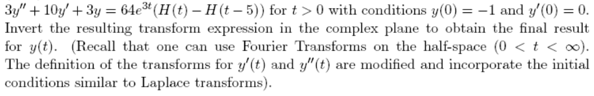
$$
\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} -1+i & -1-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1+i)t}C_1 \\ \frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1-i)t}C_2 \end{pmatrix} \\
&= \begin{pmatrix} (-1+i)\left(-\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1+i)t}C_1\right) + (-1-i)\,\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1-i)t}C_2 \\ \left(-\frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1+i)t}C_1\right) + \frac{1}{2}ie^{-t} + \frac{1}{2}e^{-t} + e^{-(1-i)t}C_2 \end{pmatrix} \\
&= \begin{pmatrix} \left(1 + \frac{1}{2}i\right)e^{-t} - (1-i)\,C_1e^{-(1+i)t} + C_2e^{-(1-i)t} \\ e^{-t} + C_1e^{-(1+i)t} + C_2e^{-(1-i)t} \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} + \frac{1}{2}ie^{-t} - (1-i)\,C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \\ e^{-t} + C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \end{pmatrix} \\
&= \begin{pmatrix} e^{-t} + \frac{1}{2}ie^{-t} - C_1e^{-t}e^{-it} + iC_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \\ e^{-t} + C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \end{pmatrix}
\end{aligned}
$$

The above can be simplified more if needed using Euler relation.

$$
\begin{aligned}
x(t) &= e^{-t} + \frac{1}{2}ie^{-t} - C_1e^{-t}e^{-it} + iC_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \\
y(t) &= e^{-t} + C_1e^{-t}e^{-it} + C_2e^{-t}e^{it}
\end{aligned}
$$

$C_1, C_2$ can be found from I.C.

### 2.14.11 problem 9



(4pts) Solve the following differential equation using Fourier Transforms:
$3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions $y(0) = -1$ and $y'(0) = 0$. Invert the resulting transform expression in the complex plane to obtain the final result for $y(t)$. (Recall that one can use Fourier Transforms on the half-space $(0 < t < \infty)$. The definition of the transforms for $y'(t)$ and $y''(t)$ are modified and incorporate the initial conditions similar to Laplace transforms).

**Solution**

Using the following relations for $0 < t < \infty$, where $\mathscr{F}\big(f(t)\big) \equiv F(\omega)$

$$
\begin{aligned}
\mathscr{F}\big(f'(t)\big) &= i\omega F(\omega) - f(0) \\
\mathscr{F}\big(f''(t)\big) &= (i\omega)^2 F(\omega) - i\omega f(0) - f'(0)
\end{aligned}
$$

Now we need to take the Fourier transform of $3y''(t) + 10y'(t) + 3y(t) = 64e^{3t}(H(t) - H(t-5))$. The first step is to normalize the leading term, this will make it easier later. Hence the ODE becomes

$$
y''(t) + \frac{10}{3}y'(t) + y(t) = \frac{64}{3}e^{3t}(H(t) - H(t-5))
$$

But

$$
\begin{aligned}
\mathscr{F}\left(\frac{64}{3}e^{3t}(H(t) - H(t-5))\right) &= \frac{64}{3}\int_0^5 e^{3t}e^{-i\omega t}dt \\
&= \frac{64}{3}\int_0^5 e^{-t(i\omega-3)}dt \\
&= \frac{64}{3}\left[\frac{e^{-t(i\omega-3)}}{-(i\omega-3)}\right]_0^5 = \frac{64}{3}\frac{e^{-5(i\omega-3)} - 1}{-(i\omega-3)}
\end{aligned}
$$

Hence the Fourier transform of the ODE becomes

$$
\Big((i\omega)^2 F(\omega) - i\omega f(0) - f'(0)\Big) + \frac{10}{3}\Big(i\omega F(\omega) - f(0)\Big) + F(\omega) = \frac{64}{3}\frac{e^{-5(i\omega-3)}}{-(i\omega-3)} + \frac{64}{3}\frac{1}{(i\omega-3)}
$$

Applying initial conditions

$$\begin{aligned}
\left(-\omega^2 F(\omega) + i\omega\right) + \frac{10}{3}\left(i\omega F(\omega) + 1\right) + F(\omega) &= \frac{64}{3}\frac{e^{-5(i\omega-3)}}{-(i\omega-3)} + \frac{64}{3}\frac{1}{(i\omega-3)} \\
-\omega^2 F(\omega) + i\omega + \frac{10}{3} i\omega F(\omega) + \frac{10}{3} + F(\omega) &= \frac{64}{3}\frac{e^{-5(i\omega-3)}}{-(i\omega-3)} + \frac{64}{3}\frac{1}{(i\omega-3)} \\
F(\omega)\left(-\omega^2 + \frac{10}{3} i\omega + 1\right) &= \frac{64}{3}\frac{e^{-5(i\omega-3)}}{-(i\omega-3)} + \frac{64}{3}\frac{1}{(i\omega-3)} - i\omega - \frac{10}{3} \\
F(\omega)\left(\omega^2 - \frac{10}{3} i\omega - 1\right) &= \frac{64}{3}\frac{e^{-5(i\omega-3)}}{(i\omega-3)} - \frac{64}{3}\frac{1}{(i\omega-3)} + i\omega + \frac{10}{3}
\end{aligned}$$

Solving for $F(\omega)$

$$\begin{aligned}
F(\omega) = &\frac{64}{3}\frac{e^{-5(i\omega-3)}}{(i\omega-3)\left(\omega^2 - \frac{10}{3}i\omega - 1\right)} - \frac{64}{3}\frac{1}{(i\omega-3)\left(\omega^2 - \frac{10}{3}i\omega - 1\right)} \\
&+ \frac{i\omega}{\left(\omega^2 - \frac{10}{3}i\omega - 1\right)} + \frac{10}{3\left(\omega^2 - \frac{10}{3}i\omega - 1\right)}
\end{aligned}$$

But $\left(\omega^2 - \frac{10}{3}i\omega - 1\right) = \left(\omega - \frac{1}{3}i\right)(\omega - 3i)$ hence the above becomes

$$\begin{aligned}
F(\omega) = &\frac{64e^{15}}{3}\frac{e^{-5i\omega}}{(i\omega-3)\left(\omega - \frac{1}{3}i\right)(\omega - 3i)} - \frac{64}{3}\frac{1}{(i\omega-3)\left(\omega - \frac{1}{3}i\right)(\omega - 3i)} \\
&+ \frac{i\omega}{\left(\omega - \frac{1}{3}i\right)(\omega - 3i)} + \frac{10}{3}\frac{1}{\left(\omega - \frac{1}{3}i\right)(\omega - 3i)}
\end{aligned}$$

We now start to do inverse Fourier transform. For the first integral, using the Fourier shift in time property

$$e^{-i5\omega}F(\omega) \iff f(t-5)$$

Hence we just need to find $f(t)$ for the first term and then shift it in time. The first term now becomes

$$T_1 = \frac{1}{(i\omega-3)\left(\omega - \frac{1}{3}i\right)(\omega - 3i)}$$

Partial fractions give

$$T_1 = \frac{1}{16}\frac{i}{\omega - 3i} + \frac{1}{20}\frac{i}{\omega + 3i} - \frac{9}{80}\frac{i}{\omega - \frac{1}{3}i}$$

Using property $\mathscr{F}^{-1}\left(\frac{1}{i\omega+a}\right) = e^{-at}H(t)$ and $\mathscr{F}^{-1}\left(\frac{1}{i\omega-a}\right) = -e^{at}H(-t)$

$$\begin{aligned}
\frac{1}{16}\mathscr{F}^{-1}\left(\frac{i}{\omega-3i}\right) &= \frac{1}{16}\mathscr{F}^{-1}\left(\frac{-1}{\omega i+3}\right) = -\frac{1}{16}e^{-3t}H(t) \\
\frac{1}{20}\mathscr{F}^{-1}\left(\frac{i}{\omega+3i}\right) &= \frac{1}{20}\mathscr{F}^{-1}\left(\frac{-1}{\omega i-3}\right) = -\frac{1}{20}e^{3t}H(-t) \\
-\frac{9}{80}\mathscr{F}^{-1}\left(\frac{i}{\omega-\frac{1}{3}i}\right) &= -\frac{9}{80}\mathscr{F}^{-1}\left(\frac{-1}{\omega i+\frac{1}{3}}\right) = +\frac{9}{80}e^{-\frac{t}{3}}H(t)
\end{aligned}$$

Terms with $H(-t)$ do not count, since these are only defined for negative time. The solution is only valid for $t > 0$, therefore these terms are ignored in the final solution. Hence

$$\begin{aligned}
\mathscr{F}^{-1}\left(\frac{64e^{15}}{3}\frac{e^{-5i\omega}}{(i\omega-3)\left(\omega - \frac{1}{3}i\right)(\omega - 3i)}\right) &= \frac{64e^{15}}{3}\left(-\frac{1}{16}e^{-3t}H(t) + \frac{9}{80}e^{-\frac{t}{3}}H(t)\right)_{t=t-5} \\
&= \frac{64e^{15}}{3}\left(-\frac{1}{16}e^{-3(t-5)}H(t-5) + \frac{9}{80}e^{-\frac{t-5}{3}}H(t-5)\right)
\end{aligned}$$

Ok, one term is done. Four more to go. Looking at $T_2 = -\frac{64}{3}\frac{1}{(i\omega-3)\left(\omega-\frac{1}{3}i\right)(\omega-3i)}$ this is the same as above, but with no time shift, hence

$$\mathscr{F}^{-1}\left(-\frac{64}{3}\frac{1}{(i\omega-3)\left(\omega-\frac{1}{3}i\right)(\omega-3i)}\right)=\frac{64}{3}\left(-\frac{1}{16}e^{-3t}H(t)+\frac{9}{80}e^{-\frac{t}{3}}H(t)\right)$$

Looking at $T_3=\frac{i\omega}{\left(\omega-\frac{1}{3}i\right)(\omega-3i)}$ partial fractions gives $T_2=\frac{9}{8}\frac{i}{\omega-3i}-\frac{1}{8}\frac{i}{\omega-\frac{1}{3}i}$ hence

$$\mathscr{F}^{-1}\left(\frac{9}{8}\frac{i}{\omega-3i}-\frac{1}{8}\frac{i}{\omega-\frac{1}{3}i}\right)=\frac{-9}{8}e^{-3t}H(t)+\frac{1}{8}e^{-\frac{t}{3}}H(t)$$

And finally, $T_4=\frac{10}{3}\frac{1}{\left(\omega-\frac{1}{3}i\right)(\omega-3i)}$ and partial fractions gives $T_4=\frac{10}{3}\left(\frac{3}{8}\frac{i}{\omega-\frac{1}{3}i}-\frac{3}{8}\frac{i}{\omega-3i}\right)$, hence

$$\mathscr{F}^{-1}\left(\frac{10}{8}\frac{i}{\omega-\frac{1}{3}i}-\frac{10}{8}\frac{i}{\omega-3i}\right)=\frac{-10}{8}e^{-\frac{t}{3}}H(t)+\frac{10}{8}e^{-3t}H(t)$$

Collecting all terms, the final result is

$$\begin{aligned}y(t)=&\frac{64e^{15}}{3}\left(-\frac{1}{16}e^{-3(t-5)}H(t-5)+\frac{9}{80}e^{-\frac{t-5}{3}}H(t-5)\right)\\&+\frac{64}{3}\left(-\frac{1}{16}e^{-3t}H(t)+\frac{9}{80}e^{-\frac{t}{3}}H(t)\right)\\&\frac{-9}{8}e^{-3t}H(t)+\frac{1}{8}e^{-\frac{t}{3}}H(t)\\&\frac{-10}{8}e^{-\frac{t}{3}}H(t)+\frac{10}{8}e^{-3t}H(t)\end{aligned}$$

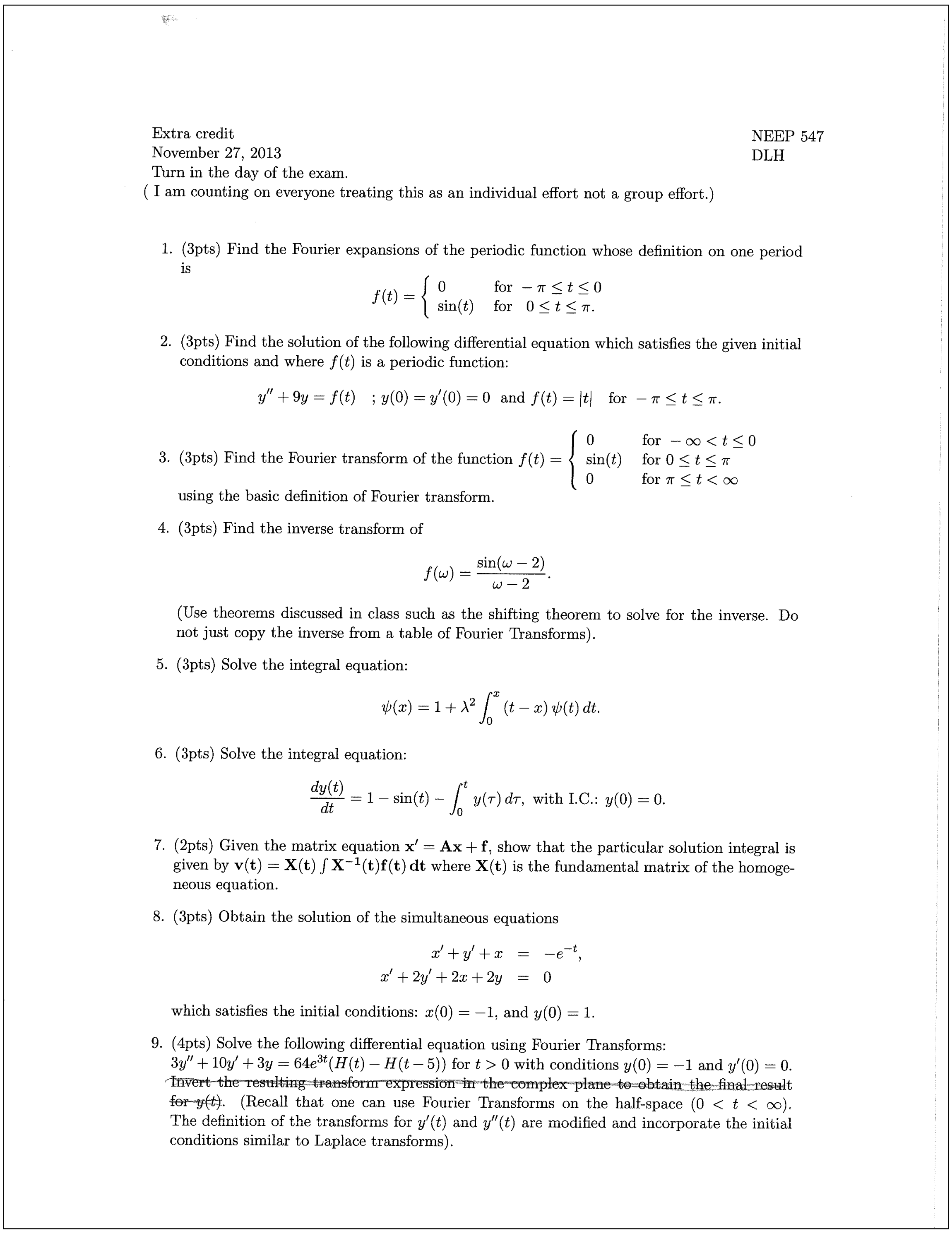
Expanding and collecting

$$\begin{aligned}y(t)=&\frac{64e^{15}}{3}H(t-5)\left(-\frac{1}{16}e^{-3(t-5)}+\frac{9}{80}e^{-\frac{t-5}{3}}\right)\\&+H(t)\left(-\frac{1}{16}e^{-3t}+\frac{9}{80}e^{-\frac{t}{3}}-\frac{9}{8}e^{-3t}+\frac{1}{8}e^{-\frac{t}{3}}-\frac{10}{8}e^{-\frac{t}{3}}+\frac{10}{8}e^{-3t}\right)\\\\=&\frac{64e^{15}}{3}H(t-5)\left(-\frac{1}{16}e^{-3(t-5)}+\frac{9}{80}e^{-\frac{t-5}{3}}\right)\\&+H(t)\left(\frac{1}{16}e^{-3t}-\frac{81}{80}e^{-\frac{1}{3}t}\right)\end{aligned}$$

Therefore the solution is

$$y(t)=\begin{cases}\frac{64e^{15}}{3}\left(-\frac{1}{16}e^{-3(t-5)}+\frac{9}{80}e^{-\frac{t-5}{3}}\right)+\left(\frac{1}{16}e^{-3t}-\frac{81}{80}e^{-\frac{1}{3}t}\right) & 0<t\leq 5\\ \frac{1}{16}e^{-3t}-\frac{81}{80}e^{-\frac{1}{3}t} & 5<t<\infty\end{cases}$$

### 2.14.12 key solution



Extra credit
November 27, 2013
Turn in the day of the exam.
( I am counting on everyone treating this as an individual effort not a group effort.)

NEEP 547
DLH

1. (3pts) Find the Fourier expansions of the periodic function whose definition on one period is

$$f(t) = \begin{cases} 0 & \text{for } -\pi \le t \le 0 \\ \sin(t) & \text{for } \ 0 \le t \le \pi. \end{cases}$$

2. (3pts) Find the solution of the following differential equation which satisfies the given initial conditions and where $f(t)$ is a periodic function:

$$y'' + 9y = f(t) \quad ; y(0) = y'(0) = 0 \ \text{ and } f(t) = |t| \ \text{ for } -\pi \le t \le \pi.$$

3. (3pts) Find the Fourier transform of the function $f(t) = \begin{cases} 0 & \text{for } -\infty < t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi \le t < \infty \end{cases}$ using the basic definition of Fourier transform.

4. (3pts) Find the inverse transform of

$$f(\omega) = \frac{\sin(\omega - 2)}{\omega - 2}.$$

(Use theorems discussed in class such as the shifting theorem to solve for the inverse. Do not just copy the inverse from a table of Fourier Transforms).

5. (3pts) Solve the integral equation:

$$\psi(x) = 1 + \lambda^2 \int_0^x (t - x)\,\psi(t)\,dt.$$

6. (3pts) Solve the integral equation:

$$\frac{dy(t)}{dt} = 1 - \sin(t) - \int_0^t y(\tau)\,d\tau, \text{ with I.C.: } y(0) = 0.$$

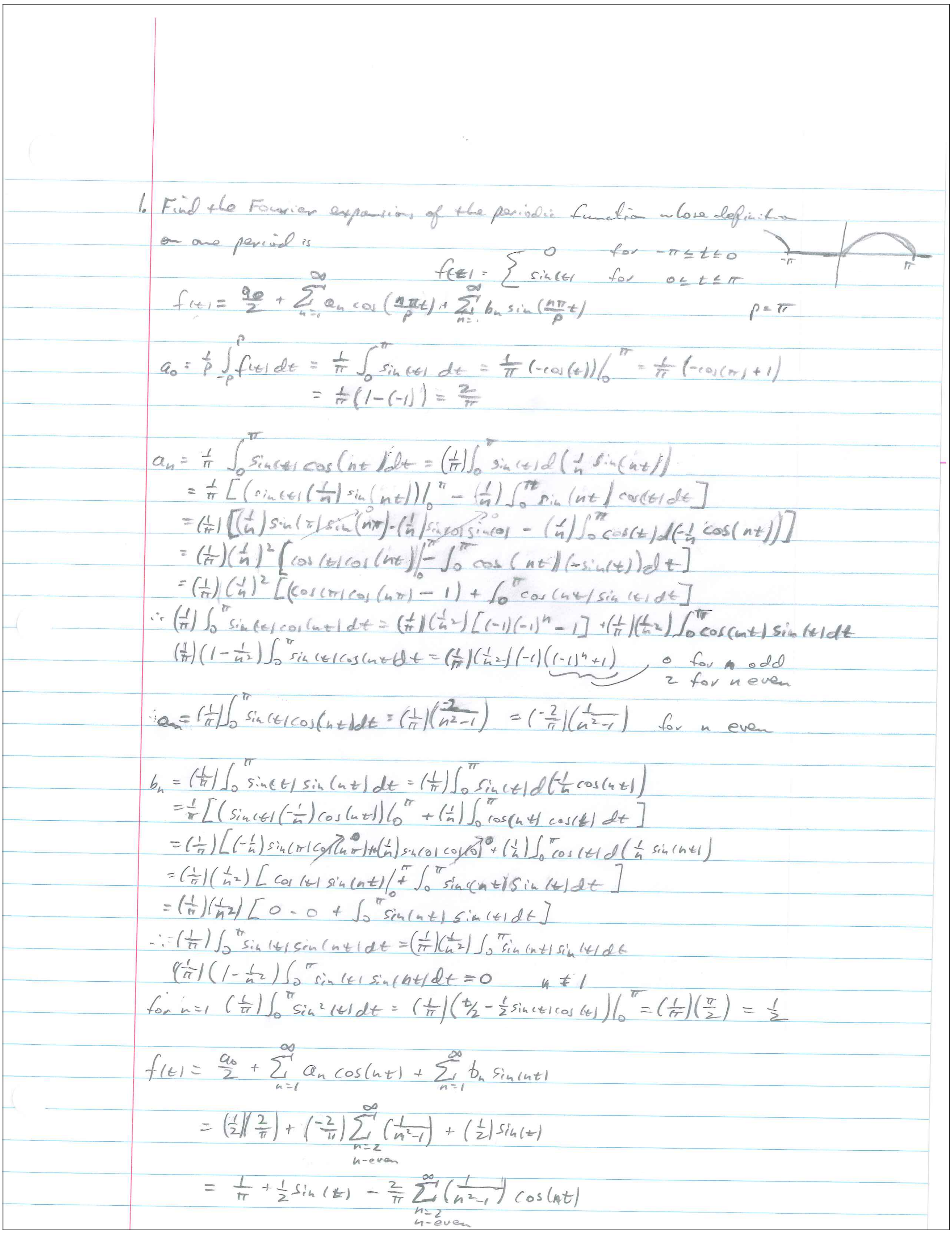
7. (2pts) Given the matrix equation $\mathbf{x}' = \mathbf{Ax} + \mathbf{f}$, show that the particular solution integral is given by $\mathbf{v(t)} = \mathbf{X(t)} \int \mathbf{X^{-1}(t)f(t)\,dt}$ where $\mathbf{X(t)}$ is the fundamental matrix of the homogeneous equation.

8. (3pts) Obtain the solution of the simultaneous equations

$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$

which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.

9. (4pts) Solve the following differential equation using Fourier Transforms: $3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions $y(0) = -1$ and $y'(0) = 0$. ~~Invert the resulting transform expression in the complex plane to obtain the final result for $y(t)$~~. (Recall that one can use Fourier Transforms on the half-space ($0 < t < \infty$). The definition of the transforms for $y'(t)$ and $y''(t)$ are modified and incorporate the initial conditions similar to Laplace transforms).



1. Find the Fourier expansion of the periodic function whose definition on one period is

$$f(t) = \begin{cases} 0 & \text{for } -\pi \le t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi \end{cases}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p} t\right) \qquad p = \pi$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{1}{\pi}(-\cos(t))\Big|_0^{\pi} = \frac{1}{\pi}(-\cos(\pi)+1)$$
$$= \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\,d\left(\frac{1}{n}\sin(nt)\right)$$
$$= \frac{1}{\pi}\left[\left(\sin(t)\left(\frac{1}{n}\right)\sin(nt)\right)\Big|_0^{\pi} - \left(\frac{1}{n}\right)\int_0^{\pi} \sin(nt)\cos(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left[\left(\frac{1}{n}\right)\sin(\pi)\cancelto{0}{\sin(n\pi)} - \left(\frac{1}{n}\right)\cancelto{0}{\sin(0)\sin(0)} - \left(\frac{1}{n}\right)\int_0^{\pi} \cos(t)\,d\left(-\frac{1}{n}\cos(nt)\right)\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n}\right)^2\left[\cos(t)\cos(nt)\Big|_0^{\pi} - \int_0^{\pi} \cos(nt)(-\sin(t))\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n}\right)^2\left[(\cos(\pi)\cos(n\pi) - 1) + \int_0^{\pi} \cos(nt)\sin(t)\,dt\right]$$
$$\therefore \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[(-1)(-1)^n - 1\right] + \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\int_0^{\pi} \cos(nt)\sin(t)\,dt$$
$$\left(\frac{1}{\pi}\right)\left(1-\frac{1}{n^2}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)(-1)\underbrace{((-1)^n+1)}_{\substack{0 \text{ for } n \text{ odd} \\ 2 \text{ for } n \text{ even}}}$$

$$\therefore a_n = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{-2}{n^2-1}\right) = \left(-\frac{2}{\pi}\right)\left(\frac{1}{n^2-1}\right) \quad \text{for } n \text{ even}$$

$$b_n = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\,d\left(-\frac{1}{n}\cos(nt)\right)$$
$$= \frac{1}{\pi}\left[\left(\sin(t)\left(-\frac{1}{n}\right)\cos(nt)\right)\Big|_0^{\pi} + \left(\frac{1}{n}\right)\int_0^{\pi} \cos(nt)\cos(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left[\left(-\frac{1}{n}\right)\sin(\pi)\cancelto{0}{\cos(n\pi)} + \left(\frac{1}{n}\right)\sin(0)\cancelto{0}{\cos(0)} + \left(\frac{1}{n}\right)\int_0^{\pi} \cos(t)\,d\left(\frac{1}{n}\sin(nt)\right)\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[\cos(t)\sin(nt)\Big|_0^{\pi} + \int_0^{\pi} \sin(nt)\sin(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[0 - 0 + \int_0^{\pi} \sin(nt)\sin(t)\,dt\right]$$
$$\therefore \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\int_0^{\pi} \sin(nt)\sin(t)\,dt$$
$$\left(\frac{1}{\pi}\right)\left(1-\frac{1}{n^2}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = 0 \qquad n \neq 1$$

$$\text{for } n=1 \quad \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin^2(t)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{t}{2} - \frac{1}{2}\sin(t)\cos(t)\right)\Big|_0^{\pi} = \left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt)$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right) + \left(\frac{-2}{\pi}\right)\sum_{\substack{n=2 \\ n\text{-even}}}^{\infty}\left(\frac{1}{n^2-1}\right) + \left(\frac{1}{2}\right)\sin(t)$$
$$= \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{\pi}\sum_{\substack{n=2 \\ n\text{-even}}}^{\infty}\left(\frac{1}{n^2-1}\right)\cos(nt)$$

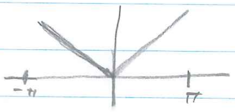
2. Find the solution of the following differential equation which satisfies the given initial conditions and where $f(t)$ is a periodic function:

$y''+9y=f(t)$; $y(0)=y'(0)=0$ and $f(t)=|t|$ for $-\pi\le t\le\pi$

note $y=y_h+y_p$. we will find $y_h$ by regular means and $y_p$ by fourier series expansion

$y_h''+9y_h=0 \Rightarrow y_h(x)=A\cos(3t)+B\sin(3t)$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nt)+\sum_{n=1}^{\infty}b_n\sin(nt)$$



$f(t)$ is an even function

$p=\pi$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt=\frac{1}{\pi}\underbrace{\int_{-\pi}^{0}(-t)dt}+\frac{1}{\pi}\int_0^{\pi}(t)dt=\frac{1}{\pi}\int_0^{\pi}zdz+\frac{1}{\pi}\int_0^{\pi}tdt=\frac{2}{\pi}\int_0^{\pi}tdt$$

let $z=-t$, $dz=-dt$
$t=-\pi$, $z=\pi$
$t=0$ $z=0$

$$=\frac{2}{\pi}\left(\frac{t^2}{2}\right)\Big|_0^{\pi}=\left(\frac{2}{\pi}\right)\left(\frac{\pi^2}{2}\right)=\pi$$

$$a_0=\underline{\pi}$$

let $z=-t$, $dz=-dt$
$t=-\pi$, $z=\pi$
$t=0$, $z=0$
$\cos(nz)=\cos(-nz)$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)dt=\frac{1}{\pi}\int_{-\pi}^{0}(-t)\cos(nt)dt+\frac{1}{\pi}\int_0^{\pi}(t)\cos(nt)dt$$
$$=\frac{1}{\pi}\int_0^{\pi}z\cos(nz)dz+\frac{1}{\pi}\int_0^{\pi}t\cos(nt)dt$$
$$=\frac{2}{\pi}\int_0^{\pi}t\cos(nt)dt=\frac{2}{\pi}\int_0^{\pi}t\,d\left(\frac{1}{n}\sin(nt)\right)=\frac{2}{\pi}\left(\frac{t}{n}\sin(nt)\Big|_0^{\pi}-\int_0^{t}\frac{1}{n}\sin(nt)dt\right)$$
$$=\frac{2}{\pi}\left(0-\frac{1}{n}\int_0^{t}\sin(nt)dt\right)\Rightarrow\frac{2}{\pi}\left(-\frac{1}{n}\right)\left(-\frac{1}{n}\cos(nt)\Big|_0^{\pi}\right)=\left(\frac{2}{\pi}\right)\left(\frac{1}{n^2}\right)(\cos(n\pi)-1)$$
$$=\left(\frac{2}{\pi}\right)\left(\frac{1}{n^2}\right)((-1)^n-1)=\left(\frac{-4}{\pi}\right)\left(\frac{1}{n^2}\right)\text{ for } n=\text{odd}$$

let $z=-t$ $dz=-dt$
$t=-\pi$, $z=\pi$
$t=0$, $z=0$
$\sin(nz)=-\sin(nz)$

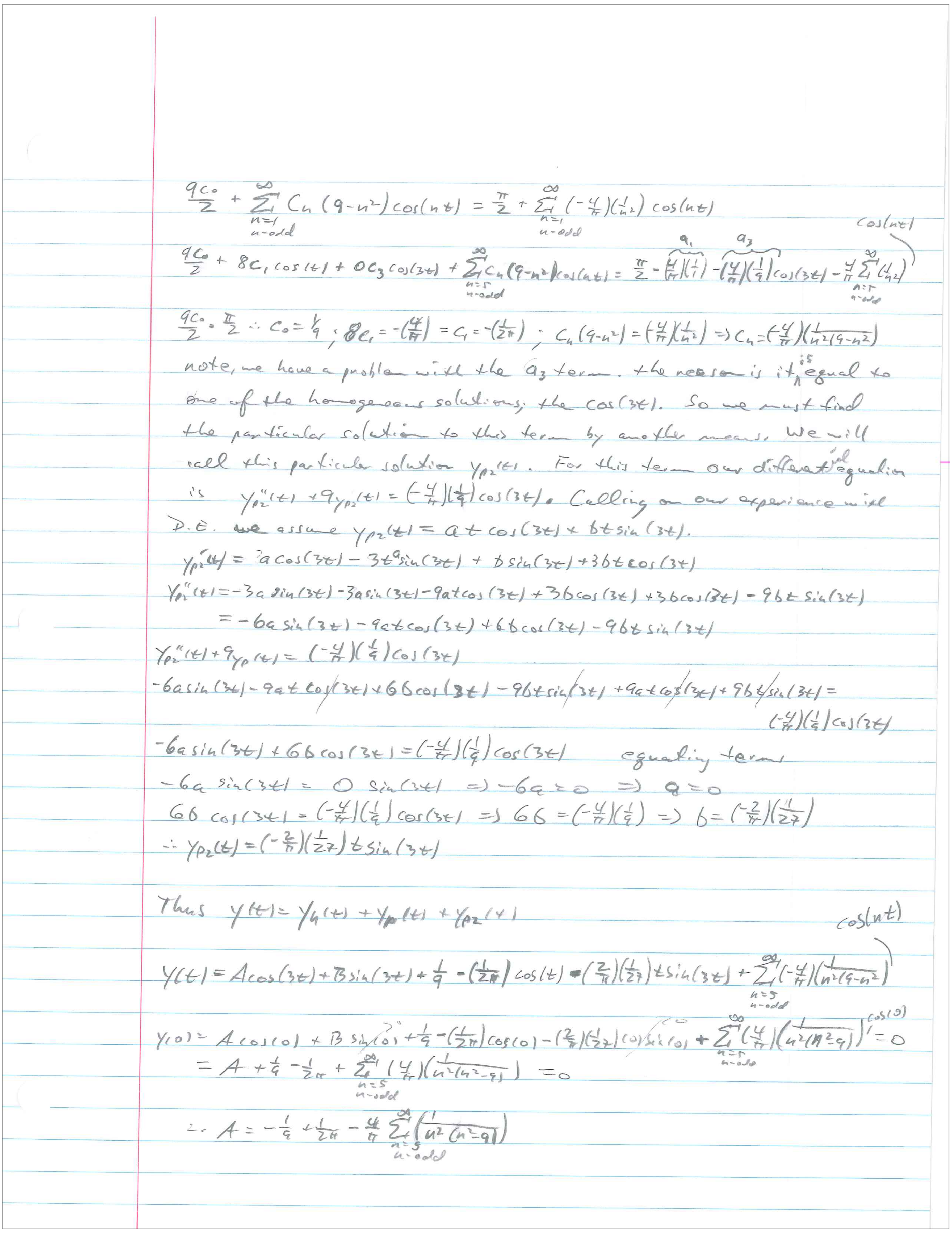
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)dt=\frac{1}{\pi}\int_{-\pi}^{0}(-t)\sin(nt)dt+\frac{1}{\pi}\int_0^{\pi}(t)\sin(nt)dt$$
$$=\frac{1}{\pi}\int_{\pi}^{0}z\sin(nz)dz+\frac{1}{\pi}\int_0^{\pi}t\sin(nt)dt$$
$$=-\frac{1}{\pi}\int_0^{\pi}t\sin(nt)dt+\frac{1}{\pi}\int_0^{\pi}t\sin(nt)dt=0\quad\text{as it should be}$$

we now assume $y_p(t)=\dfrac{c_0}{2}+\displaystyle\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}c_n\cos(nt)$, $y_p'(t)=\displaystyle\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}c_n(-n)\sin(nt)$

$$y_p''(t)=-\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}c_n(n^2)\cos(nt)$$

$$\therefore y_p''(t)+9\cdot y_p(t)=\frac{a_0}{2}+\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}a_n\cos(nt)$$

$$-\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}c_n(n^2)\cos(nt)+9\left(\frac{c_0}{2}+\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}c_n\cos(nt)\right)=\frac{\pi}{2}+\sum_{\substack{n=1\\n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right)\cos(nt)$$



$$\frac{9c_0}{2} + \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n(9-n^2)\cos(nt) = \frac{\pi}{2} + \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right)\cos(nt)$$

$$\frac{9c_0}{2} + 8c_1\cos(t) + 0c_3\cos(3t) + \sum_{\substack{n=5\\ n\text{-odd}}}^{\infty} c_n(9-n^2)\cos(nt) = \frac{\pi}{2} - \overbrace{\left(\frac{4}{\pi}\right)\left(\frac{1}{1}\right)}^{a_1} - \overbrace{\left(\frac{4}{\pi}\right)\left(\frac{1}{9}\right)}^{a_3}\cos(3t) - \frac{4}{\pi}\sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}\left(\frac{1}{n^2}\right)\cos(nt)$$

$\frac{9c_0}{2} = \frac{\pi}{2} \therefore c_0 = \frac{1}{9}$; $8c_1 = -\left(\frac{4}{\pi}\right) = c_1 = -\left(\frac{1}{2\pi}\right)$; $c_n(9-n^2) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right) \Rightarrow c_n = \left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2(9-n^2)}\right)$

note, we have a problem with the $a_3$ term. the reason is it is equal to one of the homogeneous solutions, the $\cos(3t)$. So we must find the particular solution to this term by another means. We will call this particular solution $y_{p_2}(t)$. For this term our differential equation is $y_{p_2}''(t) + 9y_{p_2}(t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$. Calling on our experience with D.E. ~~we~~ assume $y_{p_2}(t) = at\cos(3t) + bt\sin(3t)$.

$$y_{p_2}'(t) = a\cos(3t) - 3ta\sin(3t) + b\sin(3t) + 3bt\cos(3t)$$

$$y_{p_2}''(t) = -3a\sin(3t) - 3a\sin(3t) - 9at\cos(3t) + 3b\cos(3t) + 3b\cos(3t) - 9bt\sin(3t)$$

$$= -6a\sin(3t) - 9at\cos(3t) + 6b\cos(3t) - 9bt\sin(3t)$$

$$y_{p_2}''(t) + 9y_p(t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$$

$$-6a\sin(3t) - \cancel{9at\cos(3t)} + 6b\cos(3t) - \cancel{9bt\sin(3t)} + \cancel{9at\cos(3t)} + \cancel{9bt\sin(3t)} = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$$

$-6a\sin(3t) + 6b\cos(3t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$ equating terms

$$-6a\sin(3t) = 0\sin(3t) \Rightarrow -6a = 0 \Rightarrow a = 0$$

$$6b\cos(3t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t) \Rightarrow 6b = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right) \Rightarrow b = \left(-\frac{2}{\pi}\right)\left(\frac{1}{27}\right)$$

$$\therefore y_{p_2}(t) = \left(-\frac{2}{\pi}\right)\left(\frac{1}{27}\right)t\sin(3t)$$

Thus $y(t) = y_h(t) + y_p(t) + y_{p_2}(t)$

$$y(t) = A\cos(3t) + B\sin(3t) + \frac{1}{9} - \left(\frac{1}{2\pi}\right)\cos(t) - \left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)t\sin(3t) + \sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2(9-n^2)}\right)\cos(nt)$$

$$y(0) = A\cos(0) + B\cancelto{0}{\sin(0)} + \frac{1}{9} - \left(\frac{1}{2\pi}\right)\cos(0) - \cancelto{0}{\left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)(0)\sin(0)} + \sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}\left(\frac{4}{\pi}\right)\left(\frac{1}{n^2(n^2-9)}\right)\overset{\cos(0)}{1} = 0$$

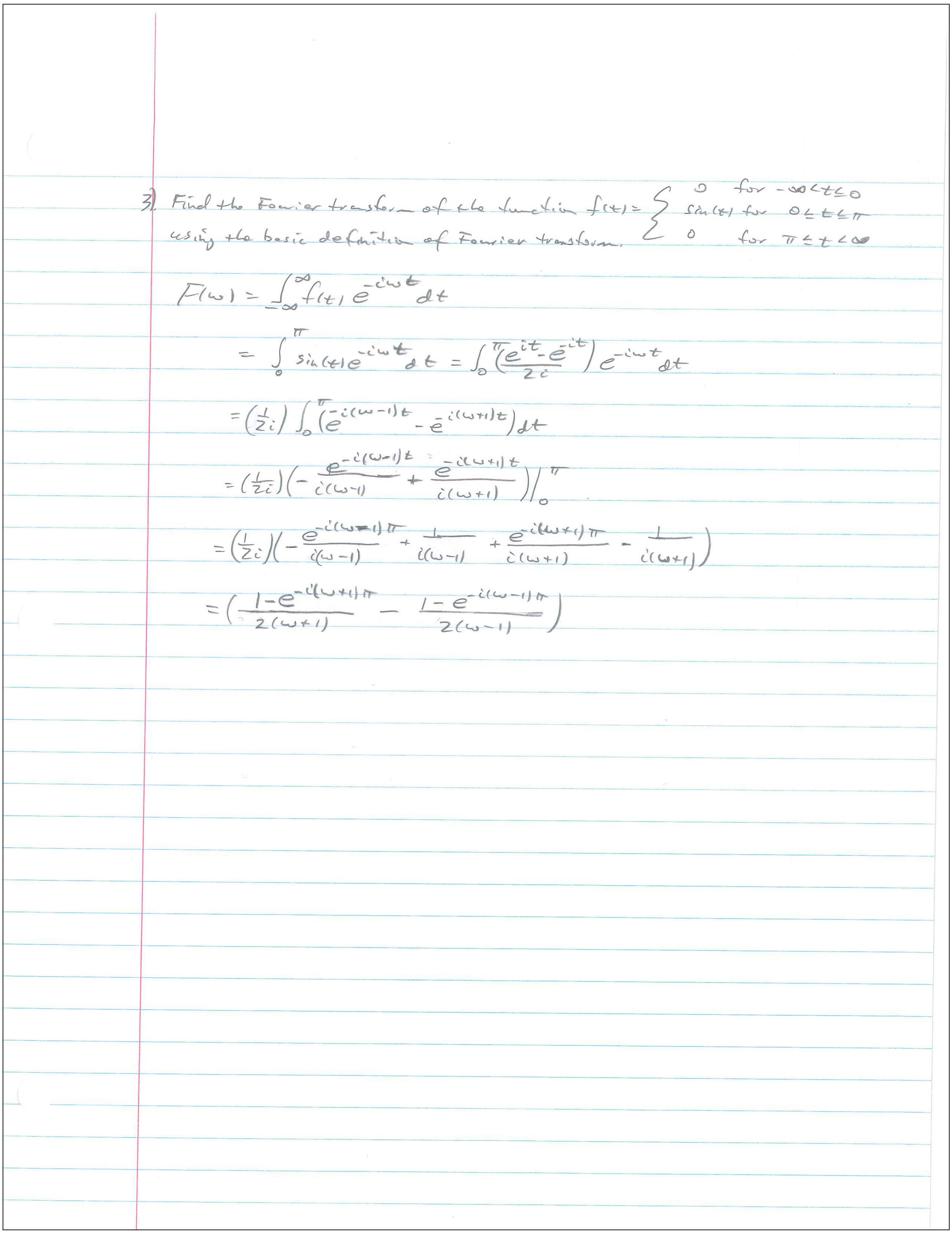
$$= A + \frac{1}{9} - \frac{1}{2\pi} + \sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}\left(\frac{4}{\pi}\right)\left(\frac{1}{n^2(n^2-9)}\right) = 0$$

$$\therefore A = -\frac{1}{9} + \frac{1}{2\pi} - \frac{4}{\pi}\sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}\left(\frac{1}{n^2(n^2-9)}\right)$$

$$y'(0) = -3A\overset{\to 0}{\cancel{\sin(0)}} + 3B\cos(0) + \frac{1}{2\pi}\overset{\to 0}{\cancel{\sin(0)}} - \left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)\overset{\to 0}{\cancel{\sin(0)}} - \left(\frac{2}{\pi}\right)\left(\frac{3}{27}\right)\overset{\to 0}{\cancel{(0)\cos(0)}} + \sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{4}{\pi}\right)\left(\frac{-n}{n^2(n^2-9)}\overset{\to 0}{\cancel{\sin(0)}}\right) = 0$$

$$= 3B = 0$$

$$\therefore y(t) = \left(-\frac{1}{9} + \frac{1}{2\pi} - \frac{4}{\pi}\sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{1}{n^2(n^2-9)}\right)\right)\cos(3t) + \frac{1}{9} - \frac{1}{2\pi}\cos(t) - \left(\frac{2}{\pi}\right)\left(\frac{t}{27}\right)\sin(3t) + \left(\frac{4}{\pi}\right)\sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{1}{n^2(n^2-9)}\right)\cos(nt)$$



3) Find the Fourier transform of the function $f(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0 \\ \sin(t) & \text{for } 0 \leq t \leq \pi \\ 0 & \text{for } \pi \leq t < \infty \end{cases}$ using the basic definition of Fourier transform.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

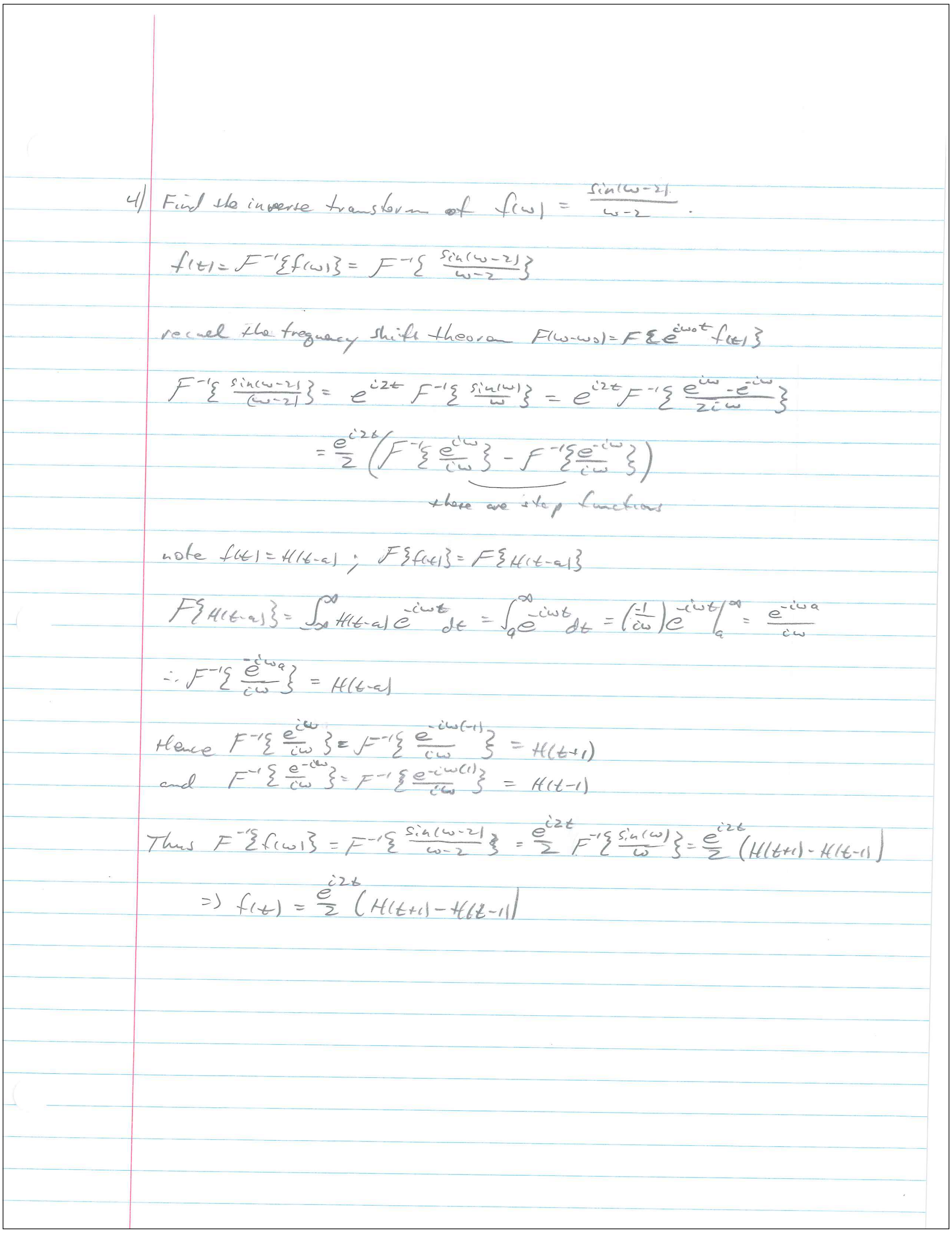
$$= \int_0^\pi \sin(t) e^{-i\omega t}\, dt = \int_0^\pi \left(\frac{e^{it} - e^{-it}}{2i}\right) e^{-i\omega t}\, dt$$

$$= \left(\frac{1}{2i}\right) \int_0^\pi \left(e^{-i(\omega-1)t} - e^{-i(\omega+1)t}\right) dt$$

$$= \left(\frac{1}{2i}\right)\left(-\frac{e^{-i(\omega-1)t}}{i(\omega-1)} + \frac{e^{-i(\omega+1)t}}{i(\omega+1)}\right)\Bigg|_0^\pi$$

$$= \left(\frac{1}{2i}\right)\left(-\frac{e^{-i(\omega-1)\pi}}{i(\omega-1)} + \frac{1}{i(\omega-1)} + \frac{e^{-i(\omega+1)\pi}}{i(\omega+1)} - \frac{1}{i(\omega+1)}\right)$$

$$= \left(\frac{1-e^{-i(\omega+1)\pi}}{2(\omega+1)} - \frac{1-e^{-i(\omega-1)\pi}}{2(\omega-1)}\right)$$



4) Find the inverse transform of $f(\omega) = \frac{\sin(\omega-2)}{\omega-2}$.

$$f(t) = F^{-1}\{f(\omega)\} = F^{-1}\left\{\frac{\sin(\omega-2)}{\omega-2}\right\}$$

recall the frequency shift theorem $F(\omega-\omega_0) = F\{e^{i\omega_0 t} f(t)\}$

$$F^{-1}\left\{\frac{\sin(\omega-2)}{(\omega-2)}\right\} = e^{i2t} F^{-1}\left\{\frac{\sin(\omega)}{\omega}\right\} = e^{i2t} F^{-1}\left\{\frac{e^{i\omega}-e^{-i\omega}}{2i\omega}\right\}$$

$$= \frac{e^{i2t}}{2}\left(\underbrace{F^{-1}\left\{\frac{e^{i\omega}}{i\omega}\right\} - F^{-1}\left\{\frac{e^{-i\omega}}{i\omega}\right\}}_{\text{these are step functions}}\right)$$

note $f(t) = H(t-a)$ ; $F\{f(t)\} = F\{H(t-a)\}$

$$F\{H(t-a)\} = \int_{-\infty}^{\infty} H(t-a)\, e^{-i\omega t}\, dt = \int_a^{\infty} e^{-i\omega t}\, dt = \left(\frac{-1}{i\omega}\right) e^{-i\omega t}\Big|_a^{\infty} = \frac{e^{-i\omega a}}{i\omega}$$

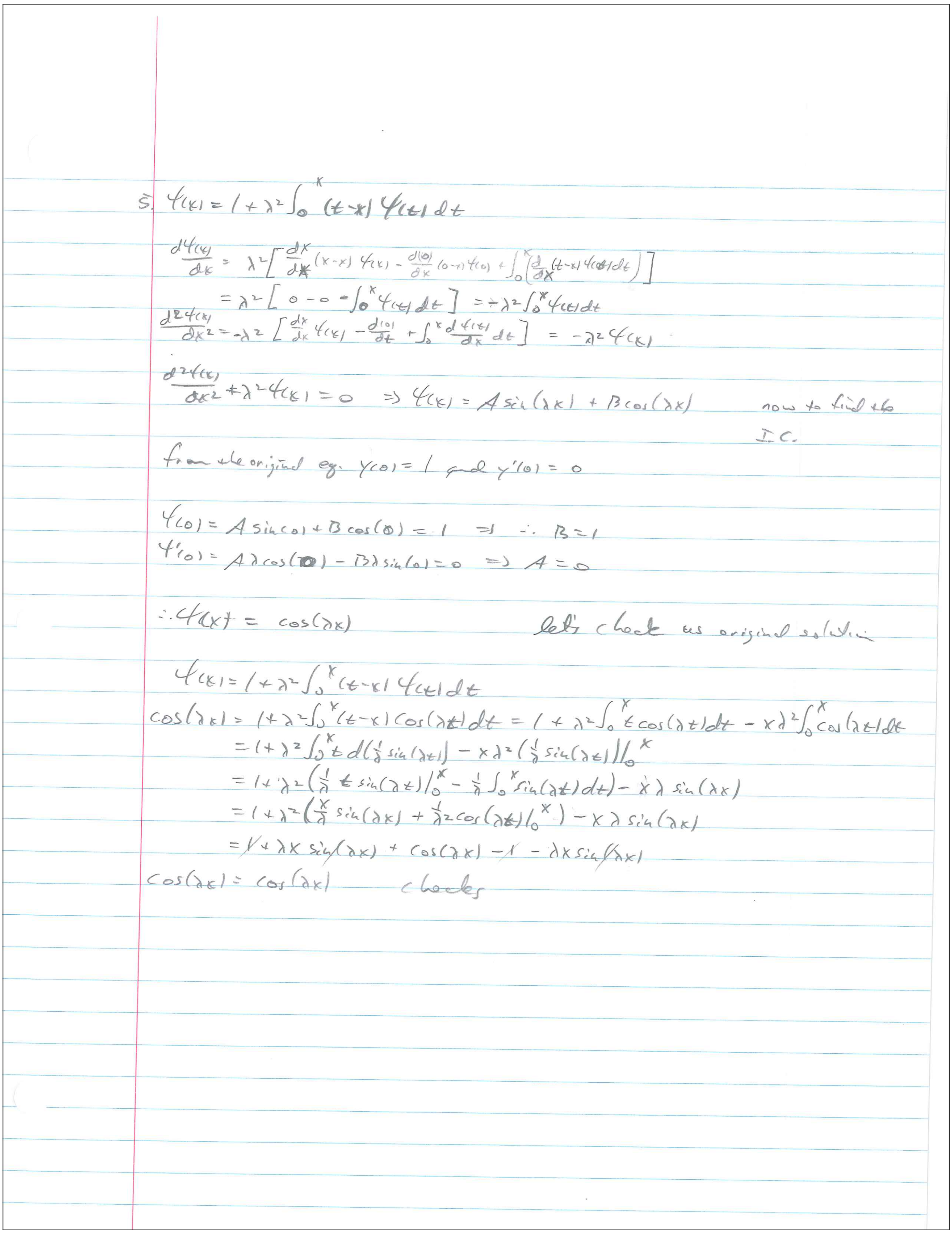
$$\therefore F^{-1}\left\{\frac{e^{-i\omega a}}{i\omega}\right\} = H(t-a)$$

Hence $F^{-1}\left\{\frac{e^{i\omega}}{i\omega}\right\} = F^{-1}\left\{\frac{e^{-i\omega(-1)}}{i\omega}\right\} = H(t+1)$

and $F^{-1}\left\{\frac{e^{-i\omega}}{i\omega}\right\} = F^{-1}\left\{\frac{e^{-i\omega(1)}}{i\omega}\right\} = H(t-1)$

Thus $F^{-1}\{f(\omega)\} = F^{-1}\left\{\frac{\sin(\omega-2)}{\omega-2}\right\} = \frac{e^{i2t}}{2} F^{-1}\left\{\frac{\sin(\omega)}{\omega}\right\} = \frac{e^{i2t}}{2}\left(H(t+1) - H(t-1)\right)$

$$\Rightarrow f(t) = \frac{e^{i2t}}{2}\left(H(t+1) - H(t-1)\right)$$



5. $\phi(x) = 1 + \lambda^2 \int_0^x (t-x)\,\phi(t)\,dt$

$$\frac{d\phi(x)}{dx} = \lambda^2\left[\frac{dx}{dx}(x-x)\,\phi(x) - \frac{d(0)}{dx}(0-x)\,\phi(0) + \int_0^x \left(\frac{d}{dx}(t-x)\,\phi(t)\,dt\right)\right]$$

$$= \lambda^2\left[0 - 0 - \int_0^x \phi(t)\,dt\right] = -\lambda^2 \int_0^x \phi(t)\,dt$$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda^2\left[\frac{dx}{dx}\phi(x) - \frac{d(0)}{dt} + \int_0^x \frac{d\,\phi(t)}{dx}\,dt\right] = -\lambda^2\phi(x)$$

$$\frac{d^2\phi(x)}{dx^2} + \lambda^2\phi(x) = 0 \Rightarrow \phi(x) = A\sin(\lambda x) + B\cos(\lambda x)$$

now to find the I.C.

from the original eq. $y(0) = 1$ and $y'(0) = 0$

$$\phi(0) = A\sin(0) + B\cos(0) = 1 \Rightarrow \therefore B = 1$$

$$\phi'(0) = A\lambda\cos(0) - B\lambda\sin(0) = 0 \Rightarrow A = 0$$

$$\therefore \phi(x) = \cos(\lambda x)$$

let's check us original solution

$$\phi(x) = 1 + \lambda^2 \int_0^x (t-x)\,\phi(t)\,dt$$

$$\cos(\lambda x) = 1 + \lambda^2 \int_0^x (t-x)\cos(\lambda t)\,dt = 1 + \lambda^2 \int_0^x t\cos(\lambda t)\,dt - x\lambda^2 \int_0^x \cos(\lambda t)\,dt$$

$$= 1 + \lambda^2 \int_0^x t\,d\left(\frac{1}{\lambda}\sin(\lambda t)\right) - x\lambda^2\left(\frac{1}{\lambda}\sin(\lambda t)\right)\Big|_0^x$$

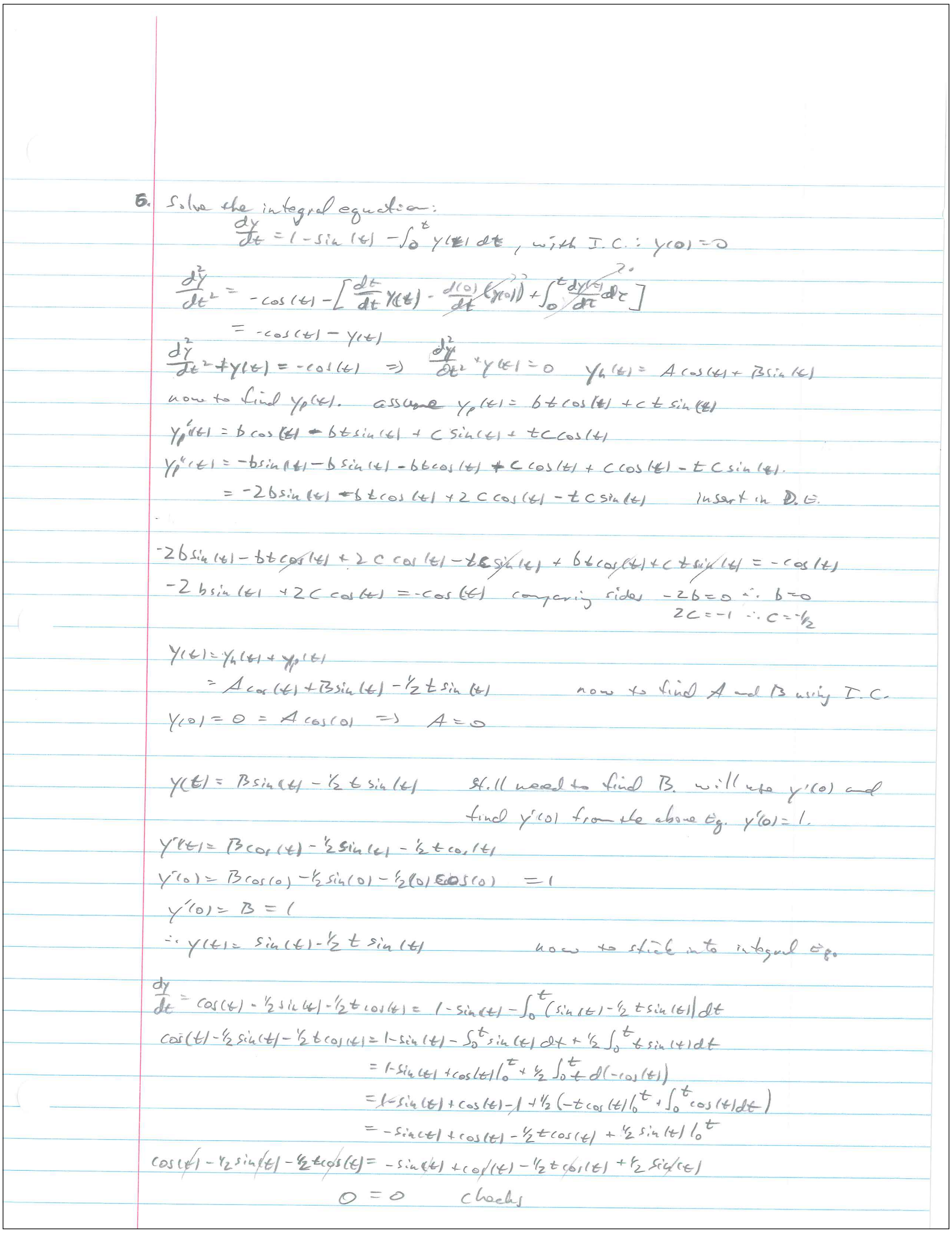
$$= 1 + \lambda^2\left(\frac{1}{\lambda}t\sin(\lambda t)\Big|_0^x - \frac{1}{\lambda}\int_0^x \sin(\lambda t)\,dt\right) - x\lambda\sin(\lambda x)$$

$$= 1 + \lambda^2\left(\frac{x}{\lambda}\sin(\lambda x) + \frac{1}{\lambda^2}\cos(\lambda t)\Big|_0^x\right) - x\lambda\sin(\lambda x)$$

$$= \cancel{1} + \cancel{\lambda x\sin(\lambda x)} + \cos(\lambda x) - \cancel{1} - \cancel{\lambda x\sin(\lambda x)}$$

$$\cos(\lambda x) = \cos(\lambda x)$$

checks



6. Solve the integral equation:

$$\frac{dy}{dt} = 1 - \sin(t) - \int_0^t y(t)\,dt \quad , \text{ with I.C.: } y(0) = 0$$

$$\frac{d^2y}{dt^2} = -\cos(t) - \left[\frac{dt}{dt} y(t) - \cancel{\frac{d(0)}{dt}}\cancel{(y(0))} + \cancel{\int_0^t \frac{dy(\tau)}{dt} d\tau}\right]$$

$$= -\cos(t) - y(t)$$

$$\frac{d^2y}{dt^2} + y(t) = -\cos(t) \quad \Rightarrow \quad \frac{d^2y}{dt^2} + y(t) = 0 \qquad y_h(t) = A\cos(t) + B\sin(t)$$

now to find $y_p(t)$. assume $y_p(t) = b\,t\cos(t) + c\,t\sin(t)$

$$y_p'(t) = b\cos(t) - bt\sin(t) + c\sin(t) + tc\cos(t)$$

$$y_p''(t) = -b\sin(t) - b\sin(t) - bt\cos(t) + c\cos(t) + c\cos(t) - tc\sin(t).$$

$$= -2b\sin(t) - bt\cos(t) + 2c\cos(t) - tc\sin(t) \qquad \text{insert in D.E.}$$

$$-2b\sin(t) - \cancel{bt\cos(t)} + 2c\cos(t) - \cancel{tc\sin(t)} + \cancel{bt\cos(t)} + \cancel{ct\sin(t)} = -\cos(t)$$

$$-2b\sin(t) + 2c\cos(t) = -\cos(t) \qquad \text{comparing sides} \quad -2b = 0 \therefore b = 0$$

$$2c = -1 \therefore c = -\tfrac{1}{2}$$

$$y(t) = y_h(t) + y_p(t)$$

$$= A\cos(t) + B\sin(t) - \tfrac{1}{2}t\sin(t) \qquad \text{now to find } A \text{ and } B \text{ using I.C.}$$

$$y(0) = 0 = A\cos(0) \Rightarrow A = 0$$

$$y(t) = B\sin(t) - \tfrac{1}{2}t\sin(t)$$

Still need to find $B$. will use $y'(0)$ and find $y'(0)$ from the above eq. $y'(0) = 1$.

$$y'(t) = B\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t)$$

$$y'(0) = B\cos(0) - \tfrac{1}{2}\sin(0) - \tfrac{1}{2}(0)\cos(0) = 1$$

$$y'(0) = B = 1$$

$$\therefore y(t) = \sin(t) - \tfrac{1}{2}t\sin(t) \qquad \text{now to stick into integral eq.}$$

$$\frac{dy}{dt} = \cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t) = 1 - \sin(t) - \int_0^t (\sin(t) - \tfrac{1}{2}t\sin(t))\,dt$$

$$\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t) = 1 - \sin(t) - \int_0^t \sin(t)\,dt + \tfrac{1}{2}\int_0^t t\sin(t)\,dt$$

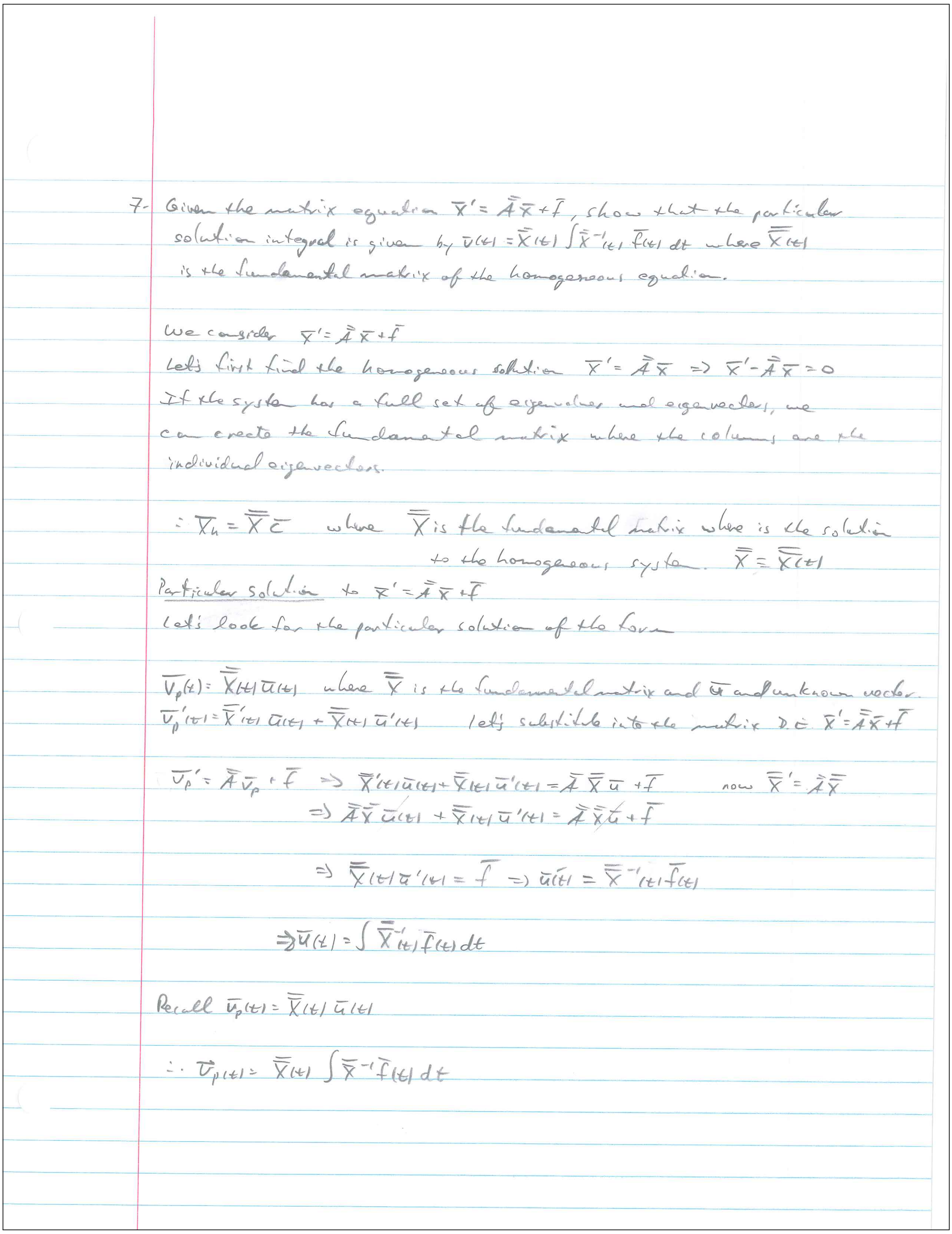
$$= 1 - \sin(t) + \cos(t)\Big|_0^t + \tfrac{1}{2}\int_0^t t\,d(-\cos(t))$$

$$= \cancel{1} - \sin(t) + \cos(t) - \cancel{1} + \tfrac{1}{2}\left(-t\cos(t)\Big|_0^t + \int_0^t \cos(t)\,dt\right)$$

$$= -\sin(t) + \cos(t) - \tfrac{1}{2}t\cos(t) + \tfrac{1}{2}\sin(t)\Big|_0^t$$

$$\cancel{\cos(t)} - \cancel{\tfrac{1}{2}\sin(t)} - \cancel{\tfrac{1}{2}t\cos(t)} = -\sin(t) + \cancel{\cos(t)} - \cancel{\tfrac{1}{2}t\cos(t)} + \tfrac{1}{2}\cancel{\sin(t)}$$

$$0 = 0 \qquad \text{checks}$$



7- Given the matrix equation $\overline{x}' = \overline{\overline{A}}\,\overline{x} + \overline{f}$, show that the particular solution integral is given by $\overline{v}(t) = \overline{\overline{X}}(t) \int \overline{\overline{X}}^{-1}(t)\, \overline{f}(t)\, dt$ where $\overline{\overline{X}}(t)$ is the fundamental matrix of the homogeneous equation.

We consider $\overline{x}' = \overline{\overline{A}}\,\overline{x} + \overline{f}$

Let's first find the homogeneous solution $\overline{x}' = \overline{\overline{A}}\,\overline{x} \Rightarrow \overline{x}' - \overline{\overline{A}}\,\overline{x} = 0$

If the system has a full set of eigenvalues and eigenvectors, we can create the fundamental matrix where the columns are the individual eigenvectors.

$\therefore \overline{x}_h = \overline{\overline{X}}\,\overline{c}$ where $\overline{\overline{X}}$ is the fundamental matrix where is the solution to the homogeneous system. $\overline{\overline{X}} = \overline{\overline{X}}(t)$

Particular Solution to $\overline{x}' = \overline{\overline{A}}\,\overline{x} + \overline{f}$

Let's look for the particular solution of the form

$\overline{v}_p(t) = \overline{\overline{X}}(t)\,\overline{u}(t)$ where $\overline{\overline{X}}$ is the fundamental matrix and $\overline{u}$ and unknown vector.

$\overline{v}_p'(t) = \overline{\overline{X}}'(t)\,\overline{u}(t) + \overline{\overline{X}}(t)\,\overline{u}'(t)$ let's substitute into the matrix D.E. $\overline{x}' = \overline{\overline{A}}\,\overline{x} + \overline{f}$

$$\overline{v}_p' = \overline{\overline{A}}\,\overline{v}_p + \overline{f} \Rightarrow \overline{\overline{X}}'(t)\,\overline{u}(t) + \overline{\overline{X}}(t)\,\overline{u}'(t) = \overline{\overline{A}}\,\overline{\overline{X}}\,\overline{u} + \overline{f} \qquad \text{now } \overline{\overline{X}}' = \overline{\overline{A}}\,\overline{\overline{X}}$$

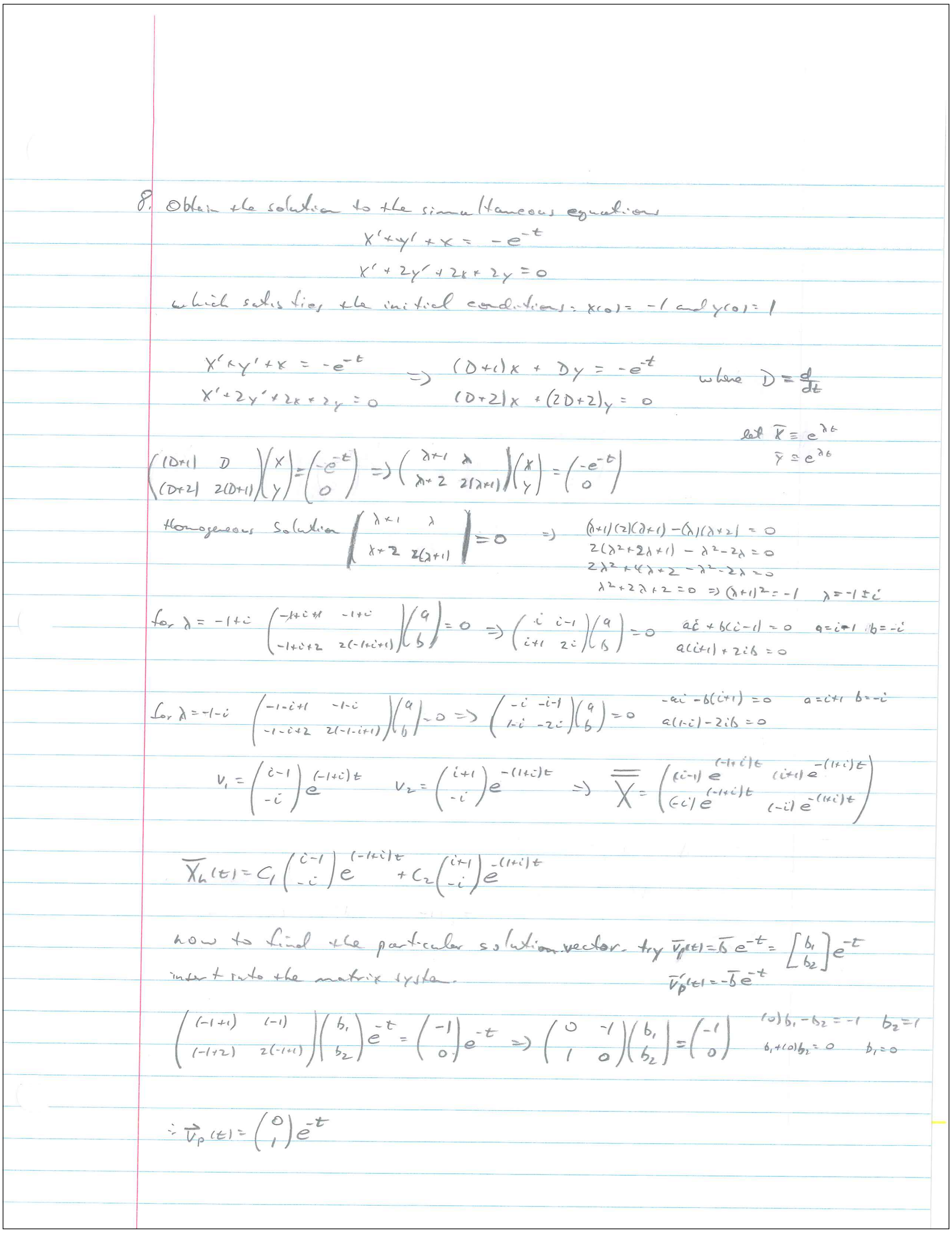
$$\Rightarrow \cancel{\overline{\overline{A}}\,\overline{\overline{X}}\,\overline{u}(t)} + \overline{\overline{X}}(t)\,\overline{u}'(t) = \cancel{\overline{\overline{A}}\,\overline{\overline{X}}\,\overline{u}} + \overline{f}$$

$$\Rightarrow \overline{\overline{X}}(t)\,\overline{u}'(t) = \overline{f} \Rightarrow \overline{u}'(t) = \overline{\overline{X}}^{-1}(t)\,\overline{f}(t)$$

$$\Rightarrow \overline{u}(t) = \int \overline{\overline{X}}^{-1}(t)\,\overline{f}(t)\,dt$$

Recall $\overline{v}_p(t) = \overline{\overline{X}}(t)\,\overline{u}(t)$

$$\therefore \overline{v}_p(t) = \overline{\overline{X}}(t) \int \overline{\overline{X}}^{-1}\,\overline{f}(t)\,dt$$



8. Obtain the solution to the simultaneous equations

$$x'+y'+x=-e^{-t}$$

$$x'+2y'+2x+2y=0$$

which satisfies the initial conditions: $x(0)=-1$ and $y(0)=1$

$$\begin{matrix} x'+y'+x=-e^{-t} \\ x'+2y'+2x+2y=0 \end{matrix} \Rightarrow \begin{matrix} (D+1)x+Dy=-e^{-t} \\ (D+2)x+(2D+2)y=0 \end{matrix} \quad \text{where } D=\frac{d}{dt}$$

let $\bar{x}\equiv e^{\lambda t}$

$\bar{y}\equiv e^{\lambda t}$

$$\begin{pmatrix} (D+1) & D \\ (D+2) & 2(D+1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda+1 & \lambda \\ \lambda+2 & 2(\lambda+1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}$$

Homogeneous Solution $\begin{vmatrix} \lambda+1 & \lambda \\ \lambda+2 & 2(\lambda+1) \end{vmatrix}=0 \Rightarrow$

$$(\lambda+1)(2)(\lambda+1)-(\lambda)(\lambda+2)=0$$

$$2(\lambda^2+2\lambda+1)-\lambda^2-2\lambda=0$$

$$2\lambda^2+4\lambda+2-\lambda^2-2\lambda=0$$

$$\lambda^2+2\lambda+2=0 \Rightarrow (\lambda+1)^2=-1 \quad \lambda=-1\pm i$$

for $\lambda=-1+i$ $\begin{pmatrix} \cancel{-1}+i\cancel{+1} & -1+i \\ -1+i+2 & 2(-1+i+1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=0 \Rightarrow \begin{pmatrix} i & i-1 \\ i+1 & 2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=0$

$ai+b(i-1)=0$

$a(i+1)+2ib=0$

$a=i+1 \quad b=-i$

for $\lambda=-1-i$ $\begin{pmatrix} -1-i+1 & -1-i \\ -1-i+2 & 2(-1-i+1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=0 \Rightarrow \begin{pmatrix} -i & -i-1 \\ 1-i & -2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=0$

$-ai-b(i+1)=0$

$a(1-i)-2ib=0$

$a=i+1 \quad b=-i$

$$v_1=\begin{pmatrix} i-1 \\ -i \end{pmatrix}e^{(-1+i)t} \quad v_2=\begin{pmatrix} i+1 \\ -i \end{pmatrix}e^{-(1+i)t} \Rightarrow \overline{\overline{X}}=\begin{pmatrix} (i-1)e^{(-1+i)t} & (i+1)e^{-(1+i)t} \\ (-i)e^{(-1+i)t} & (-i)e^{-(1+i)t} \end{pmatrix}$$

$$\overline{X}_h(t)=C_1\begin{pmatrix} i-1 \\ -i \end{pmatrix}e^{(-1+i)t}+C_2\begin{pmatrix} i+1 \\ -i \end{pmatrix}e^{-(1+i)t}$$

now to find the particular solution vector. try $\bar{v}_p(t)=\bar{b}e^{-t}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}e^{-t}$

insert into the matrix system.

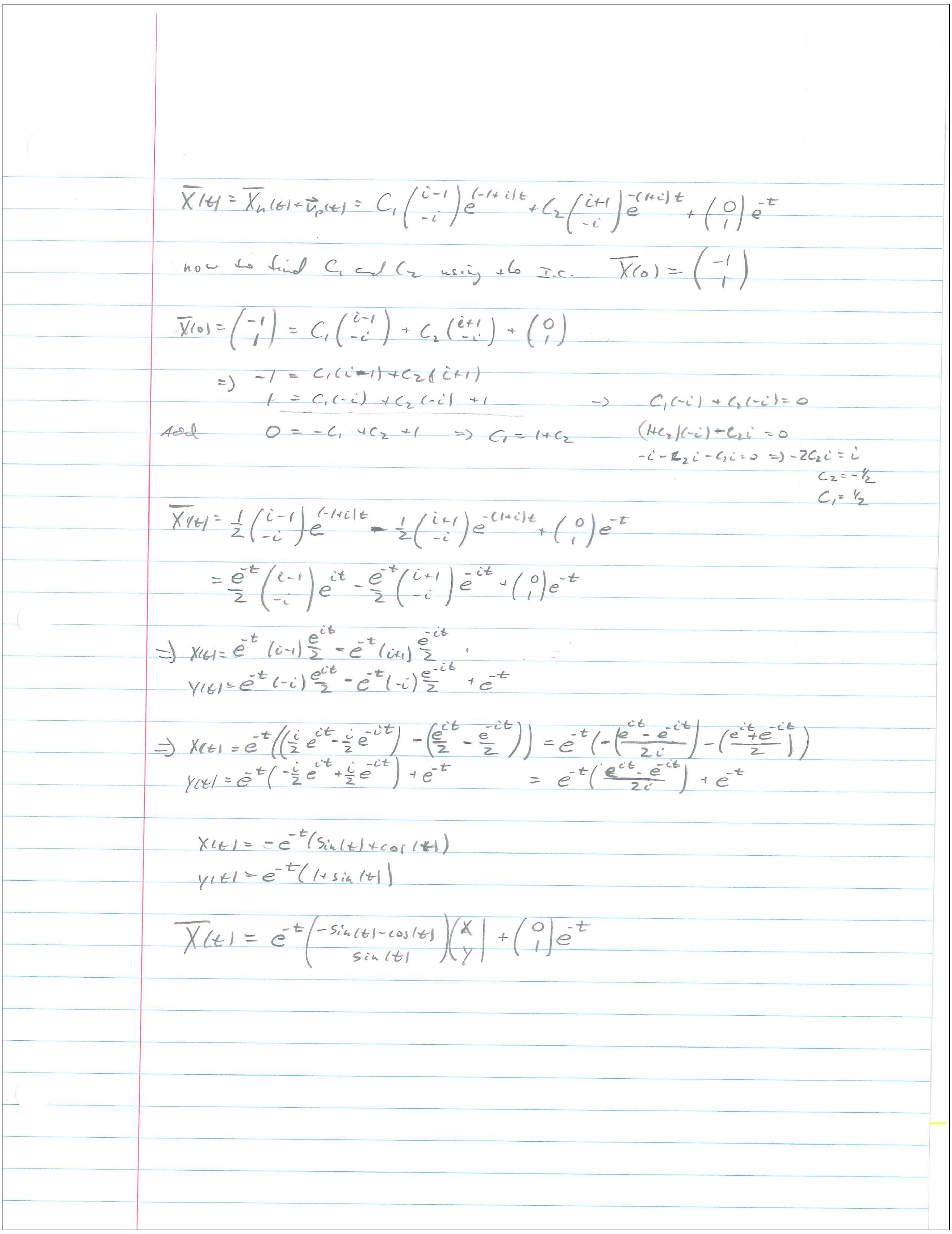
$\bar{v}_p'(t)=-\bar{b}e^{-t}$

$$\begin{pmatrix} (-1+1) & (-1) \\ (-1+2) & 2(-1+1) \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}e^{-t}=\begin{pmatrix} -1 \\ 0 \end{pmatrix}e^{-t} \Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$(0)b_1-b_2=-1 \quad b_2=1$

$b_1+(0)b_2=0 \quad b_1=0$

$$\therefore \vec{v}_p(t)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$$



$$\bar{X}(t) = \bar{X}_h(t) + \vec{v}_p(t) = C_1\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} + C_2\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

now to find $C_1$ and $C_2$ using the I.C. $\bar{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\bar{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} i-1 \\ -i \end{pmatrix} + C_2\begin{pmatrix} i+1 \\ -i \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad -1 = C_1(i-1) + C_2(i+1)$$

$$1 = C_1(-i) + C_2(-i) + 1 \quad \rightarrow \quad C_1(-i) + C_2(-i) = 0$$

Add $\quad 0 = -C_1 + C_2 + 1 \Rightarrow C_1 = 1 + C_2$

$$(1+C_2)(-i) + C_2 i = 0$$

$$-i - C_2 i - C_2 i = 0 \Rightarrow -2C_2 i = i$$

$$C_2 = -1/2$$

$$C_1 = 1/2$$

$$\bar{X}(t) = \frac{1}{2}\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} - \frac{1}{2}\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$= \frac{e^{-t}}{2}\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{it} - \frac{e^{-t}}{2}\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-it} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow x(t) = e^{-t}(i-1)\frac{e^{it}}{2} - e^{-t}(i+1)\frac{e^{-it}}{2},$$

$$y(t) = e^{-t}(-i)\frac{e^{it}}{2} - e^{-t}(-i)\frac{e^{-it}}{2} + e^{-t}$$

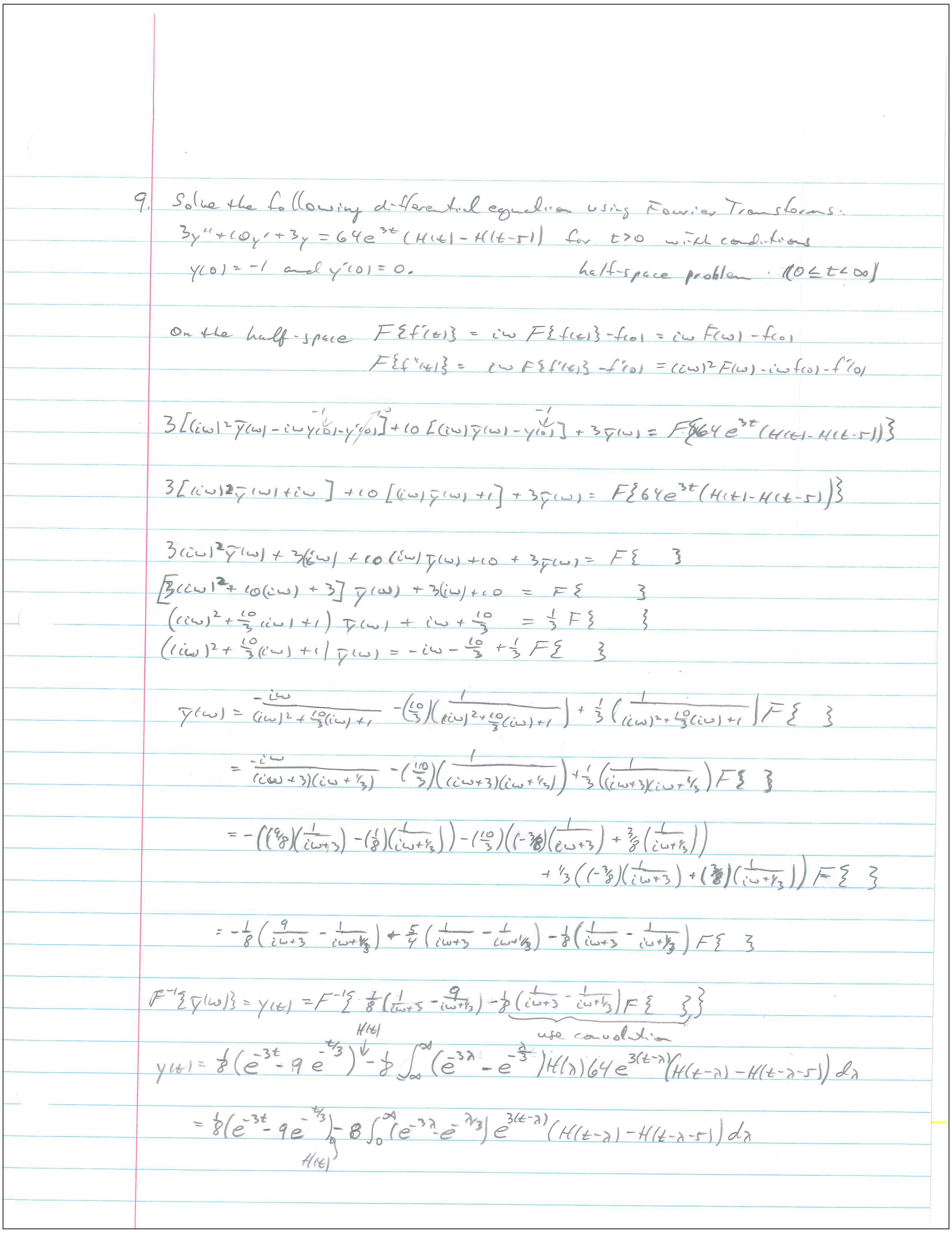
$$\Rightarrow x(t) = e^{-t}\left(\left(\frac{i}{2}e^{it} - \frac{i}{2}e^{-it}\right) - \left(\frac{e^{it}}{2} - \frac{e^{-it}}{2}\right)\right) = e^{-t}\left(-\left(\frac{e^{it} - e^{-it}}{2i}\right) - \left(\frac{e^{it} + e^{-it}}{2}\right)\right)$$

$$y(t) = e^{-t}\left(-\frac{i}{2}e^{it} + \frac{i}{2}e^{-it}\right) + e^{-t} \quad = e^{-t}\left(\frac{e^{it} - e^{-it}}{2i}\right) + e^{-t}$$

$$x(t) = -e^{-t}(\sin(t) + \cos(t))$$

$$y(t) = e^{-t}(1 + \sin(t))$$

$$\bar{X}(t) = e^{-t}\begin{pmatrix} -\sin(t) - \cos(t) \\ \sin(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$



9. Solve the following differential equation using Fourier Transforms:
$3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions
$y(0) = -1$ and $y'(0) = 0.$ half-space problem $(0 \le t < \infty)$

On the half-space $F\{f'(t)\} = i\omega F\{f(t)\} - f(0) = i\omega F(\omega) - f(0)$
$F\{f''(t)\} = i\omega F\{f'(t)\} - f'(0) = (i\omega)^2 F(\omega) - i\omega f(0) - f'(0)$

$$3[(i\omega)^2\bar{y}(\omega) - i\omega \overset{-1}{y(0)} - \overset{0}{\cancel{y'(0)}}] + 10[(i\omega)\bar{y}(\omega) - \overset{-1}{y(0)}] + 3\bar{y}(\omega) = F\{64e^{3t}(H(t) - H(t-5))\}$$

$$3[(i\omega)^2\bar{y}(\omega) + i\omega] + 10[(i\omega)\bar{y}(\omega) + 1] + 3\bar{y}(\omega) = F\{64e^{3t}(H(t) - H(t-5))\}$$

$$3(i\omega)^2\bar{y}(\omega) + 3(i\omega) + 10(i\omega)\bar{y}(\omega) + 10 + 3\bar{y}(\omega) = F\{\quad\}$$
$$[3(i\omega)^2 + 10(i\omega) + 3]\,\bar{y}(\omega) + 3(i\omega) + 10 = F\{\quad\}$$
$$((i\omega)^2 + \tfrac{10}{3}(i\omega) + 1)\,\bar{y}(\omega) + i\omega + \tfrac{10}{3} = \tfrac{1}{3}F\{\quad\}$$
$$((i\omega)^2 + \tfrac{10}{3}(i\omega) + 1)\,\bar{y}(\omega) = -i\omega - \tfrac{10}{3} + \tfrac{1}{3}F\{\quad\}$$

$$\bar{y}(\omega) = \frac{-i\omega}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1} - \left(\frac{10}{3}\right)\left(\frac{1}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1}\right) + \frac{1}{3}\left(\frac{1}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1}\right)F\{\quad\}$$

$$= \frac{-i\omega}{(i\omega+3)(i\omega+1/3)} - \left(\frac{10}{3}\right)\left(\frac{1}{(i\omega+3)(i\omega+1/3)}\right) + \frac{1}{3}\left(\frac{1}{(i\omega+3)(i\omega+1/3)}\right)F\{\quad\}$$

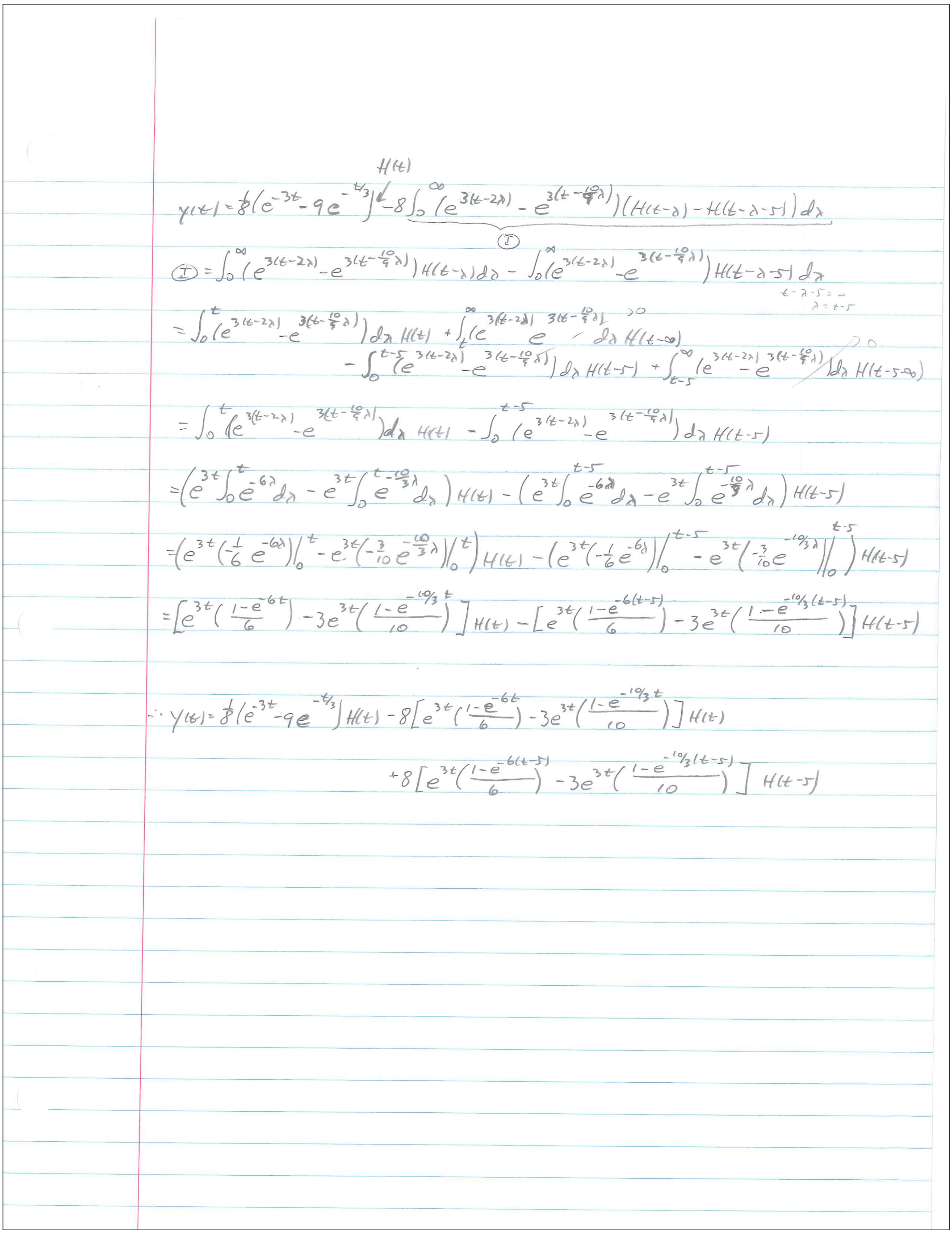
$$= -\left(\left(\tfrac{9}{8}\right)\left(\frac{1}{i\omega+3}\right) - \left(\tfrac{1}{8}\right)\left(\frac{1}{i\omega+1/3}\right)\right) - \left(\tfrac{10}{3}\right)\left(\left(-\tfrac{3}{8}\right)\left(\frac{1}{i\omega+3}\right) + \tfrac{3}{8}\left(\frac{1}{i\omega+1/3}\right)\right)$$
$$+ \tfrac{1}{3}\left(\left(-\tfrac{3}{8}\right)\left(\frac{1}{i\omega+3}\right) + \left(\tfrac{3}{8}\right)\left(\frac{1}{i\omega+1/3}\right)\right)F\{\quad\}$$

$$= -\frac{1}{8}\left(\frac{9}{i\omega+3} - \frac{1}{i\omega+1/3}\right) + \frac{5}{4}\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+1/3}\right) - \frac{1}{8}\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+1/3}\right)F\{\quad\}$$

$$F^{-1}\{\bar{y}(\omega)\} = y(t) = F^{-1}\left\{\frac{1}{8}\left(\frac{1}{i\omega+3} - \frac{9}{i\omega+1/3}\right) - \underbrace{\frac{1}{8}\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+1/3}\right)F\{\quad\}}_{\text{use convolution}}\right\}$$

$$y(t) = \frac{1}{8}\left(e^{-3t} + 9e^{-t/3}\right)\overset{H(t)}{\downarrow} - \frac{1}{8}\int_{\infty}^{\infty}\left(e^{-3\lambda} - e^{-\frac{\lambda}{3}}\right)H(\lambda)\,64e^{3(t-\lambda)}\left(H(t-\lambda) - H(t-\lambda-5)\right)d\lambda$$

$$= \frac{1}{8}\left(e^{-3t} - 9e^{-t/3}\right)\underset{H(t)}{\uparrow} - 8\int_{0}^{\infty}\left(e^{-3\lambda} - e^{-\lambda/3}\right)e^{3(t-\lambda)}\left(H(t-\lambda) - H(t-\lambda-5)\right)d\lambda$$



$$y(t)=\frac{1}{8}\left(e^{-3t}-9e^{-t/3}\right)\overset{H(t)}{\swarrow}-8\underbrace{\int_0^\infty\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)\left(H(t-\lambda)-H(t-\lambda-5)\right)d\lambda}_{①}$$

$$① = \int_0^\infty\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)H(t-\lambda)d\lambda-\int_0^\infty\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)H(t-\lambda-5)\,d\lambda$$

$t-\lambda-5=0$

$\lambda=t-5$

$$=\int_0^t\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t)+\int_t^\infty\left(e^{3(t-2\lambda)}\;e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t-\infty)\to 0$$

$$-\int_0^{t-5}\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t-5)+\cancel{\int_{t-5}^\infty\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t-5-\infty)}\to 0$$

$$=\int_0^t\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t)-\int_0^{t-5}\left(e^{3(t-2\lambda)}-e^{3(t-\frac{10}{9}\lambda)}\right)d\lambda\,H(t-5)$$

$$=\left(e^{3t}\int_0^t e^{-6\lambda}d\lambda-e^{3t}\int_0^t e^{-\frac{10}{3}\lambda}d\lambda\right)H(t)-\left(e^{3t}\int_0^{t-5}e^{-6\lambda}d\lambda-e^{3t}\int_0^{t-5}e^{-\frac{10}{3}\lambda}d\lambda\right)H(t-5)$$

$$=\left(e^{3t}\left(-\frac{1}{6}e^{-6\lambda}\right)\Big|_0^t-e^{3t}\left(-\frac{3}{10}e^{-\frac{10}{3}\lambda}\right)\Big|_0^t\right)H(t)-\left(e^{3t}\left(-\frac{1}{6}e^{-6\lambda}\right)\Big|_0^{t-5}-e^{3t}\left(-\frac{3}{10}e^{-10/3\lambda}\right)\Big|_0^{t-5}\right)H(t-5)$$

$$=\left[e^{3t}\left(\frac{1-e^{-6t}}{6}\right)-3e^{3t}\left(\frac{1-e^{-10/3\,t}}{10}\right)\right]H(t)-\left[e^{3t}\left(\frac{1-e^{-6(t-5)}}{6}\right)-3e^{3t}\left(\frac{1-e^{-10/3(t-5)}}{10}\right)\right]H(t-5)$$

$$\therefore y(t)=\frac{1}{8}\left(e^{-3t}-9e^{-t/3}\right)H(t)-8\left[e^{3t}\left(\frac{1-e^{-6t}}{6}\right)-3e^{3t}\left(\frac{1-e^{-10/3\,t}}{10}\right)\right]H(t)$$

$$+8\left[e^{3t}\left(\frac{1-e^{-6(t-5)}}{6}\right)-3e^{3t}\left(\frac{1-e^{-10/3(t-5)}}{10}\right)\right]H(t-5)$$

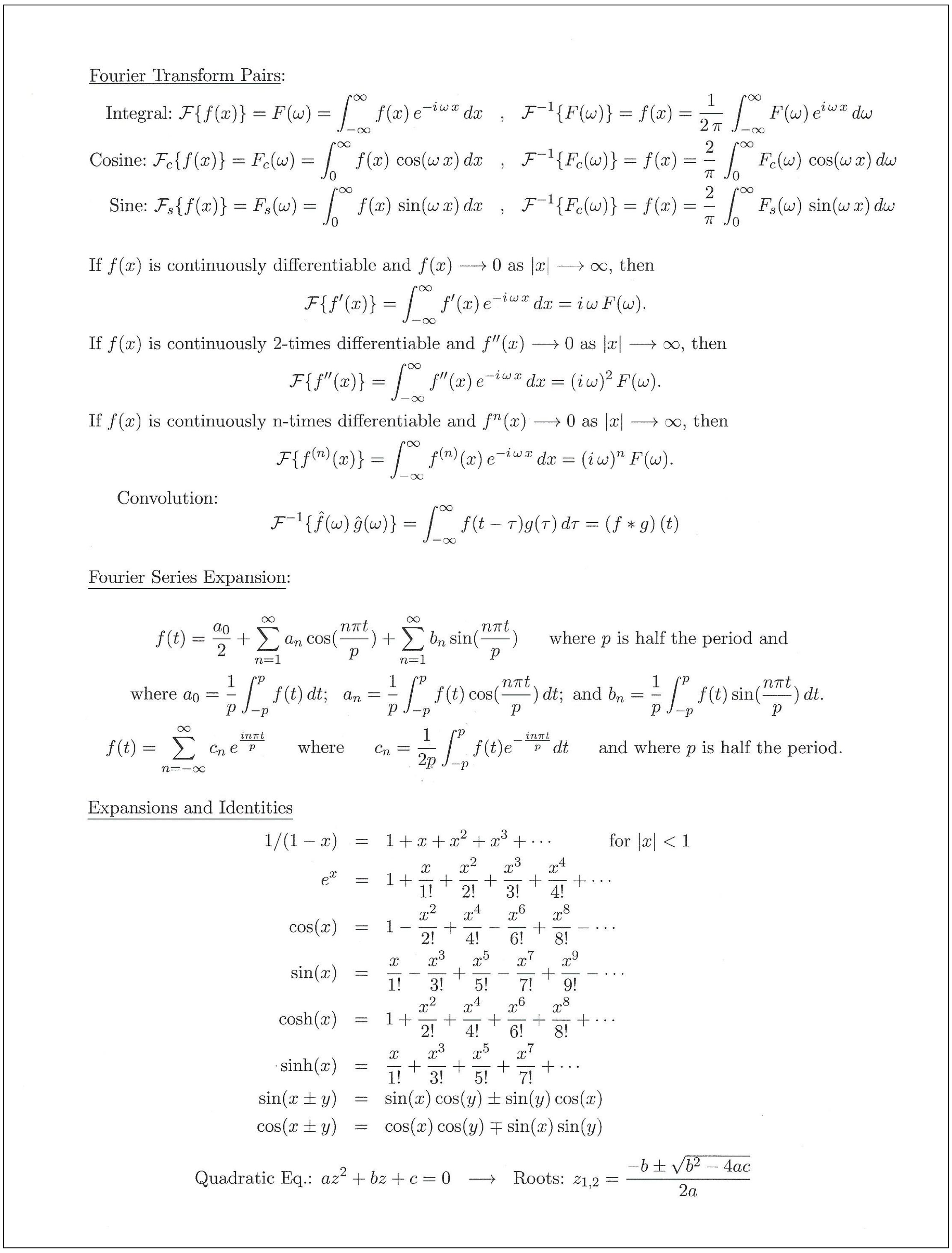
# Chapter 3

# cheat sheets, study notes

## Local contents

## 3.1 Integrating factors sheet and Transform tables

These are few reference sheets we used in course from difference references. (integrating factors, and transform tables)



Fourier Transform Pairs:

Integral: $\mathcal{F}\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\,\omega\, x}\, dx \quad , \quad \mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\,\omega\, x}\, d\omega$

Cosine: $\mathcal{F}_c\{f(x)\} = F_c(\omega) = \int_{0}^{\infty} f(x)\, \cos(\omega\, x)\, dx \quad , \quad \mathcal{F}^{-1}\{F_c(\omega)\} = f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_c(\omega)\, \cos(\omega\, x)\, d\omega$

Sine: $\mathcal{F}_s\{f(x)\} = F_s(\omega) = \int_{0}^{\infty} f(x)\, \sin(\omega\, x)\, dx \quad , \quad \mathcal{F}^{-1}\{F_c(\omega)\} = f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_s(\omega)\, \sin(\omega\, x)\, d\omega$

If $f(x)$ is continuously differentiable and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, then

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\, e^{-i\,\omega\, x}\, dx = i\,\omega\, F(\omega).$$

If $f(x)$ is continuously 2-times differentiable and $f''(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, then

$$\mathcal{F}\{f''(x)\} = \int_{-\infty}^{\infty} f''(x)\, e^{-i\,\omega\, x}\, dx = (i\,\omega)^2\, F(\omega).$$

If $f(x)$ is continuously n-times differentiable and $f^n(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, then

$$\mathcal{F}\{f^{(n)}(x)\} = \int_{-\infty}^{\infty} f^{(n)}(x)\, e^{-i\,\omega\, x}\, dx = (i\,\omega)^n\, F(\omega).$$

Convolution:

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\, \hat{g}(\omega)\} = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\, d\tau = (f * g)\,(t)$$

Fourier Series Expansion:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{p}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{p}) \qquad \text{where } p \text{ is half the period and}$$

$$\text{where } a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\, dt; \quad a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos(\frac{n\pi t}{p})\, dt; \text{ and } b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin(\frac{n\pi t}{p})\, dt.$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi t}{p}} \qquad \text{where} \qquad c_n = \frac{1}{2p}\int_{-p}^{p} f(t) e^{-\frac{in\pi t}{p}} dt \qquad \text{and where } p \text{ is half the period.}$$

Expansions and Identities

$$\begin{aligned}
1/(1-x) &= 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1 \\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
\sin(x) &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\
\cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\
\sinh(x) &= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \\
\sin(x \pm y) &= \sin(x)\cos(y) \pm \sin(y)\cos(x) \\
\cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y)
\end{aligned}$$

Quadratic Eq.: $az^2 + bz + c = 0 \longrightarrow$ Roots: $z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

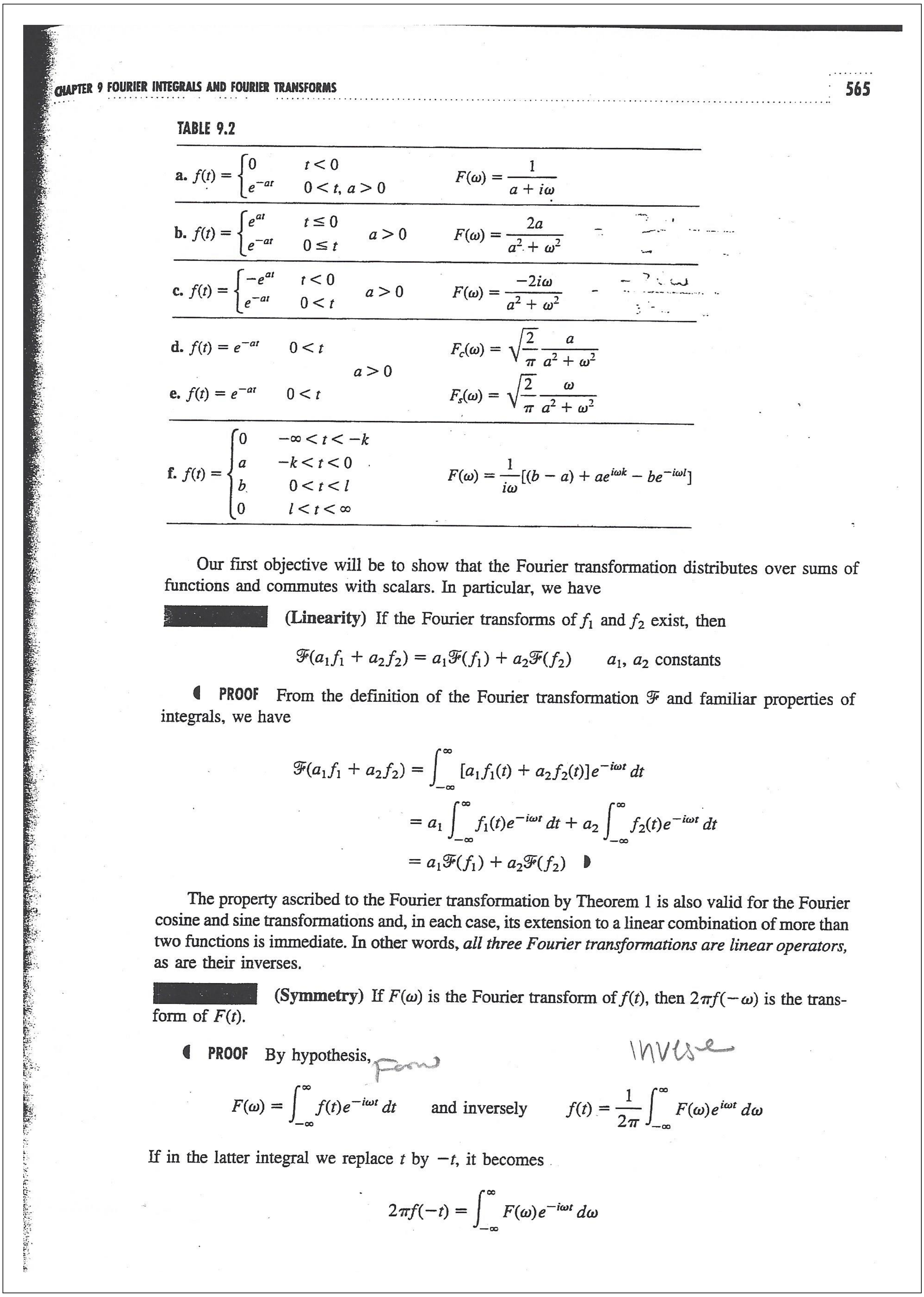


**TABLE 9.2**

| | | |
|---|---|---|
| **a.** $f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & 0 < t,\ a > 0 \end{cases}$ | | $F(\omega) = \dfrac{1}{a + i\omega}$ |
| **b.** $f(t) = \begin{cases} e^{at} & t \le 0 \\ e^{-at} & 0 \le t \end{cases}$ | $a > 0$ | $F(\omega) = \dfrac{2a}{a^2 + \omega^2}$ |
| **c.** $f(t) = \begin{cases} -e^{at} & t < 0 \\ e^{-at} & 0 < t \end{cases}$ | $a > 0$ | $F(\omega) = \dfrac{-2i\omega}{a^2 + \omega^2}$ |
| **d.** $f(t) = e^{-at}$ $\quad 0 < t$ | $a > 0$ | $F_c(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{a}{a^2 + \omega^2}$ |
| **e.** $f(t) = e^{-at}$ $\quad 0 < t$ | | $F_s(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{\omega}{a^2 + \omega^2}$ |
| **f.** $f(t) = \begin{cases} 0 & -\infty < t < -k \\ a & -k < t < 0 \\ b & 0 < t < l \\ 0 & l < t < \infty \end{cases}$ | | $F(\omega) = \dfrac{1}{i\omega}[(b - a) + ae^{i\omega k} - be^{-i\omega l}]$ |

Our first objective will be to show that the Fourier transformation distributes over sums of functions and commutes with scalars. In particular, we have

[redacted] **(Linearity)** If the Fourier transforms of $f_1$ and $f_2$ exist, then

$$\mathcal{F}(a_1 f_1 + a_2 f_2) = a_1\mathcal{F}(f_1) + a_2\mathcal{F}(f_2) \qquad a_1, a_2 \text{ constants}$$

◖ **PROOF** From the definition of the Fourier transformation $\mathcal{F}$ and familiar properties of integrals, we have

$$\begin{aligned} \mathcal{F}(a_1 f_1 + a_2 f_2) &= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)]e^{-i\omega t}\,dt \\ &= a_1 \int_{-\infty}^{\infty} f_1(t)e^{-i\omega t}\,dt + a_2 \int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}\,dt \\ &= a_1\mathcal{F}(f_1) + a_2\mathcal{F}(f_2) \end{aligned}$$ ◗

The property ascribed to the Fourier transformation by Theorem 1 is also valid for the Fourier cosine and sine transformations and, in each case, its extension to a linear combination of more than two functions is immediate. In other words, *all three Fourier transformations are linear operators,* as are their inverses.

[redacted] **(Symmetry)** If $F(\omega)$ is the Fourier transform of $f(t)$, then $2\pi f(-\omega)$ is the transform of $F(t)$.

◖ **PROOF** By hypothesis, Form

Inverse

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \qquad \text{and inversely} \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

If in the latter integral we replace $t$ by $-t$, it becomes

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\,d\omega$$

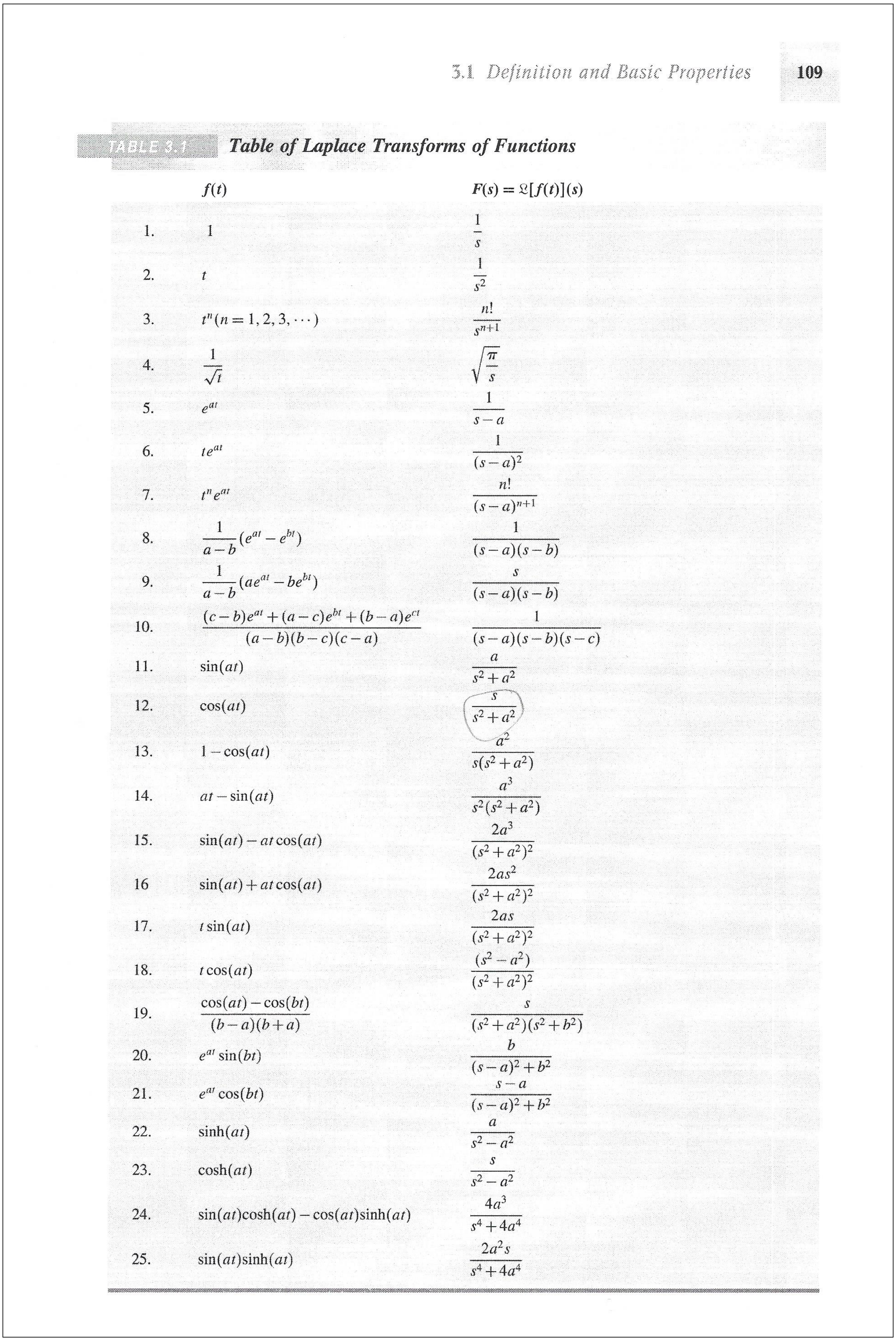


**TABLE 3.1** **Table of Laplace Transforms of Functions**

| | $f(t)$ | $F(s) = \mathfrak{L}[f(t)](s)$ |
|---|---|---|
| 1. | $1$ | $\frac{1}{s}$ |
| 2. | $t$ | $\frac{1}{s^2}$ |
| 3. | $t^n (n = 1, 2, 3, \cdots)$ | $\frac{n!}{s^{n+1}}$ |
| 4. | $\frac{1}{\sqrt{t}}$ | $\sqrt{\frac{\pi}{s}}$ |
| 5. | $e^{at}$ | $\frac{1}{s-a}$ |
| 6. | $te^{at}$ | $\frac{1}{(s-a)^2}$ |
| 7. | $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| 8. | $\frac{1}{a-b}(e^{at} - e^{bt})$ | $\frac{1}{(s-a)(s-b)}$ |
| 9. | $\frac{1}{a-b}(ae^{at} - be^{bt})$ | $\frac{s}{(s-a)(s-b)}$ |
| 10. | $\frac{(c-b)e^{at} + (a-c)e^{bt} + (b-a)e^{ct}}{(a-b)(b-c)(c-a)}$ | $\frac{1}{(s-a)(s-b)(s-c)}$ |
| 11. | $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| 12. | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| 13. | $1 - \cos(at)$ | $\frac{a^2}{s(s^2+a^2)}$ |
| 14. | $at - \sin(at)$ | $\frac{a^3}{s^2(s^2+a^2)}$ |
| 15. | $\sin(at) - at\cos(at)$ | $\frac{2a^3}{(s^2+a^2)^2}$ |
| 16 | $\sin(at) + at\cos(at)$ | $\frac{2as^2}{(s^2+a^2)^2}$ |
| 17. | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ |
| 18. | $t\cos(at)$ | $\frac{(s^2-a^2)}{(s^2+a^2)^2}$ |
| 19. | $\frac{\cos(at) - \cos(bt)}{(b-a)(b+a)}$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}$ |
| 20. | $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| 21. | $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| 22. | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ |
| 23. | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| 24. | $\sin(at)\cosh(at) - \cos(at)\sinh(at)$ | $\frac{4a^3}{s^4+4a^4}$ |
| 25. | $\sin(at)\sinh(at)$ | $\frac{2a^2 s}{s^4+4a^4}$ |

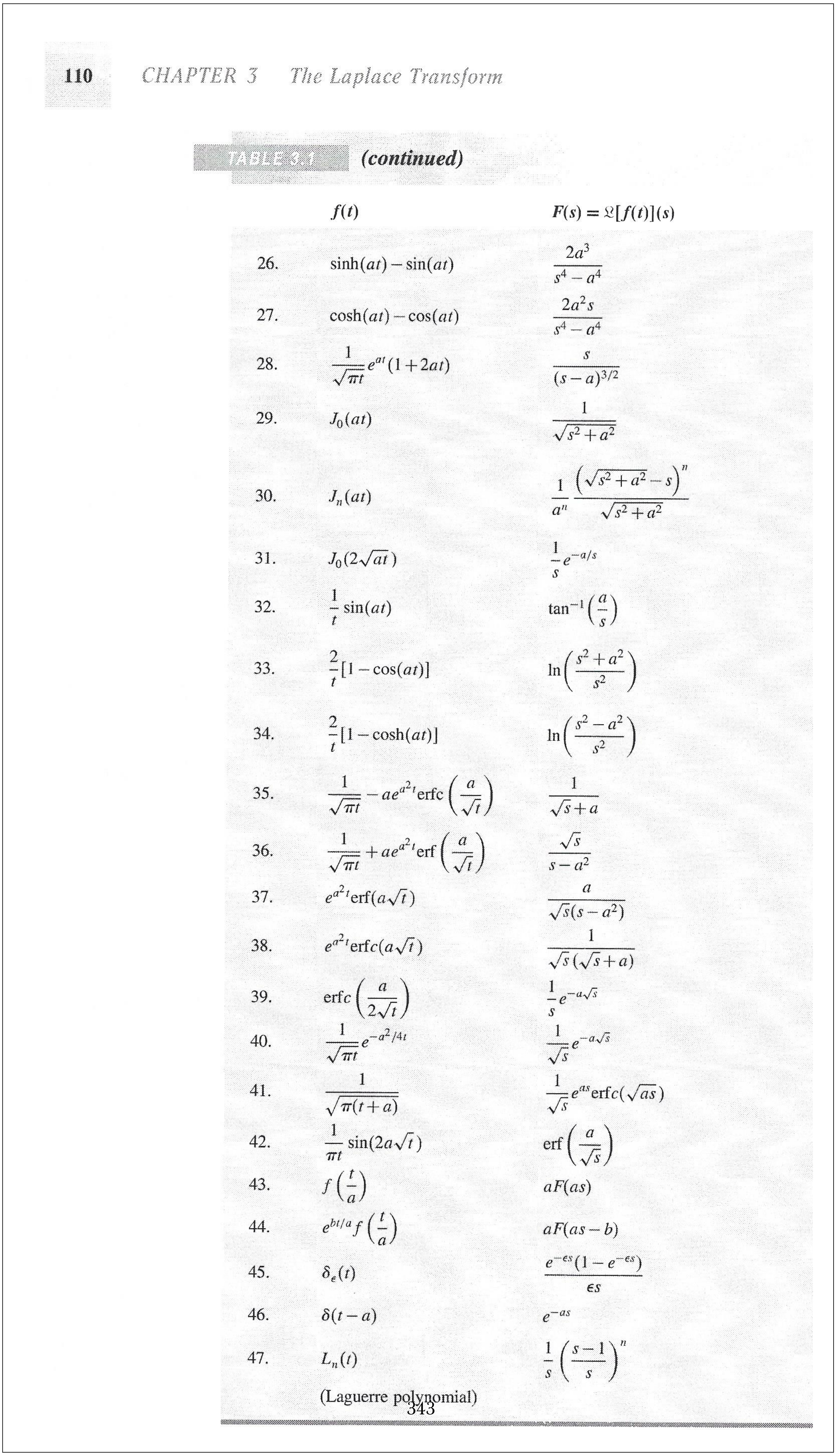
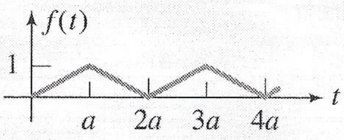
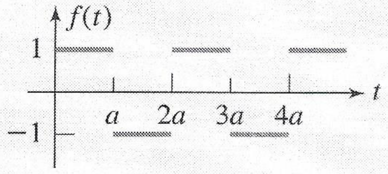
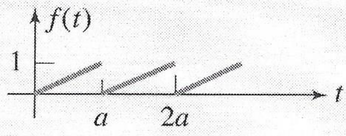


**TABLE 3.1 (continued)**

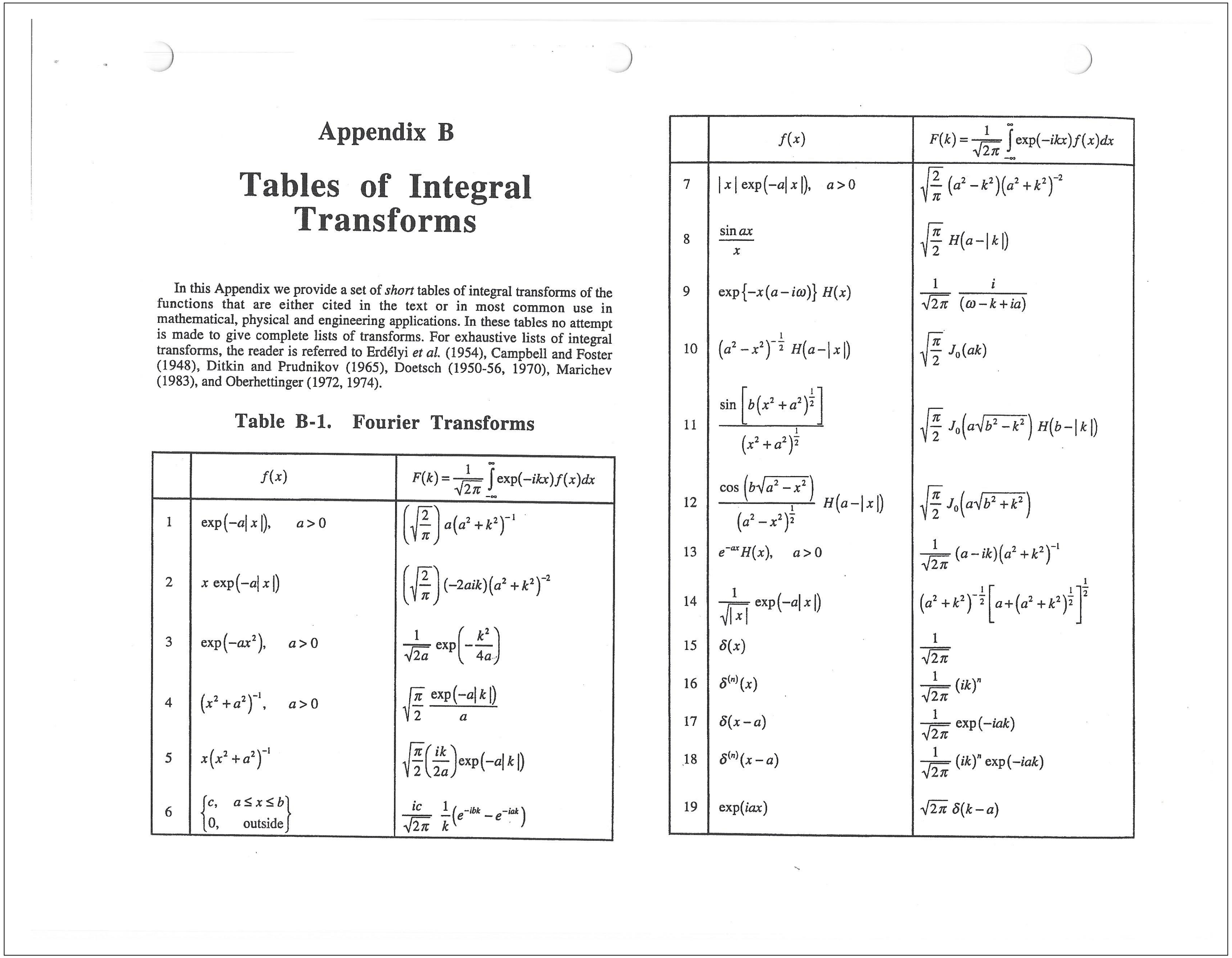
| | $f(t)$ | $F(s) = \mathfrak{L}[f(t)](s)$ |
|---|---|---|
| 26. | $\sinh(at) - \sin(at)$ | $\frac{2a^3}{s^4 - a^4}$ |
| 27. | $\cosh(at) - \cos(at)$ | $\frac{2a^2 s}{s^4 - a^4}$ |
| 28. | $\frac{1}{\sqrt{\pi t}} e^{at}(1+2at)$ | $\frac{s}{(s-a)^{3/2}}$ |
| 29. | $J_0(at)$ | $\frac{1}{\sqrt{s^2+a^2}}$ |
| 30. | $J_n(at)$ | $\frac{1}{a^n} \frac{\left(\sqrt{s^2+a^2} - s\right)^n}{\sqrt{s^2+a^2}}$ |
| 31. | $J_0(2\sqrt{at})$ | $\frac{1}{s} e^{-a/s}$ |
| 32. | $\frac{1}{t} \sin(at)$ | $\tan^{-1}\left(\frac{a}{s}\right)$ |
| 33. | $\frac{2}{t}[1 - \cos(at)]$ | $\ln\left(\frac{s^2+a^2}{s^2}\right)$ |
| 34. | $\frac{2}{t}[1 - \cosh(at)]$ | $\ln\left(\frac{s^2-a^2}{s^2}\right)$ |
| 35. | $\frac{1}{\sqrt{\pi t}} - ae^{a^2 t}\operatorname{erfc}\left(\frac{a}{\sqrt{t}}\right)$ | $\frac{1}{\sqrt{s}+a}$ |
| 36. | $\frac{1}{\sqrt{\pi t}} + ae^{a^2 t}\operatorname{erf}\left(\frac{a}{\sqrt{t}}\right)$ | $\frac{\sqrt{s}}{s-a^2}$ |
| 37. | $e^{a^2 t}\operatorname{erf}(a\sqrt{t})$ | $\frac{a}{\sqrt{s}(s-a^2)}$ |
| 38. | $e^{a^2 t}\operatorname{erfc}(a\sqrt{t})$ | $\frac{1}{\sqrt{s}\,(\sqrt{s}+a)}$ |
| 39. | $\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$ | $\frac{1}{s} e^{-a\sqrt{s}}$ |
| 40. | $\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$ | $\frac{1}{\sqrt{s}} e^{-a\sqrt{s}}$ |
| 41. | $\frac{1}{\sqrt{\pi(t+a)}}$ | $\frac{1}{\sqrt{s}} e^{as}\operatorname{erfc}(\sqrt{as})$ |
| 42. | $\frac{1}{\pi t} \sin(2a\sqrt{t})$ | $\operatorname{erf}\left(\frac{a}{\sqrt{s}}\right)$ |
| 43. | $f\left(\frac{t}{a}\right)$ | $aF(as)$ |
| 44. | $e^{bt/a} f\left(\frac{t}{a}\right)$ | $aF(as-b)$ |
| 45. | $\delta_\epsilon(t)$ | $\frac{e^{-\epsilon s}(1-e^{-\epsilon s})}{\epsilon s}$ |
| 46. | $\delta(t-a)$ | $e^{-as}$ |
| 47. | $L_n(t)$<br>(Laguerre polynomial) | $\frac{1}{s}\left(\frac{s-1}{s}\right)^n$ |

**TABLE 3.1** ***(continued)***

| | $f(t)$ | $F(s) = \mathfrak{L}[f(t)](s)$ |
|---|---|---|
| 48. | $\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(t)$<br>(Hermite polynomial) | $\frac{(1-s)^n}{s^{n+1/2}}$ |
| 49. | $\frac{-n!}{\sqrt{\pi}(2n+1)!}H_{2n+1}(t)$<br>(Hermite polynomial) | $\frac{(1-s)^n}{s^{n+3/2}}$ |
| 50. | triangular wave<br><br> | $\frac{1}{as^2}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]\left(=\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)\right)$ |
| 51. | square wave<br><br> | $\frac{1}{s}\tanh\left(\frac{as}{2}\right)$ |
| 52. | sawtooth wave<br><br> | $\frac{1}{as^2}-\frac{e^{-as}}{s(1-e^{-as})}$ |

### *Operational Formulas*

| $f(t)$ | $F(s)$ |
|---|---|
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ |
| $f'(t)$ | $sF(s)-f(0+)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$ |
| $\int_0^t f(\tau)\,d\tau$ | $\frac{1}{s}F(s)$ |
| $tf(t)$ | $-F'(s)$ |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ |
| $\frac{1}{t}f(t)$ | $\int_s^\infty F(\sigma)\,d\sigma$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $f(t-a)H(t-a)$ | $e^{-as}F(s)$ |
| $f(t+\tau)=f(t)$<br>(periodic) | $\frac{1}{1-e^{-\tau s}}\int_0^\tau e^{-st}f(t)\,dt$ |



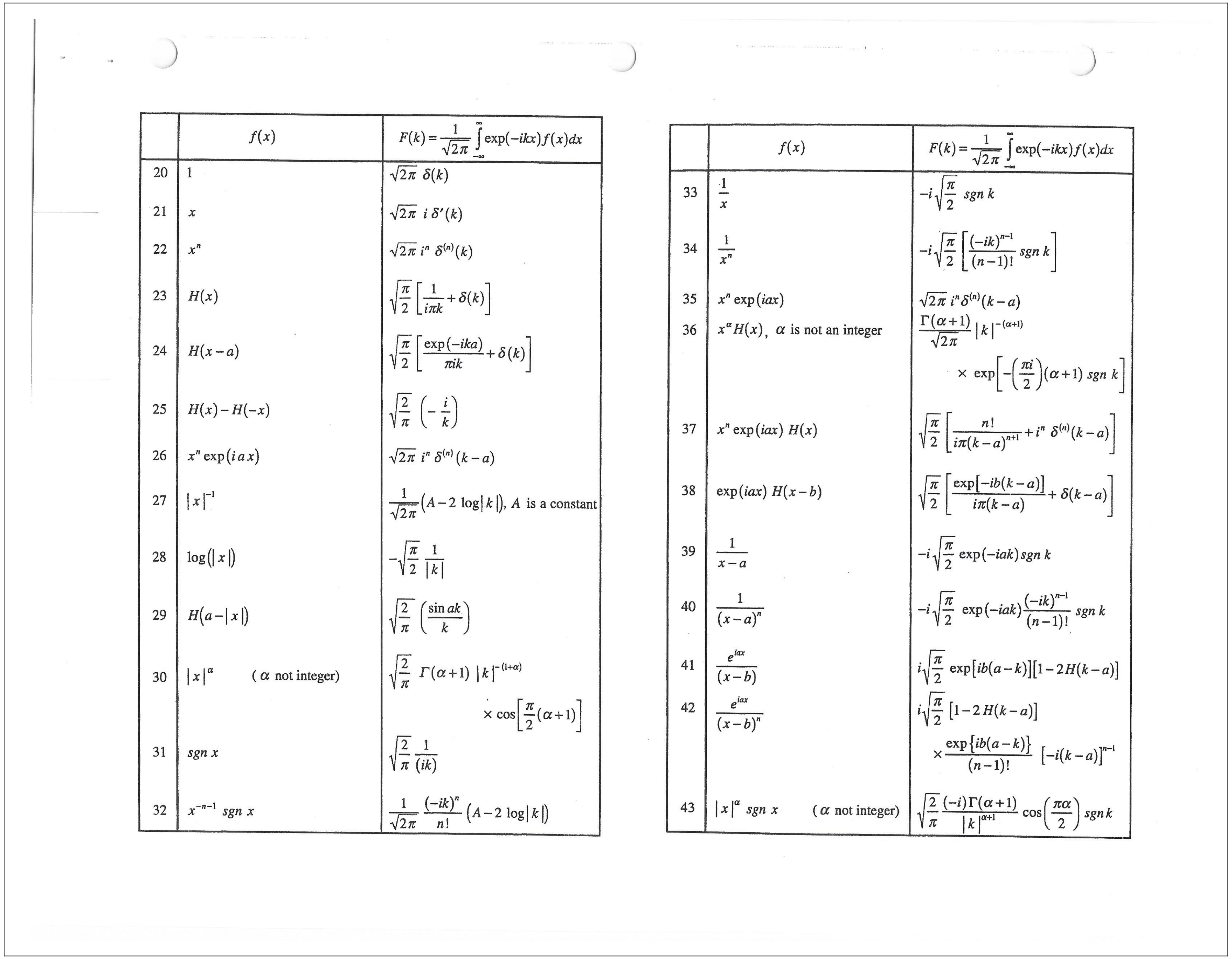
## Appendix B

# Tables of Integral Transforms

In this Appendix we provide a set of *short* tables of integral transforms of the functions that are either cited in the text or in most common use in mathematical, physical and engineering applications. In these tables no attempt is made to give complete lists of transforms. For exhaustive lists of integral transforms, the reader is referred to Erdélyi *et al.* (1954), Campbell and Foster (1948), Ditkin and Prudnikov (1965), Doetsch (1950-56, 1970), Marichev (1983), and Oberhettinger (1972, 1974).

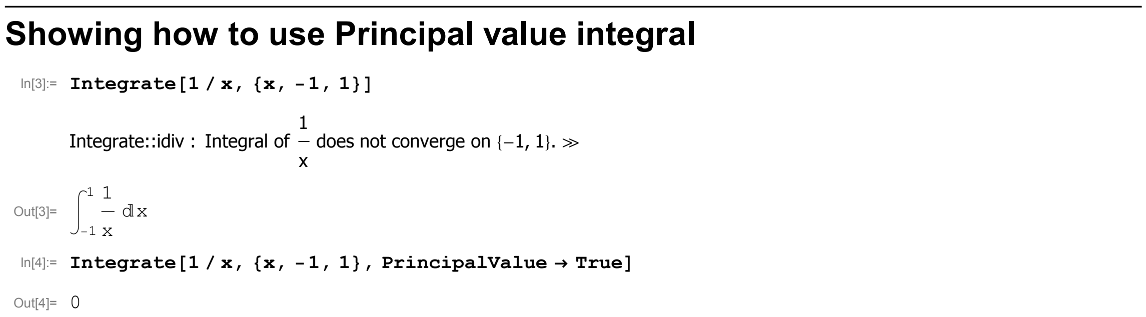
### Table B-1. Fourier Transforms

| | $f(x)$ | $F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)f(x)dx$ |
|---|---|---|
| 1 | $\exp(-a\|x\|), \quad a>0$ | $\left(\sqrt{\frac{2}{\pi}}\right) a\left(a^2+k^2\right)^{-1}$ |
| 2 | $x\exp(-a\|x\|)$ | $\left(\sqrt{\frac{2}{\pi}}\right)(-2aik)\left(a^2+k^2\right)^{-2}$ |
| 3 | $\exp(-ax^2), \quad a>0$ | $\frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right)$ |
| 4 | $\left(x^2+a^2\right)^{-1}, \quad a>0$ | $\sqrt{\frac{\pi}{2}}\frac{\exp(-a\|k\|)}{a}$ |
| 5 | $x\left(x^2+a^2\right)^{-1}$ | $\sqrt{\frac{\pi}{2}}\left(\frac{ik}{2a}\right)\exp(-a\|k\|)$ |
| 6 | $\begin{Bmatrix} c, & a\le x\le b \\ 0, & \text{outside} \end{Bmatrix}$ | $\frac{ic}{\sqrt{2\pi}}\frac{1}{k}\left(e^{-ibk}-e^{-iak}\right)$ |
| 7 | $\|x\|\exp(-a\|x\|), \quad a>0$ | $\sqrt{\frac{2}{\pi}}\left(a^2-k^2\right)\left(a^2+k^2\right)^{-2}$ |
| 8 | $\frac{\sin ax}{x}$ | $\sqrt{\frac{\pi}{2}}\,H(a-\|k\|)$ |
| 9 | $\exp\{-x(a-i\omega)\}\,H(x)$ | $\frac{1}{\sqrt{2\pi}}\frac{i}{(\omega-k+ia)}$ |
| 10 | $\left(a^2-x^2\right)^{-\frac{1}{2}}H(a-\|x\|)$ | $\sqrt{\frac{\pi}{2}}\,J_0(ak)$ |
| 11 | $\frac{\sin\left[b\left(x^2+a^2\right)^{\frac{1}{2}}\right]}{\left(x^2+a^2\right)^{\frac{1}{2}}}$ | $\sqrt{\frac{\pi}{2}}\,J_0\left(a\sqrt{b^2-k^2}\right)H(b-\|k\|)$ |
| 12 | $\frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\left(a^2-x^2\right)^{\frac{1}{2}}}H(a-\|x\|)$ | $\sqrt{\frac{\pi}{2}}\,J_0\left(a\sqrt{b^2+k^2}\right)$ |
| 13 | $e^{-ax}H(x), \quad a>0$ | $\frac{1}{\sqrt{2\pi}}(a-ik)\left(a^2+k^2\right)^{-1}$ |
| 14 | $\frac{1}{\sqrt{\|x\|}}\exp(-a\|x\|)$ | $\left(a^2+k^2\right)^{-\frac{1}{2}}\left[a+\left(a^2+k^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ |
| 15 | $\delta(x)$ | $\frac{1}{\sqrt{2\pi}}$ |
| 16 | $\delta^{(n)}(x)$ | $\frac{1}{\sqrt{2\pi}}(ik)^n$ |
| 17 | $\delta(x-a)$ | $\frac{1}{\sqrt{2\pi}}\exp(-iak)$ |
| 18 | $\delta^{(n)}(x-a)$ | $\frac{1}{\sqrt{2\pi}}(ik)^n\exp(-iak)$ |
| 19 | $\exp(iax)$ | $\sqrt{2\pi}\,\delta(k-a)$ |



| | $f(x)$ | $F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)f(x)dx$ |
|---|---|---|
| 20 | $1$ | $\sqrt{2\pi}\ \delta(k)$ |
| 21 | $x$ | $\sqrt{2\pi}\ i\,\delta'(k)$ |
| 22 | $x^n$ | $\sqrt{2\pi}\ i^n\ \delta^{(n)}(k)$ |
| 23 | $H(x)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{1}{i\pi k}+\delta(k)\right]$ |
| 24 | $H(x-a)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{\exp(-ika)}{\pi ik}+\delta(k)\right]$ |
| 25 | $H(x)-H(-x)$ | $\sqrt{\frac{2}{\pi}}\left(-\frac{i}{k}\right)$ |
| 26 | $x^n\exp(iax)$ | $\sqrt{2\pi}\ i^n\ \delta^{(n)}(k-a)$ |
| 27 | $\lvert x\rvert^{-1}$ | $\frac{1}{\sqrt{2\pi}}(A-2\log\lvert k\rvert)$, $A$ is a constant |
| 28 | $\log(\lvert x\rvert)$ | $-\sqrt{\frac{\pi}{2}}\frac{1}{\lvert k\rvert}$ |
| 29 | $H(a-\lvert x\rvert)$ | $\sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right)$ |
| 30 | $\lvert x\rvert^{\alpha}$ ( $\alpha$ not integer) | $\sqrt{\frac{2}{\pi}}\ \Gamma(\alpha+1)\ \lvert k\rvert^{-(1+\alpha)}$<br>$\times\cos\left[\frac{\pi}{2}(\alpha+1)\right]$ |
| 31 | $sgn\ x$ | $\sqrt{\frac{2}{\pi}}\frac{1}{(ik)}$ |
| 32 | $x^{-n-1}\ sgn\ x$ | $\frac{1}{\sqrt{2\pi}}\frac{(-ik)^n}{n!}(A-2\log\lvert k\rvert)$ |
| 33 | $\frac{1}{x}$ | $-i\sqrt{\frac{\pi}{2}}\ sgn\ k$ |
| 34 | $\frac{1}{x^n}$ | $-i\sqrt{\frac{\pi}{2}}\left[\frac{(-ik)^{n-1}}{(n-1)!}\ sgn\ k\right]$ |
| 35 | $x^n\exp(iax)$ | $\sqrt{2\pi}\ i^n\delta^{(n)}(k-a)$ |
| 36 | $x^{\alpha}H(x)$, $\alpha$ is not an integer | $\frac{\Gamma(\alpha+1)}{\sqrt{2\pi}}\lvert k\rvert^{-(\alpha+1)}$<br>$\times\ \exp\left[-\left(\frac{\pi i}{2}\right)(\alpha+1)\ sgn\ k\right]$ |
| 37 | $x^n\exp(iax)\ H(x)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{n!}{i\pi(k-a)^{n+1}}+i^n\ \delta^{(n)}(k-a)\right]$ |
| 38 | $\exp(iax)\ H(x-b)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{\exp[-ib(k-a)]}{i\pi(k-a)}+\delta(k-a)\right]$ |
| 39 | $\frac{1}{x-a}$ | $-i\sqrt{\frac{\pi}{2}}\exp(-iak)sgn\ k$ |
| 40 | $\frac{1}{(x-a)^n}$ | $-i\sqrt{\frac{\pi}{2}}\ \exp(-iak)\frac{(-ik)^{n-1}}{(n-1)!}\ sgn\ k$ |
| 41 | $\frac{e^{iax}}{(x-b)}$ | $i\sqrt{\frac{\pi}{2}}\ \exp[ib(a-k)][1-2H(k-a)]$ |
| 42 | $\frac{e^{iax}}{(x-b)^n}$ | $i\sqrt{\frac{\pi}{2}}\ [1-2H(k-a)]$<br>$\times\frac{\exp\{ib(a-k)\}}{(n-1)!}\ [-i(k-a)]^{n-1}$ |
| 43 | $\lvert x\rvert^{\alpha}\ sgn\ x$ ( $\alpha$ not integer) | $\sqrt{\frac{2}{\pi}}\frac{(-i)\Gamma(\alpha+1)}{\lvert k\rvert^{\alpha+1}}\cos\left(\frac{\pi\alpha}{2}\right)sgn\,k$ |

## 3.2 command for using principal value integration in Mathematica



### Showing how to use Principal value integral

```
In[3]:= Integrate[1 / x, {x, -1, 1}]

        Integrate::idiv : Integral of 1/x does not converge on {-1, 1}. >>

Out[3]= ∫_{-1}^{1} 1/x dx

In[4]:= Integrate[1 / x, {x, -1, 1}, PrincipalValue → True]

Out[4]= 0
```

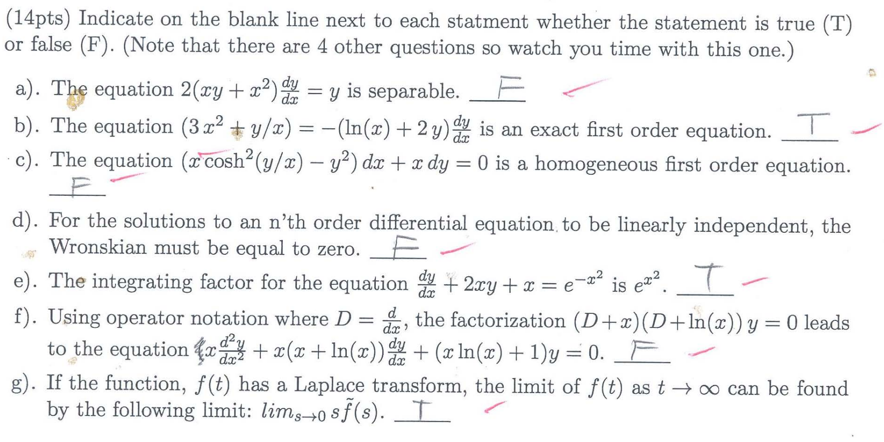
## 3.3 note on finding integrating factor

```
Given  N(x,y) y' + M(x,y) = 0, then

IF -- can write it as N(y) y' + M(x) = 0
   -- where N is function of y only and M function of x only
   THEN -- separable, i.e. @N/@x = @M/@y = 0
      -- example:   y' + x = 0  or y*y' + x+sin(x)=0 (linear)
      -- solve as separable, but some non-linear separable have no solution
      -- example: (1/ln(y)) y' + x = 0
ELSE
   IF @N/@x = @M/@y THEN  -- exact
      -- example  (2 x y +1) y' + (x +y^2) = 0
      -- solve using phi(x,y) potential function method
   ELSE -- not exact
     IF bernulli  -- check for special case
        --
     ELSE -- use generalized integrating factor

     END IF
   END IF
END IF
```

## 3.4 My write up of first exam

This is my post-mortem solution of the first exam, and after seeing the exam review given in class on Oct. 25, 2013 by professor Henderson. Made an algebra error in problem 3 since I did not simplify it before doing the derivation. Problem 5 needed the use of convolution to finish.

### 3.4.1 problem 1



(14pts) Indicate on the blank line next to each statment whether the statement is true (T) or false (F). (Note that there are 4 other questions so watch you time with this one.)

a). The equation $2(xy + x^2)\frac{dy}{dx} = y$ is separable. F

b). The equation $(3\,x^2 + y/x) = -(\ln(x) + 2\,y)\frac{dy}{dx}$ is an exact first order equation. T

c). The equation $(x\cosh^2(y/x) - y^2)\,dx + x\,dy = 0$ is a homogeneous first order equation. F

d). For the solutions to an n'th order differential equation to be linearly independent, the Wronskian must be equal to zero. F

e). The integrating factor for the equation $\frac{dy}{dx} + 2xy + x = e^{-x^2}$ is $e^{x^2}$. T

f). Using operator notation where $D = \frac{d}{dx}$, the factorization $(D+x)(D+\ln(x))\,y = 0$ leads to the equation $x\frac{d^2y}{dx^2} + x(x+\ln(x))\frac{dy}{dx} + (x\ln(x)+1)y = 0$. F

g). If the function, $f(t)$ has a Laplace transform, the limit of $f(t)$ as $t \to \infty$ can be found by the following limit: $lim_{s\to 0}\, s\tilde{f}(s)$. T

$$\{a \to false, b \to true, c \to false, d \to false, e \to true, f \to false, g \to true\}$$

### 3.4.2 problem 2

Solve the initial value problem $\frac{dy}{dx} + y = xy^3; y(0) = 1$

This is a Bernoulli differential equation. Dividing by $y^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

Let $u = \frac{1}{y^2}$, hence $\frac{du}{dx} = -2\frac{1}{y^3}\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{-y^3}{2}\frac{du}{dx}$ then the differential equation becomes

$$\begin{aligned}\frac{1}{y^3}\left(\frac{-y^3}{2}\frac{du}{dx}\right) + u &= x\\ \frac{-1}{2}\frac{du}{dx} + u &= x\\ \frac{du}{dx} - 2u &= -2x\end{aligned}$$

Integrating factor $I_f = e^{-2x}$, hence

$$\begin{aligned}d\left(e^{-2x}u\right) &= -2e^{-2x}x\\ e^{-2x}u &= -2\int xe^{-2x}dx\\ &= -2\left(e^{-2x}\left(-\frac{x}{2} - \frac{1}{4}\right)\right) + c\end{aligned}$$

Hence

$$\begin{aligned}
u &= -2e^{2x}\left(e^{-2x}\left(-\frac{x}{2}-\frac{1}{4}\right)\right)+ce^{2x}\\
&= x+\frac{1}{2}+ce^{2x}
\end{aligned}$$

But $u=\frac{1}{y^2}$, therefore

$$\begin{aligned}
y^2 &= \frac{1}{u}\\
&= \frac{1}{x+\frac{1}{2}+ce^{2x}}
\end{aligned}$$

or

$$y=\pm\frac{1}{\sqrt{x+\frac{1}{2}+ce^{2x}}}$$

Applying initial conditions give

$$\begin{aligned}
1 &= \frac{1}{\frac{1}{2}+c}=\frac{2}{1+2c}\\
1+2c &= 2\\
c &= \frac{1}{2}
\end{aligned}$$

The final answer is

$$y=\pm\frac{1}{\sqrt{x+\frac{1}{2}+\frac{1}{2}e^{2x}}}$$

### 3.4.3 problem 3

Find the general solution to the homogenous differential equation using the reduction in order method given that one of the solutions is $y_1=x$

$$x^3y'''-3x^2y''+x\left(6-x^2\right)y'-\left(6-x^2\right)y=0$$

Let $y_2=uy_1$ where $u(x)$ is a function of $x$ to be determined. Therefore

$$\begin{aligned}
y_2 &= ux\\
y_2' &= u'x+u\\
y_2'' &= u''x+u'+u'=u''x+2u'\\
y_2''' &= u'''x+u''+2u''=u'''x+3u''
\end{aligned}$$

Hence the original ODE becomes

$$\begin{aligned}
x^3\left(u'''x+3u''\right)-3x^2\left(u''x+2u'\right)+x\left(6-x^2\right)\left(u'x+u\right)-\left(6-x^2\right)ux &= 0\\
u'''\left(x^4\right)+u''\left(3x^3-3x^2\right)+u'\left(x^2\left(6-x^2\right)-6x^2\right)+u\left(x\left(6-x^2\right)-\left(6-x^2\right)x\right) &= 0\\
u'''\left(x^4\right)-u'\left(x^4\right) &= 0\\
u'''-u' &= 0
\end{aligned}$$

Let $u'=v$

$$\begin{aligned} v'' - v &= 0 \\ \left(D^2 - 1\right) v &= 0 \end{aligned}$$

Hence the roots are $\lambda = \pm 1$ and the solution is $v(x) = c_1 e^x + c_2 e^{-x}$, but since $u' = v$ then

$$u' = c_1 e^x + c_2 e^{-x}$$

Integrating

$$u = c_1 e^x - c_2 e^{-x} + c_3$$

Therefore

$$\begin{aligned} y_2 &= ux \\ &= c_1 x e^x - c_2 x e^{-x} + c_3 x \end{aligned}$$

And the final solution is

$$\begin{aligned} y &= Ay_1 + By_2 \\ &= Ax + B\left(c_1 x e^x - c_2 x e^{-x} + c_3 x\right) \\ &= Ax + Bc_1 x e^x - Bc_2 x e^{-x} + Bc_3 x \end{aligned}$$

Combining constants and renaming them gives

$$y = c_1 x + c_2 x e^x - c_3 x e^{-x}$$

The three constants can be found from initial conditions.

### 3.4.4 problem 4

Find the general solution to the following

$$x^2 y'' + 2xy' - 2y = 6x$$

Let $x = e^z$ or $z = \ln(x)$, hence $\frac{dz}{dx} = \frac{1}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \end{aligned}$$

Substituting the above in the original ODE gives

$$\begin{aligned} x^2\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) + 2x\left(\frac{1}{x}\frac{dy}{dz}\right) - 2y &= 6e^z \\ -\frac{dy}{dz} + \frac{d^2y}{dz^2} + 2\frac{dy}{dz} - 2y &= 6e^z \\ \frac{d^2y}{dz^2} + \frac{dy}{dz} - 2y &= 6e^z \qquad (1) \end{aligned}$$

Characteristic equation is $\left(\lambda^2 + \lambda - 2\right) = 0$ hence $(\lambda - 1)(\lambda + 2) = 0$ hence the roots are $\{1, -2\}$ and the solution is

$$y_h = c_1 e^z + c_2 e^{-2z}$$

To find $y_p$ guessing $y_p = Ae^z + Bze^z$ then $y_p' = Ae^z + Be^z + Bze^z$ and $y_p'' = Ae^z + Be^z + Be^z + Bze^z$ and substituting this in the Eq. (1) and matching terms gives

$$\begin{aligned} Ae^z + Be^z + Be^z + Bze^z + Ae^z + Be^z + Bze^z - 2\left(Ae^z + Bze^z\right) &= 6e^z \\ 3B &= 6 \\ B &= 2 \end{aligned}$$

Hence $y_p = 2ze^z$ and the solution is

$$y(z) = c_1 e^z + c_2 e^{-2z} + 2ze^z$$

But $z = \ln(x)$ hence

$$\begin{aligned} y(x) &= c_1 e^{\ln x} + c_2 e^{-2\ln x} + 2\ln|x|\, e^{\ln x} \\ &= c_1 x + \frac{c_2}{x^2} + 2x\ln|x| \end{aligned}$$

### 3.4.5 problem 5

Solve for $y(t)$ using the Laplace transform. Initial conditions are $y(0) = 1; y'(0) = 0$

$$y'' = f(t) + 1 + \int_0^t (t-\tau)\, y(\tau)\, d\tau$$

Taking Laplace transform, and using $Y = \mathscr{L}\big(y(t)\big)$ and $F = \mathscr{L}\big(f(t)\big)$ gives

$$\begin{aligned} s^2 Y - sy(0) - y'(0) &= F + \frac{1}{s} + \mathscr{L}(t)\, \mathscr{L}\big(y\big) \\ s^2 Y - s &= F + \frac{1}{s} + \frac{1}{s^2} Y \\ s^4 Y - s^3 - Y &= s^2 F + s \\ Y &= \frac{Fs^2 + s + s^3}{s^4 - 1} \\ &= \frac{Fs^2}{s^4-1} + \frac{s+s^3}{s^4-1} \\ &= \frac{Fs^2}{\left(s^2+1\right)\left(s^2-1\right)} + \frac{s\left(s^2+1\right)}{\left(s^2+1\right)\left(s^2-1\right)} \\ &= \frac{Fs^2}{\left(s^2+1\right)\left(s^2-1\right)} + \frac{s}{\left(s^2-1\right)} \\ &= \frac{1}{2}\frac{1}{\left(s^2+1\right)}F + \frac{1}{2}\frac{1}{\left(s^2-1\right)}F + \frac{s}{\left(s^2-1\right)} \end{aligned}$$

now $\mathscr{L}^{-1}\frac{s}{\left(s^2-1\right)} = \cosh(t)$ and

$$\mathscr{L}^{-1}\left(\frac{1}{2}\frac{1}{\left(s^2+1\right)}F\right) = \frac{1}{2}\int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

and

$$\mathscr{L}^{-1}\left(\frac{1}{2}\frac{1}{\left(s^2-1\right)}F\right)=\frac{1}{2}\int_0^t \sinh\left(t-\tau\right)f\left(\tau\right)d\tau$$

Hence the solution is

$$y\left(t\right)=\cosh\left(t\right)+\frac{1}{2}\left(\int_0^t \sin\left(t-\tau\right)f\left(\tau\right)d\tau+\int_0^t \sinh\left(t-\tau\right)f\left(\tau\right)d\tau\right)$$