HW 1

Math 5525
Introduction to Ordinary Differential Equations

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Contents

1 Problem 1

The logistic population growth model is given by the first order, nonlinear differential equations

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

Where $x = x(t)$ denotes the number of individuals of a population group at time $t \geq 0$. $N > 0$, integer, is the carrying capacity, that is, the maximum number of individuals that the environment allows (e.g. based on available resources, such as food, access to water,...). The positive number a represents the growth rate. (1) Obtain the exact solution of the equation (1) corresponding to the initial data $x(0) = x_0$ where $0 < x_0 < N$. (2) Obtain the equilibrium solutions of the problem. (3) Determine the stability of the equilibrium solutions. (4) Let $N = 100$. Plot the solution corresponding to initial data $x(0) = 50$.

Solution

1.1 part 1

$$\begin{aligned}\frac{dx}{dt} &= ax\left(1 - \frac{x}{N}\right)\\ x(0) &= x_0\end{aligned}$$

This is separable first order ODE. Therefore

$$\frac{dx}{ax\left(1 - \frac{x}{N}\right)} = dt$$

Integrating both sides gives

$$\begin{aligned}\int_{x_0}^{x} \frac{dz}{az\left(1 - \frac{z}{N}\right)} dz &= \int_0^t d\tau\\ \frac{1}{a}\int_{x_0}^{x} \frac{dz}{z\left(1 - \frac{z}{N}\right)} dz &= t \qquad (1)\end{aligned}$$

Applying partial fractions to $\frac{1}{z\left(1-\frac{z}{N}\right)}$ gives

$$\frac{1}{z\left(1 - \frac{z}{N}\right)} = \frac{A}{z} + \frac{B}{1 - \frac{z}{N}}$$

Hence $A = \left.\frac{1}{\left(1-\frac{z}{N}\right)}\right|_{z=0} = 1$ and $B = \left.\frac{1}{z}\right|_{z=N} = \frac{1}{N}$. Therefore $\frac{1}{z\left(1-\frac{z}{N}\right)} = \frac{1}{z} + \frac{1}{N}\frac{1}{1-\frac{z}{N}} = \frac{1}{z} + \frac{1}{N-z}$ and (1) now becomes

$$\begin{aligned}\frac{1}{a}\int_{x_0}^{x} \frac{1}{z} + \frac{1}{N-z} dz &= t\\ \int_{x_0}^{x} \frac{1}{z} dz + \int_{x_0}^{x} \frac{1}{N-z} dz &= at\end{aligned}$$

But $\int \frac{1}{z} dz = \ln|z|$ and $\int \frac{1}{N-z} dz = -\ln|N-z|$ and the above becomes

$$\begin{aligned}\ln\left|\frac{x}{x_0}\right| - \ln\left|\frac{N-x}{N-x_0}\right| &= at\\ \ln\left|\frac{\frac{x}{x_0}}{\frac{N-x}{N-x_0}}\right| &= at\\ \ln\left|\frac{x(N-x_0)}{x_0(N-x)}\right| &= at\\ \ln\left|\frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right)\right| &= at\end{aligned}$$

Since $N>0$ and $N>x_0$ and $x_0>0$ and since N is the carrying capacity, then hence $N-x>0$), therefore $\left|\frac{N-x_0}{N-x}\right|$ is positive. The absolute sign can be removed and the above simplifies to

$$\ln\frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right)=at$$

Taking the exponential of both sides gives

$$\begin{aligned}\frac{x}{x_0}\left(\frac{N-x_0}{N-x}\right)&=e^{at}\\ x\left(N-x_0\right)&=\left(N-x\right)x_0e^{at}\\ x\left(N-x_0\right)&=Nx_0e^{at}-xx_0e^{at}\\ x\left(N-x_0\right)+xx_0e^{at}&=Nx_0e^{at}\\ x\left(N-x_0+x_0e^{at}\right)&=Nx_0e^{at}\\ x\left(t\right)&=\frac{Nx_0e^{at}}{N-x_0+x_0e^{at}}\end{aligned}$$

Dividing RHS numerator and denominator by e^{at} gives the analytical solution as

$$x\left(t\right)=\frac{Nx_0}{x_0+\left(N-x_0\right)e^{-at}}\qquad a>0$$

1.2 part 2

Equilibrium solution is when $\frac{dx}{dt}=0$, which implies $ax\left(1-\frac{x}{N}\right)=0$. This gives $\underline{x=0}$ or $1-\frac{x}{N}=0$ which gives $\underline{x=N}$.

1.3 part 3

Let

$$\begin{aligned}\frac{dx}{dt}&=f\left(x\right)\\ &=ax\left(1-\frac{x}{N}\right)\end{aligned}$$

Hence

$$\begin{aligned}f'\left(x\right)&=a\left(1-\frac{x}{N}\right)+ax\left(-\frac{1}{N}\right)\\ &=a-a\frac{x}{N}-a\frac{x}{N}\\ &=a-2a\frac{x}{N}\end{aligned}\tag{1}$$

When $x=0$ the above shows that $f'\left(x\right)=a>0$ since a is always positive. Since the slope of $f\left(x\right)$ is positive then $x=0$ is <u>unstable</u> equilibrium.

At $x=N$ then (1) becomes $f'\left(x\right)=a-2a=-a$. Since the slope of $f\left(x\right)$ is negative then $x=N$ is <u>stable</u> equilibrium.

1.4 part 4

When $N=100$,$x\left(0\right)=50$, the solution found above $x\left(t\right)=\frac{Nx_0}{x_0+\left(N-x_0\right)e^{-at}}$ now becomes

$$\begin{aligned}x\left(t\right)&=\frac{\left(100\right)\left(50\right)}{50+\left(100-50\right)e^{-at}}\\ &=\frac{5000}{50\left(1+e^{-at}\right)}\\ &=\frac{100}{1+e^{-at}}\end{aligned}$$

The above shows that as $t \to \infty$ and since $a > 0$ then $x(t) \to 100$ which is N, the limiting capacity as expected. This is plot of the above for different $a > 0$ numerical values.

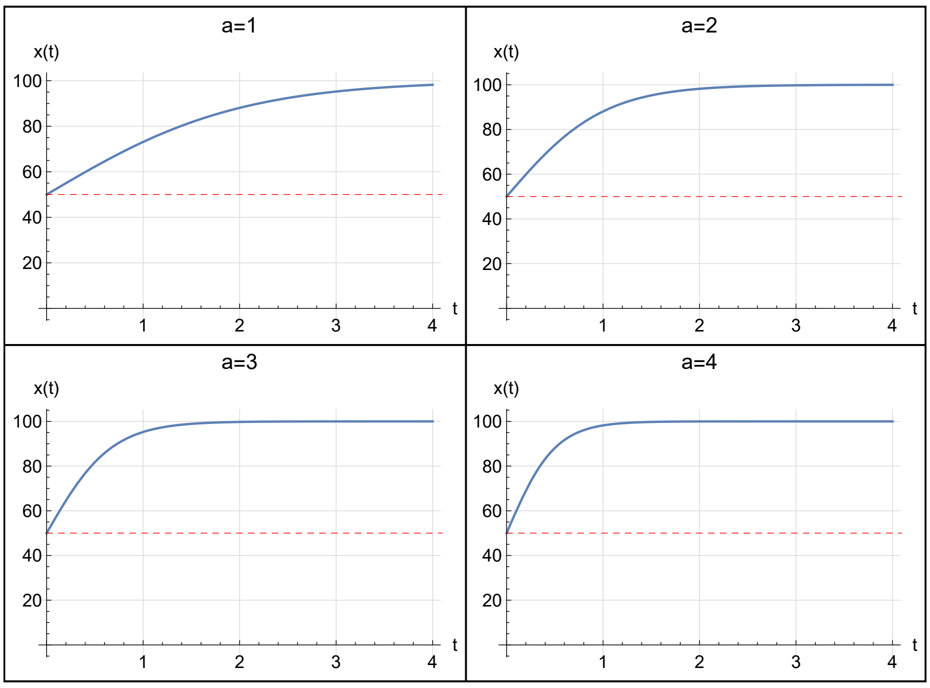

Figure 1: Solution $x(t)$ for different a values

```
x[t_, a_] := 100/(1 + Exp[-a t])
p = Table[Plot[x[t, a], {t, 0, 4}, AxesOrigin → {0, 0},
            PlotLabel → Row[{"a=", a}],
            ImageSize → 300,
            AxesLabel → {"t", "x(t)"},
            BaseStyle → 12,
            GridLines → Automatic,
            GridLinesStyle → LightGray,
            Epilog → {Red, Dashed, Line[{{0, 50}, {5, 50}}]}
          ], {a, {1, 2, 3, 4}}];
p = Grid[Partition[p, 2], Frame → All];
```

Figure 2: Code used for the above plot

<u>Observations</u> As the growth rate a increases in value, the population $x(t)$ reaches its limiting value $N = 100$ more rapidly as expected. The line shown in dashed red is the initial population size of 50. Once limiting population size if reached, the population size do not change any more with time.

2 Problem 2

(1) Solve exercise 2.5, page 24, of the textbook: Find the critical points of the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - 2x^3\end{aligned}$$

Characterize the critical points by linear analysis and determine their attraction properties.
(2) Plot the phase plane of the system.

Solution

2.1 Part 1

This is non-linear second order system.

$$\begin{aligned}\dot{x} &= f_1 = y\\ \dot{y} &= f_2 = x\left(1 - 2x^2\right)\end{aligned}$$

The critical points are $y = 0$ and $x\left(1 - 2x^2\right) = 0$ or $x = 0$ and $1 - 2x^2 = 0$ which gives $x^2 = \frac{1}{2}$ or $x = \pm\frac{1}{\sqrt{2}}$. Hence there are 3 critical points are

$$\left(x, y\right) = \left\{(0,0), \left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)\right\}$$

To find the if critical points are stable or not, the system is linearized the system and each eigenvalue is examined. The Jacobian matrix of linearized system is

$$\begin{aligned}J &= \begin{pmatrix}\frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y}\\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y}\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1 - 6x^2 & 0\end{pmatrix}\end{aligned}$$

Point $(0,0)$ At this point the Jacobian matrix becomes $J = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Its eigenvalues are found from $|\det(A) - \lambda I| = 0$ which gives

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= 0\\ \lambda^2 - 1 &= 0\end{aligned}$$

Hence $\lambda = \pm 1$. Since one of the eigenvalues is positive, then $(0,0)$ is unstable and the whole system is considered unstable. The second (negative) eigenvalue is stable, which leads to $(0,0)$ being saddle point. (which is considered unstable)

Point $\left(\frac{1}{\sqrt{2}}, 0\right)$ At this point $J = \begin{pmatrix}0 & 1\\ 1 - 6x^2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 - \frac{6}{2} & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}$. The eigenvalues are

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ -2 & -\lambda\end{vmatrix} &= 0\\ \lambda^2 + 2 &= 0\\ \lambda^2 &= -2\end{aligned}$$

The solution is $\lambda_1 = -\sqrt{2}i, \lambda_2 = \sqrt{2}i$. Since this is pure complex conjugate (zero real part) then the critical point is central point considered stable point (sometimes also called marginally stable). The solutions around this point are periodic.

Point $\left(\frac{-1}{\sqrt{2}},0\right)$ At this point $J=\begin{pmatrix}0 & 1\\ 1-\frac{6}{2} & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1-6x^2 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}$. This is the same as the above.

The eigenvalues are $\lambda_1=-\sqrt{2}i,\lambda_2=\sqrt{2}i$. Which lead to a central point. The solutions around this point are periodic.

2.2 Part 2

Writing

$$\begin{aligned}\dot{x}&=f_1=y\\ \dot{y}&=f_2=x-2x^3\end{aligned}$$

The actual phase plane orbit equation can be found by solving $\frac{dy}{dx}=\frac{f_2}{f_1}=\frac{x-2x^3}{y}$ or $ydy=\left(x-2x^3\right)dx$. Integrating gives

$$\begin{aligned}\frac{1}{2}y^2&=\left(\frac{1}{2}x^2-\frac{2}{3}x^4\right)+C\\ \frac{1}{2}y^2-\frac{1}{2}x^2+\frac{2}{3}x^4&=C\end{aligned}$$

For different constant C, different orbit result. But instead of plotting the above equation for different C, the phase plot is generated using two methods as it was not clear which method to use.

First method This is the manual method. The system is linearized as above, and for each critical point, the eigenvectors are found. From first part we found that at Point $(0,0)$. The eigenvalues are $\lambda=\pm1$. Hence for $\lambda=1$ the system $(A-\lambda I)\,v=0$ where v is the eigenvector corresponding to λ becomes

$$\begin{aligned}\begin{pmatrix}0-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

Which gives from first equation $-v_1+v_2=0$ or $v_1=v_2$. By assuming $v_1=1$, the first eigenvector is $\begin{pmatrix}1\\ 1\end{pmatrix}$. For $\lambda=-1$ the system becomes

$$\begin{aligned}\begin{pmatrix}0-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

Which gives $v_1+v_2=0$ or $v_2=-v_1$. By assuming $v_1=1$, the second eigenvector is $\begin{pmatrix}1\\ -1\end{pmatrix}$. Now the direction along along each eigenvector is found. Starting with the first eigenvector $\begin{pmatrix}1\\ 1\end{pmatrix}$ which spans from first quadrant to 3rd quadrant. We recall that the system is

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=x\left(1-2x^2\right)\end{aligned}$$

In first quadrant, $y>0$. This means from above that $\dot{x}>0$ which means x is increasing in first quadrant. Also in first quadrant, $x>0$ which means when x is close to zero such that $1-2x^2$ is positive, then $\dot{y}>0$ from the second equation above, which means y is increasing also in first quadrant. In the 3rd quadrant, $y<0$ which means $\dot{x}<0$ and hence x

is decreasing. Also, in 3rd quadrant $y < 0$ which means for x close to zero $\dot{y} < 0$ and hence y is decreasing as well. This means that the first eigenvector points away from the origin in first and third quadrant.

The second eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ extends from second quadrant to 4th quadrant. In 4th quadrant, $y < 0$ hence $\dot{x} < 0$ which means x is decreasing (getting closer to the origin). In the 4th quadrant, $x > 0$ which for x close to zero such that $1 - 2x^2$ remain positive, $\dot{y} > 0$ and hence y is increasing (getting closer to origin). Now, in the second quadrant, $y > 0$ which means $\dot{x} > 0$ which means x is increasing (getting closer to origin) and in the second quadrant $x < 0$ hence for values of x near zero, $\dot{y} < 0$ which means y is decreasing (getting closer to origin). Therefore on the second eigenvector all solutions move closer to origin (this is stable eigenvector). This is how the phase plot looks now around $(0, 0)$

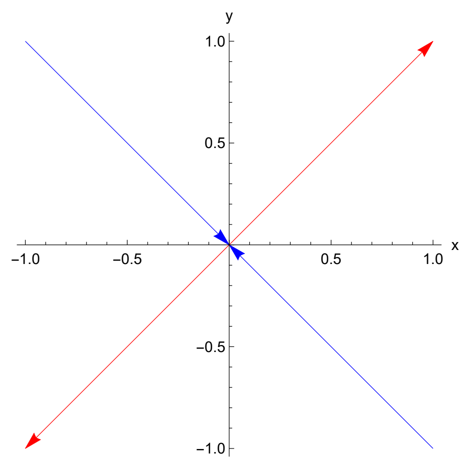

Figure 3: Manually making phase plot around $(0, 0)$

The next step is to manually draw the rest of the phase plot lines in each of the four regions as follows by continuity

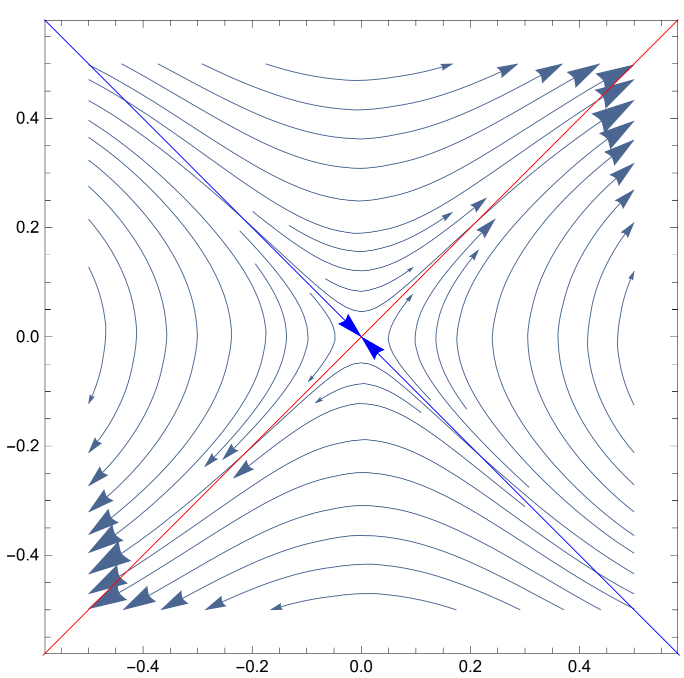

Figure 4: Manually making phase plot around $(0, 0)$

The same thing is done for each remaining critical point $\left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)$ in the manual method and will not be repeated as the same steps as above.

Now the <u>second method</u> is applied, which is to numerically generate phase plot directly for the non-linear system.

The 3 critical points are marked on the following plot. Unable point is colored in red and the stable critical points are colored in green. The plot below shows that $(0,0)$ is unstable (saddle) as shown above using the manual method and the points $\left(\pm\frac{-1}{\sqrt{2}},0\right)$ are central since solutions move around it in circular orbits. (Periodic solutions).

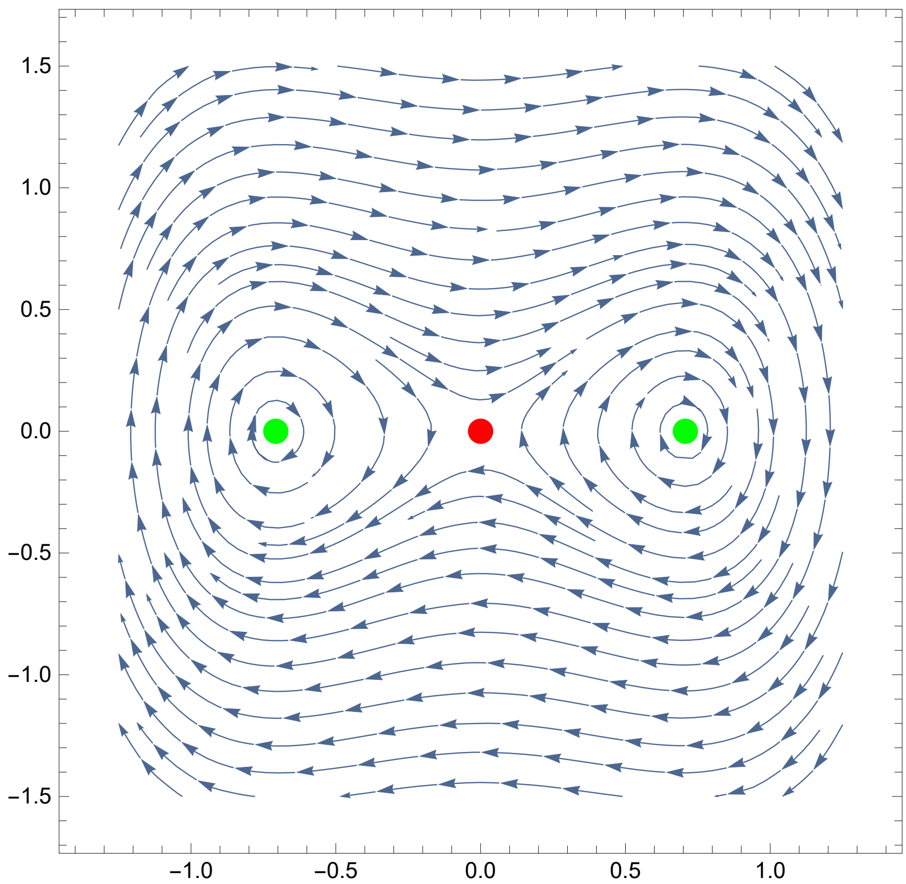

Figure 5: Phase plot

```
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Green, PointSize[0.03], Point[{1/Sqrt[2], 0}]};
p3 = {Green, PointSize[0.03], Point[{-1/Sqrt[2], 0}]};
p = StreamPlot[{y, x - 2 x^3}, {x, -1.25, 1.25}, {y, -1.5, 1.5},
        Epilog → {p1, p2, p3}
     ];
```

Figure 6: Code used for the above plot

3 Problem 3

(1) Solve exercise 2.3, page 23, of the textbook: We are studying the three-dimensional system

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right) \\ \dot{x}_2 &= x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right) \\ \dot{x}_3 &= \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)\end{aligned} \tag{A}$$

(a) Determine the critical points of this system. (b) Show that the planes $x_3 = 0$ and $x_3 = 1$ are invariant sets. (c) Consider the invariant set $x_3 = 1$. Does this set contain periodic solutions?

(2) Plot the phase plane of the system

Solution

3.1 Part 1.a

From the third equation, let $\dot{x}_3 = 0$, then

$$\left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right) = 0$$

Hence $\underline{x_3 = 1}$ or $x_3\left(1 + 2x_2^2 + x_3^2\right) = 0$, which gives additional solutions $\underline{x_3 = 0}$ or $\left(1 + 2x_2^2 + x_3^2\right) = 0$. When $x_3 = 0$ this becomes $1 + 2x_2^2 = 0$ which does not give real solution in x_2. Now, when $x_3 = 1$ then $\left(1 + 2x_2^2 + x_3^2\right) = 0$ gives $2 + 2x_2^2 = 0$ which also do not give real solution.

Considering now the first and second equations in (A) and set each to zero which gives

$$\begin{aligned}x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right) &= 0 \\ x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right) &= 0\end{aligned} \tag{1}$$

When $x_3 = 0$ the above becomes

$$\begin{aligned}x_1 - x_1x_2 - x_2^3 &= 0 \\ x_1 &= 0\end{aligned}$$

Which gives solutions $x_1 = 0$ from the second equation. This results in $x_2 = 0$ from the first equation. Hence the point $\underline{(0,0,0)}$ is the first critical point.

Now, when $x_3 = 1$ EQ (1) becomes

$$\begin{aligned}x_1 - x_1x_2 - x_2^3 + x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3 &= 0 \\ x_1 - \left(x_1 - x_2 + 2x_1x_2\right) &= 0\end{aligned}$$

Or

$$\begin{aligned}x_1^2 + x_2^2 - 1 &= 0 \\ x_2 - 2x_1x_2 &= 0\end{aligned}$$

Or

$$\begin{aligned}x_1^2 + x_2^2 - 1 &= 0 \\ x_2\left(1 - 2x_1\right) &= 0\end{aligned}$$

From the second equation above $x_2 = 0$ or $x_1 = \frac{1}{2}$. When $x_2 = 0$ the first equation above gives $x_1 = \pm 1$. Hence second critical point is $\underline{(\pm 1, 0, 1)}$. And when $x_1 = \frac{1}{2}$ the first equation gives

$$\begin{aligned}\frac{1}{4} + x_2^2 - 1 &= 0 \\ x_2^2 &= \frac{3}{4} \\ x_2 &= \pm\frac{\sqrt{3}}{2}\end{aligned}$$

Hence critical point is $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 1\right)$

In summary, the critical points are

$$(0,0,0), (\pm 1, 0, 1), \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 1\right)$$

3.2 Part 1.b

A set S is invariant, if when initial conditions are in S, then the overall solution remain in S for all time. From the third equation in (A) for $\dot{x}_3$

$$\begin{aligned}\dot{x}_3 &= (x_3 - 1)\left(x_3 + 2x_3x_2^2 + x_3^3\right) \\ &= (x_3 - 1)\, x_3 \left(1 + 2x_2^2 + x_3^2\right)\end{aligned}$$

The above shows that when $\dot{x}_3 = 0$, then $x_3 = 1$ or $x_3 = 0$ are the solutions. The term $1 + 2x_2^2 + x_3^2 = 0$ does not give real solutions hence not used. So only $x_3 = 1, x_3 = 0$ are only possible solutions. Therefore these are invariant sets. Any solution with initial conditions $x_3 = 0$ or $x_3 = 0$ will remain in the set $x_3 = 0$ or $x_3 = 0$ respectively.

3.3 Part 1.c

When $x_3 = 1$ the system reduces to

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3 \\ \dot{x}_2 &= x_1 - (x_1 - x_2 + 2x_1x_2)\end{aligned}$$

Or

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2^2 - 1 \\ \dot{x}_2 &= x_2\,(1 - 2x_1)\end{aligned} \tag{1}$$

To see if there are periodic solutions, the phase plot is drawn around the critical points $(\pm 1, 0), \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ to see if there are closed orbits or not. Here is the result for the set $x_3 = 1$

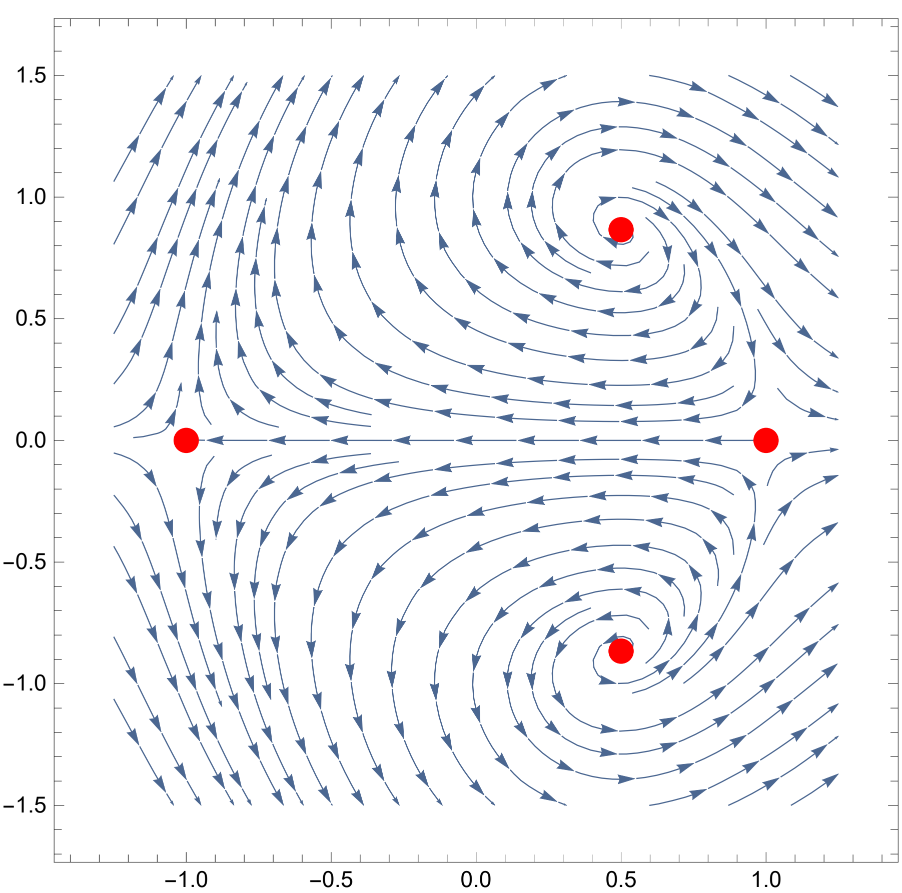

Figure 7: Phase plot when $x_3 = 1$

The above shows that there are no closed orbits. This implies no periodic solutions exist.

Another way to find this without using the computer, is to do the following: We linearize the system (1) and then determine the eigenvalues for each critical point. Since this is second order system, then only eigenvalues that pair of complex conjugate will indicate a periodic solution (which is consider to be stable). To linearize (1), we first find the Jacobian matrix, which is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 & 2x_2 \\ -2x_2 & 1-2x_1 \end{pmatrix} \end{aligned}$$

Critical point $(1,0)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which has the $-1,1$. Not stable. No periodic solutions around this point.

Critical point $(-1,0)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

Which has eigenvalues $3,-2$. Not stable. No periodic solutions around this point.

Critical point $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}$$

Which has eigenvalues: $\frac{1}{2}+\frac{1}{2}i\sqrt{11},\frac{1}{2}-\frac{1}{2}i\sqrt{11}$. This is not not stable because the real part is positive, hence can not be periodic.

Critical point $\left(\frac{1}{2},\frac{-\sqrt{3}}{2}\right)$

The Jacobian matrix at this point becomes

$$\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$$

Which has same eigenvalues as above $\frac{1}{2}+\frac{1}{2}i\sqrt{11},\frac{1}{2}-\frac{1}{2}i\sqrt{11}$. This is not not stable because the real part is positive, hence can not be periodic. Therefore we see that no periodic solutions exist.

3.4 Part 2

In part 1 above, the phase plot for the set $x_3 = 1$ is already given. The following is the phase plot for the set $x_3 = 0$. When $x_3 = 0$ the system reduces to

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1 x_2 - x_2^3 \\ \dot{x}_2 &= x_1 \end{aligned}$$

The critical points are $x_1 = 0$ from the second equation and from the first equation this gives $x_2 = 0$. Hence $(0, 0)$ is the only critical point. To determine if stable or not, the Jacobian is found

$$
\begin{aligned}
J &= \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} \\
&= \begin{pmatrix} 1 - x_2 & -x_1 - 3x_2 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

Evaluated at $x_1 = 0, x_2 = 0$ the above becomes

$$
J = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}
$$

Eigenvalues are

$$
\begin{aligned}
\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} &= 0 \\
\lambda^2 &= 0 \\
\lambda &= 0
\end{aligned}
$$

Double root. Since the system is nonlinear, and since $\lambda = 0$, then unable to determine stability of the non-linear system from the linearized system. Will have to use the phase plot to check stability of $(0, 0)$ as given below.

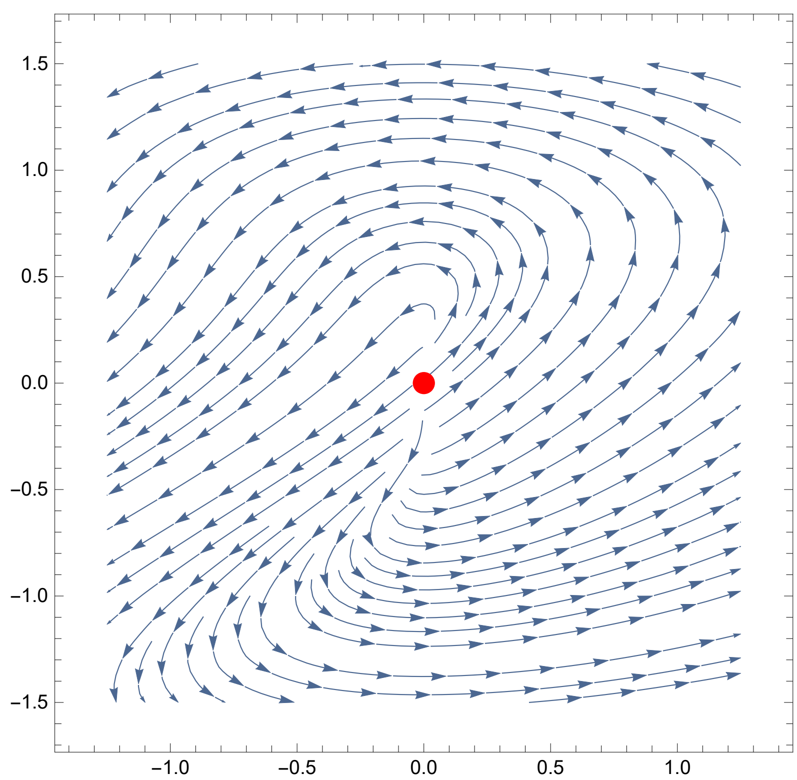

Figure 8: Phase plot when $x_3 = 0$

```
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p =
    StreamPlot[{x1 - x1 * x2 - x2^3, x1}, {x1, -1.25, 1.25}, {x2, -1.5, 1.5}, Epilog → {p1}];
```

Figure 9: Code used for the above

From the above phase plot, it shows that $(0, 0)$ critical point is <u>not stable</u> because solutions that starts near $(0, 0)$ move away from equilibrium.