University Course

PHYSICS 3041
Mathematical Methods for Physicists

University of Minnesota, Twin Cities
Spring 2021

My Class Notes

Nasser M. Abbasi

Spring 2021

Contents

Chapter 1

Introduction

Local contents

1.1 Links

1. Instructor web page `https://cse.umn.edu/physics/yong-zhong-qian`
2. Canvas web page `https://canvas.umn.edu/courses/219359`

1.2 Schedule

Class Detail

PHYS 3041 - 001 Mathematical Methods for Physicists
Twin Cities/Rochester | Spring 2021 | Lecture

Class Details

Status	Open	**Career**	Undergraduate
Class Number	56625	**Dates**	1/19/2021 - 5/3/2021
Session	001 Regular Academic Session	**Grading**	Student Option
Units	3 units	**Location**	Off Campus
Instruction Mode	Completely Online	**Campus**	Twin Cities
Class Components	Lecture Required		

Meeting Information

Days & Times	Room	Instructor	Meeting Dates
MoWeFr 1:25PM - 2:15PM	Twin Cities Remote	Yongzhong Qian	01/19/2021 - 05/03/2021

Enrollment Information

Class Attributes Remote - set time and days

Class Availability

Class Capacity	90	**Wait List Capacity**	0
Enrollment Total	80	**Wait List Total**	0
Available Seats	10		

Notes

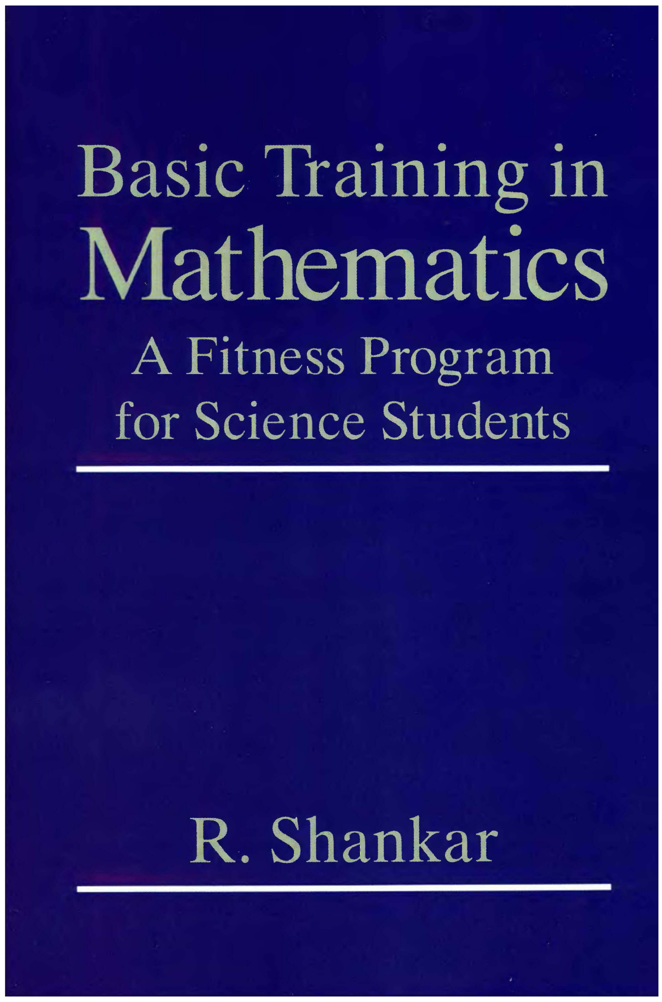
Class Notes Students and instructors must be online at the same time, at scheduled days and times. 100% of instruction is online with no in-person meetings. Exams are also all online.

Description

This course introduces additional mathematical topics that physics majors need to properly handle upper division physics classes.

prereq: PHYS 1302, MATH 2373 (or equivalent courses)

1.3 Text book


Basic Training in
Mathematics
A Fitness Program
for Science Students
R. Shankar

1.4 syllabus

Physics 3041 (Spring 2021) **Mathematical Methods for Physicists**

Lectures: M W F 1:25–2:15 pm

Quizzes: F **(2/12, 3/19, 4/23)** 1:25–2:15 pm

Final exam: TBA

In order to take the makeup final exam, you must report conflict with the regular final exam by the time to be announced.

Instructor: Yong-Zhong Qian (qianx007@umn.edu) **Office hour:** Tu 1–2 pm

TA Office hours:

Kivanc Bugan (bugan004@umn.edu)	M 11 am – 12 noon, Tu 3:30–4:30 pm
Dan Cronin-Hennessy (croni028@umn.edu)	F 11 am – 1 pm

Outline of lectures & tentative schedule

1/20, 22, 25 Taylor series (Chapters 1, 4)
1/27, 29; 2/1 Gaussian and exponential integrals (Chapter 2)
2/3, 5, 8 Complex numbers & functions (Chapter 5)
2/12 Quiz 1: Chapters 1, 2, 4, 5
2/10, 15, 17 Matrices & determinants (Chapter 8)
2/19, 22, 24, 26; 3/1, 3, 5, 8, 10 Linear vector spaces (Chapter 9)
3/19 Quiz 2: Chapters 8, 9
3/12, 15, 17, 22, 24, 26 Ordinary differential equations (Chapter 10.1–10.4)
3/29, 31; 4/2, 12, 14, 16 Multivariable & vector calculus (Chapters 3, 7)
4/23 Quiz 3: Chapters 3, 7, 10.1–10.4
4/19, 21, 26, 28, 30; 5/3 Partial differential equations (Chapter 10.5)
TBA: Final Exam

Materials: The required textbook is *Basic Training in Mathematics: A Fitness Program for Science Students* by R. Shankar. This book is rather concise and should be read before the relevant materials are covered in lectures.

If you would like another book for more detailed exposition of the materials, *Mathematical Methods in the Physical Sciences* by Mary L. Boas or *Mathematical Methods for Physics and Engineering* by K. F. Riley, M. P. Hobson, and S. J. Bence is recommended.

Other materials, such as lecture notes, homework, and solutions to homework and quizzes, will be posted on Canvas.

Online classroom courtesy: All lectures are given through Zoom. Please follow these rules of etiquette: (1) When joining class, choose as quite an environment as possible. (2) Mute yourself. Remember to unmute when asking a question, or when participating in other ways. (3) Make sure to maintain and project a professional environment if you use the camera. (4) When you show up you are joining a community intent on learning. Participate and engage. No distracting activities. As with in-person classes: no eating, drinking, newspaper reading, or other non-learning-related activities.

1

The course: The goal of this course is to review and present the mathematical tools for upper-division undergraduate physics courses. Particular emphasis will be given to physics applications through examples. Topics include single and multivariable calculus, complex numbers and functions, linear algebra, vector calculus, and ordinary and partial differential equations.

This course requires that you have completed Math 1271 (Calculus I), 1272 (Calculus II), and 2373 (CSE Linear Algebra & Differential Equations). It is highly recommended that you take Math 2374 (CSE Multivariable Calculus & Vector Analysis) concurrently. Therefore, we will NOT repeat what you should have learned from those or equivalent courses.

To get the most out of this course, you should read the relevant part of the textbook before it is covered in lectures (see the tentative schedule on the preceding page). If you find that the textbook is insufficient for you to understand a topic or that you need brush up on the materials, please consult your textbooks for Math 1271, 1272, 2373, and 2374 or read one of the other textbooks recommended on the preceding page.

In addition, you must actively work at problem solving to know whether you fully understand the concepts involved. Don't fall behind! It is extremely difficult to catch up and the longer you leave it the harder it gets. What you get out of the course will depend on the productive effort and quality time you put into it. If you are experiencing difficulties, contact me or a TA as soon as possible.

Lectures: Reading assignments will be announced on Canvas before lectures. You are expected to read the relevant material before coming to class so that the lectures reinforce the concepts, rather than presenting them for the first time. You are always encouraged to think critically about the material presented and ask questions.

Announcements: It is occasionally necessary to change schedules, including the dates of quizzes. Students are responsible for receiving ALL announcements made during the lecture, by email, or on Canvas. I will try to post the most important announcements on Canvas. It is crucial to have the correct Canvas settings so that you will receive appropriate notifications (e.g., by email) for announcements. Missing an announcement is not an acceptable excuse for missing a quiz or a course-related deadline. It is the sole responsibility of any student missing a lecture to determine what course material and announcements are missed.

App for making pdf files: Homework, all the quizzes, and the final exam require submission of clearly readable pdf files. Submission that is hard to read will receive NO credit! Please make sure that you are able to make clearly readable pdf files by using a free app such as Adobe Scan (`https://acrobat.adobe.com/us/en/mobile/scanner-app.html`) or CamScanner (`https://www.camscanner.com`).

Phone holder: For communications during a Zoom meeting, it is very helpful to show through the camera how you are solving a problem with pencil and paper. You may wish to do a Google search for a goose-neck phone mount, which is a phone holder that can be mounted to a desk, table, or bookshelf and be bent or twisted into different shapes to position the phone. By twisting it to look down at a sheet of paper and connecting to Zoom on your phone, it's straightforward to show your writing and drawing.

2

In-class discussion: To encourage interaction during the Zoom lectures, breakout rooms will be used for groups to discuss some problems, usually in the form of multiple-choice (MC) questions. Each student should submit the answers individually following the discussion. Credits will be recorded mostly based on participation. Group members are encouraged to interact outside the lectures as well.

Homework: Homework will be posted on Canvas each Wednesday and due the following Wednesday before the start of the lecture. Late homework will not be accepted. The graded homework will be returned to you within a week. Solutions will be posted on Canvas.

You can discuss with other students in the class about the homework problems, but should then solve them on your own. Infraction of this rule will be considered, and dealt with, as academic dishonesty, i.e., cheating.

Quizzes & final exam: Three quizzes and a final exam will be given on the dates specified at the beginning of this syllabus.

Grades will be determined based on the better of the two options:

Option 1: MC (4%), homework (16%), three quizzes (16% each), final exam (32%)

Option 2: MC (4%), homework (16%), best two quizzes (16% each), final exam (48%)

Division between grades is approximately: A (85–100), B (70–84), C (55–69), D (40–54), F ($<$ 40). Dividing lines will not be adjusted upwards, but may be adjusted a few points downwards. Subdivision within each grade level will be specified at the end of the course.

Regrading: If you have a dispute about your homework or quiz score, please first discuss this with the TA who graded the problem. If you are still not satisfied, please contact me. Regrading should be resolved within one week of receiving the graded work.

Makeup quizzes: There will be no early quizzes for any reason. As soon as you know that you have an acceptable excuse for not taking a scheduled quiz with the class, please contact me to discuss options and consequences.

Makeup final: There will be no early finals for any reason. To get a makeup final you must have two finals scheduled at the same time, 3 finals scheduled on the same day, or a University sanctioned excuse, and must submit a request form by the date to be announced.

Students with disabilities that affect their ability to participate fully in class or to meet all course requirements are encouraged to discuss these matters with the Disability Resource Center so that appropriate accommodations can be arranged. Please provide a copy of your accommodation letter for the current semester to the instructor and the physics front office (physics@umn.edu).

Policy & resource information can be found on the next page.

Responsibilities: The U of M assumes that all students enroll in its programs with a serious learning purpose and expects them to be responsible individuals who demand of themselves high standards of honesty and personal conduct. All students are expected to behave at all times with respect and courtesy toward their fellow students and instructors and to have the highest standards of honesty and integrity in their academic performance. Any behavior which disrupts the classroom learning environment or any attempt to present work that the student has not actually prepared on his/her own, or to pass an examination by improper means, is regarded as a serious offense which may result in the expulsion of the student from the University. The minimum penalty for such an offense is a failing grade for this course. Aiding and abetting the above behavior is also considered a serious offense resulting in equally severe penalties.

- Student Conduct Code

`http://regents.umn.edu/sites/regents.umn.edu/files/policies/Student_Conduct_Code.pdf`

- Scholastic Dishonesty

See Student Conduct Code

- Use of Personal Electronic Devices in the Classroom

`http://policy.umn.edu/education/studentresp`

- Makeup Work for Legitimate Absences

`http://policy.umn.edu/education/makeupwork`

- Appropriate Student Use of Class Notes and Course Materials

`http://policy.umn.edu/education/studentresp`

- Grading and Transcripts

`http://policy.umn.edu/education/gradingtranscripts`

- Sexual Harassment

`https://policy.umn.edu/hr/sexharassassault`

- Equity, Diversity, Equal Opportunity, and Affirmative Action

`http://regents.umn.edu/sites/regents.umn.edu/files/policies/Equity_Diversity_EO_AA.pdf`

- Disability Accommodations

`https://diversity.umn.edu/disability`

- Mental Health and Stress Management

`http://www.mentalhealth.umn.edu`

Chapter 2

MC (class 5 minutes quizz)

Local contents

2.1 number 1

A person of 1.8 m in height stands in a wide open field on a clear day. The earth's radius is approximately 6400 km. How far away is the horizon from the position of this person?

- ○ 0.6 km
- ○ 1.2 km
- ○ 2.4 km
- ◉ 4.8 km
- ○ 9.6 km

2.2 number 2

The x-axis points downward. The upper end of a vertical spring is fixed on this axis and its lower end is at the origin when it is relaxed. Then a block of mass m is attached to the lower end and released from rest at the origin. The spring constant is k and the acceleration of gravity is g. Let x be the position of the block, V(x) be the potential energy of the system, and omega be the angular frequency for the simple harmonic oscillations (SHO) of the block. Which of the following statements is correct?

- V(x) can be chosen as (kx^2)/2 + mgx.
- V(x) can be chosen as -(kx^2)/2 - mgx.
- omega = sqrt(k/m).
- The equilibrium position of the block is at x = 0.
- The amplitude of SHO is independent of g.

2.3 number 3

Given that a has the units of (Joule x meter), m is the electron mass, and h is the Planck constant, the quantities having the dimensions of energy and length, respectively, are

- ○ am/h, h/m
- ○ (a^2)m/h, h/(am)
- ○ hm/a, (a^2)/(hm)
- ○ (m^2)/h, ah/(m^2)
- ◉ (a^2)m/(h^2), (h^2)/(am)
- ○ none of the above

2.4 number 4

Consider a linear vector space of all 2x2 matrices. Which of the following statement is correct?

- This vector space is 4-dimensional.
- The Pauli matrices and the identity matrix are linearly independent.
- The matrices in B are orthogonal to each other.
- Any 2x2 matrix can be represented by a column matrix in an infinite number of ways.
- A, B.
- A, B, C.
- A, B, C, D.
- None of the above.

2.5 number 5

We have dealt with the cases of spin 1/2 (electron, Pauli matrices) and spin 1 (homework, 3x3 matrices). Based on these cases, which of the following statements is NOT correct?

- ○ Spin operators in different directions do not commute.
- ○ The square of the spin operator in each direction, (S_i)^2, is proportional to the identity matrix.
- ○ The sum of (S_i)^2 over all directions, S^2, is proportional to the identity matrix.
- ○ The operator S^2 commutes with S_i.
- ◉ Spin operators in different directions share the same eigenvalues.
- ○ The operator S^2 has a single eigenvalue (i.e., fully degenerate eigenvalues).

2.6 number 6

Which of the following statements is NOT correct?

- ○ The eigenvalues of the position operator are all real numbers with proper units.
- ○ The eigenvalues of the momentum operator are all real numbers with proper units.
- ○ The function delta(x-x') is an eigenfunction of the position operator X with eigenvalue x'.
- ◉ The matrix of the momentum operator is diagonal in the eigenbasis of the position operator.
- ○ Following the measurement of the position of a particle, the particle is in a state for which a subsequent momentum measurement would yield any value with the same probability.
- ○ Following the measurement of the momentum of a particle, the particle is in a state for which a subsequent position measurement would yield any value with the same probability.

2.7 number 7

Consider a spherical body of uniform density with radius R. Which of the following statements regarding its gravitational field is NOT correct?

- The field vanishes at the center.
- The field vanishes at all radii $r < R$.
- The field strength increases linearly with r at $r < R$.
- The field strength decreases as $1/r^2$ at $r > R$.
- The field is always directed towards the center.

2.8 number 8

Consider a uniform and isotropic gas for which the number density of gas particles with velocity between **v** and **v** + d**v** is f(v)$dv_x dv_y dv_z$. Focus on an area element dA that is perpendicular to the z-axis and on the wall of the gas container. Let theta be the angle between **v** and the +z-direction and phi be the associated azimuthal angle. Which of the following is the correct expression for the number of particles with velocity between **v** and **v** + d**v** that hit the area element over a time interval dt?

- f(v)(v_z)(dv_z)(dA)(dt)
- f(v)(v)(dv)(dA)(dt)
- f(v)(v)(dv_x)(dv_y)(dv_z)(dA)(dt)
- f(v)(v^3)(cos theta)(sin theta)(dv)(d theta)(d phi)(dA)(dt)
- f(v)(v^3)(sin theta)(dv)(d theta)(d phi)(dA)(dt)

Chapter 3

Exams

Local contents

3.1 First exam

Local contents

3.1.1 Questions

3.1.1.1 First question

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Feb 12 at 1:25pm - Feb 12 at 2:15pm about 1 hour

Two positive charges of magnitude Q are fixed at $x = -a$ and a, respectively, on the x axis $(a > 0)$. A negative charge of magnitude q and mass m is free to move along the y axis only.

(a) Find the electric potential energy of the negative charge as a function of y. Use this result to find the net force acting on it. (10 points)

(b) By deriving a differential equation, show that for a small neighborhood around the equilibrium position, the negative charge executes simple harmonic oscillations. Specify the equilibrium position and the angular frequency of oscillations. (30 points)

(c) What is the limit on the size of the small neighborhood in (b) and why? (10 points)

3.1.1.2 Second question

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Feb 12 at 1:25pm - Feb 12 at 2:15pm about 1 hour

Consider a real function $f(x) = A\exp\left(-\frac{x^2}{4\sigma^2}\right)$, where A, σ are positive parameters, x is the position variable, and $\int_{-\infty}^{+\infty} [f(x)]^2 dx = 1$. Show detailed steps leading to your results for parts (a) and (b) below.

(a) Find A in terms of σ. (10 points)

(b) Evaluate $g(k) = \int_{-\infty}^{+\infty} f(x)\exp(-ikx)dx$, where k is a real parameter. (25 points)

(c) What are the units of σ, k, A, $f(x)$, $g(k)$ and why? (15 points)

3.2 Second exam

Local contents

3.2.1 Questions

3.2.1.1 First question

3/19/2021 Q2P1

Q2P1

Submit Assignment

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Mar 19 at 1:25pm - Mar 19 at 2:25pm about 1 hour

Show detailed steps for the parts below.

(a) An explosion of energy E sends out a spherical shock wave into the surrounding air of mass density ρ. Use dimensional analysis to derive the shock front radius R as a function of time t since the setoff of the explosion. A result without any dimensionless factor is sufficient. (25 points)

(b) Consider the Hermitian operators S_1, S_2, S_3 that satisfy $[S_i, S_j] = i\sum_{k=1}^{3} \epsilon_{ijk} S_k$. Show that $S^2 = S_1^2 + S_2^2 + S_3^2$ commutes with each of these three operators. (25 points)

3.2.1.2 Second question

3/19/2021 Q2P2

Q2P2

Submit Assignment

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Mar 19 at 1:25pm - Mar 19 at 2:25pm about 1 hour

An electron is in a uniform magnetic field $\vec{B} = B\hat{e}_y$. Its Hamiltonian is $H = \frac{e\hbar B}{2m_e}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in the basis where its spin operator S_z is represented by $\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. At time $t = 0$, the electron is in the eigenstate of S_z with the eigenvalue $\frac{\hbar}{2}$. Show detailed steps for the parts below.

(a) Find the energy eigenvalues and eigenstates. (25 points)

(b) Find the spin state of the electron for $t > 0$. (25 points)

3.2.2 my solution to second problem (post exam)

$$H = \frac{e\hbar B}{2m}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The eigenvalues can be found to be $E_+ = \frac{e\hbar B}{2m} = \hbar\lambda, E_- = -\frac{e\hbar B}{2m} = -\hbar\lambda$ where $\lambda = \frac{eB}{2m}$. The associated normalized eigenvectors are

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

To find the spin state for $t > 0$, we need to solve the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

In the basis of H the above becomes

$$\begin{aligned} i\hbar\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} &= \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} \hbar\lambda & 0 \\ 0 & -\hbar\lambda \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} i\hbar x_1'(t) &= \hbar\lambda x_1(t) \\ i\hbar x_2'(t) &= -\hbar\lambda x_2(t) \end{aligned}$$

Or

$$\begin{aligned} x_1'(t) &= -i\lambda x_1(t) \\ x_2'(t) &= i\lambda x_2(t) \end{aligned}$$

The solution is

$$\begin{aligned} x_1(t) &= x_1(0)e^{-i\lambda t} \\ x_2(t) &= x_2(0)e^{i\lambda t} \end{aligned}$$

In the original basis, this becomes

$$\begin{aligned} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} &= x_1(t)\vec{v}_1 + x_2(t)\vec{v}_2 \\ &= \frac{1}{\sqrt{2}}x_1(0)e^{-i\lambda t}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}}x_2(0)e^{i\lambda t}\begin{pmatrix} i \\ 1 \end{pmatrix} \qquad (1) \end{aligned}$$

What is left is to determine $x_1(0), x_2(0)$. We are told that at $t = 0$, $\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenstate associated with $\frac{\hbar}{2}$ of S_z. Therefore the above becomes (at $t = 0$)

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{\sqrt{2}}x_1(0)\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}}x_2(0)\begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned}
\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} &= \sqrt{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \sqrt{2}\,\frac{\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{1-\left(i^2\right)} \\
&= \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}
\end{aligned}$$

Therefore

$$\begin{aligned}
x_1(0) &= \frac{1}{\sqrt{2}} \\
x_2(0) &= \frac{-i}{\sqrt{2}}
\end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned}
\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{-i\lambda t}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{-i}{\sqrt{2}}e^{i\lambda t}\begin{pmatrix} i \\ 1 \end{pmatrix} \\
&= \frac{1}{2}e^{-i\lambda t}\begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{i}{2}e^{i\lambda t}\begin{pmatrix} i \\ 1 \end{pmatrix}
\end{aligned}$$

Therefore

$$\begin{aligned}
X_1(t) &= \frac{1}{2}e^{-i\lambda t} - \frac{i^2}{2}e^{i\lambda t} \\
X_2(t) &= i\frac{1}{2}e^{-i\lambda t} - \frac{i}{2}e^{i\lambda t}
\end{aligned}$$

Or

$$\begin{aligned}
X_1(t) &= \frac{1}{2}e^{-i\lambda t} + \frac{1}{2}e^{i\lambda t} \\
X_2(t) &= -\frac{1}{2i}e^{-i\lambda t} + \frac{1}{2i}e^{i\lambda t}
\end{aligned}$$

Or

$$\begin{aligned}
X_1(t) &= \cos(\lambda t) \\
X_2(t) &= \sin(\lambda t)
\end{aligned}$$

Or

$$\begin{aligned}
X_1(t) &= \cos\left(\frac{eB}{2m}t\right) \\
X_2(t) &= \sin\left(\frac{eB}{2m}t\right)
\end{aligned}$$

Hence

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{eB}{2m}t\right) \\ \sin\left(\frac{eB}{2m}t\right) \end{pmatrix}$$

3.3 Third exam

Local contents

3.3.1 Questions

3.3.1.1 First question

Q3P1

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Apr 23 at 1:25pm - Apr 23 at 2:18pm about 1 hour

This assignment was locked Apr 23 at 2:18pm.

The normalized wave function of an energy eigenstate of the harmonic oscillator is $\psi(x) = \left(\frac{4}{\pi}\right)^{1/4}\left(\frac{\mu\omega}{\hbar}\right)^{3/4} x \exp\left(-\frac{\mu\omega x^2}{2\hbar}\right)$

where μ, ω are the mass and oscillation frequency, respectively. When momentum is measured for this state,

(a) what are the possible values (5 points) and

(b) what is the probability of measuring a momentum between p and $p + dp$? (45 points)

3.3.1.2 Second question

Q3P2

Due Friday by 2:15pm **Points** 50 **Submitting** a file upload **File Types** pdf
Available Apr 23 at 1:25pm - Apr 23 at 2:18pm about 1 hour

This assignment was locked Apr 23 at 2:18pm.

Consider $\frac{dN_1}{dt} = p - \lambda_1 N_1, \frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2,$

where p, λ_1, λ_2 are positive constants.

Given $N_1(0) = N_2(0) = 0,$

(a) find $N_1(t), N_2(t)$ for $t > 0$ (40 points) and

(b) justify the limiting values of N_1, N_2 as $t \to \infty$ (10 points)

3.4 Final exam

Local contents

3.4.1 Questions

Physics 3041 (Spring 2021) **Final Exam**

1. The electron spin is represented by the operator $\vec{s} = (\hbar/2)\vec{\sigma}$, where $\vec{\sigma}$ corresponds to the Pauli matrices. Consider the operator $s_n = \vec{s} \cdot \hat{n}$, where $\hat{n}$ is the unit vector with polar angle θ and azimuthal angle ϕ.

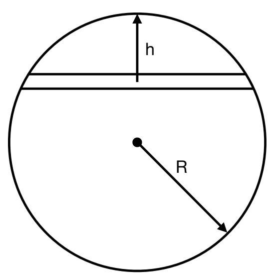
(1) Find the eigenvalues and eigenvectors of s_n. (20 points)

(2) If an electron is in the spin state of $s_z = \hbar/2$, what are the possible results and the corresponding probabilities when s_n is measured? (5 points)

2. A straight tunnel is dug between two cities on a planet of uniform mass density ρ (see cross sectional view below). The effect of the tunnel on the planet's gravity can be ignored. A train with no engine moves on the frictionless rail in the tunnel.

(1) Derive a differential equation that governs the position of the train as a function of time. There is no need to solve the equation for this part. (15 points)

(2) Assume that the train starts from rest at one city. Find the time required for the train to complete a round trip between the two cities. (10 points)



3. In a photon gas, the number density of photons with momentum between $\vec{p}$ and $\vec{p}+d\vec{p}$ is

$$dn = \frac{2}{(2\pi\hbar)^3}\frac{dp_x dp_y dp_z}{\exp[pc/(kT)]-1},$$

where c is the speed of light, k is the Boltzmann constant, and T is the temperature of the photon gas.

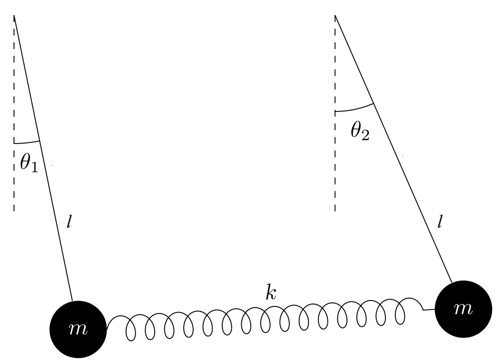
(1) Use the force law $\vec{F} = d\vec{p}/dt$ to derive an expression of the pressure exerted by the photon gas on its container. Your result should be in terms of a dimensional factor multiplied by a dimensionless integral. (20 points)

(2) Evaluate the dimensionless integral in part (1) in terms of a numerical series. (5 points)

4. Two identical pendulums are hung from the same height and coupled with a spring. Each pendulum consists of a string of length l and a bob of mass m. The entire system is in a fixed vertical plane (see figure below). The spring has a spring constant k and is relaxed when $\theta_1 = \theta_2 = 0$. The masses of the strings and the spring can be ignored. The acceleration of gravity is g.

(1) Derive the Lagrangian of the system to the second order in θ_1 and θ_2 (i.e., including terms up to θ_1^2, θ_2^2, and $\theta_1\theta_2$). (10 points)

(2) Find the normal modes and the corresponding oscillation frequencies. (15 points)



Chapter 4

HWs

Local contents

4.1 HW 1

Local contents

4.1.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 1 **(Due 1/27)**

1. Problem 1.6.1. (20 points)

2. Consider $f(x) = (1+x)^p$ for (a) $p = 1/3$ and (b) $p = -2$, respectively.

(1) Find the Taylor series of $f(x)$ around $x = 0$. (10 points)

(2) From the form of the general term, find the interval of convergence for the series. (10 points)

(3) How many terms in the series do you need to estimate $f(0.1)$ to within 1%? Check that the difference between your estimate and the actual result has approximately the same magnitude as the next term in the series. (10 points)

3. Expand $f(x) = \tan x^2$ to oder x^6 using (a) direct Taylor expansion of $\tan x$ with a substitution (20 points), and (b) the Taylor series of $\sin x$ and $\cos x$ along with appropriate substitutions (20 points).

4. A particle of mass m moves along the $+x$-axis (i.e., $x > 0$) with a potential energy

$$V(x) = \frac{a}{2x^2} - \frac{b}{x},$$

where a and b are positive parameters.

(a) Find the equilibrium position x_0. (3 points)

(b) Show that the particle executes harmonic oscillations near $x = x_0$. (5 points)

(c) Find the angular frequency of oscillations. (2 points)

4.1.2 Problem 1.6.1

Expand the function $f(x)=\frac{\sin(x)}{\cosh(x)+2}$ in Taylor series around the origin going up to x^3. Calculate $f(0.1)$ from this series and compare to the exact answer obtained by using a calculator

Solution

The Taylor series of function $f(x)$ around origin is given by (1.3.16) ($\approx$ is used throughout this HW to mean that the left side is the Taylor series approximation of $f(x)$).

$$f(x)\approx\sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0)$$

Where $f^{(n)}(0)$ is the n^{th} derivative of $f(x)$ evaluated at $x=0$.

For $n=0$, $f^{(0)}(x)=f(x)=\frac{\sin(x)}{\cosh(x)+2}$, therefore $f(0)=0$.

For $n=1$

$$\begin{aligned}
f^{(1)}(x)&=\frac{d}{dx}\left(\frac{\sin(x)}{\cosh(x)+2}\right)\\
&=\frac{\cos(x)(\cosh(x)+2)-\sin(x)\sinh(x)}{(\cosh(x)+2)^2}\\
&=\frac{\cos(x)(\cosh(x)+2)}{(\cosh(x)+2)^2}-\frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2}\\
&=\frac{\cos(x)}{\cosh(x)+2}-\frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2}
\end{aligned}$$

The above evaluated at $x=0$ becomes

$$\begin{aligned}
f^{(1)}(0)&=\frac{1}{1+2}-\frac{0}{(1+2)^2}\\
&=\frac{1}{3}
\end{aligned}$$

For $n=2$

$$\begin{aligned}
f^{(2)}(x)&=\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{\sin(x)}{\cosh(x)+2}\right)\right)\\
&=\frac{d}{dx}\left(\frac{\cos(x)}{\cosh(x)+2}-\frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2}\right)\\
&=\frac{-\sin(x)(\cosh(x)+2)-\cos(x)\sinh(x)}{(\cosh(x)+2)^2}\\
&\quad-\frac{(\cos(x)\sinh(x)+\sin(x)\cosh(x))(\cosh(x)+2)^2-\sin(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4}\\
&=\frac{-\sin(x)(\cosh(x)+2)}{(\cosh(x)+2)^2}-\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2}-\frac{\cos(x)\sinh(x)(\cosh(x)+2)^2}{(\cosh(x)+2)^4}\\
&\quad-\frac{\sin(x)\cosh(x)(\cosh(x)+2)^2}{(\cosh(x)+2)^4}+\frac{\sin(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4}\\
&=\frac{-\sin(x)}{\cosh(x)+2}-\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2}-\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2}-\frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2}+\frac{2\sin(x)\sinh(x)\sinh(x)}{(\cosh(x)+2)^3}\\
&=\frac{-\sin(x)}{\cosh(x)+2}-2\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2}-\frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2}+\frac{2\sin(x)\sinh^2(x)}{(\cosh(x)+2)^3}
\end{aligned}$$

The above evaluated at $x=0$ becomes

$$\begin{aligned}
f^{(2)}(0)&=\frac{-0}{1+2}-2\frac{0}{(1+2)^2}-\frac{0}{(1+2)^2}+\frac{0}{(1+2)^3}\\
&=0
\end{aligned}$$

For $n = 3$

$$
\begin{aligned}
f^{(3)}(x) &= \frac{d}{dx}\left(\frac{d^2}{dx^2}\left(\frac{\sin(x)}{\cosh(x)+2}\right)\right)\\
&= \frac{d}{dx}\left(\frac{-\sin(x)}{\cosh(x)+2} - 2\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2} + \frac{2\sin(x)\sinh^2(x)}{(\cosh(x)+2)^3}\right)\\
&= \frac{-\cos(x)(\cosh(x)+2)+\sin(x)\sinh(x)}{(\cosh(x)+2)^2}\\
&\quad - 2\frac{(-\sin(x)\sinh(x)+\cos(x)\cosh(x))(\cosh(x)+2)^2 - \cos(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4}\\
&\quad - \frac{(\cos(x)\cosh(x)+\sin(x)\sinh(x))(\cosh(x)+2)^2 - \sin(x)\cosh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4}\\
&\quad + 2\frac{\left(\cos(x)\sinh^2(x)+2\sin(x)\cosh(x)\right)(\cosh(x)+2)^3 - \left(\sin(x)\sinh^2(x)\right)\left(3(\cosh(x)+2)^2\sinh(x)\right)}{(\cosh(x)+2)^6}
\end{aligned}
$$

The above evaluated at $x = 0$ gives

$$
\begin{aligned}
f^{(3)}(0) &= \frac{-1(1+2)+0}{(1+2)^2} - 2\frac{(-0+1)(1+2)^2-0}{(1+2)^4} - \frac{(1+0)(1+2)^2-0}{(1+2)^4} + 2\frac{(0+0)(1+2)^3-(0)\left(3(1+2)^2 0\right)}{(1+2)^6}\\
&= \frac{-(3)}{(3)^2} - 2\frac{(1)(3)^2}{(3)^4} - \frac{(1)(3)^2}{(3)^4} + 2\frac{0}{(3)^6}\\
&= \frac{-1}{3} - 2\frac{1}{3^2} - \frac{1}{3^2}\\
&= -\frac{2}{3}
\end{aligned}
$$

The process stops here, because the problem is asking for $n = 3$. Substituting all the derivatives $f^{(n)}(0)$ values above into

$$
f(x) \approx \sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0)
$$

For up to $n = 3$ gives the following

$$
\begin{aligned}
f(x) &\approx f(0) + xf^{(1)}(0) + \frac{x^2}{2}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots\\
&\approx 0 + x\frac{1}{3} + \frac{x^2}{2}(0) + \frac{x^3}{3!}\left(-\frac{2}{3}\right)\\
&\approx x\frac{1}{3} - \frac{2}{3}\frac{x^3}{6}\\
&\approx \frac{x}{3} - \frac{x^3}{9}
\end{aligned}
$$

When $x = \frac{1}{10}$ the above becomes

$$
\begin{aligned}
f_{n=3}\left(\frac{1}{10}\right) &\approx \frac{1}{30} - \frac{1}{(1000)9}\\
&\approx \frac{1}{30} - \frac{1}{9000}\\
&\approx \frac{300-1}{9000}\\
&\approx \frac{299}{9000}
\end{aligned}
$$

From the calculator

$$
\frac{299}{9000} \approx 0.0332222
$$

And from the exact expression

$$\frac{\sin(x)}{\cosh(x)+2} = \frac{\sin(0.1)}{\cosh(0.1)+2}$$
$$= 0.0332224$$

The error is about 1.67×10^{-7}.

4.1.3 Problem 2

Consider $f(x) = (1+x)^p$ for (a) $p = \frac{1}{3}$ and (b) $p = -2$, respectively. (1) Find the Taylor series of $f(x)$ around $x = 0$. (2) From the form of the general term, find the interval of convergence of the series. (3) How many terms in the series do you need to estimate $f(0.1)$ to within 1% ? Check that the difference between your estimate and the actual result has approximately the same magnitude as the next term in the series.

Solution

4.1.3.1 Case $p = \frac{1}{3}$

$$f(x) = (1+x)^{\frac{1}{3}}$$

Part (1) The Taylor series is given by

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots \tag{1}$$

Where $f(0) = 1$ and $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$. Hence $f'(0) = \frac{1}{3}$ and $f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)(1+x)^{-\frac{5}{3}}$. Hence $f''(0) = -\frac{(2)}{3^2}$, and $f'''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+x)^{-\frac{8}{3}}$, hence $f'''(0) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right) = \frac{(2)(5)}{3^3}$, and $f^{(4)}(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(1+x)^{-\frac{11}{3}}$, hence $f^{(4)}(0) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right) = -\frac{1}{3^4}((2)(5)(8))$ and on. The series in (1) becomes

$$\begin{aligned}
f(x) &\approx 1 + \frac{1}{3}x - \frac{(2)}{3^2}\frac{x^2}{2!} + \frac{(2)(5)}{3^3}\frac{x^3}{3!} - \frac{(2)(5)(8)}{3^4}\frac{x^4}{4!} + \frac{(2)(5)(8)(11)}{3^5}\frac{x^5}{5!} - \frac{(2)(5)(8)(11)(14)}{3^6}\frac{x^6}{6!} - \cdots \\
&\approx 1 + \frac{1}{3}x - \frac{1}{3^2}x^2 + \frac{5}{3^4}x^3 - \frac{10}{3^5}x^4 + \frac{22}{3^6}x^5 - \frac{154}{3^8}x^6 + \cdots
\end{aligned} \tag{2}$$

The general term is found by comparing the above to the general term obtained from binomial expansion. Since

$$(1+x)^p = \binom{p}{0}x^0 + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots \tag{3}$$

Comparing (2,3) shows that the general term is the binomial coefficient $\binom{\frac{1}{3}}{n}$. Therefore the Taylor series for $(1+x)^{\frac{1}{3}}$ can be written as

$$f(x) \approx \sum_{n=0}^{\infty}\binom{p}{n}x^n$$

For $p = \frac{1}{3}$ the above becomes

$$f(x) \approx \sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$$

Part(2)

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\binom{p}{n}}{\binom{p}{n+1}}\right| \end{aligned}$$

The Binomial coefficient $\binom{p}{n} = \frac{p!}{n!(p-n)!}$, for when p is integer. This is not the case here.

For non-integer p The Binomial coefficient becomes $\binom{p}{n} = \frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)}$ where $\Gamma(p)$ is the Gamma function. The above ratio now becomes

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{\frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)}}{\frac{\Gamma(p+1)}{\Gamma(n+2)\Gamma(p-n)}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\Gamma(n+2)\Gamma(p-n)}{\Gamma(n+1)\Gamma(p-n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n\Gamma(p-n)}{\Gamma(p-n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n}{p-n}\right| \end{aligned}$$

But $p = \frac{1}{3}$, hence the above becomes

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{n}{\frac{1}{3}-n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n}{n-\frac{1}{3}}\right| \\ &= 1 \end{aligned}$$

Therefore the radius of convergence is 1. This means the Taylor series found above converges to $f(x)$ for $|x| < 1$.

Part 3

$$f(x) = (1+x)^{\frac{1}{3}}$$

When $x = 0.1$

$$\begin{aligned} f(0.1) &= (1.1)^{\frac{1}{3}} \\ &= 1.032280115 \end{aligned}$$

one percent of the above is

$$\frac{1}{100}(1.032280115) = 0.01032280115$$

The value n is now found such that

$$|R_n(x)| \leq M\frac{(0.1)^{n+1}}{(n+1)!} \leq 0.01032280115$$

Where $R_n(x)$ is the Taylor series remainder using n terms. M is the upper bound for the $n+1$ derivative of $f(x)$ any where between $[0, 0.1]$. Instead of trying to find M, few calculations are used to find how many terms are needed.

For $n = 0$, $\tilde{f}(0.1) = 1$ and the error is $1.032280115 - 1 = 032280115$.

For $n = 1$, $\tilde{f}(0.1) = 1 + \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} 0.1 = 1.0333333$, and the error is $|1.032280115 - 1.0333333| = 0.001053218$. Because this is smaller than $R_n(x)$ then only two terms are needed in the Taylor series to obtained the required accuracy. Therefore

$$f(x) \approx 1 + \frac{1}{3}x$$

4.1.3.2 Case $p = -2$

$$f(x) = (1+x)^{-2}$$

Part (1) The Taylor series is

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots$$

But $f(0) = 1$ and $f'(x) = (-2)(1+x)^{-3}$. Hence $f'(0) = -2$ and $f''(x) = (-2)(-3)(1+x)^{-4}$. Hence $f''(0) = (-2)(-3)$, and $f'''(x) = -2(-3)(-4)(1+x)^{-5}$, hence $f'''(0) = (-2)(-3)(-4)(-5)$ and so on. The above becomes

$$\begin{aligned} f(x) &\approx 1 + (-2)x - (-2)(-3)\frac{x^2}{2!} + (-2)(-3)(-4)\frac{x^3}{3!} + \cdots \\ &\approx 1 - 2x + (2)(3)\frac{x^2}{2!} - (2)(3)(4)\frac{x^3}{3!} + \cdots \\ &\approx 1 - 2x + 3x^2 - 4x^3 + \cdots \end{aligned}$$

The general term is therefore

$$f(x) \approx \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

Part(2)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} \right| \\ &= 1 \end{aligned}$$

Hence the series converges to $f(x)$ for $|x| < 1$.

Part 3

$$f(x) = (1+x)^{-2}$$

For $x = 0.1$

$$\begin{aligned} f(0.1) &= (1.1)^{-2} \\ &= \frac{1}{1.1^2} \\ &= 0.82644628 \end{aligned}$$

One percent of the above is

$$\frac{1}{100}(0.8264462810) = 0.0082644628$$

The value n is now found such that

$$|R_n(x)| \leq M\frac{(0.1)^{n+1}}{(n+1)!} \leq 0.0082644628$$

Where $R_n(x)$ is the Taylor series remainder using n terms. M is the upper bound for the $n+1$ derivative of $f(x)$ any where between $[0, 0.1]$. Doing few calculations gives

For $n = 0$, $\tilde{f}(0.1) = 1$, the error is $|0.82644628 - 1| = 0.1735537190$.

For $n = 1$, $\tilde{f}(0.1) = 0.8$, the error is $|0.82644628 - 0.8| = 0.02644628$.

For $n = 2$, $\tilde{f}(0.1) = 0.83$, the error is $|0.82644628 - 0.83| = 0.0035537190$. Because this is within 1% then only <u>three terms</u> are needed. Therefore

$$f(x) \approx 1 - 2x + 3x^2$$

4.1.4 Problem 3

Expand $f(x) = \tan\left(x^2\right)$ to order x^6 using (a) direct Taylor expansion. (b) The Taylor series for $\sin(x)$ and $\cos x$ with appropriate substitution.

<u>Solution</u>

4.1.4.1 Part a

Using Taylor series

$$f(x) \approx \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Where $f(x) = \tan\left(x^2\right)$ and the expansion is around $x = 0$. The Taylor series for $f(u) = \tan(u)$ is found instead of $\tan\left(x^2\right)$, and then at the end u is replaced by x^2 . This is called the substitution method. This simplifies the derivations. Therefore $f(0) = 0$. The first derivative is

$$\begin{aligned} f'(u) &= \frac{d}{du}\tan(u) \\ &= \frac{d}{du}\left(\frac{\sin u}{\cos u}\right) \\ &= \frac{\cos^2 u + \cos^2 u}{\cos^2 u} \\ &= \frac{1}{\cos^2 u} \end{aligned}$$

At $u = 0$ this gives $f'(0) = 1$.

The next derivative using the above result gives

$$\begin{aligned} f''(u) &= \frac{d}{du}\left(\frac{1}{\cos^2 u}\right) \\ &= \frac{2\cos u \sin u}{\cos^4 u} \\ &= \frac{2\sin u}{\cos^3 u} \end{aligned}$$

At $u = 0$ this gives $f^{(2)}(0) = 0$. The next derivative gives

$$
\begin{aligned}
f^{(3)}(u) &= 2\frac{d}{du}\left(\frac{\sin u}{\cos^3 u}\right) \\
&= 2\frac{\cos u \cos^3 u - \sin u\left(3\cos^2 u(-\sin u)\right)}{\cos^6 u} \\
&= 2\frac{\cos^4 u + 3\sin^2 u \cos^2 u}{\cos^6 u} \\
&= \frac{2\cos^4 u}{\cos^6 u} + \frac{6\sin^2 u \cos^2 u}{\cos^6 u} \\
&= \frac{2}{\cos^2 u} + \frac{6\sin^2 u}{\cos^4 u} \\
&= \frac{2}{\cos^2 u} + \frac{6\left(1-\cos^2 u\right)}{\cos^4 u} \\
&= \frac{2}{\cos^2 u} + \frac{6}{\cos^4 u} - \frac{6}{\cos^2 u} \\
&= -\frac{4}{\cos^2 u} + \frac{6}{\cos^4 u}
\end{aligned}
$$

At $u = 0$ this gives $f^{(3)}(0) = -\frac{4}{1} + \frac{6}{1} = 2$. Since the problem is asking for order x^6 the process stops here, as this is the same as order u^3 when u is replaced by x^2.

Therefore the Taylor series for $\tan(u)$ is (for up to $n = 3$)

$$
\begin{aligned}
f(u) &\approx f(0) + uf'(0) + \frac{u^2}{2!}f^{(2)}(0) + \frac{u^3}{3!}f^{(3)}(0) + \cdots \\
&\approx 0 + u + 0 + 2\frac{u^3}{3!} \\
&\approx u + \frac{1}{3}u^3
\end{aligned}
$$

Replacing $u = x^2$, gives the Taylor series for $\tan\left(x^2\right)$ for up to x^6 term as

$$
\tan\left(x^2\right) \approx x^2 + \frac{1}{3}x^6
$$

4.1.4.2 Part b

To obtain the above result using the Taylor series for $\sin\left(x^2\right), \cos\left(x^2\right)$, the Taylor series for $\sin\left(x^2\right)$ and $\cos\left(x^2\right)$ is found, and long division is applied using the definition of $\tan\left(x^2\right) = \frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}$. Terms with order larger than x^6 are ignored. The Taylor series for $\sin(x)$ is

$$
\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
$$

Using the substitution method, the Taylor series for $\sin\left(x^2\right)$ becomes

$$
\begin{aligned}
\sin\left(x^2\right) &\approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \\
&\approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots \qquad (1)
\end{aligned}
$$

The Taylor series for $\cos(x)$ is

$$
\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
$$

Using the substitution method, the Taylor series for $\cos\left(x^2\right)$ becomes

$$
\begin{aligned}
\cos\left(x^2\right) &\approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots \\
&\approx 1 - \frac{x^4}{2} + \frac{x^8}{24} - \cdots \qquad (2)
\end{aligned}
$$

Since $\tan\left(x^2\right) = \frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}$ then the Taylor series for $\tan\left(x^2\right)$ is

$$\tan\left(x^2\right) \approx \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots}{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots}$$

Performing long division and stopping when the remainder has powers larger than x^6 gives

$$\tan\left(x^2\right) \approx x^2 + \frac{1}{3}x^6 + \cdots$$

Which is same result as part(a).

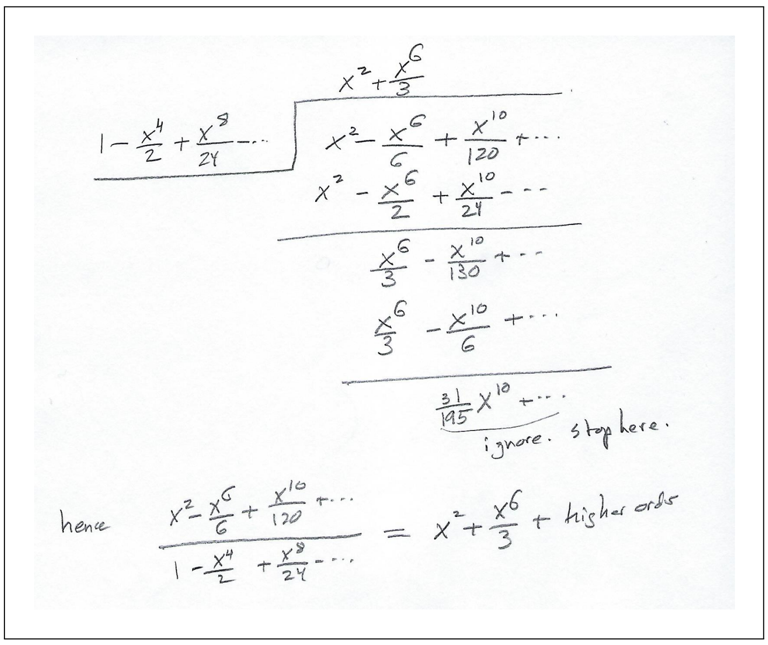


Figure 4.1: Polynomals long division

4.1.5 Problem 4

A particle of mass m moves along the $+x$ axis (i.e. $x > 0$) with potential energy

$$V(x) = \frac{a}{2x^2} - \frac{b}{x}$$

Where a and b are positive parameters. (a) Find the equilibrium position x_0. (b) Show that the particle executes harmonic oscillations near $x = x_0$. (c) Find the angular frequency of oscillations.

Solution

4.1.5.1 Part a

Equilibrium position is where the slope of the potential energy is zero. This position x_0 is found by solving for x from

$$\frac{dV}{dx} = 0$$

But

$$\begin{aligned}\frac{dV}{dx} &= \frac{a}{2}\left(-2x^{-3}\right) - b\left(-x^{-2}\right)\\ &= \frac{-a}{x^3} + \frac{b}{x^2}\\ &= \frac{-a + bx}{x^3}\end{aligned}$$

Hence

$$\frac{-a+bx}{x^3}=0$$
$$bx=a$$

Therefore

$$x_0=\frac{a}{b}$$

4.1.5.2 Part b

Approximating $V(x)$ around x_0 using Taylor series gives

$$V(x)\approx V(x_0)+(x-x_0)V'(x_0)+\frac{(x-x_0)^2}{2!}V''(x_0)+\cdots$$

But $\frac{dV}{dx}$ evaluated at x_0 is zero, since this the equilibrium point. The above simplifies to

$$V(x)\approx V(x_0)+\frac{(x-x_0)^2}{2!}V''(x_0)+\cdots \tag{A}$$

Higher terms are ignored, because $(x-x_0)$ is assumed small and mass remain close to x_0. But

$$V(x_0)=\frac{a}{2x_0^2}-\frac{b}{x_0}$$

And since $x_0=\frac{a}{b}$ from part (a), the above simplifies to

$$\begin{aligned}V(x_0)&=\frac{a}{2\left(\frac{a}{b}\right)^2}-\frac{b}{\left(\frac{a}{b}\right)}\\&=\frac{ab^2}{2a^2}-\frac{b^2}{a}\\&=\frac{b^2}{2a}-\frac{b^2}{a}\\&=-\frac{1}{2}\frac{b^2}{a}\end{aligned} \tag{A1}$$

And

$$\begin{aligned}\frac{d^2V}{dx^2}&=\frac{d}{dx}\left(\frac{-a}{x^3}+\frac{b}{x^2}\right)\\&=\frac{3a}{x^4}-\frac{b}{x^3}\end{aligned}$$

At $x=x_0$ the above becomes

$$\begin{aligned}V''(x_0)&=\frac{3a}{\left(\frac{a}{b}\right)^4}-\frac{b}{\left(\frac{a}{b}\right)^3}\\&=\frac{b^4}{a^3}\end{aligned} \tag{A2}$$

Using (A1,A2) into A gives

$$\begin{aligned}V(x)&\approx-\frac{1}{2}\frac{b^2}{a}+\frac{(x-x_0)^2}{2!}\frac{b^4}{a^3}+\cdots\\&\approx-\frac{1}{2}\frac{b^2}{a}+\frac{\left(x-\frac{a}{b}\right)^2}{2}\frac{b^4}{a^3}+\cdots\\&\approx-\frac{1}{2}\frac{b^2}{a}+\frac{1}{2}\left(x^2+\frac{a^2}{b^2}-2x\frac{a}{b}\right)\frac{b^4}{a^3}+\cdots\\&\approx-\frac{1}{2}\frac{b^2}{a}+\frac{1}{2a}b^2+\frac{1}{2a^3}b^4x^2-\frac{1}{a^2}b^3x+\cdots\\&\approx\frac{b^4}{2a^3}x^2-\frac{b^3}{a^2}x+\cdots\end{aligned}$$

Therefore near x_0 the potential energy is approximated as

$$V(x) \approx \frac{b^4}{2a^3}x^2 - \frac{b^3}{a^2}x \tag{1}$$

The force on the mass is given by

$$F = -\frac{dV}{dx}$$

Using $V(x)$ in (1) the force becomes

$$F = -\frac{b^4}{a^3}x - \frac{b^3}{a^2}$$

But $F = m\frac{d^2x}{dt^2}$. Hence we obtain the equation of motion as

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ &= -\frac{b^4}{a^3}x - \frac{b^3}{a^2} \end{aligned}$$

Therefore

$$\begin{aligned} m\frac{d^2x(t)}{dt^2} + \frac{b^4}{a^3}x(t) &= -\frac{b^3}{a^2} \\ \frac{d^2x(t)}{dt^2} + \left(\frac{b^4}{ma^3}\right)x(t) &= -\frac{b^3}{ma^2} \end{aligned} \tag{B}$$

Let

$$\frac{b^4}{ma^3} = \omega^2$$

The equation of motion (B) becomes

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = -\frac{b^3}{ma^2}$$

But this is standard second order ode whose solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + x_p(t)$$

Where $x_p(t)$ is the particular solution due to the forcing function $-\frac{b^3}{ma^2}$ and A, B are constants of integrations found from initial conditions. Since the forcing function is just constant, and not function function of time, the above becomes

$$\begin{aligned} x(t) &= A\cos(\omega t) + B\sin(\omega t) + F_p \\ &= A\cos\left(\omega t + \phi\right) + F_p \end{aligned}$$

Therefore the motion is simple harmonic motion since $\cos\left(\omega t + \phi\right)$ is harmonic. The forcing function F_p has no effect on the nature of the harmonic motion, other than adding an extra constant displacement shift to $x(t)$ for all time. Since there is no damping, the particle will continue this motion forever.

The following is a plot of the solution for 10 seconds using arbitrary values for a, b, m and with initial conditions $x(0) = 1, x'(0) = 0$. The solution shows the motion is harmonic as expected.

```
In[•]:= ClearAll[x, t, w, a, b, m];
ode = x''[t] + w x[t] == -b^3/(m a^2);
ic = {x[0] == 0, x'[0] == 0};
sol = x[t] /. First@DSolve[{ode, ic}, x[t], t]
```

Out[•]= $\frac{-b^3 + b^3 \operatorname{Cos}[t \sqrt{w}]}{a^2 m w}$

```
In[•]:= parms = {m → 1, a → 1, b → 3, w → Sqrt[b^4/(m a^3)]};
sol //. parms
```

Out[•]= $\frac{1}{9}(-27 + 27 \operatorname{Cos}[3 t])$

```
In[•]:= Plot[sol //. parms, {t, 0, 10}, AxesLabel → {"t", "x(t)"}, BaseStyle → 14, PlotStyle → Blue]
```

Out[•]=

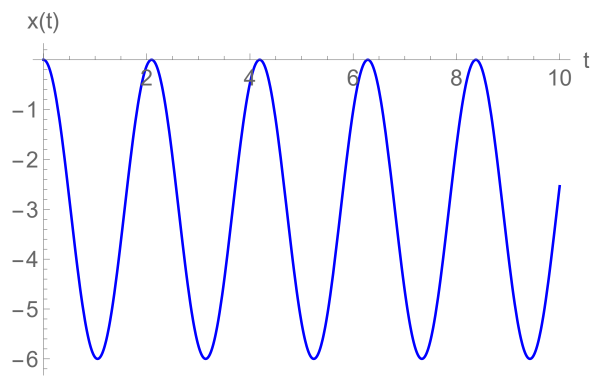


Figure 4.2: Plot of solution

4.1.5.3 Part c

The angular frequency of oscillation is

$$\omega = \sqrt{\frac{b^4}{ma^3}}$$

In radians per second. The quantity $\frac{b^4}{a^3}$ can be called the stiffness k (Newton per meter). Hence $\omega = \sqrt{\frac{k}{m}}$.

4.1.5.4 Appendix

An easier way to do part b, is to keep $(x - x_0)$ intact and replace this with y at the end. Like this

Using (A1,A2) into A gives

$$V(x) \approx -\frac{1}{2}\frac{b^2}{a} + \frac{(x - x_0)^2}{2!}\frac{b^4}{a^3} + \cdots$$

The force on the mass is given by

$$\begin{aligned} F &= -\frac{dV}{dx} \\ &= -(x - x_0)\frac{b^4}{a^3} \end{aligned}$$

But $F = m\frac{d^2x}{dt^2}$. Hence we obtain the equation of motion as

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ &= -(x - x_0)\frac{b^4}{a^3} \end{aligned}$$

Now let $y = x - x_0$. the above becomes

$$
\begin{aligned}
m\frac{d^2y}{dt^2} &= -y\frac{b^4}{a^3} \\
m\frac{d^2y}{dt^2} + y\frac{b^4}{a^3} &= 0 \\
\frac{d^2y}{dt^2} + y\frac{b^4}{ma^3} &= 0
\end{aligned}
$$

Which is SHM. Using this method, it is faster to show.

4.1.6 key solution for HW 1

Physics 3041 (Spring 2021) Solutions to Homework Set 1

1. Problem 1.6.1. (20 points)

Let's first try the most straightforward way:

$$f(x)=\frac{\sin x}{\cosh x+2}\Rightarrow f(0)=0$$

$$f'(x)=\frac{\cos x}{\cosh x+2}-\frac{\sin x\sinh x}{(\cosh x+2)^2}\Rightarrow f'(0)=\frac{1}{3}$$

$$\begin{aligned}
f''(x) &= -\frac{\sin x}{\cosh x+2}-\frac{\cos x\sinh x}{(\cosh x+2)^2}-\frac{\cos x\sinh x+\sin x\cosh x}{(\cosh x+2)^2}+\frac{2\sin x\sinh^2 x}{(\cosh x+2)^3}\\
&= -\frac{\sin x}{\cosh x+2}-\frac{2\cos x\sinh x+\sin x\cosh x}{(\cosh x+2)^2}+\frac{2\sin x\sinh^2 x}{(\cosh x+2)^3}\Rightarrow f''(0)=0
\end{aligned}$$

$$\begin{aligned}
f'''(x) &= -\frac{\cos x}{\cosh x+2}+\frac{\sin x\sinh x}{(\cosh x+2)^2}-\frac{2(-\sin x\sinh x+\cos x\cosh x)+\cos x\cosh x+\sin x\sinh x}{(\cosh x+2)^2}\\
&+ \frac{2(2\cos x\sinh x+\sin x\cosh x)\sinh x}{(\cosh x+2)^3}+\frac{2(\cos x\sinh^2 x+2\sin x\sinh x\cosh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\\
&= -\frac{\cos x}{\cosh x+2}+\frac{\sin x\sinh x}{(\cosh x+2)^2}-\frac{-\sin x\sinh x+3\cos x\cosh x}{(\cosh x+2)^2}\\
&+ \frac{6(\cos x\sinh^2 x+\sin x\cosh x\sinh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\\
&= -\frac{\cos x}{\cosh x+2}+\frac{2\sin x\sinh x-3\cos x\cosh x}{(\cosh x+2)^2}\\
&+ \frac{6(\cos x\sinh^2 x+\sin x\cosh x\sinh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\Rightarrow f'''(0)=-\frac{1}{3}-\frac{3}{9}=-\frac{2}{3}
\end{aligned}$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots=\frac{x}{3}-\frac{2x^3}{3\times 6}+\cdots=\frac{x}{3}-\frac{x^3}{9}+\cdots$$

$$f(0.1)\approx\frac{0.1}{3}-\frac{0.1^3}{9}\approx 0.0332222 \text{ to be compared with } f(0.1)=\frac{\sin 0.1}{\cosh 0.1+2}=0.0332224$$

Now consider a simpler way starting with

$$\sin x=x-\frac{x^3}{6}+\cdots,\ \cosh x=1+\frac{x^2}{2}+\cdots,$$

where we have ignored terms of orders higher than x^3.

$$\begin{aligned}
f(x) &= \frac{\sin x}{\cosh x+2}=\frac{x-x^3/6+\cdots}{(1+x^2/2+\cdots)+2}=\frac{x-x^3/6+\cdots}{3+x^2/2+\cdots}=\frac{x-x^3/6+\cdots}{3(1+x^2/6+\cdots)}\\
&= \frac{x-x^3/6+\cdots}{3}(1-x^2/6+\cdots)=\frac{1}{3}(x-x^3/6-x^3/6+\cdots)=\frac{x}{3}-\frac{x^3}{9}+\cdots,
\end{aligned}$$

where we have used $(1+y)^{-1}=1-y+\cdots$ with $y=x^2/6+\cdots$.

2. Consider $f(x) = (1+x)^p$ for (a) $p = 1/3$ and (b) $p = -2$, respectively.

(1) Find the Taylor series of $f(x)$ around $x = 0$. (10 points)

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

For $p = 1/3$,

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} \cdots + (-1)^{n-1}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{3^n n!}x^n + \cdots,$$

and for $p = -2$,

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots + (-1)^n(n+1)x^n + \cdots.$$

(2) From the form of the general term, find the interval of convergence for the series. (10 points)

For $p = 1/3$, the general term is

$$a_n x^n = (-1)^{n-1}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{3^n n!}x^n.$$

So the interval of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{2\cdot 5\cdot 8\cdot (3n-4)(3n-1)} \times \frac{3^{n+1}(n+1)!}{3^n n!} = \lim_{n\to\infty}\frac{3(n+1)}{3n-1} = 1.$$

For $p = -2$, the general term is

$$a_n x^n = (-1)^n(n+1)x^n \Rightarrow R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n+1}{n+2} = 1.$$

(3) How many terms in the series do you need to estimate $f(0.1)$ to within 1%? Check that the difference between your estimate and the actual result has approximately the same magnitude as the next term in the series. (10 points)

For $p = 1/3$, the second term is $0.1/3 \approx 0.033$ and the third term is $-0.1^2/9 \approx -1.1\times 10^{-3}$. So within 1% we only need to keep the first two terms: $1.1^{1/3} \approx 1 + 0.1/3 \approx 1.0333$. The difference between the actual result and this estimate is $1.1^{1/3} - 1.0333 \approx 1.0323 - 1.0333 = -10^{-3}$, which indeed has the same magnitude and sign as the third term.

For $p = -2$, the third term is $3\times 0.1^2 = 0.03$ and the fourth term is $-4\times 0.1^3 = -4\times 10^{-3}$. So within 1% we only need to keep the first three terms: $1.1^{-2} \approx 1 - 2\cdot 0.1 + 3\cdot 0.1^2 = 0.83$. The difference between the actual result and this estimate is $1.1^{-2} - 0.83 \approx 0.82645 - 0.83 = -3.55\times 10^{-3}$, which indeed has the same magnitude and sign as the fourth term.

3. Expand $f(x) = \tan x^2$ to oder x^6 using (a) direct Taylor expansion of $\tan x$ with a substitution (20 points), and (b) the Taylor series of $\sin x$ and $\cos x$ along with appropriate substitutions (20 points).

(a) Direct Taylor expansion of $g(y) = \tan y$

$$g(0) = 0,\ g'(y) = \frac{1}{\cos^2 y} \Rightarrow g'(0) = 1$$

$$g''(y) = \frac{2\sin y}{\cos^3 y} \Rightarrow g''(0) = 0$$

$$g'''(y) = \frac{2\cos y}{\cos^3 y} + \frac{2\sin y(3\sin y)}{\cos^4 y} = \frac{2}{\cos^2 y} + \frac{6\sin^2 y}{\cos^4 y} \Rightarrow g'''(0) = 2$$

$$\begin{aligned} g(y) &= g(0) + g'(0)y + \frac{g''(0)}{2!}y^2 + \frac{g'''(0)}{3!}y^3 + \cdots = y + \frac{2}{6}y^3 + \cdots = y + \frac{y^3}{3} + \cdots \\ &\Rightarrow f(x) = g(x^2) = x^2 + \frac{x^6}{3} + \cdots \end{aligned}$$

(b) Use Taylor series of $\sin y$ and $\cos y$

$$\sin y = y - \frac{y^3}{6} + \cdots,\ \cos y = 1 - \frac{y^2}{2} + \cdots,$$

where we have ignored terms of orders higher than y^3.

$$\tan y = \frac{\sin y}{\cos y} = \frac{y - y^3/6 + \cdots}{1 - y^2/2 + \cdots} = (y - y^3/6 + \cdots)(1 + y^2/2 + \cdots) = y - \frac{y^3}{6} + \frac{y^3}{2} + \cdots = y + \frac{y^3}{3} + \cdots,$$

where we have used $(1+z)^{-1} = 1 - z + \cdots$ with $z = -y^2/2 + \cdots$.

$$y = x^2 \Rightarrow \tan x^2 = x^2 + \frac{x^6}{3} + \cdots$$

4. A particle of mass m moves along the $+x$-axis (i.e., $x > 0$) with a potential energy

$$V(x) = \frac{a}{2x^2} - \frac{b}{x},$$

where a and b are positive parameters.

(a) Find the equilibrium position x_0. (3 points)

$$V'(x) = -\frac{a}{x^3} + \frac{b}{x^2},\ V'(x_0) = -\frac{a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{a}{b}$$

(b) Show that the particle executes harmonic oscillations near $x = x_0$. (5 points)

$$\begin{aligned}
V''(x) &= \frac{3a}{x^4} - \frac{2b}{x^3} \Rightarrow V''(x_0) = \frac{3a}{x_0^4} - \frac{2b}{x_0^3} = \frac{b^4}{a^3} > 0 \\
V(x) &\approx V(x_0) + V'(x_0)(x - x_0) + \frac{V''(x_0)}{2}(x - x_0)^2 = -\frac{b^2}{2a} + \frac{b^4}{2a^3}(x - x_0)^2 \\
F &= -V'(x) = -\frac{b^4}{a^3}(x - x_0) = m\ddot{x}(t),\ y \equiv x - x_0 \Rightarrow m\ddot{y} = -\frac{b^4}{a^3}y \\
\ddot{y} &= -\frac{b^4}{ma^3}y \equiv -\omega^2 y \Rightarrow y(t) = A\sin(\omega t + \phi_0),\ x(t) = x_0 + A\sin(\omega t + \phi_0)
\end{aligned}$$

(c) Find the angular frequency of oscillations. (2 points)

$$\omega^2 \equiv \frac{b^4}{ma^3} \Rightarrow \omega = \sqrt{\frac{b^4}{ma^3}}$$

4.2 HW 2

Local contents

4.2.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 2 **(Due 2/3)**

1. Problem 2.2.3. (10 points)

2. (a) Problem 2.2.10. (10 points)

(b) Problem 2.2.11. (10 points)

3. The probability to find a particle at position between x and $x + dx$ is

$$P(x)dx = A\exp(-\alpha x^2 + \beta x^3)dx,$$

where A, α, and β are positive parameters. By the definition of probability,

$$\int_{-\infty}^{\infty} P(x)dx = 1.$$

Treat β as a small parameter, i.e., for any given x, you can view $P(x)$ as a function of β and expand it around $\beta = 0$.

(a) Find A to the first order of β. (15 points)

(b) Find the average position

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

to the first order of β. (25 points)

4. A container of volume V encloses a neutrino gas of temperature T. The number of neutrinos with energy between E and $E + dE$ is

$$dN = \left(\frac{4\pi V}{h^3c^3}\right)\frac{E^2}{\exp[E/(kT)] + 1}dE,$$

where h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(a) Express the total energy density of the neutrino gas in terms of a dimensional factor multiplying a dimensionless integral. Show that the factor has the correct dimension. (10 points).

(b) Follow the discussion of a photon gas and evaluate the dimensionless integral. (20 points).

4.2.2 Problem 2.2.3

Evaluate $\int_0^1 e^{\sqrt{x}}dx$. Show that $\int_0^\infty e^{-x^4}dx = \Gamma\left(\frac{5}{4}\right)$

Solution

Let $y = \sqrt{x}$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\frac{1}{\sqrt{x}}\\ &= \frac{1}{2}\frac{1}{y}\end{aligned}$$

And

$$dx = 2ydy$$

When $x = 0$, $y = 0$ and when $x = 1$,$y = 1$. Substituting this back into $\int_0^1 e^{\sqrt{x}}dx$ gives $\int_0^1 e^y(2ydy) = 2\int_0^1 ye^y dy$. This integral is evaluated using integration by parts.

$$udv = uv|_0^1 - \int_0^1 vdu$$

Let $u = y$ and $dv = e^y$, then $du = dy$ and $v = e^y$. The above becomes

$$\begin{aligned}2\left(\int_0^1 ye^y dy\right) &= 2\left(uv|_0^1 - \int_0^1 vdu\right)\\ &= 2\left(ye^y|_0^1 - \int_0^1 e^y dy\right)\\ &= 2\left(\left(e^1 - 0\right) - e^y|_0^1\right)\\ &= 2(e-(e-1))\\ &= 2(e-e+1)\\ &= 2\end{aligned}$$

Hence

$$\int_0^1 e^{\sqrt{x}}dx = 2$$

For the second part of the question asking to evaluate $\int_0^\infty e^{-x^4}dx$, let

$$x = y^{\frac{1}{4}}$$

Then

$$\frac{dx}{dy} = \frac{1}{4}y^{\left(\frac{1}{4}-1\right)}$$

When $x = 0, y = 0$ and when $x = \infty, y = \infty$. Hence the above integral becomes

$$\begin{aligned}\int_0^\infty e^{-x^4}dx &= \int_0^\infty e^{-y}\left(\frac{1}{4}y^{\left(\frac{1}{4}-1\right)}dy\right)\\ &= \frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy \qquad (1)\end{aligned}$$

Comparing the above to integral (2.1.39) in the book which says

$$F(n) = \int_0^\infty y^n e^{-y}dy \qquad (2)$$

$$\Gamma(n) = F(n-1) \qquad (3)$$

Then putting $n = \frac{1}{4}$ in (3) gives

$$\begin{aligned}\Gamma\left(\frac{1}{4}\right) &= F\left(\frac{1}{4}-1\right)\\ &= \int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy\end{aligned}$$

Which is (1). This means that

$$\int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy = \Gamma\left(\frac{1}{4}\right)$$

Hence

$$\frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \tag{4}$$

To obtain the final form, the following property of Gamma functions is used

$$\Gamma(n+1) = n\Gamma(n)$$

Which means that when $n = \frac{1}{4}$, the above becomes

$$\begin{aligned}\Gamma\left(\frac{1}{4}+1\right) &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\\ \Gamma\left(\frac{5}{4}\right) &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\end{aligned}$$

Using this in (4) shows that

$$\frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy = \Gamma\left(\frac{5}{4}\right)$$

Which implies

$$\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$$

Which is what we are asked to show.

4.2.3 Problem 2.2.10 (or part a of problem 2)

Problem 2.2.10. *Consider*

$$I = \int_0^1 \frac{t-1}{\ln t}.$$

Think of the t in $t-1$ as the $a=1$ limit of t^a. Let $I(a)$ be the corresponding integral. Take the a derivative of both sides (using $t^a = e^{a\ln t}$) and evaluate dI/da by evaluating the corresponding integral by inspection. Given dI/da obtain I by performing the indefinite integral *of both sides with respect to a. Determine the constant of integration using your knowledge of $I(0)$. Show that the original integral equals* $\ln 2$.

Figure 4.3: Problem statment

Solution

Let

$$I(a)=\int_0^1 \frac{t^a-1}{\ln t}dt$$

Where $a=1$ for the specific integral in this problem. The above is the parametrized general form. Taking derivative w.r.t a gives

$$\begin{aligned}\frac{dI(a)}{da}&=\frac{d}{da}\left(\int_0^1 \frac{t^a-1}{\ln t}dt\right)\\&=\int_0^1 \frac{d}{da}\left(\frac{t^a-1}{\ln t}\right)dt\\&=\int_0^1 \frac{1}{\ln t}\frac{d}{da}(t^a-1)dt \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\frac{d}{da}(t^a-1)&=\frac{d}{da}\left(e^{a\ln t}-1\right)\\&=\ln(t)\left(e^{a\ln t}\right) \qquad (2)\end{aligned}$$

Substituting (2) into (1) gives

$$\begin{aligned}\frac{dI(a)}{da}&=\int_0^1 \frac{1}{\ln t}\left(\ln(t)\left(e^{a\ln t}\right)\right)dt\\&=\int_0^1 e^{a\ln t}dt\\&=\int_0^1 t^a dt\\&=\left.\frac{t^{a+1}}{a+1}\right|_0^1\\&=\frac{1}{1+a}\qquad a\neq -1 \qquad (3)\end{aligned}$$

Integrating the above is used to $I(a)$ gives

$$\begin{aligned}I(a)&=\int_0^a \frac{1}{1+\tau}d\tau\\&=\ln(1+\tau)|_0^a\\&=\ln(1+a)-\ln(1)\\&=\ln(1+a)\qquad a\neq -1\end{aligned}$$

When $a=1$ the above becomes

$$\begin{aligned}I(1)&=\int_0^1 \frac{t-1}{\ln t}dt\\&=\ln(1+1)\\&=\ln(2)\end{aligned}$$

Hence

$$\int_0^1 \frac{t-1}{\ln t}dt=\ln(2)$$

4.2.4 Problem 2.2.11 (or part b of problem 2)

Problem 2.2.11. *Given*

$$\int_0^\infty e^{-ax}\sin kx dx = \frac{k}{a^2+k^2},$$

evaluate $\int_0^\infty xe^{-ax}\sin kx dx$ *and* $\int_0^\infty xe^{-ax}\cos kx dx$.

Figure 4.4: Problem statment

Solution

4.2.4.1 part (1)

$$I = \int_0^\infty e^{-ax}\sin kx dx$$

Taking derivative w.r.t a gives

$$\begin{aligned}
\frac{dI}{da} &= \frac{d}{da}\left(\int_0^\infty e^{-ax}\sin kx dx\right)\\
&= \int_0^\infty \frac{d}{da}(e^{-ax}\sin kx)dx\\
&= \int_0^\infty -xe^{-ax}\sin kx dx\\
&= -\int_0^\infty xe^{-ax}\sin kx dx
\end{aligned}$$

Which is the integral the problem is asking to find. Therefore, since I is also given as $\frac{k}{a^2+k^2}$ then

$$\begin{aligned}
-\int_0^\infty xe^{-ax}\sin kx dx &= \frac{d}{da}\left(\frac{k}{a^2+k^2}\right)\\
&= k\frac{d}{da}\left(\frac{1}{a^2+k^2}\right)\\
&= k(-1)\left(a^2+k^2\right)^{-2}(2a)\\
&= -\frac{2ak}{\left(a^2+k^2\right)^2}
\end{aligned}$$

Therefore

$$\int_0^\infty xe^{-ax}\sin kx dx = \frac{2ak}{\left(a^2+k^2\right)^2}$$

4.2.4.2 part (2)

$$I = \int_0^\infty e^{-ax}\sin kx dx$$

Taking derivative w.r.t. k gives

$$\begin{aligned}\frac{dI}{dk} &= \frac{d}{dk}\left(\int_0^\infty e^{-ax}\sin kx dx\right)\\ &= \int_0^\infty \frac{d}{dk}(e^{-ax}\sin kx)dx\\ &= \int_0^\infty e^{-ax}\frac{d}{dk}(\sin kx)dx\\ &= \int_0^\infty xe^{-ax}\cos kx dx\end{aligned}$$

Which is the integral the problem is asking to find. Therefore, since I is also given as $\frac{k}{a^2+k^2}$ then

$$\begin{aligned}\int_0^\infty xe^{-ax}\cos kx dx &= \frac{d}{dk}\left(\frac{k}{a^2+k^2}\right)\\ &= \frac{\left(a^2+k^2\right)-k(2k)}{\left(a^2+k^2\right)^2}\\ &= \frac{a^2+k^2-2k^2}{\left(a^2+k^2\right)^2}\\ &= \frac{a^2-k^2}{\left(a^2+k^2\right)^2}\end{aligned}$$

Hence

$$\int_0^\infty xe^{-ax}\cos kx dx = \frac{a^2-k^2}{\left(a^2+k^2\right)^2}$$

4.2.5 Problem 3

3. The probability to find a particle at position between x and $x+dx$ is

$$P(x)dx = A\exp(-\alpha x^2+\beta x^3)dx,$$

where A, α, and β are positive parameters. By the definition of probability,

$$\int_{-\infty}^{\infty} P(x)dx = 1.$$

Treat β as a small parameter, i.e., for any given x, you can view $P(x)$ as a function of β and expand it around $\beta = 0$.

(a) Find A to the first order of β. (15 points)

(b) Find the average position

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

to the first order of β. (25 points)

Figure 4.5: Problem statment

Solution

4.2.5.1 Part (a)

$$P(x,\beta) = Ae^{-\alpha x^2+\beta x^3}$$

Expanding around $\beta = 0$ by fixing x, gives

$$P\left(x,\beta\right) = P(x,0) + \beta \left.\frac{\partial P}{\partial \beta}\right|_{\beta=0} + \frac{\beta^2}{2!}\left.\frac{\partial^2 P}{\partial \beta^2}\right|_{\beta=0} + \cdots \tag{1}$$

But

$$P(x,0) = Ae^{-\alpha x^2} \tag{2}$$

And

$$\frac{\partial P}{\partial \beta} = Ax^3 e^{-\alpha x^2 + \beta x^3} \tag{3}$$

No need to take more derivatives since the problem is asking for first order of β. Substituting (2,3) into (1) gives

$$\begin{aligned} P\left(x,\beta\right) &= Ae^{-\alpha x^2} + \beta A x^3 e^{-\alpha x^2 + \beta x^3}\Big|_{\beta=0} + \cdots \\ &= Ae^{-\alpha x^2} + \beta A x^3 e^{-\alpha x^2} + \cdots \end{aligned} \tag{4}$$

Using the above in the definition $\int_{-\infty}^{\infty} P(x)dx = 1$ gives

$$\begin{aligned} \int_{-\infty}^{\infty}\left(Ae^{-\alpha x^2} + \beta A x^3 e^{-\alpha x^2}\right)dx &= 1 \\ A\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}dx + \beta \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2}dx\right) &= 1 \end{aligned} \tag{5}$$

But

$$\int_{-\infty}^{\infty} x^3 e^{-\alpha x^2}dx = 0$$

This is because $e^{-\alpha x^2}$ is an even function over $(-\infty, +\infty)$ and x^3 is odd. Eq (5) now simplifies to

$$A\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = 1$$

But $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$ $(\alpha > 0)$ because it is standard Gaussian integral. The above now becomes

$$\begin{aligned} A\sqrt{\frac{\pi}{\alpha}} &= 1 \\ A &= \sqrt{\frac{\alpha}{\pi}} \qquad \alpha > 0 \end{aligned}$$

4.2.5.2 Part b

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

Using Eq. (4) from part (a), the above becomes

$$\begin{aligned} \bar{x} &= \int_{-\infty}^{\infty} x\left(Ae^{-\alpha x^2} + \beta A x^3 e^{-\alpha x^2}\right)dx \\ &= A\int_{-\infty}^{\infty} xe^{-\alpha x^2}dx + A\int_{-\infty}^{\infty} \beta x^4 e^{-\alpha x^2}dx \end{aligned}$$

But $\int_{-\infty}^{\infty} xe^{-\alpha x^2}dx = 0$ since $e^{-\alpha x^2}$ is an even function over $(-\infty, +\infty)$ and x is an odd function. The above simplifies to

$$\bar{x} = A\beta\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2}dx \tag{6}$$

To evaluate the above, starting from the standard Gaussian integral given by

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Taking derivative w.r.t α of both sides of the above results in

$$\begin{aligned}
I'(\alpha) &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} e^{-\alpha x^2} dx = \frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} \\
&= \int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2} dx = \sqrt{\pi}\left(-\frac{1}{2}\right)\alpha^{-\frac{3}{2}} \\
&= \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\alpha^{-\frac{3}{2}}
\end{aligned}$$

Taking one more derivative w.r.t α gives

$$\begin{aligned}
I''(\alpha) &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} x^2 e^{-\alpha x^2} dx = \frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2}\alpha^{-\frac{3}{2}}\right) \\
&= \int_{-\infty}^{\infty} -x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\left(-\frac{3}{2}\alpha^{-\frac{5}{2}}\right) \\
&= \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\left(\frac{3}{2}\alpha^{-\frac{5}{2}}\right)
\end{aligned}$$

Now the integrand is the one we want. This shows that

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}$$

Using the above result in (6) gives

$$\bar{x} = A\beta\left(\frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}\right)$$

But $A = \sqrt{\frac{\alpha}{\pi}}$ from part(a). Hence the above becomes

$$\begin{aligned}
\bar{x} &= \sqrt{\frac{\alpha}{\pi}}\beta\left(\frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}\right) \\
&= \alpha^{\frac{1}{2}}\beta\frac{3}{4\alpha^{\frac{5}{2}}} \\
&= \beta\frac{3}{4\alpha^{\frac{5}{2}-\frac{1}{2}}} \\
&= \frac{3}{4}\frac{\beta}{\alpha^2} \qquad \alpha > 0
\end{aligned}$$

4.2.6 Problem 4

4. A container of volume V encloses a neutrino gas of temperature T. The number of neutrinos with energy between E and $E + dE$ is

$$dN = \left(\frac{4\pi V}{h^3 c^3}\right)\frac{E^2}{\exp[E/(kT)] + 1} dE,$$

where h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(a) Express the total energy density of the neutrino gas in terms of a dimensional factor multiplying a dimensionless integral. Show that the factor has the correct dimension. (10 points).

(b) Follow the discussion of a photon gas and evaluate the dimensionless integral. (20 points).

Figure 4.6: Problem statment

Solution

4.2.6.1 Part a

$$dN = \left(\frac{4\pi V}{h^3c^3}\right)\frac{E^2}{1+e^{\frac{E}{kT}}}dE$$

The total energy is therefore

$$E_{total} = \int EdN$$

Hence the energy density ρ is

$$\begin{aligned}
\rho &= \frac{1}{V}\int EdN \\
&= \frac{1}{V}\int_0^\infty \left(\frac{4\pi V}{h^3c^3}\right)\frac{EE^2}{1+e^{\frac{E}{kT}}}dE \\
&= \left(\frac{1}{V}\right)\left(\frac{4\pi V}{h^3c^3}\right)\int_0^\infty \frac{E^3}{1+e^{\frac{E}{kT}}}dE \\
&= \frac{4\pi}{h^3c^3}\int_0^\infty \frac{E^3}{1+e^{\frac{E}{kT}}}dE \qquad (1)
\end{aligned}$$

k (Boltzmann constant) has units of $\frac{[J]}{[K]}$ where J is joule and K is temperature in Kelvin. Hence units of $\frac{E}{kT}$ is $\frac{[J]}{\frac{[J]}{[K]}[K]}$ which is dimensionless. Let

$$x = \frac{E}{kT}$$

Therefore $\frac{dx}{dE} = \frac{1}{kT}$ When $E = 0, x = 0$ and when $E = \infty, x = \infty$. Substituting this into the integral in (1) gives

$$\begin{aligned}
\int_0^\infty \frac{E^3}{1+e^{\frac{E}{kT}}}dE &= \int_0^\infty \frac{(xkT)^3}{1+e^x}(kTdx) \\
&= (kT)^4\int_0^\infty \frac{x^3}{1+e^x}dx \qquad (2)
\end{aligned}$$

Substituting (2) into (1) gives

$$\rho = \left(\frac{4\pi}{h^3c^3}\right)(kT)^4\int_0^\infty \frac{x^3}{1+e^x}dx \qquad (3)$$

Units of c (speed of light) is $\frac{[L]}{[T]}$ where $[L]$ is length in meters and $[T]$ is time in seconds. Units for Planck constant h is $[J][T]$ (Joule-second). Therefore the factor $\left(\frac{4\pi}{h^3c^3}\right)(kT)^4$ above in (3) in front of the integral has units

$$\begin{aligned}
\left(\frac{4\pi}{h^3c^3}\right)(kT)^4 &= \frac{1}{([J][T])^3\left(\frac{[L]}{[T]}\right)^3}\left(\frac{[J]}{[K]}[K]\right)^4 \\
&= \frac{1}{[J]^3[L]^3}([J])^4 \\
&= \frac{[J]}{[L]^3}
\end{aligned}$$

Which has the correct units of energy density. Let this factor be called $\Phi = \left(\frac{4\pi}{h^3c^3}\right)(kT)^4$. Then (3) can be written as

$$\rho = \Phi\int_0^\infty \frac{x^3}{1+e^x}dx$$

4.2.6.2 Part b

The dimensionless integral found in part (a) is

$$I = \int_0^\infty \frac{x^3}{e^x + 1} dx \tag{1}$$

But

$$\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - 2\frac{1}{e^{2x} - 1}$$

We did the above, to make the denominator has the form $e^x - 1$, which is easier to work with following the lecture notes than working with $e^x + 1$. Eq (1) now becomes

$$I = \int_0^\infty \frac{x^3}{e^x - 1} dx - 2\int_0^\infty \frac{x^3}{e^{2x} - 1} dx \tag{2}$$

The first integral has the standard form $\int_0^\infty \frac{x^n}{e^x - 1} dx$. Hence

$$\int_0^\infty \frac{x^3}{e^x - 1} = (3!)\xi(4)$$

(Derivations of the above is given at the end of this problem). Now we evaluate on the second integral in (2). Let $y = 2x$, then $\frac{dy}{dx} = 2$. The limits do not change. The integral becomes

$$\int_0^\infty \frac{\frac{y^3}{8}}{e^y - 1} \frac{dy}{2} = \frac{1}{16} \int_0^\infty \frac{y^3}{e^y - 1} dy$$

We see that $\int_0^\infty \frac{y^3}{e^y - 1} dy$ now has the same form as the first integral. Hence $\int_0^\infty \frac{y^3}{e^y - 1} dy = (3!)\xi(4)$. Putting these two results back into (2) gives the final result

$$\begin{aligned}
I &= (3!)\xi(4) - 2\left(\frac{1}{16}(3!)\xi(4)\right) \\
&= (3!)\xi(4)\left(1 - 2\left(\frac{1}{16}\right)\right) \\
&= (6)\xi(4)\left(1 - \frac{1}{8}\right) \\
&= (6)\xi(4)\frac{7}{8} \\
&= \frac{21}{4}\xi(4)
\end{aligned}$$

But from class handout, $\xi(4) = \frac{\pi^4}{90}$. Hence

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x + 1} dx &= \frac{21}{4}\left(\frac{\pi^4}{90}\right) \\
&= \frac{7}{4}\left(\frac{\pi^4}{30}\right) \\
&= \frac{7}{120}\pi^4 \\
&\approx 5.6822
\end{aligned}$$

Using this in the result obtained in part (a) gives the energy density as

$$\begin{aligned}
\rho &= \Phi \int_0^\infty \frac{x^3}{1 + e^x} dx \\
&= \left(\frac{7\pi^4}{120}\right)\left(\frac{4\pi}{h^3 c^3}\right)(kT)^4
\end{aligned}$$

Derivation of the integral

In the above, we used the result that $\int_0^\infty \frac{x^n}{e^x-1}dx = (n!)\xi(n+1)$. For $n=3$ this becomes $(3!)\xi(4)$.

To show how this came above, we start by multiplying the numerator and denominator of the integrand by e^{-x}. This gives

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx \tag{3}$$

Let $y=e^{-x}$ then

$$\begin{aligned}
\frac{e^{-x}}{1-e^{-x}} &= \frac{y}{1-y}\\
&= y\left(1+y+y^2+y^3+\cdots\right)\\
&= y+y^2+y^3+\cdots\\
&= \sum_{k=1}^{\infty} y^k\\
&= \sum_{k=1}^{\infty} e^{-kx}
\end{aligned}$$

Using the above sum in Eq (3) gives

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx &= \int_0^\infty x^n \sum_{k=1}^{\infty} e^{-kx}dx\\
&= \sum_{k=1}^{\infty}\int_0^\infty x^n e^{-kx}dx
\end{aligned}$$

Let $z=kx$. Then $\frac{dz}{dx}=k$. When $x=0, z=0$ and when $x=\infty, z=\infty$. The above becomes

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx &= \sum_{k=1}^{\infty}\int_0^\infty \left(\frac{z}{k}\right)^n e^{-z}\left(\frac{dz}{k}\right)\\
&= \sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\int_0^\infty z^n e^{-z}dz\\
&= \sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\left(\int_0^\infty x^n e^{-x}dx\right)
\end{aligned}$$

But $\int_0^\infty x^n e^{-x}dx = n!$, which can be shown by integration by parts repeatedly n times. The above integral now becomes

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = (n!)\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}$$

The sum $\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}$ is called the Zeta function $\zeta(n+1)$. When $n=3$ the above result becomes

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x-1}dx &= (3!)\sum_{k=1}^{\infty}\frac{1}{k^4}\\
&= (3!)\zeta(4)
\end{aligned}$$

Which is the result used earlier.

4.2.7 key solution for HW 2

Physics 3041 (Spring 2021) Solutions to Homework Set 2

1. Problem 2.2.3. (10 points)

Let $y = \sqrt{x}$. So $\int_0^1 e^{\sqrt{x}}dx = \int_0^1 e^y dy^2 = 2\int_0^1 ye^y dy = 2\int_0^1 yde^y = 2(ye^y|_0^1 - \int_0^1 e^y dy) = 2(e - e^y|_0^1) = 2[e - (e-1)] = 2$.

Let $y = x^4$. So $\int_0^\infty e^{-x^4}dx = xe^{-x^4}|_0^\infty - \int_0^\infty xde^{-x^4} = 4\int_0^\infty x^4 e^{-x^4}dx = 4\int_0^\infty ye^{-y}dy^{1/4} = \int_0^\infty y^{1/4}e^{-y}dy = \Gamma(5/4)$.

2. (a) Problem 2.2.10. (10 points)

Consider

$$I(a) = \int_0^1 \frac{t^a - 1}{\ln t}dt \Rightarrow \frac{dI}{da} = \int_0^1 \frac{de^{\ln t^a}}{da}\frac{dt}{\ln t} = \int_0^1 \frac{de^{a\ln t}}{da}\frac{dt}{\ln t} = \int_0^1 e^{a\ln t}dt = \int_0^1 t^a dt = \frac{1}{1+a}.$$

$I(1) - I(0) = \int_0^1 (dI/da)da = \int_0^1 da/(1+a) = \ln(1+a)|_0^1 = \ln 2$, where $I(1)$ is the original integral to be evaluated. Because $I(0) = 0$, we obtain $I(1) = \ln 2$.

(b) Problem 2.2.11. (10 points)

Let $I(a) = J(k) = \int_0^\infty e^{-ax}\sin kx dx = k/(a^2+k^2)$. We obtain $dI/da = -\int_0^\infty xe^{-ax}\sin kx dx = -2ak/(a^2+k^2)^2$, or $\int_0^\infty xe^{-ax}\sin kx dx = 2ak/(a^2+k^2)^2$. Likewise, $dJ/dk = \int_0^\infty xe^{-ax}\cos kx dx = 1/(a^2+k^2) - 2k^2/(a^2+k^2)^2 = (a^2-k^2)/(a^2+k^2)^2$.

3. The probability to find a particle at position between x and $x + dx$ is

$$P(x)dx = A\exp(-\alpha x^2 + \beta x^3)dx,$$

where A, α, and β are positive parameters. By the definition of probability,

$$\int_{-\infty}^{\infty} P(x)dx = 1.$$

Treat β as a small parameter, i.e., for any given x, you can view $P(x)$ as a function of β and expand it around $\beta = 0$.

(a) Find A to the first order of β. (15 points)

$$\int_{-\infty}^{\infty} P(x)dx = A\int_{-\infty}^{\infty} \exp(-\alpha x^2 + \beta x^3)dx = 1$$

$$\begin{aligned}
\Rightarrow A &= \frac{1}{\int_{-\infty}^{\infty}\exp(-\alpha x^2 + \beta x^3)dx} \approx \frac{1}{\int_{-\infty}^{\infty}(1+\beta x^3)\exp(-\alpha x^2)dx} \\
&= \frac{1}{\int_{-\infty}^{\infty}\exp(-\alpha x^2)dx} = \frac{\sqrt{\alpha}}{\int_{-\infty}^{\infty}\exp(-y^2)dy} = \sqrt{\frac{\alpha}{\pi}},
\end{aligned}$$

where $y = x\sqrt{\alpha}$ and we have used symmetry to obtain

$$\int_{-\infty}^{\infty} x^3\exp\left(-\alpha x^2\right)dx = 0.$$

(b) Find the average position

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

to the first order of β. (25 points)

$$\begin{aligned}
\bar{x} &= \int_{-\infty}^{\infty} xP(x)dx = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x\exp\left(-\alpha x^2 + \beta x^3\right)dx \\
&\approx \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x(1+\beta x^3)\exp(-\alpha x^2)dx \\
&= \beta\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x^4\exp(-\alpha x^2)dx = \frac{\beta}{\alpha^2\sqrt{\pi}}\int_{-\infty}^{\infty} y^4\exp(-y^2)dy,
\end{aligned}$$

where we have used symmetry to obtain

$$\int_{-\infty}^{\infty} x\exp(-\alpha x^2)dx = 0.$$

Noting that

$$I(a) = \int_{-\infty}^{\infty}\exp(-ay^2)dy = \sqrt{\frac{\pi}{a}} \Rightarrow \frac{d^2 I}{da^2} = \int_{-\infty}^{\infty} y^4\exp(-ay^2)dy = \frac{3}{4}\frac{\sqrt{\pi}}{a^{5/2}},$$

we obtain

$$\bar{x} = \frac{\beta}{\alpha^2\sqrt{\pi}}\left(\frac{3}{4}\sqrt{\pi}\right) = \frac{3\beta}{4\alpha^2}.$$

4. A container of volume V encloses a neutrino gas of temperature T. The number of neutrinos with energy between E and $E + dE$ is

$$dN = \left(\frac{4\pi V}{h^3c^3}\right)\frac{E^2}{\exp[E/(kT)]+1}dE,$$

where h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(a) Express the total energy density of the neutrino gas in terms of a dimensional factor multiplying a dimensionless integral. Show that the factor has the correct dimension. (10 points)

The total energy density is

$$\varepsilon = \frac{\int E dN}{V} = \frac{4\pi}{h^3c^3}\int_0^\infty \frac{E^3 dE}{\exp[E/(kT)]+1} = \frac{4\pi(kT)^4}{(hc)^3}\int_0^\infty \frac{x^3 dx}{\exp(x)+1},$$

where we have made the substitution of variable $x = E/(kT)$. The dimensional factor is in units of $\mathrm{J}^4/(\mathrm{J}\cdot\mathrm{s}\cdot\mathrm{m/s})^3 = \mathrm{J/m}^3$, as should be for the energy density.

(b) Follow the discussion of a photon gas and evaluate the dimensionless integral. (20 points)

Using $1 + y + y^2 + y^3 + \cdots = (1-y)^{-1}$ for $|y| < 1$, we obtain

$$\frac{1}{\exp(x)+1} = \frac{\exp(-x)}{1+\exp(-x)} = \exp(-x)\sum_{n'=0}^{\infty}(-1)^{n'}\exp(-n'x) = \sum_{n=1}^{\infty}(-1)^{n-1}\exp(-nx),$$

where we have set $y = -\exp(-x)$ and $n = n' + 1$.

Using $\int_0^\infty x^k \exp(-\alpha x)dx = k!/\alpha^{k+1}$, we obtain

$$\begin{aligned}
\int_0^\infty \frac{x^3 dx}{\exp(x)+1} &= \int_0^\infty x^3\sum_{n=1}^{\infty}(-1)^{n-1}\exp(-nx)dx = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^\infty x^3\exp(-nx)dx \\
&= 3!\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = 6\left(\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots\right) \\
&= 6\left[\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)-2\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\cdots\right)\right] \\
&= 6\left[\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)-\frac{2}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)\right] \\
&= 6\left(1-\frac{1}{8}\right)\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right) = \frac{21}{4}\zeta(4) = \frac{21}{4}\left(\frac{\pi^4}{90}\right) = \frac{7\pi^4}{120}.
\end{aligned}$$

4.3 HW 3

Local contents

4.3.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 3 **(Due 2/10)**

1. (a) Problem 5.2.3. (5 points)

(b) Problem 5.2.4.(iv). (5 points)

(c) Problem 5.2.5. (10 points)

(d) Problem 5.3.2. (20 points)

2. (a) Problem 5.3.5. (10 points)

(b) Problem 5.3.6. (10 points)

(c) Find $\int_0^\infty xe^{-ax}\cos kx dx$ using Euler's formula. (10 points)

3. Given the intensity pattern for the N-slit interference with separation d between adjacent slits, show that the pattern becomes that for the single-slit diffraction with slit width a when d goes to zero but with a fixed value of $Nd = a$. (10 points)

4. (1) Find the roots z_n ($n = 1, 2, \cdots, N$) of the complex equation $z^N = 1$. (5 points)

(2) Find $S_N = \sum_{n=1}^{N} z_n$ and give a geometric interpretation of the result. (10 points)

(3) Note that $1 - z^N = (1 - z)(1 + z + z^2 + \cdots + z^{N-1})$. Relate this result and the roots z_n to the conditions for destructive interference among N slits. (5 points)

4.3.2 Problem 1(a) (problem 5.2.3)

Solve for x and y given

$$\frac{2+3i}{6+7i}+\frac{2}{x+iy}=2+9i$$

solution

Let $z=x+iy$ be the complex number to solve for. The above becomes

$$\begin{aligned}
\frac{2}{z} &= 2+9i-\frac{2+3i}{6+7i}\\
&= 2+9i-\frac{(2+3i)(6-7i)}{(6+7i)(6-7i)}\\
&= 2+9i-\frac{12-14i+18i+21}{36+49}\\
&= 2+9i-\frac{33+4i}{85}\\
&= \frac{85(2+9i)-33-4i}{85}\\
&= \frac{170+765i-33-4i}{85}\\
&= \frac{137+761i}{85}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{2}{z} &= \frac{137+761i}{85}\\
z &= \frac{170}{137+761i}\\
&= \frac{170(137-761i)}{(137+761i)(137-761i)}\\
&= \frac{23290-129370i}{597\,890}\\
&= \frac{23290}{597890}-\frac{129370}{597890}i\\
&= \frac{137}{3517}-\frac{761}{3517}i
\end{aligned}$$

But $z=x+iy$. Hence

$$x+iy=\frac{137}{3517}-\frac{761}{3517}i$$

Comparing real and imaginary parts shows that

$$\begin{aligned}
x &= \frac{137}{3517}\\
y &= -\frac{761}{3517}
\end{aligned}$$

4.3.3 Problem 1(b) (problem 5.2.4(iv))

Find the real part, imaginary part, modulus, complex conjugate and inverse of the following (iv) $\frac{1+\sqrt{2}i}{1-\sqrt{3}i}$

solution

$$\begin{aligned}
z &= \frac{1+\sqrt{2}\,i}{1-\sqrt{3}\,i}\\
&= \frac{\left(1+\sqrt{2}\,i\right)\left(1+\sqrt{3}\,i\right)}{\left(1-\sqrt{3}\,i\right)\left(1+\sqrt{3}\,i\right)}\\
&= \frac{1+\sqrt{3}\,i+\sqrt{2}\,i-\sqrt{2}\sqrt{3}}{4}\\
&= \frac{1-\sqrt{6}}{4}+i\frac{\sqrt{3}+\sqrt{2}}{4}
\end{aligned}$$

Hence the real part is $\frac{1-\sqrt{6}}{4}$ and the imaginary part is $\frac{\sqrt{3}+\sqrt{2}}{4}$. Therefore we can now write

$$\begin{aligned}
z &= x+iy\\
&= \left(\frac{1-\sqrt{6}}{4}\right)+i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)
\end{aligned}$$

The modulus is

$$\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{\left(\frac{1-\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)^2}\\
&= \sqrt{\frac{7}{16}-\frac{1}{8}\sqrt{6}+\frac{1}{8}\sqrt{6}+\frac{5}{16}}\\
&= \sqrt{\frac{3}{4}}
\end{aligned}$$

The complex conjugate of z is z^*. Hence

$$\begin{aligned}
z^* &= x-iy\\
&= \left(\frac{1-\sqrt{6}}{4}\right)-i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)
\end{aligned}$$

The inverse is

$$\begin{aligned}
\frac{1}{z} &= \frac{z^*}{zz^*}\\
&= \frac{z^*}{|z|^2}\\
&= \frac{\left(\frac{1-\sqrt{6}}{4}\right)-i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)}{\frac{3}{4}}\\
&= \frac{1-\sqrt{6}}{3}-i\frac{\sqrt{3}+\sqrt{2}}{3}
\end{aligned}$$

4.3.4 Problem 1(c) (problem 5.2.5)

Show that a polynomial with real coefficients has only real roots or complex roots that come in complex conjugate pairs.

solution

Let

$$p(z) = a_0+a_1z+a_2z^2+\cdots+a_nz^n$$

be polynomial in z where a_i are all real. We just need to show now that if λ is a root, then its complex conjugate λ^* must also a root. If the root happened to be real, then its complex conjugate is itself. Hence nothing to do in this case. We only need to worry about the case when the root is complex and show that its complex conjugate must also be root.

Assuming λ is a root, then by definition of a root we have

$$\begin{aligned} p(\lambda) &= 0 \\ &= a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \\ &= \sum_{k=0}^{n} a_k\lambda^k \end{aligned} \tag{1}$$

Therefore, replacing λ by λ^* on both sides of (1) gives

$$p(\lambda^*) = \sum_{k=0}^{n} a_k(\lambda^*)^k$$

But $(\lambda^*)^k = \left(\lambda^k\right)^*$ from complex numbers properties (equation 5.2.20 in book). The above becomes

$$p(\lambda^*) = \sum_{k=0}^{n} a_k\left(\lambda^k\right)^*$$

Since a_k are real coefficients, then $a_k^* = a_k$ and the above can be written as

$$p(\lambda^*) = \sum_{k=0}^{n} \left(a_k\lambda^k\right)^*$$

Using property that $A^*B^* = (AB)^*$ where $A = a_k, B = \lambda^k$ in the above. Now we can move the complex conjugate outside the sum, using property that $A^* + B^* = (A + B)^*$. Hence the above becomes

$$p(\lambda^*) = \left(\sum_{k=0}^{n} a_k\lambda^k\right)^*$$

But from (1), we know that $\sum_{k=0}^{n} a_k\lambda^k = 0$, this is because λ is assumed to be a root. Therefore the above gives

$$\begin{aligned} p(\lambda^*) &= 0^* \\ &= 0 \end{aligned}$$

The above shows that λ^* is also a root if λ is a root. Therefore, the root can be either real, or complex. If the root is complex, its complex conjugate is also a root. A real root is just special case of complex root. QED.

4.3.5 Problem 1(d) (problem 5.3.2)

For the following pairs of numbers, give their polar form, their complex conjugate, moduli, product, the quotient $\frac{z_1}{z_2}$, and the complex conjugate of the quotient

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad z_2 = \sqrt{3} - i$$

$$z_1 = \frac{3+4i}{3-4i} \qquad z_2 = \left(\frac{1+2i}{1-3i}\right)^2$$

solution

4.3.5.1 First pair

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad z_2 = \sqrt{3} - i$$

The polar form of z is $re^{i\theta}$ where $r = |z|$ and $\theta = \arctan\left(\frac{y}{x}\right)$ when $z = x + iy$. The first step is to write $z = x + iy$

For z_1

$$\begin{aligned} z_1 &= \frac{1+i}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \end{aligned}$$

Hence $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$. Therefore $|z_1| = \sqrt{x^2+y^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$. And $\theta = \arctan(1) = 45^0$. Therefore in polar

$$\begin{aligned} z_1 &= re^{i\theta} \\ &= e^{i\left(45^0\right)} \\ &= e^{i\frac{\pi}{4}} \end{aligned}$$

For z_2

$$z_2 = \sqrt{3} - i$$

Hence $x = \sqrt{3}, y = -1$. Therefore $|z_1| = \sqrt{x^2+y^2} = \sqrt{3+1} = 2$. And $\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -30^0$. Therefore in polar

$$\begin{aligned} z_2 &= re^{i\theta} \\ &= 2e^{i\left(-30^0\right)} \\ &= 2e^{-i\frac{\pi}{6}} \end{aligned}$$

The complex conjugate is

$$\begin{aligned} z_1^* &= re^{-i\theta} \\ &= e^{-i\frac{\pi}{4}} \end{aligned}$$

And

$$\begin{aligned} z_2^* &= re^{-i\theta} \\ &= 2e^{i\frac{\pi}{6}} \end{aligned}$$

And moduli is

$$\begin{aligned} |z_1| &= r \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} |z_2| &= r \\ &= 2 \end{aligned}$$

And product

$$\begin{aligned} z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

But $r_1 = 1, r_2 = 2, \theta_1 = 45^0, \theta_2 = -30^0$. The above becomes

$$\begin{aligned} z_1 z_2 &= 2e^{i\left(45^0 - 30^0\right)} \\ &= 2e^{i\left(15^0\right)} \\ &= 2e^{i\frac{\pi}{12}} \end{aligned}$$

And the <u>quotient $\frac{z_1}{z_2}$</u> is

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\ &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \end{aligned}$$

But $r_1 = 1, r_2 = 2, \theta_1 = 45^0, \theta_2 = -30^0$. The above becomes

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{1}{2} e^{i\left(45^0 + 30^0\right)} \\ &= \frac{1}{2} e^{i\left(75^0\right)} \\ &= \frac{1}{2} e^{i\frac{5\pi}{12}} \end{aligned}$$

And the <u>complex conjugate of the quotient</u> is

$$\begin{aligned} \left(\frac{z_1}{z_2}\right)^* &= \left(\frac{1}{2} e^{i\frac{5\pi}{12}}\right)^* \\ &= \frac{1}{2} e^{-i\frac{5\pi}{12}} \end{aligned}$$

4.3.5.2 Second pair

$$z_1 = \frac{3+4i}{3-4i} \qquad z_2 = \left(\frac{1+2i}{1-3i}\right)^2$$

The polar form of z is $re^{i\theta}$ where $r = |z|$ and $\theta = \arctan\left(\frac{y}{x}\right)$ where $z = x + iy$. Hence

<u>For z_1</u>

$$\begin{aligned} z_1 &= \frac{3+4i}{3-4i} \\ &= \frac{\sqrt{3^2+4^2}\, e^{i\arctan\left(\frac{4}{3}\right)}}{\sqrt{3^2+4^2}\, e^{i\arctan\left(-\frac{4}{3}\right)}} \\ &= \frac{e^{i\arctan\left(\frac{4}{3}\right)}}{e^{-i\arctan\left(\frac{4}{3}\right)}} \\ &= e^{i\arctan\left(\frac{4}{3}\right)+\arctan\left(\frac{4}{3}\right)} \\ &= e^{i\left(2\arctan\left(\frac{4}{3}\right)\right)} \\ &= e^{i\left(106.26^0\right)} \end{aligned}$$

For z_2

$$\begin{aligned} z_2 &= \left(\frac{1+2i}{1-3i}\right)^2 \\ &= \left(\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}\right)^2 \\ &= \left(\frac{-5+5i}{10}\right)^2 \\ &= \frac{25-25-50i}{100} \\ &= \frac{-1}{2}i \end{aligned}$$

Hence $x=0, y=-\frac{1}{2}$. Therefore $|z_1| = \sqrt{x^2+y^2} = \sqrt{0+\frac{1}{4}} = \frac{1}{2}$. And $\theta = \arctan(-\infty) = -90^0$. Therefore in polar

$$\begin{aligned} z_2 &= re^{i\theta} \\ &= \frac{1}{2}e^{i\left(-90^0\right)} \\ &= \frac{1}{2}e^{-i\frac{\pi}{2}} \end{aligned}$$

The complex conjugate is

$$\begin{aligned} z_1^* &= r_1 e^{-i\theta_1} \\ &= e^{-i\left(2\arctan\left(\frac{4}{3}\right)\right)} \\ &= e^{i\left(-106.26^0\right)} \end{aligned}$$

And

$$\begin{aligned} z_2^* &= r_2 e^{-i\theta_2} \\ &= \frac{1}{2}e^{i\frac{\pi}{2}} \end{aligned}$$

And moduli is

$$\begin{aligned} |z_1| &= r_1 \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} |z_2| &= r_2 \\ &= \frac{1}{2} \end{aligned}$$

And product

$$\begin{aligned} z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

But $r_1 = 1, r_2 = \frac{1}{2}, \theta_1 = 106.26^0, \theta_2 = -90^0$. The above becomes

$$\begin{aligned} z_1 z_2 &= \frac{1}{2}e^{i\left(106.26^0 - 90^0\right)} \\ &= \frac{1}{2}e^{i\left(16.26^0\right)} \end{aligned}$$

And the quotient $\frac{z_1}{z_2}$ is

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\ &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\end{aligned}$$

But $r_1 = 1, r_2 = \frac{1}{2}, \theta_1 = 106.26^0, \theta_2 = -90^0$. The above becomes

$$\begin{aligned}\frac{z_1}{z_2} &= 2e^{i\left(106.26^0 + 90^0\right)} \\ &= 2e^{i\left(196.26^0\right)}\end{aligned}$$

And the complex conjugate of the quotient is

$$\begin{aligned}\left(\frac{z_1}{z_2}\right)^* &= \left(2e^{i\left(196.26^0\right)}\right)^* \\ &= 2e^{-i\left(196.26^0\right)}\end{aligned}$$

4.3.6 Problem 2(a) (problem 5.3.5)

Consider series

$$e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta}$$

Sum this geometric series, take the real and imaginary parts of both sides and show that

$$\cos\theta + \cos(3\theta) + \cdots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2\sin\theta}$$

And that a similar sum with sines adds up to $\frac{\sin^2(n\theta)}{\sin\theta}$

solution

Let

$$S = e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta} \tag{1}$$

Then

$$\begin{aligned}e^{2i\theta} S &= e^{2i\theta}\left(e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta}\right) \\ &= e^{i3\theta} + e^{5i\theta} + \cdots + e^{(2n-1)i\theta + 2i\theta} \\ &= e^{i3\theta} + e^{5i\theta} + \cdots + e^{(2n+1)i\theta} \qquad (2)\end{aligned}$$

Hence (2-1) gives

$$\begin{aligned}e^{2i\theta} S - S &= e^{(2n+1)i\theta} - e^{i\theta} \\ S\left(e^{2i\theta} - 1\right) &= e^{(2n+1)i\theta} - e^{i\theta} \\ S &= \frac{e^{(2n+1)i\theta} - e^{i\theta}}{e^{2i\theta} - 1}\end{aligned}$$

Hence

$$\begin{aligned}
S &= \frac{e^{i\theta}\left(e^{i2n\theta}-1\right)}{e^{2i\theta}-1}\\
&= e^{i\theta}\frac{\left(e^{in\theta}\left(e^{in\theta}-e^{-in\theta}\right)\right)}{e^{i\theta}\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= \frac{e^{in\theta}\left(e^{in\theta}-e^{-in\theta}\right)}{\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= e^{in\theta}\frac{\left(e^{in\theta}-e^{-in\theta}\right)}{\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= e^{in\theta}\frac{\sin n\theta}{\sin\theta}\\
&= \cos(n\theta+i\sin n\theta)\frac{\sin n\theta}{\sin\theta}\\
&= \frac{\cos(n\theta)\sin(n\theta)}{\sin\theta}+i\frac{\sin^2(n\theta)}{\sin\theta}
\end{aligned}$$

But $\cos(n\theta)\sin(n\theta)=\frac{1}{2}\sin(2n\theta)$. Therefore the above becomes

$$S=\frac{\sin(2n\theta)}{2\sin\theta}+i\frac{\sin^2(n\theta)}{\sin\theta}$$

Hence

$$\operatorname{Re}(S)=\frac{\sin(2n\theta)}{2\sin\theta} \tag{3}$$

$$\operatorname{Im}(S)=\frac{\sin^2(n\theta)}{\sin\theta} \tag{4}$$

Now we look at the LHS. Since $S=e^{i\theta}+e^{3i\theta}+\cdots+e^{(2n-1)i\theta}$, then

$$\begin{aligned}
S &= (\cos\theta+i\sin\theta)+(\cos 3\theta+i\sin 3\theta)+\cdots+(\cos(2n-1)\theta+i\sin(2n-1)\theta)\\
&= (\cos\theta+\cos 3\theta+\cdots+\cos(2n-1)\theta)+i(\sin\theta+\sin 3\theta+\cdots+\sin(2n-1)\theta)
\end{aligned} \tag{5}$$

Comparing (5) and (3,4) shows that

$$\begin{aligned}
\cos\theta+\cos 3\theta+\cdots+\cos(2n-1)\theta &= \operatorname{Re}(S)\\
&= \frac{\sin(2n\theta)}{2\sin\theta}
\end{aligned}$$

And

$$\begin{aligned}
\sin\theta+\sin 3\theta+\cdots+\sin(2n-1)\theta &= \operatorname{Im}(S)\\
&= \frac{\sin^2(n\theta)}{\sin\theta}
\end{aligned}$$

Which is the result we are asked to show.

4.3.7 Problem 2(b) (problem 5.3.6)

(1) Consider De Moivre's theorem, which states that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$. This follows from taking the *nth* power of both sides of Euler's theorem. Find the formula for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(2) Given $e^{iA}e^{iB}=e^{i(A+B)}$ deduce $\cos(A+B)$ and $\sin(A+B)$

solution

4.3.7.1 Part 1

Let $n = 4$, therefore, using De Moivre's theorem gives

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \tag{1}$$

We now expand the LHS of the above directly as follows

$$(\cos\theta + i\sin\theta)^4 = (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^2 \tag{2}$$

But

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

Substituting the above into (2) gives

$$\begin{aligned}(\cos\theta + i\sin\theta)^4 &= \left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\ &= \cos^2\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\ &- \sin^2\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\ &+ 2i\cos\theta\sin\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\end{aligned}$$

Expanding the RHS above more, then the above becomes

$$\begin{aligned}(\cos\theta + i\sin\theta)^4 &= \left(\cos^4\theta - \cos^2\theta\sin^2\theta + 2i\cos^3\theta\sin\theta\right)\\ &- \left(\sin^2\theta\cos^2\theta - \sin^4\theta + 2i\cos\theta\sin^3\theta\right)\\ &+ \left(2i\cos^3\theta\sin\theta - 2i\cos\theta\sin^3\theta - 4\cos^2\theta\sin^2\theta\right)\end{aligned}$$

Simplifying gives

$$\begin{aligned}(\cos\theta + i\sin\theta)^4 &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + 4i\cos^3\theta\sin\theta + \sin^4\theta - 4i\cos\theta\sin^3\theta\\ &= \left(\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)\end{aligned} \tag{3}$$

Comparing the real and imaginary parts of (3) with the real and imaginary parts of (1) shows that

$$\begin{aligned}\cos 4\theta &= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta\\ \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\end{aligned}$$

4.3.7.2 Part 2

Given

$$e^{iA}e^{iB} = e^{i(A+B)}$$

Applying Euler's formula $e^{ix} = \cos x + i\sin x$, on both sides of the above results in

$$\begin{aligned}(\cos A + i\sin A)(\cos B + i\sin B) &= \cos(A+B) + i\sin(A+B)\\ \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin B\sin A &= \cos(A+B) + i\sin(A+B)\\ (\cos A\cos B - \sin B\sin A) + i(\cos A\sin B + \sin A\cos B) &= \cos(A+B) + i\sin(A+B)\end{aligned}$$

Comparing the real parts and the imaginary parts in the above shows that

$$\cos A\cos B - \sin B\sin A = \cos(A+B)$$

And

$$\cos A\sin B + \sin A\cos B = \sin(A+B)$$

4.3.8 Problem 2(c)

Find $\int_0^\infty xe^{-ax}\cos(kx)dx$ using Euler's formula.

solution

Let

$$I=\int_0^\infty xe^{-ax}\cos(kx)dx$$

Then, we replace $\cos(kx)$ by e^{ikx}, evaluate the integral, and then take the real part of the result. Therefore

$$\begin{aligned}I&=\operatorname{Re}\left(\int_0^\infty xe^{-ax}e^{ikx}dx\right)\\&=\operatorname{Re}\left(\int_0^\infty xe^{x(-a+ik)}dx\right)\end{aligned}$$

Integration by parts. Let $u=x, du=dx$ and $dv=e^{x(-a+ik)}, v=\frac{e^{x(-a+ik)}}{-a+ik}$. The above now becomes

$$\begin{aligned}I&=\operatorname{Re}\left(uv|_0^\infty-\int_0^\infty vdu\right)\\&=\operatorname{Re}\left(\frac{1}{-a+ik}xe^{x(-a+ik)}\Big|_0^\infty-\int_0^\infty\frac{e^{x(-a+ik)}}{-a+ik}dx\right)\end{aligned}\tag{1}$$

But

$$xe^{x(-a+ik)}\Big|_0^\infty=0$$

With the assumption that $\operatorname{Re}(a)>0$. To see this more clearly, let us write $e^{x(-a+ik)}=e^{-ax}e^{ikx}$. e^{ikx} is bounded since it is a complex exponential. So the contribution comes from e^{-ax}. Hence when $a>0$, and $x\to\infty$ then the exponential will go to zero, and the whole term $xe^{x(-a+ik)}\to 0$, even though $x\to\infty$, since exponential subdues any polynomial order. When $x=0$, it is clear that $xe^{x(-a+ik)}=0$. Therefore (1) now simplifies to

$$\begin{aligned}I&=\operatorname{Re}\left(-\int_0^\infty\frac{e^{x(-a+ik)}}{-a+ik}dx\right)\\&=\operatorname{Re}\left(-\frac{1}{-a+ik}\int_0^\infty e^{x(-a+ik)}dx\right)\\&=\operatorname{Re}\left(-\frac{1}{-a+ik}\frac{e^{x(-a+ik)}}{-a+ik}\Bigg|_0^\infty\right)\\&=\operatorname{Re}\left(-\frac{1}{(-a+ik)^2}e^{x(-a+ik)}\Big|_0^\infty\right)\end{aligned}$$

But $e^{x(-a+ik)}\Big|_0^\infty = 0 - 1 = -1$. The above becomes

$$
\begin{aligned}
I &= \operatorname{Re}\left(\frac{1}{(-a+ik)^2}\right)\\
&= \operatorname{Re}\left(\frac{1}{a^2-k^2-2aik}\right)\\
&= \operatorname{Re}\left(\frac{\left(a^2-k^2+2aik\right)}{\left(a^2-k^2-2aik\right)\left(a^2-k^2+2aik\right)}\right)\\
&= \operatorname{Re}\left(\frac{a^2-k^2+2aik}{\left(a^2-k^2\right)^2+4a^2k^2}\right)\\
&= \operatorname{Re}\left(\frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2}+i\frac{2ak}{\left(a^2-k^2\right)^2+4a^2k^2}\right)\\
&= \frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^\infty xe^{-ax}\cos(kx)dx &= \frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2}\\
&= \frac{a^2-k^2}{a^4+k^4-2a^2k^2+4a^2k^2}\\
&= \frac{a^2-k^2}{a^4+k^4+2a^2k^2}\\
&= \frac{a^2-k^2}{\left(a^2+k^2\right)^2} \qquad a>0
\end{aligned}
$$

4.3.9 Problem 3

Given the intensity pattern for the N-slit interference with separation d between adjacent slits, show that the pattern becomes that for the single-slit diffraction with slit width a when d goes to zero but with a fixed value of $Nd = a$. (10 points)

Solution

Short version: In this version, The result for N slit $\bar{I}_N(\theta)$ will be used as given in lecture notes without deriving it again, and will also use the single slit $\bar{I}_1(\theta)$ from the lecture notes, then show that $\bar{I}_N(\theta)$ becomes $\bar{I}_1(\theta)$ as $d \to 0$ but with $Nd = a$.

Here $\bar{I}_N(\theta)$ is the average intensity for N slits at location on the screen at angle θ and similarly $\bar{I}_1(\theta)$ is the average intensity for one slit at same location on the screen at angle θ. From lecture notes (lecture 3, pages 6,7) we have the expressions for $\bar{I}_N(\theta), \bar{I}_1(\theta)$ given as

$$
\bar{I}_N(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2 \tag{1}
$$

$$
\bar{I}_1(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\frac{\pi a\sin\theta}{\lambda}}\right)^2 \tag{2}
$$

Now we need to show that (1) gives same result as (2) when d goes to zero in the limit, but with a fixed value of $Nd = a$. Replacing $Nd = a$ in the numerator of (1) and taking the limit gives

$$\lim_{d\to 0}\bar{I}_N(\theta)=\bar{I}(0)\left(\lim_{d\to 0}\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)}\right)^2$$
$$=\bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)}\right)^2 \tag{3}$$

But

$$\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)\approx\frac{\pi d\sin(\theta)}{\lambda}+\cdots \tag{4}$$

In the above we used that $\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)\approx\frac{\pi d\sin(\theta)}{\lambda}$. This comes from Taylor series expansion of sin function, for small angle approximation by keeping only the linear term in the Taylor series expansion since $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$.

Substituting (4) back into (3) gives

$$\lim_{d\to 0}\bar{I}_N(\theta)=\bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\frac{\pi d\sin(\theta)}{\lambda}}\right)^2$$

But $Nd=a$. The above simplifies to

$$\lim_{d\to 0}\bar{I}_N(\theta)=\bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\frac{\pi a\sin(\theta)}{\lambda}}\right)^2 \tag{5}$$

Comparing (5) with (2) shows that are the same. Hence

$$\lim_{d\to 0}\bar{I}_N(\theta)=\bar{I}_1(\theta)$$

Which is what we are asked to show.

4.3.9.1 Appendix

Here, the derivation of

$$\bar{I}_N(\theta)=\bar{I}(0)\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2 \tag{1}$$

is given. First, let us consider a slit located at y_n relative to the origin as show in the diagram below

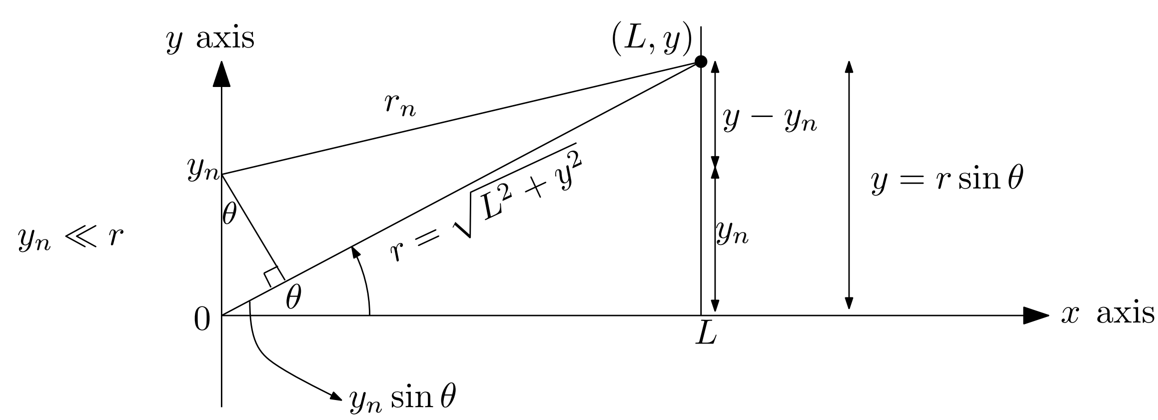


Figure 4.7: Geometry for slit at location y_n

Therefore

$$\begin{aligned}
r_n &= \sqrt{L^2 + \left(y - y_n\right)^2} \\
&= \sqrt{L^2 + \left(y^2 + y_n^2 - 2yy_n\right)} \\
&= \sqrt{L^2 + y^2 + y_n^2 - 2yy_n} \\
&= \sqrt{\left(L^2 + y^2\right)\left(1 + \frac{y_n^2 - 2yy_n}{\left(L^2 + y^2\right)}\right)} \\
&= \sqrt{\left(L^2 + y^2\right)}\sqrt{1 + \frac{y_n^2 - 2yy_n}{\left(L^2 + y^2\right)}} \\
&= \sqrt{\left(L^2 + y^2\right)}\sqrt{1 - \frac{2yy_n}{\left(L^2 + y^2\right)} + \frac{y_n^2}{\left(L^2 + y^2\right)}}
\end{aligned}$$

Since y_n is very small compared to $\left(L^2 + y^2\right)$ and it is also of order 2, then we can ignore the term $\frac{y_n^2}{\left(L^2+y^2\right)}$ above, giving

$$\begin{aligned}
r_n &\approx \sqrt{\left(L^2 + y^2\right)}\sqrt{1 - \frac{2yy_n}{\left(L^2 + y^2\right)}} \\
&= \sqrt{\left(L^2 + y^2\right)}\left(1 - \frac{2yy_n}{\left(L^2 + y^2\right)}\right)^{\frac{1}{2}}
\end{aligned}$$

But since y_n is very small compared to $\left(L^2 + y^2\right)$, then the term $\frac{2yy_n}{\left(L^2+y^2\right)}$ is very small. So we can use $(1 + x)^p = 1 + px$ and ignore higher order terms. Hence the above becomes

$$\begin{aligned}
r_n &\approx \sqrt{\left(L^2 + y^2\right)}\left(1 + \frac{1}{2}\left(\frac{-2yy_n}{\left(L^2 + y^2\right)}\right)\right) \\
&= \sqrt{\left(L^2 + y^2\right)} - \frac{yy_n}{\sqrt{\left(L^2 + y^2\right)}} \\
&= r - y_n\frac{y}{r}
\end{aligned}$$

But $\frac{y}{r} = \sin\theta$, therefore

$$r_n = r - y_n \sin\theta \tag{2}$$

The electric field E_n measured at point $\left(L, y\right)$ due to slit at y_n is

$$E_n = E_0 \sin(kr_n - \omega t)$$

Where k is the wave number $k = \frac{2\pi}{\lambda}$. Therefore for N slits, the total E is

$$\begin{aligned}
E &= \sum_{n=1}^{N} E_n \\
&= \sum_{n=1}^{N} E_0 \sin(kr_n - \omega t) \\
&= E_0\left(\operatorname{Im}\sum_{n=1}^{N} e^{i(kr_n - \omega t)}\right) \\
&= E_0\left(\operatorname{Im}\sum_{n=1}^{N} e^{i(kr_N - \omega t)} e^{ik(r_n - r_N)}\right) \\
&= E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{n=1}^{N} e^{ik(r_n - r_N)}\right)\right)
\end{aligned} \tag{3}$$

But

$$\begin{aligned}
r_n - r_N &= \left(r - y_n \sin\theta\right) - \left(r - y_N \sin\theta\right) \\
&= r - y_n \sin\theta - r + y_N \sin\theta \\
&= \left(y_N - y_n\right)\sin\theta \\
&= (Nd - nd)\sin\theta \\
&= (N-n)d\sin\theta \qquad (4)
\end{aligned}$$

Substituting (4) in (3) gives

$$E = E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{n=1}^{N} e^{ik(N-n)d\sin\theta}\right)\right)$$

Let $m = N - n$. When $n = 1$ then $m = N - 1$. When $n = N$ then $m = 0$. The above now becomes

$$\begin{aligned}
E &= E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{m=N-1}^{0} e^{ikmd\sin\theta}\right)\right) \\
&= E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{m=0}^{N-1} e^{ikmd\sin\theta}\right)\right) \\
&= E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{n=0}^{N-1} e^{iknd\sin\theta}\right)\right)
\end{aligned}$$

Let $\phi = kd\sin\theta$. The above becomes

$$E = E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}\sum_{n=0}^{N-1} e^{in\phi}\right)\right) \qquad (5)$$

But

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{in\phi} &= \frac{1 - e^{iN\phi}}{1 - e^{i\phi}} \\
&= \frac{e^{i\frac{N}{2}\phi}\left(e^{-i\frac{N}{2}\phi} - e^{i\frac{N}{2}\phi}\right)}{e^{i\frac{\phi}{2}}\left(e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}}\right)} \\
&= \frac{e^{i\frac{N}{2}\phi}}{e^{i\frac{\phi}{2}}}\frac{\left(e^{-i\frac{N}{2}\phi} - e^{i\frac{N}{2}\phi}\right)}{\left(e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}}\right)} \\
&= \frac{-e^{i\frac{N}{2}\phi}}{-e^{i\frac{\phi}{2}}}\frac{\left(e^{i\frac{N}{2}\phi} - e^{-i\frac{N}{2}\phi}\right)}{\left(e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}}\right)} \\
&= \frac{e^{i\frac{N}{2}\phi}}{e^{i\frac{\phi}{2}}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)} \\
&= e^{i\frac{(N-1)\phi}{2}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)} \qquad (6)
\end{aligned}$$

Substituting (6) in (5) gives

$$\begin{aligned}
E &= E_0\left(\operatorname{Im}\left(e^{i(kr_N - \omega t)}e^{i\frac{(N-1)\phi}{2}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\right)\right) \\
&= E_0\left(\operatorname{Im}\left(e^{i\left(kr_N - \omega t + \frac{(N-1)\phi}{2}\right)}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\right)\right) \qquad (7)
\end{aligned}$$

Let

$$t_0 = kr_N + \frac{(N-1)\phi}{2}$$

Substituting this in (7) gives

$$\begin{aligned}
E &= E_0\left(\operatorname{Im}\left(e^{i\omega(t_0-t)}\right)\right)\\
&= E_0\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\left(\operatorname{Im} e^{i\omega(t_0-t)}\right)\\
&= E_0\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\sin(\omega(t_0-t)) \qquad (8)
\end{aligned}$$

The electric field intensity is

$$\begin{aligned}
I &= c\varepsilon_0 E^2\\
&= c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)}\sin^2(\omega(t_0-t))
\end{aligned}$$

The time (period) averaged intensity is therefore

$$\begin{aligned}
I_{av} &= \frac{1}{T}\int_0^T I dt\\
&= \frac{1}{T}c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)}\int_0^T \sin^2(\omega(t_0-t))dt\\
&= \frac{1}{2}c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)}
\end{aligned}$$

But $\phi = kd\sin\theta$ and $k = \frac{2\pi}{\lambda}$, then the above becomes

$$\begin{aligned}
I(\theta)_{av} &= \frac{1}{2}c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d\sin\theta}{\lambda}\right)}\\
&= \frac{1}{2}c\varepsilon_0 E_0^2\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2
\end{aligned}$$

At $\theta = 0$, we have

$$\begin{aligned}
I(0)_{av} &= \lim_{\theta\to 0}\frac{1}{2}c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d\sin\theta}{\lambda}\right)}\\
&= N^2\frac{1}{2}c\varepsilon_0 E_0^2
\end{aligned}$$

Hence

$$\frac{I(\theta)_{av}}{I(0)_{av}} = \frac{\frac{1}{2}c\varepsilon_0 E_0^2\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2}{N^2\frac{1}{2}c\varepsilon_0 E_0^2}$$

$$= \frac{1}{N^2}\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2$$

$$= \left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2$$

Therefore

$$I(\theta)_{av} = I(0)_{av}\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2$$

Which is the formula used in the earlier derivation.

4.3.10 Problem 4

(1) Find the roots $z_n(n = 1, 2, \cdots, N)$ of the complex equation $z^N = 1$. (2) Find $S_N = \sum_{n=1}^{N} z_n$ and give a geometric interpretation of the result. (3) Note that $1-z^N = (1-z)\left(1+z+z^2+\cdots+z^{l}\right.$ Relate this result and the roots z_n to the conditions for destructive interference among N slits.

solution

4.3.10.1 Part 1

$$Z^N = 1$$
$$Z = 1^{\frac{1}{N}}$$

But $1 = e^{i(2\pi)}$ and the above becomes

$$Z = \left(e^{i(2\pi)}\right)^{\frac{1}{N}}$$
$$Z_n = (\cos(2\pi + (2\pi)n) + i\sin(2\pi + (2\pi)n))^{\frac{1}{N}} \qquad n = 0, 1, 2, \cdots, N-1$$

Since cos and sin are periodic with period 2π. Using De Moivre's theorem the above becomes

$$Z_n = \left(\cos\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right) + i\sin\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right)\right)$$
$$= e^{i\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right)}$$
$$= e^{i\left(\frac{2\pi(n+1)}{N}\right)} \qquad n = 0, 1, 2, \cdots, N-1$$

Which is the same as

$$Z_n = e^{i\left(\frac{2\pi n}{N}\right)} \qquad n = 1, 2, \cdots, N$$

For an example, let $N = 3$. Therefore we have 3 roots, given by $n = 1, 2, 3$. They are

$$Z_1 = e^{i\left(\frac{2\pi}{3}\right)} = e^{i(120^0)}$$
$$Z_2 = e^{i\left(\frac{2\pi(2)}{3}\right)} = e^{i\left(\frac{4\pi}{3}\right)} = e^{i\left(240^0\right)}$$
$$Z_3 = e^{i\left(\frac{2\pi(3)}{3}\right)} = e^{i(2\pi)} = e^{i360^0}$$

The roots are 120^0 degrees apart on the unit circle. First root has phase 0^0 (or 360^0), second at 120^0 and the third at 240^0. There are only 3 unique roots, since after that, they repeat. Here is a diagram showing the roots for $N = 3$ for illustration. The root with phase 0^0 is the real root 1 since $e^{i0^0} = 1$, the other two roots are complex valued, and complex conjugate of each others.

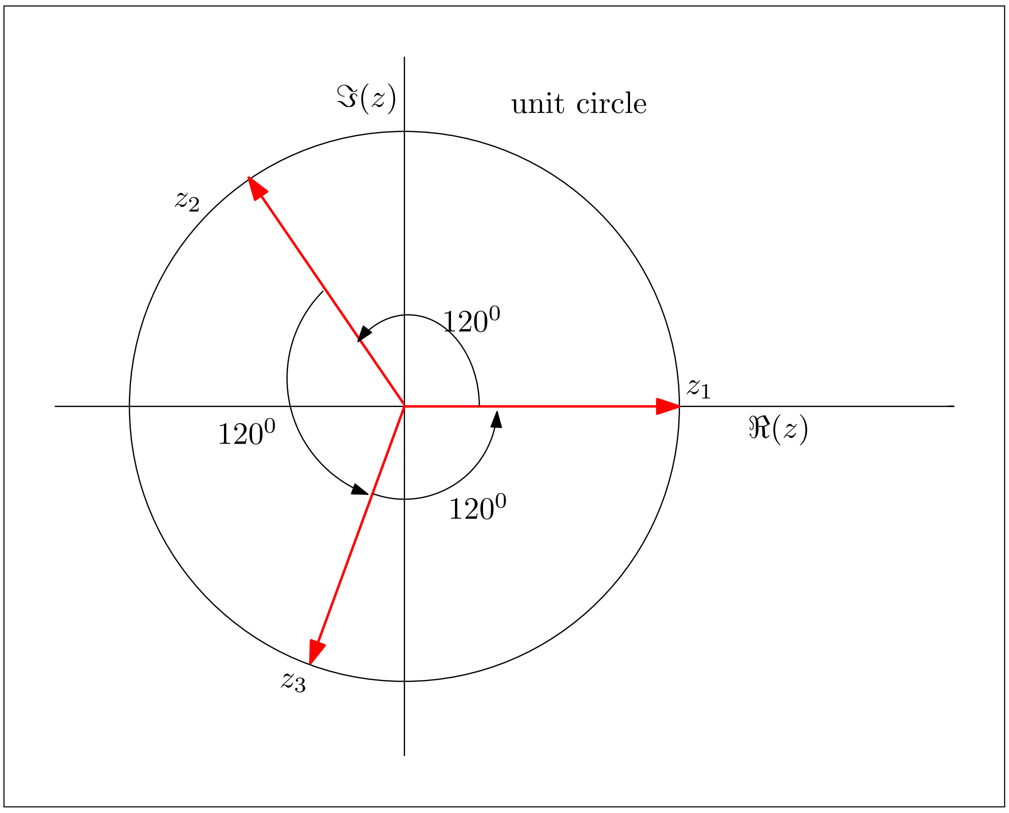


Figure 4.8: Roots of z^N for case of $N = 3$

There are only 3 unique roots, since after moving around the unit circle once, the roots repeat.

4.3.10.2 Part 2

$$S_N = \sum_{n=1}^{N} z_n \tag{1}$$

It is assumed that z_n above are all the roots of Z^N from part(a), even though the problem did not say that. Hence all roots have same modulus. But differ by the phase as found in part 1.

Let $z = x + iy = e^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. The above becomes

$$\begin{aligned} S_N &= z_1 + z_2 + \cdots + z_N \\ &= re^{i\theta_1} + re^{i\theta_2} + \cdots + re^{i\theta_N} \\ &= r\left(e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_N}\right) \end{aligned}$$

But $r = 1$, hence

$$S_N = e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_N}$$

From part 1, we found that

$$\theta_n = \frac{2\pi n}{N} \qquad n = 1, 2, 3, \cdots, N \tag{2}$$

Using (2) in (1), now the sum can be written as

$$S_N = \sum_{n=1}^{N} e^{i\frac{2\pi n}{N}} \tag{3}$$

If $N=1$, then the sum is just $e^{i2\pi}=1$. But if $N>1$ then to find the partial sum, let

$$S_N = e^{i\frac{2\pi}{N}} + e^{i\left(\frac{4\pi}{N}\right)} + e^{i\frac{6\pi}{N}} + \cdots + e^{i\frac{N(2\pi)}{N}} \tag{4}$$

$$e^{i\frac{2\pi}{N}} S_N = e^{i\frac{4\pi}{N}} + e^{i\frac{6\pi}{N}} + \cdots + e^{i\frac{N(2\pi)+2\pi}{N}} \tag{5}$$

(4-5) gives

$$S_N - e^{i\frac{2\pi}{N}} S_N = e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}$$
$$S_N\left(1 - e^{i\frac{2\pi}{N}}\right) = e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}$$
$$S_N = \frac{e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}}{1 - e^{i\frac{2\pi}{N}}}$$

But $e^{i\frac{(N+1)(2\pi)}{N}} = e^{i\frac{N(2\pi)+2\pi}{N}} = e^{i2\pi} e^{i\frac{2\pi}{N}} = e^{i\frac{2\pi}{N}}$. The above becomes

$$S_N = \frac{e^{i\frac{2\pi}{N}} - e^{i\frac{2\pi}{N}}}{1 - e^{i\frac{2\pi}{N}}} = 0$$

Therefore, the final result is

$$S_N = \begin{cases} 1 & N=1 \\ 0 & N>1 \end{cases}$$

For geometric interpretation. Each root z_n is a unit vector, where the angle between each root is the same. it is $\frac{2\pi}{N}$. Looking at each root as a vector in the complex plane, these vectors originate from the origin and end up at the unit circle, each with phase which is $\frac{2\pi}{N}$ more than the vector just to the right of it as we go anticlockwise around the circle. The first vector starts with phase 0.

The sum $\sum_{n=1}^{N} z_n$ is therefore the a vector sum of these N root. The easiest way to see that this sum is zero geometrically, is to add these vectors, by putting each vector tail, at the tip of the previous vector. To illustrate this, we will look at the case of $N=3$ where the angle between each vector is 120^0. This is because $\frac{2\pi}{3} = 120^0$. Using this method to add the roots gives this

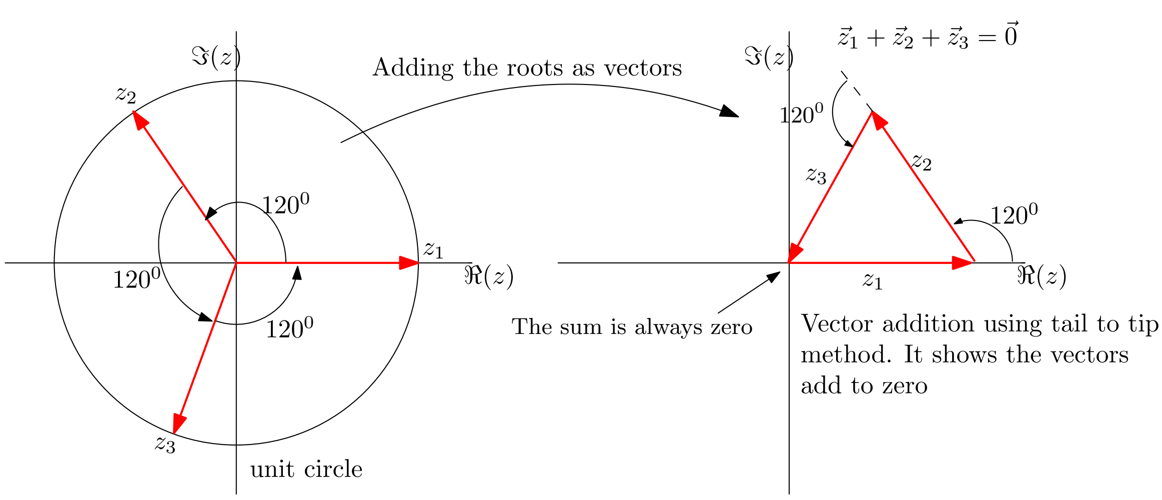


Figure 4.9: Geometric interpretation of adding the roots. Example for $N=3$

The above generalizes for any N. If the vector sum using the tail to tip method gives a closed shape which in this case ends up back at the origin, then the vector sum is zero.

4.3.10.3 Part 3

Looking at the Electric field E at an observation point at angle θ we obtain the following diagram

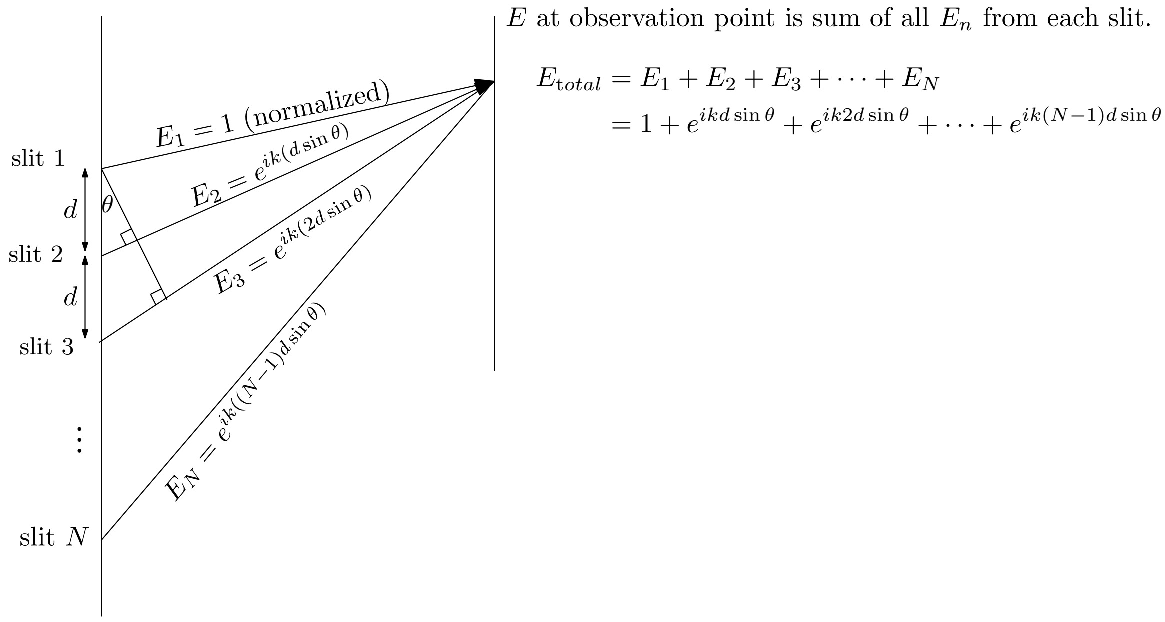


Figure 4.10: Contribution of E from each slit

In the above, the E contribution from slit 1 was normalized to be $E_1 = 1$. Therefore, the contribution of E_2 from the second slit will have a phase shift relative to the first slit. This is given by $d\sin\theta$ as seen in the diagram. For each addition slit, the phase will increase by $d\sin\theta$. Hence the E_3 will have phase of $2d\sin\theta$ and so on until the last slit N which will have phase shift of $(N-1)d\sin\theta$.

Therefore we see that electric field at the observation point is the sum of all E_n from each slit, and given by

$$\begin{aligned} E &= E_1 + E_2 + \cdots + E_N \\ &= 1 + e^{ikd\sin\theta} + e^{ik(2d\sin\theta)} + e^{ik(3d\sin\theta)} + \cdots + e^{ik((N-1)d\sin\theta)} \end{aligned} \tag{1}$$

Now, from lecture notes, we are given the conditions for minima (i.e. destructive interference) as

$$d\sin\theta = \frac{k}{N}\lambda \qquad k = \pm 1, \pm 2, \cdots \tag{2}$$

Substituting (2) into (1) gives

$$E = 1 + e^{ik\left(\frac{k}{N}\lambda\right)} + e^{ik(2\left(\frac{k}{N}\lambda\right))} + e^{ik(3\left(\frac{k}{N}\lambda\right))} + \cdots + e^{ik((N-1)\left(\frac{k}{N}\lambda\right))}$$

Replacing the first k in each term by $\frac{2\pi}{\lambda}$ since k is wave number, then the above becomes

$$\begin{aligned} E &= 1 + e^{i\left(\frac{2\pi}{\lambda}\right)\left(\frac{k}{N}\lambda\right)} + e^{i\left(\frac{2\pi}{\lambda}\right)(2\left(\frac{k}{N}\lambda\right))} + e^{i\left(\frac{2\pi}{\lambda}\right)(3\left(\frac{k}{N}\lambda\right))} + \cdots + e^{i\left(\frac{2\pi}{\lambda}\right)((N-1)\left(\frac{k}{N}\lambda\right))} \\ &= 1 + e^{i\frac{2\pi k}{N}} + e^{i2\left(\frac{2\pi k}{N}\right)} + e^{i3\left(\frac{2\pi k}{N}\right)} + \cdots + e^{i(N-1)\left(\frac{2\pi k}{N}\right)} \end{aligned}$$

Let $\phi = \frac{2\pi}{N}$. The above becomes

$$E = 1 + e^{ik\phi} + e^{i2k\phi} + e^{i3k\phi} + \cdots + e^{ik(N-1)\phi} \tag{3}$$

Comparing the above to the result obtain in part 1 we found that the sum of the roots for $Z^n = 1$ to be

$$\begin{aligned} S_N &= z_0 + z_1 + \cdots + z_{N-1} \\ &= e^{i\theta_0} + e^{i\theta_1} + \cdots + e^{i\theta_{N-1}} \end{aligned} \tag{4}$$

Where

$$\theta_n = \frac{2\pi n}{N} \qquad n = 0, 1, 2, \cdots, N-1 \tag{5}$$

Hence (4) becomes

$$S_N = 1 + e^{i\frac{2\pi}{N}} + e^{i2\frac{2\pi}{N}} + \cdots + e^{i(N-1)\frac{2\pi}{N}} \tag{6}$$

Therefore, for each different k Eq(3) is the same as (6). So (3) can be written as

$$E = 1 + z + z^2 + \cdots + z^{N-1} \tag{7}$$

Where now $z = e^{ik\phi}$ with $\phi = \frac{2\pi}{N}$. But we know that

$$\left(1 - z^N\right) = (1-z)\left(1 + z + z^2 + \cdots + z^{N-1}\right) \tag{8}$$

But $\left(1 - z^N\right) = 0$ since 1 is root of z^N. Hence the above becomes

$$0 = (1-z)\left(1 + z + z^2 + \cdots + z^{N-1}\right)$$

Since $z \neq 1$ (unless $\frac{2\pi}{N}k$ happened to be exact multiple of 2π), then we conclude that $1 + z + z^2 + \cdots + z^{N-1}$ must be zero. This implies that

$$\begin{aligned} E &= 1 + e^{ik\phi} + e^{i2k\phi} + e^{i3k\phi} + \cdots + e^{ik(N-1)\phi} \\ &= 0 \end{aligned}$$

Under the condition of destructive interference. This says the total Electric field from the N slits will vanish at the observation point when destructive interference condition is applied. Which is what we are asked to show.

4.3.11 key solution for HW 3

Physics 3041 (Spring 2021) Solutions to Homework Set 3

1. (a) Problem 5.2.3. (5 points)

$$\begin{aligned}
\frac{2+3i}{6+7i}+\frac{2}{x+iy} &= \frac{(2+3i)(6-7i)}{(6+7i)(6-7i)}+\frac{2}{x+iy}=\frac{33+4i}{85}+\frac{2}{x+iy}=2+9i,\\
\frac{2}{x+iy} &= 2+9i-\frac{33+4i}{85}=\frac{85(2+9i)-(33+4i)}{85}=\frac{137+761i}{85},\\
x+iy &= \frac{170}{137+761i}=\frac{170(137-761i)}{(137+761i)(137-761i)}=\frac{170(137-761i)}{137^2+761^2},
\end{aligned}$$

which gives

$$\begin{aligned}
x &= \frac{170\times 137}{137^2+761^2}=\frac{137}{3517},\\
y &= -\frac{170\times 761}{137^2+761^2}=-\frac{761}{3517}.
\end{aligned}$$

(b) Problem 5.2.4.(iv). (5 points)

We first show that $|z_1/z_2| = |z_1|/|z_2|$.

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{x_1+iy_1}{x_2+iy_2}\right| = \left|\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}\right| = \frac{|(x_1+iy_1)(x_2-iy_2)|}{x_2^2+y_2^2}\\
&= \frac{|(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)|}{x_2^2+y_2^2} = \frac{\sqrt{(x_1x_2+y_1y_2)^2+(x_2y_1-x_1y_2)^2}}{x_2^2+y_2^2}\\
&= \frac{\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_2^2y_1^2+x_1^2y_2^2}}{x_2^2+y_2^2} = \frac{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}}{x_2^2+y_2^2} = \frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}} = \frac{|z_1|}{|z_2|}.
\end{aligned}$$

$$\begin{aligned}
z &= \frac{1+i\sqrt{2}}{1-i\sqrt{3}} = \frac{(1+i\sqrt{2})(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})} = \frac{1-\sqrt{6}+i(\sqrt{2}+\sqrt{3})}{4} \Rightarrow \mathrm{Re}(z) = \frac{1-\sqrt{6}}{4},\\
\mathrm{Im}(z) &= \frac{\sqrt{2}+\sqrt{3}}{4},\ |z| = \frac{|1+i\sqrt{2}|}{|1-i\sqrt{3}|} = \frac{\sqrt{1+2}}{\sqrt{1+3}} = \frac{\sqrt{3}}{2} = \sqrt{[\mathrm{Re}(z)]^2+[\mathrm{Im}(z)]^2}\ ,\\
z^* &= \frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{4},\ \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{4(\sqrt{3}/2)^2} = \frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{3}.
\end{aligned}$$

(c) Problem 5.2.5. (10 points)

We first show that $(z_1z_2)^* = z_1^*z_2^*$.

$$\begin{aligned}
(z_1z_2)^* &= [(x_1+iy_1)(x_2+iy_2)]^* = [x_1x_2-y_1y_2+i(x_1y_2+y_1x_2)]^* = x_1x_2-y_1y_2-i(x_1y_2+y_1x_2)\\
z_1^*z_2^* &= (x_1+iy_1)^*(x_2+iy_2)^* = (x_1-iy_1)(x_2-iy_2) = x_1x_2-y_1y_2-i(x_1y_2+y_1x_2) = (z_1z_2)^*
\end{aligned}$$

It is straightforward to generalize the above result to $(z^m)^* = (z^*)^m$ for integers of $m \geq 2$.

Now if z satifies

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0,$$

where the coefficients a_0, a_1, $\cdots$, a_n are real, then taking complex conjugation of both sides of the equation gives

$$a_0 + a_1 z^* + a_2 (z^2)^* + \cdots + a_n (z^n)^* = 0 \Rightarrow a_0 + a_1 z^* + a_2 (z^*)^2 + \cdots + a_n (z^*)^n = 0.$$

Therefore, both z and z^* are the roots of the above polynomial equation, which means the roots are either real ($z = z^*$) or pairs of complex conjugates.

(d) Problem 5.3.2. (20 points)

$$\begin{aligned}
\text{(i) } z_1 &= \frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\pi/4} \Rightarrow z_1^* = e^{-i\pi/4},\ |z_1| = 1, \\
z_2 &= \sqrt{3} - i = 2 \times \frac{\sqrt{3}-i}{2} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6} \Rightarrow z_2^* = 2e^{i\pi/6},\ |z_2| = 2, \\
z_1 z_2 &= e^{i\pi/4} \times 2e^{-i\pi/6} = 2e^{i\pi/12},\ \frac{z_1}{z_2} = \frac{e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{1}{2}e^{5i\pi/12},\ \left(\frac{z_1}{z_2}\right)^* = \frac{1}{2}e^{-5i\pi/12}.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } z_1 &= \frac{3+4i}{3-4i} = \frac{5e^{i\tan^{-1}(4/3)}}{5e^{-i\tan^{-1}(4/3)}} = e^{2i\tan^{-1}(4/3)} \Rightarrow z_1^* = e^{-2i\tan^{-1}(4/3)},\ |z_1| = 1, \\
z_2 &= \left[\frac{1+2i}{1-3i}\right]^2 = \left[\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}\right]^2 = \left[\frac{1-6+5i}{10}\right]^2 = \left[\frac{-1+i}{2}\right]^2 = \frac{1-1-2i}{4} = -\frac{i}{2} \\
&= \frac{e^{-i\pi/2}}{2} \Rightarrow z_2^* = \frac{e^{i\pi/2}}{2},\ |z_2| = \frac{1}{2}, \\
z_1 z_2 &= e^{2i\tan^{-1}(4/3)} \times \frac{e^{-i\pi/2}}{2} = \frac{e^{i[2\tan^{-1}(4/3)-(\pi/2)]}}{2},\ \frac{z_1}{z_2} = \frac{e^{2i\tan^{-1}(4/3)}}{(1/2)e^{-i\pi/2}} = 2e^{i[2\tan^{-1}(4/3)+(\pi/2)]}, \\
\left(\frac{z_1}{z_2}\right)^* &= 2e^{-i[2\tan^{-1}(4/3)+(\pi/2)]}.
\end{aligned}$$

2. (a) Problem 5.3.5. (10 points)

$$S = e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta},\ e^{2i\theta}S = e^{3i\theta} + \cdots + e^{(2n-1)i\theta} + e^{(2n+1)i\theta}$$

$$(1 - e^{2i\theta})S = e^{i\theta} - e^{(2n+1)i\theta}$$

$$S = \frac{e^{i\theta} - e^{(2n+1)i\theta}}{1 - e^{2i\theta}} = \frac{1 - e^{2ni\theta}}{e^{-i\theta} - e^{i\theta}} = \frac{i(1 - e^{2ni\theta})}{2\sin\theta} = \frac{i(1 - \cos 2n\theta) + \sin 2n\theta}{2\sin\theta}$$

$$\begin{aligned}
\mathrm{Re}(S) &= \cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}, \\
\mathrm{Im}(S) &= \sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}.
\end{aligned}$$

(b) Problem 5.3.6. (10 points)

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + 4i(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta), \\
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\
\sin 4\theta &= 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta).
\end{aligned}$$

$$\begin{aligned}
e^{i(A+B)} &= \cos(A+B) + i\sin(A+B) \\
&= e^{iA}e^{iB} = (\cos A + i\sin A)(\cos B + i\sin B) \\
&= \cos A\cos B - \sin A\sin B + i(\sin A\cos B + \cos A\sin B), \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B, \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B.
\end{aligned}$$

(c) Find $\int_0^\infty xe^{-ax}\cos kx dx$ using Euler's formula. (10 points)

$$\begin{aligned}
\int_0^\infty xe^{-ax}\cos kx dx &= \int_0^\infty (xe^{-ax})\frac{e^{ikx}+e^{-ikx}}{2}dx = \int_0^\infty x\times\frac{e^{-(a-ik)x}+e^{-(a+ik)x}}{2}dx \\
&= \frac{1}{2}\left[\frac{1}{(a-ik)^2} + \frac{1}{(a+ik)^2}\right] = \frac{(a+ik)^2+(a-ik)^2}{2(a-ik)^2(a+ik)^2} = \frac{a^2-k^2}{(a^2+k^2)^2},
\end{aligned}$$

where we have made the substitutions $z = (a \pm ik)x$ and used $\int_0^\infty ze^{-z}dz = 1$ for $a > 0$.

3. Given the intensity pattern for the N-slit interference with separation d between adjacent slits, show that the pattern becomes that for the single-slit diffraction with slit width a when d goes to zero but with a fixed value of $Nd = a$. (10 points)

From the lectures, the intensity pattern for the N-slit interference is described by

$$\bar{I}(\theta) = \bar{I}(0)\left[\frac{\sin(N\pi d\sin\theta/\lambda)}{N\sin(\pi d\sin\theta/\lambda)}\right]^2.$$

In the limit of $d \to 0$, $\sin(\pi d\sin\theta/\lambda) \to \pi d\sin\theta/\lambda$, so we have

$$\bar{I}(\theta) \to \bar{I}(0)\left[\frac{\sin(N\pi d\sin\theta/\lambda)}{N\pi d\sin\theta/\lambda}\right]^2 = \bar{I}(0)\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2,$$

where we have used $Nd = a$. The above limiting result is the intensity pattern for the single-slit diffraction.

4. (1) Find the roots z_n ($n = 1, 2, \cdots, N$) of the complex equation $z^N = 1$. (5 points)

$$z^N = 1 = e^{i2k\pi},\ k = 0,\ 1,\ 2,\ \cdots \Rightarrow z = e^{i2k\pi/N}.$$

So we can take $z_n = e^{i2(n-1)\pi/N}$. Note that values of $n \geq N+1$ do not give new roots as $e^{i2k\pi} = 1$.

(2) Find $S_N = \sum_{n=1}^{N} z_n$ and give a geometric interpretation of the result. (10 points)

Let $\phi = 2\pi/N$. So $z_n = e^{i(n-1)\phi}$.

$$S_N = 1 + e^{i\phi} + e^{i2\phi} + \cdots + e^{i(N-1)\phi},\ e^{i\phi}S_N = e^{i\phi} + e^{i2\phi} + \cdots + e^{i(N-1)\phi} + e^{iN\phi}$$

$$(1 - e^{i\phi})S_N = 1 - e^{iN\phi} \Rightarrow S_N = \frac{1 - e^{iN\phi}}{1 - e^{i\phi}} = \frac{1 - e^{i2\pi}}{1 - e^{i2\pi/N}} = 0.$$

Recall that the complex number $e^{i\phi}$ corresponds to a unit vector making an angle ϕ with respect to the x-axis and that counterclockwise rotation of a vector by an angle ϕ corresponds to multiplication by $e^{i\phi}$. So S_N represents the sum of N unit vectors that form the sides of a regular polygon. This vectorial sum vanishes because the vectors form a closed figure.

(3) Note that $1 - z^N = (1 - z)(1 + z + z^2 + \cdots + z^{N-1})$. Relate this result and the roots z_n to the conditions for destructive interference among N slits. (5 points)

The net electric field at a point on the observational screen for N-slit interference can be represented by a sum $1+z+z^2+\cdots+z^{N-1}$, where $z = e^{i\phi'}$ with $\phi' = 2\pi d \sin\theta/\lambda$ being the phase difference between the contributions from adjacent slits. When $\phi' = 2k\pi/N$ or $d\sin\theta = k\lambda/N$ ($k = 1, 2, \cdots, N-1$), z becomes the roots z_n ($n \geq 2$) and the sum vanishes because $1 - z^N = 0$ but $1 - z \neq 0$. For values of $k \geq N+1$ that are not integer multiples of N, the same roots are repeated due to $e^{i2\pi} = 1$.

4.4 HW 4

Local contents

4.4.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 4 **(Due 2/24)**

1. (a) Problem 8.1.1. (5 points)

(b) Problem 8.1.2, and find the expression of θ in terms of the relative velocity. (10 points)

(c) Problem 8.2.4, for Lorentz transformation only. (5 points)

2. (a) Problem 8.3.4, but using Cramer's rule to solve the first set of equations only. (5 points)

(b) Problem 8.3.5. (5 points)

3. (a) Problem 8.4.3. (5 points)

(b) Problem 8.4.19, proving the first result only. (5 points)

(c) Problem 8.4.20. (10 points)

4. (a) Problem 8.4.5. (10 points)

(b) Problem 8.4.8. (10 points)

(c) Problem 8.4.10. (5 points)

5. Problem 8.4.17. (5 points)

6. (a) Consider a horizontal spring-mass system. The spring has a spring constant k and is fixed at one end. The other end is attached to a block of mass m that can move without friction on a horizontal surface. The spring is stretched a length a beyond its rest length and let go. Without solving the problem using Newton's second law, find the angular frequency of oscillations and show that it is independent of a. (5 points)

(b) Derive the Planck mass, length, and time in terms of Planck's constant $\hbar$, Newton's constant G, and speed of light c. Evaluate these quantities in SI units. (10 points)

(c) Identify the relevant physical quantities and use dimensional analysis to find the characteristic length for a black hole of mass M. (5 points)

4.4.2 Problem 1(a) (8.1.1)

Given rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ verify that $R_{\theta+\theta'} = R_{\theta'}R_\theta$

Solution

$$R_{\theta+\theta'} = \begin{bmatrix} \cos(\theta+\theta') & \sin(\theta+\theta') \\ -\sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} \tag{1}$$

But

$$\begin{aligned} R_{\theta'}R_\theta &= \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta'\cos\theta - \sin\theta'\sin\theta & \cos\theta'\sin\theta + \sin\theta'\cos\theta \\ -\sin\theta'\cos\theta - \cos\theta'\sin\theta & -\sin\theta'\sin\theta + \cos\theta'\cos\theta \end{bmatrix} \end{aligned} \tag{2}$$

But from trig identities we know that

$$\begin{aligned} \cos\theta'\cos\theta - \sin\theta'\sin\theta &= \cos(\theta+\theta') && (3) \\ \cos\theta'\sin\theta + \sin\theta'\cos\theta &= \sin(\theta+\theta') && (4) \\ -\sin\theta'\cos\theta - \cos\theta'\sin\theta &= -(\sin\theta'\cos\theta + \cos\theta'\sin\theta) \\ &= -\sin(\theta+\theta') && (5) \\ -\sin\theta'\sin\theta + \cos\theta'\cos\theta &= \cos(\theta+\theta') && (6) \end{aligned}$$

Substituting (3,4,5,6) into (2) gives

$$R_{\theta'}R_\theta = \begin{bmatrix} \cos(\theta+\theta') & \sin(\theta+\theta') \\ -\sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix}$$

Which is the same as (1). Hence

$$R_{\theta+\theta'} = R_{\theta'}R_\theta$$

4.4.3 Problem 1(b) (8.1.2)

Part 1

Recall from problem 1.6.4 in chapter 1, that the relativistic transformation of coordinates when we go from frame of reference to another is

$$\begin{aligned} x' &= x\cosh\theta - ct\sinh\theta \\ ct' &= -x\sinh\theta + ct\cosh\theta \end{aligned}$$

(ps. I added c to the formula as book assumes it is 1. This makes it more clear).

Where θ is the rapidity difference between the two frames. Write this in matrix form. Say we go to a third frame with coordinates x'', t'', moving with rapidity θ' with respect to the one with primed coordinates. Show that the matrix relating the doubly primed coordinates to the unprimed ones corresponds to rapidity $\theta+\theta'$.

Part 2. Find the expression of θ in terms of the relative velocity.

Solution

4.4.3.1 Part 1

In Matrix form Lorentz transformation becomes

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{1}$$

In the third frame (double primed), we have

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{2}$$

Substituting (1) in the RHS of (2) gives

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{3}$$

But

$$\begin{aligned} \begin{pmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{pmatrix} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} &= \begin{pmatrix} \cosh\theta'\cosh\theta + \sinh\theta'\sinh\theta & -\cosh\theta'\sinh\theta - \sinh\theta'\, c \\ -\sinh\theta'\cosh\theta - \cosh\theta'\sinh\theta & \sinh\theta'\sinh\theta + \cosh\theta'\, co \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{pmatrix} \end{aligned}$$

Substituting the above in (3) gives

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \begin{pmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{4}$$

Therefore the matrix

$$\begin{pmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{pmatrix}$$

Relates the unprimed frame to the doubly primed by rapidity $\theta + \theta'$, which is what we are asked to show.

4.4.3.2 Part 2

Need to find the expression of θ in terms of the relative velocity. The relative velocity it taken as that between the unprimed (x, ct) and the one primed frame (x', ct').

The Lorentz transformation can also be written as

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

But we also can write the above in terms of rapidity θ as given in the text book as

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{3}$$

Or

$$\begin{aligned} x' &= x\cosh\theta - ct\sinh\theta \quad (4) \\ ct' &= -x\sinh\theta + ct\cosh\theta \\ t' &= -\frac{x}{c}\sinh\theta + t\cosh\theta \quad (5) \end{aligned}$$

Equating (1,4) and (2,5) gives the following two equations

$$\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = x\cosh\theta - ct\sinh\theta \tag{6}$$

$$\frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{x}{c}\sinh\theta + t\cosh\theta \tag{7}$$

Dividing Eq (6) by Eq (7) to get rid of the root term gives

$$\frac{x - vt}{t - \frac{vx}{c^2}} = \frac{x\cosh\theta - ct\sinh\theta}{-\frac{x}{c}\sinh\theta + t\cosh\theta} \tag{8}$$

Dividing the numerator and the denominator of RHS of the above by $\cosh\theta$ gives

$$\frac{x - vt}{t - \frac{vx}{c^2}} = \frac{x - ct\tanh\theta}{t - \frac{x}{c}\tanh\theta}$$

Now we solve for v, the relative velocity from the above by simplifying the above. This results in

$$\begin{aligned}
(x - vt)\left(t - \frac{x}{c}\tanh\theta\right) &= \left(t - \frac{vx}{c^2}\right)(x - ct\tanh\theta)\\
xt - \frac{x^2}{c}\tanh\theta - vt^2 + vt\frac{x}{c}\tanh\theta &= tx - ct^2\tanh\theta - \frac{vx^2}{c^2} + \frac{vx}{c^2}ct\tanh\theta\\
v\left(-t^2 + t\frac{x}{c}\tanh\theta + \frac{x^2}{c^2} - \frac{x}{c}t\tanh\theta\right) &= tx - xt + \frac{x^2}{c}\tanh\theta - ct^2\tanh\theta\\
v\left(-t^2 + \frac{x^2}{c^2}\right) &= \left(\frac{x^2}{c} - ct^2\right)\tanh\theta\\
v &= \frac{\frac{x^2}{c} - ct^2}{\frac{x^2}{c^2} - t^2}\tanh\theta\\
&= \frac{\frac{x^2 - c^2t^2}{c}}{\frac{x^2 - c^2t^2}{c^2}}\tanh\theta\\
&= \frac{c^2\left(x^2 - c^2t^2\right)}{c\left(x^2 - c^2t^2\right)}\tanh\theta\\
&= \frac{c\left(x^2 - c^2t^2\right)}{x^2 - c^2t^2}\tanh\theta\\
&= c\tanh\theta
\end{aligned}$$

Therefore, the relative velocity is

$$v = c\tanh\theta$$

4.4.4 Problem 1(c) (8.2.4)

Find the inverse of Lorentz Transformation matrix from problem 8.1.2 and the rotation matrix R_θ. Does the answer makes sense? (You must be on top of the identities for hyperbolic and trigonometric functions to do this. Remember: when in trouble go back to the definitions in terms of exponential).

Solution

The Lorentz Transformation matrix from problem 8.1.2 above is

$$\begin{aligned}
\begin{pmatrix} x' \\ t' \end{pmatrix} &= \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}\\
&= L_\theta\begin{pmatrix} x \\ t \end{pmatrix}
\end{aligned}$$

Where

$$L_\theta = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

While the rotation matrix is

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The question is asking to find the L_θ^{-1} and R_θ^{-1}.

$$\begin{aligned} L_\theta^{-1} &= \frac{1}{\det(L_\theta)}\begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} \\ &= \frac{1}{\cosh^2\theta - \sinh^2\theta}\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \end{aligned} \tag{1}$$

The inverse of the matrix undoes whatever the matrix does. Let us check this on the above result.

$$L_{(-\theta)} = \begin{pmatrix} \cosh(-\theta) & -\sinh(-\theta) \\ -\sinh(-\theta) & \cosh(-\theta) \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \tag{2}$$

We see that (2) is the same as (1). Hence the result of (1) makes sense. For the transformation matrix, we have

$$\begin{aligned} R_\theta^{-1} &= \frac{1}{\det(R_\theta)}\begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix} \\ &= \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{3}$$

The inverse of the matrix undoes whatever the matrix does. Let us check this on the above result.

$$R_{(-\theta)} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{4}$$

We see that (4) is the same as (2). Hence the result of (3) makes sense.

4.4.5 Problem 2(a) (8.3.4)

(1) Solve the following simultaneous equations using Crammer rule.

$$\begin{aligned} 3x - y - z &= 2 \\ x - 2y - 3z &= 0 \\ 4x + y + 2z &= 4 \end{aligned}$$

solution

In Matrix form

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

Then, using Crammer rule

$$x=\frac{\begin{vmatrix}2&-1&-1\\0&-2&-3\\4&1&2\end{vmatrix}}{\begin{vmatrix}3&-1&-1\\1&-2&-3\\4&1&2\end{vmatrix}},y=\frac{\begin{vmatrix}3&2&-1\\1&0&-3\\4&4&2\end{vmatrix}}{\begin{vmatrix}3&-1&-1\\1&-2&-3\\4&1&2\end{vmatrix}},z=\frac{\begin{vmatrix}3&-1&2\\1&-2&0\\4&1&4\end{vmatrix}}{\begin{vmatrix}3&-1&-1\\1&-2&-3\\4&1&2\end{vmatrix}} \tag{A}$$

But det(A) is (using expansion along the first row)

$$\begin{aligned}\begin{vmatrix}3&-1&-1\\1&-2&-3\\4&1&2\end{vmatrix}&=3\begin{vmatrix}-2&-3\\1&2\end{vmatrix}-(-1)\begin{vmatrix}1&-3\\4&2\end{vmatrix}+(-1)\begin{vmatrix}1&-2\\4&1\end{vmatrix}\\&=3(-4+3)+(2+12)-(1+8)\\&=2\end{aligned} \tag{1}$$

And

$$\begin{aligned}\begin{vmatrix}2&-1&-1\\0&-2&-3\\4&1&2\end{vmatrix}&=2\begin{vmatrix}-2&-3\\1&2\end{vmatrix}-(-1)\begin{vmatrix}0&-3\\4&2\end{vmatrix}+(-1)\begin{vmatrix}0&-2\\4&1\end{vmatrix}\\&=2(-4+3)+(12)-(8)\\&=2\end{aligned} \tag{2}$$

And

$$\begin{aligned}\begin{vmatrix}3&2&-1\\1&0&-3\\4&4&2\end{vmatrix}&=3\begin{vmatrix}0&-3\\4&2\end{vmatrix}-(2)\begin{vmatrix}1&-3\\4&2\end{vmatrix}+(-1)\begin{vmatrix}1&0\\4&4\end{vmatrix}\\&=3(12)-2(2+12)-(4)\\&=4\end{aligned} \tag{3}$$

And

$$\begin{aligned}\begin{vmatrix}3&-1&2\\1&-2&0\\4&1&4\end{vmatrix}&=3\begin{vmatrix}-2&0\\1&4\end{vmatrix}-(-1)\begin{vmatrix}1&0\\4&4\end{vmatrix}+(2)\begin{vmatrix}1&-2\\4&1\end{vmatrix}\\&=3(-8)+(4)+2(1+8)\\&=-2\end{aligned} \tag{4}$$

Substituting (1,2,3,4) into (A) gives the solution

$$\begin{aligned}x&=\frac{2}{2}=1\\y&=\frac{4}{2}=2\\z&=\frac{-2}{2}=-1\end{aligned}$$

4.4.6 Problem 2(a) (8.3.4)

(Done again using Gaussian elimination method)

(1) Solve the following simultaneous equations by matrix inversion

$$\begin{aligned} 3x - y - z &= 2 \\ x - 2y - 3z &= 0 \\ 4x + y + 2z &= 4 \end{aligned}$$

(2)

$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -2 \\ x + y + z &= 1 \end{aligned}$$

Solution

4.4.6.1 Part 1

In Matrix form

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \tag{1}$$

To find the matrix inverse, the method of Gaussian elimination is used.

$$\begin{pmatrix} 3 & -1 & -1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Swapping R_2 and R_1

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 3 & -1 & -1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$R_2 = R_2 - 3R_1$

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 5 & 8 & 1 & -3 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 = R_3 - 4R_1$

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 5 & 8 & 1 & -3 & 0 \\ 0 & 9 & 14 & 0 & -4 & 1 \end{pmatrix}$$

$R_2 = 9R_2$ and $R_3 = 5R_3$ gives

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 45 & 72 & 9 & -27 & 0 \\ 0 & 45 & 70 & 0 & -20 & 5 \end{pmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 45 & 72 & 9 & -27 & 0 \\ 0 & 0 & -2 & -9 & 7 & 5 \end{pmatrix}$$

$R_2 = \frac{R_2}{45}, R_3 = \frac{R_3}{-2}$

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 1 & \frac{72}{45} & \frac{9}{45} & -\frac{27}{45} & 0 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{7}{-2} & \frac{5}{-2} \end{pmatrix}$$

$R_2 = R_2 - \frac{72}{45} R_3$

$$\begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{9}{45} - \left(\frac{72}{45}\right)\left(\frac{9}{2}\right) & -\frac{27}{45} - \left(\frac{72}{45}\right)\left(-\frac{7}{2}\right) & -\left(\frac{72}{45}\right)\left(-\frac{5}{2}\right) \\ 0 & 0 & 1 & \frac{9}{2} & \frac{7}{-2} & \frac{5}{-2} \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -7 & 5 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{7}{-2} & \frac{5}{-2} \end{pmatrix}$$

$R_1 = R_1 + 3R_3$

$$\begin{pmatrix} 1 & -2 & 0 & 3\left(\frac{9}{2}\right) & 1 + 3\left(-\frac{7}{2}\right) & 3\left(-\frac{5}{2}\right) \\ 0 & 1 & 0 & -7 & 5 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{7}{-2} & \frac{5}{-2} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 & \frac{27}{2} & -\frac{19}{2} & -\frac{15}{2} \\ 0 & 1 & 0 & -7 & 5 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{7}{-2} & \frac{5}{-2} \end{pmatrix}$$

$R_1 = R_1 + 2R_2$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{27}{2} + 2(-7) & -\frac{19}{2} + 2(5) & -\frac{15}{2} + 2(4) \\ 0 & 1 & 0 & -7 & 5 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -7 & 5 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} \end{pmatrix}$$

Since now the LHS matrix is I, then the RHS is the inverse. Therefore

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -7 & 5 & 4 \\ \frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} \end{pmatrix}$$

Using the above in (1) gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -7 & 5 & 4 \\ \frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Hence $x = 1, y = 2, z = -1$.

4.4.6.2 Part 2

In Matrix form

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{1}$$

To find the matrix inverse, the method of Gaussian elimination is used.

$$\begin{pmatrix} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Swapping R_3 and R_1

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -5 & -3 & 0 & 1 & -2 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

$R_3 = R_3 - 3R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -5 & -3 & 0 & 1 & -2 \\ 0 & -2 & -1 & 1 & 0 & -3 \end{pmatrix}$$

$R_2 = 2R_2, R_3 = 5R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -10 & -6 & 0 & 2 & -4 \\ 0 & -10 & -5 & 5 & 0 & -15 \end{pmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -10 & -6 & 0 & 2 & -4 \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix}$$

$R_2 = \frac{R_2}{-10}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{3}{5} & 0 & \frac{1}{-5} & \frac{2}{5} \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix}$$

$R_2 = R_2 - \frac{3}{5}R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{5}(5) & \frac{1}{-5} - \frac{3}{5}(-2) & \frac{2}{5} - \frac{3}{5}(-11) \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & 7 \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix}$$

$R_1 = R_1 - R_3$

$$\begin{pmatrix} 1 & 1 & 0 & -5 & 2 & 12 \\ 0 & 1 & 0 & -3 & 1 & 7 \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix}$$

$R_1 = R_1 - R_2$

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 1 & 5 \\ 0 & 1 & 0 & -3 & 1 & 7 \\ 0 & 0 & 1 & 5 & -2 & -11 \end{pmatrix}$$

Since now the LHS matrix is I, then the RHS is the inverse. Therefore

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 & 5 \\ -3 & 1 & 7 \\ 5 & -2 & -11 \end{pmatrix}$$

Using the above in (1) gives

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} -2 & 1 & 5 \\ -3 & 1 & 7 \\ 5 & -2 & -11 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -4 \\ 8 \end{pmatrix} \end{aligned}$$

Hence $x = -3, y = -4, z = 8$.

4.4.7 Problem 2(b) (8.3.5)

For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Find the cofactor and the inverse. Verify that your inverse does the job.

Solution

The cofactor matrix A_C has elements $(A_C)_{ij} = (-1)^{i+j} |A|_{ij}$ where $|A|_{ij}$ is determinant of A with row i and column j removed. Hence

$$A_C = \begin{bmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{bmatrix} \tag{1}$$

Where

$$A_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} = 2$$

$$A_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} = -2$$

$$A_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} = -4$$

$$A_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} = -11$$

$$A_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$$

$$A_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6$$

$$A_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$$

Substituting all the above into (1) gives the cofactor matrix

$$\begin{aligned} A_C &= \begin{bmatrix} +2 & -(-2) & +(-3) \\ -(-4) & +(-11) & -(-6) \\ +(-3) & -(-6) & +(-3) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix} \end{aligned}$$

The inverse of A is

$$A^{-1} = \frac{1}{\det(A)} A_C^T \tag{2}$$

So we just need to find $\det(A)$ and transpose the cofactor matrix. But

$$\det(A) = A_{11} - 2A_{12} + 3A_{13}$$

By expanding along the first row. Hence

$$\begin{aligned} \det(A) &= (2) - 2(-2) + 3(-3) \\ &= -3 \end{aligned}$$

Hence (2) becomes

$$A^{-1} = \frac{-1}{3}\begin{bmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T$$
$$= \frac{-1}{3}\begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

To verify

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$A^{-1}A = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verified. It does the job.

4.4.8 Problem 3(a) (8.4.3)

Show that

$$(MN)^{\dagger} = N^{\dagger}M^{\dagger}$$

Consequently the product of two Hermitian matrices is not generally Hermitian unless they commute.

solution

$A^{\dagger}$ is called the adjoint of matrix. It is the transpose of A followed by taking the complex conjugate of each entry in the result. Hence for a real matrix A the adjoint is the same as transpose, since complex conjugate of real value is itself. So we start by finding the transpose $(MN)^T$ then at the end apply conjugate.

$$\begin{aligned}(MN)^T_{ij} &= (MN)_{ji}\\ &= \sum_k M_{jk}N_{ki}\\ &= \sum_k M^T_{kj}N^T_{ik}\\ &= \sum_k N^T_{ik}M^T_{kj}\\ &= \left(N^TM^T\right)_{ij}\end{aligned}$$

The sum above over k, where k goes from 1 to the number of columns in M (which must be the same as the number of rows in N for the product to be possible). The above shows that

$$(MN)^T = N^TM^T$$

Therefore

$$\begin{aligned}(MN)^{\dagger} &= \left(N^TM^T\right)^*\\ &= \left(N^T\right)^*\left(M^T\right)^*\\ &= N^{\dagger}M^{\dagger}\end{aligned}$$

A matrix A is called Hermitian if $A^{\dagger} = A$ or $A^{\dagger} = -A$. Also, any real matrix A is always Hermitian.

Assuming M, N are Hermitian, and assuming for now that we look at the positive case. i.e. $M^{\dagger} = M, N^{\dagger} = N$. Hence

$$M^{\dagger}N^{\dagger} = (NM)^{\dagger}$$

Now, if N, M commute, then $NM = MN$ and the above becomes

$$M^{\dagger}N^{\dagger} = (MN)^{\dagger}$$

Hence the product $M^{\dagger}N^{\dagger}$ is Hermitian. But if N, M do not commute, then we can not say that.

4.4.9 Problem 3(b) (8.4.19)

(1) Show that

$$\mathrm{Tr}(MN) = \mathrm{Tr}(NM) \tag{8.4.53}$$

(First part only).

<u>solution</u>

The trace of a matrix A is the sum of elements on the diagonal. The matrix must be square for this to apply. Hence

$$Tr(A) = \sum_v A_{vv}$$

Where the sum v is over the number of rows or columns (since they are the same, since matrix is square)

In the following, we will use the definition of matrix product given by $(MN)_{ij} = \sum_k M_{jk}N_{ki}$ where the sum k is over the number of columns of M. Now we can write

$$\begin{aligned}\mathrm{Tr}(MN) &= \sum_v (MN)_{vv}\\ &= \sum_v\left(\sum_k M_{vk}N_{kv}\right)\\ &= \sum_v\left(\sum_k N_{kv}M_{vk}\right)\end{aligned}$$

Assuming N, M are square matrices, then we can replace the inner sum to be over v instead of k, since these will be the same for square N, M. Hence the above becomes

$$\text{Tr}(MN) = \sum_v \left(\sum_v N_{vv} M_{vv} \right) \tag{1}$$

Now we do the same for product NM.

$$\begin{aligned}\text{Tr}(NM) &= \sum_v (NM)_{vv} \\ &= \sum_v \left(\sum_k N_{vk} M_{kv} \right) \\ &= \sum_v \left(\sum_k M_{kv} N_{vk} \right)\end{aligned}$$

Assuming N, M are square matrices, then we can replace the inner sum to be over v instead of k, since these will be the same for square N, M. Hence the above becomes

$$\text{Tr}(NM) = \sum_v \left(\sum_v M_{vv} N_{vv} \right) \tag{2}$$

Comparing (1,2) shows they are the same. Hence $\text{Tr}(MN) = \text{Tr}(NM)$. Note that this solution assumed that M, N are both square matrices of the same size.

4.4.10 Problem 3(c) (8.4.20)

Consider four Dirac matrices that obey

$$M_i M_j + M_j M_i = 2\delta_{ij} I \tag{8.4.56}$$

Where the Kronecker delta symbol is defined as follows

$$\delta_{ij} = 1 \qquad \text{if } i = j, 0 \text{ if } i \neq j \tag{8.4.57}$$

Thus the square of each Dirac matrix is the unit matrix and any two distinct Dirac matrices anticommute. Using the latter property show that the matrices are traceless. (Use equation (8.4.54)

solution

Eq (8.4.54) from the book says

$$\text{Tr}(ABC) = \text{Tr}(BCM) = \text{Tr}(CAB) \tag{8.4.54}$$

Some definitions first. Two matrices A, B anticommute means $AB = -BA$. A matrix is traceless means the trace of the matrix (the sum of the diagonal elements) is zero.

There are Four Dirac matrices M_1, M_2, M_3, M_4. Each is 4×4 matrix.

From $M_i M_j + M_j M_i = 2\delta_{ij} I$ then

$$\begin{aligned}2M_i M_i &= 2\delta_{ii} I \\ M_i M_i &= \delta_{ii} I \\ \frac{M_i M_i}{\delta_{ii}} &= I\end{aligned}$$

Premultiplying both sides by M_j gives

$$\frac{M_j M_i M_i}{\delta_{ii}} = M_j$$

Taking trace of both sides

$$\begin{aligned}\text{Tr}\left(M_j\right) &= \text{Tr}\left(\frac{M_j M_i M_i}{\delta_{ii}}\right) \\ &= \frac{1}{\delta_{ii}} \text{Tr}\left(M_j M_i M_i\right)\end{aligned}$$

But Dirac matrices anticommute. Hence $M_jM_i = -M_iM_j$. The above becomes

$$\operatorname{Tr}\left(M_j\right) = -\frac{1}{\delta_{ii}}\operatorname{Tr}\left(M_iM_jM_i\right)$$

Using property $\operatorname{Tr}\left(M_iM_jM_i\right) = \operatorname{Tr}\left(M_jM_iM_i\right)$ the above becomes

$$\begin{aligned}\operatorname{Tr}\left(M_j\right) &= -\frac{1}{\delta_{ii}}\operatorname{Tr}\left(M_jM_iM_i\right)\\ &= -\frac{1}{\delta_{ii}}\operatorname{Tr}\left(M_jM_i^2\right)\end{aligned}$$

But $M_i^2 = I$, therefore

$$\operatorname{Tr}\left(M_j\right) = -\frac{1}{\delta_{ii}}\operatorname{Tr}\left(M_j\right)$$

The above is possible only if $\operatorname{Tr}\left(M_j\right) = 0$ since $\frac{1}{\delta_{ii}}$ is just a number. The above is like saying $n = -3n$ which is only possible if $n = 0$. Hence the trace of any Dirac matrix is zero, which means it is traceless.

4.4.11 Problem 4(a) (8.4.5)

Show that the following matrix U is unitary. Argue that the determinant of a unitary matrix must be unimodular complex number. What is it for this example?

$$U = \begin{bmatrix}\frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}}\\ \frac{-\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4}\end{bmatrix}$$

solution

A matrix U is unitary if $U^\dagger = U^{-1}$. Where $U^\dagger$ means to take the transpose followed by complex conjugate. For the above

$$U^{-1} = \frac{1}{\det(U)}\begin{bmatrix}U_{22} & -U_{12}\\ -U_{21} & U_{11}\end{bmatrix} \tag{1}$$

But

$$\begin{aligned}\det(U) &= \begin{vmatrix}\frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}}\\ \frac{-\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4}\end{vmatrix}\\ &= \left(\frac{1+i\sqrt{3}}{4}\right)\left(\frac{i+\sqrt{3}}{4}\right) - \left(\frac{\sqrt{3}(1+i)}{2\sqrt{2}}\right)\left(\frac{-\sqrt{3}(1+i)}{2\sqrt{2}}\right)\\ &= \frac{1}{4}i - \left(-\frac{3}{4}i\right)\\ &= i\end{aligned}$$

Hence (1) becomes

$$
\begin{aligned}
U^{-1} &= \frac{1}{i}\begin{bmatrix} \frac{i+\sqrt{3}}{4} & -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{1+i\sqrt{3}}{4} \end{bmatrix} \\
&= -i\begin{bmatrix} \frac{i+\sqrt{3}}{4} & -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{1+i\sqrt{3}}{4} \end{bmatrix} \\
&= \begin{bmatrix} -i\left(\frac{i+\sqrt{3}}{4}\right) & (-i)\left(-\frac{\sqrt{3}(1+i)}{2\sqrt{2}}\right) \\ (-i)\left(\frac{\sqrt{3}(1+i)}{2\sqrt{2}}\right) & (-i)\left(\frac{1+i\sqrt{3}}{4}\right) \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1-i\sqrt{3}}{4}\right) & \frac{\sqrt{3}(i-1)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix} \qquad (1)
\end{aligned}
$$

Now $U^{\dagger}$ is found.

$$
\begin{aligned}
U^{\dagger} &= \left(U^{T}\right)^{*} \\
&= \left(\begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ \frac{-\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix}^{T}\right)^{*} \\
&= \begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{-\sqrt{3}(1+i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix}^{*} \\
&= \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & \frac{-\sqrt{3}(1-i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & \frac{\sqrt{3}(i-1)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix} \qquad (2)
\end{aligned}
$$

Comparing (1,2) shows they are the same. Hence U is unitary.

A unimodular complex number z is one whose $|z| = 1$. For this example, we found above that $|U| = i$. But $|i| = 1$. Verified.

4.4.12 Problem 4(b) (8.4.8)

Show that if

$$L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then

$$L^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now consider $F(L) = e^{\theta L}$ and show by writing out the series and using $L^2 = -I$, that the series converges to a familiar matrix discussed earlier in the chapter.

Solution

$$
\begin{aligned}
e^{\theta L} &= I + \theta L + \frac{(\theta L)^2}{2!} + \frac{(\theta L)^3}{3!} + \cdots \\
&= I + \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2!}\theta^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 + \frac{1}{3!}\theta^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 + \cdots \qquad (1)
\end{aligned}
$$

But

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = -I$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -IL = -L$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = (-I)(-I) = I$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = IL = L$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^6 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = -I$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^7 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^6 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -L$$

And so on. Hence (1) becomes

$$
\begin{aligned}
e^{\theta L} &= I + \theta L - \frac{1}{2!}\theta^2 I - \frac{1}{3!}\theta^3 L + \frac{1}{4!}\theta^4 I + \frac{1}{5!}\theta^5 L - \frac{1}{6!}\theta^6 I - \frac{1}{7!}\theta^7 L + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\theta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3!}\theta^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4!}\theta^4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{5!}\theta^5 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{6!}\theta^6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \theta^2 & 0 \\ 0 & \theta^2 \end{bmatrix} - \frac{1}{3!} \begin{bmatrix} 0 & \theta^3 \\ \theta^3 & 0 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix} + \frac{1}{5!} \begin{bmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{bmatrix} - \frac{1}{6!} \begin{bmatrix} \theta^6 & 0 \\ 0 & \theta^6 \end{bmatrix} - \cdots \\
&= \begin{bmatrix} 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots & -\theta + \frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \cdots \\ \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots & 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
\end{aligned}
$$

Hence

$$e^{\theta L} = R_\theta^T$$

Where R_θ is the rotation matrix in 2D.

4.4.13 Problem 4(c) (8.4.10)

Show that if H is Hermitian, then $U = e^{iH}$ is unitary. (Write the exponential as a series and take the adjoint of each term and sum and re-exponentiate. Use the fact that exponents can be combined if only one matrix is in the picture).

solution

A matrix H is Hermitian if $H^{\dagger} = H$. Where the dagger means to take the transpose followed by conjugate. If H is real, then this implies the same as saying H is symmetric. A unitary matrix U means one whose dagger is same as its inverse. i.e.

$$U^{\dagger} = U^{-1}$$

Starting from the input given, expanding in Taylor series gives

$$\begin{aligned} U &= e^{iH} \\ &= I + iH + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \frac{(iH)^4}{4!} + \frac{(iH)^5}{5!} + \frac{(iH)^6}{6!} + \cdots \\ &= I + iH - \frac{H^2}{2!} - i\frac{H^3}{3!} + \frac{H^4}{4!} + i\frac{H^5}{5!} - \frac{H^6}{6!} \cdots \\ &= \left(I - \frac{H^2}{2!} + \frac{H^4}{4!} - \frac{H^6}{6!} \cdots\right) + i\left(H - \frac{H^3}{3!} + \frac{H^5}{5!} - \cdots\right) \end{aligned}$$

Hence

$$U^{\dagger} = \left(I^{\dagger} - \frac{H^{\dagger 2}}{2!} + \frac{H^{\dagger 4}}{4!} - \frac{H^{\dagger 6}}{6!} \cdots\right) - i\left(H^{\dagger} - \frac{H^{\dagger 3}}{3!} + \frac{H^{\dagger 5}}{5!} - \cdots\right)$$

Where the $+i$ changed to $-i$ in the above since we are taking complex conjugate. But $H^{\dagger} = H$ since matrix H is Hermitian. The above becomes

$$\begin{aligned} U^{\dagger} &= \left(I - \frac{H^2}{2!} + \frac{H^4}{4!} - \frac{H^6}{6!} \cdots\right) - i\left(H - \frac{H^3}{3!} + \frac{H^5}{5!} - \cdots\right) \\ &= e^{-iH} \end{aligned}$$

But $e^{-iH} = U^{-1}$ from definition of $U = e^{iH}$. Therefore

$$U^{\dagger} = U^{-1}$$

Hence U is unitary.

4.4.14 Problem 5 (8.4.17)

Show that

$$\left[\vec{\sigma} \cdot \vec{a}\right]\left[\vec{\sigma} \cdot \vec{b}\right] = \vec{a} \cdot \vec{b} I + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right) \tag{1}$$

Where $\vec{a}, \vec{b}$ are ordinary three dimensional vectors and

$$\vec{\sigma} = \vec{i}\sigma_x + \vec{j}\sigma_y + \vec{k}\sigma_z$$

solution

The LHS of (1) is

$$\begin{aligned} \left[\vec{\sigma} \cdot \vec{a}\right]\left[\vec{\sigma} \cdot \vec{b}\right] &= \left(\sigma_x a_x + \sigma_y a_y + \sigma_z a_z\right)\left(\sigma_x b_x + \sigma_y b_y + \sigma_z b_z\right) \\ &= \sigma_x^2 a_x b_x + \sigma_x \sigma_y a_x b_y + \sigma_x \sigma_z a_x b_z \\ &\quad + \sigma_y \sigma_x a_y b_x + \sigma_y^2 a_y b_y + \sigma_y \sigma_z a_y b_z \\ &\quad + \sigma_z \sigma_x a_z b_x + \sigma_z \sigma_y a_z b_y + \sigma_z^2 a_z b_z \end{aligned}$$

But for Pauli matrix $\sigma_i^2 = I$. Hence the above becomes

$$\left[\vec{\sigma} \cdot \vec{a}\right]\left[\vec{\sigma} \cdot \vec{b}\right] = I\left(a_x b_x + a_y b_y + a_z b\right) + \sigma_x \sigma_y a_x b_y + \sigma_x \sigma_z a_x b_z + \sigma_y \sigma_x a_y b_x + \sigma_y \sigma_z a_y b_z + \sigma_z \sigma_x a_z b_x + \sigma_z \sigma_y a_z b_y$$

But $\sigma_y\sigma_x = -\sigma_x\sigma_y$ and $\sigma_x\sigma_z = -\sigma_z\sigma_x$ and $\sigma_z\sigma_y = -\sigma_y\sigma_z$. (I verified these by working them out). Hence the above becomes

$$\begin{aligned}\left[\vec{\sigma}\cdot\vec{a}\right]\left[\vec{\sigma}\cdot\vec{b}\right] &= I\left(a_xb_x + a_yb_y + a_zb\right) + \sigma_x\sigma_ya_xb_y + \sigma_x\sigma_za_xb_z - \sigma_x\sigma_ya_yb_x + \sigma_y\sigma_za_yb_z - \sigma_x\sigma_za_zb_x - \sigma_y\sigma_za_zb_y\\ &= I\left(a_xb_x + a_yb_y + a_zb\right) + \left(\sigma_x\sigma_y\right)\left(a_xb_y - a_yb_x\right) + (\sigma_x\sigma_z)(a_xb_z - a_zb_x) + \left(\sigma_y\sigma_z\right)\left(a_yb_z - a_zb_y\right)\end{aligned} \tag{2}$$

Now we will simplify RHS of (1) and see if we get the same result as above.

$$\begin{aligned}\vec{a}\cdot\vec{b}I + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right) &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right)\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\begin{pmatrix}\sigma_x & \sigma_y & \sigma_z\end{pmatrix}\cdot\begin{vmatrix}e_i & e_j & e_k\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\begin{pmatrix}\sigma_x & \sigma_y & \sigma_z\end{pmatrix}\cdot\begin{pmatrix}a_yb_z - a_zb_y & -(a_xb_z - a_zb_x) & a_xb_y - a_yb_x\end{pmatrix}\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\begin{pmatrix}\sigma_x & \sigma_y & \sigma_z\end{pmatrix}\cdot\begin{pmatrix}a_yb_z - a_zb_y & a_zb_x - a_xb_z & a_xb_y - a_yb_x\end{pmatrix}\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\left(\sigma_x\left(a_yb_z - a_zb_y\right) + \sigma_y(a_zb_x - a_xb_z) + \sigma_z\left(a_xb_y - a_yb_x\right)\right)\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + i\sigma_x\left(a_yb_z - a_zb_y\right) + i\sigma_y(a_zb_x - a_xb_z) + i\sigma_z\left(a_xb_y - a_yb_x\right)\end{aligned} \tag{3}$$

But from property of Pauli matrices (eq 8.4.48) in text, we have (Verified these by working them out)

$$i\sigma_z = \sigma_x\sigma_y \tag{4}$$
$$i\sigma_x = \sigma_y\sigma_z \tag{5}$$
$$-i\sigma_y = \sigma_x\sigma_z \tag{6}$$

Substituting (4,5,6) into (3) gives

$$\begin{aligned}\vec{a}\cdot\vec{b}I + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right) &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + \left(\sigma_y\sigma_z\right)\left(a_yb_z - a_zb_y\right) - (\sigma_x\sigma_z)(a_zb_x - a_xb_z) + \left(\sigma_x\sigma_y\right)\left(a_xb_y - a_yb_x\right)\\ &= I\left(a_xb_x + a_yb_y + a_zb_z\right) + \left(\sigma_y\sigma_z\right)\left(a_yb_z - a_zb_y\right) + (\sigma_x\sigma_z)(a_xb_z - a_zb_x) + \left(\sigma_x\sigma_y\right)\left(a_xb_y - a_yb_x\right)\end{aligned} \tag{7}$$

Comparing (2,7) shows they are the same. Hence

$$\left[\vec{\sigma}\cdot\vec{a}\right]\left[\vec{\sigma}\cdot\vec{b}\right] = \vec{a}\cdot\vec{b}I + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right)$$

4.4.15 Problem 6

(a) Consider a horizontal spring-mass system. The spring has a spring constant k and is fixed at one end. The other end is attached to a block of mass m that can move without friction on a horizontal surface. The spring is stretched a length a beyond its rest length and let go. Without solving the problem using Newton's second law, find the angular frequency of oscillations and show that it is independent of a.

(b) Derive the Planck mass, length, and time in terms of Planck's constant $\hbar$, Newton's constant G, and speed of light c. Evaluate these quantities in SI units. (10 points)

(c) Identify the relevant physical quantities and use dimensional analysis to find the characteristic length for a black hole of mass M.

solution

4.4.15.1 Part (a)

I was not sure if we are supposed to solve this using dimensional analysis or using Physics. So I solved it both ways. Please select the method that we are supposed to have used.

Using physics

Taking the relaxed position (which is also the equilibrium position as $x = 0$) and spring extension is measured relative to this, then spring potential energy is given by $V(x) = \frac{1}{2}kx^2$ and the Force the spring exerts on the mass is $F = -kx$. Using the relation

$$V'(x) = m\omega^2 x$$

Then

$$\begin{aligned} kx &= m\omega^2 x \\ k &= m\omega^2 \end{aligned}$$

Hence

$$\omega = \sqrt{\frac{k}{m}}$$

Where m is the mass of the block attached to the spring. We see the angular frequency of oscillations ω is independent of a. The mass will oscillated around $x = 0$ from $x = +a$ to $x = -a$. When it is at $x = \pm a$ the force on the mass will be maximum of $F = -ka$ and the velocity will be zero there. When the mass at $x = 0$, the force is zero, but the velocity of mass will be largest there. The maximum amplitude of the mass from equilibrium is a.

Using dimensional analysis

Let us assume that the angular frequency of the spring depends on the attached mass m and on the spring constant k and on the initial displacement a (we will find later that it does not depend on a).

The units of angular frequency ω is radians per second or T^{-1}. Units of mass m is M. Units of k are MT^{-2} (force per unit length). And initial extension is length with units L. Hence assuming

$$\omega = m^x k^y a^z \tag{1}$$

Using dimensional analysis we replace the above with the units of each physical quantity which gives

$$\begin{aligned} T^{-1} &= [M]^x \left[MT^{-2}\right]^y [L]^z \\ &= M^{x+y} T^{-2y} L^z \end{aligned}$$

Comparing exponents gives

$$\begin{aligned} -2y &= -1 \\ x + y &= 0 \\ z &= 0 \end{aligned}$$

Hence $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ and $z = 0$. Therefore Eq. (1) becomes

$$\begin{aligned} \omega &= m^{-\frac{1}{2}} k^{\frac{1}{2}} \\ &= \sqrt{\frac{k}{m}} \end{aligned}$$

Which is the same result obtained above. This shows that ω does not depend on a, because $z = 0$.

4.4.15.2 Part (b)

Plank mass

Using dimensional analysis, let m_p be the Planck mass. Using units M, L, T for mass, length and time respectively, then the units of m_p is M. Since we want m_p to be expressed in terms of $\hbar, G, c$, then we write

$$m_p = \hbar^x G^y c^z \tag{1}$$

And then solve for x, y, z exponents such that RHS gives units of M. We know that units of $\hbar = ML^2T^{-1}$ and units of $G = M^{-1}L^3T^{-2}$ and units of $c = LT^{-1}$. The above becomes

$$\begin{aligned} M &= \left(ML^2T^{-1}\right)^x \left(M^{-1}L^3T^{-2}\right)^y \left(LT^{-1}\right)^z \\ &= M^x L^{2x} T^{-x} M^{-y} L^{3y} T^{-2y} L^z T^{-z} \\ &= M^{x-y} L^{2x+3y+z} T^{-x-2y-z} \end{aligned}$$

Therefore we need to satisfy the following equations

$$\begin{aligned} x - y &= 1 \\ 2x + 3y + z &= 0 \\ -x - 2y - z &= 0 \end{aligned}$$

Or

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & -2 & -1 & 0 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 5 & 1 & -2 \\ -1 & -2 & -1 & 0 \end{pmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 5 & 1 & -2 \\ 0 & -3 & -1 & 1 \end{pmatrix}$$

$R_2 = 3R_2, R_3 = 5R_3$

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 15 & 3 & -6 \\ 0 & -15 & -5 & 5 \end{pmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 15 & 3 & -6 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

System is now in echelon form, so no more transformations are needed. The system becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 15 & 3 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$$

Last row give $-2z=-1$ or $z=\frac{1}{2}$. Second row gives $15y+3z=-6$ or $15y+3\left(\frac{1}{2}\right)=-6$, or $y=-\frac{1}{2}$ and first row gives $x-y=1$ or $x+\frac{1}{2}=1$, hence $x=\frac{1}{2}$. The solution is

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2} \end{pmatrix} \tag{2}$$

Using (2) in (1) gives

$$\begin{aligned} m_p &= \hbar^x G^y c^z\\ &= \hbar^{\frac{1}{2}} G^{-\frac{1}{2}} c^{\frac{1}{2}}\\ &= \sqrt{\frac{\hbar c}{G}} \end{aligned}$$

Units in SI Using $c = 299792458$ m/s and $\hbar = 1.054571817\times 10^{-34}$ J.s, and $G = 6.6743015\times 10^{-11}$ $\text{m}^3\text{kg}^{-1}\text{s}^{-2}$, the above gives

$$\begin{aligned} m_p &= \sqrt{\frac{\left(1.054571817\times 10^{-34}\right)(299792458)}{\left(6.6743015\times 10^{-11}\right)}}\\ &= 2.1764\times 10^{-8}\text{ kg} \end{aligned}$$

Planck length

We now repeat the above method, but for Planck length which has units L. Therefore the equation is

$$l_p = \hbar^x G^y c^z \tag{3}$$

And now we solve for x, y, z exponents such that RHS gives units of L. We know that units of $\hbar = ML^2T^{-1}$ and units of $G = M^{-1}L^3T^{-2}$ and units of $c = LT^{-1}$. Using dimensional analysis, the above becomes

$$\begin{aligned} L &= \left(ML^2T^{-1}\right)^x \left(M^{-1}L^3T^{-2}\right)^y \left(LT^{-1}\right)^z\\ &= M^x L^{2x} T^{-x} M^{-y} L^{3y} T^{-2y} L^z T^{-z}\\ L &= M^{x-y} L^{2x+3y+z} T^{-x-2y-z} \end{aligned}$$

Therefore we need to satisfy the following equations

$$\begin{aligned} x-y &= 0\\ 2x+3y+z &= 1\\ -x-2y-z &= 0 \end{aligned}$$

Similar steps using augmented matrix will now be done. No need to duplicate these again. The final solution is

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\\ \frac{1}{2}\\ -\frac{3}{2} \end{pmatrix} \tag{4}$$

Using (4) in (3) gives

$$\begin{aligned} l_p &= \hbar^x G^y c^z\\ &= \hbar^{\frac{1}{2}} G^{\frac{1}{2}} c^{-\frac{3}{2}}\\ &= \sqrt{\frac{\hbar G}{c^3}} \end{aligned}$$

Units in SI Using $c = 299792458$ m/s and $\hbar = 1.054571817 \times 10^{-34}$ J.s, and $G = 6.6743015 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, the above gives

$$\begin{aligned}
l_p &= \sqrt{\frac{\left(1.054571817 \times 10^{-34}\right)\left(6.6743015 \times 10^{-11}\right)}{(299792458)^3}} \\
&= 1.6163 \times 10^{-35} \text{ meter}
\end{aligned}$$

Planck time

We now repeat the above method, but for Planck time which has units T. Therefore the equation is

$$t_p = \hbar^x G^y c^z \tag{5}$$

And now solve for x, y, z exponents such that RHS gives units of T. We know that units of $\hbar = ML^2T^{-1}$ and units of $G = M^{-1}L^3T^{-2}$ and units of $c = LT^{-1}$. The above becomes

$$\begin{aligned}
T &= \left(ML^2T^{-1}\right)^x \left(M^{-1}L^3T^{-2}\right)^y \left(LT^{-1}\right)^z \\
&= M^x L^{2x} T^{-x} M^{-y} L^{3y} T^{-2y} L^z T^{-z} \\
T &= M^{x-y} L^{2x+3y+z} T^{-x-2y-z}
\end{aligned}$$

Therefore we need to satisfy the following equations

$$\begin{aligned}
x - y &= 0 \\
2x + 3y + z &= 0 \\
-x - 2y - z &= 1
\end{aligned}$$

Similar steps using augmented matrix will now be done. No need to duplicate these again. The final solution came out to be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{6}$$

Using (5) in (6) gives

$$\begin{aligned}
t_p &= \hbar^x G^y c^z \\
&= \hbar^{\frac{1}{2}} G^{\frac{1}{2}} c^{-\frac{5}{2}} \\
&= \sqrt{\frac{\hbar G}{c^5}}
\end{aligned}$$

Units in SI Using $c = 299792458$ m/s and $\hbar = 1.054571817 \times 10^{-34}$ J.s, and $G = 6.6743015 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, the above gives

$$\begin{aligned}
t_p &= \sqrt{\frac{\left(1.054571817 \times 10^{-34}\right)\left(6.6743015 \times 10^{-11}\right)}{(299792458)^5}} \\
&= 5.3912 \times 10^{-44} \text{ second}
\end{aligned}$$

4.4.15.3 Part (c)

The characteristic length of a black hole should depend on its mass M and universal gravitational constant G and c. Therefore

$$L_c = M^x G^y c^z$$

The units of $G = M^{-1}L^3T^{-2}$ and units of $c = LT^{-1}$. The above becomes

$$\begin{aligned} L_c &= M^x \left(M^{-1}L^3T^{-2}\right)^y \left(LT^{-1}\right)^z \\ &= M^{x-y}L^{3y+z}T^{-2y-z} \end{aligned}$$

Hence

$$\begin{aligned} x - y &= 0 \\ 3y + z &= 1 \\ 2y + z &= 0 \end{aligned}$$

Solving gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Hence

$$L_c = \frac{MG}{c^2}$$

4.4.16 key solution for HW 4

Physics 3041 (Spring 2021) Solutions to Homework Set 4

1. (a) Problem 8.1.1. (5 points)

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \;\Rightarrow\; R_{\theta'}R_\theta = \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta'\cos\theta - \sin\theta'\sin\theta & \cos\theta'\sin\theta + \sin\theta'\cos\theta \\ -\sin\theta'\cos\theta - \cos\theta'\sin\theta & -\sin\theta'\sin\theta + \cos\theta'\cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta+\theta') & \sin(\theta+\theta') \\ -\sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} = R_{\theta+\theta'}.$$

(b) Problem 8.1.2, and find the expression of θ in terms of the relative velocity. (10 points)

From the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1-v^2}} = x\cosh\theta - t\sinh\theta,$$
$$t' = \frac{t - vx}{\sqrt{1-v^2}} = t\cosh\theta - x\sinh\theta,$$

we obtain

$$\cosh\theta = \frac{1}{\sqrt{1-v^2}},\ \sinh\theta = \frac{v}{\sqrt{1-v^2}} \Rightarrow \frac{\sinh\theta}{\cosh\theta} = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}} = \frac{\eta - \eta^{-1}}{\eta + \eta^{-1}} = \frac{\eta^2 - 1}{\eta^2 + 1} = v,$$
$$\Rightarrow \eta^2 - 1 = (\eta^2+1)v,\ \eta = e^\theta = \sqrt{\frac{1+v}{1-v}},$$
$$\Rightarrow \theta = \ln\sqrt{\frac{1+v}{1-v}} = \frac{1}{2}\ln\frac{1+v}{1-v},$$

where v is in units of the speed of light c.

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \begin{bmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{bmatrix}\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{bmatrix}\begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix}$$
$$= \begin{bmatrix} \cosh\theta'\cosh\theta + \sinh\theta'\sinh\theta & -\cosh\theta'\sinh\theta - \sinh\theta'\cosh\theta \\ -\sinh\theta'\cosh\theta - \cosh\theta'\sinh\theta & \sinh\theta'\sinh\theta + \cosh\theta'\cosh\theta \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix}$$
$$= \begin{bmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix},$$

where we have used

$$\cosh\theta'\cosh\theta + \sinh\theta'\sinh\theta = \frac{(e^{\theta'} + e^{-\theta'})(e^\theta + e^{-\theta})}{4} + \frac{(e^{\theta'} - e^{-\theta'})(e^\theta - e^{-\theta})}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{\theta'-\theta} + e^{-\theta'+\theta} + e^{-\theta'-\theta}}{4} + \frac{e^{\theta'+\theta} - e^{\theta'-\theta} - e^{-\theta'+\theta} + e^{-\theta'-\theta}}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{-\theta'-\theta}}{2} = \cosh(\theta+\theta'),$$
$$\sinh\theta'\cosh\theta + \cosh\theta'\sinh\theta = \frac{(e^{\theta'} - e^{-\theta'})(e^\theta + e^{-\theta})}{4} + \frac{(e^{\theta'} + e^{-\theta'})(e^\theta - e^{-\theta})}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{\theta'-\theta} - e^{-\theta'+\theta} - e^{-\theta'-\theta}}{4} + \frac{e^{\theta'+\theta} - e^{\theta'-\theta} + e^{-\theta'+\theta} - e^{-\theta'-\theta}}{4}$$
$$= \frac{e^{\theta'+\theta} - e^{-\theta'-\theta}}{2} = \sinh(\theta+\theta').$$

(c) Problem 8.2.4, for Lorentz transformation only. (5 points)

For the Lorentz transformation,

$$L_\theta = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}, \; |L_\theta| = \cosh^2\theta - \sinh^2\theta = \frac{(e^\theta + e^{-\theta})^2}{4} - \frac{(e^\theta - e^{-\theta})^2}{4} = 1,$$

$$L_{\theta,C} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \Rightarrow L_\theta^{-1} = \frac{L_{\theta,C}^T}{|L_\theta|} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = L_{-\theta}.$$

The above result makes sense as the inverse Lorentz transformation corresponds to changing the sign of the relative velocity $v \to -v$, which in turn changes the sign of the rapidity $\theta = \frac{1}{2}\ln\frac{1+v}{1-v} \to -\theta = \frac{1}{2}\ln\frac{1-v}{1+v}$.

2. (a) Problem 8.3.4, but using Cramer's rule to solve the first set of equations only. (5 points)

$$\begin{aligned} 3x - y - z &= 2 \\ x - 2y - 3z &= 0 \\ 4x + y + 2z &= 4 \end{aligned}$$

$$\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} = 3 \times (-1) + 14 - 9 = 2$$

$$x = \frac{\begin{vmatrix} 2 & -1 & -1 \\ 0 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(2\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} -1 & -1 \\ -2 & -3 \end{vmatrix}\right) = \frac{-2+4}{2} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -3 \\ 4 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(-2\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}\right) = \frac{-2 \times 14 - 4 \times (-8)}{2} = 2$$

$$z = \frac{\begin{vmatrix} 3 & -1 & 2 \\ 1 & -2 & 0 \\ 4 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}\right) = \frac{2 \times 9 + 4 \times (-5)}{2} = -1$$

(b) Problem 8.3.5. (5 points)

$$M=\begin{bmatrix}1&2&3\\4&5&6\\7&8&10\end{bmatrix}\Rightarrow M_C=\begin{bmatrix}2&2&-3\\4&-11&6\\-3&6&-3\end{bmatrix},\ |M|=1\times 2+2\times 2+3\times(-3)=-3,$$

$$M^{-1}=\frac{M_C^T}{|M|}=-\frac{1}{3}\begin{bmatrix}2&4&-3\\2&-11&6\\-3&6&-3\end{bmatrix}=\begin{bmatrix}-2/3&-4/3&1\\-2/3&11/3&-2\\1&-2&1\end{bmatrix},$$

$$M^{-1}M=\begin{bmatrix}-2/3&-4/3&1\\-2/3&11/3&-2\\1&-2&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&10\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

3. (a) Problem 8.4.3. (5 points)

$$(MN)_{ij}=\sum_k M_{ik}N_{kj}$$

$$\Rightarrow (MN)^\dagger_{ij}=(MN)^*_{ji}=\sum_k M^*_{jk}N^*_{ki}=\sum_k M^\dagger_{kj}N^\dagger_{ik}=\sum_k N^\dagger_{ik}M^\dagger_{kj}=(N^\dagger M^\dagger)_{ij}$$

$$\Rightarrow (MN)^\dagger=N^\dagger M^\dagger$$

For $M^\dagger=M$ and $N^\dagger=N$, we have

$$(MN)^\dagger=N^\dagger M^\dagger=NM.$$

If $NM=MN$, i.e., M and N commute, then $(MN)^\dagger=MN$, i.e., MN is Hermitian. Otherwise, MN is not Hermitian.

(b) Problem 8.4.19, proving the first result only. (5 points)

$$\begin{aligned}\text{Tr } MN&=\sum_i (MN)_{ii}=\sum_i\sum_k M_{ik}N_{ki}\\&=\sum_i\sum_k N_{ki}M_{ik}=\sum_k\sum_i N_{ki}M_{ik}\\&=\sum_k (NM)_{kk}=\text{Tr } NM\end{aligned}$$

(c) Problem 8.4.20. (10 points)

From the given properties of the Dirac matrices, $M_i^2=I$, and for $i\neq j$,

$$M_iM_j+M_jM_i=0\Rightarrow M_i^2M_j+M_iM_jM_i=M_j+M_iM_jM_i=0,$$

where we have multiplied both sides of the first equality by M_i. Taking trace of the two sides of the last equality, we obtain

$$\begin{aligned}\text{Tr }(M_j+M_iM_jM_i)&=\text{Tr } M_j+\text{Tr } M_iM_jM_i=\text{Tr } M_j+\text{Tr } M_jM_i^2\\&=\text{Tr } M_j+\text{Tr } M_j=2\text{Tr } M_j=0\Rightarrow \text{Tr } M_j=0,\end{aligned}$$

where we have used $\text{Tr } ABC=\text{Tr } BCA$.

4. (a) Problem 8.4.5. (10 points)

$$U = \begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix} \Rightarrow U^\dagger = \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix}$$

$$UU^\dagger = \begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As $(U^\dagger)_{ij} = [(U^T)_{ij}]^*$ and $|U| = |U^T|$, we have $|U^\dagger| = |U|^*$. Let $|U| = re^{i\theta}$, so $|U^\dagger| = re^{-i\theta}$.

$$|UU^\dagger| = |U||U^\dagger| = r^2 = |I| = 1 \Rightarrow r = 1 \Rightarrow |U| = e^{i\theta}$$

Note here $|\ |$ means the determinant of a matrix, **NOT** the modulus of a complex number. For the above example,

$$\begin{aligned} |U| &= \left(\frac{1+i\sqrt{3}}{4}\right)\frac{i+\sqrt{3}}{4} - \frac{\sqrt{3}(1+i)}{2\sqrt{2}}\left[-\frac{\sqrt{3}(1+i)}{2\sqrt{2}}\right] = i\left(\frac{-i+\sqrt{3}}{4}\right)\frac{i+\sqrt{3}}{4} + \frac{3(1+i)^2}{8} \\ &= \frac{i}{4} + \frac{3i}{4} = i = e^{i\pi/2} \end{aligned}$$

$$\begin{aligned} |U^\dagger| &= \left(\frac{1-i\sqrt{3}}{4}\right)\frac{-i+\sqrt{3}}{4} - \frac{\sqrt{3}(1-i)}{2\sqrt{2}}\left[-\frac{\sqrt{3}(1-i)}{2\sqrt{2}}\right] = -i\left(\frac{i+\sqrt{3}}{4}\right)\frac{-i+\sqrt{3}}{4} + \frac{3(1-i)^2}{8} \\ &= -\frac{i}{4} - \frac{3i}{4} = -i = e^{-i\pi/2} = |U|^* \end{aligned}$$

(b) Problem 8.4.8. (10 points)

$$L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow L^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$\begin{aligned} F(L) = e^{\theta L} &= \sum_{n=0}^{\infty} \frac{(\theta L)^n}{n!} = I - \frac{\theta^2}{2!}I + \frac{\theta^4}{4!}I - \cdots + \left(\theta L - \frac{\theta^3}{3!}L + \frac{\theta^5}{5!}L - \cdots\right) \\ &= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + L\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = I\cos\theta + L\sin\theta \\ &= \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$

(c) Problem 8.4.10. (5 points)

$$\begin{aligned} e^{iH} &= I + iH + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \cdots + \frac{(iH)^n}{n!} + \cdots, \quad H^\dagger = H \\ \left(e^{iH}\right)^\dagger &= I - iH^\dagger + \frac{(-iH^\dagger)^2}{2!} + \frac{(-iH^\dagger)^3}{3!} + \cdots + \frac{(-iH^\dagger)^n}{n!} + \cdots \\ &= I - iH + \frac{(-iH)^2}{2!} + \frac{(-iH)^3}{3!} + \cdots + \frac{(-iH)^n}{n!} + \cdots = e^{-iH} \\ e^{iH}\left(e^{iH}\right)^\dagger &= e^{iH}e^{-iH} = I \Rightarrow e^{iH} \text{ is unitary} \end{aligned}$$

5. Problem 8.4.17. (5 points)

$$\begin{aligned}(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= \left(\sum_{i=1}^{3}\sigma_i a_i\right)\left(\sum_{j=1}^{3}\sigma_j b_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \sigma_i \sigma_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \left(I\delta_{ij} + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k\right) \\ &= I\sum_{i=1}^{3} a_i b_i + i\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk} a_i b_j \left(\hat{e}_k\cdot\vec{\sigma}\right) \\ &= \vec{a}\cdot\vec{b}I + i\left(\vec{a}\times\vec{b}\right)\cdot\vec{\sigma} = \vec{a}\cdot\vec{b}I + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right),\end{aligned}$$

where we have used

$$\vec{a}\times\vec{b} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk} a_i b_j \hat{e}_k.$$

6. (a) Consider a horizontal spring-mass system. The spring has a spring constant k and is fixed at one end. The other end is attached to a block of mass m that can move without friction on a horizontal surface. The spring is stretched a length a beyond its rest length and let go. Without solving the problem using Newton's second law, find the angular frequency of oscillations and show that it is independent of a. (5 points)

Using units to indicate dimensions, we have

$$\begin{aligned}[k] &= \mathrm{N/meter} = \mathrm{kg}\cdot(\mathrm{meter/s^2})/\mathrm{meter} = \mathrm{kg/s^2}, \\ [m] &= \mathrm{kg},\ \ [a] = \mathrm{meter}.\end{aligned}$$

$$\begin{aligned}[\omega] = 1/\mathrm{s} &= [k]^\alpha[m]^\beta[a]^\gamma = \mathrm{kg}^{\alpha+\beta}\mathrm{meter}^\gamma/\mathrm{s}^{2\alpha} \\ \alpha+\beta = 0,\ \ 2\alpha &= 1, \gamma = 0 \Rightarrow \alpha = 1/2,\ \ \beta = -1/2,\ \ \gamma = 0 \\ [\omega] &= [k]^{1/2}[m]^{-1/2} = \left[\sqrt{k/m}\right].\end{aligned}$$

Therefore, the angular frequency ω is independent of a.

(b) Derive the Planck mass, length, and time in terms of Planck's constant $\hbar$, Newton's constant G, and speed of light c. Evaluate these quantities in SI units. (10 points)

Using units to indicate dimensions, we have

$$\begin{aligned}[\hbar] &= \mathrm{J}\cdot\mathrm{s} = \mathrm{kg}\cdot(\mathrm{m/s})^2\cdot\mathrm{s} = \mathrm{kg}\cdot\mathrm{m^2/s}, \\ [G] &= \mathrm{N}\cdot\mathrm{m^2/kg^2} = \mathrm{kg}\cdot(\mathrm{m/s^2})\cdot\mathrm{m^2/kg^2} = \mathrm{m^3/(kg\cdot s^2)}, \\ [c] &= \mathrm{m/s}.\end{aligned}$$

$$\begin{aligned}[M_{\mathrm{Pl}}] &= \mathrm{kg} = [\hbar]^\alpha[G]^\beta[c]^\gamma = \mathrm{kg}^{\alpha-\beta}\cdot\mathrm{m}^{2\alpha+3\beta+\gamma}/\mathrm{s}^{\alpha+2\beta+\gamma}, \\ \alpha-\beta &= 1,\ \ 2\alpha+3\beta+\gamma = 0,\ \ \alpha+2\beta+\gamma = 0 \Rightarrow \alpha = 1/2,\ \ \beta = -1/2,\ \ \gamma = 1/2.\end{aligned}$$

So the Planck mass is

$$M_{\rm Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-8}\ {\rm kg}.$$

With $[M_{\rm Pl}c^2] = {\rm J}$, it is straightforward to obtain the Planck time and length

$$T_{\rm Pl} = \frac{\hbar}{M_{\rm Pl}c^2} = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44}\ {\rm s},$$
$$L_{\rm Pl} = cT_{\rm Pl} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-35}\ {\rm m}.$$

(c) Identify the relevant physical quantities and use dimensional analysis to find the characteristic length for a black hole of mass M. (5 points)

The relevant physical quantities are Newton's constant G, the speed of light c, and the black-hole mass M, the first two of which are fundamental to general relativity and the last of which specifies the macroscopic property of the black hole.

Using units to denote the dimenions, we have

$$[G] = {\rm N} \cdot {\rm m}^2/{\rm kg}^2 = {\rm kg} \cdot ({\rm m/s}^2) \cdot {\rm m}^2/{\rm kg}^2 = {\rm m}^3/({\rm s}^2 \cdot {\rm kg}),$$
$$[c] = {\rm m/s},\ [M] = {\rm kg}.$$

$$[{\rm length}] = {\rm m} = [G]^\alpha [c]^\beta [M]^\gamma = {\rm m}^{3\alpha+\beta} \cdot {\rm s}^{-2\alpha-\beta} \cdot {\rm kg}^{-\alpha+\gamma},$$
$$3\alpha + \beta = 1,\ -2\alpha - \beta = 0,\ -\alpha + \gamma = 0 \Rightarrow \alpha = 1,\ \beta = -2,\ \gamma = 1.$$

So we obtain

$$[{\rm length}] = \frac{GM}{c^2}.$$

4.5 HW 5

Local contents

4.5.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 5 **(Due 3/3)**

1. (a) Problem 9.1.6. (5 points)

(b) Problem 9.2.1.(ii). (10 points)

(c) Problem 9.2.3. (10 points)

2. Use Tr $\sigma_i = 0$, $\sigma_i^2 = I$, and $\sigma_i \sigma_j = i \sum_k \epsilon_{ijk} \sigma_k$ to obtain the components of a general 2×2 matrix in the basis of $\{\sigma_1, \sigma_2, \sigma_3, I\}$, where σ_i represents the Pauli matrices and I is the identity matrix. (15 points)

3. Problem 9.2.5. (10 points)

4. Problem 9.3.5. (20 points)

5. Problem 9.5.6, but only for the proof without doing the inverse matrix part. (10 points)

6. Problem 9.5.10. (20 points)

4.5.2 Problem 1 a (9.1.6)

Show that the following row vectors are linearly dependent. $\begin{pmatrix}1 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\end{pmatrix}, \begin{pmatrix}3 & 2 & 1\end{pmatrix}$. Show the opposite for $\begin{pmatrix}1 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 1\end{pmatrix}$.

Solution

4.5.2.1 Part 1

Vectors $\vec{V}_1, \vec{V}_2, \vec{V}_2$ are Linearly dependent if we can find a, b, c not all zero, such that

$$a\vec{V}_1 + b\vec{V}_2 + c\vec{V}_2 = \vec{0}$$

Applying the above to the vectors we are given gives

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\begin{pmatrix}1 & 1 & 3\\1 & 0 & 2\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$Ax = 0 \tag{1}$$

One way is to find $\det(A)$. If $\det(A) = 0$ then there exists non-trivial solution x. Which means linearly dependent, otherwise linearly independent.

$$\det(A) = 1\begin{vmatrix}0 & 2\\1 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix} + 3\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = -2 - 1 + 3 = 0$$

Since $\det(A) = 0$ then linearly dependent.

Another method is to actually solve for $\begin{pmatrix}a\\b\\c\end{pmatrix}$ to see if we can obtain non zero solution or not. Using Gaussian elimination

$R_2 = R_2 - R_1$

$$\begin{pmatrix}1 & 1 & 3\\0 & -1 & -1\\0 & 1 & 1\end{pmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{pmatrix}1 & 1 & 3\\0 & -1 & -1\\0 & 0 & 0\end{pmatrix}$$

Hence the system becomes

$$\begin{pmatrix}1 & 1 & 3\\0 & -1 & -1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Last row show that c is free variable. Hence it can be any value. Second row gives $-b-c=0$ or $b=-c$. First row gives $a+b+3c=0$ or $a=-b-3c=c-3c=-2c$. Therefore the solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ -c \\ c \end{pmatrix}$$
$$= c\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

There are infinite number of solutions. Let $c=1$. Hence one solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

Since we found a,b,c not all zero which makes $a\vec{V}_1 + b\vec{V}_2 + c\vec{V}_2 = \vec{0}$, then the vectors are Linearly dependent .

4.5.2.2 Part 2

Vectors $\vec{V}_1, \vec{V}_2, \vec{V}_2$ are Linearly independent if the only solution to

$$a\vec{V}_1 + b\vec{V}_2 + c\vec{V}_2 = \vec{0}$$

is when $a=b=c=0$. As in part 1, we setup $Ax=0$ system and solve it to find out.

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$Ax = 0 \tag{2}$$

One way is to find $\det(A)$. If $\det(A)=0$ then there exists non-trivial solution x. Which means linearly dependent, otherwise linearly independent.

$$\det(A) = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1-1 = -2$$

Since $\det(A) \neq 0$ then linearly independent

Another method is to solve (2) directly. Using Gaussian elimination gives

$R_2 = R_2 - R_1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence the system becomes

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row gives $c = 0$. Second row gives $-b + c = 0$ or $b = 0$. First row gives $a + b = 0$ or $a = 0$. Hence the solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore $a\vec{V}_1 + b\vec{V}_2 + c\vec{V}_2 = \vec{0}$ implies that $a = b = c = 0$, then the vectors are Linearly independent .

4.5.3 Problem 1 b (9.2.1 (ii))

Repeat the above calculation of expanding the vector in Eqn (9.2.32) but in the following basis, after first demonstrating its orthonormality. At the end check that the norm squared of the vector comes out to be 6.

$$|I\rangle = \begin{pmatrix} \frac{1+i\sqrt{3}}{4} \\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix}$$

$$|II\rangle = \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix}$$

The vector is

$$|V\rangle = \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} \tag{9.2.32}$$

Solution

First we need to check the basis given are orthogonal to each others, and each have norm of 1 each. To check orthogonality

$$\begin{aligned}
\langle I|II\rangle &= \begin{pmatrix} \frac{1+i\sqrt{3}}{4} & -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix}^* \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix} \\
&= \left(\frac{\left(1-i\sqrt{3}\right)}{4}\right)\left(\frac{\sqrt{3}(1+i)}{\sqrt{8}}\right) + \left(-\frac{\sqrt{3}(1-i)}{\sqrt{8}}\right)\left(\frac{\sqrt{3}+i}{4}\right) \\
&= \frac{\left(1-i\sqrt{3}\right)\left(\sqrt{3}+i\sqrt{3}\right)}{4\sqrt{8}} - \frac{\left(\sqrt{3}-\sqrt{3}i\right)\left(\sqrt{3}+i\right)}{4\sqrt{8}} \\
&= \frac{\sqrt{3}+i\sqrt{3}-3i+3}{4\sqrt{8}} - \frac{3+\sqrt{3}i-3i+\sqrt{3}}{4\sqrt{8}} \\
&= 0
\end{aligned}$$

Since dot product is zero, then they are orthogonal to each others. To check the norm

$$
\begin{aligned}
\langle I|I\rangle &= \begin{pmatrix} \frac{1+i\sqrt{3}}{4} & -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix}^* \begin{pmatrix} \frac{1+i\sqrt{3}}{4} \\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}} \end{pmatrix} \begin{pmatrix} \frac{1+i\sqrt{3}}{4} \\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix} \\
&= \left(\frac{\left(1-i\sqrt{3}\right)}{4}\right)\left(\frac{\left(1+i\sqrt{3}\right)}{4}\right) + \left(-\frac{\sqrt{3}(1-i)}{\sqrt{8}}\right)\left(-\frac{\sqrt{3}(1+i)}{\sqrt{8}}\right) \\
&= \frac{\left(1-i\sqrt{3}\right)\left(1+i\sqrt{3}\right)}{16} + \frac{\left(\sqrt{3}-i\sqrt{3}\right)\left(\sqrt{3}+i\sqrt{3}\right)}{8} \\
&= \frac{1+3}{16} + \frac{3+3}{8} \\
&= \frac{4}{16} + \frac{6}{8} \\
&= 1
\end{aligned}
$$

Since $\langle I|I\rangle = \|I\|^2$ then $\|I\|^2 = 1$ which means $\|I\| = 1$. Now we do the same for the second basis

$$
\begin{aligned}
\langle II|II\rangle &= \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} & \frac{\sqrt{3}+i}{4} \end{pmatrix}^* \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}-i\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}-i}{4} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix} \\
&= \left(\frac{\sqrt{3}-i\sqrt{3}}{\sqrt{8}}\right)\left(\frac{\sqrt{3}(1+i)}{\sqrt{8}}\right) + \left(\frac{\sqrt{3}-i}{4}\right)\left(\frac{\sqrt{3}+i}{4}\right) \\
&= \frac{\left(\sqrt{3}-i\sqrt{3}\right)\left(\sqrt{3}+i\sqrt{3}\right)}{8} + \frac{\left(\sqrt{3}-i\right)\left(\sqrt{3}+i\right)}{16} \\
&= \frac{3+3}{8} + \frac{3+1}{16} \\
&= \frac{6}{8} + \frac{4}{16} \\
&= 1
\end{aligned}
$$

which means $\|II\| = 1$. We finished showing the basis are orthonormal. Now we express the vector $|V\rangle = \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix}$ in these basis. Let

$$
|V\rangle = v_1|I\rangle + v_2|II\rangle
$$

To find v_1, we take dot product of both sides w.r.t $|I\rangle$. This gives

$$
\langle I|V\rangle = v_1\langle I|I\rangle
$$

But $\langle I|I\rangle = 1$. Hence

$$
\begin{aligned}
v_1 &= \langle I|V\rangle \\
&= \begin{pmatrix} \frac{1+i\sqrt{3}}{4} & -\frac{\sqrt{3}(1+i)}{\sqrt{8}} \end{pmatrix}^* \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i\sqrt{3}}{4} & \frac{-\sqrt{3}+i\sqrt{3}}{\sqrt{8}} \end{pmatrix} \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} \\
&= \frac{\left(1-i\sqrt{3}\right)(1+i)}{4} + \left(\frac{-\sqrt{3}+i\sqrt{3}}{\sqrt{8}}\right)\left(\sqrt{3}+i\right) \\
&= \frac{1+i-i\sqrt{3}+\sqrt{3}}{4} + \frac{-3-\sqrt{3}\,i+3i-\sqrt{3}}{\sqrt{8}} \\
&= \frac{\sqrt{8}\left(1+i-i\sqrt{3}+\sqrt{3}\right)+4\left(-3-\sqrt{3}\,i+3i-\sqrt{3}\right)}{4\sqrt{8}} \\
&= \frac{\sqrt{8}+\sqrt{8}\,i-i\sqrt{24}+\sqrt{24}-12-4\sqrt{3}\,i+12i-4\sqrt{3}}{4\sqrt{8}} \\
&= \frac{\sqrt{8}+\sqrt{24}-12-4\sqrt{3}}{4\sqrt{8}} + i\frac{\sqrt{8}-\sqrt{24}-4\sqrt{3}+12}{4\sqrt{8}} \\
&= \frac{1}{4}\left(1+\sqrt{3}-\frac{12}{\sqrt{8}}-\frac{4\sqrt{3}}{\sqrt{8}}\right) + i\frac{1}{4}\left(1-\sqrt{3}-\frac{4\sqrt{3}}{\sqrt{8}}+\frac{12}{\sqrt{8}}\right) \\
&= \frac{1}{4}\left(1+\sqrt{3}-\frac{12}{2\sqrt{2}}-\frac{4\sqrt{3}}{2\sqrt{2}}\right) + i\frac{1}{4}\left(1-\sqrt{3}-\frac{4\sqrt{3}}{2\sqrt{2}}+\frac{12}{2\sqrt{2}}\right) \\
&= \frac{1}{4}\left(1+\sqrt{3}-\frac{6}{\sqrt{2}}-\frac{2\sqrt{3}}{\sqrt{2}}\right) + i\frac{1}{4}\left(1-\sqrt{3}-\frac{2\sqrt{3}}{\sqrt{2}}+\frac{6}{\sqrt{2}}\right) \\
&= \frac{1}{4}\left(1+\sqrt{3}-3\sqrt{2}-\sqrt{6}\right) + i\frac{1}{4}\left(1-\sqrt{3}-\sqrt{6}+3\sqrt{2}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
v_2 &= \langle II|V\rangle \\
&= \begin{pmatrix} \frac{\sqrt{3}(1+i)}{\sqrt{8}} & \frac{\sqrt{3}+i}{4} \end{pmatrix}^* \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}(1-i)}{\sqrt{8}} & \frac{\sqrt{3}-i}{4} \end{pmatrix} \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} \\
&= \frac{\left(\sqrt{3}-i\sqrt{3}\right)(1+i)}{\sqrt{8}} + \left(\frac{\sqrt{3}-i}{4}\right)\left(\sqrt{3}+i\right) \\
&= \frac{\sqrt{3}+i\sqrt{3}-i\sqrt{3}+\sqrt{3}}{\sqrt{8}} + \frac{3+\sqrt{3}\,i-i\sqrt{3}+1}{4} \\
&= \frac{2\sqrt{3}}{\sqrt{8}} + \frac{3+1}{4} \\
&= \frac{2\sqrt{3}}{\sqrt{8}} + 1 \\
&= \frac{2\sqrt{3}}{2\sqrt{2}} + 1 \\
&= 1 + \sqrt{\frac{3}{2}}
\end{aligned}
$$

Hence

$$|V\rangle = \left(\frac{1}{4}\left(1+\sqrt{3}-3\sqrt{2}-\sqrt{6}\right)+i\frac{1}{4}\left(1-\sqrt{3}-\sqrt{6}+3\sqrt{2}\right)\right)|I\rangle + \left(1+\sqrt{\frac{3}{2}}\right)|II\rangle$$

$$\equiv \begin{pmatrix} \frac{1}{4}\left(1+\sqrt{3}-3\sqrt{2}-\sqrt{6}\right)+i\frac{1}{4}\left(1-\sqrt{3}-\sqrt{6}+3\sqrt{2}\right) \\ 1+\sqrt{\frac{3}{2}} \end{pmatrix}$$

Now we check the square of the of norm of $|V\rangle$

$$\|V\|^2 = \left\|\frac{1}{4}\left(1+\sqrt{3}-3\sqrt{2}-\sqrt{6}\right)+i\frac{1}{4}\left(1-\sqrt{3}-\sqrt{6}+3\sqrt{2}\right)\right\|^2 + \left\|1+\sqrt{\frac{3}{2}}\right\|^2$$

$$= \left(\frac{1}{4}\left(1+\sqrt{3}-3\sqrt{2}-\sqrt{6}\right)\right)^2 + \left(\frac{1}{4}\left(1-\sqrt{3}-\sqrt{6}+3\sqrt{2}\right)\right)^2 + \left(1+\sqrt{\frac{3}{2}}\right)^2$$

$$= 6$$

Verified.

4.5.4 Problem 1 c (9.2.3)

Show how to go from the basis

$$|I\rangle = \begin{bmatrix}3\\0\\0\end{bmatrix} \qquad |II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} \qquad |III\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix}$$

To the orthonormal basis

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad |2\rangle = \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix} \qquad |3\rangle = \begin{bmatrix}0\\\frac{-2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{bmatrix}$$

Solution

Using Gram-Schmidt method, let $|1\rangle = \frac{|I\rangle}{\|I\|} = \begin{bmatrix}3\\0\\0\end{bmatrix}\frac{1}{3} = \begin{bmatrix}1\\0\\0\end{bmatrix}$. Now

$$\begin{aligned}
|2'\rangle &= |II\rangle - |1\rangle\langle 1|II\rangle \\
&= \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}^*\begin{bmatrix}0\\1\\2\end{bmatrix}\right) \\
&= \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}(0) \\
&= \begin{bmatrix}0\\1\\2\end{bmatrix}
\end{aligned}$$

Hence

$$|2\rangle = \frac{|2'\rangle}{\||2'\rangle\|} = \begin{bmatrix}0\\1\\2\end{bmatrix}\frac{1}{\sqrt{1+4}} = \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}$$

And

$$\begin{aligned}
|3'\rangle &= |III\rangle - (|1\rangle\langle 1|III\rangle + |2\rangle\langle 2|III\rangle)\\
&= \begin{bmatrix}0\\2\\5\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}^*\begin{bmatrix}0\\2\\5\end{bmatrix}\right) + \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}\left(\begin{bmatrix}0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}^*\begin{bmatrix}0\\2\\5\end{bmatrix}\right)\right)\\
&= \begin{bmatrix}0\\2\\5\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix}(0) + \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}\frac{12}{\sqrt{5}}\right)\\
&= \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\\frac{12}{5}\\\frac{24}{5}\end{bmatrix}\\
&= \begin{bmatrix}0\\2-\frac{12}{5}\\5-\frac{24}{5}\end{bmatrix}\\
&= \begin{bmatrix}0\\-\frac{2}{5}\\\frac{1}{5}\end{bmatrix}
\end{aligned}$$

Hence

$$
\begin{aligned}
3\rangle &= \frac{|3'\rangle}{\| |3'\rangle \|} \\
&= \begin{bmatrix} 0 \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix} \frac{1}{\sqrt{\frac{4}{25}+\frac{1}{25}}} \\
&= \begin{bmatrix} 0 \\ -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
\end{aligned}
$$

Therefore the orthonormal basis are

$$
|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad |2\rangle = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \qquad |3\rangle = \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
$$

4.5.5 Problem 2

Use $\operatorname{Tr}\sigma_i = 0, \sigma_i^2 = I$ and $\sigma_i\sigma_j = i\sum_k \epsilon_{ijk}\sigma_k$ to obtain the components of a general 2×2 matrix in the basis of $\{\sigma_1,\sigma_2,\sigma_3,I\}$, where σ_i represents the Pauli matrixes and I is the identity matrix.

Solution

The Pauli matrices are

$$
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

And

$$
\sigma_i\sigma_j = \begin{cases} i\sum_k \epsilon_{ijk}\sigma_k & i \neq j \\ I & i = j \end{cases}
$$

We are given basis $\{\sigma_1,\sigma_2,\sigma_3,I\}$ to use to express general 2×2 with. This implies that, we want

$$
\begin{aligned}
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 + c_4 I \\
&= c_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1)
\end{aligned}
$$

Where c_i are weights to be found and $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is any general matrix.

Taking the trace of the LHS and RHS of (1) gives

$$\operatorname{Tr}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \operatorname{Tr}(c_1\sigma_1) + \operatorname{Tr}(c_2\sigma_2) + \operatorname{Tr}(c_3\sigma_3) + \operatorname{Tr}(c_4 I)$$
$$A_{11} + A_{22} = c_1 \operatorname{Tr}(\sigma_1) + c_2 \operatorname{Tr}(\sigma_2) + c_3 \operatorname{Tr}(\sigma_3) + c_4 \operatorname{Tr}(I)$$

But $\operatorname{Tr}(\sigma_i) = 0, i = 1,2,3$ and $\operatorname{Tr}(I) = 2$. The above becomes

$$A_{11} + A_{22} = 2c_4$$
$$c_4 = \frac{A_{11} + A_{22}}{2} \tag{2}$$

We have found one of the weights. Now we need to find the remaining.

Pre multiplying both sides of (1) by σ_1 gives

$$\sigma_1\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = c_1\sigma_1^2 + c_2\sigma_1\sigma_2 + c_3\sigma_1\sigma_3 + c_4\sigma_1 I$$

But from properties of Pauli matrix, $\sigma_1^2 = I$ and $\sigma_1\sigma_2 = i\sum_k \epsilon_{12k}\sigma_k = i\left(\overbrace{\epsilon_{121}}^{0}\sigma_1 + \overbrace{\epsilon_{122}}^{0}\sigma_2 + \overbrace{\epsilon_{123}}^{+1}\sigma_3\right) = i\sigma_3$ and $\sigma_1\sigma_3 = i\sum_k \epsilon_{13k}\sigma_k = i\left(\overbrace{\epsilon_{131}}^{0}\sigma_1 + \overbrace{\epsilon_{132}}^{-1}\sigma_2 + \overbrace{\epsilon_{133}}^{0}\sigma_3\right) = -i\sigma_2$ and $\sigma_1 I = \sigma_1$, Hence the above becomes

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = c_1 I + ic_2\sigma_3 - ic_3\sigma_2 + c_4\sigma_1$$
$$\begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + ic_2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - ic_3\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Taking the trace again of both sides gives

$$A_{21} + A_{12} = 2c_1$$
$$c_1 = \frac{A_{21} + A_{12}}{2} \tag{3}$$

We now repeat the above process.

Pre multiplying both sides of (1) by σ_2 gives

$$\sigma_2\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = c_1\sigma_2\sigma_1 + c_2\sigma_2^2 + c_3\sigma_2\sigma_3 + c_4\sigma_2 I$$

But from properties of Pauli matrix, $\sigma_2^2 = I$ and $\sigma_2\sigma_1 = i\sum_k \epsilon_{21k}\sigma_k = i\left(\overbrace{\epsilon_{211}}^{0}\sigma_1 + \overbrace{\epsilon_{212}}^{0}\sigma_2 + \overbrace{\epsilon_{213}}^{-1}\sigma_3\right) = -i\sigma_3$ and $\sigma_2\sigma_3 = i\sum_k \epsilon_{23k}\sigma_k = i\left(\overbrace{\epsilon_{231}}^{-1}\sigma_1 + \overbrace{\epsilon_{232}}^{0}\sigma_2 + \overbrace{\epsilon_{233}}^{0}\sigma_3\right) = -\sigma_1$ and $\sigma_4 I = \sigma_4$, Hence the above becomes

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = -c_1 i\sigma_3 + c_2 I - c_3\sigma_1 + c_4\sigma_2$$
$$\begin{bmatrix} -iA_{21} & -iA_{22} \\ iA_{11} & iA_{12} \end{bmatrix} = -c_1 i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - c_3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Taking the trace of both sides of the above gives

$$
\begin{aligned}
-iA_{21} + iA_{12} &= 2c_2 \\
c_2 &= i\left(\frac{A_{12} - A_{21}}{2}\right) \qquad (4)
\end{aligned}
$$

And finally, we repeat one more time to find final coefficient c_3.

Pre multiplying both sides of (1) by σ_3 gives

$$
\sigma_3 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = c_1\sigma_3\sigma_1 + c_2\sigma_3\sigma_2 + c_3\sigma_3^2 + c_4\sigma_3 I
$$

But from properties of Pauli matrix, $\sigma_3^2 = I$ and $\sigma_3\sigma_1 = i\sum_k \epsilon_{31k}\sigma_k = i\left(\overbrace{\epsilon_{311}}^{0}\sigma_1 + \overbrace{\epsilon_{312}}^{-1}\sigma_2 + \overbrace{\epsilon_{313}}^{0}\sigma_3\right) = i\sigma_2$ and $\sigma_3\sigma_2 = i\sum_k \epsilon_{32k}\sigma_k = i\left(\overbrace{\epsilon_{321}}^{-1}\sigma_1 + \overbrace{\epsilon_{322}}^{0}\sigma_2 + \overbrace{\epsilon_{323}}^{0}\sigma_3\right) = -i\sigma_1$ and $\sigma_3 I = \sigma_3$, Hence the above becomes

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= c_1 i\sigma_2 - ic_2\sigma_1 + c_3 I + c_4\sigma_3 I \\
\begin{bmatrix} A_{11} & -A_{22} \\ -A_{21} & -A_{22} \end{bmatrix} &= c_1 i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - ic_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Taking the trace of both sides of the above gives

$$
\begin{aligned}
A_{11} - A_{22} &= 2c_3 \\
c_3 &= \frac{A_{11} - A_{22}}{2} \qquad (5)
\end{aligned}
$$

Hence the weights are from Eq. (2,3,4,5) are

$$
\begin{aligned}
c_1 &= \frac{A_{21} + A_{12}}{2} \\
c_2 &= \frac{i}{2}(A_{12} - A_{21}) \\
c_3 &= \frac{A_{11} - A_{22}}{2} \\
c_4 &= \frac{A_{11} + A_{22}}{2}
\end{aligned}
$$

Therefore we can now write any A matrix as

$$
\begin{aligned}
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 + c_4 I \\
&= c_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{A_{21} + A_{12}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{i}{2}(A_{12} - A_{21})\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{A_{11} - A_{22}}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{A_{11} + A_{22}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (8)
\end{aligned}
$$

Verification

As an example, let us try the above on some random matrix A say

$$
A = \begin{bmatrix} 1 & 2i \\ 5 & 99 \end{bmatrix}
$$

Using (8) gives

$$A = \frac{A_{21}+A_{12}}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \frac{i}{2}(A_{12}-A_{21})\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \frac{A_{11}-A_{22}}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \frac{A_{11}+A_{22}}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

But $A_{11} = 1, A_{12} = 2i, A_{21} = 2, A_{22} = 99$. Hence the above becomes

$$\begin{aligned}
A &= \frac{2+2i}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \frac{i}{2}(2i-2)\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \frac{1-99}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \frac{1+99}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}0 & \frac{2+2i}{2}\\ \frac{2+2i}{2} & 0\end{bmatrix} + \begin{bmatrix}0 & -i\left(\frac{i}{2}(2i-2)\right)\\ i\left(\frac{i}{2}(2i-2)\right) & 0\end{bmatrix} + \begin{bmatrix}\frac{-98}{2} & 0\\ 0 & \frac{98}{2}\end{bmatrix} + \begin{bmatrix}50 & 0\\ 0 & 50\end{bmatrix}\\
&= \begin{bmatrix}\frac{-98}{2}+50 & \frac{2+2i}{2} - i\left(\frac{i}{2}(2i-2)\right)\\ \frac{2+2i}{2} + i\left(\frac{i}{2}(2i-2)\right) & \frac{98}{2}+50\end{bmatrix}\\
&= \begin{bmatrix}1 & 2i\\ 2 & 99\end{bmatrix}
\end{aligned}$$

Which is the correct A matrix.

4.5.6 Problem 9.2.5

Prove the triangle inequality starting with $\|V+W\|^2$. You must use $\text{Re}\langle V|W\rangle \leq |\langle V|W\rangle|$ and the Schwarz inequality. Show that the final inequality becomes an equality only if $|V\rangle = a|W\rangle$ where a is real positive scalar.

Solution

Note: I am using $\|V\|$ to mean the norm or magnitude of a Vector and $|a|$ for absolute value.

The Schwarz inequality is given in 9.2.44 as

$$|\langle V|W\rangle| \leq \|V\|\|W\| \tag{9.2.44}$$

The triangle inequality we need to prove is given in (9.2.45)

$$\|V+W\| \leq \|V\| + \|W\| \tag{9.2.44}$$

Starting with

$$\begin{aligned}
\|V+W\|^2 &= \langle (V+W)|(V+W)\rangle\\
&= \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle\\
&= \langle V|V\rangle + \langle V|W\rangle + \langle V|W\rangle^* + \langle W|W\rangle\\
&= \|V\|^2 + 2\,\text{Re}\langle V|W\rangle + \|W\|^2
\end{aligned}$$

Applying Schwarz inequality $|\langle V|W\rangle| \leq \|V\|\|W\|$ to the above gives

$$\|V+W\|^2 \leq \|V\|^2 + 2\|V\|\|W\| + \|W\|^2$$

Hence the above becomes

$$\|V+W\|^2 \leq (\|V\| + \|W\|)^2$$

Which means the same as

$$\|V+W\| \leq \|V\| + \|W\|$$

Which is the Schwarz inequality.

4.5.7 Problem 9.3.5

You have seen above the matrix R_z (9.3.19) that rotates by $\frac{\pi}{2}$ about the z axis. Construct a matrix that rotates by an arbitrary angle about the z axis. Repeat for a rotation around the x axis by some other angle. Verify that each matrix is orthogonal. Take their products and verify that it is also orthogonal. Show in general that the product of two orthogonal matrices is orthogonal. (Remember the rule for the transpose of a product).

Solution

Equation 9.3.19 is

$$R_z\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To construct rotation matrix Ω, we follow this guideline.

$$\Omega_z(\theta) = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}$$

The first column of Ω is the representation (components) of $|1'\rangle$ in terms of the original basis vectors $|1\rangle, |2\rangle, |3\rangle$ before rotation.

Using normal notation, this is the same as saying first column gives the components of e'_x in terms of unit original basis e_x, e_y, e_z. The second column of Ω is the components of $|2'\rangle$ in terms of the original basis vectors $|1\rangle, |2\rangle, |3\rangle$ and third column is components of $|3'\rangle$ in terms of the original basis vectors $|1\rangle, |2\rangle, |3\rangle$.

The representation is found using dot product. For example, first column of Ω is

$$\begin{aligned} \Omega_{11} &= \langle 1|1'\rangle \\ \Omega_{21} &= \langle 2|1'\rangle \\ \Omega_{31} &= \langle 3|1'\rangle \end{aligned}$$

And so on for the rest of the columns. For an angle θ, a diagram helps to see the representation. Since the dot product is the projection of $|1'\rangle$ on the original basis. In other words $\langle 1|1'\rangle$ is the projection of $|1'\rangle$ on $|1\rangle$ and $\langle 2|1'\rangle$ is the projection of $|1'\rangle$ on $|2\rangle$ and so on. So we can read the components directly from the diagram.

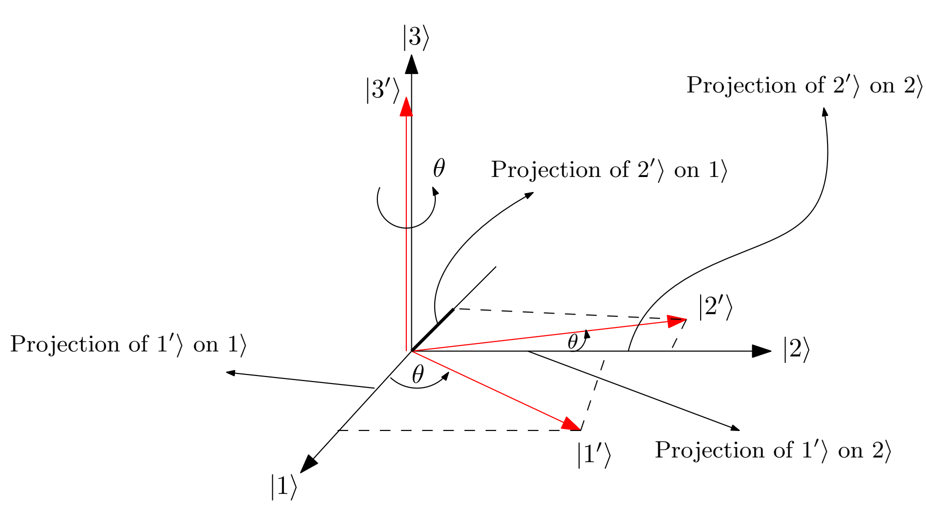


Figure 4.11: Rotation around z by arbitrary angle θ

We see from the diagram that

$$\begin{aligned} \langle 1|1'\rangle &= \||1\rangle\|\||1'\rangle\| \cos\theta \\ &= \cos\theta \end{aligned}$$

Since basis vectors have norm of 1. And

$$\langle 2|1'\rangle = \|2\|\|1'\|\sin\theta$$

and $\langle 3|1'\rangle = 0$ since the projection of $|1'\rangle$ on $|3\rangle$ is zero, since rotation is around z axis, hence vectors on xy plane remain in the xy plane. The above gives us the first column of Ω. So now we have

$$\Omega_z(\theta) = \begin{bmatrix} \cos\theta & \Omega_{12} & \Omega_{13} \\ \sin\theta & \Omega_{22} & \Omega_{23} \\ 0 & \Omega_{32} & \Omega_{33} \end{bmatrix}$$

The second column of Ω are the projections of $|2'\rangle$ on $|1\rangle, |2\rangle, |3\rangle$ which are

$$\begin{aligned} \langle 1|2'\rangle &= \|1\|\|2'\|\sin\theta \\ &= \sin\theta \end{aligned}$$

But this is in the direction of negative $1\rangle$ so we need to add a negative sign. Hence $\langle 1|2'\rangle = -\sin\theta$.

$$\begin{aligned} \langle 2|2'\rangle &= \|2\|\|2'\|\cos\theta \\ &= \cos\theta \end{aligned}$$

and $\langle 3|2'\rangle = 0$ since rotation in only in the xy plane. For the third column, we see that $3'\rangle$ remains the same as original $3\rangle$. Hence no change here. Therefore

$$\Omega_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We now do the rotation around x axis to find $\Omega_x(\phi)$.

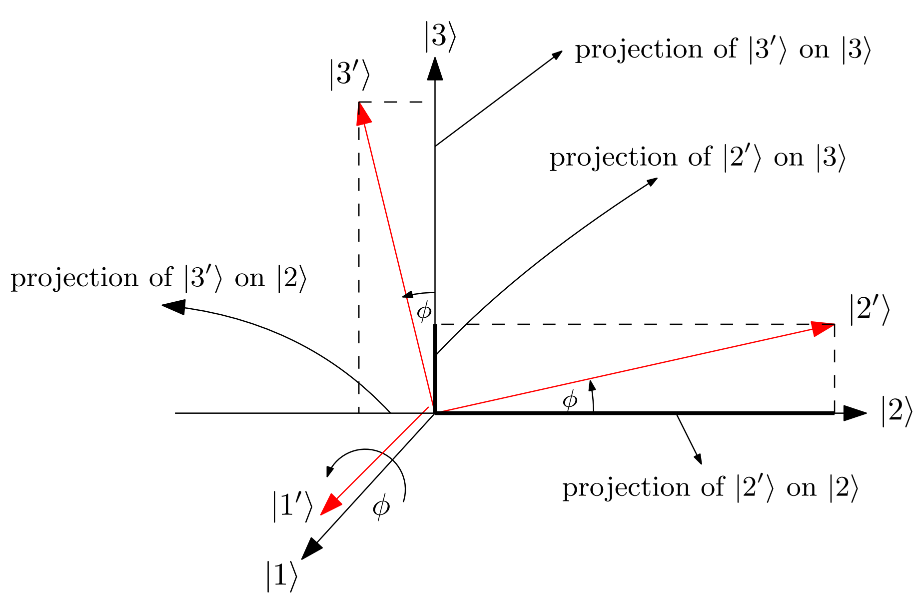


Figure 4.12: Rotation around x by arbitrary angle π

We see from the diagram that

$$1'\rangle = 1\rangle$$

And

$$2'\rangle = (\cos\phi)2\rangle + (\sin\phi)3\rangle$$

And

$$3'\rangle = -(\sin\phi)2\rangle + (\cos\phi)3\rangle$$

Therefore

$$\Omega_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Where the first column of the above matrix, is the components of $1'\rangle$ expressed in terms of $1\rangle, 2\rangle, 3\rangle$ and the second column is the components of $2'\rangle$ expressed in terms of $1\rangle, 2\rangle, 3\rangle$ and third column is the components of $3'\rangle$ expressed in terms of $1\rangle, 2\rangle, 3\rangle$.

Now we need to verify that $\Omega_z(\theta)$ and $\Omega_x(\phi)$ are orthogonal. What this means is that each column of the matrix is orthogonal to each other column in the same matrix. One to way to do that is to multiply the matrix by its transpose. If we get the identity matrix as a result, then the matrix is orthogonal.

Verify $\Omega_z(\theta)$ is orthogonal

$$\begin{aligned}
\Omega_z(\theta)\Omega_z^T(\theta) &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Verified.

Verify $\Omega_x(\phi)$ is orthogonal

$$\begin{aligned}
\Omega_x(\phi)\Omega_x^T(\phi) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ 0 & \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Verified.

The product is

$$\Omega_x(\phi)\Omega_z(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\theta\cos\phi & -\sin\phi \\ \sin\phi\sin\theta & \cos\theta\sin\phi & \cos\phi \end{bmatrix}$$

To show that the is also orthogonal, then, using $\Delta = \left(\Omega_x(\phi)\Omega_z(\phi)\right)\left(\Omega_x(\phi)\Omega_z(\phi)\right)^T$ then

$$\Delta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\theta\cos\phi & -\sin\phi \\ \sin\phi\sin\theta & \cos\theta\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\theta\cos\phi & -\sin\phi \\ \sin\phi\sin\theta & \cos\theta\sin\phi & \cos\phi \end{bmatrix}^T$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\theta\cos\phi & -\sin\phi \\ \sin\phi\sin\theta & \cos\theta\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & \cos\phi\sin\theta & \sin\phi\sin\theta \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

Expanding gives

$$\Delta = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\cos\phi\sin\theta - \sin\theta\cos\theta\cos\phi & \cos\theta\sin\phi\sin\theta - \sin \\ \cos\phi\sin\theta\cos\theta - \sin\theta\cos\theta\cos\phi & \cos^2\phi\sin^2\theta + \cos^2\theta\cos^2\phi + \sin^2\phi & \cos\phi\sin^2\theta\sin\phi + \cos^2\theta\cos \\ \sin\phi\sin\theta\cos\theta - \sin\theta\cos\theta\sin\phi & \sin\phi\sin^2\theta\cos\phi + \cos^2\theta\sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi\sin^2\theta + \cos^2\theta\,\mathrm{si} \end{bmatrix}$$

Simplifying

$$\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi\left(\sin^2\theta + \cos^2\theta\right) + \sin^2\phi & \cos\phi\sin^2\theta\sin\phi + \cos^2\theta\cos\phi\sin\phi - \sin\phi\cos\phi \\ 0 & \sin\phi\sin^2\theta\cos\phi + \cos^2\theta\sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi\left(\sin^2\theta + \cos^2\theta\right) + \cos^2\phi \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi\left(\sin^2\theta + \cos^2\theta\right) - \sin\phi\cos\phi \\ 0 & \sin\phi\cos\phi\left(\sin^2\theta + \cos^2\theta\right) - \cos\phi\sin\phi & \sin^2\phi\left(\sin^2\theta + \cos^2\theta\right) + \cos^2\phi \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ 0 & \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the result is identity matrix, then the product $\Omega_x(\phi)\Omega_z(\phi)$ is an orthogonal matrix.

Now need to show in general that the product of two orthogonal matrices is orthogonal. Let A, B be both orthogonal. Hence $AA^T = I$ and $BB^T = I$. Now

$$\begin{aligned}(AB)(AB)^T &= (AB)\left(B^T A^T\right)\\ &= ABB^T A^T\end{aligned}$$

But $BB^T = I$. Therefore

$$\begin{aligned}(AB)(AB)^T &= AIA^T\\ &= AA^T\end{aligned}$$

But also $AA^T = I$. Therefore

$$(AB)(AB)^T = I$$

Therefore AB is orthogonal. QED.

4.5.8 Problem 9.5.6

The Cayley-Hamilton theorem states that every matrix obeys its characteristic equation. In other words, if $P(\omega)$ is the characteristic polynomial for the matrix Ω, then $P(\Omega)$ vanishes as a matrix. This means that it will annihilate any vector. First prove the theorem for a Hermitian Ω with nondegenerate eigenvectors by starting with the action of $P(\Omega)$ on the eigenvectors.

(Verified from the instructor that the above is the only part required to prove).

Solution

A matrix Ω with nondegenerate eigenvector is diagonalizable. This is by definition, as it implies that for the matrix with n eigenvalues, it is possible to find n orthonormal eigenvectors associated with the eigenvalues. What this means is that we can write

$$\Omega = RDR^{-1}$$

Where R is $n \times n$ matrix, whose columns are the n eigenvectors of Ω and D is a diagonal matrix which has the corresponding eigenvalues $\omega_1, \omega_2, \cdots, \omega_n$ on the diagonal of D. Since $P(\Omega)$ is polynomial in Ω, then we can write

$$\begin{aligned}P(\Omega) &= \sum_{k=0}^{n} a_k \Omega^k\\ &= \sum_{k=0}^{n} a_k \left(RDR^{-1}\right)^k \qquad (1)\end{aligned}$$

But

$$\left(RDR^{-1}\right)^k = RD^k R^{-1}$$

To show the above, consider $\left(RDR^{-1}\right)^2 = \left(RDR^{-1}\right)\left(RDR^{-1}\right) = RD\overbrace{R^{-1}R}^{I}DR^{-1} = RD^2R^{-1}$ and similarly for any higher powers. Eq. (1) now becomes

$$\begin{aligned}P(\Omega) &= \sum_{k=0}^{n} a_k RD^k R^{-1}\\ &= R\left(\sum_{k=0}^{n} a_k D^k\right)R^{-1}\end{aligned}$$

But $\sum_{k=0}^{n} a_k D^k = P(D)$, which means applying operator on D only. Hence the above becomes

$$P(\Omega) = R\, P(D)\, R^{-1} \qquad (2)$$

But since D is a diagonal matrix, having the structure $D = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega_n \end{bmatrix}$, then $P(D) = \begin{bmatrix} P(\omega_1) & 0 & 0 & 0 \\ 0 & p(\omega_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & p(\omega_n) \end{bmatrix}$. Eq (2) now becomes

$$P(\Omega) = R \begin{bmatrix} P(\omega_1) & 0 & 0 & 0 \\ 0 & p(\omega_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & p(\omega_n) \end{bmatrix} R^{-1}$$

But $P(\omega_1) = p(\omega_2) = \cdots = p(\omega_n) = 0$, since each ω_i is a root of the characteristic polynomial of matrix Ω. Therefore the above reduces to

$$\begin{aligned} P(\Omega) &= R \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R^{-1} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

This proves the Cayley-Hamilton for the case of Ω with nondegenerate eigenvectors, which is what we are asked to show.

4.5.8.1 Appendix

(We are not asked to do the matrix inverse part only, but I did it for practice. Not for grading).

Show that $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{11}{8} & -\frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$ by using Cayley-Hamilton theorem. Also show that $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Solution

Cayley-Hamilton theorem says that a matrix Ω obeys its characteristic equation. In other words

$$\begin{aligned} P(\Omega) &= 0 \\ a_n\Omega^n + a_{n-1}\Omega^{n-1} + \cdots + a_1\Omega + a_0 &= 0 \end{aligned}$$

Multiplying both sides of the above by the inverse Ω^{-1} gives

$$a_n\Omega^{n-1} + a_{n-1}\Omega^{n-2} + \cdots + a_1 + a_0\Omega^{-1} = 0$$

$$\Omega^{-1} = \frac{a_n\Omega^{n-1} + a_{n-1}\Omega^{n-2} + \cdots + a_1}{a_0} \tag{1}$$

We now apply the above to the first matrix. For $\Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, we first need to find the characteristic equation.

$$\det\left(\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} - 3\begin{vmatrix} 0 & 0 \\ 0 & 4-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 2-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)((2-\lambda)(4-\lambda)) = 0$$

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Therefore, using Cayley-Hamilton, the above becomes

$$\Omega^3 - 7\Omega^2 + 14\Omega - 8 = 0$$

Where now Ω is the matrix itself. Multiplying both sides by Ω^{-1} gives

$$\Omega^2 - 7\Omega + 14I - 8\Omega^{-1} = 0$$

$$-8\Omega^{-1} = -\Omega^2 + 7\Omega - 14I$$

$$-\Omega^{-1} = \frac{1}{8}\left(-\Omega^2 + 7\Omega - 14I\right)$$

$$\Omega^{-1} = \frac{1}{8}\left(\Omega^2 - 7\Omega + 14I\right) \tag{2}$$

So to find matrix inverse Ω^{-1} we just need to calculate Ω^2 and then simplify the result. But

$$\Omega^2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & 5 \\ 0 & 4 & 0 \\ 0 & 6 & 16 \end{bmatrix}$$

Substituting the above in Eq. (2) gives

$$
\begin{aligned}
\Omega^{-1} &= \frac{1}{8}\left(\begin{bmatrix}1 & 10 & 5\\ 0 & 4 & 0\\ 0 & 6 & 16\end{bmatrix} - 7\begin{bmatrix}1 & 3 & 1\\ 0 & 2 & 0\\ 0 & 1 & 4\end{bmatrix} + 14\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\\
&= \frac{1}{8}\left(\begin{bmatrix}1 & 10 & 5\\ 0 & 4 & 0\\ 0 & 6 & 16\end{bmatrix} - 7\begin{bmatrix}1 & 3 & 1\\ 0 & 2 & 0\\ 0 & 1 & 4\end{bmatrix} + 14\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\\
&= \frac{1}{8}\begin{bmatrix}8 & -11 & -2\\ 0 & 4 & 0\\ 0 & -1 & 2\end{bmatrix}\\
&= \begin{bmatrix}1 & -\frac{11}{8} & -\frac{1}{4}\\ 0 & \frac{1}{2} & 0\\ 0 & -\frac{1}{8} & \frac{1}{4}\end{bmatrix}
\end{aligned}
$$

4.5.9 Problem 9.5.10

Show that the following matrices commute and find a common eigenbasis

$$
M = \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix} \qquad N = \begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix}
$$

Solution

The matrices commute if $MN = NM$. But

$$
\begin{aligned}
MN &= \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix}\\
&= \begin{bmatrix}3 & 0 & 3\\ 0 & 0 & 0\\ 3 & 0 & 3\end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
NM &= \begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}3 & 0 & 3\\ 0 & 0 & 0\\ 3 & 0 & 3\end{bmatrix}
\end{aligned}
$$

We see that $MN = NM$ therefore they commute.

Now we need to find the common eigenbasis. To do this, the eigenvalues and corresponding eigenvectors for M and N are now found.

We start with matrix M.

To find eigenvalues for M, we solve the equation

$$\det(M - \lambda I) = 0$$

Where λ represent the eigenvalues. The above becomes

$$\det\left(\begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\begin{vmatrix}1-\lambda & 0 & 1\\0 & -\lambda & 0\\1 & 0 & 1-\lambda\end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix}-\lambda & 0\\0 & 1-\lambda\end{vmatrix} + \begin{vmatrix}0 & -\lambda\\1 & 0\end{vmatrix} = 0$$

$$(1-\lambda)(-\lambda(1-\lambda)) + \lambda = 0$$

$$2\lambda^2 - \lambda^3 = 0$$

$$\lambda^2(2-\lambda) = 0$$

Hence the roots (eigenvalues) are $\lambda = 0$ with multiplicity 2 and $\lambda = 2$. For each λ_i now we find the corresponding eigenvector $|v_i\rangle$.

$\underline{\lambda = 2}$

We now need to solve $Mv = \lambda v$ for v. This implies

$$(M - \lambda I)v = 0$$

$$\begin{bmatrix}1-\lambda & 0 & 1\\0 & -\lambda & 0\\1 & 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

But $\lambda = 2$ and the above becomes

$$\begin{bmatrix}-1 & 0 & 1\\0 & -2 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{bmatrix}-1 & 0 & 1\\0 & -2 & 0\\0 & 0 & 0\end{bmatrix}$$

The system becomes

$$\begin{bmatrix}-1 & 0 & 1\\0 & -2 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since last row is zero, then we have one free variable v_3 and two leading variables v_1, v_2. Let $v_3 = s$. Second row gives $v_2 = 0$ and first row gives $-v_1 + s = 0$ or $v_1 = s$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since s is free variable, we can pick any non-zero value for it. Let $s = 1$ and the above becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The above is the eigenvector that corresponds to $\lambda = 2$. Now we find the eigenvectors that correspond to $\lambda = 0$. Hopefully we will be able to find two of them.

$\underline{\lambda = 0}$

We now need to solve $Mv = \lambda v$ for v. This implies

$$(M - \lambda I)v = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But $\lambda = 0$ and the above becomes

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the system becomes

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that v_3, v_2 are free variables and v_1 is leading variables. Let $v_3 = s, v_2 = t$. From first row, $v_1 + s = 0$ or $v_1 = -s$. Therefore the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Picking $s = 1, t = 0$ gives one eigenvector as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Picking $s = 0, t = 1$ gives second eigenvector as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So we were able to find two eigenvectors from one eigenvalue $\lambda = 0$, which is good. This table summarizes the result we have found so far for the matrix M

eigenvalue	multiplicity	corresponding eigenvector(s)
$\lambda = 2$	1	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
$\lambda = 0$	2	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now we normalized them. This gives

eigenvalue	multiplicity	corresponding normalized eigenvector(s)
$\lambda = 2$	1	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
$\lambda = 0$	2	$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For the matrix N

To find eigenvalues for M, we solve the equation

$$\det(N - \lambda I) = 0$$

Where λ represent the eigenvalues. The above becomes

$$\det\left(\begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\begin{vmatrix}2-\lambda & 1 & 1\\ 1 & -\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix}-\lambda & -1\\ -1 & 2-\lambda\end{vmatrix} - \begin{vmatrix}1 & -1\\ 1 & 2-\lambda\end{vmatrix} + \begin{vmatrix}1 & -\lambda\\ 1 & -1\end{vmatrix} = 0$$

$$(2-\lambda)(-\lambda(2-\lambda)-1) - (2-\lambda+1) + (-1+\lambda) = 0$$

$$-\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Lets guess $\lambda = -1$ is a root. Then the above becomes $-1-4-1+6=0$. Good. So $(\lambda+1)$ is a factor. Doing long division

$$\frac{\lambda^3 - 4\lambda^2 + \lambda + 6}{(\lambda+1)} = \lambda^2 - 5\lambda + 6$$

Therefore the polynomial becomes

$$\left(\lambda^2 - 5\lambda + 6\right)(\lambda+1) = 0$$

$$(\lambda-2)(\lambda-3)(\lambda+1) = 0$$

Hence the roots (eigenvalues) are $\lambda = 2, \lambda = 3, \lambda = -1$. For each λ_i now we find the corresponding eigenvector $|v_i\rangle$.

$\underline{\lambda = 2}$

We now need to solve $Nv = \lambda v$ for v. This implies

$$(N - \lambda I)v = 0$$

$$\begin{bmatrix}2-\lambda & 1 & 1\\ 1 & -\lambda & -1\\ 1 & -1 & 2-\lambda\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

But $\lambda = 2$ and the above becomes

$$\begin{bmatrix}0 & 1 & 1\\ 1 & -2 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Swapping R_1 with R_3 so that pivot is not zero gives

$$\begin{bmatrix}1 & -1 & 0\\ 1 & -2 & -1\\ 0 & 1 & 1\end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix}1 & -1 & 0\\ 0 & -1 & -1\\ 0 & 1 & 1\end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence system becomes

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_3 and leading variables are v_1, v_2. Let $v_3 = s$. Second row gives $-v_2 - s = 0$ or $v_2 = -s$. First row gives $v_1 - v_2 = 0$ or $v_1 = v_2 = -s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix}$$

$$= s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $s = 1$ therefore

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 3}$

We now need to solve $Nv = \lambda v$ for v. This implies

$$(N - \lambda I)v = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But $\lambda = 3$ and the above becomes

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence system becomes

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable and v_1, v_2 are leading variables. Let $v_3 = s$. Second row gives $-2v_2 = 0$ or $v_2 = 0$. First row gives $-v_1 + s = 0$ or $v_1 = s$. Solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix}$$
$$= s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $s = 1$. The solution becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda = -1}$

We now need to solve $Nv = \lambda v$ for v. This implies

$$(N - \lambda I)v = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But $\lambda = -1$ and the above becomes

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Swapping R_2 and R_1 to keep pivot 1 gives

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence system becomes

$$\begin{bmatrix}1 & 1 & -1\\ 0 & -2 & 4\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

v_3 is free variable and v_1, v_2 are leading variables. Let $v_3 = s$. Second row gives $-2v_2+4s = 0$ or $v_2 = 2s$. First row gives $v_1 + v_2 - s = 0$ or $v_1 = -v_2 + s = -2s + s = -s$. Solution is

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}-s\\ 2s\\ s\end{bmatrix}$$
$$= s\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}$$

Let $s = 1$, the solution becomes

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}$$

This table summarizes the result we have found so far for the matrix N

eigenvalue	multiplicity	corresponding eigenvector(s)
$\lambda = 2$	1	$\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$
$\lambda = 3$	1	$\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$
$\lambda = -1$	1	$\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}$

Now we normalized them. This gives

eigenvalue	multiplicity	corresponding normalized eigenvector(s)
$\lambda = 2$	1	$\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$
$\lambda = 3$	1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$
$\lambda = -1$	1	$\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}$

Now we compare the eigenbasis for M and N. This table shows the final result

Operator	eigenvalues	eigenbases
$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$2, 0, 0$	$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
$N = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$	$2, 3, -1$	$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Looking at the above, we see that all basis are common (linear combinations of M eigenvectors associated with zero eigenvalue can be used to generate two of N eigenvectors).

4.5.10 key solution for HW 5

Physics 3041 (Spring 2021) Solutions to Homework Set 5

1. (a) Problem 9.1.6. (5 points)

$$a(1,1,0)+b(1,0,1)+c(3,2,1)=(a+b+3c,a+2c,b+c)=(0,0,0),$$
$$\Rightarrow a=-2c,\ b=-c,\ a+b+3c=0.$$

The above linear combination of the three row vectors is a null row vector for any nonzero value of c with $a=-2c$ and $b=-c$. Therefore, the three row vectors are linearly dependent.

$$a(1,1,0)+b(1,0,1)+c(0,1,1)=(a+b,a+c,b+c)=(0,0,0),$$
$$\Rightarrow a=-b=-c,\ b=-c\Rightarrow b=c=-c=0\Rightarrow a=0.$$

Therefore, the above three row vectors are linearly independent.

(b) Problem 9.2.1.(ii). (10 points)

$$|V\rangle=\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}$$

$$|I\rangle=\begin{bmatrix}\frac{1+i\sqrt{3}}{4}\\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}}\end{bmatrix},\quad |II\rangle=\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}$$

$$\langle I|I\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}\frac{1+i\sqrt{3}}{4}\\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}}\end{bmatrix}=\frac{1+3}{16}+\frac{3(1+1)}{8}=1$$

$$\langle II|II\rangle=\begin{bmatrix}\frac{\sqrt{3}(1-i)}{\sqrt{8}} & \frac{\sqrt{3}-i}{4}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}=\frac{3(1+1)}{8}+\frac{3+1}{16}=1$$

$$\langle I|II\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}=\frac{(1-i\sqrt{3})\sqrt{3}(1+i)-\sqrt{3}(1-i)(\sqrt{3}+i)}{4\sqrt{8}}$$
$$=\frac{-i(i+\sqrt{3})\sqrt{3}(1+i)+i\sqrt{3}(1+i)(\sqrt{3}+i)}{4\sqrt{8}}=0\Rightarrow\langle II|I\rangle=\langle I|II\rangle^{*}=0$$

$$v_I=\langle I|V\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}=\frac{(1-i\sqrt{3})(1+i)}{4}-\frac{\sqrt{3}(1-i)(\sqrt{3}+i)}{\sqrt{8}}$$
$$=\frac{1+\sqrt{3}+i(1-\sqrt{3})}{4}-\frac{\sqrt{3}[1+\sqrt{3}+i(1-\sqrt{3})]}{\sqrt{8}}=\left(\frac{1}{4}-\sqrt{\frac{3}{8}}\right)[1+\sqrt{3}+i(1-\sqrt{3})]$$

$$v_{II}=\langle II|V\rangle=\begin{bmatrix}\frac{\sqrt{3}(1-i)}{\sqrt{8}} & \frac{\sqrt{3}-i}{4}\end{bmatrix}\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}=\frac{\sqrt{3}(1-i)(1+i)}{\sqrt{8}}+\frac{(\sqrt{3}-i)(\sqrt{3}+i)}{4}$$
$$=\frac{2\sqrt{3}}{\sqrt{8}}+1=1+\sqrt{\frac{3}{2}}$$

$$
\begin{aligned}
|v_I|^2 + |v_{II}|^2 &= \left| \left(\frac{1}{4} - \sqrt{\frac{3}{8}} \right) [1 + \sqrt{3} + i(1 - \sqrt{3}\,)] \right|^2 + \left| 1 + \sqrt{\frac{3}{2}} \right|^2 \\
&= \left(\frac{1}{16} - \frac{1}{2}\sqrt{\frac{3}{8}} + \frac{3}{8} \right) [(1 + \sqrt{3}\,)^2 + (1 - \sqrt{3}\,)^2] + 1 + 2\sqrt{\frac{3}{2}} + \frac{3}{2} \\
&= 8\left(\frac{7}{16} - \frac{1}{4}\sqrt{\frac{3}{2}} \right) + \frac{5}{2} + 2\sqrt{\frac{3}{2}} = 6
\end{aligned}
$$

(c) Problem 9.2.3. (10 points)

$$
|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}
$$

$$
|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\langle 1|II\rangle = 0 \Rightarrow |2'\rangle = |II\rangle,\ \langle 2'|2'\rangle = 5 \Rightarrow |2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}
$$

$$
\langle 1|III\rangle = 0,\ \langle 2|III\rangle = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \frac{2+10}{\sqrt{5}} = \frac{12}{\sqrt{5}}
$$

$$
|3'\rangle = |III\rangle - |2\rangle\langle 2|III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{12}{\sqrt{5}}\begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - \frac{12}{5} \\ 5 - \frac{24}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix}
$$

$$
\langle 3'|3'\rangle = \frac{4+1}{25} = \frac{1}{5} \Rightarrow |3\rangle = \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \sqrt{5}\begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}
$$

2. Use Tr $\sigma_i = 0$, $\sigma_i^2 = I$, and $\sigma_i\sigma_j = i\sum_k \epsilon_{ijk}\sigma_k$ to obtain the components of a general 2×2 matrix in the basis of $\{\sigma_1, \sigma_2, \sigma_3, I\}$, where σ_i represents the Pauli matrices and I is the identity matrix. (15 points)

$$
\begin{aligned}
M &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 + \delta I,\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathrm{Tr}(I) = 2 \\
\mathrm{Tr}(M) &= a + d = \mathrm{Tr}(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 + \delta I) = \alpha\mathrm{Tr}(\sigma_1) + \beta\mathrm{Tr}(\sigma_2) + \gamma\mathrm{Tr}(\sigma_3) + \delta\mathrm{Tr}(I) = 2\delta \\
\delta &= \frac{a+d}{2} \\
\mathrm{Tr}(M\sigma_1) &= \mathrm{Tr}(\alpha\sigma_1^2 + \beta\sigma_2\sigma_1 + \gamma\sigma_3\sigma_1 + \delta I\sigma_1) = \alpha\mathrm{Tr}(\sigma_1^2) + \beta\mathrm{Tr}(\sigma_2\sigma_1) + \gamma\mathrm{Tr}(\sigma_3\sigma_1) + \delta\mathrm{Tr}(I\sigma_1) \\
&= \alpha\mathrm{Tr}(I) + \beta\mathrm{Tr}(-i\sigma_3) + \gamma\mathrm{Tr}(i\sigma_2) + \delta\mathrm{Tr}(\sigma_1) = 2\alpha \\
\alpha &= \frac{1}{2}\mathrm{Tr}(M\sigma_1) = \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} b & a \\ d & c \end{bmatrix} = \frac{b+c}{2}
\end{aligned}
$$

Similarly, we obtain

$$\beta = \frac{1}{2}\text{Tr}(M\sigma_2) = \frac{1}{2}\text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = \frac{1}{2}\text{Tr}\begin{bmatrix} ib & -ia \\ id & -ic \end{bmatrix} = \frac{i(b-c)}{2}$$
$$\gamma = \frac{1}{2}\text{Tr}(M\sigma_3) = \frac{1}{2}\text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \frac{1}{2}\text{Tr}\begin{bmatrix} a & -b \\ c & -d \end{bmatrix} = \frac{a-d}{2}$$

3. Problem 9.2.5. (10 points)

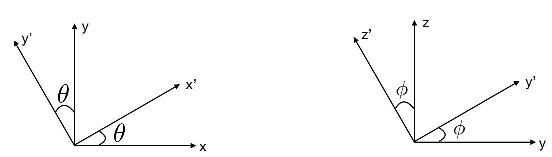
$$\begin{aligned} |V+W|^2 &= \langle V+W|V+W\rangle = \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle \\ &= |V|^2 + \langle V|W\rangle + \langle V|W\rangle^* + |W|^2 = |V|^2 + 2\text{Re}\langle V|W\rangle + |W|^2 \\ &\le |V|^2 + 2|\langle V|W\rangle| + |W|^2 \le |V|^2 + 2|V||W| + |W|^2 = (|V|+|W|)^2 \\ \Rightarrow |V+W| &\le |V| + |W|. \end{aligned}$$

For the equality $|V+W| = |V| + |W|$ to hold, we must have $\text{Re}\langle V|W\rangle = |\langle V|W\rangle|$ and $|\langle V|W\rangle| = |V||W|$. From the first condition, $\langle V|W\rangle$ is real and positive. From the second condition (see proof of the Schwarz inequality in the textbook),

$$|V\rangle = \frac{\langle W|V\rangle}{|W|^2}|W\rangle = \frac{\langle V|W\rangle^*}{|W|^2}|W\rangle = a|W\rangle,\ a = \frac{\langle V|W\rangle^*}{|W|^2}.$$

Because $\langle V|W\rangle$ is real and positive, a is also real and positive.

4. Problem 9.3.5. (20 points)



From the above figures, we have

$$R_z(\theta) = \begin{bmatrix} \vec{i}\cdot\vec{i'} & \vec{i}\cdot\vec{j'} & \vec{i}\cdot\vec{k'} \\ \vec{j}\cdot\vec{i'} & \vec{j}\cdot\vec{j'} & \vec{j}\cdot\vec{k'} \\ \vec{k}\cdot\vec{i'} & \vec{k}\cdot\vec{j'} & \vec{k}\cdot\vec{k'} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for rotation around the z-axis by an angle θ, and

$$R_x(\phi) = \begin{bmatrix} \vec{i}\cdot\vec{i'} & \vec{i}\cdot\vec{j'} & \vec{i}\cdot\vec{k'} \\ \vec{j}\cdot\vec{i'} & \vec{j}\cdot\vec{j'} & \vec{j}\cdot\vec{k'} \\ \vec{k}\cdot\vec{i'} & \vec{k}\cdot\vec{j'} & \vec{k}\cdot\vec{k'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

for rotation around the x-axis by an angle ϕ.

$$R_z(\theta)^T R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$R_x(\phi)^T R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$R_x(\phi) R_z(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix},$$

$$\begin{aligned} &[R_x(\phi)R_z(\theta)]^T R_x(\phi)R_z(\theta) \\ &= \begin{bmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$

In general, if $M^T M = I$ and $N^T N = I$, then we have $(NM)^T (NM) = M^T N^T N M = M^T M = I$.

5. Problem 9.5.6, but only for the proof without doing the inverse matrix part. (10 points)

For a Hermitian operator Ω with non-degenerate eigenvalues, we have

$$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle, \quad i = 1,\ 2,\ \cdots,\ n.$$

We can then expand an arbitrary vector as

$$|V\rangle = \sum_{i=1}^{n} v_i |\omega_i\rangle.$$

The characteristic polynomial satisfies $P(\omega_i) = 0$, so

$$P(\Omega)|V\rangle = \sum_{i=1}^{n} v_i P(\Omega)|\omega_i\rangle = \sum_{i=1}^{n} v_i P(\omega_i)|\omega_i\rangle = \sum_{i=1}^{n} 0|\omega_i\rangle = |0\rangle \Rightarrow P(\Omega) = 0.$$

6. Problem 9.5.10. (20 points)

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$MN = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$NM = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = MN$$

$$M \Rightarrow \begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = -\omega(1-\omega)^2 + \omega = \omega^2(2-\omega) = 0 \Rightarrow \omega_M = 0,\ 0,\ 2.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_0 + c_0 \\ 0 \\ a_0 + c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |\omega_M = 0\rangle = \begin{bmatrix} a_0 \\ b_0 \\ -a_0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -a_2 + c_2 \\ -2b_2 \\ a_2 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |\omega_M = 2\rangle = \begin{bmatrix} a_2 \\ 0 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$N \Rightarrow \begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} = (2-\omega)[-\omega(2-\omega) - 1] - (2 - \omega + 1) - 1 + \omega$$

$$= (2-\omega)(\omega - 3)(\omega + 1) = 0 \Rightarrow \omega_N = -1,\ 2,\ 3.$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ e_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 3d_1 + e_1 + f_1 \\ d_1 + e_1 - f_1 \\ d_1 - e_1 + 3f_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = -1\rangle = d_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} d_2 \\ e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} e_2 + f_2 \\ d_2 - 2e_2 - f_2 \\ d_2 - e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = 2\rangle = d_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} d_3 \\ e_3 \\ f_3 \end{bmatrix} = \begin{bmatrix} -d_3 + e_3 + f_3 \\ d_3 - 3e_3 - f_3 \\ d_3 - e_3 - f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = 3\rangle = d_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

It is clear from the above results that M and N share a common eigenbasis

$$\{|\omega_N = -1\rangle,\ |\omega_N = 2\rangle,\ |\omega_N = 3\rangle\}.$$

4.6 HW 6

Local contents

4.6.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 6 **(Due 3/10)**

1. Problem 9.5.11. (40 points)

2. Prove the following results on the commutators: $[A, B + C] = [A, B] + [A, C]$, $[A + B, C] = [A, C] + [B, C]$, $[A, BC] = B[A, C] + [A, B]C$, $[AB, C] = A[B, C] + [A, C]B$. (10 points)

3. Follow the discussion of $s_+ = s_x + is_y$ for the electron spin to derive the matrix representation of $s_- = s_x - is_y$. (20 points)

4. Problem 9.6.2, and find the solutions for $x_1(t)$ and $x_2(t)$ with the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = v_1$ and $\dot{x}_2(0) = v_2$. (30 points)

4.6.2 Problem 1 (9.5.11)

Problem 9.5.11. Important quantum problem. *Consider the three* spin-1 matrices:

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{9.5.55}$$

which represent the components of the internal angular momentum *of some elementary particle at rest. That is to say, the particle has some angular momentum unrelated to* $\vec{r} \times \vec{p}$. *The operator* $S^2 = S_x^2 + S_y^2 + S_z^2$ *represents the total angular momentum squared. The dynamical state of the system is given by a* state vector *in the complex three dimensional space on which these spin matrices act. By this we mean that all available information on the particle is stored in this vector. According to the laws of quantum mechanics*

- *A measurement of the angular momentum along any direction will give only one of the eigenvalues of the corresponding spin operator.*
- *The probability that a given eigenvalue will result is equal to the absolute value squared of the inner product of the state vector with the corresponding eigenvector. (The state vector and all eigenvectors are all normalized.)*
- *The state of the system immediately following this measurement will be the corresponding eigenvector.*

(a) What are the possible values we can get if we measure spin along the z-axis?
(b) What are the possible values we can get if we measure spin along the x or y-axis?
(c) Say we got the largest possible value for S_x. *What is the state vector immediately afterwards?*
(d) If S_z *is now measured what are the odds for the various outcomes? Say we got the largest value. What is the state just after the measurement? If we remeasure* S_x *at once, will we once again get the largest value?*
(e) What are the outcomes when S^2 *is measured?*
(f) From the four operators S_x, S_y, S_z, S^2, *what is the largest number of commuting operators we can pick at a time?*
(g) A particle is in a state given by a column vector

$$|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

First rescale the vector to normalize it. What are the odds for getting the three possible eigenvalues of S_z? *What is the statistical or weighted average of these values? Compare this to* $\langle V|S_z|V\rangle$.
(h) Repeat all this for S_x.

Figure 4.13: Problem statement

Solution

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4.6.2.1 Part a

The first step is to find the eigenvalues of S_z. These are the possible values that can be obtained when measuring the spin along the z axis. Because S_z is a diagonal matrix, its eigenvalues are on the diagonal. Hence the eigenvalues are $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. Because the eigenvalues are different, S_z is not degenerate. The values are

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= 1 \\ \omega_3 &= -1 \end{aligned}$$

4.6.2.2 Part b

Now we need to find the eigenvalues for S_y and S_x. The factor $\frac{1}{\sqrt{2}}$ is not included in the following calculation, but added again at the end. This is to simplify the algebra.

For S_y

$$\begin{aligned}
\left|S_y - \omega I\right| &= 0\\
\begin{vmatrix} -\omega & -i & 0\\ i & -\omega & -i\\ 0 & i & -\omega \end{vmatrix} &= 0\\
-\omega\begin{vmatrix} -\omega & -i\\ i & -\omega \end{vmatrix} + i\begin{vmatrix} i & -i\\ 0 & -\omega \end{vmatrix} &= 0\\
(-\omega)\left(\omega^2 + i^2\right) + i(-\omega i) &= 0\\
(-\omega)\left(\omega^2 - 1\right) - \omega i^2 &= 0\\
-\omega^3 + \omega + \omega &= 0\\
-\omega^3 + 2\omega &= 0\\
\omega\left(-\omega^2 + 2\right) &= 0
\end{aligned}$$

The eigenvalues are the roots of the above polynomial. They are

$$\begin{aligned}
\omega_1 &= 0\\
\omega_2 &= \sqrt{2}\\
\omega_3 &= -\sqrt{2}
\end{aligned}$$

Adding back the factor $\frac{1}{\sqrt{2}}$ which was in front of S_y by multiplying the above results with it gives

$$\begin{aligned}
\omega_1 &= 0\\
\omega_2 &= 1\\
\omega_3 &= -1
\end{aligned}$$

For S_x

$$\begin{aligned}
\left|S_x - \omega I\right| &= 0\\
\begin{vmatrix} -\omega & 1 & 0\\ 1 & -\omega & 1\\ 0 & 1 & -\omega \end{vmatrix} &= 0\\
-\omega\begin{vmatrix} -\omega & 1\\ 1 & -\omega \end{vmatrix} - 1\begin{vmatrix} 1 & 1\\ 0 & -\omega \end{vmatrix} &= 0\\
(-\omega)\left(\omega^2 - 1\right) - 1(-\omega) &= 0\\
-\omega^3 + \omega + \omega &= 0\\
-\omega^3 + 2\omega &= 0\\
\omega\left(2 - \omega^2\right) &= 0
\end{aligned}$$

The eigenvalues are the roots of the above polynomial. They are

$$\begin{aligned}
\omega_1 &= 0\\
\omega_2 &= \sqrt{2}\\
\omega_3 &= -\sqrt{2}
\end{aligned}$$

Adding back the factor $\frac{1}{\sqrt{2}}$ which was in front of S_y by multiplying the above results with it gives

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= 1 \\ \omega_3 &= -1 \end{aligned}$$

This table gives a summary of result found so far before going to the next part.

Spin matrix	Eigenvalues found
$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$
$S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$
$S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$

The above table shows that the possible values if we measure the spin along the x or y axis are $\{0, 1, -1\}$.

4.6.2.3 Part c

From part (b) and taking the largest eigenvalue of S_x as $\omega_2 = +1$, the question is asking us to find the associated eigenvector $|S_x = \omega_2\rangle$. This is found by solving

$$\begin{bmatrix} -\omega_2 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In the above $\omega_2 = 1$. Therefore

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_2 = R_2 + \frac{1}{\sqrt{2}} R_1$ gives

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}$$

$R_3 = R_3 + \frac{2}{\sqrt{2}} R_2$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

The above is now in Echelon form. The system becomes

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is a free variable, and v_2, v_1 are the leading variables. Let $v_3 = s$. Second row gives $-\frac{1}{2} v_2 + \frac{1}{\sqrt{2}} s = 0$ or $v_2 = \frac{2}{\sqrt{2}} s$. First row gives $-v_1 + \frac{1}{\sqrt{2}} v_2 = 0$ or $v_1 = \frac{1}{\sqrt{2}} v_2$ or $v_1 = \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} s \right) = s$. Hence the solution (the eigenvector) is

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} s \\ \frac{2}{\sqrt{2}} s \\ s \end{bmatrix} \\ &= s \begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} \end{aligned}$$

Since s is a free variable, we will choose it so that the norm is 1. Therefore

$$\begin{aligned} s\sqrt{1+2+1} &= 1 \\ s\sqrt{4} &= 1 \\ s &= \frac{1}{2} \end{aligned}$$

Hence the <u>state vector</u> for the largest value of S_x is

$$|S_x = 1\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

4.6.2.4 Part d

$$S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We first need to find the eigenvectors $|S_z = \omega_i\rangle$ for S_z. From part (a), the eigenvalues are

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= 1 \\ \omega_3 &= -1 \end{aligned}$$

For $\underline{\omega_1 = 0}$ the associated eigenvector is found by solving

$$\begin{bmatrix} 1-\omega_1 & 0 & 0 \\ 0 & -\omega_1 & 0 \\ 0 & 0 & -1-\omega_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_2 is a free variable, and v_1, v_3 are the leading variables. Let $v_2 = s$. Last row gives $v_3 = 0$. First row gives $v_1 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$$

$$= s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Choosing $s = 1$ gives

$$|S_z = \omega_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\underline{\omega_2 = 1}$ we need to solve

$$\begin{bmatrix} 1-\omega_2 & 0 & 0 \\ 0 & -\omega_2 & 0 \\ 0 & 0 & -1-\omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_1 is a free variable, and v_2, v_3 are the leading variables. Let $v_1 = s$. Last row gives $v_3 = 0$. Second row gives $v_2 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$$

$$= s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Choosing $s = 1$ then

$$|S_z = \omega_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\omega_3 = -1$ the associated eigenvector is found by solving

$$\begin{bmatrix} 1-\omega_3 & 0 & 0 \\ 0 & -\omega_3 & 0 \\ 0 & 0 & -1-\omega_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is a free variable, and v_1, v_2 are the leading variables. Let $v_3 = s$. Second row gives $v_2 = 0$. First row gives $v_1 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}$$

$$= s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Choosing $s = 1$ gives

$$|S_z = \omega_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Summary table for S_z

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$\omega_2 = 1$	$\lvert S_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\omega_3 = -1$	$\lvert S_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Calculating $|\langle S_z = \omega_1|\Psi\rangle|^2$ gives the odds of $|S_z = \omega_1\rangle$. Ψ is the initial state vector. Similarly, calculating $|\langle S_x = \omega_1|\Psi\rangle|^2$ gives find the odds of $|S_x = \omega_1\rangle$ and similarly for $|S_z = \omega_3\rangle$.

Ψ is the state vector from part (c), which is

$$|\Psi\rangle = |S_x = 1\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

Hence the odds of $|S_z = 0\rangle$ is

$$|\langle S_z = \omega_1|\Psi\rangle|^2 = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

And the odds for $|S_z = 1\rangle$ is

$$|\langle S_z = \omega_2|\Psi\rangle|^2 = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

And the odds for $|S_z = -1\rangle$ is

$$\left|\langle S_z = \omega_3|\Psi\rangle\right|^2 = \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

The odds for $|S_z = 0\rangle$ is 50%, the odds for $|S_z = 1\rangle$ is 25% and odds for $|S_z = -1\rangle$ is 25%. The total is 100% as expected.

Summary table of results so far S_z

eigenvalue	eigenvector	probability of this outcome
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$P(0) = 50\%$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$P(1) = 25\%$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$P(-1) = 25\%$

The state just after the measurement is $|S_z = 1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$ since that is the state associated with the largest eigenvalue $\omega_2 = 1$. This now becomes the initial state

$$|\Psi\rangle = |S_z = 1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

We know that $S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$ with the eigenvalues found earlier as $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. In part (c) we found that $|S_x = 1\rangle = \begin{bmatrix}\frac{1}{2}\\\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$ for S_x associated with its largest eigenvalue which is $\omega_2 = 1$. Therefore the odds of this is

$$|\langle S_x = 1|\Psi\rangle|^2 = \left(\begin{bmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}^* \begin{bmatrix}1\\0\\0\end{bmatrix}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 25\%$$

This says the odds of getting again the largest value (which is 1) is not likely since it is not the highest possible odd being only 25% with 3 possible values.

4.6.2.5 Part e

$$\begin{aligned}S^2 &= S_x^2 + S_y^2 + S_z^3\\ &= \frac{1}{2}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}^2 + \frac{1}{2}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}^2 + \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}^2\end{aligned} \tag{1}$$

But

$$\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}^2 = \begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}^2 = \begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix} = \begin{bmatrix}1&0&-1\\0&2&0\\-1&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}^2 = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Hence (1) becomes

$$
\begin{aligned}
S^2 &= \frac{1}{2}\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}
\end{aligned}
$$

Since S^2 is diagonal, then its eigenvalues are on the diagonal. They are all $\omega = 2$ with multiplicity 3. It is a degenerate matrix. Since the outcome is the eigenvalue (it is a measure of the spin angular momentum), then we see that the outcome is always 2, since that is the only possible eigenvalue.

4.6.2.6 Part f

The operators are

$$
S^2 = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} \qquad S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix} \qquad S_z = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}
$$

Commutator is defined as

$$[M, N] = MN - NM$$

If $[M, N] = 0$ then they commute. We know that S_x, S_y, S_z do not commute with each others per lecture notes. So we only need to check if S^2 commutes with S_x, S_y, S_z or not.

$$
\begin{aligned}
\left[S^2, S_x\right] &= S^2 S_x - S_x S^2\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 2 & 0\\ 2 & 0 & 2\\ 0 & 2 & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 2 & 0\\ 2 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}\\
&= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

Hence S^2, S_x commute. And

$$
\begin{aligned}
\left[S^2, S_y\right] &= S^2 S_y - S_y S^2 \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Hence S^2, S_y commute. And

$$
\begin{aligned}
\left[S^2, S_z\right] &= S^2 S_z - S_z S^2 \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Hence S^2, S_z commute. Therefore there are three sets of commuting operators. They are $\left\{S^2, S_x\right\}, \left\{S^2, S_y\right\}, \left\{S^2, S_z\right\}$. So the maximum number of operators such that they all commute with each others is two.

4.6.2.7 Part g

$$
|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

The norm is $\sqrt{1+4+9} = \sqrt{14}$. Hence the normalized state is

$$
|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

In part (c) we found the eigenvalues and associated eigenvector for S_z. Here they are again

Summary table of results so far S_z

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$

We will now find the odds of getting $|S_z = \omega_1\rangle$ given the current state vector is $|V\rangle$ (after normalizing). The odds are

$$|\langle S_z = 0|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{2}{\sqrt{14}}\right)^2 = \frac{4}{14} = 28.571\%$$

And

$$|\langle S_z = +1|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{1}{\sqrt{14}}\right)^2 = \frac{1}{14} = 7.143\%$$

And

$$|\langle S_z = -1|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{3}{\sqrt{14}}\right)^2 = \frac{9}{14} = 64.285\%$$

Updated summary table of results so far S_z

eigenvalue	eigenvector	odd of getting this eigenvalue
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$	$P(0) = \frac{4}{14} = 28.571\%$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$	$P(1) = \frac{1}{14} = 7.143\%$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$	$P(-1) = \frac{9}{14} = 64.285\%$

The statistical average is

$$\begin{aligned}\omega_1\left(\frac{4}{14}\right)+\omega_2\left(\frac{1}{14}\right)+\omega_3\left(\frac{9}{14}\right) &= 0\left(\frac{4}{14}\right)+1\left(\frac{1}{14}\right)-1\left(\frac{9}{14}\right)\\ &= -\frac{4}{7}\\ &= -0.571\,43 \end{aligned} \tag{1}$$

The above is now compared to Now we compare $\langle V|S_z|V\rangle$

$$\langle V|(S_z|V\rangle) = \frac{1}{\sqrt{14}}\langle V|\overbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}}^{S_z|V\rangle}\rangle \tag{2}$$

But

$$S_z|V\rangle = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\-3\end{bmatrix}$$

Hence Eq. (2) becomes

$$\begin{aligned}\langle V|(S_z|V\rangle) &= \frac{1}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\begin{bmatrix}1&2&3\end{bmatrix}^*\begin{bmatrix}1\\0\\-3\end{bmatrix}\right)\\ &= \frac{1}{14}(1-9)\\ &= \frac{-8}{14}\\ &= -0.571\,43\end{aligned} \tag{3}$$

Comparing (1) and (3) shows it is the <u>same value</u>. This is the expectation value when measuring S_z.

4.6.2.8 Part h

Part (g) is now repeated, but using S_x. We found from the above part that

$$|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

From part(b), we found the eigenvalues for $S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$ to be $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. But we did not find the associated eigenvectors yet in order to repeat part g as was done for S_z. So we need now to find the eigenvectors for S_x before being able to answer this part for S_x.

For <u>$\omega_1 = 0$</u>

$$\begin{bmatrix}0&\frac{1}{\sqrt{2}}&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Swapping R_2, R_1

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $v_2 = 0$. First row gives $\frac{1}{\sqrt{2}}v_1 + \frac{1}{\sqrt{2}}s = 0$ or $v_1 = -s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$. Therefore

$$|S_x = \omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\underline{\omega_2 = 1}$

$$\begin{bmatrix} -\omega_2 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{\sqrt{2}} R_1$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}$$

$R_3 = R_3 + \frac{2}{\sqrt{2}} R_2$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $-\frac{1}{2} v_2 + \frac{1}{\sqrt{2}} s = 0$ or $v_2 = \frac{2}{\sqrt{2}} s$. First row gives $-v_1 + \frac{1}{\sqrt{2}} v_2 = 0$ or $v_1 = \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} s \right) = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ \frac{2}{\sqrt{2}} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{2}$. Therefore

$$|S_x = \omega_2\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

For $\underline{\omega_3 = -1}$

$$\begin{bmatrix} -\omega_3 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_3 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{\sqrt{2}} R_1$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{2}{\sqrt{2}} R_2$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $\frac{1}{2} v_2 + \frac{1}{\sqrt{2}} s = 0$ or $v_2 = -\frac{2}{\sqrt{2}} s$. First row gives $v_1 + \frac{1}{\sqrt{2}} v_2 = 0$ or $v_1 = -\frac{1}{\sqrt{2}} \left(-\frac{2}{\sqrt{2}} s \right) = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ -\frac{2}{\sqrt{2}} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ -\frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{2}$. Therefore

$$|S_x = \omega_3\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

Summary table of results so far S_x

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_x = \omega_1\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$\omega_2 = 1$	$\lvert S_x = \omega_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$
$\omega_3 = -1$	$\lvert S_x = \omega_3\rangle = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$

The odds of getting $|S_x = \omega_i\rangle$ given the current state vector is $|V\rangle$ are now found. Expressing $|V\rangle$ in the eigenbasis of S_x gives

$$\begin{aligned} |V\rangle &= c_1\,|S_x = \omega_1\rangle + c_2\,|S_x = \omega_2\rangle + c_3\,|S_x = \omega_3\rangle \\ &= c_1\,|S_x = 0\rangle + c_2\,|S_x = 1\rangle + c_3\,|S_x = -1\rangle \end{aligned} \tag{1}$$

Where

$$c_1 = \langle S_x = 0|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\left(\frac{-1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right) = \frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)$$

$$c_2 = \langle S_x = 1|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\left(\frac{1}{2} + \frac{2}{\sqrt{2}} + \frac{3}{2}\right) = \frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)$$

$$c_3 = \langle S_x = -1|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\left(\frac{1}{2} - \frac{2}{\sqrt{2}} + \frac{3}{2}\right) = \frac{1}{\sqrt{14}}\left(2 - \sqrt{2}\right)$$

Eq. (1) becomes

$$|V\rangle = \frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{14}}\left(\sqrt{2} + 2\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{\sqrt{14}}\left(2 - \sqrt{2}\right)\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

The above is the representation of $|V\rangle$ in the eigenbasis of S_x. The odds of each eigenvalue is the square of the coefficients $|c_1|^2, |c_2|^2, |c_3|^2$ above. Therefore

$$P(0) = \left(\frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)\right)^2 = \frac{2}{14} = 14.286\%$$
$$P(+1) = \left(\frac{1}{\sqrt{14}}\left(\sqrt{2}+2\right)\right)^2 = \frac{1}{14}\left(6+4\sqrt{2}\right) = 83.263\%$$
$$P(-1) = \left(\frac{1}{\sqrt{14}}\left(2-\sqrt{2}\right)\right)^2 = \frac{1}{14}\left(6-4\sqrt{2}\right) = 24.51\%$$

Updated summary table for S_x

eigenvalue	eigenvector	Odds of getting this eigenvalue
$\omega_1 = 0$	$\|S_x = \omega_1\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$P(0) = \frac{2}{14} = 14.286\%$
$\omega_2 = 1$	$\|S_x = \omega_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$	$P(1) = \frac{1}{14}\left(6+4\sqrt{2}\right) = 83.263\%$
$\omega_3 = -1$	$\|S_x = \omega_3\rangle = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$	$P(-1) = \frac{1}{14}\left(6-4\sqrt{2}\right) = 24.51\%$

The statistical average is

$$\begin{aligned}\omega_1\left(\frac{2}{14}\right) + \omega_2\left(\frac{1}{14}\left(6+4\sqrt{2}\right)\right) + \omega_3\left(\frac{1}{14}\left(6-4\sqrt{2}\right)\right) &= 0\left(\frac{2}{14}\right) + 1\left(\frac{1}{14}\left(6+4\sqrt{2}\right)\right) - 1\left(\frac{1}{14}\left(6-4\sqrt{2}\right)\right) \\ &= \frac{4}{7}\sqrt{2} \\ &= 0.80812 \end{aligned} \tag{1}$$

The above is now compared to

$$\langle V|S_x|V\rangle = \frac{1}{\sqrt{14}}\langle V|\overbrace{\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}}^{S_x|V\rangle}\rangle \tag{2}$$

But

$$S_x|V\rangle = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\2\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

Hence Eq. (2) becomes

$$\begin{aligned}
\langle V|S_x|V\rangle &= \frac{1}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & 3\end{bmatrix}^{*}\begin{bmatrix}\sqrt{2}\\ 2\sqrt{2}\\ \sqrt{2}\end{bmatrix}\right)\\
&= \frac{1}{14}\left(\sqrt{2}+4\sqrt{2}+3\sqrt{2}\right)\\
&= 0.80812 \qquad (3)
\end{aligned}$$

Comparing (1) and (3) shows it is the same value.This is the expectation value when measuring S_x

4.6.3 Problem 2

Prove the following results on commutators:

$$\begin{aligned}
[A, B+C] &= [A, B] + [A, C]\\
[A+B, C] &= [A, C] + [B, C]\\
[A, BC] &= B[A, C] + [A, B]C\\
[AB, C] &= A[B, C] + [A, C]B
\end{aligned}$$

Solution

4.6.3.1 Part 1

By definition of commutator, which is $[A, B] = AB - BA$, then

$$\begin{aligned}
[A, B+C] &= A(B+C) - (B+C)A\\
&= AB + AC - BA - CA\\
&= (AB - BA) + (AC - CA)\\
&= [A, B] + [A, C]
\end{aligned}$$

4.6.3.2 Part 2

By definition of commutator, which is $[A, B] = AB - BA$, then

$$\begin{aligned}
[A+B, C] &= (A+B)C - C(A+B)\\
&= AC + BC - CA - CB\\
&= (AC - CA) + (BC - CB)\\
&= [A, C] + [B, C]
\end{aligned}$$

4.6.3.3 Part 3

By definition of commutator, which is $[A, B] = AB - BA$, then

$$[A, BC] = A(BC) - (BC)A$$

Adding and subtracting BAC on the RHS gives

$$\begin{aligned}
[A, BC] &= \widehat{BAC} + ABC - BCA - \widehat{BAC}\\
&= (BAC - BCA) + (ABC - BAC)\\
&= B(AC - CA) + (AB - BA)C\\
&= B[A, C] + [A, B]C
\end{aligned}$$

4.6.3.4 Part 4

By definition of commutator, which is $[A,B]=AB-BA$, then

$$\begin{aligned}[AB,C] &= (AB)C - C(AB)\\ &= ABC - CAB\end{aligned}$$

Adding and subtracting ACB on the RHS gives

$$\begin{aligned}[AB,C] &= \widehat{ACB} + ABC - CAB - \widehat{ACB}\\ &= (ABC - ACB) + (ACB - CAB)\\ &= A(BC - CB) + (AC - CA)B\\ &= A[B,C] + [A,C]B\end{aligned}$$

4.6.4 Problem 3

Follow the discussion of $s_+ = s_x + is_y$ for the electron spin to derive the matrix representation of $s_- = s_x - is_y$

Solution

Experiments show that S_z has two possible values (eigenvalues) of $\frac{\hbar}{2},-\frac{\hbar}{2}$. Using eigenbasis of S_z

$$|S_z = \frac{\hbar}{2}\rangle = \begin{bmatrix}1\\0\end{bmatrix} = |1\rangle$$

$$|S_z = -\frac{\hbar}{2}\rangle = \begin{bmatrix}0\\1\end{bmatrix} = |2\rangle$$

Gives

$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Consider $S_- = S_x - iS_y$. Then

$$\begin{aligned}[S_z,S_-] &= \left[S_z, S_x - iS_y\right]\\ &= [S_z,S_x] - i\left[S_z,S_y\right] \qquad (1)\end{aligned}$$

But, using $\left[S_i,S_j\right] = i\hbar\sum_k \epsilon_{ijk}S_k$. Hence

$$[S_z,S_x] = i\hbar S_y \qquad (2)$$

$$\left[S_z,S_y\right] = -i\hbar S_x \qquad (3)$$

Substituting (2,3) into (1) gives

$$\begin{aligned}[S_z,S_-] &= i\hbar S_y - i(-i\hbar S_x)\\ &= i\hbar S_y + i^2(\hbar S_x)\\ &= i\hbar S_y - \hbar S_x\\ &= \hbar\left(iS_y - S_x\right)\\ &= -\hbar\left(S_x - iS_y\right)\\ &= -\hbar S_-\end{aligned}$$

Therefore we see that

$$[S_z,S_-] = S_zS_- - S_-S_z = -\hbar S_-$$

This implies

$$S_zS_- = S_-S_z - \hbar S_-$$

Therefore

$$\begin{aligned} S_z S_- |1\rangle &= (S_- S_z - \hbar S_-)|1\rangle \\ &= S_- S_z |1\rangle - \hbar S_- |1\rangle \end{aligned}$$

But $S_z|1\rangle = \frac{\hbar}{2}|1\rangle$ then the above becomes

$$\begin{aligned} S_z S_- |1\rangle &= S_- \frac{\hbar}{2}|1\rangle - \hbar S_- |1\rangle \\ &= \left(\frac{\hbar}{2} - \hbar\right) S_- |1\rangle \\ &= -\frac{\hbar}{2} S_- |1\rangle \end{aligned}$$

The above shows that $S_-|1\rangle$ is eigenvector (eigenstate) of S_z with eigenvalue $-\frac{\hbar}{2}$ which is compatible with experiments. Because $S_z|2\rangle = -\frac{\hbar}{2}|2\rangle$ then let

$$S_-|1\rangle = c|2\rangle \tag{4}$$

We now need to find c. Taking the adjoint of both sides of (4) gives

$$\langle 1|S_-^{\dagger} = c^* \langle 2|$$

Therefore

$$\begin{aligned} \langle 1|S_-^{\dagger}\, S_-|1\rangle &= c^* c \langle 2|2\rangle \\ &= |c|^2 \langle 2|2\rangle \end{aligned}$$

Since c is real. But $\langle 2|2\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$. The above becomes

$$\langle 1|S_-^{\dagger}\, S_-|1\rangle = |c|^2 \tag{5}$$

To find c, we need now to calculate $\langle 1|S_-^{\dagger}\, S_-|1\rangle$. But

$$\begin{aligned} S_-^{\dagger}\, S_- &= \left(S_x - iS_y\right)^{\dagger} \left(S_x - iS_y\right) \\ &= \left(S_x^{\dagger} + iS_y^{\dagger}\right)\left(S_x - iS_y\right) \end{aligned}$$

Since S_x, S_y are Hermitian operators then $S_x^{\dagger} = S_x$ and $S_y^{\dagger} = S_y$. The above now becomes

$$\begin{aligned} S_-^{\dagger}\, S_- &= \left(S_x + iS_y\right)\left(S_x - iS_y\right) \\ &= S_x^2 - iS_x S_y + iS_y S_x + S_y^2 \\ &= S_x^2 + S_y^2 - i\left(S_x S_y - S_y S_x\right) \\ &= S_x^2 + S_y^2 - i[S_x, S_y] \end{aligned}$$

Where $[S_x, S_y]$ is the commutator. But $[S_x, S_y] = i\hbar \sum_k \epsilon_{ijk} S_k$. Using $i = 1, j = 2$ for x, y, then $[S_i, S_i] = i\hbar(\epsilon_{121} S_1 + \epsilon_{122} S_2 + \epsilon_{123} S_3) = i\hbar S_3 = i\hbar S_z$. Therefore the above now becomes

$$\begin{aligned} S_-^{\dagger}\, S_- &= S_x^2 + S_y^2 - i(i\hbar S_z) \\ &= S_x^2 + S_y^2 + \hbar S_z \end{aligned} \tag{6}$$

Substituting (6) in (5) gives

$$\langle 1|\left(S_x^2 + S_y^2 + \hbar S_z\right)|1\rangle = |c|^2$$

But $S^2 = S_x^2 + S_y^2 + S_z^2$. Hence $S_x^2 + S_y^2 = S^2 - S_z^2$. Using this in the above gives

$$\langle 1|\left(S^2 - S_z^2 + \hbar S_z\right)|1\rangle = |c|^2 \tag{7}$$

But $S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. Hence $S_z^2 = \frac{\hbar^2}{4}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \frac{\hbar^2}{4}I$. And since there is nothing special about the z direction, then $S_x^2 = S_y^2 = S_z^2$. Therefore $S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}I + \frac{\hbar^2}{4}I + \frac{\hbar^2}{4}I = \frac{3}{4}\hbar^2 I$. Eq. (7) now becomes

$$\langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \hbar S_z|1\rangle = |c|^2$$

But $S_z|1\rangle = \frac{\hbar}{2}|1\rangle$. This is because $|1\rangle$ is an eigenvector for S_z with an eigenvalue $\frac{\hbar}{2}$. The above becomes

$$\begin{aligned}
\langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \hbar\frac{\hbar}{2}|1\rangle &= |c|^2\\
\langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \frac{\hbar^2}{2}|1\rangle &= |c|^2\\
\langle 1|\hbar^2|1\rangle &= |c|^2\\
\hbar^2\langle 1|1\rangle &= |c|^2\\
\hbar^2 &= |c|^2
\end{aligned}$$

We pick

$$c = \hbar$$

Now that c is found, then Eq. (4) above becomes

$$S_-|1\rangle = \hbar|2\rangle \tag{8}$$

The same method is now repeated for finding $S_-|2\rangle$

$$\begin{aligned}
S_z S_-|2\rangle &= (S_- S_z - \hbar S_-)|2\rangle\\
&= S_- S_z|2\rangle - \hbar S_-|2\rangle
\end{aligned}$$

But $S_z|2\rangle = -\frac{\hbar}{2}|2\rangle$. The above becomes

$$\begin{aligned}
S_z S_-|2\rangle &= -S_-\frac{\hbar}{2}|2\rangle - \hbar S_-|2\rangle\\
&= \left(-\frac{\hbar}{2} - \hbar\right)S_-|2\rangle\\
&= \left(-\frac{3\hbar}{2}\right)S_-|2\rangle
\end{aligned}$$

The above shows that $S_-|2\rangle$ is eigenvector (eigenstate) of S_z with eigenvalue $-\frac{3\hbar}{2}$ which conflicts with experiments. This means

$$S_-|2\rangle = 0|2\rangle \tag{9}$$

is the only logical result. Therefore now we have all the information to find matrix representation of S_- using (8,9), which is

$$\begin{aligned} S_- &= \begin{bmatrix} \langle 1|S_-|1\rangle & \langle 1|S_-|2\rangle \\ \langle 2|S_-|1\rangle & \langle 2|S_-|2\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 1|\hbar|2\rangle & \langle 1|0|2\rangle \\ \langle 2|\hbar|2\rangle & \langle 2|0|2\rangle \end{bmatrix} \\ &= \hbar \begin{bmatrix} \langle 1|2\rangle & 0 \\ \langle 2|2\rangle & 0 \end{bmatrix} \\ &= \hbar \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 0 \end{bmatrix} \\ &= \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} S_- &= S_x - iS_y \\ &= \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Which is what we are asked to show.

4.6.5 Problem 4 (9.6.2)

Find the solutions $x_1(t), x_2(t)$ with initial conditions $x_1(0) = 0, x_2(0) = 0$ and $\dot{x}_1(0) = v_1, \dot{x}_2(0) = v_2$.

Solution

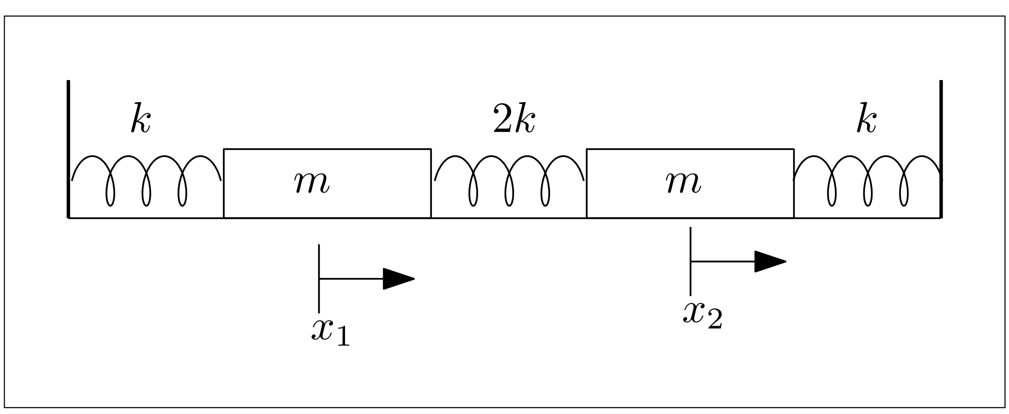


Figure 4.14: Coupled system to solve

The first step is to draw the free body diagram for each mass. Let us assume that first mass is at some positive distance $x_1 > 0$ so that the first string is in tension, and that $x_2 > x_1 > 0$ so that the middle spring is in tension also, and the third spring is in compression. Any other configuration will also work as well. Based on this, the free body diagrams are

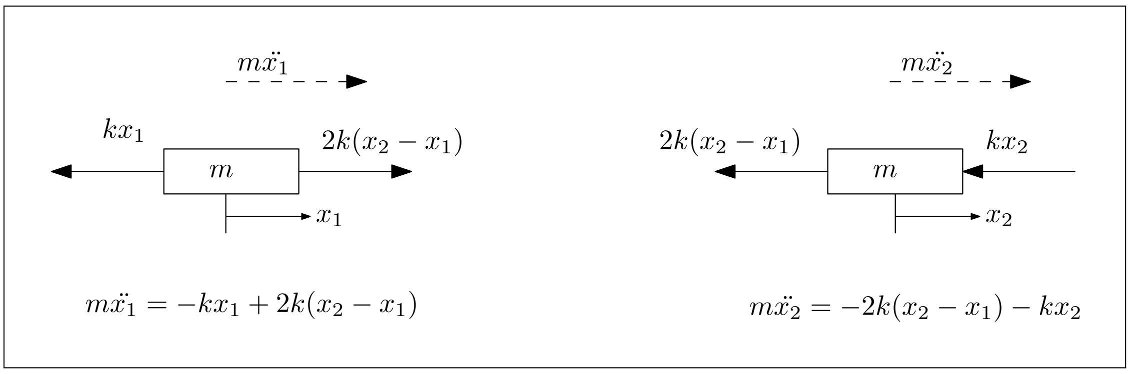


Figure 4.15: Free body diagram

From the free body diagram, we can now write the equation of motion based on $F = ma$ from each mass. This gives

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + 2k(x_2 - x_1) \\ m\ddot{x}_2 &= -2k(x_2 - x_1) - kx_2 \end{aligned}$$

or

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + 2kx_2 - 2kx_1 \\ m\ddot{x}_2 &= -2kx_2 + 2kx_1 - kx_2 \end{aligned}$$

or

$$\begin{aligned} m\ddot{x}_1 &= x_1(-k - 2k) + x_2(2k) \\ m\ddot{x}_2 &= x_1(2k) + x_2(-2k - k) \end{aligned}$$

or

$$\begin{aligned} \ddot{x}_1 &= -\frac{3k}{m}x_1 + \frac{2k}{m}x_2 \\ \ddot{x}_2 &= \frac{2k}{m}x_1 - 3\frac{k}{m}x_2 \end{aligned}$$

In matrix form the above becomes

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{k}{m}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|\ddot{x}\rangle + M|x\rangle = |0\rangle \tag{1}$$

Where the operator M is

$$M = \frac{k}{m}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \tag{1A}$$

In (1), the state vector is $|x\rangle$ is represented using basis $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, since we can write

$$|x\rangle = x_1|1\rangle + x_2|2\rangle$$

In these basis, called the natural coordinates, we see than operator M is not diagonal. This makes solving (1) harder, since it is now a coupled system of ODE's.

We would like to decouple (1) to make solving each ODE separate and easier. To do this, we change the basis of M. The new basis are $|I\rangle, |II\rangle$. These are the eigenvectors of M. Since M is Hermitian, then its eigenvalues will be real, and its eigenvectors are orthogonal. So now we need to first find the eigenvalues of M given in (1A) by solving

$$\det(M - \omega I) = 0$$
$$\det\left(\frac{k}{m}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} - \omega\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\left(\frac{k}{m}\begin{bmatrix} 3-\omega & -2 \\ -2 & 3-\omega \end{bmatrix}\right) = 0$$

This gives (we remove the factor $\frac{k}{m}$ for now, then add it at the end to simplify the computation)

$$(3-\omega)(3-\omega) - 4 = 0$$
$$\omega^2 - 6\omega + 5 = 0$$
$$(\omega - 5)(\omega - 1) = 0$$

Hence the eigenvalues are (now we add back the factor $\frac{k}{m}$)

$$\omega_1 = \frac{5k}{m} \qquad \omega_1 = \frac{k}{m}$$

For $\omega_1 = \frac{k}{m}$

We need to solve

$$\begin{bmatrix} \frac{3k}{m} - \omega_1 & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \omega_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3k}{m} - \frac{k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \frac{k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{2k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_2 is free variable. Let $v_2 = s$. First row gives $\frac{2k}{m}v_1 - \frac{2k}{m}s = 0$ or $v_1 = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$ then

$$|I\rangle = |M = \omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\omega_1 = 5\frac{k}{m}$

We need to solve

$$\begin{bmatrix} \frac{3k}{m} - \omega_2 & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3k}{m} - 5\frac{k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - 5\frac{k}{m} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_2 is free variable. Let $v_2 = s$. First row gives $-\frac{2k}{m}v_1 - \frac{2k}{m}s = 0$ or $v_1 = -s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$ then

$$|II\rangle = |M = \omega_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>Summary table of results so far M</u>

eigenvalue	eigenfrequncy	eigenvector
$\omega_1 = \frac{k}{m}$	$\sqrt{\frac{k}{m}}$	$\lvert I\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\omega_2 = 5\frac{k}{m}$	$\sqrt{\frac{5k}{m}}$	$\lvert II\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The transformation matrix Φ becomes

$$\begin{aligned} \Phi &= \begin{bmatrix} |I\rangle & |II\rangle \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{aligned} \tag{2}$$

Now that we found the transformation matrix Φ we can use it to transform $|\ddot{x}\rangle + M|x\rangle = 0$ which is the natural coordinates basis $|1\rangle, |2\rangle$, to the <u>modal coordinates</u> based on basis $|I\rangle, |II\rangle$ as follows

$$|x\rangle = \Phi|X\rangle \tag{3}$$

$$|\ddot{x}\rangle = \Phi|\ddot{X}\rangle \tag{4}$$

Where

$$|X\rangle = X_1|I\rangle + X_2|II\rangle$$

is the state vector in the modal coordinate and $|\ddot{X}\rangle$ is the acceleration of the state vector in modal coordinates. Applying Eq. (3,4) to $|\ddot{x}\rangle = M|x\rangle$ gives the system in the modal coordinates as

$$\Phi|\ddot{X}\rangle + M\Phi|X\rangle = 0$$

Premultiplying both sides by Φ^T (since Φ is real, then transpose is same as dagger), gives

$$\Phi^T\Phi|\ddot{X}\rangle + \Phi^T M\Phi|X\rangle = 0 \tag{5}$$

But by definition of the modal transformation matrix[1]

$$\Phi^T\Phi = I \tag{6}$$

And by definition of the transformation matrix [2]

$$\begin{aligned}\Phi^T M\Phi &= \begin{bmatrix}\omega_1 & 0\\ 0 & \omega_2\end{bmatrix}\\ &= \begin{bmatrix}\frac{k}{m} & 0\\ 0 & \frac{5k}{m}\end{bmatrix}\end{aligned} \tag{7}$$

Using (6,7) in (5) gives the system in modal coordinates

$$\begin{aligned}|\ddot{X}\rangle + \begin{bmatrix}\frac{k}{m} & 0\\ 0 & \frac{5k}{m}\end{bmatrix}|X\rangle &= 0\\ \begin{bmatrix}\ddot{X}_1\\ \ddot{X}_2\end{bmatrix} + \begin{bmatrix}\frac{k}{m} & 0\\ 0 & \frac{5k}{m}\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned} \tag{8}$$

The above is what solve, since it is now decoupled. Comparing (8) to (1) which is repeated below

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}\frac{3k}{m} & -\frac{2k}{m}\\ -\frac{2k}{m} & \frac{3k}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \tag{1}$$

Shows clearly why (8) is much simpler to solve in the modal coordinates basis $|I\rangle, |II\rangle$ since it is now decoupled, while Eq (1) which is in natural coordinates basis $|1\rangle, |2\rangle$ is coupled.

Eq (8) is now solved for $|X\rangle$, and at the end transformed back to $|x\rangle$ using Eq. (3). Eq (8) above can be written as two separate ODE's

$$\begin{aligned}\ddot{X}_1 + \frac{k}{m}X_1 &= 0\\ \ddot{X}_2 + \frac{5k}{m}X_2 &= 0\end{aligned}$$

[1]This can also be shown for $\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ by working it out. $\Phi^T\Phi = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

[2]This can also be shown by working it out as follows. $\Phi^T M\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^T \frac{k}{m}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ which becomes $\Phi^T M\Phi = \frac{k}{2m}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}\frac{k}{m} & 0\\ 0 & \frac{5k}{m}\end{bmatrix}$

Before solving the above, the initial conditions, given in the natural coordinates, needs to be transformed to modal coordinates. It is not clear which initial conditions we should use, since book uses

$$\begin{aligned} x_1(t=0) &= x_1(0) & \dot{x}_1(t=0) &= 0 \\ x_2(t=0) &= x_2(0) & \dot{x}_2(t=0) &= 0 \end{aligned}$$

And in the HW pdf, we are also asked to use the following initial conditions

$$\begin{aligned} x_1(t=0) &= 0 & \dot{x}_1(t=0) &= v_1 \\ x_2(t=0) &= 0 & \dot{x}_2(t=0) &= v_2 \end{aligned}$$

Should we solve it for both cases, or just the second case? I will solve the problem for both cases, since I am not sure which to pick.

4.6.5.1 Part 1

Solving using book initial conditions

$$\begin{aligned} x_1(t=0) &= x_1(0) & \dot{x}_1(0) &= 0 \\ x_2(t=0) &= x_2(0) & \dot{x}_2(0) &= 0 \end{aligned}$$

Since $|x\rangle = \Phi|X\rangle$ then the inverse is

$$|X\rangle = \Phi^{-1}|x\rangle$$

But $\Phi^{-1} = \Phi^T$ therefore

$$\begin{aligned} |X(0)\rangle &= \Phi^T|x_1(0)\rangle \\ \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} x_1(0) + x_2(0) \\ -x_1(0) + x_2(0) \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} |\dot{X}(0)\rangle &= \Phi^T|\dot{x}_1(0)\rangle \\ \begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Since $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now that we found the initial conditions in modal coordinates, we can solve Eq. (8). Here it is again

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1(0) + x_2(0) \\ -x_1(0) + x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation of (9) becomes

$$\begin{aligned}\ddot{X}_1 + \frac{k}{m}X_1 &= 0\\ X_1(0) &= \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\\ \dot{X}_1(0) &= 0\end{aligned}$$

The solution is

$$X_1(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{10}$$

Where A, B are the constants of integrations. At $t = 0$ and from the initial conditions, the above becomes

$$\frac{1}{\sqrt{2}}(x_1(0)+x_2(0)) = A$$

Taking time derivative of (10) gives

$$\dot{X}_1 = -A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + B\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Since $\dot{X}_1(0) = 0$ then the above becomes

$$0 = B\sqrt{\frac{k}{m}}$$

Hence $B = 0$. Therefore the solution of Eq (10) is

$$X_1 = \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}t\right) \tag{11}$$

Tthe second ODE in (9) is now solved.

$$\begin{aligned}\ddot{X}_2 + \frac{5k}{m}X_1 &= 0\\ X_2(0) &= \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\\ \dot{X}_2(0) &= 0\end{aligned}$$

The solution is

$$X_2 = A\cos\left(\sqrt{\frac{5k}{m}}t\right) + B\sin\left(\sqrt{\frac{5k}{m}}t\right) \tag{12}$$

Where A, B are the constants of integrations. At $t = 0$,

$$\frac{1}{\sqrt{2}}(-x_1(0)+x_2(0)) = A$$

Taking time derivative of (12) gives

$$\dot{X}_2 = -A\sqrt{\frac{5k}{m}}\sin\left(\sqrt{\frac{5k}{m}}t\right) + B\sqrt{\frac{5k}{m}}\cos\left(\sqrt{\frac{5k}{m}}t\right)$$

At $t = 0$ the above becomes

$$0 = B\sqrt{\frac{5k}{m}}$$

Hence $B = 0$. The solution of Eq (12) becomes

$$X_2 = \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}t\right) \tag{13}$$

Therefore the solution is

$$|X\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix} \tag{14}$$

This is the final solution. But it is in modal coordinates. This is transformed back to natural coordinates using Eq (3)

$$|x\rangle = \Phi|X\rangle$$

Therefore

$$\begin{aligned}
|x\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} (x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) + (x_1(0)-x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \\ (x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) - (x_1(0)-x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix}
\end{aligned} \tag{15}$$

Hence

$$x_1(t) = \frac{x_1(0)+x_2(0)}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{x_1(0)-x_2(0)}{2}\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{16}$$

$$x_2(t) = \frac{x_1(0)+x_2(0)}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{x_1(0)-x_2(0)}{2}\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{17}$$

The above is the final solution in the natural coordinates. The above is repeated using the other initial conditions given in the PDF file.

4.6.5.2 Part 2

Solving using book initial conditions

$$\begin{aligned} x_1(t=0) &= 0 & \dot{x}_1(t=0) &= v_1 \\ x_2(t=0) &= 0 & \dot{x}_2(t=0) &= v_2 \end{aligned}$$

Using $|x\rangle = \Phi|X\rangle$ then

$$|X\rangle = \Phi^{-1}|x\rangle$$

But $\Phi^{-1} = \Phi^T$ then

$$\begin{aligned}
|X(0)\rangle &= \Phi^T|x_1(0)\rangle \\
\begin{bmatrix}X_1(0)\\ X_2(0)\end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^T\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}\\
&= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}$$

And

$$
\begin{aligned}
|\dot{X}(0)\rangle &= \Phi^T |\dot{x}_1(0)\rangle \\
\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_2 \end{bmatrix}
\end{aligned}
$$

Now that we found initial conditions in modal coordinates, we can finally solve the (8). Here it is again

$$
\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}
$$

$$
\begin{aligned}
\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_2 \end{bmatrix}
\end{aligned}
$$

The first equation of (18) becomes

$$
\begin{aligned}
\ddot{X}_1 + \frac{k}{m} X_1 &= 0 \\
X_1(0) &= 0 \\
\dot{X}_1(0) &= \frac{1}{\sqrt{2}} (v_1 + v_2)
\end{aligned}
$$

The solution is (since SHM)

$$
X_1 = A \cos\left(\sqrt{\frac{k}{m}} t \right) + B \sin\left(\sqrt{\frac{k}{m}} t \right) \tag{19}
$$

Where A, B are the constants of integrations. At $t = 0$,

$$
0 = A
$$

The solution (19) becomes

$$
X_1 = B \sin\left(\sqrt{\frac{k}{m}} t \right)
$$

Taking time derivative of the above gives

$$
\dot{X}_1 = B \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}} t \right)
$$

At $t = 0$ the above becomes

$$
\begin{aligned}
\frac{1}{\sqrt{2}} (v_1 + v_2) &= B \sqrt{\frac{k}{m}} \\
B &= \sqrt{\frac{m}{k}} \frac{1}{\sqrt{2}} (v_1 + v_2)
\end{aligned}
$$

Therefore the solution of Eq (19) is

$$
X_1 = \sqrt{\frac{m}{2k}} (v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}} t \right) \tag{20}
$$

The second ODE in (18) is now solved.

$$\begin{aligned}\ddot{X}_2 + \frac{5k}{m}X_1 &= 0\\ X_2(0) &= 0\\ \dot{X}_2(0) &= \frac{1}{\sqrt{2}}(-v_1 + v_2)\end{aligned}$$

The solution is

$$X_2 = A\cos\left(\sqrt{\frac{5k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{21}$$

Where A, B are the constants of integrations. At $t = 0$,

$$0 = A$$

The solution (21) becomes

$$X_2 = B\sin\left(\sqrt{\frac{5k}{m}}\,t\right)$$

Taking time derivative gives

$$\dot{X}_2 = B\sqrt{\frac{5k}{m}}\cos\left(\sqrt{\frac{5k}{m}}\,t\right)$$

At $t = 0$ the above becomes

$$\begin{aligned}\frac{1}{\sqrt{2}}(-v_1 + v_2) &= B\sqrt{\frac{5k}{m}}\\ B &= \sqrt{\frac{m}{10k}}(-v_1 + v_2)\end{aligned}$$

Therefore the solution of Eq (21) is

$$X_2 = \sqrt{\frac{m}{10k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{22}$$

Therefore the solution state vector is

$$|X\rangle = \begin{bmatrix}\sqrt{\frac{m}{2k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right)\\ \sqrt{\frac{m}{10k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\end{bmatrix} \tag{23}$$

This is the final solution. But it is in modal coordinates. It is now transformed back to natural coordinates using Eq (3)

$$|x\rangle = \Phi|X\rangle$$

Therefore

$$\begin{aligned}|x\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{\frac{m}{2k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right)\\ \sqrt{\frac{m}{10k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{\frac{m}{2k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) - \sqrt{\frac{m}{10k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\\ \sqrt{\frac{m}{2k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \sqrt{\frac{m}{10k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\end{bmatrix}\\ &= \begin{bmatrix}\sqrt{\frac{m}{4k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) - \sqrt{\frac{m}{20k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\\ \sqrt{\frac{m}{4k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \sqrt{\frac{m}{20k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right)\end{bmatrix} \qquad (24)\end{aligned}$$

Hence

$$x_1(t) = \frac{1}{2}\sqrt{\frac{m}{k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{\frac{m}{20k}}(v_1 - v_2)\sin\left(\sqrt{\frac{5k}{m}}t\right) \tag{25}$$

$$x_2(t) = \frac{1}{2}\sqrt{\frac{m}{k}}(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{\frac{m}{20k}}(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}t\right) \tag{26}$$

The above is the final solution in the natural coordinates.

4.6.6 key solution for HW 6

Physics 3041 (Spring 2021) Solutions to Homework Set 6

1. Problem 9.5.11. (40 points)

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) From the diagonal form of S_z, we know that its eigenvalues are $s_z = 1,\ 0,\ -1$ corresponding to eigenvectors

$$|s_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ |s_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ |s_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the possible measured values for S_z are 1, 0, −1.

(b) For S_x,

$$\begin{vmatrix} -s_x & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -s_x & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -s_x \end{vmatrix} = -s_x(s_x^2 - \frac{1}{2}) - \frac{1}{\sqrt{2}}\frac{(-s_x)}{\sqrt{2}} = s_x(1 - s_x^2) = s_x(1 - s_x)(1 + s_x)$$

$$\Rightarrow s_x = 1,\ 0,\ -1.$$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} -a_1 + b_1/\sqrt{2} \\ (a_1 + c_1)/\sqrt{2} - b_1 \\ b_1/\sqrt{2} - c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = 1\rangle = a_1\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_2/\sqrt{2} \\ (a_2 + c_2)/\sqrt{2} \\ b_2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = 0\rangle = a_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_3 + b_3/\sqrt{2} \\ (a_3 + c_3)/\sqrt{2} + b_3 \\ b_3/\sqrt{2} + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = -1\rangle = a_3\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

For S_y,

$$\begin{vmatrix} -s_y & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -s_y & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -s_y \end{vmatrix} = -s_y(s_y^2 - \frac{1}{2}) + \frac{i}{\sqrt{2}}\frac{(-is_y)}{\sqrt{2}} = s_y(1-s_y^2) = s_y(1-s_y)(1+s_y)$$

$$\Rightarrow s_y = 1,\ 0,\ -1.$$

So for both S_x and S_y, the possible measured values are 1, 0, and -1.

(c) After measuring the largest possible value of $s_x = 1$, the state vector is

$$|s_x = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) If S_z is measured, the possible values are 1, 0, and -1 with probabilities of $(1/2)^2 = 1/4$, $(1/\sqrt{2})^2 = 1/2$, and $(1/2)^2 = 1/4$, respectively.

If the largest possible value of $s_z = 1$ is measured, the state vector becomes

$$|s_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Because $|s_z = 1\rangle$ differs from $|s_x = 1\rangle$, the probability of measuring $s_x = 1$ is

$$|\langle s_x = 1|s_z = 1\rangle|^2 = \left|\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right|^2 = \frac{1}{4}.$$

(e) From

$$\begin{aligned} S^2 &= S_x^2 + S_y^2 + S_z^2 \\ &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \end{aligned}$$

the measured S^2 value is always 2 for any state vector because $S^2 = 2I$.

(f) From the matrix representation and the results in (a) and (b), S_x, S_y, and S_z are Hermitian operators with non-degenerate eigenvalues. So if S_z commutes with S_x or S_y, they would share the same eigenvectors and be diagonal in the corresponding eigenbasis. However, because S_x and S_y are not diagonal in the basis where S_z is diagonal, we conclude S_z does not commute with either S_x or S_y.

Although we did not solve for the eigenvectors of S_y, it is clear that they are distinct from those of S_x because S_x differs from S_y in the structure of matrix elements but both operators have the same eigenvalues. So S_x and S_y do not commute, either.

On the other hand, S^2 commutes with S_x, S_y, and S_z. Therefore, the maximum number of commuting operators is 2, which corresponds to S^2 and any one of the other three (i.e., S_x, or S_y, or S_z).

(g) From

$$|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \langle V|V\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1+4+9 = 14,$$

the normalized state vector is

$$|V'\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{\sqrt{14}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{\sqrt{14}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the probabilities for measuring $s_z = 1$, 0, and -1 are 1/14, 2/7, and 9/14, respectively. The statistical average of the measured values is $\langle s_z\rangle = 1\times(1/14)+0\times(2/7)+(-1)\times(9/14) = -4/7$, which is the same as

$$\langle V'|S_z|V'\rangle = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = -\frac{4}{7}.$$

(h) The probabilities for measuring $s_x = 1$, 0, and -1 are

$$|\langle s_x = 1|V'\rangle|^2 = \frac{1}{4\times 14} \left| \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1+2\sqrt{2}+3)^2}{56} = \frac{3+2\sqrt{2}}{7}$$

$$|\langle s_x = 0|V'\rangle|^2 = \frac{1}{2\times 14} \left| \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1-3)^2}{28} = \frac{1}{7}$$

$$|\langle s_x = -1|V'\rangle|^2 = \frac{1}{4\times 14} \left| \begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1-2\sqrt{2}+3)^2}{56} = \frac{3-2\sqrt{2}}{7},$$

respectively. The statistical average of the measured values is $\langle s_x\rangle = 1\times(3+2\sqrt{2})/7 + 0\times(1/7) + (-1)\times(3-2\sqrt{2})/7 = 4\sqrt{2}/7$, which is the same as

$$\langle V'|S_s|V'\rangle = \frac{1}{14\sqrt{2}} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14\sqrt{2}} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \frac{4\sqrt{2}}{7}.$$

2. Prove the following results on the commutators: $[A, B + C] = [A, B] + [A, C]$, $[A + B, C] = [A, C] + [B, C]$, $[A, BC] = B[A, C] + [A, B]C$, $[AB, C] = A[B, C] + [A, C]B$. (10 points)

$$
\begin{aligned}
[A, B + C] &= A(B + C) - (B + C)A = AB + AC - BA - CA = [A, B] + [A, C] \\
[A + B, C] &= (A + B)C - C(A + B) = AC + BC - CA - CB = [A, C] + [B, C] \\
[A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C] \\
[AB, C] &= ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B
\end{aligned}
$$

3. Follow the discussion of $s_+ = s_x + is_y$ for the electron spin to derive the matrix representation of $s_- = s_x - is_y$. (20 points)

$$
\begin{aligned}
[s_z, s_-] &= [s_z, s_x - is_y] = [s_z, s_x] - i[s_z, s_y] = i\hbar s_y - i(-i\hbar s_x) = -\hbar(s_x - is_y) = -\hbar s_- \\
[s_z, s_-] &= s_z s_- - s_- s_z = -\hbar s_- \Rightarrow s_z s_- = s_- s_z - \hbar s_- \\
s_z|1\rangle &= \frac{\hbar}{2}|1\rangle \Rightarrow s_z s_-|1\rangle = (s_- s_z - \hbar s_-)|1\rangle = (s_-\frac{\hbar}{2} - \hbar s_-)|1\rangle = -\frac{\hbar}{2}s_-|1\rangle \\
s_z|2\rangle &= -\frac{\hbar}{2}|2\rangle \Rightarrow s_-|1\rangle = c|2\rangle \\
(s_-|1\rangle)^\dagger &= \langle 1|s_-^\dagger = \langle 1|(s_x - is_y)^\dagger = \langle 1|(s_x^\dagger + is_y^\dagger) = \langle 1|(s_x + is_y) = \langle 1|s_+ = c^*\langle 2| \\
\langle 1|s_+s_-|1\rangle &= c^*c\langle 2|2\rangle = |c|^2 \\
\langle 1|s_+s_-|1\rangle &= \langle 1|(s_x + is_y)(s_x - is_y)|1\rangle = \langle 1|s_x^2 + s_y^2 - i(s_x s_y - s_y s_x)|1\rangle = \langle 1|s^2 - s_z^2 - i(i\hbar s_z)|1\rangle \\
&= \langle 1|s^2 - s_z^2 + \hbar s_z|1\rangle = \frac{3}{4}\hbar^2 - \frac{\hbar}{2} \times \frac{\hbar}{2} + \frac{\hbar^2}{2} = \hbar^2 = |c|^2
\end{aligned}
$$

pick $c = \hbar \Rightarrow s_-|1\rangle = \hbar|2\rangle$, $\langle 1|s_-|1\rangle = \langle 1|\hbar|2\rangle = 0$, $\langle 2|s_-|1\rangle = \langle 2|\hbar|2\rangle = \hbar$

$$
s_z|2\rangle = -\frac{\hbar}{2}|2\rangle \Rightarrow s_z s_-|2\rangle = (s_- s_z - \hbar s_-)|2\rangle = [s_-(-\frac{\hbar}{2}) - \hbar s_-]|2\rangle = -\frac{3\hbar}{2}s_-|2\rangle
$$

The above result appears to imply that $s_-|2\rangle$ is an eigenstate of s_z with an eigenvalue of $-3\hbar/2$, which is in conflict with experiments. So the only logical result is

$$
s_-|2\rangle = 0|2\rangle \Rightarrow \langle 1|s_-|2\rangle = 0,\ \langle 2|s_-|2\rangle = 0.
$$

Finally, we obtain

$$
s_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
$$

4. Problem 9.6.2, and find the solutions for $x_1(t)$ and $x_2(t)$ with the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = v_1$ and $\dot{x}_2(0) = v_2$. (30 points)

$$m\ddot{x}_1 = -kx_1 + 2k(x_2 - x_1) = -3kx_1 + 2kx_2,\ \ddot{x}_1 = -\frac{3k}{m}x_1 + \frac{2k}{m}x_2$$
$$m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2 = -3kx_2 + 2kx_1,\ \ddot{x}_2 = -\frac{3k}{m}x_2 + \frac{2k}{m}x_1$$
$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3k/m & 2k/m \\ 2k/m & -3k/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{vmatrix} -3k/m - \lambda & 2k/m \\ 2k/m & -3k/m - \lambda \end{vmatrix} = \left(-\frac{3k}{m} - \lambda\right)^2 - \left(\frac{2k}{m}\right)^2 = 0 \Rightarrow \lambda_I = -\frac{k}{m},\ \lambda_{II} = -\frac{5k}{m}$$
$$\begin{bmatrix} -2k/m & 2k/m \\ 2k/m & -2k/m \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \frac{2k}{m}\begin{bmatrix} -a_1 + b_1 \\ a_1 - b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |I\rangle = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{vmatrix} 2k/m & 2k/m \\ 2k/m & 2k/m \end{vmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \frac{2k}{m}\begin{bmatrix} a_2 + b_2 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |II\rangle = a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|x(t)\rangle = x_I(t)|I\rangle + x_{II}(t)|II\rangle$$
$$\Rightarrow \frac{d^2}{dt^2}\begin{bmatrix} x_I \\ x_{II} \end{bmatrix} = \begin{bmatrix} -\frac{k}{m} & 0 \\ 0 & -\frac{5k}{m} \end{bmatrix}\begin{bmatrix} x_I \\ x_{II} \end{bmatrix} = \begin{bmatrix} -(k/m)x_I \\ -(5k/m)x_{II} \end{bmatrix} = \begin{bmatrix} -\omega_I^2 x_I \\ -\omega_{II}^2 x_{II} \end{bmatrix}$$

So the normal modes are

$$x_I(t) = \langle I|x(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_1(t) + x_2(t)}{\sqrt{2}},$$
$$x_{II}(t) = \langle II|x(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_1(t) - x_2(t)}{\sqrt{2}},$$

with eigenfrequencies $\omega_I = \sqrt{k/m}$ and $\omega_{II} = \sqrt{5k/m}$, respectively.

Applying the initial conditions $x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = v_1$, and $\dot{x}_2(0) = v_2$, we obtain $x_I(0) = x_{\text{II}}(0) = 0$, $\dot{x}_I(0) = (v_1 + v_2)/\sqrt{2}$, $\dot{x}_{II}(0) = (v_1 - v_2)/\sqrt{2}$, and the solutions

$$\begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} (v_1 + v_2)\omega_I^{-1}\sin\omega_I t \\ (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \end{bmatrix}.$$

Going back to the original basis,

$$|x(t)\rangle = x_I(t)|I\rangle + x_{II}(t)|II\rangle = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_I(t)}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_{II}(t)}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} (v_1 + v_2)\omega_I^{-1}\sin\omega_I t + (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \\ (v_1 + v_2)\omega_I^{-1}\sin\omega_I t - (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \end{bmatrix}.$$

4.7 HW 7

Local contents

4.7.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 7 **(Due 3/24)**

1. Problem 9.7.3. (15 points)

2. Problem 9.7.8. (35 points)

3. Perform appropriate integration to show the following results regarding the Dirac delta function (25 points):

$$\delta(ax) = \delta(x)/|a|, \text{ where } a \text{ is a real number,}$$
$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x_i}}, \text{ where } x_i \text{ satisfies } f(x_i) = 0,$$
$$\frac{d}{dx}\delta(x - x') = \delta(x - x')\frac{d}{dx'}\,.$$

4. For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured. (25 points)

4.7.2 Problem 1 (9.7.3)

(a) Expand $f(x)=\begin{cases}\frac{2xh}{L} & 0\leq x\leq\frac{L}{2}\\ \frac{2h(L-x)}{L} & \frac{L}{2}\leq x\leq L\end{cases}$ in an exponential Fourier series. (b) What do you think is the value of $\sum_{odd}\frac{1}{m^2}$ where sum is over all positive odd integers?

Solution

4.7.2.1 Part a

The period is L. The function $f(x)$ looks like the following (where L is choosing to be 2 and $h=1$, for illustration only)

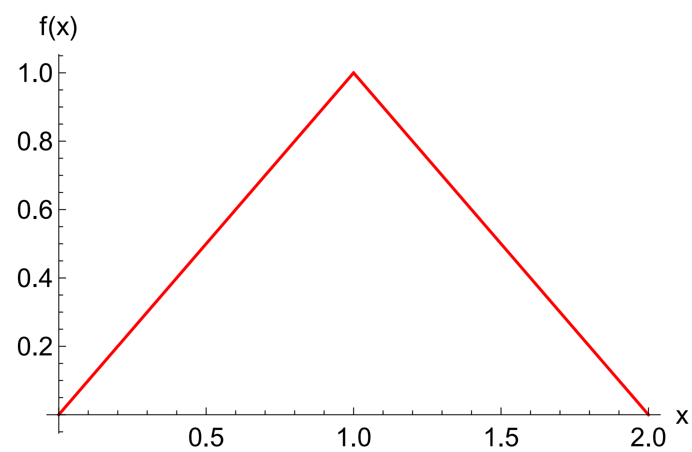


Figure 4.16: Plot of $f(x)$

```
L = 2; h = 1;
f[x_] := Piecewise[{{2 x h/L, 0 ≤ x ≤ L/2}, {2 h (L - x)/L, L/2 ≤ x ≤ L}}];
p = Plot[f[x], {x, 0, L}, AxesLabel → {"x", "f(x)"}, BaseStyle → 14, PlotStyle → Red];
```

Figure 4.17: Code used to generate the plot

The period is L. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle=\sum_m f_m|m\rangle \tag{1}$$

Where

$$\begin{aligned}f_m&=\langle m|f\rangle\\ &=\int_0^L\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}f(x)dx\end{aligned} \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle=\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x}\qquad m=0,\pm1,\pm2,\cdots$$

Putting these together gives

$$f(x)\sim\sum_{m=-\infty}^{\infty}f_m\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found

$$\begin{aligned} f_m &= \int_0^L \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x) dx \\ &= \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx \end{aligned} \tag{4}$$

For $m = 0$

$$\begin{aligned} f_0 &= \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} dx \\ &= \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x dx + \frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^L (L-x) dx \\ &= \frac{2h}{L\sqrt{L}} \left(\int_0^{\frac{L}{2}} x dx + \int_{\frac{L}{2}}^L (L-x) dx \right) \\ &= \frac{2h}{L\sqrt{L}} \left(\left(\frac{x^2}{2} \right)_0^{\frac{L}{2}} + \left(Lx - \frac{x^2}{2} \right)_{\frac{L}{2}}^L \right) \\ &= \frac{2h}{L\sqrt{L}} \left(\frac{1}{2} \frac{L^2}{4} + \left(L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{1}{2} \frac{L^2}{4} \right) \right) \\ &= \frac{\sqrt{L} h}{2} \end{aligned} \tag{5}$$

And for $m \neq 0$, the first integral in (4) is

$$\int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x e^{-i\frac{2\pi m}{L}x} dx$$

Integration by parts. Let $u = x, dv = e^{-i\frac{2\pi m}{L}x}$, then $du = 1, v = \frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}} = \frac{iL}{2\pi m} e^{-i\frac{2\pi m}{L}x}$. The above now becomes

$$
\begin{aligned}
\int_0^{\frac{L}{2}} \frac{2xh}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx &= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[xe^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}} - \int_0^{\frac{L}{2}}\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}dx\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}e^{-i\frac{2\pi m}{L}\frac{L}{2}}\right] - \frac{iL}{2\pi m}\int_0^{\frac{L}{2}} e^{-i\frac{2\pi m}{L}x}dx\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}e^{-i\pi m}\right] - \frac{iL}{2\pi m}\left[\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}(-1)^m\right] - \frac{i^2L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}\frac{L}{2}} - 1\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left[e^{-i2\pi m} - 1\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left((-1)^m - 1\right)\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}(-1)^m - \frac{L^2}{4\pi^2m^2}\right)\\
&= \frac{i2hL^2}{L\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL^2}{L\sqrt{L}\,4\pi^2m^2}(-1)^m - \frac{2hL^2}{L\sqrt{L}\,4\pi^2m^2}\\
&= \frac{i2hL}{\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL}{\sqrt{L}\,4\pi^2m^2}(-1)^m - \frac{2hL}{\sqrt{L}\,4\pi^2m^2}\\
&= \frac{i2h\sqrt{L}}{4\pi m}(-1)^m + \frac{2h\sqrt{L}}{4\pi^2m^2}(-1)^m - \frac{2h\sqrt{L}}{4\pi^2m^2}\\
&= \frac{ih\sqrt{L}}{2\pi m}(-1)^m + \frac{h\sqrt{L}}{2\pi^2m^2}(-1)^m - \frac{h\sqrt{L}}{2\pi^2m^2}\\
&= \frac{ih\sqrt{L}\,\pi m(-1)^m}{2\pi^2m^2} + \frac{h\sqrt{L}\,(-1)^m}{2\pi^2m^2} - \frac{h\sqrt{L}}{2\pi^2m^2}\\
&= \frac{ih\sqrt{L}\,\pi m(-1)^m + h\sqrt{L}\,(-1)^m - h\sqrt{L}}{2\pi^2m^2}
\end{aligned}
$$

Hence

$$
\int_0^{\frac{L}{2}} \frac{2xh}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx = h\sqrt{L}\,\frac{\left(i\pi m(-1)^m + (-1)^m - 1\right)}{2\pi^2m^2} \tag{6}
$$

Now the second integral in (4) is evaluated

$$
\int_{\frac{L}{2}}^{L} \frac{2h(L-x)}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx = \frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx
$$

Integration by parts. Let $u = L - x, dv = e^{-i\frac{2\pi m}{L}x}, du = -1, v = v = \frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}} = \frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}$. The integral becomes

$$
\begin{aligned}
\frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx &= \frac{2h}{L\sqrt{L}}\left(\left[(L-x)e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}+\int_{\frac{L}{2}}^{L}\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[0-\frac{L}{2}e^{-i\frac{2\pi m}{L}\frac{L}{2}}\right]+\frac{iL}{2\pi m}\int_{\frac{L}{2}}^{L}e^{-i\frac{2\pi m}{L}x}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}\left[e^{-i\pi m}\right]+\frac{iL}{2\pi m}\left[\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m+\frac{i^2L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m-\frac{L^2}{4\pi^2m^2}\left[e^{-i2\pi m}-e^{-i\pi m}\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m-\frac{L^2}{4\pi^2m^2}\left[1-(-1)^m\right]\right)\\
&= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m-\frac{2hL}{4\sqrt{L}\,\pi^2m^2}\left[1-(-1)^m\right]\\
&= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m-\frac{2hL}{4\sqrt{L}\,\pi^2m^2}-(-1)^m\frac{2hL}{4\sqrt{L}\,\pi^2m^2}\\
&= \frac{-i2h\pi mL}{4\pi^2m^2}(-1)^m-\frac{2hL}{4\pi^2m^2}-(-1)^m\frac{2h\sqrt{L}}{4\pi^2m^2}\\
&= \frac{-i2h\pi mL(-1)^m-2h\sqrt{L}-2h\sqrt{L}\,(-1)^m}{4\pi^2m^2}
\end{aligned}
$$

Therefore

$$
\frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx = h\sqrt{L}\,\frac{\left(-i\pi m(-1)^m-(-1)^m-1\right)}{2\pi^2m^2} \tag{7}
$$

Therefore, using (5,6,7) gives

$$
f_m=\begin{cases}\frac{\sqrt{L}h}{2} & m=0\\ h\sqrt{L}\,\frac{\left(i\pi m(-1)^m+(-1)^m-1\right)}{2\pi^2m^2}+h\sqrt{L}\,\frac{\left(-i\pi m(-1)^m-(-1)^m-1\right)}{2\pi^2m^2} & m\neq 0\end{cases}
$$

The above can be simplified more to

$$
f_m=\begin{cases}\frac{\sqrt{L}h}{2} & m=0\\ h\sqrt{L}\,\frac{(-1)^m-1-(-1)^m-1}{2\pi^2m^2} & m\neq 0\end{cases}
$$

Now, for $m=\pm 2,\pm 4,\cdots$ even, the above becomes

$$
f_{m_{even}}=\begin{cases}\frac{\sqrt{L}h}{2} & m=0\\ 0 & m\neq 0, even\end{cases}
$$

And for or $m=\pm 1,\pm 3,\cdots$ odd, it becomes

$$
f_{m_{odd}}=\begin{cases}\frac{\sqrt{L}h}{2} & m=0\\ -h\sqrt{L}\,\frac{2}{\pi^2m^2} & m\neq 0, odd\end{cases}
$$

Therefore only the odd terms survive. From (3)

$$\begin{aligned}
f(x) &\sim \sum_{odd} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{\sqrt{L}h}{2}\frac{1}{\sqrt{L}} + \sum_{odd} \frac{-2h\sqrt{L}}{\pi^2 m^2}\frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{h}{2} - h\sum_{odd} \frac{2}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x} \\
&= \frac{h}{2} - \frac{h}{2}\sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x}
\end{aligned}$$

Or

$$f(x) \sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x}\right) \tag{8}$$

4.7.2.2 Part b

From Eq (8), by letting $x = \frac{L}{2}$, it becomes

$$\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\pi m}\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{n=-\infty,odd}^{-1} \frac{4}{\pi^2 n^2} e^{i\pi n} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)
\end{aligned}$$

Replacing $m = -n$ in the first sum above gives

$$\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \left(\sum_{m=\infty,odd}^{1} \frac{4}{\pi^2 (-m)^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{m=1,odd}^{\infty} \frac{4}{\pi^2 m^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)
\end{aligned}$$

Combining the terms and calling the common index n gives

$$\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{4}{\pi^2 m^2}\left(e^{i\pi n} + e^{-i\pi n}\right)\right)\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2} \cos(n\pi)\right)\right)
\end{aligned}$$

But $\cos(\pi n) = -1$ since n and odd. The above becomes

$$f\left(x = \frac{L}{2}\right) \sim \frac{h}{2}\left(1 + \sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2}\right)$$

but $f\left(x = \frac{L}{2}\right) = \left[\frac{2xh}{L}\right]_{x=\frac{L}{2}} = h$. Hence the above becomes

$$\begin{aligned}
h &= \frac{h}{2}\left(1 + \sum_{odd} \frac{8}{\pi^2 m^2}\right) \\
2 &= 1 + \sum_{odd} \frac{8}{\pi^2 m^2} \\
1 &= \frac{8}{\pi^2}\sum_{odd} \frac{1}{m^2}
\end{aligned}$$

Therefore

$$\sum_{odd} \frac{1}{m^2} = \frac{\pi^2}{8}$$

4.7.3 Problem 2 (9.7.8)

(i) Obtain the series in terms of sines and cosine for $f(x)=e^{-|x|}$ in the interval $-1\leq x\leq 1$.
(ii) repeat for the case $f(x)=\cosh x$. Show that

$$f(x)\sim\frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx-n\sin x)\right)$$

represents e^x in the interval $-\pi\leq x\leq\pi$ (and its periodicized version outside). Show how you can get the series for $\sinh x$ and $\cosh x$ from the above.

solution

4.7.3.1 Part 1

The period is $L=2$ in this case. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle=\sum_m f_m|m\rangle \tag{1}$$

Where

$$\begin{aligned} f_m &= \langle m|f\rangle \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}f(x)dx \end{aligned} \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle=\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x}\qquad m=0,\pm1,\pm2,\cdots$$

Putting these together gives

$$f(x)\sim\sum_{m=-\infty}^{\infty}f_m\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found, using $L=2$

$$\begin{aligned} f_m &= \int_{-1}^{1}\frac{1}{\sqrt{2}}e^{-i\pi mx}f(x)dx \\ &= \frac{1}{\sqrt{2}}\int_{-1}^{0}e^{-i\pi mx}e^{x}dx+\frac{1}{\sqrt{2}}\int_{0}^{1}e^{-i\pi mx}e^{-x}dx \end{aligned} \tag{44}$$

The first integral in (4) gives

$$\begin{aligned} \frac{1}{\sqrt{2}}\int_{-1}^{0}e^{-i\pi mx}e^{x}dx &= \frac{1}{\sqrt{2}}\int_{-1}^{0}e^{(-i\pi m+1)x}dx \\ &= \frac{1}{\sqrt{2}}\left[\frac{e^{(-i\pi m+1)x}}{-i\pi m+1}\right]_{-1}^{0} \\ &= \frac{1}{\sqrt{2}(-i\pi m+1)}\left[e^{(-i\pi m+1)x}\right]_{-1}^{0} \\ &= \frac{1}{\sqrt{2}(-i\pi m+1)}\left[1-e^{-(-i\pi m+1)}\right] \\ &= \frac{1}{\sqrt{2}}\frac{e^{i\pi m-1}-1}{i\pi m-1} \end{aligned} \tag{5}$$

And the second integral in (4) gives

$$
\begin{aligned}
\frac{1}{\sqrt{2}}\int_0^1 e^{-i\pi m x}e^{-x}dx &= \frac{1}{\sqrt{2}}\int_0^1 e^{(-i\pi m-1)x}dx\\
&= \frac{1}{\sqrt{2}}\left[\frac{e^{(-i\pi m-1)x}}{-i\pi m-1}\right]_0^1\\
&= \frac{1}{\sqrt{2}\,(-i\pi m-1)}\left[e^{(-i\pi m-1)x}\right]_0^1\\
&= \frac{-1}{\sqrt{2}\,(i\pi m+1)}\left[e^{(-i\pi m-1)}-1\right]\\
&= \frac{1}{\sqrt{2}}\frac{1-e^{-i\pi m-1}}{i\pi m+1} \qquad (6)
\end{aligned}
$$

Putting (5,6) together gives

$$
f_m = \frac{1}{\sqrt{2}}\frac{e^{i\pi m-1}-1}{i\pi m-1}+\frac{1}{\sqrt{2}}\frac{1-e^{-i\pi m-1}}{i\pi m+1} \qquad (7)
$$

For $m=0$, eq (7) becomes

$$
\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{e^{-1}-1}{-1}\right)+\frac{1}{\sqrt{2}}\left(1-e^{-1}\right)\\
&= \frac{1}{\sqrt{2}}\left(1-e^{-1}\right)+\frac{1}{\sqrt{2}}\left(1-e^{-1}\right)\\
&= \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}\,e}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}\,e}\\
&= \frac{1}{\sqrt{2}}\left(2-\frac{2}{e}\right)\\
&= \frac{1}{\sqrt{2}}\left(\frac{2e-2}{e}\right)\\
&= \frac{2}{\sqrt{2}}\left(\frac{e-1}{e}\right)
\end{aligned}
$$

And for $m\neq 0$, eq (7) becomes

$$
\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{e^{i\pi m-1}-1}{i\pi m-1}+\frac{1-e^{-i\pi m-1}}{i\pi m+1}\right)\\
&= \frac{1}{\sqrt{2}}\left(\frac{e^{i\pi m}e^{-1}-1}{i\pi m-1}+\frac{1-e^{-i\pi m}e^{-1}}{i\pi m+1}\right)
\end{aligned}
$$

Since m is integer, then the above becomes

$$
\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{\cos(\pi m)e^{-1}-1}{i\pi m-1}+\frac{1-\cos(\pi m)e^{-1}}{i\pi m+1}\right)\\
&= \frac{1}{\sqrt{2}}\left(\frac{(-1)^m e^{-1}-1}{i\pi m-1}+\frac{1-(-1)^m e^{-1}}{i\pi m+1}\right)\\
&= \frac{1}{\sqrt{2}}\frac{\left((-1)^m e^{-1}-1\right)(i\pi m+1)+\left(1-(-1)^m e^{-1}\right)(i\pi m-1)}{(i\pi m-1)(i\pi m+1)}\\
&= \frac{1}{\sqrt{2}}\frac{2(-1)^m e^{-1}-2}{-\pi^2 m^2-1}\\
&= \frac{1}{\sqrt{2}}\frac{2-2(-1)^m e^{-1}}{1+\pi^2 m^2}\\
&= \frac{2}{\sqrt{2}}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}
\end{aligned}
$$

Hence (3) becomes

$$\begin{aligned}
f(x) &\sim \frac{2}{\sqrt{2}}\left(\frac{e-1}{e}\right)\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{2}{\sqrt{2}}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{i\pi m x}\\
&= \frac{e-1}{e} + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{i\pi m x}\\
&= \frac{e-1}{e} + \sum_{n=-\infty}^{-1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}e^{i\pi n x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}
\end{aligned}$$

Let $m=-n$ in the first sum above. This gives

$$\begin{aligned}
f(x) &= \frac{e-1}{e} + \sum_{m=\infty}^{1}\frac{1-(-1)^{(-m)}e^{-1}}{1+\pi^2(-m)^2}e^{i\pi(-m)x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}\\
&= \frac{e-1}{e} + \sum_{m=1}^{\infty}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{-i\pi m x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}
\end{aligned}$$

Now the two sums can be combined using one index, say n, since they sum over the same interval

$$f(x) = \frac{e-1}{e} + \sum_{n=\infty}^{1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}\left(e^{-i\pi n x} + e^{i\pi n x}\right)$$

But $\left(e^{-i\pi n x} + e^{i\pi n x}\right) = 2\cos \pi n x$. The above becomes

$$\begin{aligned}
f(x) &= \frac{e-1}{e} + 2\sum_{n=\infty}^{1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}\cos \pi n x\\
&= \frac{e-1}{e} + 2\sum_{n=\infty}^{1}\frac{e-(-1)^n}{e\left(1+\pi^2 n^2\right)}\cos \pi n x
\end{aligned}$$

4.7.3.2 Part 2

Now $f(x) = \cosh x$. Therefore

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Where now f_m is found, using $L=2$

$$f_m = \int_{-1}^{1}\frac{1}{\sqrt{2}}e^{-i\pi m x}\cosh(x)dx$$

But $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The above becomes

$$\begin{aligned}
f_m &= \int_{-1}^{1}\frac{1}{\sqrt{2}}e^{-i\pi m x}\frac{1}{2}(e^x + e^{-x})dx\\
&= \frac{1}{2\sqrt{2}}\left(\int_{-1}^{1}e^{-i\pi m x}e^x dx + \int_{-1}^{1}e^{-i\pi m x}e^{-x}dx\right)\\
&= \frac{1}{2\sqrt{2}}\left(\int_{-1}^{1}e^{(-i\pi m+1)x}dx + \int_{-1}^{1}e^{(-i\pi m-1)x}dx\right)
\end{aligned} \tag{4}$$

The first integral is

$$\begin{aligned}
\int_{-1}^{1}e^{(-i\pi m+1)x}dx &= \frac{1}{-i\pi m+1}\left[e^{(-i\pi m+1)x}\right]_{-1}^{1}\\
&= \frac{1}{1-i\pi m}\left(e^{(-i\pi m+1)} - e^{-(-i\pi m+1)}\right)\\
&= \frac{1}{1-i\pi m}\left(e^{-i\pi m}e - e^{i\pi m}e^{-1}\right)
\end{aligned}$$

Since m is integer, then $e^{-i\pi m}=(-1)^m$ and $e^{i\pi m}=(-1)^m$. The above becomes

$$\int_{-1}^{1} e^{(-i\pi m+1)x} = \frac{(-1)^m}{1-i\pi m}\left(e-e^{-1}\right) \tag{5}$$

The second integral in (4) becomes

$$\begin{aligned}\int_{-1}^{1} e^{(-i\pi m-1)x}dx &= \frac{1}{-i\pi m-1}\left[e^{(-i\pi m-1)x}\right]_{-1}^{1}\\ &= \frac{-1}{1+i\pi m}\left(e^{(-i\pi m-1)}-e^{-(-i\pi m-1)}\right)\\ &= \frac{-1}{1+i\pi m}\left(e^{-i\pi m}e^{-1}-e^{i\pi m}e\right)\end{aligned}$$

Since m is integer, the above becomes

$$\int_{-1}^{1} e^{(-i\pi m-1)x}dx = \frac{-(-1)^m}{1+i\pi m}\left(e^{-1}-e\right) \tag{6}$$

Substituting (5,6) in (4) gives

$$\begin{aligned}f_m &= \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1-i\pi m}\left(e-e^{-1}\right)+\frac{-(-1)^m}{1+i\pi m}\left(e^{-1}-e\right)\right)\\ &= \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1-i\pi m}\left(e-e^{-1}\right)+\frac{(-1)^m}{1+i\pi m}\left(e-e^{-1}\right)\right)\\ &= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{1}{1-i\pi m}+\frac{1}{1+i\pi m}\right)\\ &= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{(1+i\pi m)+(1-i\pi m)}{(1-i\pi m)(1+i\pi m)}\right)\\ &= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{2}{\pi^2m^2+1}\right)\\ &= \frac{(-1)^m\left(e-e^{-1}\right)}{\sqrt{2}\left(\pi^2m^2+1\right)}\end{aligned}$$

Hence for $m=0$,

$$f_0 = \frac{\left(e-e^{-1}\right)}{\sqrt{2}}$$

Therefore $f(x)\sim\sum_{m=-\infty}^{\infty} f_m\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x}$ becomes

$$\begin{aligned}\cosh(x) &\sim \frac{\left(e-e^{-1}\right)}{\sqrt{2}}\frac{1}{\sqrt{2}}+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{(-1)^m\left(e-e^{-1}\right)}{\sqrt{2}\left(\pi^2m^2+1\right)}\frac{1}{\sqrt{2}}e^{i\pi mx}\\ &= \frac{\left(e-e^{-1}\right)}{2}+\frac{1}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{(-1)^m\left(e-e^{-1}\right)}{\left(\pi^2m^2+1\right)}e^{i\pi mx}\end{aligned}$$

As was done in part(i), $\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} e^{i\pi mx}$ can be rewritten as $\sum_{n=1}^{\infty} 2\cos(n\pi x)$. The above reduces to

$$\cosh(x)\sim\frac{\left(e-e^{-1}\right)}{2}+\sum_{n=1}^{\infty}2\frac{(-1)^n\left(e-e^{-1}\right)}{\left(\pi^2n^2+1\right)}\cos(n\pi x)$$

But $\frac{\left(e-e^{-1}\right)}{2}=\sinh 1$. Therefore the above becomes

$$\cosh(x)\sim\sinh(1)\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+\pi^2n^2\right)}\cos(n\pi x)\right)$$

4.7.3.3 Part 3

I think now the book is asking to find the Fourier series for e^x over $-\pi \leq x \leq \pi$ in this last part. Therefore, as before, starting with

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{1}$$

Where now, using $L = 2\pi$ as the period, then

$$\begin{aligned}
f_m &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{-i\frac{2\pi m}{2\pi}x} e^x dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{(-im+1)x} dx \\
&= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{(-im+1)x}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{(-im+1)\pi} - e^{-(-im+1)\pi}\right] \\
&= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{-im\pi}e^{\pi} - e^{im\pi}e^{-\pi}\right]
\end{aligned}$$

But $e^{im\pi} = (-1)^m$ and $e^{-im\pi} = (-1)^m$ since m is integer. The above becomes

$$\begin{aligned}
f_m &= \frac{(-1)^m}{\sqrt{2\pi}(1-im)} [e^{\pi} - e^{-\pi}] \\
&= \frac{(-1)^m (2\sinh\pi)}{\sqrt{2\pi}(1-im)} \\
&= \frac{2}{\sqrt{2\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi
\end{aligned}$$

For $m = 0$ the above gives

$$f_0 = \frac{\sqrt{2}}{\sqrt{\pi}} \sinh\pi$$

Therefore $f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x}$ becomes, where $L = 2\pi$ now,

$$\begin{aligned}
e^x &\sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \sinh\pi \left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{m=-\infty,\neq 0}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi\right) \frac{1}{\sqrt{2\pi}} e^{imx} \\
&= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{(1-im)} e^{imx} \\
&= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m (1+im)}{(1-im)(1+im)} e^{imx} \\
&= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m + i(-1)^m m}{1+m^2} e^{imx}\right) \\
&= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \frac{i(-1)^m m}{1+m^2} e^{imx}\right) \\
&= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx}\right)
\end{aligned} \tag{2}$$

The first sum above becomes

$$\sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+m^2} \cos(nx) \tag{3}$$

And the second sum in (2) becomes

$$\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)(-1)^m m\, e^{imx}}{1+m^2} \frac{}{i} \\
&= \sum_{k=-\infty}^{-1} \frac{(-1)(-1)^k k\, e^{ikx}}{1+k^2} \frac{}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r\, e^{irx}}{1+r^2} \frac{}{i}
\end{aligned}$$

Letting $m=-k$ in the first sum above gives

$$\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{m=\infty}^{1} \frac{(-1)(-1)^{-m}(-m)\, e^{-imx}}{1+(-m)^2} \frac{}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r\, e^{irx}}{1+r^2} \frac{}{i} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m m\, e^{-imx}}{1+m^2} \frac{}{i} - \sum_{r=1}^{\infty} \frac{(-1)^r r\, e^{irx}}{1+r^2} \frac{}{i}
\end{aligned}$$

Merging the two sums back together since now on same interval, and using n for the common index

$$\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{n=1}^{\infty} \frac{(-1)^n n}{1+n^2}\left(\frac{e^{-inx}}{i} - \frac{e^{inx}}{i}\right) \\
&= -\sum_{n=1}^{\infty} 2\frac{(-1)^n n}{1+n^2} (\sin(nx)) \qquad (4)
\end{aligned}$$

Substituting (3,4) back in (2) gives

$$\begin{aligned}
e^x &\sim \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\cos(nx) - \sum_{n=1}^{\infty} 2\frac{(-1)^n n}{1+n^2}(\sin(nx))\right) \\
&= \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) \qquad (5)
\end{aligned}$$

The question is now asking to show how to use (5) to obtain the series for $\sinh x$ and $\cosh x$. Since

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$\begin{aligned}
\sinh x &\sim \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(n(-x)) - n\sin(n(-x))\right.\right. \\
&= \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right)\right) \\
&= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(\right.\right. \\
&= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \frac{\sinh \pi}{\pi} - 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(n\right. \\
&= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right) \\
&= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx) - \cos(nx) - n\sin(nx))\right) \\
&= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-n\sin(nx) - n\sin(nx))\right) \\
&= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-2n\sin(nx))\right)
\end{aligned}$$

Hence

$$\sinh x \sim \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{1+n^2}n\sin(nx) \tag{6}$$

Similarly for

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$\begin{aligned}
\cosh x &\sim \frac{1}{2}\left(\frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))\right)+\frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(n(-x))-n\sin(n(-\right.\right.\\
&= \frac{1}{2}\left(\frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))\right)+\frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\sin(nx))\right)\right)\\
&= \frac{1}{2}\left(\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))+\left(\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\right.\right.\\
&= \frac{1}{2}\left(\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))+\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\sin\right.\\
&= \frac{1}{2}\left(2\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx)+\cos(nx)+n\sin(nx))\right)\\
&= \frac{1}{2}\left(2\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+\cos(nx))\right)\\
&= \frac{\sinh\pi}{\pi}+\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}2\cos(nx)
\end{aligned}$$

Hence

$$\cosh x \sim \frac{\sinh\pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(nx)\right) \tag{7}$$

4.7.4 Problem 3

Perform appropriate integration to show the following results regarding the Dirac delta function

(1) $\delta(ax) = \frac{\delta(x)}{|a|}$ where a is real number. (2) $\delta\big(f(x)\big) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_i}$ where x_i satisfies $f(x_i) = 0$

(3) $\frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'}$

Solution

4.7.4.1 Part (1)

Using the integral definition of delta function given by

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \tag{1}$$

Then

$$\delta(ax) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikax}dk$$

Case $a > 0$. Let $u = ak$. Then $du = adk$. The above becomes

$$\delta(ax) = \frac{1}{a}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}du\right)$$

But $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}du = \delta(x)$ by definition. Hence the above becomes

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{2}$$

Case $a<0$. Let $u=ak$. Then $du=adk$. When $k=\infty, u=-\infty$ and when $k=-\infty, u=+\infty$. The integral becomes

$$
\begin{aligned}
\delta(ax) &= \frac{1}{a}\left(\frac{1}{2\pi}\int_{\infty}^{-\infty} e^{iux}du\right)\\
&= \frac{1}{-a}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}du\right)\\
&= \frac{1}{-a}\delta(x) \qquad (3)
\end{aligned}
$$

Combining (2,3) gives

$$
\delta(ax)=\frac{1}{|a|}\delta(x)
$$

4.7.4.2 Part (2)

Using

$$
\int_{-\infty}^{\infty}\delta\big(f(x)\big)dx=\sum_i\int_{x_i-\varepsilon}^{x_i+\varepsilon}\delta\big(f(x)\big)dx \qquad (1)
$$

Where in the RHS, the sum is over the roots of $f(x)$, where $f(x_i)=0$ where x_i is root of $f(x)$ since $\delta(u)$ is nonzero only when its argument is zero, which is at the roots of $f(x)$. Now, expanding $f(x)$ near each one of its roots using Taylor series

$$
f(x)=f(x_i)+(x-x_i)f'(x_i)+O\big(x^2\big)
$$

But $f(x_i)=0$ since x_i is root, and keeping only linear terms, then (1) now becomes

$$
\int_{-\infty}^{\infty}\delta\big(f(x)\big)dx=\sum_i\int_{x_i-\varepsilon}^{x_i+\varepsilon}\delta\big((x-x_i)f'(x_i)\big)dx
$$

But from part (1), we found that $\delta(a(x-x_i))=\frac{1}{|a|}\delta(x-x_i)$, where now $a=f'(x_i)$. Using this relation in the above gives

$$
\int_{-\infty}^{\infty}\delta\big(f(x)\big)dx=\int_{-\infty}^{\infty}\sum_i\frac{1}{\left|f'(x_i)\right|}\delta(x-x_i)
$$

Therefore the integrands on each side is the same. This implies

$$
\delta\big(f(x)\big)=\sum_i\frac{\delta(x-x_i)}{\left|f'(x_i)\right|}
$$

4.7.4.3 Part (3)

Starting from property of delta function which is

$$
\int\delta(x-x')f(x')dx'=f(x)
$$

Taking derivative of both sides w.r.t. x gives

$$
\begin{aligned}
\frac{d}{dx}\int\delta(x-x')f(x')dx' &= \frac{d}{dx}f(x)\\
\int\frac{d\delta(x-x')}{dx}f(x')dx' &= \frac{d}{dx}f(x)
\end{aligned}
$$

Integration by part. Let $\frac{d\delta(x-x')}{dx}=dv, u=f(x')$, then $v=(x-x'), du=\frac{d}{dx'}f(x')$. The above becomes

$$
\int\delta(x-x')\frac{d}{dx'}f(x')dx'=\frac{d}{dx}f(x)
$$

Therefore

$$\int \frac{d\delta(x-x')}{dx} f(x')dx' = \int \delta(x-x') \frac{d}{dx'} f(x')dx'$$

or

$$\frac{d\delta(x-x')}{dx} f(x') = \delta(x-x') \frac{d}{dx'} f(x')$$

or

$$\frac{d\delta(x-x')}{dx} = \delta(x-x') \frac{d}{dx'}$$

4.7.5 Problem 4

For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured.

Solution

The goal is to determine $\left|\langle\phi_p|\psi\rangle\right|^2$ which will give the probability of measuring momentum p. But

$$\begin{aligned}\langle\phi_p|\psi_n\rangle &= \int_0^\infty \langle\phi_p|x\rangle\langle x|\psi_n\rangle dx \\ &= \int_0^\infty \langle x|\phi_p\rangle^* \langle x|\psi_n\rangle dx \qquad (1)\end{aligned}$$

But $\langle x|\phi_p\rangle = \phi_p(x)$ and $\langle x|\psi_n\rangle = \psi_n(x)$. From lecture notes,

$$\begin{aligned}\phi_p(x) &= \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}} \\ \psi_n(x) &= \begin{Bmatrix} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{otherwise} \end{Bmatrix}\end{aligned}$$

For $n = 1, 2, 3, \cdots$. Substituting the above in (1) gives

$$\begin{aligned}\langle\phi_p|\psi_n\rangle &= \int_0^\infty \phi_p^*(x)\psi_n(x)dx \\ &= \int_0^L \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{px}{\hbar}} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L e^{-i\frac{px}{\hbar}} \sin\left(\frac{n\pi x}{L}\right) dx \qquad (2)\end{aligned}$$

To evaluate $I = \int_0^L e^{-i\frac{px}{\hbar}} \sin\left(\frac{n\pi x}{L}\right)dx$ we do Integration by parts twice. Let $u = \sin\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)dx$ and $v = \hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}$. Hence

$$\begin{aligned}I &= [uv]_0^L - \int_0^L v du \\ &= \left[\sin\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}\right]_0^L - \int_0^L \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}dx \\ &= \left[\sin\left(\frac{n\pi L}{L}\right)\hbar\frac{e^{-i\frac{pL}{\hbar}}}{-ip} - 0\right] + \frac{\hbar n\pi}{ipL}\int_0^L \cos\left(\frac{n\pi x}{L}\right)e^{-i\frac{px}{\hbar}}dx\end{aligned}$$

Since n is integer, then boundary terms are zero.

$$I = \frac{\hbar n\pi}{ipL}\int_0^L \cos\left(\frac{n\pi x}{L}\right)e^{-i\frac{px}{\hbar}}dx$$

Doing integration by parts one more time. Let $u = \cos\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx$, then the above becomes

$$\begin{aligned}
I &= \frac{\hbar n\pi}{ipL}\left(\left[\cos\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}\right]_0^L + \int_0^L \frac{\hbar e^{-i\frac{px}{\hbar}}}{-ip}\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)\\
&= \frac{\hbar n\pi}{ipL}\left(\frac{\hbar}{-ip}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right] - \frac{\hbar n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx\right)\\
&= \frac{\hbar^2}{-ip}\frac{n\pi}{ipL}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right] - \frac{n\pi}{ipL}\frac{\hbar^2 n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx\\
&= \frac{\hbar^2 n\pi}{p^2 L}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right] + \frac{\hbar^2 n^2\pi^2}{p^2L^2}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

But $\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx = I$. Therefore the above becomes

$$I = \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right) + \frac{\hbar^2n^2\pi^2}{p^2L^2}I$$

Solving for I

$$\begin{aligned}
I - \frac{\hbar^2n^2\pi^2}{p^2L^2}I &= \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right)\\
I\left(1-\frac{\hbar^2n^2\pi^2}{p^2L^2}\right) &= \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right)\\
I &= \frac{\hbar^2 n\pi}{p^2L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right)}{\left(1-\frac{\hbar^2n^2\pi^2}{p^2L^2}\right)}\\
&= p^2L^2\frac{\hbar^2 n\pi}{p^2L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}}-1\right)}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}\\
&= \frac{\hbar^2 n\pi L}{p^2L^2-\hbar^2n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}}-1\right)
\end{aligned}$$

Substituting the above in (2) gives

$$\begin{aligned}
\langle\phi_p|\psi_n\rangle &= \frac{1}{\sqrt{\pi\hbar L}}\left(\frac{\hbar^2 n\pi L}{p^2L^2-\hbar^2n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}}-1\right)\right)\\
&= \frac{\hbar^2 n\pi L\left(\sqrt{\pi\hbar L}\right)}{(\pi\hbar L)\left(p^2L^2-\hbar^2n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)\\
&= \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)
\end{aligned}$$

Let $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}$, then

$$\begin{aligned}
\langle\phi_p|\psi_n\rangle &= k_n\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)\\
&= (-1)^n k_n e^{-\frac{ipL}{\hbar}} - k_n\\
&= (-1)^n k_n\left(\cos\frac{pL}{\hbar} - i\sin\frac{pL}{\hbar}\right) - k_n\\
&= (-1)^n k_n\cos\frac{pL}{\hbar} - i(-1)^n k_n\sin\frac{pL}{\hbar} - k_n\\
&= \left((-1)^n k_n\cos\frac{pL}{\hbar} - k_n\right) - i\left((-1)^n k_n\sin\frac{pL}{\hbar}\right)
\end{aligned}$$

Hence

$$\left|\langle\phi_p|\psi_n\rangle\right| = \sqrt{\left((-1)^n k_n \cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n \sin\frac{pL}{\hbar}\right)^2}$$

And

$$\begin{aligned}
\left|\langle\phi_p|\psi_n\rangle\right|^2 &= \left((-1)^n k_n \cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n \sin\frac{pL}{\hbar}\right)^2\\
&= (-1)^{2n} k_n^2 \cos^2\frac{pL}{\hbar} + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar} + (-1)^{2n}k_n^2\sin^2\frac{pL}{\hbar}\\
&= (-1)^{2n} k_n^2\left(\cos^2\frac{pL}{\hbar} + \sin^2\frac{pL}{\hbar}\right) + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar}\\
&= (-1)^{2n} k_n^2 + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar}\\
&= k_n^2\left(1 + (-1)^{2n} - 2(-1)^n\cos\frac{pL}{\hbar}\right)
\end{aligned}$$

But $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)}$, therefore the above becomes

$$\begin{aligned}
\left|\langle\phi_p|\psi_n\rangle\right|^2 &= \left(\frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)}\right)^2\left(1 + (-1)^{2n} - 2(-1)^n\cos\frac{pL}{\hbar}\right)\\
&= \frac{n^2\hbar^3\pi L}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)^2}\left(1 + (-1)^{2n} - 2(-1)^n\cos\left(\frac{pL}{\hbar}\right)\right)
\end{aligned}$$

The above gives the probability of measurement of p, where $n = 1, 2, 3, \cdots$. For illustration, the following two tables are generated to see how the probability of measuring say $p = 1$ and $p = 2$ changes as function of n. To generate this, L is taken as 1 and $\hbar = 1$ for simplicity.

n	Probability of measuring p=1
1	0.12302
2	0.00780329
3	0.0112922
4	0.00187694
5	0.00400658
6	0.00082832
7	0.00203605
8	0.000464782
9	0.00122967
10	0.000297121

Figure 4.18: Probability to measure $p = 1$

n	Probability of measuring p=2
1	0.106479
2	0.0282761
3	0.00458842
4	0.00600971
5	0.00155647
6	0.00259549
7	0.000781451
8	0.00144553
9	0.00046963
10	0.000920908

Figure 4.19: Probability to measure $p = 2$

```
p = 1; h = 1;  L = 1;
f[p_, n_] := (n^2 h^3 Pi L)/(p^2 L^2 - h^2 n^2 Pi^2)^2 (1 + (-1)^(2 n) - 2 (-1)^n Cos[p L/h])
data = Table[{n, N@f[p, n]}, {n, 1, 10}];
data = PrependTo[data, {"n", "Probability of measuring p=1"}];
Grid[data, Frame → All];
```

Figure 4.20: Code used

4.7.6 key solution for HW 7

Physics 3041 (Spring 2021) Solutions to Homework Set 7

1. Problem 9.7.3. (15 points)

(a) Using the orthonormal basis $\{|m\rangle \to \frac{e^{i2m\pi x/L}}{\sqrt{L}},\ m = 0,\ \pm 1,\ \pm 2,\ \cdots\}$, we can expand

$$f(x) = \begin{cases} 2xh/L, & 0 \le x \le L/2, \\ 2(L-x)h/L, & L/2 \le x \le L \to 2x'h/L,\ 0 \le x' \le L/2, \end{cases}$$

as follows. Note that a change of variable $x' = L - x$ is made to simplify the integration over $L/2 \le x \le L$.

$$f_{m=0} = \langle m=0|f\rangle = \frac{1}{\sqrt{L}}\int_0^L f(x)dx = \frac{1}{\sqrt{L}}\left[\int_0^{L/2}\frac{2xh}{L}dx + \int_0^{L/2}\frac{2x'h}{L}dx'\right] = \frac{2}{\sqrt{L}}\frac{h}{L}\left(\frac{L}{2}\right)^2 = \frac{h\sqrt{L}}{2},$$

$$\begin{aligned} f_{m\neq 0} = \langle m|f\rangle &= \frac{1}{\sqrt{L}}\int_0^L e^{-\frac{i2m\pi x}{L}} f(x)dx = \frac{2h}{L\sqrt{L}}\left[\int_0^{L/2} x e^{-\frac{i2m\pi x}{L}}dx + \int_0^{L/2} x' e^{-\frac{i2m\pi(L-x')}{L}}dx'\right] \\ &= \frac{2h}{L\sqrt{L}}\int_0^{L/2} x\left[e^{-\frac{i2m\pi x}{L}} + e^{\frac{i2m\pi x}{L}}\right]dx = \frac{4h}{L\sqrt{L}}\int_0^{L/2} x\cos\frac{2m\pi x}{L}dx \\ &= \frac{4h}{L\sqrt{L}}[(-1)^m - 1]\left(\frac{L}{2m\pi}\right)^2 = [(-1)^m - 1]\frac{h\sqrt{L}}{(m\pi)^2} = \begin{cases} -2h\sqrt{L}/(m\pi)^2, & m = \text{odd}, \\ 0, & m = \text{even}. \end{cases} \end{aligned}$$

$$\begin{aligned} f(x) &= \sum_m \frac{f_m}{\sqrt{L}} e^{\frac{i2m\pi x}{L}} = \frac{h}{2} - \frac{2h}{\pi^2}\sum_{m=\text{odd}} \frac{e^{\frac{i2m\pi x}{L}}}{m^2} = \frac{h}{2} - \frac{2h}{\pi^2}\sum_{n=0}^{\infty}\frac{e^{\frac{i2(2n+1)\pi x}{L}} + e^{\frac{-i2(2n+1)\pi x}{L}}}{(2n+1)^2} \\ &= \frac{h}{2} - \frac{4h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{2(2n+1)\pi x}{L} \end{aligned}$$

(b) For $x = 0$,

$$f(x) = 0 = \frac{h}{2} - \frac{4h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

2. Problem 9.7.8. (35 points)

(i) We use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{L}},\ |m,\alpha=1\rangle \to \sqrt{\frac{2}{L}}\cos\frac{2m\pi x}{L},\ |m,\alpha=2\rangle \to \sqrt{\frac{2}{L}}\sin\frac{2m\pi x}{L},\ m=1,\ 2,\ \cdots\}$ to expand $f(x),\ -L/2 \le x \le L/2$.

For $f(x)=e^{-|x|},\ -1 \le x \le 1$, we use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{2}},\ |m,\alpha=1\rangle \to \cos m\pi x,\ |m,\alpha=2\rangle \to \sin m\pi x,\ m=1,\ 2,\ \cdots\}$ for $L=2$. Note that $f(-x)=f(x)$.

$$\begin{aligned}
a_0 &= \langle m=0|f\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^{-|x|}dx = \frac{2}{\sqrt{2}}\int_0^1 e^{-x}dx = (1-e^{-1})\sqrt{2},\\
a_m &= \langle m,\alpha=1|f\rangle = \int_{-1}^{1} e^{-|x|}\cos m\pi x\,dx = 2\int_0^1 e^{-x}\cos m\pi x\,dx = \int_0^1 e^{-x}(e^{im\pi x}+e^{-im\pi x})dx\\
&= \frac{1-e^{-1+im\pi}}{1-im\pi} + \frac{1-e^{-1-im\pi}}{1+im\pi} = \frac{2[1-(-1)^m e^{-1}]}{1+(m\pi)^2},\\
b_m &= \langle m,\alpha=2|f\rangle = \int_{-1}^{1} e^{-|x|}\sin m\pi x\,dx = 0,\\
f(x) &= \frac{a_0}{\sqrt{2}} + \sum_{m=1}^{\infty} a_m\cos m\pi x + \sum_{m=1}^{\infty} b_m \sin m\pi x = 1-e^{-1} + 2\sum_{m=1}^{\infty}\frac{1-(-1)^m e^{-1}}{1+(m\pi)^2}\cos m\pi x
\end{aligned}$$

(ii) $f(x)=\cosh x,\ -1 \le x \le 1$. Note that $f(-x)=f(x)$.

$$\begin{aligned}
a_0 &= \langle m=0|f\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1}\cosh x\,dx = \frac{2}{\sqrt{2}}\int_0^1 \cosh x\,dx = (\sinh 1)\sqrt{2}\,,\\
a_m &= \langle m,\alpha=1|f\rangle = \int_{-1}^{1}\cosh x\cos m\pi x\,dx = 2\int_0^1 \cosh x\cos m\pi x\,dx\\
&= \frac{1}{2}\int_0^1 (e^x+e^{-x})(e^{im\pi x}+e^{-im\pi x})dx\\
&= \frac{1}{2}\left(\frac{e^{1+im\pi}-1}{1+im\pi} + \frac{e^{1-im\pi}-1}{1-im\pi} + \frac{1-e^{-1+im\pi}}{1-im\pi} + \frac{1-e^{-1-im\pi}}{1+im\pi}\right)\\
&= \frac{(-1)^m(e-e^{-1})}{1+(m\pi)^2} = \frac{2(-1)^m(\sinh 1)}{1+(m\pi)^2},\\
b_m &= \langle m,\alpha=2|f\rangle = \int_{-1}^{1}\cosh x\sin m\pi x\,dx = 0,\\
f(x) &= \sinh 1 + \sum_{m=1}^{\infty}\frac{2(-1)^m(\sinh 1)}{1+(m\pi)^2}\cos m\pi x = (\sinh 1)\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m\cos m\pi x}{1+(m\pi)^2}\right]
\end{aligned}$$

(iii) For $f(x)=e^x,\ -\pi < x < \pi$, we use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{2\pi}},\ |m,\alpha=1\rangle \to \frac{\cos mx}{\sqrt{\pi}},\ |m,\alpha=2\rangle \to \frac{\sin mx}{\sqrt{\pi}},\ m=1,\ 2,\ \cdots\}$ for $L=2\pi$. Note that to enforce $f(-\pi)=f(\pi)$, we must allow two discontinuities at $x=\pm\pi$, which does not affect the convergence of the expansion within the interval $-\pi < x < \pi$.

$$
\begin{aligned}
a_0 &= \langle m=0|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^x\,dx = \frac{e^\pi - e^{-\pi}}{\sqrt{2\pi}} = (\sinh\pi)\sqrt{\frac{2}{\pi}}\,,\\
a_m &= \langle m,\alpha=1|f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} e^x\cos mx\,dx = \frac{1}{2\sqrt{\pi}}\int_{-\pi}^{\pi} e^x(e^{imx}+e^{-imx})dx\\
&= \frac{1}{2\sqrt{\pi}}\left[\frac{e^{(1+im)\pi}-e^{-(1+im)\pi}}{1+im} + \frac{e^{(1-im)\pi}-e^{-(1-im)\pi}}{1-im}\right]\\
&= \frac{(-1)^m(e^\pi - e^{-\pi})}{(1+m^2)\sqrt{\pi}} = \frac{2(-1)^m(\sinh\pi)}{(1+m^2)\sqrt{\pi}},\\
b_m &= \langle m,\alpha=2|f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} e^x\sin mx\,dx = \frac{1}{2i\sqrt{\pi}}\int_{-\pi}^{\pi} e^x(e^{imx}-e^{-imx})dx\\
&= \frac{1}{2i\sqrt{\pi}}\left[\frac{e^{(1+im)\pi}-e^{-(1+im)\pi}}{1+im} - \frac{e^{(1-im)\pi}-e^{-(1-im)\pi}}{1-im}\right]\\
&= -\frac{m(-1)^m(e^\pi - e^{-\pi})}{(1+m^2)\sqrt{\pi}} = -\frac{2m(-1)^m(\sinh\pi)}{(1+m^2)\sqrt{\pi}},\\
f(x) &= \frac{\sinh\pi}{\pi} + \sum_{m=1}^{\infty}\frac{2(-1)^m(\sinh\pi)}{(1+m^2)\pi}\cos mx - \sum_{m=1}^{\infty}\frac{2m(-1)^m(\sinh\pi)}{(1+m^2)\pi}\sin mx\\
&= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m(\cos mx - m\sin mx)}{1+m^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} = \frac{f(x)-f(-x)}{2} = \frac{\sinh\pi}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m(-2m\sin mx)}{1+m^2}\\
&= \frac{2\sinh\pi}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}m\sin mx}{1+m^2}\\
\cosh x &= \frac{e^x + e^{-x}}{2} = \frac{f(x)+f(-x)}{2} = \frac{\sinh\pi}{\pi}\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m(2\cos mx)}{1+m^2}\right]\\
&= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m\cos mx}{1+m^2}\right]
\end{aligned}
$$

Note that $\cosh(-\pi) = \cosh\pi$, so we expect that its expansion is continuous at $x = \pm\pi$. We obtain

$$
\begin{aligned}
\cosh\pi &= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m\cos m\pi}{1+m^2}\right] = \frac{\sinh\pi}{\pi}\left(1+2\sum_{m=1}^{\infty}\frac{1}{1+m^2}\right)\\
\Rightarrow \sum_{m=1}^{\infty}\frac{1}{1+m^2} &= \frac{\pi\cosh\pi}{2\sinh\pi} - \frac{1}{2}\,.
\end{aligned}
$$

Clearly, the expansion of $\sinh x$ is discontinuous at $x = \pm\pi$ as it gives zero. Similarly, the expansion of e^x is also discontinuous at $x = \pm\pi$, as it gives $\cosh\pi$, i.e., the average of the actual values at $x = \pm\pi$.

3. Perform appropriate integration to show the following results regarding the Dirac delta function (25 points):

$$\begin{aligned}
\delta(ax) &= \delta(x)/|a|, \text{ where } a \text{ is a real number,}\\
\delta(f(x)) &= \sum_i \frac{\delta(x-x_i)}{|df/dx|_{x_i}}, \text{ where } x_i \text{ satisfies } f(x_i)=0,\\
\frac{d}{dx}\delta(x-x') &= \delta(x-x')\frac{d}{dx'}\,.
\end{aligned}$$

Consider $a>0$ and let $y=ax$. We have

$$\int_{-\epsilon}^{\epsilon}\delta(ax)dx = \frac{1}{a}\int_{-a\epsilon}^{a\epsilon}\delta(y)dy = \frac{1}{a} = \int_{-\epsilon}^{\epsilon}\frac{\delta(x)}{a}dx \Rightarrow \delta(ax)=\delta(x)/a.$$

For $a<0$, we have $\delta(ax)=\delta(-|a|x)=\delta(|a|x)=\delta(x)/|a|$, where the last equality follows from the result for $a>0$. Therefore, $\delta(ax)=\delta(x)/|a|$ in general.

Consider Taylor expansion near a specific root x_i of $f(x)$:

$$\begin{aligned}
f(x) &= f(x_i)+f'(x_i)(x-x_i)+\cdots = f'(x_i)(x-x_i)+\cdots\,,\\
\Rightarrow \int_{x_i-\epsilon}^{x_i+\epsilon}\delta(f(x))dx &= \int_{x_i-\epsilon}^{x_i+\epsilon}\delta(f'(x_i)(x-x_i))dx = \frac{1}{|f'(x_i)|},
\end{aligned}$$

where $f'(x_i)=(df/dx)_{x_i}$ and the last equality follows from $\delta(ax)=\delta(x)/|a|$. Including all roots of $f(x)$, we have

$$\sum_i\int_{x_i-\epsilon}^{x_i+\epsilon}\delta(f(x))dx = \sum_i\frac{1}{|f'(x_i)|} = \sum_i\int_{x_i-\epsilon}^{x_i+\epsilon}\frac{\delta(x-x_i)}{|f'(x_i)|}dx \Rightarrow \delta(f(x)) = \sum_i\frac{\delta(x-x_i)}{|f'(x_i)|}\,.$$

Note that

$$\begin{aligned}
\frac{d}{dx}\int_{x-\epsilon}^{x+\epsilon}\delta(x-x')f(x')dx' &= \int_{x-\epsilon}^{x+\epsilon}\frac{d}{dx}\delta(x-x')f(x')dx' + [\delta(x-x')f(x')]_{x-\epsilon}^{x+\epsilon}\\
&= \int_{x-\epsilon}^{x+\epsilon}\frac{d}{dx}\delta(x-x')f(x')dx',
\end{aligned}$$

where we have used $\delta(x-x')=0$ for $x-x'\neq 0$.

$$\begin{aligned}
&\int_{x-\epsilon}^{x+\epsilon}\frac{d}{dx}\delta(x-x')f(x')dx' = \frac{d}{dx}\int_{x-\epsilon}^{x+\epsilon}\delta(x-x')f(x')dx' = \frac{d}{dx}f(x),\\
&\int_{x-\epsilon}^{x+\epsilon}\delta(x-x')\frac{d}{dx'}f(x')dx' = \frac{d}{dx}f(x) \Rightarrow \frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'}\,.
\end{aligned}$$

4. For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured. (25 points)

The wavefunctions of the energy eigenstates with eigenvalues $E_n = n^2\pi^2\hbar^2/(2mL^2)$ for $n = 1, 2, \cdots$ are

$$\langle x|\psi_n\rangle = \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, & 0 < x < L, \\ 0, & \text{elsewhere.} \end{cases}$$

The wave function of the momentum eigenstate with eigenvalue p is

$$\langle x|\phi_p\rangle = \phi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}.$$

The probability amplitude for measuring momentum p is

$$\begin{aligned}
\langle\phi_p|\psi_n\rangle &= \int_{-\infty}^{\infty}\langle\phi_p|x\rangle\langle x|\psi_n\rangle dx = \int_{-\infty}^{\infty}\phi_p^*(x)\psi_n(x)dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\int_0^L e^{-ipx/\hbar}\sin\frac{n\pi x}{L}dx = \frac{1}{2i\sqrt{\pi\hbar L}}\int_0^L e^{-ipx/\hbar}(e^{in\pi x/L} - e^{-in\pi x/L})dx \\
&= \frac{1}{2i\sqrt{\pi\hbar L}}\left(\frac{e^{-ipL/\hbar+in\pi}-1}{-ip/\hbar+in\pi/L} - \frac{e^{-ipL/\hbar-in\pi}-1}{-ip/\hbar-in\pi/L}\right) \\
&= \frac{(-1)^n e^{-ipL/\hbar}-1}{2\sqrt{\pi\hbar L}}\left(\frac{1}{p/\hbar-n\pi/L} - \frac{1}{p/\hbar+n\pi/L}\right) \\
&= \frac{(-1)^n e^{-ipL/\hbar}-1}{2\sqrt{\pi\hbar L}}\left[\frac{2n\pi/L}{(p/\hbar)^2-(n\pi/L)^2}\right] = \frac{1-(-1)^n e^{-ipL/\hbar}}{\sqrt{\pi\hbar L}}\left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right],
\end{aligned}$$

which corresponds to the probability distribution

$$\begin{aligned}
P_n(p) = |\langle\phi_p|\psi_n\rangle|^2 &= \left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right]^2\frac{[1-(-1)^n e^{-ipL/\hbar}][1-(-1)^n e^{ipL/\hbar}]}{\pi\hbar L} \\
&= \left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right]^2\frac{2[1-(-1)^n\cos(pL/\hbar)]}{\pi\hbar L} \\
&= \frac{2L}{\pi\hbar}\left[\frac{n\pi}{(n\pi)^2-(pL/\hbar)^2}\right]^2[1-\cos(n\pi-pL/\hbar)] \\
&= \frac{4L}{\pi\hbar}\left[\frac{n\pi}{(n\pi)^2-(pL/\hbar)^2}\right]^2\sin^2\left(\frac{n\pi}{2}-\frac{pL}{2\hbar}\right) \\
&= \frac{L}{\pi\hbar}\left(\frac{n\pi}{n\pi+pL/\hbar}\right)^2\frac{\sin^2[(n\pi-pL/\hbar)/2]}{[(n\pi-pL/\hbar)/2]^2}.
\end{aligned}$$

Note that $-\infty < p < \infty$.

4.8 HW 8

Local contents

4.8.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 8 **(Due 3/31)**

1. Problem 10.4.3. (20 points)
2. Problem 10.4.4. (30 points)
3. Problem 10.4.5. (20 points)
4. Problem 10.4.10. (30 points)

4.8.2 Problem 1 (10.4.3)

Problem 10.4.3. *Show that the first four Hermite polynomials are*

$$H_0 = 1 \tag{10.4.35}$$
$$H_1 = 2y \tag{10.4.36}$$
$$H_2 = -2(1-2y^2) \tag{10.4.37}$$
$$H_3 = -12(y - \frac{2}{3}y^3) \tag{10.4.38}$$

where the overall normalization (choice of a_0 or a_1) is as per some convention we need not get into. To compare your answers to the above, choose the starting coefficients to agree with the above. Show that

$$\int_{-\infty}^{\infty} e^{-y^2} H_n(y) H_m(y) dy = \delta_{nm}(\sqrt{\pi}2^n n!) \tag{10.4.39}$$

for the cases $m, n \leq 2$. Notice that the Hermite polynomials are not themselves orthogonal or even normalizable, we need the weight function e^{-y^2} *in the integration measure. We understand this is follows: the exponential factor converts u's to ψ's, which are the eigenfunctions of a hermitian operator (hermitian with respect to normalizable function that vanished at infinity) and hence orthogonal for different eigenvalues.*

Figure 4.21: Problem statement

<u>Solution</u>

4.8.2.1 Part 1

Starting with ode (10.4.12) which is

$$\psi''(y) - y^2\psi(y) = -2\epsilon\psi(y) \tag{10.4.12}$$

Where $\epsilon = \frac{E}{\hbar\omega}$ the energy of the particle. Let the solution be

$$\begin{aligned} \psi(y) &= u(y)e^{\frac{-y^2}{2}} \\ &= e^{\frac{-y^2}{2}} \sum_{m=0}^{n} a_m y^m \end{aligned} \tag{1}$$

Where

$$u(y) = \sum_{m=0}^{n} a_m y^m \tag{1A}$$

Eq. (1) can be written as

$$\psi(y) = \begin{cases} e^{\frac{-y^2}{2}} \sum_{m=0}^{n} a_m y^m & \lim_{y\to 0} \\ a_m y^m e^{\frac{-y^2}{2}} & \lim_{y\to\infty} \end{cases}$$

Substituting (1) in 10.4.12 gives

$$\begin{aligned}
\frac{d^2}{dy^2}\left(ue^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}}\\
\frac{d}{dy}\left(u'e^{\frac{-y^2}{2}} - uye^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}}\\
\left(u''e^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u\left(e^{\frac{-y^2}{2}} - y^2e^{\frac{-y^2}{2}}\right)\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}}\\
\left(u''e^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - ue^{\frac{-y^2}{2}} + y^2ue^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}}
\end{aligned}$$

Dividing by $e^{\frac{-y^2}{2}} \neq 0$ gives

$$\begin{aligned}
u'' - u'y - u'y - u + y^2u - y^2u &= -2\epsilon u\\
u'' - 2u'y - u &= -2\epsilon u
\end{aligned}$$

Which becomes the Hermite ODE as given in 10.4.24

$$u''(y) - 2yu'(y) + (2\varepsilon - 1)u(y) = 0 \tag{10.4.24}$$

From (1A)

$$\begin{aligned}
u' &= \sum_{m=0}^{n} ma_m y^{m-1}\\
u'' &= \sum_{m=0}^{n} m(m-1)a_m y^{m-2}
\end{aligned}$$

Substituting the above in (10.4.24) gives

$$\begin{aligned}
\sum_{m=0}^{n} m(m-1)a_m y^{m-2} - 2y\sum_{m=0}^{n} ma_m y^{m-1} + (2\varepsilon-1)\sum_{m=0}^{n} a_m y^m &= 0\\
\sum_{m=0}^{n} m(m-1)a_m y^{m-2} - \sum_{m=0}^{n} 2ma_m y^m + \sum_{m=0}^{n}(2\varepsilon-1)a_m y^m &= 0\\
\sum_{m=0}^{n} m(m-1)a_m y^{m-2} + \sum_{m=0}^{n}(2\varepsilon-1-2m)a_m y^m &= 0
\end{aligned}$$

The first sum can start from $m = 2$ without affecting the sum, hence the above becomes

$$\sum_{m=2}^{n} m(m-1)a_m y^{m-2} + \sum_{m=0}^{n}(2\varepsilon-1-2m)a_m y^m = 0$$

Let $m' = m - 2$ in the first sum, it becomes

$$\sum_{m'=0}^{n-2}(m'+2)(m'+1)a_{m'+2}y^{m'} + \sum_{m=0}^{n}(2\varepsilon-1-2m)a_m y^m = 0$$

Changing the index in the first sum from m' back to m gives

$$\sum_{m=0}^{n-2}(m+2)(m+1)a_{m+2}y^{m} + \sum_{m=0}^{n}(2\varepsilon-1-2m)a_m y^m = 0$$

Combining terms gives

$$\sum_{m=0}^{n-2}\left((m+2)(m+1)a_{m+2} + (2\varepsilon-1-2m)a_m\right)y^m + \sum_{m=n-1}^{n}(2\varepsilon-1-2m)a_m y^m = 0 \tag{1B}$$

Considering the second term above for now.

$$\begin{aligned}
\sum_{m=n-1}^{n}(2\varepsilon-1-2m)a_m y^m &= 0\\
(2\varepsilon-1-2m)a_m &= 0 \qquad m = n, m = n-1
\end{aligned}$$

Looking at case $\underline{m = n}$

$$(2\varepsilon - 1 - 2n)a_n = 0$$

but $a_n \neq 0$ since that is the highest order of the power series. If $a_n = 0$ then the dominant term of the power series is lost. This means $(2\varepsilon - 1 - 2n) = 0$ or

$$\varepsilon = n + \frac{1}{2} \tag{10.4.34}$$

Looking at case $\underline{m = n - 1}$

$$\begin{aligned}(2\varepsilon - 1 - 2(n-1))a_{n-1} &= 0\\ (2\varepsilon - 1 - 2n + 2)a_{n-1} &= 0\\ (2\varepsilon + 1 - 2n)a_{n-1} &= 0\end{aligned}$$

But $\varepsilon = n + \frac{1}{2}$, hence the above becomes

$$\begin{aligned}\left(2\left(n + \frac{1}{2}\right) + 1 - 2n\right)a_{n-1} &= 0\\ (2n + 1 + 1 - 2n)a_{n-1} &= 0\\ 2a_{n-1} &= 0\end{aligned}$$

This means

$$a_{n-1} = 0 \tag{2}$$

Now looking at case $\underline{m \leq n - 2}$ from Eq. (1C) above

$$\begin{aligned}\sum_{m=0}^{n-2}((m+2)(m+1)a_{m+2} + (2\varepsilon - 1 - 2m)a_m)y^m &= 0\\ (m+2)(m+1)a_{m+2} + (2\varepsilon - 1 - 2m)a_m &= 0\\ a_{m+2} &= \frac{-(2\varepsilon - 1 - 2m)}{(m+2)(m+1)}a_m\end{aligned}$$

But $\varepsilon = n + \frac{1}{2}$, therefore the above becomes

$$\begin{aligned}a_{m+2} &= \frac{-\left(2\left(n + \frac{1}{2}\right) - 1 - 2m\right)}{(m+2)(m+1)}a_m\\ &= \frac{-(2n - 2m)}{(m+2)(m+1)}a_m\\ &= -\frac{2(n-m)}{(m+2)(m+1)}a_m \qquad (3)\end{aligned}$$

If n is even then $n-1$ is odd. Then $a_{n-1} = 0$ from (2). But due to the recursive formula (3), this implies $a_1 = a_3 = a_5 \cdots = 0$. Which means all odd terms in the solution polynomial vanish. And if n is odd, then $n - 1$ is even. Therefore $a_{n-1} = 0$, But due to the recursive formula (3), this implies $a_0 = a_2 = a_4 \cdots = 0$. Which means all even terms in the solution polynomial vanish.

Now Eq. (3) is the recursive relation used to determine all coefficients a_i. For $m = 0$, (3) gives

$$a_2 = -na_0 \tag{4}$$

For $m = 1$, (3) gives

$$a_3 = \frac{-2(n-1)}{3!}a_1 \tag{5}$$

For $m=2$, (3) gives

$$\begin{aligned} a_4 &= \frac{-2(n-2)}{(4)(3)} a_2 \\ &= \frac{-2^2(n-2)}{4!} a_2 \\ &= \frac{2^2(n-2)n}{4!} a_0 \end{aligned} \tag{6}$$

For $m=3$, (3) gives

$$\begin{aligned} a_5 &= \frac{-2(n-3)}{(3+2)(3+1)} a_3 \\ &= \frac{-2(n-3)}{(5)(4)} \frac{-2^2(n-1)}{3!} a_1 \\ &= \frac{2^3(n-3)(n-1)}{5!} a_1 \end{aligned} \tag{7}$$

For $m=4$, (3) gives

$$\begin{aligned} a_6 &= \frac{-2(n-4)}{(4+2)(4+1)} a_4 \\ &= \frac{-2(n-4)}{(6)(5)} \frac{2^2(n-2)n}{4!} a_0 \\ &= \frac{-2^3(n-4)(n-2)n}{6!} a_0 \end{aligned} \tag{8}$$

And so on. Therefore the solution to the Hermite ODE (2) is

$$\begin{aligned} u &= \sum_{m=0}^{n} a_m y^m \\ &= a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 + \cdots \\ &= a_0 + a_1 y - n a_0 y^2 - \frac{2(n-1)}{3!} a_1 y^3 + \frac{2^2(n-2)n}{4!} a_0 y^4 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 - \frac{2^3(n-4)(n-2)n}{6!} a_0 y^6 + \cdots \end{aligned} \tag{9}$$

Which can be written as

$$\begin{aligned} u(y) = a_0 &\left(1 - n y^2 + \frac{2^2(n-2)n}{4!} y^4 - \frac{2^3(n-4)(n-2)n}{6!} y^6 + \cdots\right) \\ &+ a_1 \left(y - \frac{2(n-1)}{3!} y^3 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 + \cdots\right) \end{aligned}$$

Or

$$u(y) = a_0 u_0 + a_1 u_1$$

Where u_1, u_2 are two linearly independent solutions for the second order Hermite ODE where

$$\begin{aligned} u_0 &= 1 - n y^2 + \frac{2^2(n-2)n}{4!} y^4 - \frac{2^3(n-4)(n-2)n}{6!} y^6 + \cdots \\ u_1 &= y - \frac{2^2(n-1)}{3!} y^3 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 + \cdots \end{aligned}$$

For even n the solution $u_0(y)$ will eventually terminates, and for odd n the solution $u_1(y)$ eventually terminates. The even Hermite polynomials $H_0, H_2, H_4, \cdots$ are found from $u_0(y)$ for $n = 0, 2, 4, \cdots$ and the odd Hermite polynomials $H_1, H_3, H_5, \cdots$ are found from $u_1(y)$ for $n = 1, 3, 5, \cdots$. The Hermite polynomials need to also be normalize at the end. The even Hermite polynomials are the following

For $n = 0$

$$\begin{aligned} u_0(y) &= a_0\left(1 - ny^2 - \frac{2^2(n-2)n}{4!}y^4 - \frac{2^3(n-4)(n-2)n}{6!}y^6 + \cdots\right)_{n=0} \\ &= a_0 \end{aligned}$$

Therefore

$$H_0(y) = a_0$$

To find a_0, the normalization $\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) H_n(y) dy = 2^n n! \sqrt{\pi}\, \delta_{n,n'}$ is used, where $H_0(y) = a_0$ in this case. This gives

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_0(y) dy &= \sqrt{\pi} \\ \int_{-\infty}^{\infty} e^{-y^2} a_0^2 dy &= \sqrt{\pi} \\ a_0^2 \int_{-\infty}^{\infty} e^{-y^2} dy &= \sqrt{\pi} \\ a_0^2 \sqrt{\pi} &= \sqrt{\pi} \\ a_0 &= 1 \end{aligned}$$

Hence $a_0 = 1$ and

$$H_0(y) = 1$$

For $n = 2$

$$\begin{aligned} u_0(y) &= a_0\left(1 - ny^2 + \frac{2^2(n-2)n}{4!}y^4 - \frac{2^3(n-4)(n-2)n}{6!}y^6 + \cdots\right)_{n=2} \\ &= a_0\left(1 - 2y^2\right) \end{aligned}$$

Therefore

$$H_2(y) = a_0\left(1 - 2y^2\right)$$

To find a_0, There is an easier way to normalize $H_n(x)$ than using the normalization integral equation as was done above. This method will be used for the rest of the problem as it is simpler. It works as follows. $H_n(y) = \left(1 - 2y^2\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^2. Hence $n = 2$. Therefore the coefficient is $2^2 = 4$. But the coefficient is -2. Therefore the whole expression is multiplied by -2. This means $a_0 = -2$. Hence

$$H_2(y) = -2\left(1 - 2y^2\right)$$

For $H_4(y)$ (This is not required to find, but found for verification)

For $n = 4$

$$\begin{aligned} u_0(y) &= a_0\left(1 - ny^2 + \frac{2^2(n-2)n}{4!}y^4 - \frac{2^3(n-4)(n-2)n}{6!}y^6 + \cdots\right)_{n=4} \\ &= a_0\left(1 - 4y^2 + \frac{2^2(4-2)4}{4!}y^4\right) \\ &= a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right) \end{aligned}$$

Therefore

$$H_4(y) = a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right)$$

$H_4(y) = a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^4. Hence $n = 4$. Therefore the coefficient is $2^4 = 16$. But the coefficient is $\frac{4}{3}$. Therefore the whole expression is multiplied by 12. This means $a_0 = 12$. Hence

$$H_4(y) = 12\left(1 - 4y^2 + \frac{4}{3}y^4\right)$$

Now the odd Hermite polynomials are found. These are found from $u_1(y)$.

For $n = 1$

$$\begin{aligned} u_1(y) &= a_1\left(y - \frac{2(n-1)}{3!}y^3 + \frac{2^3(n-3)(n-1)}{5!}a_1y^5 + \cdots\right)_{n=1} \\ &= a_1 y \end{aligned}$$

Hence

$$H_1(y) = a_1 y$$

$H_1(y) = a_1 y$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^1. Hence $n = 1$. Therefore the coefficient is $2^1 = 2$. But the coefficient is 1. Therefore the whole expression is multiplied by 2. This means $a_1 = 2$. Hence

$$H_1(y) = 2y$$

For $n = 3$

$$\begin{aligned} u_2(y) &= a_1\left(y - \frac{2(n-1)}{3!}y^3 + \frac{2^3(n-3)(n-1)}{5!}a_1y^5 + \cdots\right)_{n=3} \\ &= a_1\left(y - \frac{2(3-1)}{3!}y^3\right) \end{aligned}$$

Hence

$$H_3(y) = a_1\left(y - \frac{2}{3}y^3\right)$$

$H_3(y) = a_1\left(y - \frac{2}{3}y^3\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^3. Hence $n = 3$. Therefore the coefficient is $2^3 = 8$. But the coefficient is $-\frac{2}{3}$. Therefore the whole expression is multiplied by -12. This means $a_1 = -12$. Hence

$$H_3(y) = -12\left(y - \frac{2}{3}y^3\right)$$

The following gives the final results

$$\begin{aligned} H_0(y) &= 1 \\ H_1(y) &= 2y \\ H_2(y) &= -2\left(1 - 2y^2\right) \\ H_3(y) &= -12\left(y - \frac{2}{3}y^3\right) \\ H_4(y) &= 12\left(1 - 4y^2 + \frac{4}{3}y^4\right) \end{aligned}$$

4.8.2.2 Part 2

This part verifies the results obtained in part 1 above for $m, n \leq 2$ using

$$\int_{-\infty}^{\infty} e^{-y^2} H_n(y) H_m(y) dy = 2^n n! \sqrt{\pi}\, \delta_{n,m} \tag{1}$$

For $n = 0, m = 0$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_0(y) dy = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

But $\int_{-\infty}^{\infty} e^{-y^2} dy$ is the Gaussian integral which is $\sqrt{\pi}$. Hence

$$\sqrt{\pi} = \sqrt{\pi}$$

Verified.

For $n = 0, m = 1$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_1(y) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} (2y) dy = 0$$
$$2 \int_{-\infty}^{\infty} y e^{-y^2} dy = 0$$

But y is odd, and e^{-y^2} is even. Hence the LHS is integral over odd function. Hence it must be zero. Therefore

$$0 = 0$$

Verified.

For $n = 0, m = 2$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_2(y) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(-2\left(1 - 2y^2\right)\right) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(-2 + 4y^2\right) dy = 0$$
$$-2 \int_{-\infty}^{\infty} e^{-y^2} dy + 4 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy = 0$$

But $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$, therefore the above becomes

$$-2\sqrt{\pi} + 4\left(\frac{\sqrt{\pi}}{2}\right) = 0$$
$$-2\sqrt{\pi} + 2\sqrt{\pi} = 0$$
$$0 = 0$$

Verified.

For $n = 1, m = 1$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_1(y) H_1(y) dy = 2\sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-y^2} (2y)(2y) dy = 2\sqrt{\pi}$$
$$4 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy = 2\sqrt{\pi}$$

But $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$. The above becomes

$$\begin{aligned} 4\frac{\sqrt{\pi}}{2} &= 2\sqrt{\pi} \\ 2\sqrt{\pi} &= 2\sqrt{\pi} \end{aligned}$$

Verified.

<u>For $n = 1, m = 2$</u>

Eq (1) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} H_1(y) H_2(y) dy &= 0 \\ \int_{-\infty}^{\infty} e^{-y^2} \left(2y\right)\left(-2\left(1 - 2y^2\right)\right) dy &= 0 \\ \int_{-\infty}^{\infty} e^{-y^2} \left(8y^3 - 4y\right) dy &= 0 \\ 8\int_{-\infty}^{\infty} y^3 e^{-y^2} dy - 4\int_{-\infty}^{\infty} y e^{-y^2} dy &= 0 \end{aligned}$$

Both integrals in the LHS are zero, since both are odd functions. Therefore

$$0 = 0$$

Verified.

<u>For $n = 2, m = 2$</u>

Eq (1) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} H_2(y) H_2(y) dy &= (4)2!\sqrt{\pi} \\ \int_{-\infty}^{\infty} e^{-y^2} \left(\left(-2\left(1 - 2y^2\right)\right)\right)\left(-2\left(1 - 2y^2\right)\right) dy &= 8\sqrt{\pi} \\ \int_{-\infty}^{\infty} \left(16y^4 - 16y^2 + 4\right) e^{-y^2} dy &= 8\sqrt{\pi} \\ 16\int_{-\infty}^{\infty} y^4 e^{-y^2} dy - 16\int_{-\infty}^{\infty} y^2 e^{-y^2} dy + 4\int_{-\infty}^{\infty} e^{-y^2} dy &= 8\sqrt{\pi} \end{aligned}$$

But $\int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3}{4}\sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{1}{2}\sqrt{\pi}$ and $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$. The above becomes

$$\begin{aligned} 16\left(\frac{3}{4}\sqrt{\pi}\right) - 16\left(\frac{1}{2}\sqrt{\pi}\right) + 4\sqrt{\pi} &= 8\sqrt{\pi} \\ 12\sqrt{\pi} - 8\sqrt{\pi} + 4\sqrt{\pi} &= 8\sqrt{\pi} \\ 8\sqrt{\pi} &= 8\sqrt{\pi} \end{aligned}$$

Verified. This completes the solution.

4.8.3 Problem 3 (10.4.4)

Problem 10.4.4. *Consider the* Legendre Equation

$$(1-x^2)y'' - 2xy + l(l+1)y = 0 \qquad (10.4.40)$$

Argue that the power series method will lead to a two term recursion relation and find the latter. Show that if l is an even (odd) integer, the even(odd) series will reduce to polynomials, called P_l, the Legendre polynomials *of order l. Show that*

$$\begin{aligned} P_0 &= 1 && (10.4.41)\\ P_1 &= x && (10.4.42)\\ P_2 &= \frac{1}{2}(3x^2-1) && (10.4.43)\\ P_3 &= \frac{1}{2}(5x^3-3x) && (10.4.44) \end{aligned}$$

(The overall scale of these functions is not defined by the equation, but by convention as above.) Pick any two of the above and show that they are orthogonal over the interval $-1 \le x \le 1$.

Figure 4.22: Problem statement

Solution

4.8.3.1 Part 1

The Legendre ODE is given by 10.4.40 as (L is used instead of l as it is more clear because l looks like 1, depending on font used.)

$$\left(1-x^2\right)y'' - 2xy + L(L+1)y = 0 \qquad (10.4.40)$$

Let the solution be

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1}\\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned}$$

And

$$\begin{aligned} y'' &= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}\\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above results back in (10.4.40) gives

$$\begin{aligned} \left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + L(L+1)\sum_{n=0}^{\infty} a_n x^n &= 0\\ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} L(L+1)a_n x^n &= 0\\ \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} L(L+1)a_n x^n &= 0 \qquad (1) \end{aligned}$$

For $n=0$ only the above gives

$$\begin{aligned}(n+2)(n+1)a_{n+2}x^n + L(L+1)a_nx^n &= 0\\ 2a_2 + L(L+1)a_0 &= 0\\ a_2 &= -\frac{L(L+1)}{2}a_0\end{aligned}$$

For $n=1$ only Eq (1) gives

$$\begin{aligned}(n+2)(n+1)a_{n+2}x^n - 2na_nx^n + L(L+1)a_nx^n &= 0\\ (3)(2)a_3 - 2a_1 + L(L+1)a_1 &= 0\\ a_3 &= \frac{2a_1 - L(L+1)a_1}{6}\\ &= \frac{2-L(L+1)}{6}a_1\end{aligned}$$

And for $n \geq 2$, Eq(1) gives the recusive relation

$$\begin{aligned}((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + L(L+1)a_n)x^n &= 0\\ (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + L(L+1)a_n &= 0\\ (n+2)(n+1)a_{n+2} &= (n(n-1) + 2n - L(L+1))a_n\end{aligned}$$

Hence the two term recursive is

$$a_{n+2} = \frac{n(n-1)+2n-L(L+1)}{(n+2)(n+1)}a_n \tag{1}$$

For $n=2$

$$\begin{aligned}a_4 &= \frac{n(n-1)+2n-L(L+1)}{(n+2)(n+1)}a_2\\ &= \frac{2(2-1)+4-L(L+1)}{(4)(3)}a_2\\ &= \frac{6-L(L+1)}{12}a_2\end{aligned}$$

But $a_2 = \frac{-L(L+1)}{2}a_0$ hence the above becomes

$$a_4 = \frac{6-L(L+1)}{12}\left(\frac{-L(L+1)}{2}a_0\right)$$

For $n=3$

$$\begin{aligned}a_5 &= \frac{n(n-1)+2n-L(L+1)}{(n+2)(n+1)}a_3\\ &= \frac{3(3-1)+6-L(L+1)}{(3+2)(3+1)}a_3\\ &= \frac{12-L(L+1)}{20}a_3\end{aligned}$$

But $a_3 = \frac{2-L(L+1)}{6}a_1$, hence the above becomes

$$a_5 = \frac{12-L(L+1)}{20}\left(\frac{2-L(L+1)}{6}a_1\right)$$

And so on. The solution becomes

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\
&= a_0 + a_1 x - \frac{L(L+1)}{2} a_0 x^2 + \frac{2-L(L+1)}{6} a_1 x^3 - \left(\frac{6-L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right) a_0 x^4 + \left(\frac{12-L(L+1)}{20}\right)\left(\frac{2-L(L+1)}{6}\right) a_1 x^5 + \cdots \\
&= a_0\left(1 - \frac{L(L+1)}{2} x^2 - \left(\frac{6-L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right) x^4 + \cdots\right) + a_1\left(x + \frac{2-L(L+1)}{6} x^3 + \left(\frac{12-L(L+1)}{20}\right)\left(\frac{2-L(L+1)}{6}\right) x^5 + \cdots\right)
\end{aligned}
$$

Or

$$y(x) = a_0 y_0(x) + a_1 y_1(x)$$

Where

$$
\begin{aligned}
y_0(x) &= 1 - \frac{L(L+1)}{2} x^2 - \left(\frac{6-L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right) x^4 + \cdots \\
y_1(x) &= x + \frac{2-L(L+1)}{6} x^3 + \left(\frac{12-L(L+1)}{20}\right)\left(\frac{2-L(L+1)}{6}\right) x^5 + \cdots
\end{aligned}
$$

Where y_0, y_1 are two linearly independent solutions. The even Legendre polynomials are obtained from $y_0(x)$ for integer $L = 0, 2, 4, \cdots$ and the odd Legendre polynomials are obtained from $y_1(x)$ for integer $L = 1, 3, 5, \cdots$.

For $L = 0$

$$y(x) = a_0(1)$$

Since all higher terms vanish. Choosing $a_0 = 1$ then

$$P_0(x) = 1$$

For $L = 2$

$$
\begin{aligned}
y(x) &= a_0\left(1 - \frac{L(L+1)}{2} x^2 - \left(\frac{6-L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right) x^4 + \cdots\right) \\
&= a_0\left(1 - \frac{2(2+1)}{2} x^2 - \left(\frac{6-2(2+1)}{12}\right)\left(\frac{2(2+1)}{2}\right) x^4 + \cdots\right) \\
&= a_0\left(1 - 3x^2\right)
\end{aligned}
$$

Since all higher terms vanish. Choosing $a_0 = -\frac{1}{2}$ then

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

For $L = 1$

Since L is odd, then $y_1(x)$ is used now.

$$
\begin{aligned}
y(x) &= a_1\left(x + \frac{2-L(L+1)}{6} x^3 + \left(\frac{12-L(L+1)}{20}\right)\left(\frac{2-L(L+1)}{6}\right) x^5 + \cdots\right) \\
&= a_1\left(x + \frac{2-(1+1)}{6} x^3 + \left(\frac{12-(1+1)}{20}\right)\left(\frac{2-(1+1)}{6}\right) x^5 + \cdots\right) \\
&= a_1 x
\end{aligned}
$$

Since all higher terms vanish. Choosing $a_1 = 1$ then

$$P_1(x) = x$$

For $L = 3$

$$
\begin{aligned}
y(x) &= a_1\left(x + \frac{2-L(L+1)}{6} x^3 + \left(\frac{12-L(L+1)}{20}\right)\left(\frac{2-L(L+1)}{6}\right) x^5 + \cdots\right) \\
&= a_1\left(x + \frac{2-3(3+1)}{6} x^3 + \left(\frac{12-3(3+1)}{20}\right)\left(\frac{2-3(3+1)}{6}\right) x^5 + \cdots\right) \\
&= a_1\left(x - \frac{5}{3} x^3\right)
\end{aligned}
$$

Since all higher terms vanish. Choosing $a_1 = -\frac{3}{2}$ then

$$\begin{aligned} P_3(x) &= -\frac{3}{2}\left(x - \frac{5}{3}x^3\right) \\ &= \frac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

Summary

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

4.8.3.2 Part 2

To show any two are orthogonal over $-1 \leq x \leq 1$. Selecting $P_0(x)$ and $P_1(x)$, then

$$\begin{aligned} \int_{-1}^{1} P_0(x)P_1(x)dx &= \int_{-1}^{1} xdx \\ &= \frac{1}{2}\left[x^2\right]_{-1}^{1} \\ &= \frac{1}{2}(1-1) \\ &= 0 \end{aligned}$$

Hence $P_0(x)$ and $P_1(x)$ are orthogonal to each others. Verified.

4.8.4 Problem 3 (10.4.5)

Problem 10.4.5. *The functions* $1,\ x,\ x^2, \cdots$ *are linearly independent—there is no way, for example, to express* x^3 *in terms of sums of other powers. Use the Gram–Schmidt procedure to extract from this set the first four Legendre polynomials (up to normalization) known to be orthonormal in the interval* $-1 \leq x \leq 1$.

Figure 4.23: Problem statement

Solution

Let

$$\{|x\rangle\} = \left\{1, x, x^2, x^3, \cdots\right\}$$

Where $|x_1\rangle = 1, |x_2\rangle = x, |x_3\rangle = x^2$ and so on. Let

$$\begin{aligned} P_0 &= |x_1\rangle \\ &= 1 \end{aligned}$$

Normalizing gives

$$P_0 = \frac{P_0}{\left\|P_0\right\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \sqrt{\frac{1}{2}}$$

And

$$\begin{aligned}
P_1 &= |x_2\rangle - P_0\langle P_0|x_2\rangle \\
&= x - \sqrt{\frac{1}{2}}\langle\sqrt{\frac{1}{2}}|x_2\rangle \\
&= x - \frac{1}{2}\int_{-1}^{1} x dx \\
&= x - 0 \\
&= x
\end{aligned}$$

Normalizing gives

$$P_1 = \frac{P_1}{||P_1||} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$$

And

$$\begin{aligned}
P_2 &= |x_3\rangle - (P_0\langle P_0|x_3\rangle + P_1\langle P_1|x_3\rangle) \\
&= x^2 - \left(\sqrt{\frac{1}{2}}\langle\sqrt{\frac{1}{2}}|x_3\rangle + \sqrt{\frac{3}{2}}x\langle\sqrt{\frac{3}{2}}x|x_3\rangle\right) \\
&= x^2 - \left(\frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{3}{2}x\int_{-1}^{1} x x^2 dx\right) \\
&= x^2 - \left(\frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{3}{2}x\int_{-1}^{1} x^3 dx\right) \\
&= x^2 - \left(\frac{1}{2}\frac{1}{3}\left[1-(-1)^3\right] + 0\right) \\
&= x^2 - \left(\frac{1}{2}\frac{1}{3}(2)\right) \\
&= x^2 - \frac{1}{3}
\end{aligned}$$

Normalizing

$$\begin{aligned}
P_2 &= \frac{P_2}{||P_2||} \\
&= \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)dx}} \\
&= \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} \\
&= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{5}{8}}3\left(x^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{5}{8}}\left(3x^2 - 1\right)
\end{aligned}$$

And

$$
\begin{aligned}
P_3 &= |x_4\rangle - \left(P_0\langle P_0|x_4\rangle + P_1\langle P_1|x_4\rangle + P_2\langle P_2|x_4\rangle\right)\\
&= x^3 - \left(\sqrt{\frac{1}{2}}\langle\sqrt{\frac{1}{2}}|x_4\rangle + \sqrt{\frac{3}{2}}x\langle\sqrt{\frac{3}{2}}x|x_4\rangle + \sqrt{\frac{5}{8}}\left(3x^2-1\right)\langle\sqrt{\frac{5}{8}}\left(3x^2-1\right)|x_4\rangle\right)\\
&= x^3 - \left(\frac{1}{2}\int_{-1}^{1} x^3 dx + \frac{3}{2}x\int_{-1}^{1} xx^3 dx + \frac{5}{8}\left(3x^2-1\right)\int_{-1}^{1}\left(3x^2-1\right)x^3 dx\right)\\
&= x^3 - \left(\frac{1}{2}\int_{-1}^{1} x^3 dx + \frac{3}{2}x\int_{-1}^{1} x^4 dx + \frac{5}{8}\left(3x^2-1\right)\int_{-1}^{1}\left(3x^5-x^3\right)dx\right)\\
&= x^3 - \left(\frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} + \frac{3}{2}x\left[\frac{x^5}{5}\right]_{-1}^{1} + \frac{5}{8}\left(3x^2-1\right)(0)\right)\\
&= x^3 - \left(\frac{1}{8}\left[x^4\right]_{-1}^{1} + \frac{3}{10}x\left[x^5\right]_{-1}^{1}\right)\\
&= x^3 - \left(\frac{1}{8}\left[1-(-1)^4\right] + \frac{3}{10}x\left[1-(-1)^5\right]\right)\\
&= x^3 - \left(\frac{1}{8}[0] + \frac{3}{10}x[2]\right)\\
&= x^3 - \frac{3}{5}x
\end{aligned}
$$

Normalizing

$$
\begin{aligned}
P_3 &= \frac{P_3}{\|P_3\|}\\
&= \frac{x^3-\frac{3}{5}x}{\sqrt{\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)\left(x^3-\frac{3}{5}x\right)dx}}\\
&= \frac{x^3-\frac{3}{5}x}{\sqrt{\frac{8}{175}}}\\
&= \sqrt{\frac{175}{8}}\left(x^3-\frac{3}{5}x\right)\\
&= \sqrt{\frac{(25)(7)}{8}}\left(x^3-\frac{3}{5}x\right)\\
&= \sqrt{\frac{7}{8}}\left(5x^3-3x\right)
\end{aligned}
$$

These are the first 4 Legenrdre polynomials. The scaling is different from the last problem due to difference in method used to normalize them. The following table shows the final result and difference in scaling.

P_n	Problem 10.4.5 result	Problem 10.4.4 result
$P_0(x)$	$\sqrt{\frac{1}{2}}$	1
$P_1(x)$	$\sqrt{\frac{3}{2}}x$	x
$P_2(x)$	$\sqrt{\frac{5}{8}}\left(3x^2-1\right)$	$\frac{1}{2}\left(3x^2-1\right)$
$P_3(x)$	$\sqrt{\frac{7}{8}}\left(5x^3-3x\right)$	$\frac{1}{2}\left(5x^3-3x\right)$

4.8.5 Problem 4 (10.4.10)

Problem 10.4.10. *Solve* Laguerre's Equation *which enters the solution of the hydrogen atom problem in quantum mechanics*

$$xy'' + (1-x)y' + my = 0 \qquad (10.4.65)$$

by the power series method. Show that there is a repeated root and focus on the solution which is regular at the origin. Show that this reduces to a polynomial when m is an integer. These are the Laguerre polynomials L_m. *Find the first four polynomials choosing $c_0 = 1$. Show that L_1 and L_2 are orthogonal in the interval $0 \leq x \leq \infty$ with a weight function e^{-x}. (Recall the gamma function.)*

Figure 4.24: Problem statement

Solution

Since the ODE is singular at $x = 0$ then Frobenius series is used. Let

$$\begin{aligned} y &= x^s \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_n x^{n+s} \qquad c_0 \neq 0 \end{aligned}$$

Hence

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} \\ y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-2} \end{aligned}$$

Substituting this in the ODE (10.4.65) gives

$$\begin{aligned} x \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-2} + (1-x) \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} + m \sum_{n=0}^{\infty} c_n x^{n+s} &= 0 \\ \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s) c_n x^{n+s} + m \sum_{n=0}^{\infty} c_n x^{n+s} &= 0 \\ \sum_{n=0}^{\infty} ((n+s)(n+s-1) + (n+s)) c_n x^{n+s-1} + \sum_{n=0}^{\infty} (m - (n+s)) c_n x^{n+s} &= 0 \end{aligned}$$

To make all power on x the same, the second sum is rewritten by shifting the index. This gives

$$\sum_{n=0}^{\infty} ((n+s)(n+s-1) + (n+s)) c_n x^{n+s-1} + \sum_{n=1}^{\infty} (m - (n-1+s)) c_{n-1} x^{n+s-1} = 0$$

For $n = 0$

$$\begin{aligned} ((n+s)(n+s-1) + (n+s)) c_n x^{n+s-1} &= 0 \\ ((n+s)(n+s-1) + (n+s)) c_0 &= 0 \end{aligned}$$

But by definition $c_0 \neq 0$. Therefore the indicial equation is

$$(n+s)(n+s-1) + (n+s) = 0$$

But $n = 0$. This becomes

$$\begin{aligned} s(s-1) + s &= 0 \\ s^2 - s + s &= 0 \\ s^2 &= 0 \end{aligned}$$

Hence root is $s = 0$ (repeated root). Since there is a repeated root, then this is degenerate case. First solution $y_1(x)$ is the assumed form but with $s = 0$. This means

$$\begin{aligned} y_1(x) &= x^0 \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

And the second solution is

$$\begin{aligned} y_2(x) &= y_1 \ln x + x^s \sum_{n=0}^{\infty} b_n x^n \\ &= y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

But this solution $y_2(x)$ is not bounded at $x = 0$ due to $\ln x$ blowing up at origin. The regular solution is only $y_1(x)$. So $y_1(x)$ will be used from now on and not $y_2(x)$. Therefore

$$\begin{aligned} y_1'(x) &= \sum_{n=0}^{\infty} n c_n x^{n-1} \\ y_1''(x) &= \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} \end{aligned}$$

Substituting the above in ODE (10.4.65) gives

$$\begin{aligned} x \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + (1-x) \sum_{n=0}^{\infty} n c_n x^{n-1} + m \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=0}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} m c_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n(n-1)+n) c_n x^{n-1} + \sum_{n=0}^{\infty} (m-n) c_n x^n &= 0 \end{aligned}$$

To make powers on x the same, the index of the first sum is shifted to give

$$\sum_{n=-1}^{\infty} ((n+1)n + (n+1)) c_{n+1} x^n + \sum_{n=0}^{\infty} (m-n) c_n x^n = 0$$

But when $n = -1$ the first sum is zero. So the first sum index can start $n = 0$ which gives

$$\sum_{n=0}^{\infty} ((n+1)n + (n+1)) c_{n+1} x^n + \sum_{n=0}^{\infty} (m-n) c_n x^n = 0$$

Now the sums are combined to give

$$\sum_{n=0}^{\infty} [((n+1)n + (n+1)) c_{n+1} + (m-n) c_n] x^n = 0$$

Hence recursive relation is

$$\begin{aligned} ((n+1)n + (n+1)) c_{n+1} + (m-n) c_n &= 0 \\ c_{n+1} &= \frac{n-m}{((n+1)n + (n+1))} c_n \\ &= \frac{n-m}{n^2 + 2n + 1} c_n \end{aligned}$$

For $n = 0$

$$c_1 = -m c_0$$

For $n = 1$

$$\begin{aligned} c_2 &= \frac{1-m}{1+2+1} c_1 \\ &= \frac{1-m}{4} c_1 \\ &= \frac{1-m}{4} (-m c_0) \\ &= \frac{m^2 - m}{4} c_0 \end{aligned}$$

<u>For $n = 2$</u>

$$\begin{aligned}
c_3 &= \frac{2-m}{2^2+4+1}c_2\\
&= \frac{2-m}{9}c_2\\
&= \frac{2-m}{9}\left(\frac{m^2-m}{4}c_0\right)\\
&= \frac{(2-m)\left(m^2-m\right)}{36}c_0\\
&= \frac{-m^3+3m^2-2m}{36}c_0
\end{aligned}$$

<u>For $n = 3$</u>

$$\begin{aligned}
c_4 &= \frac{n-m}{n^2+2n+1}c_3\\
&= \frac{3-m}{9+6+1}c_3\\
&= \frac{3-m}{16}\left(\frac{-m^3+3m^2-2m}{36}c_0\right)\\
&= \frac{(3-m)\left(-m^3+3m^2-2m\right)}{(16)(36)}c_0\\
&= \frac{m^4-6m^3+11m^2-6m}{576}c_0
\end{aligned}$$

And so on. The solution becomes

$$\begin{aligned}
y_1(x) &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\\
&= c_0 - mc_0x + \frac{m^2-m}{4}c_0x^2 + \frac{-m^3+3m^2-2m}{36}c_0x^3 + \frac{m^4-6m^3+11m^2-6m}{576}c_0x^4 + \cdots\\
&= c_0\left(1 - mx + \frac{m^2-m}{4}x^2 + \frac{-m^3+3m^2-2m}{36}x^3 + \frac{m^4-6m^3+11m^2-6m}{576}x^4 + \cdots\right)
\end{aligned}$$

Setting $c_0 = 1$, the solution is

$$y_1(x) = 1 - mx + \frac{m^2-m}{4}x^2 + \frac{-m^3+3m^2-2m}{36}x^3 + \frac{m^4-6m^3+11m^2-6m}{576}x^4 + \cdots$$

For integer m these are polynomials given by

<u>For $m = 0$</u>

$$L_0(x) = 1$$

Since rest of terms are zero.

<u>For $m = 1$</u>

$$L_1(x) = 1 - x$$

Since rest of terms are zero.

<u>For $m = 2$</u>

$$L_2(x) = 1 - 2x + \frac{1}{2}x^2$$

Since rest of terms are zero.

<u>For $m = 3$</u>

$$\begin{aligned}
L_3(x) &= 1 - 3x + \frac{3^2-3}{4}x^2 + \frac{-3^3+3\left(3^2\right)-6}{36}x^3\\
&= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3
\end{aligned}$$

Since rest of terms are zero. Hence

$$\begin{aligned}L_0(x) &= 1\\ L_1(x) &= 1 - x\\ L_2(x) &= 1 - 2x + \frac{1}{2}x^2\\ L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3\end{aligned}$$

Or

$$\begin{aligned}L_0(x) &= 1\\ L_1(x) &= 1 - x\\ L_2(x) &= \frac{1}{2}\left(2 - 4x + x^2\right)\\ L_3(x) &= \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)\end{aligned}$$

The following shows that $L_1(x), L_2(x)$ are orthogonal on $0 \leq x \leq \infty$ with weight e^{-x}

$$\begin{aligned}\int_0^\infty L_1(x)L_2(x)e^{-x}dx &= \int_0^\infty (1-x)\left(\frac{1}{2}\left(2 - 4x + x^2\right)\right)e^{-x}dx\\ &= \int_0^\infty \left(-\frac{1}{2}x^3 + \frac{5}{2}x^2 - 3x + 1\right)e^{-x}dx\\ &= -\frac{1}{2}\int_0^\infty x^3e^{-x}dx + \frac{5}{2}\int_0^\infty x^2e^{-x}dx - 3\int_0^\infty xe^{-x}dx + \int_0^\infty e^{-x}dx\end{aligned}$$

To evaluate these integrals the following relation will be used

$$\int_0^\infty x^n e^{-x} = n!$$

Therefore

$$\begin{aligned}\int_0^\infty x^3e^{-x}dx &= 3! = 6\\ \int_0^\infty x^2e^{-x}dx &= 2! = 2\\ \int_0^\infty xe^{-x}dx &= 1! = 1\end{aligned}$$

And

$$\int_0^\infty e^{-x}dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1$$

Using these results gives

$$\begin{aligned}\int_0^\infty L_1(x)L_2(x)e^{-x}dx &= -\frac{1}{2}(6) + \frac{5}{2}(2) - 3(1) + 1\\ &= 0\end{aligned}$$

This shows that $L_1(x), L_2(x)$ are orthogonal on $0 \leq x \leq \infty$ with weight e^{-x}. This complete the solution.

4.8.6 key solution for HW 8

Physics 3041 (Spring 2021) Solutions to Homework Set 8

1. Problem 10.4.3. (20 points)

$$H_n(y) = \sum_{m=0}^{n} a_m y^m,\ a_{n-1} = 0,\ \epsilon = n + \frac{1}{2},\ a_{m+2} = \frac{1+2m-2\epsilon}{(m+2)(m+1)} a_m = \frac{2(m-n)}{(m+2)(m+1)} a_m$$

$$n = 0 \Rightarrow H_0 = a_0 \to H_0 = 1,\ n = 1 \Rightarrow H_1 = a_1 y \to H_1 = 2y$$

$$n = 2 \Rightarrow a_2 = \frac{2(-2)}{2} a_0 = -2a_0,\ H_2 = (1-2y^2)a_0 \to H_2 = -2(1-2y^2)$$

$$n = 3 \Rightarrow a_3 = \frac{2(1-3)}{3\cdot 2} a_1 = -\frac{2}{3} a_1,\ H_3 = (y - \frac{2}{3}y^3)a_1 \to H_3 = -12(y - \frac{2}{3}y^3)$$

$$\int_{-\infty}^{\infty} [H_0(y)]^2 e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} [H_1(y)]^2 e^{-\alpha y^2} dy = 4\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2} dy = -4\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} e^{-\alpha y^2} dy = -4\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} e^{-z^2} d\frac{z}{\sqrt{\alpha}}$$

$$= -4\frac{\partial}{\partial\alpha}\frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{4\sqrt{\pi}}{2\alpha^{3/2}} = \frac{2\sqrt{\pi}}{\alpha^{3/2}} \Rightarrow \int_{-\infty}^{\infty} [H_1(y)]^2 e^{-y^2} dy = 2\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} [H_2(y)]^2 e^{-y^2} dy = 4\int_{-\infty}^{\infty} (1-2y^2)^2 e^{-y^2} dy = 4\int_{-\infty}^{\infty} (1-4y^2+4y^4) e^{-y^2} dy$$

$$= 4[\sqrt{\pi} - 2\sqrt{\pi} + 4(\frac{\partial^2}{\partial\alpha^2}\frac{\sqrt{\pi}}{\sqrt{\alpha}})_{\alpha=1}] = 4\sqrt{\pi}(1 - 2 + \frac{4\cdot 3}{2\cdot 2}) = 8\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} H_0(y)H_2(y) e^{-y^2} dy = -2\int_{-\infty}^{\infty} (1-2y^2) e^{-y^2} dy = -2(\sqrt{\pi} - \frac{2\sqrt{\pi}}{2}) = 0$$

$$\int_{-\infty}^{\infty} H_0(y)H_1(y) e^{-y^2} dy = 0,\ \int_{-\infty}^{\infty} H_1(y)H_2(y) e^{-y^2} dy = 0$$

The last two results are straightforward because the integrands are odd functions of y. The general result is $\int_{-\infty}^{\infty} H_m(y)H_n(y)e^{-y^2}dy = \delta_{mn}(2^n n!\sqrt{\pi})$.

2. Problem 10.4.4. (30 points)

$$y = \sum_{m=0}^{n} c_m x^m,\ y' = \sum_{m=0}^{n} m c_m x^{m-1},\ y'' = \sum_{m=0}^{n} m(m-1) c_m x^{m-2}$$

$$\begin{aligned}
(1-x^2)y'' - 2xy' + l(l+1)y &= \sum_{m=0}^{n} m(m-1)c_m x^{m-2} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m=2}^{n} m(m-1)c_m x^{m-2} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m'=0}^{n-2} (m'+2)(m'+1)c_{m'+2} x^{m'} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m=0}^{n-2} \{(m+2)(m+1)c_{m+2} + [l(l+1) - m(m+1)]c_m\} x^m \\
&+ \sum_{m=n-1}^{n} [l(l+1) - m(m+1)]c_m x^m = 0
\end{aligned}$$

For $m = n$, $[l(l+1) - n(n+1)]c_n = 0 \Rightarrow n = l$ because $c_n \neq 0$ by the definition of x^n as the term of the highest power. The other possibility of $n = -l-1$ is discarded because we are looking for solutions of $n \geq 0$.

For $m = n-1$, $[l(l+1) - n(n-1)]c_{n-1} = [l(l+1) - l(l-1)]c_{n-1} = 2lc_{n-1} = 0 \Rightarrow c_{n-1} = 0$.

For $0 \leq m \leq n-2$,

$$(m+2)(m+1)c_{m+2} + [l(l+1) - m(m+1)]c_m = 0 \Rightarrow c_{m+2} = \frac{m(m+1) - l(l+1)}{(m+2)(m+1)} c_m.$$

Therefore, for an even (odd) l, all the odd (even) terms vanish because $c_{l-1} = 0$ and the solution is a polynomial of the lth order with even (odd) terms only, defined as P_l.

$$\begin{aligned}
&l = 0 \Rightarrow P_0 = c_0 \rightarrow P_0 = 1 \\
&l = 1 \Rightarrow P_1 = c_1 x \rightarrow P_1 = x \\
&l = 2 \Rightarrow c_2 = \frac{-2\cdot 3}{2} c_0 = -3c_0,\ P_2 = c_0(1 - 3x^2) \rightarrow P_2 = \frac{1}{2}(3x^2 - 1) \\
&l = 3 \Rightarrow c_3 = \frac{2 - 3\cdot 4}{3\cdot 2} c_1 = -\frac{5}{3} c_1,\ P_3 = c_1(x - \frac{5}{3}x^3) \rightarrow P_3 = \frac{1}{2}(5x^3 - 3x)
\end{aligned}$$

Clearly, $\int_{-1}^{1} P_l(x) P_{l'}(x) dx = 0$ for even-odd or odd-even pairs of l and l'.

$$\begin{aligned}
\int_{-1}^{1} P_0(x)P_2(x)dx &= \frac{1}{2}\int_{-1}^{1}(3x^2 - 1)dx = \int_{0}^{1}(3x^2 - 1)dx = x^3|_0^1 - x|_0^1 = 0 \\
\int_{-1}^{1} P_1(x)P_3(x)dx &= \frac{1}{2}\int_{-1}^{1} x(5x^3 - 3x)dx = \int_{0}^{1} x(5x^3 - 3x)dx = x^5|_0^1 - x^3|_0^1 = 0
\end{aligned}$$

3. Problem 10.4.5. (20 points)

$$
\begin{aligned}
|I\rangle &= 1 \Rightarrow \langle I|I\rangle = \int_{-1}^{1} dx = 2,\ |1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{\sqrt{2}} \rightarrow P_0 = 1 \\
|II\rangle &= x \Rightarrow |II'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle = x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x}{\sqrt{2}}dx = x \\
\langle II'|II'\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3},\ |2\rangle = \frac{|II'\rangle}{\sqrt{\langle II'|II'\rangle}} = x\sqrt{\frac{3}{2}} \rightarrow P_1 = x \\
|III\rangle &= x^2 \Rightarrow |III'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle \\
&= x^2 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^2}{\sqrt{2}}dx - x\sqrt{\frac{3}{2}}\int_{-1}^{1} x^3\sqrt{\frac{3}{2}}\,dx = x^2 - \frac{1}{3} \\
\langle III'|III'\rangle &= \int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45} \\
|3\rangle &= \frac{|III'\rangle}{\sqrt{\langle III'|III'\rangle}} = (x^2 - \frac{1}{3})\sqrt{\frac{45}{8}} = (3x^2 - 1)\sqrt{\frac{5}{8}} \rightarrow P_2 = \frac{1}{2}(3x^2 - 1) \\
|IV\rangle &= x^3 \Rightarrow |IV'\rangle = |IV\rangle - |1\rangle\langle 1|IV\rangle - |2\rangle\langle 2|IV\rangle - |3\rangle\langle 3|IV\rangle \\
&= x^3 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^3}{\sqrt{2}}dx - x\sqrt{\frac{3}{2}}\int_{-1}^{1} x^4\sqrt{\frac{3}{2}}\,dx - (3x^2 - 1)\sqrt{\frac{5}{8}}\int_{-1}^{1} x^3(3x^2 - 1)\sqrt{\frac{5}{8}}\,dx \\
&= x^3 - x\frac{3}{2}\frac{2}{5} = x^3 - \frac{3}{5}x \\
\langle IV'|IV'\rangle &= \int_{-1}^{1}(x^3 - \frac{3}{5}x)^2 dx = \int_{-1}^{1}(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2)dx = \frac{2}{7} - \frac{12}{25} + \frac{6}{25} = \frac{8}{175} \\
|4\rangle &= \frac{|IV'\rangle}{\sqrt{\langle IV'|IV'\rangle}} = (x^3 - \frac{3}{5}x)\sqrt{\frac{175}{8}} = (5x^3 - 3x)\sqrt{\frac{7}{8}} \rightarrow P_3 = \frac{1}{2}(5x^3 - 3x)
\end{aligned}
$$

Following the above procedure, the general result is $|i = l + 1\rangle = P_l\sqrt{\frac{2l+1}{2}}$.

4. Problem 10.4.10. (30 points)

$$y = x^s \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=0}^{\infty}(n+s)c_n x^{n+s-1},\ y'' = \sum_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s-2}$$

$$\begin{aligned} xy'' + (1-x)y' + my &= \sum_{n=0}^{\infty}[(n+s)(n+s-1)+(n+s)]c_n x^{n+s-1} + \sum_{n=0}^{\infty}[-(n+s)+m]c_n x^{n+s} \\ &= \sum_{n=0}^{\infty}(n+s)^2 c_n x^{n+s-1} + \sum_{n=0}^{\infty}[m-(n+s)]c_n x^{n+s} = 0 \end{aligned}$$

For $n = 0$, $c_n = c_0 \neq 0$ for the lowest term x^{s-1}, so $(n+s)^2 = s^2 = 0 \Rightarrow s = 0$ (repeated). The regular solution is

$$\begin{aligned} y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty}(m-n)c_n x^n &= \sum_{n=1}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty}(m-n)c_n x^n \\ = \sum_{n'=0}^{\infty}(n'+1)^2 c_{n'+1} x^{n'} + \sum_{n=0}^{\infty}(m-n)c_n x^n &= \sum_{n=0}^{\infty}[(n+1)^2 c_{n+1} + (m-n)c_n]x^n = 0 \end{aligned}$$

So we have

$$c_{n+1} = \frac{n-m}{(n+1)^2}c_n,$$

which indicates that the series stops at $n = m$ with the highest term x^m for integer m.

$$\begin{aligned} & m = 0 \Rightarrow y = c_0,\ \int_0^\infty y^2 e^{-x}dx = c_0^2 = 1,\ c_0 = 1 \to L_0 = 1 \\ & m = 1 \Rightarrow c_1 = -c_0,\ y = c_0(1-x) \\ \int_0^\infty y^2 e^{-x}dx &= c_0^2 \int_0^\infty (1-2x+x^2)e^{-x}dx = c_0^2(1-2+2) = c_0^2 = 1,\ c_0 = 1 \to L_1 = 1-x \\ & m = 2 \Rightarrow c_1 = -2c_0,\ c_2 = -\frac{1}{4}c_1 = \frac{1}{2}c_0,\ y = c_0(1-2x+\frac{1}{2}x^2) \\ \int_0^\infty y^2 e^{-x}dx &= c_0^2 \int_0^\infty (1-2x+\frac{1}{2}x^2)^2 e^{-x}dx = c_0^2 \int_0^\infty (1-4x+5x^2-2x^3+\frac{1}{4}x^4)e^{-x}dx \\ &= c_0^2(1-4+5\cdot 2-2\cdot 6+\frac{24}{4}) = c_0^2 = 1,\ c_0 = 1 \to L_2 = 1-2x+\frac{1}{2}x^2 \\ & m = 3 \Rightarrow c_1 = -3c_0,\ c_2 = -\frac{1}{2}c_1 = \frac{3}{2}c_0,\ c_3 = -\frac{1}{9}c_2 = -\frac{1}{6}c_0,\ y = c_0(1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3) \\ \int_0^\infty y^2 e^{-x}dx &= c_0^2 \int_0^\infty (1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3)^2 e^{-x}dx \\ &= c_0^2 \int_0^\infty (1-6x+12x^2-\frac{28}{3}x^3+\frac{13}{4}x^4-\frac{1}{2}x^5+\frac{1}{36}x^6)e^{-x}dx \\ &= c_0^2(1-6+12\cdot 2-\frac{28}{3}\cdot 6+\frac{13}{4}\cdot 24-\frac{1}{2}\cdot 120+\frac{1}{36}\cdot 720) = c_0^2 = 1,\ c_0 = 1 \\ &\to L_3 = 1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3 \end{aligned}$$

$$\begin{aligned} \int_0^\infty L_1 L_2 e^{-x}dx &= \int_0^\infty (1-x)(1-2x+\frac{1}{2}x^2)e^{-x}dx = \int_0^\infty (1-3x+\frac{5}{2}x^2-\frac{1}{2}x^3)e^{-x}dx \\ &= 1-3+\frac{5}{2}\cdot 2-\frac{1}{2}\cdot 6 = 0 \end{aligned}$$

4.9 HW 9

Local contents

4.9.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 9 **(Due 4/16)**

1. Problem 10.2.8. (20 points)
2. Problem 10.2.11. (40 points)
3. Problem 10.3.5. (10 points)
4. Problem 10.3.8. (10 points)
5. Problem 10.3.9. (20 points)

4.9.2 Problem 1 (10.2.8)

Problem 10.2.8. *Find the solutions to*

(i)	$(D^2+2D+1)x(t)=0$	*with* $x(0)=1,\ \dot{x}(0)=0$
(ii)	$(D^4+1)x(t)=0$	
(iii)	$(D^3-3D^2-9D-5)x(t)=0$	*(5 is a root)*
(iv)	$(D+1)^2(D^4-256)x(t)=0$	

Figure 4.25: Problem statement

<u>Solution</u>

4.9.2.1 Part 1

The ode to solve is

$$\begin{aligned} x''(t)+2x'(t)+x(t)&=0 \qquad (1)\\ x(0)&=1\\ x'(0)&=0 \end{aligned}$$

This is a constant coefficient ODE. Assuming the solution has the form $x=Ae^{\lambda t}$ and substituting this back in (1) gives the characteristic equation (the constant A drops out)

$$\begin{aligned} \lambda^2e^{\lambda t}+2\lambda e^{\lambda t}+e^{\lambda t}&=0\\ \left(\lambda^2+42\lambda+1\right)e^{\lambda t}&=0 \end{aligned}$$

Since $e^{\lambda t}\neq 0$, the above gives

$$\begin{aligned} \lambda^2+2\lambda+1&=0\\ (\lambda+1)^2&=0 \end{aligned}$$

Therefore $\lambda=-1$. (double root). Since the root is double, then the basis solutions are $x_1(t)=e^{\lambda t},x_2(t)=te^{\lambda t}$ and the general solution is a linear combination of these basis solutions. Therefore the general solution is

$$x(t)=Ae^{-t}+Bte^{-t} \qquad (2)$$

The constants A,B are found from initial conditions. At $t=0$ and using $x(0)=1$ gives

$$1=A \qquad (3)$$

Solution (2) becomes

$$x(t)=e^{-t}+Bte^{-t} \qquad (4)$$

Taking derivative of (4) gives

$$x'(t)=-e^{-t}+Be^{t}-Bte^{-t}$$

Using $x'(0)=0$ on the above gives

$$\begin{aligned} 0&=-1+B\\ B&=1 \qquad (5) \end{aligned}$$

Substituting (3,5) in (4) gives the final solution

$$\begin{aligned} x(t)&=e^{-t}+te^{-t}\\ &=(1+t)e^{-t} \end{aligned}$$

4.9.2.2 Part 2

The ode to solve is

$$x''''(t) + x(t) = 0$$

As was done in the above part, substituting $x = Ae^{\lambda t}$ in the above and simplifying gives the characteristic equation

$$\lambda^4 + 1 = 0$$

Hence the roots are $\lambda^4 = -1$ or $\lambda^4 = e^{-i\frac{\pi}{2}}$. There are 4 roots that divide the unit circle equally, each is 90 degrees phase shifted (anti clockwise)from the other, starting from first root at phase $-\frac{\pi}{2} = -45$ degrees. Hence the roots are

$$\begin{aligned}
\lambda_1 &= \cos(-45) + i\sin(-45)\\
\lambda_2 &= \cos(45) + i\sin(45)\\
\lambda_3 &= \cos(135) + i\sin(135)\\
\lambda_4 &= \cos(225) + i\sin(225)
\end{aligned}$$

or

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\\
\lambda_2 &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\\
\lambda_3 &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\\
\lambda_4 &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}
\end{aligned}$$

Therefore the basis solutions are

$$\begin{aligned}
x_1(t) &= e^{\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}\\
x_2(t) &= e^{\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t}\\
x_3(t) &= e^{\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t}\\
x_4(t) &= e^{\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}
\end{aligned}$$

The general solution is linear combination of the above basis solutions, which becomes

$$\begin{aligned}
x(t) &= c_1 e^{\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t} + c_2 e^{\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t} + c_3 e^{\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t} + c_4 e^{\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}\\
&= c_1 e^{\frac{\sqrt{2}}{2}t} e^{-i\frac{\sqrt{2}}{2}t} + c_2 e^{\frac{\sqrt{2}}{2}t} e^{i\frac{\sqrt{2}}{2}t} + c_3 e^{-\frac{\sqrt{2}}{2}t} e^{i\frac{\sqrt{2}}{2}t} + c_4 e^{-\frac{\sqrt{2}}{2}t} e^{-i\frac{\sqrt{2}}{2}t}\\
&= e^{\frac{\sqrt{2}}{2}t}\left(c_1 e^{-i\frac{\sqrt{2}}{2}t} + c_2 e^{i\frac{\sqrt{2}}{2}t}\right) + e^{-\frac{\sqrt{2}}{2}t}\left(c_3 e^{i\frac{\sqrt{2}}{2}t} + c_4 e^{-i\frac{\sqrt{2}}{2}t}\right)
\end{aligned}$$

Using Euler relation, the above can be rewritten as

$$x(t) = e^{\frac{\sqrt{2}}{2}t}\left(c_1 \sin\left(\frac{\sqrt{2}}{2}t\right) + c_2 \cos\left(\frac{\sqrt{2}}{2}t\right)\right) + e^{-\frac{\sqrt{2}}{2}t}\left(c_3 \sin\left(\frac{\sqrt{2}}{2}t\right) + c_4 \cos\left(\frac{\sqrt{2}}{2}t\right)\right)$$

4.9.2.3 Part 3

The ode to solve is

$$x'''(t) - 3x''(t) - 9x'(t) - 5x(t) = 0$$

As was done in the above part, substituting $x = Ae^{\lambda t}$ in the above and simplifying gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$

Since one root is 5, then the above can be written as

$$(\lambda - 5)(\Delta) = 0$$

Where

$$\Delta = \frac{\lambda^3 - 3\lambda^2 - 9\lambda - 5}{\lambda - 5}$$

Using long division gives

$$\Delta = (\lambda + 1)^2$$

Therefore the roots of the characteristic equation are

$$\begin{aligned} \lambda_1 &= 5 \\ \lambda_2 &= -1 \\ \lambda_3 &= -1 \end{aligned}$$

roots λ_2, λ_3 are the same. $\lambda = -1$ is a double root. Therefore the basis solutions are

$$\begin{aligned} x_1(t) &= e^{5t} \\ x_2(t) &= e^{t} \\ x_3(t) &= te^{t} \end{aligned}$$

Where t multiplies the last basis $x_3(t)$ due to the double root. The general solution is linear combination of the above basis solutions, which gives

$$\begin{aligned} x(t) &= c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) \\ &= c_1 e^{5t} + c_2 e^{t} + c_3 t e^{t} \end{aligned}$$

4.9.2.4 Part 4

The ode to solve is

$$(D+1)^2 \left(D^4 - 256\right) x(t) = 0$$

This has the characteristic equation equation $(\lambda + 1)^2 \left(\lambda^4 - 256\right) = 0$. The roots of $\left(\lambda^4 - 256\right)$ are given by $\lambda^4 = 256$. Let $\lambda^2 = \omega$. Therefore $\omega^2 = 256$ which gives $\omega = \pm 16$.

When $\omega = 16$, then $\lambda^2 = 16$ which gives $\lambda = \pm 4$ and when $\omega = -16$, then $\lambda^2 = -16$ which gives $\lambda = \pm 4i$.

The other part $(\lambda + 1)^2 = 0$ gives $\lambda = -1$, double root. Therefore the roots of the characteristic equation are

$$\begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= -4 \\ \lambda_3 &= 4i \\ \lambda_4 &= -4i \\ \lambda_5 &= -1 \\ \lambda_6 &= -1 \end{aligned}$$

Root $\lambda = -1$ is a double root. Therefore the basis solutions as

$$\begin{aligned} x_1(t) &= e^{4t} \\ x_2(t) &= e^{-4t} \\ x_3(t) &= e^{4it} \\ x_4(t) &= e^{-4it} \\ x_5(t) &= e^{-t} \\ x_6(t) &= te^{-t} \end{aligned}$$

Where t was multiplied by e^{-t} in $x_6(t)$ since the root is double. The solution is linear combination of the above basis solutions, which gives

$$\begin{aligned} x(t) &= c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + c_4 x_4(t) + c_5 x_5(t) + c_6 x_6(t) \\ &= c_1 e^{4t} + c_2 e^{-4t} + c_3 e^{4it} + c_4 e^{-4it} + c_5 e^{-t} + c_6 t e^{-t} \\ &= e^{-t}(c_5 + t c_6) + c_1 e^{4t} + c_2 e^{-4t} + c_3 \sin(4t) + c_4 \cos(4t) \end{aligned}$$

Where Euler relation was used in the last step above to rewrite $c_3 e^{4it} + c_4 e^{-4it}$.

4.9.3 Problem 2 (10.2.11)

Problem 10.2.11. *Solve the following subject to* $y(0) = 1,\ \dot{y}(0) = 0$

$$
\begin{aligned}
&(i) && \ddot{y} - \dot{y} - 2y = e^{2x} \\
&(ii) && (D^2 - 2D + 1)y = 2\cos x \\
&(iii) && y'' + 16y = 16\cos 4x \\
&(iv) && y'' - y = \cosh x
\end{aligned}
$$

Figure 4.26: Problem statement

Solution

4.9.3.1 Part 1

The ode to solve is

$$y'' - y' - 2y = e^{2x} \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. The general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' - y' - 2y = 0$ and $y_p(x)$ is any particular solution to $y'' - y' - 2y = e^{2x}$. The homogenous solution is found using the characteristic polynomial method as was done in the above problems. Substituting $y = Ae^{\lambda x}$ in $y'' - y' - 2y = 0$ and simplifying gives

$$
\begin{aligned}
\lambda^2 - \lambda - 2 &= 0 \\
(\lambda + 1)(\lambda - 2) &= 0
\end{aligned}
$$

The roots are $\lambda_1 = -1, \lambda_2 = 2$. Therefore the basis solutions are

$$
\begin{aligned}
y_1(x) &= e^{-x} \\
y_1(x) &= e^{2x}
\end{aligned} \tag{3}
$$

Hence $y_h(x)$ is linear combination of the above, which gives

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x}$$

The particular solution is now found. Assuming $y_p = Ae^{2x}$. But e^{2x} is a basis solution of the homogeneous ode. Therefore y_p is multiplied by x giving

$$y_p = Axe^{2x}$$

Substituting this back in (1) and solving for A gives

$$
\begin{aligned}
y_p' &= Ae^{2x} + 2Axe^{2x} \\
y_p'' &= 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} \\
&= 4Ae^{2x} + 4Axe^{2x}
\end{aligned}
$$

Eq (1) becomes

$$
\begin{aligned}
\left(4Ae^{2x} + 4Axe^{2x}\right) - \left(Ae^{2x} + 2Axe^{2x}\right) - 2\left(Axe^{2x}\right) &= e^{2x} \\
4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x} &= e^{2x} \\
4A + 4Ax - A - 2Ax - 2Ax &= 1 \\
3A &= 1 \\
A &= \frac{1}{3}
\end{aligned}
$$

Hence the particular solution is

$$y_p(x) = \frac{1}{3}xe^{2x}$$

Therefore from (2) the general solution is

$$y(x) = c_1e^{-x} + c_2e^{2x} + \frac{1}{3}xe^{2x} \tag{4}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (4) becomes

$$1 = c_1 + c_2 \tag{5}$$

Taking derivative of (4) gives

$$y'(x) = -c_1e^{-x} + 2c_2e^{2x} + \frac{1}{3}e^{2x} + \frac{2}{3}xe^{2x}$$

At $x = 0$ the above gives

$$0 = -c_1 + 2c_2 + \frac{1}{3} \tag{6}$$

Eq (5,6) are now solved for c_1, c_2. From (5)

$$c_1 = 1 - c_2$$

Substituting this back in (6) gives

$$\begin{aligned} 0 &= -(1 - c_2) + 2c_2 + \frac{1}{3} \\ c_2 &= \frac{2}{9} \end{aligned}$$

Therefore $c_1 = 1 - \frac{2}{9} = \frac{7}{9}$. The final solution (4) becomes

$$y(x) = \frac{7}{9}e^{-x} + \frac{2}{9}e^{2x} + \frac{1}{3}xe^{2x}$$

4.9.3.2 Part 2

The ode to solve is

$$y'' - 2y' + y = 2\cos x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' - 2y' + y = 0$ and $y_p(x)$ is any particular solution to $y'' - 2y' + y = 2\cos x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' - 2y' + y = 0$ and simplifying gives

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)(\lambda - 1) &= 0 \end{aligned}$$

roots are $\lambda_1 = 1, \lambda_2 = 1$. (double root). The basis solutions are therefore

$$\begin{aligned} y_1(x) &= e^x \\ y_1(x) &= xe^x \end{aligned} \tag{3}$$

$y_h(x)$ is linear combination of the the above which gives

$$y_h(x) = c_1e^x + c_2xe^x$$

The particular solution is now found. Assuming $y_p = A\cos x$. Taking all derivatives of this solution gives the set $\{\cos x, \sin x\}$. Therefore

$$y_p = A\cos x + B\sin x$$

Substituting this back in (1) to solve for A, B gives

$$\begin{aligned} y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2\cos x \\ (-A\cos x - B\sin x) - 2(-A\sin x + B\cos x) + (A\cos x + B\sin x) &= 2\cos x \\ -A\cos x - B\sin x + 2A\sin x - 2B\cos x + A\cos x + B\sin x &= 2\cos x \\ \cos x(-A - 2B + A) + \sin x(-B + 2A + B) &= 2\cos x \\ -2B\cos x + 2A\sin x &= 2\cos x \end{aligned}$$

Hence $A = 0$ and $B = -1$. Therefore the particular solution is

$$y_p(x) = -\sin x$$

Eq (2) becomes

$$y(x) = c_1 e^x + c_2 x e^x - \sin x \tag{4}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (4) becomes

$$1 = c_1 \tag{5}$$

The solution (4) becomes

$$y(x) = e^x + c_2 x e^x - \sin x \tag{6}$$

Taking derivative of (6) gives

$$y'(x) = e^x + c_2 e^x + c_2 x e^x - \cos x$$

At $x = 0$ the above gives

$$0 = 1 + c_2 - 1 \tag{6}$$

Therefore $c_2 = 0$ and now Eq (6) gives the final solution as

$$y(x) = e^x - \sin x$$

4.9.3.3 Part 3

The ode to solve is

$$y'' + 16y = 16\cos 4x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' + 16y = 0$ and $y_p(x)$ is any particular solution to $y'' + 16y = 16\cos 4x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' + 16y = 0$ and simplifying gives

$$\begin{aligned} \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

The roots are $\lambda_1 = 4i, \lambda_2 = -4i$. The basis solutions are therefore

$$\begin{aligned} y_1(x) &= e^{i4x} \\ y_2(x) &= e^{-i4x} \end{aligned} \tag{3}$$

Therefore $y_h(x)$ is linear combination of the the above.

$$y_h(x) = c_1 e^{i4x} + c_2 e^{-i4x}$$

Which can be written, using Euler formula as

$$y_h(x) = c_1 \cos 4x + c_2 \sin 4x$$

The particular solution is now found. Assuming $y_p = A\cos 4x$. Taking all derivatives of this, the basis for y_p becomes $\{\cos 4x, \sin 4x\}$. But $\cos 4x$ is a basis of y_h. Therefore this set is multiplied by x. The whole set is multiplied by x and not just $\cos 4x$ because the set was generated by taking derivative of $\cos 4x$.

The basis set for y_p now becomes $\{x\cos 4x, x\sin 4x\}$. Hence y_p is linear combination of these basis, giving trial y_p as

$$y_p = Ax\cos 4x + Bx\sin 4x \tag{4}$$

Therefore

$$\begin{aligned} y_p' &= (A\cos 4x - 4Ax\sin 4x) + (B\sin 4x + 4Bx\cos 4x) \\ y_p'' &= (-4A\sin 4x - 4A\sin 4x - 16Ax\cos 4x) + (4B\cos 4x + 4B\cos 4x - 16Bx\sin 4x) \\ &= -8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x \end{aligned}$$

Substituting the above back in (1) gives

$$\begin{aligned} (-8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x) + 16(Ax\cos 4x + Bx\sin 4x) &= 16\cos 4x \\ \sin 4x(-8A - 16Bx + 16Bx) + \cos 4x(-16Ax + 8B + 16Ax) &= 16\cos 4x \end{aligned}$$

Hence

$$\begin{aligned} -16Ax + 8B + 16Ax &= 16 \\ -8A - 16Bx + 16Bx &= 0 \end{aligned}$$

Or

$$\begin{aligned} 8B &= 16 \\ -8A &= 0 \end{aligned}$$

First equation gives $B = 2$. Second equation gives $A = 0$. Therefore the particular solution (4) becomes

$$y_p = 2x\sin 4x$$

From (2), the general solution becomes

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1\cos 4x + c_2\sin 4x + 2x\sin 4x \end{aligned} \tag{5}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (5) becomes

$$1 = c_1$$

Hence the solution (5) becomes

$$y(x) = \cos 4x + c_2\sin 4x + 2x\sin 4x \tag{6}$$

Taking derivative of the above

$$y'(x) = -4\sin 4x + 4c_2\cos 4x + 2\sin 4x + 8x\cos 4x$$

At $t = 0$ the above gives

$$0 = 4c_2$$

Hence $c_2 = 0$ and the final solution (6) becomes

$$y(x) = \cos 4x + 2x\sin 4x$$

4.9.3.4 Part 4

The ode to solve is

$$y'' - y = \cosh x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' + y = 0$ and $y_p(x)$ is any particular solution to $y'' + y = \cosh x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' + y = 0$ and simplifying gives

$$\begin{aligned} \lambda^2 - 1 &= 0 \\ \lambda &= \pm 1 \end{aligned}$$

roots are $\lambda_1 = 1, \lambda_2 = -1$. The basis solutions are therefore

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= e^{-x} \end{aligned} \tag{3}$$

Therefore $y_h(x)$ is linear combination of the the above.

$$y_h(x) = c_1 e^x + c_2 e^{-x}$$

Which can be written, using Euler formula as

$$y_h(x) = c_1 \cosh x + c_2 \sinh x$$

The particular solution is now found. Assuming $y_p = A \cosh x$. Taking all derivatives of this, the basis for y_p becomes $\{\cosh x, \sinh x\}$. But $\cosh x$ is basis of y_h. Therefore this set is multiplied by x. The whole set is multiplied by x and not just $\cosh x$ because the set was generated by taking derivative of $\cosh x$.

The basis set for y_p becomes $\{x \cosh x, x \sinh x\}$. Hence y_p is linear combination of these basis, giving trial y_p as

$$y_p = Ax \cosh x + Bx \sinh x \tag{4}$$

Therefore

$$\begin{aligned} y_p' &= A \cosh x + Ax \sinh x + B \sinh x + Bx \cosh x \\ y_p'' &= A \sinh x + A \sinh x + Ax \cosh x + B \cosh x + B \cosh x + Bx \sinh x \\ &= 2A \sinh x + Ax \cosh x + 2B \cosh x + Bx \sinh x \end{aligned}$$

Substituting the above back in (1) gives

$$\begin{aligned} (2A \sinh x + Ax \cosh x + 2B \cosh x + Bx \sinh x) - (Ax \cosh x + Bx \sinh x) &= \cosh x \\ \sinh x(2A + Bx - Bx) + \cosh x(Ax + 2B - Ax) &= \cosh x \end{aligned}$$

Hence

$$\begin{aligned} 2B &= 1 \\ 2A &= 0 \end{aligned}$$

Therefore $B = \frac{1}{2}, A = 0$ and (4) becomes

$$y_p = \frac{1}{2} x \sinh x$$

From (2), the general solution becomes

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 \cosh x + c_2 \sinh x + \frac{1}{2} x \sinh x \end{aligned} \tag{5}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (5) becomes

$$1 = c_1$$

Hence the solution (5) becomes

$$y(x) = \cosh x + c_2 \sinh x + \frac{1}{2} x \sinh x \tag{6}$$

Taking derivative of the above

$$y'(x) = \sinh x + c_2 \cosh x + \frac{1}{2} \sinh x + \frac{1}{2} x \cosh x$$

At $t = 0$ the above gives

$$0 = c_2 \cosh x$$

Hence $c_2 = 0$ and the final solution (6) becomes

$$y(x) = \cosh x + \frac{1}{2} x \sinh x$$

4.9.4 Problem 3 (10.3.5)

Solve $x^2 y' + 2xy = \sinh x$ with $y(1) = 2$

Solution

Dividing by $x \neq 0$

$$y' + 2\frac{y}{x} = \frac{\sinh x}{x^2}$$

The integrating factor is $I = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$. Multiplying both sides by this integration factor makes the left side a complete differential

$$\begin{aligned} \frac{d}{dx}\left(yx^2\right) &= x^2 \frac{\sinh x}{x^2} \\ \frac{d}{dx}\left(yx^2\right) &= \sinh x \end{aligned}$$

Integrating gives

$$\begin{aligned} yx^2 &= \int \sinh x dx + C \\ yx^2 &= \cosh x + C \\ y &= \frac{\cosh x}{x^2} + \frac{C}{x^2} \end{aligned} \tag{1}$$

At $x = 1$ the above becomes

$$\begin{aligned} 2 &= \cosh 1 + C \\ C &= 2 - \cosh 1 \end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned} y(x) &= \frac{\cosh x}{x^2} + \frac{2 - \cosh 1}{x^2} \\ &= \frac{1}{x^2}(\cosh x + 2 - \cosh 1) \end{aligned}$$

Where $x \neq 0$

4.9.5 Problem 4 (10.3.8)

Solve

$$\left(1+x^2\right)y' = 1 + xy$$

Solution

$$\begin{aligned} y' &= \frac{1+xy}{1+x^2} \\ &= \frac{1}{1+x^2} + \frac{xy}{1+x^2} \end{aligned}$$

Therefore

$$y' - y\frac{x}{1+x^2} = \frac{1}{1+x^2} \tag{1}$$

This is linear in y first order ODE. It has the form $y' + p(x)y = q(x)$. The integration factor is

$$\begin{aligned} I &= e^{\int p(x)dx} \\ &= e^{-\int \frac{x}{1+x^2}dx} \end{aligned}$$

But $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln\left(1+x^2\right)$. Therefore

$$\begin{aligned} I &= e^{-\frac{1}{2}\ln\left(1+x^2\right)} \\ &= e^{\ln\left(1+x^2\right)^{-\frac{1}{2}}} \\ &= \left(1+x^2\right)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Multiplying both sides of (1) by this integrating factor makes the left side a complete differential

$$\begin{aligned} \frac{d}{dx}\left(y\frac{1}{\sqrt{1+x^2}}\right) &= \frac{1}{\sqrt{1+x^2}}\frac{1}{1+x^2} \\ \frac{d}{dx}\left(y\frac{1}{\sqrt{1+x^2}}\right) &= \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}} \\ &= \left(1+x^2\right)^{-\frac{3}{2}} \end{aligned}$$

Integrating gives

$$y\frac{1}{\sqrt{1+x^2}} = \int \left(1+x^2\right)^{-\frac{3}{2}}dx + C \tag{2}$$

To integrate $\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}dx$, let $x = \tan u$, then $dx = \left(1+\tan^2 u\right)du$. Hence

$$\begin{aligned}\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}dx &= \int \frac{1}{\left(1+\tan^2 u\right)^{\frac{3}{2}}}\left(1+\tan^2 u\right)du\\ &= \int \frac{1}{\left(1+\tan^2 u\right)^{\frac{1}{2}}}du\\ &= \int \frac{1}{\left(1+\frac{\sin^2 u}{\cos^2 u}\right)^{\frac{1}{2}}}du\\ &= \int \frac{\cos u}{\left(\cos^2 u+\sin^2 u\right)^{\frac{1}{2}}}du\\ &= \int \cos u\, du\\ &= \sin u\end{aligned}$$

But $\sin u = \frac{\frac{\sin u}{\cos u}}{\sqrt{1+\frac{\sin^2 u}{\cos^2 u}}} = \frac{\tan u}{\sqrt{1+\tan^2 u}} = \frac{x}{\sqrt{1+x^2}}$. Hence

$$\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}dx = \frac{x}{\sqrt{1+x^2}}$$

Therefore the final solution (2) becomes

$$\begin{aligned}y\frac{1}{\sqrt{1+x^2}} &= \frac{x}{\sqrt{1+x^2}}+C\\ y &= x + C\sqrt{1+x^2}\end{aligned}\tag{3}$$

4.9.6 Problem 5 (10.3.9)

Solve (a) $y' + xy = xy^2$ (b) $3xy' + y + x^2y^4 = 0$

Solution

4.9.6.1 Part a

The ode has the form

$$y' + p(x)y = q(x)y^m$$

Where $p(x) = x, q(x) = x$ and $m = 2$. Therefore this is Bernoulli ODE. The first step is to divide throughout by $y^m = y^2$ which gives

$$\frac{y'}{y^2} + p(x)y^{-1} = q(x)\tag{1}$$

Setting

$$v(x) = y^{-1}\tag{2}$$

Taking derivatives of the above w.r.t. x gives

$$v'(x) = \frac{-1}{y^2}y'(x)\tag{3}$$

Substituting (2,3) into (1) gives

$$-v'(x) + p(x)v(x) = q(x)$$

But here $p(x) = x$ and $q(x) = x$. The above becomes

$$\begin{aligned} -v'(x) + xv(x) &= x \\ v'(x) - xv(x) &= -x \end{aligned}$$

This is linear ODE in $v(x)$. The integrating factor is $e^{\int -x dx} = e^{-\frac{x^2}{2}}$. Multiplying both sides of the above by this integrating factor makes the left side a complete differential

$$\frac{d}{dx}\left(ve^{-\frac{x^2}{2}}\right) = -xe^{-\frac{x^2}{2}}$$

Integrating gives

$$ve^{-\frac{x^2}{2}} = -\int xe^{-\frac{x^2}{2}} dx + C \tag{4}$$

To integrate $\int xe^{-\frac{x^2}{2}} dx$, let $u = x^2$. Then $du = 2xdx$. Substituting gives

$$\begin{aligned} \int xe^{-\frac{x^2}{2}} dx &= \int xe^{-\frac{u}{2}} \frac{du}{2x} \\ &= \frac{1}{2} \int e^{-\frac{u}{2}} du \\ &= \frac{1}{2} \frac{e^{-\frac{u}{2}}}{-\frac{1}{2}} \\ &= -e^{-\frac{u}{2}} \end{aligned}$$

But $u = x^2$. Therefore

$$\int xe^{-\frac{x^2}{2}} dx = -e^{-\frac{x^2}{2}}$$

Substituting the above in (4) gives

$$\begin{aligned} ve^{-\frac{x^2}{2}} &= e^{-\frac{x^2}{2}} + C \\ v &= 1 + e^{\frac{x^2}{2}} C \end{aligned}$$

But $v = y^{-1}$, therefore

$$\begin{aligned} y^{-1} &= 1 + e^{\frac{x^2}{2}} C \\ y(x) &= \frac{1}{1 + e^{\frac{x^2}{2}} C} \end{aligned}$$

Where C is constant of integration.

4.9.6.2 Part b

The ode is

$$3xy' + y + x^2 y^4 = 0$$

Dividing by $3x$ for $x \neq 0$ gives

$$\begin{aligned} y' + \frac{y}{3x} + \frac{x}{3} y^4 &= 0 \\ y' + \frac{1}{3x} y &= -\frac{x}{3} y^4 \end{aligned}$$

Now this ODE has the Bernoulli form,

$$y' + p(x)y = q(x)y^m$$

Where $p(x) = \frac{1}{3x}, q(x) = -\frac{x}{3}$ and $m = 4$. Therefore this is Bernoulli ODE. The first step is to divide throughout by $y^m = y^4$ which gives

$$\frac{y'}{y^4} + p(x)y^{-3} = q(x) \tag{1}$$

Setting

$$v(x) = y^{-3} \tag{2}$$

Taking derivatives of the above w.r.t. x gives

$$v'(x) = \frac{-3}{y^4}y'(x) \tag{3}$$

Substituting (2,3) into (1) gives

$$-\frac{1}{3}v'(x) + p(x)v(x) = q(x)$$

But here $p(x) = \frac{1}{3x}, q(x) = -\frac{x}{3}$. The above becomes

$$\begin{aligned} -\frac{1}{3}v'(x) + \frac{1}{3x}v(x) &= -\frac{x}{3} \\ v'(x) - \frac{1}{x}v(x) &= x \end{aligned}$$

This is linear in $v(x)$. The integrating factor is $e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$. Multiplying both sides of the above by this integrating factor make the left side a complete differential

$$\frac{d}{dx}\left(v\frac{1}{x}\right) = 1$$

Integrating gives

$$\begin{aligned} v\frac{1}{x} &= x + C \\ v &= x^2 + xC \end{aligned} \tag{4}$$

But $v(x) = y^{-3}$. Therefore the above becomes

$$\begin{aligned} y^{-3} &= x^2 + xC \\ y^3(x) &= \frac{1}{x^2 + xC} \end{aligned}$$

Or

$$y(x) = \left(x^2 + xC\right)^{-\frac{1}{3}}$$

4.9.7 key solution for HW 9

Physics 3041 (Spring 2021) Solutions to Homework Set 9

1. Problem 10.2.8. (20 points)

(i) $x(0) = 1$, $\dot{x}(0) = 0$

$$(D^2 + 2D + 1)x(t) = 0 \Rightarrow \alpha^2 + 2\alpha + 1 = (\alpha + 1)^2 = 0,\ \alpha = -1 \text{ (repeated)}$$
$$x(t) = (A + Bt)e^{-t} \Rightarrow x(0) = A = 1$$
$$\dot{x} = Be^{-t} - (A + Bt)e^{-t} \Rightarrow \dot{x}(0) = B - A = 0,\ B = A = 1$$
$$x(t) = (1 + t)e^{-t}$$

(ii) $(D^4 + 1)x(t) = 0$

$$\alpha^4 + 1 = 0 \Rightarrow \alpha^4 = -1 = e^{i(2n+1)\pi},\ \alpha = e^{i(2n+1)\pi/4},\ n = 0,\ 1,\ 2,\ 3$$
$$\alpha_0 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}},\ \alpha_1 = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}},\ \alpha_2 = e^{i5\pi/4} = -\frac{1+i}{\sqrt{2}},\ \alpha_3 = e^{i7\pi/4} = \frac{1-i}{\sqrt{2}}$$
$$x(t) = A_0e^{\alpha_0 t} + A_1e^{\alpha_1 t} + A_2e^{\alpha_2 t} + A_3e^{\alpha_3 t} = A_0e^{\frac{1+i}{\sqrt{2}}t} + A_1e^{\frac{-1+i}{\sqrt{2}}t} + A_2e^{-\frac{1+i}{\sqrt{2}}t} + A_3e^{\frac{1-i}{\sqrt{2}}t}$$

(iii) $(D^3 - 3D^2 - 9D - 5)x(t) = 0$

$$\alpha^3 - 3\alpha^2 - 9\alpha - 5 = \alpha^3 + \alpha^2 - (4\alpha^2 + 9\alpha + 5) = \alpha^2(\alpha + 1) - (4\alpha + 5)(\alpha + 1) = 0$$
$$(\alpha + 1)(\alpha^2 - 4\alpha - 5) = (\alpha + 1)^2(\alpha - 5) = 0 \Rightarrow \alpha = -1 \text{ (repeated)},\ 5$$
$$x(t) = (A + Bt)e^{-t} + Ce^{5t}$$

(iv) $(D + 1)^2(D^4 - 256)x(t) = 0$

$$(\alpha + 1)^2(\alpha^4 - 256) = (\alpha + 1)^2(\alpha^2 - 16)(\alpha^2 + 16) = (\alpha + 1)^2(\alpha + 4)(\alpha - 4)(\alpha + 4i)(\alpha - 4i) = 0$$
$$\alpha = -1 \text{ (repeated)},\ -4,\ 4,\ -4i,\ 4i \Rightarrow x(t) = (A + Bt)e^{-t} + Ce^{-4t} + Fe^{4t} + Ge^{-i4t} + He^{i4t}$$

2. Problem 10.2.11. (40 points)

$y(0) = 1$, $\dot{y}(0) = 0$

(i) $\ddot{y} - \dot{y} - 2y = e^{2x}$

$$\alpha^2 - \alpha - 2 = (\alpha + 1)(\alpha - 2) = 0 \Rightarrow \alpha = -1,\ 2,\ y_c(x) = Ae^{-x} + Be^{2x}$$
$$y_p(x) = Cxe^{2x},\ (D^2 - D - 2)y_p(t) = C(D + 1)(D - 2)xe^{2x} = C(D + 1)e^{2x} = 3Ce^{2x} = e^{2x},\ C = \frac{1}{3}$$
$$y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{2x} + \frac{xe^{2x}}{3} \Rightarrow y(0) = A + B = 1$$
$$y'(x) = -Ae^{-x} + 2Be^{2x} + \frac{(1 + 2x)e^{2x}}{3} \Rightarrow y'(0) = -A + 2B + \frac{1}{3} = 0$$
$$A = \frac{7}{9},\ B = \frac{2}{9},\ y(x) = \frac{7e^{-x} + 2e^{2x}}{9} + \frac{xe^{2x}}{3}$$

(ii) $(D^2 - 2D + 1)y = 2\cos x = 2\mathrm{Re}(e^{ix})$

$$\alpha^2 - 2\alpha + 1 = (\alpha - 1)^2 = 0 \Rightarrow \alpha = 1 \text{ (repeated)},\ y_c(x) = (A + Bx)e^x$$
$$y_p(x) = Ce^{ix},\ (D^2 - 2D + 1)y_p(x) = C(-1 - 2i + 1)e^{ix} = -2iCe^{ix} = 2e^{ix} \Rightarrow C = i$$
$$y(x) = y_c(x) + \mathrm{Re}[y_p(x)] = (A + Bx)e^x + \mathrm{Re}(ie^{ix}) = (A + Bx)e^x + \mathrm{Re}(i\cos x - \sin x)$$
$$= (A + Bx)e^x - \sin x \Rightarrow y(0) = A = 1$$
$$y'(x) = Be^x + (A + Bx)e^x - \cos x \Rightarrow y'(0) = B + A - 1 = 0,\ B = 0$$
$$y(x) = e^x - \sin x$$

(iii) $y'' + 16y = 16\cos 4x = 16\mathrm{Re}(e^{i4x})$

$$\alpha^2 + 16 = (\alpha + 4i)(\alpha - 4i) = 0 \Rightarrow \alpha = 4i,\ -4i,\ y_c(x) = Ae^{i4x} + Be^{-i4x}$$
$$y_p(x) = Cxe^{i4x},\ (D^2 + 16)y_p(x) = C(D + 4i)(D - 4i)xe^{i4x}$$
$$= C(D + 4i)e^{i4x} = i8Ce^{i4x} = 16e^{i4x},\ C = -2i$$
$$y(x) = y_c(x) + \mathrm{Re}[y_p(x)] = Ae^{i4x} + Be^{-i4x} + \mathrm{Re}(-2ixe^{i4x})$$
$$= Ae^{i4x} + Be^{-i4x} + \mathrm{Re}(-2ix\cos 4x + 2x\sin 4x)$$
$$= Ae^{i4x} + Be^{-i4x} + 2x\sin 4x \Rightarrow y(0) = A + B = 1$$
$$y'(x) = i4Ae^{i4x} - i4Be^{-i4x} + 2\sin 4x + 8x\cos 4x \Rightarrow y'(0) = i4A - i4B = 0,\ A = B = \frac{1}{2}$$
$$y(x) = \frac{e^{i4x} + e^{-i4x}}{2} + 2x\sin 4x = \cos 4x + 2x\sin 4x$$

(iv) $y'' - y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$\alpha^2 - 1 = (\alpha + 1)(\alpha - 1) \Rightarrow \alpha = -1,\ 1,\ y_c(x) = Ae^{-x} + Be^x$$
$$y_{p+}(x) = C_+xe^x,\ (D^2 - 1)y_{p+}(x) = C_+(D + 1)(D - 1)xe^x$$
$$= C_+(D + 1)e^x = 2C_+e^x = \frac{e^x}{2},\ C_+ = \frac{1}{4}$$
$$y_{p-}(x) = C_-xe^{-x},\ (D^2 - 1)y_{p-}(x) = C_-(D - 1)(D + 1)xe^{-x}$$
$$= C_-(D - 1)e^{-x} = -2C_-e^{-x} = \frac{e^{-x}}{2},\ C_- = -\frac{1}{4}$$
$$y(x) = y_c(x) + y_{p+}(x) + y_{p-}(x) = Ae^{-x} + Be^x + \frac{x(e^x - e^{-x})}{4}$$
$$= Ae^{-x} + Be^x + \frac{x\sinh x}{2},\ y(0) = A + B = 1$$
$$y'(x) = -Ae^{-x} + Be^x + \frac{\sinh x + x\cosh x}{2},\ y'(0) = -A + B = 0,\ A = B = \frac{1}{2}$$
$$y(x) = \frac{e^{-x} + e^x}{2} + \frac{x\sinh x}{2} = \cosh x + \frac{x\sinh x}{2}$$

3. Problem 10.3.5. (10 points)

$$x^2y' + 2xy = \sinh x,\ y(1) = 2$$
$$x^2y' + 2xy = (x^2y)' = \sinh x \Rightarrow d(x^2y) = \sinh x\,dx,\ \int_{x=1}^{x=x} d(x^2y) = \int_1^x \sinh x\,dx$$
$$x^2y - (x^2y)_{x=1} = \cosh x - \cosh 1,\ x^2y - 2 = \cosh x - \cosh 1,\ y(x) = \frac{\cosh x - \cosh 1 + 2}{x^2}$$

4. Problem 10.3.8. (10 points)

$$(1+x^2)y' = 1+xy$$

$$y' - \frac{x}{1+x^2}y = \frac{1}{1+x^2},\ P(x) = -\int \frac{xdx}{1+x^2} = -\frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = -\frac{\ln(1+x^2)}{2} + C'$$

$$e^{-P(x)}\frac{d}{dx}[e^{P(x)}y(x)] = e^{\frac{1}{2}\ln(1+x^2)}\frac{d}{dx}[e^{-\frac{1}{2}\ln(1+x^2)}y(x)]$$

$$= \sqrt{1+x^2}\frac{d}{dx}\frac{y(x)}{\sqrt{1+x^2}} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\frac{y(x)}{\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{3/2}},\ \frac{y(x)}{\sqrt{1+x^2}} = \int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{dx}{x^3(1+x^{-2})^{3/2}}$$

$$= -\frac{1}{2}\int \frac{d(1+x^{-2})}{(1+x^{-2})^{3/2}} = \frac{1}{\sqrt{1+x^{-2}}} + C$$

$$y(x) = \frac{\sqrt{1+x^2}}{\sqrt{1+x^{-2}}} + C\sqrt{1+x^2} = x + C\sqrt{1+x^2}$$

5. Problem 10.3.9. (20 points)

$$y' + p(x)y = q(x)y^m \Rightarrow y^{-m}y' + p(x)y^{1-m} = q(x),\ \frac{(y^{1-m})'}{1-m} + p(x)y^{1-m} = q(x)$$

$$v = y^{1-m} \Rightarrow v' + (1-m)p(x)v = (1-m)q(x)$$

(a) $y' + xy = xy^2$

$$\frac{y'}{y^2} + \frac{x}{y} = x,\ -\frac{d}{dx}\frac{1}{y} + \frac{x}{y} = x,\ v = \frac{1}{y} \Rightarrow v' - xv = -x$$

$$P(x) = -\int xdx = -\frac{x^2}{2} + C',\ e^{-P(x)}\frac{d}{dx}[e^{P(x)}v(x)] = e^{\frac{x^2}{2}}\frac{d}{dx}[e^{-\frac{x^2}{2}}v(x)] = -x$$

$$\frac{d}{dx}[e^{-\frac{x^2}{2}}v(x)] = -xe^{-\frac{x^2}{2}},\ e^{-\frac{x^2}{2}}v(x) = -\int xe^{-\frac{x^2}{2}}dx = \int e^{-\frac{x^2}{2}}d(-\frac{x^2}{2}) = e^{-\frac{x^2}{2}} + C$$

$$v(x) = 1 + Ce^{\frac{x^2}{2}} \Rightarrow y(x) = \frac{1}{v(x)} = \frac{1}{1+Ce^{\frac{x^2}{2}}}$$

(b) $3xy' + y + x^2y^4 = 0$

$$3x\frac{y'}{y^4} + \frac{1}{y^3} = -x^2,\ -x\frac{d}{dx}\frac{1}{y^3} + \frac{1}{y^3} = -x^2,\ v(x) = \frac{1}{y^3} \Rightarrow v' - \frac{v}{x} = x$$

$$P(x) = -\int \frac{dx}{x} = -\ln x + C',\ e^{-P(x)}\frac{d}{dx}[e^{P(x)}v(x)] = e^{\ln x}\frac{d}{dx}[e^{-\ln x}v(x)] = x\frac{d}{dx}\frac{v(x)}{x} = x$$

$$\frac{d}{dx}\frac{v(x)}{x} = 1,\ \frac{v(x)}{x} = \int dx = x + C,\ v(x) = x^2 + Cx \Rightarrow y(x) = \frac{1}{v(x)^{1/3}} = \frac{1}{(x^2+Cx)^{1/3}}$$

4.10 HW 10

Local contents

4.10.1 Problems listing

Physics 3041 (Spring 2021) Homework Set 10 **(Due 4/30)**

1. (10 points) Given

$$\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}\,,$$

make a 3D integral and use the transformation from Cartesian to spherical coordinates to evaluate

$$\int_0^{\infty} x^2 \exp(-x^2)dx.$$

2. Follow the lecture example of deriving the gravitational field of a thin shell and calculate the gravitational potential of such a shell over all space. (10 points)

3. Follow the lecture example of deriving the gas pressure and calculate the number of gas particles hitting the container per unit area per unit time. Give your answer in terms of the net number density and the average speed of these particles. (10 points)

4. Derive the expressions of the quantum mechanical orbital angular momentum operators L_x, L_y, L_z in spherical coordinates. Show that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2}$$

in spherical coordinates. (40 points)

5. Consider $\psi(x,t)$ for $0 \leq x \leq L$. Given that $\psi(0,t) = \psi(L,t) = 0$ and

$$\psi(x,0) = \begin{cases} A\sin(2\pi x/L), & 0 \leq x \leq L/2, \\ 0, & L/2 < x \leq L, \end{cases}$$

find $\psi(x,t)$ that satisfies the following partial differential equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial x^2},$$

where A, L, $\hbar$, and μ are positive constants. (30 points)

4.10.2 Problem 1

Given

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Make a 3D integral and use the transformation from Cartesian to spherical coordinates to evaluate $\int_0^\infty x^2 e^{-x^2} dx$.

Solution

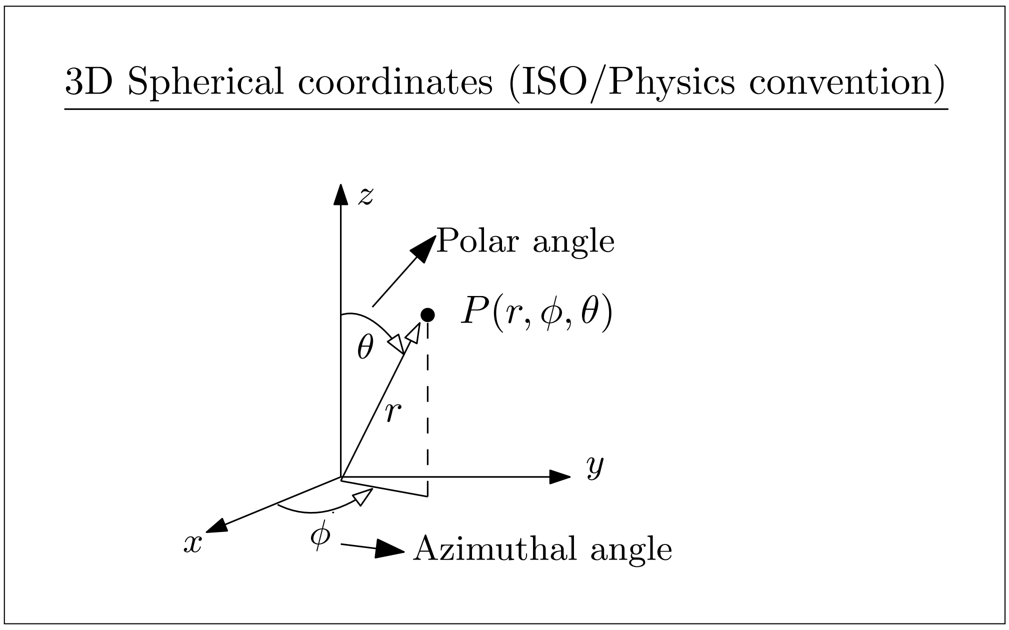


Figure 4.27: Spherical coordinates

The relation between the Cartesian and spherical coordinates is

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{1}$$

The 3D integral in Cartesian coordinates is

$$\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}\int_{z=-\infty}^{z=\infty} e^{-x^2-y^2-z^2} dxdydz = \left(\sqrt{\pi}\right)^3$$

But $x^2 + y^2 + z^2 = r^2$ in spherical coordinates. The above now simplifies to

$$\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}\int_{z=-\infty}^{z=\infty} e^{-r^2} dxdydz = \pi^{\frac{3}{2}}$$

Changing integration from Cartesian to spherical and changing the limits accordingly. the above becomes

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\phi=2\pi} e^{-r^2}\, J\, drd\theta d\phi = \pi^{\frac{3}{2}} \tag{2}$$

The Jacobian J is

$$J = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{d\phi} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{d\phi} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{d\phi} \end{vmatrix} \tag{3}$$

The relation between Cartesian and spherical in (1) shows that

$$
\begin{aligned}
\frac{dx}{dr} &= \sin\theta\cos\phi \\
\frac{dx}{d\theta} &= r\cos\theta\cos\phi \\
\frac{dx}{d\phi} &= -r\sin\theta\sin\phi \\
\frac{dy}{dr} &= \sin\theta\sin\phi \\
\frac{dy}{d\theta} &= r\cos\theta\sin\phi \\
\frac{dy}{d\phi} &= r\sin\theta\cos\phi \\
\frac{dz}{dr} &= \cos\theta \\
\frac{dz}{d\theta} &= -r\sin\theta \\
\frac{dz}{d\phi} &= 0
\end{aligned}
$$

Substituting the above in (3) gives

$$
J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \tag{3}
$$

Expanding along the last row to find the determinant (since last row has most number of zeros in it) gives the determinant as

$$
\begin{aligned}
J &= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} \\
&= \cos\theta\Big(\big(r\cos\theta\cos\phi\big)\big(r\sin\theta\cos\phi\big) + \big(r\sin\theta\sin\phi\big)\big(r\cos\theta\sin\phi\big)\Big) + r\sin\theta\Big(\big(\sin\theta\cos\phi\big)\big(r\sin\theta\cos\phi\big) + \\
&= \cos\theta\big(r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\big) + r\sin\theta\big(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\big) \\
&= r^2\sin\theta\cos^2\theta\big(\cos^2\phi + \sin^2\phi\big) + r^2\sin^3\theta\big(\cos^2\phi + \sin^2\phi\big) \\
&= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta \\
&= r^2\sin\theta\big(\cos^2\theta + \sin^2\theta\big)
\end{aligned}
$$

Therefore

$$
\boxed{J = r^2\sin\theta}
$$

Substituting the Jacobian in integral (2) gives

$$\begin{aligned}
\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\phi=2\pi} e^{-r^2}\left(r^2\sin\theta\right)drd\theta d\phi &= \pi^{\frac{3}{2}}\\
\int_{\phi=0}^{\phi=2\pi} d\phi\int_{\theta=0}^{\pi}\sin\theta d\theta\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
2\pi\int_{\theta=0}^{\pi}\sin\theta d\theta\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
-2\pi[\cos\theta]_0^{\pi}\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
-2\pi[(-1)-1]\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
-2\pi[-2]\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
4\pi\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \pi^{\frac{3}{2}}\\
\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \frac{\pi^{\frac{3}{2}}}{4\pi}\\
\int_{r=0}^{\infty} r^2e^{-r^2}dr &= \frac{1}{4}\sqrt{\pi}
\end{aligned}$$

Since r is just an integration variable, changing it to x gives

$$\int_0^{\infty} x^2e^{-x^2}dx = \frac{1}{4}\sqrt{\pi}$$

Which is what we asked to show.

4.10.3 Problem 2

Follow the lecture example of deriving the gravitational field of a thin shell and calculate the gravitational *potential* of such a shell over all space

Solution

4.10.3.1 Field outside the shell

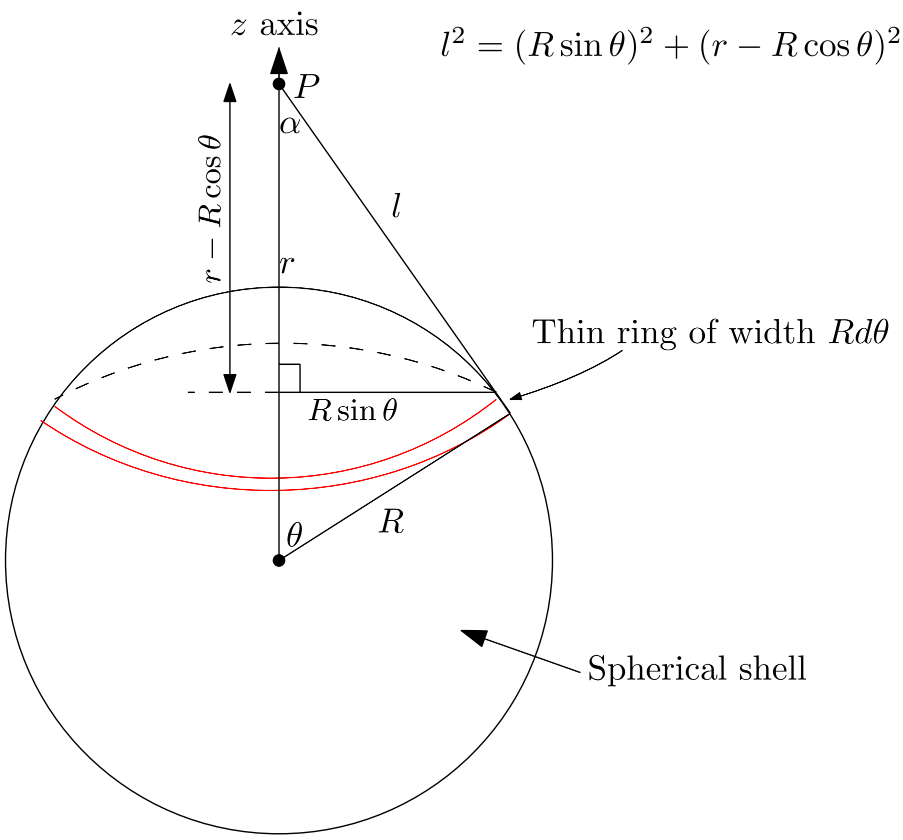


Figure 4.28: Problem setup

The gravitational field at point p as shown in the diagram will be determined. The point p is at distance r from the center of the shell. Due to symmetry any radial direction can be used as z axis.

A small ring is considered as shown. The field due to this at point p is due to vertical contribution only, since horizontal contribution cancel out. This means field due to this ring is given by

$$dg = G\frac{dm}{l^2}\cos\alpha \tag{1}$$

Where dm is the mass of the ring. But $dm = \sigma dA$, where dA is the surface area of the ring between θ and $\theta + d\theta$.

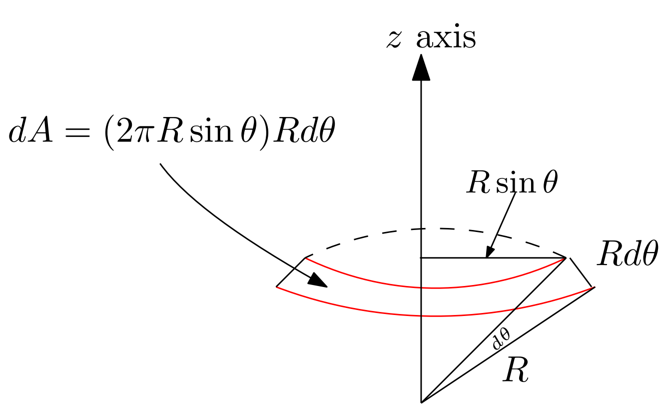


Figure 4.29: Surface area of ring

Hence

$$dA = (2\pi R\sin\theta)Rd\theta$$

Where $2\pi R\sin\theta$ is the circumference. Hence (1) becomes

$$\begin{aligned} dg &= G\frac{\sigma dA}{l^2}\cos\alpha \\ &= G\frac{\sigma(2\pi R\sin\theta)Rd\theta}{l^2}\cos\alpha \end{aligned} \tag{2}$$

Where σ is the surface mass density of the shell. But from the above diagram

$$\cos\alpha = \frac{r-R\cos\theta}{l}$$

Using this in (2) gives

$$\begin{aligned} dg &= G\frac{\sigma(2\pi R\sin\theta)Rd\theta}{l^2}\left(\frac{r-R\cos\theta}{l}\right) \\ &= G\sigma\left(2\pi R^2\sin\theta\right)(r-R\cos\theta)\frac{1}{l^3}d\theta \end{aligned} \tag{3}$$

l is now found from Pythagoras theorem (another option would have been to use the cosine angle rule)

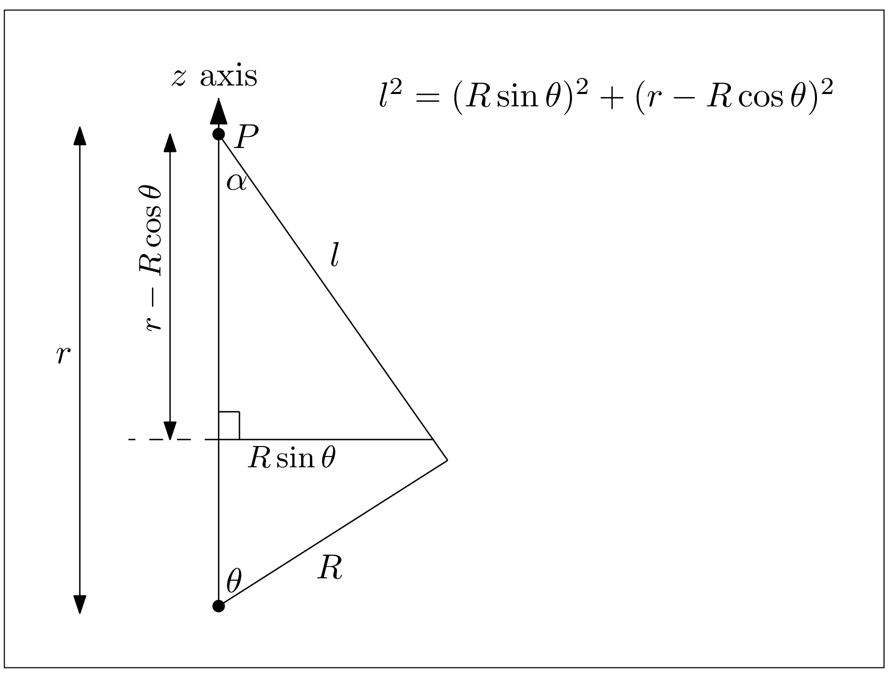


Figure 4.30: Finding l

$$\begin{aligned} l^2 &= (r-R\cos\theta)^2 + (R\sin\theta)^2 \\ &= r^2 + R^2\cos^2\theta - 2rR\cos\theta + R^2\sin^2\theta \\ &= r^2 + R^2 - 2rR\cos\theta \end{aligned}$$

Therefore

$$\boxed{l = \sqrt{r^2 + R^2 - 2rR\cos\theta}}$$

Substituting this in (3) gives

$$dg = G\sigma\frac{\left(2\pi R^2\sin\theta\right)(r-R\cos\theta)}{\left(r^2+R^2-2rR\cos\theta\right)^{\frac{3}{2}}}d\theta$$

The above is the field at point p due to the small ring shown. To find the contribution from all of the shell, we need to integrate the above, which gives

$$\begin{aligned} g &= \int_{\theta=0}^{\theta=\pi} G\sigma\frac{\left(2\pi R^2\sin\theta\right)(r-R\cos\theta)}{\left(r^2+R^2-2rR\cos\theta\right)^{\frac{3}{2}}}d\theta \\ &= G\sigma\left(2\pi R^2\right)\int_0^{\pi}\frac{\sin\theta(r-R\cos\theta)}{\left(r^2+R^2-2rR\cos\theta\right)^{\frac{3}{2}}}d\theta \end{aligned} \tag{4}$$

Let $u=\cos\theta$, then $du=-\sin\theta d\theta$. When $\theta=0,u=1$ and when $\theta=\pi,u=-1$. Hence the integral (4) becomes

$$
\begin{aligned}
g &= G\sigma\left(2\pi R^2\right)\int_1^{-1}\frac{\sin\theta(r-Ru)}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}\frac{du}{-\sin\theta}\\
&= G\sigma\left(2\pi R^2\right)\int_{-1}^{1}\frac{r-Ru}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\\
&= G\sigma\left(2\pi R^2\right)\left(\int_{-1}^{1}\frac{r}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du-\int_{-1}^{1}\frac{Ru}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\right)\\
&= G\sigma\left(2\pi R^2\right)\left(r\int_{-1}^{1}\frac{1}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du-R\int_{-1}^{1}\frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\right)\\
&= G\sigma\left(2\pi R^2\right)(rI_1-RI_2) \qquad (5)
\end{aligned}
$$

Where

$$I_1=\int_{-1}^{1}\frac{1}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du \qquad (6)$$

$$I_2=\int_{-1}^{1}\frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du \qquad (7)$$

To evaluate I_1. Let

$$v^2=r^2+R^2-2rRu$$

Hence

$$
\begin{aligned}
\frac{d}{dv}\left(v^2\right) &= \frac{d}{du}\left(r^2+R^2-2rRu\right)\\
2vdv &= -2rRdu
\end{aligned}
$$

Therefore

$$
\begin{aligned}
du &= \frac{2v}{-2rR}dv\\
&= \frac{-v}{rR}dv
\end{aligned}
$$

When $u=-1,v=\sqrt{r^2+R^2+2rR}$ and when $u=1,v=\sqrt{r^2+R^2-2rR}$. Hence I_1 becomes

$$
\begin{aligned}
I_1 &= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\frac{1}{v^3}\left(\frac{-v}{rR}dv\right)\\
&= -\frac{1}{rR}\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\frac{1}{v^2}dv\\
&= -\frac{1}{rR}\left(\frac{-1}{v}\right)_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\\
&= \frac{1}{rR}\left(\frac{1}{v}\right)_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\\
&= \frac{1}{rR}\left(\frac{1}{\sqrt{r^2+R^2-2rR}}-\frac{1}{\sqrt{r^2+R^2+2rR}}\right)\\
&= \frac{1}{rR}\left(\frac{\sqrt{r^2+R^2+2rR}-\sqrt{r^2+R^2-2rR}}{\sqrt{r^2+R^2-2rR}\sqrt{r^2+R^2+2rR}}\right)
\end{aligned}
$$

Since $r > R$, the above can be written as

$$\begin{aligned} I_1 &= \frac{1}{rR}\left(\frac{\sqrt{(r+R)^2}-\sqrt{(r-R)^2}}{\sqrt{R^4-2R^2r^2+r^4}}\right)\\ &= \frac{1}{rR}\left(\frac{(r+R)-(r-R)}{\sqrt{\left(r^2-R^2\right)^2}}\right)\\ &= \frac{1}{rR}\left(\frac{2R}{\left(r^2-R^2\right)}\right)\\ &= \frac{1}{r}\frac{2}{\left(r^2-R^2\right)} \end{aligned} \tag{8}$$

Now that I_1 is found, similar calculation is made to evaluate I_2 from (7)

$$I_2 = \int_{-1}^{1} \frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}} du$$

Similar to I_1, Let

$$v^2 = r^2 + R^2 - 2rRu$$

Hence

$$u = \frac{v^2 - r^2 - R^2}{-2rR}$$

Hence I_2 becomes

$$\begin{aligned} I_2 &= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{\left(\frac{v^2-r^2-R^2}{-2rR}\right)}{v^3}\left(\frac{-v}{rR}dv\right)\\ &= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^3(-2rR)}\left(\frac{-v}{rR}dv\right)\\ &= \left(\frac{1}{-2rR}\right)\left(\frac{1}{-rR}\right)\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^3} v dv\\ &= \frac{1}{2r^2R^2}\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^2} dv\\ &= \frac{1}{2r^2R^2}\left(\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2}{v^2}dv - \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{r^2+R^2}{v^2}dv\right)\\ &= \frac{1}{2r^2R^2}\left(\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} dv - \left(r^2+R^2\right)\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv\right) \end{aligned} \tag{9}$$

The first integral in above is, and since $r > R$

$$\begin{aligned} \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} dv &= \sqrt{r^2+R^2-2rR} - \sqrt{r^2+R^2+2rR}\\ &= \sqrt{(r-R)^2} - \sqrt{(r+R)^2}\\ &= (r-R)-(r+R)\\ &= -2R \end{aligned} \tag{10}$$

The second integral in (9) is

$$\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv = -\left[\frac{1}{v}\right]_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}$$

As was done for I_1, the above simplifies to

$$\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2} dv = -\frac{2R}{\left(r^2-R^2\right)} \tag{11}$$

Substituting (10,11) back in (9) gives I_2

$$\begin{aligned}
I_2 &= \frac{1}{2r^2R^2}\left(-2R-\left(r^2+R^2\right)\left(-\frac{2R}{\left(r^2-R^2\right)}\right)\right)\\
&= \frac{1}{2r^2R^2}\left(-2R+2R\frac{r^2+R^2}{\left(r^2-R^2\right)}\right)\\
&= \frac{2R}{2r^2R^2}\left(-1+\frac{r^2+R^2}{\left(r^2-R^2\right)}\right)\\
&= \frac{2}{2r^2R}\left(\frac{-\left(r^2-R^2\right)+\left(r^2+R^2\right)}{r^2-R^2}\right)\\
&= \frac{2}{2r^2R}\left(\frac{-r^2+R^2+r^2+R^2}{r^2-R^2}\right)\\
&= \frac{1}{r^2}\frac{2R}{r^2-R^2} \qquad (12)
\end{aligned}$$

Now that I_1 and I_2 are found in (8) and (12), then substituting these in (5) gives

$$\begin{aligned}
g &= G\sigma\left(2\pi R^2\right)(rI_1 - RI_2)\\
&= G\sigma\left(2\pi R^2\right)\left(r\left(\frac{1}{r}\frac{2}{\left(r^2-R^2\right)}\right)-R\left(\frac{1}{r^2}\frac{2R}{r^2-R^2}\right)\right)\\
&= G\sigma\left(2\pi R^2\right)\left(\frac{2}{\left(r^2-R^2\right)}-\frac{1}{r^2}\frac{2R^2}{r^2-R^2}\right)\\
&= G\sigma\left(2\pi R^2\right)\left(\frac{2r^2-2R^2}{r^2\left(r^2-R^2\right)}\right)\\
&= \frac{G\sigma\left(2\pi R^2\right)}{r^2}\left(\frac{2\left(r^2-R^2\right)}{\left(r^2-R^2\right)}\right)\\
&= \frac{G\sigma\left(4\pi R^2\right)}{r^2}\left(\frac{\left(r^2-R^2\right)}{\left(r^2-R^2\right)}\right)\\
&= \frac{G\sigma\left(4\pi R^2\right)}{r^2}
\end{aligned}$$

But $\sigma\left(4\pi R^2\right) = M$, which is the mass of the shell. Hence the above becomes

$$\boxed{g = \frac{GM}{r^2}}$$

This is the field strength at distance r from the center of the shell, where $r > R$. This shows that the field strength is the same as if the total mass of the shell was concentrated at a point in its center.

Now we need to obtain the potential energy of a particle of mass m located at distance r from the center of the shell. Taking potential energy of m to be zero at $r = \infty$, the potential energy is the work needed to move m from ∞ to distance r from center of shell. But work done is $U = -\int_\infty^r F dr'$ where F is the wight of m which is mg. Hence

$$U = -\int_\infty^r -mg dr'$$

The minus sign inside the integral is because the weight acts down, which is in the negative direction. The minus sign outside the integral is because work is done being done to increase the U of the mass. The rule is that, if work increases the potential energy of m, then it is negative. Since U is zero at infinity, then this work is negative. Therefore the above becomes

$$\begin{aligned} U &= \int_{\infty}^{r} mgdr' \\ &= \int_{\infty}^{r} \frac{GMm}{r'^2} dr' \\ &= -\left[\frac{GMm}{r'}\right]_{\infty}^{r} \\ &= -\left[\frac{GMm}{r} - 0\right] \end{aligned}$$

Therefore the gravitational potential energy of mass m at distance r from center of shell is

$$\boxed{U = -\frac{GMm}{r}}$$

4.10.3.2 Field inside the shell

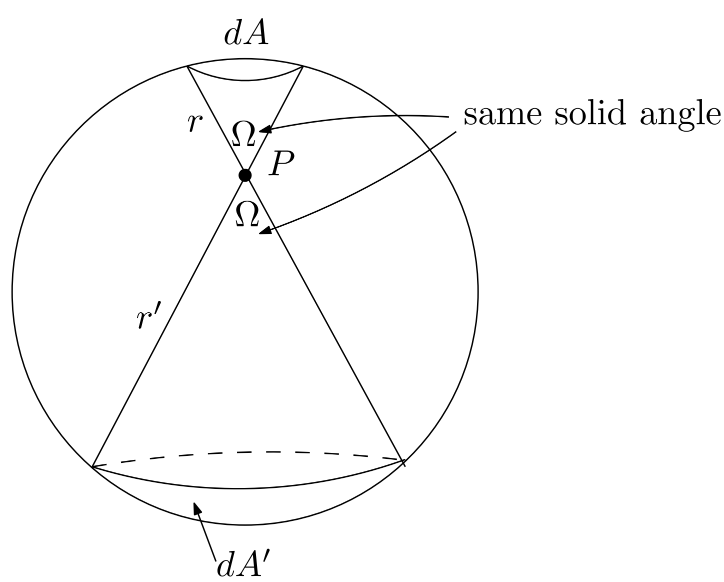


Figure 4.31: Problem setup

Let P be any arbitrary location inside the shell. Then the field at P due to contribution from dA only is

$$dg_A = G\frac{\sigma dA}{r^2}$$

And the field at P due to contribution from dA' only is

$$dg_{A'} = G\frac{\sigma dA'}{(r')^2}$$

The mass due to dA is pulling P upwards and the mass due to dA' is pulling P down. If we can show that these fields are of equal strength, then this shows the net gravitational field will be zero at P.

But $\frac{dA}{r^2} = \Omega$ where Ω is the solid angle made by the area dA as shown above. By symmetry, this is the same solid angle made by dA'. Therefore

$$\frac{dA}{r^2} = \frac{dA'}{(r')^2}$$

Therefore the net gravitational field is zero at P. Since P is arbitrary point. Then any point inside the shell will have zero net gravitational field.

Potential energy of a particle of mass m inside the shell is the same as the potential energy at surface of the shell, this is because $g = 0$ inside the shell.

Using the same derivation of potential energy in part 1 above gives

$$\begin{aligned} U &= \int_{\infty}^{R} mgdr' \\ &= \int_{\infty}^{R} \frac{GMm}{r'^2} dr' \\ &= -\left[\frac{GMm}{r'}\right]_{\infty}^{R} \\ &= -\left[\frac{GMm}{R} - 0\right] \end{aligned}$$

Therefore the gravitational potential energy of mass m anywhere inside the shell is

$$\boxed{U = -\frac{GMm}{R}}$$

4.10.4 Problem 3

Follow the lecture example of deriving the gas pressure and calculate the number of gas particles hitting the container per unit area per unit time. Give your answer in terms of the net number density and the average speed of these particles.

Solution

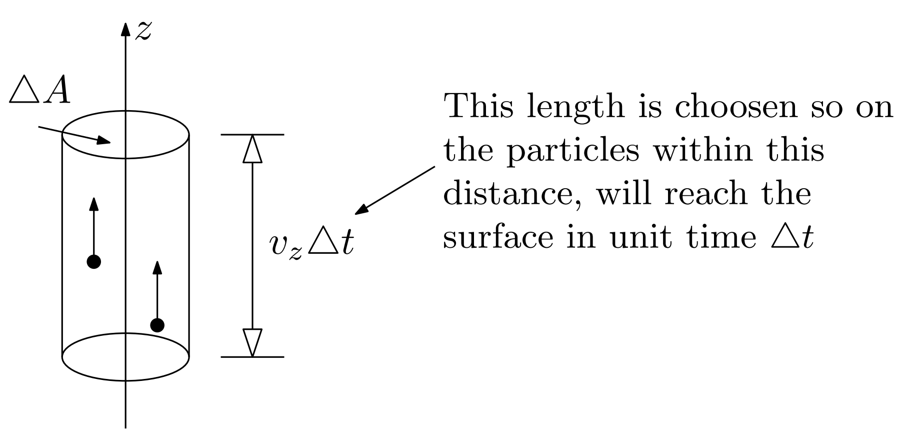


Figure 4.32: Problem setup

In the above diagram v_z is the average speed of particles in the z direction within Δt time from hitting ΔA. The number of particles per unit volume with velocity $\vec{v}$ and $\vec{v} + d\vec{v}$ is given by

$$dn = f(v)dv_x dv_y dv_z$$

Where v above is the magnitude (speed) of $\vec{v}$. During interval Δt, the number of particles hitting the wall is dN which is therefore given by

$$dN = dn(\Delta V) \tag{1}$$

Where dV is the unit volume shown in the diagram. But

$$dV = (v_z \Delta t)\Delta A$$

Therefore (1) becomes

$$\begin{aligned} dN &= dn(v_z \Delta t)\Delta A \\ &= f(v) dv_x dv_y dv_z (v_z \Delta t)\Delta A \end{aligned}$$

The above is the number of particles hitting ΔA of the wall in interval Δt.

4.10.5 Problem 4

Derive the expressions of the orbital angular momentum operators L_x, L_y, L_z in spherical coordinates. Show that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2}$$

<u>Solution</u>

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{L}$ is vector whose components are the orbital angular momentum operators L_x, L_y, L_z and $\vec{r}$ is a vector whose components are the position operators and $\vec{p}$ is a vector whose components are the momentum operators and $\times$ is the vector cross product. In Cartesian coordinates, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are the orthonormal basis. Hence

$$\begin{aligned} \vec{L} &= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \\ &= \hat{e}_x\left(y p_z - z p_y\right) - \hat{e}_y\left(x p_z - z p_x\right) + \hat{e}_z\left(x p_y - y p_x\right) \end{aligned}$$

Hence the corresponding components of $\vec{L} = \left\{L_x, L_y, L_z\right\}$ are

$$\begin{aligned} L_x &= y p_z - z p_y \\ L_y &= z p_x - x p_z \\ L_z &= x p_y - y p_x \end{aligned} \tag{1}$$

But in Quantum mechanics, the operators p_x, p_y, p_z are

$$\begin{aligned} p_x &= -i\hbar\left(\frac{\partial}{\partial x}\right) \\ p_y &= -i\hbar\left(\frac{\partial}{\partial y}\right) \\ p_z &= -i\hbar\left(\frac{\partial}{\partial z}\right) \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} L_x &= y\left(-i\hbar\left(\frac{\partial}{\partial z}\right)\right) - z\left(-i\hbar\left(\frac{\partial}{\partial y}\right)\right) \\ L_y &= z\left(-i\hbar\left(\frac{\partial}{\partial x}\right)\right) - x\left(-i\hbar\left(\frac{\partial}{\partial z}\right)\right) \\ L_z &= x\left(-i\hbar\left(\frac{\partial}{\partial y}\right)\right) - y\left(-i\hbar\left(\frac{\partial}{\partial x}\right)\right) \end{aligned}$$

Or

$$\begin{aligned} L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{1A}$$

Hence in Cartesian coordinates

$$\vec{L} = \begin{bmatrix} -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{bmatrix}$$

Now the above is converted to spherical coordinates. The relation between the Cartesian and spherical coordinates is

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2}$$

We also need expression for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. But by chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial}{\partial \phi}\frac{d\phi}{dx} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial}{\partial \phi}\frac{d\phi}{dy} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial r}\frac{dr}{dz} + \frac{\partial}{\partial \theta}\frac{d\theta}{dz} + \frac{\partial}{\partial \phi}\frac{d\phi}{dz} \end{aligned}$$

To evaluate the above, we need to do the reverse of (2), which is to relate r, θ, ϕ to x, y, z. From the geometry we see that

$$r = \sqrt{x^2 + y^2 + z^2} \tag{3}$$

$$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{4}$$

$$\tan\phi = \frac{y}{x} \tag{5}$$

Therefore, from (3)

$$dr = \frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2 + z^2}}dx$$

But $x = r\sin\theta\cos\phi$ and $r = \sqrt{x^2 + y^2 + z^2}$. The above becomes

$$\begin{aligned} \frac{dr}{dx} &= \frac{r\sin\theta\cos\phi}{r} \\ &= \sin\theta\cos\phi \end{aligned} \tag{6}$$

And from (4)

$$\begin{aligned} \frac{d}{d\theta}(\cos\theta) &= \frac{d}{dx}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ -\sin\theta d\theta &= -\frac{1}{2}\frac{z(2x)}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}dx \end{aligned}$$

But $x^2 + y^2 + z^2 = r^2$ and $z = r\cos\theta$ and $x = r\sin\theta\cos\phi$. The above becomes

$$\begin{aligned} -\sin\theta d\theta &= -\frac{1}{2}\frac{r\cos\theta\left(2r\sin\theta\cos\phi\right)}{\left(r^2\right)^{\frac{3}{2}}}dx \\ &= \frac{-r^2\cos\theta\sin\theta\cos\phi}{r^3}dx \\ &= \frac{-\cos\theta\sin\theta\cos\phi}{r}dx \end{aligned}$$

Hence

$$
\begin{aligned}
d\theta &= \frac{\cos\theta\cos\phi}{r}dx \\
\frac{d\theta}{dx} &= \frac{1}{r}\cos\theta\cos\phi \qquad (7)
\end{aligned}
$$

And from (5)

$$
\begin{aligned}
\frac{d}{d\phi}\left(\tan\phi\right) &= \frac{d}{dx}\left(\frac{y}{x}\right) \\
\frac{1}{\cos^2\phi}d\phi &= y\left(\frac{-1}{x^2}\right)dx
\end{aligned}
$$

But $y = r\sin\theta\sin\phi$ and $x = r\sin\theta\cos\phi$. Therefore

$$
\begin{aligned}
\frac{1}{\cos^2\phi}d\phi &= \frac{-r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi}dx \\
\frac{d\phi}{dx} &= \frac{-r\sin\theta\sin\phi\cos^2\phi}{r^2\sin^2\theta\cos^2\phi} \\
&= \frac{-\sin\phi}{r\sin\theta} \qquad (8)
\end{aligned}
$$

The above completes all the terms needed to find $\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial\theta}\frac{d\theta}{dx} + \frac{\partial}{\partial\phi}\frac{d\phi}{dx}$. Hence, using (6,7,8) above gives

$$
\boxed{\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}} \qquad (9)
$$

Now the same thing is repeated to find $\frac{\partial}{\partial y}$ in spherical coordinates. From (3)

$$
dr = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}dy
$$

But $y = r\sin\theta\sin\phi$ and $r = \sqrt{x^2+y^2+z^2}$. The above becomes

$$
\begin{aligned}
\frac{dr}{dy} &= \frac{r\sin\theta\sin\phi}{r} \\
&= \sin\theta\sin\phi \qquad (10)
\end{aligned}
$$

And from (4)

$$
\begin{aligned}
\frac{d}{d\theta}\cos\theta &= \frac{d}{dy}\frac{z}{\sqrt{x^2+y^2+z^2}} \\
-\sin\theta d\theta &= -\frac{1}{2}\frac{z\left(2y\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dy
\end{aligned}
$$

But $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$ and $y = r\sin\theta\sin\phi$. The above becomes

$$
\begin{aligned}
-\sin\theta d\theta &= -\frac{1}{2}\frac{r\cos\theta\left(2r\sin\theta\sin\phi\right)}{\left(r^2\right)^{\frac{3}{2}}}dy \\
&= \frac{-r^2\cos\theta\sin\theta\sin\phi}{r^3}dy \\
&= \frac{-\cos\theta\sin\theta\sin\phi}{r}dy
\end{aligned}
$$

Hence

$$\begin{aligned} d\theta &= \frac{\cos\theta\sin\phi}{r}dy \\ \frac{d\theta}{dy} &= \frac{1}{r}\cos\theta\sin\phi \end{aligned} \tag{11}$$

And from (5)

$$\begin{aligned} \frac{d}{d\phi}\left(\tan\phi\right) &= \frac{d}{dy}\left(\frac{y}{x}\right) \\ \frac{1}{\cos^2\phi}d\phi &= \left(\frac{1}{x}\right)dy \end{aligned}$$

But $x = r\sin\theta\cos\phi$. Therefore

$$\begin{aligned} \frac{1}{\cos^2\phi}d\phi &= \frac{1}{r\sin\theta\cos\phi}dy \\ \frac{d\phi}{dy} &= \frac{\cos^2\phi}{r\sin\theta\cos\phi} \\ &= \frac{1}{r}\frac{\cos\phi}{\sin\theta} \end{aligned} \tag{12}$$

The above completes all the terms needed to find $\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial}{\partial \phi}\frac{d\phi}{dy}$. Hence, using (10,11,12) above gives

$$\boxed{\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}} \tag{13}$$

Now the same thing is repeated to find $\frac{\partial}{\partial z}$ in spherical coordinates. From (3)

$$dr = \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}dz$$

But $z = r\cos\theta$ and $r = \sqrt{x^2+y^2+z^2}$. The above becomes

$$\begin{aligned} \frac{dr}{dz} &= \frac{r\cos\theta}{r} \\ &= \cos\theta \end{aligned} \tag{14}$$

And from (4)

$$\begin{aligned} \frac{d}{d\theta}\cos\theta &= \frac{d}{dz}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\sin\theta d\theta &= \left(\frac{1}{\sqrt{x^2+y^2+z^2}} + z\left(-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2z)\right)\right)dz \\ &= \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)dz \\ &= \left(\frac{\left(x^2+y^2+z^2\right)-z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)dz \end{aligned}$$

But $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$. The above becomes

$$\begin{aligned} -\sin\theta d\theta &= \left(\frac{r^2 - r^2\cos^2\theta}{r^3}\right)dz \\ &= \frac{1-\cos^2\theta}{r}dz \end{aligned}$$

Hence

$$\begin{aligned}\frac{d\theta}{dz} &= -\frac{1-\cos^2\theta}{r\sin\theta}\\ &= -\frac{\sin^2\theta}{r\sin\theta}\\ &= -\frac{1}{r}\sin\theta \end{aligned} \tag{15}$$

And from (5)

$$\frac{d}{d\phi}\left(\tan\phi\right) = \frac{d}{dz}\left(\frac{y}{x}\right)$$

Hence, since RHS does not depend on z then

$$\frac{d\phi}{dz} = 0 \tag{16}$$

The above completes all the terms needed to find $\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{dr}{dz} + \frac{\partial}{\partial\theta}\frac{d\theta}{dz} + \frac{\partial}{\partial\phi}\frac{d\phi}{dz}$. Hence, using (14,15,16) above gives

$$\boxed{\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}} \tag{17}$$

The above completes all derivations needed to find L_x, L_y, L_z in Spherical coordinates. Eqs (9,13,17). Here they are in one place.

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{9}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{13}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \tag{17}$$

Given Eq(1A) found earlier (repeated below)

$$\begin{aligned} L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\\ L_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\\ L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{1A}$$

And given (9,13,17), then (1A) becomes

$$\begin{aligned} L_x &= -i\hbar\left(y\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - z\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\\ L_y &= -i\hbar\left(z\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - x\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right)\\ L_z &= -i\hbar\left(x\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\\ &\quad + i\hbar\left(y\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right) \end{aligned}$$

But $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$. The above becomes

$$\begin{aligned} L_x &= -i\hbar\left(r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\\ L_y &= -i\hbar\left(r\cos\theta\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\cos\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right)\\ L_z &= -i\hbar\left(r\sin\theta\cos\phi\right)\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ &\quad + i\hbar\left(r\sin\theta\sin\phi\right)\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \end{aligned}$$

Simplifying gives

$$
\begin{aligned}
L_x &= -i\hbar\left(\left(r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta}\right) - \left(r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r} + \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} + \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\\
L_y &= -i\hbar\left(r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&\quad + i\hbar\left(r\sin\theta\cos\phi\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}r\sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)\\
L_z &= -i\hbar\left(r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r} + \sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cos^2\phi\frac{\partial}{\partial\phi}\right)\\
&\quad + i\hbar\left(r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - r\sin\theta\cos\phi\cos\theta\frac{\partial}{\partial r} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)\\
L_z &= -i\hbar\left(-r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r} + \cos^2\phi\frac{\partial}{\partial\phi} - r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)\\
L_z &= -i\hbar\left(\sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cos^2\phi\frac{\partial}{\partial\phi} - \sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta} + \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\left(\sin^2\theta + \cos^2\theta\right)\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\left(\cos^2\theta + \sin^2\theta\right)\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
L_z &= -i\hbar\left(\cos^2\phi\frac{\partial}{\partial\phi} + \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\\
L_z &= -i\hbar\left(\frac{\partial}{\partial\phi}\right)
\end{aligned}
\tag{18}
$$

The above are L_x, L_y, L_z in spherical coordinates. Therefore

$$
\begin{aligned}
\vec{L}\cdot\vec{L} &= L^2\\
&= L_x^2 + L_y^2 + L_z^2
\end{aligned}
$$

But

$$
\begin{aligned}
L_x^2 &= -\hbar^2\left(\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\right)\\
&= -\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta} + \sin\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right) + \frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&= -\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta} - \frac{\sin\phi\cos\phi}{\sin^2\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
L_y^2 &= -\hbar^2\left(\left(\cos\phi\frac{\partial}{\partial\theta}-\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\theta}-\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\right)\\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta}-\cos\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)-\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right)+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta}-\cos\phi\sin\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)-\frac{\cos\theta}{\sin\theta}\sin\phi\left(-\sin\phi\frac{\partial}{\partial\theta}\right)+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta}+\cos\phi\sin\phi\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\right)+\frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta}+\frac{\cos\phi\sin\phi}{\sec^2\theta}\frac{\partial}{\partial\phi}+\frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

And

$$
L_z^2 = -\hbar^2\left(\frac{\partial^2}{\partial^2\phi}\right)
$$

Hence

$$
\begin{aligned}
L^2 = &-\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta}-\frac{\sin\phi\cos\phi}{\sec^2\theta}\frac{\partial}{\partial\phi}+\frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&-\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta}+\frac{\cos\phi\sin\phi}{\sec^2\theta}\frac{\partial}{\partial\phi}+\frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&-\hbar^2\left(\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L^2 = -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}\left(\sin^2\phi+\cos^2\phi\right)+\frac{\partial^2}{\partial^2\phi}\left(\frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi+\frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi+1\right)\right)\\
-\hbar^2\frac{\partial}{\partial\phi}\left(-\frac{\sin\phi\cos\phi}{\sec^2\theta}+\frac{\cos\phi\sin\phi}{\sec^2\theta}\right)\\
-\hbar^2\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi+\frac{\cos\theta}{\sin\theta}\sin^2\phi\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
L^2 &= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\left(\cos^2\phi+\sin^2\phi\right)+\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left(\cos^2\phi+\sin^2\phi\right)\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}+\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\left(1+\frac{\cos^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\left(\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2} &= \frac{1}{\hbar^2 r^2}\left(-\hbar^2\left(\frac{\partial^2}{\partial^2\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)\\
&= -\frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} \qquad (20)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2} &= \frac{1}{r^2}\left(2r\frac{\partial}{\partial r}+r^2\frac{\partial^2}{\partial r^2}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\\
&= \frac{2}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial^2\theta}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}
\end{aligned}
$$

But the term in the RHS above is indeed the Laplacian in spherical coordinates. Therefore in spherical coordinates

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2}}$$

Which is what we are asked to show.

4.10.6 Problem 5

Consider $\psi(x,t)$ for $0 \le x \le L$. Given $\psi(0,t) = \psi(L,t) = 0$ and

$$\psi(x,0) = \begin{cases} A\sin\left(\frac{2\pi x}{L}\right) & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases}$$

Find $\psi(x,t)$ that satisfies the following partial differential equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

Where $A, L, \hbar, \mu$ are positive constants.

Solution

Using separation of variables, assuming the solution is

$$\psi(x,t) = X(x)T(t)$$

Where $X(x)$ is function that depends on space only and $T(t)$ is function that depends on t only. Substituting the above into the PDE (1) gives

$$i\hbar XT' = -\frac{\hbar^2}{2\mu}X''T$$

Diving both sides by $XT \neq 0$ gives

$$\begin{aligned} i\hbar\frac{T'}{T} &= -\frac{\hbar^2}{2\mu}\frac{X''}{X} \\ -\frac{2\mu i}{\hbar}\frac{T'}{T} &= \frac{X''}{X} \end{aligned}$$

Since both sides are equal, and left side depends on t only and right side depends on x only, then both must be equal to a constant. Let this constant be $-\lambda$. This gives the following two ODE's to solve

$$-\frac{2\mu i}{\hbar}\frac{T'}{T} = -\lambda \tag{2}$$

$$\frac{X''}{X} = -\lambda \tag{3}$$

Starting with the spatial ODE in order to determine the eigenvalues λ

$$X''(x) + \lambda X(x) = 0 \tag{4}$$

With the boundary conditions transferred from the PDE as

$$\begin{aligned} X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

There are three cases to consider. $\lambda < 0, \lambda = 0, \lambda > 0$.

case $\lambda < 0$

Let $\lambda = -\mu^2$ for some real μ. Then the ODE (4) becomes $X''(x) - \mu^2 X(x) = 0$. The roots of the characteristic equation are $\pm\mu$. Hence the solution is

$$\begin{aligned} X(x) &= Ae^{\mu x} + Be^{-\mu x} \\ &= A\cosh\left(\mu x\right) + B\sinh\left(\mu x\right) \end{aligned}$$

At $x = 0$, the above becomes

$$0 = A$$

Hence the solution now reduces to

$$X(x) = B\sinh\left(\mu x\right)$$

At $x = L$, this becomes

$$0 = B\sinh\left(\mu L\right)$$

But $\mu L \neq 0$ since $L > 0$ and $\mu \neq 0$. Therefore the only option is that $B = 0$. But this gives trivial solution $X(x) = 0$. Therefore $\lambda < 0$ is not possible.

case $\lambda = 0$

The ODE (4) now becomes

$$X''(x) = 0$$

This has solution $X = Ax + B$. At $x = 0$ this gives $0 = B$. Therefore the solution now reduces to $X(x) = Ax$. At $x = L$ this gives $0 = AL$, which implies $A = 0$. But this gives trivial solution $X(x) = 0$. Therefore $\lambda = 0$ is not possible.

case $\lambda > 0$

In this case, the roots of the characteristic equation of ODE (4) are $\pm i\sqrt{\lambda}$. Hence the solution can written as (by using Euler relation to convert complex exponentials to trigonometric functions) as

$$X(x) = A\cos\left(\sqrt{\lambda}\,x\right) + B\sin\left(\sqrt{\lambda}\,x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution now reduces to

$$X(x) = B\sin\left(\sqrt{\lambda}\,x\right)$$

At $x = L$

$$0 = B\sin\left(\sqrt{\lambda}\,L\right)$$

For non-trivial solution this requires that $\sin\left(\sqrt{\lambda}\,L\right) = 0$ or $\sqrt{\lambda}\,L = n\pi$ for $n = 1, 2, \cdots$. Therefore the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

This completes the solution to the spatial part. The eigenfunctions are therefore

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, \cdots \tag{5}$$

Now the time domain part ODE is solved. This is ODE (2) above. Now that the eigenvalues are known, ODE (2) becomes

$$\begin{aligned} -\frac{2\mu i}{\hbar}\frac{T_n'}{T_n} &= -\lambda_n \\ T_n' &= \frac{T_n \hbar}{2\mu i}\lambda_n \\ T_n' - \frac{\hbar}{2\mu i}\lambda_n T_n &= 0 \end{aligned}$$

This is linear first order ODE. The integrating factor is $I = e^{\frac{-\lambda_n \hbar}{2\mu i}t}$. The above now becomes

$$\frac{d}{dt}\left(T_n e^{\frac{-\lambda_n \hbar}{2\mu i}t}\right) = 0$$

Integrating gives

$$\begin{aligned}
T_n e^{\frac{-\lambda_n \hbar}{2\mu i}t} &= C_n \\
T_n(t) &= C_n e^{\frac{\lambda_n \hbar}{2\mu i}t} \\
&= C_n e^{-\frac{i}{2}\frac{\lambda_n \hbar}{\mu}t}
\end{aligned}$$

But λ_n are the eigenvalues, given by $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \cdots$. Rewriting the above gives

$$T_n(t) = C_n e^{-\frac{i}{2}\frac{\hbar n^2 \pi^2}{\mu L^2}t} \tag{6}$$

But since the solution was assumed to be $\psi(x,t) = X(x)T(t)$, then

$$\psi_n(x,t) = X_n(x)T_n(t)$$

But the general solution is a linear combination of all the solutions $\psi_n(x,t)$. Therefore

$$\begin{aligned}
\psi(x,t) &= \sum_{n=1}^{\infty} \psi_n(x,t) \\
&= \sum_{n=1}^{\infty} X_n(x)T_n(t)
\end{aligned}$$

And using (5,6) in the above, gives

$$\psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) C_n e^{-\frac{i}{2}\frac{\hbar n^2 \pi^2}{\mu L^2}t}$$

But the two constants $B_n C_n$ can be merged into one, say D_n. Therefore the above becomes

$$\psi(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{i}{2}\frac{\hbar n^2 \pi^2}{\mu L^2}t} \tag{7}$$

The above is the general solution. What is left is to determine D_n. This is done from initial conditions. At $t = 0$ the above becomes

$$\begin{cases} A\sin\left(\frac{2\pi x}{L}\right) & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases} = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)$$

The above says that D_n are the Fourier sine series coefficients of the initial conditions. To determine D_n, orthogonality of eigenfunctions $\sin\left(\frac{n\pi}{L}x\right)$ is used.

Multiplying both sides of the above by $\sin\left(\frac{m\pi}{L}x\right)$ and integration both sides from $x = 0$ to $x = L$ gives

$$\int_0^L \begin{cases} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases} = \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)dx$$

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty} D_n \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

Case $m = 1$

The sum above now collapses to one term only when $m = n = 1$, since the sin functions are orthogonal to each others, which gives

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx = D_1\int_0^L \sin^2\left(\frac{\pi}{L}x\right)dx$$
$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx = D_1\frac{L}{2}$$
$$D_1 = \frac{2}{L}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx \tag{8}$$

The integral $\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx$, is evaluated using the relation

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

the integral becomes

$$\begin{aligned}
\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx &= A\int_0^{\frac{L}{2}} \frac{1}{2}\left(\cos\left(\frac{2\pi x}{L} - \frac{\pi}{L}x\right) - \cos\left(\frac{2\pi x}{L} + \frac{\pi}{L}x\right)\right)dx\\
&= \frac{A}{2}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi x}{L} - \frac{\pi}{L}x\right)dx - \int_0^{\frac{L}{2}} \cos\left(\frac{2\pi x}{L} + \frac{\pi}{L}x\right)dx\right)\\
&= \frac{A}{2}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{\pi x}{L}\right)dx - \int_0^{\frac{L}{2}} \cos\left(\frac{3\pi x}{L}\right)dx\right)\\
&= \frac{A}{2}\left(\left[\frac{\sin\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}}\right]_0^{\frac{L}{2}} - \left[\frac{\sin\left(\frac{3\pi x}{L}\right)}{\frac{3\pi}{L}}\right]_0^{\frac{L}{2}}\right)\\
&= \frac{A}{2}\left(\frac{L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right)\right]_0^{\frac{L}{2}} - \frac{L}{3\pi}\left[\sin\left(\frac{3\pi x}{L}\right)\right]_0^{\frac{L}{2}}\right)\\
&= \frac{A}{2}\left(\frac{L}{\pi}\sin\left(\frac{\pi\frac{L}{2}}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi\frac{L}{2}}{L}\right)\right)\\
&= \frac{A}{2}\left(\frac{L}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{L}{3\pi}\sin\left(\frac{3}{2}\pi\right)\right)\\
&= \frac{A}{2}\left(\frac{L}{\pi} + \frac{L}{3\pi}\right)\\
&= \frac{L}{\pi}\frac{A}{2}\left(1 + \frac{1}{3}\right)\\
&= \frac{L}{\pi}\frac{A}{2}\left(\frac{4}{3}\right)\\
&= \frac{L}{\pi}\frac{2A}{3}
\end{aligned}$$

Hence Eq. (8) becomes

$$\begin{aligned}
D_1 &= \frac{2}{L}\left(\frac{L}{\pi}\frac{2A}{3}\right)\\
&= \frac{4A}{3\pi}
\end{aligned}$$

Case $m = 2$

The sum above now collapses to one term only, since the sin functions are orthogonal to each others, so only for $n = 2$ the sum gives a result. Hence

$$\begin{aligned}\int_0^{\frac{L}{2}} A\sin^2\left(\frac{2\pi x}{L}\right)dx &= D_2\int_0^L \sin^2\left(\frac{2\pi}{L}x\right)dx\\ A\frac{L}{4} &= D_2\frac{L}{2}\\ D_2 &= \frac{1}{2}A\end{aligned}$$

<u>case $m \geq 3$</u>

The sum now collapses to case when $m = n$, since the sin functions are orthogonal to each others. Hence

$$\begin{aligned}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx &= D_m\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx\\ &= D_m\frac{L}{2}\end{aligned}$$

Therefore (now calling $m = n$ since a dummy index)

$$D_n = \frac{2}{L}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx \tag{9}$$

The integral $I = \int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx$, is evaluated using the relation

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

The integral I becomes, where here $A=\frac{2\pi x}{L},B=\frac{n\pi}{L}x$

$$
\begin{aligned}
I &= \frac{A}{2}\int_0^{\frac{L}{2}}\cos\left(\frac{2\pi x}{L}-\frac{n\pi x}{L}\right)-\cos\left(\frac{2\pi x}{L}+\frac{n\pi x}{L}\right)dx\\
&= \frac{A}{2}\left(\int_0^{\frac{L}{2}}\cos\left(\frac{(2-n)\pi x}{L}\right)dx-\int_0^{\frac{L}{2}}\cos\left(\frac{(2+n)\pi x}{L}\right)dx\right)\\
&= \frac{A}{2}\left(\left[\frac{\sin\left(\frac{(2-n)\pi x}{L}\right)}{\frac{(2-n)\pi}{L}}\right]_0^{\frac{L}{2}}-\left[\sin\left(\frac{\frac{(2+n)\pi x}{L}}{\frac{(2+n)\pi}{L}}\right)\right]_0^{\frac{L}{2}}\right)\\
&= \frac{A}{2}\left(\frac{L}{(2-n)\pi}\left[\sin\left(\frac{(2-n)\pi x}{L}\right)\right]_0^{\frac{L}{2}}-\frac{L}{(2+n)\pi}\left[\sin\left(\frac{(2+n)\pi x}{L}\right)\right]_0^{\frac{L}{2}}\right)\\
&= \frac{A}{2}\left(\frac{L\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)}{(2-n)\pi}-\frac{L\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2+n)\pi}\right)\\
&= \frac{LA}{2\pi}\left(\frac{\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)}{(2-n)}-\frac{\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2+n)}\right)\\
&= \frac{LA}{2\pi}\left(\frac{(2+n)\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)-(2-n)\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2-n)(2+n)}\right)\\
&= \frac{LA}{2\pi(2-n)(2+n)}\left((2+n)\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)-(2-n)\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left((2+n)\sin\left(\frac{2\pi\frac{L}{2}-n\pi\frac{L}{2}}{L}\right)-(2-n)\sin\left(\frac{2\pi\frac{L}{2}+n\pi\frac{L}{2}}{L}\right)\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
I &= \frac{LA}{2\pi\left(4-n^2\right)}\left((2+n)\sin\left(\pi-\frac{n}{2}\pi\right)-(2-n)\sin\left(\pi+\frac{n}{2}\pi\right)\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(2\sin\left(\pi-\frac{n}{2}\pi\right)+n\sin\left(\pi-\frac{n}{2}\pi\right)-2\sin\left(\pi+\frac{n}{2}\pi\right)+n\sin\left(\pi+\frac{n}{2}\pi\right)\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(2\left[\sin\left(\pi-\frac{n}{2}\pi\right)-\sin\left(\pi+\frac{n}{2}\pi\right)\right]+n\left[\sin\left(\pi-\frac{n}{2}\pi\right)+\sin\left(\pi+\frac{n}{2}\pi\right)\right]\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[\sin\left(\pi+\frac{n}{2}\pi\right)-\sin\left(\pi-\frac{n}{2}\pi\right)\right]+n\left[\sin\left(\pi+\frac{n}{2}\pi\right)+\sin\left(\pi-\frac{n}{2}\pi\right)\right]\right)
\end{aligned}
$$

Using $\sin\left(x+y\right)-\sin\left(x-y\right)=2\cos x\sin y$ and $\sin\left(x+y\right)+\sin\left(x-y\right)=2\sin x\cos y$ on the above gives (where $x=\pi,y=\frac{n}{2}\pi$ in this case)

$$
\begin{aligned}
I &= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[2\cos\pi\sin\frac{n}{2}\pi\right]+n\left[2\sin\pi\cos\frac{n}{2}\pi\right]\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[-2\sin\frac{n}{2}\pi\right]\right)\\
&= \frac{2LA}{\pi\left(4-n^2\right)}\left(\sin\frac{n}{2}\pi\right)
\end{aligned}
$$

Hence (9) becomes

$$\begin{aligned}
D_n &= \frac{2}{L}\left(\frac{2LA}{\pi\left(4-n^2\right)}\left(\sin\frac{n}{2}\pi\right)\right)\\
&= \frac{4A}{\pi\left(4-n^2\right)}\sin\frac{n}{2}\pi\\
&= \frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n}{2}\pi\right)
\end{aligned}$$

Now all coefficients of the Fourier sine series are found. Therefore the solution (7) becomes

$$\begin{aligned}
\psi(x,t) &= \psi_1(x,t)+\psi_2(x,t)+\sum_{n=3}^{\infty}D_n\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}\\
&= D_1\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+D_2\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}+\sum_{n=3}^{\infty}\left(\frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}\\
&= \frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+\frac{1}{2}A\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}+\sum_{n=3}^{\infty}\left(\frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}
\end{aligned}$$

Therefore the final solution is

$$\psi(x,t)=\frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+\frac{1}{2}A\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}-\frac{4A}{\pi}\sum_{n=3}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(n^2-4\right)}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}$$

When $n=4,6,8,\cdots$ then $\sin\left(\frac{n\pi}{2}\right)=0$. Therefore only odd terms survive

$$\psi(x,t)=\frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+\frac{A}{2}\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}-\frac{4A}{\pi}\sum_{n=3,5,7,\cdots}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(n^2-4\right)}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}$$

4.10.7 key solution for HW 10

Physics 3041 (Spring 2021) Solutions to Homework Set 10

1. (10 points) Given

$$\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}\,,$$

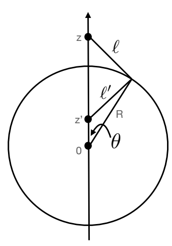
make a 3D integral and use the transformation from Cartesian to spherical coordinates to evaluate

$$\int_0^{\infty} x^2 \exp(-x^2)dx.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2-z^2}dxdydz = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \int_{-\infty}^{\infty} e^{-z^2}dz = (\sqrt{\pi}\,)^3$$

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2-z^2}dxdydz &= \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-r^2}r^2\sin\theta\, drd\theta d\phi \\ &= \int_0^{\infty} r^2e^{-r^2}dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= (\int_0^{\infty} r^2e^{-r^2}dr)\times 2\times(2\pi) = 4\pi\int_0^{\infty} r^2e^{-r^2}dr \\ \Rightarrow \int_0^{\infty} r^2e^{-r^2}dr = \frac{(\sqrt{\pi}\,)^3}{4\pi} = \frac{\sqrt{\pi}}{4} &= \int_0^{\infty} x^2e^{-x^2}dx\end{aligned}$$

2. Follow the lecture example of deriving the gravitational field of a thin shell and calculate the gravitational potential of such a shell over all space. (10 points)



Due to spherical symmetry, we only need to consider the potential along the z-axis. For a point at $z = r > R$, the potential is

$$\begin{aligned}\phi(r) = -G\sigma\int_0^{\pi}\frac{2\pi R^2\sin\theta d\theta}{\ell} \\ &= -2\pi G\sigma R^2\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{\pi G\sigma R}{r}\int_0^{\pi}\frac{d(r^2+R^2-2rR\cos\theta)}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{2\pi G\sigma R}{r}\sqrt{r^2+R^2-2rR\cos\theta}\Big|_0^{\pi} \\ &= -\frac{2\pi G\sigma R}{r}[r+R-(r-R)] = -\frac{4\pi G\sigma R^2}{r} = -\frac{Gm}{r}\end{aligned}$$

Similarly, for a point at $z' = r < R$, the potential is

$$\begin{aligned}\phi(r) &= -G\sigma\int_0^{\pi}\frac{2\pi R^2\sin\theta d\theta}{\ell'} = -2\pi G\sigma R^2\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{\pi G\sigma R}{r}\int_0^{\pi}\frac{d(r^2+R^2-2rR\cos\theta)}{\sqrt{r^2+R^2-2rR\cos\theta}} = -\frac{2\pi G\sigma R}{r}\sqrt{r^2+R^2-2rR\cos\theta}\Big|_0^{\pi} \\ &= -\frac{2\pi G\sigma R}{r}[r+R-(R-r)] = -4\pi G\sigma R = -\frac{Gm}{R}\end{aligned}$$

3. Follow the lecture example of deriving the gas pressure and calculate the number of gas particles hitting the container per unit area per unit time. Give your answer in terms of the net number density and the average speed of these particles. (10 points)

Consider an area element ΔA perpendicular to the z-axis. The number of particles with velocity between $\vec{v}$ and $\vec{v}+d\vec{v}$ that hit ΔA during an interval Δt is

$$\Delta N = dn \cdot \Delta A \cdot (v_z \Delta t) = f(v) v_z dv_x dv_y dv_z (\Delta A \Delta t),$$

so the net number of particles hitting the container per unit area per unit time is

$$\begin{aligned}
\frac{\Delta N}{\Delta A \Delta t} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) v_z dv_x dv_y dv_z \\
&= \int_0^{2\pi} \int_0^{\pi/2} \int_0^\infty f(v)(v\cos\theta) v^2 \sin\theta dv d\theta d\phi \\
&= \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^\infty f(v) v^3 dv = \pi \int_0^\infty f(v) v^3 dv
\end{aligned}$$

The average speed of the particles is

$$\begin{aligned}
\bar{v} &= \frac{\int v dn}{\int dn} = \frac{1}{n} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) v dv_x dv_y dv_z \\
&= \frac{1}{n} \int_0^{2\pi} \int_0^{\pi} \int_0^\infty f(v) v^3 \sin\theta dv d\theta d\phi = \frac{4\pi}{n} \int_0^\infty f(v) v^3 dv,
\end{aligned}$$

where n is the net number density. So we obtain

$$\frac{\Delta N}{\Delta A \Delta t} = \frac{1}{4} n \bar{v}.$$

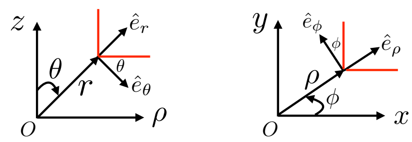
4. Derive the expressions of the quantum mechanical orbital angular momentum operators L_x, L_y, L_z in spherical coordinates. Show that

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L} \cdot \vec{L}}{\hbar^2 r^2}$$

in spherical coordinates. (40 points)

The orbital angular momentum operator is

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i} \vec{r} \times \nabla = \frac{\hbar}{i} r \hat{e}_r \times \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right) = \frac{\hbar}{i} \left(\hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right).$$



It is clear from the above figure that

$$\hat{e}_\theta = \hat{e}_\rho \cos\theta - \hat{e}_z \sin\theta,\ \hat{e}_\rho = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi,\ \hat{e}_\phi = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi.$$

So we obtain

$$\begin{aligned}
\vec{L} &= \frac{\hbar}{i}\left\{(-\hat{e}_x \sin\phi + \hat{e}_y \cos\phi)\frac{\partial}{\partial\theta} - [(\hat{e}_x \cos\phi + \hat{e}_y \sin\phi)\cos\theta - \hat{e}_z \sin\theta]\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\} \\
&= \frac{\hbar}{i}\left[-\hat{e}_x\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) + \hat{e}_y\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) + \hat{e}_z\frac{\partial}{\partial\phi}\right] \\
\Rightarrow L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\
L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\
L_z &= -i\hbar\frac{\partial}{\partial\phi} \\
\vec{L}\cdot\vec{L} &= L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\
&- \hbar^2\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi} \\
&= -\hbar^2\left[\frac{\partial^2}{\partial\phi^2} + \sin^2\phi\frac{\partial^2}{\partial\theta^2} + \sin\phi\frac{\partial\cot\theta}{\partial\theta}\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\cot\theta\cos\phi\frac{\partial^2}{\partial\theta\partial\phi}\right. \\
&+ \cot\theta\cos\phi\frac{\partial\sin\phi}{\partial\phi}\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\sin\phi\frac{\partial^2}{\partial\phi\partial\theta} + \cot^2\theta\cos\phi\frac{\partial\cos\phi}{\partial\phi}\frac{\partial}{\partial\phi} + \cot^2\theta\cos^2\phi\frac{\partial^2}{\partial\phi^2} \\
&+ \cos^2\phi\frac{\partial^2}{\partial\theta^2} - \cos\phi\frac{\partial\cot\theta}{\partial\theta}\sin\phi\frac{\partial}{\partial\phi} - \cos\phi\cot\theta\sin\phi\frac{\partial^2}{\partial\theta\partial\phi} \\
&\left.- \cot\theta\sin\phi\frac{\partial\cos\phi}{\partial\phi}\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\phi\partial\theta} + \cot^2\theta\sin\phi\frac{\partial\sin\phi}{\partial\phi}\frac{\partial}{\partial\phi} + \cot^2\theta\sin^2\phi\frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right) = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right). \\
\nabla^2 &= \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\
&= \frac{\partial^2}{\partial r^2} + \hat{e}_\theta\cdot\frac{\partial\hat{e}_r}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \hat{e}_\phi\cdot\frac{\partial\hat{e}_r}{\partial\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial r} + \hat{e}_\phi\cdot\frac{\partial\hat{e}_\theta}{\partial\phi}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2},
\end{aligned}$$

where we have used

$$d\hat{e}_r = \hat{e}_\phi\sin\theta d\phi + \hat{e}_\theta d\theta,\ d\hat{e}_\theta = \hat{e}_\phi\cos\theta d\phi - \hat{e}_r d\theta,\ d\hat{e}_\phi = -\hat{e}_r\sin\theta d\phi - \hat{e}_\theta\cos\theta d\phi.$$

5. Consider $\psi(x,t)$ for $0 \le x \le L$. Given that $\psi(0,t) = \psi(L,t) = 0$ and

$$\psi(x,0) = \begin{cases} A\sin(2\pi x/L), & 0 \le x \le L/2, \\ 0, & L/2 < x \le L, \end{cases}$$

find $\psi(x,t)$ that satisfies the following partial differential equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial x^2},$$

where A, L, $\hbar$, and μ are positive constants. (30 points)

$$\begin{aligned}
\psi(x,t) &\to X(x)T(t) \Rightarrow i\hbar X(x)\dot{T} = -\frac{\hbar^2}{2\mu}T(t)X'',\ i\hbar\frac{\dot{T}}{T(t)} = -\frac{\hbar^2}{2\mu}\frac{X''}{X(x)} = \frac{\hbar^2k^2}{2\mu} \\
X'' &= -k^2X(x) \Rightarrow X(x) = A'\sin kx + B'\cos kx \\
X(0) &= X(L) = 0 \Rightarrow B' = 0,\ kL = n\pi,\ k = \frac{n\pi}{L},\ X_n(x) = A'\sin\frac{n\pi x}{L},\ n = 1,2,\cdots \\
\int_0^L [X_n(x)]^2dx &= (A')^2\int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{(A')^2L}{2} = 1 \Rightarrow A' = \sqrt{\frac{2}{L}},\ X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \\
\dot{T} &= -\frac{i\hbar k^2}{2\mu}T(t) \Rightarrow T_n(t) = C_ne^{-in^2\pi^2\hbar t/(2\mu L^2)} \\
\psi(x,t) &= \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} C_ne^{-in^2\pi^2\hbar t/(2\mu L^2)}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \\
\psi(x,0) &= \sum_{n=1}^{\infty} C_n\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \\
C_n &= \sqrt{\frac{2}{L}}\int_0^L \psi(x,0)\sin\frac{n\pi x}{L}\,dx = A\sqrt{\frac{2}{L}}\int_0^{L/2}\sin\frac{2\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\
C_2 &= A\sqrt{\frac{2}{L}}\int_0^{L/2}\sin^2\frac{2\pi x}{L}\,dx = \frac{AL}{4}\sqrt{\frac{2}{L}} = \frac{A}{2}\sqrt{\frac{L}{2}} \\
\sin\alpha\sin\beta &= \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\left(\frac{e^{i\beta}-e^{-i\beta}}{2i}\right) = -\frac{e^{i(\alpha+\beta)}-e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}+e^{-i(\alpha+\beta)}}{4} \\
&= \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \\
C_{n\neq 2} &= \frac{A}{2}\sqrt{\frac{2}{L}}\int_0^{L/2}\left[\cos\frac{(n-2)\pi x}{L} - \cos\frac{(n+2)\pi x}{L}\right]dx \\
&= \frac{A}{2}\sqrt{\frac{2}{L}}\left[\frac{L}{(n-2)\pi}\sin\frac{(n-2)\pi}{2} - \frac{L}{(n+2)\pi}\sin\frac{(n+2)\pi}{2}\right] \\
&= \frac{A}{\pi}\sqrt{\frac{L}{2}}\left[-\frac{\sin(n\pi/2)}{n-2} + \frac{\sin(n\pi/2)}{n+2}\right] = -\frac{4A\sin(n\pi/2)}{(n^2-4)\pi}\sqrt{\frac{L}{2}} \\
C_{2m+1} &= \frac{4A(-1)^{m+1}}{(2m-1)(2m+3)\pi}\sqrt{\frac{L}{2}},\ C_{2m+4} = 0,\ m = 0,1,\cdots \\
\psi(x,t) &= \frac{Ae^{-i2\pi^2\hbar t/(\mu L^2)}}{2}\sin\frac{2\pi x}{L} \\
&+ \frac{4A}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}e^{-i(2m+1)^2\pi^2\hbar t/(2\mu L^2)}}{(2m-1)(2m+3)}\sin\frac{(2m+1)\pi x}{L}
\end{aligned}$$

Chapter 5

study notes

5.1 Using potential energy

There are two types of problems related to using potential energy. We can be given $V(x)$ but not at the equilibrium point, or given $V(x)$ at the equilibrium point. If $V(x)$ given is not at the equilibrium point, then we first need to find x_0 which is the equilibrium point. This is done by solving $V'(x) = 0$. Then expand $V(x)$ near x_0 using Taylor series and obtain new $V(x)$ which is now centered around x_0.

The other type of problem, is where we need to find $V(x)$ at equilibrium, from the physics of the problem. See MC2 as example. For the vertical pendulum problem $V(x) = \frac{1}{2}kx^2 - mgx$. This is the potential energy at equilibrium.

We need to convert the above to $V(y) = \frac{1}{2}ky^2 + V(0)$ and only now we can write

$$F = -V'(y) = -m\omega^2 y$$

From the above, ω can be found.

$$\begin{aligned} ky &= m\omega^2 y \\ \omega^2 &= \frac{k}{m} \end{aligned}$$

Remember, we can only use $F = -V'(y) = -m\omega^2 y$ when $V(y)$ has form $\frac{1}{2}ky^2 + V(0)$. Do not use $\frac{1}{2}kx^2 - mgx$. There should not be linear term in $V(x)$.

$V(y)$ should always be 0 at equilibrium. And $V\left(y\right) = \frac{1}{2}m\omega^2 y^2$ so $V'(y) = m\omega^2 y$

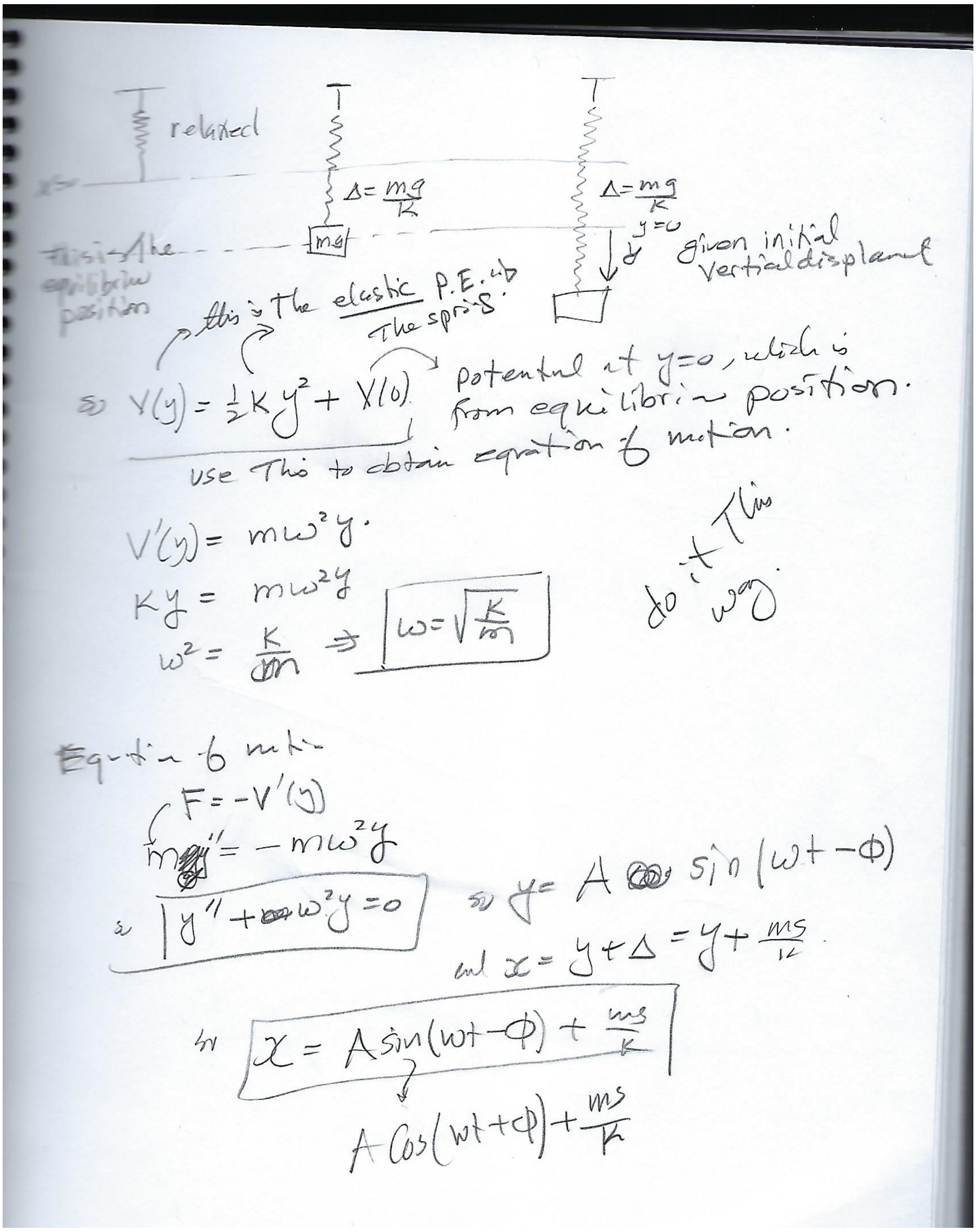


Figure 5.1: How to do the Vibration problems

5.2 Sterling approximation

$$
\begin{aligned}
\int_0^\infty t^n e^{-t} dt &= n! \\
\int_0^\infty t^n e^{-t} dt &= \int_0^\infty e^{n\ln t} e^{-t} dt \\
&= \int_0^\infty e^{(n\ln(t)-t)} dt \\
&= \int_0^\infty e^{f(t)} dt
\end{aligned} \tag{1}
$$

Where $f(t) = n\ln(t) - t$. Contribution to integral comes mostly from where $f(t)$ is maximum.

$$
\begin{aligned}
f'(t) &= 0 \\
\frac{n}{t} - 1 &= 0 \\
t_{\max} &= n
\end{aligned}
$$

Approximating $f(t)$ around t_0

$$
f(t) = f(t_{\max}) + (t - t_{\max})f'(t_{\max}) + \frac{1}{2}(t - t_{\max})^2 f''(t_{\max}) + \cdots
$$

But $f'(t_{\max}) = 0$ and $f''(t) = -\frac{n}{t^2}$. Hence the above becomes

$$
f(t) = f(t_{\max}) - \frac{1}{2}(t - t_{\max})^2 \frac{n}{t_{\max}^2} + \cdots
$$

Replacing $t_{\max} = n$ in the above gives

$$
\begin{aligned}
f(t) &= (n\ln(n) - n) - \frac{1}{2}(t-n)^2 \frac{n}{n^2} + \cdots \\
&= (n\ln(n) - n) - \frac{1}{2}(t-n)^2 \frac{1}{n} + \cdots \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) gives

$$
\begin{aligned}
n! &\approx \int_0^\infty e^{(n\ln(n)-n) - \frac{1}{2}(t-n)^2\frac{1}{n}} dt \\
&\approx e^{(n\ln(n)-n)} \int_0^\infty e^{-\frac{1}{2}(t-n)^2\frac{1}{n}} dt \\
&\approx n^n e^{-n} \int_0^\infty e^{-\frac{1}{2}(t-n)^2\frac{1}{n}} dt
\end{aligned}
$$

Let $u = \frac{t-n}{\sqrt{2n}}$. When $t = 0, u = -\frac{n}{\sqrt{2n}}$ and when $t = \infty, u = \infty$. And $du = \frac{1}{\sqrt{2n}} dt$. The above now becomes

$$
n! \approx n^n e^{-n} \int_{-\frac{n}{\sqrt{2n}}}^\infty e^{-u^2} \sqrt{2n}\, du
$$

When $n \gg 1$, the lower limit of the integral $\to -\infty$. Hence

$$
\begin{aligned}
n! &\approx n^n e^{-n} \int_{-\infty}^\infty e^{-u^2} \sqrt{2n}\, du \\
&\approx \sqrt{2n}\, n^n e^{-n} \sqrt{\pi} \\
&\approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}
\end{aligned}
$$

5.3 Taylor series, convergence

Used to approximate function $f(x)$ at some x knowing its values and all its derivatives at some point x_0, called the expansion point.

$$
f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots
$$

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\end{aligned}
$$

To find series for $\ln(1+x)$, do this

$$
\begin{aligned}
\int \frac{1}{1+x}dx &= \ln(1+x)+C\\
\int (1-x+x^2-x^3+\cdots)dx &= \ln(1+x)+C\\
x-\frac{x^2}{2}+\frac{x^3}{3!}-\cdots &= \ln(1+x)+C \qquad |x|<1
\end{aligned}
$$

To find C, let $x=0$. Hence $0=\ln(1)+C$. So $C=-\ln(1)$. Therefore

$$
\ln(1+x)=\ln(1)+x-\frac{x^2}{2}+\frac{x^3}{3!}-\cdots \qquad |x|<1
$$

And

$$
\begin{aligned}
\int \frac{1}{1-x}dx &= -\ln(1-x)+C\\
-\int (1+x+x^2+x^3+\cdots)dx &= \ln(1-x)+C\\
-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right) &= \ln(1-x)+C\\
-x-\frac{x^2}{2}-\frac{x^3}{3}+\cdots &= \ln(1-x)+C \qquad |x|<1
\end{aligned}
$$

To find C, let $x=0$. Hence $0=\ln(1)+C$. So $C=-\ln(1)$. Therefore

$$
\ln(1-x)=\ln(1)-x-\frac{x^2}{2}-\frac{x^3}{3!}+\cdots
$$

And $\ln(1+2x)$ series is found as follows

$$
\begin{aligned}
\int \frac{1}{1+2x}dx &= \frac{1}{2}\ln(1+2x)+C\\
\int (1-2x+(2x)^2-(2x)^3+\cdots)dx &= \frac{1}{2}\ln(1+2x)+C\\
\left(x-\frac{2x^2}{2}+\frac{4x^3}{3}-\frac{8x^4}{4}\cdots\right) &= \frac{1}{2}\ln(1+2x)+C \qquad |x|<1
\end{aligned}
$$

To find C, let $x=0$. Hence $0=\ln(1)+C$. So $C=-\ln(1)$. Therefore

$$
\ln(1+2x)=2\ln(1)+2\left(x-\frac{2x^2}{2}+\frac{4x^3}{3}-\frac{8x^4}{4}\cdots\right)
$$

And

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
\tan x &= x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots
\end{aligned}
$$

Some others

$$
\begin{aligned}
\frac{1}{1+x} &= 1-x+x^2-x^3+\cdots \qquad |x|<1\\
\frac{1}{1-x} &= 1+x+x^2+x^3+\cdots \qquad |x|<1\\
(1+x)^a &= \sum \binom{a}{n}x^n
\end{aligned}
$$

Where $\binom{a}{n}$ is binomial coefficient $\binom{a}{n}=\frac{a!}{n!(a-n)!}$. General Binomial

$$
(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots
$$

This works for positive and negative n, rational or not. The sum converges only for $|x| < 1$. So, for $n = -1$ the above becomes

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots$$

And

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

And

$$(1+x)^p = 1 + px + p\left(p-1\right)x^2 \cdots$$

For small x the above approximates to

$$(1+x)^p = 1 + px$$

5.3.1 Convergence

First test, check if $\lim_{n\to\infty} a_n$ goes to zero. If not, then no need to do anything. Series does not converge. Then use ratio test. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

Then converges. if result is > 1 then diverges. If result is one, then more testing is needed. If converges, then radius of convergence R is

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ |x| &< R \end{aligned}$$

5.3.2 Closed sums

$$\begin{aligned} \sum_{n=1}^{N} n &= \frac{1}{2}N(N+1) \\ \sum_{n=1}^{N} a_n &= N\left(\frac{a_1 + a_N}{2}\right) \end{aligned}$$

i.e. the sum is N times the arithmetic mean.

Geometric series.

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ &= \sum_{k=0}^{N} ar^k \\ &= a\left(\frac{1 - r^{N+1}}{1 - r}\right) \end{aligned}$$

For $|r| < 1$

$$S = \frac{a}{1-r}$$

5.4 Derivatives of inverse trig functions

To find $y = \arcsin(x)$, always write as $x = \sin(y)$. Then $\frac{dx}{dy} = \cos(y) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$. Then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, Hence

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

To find $\underline{y = \arccos(x)}$, write as $x = \cos\left(y\right)$. Then $\frac{dx}{dy} = -\sin(y) = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$. Then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$, Hence

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

To find $\underline{y = \arctan(x)}$, write as $x = \tan\left(y\right)$. Then $\frac{dx}{dy} = \frac{1}{\cos^2 y}$, now need to use trick that $\cos^2 y + \sin^2 y = 1$ and divide both sides by $\cos^2 y$, hence $1 + \tan^2 y = \frac{1}{\cos^2 y}$. Then $\frac{dx}{dy} = 1 + \tan^2 y$. Hence $\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$. Therefore

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

5.5 Slit interference formulas

k is wave number.

$$k = \frac{2\pi}{\lambda}$$

5.6 Identities

5.6.0.1 trig and Hyper trig identities

$$\begin{aligned}\cos(i\theta) &= \cosh(\theta)\\ \sin(i\theta) &= i\sinh(\theta)\end{aligned}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\begin{aligned}\tan^2(\theta) &= \frac{1}{\cos^2(\theta)} - 1\\ &= \sec^2(\theta) - 1\\ \frac{\cos^2(\theta)}{\sin^2(\theta)} + 1 &= \frac{1}{\sin^2(\theta)}\\ \frac{1}{\tan^2(\theta)} &= \frac{1}{\sin^2(\theta)} - 1\\ \cot^2(\theta) &= \csc^2(\theta) - 1\end{aligned}$$

$$\cosh^2(\theta) - \sinh^2(\theta) = 1$$

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\\ \tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\\ \sinh(2\theta) &= 2\sinh(\theta)\cosh(\theta)\\ \cosh(2\theta) &= 2\cosh^2(\theta) - 1\\ \tanh(2\theta) &= \frac{2\tanh(\theta)}{1+\tanh^2(\theta)}\end{aligned}$$

$$\begin{aligned}\sin(\theta) &= \cos\left(\frac{\pi}{2} - \theta\right)\\ \cos(\theta) &= \sin\left(\frac{\pi}{2} - \theta\right)\end{aligned}$$

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\\ \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B}\\ \tan(A-B) &= \frac{\tan A + \tan B}{1 + \tan A\tan B}\end{aligned}$$

$$\begin{aligned}\sin^2(\theta) &= \frac{1}{2}(1 - \cos(2\theta))\\ \cos^2(\theta) &= \frac{1}{2}(1 + \cos(2\theta))\\ \tan^2(\theta) &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

$$\begin{aligned}\sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \sin A - \sin B &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\\ \cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \cos A - \cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\end{aligned}$$

$$\begin{aligned}\sin A\sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B))\\ \cos A\cos B &= \frac{1}{2}(\cos(A-B) + \cos(A+B))\\ \sin A\cos B &= \frac{1}{2}(\sin(A+B) + \sin(A-B))\\ \cos A\sin B &= \frac{1}{2}(\sin(A+B) - \sin(A-B))\end{aligned}$$

$$
\begin{aligned}
a\cos(\omega t) + b\sin(\omega t) &= A\sin\left(\omega t + \phi\right) \\
&= A\cos\left(\omega t - \phi\right) \\
A &= \sqrt{a^2 + b^2} \\
\phi &= \arctan\left(\frac{B}{A}\right) \\
\cos x + \sin x &= \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right) \\
\cos x + \sin x &= \sqrt{2}\,\cos\left(x - \frac{\pi}{4}\right)
\end{aligned}
$$

Laws of sines (a, b, c) are lengths of triangle sides and A, B, C are facing angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

laws of cosine

$$a^2 = b^2 + c^2 - 2bc\cos A$$

5.6.0.2 GAMMA function

$$
\begin{aligned}
\Gamma(n) &= (n-1)! \\
\Gamma(n+1) &= n(n-1)! \\
&= n\Gamma(n)
\end{aligned}
$$

5.6.0.3 Sterling

For $n \gg 1$

$$\Gamma(n+1) = n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$$

5.7 Integrals

5.7.1 Integrals from 0 to infinity

$$
\begin{aligned}
\int_0^\infty x^n e^{-x} dx &= n! \\
\int_0^\infty x^n e^{-ax} dx &= n! \frac{1}{a^{n+1}} \qquad \text{use } y = ax \\
\int_0^\infty x^3 e^{-x} dx &= 3! \\
\int_0^\infty \frac{x^3}{e^x - 1} dx &= 3!\xi(4)
\end{aligned}
$$

Start by multiplying numerator and denominator by e^{-x} using $\frac{1}{1-y} = 1 + y + y^2 + \cdots$ which becomes $\int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx} dx$ or $\sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx} dx$, then use $z = nx$, this gives $\sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty z^3 e^{-z} dx$ or $(3!) \sum_{n=1}^\infty \frac{1}{n^4}$ or $3!\xi(4)$

$$\int_0^\infty e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

Start by using $x^4 = y$ or $x = y^{\frac{1}{4}}$. then $\frac{dy}{dx} = \frac{1}{4} y^{\left(\frac{1}{4}-1\right)}$, now the integral becomes $\frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy$ and compare this to $\int_0^\infty y^{(s-1)} e^{-x} dx = \Gamma(s)$

$$\int_0^\infty e^{-\sqrt{x}} dx = \int_0^\infty e^{-x^{\frac{1}{2}}} dx$$

Use same method as above. Will get $2\Gamma(2) = 2$

$$
\begin{aligned}
\zeta(s)\Gamma(s) &= \int_0^\infty \frac{x^{(s-1)}}{e^x - 1} dx \qquad s > 1 \\
\zeta(n+1)(n!) &= \int_0^\infty \frac{x^n}{e^x - 1} dx \qquad n > 0 \\
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(4) &= \frac{\pi^4}{90} \\
\zeta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} \qquad s > 1 \\
\zeta(4) &= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots
\end{aligned}
$$

So given $\int_0^\infty \frac{x^3}{e^x-1}dx$, write as $\int_0^\infty \frac{x^{(4-1)}}{e^x-1}dx = \zeta(4)\Gamma(4)$ or $(3!)\zeta(4)$

$$\int_0^\infty x^n e^{-x} dx = n!$$
$$\int_0^\infty x^{1-n} e^{-x} dx = \Gamma(n) = (n-1)!$$

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}} \qquad \text{use } x = a\sin\theta$$
$$I = \int \frac{dx}{x^2 + a^2} \qquad \text{use } x = a\tan\theta$$
$$I = \int_0^\infty x e^{-ax^2} dx \qquad \text{use } u = x^2$$
$$I = \int_0^\infty e^{-ax^2} dx \qquad \text{use } I = \frac{1}{2}\int_{-\infty}^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

For $I = \int_0^\infty x^n e^{-ax^2} dx$ or $I = \int_{-\infty}^\infty x^n e^{-ax^2} dx$. If n is even, use the trick of $I(a) = \int_{-\infty}^\infty e^{-ax^2} dx$ and repeated $I'(a)$. if n is odd, use $I(a) = \int_{-\infty}^\infty x e^{-ax^2} dx = \frac{1}{2a}$ (integration by parts) and then repeated $I'(a)$.

GAMMA:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$
$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{-1}{2}} e^{-x} dx$$

use $u = x^{\frac{1}{2}}$, then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ and the integral becomes $\int_0^\infty x^{\frac{-1}{2}} e^{-x} dx = \int_0^\infty \frac{1}{u} e^{-u^2}(2u du) = 2\int_0^\infty e^{-u^2} du = \sqrt{\pi}$

$$I = \int_0^\infty x e^{-ax} \sin kx \, dx$$
$$I = \int_0^\infty x e^{-ax} \cos kx \, dx$$

For these, we will be given $I = \int_0^\infty e^{-ax} \sin kx \, dx$ and then use $I(a) = \int_0^\infty e^{-ax} \sin kx \, dx$ and then do the $I'(a)$ method.

5.7.2 Integrals from -infinity to infinity

$$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^\infty e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^\infty e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^\infty x^n e^{-ax^2} dx = I \qquad \text{for } n \text{ even, use the } I'(a) \text{ method}$$

5.8 Lorentz transformation

Lorentz transformation is given by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Where θ is called the rapidity. Also

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And

$$v = c \tanh \theta$$

5.9 Rotation matrices and coordinates transformations

Rotation matrix 2D

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation matrix 3D

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is how to find the above. First row, is the projection of x', y', z' on x. Second row is projection of x', y', z' on y and so on.

Spherical coordinates

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

5.10 Matrices and linear algebra

Commutator is defined as

$$[M, N] = MN - NM$$

Where N, M are matrices.

Anti-commutator is when

$$[M, N]_+ = MN + NM$$

Two matrices commute means $MN - NM = 0$. Matrices that commute share an eigenbasis.

Properties of commutators

$$
\begin{aligned}
[A+B,C] &= [A,C]+[B,C] \\
[A,B+C] &= [A,B]+[B,C] \\
[A,A] &= 0 \\
\left[A^2,B\right] &= A[A,B]+[A,B]A \\
[AB,C] &= A[B,C]+[A,C]B \\
[A,BC] &= [A,B]C+B[A,C]
\end{aligned}
$$

Matrices are generally noncommutative. i.e.

$$MN \neq NM$$

Matrix Inverse

$$A^{-1} = \frac{1}{|A|}A_c^T$$

Where A_c is the cofactor matrix.

Matrix inverse satisfies

$$A^{-1}A = I = AA^{-1}$$

Matrix adjoint is same as Transpose for real matrix. If Matrix is complex, then Matrix adjoint does conjugate in addition to transposing. This is also called dagger.

$$A_{ij}^{\dagger} = A_{ji}^{*}$$

So dagger is just transpose but for complex, we also do conjugate after transposing. That is all.

If $A_{ij} = A_{ji}$ then matrix is symmetric. If $A_{ij} = -A_{ji}$ then antisymmetric.

Hermitian matrix is one which $A^{\dagger} = A$. If $A^{\dagger} = -A$ then it is antiHermitian.

Any real symmetric matrix is always Hermitian. But for complex matrix, non-symmetric can still be Hermitian. An example is $\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$.

Unitary matrix Is one whose dagger is same as its inverse. i.e.

$$
\begin{aligned}
A^{\dagger} &= A^{-1} \\
A^{\dagger}A &= I
\end{aligned}
$$

Remember, dagger is just transpose followed by conjugate if complex. Example of unitary matrix is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Determinant of a unitrary matrix must be complex number whose magnitude is 1.

Also $|Av| = |v|$ if A is unitary. This means A maps vector of some norm, to vector which must have same length as the original vector.

A unitary operator looks the same in any basis.

Orthogonal matrix One which satisfies

$$
\begin{aligned}
AA^T &= I \\
A^TA &= I \\
A^{-1} &= A^T
\end{aligned}
$$

commute means $[MN] = MN - NM$. Also $[MN]_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Another property is that $\det(\alpha_i) = -1$. Since they are Hermitian and unitary, then $\alpha_i^{-1} = \alpha_i$.

If H is Hermitian, then $U = e^{iH}$ is unitary.

When moving a number out of a BRA, make sure to complex conjugate it. For example $\langle 3v_1|v_2\rangle = 3^*\langle v_1|v_2\rangle$. But for the ket, no need to. For example $\langle v_1|3v_2\rangle = 3\langle v_1|v_2\rangle$

item $\langle f|\Omega|g\rangle^* = \langle (\Omega|g)^*|f\rangle = \langle g|\Omega^\dagger|f\rangle$

item when moving operator from ket to bra, remember to dagger it. $\langle u|Tv\rangle = \langle T^\dagger u|v\rangle$

item if given set of vectors and asked to show L.I., then set up $Ax = 0$ system, and check $|A|$. If determinant is zero, then there exist non-trivial solution, which means Linearly dependent. Otherwise, L.I.

item if given A, then to represent it in say basis e_i, we say $A_{ki}^{(e)} = \langle e_k, Ae_i\rangle = \langle e_k|A|e_i\rangle$. i.e $A_{1,1} = \langle e_1, Ae_1\rangle$ and $A_{1,2} = \langle e_1, Ae_2\rangle$ and so on.

5.11 Gram-Schmidt

Let the input $V_1, V_2, \cdots, V_n$ be a set of n linearly independent vectors. We want to use Grame-Schmidt to obtain set of n orthonormal vectors, called $v_1, v_2, \cdots, v_n$. The notation $\langle V_1, V_2\rangle$ is used to mean the inner product between any two vectors. The first vector v_1 is easy to find777

$$v_1 = \frac{V_1}{\sqrt{\langle V_1, V_1\rangle}} \tag{1}$$

The second

$$v_2' = V_2 - v_1\langle v_1, V_2\rangle$$

Where v_2' means v_2 but not yet normalized. Before we normalize v_2', we need to show that $\langle v_1, v_2'\rangle = 0$. But

$$\langle v_1, v_2'\rangle = \langle v_1, (V_2 - v_1\langle v_1, V_2\rangle)\rangle$$

Expanding the above gives

$$\langle v_1, v_2'\rangle = \langle v_1, V_2\rangle - \langle v_1, v_1\langle v_1, V_2\rangle\rangle$$

But $\langle v_1, V_2\rangle$ above is just a number. We can take it out of the second inner product term above. The above becomes

$$\langle v_1, v_2'\rangle = \langle v_1, V_2\rangle - \langle v_1, V_2\rangle\langle v_1, v_1\rangle$$

But $\langle v_1, v_1\rangle = 1$, since v_1 is normalized vector. The above becomes

$$\begin{aligned}\langle v_1, v_2'\rangle &= \langle v_1, V_2\rangle - \langle v_1, V_2\rangle \\ &= 0\end{aligned}$$

Now we normalized v_2'

$$v_2 = \frac{v_2'}{\sqrt{\langle v_2', v_2'\rangle}}$$

Now we find v_3

$$\begin{aligned}v_3' &= V_3 - (v_1\langle v_1, V_3\rangle + v_2\langle v_2, V_3\rangle) \\ v_3 &= \frac{v_3'}{\sqrt{\langle v_3', v_3'\rangle}}\end{aligned}$$

And so on.

5.12 Modal analysis

given $|\ddot{x}(t)\rangle + M|x(t)\rangle = 0$, find the eigenvectors and eigenvalues of M. Then $\Phi = [V_2, V_2]$ is 2×2 matrix, transformation matrix. where each column is the eigenvector of M. Then

$|X(t)\rangle = \Phi^T|x(t)\rangle$ and $|x(t)\rangle = \Phi\,|X(t)\rangle$. The new system becomes $|\ddot{X}(t)\rangle + \Omega|X(t)\rangle = 0$ where Ω is now diagonal matrix with eigenvalues of M on the diagonal. Solve using this. First transform initial conditions to $X(t)$. Then trandform solution back to $|x(t)\rangle$ using $|x(t)\rangle = \Phi\,|X(t)\rangle$.

5.13 Complex Fourier series and Fourier transform

Given $f(x)$ which is periodic on $0 < x < L$, so period is L, then Fourier series is

$$f(x) \sim \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{L}x}$$

Where

$$\begin{aligned} c_n &= \langle n|f\rangle \\ &= \frac{1}{\sqrt{L}} \int_0^L f(x) e^{-in\frac{2\pi}{L}x} dx \end{aligned}$$

The basis are $|n\rangle = \frac{1}{\sqrt{L}} e^{-in\frac{2\pi}{L}x}$ and L is the period.

Fourier transform for non periodic $f(x)$ is (sum above becomes integral)

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c_k e^{ikx} dk \\ c_k &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \end{aligned}$$

This gives rise to

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk$$

5.14 RLC circuit

$$\begin{aligned} V(s) &= I(s)\left(R + Ls + \frac{1}{Cs}\right) \\ I(s) &= \frac{1}{R + Ls + \frac{1}{Cs}} V(s) \end{aligned}$$

As differential equation for current

$$I''(t) + 2\frac{R}{2L} I'(t) + \frac{1}{LC} I(t) = 0$$

5.15 Time evaluation of spin state

$$\begin{aligned} H &= -\mu \cdot B \\ &= \frac{eB}{m_e} S_z \\ &= \frac{eB\hbar}{2m_e} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

The eigenvalues are $E_+ = \frac{eB\hbar}{2m_e} m, E_- = -\frac{eB\hbar}{2m_e} m$

$$\begin{aligned} i\hbar \frac{d}{dt}|X\rangle &= H|X\rangle \\ &= \frac{eB\hbar}{2m_e} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |X\rangle \end{aligned}$$

Hence

$$
\begin{aligned}
i\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \frac{eB}{2m_e}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\
\hbar\dot{x}_1(t) &= \frac{eB}{2m_e}x_1(t)\\
\hbar\dot{x}_2(t) &= -\frac{eB}{2m_e}x_2(t)
\end{aligned}
$$

The solution is

$$
\begin{aligned}
x_1(t) &= \frac{1}{\sqrt{2}}e^{-i\gamma t}\\
x_2(t) &= \frac{1}{\sqrt{2}}e^{i\gamma t}
\end{aligned}
$$

Or

$$
|X\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}e^{-i\gamma t}\\ e^{i\gamma t}\end{bmatrix}
$$

Where $\gamma = \frac{eB}{2m_e}$

$$
\begin{aligned}
|X\rangle &= c_+|S_x = \frac{\hbar}{2}\rangle + c_-|S_x = -\frac{\hbar}{2}\rangle\\
c_+ &= \langle S_x = \frac{\hbar}{2}|X\rangle\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}e^{-i\gamma t}\\ e^{i\gamma t}\end{bmatrix}\\
&= \frac{1}{2}\left(e^{i\gamma t}+e^{-i\gamma t}\right)\\
&= \cos\gamma t
\end{aligned}
$$

Probability to measure $S_x = \frac{\hbar}{2}$ at $t>0$ is $P(t) = |c_+|^2 = \cos^2\gamma t$. And

$$
\begin{aligned}
|X\rangle &= c_+|S_x = \frac{\hbar}{2}\rangle + c_-|S_x = -\frac{\hbar}{2}\rangle\\
c_- &= \langle S_x = -\frac{\hbar}{2}|X\rangle\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}e^{-i\gamma t}\\ e^{i\gamma t}\end{bmatrix}\\
&= \frac{1}{2}\left(e^{i\gamma t}-e^{i\gamma t}\right)\\
&= i\sin\gamma t
\end{aligned}
$$

Probability to measure $S_x = -\frac{\hbar}{2}$ at $t>0$ is $P(t) = |c_-|^2 = \sin^2\gamma t$

5.16 Pauli matrices, Spin matrices

Pauli matrices There are 3 of these. They are

$$
\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}
$$

There are also sometimes called $\alpha_x, \alpha_y, \alpha_z$. Not to be confused by component x, y, z of an ordinary vector. Important property is that $\sigma_i^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$. Also they are all Hermitians (i.e. $A^\dagger = A$). This is obvious for the first and last matrix, since there are symmetric and real (we know if a matrix is real and also symmetric, it is also Hermitian.). Another important property is that they are unitary. i.e. $A^\dagger = A^{-1}$. Also any two anticommute. This means $[M,N]_+ = MN + NM$.

$$\left[\sigma_x, \sigma_y\right] = 2i\sigma_z$$

For Pauli matrices, $\left[\sigma_i, \sigma_j\right] = 2i \sum \epsilon_{ijk}\sigma_k$. Hence

$$\begin{aligned}
[\sigma_1, \sigma_2] &= 2i\sigma_3\\
[\sigma_2, \sigma_1] &= -2i\sigma_3\\
[\sigma_1, \sigma_3] &= -2i\sigma_2\\
[\sigma_3, \sigma_1] &= 2i\sigma_2\\
[\sigma_2, \sigma_3] &= 2i\sigma_1\\
[\sigma_3, \sigma_2] &= -2i\sigma_1
\end{aligned}$$

Eigenvalues of Pauli matrices can be only $1, -1$.

$$\text{Tr}(\sigma_i) = 0$$

And Pauli matrices do not commute. This means $\sigma_x\sigma_y \neq \sigma_y\sigma_x$.

Electron $\frac{1}{2}$ spin matrices

Spin matrix	Eigenvalues	Eigenvectors
$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$
$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

And using $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk}S_k$. Hence $[S_1, S_2] = i\hbar S_3$ and $[S_1, S_3] = -i\hbar S_2$ and $[S_2, S_1] = -i\hbar S_3$ and $[S_2, S_3] = i\hbar S_1$ and $[S_3, S_1] = -i\hbar S_2$ and $[S_3, S_2] = -i\hbar S_1$. Hence

$$\begin{aligned}
\left[S_x, S_y\right] &= i\hbar S_z\\
\left[S_y, S_x\right] &= -i\hbar S_z\\
[S_x, S_z] &= -i\hbar S_y\\
[S_z, S_x] &= i\hbar S_y\\
\left[S_y, S_z\right] &= i\hbar S_x\\
\left[S_z, S_y\right] &= -i\hbar S_x
\end{aligned}$$

And

$$S_i = \frac{\hbar}{2}\sigma_i$$

And

$$\sigma_i^2 = I$$

And

$$\begin{aligned}
S_+^\dagger S_+ &= S^2 - S_z^2 - \hbar S_z\\
&= \hbar^2\\
S_-^\dagger S_- &= S^2 - S_z^2 + \hbar S_z\\
&= \hbar^2
\end{aligned}$$

Where $S^2 = \frac{3}{4}\hbar^2 I$.

Electron 1 spin matrices

Spin matrix	Eigenvalues	Eigenvectors
$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$	$1, 0, -1$	$\begin{bmatrix}\frac{1}{2}\\\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$ $\begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\begin{bmatrix}\frac{1}{2}\\\frac{-1}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$
$S_y = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{bmatrix}$	$1, 0, -1$	$\begin{bmatrix}-\frac{1}{2}\\-\frac{i}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$ $\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\begin{bmatrix}-\frac{1}{2}\\\frac{i}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$
$S_z = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{bmatrix}$	$\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

And

$$\begin{aligned}S_+^\dagger S_+ &= S^2 - S_z^2 - \hbar S_z\\&= \hbar^2\\S_-^\dagger S_- &= S^2 - S_z^2 + \hbar S_z\\&= \hbar^2\end{aligned}$$

Where $S^2 = 2\hbar^2 I = \hbar^2\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$.

If we are given state vector V and asked to find expectation value when measuring along x axis, then do $\langle V|S_x|V\rangle$

5.17 Quantum mechanics cheat sheet

5.17.1 Hermitian operator in function spaces

If Ω is Hermitian operator, then it satisfies

$$\begin{aligned}\langle u|\Omega|v\rangle^* &= \langle v|\Omega|u\rangle\\\left(\int u^*(x)\Omega[v(x)]dx\right)^* &= \int v^*(x)\Omega[u(x)]dx\\\int u(x)\Omega[v^*(x)]dx &= \int v^*(x)\Omega[u(x)]dx\end{aligned}$$

For this, the boundary terms must vanish. For example, for the operator $\Omega = -i\frac{d}{dx}$

5.17.2 Dirac delta relation to integral

$$\delta(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}dx$$

5.17.3 Normalization condition

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)dt = 1$$

5.17.4 Expectation (or average value)

If a system is in state of Ψ , then we apply operator $\hat{A}$, then the average value of the observable quantity is the expectation integral

$$\begin{aligned}\left\langle \hat{A} \right\rangle &= \langle \psi|\hat{A}|\psi\rangle \\ &= \frac{\int_{-\infty}^{\infty} \Psi^* \hat{A}\Psi dx}{\int_{-\infty}^{\infty} \Psi\Psi dx}\end{aligned}$$

Note that $\int_{-\infty}^{\infty} \Psi(x)\Psi(x)dx = 1$ if the state wave function is already normalized.

Given an operator $\hat{X}$, acting on $\Psi(x,t)$ then

$$\hat{X}\Psi(x,t) = x\Psi(x,t)$$

The expectation of measuring x is (assuming everything is normalized)

$$\begin{aligned}\left\langle \hat{X} \right\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{X}\Psi(x,t)dx \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)x\Psi(x,t)dx \\ &= \langle x \rangle\end{aligned}$$

Given system is in state $\psi(x)$. What is the expectation value for x measurement. Is this same as writing $\langle X\rangle$. Yes. it is

$$\langle \psi|x|\psi\rangle$$

5.17.5 Probability

The probability that position x of particle is between x and $x + dx$ is $|\Psi(x,t)|^2 dx$. Hence $|\Psi(x,t)|^2$ is the probability density.

Note that

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \int_{-\infty}^{\infty} |\Psi(x)|^2 dx \\ \langle\Psi_1|\Psi_2\rangle &= \int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx\end{aligned}$$

Given $|\Psi\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle$ then the probabilities to measure a or b are

$$\begin{aligned}P(a) &= \frac{|a|^2}{|a|^2 + |b|^2} \\ P(b) &= \frac{|b|^2}{|a|^2 + |b|^2}\end{aligned}$$

5.17.6 Position operator $\hat{x}$

eigenvalue/eigenfunction	$\hat{x}\|x\rangle = x\|x\rangle$ Where x is eigenvalue and $\|x\rangle$ is position vector.
orthonormal eigenbasis	$\{\|x\rangle\} \to \begin{cases} \langle x\|x'\rangle = \delta(x-x') \\ \int_{-\infty}^{\infty} \|x\rangle\langle x\|\, dx = 1 \end{cases}$ for $-\infty < x < \infty$
Vector form to function form	$\langle x\|\psi\rangle \equiv \psi(x)$ probability at position x
Expansion of state vector $\|\psi\rangle$	$\|\psi\rangle = \int_{-\infty}^{\infty} \|x'\rangle\langle x'\|\psi\rangle dx' = \int_{-\infty}^{\infty} \|x'\rangle\psi(x')dx'$
Eigenfunctions in deep well	Not defined for position operator
Operator matrix elements	$\langle x\|\hat{x}\|x'\rangle = x'\delta(x-x')$ Operator is diagonal matrix.

5.17.7 Momentum operator $\hat{p}$

eigenvalue/eigenfunction	$\hat{p}\|\phi_p\rangle = p\|\phi_p\rangle$ Where p is eigenvalue and $\|\phi_p\rangle$ is momentum eigenstate
orthonormal eigenbasis	$\{\|\phi_p\rangle\} \to \begin{cases} \langle \phi_p\|\phi_{p'}\rangle = \delta(p-p') \\ \int_{-\infty}^{\infty} \|\phi_p\rangle\langle \phi_p\|\, dp = 1 \end{cases}$ for $-\infty < p < \infty$
Vector form to function form	$\langle x\|\phi_p\rangle \equiv \phi_p(x)$
Expansion of state vector $\|\psi\rangle$	$\|\psi\rangle = \int_{-\infty}^{\infty} \|\phi_p\rangle\langle\phi_p\|\psi\rangle dp$
General Eigenfunction	$\langle x\|\phi_p\rangle \equiv \phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$
Operator matrix elements	$\langle x\|\hat{p}\|x'\rangle = -i\hbar\delta(x-x')\frac{d}{dx'}$ Operator is not diagonal matrix.

5.17.8 Hamilitonian operator $\hat{H}$

$$\hat{H} = \hat{T} + \hat{V}$$

Where $\hat{T}$ is K.E. operator and $\hat{V}$ is P.E. operator. Recall that $p = mv$ and $T = \frac{1}{2}mv^2$. Hence $\hat{T} = \frac{\hat{p}^2}{2m}$.

eigenvalue/eigenfunction	$\hat{H}\|\psi_{E_n}\rangle = E_n\|\psi_E\rangle$ Where E_n is eigenvalue (energy level)
Orthonormal basis of operator	$\{\|\psi_{E_n}\rangle\} \to \begin{cases} \langle \psi_{E_n}(x)\|\psi_{E_m}(x)\rangle = \delta(E_n - E_m) \\ \int_{-\infty}^{\infty} \|\psi_{E_n}\rangle\langle\psi_{E_n}\|\, dE = 1 \end{cases}$ for $n = 1, 2, \cdots$ (check)
Vector form to function form	$\langle x\|\psi_{E_n}\rangle \equiv \psi_{E_n}(x)$
Expansion of state vector $\|\psi\rangle$	$\|\psi_E\rangle = \sum_n \|\psi_{E_n}\rangle\langle\psi_{E_n}\|\psi\rangle$
Eigenfunctions for deep well problem	$\langle x\|\psi_E\rangle = \psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & \text{otherwise} \end{cases}, E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$
Operator matrix elements	$\langle x\|\hat{H}\|x'\rangle = \frac{1}{2}mv^2 + V(x) = \delta(x-x')\left(\frac{\hat{p}^2}{2m} + \hat{V}(x')\right) = \delta(x-x')\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx'^2} + \hat{V}(x')\right)$

The ODE for deep well is derived as follows.

$$\hat{H}\psi = E_n\psi$$
$$\left(\hat{T} + \hat{V}\right)\psi = E_n\psi$$

But $\hat{V}=0$ inside and $\hat{T}=\frac{\hat{p}^2}{2m}=\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$. Hence the above becomes

$$\begin{aligned}\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)&=E\psi(x)\\ \frac{d^2}{dx^2}\psi(x)+\frac{2mE}{\hbar^2}\psi(x)&=0\\ \frac{d^2}{dx^2}\psi(x)+k^2\psi(x)&=0\end{aligned}$$

Where $k=\sqrt{\frac{2mE}{\hbar^2}}$. The eigenvalues are k_n from solving for boundary conditions at $x=L$. Now solve as standard second order ODE, with BC $\psi(0)=0,\psi(L)=0$. The solution becomes

$$\psi(x)=\psi(x)=\begin{cases}\sqrt{\frac{2}{L}}\sin(k_nx) & 0<x<L\\ 0 & \text{otherwise}\end{cases}$$

Where eigenvalues are $k_n=\frac{n\pi}{L}, n=1,2,3,\cdots$

5.18 Questions and answers

5.18.1 Question 1

Problem says that the system is in some general state $\psi(x)$ and asks what is the probability distribution to measure momentum p ?

solution

The probability is $\left|\langle\phi_p|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current state*.

$$\begin{aligned}\langle\phi_p|\psi\rangle&=\int_{-\infty}^{\infty}\langle\phi_p|x\rangle\langle x|\psi\rangle dx\\ &=\int_{-\infty}^{\infty}\langle x|\phi_p\rangle^*\langle x|\psi\rangle dx\\ &=\int_{-\infty}^{\infty}\phi_p^*(x)\psi(x)dx\end{aligned}$$

Now, for the deep well problem for $0<x<L$, we should know that $\phi_p(x)=\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}$ and $\psi(x)$ will be given. For example $\psi_E(x)=\begin{cases}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & 0<x<L\\ 0 & \text{otherwise}\end{cases}$. Hence

$$\langle\phi_p|\psi\rangle=\int_0^L\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-ipx}{\hbar}}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx$$

Now evaluate this integral and at the end take the square of the modulus. This will give the probability distribution to measure p. The above was problem 4, in HW7.

5.18.2 Question 2

Problem says that the system is in some general state $\psi(x)$ and asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current state*.

$$
\begin{aligned}
\langle x|\psi\rangle &= \int_{-\infty}^{\infty} \langle x|x'\rangle\langle x'|\psi\rangle dx' \\
&= \int_{-\infty}^{\infty} \delta(x-x')\psi(x'))dx \\
&= \psi(x)
\end{aligned}
$$

Hence $prob(x) = \left|\langle x|\psi\rangle\right|^2 = |\psi(x)|^2$

5.18.3 Question 3

Problem says that the system is in some general state $\psi_E(x)$ and asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current or given eigenstate*.

$$
\begin{aligned}
\langle x|\psi\rangle &= \int_0^L \langle x|x'\rangle\langle x'|\psi\rangle dx' \\
&= \int_0^L \delta(x-x')\psi(x')dx' \\
&= \psi(x)
\end{aligned}
$$

Hence the probability is $\left|\psi(x)\right|^2$. Now, for the deep well problem for $0<x<L$, we know that $\psi_{E_n}(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & 0<x<L \\ 0 & \text{otherwise} \end{cases}$ then

$$
\begin{aligned}
\left|\psi_{E_n}(x)\right|^2 &= \left|\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right|^2 \\
&= \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)
\end{aligned}
$$

Is this correct? Checked, yes correct.

5.18.4 Question 4

Problem gives that the system is in some general state $\phi_p(x)$ (i.e. momentum eigenstate, not energy eigenstate as above, due to having done momentum measurement done before) and then problem asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\phi_p\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current eigenstate*.

$$
\begin{aligned}
\langle x|\phi_p\rangle &= \int_0^L \langle x|x'\rangle\langle x'|\phi_p\rangle dx' \\
&= \int_0^L \delta(x-x')\phi_p(x')dx' \\
&= \phi_p(x)
\end{aligned}
$$

Hence the probability is $\left|\phi_p(x)\right|^2$. we know that $\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$ then

$$
\begin{aligned}
\left|\phi_p(x)\right|^2 &= \left|\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}\right|^2 \\
&= \frac{1}{2\pi\hbar}
\end{aligned}
$$

Which is constant. So if we measure momentum first, then ask for probability of measuring position x next, it will be the above. Same probability to measure any position? Is this correct? yes.

5.18.5 Question 5

Problem gives that the system is in some general state $\phi_p(x)$ and asks what is the probability to measure momentum p'?

The probability of measuring momentum p' given that system is already in state $|\psi_p\rangle \equiv |\phi_p\rangle$ is $\left|\langle\phi_{p'}|\phi_p\rangle\right|^2$ where

$$
\begin{aligned}
\langle\phi_{p'}|\phi_p\rangle &= \int_{-\infty}^{\infty} \langle\phi_{p'}|x\rangle\langle x|\phi_p\rangle dx \\
&= \int_{-\infty}^{\infty} \langle x|\phi_{p'}\rangle^* \langle x|\phi_p\rangle dx \\
&= \int_{-\infty}^{\infty} \phi_{p'}^*(x)\phi_p(x) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{-ip'x}{\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) dx \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{i\left(p-p'\right)x}{\hbar}\right) dx
\end{aligned}
$$

but $\delta(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} dx$, therefore $\delta\left(p-p'\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(p-p'\right)x} dx$.

Let $u = \frac{x}{\hbar}$, then $du = \frac{1}{\hbar}dx$. The integral becomes

$$
\begin{aligned}
\langle\phi_{p'}|\phi_p\rangle &= \frac{\hbar}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i\left(p-p'\right)u} du \\
&= \frac{1}{2\pi}\left(2\pi\delta\left(p-p'\right)\right) \\
&= \delta\left(p-p'\right)
\end{aligned}
$$

5.19 Position, velocity and acc in different coordinates system

In polar, just remember these

$$
\begin{aligned}
\vec{r} &= \rho\hat{e}_\rho \\
d\vec{r} &= \hat{e}_\rho d\rho + \hat{e}_\phi \rho d\phi \\
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \hat{e}_\rho \frac{d\rho}{dt} + \hat{e}_\phi \rho \frac{d\phi}{dt} \\
\frac{d}{dt}\hat{e}_\rho &= \dot{\phi}\hat{e}_\phi \\
\frac{d}{dt}\hat{e}_\phi &= -\dot{\phi}\hat{e}_\rho
\end{aligned}
$$

Given $\vec{r} = \rho\hat{e}_\rho$, then

$$\begin{aligned}\vec{v} &= \dot{\rho}\hat{e}_\rho + \rho\frac{d}{dt}\hat{e}_\rho \\ &= \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi\end{aligned}$$

And similarly for $\vec{a}$.

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{e}_\rho + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{e}_\phi$$

This is much better than the alternatives.

In Cylindrical

$$\begin{aligned}d\hat{e}_\rho &= \hat{e}_\phi d\phi \\ d\hat{e}_\phi &= -\hat{e}_\rho d\phi \\ d\hat{e}_z &= 0\end{aligned}$$

dr is different coordinates

Cartesian

$$dr = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz$$

Cylindrical

$$dr = \hat{e}_\rho d\rho + \hat{e}_\phi \rho d\phi + \hat{e}_z dz$$

Spherical

$$dr = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r\sin\theta d\phi$$

v is different coordinates

Use these for finding Lagrangian.

In Cartesian

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

Polar

$$\vec{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi$$

Spherical

$$\vec{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\theta}\hat{e}_\theta + \rho\sin\theta\dot{\phi}\hat{e}_\phi$$

$$\nabla V\left(\rho,\theta,\phi\right) = \hat{e}_\rho V_r + \hat{e}_\theta\frac{1}{\rho}V_\theta + \hat{e}_\phi\frac{1}{\rho\sin\theta}V_\phi$$

5.20 Gradient, Curl, divergence, Gauss flux law, Stokes

The gradient ∇ is vector operator. In Cartesian

$$\nabla = \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}$$

$$\nabla f = \begin{pmatrix}\frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z}\end{pmatrix}$$

In Cylindrical

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \rho \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \rho \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

In spherical

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\theta \frac{1}{\rho} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{\rho \sin\theta} \frac{\partial}{\partial \phi}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{1}{\rho} \frac{\partial f}{\partial \theta} \\ \frac{1}{\rho \sin\theta} \frac{\partial f}{\partial \phi} \end{pmatrix}$$

For conservative force

$$F = -\nabla V$$

Notice that $-\int \bar{F} \cdot d\bar{r} = \int \nabla V \cdot d\bar{r} = \int_{from}^{to} dV = V(to) - V\big(from\big)$ also $\oint \bar{F} \cdot d\bar{r} = 0$ for conservative force.

The curl in Cartesian

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

In Cylinderical

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & F_\phi & F_z \end{vmatrix}$$

In Spherical

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_\theta \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho \sin\theta} \frac{\partial}{\partial \phi} & \frac{1}{\rho} \frac{\partial}{\partial \theta} \\ F_\rho & F_\phi & F_\theta \end{vmatrix}$$

Divergence This is scalar. see cha7b.pdf

$$\nabla \cdot \bar{F}$$

Gauss law

From Wiki

It states that the flux of the electric field out of an arbitrary closed surface is proportional to the electric charge enclosed by the surface.

Gauss's law can be used in its differential form, which states that the divergence of the electric field is proportional to the local density of charge.

$$\overbrace{\int\int \bar{F}\cdot d\bar{s}}^{\text{surface integral}} = \int_V \left(\nabla\cdot\bar{F}\right) dV$$

Stoke's theorem

$$\overbrace{\oint \bar{F}\cdot d\bar{r}}^{\text{line integral}} = \int_S \left(\nabla\times\bar{F}\right)\cdot d\bar{s}$$

Also divergence of the curl is zero.

$$\nabla\cdot\left(\nabla\times\bar{F}\right) = 0$$

From the net

The characteristic of a conservative field is that the line integral around every simple closed contour is zero. Since the curl is defined as a particular closed contour line integral, it follows that curl(gradF) equals zero.

And curl of a gradient is the zero vector.

$$\nabla\times\left(\nabla\bar{F}\right) = \bar{0}$$

5.21 Gas pressure

average speed of gas particles is v_{rms} or take avergae of the squares of each particle velocity and then take the square root at end. Or

$$\bar{v} = \sqrt{\frac{3RT}{m}}$$

Where R is the gas constant, T is gas absolute temperature and m is molar mass of each gas particle in kg/mol.

dn

$$dn = f(v)dv_x dv_y dv_z$$

Where dn is the number denity of gas particles (how many particles per unit volume with velocity between v and $v+dv$)

Average speed of particles

$$\begin{aligned}
\bar{v} &= \frac{\int v dn}{\int dn}\\
&= \frac{\int\int\int v f(v) dv_x dv_y dv_z}{n}\\
&= \frac{1}{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v f(v) dv_x dv_y dv_z\\
&= \frac{1}{n}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{v=0}^{\infty} v f(v)\left(v^2\sin\theta\right) dv d\theta d\phi\\
&= \frac{1}{n}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\sin\theta d\theta \int_{v=0}^{\infty} f(v) v^3 dv\\
&= \frac{1}{n}(2\pi)(-\cos\theta)_0^{\pi}\int_{v=0}^{\infty} f(v) v^3 dv\\
&= -\frac{1}{n}(2\pi)(-1-1)\int_{v=0}^{\infty} f(v) v^3 dv\\
&= \frac{4\pi}{n}\int_{v=0}^{\infty} f(v) v^3 dv
\end{aligned}$$

Pressure

$$\begin{aligned} dF &= F_1 dN \\ &= \left(\frac{2mv_z}{\Delta t}\right) dn \Delta A v_z \Delta t \\ &= 2mv_z^2 dn \Delta A \end{aligned}$$

Hence

$$\begin{aligned} P &= \frac{\int dF}{\Delta A} \\ &= 2m \int v_z^2 dn \\ &= 2m \int dv_x \int dv_y \int f(v) v_z^2 dv_z \end{aligned}$$

This integral can be evaluated in spherical coordinates.

net energy density of gas

$$\begin{aligned} E &= \int \frac{1}{2} m v^2 dn \\ &= \frac{1}{2} m \int \int \int v^2 dn \\ &= \frac{1}{2} m \int \int \int \left(v_x^2 + v_y^2 + v_z^2\right) dn \\ &= \frac{3}{2} m \int \int \int v_z^2 dn \\ &= \frac{3}{2} m \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} v_z^2 f(v) dv_z \\ &= 3m \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{0}^{\infty} v_z^2 f(v) dv_z \end{aligned}$$

Hence

$$P = \frac{2}{3} E$$

And $E = \frac{3}{2} nKT \rightarrow P = nKT$ for ideal gas.

5.22 Table of study guide

chapter	topics
ch7c.pdf	PDE's, seperation of variables, Lagrangian method
ch7b.pdf	Position, velocity and acc in different coordinates. Gradient, Curl and Div.
ch7a.pdf	Multivariable calculus. Jacobian. Gravitional field for shell, Pressure and energy of gas
ch6b.pdf	First order ODE's. Second order Constant coefficients. under,over and critical damping
ch6a.pdf	Second order ODE's. Variable coefficient. Power series methods. Hermite ODE.
ch5c.pdf	Function spaces. Hermitian operators. Complex Fourier series. Fourier transform. Deep well probem

ch5b.pdf	Linear vector spaces and QM. Probability when making measurements. Commutation. Schrodinger equation. Spin operators S_x, S_y, S_z. Pauli matrices. Time evolution of spin state. Solving mass/spring problem using normal modes.
ch5a.pdf	Linear vector spaces. Linear independence. Gram-Schmidt. Linear operators. Finding eigenvalues and eigenvectors for matrices. Coordinates transformation between orthonormal basis.
ch4.pdf	Matrices and Determinants. 2D rotation matrix. Lorentz transformation. Pauli matrices. Levi-civita. Properties of determinants. Solution to linear equations. Cramer rule. Dimensional analysis.
ch3.pdf	Complex numbers. Taylor series expansion. Solving $x^n = 1$. Integrals. Completing the squares for $\int_{-\infty}^{\infty} e^{(x+ia)^2}\,dx$. Gaussian integral, N slit intererence. Single slit diffraction.
ch2.pdf	Gaussian and exponential integrals. Evaluating Gaussian integral. Evaluating $\int_0^{\infty} x^n e^{-x}\,dx = n!$. Zeta function. Gamma function. Sterling formula.
ch1.pdf	Taylor series. Convergence test. Taylor series of common functions. Using Taylor series to find equilibrium point for small oscillations. Pendulum.

5.23 Questions

1. Do all spin matrices always have same eigenvalues? this is the case for S_x, S_y, S_z for electron. NO. depends spin number.

2. How do we get the probability of measuring $S_y = -\frac{h}{2}$ or $S_y = \frac{h}{2}$ to be $\frac{1}{2}$? is it because there are two eigenvalues, and it is 50% each? see class notes lecture 5b. page 9. Answer: Current state vector is $|S_x = \frac{h}{2}\rangle$.

3. Does the order matter? In page 5, lecture 5B, could we do $C_+ = \langle S_x = \frac{h}{2}|S_z = \frac{h}{2}\rangle$ or $C_+ = \langle S_z = \frac{h}{2}|S_x = \frac{h}{2}\rangle$? Resolved.

4. Why is $\langle V|S_x|V\rangle$ gives the The statistical average of measuring S_x given current state vector is $|V\rangle$? Resolved.

5. Can we just move the H operator to RHS, as in $x'' + Mx = 0$ instead of $x'' = -Mx$. This way no need to work with negative eigenvalues? Yes.

6. HW 5, last problem, I do not see how M, N share all the 3 eigenvectors. I get only one common eigenvector. I also do not understand the comment in my solution to refer to set of vectors as basis? What does this mean? Also, we know M, N commute, and so they share a common basis, but the question is asking which ones they share? Resolved.

7. For Pauli matrices, $\left[\sigma_i, \sigma_j\right] = 2i \sum \epsilon_{ijk}\sigma_k$. and for spin $\frac{1}{2}$ it is $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$. So what is it for spin 1? is it still $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$? Yes.

8. I think $\Psi(x,t)$ is just the eigenfunction corresponding to the eigenvalue just measured. So if the operator used is the position operator X, then it is called $\Psi(x)$. If

the operator used is momentum operator P, we call it $\phi_p(x)$, but should it be really be $\Psi_p(x)$? If the operator is Hamiltonian $\hat{H}$, then the eigenvalue is the energy level E and the Ψ is called $\Psi_E(x)$. Any of these are also called the wave function $\Psi(x)$. Is this correct? I think so.

5.24 Appendix