HW 7

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 1 (9.7.3)

(a) Expand $f(x) = \begin{cases} \frac{2xh}{L} & 0 \le x \le \frac{L}{2} \\ \frac{2h(L-x)}{L} & \frac{L}{2} \le x \le L \end{cases}$ in an exponential Fourier series. (b) What do you think is the value of $\sum_{odd} \frac{1}{m^2}$ where sum is over all positive odd integers?

<u>Solution</u>

1.1 Part a

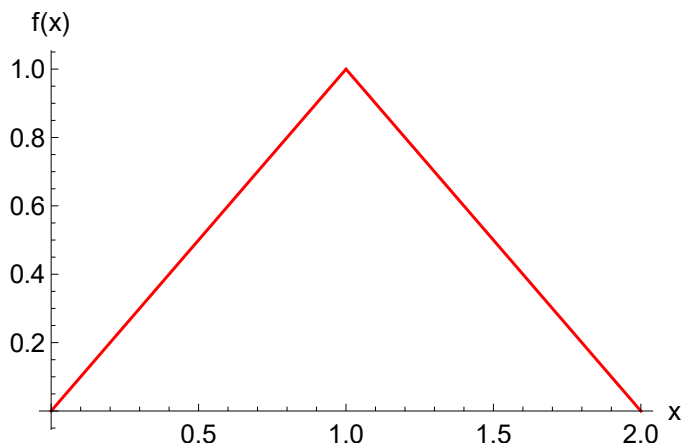The period is L. The function $f(x)$ looks like the following (where L is choosing to be 2 and $h = 1$, for illustration only)

Figure 1: Plot of $f(x)$

```
L = 2; h = 1;
f[x_] := Piecewise[{{(2 x h)/L, 0 ≤ x ≤ L/2}, {(2 h (L - x))/L, L/2 ≤ x ≤ L}}];
p = Plot[f[x], {x, 0, L}, AxesLabel → {"x", "f(x)"}, BaseStyle → 14, PlotStyle → Red];
```

Figure 2: Code used to generate the plot

The period is L. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle = \sum_m f_m |m\rangle \tag{1}$$

Where

$$f_m = \langle m | f \rangle$$
$$= \int_0^L \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x) dx \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle = \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \qquad m = 0, \pm 1, \pm 2, \cdots$$

Putting these together gives

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found

$$f_m = \int_0^L \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x) dx$$

$$= \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx \tag{4}$$

For $m = 0$

$$f_0 = \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} dx$$

$$= \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x dx + \frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^L (L-x) dx$$

$$= \frac{2h}{L\sqrt{L}} \left(\int_0^{\frac{L}{2}} x dx + \int_{\frac{L}{2}}^L (L-x) dx \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\left(\frac{x^2}{2} \right)_0^{\frac{L}{2}} + \left(Lx - \frac{x^2}{2} \right)_{\frac{L}{2}}^L \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{1}{2}\frac{L^2}{4} + \left(L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{1}{2}\frac{L^2}{4} \right) \right)$$

$$= \frac{\sqrt{L}h}{2} \tag{5}$$

And for $m \neq 0$, the first integral in (4) is

$$\int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x e^{-i\frac{2\pi m}{L}x} dx$$

Integration by parts. Let $u = x, dv = e^{-i\frac{2\pi m}{L}x}$, then $du = 1, v = \frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}} = \frac{iL}{2\pi m} e^{-i\frac{2\pi m}{L}x}$. The

above now becomes

$$\int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = \frac{2h}{L\sqrt{L}} \left(\frac{iL}{2\pi m} \left[xe^{-i\frac{2\pi m}{L}x} \right]_0^{\frac{L}{2}} - \int_0^{\frac{L}{2}} \frac{iL}{2\pi m} e^{-i\frac{2\pi m}{L}x} dx \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL}{2\pi m} \left[\frac{L}{2} e^{-i\frac{2\pi m}{L}\frac{L}{2}} \right] - \frac{iL}{2\pi m} \int_0^{\frac{L}{2}} e^{-i\frac{2\pi m}{L}x} dx \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL}{2\pi m} \left[\frac{L}{2} e^{-i\pi m} \right] - \frac{iL}{2\pi m} \left[\frac{iL}{2\pi m} e^{-i\frac{2\pi m}{L}x} \right]_0^{\frac{L}{2}} \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL}{2\pi m} \left[\frac{L}{2}(-1)^m \right] - \frac{i^2 L^2}{4\pi^2 m^2} \left[e^{-i\frac{2\pi m}{L}x} \right]_0^{\frac{L}{2}} \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2 m^2} \left[e^{-i\frac{2\pi m}{L}\frac{L}{2}} - 1 \right] \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2 m^2} \left[e^{-i2\pi m} - 1 \right] \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2 m^2} \left((-1)^m - 1 \right) \right)$$

$$= \frac{2h}{L\sqrt{L}} \left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2 m^2}(-1)^m - \frac{L^2}{4\pi^2 m^2} \right)$$

$$= \frac{i2hL^2}{L\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL^2}{L\sqrt{L}\,4\pi^2 m^2}(-1)^m - \frac{2hL^2}{L\sqrt{L}\,4\pi^2 m^2}$$

$$= \frac{i2hL}{\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL}{\sqrt{L}\,4\pi^2 m^2}(-1)^m - \frac{2hL}{\sqrt{L}\,4\pi^2 m^2}$$

$$= \frac{i2h\sqrt{L}}{4\pi m}(-1)^m + \frac{2h\sqrt{L}}{4\pi^2 m^2}(-1)^m - \frac{2h\sqrt{L}}{4\pi^2 m^2}$$

$$= \frac{ih\sqrt{L}}{2\pi m}(-1)^m + \frac{h\sqrt{L}}{2\pi^2 m^2}(-1)^m - \frac{h\sqrt{L}}{2\pi^2 m^2}$$

$$= \frac{ih\sqrt{L}\,\pi m(-1)^m}{2\pi^2 m^2} + \frac{h\sqrt{L}\,(-1)^m}{2\pi^2 m^2} - \frac{h\sqrt{L}}{2\pi^2 m^2}$$

$$= \frac{ih\sqrt{L}\,\pi m(-1)^m + h\sqrt{L}\,(-1)^m - h\sqrt{L}}{2\pi^2 m^2}$$

Hence

$$\int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = h\sqrt{L} \frac{\left(i\pi m(-1)^m + (-1)^m - 1 \right)}{2\pi^2 m^2} \tag{6}$$

Now the second integral in (4) is evaluated

$$\int_{\frac{L}{2}}^{L} \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = \frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^{L} (L-x) e^{-i\frac{2\pi m}{L}x} dx$$

Integration by parts. Let $u = L - x, dv = e^{-i\frac{2\pi m}{L}x}, du = -1, v = v = \frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}} = \frac{iL}{2\pi m} e^{-i\frac{2\pi m}{L}x}$. The

integral becomes

$$\frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^{L} (L-x)e^{-i\frac{2\pi m}{L}x}dx = \frac{2h}{L\sqrt{L}}\left(\left[(L-x)e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L} + \int_{\frac{L}{2}}^{L}\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right)$$

$$= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[0 - \frac{L}{2}e^{-i\frac{2\pi m}{L}\frac{L}{2}}\right] + \frac{iL}{2\pi m}\int_{\frac{L}{2}}^{L}e^{-i\frac{2\pi m}{L}x}\right)$$

$$= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}\left[e^{-i\pi m}\right] + \frac{iL}{2\pi m}\left[\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)$$

$$= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m + \frac{i^2L^2}{4\pi^2 m^2}\left[e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)$$

$$= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m - \frac{L^2}{4\pi^2 m^2}\left[e^{-i2\pi m} - e^{-i\pi m}\right]\right)$$

$$= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m - \frac{L^2}{4\pi^2 m^2}\left[1 - (-1)^m\right]\right)$$

$$= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m - \frac{2hL}{4\sqrt{L}\,\pi^2 m^2}\left[1 - (-1)^m\right]$$

$$= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m - \frac{2hL}{4\sqrt{L}\,\pi^2 m^2} - (-1)^m\frac{2hL}{4\sqrt{L}\,\pi^2 m^2}$$

$$= \frac{-i2h\pi mL}{4\pi^2 m^2}(-1)^m - \frac{2hL}{4\pi^2 m^2} - (-1)^m\frac{2h\sqrt{L}}{4\pi^2 m^2}$$

$$= \frac{-i2h\pi mL(-1)^m - 2h\sqrt{L} - 2h\sqrt{L}(-1)^m}{4\pi^2 m^2}$$

Therefore

$$\frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^{L} (L-x)e^{-i\frac{2\pi m}{L}x}dx = h\sqrt{L}\,\frac{\left(-i\pi m(-1)^m - (-1)^m - 1\right)}{2\pi^2 m^2} \tag{7}$$

Therefore, using (5,6,7) gives

$$f_m = \begin{cases} \dfrac{\sqrt{L}h}{2} & m = 0 \\[2mm] h\sqrt{L}\,\dfrac{\left(i\pi m(-1)^m + (-1)^m - 1\right)}{2\pi^2 m^2} + h\sqrt{L}\,\dfrac{\left(-i\pi m(-1)^m - (-1)^m - 1\right)}{2\pi^2 m^2} & m \neq 0 \end{cases}$$

The above can be simplified more to

$$f_m = \begin{cases} \dfrac{\sqrt{L}h}{2} & m = 0 \\[2mm] h\sqrt{L}\,\dfrac{(-1)^m - 1 - (-1)^m - 1}{2\pi^2 m^2} & m \neq 0 \end{cases}$$

Now, for $m = \pm 2, \pm 4, \cdots$ even, the above becomes

$$f_{m_{even}} = \begin{cases} \dfrac{\sqrt{L}h}{2} & m = 0 \\[2mm] 0 & m \neq 0, even \end{cases}$$

And for or $m = \pm 1, \pm 3, \cdots$ odd, it becomes

$$f_{m_{odd}} = \begin{cases} \dfrac{\sqrt{L}h}{2} & m = 0 \\[2mm] -h\sqrt{L}\,\dfrac{2}{\pi^2 m^2} & m \neq 0, odd \end{cases}$$

Therefore <u>only the odd terms</u> survive. From (3)

$$
\begin{aligned}
f(x) &\sim \sum_{odd} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{\sqrt{L}\,h}{2}\frac{1}{\sqrt{L}} + \sum_{odd} \frac{-2h\sqrt{L}}{\pi^2 m^2}\frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{h}{2} - h\sum_{odd} \frac{2}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x} \\
&= \frac{h}{2} - \frac{h}{2}\sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x}
\end{aligned}
$$

Or

$$
f(x) \sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x}\right) \tag{8}
$$

1.2   Part b

From Eq (8), by letting $x = \frac{L}{2}$, it becomes

$$
\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\pi m}\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{n=-\infty,odd}^{-1} \frac{4}{\pi^2 n^2} e^{i\pi n} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)
\end{aligned}
$$

Replacing $m = -n$ in the first sum above gives

$$
\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \left(\sum_{m=\infty,odd}^{1} \frac{4}{\pi^2 (-m)^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{m=1,odd}^{\infty} \frac{4}{\pi^2 m^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)
\end{aligned}
$$

Combining the terms and calling the common index n gives

$$
\begin{aligned}
f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{4}{\pi^2 m^2}\left(e^{i\pi n} + e^{-i\pi n}\right)\right)\right) \\
&= \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2} \cos(n\pi)\right)\right)
\end{aligned}
$$

But $\cos(\pi n) = -1$ since n and odd. The above becomes

$$
f\left(x = \frac{L}{2}\right) \sim \frac{h}{2}\left(1 + \sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2}\right)
$$

but $f\left(x = \frac{L}{2}\right) = \left[\frac{2xh}{L}\right]_{x=\frac{L}{2}} = h$. Hence the above becomes

$$
h = \frac{h}{2}\left(1 + \sum_{odd} \frac{8}{\pi^2 m^2}\right)
$$

$$
2 = 1 + \sum_{odd} \frac{8}{\pi^2 m^2}
$$

$$
1 = \frac{8}{\pi^2}\sum_{odd} \frac{1}{m^2}
$$

Therefore

$$
\sum_{odd} \frac{1}{m^2} = \frac{\pi^2}{8}
$$

2   Problem 2 (9.7.8)

(i) Obtain the series in terms of sines and cosine for $f(x) = e^{-|x|}$ in the interval $-1 \leq x \leq 1$.
(ii) repeat for the case $f(x) = \cosh x$. Show that

$$f(x) \sim \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin x)\right)$$

represents e^x in the interval $-\pi \leq x \leq \pi$ (and its periodicized version outside). Show how you can get the series for $\sinh x$ and $\cosh x$ from the above.

solution

2.1   Part 1

The period is $L = 2$ in this case. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle = \sum_m f_m|m\rangle \tag{1}$$

Where

$$f_m = \langle m|f\rangle$$
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}f(x)dx \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle = \frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x} \qquad m = 0, \pm 1, \pm 2, \cdots$$

Putting these together gives

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found, using $L = 2$

$$f_m = \int_{-1}^{1} \frac{1}{\sqrt{2}}e^{-i\pi mx}f(x)dx$$
$$= \frac{1}{\sqrt{2}}\int_{-1}^{0} e^{-i\pi mx}e^x dx + \frac{1}{\sqrt{2}}\int_{0}^{1} e^{-i\pi mx}e^{-x}dx \tag{44}$$

The first integral in (4) gives

$$\frac{1}{\sqrt{2}}\int_{-1}^{0} e^{-i\pi mx}e^x dx = \frac{1}{\sqrt{2}}\int_{-1}^{0} e^{(-i\pi m+1)x}dx$$
$$= \frac{1}{\sqrt{2}}\left[\frac{e^{(-i\pi m+1)x}}{-i\pi m + 1}\right]_{-1}^{0}$$
$$= \frac{1}{\sqrt{2}(-i\pi m + 1)}\left[e^{(-i\pi m+1)x}\right]_{-1}^{0}$$
$$= \frac{1}{\sqrt{2}(-i\pi m + 1)}\left[1 - e^{-(-i\pi m+1)}\right]$$
$$= \frac{1}{\sqrt{2}}\frac{e^{i\pi m-1} - 1}{i\pi m - 1} \tag{5}$$

And the second integral in (4) gives

$$\frac{1}{\sqrt{2}} \int_0^1 e^{-i\pi mx} e^{-x} dx = \frac{1}{\sqrt{2}} \int_0^1 e^{(-i\pi m - 1)x} dx$$

$$= \frac{1}{\sqrt{2}} \left[\frac{e^{(-i\pi m - 1)x}}{-i\pi m - 1} \right]_0^1$$

$$= \frac{1}{\sqrt{2}(-i\pi m - 1)} \left[e^{(-i\pi m - 1)x} \right]_0^1$$

$$= \frac{-1}{\sqrt{2}(i\pi m + 1)} \left[e^{(-i\pi m - 1)} - 1 \right]$$

$$= \frac{1}{\sqrt{2}} \frac{1 - e^{-i\pi m - 1}}{i\pi m + 1} \tag{6}$$

Putting (5,6) together gives

$$f_m = \frac{1}{\sqrt{2}} \frac{e^{i\pi m - 1} - 1}{i\pi m - 1} + \frac{1}{\sqrt{2}} \frac{1 - e^{-i\pi m - 1}}{i\pi m + 1} \tag{7}$$

For $m = 0$, eq (7) becomes

$$f_m = \frac{1}{\sqrt{2}} \left(\frac{e^{-1} - 1}{-1} \right) + \frac{1}{\sqrt{2}} \left(1 - e^{-1} \right)$$

$$= \frac{1}{\sqrt{2}} \left(1 - e^{-1} \right) + \frac{1}{\sqrt{2}} \left(1 - e^{-1} \right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}e} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}e}$$

$$= \frac{1}{\sqrt{2}} \left(2 - \frac{2}{e} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{2e - 2}{e} \right)$$

$$= \frac{2}{\sqrt{2}} \left(\frac{e - 1}{e} \right)$$

And for $m \neq 0$, eq (7) becomes

$$f_m = \frac{1}{\sqrt{2}} \left(\frac{e^{i\pi m - 1} - 1}{i\pi m - 1} + \frac{1 - e^{-i\pi m - 1}}{i\pi m + 1} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{e^{i\pi m} e^{-1} - 1}{i\pi m - 1} + \frac{1 - e^{-i\pi m} e^{-1}}{i\pi m + 1} \right)$$

Since m is integer, then the above becomes

$$f_m = \frac{1}{\sqrt{2}} \left(\frac{\cos(\pi m) e^{-1} - 1}{i\pi m - 1} + \frac{1 - \cos(\pi m) e^{-1}}{i\pi m + 1} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{(-1)^m e^{-1} - 1}{i\pi m - 1} + \frac{1 - (-1)^m e^{-1}}{i\pi m + 1} \right)$$

$$= \frac{1}{\sqrt{2}} \frac{\left((-1)^m e^{-1} - 1 \right)(i\pi m + 1) + \left(1 - (-1)^m e^{-1} \right)(i\pi m - 1)}{(i\pi m - 1)(i\pi m + 1)}$$

$$= \frac{1}{\sqrt{2}} \frac{2(-1)^m e^{-1} - 2}{-\pi^2 m^2 - 1}$$

$$= \frac{1}{\sqrt{2}} \frac{2 - 2(-1)^m e^{-1}}{1 + \pi^2 m^2}$$

$$= \frac{2}{\sqrt{2}} \frac{1 - (-1)^m e^{-1}}{1 + \pi^2 m^2}$$

Hence (3) becomes

$$f(x) \sim \frac{2}{\sqrt{2}} \left(\frac{e-1}{e} \right) \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{2}{\sqrt{2}} \frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2} e^{i\pi m x}$$

$$= \frac{e-1}{e} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2} e^{i\pi m x}$$

$$= \frac{e-1}{e} + \sum_{n=-\infty}^{-1} \frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2} e^{i\pi n x} + \sum_{k=1}^{\infty} \frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2} e^{i\pi k x}$$

Let $m = -n$ in the first sum above. This gives

$$f(x) = \frac{e-1}{e} + \sum_{m=\infty}^{1} \frac{1-(-1)^{(-m)} e^{-1}}{1+\pi^2 (-m)^2} e^{i\pi(-m)x} + \sum_{k=1}^{\infty} \frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2} e^{i\pi k x}$$

$$= \frac{e-1}{e} + \sum_{m=1}^{\infty} \frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2} e^{-i\pi m x} + \sum_{k=1}^{\infty} \frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2} e^{i\pi k x}$$

Now the two sums can be combined using one index, say n, since they sum over the same interval

$$f(x) = \frac{e-1}{e} + \sum_{n=\infty}^{1} \frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2} \left(e^{-i\pi n x} + e^{i\pi n x} \right)$$

But $\left(e^{-i\pi n x} + e^{i\pi n x} \right) = 2\cos \pi n x$. The above becomes

$$f(x) = \frac{e-1}{e} + 2\sum_{n=\infty}^{1} \frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2} \cos \pi n x$$

$$= \frac{e-1}{e} + 2\sum_{n=\infty}^{1} \frac{e-(-1)^n}{e\left(1+\pi^2 n^2\right)} \cos \pi n x$$

2.2 Part 2

Now $f(x) = \cosh x$. Therefore

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Where now f_m is found, using $L = 2$

$$f_m = \int_{-1}^{1} \frac{1}{\sqrt{2}} e^{-i\pi m x} \cosh(x) dx$$

But $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The above becomes

$$f_m = \int_{-1}^{1} \frac{1}{\sqrt{2}} e^{-i\pi m x} \frac{1}{2}(e^x + e^{-x}) dx$$

$$= \frac{1}{2\sqrt{2}} \left(\int_{-1}^{1} e^{-i\pi m x} e^x dx + \int_{-1}^{1} e^{-i\pi m x} e^{-x} dx \right)$$

$$= \frac{1}{2\sqrt{2}} \left(\int_{-1}^{1} e^{(-i\pi m+1)x} dx + \int_{-1}^{1} e^{(-i\pi m-1)x} dx \right) \tag{4}$$

The first integral is

$$\int_{-1}^{1} e^{(-i\pi m+1)x} dx = \frac{1}{-i\pi m+1} \left[e^{(-i\pi m+1)x} \right]_{-1}^{1}$$

$$= \frac{1}{1-i\pi m} \left(e^{(-i\pi m+1)} - e^{-(-i\pi m+1)} \right)$$

$$= \frac{1}{1-i\pi m} \left(e^{-i\pi m} e - e^{i\pi m} e^{-1} \right)$$

Since m is integer, then $e^{-i\pi m} = (-1)^m$ and $e^{i\pi m} = (-1)^m$. The above becomes

$$\int_{-1}^{1} e^{(-i\pi m+1)x} = \frac{(-1)^m}{1 - i\pi m}\left(e - e^{-1}\right) \tag{5}$$

The second integral in (4) becomes

$$\int_{-1}^{1} e^{(-i\pi m-1)x}dx = \frac{1}{-i\pi m - 1}\left[e^{(-i\pi m-1)x}\right]_{-1}^{1}$$

$$= \frac{-1}{1 + i\pi m}\left(e^{(-i\pi m-1)} - e^{-(-i\pi m-1)}\right)$$

$$= \frac{-1}{1 + i\pi m}\left(e^{-i\pi m}e^{-1} - e^{i\pi m}e\right)$$

Since m is integer, the above becomes

$$\int_{-1}^{1} e^{(-i\pi m-1)x}dx = \frac{-(-1)^m}{1 + i\pi m}\left(e^{-1} - e\right) \tag{6}$$

Substituting (5,6) in (4) gives

$$f_m = \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1 - i\pi m}\left(e - e^{-1}\right) + \frac{-(-1)^m}{1 + i\pi m}\left(e^{-1} - e\right)\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1 - i\pi m}\left(e - e^{-1}\right) + \frac{(-1)^m}{1 + i\pi m}\left(e - e^{-1}\right)\right)$$

$$= \frac{(-1)^m\left(e - e^{-1}\right)}{2\sqrt{2}}\left(\frac{1}{1 - i\pi m} + \frac{1}{1 + i\pi m}\right)$$

$$= \frac{(-1)^m\left(e - e^{-1}\right)}{2\sqrt{2}}\left(\frac{(1 + i\pi m) + (1 - i\pi m)}{(1 - i\pi m)(1 + i\pi m)}\right)$$

$$= \frac{(-1)^m\left(e - e^{-1}\right)}{2\sqrt{2}}\left(\frac{2}{\pi^2 m^2 + 1}\right)$$

$$= \frac{(-1)^m\left(e - e^{-1}\right)}{\sqrt{2}\left(\pi^2 m^2 + 1\right)}$$

Hence for $m = 0$,

$$f_0 = \frac{\left(e - e^{-1}\right)}{\sqrt{2}}$$

Therefore $f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x}$ becomes

$$\cosh(x) \sim \frac{\left(e - e^{-1}\right)}{\sqrt{2}}\frac{1}{\sqrt{2}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(-1)^m\left(e - e^{-1}\right)}{\sqrt{2}\left(\pi^2 m^2 + 1\right)}\frac{1}{\sqrt{2}}e^{i\pi mx}$$

$$= \frac{\left(e - e^{-1}\right)}{2} + \frac{1}{2}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(-1)^m\left(e - e^{-1}\right)}{\left(\pi^2 m^2 + 1\right)}e^{i\pi mx}$$

As was done in part(i), $\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} e^{i\pi mx}$ can be rewritten as $\sum_{n=1}^{\infty} 2\cos(n\pi x)$. The above reduces to

$$\cosh(x) \sim \frac{\left(e - e^{-1}\right)}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n\left(e - e^{-1}\right)}{\left(\pi^2 n^2 + 1\right)}\cos(n\pi x)$$

But $\frac{\left(e - e^{-1}\right)}{2} = \sinh 1$. Therefore the above becomes

$$\cosh(x) \sim \sinh(1)\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(1 + \pi^2 n^2\right)}\cos(n\pi x)\right)$$

2.3 Part 3

I think now the book is asking to find the Fourier series for e^x over $-\pi \leq x \leq \pi$ in this last part. Therefore, as before, starting with

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{1}$$

Where now, using $L = 2\pi$ as the period, then

$$f_m = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{-i\frac{2\pi m}{2\pi}x} e^x dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{(-im+1)x} dx$$

$$= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{(-im+1)x} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{(-im+1)\pi} - e^{-(-im+1)\pi} \right]$$

$$= \frac{1}{\sqrt{2\pi}(1-im)} \left[e^{-im\pi} e^{\pi} - e^{im\pi} e^{-\pi} \right]$$

But $e^{im\pi} = (-1)^m$ and $e^{-im\pi} = (-1)^m$ since m is integer. The above becomes

$$f_m = \frac{(-1)^m}{\sqrt{2\pi}(1-im)} [e^{\pi} - e^{-\pi}]$$

$$= \frac{(-1)^m (2\sinh\pi)}{\sqrt{2\pi}(1-im)}$$

$$= \frac{2}{\sqrt{2\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi$$

For $m = 0$ the above gives

$$f_0 = \frac{\sqrt{2}}{\sqrt{\pi}} \sinh\pi$$

Therefore $f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x}$ becomes, where $L = 2\pi$ now,

$$e^x \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \sinh\pi \left(\frac{1}{\sqrt{2\pi}} \right) + \sum_{m=-\infty,\neq 0}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi \right) \frac{1}{\sqrt{2\pi}} e^{imx}$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{(1-im)} e^{imx}$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m(1+im)}{(1-im)(1+im)} e^{imx}$$

$$= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m + i(-1)^m m}{1+m^2} e^{imx} \right)$$

$$= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \frac{i(-1)^m m}{1+m^2} e^{imx} \right)$$

$$= \frac{\sinh\pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} \right) \tag{2}$$

The first sum above becomes

$$\sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+m^2}\cos(nx) \tag{3}$$

And the second sum in (2) becomes

$$\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} = \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)(-1)^m m}{1+m^2} \frac{e^{imx}}{i}$$

$$= \sum_{k=-\infty}^{-1} \frac{(-1)(-1)^k k}{1+k^2} \frac{e^{ikx}}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r}{1+r^2} \frac{e^{irx}}{i}$$

Letting $m = -k$ in the first sum above gives

$$\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} = \sum_{m=\infty}^{1} \frac{(-1)(-1)^{-m}(-m)}{1+(-m)^2} \frac{e^{-imx}}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r}{1+r^2} \frac{e^{irx}}{i}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m m}{1+m^2} \frac{e^{-imx}}{i} - \sum_{r=1}^{\infty} \frac{(-1)^r r}{1+r^2} \frac{e^{irx}}{i}$$

Merging the two sums back together since now on same interval, and using n for the common index

$$\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{1+n^2}\left(\frac{e^{-inx}}{i} - \frac{e^{inx}}{i}\right)$$

$$= -\sum_{n=1}^{\infty} 2\frac{(-1)^n n}{1+n^2}(\sin(nx)) \tag{4}$$

Substituting (3,4) back in (2) gives

$$e^x \sim \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\cos(nx) - \sum_{n=1}^{\infty} 2\frac{(-1)^n n}{1+n^2}(\sin(nx))\right)$$

$$= \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) \tag{5}$$

The question is now asking to show how to use (5) to obtain the series for $\sinh x$ and $\cosh x$. Since

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$\sinh x \sim \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(n(-x)) - n\sin(n(-$$

$$= \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\text{ s}\right.\right.$$

$$= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \frac{\sinh \pi}{\pi} - 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\text{ si}\right.$$

$$= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right)$$

$$= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx) - \cos(nx) - n\sin(nx))\right)$$

$$= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-n\sin(nx) - n\sin(nx))\right)$$

$$= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-2n\sin(nx))\right)$$

Hence

$$\sinh x \sim \frac{2\sinh\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+n^2} n \sin(nx) \tag{6}$$

Similarly for

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$\cosh x \sim \frac{1}{2}\left(\frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) + \frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(n(-x)) - n\sin(n(-$$

$$= \frac{1}{2}\left(\frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) + \frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) + \left(\frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\,$$

$$= \frac{1}{2}\left(\frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) + \frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\,\text{si}$$

$$= \frac{1}{2}\left(2\frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx) + \cos(nx) + n\sin(nx))\right)$$

$$= \frac{1}{2}\left(2\frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + \cos(nx))\right)$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}2\cos(nx)$$

Hence

$$\cosh x \sim \frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\cos(nx)\right) \tag{7}$$

3 Problem 3

Perform appropriate integration to show the following results regarding the Dirac delta function

(1) $\delta(ax) = \frac{\delta(x)}{|a|}$ where a is real number. (2) $\delta\big(f(x)\big) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_i}$ where x_i satisfies $f(x_i) = 0$

(3) $\frac{d}{dx}\delta(x - x') = \delta(x - x')\frac{d}{dx'}$

Solution

3.1 Part (1)

Using the integral definition of delta function given by

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \tag{1}$$

Then

$$\delta(ax) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikax} dk$$

<u>Case $a > 0$.</u> Let $u = ak$. Then $du = adk$. The above becomes

$$\delta(ax) = \frac{1}{a}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} du\right)$$

But $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} du = \delta(x)$ by definition. Hence the above becomes

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{2}$$

<u>Case $a < 0$.</u> Let $u = ak$. Then $du = adk$. When $k = \infty, u = -\infty$ and when $k = -\infty, u = +\infty$. The integral becomes

$$\begin{aligned}
\delta(ax) &= \frac{1}{a}\left(\frac{1}{2\pi} \int_{\infty}^{-\infty} e^{iux} du\right) \\
&= \frac{1}{-a}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} du\right) \\
&= \frac{1}{-a}\delta(x) \tag{3}
\end{aligned}$$

Combining (2,3) gives

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

3.2 Part (2)

Using

$$\int_{-\infty}^{\infty} \delta\big(f(x)\big) dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\big(f(x)\big) dx \tag{1}$$

Where in the RHS, the sum is over the roots of $f(x)$, where $f(x_i) = 0$ where x_i is root of $f(x)$ since $\delta(u)$ is nonzero only when its argument is zero, which is at the roots of $f(x)$. Now, expanding $f(x)$ near each one of its roots using Taylor series

$$f(x) = f(x_i) + (x - x_i)f'(x_i) + O\big(x^2\big)$$

But $f(x_i) = 0$ since x_i is root, and keeping only linear terms, then (1) now becomes

$$\int_{-\infty}^{\infty} \delta\big(f(x)\big) dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\big((x - x_i)f'(x_i)\big) dx$$

But from part (1), we found that $\delta(a(x - x_i)) = \frac{1}{|a|}\delta(x - x_i)$, where now $a = f'(x_i)$. Using this relation in the above gives

$$\int_{-\infty}^{\infty} \delta\big(f(x)\big)dx = \int_{-\infty}^{\infty} \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$$

Therefore the integrands on each side is the same. This implies

$$\delta\big(f(x)\big) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

3.3 Part (3)

Starting from property of delta function which is

$$\int \delta(x - x')f(x')dx' = f(x)$$

Taking derivative of both sides w.r.t. x gives

$$\frac{d}{dx}\int \delta(x - x')f(x')dx' = \frac{d}{dx}f(x)$$
$$\int \frac{d\delta(x - x')}{dx}f(x')dx' = \frac{d}{dx}f(x)$$

Integration by part. Let $\frac{d\delta(x-x')}{dx} = dv, u = f(x')$, then $v = (x - x'), du = \frac{d}{dx'}f(x')$. The above becomes

$$\int \delta(x - x')\frac{d}{dx'}f(x')dx' = \frac{d}{dx}f(x)$$

Therefore

$$\int \frac{d\delta(x - x')}{dx}f(x')dx' = \int \delta(x - x')\frac{d}{dx'}f(x')dx'$$

or

$$\frac{d\delta(x - x')}{dx}f(x') = \delta(x - x')\frac{d}{dx'}f(x')$$

or

$$\frac{d\delta(x - x')}{dx} = \delta(x - x')\frac{d}{dx'}$$

4 Problem 4

For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured.

<u>Solution</u>

The goal is to determine $\left|\langle \phi_p | \psi \rangle\right|^2$ which will give the probability of measuring momentum p. But

$$\langle \phi_p | \psi_n \rangle = \int_0^\infty \langle \phi_p | x \rangle \langle x | \psi_n \rangle dx$$

$$= \int_0^\infty \langle x | \phi_p \rangle^* \langle x | \psi_n \rangle dx \tag{1}$$

But $\langle x | \phi_p \rangle = \phi_p(x)$ and $\langle x | \psi_n \rangle = \psi_n(x)$. From lecture notes,

$$\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

For $n = 1, 2, 3, \cdots$. Substituting the above in (1) gives

$$\langle \phi_p | \psi_n \rangle = \int_0^\infty \phi_p^*(x) \psi_n(x) dx$$

$$= \int_0^L \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{px}{\hbar}} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} dx$$

$$= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L e^{-i\frac{px}{\hbar}} \sin\left(\frac{n\pi x}{L}\right) dx \tag{2}$$

To evaluate $I = \int_0^L e^{-i\frac{px}{\hbar}} \sin\left(\frac{n\pi x}{L}\right) dx$ we do Integration by parts twice. Let $u = \sin\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du = \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) dx$ and $v = \hbar \frac{e^{-i\frac{px}{\hbar}}}{-ip}$. Hence

$$I = [uv]_0^L - \int_0^L v du$$

$$= \left[\sin\left(\frac{n\pi x}{L}\right) \hbar \frac{e^{-i\frac{px}{\hbar}}}{-ip} \right]_0^L - \int_0^L \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) \hbar \frac{e^{-i\frac{px}{\hbar}}}{-ip} dx$$

$$= \left[\sin\left(\frac{n\pi L}{L}\right) \hbar \frac{e^{-i\frac{pL}{\hbar}}}{-ip} - 0 \right] + \frac{\hbar n\pi}{ipL} \int_0^L \cos\left(\frac{n\pi x}{L}\right) e^{-i\frac{px}{\hbar}} dx$$

Since n is integer, then boundary terms are zero.

$$I = \frac{\hbar n\pi}{ipL} \int_0^L \cos\left(\frac{n\pi x}{L}\right) e^{-i\frac{px}{\hbar}} dx$$

Doing integration by parts one more time. Let $u = \cos\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du =$

$-\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx$, then the above becomes

$$I = \frac{\hbar n\pi}{ipL}\left(\left[\cos\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}\right]_0^L + \int_0^L \frac{\hbar e^{-i\frac{px}{\hbar}}}{-ip}\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{\hbar n\pi}{ipL}\left(\frac{\hbar}{-ip}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] - \frac{\hbar n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{\hbar^2}{-ip}\frac{n\pi}{ipL}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] - \frac{n\pi}{ipL}\frac{\hbar^2 n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{\hbar^2 n\pi}{p^2 L}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] + \frac{\hbar^2 n^2\pi^2}{p^2 L^2}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx$$

But $\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx = I$. Therefore the above becomes

$$I = \frac{\hbar^2 n\pi}{p^2 L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right) + \frac{\hbar^2 n^2\pi^2}{p^2 L^2}I$$

Solving for I

$$I - \frac{\hbar^2 n^2\pi^2}{p^2 L^2}I = \frac{\hbar^2 n\pi}{p^2 L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)$$

$$I\left(1 - \frac{\hbar^2 n^2\pi^2}{p^2 L^2}\right) = \frac{\hbar^2 n\pi}{p^2 L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)$$

$$I = \frac{\hbar^2 n\pi}{p^2 L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)}{\left(1 - \frac{\hbar^2 n^2\pi^2}{p^2 L^2}\right)}$$

$$= p^2 L^2\frac{\hbar^2 n\pi}{p^2 L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)}{\left(p^2 L^2 - \hbar^2 n^2\pi^2\right)}$$

$$= \frac{\hbar^2 n\pi L}{p^2 L^2 - \hbar^2 n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}} - 1\right)$$

Substituting the above in (2) gives

$$\langle\phi_p|\psi_n\rangle = \frac{1}{\sqrt{\pi\hbar L}}\left(\frac{\hbar^2 n\pi L}{p^2 L^2 - \hbar^2 n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}} - 1\right)\right)$$

$$= \frac{\hbar^2 n\pi L\left(\sqrt{\pi\hbar L}\right)}{(\pi\hbar L)\left(p^2 L^2 - \hbar^2 n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}} - 1\right)$$

$$= \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2 L^2 - \hbar^2 n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}} - 1\right)$$

Let $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2 L^2 - \hbar^2 n^2\pi^2\right)}$, then

$$\langle\phi_p|\psi_n\rangle = k_n\left((-1)^n e^{-\frac{ipL}{\hbar}} - 1\right)$$

$$= (-1)^n k_n e^{-\frac{ipL}{\hbar}} - k_n$$

$$= (-1)^n k_n\left(\cos\frac{pL}{\hbar} - i\sin\frac{pL}{\hbar}\right) - k_n$$

$$= (-1)^n k_n\cos\frac{pL}{\hbar} - i(-1)^n k_n\sin\frac{pL}{\hbar} - k_n$$

$$= \left((-1)^n k_n\cos\frac{pL}{\hbar} - k_n\right) - i\left((-1)^n k_n\sin\frac{pL}{\hbar}\right)$$

Hence

$$\left|\langle\phi_p|\psi_n\rangle\right| = \sqrt{\left((-1)^n k_n \cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n \sin\frac{pL}{\hbar}\right)^2}$$

And

$$\left|\langle\phi_p|\psi_n\rangle\right|^2 = \left((-1)^n k_n \cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n \sin\frac{pL}{\hbar}\right)^2$$

$$= (-1)^{2n} k_n^2 \cos^2\frac{pL}{\hbar} + k_n^2 - 2k_n^2(-1)^n \cos\frac{pL}{\hbar} + (-1)^{2n} k_n^2 \sin^2\frac{pL}{\hbar}$$

$$= (-1)^{2n} k_n^2 \left(\cos^2\frac{pL}{\hbar} + \sin^2\frac{pL}{\hbar}\right) + k_n^2 - 2k_n^2(-1)^n \cos\frac{pL}{\hbar}$$

$$= (-1)^{2n} k_n^2 + k_n^2 - 2k_n^2(-1)^n \cos\frac{pL}{\hbar}$$

$$= k_n^2\left(1 + (-1)^{2n} - 2(-1)^n \cos\frac{pL}{\hbar}\right)$$

But $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)}$, therefore the above becomes

$$\left|\langle\phi_p|\psi_n\rangle\right|^2 = \left(\frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)}\right)^2 \left(1 + (-1)^{2n} - 2(-1)^n \cos\frac{pL}{\hbar}\right)$$

$$= \frac{n^2\hbar^3\pi L}{\left(p^2L^2 - \hbar^2 n^2\pi^2\right)^2}\left(1 + (-1)^{2n} - 2(-1)^n \cos\left(\frac{pL}{\hbar}\right)\right)$$

The above gives the probability of measurement of p, where $n = 1, 2, 3, \cdots$. For illustration, the following two tables are generated to see how the probability of measuring say $p = 1$ and $p = 2$ changes as function of n. To generate this, L is taken as 1 and $\hbar = 1$ for simplicity.

n	Probability of measuring p=1
1	0.12302
2	0.00780329
3	0.0112922
4	0.00187694
5	0.00400658
6	0.00082832
7	0.00203605
8	0.000464782
9	0.00122967
10	0.000297121

Figure 3: Probability to measure $p = 1$

n	Probability of measuring p=2
1	0.106479
2	0.0282761
3	0.00458842
4	0.00600971
5	0.00155647
6	0.00259549
7	0.000781451
8	0.00144553
9	0.00046963
10	0.000920908

Figure 4: Probability to measure $p = 2$

```
p = 1; h = 1; L = 1;
```

$$f[p\_, n\_] := \frac{n^2\, h^3\, Pi\, L}{\left(p^2\, L^2 - h^2\, n^2\, Pi^2\right)^2} \left(1 + (-1)^{2n} - 2\,(-1)^n \cos\left[\frac{p\, L}{h}\right]\right)$$

```
data = Table[{n, N@f[p, n]}, {n, 1, 10}];
data = PrependTo[data, {"n", "Probability of measuring p=1"}];
Grid[data, Frame → All];
```

Figure 5: Code used