# Assignment 4: Solutions

Date (April 23, 2020)

**Problem 1.** Consider the planar system given in polar coordinates by

$$\dot{r} = r(1-r), \quad \dot{\theta} = 1.$$

Note that $x = \cos t$, $y = \sin t$ is a periodic solution of the system (given in rectangular coordinates), satisfying $x = 1, y = 0$ at time $t = 0$. This solution belongs to the orbit $r = 1$.

1. Find the Poincaré map of the periodic solution $\phi(t) := (\cos t, \sin t)$.

2. Determine the stability of $\phi$.

**Solution.**
**1.** Let us first integrate the differential equations.

$$\begin{aligned} \int \frac{dr}{r(1-r)} =& t + C, \quad \log \frac{r}{|r-1|} = t + C, \\ \frac{r}{r-1} =& C_1 e^t, \quad r \neq 1. \end{aligned}$$

At $t = 0$, $\frac{r_0}{r_0 - 1} = C_1$. (So, $C_1 > 0$, for $r_0 > 1$, and negative otherwise.) Substituting $C_1$ as given in terms of $r_0$ into the solution, we get

$$r(t) = \frac{r_0 e^t}{r_0(e^t - 1) + 1}, \quad t > 0, \quad \theta = t + c.$$

The Poincaré map associated with the given $2\pi$-periodic solution is given as

$$P(r_0) := r(2\pi) = \frac{r_0 e^{2\pi}}{r_0(e^{2\pi} - 1) + 1} = \frac{r_0}{r_0 - (r_0 - 1)e^{-2\pi}}.$$

**2.** To determine the stability of the solution $x = \cos t$, $y = sint$ associated with the orbit $r = 1$, we calculate the iterations of the Poincaré map.

$$P^2(r_0) = P(P(r_0)) = \frac{P(r_0)}{P(r_0) - (P(r_0) - 1)e^{-2\pi}} = \frac{r_0}{r_0 + e^{-4\pi} - r_0 e^{-4\pi}}.$$

$$P^3(r_0) = P(P^2(r_0)) = \frac{P(r_0)}{P(r_0) + e^{-4\pi} - P(r_0)e^{-4\pi}} = \frac{r_0}{r_0 + e^{-6\pi} - r_0 e^{-6\pi}} = \frac{r_0}{r_0 + e^{-3.(2\pi)} - r_0 e^{-3.(2\pi)}}.$$

So,

$$P^n(r_0) = \frac{r_0}{r_0 + e^{-n.(2\pi)} - r_0 e^{-n.(2\pi)}}.$$

Hence

$$\lim_{n\to\infty} P^n(r_0) = 1.$$

That is, the Poincaré map tends to the periodic solution $r = 1$, at the limit of infinitely many iterations.

So, the solution $x = \cos t$, $y = \sin t$, corresponding to the orbit $r = 1$ is asymptotically stable.

**Problem 2.** The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. We now study a model that is more complicated than the basic SIR model (Problem 2, Assignment 3) where we now allow for the recovered population to become reinfected again (SIRS model).

Again, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population.

We still assume that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0, \tag{1}$$

that is, $S + I + R = \tau > 0$, for a given $\tau > 0$.

The SIRS model is stated as

$$\dot{S} = -\beta SI + \mu R, \tag{2}$$
$$\dot{I} = \beta SI - \nu I, \tag{3}$$
$$\dot{R} = \nu I - \mu R, \tag{4}$$

where $\nu, \mu > 0$ and $\beta > 0$ are parameters.

**1.** Use the restriction (1) to reduce the system to one with two equations and two unknown fields.

**Solution.** Integrating equation (1) we get

$$S(t) + I(t) + R(t) = \tau,$$

where $\tau > 0$, constant, is the total population of the community prior to the onset of the virus. Let us solve it for $R$ and substitute it into equation (2), which together with equation (3) give the system with two equations and two unknowns $S$ and $I$:

$$\begin{aligned}\dot{S} &= -\beta SI - \mu S - \mu I + \mu\tau,\\ \dot{I} &= \beta SI - \nu I.\end{aligned}$$

**2.** Find the critical points in two different cases: (a) $\tau \geq \frac{\nu}{\beta}$ and (b) $\tau \leq \frac{\nu}{\beta}$.

**Solution.**

We find the equilibrium solutions

$$\{S = \tau,\, I = 0\}, \quad \text{and} \quad \{S_0 := \frac{\nu}{\mu},\, I_0 := \frac{\mu}{\mu + \nu}(\tau - \frac{\nu}{\beta})\}.$$

Of course, the second solution is only valid in the case $\tau \geq \frac{\nu}{\beta}$, for which $I > 0$.

**3.** Calculate the eigenvalues associated to the critical points in each case, (a) and (b).

**Solution.**

Let $u := S - \tau$ and linearize the system about the first equilibrium point $\{I = 0,\ S = \tau\}$, giving

$$\begin{bmatrix} \dot{u} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\mu & -(\beta\tau + \mu) \\ 0 & (\beta\tau - \nu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = -\mu < 0, \quad \lambda_2 = \beta\tau - \nu.$$

Note that the solution $\{I = 0,\ S = \tau\}$ is stable if $\tau < \frac{\nu}{\beta}$ (case (b)) and unstable if $\tau > \frac{\nu}{\beta}$ (case (a)).

To examine the stability of the second equilibrium point, we call $u := S - S_0$ and $v = I - I_0$, and linearize the system about $(S_0,\ I_0)$.

The linearized system is given by

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\beta I_0 - \mu & -\beta S_0 - \mu \\ \beta I_0 & \beta S_0 - \nu \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} := A \begin{bmatrix} u \\ v \end{bmatrix}.$$

A calculation of the eigenvalues of the previous matrix shows that both eigenvalues have negative real parts, and so the equilibrium solution is *asymptotically stable.* A simple way to arrive at the latter result comes from the fact that

$$\mathrm{trace} A < 0, \quad \det A > 0.$$

**4.** In case (a), give an interpretation of the biological significance of the stability properties of the critical points.

**Solution.** Note that the equilibrium solution with $I = 0$ is stable below the population threshold $\frac{\nu}{\beta}$, in which case the disease may day out. Whereas about the population threshold, the infection will prevail and the disease may get established in the community.

*From now on, we restrict ourselves to case (a).*

**5.** Note that the SIRS system is only of interest in the region

$$\Delta := \{(I, S) :\ I, S \geq 0, \text{and}\, S + I \leq \tau\}.$$

Why?

**Solution.** $I$ and $S$ represent population groups, so they are positive, and also their sum, $I + S$, cannot be greater than the total population $\tau$.

Determine whether the $I$ axis is invariant. What about the $S$-axis? What is the behavior of the solutions restricted to the $S$-axis.

**Solution.** It is easy to see that the line $I = 0$ is invariant, since $\dot{I} = 0$ when $I = 0$, and so, a solution starting with $I(0) = 0$ will satisfy $I(t) = 0$ for all time.

Biologically, it means that, for the disease to get hold in the population, there has to be some initial infection, even if just very small.

However, the line $S = 0$ is not invariant: starting with $S(0) = 0$, it does not follow that $S(t) = 0$ for all time.

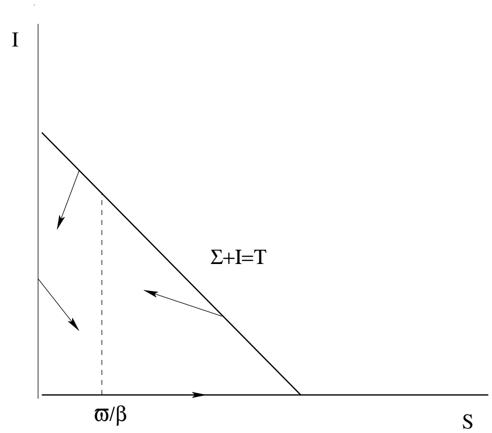


Figure 1: Invariant region $\Delta$. Note that the vector field points inwards along the three sides of the triangle.

**6.** Show that the region $\Delta$ is positively invariant.

**Solution.** We need to show that, on all three sides of the triangle $\Delta$, the vector field of the system points inwards (or tangent to the side as it is the case along $I = 0$) (figure 1).

1. Along $S = 0$: $\dot{S} > 0$ and $\dot{I} = -\nu I < 0$.
2. Along $I = 0$: $\dot{I} = 0$, $\dot{S} > 0$, with solutions tending to the stable equilibrium $\{S = \tau,\, I = 0\}$.
3. On the line $S + I = \tau$, we see that $\dot{S} = -\beta S(\tau - S) < 0$.

   We split the remaining part of the proof into two parts. First, let us consider $0 < S < \frac{\nu}{\beta}$. We see that $\dot{I} = I(\beta S - \nu) < 0$. That is, the vector field points to the left and downwards, so it is pointing towards the interior of $\Delta$. Next, let us consider $\tau > S > \frac{\nu}{\beta}$, and calculate

   $$\frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S}.$$

   Hence

   $$-1 \leq \frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S} < 0.$$

   Again, the vector field points inwards.

**Problem 3.** For each of the following systems, identify all points that lie in either an $\omega-$limit or an $\alpha$-limit set:

- $\dot{r} = r - r^3$, $\dot{\theta} = 1$.
- $\dot{r} = r^3 - 3r^2 + 2r$, $\dot{\theta} = 1$.

**Solution, part I.** First, note that the equilibrium solutions of the equation are $r = 0$ and $r = 1$, the latter is the circle of radius 1 around the origin.

The linearized equation about $r = 0$ is

$$\dot{r} = r, \quad r(t) = Ce^t,$$

so $r = 0$ is an unstable equilibrium point. *$r = 0$ is an $\alpha$-limit set.*

The linearized equation about $r = 1$ is

$$\dot{r} = 2(1 - r), \quad r(t) = 1 - Ce^{-2t}, \quad |C| \leq 1.$$

Hence, $r = 1$ is a limit cycle that positively attracts orbits. That is, *$r = 1$ is an $\omega$-limit set.*

**Solution, part II.** The equilibrium solutions of the second equation are

$$r = 0, \ r = 1 \text{ and } r = 2.$$

The linearized equation about $r = 0$ is $\dot{r} = 2r$ showing that $r = 0$ is an unstable equilibrium point, and so *$r = 0$ is an $\alpha$-limit set of the solutions.*

The linearized equation about $r = 1$ is $\dot{r} = -(r - 1)$, and so *the limit cycle $r = 1$ an $\omega$-limit set.*

The linearized equation about $r = 2$ is $\dot{r} = 2(r - 2)$, and so *the limit cycle $r = 2$ an $\alpha$-limit set.*