HW 5

Math 5525
Introduction to Ordinary Differential Equations

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Contents

1 Problem 6.6

Consider the equation $\dot{x} = A(t)x$ with $x \in \mathbb{R}^2$ and

$$A(t) = \begin{pmatrix} \frac{1}{2} - \cos t & b \\ a & \frac{3}{2} + \sin t \end{pmatrix}$$

And a, b constants. Show that that there exists at least a one-parameter family of solutions which becomes unbounded as $t \to \infty$

solution

$\dot{x} = A(t)x$ has characteristic multiplies ρ_i and exponents λ_i. Where $\rho_i = e^{\lambda_i T}$ and T is the period of the coefficients of $A(t)$ which is

$$T = 2\pi$$

To answer this question we need to show that there is at least one characteristic exponent λ_i with real part strictly positive.

Using theorem 6.6, which applies here because $A(t)$ is periodic, it says that

$$\rho_1 \rho_2 = e^{\int_0^T \operatorname{trace}(A(\tau))d\tau} \tag{1}$$

$$\lambda_1 + \lambda_2 = \frac{1}{T}\left(\int_0^T \operatorname{trace}(A(\tau))\,d\tau\right) \operatorname{mod} \frac{2\pi i}{T} \tag{2}$$

We only need to use (2) in the above to answer this question. Trace of $A(t)$ is sum of diagonal elements of $A(t)$ which is

$$\begin{aligned} \operatorname{trace}(A(\tau)) &= \frac{1}{2} - \cos t + \frac{3}{2} + \sin t \\ &= 2 - \cos t + \sin t \end{aligned}$$

Then

$$\begin{aligned} \int_0^T \operatorname{trace}(A(\tau))\,d\tau &= \int_0^{2\pi} (2 - \cos\tau + \sin\tau)\,d\tau \\ &= [2\tau - \sin\tau - \cos\tau]_0^{2\pi} \\ &= (2(2\pi) - \sin 2\pi - \cos 2\pi) - (-\cos 0) \\ &= (4\pi - 1) - (-1) \\ &= 4\pi \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \lambda_1 + \lambda_2 &= \left(\frac{1}{2\pi} 4\pi\right) \operatorname{mod} \frac{2\pi i}{2\pi} \\ &= 2 \end{aligned}$$

Since $\Phi(t) = P(t)e^{Bt}$ where $\Phi(t)$ is the fundamental matrix and since λ_1, λ_2 are the eigenvalues of the B matrix, then we see that one solution exist which blows up. This shows there exists at least a one-parameter family of solutions which becomes unbounded as $t \to \infty$

2 Problem 8.4

Consider the system

$$\dot{x} = 2y(z-1) \tag{1}$$
$$\dot{y} = -x(z-1) \tag{2}$$
$$\dot{z} = xy \tag{3}$$

a Show that the solution $(0,0,0)$ is stable

b Is this solution asymptotically stable?

solution

2.1 Part a

Setting $x = 0, y = 0, z = 0$ gives

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \\ \dot{z} &= 0 \end{aligned}$$

Therefore $(0,0,0)$ is critical point. Eq(1)/Eq(2) gives

$$\begin{aligned} \frac{\frac{dx}{dt}}{\frac{dy}{dt}} &= \frac{2y(z-1)}{-x(z-1)} \\ \frac{dx}{dy} &= \frac{-2y}{x} \end{aligned}$$

Hence

$$-2ydy = xdx$$

Integrating gives

$$-y^2 = \frac{x^2}{2} + V_1$$

Where V_1 is integration constant. Therefore

$$\begin{aligned} V_1 &= -y^2 - \frac{x^2}{2} \\ V_1 &= y^2 + \frac{x^2}{2} \end{aligned}$$

Where the sign was absorbed in the constant. The above can be written as

$$V_1 = 2y^2 + x^2 \tag{4}$$

Where the 2 factor was absorbed in the constant.

Now solving Eq (2) for x gives, $x = \frac{\dot{y}}{-(z-1)}$ and substituting this into Eq (3) gives

$$\begin{aligned} \dot{z} &= \frac{\dot{y}}{1-z} y \\ \frac{\dot{z}}{\dot{y}} &= \frac{y}{1-z} \\ \frac{\frac{dz}{dt}}{\frac{dy}{dt}} &= \frac{y}{1-z} \\ \frac{dz}{dy} &= \frac{y}{1-z} \end{aligned}$$

Hence

$$ydy = (1-z)\,dz$$

Integrating gives

$$\frac{y^2}{2} = \left(z - \frac{z^2}{2}\right) + V_2$$

Where V_2 is the constant of integration. Therefore

$$\begin{aligned} V_2 &= \frac{y^2}{2} - z + \frac{z^2}{2} \\ V_2 &= z^2 - 2z + y^2 \end{aligned} \tag{5}$$

Let the candidate Lyapunov function (we still have to check it is indeed a Lyapunov function) be the following (per the hint given)

$$\begin{aligned} V\left(x,y,z\right) &= V_1 + (V_2 - 1)^2 \\ &= 2y^2 + x^2 + \left(z^2 - 2z + y^2 - 1\right)^2 \\ &= x^2 + y^4 + 2y^2z^2 - 4y^2z + z^4 - 4z^3 + 2z^2 + 4z + 1 \end{aligned} \tag{6}$$

We will now verify it is a Lyapunov function. The function $V\left(x,y,z\right)$ is Lyapunov function for the system if the following conditions are all met

1. $V\left(x.y,z\right)$ is continuously differentiable function in $\mathbb{R}^3$ and $V\left(x.y,z\right) \geq 0$ (positive definite or positive semidefinite) for all x,y,z away from the origin, or everywhere inside some fixed region around the origin. This function represents the total energy of the system (For Hamiltonian systems).

2. $V(0,0,0) = 0$. This says the system has no energy when it is at the equilibrium point. (rest state).

3. The orbital derivative $\frac{dV}{dt} \leq 0$ (i.e. negative definite or negative semi-definite) for all x,y,z, or inside some fixed region around the origin. The orbital derivative is same as $\frac{dV}{dt}$ along any solution trajectory. This condition says that the total energy is either constant in time (the zero case) or the total energy is decreasing in time (the negative definite case). Both of which indicate that the origin is a stable equilibrium point.

If $\frac{dV}{dt}$ is negative semi-definite then the origin is stable in Lyapunov sense. If $\frac{dV}{dt}$ is negative definite then the origin is asymptotically stable equilibrium. Negative semi-definite means the system, when perturbed away from the origin, a trajectory will remain around the origin since its energy do not increase nor decrease. So it is stable. But asymptotically stable equilibrium is a stronger stability. It means when perturbed from the origin the solution will eventually return back to the origin since the energy is decreasing. Global stability means $\frac{dV}{dt} \leq 0$ everywhere, and not just in some closed region around the origin. Local stability means $\frac{dV}{dt} \leq 0$ in some closed region around the origin. Global stability is stronger stability than local stability.

Condition (1) is satisfied $V\left(x.y,z\right) \geq 0$ (since of squares) and $V\left(0,0,\pm\left(1+\sqrt{2}\right)\right) = 0$. Hence $V\left(x,y,z\right)$ is positive semidefinite (not positive definite).

Condition (2) is easily checked is valid. Since $V = V_1 + (V_2 - 1)^2 = 0$ at $(0,0,0)$.

To check for condition (1), we see from looking at (6) that V can not be negative since $V_1 = 2y^2 + x^2$ is square quantity and $(V_2 - 1)^2$ is also square. So we need to check if $V\left(x,y,z\right)$ is always positive away from the origin. One way to do this is to find its Hessian and check if its eigenvalues. If the eigenvalues of the Hessian are all positive everywhere, then this implies V is positive definite. But we can do a short cut here. Since V is the sum of 2 square quantities, we just need to check if one of these two quantities is always positive. We do not have to check the whole V.. Let us check if V_1 is positive definite or not first. Since V_1

depends on x, y only, then

$$
\nabla V_1 = \begin{pmatrix} \frac{\partial V_1}{\partial x} \\ \frac{\partial V_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \end{pmatrix}
$$
$$
\nabla^2 V_1 = \begin{pmatrix} \frac{\partial^2 V_1}{\partial x \partial x} & \frac{\partial^2 V_1}{\partial x \partial y} \\ \frac{\partial^2 V_1}{\partial y \partial x} & \frac{\partial^2 V_1}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}
$$

Hence the eigenvalues are $2, 4$. Since these are positive everywhere, then we conclude that $V_1(x, y)$ is concave up. This means the minimum is at zero and it is positive everywhere else away from the origin. This implies that $V(x, y, z)$ is positive definite everywhere away from zero, which is what we wanted to show. Now we check the third condition $\frac{dV}{dt} \leq 0$.

The orbital derivative $\frac{dV}{dt}$ is

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} \\
&= 2x\dot{x} + 6y\dot{y} + (2z - 2)\dot{z}
\end{aligned}
$$

But $\frac{\partial V}{\partial x} = 2x$ and $\frac{\partial V}{\partial y} = 4y\left(y^2 - 2z + z^2\right)$ and $\frac{\partial V}{\partial z} = 4(z-1)\left(z^2 + y^2 - 1 - 2z\right)$. Therefore using (1,2,3) the above becomes

$$
\begin{aligned}
L_t V &= 2x\left(2y(z-1)\right) + 4y\left(y^2 - 2z + z^2\right)(-x(z-1)) + \left(4(z-1)\left(z^2 + y^2 - 1 - 2z\right)\right)xy \\
&= 0
\end{aligned}
$$

Therefore conditions 3 is also satisfied. Hence $V(x, y, z)$ is a Lyapunov function for the system and $(0, 0, 0)$ is stable equilibrium point since $\frac{dV}{dt}$ is zero. (by theorem 8.1)

2.2 Part b

By theorem 8.2, since we found from part a that $\frac{dV}{dt}$ is zero, therefore it is not negative definite but negative semi-definite, hence $(0, 0, 0)$ is not asymptotically stable (for this specific $V(x, y, z)$).

3 Problem 8.9

Determine the stability of the trivial solution of

$$
\begin{aligned}
\dot{x} &= xy^2 - \frac{1}{2}x^3 \\
\dot{y} &= -\frac{1}{2}y^3 + \frac{1}{5}x^2 y
\end{aligned}
$$

solution

Setting $x = 0, y = 0$ gives

$$
\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0
\end{aligned}
$$

Therefore $(0,0)$ is critical point. We need to find Lyapunov function. Let $V(x,y) = ax^2 + by^2$. A quadratic function. The function $V(x,y,z)$ is Lyapunov function for the system if the three conditions given in the above problem are met.

Condition (2) is clearly satisfied. Condition (1) is also satisfied since both terms are squared if we choose $a, b > 0$. Hence $V(x,y) > 0$ for non zero x, y. We now need to check the third condition. The orbital derivative $\frac{dV}{dt}$ is

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\
&= 2ax\dot{x} + 2by\dot{y} \\
&= 2ax\left(xy^2 - \frac{1}{2}x^3\right) + 2by\left(-\frac{1}{2}y^3 + \frac{1}{5}x^2 y\right) \\
&= 2ax^2y^2 - by^4 - ax^4 + \frac{2}{5}bx^2y^2 \\
&= \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right) - \left(by^4 + ax^4\right) \\
&= -\left[\left(by^4 + ax^4\right) - \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right)\right] \qquad (1)
\end{aligned}
$$

Completing the squares

$$
\begin{aligned}
\frac{dV}{dt} &= -\left[\left(\sqrt{b}y^2 - \sqrt{a}x^2\right)^2 + 2\sqrt{a}\sqrt{b}x^2y^2 - \left(2a + \frac{2}{5}b\right)\left(x^2y^2\right)\right] \\
&= -\left[\left(\sqrt{b}y^2 - \sqrt{a}x^2\right)^2 + \left(2\sqrt{a}\sqrt{b} - 2a - \frac{2}{5}b\right)\left(x^2y^2\right)\right]
\end{aligned}
$$

The above is negative definite if we can find $a, b > 0$ such that

$$
2\sqrt{a}\sqrt{b} - 2a - \frac{2}{5}b > 0
$$

Picking $a = 1, b = 2$ then left side above is

$$
2\sqrt{2} - 2 - \frac{2}{5}(2) = 0.028
$$

Hence $a = 1, b = 2$ is one choice that makes $V(x,y) = ax^2 + by^2$ a Lyapunov function. This shows that $(0,0)$ is asymptotically stable. The following is a plot of $\frac{dV}{dt}$ given in (1) to confirm it is negative definite (it is zero only at the origin, but negative everywhere else).

```
In[]:= a = 1; b = 2;
       V = a x^2 + b y^2;
       orbital = D[V, x] * (x y^2 - 1 / 2 x^3) + D[V, y] (-1 / 2 y^3 + 1 / 5 x^2 y)
```

Out[]= $2\,x\left(-\frac{x^3}{2}+x\,y^2\right)+4\,y\left(\frac{x^2\,y}{5}-\frac{y^3}{2}\right)$

```
In[]:= Plot3D[orbital, {x, -4, 4}, {y, -4, 4}]
```

Out[]=

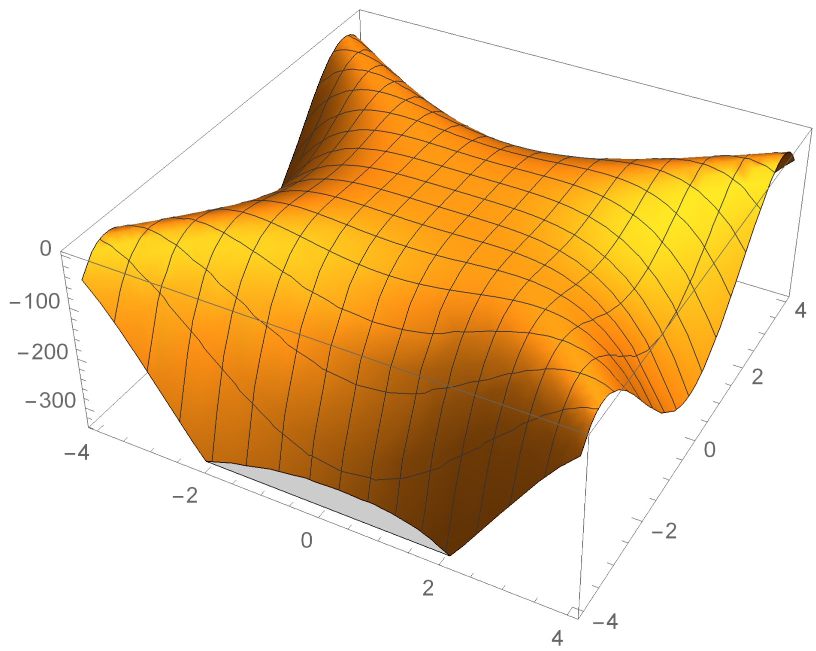

Figure 1: Showing the Orbtial derivative negative everywhere around the origin