HW 7

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 1 (9.7.3)

(a) Expand $f(x) = \begin{cases} \frac{2xh}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2h(L-x)}{L} & \frac{L}{2} \leq x \leq L \end{cases}$ in an exponential Fourier series. (b) What do you think is the value of $\sum_{odd} \frac{1}{m^2}$ where sum is over all positive odd integers?

Solution

1.1 Part a

The period is L. The function $f(x)$ looks like the following (where L is choosing to be 2 and $h = 1$, for illustration only)

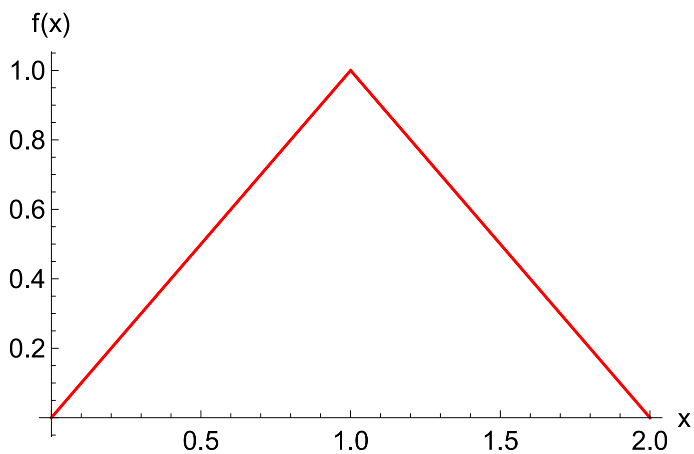

Figure 1: Plot of $f(x)$

```
L = 2; h = 1;
f[x_] := Piecewise[{{(2 x h)/L, 0 ≤ x ≤ L/2}, {(2 h (L - x))/L, L/2 ≤ x ≤ L}}];
p = Plot[f[x], {x, 0, L}, AxesLabel → {"x", "f(x)"}, BaseStyle → 14, PlotStyle → Red];
```

Figure 2: Code used to generate the plot

The period is L. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle = \sum_m f_m |m\rangle \tag{1}$$

Where

$$\begin{aligned} f_m &= \langle m|f\rangle \\ &= \int_0^L \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x)dx \end{aligned} \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle = \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \qquad m = 0, \pm 1, \pm 2, \cdots$$

Putting these together gives

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found

$$\begin{aligned} f_m &= \int_0^L \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x)dx \\ &= \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx \end{aligned} \tag{4}$$

For $m = 0$

$$\begin{aligned} f_0 &= \int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} dx + \int_{\frac{L}{2}}^L \frac{2h(L-x)}{L} \frac{1}{\sqrt{L}} dx \\ &= \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x dx + \frac{2h}{L\sqrt{L}} \int_{\frac{L}{2}}^L (L-x) dx \\ &= \frac{2h}{L\sqrt{L}} \left(\int_0^{\frac{L}{2}} x dx + \int_{\frac{L}{2}}^L (L-x) dx \right) \\ &= \frac{2h}{L\sqrt{L}} \left(\left(\frac{x^2}{2} \right)_0^{\frac{L}{2}} + \left(Lx - \frac{x^2}{2} \right)_{\frac{L}{2}}^L \right) \\ &= \frac{2h}{L\sqrt{L}} \left(\frac{1}{2}\frac{L^2}{4} + \left(L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{1}{2}\frac{L^2}{4} \right) \right) \\ &= \frac{\sqrt{L}h}{2} \end{aligned} \tag{5}$$

And for $m \neq 0$, the first integral in (4) is

$$\int_0^{\frac{L}{2}} \frac{2xh}{L} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} dx = \frac{2h}{L\sqrt{L}} \int_0^{\frac{L}{2}} x e^{-i\frac{2\pi m}{L}x} dx$$

Integration by parts. Let $u = x, dv = e^{-i\frac{2\pi m}{L}x}$, then $du = 1, v = \frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}} = \frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}$. The above now becomes

$$
\begin{aligned}
\int_0^{\frac{L}{2}} \frac{2xh}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx &= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[xe^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}} - \int_0^{\frac{L}{2}} \frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}dx\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}e^{-i\frac{2\pi m}{L}\frac{L}{2}}\right] - \frac{iL}{2\pi m}\int_0^{\frac{L}{2}} e^{-i\frac{2\pi m}{L}x}dx\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}e^{-i\pi m}\right] - \frac{iL}{2\pi m}\left[\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[\frac{L}{2}(-1)^m\right] - \frac{i^2L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}x}\right]_0^{\frac{L}{2}}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}\frac{L}{2}} - 1\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left[e^{-i2\pi m} - 1\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}\left((-1)^m - 1\right)\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL^2}{4\pi m}(-1)^m + \frac{L^2}{4\pi^2m^2}(-1)^m - \frac{L^2}{4\pi^2m^2}\right)\\
&= \frac{i2hL^2}{L\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL^2}{L\sqrt{L}\,4\pi^2m^2}(-1)^m - \frac{2hL^2}{L\sqrt{L}\,4\pi^2m^2}\\
&= \frac{i2hL}{\sqrt{L}\,4\pi m}(-1)^m + \frac{2hL}{\sqrt{L}\,4\pi^2m^2}(-1)^m - \frac{2hL}{\sqrt{L}\,4\pi^2m^2}\\
&= \frac{i2h\sqrt{L}}{4\pi m}(-1)^m + \frac{2h\sqrt{L}}{4\pi^2m^2}(-1)^m - \frac{2h\sqrt{L}}{4\pi^2m^2}\\
&= \frac{ih\sqrt{L}}{2\pi m}(-1)^m + \frac{h\sqrt{L}}{2\pi^2m^2}(-1)^m - \frac{h\sqrt{L}}{2\pi^2m^2}\\
&= \frac{ih\sqrt{L}\,\pi m(-1)^m}{2\pi^2m^2} + \frac{h\sqrt{L}\,(-1)^m}{2\pi^2m^2} - \frac{h\sqrt{L}}{2\pi^2m^2}\\
&= \frac{ih\sqrt{L}\,\pi m(-1)^m + h\sqrt{L}\,(-1)^m - h\sqrt{L}}{2\pi^2m^2}
\end{aligned}
$$

Hence

$$
\int_0^{\frac{L}{2}} \frac{2xh}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx = h\sqrt{L}\,\frac{\left(i\pi m(-1)^m + (-1)^m - 1\right)}{2\pi^2m^2} \tag{6}
$$

Now the second integral in (4) is evaluated

$$\int_{\frac{L}{2}}^{L}\frac{2h(L-x)}{L}\frac{1}{\sqrt{L}}e^{-i\frac{2\pi m}{L}x}dx=\frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx$$

Integration by parts. Let $u=L-x, dv=e^{-i\frac{2\pi m}{L}x}, du=-1, v=v=\frac{e^{-i\frac{2\pi m}{L}x}}{-i\frac{2\pi m}{L}}=\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}$. The integral becomes

$$\begin{aligned}
\frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx &= \frac{2h}{L\sqrt{L}}\left(\left[(L-x)e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}+\int_{\frac{L}{2}}^{L}\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{iL}{2\pi m}\left[0-\frac{L}{2}e^{-i\frac{2\pi m}{L}\frac{L}{2}}\right]+\frac{iL}{2\pi m}\int_{\frac{L}{2}}^{L}e^{-i\frac{2\pi m}{L}x}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}\left[e^{-i\pi m}\right]+\frac{iL}{2\pi m}\left[\frac{iL}{2\pi m}e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m+\frac{i^2L^2}{4\pi^2m^2}\left[e^{-i\frac{2\pi m}{L}x}\right]_{\frac{L}{2}}^{L}\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m-\frac{L^2}{4\pi^2m^2}\left[e^{-i2\pi m}-e^{-i\pi m}\right]\right)\\
&= \frac{2h}{L\sqrt{L}}\left(\frac{-iL^2}{4\pi m}(-1)^m-\frac{L^2}{4\pi^2m^2}\left[1-(-1)^m\right]\right)\\
&= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m-\frac{2hL}{4\sqrt{L}\,\pi^2m^2}\left[1-(-1)^m\right]\\
&= \frac{-i2hL}{\sqrt{L}\,4\pi m}(-1)^m-\frac{2hL}{4\sqrt{L}\,\pi^2m^2}-(-1)^m\frac{2hL}{4\sqrt{L}\,\pi^2m^2}\\
&= \frac{-i2h\pi mL}{4\pi^2m^2}(-1)^m-\frac{2hL}{4\pi^2m^2}-(-1)^m\frac{2h\sqrt{L}}{4\pi^2m^2}\\
&= \frac{-i2h\pi mL(-1)^m-2h\sqrt{L}-2h\sqrt{L}\,(-1)^m}{4\pi^2m^2}
\end{aligned}$$

Therefore

$$\frac{2h}{L\sqrt{L}}\int_{\frac{L}{2}}^{L}(L-x)e^{-i\frac{2\pi m}{L}x}dx=h\sqrt{L}\,\frac{\left(-i\pi m(-1)^m-(-1)^m-1\right)}{2\pi^2m^2}\tag{7}$$

Therefore, using (5,6,7) gives

$$f_m=\begin{cases}\dfrac{\sqrt{L}h}{2} & m=0\\ h\sqrt{L}\,\dfrac{\left(i\pi m(-1)^m+(-1)^m-1\right)}{2\pi^2m^2}+h\sqrt{L}\,\dfrac{\left(-i\pi m(-1)^m-(-1)^m-1\right)}{2\pi^2m^2} & m\neq 0\end{cases}$$

The above can be simplified more to

$$f_m = \begin{cases} \frac{\sqrt{L}h}{2} & m = 0 \\ h\sqrt{L}\,\frac{(-1)^m - 1 - (-1)^m - 1}{2\pi^2 m^2} & m \neq 0 \end{cases}$$

Now, for $m = \pm 2, \pm 4, \cdots$ even, the above becomes

$$f_{m_{even}} = \begin{cases} \frac{\sqrt{L}h}{2} & m = 0 \\ 0 & m \neq 0, even \end{cases}$$

And for or $m = \pm 1, \pm 3, \cdots$ odd, it becomes

$$f_{m_{odd}} = \begin{cases} \frac{\sqrt{L}h}{2} & m = 0 \\ -h\sqrt{L}\,\frac{2}{\pi^2 m^2} & m \neq 0, odd \end{cases}$$

Therefore <u>only the odd terms</u> survive. From (3)

$$\begin{aligned} f(x) &\sim \sum_{odd} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\ &= \frac{\sqrt{L}h}{2}\frac{1}{\sqrt{L}} + \sum_{odd} \frac{-2h\sqrt{L}}{\pi^2 m^2}\frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\ &= \frac{h}{2} - h\sum_{odd} \frac{2}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x} \\ &= \frac{h}{2} - \frac{h}{2}\sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x} \end{aligned}$$

Or

$$f(x) \sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\frac{2\pi m}{L}x}\right) \tag{8}$$

1.2 Part b

From Eq (8), by letting $x = \frac{L}{2}$, it becomes

$$\begin{aligned} f\left(x = \frac{L}{2}\right) &\sim \frac{h}{2}\left(1 - \sum_{odd} \frac{4}{\pi^2 m^2} e^{i\pi m}\right) \\ &= \frac{h}{2}\left(1 - \left(\sum_{n=-\infty,odd}^{-1} \frac{4}{\pi^2 n^2} e^{i\pi n} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right) \end{aligned}$$

Replacing $m = -n$ in the first sum above gives

$$f\left(x = \frac{L}{2}\right) \sim \frac{h}{2}\left(1 - \left(\sum_{m=\infty,odd}^{1} \frac{4}{\pi^2(-m)^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)$$

$$= \frac{h}{2}\left(1 - \left(\sum_{m=1,odd}^{\infty} \frac{4}{\pi^2 m^2} e^{-i\pi m} + \sum_{k=1,odd}^{\infty} \frac{4}{\pi^2 k^2} e^{i\pi k}\right)\right)$$

Combining the terms and calling the common index n gives

$$f\left(x = \frac{L}{2}\right) \sim \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{4}{\pi^2 m^2} \left(e^{i\pi n} + e^{-i\pi n}\right)\right)\right)$$

$$= \frac{h}{2}\left(1 - \left(\sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2} \cos(n\pi)\right)\right)$$

But $\cos(\pi n) = -1$ since n and odd. The above becomes

$$f\left(x = \frac{L}{2}\right) \sim \frac{h}{2}\left(1 + \sum_{n=1,odd}^{\infty} \frac{8}{\pi^2 m^2}\right)$$

but $f\left(x = \frac{L}{2}\right) = \left[\frac{2xh}{L}\right]_{x=\frac{L}{2}} = h$. Hence the above becomes

$$h = \frac{h}{2}\left(1 + \sum_{odd} \frac{8}{\pi^2 m^2}\right)$$

$$2 = 1 + \sum_{odd} \frac{8}{\pi^2 m^2}$$

$$1 = \frac{8}{\pi^2} \sum_{odd} \frac{1}{m^2}$$

Therefore

$$\sum_{odd} \frac{1}{m^2} = \frac{\pi^2}{8}$$

2 Problem 2 (9.7.8)

(i) Obtain the series in terms of sines and cosine for $f(x) = e^{-|x|}$ in the interval $-1 \leq x \leq 1$.
(ii) repeat for the case $f(x) = \cosh x$. Show that

$$f(x) \sim \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n \sin x)\right)$$

represents e^x in the interval $-\pi \leq x \leq \pi$ (and its periodicized version outside). Show how you can get the series for $\sinh x$ and $\cosh x$ from the above.

solution

2.1 Part 1

The period is $L = 2$ in this case. The exponential Fourier series for periodic $f(x)$ is given by the expansion

$$|f\rangle = \sum_m f_m |m\rangle \tag{1}$$

Where

$$\begin{aligned} f_m &= \langle m|f\rangle \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{\sqrt{L}} e^{-i\frac{2\pi m}{L}x} f(x)dx \end{aligned} \tag{2}$$

And $|m\rangle$ are the basis functions given by

$$|m\rangle = \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \qquad m = 0, \pm 1, \pm 2, \cdots$$

Putting these together gives

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Now f_m is found, using $L = 2$

$$\begin{aligned} f_m &= \int_{-1}^{1} \frac{1}{\sqrt{2}} e^{-i\pi m x} f(x)dx \\ &= \frac{1}{\sqrt{2}} \int_{-1}^{0} e^{-i\pi m x} e^{x} dx + \frac{1}{\sqrt{2}} \int_{0}^{1} e^{-i\pi m x} e^{-x} dx \end{aligned} \tag{44}$$

The first integral in (4) gives

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_{-1}^{0} e^{-i\pi m x}e^{x}dx &= \frac{1}{\sqrt{2}}\int_{-1}^{0} e^{(-i\pi m+1)x}dx\\ &= \frac{1}{\sqrt{2}}\left[\frac{e^{(-i\pi m+1)x}}{-i\pi m+1}\right]_{-1}^{0}\\ &= \frac{1}{\sqrt{2}\,(-i\pi m+1)}\left[e^{(-i\pi m+1)x}\right]_{-1}^{0}\\ &= \frac{1}{\sqrt{2}\,(-i\pi m+1)}\left[1-e^{-(-i\pi m+1)}\right]\\ &= \frac{1}{\sqrt{2}}\frac{e^{i\pi m-1}-1}{i\pi m-1} \qquad (5)\end{aligned}$$

And the second integral in (4) gives

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_{0}^{1} e^{-i\pi m x}e^{-x}dx &= \frac{1}{\sqrt{2}}\int_{0}^{1} e^{(-i\pi m-1)x}dx\\ &= \frac{1}{\sqrt{2}}\left[\frac{e^{(-i\pi m-1)x}}{-i\pi m-1}\right]_{0}^{1}\\ &= \frac{1}{\sqrt{2}\,(-i\pi m-1)}\left[e^{(-i\pi m-1)x}\right]_{0}^{1}\\ &= \frac{-1}{\sqrt{2}\,(i\pi m+1)}\left[e^{(-i\pi m-1)}-1\right]\\ &= \frac{1}{\sqrt{2}}\frac{1-e^{-i\pi m-1}}{i\pi m+1} \qquad (6)\end{aligned}$$

Putting (5,6) together gives

$$f_m = \frac{1}{\sqrt{2}}\frac{e^{i\pi m-1}-1}{i\pi m-1} + \frac{1}{\sqrt{2}}\frac{1-e^{-i\pi m-1}}{i\pi m+1} \qquad (7)$$

For $m = 0$, eq (7) becomes

$$\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{e^{-1}-1}{-1}\right) + \frac{1}{\sqrt{2}}\left(1-e^{-1}\right) \\
&= \frac{1}{\sqrt{2}}\left(1-e^{-1}\right) + \frac{1}{\sqrt{2}}\left(1-e^{-1}\right) \\
&= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}\,e} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}\,e} \\
&= \frac{1}{\sqrt{2}}\left(2-\frac{2}{e}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{2e-2}{e}\right) \\
&= \frac{2}{\sqrt{2}}\left(\frac{e-1}{e}\right)
\end{aligned}$$

And for $m \neq 0$, eq (7) becomes

$$\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{e^{i\pi m-1}-1}{i\pi m-1} + \frac{1-e^{-i\pi m-1}}{i\pi m+1}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{e^{i\pi m}e^{-1}-1}{i\pi m-1} + \frac{1-e^{-i\pi m}e^{-1}}{i\pi m+1}\right)
\end{aligned}$$

Since m is integer, then the above becomes

$$\begin{aligned}
f_m &= \frac{1}{\sqrt{2}}\left(\frac{\cos(\pi m)e^{-1}-1}{i\pi m-1} + \frac{1-\cos(\pi m)e^{-1}}{i\pi m+1}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{(-1)^m e^{-1}-1}{i\pi m-1} + \frac{1-(-1)^m e^{-1}}{i\pi m+1}\right) \\
&= \frac{1}{\sqrt{2}}\frac{\left((-1)^m e^{-1}-1\right)(i\pi m+1) + \left(1-(-1)^m e^{-1}\right)(i\pi m-1)}{(i\pi m-1)(i\pi m+1)} \\
&= \frac{1}{\sqrt{2}}\frac{2(-1)^m e^{-1}-2}{-\pi^2 m^2-1} \\
&= \frac{1}{\sqrt{2}}\frac{2-2(-1)^m e^{-1}}{1+\pi^2 m^2} \\
&= \frac{2}{\sqrt{2}}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}
\end{aligned}$$

Hence (3) becomes

$$\begin{aligned}
f(x) &\sim \frac{2}{\sqrt{2}}\left(\frac{e-1}{e}\right)\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{2}{\sqrt{2}}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{i\pi m x}\\
&= \frac{e-1}{e} + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{i\pi m x}\\
&= \frac{e-1}{e} + \sum_{n=-\infty}^{-1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}e^{i\pi n x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}
\end{aligned}$$

Let $m=-n$ in the first sum above. This gives

$$\begin{aligned}
f(x) &= \frac{e-1}{e} + \sum_{m=\infty}^{1}\frac{1-(-1)^{(-m)} e^{-1}}{1+\pi^2 (-m)^2}e^{i\pi(-m)x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}\\
&= \frac{e-1}{e} + \sum_{m=1}^{\infty}\frac{1-(-1)^m e^{-1}}{1+\pi^2 m^2}e^{-i\pi m x} + \sum_{k=1}^{\infty}\frac{1-(-1)^k e^{-1}}{1+\pi^2 k^2}e^{i\pi k x}
\end{aligned}$$

Now the two sums can be combined using one index, say n, since they sum over the same interval

$$f(x) = \frac{e-1}{e} + \sum_{n=\infty}^{1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}\left(e^{-i\pi n x}+e^{i\pi n x}\right)$$

But $\left(e^{-i\pi n x}+e^{i\pi n x}\right) = 2\cos \pi n x$. The above becomes

$$\begin{aligned}
f(x) &= \frac{e-1}{e} + 2\sum_{n=\infty}^{1}\frac{1-(-1)^n e^{-1}}{1+\pi^2 n^2}\cos \pi n x\\
&= \frac{e-1}{e} + 2\sum_{n=\infty}^{1}\frac{e-(-1)^n}{e\left(1+\pi^2 n^2\right)}\cos \pi n x
\end{aligned}$$

2.2 Part 2

Now $f(x)=\cosh x$. Therefore

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{3}$$

Where now f_m is found, using $L=2$

$$f_m = \int_{-1}^{1}\frac{1}{\sqrt{2}}e^{-i\pi m x}\cosh(x)dx$$

But $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The above becomes

$$
\begin{aligned}
f_m &= \int_{-1}^{1} \frac{1}{\sqrt{2}} e^{-i\pi m x} \frac{1}{2}(e^x + e^{-x}) dx \\
&= \frac{1}{2\sqrt{2}} \left(\int_{-1}^{1} e^{-i\pi m x} e^x dx + \int_{-1}^{1} e^{-i\pi m x} e^{-x} dx \right) \\
&= \frac{1}{2\sqrt{2}} \left(\int_{-1}^{1} e^{(-i\pi m+1)x} dx + \int_{-1}^{1} e^{(-i\pi m-1)x} dx \right) \qquad (4)
\end{aligned}
$$

The first integral is

$$
\begin{aligned}
\int_{-1}^{1} e^{(-i\pi m+1)x} dx &= \frac{1}{-i\pi m+1} \left[e^{(-i\pi m+1)x} \right]_{-1}^{1} \\
&= \frac{1}{1-i\pi m} \left(e^{(-i\pi m+1)} - e^{-(-i\pi m+1)} \right) \\
&= \frac{1}{1-i\pi m} \left(e^{-i\pi m} e - e^{i\pi m} e^{-1} \right)
\end{aligned}
$$

Since m is integer, then $e^{-i\pi m} = (-1)^m$ and $e^{i\pi m} = (-1)^m$. The above becomes

$$
\int_{-1}^{1} e^{(-i\pi m+1)x} = \frac{(-1)^m}{1-i\pi m} \left(e - e^{-1} \right) \qquad (5)
$$

The second integral in (4) becomes

$$
\begin{aligned}
\int_{-1}^{1} e^{(-i\pi m-1)x} dx &= \frac{1}{-i\pi m-1} \left[e^{(-i\pi m-1)x} \right]_{-1}^{1} \\
&= \frac{-1}{1+i\pi m} \left(e^{(-i\pi m-1)} - e^{-(-i\pi m-1)} \right) \\
&= \frac{-1}{1+i\pi m} \left(e^{-i\pi m} e^{-1} - e^{i\pi m} e \right)
\end{aligned}
$$

Since m is integer, the above becomes

$$
\int_{-1}^{1} e^{(-i\pi m-1)x} dx = \frac{-(-1)^m}{1+i\pi m} \left(e^{-1} - e \right) \qquad (6)
$$

Substituting (5,6) in (4) gives

$$\begin{aligned}
f_m &= \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1-i\pi m}\left(e-e^{-1}\right)+\frac{-(-1)^m}{1+i\pi m}\left(e^{-1}-e\right)\right)\\
&= \frac{1}{2\sqrt{2}}\left(\frac{(-1)^m}{1-i\pi m}\left(e-e^{-1}\right)+\frac{(-1)^m}{1+i\pi m}\left(e-e^{-1}\right)\right)\\
&= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{1}{1-i\pi m}+\frac{1}{1+i\pi m}\right)\\
&= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{(1+i\pi m)+(1-i\pi m)}{(1-i\pi m)(1+i\pi m)}\right)\\
&= \frac{(-1)^m\left(e-e^{-1}\right)}{2\sqrt{2}}\left(\frac{2}{\pi^2m^2+1}\right)\\
&= \frac{(-1)^m\left(e-e^{-1}\right)}{\sqrt{2}\left(\pi^2m^2+1\right)}
\end{aligned}$$

Hence for $m=0$,

$$f_0 = \frac{\left(e-e^{-1}\right)}{\sqrt{2}}$$

Therefore $f(x)\sim\sum_{m=-\infty}^{\infty} f_m\frac{1}{\sqrt{L}}e^{i\frac{2\pi m}{L}x}$ becomes

$$\begin{aligned}
\cosh(x) &\sim \frac{\left(e-e^{-1}\right)}{\sqrt{2}}\frac{1}{\sqrt{2}}+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{(-1)^m\left(e-e^{-1}\right)}{\sqrt{2}\left(\pi^2m^2+1\right)}\frac{1}{\sqrt{2}}e^{i\pi mx}\\
&= \frac{\left(e-e^{-1}\right)}{2}+\frac{1}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{(-1)^m\left(e-e^{-1}\right)}{\left(\pi^2m^2+1\right)}e^{i\pi mx}
\end{aligned}$$

As was done in part(i), $\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} e^{i\pi mx}$ can be rewritten as $\sum_{n=1}^{\infty} 2\cos(n\pi x)$. The above reduces to

$$\cosh(x)\sim\frac{\left(e-e^{-1}\right)}{2}+\sum_{n=1}^{\infty}2\frac{(-1)^n\left(e-e^{-1}\right)}{\left(\pi^2n^2+1\right)}\cos(n\pi x)$$

But $\frac{\left(e-e^{-1}\right)}{2}=\sinh 1$. Therefore the above becomes

$$\cosh(x)\sim\sinh(1)\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+\pi^2n^2\right)}\cos(n\pi x)\right)$$

2.3 Part 3

I think now the book is asking to find the Fourier series for e^x over $-\pi \leq x \leq \pi$ in this last part. Therefore, as before, starting with

$$f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \tag{1}$$

Where now, using $L = 2\pi$ as the period, then

$$\begin{aligned}
f_m &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{-i\frac{2\pi m}{2\pi}x} e^x dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{(-im+1)x} dx \\
&= \frac{1}{\sqrt{2\pi}\,(1-im)} \left[e^{(-im+1)x}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\sqrt{2\pi}\,(1-im)} \left[e^{(-im+1)\pi} - e^{-(-im+1)\pi}\right] \\
&= \frac{1}{\sqrt{2\pi}\,(1-im)} \left[e^{-im\pi}e^{\pi} - e^{im\pi}e^{-\pi}\right]
\end{aligned}$$

But $e^{im\pi} = (-1)^m$ and $e^{-im\pi} = (-1)^m$ since m is integer. The above becomes

$$\begin{aligned}
f_m &= \frac{(-1)^m}{\sqrt{2\pi}\,(1-im)} \left[e^{\pi} - e^{-\pi}\right] \\
&= \frac{(-1)^m (2\sinh\pi)}{\sqrt{2\pi}\,(1-im)} \\
&= \frac{2}{\sqrt{2\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh\pi
\end{aligned}$$

For $m = 0$ the above gives

$$f_0 = \frac{\sqrt{2}}{\sqrt{\pi}} \sinh\pi$$

Therefore $f(x) \sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x}$ becomes, where $L = 2\pi$ now,

$$
\begin{aligned}
e^x &\sim \sum_{m=-\infty}^{\infty} f_m \frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}x} \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \sinh \pi \left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{m=-\infty,\neq 0}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^m}{(1-im)} \sinh \pi\right) \frac{1}{\sqrt{2\pi}} e^{imx} \\
&= \frac{\sinh \pi}{\pi} + \frac{\sinh \pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{(1-im)} e^{imx} \\
&= \frac{\sinh \pi}{\pi} + \frac{\sinh \pi}{\pi} \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m (1+im)}{(1-im)(1+im)} e^{imx} \\
&= \frac{\sinh \pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m + i(-1)^m m}{1+m^2} e^{imx}\right) \\
&= \frac{\sinh \pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \frac{i(-1)^m m}{1+m^2} e^{imx}\right) \\
&= \frac{\sinh \pi}{\pi} \left(1 + \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} + \sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx}\right) \qquad (2)
\end{aligned}
$$

The first sum above becomes

$$
\sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)^m}{1+m^2} e^{imx} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1+m^2} \cos(nx) \qquad (3)
$$

And the second sum in (2) becomes

$$
\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{m=-\infty,\neq 0}^{\infty} \frac{(-1)(-1)^m m}{1+m^2} \frac{e^{imx}}{i} \\
&= \sum_{k=-\infty}^{-1} \frac{(-1)(-1)^k k}{1+k^2} \frac{e^{ikx}}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r}{1+r^2} \frac{e^{irx}}{i}
\end{aligned}
$$

Letting $m = -k$ in the first sum above gives

$$
\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{m=\infty}^{1} \frac{(-1)(-1)^{-m}(-m)}{1+(-m)^2} \frac{e^{-imx}}{i} + \sum_{r=1}^{\infty} \frac{(-1)(-1)^r r}{1+r^2} \frac{e^{irx}}{i} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m m}{1+m^2} \frac{e^{-imx}}{i} - \sum_{r=1}^{\infty} \frac{(-1)^r r}{1+r^2} \frac{e^{irx}}{i}
\end{aligned}
$$

Merging the two sums back together since now on same interval, and using n for the common index

$$
\begin{aligned}
\sum_{m=-\infty,\neq 0}^{\infty} \frac{i(-1)^m m}{1+m^2} e^{imx} &= \sum_{n=1}^{\infty} \frac{(-1)^n n}{1+n^2} \left(\frac{e^{-inx}}{i} - \frac{e^{inx}}{i}\right) \\
&= -\sum_{n=1}^{\infty} 2 \frac{(-1)^n n}{1+n^2} (\sin(nx)) \qquad (4)
\end{aligned}
$$

Substituting (3,4) back in (2) gives

$$\begin{aligned} e^x &\sim \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\cos(nx) - \sum_{n=1}^{\infty} 2\frac{(-1)^n n}{1+n^2}(\sin(nx))\right) \\ &= \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) \end{aligned} \tag{5}$$

The question is now asking to show how to use (5) to obtain the series for $\sinh x$ and $\cosh x$. Since

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$\begin{aligned} \sinh x &\sim \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(n(-x)) - n\sin(n(-x \right.\right. \\ &= \frac{1}{2}\left(\frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right) - \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right)\right) \\ &= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin \right.\right. \\ &= \frac{1}{2}\left(\frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \frac{\sinh \pi}{\pi} - 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(\right. \\ &= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)) - \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) + n\sin(nx))\right) \\ &= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx) - \cos(nx) - n\sin(nx))\right) \\ &= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-n\sin(nx) - n\sin(nx))\right) \\ &= \frac{\sinh \pi}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(-2n\sin(nx))\right) \end{aligned}$$

Hence

$$\sinh x \sim \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+n^2} n\sin(nx) \tag{6}$$

Similarly for

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Then substituting (5) in the RHS of the above gives

$$
\begin{aligned}
\cosh x &\sim \frac{1}{2}\Bigg(\frac{\sinh \pi}{\pi}\Bigg(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))\Bigg)+\frac{\sinh \pi}{\pi}\Bigg(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(n(-x))-n\sin(n(-x \\
&= \frac{1}{2}\Bigg(\frac{\sinh \pi}{\pi}\Bigg(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))\Bigg)+\frac{\sinh \pi}{\pi}\Bigg(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\sin(nx))\Bigg)\Bigg) \\
&= \frac{1}{2}\Bigg(\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))+\Bigg(\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\,\mathrm{si} \\
&= \frac{1}{2}\Bigg(\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx))+\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+n\,\mathrm{sin} \\
&= \frac{1}{2}\Bigg(2\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)-n\sin(nx)+\cos(nx)+n\sin(nx))\Bigg) \\
&= \frac{1}{2}\Bigg(2\frac{\sinh \pi}{\pi}+2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx)+\cos(nx))\Bigg) \\
&= \frac{\sinh \pi}{\pi}+\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}2\cos(nx)
\end{aligned}
$$

Hence

$$
\cosh x \sim \frac{\sinh \pi}{\pi}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(nx)\right) \tag{7}
$$

3 Problem 3

Perform appropriate integration to show the following results regarding the Dirac delta function

(1) $\delta(ax) = \frac{\delta(x)}{|a|}$ where a is real number. (2) $\delta\left(f(x)\right) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_i}$ where x_i satisfies $f(x_i) = 0$

(3) $\frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'}$

Solution

3.1 Part (1)

Using the integral definition of delta function given by

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \tag{1}$$

Then

$$\delta(ax) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikax} dk$$

Case $a > 0$. Let $u = ak$. Then $du = adk$. The above becomes

$$\delta(ax) = \frac{1}{a}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux} du\right)$$

But $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux} du = \delta(x)$ by definition. Hence the above becomes

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{2}$$

Case $a < 0$. Let $u = ak$. Then $du = adk$. When $k = \infty, u = -\infty$ and when $k = -\infty, u = +\infty$. The integral becomes

$$\begin{aligned}\delta(ax) &= \frac{1}{a}\left(\frac{1}{2\pi}\int_{\infty}^{-\infty} e^{iux} du\right) \\ &= \frac{1}{-a}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux} du\right) \\ &= \frac{1}{-a}\delta(x) \end{aligned}\tag{3}$$

Combining (2,3) gives

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

3.2 Part (2)

Using

$$\int_{-\infty}^{\infty} \delta\Big(f(x)\Big)dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\Big(f(x)\Big)dx \tag{1}$$

Where in the RHS, the sum is over the roots of $f(x)$, where $f(x_i) = 0$ where x_i is root of $f(x)$ since $\delta(u)$ is nonzero only when its argument is zero, which is at the roots of $f(x)$. Now, expanding $f(x)$ near each one of its roots using Taylor series

$$f(x) = f(x_i) + (x - x_i)f'(x_i) + O\Big(x^2\Big)$$

But $f(x_i) = 0$ since x_i is root, and keeping only linear terms, then (1) now becomes

$$\int_{-\infty}^{\infty} \delta\Big(f(x)\Big)dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\Big((x - x_i)f'(x_i)\Big)dx$$

But from part (1), we found that $\delta(a(x - x_i)) = \frac{1}{|a|}\delta(x - x_i)$, where now $a = f'(x_i)$. Using this relation in the above gives

$$\int_{-\infty}^{\infty} \delta\Big(f(x)\Big)dx = \int_{-\infty}^{\infty} \sum_i \frac{1}{\left|f'(x_i)\right|}\delta(x - x_i)$$

Therefore the integrands on each side is the same. This implies

$$\delta\Big(f(x)\Big) = \sum_i \frac{\delta(x - x_i)}{\left|f'(x_i)\right|}$$

3.3 Part (3)

Starting from property of delta function which is

$$\int \delta(x - x')f(x')dx' = f(x)$$

Taking derivative of both sides w.r.t. x gives

$$\begin{aligned} \frac{d}{dx}\int \delta(x - x')f(x')dx' &= \frac{d}{dx}f(x) \\ \int \frac{d\delta(x - x')}{dx}f(x')dx' &= \frac{d}{dx}f(x) \end{aligned}$$

Integration by part. Let $\frac{d\delta(x-x')}{dx} = dv, u = f(x')$, then $v = (x - x'), du = \frac{d}{dx'}f(x')$. The above becomes

$$\int \delta(x - x')\frac{d}{dx'}f(x')dx' = \frac{d}{dx}f(x)$$

Therefore

$$\int \frac{d\delta(x - x')}{dx} f(x')dx' = \int \delta(x - x') \frac{d}{dx'} f(x')dx'$$

or

$$\frac{d\delta(x - x')}{dx} f(x') = \delta(x - x') \frac{d}{dx'} f(x')$$

or

$$\frac{d\delta(x - x')}{dx} = \delta(x - x') \frac{d}{dx'}$$

4 Problem 4

For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured.

Solution

The goal is to determine $\left|\langle\phi_p|\psi\rangle\right|^2$ which will give the probability of measuring momentum p. But

$$\begin{aligned}\langle\phi_p|\psi_n\rangle &= \int_0^\infty \langle\phi_p|x\rangle\langle x|\psi_n\rangle dx \\ &= \int_0^\infty \langle x|\phi_p\rangle^*\langle x|\psi_n\rangle dx \qquad (1)\end{aligned}$$

But $\langle x|\phi_p\rangle = \phi_p(x)$ and $\langle x|\psi_n\rangle = \psi_n(x)$. From lecture notes,

$$\begin{aligned}\phi_p(x) &= \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{px}{\hbar}} \\ \psi_n(x) &= \left\{\begin{array}{cc}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{otherwise}\end{array}\right\}\end{aligned}$$

For $n = 1, 2, 3, \cdots$. Substituting the above in (1) gives

$$\begin{aligned}\langle\phi_p|\psi_n\rangle &= \int_0^\infty \phi_p^*(x)\psi_n(x)dx \\ &= \int_0^L \frac{1}{\sqrt{2\pi\hbar}}e^{-i\frac{px}{\hbar}}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx \\ &= \frac{1}{\sqrt{\pi\hbar L}}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx \qquad (2)\end{aligned}$$

To evaluate $I = \int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx$ we do Integration by parts twice. Let $u = \sin\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)dx$ and $v = \hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}$. Hence

$$\begin{aligned}I &= [uv]_0^L - \int_0^L vdu \\ &= \left[\sin\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}\right]_0^L - \int_0^L \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}dx \\ &= \left[\sin\left(\frac{n\pi L}{L}\right)\hbar\frac{e^{-i\frac{pL}{\hbar}}}{-ip} - 0\right] + \frac{\hbar n\pi}{ipL}\int_0^L \cos\left(\frac{n\pi x}{L}\right)e^{-i\frac{px}{\hbar}}dx\end{aligned}$$

Since n is integer, then boundary terms are zero.

$$I = \frac{\hbar n\pi}{ipL}\int_0^L \cos\left(\frac{n\pi x}{L}\right)e^{-i\frac{px}{\hbar}}dx$$

Doing integration by parts one more time. Let $u = \cos\left(\frac{n\pi x}{L}\right), dv = e^{-i\frac{px}{\hbar}}$ then $du = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx$, then the above becomes

$$\begin{aligned}
I &= \frac{\hbar n\pi}{ipL}\left(\left[\cos\left(\frac{n\pi x}{L}\right)\hbar\frac{e^{-i\frac{px}{\hbar}}}{-ip}\right]_0^L + \int_0^L \frac{\hbar e^{-i\frac{px}{\hbar}}}{-ip}\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)\\
&= \frac{\hbar n\pi}{ipL}\left(\frac{\hbar}{-ip}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] - \frac{\hbar n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx\right)\\
&= \frac{\hbar^2}{-ip}\frac{n\pi}{ipL}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] - \frac{n\pi}{ipL}\frac{\hbar^2 n\pi}{ipL}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx\\
&= \frac{\hbar^2 n\pi}{p^2 L}\left[\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right] + \frac{\hbar^2 n^2\pi^2}{p^2L^2}\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

But $\int_0^L e^{-i\frac{px}{\hbar}}\sin\left(\frac{n\pi x}{L}\right)dx = I$. Therefore the above becomes

$$I = \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right) + \frac{\hbar^2n^2\pi^2}{p^2L^2}I$$

Solving for I

$$\begin{aligned}
I - \frac{\hbar^2n^2\pi^2}{p^2L^2}I &= \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)\\
I\left(1 - \frac{\hbar^2n^2\pi^2}{p^2L^2}\right) &= \frac{\hbar^2 n\pi}{p^2L}\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)\\
I &= \frac{\hbar^2 n\pi}{p^2L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)}{\left(1 - \frac{\hbar^2n^2\pi^2}{p^2L^2}\right)}\\
&= p^2L^2\frac{\hbar^2 n\pi}{p^2L}\frac{\left(\cos(n\pi)e^{-i\frac{pL}{\hbar}} - 1\right)}{\left(p^2L^2 - \hbar^2n^2\pi^2\right)}\\
&= \frac{\hbar^2 n\pi L}{p^2L^2 - \hbar^2n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}} - 1\right)
\end{aligned}$$

Substituting the above in (2) gives

$$
\begin{aligned}
\langle\phi_p|\psi_n\rangle &= \frac{1}{\sqrt{\pi\hbar L}}\left(\frac{\hbar^2 n\pi L}{p^2L^2-\hbar^2n^2\pi^2}\left((-1)^n e^{-i\frac{pL}{\hbar}}-1\right)\right)\\
&= \frac{\hbar^2 n\pi L\left(\sqrt{\pi\hbar L}\,\right)}{(\pi\hbar L)\left(p^2L^2-\hbar^2n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)\\
&= \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)
\end{aligned}
$$

Let $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}$, then

$$
\begin{aligned}
\langle\phi_p|\psi_n\rangle &= k_n\left((-1)^n e^{-\frac{ipL}{\hbar}}-1\right)\\
&= (-1)^n k_n e^{-\frac{ipL}{\hbar}} - k_n\\
&= (-1)^n k_n\left(\cos\frac{pL}{\hbar} - i\sin\frac{pL}{\hbar}\right) - k_n\\
&= (-1)^n k_n\cos\frac{pL}{\hbar} - i(-1)^n k_n\sin\frac{pL}{\hbar} - k_n\\
&= \left((-1)^n k_n\cos\frac{pL}{\hbar} - k_n\right) - i\left((-1)^n k_n\sin\frac{pL}{\hbar}\right)
\end{aligned}
$$

Hence

$$
\left|\langle\phi_p|\psi_n\rangle\right| = \sqrt{\left((-1)^n k_n\cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n\sin\frac{pL}{\hbar}\right)^2}
$$

And

$$
\begin{aligned}
\left|\langle\phi_p|\psi_n\rangle\right|^2 &= \left((-1)^n k_n\cos\frac{pL}{\hbar} - k_n\right)^2 + \left((-1)^n k_n\sin\frac{pL}{\hbar}\right)^2\\
&= (-1)^{2n}k_n^2\cos^2\frac{pL}{\hbar} + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar} + (-1)^{2n}k_n^2\sin^2\frac{pL}{\hbar}\\
&= (-1)^{2n}k_n^2\left(\cos^2\frac{pL}{\hbar} + \sin^2\frac{pL}{\hbar}\right) + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar}\\
&= (-1)^{2n}k_n^2 + k_n^2 - 2k_n^2(-1)^n\cos\frac{pL}{\hbar}\\
&= k_n^2\left(1 + (-1)^{2n} - 2(-1)^n\cos\frac{pL}{\hbar}\right)
\end{aligned}
$$

But $k_n = \frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}$, therefore the above becomes

$$
\begin{aligned}
\left|\langle\phi_p|\psi_n\rangle\right|^2 &= \left(\frac{n\hbar\sqrt{\pi\hbar L}}{\left(p^2L^2-\hbar^2n^2\pi^2\right)}\right)^2\left(1+(-1)^{2n}-2(-1)^n\cos\frac{pL}{\hbar}\right)\\
&= \frac{n^2\hbar^3\pi L}{\left(p^2L^2-\hbar^2n^2\pi^2\right)^2}\left(1+(-1)^{2n}-2(-1)^n\cos\left(\frac{pL}{\hbar}\right)\right)
\end{aligned}
$$

The above gives the probability of measurement of p, where $n = 1, 2, 3, \cdots$. For illustration, the following two tables are generated to see how the probability of measuring say $p = 1$ and $p = 2$ changes as function of n. To generate this, L is taken as 1 and $\hbar = 1$ for simplicity.

n	Probability of measuring p=1
1	0.12302
2	0.00780329
3	0.0112922
4	0.00187694
5	0.00400658
6	0.00082832
7	0.00203605
8	0.000464782
9	0.00122967
10	0.000297121

Figure 3: Probability to measure $p = 1$

n	Probability of measuring p=2
1	0.106479
2	0.0282761
3	0.00458842
4	0.00600971
5	0.00155647
6	0.00259549
7	0.000781451
8	0.00144553
9	0.00046963
10	0.000920908

Figure 4: Probability to measure $p = 2$

```
p = 1; h = 1;  L = 1;
```

$$f[p\_, n\_] := \frac{n^2 h^3 Pi L}{\left(p^2 L^2 - h^2 n^2 Pi^2\right)^2} \left(1 + (-1)^{2n} - 2 (-1)^n Cos\left[\frac{p L}{h}\right]\right)$$

```
data = Table[{n, N@f[p, n]}, {n, 1, 10}];
data = PrependTo[data, {"n", "Probability of measuring p=1"}];
Grid[data, Frame → All];
```

Figure 5: Code used