HW 2

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 2.2.3

Evaluate $\int_0^1 e^{\sqrt{x}} dx$. Show that $\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$

<u>Solution</u>

Let $y = \sqrt{x}$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\frac{1}{\sqrt{x}} \\
&= \frac{1}{2}\frac{1}{y}
\end{aligned}
$$

And

$$dx = 2ydy$$

When $x = 0$, $y = 0$ and when $x = 1$,$y = 1$. Substituting this back into $\int_0^1 e^{\sqrt{x}} dx$ gives $\int_0^1 e^y\left(2ydy\right) = 2\int_0^1 ye^y dy$. This integral is evaluated using integration by parts.

$$udv = uv|_0^1 - \int_0^1 vdu$$

Let $u = y$ and $dv = e^y$, then $du = dy$ and $v = e^y$. The above becomes

$$
\begin{aligned}
2\left(\int_0^1 ye^y dy\right) &= 2\left(uv|_0^1 - \int_0^1 vdu\right) \\
&= 2\left(ye^y|_0^1 - \int_0^1 e^y dy\right) \\
&= 2\left(\left(e^1 - 0\right) - e^y|_0^1\right) \\
&= 2(e - (e - 1)) \\
&= 2(e - e + 1) \\
&= 2
\end{aligned}
$$

Hence

$$\int_0^1 e^{\sqrt{x}} dx = 2$$

For the second part of the question asking to evaluate $\int_0^\infty e^{-x^4} dx$, let

$$x = y^{\frac{1}{4}}$$

Then

$$\frac{dx}{dy} = \frac{1}{4}y^{\left(\frac{1}{4}-1\right)}$$

When $x=0, y=0$ and when $x=\infty, y=\infty$. Hence the above integral becomes

$$\begin{aligned}\int_0^\infty e^{-x^4}dx &= \int_0^\infty e^{-y}\left(\frac{1}{4}y^{\left(\frac{1}{4}-1\right)}dy\right)\\ &= \frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy \end{aligned} \tag{1}$$

Comparing the above to integral (2.1.39) in the book which says

$$F(n) = \int_0^\infty y^n e^{-y}dy \tag{2}$$

$$\Gamma(n) = F(n-1) \tag{3}$$

Then putting $n=\frac{1}{4}$ in (3) gives

$$\begin{aligned}\Gamma\left(\frac{1}{4}\right) &= F\left(\frac{1}{4}-1\right)\\ &= \int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy\end{aligned}$$

Which is (1). This means that

$$\int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy = \Gamma\left(\frac{1}{4}\right)$$

Hence

$$\frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \tag{4}$$

To obtain the final form, the following property of Gamma functions is used

$$\Gamma(n+1) = n\Gamma(n)$$

Which means that when $n=\frac{1}{4}$, the above becomes

$$\begin{aligned}\Gamma\left(\frac{1}{4}+1\right) &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\\ \Gamma\left(\frac{5}{4}\right) &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\end{aligned}$$

Using this in (4) shows that

$$\frac{1}{4}\int_0^\infty y^{\left(\frac{1}{4}-1\right)}e^{-y}dy = \Gamma\left(\frac{5}{4}\right)$$

Which implies

$$\int_0^\infty e^{-x^4}dx = \Gamma\left(\frac{5}{4}\right)$$

Which is what we are asked to show.

2 Problem 2.2.10 (or part a of problem 2)

Problem 2.2.10. *Consider*

$$I = \int_0^1 \frac{t-1}{\ln t}.$$

Think of the t in $t-1$ as the $a=1$ limit of t^a. Let $I(a)$ be the corresponding integral. Take the a derivative of both sides (using $t^a = e^{a\ln t}$) and evaluate dI/da by evaluating the corresponding integral by inspection. Given dI/da obtain I by performing the indefinite integral *of both sides with respect to a. Determine the constant of integration using your knowledge of $I(0)$. Show that the original integral equals* $\ln 2$.

Figure 1: Problem statment

Solution

Let

$$I(a) = \int_0^1 \frac{t^a - 1}{\ln t} dt$$

Where $a = 1$ for the specific integral in this problem. The above is the parametrized general form. Taking derivative w.r.t a gives

$$\begin{aligned} \frac{dI(a)}{da} &= \frac{d}{da}\left(\int_0^1 \frac{t^a-1}{\ln t}dt\right) \\ &= \int_0^1 \frac{d}{da}\left(\frac{t^a-1}{\ln t}\right)dt \\ &= \int_0^1 \frac{1}{\ln t}\frac{d}{da}(t^a-1)dt \end{aligned} \tag{1}$$

But

$$\begin{aligned} \frac{d}{da}(t^a-1) &= \frac{d}{da}\left(e^{a\ln t}-1\right) \\ &= \ln(t)\left(e^{a\ln t}\right) \end{aligned} \tag{2}$$

Substituting (2) into (1) gives

$$
\begin{aligned}
\frac{dI(a)}{da} &= \int_0^1 \frac{1}{\ln t}\Big(\ln(t)\big(e^{a\ln t}\big)\Big)dt \\
&= \int_0^1 e^{a\ln t}dt \\
&= \int_0^1 t^a dt \\
&= \left.\frac{t^{a+1}}{a+1}\right|_0^1 \\
&= \frac{1}{1+a} \qquad a \neq -1 \qquad (3)
\end{aligned}
$$

Integrating the above is used to $I(a)$ gives

$$
\begin{aligned}
I(a) &= \int_0^a \frac{1}{1+\tau}d\tau \\
&= \ln(1+\tau)|_0^a \\
&= \ln(1+a) - \ln(1) \\
&= \ln(1+a) \qquad a \neq -1
\end{aligned}
$$

When $a = 1$ the above becomes

$$
\begin{aligned}
I(1) &= \int_0^1 \frac{t-1}{\ln t}dt \\
&= \ln(1+1) \\
&= \ln(2)
\end{aligned}
$$

Hence

$$
\int_0^1 \frac{t-1}{\ln t}dt = \ln(2)
$$

3 Problem 2.2.11 (or part b of problem 2)

Problem 2.2.11. *Given*

$$\int_0^\infty e^{-ax} \sin kx dx = \frac{k}{a^2+k^2},$$

evaluate $\int_0^\infty x e^{-ax} \sin kx dx$ *and* $\int_0^\infty x e^{-ax} \cos kx dx$.

Figure 2: Problem statment

Solution

3.1 part (1)

$$I = \int_0^\infty e^{-ax} \sin kx dx$$

Taking derivative w.r.t a gives

$$\begin{aligned}
\frac{dI}{da} &= \frac{d}{da}\left(\int_0^\infty e^{-ax} \sin kx dx\right) \\
&= \int_0^\infty \frac{d}{da}(e^{-ax} \sin kx) dx \\
&= \int_0^\infty -x e^{-ax} \sin kx dx \\
&= -\int_0^\infty x e^{-ax} \sin kx dx
\end{aligned}$$

Which is the integral the problem is asking to find. Therefore, since I is also given as $\frac{k}{a^2+k^2}$ then

$$\begin{aligned}
-\int_0^\infty xe^{-ax}\sin kx dx &= \frac{d}{da}\left(\frac{k}{a^2+k^2}\right)\\
&= k\frac{d}{da}\left(\frac{1}{a^2+k^2}\right)\\
&= k(-1)\left(a^2+k^2\right)^{-2}(2a)\\
&= -\frac{2ak}{\left(a^2+k^2\right)^2}
\end{aligned}$$

Therefore

$$\int_0^\infty xe^{-ax}\sin kx dx = \frac{2ak}{\left(a^2+k^2\right)^2}$$

3.2 part (2)

$$I = \int_0^\infty e^{-ax}\sin kx dx$$

Taking derivative w.r.t. k gives

$$\begin{aligned}
\frac{dI}{dk} &= \frac{d}{dk}\left(\int_0^\infty e^{-ax}\sin kx dx\right)\\
&= \int_0^\infty \frac{d}{dk}(e^{-ax}\sin kx)dx\\
&= \int_0^\infty e^{-ax}\frac{d}{dk}(\sin kx)dx\\
&= \int_0^\infty xe^{-ax}\cos kx dx
\end{aligned}$$

Which is the integral the problem is asking to find. Therefore, since I is also given as $\frac{k}{a^2+k^2}$ then

$$\begin{aligned}
\int_0^\infty xe^{-ax}\cos kx dx &= \frac{d}{dk}\left(\frac{k}{a^2+k^2}\right)\\
&= \frac{\left(a^2+k^2\right)-k(2k)}{\left(a^2+k^2\right)^2}\\
&= \frac{a^2+k^2-2k^2}{\left(a^2+k^2\right)^2}\\
&= \frac{a^2-k^2}{\left(a^2+k^2\right)^2}
\end{aligned}$$

Hence

$$\int_0^\infty xe^{-ax}\cos kx dx = \frac{a^2 - k^2}{\left(a^2 + k^2\right)^2}$$

4 Problem 3

3. The probability to find a particle at position between x and $x + dx$ is

$$P(x)dx = A\exp(-\alpha x^2 + \beta x^3)dx,$$

where A, α, and β are positive parameters. By the definition of probability,

$$\int_{-\infty}^{\infty} P(x)dx = 1.$$

Treat β as a small parameter, i.e., for any given x, you can view $P(x)$ as a function of β and expand it around $\beta = 0$.

(a) Find A to the first order of β. (15 points)

(b) Find the average position

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

to the first order of β. (25 points)

Figure 3: Problem statment

Solution

4.1 Part (a)

$$P\left(x,\beta\right) = Ae^{-\alpha x^2+\beta x^3}$$

Expanding around $\beta = 0$ by fixing x, gives

$$P\left(x,\beta\right) = P(x,0) + \beta \left.\frac{\partial P}{\partial \beta}\right|_{\beta=0} + \frac{\beta^2}{2!}\left.\frac{\partial^2 P}{\partial \beta^2}\right|_{\beta=0} + \cdots \tag{1}$$

But

$$P(x,0) = Ae^{-\alpha x^2} \tag{2}$$

And

$$\frac{\partial P}{\partial \beta} = Ax^3 e^{-\alpha x^2+\beta x^3} \tag{3}$$

No need to take more derivatives since the problem is asking for first order of β. Substituting (2,3) into (1) gives

$$\begin{aligned} P(x,\beta) &= Ae^{-\alpha x^2} + \beta\, Ax^3 e^{-\alpha x^2+\beta x^3}\Big|_{\beta=0} + \cdots \\ &= Ae^{-\alpha x^2} + \beta Ax^3 e^{-\alpha x^2} + \cdots \end{aligned} \tag{4}$$

Using the above in the definition $\int_{-\infty}^{\infty} P(x)dx = 1$ gives

$$\begin{aligned} \int_{-\infty}^{\infty} \left(Ae^{-\alpha x^2} + \beta Ax^3 e^{-\alpha x^2}\right)dx &= 1 \\ A\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}dx + \beta \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2}dx\right) &= 1 \end{aligned} \tag{5}$$

But

$$\int_{-\infty}^{\infty} x^3 e^{-\alpha x^2}dx = 0$$

This is because $e^{-\alpha x^2}$ is an even function over $(-\infty,+\infty)$ and x^3 is odd. Eq (5) now simplifies to

$$A\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = 1$$

But $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$ $(\alpha > 0)$ because it is standard Gaussian integral. The above now becomes

$$\begin{aligned} A\sqrt{\frac{\pi}{\alpha}} &= 1 \\ A &= \sqrt{\frac{\alpha}{\pi}} \qquad \alpha > 0 \end{aligned}$$

4.2 Part b

$$\bar{x} = \int_{-\infty}^{\infty} xP(x)dx$$

Using Eq. (4) from part (a), the above becomes

$$\begin{aligned} \bar{x} &= \int_{-\infty}^{\infty} x\left(Ae^{-\alpha x^2} + \beta Ax^3 e^{-\alpha x^2}\right)dx \\ &= A\int_{-\infty}^{\infty} xe^{-\alpha x^2}dx + A\int_{-\infty}^{\infty} \beta x^4 e^{-\alpha x^2}dx \end{aligned}$$

But $\int_{-\infty}^{\infty} xe^{-\alpha x^2}dx = 0$ since $e^{-\alpha x^2}$ is an even function over $(-\infty,+\infty)$ and x is an odd function. The above simplifies to

$$\bar{x} = A\beta\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2}dx \tag{6}$$

To evaluate the above, starting from the standard Gaussian integral given by

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Taking derivative w.r.t α of both sides of the above results in

$$\begin{aligned}
I'(\alpha) &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} e^{-\alpha x^2} dx = \frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} \\
&= \int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2} dx = \sqrt{\pi}\left(-\frac{1}{2}\right)\alpha^{-\frac{3}{2}} \\
&= \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\alpha^{-\frac{3}{2}}
\end{aligned}$$

Taking one more derivative w.r.t α gives

$$\begin{aligned}
I''(\alpha) &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} x^2 e^{-\alpha x^2} dx = \frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2}\alpha^{-\frac{3}{2}}\right) \\
&= \int_{-\infty}^{\infty} -x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\left(-\frac{3}{2}\alpha^{-\frac{5}{2}}\right) \\
&= \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2}\left(\frac{3}{2}\alpha^{-\frac{5}{2}}\right)
\end{aligned}$$

Now the integrand is the one we want. This shows that

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}$$

Using the above result in (6) gives

$$\bar{x} = A\beta\left(\frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}\right)$$

But $A = \sqrt{\frac{\alpha}{\pi}}$ from part(a). Hence the above becomes

$$\begin{aligned}
\bar{x} &= \sqrt{\frac{\alpha}{\pi}}\beta\left(\frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}\right) \\
&= \alpha^{\frac{1}{2}}\beta\frac{3}{4\alpha^{\frac{5}{2}}} \\
&= \beta\frac{3}{4\alpha^{\frac{5}{2}-\frac{1}{2}}} \\
&= \frac{3}{4}\frac{\beta}{\alpha^2} \qquad \alpha > 0
\end{aligned}$$

5 Problem 4

4. A container of volume V encloses a neutrino gas of temperature T. The number of neutrinos with energy between E and $E + dE$ is

$$dN = \left(\frac{4\pi V}{h^3c^3}\right)\frac{E^2}{\exp[E/(kT)] + 1}dE,$$

where h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(a) Express the total energy density of the neutrino gas in terms of a dimensional factor multiplying a dimensionless integral. Show that the factor has the correct dimension. (10 points).

(b) Follow the discussion of a photon gas and evaluate the dimensionless integral. (20 points).

Figure 4: Problem statment

<u>Solution</u>

5.1 Part a

$$dN = \left(\frac{4\pi V}{h^3c^3}\right)\frac{E^2}{1 + e^{\frac{E}{kT}}}dE$$

The total energy is therefore

$$E_{total} = \int E dN$$

Hence the energy density ρ is

$$\begin{aligned}
\rho &= \frac{1}{V}\int E dN \\
&= \frac{1}{V}\int_0^\infty \left(\frac{4\pi V}{h^3c^3}\right)\frac{EE^2}{1 + e^{\frac{E}{kT}}}dE \\
&= \left(\frac{1}{V}\right)\left(\frac{4\pi V}{h^3c^3}\right)\int_0^\infty \frac{E^3}{1 + e^{\frac{E}{kT}}}dE \\
&= \frac{4\pi}{h^3c^3}\int_0^\infty \frac{E^3}{1 + e^{\frac{E}{kT}}}dE \qquad (1)
\end{aligned}$$

k (Boltzmann constant) has units of $\frac{[J]}{[K]}$ where J is joule and K is temperature in Kelvin. Hence units of $\frac{E}{kT}$ is $\frac{[J]}{\frac{[J]}{[K]}[K]}$ which is dimensionless. Let

$$x = \frac{E}{kT}$$

Therefore $\frac{dx}{dE} = \frac{1}{kT}$ When $E = 0, x = 0$ and when $E = \infty, x = \infty$. Substituting this into the integral in (1) gives

$$\begin{aligned}\int_0^\infty \frac{E^3}{1+e^{\frac{E}{kT}}} dE &= \int_0^\infty \frac{(xkT)^3}{1+e^x}(kTdx) \\ &= (kT)^4 \int_0^\infty \frac{x^3}{1+e^x} dx \qquad (2)\end{aligned}$$

Substituting (2) into (1) gives

$$\rho = \left(\frac{4\pi}{h^3 c^3}\right)(kT)^4 \int_0^\infty \frac{x^3}{1+e^x} dx \qquad (3)$$

Units of c (speed of light) is $\frac{[L]}{[T]}$ where $[L]$ is length in meters and $[T]$ is time in seconds. Units for Planck constant h is $[J][T]$ (Joule-second). Therefore the factor $\left(\frac{4\pi}{h^3c^3}\right)(kT)^4$ above in (3) in front of the integral has units

$$\begin{aligned}\left(\frac{4\pi}{h^3 c^3}\right)(kT)^4 &= \frac{1}{([J][T])^3 \left(\frac{[L]}{[T]}\right)^3} \left(\frac{[J]}{[K]}[K]\right)^4 \\ &= \frac{1}{[J]^3 [L]^3}([J])^4 \\ &= \frac{[J]}{[L]^3}\end{aligned}$$

Which has the correct units of energy density. Let this factor be called $\Phi = \left(\frac{4\pi}{h^3c^3}\right)(kT)^4$. Then (3) can be written as

$$\rho = \Phi \int_0^\infty \frac{x^3}{1+e^x} dx$$

5.2 Part b

The dimensionless integral found in part (a) is

$$I = \int_0^\infty \frac{x^3}{e^x+1} dx \qquad (1)$$

But

$$\frac{1}{e^x+1} = \frac{1}{e^x-1} - 2\frac{1}{e^{2x}-1}$$

We did the above, to make the denominator has the form $e^x - 1$, which is easier to work with following the lecture notes than working with $e^x + 1$. Eq (1) now becomes

$$I = \int_0^\infty \frac{x^3}{e^x-1} dx - 2\int_0^\infty \frac{x^3}{e^{2x}-1} dx \qquad (2)$$

The first integral has the standard form $\int_0^\infty \frac{x^n}{e^x-1}dx$. Hence

$$\int_0^\infty \frac{x^3}{e^x-1} = (3!)\xi(4)$$

(Derivations of the above is given at the end of this problem). Now we evaluate on the second integral in (2). Let $y = 2x$, then $\frac{dy}{dx} = 2$. The limits do not change. The integral becomes

$$\int_0^\infty \frac{\frac{y^3}{8}}{e^y-1}\frac{dy}{2} = \frac{1}{16}\int_0^\infty \frac{y^3}{e^y-1}dy$$

We see that $\int_0^\infty \frac{y^3}{e^y-1}dy$ now has the same form as the first integral. Hence $\int_0^\infty \frac{y^3}{e^y-1}dy = (3!)\xi(4)$. Putting these two results back into (2) gives the final result

$$\begin{aligned}
I &= (3!)\xi(4) - 2\left(\frac{1}{16}(3!)\xi(4)\right) \\
&= (3!)\xi(4)\left(1 - 2\left(\frac{1}{16}\right)\right) \\
&= (6)\xi(4)\left(1 - \frac{1}{8}\right) \\
&= (6)\xi(4)\frac{7}{8} \\
&= \frac{21}{4}\xi(4)
\end{aligned}$$

But from class handout, $\xi(4) = \frac{\pi^4}{90}$. Hence

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x+1}dx &= \frac{21}{4}\left(\frac{\pi^4}{90}\right) \\
&= \frac{7}{4}\left(\frac{\pi^4}{30}\right) \\
&= \frac{7}{120}\pi^4 \\
&\approx 5.6822
\end{aligned}$$

Using this in the result obtained in part (a) gives the energy density as

$$\begin{aligned}
\rho &= \Phi\int_0^\infty \frac{x^3}{1+e^x}dx \\
&= \left(\frac{7\pi^4}{120}\right)\left(\frac{4\pi}{h^3c^3}\right)(kT)^4
\end{aligned}$$

Derivation of the integral

In the above, we used the result that $\int_0^\infty \frac{x^n}{e^x-1}dx = (n!)\xi(n+1)$. For $n = 3$ this becomes $(3!)\xi(4)$.

To show how this came above, we start by multiplying the numerator and denominator of the integrand by e^{-x}. This gives

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx \tag{3}$$

Let $y = e^{-x}$ then

$$\begin{aligned}
\frac{e^{-x}}{1-e^{-x}} &= \frac{y}{1-y} \\
&= y\left(1 + y + y^2 + y^3 + \cdots\right) \\
&= y + y^2 + y^3 + \cdots \\
&= \sum_{k=1}^{\infty} y^k \\
&= \sum_{k=1}^{\infty} e^{-kx}
\end{aligned}$$

Using the above sum in Eq (3) gives

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx &= \int_0^\infty x^n \sum_{k=1}^{\infty} e^{-kx} dx \\
&= \sum_{k=1}^{\infty} \int_0^\infty x^n e^{-kx} dx
\end{aligned}$$

Let $z = kx$. Then $\frac{dz}{dx} = k$. When $x = 0, z = 0$ and when $x = \infty, z = \infty$. The above becomes

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx &= \sum_{k=1}^{\infty} \int_0^\infty \left(\frac{z}{k}\right)^n e^{-z} \left(\frac{dz}{k}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} \int_0^\infty z^n e^{-z} dz \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} \left(\int_0^\infty x^n e^{-x} dx\right)
\end{aligned}$$

But $\int_0^\infty x^n e^{-x} dx = n!$, which can be shown by integration by parts repeatedly n times. The above integral now becomes

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = (n!) \sum_{k=1}^{\infty} \frac{1}{k^{n+1}}$$

The sum $\sum_{k=1}^{\infty} \frac{1}{k^{n+1}}$ is called the Zeta function $\zeta(n+1)$. When $n = 3$ the above result becomes

$$\begin{aligned}\int_0^{\infty} \frac{x^3}{e^x - 1} dx &= (3!) \sum_{k=1}^{\infty} \frac{1}{k^4} \\ &= (3!)\zeta(4)\end{aligned}$$

Which is the result used earlier.