Math 5525 - Solution set 1.

1. $\dot{x} = ax\left(1 - \frac{x}{N}\right)$, $N > 0$, integer, $a > 0$

(1) $\iff \dot{x} = \alpha x (N - x)$, $\alpha \equiv \frac{a}{N}$.

$$\frac{1}{x(N-x)} = \frac{A}{x} + \frac{B}{N-x} = \frac{A(N-x) + Bx}{x(N-x)} \implies A = B = 1/N$$

(partial fractions)

$$\frac{1}{N}\int^{x} \frac{dx}{x} + \frac{dx}{N-x} = \frac{1}{N} \ln\left|\frac{x}{N-x}\right| = \alpha t + K_0$$

K_0 arbitrary const.

$$\ln\left(\frac{x}{N-x}\right) = at + K, \quad \frac{x}{N-x} = Ce^{at}$$

$0 < x < N$

$$t = 0: \quad C = \frac{1}{\frac{N}{x_0} - 1} = \frac{x_0}{N - x_0}$$

$$\therefore \quad x = \frac{NCe^{at}}{1 + Ce^{at}} = \frac{Ne^{at}}{1/C + e^{at}} = \frac{N}{\frac{1}{C}e^{-at} + 1}$$

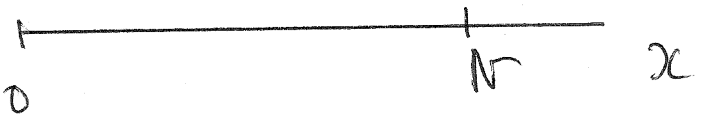
$$= \frac{N}{\frac{N - x_0}{x_0} e^{-at} + 1}$$

$x_0 < N$

Note that $x(t) \to N$ as $t \to \infty$.

(2) $x = 0$, $x = N$ are equilibrium solutions.

(3)



Note that $\dot{x}(t) > 0$ for $0 < x < N$

Hence, solutions with initial data $0 < x_0 < N$ satisfy $\dot{x}(t) > 0$, $\forall t$, and since $x = N$ is an equilibrium point $x(t) \xrightarrow[t \to \infty]{} N$.

$\therefore$ $x = N$ is a stable equilibria

$x = 0$ " an unstable " .

(4) Plot.

2.
$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

Equilibrium points satisfy $y = 0$ and $x - x^3 = 0$

$0 = x - x^3 = x(1 - x^2) \implies x = 0,\ x = \pm 1$

$(0, 0)$, $(1, 0)$, $(-1, 0)$.

To classify the equilibrium points, we study the linearized system about equilibrium.

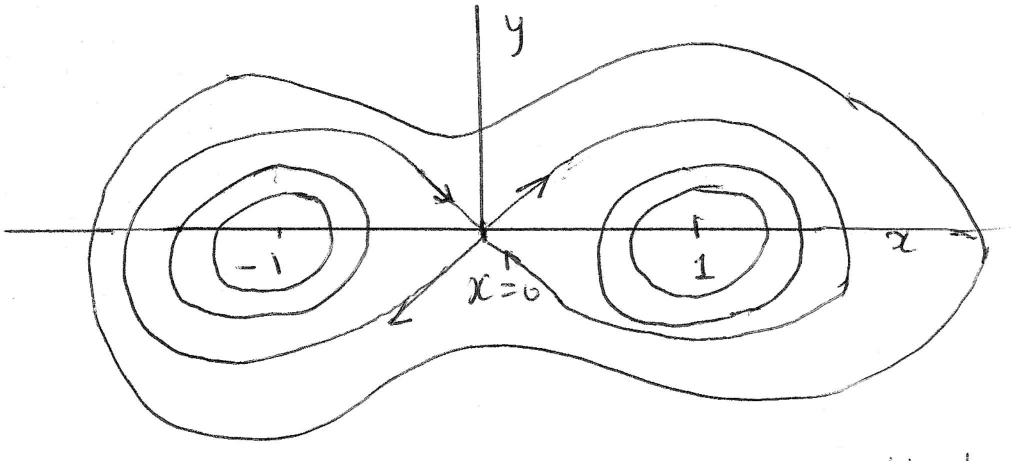
First, denote $g(x) = -x^3 + x$ and $x = x^*$ a value of x corresponding to equilibrium.

$x^* = \pm 1 \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$\lambda = \pm\sqrt{2}i$ (eigenvalues)

$\begin{bmatrix} 1 \\ \pm\sqrt{2}i \end{bmatrix}$ (eigenvectors)

$(1, 0)$ and $(-1, 0)$ are centers.



To plot the phase plane, note that the system has a first integral (energy integral)
Indeed, it can be written as

$$\ddot{x} = \dot{y} = x - x^3$$

$$\dot{x}\,\ddot{x} = (x - x^3)\,\dot{x} \implies \frac{1}{2}(\dot{x})^2 + \frac{1}{2}\left(\frac{x^2}{2} - 1\right)x^2 = E$$

constant.

Equivalently, the equations of the orbits are $\frac{1}{2}y^2 + \frac{1}{2}\left(\frac{x^2}{2} - 1\right)x^2 = E$.

Taylor expansion of $g(x)$ about $x = x^*$:

$$g(x) = \underbrace{g(x^*) + g'(x^*)\,(x - x^*)}_{\text{Linearization}} + o(x - x^*)$$

Moreover, since $g(x^*) = 0$, we have

$$g(x) = g'(x^*)\,(x - x^*) + o(x - x^*)$$

Linearized system about equilibrium:

[Notation $u_1 = x - x^*$
$u_2 = y$].

$$\dot{u}_1 = u_2$$
$$\dot{u}_2 = g'(x^*)\,u_1$$

$$\to \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g'(x^*) & 0 \end{bmatrix}}_{\equiv A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$x^* = 0$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Eigenvalues $\lambda = \pm 1$

Eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
$(\lambda = 1)$ $(\lambda = -1)$

$\therefore$ $(0,0)$ is a saddle point of the system.

$x^* = \pm 1$, $g'(x^*) = 1 - 3(x^*)^2 = 1 - 3(\pm 1)^2 = -2$.

3. $$\dot{x}_1 = x_1 - x_1 x_2 - x_2^3 + x_3(x_1^2 + x_2^2 - 1 - x_1 + x_1 x_2 + x_2^3)$$
$$\dot{x}_2 = x_1 - x_3(x_1 - x_2 + 2x_1 x_2)$$
$$\dot{x}_3 = (x_3 - 1)(x_3 + 2x_3 x_2^2 + x_3^3)$$

Equilibrium points: $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3$ the system are (Right hand side expressions of the system are identically 0).

Start with 3th equation: $x_3 = 0$ and $x_3 = 1$ make the RHS equals to 0.

$\underline{x_3 = 0}$:
$$\dot{x}_1 = x_1 - x_1 x_2 - x_2^3$$
$$\dot{x}_2 = x_1$$

$x_1 = 0$, $x_2 = 0$

(0, 0, 0) equilibrium point

$\underline{x_3 = 1}$: (a) $\dot{x}_1 = x_1 - x_1 x_2 - x_2^3 + (x_1^2 + x_2^2 - 1 - x_1 + x_1 x_2 + x_2^3)$
$$= x_1^2 + x_2^2 - 1$$

(b) $\dot{x}_2 = x_1 - (x_1 - x_2 + 2x_1 x_2) = x_2 - 2x_1 x_2$

$\dot{x}_2 = 0 \Rightarrow x_2 = 0$ or $x_1 = 1/2$

$x_2 = 0 \xrightarrow{\text{eqn (a)}} x_1^2 - 1 = 0$, $x_1 = \pm 1$

$x_1 = 1/2 \xrightarrow{\text{eqn (a)}} x_2^2 = 1 - \frac{1}{4} = \frac{3}{4}$, $x_2 = \pm\frac{\sqrt{3}}{2}$

$\boxed{(\pm 1, 0, 1)}$

$\boxed{(1/2, \pm\frac{\sqrt{3}}{2}, 1)}$

Invariant sets:

$x_3 = 0$ or $x_3 = 1$

Solutions with initial data $(x_1^0, x_2^0, 0)$ and $(x_1^0, x_2^0, 1)$ will satisfy $x_3(t) = 0$ and $x_3(t) = 1$, respectively, for all $t \geq 0$.

$\therefore$ The planes $x_3 = 0$ and $x_3 = 1$ are invariant sets of the system.

The system restricted to the plane $x_3 = 1$ is

$$\dot{x}_1 = x_1^2 + x_2^2 - 1$$
$$\dot{x}_2 = x_2 - 2x_1 x_2$$

Equilibrium solutions of the plane system:

$$(1/2, 0), \quad \left(\pm 1, \pm \frac{\sqrt{3}}{2}\right)$$

$(1/2, 0)$: unstable (special case of one zero eigenvalue)

$\left(+1, \frac{\sqrt{3}}{2}\right)$: unstable spiral

$\left(-1, -\frac{\sqrt{3}}{2}\right)$: saddle point

The absence of centers indicates that the system does not have any closed orbits.