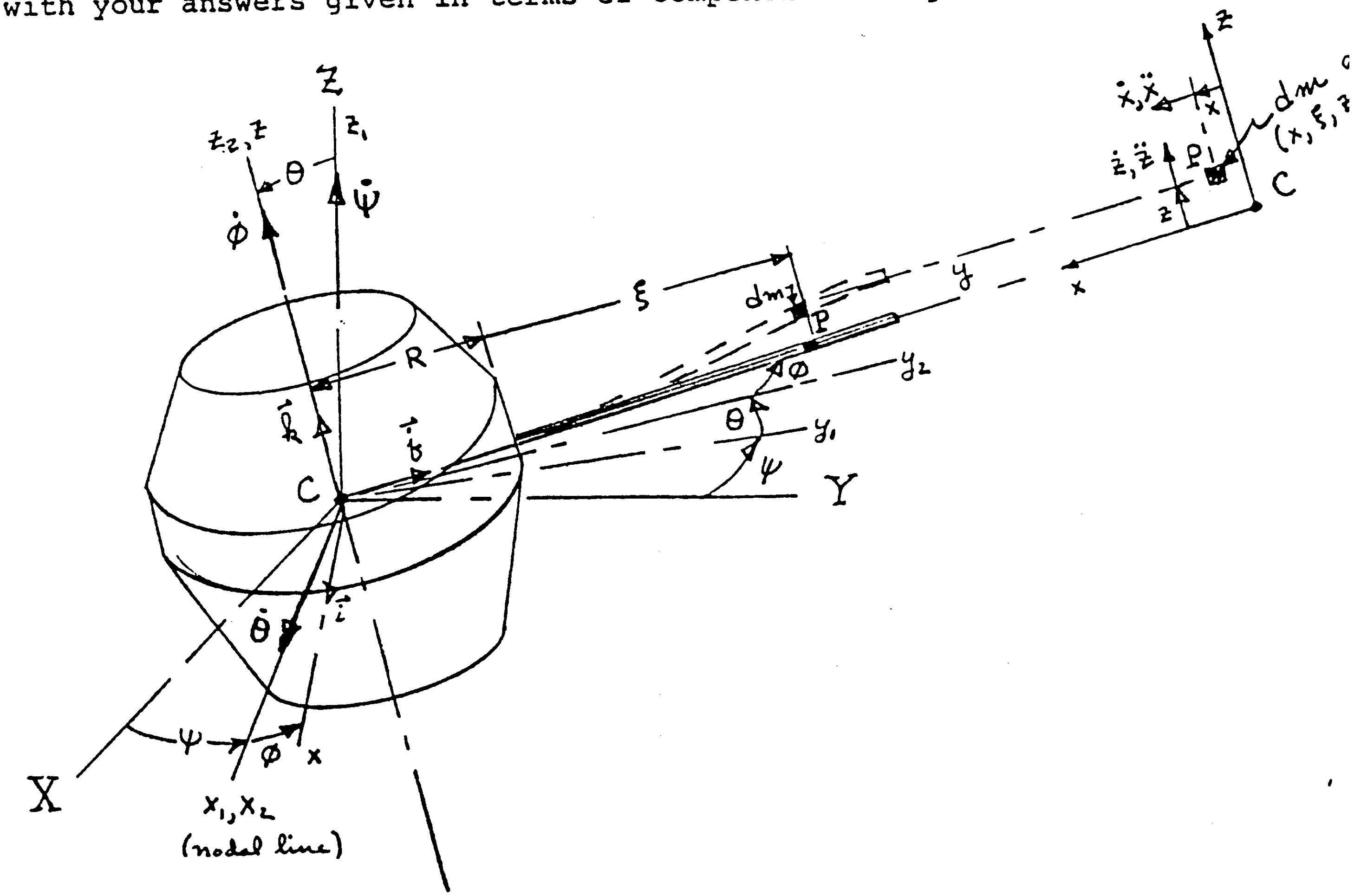
(19) The mass center C of the satellite shown moves with a velocity $\vec{v}_C = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ and an acceleration $\vec{a}_C = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ relative to inertial space denoted by axes XYZ. The satellite precesses at a constant rate $\dot{\psi}$ and spins at a constant rate $\dot{\phi}$ and the angle of nutation θ is also constant. An antenna, modeled by a slender rod cantilevered from the satellite along the y axis of the xyz set of body axes, is free to vibrate transversely so that each element dm of the antenna is assumed to move parallel to the xz body plane. Considering the general position of an element P at a distance ξ from the outer perimeter of the satellite to be at the instantaneous location (x, ξ, z), determine:

1. the inertial velocity of the elemental mass dm;

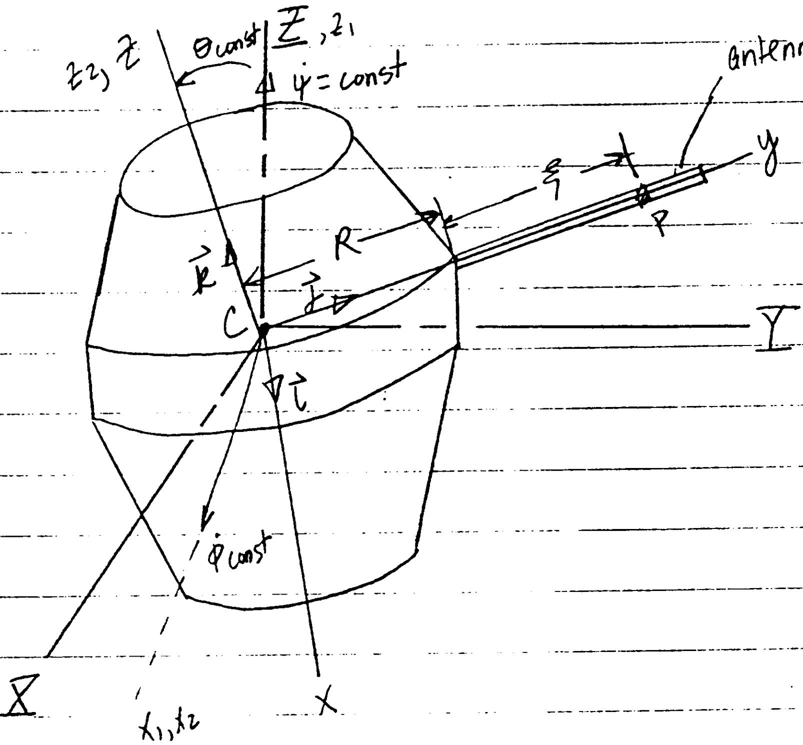
2. the inertial acceleration of dm.

Give answers in terms of $x, y, \xi, \phi, \theta, \psi$ and related time derivitives with your answers given in terms of components along the body axes.



Problem #19

10/19/91



$$\vec{v}_c = v_X \vec{i} + v_Y \vec{j} + v_Z \vec{k}$$
$$\vec{a}_c = a_X \vec{i} + a_Y \vec{j} + a_Z \vec{k}$$
relative to inertial XYZ frame

precesses at constant $\dot{\psi}$
spins " " $\dot{\phi}$
angle of nutation ... θ, is constant, $\dot{\theta} = 0$

antenna, modeled as a slender rod, is along y axis & is free to vibrate transversely so that each element, dm, of the antenna is assumed to move parallel to the xz body plane.

Determine: 1) the inertial velocity of the elemental mass, dm.
2) the " acceleration " " ".

#19 cont'd... G.H.

RELATIVE PROBLEM:

(1) $\vec{v}_p = \vec{v}_c + \vec{\omega}_{cs} \times \vec{\rho}_{P/c} + \vec{\dot{\rho}}_r$

where $\vec{v}_c = v_X \vec{i} + v_Y \vec{j} + v_Z \vec{k}$

$\vec{\omega}_{cs} = \omega_X \vec{i} + \omega_Y \vec{j} + \omega_Z \vec{k}$ (body axes)

$$= \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = \begin{bmatrix} \sin\theta \sin\phi & \cos\phi & 0 \\ \sin\theta \cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \nearrow 0 \\ \dot{\phi} \end{bmatrix}$$

$$\vec{\omega}_{cs} = (\sin\theta \sin\phi)\dot{\psi}\, \vec{i} + (\sin\theta \cos\phi)\dot{\psi}\, \vec{j} + (\dot{\psi}\cos\theta + \dot{\phi})\, \vec{k}$$

$$\vec{\rho}_{P/c} = x\vec{i} + (R+\xi)\vec{j} + z\vec{k}$$

$$\vec{\omega}_{cs} \times \vec{\rho}_{P/c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{\psi}\sin\theta\sin\phi & \dot{\psi}\sin\theta\cos\phi & \dot{\psi}\cos\theta + \dot{\phi} \\ x & R+\xi & z \end{vmatrix}$$

$$\vec{\omega}_{cs} \times \vec{\rho}_{P/c} = \left\{ \dot{\psi} z \sin\theta\cos\phi - (R+\xi)(\dot{\psi}\cos\theta + \dot{\phi}) \right\} \vec{i}$$

$$- \left\{ \dot{\psi} z \sin\theta \sin\phi - x(\dot{\psi}\cos\theta + \dot{\phi}) \right\} \vec{j}$$

$$+ \left\{ (R+\xi)\dot{\psi}\sin\theta\sin\phi - x\dot{\psi}\sin\theta\cos\phi \right\} \vec{k}$$

$\vec{\dot{\rho}}_r = \dot{x}\vec{i} + \dot{z}\vec{k}$ (dm moves parallel to xz body plane)

$$\therefore \underline{\vec{v}_p} = \left(V_X + \dot{x} + \dot{\psi} z \sin\theta\cos\phi - (R+\xi)(\dot{\psi}\cos\theta + \dot{\phi}) \right) \vec{i}$$

$$+ \left(V_Y + x(\dot{\psi}\cos\theta + \dot{\phi}) - \dot{\psi} z \sin\theta\sin\phi \right) \vec{j}$$

$$+ \left(V_Z + \dot{z} - \dot{\psi} x \sin\theta\cos\phi + (R+\xi)\dot{\psi}\sin\theta\sin\phi \right) \vec{k}$$

(2) $\vec{a}_P = \vec{a}_C + \vec{\omega}_{CS} \times (\vec{\omega}_{CS} \times \vec{\rho}_{P/C}) + \dot{\vec{\omega}}_{CS} \times \vec{\rho}_{P/C} + \ddot{\vec{\rho}}_{P/C} + 2\vec{\omega}_{CS} \times \dot{\vec{\rho}}_{P/C}$

where $\vec{a}_C = a_X \vec{i} + a_Y \vec{j} + a_Z \vec{k}$

$$\vec{\omega} \times \vec{\omega} \times \vec{\rho} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{\psi}\sin\theta\sin\phi & \dot{\psi}\sin\theta\cos\phi & \dot{\psi}\cos\theta + \dot{\phi} \\ \dot{\psi} z\sin\theta\cos\phi - (R+\xi)(\dot{\psi}\cos\theta + \dot{\phi}) & x(\dot{\psi}\cos\theta + \dot{\phi}) - \dot{\psi} z\sin\theta\sin\phi & (R+\xi)(\dot{\psi}\sin\theta\sin\phi) - x\dot{\psi}\sin\theta\cos\phi \end{vmatrix}$$

$$\vec{\omega} \times \vec{\omega} \times \vec{\rho} = \Big\{ \dot{\psi}\sin\theta\cos\phi\left[(R+\xi)(\dot{\psi}\sin\theta\sin\phi) - x\dot{\psi}\sin\theta\cos\phi\right] - (\dot{\psi}\cos\theta + \dot{\phi})\left[x(\dot{\psi}\cos\theta + \dot{\phi}) - \dot{\psi} z\sin\theta\sin\phi\right] \Big\} \vec{i}$$

$$- \Big\{ \dot{\psi}\sin\theta\sin\phi\left[(R+\xi)(\dot{\psi}\sin\theta\sin\phi) - x\dot{\psi}\sin\theta\cos\phi\right] - (\dot{\psi}\cos\theta + \dot{\phi})\left[\dot{\psi} z\sin\theta\cos\phi - (R+\xi)(\dot{\psi}\cos\theta + \dot{\phi})\right] \Big\} \vec{j}$$

$$+ \Big\{ \dot{\psi}\sin\theta\sin\phi\left[x(\dot{\psi}\cos\theta + \dot{\phi}) - \dot{\psi} z\sin\theta\sin\phi\right] - \dot{\psi}\sin\theta\cos\phi\left[\dot{\psi} z\sin\theta\cos\phi - (R+\xi)(\dot{\psi}\cos\theta + \dot{\phi})\right] \Big\} \vec{k}$$

$$\dot{\vec{\omega}}_{CS} = \dot{\psi}(\sin\theta\cos\phi)\dot{\phi}\, \vec{i} - \dot{\psi}(\sin\theta\sin\phi)\dot{\phi}\, \vec{j} + 0\, \vec{k}$$

$$\dot{\vec{\omega}}_{CS} \times \vec{\rho}_{P/C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{\psi}\dot{\phi}\sin\theta\cos\phi & -\dot{\psi}\dot{\phi}\sin\theta\sin\phi & 0 \\ X & R+\xi & Z \end{vmatrix}$$

$$= (-\dot{\psi}\dot{\phi} z\sin\theta\sin\phi)\vec{i} - (\dot{\psi}\dot{\phi} z\sin\theta\cos\phi)\vec{j} + \left[(R+\xi)(\dot{\psi}\dot{\phi}\sin\theta\cos\phi) + \dot{\psi}\dot{\phi} x\sin\theta\sin\phi\right]\vec{k}$$

$$\ddot{\vec{\rho}}_{P/C} = \ddot{x}\vec{i} + \ddot{z}\vec{k}$$

$$2\vec{\omega}_{CS} \times \dot{\vec{\rho}}_{P/C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\dot{\psi}\sin\theta\sin\phi & 2\dot{\psi}\sin\theta\cos\phi & 2(\dot{\psi}\cos\theta + \dot{\phi}) \\ \dot{x} & 0 & \dot{z} \end{vmatrix}$$

#19 cont'd... 3 G.H.

$$2\vec{\omega}_{cs} \times \dot{\vec{r}}_{P/c} = (2\dot{\psi}\dot{z}\sin\theta\cos\phi)\vec{i} - \left(2\dot{\psi}\dot{z}\sin\theta\sin\phi - 2\dot{x}(\dot{\psi}\cos\theta+\dot{\phi})\right)\vec{j} + (-2\dot{\psi}\dot{x}\sin\theta\cos\phi)\vec{k}$$

$$\therefore \underline{\vec{a}_P} = \Big\{ a_x + (R+\xi)(\dot{\psi}^2\sin^2\theta\sin\phi\cos\phi) - (\dot{\psi}^2 x\sin^2\theta\cos^2\phi) - (x(\dot{\psi}\cos\theta+\dot{\phi})^2 - \dot{\psi}z\sin\theta\sin\phi(\dot{\psi}\cos\theta+\dot{\phi})) - \dot{\psi}\dot{\phi}z\sin\theta\sin\phi + 2\dot{\psi}\dot{z}\sin\theta\cos\phi + \ddot{x} \Big\}\vec{i}$$

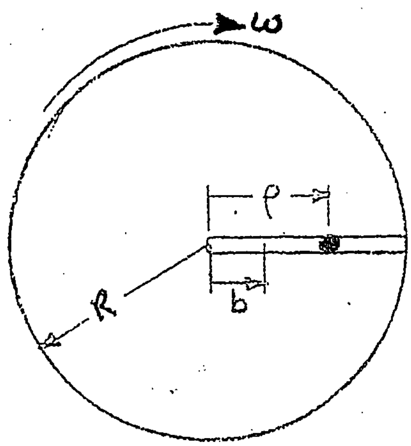
$$+ \Big\{ a_y - (R+\xi)(\dot{\psi}^2\sin^2\theta\sin^2\phi) + \dot{\psi}^2 x\sin^2\theta\sin\phi\cos\phi + \dot{\psi}z\sin\theta\cos\phi(\dot{\psi}\cos\theta+\dot{\phi}) - (R+\xi)(\dot{\psi}\cos\theta+\dot{\phi})^2 - \dot{\psi}\dot{\phi}z\sin\theta\cos\phi - 2\dot{\psi}\dot{z}\sin\theta\sin\phi + 2\dot{x}(\dot{\psi}\cos\theta+\dot{\phi}) \Big\}\vec{j}$$

$$+ \Big\{ a_z + \dot{\psi}x\sin\theta\sin\phi(\dot{\psi}\cos\theta+\dot{\phi}) - \dot{\psi}^2 z\sin^2\theta\sin^2\phi - \dot{\psi}^2\sin^2\theta\cos^2\phi - (R+\xi)(\dot{\psi}\cos\theta+\dot{\phi})\dot{\psi}\sin\theta\cos\phi + (R+\xi)\dot{\psi}\dot{\phi}\sin\theta\cos\phi + \dot{\psi}\dot{\phi}x\sin\theta\sin\phi - 2\dot{\psi}\dot{x}\sin\theta\cos\phi + \ddot{z} \Big\}\vec{k}$$

22b

A marble represented by the particle of mass m is constrained to move along a frictionless groove cut in a circular rotating platform of outer radius R. The platform rotates about a vertical axis at a constant rate ω. Considering that the marble is released at a radius b with zero velocity relative to the platform,

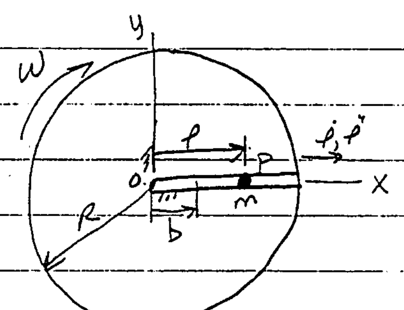
[a] determine the time for the marble to reach the outer edge of the platform by applying Newton's laws directly



PROBLEM # 22b)   11/12/91 G.H.

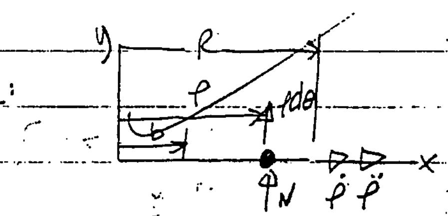
A marble represented by the particle of mass m is constrained to move along a frictionless groove cut in a circular rotating platform of outer radius R. The platform rotates about a vertical axis at a constant rate ω. Considering that the marble is released at a radius b with zero velocity relative to the platform,

(a) determine the time for the marble to reach the outer edge of the platform by applying Newton's laws directly.



$\vec{F} = m\vec{a}$

FBD:



(horizontal plane → mg = 0)

$\vec{v}_P = \vec{v}_O + \vec{\omega} \times \vec{\rho} \times \dot{\vec{\rho}}_r$

where $\vec{v}_O = 0$

$\vec{\omega} \times \vec{\rho} = -\omega \vec{k} \times \rho \vec{i} = -\rho\omega \vec{j}$

$\dot{\vec{\rho}} = \dot{\rho} \vec{i}$

$\therefore \vec{v}_P = -\rho\omega\vec{j} + \dot{\rho}\vec{i}$

$\vec{a}_P = \vec{a}_O + \vec{\omega} \times \vec{\omega} \times \vec{\rho} + \dot{\vec{\omega}} \times \vec{\rho} + \ddot{\vec{\rho}}_r + 2\vec{\omega} \times \dot{\vec{\rho}}_r$

where $\vec{a}_O = 0$

$\vec{\omega} \times \vec{\omega} \times \vec{\rho} = -\omega\vec{k} \times -\rho\omega\vec{j} = -\rho\omega^2 \vec{i}$

$\dot{\vec{\omega}} \times \vec{\rho} = 0$

$\ddot{\vec{\rho}}_r = \ddot{\rho}\vec{i}$

$2\vec{\omega} \times \dot{\vec{\rho}}_r = 2\omega\vec{k} \times \dot{\rho}\vec{i} = 2\omega\dot{\rho}\vec{j}$

$\therefore \vec{a}_P = (\ddot{\rho} - \omega^2\rho)\vec{i} + 2\omega\dot{\rho}\vec{j}$

22b cont'd'...   11/12/91 B.H.

$\Sigma \vec{F} = N\vec{j}$

$\therefore \quad N\vec{j} = [(\ddot{\rho} - \omega^2\rho)\vec{i} + 2\omega\dot{\rho}\vec{j}]\, m$

equating terms:

$$\left.\begin{array}{l} \ddot{\rho} - \omega^2\rho = 0 \\ 2\omega\dot{\rho} = N/m \end{array}\right\} \quad \therefore \quad \boxed{\begin{array}{l} \ddot{\rho} - \omega^2\rho = 0 \\ \vec{N} = 2m\omega\dot{\rho}\,\vec{j} \end{array}}$$

$\ddot{\rho} = \omega^2\rho$

where $\ddot{\rho} = \frac{d\dot{\rho}}{dt} = \frac{d\dot{\rho}}{d\rho}\frac{d\rho}{dt} = \frac{d\dot{\rho}}{d\rho}\dot{\rho} = \rho\omega^2$ by chain rule

$\int \dot{\rho}\, d\dot{\rho} = \int \rho\omega^2\, d\rho$

$\frac{1}{2}\dot{\rho}^2 = \frac{1}{2}\rho^2\omega^2 + C_1$

@ $t = 0$, $\dot{\rho} = 0$, $\rho = b$ (initial conditions)

$\therefore \quad C = -\frac{1}{2}\omega^2 b^2$

$\therefore \cancel{\tfrac{1}{2}}\dot{\rho}^2 = \frac{1}{2}\omega^2\rho^2 - \frac{1}{2}\omega^2 b^2 = \cancel{\tfrac{1}{2}}\omega^2(\rho^2 - b^2)$

$\dot{\rho}^2 = \omega^2(\rho^2 - b^2)$

$\dot{\rho} = \omega\sqrt{\rho^2 - b^2} = d\rho/dt$

$$\int_0^t dt = \int_b^R \frac{1}{\omega\sqrt{\rho^2 - b^2}}\, d\rho$$

$$t = \frac{1}{\omega}\left[\ln\left(\rho + \sqrt{\rho^2 - b^2}\right)\right]_b^R$$

$$\therefore \quad \underline{t = \frac{1}{\omega}\left[\ln\left(R + \sqrt{R^2 - b^2}\right) - \ln b\right]} = \underline{\frac{1}{\omega}\ln\left[\frac{R + \sqrt{R^2 - b^2}}{b}\right]}$$

#22b cont'd^2 ... 11/12/91 G.H.

(b) Check answer (a) by applying Work-Energy principle.

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{v_1}^{v_2} m v\, dv \quad \text{or} \quad \underline{W_k = \Delta T}$$

$$dW_k = \vec{F}\cdot ds = \vec{F}\cdot \rho\, d\theta \qquad s = r\theta$$

$$= (2\omega \dot{\rho} m \vec{j}) \cdot (d\rho \vec{i} + \rho d\theta \vec{j})$$

$$dW_k = 2\omega \dot{\rho} \rho m\, d\theta = 2m\omega \frac{d\rho}{dt} \rho\, d\theta$$

where $\frac{d\theta}{dt} = \omega \rightarrow d\theta = \omega dt$

$$\therefore dW_k = 2m\omega \frac{d\rho}{dt} \rho\, \omega dt = \underline{2m\omega^2 \rho\, d\rho}$$

$$W_k = \int_b^R 2m\omega^2 \rho\, d\rho = \frac{2m\omega^2 \rho^2}{2}\Big|_{\rho=b}^{\rho=R}$$

$$\underline{W_k = m\omega^2 (R^2 - b^2)}$$

$$\Delta T = T_2 - T_1$$

where $v_p = (\rho^2\omega^2 + \dot{\rho}^2)^{1/2}$

@1, $\dot{\rho} = 0$, $\rho = b$ $\quad v_{p_1} = (b^2\omega^2)^{1/2} = b\omega$

@2, $\rho = R$

$$T_2 - T_1 = \tfrac{1}{2} m v_p^2 - \tfrac{1}{2} m v_{p_1}^2$$

$$= \underline{\tfrac{1}{2} m (R^2\omega^2 + \dot{\rho}^2) - \tfrac{1}{2} m (b\omega)^2}$$

$$\therefore m\omega^2 (R^2 - b^2) = \tfrac{1}{2} m \left[R^2\omega^2 + \dot{\rho}^2 - (b\omega)^2 \right]$$

$$m\omega^2 R^2 - m\omega^2 b^2 - \tfrac{1}{2} m\omega^2 R^2 + \tfrac{1}{2} m\omega^2 b^2 = \tfrac{1}{2} m \dot{\rho}^2$$

#22 b cont'd[3]...

11/12/91 Q.H.

$$\cancel{\tfrac{1}{2}} m\omega^2 R^2 - \cancel{\tfrac{1}{2}} m\omega^2 b^2 = \cancel{\tfrac{1}{2}} m\dot{\rho}^2$$

$$\cancel{m}\omega^2 (R^2 - b^2) = \cancel{m}\dot{\rho}^2$$

$$\omega\sqrt{R^2 - b^2} = \dot{\rho} = d\rho/dt$$

$$\int_0^t dt = \int_b^d \frac{1}{\omega\sqrt{R^2-b^2}}\, d\rho$$

$$\therefore \quad t = \frac{1}{\omega} \ln\left[\frac{R + \sqrt{R^2-b^2}}{b}\right]$$

same result as (a)

SOLUTION TO 22b

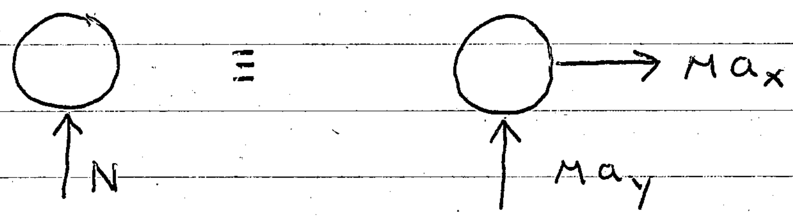
$$\vec{a}_p = \ddot{\vec{R}} + \vec{\omega}\times(\vec{\omega}\times\vec{\rho}) + \dot{\vec{\omega}}\times\vec{\rho} + 2\vec{\omega}\times\dot{\vec{\rho}}_r + \ddot{\vec{\rho}}_r$$

$$\ddot{\vec{R}} = 0 \qquad \vec{\omega} = \omega\bar{k} \qquad \vec{\rho} = \rho\bar{\imath} \qquad \dot{\vec{\omega}} = 0$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{\rho}) = -\omega^2\rho\,\bar{\imath} \qquad \ddot{\vec{\rho}}_r = \ddot{\rho}$$

$$2\vec{\omega}\times\dot{\vec{\rho}}_r = 2\omega\bar{k}\times\dot{\rho}\,\bar{\imath} = 2\omega\dot{\rho}\,\bar{\jmath}$$

$$\Rightarrow \quad \vec{a}_p = (\ddot{\rho} - \omega^2\rho)\,\bar{\imath} + 2\omega\dot{\rho}\,\bar{\jmath}$$



$$\Sigma F_x \Rightarrow \quad 0 = M(\ddot{\rho} - \omega^2\rho)$$

$$\Rightarrow \quad \ddot{\rho} - \omega^2\rho = 0 \qquad \rho(0) = b \qquad \dot{\rho}(0) = 0$$

ASSUME $\quad \rho = Ae^{\lambda t} \quad \Rightarrow \quad \lambda^2 - \omega^2 = 0$

$$\Rightarrow \quad \lambda = \pm\omega \quad \Rightarrow \quad \rho = Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{\rho} = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

$\dot{\rho}(0) = 0 \Rightarrow \omega(A - B) = 0$ OR $A = B$ ①

$\rho(0) = b \Rightarrow b = A + B = 2A$ ②

$$\Rightarrow A = \frac{b}{2}$$

$$\Rightarrow \rho = \frac{b}{2} e^{\omega t} + \frac{b}{2} e^{-\omega t}$$

AT OUTER EDGE $\rho = R \quad t = T$

$$\Rightarrow R = \frac{b}{2}\left[e^{\omega T} + e^{-\omega T}\right]$$

$$= \frac{b}{2}\left[e^{\theta} + e^{-\theta}\right] \qquad \theta = \omega T$$

$$\Rightarrow \frac{R}{b} = \frac{1}{2}\left[e^{\theta} + e^{-\theta}\right] = \cosh\theta$$

$$\Rightarrow \theta = \cosh^{-1}\left(\frac{R}{b}\right) = \ln\left[\frac{R}{b} + \sqrt{\frac{R^2}{b^2} - 1}\right]$$

$$\Rightarrow T = \frac{1}{\omega} \ln\left[\frac{R + \sqrt{R^2 - b^2}}{b}\right]$$