University Course

Math 5525
Introduction to Ordinary Differential Equations

University of Minnesota, Twin Cities
Spring 2020

My Class Notes

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Spring 2020

Contents

Chapter 1

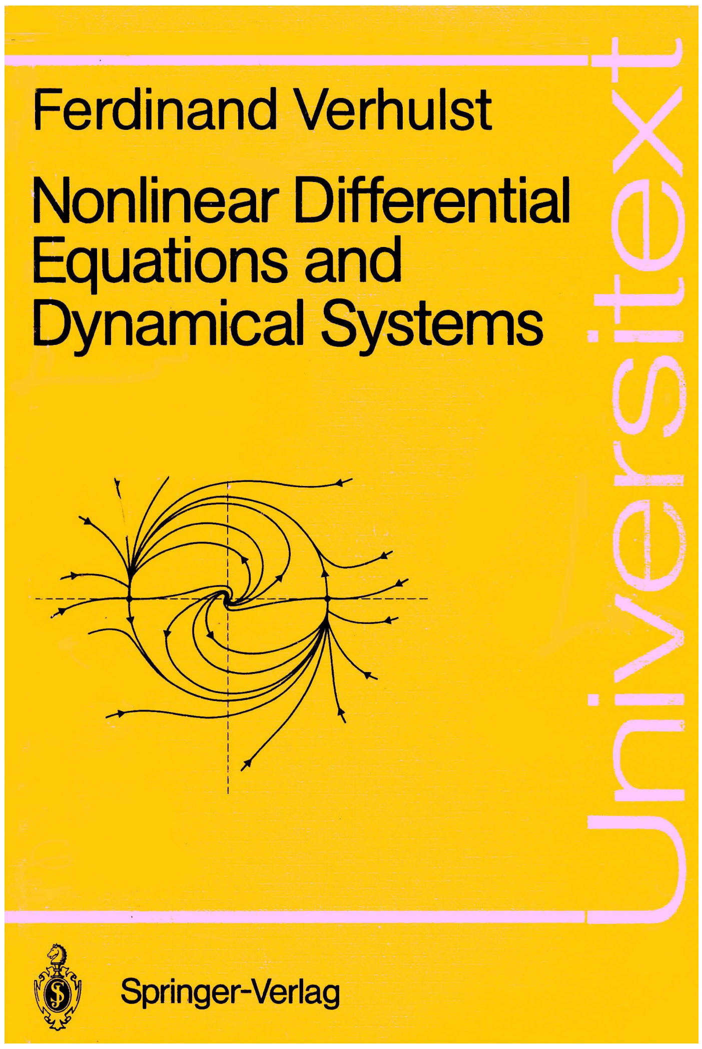
Introduction

Local contents

1.1 Links

1. Instructor web page http://www-users.math.umn.edu/~calde014/courses.html

1.2 Text book



1.3 syllabus

MATH 5525: COURSE SYLLABUS
Class schedule and location: MWF 11:15-12:05, VinH 113

Instructor: M. Carme Calderer

Office: VinH 507

email: calde014@umn.edu

http: http://www-users.math.umn.edu/ calde014/

Office Hours: Tuesday, 3:30-5 pm; Friday, 1:30-3:00 pm.

Textbook: *Nonlinear Differential Equations and Dynamical Systems* by F.Verhulst, Springer.
It may be helpful to consult other reference books on differential equations, available in the University Library. For instance, *Differential Equations, Dynamical Systems and Introduction to Chaos*, by Hirsh, Smale and Devaney.
For issues related to Matlab, I highly recommend the book *Introduction to Matlab* by D.Higham and N.Higham. Students are also encouraged to work through Matlab tutorials (online and from YouTube).

Course Prerequisites: Math 2243 or 2373 (Linear Algebra and Differential Equations) or 2573 (Honors Calculus III), and Math 2263 or 2374 (Multivariable Calculus and Vector Analysis) or 2574 (Honors Calculus IV). Knowledge of Matlab or other programming resource will be needed to solve homework problems.

Course description: One of the main focus of the course is the application of the methods and principles of linear algebra to the study of linear systems of differential equations. Another part of the course deals with second order systems of nonlinear equations, and application of phase plane methods. For both, linear as well as nonlinear systems, we will develop the concepts of stability of fixed points and existence of periodic solutions . The course will also involve theorems and proofs, including , the theory of existence of solutions (sections 1.2 and 1.3) and the Poincare-Bendixon theory (section 4.3). We will study applications relevant to physics and biology. Matlab will be the computational software used in the course (in class, and also needed to complete the homework assignments.)

Course content: We will (approximately) cover the following sections of the book:

1.1, 1.2; 2.1–2.3; 3.1–3.4; 4.1–4.3; 5.1-5.5; 6.1-6.2; 7.1-7.2; 8.1-8.3.

Examples and applications from other bibliographic resources will also be presented in class.

Homework and Assignments: There will be a homework assignment every other week, two midterm and the final examinations. Students are encouraged to work in group for the homework assignments; however each student should individually submit the completed work. Some problems will require the use of Matlab. I strongly encourage to take notes during the class; the pace of the lecture should allow for it.
Class notes will be allowed during the tests.

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The grade of the course will be based upon a weighted average of homeworks and examinations:

Homework: 25 % of the final grade.

Midterm Test 1 (Friday, February 21): 20 %.

Midterm Test 2 (Monday, March 30): 20 %.
Final examination (Comprehensive. Monday, May 11, 10:30-12:30; VinH 113): 35%.

Course policies.

- The final examination is compulsory. Failure to take it will automatically result in the grade 'F' in the course.
- Missing homework assignments and midterm tests will count as 0.
- Please, do not wait to the day before a test or assignment deadline to ask for help. Otherwise, I cannot guarantee to provide the appropriate individual assistance to students.
- All University of Minnesota policies regarding teaching and instruction apply to the course.

Chapter 2

HWs

Local contents

2.1 HW 1

Local contents

2.1.1 Problem 1

The logistic population growth model is given by the first order, nonlinear differential equations

$$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right)$$

Where $x = x(t)$ denotes the number of individuals of a population group at time $t \geq 0$. $N > 0$, integer, is the carrying capacity, that is, the maximum number of individuals that the environment allows (e.g. based on available resources, such as food, access to water,...). The positive number a represents the growth rate. (1) Obtain the exact solution of the equation (1) corresponding to the initial data $x(0) = x_0$ where $0 < x_0 < N$. (2) Obtain the equilibrium solutions of the problem. (3) Determine the stability of the equilibrium solutions. (4) Let $N = 100$. Plot the solution corresponding to initial data $x(0) = 50$.

Solution

2.1.1.1 part 1

$$\begin{aligned}\frac{dx}{dt} &= ax\left(1-\frac{x}{N}\right)\\ x(0) &= x_0\end{aligned}$$

This is separable first order ODE. Therefore

$$\frac{dx}{ax\left(1-\frac{x}{N}\right)} = dt$$

Integrating both sides gives

$$\begin{aligned}\int_{x_0}^{x} \frac{dz}{az\left(1-\frac{z}{N}\right)}dz &= \int_0^t d\tau\\ \frac{1}{a}\int_{x_0}^{x} \frac{dz}{z\left(1-\frac{z}{N}\right)}dz &= t \qquad (1)\end{aligned}$$

Applying partial fractions to $\frac{1}{z\left(1-\frac{z}{N}\right)}$ gives

$$\frac{1}{z\left(1-\frac{z}{N}\right)} = \frac{A}{z} + \frac{B}{1-\frac{z}{N}}$$

Hence $A = \left.\frac{1}{\left(1-\frac{z}{N}\right)}\right|_{z=0} = 1$ and $B = \left.\frac{1}{z}\right|_{z=N} = \frac{1}{N}$. Therefore $\frac{1}{z\left(1-\frac{z}{N}\right)} = \frac{1}{z} + \frac{1}{N}\frac{1}{1-\frac{z}{N}} = \frac{1}{z} + \frac{1}{N-z}$ and (1) now becomes

$$\begin{aligned}\frac{1}{a}\int_{x_0}^{x} \frac{1}{z} + \frac{1}{N-z}dz &= t\\ \int_{x_0}^{x} \frac{1}{z}dz + \int_{x_0}^{x} \frac{1}{N-z}dz &= at\end{aligned}$$

But $\int \frac{1}{z}dz = \ln|z|$ and $\int \frac{1}{N-z}dz = -\ln|N-z|$ and the above becomes

$$\begin{aligned}\ln\left|\frac{x}{x_0}\right| - \ln\left|\frac{N-x}{N-x_0}\right| &= at\\ \ln\left|\frac{\frac{x}{x_0}}{\frac{N-x}{N-x_0}}\right| &= at\\ \ln\left|\frac{x(N-x_0)}{x_0(N-x)}\right| &= at\\ \ln\left|\frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right)\right| &= at\end{aligned}$$

Since $N > 0$ and $N > x_0$ and $x_0 > 0$ and since N is the carrying capacity, then hence $N - x > 0$), therefore $\left|\frac{N-x_0}{N-x}\right|$ is positive. The absolute sign can be removed and the above simplifies to

$$\ln \frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right) = at$$

Taking the exponential of both sides gives

$$\begin{aligned}
\frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right) &= e^{at} \\
x(N-x_0) &= (N-x)\,x_0e^{at} \\
x(N-x_0) &= Nx_0e^{at} - xx_0e^{at} \\
x(N-x_0) + xx_0e^{at} &= Nx_0e^{at} \\
x\left(N - x_0 + x_0e^{at}\right) &= Nx_0e^{at} \\
x(t) &= \frac{Nx_0e^{at}}{N - x_0 + x_0e^{at}}
\end{aligned}$$

Dividing RHS numerator and denominator by e^{at} gives the analytical solution as

$$x(t) = \frac{Nx_0}{x_0 + (N-x_0)\,e^{-at}} \qquad a > 0$$

2.1.1.2 part 2

Equilibrium solution is when $\frac{dx}{dt} = 0$, which implies $ax\left(1 - \frac{x}{N}\right) = 0$. This gives $\underline{x = 0}$ or $1 - \frac{x}{N} = 0$ which gives $\underline{x = N}$.

2.1.1.3 part 3

Let

$$\begin{aligned}
\frac{dx}{dt} &= f(x) \\
&= ax\left(1 - \frac{x}{N}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
f'(x) &= a\left(1 - \frac{x}{N}\right) + ax\left(-\frac{1}{N}\right) \\
&= a - a\frac{x}{N} - a\frac{x}{N} \\
&= a - 2a\frac{x}{N} \qquad (1)
\end{aligned}$$

When $x = 0$ the above shows that $f'(x) = a > 0$ since a is always positive. Since the slope of $f(x)$ is positive then $x = 0$ is unstable equilibrium.

At $x = N$ then (1) becomes $f'(x) = a - 2a = -a$. Since the slope of $f(x)$ is negative then $x = N$ is stable equilibrium.

2.1.1.4 part 4

When $N = 100, x(0) = 50$, the solution found above $x(t) = \frac{Nx_0}{x_0 + (N-x_0)e^{-at}}$ now becomes

$$\begin{aligned}
x(t) &= \frac{(100)(50)}{50 + (100-50)\,e^{-at}} \\
&= \frac{5000}{50\left(1 + e^{-at}\right)} \\
&= \frac{100}{1 + e^{-at}}
\end{aligned}$$

The above shows that as $t \to \infty$ and since $a > 0$ then $x(t) \to 100$ which is N, the limiting capacity as expected. This is plot of the above for different $a > 0$ numerical values.

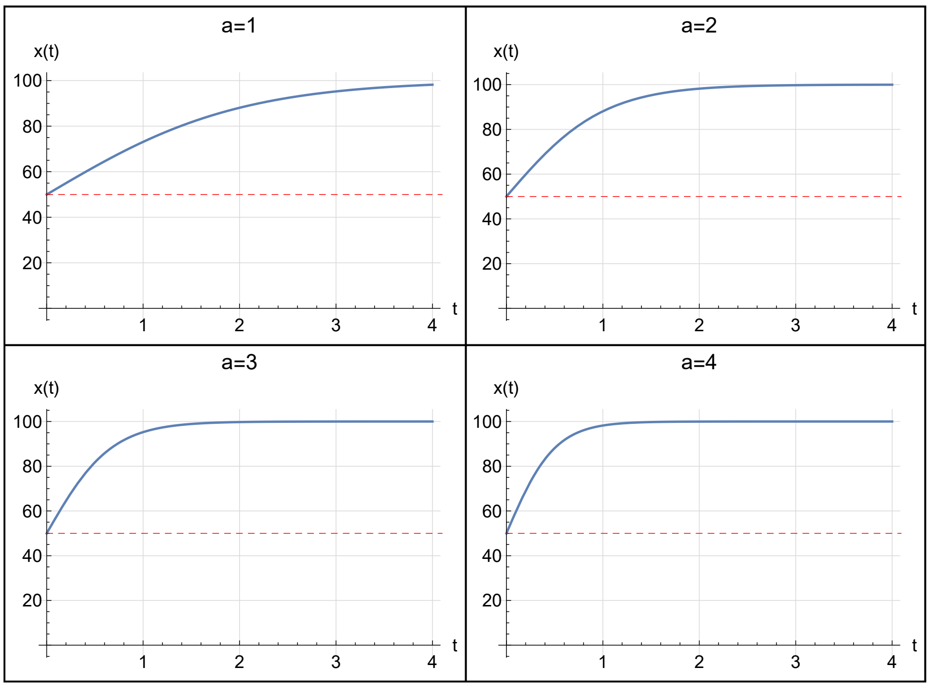


Figure 2.1: Solution $x(t)$ for different a values

```
x[t_, a_] := 100/(1 + Exp[-a t])
p = Table[Plot[x[t, a], {t, 0, 4}, AxesOrigin → {0, 0},
               PlotLabel → Row[{"a=", a}],
               ImageSize → 300,
               AxesLabel → {"t", "x(t)"},
               BaseStyle → 12,
               GridLines → Automatic,
               GridLinesStyle → LightGray,
               Epilog → {Red, Dashed, Line[{{0, 50}, {5, 50}}]}
            ], {a, {1, 2, 3, 4}}];
p = Grid[Partition[p, 2], Frame → All];
```

Figure 2.2: Code used for the above plot

Observations As the growth rate a increases in value, the population $x(t)$ reaches its limiting value $N = 100$ more rapidly as expected. The line shown in dashed red is the initial population size of 50. Once limiting population size if reached, the population size do not change any more with time.

2.1.2 Problem 2

(1) Solve exercise 2.5, page 24, of the textbook: Find the critical points of the system

$$\dot{x} = y$$
$$\dot{y} = x - 2x^3$$

Characterize the critical points by linear analysis and determine their attraction properties.
(2) Plot the phase plane of the system.

Solution

2.1.2.1 Part 1

This is non-linear second order system.

$$\begin{aligned}\dot{x} &= f_1 = y\\ \dot{y} &= f_2 = x\left(1-2x^2\right)\end{aligned}$$

The critical points are $y=0$ and $x\left(1-2x^2\right)=0$ or $x=0$ and $1-2x^2=0$ which gives $x^2=\frac{1}{2}$ or $x=\pm\frac{1}{\sqrt{2}}$. Hence there are 3 critical points are

$$(x,y)=\left\{(0,0),\left(\frac{1}{\sqrt{2}},0\right),\left(-\frac{1}{\sqrt{2}},0\right)\right\}$$

To find the if critical points are stable or not, the system is linearized the system and each eigenvalue is examined. The Jacobian matrix of linearized system is

$$\begin{aligned}J &= \begin{pmatrix}\frac{\partial\dot{x}}{\partial x} & \frac{\partial\dot{x}}{\partial y}\\ \frac{\partial\dot{y}}{\partial x} & \frac{\partial\dot{y}}{\partial y}\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1-6x^2 & 0\end{pmatrix}\end{aligned}$$

Point $(0,0)$ At this point the Jacobian matrix becomes $J=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Its eigenvalues are found from $|\det(A)-\lambda I|=0$ which gives

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= 0\\ \lambda^2-1 &= 0\end{aligned}$$

Hence $\lambda=\pm1$. Since one of the eigenvalues is positive, then $(0,0)$ is unstable and the whole system is considered unstable. The second (negative) eigenvalue is stable, which leads to $(0,0)$ being saddle point. (which is considered unstable)

Point $\left(\frac{1}{\sqrt{2}},0\right)$ At this point $J=\begin{pmatrix}0&1\\1-6x^2&0\end{pmatrix}=\begin{pmatrix}0&1\\1-\frac{6}{2}&0\end{pmatrix}=\begin{pmatrix}0&1\\-2&0\end{pmatrix}$. The eigenvalues are

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ -2 & -\lambda\end{vmatrix} &= 0\\ \lambda^2+2 &= 0\\ \lambda^2 &= -2\end{aligned}$$

The solution is $\lambda_1=-\sqrt{2}i,\lambda_2=\sqrt{2}i$. Since this is pure complex conjugate (zero real part) then the critical point is central point considered stable point (sometimes also called marginally stable). The solutions around this point are periodic.

Point $\left(\frac{-1}{\sqrt{2}},0\right)$ At this point $J=\begin{pmatrix}0&1\\1-\frac{6}{2}&0\end{pmatrix}=\begin{pmatrix}0&1\\1-6x^2&0\end{pmatrix}=\begin{pmatrix}0&1\\-2&0\end{pmatrix}$. This is the same as the above.

The eigenvalues are $\lambda_1=-\sqrt{2}i,\lambda_2=\sqrt{2}i$. Which lead to a central point. The solutions around this point are periodic.

2.1.2.2 Part 2

Writing

$$\begin{aligned}\dot{x} &= f_1 = y\\ \dot{y} &= f_2 = x-2x^3\end{aligned}$$

The actual phase plane orbit equation can be found by solving $\frac{dy}{dx}=\frac{f_2}{f_1}=\frac{x-2x^3}{y}$ or $ydy=\left(x-2x^3\right)dx$. Integrating gives

$$\frac{1}{2}y^2=\left(\frac{1}{2}x^2-\frac{2}{3}x^4\right)+C$$
$$\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{2}{3}x^4=C$$

For different constant C, different orbit result. But instead of plotting the above equation for different C, the phase plot is generated using two methods as it was not clear which method to use.

First method This is the manual method. The system is linearized as above, and for each critical point, the eigenvectors are found. From first part we found that at Point $(0,0)$. The eigenvalues are $\lambda=\pm 1$. Hence for $\lambda=1$ the system $(A-\lambda I)v=0$ where v is the eigenvector corresponding to λ becomes

$$\begin{pmatrix}0-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Which gives from first equation $-v_1+v_2=0$ or $v_1=v_2$. By assuming $v_1=1$, the first eigenvector is $\begin{pmatrix}1\\ 1\end{pmatrix}$. For $\lambda=-1$ the system becomes

$$\begin{pmatrix}0-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Which gives $v_1+v_2=0$ or $v_2=-v_1$. By assuming $v_1=1$, the second eigenvector is $\begin{pmatrix}1\\ -1\end{pmatrix}$. Now the direction along along each eigenvector is found. Starting with the first eigenvector $\begin{pmatrix}1\\ 1\end{pmatrix}$ which spans from first quadrant to 3rd quadrant. We recall that the system is

$$\dot{x}=y$$
$$\dot{y}=x\left(1-2x^2\right)$$

In first quadrant, $y>0$. This means from above that $\dot{x}>0$ which means x is increasing in first quadrant. Also in first quadrant, $x>0$ which means when x is close to zero such that $1-2x^2$ is positive, then $\dot{y}>0$ from the second equation above, which means y is increasing also in first quadrant. In the 3rd quadrant, $y<0$ which means $\dot{x}<0$ and hence x is decreasing. Also, in 3rd quadrant $y<0$ which means for x close to zero $\dot{y}<0$ and hence y is decreasing as well. This means that the first eigenvector points away from the origin in first and third quadrant.

The second eigenvector $\begin{pmatrix}1\\ -1\end{pmatrix}$ extends from second quadrant to 4th quadrant. In 4th quadrant, $y<0$ hence $\dot{x}<0$ which means x is decreasing (getting closer to the origin). In the 4th quadrant, $x>0$ which for x close to zero such that $1-2x^2$ remain positive, $\dot{y}>0$ and hence y is increasing (getting closer to origin). Now, in the second quadrant, $y>0$ which means $\dot{x}>0$ which means x is increasing (getting closer to origin) and in the second quadrant $x<0$ hence for values of x near zero, $\dot{y}<0$ which means y is decreasing (getting closer to origin). Therefore on the second eigenvector all solutions move closer to origin (this is stable eigenvector). This is how the phase plot looks now around $(0,0)$

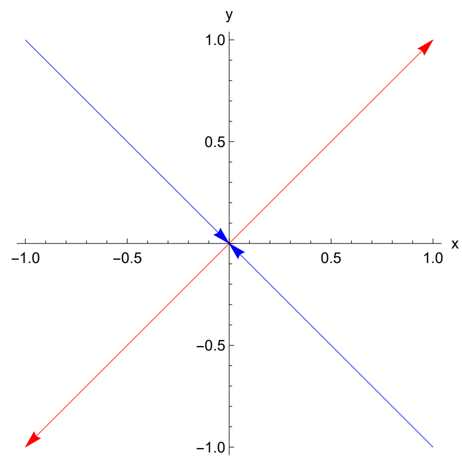


Figure 2.3: Manually making phase plot around $(0, 0)$

The next step is to manually draw the rest of the phase plot lines in each of the four regions as follows by continuity

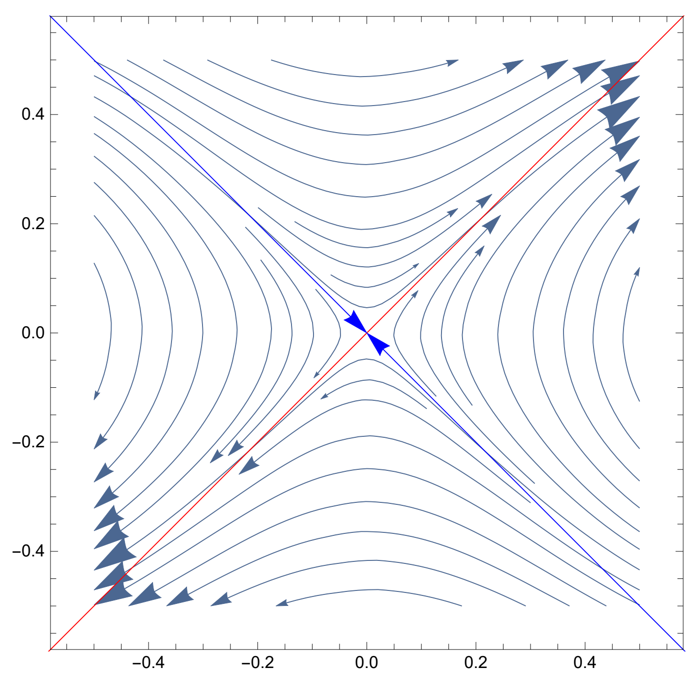


Figure 2.4: Manually making phase plot around $(0, 0)$

The same thing is done for each remaining critical point $\left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)$ in the manual method and will not be repeated as the same steps as above.

Now the second method is applied, which is to numerically generate phase plot directly for the non-linear system.

The 3 critical points are marked on the following plot. Unable point is colored in red and the stable critical points are colored in green. The plot below shows that $(0, 0)$ is unstable (saddle) as shown above using the manual method and the points $\left(\pm\frac{-1}{\sqrt{2}}, 0\right)$ are central since solutions move around it in circular orbits. (Periodic solutions).

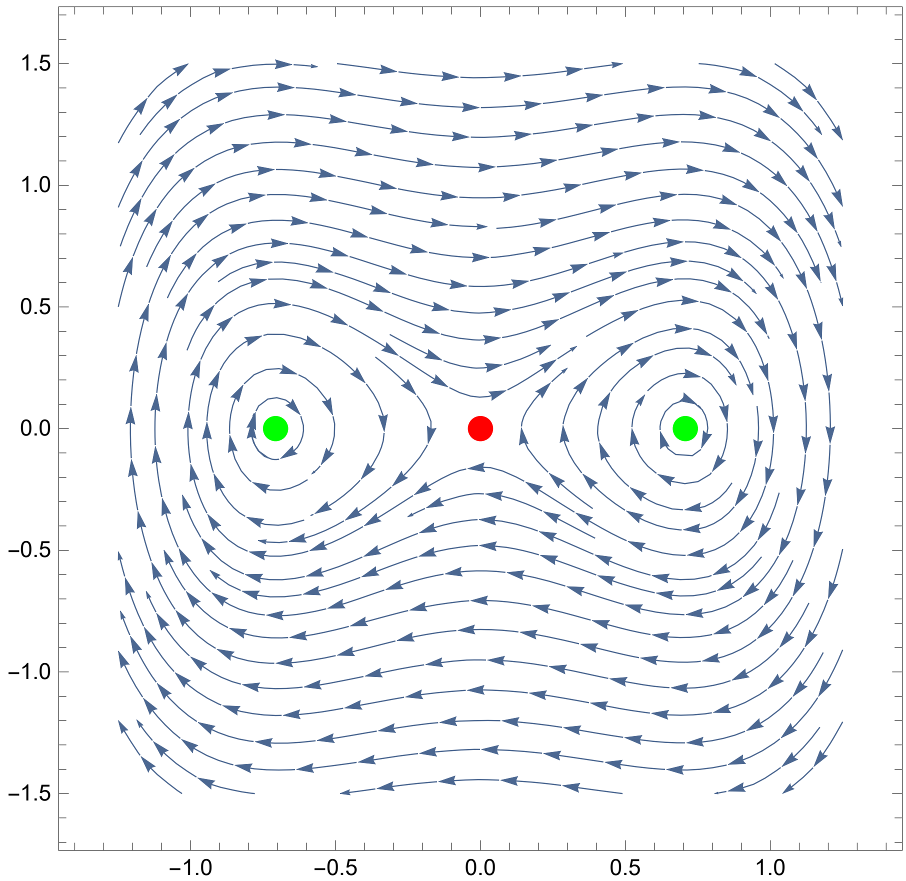


Figure 2.5: Phase plot

```
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Green, PointSize[0.03], Point[{1/Sqrt[2], 0}]};
p3 = {Green, PointSize[0.03], Point[{-1/Sqrt[2], 0}]};
p = StreamPlot[{y, x - 2 x^3}, {x, -1.25, 1.25}, {y, -1.5, 1.5},
        Epilog → {p1, p2, p3}
    ];
```

Figure 2.6: Code used for the above plot

2.1.3 Problem 3

(1) Solve exercise 2.3, page 23, of the textbook: We are studying the three-dimensional system

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right) \\ \dot{x}_2 &= x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right) \\ \dot{x}_3 &= \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)\end{aligned} \tag{A}$$

(a) Determine the critical points of this system. (b) Show that the planes $x_3 = 0$ and $x_3 = 1$ are invariant sets. (c) Consider the invariant set $x_3 = 1$. Does this set contain periodic solutions?

(2) Plot the phase plane of the system

<u>Solution</u>

2.1.3.1 Part 1.a

From the third equation, let $\dot{x}_3 = 0$, then

$$\left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right) = 0$$

Hence $\underline{x_3 = 1}$ or $x_3\left(1 + 2x_2^2 + x_3^2\right) = 0$, which gives additional solutions $\underline{x_3 = 0}$ or $\left(1 + 2x_2^2 + x_3^2\right) = 0$. When $x_3 = 0$ this becomes $1 + 2x_2^2 = 0$ which does not give real solution in x_2. Now, when $x_3 = 1$ then $\left(1 + 2x_2^2 + x_3^2\right) = 0$ gives $2 + 2x_2^2 = 0$ which also do not give real solution.

Considering now the first and second equations in (A) and set each to zero which gives

$$\begin{aligned} x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right) &= 0 \qquad (1)\\ x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right) &= 0 \end{aligned}$$

When $x_3 = 0$ the above becomes

$$\begin{aligned} x_1 - x_1x_2 - x_2^3 &= 0\\ x_1 &= 0 \end{aligned}$$

Which gives solutions $x_1 = 0$ from the second equation. This results in $x_2 = 0$ from the first equation. Hence the point $\underline{(0,0,0)}$ is the first critical point.

Now, when $x_3 = 1$ EQ (1) becomes

$$\begin{aligned} x_1 - x_1x_2 - x_2^3 + x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3 &= 0\\ x_1 - \left(x_1 - x_2 + 2x_1x_2\right) &= 0 \end{aligned}$$

Or

$$\begin{aligned} x_1^2 + x_2^2 - 1 &= 0\\ x_2 - 2x_1x_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} x_1^2 + x_2^2 - 1 &= 0\\ x_2\left(1 - 2x_1\right) &= 0 \end{aligned}$$

From the second equation above $x_2 = 0$ or $x_1 = \frac{1}{2}$. When $x_2 = 0$ the first equation above gives $x_1 = \pm 1$. Hence second critical point is $\underline{(\pm 1,0,1)}$. And when $x_1 = \frac{1}{2}$ the first equation gives

$$\begin{aligned} \frac{1}{4} + x_2^2 - 1 &= 0\\ x_2^2 &= \frac{3}{4}\\ x_2 &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Hence critical point is $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 1\right)$

In summary, the critical points are

$$(0,0,0), (\pm 1,0,1), \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 1\right)$$

2.1.3.2 Part 1.b

A set S is invariant, if when initial conditions are in S, then the overall solution remain in S for all time. From the third equation in (A) for $\dot{x}_3$

$$\begin{aligned} \dot{x}_3 &= \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)\\ &= \left(x_3 - 1\right)x_3\left(1 + 2x_2^2 + x_3^2\right) \end{aligned}$$

The above shows that when $\dot{x}_3 = 0$, then $x_3 = 1$ or $x_3 = 0$ are the solutions. The term $1 + 2x_2^2 + x_3^2 = 0$ does not give real solutions hence not used. So only $x_3 = 1, x_3 = 0$ are only possible solutions. Therefore these are invariant sets. Any solution with initial conditions $x_3 = 0$ or $x_3 = 0$ will remain in the set $x_3 = 0$ or $x_3 = 0$ respectively.

2.1.3.3 Part 1.c

When $x_3 = 1$ the system reduces to

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\\ \dot{x}_2 &= x_1 - (x_1 - x_2 + 2x_1x_2)\end{aligned}$$

Or

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2^2 - 1\\ \dot{x}_2 &= x_2(1 - 2x_1)\end{aligned} \tag{1}$$

To see if there are periodic solutions, the phase plot is drawn around the critical points $(\pm 1, 0), \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ to see if there are closed orbits or not. Here is the result for the set $x_3 = 1$

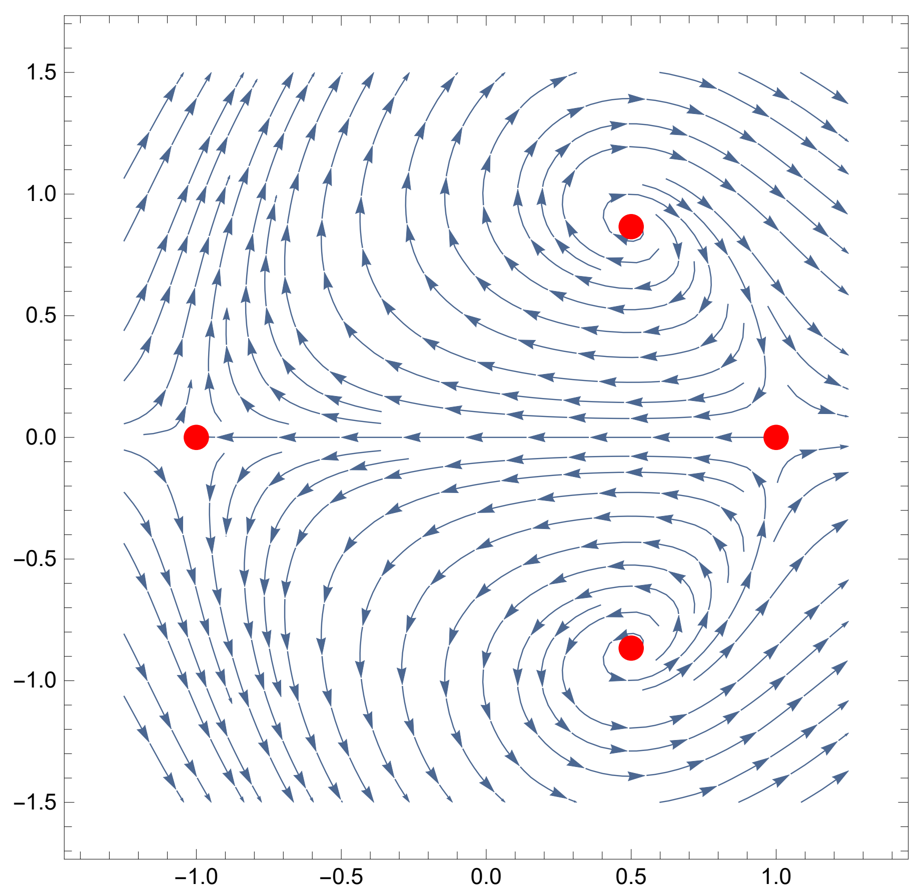


Figure 2.7: Phase plot when $x_3 = 1$

The above shows that there are no closed orbits. This implies no periodic solutions exist.

Another way to find this without using the computer, is to do the following: We linearize the system (1) and then determine the eigenvalues for each critical point. Since this is second order system, then only eigenvalues that pair of complex conjugate will indicate a periodic solution (which is consider to be stable). To linearize (1), we first find the Jacobian matrix, which is

$$\begin{aligned}J &= \begin{pmatrix}\frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2}\\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2}\end{pmatrix}\\ &= \begin{pmatrix}2x_1 & 2x_2\\ -2x_2 & 1 - 2x_1\end{pmatrix}\end{aligned}$$

Critical point $(1, 0)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which has the $-1, 1$. Not stable. No periodic solutions around this point.

Critical point $(-1, 0)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

Which has eigenvalues $3, -2$. Not stable. No periodic solutions around this point.

Critical point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}$$

Which has eigenvalues: $\frac{1}{2} + \frac{1}{2} i\sqrt{11}, \frac{1}{2} - \frac{1}{2} i\sqrt{11}$. This is not not stable because the real part is positive, hence can not be periodic.

Critical point $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$$

Which has same eigenvalues as above $\frac{1}{2} + \frac{1}{2} i\sqrt{11}, \frac{1}{2} - \frac{1}{2} i\sqrt{11}$. This is not not stable because the real part is positive, hence can not be periodic. Therefore we see that no periodic solutions exist.

2.1.3.4 Part 2

In part 1 above, the phase plot for the set $x_3 = 1$ is already given. The following is the phase plot for the set $x_3 = 0$. When $x_3 = 0$ the system reduces to

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1 x_2 - x_2^3 \\ \dot{x}_2 &= x_1 \end{aligned}$$

The critical points are $x_1 = 0$ from the second equation and from the first equation this gives $x_2 = 0$. Hence $(0, 0)$ is the only critical point. To determine if stable or not, the Jacobian is found

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 1 - x_2 & -x_1 - 3x_2 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Evaluated at $x_1 = 0, x_2 = 0$ the above becomes

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues are

$$\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 = 0$$
$$\lambda = 0$$

Double root. Since the system is nonlinear, and since $\lambda = 0$, then unable to determine stability of the non-linear system from the linearized system. Will have to use the phase plot to check stability of $(0,0)$ as given below.

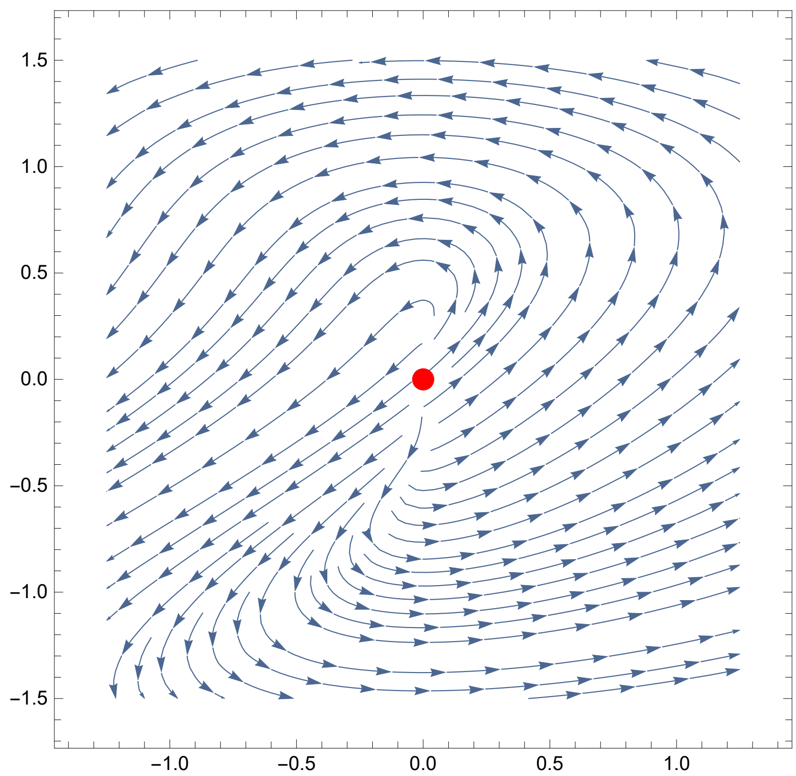


Figure 2.8: Phase plot when $x_3 = 0$

```
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p =
    StreamPlot[{x1 - x1 * x2 - x2^3, x1}, {x1, -1.25, 1.25}, {x2, -1.5, 1.5}, Epilog → {p1}];
```

Figure 2.9: Code used for the above

From the above phase plot, it shows that $(0,0)$ critical point is <u>not stable</u> because solutions that starts near $(0,0)$ move away from equilibrium.

2.1.4 key solution for HW 1

Page 1

Math 5525 - Solution set 1.

1. $\dot{x} = ax\left(1 - \frac{x}{N}\right)$, $N > 0$, integer, $a > 0$

(1) $\iff \dot{x} = \alpha x (N - x)$, $\alpha \equiv \frac{a}{N}$.

$$\frac{1}{x(N-x)} = \frac{A}{x} + \frac{B}{N-x} = \frac{A(N-x) + Bx}{x(N-x)} \Rightarrow A = B = 1/N$$

(partial fractions)

$$\frac{1}{N}\int^{x} \frac{dx}{x} + \frac{dx}{N-x} = \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = \alpha t + K_0$$

K_0 arbitrary const.

$$\ln\left(\frac{x}{N-x}\right) = at + K, \quad \frac{x}{N-x} = Ce^{at}$$

$0 < x < N$

$$t = 0: \quad C = \frac{1}{\frac{N}{x_0} - 1} = \frac{x_0}{N - x_0}$$

$$\therefore \quad x = \frac{NCe^{at}}{1 + Ce^{at}} = \frac{Ne^{at}}{1/C + e^{at}} = \frac{N}{\frac{1}{C}e^{-at} + 1}$$

$$= \frac{N}{\frac{N - x_0}{x_0}e^{-at} + 1}$$

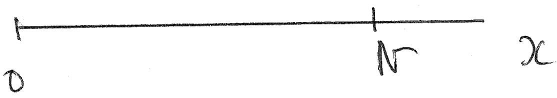
$x_0 < N$

Note that $x(t) \to N$ as $t \to \infty$.

(2) $x = 0$, $x = N$ are equilibrium solutions.

Page 2

(3)



Note that $\dot{x}(t) > 0$ for $0 < x < N$

Hence, solutions with initial data $0 < x_0 < N$ satisfy $\dot{x}(t) > 0$, $\forall t$, and since $x = N$ is an equilibrium point $x(t) \underset{t \to \infty}{\longrightarrow} N$.

$\therefore$ $x = N$ is a stable equilibria
$x = 0$ " an unstable " .

(4) Plot.

2.
$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

Equilibrium points satisfy $y = 0$ and $x - x^3 = 0$

$0 = x - x^3 = x(1 - x^2) \Rightarrow x = 0,\ x = \pm 1$

$(0, 0)$, $(1, 0)$, $(-1, 0)$.

To classify the equilibrium points, we study the linearized system about equilibrium.

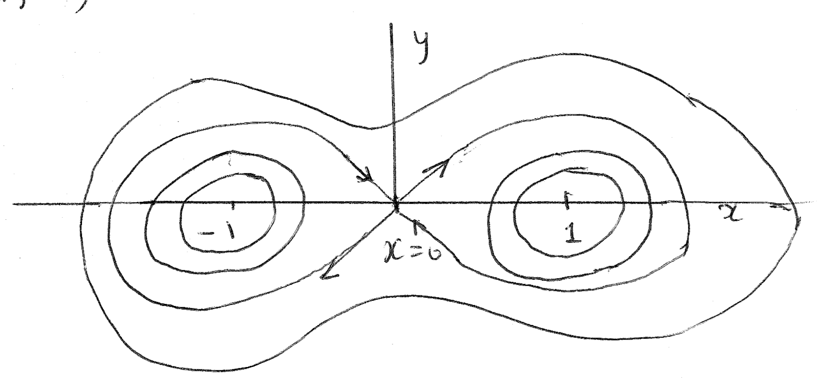
First, denote $g(x) = -x^3 + x$ and $x = x^*$ a value of x corresponding to equilibrium.

Page 3

$x^* = \pm 1$ $\quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\quad \lambda = \pm\sqrt{2}i$ (eigenvalues)

$\begin{bmatrix} 1 \\ \pm\sqrt{2}i \end{bmatrix}$ (eigenvectors)

(1, 0) and (-1, 0) are centers.



To plot the phase plane, note that the system has a first integral (energy integral). Indeed, it can be written as

$$\ddot{x} = \dot{y} = x - x^3$$

$$\dot{x}\ddot{x} = (x - x^3)\dot{x} \Rightarrow \frac{1}{2}(\dot{x})^2 + \frac{1}{2}\left(\frac{x^2}{2} - 1\right)x^2 = E$$

constant.

Equivalently, the equations of the orbits are $\frac{1}{2}y^2 + \frac{1}{2}\left(\frac{x^2}{2} - 1\right)x^2 = E$.

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Taylor expansion of $g(x)$ about $x = x^*$:

$$g(x) = \underbrace{g(x^*) + g'(x^*)(x - x^*)}_{\text{Linearization}} + o(x - x^*)$$

Moreover, since $g(x^*) = 0$, we have

$$g(x) = g'(x^*)(x - x^*) + o(x - x^*)$$

Linearized system about equilibrium:

[Notation $u_1 = x - x^*$, $u_2 = y$].

$$\dot{u}_1 = u_2$$

$$\dot{u}_2 = g'(x^*) u_1$$

$$\rightarrow \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g'(x^*) & 0 \end{bmatrix}}_{\equiv A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$x^* = 0$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Eigenvalues $\lambda = \pm 1$

Eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(\lambda = 1)$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $(\lambda = -1)$.

$\therefore$ $(0,0)$ is a saddle point of the system.

$x^* = \pm 1$, $g'(x^*) = 1 - 3(x^*)^2 = 1 - 3(\pm 1)^2 = -2$.

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3. $\dot{x}_1 = x_1 - x_1 x_2 - x_2^3 + x_3(x_1^2 + x_2^2 - 1 - x_1 + x_1 x_2 + x_2^3)$
$\dot{x}_2 = x_1 - x_3(x_1 - x_2 + 2x_1 x_2)$
$\dot{x}_3 = (x_3 - 1)(x_3 + 2x_3 x_2^2 + x_3^3)$

Equilibrium points: $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3$
(Right hand side expressions of the system are identically 0).

Start with 3th equation: make the RHS equals to 0. $x_3 = 0$ and $x_3 = 1$

$\underline{x_3 = 0}$: $\dot{x}_1 = x_1 - x_1 x_2 - x_2^3$
$\dot{x}_2 = x_1$

$x_1 = 0$, $x_2 = 0$

$(0,0,0)$ equilibrium point

$\underline{x_3 = 1}$: (a) $\dot{x}_1 = x_1 - x_1 x_2 - x_2^3 + (x_1^2 + x_2^2 - 1 - x_1 + x_1 x_2 + x_2^3)$
$= x_1^2 + x_2^2 - 1$

(b) $\dot{x}_2 = x_1 - (x_1 - x_2 + 2x_1 x_2) = x_2 - 2x_1 x_2$

$\dot{x}_2 = 0 \Rightarrow x_2 = 0$ or $x_1 = 1/2$

eqn (a) ↓ : $x_1^2 - 1 = 0$, $x_1 = \pm 1$

↓ eqn (a): $x_2^2 = 1 - \frac{1}{4} = \frac{3}{4}$, $x_2 = \pm \frac{\sqrt{3}}{2}$

$\boxed{(\pm 1, 0, 1)}$

$\boxed{(1/2, \pm\frac{\sqrt{3}}{2}, 1)}$

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Invariant sets:

$x_3 = 0$ or $x_3 = 1$

Solutions with initial data $(x_1^0, x_2^0, 0)$ and $(x_1^0, x_2^0, 1)$ will satisfy $x_3(t) = 0$ and $x_3(t) = 1$, respectively, for all $t \geq 0$.

$\therefore$ The planes $x_3 = 0$ and $x_3 = 1$ are invariant sets of the system.

The system restricted to the plane $x_3 = 1$ is

$$\dot{x}_1 = x_1^2 + x_2^2 - 1$$
$$\dot{x}_2 = x_2 - 2x_1x_2$$

Equilibrium solutions of the plane system:

$(1/2, 0)$, $(\pm 1, \pm \frac{\sqrt{3}}{2})$

$(1/2, 0)$ → unstable (special case of one zero eigenvalue)

$(+1, \frac{\sqrt{3}}{2})$: unstable spiral

$(-1, -\frac{\sqrt{3}}{2})$: saddle point

The absence of centers indicates that the system does not have any closed orbits.

2.2 HW 2

Local contents

2.2.1 Problem 3.1

Consider the two-dimensional system in R^2

$$\dot{x} = y\left(1 + x - y^2\right)$$
$$\dot{y} = x\left(1 + y - x^2\right)$$

Determine the critical points and characterize the linearized flow in a neighborhood of these points.

solution

Let

$$\dot{x} = y\left(1 + x - y^2\right) = f_1\left(x, y\right) \tag{1}$$
$$\dot{y} = x\left(1 + y - x^2\right) = f_2\left(x, y\right) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. Equation $f_1 = 0$ gives the following solutions

$$y = 0 \tag{3}$$
$$1 + x - y^2 = 0 \tag{4}$$

Starting with (3). Substituting in (2) the solution $y = 0$ gives

$$x\left(1 - x^2\right) = 0$$

This gives solutions $x = 0$ or $x = \pm 1$. The first set of critical points generated from (3) is $(0, 0), (1, 0), (-1, 0)$. Now we do the same starting from (4). Solving (4) for x gives

$$x = y^2 - 1 \tag{5}$$

Substituting for x from above back into (2) gives

$$\left(y^2 - 1\right)\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$$

This gives solutions $y^2 - 1 = 0$ or $\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$. Starting $y^2 - 1 = 0$. This gives $y = \pm 1$. From (5) this gives $x = 0$ for both cases. So now we can add the next set of critical points found so far $(0, 1), (0, -1)$.

When $\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$, or $1 + y - y^4 - 1 + 2y^2 = 0$ or $y^4 - 2y^2 - y = 0$ or $y\left(y^3 - 2y - 1\right) = 0$. Hence $y = 0$ which from (5) gives another critical point $x = -1$. Hence $(-1, 0)$. This critical point is one already found earlier. For $y^3 - 2y - 1 = 0$, this gives solutions $y = -1, y = \frac{1}{2}\left(1 - \sqrt{5}\right), y = \frac{1}{2}\left(1 + \sqrt{5}\right)$. From each one of these solutions, using EQ. (5) gives x. When $y = -1$, then (5) gives $x = 0$ and when $y = \frac{1}{2}\left(1 - \sqrt{5}\right)$ then (5) gives

$$x = \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^2 - 1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$$

And when $y = \frac{1}{2}\left(1 + \sqrt{5}\right)$ then (5) gives

$$x = \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^2 - 1 = \frac{1}{2}\sqrt{5} + \frac{1}{2}$$

Therefore we have found the following 3 extra critical points

$$(0, -1), \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}, \frac{1}{2}\left(1 - \sqrt{5}\right)\right), \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}, \frac{1}{2}\left(1 + \sqrt{5}\right)\right)$$

In summary, the following are all the critical points found. There are 7 of them

$$
\begin{aligned}
(x,y)^* &= (0,0)\\
&= (1,0)\\
&= (-1,0)\\
&= (0,1)\\
&= (0,-1)\\
&= \left(\frac{1}{2}\left(1-\sqrt{5}\right),\frac{1}{2}\left(1-\sqrt{5}\right)\right)\\
&= \left(\frac{1}{2}\left(1+\sqrt{5}\right),\frac{1}{2}\left(1+\sqrt{5}\right)\right)
\end{aligned}
$$

To characterize the linearized flow in a neighborhood of these points, the Jacobian matrix is evaluated at each of the critical points and from its eigenvalues, the type of critical point is determined.

$$
\begin{aligned}
J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\\
&= \begin{pmatrix} y & \left(1+x-y^2\right)+y\left(-2y\right)\\ \left(1+y-x^2\right)+x\left(-2x\right) & x\end{pmatrix}\\
&= \begin{pmatrix} y & -3y^2+x+1\\ -3x^2+y+1 & x\end{pmatrix}
\end{aligned}
$$

At Point $(0,0)$ the Jacobian matrix becomes

$$J = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$$

Hence $|J-\lambda I| = 0$ gives

$$
\begin{aligned}
\begin{vmatrix} -\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= 0\\
\lambda^2 - 1 &= 0\\
\lambda &= \pm 1
\end{aligned}
$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable).

At Point $(1,0)$ the Jacobian is

$$J = \begin{pmatrix} 0 & 2\\ -2 & 1\end{pmatrix}$$

Hence $|J-\lambda I| = 0$ gives

$$
\begin{aligned}
\begin{vmatrix} -\lambda & 2\\ -2 & 1-\lambda\end{vmatrix} &= 0\\
-\lambda\left(1-\lambda\right)+4 &= 0\\
\lambda^2-\lambda+4 &= 0\\
\lambda &= \frac{1}{2} \pm \frac{1}{2} i\sqrt{15}
\end{aligned}
$$

Since the real part is positive, then this is unstable point. Spiral out. (book calls this focus with negative attraction).

At Point $(-1,0)$ the Jacobian is

$$J = \begin{pmatrix} 0 & 0\\ -2 & -1\end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -\lambda & 0 \\ -2 & -1-\lambda \end{vmatrix} = 0$$
$$-\lambda(-1-\lambda) = 0$$
$$\lambda(\lambda+1) = 0$$
$$\lambda = 0, -1$$

Since this is nonlinear system, and one eigenvalue is zero, then unable to decide on stability of this critical point.

Note: In back of text book, it says that this degenerate. But it is not clear why that is. Because to determine if a critical point is degenerate, the determinant of Hessian $\det\left(\nabla^2 F\left(x,y\right)\right)$ must be zero at that point, where $F\left(x,y\right)$ is the first integral (or energy of system). I could not find $F\left(x,y\right)$ for this system, and so I could not check if this was the case. Will follow the book for now and call this point degenerate, but it will be useful to find out how or why the book calls this degenerate.

At Point $(0,1)$ the Jacobian is

$$J = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = 0$$
$$-\lambda(1-\lambda) + 4 = 0$$
$$\lambda^2 - \lambda + 4 = 0$$
$$\lambda = \frac{1}{2} \pm \frac{1}{2} i\sqrt{15}$$

This is the same $(1,0)$. Since the real part is positive, then this is unstable point. Spiral out. (focus with negative attraction).

At Point $(0,-1)$ the Jacobian is

$$J = \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & -2 \\ 0 & -\lambda \end{vmatrix} = 0$$
$$-\lambda(-1-\lambda) = 0$$
$$\lambda(1+\lambda) = 0$$
$$\lambda = 0, -1$$

This is the same as point $(-1,0)$ above. Since this is nonlinear system, and one eigenvalue is zero, then unable to decide on stability of this critical point. degenerate.

Point $\left(\frac{1}{2}\left(1-\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right)$

At this point Jacobian becomes

$$\begin{aligned} J &= \begin{pmatrix} y & -3y^2 + x + 1 \\ -3x^2 + y + 1 & x \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(1-\sqrt{5}\right) & -3\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^2 + \frac{1}{2}\left(1-\sqrt{5}\right) + 1 \\ -3\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^2 + \frac{1}{2}\left(1-\sqrt{5}\right) + 1 & \frac{1}{2}\left(1-\sqrt{5}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(1-\sqrt{5}\right) & \sqrt{5}-3 \\ \sqrt{5}-3 & \frac{1}{2}\left(1-\sqrt{5}\right) \end{pmatrix} \end{aligned}$$

Hence $|J-\lambda I|=0$ gives

$$
\begin{aligned}
\begin{vmatrix} \frac{1}{2}\left(1-\sqrt{5}\right)-\lambda & \sqrt{5}-3 \\ \sqrt{5}-3 & \frac{1}{2}\left(1-\sqrt{5}\right)-\lambda \end{vmatrix} &= 0 \\
\left(\frac{1}{2}\left(1-\sqrt{5}\right)-\lambda\right)\left(\frac{1}{2}\left(1-\sqrt{5}\right)-\lambda\right)-\left(\sqrt{5}-3\right)^2 &= 0 \\
\lambda^2+\lambda\left(\sqrt{5}-1\right)+\frac{11}{2}\sqrt{5}-\frac{25}{2} &= 0 \\
\lambda &= \frac{7}{2}-\frac{3}{2}\sqrt{5}, \frac{1}{2}\sqrt{5}-\frac{5}{2} \\
&= 0.146, -1.382
\end{aligned}
$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable).

Point $\left(\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1+\sqrt{5}\right)\right)$

At this point Jacobian becomes

$$
\begin{aligned}
J &= \begin{pmatrix} y & -3y^2+x+1 \\ -3x^2+y+1 & x \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & -3\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^2+\frac{1}{2}\left(1+\sqrt{5}\right)+1 \\ -3\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^2+\frac{1}{2}\left(1+\sqrt{5}\right)+1 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & -\sqrt{5}-3 \\ -\sqrt{5}-3 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{pmatrix}
\end{aligned}
$$

Hence $|J-\lambda I|=0$ gives

$$
\begin{aligned}
\begin{vmatrix} \frac{1}{2}\left(1+\sqrt{5}\right)-\lambda & -\sqrt{5}-3 \\ -\sqrt{5}-3 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{vmatrix} &= 0 \\
\left(\frac{1}{2}\left(1+\sqrt{5}\right)-\lambda\right)\left(\frac{1}{2}\left(1+\sqrt{5}\right)-\lambda\right)-\left(-\sqrt{5}-3\right)^2 &= 0 \\
\lambda^2-\lambda\left(\sqrt{5}+1\right)-\frac{11}{2}\sqrt{5}-\frac{25}{2} &= 0 \\
\lambda &= \frac{3}{2}\sqrt{5}+\frac{7}{2}, -\frac{1}{2}\sqrt{5}-\frac{5}{2} \\
&= 6.854, -3.618
\end{aligned}
$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable).

The following is summary of result of the above

critical point	stable/unstable	λ_1,λ_2	type
$(0,0)$	unstable	$1,-1$	Saddle
$(1,0)$	unstable	$\frac{1}{2}\pm\frac{1}{2}i\sqrt{15}$	Spiral out (focus, negative attraction)
$(-1,0)$	unable to decide	$0,-1$	Degenerate
$(0,1)$	unstable	$\frac{1}{2}\pm\frac{1}{2}i\sqrt{15}$	Spiral out (focus, negative attraction)
$(0,-1)$	unable to decide	$0,-1$	Degenerate
$\left(\frac{1}{2}\left(1-\sqrt{5}\right),\frac{1}{2}\left(1-\sqrt{5}\right)\right)$	unstable	$0.146,-1.382$	Saddle
$\left(\frac{1}{2}\left(1+\sqrt{5}\right),\frac{1}{2}\left(1+\sqrt{5}\right)\right)$	unstable	$6.854,-3.618$	Saddle

The following is phase plot, generated from the nonlinear system numerically using the computer. Red dots are the unstable points. Blue points are the degenerate points.

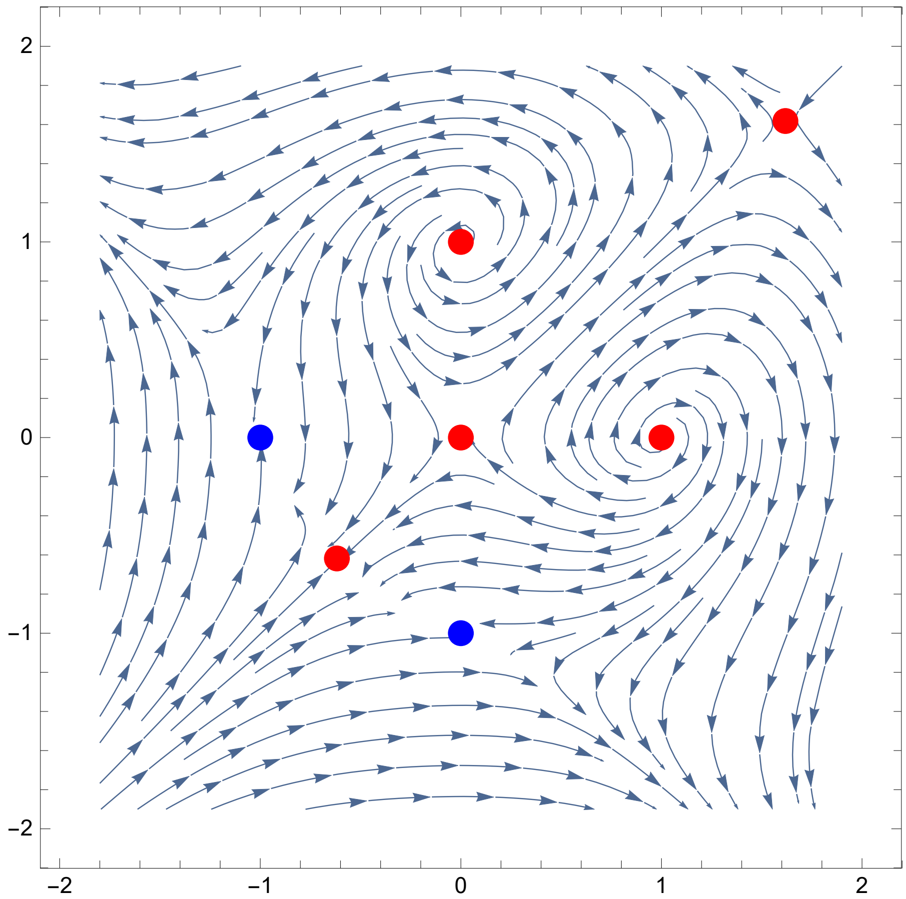


Figure 2.10: Phase plot of the nonlinear system

```
f1 = y (1 + x - y^2);
f2 = x (1 + y - x^2);
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Red, PointSize[0.03], Point[{1, 0}]};
p3 = {Blue, PointSize[0.03], Point[{-1, 0}]};
p4 = {Red, PointSize[0.03], Point[{0, 1}]};
p5 = {Blue, PointSize[0.03], Point[{0, -1}]};
p6 = {Red, PointSize[0.03], Point[{1/2 (1 - Sqrt[5]), 1/2 (1 - Sqrt[5])}]};
p7 = {Red, PointSize[0.03], Point[{1/2 (1 + Sqrt[5]), 1/2 (1 + Sqrt[5])}]};
p = StreamPlot[{f1, f2}, {x, -1.8, 1.9}, {y, -1.9, 1.9}, Epilog → {p1, p2, p3, p4, p5, p6, p7}];
```

Figure 2.11: Code used for the above plot

2.2.2 Problem 3.3

Consider the system

$$\dot{x} = 16x^2 + 9y^2 - 25$$
$$\dot{y} = 16x^2 - 16y^2$$

(a) Determine the critical points and characterize them by linearization. (b) Sketch the phase-flow.

solution

2.2.2.1 Part a

Let

$$\dot{x} = 16x^2 + 9y^2 - 25 = f_1\left(x, y\right) \tag{1}$$
$$\dot{y} = 16x^2 - 16y^2 = f_2\left(x, y\right) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. The equation $f_2 = 0$ gives solutions

$$\begin{aligned} 16x^2 - 16y^2 &= 0 \\ y &= \pm x \end{aligned} \tag{3}$$

When $y = x$, substitution into $f_1 = 0$ gives

$$\begin{aligned} 16x^2 + 9x^2 - 25 &= 0 \\ x &= \pm 1 \end{aligned}$$

Hence the first set of critical points is $(1,1), (-1,-1)$. When $y = -x$ then $x = \pm 1$ also. Therefore the next set of critical points is $(1,-1), (-1,1)$

In summary, the following are the critical points found. There are 4 of them

$$\begin{aligned} (x,y)^* &= (1,1) \\ &= (-1,-1) \\ &= (1,-1) \\ &= (-1,1) \end{aligned}$$

To characterize the linearized system at these points, the Jacobian matrix is evaluated at each of point and from the nature of eigenvalues, the type of critical point is determined. Since $f_1 = 16x^2 + 9y^2 - 25, f_2 = 16x^2 - 16y^2$ then the Jacobian matrix is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix} \end{aligned}$$

At Point $(1,1)$ the Jacobian is

$$J = \begin{pmatrix} 32 & 18 \\ 32 & -32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} 32-\lambda & 18 \\ 32 & -32-\lambda \end{vmatrix} &= 0 \\ (32-\lambda)(-32-\lambda) - (18)(32) &= 0 \\ \lambda^2 - 1600 &= 0 \\ \lambda &= \pm\sqrt{1600} \\ &= \pm 40 \end{aligned}$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable). Since the problem asks also to sketch the phase plot, then the eigenvectors are now found as well. For $\lambda = 40$

$$\begin{pmatrix} 32-40 & 18 \\ 32 & -32-40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-8v_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{4}{9}$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

For $\lambda = -40$

$$\begin{pmatrix} 32+40 & 18 \\ 32 & -32+40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $72v_1 + 18v_2 = 0$ or $4v_1 + v_2 = 0$. Let $v_1 = 1$ then $v_2 = -4$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Summary for point $(1,1)$ (Saddle)

λ_i	$\vec{v}_i$	direction
40	$\begin{pmatrix} 9 \\ 4 \end{pmatrix}$	not stable (move away from (1,1))
−40	$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$	stable (move towards from (1,1))

Now that we know the eigenvectors, we can sketch them at (1,1) as follows

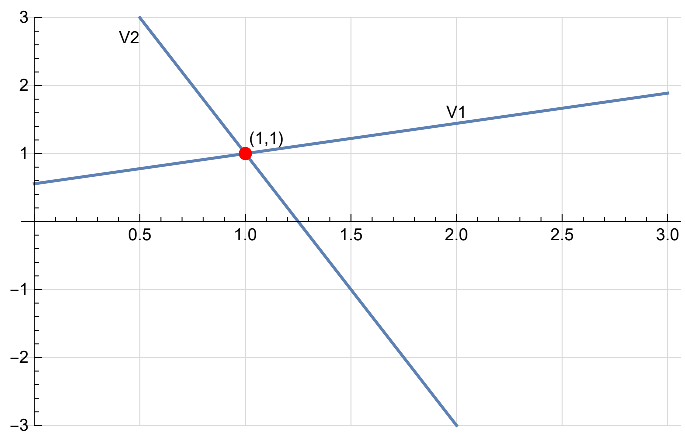


Figure 2.12: Eigenvectors around (1,1)

But we still do not know the directions along the eigenvectors. But we know that for negative eigenvalue, the solution is stable and for positive eigenvalue the solution is not stable. Hence along $\vec{v}_1$ the solution must move away from (1,1) since $\vec{v}_1$ is associated with an unstable λ.

For $\vec{v}_2$ the solution must be stable, therefore on $\vec{v}_2$ the solution must move towards (1,1). Now that we know the directions, we can update the above plot sketch by addition directions.

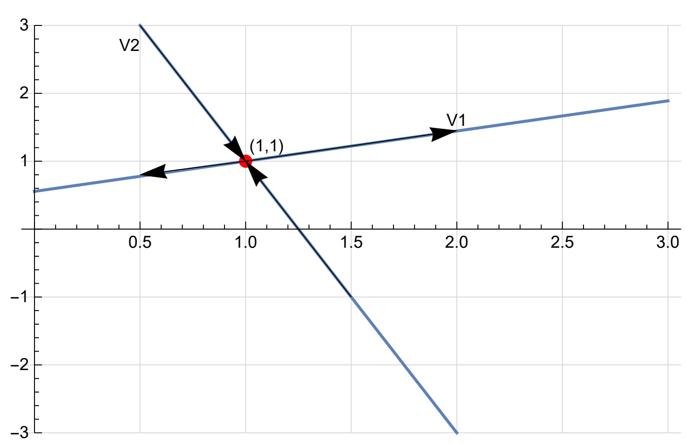


Figure 2.13: Eigenvectors around (1,1) with directions

Now the sketch is finished by adding stream lines that follow along the directions of the eigenvector due to continuity and because solution lines can not cross each others (due to uniqueness). This gives the phase plot around (1,1) found by linearization as follows

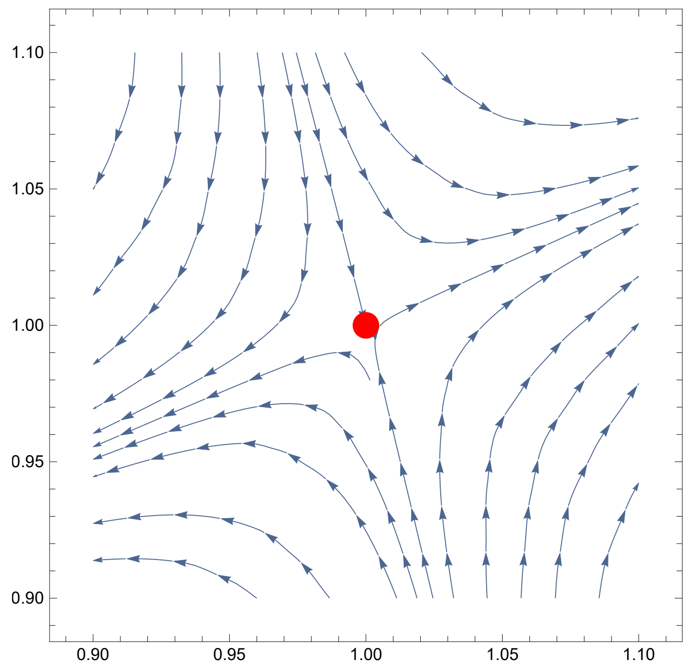


Figure 2.14: Adding more stream lines around (1,1)

The same steps above are now repeated for the next critical point $(-1,-1)$

At Point $(-1,-1)$ the Jacobian is

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix} = \begin{pmatrix} -32 & -18 \\ -32 & 32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{vmatrix} = 0$$

$$(-32-\lambda)(32-\lambda) - (18)(32) = 0$$

$$\lambda^2 - 1600 = 0$$

$$\lambda = \pm\sqrt{1600} = \pm 40$$

This is the same result as the earlier point. This is a saddle point because one eigenvalue is positive (not stable) and one is negative (stable). Now the eigenvectors are found. For $\lambda = 40$

$$\begin{pmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -32-40 & -18 \\ -32 & 32-40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -72 & -18 \\ -32 & -8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-72v_1 - 18v_2 = 0$ or $-4v_1 - v_2 = 0$ Let $v_1 = 1$ then $v_2 = -4$. The eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

For $\lambda = -40$

$$\begin{pmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -32+40 & -18 \\ -32 & 32+40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 8 & -18 \\ -32 & 72 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $8v_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{4}{9}$. The eigenvector is $\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

Summary for point $(-1,-1)$ (Saddle, not stable).

λ_i	$\vec{v}_i$	direction
40	$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$	Not stable. Move away from $(-1,-1)$
-40	$\begin{pmatrix} 9 \\ 4 \end{pmatrix}$	Stable. Move towards $(-1,-1)$

Now that we know the eigenvectors, we sketch them at $(-1,-1)$ as follows

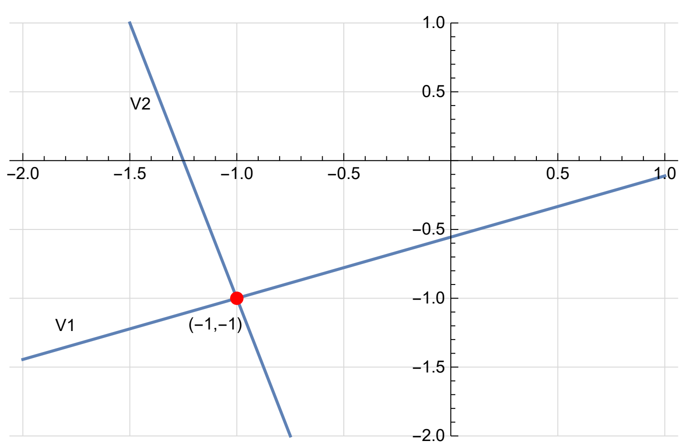


Figure 2.15: Eigenvectors around $(-1,-1)$

But we still do not know the directions along the eigenvectors. As was mentioned above, for negative eigenvalue, the solution is stable and for positive eigenvalue the solution is not stable. This means on $\vec{v}_1$ the solution moves away from $(-1,-1)$ since $\vec{v}_1$ is associated with unstable λ. For $\vec{v}_2$, since the solution is stable then on $\vec{v}_2$ the solution moves towards $(-1,-1)$. Now that the directions are known, the above sketch is updated giving

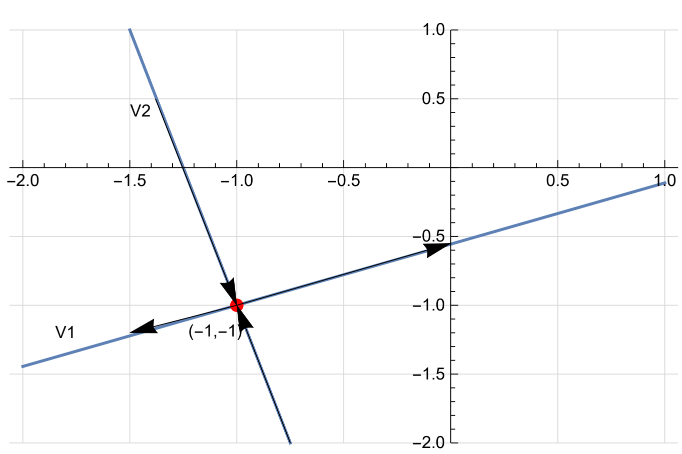


Figure 2.16: Eigenvectors around $(-1,-1)$ with directions

The sketch is finished by adding stream lines that follow along the directions of the eigenvector by continuity. This gives the phase plot around $(-1,-1)$ found by linearization as follows

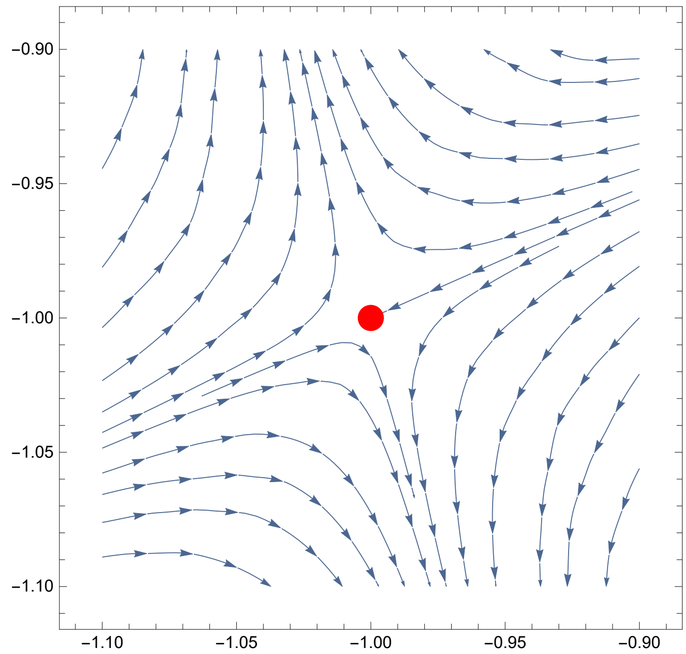


Figure 2.17: Adding more stream lines around $(-1,-1)$

The same steps are now repeated for the next critical point $(1,-1)$

At Point $(1,-1)$ the Jacobian is

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix} = \begin{pmatrix} 32 & -18 \\ 32 & 32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{vmatrix} = 0$$
$$(32-\lambda)^2 + (18)(32) = 0$$
$$\lambda^2 - 64\lambda + 1600 = 0$$
$$\lambda = 32 \pm 24i$$

This is unstable critical point, since since real part of the complex number is positive. This is spiral out point. Also called focus, with negative attraction. For $\lambda = 32 + 24i$ the eigenvector is

$$\begin{pmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 32-(32+24i) & -18 \\ 32 & 32-(32+24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -24i & -18 \\ 32 & -24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-24iv_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = -\frac{24}{18}i$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ -4i \end{pmatrix} = \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

For $\lambda = 32 - 24i$

$$\begin{pmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 32-(32-24i) & -18 \\ 32 & 32-(32-24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 24i & -18 \\ 32 & 24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $24iv_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{24}{18}i$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ 4i \end{pmatrix} = \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \begin{pmatrix} -3i \\ 4 \end{pmatrix}$. What is left to find out is to determine if the spiral is clockwise or anti-clockwise. One way to find this is to select a point (x, y) to the right of the critical point and then find if x is increasing or decreasing there and find out also if y is increasing or decreasing there. This gives the slop. Since the critical point is $(1, -1)$, let us pick point $(2, -1)$ to its right. From

$$\begin{aligned} \dot{x} &= 16x^2 + 9y^2 - 25 \\ \dot{y} &= 16x^2 - 16y^2 \end{aligned}$$

Then at $(2, -1)$ the above gives

$$\begin{aligned} \dot{x} &= 64 + 9 - 25 = 48 \\ \dot{y} &= 64 - 16 = 48 \end{aligned}$$

Hence $\dot{x} > 0$, then x is increasing and $\dot{y} > 0$, then y also increasing. This means the solution curve is moving in the NE direction (↗). Hence the spiral is anti-clockwise direction around $(1, -1)$.

Summary for $(1, -1)$ (not stable, spiral out)

λ_i	$\vec{v}_i$	direction
$32 + 24i$	$\begin{pmatrix} 3i \\ 4 \end{pmatrix}$	Not stable. focus, negative attraction. Anti-clockwise direction
$32 - 24i$	$\begin{pmatrix} -3i \\ 4 \end{pmatrix}$	Not stable. focus, negative attraction. Anti-clockwise direction

Now that we know the eigenvectors, we can sketch them at $(1, -1)$ as follows

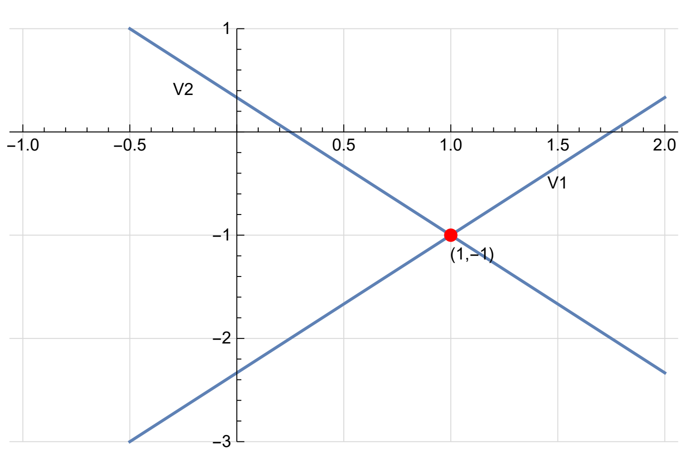


Figure 2.18: Eigenvectors around $(1, -1)$

Since both eigenvector are not stable, direction of solution near $(1, -1)$ is moving away from $(1, -1)$. Now that we know the directions, we can update the above plot sketch.

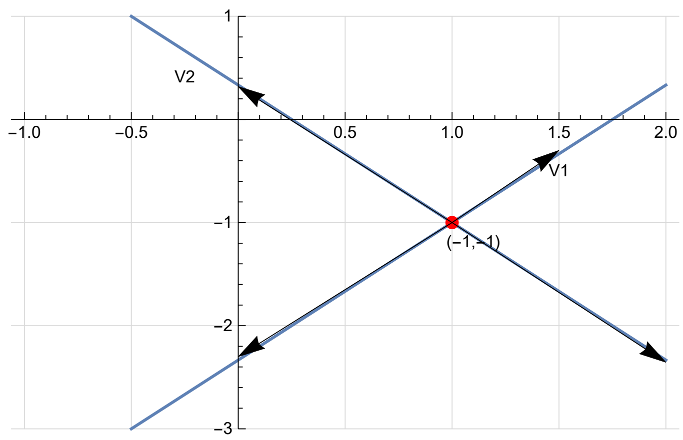


Figure 2.19: Eigenvectors around $(1,-1)$ with directions

Now the sketch is finished by adding the spiral out stream lines. This gives the phase plot around $(1,-1)$ found by linearization as follows

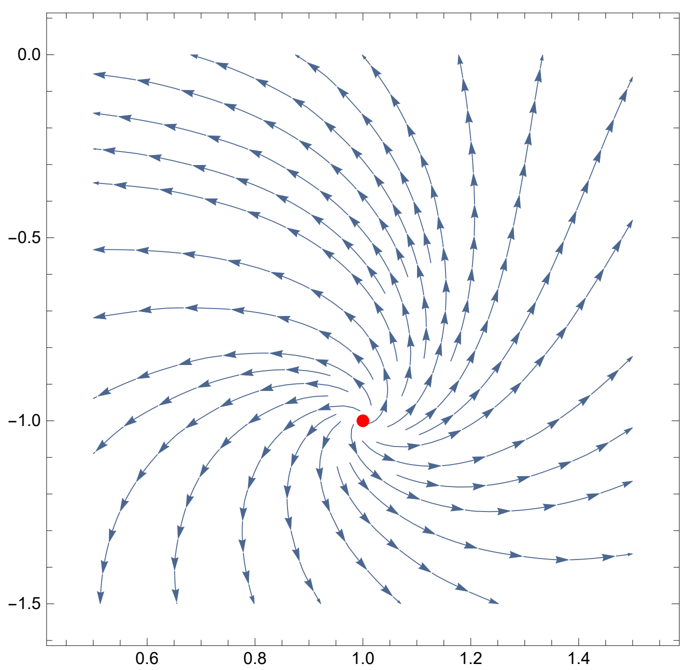


Figure 2.20: Adding more stream lines around $(1,-1)$

The same steps are now repeated for final critical point $(-1,1)$

At Point $(-1,1)$ the Jacobian is

$$\begin{aligned} J &= \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix} \\ &= \begin{pmatrix} -32 & 18 \\ -32 & -32 \end{pmatrix} \end{aligned}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{vmatrix} = 0$$

$$(-32-\lambda)^2 + (18)(32) = 0$$

$$\lambda^2 + 64\lambda + 1600 = 0$$

$$\lambda = -32 \pm 24i$$

This is stable point, both eigenvalues has negative real part. The type is spiral in (focus, with positive attraction). For $\lambda_1 = -32 + 24i$ the eigenvector is

$$\begin{pmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -32-(-32+24i) & 18 \\ -32 & -32-(-32+24i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -24i & 18 \\ -32 & -24i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-24iv_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{24}{18}i$ and the eigenvector becomes $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ 4i \end{pmatrix} = \begin{pmatrix} 3i \\ -4 \end{pmatrix}$

For $\lambda_2 = -32 - 24i$ the eigenvector is

$$\begin{pmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -32-(-32-24i) & 18 \\ -32 & -32-(-32-24i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 24i & 18 \\ -32 & 24i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $24iv_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = -\frac{24}{18}i$ and the second eigenvector becomes $\vec{v}_2 = \begin{pmatrix} 1 \\ -\frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ -4i \end{pmatrix} = \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

The only thing left is to determine if the spiral is clockwise or anti-clockwise. One way to find out is to pick a point (x, y) to the right of the critical point and find if x is increasing or decreasing there and also find out if y is increasing or decreasing there. This gives the slope. Since the critical point is $(-1, 1)$, let us pick point $(0, 1)$ to its right. From

$$\dot{x} = 16x^2 + 9y^2 - 25$$
$$\dot{y} = 16x^2 - 16y^2$$

Then at $(0, 1)$ the above gives

$$\dot{x} = 9 - 25 = -16$$
$$\dot{y} = -16 = -16$$

Hence $\dot{x} < 0$, then x is decreasing and $\dot{y} < 0$, then y also decreasing. This means the solution curve is moving in the SW direction (↙). Hence the spiral is in the clockwise direction around $(-1, 1)$.

Summary for $(-1, 1)$ (Stable)

λ_i	$\vec{v}_i$	direction
$-32 + 24i$	$\begin{pmatrix} 3i \\ -4 \end{pmatrix}$	Stable. Focus, positive attraction. Clockwise direction
$-32 - 24i$	$\begin{pmatrix} 3i \\ 4 \end{pmatrix}$	Stable. Focus, positive attraction. Clockwise direction

Now that we know the eigenvectors, we can sketch them at $(-1, 1)$ as follows

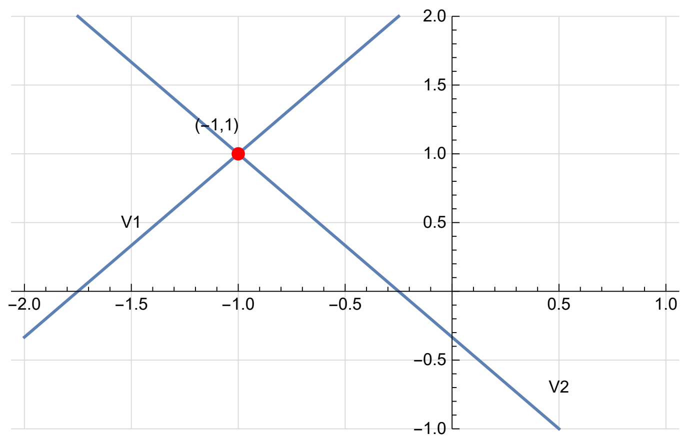


Figure 2.21: Eigenvectors around (−1,1)

Since both eigenvectors are stable, the direction along each is towards (−1,1). Now that we know the directions, we can update the above plot sketch.

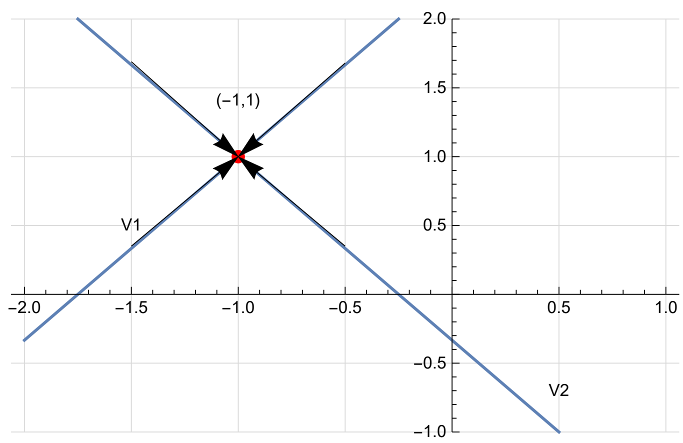


Figure 2.22: Eigenvectors around (−1,1) with directions

The sketch is finished by adding the spiral stream lines. This gives the phase plot around (−1,1) found by linearization as follows

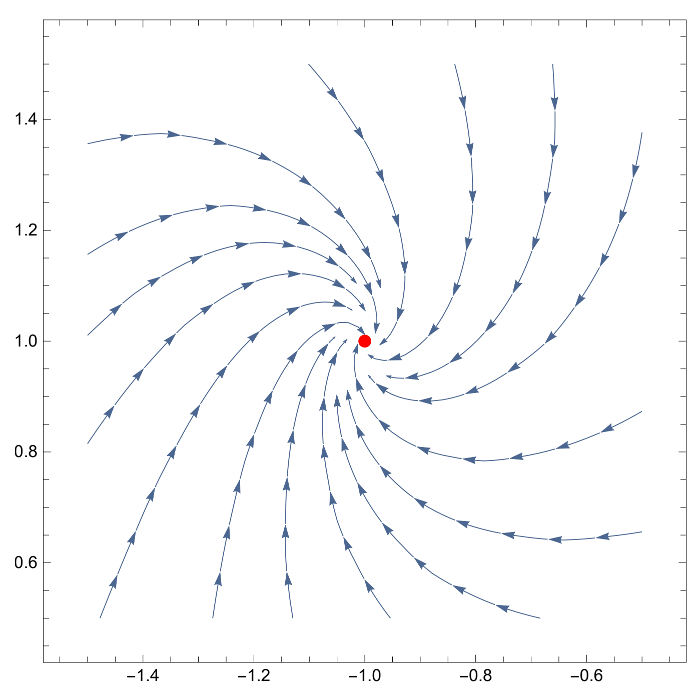


Figure 2.23: Adding more stream lines around (−1,1)

putting all the above result together gives the final sketch of phase plot as

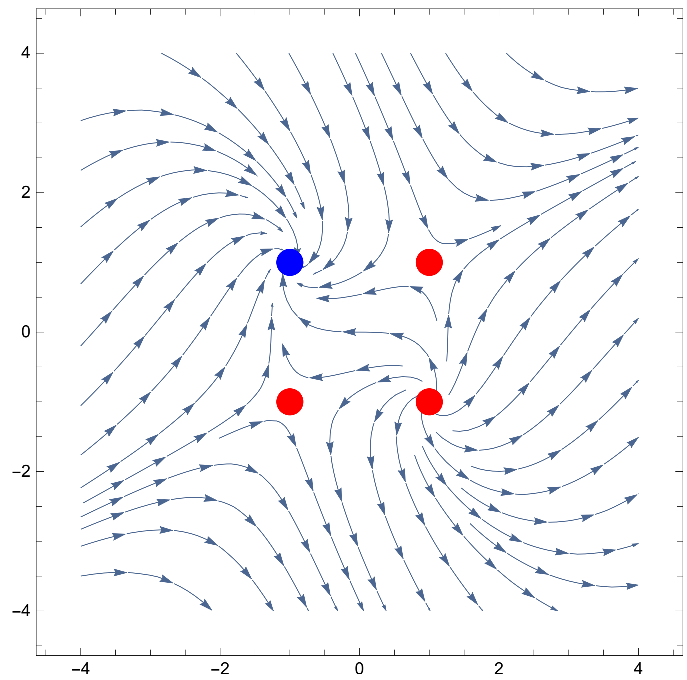


Figure 2.24: Final phase plot

The following is summary of result

critical point	stable/unstable	λ_1, λ_2	type
(1,1)	unstable	$40, -40$	Saddle node
(−1,−1)	unstable	$40, -40$	Saddle node
(1,−1)	unstable	$32 \pm 24i$	Spiral out (focus, negative attraction)
(−1,1)	Stable	$-32 \pm 24i$	Spiral in (focus, positive attraction)

2.2.3 Problem 3.5

In certain applications one studies the equation

$$\ddot{x} + c\dot{x} - x(1-x) = 0$$

with a special interest in solutions with the properties:

$$\lim_{t\to-\infty} x(t) = 0, \lim_{t\to\infty} x(t) = 1, \dot{x}(t) > 0 \text{ for } -\infty < t < \infty$$

Derive a necessary condition for the parameter c for such solutions to exist

solution

Using $x_1 = x, x_2 = \dot{x}$, the first step is to determine the critical points. Hence $\dot{x}_1 = x_2, \dot{x}_2 = \ddot{x} = -c\dot{x} + x(1-x) = -cx_2 + x_1(1-x_1)$. In state space the system becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 + x_1(1-x_1) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

The first equation gives solution $x_2 = 0$. When $x_2 = 0$ the second equation gives $x_1(1-x_1) = 0$ or $x_1 = 0, x_1 = 1$. Hence the critical points are $(0,0), (1,0)$.

From the properties of the solutions, it shows that solutions that start with $x_1 = 0$ eventually go to $x_1 = 1$. Also, since $\dot{x}(t) > 0$ for $-\infty < t < \infty$ then this means $x_2 > 0$ for all time. Hence solution curves are in upper half of phase plane. Here is sketch of what phase plane should look like (I am taking $x_1 = 0$ as initial condition, at $t = -\infty$.)

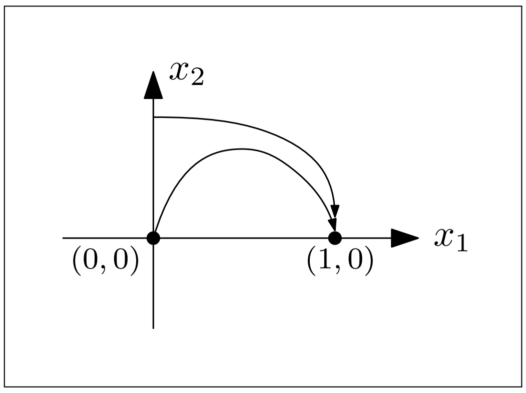


Figure 2.25: possible solution curves in phase plane

The Jacobian of the linearlized system is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1-2x_1 & -c \end{pmatrix} \end{aligned}$$

At $(0,0)$ the above becomes

$$J = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ 1 & -c-\lambda \end{vmatrix} &= 0 \\ (-\lambda)(-c-\lambda) - 1 &= 0 \\ \lambda^2 + c\lambda - 1 &= 0 \\ \lambda &= -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2+4} \end{aligned} \tag{1}$$

At $(1,0)$ the Jacobian becomes

$$\begin{aligned} J &= \begin{pmatrix} 0 & 1 \\ 1-2x_1 & -c \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \end{aligned}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} &= 0 \\ (-\lambda)(-c-\lambda) + 1 &= 0 \\ \lambda^2 + c\lambda + 1 &= 0 \\ \lambda &= -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2-4} \end{aligned} \tag{2}$$

We know that (2) must give stable solution, because we want the solution to eventually move to that critical point $(1,0)$. Also, since we do not want to move into negative half plane because $x_2 > 0$ for all time, then this mean that we can not have spiral solution around $(1,0)$. Therefore $\sqrt{c^2-4}$ must be positive to avoid complex eigenvalue which gives spiral solutions. This means $c^2 \geq 4$ or

$$c \geq 2$$

Here is the phase plot for $c = 2.5$

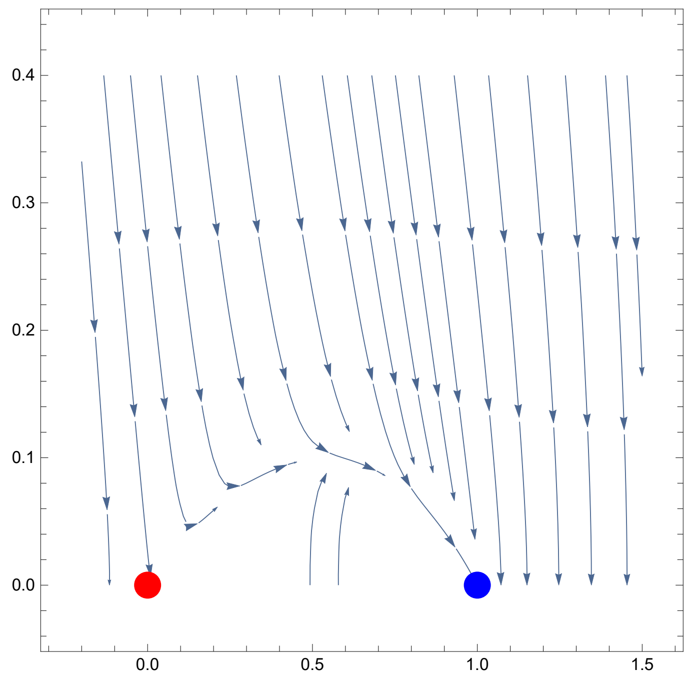


Figure 2.26: Phase plot for $c = 2.5$

We can now check that for such c value, periodic solutions do not exist. The gradient of the vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 + x_1(1 - x_1) \end{pmatrix}$ is

$$\begin{aligned} \nabla \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &= 0 + -c \\ &= -c \end{aligned}$$

And since we determined that c must be positive, then $\nabla \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -c$ do not change sign and remain negative. Hence by Bendixson's criterion (4.1 in book) no periodic solution is possible.

2.2.4 Problem 3.7

Determine the critical points of the system

$$\begin{aligned} \dot{x} &= x\left(1 - x^2 - 6y^2\right) \\ \dot{y} &= y\left(1 - 3x^2 - 3y^2\right) \end{aligned}$$

And characterize them by linear analysis.

solution

Let

$$\dot{x} = x\left(1 - x^2 - 6y^2\right) = f_1(x, y) \tag{1}$$
$$\dot{y} = y\left(1 - 3x^2 - 3y^2\right) = f_2(x, y) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. Solving for x from $f_1 = 0$ gives

$$x = 0 \tag{3}$$
$$1 - 6y^2 = x^2 \tag{4}$$

From each solution above, we go to EQ (2) and solve for y. When $x = 0$ then (2) gives

$$y\left(1 - 3y^2\right) = 0$$

Hence $y=0, y=\pm\sqrt{\frac{1}{3}}$. Therefore the first set of critical points is $(0,0), \left(0,\sqrt{\frac{1}{3}}\right), \left(0,-\sqrt{\frac{1}{3}}\right)$.

Now, when $1-6y^2=x^2$ then (2) gives

$$\begin{aligned} y\left(1-3\left(1-6y^2\right)-3y^2\right) &= 0 \\ y\left(15y^2-2\right) &= 0 \end{aligned}$$

Hence $y=0, y=\pm\sqrt{\frac{2}{15}}$. When $y=0$ then $1-6y^2=x^2$ gives $x=\pm1$ and when $y=\sqrt{\frac{2}{15}}$ then $1-6y^2=x^2$ gives $1-6\left(\frac{2}{15}\right)=x^2$, or $x=\pm\frac{1}{5}\sqrt{5}=\pm\frac{1}{\sqrt{5}}$ and when $y=-\sqrt{\frac{2}{15}}$ then $1-6y^2=x^2$ gives same solution $x=\pm\frac{1}{\sqrt{5}}$. Therefore the second set of critical points is $(\pm1,0), \left(\pm\frac{1}{\sqrt{5}},\sqrt{\frac{2}{15}}\right), \left(\pm\frac{1}{\sqrt{5}},-\sqrt{\frac{2}{15}}\right)$. In summary, these are the critical points (9 in total)

$$(0,0), \left(0,\pm\sqrt{\frac{1}{3}}\right), (\pm1,0), \left(\pm\frac{1}{\sqrt{5}},\sqrt{\frac{2}{15}}\right), \left(\pm\frac{1}{\sqrt{5}},-\sqrt{\frac{2}{15}}\right)$$

Now that critical points are found, they are classified by linearizing the system and finding the eigenvalues of the Jacobian matrix which acts as the A matrix in $\dot{u}=Au$ of the linearized system. The Jacobian of the linearlized system is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \left(1-x^2-6y^2\right)-2x^2 & -12xy \\ -6xy & \left(1-3x^2-3y^2\right)-y\left(6y\right) \end{pmatrix} \\ &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \end{aligned}$$

At point $(0,0)$

$$J=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)^2 &= 0 \\ \lambda &= 1,1 \end{aligned}$$

A repeated root. Since $\lambda>1$ then this is unstable point. Negative attractor node. It is not spiral since pure real eigenvalues.

At point $\left(0,\sqrt{\frac{1}{3}}\right)$

$$\begin{aligned} J &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1-6\left(\frac{1}{3}\right) & 0 \\ 0 & 1-9\left(\frac{1}{3}\right) \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} &= 0 \\ (-1-\lambda)(-2-\lambda) &= 0 \\ (1+\lambda)(2+\lambda) &= 0 \\ \lambda &= -2,-1 \end{aligned}$$

Since both eigenvalues are negative then this is table point. Positive attractor node. Not a spiral node since pure real eigenvalues.

point $\left(0,-\sqrt{\frac{1}{3}}\right)$

This gives same result as above. Positive attractor node.

point $(1,0)$

$$\begin{aligned} J &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1-3 & 0 \\ 0 & 1-3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned} \begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} &= 0 \\ (-2-\lambda)^2 &= 0 \\ \lambda &= -2,-2 \end{aligned}$$

Repeated root. Since eigenvalue is negative then this is table point. Positive attractor node. Not a spiral node since pure real eigenvalues.

point $(-1,0)$

This gives same result as above. Positive attractor node.

point $\left(\frac{1}{\sqrt{5}},\sqrt{\frac{2}{15}}\right)$

$$\begin{aligned} J &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1-3\left(\frac{1}{\sqrt{5}}\right)^2-6\left(\sqrt{\frac{2}{15}}\right)^2 & -12\left(\frac{1}{\sqrt{5}}\right)\left(\sqrt{\frac{2}{15}}\right) \\ -6\left(\frac{1}{\sqrt{5}}\right)\left(\sqrt{\frac{2}{15}}\right) & 1-3\left(\frac{1}{\sqrt{5}}\right)^2-9\left(\sqrt{\frac{2}{15}}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{5} & -\frac{4}{5}\sqrt{2}\sqrt{3} \\ -\frac{2}{5}\sqrt{2}\sqrt{3} & -\frac{4}{5} \end{pmatrix} \end{aligned}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned} \begin{vmatrix} -\frac{2}{5}-\lambda & -\frac{4}{5}\sqrt{2}\sqrt{3} \\ -\frac{2}{5}\sqrt{2}\sqrt{3} & -\frac{4}{5}-\lambda \end{vmatrix} &= 0 \\ \left(-\frac{2}{5}-\lambda\right)\left(-\frac{4}{5}-\lambda\right)-\left(-\frac{4}{5}\sqrt{2}\sqrt{3}\right)\left(-\frac{2}{5}\sqrt{2}\sqrt{3}\right) &= 0 \\ \lambda^2+\frac{6}{5}\lambda-\frac{8}{5} &= 0 \\ \lambda &= \frac{4}{5},-2 \end{aligned}$$

Since one eigenvalue is negative (stable) but the other is positive (unstable), then this is saddle node. (considered unstable node).

point $\left(-\frac{1}{5}\sqrt{5},\sqrt{\frac{2}{15}}\right)$

Same as above.

point $\left(\frac{1}{5}\sqrt{5},\sqrt{\frac{2}{15}}\right)$

Same as above.

point $\left(-\frac{1}{5}\sqrt{5}, -\sqrt{\frac{2}{15}}\right)$

Same as above.

2.2.4.1 Summary of results

	critical point	stable/unstable	λ_1, λ_2	type
1	$(0,0)$	unstable	$1,1$	node, negative attractor
2	$(1,0)$	Stable	$-2,-2$	node, positive attractor
3	$(-1,0)$	Stable	$-2,-2$	node, positive attractor
4	$\left(0, \frac{1}{\sqrt{3}}\right)$	Stable	$-2,-1$	node, positive attractor
5	$\left(0, \frac{-1}{\sqrt{3}}\right)$	Stable	$-2,-1$	node, positive attractor
6	$\left(\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
7	$\left(-\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
8	$\left(\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
9	$\left(-\frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle

The following is phase plot, generated numerically directly from the non-linear system. A red dot indicates an unstable node and blue colored node is a stable node.

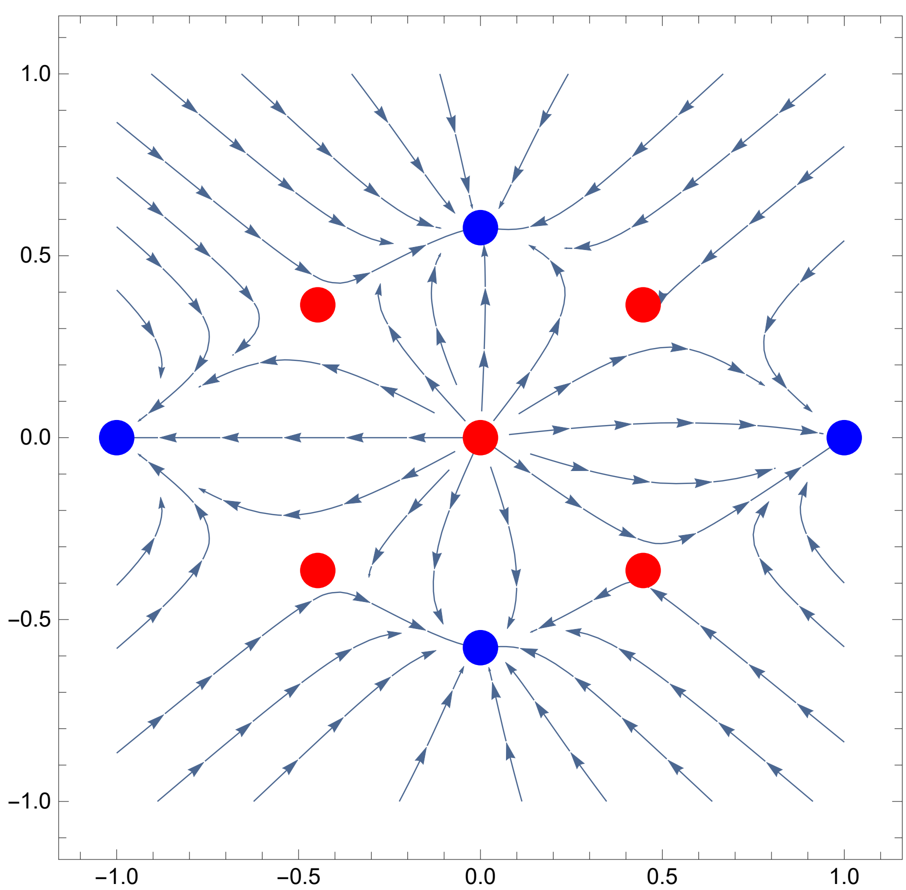


Figure 2.27: Phase plot

2.2.5 key solution for HW 2

MATH 5525: HOMEWORK ASSIGNMENT 2

Date (March 4, 2020)

3.1, page 36, textbook. Consider the system of ODEs

$$\dot{x} = y(1 + x - y^2) := f(x, y), \quad \dot{y} = x(1 + y - x^2) := g(x, y). \tag{1}$$

Find the critical points and characterize their stability properties.

1. To find the critical points, we solve the algebraic equations

$$y(1 + x - y^2) = 0, \quad \text{and} \quad x(1 + y - x^2) = 0.$$

The solutions satisfy the relations:

$$y = 0 \quad \text{or} \quad 1 + x - y^2 = 0, \quad \text{and} \quad x = 0 \quad \text{or} \quad 1 + y - x^2 = 0 \tag{2}$$

This results in the following pairs:

$$(0, 0),\ (\pm 1, 0),\ (0, \pm 1),\ (\frac{1}{2} \pm \frac{\sqrt{5}}{2}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}).$$

The two last solution pairs, result from searching solutions such that $x = y$, in which case $1 + x - x^2 = 0$ must hold.

2. The Jacobian matrix (that is, the matrix of the partial derivatives of $f(x, y)$ and $g(x, y)$) is given by

$$\begin{bmatrix} y & 1 + x - 3y^2 \\ 1 + y - 3x^2 & x \end{bmatrix}.$$

The corresponding matrix near the equilibrium points is:

$$(0,0): \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \text{ Eigenvalues } \pm 1. \quad \text{Saddle point.}$$

$$(0,1): \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}. \text{ Eigenvalues } \frac{1}{2} \pm \frac{\sqrt{3}}{2} i. \quad \text{Focus, negative attractor.}$$

$$(0,-1): \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}. \text{ Eigenvalues } 0, -1. \quad \text{Degenerate.}$$

$$(1,0): \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}. \text{ Eigenvalues } \frac{1}{2}(1 \pm \sqrt{23} i). \quad \text{Focus, negative atttractor.}$$

$$(-1,0): \begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix}. \text{ Eigenvalues } 0, -1. \quad \text{Degenerate.}$$

$$(\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{1}{2} + \frac{\sqrt{5}}{2}): \quad \text{saddle point}$$

$$(\frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{1}{2} - \frac{\sqrt{5}}{2}): \quad \text{saddle point.}$$

1

3.3, page 36, textbook. Consider the system

$$\dot{x} = 16x^2 + 9y^2 - 25 := f(x,y), \quad \dot{y} = 16x^2 - 16y^2 := g(x,y).$$

1. To find the critical points, we solve the algebraic equations

$$16x^2 + 9y^2 - 25 = 0 \quad \text{and} \quad 16x^2 - 16y^2 = 0.$$

Solutions, equilibrium points, are

$$(1,1), \ (1,-1), \ (-1,1), \ (-1,-1).$$

2. The Jacobian matrix is given by

$$\begin{bmatrix} 32x & 18y \\ 32x & -32y \end{bmatrix}.$$

The corresponding matrix near the equilibrium point is:

$(1,1)$: $\begin{bmatrix} 32 & 18 \\ 32 & -32 \end{bmatrix}$. Eigenvalues ± 40. Saddle point.

$(1,-1)$: $\begin{bmatrix} 32 & -18 \\ 32 & 32 \end{bmatrix}$. Eigenvalues $32 \pm 24i$. Unstable spiral (or negative focus).

$(-1,1)$: $\begin{bmatrix} -32 & 18 \\ -32 & 32 \end{bmatrix}$. Eigenvalues $-32 \pm 24i$. Stable spiral (or positive focus).

$(-1,-1)$: $\begin{bmatrix} -32 & 18 \\ -32 & 32 \end{bmatrix}$. Eigenvalues ± 40. Saddle point.

3. Sketch the phase-plane of the system.

3.5, page 36, textbook. Consider the second order ODE:

$$x'' + cx' - x(1-x) = 0.$$

The limiting conditions imply that $x = 0, x' = 0$ and $x = 1, x' = 0$ are critical points of the equation. Moreover, $x = 0$ is unstable and $x = 1$ is stable (because the limits correspond to $t \to -\infty$ and $t \to \infty$, respectively.)

2

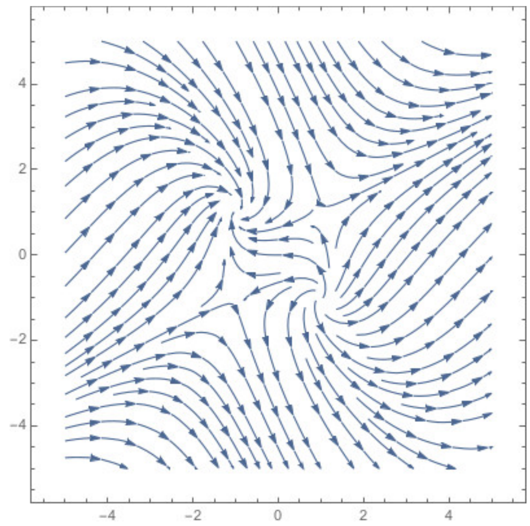


Figure 1: Phase-plane exercise 3.3.

The matrix of the linear system about $(0,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$. Its eigenvalues are

$$2\lambda = -c \pm \sqrt{c^2+4}$$

.

The matrix of the linear system about $(1,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$. Its eigenvalues are

$$2\lambda = -c \pm \sqrt{c^2-4}.$$

Note that the stability of $x = 1$ requires $c > 0$. Moreover, if $c < 2$ the eigenvalues are complex, in which case the equilibrium point is a stable focus. It is easy to see that, in such as case, $x' < 0$ for some values of t. Hence, we require $c \geqslant 2$, so that both eigenvalues are real and negative and $x = 1$ is a stable node.

With $c \geqslant 2$, then $x = 0$ is a saddle point.

3.5, page 36, textbook. Consider the system

$$\dot{x} = x(1-x^2-6y^2), \quad \dot{y} = y(1-3x^2-3y^2).$$

1. To find the critical points, we solve the algebraic equations

$$x(1-x^2-6y^2) = 0 \quad \text{and} \quad y(1-3x^2-3y^2) = 0.$$

The critical points are:

- $(0,0)$. Unstable node.
- $(0, \pm\frac{1}{\sqrt{3}})$. 2 stable nodes.
- $(\pm 1, 0)$. 2 stable nodes.
- $(\pm\frac{1}{\sqrt{5}}, \pm\frac{\sqrt{2}}{\sqrt{15}})$. These are 4 saddle points

3

2.3 HW 3

Local contents

2.3.1 Problem 1

A modification of the predator-prey system is given by

$$\dot{x} = x(1-x) - \frac{axy}{x+1}$$
$$\dot{y} = y(1-y)$$

Where $a > 0$ is a parameter.

1. Find all equilibrium points, in the following two cases: $0 < a < 1$ and $a > 1$. (You may select specific values of a, if you wish.)
2. Classify the equilibrium points in each case.
3. Sketch the nullclines and the phase portraits for different values of a.
4. What is special about the parameter value $a = 1$? (It is called a bifurcation value, why?)

solution

2.3.1.1 Part 1

case $0 < a < 1$

Equilibrium points are found by solving

$$x(1-x) - \frac{axy}{x+1} = 0 \tag{1}$$
$$y(1-y) = 0 \tag{2}$$

EQ (2) gives $y = 0$ or $y = 1$. When $y = 0$ then EQ(1) becomes $x(1-x) = 0$ which has solutions $x = 0, x = 1$. Hence the critical points found so far are $(0,0), (1,0)$.

When $y = 1$ then EQ(1) becomes

$$\begin{aligned} x(1-x) - \frac{ax}{x+1} &= 0 \\ x(1-x)(1+x) - ax &= 0 \\ x - x^3 - ax &= 0 \\ x\left(1 - x^2 - a\right) &= 0 \end{aligned}$$

Hence $x = 0$ or $1 - x^2 - a = 0$ or $x^2 - 1 + a = 0$. Therefore $x^2 = 1 - a$ or $x = \pm\sqrt{1-a}$. Since we are in the case $0 < a < 1$ then $\sqrt{1-a}$ is positive. Let $\sqrt{1-a} = n^2$. Hence $x = \pm n$. Therefore the critical points found for this case are $(0,1), \left(\sqrt{1-a}, 1\right), \left(-\sqrt{1-a}, 1\right)$.

Hence all the critical points for $0 < a < 1$ are

$$\left(x_i, y_i\right) = \left\{(0,0), (1,0), (0,1), \left(\sqrt{1-a}, 1\right), \left(-\sqrt{1-a}, 1\right)\right\}$$

For say $a = \frac{1}{2}$, these critical points become

$$\begin{aligned} \left(x_i, y_i\right) &= \left\{(0,0), (1,0), (0,1), \left(\sqrt{\frac{1}{2}}, 1\right), \left(-\sqrt{\frac{1}{2}}, 1\right)\right\} \\ &= \{(0,0), (1,0), (0,1), (0.707107, 1), (-0.707107, 1)\} \end{aligned}$$

case $a > 1$

Equilibrium points are found by solving

$$x(1-x)-\frac{axy}{x+1}=0 \tag{1}$$
$$y(1-y)=0 \tag{2}$$

EQ (2) gives $y=0$ or $y=1$. When $y=0$ then EQ(1) becomes $x(1-x)=0$ which has solutions $x=0,x=1$. Hence the critical points found so far are $(0,0),(1,0)$.

When $y=1$ then EQ(1) reduces to (as was done above)

$$x\left(1-x^2-a\right)=0$$

Hence $x=0$ or $x=\pm\sqrt{1-a}$. Since we are in the case $a>1$ then $\sqrt{1-a}$ is negative. which means $\sqrt{1-a}$ is complex. We are assuming real domain, then these solutions are rejected. This leaves the critical points for $a>1$ as only the following

$$\left(x_i,y_i\right)=\{(0,0),(1,0),(0,1)\}$$

2.3.1.2 Part 2

The Jacobian matrix for the system

$$\dot{x}=x(1-x)-\frac{axy}{x+1}$$
$$\dot{y}=y\left(1-y\right)$$

Is given by the following, where the rule of derivative $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{gf'-fg'}{g^2}$ is used

$$\begin{aligned}
J=\begin{pmatrix}\frac{\partial\dot{x}}{\partial x} & \frac{\partial\dot{x}}{\partial y}\\ \frac{\partial\dot{y}}{\partial x} & \frac{\partial\dot{y}}{\partial y}\end{pmatrix} &= \begin{pmatrix}(1-x)-x-\frac{(x+1)ay-\left(axy\right)}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & \left(1-y\right)-y\end{pmatrix}\\
&=\begin{pmatrix}1-2x-\frac{axy+ay-axy}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}\\
&=\begin{pmatrix}1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}
\end{aligned}$$

case $0<a<1$

The critical points for this case from part (1) are

$$\left(x_i,y_i\right)=\left\{(0,0),(1,0),(0,1),\left(\sqrt{1-a},1\right),\left(-\sqrt{1-a},1\right)\right\}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned}
A&=\begin{pmatrix}1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}\\
&=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}
\end{aligned}$$

Hence $|A-\lambda I|=0$ becomes

$$\begin{aligned}
\begin{vmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix}&=0\\
(1-\lambda)^2&=0\\
\lambda&=1 \text{ double root}
\end{aligned}$$

Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-2 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Hence $|A-\lambda I|=0$ becomes

$$
\begin{aligned}
\begin{vmatrix} -1-\lambda & -\frac{a}{2} \\ 0 & 1-\lambda \end{vmatrix} &= 0 \\
(-1-\lambda)(1-\lambda) &= 0 \\
\lambda^2-1 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

This means this critical points is a saddle point (unstable) since one eigenvalue is negative and one is negative.

At point $(0,1)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-a & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

Hence $|A-\lambda I|=0$ becomes

$$
\begin{aligned}
\begin{vmatrix} (1-a)-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
((1-a)-\lambda)(-1-\lambda) &= 0 \\
\lambda^2+a\lambda+a-1 &= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} \\
&= -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2-4(a-1)} \\
&= -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2-4a+4}
\end{aligned}
$$

Since $0<a<1$ then $-3<a^2-4a<0$ which means the term under the root will remain positive for all a values between 0 and 1. This means this critical points is a saddle point (unstable) since one eigenvalue will be negative and one is positive.

At point $\left(\sqrt{1-a},1\right)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-2\sqrt{1-a}-\frac{a}{\left(1+\sqrt{1-a}\right)^2} & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

Hence $|A - \lambda I| = 0$ becomes

$$\begin{vmatrix} \left(1 - 2\sqrt{1-a} - \frac{a}{\left(1+\sqrt{1-a}\right)^2}\right) - \lambda & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\left(\left(1 - 2\sqrt{1-a} - \frac{a}{\left(1+\sqrt{1-a}\right)^2}\right) - \lambda\right)(-1-\lambda) = 0$$

To simplify this, let us pick $a = \frac{1}{2}$. Hence the point $\left(\sqrt{1-a}, 1\right)$ becomes $(0.707, 1)$. The above becomes

$$\left(\left(1 - 2\sqrt{\frac{1}{2}} - \frac{\frac{1}{2}}{\left(1+\sqrt{\frac{1}{2}}\right)^2}\right) - \lambda\right)(-1-\lambda) = 0$$

$$(\lambda+1)\left(\lambda - \sqrt{2} + 2\right) = 0$$

Hence $\lambda = -1$ and $\lambda = \sqrt{2} - 2$. So one eigenvalue is negative and also the second is negative. This means this is stable point (positive attraction). Even though we used specific a value here, this result is value for all $0 < a < 1$.

At point $\left(-\sqrt{1-a}, 1\right)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$A = \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2\sqrt{1-a} - \frac{a}{\left(1-\sqrt{1-a}\right)^2} & \frac{a\sqrt{1-a}}{-\sqrt{1-a}+1} \\ 0 & -1 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ becomes

$$\begin{vmatrix} \left(1 + 2\sqrt{1-a} - \frac{a}{\left(1-\sqrt{1-a}\right)^2}\right) - \lambda & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\left(\left(1 + 2\sqrt{1-a} - \frac{a}{\left(1-\sqrt{1-a}\right)^2}\right) - \lambda\right)(-1-\lambda) = 0$$

To simplify this, let us pick $a = \frac{1}{2}$. Hence the point $\left(-\sqrt{1-a}, 1\right)$ becomes $(-0.707, 1)$. The above becomes

$$\left(\left(1 + 2\sqrt{\frac{1}{2}} - \frac{\frac{1}{2}}{\left(1-\sqrt{\frac{1}{2}}\right)^2}\right) - \lambda\right)(-1-\lambda) = 0$$

$$(\lambda+1)\left(\lambda + \sqrt{2} + 2\right) = 0$$

Hence $\lambda = -1$ and $\lambda = -\sqrt{2} - 2$. So one eigenvalue is negative and also the second is negative. This means this is stable point (positive attraction). Even though we used specific a value here, this result is value for all $0 < a < 1$.

The following table is a summary of the above results, all for $0 < a < 1$. To obtain numerical values below for eigenvalues, $a = \frac{1}{2}$ was used as an example.

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 1, \lambda_2 = 1$	negative attraction, unstable
$(1,0)$	$\lambda = \pm 1$	saddle point, unstable.
$(0,1)$	$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4a + 4}$	saddle point, unstable.
$\left(\sqrt{1-a}, 1\right)$	$\lambda_1 = -1, \lambda_2 = \sqrt{2} - 2$	positive attraction, stable
$\left(-\sqrt{1-a}, 1\right)$	$\lambda_1 = -1, \lambda_2 = -\sqrt{2} - 2$	positive attraction, stable

For specific value $a = \frac{1}{2}$ the above table becomes

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 1, \lambda_2 = 1$	negative attraction, unstable
$(1,0)$	$\lambda = \pm 1$	saddle point, unstable.
$(0,1)$	$\lambda_1 = -1, \lambda_2 = \frac{1}{2}$	saddle point, unstable.
$(0.707,1)$	$\lambda_1 = -1, \lambda_2 = -0.5857$	positive attraction, stable
$(-0.707,1)$	$\lambda_1 = -1, \lambda_2 = -3.41421$	positive attraction, stable

case $a > 1$

The critical points for this case from part(1) are

$$\left(x_i, y_i\right) = \{(0,0), (1,0), (0,1)\}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as with part 1. Which gives $\lambda = 1$ double root. Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as with part 1. Which gives $\lambda = 1$ double root. Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as $0 < a < 1$, since a is not involved and cancels out. Hence $\lambda = \pm 1$. This means this critical points is a saddle point (unstable) since one eigenvalue is negative and one is negative.

At point $(0,1)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 - a & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

From case $0 < a < 1$ we found

$$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4a + 4}$$

Let us pick $a = 3$. Hence

$$\begin{aligned}\lambda &= -\frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 12 + 4}\\ &= -\frac{3}{2} \pm \frac{1}{2}\\ &= -2, -1\end{aligned}$$

Therefore this is stable. This is different from case $0 < a < 1$ where this point was unstable.

The following table is a summary of the above results, all for $a > 1$. To obtain numerical values below for eigenvalues, $a = 3$ was used as an example.

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 1, \lambda_2 = 1$ (same as $0 < a < 1$)	negative attraction, unstable
$(1,0)$	$\lambda = \pm 1$ (same as $0 < a < 1$)	saddle point, unstable.
$(0,1)$	$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4a + 4}$ or $\lambda_1 = -1, \lambda_2 = -2$	positive attraction, stable

From the above, we notice that when a changes from $0 < a < 1$ to $a > 1$ then one critical point $(0,1)$ switches from being unstable to stable. This implies solution encountered bifurcation value.

2.3.1.3 Part 3

$$\begin{aligned}\dot{x} &= x(1 - x) - \frac{axy}{x + 1}\\ \dot{y} &= y\left(1 - y\right)\end{aligned}$$

The x nullclines are the solution of $x(1 - x) - \frac{axy}{x+1} = 0$ and the y nullclines are solutions of $y\left(1 - y\right) = 0$. Therefore y nullclines are $y = 0$ (which is the x axis) and $y = 1$. These are both straight lines. To find the x nullclines

$$\begin{aligned}x(1 - x) - \frac{axy}{x + 1} &= 0\\ x(1 - x)(x + 1) - axy &= 0\\ x - x^3 - axy &= 0\\ x\left(1 - x^2 - ay\right) &= 0\end{aligned}$$

Hence $x = 0$ (which is the y axis) and $x^2 = 1 - ay$ are the x nullclines.

x nullclines	y nullclines
$x = 0$	$y = 0$
$x^2 = 1 - ay$	$y = 1$

The following is a plot of the nullclines for the case of $0 < a < 1$, using $a = \frac{1}{2}$

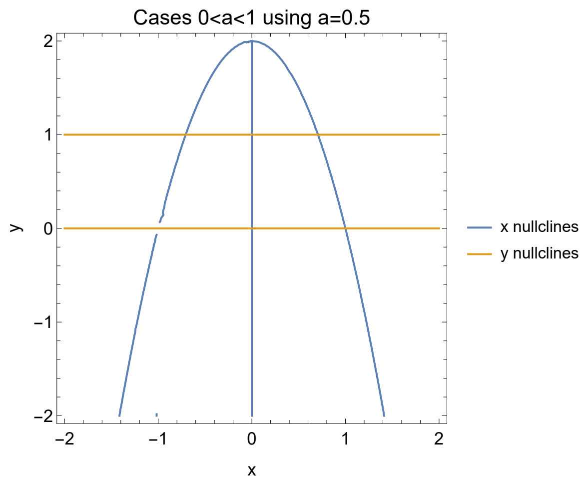


Figure 2.28: nullclines for case $0 < a < 1$

```
ClearAll[x, y, a]
a = 1/2;
f1 = x (1 - x) - a x y/ (x + 1);
f2 = y (1 - y);
p = ContourPlot[{f1 == 0, f2 == 0}, {x, -2, 2}, {y, -2, 2},
        PlotLegends → {"x nullclines", "y nullclines"},
        PlotLabel → "Cases 0<a<1 using a=0.5",
        BaseStyle → 14, FrameLabel → {{"y", None}, {"x", None}}];
```

Figure 2.29: code used for the above plot

The following is a plot of the nullclines for the case of $a > 1$, using $a = 3$

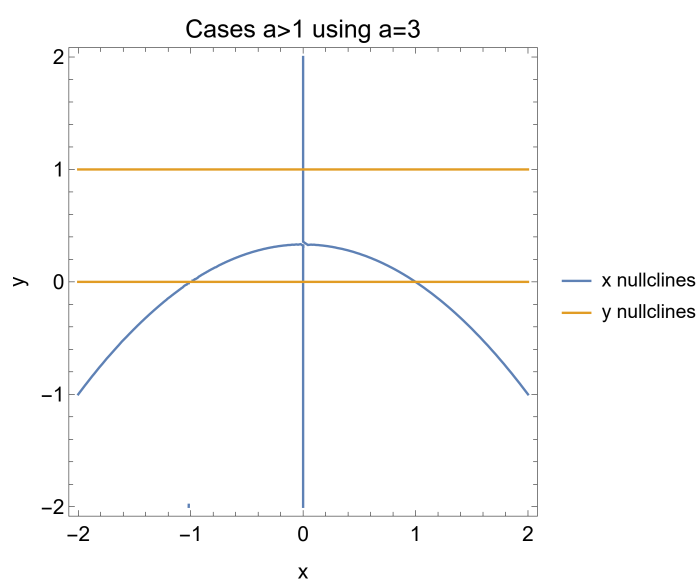


Figure 2.30: nullclines for case $a > 1$

We notice that the case of $a = 1$ is the following

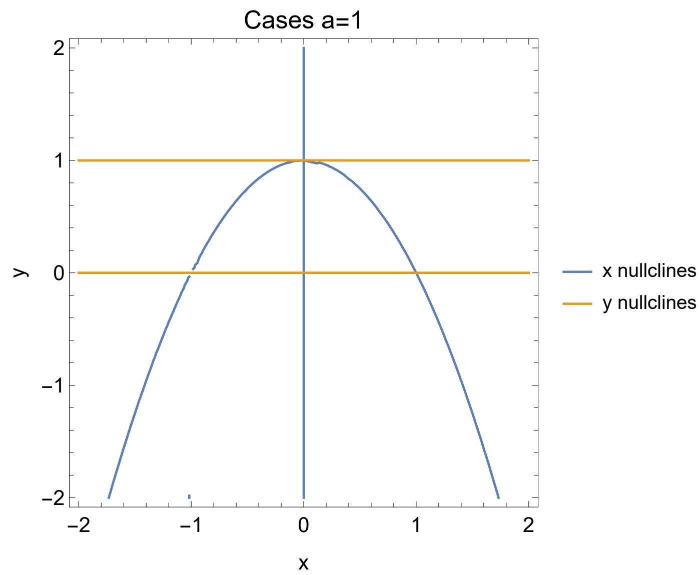


Figure 2.31: nullclines for case $a > 1$

The following is a phase plot for the case of $0 < a < 1$, using $a = \frac{1}{2}$ with the critical points highlighted. Red dot indicates unstable point and green dot color indicates stable point.

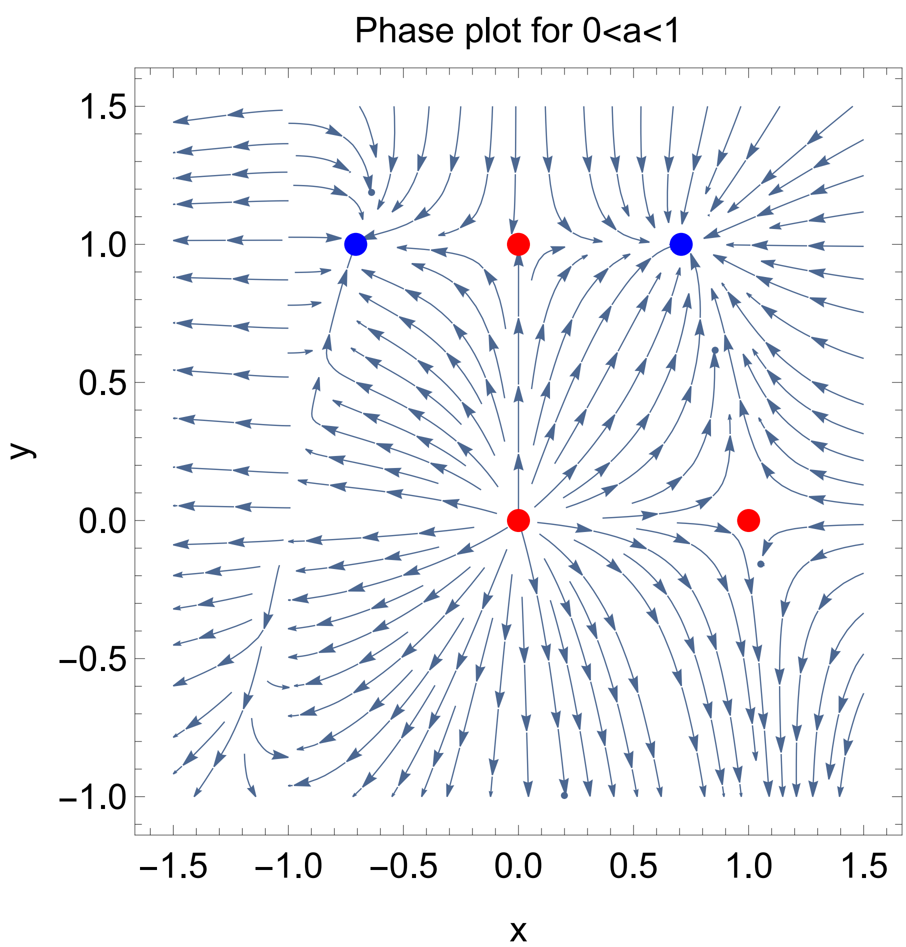


Figure 2.32: Phase plot $0 < a < 1$ using $a = 0.5$

```
ClearAll[x, y, a]
a = 1/2;
f1 = x (1 - x) - a x y/ (x + 1);
f2 = y (1 - y);
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Red, PointSize[0.03], Point[{1, 0}]};
p3 = {Red, PointSize[0.03], Point[{0, 1}]};
p4 = {Blue, PointSize[0.03], Point[{Sqrt[1 - a], 1}]};
p5 = {Blue, PointSize[0.03], Point[{-Sqrt[1 - a], 1}]};
ps = StreamPlot[{f1, f2}, {x, -1.5, 1.5}, {y, -1, 1.5}, Epilog → {p1, p2, p3, p4, p5},
         FrameLabel → {{"y", None}, {"x", "Phase plot for 0<a<1"}}, BaseStyle → 14];
```

Figure 2.33: Code for the above plot

The following is a phase plot for the case of $a > 1$, using $a = 3$.

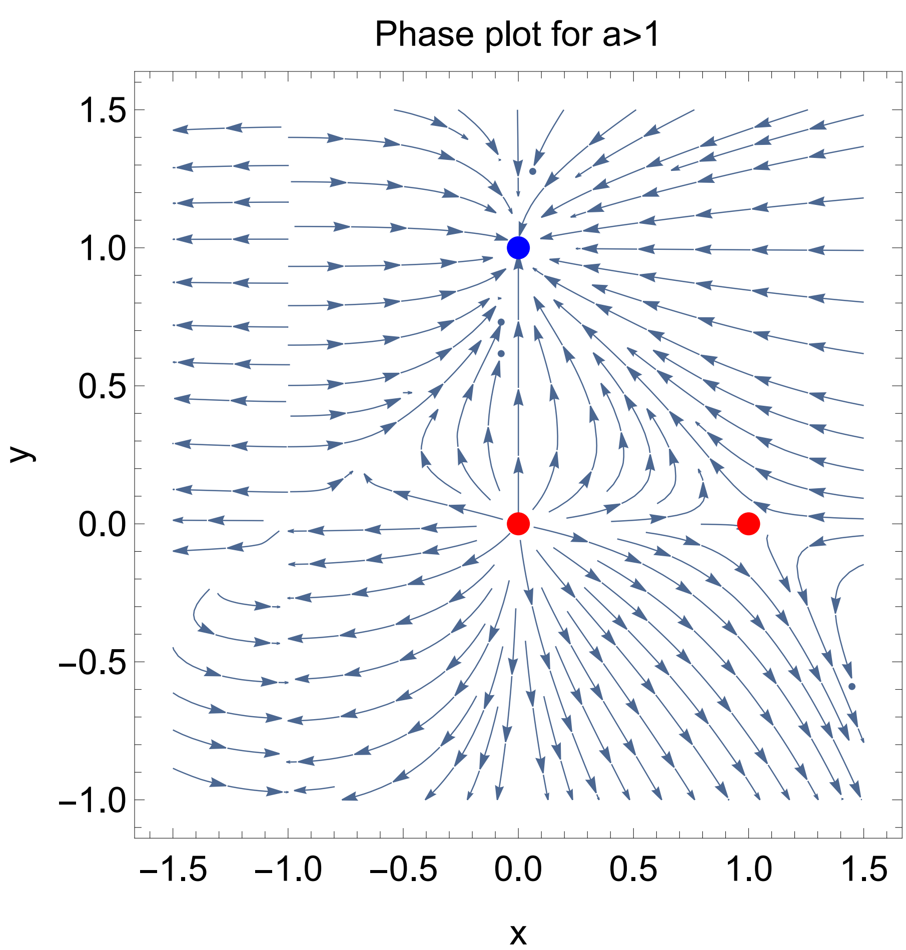


Figure 2.34: Phase plot $a > 1$ using $a = 3$

2.3.1.4 Part 4

When $a = 1$ the solution of the system changes abruptly. To see this, the following is the two phase plots about side by side, one for $a = 0.99$ and one for $a = 1.01$.

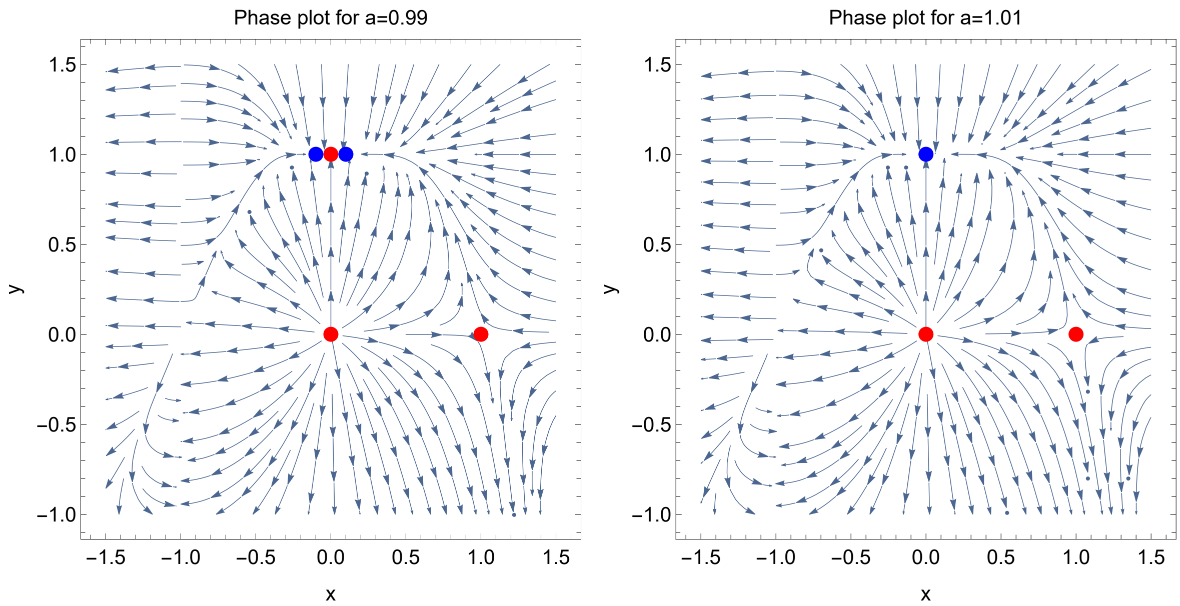


Figure 2.35: Phase plots changes as a cross over 1

We notice the following. As a changes from $0 < a < 1$ to $a > 1$, the equilibrium point $(0,1)$ changes from being unstable to stable (this is in addition to now having 3 equilibrium points instead of 5). This is exactly what bifurcation is. So $a = 1$ is a bifurcation value. It is a parameter in the system, which cause sudden change in the solution trajectories when its value crosses over some specific value, which is 1 in this problem.

2.3.2 Problem 2

The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. In the simplest form of the model, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population. We assume for simplicity, that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0$$

The SIR model, in its simplest form, is stated as

$$\dot{S} = -\beta SI \tag{1}$$
$$\dot{I} = \beta SI - \upsilon I \tag{2}$$
$$\dot{R} = \upsilon I \tag{3}$$

Where $\upsilon > 0$ and $\beta > 0$ are parameters.

1. Show that the line $I = 0$ is an equilibrium line.
2. Find the matrix that results from linearizing the system about $I = 0$.
3. Calculate the eigenvalues of the resulting matrix.
4. Find the nullclines of the system.
5. What can we infer about the prospects of full recovery of the population?

solution

This is a diagram of the SIR model

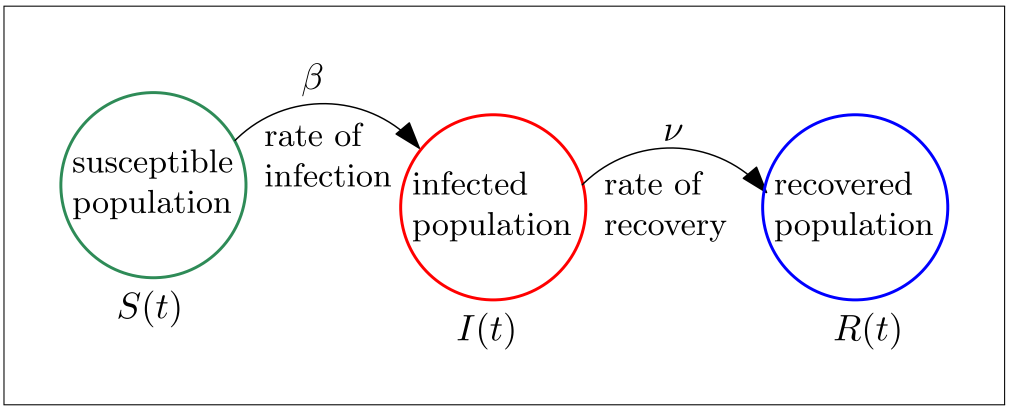


Figure 2.36: SIR model of infection

Hence total population is

$$N = S + I + R$$

Which is a constant (This assumes no death occurs, only infection and recovery). In this model, it is assumed that recovered population $R(t)$ can not become infected again.

2.3.2.1 Part 1

Critical points are solutions to

$$0 = -\beta SI \tag{1A}$$
$$0 = \beta SI - vI \tag{2A}$$
$$0 = vI \tag{3A}$$

Eq. (3) shows that, since v can not be zero, then $I = 0$ is equilibrium. This says that if there are no infected individuals, then the population N do not change which is to be expected since $N = S + I + R$ and hence if $I = 0$ then this implies also that $R = 0$ and hence N remains constant.

2.3.2.2 Part 2

The Jacobian Matrix is

$$J = \begin{pmatrix} \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial I} & \frac{\partial \dot{S}}{\partial R} \\ \frac{\partial \dot{I}}{\partial S} & \frac{\partial \dot{I}}{\partial I} & \frac{\partial \dot{I}}{\partial R} \\ \frac{\partial \dot{R}}{\partial S} & \frac{\partial \dot{R}}{\partial I} & \frac{\partial \dot{R}}{\partial R} \end{pmatrix} = \begin{pmatrix} -\beta I & -\beta & 0 \\ \beta I & \beta S - v & 0 \\ 0 & v & 0 \end{pmatrix}$$

Evaluating the above at $I = 0$ gives

$$A = \begin{pmatrix} 0 & -\beta & 0 \\ 0 & \beta S - v & 0 \\ 0 & v & 0 \end{pmatrix}$$

2.3.2.3 Part 3

To find the eigenvalues

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -\lambda & -\beta & 0 \\ 0 & (\beta S - v) - \lambda & 0 \\ 0 & v & -\lambda \end{vmatrix} = 0$$
$$-\lambda \begin{vmatrix} (\beta S - v) - \lambda & 0 \\ v & -\lambda \end{vmatrix} + \beta \begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix} + 0 = 0$$
$$\lambda^2 (\beta S - v - \lambda) = 0$$

Hence $\lambda = 0$ (double root) and $\lambda = \beta S - v$ are the eigenvalues.

2.3.2.4 Part 4

The S nullclines are the solution of $-\beta SI = 0$ and the I nullclines are solutions of $\beta SI - vI = 0$ and R nullclines are solutions of vI

S nullclines are therefore $I = 0$ and $S = 0$.

I nullclines are therefore $I = 0$ and $S = \frac{v}{\beta}$.

R nullclines are $I = 0$.

2.3.2.5 Part 5

To answer this, we need to assume that $I(0)$ is not zero. This means initial condition such that some infection exist, otherwise $S(t)$ will not change.

Since $\dot{I} = \beta SI - vI$ then as I increases (infected population increases) and because $v > 0$ then the term vI becomes more negative. Since $S(t)$ also at the same time becomes smaller during this process (because more people are infected), then we see that $\dot{I}$ will eventually starts to decrease as I increases and this happens when vI becomes larger than βSI. (This is the peak infection).

This means infected population size eventually decreases as $I(t)$ passes some peak value. Since population is assumed constant, this implies the recovered population size will also increase and eventually all susceptible population that became infected will recover and infected population will go to zero with time.

2.3.3 Problem 3 (exercise 4.1, page 57)

In exercise 2.3 of chapter 2 we analyzed the existence of periodic solutions in an invariant set of a three-dimensional system. Obtain this result in a more straightforward manner. The following is 2.3 of chapter 2

We are studying the three-dimensional system

$$\begin{aligned}
\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right)\\
\dot{x}_2 &= x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right)\\
\dot{x}_3 &= \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)
\end{aligned}$$

Consider the invariant set $x_3 = 1$. Does this set contain periodic solutions?

solution

Writing the above as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{F}(x_1, x_2, x_2)\\
\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} &= \begin{pmatrix}f_1(x_1,x_2,x_3)\\ f_2(x_1,x_2,x_3)\\ f_3(x_1,x_2,x_3)\end{pmatrix} = \begin{pmatrix}x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right)\\ x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right)\\ \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)\end{pmatrix}
\end{aligned}$$

We now need to set $x_3 = 1$ before taking the divergence. The above simplifies to

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}f_1(x_1,x_2,x_3)\\ f_2(x_1,x_2,x_3)\\ f_3(x_1,x_2,x_3)\end{pmatrix} = \begin{pmatrix}x_1 - x_1x_2 - x_2^3 + \left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + 1\right)\\ x_1 - \left(x_1 - x_2 + 2x_1x_2\right)\\ 0\end{pmatrix}$$

Then using Bendixson's criterion, periodic solution exist only if divergence of $\boldsymbol{F}(x_1, x_2, x_2)$ changes sign in the domain D or if the divergence is identically zero. D is taken as the whole of $\mathbb{R}^3$.

$$\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \tag{1}$$

But

$$\frac{\partial f_1}{\partial x_1} = 1 - x_2 + (2x_1 - 1 + x_2) = 2x_1$$
$$\frac{\partial f_2}{\partial x_2} = -(-1 + 2x_1) = 1 - 2x_1$$
$$\frac{\partial f_3}{\partial x_3} = 0$$

Hence (1) now becomes

$$\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2) = 2x_1 + 1 - 2x_1 = 1$$

Therefore $\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2)$ does not change sign. Therefore no periodic solution exist.

2.3.4 key solution for HW 3

MATH 5525: HOMEWORK ASSIGNMENT 3

Date (April 3, 2020)

Problem 1. A modification of the predator-prey system is given by

$$\dot{x} = x(1-x) - \frac{axy}{x+1}, \quad \dot{y} = y(1-y), \tag{1}$$

where $a > 0$ is a parameter.

1. Find all equilibrium points, in the following two cases: $0 < a < 1$ and $a > 1$. (You may select spefic values of a, if you wish.)
2. Classify the equilibrium points in each case.
3. Sketch the nullclines and the phase portraits of for different values of a.
4. What is special about the parameter value $a = 1$? (It is called a bifurcation value, why?)

Solution.

1. Equilibrium points:

- For $0 < a < 1$: $(0,0)$, $(1,0)$, $(0,1)$, $(\sqrt{1-a}, 1)$.
- For $a > 1$: $(0,0)$, $(1,0)$, $(0,1)$.

2. Classification of equilibrium points:

- Linearized system about $(0,0)$: $\dot{x} = x$, $\dot{y} = y$. So the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$. *Unstable (or negatively attractive) node critical point.*
- Linearized system about $(0,1)$: $\dot{x} = x(1-a)$, $\dot{y} = (1-y)$. To calculate the eigenvalues, we change variables to $v = y - 1$, with $\dot{v} = \dot{y}$, leave x unchanged. The new linear system is $\dot{x} = x(1-a)$, $\dot{v} = -v$. The eigenvalues are $\lambda_1 = 1 - a$, $\lambda_2 = -1$. Hence, the critical point $(0,1)$ is a:
  1. A *stable (or positively attractive) node* for $a > 1$.
  2. A *saddle point* for $a < 1$.
- Linearized system about $(1,0)$: To calculate the eigenvalues, we change variables to $u = x - 1$, with $\dot{u} = \dot{x}$, leave y unchanged. The new linear system is $\dot{u} = -u - \frac{ay}{2}$, $\dot{y} = y$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Hence, the critical point $(1,0)$ is a *saddle point.*
- Linearized system about $(\sqrt{1-a}, 1)$, in the case $a < 1$. Let us denote $x_0 = \sqrt{1-a}$ and $u = x - x_0$ and $v = y - 1$. The linear system about $(x = x_0, y_0 = 1)$ (equivalently, about $(u = 0, v = 0)$) in terms of the new variables (u, v) is:

$$\dot{u} = x_0\big(-1 + \frac{a}{(1+x_0)^2}\big)u - \frac{ax_0}{1+x_0}v, \qquad \dot{v} = -v.$$

  The eigenvalues of the system are:

$$\lambda_1 = x_0\big(-1 + \frac{a}{(1+x_0)^2}\big) \quad \text{and} \quad \lambda_2 = -1.$$

  Since $0 < a < 1$, it immediately follows that $\lambda_1 < 0$. Since both eigenvalues are real and negative, we conclude that the equilibrium point $(\sqrt{1-a}, 1)$ is a *stable (or positively attractive) node.*

1

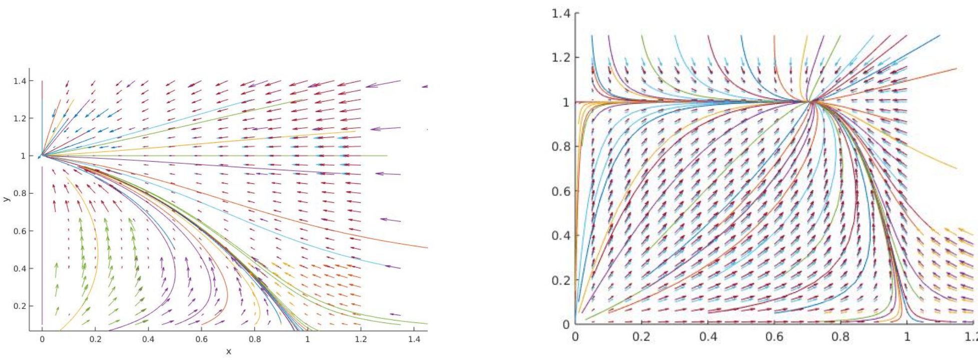


Figure 1: Left: phase plane for $a = 2$. Right: a=0.5. The parameter value $a = 1$ is called a bifurcation value due to the change of structure of the solutions as a increases from $a < 1$ to $a > 1$.

3. The nullclines are obtained by alternatively setting the right-hand sides of the governing equations equal to 0. These gives the following cases:

1. $x = 0, \quad \dot{y} = y(1 - y)$. *The y-axis is a nullcline.*
2. $\dot{x} = x(1 - x), \quad y = 0$. *The x-axis is a nullcline.*
3. $\dot{x} = x(1 - x) - \frac{ax}{x+1}, \quad y = 1$. *The line $y = 1$ is a nullcline.*

Problem 2. The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. In the simplest form of the model, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population.

We assume for simplicity, that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0.$$

The SIR model, in its simplest form, is stated as

$$\begin{aligned}\dot{S} =& -\beta SI,\\ \dot{I} =& \beta SI - \nu I,\\ \dot{R} =& \nu I,\end{aligned}$$

where $\nu > 0$ and $\beta > 0$ are parameters.

1. Show that the line $I = 0$ is an equilibrium line.
2. Find the matrix that results from linearizing the system about $I = 0$.
3. Calculate the eigenvalues of the resulting matrix.
4. Find the nullclines of the system.
5. What can we infer about the prospects of full recovery of the population?

Solution.

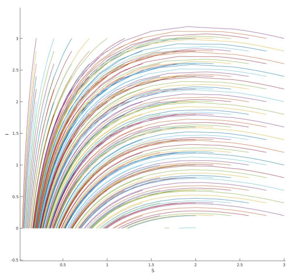


Figure 2: Phase plane for $\beta = 1$.and $\nu = 2$. Note that, in each orbit, the maximum of I is reached at $S = \frac{\nu}{\beta} = 2$.

First, note that the unknown fields $S, I, R \geq 0$, since they represent population figures. Let us consider the constraint $\frac{d}{dt}(S + I + R) = 0$ and its integrated form

$$S + I + R = C,$$

where $C \geq 0$ is a constant. We can solve it for $R = C - S - I$ and use it to calculate R, once we have found S and I. For this, we only need to consider the first two equations:

$$\dot{S} = -\beta SI, \quad \dot{I} = \beta SI - \nu I.$$

(In particular, note that they are independent of R.)

1. Note that the line $I = 0$ (the S-axis) is an equilibrium line since it makes $\dot{S} = 0$ and $\dot{I} = 0$, regardless the value of S.
2. Linearization at $S = 0$ yields the matrix $\begin{bmatrix} 0 & -\beta S \\ 0 & \beta S - \nu \end{bmatrix}$.
3. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \beta S - \nu$. This second eigenvalue is negative if $0 < S < \frac{\nu}{\beta}$ and positive $S > \frac{\nu}{\beta}$.
4. The nullclines of the system are the lines $S = 0$, $I = 0$ and $S = \frac{\nu}{\beta}$.
5. From the graph, we can see that given an initial population (I_0, S_0) with $S_0 > \frac{\nu}{\beta}$ and $I_0 > 0$, the susceptible population decreases monotonically, while the infected population at first rises, but eventually reaches a maximum and then declines to 0.

   [Note: We can analytically reach such a conclusion by examining the equation $\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S}$.]

Problem 3. Solve exercise 4.1, page 57 (textbook, second edition).

Solution.

1. The critical points of the system are:

$$(0, 0, 0), \quad (\pm 1, 0, 1), \quad (\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, 1).$$

2. Setting $x_3 = 0$ in the governing equations, we get

$$\dot{x}_1 = x_1 - x_1 x_2 - x_2^3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = 0.$$

Consider a solution with initial data $(a, b, 0)$. The third equation implies that x_3 is constant. So, $x_3 = 0$ for all time, and the system reduces to the firs two equations above. Hence, a solution with initial data $x_1 = a$ and $x_2 = b$ will remain in the $x_1 - x_2$-plane for all time.

The system obtained from the original one setting $x_3 = 1$ is

$$\dot{x}_1 = x_1^2 + x_2^2 - 1, \quad \dot{x}_2 = x_2(1 - 2x_1), \quad \dot{x}_3 = 0.$$

Arguing as in the previous case ($x_3 = 0$), we conclude that $x_3 = 1$ is an invariant set.

3. Let us consider the system

$$\dot{x}_1 = x_1^2 + x_2^2 - 1, \quad \dot{x}_2 = x_2(1 - 2x_1).$$

Calculate

$$\nabla \cdot \big(x_1^2 + x_2^2 - 1, x_2(1 - 2x_1)\big) = 2x_1 + 1 - 2x_1 = 1.$$

Since the divergence of the vector field does not change sign anywhere in the plane, by the Bendixon criterion, we conclude that the 2-d system does not have periodic solutions. That is, there are no periodic solutions in the plane $x_3 = 1$.

2.4 HW 4

Local contents

2.4.1 Problem 1

Problem 1. Consider the planer system given in polar coordinates by

$$\dot{r} = r(1-r), \quad \dot{\theta} = 1.$$

Note that $x = \cos t$, $y = \sin t$ is a periodic solution of the system (given in rectangular coordinates), satisfying $x = 1, y = 0$ at time $t = 0$. This solution belongs to the orbit $r = 1$.

1. Find the Poincaré map of the periodic solution $\phi(t) := (\cos t, \sin t)$.
2. Determine the stability of ϕ.

Figure 2.37: Problem description

solution

2.4.1.1 Part 1

$$\begin{aligned}\dot{r} &= r(1-r)\\ \dot{\theta} &= 1\end{aligned}$$

The Poincare section (line) is taken as the $\theta = 0$ axis (this means the x-axis in the Phase plane). The Poincare section can be any other line which is orthogonal to the orbits, but this line is the easiest to use because $\theta = 0$ on it.

Let $\phi(r_0)$ be the Poincare map. This means given an initial conditions point $r = r_0$ on the Poincare section then $\phi(r_0)$ will return the location of the solution $r(t)$ on this line after the solution $r(t)$ has run for one period T of time. i.e. $r(t+T)$, where the period is T. The following diagram helps illustrate this.

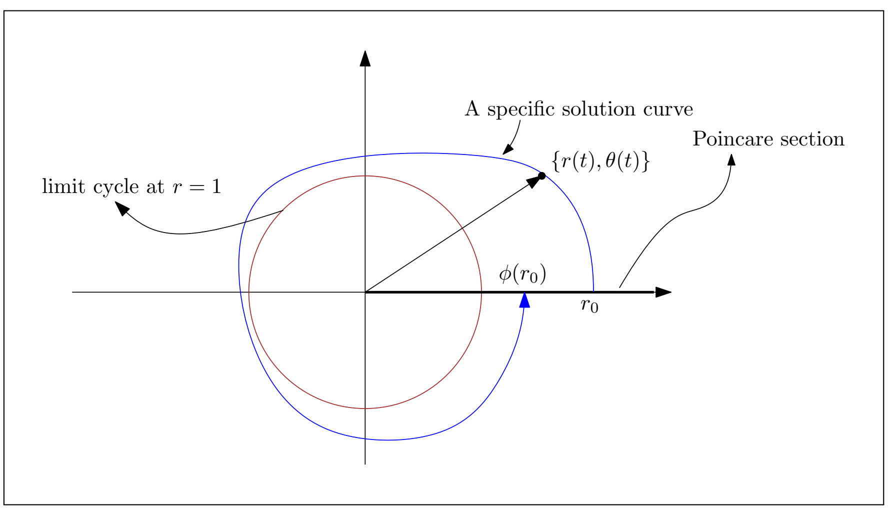


Figure 2.38: Showing Poincare section used

The first thing to determine when finding the Poincare map $\phi(r_0)$ is to solve the ODE. In this case, only the $r(t)$ ODE needs to be solved because there is no dependency on $\theta(t)$ in the $r(t)$ ODE. The following solves the ODE $\dot{r} = r(1-r)$.

$$\begin{aligned}\frac{dr}{dt} &= r(1-r)\\ \frac{dr}{r(1-r)} &= dt\\ \left(\frac{1}{r}-\frac{1}{r-1}\right)dr &= dt\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\int\left(\frac{1}{r}-\frac{1}{r-1}\right)dr &= \int dt\\ \ln|r|-\ln|r-1| &= t+C_1\\ \ln\left|\frac{r}{r-1}\right| &= t+C_1\\ \frac{r}{r-1} &= C_2e^t\end{aligned}$$

At initial conditions, $t=0, r=r_0$ the above becomes

$$\frac{r_0}{r_0-1} = C_2$$

Hence the solution is

$$\begin{aligned}\frac{r}{r-1} &= \frac{r_0}{r_0-1}e^t\\ r &= (r-1)\left(\frac{r_0}{r_0-1}e^t\right)\\ r &= r\left(\frac{r_0}{r_0-1}e^t\right)-\left(\frac{r_0}{r_0-1}e^t\right)\\ r-r\left(\frac{r_0}{r_0-1}e^t\right) &= -\left(\frac{r_0}{r_0-1}e^t\right)\\ r\left(1-\frac{r_0}{r_0-1}e^t\right) &= -\left(\frac{r_0}{r_0-1}e^t\right)\end{aligned}$$

Simplifying gives

$$\begin{aligned}r(t) &= \frac{-\left(\frac{r_0}{r_0-1}e^t\right)}{1-\frac{r_0}{r_0-1}e^t}\\ &= \frac{-1}{\frac{1}{\frac{r_0}{r_0-1}e^t}-1}\\ &= \frac{-1}{\frac{(r_0-1)e^{-t}}{r_0}-1}\\ &= \frac{-r_0}{(r_0-1)e^{-t}-r_0}\\ &= \frac{r_0}{r_0-(r_0-1)e^{-t}}\end{aligned}$$

At $t=T$ the above solution becomes

$$r(T) = \frac{r_0}{r_0-(r_0-1)e^{-T}}$$

Therefore the Poincare map is

$$\phi(r_0) = \frac{r_0}{r_0-(r_0-1)e^{-T}}$$

Taking the period as 2π, then the above becomes

$$\phi(r_0) = \frac{r_0}{r_0-(r_0-1)e^{-2\pi}} \tag{1}$$

What the above function does, is that given a specific value of r_0 on the Poincare section (the x-axis), it returns where the solution location will be on this line after one period.

For an example, taking initial conditions $r_0 = 1.5, \theta = 0$ then (1) gives

$$\begin{aligned}\phi(1.5) &= \frac{1.5}{1.5-(1.5-1)e^{-2\pi}}\\ &= 1.00062\end{aligned}$$

This shows the solution $r(t)$ after one period, has reached the Poincare section again at a point which is closer to the limit cycle at $r = 1$. Initially $r(0) = 1.5$ and after 2π seconds, $r(2\pi) = 1.00062$.

2.4.1.2 Part 2

Let $\phi(r_n)$ be Poincare map at iteration n.

$$\begin{aligned}
\phi(r_0) &= \frac{r_0}{r_0-(r_0-1)e^{-2\pi}}\\
\phi(r_1) &= \frac{\left(\frac{r_0}{r_0-(r_0-1)e^{-2\pi}}\right)}{\left(\frac{r_0}{r_0-(r_0-1)e^{-2\pi}}\right)-\left(\left(\frac{r_0}{r_0-(r_0-1)e^{-2\pi}}\right)-1\right)e^{-2\pi}} = \frac{r_0}{r_0+e^{-4\pi}-r_0e^{-4\pi}}\\
\phi(r_2) &= \frac{\left(\frac{r_0}{r_0+e^{-4\pi}-r_0e^{-4\pi}}\right)}{\left(\frac{r_0}{r_0+e^{-4\pi}-r_0e^{-4\pi}}\right)-\left(\left(\frac{r_0}{r_0+e^{-4\pi}-r_0e^{-4\pi}}\right)-1\right)e^{-2\pi}} = \frac{r_0}{r_0+e^{-6\pi}-r_0e^{-6\pi}}\\
\phi(r_3) &= \frac{\frac{r_0}{r_0+e^{-6\pi}-r_0e^{-6\pi}}}{\left(\frac{r_0}{r_0+e^{-6\pi}-r_0e^{-6\pi}}\right)-\left(\left(\frac{r_0}{r_0+e^{-6\pi}-r_0e^{-6\pi}}\right)-1\right)e^{-2\pi}} = \frac{r_0}{r_0+e^{-8\pi}-r_0e^{-8\pi}}\\
&\vdots\\
\phi(r_n) &= \frac{r_0}{r_0+e^{-(2+2n)\pi}-r_0e^{-(2+2n)\pi}}
\end{aligned}$$

As $n\rightarrow\infty$ the above gives

$$\lim_{n\rightarrow\infty}\phi(r_n)\rightarrow\frac{r_0}{r_0}=1$$

Therefore $\phi(r_0)$ is stable. The limit cycle at $r = 1$ is stable because any solution that starts always from $r = 1$ will eventually reach $r = 1$. The following also is a plot of $\phi(r_0)$

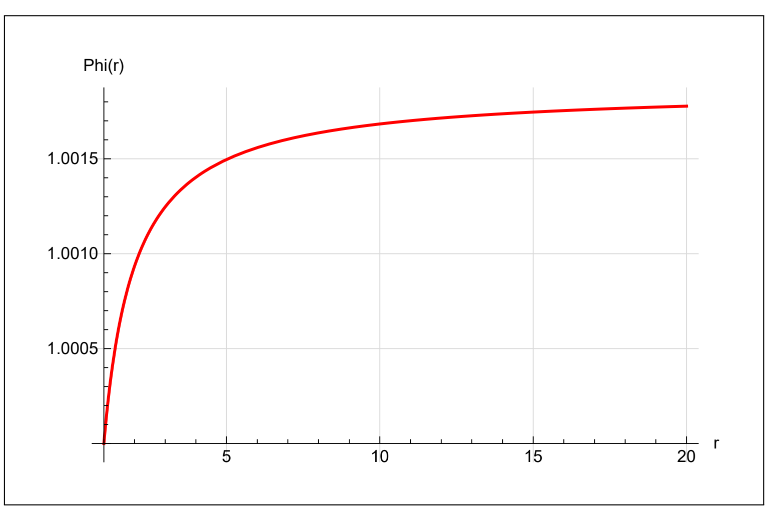


Figure 2.39: Plot of Poincare map function

2.4.2 Problem 2

Problem 2. The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. We now study a model that is more complicated than the basic SIR model (Problem 2, Assignment 3) where we now allow for the recovered population to become reinfected again (SIRS model).

Again, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population.

We still assume that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0, \tag{1}$$

that is, $S + I + R = \tau > 0$, for a given $\tau > 0$.

The SIRS model is stated as

$$\begin{aligned} \dot{S} &= -\beta SI + \mu R, && (2)\\ \dot{I} &= \beta SI - \nu I, && (3)\\ \dot{R} &= \nu I - \mu R, && (4) \end{aligned}$$

where $\nu, \mu > 0$ and $\beta > 0$ are parameters.

1. Use the restriction (1) to reduce the system to one with two equations and two unknown fields.
2. Find the critical points in two different cases: (a) $\tau \geq \frac{\nu}{\beta}$ and (b) $\tau \leq \frac{\nu}{\beta}$.
3. Calculate the eigenvalues associated to the critical points in each case, (a) and (b).
4. In case (a), give an interpretation of the biological significance of the stability properties of the critical points.

   From now on, we restrict ourselves to case (a).

5. Note that the SIRS system is only of interest in the region

   $$\Delta := \{(I, S) : I, S \geq 0, \text{and } S + I \leq \tau\}.$$

   Why?

   Determine whether the I axis is invariant. What about the S-axis? What is the behavior of the solutions restricted to the S-axis.
6. Show that the region Δ is positively invariant.

Figure 2.40: Problem description

solution

The SIRS model is

$$\begin{aligned} \dot{S} &= -\beta SI + \mu R \qquad (1)\\ \dot{I} &= \beta SI - \upsilon I\\ \dot{R} &= \upsilon I - \mu R \end{aligned}$$

Where $S = S(t)$ is the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population. This diagram shows the model where now some of the recovered population can become susceptible again and later become infected. The parameter μ indicates how much of the recovered population could become susceptible again.

The units of S, R, I are population measured in ***person***. These values can not be negative since they are population amount. The units of μ is $\frac{1}{\text{time}}$ where time can be day or week or any other unit of time. The units of v is also $\frac{1}{\text{time}}$. The units of β is $\frac{1}{(\text{time})(\text{person})}$

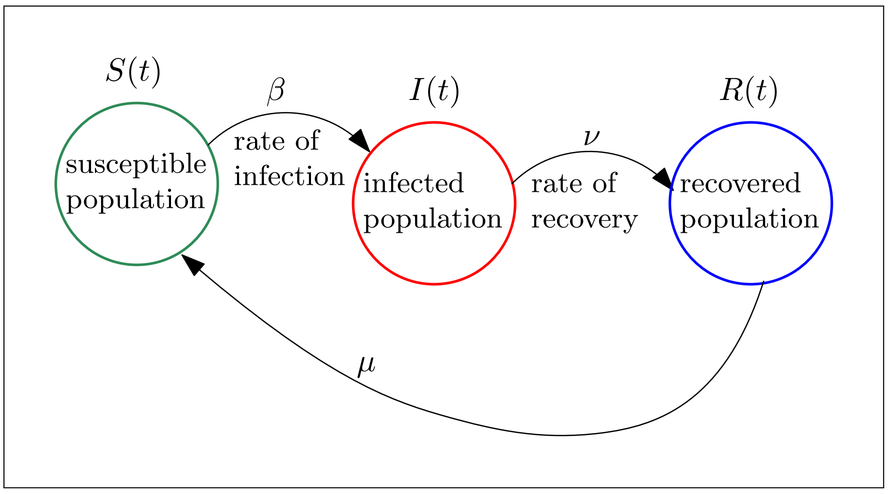


Figure 2.41: SIRS model

2.4.2.1 Part 1

Since $\frac{d}{dt}(S+I+R)=0$ then $S+I+R=\tau$, where τ is constant (the total size of the population). Therefore $R=\tau-I-S$. This implies that, since τ is known, then finding I and S allows finding R automatically from this algebraic equation. Replacing R by $\tau-I-S$ in the first and the second equations in (1) eliminates R and gives (the restricted system) as

$$\begin{aligned}\dot{S}&=-\beta SI+\mu(\tau-I-S)\\ \dot{I}&=\beta SI-vI\end{aligned}\tag{2}$$

This is the new system of equations. The third ODE is not needed. Once $S(t), I(t)$ are solved for, then $R(t)$ is found from $R=\tau-I-S$ without the need to solve a third ODE.

2.4.2.2 Part 2

From (2), the critical points are solutions to restricted system

$$-\beta SI+\mu(\tau-I-S)=0 \tag{2}$$
$$\beta SI-vI=0 \tag{4}$$

From (4) $I(\beta S-v)=0$. This gives solutions $I=0, S=\frac{v}{\beta}$. When $I=0$ then (2) gives $\mu(\tau-S)=0$ or $S=\tau$ since μ is not zero. The first critical point is therefore $\{I=0,S=\tau\}$.

Now, when $S=\frac{v}{\beta}$ Eq. (2) gives $-\beta\frac{v}{\beta}I+\mu\left(\tau-I-\frac{v}{\beta}\right)=0$ or $-\beta\frac{v}{\beta}I+\mu\tau-\mu I-\mu\frac{v}{\beta}=0$, hence

$$\begin{aligned}I\left(-\beta\frac{v}{\beta}-\mu\right)&=-\mu\tau+\mu\frac{v}{\beta}\\ I&=-\frac{\mu\left(\frac{v}{\beta}-\tau\right)}{\beta\frac{v}{\beta}+\mu}\\ &=\frac{\mu\left(\tau-\frac{v}{\beta}\right)}{\beta\frac{v}{\beta}+\mu}\end{aligned}$$

Therefore the second critical point is $\left\{I=\frac{\mu\left(\tau-\frac{v}{\beta}\right)}{\beta\frac{v}{\beta}+\mu}, S=\frac{v}{\beta}\right\}$ or $\left\{I=\frac{\tau\mu\beta-v\mu}{\beta(v+\mu)}, S=\frac{v}{\beta}\right\}$. These critical points are for the reduced system.

For each one of these critical points found above, the critical value for R can now be found from the relation $R=\tau-I-S$. For the first critical point $\{I=0,S=\tau\}$ then $R=\tau-0-\tau=0$. Therefore this critical point becomes $\{I=0,S=\tau,R=0\}$.

For the second critical point found above then

$$\begin{aligned}R &= \tau - I - S\\&= \tau - \frac{\tau\mu\beta - v\mu}{\beta\left(v+\mu\right)} - \frac{v}{\beta}\\&= v\frac{\beta\tau - v}{\beta\left(\mu+v\right)}\end{aligned}$$

Hence this critical point becomes $\left\{I = \frac{\tau\mu\beta - v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\frac{\beta\tau - v}{\beta\left(\mu+v\right)}\right\}$. Therefore the two critical points for the full system are

$$\{I=0, S=\tau, R=0\}$$
$$\left\{I = \frac{\tau\mu\beta - v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\frac{\beta\tau - v}{\beta\left(\mu+v\right)}\right\}$$

case (a) Under the case that $\tau > \frac{v}{\beta}$ then the second critical point above becomes

$$I = \frac{\tau\mu\beta}{\beta\left(v+\mu\right)} - \frac{v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\left(\frac{\tau}{\left(\mu+v\right)} - \frac{v}{\beta\left(\mu+v\right)}\right)$$

Replacing τ by $\frac{v}{\beta} + \delta$ where $\delta > 0$ then the above becomes

$$\begin{aligned}I &= \frac{v}{\beta}\frac{\mu}{\left(v+\mu\right)} + \delta\frac{\mu}{\left(v+\mu\right)} - \frac{v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\left(\frac{v}{\beta}\frac{1}{\left(\mu+v\right)} + \frac{\delta}{\mu+v} - \frac{v}{\beta\left(\mu+v\right)}\right)\\I &= \delta\frac{\mu}{\left(v+\mu\right)}, S = \frac{v}{\beta}, R = \frac{v\delta}{\mu+v}\end{aligned}$$

Since all $\mu, v, \beta, \delta > 0$ then the above is a valid critical point. Hence under this case the critical points are

$$\{I=0, S=\tau, R=0\}$$
$$\left\{I = \delta\frac{\mu}{\left(v+\mu\right)}, S = \frac{v}{\beta}, R = \frac{v\delta}{\mu+v}\right\}$$

Where $\tau = \frac{v}{\beta} + \delta$ where $\delta > 0$.

case (b) Under the case that $\tau < \frac{v}{\beta}$ the second critical point above becomes

$$I = \frac{\tau\mu\beta}{\beta\left(v+\mu\right)} - \frac{v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\left(\frac{\tau}{\left(\mu+v\right)} - \frac{v}{\beta\left(\mu+v\right)}\right)$$

Replacing τ by $\frac{v}{\beta} - \delta$ where $\delta > 0$ the above becomes

$$\begin{aligned}I &= \frac{v}{\beta}\frac{\mu}{\left(v+\mu\right)} - \delta\frac{\mu}{\left(v+\mu\right)} - \frac{v\mu}{\beta\left(v+\mu\right)}, S = \frac{v}{\beta}, R = v\left(\frac{v}{\beta}\frac{1}{\left(\mu+v\right)} - \frac{\delta}{\mu+v} - \frac{v}{\beta\left(\mu+v\right)}\right)\\I &= -\delta\frac{\mu}{\left(v+\mu\right)}, S = \frac{v}{\beta}, R = -\frac{v\delta}{\mu+v}\end{aligned}$$

Since all $\mu, v, \beta, \delta > 0$ then the above is a not a valid critical point, because it says I, R are negative and this is not possible, as these are population size. Hence under this case the critical point is only

$$\{I=0, S=\tau, R=0\}$$

case $\tau = \frac{v}{\beta}$

Under this case then the second critical point above becomes

$$I=\frac{\tau\mu\beta}{\beta\left(v+\mu\right)}-\frac{v\mu}{\beta\left(v+\mu\right)},S=\frac{v}{\beta},R=v\left(\frac{\tau}{\left(\mu+v\right)}-\frac{v}{\beta\left(\mu+v\right)}\right)$$

Replacing τ by $\frac{v}{\beta}$ then the above becomes

$$\begin{aligned}I&=\frac{v}{\beta}\frac{\mu}{\left(v+\mu\right)}-\frac{v\mu}{\beta\left(v+\mu\right)},S=\frac{v}{\beta},R=v\left(\frac{v}{\beta}\frac{1}{\left(\mu+v\right)}-\frac{v}{\beta\left(\mu+v\right)}\right)\\I&=0,S=\frac{v}{\beta},R=0\end{aligned}$$

But this is the same as the first critical point. Hence only one critical point exist under this condition. This is summary of result.

case	critical points
case (a) $\tau\geq\frac{v}{\beta}$	$\{I=0,S=\tau,R=0\},\left\{I=\frac{\mu}{v+\mu}\delta,S=\frac{v}{\beta},R=\frac{v}{\mu+v}\delta\right\}$ where $\delta=\tau-\frac{v}{\beta}$
case (b) $\tau\leq\frac{v}{\beta}$	$\{I=0,S=\tau,R=0\}$

2.4.2.3 Part 3

The full system (1) is (I assumed we need to find eigenvalues for full system. The question was not clear on this).

$$\begin{aligned}\dot{S}&=-\beta SI+\mu R\\\dot{I}&=\beta SI-vI\\\dot{R}&=vI-\mu R\end{aligned}\tag{1}$$

The Jacobian matrix is

$$J=\begin{pmatrix}\frac{\partial\dot{S}}{\partial S}&\frac{\partial\dot{S}}{\partial I}&\frac{\partial\dot{S}}{\partial R}\\\frac{\partial\dot{I}}{\partial S}&\frac{\partial\dot{I}}{\partial I}&\frac{\partial\dot{I}}{\partial R}\\\frac{\partial\dot{R}}{\partial S}&\frac{\partial\dot{R}}{\partial I}&\frac{\partial\dot{R}}{\partial R}\end{pmatrix}=\begin{pmatrix}-\beta I&-\beta S&\mu\\\beta I&\beta S-v&0\\0&v&-\mu\end{pmatrix}$$

2.4.2.3.1 Case a

At the first critical point $\{I=0,S=\tau,R=0\}$ the above becomes

$$A=\begin{pmatrix}0&-\beta\tau&\mu\\0&\beta\tau-v&0\\0&v&-\mu\end{pmatrix}$$

The eigenvalues are from $|A-\lambda I|=0$ or

$$\begin{vmatrix}-\lambda&-\beta\tau&\mu\\0&\beta\tau-v-\lambda&0\\0&v&-\mu-\lambda\end{vmatrix}=0$$

$$\begin{aligned}-\lambda\begin{vmatrix}\beta\tau-v-\lambda&0\\v&-\mu-\lambda\end{vmatrix}+\beta\tau\begin{vmatrix}0&0\\0&-\mu-\lambda\end{vmatrix}+\mu\begin{vmatrix}0&\beta\tau-v-\lambda\\0&v\end{vmatrix}&=0\\-\lambda\left(\beta\tau-v-\lambda\right)\left(-\mu-\lambda\right)&=0\\-\lambda\left(\lambda+\mu\right)\left(\lambda+v-\beta\tau\right)&=0\end{aligned}$$

Hence $\lambda=0,\lambda=-\mu,\lambda=\beta\tau-v$ are the eigenvalues.

At the second critical point $\left\{I = \frac{\mu}{v+\mu}\delta, S = \frac{v}{\beta}, R = \frac{v}{\mu+v}\delta\right\}$ the Jacobian matrix becomes

$$\begin{aligned} A &= \begin{pmatrix} -\beta I & -\beta S & \mu \\ \beta I & \beta S - v & 0 \\ 0 & v & -\mu \end{pmatrix} \\ &= \begin{pmatrix} -\beta\frac{\mu}{v+\mu}\delta & -\beta\frac{v}{\beta} & \mu \\ \beta\frac{\mu}{v+\mu}\delta & \beta\frac{v}{\beta} - v & 0 \\ 0 & v & -\mu \end{pmatrix} \\ &= \begin{pmatrix} -\beta\frac{\mu}{v+\mu}\delta & -v & \mu \\ \beta\frac{\mu}{v+\mu}\delta & 0 & 0 \\ 0 & v & -\mu \end{pmatrix} \end{aligned}$$

The eigenvalues are from $|A - \lambda I| = 0$ or

$$\begin{vmatrix} -\beta\frac{\mu}{v+\mu}\delta - \lambda & -v & \mu \\ \beta\frac{\mu}{v+\mu}\delta & -\lambda & 0 \\ 0 & v & -\mu - \lambda \end{vmatrix} = 0$$

$$\left(-\beta\frac{\mu}{v+\mu}\delta - \lambda\right)\begin{vmatrix} -\lambda & 0 \\ v & -\mu - \lambda \end{vmatrix} + v\begin{vmatrix} \frac{\tau\mu\beta - v\mu}{(v+\mu)} & 0 \\ 0 & -\mu - \lambda \end{vmatrix} + \mu\begin{vmatrix} \beta\frac{\mu}{v+\mu}\delta & -\lambda \\ 0 & v \end{vmatrix} = 0$$

$$\left(-\beta\frac{\mu}{v+\mu}\delta - \lambda\right)(-\lambda)\left(-\mu - \lambda\right) + v\left(\frac{\tau\mu\beta - v\mu}{\left(v+\mu\right)}\right)\left(-\mu - \lambda\right) + \mu v\left(\beta\frac{\mu}{v+\mu}\delta\right) = 0$$

$$\lambda^3 + \lambda^2\left(\beta\frac{\mu}{v+\mu}\delta + \mu\right) + \lambda\left(\beta\frac{\mu v}{v+\mu}\delta + \beta\frac{\mu^2}{v+\mu}\delta\right) = 0$$

$$\lambda\left(\lambda^2 + \lambda\left(\beta\frac{\mu}{v+\mu}\delta + \mu\right) + \left(\beta\frac{\mu v}{v+\mu}\delta + \beta\frac{\mu^2}{v+\mu}\delta\right)\right) = 0$$

Hence one eigenvalue is $\lambda_1 = 0$ and the other two are the solution to

$$\lambda^2 + \lambda\left(\beta\delta\frac{\mu}{v+\mu} + \mu\right) + \frac{\beta\delta\mu}{v+\mu}\left(\mu + v\right) = 0$$

Which is quadratic. Using $\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ then the second and third eigenvalues are

$$\lambda_{2,3} = -\frac{\beta\delta\frac{\mu}{v+\mu} + \mu}{2} \pm \frac{1}{2}\sqrt{\left(\beta\delta\frac{\mu}{v+\mu} + \mu\right)^2 - 4\left(\frac{\beta\delta\mu}{v+\mu}\left(\mu + v\right)\right)}$$

2.4.2.3.2 Case b Only one critical point $\{I = 0, S = \tau, R = 0\}$. This was found above as $\lambda = 0, \lambda = -\mu, \lambda = \beta\tau - v$.

Summary of result

In the following, for case a, $\tau - \frac{v}{\beta} = \delta$ where $\delta > 0$

<table>
<tr><td>case</td><td>result</td></tr>
<tr><td>case (a) $\tau \geq \frac{v}{\beta}$</td><td>
<table>
<tr><td>Critical point</td><td>Eigenvalues</td></tr>
<tr><td>$\{I = 0, S = \tau, R = 0\}$</td><td>$\lambda_1 = 0, \lambda_2 = -\mu, \lambda_2 = \beta\tau - v$</td></tr>
<tr><td>$I = \frac{\mu}{v+\mu}\delta, S = \frac{v}{\beta}, R = \frac{v}{\mu+v}\delta$</td><td>$\lambda_1 = 0, \lambda_{2,3} = -\frac{\beta\delta\frac{\mu}{v+\mu} + \mu}{2} \pm \frac{1}{2}\sqrt{\left(\beta\delta\frac{\mu}{v+\mu} + \mu\right)^2 - 4\left(\frac{\beta\delta\mu}{v+\mu}\left(\mu + v\right)\right)}$</td></tr>
</table>
</td></tr>
<tr><td>case (b) $\tau \leq \frac{v}{\beta}$</td><td>
<table>
<tr><td>Critical point</td><td>Eigenvalues</td></tr>
<tr><td>$\{I = 0, S = \tau, R = 0\}$</td><td>$\lambda_1 = 0, \lambda_2 = -\mu, \lambda_2 = \beta\tau - v$</td></tr>
</table>
</td></tr>
</table>

2.4.2.4 Part 4

For case (a), the critical points are found to be $\{I = 0, S = \tau, R = 0\}$ and $\left\{I = \frac{\mu}{v+\mu}\delta, S = \frac{v}{\beta}, R = \frac{v}{\mu+v}\delta\right\}$ where $\tau - \frac{v}{\beta} = \delta$ and $\delta > 0$.

For the first critical point, it says that there are no infected population and no recovered population. Hence we expect this to be stable, since no infection will occur, and the population τ is all of type $S(t)$ and will remain so for all time.

For the second critical point $\left\{I = \frac{\mu}{v+\mu}\delta, S = \frac{v}{\beta}, R = \frac{v}{v+\mu}\delta\right\}$, it implies that amount of people who become infected each day is offset by the amount of people who are recovered but become again susceptible. And the amount of people who recover each day is offset by the same amount of people who were recovered but become susceptible. This keeps the size of infected people each day the same not changing, as well as the size of susceptible and recovered population the same. This will only happen when $\tau \geq \frac{v}{\beta}$. A simulation is made to show this. This below shows, using $\tau = 1000$, the second critical point when $\frac{v}{\beta} = 707$, showing the current size of $S(t), I(t), R(t)$ which as be seen, remain the same for all time.

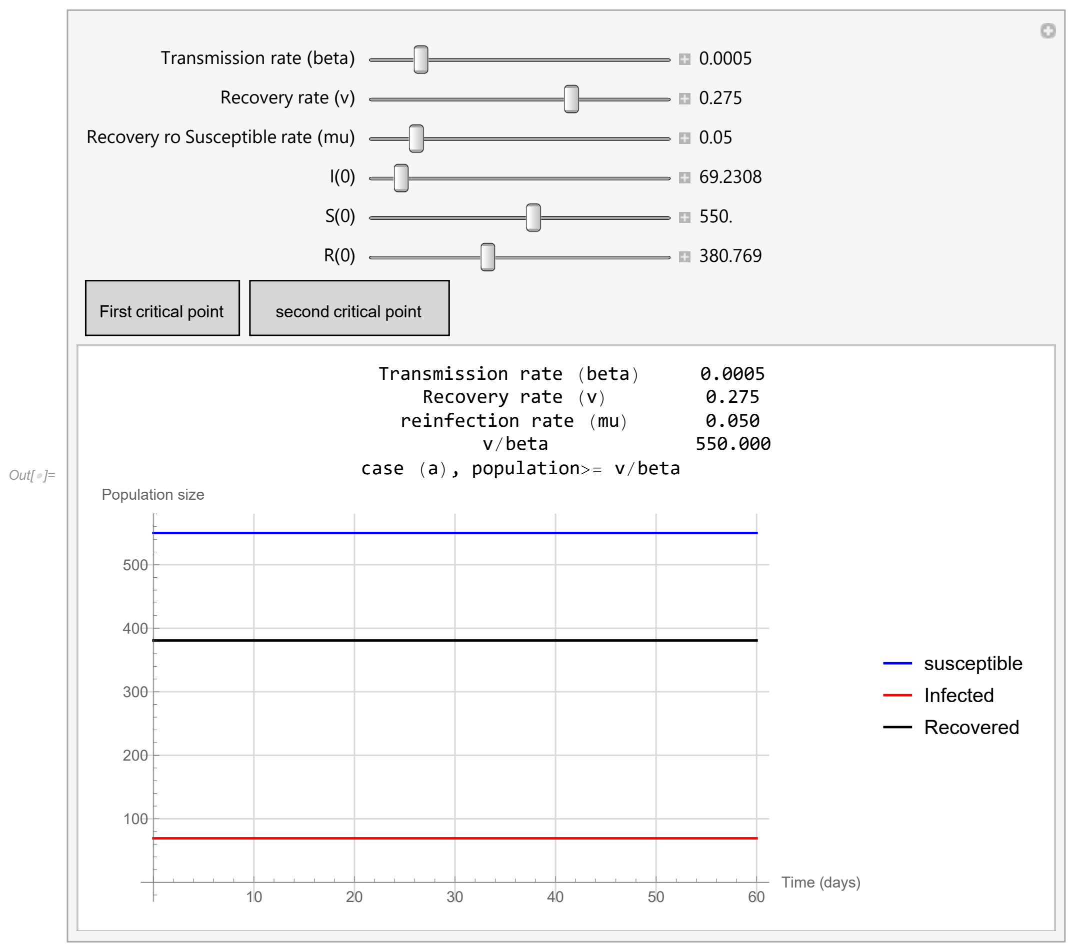


Figure 2.42: Second critial point example

This shows what happens when the infection rate β is relatively high at $\beta = 0.0013$

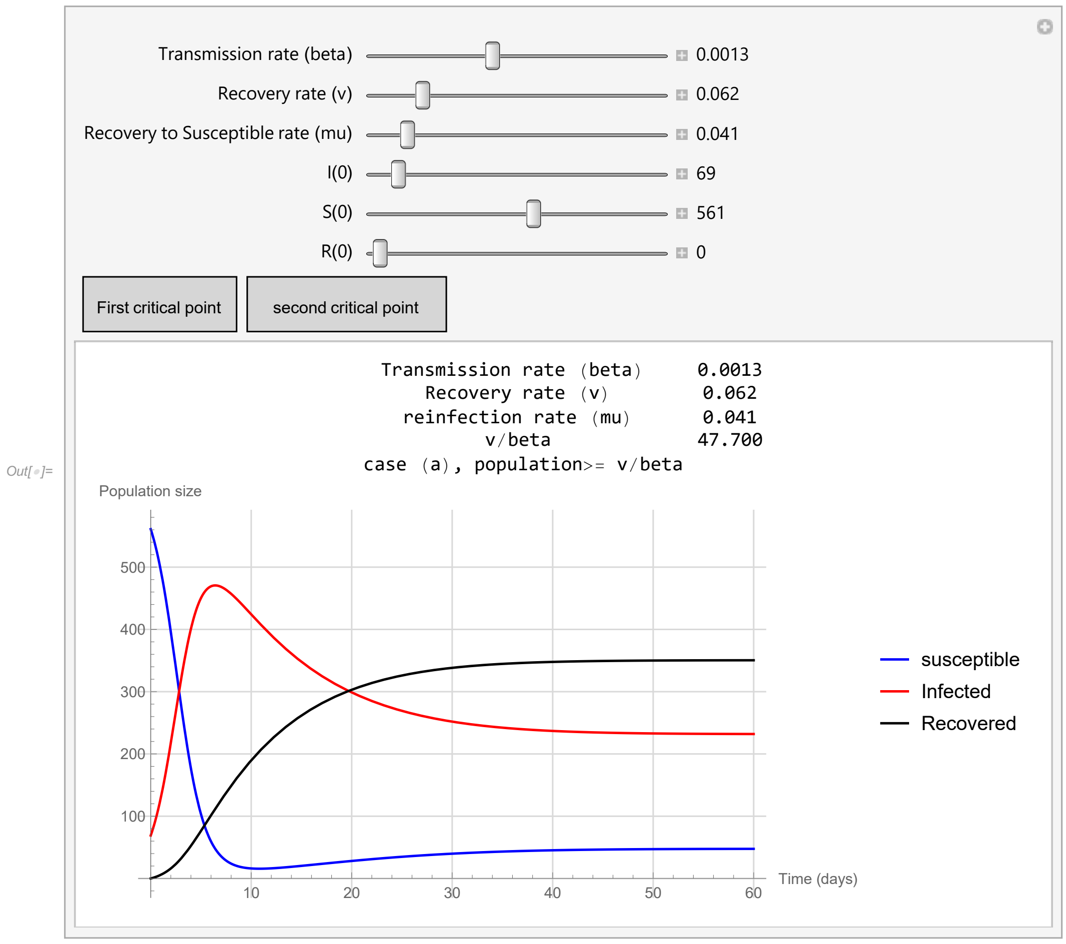


Figure 2.43: Example for large β

This shows what happens when the infection rate β has been decreased down to $\beta = 0.0003$. Notice the flattening of the red curve (infected population).

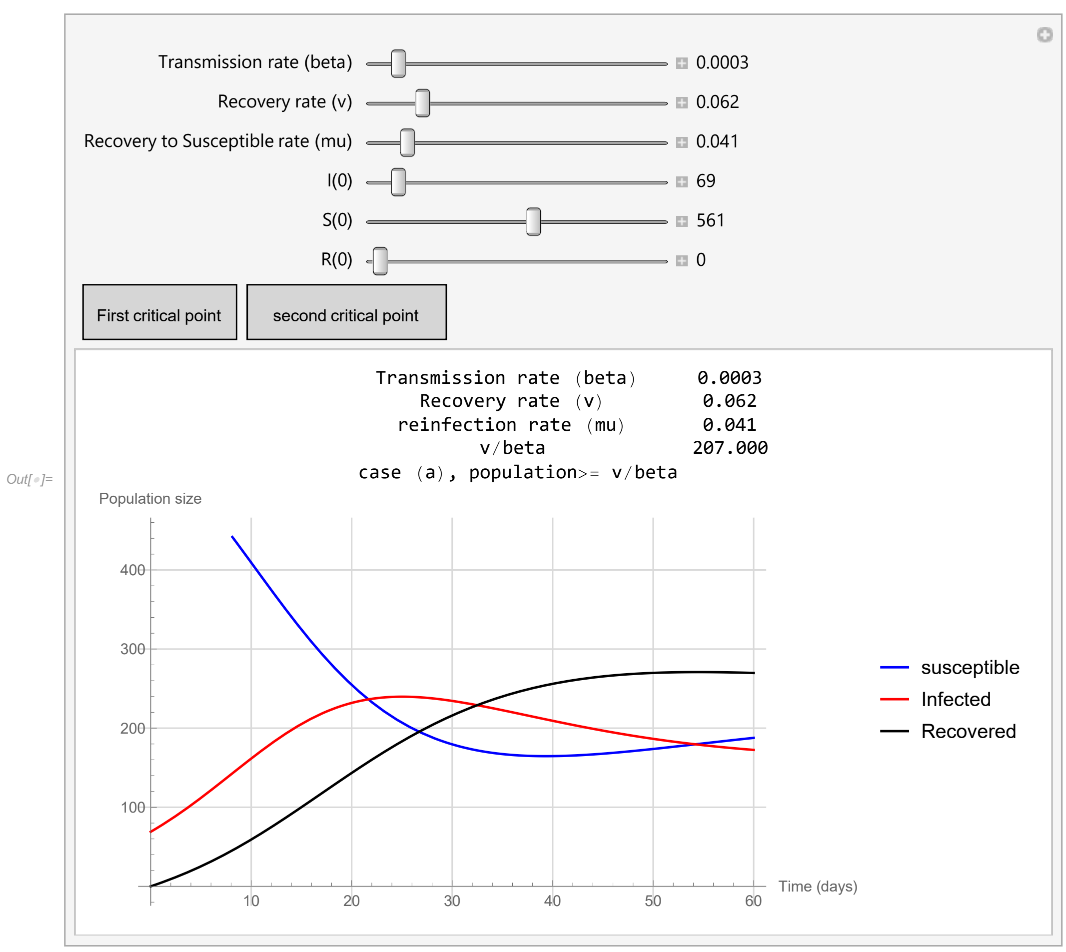


Figure 2.44: Example for small β

2.4.2.5 Part 5

Considering now only the restricted system found in part 1, which is

$$\dot{S} = -\beta SI + \mu(\tau - I - S)$$
$$\dot{I} = \beta SI - \upsilon I$$

The condition $S + I \leq \tau$ is always true since total size of population is fixed $\tau = S + I + R$.

For condition $I, S \geq 0$, it means that there are some infected population and some susceptible population present. This is of interest, since it implies dynamics of infection is still in place and taking effect.

Going back now to the full model

$$\dot{S} = -\beta SI + \mu R$$
$$\dot{I} = \beta SI - \upsilon I$$
$$\dot{R} = \upsilon I - \mu R$$

Considering the line $I(t)$. To check if it is invariant of not. Then assuming initial conditions starts on this line only, which means $S(0) = 0, R(0) = 0, I(0) = \tau$. When this is the case, then we see that from third equation that $R(t)$ will change with time, since it depends on $I(t)$ which is not zero. This in turn causes $S(t)$ to change from zero. Therefore the line $I(t)$ is not invariant. In other words, if $I(0) = \tau$, then solution will not remain on this line for all time and eventually both $R(t)$ and $S(t)$ will be non-zero.

To verify this, I run the simulation program with $I(0) = \tau = 1000$ and it shows that solution will not remain on axis $I(t)$. This is because some infected population will recover, and some of those who recovered will become susceptible again.

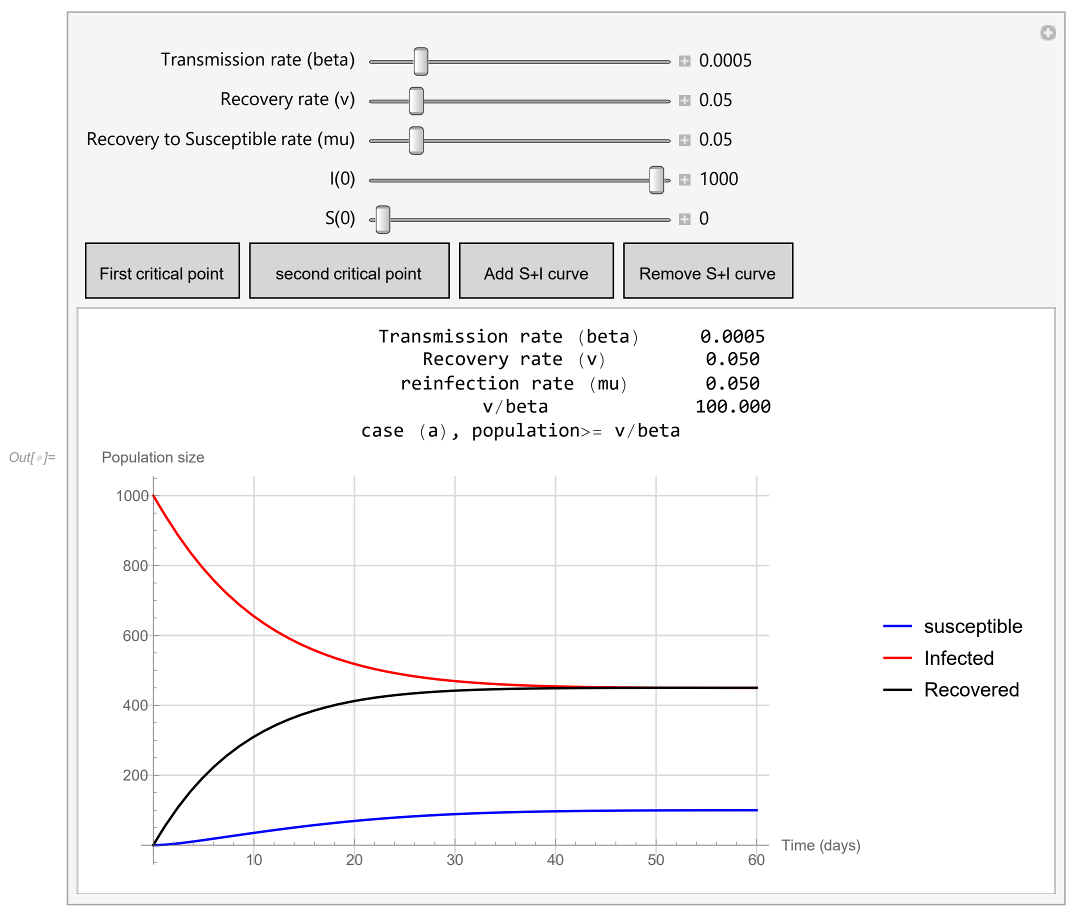


Figure 2.45: Showing that $I(t)$ line is not invariant

Now let us consider the $S(t)$ axis. This means $S(0) = \tau$ and now we check if the solution remains on $S(t)$ axis or not. In this case we would expect S axis to be invariant, since there are no infected population. This means $I(0) = 0, R(0) = 0$. From (1) this shows that $\dot{R}(0) = 0$, hence $R(t)$ is constant and remain zero for all time. And since $I(0) = 0$ then second equation in (1) also shows that $\dot{I}(0) = 0$, hence $I(t)$ is constant and remain zero for all time. This means first equation of (1) gives $\dot{S}(t) = 0$ and $S(t)$ is constant for all time which is τ. So the solution remains on the S axis. Hence S axis is invariant.

2.4.2.6 Part 6

Considering now only the restricted system found in part 1, which is

$$\dot{S} = -\beta SI + \mu(\tau - I - S)$$
$$\dot{I} = \beta SI - vI$$

Starting in region Δ, the solution will remain in Δ, because total population is fixed and there is no death. Hence starting with any combination of initial conditions $S(0), I(0)$ the solution will always be in this region. Hence positive invariant.

2.4.3 Problem 3

Problem 3. For each of the following systems, identify all points that lie in either an $\omega-$limit or an α-limit set:

- $r' = r - r^3$, $\theta' = 1$.
- $r' = r^3 - 3r^2 + 2r$, $\theta' = 1$.

Figure 2.46: Problem description

solution

ω limit set, is the set of all points that are the limit of all positive orbits $\gamma^+(x)$. In other words, given a specific orbit $\gamma^+(x)$ that starts at some initial conditions point x_0 and if as $t \to \infty$ this orbit terminates at point p then p is in the ω limit set of such orbit.

Similarly, the α limit set, is the set of all points that are the limit of the negative orbits $\gamma^-(x)$. These are orbits where as $t \to -\infty$ the orbit would have originated from the point p. Then p is in the α limit set of such orbit.

To find the ω limit set, we need to find the points where solutions terminate at them eventually (attractive or saddle points) and the α limit set are points that solutions do not terminate at them, but move away from them (negative attractions).

2.4.3.1 Part a

$$r' = r\left(1 - r^2\right)$$
$$\theta' = 1$$

The critical points are $r = 0, r = -1, r = 1$. But since this is polar coordinates and r is the radius, then r can not be negative, Hence only $r = 0, r = 1$ are the critical points (with θ any value). This implies that all points on circle of radius 1 and the point $r = 0$. To determine ω limit set and α limit set we have to check the eigenvalues. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial \theta} \\ \frac{\partial \theta'}{\partial r} & \frac{\partial \theta'}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 1 - 3r^2 & 0 \\ 0 & 0 \end{pmatrix}$$

At $r = 0$

$$|J - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

Therefore $(1 - \lambda)(-\lambda) = 0$ or $\lambda = 0, \lambda = 1$. Hence the point $r = 0$ is unstable. Therefore the α limit set is the origin $\{r = 0, \theta\}$ for all orbits that start with initial conditions $r(0) < 1$. There is no α limit set for orbits that start outside $r > 1$ since there is no critical point outside the limit cycle.

At $r = 1$ the determinant becomes

$$|J - \lambda I| = \begin{vmatrix} (1 - 3) - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

Therefore $(-2-\lambda)(-\lambda)=0$. Hence $\lambda=0,\lambda=-2$. Hence $r=1$ is stable. What this means is that ω limit set is all points on the unit circle corresponding to all orbits that start anywhere, including on the limit circle itself, (except orbits that have initial conditions $r=0$). The following diagram illustrates this result

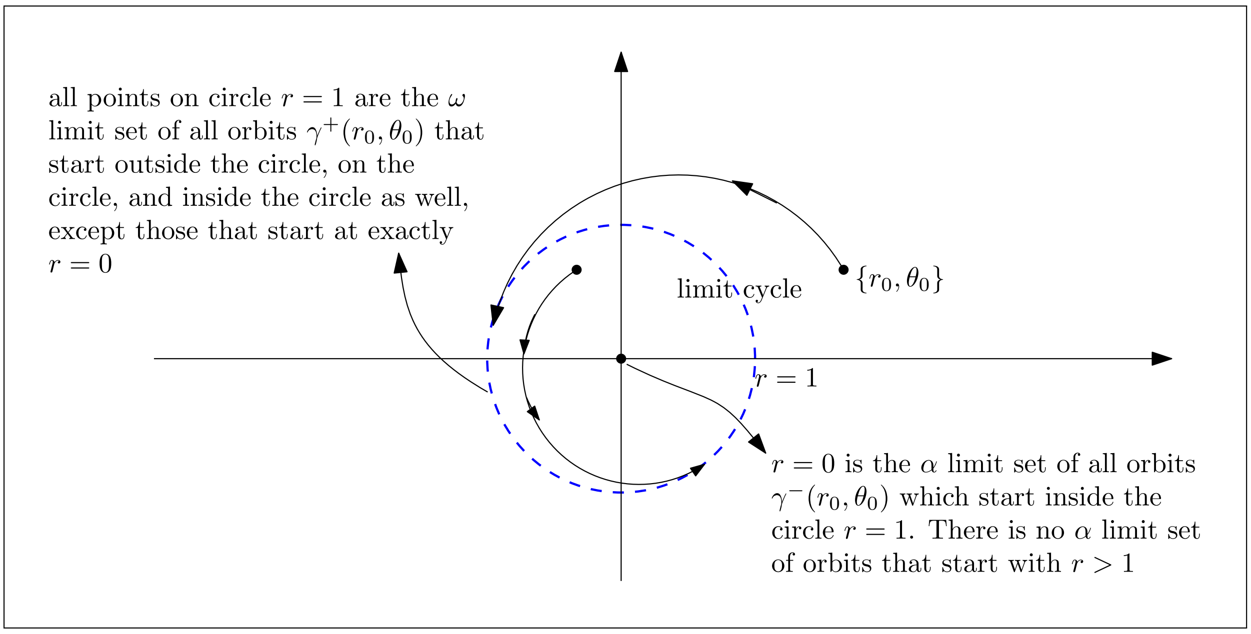


Figure 2.47: Limit sets for part a

2.4.3.2 Part b

$$
\begin{aligned}
r' &= r\left(r^2-3r+2\right)\\
\theta' &= 1
\end{aligned}
$$

The critical points are $r=0$, all points on circle of radius $r=2$ and all points on circle of radius $r=1$.

To determine ω limit set and α limit set we have to check the eigenvalues at each set of critial points fround above. The Jacobian matrix is

$$
J=\begin{pmatrix}\frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial \theta}\\ \frac{\partial \theta'}{\partial r} & \frac{\partial \theta'}{\partial \theta}\end{pmatrix}=\begin{pmatrix}3r^2-6r+2 & 0\\ 0 & 0\end{pmatrix}
$$

At $r=0$

$$
|J-\lambda I|=\begin{vmatrix}2-\lambda & 0\\ 0 & -\lambda\end{vmatrix}=0
$$

Therefore $(2-\lambda)(-\lambda)=0$ or $\lambda=0,\lambda=2$. Hence the point $r=0$ is unstable.

At $r=1$

$$
\begin{aligned}
|J-\lambda I| &= \begin{vmatrix}(3-6+2)-\lambda & 0\\ 0 & -\lambda\end{vmatrix}\\
&= \begin{vmatrix}-1-\lambda & 0\\ 0 & -\lambda\end{vmatrix}=0
\end{aligned}
$$

Therefore $(-1-\lambda)(-\lambda)=0$. Hence $\lambda=0,\lambda=-1$. Hence all points on circle $r=1$ are stable critical points.

At $r=2$,

$$
\begin{aligned}
|J-\lambda I| &= \begin{vmatrix}(12-12+2)-\lambda & 0\\ 0 & -\lambda\end{vmatrix}\\
&= \begin{vmatrix}2-\lambda & 0\\ 0 & -\lambda\end{vmatrix}=0
\end{aligned}
$$

Therefore $(2-\lambda)(-\lambda) = 0$. Hence $\lambda = 0, \lambda = 2$. Hence $r = 2$ is not stable. The following diagram illustrates the result of what was found above showing the ω limit set and the α limit sets found.

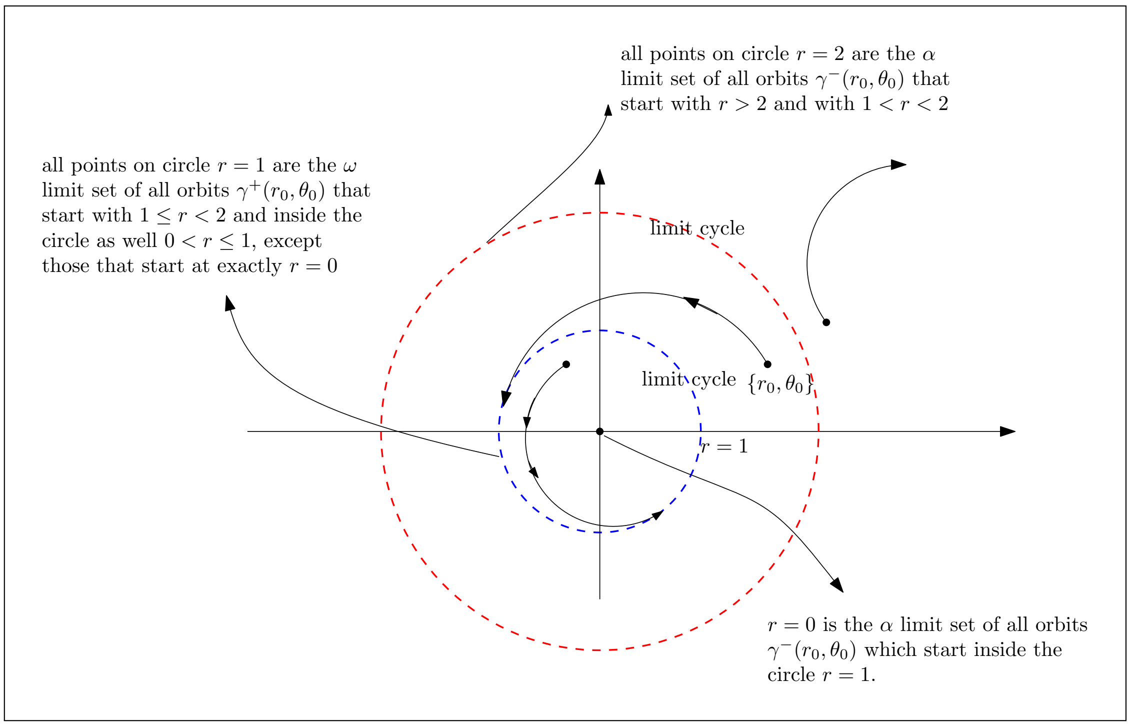


Figure 2.48: Limit sets for part b

2.4.4 key solution for HW 4

Assignment 4: Solutions

Date (April 23, 2020)

Problem 1. Consider the planar system given in polar coordinates by

$$\dot{r} = r(1-r), \quad \dot{\theta} = 1.$$

Note that $x = \cos t$, $y = \sin t$ is a periodic solution of the system (given in rectangular coordinates), satisfying $x = 1, y = 0$ at time $t = 0$. This solution belongs to the orbit $r = 1$.

1. Find the Poincaré map of the periodic solution $\phi(t) := (\cos t, \sin t)$.
2. Determine the stability of ϕ.

Solution.
1. Let us first integrate the differential equations.

$$\int \frac{dr}{r(1-r)} = t + C, \quad \log \frac{r}{|r-1|} = t + C,$$
$$\frac{r}{r-1} = C_1 e^t, \quad r \neq 1.$$

At $t = 0$, $\frac{r_0}{r_0 - 1} = C_1$. (So, $C_1 > 0$, for $r_0 > 1$, and negative otherwise.) Substituting C_1 as given in terms of r_0 into the solution, we get

$$r(t) = \frac{r_0 e^t}{r_0(e^t - 1) + 1}, \quad t > 0, \quad \theta = t + c.$$

The Poincaré map associated with the given 2π-periodic solution is given as

$$P(r_0) := r(2\pi) = \frac{r_0 e^{2\pi}}{r_0(e^{2\pi} - 1) + 1} = \frac{r_0}{r_0 - (r_0 - 1)e^{-2\pi}}.$$

2. To determine the stability of the solution $x = \cos t$, $y = sint$ associated with the orbit $r = 1$, we calculate the iterations of the Poincaré map.

$$P^2(r_0) = P(P(r_0)) = \frac{P(r_0)}{P(r_0) - (P(r_0) - 1)e^{-2\pi}} = \frac{r_0}{r_0 + e^{-4\pi} - r_0 e^{-4\pi}}.$$

$$P^3(r_0) = P(P^2(r_0)) = \frac{P(r_0)}{P(r_0) + e^{-4\pi} - P(r_0)e^{-4\pi}} = \frac{r_0}{r_0 + e^{-6\pi} - r_0 e^{-6\pi}} = \frac{r_0}{r_0 + e^{-3.(2\pi)} - r_0 e^{-3.(2\pi)}}.$$

So,

$$P^n(r_0) = \frac{r_0}{r_0 + e^{-n.(2\pi)} - r_0 e^{-n.(2\pi)}}.$$

Hence

$$\lim_{n \to \infty} P^n(r_0) = 1.$$

1

That is, the Poincaré map tends to the periodic solution $r = 1$, at the limit of infinitely many iterations.

So, the solution $x = \cos t$, $y = \sin t$, corresponding to the orbit $r = 1$ is asymptotically stable.

Problem 2. The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. We now study a model that is more complicated than the basic SIR model (Problem 2, Assignment 3) where we now allow for the recovered population to become reinfected again (SIRS model).

Again, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population.

We still assume that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0, \tag{1}$$

that is, $S + I + R = \tau > 0$, for a given $\tau > 0$.

The SIRS model is stated as

$$\dot{S} = -\beta SI + \mu R, \tag{2}$$
$$\dot{I} = \beta SI - \nu I, \tag{3}$$
$$\dot{R} = \nu I - \mu R, \tag{4}$$

where $\nu, \mu > 0$ and $\beta > 0$ are parameters.

1. Use the restriction (1) to reduce the system to one with two equations and two unknown fields.

Solution. Integrating equation (1) we get

$$S(t) + I(t) + R(t) = \tau,$$

where $\tau > 0$, constant, is the total population of the community prior to the onset of the virus. Let us solve it for R and substitute it into equation (2), which together with equation (3) give the system with two equations and two unknowns S and I:

$$\dot{S} = -\beta SI - \mu S - \mu I + \mu\tau,$$
$$\dot{I} = \beta SI - \nu I.$$

2. Find the critical points in two different cases: (a) $\tau \geq \frac{\nu}{\beta}$ and (b) $\tau \leq \frac{\nu}{\beta}$.

Solution.

We find the equilibrium solutions

$$\{S = \tau,\, I = 0\}, \quad \text{and} \quad \{S_0 := \frac{\nu}{\mu},\, I_0 := \frac{\mu}{\mu+\nu}(\tau - \frac{\nu}{\beta})\}.$$

Of course, the second solution is only valid in the case $\tau \geq \frac{\nu}{\beta}$, for which $I > 0$.

3. Calculate the eigenvalues associated to the critical points in each case, (a) and (b).

2

Solution.

Let $u := S - \tau$ and linearize the system about the first equilibrium point $\{I = 0,\ S = \tau\}$, giving

$$\begin{bmatrix} \dot{u} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\mu & -(\beta\tau + \mu) \\ 0 & (\beta\tau - \nu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = -\mu < 0, \quad \lambda_2 = \beta\tau - \nu.$$

Note that the solution $\{I = 0,\ S = \tau\}$ is stable if $\tau < \frac{\nu}{\beta}$ (case (b)) and unstable if $\tau > \frac{\nu}{\beta}$ (case (a)).

To examine the stability of the second equilibrium point, we call $u := S - S_0$ and $v = I - I_0$, and linearize the system about $(S_0,\, I_0)$.

The linearized system is given by

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\beta I_0 - \mu & -\beta S_0 - \mu \\ \beta I_0 & \beta S_0 - \nu \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} := A \begin{bmatrix} u \\ v \end{bmatrix}.$$

A calculation of the eigenvalues of the previous matrix shows that both eigenvalues have negative real parts, and so the equilibrium solution is *asymptotically stable.* A simple way to arrive at the latter result comes from the fact that

$$\text{trace} A < 0, \quad \det A > 0.$$

4. In case (a), give an interpretation of the biological significance of the stability properties of the critical points.

Solution. Note that the equilibrium solution with $I = 0$ is stable below the population threshold $\frac{\nu}{\beta}$, in which case the disease may day out. Whereas about the population threshold, the infection will prevail and the disease may get established in the community.

From now on, we restrict ourselves to case (a).

5. Note that the SIRS system is only of interest in the region

$$\Delta := \{(I, S) :\ I, S \geq 0, \text{and}\, S + I \leq \tau\}.$$

Why?

Solution. I and S represent population groups, so they are positive, and also their sum, $I + S$, cannot be greater than the total population τ.

Determine whether the I axis is invariant. What about the S-axis? What is the behavior of the solutions restricted to the S-axis.

Solution. It is easy to see that the line $I = 0$ is invariant, since $\dot{I} = 0$ when $I = 0$, and so, a solution starting with $I(0) = 0$ will satisfy $I(t) = 0$ for all time.

Biologically, it means that, for the disease to get hold in the population, there has to be some initial infection, even if just very small.

However, the line $S = 0$ is not invariant: starting with $S(0) = 0$, it does not follow that $S(t) = 0$ for all time.

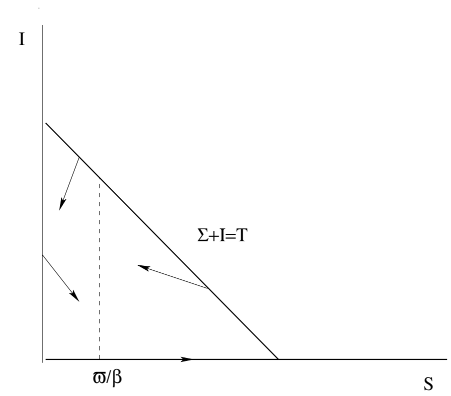


Figure 1: Invariant region Δ. Note that the vector field points inwards along the three sides of the triangle.

6. Show that the region Δ is positively invariant.

Solution. We need to show that, on all three sides of the triangle Δ, the vector field of the system points inwards (or tangent to the side as it is the case along $I = 0$) (figure 1).

1. Along $S = 0$: $\dot{S} > 0$ and $\dot{I} = -\nu I < 0$.
2. Along $I = 0$: $\dot{I} = 0$, $\dot{S} > 0$, with solutions tending to the stable equilibrium $\{S = \tau, I = 0\}$.
3. On the line $S + I = \tau$, we see that $\dot{S} = -\beta S(\tau - S) < 0$.

   We split the remaining part of the proof into two parts. First, let us consider $0 < S < \frac{\nu}{\beta}$. We see that $\dot{I} = I(\beta S - \nu) < 0$. That is, the vector field points to the left and downwards, so it is pointing towards the interior of Δ. Next, let us consider $\tau > S > \frac{\nu}{\beta}$, and calculate

   $$\frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S}.$$

   Hence

   $$-1 \leq \frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S} < 0.$$

   Again, the vector field points inwards.

Problem 3. For each of the following systems, identify all points that lie in either an $\omega-$limit or an α-limit set:

- $\dot{r} = r - r^3$, $\dot{\theta} = 1$.
- $\dot{r} = r^3 - 3r^2 + 2r$, $\dot{\theta} = 1$.

Solution, part I. First, note that the equilibrium solutions of the equation are $r = 0$ and $r = 1$, the latter is the circle of radius 1 around the origin.

The linearized equation about $r = 0$ is

$$\dot{r} = r, \quad r(t) = Ce^t,$$

so $r = 0$ is an unstable equilibrium point. $r = 0$ *is an α-limit set.*

The linearized equation about $r = 1$ is

$$\dot{r} = 2(1 - r), \quad r(t) = 1 - Ce^{-2t}, \quad |C| \leq 1.$$

Hence, $r = 1$ is a limit cycle that positively attracts orbits. That is, $r = 1$ *is an ω-limit set.*

Solution, part II. The equilibrium solutions of the second equation are

$$r = 0,\ r = 1 \text{ and } r = 2.$$

The linearized equation about $r = 0$ is $\dot{r} = 2r$ showing that $r = 0$ is an unstable equilibrium point, and so $r = 0$ *is an α-limit set of the solutions.*

The linearized equation about $r = 1$ is $\dot{r} = -(r - 1)$, and so *the limit cycle* $r = 1$ *an ω-limit set.*

The linearized equation about $r = 2$ is $\dot{r} = 2(r - 2)$, and so *the limit cycle* $r = 2$ *an α-limit set.*

2.5 HW 5

Local contents

2.5.1 Problem 6.6

Consider the equation $\dot{x} = A(t)x$ with $x \in \mathbb{R}^2$ and

$$A(t) = \begin{pmatrix} \frac{1}{2} - \cos t & b \\ a & \frac{3}{2} + \sin t \end{pmatrix}$$

And a, b constants. Show that that there exists at least a one-parameter family of solutions which becomes unbounded as $t \to \infty$

<u>solution</u>

$\dot{x} = A(t)x$ has characteristic multiplies ρ_i and exponents λ_i. Where $\rho_i = e^{\lambda_i T}$ and T is the period of the coefficients of $A(t)$ which is

$$T = 2\pi$$

To answer this question we need to show that there is at least one characteristic exponent λ_i with real part strictly positive.

Using theorem 6.6, which applies here because $A(t)$ is periodic, it says that

$$\rho_1 \rho_2 = e^{\int_0^T \text{trace}(A(\tau))d\tau} \tag{1}$$

$$\lambda_1 + \lambda_2 = \frac{1}{T}\left(\int_0^T \text{trace}(A(\tau))\,d\tau\right) \bmod \frac{2\pi i}{T} \tag{2}$$

We only need to use (2) in the above to answer this question. Trace of $A(t)$ is sum of diagonal elements of $A(t)$ which is

$$\begin{aligned} \text{trace}(A(\tau)) &= \frac{1}{2} - \cos t + \frac{3}{2} + \sin t \\ &= 2 - \cos t + \sin t \end{aligned}$$

Then

$$\begin{aligned} \int_0^T \text{trace}(A(\tau))\,d\tau &= \int_0^{2\pi} (2 - \cos\tau + \sin\tau)\,d\tau \\ &= [2\tau - \sin\tau - \cos\tau]_0^{2\pi} \\ &= (2(2\pi) - \sin 2\pi - \cos 2\pi) - (-\cos 0) \\ &= (4\pi - 1) - (-1) \\ &= 4\pi \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \lambda_1 + \lambda_2 &= \left(\frac{1}{2\pi} 4\pi\right) \bmod \frac{2\pi i}{2\pi} \\ &= 2 \end{aligned}$$

Since $\Phi(t) = P(t)e^{Bt}$ where $\Phi(t)$ is the fundamental matrix and since λ_1, λ_2 are the eigenvalues of the B matrix, then we see that one solution exist which blows up. This shows there exists at least a one-parameter family of solutions which becomes unbounded as $t \to \infty$

2.5.2 Problem 8.4

Consider the system

$$\dot{x} = 2y(z-1) \tag{1}$$

$$\dot{y} = -x(z-1) \tag{2}$$

$$\dot{z} = xy \tag{3}$$

a Show that the solution $(0,0,0)$ is stable

b Is this solution asymptotically stable?

solution

2.5.2.1 Part a

Setting $x=0, y=0, z=0$ gives

$$\begin{aligned}\dot{x}&=0\\ \dot{y}&=0\\ \dot{z}&=0\end{aligned}$$

Therefore $(0,0,0)$ is critical point. Eq(1)/Eq(2) gives

$$\begin{aligned}\frac{\frac{dx}{dt}}{\frac{dy}{dt}}&=\frac{2y(z-1)}{-x(z-1)}\\ \frac{dx}{dy}&=\frac{-2y}{x}\end{aligned}$$

Hence

$$-2ydy=xdx$$

Integrating gives

$$-y^2=\frac{x^2}{2}+V_1$$

Where V_1 is integration constant. Therefore

$$\begin{aligned}V_1&=-y^2-\frac{x^2}{2}\\ V_1&=y^2+\frac{x^2}{2}\end{aligned}$$

Where the sign was absorbed in the constant. The above can be written as

$$V_1=2y^2+x^2 \tag{4}$$

Where the 2 factor was absorbed in the constant.

Now solving Eq (2) for x gives, $x=\frac{\dot{y}}{-(z-1)}$ and substituting this into Eq (3) gives

$$\begin{aligned}\dot{z}&=\frac{\dot{y}}{1-z}y\\ \frac{\dot{z}}{\dot{y}}&=\frac{y}{1-z}\\ \frac{\frac{dz}{dt}}{\frac{dy}{dt}}&=\frac{y}{1-z}\\ \frac{dz}{dy}&=\frac{y}{1-z}\end{aligned}$$

Hence

$$ydy=(1-z)\,dz$$

Integrating gives

$$\frac{y^2}{2}=\left(z-\frac{z^2}{2}\right)+V_2$$

Where V_2 is the constant of integration. Therefore

$$\begin{aligned}V_2&=\frac{y^2}{2}-z+\frac{z^2}{2}\\ V_2&=z^2-2z+y^2 \end{aligned}\tag{5}$$

Let the candidate Lyapunov function (we still have to check it is indeed a Lyapunov function) be the following (per the hint given)

$$
\begin{aligned}
V\left(x,y,z\right) &= V_1 + (V_2-1)^2 \qquad (6)\\
&= 2y^2 + x^2 + \left(z^2 - 2z + y^2 - 1\right)^2\\
&= x^2 + y^4 + 2y^2z^2 - 4y^2z + z^4 - 4z^3 + 2z^2 + 4z + 1
\end{aligned}
$$

We will now verify it is a Lyapunov function. The function $V\left(x,y,z\right)$ is Lyapunov function for the system if the following conditions are all met

1. $V\left(x.y,z\right)$ is continuously differentiable function in $\mathbb{R}^3$ and $V\left(x.y,z\right) \geq 0$ (positive definite or positive semidefinite) for all x,y,z away from the origin, or everywhere inside some fixed region around the origin. This function represents the total energy of the system (For Hamiltonian systems).
2. $V(0,0,0) = 0$. This says the system has no energy when it is at the equilibrium point. (rest state).
3. The orbital derivative $\frac{dV}{dt} \leq 0$ (i.e. negative definite or negative semi-definite) for all x,y,z, or inside some fixed region around the origin. The orbital derivative is same as $\frac{dV}{dt}$ along any solution trajectory. This condition says that the total energy is either constant in time (the zero case) or the total energy is decreasing in time (the negative definite case). Both of which indicate that the origin is a stable equilibrium point.

If $\frac{dV}{dt}$ is negative semi-definite then the origin is stable in Lyapunov sense. If $\frac{dV}{dt}$ is negative definite then the origin is asymptotically stable equilibrium. Negative semi-definite means the system, when perturbed away from the origin, a trajectory will remain around the origin since its energy do not increase nor decrease. So it is stable. But asymptotically stable equilibrium is a stronger stability. It means when perturbed from the origin the solution will eventually return back to the origin since the energy is decreasing. Global stability means $\frac{dV}{dt} \leq 0$ everywhere, and not just in some closed region around the origin. Local stability means $\frac{dV}{dt} \leq 0$ in some closed region around the origin. Global stability is stronger stability than local stability.

Condition (1) is satisfied $V\left(x.y,z\right) \geq 0$ (since of squares) and $V\left(0,0,\pm\left(1+\sqrt{2}\right)\right) = 0$. Hence $V\left(x,y,z\right)$ is positive semidefinite (not positive definite).

Condition (2) is easily checked is valid. Since $V = V_1 + (V_2-1)^2 = 0$ at $(0,0,0)$.

To check for condition (1), we see from looking at (6) that V can not be negative since $V_1 = 2y^2 + x^2$ is square quantity and $(V_2-1)^2$ is also square. So we need to check if $V\left(x,y,z\right)$ is always positive away from the origin. One way to do this is to find its Hessian and check if its eigenvalues. If the eigenvalues of the Hessian are all positive everywhere, then this implies V is positive definite. But we can do a short cut here. Since V is the sum of 2 square quantities, we just need to check if one of these two quantities is always positive. We do not have to check the whole V.. Let us check if V_1 is positive definite or not first. Since V_1 depends on x,y only, then

$$
\nabla V_1 = \begin{pmatrix} \frac{\partial V_1}{\partial x} \\ \frac{\partial V_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \end{pmatrix}
$$

$$
\nabla^2 V_1 = \begin{pmatrix} \frac{\partial^2 V_1}{\partial x \partial x} & \frac{\partial^2 V_1}{\partial x \partial y} \\ \frac{\partial^2 V_1}{\partial y \partial x} & \frac{\partial^2 V_1}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}
$$

Hence the eigenvalues are $2,4$. Since these are positive everywhere, then we conclude that $V_1\left(x,y\right)$ is concave up. This means the minimum is at zero and it is positive everywhere else away from the origin. This implies that $V\left(x,y,z\right)$ is positive definite everywhere away from zero, which is what we wanted to show. Now we check the third condition $\frac{dV}{dt} \leq 0$.

The orbital derivative $\frac{dV}{dt}$ is

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} \\ &= 2x\dot{x} + 6y\dot{y} + (2z-2)\,\dot{z}\end{aligned}$$

But $\frac{\partial V}{\partial x} = 2x$ and $\frac{\partial V}{\partial y} = 4y\left(y^2 - 2z + z^2\right)$ and $\frac{\partial V}{\partial z} = 4(z-1)\left(z^2 + y^2 - 1 - 2z\right)$. Therefore using (1,2,3) the above becomes

$$\begin{aligned}L_t V &= 2x\left(2y(z-1)\right) + 4y\left(y^2 - 2z + z^2\right)(-x(z-1)) + \left(4(z-1)\left(z^2 + y^2 - 1 - 2z\right)\right)xy \\ &= 0\end{aligned}$$

Therefore conditions 3 is also satisfied. Hence $V(x,y,z)$ is a Lyapunov function for the system and $(0,0,0)$ is stable equilibrium point since $\frac{dV}{dt}$ is zero. (by theorem 8.1)

2.5.2.2 Part b

By theorem 8.2, since we found from part a that $\frac{dV}{dt}$ is zero, therefore it is not negative definite but negative semi-definite, hence $(0,0,0)$ is not asymptotically stable (for this specific $V(x,y,z)$).

2.5.3 Problem 8.9

Determine the stability of the trivial solution of

$$\begin{aligned}\dot{x} &= xy^2 - \frac{1}{2}x^3 \\ \dot{y} &= -\frac{1}{2}y^3 + \frac{1}{5}x^2 y\end{aligned}$$

solution

Setting $x = 0, y = 0$ gives

$$\begin{aligned}\dot{x} &= 0 \\ \dot{y} &= 0\end{aligned}$$

Therefore $(0,0)$ is critical point. We need to find Lyapunov function. Let $V(x,y) = ax^2 + by^2$. A quadratic function. The function $V(x,y,z)$ is Lyapunov function for the system if the three conditions given in the above problem are met.

Condition (2) is clearly satisfied. Condition (1) is also satisfied since both terms are squared if we choose $a, b > 0$. Hence $V(x,y) > 0$ for non zero x, y. We now need to check the third condition. The orbital derivative $\frac{dV}{dt}$ is

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= 2ax\dot{x} + 2by\dot{y} \\ &= 2ax\left(xy^2 - \frac{1}{2}x^3\right) + 2by\left(-\frac{1}{2}y^3 + \frac{1}{5}x^2y\right) \\ &= 2ax^2y^2 - by^4 - ax^4 + \frac{2}{5}bx^2y^2 \\ &= \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right) - \left(by^4 + ax^4\right) \\ &= -\left[\left(by^4 + ax^4\right) - \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right)\right] \qquad (1)\end{aligned}$$

Completing the squares

$$\begin{aligned}\frac{dV}{dt} &= -\left[\left(\sqrt{b}y^2 - \sqrt{a}x^2\right)^2 + 2\sqrt{a}\sqrt{b}x^2y^2 - \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right)\right] \\ &= -\left[\left(\sqrt{b}y^2 - \sqrt{a}x^2\right)^2 + \left(2\sqrt{a}\sqrt{b} - 2a - \frac{2}{5}b\right)\left(x^2y^2\right)\right]\end{aligned}$$

The above is negative definite if we can find $a, b > 0$ such that

$$2\sqrt{a}\sqrt{b} - 2a - \frac{2}{5}b > 0$$

Picking $a = 1, b = 2$ then left side above is

$$2\sqrt{2} - 2 - \frac{2}{5}(2) = 0.028$$

Hence $a = 1, b = 2$ is one choice that makes $V\left(x, y\right) = ax^2 + by^2$ a Lyapunov function. This shows that $(0, 0)$ is asymptotically stable. The following is a plot of $\frac{dV}{dt}$ given in (1) to confirm it is negative definite (it is zero only at the origin, but negative everywhere else).

```
In[]:= a = 1; b = 2;
       V = a x^2 + b y^2;
       orbital = D[V, x] * (x y^2 - 1 / 2 x^3) + D[V, y] (-1 / 2 y^3 + 1 / 5 x^2 y)
```

Out[]= $2\,x\left(-\frac{x^3}{2} + x\,y^2\right) + 4\,y\left(\frac{x^2\,y}{5} - \frac{y^3}{2}\right)$

```
In[]:= Plot3D[orbital, {x, -4, 4}, {y, -4, 4}]
```

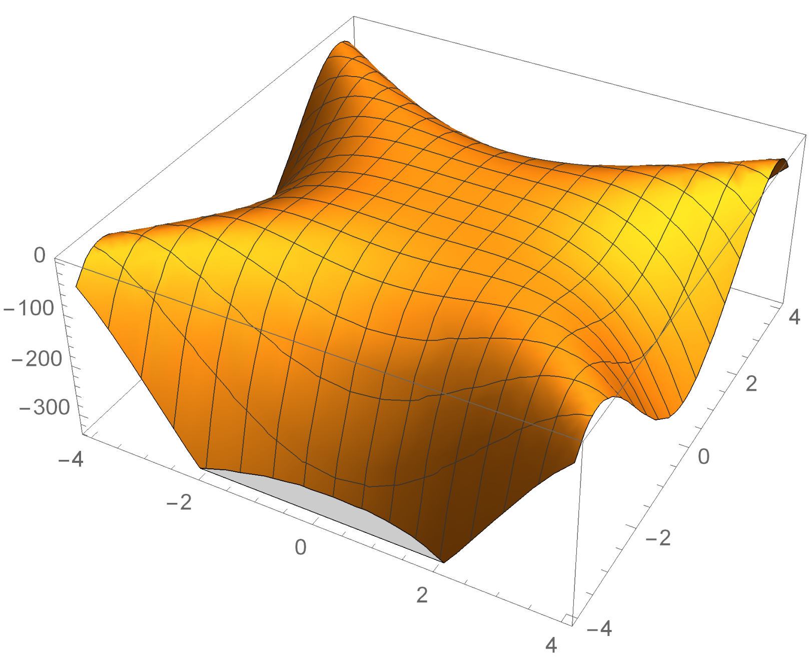
Out[]=


Figure 2.49: Showing the Orbtial derivative negative everywhere around the origin

2.5.4 key solution for HW 5

MATH 5525: HOMEWORK ASSIGNMENT 4

Date (May 8, 2020)

Problem 1. Exercise 6.6 of the textbook (page 81, second edition).

Hint: Apply theorems 6.5 and 6.6 of the textbook.

Solution. According to Theorem 6.6, we have that the exponents λ_1 and λ_2 satisfy

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi}\int_0^{2\pi} \text{trace}A(t)\,dt\,(\text{mod}2\pi i).$$

(Note that we take $T = 2\pi$ since the period of the matrix coefficients is 2π.) Here

$$\text{trace}A(t) = \frac{1}{2} - \cos t + \frac{3}{2} + \sin t = 2 + \sin t - \cos t.$$

So,

$$\lambda_1 + \lambda_2 = 2.$$

Consequently at least one of the λ_1 or λ_2 is positive.

By Theorem 2.5, the fundamental matrix solution is

$$\Phi(t) = P(t)e^{Bt},$$

where λ_1, λ_2 are the eigenvalues of B. Therefore, at least one of the linearly independent solutions of the system (that is, a column of Φ) involves a positive exponential function of t, and therefore solutions of the given system become unbounded at $t \to \infty$.

Problem 2. Exercise 8.4 of the textbook (page 109, second edition).

Hint: Eliminating $(z-1)$ from equation (1) and equation (2) gives a system of two equations for x and y. Find a Lyapunov function for the latter system; call it $V_1(x,y)$. Likewise, eliminating x from equations (2) and (3) you get another system for the variables y and z. Find a Lyapunov function, $V_2(y,z)$. Then, show that $V(x,y,z) := V_1(x,y) + (V_2(y,z) - 1)^2$ is a Lyapunov function for the original system. Subsequent application of the Lyapunov theorem on stability (and/or asymptotic stability) gives the result.

Solution. An easy calculation shows that

$$V_1(x,y) = x^2 + 2y^2, \quad V_2(y,z) = y^2 + (z-1)^2.$$

Let us start showing that

$$V(x,y,z) = V_1 + (V_2 - 1)^2 = x^2 + 2y^2 + (y^2 + z^2 - 2z)^2$$

is a Lyapunov function of the system. For this, we need to investigate two properties,

1

- $V(0,0,0) = 0$, which is satisfied.
- $V(x,y,z)$ is positive semidefinite. Indeed $V(x,y,z) \geq 0$. Note that $V(0,0,2) = 0$, so V is not positive definite.
- $\dot{V}$, the orbital derivative also denoted as $L_t V$, is negative semidefinite. Indeed, $\dot{V} = V_x\dot{x} + V_y\dot{y} + V_z\dot{z} = 0$. So, $\dot{V}$ is positive semidefinite.

By Theorem 8.1, the equilibruim solution $(0,0,0)$ is stable. However, we cannot conclude asymptotic stability since $\dot{V}$ is not negative definite.

Problem 3. Exercise 8.9 of the textbook (page 109, second edition).

Hint: Consider a quadratic function of the form $V(x,y) = ax^2 + by^2$. Choose particular values of a and b so that V is a Lyapunov function.

Solution. Let us determine a and b so that $V(xy) = ax^2 + by^2$ is a Lyapunov function.

1. First of all, note that $a > 0$ and $b > 0$ imply that $V(x,y) > 0$, if $(x,y) \neq (0,0)$ and $V(0,0) = 0$. So V is positive definite.
2. Calculate $\dot{V} = V_x\dot{x} + V_y\dot{y} = 2(a - \frac{b}{5})x^2y^2 - ax^4 - by^4$. Choosing $a = \frac{b}{5}$ we see that $\dot{V} = -a(x^4 + 5y^4)$. So, it is negative definite.

Therefore, the equilibrium solution $(0,0)$ of the system is asymptotically stable. Indeed, we could find a Lyapunov function of the system with $\dot{V}$ negative definite.

2

Chapter 3

study notes, cheat sheet

3.1 cheat sheet

Notes by Nasser M Abbasi

■ Linear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = [A]_{n\times n} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For nonlinear,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

To find First integral $F(x_1, x_2)$, (also the orbit equation in phase plane) solve $\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$. This gives an ODE to solve. For example, if

$$x'' + x - \frac{1}{2}x^2 = 0$$

Then $x_1 = x, x_2 = x'$ and $\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + \frac{1}{2}x_1^2$. Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + \frac{1}{2}x_1^2 \end{pmatrix}$$

And $\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$ gives $\frac{dx_2}{dx_1} = \frac{-x_1 + \frac{1}{2}x_1^2}{x_2}$ or $x_2 dx_2 = \left(-x_1 + \frac{1}{2}x_1^2\right) dx_1$. Integrating $\frac{1}{2}x_2 = -\frac{1}{2}x_1^2 + \frac{1}{6}x_1^3 + E$ or $\frac{1}{2}x_2 + \frac{1}{2}x_1^2 - \frac{1}{6}x_1^3 = E$. Hence

$$F(x_1, x_2) = \frac{1}{2}x_2 + \frac{1}{2}x_1^2 - \frac{1}{6}x_1^3$$

Is the first integral.

■ Hamiltonian $H(x_1, x_2)$ is first integral. It is the energy of the system. For second order ODE, the equation of motion using H becomes

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

But $p = \dot{x}$ or in state space notation

$$p = \dot{x}_1 = x_2$$

And $q = x$ or in state space notation

$$q = x_1$$

Hence the above becomes, in state space as

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}$$

Therefore, state space can be written as (for second order ODE)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}\frac{\partial H}{\partial x_2}\\ -\frac{\partial H}{\partial x_1}\end{pmatrix}$$

For example, in the last example, we had $F(x_1,x_2)=\frac{1}{2}x_2+\frac{1}{2}x_1^2-\frac{1}{6}x_1^3$ which is also H. Hence applying the above gives

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}x_2\\ -\left(x_1-\frac{1}{2}x_1^2\right)\end{pmatrix}=\begin{pmatrix}x_2\\ -x_1+\frac{1}{2}x_1^2\end{pmatrix}$$

Which is the original system equation of motion. Hence, if we know H or F, we can find the equation of motion. (For constant H) which is the normal case.

■ Hessian. Given $F(x_1,x_2)$, as first integral, then ∇F is the gradient, written as $\begin{pmatrix}\frac{\partial F}{\partial x_1}\\ \frac{\partial F}{\partial x_2}\end{pmatrix}$. ∇ is called **del** operator. and the Hessian is $\nabla^2 F=\begin{pmatrix}\frac{\partial^2 F}{\partial x_1\partial x_1} & \frac{\partial^2 F}{\partial x_1\partial x_2}\\ \frac{\partial^2 F}{\partial x_2\partial x_1} & \frac{\partial^2 F}{\partial x_2\partial x_2}\end{pmatrix}=\begin{pmatrix}\frac{\partial^2 F}{\partial^2 x_1} & \frac{\partial^2 F}{\partial x_1\partial x_2}\\ \frac{\partial^2 F}{\partial x_2\partial x_1} & \frac{\partial^2 F}{\partial^2 x_2}\end{pmatrix}$. A critical point is called non-degnerate if $\det\nabla^2 F$ evaluates at the critical point is non-zero. This means the linearization is non-degenerate around that critical point.

■ For scalar function, say $f(x,y)$ its gradient is $\nabla F=\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{pmatrix}$ which represent the tangent vector when evaluated at some point $p=(x_0,y_0)$. This can also be written as $\nabla F=\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}$ in vector notation.

For a vector of functions, say $\vec{F}=\begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix}$ then its gradient is matrix given by $\nabla\vec{F}=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}$. This is called the Jacobian also. Normally this is evaluated at point (equilibrium point) and its eigenvalues indicate if the linearized system is stable or not.

■ Directional derivative at a point is given by $\hat{n}\cdot\nabla f(x,y)$ where $\nabla f(x,y)$ is the gradient (a vector field), and the result is evalulated at some specific point. For example, if $f=\sqrt{x^2+y^2}$ and we want the directional derivative in direction defined by vector $2\hat{\imath}+2\hat{\jmath}+\hat{k}$, then $\hat{n}=\frac{1}{3}\left(2\hat{\imath}+2\hat{\jmath}+\hat{k}\right)$ and $\nabla f(x,y)=\frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\imath}+y\hat{\jmath}\right)$. Hence $\frac{1}{3}\left(2\hat{\imath}+2\hat{\jmath}+\hat{k}\right)\cdot\frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\imath}+y\hat{\jmath}\right)=\frac{2}{3}\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)$. At the point $(0,-2,1)$ this gives $-\frac{2}{3}$.

This all can also be written using vector notation. if $f=\sqrt{x^2+y^2}$ and we want the directional derivative in direction defined by vector $2\hat{\imath}+2\hat{\jmath}+\hat{k}$, then $\hat{n}=\frac{1}{3}\left(2\hat{\imath}+2\hat{\jmath}+\hat{k}\right)=\begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$ and

$\nabla f\left(x,y\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\imath}+y\hat{\jmath}\right) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$. Hence

$$\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix} = \frac{2}{3}\frac{x}{\sqrt{x^2+y^2}} + \frac{2}{3}\frac{y}{\sqrt{x^2+y^2}}$$
$$= \frac{2}{3}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

At the point $(0,-2,1)$ this gives $-\frac{2}{3}$.

■ For vector function, say $F\left(x,y\right) = \begin{pmatrix} f\left(x,y\right) \\ g\left(x,y\right) \end{pmatrix}$ its divergence is scalar, given by

$$\begin{aligned} \nabla \cdot \vec{F} &= \operatorname{div}(F) \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \end{aligned}$$

■ If in the above, $F(x_1,x_2)$ was the state space vector in $\dot{x} = F(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$, then there is a theory which says if $\nabla \cdot \vec{F}$ do not change sign over the whole domain D, then the system can only have periodic solutions. This assumes D is simply connected (i.e. no holes in it) and that $f_1(x_1,x_2), f_2(x_1,x_2)$ are smooth functions.

■ Morse function. If $F(x_1,x_2)$ is non-degenerate around critical point $x = a$, then $F(x_1,x_2)$ is called Morse function around $x = a$. To find Morse function, expand $F(x)$ in Taylor series around the critical point.

$$F(x_1,x_2) = \bar{F}(a) + (x-a)\nabla F(a) + \frac{1}{2}(x-a)\nabla^2 F(a)(x-a)^T$$

But $\nabla F(a) = 0$ since critical point, hence

$$F(x_1,x_2) = \bar{F}(a) + \frac{1}{2}(x-a)\nabla^2 F(a)(x-a)^T$$

The above should come out as quadratic in x_1,x_2 if F is non-degenerate.

■ Jordan form

Given linearized system $\dot{x} = Ax$, find T such that $z = T^{-1}x$ which makes system $\dot{z} = Bz$ where it is now decoupled. B is the Jordan form of A. For case of non-zero eigenvalue of A at each critical point, $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $B = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ depending if eigenvalues of A are distinct or repeated. Now solve $\dot{z} = Bz$ since decoupled and then convert back to x space when done using $x = Tz$. The matrix T is the matrix of the eigenvectors of A. Each eigenvector is column of T. Note that A is constant, since it is evaluated at critical point.

■ Eigenvalues of A can also be found using $\lambda = \frac{1}{2}\left(\operatorname{trace}(A)^2 - 4\det(A)\right)$

■ From the book "a critical point which, after linearisation, corresponds with a positive attractor, turns out to be asymptotically stable". This means if all eigenvalues (of the Jacobian at that point) are negative, then asymptotically stable. But if one eigenvalue is zero, we can not say that. Normally we just say unable to decide (if the system is non-linear).

Chapter 4

Exams

Local contents

4.1 First exam, Friday Feb 21 2020

Local contents

4.1.1 Questions

MATH 5525- Test 1

February 19, 2020

You may use class notes at your convenience. Please, show all your work.

Problem 1. (50 points). Consider the system of differential equations

$$\dot{y} = v, \quad \dot{v} = f(y),$$

where f is a continuous function $f : \mathbf{R} \to \mathbf{R}$.

- Find a first integral of the system.
- Find the equilibrium points of the system in the case that $f(y) = \sin y$. From now on, consider $f(y) = \sin(y)$.
- Find the Jacobian matrix of the system at the equilibrium points. (That is, write the linearized system about the equilibrium points).
- Determine the nature of the equilibrium points.
- Sketch the phase plane of the system in the interval $-\pi \leq y \leq \pi$.

Problem 2. (40 points). Consider the predator-pray system governing the number of individuals x y of the two species at time $t > 0$:

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(-2 + x).$$

- Find the equilibrium points of the system;
- Find two invariant sets.
- Sketch the phase plane.

4.1.2 key solution exam 1

MATH 5525- Test 1-Solutions

March 2, 2020

Problem 1. (50 points). Consider the system of differential equations

$$\dot{y} = v, \quad \dot{v} = f(y),$$

where f is a continuous function $f : \mathbf{R} \to \mathbf{R}$.

1. *Find a first integral of the system.* Let us rewrite the system as a single, second order ordinary differential equation:

$$\frac{d^2y}{dt^2} = f(y).$$

Now, multiply both sides by $\dot{y}$ to get:

$$\dot{y}\frac{d^2y}{dt^2} = \dot{y}f(y), \quad \text{or, equivalently} \quad \frac{1}{2}\frac{d\dot{y}^2}{dt} - \frac{dF(y)}{dt} = 0,$$

where $F(y)$ is the antiderivative of $f(y)$, that is, it satisfies the relation

$$F'(y) = f(y).$$

Hence, the first integral of the system is

$$\frac{1}{2}(\dot{y})^2 - F(y) = E,$$

where E is an arbitrary constant.

2. *Find the equilibrium points of the system in the case that $f(y) = \sin y$.* From now on, consider $f(y) = \sin(y)$. The equilibrium points satisfy the equations $\sin y = 0$ and $\dot{y} = 0$. That is,

$$y = 0, \pm\pi, \pm 2\pi, \ldots \quad \text{and} \quad \dot{y} = 0.$$

1

3. *Find the Jacobian matrix of the system at the equilibrium points.* (That is, write the linearized system about the equilibrium points).

We first express the function $f(y)$ in a Taylor series about $y = y_0$, an equilibrium value, taking into account that $f(y_0) = 0$:

$$f(y) = f'(y_0)(y - y_0) + o(|y - y_0|).$$

In particular, for $f(y) = \sin y$, the matrix of the linear system about $(y_0, 0)$ becomes:

$$A = \begin{bmatrix} 0 & 1 \\ \cos y_0 & 0 \end{bmatrix}.$$

A. For $y_0 = 0, \pm 2\pi, \pm 4\pi, \ldots,\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

B. For $y_0 = 0, \pm \pi, \pm 3\pi, \ldots,\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

4. *Determine the nature of the equilibrium points.*

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda = \pm 1$. Hence the equilibrium points $(0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots$ are saddle points.

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\lambda = \pm i$. Hence the equilibrium points $(0,0), (\pm \pi, 0), (\pm 3\pi, 0), \ldots$ are centers.

5. *Sketch the phase plane of the system in the interval $-\pi \le y \le \pi$.*

Problem 2. (40 points). Consider the predator-pray system governing the number of individuals x y of the two species at time $t > 0$:

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(-2 + x).$$

1. *Find the equilibrium points of the system.* For this, we need to solve the equations

$$x(1 - x - y) = 0 \quad \text{and} \quad y(-2 + x) = 0.$$

The solutions are given by

$$x = 0 \text{ and } y = 0; \quad x + y = 1 \text{ and } y = 0; \quad x + y = 1 \text{ and } \; -2 + x = 0.$$

Consequently, the equilibrium points (all of them) are

$$(0,0),\ (1,0),\ (2,-1).$$

2. *Find two invariant sets.*

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- $x = 0$ is an invariant set. Note that a solution such that $x(0) = 0$ will satisfy $x(t) = 0$, for all $t \geq 0$. The variable y is obtained by solving the second equation, now given by $\dot{y} = -2y$.
- $y = 0$ is another invariant set. A solution such that $y(0) = 0$ will satisfy $y(t) = 0$, for all $t \geq 0$. The variable x is obtained by solving the equation $\dot{x} = x(1 - x)$.

3. *Sketch the phase plane.* Let us classify the equilibrium points. For this, denote

$$f(x, y) = x - x^2 - xy, \quad g(x, y) = -2y + xy.$$

Calculate

$$\frac{\partial f}{\partial x} = 1 - 2x - y; \quad \frac{\partial f}{\partial y} = -x; \quad \frac{\partial g}{\partial x} = y; \quad \frac{\partial g}{\partial y} = -2 + x.$$

• The matrix of the linearized system about $(0, 0)$ is $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. Its eigenvalues are 1 and -2; the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. $(0, 0)$ is a *saddle point.*

• The matrix of the linearized system about $(2, -1)$ is $A = \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}$. Its eigenvalues are $\lambda = -1 \pm \sqrt{3}$; the corresponding eigenvectors are $\begin{bmatrix} 1 \pm \sqrt{3} \\ 1 \end{bmatrix}$. $(2, -1)$ is a *saddle point.*

• The matrix of the linearized system about $(1, 0)$ is $A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$. It has a (double) eigenvalue $\lambda = -1$ with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $(1, 0)$ is a degenerate stable node.

Phase plane sketches will be given in separate handout.

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4.2 Second exam, Monday March 30 2020

Local contents

4.2.1 Questions

MATH 5525- MIDTERM EXAMINATION II

March 30, 2020

Problem 1.

1. State the Bendixon criterion of non-existence of periodic orbits of a differential equation.
2. Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + x = 0,$$

where $f(x) = x^2 + x + a,\ a \in \mathbb{R}$. Determine the range of values of a for which the equation does not have any periodic orbits.

Problem 2.
Consider the system of differential equations that models the growth of two competing species with populations $x \geq 0$ and $y \geq 0$:

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(3 - 2x - y).$$

1. Find all equilibruium points and determine their stability type.
2. Determine the nullclines of the system.
3. Find the invariant regions of the xy-plane.
4. Draw the phase-plane using your favorite sofware (Matlab, Mathematica, ...).
5. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

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Guidelines:

1. You may use books, notes and internet resources as you wish.
2. The work has to be personal, that is, you may not consult with anyone or receive any help. (You may always email me, if you have questions or difficulties.)
3. The exam should be back tonight, by midnight.
4. Upload the complete work on canvas. If you experience difficulties, please email it directly to me.

Please, sign the following statement:
I hereby certify that I have not received help from anyone in the completion of this test.

Signature:

____

Minneaplis, March 30, 2020.

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4.2.2 Key solution

MATH 5525- MIDTERM EXAMINATION II

April 4, 2020

Problem 1.

1. State the Bendixon criterion of non-existence of periodic orbits of a differential equation.
2. Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + x = 0,$$

where $f(x) = x^2 + x + a,\ a \in \mathbb{R}$. Determine the range of values of a for which the equation does not have any periodic orbits.

Solution 1. Let

$$\dot{x} = f(x,y),\, \dot{y} = g(x,y), \quad (x,y) \in D \subset \mathbb{R}^2.$$

1. *Criterion of Bendixon.* Suppose that D is simply connected and (f,g) continuously differentiable in D. The equation can only have periodic solutions if $\nabla \cdot (f,g)$ changes sign in D or if $\nabla \cdot (f,g) = 0$ in D.
2. For the given second order equation, which written as a system takes the form $\dot{x} = y, \quad \dot{y} = -f(x)y - x,$

the divergence of the vector field is

$$\nabla \cdot (y,\, -f(x)y - x) = -f(x) = -(x^2 + x + a).$$

Note that the zeros of the quadratic polynomial $f(x)$ are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$. Therefore, for $f(x)$ not to change sign, we must require its zeros to be complex, that is,

$$1 - 4a < 0, \quad a > \frac{1}{4}.$$

1

Problem 2. Consider the system of differential equations that models the growth of two competing species with populations $x \geq 0$ and $y \geq 0$:

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(3 - 2x - y).$$

1. Find all equilibrium points and determine their stability type.
2. Determine the nullclines of the system.
3. Find the invariant regions of the xy-plane.
4. Draw the phase-plane using your favorite software (Matlab, Mathematica, ...).
5. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

Solution 2.

1. $(0,0)$, $(0,3)$, $(2,0)$, $(1,1)$.
   - $(0,0)$: associated matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$; eigenvalues: $(2,3)$. *Unstable node.*
   - $(0,3)$: associated matrix $A = \begin{bmatrix} -1 & 0 \\ -6 & -3 \end{bmatrix}$; eigenvalues: $(-3,-1)$. *Stable node.*
   - $(2,0)$: associated matrix $A = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$; eigenvalues: $(-2,-1)$. *Stable node.*
   - $(1,1)$: associated matrix $A = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$; eigenvalues: $(0.4142, -2.4142)$. *Saddle point.*
2. The nullclines of the system are the lines

$$x = 0; \quad \text{on this line} \quad \dot{y} = y(3 - y). \tag{1}$$
$$x + y = 2; \quad \text{on this line} \quad \dot{y} = (-1 + y). \tag{2}$$
$$y = 0; \quad \text{on this line} \quad \dot{x} = x(2 - x). \tag{3}$$
$$2x + y = 3; \quad \text{on this line} \quad \dot{x} = (-1 + x). \tag{4}$$

   The vector field of the system is $\mathbf{f} := (x(2 - x - y), y(3 - 2x - y))$.

2

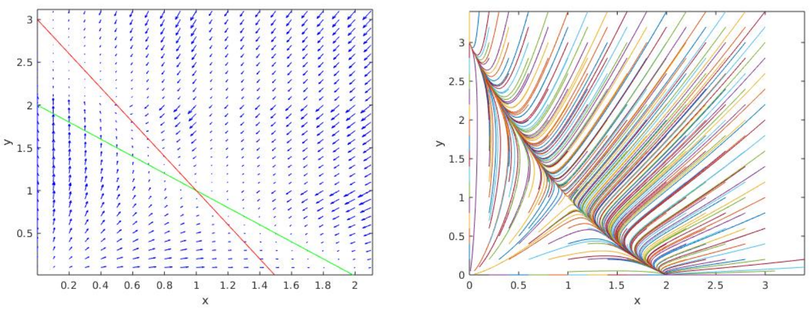

Figure 1: Part 4. The left figure shows vector field and the nullclines (2 green) and (4 red). The equilibrium points are $(0,0), (3,0), (1,1), (2,0)$. The right figure shows some orbit plots.

- On the points of the nullcline (2), we have $\mathbf{f} = (0, y(1-y))$. Therefore, the vector field is vertical and points *up* for $0 < y < 1$, and *down* for $y > 1$.
- On the points of the nullcline (4), we have $\mathbf{f} = (-1+3x, 0)$. Therefore, the vector field is *horizontal* and points to the *left* for $0 < x < 1$, and to the *right* for $x > 1$.

3. Note that the lines $x = 0$ and $y = 0$, that is the axes, are *invariant.*

   The first quadrant is also *invariant*, since, $\dot{x} < 0$ for $x+y > 2$ and $\dot{y} < 0$ for $2x + y > 3$. That is, the vector field points down and towards the left above the nullcline (4), for $0 < x < 1$; the vector field also points down and towards the left above the nullcline (2), for $1 < x$.

   Within the first quadrant there are two *invariant* regions, the triangle with vertices $(1,1), (0,2)$ and $(0,3)$, and the triangle $(2,0), (\frac{3}{2}, 0)$ and $(1,1)$. (You only need to verify that the vector field on the sides of the triangles always points towards the interior.)

   The consequence of the previous statements is that, for sufficiently large times, the solutions will either enter the *top* triangle or the *lower* one. Moreover, at $t \to \infty$, the former of will tend to the stable node $(0,3)$ and the latter tend to the other stable node $(2,0)$.

4. Phase-plane, nullclines and vector field. Figure1.

5. We have shown that, as $t \to \infty$, the solutions either converge to $(0,3)$ or $(2,0)$. So, in each case, only one of the two species survives.

3

4.2.3 My written exam 2

4.2.3.1 Problem 1

1. State the Bendixon criterion of non-existence of periodic orbits of a differential equation.

2. Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + x = 0$$

Where $f(x) = x^2 + x + a,\ a \in \mathbb{R}$. Determine the range of values of a for which the equation does not have any periodic orbits

solution

4.2.3.1.1 Part 1

The Bendixon criterion gives a condition to check if a system of first order ODE's defined in region $D \subset \mathbb{R}^2$ has only periodic solutions. It uses the state space form of the system given as $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$. The domain D has to be simply connected and f_1, f_2 are continuously differentiable functions in D. Given the above, then if the divergence of $\boldsymbol{F}$ is never zero at any point in the domain D then the system has no periodic solutions . In other words, Bendixon criterion says that

$$\text{if } \nabla \cdot F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0 \text{ at any point in } D\text{, then system has no periodic solutions.}$$

No periodic solution is the same as saying phase plot contains no closed orbits.

This criterion can be also stated in another way. The condition given above that $\nabla \cdot F \neq 0$ anywhere, just means that the sign of the divergence do not change in D. (For the sign to change, $\nabla \cdot F$ has to cross the value zero somewhere. This is from calculus). Therefore we can also say the criterion as follows. If $\nabla \cdot F$ changes sign somewhere in D or if it attains the value zero in D then system has only periodic solutions.

4.2.3.1.2 Part 2

$$\ddot{x} + f(x)\dot{x} + x = 0$$

We first observe immediately that the above ODE is a harmonic oscillator with damping coefficient $f(x)$ present, so we can make some observations before applying Bendixon criterion. We know from Physics that if the damping coefficient is positive, then the solution is stable and will have the form of exponentially decaying damped oscillations (this is because damping now takes energy away). But if the damping is negative then the solution becomes unstable and has the form of exponentially increasing damped oscillations (because now damping adds energy to the system).

If the damping coefficient is zero, then the ODE becomes pure harmonic oscillator $\ddot{x} + x = 0$ and the solution is pure oscillatory (periodic) that lasts for all time (given non-zero initial conditions).

Therefore, just from Physics considerations only, we expect that only if $f(x) = 0$ everywhere then closed trajectory (or periodic solution) will result. Or if $f(x) \neq 0$ then no periodic solutions exist. In real physical mechanical systems the damping coefficient, if present, is always positive.

Now we can apply Bendixon criterion. Let $x_1 = x, x_2 = \dot{x}$. Taking time derivatives gives $\dot{x}_1 = x_2, \dot{x}_2 = -f(x)\dot{x} - x = -f(x_1)x_2 - x_1$. In state space the system becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f(x_1)x_2 - x_1 \end{pmatrix}$$

Therefore the gradient is

$$\begin{aligned} \nabla \cdot F &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &= 0 - f(x_1) \\ &= -f(x_1) \end{aligned}$$

Using Bendixon criterion which says that if $\nabla \cdot F \neq 0$ at every point, then no periodic solutions exist, shows right away that the condition for no periodic solution is that $f(x_1) \neq 0$ for all x_1 domain, which is what Physics tells us. This means that if (where now we write x_1 as x since they are the same and for simplicity)

$$x^2 + x + a \neq 0 \qquad \text{For all } x$$

Or if

$$a \neq -\left(x^2 + x\right)$$

Then no periodic solutions exist.

4.2.3.2 Problem 2

Consider the system of differential equations that models the growth of two competing species with populations $x \geq 0$ and $y \geq 0$

$$\dot{x} = x\left(2 - x - y\right)$$
$$\dot{y} = y\left(3 - 2x - y\right)$$

1. Find all equilibrium points and determine their stability type.
2. Determine the nullclines of the system.
3. Find the invariant regions of the xy-plane.
4. Draw the phase-plane using your favorite software.
5. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

solution

4.2.3.2.1 Part 1

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \boldsymbol{F} = \begin{pmatrix}f_1\left(x,y\right)\\ f_2\left(x,y\right)\end{pmatrix} = \begin{pmatrix}x\left(2-x-y\right)\\ y\left(3-2x-y\right)\end{pmatrix}$$

Equilibrium points are solutions of $f_1 = 0, f_2 = 0$. When $f_1 = 0$ then $x = 0$ or $x = 2 - y$. Now, when $x = 0$, then $f_2 = 0 = y\left(3 - y\right)$ which has solution as $y = 0$ and $y = 3$. Therefore the critical points found so far are $\{0,0\},\{0,3\}$.

Now when $x = 2 - y$ then $f_2 = 0 = y\left(3 - 2\left(2 - y\right) - y\right) = y\left(y - 1\right)$, which has solution as $y = 0$ and $y = 1$. This means that $x = 2$ and $x = 1$ respectively. This adds the points $\{2,0\},\{1,1\}$ to what was found above. Therefore the list of critical points are

$$\left(x_i, y_i\right) = \{0,0\},\{0,3\},\{2,0\},\{1,1\}$$

The Jacobian matrix for the system is given by gradient of $\boldsymbol{F}$

$$\begin{aligned}\nabla\boldsymbol{F} = J &= \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial}{\partial x}\left(2x - x^2 - xy\right) & \frac{\partial}{\partial y}\left(2x - x^2 - xy\right)\\ \frac{\partial}{\partial x}\left(3y - 2xy - y^2\right) & \frac{\partial}{\partial y}\left(3y - 2xy - y^2\right)\end{pmatrix}\\ &= \begin{pmatrix}2 - 2x - y & -x\\ -2y & 3 - 2x - 2y\end{pmatrix} \qquad (1)\end{aligned}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$A = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix}2-\lambda & 0\\ 0 & 3-\lambda\end{vmatrix} = 0$$
$$(2-\lambda)(3-\lambda) = 0$$

Therefore $\lambda_1 = 2, \lambda_2 = 3$. Since both eigenvalues are positive, then this is unstable critical point. It is a negative attractor.

At point $(0,3)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix}-1 & 0\\ -6 & -3\end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & 0 \\ -6 & -3-\lambda \end{vmatrix} = 0$$
$$(-1-\lambda)(-3-\lambda) = 0$$

Therefore $\lambda_1 = -1, \lambda_2 = -3$. Since both eigenvalues are negative, then this is a stable critical point. It is a positive attractor.

At point $(2,0)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -2-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix} = 0$$
$$(-2-\lambda)(-1-\lambda) = 0$$

Therefore $\lambda_1 = -2, \lambda_2 = -1$. Since both eigenvalues are negative, then this is a stable critical point. It is a positive attractor.

At point $(1,1)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & -1 \\ -2 & -1-\lambda \end{vmatrix} = 0$$
$$(-1-\lambda)^2 - 2 = 0$$
$$\lambda^2 + 2\lambda - 1 = 0$$

Using quadratic formula, $\lambda = -\frac{b}{2a} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4+4} = -1 \pm \sqrt{2}$. Hence $\lambda_1 = -1 + \sqrt{2}, \lambda_2 = -1 - \sqrt{2}$. Since $\sqrt{2} > 1$ then $\lambda_1 > 0$ and $\lambda_2 < 0$. Because one eigenvalue is positive and one is negative, then this is a saddle point (unstable).

The following table is a summary of the above results

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 2, \lambda_2 = 3$	negative attraction, unstable
$(0,3)$	$\lambda_1 = -1, \lambda_2 = -3$	positive attraction, stable
$(2,0)$	$\lambda_1 = -2, \lambda_2 = -1$	positive attraction, stable
$(1,1)$	$\lambda = -1 \pm \sqrt{2}$	Saddle, unstable

4.2.3.2.2 Part 2

Since the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x(2-x-y) \\ y(3-2x-y) \end{pmatrix} \tag{1}$$

Then the x nullclines are the solution of $x(2-x-y) = 0$ and the y nullclines are solutions of $y(3-2x-y)$. This shows that the x nullclines are given by $x = 0$ (the y axis) line and by $y = 2 - x$ line. Similarly the y nullclines are $y = 0$ line (the x axis) and $y = 3 - 2x$ line. This is plot of nullclines for $x \geq 0, y \geq 0$

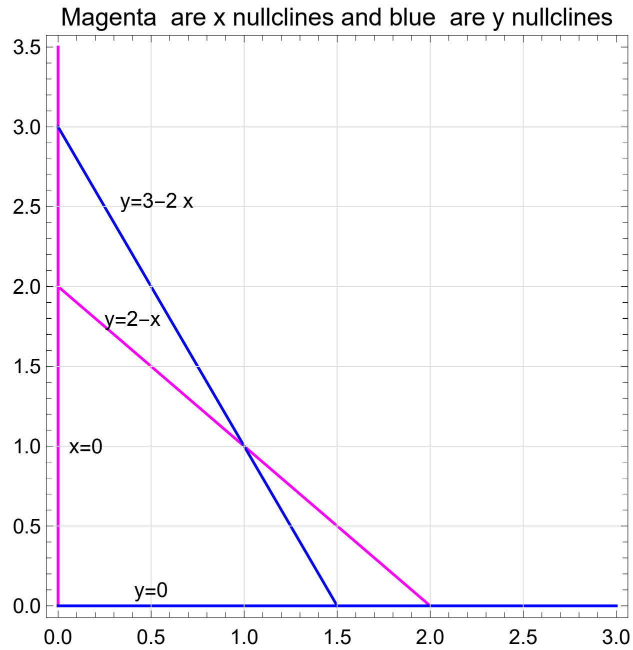

Figure 4.1: nullclines lines

This plot adds the critical points on the above plot to make it more clear. Red points are unstable and Blue points are stable. The location of the critical points are from part 1.

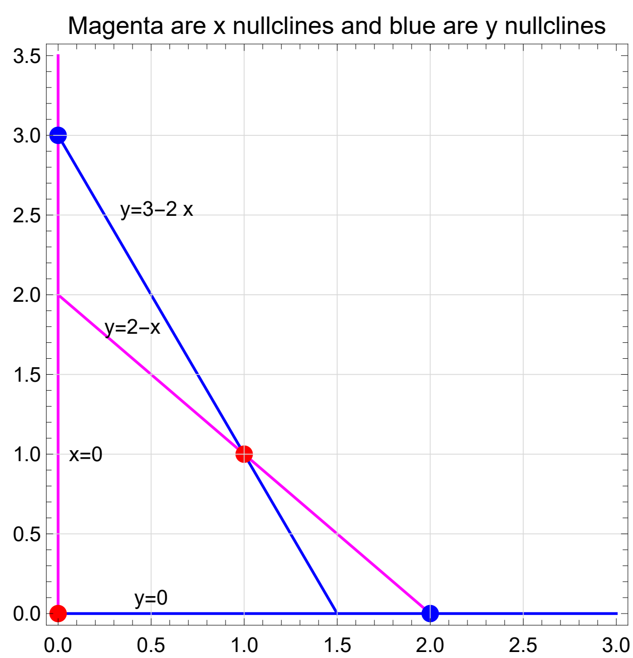

Figure 4.2: nullclines lines with critical points added

4.2.3.2.3 Part 3 invariant regions are those regions where solutions always remain inside the region for all time, given that initial conditions are inside the region. We know that $x = 0$ is invariant (which means solutions that start on this line remain on this line). This is because when $x = 0$ then $\dot{x} = 0$ and $\dot{y} = 3y - 3y^2$. So solution remain on $x = 0$ line. We also know that $y = 0$ is invariant line. This is because when $y = 0$ then $\dot{y} = 0$ and $\dot{x} = 2x - x^2$. So solution remain on $y = 0$ line. This means orbits can not cross these two lines. There are four main regions. These are shown in this plot. We also know that orbits can not cross over invariant lines.

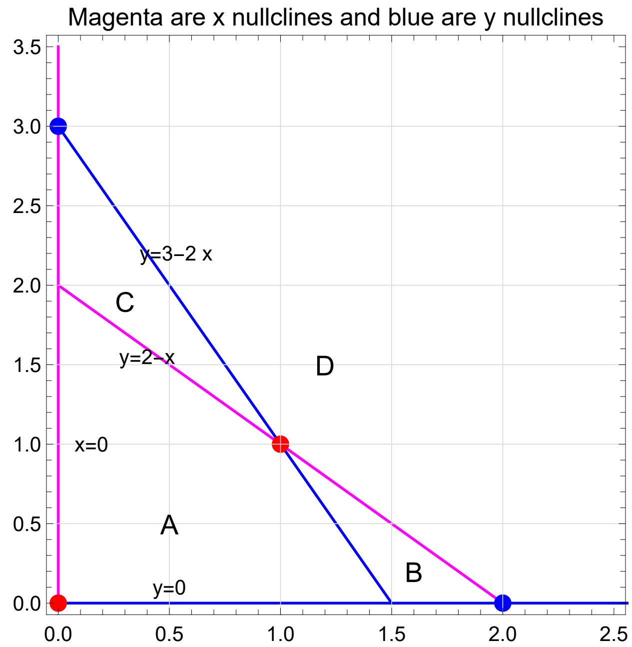

Figure 4.3: Four regions to examin

Starting with x nullcline ($x = 2 - y$) where $\dot{x} = 0$, substituting this in the second equation in (1) gives $\dot{y} = y\left(3 - 2\left(2 - y\right) - y\right) = y\left(y - 1\right)$. When $0 < y < 1$ then we see that $\dot{y} < 0$. Hence vector field is pointing downwards. When $1 < y < 3$ then $\dot{y} > 0$. Hence vector field is pointing upwards. The above plot is now updated with this new information.

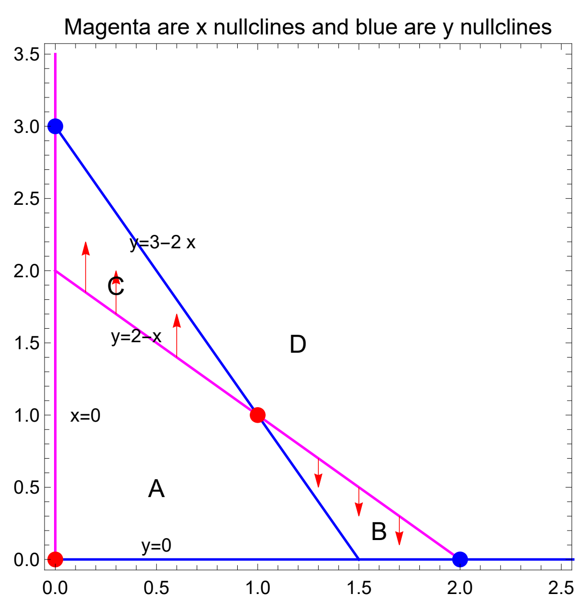

Figure 4.4: Direction fields based on x nullcline analysis

Now we do the same for the y nullcline line ($y = 3 - 2x$). substituting this in the first equation in (1) gives $\dot{x} = x(2 - x - (3 - 2x)) = x(x - 1)$. When $x > 1$ then $\dot{x} > 1$ hence vector field is pointing to the right. When $0 < x < 1$ then $\dot{x} < 0$ and vector field is pointing to the left. The above plot is now updated with this new information.

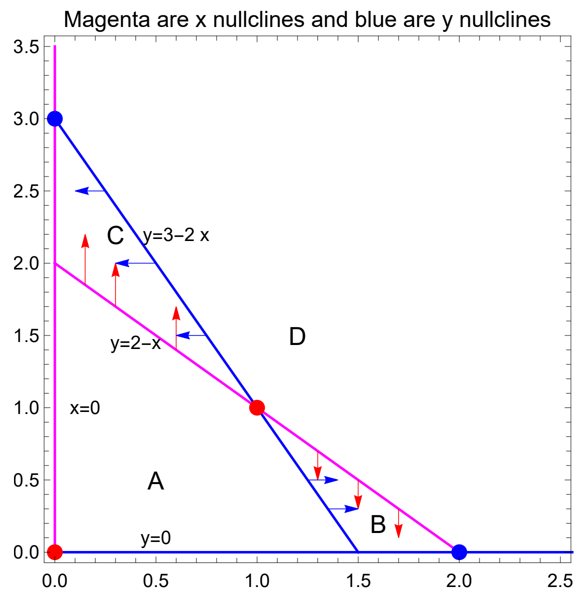

Figure 4.5: Direction fields based on y nullcline analysis

We see from the above that solutions that starts in region B will remain in B and eventually as $t \to \infty$ reach the stable point $(2,0)$. Hence region B is an invariant region.

Also solutions that start in C remains in C and eventually as $t \to \infty$ reach the stable point $(0,3)$. Hence C is invariant region.

A solution that starts in region A can either go to the critical stable point $(2,0)$ or towards the critical stable point $(0,3)$ or enter regions C or B first and eventually reach $(2,0)$ or $(0,3)$.[1]

We could also consider the union of regions A, C, B as new region say E. Then region E is invariant, since any solution that starts in E remains in E. Similarly, we could also consider the union of regions D, C, B as new region say F. Then region F is also invariant.

4.2.3.2.4 Part 4 The phase plot was generated numerically on the computer. The following is the result

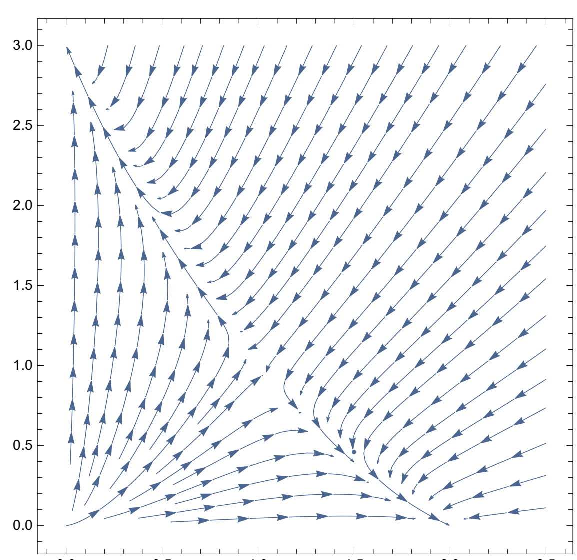

Figure 4.6: Phase plot

In the following plot, the nullclines are plotted on top of the phase plot to better see the invariant sets found in part 3.

[1]If solution starts in region A on exactly the stable eigenvector associated with saddle point $(1,1)$ then it will remain in A and reach $(1,1)$ at $t \to \infty$ but this is very unlikely to happen. More on this in part 5 below

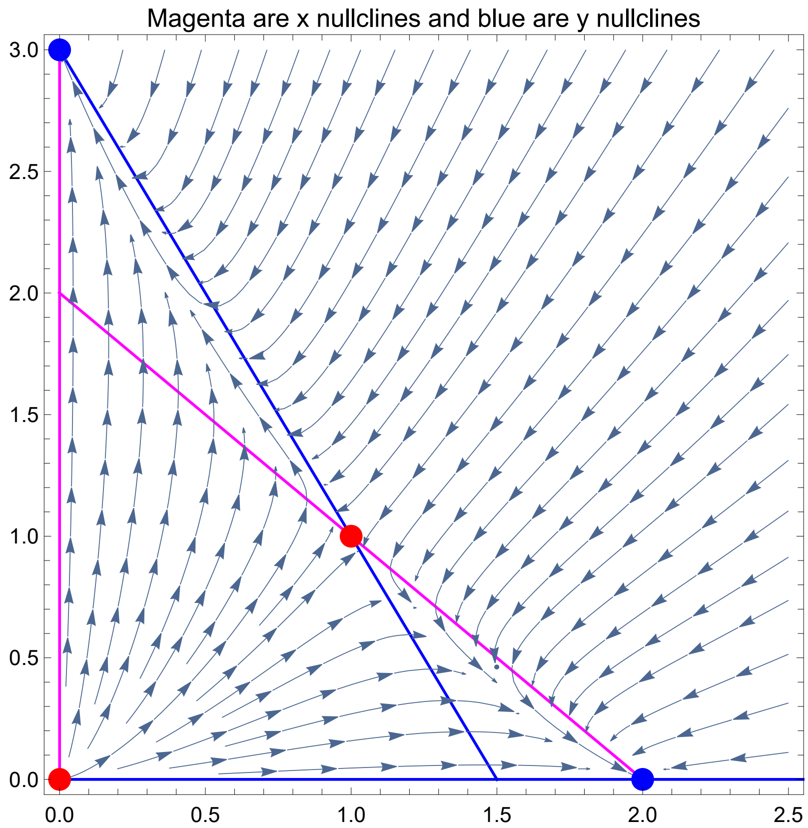

Figure 4.7: Phase plot with nullclines

4.2.3.2.5 Part 5 The point (1,1) is the only saddle point in the system. With the right exact initial conditions, it is possible for the solution to reach this saddle point instead of the other two stable points, if the initial conditions are on the stable eigenvector associated with this saddle point. This saddle equilibrium point have one stable eigenvector and one unstable eigenvector.

This is the only possibility for both species to survive, since at this saddle point both x,y are not zero and remain so for all time. But the probability of the initial conditions being exactly on the stable eigenvector direction for this saddle point is very low compared to having initial conditions being any where else in $\mathbb{R}^2$ where $x \geq 0, y \geq 0$ as any very small deviation in initial conditions will make the solution go to one of the two stable points.

Any other initial conditions location anywhere else in the first quadrant, the solution will reach as $t \to \infty$ either the stable point $(2,0)$ where x only survives or will reach the other stable point $(0,3)$ where now y only survives. This shows that it is extremely unlikely, for both species to survive.

4.3 Final exam, May 10, 2020

Local contents

4.3.1 What will be covered

MATH 5525: Final Examination Guide

Date. The date of the *take-home* final examination is Monday, May 11, 2020. The test will be posted on Canvas by 8 am of Monday, May 11. The completed test should be uploaded back into canvas by midnight of the same day. Please, post only single .pdf files.

Topics to be covered in the final exam. They include the following sections of the textbook:

- 2.1–2.4.
- 3.1–3.4.
- 4.1–4.4 (Proofs of the Poincaré-Bendixon theory are not required).
- 5.1-5.4.
- 6.1, 6.2 (no proofs required), 6.3.
- 7.1-7.3.
- 8.1-8.4.
- 14.4-14.5.

In addition, the exam will also cover topics of nonlinear systems such as *invariant sets*, *nullclines* and also the *SIR and SIRS models.* These topics will be treated at the level of the examples worked out in class, in assignments and in the midterm tests.

Grading policy. The grade of the course will be based upon a weighted average of homeworks and examinations:

Homework: 25 % of the final grade.
Midterm Test 1 (Friday, February 21): 20 %.
Midterm Test 2 (Monday, March 30): 20 %.
Final examination (Comprehensive. Monday, May 11): 35%.

Final Examination policies.

- The final examination is compulsory. Failure to take it will automatically result in the grade 'F' in the course.
- Books, notes and computers are allowed during the exam.
- The final exam has to be fully completed by the student. Students are not allowed to receive help from anyone.
- All University of Minnesota policies regarding take-home and final examinations apply to the course.

1

4.3.2 Questions

Final Exam for Dynamical Systems, Math 5525

May 10, 2020

Problem 1

Consider the following system of ordinary differential equations

$$\begin{aligned} \dot{x} =& -x + y + xy \\ \dot{y} =& x - y - x^2 - y^3. \end{aligned} \quad (1)$$

1. Find the (unique) equilibrium point (x^*, y^*) of the system (1).
2. Linearize the system about about (x^*, y^*) and write the corresponding Jacobian matrix A (that is, the matrix of the linear system.)
3. Find the eigenvalues and eigenvectors of A.
4. Can you reach any conclusions about the stability of (x^*, y^*)?
5. Write down the definition of Lyapunov stability of an equilibrium point.
6. Write down the (Lyapunov) theorem that gives sufficient conditions for the stability of an equilibrium point.
7. Apply the previous theorem to show that the equilibrium solution (x^*, y^*) is, indeed, stable. For this, choose $a \in \mathbb{R}$, so that $V(x, y) = ax^2 + 2y^2$, is a Lyapunov function of the system.
8. Determine an ω-limit set of the system.
9. Write down the Poincaré-Bendixon theorem for two dimensional systems.
10. Taking into account the Poincaré-Bendixon theorem, would you say that a limit cycle is possible for system (1)?

1

Problem 2.

This problem is about discrete dynamical systems/one-dimensional dynamics. (Class notes, pages 63-69, *Lecture notes, 5525-May4-2020* and section 14.4 of textbook.)

Consider the map

$$f(x) = \frac{x}{1+x^2} - ax, \quad a \in \mathbb{R}, \tag{2}$$

and the discrete *orbit* defined by the sequence

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ \ldots x_n = f(x_{n-1}) = f^n(x_0), \ldots$$

11. Define a *fixed-point* of a map.

12. Find all the fixed points x^* of the map (2) and determine in which intervals of a they exist.

13. Determine the stability of the nonzero fixed point in the parameter interval $a \in (-1, 0)$. Hint: Use the proposition in page 68 of the notes.

14. For $x = \epsilon > 0$, ϵ very small, consider the approximate map $g(\cdot)$ given by
$$g(x) := (1-a)x.$$
Show that the map $g(\cdot)$ has a *two-cycle*, that is a discrete periodic orbit of period 2.

2

Guidelines:

1. All questions are equally weighted.
2. You may use books, notes and internet resources as you wish.
3. The class notes are posted on Canvas, with the last set of *lecture notes* labelled as *5525-May4-2020.pdf.*
4. The work has to be personal, that is, you may not consult with anyone or receive any help. (You may always email me, if you have questions or difficulties.)
5. The exam should be back tonight (Monday, May 11), by midnight.
6. Upload the complete work on canvas. If you experience difficulties, please email it directly to me.

Please, sign the following statement:
I hereby certify that I have not received help from anyone in the completion of this test.

Signature:

____

Minneapolis, May 11, 2020.

3

4.3.3 My exam

Local contents

4.3.3.1 Problem 1

Problem 1

Consider the following system of ordinary differential equations

$$\begin{aligned} \dot{x} =& -x + y + xy \\ \dot{y} =& x - y - x^2 - y^3. \end{aligned} \tag{1}$$

1. Find the (unique) equilibrium point (x^*, y^*) of the system (1).
2. Linearize the system about about (x^*, y^*) and write the corresponding Jacobian matrix A (that is, the matrix of the linear system.)
3. Find the eigenvalues and eigenvectors of A.
4. Can you reach any conclusions about the stability of (x^*, y^*)?
5. Write down the definition of Lyapunov stability of an equilibrium point.
6. Write down the (Lyapunov) theorem that gives sufficient conditions for the stability of an equilibrium point.
7. Apply the previous theorem to show that the equilibrium solution (x^*, y^*) is, indeed, stable. For this, choose $a \in \mathbb{R}$, so that $V(x, y) = ax^2 + 2y^2$, is a Lyapunov function of the system.
8. Determine an ω-limit set of the system.
9. Write down the Poincaré-Bendixon theorem for two dimensional systems.
10. Taking into account the Poincaré-Bendixon theorem, would you say that a limit cycle is possible for system (1)?

Figure 4.8: Problem description

$$
\begin{aligned}
\dot{x} &= -x + y + xy \\
\dot{y} &= x - y - x^2 - y^3
\end{aligned}
$$

4.3.3.1.1 Part 1 Equilibrium points are found by solving for x, y in

$$
\begin{aligned}
-x + y + xy &= 0 \qquad (1)\\
x - y - x^2 - y^3 &= 0 \qquad (2)
\end{aligned}
$$

The first obvious solution is $x = 0, y = 0$. To find other solutions, then from (1) and solving for x gives

$$
\begin{aligned}
y + x\left(y - 1\right) &= 0 \\
x &= \frac{-y}{y-1} = \frac{y}{1-y} \qquad (3)
\end{aligned}
$$

Substituting (3) into (2) results in

$$
\begin{aligned}
\left(\frac{y}{1-y}\right) - y - \left(\frac{y}{1-y}\right)^2 - y^3 &= 0 \\
\frac{y}{1-y} - y - \frac{y^2}{\left(1-y\right)^2} - y^3 &= 0 \\
y\left(1-y\right) - y\left(1-y\right)^2 - y^2 - y^3\left(1-y\right)^2 &= 0 \\
y\left(\left(1-y\right) - \left(1-y\right)^2 - y - y^2\left(1-y\right)^2\right) &= 0
\end{aligned}
$$

The above shows that $y = 0$ is a solution and $\left(1-y\right)^2 - \left(1-y\right)^2 - y - y^2\left(1-y\right)^2 = 0$ is the second solution. But $y = 0$ gives $x = 0$ which we already found earlier. So now we look at the solution for y from the second case which gives the following

$$
\begin{aligned}
\left(1-y\right) - \left(1-y\right)^2 - y - y^2\left(1-y\right)^2 &= 0 \\
\left(1-y\right) - \left(1+y^2-2y\right) - y - y^2\left(1+y^2-2y\right) &= 0 \\
1 - y - 1 - y^2 + 2y - y - \left(y^2 + y^4 - 2y^3\right) &= 0 \\
1 - y - 1 - y^2 + 2y - y - y^2 - y^4 + 2y^3 &= 0 \\
-y^2 - y^2 - y^4 + 2y^3 &= 0 \\
-2y^2 - y^4 + 2y^3 &= 0 \\
y^2\left(-2 - y^2 + 2y\right) &= 0
\end{aligned}
$$

This gives solutions $y = 0$ or $-2 - y^2 + 2y = 0$. But $y = 0$ gives $x = 0$ from (3) which we already found earlier. So now we look at the solution for y from the second solution which gives

$$
\begin{aligned}
-2 - y^2 + 2y &= 0 \\
y^2 - 2y + 2 &= 0
\end{aligned}
$$

Therefore the roots are, by using the quadratic formula $y = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ or

$$
\begin{aligned}
y &= \frac{2}{2} \pm \frac{1}{2}\sqrt{4-8} \\
&= 1 \pm \frac{1}{2}\sqrt{-4} \\
&= 1 \pm i
\end{aligned}
$$

Since we are looking for real solutions, then the above is not a solution that we can accept. This shows that there is only one equilibrium point

$$
\left(x^*, y^*\right) = \{0, 0\}
$$

Using the computer, the phase plot for the non-linear is given below. The red point is the equilibrium point $\{0,0\}$

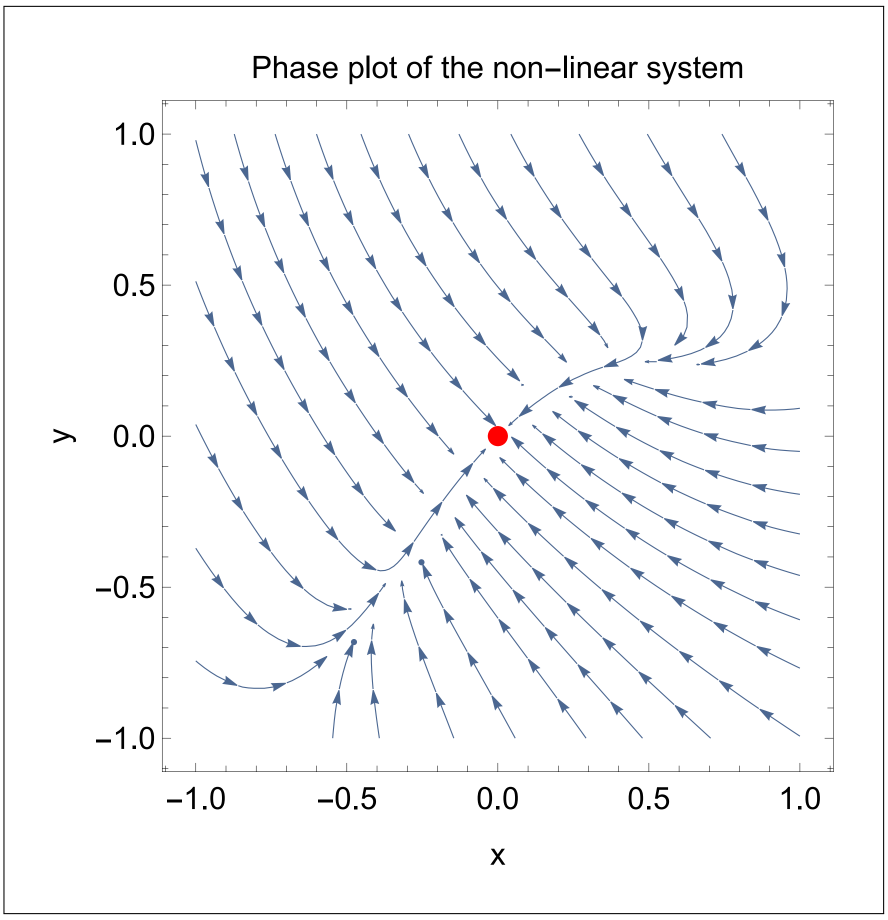

Figure 4.9: Phase plot

The following is the same phase plot, but made for a much larger domain of the state variables x, y.

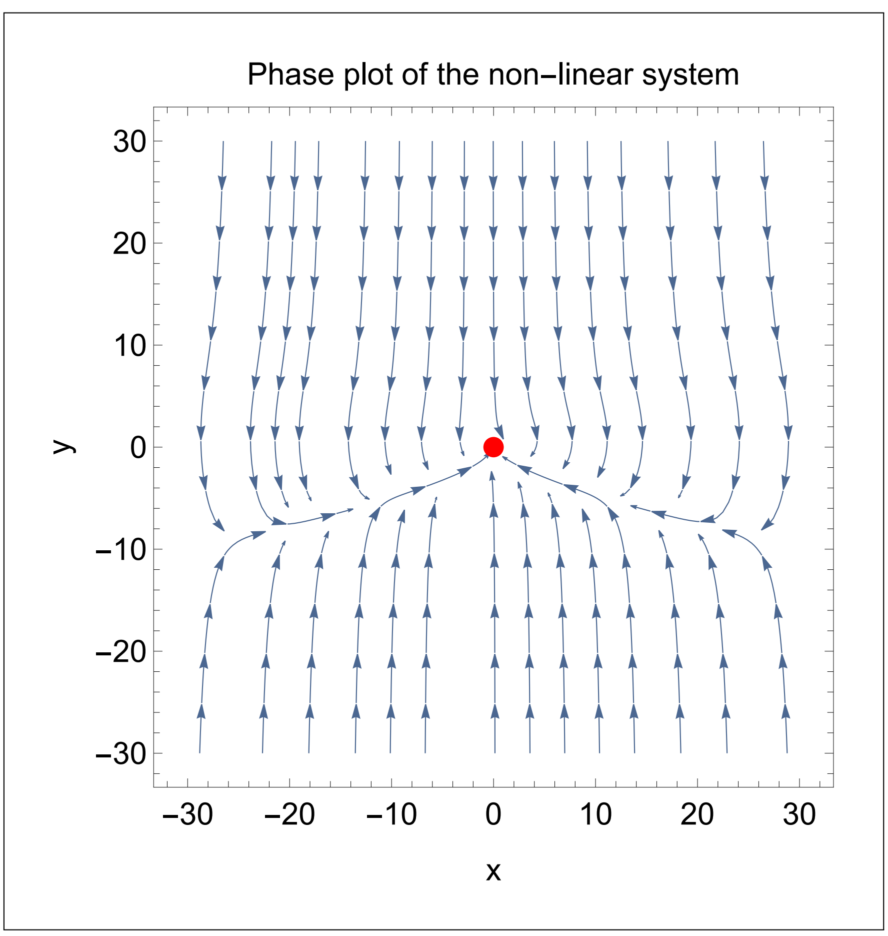

Figure 4.10: Phase plot using larger domain

```
ClearAll[x, y];
eq1 = -x + y + x y;
eq2 = x - y - x^2 - y^3;
p = StreamPlot[{eq1, eq2}, {x, -1, 1}, {y, -1, 1},
        Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        StreamPoints → 30,
        FrameLabel → {{"y", None}, {"x", "Phase plot of the non-linear system"}},
        BaseStyle → 14];
```

Figure 4.11: code used for the above plot

4.3.3.1.2 Part 2 The linearized system at the equilibrium point is given by

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = [A]\begin{pmatrix}x\\ y\end{pmatrix} \tag{1}$$

Where the matrix A is the Jacobian matrix J when evaluated at the equilibrium point. The Jacobian matrix is given by

$$J = \begin{pmatrix}\frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y}\\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y}\end{pmatrix} \tag{2}$$

Where $\dot{x} = -x + y + xy, \dot{y} = x - y - x^2 - y^3$. Therefore

$$\begin{aligned}\frac{\partial \dot{x}}{\partial x} &= -1 + y\\ \frac{\partial \dot{x}}{\partial y} &= 1 + x\\ \frac{\partial \dot{y}}{\partial x} &= 1 - 2x\\ \frac{\partial \dot{y}}{\partial y} &= -1 - 3y^2\end{aligned}$$

Using the above in (2) gives the Jacobian matrix as

$$J = \begin{pmatrix}-1 + y & 1 + x\\ 1 - 2x & -1 - 3y^2\end{pmatrix}$$

Then the linearized system around $x = 0, y = 0$ now is found as

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} &= A\begin{pmatrix}x\\ y\end{pmatrix}\\ &= \begin{pmatrix}-1 + y & 1 + x\\ 1 - 2x & -1 - 3y^2\end{pmatrix}_{\substack{x=0\\ y=0}}\begin{pmatrix}x\\ y\end{pmatrix}\\ &= \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\end{aligned}$$

4.3.3.1.3 Part 3 From part (2) above, we found the linearized system around $x = 0, y = 0$ to be

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

Now we find the eigenvalues of A. Solving

$$\begin{aligned}|A - \lambda I| &= 0\\ \begin{vmatrix}-1 - \lambda & 1\\ 1 & -1 - \lambda\end{vmatrix} &= 0\\ (-1 - \lambda)(-1 - \lambda) - 1 &= 0\\ \lambda^2 + 2\lambda &= 0\\ \lambda(\lambda + 2) &= 0\end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = 0, \lambda_2 = -2$. Now we find the corresponding eigenvectors of A.

For $\lambda_1 = 0$ we solve for v from

$$\begin{pmatrix} -1-\lambda_1 & 1 \\ 1 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First equation gives $-v_1 + v_2 = 0$. Let $v_1 = 1$ then $v_2 = 1$. Hence the eigenvector associated with $\lambda_1 = 0$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -2$ we solve for v from

$$\begin{pmatrix} -1-\lambda_2 & 1 \\ 1 & -1-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1+2 & 1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First equation gives $v_1 + v_2 = 0$. Let $v_1 = 1$ then $v_2 = -1$. Hence the eigenvector associated with $\lambda_2 = -2$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Summary of results for part 3

x^*	Linearized system at x^*	Eigenvalues	Eigenvectors
$(0,0)$	$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$	$\lambda_1 = 0, \lambda_2 = -2$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4.3.3.1.4 Part 4 Since the system is non-linear, and one of the eigenvalues is zero, then the equilibrium point is called defective. What this means is that it is not possible to conclude that the origin is stable or not. Even though the second eigenvalue is negative, we can not conclude that the non-linear system is stable at the origin since one eigenvalue is zero.

This only happens for non-linear systems. If the actual system was linear, then we could have concluded it is stable. But not for non-linear systems.

4.3.3.1.5 Part 5 Considering system $\dot{x} = f(x,t)$ and neighborhood $D \subset \mathbb{R}^n$ around the origin point $x = 0$. Here the origin is always taken as the equilibrium point. But any other equilibrium point will also work in this definition, since we can always translate the system to make the equilibrium point as the origin. So it is easier to always take the equilibrium point as the origin.

Now, let solution that start at time $t = t_0$ from point $x_0 \in D$ be called $x(t; t_0, x_0)$. Then we say that the the solution at $x = 0$ is stable in the sense of Lyapunov if for each $\epsilon > 0$ and t_0 we can find $\delta(\epsilon, t_0)$ such that $\|x_0\| \leq \delta$ implies $\|x(t; t_0, x_0)\| \leq \epsilon$ for all $t \geq t_0$.

The above is basically what the book gives as the definition of Lyapunov stability.

The following is a diagram made to help explain what the above means, and also I give may be a little simpler definition as follows.

Lyapunov stability intuitively says that if we start with initial conditions $x_0(t_0)$ at time t_0 somewhere near the equilibrium point (this is the domain D) then if the solution $x(t)$ is always bounded from above for any future time t by some limit (which depends on how far the initial conditions are from the origin, and the time t_0 the solution started), then that the origin is called a stable equilibrium point in the sense of Lyapunov.

This basically says that solutions that starts near the equilibrium point will never go too far away from the origin for all time.

To make this more mathematically precise[2], we say that for any $\|x_0\| \leq \delta(t_0)$ we can find $\epsilon(\delta)$ such that $\|x(t)\| \leq \epsilon$ for any $t \geq t_0$. In this both δ and ϵ are some positive quantities and ϵ depends on choice of δ and δ depends on t_0.

This diagram helps illustrate the above definition.

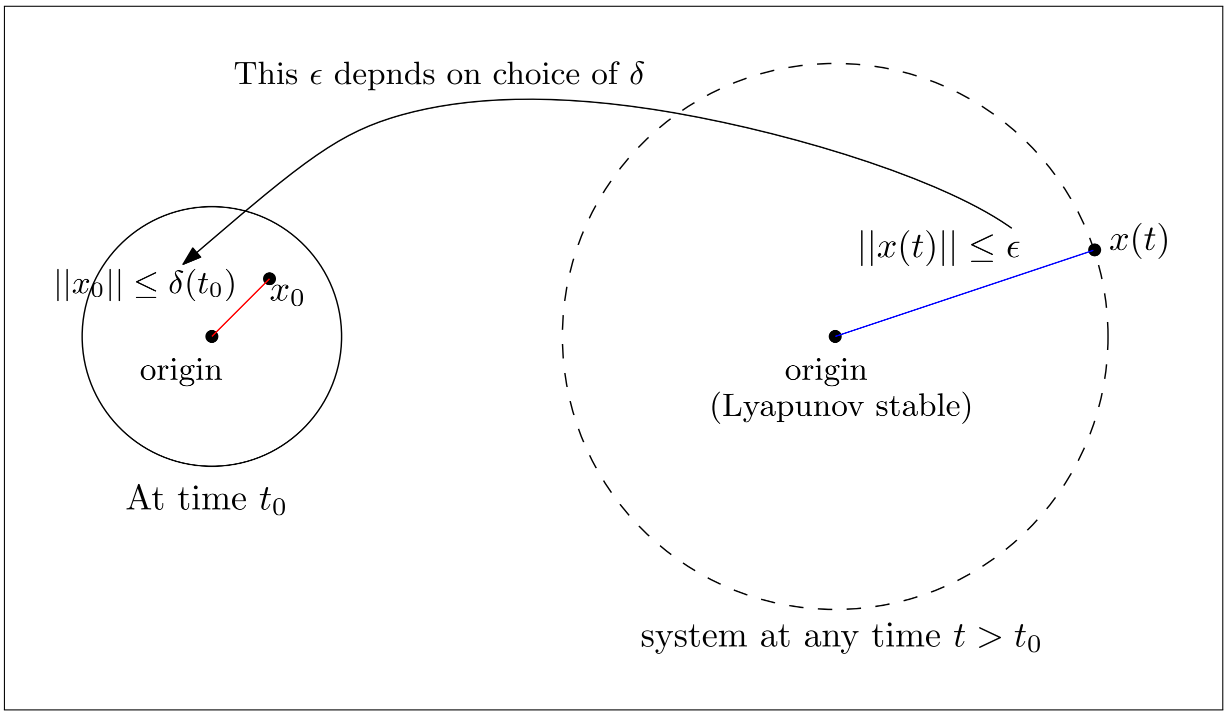

Figure 4.12: Graphical representation of Lyapunov stability

In the above diagram, we start with the system in some initial state shown on the left where we have the norm $\|x_0\| \leq \delta$ where δ depends on t_0. Now, if we can always find ϵ such that the solution norm $\|x(t)\| \leq \epsilon$ for any time in the future $t > t_0$ where ϵ depends on δ, then we say the equilibrium point is stable in the sense of Lyapunov.

4.3.3.1.6 Part 6 The theorem that gives the conditions for Lyapunov stability is given in theorem 8.8 in the book. This is what it basically says. If given the system $\dot{x} = f(x,t)$ with $f(0,t) = 0$ and $x \in D \subset \mathbb{R}^n, t \geq t_0$, then assuming we can find what is called the Lyapunov function $V(x)$ for this system with the following three conditions

1. $V(x)$ is continuously differentiable function in $\mathbb{R}^n$ and $V(x) \geq 0$ (positive definite or positive semidefinite) for all points away from the origin, or everywhere inside some fixed region around the origin. This function represents the total energy of the system (For Hamiltonian systems). For non-Hamiltonian systems we have to work harder to find it.

2. $V(0) = 0$. This condition says the system has no energy when it is at the equilibrium point. (rest state).

3. The orbital derivative along any solution trajectory is $\frac{dV}{dt} \leq 0$ (negative definite or negative semi-definite) for all points, or inside some fixed region around the origin. This condition says that the total energy is either constant in time (the zero case) or the total energy is decreasing in time (the negative definite case). Both of which indicate that the origin is a stable equilibrium point.

[2] notice that this definition is slightly different from book definition, which I modified slightly in the hope to make it more clear, at least to me it does.

If such $V(x)$ could be found, then these are sufficient conditions for the stability of equilibrium point. If $\frac{dV}{dt}$ is strictly negative definite, then we say the equilibrium point is asymptotically stable. If $\frac{dV}{dt}$ is negative semidefinite, then the equilibrium point is stable in the sense of Lyapunov. asymptotically stable have stronger stability.

Negative semi-definite means the system, when perturbed away from the origin, a solution trajectory remains around the origin since its energy do not increase nor decrease. So it is stable. But asymptotically stable equilibrium is a stronger stability. It means when perturbed from the origin the solution will eventually return back to the origin since the energy is always decreasing. Global stability means $\frac{dV}{dt} \leq 0$ everywhere, and not just in some closed region around the origin. Local stability means $\frac{dV}{dt} \leq 0$ in some closed region around the origin. Global stability is stronger stability than local stability. Sometimes it is easier to determine local stability than global stability.

4.3.3.1.7 Part 7 Let $V(x,y) = ax^2 + 2y^2$. Condition (2) $V(0) = 0$ is satisfied, since when $x = 0, y = 0$ then $V(x,y) = 0$.

Condition (1) is also satisfied since both terms are positive <u>if we choose</u> $a > 0$. This makes $V(x,y) > 0$ for non zero x, y. We now need to check the third condition. This condition is always the hardest one to check. The orbital derivative $\frac{dV}{dt}$ is

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= 2ax\dot{x} + 4y\dot{y} \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\dot{x} &= -x + y + xy \\ \dot{y} &= x - y - x^2 - y^3\end{aligned}$$

Eq(1) now becomes

$$\begin{aligned}\frac{dV}{dt} &= 2ax\left(-x + y + xy\right) + 4y\left(x - y - x^2 - y^3\right) \\ &= -2ax^2 + 2axy + 2ax^2y + 4yx - 4y^2 - 4yx^2 - 4y^4 \\ &= -\left(2ax^2 + 4y^2 + 4y^4\right) + 2axy + 2ax^2y + 4yx - 4yx^2 \\ &= -\left(2ax^2 + 4y^2 + 4y^4\right) + xy\left(2a + 4\right) + 2ax^2y - 4yx^2 \\ &= -\left(2ax^2 + 4y^2 + 4y^4\right) - \left(-xy\left(2a + 4\right) - 2ax^2y + 4yx^2\right)\end{aligned}$$

We see that is we choose $a \geq 0$ then the first term above which is $-\left(2ax^2 + 4y^2 + 4y^4\right)$ is always negative (or negative semidefinite for $x = 0, y = 0$) and can not be positive.

Let us try $a = 2$, (we only need to find one a value to make it valid Lyapunov function). This means our choice of Lyapunov function becomes

$$\boxed{V(x,y) = 2x^2 + 2y^2}$$

The above $\frac{dV}{dt}$ now becomes

$$\begin{aligned}\frac{dV}{dt} &= -\left(4x^2 + 4y^2 + 4y^4\right) - \left(-xy\left(4 + 4\right) - 4x^2y + 4yx^2\right) \\ &= -4x^2 + 8xy - 4y^4 - 4y^2 \\ &= -\left(4x^2 - 8xy + 8y^4\right) \\ &= -\left(\left(2x - 2y\right)^2 + 4y^2\right)\end{aligned}$$

Since the terms inside are all squares, then this shows $\frac{dV}{dt} \leq 0$. It can not be positive. The maximum it can be is zero and this is at the origin only. This shows origin is indeed stable in the sense of Lyapunov because now all the three conditions given above are satisfied. Plotting the Lyapunov $2x^2 + 2y^2$ for some region around the origin gives

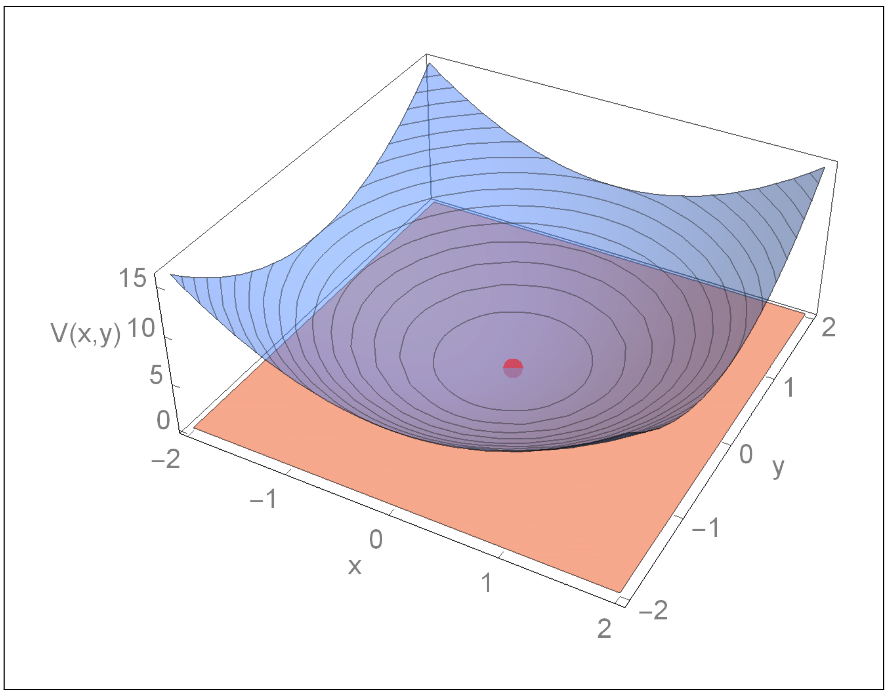

Figure 4.13: Graphical representation of Lyapunov function used

```
V[x_, y_] := 2 x^2 + 2 y^2;
p = Show[
          Plot3D[{0, V[x, y]}, {x, -2, 2}, {y, -2, 2},
             PerformanceGoal → "Quality",
             PlotRange → All,
             PlotTheme -> "Web",
             BaseStyle → {Opacity[.5], 14},
             AxesLabel → {"x", "y", "V(x,y)"},
             ImageSize → 400
          ],
          Graphics3D[{Red, PointSize[0.03], Point[{0, 0, 0}]}]
       ];
```

Figure 4.14: Code used for the above

The following shows the orbital derivative $\frac{dV}{dt}$ plot also in a region around the origin showing it is indeed negative definite.

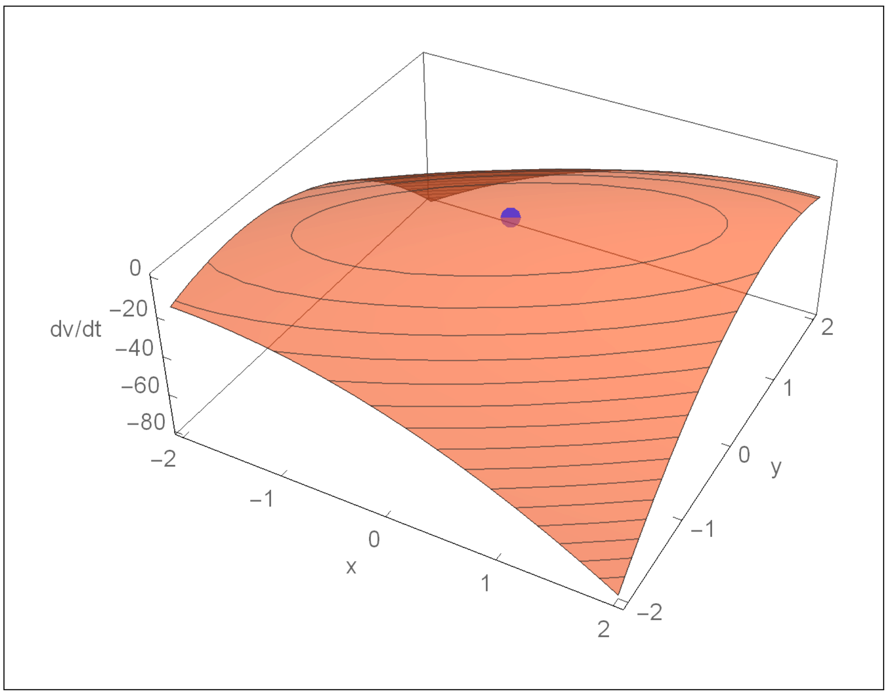

Figure 4.15: Graphical representation of $\frac{dV}{dt}$

```
orbitalDerivative[x_, y_] := -((2 x - 2 y)^2 + 4 y^2)
p = Show[
        Plot3D[{orbitalDerivative[x, y]}, {x, -2, 2}, {y, -2, 2},
          PerformanceGoal → "Quality",
          PlotRange → All,
          PlotTheme -> "Web",
          BaseStyle → {Opacity[.6], 12},
          AxesLabel → {"x", "y", "dv/dt"},
          ImageSize → 400
        ],
        Graphics3D[{Blue, PointSize[0.03], Point[{0, 0, 0}]}]
    ];
```

Figure 4.16: Code used for the above

The following plot shows Lyapunov function and the orbital derivative function found above in the same plot. These functions can only meet at the equilibrium point which is the origin in this case if the system is stable in the sense of Lyapunov.

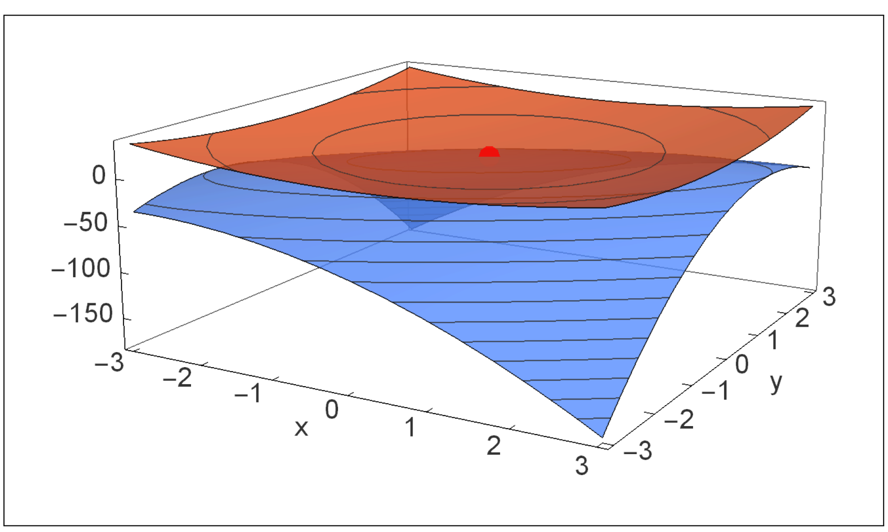

Figure 4.17: Combined Graphical representation of $\frac{dV}{dt}$ and Lapunov function

```
V[x_, y_] := 2 x^2 + 2 y^2;
orbitalDerivative[x_, y_] := - ((2 x - 2 y) ^2 + 4 y^2)
p = Show[
        Plot3D[{V[x, y], orbitalDerivative[x, y]}, {x, -3, 3}, {y, -3, 3},
           PerformanceGoal → "Quality",
           PlotRange → All,
           PlotTheme -> "Web",
           BaseStyle → {Opacity[.8], 14},
           AxesLabel → {"x", "y", None},
           ImageSize → 400
        ],
        Graphics3D[{Red, PointSize[0.03], Point[{0, 0, 0}]}]
      ];
```

Figure 4.18: Code used for the above

4.3.3.1.8 Part 8 ω limit set, is the set of all points that are the limit of all positive orbits $\gamma^+(x)$. In other words, given a specific orbit $\gamma^+(x)$ that starts at some initial conditions point x_0 and if as $t \to \infty$ this orbit terminates at point p then p is in the ω limit set of such orbit.

To find the ω limit set, we need to find the points where solutions terminate at them eventually (attractive or saddle points). But from above, we found that there is only one critical point, which is the origin, and that this point was stable. And since $\frac{dV}{dt} < 0$ for all points away from the origin and zero only at the origin, then the origin is asymptotically stable equilibrium. This means all orbits $\gamma^+(x)$ have their limit as the origin. Hence ω limit set is the origin point.

4.3.3.1.9 Part 9 Poincare-Bedixon theorem for $\mathbb{R}^2$, says that having positive, bounded, non-periodic orbit γ^+ of the system $\dot{x} = f(x)$, then the ω limit set $\omega(\gamma^+)$ contains <u>either</u> a critical point <u>or</u> consists of closed orbit. In this, γ^+ means a solution orbit which as $t \to \infty$ goes to or terminates at a point in the ω limit set. In this we also require that $f : \mathbb{R}^2 \to \mathbb{R}^2$ has continuous first partial derivatives and that solutions exist for all time $-\infty < t < \infty$.

4.3.3.1.10 Part 10 Since a limit cycle implies closed orbit, and since we found that in part 8 the ω limit set contains a critical point (the origin), then by Poincare-Bedixon, <u>it is not possible</u> for the system to have a limit cycle in its ω limit set.

4.3.3.2 Problem 2

Problem 2.

This problem is about discrete dynamical systems/one-dimensional dynamics. (Class notes, pages 63-69, *Lecture notes, 5525-May4-2020* and section 14.4 of textbook.)

Consider the map

$$f(x) = \frac{x}{1+x^2} - ax, \quad a \in \mathbb{R}, \tag{2}$$

and the discrete *orbit* defined by the sequence

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ \ldots x_n = f(x_{n-1}) = f^n(x_0), \ldots$$

11. Define a *fixed-point* of a map.
12. Find all the fixed points x^* of the map (2) and determine in which intervals of a they exist.
13. Determine the stability of the nonzero fixed point in the parameter interval $a \in (-1, 0)$. Hint: Use the proposition in page 68 of the notes.
14. For $x = \epsilon > 0$, ϵ very small, consider the approximate map $g(\cdot)$ given by
$$g(x) := (1-a)x.$$
Show that the map $g(\cdot)$ has a *two-cycle*, that is a discrete periodic orbit of period 2.

Figure 4.19: Problem description

4.3.3.2.1 Part 11 A fixed point of a map $f(x)$ is one which is mapped to itself. In other words, all points x^* that satisfy $f(x^*) = x^*$ where $f(x)$ is the map.

4.3.3.2.2 Part 12 From (2)

$$f(x) = \frac{x}{1+x^2} - ax$$

Hence we need to solve for x in the following

$$\begin{aligned}
\frac{x}{1+x^2} - ax &= x \\
x - ax\left(1+x^2\right) &= x\left(1+x^2\right) \\
x\left(1+x^2\right) + ax\left(1+x^2\right) - x &= 0 \\
ax^3 + ax + x^3 &= 0 \\
x\left(ax^2 + a + x^2\right) &= 0 \\
x\left(x^2(1+a) + a\right) &= 0
\end{aligned}$$

Hence $x = 0$ is a fixed point, and $x^2(1+a) + a = 0$. Or

$$x^2 = \frac{-a}{1+a}$$

For real x, the RHS must be positive and also $a \neq -1$. Hence we need $-1 < a < 0$. And now the remaining fixed points are given by $x = \pm\sqrt{\frac{-a}{1+a}}$. Hence the fixed points are

$$\begin{aligned}
x_1^* &= 0 && \text{for all } a \\
x_2^* &= \sqrt{\frac{-a}{1+a}} && -1 < a < 0 \\
x_3^* &= -\sqrt{\frac{-a}{1+a}} && -1 < a < 0
\end{aligned}$$

4.3.3.2.3 Part 13 By definition, for a map $f(x)$ with fixed point x^* then

1. x^* is sink of $|f'(x^*)| < 1$
2. x^* is source if $|f'(x^*)| > 1$
3. Unable to decide if $|f'(x^*)| = 1$

Therefore, we now apply the above on the two non-zero fixed points found in part 12.

For $x_2^* = \sqrt{\frac{-a}{1+a}}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x}{1+x^2} - ax\right) \\
&= \frac{d}{dx}\frac{x}{1+x^2} - \frac{d}{dx}ax \\
&= \frac{\left(1+x^2\right) - x(2x)}{\left(1+x^2\right)^2} - a \\
&= \frac{1+x^2-2x^2}{\left(1+x^2\right)^2} - a \\
&= \frac{1-x^2}{\left(1+x^2\right)^2} - a \qquad (1)
\end{aligned}$$

Evaluating the above at $x = x_2^* = \sqrt{\frac{-a}{1+a}}$ gives

$$
\begin{aligned}
f'\left(x_2^*\right) &= \frac{1-\left(\sqrt{\frac{-a}{1+a}}\right)^2}{\left(1+\left(\sqrt{\frac{-a}{1+a}}\right)^2\right)^2} - a\\
&= \frac{1-\frac{-a}{1+a}}{\left(1+\frac{-a}{1+a}\right)^2} - a\\
&= \frac{\frac{1+a+a}{1+a}}{\left(\frac{1+a-a}{1+a}\right)^2} - a\\
&= \frac{\frac{1+2a}{1+a}}{\left(\frac{1}{1+a}\right)^2} - a\\
&= \frac{\frac{1+2a}{1+a}}{\frac{1}{(1+a)^2}} - a\\
&= \frac{1+2a}{\frac{1}{1+a}} - a\\
&= (1+2a)(1+a) - a\\
&= 2a^2 + 2a + 1\\
&= 1 + 2a(1+a)
\end{aligned}
$$

Since $-1 < a < 0$ then $0 < 1+a < 1$ and $-1 < 2a < 0$. Hence $0 < 1 + 2a(1+a) < 1$. This means $\left|f'\left(x_2^*\right)\right| < 1$ which implies that x_2^* is a sink. To verify this, the $f'\left(x_2^*\right) = 1 + 2a(1+a)$ was plotted for $-1 < a < 0$ which shows it is indeed smaller than one over this range of a.

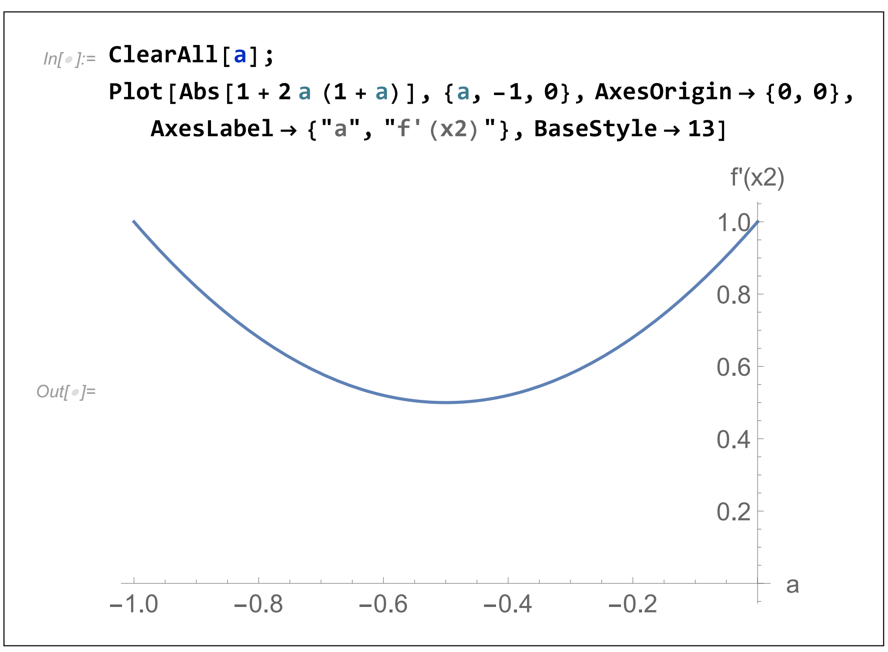

Figure 4.20: Plot of $f'(x_2^*)$ showing it is less than 1

For $x_3^* = -\sqrt{\frac{-a}{1+a}}$, evaluating $f'(x) = \frac{1-x^2}{\left(1+x^2\right)^2} - a$ found in Eq (1) above, at this fixed point gives

$$
\begin{aligned}
f'\left(x_3^*\right) &= \frac{1-\left(-\sqrt{\frac{-a}{1+a}}\right)^2}{\left(1+\left(-\sqrt{\frac{-a}{1+a}}\right)^2\right)^2} - a\\
&= \frac{1-\frac{-a}{1+a}}{\left(1+\frac{-a}{1+a}\right)^2} - a
\end{aligned}
$$

Which gives the result above x_2^* which is $f'\left(x_3^*\right) = 1 + 2a\left(1 + a\right)$. This means This means x_3^* is also a sink.

What the above analysis means, is that if we start near one of these fixed points, then map iteration (the discrete orbit sequence) will converge to the sink fixed point. For illustration, let us choose $a = -\frac{1}{2}$. For this a the fixed point is $x_2^* = \sqrt{\frac{-a}{1+a}} = \sqrt{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = 1$. We expect if we start the sequence near 1, say at 1.2, then the discrete orbit will approach 1 as more iterations of the map are made. Let us find out.

$$
\begin{aligned}
x_0 &= 1.2\\
x_1 &= f\left(x_0\right)\\
x_2 &= f\left(x_1\right)\\
x_3 &= f\left(x_2\right)\\
x_4 &= f\left(x_3\right)\\
x_5 &= f\left(x_4\right)\\
&\vdots
\end{aligned}
$$

Plugging in numerical values gives

$$
\begin{aligned}
x_0 &= 1.2\\
x_1 &= f\left(x_0\right) = \frac{x_0}{1+x_0^2} - \left(-\frac{1}{2}\right)x_0 = \frac{1.2}{1+(1.2)^2} - \left(-\frac{1}{2}\right)(1.2) = 1.0918\\
x_2 &= f\left(x_1\right) = \frac{x_1}{1+x_1^2} - \left(-\frac{1}{2}\right)x_1 = \frac{1.0918}{1+(1.0918)^2} - \left(-\frac{1}{2}\right)(1.0918) = 1.044\\
x_3 &= f\left(x_2\right) = \frac{x_2}{1+x_2^2} - \left(-\frac{1}{2}\right)x_2 = \frac{1.044}{1+(1.044)^2} - \left(-\frac{1}{2}\right)(1.044) = 1.0215\\
x_4 &= f\left(x_3\right) = \frac{x_3}{1+x_3^2} - \left(-\frac{1}{2}\right)x_3 = \frac{1.0215}{1+(1.0215)^2} - \left(-\frac{1}{2}\right)(1.0215) = 1.0106\\
x_5 &= f\left(x_4\right) = \frac{x_4}{1+x_4^2} - \left(-\frac{1}{2}\right)x_4 = \frac{1.0106}{1+(1.0106)^2} - \left(-\frac{1}{2}\right)(1.0106) = 1.0053\\
&\vdots
\end{aligned}
$$

We see that the map discrete orbit is given by

$$
1.2, 1.0918, 1.044, 1.0215, 1.0106, 1.0053, \cdots, x_2^*
$$

Where $x_2^* = 1$ in this case. The same thing will happen if we choose to start near the other fixed point x_3^* using the same a used in this example. This will now give $x_3^* = -\sqrt{\frac{-a}{1+a}} = -\sqrt{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = -1$. If we start the sequence now near -1, say at -1.2, then the discrete orbit will approach -1 as more iterations of the map are made. Let us find out.

$$
\begin{aligned}
x_0 &= -1.2\\
x_1 &= f\left(x_0\right)\\
x_2 &= f\left(x_1\right)\\
x_3 &= f\left(x_2\right)\\
x_4 &= f\left(x_3\right)\\
x_5 &= f\left(x_4\right)\\
&\vdots
\end{aligned}
$$

Plugging in numerical values gives

$$
\begin{aligned}
x_0 &= -1.2\\
x_1 &= f(x_0) = \frac{x_0}{1+x_0^2} - \left(-\frac{1}{2}\right)x_0 = \frac{-1.2}{1+(-1.2)^2} - \left(-\frac{1}{2}\right)(-1.2) = -1.0918\\
x_2 &= f(x_1) = \frac{x_1}{1+x_1^2} - \left(-\frac{1}{2}\right)x_1 = \frac{-1.0918}{1+(-1.0918)^2} - \left(-\frac{1}{2}\right)(-1.0918) = -1.044\\
x_3 &= f(x_2) = \frac{x_2}{1+x_2^2} - \left(-\frac{1}{2}\right)x_2 = \frac{-1.044}{1+(-1.044)^2} - \left(-\frac{1}{2}\right)(-1.044) = -1.0215\\
x_4 &= f(x_3) = \frac{x_3}{1+x_3^2} - \left(-\frac{1}{2}\right)x_3 = \frac{-1.0215}{1+(-1.0215)^2} - \left(-\frac{1}{2}\right)(-1.0215) = -1.0106\\
x_5 &= f(x_4) = \frac{x_4}{1+x_4^2} - \left(-\frac{1}{2}\right)x_4 = \frac{-1.0106}{1+(-1.0106)^2} - \left(-\frac{1}{2}\right)(-1.0106) = -1.0053\\
&\vdots
\end{aligned}
$$

We see that the map discrete orbit is given by

$$-1.2, -1.0918, -1.044, -1.0215, -1.0106, -1.0053, \cdots, x_3^*$$

where $x_3^* = -1$ in this case. The above verifies that x_3^*, x_2^* are fixed point of type sink.

4.3.3.2.4 Part 14

$$g(x) = (1-a)x$$

The fixed point is given by solving $(1-a)x = x$ which gives

$$x^* = 0$$

Let us apply $g(x)$. Using seed $x = \epsilon > 0$, a very small value. Therefore

$$
\begin{aligned}
x_0 &= \epsilon\\
x_1 &= g(x_0) = (1-a)x_0 = (1-a)\epsilon\\
x_2 &= g(x_1) = (1-a)x_1 = (1-a)(1-a)\epsilon = (1-a)^2\epsilon\\
x_3 &= g(x_2) = (1-a)x_2 = (1-a)(1-a)(1-a)\epsilon = (1-a)^3\epsilon\\
&\vdots\\
x_n &= g(x_n) = (1-a)x_{n-1} = (1-a)^n\epsilon
\end{aligned}
$$

Choosing $a = 2$ this results in

$$
\begin{aligned}
x_n &= g(x_n)\\
&= (1-a)^n x_{n-1}\\
&= (-1)^n\epsilon
\end{aligned}
$$

We now see that for $n = 0, x_0 = \epsilon > 0$ and for $n = 1, x_1 = -\epsilon$ and for $n = 2, x_2 = \epsilon$ and for $n = 3, x_3 = -\epsilon$ and so on. In other words, the sequence is

$$\{\epsilon, -\epsilon, \epsilon, -\epsilon, \cdots\}$$

Hence, from the above we see that the map $g(\cdot)$ has discrete period of 2. We notice also that the orbit is switching back and forth around $x^* = 0$, the fixed point found for above for $g(x)$.