Physics 3041 (Spring 2021) Solutions to Homework Set 2

1. Problem 2.2.3. (10 points)

Let $y = \sqrt{x}$. So $\int_0^1 e^{\sqrt{x}} dx = \int_0^1 e^y dy^2 = 2\int_0^1 ye^y dy = 2\int_0^1 y de^y = 2(ye^y|_0^1 - \int_0^1 e^y dy) = 2(e - e^y|_0^1) = 2[e - (e-1)] = 2$.

Let $y = x^4$. So $\int_0^\infty e^{-x^4} dx = xe^{-x^4}|_0^\infty - \int_0^\infty x de^{-x^4} = 4\int_0^\infty x^4 e^{-x^4} dx = 4\int_0^\infty ye^{-y} dy^{1/4} = \int_0^\infty y^{1/4} e^{-y} dy = \Gamma(5/4)$.

2. (a) Problem 2.2.10. (10 points)

Consider

$$I(a) = \int_0^1 \frac{t^a - 1}{\ln t} dt \Rightarrow \frac{dI}{da} = \int_0^1 \frac{de^{\ln t^a}}{da} \frac{dt}{\ln t} = \int_0^1 \frac{de^{a\ln t}}{da} \frac{dt}{\ln t} = \int_0^1 e^{a\ln t} dt = \int_0^1 t^a dt = \frac{1}{1+a}.$$

$I(1) - I(0) = \int_0^1 (dI/da) da = \int_0^1 da/(1+a) = \ln(1+a)|_0^1 = \ln 2$, where $I(1)$ is the original integral to be evaluated. Because $I(0) = 0$, we obtain $I(1) = \ln 2$.

(b) Problem 2.2.11. (10 points)

Let $I(a) = J(k) = \int_0^\infty e^{-ax} \sin kx dx = k/(a^2+k^2)$. We obtain $dI/da = -\int_0^\infty xe^{-ax} \sin kx dx = -2ak/(a^2+k^2)^2$, or $\int_0^\infty xe^{-ax} \sin kx dx = 2ak/(a^2+k^2)^2$. Likewise, $dJ/dk = \int_0^\infty xe^{-ax} \cos kx dx = 1/(a^2+k^2) - 2k^2/(a^2+k^2)^2 = (a^2-k^2)/(a^2+k^2)^2$.

3. The probability to find a particle at position between x and $x + dx$ is

$$P(x)dx = A\exp(-\alpha x^2 + \beta x^3)dx,$$

where A, α, and β are positive parameters. By the definition of probability,

$$\int_{-\infty}^{\infty} P(x)dx = 1.$$

Treat β as a small parameter, i.e., for any given x, you can view $P(x)$ as a function of β and expand it around $\beta = 0$.

(a) Find A to the first order of β. (15 points)

$$\int_{-\infty}^{\infty} P(x)dx = A\int_{-\infty}^{\infty} \exp(-\alpha x^2 + \beta x^3)dx = 1$$

$$\begin{aligned} \Rightarrow A &= \frac{1}{\int_{-\infty}^{\infty}\exp(-\alpha x^2 + \beta x^3)dx} \approx \frac{1}{\int_{-\infty}^{\infty}(1 + \beta x^3)\exp(-\alpha x^2)dx} \\ &= \frac{1}{\int_{-\infty}^{\infty}\exp(-\alpha x^2)dx} = \frac{\sqrt{\alpha}}{\int_{-\infty}^{\infty}\exp(-y^2)dy} = \sqrt{\frac{\alpha}{\pi}}, \end{aligned}$$

where $y = x\sqrt{\alpha}$ and we have used symmetry to obtain

$$\int_{-\infty}^{\infty} x^3 \exp\left(-\alpha x^2\right) dx = 0.$$

(b) Find the average position

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

to the first order of β. (25 points)

$$\begin{aligned} \bar{x} &= \int_{-\infty}^{\infty} xP(x)dx = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x\exp\left(-\alpha x^2 + \beta x^3\right) dx \\ &\approx \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x(1 + \beta x^3)\exp(-\alpha x^2)dx \\ &= \beta\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} x^4\exp(-\alpha x^2)dx = \frac{\beta}{\alpha^2\sqrt{\pi}}\int_{-\infty}^{\infty} y^4\exp(-y^2)dy, \end{aligned}$$

where we have used symmetry to obtain

$$\int_{-\infty}^{\infty} x\exp(-\alpha x^2)dx = 0.$$

Noting that

$$I(a) = \int_{-\infty}^{\infty}\exp(-ay^2)dy = \sqrt{\frac{\pi}{a}} \Rightarrow \frac{d^2 I}{da^2} = \int_{-\infty}^{\infty} y^4\exp(-ay^2)dy = \frac{3}{4}\frac{\sqrt{\pi}}{a^{5/2}},$$

we obtain

$$\bar{x} = \frac{\beta}{\alpha^2\sqrt{\pi}}\left(\frac{3}{4}\sqrt{\pi}\right) = \frac{3\beta}{4\alpha^2}.$$

4. A container of volume V encloses a neutrino gas of temperature T. The number of neutrinos with energy between E and $E + dE$ is

$$dN = \left(\frac{4\pi V}{h^3 c^3}\right) \frac{E^2}{\exp[E/(kT)] + 1} dE,$$

where h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(a) Express the total energy density of the neutrino gas in terms of a dimensional factor multiplying a dimensionless integral. Show that the factor has the correct dimension. (10 points)

The total energy density is

$$\varepsilon = \frac{\int E dN}{V} = \frac{4\pi}{h^3 c^3} \int_0^\infty \frac{E^3 dE}{\exp[E/(kT)] + 1} = \frac{4\pi (kT)^4}{(hc)^3} \int_0^\infty \frac{x^3 dx}{\exp(x) + 1},$$

where we have made the substitution of variable $x = E/(kT)$. The dimensional factor is in units of $\mathrm{J}^4/(\mathrm{J} \cdot \mathrm{s} \cdot \mathrm{m/s})^3 = \mathrm{J/m}^3$, as should be for the energy density.

(b) Follow the discussion of a photon gas and evaluate the dimensionless integral. (20 points)

Using $1 + y + y^2 + y^3 + \cdots = (1 - y)^{-1}$ for $|y| < 1$, we obtain

$$\frac{1}{\exp(x) + 1} = \frac{\exp(-x)}{1 + \exp(-x)} = \exp(-x) \sum_{n'=0}^{\infty} (-1)^{n'} \exp(-n'x) = \sum_{n=1}^{\infty} (-1)^{n-1} \exp(-nx),$$

where we have set $y = -\exp(-x)$ and $n = n' + 1$.

Using $\int_0^\infty x^k \exp(-\alpha x) dx = k!/\alpha^{k+1}$, we obtain

$$\begin{aligned}
\int_0^\infty \frac{x^3 dx}{\exp(x) + 1} &= \int_0^\infty x^3 \sum_{n=1}^{\infty} (-1)^{n-1} \exp(-nx) dx = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^\infty x^3 \exp(-nx) dx \\
&= 3! \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = 6 \left(\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots \right) \\
&= 6 \left[\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \right) - 2 \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots \right) \right] \\
&= 6 \left[\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \right) - \frac{2}{2^4} \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \right) \right] \\
&= 6 \left(1 - \frac{1}{8} \right) \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \right) = \frac{21}{4} \zeta(4) = \frac{21}{4} \left(\frac{\pi^4}{90} \right) = \frac{7\pi^4}{120}.
\end{aligned}$$