University Course

EE 3015
Signals and systems

University of Minnesota, Twin Cities
Spring 2020

My Class Notes

Nasser M. Abbasi

Spring 2020

Contents

Chapter 1

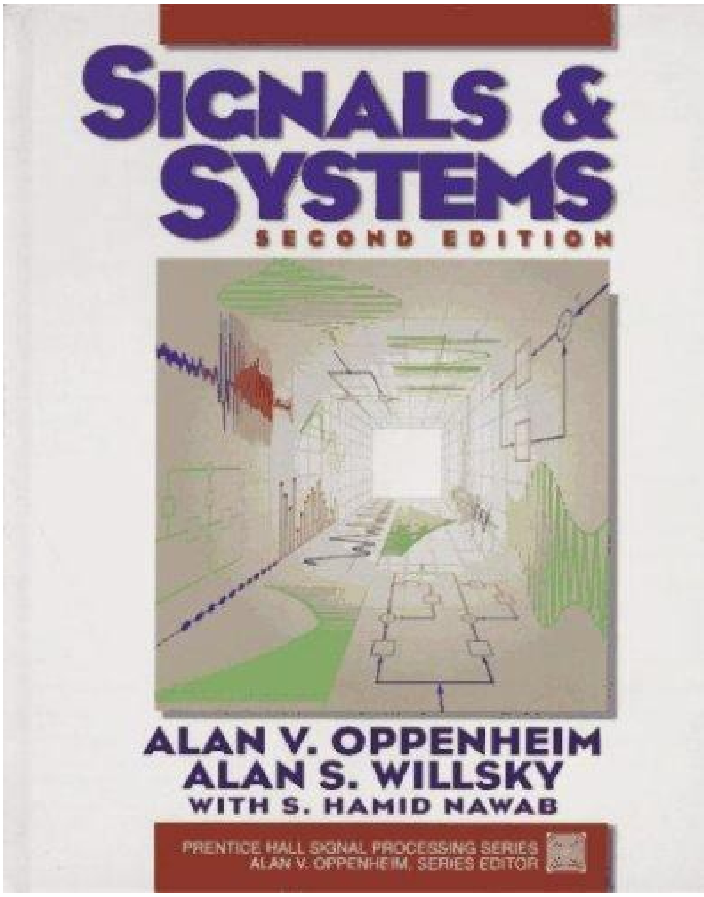
Introduction

Local contents

1.1 Links

1. class web page (needs login) `https://canvas.umn.edu/courses/158230`
2. Media page (needs login) `https://www.unite.umn.edu/secure/Spring20/EE3015/`

1.2 Text book



1.3 syllabus

EE3015: Signals & Systems, Spring 2020

Course Description

Basic techniques for analysis/design of signal processing, communications, and control systems. Time/frequency models, Fourier-domain representations, modulation. Discrete-time/digital signal/system analysis. Z transform. State models, stability, feedback.
Prereq: [EE 2011, CSE Upper Division] or dept consent

Instructor

A. B (Bob) Mahmoodi

Office:	Keller Hall 2-115
Phone:	612-625-3027
Email:	mahmo006@umn.edu
Office Hours:	M W F 9:00-10:00 am

Lecture

M W F	10:10 – 11: 05 am	Keller Hall 3-125

Discussion Sections

Wednesdays	Sec. 002: A. B. Mahmoodi	12:20 – 1:10 pm	Vincent Hall 213
Wednesdays	Sec. 003: A. B. Mahmoodi	1:25 – 2:15 pm	Akerman Hall 215

All discussion sections will start on January 29th.

Teaching Assistants

Omer Burak Demirel

Email:	demir035@umn.edu
Office Hours:	Thursday Friday 9:00-10am (Keller Hall 2-276)

Jack Erhardt

Email:	erhar057@umn.edu
Office Hours:	M Tues 12:00-1:00pm (Keller Hall 2-276)

Text Book

Signals & Systems 2nd ed. Oppenheim & Willsky & Nawab

Computer Software

MATLAB Student version, latest version (this software is available to all CSE students)

Topics covered

Chapter 1 (1/22, 1/24)

Introduction (Sections 1.1-1.7); Continuous and discrete-time signals; Operations on signals; Properties of signals; Elementary signals, Continuous- and discrete-time systems; Interconnections of systems; System Properties; Intro to Convolution

Chapter 2 (1/27 thru 1/31)

Time Domain Representations for Linear Time Invariant Systems (Sections 2.1-2.5); Convolution; Properties of convolution; Difference and differential equations (characterizing solutions, block diagrams & interconnections)

Chapter 3 & 4 (2/3 thru 2/21)

Fourier Representations of Signals (Sections 3.1-3.11); continuous time Fourier series & transform and properties (sections 4.1 – 4.7)

Midterm 1, Friday Feb 28th, Chapters 1, 2, 3, 4

Chapter 5 & 6 & 7 (2/24 thru 3/27)

Applications of Fourier Representations (discrete time); Frequency response; Fourier transform representation for discrete-time signals (sections 5.1- 5.8); Application to filters (sections (6.1 – 6.7); Sampling continuous-time signals; Reconstruction of continuous-time signals from samples (sections 7.1-7.5)

Chapter 9 (3/30 thru 4/3)

The Laplace Transform, definition and convergence properties (Sections 9.1-9.9); Inversion; Solving Differential Equations; Transform Analysis of Systems.

Midterm 2, Friday April 3rd, Chapters 5, 6, 7

Chapter 10 (4/6 thru 4/17)

Intro to Z transform (sections 10.1-10.9); inverse Z transform properties and existence of the transform; Applications in digital signal processing.

Chapter 8 (4/20 thru 4/24)

Introduction to Communication Systems (sections 8.1-8.9); Modulation application

Chapter 11 (4/27 4/29)

Intro to Feedback System (11.1-11.5) (If time permits, review otherwise. Typically this is covered in detail in your control systems course.

Review (5/1, 5/4) Last day of class is on 5/4.

Final Exam: 1:30 – 3:30 pm Saturday May 9th, Keller Hall 3-125

Homework Assignments

Homework is assigned every week and is due the following week. The grader will grade a random selection of problems out of all assigned. 50% of your homework problems will be graded for correctness of your solution, and the remaining 50% will be graded based on an attempt at a solution (please show your work). No late homework will be accepted, except in emergency situations.

Please scan your homework as a PDF file and submit it via canvas to our TA by the due date. Paper submissions will not be accepted.

Several assigned problems require the use of MATLAB. This software package is available to all CSE students, and on the CSE lab computers.

Grading Policy

Midterm I	25%	2/28, in lecture
Midterm II	25%	4/3, in lecture
Final Exam	35%	5/9, 1:30 – 3:30 pm
Homework	10%	
Discussion Quiz	5%	

Other Important Information

Student Academic Integrity and Scholastic Dishonesty:

Academic integrity is essential to a positive teaching and learning environment. All students enrolled in University courses are expected to complete coursework responsibilities with fairness and honesty. Failure to do so by seeking unfair advantage over others or misrepresenting someone else's work as your own, can result in disciplinary action. The University Student Conduct Code defines scholastic dishonesty as follows:

Scholastic Dishonesty:

Scholastic dishonesty means plagiarizing; cheating on assignments or examinations; engaging in unauthorized collaboration on academic work; taking, acquiring, or using test materials without faculty permission; submitting false or incomplete records of academic achievement; acting alone or in cooperation with another to falsify records or to obtain dishonestly grades, honors, awards, or professional endorsement; altering, forging, or misusing a University academic record; or fabricating or falsifying data, research procedures, or data analysis.

Within this course, scholastic dishonesty includes, but is not limited to, looking at and/or copying from another's exam, using unauthorized note sheets during exams, any unauthorized communication during exams (including verbal and/or electronic communications), etc. A student responsible for scholastic dishonesty can be assigned a penalty up to and including an "F" or "N" for the course. For additional information, refer to the student conduct code available here:
http://regents.umn.edu/sites/default/files/policies/Student_Conduct_Code.pdf

Disability Accommodations:

The University of Minnesota views disability as an important aspect of diversity, and is committed to providing equitable access to learning opportunities for all students. The Disability Resource Center (DRC) is the campus office that collaborates with students who have disabilities to provide and/or arrange reasonable accommodations. Additional information is available on the DRC website:
https://diversity.umn.edu/disability/

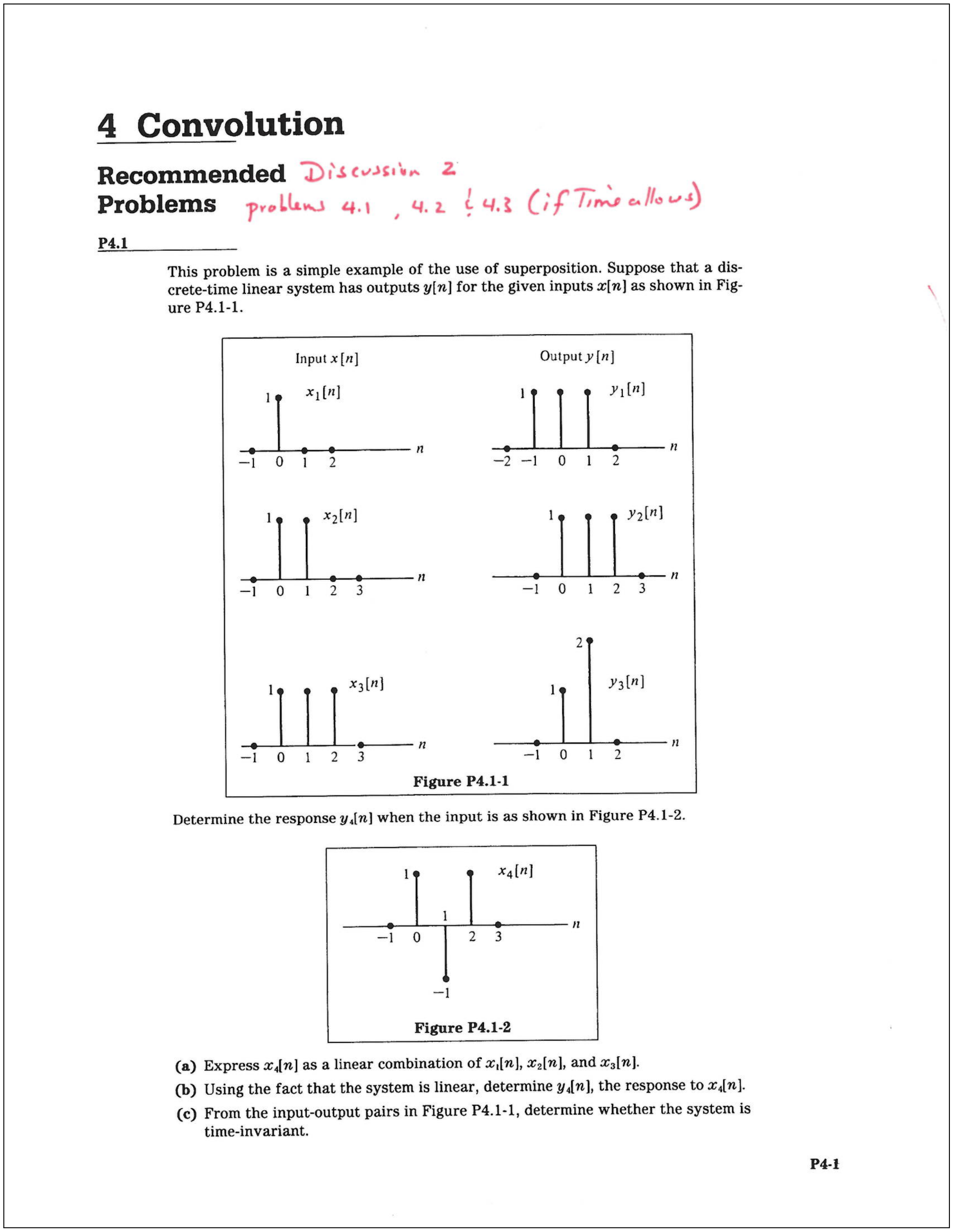
Chapter 2

My solutions to some discussion problems

Local contents

2.1 Discussion, second week

2.1.1 Questions



4 Convolution

Recommended Problems Discussion 2 problems 4.1 , 4.2 & 4.3 (if Time allows)

P4.1

This problem is a simple example of the use of superposition. Suppose that a discrete-time linear system has outputs $y[n]$ for the given inputs $x[n]$ as shown in Figure P4.1-1.

Figure P4.1-1

Determine the response $y_4[n]$ when the input is as shown in Figure P4.1-2.

Figure P4.1-2

(a) Express $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

(b) Using the fact that the system is linear, determine $y_4[n]$, the response to $x_4[n]$.

(c) From the input-output pairs in Figure P4.1-1, determine whether the system is time-invariant.

Signals and Systems
P4-2

P4.2

Determine the discrete-time convolution of $x[n]$ and $h[n]$ for the following two cases.

(a)

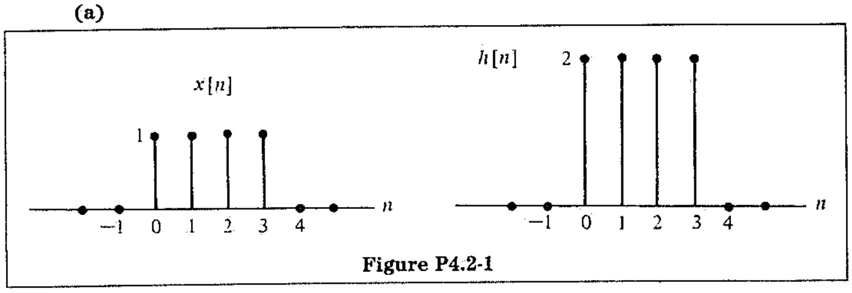


Figure P4.2-1

(b)

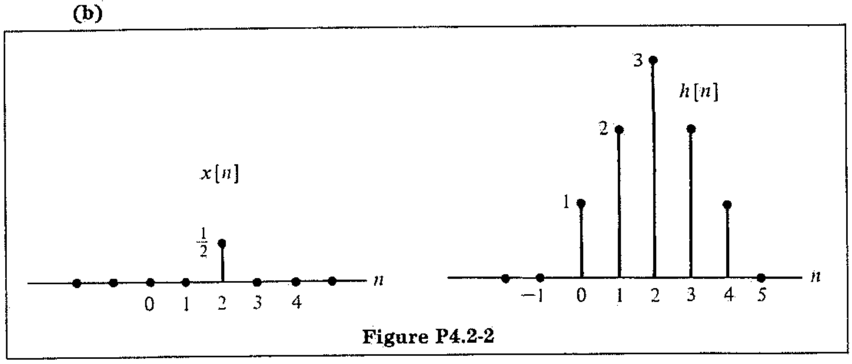


Figure P4.2-2

P4.3

Determine the continuous-time convolution of $x(t)$ and $h(t)$ for the following three cases:

(a)

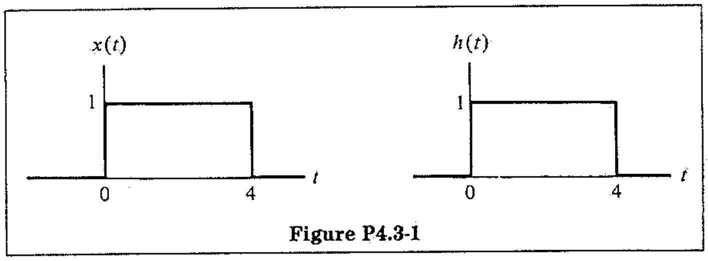


Figure P4.3-1

(b)

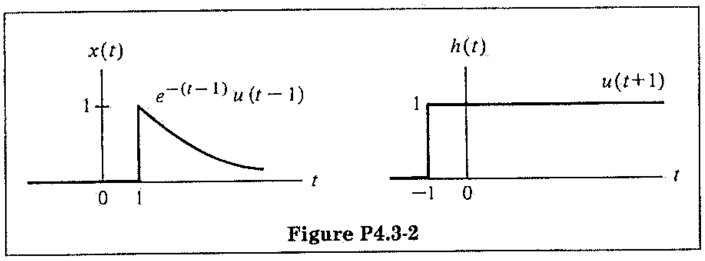


Figure P4.3-2

(c)

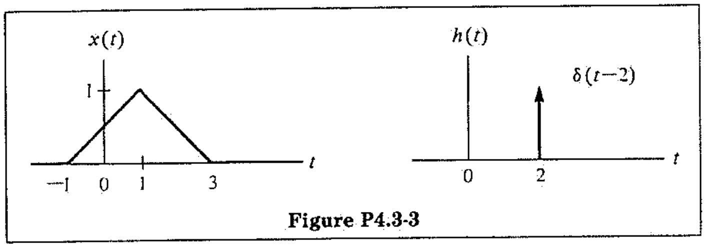


Figure P4.3-3

P4.4

Consider a discrete-time, linear, shift-invariant system that has unit sample response $h[n]$ and input $x[n]$.

(a) Sketch the response of this system if $x[n] = \delta[n - n_0]$, for some $n_0 > 0$, and $h[n] = (\frac{1}{2})^n u[n]$.

(b) Evaluate and sketch the output of the system if $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = u[n]$.

(c) Consider reversing the role of the input and system response in part (b). That is,

$$h[n] = u[n],$$
$$x[n] = (\tfrac{1}{2})^n u[n]$$

Evaluate the system output $y[n]$ and sketch.

P4.5

(a) Using convolution, determine and sketch the responses of a linear, time-invariant system with impulse response $h(t) = e^{-t/2}\,u(t)$ to each of the two inputs $x_1(t)$, $x_2(t)$ shown in Figures P4.5-1 and P4.5-2. Use $y_1(t)$ to denote the response to $x_1(t)$ and use $y_2(t)$ to denote the response to $x_2(t)$.

2.1.2 Problem 4.1

Solution

2.1.2.1 part a

We need to find linear combination of $x_1[n], x_2[n], x_3[n]$ which gives $x_4[n]$. In other words, looking at samples at $n = 0,1,2$ and adding corresponding samples gives

$$\begin{aligned} a+b+c &= 1 \\ b+c &= -1 \\ c &= 1 \end{aligned}$$

But from second equation $b = -1-1 = -2$ and from first equation $a = 1-b-c = 1+2-1 = 2$. Hence

$$2x_1[n] - 2x_2[n] + x_3[n] = x_4[n]$$

2.1.2.2 part b

Therefore by linearity

$$2y_1[n] - 2y_2[n] + y_3[n] = y_4[n]$$

Hence

$$y_4[n] = 2\delta[n+1] + \delta[n] + 2\delta[n-1] - 2\delta[n-2]$$

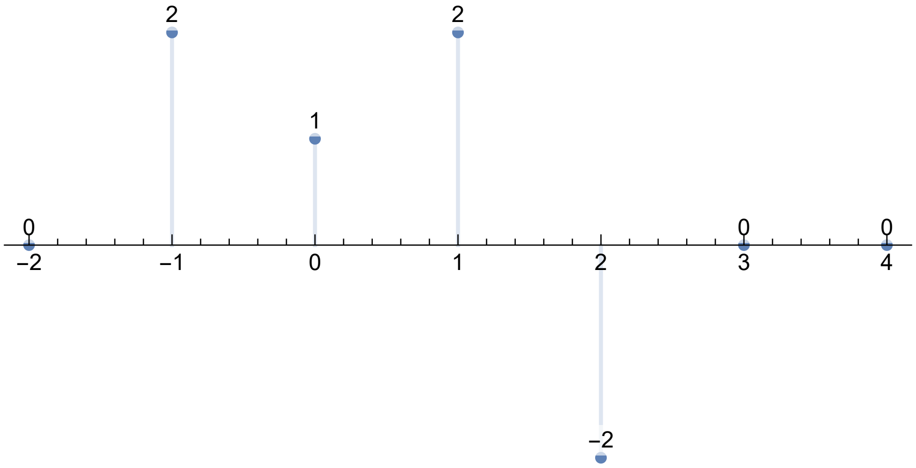


Figure 2.1: Plot of $y[n]$

2.1.2.3 part c

System is time invariant if shifted input gives same output but also shifted by the same amount as the input is shifted by. Let us consider $x_1[n]$. By shifting it to the right by one, then the output should $y_1[n]$ but shifted to the right by one which is $y_1[n-1]$

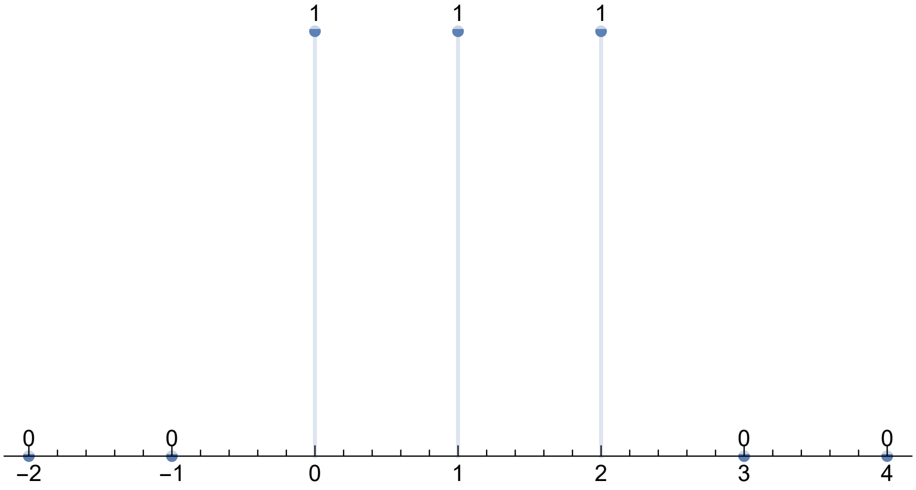


Figure 2.2: Plot of $y_1[n-1]$

Shifting $x_1\,[n]$ by 2 now the output should be $y_1\,[n-2]$

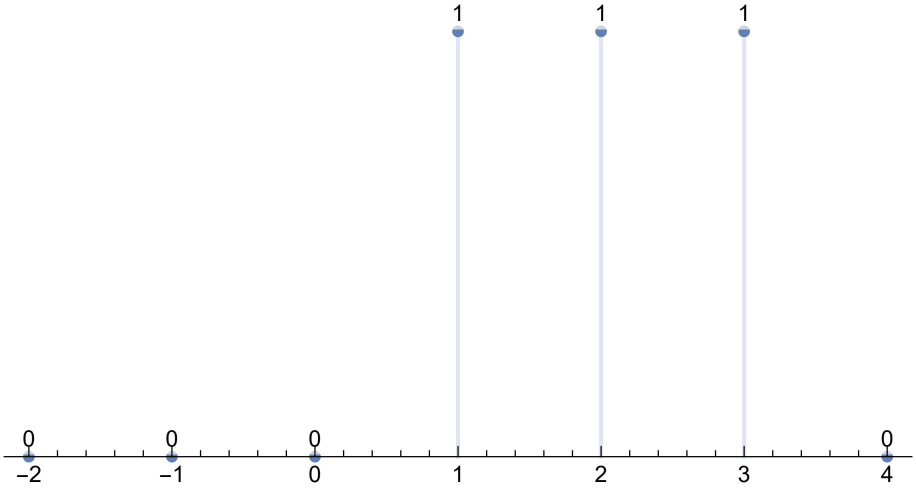


Figure 2.3: Plot of $y_1[n-2]$

But adding $x_1\,[n]+x_1\,[n-1]+x_1\,[n-2]$ gives $x_3\,[n]$. Which has the output shown. Let us now add $y_1\,[n]+y_1\,[n-1]+y_1\,[n-2]$ and see if this gives same as $y_3\,[n]$

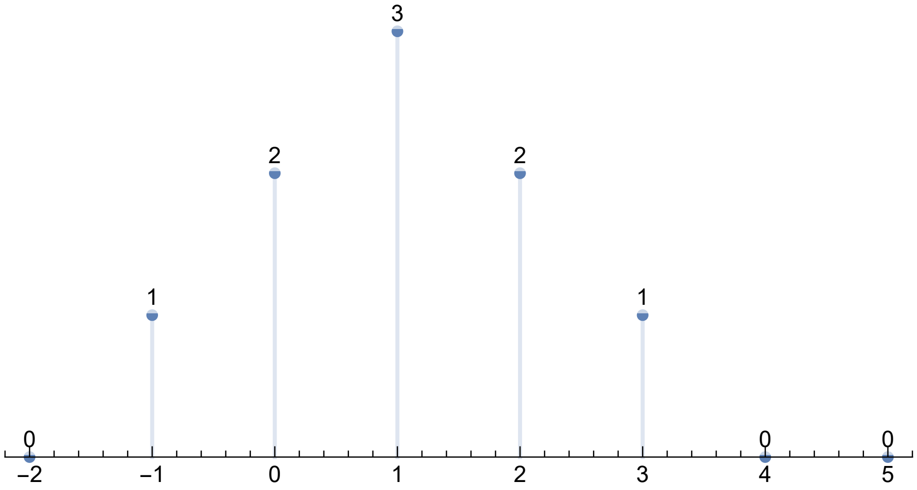


Figure 2.4: Plot of all shifted inputs of $x_1[x]$

Since the above is not the same as $y_3[n]$ then the system is not time invariant.

2.1.3 Problem 4.2

Solution

2.1.3.1 Part a

By folding $x[n]$ and shifting to the right, we see that $y[0] = 2, y[1] = 2+2 = 4, y[2] = 2+2+2 = 6, y[3] = 8, y[4] = 6, y[5] = 4, y[6] = 2, y[7] = 0$ and $y[n] = 0$ for all other values.

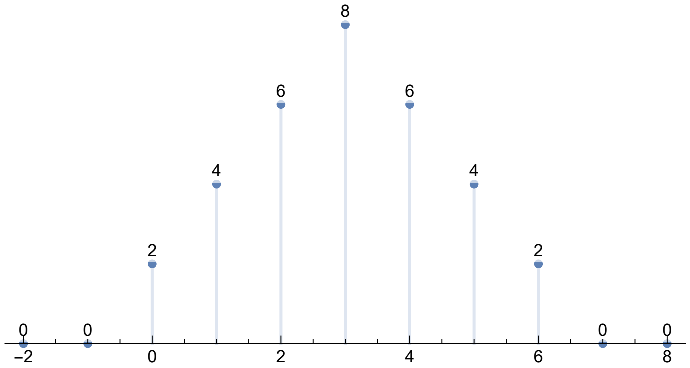


Figure 2.5: $y[n]$

2.1.3.2 Part b

By folding $x[n]$ and shifting to the right, we see that $y[0] = 0, y[1] = 0, y[2] = 0.5, y[3] = 1, y[4] = 1.5, y[5] = 1, y[6] = 0.5, y[7] = 0$ and $y[n] = 0$ for all other values.

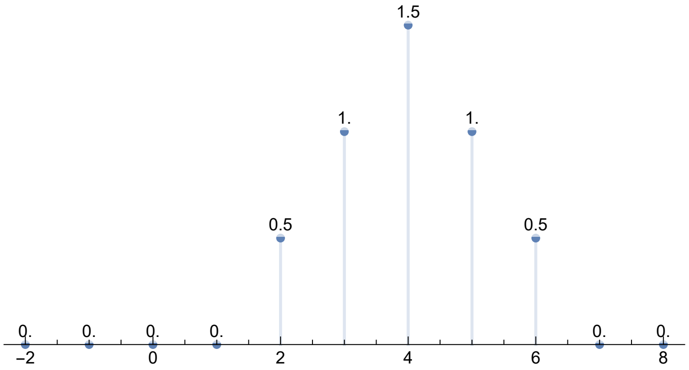


Figure 2.6: $y[n]$

2.1.4 Problem 4.3

Solution

2.1.4.1 Part a

By folding $x(t)$ and shifting, we see that for $t<0$ that $y(t)=0$. And for $0<t<4$ the integral becomes

$$\begin{aligned} y(t) &= \int_0^t h(\tau)\,d\tau \qquad 0<t<4 \\ &= \int_0^t 1 d\tau \\ &= t \end{aligned}$$

And for $0<t-4<4$ or $4<t<8$

$$\begin{aligned} y(t) &= \int_{t-4}^4 h(\tau)\,d\tau \qquad 4<t<8 \\ &= \int_{t-4}^4 1 d\tau \\ &= 4-(t-4) \\ &= 8-t \end{aligned}$$

And for $4<t-4$ or $t>8$

$$y(t)=0$$

Hence $y(t)$ is

$$y(t)=\begin{cases} 0 & t<0 \\ t & 0<t<4 \\ 8-t & 4<t<8 \\ 0 & t>8 \end{cases}$$

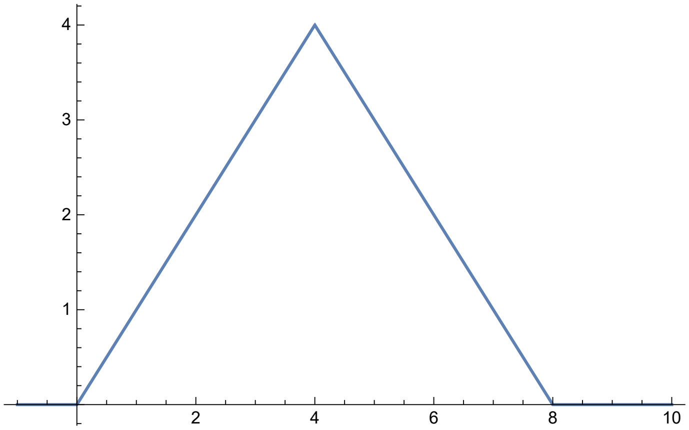


Figure 2.7: $y(t)$

2.1.4.2 Part b

By folding $h(t)$ and shifting, we see that for $t < 0$ that $y(t) = 0$. And for $t > 0$ the integral becomes

$$\begin{aligned}
y(t) &= \int_1^{1+t} h(\tau)\, d\tau \qquad t > 0\\
&= \int_1^{1+t} e^{-(\tau-1)} d\tau\\
&= \left[\frac{e^{-(\tau-1)}}{-1}\right]_1^{1+t}\\
&= -\left[e^{-(\tau-1)}\right]_1^{1+t}\\
&= -\left[e^{-((1+t)-1)} - e^{-(1-1)}\right]\\
&= -\left[e^{-t} - 1\right]\\
&= 1 - e^{-t}
\end{aligned}$$

Hence $y(t)$ is

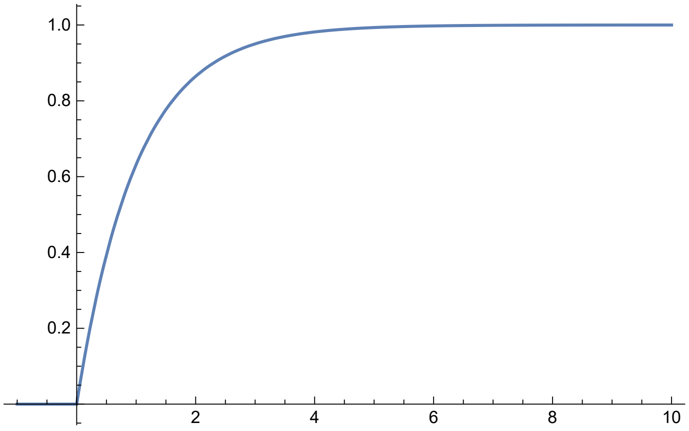


Figure 2.8: $y(t)$

2.1.4.3 Part c

By folding $h(t)$ and shifting, we see that for $-2+t<-1$ or $t<1$ that $y(t)=0$. And for $-1<-2+t<3$ or $1<t<5$ the integral becomes $x(t)$ itself (i.e. original $x(t)$ but shifted to right by 2). And for $3<-2+t$ or $t>5$ then $y(t)=0$. Hence

$$y(t)=\begin{cases} 0 & t<1 \\ x(t-2) & 1<t<5 \\ 0 & t>5 \end{cases}$$

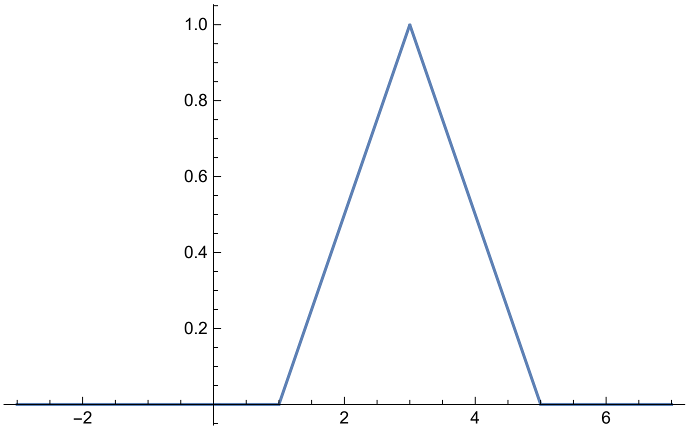


Figure 2.9: $y(t)$

2.1.5 key solution

4 Convolution

Solutions to Recommended Problems

Discussion 2 Solution

S4.1

The given input in Figure S4.1-1 can be expressed as linear combinations of $x_1[n]$, $x_2[n]$, $x_3[n]$.

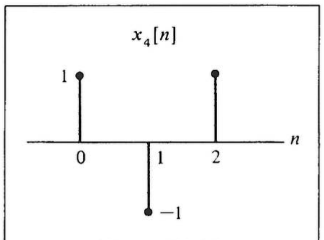


Figure S4.1-1

(a) $x_4[n] = 2x_1[n] - 2x_2[n] + x_3[n]$

(b) Using superposition, $y_4[n] = 2y_1[n] - 2y_2[n] + y_3[n]$, shown in Figure S4.1-2.

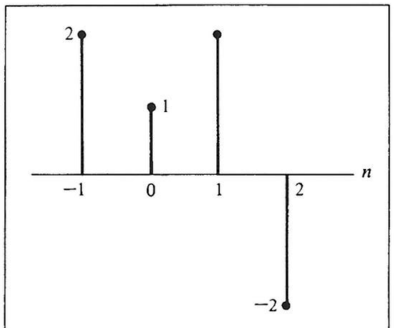


Figure S4.1-2

(c) The system is not time-invariant because an input $x_1[n] + x_1[n-1]$ does not produce an output $y_1[n] + y_1[n-1]$. The input $x_1[n] + x_1[n-1]$ is $x_1[n] + x_1[n-1] = x_2[n]$ (shown in Figure S4.1-3), which we are told produces $y_2[n]$. Since $y_2[n] \neq y_1[n] + y_1[n-1]$, this system is not time-invariant.

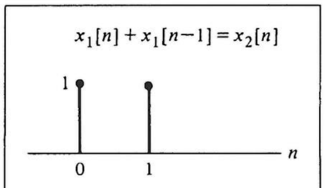


Figure S4.1-3

S4.2

The required convolutions are most easily done graphically by reflecting $x[n]$ about the origin and shifting the reflected signal.

(a) By reflecting $x[n]$ about the origin, shifting, multiplying, and adding, we see that $y[n] = x[n] * h[n]$ is as shown in Figure S4.2-1.

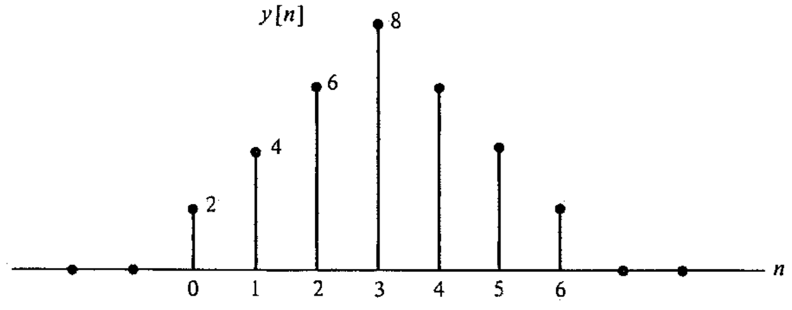


Figure S4.2-1

(b) By reflecting $x[n]$ about the origin, shifting, multiplying, and adding, we see that $y[n] = x[n] * h[n]$ is as shown in Figure S4.2-2.

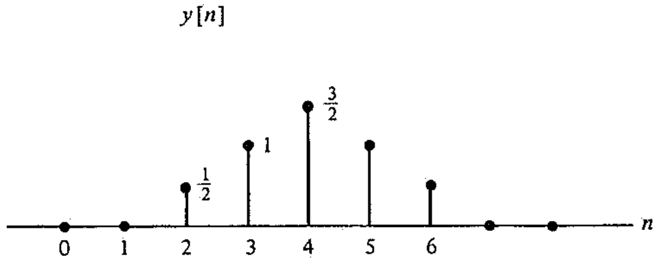


Figure S4.2-2

Notice that $y[n]$ is a shifted and scaled version of $h[n]$.

S4.3

(a) It is easiest to perform this convolution graphically. The result is shown in Figure S4.3-1.

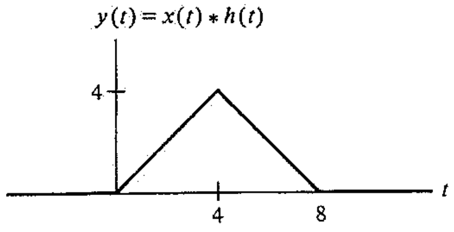


Figure S4.3-1

(b) The convolution can be evaluated by using the convolution formula. The limits can be verified by graphically visualizing the convolution.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h\,(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-(\tau-1)}u(\tau-1)u(t-\tau+1)d\tau \\ &= \begin{cases} \int_{1}^{t+1} e^{-(\tau-1)}d\tau, & t > 0, \\ 0, & t < 0, \end{cases} \end{aligned}$$

Let $\tau' = \tau - 1$. Then

$$y(t) = \begin{cases} \int_{0}^{t} e^{-\tau'}d\tau' \\ 0 \end{cases} = \begin{cases} 1 - e^{-t}, & t > 0, \\ 0, & t < 0 \end{cases}$$

(c) The convolution can be evaluated graphically or by using the convolution formula.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-2)\,d\tau = x(t-2)$$

So $y(t)$ is a shifted version of $x(t)$.

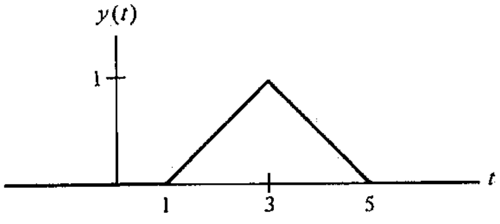
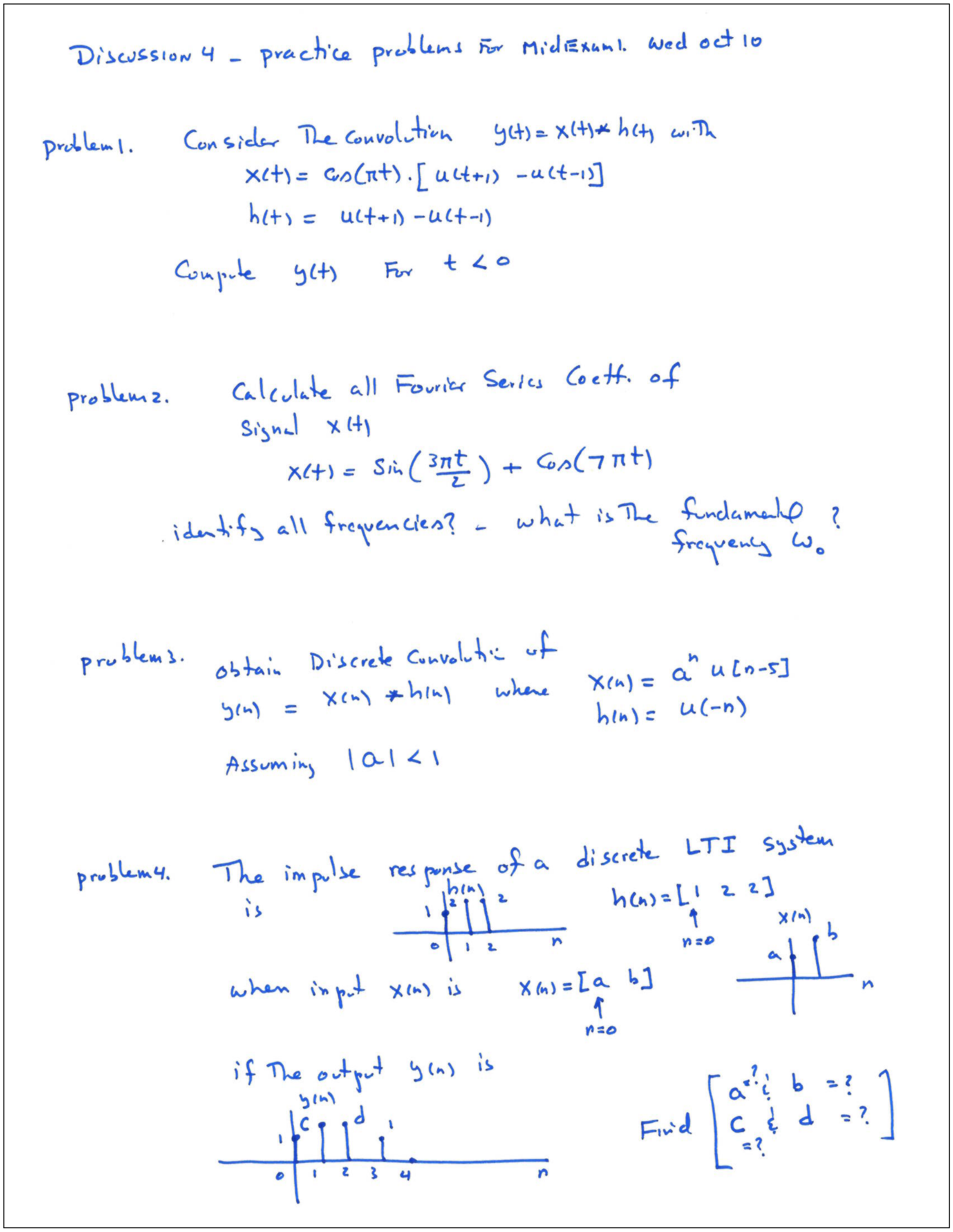


Figure S4.3-2

2.2 Discussion, week 3

2.2.1 Questions



Discussion 4 - practice problems For MidExam1. wed oct 10

problem1. Consider The convolution $y(t) = x(t) * h(t)$ with

$x(t) = \cos(\pi t) . [u(t+1) - u(t-1)]$

$h(t) = u(t+1) - u(t-1)$

Compute $y(t)$ For $t < 0$

problem2. Calculate all Fourier Series Coeff. of Signal $x(t)$

$x(t) = \sin(\frac{3\pi t}{2}) + \cos(7\pi t)$

identify all frequencies? - what is The fundamental frequency ω_0 ?

problem3. obtain Discrete Convolution of

$y(n) = x(n) * h(n)$ where $x(n) = a^n u[n-5]$

$h(n) = u(-n)$

Assuming $|a| < 1$

problem4. The impulse response of a discrete LTI system is

$h(n) = [\underset{\substack{\uparrow \\ n=0}}{1} \ 2 \ 2]$

when input $x(n)$ is $x(n) = [\underset{\substack{\uparrow \\ n=0}}{a} \ b]$

if The output $y(n)$ is

Find $\begin{bmatrix} a = ? \ \& \ b = ? \\ c \ \& \ d = ? \end{bmatrix}$

2.2.2 Problem 1

Solution

Folding $h(\tau)$ to becomes $h(-\tau)$. Therefore, when $1+t<-1$ or $t<-2$, then $y(t)=0$ since there is no overlap.

When $-1<1+t<1$, or $-2<t<0$, then there is partial overlap. In this case

$$\begin{aligned}
y(t) &= \int_{-1}^{1+t}\cos(\pi\tau)\,d\tau \qquad -2<t<0\\
&= \frac{1}{\pi}\left[\sin(\pi\tau)\right]_{-1}^{1+t}\\
&= \frac{1}{\pi}\left[\sin(\pi(1+t))-\sin(-\pi)\right]\\
&= \frac{1}{\pi}\sin(\pi(1+t))
\end{aligned}$$

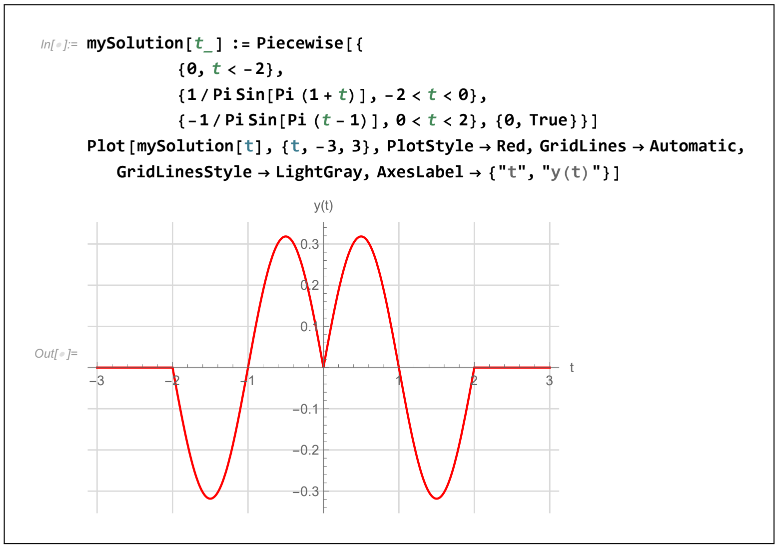
When $1<1+t<3$, or $0<t<2$, then there is partial overlap. In this case

$$\begin{aligned}
y(t) &= \int_{t-1}^{1}\cos(\pi\tau)\,d\tau \qquad 0<t<2\\
&= \frac{1}{\pi}\left[\sin(\pi\tau)\right]_{t-1}^{1}\\
&= \frac{1}{\pi}\left[\sin(\pi)-\sin(\pi(t-1))\right]\\
&= \frac{-1}{\pi}\sin(\pi(t-1))
\end{aligned}$$

When $3<1+t$ or $t>2$ then $y(t)=0$ since there is no overlap any more. Hence solution is

$$y(t)=\begin{cases}0 & t\leq -2\\ \frac{1}{\pi}\sin(\pi(1+t)) & -2<t\leq 0\\ \frac{-1}{\pi}\sin(\pi(t-1)) & 0<t\leq 2\\ 0 & t>2\end{cases}$$

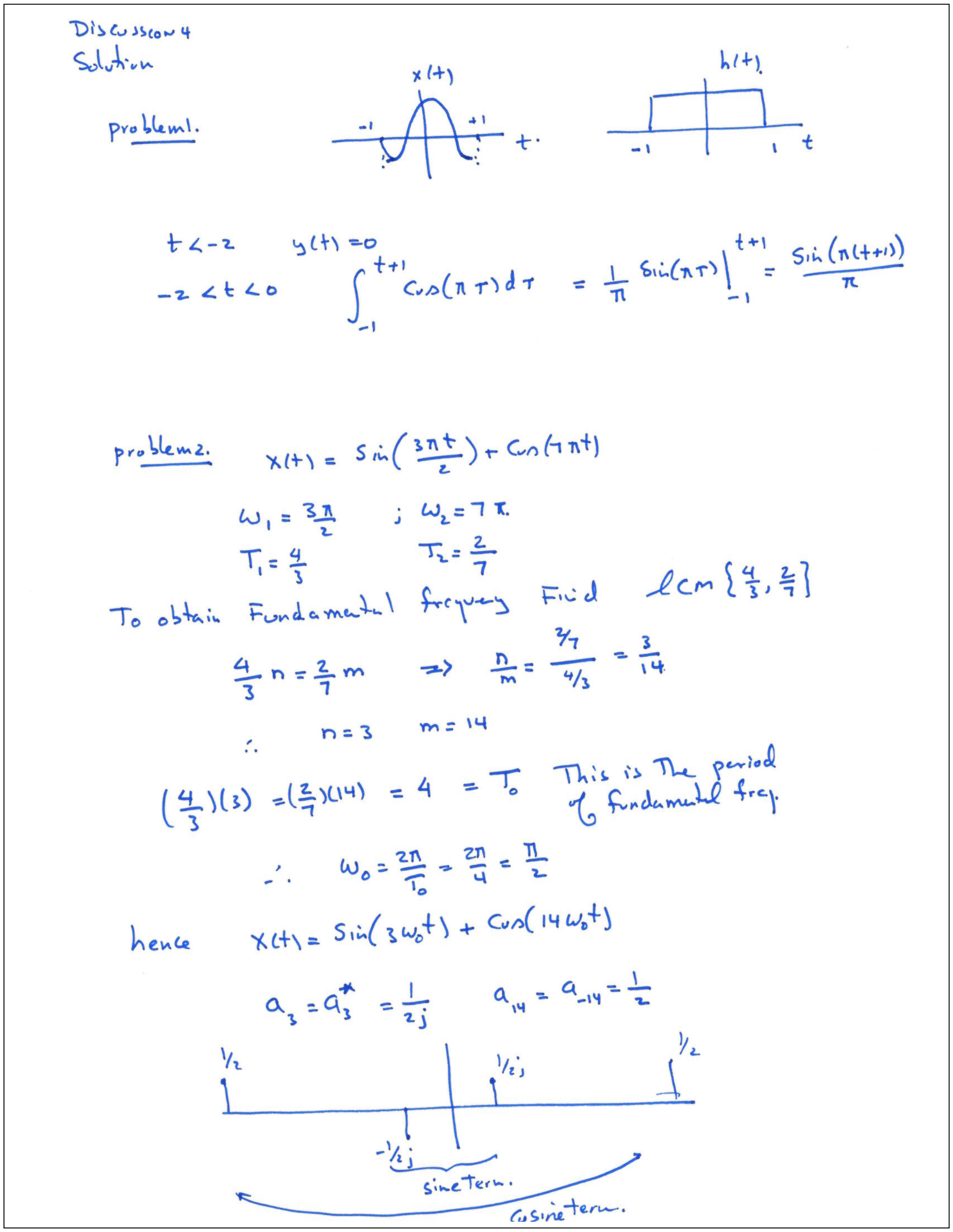
The following is a plot of $y(t)$



```
In[∘]:= mySolution[t_] := Piecewise[{
            {0, t < -2},
            {1 / Pi Sin[Pi (1 + t)], -2 < t < 0},
            {-1 / Pi Sin[Pi (t - 1)], 0 < t < 2}, {0, True}}]
        Plot[mySolution[t], {t, -3, 3}, PlotStyle → Red, GridLines → Automatic,
          GridLinesStyle → LightGray, AxesLabel → {"t", "y(t)"}]
```

Figure 2.10: Plot of $y(t)$

2.2.3 Key solution



Discussion 4
Solution

problem1.

t < -2 y(t) = 0

$-2 < t < 0$ $\int_{-1}^{t+1} \cos(\pi\tau)\,d\tau = \frac{1}{\pi}\sin(\pi\tau)\Big|_{-1}^{t+1} = \frac{\sin(\pi(t+1))}{\pi}$

problem2. $x(t) = \sin\left(\frac{3\pi t}{2}\right) + \cos(7\pi t)$

$\omega_1 = \frac{3\pi}{2}$; $\omega_2 = 7\pi$.

$T_1 = \frac{4}{3}$ $T_2 = \frac{2}{7}$

To obtain Fundamental frequency Find lcm $\{\frac{4}{3}, \frac{2}{7}\}$

$\frac{4}{3}n = \frac{2}{7}m \Rightarrow \frac{n}{m} = \frac{2/7}{4/3} = \frac{3}{14}$

∴ n = 3 m = 14

$(\frac{4}{3})(3) = (\frac{2}{7})(14) = 4 = T_0$ This is The period of fundamental freq.

∴ $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$

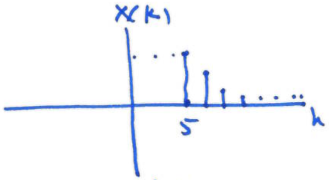
hence $x(t) = \sin(3\omega_0 t) + \cos(14\omega_0 t)$

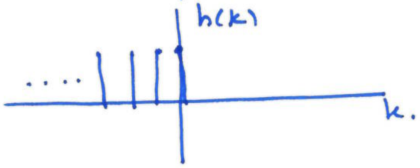
$a_3 = a_3^* = \frac{1}{2j}$ $a_{14} = a_{-14} = \frac{1}{2}$

Discussion 4
Solution

<u>problem 3.</u>

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$





$x(n) = a^n u(n-5)$

$h(n) = u(-n)$

For $n \leq 5$

$$y(n) = \sum_{k=5}^{\infty} a^k = \frac{a^5}{1-a}$$

For $n \geq 5$ $y(n) = \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}$

problem 4.

$$y(n) = x(n) * h(n)$$

perform Convolution you obtain

$$y(n) = [\underset{\substack{\uparrow \\ n=0}}{a} \quad \underset{\substack{\uparrow \\ n=1}}{2a+b} \quad \underset{\substack{\uparrow \\ n=2}}{2a+2b} \quad \underset{\substack{\uparrow \\ n=3}}{2b} \quad \underset{\substack{\uparrow \\ n=4}}{0}]$$

Then $a=1$ $2a+b=c$ $2a+2b=d$ $2b=1$

$a=1$ $c=5/2$
$b=1/2$ $d=3$

Chapter 3

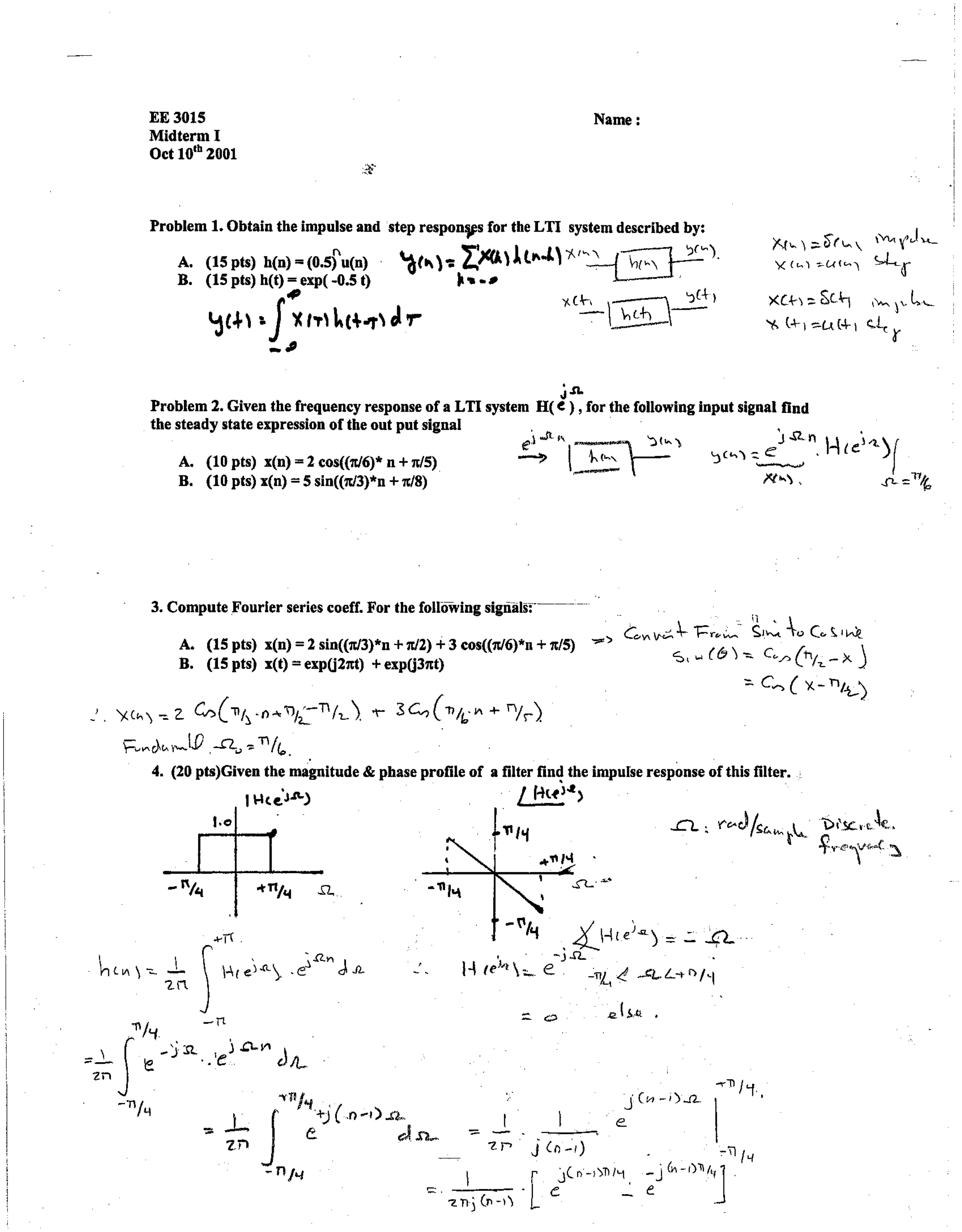
Exams

Local contents

3.1 practice exams

Local contents

3.1.1 Midterm 1, oct 2001



EE 3015
Midterm I
Oct 10th 2001

Name :

Problem 1. Obtain the impulse and step responses for the LTI system described by:

A. (15 pts) $h(n) = (0.5)^n u(n)$

B. (15 pts) $h(t) = \exp(-0.5\,t)$

$y(n) = \sum_{k=-\infty} x(k)\,h(n-k)$

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$.

$x(n) = \delta(n)$ impulse
$x(n) = u(n)$ step

$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$x(t) = \delta(t)$ impulse
$x(t) = u(t)$ step

Problem 2. Given the frequency response of a LTI system $H(e^{j\Omega})$, for the following input signal find the steady state expression of the out put signal

A. (10 pts) $x(n) = 2\cos((\pi/6)^* n + \pi/5)$

B. (10 pts) $x(n) = 5\sin((\pi/3)^* n + \pi/8)$

$e^{j\Omega n} \rightarrow \boxed{h(n)} \rightarrow y(n)$

$y(n) = \underbrace{e^{j\Omega n}}_{x(n)}.\,H(e^{j\Omega})\Big|_{\Omega=\pi/6}$

3. Compute Fourier series coeff. For the following signals:

A. (15 pts) $x(n) = 2\sin((\pi/3)^*n + \pi/2) + 3\cos((\pi/6)^*n + \pi/5)$

B. (15 pts) $x(t) = \exp(j2\pi t) + \exp(j3\pi t)$

$\Rightarrow$ Convert From Sine to Cosine

$\sin(\theta) = \cos(\pi/2 - x)$
$= \cos(x - \pi/2)$

$\therefore\ x(n) = 2\cos(\pi/3 \cdot n + \pi/2 - \pi/2) + 3\cos(\pi/6 \cdot n + \pi/5)$

Fundamental $\Omega_0 = \pi/6$.

4. (20 pts)Given the magnitude & phase profile of a filter find the impulse response of this filter.

Ω: rad/sample Discrete frequency

$\angle H(e^{j\Omega}) = -\Omega$

$\therefore\ H(e^{j\Omega}) = e^{-j\Omega} \quad -\pi/4 < \Omega < +\pi/4$
$= 0 \quad$ else.

$h(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\Omega}).e^{j\Omega n}\,d\Omega$

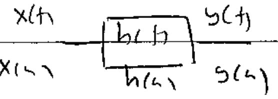
$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j\Omega}.e^{j\Omega n}\,d\Omega$

$= \frac{1}{2\pi}\int_{-\pi/4}^{+\pi/4} e^{+j(n-1)\Omega}\,d\Omega = \frac{1}{2\pi}.\frac{1}{j(n-1)}\,e^{j(n-1)\Omega}\Big|_{-\pi/4}^{+\pi/4}$

$= \frac{1}{2\pi j(n-1)}.\left[e^{j(n-1)\pi/4} - e^{-j(n-1)\pi/4}\right]$

page 1

Solution to Sample Exam



problem 1.

A. if $h(n)=0.5u(n)$

if $x(n)=\delta(n)$ Then $y(n)=\sum_{k=-\infty}^{\infty}\delta(k).h(n-k)=h(n)=0.5u(n)$

impulse. impulse response

if $x(n)=u(n)$ Then $y(n)=\sum_{k=-\infty}^{\infty}u(k).h(n-k)$ since $u(k)=\begin{cases}1 & k\geq 0\\ 0 & \text{else.}\end{cases}$

unit step

$$y(n)=\sum_{k=0}^{\infty}1.(0.5)^{n-k}u(n-k).=0.5^{n}.\sum_{k=0}^{\infty}(0.5)^{-k}.$$

For $n\geq k$.

B. if $x(t)=\delta(t)$ Then $y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}\delta(\tau).h(t-\tau)d\tau$

impulse

$$=h(t)$$
$$=e^{-0.5t}.$$

if $x(t)=u(t)$ Then $y(t)=\int_{-\infty}^{\infty}u(\tau)e^{-0.5(t-\tau)}d\tau$

$$=\int_{0}^{\infty}e^{-0.5t}.e^{0.5\tau}d\tau=e^{-0.5t}.\int_{0}^{\infty}e^{0.5\tau}d\tau.=e^{-0.5t}.\frac{e^{0.5\tau}}{0.5}\Big|_{0}^{\infty}$$

$$=e^{-0.5t}\left[e^{0.5(\infty)}-1\right]$$

problem 2.

A. $x(n)=2\cos(\pi/6 n+\pi/5)$

$$y(n)=2.|H(e^{j\Omega})|_{\Omega=\pi/6}.\cos\left(\pi/6.n+\pi/5+\angle H(e^{j\Omega})|_{\Omega=\pi/6}\right).$$

B. $x(n)=5\sin(\pi/3 n+\pi/8)=5\cos(\pi/3 n+\pi/8-\pi/2)$

$$\therefore y(n)=5.|H(e^{j\Omega})|_{\Omega=\pi/3}.\cos\left(\pi/3.n+\pi/8-\pi/2+\angle H(e^{j\Omega})|_{\Omega=\pi/3}\right.$$

Page 2

problem 3.

A. $$x(n) = 2\sin\left(\pi/3 \cdot n + \pi/2\right) + 3\cos\left(\pi/6 \cdot n + \pi/5\right)$$

Find fundamental freq. $\Omega_1 = \pi/3$ $\Omega_2 = \pi/6$ Ω_0 : fundamental freq. $= \pi/6$.

$$x(n) = 2\cos\left(\pi/3 \cdot n + \cancel{\pi/2} - \cancel{\pi/2}\right) + 3\cos\left(\pi/6 \cdot n + \pi/5\right)$$

Fourier Coeff

use Euler formula $x(n) = e^{j(\pi/3 \cdot n)} + e^{-j(\pi/3 \cdot n)} + \frac{3}{2} e^{j(\pi/6 \cdot n + \pi/5)} + \frac{3}{2} e^{-j(\pi/6 n + \pi/5)}$

if $x(n) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j\Omega_0 k n} = e^{j(2\Omega_0 n)} + e^{-j(2\Omega_0 n)} + \frac{3}{2} \cdot e^{j\pi/5} \cdot e^{j\Omega_0 n} + \frac{3}{2} e^{-j\pi/5} \cdot e^{j\Omega_0 n}$

Then by inspection

For $k=1$ $a_1 = \frac{3}{2} e^{j\pi/5}$ $a_{-1} = \frac{3}{2} e^{-j\pi/5}$

$k=2$ $a_2 = 1$ $a_{-2} = 1$

For all other k $a_k = 0$

B $x(t) = e^{j2\pi t} + e^{j3\pi t}$ use. $a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) \cdot e^{-jk\omega_0 t} dt$

$\omega_1 = 2\pi$ $\omega_2 = 3\pi$

$\omega_0 = \pi$ Fundamental freq. $\omega_0 = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\pi} = 2$.

$$\therefore a_k = \frac{1}{2} \int_0^2 e^{j2\pi t} \cdot e^{-jk\pi t} dt + \frac{1}{2} \int_0^2 e^{j3\pi t} \cdot e^{-jk\pi t} dt$$

evaluate The integral

problem 4.

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\Omega}) e^{j\Omega n} d\Omega = \frac{1}{2\pi j(n-1)} \left[e^{j(n-1)\pi/4} - e^{-j(n-1)\pi/4} \right]$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} H(e^{j\Omega}) \cdot e^{j\Omega n} d\Omega$$

3.1.2 My solution to Midterm 1, oct 2001

3.1.2.1 Problem 1

Obtain impulse and step response for LTI described by (a) $h[n] = \left(\frac{1}{2}\right)^n u[n]$ (b) $h(t) = e^{-\frac{1}{2}t} u(t)$

solution

3.1.2.1.1 Part a

Let $\underline{x[n] = \delta[n]}$, hence

$$\begin{aligned} y[n] &= \delta[n] \circledast h[n] \\ &= \sum_{k=-\infty}^{\infty} \delta[k]\, h[n-k] \end{aligned}$$

But $\delta[k] = 0$ for all k except when $k = 0$. Hence the above reduces to

$$\begin{aligned} y[n] &= h[n] \\ &= \left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

Let $\underline{x[n] = u[n]}$, hence

$$\begin{aligned} y[n] &= u[n] \circledast h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \end{aligned}$$

Folding $x[-n]$, we see that for $n < 0$ then there no overlap with $h[n]$. Hence $y[n] = 0$ for $n < 0$. As $x[-n]$ is shifted to the right, then the convolution sum becomes

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} h[k] \qquad n \geq 0 \\ &= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \end{aligned}$$

This is the partial sum, given by $\frac{a^{1+n}-1}{a-1}$ where $a = \frac{1}{2} < 1$

$$\begin{aligned} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k &= \frac{\left(\frac{1}{2}\right)^{1+n} - 1}{\frac{1}{2} - 1} \\ &= \frac{\left(\frac{1}{2}\right)^{1+n} - 1}{-\frac{1}{2}} \\ &= 2 - 2(2)^{-n-1} \\ &= 2 - 2^{-n} \end{aligned} \tag{2}$$

Therefore

$$y[n] = \begin{cases} 2 - 2^{-n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

3.1.2.1.2 Part b

Let $\underline{x(t) = \delta(t)}$, hence

$$\begin{aligned} y(t) &= u(t) \circledast h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \delta(t)\, h(t-\tau)\, d\tau \\ &= h(t) \\ &= e^{-0.5t} \end{aligned}$$

Let $\underline{x(t) = u(t)}$, hence

$$\begin{aligned} y(t) &= u(t) \circledast h(t) \\ &= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \end{aligned}$$

Folding $u(-t)$, we see that for $t < 0$ then there no overlap with $h(\tau) = e^{-0.5\tau}u(\tau)$. Hence $y(t) = 0$ for $t < 0$. As $u[-n]$ is shifted to the right, then the convolution becomes

$$\begin{aligned} y[n] &= \int_0^t h(\tau)\, d\tau \qquad t > 0 \\ &= \int_0^t e^{-0.5\tau} d\tau \\ &= \left(\frac{e^{-0.5\tau}}{-0.5}\right)_0^t \\ &= -2\left(e^{-0.5t} - 1\right) \\ &= 2 - 2e^{-0.5t} \\ &= 2\left(1 - e^{-0.5t}\right) \end{aligned}$$

Hence

$$y(t) = \begin{cases} 2\left(1 - e^{-0.5t}\right) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

3.1.2.2 Problem 2

Given the frequency response of LTI system $H(\Omega)$ for the following input signal, find the steady state expression of the output signal. (a) $x[n] = 2\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$ (b) $x[n] = 5\sin\left(\frac{\pi}{3}n + \frac{\pi}{8}\right)$

solution

3.1.2.2.1 Part a

$$x[n] = 2\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$$

To find the fundamental period, $\cos\left(\frac{\pi}{6}n+\frac{\pi}{5}\right)=\cos\left(\frac{\pi}{6}(n+N)+\frac{\pi}{5}\right)=\cos\left(\left(\frac{\pi}{6}n+\frac{\pi}{5}\right)+\frac{\pi}{6}N\right)$. Hence need $\frac{\pi}{6}N=m2\pi$ or $\frac{m}{N}=\frac{1}{12}$. Hence. $N=12$. Therefore

$$\Omega_0=\frac{2\pi}{12}$$

And the input is $x[n]=2\cos\left(\Omega_0 n+\frac{\pi}{5}\right)$. Hence the output is

$$y[n]=2\left|H(\Omega_0)\right|\cos\left(\Omega_0 n+\frac{\pi}{5}+\arg H(\Omega_0)\right)$$

3.1.2.2.2 Part b

$$x[n]=5\sin\left(\frac{\pi}{3}n+\frac{\pi}{8}\right)$$

To find the fundamental period, $\sin\left(\frac{\pi}{3}n+\frac{\pi}{8}\right)=\sin\left(\frac{\pi}{3}(n+N)+\frac{\pi}{8}\right)=\sin\left(\frac{\pi}{3}n+\frac{\pi}{8}+\frac{\pi}{3}N\right)$. Hence need $\frac{\pi}{3}N=m2\pi$ or $\frac{m}{N}=\frac{1}{6}$. Hence. $N=6$. Therefore

$$\Omega_0=\frac{2\pi}{6}$$

And the input is $x[n]=5\sin\left(\Omega_0 n+\frac{\pi}{8}\right)$. Hence the output is

$$y[n]=5\left|H(\Omega_0)\right|\sin\left(\Omega_0 n+\frac{\pi}{8}+\arg H(\Omega_0)\right)$$

3.1.2.3 Problem 3

Compute Fourier series coeff. for the following signals. (a) $x[n]=2\sin\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)+3\cos\left(\frac{\pi}{6}n+\frac{\pi}{5}\right)$. (b) $x(t)=e^{j2\pi t}+e^{j3\pi t}$

solution

3.1.2.3.1 Part a For discrete periodic signal, the Fourier series coeff. a_k is given by

$$x[n]=\sum_{k=-\infty}^{\infty}a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{1}$$
$$a_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{2}$$

In this problem

$$x[n]=2\sin\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)+3\cos\left(\frac{\pi}{6}n+\frac{\pi}{5}\right)$$

To find the common fundamental period. $\sin\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{3}(n+N)+\frac{\pi}{2}\right)=\sin\left(\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)+\frac{\pi}{3}N\right)$. Hence $\frac{\pi}{3}N=m2\pi$ or $\frac{m}{N}=\frac{1}{6}$. hence $N=6$ for first signal. For second signal $\cos\left(\frac{\pi}{6}n+\frac{\pi}{5}\right)$ we obtain $\frac{\pi}{6}N = m2\pi$ or $\frac{m}{N} = \frac{1}{12}$ or $N = 12$. hence the least common multiplier between 6 and 12 is $N = 12$. Therefore

$$\Omega_0 = \frac{2\pi}{12}$$

Hence (2) becomes

$$\begin{aligned} a_k &= \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-jk\left(\frac{2\pi}{12}\right)n} \\ &= \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-jk\Omega_0 n} \end{aligned}$$

But instead of using the above formula, an easier way is to rewrite $x[n]$ using Euler relation and use (1) to read off a_k directly from the result. Writing $x[n]$ in terms of the fundamental frequency Ω_0 gives

$$\begin{aligned} x[n] &= 2\sin\left(2\Omega_0 n + \frac{\pi}{2}\right) + 3\cos\left(\Omega_0 n + \frac{\pi}{5}\right) \\ &= 2\left(\frac{e^{j\left(2\Omega_0 n + \frac{\pi}{2}\right)} - e^{-j\left(2\Omega_0 n + \frac{\pi}{2}\right)}}{2j}\right) + 3\left(\frac{e^{j\left(\Omega_0 n + \frac{\pi}{5}\right)} + e^{-j\left(\Omega_0 n + \frac{\pi}{5}\right)}}{2}\right) \\ &= \frac{2}{2j}\left(e^{j\frac{\pi}{2}}e^{j2\Omega_0 n} - e^{-j\frac{\pi}{2}}e^{-j2\Omega_0 n}\right) + \frac{3}{2}\left(e^{j\frac{\pi}{5}}e^{j\Omega_0 n} + e^{-j\frac{\pi}{5}}e^{-j\Omega_0 n}\right) \\ &= \frac{1}{j}e^{j\frac{\pi}{2}}e^{j2\Omega_0 n} - \frac{1}{j}e^{-j\frac{\pi}{2}}e^{-j2\Omega_0 n} + \frac{3}{2}e^{j\frac{\pi}{5}}e^{j\Omega_0 n} + \frac{3}{2}e^{-j\frac{\pi}{5}}e^{-j\Omega_0 n} \end{aligned}$$

Now we can read the Fourier coefficients by comparing the above to Eq(1).

This gives for $k = 2, a_2 = \frac{1}{j}e^{j\frac{\pi}{2}}$ and for $k = -2, a_{-2} = -\frac{1}{j}e^{-j\frac{\pi}{2}}$ and for $k = 1, a_1 = \frac{3}{2}e^{j\frac{\pi}{5}}$ and for $k = -1, a_{-1} = \frac{3}{2}e^{-j\frac{\pi}{5}}$

$$\begin{aligned} a_1 &= \frac{3}{2}e^{j\frac{\pi}{5}} \\ a_{-1} &= \frac{3}{2}e^{-j\frac{\pi}{5}} \\ a_2 &= \frac{1}{j}e^{j\frac{\pi}{2}} \\ a_{-2} &= -\frac{1}{j}e^{-j\frac{\pi}{2}} \end{aligned}$$

But $e^{j\frac{\pi}{2}} = j\sin\frac{\pi}{2} = j$ and $e^{-j\frac{\pi}{2}} = -j\sin\frac{\pi}{2} = -j$. Hence the above becomes

$$\begin{aligned} a_1 &= \frac{3}{2}e^{j\frac{\pi}{5}} \\ a_{-1} &= \frac{3}{2}e^{-j\frac{\pi}{5}} \\ a_2 &= \frac{1}{j}j = 1 \\ a_{-2} &= -\frac{1}{j}\left(-j\right) = 1 \end{aligned}$$

And $a_k = 0$ for all other k.

3.1.2.3.2 Part b

For continuos time periodic signal $x(t)$, the Fourier series coeff. a_k is given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt \end{aligned}$$

In this problem

$$x(t) = e^{j2\pi t} + e^{j3\pi t}$$

The period of $e^{j2\pi t}$ is 1 and the period of $e^{j3\pi t}$ is $\frac{2}{3}$. Hence least common multiplier is $T_0 = 2$ seconds. $\omega_0 = \frac{2\pi}{2} = \pi$ rad/sec. Both of the above terms can now be written

$$\begin{aligned} x(t) &= e^{j\frac{4\pi}{T_0}t} + e^{j\frac{6\pi}{T_0}t} \\ &= e^{j2\omega_0 t} + e^{j3\omega_0 t} \end{aligned}$$

Comparing the above to

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Shows that for $k = 2, a_k = 1$ and for $k = 3, a_k = 1$ and $a_k = 0$ for all other k.

3.1.2.4 Problem 4

Given the magnitude and phase profile of this filter, find impulse response.

solution

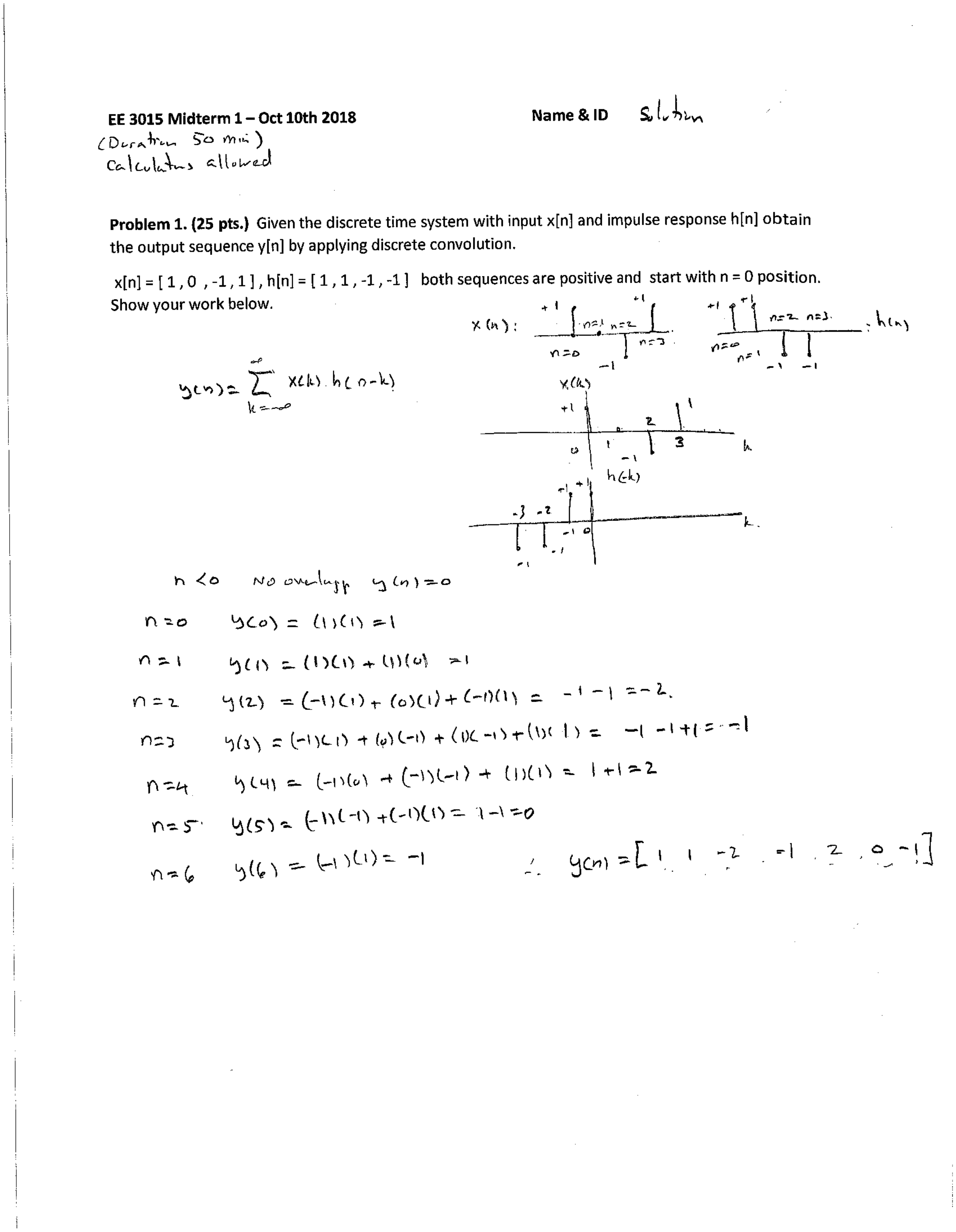
We are given $H(\Omega)$ and need to find $h[n]$. i.e. the inverse Fourier transform

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega)\, e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \end{aligned}$$

But $|H(\Omega)| = 1$ and $\arg H(\Omega) = -\Omega$ as given. The above reduces to

$$\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\Omega}e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\Omega(1-n)}d\Omega \\
&= \left(\frac{1}{2\pi}\right)\frac{1}{-j(1-n)}\left[e^{-j\Omega(1-n)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \left(\frac{1}{2\pi}\right)\frac{1}{-j(1-n)}\left(e^{-j\frac{\pi}{4}(1-n)} - e^{j\frac{\pi}{4}(1-n)}\right) \\
&= \frac{1}{\pi}\frac{1}{(1-n)}\left(\frac{e^{j\frac{\pi}{4}(1-n)} - e^{-j\frac{\pi}{4}(1-n)}}{2j}\right) \\
&= \frac{1}{\pi(1-n)}\sin\left(\frac{\pi}{4}(1-n)\right) \\
&= \frac{-1}{\pi(1-n)}\sin\left(\frac{\pi}{4}(n-1)\right) \\
&= \frac{1}{\pi(n-1)}\sin\left(\frac{\pi}{4}(n-1)\right)
\end{aligned}$$

3.1.3 Midterm 1, oct 2018



EE 3015 Midterm 1 – Oct 10th 2018 **Name & ID** Solution

(Duration 50 min)
Calculators allowed

Problem 1. (25 pts.) Given the discrete time system with input x[n] and impulse response h[n] obtain the output sequence y[n] by applying discrete convolution.

x[n] = [1 , 0 , -1 , 1] , h[n] = [1 , 1 , -1 , -1] both sequences are positive and start with n = 0 position.
Show your work below.

$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

$n<0$ No overlapp $y(n)=0$

$n=0$ $y(0) = (1)(1) = 1$

$n=1$ $y(1) = (1)(1) + (1)(0) = 1$

$n=2$ $y(2) = (-1)(1) + (0)(1) + (-1)(1) = -1-1 = -2.$

$n=3$ $y(3) = (-1)(1) + (0)(-1) + (1)(-1) + (1)(1) = -1-1+1 = -1$

$n=4$ $y(4) = (-1)(0) + (-1)(-1) + (1)(1) = 1+1 = 2$

$n=5$ $y(5) = (-1)(-1) + (-1)(1) = 1-1 = 0$

$n=6$ $y(6) = (-1)(1) = -1$

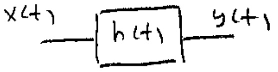
$\therefore$ $y(n) = [1, 1, -2, -1, 2, 0, -1]$

Problem 2. (35 pts.) The impulse response of a LTI system is given as

h(t) = u(t) - 2 u(t - 1) + u(t – 2)

Where u(t) is a unit step signal. Determine the output of this system y (t) where it's input x(t) is given as

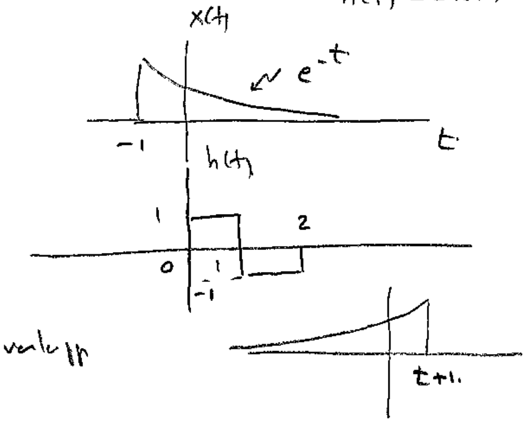
x(t) = e ^ (-t) . u(t + 1)



$x(t) = e^{-t}\, u(t+1)$

$h(t) = u(t) - 2u(t-1) + u(t-2)$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$



Región 1.

$-\infty < t+1 < 0$ No overlapp

$y(t) = 0$

Region 2:

$1 > t+1 > 0 \Rightarrow 0 > t > -1$

$$y(t) = \int_0^{t+1} e^{-(t-\tau)} d\tau = e - e^{-t}$$

Region 3.

$2 > t+1 > 1 \Rightarrow 1 > t > 0$

$$y(t) = \int_0^1 1.e^{-(t-\tau)} d\tau + \int_1^{t+1} (-1).e^{-(t-\tau)} d\tau = 2e^{(1-t)} - e^{-t} - e$$

Reg. 4

$\infty > t+1 > 2 \Rightarrow \infty > t > 1$

$$y(t) = \int_0^1 e^{-(t-\tau)} (1).d\tau + \int_1^2 (-1).e^{-(t-\tau)} d\tau$$

$$= 2e^{(1-t)} - e^{-t} - e^{(2-t)}$$

$$\therefore \quad y(t) = \begin{cases} 0 & -\infty < t < -1 \\ e - e^{-t} & -1 < t < 0 \\ 2e^{(1-t)} - e^{-t} - e & 0 < t < 1 \\ 2e^{(1-t)} - e^{-t} - e^{(2-t)} & 1 < t < \infty. \end{cases}$$

Problem 3.(40 pts.) A discrete periodic sequence is given as x[n]:

x [n] = 2 sin (3πn/8 + π/2) + cos (πn/4 + π/3) obtain the following :

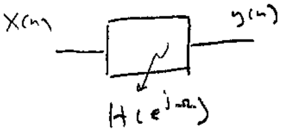
(a) (10 pts) Find the fundamental frequency and period of this signal.

(b) (15 pts) The Fourier series coefficients for x(n) sequence.

(c) (15 pts) If x(n) = cos (πn/4 + π/3) is an input to a system with frequency response H(e ^jΩ) where

H (e^jΩ) = (1 - e^(-jΩ))/ (2 + e^(-j2Ω)) obtain the expression for output y(n).

$$H(e^{j\Omega}) = \frac{1 - e^{-j\Omega}}{2 + e^{-j2\Omega}}$$



$$|H(e^{j\Omega})| =$$

$$x(n) = 2\sin(3\pi n/8 + \pi/2) + \cos(\pi n/4 + \pi/3)$$

$$\Omega_1 = \frac{3\pi}{8} \quad \Omega_2 = \pi/4 \quad \text{Fundamental freq } \Omega_0 = \frac{2\pi}{N}.$$

(a) $$\Omega_1 = \frac{3(2\pi)}{8(2)} \quad \Omega_2 = \frac{3(2\pi)}{16} \Rightarrow N = 16 \text{ period}$$

$$\therefore \quad \Omega_0 = \frac{2\pi}{16} = \pi/8 \quad \text{Fundamental freq.}$$

(b). $$x(n) = \sum_{k=-\infty}^{\infty} X(k).\, e^{jk\Omega_0 n.} = \frac{2}{2j}\left(e^{j(3\Omega_0 n + \pi/2)} - e^{-j(3\Omega_0 n + \pi/2)}\right) + \frac{1}{2}\left(e^{j(2\Omega_0 n + \pi/3)} + e^{-j(2\Omega_0 n + \pi/3)}\right)$$

$$\therefore \quad X(k) = \begin{cases} \frac{e^{-j\pi/2}}{j} = 1 & k = -3 \\ \frac{e^{-j\pi/3}}{2} & k = -2 \\ \frac{e^{j\pi/3}}{2} & k = +2 \\ \frac{e^{j\pi/2}}{j} = 1 & k = +3 \end{cases}$$

$$\frac{1 - \cos(\Omega) + j\sin\Omega}{2 - \cos(2\Omega) + j\sin(2\Omega)} = H(e^{j\Omega})$$

$$|H(e^{j\Omega})| = \frac{\sqrt{(1-\cos(j\Omega))^2 + \sin^2\Omega}}{\sqrt{(2-\cos(j2\Omega))^2 + \sin^2(2\Omega)}}$$

$\uparrow$ at $\Omega = \pi/4$.

(C) $$y(n) = |H(e^{j\Omega})|\Big|_{\Omega=\pi/4} \cdot \cos\left(\frac{\pi}{4}.n + \pi/3 + \measuredangle H(e^{j\Omega})\Big|_{\Omega=\pi/4}\right)$$

$$\measuredangle H(e^{j\Omega}) = \tan^{-1}\left(\frac{\sin\Omega}{1-\cos(\Omega)}\right) - \tan^{-1}\left(\frac{\sin(2\Omega)}{2-\cos(2\Omega)}\right)$$

$$|H(e^{j\Omega})|_{\Omega=\pi/4} = \frac{|1 - e^{j\pi/4}|}{|2 - e^{-j\pi/2}|} = \sqrt{\frac{(1-\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2}{3}} =$$

$$\measuredangle H(e^{j\Omega})\Big|_{\Omega=\pi/4} = \tan^{-1}\left(\frac{\sin(\pi/4)}{1-\cos(\pi/4)}\right) - \tan^{-1}\left(\frac{\sin(\pi/2)}{2-\cos(\pi/2)}\right)$$

3.1.4 My solution to Midterm 1, oct 2018

3.1.4.1 Problem 1

Given the discrete time system with input $x[n]$ and impulse response $h[n]$ obtain the output sequence $y[n]$ by applying discrete convolution.

$x[n] = [1,0,-1,1], h[n] = [1,1,-1,-1]$. Both sequences are positive and start with $n = 0$ position.

Solution

$$y[n] = x[n] \circledast h[n]$$

Folding $x[-n]$. When $n = 0, y[0] = 1$. When $n = 1$, then $y[n] = (0)(1) + (1)(1) = 1$. When $n = 2, y[n] = (-1)(1) + (0)(1) + (1)(-1) = -2$. When $n = 3, y[n] = (1)(1) + (-1)(1) + (0)(-1) + (1)(-1) = 1 - 1 - 1 = -1$. When $n = 4, y[n] = 1 + 1 = 2$. When $n = 5, y[n] = -1 + 1 = 0$, when $n = 6, y[6] = -1$. When $n > 6, y[n] = 0$.

Hence

$$y[n] = [1,1,-2,-1,2,0,-1]$$

3.1.4.2 Problem 2

The impulse response of LTI system is given by $h(t) = u(t) - 2u(t-1) + u(t-2)$ where $u(t)$ is unit step signal. Determine the output of this $y(t)$ where its input $x(t)$ is given as $x(t) = e^{-t}u(t+1)$.

Solution

By folding $x(t)$. See key solution. Used same method.

3.1.4.3 Problem 3

A discrete periodic sequence is given as $x[n] = 2\sin\left(\frac{3\pi}{8}n + \frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$. (a) Find fundamental frequency of this signal. (b) Fourier series coefficients for $x[n]$. (c) if $x[n] = \cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$ is an input to system with frequency response $H(\Omega) = \frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}}$, obtain expression for $y[n]$

Solution

3.1.4.3.1 Part a For $\sin\left(\frac{3\pi}{8}n + \frac{\pi}{2}\right)$, we need $\frac{3}{8}\pi N = m2\pi$ or $\frac{m}{N} = \frac{3}{16}$. Since relatively prime, hence $N = 16$. For $\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$ we need $\frac{\pi}{4}N = m2\pi$ or $\frac{m}{N} = \frac{1}{8}$. Hence $N = 8$. The least common multiplier is 16. Hence fundamental period is $N = 16$. Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}$.

3.1.4.3.2 Part b Since input is periodic, then

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 n} \tag{1}$$

By writing the input, using Euler relation, we can compare the input to the above and read off a_k. First we rewrite the input using common Ω_0 as

$$x[n] = 2\sin\left(3\Omega_0 n + \frac{\pi}{2}\right) + \cos\left(2\Omega_0 n + \frac{\pi}{3}\right)$$

Hence

$$\begin{aligned}
x[n] &= 2\left(\frac{e^{j\left(3\Omega_0 n+\frac{\pi}{2}\right)} - e^{-j\left(3\Omega_0 n+\frac{\pi}{2}\right)}}{2j}\right) + \frac{e^{j\left(2\Omega_0 n+\frac{\pi}{3}\right)} + e^{-j\left(2\Omega_0 n+\frac{\pi}{3}\right)}}{2} \\
&= \frac{1}{j}e^{j\left(3\Omega_0 n+\frac{\pi}{2}\right)} - \frac{1}{j}e^{-j\left(3\Omega_0 n+\frac{\pi}{2}\right)} + \frac{1}{2}e^{j\left(2\Omega_0 n+\frac{\pi}{3}\right)} + \frac{1}{2}e^{-j\left(2\Omega_0 n+\frac{\pi}{3}\right)} \\
&= \frac{1}{j}e^{j\frac{\pi}{2}}e^{j3\Omega_0 n} - \frac{1}{j}e^{-j\frac{\pi}{2}}e^{-j3\Omega_0 n} + \frac{1}{2}e^{j\frac{\pi}{3}}e^{j2\Omega_0 n} + \frac{1}{2}e^{-j\frac{\pi}{3}}e^{-j2\Omega_0 n}
\end{aligned}$$

But $e^{j\frac{\pi}{2}} = j\sin\frac{\pi}{2} = j$ and $e^{-j\frac{\pi}{2}} = -j\sin\frac{\pi}{2} = -j$ and $e^{j\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}\sqrt{3}j + \frac{1}{2}$ and $e^{-j\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3}j$. Hence the above simplifies to

$$x[n] = e^{j3\Omega_0 n} + e^{-j3\Omega_0 n} + \frac{1}{4}\left(1+\sqrt{3}j\right)e^{j2\Omega_0 n} - \frac{1}{4}\left(1-\sqrt{3}j\right)e^{-j2\Omega_0 n} \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned}
a_3 &= 1 \\
a_{-3} &= 1 \\
a_2 &= \frac{1}{2}e^{j\frac{\pi}{3}} = \frac{1}{4}\left(1+\sqrt{3}j\right) \\
a_{-2} &= \frac{1}{2}e^{-j\frac{\pi}{3}} = -\frac{1}{4}\left(1-\sqrt{3}j\right)
\end{aligned}$$

3.1.4.3.3 Part c

The output is

$$y[n] = \left|H(\Omega)\right|_{\Omega=\frac{\pi}{4}} \cos\left(\frac{\pi}{4}n + \frac{\pi}{3} + \arg H(\Omega)_{\Omega=\frac{\pi}{4}}\right) \tag{1}$$

But

$$\begin{aligned}
\left|H(\Omega)\right| &= \left|\frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}}\right| \\
&= \frac{\left|1-e^{-j\Omega}\right|}{\left|2+e^{-2j\Omega}\right|} \\
&= \frac{\sqrt{(1-\cos\Omega)^2 + \sin^2\Omega}}{\sqrt{(2+\cos 2\Omega)^2 + \sin^2(2\Omega)}}
\end{aligned}$$

When $\Omega = \frac{\pi}{4}$ the above becomes

$$\begin{aligned}\left|H\left(\frac{\pi}{4}\right)\right| &= \frac{\sqrt{\left(1-\cos\left(\frac{\pi}{4}\right)\right)^2+\sin^2\left(\frac{\pi}{4}\right)}}{\sqrt{\left(2+\cos\left(\frac{\pi}{2}\right)\right)^2+\sin^2\left(\frac{\pi}{2}\right)}}\\ &= \frac{\sqrt{\left(1-\cos\left(\frac{\pi}{4}\right)\right)^2+\sin^2\left(\frac{\pi}{4}\right)}}{\sqrt{4+1}}\\ &= \frac{\sqrt{\frac{3}{2}-\sqrt{2}+\frac{1}{2}}}{\sqrt{5}}\\ &= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{5}}\\ &= 0.342\,28\end{aligned}$$

And

$$\begin{aligned}\arg H(\Omega) &= \arg\frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}}\\ &= \arg\frac{(1-\cos\Omega)+j\sin\Omega}{(2+\cos(2\Omega))-j\sin(2\Omega)}\\ &= \arctan\left(\frac{\sin\Omega}{1-\cos\Omega}\right)-\arctan\left(\frac{-\sin(2\Omega)}{2+\cos(2\Omega)}\right)\end{aligned}$$

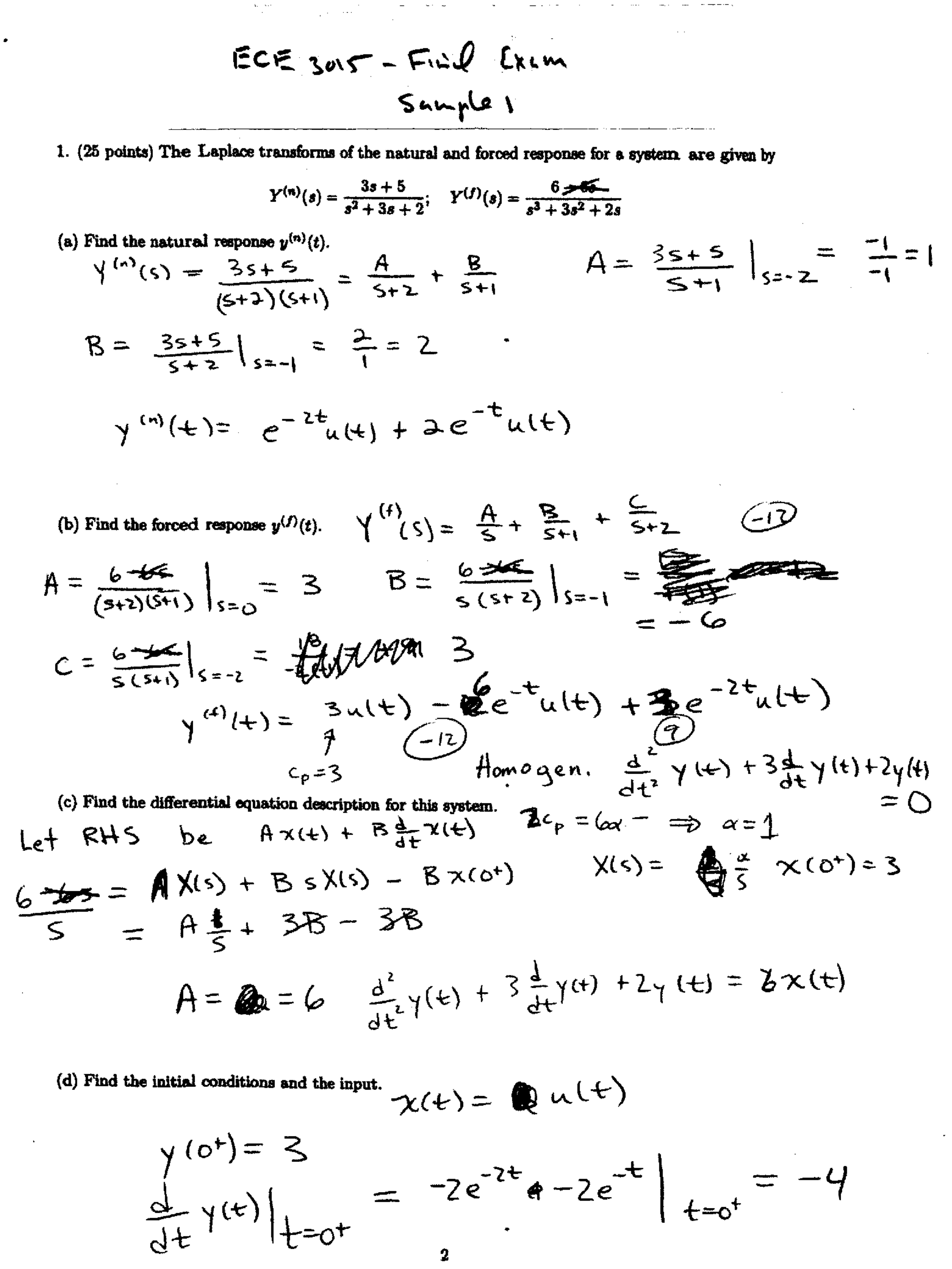
When $\Omega = \frac{\pi}{4}$ the above becomes

$$\begin{aligned}\arg H\left(\frac{\pi}{4}\right) &= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right)-\arctan\left(\frac{-\sin\left(\frac{\pi}{2}\right)}{2+\cos\left(\frac{\pi}{2}\right)}\right)\\ &= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right)-\arctan\left(\frac{-1}{2}\right)\\ &= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right)+\arctan\left(\frac{1}{2}\right)\\ &= \arctan(2.4142)+0.46365\\ &= 1.1781+0.46365\\ &= 1.6418\text{ rad}\end{aligned}$$

Hence (1) becomes

$$y[n] = 0.3423\cos\left(\frac{\pi}{4}n+\frac{\pi}{3}+1.6418\right)$$

3.1.5 Final exam practice exam 1



ECE 3015 - Final Exam
Sample 1

1. **(25 points) The Laplace transforms of the natural and forced response for a system are given by**

$$Y^{(n)}(s) = \frac{3s+5}{s^2+3s+2}; \quad Y^{(f)}(s) = \frac{6\,\cancel{+6s}}{s^3+3s^2+2s}$$

(a) Find the natural response $y^{(n)}(t)$.

$$Y^{(n)}(s) = \frac{3s+5}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} \qquad A = \left.\frac{3s+5}{s+1}\right|_{s=-2} = \frac{-1}{-1} = 1$$

$$B = \left.\frac{3s+5}{s+2}\right|_{s=-1} = \frac{2}{1} = 2$$

$$y^{(n)}(t) = e^{-2t}u(t) + 2e^{-t}u(t)$$

(b) Find the forced response $y^{(f)}(t)$. $Y^{(f)}(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$ $\textcircled{-12}$

$$A = \left.\frac{6\,\cancel{+6s}}{(s+2)(s+1)}\right|_{s=0} = 3 \qquad B = \left.\frac{6\,\cancel{+6s}}{s(s+2)}\right|_{s=-1} = [illegible] = -6$$

$$C = \left.\frac{6\,\cancel{+6s}}{s(s+1)}\right|_{s=-2} = [illegible]\ 3$$

$$y^{(f)}(t) = \underset{c_p=3}{\underset{\uparrow}{3u(t)}} - 6e^{-t}u(t) + 3e^{-2t}u(t)$$

$\textcircled{-12}$ $\textcircled{9}$

Homogen. $\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 0$

(c) Find the differential equation description for this system.

$c_p = 6\alpha \Rightarrow \alpha = 1$

Let RHS be $Ax(t) + B\frac{d}{dt}x(t)$

$$\frac{6\,\cancel{+6s}}{s} = AX(s) + BsX(s) - Bx(0^+) \qquad X(s) = \frac{\alpha}{s} \quad x(0^+) = 3$$

$$= A\frac{1}{s} + 3B - 3B$$

$$A = 6 \qquad \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 6x(t)$$

(d) Find the initial conditions and the input. $x(t) = u(t)$

$$y(0^+) = 3$$

$$\left.\frac{d}{dt}y(t)\right|_{t=0^+} = \left.-2e^{-2t} - 2e^{-t}\right|_{t=0^+} = -4$$

2

2. (15 points) Find the Fourier Transform of the signal

$$x(t) = \left(\frac{d}{dt}\left[te^{-|t-1|}\right]\right) * \left(\cos(2t)e^{-t}u(t)\right)$$

$$e^{-|t|} \xleftrightarrow{FT} \frac{2}{1+\omega^2}$$

$$e^{-|t-1|} \xleftrightarrow{FT} \frac{2e^{-j\omega}}{1+\omega^2}$$

$$t\,e^{-|t-1|} \xleftrightarrow{FT} \frac{d}{d\omega}\left\{\frac{2e^{-j\omega}}{1+\omega^2}\right\} = \frac{-2je^{-j\omega}}{1+\omega^2} - \frac{2e^{-j\omega}\,2\omega}{(1+\omega^2)^2}$$

$$\frac{d}{dt}\left[t\,e^{-|t-1|}\right] \xleftrightarrow{FT} j\omega\frac{d}{d\omega}\left\{\frac{2e^{-j\omega}}{1+\omega^2}\right\}$$

$$e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{j\omega+1}$$

$$\cos 2t\, e^{-t}u(t) \longleftrightarrow \frac{1}{2}\left(\frac{1}{j(\omega-2)+1}\right) + \frac{1}{2}\left(\frac{1}{j(\omega+2)+1}\right)$$

$$X(j\omega) = j\frac{\omega}{2}\frac{d}{d\omega}\left\{\frac{2e^{-j\omega}}{1+\omega^2}\right\}\left[\frac{1}{j(\omega-2)+1} + \frac{1}{j(\omega+2)+1}\right]$$

3. **(20 points) A system has impulse response**

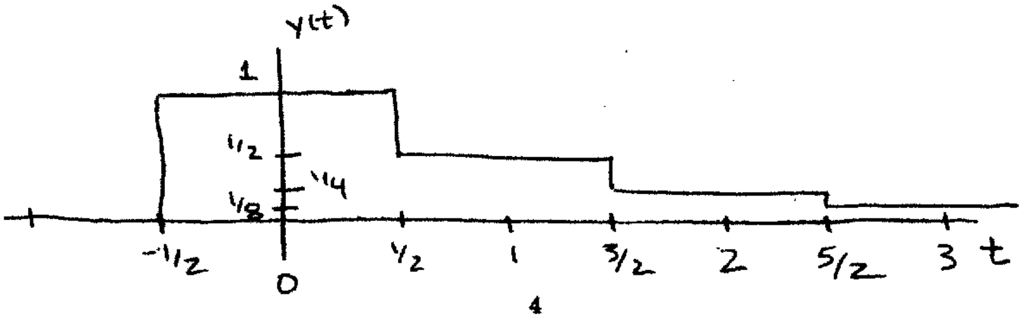
$$h(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \delta(t-n)$$

Use convolution to find the output $y(t)$ for an input $x(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$. Sketch $y(t)$ in the space provided.

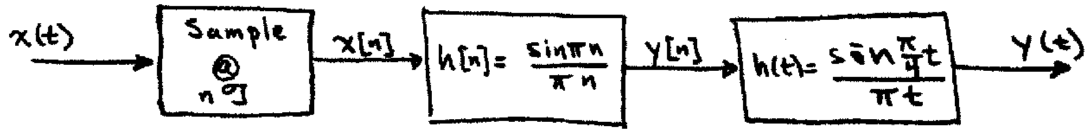
since $\delta(t-n) * x(t) = x(t-n)$

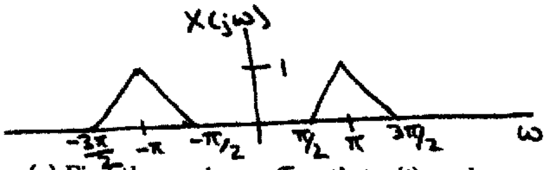
$$\sum \left(\tfrac{1}{2}\right)^n \delta(t-n) * x(t) = \sum \left(\tfrac{1}{2}\right)^n x(t-n)$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \left[u(t-n+1/2) - u(t-n+1/2)\right]$$



4. (25 points) Consider the system depicted below. We have $x[n] = x(nT)$, $y[n] = h[n] * x[n]$, and $y(t) = h(t) * y_s(t)$.

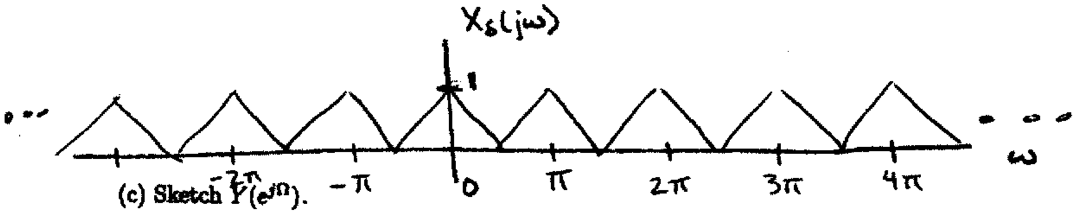




(a) Find the maximum T so that $x(t)$ can be reconstructed from $x[n]$ using a lowpass reconstruction filter.

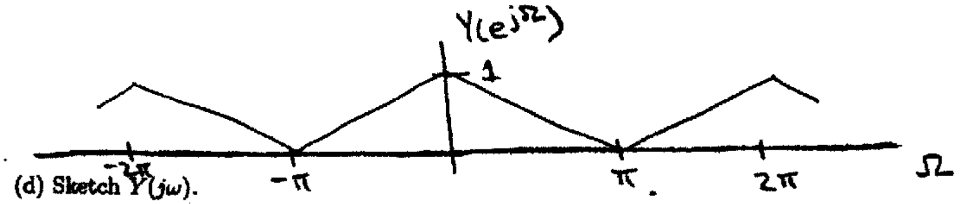
Sampling Thm $\frac{2\pi}{T} > 2\left(\frac{3\pi}{2}\right)$ $T < 2/3$

(b) Let $T = 2$. Sketch $X_s(j\omega)$ in the space provided. $X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ $\omega_s = \pi$



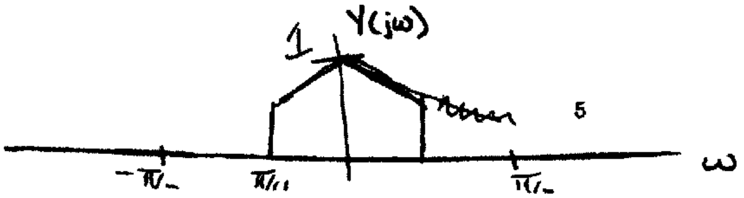
(c) Sketch $Y(e^{j\Omega})$.

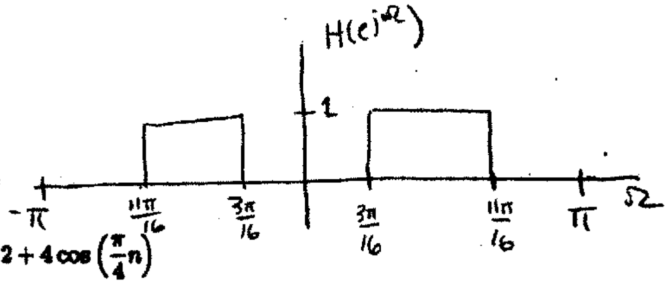
$h[n] = \delta[n]$ so $y[n] = x[n]$ $Y(e^{j\Omega}) = X_s(j\omega)\Big|_{\omega = \frac{\Omega}{T}}$



(d) Sketch $Y(j\omega)$.

$$H(j\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{4} \\ 0 & \text{other} \end{cases}$$

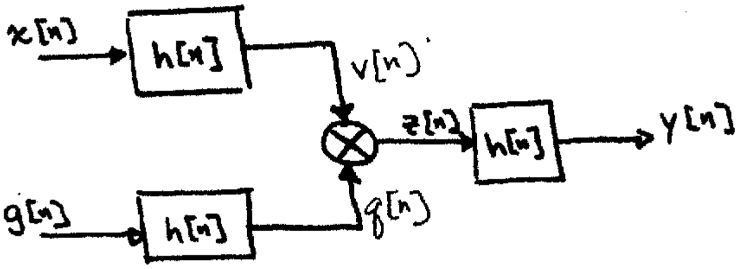




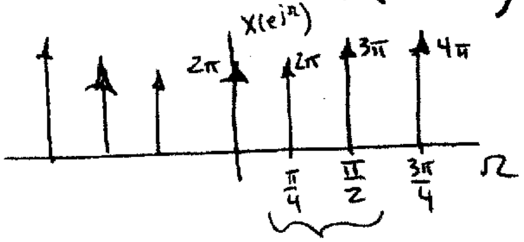
5. (15 points) A discrete-time system is depicted below. We have

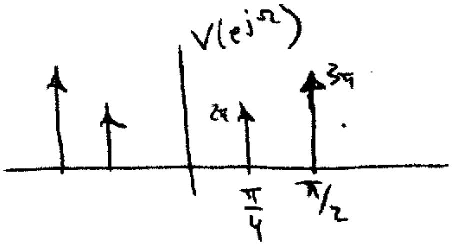
$$x[n] = \sum_{k=0}^{3}(1+k)\cos\left(k\frac{\pi}{4}n\right), \quad g[n] = 2 + 4\cos\left(\frac{\pi}{4}n\right)$$

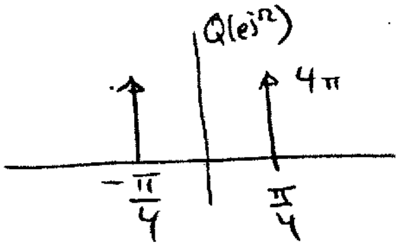
$$h[n] = \frac{2\sin\left(\frac{\pi}{4}n\right)}{\pi n}\cos\left(\frac{7\pi}{16}n\right)$$



$z[n] = (h[n] * x[n])(g[n] * h[n])$

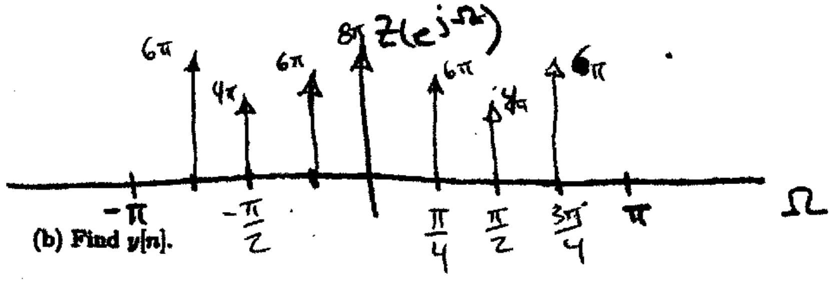




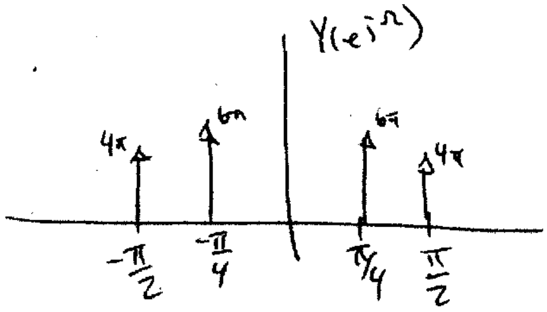


(a) Sketch $Z(e^{j\Omega})$.

$Z(e^{j\Omega}) = \frac{1}{2\pi} V(e^{j\Omega}) \circledast Q(e^{j\Omega})$



(b) Find $y[n]$.



$y[n] = 6\cos\frac{\pi}{4}n + 4\cos\frac{\pi}{2}n$

3.1.6 Final exam practice exam 2

ECE 3015 Final exam
Sample 2

1. (20 points) An LTI system is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 9\frac{d}{dt}y(t) - 10y(t) = 3\frac{d^2}{dt^2}x(t) + 7\frac{d}{dt}x(t) - 10x(t)$$

Assume the system has initial conditions $y(0^+) = 5$, $\frac{d}{dt}y(t)\Big|_{t=0^+} = -28$ and input $x(t) = u(t-1)$.

(a) Use Laplace transforms to find the natural response. $x(t) = 0$

$$s^2Y(s) - \frac{dy(t)}{dt}\Big|_{t=0^+} - sy(0^+) + 9sY(s) - 9y(0^+) - 10Y(s) = 0$$

$$Y(s) = \frac{5s + 45 - 28}{s^2 + 9s - 10} = \frac{5s + 17}{(s-1)(s+10)} = \frac{A}{s-1} + \frac{B}{s+10}$$

$$A = \frac{5s+17}{s+10}\Big|_{s=1} = \frac{22}{11} = 2 \qquad B = \frac{5s+17}{s-1}\Big|_{s=-10} = \frac{-33}{-11} = 3$$

$$y^{(n)}(t) = 2e^{t}u(t) + 3e^{-10t}u(t)$$

(b) Use Laplace transforms to find the overall output of the system.

$y(t) = y^{(n)}(t) + y^{(f)}(t)$ Note $x(0)=0$ $\frac{d}{dt}x(t)\Big|_{t=0^+}=0$

$Y^{(f)}(s) = \frac{3s^2+7s-10}{s^2+9s-10}X(s)$ $X(s) = e^{-s}\left(\frac{1}{s}\right)$

$= \frac{Ae^{-s}}{s} + \frac{Be^{-s}}{s-1} + \frac{Ce^{-s}}{s+10}$ $A = \frac{3s^2+7s-10}{s^2+9s-10}\Big|_{s=0} = 1$

$B = \frac{3s^2+7s-10}{s(s+10)}\Big|_{s=1} = \frac{0}{11} = 0$

300− $C = \frac{3s^2+7s-10}{s(s-1)}\Big|_{s=-10} = \frac{220}{110} = 2$

$-10\ (-11)$

$y^{(f)}(t) = u(t-1) + 2e^{-10(t-1)}u(t-1)$

$y(t) = y^{(f)}(t) + y^{(n)}(t)$

$= 2e^{t}u(t) + 3e^{-10t}u(t-1) + u(t-1) + 2e^{-10(t-1)}u(t-1)$

(c) Is this system stable? Justify your answer.

Not stable. Natural response blows up.

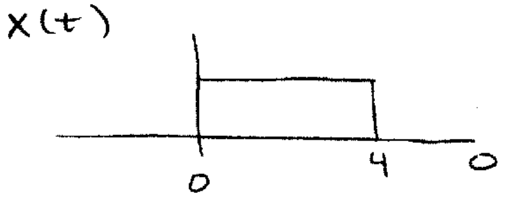
2. (20 points) Let a signal $y(t)$ have FT representation

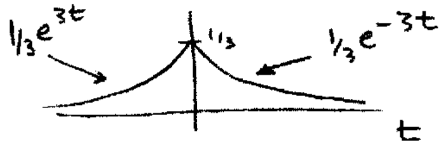
$$Y(j\omega) = e^{-j2\omega}\left(\frac{\sin(2\omega)}{\omega}\right)\left(\frac{4}{9+\omega^2}\right)$$

Find $y(t)$.

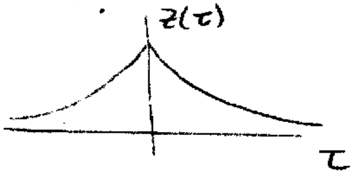
$$X(j\omega) = e^{-j2\omega}\,\frac{2\sin(2\omega)}{\omega} \qquad Z(j\omega) = \frac{2}{9+\omega^2}$$

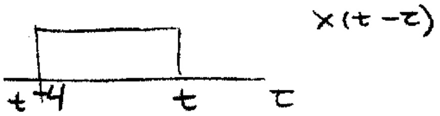
$$z(t) = \frac{1}{3}e^{-3|t|}$$





$$y(t) = x(t) * z(t)$$





1) $t<0$

$$W_t(\tau) = \begin{cases} \frac{1}{3}e^{3t} & t-4 \le \tau \le t \\ 0 & \text{other} \end{cases}$$

2) $0<t<t-4$

$$W_t(\tau) = \begin{cases} \frac{1}{3}e^{3t} & t-4<\tau<0 \\ \frac{1}{3}e^{-3t} & 0<\tau<t \\ 0 & \text{other} \end{cases}$$

3) $4<t$!

$$W_t(\tau) = \begin{cases} \frac{1}{3}e^{-3t} & t-4<\tau<t \\ 0 & \text{other} \end{cases}$$

$t<0$

$$y(t) = \int_{t-4}^{t} \frac{1}{3}e^{3\tau}\,d\tau = \frac{1}{9}e^{3\tau}\Big|_{t-4}^{t} = \underline{\frac{1}{9}\left[e^{3t} - e^{3(t-4)}\right]}$$

$4<t$

$$y(t) = -\frac{1}{9}e^{-3\tau}\Big|_{t-4}^{t} = \underline{\frac{1}{9}\left[e^{-3(t-4)} - e^{-3t}\right]}$$

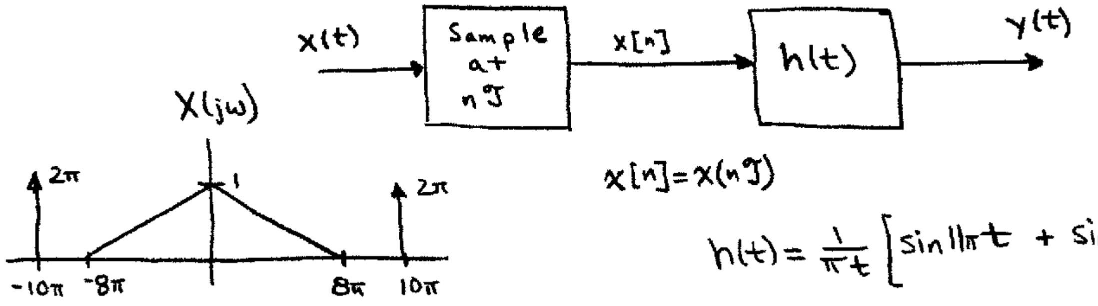
$) < t < t-4$

$$y(t) = \int_{t-4}^{0}\frac{1}{3}e^{3\tau}\,d\tau + \int_{0}^{t}\frac{1}{3}e^{-3\tau}\,d\tau$$

$$= \frac{1}{9}e^{3\tau}\Big|_{t-4}^{0} - \frac{1}{9}e^{-3\tau}\Big|_{0}^{t}$$

$$= \underline{\frac{1}{9}\left[1 - e^{3(t-4)} + 1 - e^{-3t}\right]}$$

3. (20 points) Consider the system depicted below



$x[n]=x(nT)$

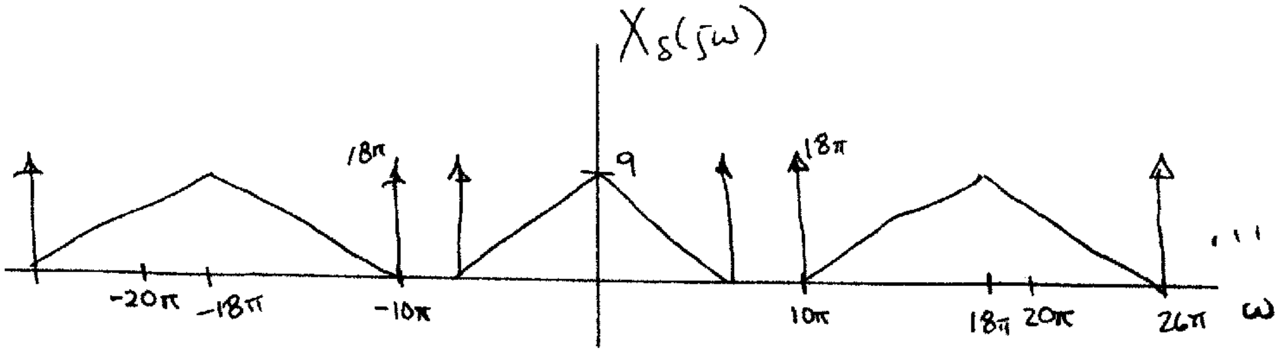
$h(t)=\frac{1}{\pi t}\left[\sin 11\pi t + \sin 24\pi t\right]$

(a) Sketch the FT representation $\underset{\wedge}{\text{in}}$ the space given [redacted] assuming $T = \frac{1}{9}$ seconds.

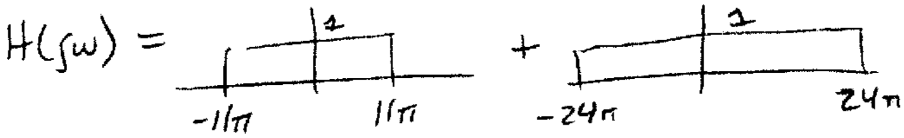
corresponding to $x[n]$

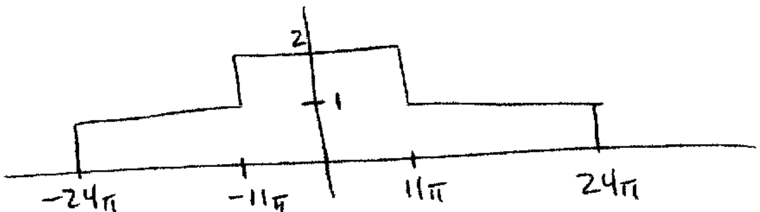
$\omega_s = 18\pi$

$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$



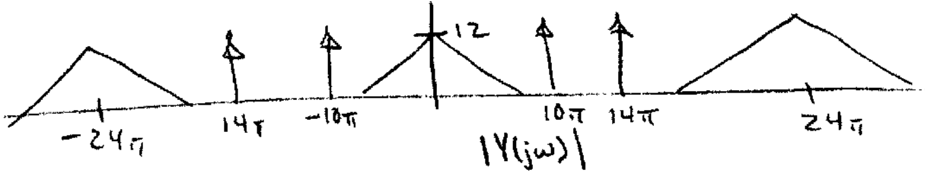
(b) Sketch $|Y(j\omega)|$ in the space provided assuming $T = \frac{1}{12}$. Hint: $y(t) = x_\delta(t) * h(t)$.



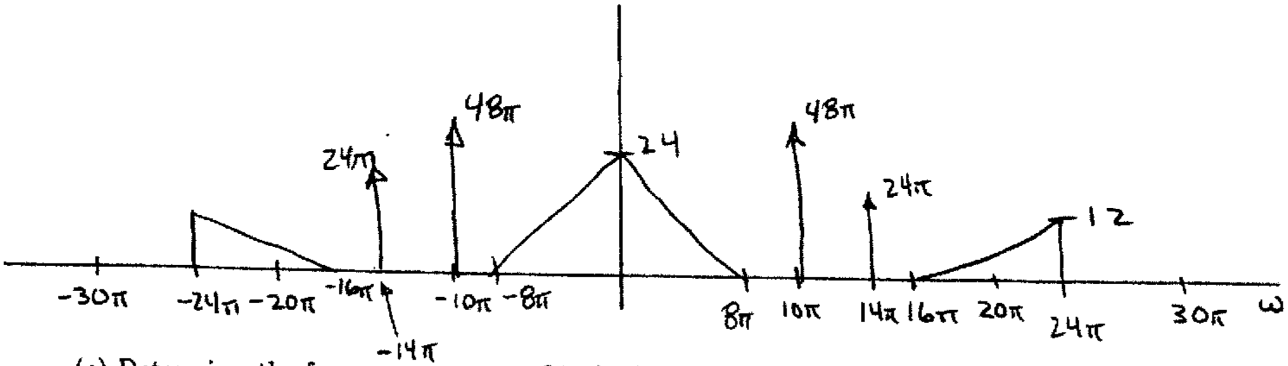


$\omega_s = 24\pi$

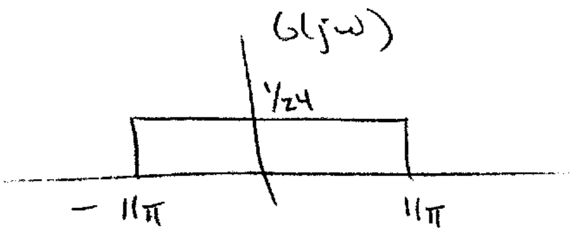
$X_\delta(j\omega)$



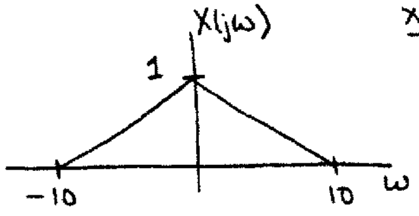
$|Y(j\omega)|$

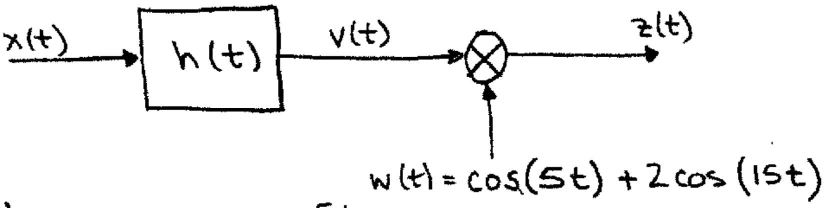


(c) Determine the frequency response $G(j\omega)$ of the inverse system for this system assuming $T = \frac{1}{12}$. Recover $X(j\omega)$ from $Y(j\omega)$



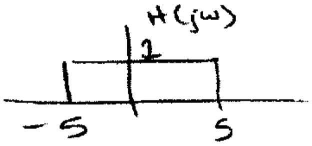
4. (20 points) Consider the system shown below.

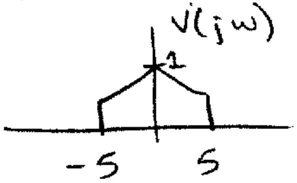




$$h(t) = \frac{\sin 5t}{\pi t}$$

(a) Find the largest sampling interval T such that $v(t)$ can be uniquely reconstructed from its samples $v[n] = v(nT)$.





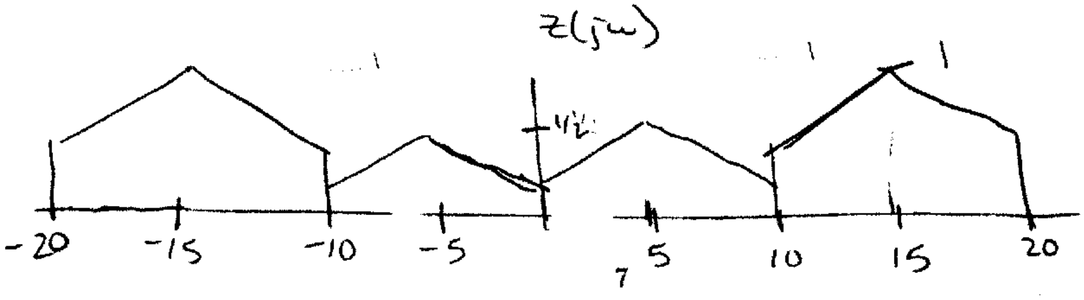
$$\omega_s = \frac{2\pi}{T} = 10 \Rightarrow T = \frac{2\pi}{10}$$

(b) Find the FT representation for $w(t)$.

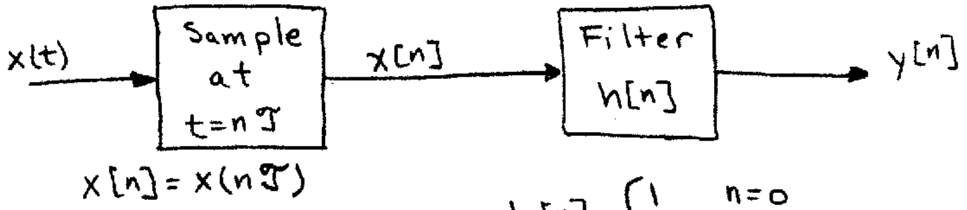
$$W(j\omega) = \pi\,\delta(\omega-5) + \pi\,\delta(\omega+5) + 2\pi\,\delta(\omega-15) + 2\pi\,\delta(\omega+15)$$

(c) Sketch the FT representation for $z(t)$ in the space provided.

$$Z(j\omega) = \frac{1}{2\pi} V(j\omega) * W(j\omega)$$



5. (20 points) The system depicted below has input $x(t)$ and output $y[n]$. The filter is LTI and has impulse response $h[n] = \delta[n] + \alpha\delta[n-1]$ with $|\alpha| < 1$.



$$h[n] = \begin{cases} 1 & n=0 \\ \alpha & n=1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Is this system stable? Prove your answer.

$$y[n] = h[n] * x[n] = x[n] + \alpha x[n-1]$$

$$|y[n]| \leq |x[n]| + |\alpha|\,|x[n-1]|$$

$$= |x(n\mathcal{T})| + |\alpha|\,|x((n-1)\mathcal{T})|$$

$$\leq M_x\,(1+|\alpha|)$$

$$< \infty$$

stable

(b) Is this system time-invariant? Prove your answer.

No, sampling looses values in between sample times. Shifting the input does not correspond to shifting the output except when the input shift is an integral multiple of T

(c) Is this system linear? Prove your answer.

$$x(t) = a x_1(t) + b x_2(t) \implies$$

$$y[n] = a x_1[n] + b x_2(n) + \alpha\left(a x_1[n-1] + b x_1[n-1]\right)$$

$$= a\left[x_1[n] + \alpha x_1[n-1]\right] + b\left[x_2[n] + \alpha x_2[n-1]\right])$$

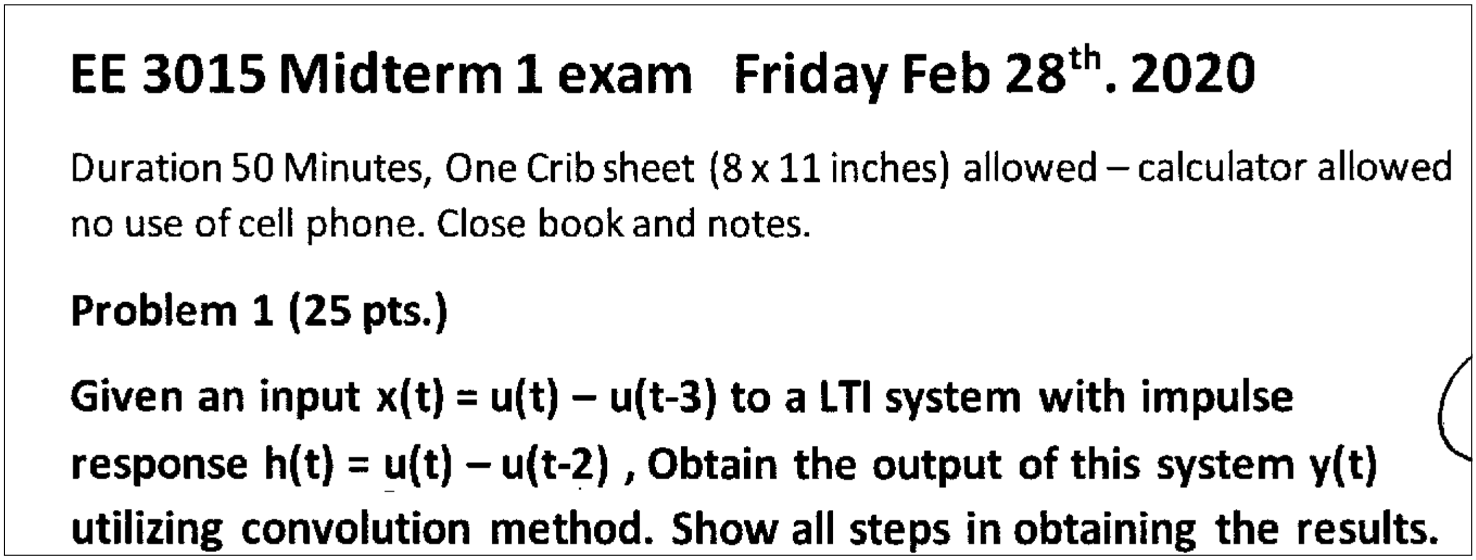
$$= a y_1[n] + b y_2(n]$$

yes, linear

3.2 Exam 1

Local contents

3.2.1 questions

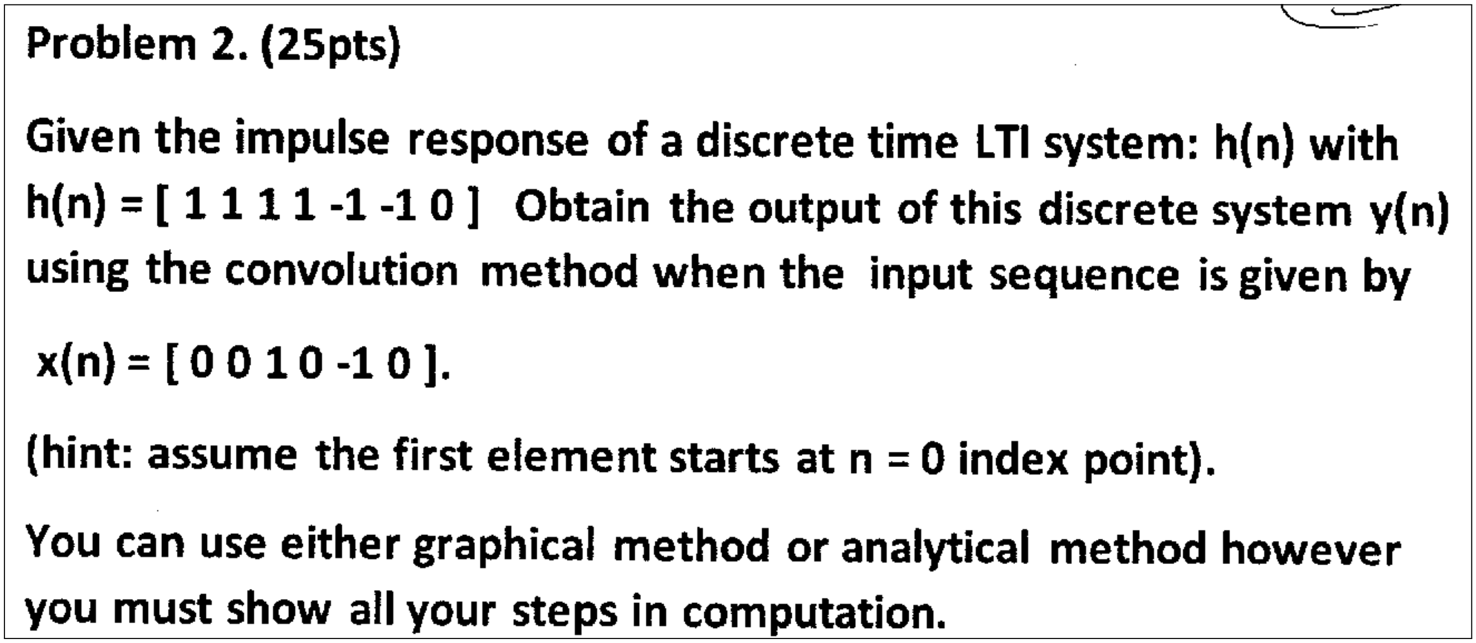


EE 3015 Midterm 1 exam Friday Feb 28^{th}. 2020

Duration 50 Minutes, One Crib sheet (8 x 11 inches) allowed – calculator allowed no use of cell phone. Close book and notes.

Problem 1 (25 pts.)

Given an input x(t) = u(t) – u(t-3) to a LTI system with impulse response h(t) = u(t) – u(t-2) , Obtain the output of this system y(t) utilizing convolution method. Show all steps in obtaining the results.



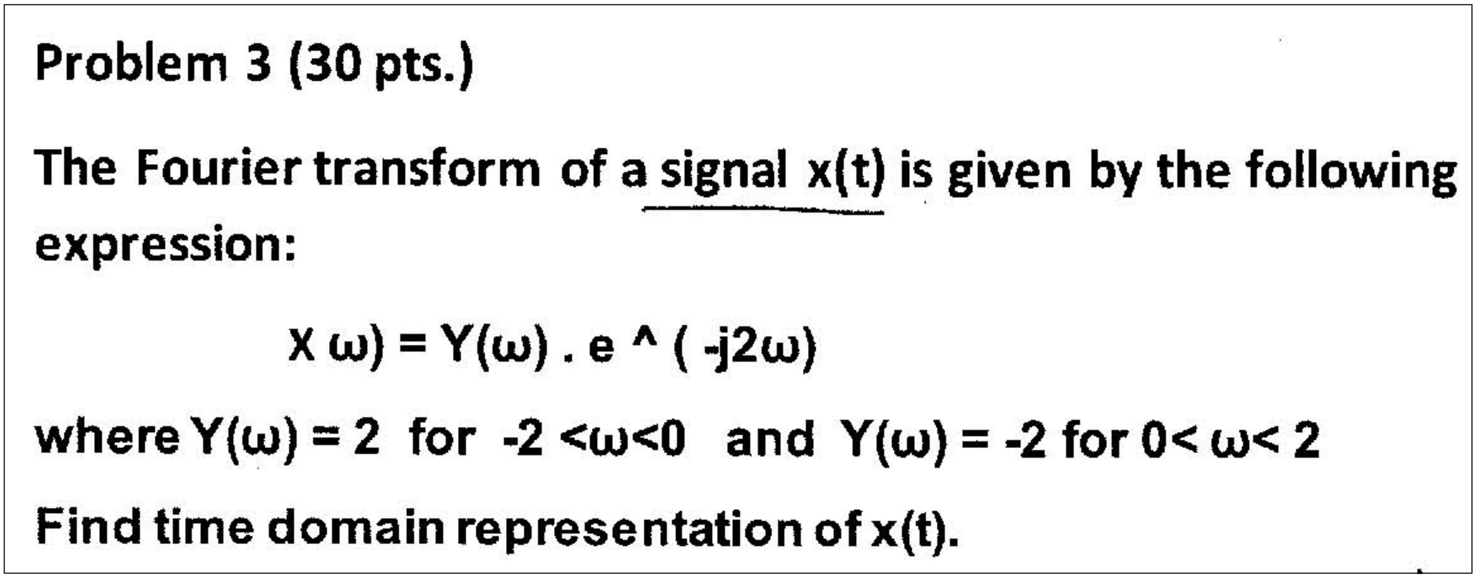
Problem 2. (25pts)

Given the impulse response of a discrete time LTI system: h(n) with h(n) = [1 1 1 1 -1 -1 0] Obtain the output of this discrete system y(n) using the convolution method when the input sequence is given by

x(n) = [0 0 1 0 -1 0].

(hint: assume the first element starts at n = 0 index point).

You can use either graphical method or analytical method however you must show all your steps in computation.



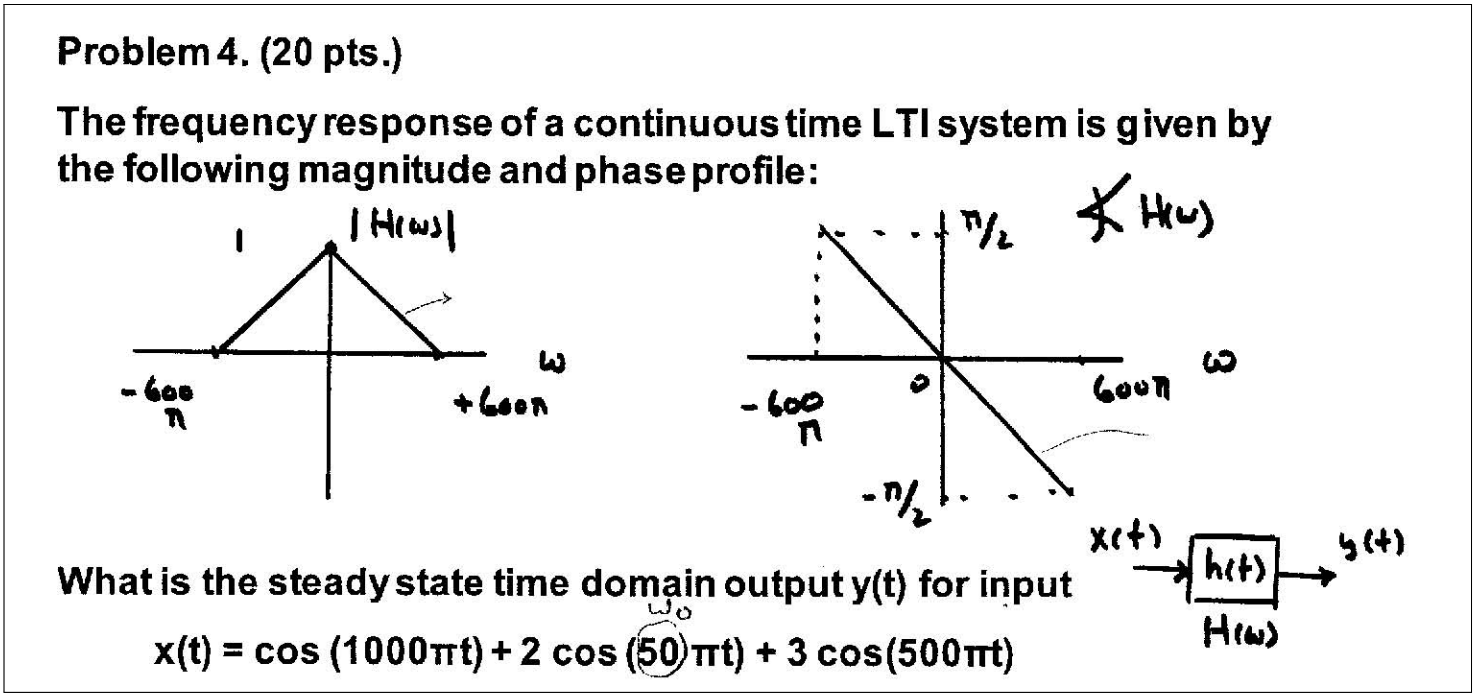
Problem 3 (30 pts.)

The Fourier transform of a signal x(t) is given by the following expression:

X ω) = Y(ω) . e ^ (-j2ω)

where Y(ω) = 2 for -2 <ω<0 and Y(ω) = -2 for 0< ω< 2

Find time domain representation of x(t).



Problem 4. (20 pts.)

The frequency response of a continuous time LTI system is given by the following magnitude and phase profile:

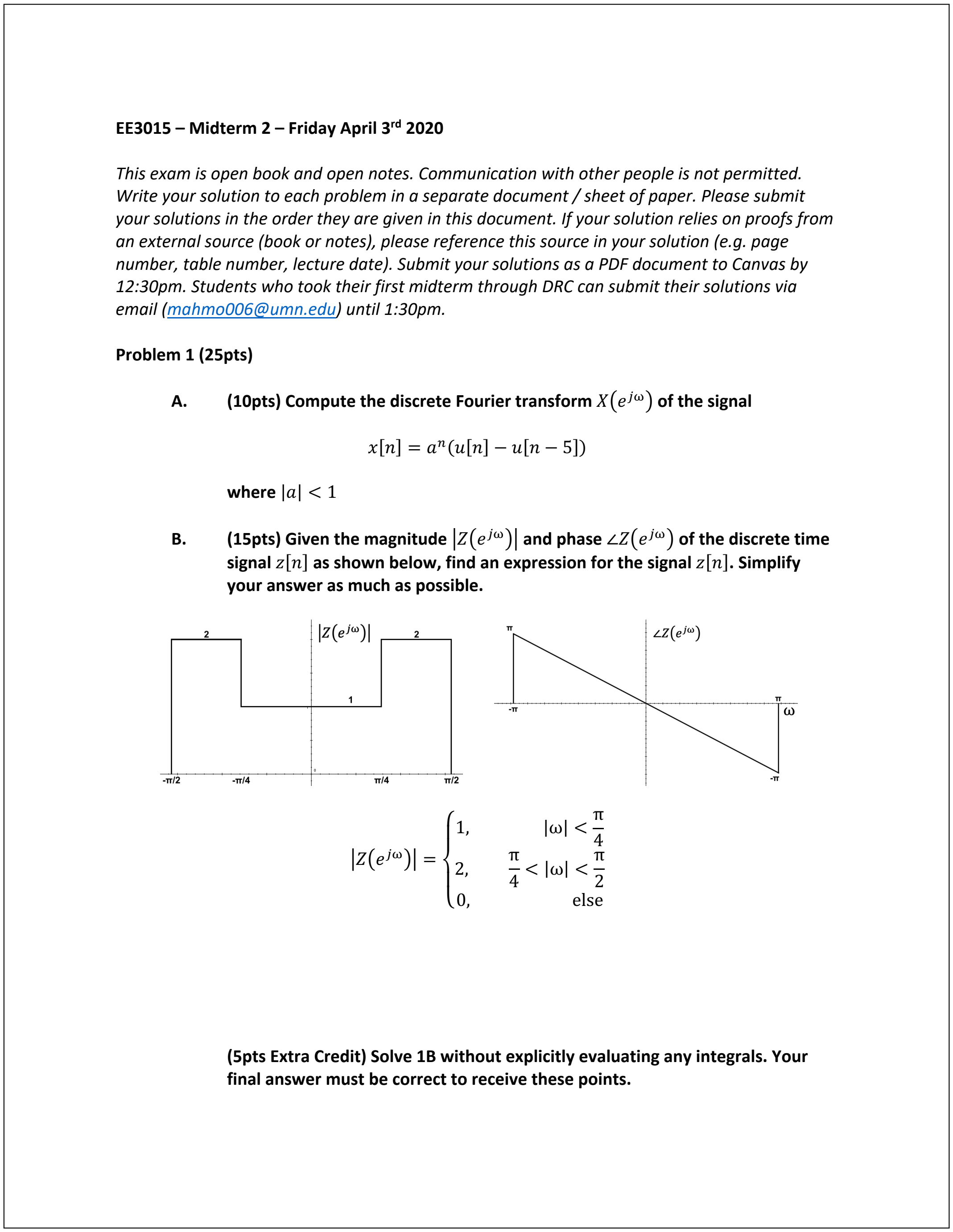
What is the steady state time domain output y(t) for input

x(t) = cos (1000πt) + 2 cos ($\overset{\omega_0}{\boxed{50}}$πt) + 3 cos(500πt)

3.3 Exam 2

Local contents

3.3.1 questions



EE3015 – Midterm 2 – Friday April 3rd 2020

This exam is open book and open notes. Communication with other people is not permitted. Write your solution to each problem in a separate document / sheet of paper. Please submit your solutions in the order they are given in this document. If your solution relies on proofs from an external source (book or notes), please reference this source in your solution (e.g. page number, table number, lecture date). Submit your solutions as a PDF document to Canvas by 12:30pm. Students who took their first midterm through DRC can submit their solutions via email (mahmo006@umn.edu) until 1:30pm.

Problem 1 (25pts)

A. **(10pts) Compute the discrete Fourier transform $X(e^{j\omega})$ of the signal**

$$x[n] = a^n(u[n] - u[n-5])$$

where $|a| < 1$

B. **(15pts) Given the magnitude $|Z(e^{j\omega})|$ and phase $\angle Z(e^{j\omega})$ of the discrete time signal $z[n]$ as shown below, find an expression for the signal $z[n]$. Simplify your answer as much as possible.**

$$|Z(e^{j\omega})| = \begin{cases} 1, & |\omega| < \frac{\pi}{4} \\ 2, & \frac{\pi}{4} < |\omega| < \frac{\pi}{2} \\ 0, & \text{else} \end{cases}$$

(5pts Extra Credit) Solve 1B without explicitly evaluating any integrals. Your final answer must be correct to receive these points.

Problem 2 (25pts)

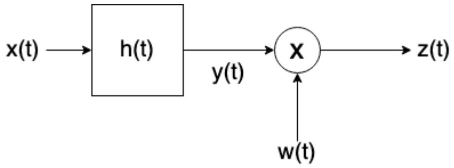
Let the signal $x(t)$ have Fourier transform

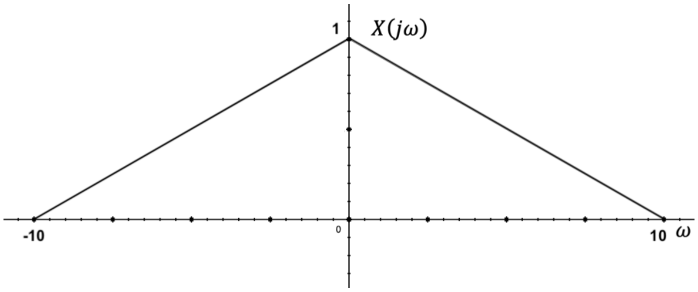
$$X(j\omega) = \left(\frac{4}{9+\omega^2}\right) * \left(e^{-j2\omega}\,\frac{sin(2\omega)}{\omega}\right)$$

where $*$ represents convolution. Use the inverse Fourier transform to determine the original signal $x(t)$. List any properties you use from notes or from the textbook.

Problem 3 (25pts)

Consider the following filtering and modulation system.





$h(t) = \frac{sin(5t)}{\pi t}$ $\qquad$ $w(t) = cos(5t) + 2\,cos(15t)$

A. **(10pts) Find the Fourier transform** $Y(j\omega)$ **of signal** $y(t)$**. Sketch** $Y(j\omega)$**.**

B. **(5pts) Find the Fourier transform** $W(j\omega)$ **of signal** $w(t)$. **Sketch** $W(j\omega)$.

C. **(10pts) Find the Fourier transform** $Z(j\omega)$ **of signal** $z(t)$**. Sketch** $Z(j\omega)$.

Problem 4 (25pts)

Consider the signals

$$x_1(t) = 3\,cos(\pi t)$$
$$x_2(t) = 2\,cos(3\pi t)$$
$$x_3(t) = 2\,sin(5\pi t)$$

and

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

A. **(5pts) Let the signals $x_1(t), x_2(t), x_3(t)$ be sampled with sampling period $T = 0.4$ to obtain sampled signals $x_1[n], x_2[n], x_3[n]$. For each of these signals, determine if the sampled signal can be used to recover the original signal without aliasing. Can $x(t)$ be recovered from these sampled signals? (e.g. can $x[n] = x_1[n] + x_2[n] + x_3[n]$ be used to recover $x(t)$ without aliasing?)**

B. **(10pts) Find the Nyquist frequency of $x(t)$.**

C. **(10pts) Let the signal $c(t)$ be given by**

$$c(t) = cos(20\pi t)$$

Now suppose the signal $x(t)$ is used to modulate the signal $c(t)$ to produce a signal $y(t)$, where

$$y(t) = x(t)c(t)$$

Find the highest frequency present in the signal $y(t)$. In other words, find the largest frequency ω where $Y(j\omega) \neq 0$.

3.3.2 key solution

1) a) i) Solution 1 (Brute force)

Using the Discrete Fourier Transform analysis equation,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Since $x[n] = a^n(u[n] - u[n-5])$, the only non-zero points of $x[n]$ are $n \in \{0, 1, 2, 3, 4\}$, so

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{4} a^n e^{-j\omega n} \\ &= \sum_{n=0}^{4} (ae^{-j\omega})^n \end{aligned}$$

Since this is a partial sum of a geometric series, we can apply the formula

$$\sum_{k=0}^{N} b^k = \frac{1 - b^{N+1}}{1 - b}, \quad |b| < 1$$

to compute

$$X(e^{j\omega}) = \frac{1 - (ae^{-j\omega})^5}{1 - ae^{-j\omega}}$$

ii) Solution 2 (Transform pairs)

Observe that

$$\begin{aligned} x[n] &= a^n u[n] - a^n u[n-5] \\ &= a^n u[n] - a^5 a^{n-5} u[n-5] \\ &= (a^n u[n]) - a^5 (a^n u[n]) * \delta[n-5] \\ &= x_1[n] - a^5 x_1[n] * \delta[n-5], \quad x_1[n] = a^n u[n] \end{aligned}$$

Using the Fourier transform pair,

$$\mathcal{F}\{x_1[n]\} = \mathcal{F}\{a^n u[n]\} = \frac{1}{1 - ae^{-j\omega}}$$

and the time shift property,

$$\mathcal{F}\{x[n] * \delta[n - n_0]\} = e^{-j\omega n_0} X(e^{j\omega})$$

we can then write

$$\begin{aligned} X(e^{j\omega}) &= \mathcal{F}\{x[n]\} \\ &= \mathcal{F}\{x_1[n]\} - a^5 \mathcal{F}\{x_1[n] * \delta[n-5]\} \\ &= X_1(e^{j\omega}) - a^5 e^{-5j\omega} X_1(e^{j\omega}) \\ &= \frac{1}{1 - ae^{-j\omega}} - \frac{a^5 e^{-5j\omega}}{a - ae^{-j\omega}} \end{aligned}$$

b) i) Solution 1 (Brute force)

I wrote all of this using $X(e^{j\omega})$ and $x[n]$ instead of $Z(e^{j\omega})$ and $z[n]$ and I don't want to go back and change all of them. Soorryyyyyyyyyyy.

We use the Continuous Fourier Transform synthesis equation,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

We can express $X(e^{j\omega})$ in terms of it's phase and magnitude. By inspecting the slope of $\angle X(e^{j\omega})$, we see that $\angle X(e^{j\omega}) = -\omega$, so

$$\begin{aligned} X(e^{j\omega}) &= |X(e^{j\omega})| * e^{\angle X(e^{j\omega})} \\ &= e^{-j\omega} * \begin{cases} 2, & -\pi/2 < \omega < -\pi/4 \\ 1, & -\pi/4 < \omega < \pi/4 \\ 2, & \pi/4 < \omega < \pi/2 \\ 0, & \text{else} \end{cases} \end{aligned}$$

We take the integral in the Continuous Fourier Transform synthesis equation to be from $-\pi$ to π so that the bounds are symmetric. Using the piecewise form of $X(e^{j\omega})$ to break up this bound into intervals, we get

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{-\pi/4} 2 * e^{-j\omega} * e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} 1 * e^{-j\omega} * e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi/2} 2 * e^{-j\omega} * e^{j\omega n}d\omega \\ &= \frac{2}{2\pi}\int_{-\pi/2}^{-\pi/4} e^{j\omega(n-1)}d\omega + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-1)}d\omega + \frac{2}{2\pi}\int_{\pi/4}^{\pi/2} *e^{j\omega(n-1)}d\omega \\ &= \frac{2}{2\pi j(n-1)}(e^{j\omega(n-1)})|_{-\pi/2}^{-\pi/4} + \frac{1}{j(n-1)}(e^{j\omega(n-1)})|_{-\pi/4}^{\pi/4} + \frac{2}{j(n-1)}(e^{j\omega(n-1)})|_{\pi/4}^{\pi/2} \\ &= \frac{1}{2\pi j(n-1)}(2e^{-j\pi(n-1)/4} - 2e^{-j\pi(n-1)/2} + e^{j\pi(n-1)/4} - e^{-j\pi(n-1)/4} + 2e^{j\pi(n-1)/2} - 2e^{j\pi(n-1)/4}) \end{aligned}$$

Rearranging terms,

$$\begin{aligned} x[n] &= \frac{1}{2\pi j(n-1)}((2e^{j\pi(n-1)/2} - 2e^{-j\pi(n-1)/2}) + (2e^{-j\pi(n-1)/4} - e^{-j\pi(n-1)/4}) + (e^{j\pi(n-1)/4} - 2e^{j\pi(n-1} \\ &= \frac{1}{2\pi j(n-1)}((2e^{j\pi(n-1)/2} - 2e^{-j\pi(n-1)/2}) + e^{-j\pi(n-1)/4} - e^{j\pi(n-1)/4}) \\ &= \frac{1}{\pi(n-1)}(2(\frac{1}{2j}e^{j\pi(n-1)/2} - \frac{1}{2j}e^{-j\pi(n-1)/2}) - (\frac{1}{2j}e^{j\pi(n-1)/4} - \frac{1}{2j}e^{-j\pi(n-1)/4}) \end{aligned}$$

Using the definition of the sin function,

$$\sin(x) = \frac{1}{2j}e^{jx} - \frac{1}{2j}e^{-jx}$$

, We can rewrite this as

$$\frac{1}{\pi(n-1)}(2\sin(\pi(n-1)/2) - \sin(\pi(n-1)/4))$$

ii) Solution 2 (Transform pairs) (Extra credit)

Consider a version $X_1(e^{j\omega}$ of $X(e^{j\omega})$ with zero phase, that is

$$|X_1(e^{j\omega})| = |X(e^{j\omega})|, \quad \angle X_1(e^{j\omega}) = 0$$

Such that

$$\begin{aligned} X_1(e^{j\omega}) &= |X_1(e^{j\omega})|e^{j\angle X_1(e^{j\omega})} \\ &= |X(e^{j\omega})|e^{j*0} \\ &= |X(e^{j\omega})| \end{aligned}$$

Then we can write $X(e^{j\omega})$ as

$$\begin{aligned} X(e^{j\omega}) &= |X(e^{j\omega})|e^{j\angle X(e^{j\omega})} \\ &= X_1(e^{j\omega})e^{-j\omega} \end{aligned}$$

When comparing this expression with the time delay property, we can see that the phase $e^{-j\omega}$ is really just shifting the signal $x_1[n]$ by 1 unit, so that

$$x[n] = x_1[n-1]$$

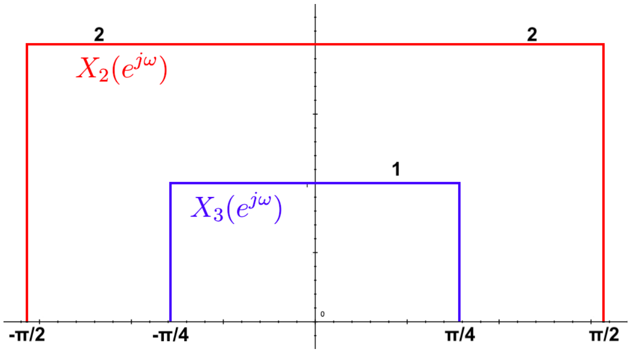
Effectively, this allows us to just find the inverse Fourier transform of the magnitude on it's own, and then apply a time shift of 1 unit at the end to account for the phase of the system. This is a nice property that we can use in general on systems that have linear phase

To determine $x_1[n]$, we consider the rectangle wave $X_1(e^{j\omega})$ as a difference of two rectangle waves. Specifically, define

$$X_2(e^{j\omega}) = \begin{cases} 2, & -\pi/2 < \omega < \pi/2 \\ 0, & \text{else} \end{cases}$$

$$X_3(e^{j\omega}) = \begin{cases} 1, & -\pi/4 < \omega < \pi/4 \\ 0, & \text{else} \end{cases}$$

or graphically,



Then it is clear that

$$X_1(e^{j\omega}) = X_2(e^{j\omega}) - X_3(e^{j\omega})$$

and therefore

$$x_1[n] = x_2[n] - x_3[n]$$

In table 5.2, we see the Fourier transform pair

$$\mathcal{F}^{-1}\{\begin{cases} 1, & -W < \omega < W \\ 0, & \text{else} \end{cases}\} = \frac{\sin(Wn)}{\pi n}$$

Since $X_2(e^{j\omega})$ and $X_3(e^{j\omega})$ are already in this form, we can easily find $x_2[n]$ and $x_3[n]$ as

$$x_2[n] = \frac{2\sin(\pi n/2)}{\pi n}$$

$$x_3[n] = \frac{\sin(\pi n/4)}{\pi n}$$

Then

$$x_1[n] = x_2[n] - x_3[n] = \frac{2\sin(\pi n/2)}{\pi n} - \frac{\sin(\pi n/4)}{\pi n}$$

and

$$x[n] = x_1[n-1] = \frac{2\sin(\pi(n-1)/2)}{\pi(n-1)} - \frac{\sin(\pi(n-1)/4)}{\pi(n-1)}$$

2) The main focus of this problem is to break the expression up fractally, handle it in small portions, and then build it back up into the final answer. From our initial expression,

$$X(j\omega) = (\frac{4}{9+\omega^2}) * (e^{-2j\omega}\frac{\sin(2\omega)}{\omega})$$

Make the definitions

$$X_1(j\omega) = \frac{4}{9+\omega^2}$$

$$X_2(j\omega) = e^{-2j\omega}\frac{\sin(2\omega)}{\omega}$$

So that

$$X(j\omega) = X_1(j\omega) * X_2(j\omega)$$

and, using the multiplication property,

$$x(t) = 2\pi x_1(t)x_2(t)$$

i) To handle $X_1(j\omega) = \frac{4}{9+\omega^2}$, we see that this expression has similar form to the Fourier Transform pair found in the provided tables,

$$\mathcal{F}\{e^{-\alpha|t|}\} = \frac{2\alpha}{\alpha^2+\omega^2}$$

In order to apply this pair to $X_1(j\omega)$, we need to have it in this exact form. Therefore, we need to say that

$$X_1(j\omega) = \frac{4}{9+\omega^2} = \frac{4}{6} * \frac{2(3)}{3^2+\omega^2}$$

to find that

$$x_1(t) = \frac{4}{6}e^{-3|t|}$$

ii) To handle $X_2(j\omega) = e^{-2j\omega}\frac{\sin(2\omega)}{\omega}$, we again need to break the problem up into smaller parts. We make the additional definition that

$$X_3(j\omega) = \frac{\sin(2\omega)}{\omega}$$

so that

$$X_2(j\omega) = e^{-2j\omega}X_3(j\omega)$$

From this, we can clearly invoke the same time shifting property that we used in problem 1(b) to show that

$$x_2(t) = x_3(t-2)$$

Then, to find $x_3(t)$, we see the Fourier transform pair

$$\mathcal{F}\{\begin{cases} 1, & -T_1 < t < T_1 \\ 0, & \text{else} \end{cases}\} = \frac{2\sin(\omega T_1)}{\omega}$$

We again need to exactly match $X_3(j\omega)$ to this form to invoke the property. By rewriting

$$X_3(j\omega) = \frac{\sin(2\omega)}{\omega} = \frac{1}{2} * \frac{2\sin(2\omega)}{\omega}$$

We can then use this property to say that

$$x_3(t) = \begin{cases} \frac{1}{2}, & -2 < t < 2 \\ 0, & \text{else} \end{cases}$$

then

$$x_2(t) = x_3(t-2) = \begin{cases} \frac{1}{2}, & -2 < t-2 < 2 \\ 0, & \text{else} \end{cases} = \begin{cases} \frac{1}{2}, & 0 < t < 4 \\ 0, & \text{else} \end{cases}$$

In general, if something says $|t| < T$, this is the same as $-T < t < T$.

Now that we know $x_1(t)$ and $x_2(t)$, we can use our original relation

$$x(t) = 2\pi x_1(t)x_2(t)$$

to write

$$x(t) = 2\pi * \frac{4}{6}e^{-3|t|} * \begin{cases} \frac{1}{2}, & 0 < t < 4 \\ 0, & \text{else} \end{cases} = \begin{cases} \frac{2\pi}{3}e^{-3|t|}, & 0 < t < 4 \\ 0, & \text{else} \end{cases}$$

4) We start by computing the Nyquist frequency for each signal $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x(t)$. This is twice the maximum frequency present in each signal. Then

$$\omega_{\text{Nyquist, } x_1} = 2 * \pi, \quad f_{\text{Nyquist, } x_1} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_1} = 1\text{Hz}$$

$$\omega_{\text{Nyquist, } x_2} = 2 * 3\pi, \quad f_{\text{Nyquist, } x_2} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_2} = 3\text{Hz}$$
$$\omega_{\text{Nyquist, } x_3} = 2 * 5\pi, \quad f_{\text{Nyquist, } x_3} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_3} = 5\text{Hz}$$
$$\omega_{\text{Nyquist, } x} = 2 * 5\pi, \quad f_{\text{Nyquist, } x} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x} = 5\text{Hz}$$

a) The sampling rate $T = 0.4$ corresponds to a sampling frequency $f_s = \frac{1}{T} = 2.5\text{Hz}$. This is less than the Nyquist frequencies of $x_2(t)$, $x_3(t)$, and $x(t)$, so these signals cannot be recovered in this sort of sampling scheme. However, $x_1(t)$ can be.

b) As we showed earlier, $f_{\text{Nyquist, } x} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x} = 5\text{Hz}$

c) By the multiplication property, for $y(t) = x(t)c(t)$,

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * C(j\omega)$$

Using the properties of cosines and sines, we can quickly find that

$$C(j\omega) = \pi\delta(\omega - 20\pi) + \pi\delta(\omega + 20\pi)$$

Then

$$Y(j\omega) = \frac{1}{2\pi}(\pi\delta(\omega - 20\pi) + \pi\delta(\omega + 20\pi)) * X(j\omega)$$

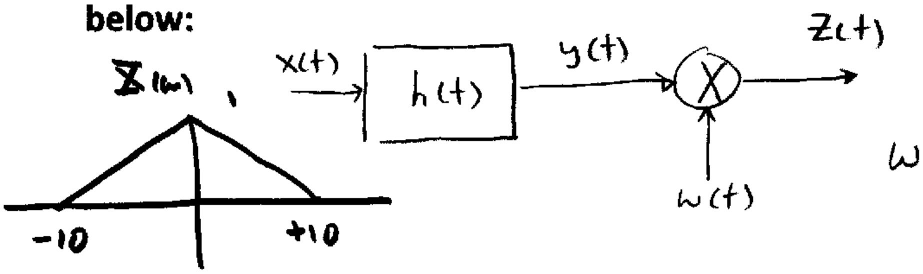
Using the properties of convolution with delta functions,

$$Y(j\omega) = \frac{1}{2}X(j(\omega - 20\pi)) + \frac{1}{2}X(j(\omega + 20\pi))$$

This means that $Y(j\omega)$ will contain all the frequencies present in $X(j\omega)$, shifted either up or down by 20π radians. As the maximum frequency in $X(j\omega)$ is 5π, the maximum frequency in $Y(j\omega)$ will be that shifted up by 20π, so $20\pi + 5\pi = 25\pi$. This can be confirmed by finding $X(j\omega)$ and substituting this into the above expression for $Y(j\omega)$, which I will not do here for sake of legibility.

page 3.

Problem 3. (25 pts.) Consider the filtering and modulation system below:



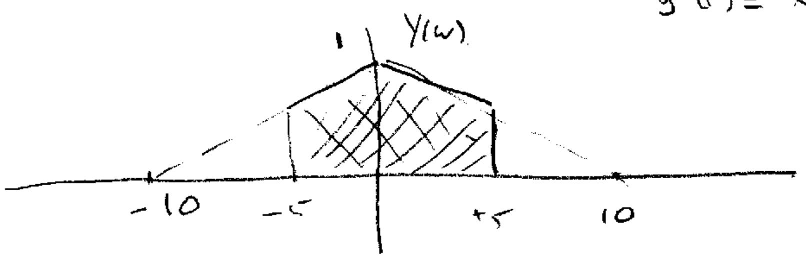
$h(t) = \frac{\sin(5t)}{\pi t}$

$w(t) = \cos(5t) + 2\cos(15t)$

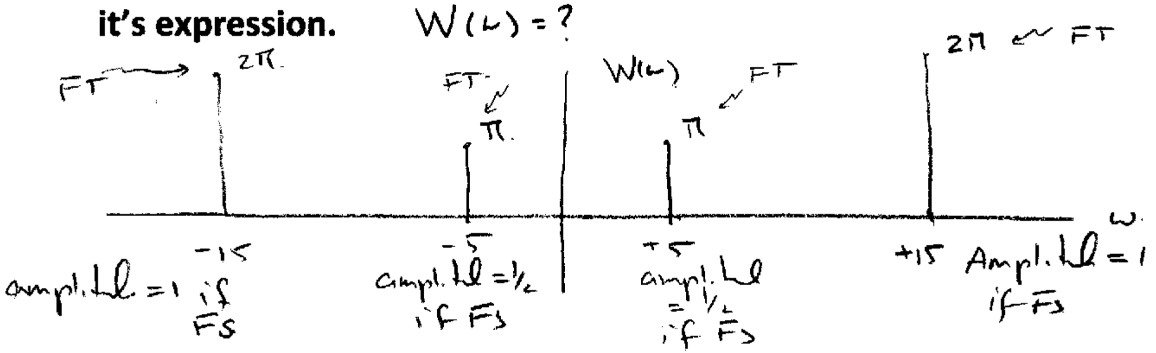
A. (10 pts.) Sketch the Fourier Transform of signal y(t) and write up it's expression. $Y(w) = ?$

$Y(w) = X(w) * H(w)$

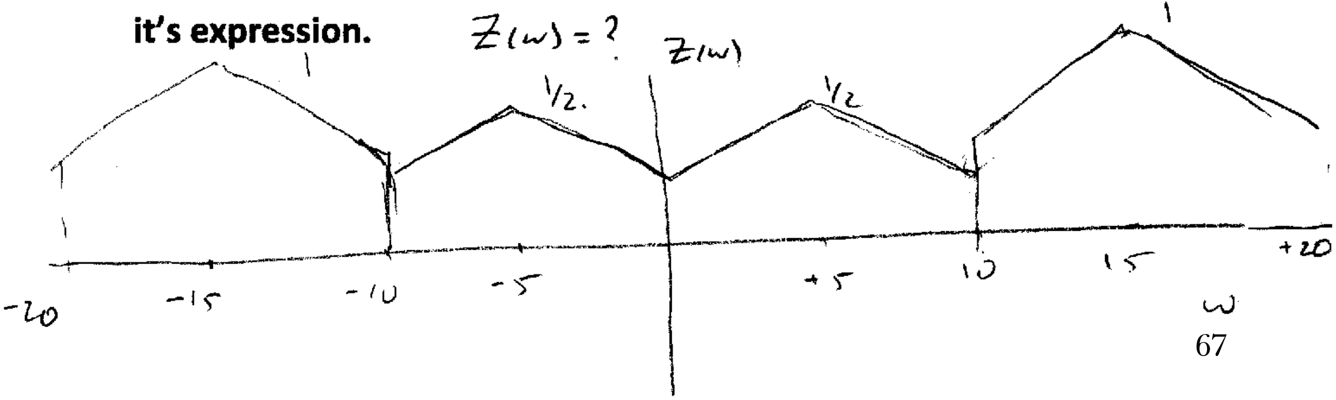
$y(t) = x(t) . h(t)$



B. (5 pts.) Sketch the Fourier transform of signal w(t) and write up it's expression. $W(w) = ?$



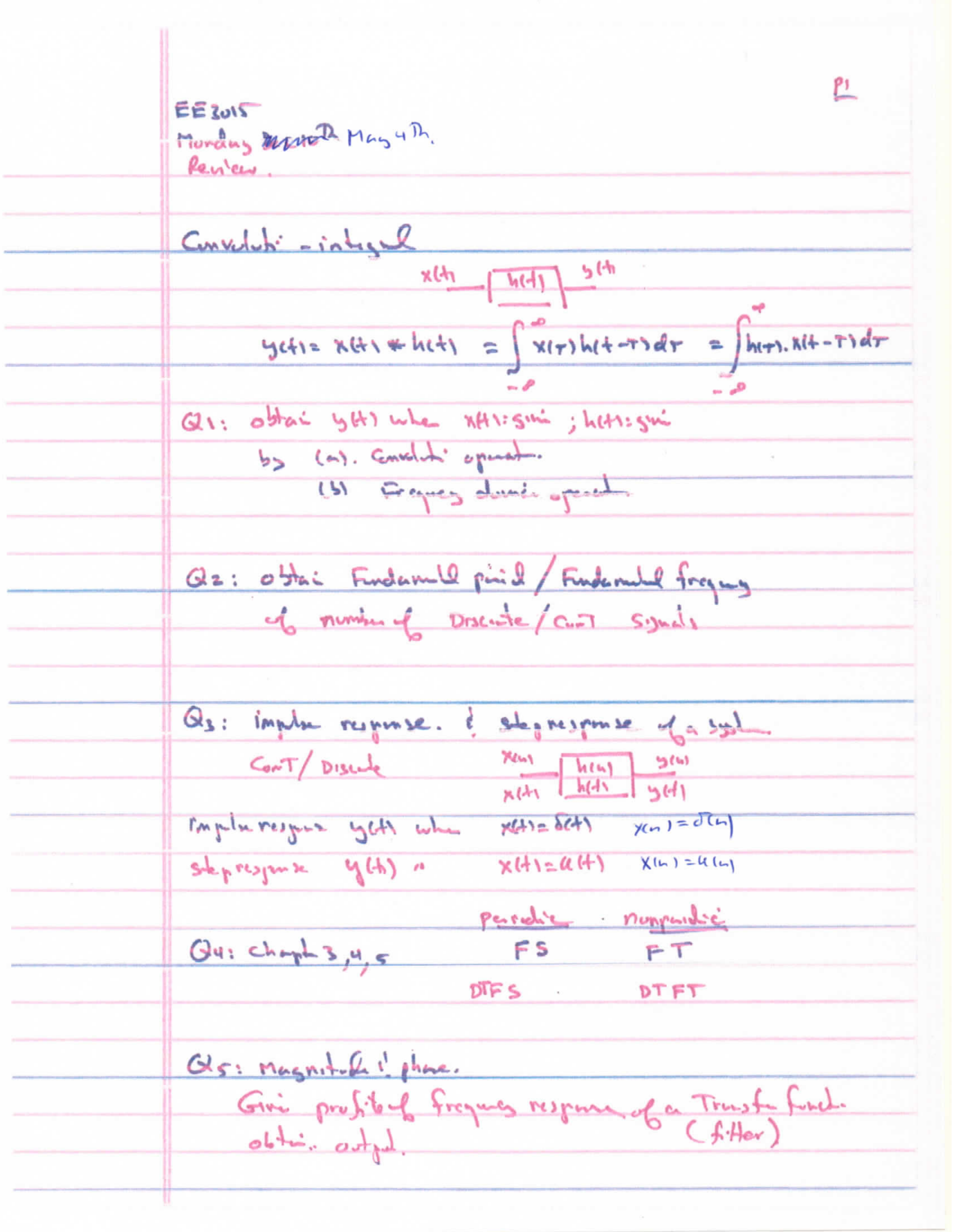
C. (10 pts.) Sketch the Fourier transform of signal z(t) and obtain it's expression. $Z(w) = ?$



3.4 final exam

Local contents

3.4.1 review for final exam



P1

EE3015
Monday May 4th.
Review.

Convolution - integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau).x(t-\tau)d\tau$$

Q1: obtain y(t) when x(t): given ; h(t): given
by (a). Convolution operation.
(b) Frequency domain operation.

Q2: obtain Fundamental period / Fundamental frequency
of number of Discrete / Cont Signals

Q3: impulse response. & step response of a system
Cont / Discrete

impulse response y(t) when $x(t) = \delta(t)$ $x(n) = \delta(n)$
step response y(t) " $x(t) = u(t)$ $X(n) = u(n)$

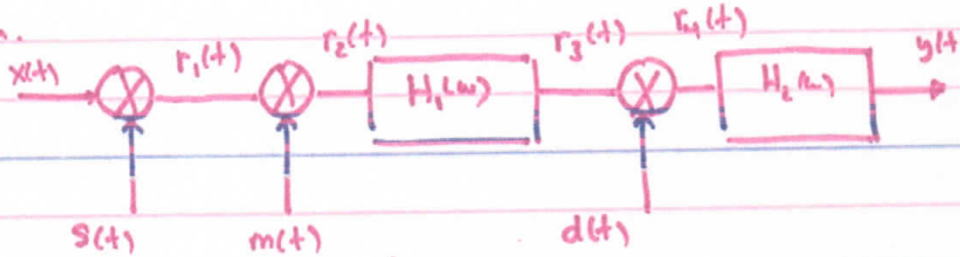
Q4: chapter 3, 4, 5

	periodic	nonperiodic
	FS	FT
	DTFS	DTFT

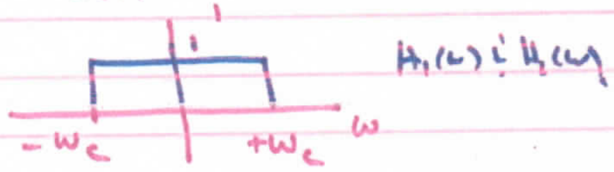
Q5: Magnitude & phase.
Given profile of frequency response of a Transfer function (filter)
obtain output.

P2

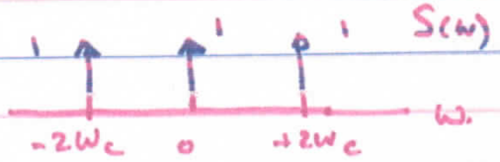
problem.



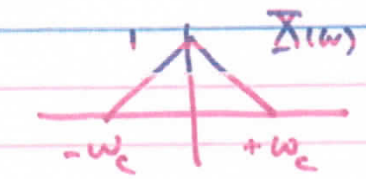
Both have $H_1(w)$ & $H_2(w)$



S(t) has frequency profile ⇒



x(t) has frequency profile



Question:? For which of the following choices for m(t) and d(t) y(t) is non zero.

(a) $m(t)=1$ $d(t)=1$ *

(b) $m(t)=\cos(w_c t)$ $d(t)=\cos(w_c t)$ *

(c) $\sin(w_c t)=m(t)$ $d(t)=\sin(w_c t)$

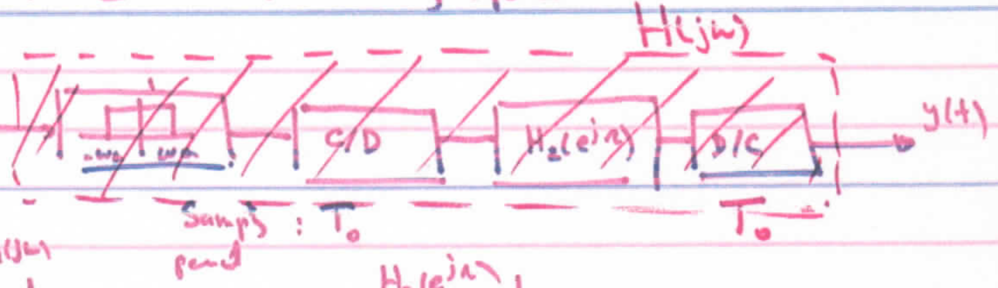
(d) $m(t)=\cos(2w_c t)$ $d(t)=\cos(2w_c t)$

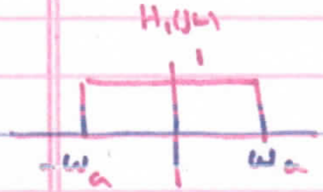
(e) $m(t)=\cos(2w_c t)$ $d(t)=\cos(w_c t)$

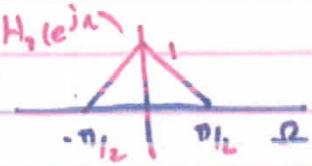
Wed Dec 12Th
EE 345

P1

problem: Given This Filtering operation.

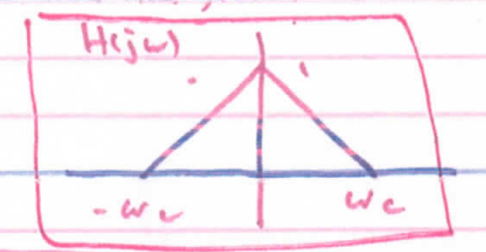






input signal x(t) has Fourier representation as

Q: Find in terms of w_c The value of The sampling period T_0 and Corresponding w_a Such That The Total continuous time filter has.



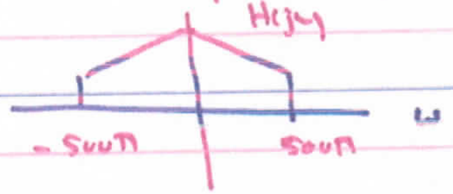
$$\boxed{T_0 = \frac{\pi/2}{w_c}} \; ?$$

$$\frac{2\pi}{T_0} = \frac{2\pi}{(\pi/2)/w_c} = 4w_c$$

$$\boxed{W_a = 2w_c} \; ?$$

P2

problem. Consider ~~that~~ The frequency response
$H(j\omega)$



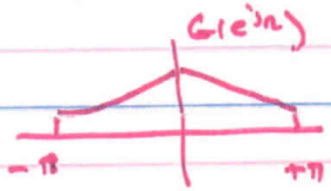
we want to implement This continuous filter using
Discrete-time-processing

(a) what is The maximum value of sampling period T
required T=?

(b) what is The required $H(e^{j\Omega})$

(c) sketch The Total system

$$T_{max} = \frac{\pi}{500\pi} = 2 \text{ msec.} \qquad \Omega = \pi$$



Laplace - Transform.

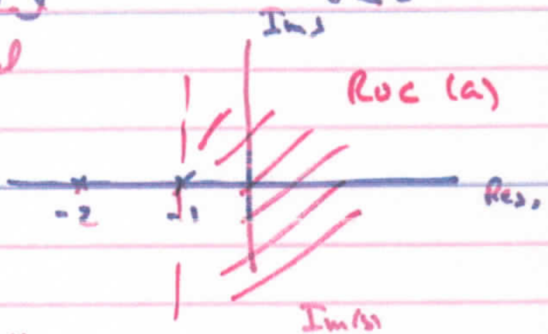
problem. Determine The following f

$$X(s) = \frac{1}{(s+1)(s+2)}$$

(a) x(t) if it is right handed   toward positive t > 0

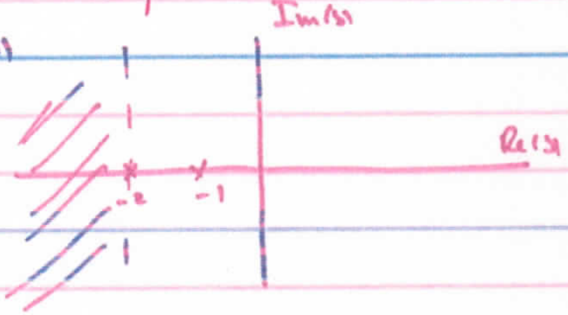
(b) x(t) is left handed   t < 0

(c) x(t) is two sided



$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

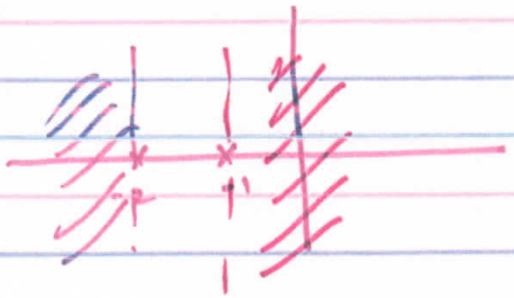
(a) $x(t) = e^{-t}u(t) - e^{-2t}u(t)$



(b)

$$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$$

(c)



$$x(t) = e^{-t}u(t) + e^{-2t}u(-t)$$

problem: Given a LTI Causal System

where

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t) + 2\frac{dx(t)}{dt}$$

84

(a) Determine The system transfer function $H_c(s)$ and obtain $h_c(t)$ Assuming all initial Cond. are at zero.

(b). Determine The system $H_d(z)$ of a discrete-Time LTI Causal system obtained from $H_c(s)$ via impulse invariant method

$$\Omega = \omega \cdot T$$

$$H_c(s) \qquad S = j\omega \qquad Z = e^{j\Omega}$$

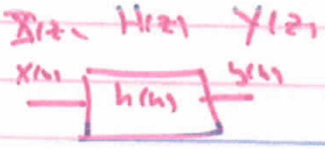
$$Z = e^{sT}$$

$$H_c(s) = \frac{2s+1}{s^2+5s+6} \quad ? \quad H_d(z)$$

$$S = \frac{1}{T}\ln(z)$$

$$H_d(z) = \frac{2\left(\frac{1}{T}\ln(z)\right)+1}{\left(\frac{1}{T}\ln(z)\right)^2+5\left(\frac{1}{T}\ln(z)\right)+6}$$

in Discrete LTI Causal system



problem Given The difference eq.

$$y(n) - 3y(n-1) + 2y(n-2) = x(n)$$

(a) Find $H(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})} = \frac{2}{1-2z^{-1}} + \frac{-1}{1-z^{-1}}$

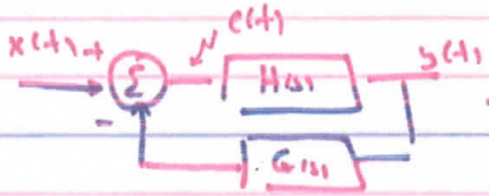
(b) Find impulse response $h(n) = ? = 2(2^n)u(n) + (-1)(1^n)u(n)$

~~(d)~~

(c) is This stable (unstable) $z_1 = 1$, $z_2 = 2$

P5

problem. Consider two feed back system

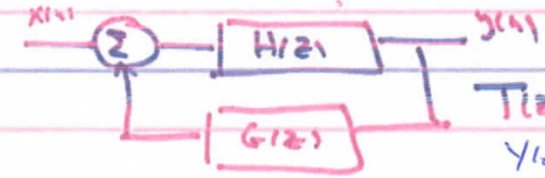


$$T(s) = \frac{H(s)}{1 + G(s)H(s)}$$

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

$x(t) = \delta(t)$ $X(s) = 1$

$y(t)$:



$$T(z) = \frac{H(z)}{1 + G(z)H(z)}$$

$$\frac{Y(z)}{X(z)}$$

$x(n) = \delta(n)$

$X(z) = 1$

if $H(s) = \frac{1}{s+3}$, $G(s) = s+1$

& $H(z) = \frac{2}{3} - \frac{1}{6}z^{-1}$ $G(z) = \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}$

Find The impulse responses of each closed loop system

3.4.2 questions

EE3015 – Final Exam – Saturday, May 9th 2020

This exam is open book and open notes. Communication with other people is not permitted. Write your solution to each problem in a separate document / sheet of paper. Please submit your solutions in the order they are given in this document. If your solution relies on proofs from an external source (book or notes), please reference this source in your solution (e.g. page number, table number, lecture date). Submit your solutions as a PDF document to Canvas by 5:00pm. If you are registered with the DRC, please ask for additional details regarding submission.

Problem 1 (25pts)

Consider a causal LTI system with transfer function

$$\boldsymbol{H(s) = \frac{-10s + 10}{(s + 10)(s + 1)}}$$

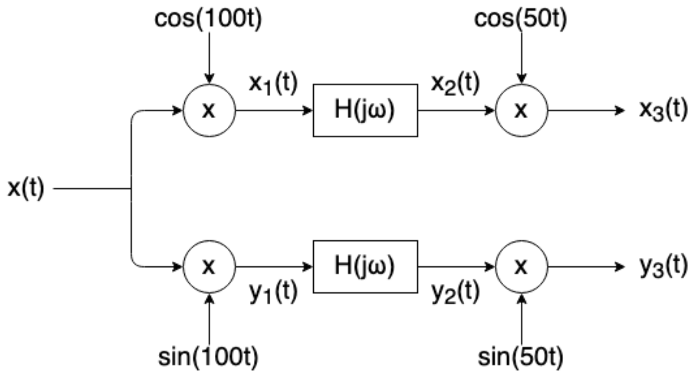
A. **(5pts) Write a differential equation in terms of $\boldsymbol{x(t), y(t)}$ (and their derivatives) realizing this system**

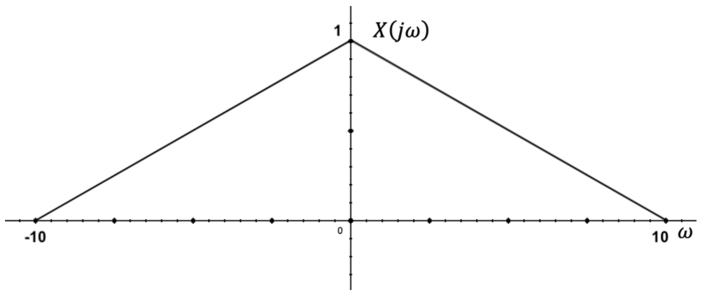
B. **(10pts) Find the impulse response $\boldsymbol{h(t)}$ of this system. Determine the ROC of the systems transfer function.**

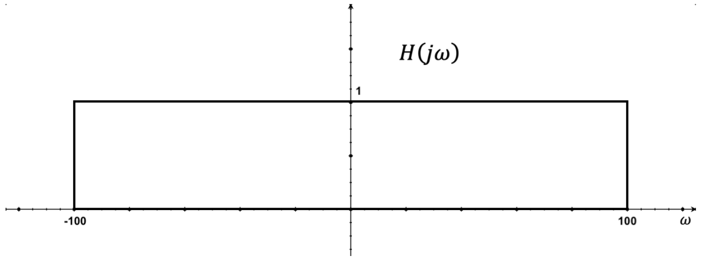
C. **(10pts) Find the unit step response of the system, i.e. compute $\boldsymbol{y(t)}$ for $\boldsymbol{x(t) = u(t)}$**

Problem 2 (30pts)

Consider the following block diagram representing a modulating system





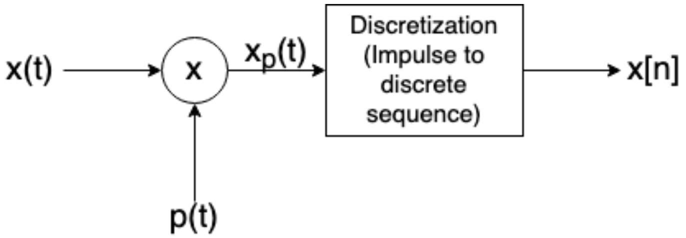


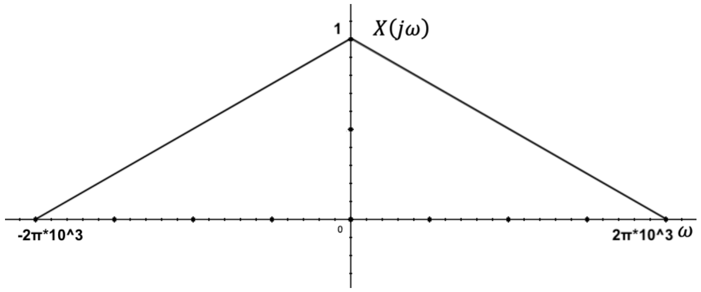
A. **(15pts) Sketch and label $X_1(j\omega), X_2(j\omega), X_3(j\omega)$**

B. **(15pts) Sketch and label $Y_1(j\omega), Y_2(j\omega), Y_3(j\omega)$**

Problem 3 (30pts)

Consider the following Analog / Digital conversion system





where

$$\boldsymbol{p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \qquad T = 0.5 \times 10^{-3}}$$

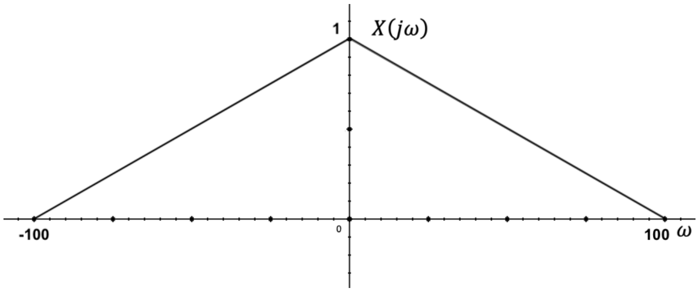
A. **(10pts) Find an expression for $X(j\omega)$, the Fourier transform of $x(t)$. Sketch and label it.**

B. **(10pts) Find an expression for $X_p(j\omega)$, the Fourier transform of $x_p(t)$. Sketch and label it.**

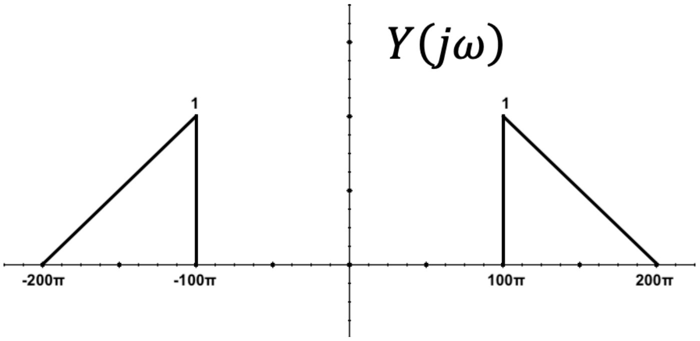
C. **(10pts) Find an expression for $X(e^{j\omega})$, the Fourier transform of $x[n]$. Sketch and label it.**

Problem 4 (25pts)

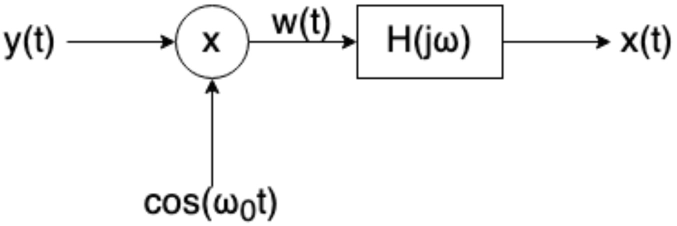
Consider a signal with Fourier domain representation $X(j\omega)$ of the form



We seek to broadcast this signal to a receiver. To achieve this, we filter and modulate $X(j\omega)$ into the signal $Y(j\omega)$, which is then transmitted. $Y(j\omega)$ is of the form



We seek to design a receiver that can recover the signal $X(j\omega)$ from the transmitted signal $Y(j\omega)$. This receiver takes the form of the following block diagram.

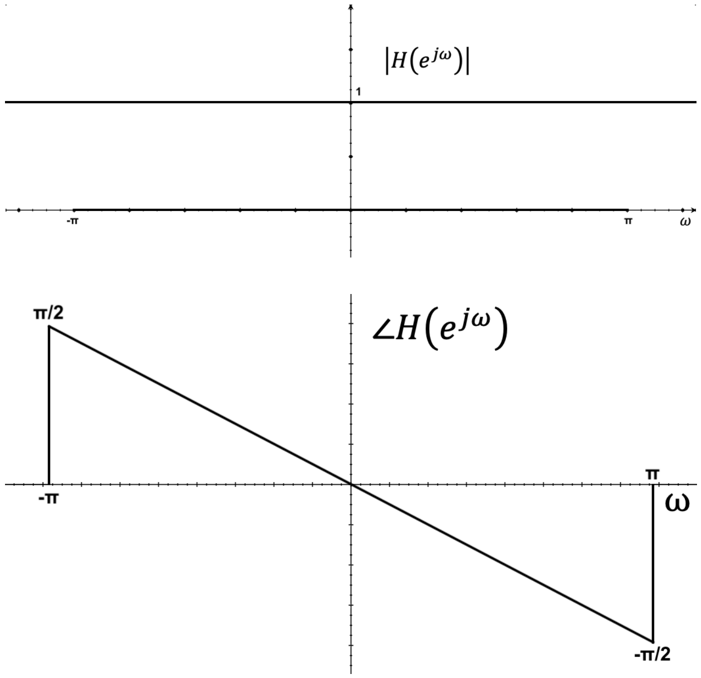


A. **(15pts) Determine ω_0. Sketch and label $W(j\omega)$.**

B. **(10pts) Assume that $H(j\omega)$ is an ideal filter (i.e. it's Fourier Transform $H(j\omega)$ is a sum of rectangle waves). Sketch and label $H(j\omega)$. Furthermore, specify what type of filter this is, and the cutoff frequency and gain of the filter.**

Problem 5 (30pts)

Consider the causal LTI system with transfer function $H(e^{j\omega})$, characterized by



A. **(10pts) For $x[n] = cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right)$, compute $X(e^{j\omega})$, the Fourier transform of $x[n]$**

B. **(20pts) Assume that $x[n]$ from part (A) is fed as an input to the LTI system characterized by $H(e^{j\omega})$ to produce an output signal $y[n]$. Compute $Y(e^{j\omega})$ and $y[n]$.**

Problem 6 (20pts)

Consider the difference equation

$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] - 2x[n-1]$$

A. **(5pts) Find the Z-domain transfer function of this system, $H(z) = \frac{Y(z)}{X(z)}$.**

B. **(5pts) Identify the poles, zeroes, and ROC of the transfer function you obtained in (A).**

C. **(10pts) Find the impulse response of this system, $h[n]$**

Chapter 4

HWs

Local contents

4.1 HW 1

Local contents

4.1.1 Problem 1.8, Chapter 1

Express the real part of each of the following signals in the form $Ae^{-at}\cos\left(\omega t+\phi\right)$, where A,a,ω,ϕ are real numbers with $A>0$ and $-\pi<\phi\leq\pi$: (a) $x_1(t)=-2$, (b) $x_2(t)=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t+2\pi)$, (c) $x_3(t)=e^{-t}\sin(3t+\pi)$, (d) $x_4(t)=je^{\left(-2+100j\right)t}$

Solution

4.1.1.1 part a

$$x_1(t)=-2$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A=2,a=0,\phi=0,\omega=0,\phi=-\pi$$

4.1.1.2 part b

$$x_2(t)=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t+2\pi)$$

Since $\cos(3t+2\pi)=\cos(3t)$, the above becomes

$$\begin{aligned}x_2(t)&=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t)\\&=\sqrt{2}\left(\cos\frac{\pi}{4}+j\sin\frac{\pi}{4}\right)\cos(3t)\\&=\sqrt{2}\left(\frac{1}{2}\sqrt{2}+j\frac{1}{2}\sqrt{2}\right)\cos(3t)\\&=\left(1+j\right)\cos(3t)\end{aligned}$$

Hence the real part of $x_2(t)$ is

$$\operatorname{Re}(x_2(t))=\cos(3t)$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A=1,a=0,\omega=3,\phi=0$$

4.1.1.3 part c

$$x_3(t) = e^{-t}\sin(3t+\pi)$$

Since $\sin(3t+\pi) = \cos\left(3t+\pi-\frac{\pi}{2}\right) = \cos\left(3t+\frac{\pi}{2}\right)$, then the above becomes

$$x_3(t) = e^{-t}\cos\left(3t+\frac{\pi}{2}\right)$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A = 1, a = 1, \omega = 3, \phi = \frac{\pi}{2}$$

4.1.1.4 part d

$$\begin{aligned} x_4(t) &= je^{\left(-2+100j\right)t} \\ &= je^{-2t}e^{j100t} \end{aligned}$$

But $j = e^{j\frac{\pi}{2}}$, hence the above becomes

$$\begin{aligned} x_4(t) &= e^{j\frac{\pi}{2}}e^{-2t}e^{j100t} \\ &= e^{-2t}e^{j\left(100t+\frac{\pi}{2}\right)} \\ &= e^{-2t}\left(\cos\left(100t+\frac{\pi}{2}\right)+j\sin\left(100t+\frac{\pi}{2}\right)\right) \end{aligned}$$

Therefore

$$\operatorname{Re}(x_4(t)) = e^{-2t}\cos\left(100t+\frac{\pi}{2}\right)$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A = 1, a = 2, \omega = 100, \phi = \frac{\pi}{2}$$

4.1.2 Problem 1.13, Chapter 1

Consider the continuous-time signal $x(t) = \delta(t+2) - \delta(t-2)$. Calculate the value of E_∞ for the signal $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

Solution

$y(t)$ is first found

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} \delta(t+2) - \delta(t-2)\,d\tau \\ &= \int_{-\infty}^{t} \delta(t+2)\,d\tau - \int_{-\infty}^{t} \delta(t-2)\,d\tau \end{aligned}$$

$\delta(t+2)$ is an impulse at $t=-2$ and $\delta(t-2)$ is an impulse at $t=2$. Hence is $t<-2$ then the above is zero. If $-2<t<2$ then only the first integral contributes giving 1 and if $t>2$ then both integral contribute 1 each, and hence cancel each others giving $y=0$. Therefore

$$y(t)=\begin{cases}0 & t<-2\\ 1 & -2<t<2\\ 0 & t>2\end{cases}$$

Now that $y(t)$ is found, its E_∞ can be calculated using the definition

$$\begin{aligned}E_\infty &= \int_{-\infty}^{\infty}|y(t)|^2\,dt\\ &= \int_{-2}^{2}1dt\\ &= [t]_{-2}^{2}\\ &= 2+2\end{aligned}$$

Hence

$$E_\infty = 4$$

4.1.3 Problem 1.17, Chapter 1

Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by $y(t)=x(\sin(t))$. (a) Is this system causal? (b) Is this system linear?

Solution

4.1.3.1 Part a

A system is causal if its output at time t depends only on current t and on past t and not on future t. Picking $t=-\pi$, then $y(-\pi)=x(\sin(-\pi))=x(0)$. This shows that $y(-\pi)=x(0)$. Hence the output depends on input at future time (since $0>-\pi$). Therefore this system is not causal.

4.1.3.2 Part b

Let input be $x(t)=a_1x_1(t)+a_2x_2(t)$. If the output when the input is $x(t)$ is given by $y(t)=a_1y_1(t)+a_2y_2(t)$ where $y_1(t)=x_1(t)$ and $y_2(t)=x_2(t)$ then the system is linear. From the definition

$$\begin{aligned}y(t)&=x(\sin(t))\\ &=a_1x_1(\sin(t))+a_2x_2(\sin(t))\end{aligned}$$

Now, $y_1(t)=x_1(\sin t)$ and $y_2(t)=x_2(\sin t)$. Hence the above becomes

$$\begin{aligned}y(t)&=x(\sin(t))\\ &=a_1y_1(t)+a_2y_2(t)\end{aligned}$$

Therefore the system is linear.

4.1.4 Problem 1.21, Chapter 1

A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals: (a) $x(t-1)$. (b) $x(2-t)$ (c) $x(2t+1)$ (d) $x\left(4-\frac{t}{2}\right)$

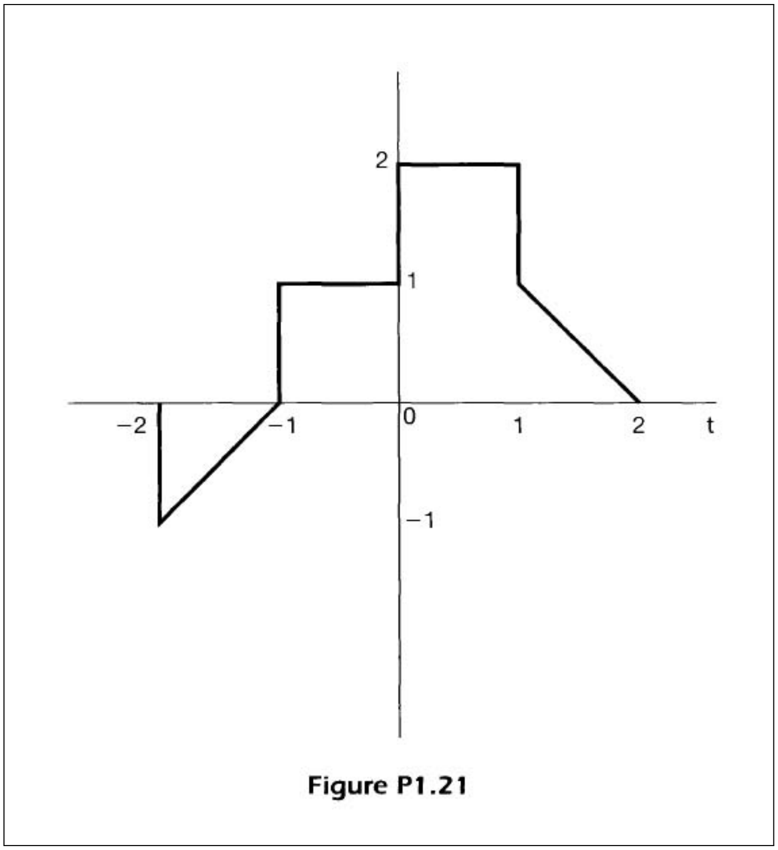


Figure 4.1: The function $x(t)$

Solution

Looking at the plot, it can be constructed from unit step $u(t)$ and ramp function $r(t)$ as follows

$$x(t) = -u(t+2) + r(t+2) - r(t+1) + u(t+1) + u(t) - u(t-1) - r(t-1) + r(t-2)$$

Here is an implementation

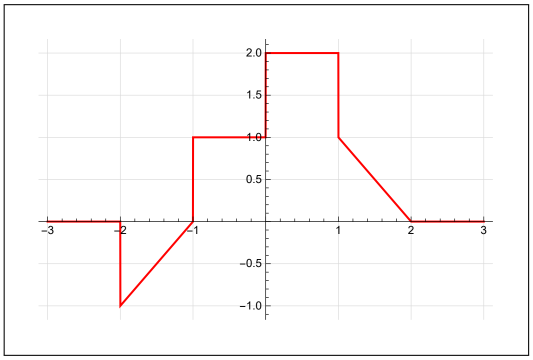
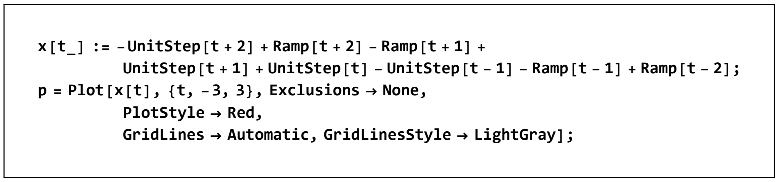


Figure 4.2: Construction the signal $x(t)$ from unit step and ramp functions



```
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[t], {t, -3, 3}, Exclusions → None,
         PlotStyle → Red,
         GridLines → Automatic, GridLinesStyle → LightGray];
```

Figure 4.3: Code for the above

4.1.4.1 Part a

$x(t-1)$ is $x(t)$ shifted to right by one unit time. Hence it becomes

$$x(t-1) = -u(t+1) + r(t+1) - r(t) + u(t) + u(t-1) - u(t-2) - r(t-2) + r(t-3)$$

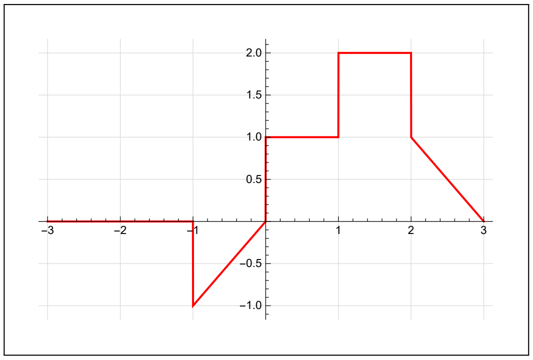
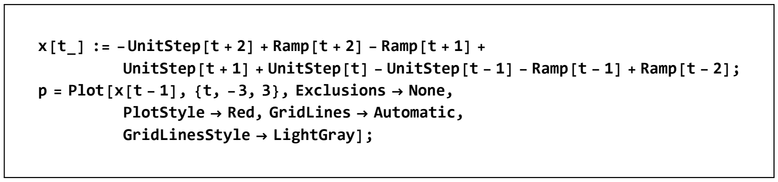


Figure 4.4: Part (a) plot



```
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[t - 1], {t, -3, 3}, Exclusions → None,
         PlotStyle → Red, GridLines → Automatic,
         GridLinesStyle → LightGray];
```

Figure 4.5: Code for the above

4.1.4.2 Part b

$$x(2-t)=x(-(t-2))$$

Hence the signal $x(t)$ is first flipped right to left (also called reflection about the $t=0$ axis) and the resulting function is then shifted to the right by 2 units. It becomes

$$\begin{aligned}x(2-t)&=-u((2-t)+2)+r((2-t)+2)-r((2-t)+1)+u((2-t)+1)+u(2-t)-u((2-t)-1)-r((2-t)-1)+r((2-t)-2)\\&=-u(4-t)+r(4-t)-r(3-t)+u(3-t)+u(2-t)-u(1-t)-r(1-t)+r(-t)\end{aligned}$$

The flipped signal is

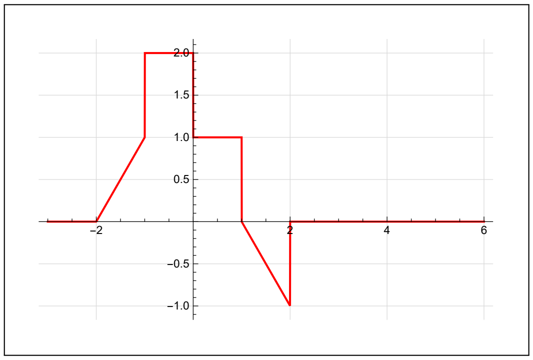


Figure 4.6: Part (b) signal after reflection

Now the above is shifted to the right by 2 units giving

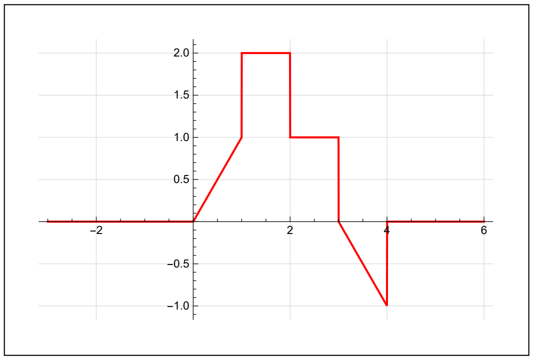


Figure 4.7: Part (b) final plot

It also possible to first do the shifting, followed by the reflection. Same output will result.

4.1.4.3 Part c

$x(2t+1)=x\left(2\left(t+\frac{1}{2}\right)\right)$. The signal is first shifted to the left by $\frac{1}{2}$ due to the $+\frac{1}{2}$ term, and then the resulting signal is squashed (contraction) by factor of 2. Since the original signal is from

-1 to 3, then after first shifting it to the left by 0.5 it becomes from -1.5 to 2.5. Hence the original ramp that went from -2 to -1 now goes from -1.5 to -1 and the line that originally went from -1 to 0 now goes from -1 to $-\frac{1}{2}$ (half the length) and so on. This is the result

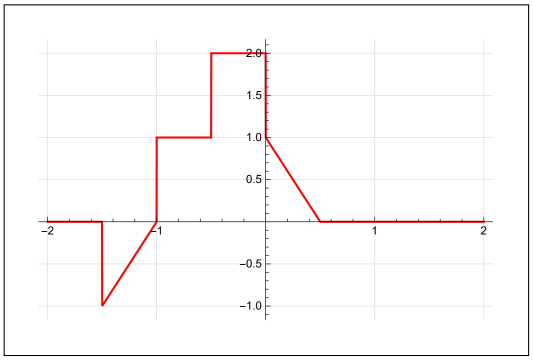
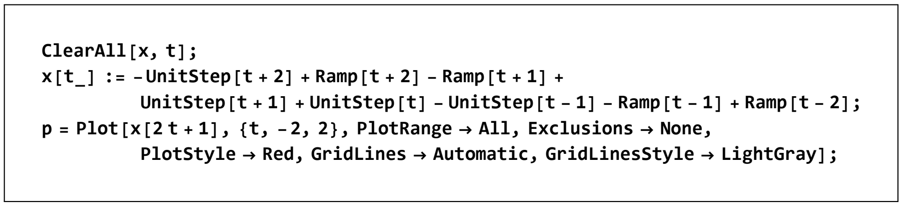


Figure 4.8: Part (c) plot



```
ClearAll[x, t];
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[2 t + 1], {t, -2, 2}, PlotRange → All, Exclusions → None,
         PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray];
```

Figure 4.9: Code for the above

4.1.4.4 Part d

$x\left(4-\frac{t}{2}\right)=x\left(-\left(\frac{t}{2}-4\right)\right)=x\left(-\frac{1}{2}(t-8)\right)$. Hence, the signal is first shifted to the right by 8 due to the -8 term, and then the resulting signal is flipped across the $t=0$ axis, and then the resulting signal is stretched (expanded) by factor of 2 due to the multiplication by $\frac{1}{2}$ term. This is the result showing each step


Shift to right by 8
Now flip left–to–right
Now expand by factor of 2

Figure 4.10: Part (d) plot

4.1.5 Problem 1.22, Chapter 1

A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals (a) $x[n-4]$ (b) $x[3-n]$ (c) $x[3n]$ (d) $x[3n+1]$

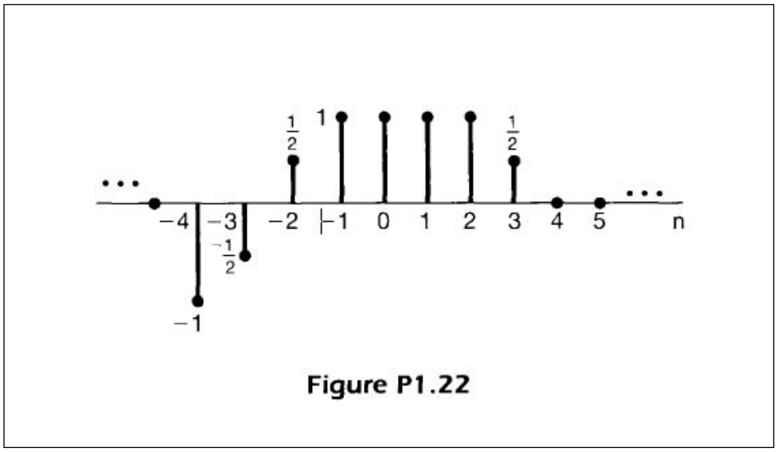


Figure 4.11: The function $x[n]$

<u>Solution</u>

4.1.5.1 Part a

$x[n-4]$ is $x[n]$ shifted to the right by 4 positions. Hence it becomes

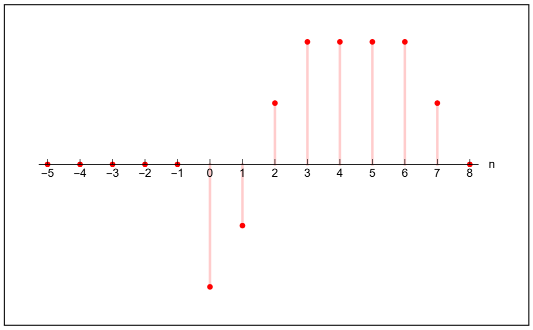
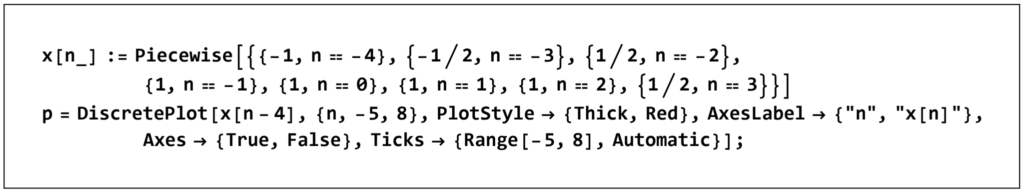


Figure 4.12: Part (a) plot



```
x[n_] := Piecewise[{{-1, n == -4}, {-1/2, n == -3}, {1/2, n == -2},
        {1, n == -1}, {1, n == 0}, {1, n == 1}, {1, n == 2}, {1/2, n == 3}}]
p = DiscretePlot[x[n - 4], {n, -5, 8}, PlotStyle → {Thick, Red}, AxesLabel → {"n", "x[n]"},
        Axes → {True, False}, Ticks → {Range[-5, 8], Automatic}];
```

Figure 4.13: Code used for the above

4.1.5.2 Part b

$x[3-n] = x[-(n-3)]$. Hence $x[n]$ is first reflected to obtain $x[-n]$ and then the result is shifted to right by 3. This is the result showing each step

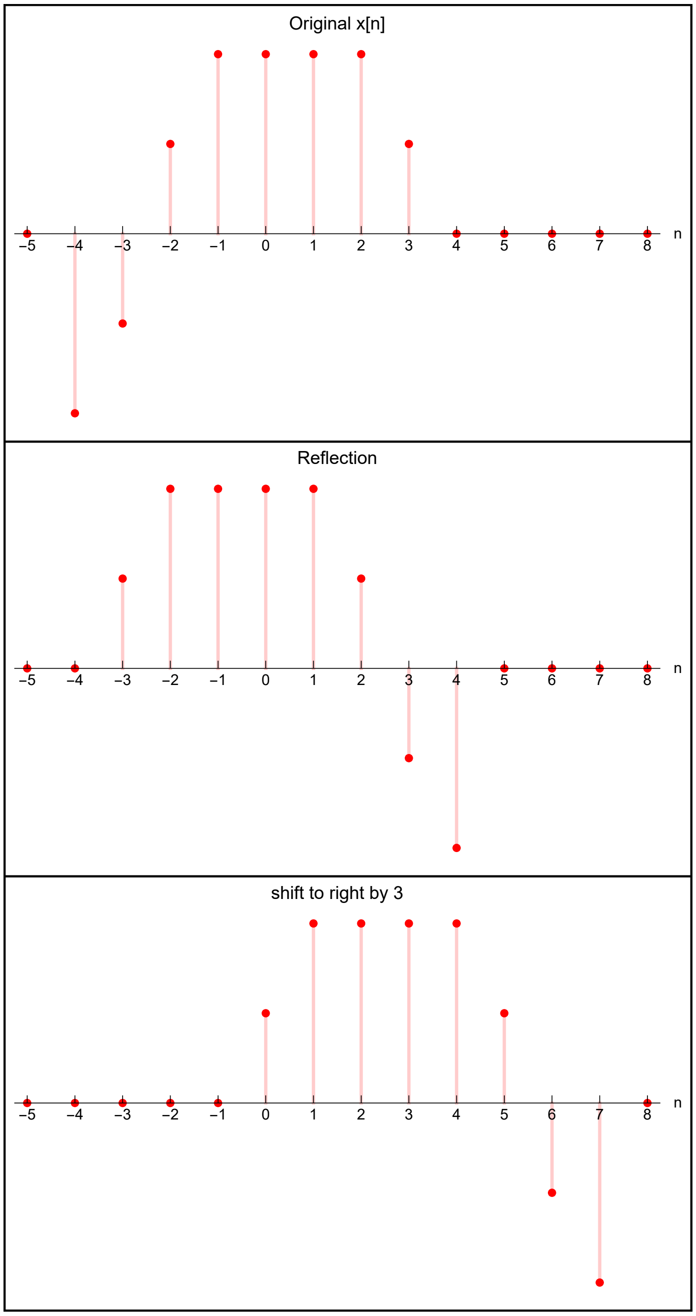


Figure 4.14: Part (b) plot

4.1.5.3 Part c

$x[3n]$. Sample at $n = 0$ remains the same. Sample at $n = -1$ gets the value of the sample that was at -3 which is $-\frac{1}{2}$. Sample at $n = -2$ gets the value of the sample that was at $n = -6$ which is zero. Hence for all n less than -1 new values are all zero. Same for the right side. The sample at $n = 1$ gets the value of the sample that was at 3 which is $\frac{1}{2}$ and sample at $n = 2$ gets the value of the sample that was at 6 which is 0 and all $n > 1$ are therefore zero. Notice that this operation causes samples to be lost from the original signal. This is the final result

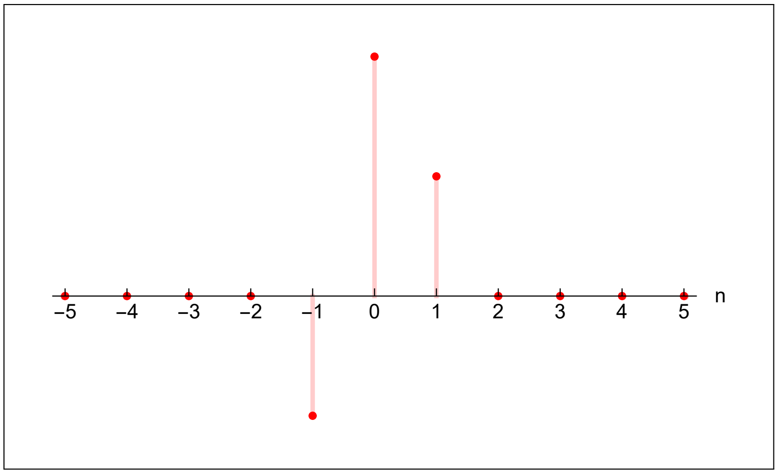


Figure 4.15: Part (c) plot

4.1.5.4 Part d

$x[3n+1]$. Sample at $n = 0$ gets the value that was at $n = 1$ which is 1. Sample at $n = -1$ gets the value of the sample that was at $-3+1 = -2$ which is $\frac{1}{2}$. Sample at $n = -2$ gets the value of the sample that was at $n = -6+1 = -5$ which is zero. Hence for all $n < -1$ all values are zero. Same for the right side. Sample at $n = 1$ gets the value of the sample that was at $3+1 = 4$ which is 0 and therefore for all $n > 1$ samples are zero. Notice that this operation causes samples to be lost from the original signal. The result is

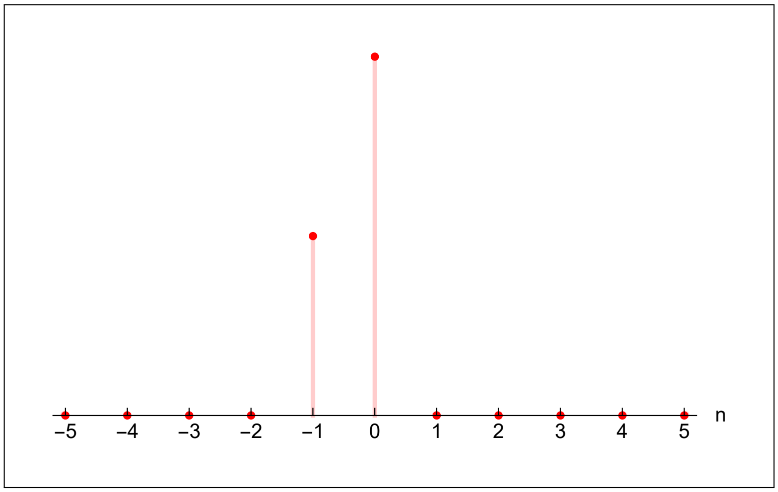


Figure 4.16: Part (d) plot

4.1.6 Problem 1.26, Chapter 1

Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period (a) $\sin\left(\frac{6}{7}\pi n+1\right)$ (b) $x[n]=\cos\left(\frac{n}{8}-\pi\right)$ (c) $x[n]=\cos\left(\frac{\pi}{8}n^2\right)$ (d) $x[n]=\cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

Solution

The signal $x[n]$ is periodic, if integer N can be found that $x[n]=x[n+N]$ for all n. Fundamental period is the smallest such integer N.

4.1.6.1 Part a

In this part, $x[n]=\sin\left(\frac{6}{7}\pi n+1\right)$. Hence the signal is periodic if

$$\begin{aligned}x[n]&=x[n+N]\\ \sin\left(\frac{6}{7}\pi n+1\right)&=\sin\left(\frac{6}{7}\pi(n+N)+1\right)\\ &=\sin\left(\left(\frac{6}{7}\pi n+1\right)+\frac{6}{7}\pi N\right)\end{aligned}$$

The above will be true if

$$\frac{6}{7}\pi N=2\pi m$$

For some integer m and N. This is because sin has 2π period. This implies that

$$\frac{3}{7}=\frac{m}{N}$$

Therefore $N=7, m=3$. Since it was possible to find n, N integers, then it is periodic. Since m, N are relatively prime then $N=7$ is the fundamental period.

4.1.6.2 Part b

In this part, $x[n] = \cos\left(\frac{n}{8} - \pi\right)$. Hence the signal is periodic if

$$\begin{aligned} x[n] &= x[n+N] \\ \cos\left(\frac{n}{8} - \pi\right) &= \cos\left(\frac{n+N}{8} - \pi\right) \\ &= \cos\left(\left(\frac{n}{8} - \pi\right) + \frac{N}{8}\right) \end{aligned}$$

The above will be true if

$$\begin{aligned} \frac{N}{8} &= 2\pi m \\ \frac{1}{16\pi} &= \frac{m}{N} \end{aligned}$$

For some integer N, m. It is not possible to find integers m, N to satisfy the above since π is an irrational number. Hence $\underline{\text{not periodic}}$.

4.1.6.3 Part c

$x[n] = \cos\left(\frac{\pi}{8}n^2\right)$. Hence the signal is periodic if

$$\begin{aligned} x[n] &= x[n+N] \\ \cos\left(\frac{\pi}{8}n^2\right) &= \cos\left(\frac{\pi}{8}(n+N)^2\right) \\ &= \cos\left(\frac{\pi}{8}\left(n^2 + N^2 + 2nN\right)\right) \\ &= \cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{8}\left(N^2 + 2nN\right)\right) \end{aligned}$$

The above will be true if

$$\frac{\pi}{8}\left(N^2 + 2nN\right) = 2\pi m$$

Need to find smallest integer N to satisfy this for all n. Choosing $N = 8$ the above becomes

$$\begin{aligned} \frac{\pi}{8}(64 + 16n) &= m(2\pi) \\ 8\pi + 2n\pi &= m(2\pi) \end{aligned}$$

Hence for all n, $N = 8$ satisfies the equation (since m is arbitrary integer). Therefore it is $\underline{\text{periodic}}$ and fundamental period $N = 8$.

4.1.6.4 Part d

Using $\cos A \cos B = \frac{1}{2}\left(\cos\left(A+B\right)+\cos\left(A-B\right)\right)$ then

$$\begin{aligned}\cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right) &= \frac{1}{2}\left(\cos\left(\frac{\pi}{2}n+\frac{\pi}{4}n\right)+\cos\left(\frac{\pi}{2}n-\frac{\pi}{4}n\right)\right)\\ &= \frac{1}{2}\left(\cos\left(\frac{3\pi}{4}n\right)+\cos\left(\frac{\pi}{4}n\right)\right)\end{aligned}$$

Considering each signal separately. $x\left[n\right]=\cos\left(\frac{3\pi}{4}n\right)$. This is periodic if

$$\begin{aligned}x\left[n\right] &= x\left[n+N\right]\\ \cos\left(\frac{3\pi}{4}n\right) &= \cos\left(\frac{3\pi}{4}\left(n+N\right)\right)\\ &= \cos\left(\left(\frac{3\pi}{4}n\right)+\frac{3\pi}{4}N\right)\end{aligned}$$

The above will be true if

$$\begin{aligned}\frac{3\pi}{4}N &= 2\pi m\\ \frac{3}{8} &= \frac{m}{N}\end{aligned}$$

It was possible to find integers N,m to satisfy this, where $\underline{\text{period } N=8}$. Considering the second signal $x\left[n\right]=\cos\left(\frac{\pi}{4}n\right)$. This is periodic if

$$\begin{aligned}x\left[n\right] &= x\left[n+N\right]\\ \cos\left(\frac{\pi}{4}n\right) &= \cos\left(\frac{\pi}{4}\left(n+N\right)\right)\\ &= \cos\left(\left(\frac{\pi}{4}n\right)+\frac{\pi}{4}N\right)\end{aligned}$$

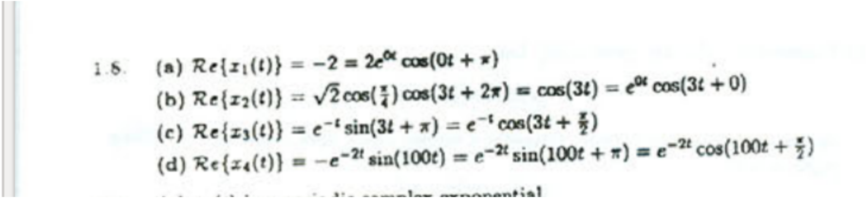
The above will be true if

$$\begin{aligned}\frac{\pi}{4}N &= 2\pi m\\ \frac{1}{8} &= \frac{m}{N}\end{aligned}$$

It was possible to find integers N,m to satisfy this, where $\underline{\text{period } N=8}$. Therefore both signals periodic with same period, the sum is therefore $\underline{\text{periodic}}$ and the fundamental period is $N=8$.

4.1.7 key solution

HW 1 Solutions

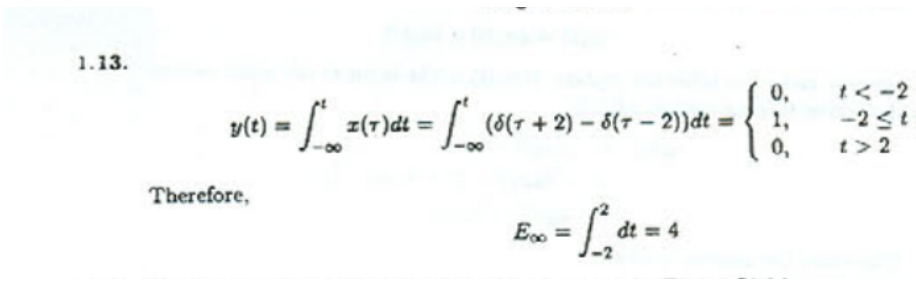


1.8. (a) $\mathcal{R}e\{x_1(t)\} = -2 = 2e^{0t}\cos(0t+\pi)$

(b) $\mathcal{R}e\{x_2(t)\} = \sqrt{2}\cos(\frac{\pi}{4})\cos(3t+2\pi) = \cos(3t) = e^{0t}\cos(3t+0)$

(c) $\mathcal{R}e\{x_3(t)\} = e^{-t}\sin(3t+\pi) = e^{-t}\cos(3t+\frac{\pi}{2})$

(d) $\mathcal{R}e\{x_4(t)\} = -e^{-2t}\sin(100t) = e^{-2t}\sin(100t+\pi) = e^{-2t}\cos(100t+\frac{\pi}{2})$

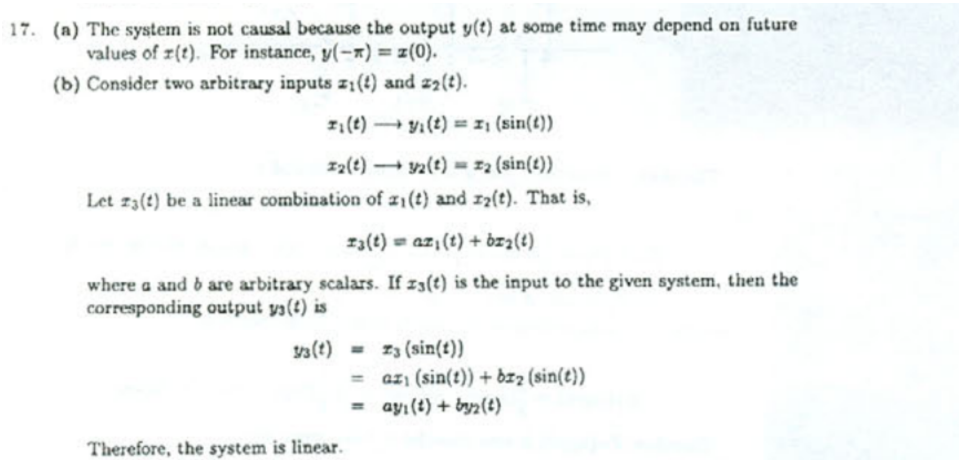


1.13.

$$y(t) = \int_{-\infty}^{t} x(\tau)dt = \int_{-\infty}^{t} (\delta(\tau+2) - \delta(\tau-2))dt = \begin{cases} 0, & t < -2 \\ 1, & -2 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

Therefore,

$$E_\infty = \int_{-2}^{2} dt = 4$$



17. (a) The system is not causal because the output $y(t)$ at some time may depend on future values of $x(t)$. For instance, $y(-\pi) = x(0)$.

(b) Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = x_1(\sin(t))$$

$$x_2(t) \longrightarrow y_2(t) = x_2(\sin(t))$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to the given system, then the corresponding output $y_3(t)$ is

$$\begin{aligned} y_3(t) &= x_3(\sin(t)) \\ &= ax_1(\sin(t)) + bx_2(\sin(t)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is linear.

1.21. The signals are sketched in Figure S1.21.

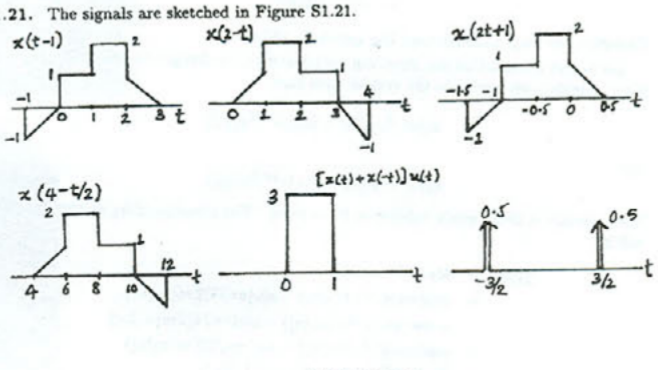


Figure S1.21

9

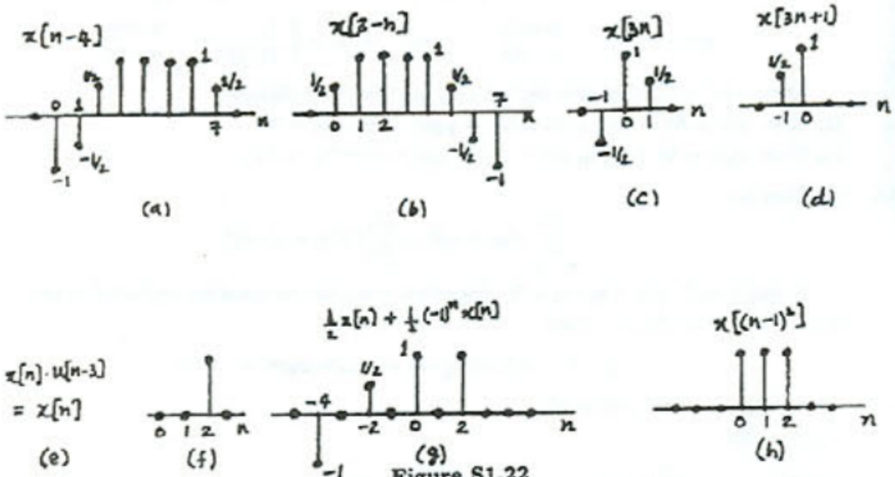


Figure S1.22

1.26. (a) Periodic, period = 7.

(b) Not periodic.

(c) Periodic, period = 8.

(d) $x[n] = (1/2)[\cos(3\pi n/4) + \cos(\pi n/4)]$. Periodic, period = 8.

(e) Periodic, period = 16.

4.2 HW 2

Local contents

4.2.1 Problem 2.1, Chapter 2

Let $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ and $h[n] = 2\delta[n+1] + 2\delta[n-1]$. Compute and plot each of the following convolutions (a) $y_1[n] = x[n] \circledast h[n]$ (b) $y_2[n] = x[n+2] \circledast h[n]$

<u>Solution</u>

4.2.1.1 Part a

The following is plot of $x[n], h[n]$

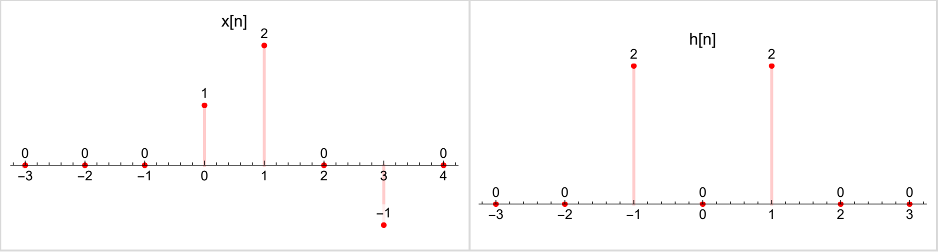
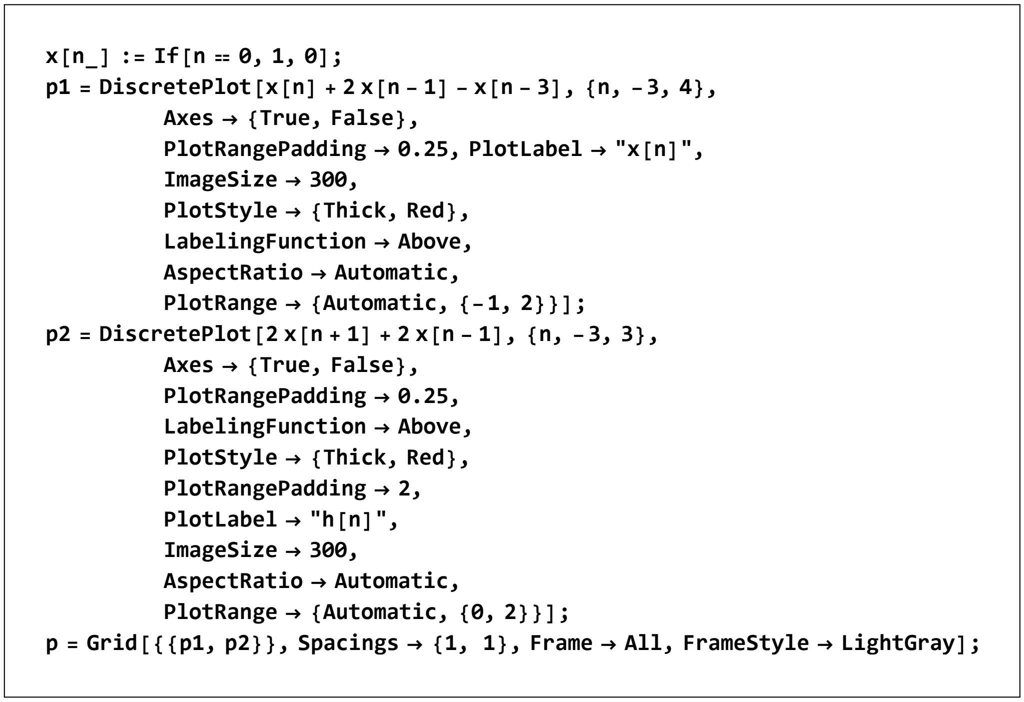


Figure 4.17: Plot of $x[n], h[n]$



```
x[n_] := If[n ⩵ 0, 1, 0];
p1 = DiscretePlot[x[n] + 2 x[n - 1] - x[n - 3], {n, -3, 4},
         Axes → {True, False},
         PlotRangePadding → 0.25, PlotLabel → "x[n]",
         ImageSize → 300,
         PlotStyle → {Thick, Red},
         LabelingFunction → Above,
         AspectRatio → Automatic,
         PlotRange → {Automatic, {-1, 2}}];
p2 = DiscretePlot[2 x[n + 1] + 2 x[n - 1], {n, -3, 3},
         Axes → {True, False},
         PlotRangePadding → 0.25,
         LabelingFunction → Above,
         PlotStyle → {Thick, Red},
         PlotRangePadding → 2,
         PlotLabel → "h[n]",
         ImageSize → 300,
         AspectRatio → Automatic,
         PlotRange → {Automatic, {0, 2}}];
p = Grid[{{p1, p2}}, Spacings → {1, 1}, Frame → All, FrameStyle → LightGray];
```

Figure 4.18: Code used for the above

Linear convolution is done by flipping $h[n]$ (reflection), then shifting the now flipped $h[n]$ one step to the right at a time. Each step the corresponding entries of $h[n]$ and $x[n]$ are multiplied and added. This is done until no overlapping between the two sequences. Mathematically this is the same as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Since $x[n]$ length is 3 and $x[n] = 0$ for $n < 0$ then the sum is

$$y[n] = \sum_{k=0}^{3} x[k]\,h[n-k]$$

For $\underline{n = -1}$

$$\begin{aligned} y[-1] &= \sum_{k=0}^{3} x[k]\,h[-1-k] \\ &= x[0]\,h[-1] + x[1]\,h[0] + x[2]\,h[1] + x[3]\,h[2] \\ &= (1)(2) + (2)(0) + (0)(2) + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n=0}$

$$\begin{aligned} y[0] &= \sum_{k=0}^{3} x[k]h[-k] \\ &= x[0]h[0] + x[1]h[-1] + x[2]h[-2] + x[3]h[-3] \\ &= 0 + (2)(2) + 0 + 0 \\ &= 4 \end{aligned}$$

For $\underline{n=1}$

$$\begin{aligned} y[1] &= \sum_{k=0}^{3} x[k]h[1-k] \\ &= x[0]h[1] + x[1]h[0] + x[2]h[-1] + x[3]h[-2] \\ &= (1)(2) + (2)(0) + (0)(1) + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n=2}$

$$\begin{aligned} y[2] &= \sum_{k=0}^{3} x[k]h[2-k] \\ &= x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[-1] \\ &= (1)(0) + (2)(2) + (0)(0) + (-1)(2) \\ &= 2 \end{aligned}$$

For $\underline{n=3}$

$$\begin{aligned} y[3] &= \sum_{k=0}^{3} x[k]h[3-k] \\ &= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] \\ &= (1)(0) + (2)(0) + (0)(2) + (-1)(2) \\ &= 0 \end{aligned}$$

For $\underline{n=4}$

$$\begin{aligned} y[4] &= \sum_{k=0}^{3} x[k]h[4-k] \\ &= x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] \\ &= (1)(0) + (2)(0) + (0)(2) + (-1)(2) \\ &= -2 \end{aligned}$$

All higher n values give $y[n] = 0$. Therefore

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

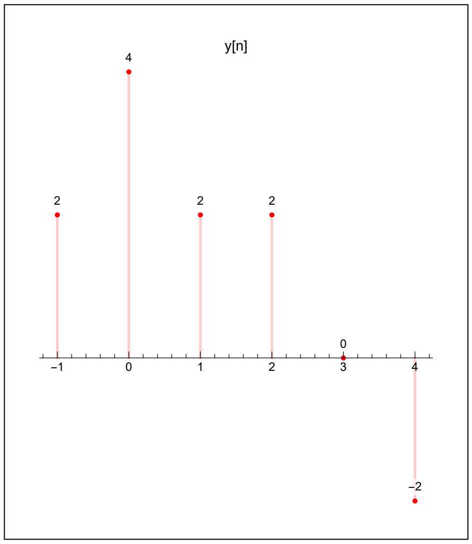


Figure 4.19: Plot of $y[n]$

4.2.1.2 Part b

First $x[n]$ is shifted to the left by 2 to obtain $x[n+2]$ and the result is convolved with $h[n]$ The following is plot of $x[n+2], h[n]$

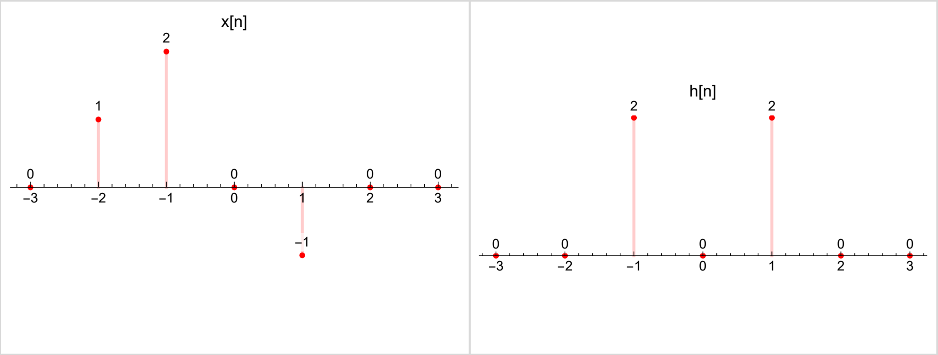


Figure 4.20: Plot of $x[n+2], h[n]$

Since Linear time invariant system, then shifted input convolved with $h[n]$ will give the shifted output found in part (a). Hence $y_2[n] = y_1[n+2]$. Hence

$$y_2[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2]$$

To show this explicitly, the convolution of shifted input is now computed directly. Linear convolution is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Since $x[n+2]$ length is 3 and $x[n]=0$ for $n<-2$ then the sum is

$$y[n]=\sum_{k=-2}^{1}x[k]h[n-k]$$

For $\underline{n=-3}$

$$\begin{aligned}y[-3]&=\sum_{k=-2}^{1}x[k]h[-3-k]\\&=x[-2]h[-1]+x[-1]h[-2]+x[0]h[-3]+x[1]h[-4]\\&=(1)(2)+(2)(0)+(0)(0)+(-1)(0)\\&=2\end{aligned}$$

For $\underline{n=-2}$

$$\begin{aligned}y[-2]&=\sum_{k=-2}^{1}x[k]h[-2-k]\\&=x[-2]h[0]+x[-1]h[-1]+x[0]h[-2]+x[1]h[-3]\\&=(1)(0)+(2)(2)+0+(-1)(0)\\&=4\end{aligned}$$

For $\underline{n=-1}$

$$\begin{aligned}y[-1]&=\sum_{k=-2}^{1}x[k]h[-1-k]\\&=x[-2]h[1]+x[-1]h[0]+x[0]h[-1]+x[1]h[-2]\\&=(1)(2)+(2)(0)+0+(-1)(0)\\&=2\end{aligned}$$

For $\underline{n=0}$

$$\begin{aligned}y[0]&=\sum_{k=-2}^{1}x[k]h[0-k]\\&=x[-2]h[2]+x[-1]h[1]+x[0]h[0]+x[1]h[-1]\\&=(1)(0)+(2)(2)+0+(-1)(2)\\&=2\end{aligned}$$

For $\underline{n=1}$

$$\begin{aligned}y[1]&=\sum_{k=-2}^{1}x[k]h[1-k]\\&=x[-2]h[3]+x[-1]h[2]+x[0]h[1]+x[1]h[0]\\&=(1)(0)+(2)(2)+0+(-1)(0)\\&=4\end{aligned}$$

For $\underline{n=2}$

$$\begin{aligned}y[2] &= \sum_{k=-2}^{1} x[k]\,h[2-k]\\ &= x[-2]\,h[4] + x[-1]\,h[3] + x[0]\,h[2] + x[1]\,h[1]\\ &= (1)(0) + (2)(0) + 0 + (-1)(2)\\ &= -2\end{aligned}$$

Hence

$$y[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2]$$

Which is the shifted output found in part (a)

4.2.2 Problem 2.6, Chapter 2

Compute and plot the convolution $y[n] = x[n] \circledast h[n]$ where $x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1]$ and $h[n] = u[n-1]$

Solution

It is easier to do this using graphical method. $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$. We could either flip and shift $x[n]$ or $h[n]$. Let us flip and shift $h[n]$. This below is the result for $n=0$ when $h[n-k]$ and $x[k]$ are plotted on top of each others

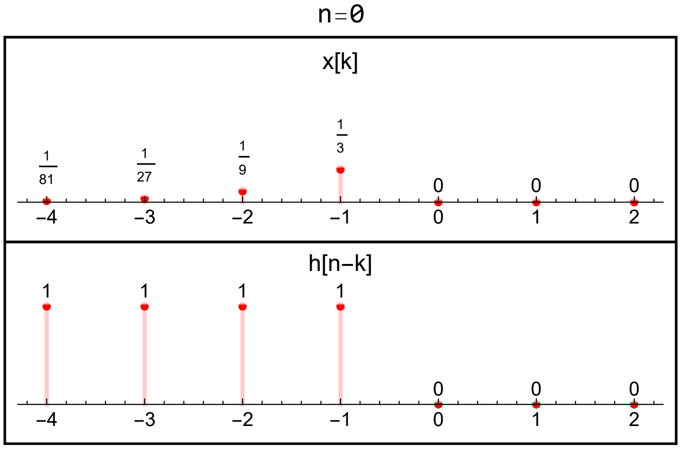


Figure 4.21: Convolution sum for $n=0$

By multiplying corresponding values and summing the result can be seen to be $\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k}$.

Let $r = \frac{1}{3}$ then this sum is $\left(\sum_{k=0}^{\infty} r^k\right) - 1$ But $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ since $r < 1$. Therefore

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k &= \frac{1}{1-\frac{1}{3}} - 1\\ &= \frac{3}{3-1} - 1\\ &= \frac{3}{2} - 1\\ &= \frac{1}{2}\end{aligned}$$

Hence $y[0] = \frac{1}{2}$. Now, the signal $h[n-k]$ is shifted to the right by 1 then 2 then 3 and so on. This gives $y[1], y[2], \cdots$. Each time, the same sum result which is $\frac{1}{2}$. Here is a diagram for $n = 1$ and $n = 2$ for illustration

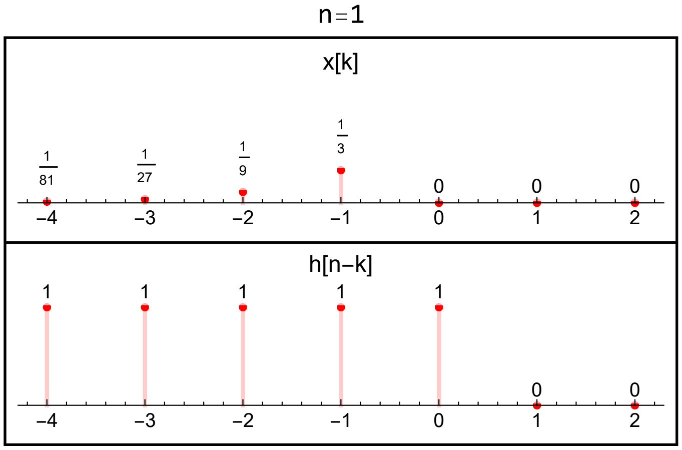


Figure 4.22: Convolution sum for $n = 1$

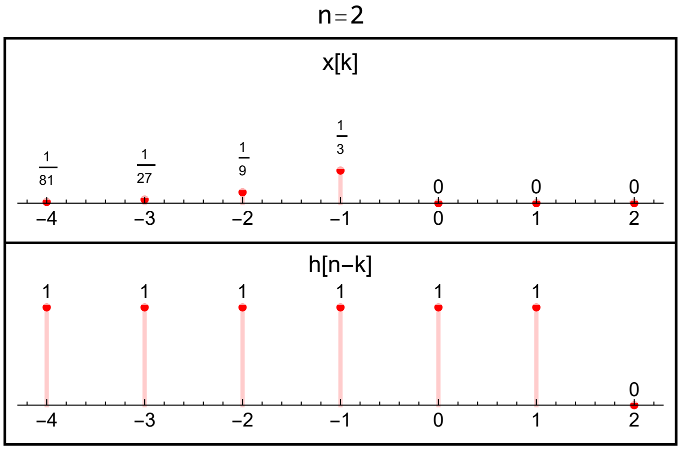


Figure 4.23: Convolution sum for $n = 2$

Therefore $y[n] = \frac{1}{2}$ for $n \geq 0$. Now we will look to see what happens when $h[-k]$ is shifted to the left. For $n = -1$ this is the result

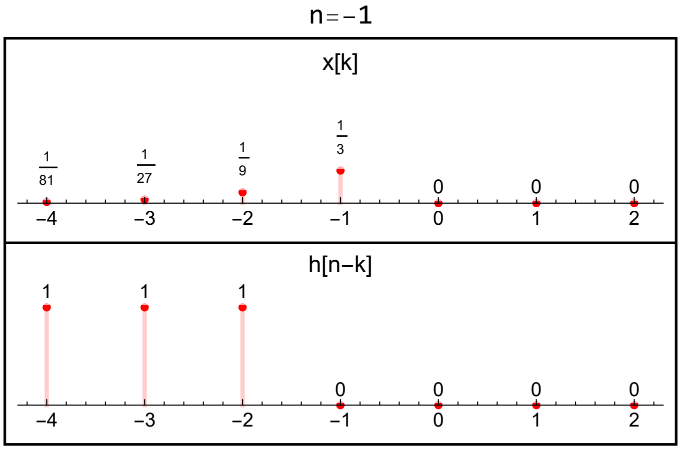


Figure 4.24: Convolution sum for $n = -1$

When multiplying the corresponding elements and adding, now the element $\frac{1}{3}$ is multiplied by a zero and not by 1. Hence the sum becomes $\left(\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k}\right) - \frac{1}{3}$ which is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$. Therefore $y[-1] = \frac{1}{6}$. When $h[-k]$ is shifted to the left one more step, it gives $y[-2]$ which is

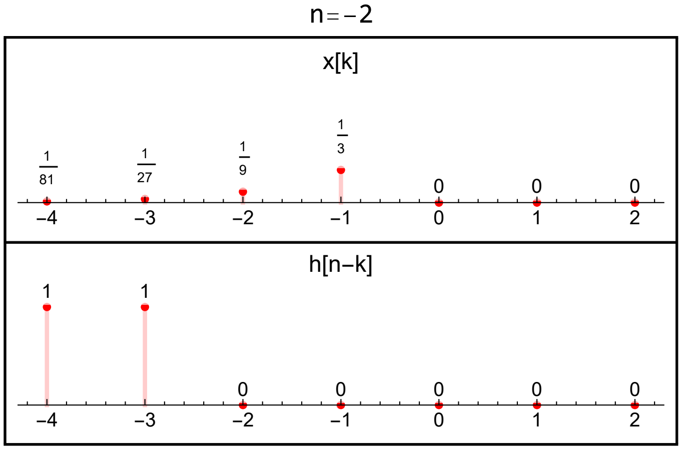


Figure 4.25: Convolution sum for $n = -2$

We see from the above diagram that now $\frac{1}{3}$ and $\frac{1}{9}$ do not contribute to the sum since both are multiplied by zero. This means $y[-2] = \left(\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k\right) - \left(\frac{1}{3}+\frac{1}{9}\right) = \frac{1}{2} - \left(\frac{1}{3}+\frac{1}{9}\right) = \frac{1}{18} = \left(\frac{1}{2}\right)\left(\frac{1}{9}\right)$. Each time $h[-k]$ is shifted to the left by one, the sum reduces. From the above we see that

$$y[-1] = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$$

$$y[-2] = \left(\frac{1}{2}\right)\left(\frac{1}{3^2}\right)$$

Hence by extrapolation the pattern is

$$y[-n] = \left(\frac{1}{2}\right)\left(\frac{1}{3^{-n}}\right) = \frac{3^n}{2}$$

Therefore the final result is

$$y[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ \frac{3^n}{2} & n < 0 \end{cases}$$

Here is plot of $y[n] = x[n] \circledast h[n]$ given by the above

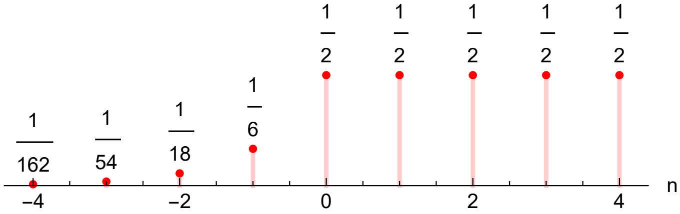


Figure 4.26: Plot of $y[n]$

4.2.3 Problem 2.11, Chapter 2

Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$. (a) compute $y(t) = x(t) \circledast h(t)$. (b) Compute $g(t) = \frac{dx}{dt} \circledast h(t)$. (c) How is $g(t)$ related to $y(t)$?

Solution

4.2.3.1 Part (a)

It is easier to do this using graphical method. This is plot of $x(t)$ and $h(t)$.

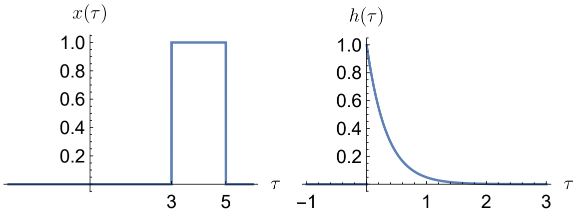
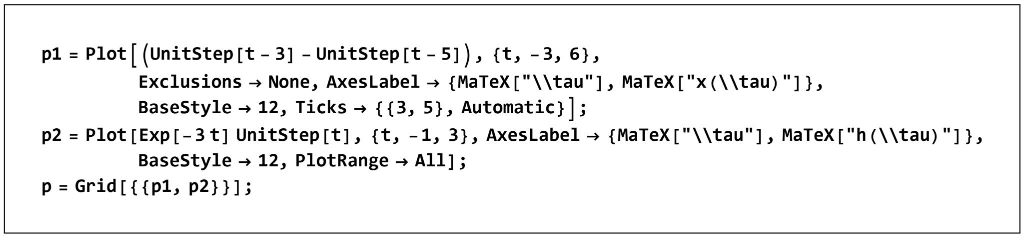


Figure 4.27: Plot $x(t)$ and $h(t)$



```
p1 = Plot[(UnitStep[t - 3] - UnitStep[t - 5]), {t, -3, 6},
        Exclusions → None, AxesLabel → {MaTeX["\\tau"], MaTeX["x(\\tau)"]},
        BaseStyle → 12, Ticks → {{3, 5}, Automatic}];
p2 = Plot[Exp[-3 t] UnitStep[t], {t, -1, 3}, AxesLabel → {MaTeX["\\tau"], MaTeX["h(\\tau)"]},
        BaseStyle → 12, PlotRange → All];
p = Grid[{{p1, p2}}];
```

Figure 4.28: Code used for the above plot

The next step is to fold one of the signals and then slide it to the right. We can folder either $x(t)$ or $h(t)$. Let us fold $x(t)$. Hence the integral is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

If we have chosen to fold $h(t)$ instead, then the integral would have been

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This is the result after folding (reflection) of $x(t)$

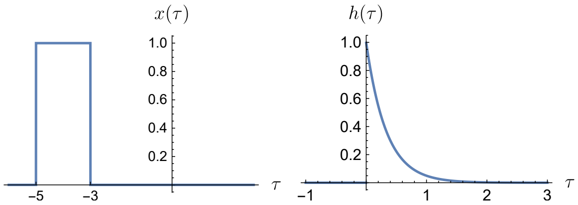


Figure 4.29: Folding $x(t)$

Next we label each edge of the folded signal before shifting it to the right as follows

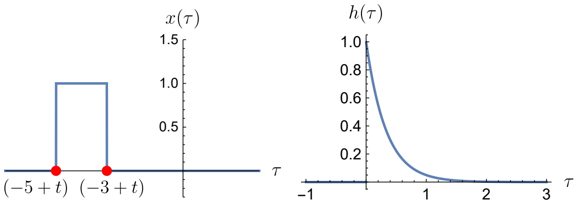


Figure 4.30: Folding $x(t)$ and labeling the edges

We see from the above that for $t-3<0$ or for $t<3$ the integral is zero since there is no overlapping between the folded $x(\tau)$ and $h(\tau)$. As we slide the folded $x(\tau)$ more to the right, we end up with $x(\tau)$ partially under $h(\tau)$ like this

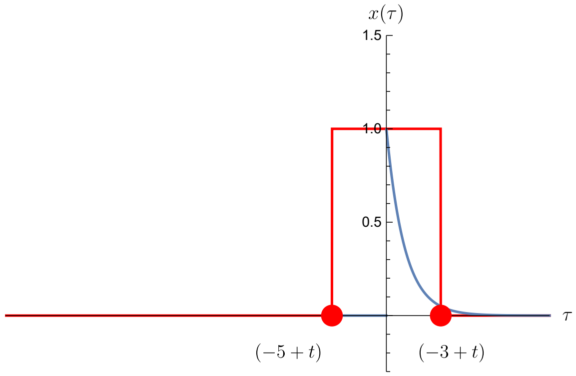


Figure 4.31: Shifting $x(\tau)$ to the right, partially inside

From the above, we see that for $0<t-3<2$ (since 2 is the width of $x(\tau)$) or for $3<t<5$, then the overlap is partial. Hence the integral now becomes

$$\begin{aligned} y(t) &= \int_0^{t-3} x(t-\tau)\,h(\tau)\,d\tau \qquad 3 < t \leq 5 \\ &= \int_0^{t-3} e^{-3\tau} d\tau \\ &= \frac{-1}{3}\left[e^{-3\tau}\right]_0^{t-3} \\ &= \frac{-1}{3}\left[e^{-3(t-3)} - 1\right] \\ &= \frac{1}{3}\left(1 - e^{-3(t-3)}\right) \end{aligned}$$

The next step is when folded $x(\tau)$ is fully inside $h(\tau)$ as follows

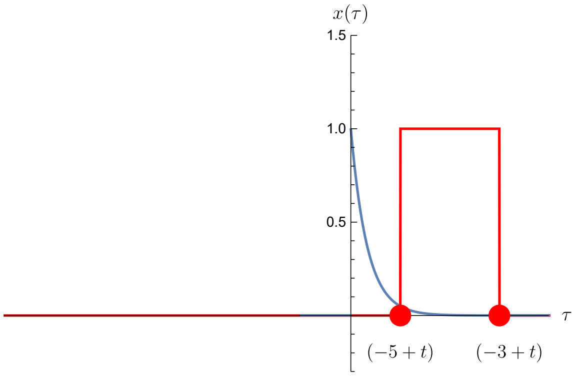


Figure 4.32: Shifting $x(\tau)$ to the right, fully inside

From the above, we see that for $0 < t - 5$ or $t > 5$, then the overlap is complete. Hence the integral now becomes

$$\begin{aligned} y(t) &= \int_{t-5}^{t-3} x(t-\tau)\,h(\tau)\,d\tau \qquad 5 < t \leq \infty \\ &= \int_{t-5}^{t-3} e^{-3\tau} d\tau \\ &= \frac{-1}{3}\left[e^{-3\tau}\right]_{t-5}^{t-3} \\ &= \frac{-1}{3}\left(e^{-3(t-3)} - e^{-3(t-5)}\right) \\ &= \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right) \end{aligned}$$

The above result $y(t) = \frac{1}{3}\left[e^{-3(t-5)} - e^{-3(t-3)}\right]$ can be rewritten as $\frac{1}{3}\left[\left(1-e^{-6}\right)e^{-3(t-5)}\right]$ if needed to match the book. Therefore the final answer is

$$y(t) = \begin{cases} 0 & -\infty < t \leq 3 \\ \frac{1}{3}\left(1 - e^{-3(t-3)}\right) & 3 < t \leq 5 \\ \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right) & 5 < t \leq \infty \end{cases}$$

Here is a plot of the above

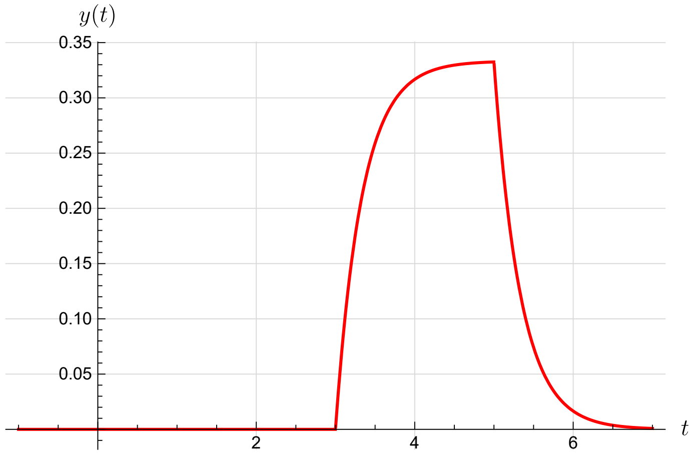
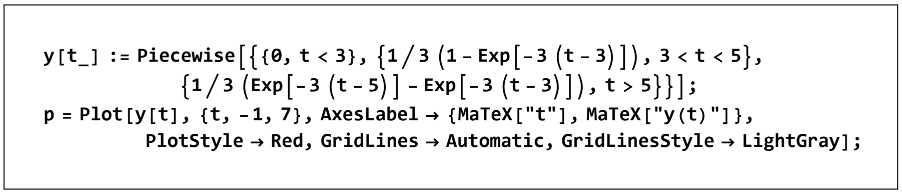


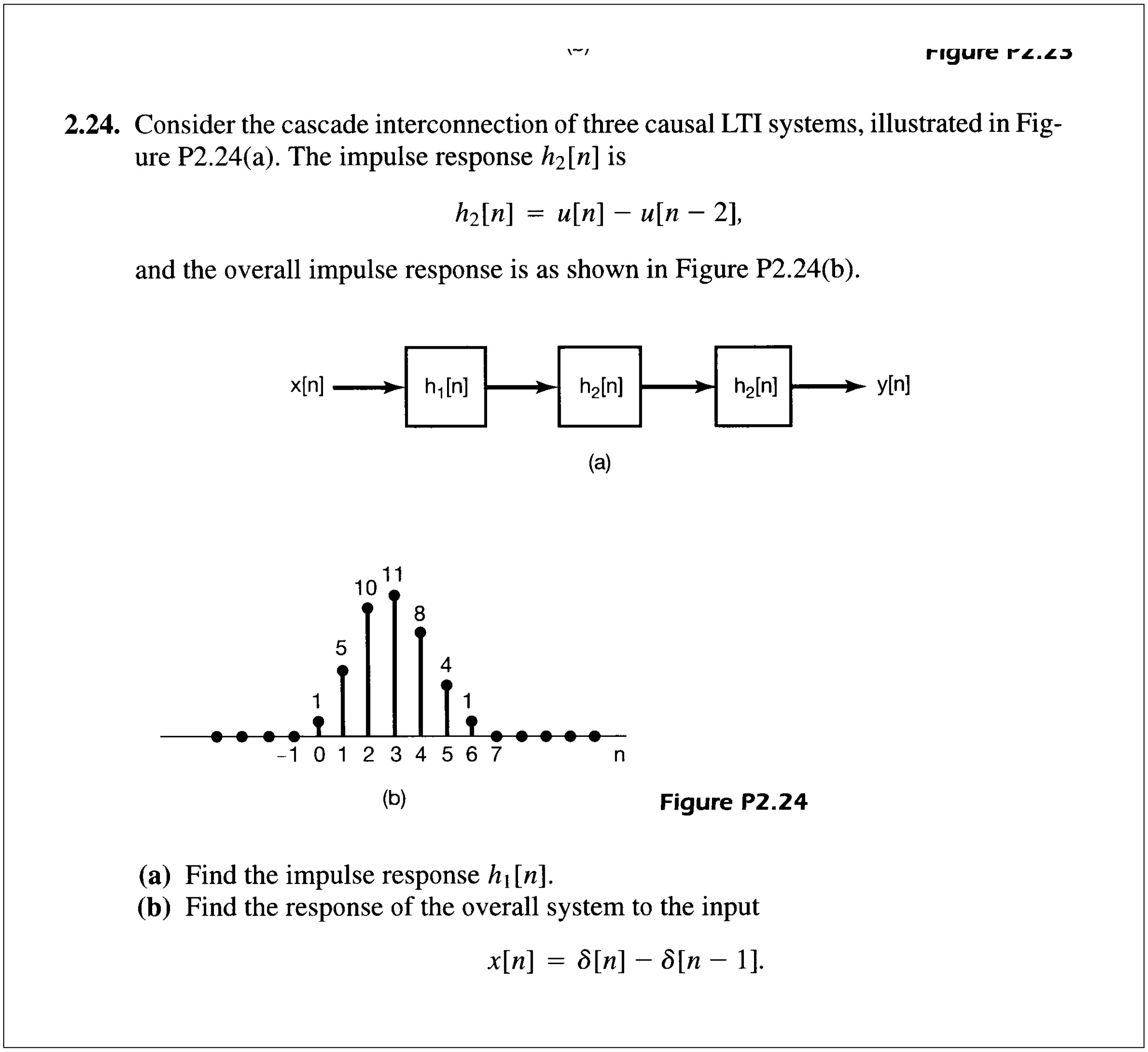
Figure 4.33: $y(t)$



```
y[t_] := Piecewise[{{0, t < 3}, {1/3 (1 - Exp[-3 (t - 3)]), 3 < t < 5},
      {1/3 (Exp[-3 (t - 5)] - Exp[-3 (t - 3)]), t > 5}}];
p = Plot[y[t], {t, -1, 7}, AxesLabel → {MaTeX["t"], MaTeX["y(t)"]},
    PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray];
```

Figure 4.34: Code for the above

4.2.4 Problem 2.24, Chapter 2



2.24. Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure P2.24(b).

Figure P2.24

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

Figure 4.35: Problem description

Solution

4.2.4.1 Part a

The impulse response $h[n]$ is given. This is the response when the input is $x[n] = \delta[0]$. Hence

$$h[n] = h_1[n] \circledast (h_2[n] \circledast h_2[n])$$

But $h_2[n]$ is given as $h_2[n] = \delta[0] + \delta[1]$. Hence, let $H[n] = h_2[n] \circledast h_2[n]$, therefore

$$\begin{aligned} H[n] &= \sum_{k=-\infty}^{\infty} h_2[k]\, h_2[n-k] \\ &= \sum_{k=-1}^{2} h_2[k]\, h_2[n-k] \end{aligned}$$

For $n = 0$.

$$\begin{aligned} H[0] &= \sum_{k=-1}^{0} h_2[k]\, h_2[-k] \\ &= h_2[-1]\, h_2[1] + h_2[0]\, h_2[0] \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

For $n = 1$.

$$\begin{aligned} H[1] &= \sum_{k=-1}^{0} h_2[k]\, h_2[1-k] \\ &= h_2[-1]\, h_2[0] + h_2[0]\, h_2[1] \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

For $n = 2$.

$$\begin{aligned} H[2] &= \sum_{k=-1}^{0} h_2[k]\, h_2[2-k] \\ &= h_2[-1]\, h_2[3] + h_2[0]\, h_2[2] \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

And zero for all other n. Hence

$$\begin{aligned} H[n] &= h_2[n] \circledast h_2[n] \\ &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

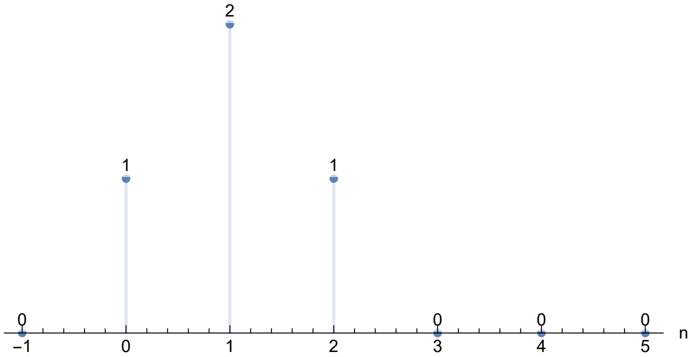
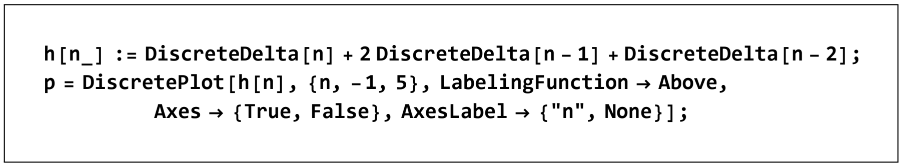


Figure 4.36: Plot of $h_2[n] \circledast h_2[n]$



```
h[n_] := DiscreteDelta[n] + 2 DiscreteDelta[n - 1] + DiscreteDelta[n - 2];
p = DiscretePlot[h[n], {n, -1, 5}, LabelingFunction → Above,
        Axes → {True, False}, AxesLabel → {"n", None}];
```

Figure 4.37: Code for the above

Now we need to find $h_1[n]$ given that $h_1[n] \circledast H[n]$ is what is shown in the problem. We do not know $h_1[n]$. so let us assume it is the sequence $\{h_1[0], h_2[0], \cdots\}$. Then by doing convolution by folding $h_1[n]$ and then sliding it to the right one step at a time, we obtain the following relations for each n.

$\underline{n=0}$ $h_1[0] H_1[0] = 1$ and since $H_1[0] = 1$ then $h_1[0] = 1$

$\underline{n=1}$ $h_1[1] H_1[0] + h_1[0] H_1[1] = 5$ and since $H_1[0] = 1, H_1[1] = 2$ then $h_1[1] + 2h_1[0] = 5$. But $h_1[0] = 1$ found above. Hence $h_1[1] + 2 = 5$ or $h_1[1] = 3$

$\underline{n=2}$ $h_1[2] H_1[0] + h_1[1] H_1[1] + h_1[0] H_1[2] = 10$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[2] + 2h_1[1] + h_1[0] = 10$. But $h_1[0] = 1, h_1[1] = 3$ found above. Hence $h_1[2] + (2)(3) + 1 = 10$ or $h_1[2] = 3$

$\underline{n=3}$ $h_1[3] H_1[0] + h_1[2] H_1[1] + h_1[1] H_1[2] = 11$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[3] + 2h_1[2] + h_1[1] = 11$. But $h_1[2] = 3, h_1[1] = 3$ found above. Hence $h_1[3] + (2)(3) + 3 = 11$ or $h_1[3] = 2$

$\underline{n=4}$ $h_1[4] H_1[0] + h_1[3] H_1[1] + h_1[2] H_1[2] = 8$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[4] + 2h_1[3] + h_1[2] = 8$. But $h_1[3] = 2, h_1[2] = 3$ found above. Hence $h_1[4] + 2(2) + 3 = 8$ or $h_1[4] = 1$

$\underline{n=5}$ $h_1[5] H_1[0] + h_1[4] H_1[1] + h_1[3] H_1[2] = 4$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[5] + 2h_1[4] + h_1[3] = 4$. But $h_1[4] = 1, h_1[3] = 2$ found above. Hence $h_1[5] + 2(1) + 2 = 4$ or $h_1[5] = 0$

And since the output is zero for $n > 5$ then $h_1[n] = 0$ for all $n > 5$. Therefore

$$h_1[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

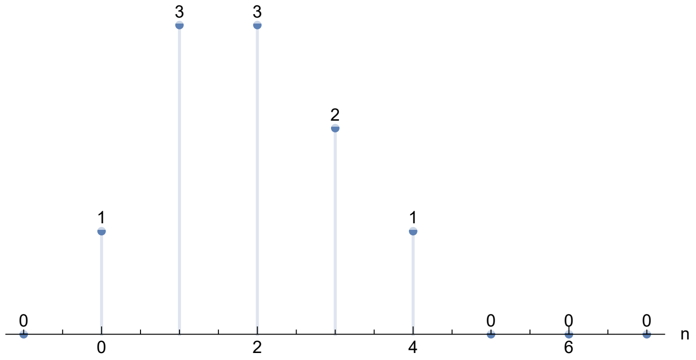
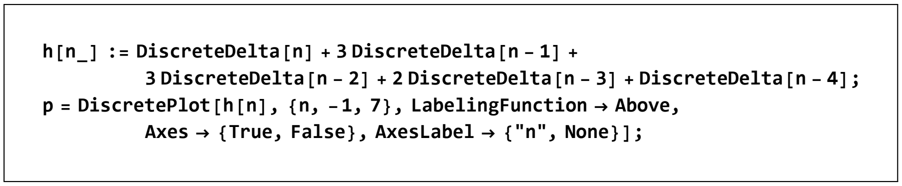


Figure 4.38: Plot of $h_1[n]$



```
h[n_] := DiscreteDelta[n] + 3 DiscreteDelta[n - 1] +
        3 DiscreteDelta[n - 2] + 2 DiscreteDelta[n - 3] + DiscreteDelta[n - 4];
p = DiscretePlot[h[n], {n, -1, 7}, LabelingFunction → Above,
        Axes → {True, False}, AxesLabel → {"n", None}];
```

Figure 4.39: Code for the above

4.2.4.2 Part b

When the input is $x[n] = \delta[n] - \delta[n-1]$ then response is given by $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ where $h[n]$ is the impulse response given in the problem P2.24 diagram. Hence we need to convolve the following two signals

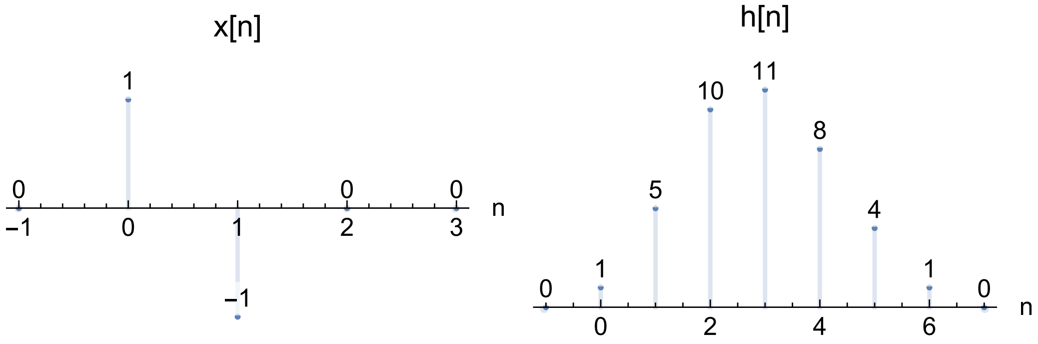


Figure 4.40: Plot of $x[n], h[n]$

By folding $x[n]$ and then shift it one step at a time, we see that we obtain the following

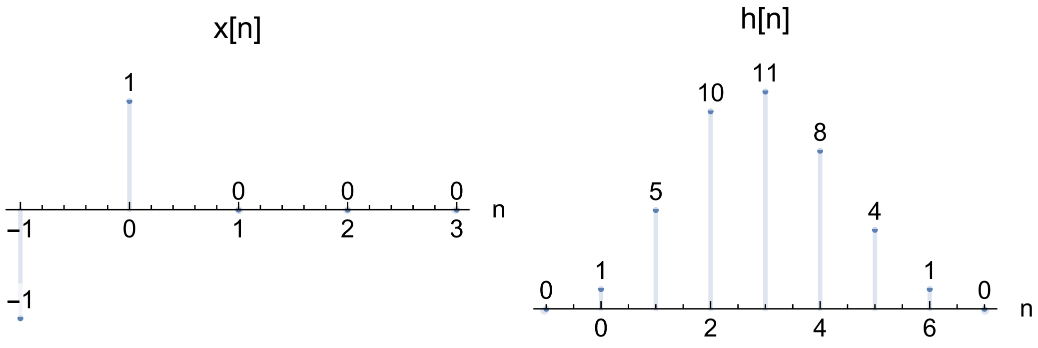


Figure 4.41: Plot of $x[n], h[n]$

$\underline{n=0}$ $(1)(1)=1$

$\underline{n=1}$ $(-1)(1)+(1)(5)=4$

$\underline{n=2}$ $(-1)(5)+(1)(10)=5$

$\underline{n=3}$ $(-1)(10)+(1)(11)=1$

$\underline{n=4}$ $(-1)(11)+(1)(8)=-3$

$\underline{n=5}$ $(-1)(8)+(1)(4)=-4$

$\underline{n=6}$ $(-1)(4)+(1)(1)=-3$

$\underline{n=7}$ $(-1)(1)+(1)(0)=-1$

$\underline{n=8}$ $(-1)(0)+(1)(0)=0$

And zero for all $n>7$. This is plot of $y[n]$

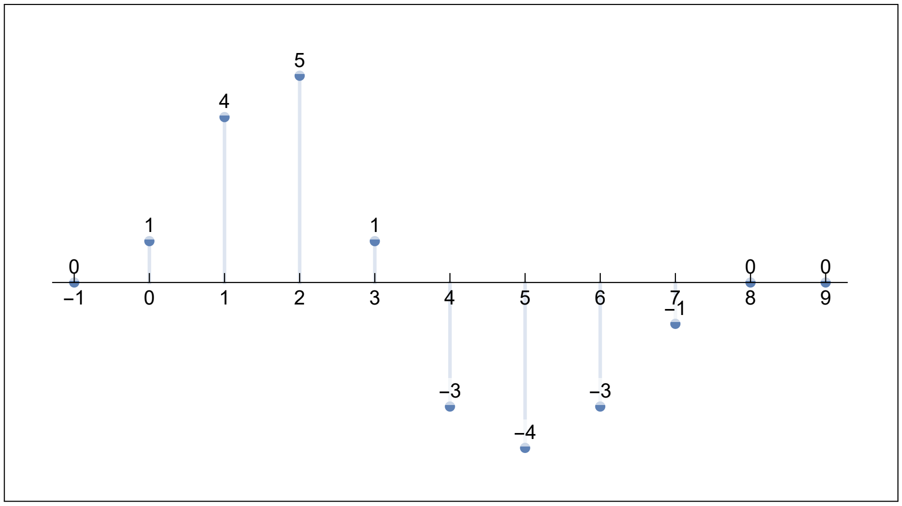
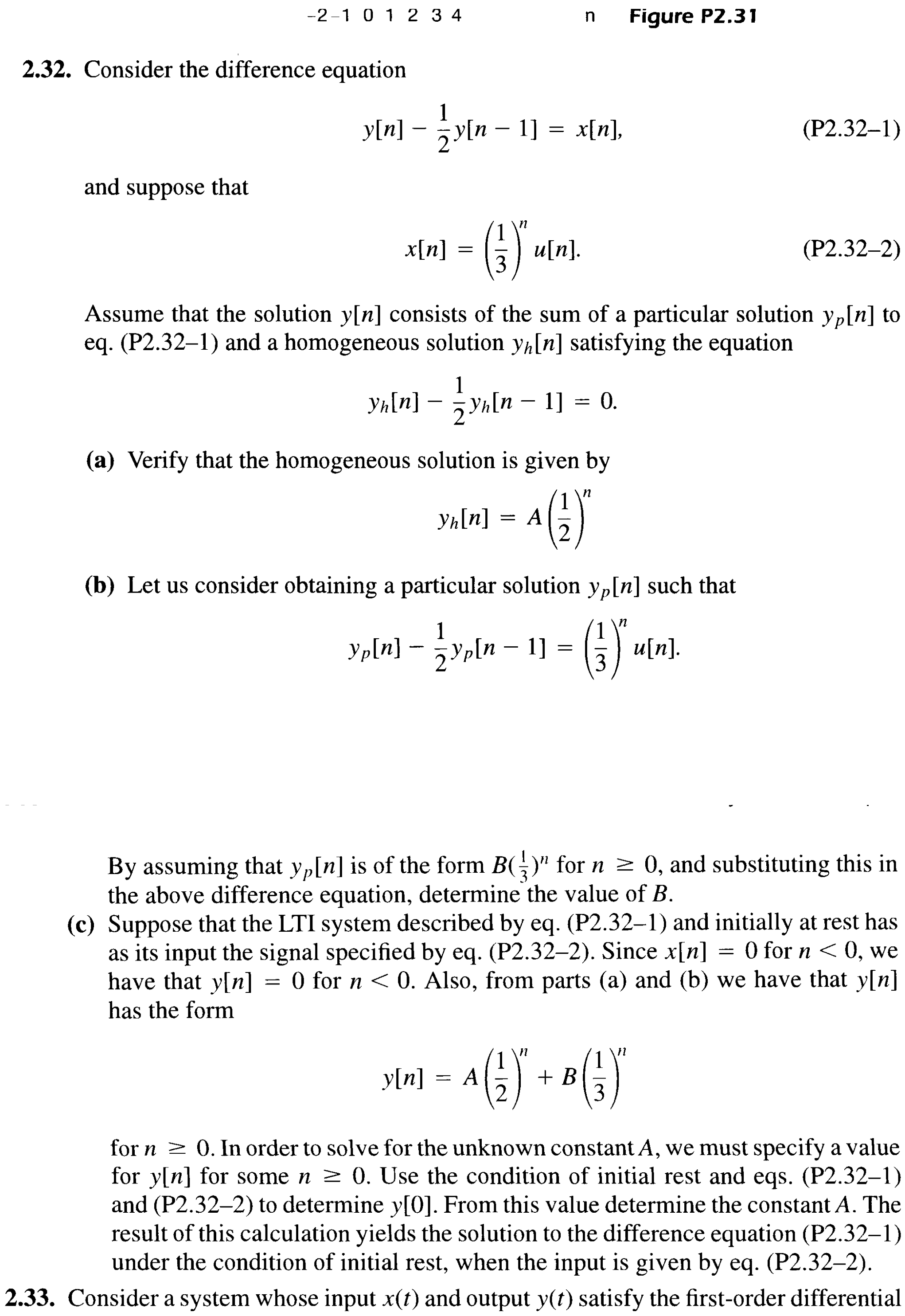


Figure 4.42: Plot of $y[n]$

4.2.5 Problem 2.32, Chapter 2

Solution



-2 -1 0 1 2 3 4 n **Figure P2.31**

2.32. Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \qquad \text{(P2.32–1)}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \qquad \text{(P2.32–2)}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P2.32–1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

(a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

(b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

(c) Suppose that the LTI system described by eq. (P2.32–1) and initially at rest has as its input the signal specified by eq. (P2.32–2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P2.32–1) and (P2.32–2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P2.32–1) under the condition of initial rest, when the input is given by eq. (P2.32–2).

2.33. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential

Figure 4.43: Problem description

4.2.5.1 Part a

Substituting $y_h[n] = A\left(\frac{1}{2}\right)^n$ into the difference equation $y_h[n] - \frac{1}{2}y_h[n-1] = 0$ gives

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0$$

Since $A \neq 0$, the above simplifies to

$$\begin{aligned}\frac{1}{2^n} - \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) &= 0\\ \frac{1}{2^n} - \frac{1}{2^n} &= 0\\ 0 &= 0\end{aligned}$$

Verified OK.

4.2.5.2 Part b

Substituting $y_p[n] = B\left(\frac{1}{3^n}\right)$ into $y_p[n] - \frac{1}{2}y_p[n-1] = \frac{1}{3^n}u[n]$ gives

$$\begin{aligned}B\left(\frac{1}{3^n}\right) - \frac{1}{2}B\left(\frac{1}{3^{n-1}}\right) &= \frac{1}{3^n}u[n]\\ B\left(\frac{1}{3^n} - \frac{1}{2}\frac{1}{3^{n-1}}\right) &= \frac{1}{3^n}u[n]\\ B\left(\frac{1}{3^n}\left(1 - \frac{1}{2}\frac{1}{3^{-1}}\right)\right) &= \frac{1}{3^n}u[n]\\ B\left(\frac{1}{3^n}\left(1 - \frac{3}{2}\right)\right) &= \frac{1}{3^n}u[n]\\ B\left(\frac{1}{3^n}\left(\frac{-1}{2}\right)\right) &= \frac{1}{3^n}u[n]\\ \frac{-1}{2}B &= u[n]\\ B &= -2u[n]\end{aligned}$$

Hence for $n \geq 0$

$$B = -2$$

Therefore

$$y_p[n] = -2\left(\frac{1}{3^n}\right)$$

4.2.5.3 Part c

The solution is given by the sum of the homogenous and particular solutions. Hence

$$\begin{aligned} y[n] &= y_h[n] + y_p[n] \\ &= A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right) \end{aligned} \tag{1}$$

Since system initially at rest, then $y[-1] = 0$. The recurrence equation is given as

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Substituting (1) into the above and using $x[n] = \frac{1}{3^n}u[n]$ gives

$$y[n] - \frac{1}{2}y[n-1] = \frac{1}{3^n}u[n]$$

At $n = 0$ the above becomes

$$y[0] - \frac{1}{2}y[-1] = 1$$

But $y[-1] = 0$ and $y[0] = \left(A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right)\right)_{n=0} = A - 2$. Hence $A - 2 = 1$ or

$$A = 3$$

Therefore the solution (1) becomes

$$y[n] = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right)$$

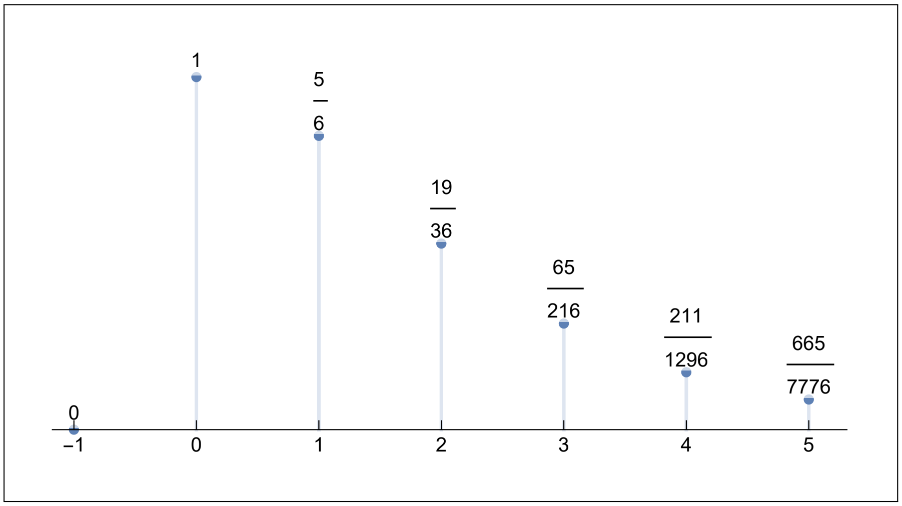
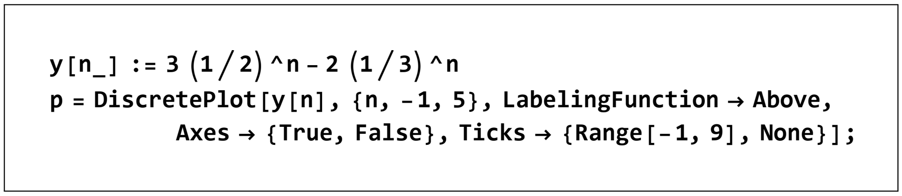


Figure 4.44: Plot of $y[n]$



```
y[n_] := 3 (1/2)^n - 2 (1/3)^n
p = DiscretePlot[y[n], {n, -1, 5}, LabelingFunction → Above,
          Axes → {True, False}, Ticks → {Range[-1, 9], None}];
```

Figure 4.45: Code used for the above

4.2.6 Problem 2.42, Chapter 2

Suppose the signal $x(t) = u\left(t+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right)$ is convolved with the signal $h(t) = e^{j\omega_0 t}$. (a) Determine the value of ω_0 which insures that $y(0) = 0$. Where $y(t) = x(t) \circledast h(t)$. (b) Is your answer to previous part unique?

Solution

4.2.6.1 Part a

$$x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Since $x(t)$ is box function from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$

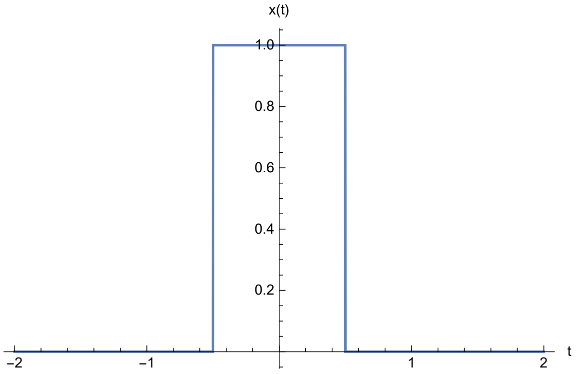
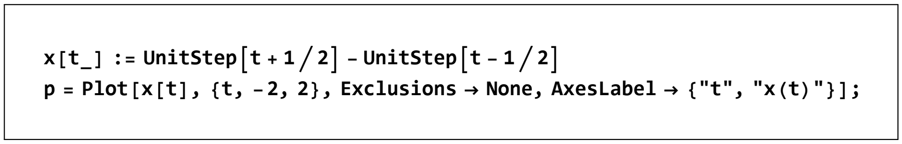


Figure 4.46: Plot of $x(n)$



```
x[t_] := UnitStep[t + 1/2] - UnitStep[t - 1/2]
p = Plot[x[t], {t, -2, 2}, Exclusions → None, AxesLabel → {"t", "x(t)"}];
```

Figure 4.47: Code used for the above

Then by folding $h(t)$ and shifting it over $x(t)$ it is clear that only the region between $\tau = -\frac{1}{2}$ to $\tau = \frac{1}{2}$ will contribute to the integral above since $x(\tau)$ is zero everywhere else. Hence the integral simplifies to

$$
\begin{aligned}
y(t) &= x(t) \circledast h(t) \\
&= \int_{\frac{-1}{2}}^{\frac{1}{2}} h(t-\tau)\, d\tau \\
&= \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{j\omega_0 (t-\tau)} d\tau \\
&= e^{j\omega_0 t} \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j\omega_0 \tau} d\tau \\
&= e^{j\omega_0 t} \left[\frac{e^{-j\omega_0 \tau}}{-j\omega_0} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= e^{j\omega_0 t} \left(\frac{e^{-\frac{1}{2} j\omega_0} - e^{\frac{1}{2} j\omega_0}}{-j\omega_0} \right) \\
&= e^{j\omega_0 t} \left(\frac{e^{\frac{1}{2} j\omega_0} - e^{-\frac{1}{2} j\omega_0}}{j\omega_0} \right) \\
&= 2\frac{e^{j\omega_0 t}}{\omega_0} \left(\frac{e^{\frac{1}{2} j\omega_0} - e^{-\frac{1}{2} j\omega_0}}{2j} \right)
\end{aligned}
$$

But $\frac{e^{\frac{1}{2} j\omega_0} - e^{-\frac{1}{2} j\omega_0}}{2j} = \sin\left(\frac{\omega_0}{2}\right)$ using Euler relation. Hence the above becomes

$$
y(t) = 2\frac{e^{j\omega_0 t}}{\omega_0} \sin\left(\frac{\omega_0}{2}\right)
$$

When $t = 0$ we are told $y(0) = 0$. The above becomes

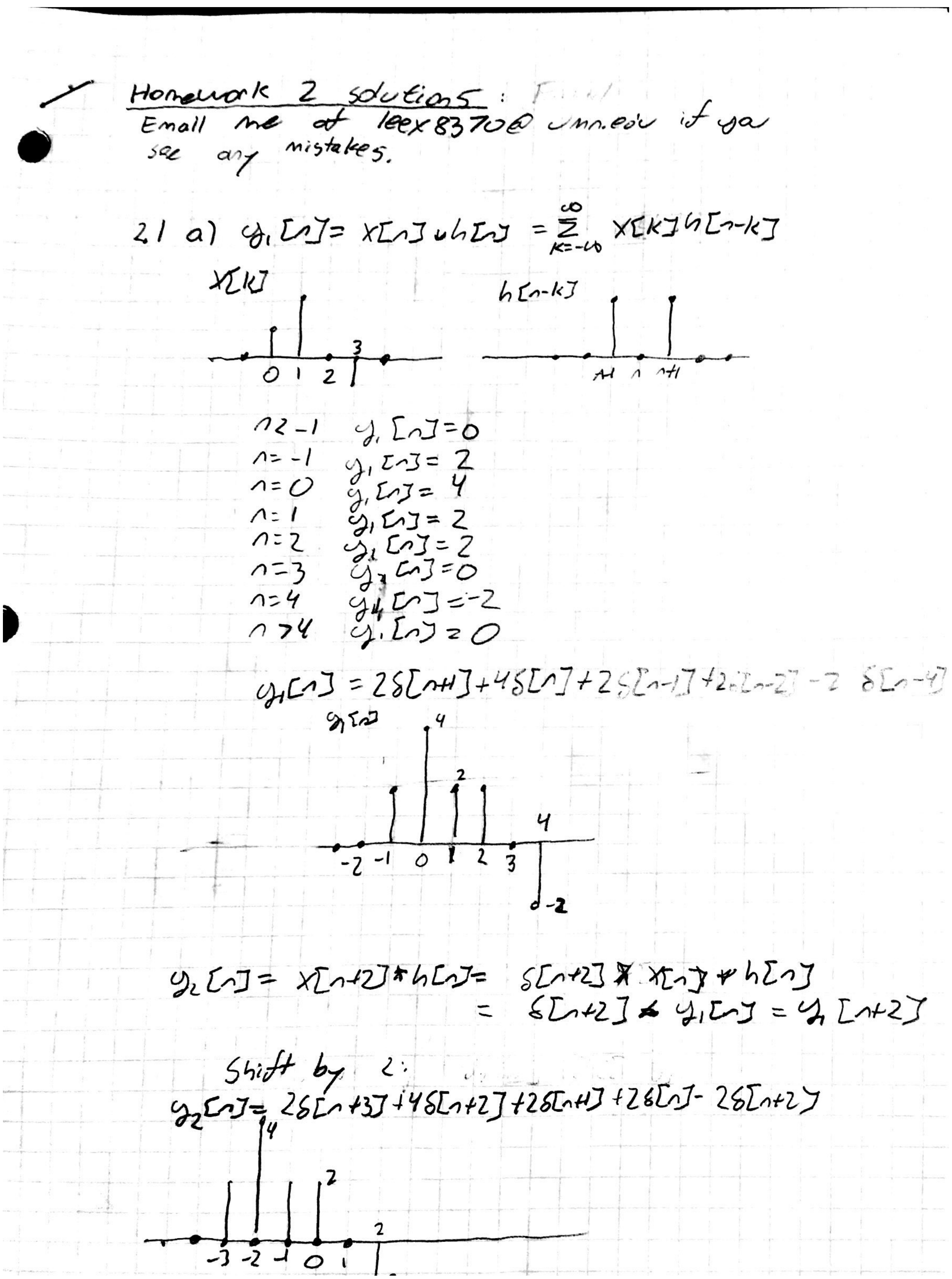
$$
0 = \frac{2}{\omega_0} \sin\left(\frac{\omega_0}{2}\right)
$$

A value of ω_0 which will satisfy the above is $\underline{\omega_0 = 2\pi}$

4.2.6.2 Part b

The value ω_0 found in part (a) is not unique, since any nonzero integer multiple of 2π will also satisfy $y(0) = 0$

4.2.7 key solution



Homework 2 solutions :

Email me at leex8370@umn.edu if you see any mistakes.

2.1 a) $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$n < -1$ $y_1[n] = 0$

$n = -1$ $y_1[n] = 2$

$n = 0$ $y_1[n] = 4$

$n = 1$ $y_1[n] = 2$

$n = 2$ $y_1[n] = 2$

$n = 3$ $y_1[n] = 0$

$n = 4$ $y_4[n] = -2$

$n > 4$ $y_1[n] = 0$

$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$

$y_2[n] = x[n+2] * h[n] = \delta[n+2] * x[n] * h[n]$

$= \delta[n+2] * y_1[n] = y_1[n+2]$

Shift by 2:

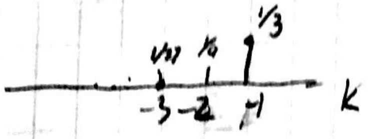
$y_2[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n+2]$

2.6 compute & plot $y[n] = x[n] * h[n]$

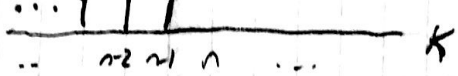
$x[n] = (1/3)^{-n} u[-n-1]$ $h[n] = u[n-1]$

0 for $n \geq +1$

$x[k] =$



$h[n-k]$



for $n-1 > -1$, we have total overlap.

Then $y[n] = \sum_{k=-\infty}^{-1} (1/3)^{-k}$

let $k = -p-1$

$$y[n] = \sum_{p=0}^{\infty} (1/3)^{(p+1)}$$

$$= \frac{1}{3} \cdot \sum_{p=0}^{\infty} (1/3)^{p} = \frac{1}{3} \cdot \frac{1}{1-1/3}$$

$$= \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

for $n < 0$

we have $y[n] = \sum_{k=-\infty}^{n-1} (1/3)^{-k}$

let $k = -p+n-1$

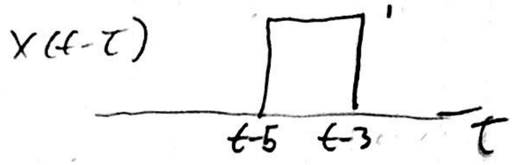
$$\sum_{p=0}^{\infty} (1/3)^{p-n+1}$$

$$= 3^n \sum_{p=0}^{\infty} (1/3)^{p+1} = 1/2 \cdot 3^n$$

2.11 $x(t) = u(t-3) - u(t-5)$

$h(t) = e^{-3t}u(t)$

a) compute $g(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$



$t < 3$ no overlap $g(t) = 0$

$3 < t < 5$ $g(t) = \int_0^{t-3} e^{-3\tau}d\tau = \frac{-e^{-3\tau}}{3}\Big|_0^{t-3} = \frac{1-e^{-3(t-3)}}{3}$

$t > 5$ $g(t) = \int_{t-5}^{t-3} e^{-3\tau}d\tau = \frac{-e^{-3\tau}}{3}\Big|_{t-5}^{t-3} = \frac{e^{-3(t-5)}-e^{-3(t-3)}}{3}$

b) $g(t) = \frac{dx(t)}{dt} * h(t)$

$$\frac{dx(t)}{dt} = \frac{du(t-3)}{dt} - \frac{du(t-5)}{dt} = \delta(t-3) - \delta(t-5)$$

$$g(t) = (\delta(t-3) - \delta(t-5)) * h(t)$$

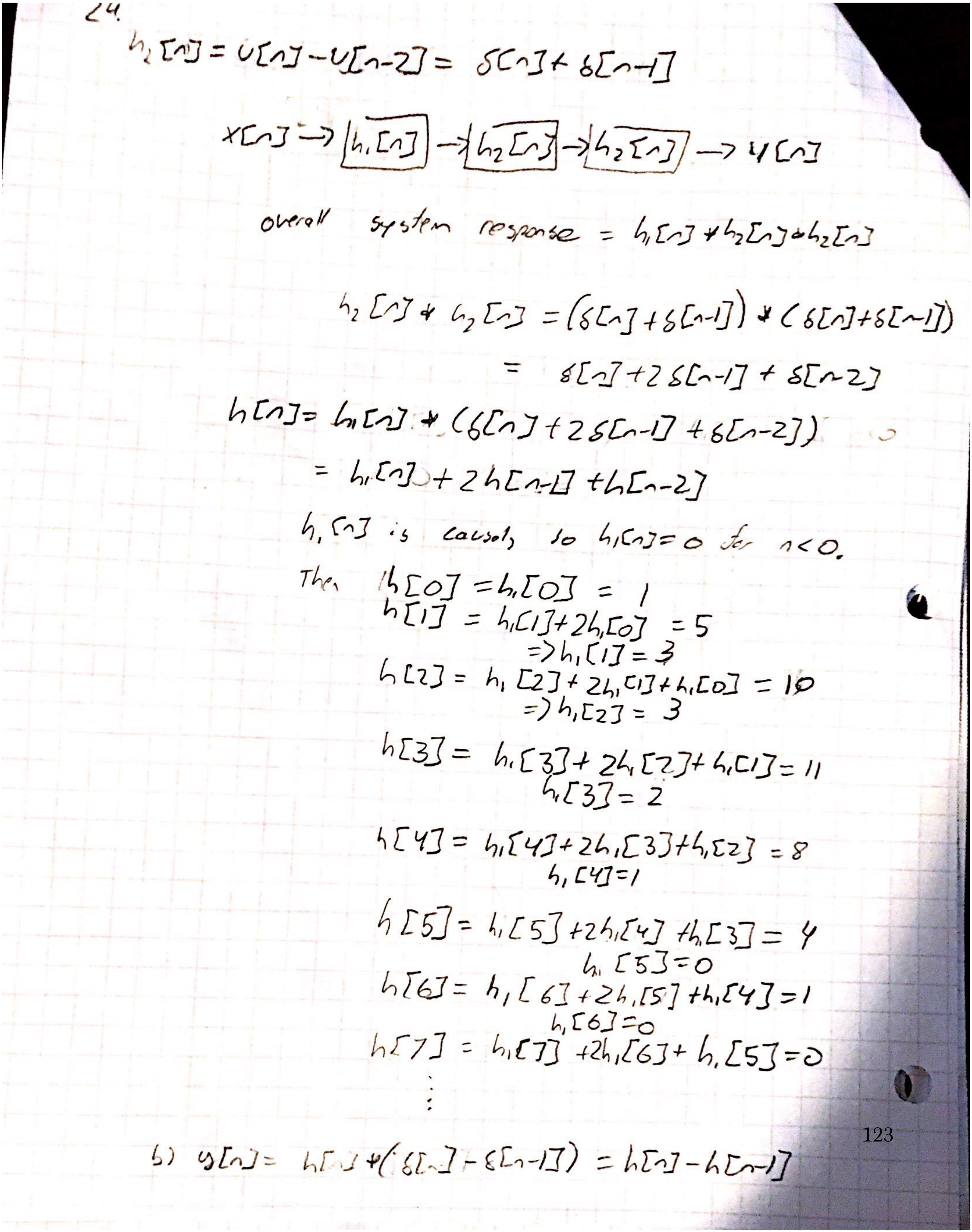
$$= h(t-3) - h(t-5)$$

↳ convolving with delta results in time shift.

$$= e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

c $\frac{dy(t)}{dt} = \begin{cases} 0 & t<0 \\ e^{-3(t-3)} & 3<t<5 \\ e^{-3(t-5)} - e^{-3(t-3)} & t>5 \end{cases} = g(t)$

So $g(t) = \frac{dy(t)}{dt}$



24.

$h_2[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$

$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$

overall system response = $h_1[n] * h_2[n] * h_2[n]$

$h_2[n] * h_2[n] = (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1])$

$= \delta[n] + 2\delta[n-1] + \delta[n-2]$

$h[n] = h_1[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2])$

$= h_1[n] + 2h[n-1] + h[n-2]$

$h_1[n]$ is causal, so $h_1[n] = 0$ for $n < 0$.

Then $h[0] = h_1[0] = 1$

$h[1] = h_1[1] + 2h_1[0] = 5$

$\Rightarrow h_1[1] = 3$

$h[2] = h_1[2] + 2h_1[1] + h_1[0] = 10$

$\Rightarrow h_1[2] = 3$

$h[3] = h_1[3] + 2h_1[2] + h_1[1] = 11$

$h_1[3] = 2$

$h[4] = h_1[4] + 2h_1[3] + h_1[2] = 8$

$h_1[4] = 1$

$h[5] = h_1[5] + 2h_1[4] + h_1[3] = 4$

$h_1[5] = 0$

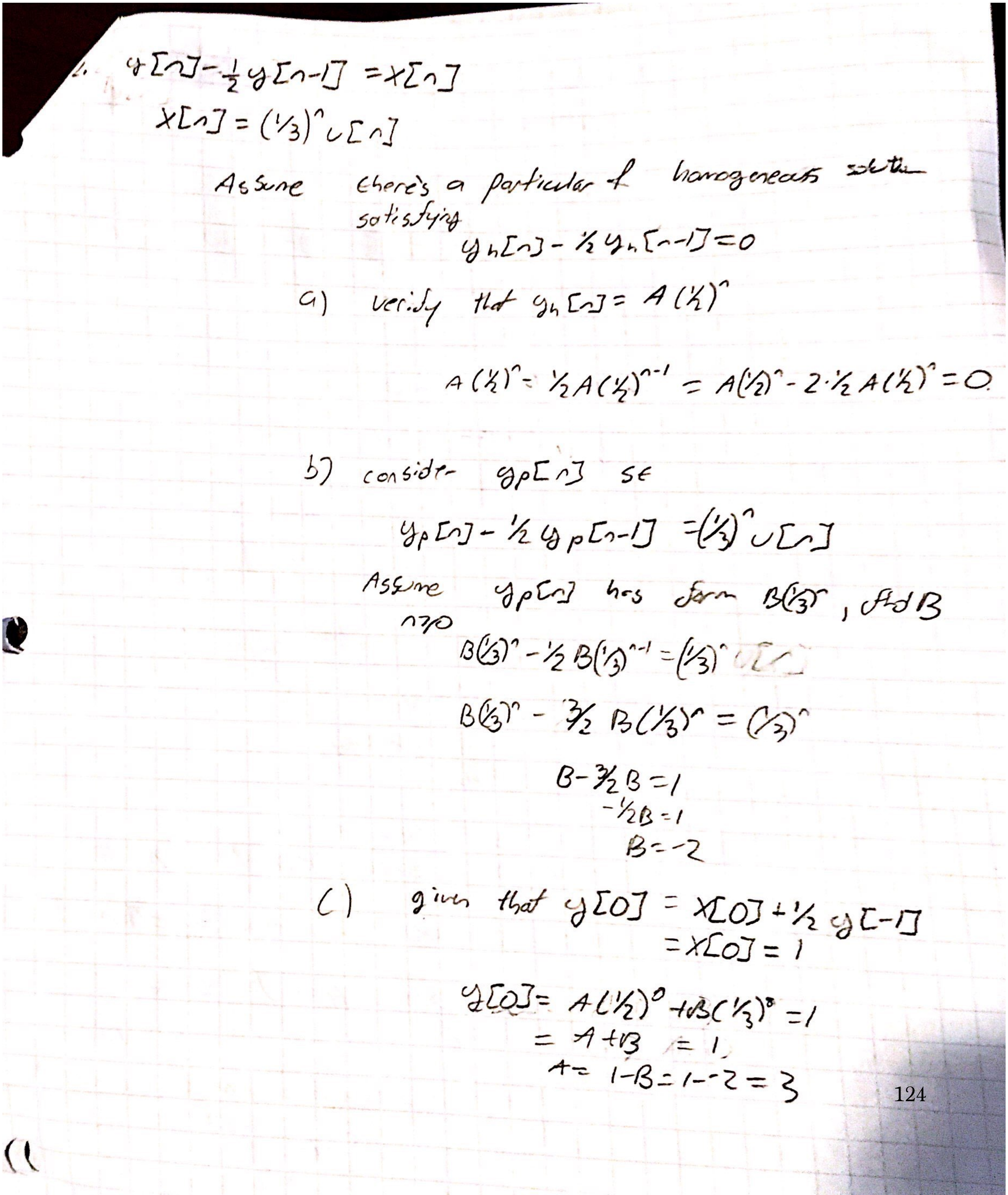
$h[6] = h_1[6] + 2h_1[5] + h_1[4] = 1$

$h_1[6] = 0$

$h[7] = h_1[7] + 2h_1[6] + h_1[5] = 0$

⋮

b) $y[n] = h[n] * (\delta[n] - \delta[n-1]) = h[n] - h[n-1]$



1. $y[n] - \frac{1}{2} y[n-1] = x[n]$

$x[n] = (1/3)^n u[n]$

Assume there's a particular & homogeneous soln satisfying

$$y_h[n] - 1/2\, y_h[n-1] = 0$$

a) verify that $y_h[n] = A(1/2)^n$

$$A(1/2)^n - 1/2 A(1/2)^{n-1} = A(1/2)^n - 2 \cdot 1/2\, A(1/2)^n = 0$$

b) consider $y_p[n]$ st

$$y_p[n] - 1/2\, y_p[n-1] = (1/3)^n u[n]$$

Assume $y_p[n]$ has form $B(1/3)^n$, find B

$n \geq 0$

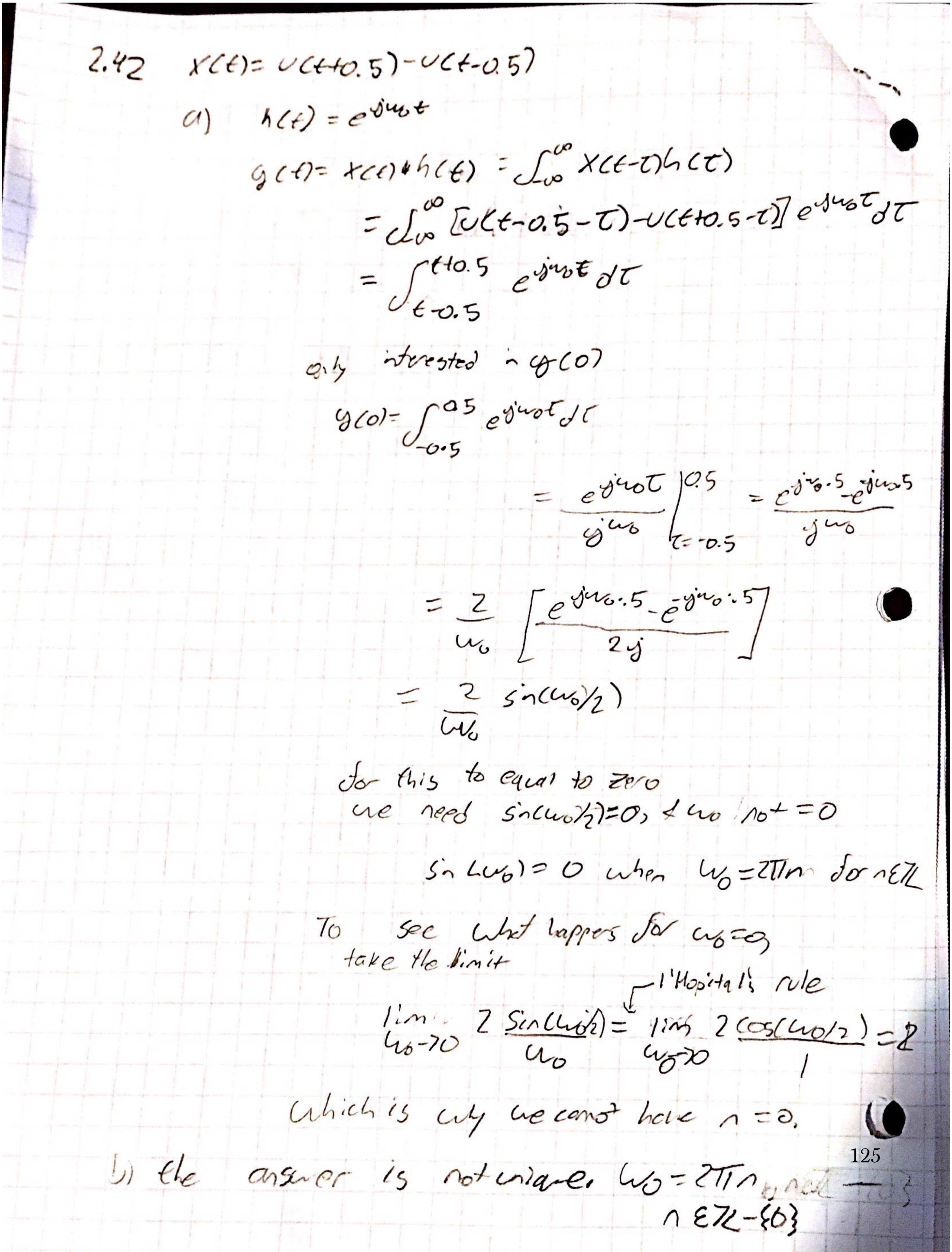
$$B(1/3)^n - 1/2 B(1/3)^{n-1} = (1/3)^n u[n]$$

$$B(1/3)^n - 3/2\, B(1/3)^n = (1/3)^n$$

$$B - 3/2 B = 1$$
$$-1/2 B = 1$$
$$B = -2$$

c) given that $y[0] = x[0] + 1/2\, y[-1]$
$= x[0] = 1$

$$y[0] = A(1/2)^0 + B(1/3)^0 = 1$$
$$= A + B = 1$$
$$A = 1 - B = 1 - -2 = 3$$



2.42 $x(t) = u(t+0.5) - u(t-0.5)$

a) $h(t) = e^{j\omega_0 t}$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau)$$

$$= \int_{-\infty}^{\infty} [u(t-0.5-\tau) - u(t+0.5-\tau)] e^{j\omega_0 \tau} d\tau$$

$$= \int_{t-0.5}^{t+0.5} e^{j\omega_0 t} d\tau$$

only interested in $y(0)$

$$y(0) = \int_{-0.5}^{0.5} e^{j\omega_0 \tau} d\tau$$

$$= \frac{e^{j\omega_0 \tau}}{j\omega_0} \Big|_{\tau=-0.5}^{0.5} = \frac{e^{j\omega_0 .5} - e^{-j\omega_0 .5}}{j\omega_0}$$

$$= \frac{2}{\omega_0} \left[\frac{e^{j\omega_0 \cdot .5} - e^{-j\omega_0 \cdot .5}}{2j} \right]$$

$$= \frac{2}{\omega_0} \sin(\omega_0/2)$$

for this to equal to zero we need $\sin(\omega_0/2) = 0$, & ω_0 not $= 0$

$\sin(\omega_0) = 0$ when $\omega_0 = 2\pi n$ for $n \in \mathbb{Z}$

To see what happens for $\omega_0 = 0$, take the limit

$$\lim_{\omega_0 \to 0} 2 \frac{\sin(\omega_0/2)}{\omega_0} \overset{\text{l'Hopital's rule}}{=} \lim_{\omega_0 \to 0} 2 \frac{\cos(\omega_0/2)}{1} = 2$$

which is why we cannot have $n = 0$.

b) the answer is not unique. $\omega_0 = 2\pi n$, where $n \in \mathbb{Z} - \{0\}$

4.3 HW 3

Local contents

4.3.1 Problem 3 Chapter 3

For the continuous-time periodic signal $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$ determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

<u>Solution</u>

The signal $\cos\left(\frac{2\pi}{3}t\right)$ has period $\frac{2\pi}{T_1} = \frac{2\pi}{3}$. Hence $T_1 = 3$ and the signal $\sin\left(\frac{5\pi}{3}t\right)$ has period $\frac{2\pi}{T_2} = \frac{5\pi}{3}$ or $T_2 = \frac{6}{5}$. Therefore the LCM of $3, \frac{6}{5}$ is

$$\begin{aligned} 3m &= \frac{6}{5}n \\ \frac{m}{n} &= \frac{2}{5} \end{aligned}$$

Hence $m = 2$ and $n = 5$. Therefore $\underline{T_0 = 6}$. Therefore

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T_0} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

Hence

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{1}$$

Where

$$a_k = \frac{1}{T_0} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t} \tag{2}$$

To find a_k for the given signal, instead of using the above integration formula, we could write the signal $x(t)$ in exponential form using Euler relation and just read the a_k coefficients directly from the result. The signal $x(t)$ can be written as

$$\begin{aligned} x(t) &= 2 + \frac{e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}}{2} + 4\frac{e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}}{2i} \\ &= 2 + \frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} + 4\frac{e^{j5\omega_0 t} - e^{-j5\omega_0 t}}{2i} \\ &= 2 + \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} + 2ie^{j5\omega_0 t} - 2ie^{-j5\omega_0 t} \end{aligned} \tag{3}$$

Comparing (3) to (1) shows that the coefficients are

$$\begin{aligned} a_0 &= 2 \\ a_2 &= \frac{1}{2} \\ a_{-2} &= \frac{1}{2} \\ a_5 &= 2j \\ a_{-5} &= -2j \end{aligned}$$

4.3.2 Problem 10 Chapter 3

Let $x[n]$ be real and odd periodic signal with period $N = 7$ and Fourier coefficients a_k. Given that $a_{15} = j, a_{16} = 2j, a_{17} = 3j$, determine the values of $a_0, a_{-1}, a_{-2}, a_{-3}$.

Solution

For discrete signal

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \\ &= \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \end{aligned}$$

Where

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \end{aligned}$$

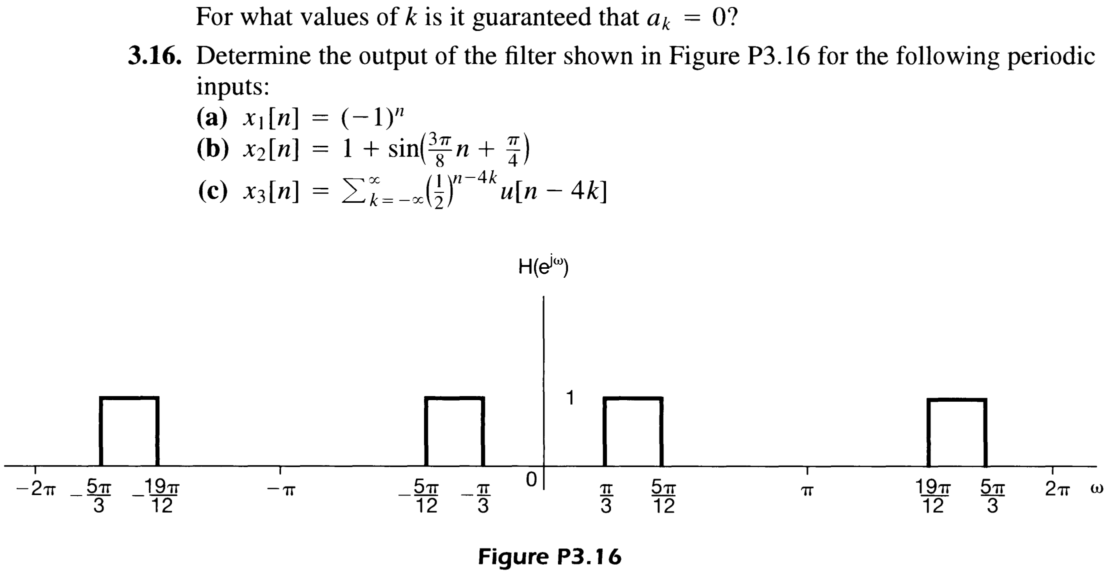
Since the signal $x[n]$ is real, then we know that $a_k = a^*_{-k}$. And since $x[n]$ is odd then we know that a_k is purely imaginary and odd. The Fourier coefficients repeat every N samples which is 7. Hence $a_{15} = a_9 = a_1$ and $a_{16} = a_9 = a_2$ and $a_{17} = a_{10} = a_3$. And since a_k is odd then

$$\begin{aligned} a_0 &= 0 \\ a_1 &= -a_{-1} \\ a_2 &= -a_{-2} \\ a_3 &= -a_{-3} \end{aligned}$$

But we know from above that $a_1 = a_{15} = j$ and $a_2 = a_{16} = 2j$ and $a_3 = a_{17} = 3j$ then the above gives

$$\begin{aligned} a_0 &= 0 \\ a_{-1} &= -j \\ a_{-2} &= -2j \\ a_{-3} &= -3j \end{aligned}$$

4.3.3 Problem 16 Chapter 3



For what values of k is it guaranteed that $a_k = 0$?

3.16. Determine the output of the filter shown in Figure P3.16 for the following periodic inputs:

(a) $x_1[n] = (-1)^n$

(b) $x_2[n] = 1 + \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{n-4k}u[n-4k]$

Figure P3.16

Figure 4.48: Problem description

Solution

The output of discrete LTI system when the input is $x[n] = a_n e^{jn\omega}$ is given by $y[n] = a_n H(e^{j\omega}) e^{jn\omega}$ where $H(e^{j\omega})$ is given to us in the problem statement. Hence, to find $y[n]$ we need to express each input in its Fourier series representation in order to determine the a_n.

4.3.3.1 Part a

Here $x_1[n] = (-1)^n = \left(e^{j\pi}\right)^n = e^{jn\pi}$. To find the period$N$, let $x_1[n] = x_1[n+N]$ or

$$\begin{aligned} e^{jn\pi} &= e^{j(n+N)\pi} \\ &= e^{jn\pi} e^{jN\pi} \end{aligned}$$

Hence $\underline{N=2}$. Therefore $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{2} = \pi$ and $x_1[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = a_0 + a_1 e^{j\pi n}$. Comparing this to $e^{jn\pi}$ shows that

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \end{aligned}$$

Now that we found the Fourier coefficients for $x_1[n]$ then the output is

$$\begin{aligned} y_1[n] &= \sum_{k=0}^{N-1} a_n H\left(jk\omega_0\right) e^{jnk\omega_0} \\ &= a_0 H(0) e^0 + a_1 H\left(j\pi\right) e^{jn\pi} \end{aligned}$$

But $a_0 = 1, a_1 = 1$ and the above becomes

$$y_1[n] = H\left(j\pi\right) e^{jn\pi}$$

From the graph of $H\left(jk\omega_0\right)$ given, we see that at $\omega = \pi, H\left(j\pi\right) = 0$. Therefore

$$y_1[n] = 0$$

4.3.3.2 Part b

Here $x_2[n] = 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right)$. The first step is to find the period N

$$\begin{aligned} x_2[n] &= x_2[n+N] \\ 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) &= 1 + \sin\left(\frac{3\pi}{8}(n+N) + \frac{\pi}{4}\right) \\ &= 1 + \sin\left(\frac{3\pi}{8}n + \frac{3\pi}{8}N + \frac{\pi}{4}\right) \\ &= 1 + \sin\left(\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) + \frac{3\pi}{8}N\right) \end{aligned}$$

Hence $\frac{3\pi}{8}N = 2\pi m$ or $\frac{N}{m} = \frac{16}{3}$. Since these are relatively prime, then $N = 16$ is the fundamental period. Therefore

$$x_2[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

where $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{16} = \frac{\pi}{8}$. The above becomes

$$x_2[n] = \sum_{k=0}^{15} a_k e^{jk\frac{\pi}{8}n} \tag{1}$$

But

$$\begin{aligned} 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) &= 1 + \frac{e^{j\left(\frac{3\pi}{8}n+\frac{\pi}{4}\right)} - e^{-j\left(\frac{3\pi}{8}n+\frac{\pi}{4}\right)}}{2j} \\ &= 1 + \frac{1}{2j}e^{j\frac{3\pi}{8}n}e^{j\frac{\pi}{4}} - \frac{1}{2j}e^{-j\frac{3\pi}{8}n}e^{-j\frac{\pi}{4}} \end{aligned} \tag{2}$$

Comparing (1) and (2) shows that $a_0 = 1, a_3 = \frac{1}{2j}e^{j\frac{\pi}{4}}, a_{-3} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$. But $a_{-3} = a_{-3+16} = a_{13}$ due to periodicity (and since we want to keep the index from 0 to 15. Therefore

$$\begin{aligned} a_0 &= 1 \\ a_3 &= \frac{1}{2j}e^{j\frac{\pi}{4}} \\ a_{13} &= -\frac{1}{2j}e^{-j\frac{\pi}{4}} \end{aligned}$$

And all other $a_k = 0$. Now that we found the Fourier coefficient, then the response $y_2[n]$ is found from

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{N-1} a_n H\left(jk\omega_0\right) e^{jkn\omega_0} \\ &= a_0 H(0) + a_3 H\left(j3\frac{\pi}{8}\right) e^{j3\frac{\pi}{8}n} + a_{13} H\left(j13\frac{\pi}{8}\right) e^{j13\frac{\pi}{8}n} \\ &= H(0) + \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) H\left(j\frac{3\pi}{8}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) H\left(j\frac{13\pi}{8}\right) e^{j\frac{13\pi}{8}n} \end{aligned}$$

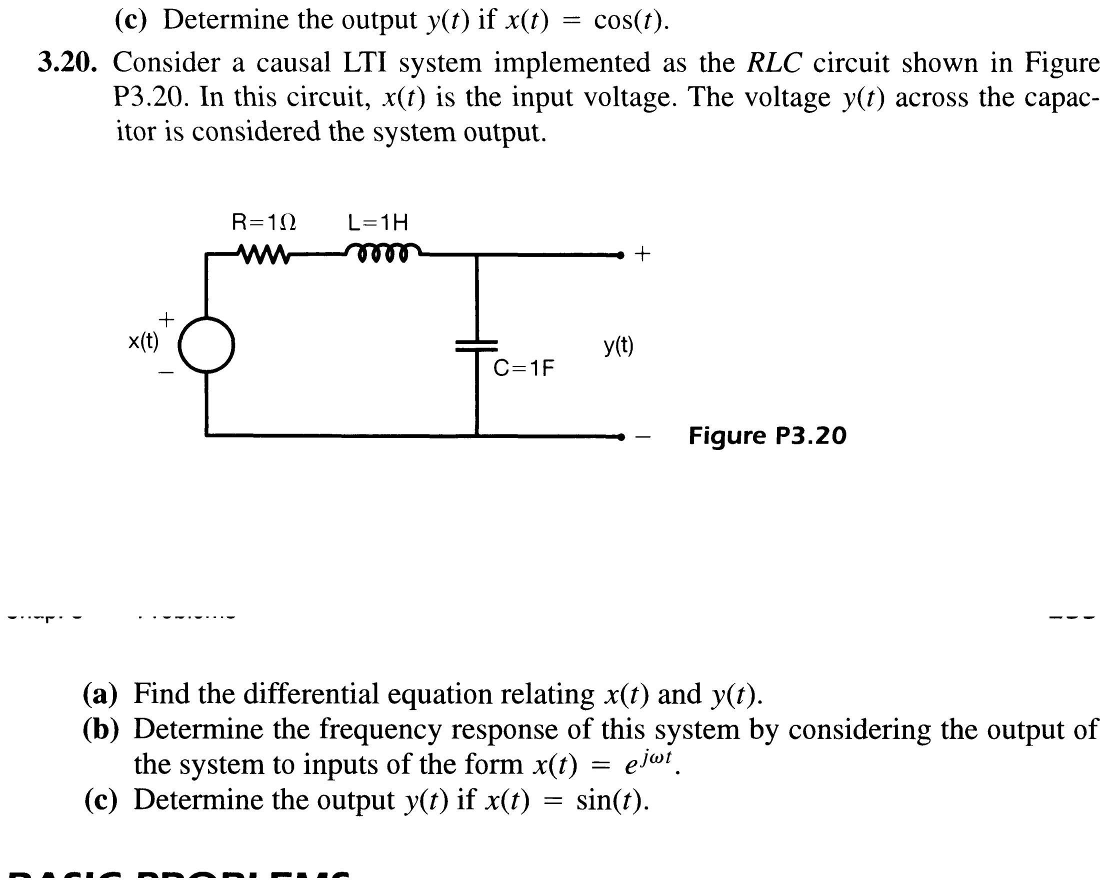
From the graph of $H\left(jk\omega_0\right)$ given, we see that at $\omega = 0, H(0) = 0$ and at $\omega = \frac{3\pi}{8}, H\left(j\frac{3\pi}{8}\right) = 1$ and that at $\omega = \frac{13\pi}{8}, H\left(j\frac{13\pi}{8}\right) = 1$. Hence the above becomes

$$y_2[n] = \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) e^{j\frac{13\pi}{8}n}$$

But $e^{j\frac{13\pi}{8}n} = e^{j\frac{-3\pi}{8}n}$ since period is $N = 16$. Therefore the above simplifies to

$$\begin{aligned} y_2[n] &= \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) e^{j\frac{-3\pi}{8}n} \\ &= \frac{e^{j\left(\frac{\pi}{4}+\frac{3\pi}{8}n\right)} - e^{-j\left(\frac{\pi}{4}+\frac{3\pi}{8}n\right)}}{2j} \\ &= \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) \end{aligned}$$

4.3.4 Problem 20 Chapter 3



(c) Determine the output $y(t)$ if $x(t) = \cos(t)$.

3.20. Consider a causal LTI system implemented as the *RLC* circuit shown in Figure P3.20. In this circuit, $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered the system output.

Figure P3.20

(a) Find the differential equation relating $x(t)$ and $y(t)$.
(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.
(c) Determine the output $y(t)$ if $x(t) = \sin(t)$.

Figure 4.49: Problem description

Solution

4.3.4.1 Part a

Input voltage is $x(t)$. Hence drop in voltage around circuit is

$$x(t) = Ri(t) + L\frac{di}{dt} + y(t)$$

Now we need to relate the current $i(t)$ to $y(t)$. Since current across the capacitor is given by $i(t) = C\frac{dy}{dt}$ then replacing $i(t)$ in the above by $C\frac{dy}{dt}$ gives the diffeential equation

$$x(t) = RC\frac{dy}{dt} + LC\frac{d^2y}{dt^2} + y(t)$$

Or

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

But $L=1,R=1,C=1$ therefore

$$y''(t)+y'(t)+y(t)=x(t)$$

4.3.4.2 Part b

Let the input $x(t)=e^{j\omega t}$. Therefore $y(t)=H(\omega)e^{j\omega t}$ where $H(\omega)$ is the frequency response (Book writes this as $H\left(e^{j\omega}\right)$ but $H(\omega)$ is simpler notation).

Hence

$$\begin{aligned}y'(t)&=H(\omega)j\omega e^{j\omega t}\\ y''(t)&=H(\omega)\left(j\omega\right)^2e^{j\omega t}\\ &=-H(\omega)\omega^2e^{j\omega t}\end{aligned}$$

Substituting the above into the ODE gives

$$-H(\omega)\omega^2e^{j\omega t}+H(\omega)j\omega e^{j\omega t}+H(\omega)e^{j\omega t}=e^{j\omega t}$$

Dividing by $e^{j\omega t}\neq0$ results in

$$-H(\omega)\omega^2+H(\omega)j\omega+H(\omega)=1$$

Solving for $H(\omega)$ gives

$$\begin{aligned}H(\omega)\left(-\omega^2+j\omega+1\right)&=1 \qquad (1)\\ H(\omega)&=\frac{1}{-\omega^2+j\omega+1}\end{aligned}$$

4.3.4.3 Part c

Since we now know $H(\omega)$ then the output $y(t)$ when the input is $x(t)=\sin(t)$ is given by

$$y(t)=\sum_{k=-\infty}^{\infty}a_kH(k\omega_0)e^{jk\omega_0t} \qquad (2)$$

Where a_k are the Fourier coefficients of $\sin(t)$ and ω_0 is the fundamental frequency of $x(t)$. Since $\sin(t)=\sin\left(\frac{2\pi}{T}t\right)$ then $\frac{2\pi}{T}=1$ and $T=2\pi$. Hence $\omega_0=1$. And since $\sin(t)=\frac{1}{2j}\left(e^{jt}-e^{-jt}\right)$ then $a_1=\frac{1}{2j},a_{-1}=-\frac{1}{2j}$. Eq. (2) becomes

$$\begin{aligned}y(t)&=a_{-1}H(-\omega_0)e^{-j\omega_0t}+a_1H(\omega_0)e^{j\omega_0t}\\ &=-\frac{1}{2j}H(-1)e^{-jt}+\frac{1}{2j}H(1)e^{jt} \qquad (3)\end{aligned}$$

Now we need to find $H(-1), H(1)$. From (1)

$$\begin{aligned} H(-1) &= \frac{1}{-(-1)^2 - j(-1) + 1} \\ &= \frac{1}{-1 + j + 1} \\ &= \frac{1}{j} \end{aligned}$$

And

$$\begin{aligned} H(+1) &= \frac{1}{-(+1)^2 - j(+1) + 1} \\ &= \frac{1}{-1 - j + 1} \\ &= \frac{1}{j} \end{aligned}$$

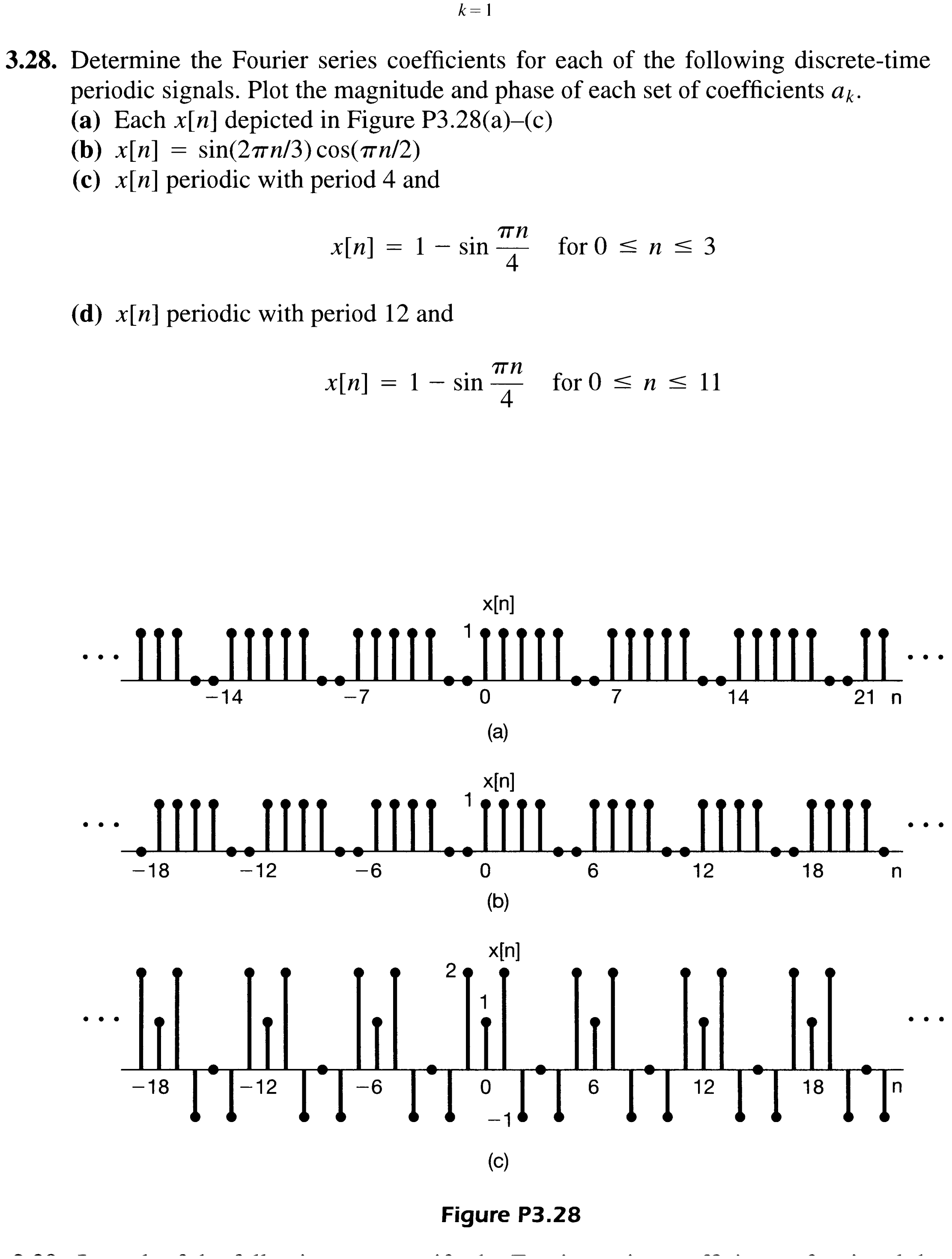
Therefore (3) becomes

$$\begin{aligned} y(t) &= -\frac{1}{2j}\frac{1}{j}e^{-jt} + \frac{1}{2j}\frac{1}{j}e^{jt} \\ &= -\frac{1}{2j^2}e^{-jt} + \frac{1}{2j^2}e^{jt} \\ &= \frac{1}{2}e^{-jt} - \frac{1}{2}e^{jt} \\ &= -\left(\frac{1}{2}e^{jt} - \frac{1}{2}e^{-jt}\right) \end{aligned}$$

Hence

$$y(t) = -\cos(t)$$

4.3.5 Problem 28 Chapter 3



$k = 1$

3.28. Determine the Fourier series coefficients for each of the following discrete-time periodic signals. Plot the magnitude and phase of each set of coefficients a_k.

(a) Each $x[n]$ depicted in Figure P3.28(a)–(c)

(b) $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$

(c) $x[n]$ periodic with period 4 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \leq n \leq 3$$

(d) $x[n]$ periodic with period 12 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \leq n \leq 11$$

Figure P3.28

Figure 4.50: Problem description

Solution

4.3.5.1 Part a

First signal

The signal in P3.28(a) has period $\underline{N = 7}$. Therefore $x[n] = \sum_{k=0}^{N-1} a_k e^{jn(k\omega_0)}$. We need to determine a_k. Since $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{7}$, then

$$\begin{aligned}a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n}\\ &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}\\ &= \frac{1}{7}\sum_{n=0}^{6} x[n]\, e^{-jk\frac{2\pi}{7}n}\end{aligned}$$

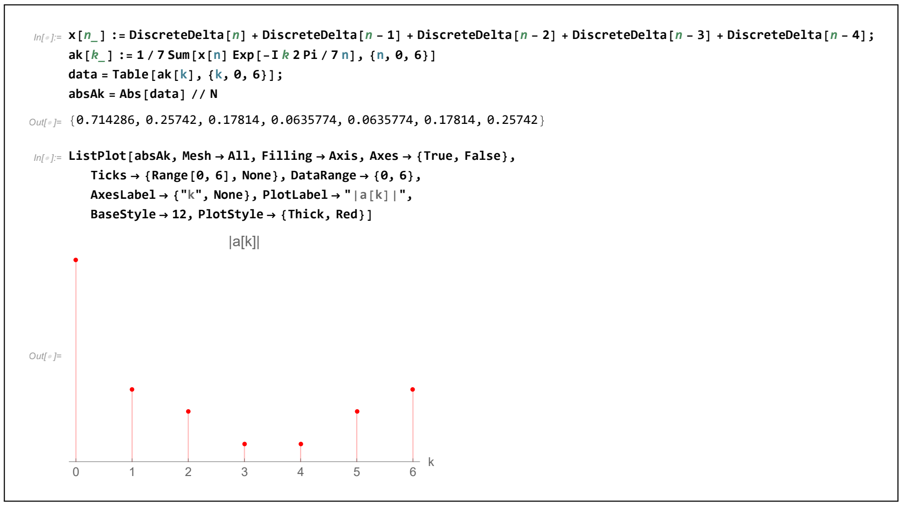
We first notice that $x[n] = 0$ for $n = 5,6$ and $x[n] = 1$ otherwise. Hence the above sum simplifies to

$$a_k = \frac{1}{7}\sum_{n=0}^{4} e^{-jk\frac{2\pi}{7}n}$$

Using the relation $\sum_{n=0}^{M-1} a^n = \begin{cases} M & a = 1 \\ \frac{1-a^M}{1-a} & a \neq 1 \end{cases}$ to simplify the above where now $M = 5$ gives

$$\begin{aligned}a_k &= \frac{1}{7}\frac{1-\left(e^{-jk\frac{2\pi}{7}}\right)^5}{1-e^{-jk\frac{2\pi}{7}}} \qquad k = 0,1,\cdots 6\\ &= \frac{1}{7}\frac{1-e^{-jk\frac{10\pi}{7}}}{1-e^{-jk\frac{2\pi}{7}}}\end{aligned}$$

This is plot of $|a_k|$

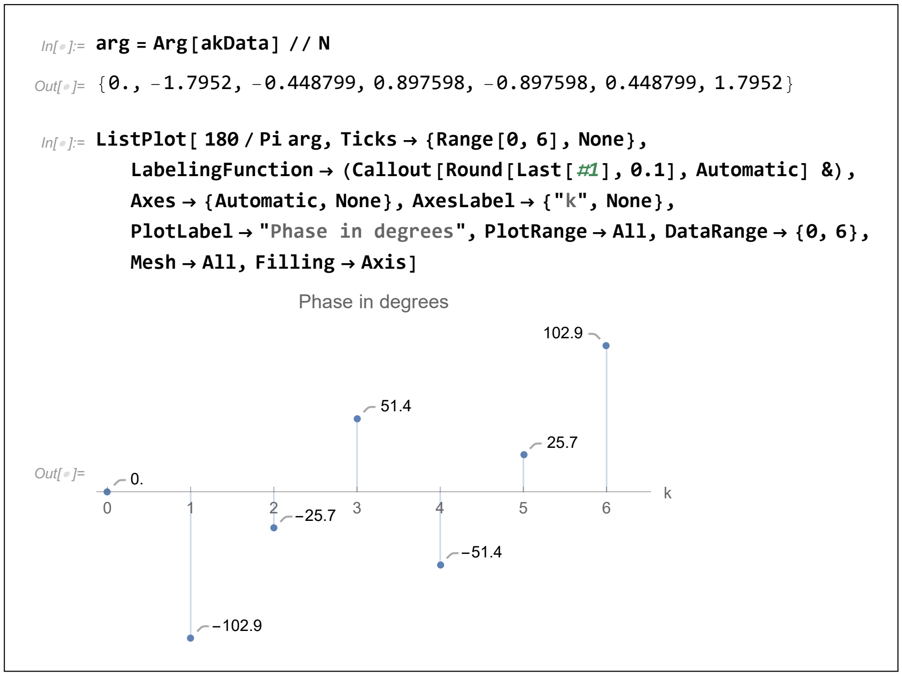


```
In[ ]:= x[n_] := DiscreteDelta[n] + DiscreteDelta[n - 1] + DiscreteDelta[n - 2] + DiscreteDelta[n - 3] + DiscreteDelta[n - 4];
        ak[k_] := 1 / 7 Sum[x[n] Exp[-I k 2 Pi / 7 n], {n, 0, 6}]
        data = Table[ak[k], {k, 0, 6}];
        absAk = Abs[data] // N
Out[ ]= {0.714286, 0.25742, 0.17814, 0.0635774, 0.0635774, 0.17814, 0.25742}

In[ ]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False},
          Ticks → {Range[0, 6], None}, DataRange → {0, 6},
          AxesLabel → {"k", None}, PlotLabel → "|a[k]|",
          BaseStyle → 12, PlotStyle → {Thick, Red}]
Out[ ]=
```

Figure 4.51: Plot of $|a_k|$

This is plot of the phase of a_k



```
In[ ]:= arg = Arg[akData] // N
Out[ ]= {0., -1.7952, -0.448799, 0.897598, -0.897598, 0.448799, 1.7952}

In[ ]:= ListPlot[ 180 / Pi arg, Ticks → {Range[0, 6], None},
          LabelingFunction → (Callout[Round[Last[#1], 0.1], Automatic] &),
          Axes → {Automatic, None}, AxesLabel → {"k", None},
          PlotLabel → "Phase in degrees", PlotRange → All, DataRange → {0, 6},
          Mesh → All, Filling → Axis]
Out[ ]=
```

Figure 4.52: Plot of phase of a_k

second signal

The signal in P3.28(b) has period $\underline{N=6}$. Therefore $x[n]=\sum_{k=0}^{N-1}a_k e^{jn(k\omega_0)}$. We need to determine a_k, where $\omega_0=\frac{2\pi}{N}=\frac{2\pi}{6}$. Hence

$$\begin{aligned}a_k &= \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\omega_0 n}\\ &= \frac{1}{6}\sum_{n=0}^{5}x[n]e^{-jk\frac{2\pi}{6}n}\end{aligned}$$

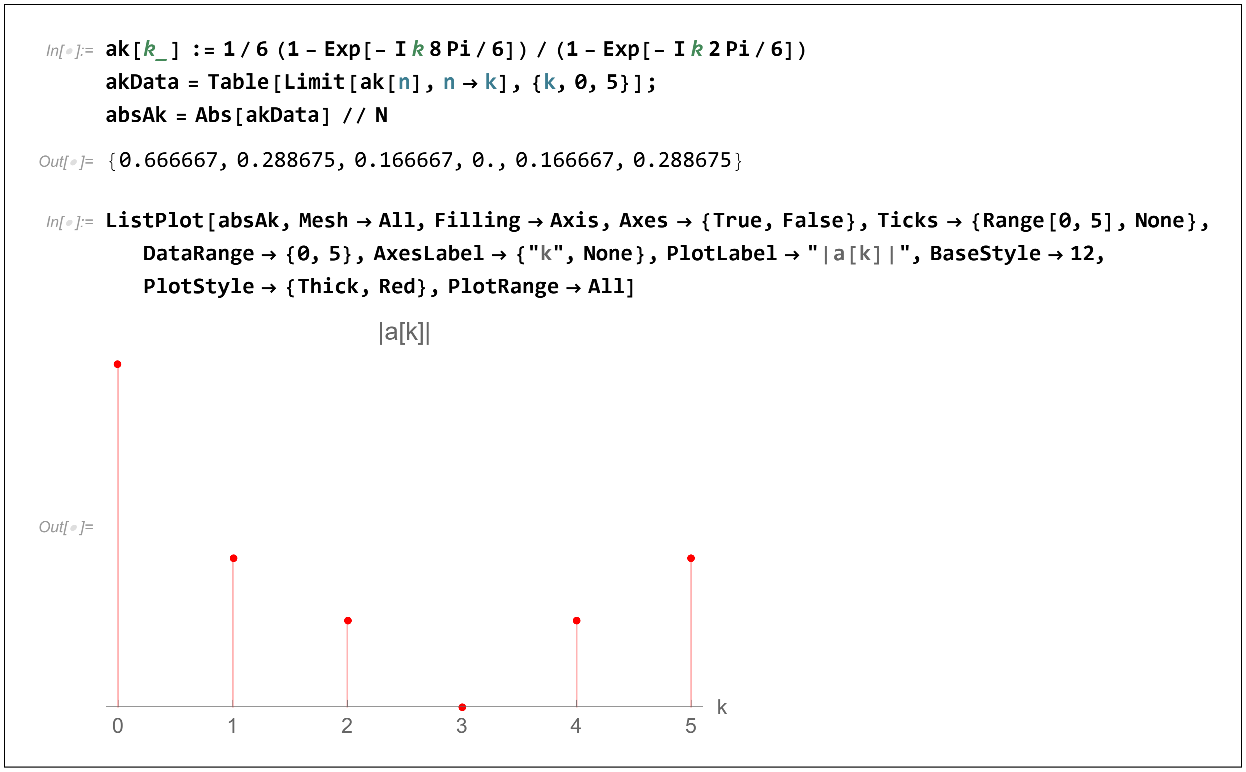
We first notice that $x[n]=0$ for $n=4,5$ and $x[n]=1$ otherwise, Hence the above sum simplifies to

$$a_k=\frac{1}{6}\sum_{n=0}^{3}e^{-jk\frac{2\pi}{6}n}$$

Using the relation $\sum_{n=0}^{M-1}a^n=\begin{cases}M & a=1\\ \frac{1-a^M}{1-a} & a\neq 1\end{cases}$ to simplify the above, where now $M=4$ gives

$$\begin{aligned}a_k &= \frac{1}{6}\frac{1-\left(e^{-jk\frac{2\pi}{6}}\right)^4}{1-e^{-jk\frac{2\pi}{6}}} \qquad k=0,1,\cdots 5\\ &= \frac{1}{6}\frac{1-e^{-jk\frac{8\pi}{6}}}{1-e^{-jk\frac{2\pi}{6}}}\end{aligned}$$

This is plot of $|a_k|$

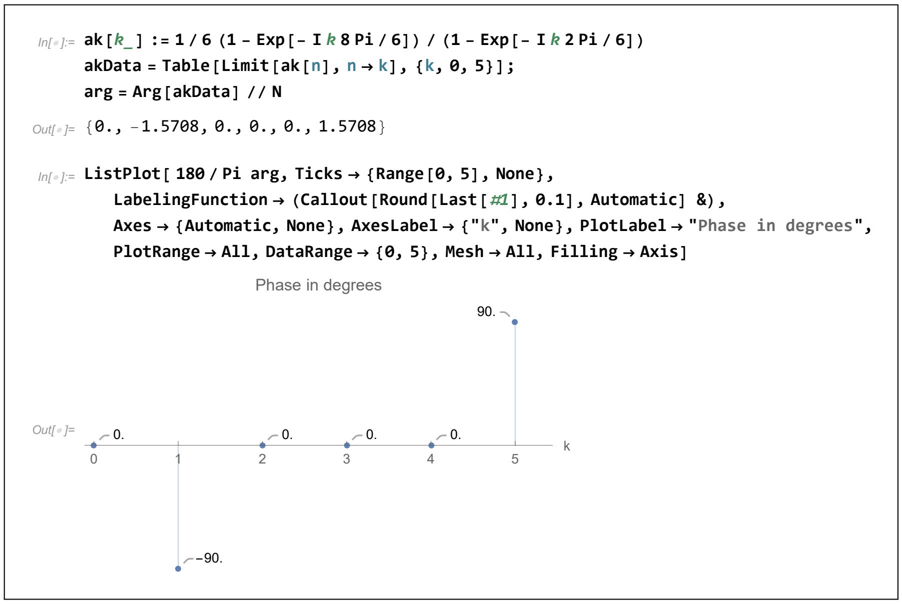


```
In[ ]:= ak[k_] := 1 / 6 (1 - Exp[- I k 8 Pi / 6]) / (1 - Exp[- I k 2 Pi / 6])
        akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
        absAk = Abs[akData] // N
Out[ ]= {0.666667, 0.288675, 0.166667, 0., 0.166667, 0.288675}

In[ ]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False}, Ticks → {Range[0, 5], None},
          DataRange → {0, 5}, AxesLabel → {"k", None}, PlotLabel → "|a[k]|", BaseStyle → 12,
          PlotStyle → {Thick, Red}, PlotRange → All]
Out[ ]=
```

Figure 4.53: Plot of $|a_k|$

This is plot of the phase of a_k



```
In[ ]:= ak[k_] := 1 / 6 (1 - Exp[- I k 8 Pi / 6]) / (1 - Exp[- I k 2 Pi / 6])
        akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
        arg = Arg[akData] // N
Out[ ]= {0., -1.5708, 0., 0., 0., 1.5708}

In[ ]:= ListPlot[ 180 / Pi arg, Ticks → {Range[0, 5], None},
          LabelingFunction → (Callout[Round[Last[#1], 0.1], Automatic] &),
          Axes → {Automatic, None}, AxesLabel → {"k", None}, PlotLabel → "Phase in degrees",
          PlotRange → All, DataRange → {0, 5}, Mesh → All, Filling → Axis]
Out[ ]=
```

Figure 4.54: Plot of phase of a_k

Third signal

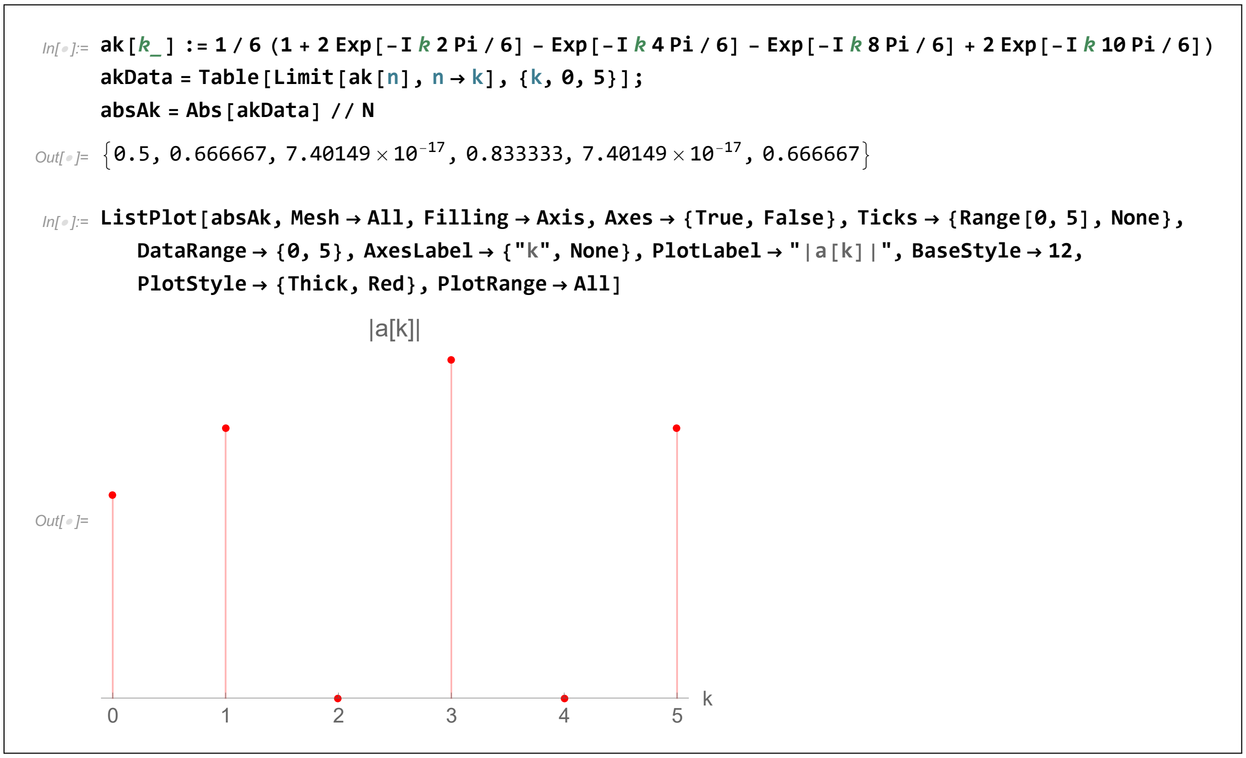
The signal in P3.28(c) also has period $N = 6$. Therefore $x[n] = \sum_{k=0}^{N-1} a_k e^{jn(k\omega_0)}$. We need to determine a_k. Given that $\omega_0 = \frac{2\pi}{N}$ then

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-jk\frac{2\pi}{6}n} \end{aligned}$$

Where $x[0] = 1, x[1] = 2, x[2] = -1, x[3] = 0, x[4] = -1, x[5] = 2$. Hence the above sum becomes

$$\begin{aligned} a_k &= \frac{1}{6}\left(1 + 2e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{2\pi}{6}2} + 0 - e^{-jk\frac{2\pi}{6}4} + 2e^{-jk\frac{2\pi}{6}5}\right) \\ &= \frac{1}{6}\left(1 + 2e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{4\pi}{6}} - e^{-jk\frac{8\pi}{6}} + 2e^{-jk\frac{10\pi}{6}}\right) \end{aligned}$$

This is plot of $|a_k|$



```
In[∘]:= ak[k_] := 1 / 6 (1 + 2 Exp[-I k 2 Pi / 6] - Exp[-I k 4 Pi / 6] - Exp[-I k 8 Pi / 6] + 2 Exp[-I k 10 Pi / 6])
        akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
        absAk = Abs[akData] // N
Out[∘]= {0.5, 0.666667, 7.40149 × 10^-17, 0.833333, 7.40149 × 10^-17, 0.666667}

In[∘]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False}, Ticks → {Range[0, 5], None},
          DataRange → {0, 5}, AxesLabel → {"k", None}, PlotLabel → "|a[k]|", BaseStyle → 12,
          PlotStyle → {Thick, Red}, PlotRange → All]
Out[∘]=
```

Figure 4.55: Plot of $|a_k|$

This is plot of the phase of a_k

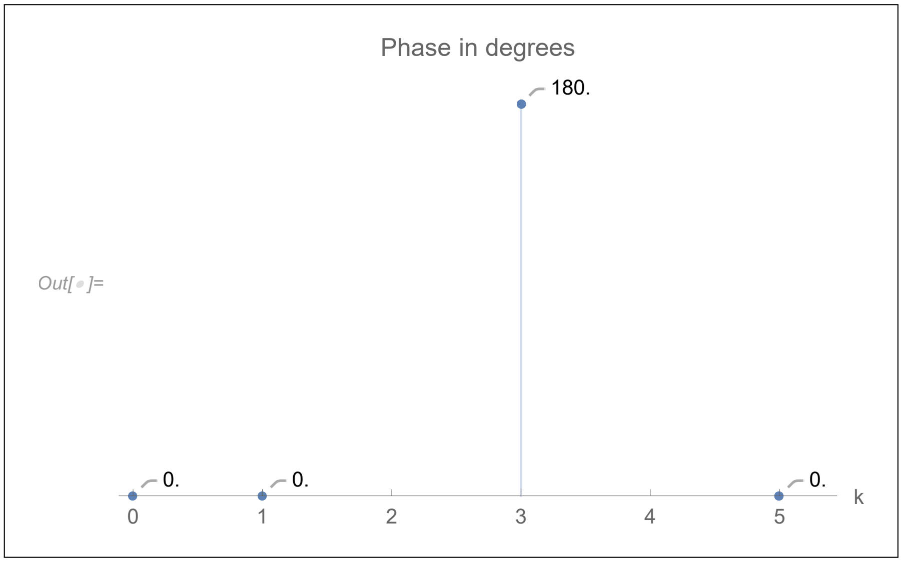


Figure 4.56: Plot of phase of a_k

4.3.5.2 Part b

$$x[n]=\sin\left(2\pi\frac{n}{3}\right)\cos\left(\pi\frac{n}{2}\right)$$

The first step is to find N, the fundamental period. Since $\sin(A)\cos(B)=\frac{1}{2}(\sin(A+B)+\sin(A-B))$ then

$$\begin{aligned}x[n]&=\frac{1}{2}\left(\sin\left(2\pi\frac{n}{3}+\pi\frac{n}{2}\right)+\sin\left(2\pi\frac{n}{3}-\pi\frac{n}{2}\right)\right)\\&=\frac{1}{2}\left(\sin\left(\frac{7}{6}\pi n\right)+\sin\left(\frac{1}{6}\pi n\right)\right)\end{aligned}$$

To find the period of $\sin\left(\frac{7}{6}\pi n\right)=\sin\left(\frac{7}{6}\pi(n+N)\right)$ or $\sin\left(\frac{7}{6}\pi n\right)=\sin\left(\frac{7}{6}\pi n+\frac{7}{6}\pi N\right)$. Hence $\frac{7}{6}\pi N=2\pi m$ which gives $\frac{m}{N}=\frac{7}{12}$. Hence $\underline{N=12}$.

The period of $\sin\left(\frac{1}{6}\pi n\right)=\sin\left(\frac{1}{6}\pi(n+N)\right)$ or $\sin\left(\frac{1}{6}\pi n\right)=\sin\left(\frac{1}{6}\pi n+\frac{1}{6}\pi N\right)$. Hence $\frac{1}{6}\pi N=2\pi m$ or $\frac{m}{N}=\frac{1}{12}$. Hence common period is $\underline{N=12}$. Now that we know N then

$$a_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\omega_0 n}$$

Where $\omega_0=\frac{2\pi}{12}$. The above becomes

$$\begin{aligned}a_k&=\frac{1}{12}\sum_{n=0}^{11}\sin\left(2\pi\frac{n}{3}\right)\cos\left(\pi\frac{n}{2}\right)e^{-jk\frac{2\pi}{12}n}\\12a_k&=0+\sin\left(2\pi\frac{1}{3}\right)\cos\left(\pi\frac{1}{2}\right)e^{-jk\frac{2\pi}{12}}+\sin\left(2\pi\frac{2}{3}\right)\cos\left(\pi\frac{2}{2}\right)e^{-jk\frac{2\pi}{12}2}+\sin\left(2\pi\frac{3}{3}\right)\cos\left(\pi\frac{3}{2}\right)e^{-jk\frac{2\pi}{12}3}\\&+\sin\left(2\pi\frac{4}{3}\right)\cos\left(\pi\frac{4}{2}\right)e^{-jk\frac{2\pi}{12}4}+\sin\left(2\pi\frac{5}{3}\right)\cos\left(\pi\frac{5}{2}\right)e^{-jk\frac{2\pi}{12}5}+\sin\left(2\pi\frac{6}{3}\right)\cos\left(\pi\frac{6}{2}\right)e^{-jk\frac{2\pi}{12}6}\\&+\sin\left(2\pi\frac{7}{3}\right)\cos\left(\pi\frac{7}{2}\right)e^{-jk\frac{2\pi}{12}7}+\sin\left(2\pi\frac{8}{3}\right)\cos\left(\pi\frac{8}{2}\right)e^{-jk\frac{2\pi}{12}8}+\sin\left(2\pi\frac{9}{3}\right)\cos\left(\pi\frac{9}{2}\right)e^{-jk\frac{2\pi}{12}9}\\&+\sin\left(2\pi\frac{10}{3}\right)\cos\left(\pi\frac{10}{2}\right)e^{-jk\frac{2\pi}{12}10}+\sin\left(2\pi\frac{11}{3}\right)\cos\left(\pi\frac{11}{2}\right)e^{-jk\frac{2\pi}{12}11}\end{aligned}$$

Which simplifies to (many terms go to zero)

$$12a_k=\frac{1}{2}\sqrt{3}e^{-jk\frac{4\pi}{12}}+\frac{1}{2}\sqrt{3}e^{-jk\frac{8\pi}{12}}-\frac{1}{2}\sqrt{3}e^{-jk\frac{2\pi}{12}8}-\frac{1}{2}\sqrt{3}e^{-jk\frac{2\pi}{12}10}$$

Hence

$$a_k=\frac{\sqrt{3}}{24}\left(e^{-jk\frac{4\pi}{12}}+e^{-jk\frac{8\pi}{12}}-e^{-jk\frac{16\pi}{12}}-e^{-jk\frac{20\pi}{12}}\right)$$

Evaluating these for $k=0\cdots N-1$ gives

k	a_k
0	0
1	$\frac{-j}{4}$
2	0
3	0
4	0
5	$\frac{j}{4}$
6	0
7	$\frac{-j}{4}$
8	0
9	0
10	0
11	$\frac{j}{4}$

Hence the $|a_k|$ and phase are

k	a_k	$|a_k|$	phase (degree)
0	0	0	0
1	$\frac{-j}{4}$	$\frac{1}{4}$	-90
2	0	0	0
3	0	0	0
4	0	0	0
5	$\frac{j}{4}$	$\frac{1}{4}$	90
6	0	0	0
7	$\frac{-j}{4}$	$\frac{1}{4}$	-90
8	0	0	0
9	0	0	0
10	0	0	0
11	$\frac{j}{4}$	$\frac{1}{4}$	90

4.3.6 Problem 47 Chapter 3

Consider the signal $x(t) = \cos(2\pi t)$ since $x(t)$ is periodic with a fundamental period of 1, it is also periodic with a period of N, where N is any positive integer. What are the Fourier series coefficients of $x(t)$ if we regard it as a periodic signal with period 3?

Solution

The Fourier series coefficients for $\cos(2\pi t)$ are found from Euler relation. Since $\omega_0 = 2\pi$

rad/sec, then

$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

Comparing the above to

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Show that $a_1 = \frac{1}{2}$ and $a_{-1} = \frac{1}{2}$ and all other $a_k = 0$.

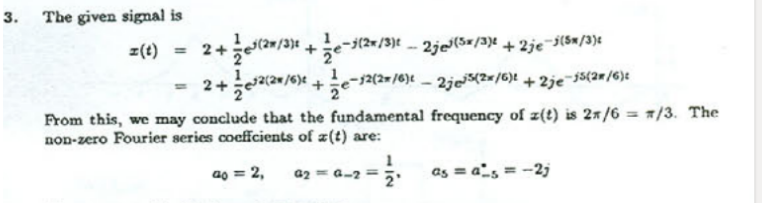
Similarly, if the period happened to be 3, then $\omega_0 = \frac{2\pi}{3}$ and now $x(t)$ can be written as $\cos(2\pi t) = \cos(3\omega_0 t)$. Therefore doing the same as above gives

$$\cos(3\omega_0 t) = \frac{1}{2}e^{j3\omega_0 t} + \frac{1}{2}e^{-j3\omega_0 t}$$

Comparing the above to $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ shows that $a_3 = \frac{1}{2}$ and $a_{-3} = \frac{1}{2}$ and all other $a_k = 0$.

4.3.7 key solution

Homework 3 solutions

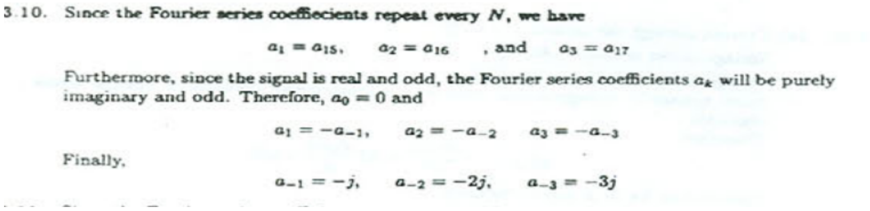


3. The given signal is

$$\begin{aligned} x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \qquad a_2 = a_{-2} = \frac{1}{2}, \qquad a_5 = a_{-5}^* = -2j$$



3.10. Since the Fourier series coeffiecients repeat every N, we have

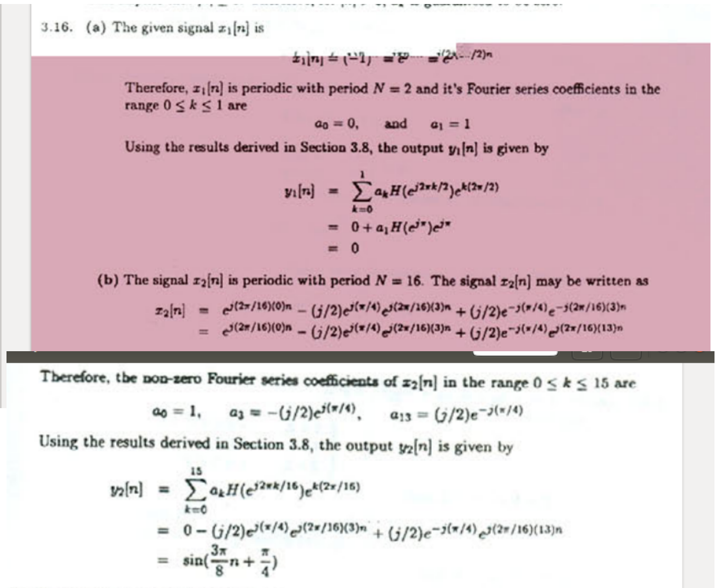
$$a_1 = a_{15}, \qquad a_2 = a_{16} \quad , \text{and} \qquad a_3 = a_{17}$$

Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd. Therefore, $a_0 = 0$ and

$$a_1 = -a_{-1}, \qquad a_2 = -a_{-2} \qquad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j, \qquad a_{-2} = -2j, \qquad a_{-3} = -3j$$



3.16. (a) The given signal $x_1[n]$ is

$$x_1[n] = (-1)^n = e^{j\pi n} = e^{j(2\pi/2)n}$$

Therefore, $x_1[n]$ is periodic with period $N = 2$ and it's Fourier series coefficients in the range $0 \leq k \leq 1$ are

$$a_0 = 0, \qquad \text{and} \qquad a_1 = 1$$

Using the results derived in Section 3.8, the output $y_1[n]$ is given by

$$\begin{aligned} y_1[n] &= \sum_{k=0}^{1} a_k H(e^{j2\pi k/2})e^{k(2\pi/2)} \\ &= 0 + a_1 H(e^{j\pi})e^{j\pi} \\ &= 0 \end{aligned}$$

(b) The signal $x_2[n]$ is periodic with period $N = 16$. The signal $x_2[n]$ may be written as

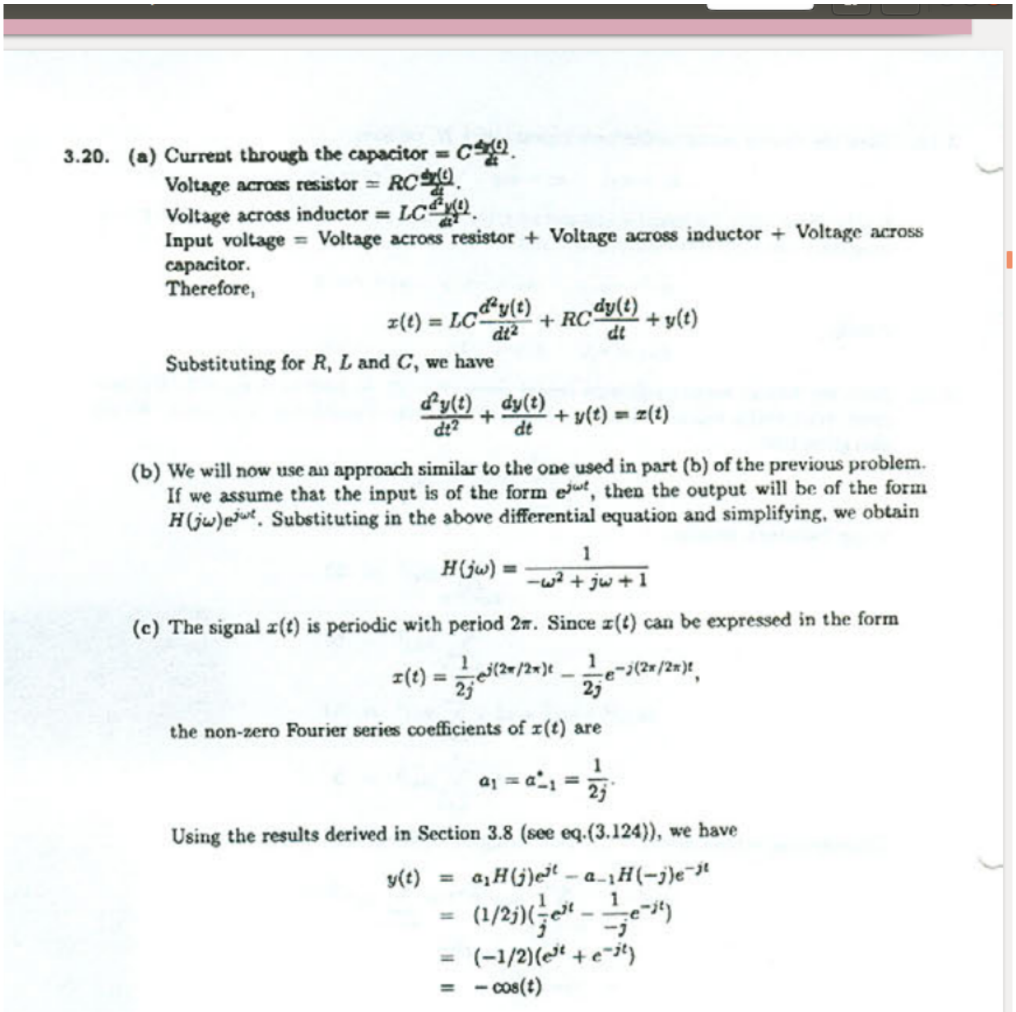
$$\begin{aligned} x_2[n] &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{-j(2\pi/16)(3)n} \\ &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n} \end{aligned}$$

Therefore, the non-zero Fourier series coefficients of $x_2[n]$ in the range $0 \leq k \leq 15$ are

$$a_0 = 1, \qquad a_3 = -(j/2)e^{j(\pi/4)}, \qquad a_{13} = (j/2)e^{-j(\pi/4)}$$

Using the results derived in Section 3.8, the output $y_2[n]$ is given by

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{15} a_k H(e^{j2\pi k/16})e^{k(2\pi/16)} \\ &= 0 - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n} \\ &= \sin(\frac{3\pi}{8}n + \frac{\pi}{4}) \end{aligned}$$



3.20. (a) Current through the capacitor $= C\frac{dy(t)}{dt}$.

Voltage across resistor $= RC\frac{dy(t)}{dt}$.

Voltage across inductor $= LC\frac{d^2y(t)}{dt^2}$.

Input voltage = Voltage across resistor + Voltage across inductor + Voltage across capacitor.

Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Substituting for R, L and C, we have

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b) We will now use an approach similar to the one used in part (b) of the previous problem. If we assume that the input is of the form $e^{j\omega t}$, then the output will be of the form $H(j\omega)e^{j\omega t}$. Substituting in the above differential equation and simplifying, we obtain

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

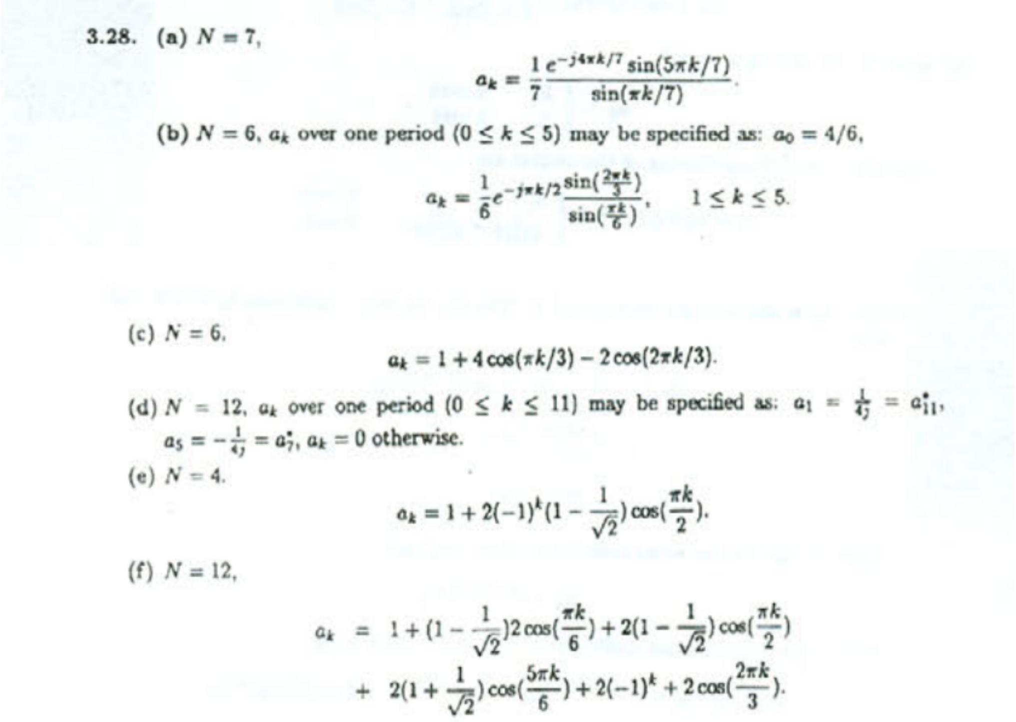
$$x(t) = \frac{1}{2j}e^{j(2\pi/2\pi)t} - \frac{1}{2j}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1}^* = \frac{1}{2j}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} - a_{-1}H(-j)e^{-jt} \\ &= (1/2j)(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt}) \\ &= (-1/2)(e^{jt} + e^{-jt}) \\ &= -\cos(t) \end{aligned}$$



3.28. (a) $N = 7$,

$$a_k = \frac{1}{7}\frac{e^{-j4\pi k/7}\sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period $(0 \leq k \leq 5)$ may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6}e^{-j\pi k/2}\frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \qquad 1 \leq k \leq 5.$$

(c) $N = 6$,

$$a_k = 1 + 4\cos(\pi k/3) - 2\cos(2\pi k/3).$$

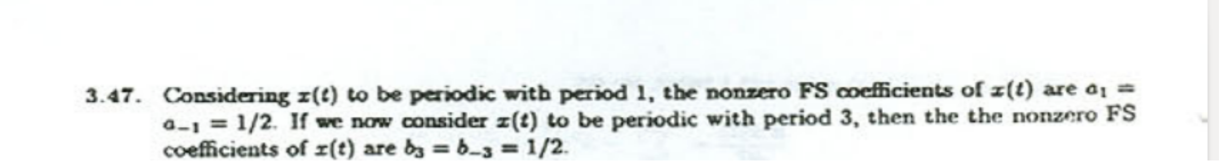
(d) $N = 12$, a_k over one period $(0 \leq k \leq 11)$ may be specified as: $a_1 = \frac{1}{4j} = a_{11}^*$, $a_5 = -\frac{1}{4j} = a_7^*$, $a_k = 0$ otherwise.

(e) $N = 4$,

$$a_k = 1 + 2(-1)^k(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}).$$

(f) $N = 12$,

$$\begin{aligned} a_k &= 1 + (1 - \frac{1}{\sqrt{2}})2\cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}) \\ &+ 2(1 + \frac{1}{\sqrt{2}})\cos(\frac{5\pi k}{6}) + 2(-1)^k + 2\cos(\frac{2\pi k}{3}). \end{aligned}$$



3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

4.4 HW 4

Local contents

4.4.1 Problem 4.1(a), Chapter 4

Find Fourier transform of (a) $e^{-2(t-1)}u(t-1)$

Solution

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \\ &= \int_{1}^{\infty} e^{-2(t-1)} e^{-i\omega t} dt \\ &= \int_{1}^{\infty} e^{-2t} e^{2} e^{-i\omega t} dt \\ &= e^{2} \int_{1}^{\infty} e^{-t(2+i\omega)} dt \\ &= \frac{e^{2}}{-(2+i\omega)} \left[e^{-t(2+i\omega)} \right]_{1}^{\infty} \end{aligned}$$

Assuming $\operatorname{Im}(\omega) < 2$ then

$$\begin{aligned} X(\omega) &= \frac{e^{2}}{-(2+i\omega)} \left[0 - e^{-2} e^{-i\omega} \right] \\ &= \frac{e^{2}}{(2+i\omega)} \left[e^{-2} e^{-i\omega} \right] \\ &= \frac{e^{-i\omega}}{(2+i\omega)} \end{aligned}$$

4.4.2 Problem 4.3, Chapter 4

Determine the Fourier transform of each of the following periodic signals (a) $\sin\left(2\pi t + \frac{\pi}{4}\right)$ (b) $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$

Solution

4.4.2.1 Part a

Since this is periodic signal, then we can not use $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$ which is for aperiodic signal. Instead we need to use 4.22 in the textbook which is

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

From $\sin\left(2\pi t + \frac{\pi}{4}\right)$ we see that $\omega_0 = 2\pi$, hence

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2k\pi)$$

Writing $\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2j} e^{j\left(\omega_0 t + \frac{\pi}{4}\right)} - \frac{1}{2j} e^{-j\left(\omega_0 t + \frac{\pi}{4}\right)} = \left(\frac{1}{2j} e^{j\frac{\pi}{4}}\right) e^{j\omega_0 t} - \left(\frac{1}{2j} e^{-j\frac{\pi}{4}}\right) e^{-j\omega_0 t}$ shows that $a_1 = \frac{1}{2j} e^{j\frac{\pi}{4}}$ and $a_{-1} = -\frac{1}{2j} e^{-j\frac{\pi}{4}}$ and $a_k = 0$ for all other k. Hence above simplifies to

$$\begin{aligned} X(\omega) &= 2\pi \left(\frac{1}{2j} e^{j\frac{\pi}{4}}\right) \delta(\omega - 2\pi) + 2\pi \left(-\frac{1}{2j} e^{-j\frac{\pi}{4}}\right) \delta(\omega + 2\pi) \\ &= \frac{\pi}{j} e^{j\frac{\pi}{4}} \delta(\omega - 2\pi) - \frac{\pi}{j} e^{-j\frac{\pi}{4}} \delta(\omega + 2\pi) \end{aligned}$$

4.4.2.2 Part b

Since this is periodic signal, then its Fourier transform is

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

From $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$ we see that $\omega_0 = 6\pi$, hence

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 6k\pi)$$

Writing $1 + \cos\left(6\pi t + \frac{\pi}{8}\right) = 1 + \left(\frac{1}{2} e^{j\left(\omega_0 t + \frac{\pi}{8}\right)} + \frac{1}{2} e^{-j\left(\omega_0 t + \frac{\pi}{8}\right)}\right) = 1 + \left(\frac{1}{2} e^{j\frac{\pi}{8}} e^{j\omega_0 t} + \frac{1}{2} e^{-j\frac{\pi}{8}} e^{-j\omega_0 t}\right)$ shows that $a_1 = \frac{1}{2} e^{j\frac{\pi}{8}}$ and $a_{-1} = \frac{1}{2} e^{-j\frac{\pi}{8}}$ and $a_0 = 1$. Therefore the above becomes

$$\begin{aligned} X(\omega) &= 2\pi a_{-1} \delta(\omega + 6\pi) + 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - 6\pi) \\ &= 2\pi \left(\frac{1}{2} e^{-j\frac{\pi}{8}}\right) \delta(\omega + 6\pi) + 2\pi \delta(\omega) + 2\pi \left(\frac{1}{2} e^{j\frac{\pi}{8}}\right) \delta(\omega - 6\pi) \\ &= \pi e^{-j\frac{\pi}{8}} \delta(\omega + 6\pi) + 2\pi \delta(\omega) + \pi e^{j\frac{\pi}{8}} \delta(\omega - 6\pi) \end{aligned}$$

4.4.3 Problem 4.5, Chapter 4

Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X\left(j\omega\right)=\left|X\left(j\omega\right)\right|e^{j\measuredangle X\left(j\omega\right)}$ where $\left|X\left(j\omega\right)\right|=2\left(u\left(\omega+3\right)-u\left(\omega-3\right)\right),\measuredangle X\left(j\omega\right)=-\frac{3}{2}\omega+\pi$

Solution

4.8 is

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(\omega\right)e^{j\omega t}d\omega \tag{4.8}$$

Hence

$$\begin{aligned}x\left(t\right)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X\left(j\omega\right)\right|e^{j\measuredangle X\left(j\omega\right)}e^{j\omega t}d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\left(u\left(\omega+3\right)-u\left(\omega-3\right)\right)e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega\end{aligned}$$

But $u\left(\omega+3\right)-u\left(\omega-3\right)$ is one over $\omega=-3\cdots3$ and zero otherwise. The above simplifies to

$$\begin{aligned}x\left(t\right)&=\frac{1}{2\pi}\int_{-3}^{3}2e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega\\&=\frac{1}{\pi}\int_{-3}^{3}e^{j\pi}e^{j-\frac{3}{2}\omega}e^{j\omega t}d\omega\\&=\frac{e^{j\pi}}{\pi}\int_{-3}^{3}e^{j\left(-\frac{3}{2}+t\right)\omega}d\omega\end{aligned}$$

But $e^{j\pi}=-1$ and $\int e^{j\left(-\frac{3}{2}+t\right)\omega}d\omega=\frac{e^{j\left(-\frac{3}{2}+t\right)\omega}}{j\left(-\frac{3}{2}+t\right)}$, hence the above becomes

$$\begin{aligned}x\left(t\right)&=\frac{-1}{\pi}\frac{1}{j\left(-\frac{3}{2}+t\right)}\left[e^{j\left(-\frac{3}{2}+t\right)\omega}\right]_{-3}^{3}\\&=\frac{-1}{\pi\left(-\frac{3}{2}+t\right)}\left(\frac{e^{j3\left(-\frac{3}{2}+t\right)}-e^{-j3\left(-\frac{3}{2}+t\right)}}{j}\right)\\&=\frac{-1}{\pi\left(-\frac{3}{2}+t\right)}2\left(\sin\left(3\left(-\frac{3}{2}+t\right)\right)\right)\\&=\frac{-2}{\pi\left(t-\frac{3}{2}\right)}\sin\left(3\left(t-\frac{3}{2}\right)\right)\end{aligned}$$

4.4.4 Problem 4.11, Chapter 4

Given the relationships $y\left(t\right)=x\left(t\right)\circledast h\left(t\right)$ and $g\left(t\right)=x\left(3t\right)\circledast h\left(3t\right)$ and given that $x\left(t\right)$ has Fourier transform $X\left(j\omega\right)$ and $h\left(t\right)$ has Fourier transform $H\left(j\omega\right)$, use Fourier transform properties to show that $g(t)$ has the form $g\left(t\right)=Ay\left(Bt\right)$. Determine the values of A and B

Solution

The main relation to use is that if $y(t) \Longleftrightarrow Y(\omega)$ then $y(at) \Longleftrightarrow \frac{1}{a}Y\left(\frac{\omega}{a}\right)$. Therefore $x(3t) \Longleftrightarrow \frac{1}{3}X\left(\frac{\omega}{3}\right)$ and $h(3t) \Longleftrightarrow \frac{1}{3}H\left(\frac{\omega}{3}\right)$. Hence since $g(t) = x(3t) \circledast h(3t)$ then

$$\begin{aligned} G(\omega) &= \frac{1}{3}X\left(\frac{\omega}{3}\right)\frac{1}{3}H\left(\frac{\omega}{3}\right) \\ &= \frac{1}{3}\left(\frac{1}{3}X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right)\right) \end{aligned}$$

But $X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right) = Y\left(\frac{\omega}{3}\right)$. Therefore the above becomes

$$G(\omega) = \frac{1}{3}\left(\frac{1}{3}Y\left(\frac{\omega}{3}\right)\right)$$

Inverse Fourier transform gives

$$g(t) = \frac{1}{3}y(3t)$$

Where in the above, we used $\frac{1}{3}Y\left(\frac{\omega}{3}\right) \Longleftrightarrow y(3t)$. Hence $A = \frac{1}{3}, B = 3$

4.4.5 Problem 4.19, Chapter 4

Consider a causal LTI system with frequency response $H\left(j\omega\right) = \frac{1}{j\omega+3}$. For a particular input $x(t)$ this system is observed to produce the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$. Determine $x(t)$.

Solution

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform gives

$$Y(\omega) = X(\omega)H(\omega)$$

Hence

$$X(\omega) = \frac{Y(\omega)}{H(\omega)}$$

But $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, therefore, from table $Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{(4+j\omega)-(3+j\omega)}{(3+j\omega)(4+j\omega)} = \frac{1}{(3+j\omega)(4+j\omega)}$ and the above becomes

$$X(\omega) = \frac{\frac{1}{(3+j\omega)(4+j\omega)}}{H(\omega)}$$

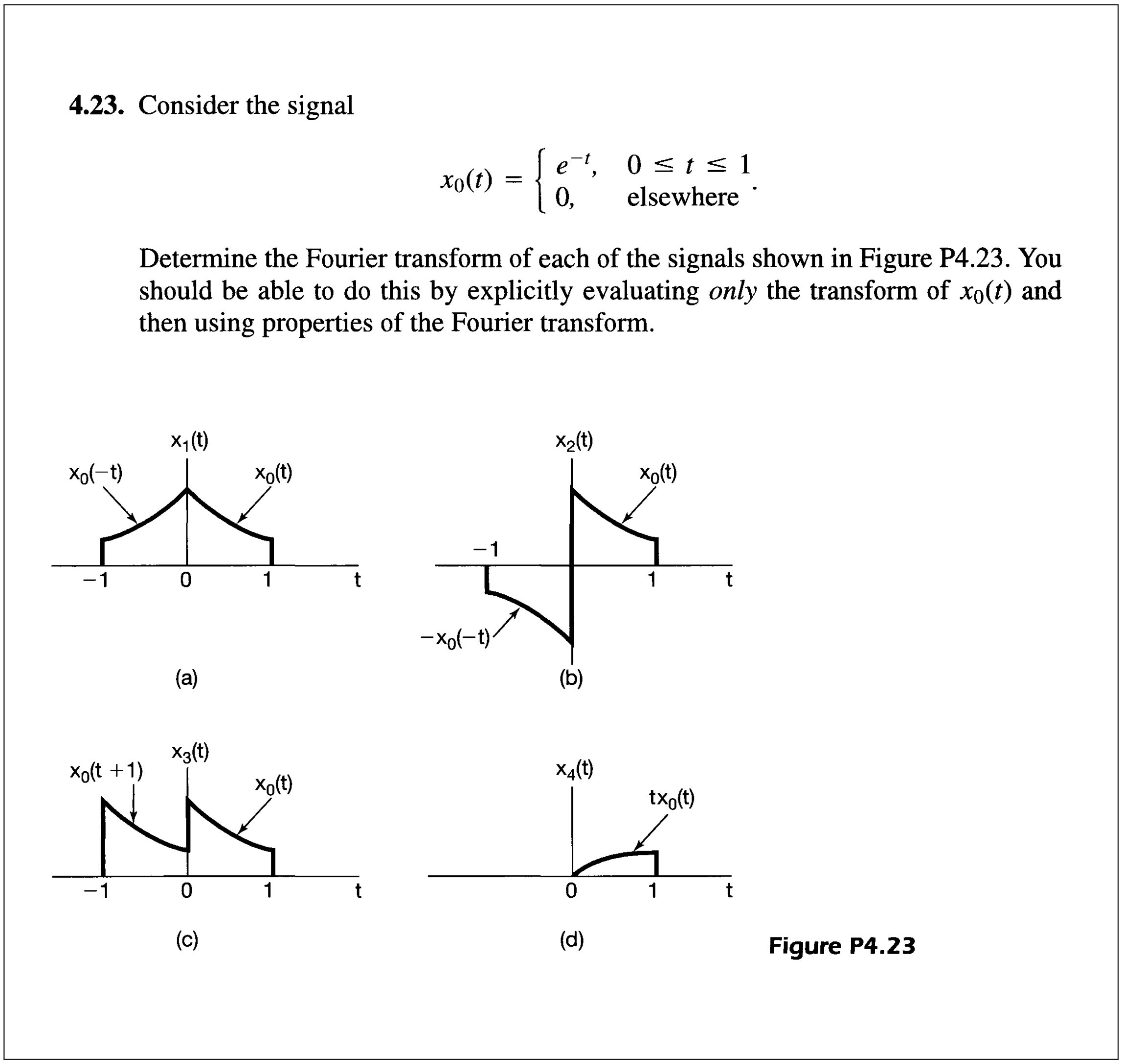
But we are given that $H\left(j\omega\right) = \frac{1}{j\omega+3}$. The above simplifies to

$$X(\omega) = \frac{\frac{1}{\left(3+j\omega\right)\left(4+j\omega\right)}}{\frac{1}{j\omega+3}} = \frac{1}{4+j\omega}$$

From tables

$$x(t) = e^{-4t}u(t)$$

4.4.6 Problem 4.23, Chapter 4



4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

Figure P4.23

Figure 4.57: Problem description

Solution

4.4.6.1 Part a

First we find the Fourier transform of $x_0(t)$. Since this is a periodic, then $x_0(t) \Longleftrightarrow X_0(\omega)$ and

$$\begin{aligned}
X_0(\omega) &= \int_{-\infty}^{\infty} x_0(t)\, e^{-j\omega t} dt \\
&= \int_0^1 e^{-t} e^{-j\omega t} dt \\
&= \int_0^1 e^{-t\left(1+j\omega\right)} dt \\
&= \frac{-1}{1+j\omega}\left[e^{-t\left(1+j\omega\right)}\right]_0^1 \\
&= \frac{-1}{1+j\omega}\left(e^{-\left(1+j\omega\right)} - 1\right) \\
&= \frac{1-e^{-\left(1+j\omega\right)}}{1+j\omega}
\end{aligned}$$

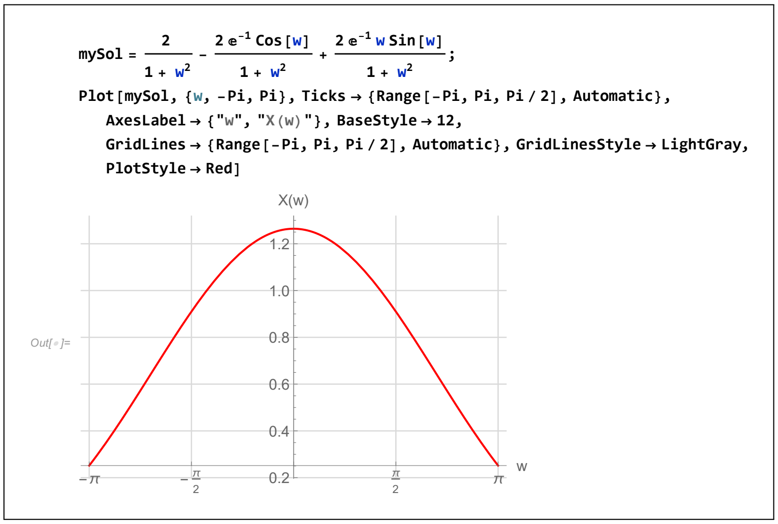
From table 4.1, property 4.3.5, Fourier transform of $x(-t) = X(-\omega)$. Hence $x_0(-t) \Longleftrightarrow X_0(-\omega)$. Therefore, using the above result and taking its complex conjugate gives

$$X_0(-\omega) = \frac{1-e^{-\left(1-j\omega\right)}}{1-j\omega}$$

Therefore the Fourier transform of $x_0(t) + x_0(-t) \iff X_1(\omega) = X_0(\omega) + X_0(-\omega)$ This is by linearity property. Hence

$$
\begin{aligned}
X_1(\omega) &= X_0(\omega) + X_0(-\omega) \\
&= \frac{1 - e^{-(1+j\omega)}}{1+j\omega} + \frac{1 - e^{-(1-j\omega)}}{1-j\omega} \\
&= \frac{(1-j\omega)\left(1 - e^{-(1+j\omega)}\right) + (1+j\omega)\left(1 - e^{-(1-j\omega)}\right)}{(1+j\omega)(1-j\omega)} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega\left(1 - e^{-(1+j\omega)}\right) + \left(1 - e^{-(1-j\omega)}\right) + j\omega\left(1 - e^{-(1-j\omega)}\right)}{1+\omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega + j\omega e^{-(1+j\omega)} + \left(1 - e^{-(1-j\omega)}\right) + j\omega - j\omega e^{-(1-j\omega)}}{1+\omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} + j\omega e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}}{1+\omega^2} \\
&= \frac{1 - e^{-1}e^{-j\omega} + j\omega e^{-1}e^{-j\omega} + 1 - e^{-1}e^{j\omega} - j\omega e^{-1}e^{j\omega}}{1+\omega^2} \\
&= \frac{1 - e^{-1}\left(e^{j\omega} + e^{-j\omega}\right) - j\omega e^{-1}\left(e^{j\omega} - e^{-j\omega}\right)}{1+\omega^2} \\
&= \frac{1}{1+\omega^2} - \frac{e^{-1}}{1+\omega^2}\left(e^{j\omega} + e^{-j\omega}\right) + \frac{\omega e^{-1}}{1+\omega^2}\frac{\left(e^{j\omega} - e^{-j\omega}\right)}{j} \\
&= \frac{1}{1+\omega^2} - \frac{2}{e\left(1+\omega^2\right)}\cos\omega + \frac{2\omega}{e\left(1+\omega^2\right)}\sin\omega
\end{aligned}
$$

The following is a plot of the above



$$\text{mySol} = \frac{2}{1+\text{w}^2} - \frac{2\, e^{-1}\,\text{Cos}[\text{w}]}{1+\text{w}^2} + \frac{2\, e^{-1}\,\text{w}\,\text{Sin}[\text{w}]}{1+\text{w}^2};$$

```
Plot[mySol, {w, -Pi, Pi}, Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
   AxesLabel → {"w", "X(w)"}, BaseStyle → 12,
   GridLines → {Range[-Pi, Pi, Pi / 2], Automatic}, GridLinesStyle → LightGray,
   PlotStyle → Red]
```

Figure 4.58: Plot of $X(\omega)$

We see that $X_1(\omega)$ is even and real. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is even, then its Fourier transform is real and even.

4.4.6.2 Part b

We found $X_0(\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$ above. Hence $-x_0(-t) \Longleftrightarrow -X_0(-\omega) = -\frac{1-e^{-(1-j\omega)}}{1-j\omega} = \frac{1-e^{-(1-j\omega)}}{j\omega-1}$. Therefore

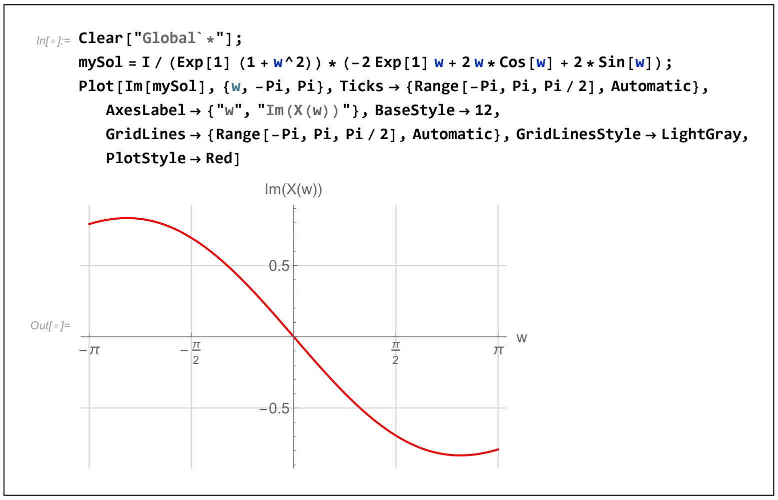
$$\begin{aligned}
X_2(\omega) &= X_0(\omega) - X_0(-\omega)\\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{1-e^{-(1-j\omega)}}{j\omega-1}\\
&= \frac{\left(j\omega-1\right)\left(1-e^{-(1+j\omega)}\right) + \left(1+j\omega\right)\left(1-e^{-(1-j\omega)}\right)}{\left(1+j\omega\right)\left(j\omega-1\right)}\\
&= \frac{j\omega\left(1-e^{-(1+j\omega)}\right) - \left(1-e^{-(1+j\omega)}\right) + \left(1-e^{-(1-j\omega)}\right) + j\omega\left(1-e^{-(1-j\omega)}\right)}{j\omega-1-\omega^2-j\omega}\\
&= \frac{j\omega - j\omega e^{-(1+j\omega)} - 1 + e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} + j\omega - j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)}\\
&= \frac{2j\omega - j\omega e^{-(1+j\omega)} + e^{-(1+j\omega)} - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)}\\
&= \frac{2j\omega - j\omega e^{-1}e^{-j\omega} + e^{-1}e^{-j\omega} - e^{-1}e^{j\omega} - j\omega e^{-1}e^{j\omega}}{j\omega-\omega^2-2}\\
&= \frac{2j\omega - j\omega e^{-1}\left(e^{j\omega}+e^{-j\omega}\right) - e^{-1}\left(e^{j\omega}-e^{-j\omega}\right)}{-\left(1+\omega^2\right)}\\
&= \frac{2j\omega}{-\left(1+\omega^2\right)} - j\omega e^{-1}\frac{\left(e^{j\omega}+e^{-j\omega}\right)}{-\left(1+\omega^2\right)} - e^{-1}\frac{e^{j\omega}-e^{-j\omega}}{-\left(1+\omega^2\right)}\\
&= \frac{2j\omega}{-\left(1+\omega^2\right)} - 2j\omega e^{-1}\frac{\cos\omega}{-\left(1+\omega^2\right)} - 2je^{-1}\frac{\sin(\omega)}{-\left(1+\omega^2\right)}
\end{aligned}$$

Hence

$$\begin{aligned}
X_2(\omega) &= j\left(\frac{-2\omega}{\left(1+\omega^2\right)} + 2\omega\frac{\cos\omega}{e\left(1+\omega^2\right)} + 2\frac{\sin(\omega)}{e\left(1+\omega^2\right)}\right)\\
&= \frac{j}{e\left(1+\omega^2\right)}\left(-2e\omega + 2\omega\cos\omega + 2\sin(\omega)\right)
\end{aligned}$$

We see that $X_2(\omega)$ is pure imaginary. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is odd, then its Fourier transform is pure imaginary and odd.

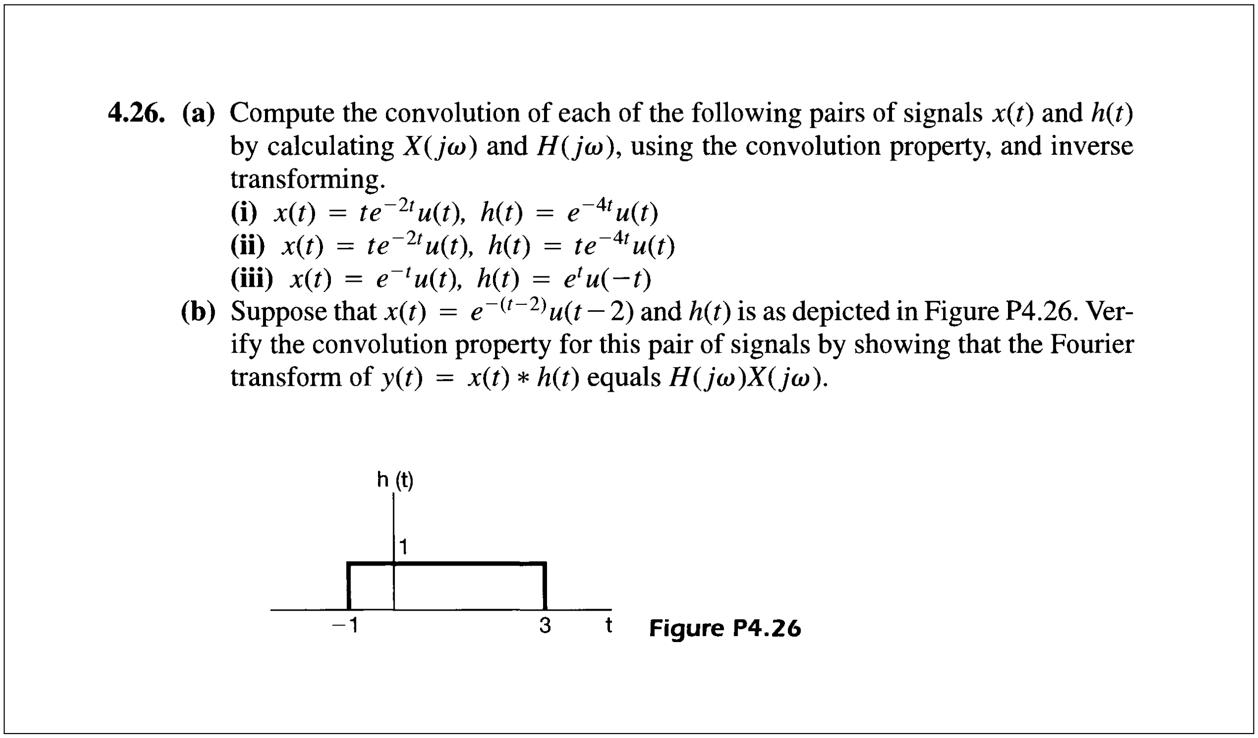
The following is a plot of the above which shows that the imaginary part of $X_2(\omega)$ is odd



```
Clear["Global`*"];
mySol = I / (Exp[1] (1 + w^2)) * (-2 Exp[1] w + 2 w * Cos[w] + 2 * Sin[w]);
Plot[Im[mySol], {w, -Pi, Pi}, Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
   AxesLabel → {"w", "Im(X(w))"}, BaseStyle → 12,
   GridLines → {Range[-Pi, Pi, Pi / 2], Automatic}, GridLinesStyle → LightGray,
   PlotStyle → Red]
```

Figure 4.59: Plot of imaginary part of $X(\omega)$

4.4.7 Problem 4.26, Chapter 4



4.26. **(a)** Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

Figure P4.26

Figure 4.60: Problem description

Solution

4.4.7.1 Part a

4.4.7.1.1 Part i

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega) H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \frac{1}{(2+j\omega)^2}$ and given that $h(t) = e^{-4t}u(t)$ then from table 4.2 $H(\omega) = \frac{1}{4+j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(2+j\omega\right)^2\left(4+j\omega\right)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\begin{aligned}\frac{1}{(2+s)^2(4+s)} &= \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{4+s}\\ &= \frac{A(4+s) + B((2+s)(4+s)) + C(2+s)^2}{(2+s)^2(4+s)}\end{aligned}$$

Expanding numerator

$$\begin{aligned}1 &= 4A + 8B + 4C + Bs^2 + Cs^2 + As + 6Bs + 4Cs\\ 1 &= (4A + 8B + 4C) + s(A + 6B + 4C) + s^2(B + C)\end{aligned}$$

Comparing coefficients

$$\begin{aligned}1 &= 4A + 8B + 4C\\ 0 &= A + 6B + 4C\\ 0 &= B + C\end{aligned}$$

Or

$$\begin{pmatrix}4 & 8 & 4\\ 1 & 6 & 4\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Gaussian elimination. Multiplying second row by 4 and subtracting result from first row gives

$$\begin{pmatrix}4 & 8 & 4\\ 0 & 16 & 12\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$$

Multiplying third row by 16 and subtracting result from second row gives

$$\begin{pmatrix} 4 & 8 & 4 \\ 0 & 16 & 12 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Backsubstitution. Last row gives $C = \frac{1}{4}$. Second row gives $16B + 12C = -1$ or $16B = -1 - 12\left(\frac{1}{4}\right)$, hence $16B = -4$, Hence $B = -\frac{1}{4}$. First row gives $4A + 8B + 4C = 1$ or $4A + 8\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) = 1$ or $4A - 2 + 1 = 1$. Hence $A = \frac{1}{2}$. Therefore partial fractions gives

$$\frac{1}{(2+s)^2(4+s)} = \frac{\left(\frac{1}{2}\right)}{(2+s)^2} + \frac{\left(-\frac{1}{4}\right)}{2+s} + \frac{\left(\frac{1}{4}\right)}{4+s}$$

Replacing s back with $j\omega$

$$Y(\omega) = \frac{1}{2}\frac{1}{\left(2+j\omega\right)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{4+j\omega}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{2}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

4.4.7.1.2 Part ii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \frac{1}{\left(2+j\omega\right)^2}$ and given that $h(t) = te^{-4t}u(t)$ then from table 4.2 $H(\omega) = \frac{1}{\left(4+j\omega\right)^2}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(2+j\omega\right)^2\left(4+j\omega\right)^2}$$

Doing partial fractions. Let $s = j\omega$ then

$$\begin{aligned}\frac{1}{(2+s)^2(4+s)^2} &= \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)} \\ &= \frac{A(4+s)^2 + B\left((2+s)(4+s)^2\right) + C(2+s)^2 + D\left((2+s)^2(4+s)\right)}{(2+s)^2(4+s)^2}\end{aligned}$$

Expanding numerator

$$1 = 16A + 32B + 4C + 16D + 20sD + As^2 + 10Bs^2 + Bs^3 + Cs^2 + 8s^2D + s^3D + 8As + 32Bs + 4Cs$$
$$1 = (16A + 32B + 4C + 16D) + s\,(20D + 8A + 32B + 4C) + s^2\,(A + 10B + C + 8D) + s^3\,(B + D)$$

Comparing coefficients

$$\begin{aligned} 1 &= 16A + 32B + 4C + 16D \\ 0 &= 8A + 32B + 4C + 20D \\ 0 &= A + 10B + C + 8D \\ 0 &= B + D \end{aligned}$$

Or

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 8 & 32 & 4 & 20 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination. replacing row 2 by result of subtracting 1/2 times row 1 from row 2

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

Replacing row 3 by result of subtracting $\frac{1}{16}$ times row 1 from row 3

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 8 & \frac{3}{4} & 7 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{16} \\ 0 \end{pmatrix}$$

Replacing row 3 by result of subtracting $\frac{1}{2}$ times row 2 from row 3

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ 0 \end{pmatrix}$$

Replacing row 4 by result of subtracting $\frac{1}{16}$ times row 2 from row 4

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & \frac{-1}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{1}{32} \end{pmatrix}$$

Replacing row 4 by result of subtracting $\frac{1}{2}$ times row 3 from row 4

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & 0 & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{-1}{16} \end{pmatrix}$$

Backsubstitution phase:

$$\frac{-1}{4}D = \frac{-1}{16}$$
$$D = \frac{1}{4}$$

Third row gives $\frac{-1}{4}C + D = \frac{3}{16}$ or $\frac{-1}{4}C + \frac{1}{4} = \frac{3}{16} \rightarrow C = \frac{1}{4}$. Second row gives $16B + 2C + 12D = -\frac{1}{2}$ or $16B + 2\left(\frac{1}{4}\right) + 12\left(\frac{1}{4}\right) = -\frac{1}{2} \rightarrow B = -\frac{1}{4}$. First row gives $16A + 32B + 4C + 16D = 1$ or $16A + 32\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 16\left(\frac{1}{4}\right) = 1, \rightarrow A = \frac{1}{4}$.

Therefore partial fractions gives

$$\begin{aligned} \frac{1}{(2+s)^2(4+s)^2} &= \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)} \\ &= \frac{1}{4}\frac{1}{(2+s)^2} - \frac{1}{4}\frac{1}{2+s} + \frac{1}{4}\frac{1}{(4+s)^2} + \frac{1}{4}\frac{1}{(4+s)} \end{aligned}$$

Replace s back with $j\omega$

$$Y(\omega) = \frac{1}{4}\frac{1}{\left(2+j\omega\right)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)^2} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-4t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

4.4.7.1.3 Part iii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = e^{-t}u(t)$, then from tables $X(\omega) = \frac{1}{\left(1+j\omega\right)}$ and given that $h(t) = e^{t}u(-t)$ then $H(\omega) = \frac{1}{1-j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(1+j\omega\right)}\frac{1}{\left(1-j\omega\right)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\frac{1}{(1+s)}\frac{1}{(1-s)} = \frac{A}{(1+s)} + \frac{B}{1-s}$$

Hence $A = \left(\frac{1}{(1-s)}\right)_{s=-1} = \frac{1}{2}$ and $B = \left(\frac{1}{(1+s)}\right)_{s=1} = \frac{1}{2}$. Hence

$$Y(\omega) = \frac{1}{2}\frac{1}{\left(1+j\omega\right)} + \frac{1}{2}\frac{1}{1-j\omega}$$

Therefore

$$y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

The above means for $t < 0, y(t) = \frac{1}{2}e^{t}$ and for $t > 0, y(t) = \frac{1}{2}e^{-t}$.

4.4.7.2 Part b

$x(t) = e^{-(t-2)}u(t-2)$, $h(t) = u(t+1) - u(t-3)$. Let us first find $X(\omega)$ and $H(\omega)$

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-(t-2)}u(t-2)\,e^{-j\omega t}dt \\
&= \int_{2}^{\infty} e^{-(t-2)}e^{-j\omega t}dt \\
&= \int_{2}^{\infty} e^{-t}e^{2}e^{-j\omega t}dt \\
&= e^{2}\int_{2}^{\infty} e^{-t\left(1+j\omega\right)}dt \\
&= -\frac{e^{2}}{1+j\omega}\left[e^{-t\left(1+j\omega\right)}\right]_{2}^{\infty} \\
&= -\frac{e^{2}}{1+j\omega}\left(0 - e^{-2\left(1+j\omega\right)}\right) \\
&= \frac{e^{2}e^{-2\left(1+j\omega\right)}}{1+j\omega} \\
&= \frac{e^{-2-2j\omega+2}}{1+j\omega} \\
&= \frac{e^{-2j\omega}}{1+j\omega}
\end{aligned}$$

And

$$
\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} u(t+1) - u(t-3)\, e^{-j\omega t} dt \\
&= \int_{-1}^{3} e^{-j\omega t} dt \\
&= \frac{1}{j\omega}\left(e^{-j\omega t}\right)_{-1}^{3} \\
&= \frac{1}{j\omega}\left(e^{-3j\omega} - e^{j\omega t}\right) \\
&= \frac{1}{j\omega} e^{-j\omega}\left(e^{-j2\omega} - e^{j2\omega t}\right) \\
&= -\frac{1}{j\omega} e^{-j\omega}\left(e^{j2\omega t} - e^{-j2\omega}\right) \\
&= -\frac{1}{\omega} e^{-j\omega}\left(\frac{e^{j2\omega t} - e^{-j2\omega}}{j}\right) \\
&= -\frac{2}{\omega} e^{-j\omega} \sin(2\omega)
\end{aligned}
$$

Hence

$$
\begin{aligned}
Y(\omega) &= H(\omega) X(\omega) \\
&= \frac{e^{-2j\omega}}{1+j\omega}\left(-\frac{2}{\omega} e^{-j\omega} \sin(2\omega)\right) \\
&= \frac{-2}{\omega\left(1+j\omega\right)} e^{-2j\omega} e^{-j\omega} \sin(2\omega) \\
&= \frac{-2}{\omega\left(1+j\omega\right)} e^{-3j\omega} \sin(2\omega) \qquad (1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
y(t) &= x(t) \circledast h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau
\end{aligned}
$$

Folding $h(t)$. For $t < 1$, then $y(t) = 0$. For $2 < 1 + t < 6$ or $1 < t < 5$

$$
\begin{aligned}
y(t) &= \int_{2}^{1+t} x(\tau)\, d\tau \\
&= \int_{2}^{1+t} e^{-(\tau-2)} d\tau \\
&= -\left(e^{-(\tau-2)}\right)_{2}^{1+t} \\
&= -\left(e^{-(1+t-2)} - e^{-(2-2)}\right) \\
&= -\left(e^{-(-1+t)} - 1\right) \\
&= 1 - e^{1-t}
\end{aligned}
$$

For $6 < 1 + t$ or $t > 5$

$$\begin{aligned}
y(t) &= \int_{t-3}^{t+1} x(\tau)\, d\tau \\
&= \int_{t-3}^{t+1} e^{-(\tau-2)} d\tau \\
&= -\left(e^{-(\tau-2)}\right)_{t-3}^{1+t} \\
&= -\left(e^{-(1+t-2)} - e^{-(t-3-2)}\right) \\
&= -\left(e^{-(-1+t)} - e^{-(t-5)}\right) \\
&= -\left(e^{1-t} - e^{5-t}\right) \\
&= e^{5-t} - e^{1-t}
\end{aligned}$$

Hence

$$y(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{1-t} & 1 < t < 5 \\ e^{5-t} - e^{1-t} & t > 5 \end{cases}$$

The above can be written as $y(t)$

$$\begin{aligned}
y(t) &= \left(1 - e^{1-t}\right)(u(t-1) - u(t-5)) + \left(e^{5-t} - e^{1-t}\right) u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t}(u(t-1) - u(t-5)) + \left(e^{5-t} - e^{1-t}\right) u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t} u(t-1) + e^{1-t} u(t-5) + e^{5-t} u(t-5) - e^{1-t} u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t} u(t-1) + e^{5-t} u(t-5) \qquad (2)
\end{aligned}$$

Taking the Fourier transform of the above from tables

$$\begin{aligned}
u(t-1) &\Longleftrightarrow \frac{1}{j\omega} e^{-j\omega} + \pi\delta(\omega) \\
u(t-5) &\Longleftrightarrow \frac{1}{j\omega} e^{-5j\omega} + \pi\delta(\omega) \\
e^{1-t} u(t-1) &\Longleftrightarrow \frac{e^{-j\omega}}{1 + j\omega} \\
e^{5-t} u(t-5) &\Longleftrightarrow \frac{e^{-5j\omega}}{1 + j\omega}
\end{aligned}$$

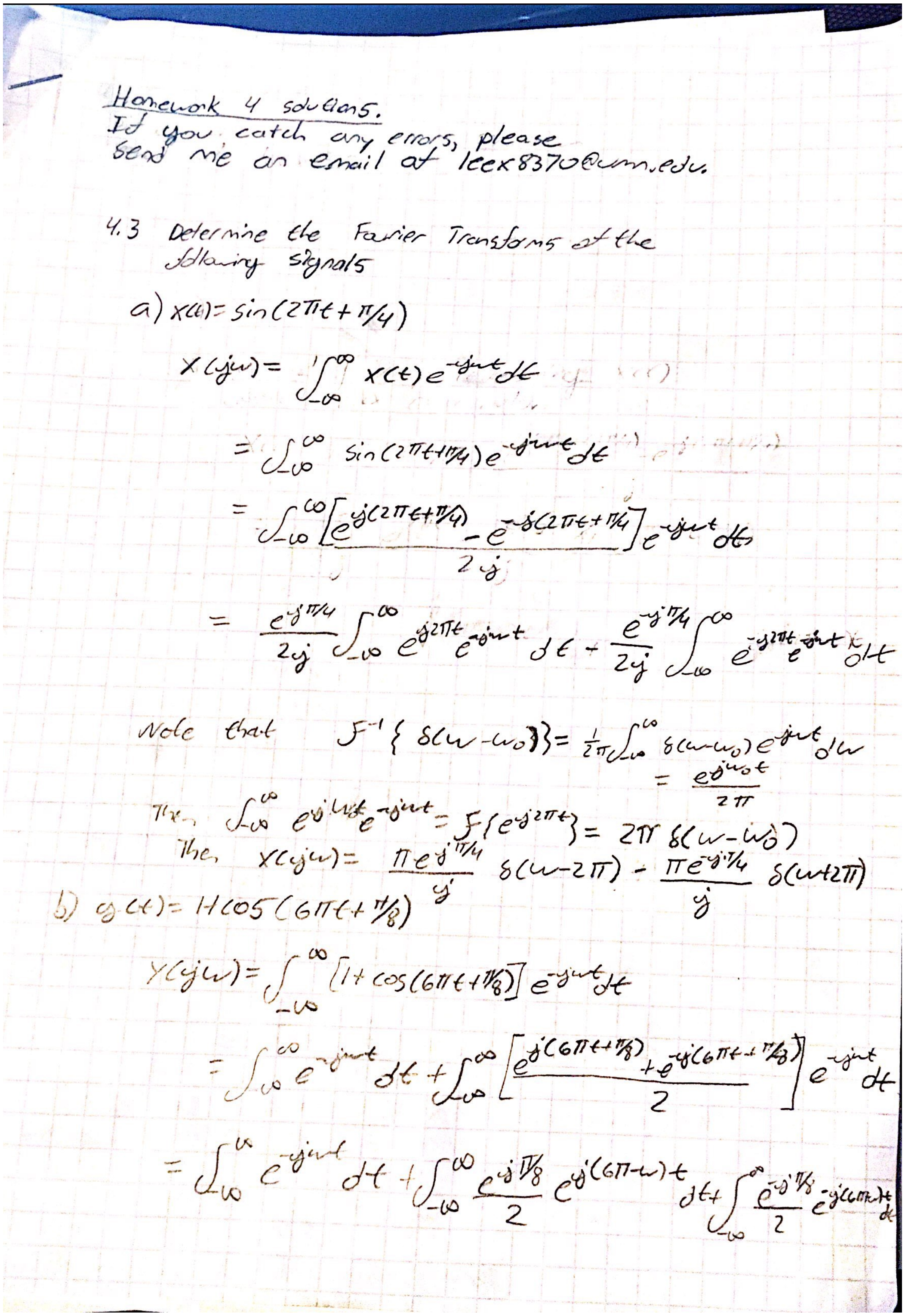
Hence $Y(\omega)$ is

$$\begin{aligned}
Y(\omega) &= \frac{1}{j\omega}e^{-j\omega} + \pi\delta(\omega) - \left(\frac{1}{j\omega}e^{-5j\omega} + \pi\delta(\omega)\right) - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{1}{j\omega}e^{-j\omega} - \frac{1}{j\omega}e^{-5j\omega} - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}\left(\frac{1}{j}e^{2j\omega} - \frac{1}{j}e^{-2j\omega}\right) - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - \frac{e^{-3j\omega}}{1+j\omega}\left(e^{2j\omega} - e^{-2j\omega}\right) \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - 2\frac{e^{-3j\omega}}{j+\omega}\sin 2\omega \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{1}{\omega} - \frac{1}{j+\omega}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{j+\omega-\omega}{\omega\left(\omega+j\right)}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\frac{j}{\omega\left(\omega+j\right)} \\
&= -2e^{-3j\omega}\sin 2\omega\frac{1}{\omega\left(1+j\omega\right)}
\end{aligned}$$

Comparing the above to (1) shows they are the same.

Hence this shows that Fourier transform of $x(t) \circledast h(t)$ gives same answer as $H(\omega)X(\omega)$

4.4.8 key solution



Homework 4 solutions.

If you catch any errors, please send me an email at leex8370@umn.edu.

4.3 Determine the Fourier Transforms of the following signals

a) $x(t) = \sin(2\pi t + \pi/4)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \sin(2\pi t + \pi/4) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{e^{j(2\pi t + \pi/4)} - e^{-j(2\pi t + \pi/4)}}{2j}\right] e^{-j\omega t} dt$$

$$= \frac{e^{j\pi/4}}{2j} \int_{-\infty}^{\infty} e^{j2\pi t} e^{-j\omega t} dt + \frac{e^{-j\pi/4}}{2j} \int_{-\infty}^{\infty} e^{-j2\pi t} e^{-j\omega t} dt$$

Note that $\mathcal{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

Then $\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} = \mathcal{F}\{e^{j2\pi t}\} = 2\pi\,\delta(\omega - \omega_0)$

Then $X(j\omega) = \frac{\pi e^{j\pi/4}}{j} \delta(\omega - 2\pi) - \frac{\pi e^{-j\pi/4}}{j} \delta(\omega + 2\pi)$

b) $y(t) = 1 + \cos(6\pi t + \pi/8)$

$$Y(j\omega) = \int_{-\infty}^{\infty} [1 + \cos(6\pi t + \pi/8)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \int_{-\infty}^{\infty} \left[\frac{e^{j(6\pi t + \pi/8)} + e^{-j(6\pi t + \pi/8)}}{2}\right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \int_{-\infty}^{\infty} \frac{e^{j\pi/8}}{2} e^{j(6\pi - \omega)t} dt + \int_{-\infty} \frac{e^{-j\pi/8}}{2} e^{-j(6\pi + \omega)t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \frac{e^{j\pi/8}}{2}\int_{-\infty}^{\infty} e^{j(6\pi-\omega)t} dt + \frac{e^{-j\pi/4}}{2}\int_{-\infty}^{\infty} e^{-j(6\pi+\omega)} dt$$

We'll again use the fact that

$$\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t} d\omega$$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

rewriting our expression for

$Y(j\omega)$ we see

$$Y(j\omega) = 2\pi \int_{-\infty}^{\infty} \frac{e^{j\cdot 0 t}}{2\pi} e^{-j\omega t} dt$$

$$+ \pi e^{j\pi/8} \int_{-\infty}^{\infty} e^{j6\pi t} e^{-j\omega t} dt$$

$$+ \pi e^{-j\pi/8} \int_{-\infty}^{\infty} e^{-6\pi t} e^{-j\omega t} dt$$

$$= 2\pi \cdot \mathcal{F}\left\{\frac{e^{j0t}}{2\pi}\right\} + \pi e^{j\pi/8} \mathcal{F}\left\{\frac{e^{j6\pi t}}{2\pi}\right\} + \pi e^{-j\pi/8} \mathcal{F}\left\{\frac{e^{-j6\pi t}}{2\pi}\right\}$$

$$= 2\pi\, \delta(\omega) + \pi e^{j\pi/8}\, \delta(\omega-6\pi) + \pi e^{-j\pi/8}\, \delta(\omega+6\pi)$$

3b2)

4.5 Use the synthesis equation to determine $X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)}$

$$\text{where } |X(j\omega)| = 2[u(\omega+3) - u(\omega-3)]$$
$$\measuredangle X(j\omega) = -\tfrac{3}{2}\omega + \pi$$

$$X(j\omega) = 2[u(\omega+3) - u(\omega-3)] e^{-j\frac{3}{2}\omega + j\pi}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2[u(\omega+3) - u(\omega-3)] e^{-j\frac{3}{2}\omega + j\pi} e^{j\omega t}\, d\omega$$

* The integral is zero for $\omega < -3$ & $\omega > 3$.

$$= \frac{1}{2\pi}\int_{-3}^{3} 2 \cdot e^{j\pi} e^{j\omega(t-\frac{3}{2})}\, d\omega$$

$2e^{j\pi}$ is indepedent of ω, and can be brought outside of the integral. Also not $e^{j\pi} = -1$.

$$\Rightarrow x(t) = \frac{-1}{\pi}\int_{-3}^{3} e^{j\omega(t-\frac{3}{2})}\, d\omega$$

$$= \frac{-1}{\pi} \cdot \left[\frac{e^{j\omega(t-\frac{3}{2})}}{j(t-\frac{3}{2})}\right]\Bigg|_{\omega=-3}^{3}$$

$$= \frac{-1}{\pi} \frac{e^{j\cdot 3(t-\frac{3}{2})} - e^{-j3(t-\frac{3}{2})}}{j(t-\frac{3}{2})}$$

$$= \frac{-2}{\pi(t-\frac{3}{2})} \cdot \left[\frac{e^{j3(t-\frac{3}{2})} - e^{-j\cdot 3(t-\frac{3}{2})}}{2j}\right]$$

$$= \frac{-2\sin(3t - \frac{9}{2})}{\pi(t-\frac{3}{2})}$$

$$x(t) = 0 \Rightarrow \frac{\sin(3(t-\frac{3}{2}))}{(t-\frac{3}{2})} = 0$$

Clearly, we need $3(t-\frac{3}{2}) = \pi n$, where $n \in \mathbb{Z}$

Then $t = \dfrac{\pi n}{3} + \frac{3}{2}$

l'Hopital's rule.

but at $n=0$, we get $\displaystyle\lim_{t\to\frac{3}{2}} \frac{\sin(3(t-\frac{3}{2}))}{t-\frac{3}{2}} \overset{\downarrow}{=} \lim_{t\to\frac{3}{2}} \frac{3\cos(3(t-\frac{3}{2}))}{1} \neq 0$

Then we have $t = \frac{\pi n}{3} + \frac{3}{2}$ for $n \in \mathbb{Z} - \{0\}$.

4.11 $y(t) = x(t) * h(t)$

$g(t) = x(3t) * h(3t)$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$
$$h(t) \longleftrightarrow H(j\omega)$$

use properties of the transform to show that $g(t) = Ay(Bt)$

The time scaling property tells us

$$x(3t) \longleftrightarrow \tfrac{1}{3} X(j\omega/3)$$
$$h(3t) \longleftrightarrow \tfrac{1}{3} H(j\omega/3)$$

$$g(t) = x(3t) * g(3t) \longleftrightarrow \tfrac{1}{3} X(j\omega/3) \cdot \tfrac{1}{3} H(j\omega/3)$$

$$y(t) = x(t) * g(t) \longleftrightarrow X(j\omega) H(j\omega)$$

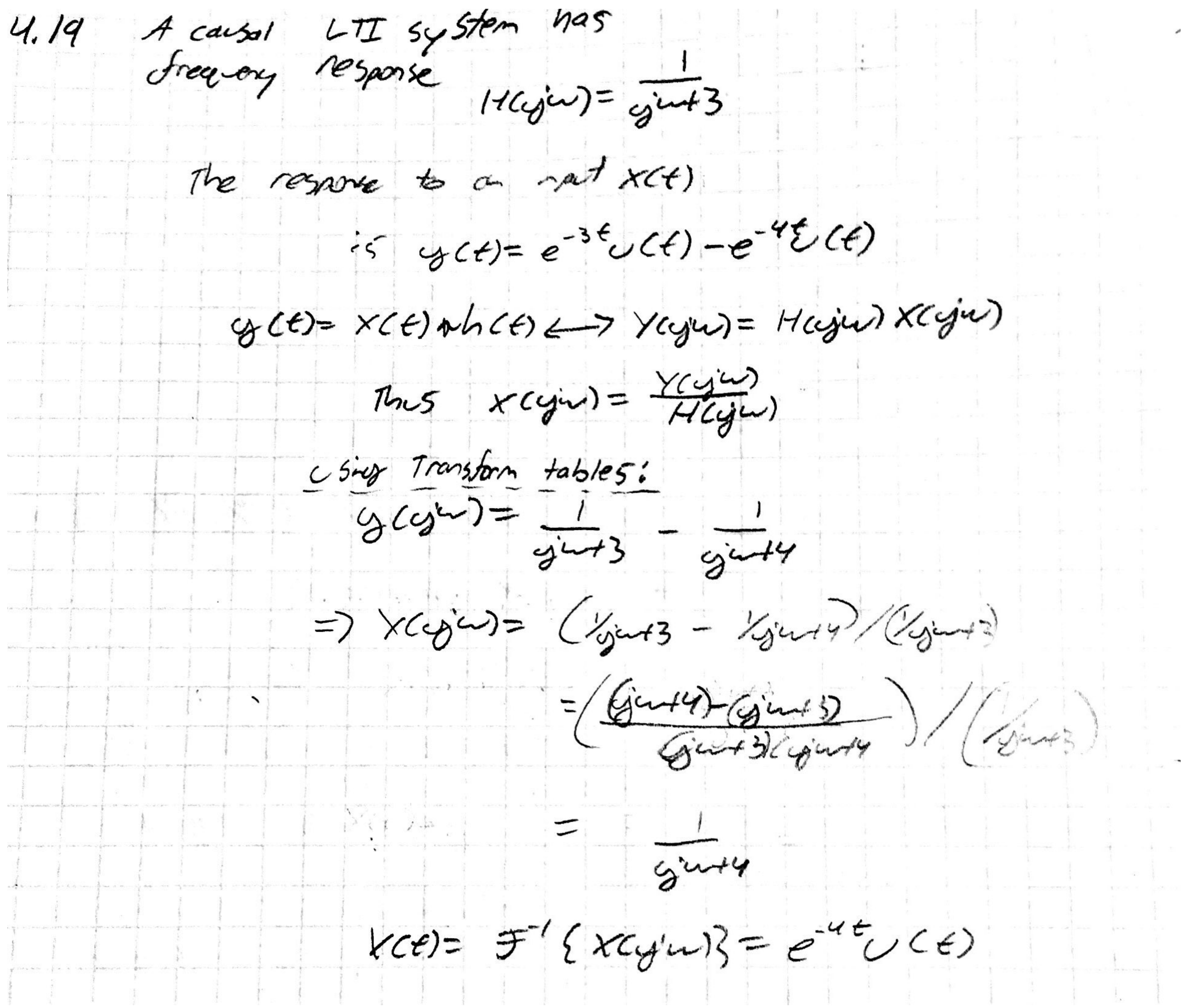
Thus $G(j\omega) = \tfrac{1}{9} Y(j\omega/3)$

$$g(t) = Ay(B(t)) \longleftrightarrow \tfrac{1}{9} Y(j\omega/3)$$

$$\Rightarrow g(t) = \tfrac{1}{3} y(3t)$$

which was found using the frequency scaling property.

Thus $A = 1/3$, $B = 3$.



4.19 A causal LTI system has frequency response $H(j\omega)=\frac{1}{j\omega+3}$

The response to an input $x(t)$ is $y(t)=e^{-3t}u(t)-e^{-4t}u(t)$

$$y(t)=x(t)*h(t)\longleftrightarrow Y(j\omega)=H(j\omega)X(j\omega)$$

Thus $X(j\omega)=\frac{Y(j\omega)}{H(j\omega)}$

<u>Using Transform tables:</u>

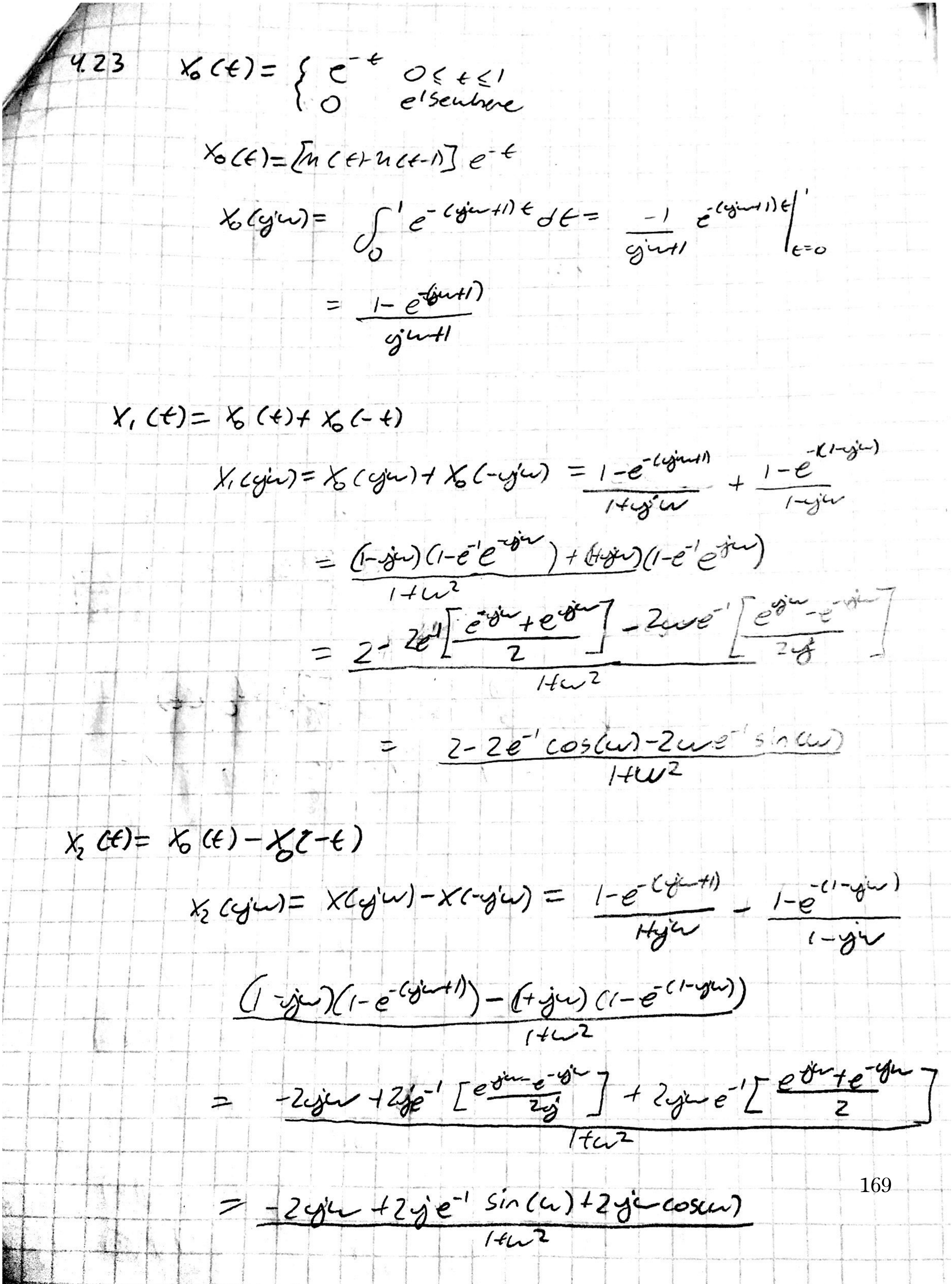
$$Y(j\omega)=\frac{1}{j\omega+3}-\frac{1}{j\omega+4}$$

$$\Rightarrow X(j\omega)=\left(1/j\omega+3-1/j\omega+4\right)/\left(1/j\omega+3\right)$$

$$=\left(\frac{(j\omega+4)-(j\omega+3)}{(j\omega+3)(j\omega+4)}\right)/\left(1/j\omega+3\right)$$

$$=\frac{1}{j\omega+4}$$

$$x(t)=\mathcal{F}^{-1}\{X(j\omega)\}=e^{-4t}u(t)$$



4.23 $x_0(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$$x_0(t) = [u(t) - u(t-1)]\, e^{-t}$$

$$X_0(j\omega) = \int_0^1 e^{-(j\omega+1)t}\, dt = \frac{-1}{j\omega+1} e^{-(j\omega+1)t}\Big|_{t=0}^{1}$$

$$= \frac{1 - e^{-(j\omega+1)}}{j\omega+1}$$

$$x_1(t) = x_0(t) + x_0(-t)$$

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{1 - e^{-(j\omega+1)}}{1+j\omega} + \frac{1 - e^{-(1-j\omega)}}{1-j\omega}$$

$$= \frac{(1-j\omega)(1 - e^{-1}e^{-j\omega}) + (1+j\omega)(1 - e^{-1}e^{j\omega})}{1+\omega^2}$$

$$= \frac{2 - 2e^{-1}\left[\frac{e^{-j\omega} + e^{j\omega}}{2}\right] - 2\omega e^{-1}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]}{1+\omega^2}$$

$$= \frac{2 - 2e^{-1}\cos(\omega) - 2\omega e^{-1}\sin(\omega)}{1+\omega^2}$$

$$x_2(t) = x_0(t) - x_0(-t)$$

$$X_2(j\omega) = X(j\omega) - X(-j\omega) = \frac{1 - e^{-(j\omega+1)}}{1+j\omega} - \frac{1 - e^{-(1-j\omega)}}{1-j\omega}$$

$$\frac{(1-j\omega)(1 - e^{-(j\omega+1)}) - (1+j\omega)(1 - e^{-(1-j\omega)})}{1+\omega^2}$$

$$= \frac{-2j\omega + 2je^{-1}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right] + 2j\omega e^{-1}\left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right]}{1+\omega^2}$$

$$= \frac{-2j\omega + 2je^{-1}\sin(\omega) + 2j\omega\cos(\omega)}{1+\omega^2}$$

$x_3(t) = x_0(t+1) + x_0(t)$

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega} X_0(-j\omega)$$

$$= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{e^{j\omega}(1-e^{-(1-j\omega)})}{1-j\omega}$$

$$= \frac{1+e^{j\omega} - e^{-1}(1+e^{-j\omega})}{1+j\omega}$$

$x_4(t) = t\, x_0(t)$

$$X_4(j\omega) = j\frac{\partial}{\partial\omega} X_0(j\omega)$$

$$= j\frac{\partial}{\partial\omega}\left[\frac{1-e^{-1}e^{-j\omega}}{1+j\omega}\right]$$

$$= j\left[\frac{(1+j\omega)(je^{-1}e^{-j\omega}) - (1-e^{-1}e^{-j\omega})j}{(1+j\omega)^2}\right]$$

$$= \frac{[(j-\omega)(je^{-1}e^{-j\omega}) + (1-e^{-1}e^{-j\omega})]}{(1+j\omega)^2}$$

$$= \frac{1-2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}$$

426

a)

i $X(t) = te^{-2t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{(j\omega+2)^2}$

$h(t) = e^{-4t}u(t) \longleftrightarrow H(j\omega) = \frac{1}{j\omega+4}$

$y(t) = X(t) * H(t) \longleftrightarrow X(j\omega)H(j\omega)$

$$= \frac{1}{(j\omega+2)^2(j\omega+4)} = \frac{1/4}{j\omega+4} - \frac{1/4}{j\omega+2} + \frac{1/2}{(2+j\omega)^2}$$

$$y(t) = \left[\tfrac{1}{4}e^{-4t} - \tfrac{1}{2}e^{-2t} + \tfrac{1}{2}te^{-2t}\right]u(t)$$

ii

$X(t) = te^{-2t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{(j\omega+2)^2}$

$h(t) = te^{-4t}u(t) \longleftrightarrow H(j\omega) = \frac{1}{(j\omega+4)^2}$

$y(t) = x(t) * h(t) \longleftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$

$$Y(j\omega) = \frac{1}{(j\omega+2)^2(j\omega+4)^2} = \frac{1/4}{2+j\omega} + \frac{1/4}{(2+j\omega)^2} - \frac{1/4}{4+j\omega} + \frac{1/4}{(4+j\omega)^2}$$

$$\Rightarrow y(t) = \left[\tfrac{1}{4}e^{-2t} + \tfrac{1}{4}te^{-2t} - \tfrac{1}{4}e^{-4t} + \tfrac{1}{4}te^{-4t}\right]u(t)$$

iii

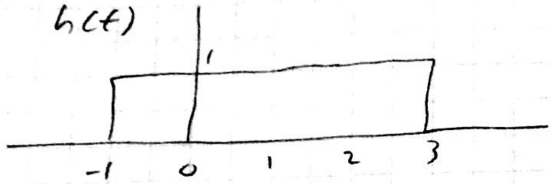
$X(t) = e^{-t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{j\omega+1}$

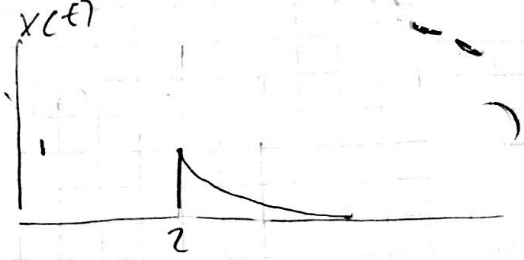
$h(t) = e^{t}u(-t) \longleftrightarrow H(j\omega) = \frac{1}{1-j\omega}$

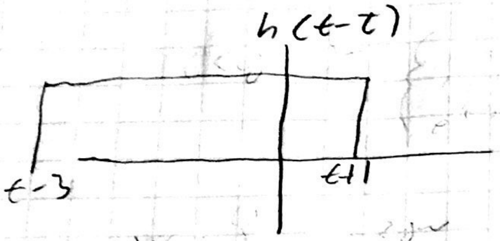
$$y(t) = X(t) * h(t) \longleftrightarrow \frac{1}{j\omega+1} \cdot \frac{1}{1-j\omega} = \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega}$$

$$\Rightarrow y(t) = \tfrac{1}{2}e^{-|t|}$$

6. $x(t) = e^{-(t-2)}u(t-2)$







$t<1 \quad x(t)*h(t)=0$

$1<t<5 \quad x(t)*h(t) = \int_{2}^{t+1} e^{-(\tau-2)}d\tau$

$= -e^{-(\tau-2)}\Big|_{\tau=2}^{t+1} = 1-e^{-(t-1)}$

$t>5 \quad x(t)*h(t) = \int_{t-3}^{t+1} e^{-(\tau-2)}d\tau$

$= -e^{-(\tau-2)}\Big|_{\tau=t-3}^{t+1} = e^{-(t-5)} - e^{-(t-1)}$

$x(t) = e^{-(t-2)}u(t-2)$ is $e^{-t}u(t)$ shifted by 2

Then $X(j\omega) = e^{-2j\omega}\cdot\frac{1}{1+j\omega}$

$h(t) = e^{-t}\cdot b(t)$, where $b(t) = \begin{cases} 1 & |t|<2 \\ 0 & |t|>2 \end{cases}$

Then $H(j\omega) = \frac{e^{-j\omega}\cdot 2\sin(\omega\cdot 2)}{\omega} = e^{-j\omega}\cdot\frac{2}{\omega}\cdot\left[\frac{e^{j2\omega}-e^{-2j\omega}}{2j}\right]$

$= \frac{e^{j\omega}-e^{-3j\omega}}{j\omega}$

$X(j\omega)H(j\omega) = \frac{e^{-2j\omega}(e^{j\omega}-e^{-3j\omega})}{(1+j\omega)(j\omega)} = \frac{(e^{-j\omega}-e^{-5j\omega})}{(1+j\omega)j\omega}$

performing partial fraction decomposition

on $\frac{1}{(1+j\omega)(j\omega)}$ gives

$$\frac{1}{(1+j\omega)(j\omega)}=\frac{A}{1+j\omega}+\frac{B}{j\omega}$$

$$A=\left.\frac{1}{j\omega}\right|_{\omega=j}=-1$$

$$B=\left.\frac{1}{(1+j\omega)}\right|_{\omega=0}=1$$

Then we get

$$y(j\omega)=X(j\omega)H(j\omega)=\left(e^{-j\omega}-e^{-5j\omega}\right)\cdot\left[\frac{-1}{1+j\omega}+\frac{1}{j\omega}\right]$$

$$=\left(e^{-j\omega}-e^{-5j\omega}\right)\left[\frac{-1}{1+j\omega}+\frac{1}{j\omega}+\pi\delta(\omega)\right]$$

$$\Rightarrow y(t)=\left(1-e^{-(t-1)}\right)u(t-1)-\left(1-e^{-(t-5)}\right)u(t-5)$$

which agrees with our convolution.

Note that the $\pi\delta(\omega)$ term was added to match the form of the Fourier transform of $u(t)$. It does not change the value of the expression because

$$\left(e^{-j\omega}-e^{-5j\omega}\right)\pi\delta(\omega)=\left(e^{0}-e^{0}\right)\pi\delta(\omega)=0$$

4.5 HW 5

Local contents

4.5.1 Problem 5.3, Chapter 5

Determine the Fourier transform for $-\pi \leq \omega < \pi$ in the case of each of the following periodic signals (a) $\sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$ (b) $2 + \cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right)$

solution

4.5.1.1 Part a

Since the signal is periodic, then the Fourier transform is given by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \tag{1}$$

Where a_k are the Fourier series coefficients of $x[n]$. To determine a_k we can expression $x[n]$ using Euler relation. To find the period, $\frac{\pi}{3}N = m2\pi$. Hence $\frac{m}{N} = \frac{1}{6}$. Hence

$$N = 6$$

Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{3}$. Now, using Euler relation

$$\begin{aligned}\sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) &= \frac{e^{\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)j} - e^{-\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)j}}{2j} \\ &= \frac{1}{2j}e^{j\frac{\pi}{4}}\left(e^{j\frac{\pi}{3}n}\right) - \frac{1}{2j}e^{-j\frac{\pi}{4}}\left(e^{-j\frac{\pi}{3}n}\right) \end{aligned} \tag{2}$$

Comparing (2) to Fourier series expansion of periodic signal given by

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=0}^{5} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=-2}^{3} a_k e^{jk\Omega_0 n} \end{aligned}$$

Since $\Omega_0 = \frac{\pi}{3}$ then the above becomes

$$x[n] = \sum_{k=-2}^{3} a_k e^{jk\frac{\pi}{3}n}$$

Comparing the above with (2) shows that $a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$ and $a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$ and all other $a_k = 0$ for $k = -2, 0, 2, 3$. Hence (1) becomes

$$\begin{aligned} X(\Omega) &= 2\pi\left(a_{-1}\delta(\Omega+\Omega_0) + a_1\delta(\Omega-\Omega_0)\right) \\ &= 2\pi\left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\delta\left(\Omega+\frac{\pi}{3}\right) + \frac{1}{2j}e^{j\frac{\pi}{4}}\delta\left(\Omega-\frac{\pi}{3}\right)\right) \\ &= \frac{\pi}{j}\left(-e^{-j\frac{\pi}{4}}\delta\left(\Omega+\frac{\pi}{3}\right) + e^{j\frac{\pi}{4}}\delta\left(\Omega-\frac{\pi}{3}\right)\right) \end{aligned}$$

4.5.1.2 Part b

Since the signal $2+\cos\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)$ is periodic, then the Fourier transform is given by

$$X(\Omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\Omega - k\Omega_0) \tag{1}$$

Where a_k are the Fourier series coefficients of $x[n]$. To determine a_k we can expression $x[n]$ using Euler relation. To find the period, $\frac{\pi}{6}N = m2\pi$. Hence $\frac{m}{N} = \frac{1}{12}$. Hence

$$N = 12$$

Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{6}$. Now, using Euler relation

$$\begin{aligned} 2+\cos\left(\frac{\pi}{6}n+\frac{\pi}{8}\right) &= 2 + \frac{e^{\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)j} + e^{-\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)j}}{2} \\ &= 2 + \frac{1}{2}e^{j\frac{\pi}{8}}\left(e^{j\frac{\pi}{6}n}\right) + \frac{1}{2}e^{-j\frac{\pi}{8}}\left(e^{-j\frac{\pi}{6}n}\right) \end{aligned} \tag{2}$$

Comparing (2) to Fourier series expansion of periodic signal given by

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=0}^{11} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=-5}^{6} a_k e^{jk\Omega_0 n} \end{aligned}$$

Since $\Omega_0 = \frac{\pi}{6}$ then the above becomes

$$x[n] = \sum_{k=-5}^{6} a_k e^{jk\frac{\pi}{6}n}$$

Comparing the above with (2) shows that $a_0 = 2, a_1 = \frac{1}{2}e^{j\frac{\pi}{8}}$ and $a_{-1} = \frac{1}{2}e^{-j\frac{\pi}{8}}$ and all other $a_k = 0$. Hence (1) becomes

$$\begin{aligned} X(\Omega) &= 2\pi\left(a_0\delta(\Omega) + a_{-1}\delta(\Omega+\Omega_0) + a_1\delta(\Omega-\Omega_0)\right) \\ &= 2\pi\left(2\delta(\Omega) + \frac{1}{2}e^{-j\frac{\pi}{8}}\delta\left(\Omega+\frac{\pi}{6}\right) + \frac{1}{2}e^{j\frac{\pi}{8}}\delta\left(\Omega-\frac{\pi}{6}\right)\right) \\ &= 4\pi\delta(\Omega) + \pi e^{-j\frac{\pi}{8}}\delta\left(\Omega+\frac{\pi}{6}\right) + \pi e^{j\frac{\pi}{8}}\delta\left(\Omega-\frac{\pi}{6}\right) \end{aligned}$$

4.5.2 Problem 5.5, Chapter 5

Use the Fourier transform synthesis equation (5.8)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega \tag{5.8}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{5.9}$$

To determine the inverse Fourier transform of $X(\Omega) = |X(\Omega)|\, e^{j\arg H(\Omega)}$, where $|X(\Omega)| = \begin{cases} 1 & 0 \leq |\Omega| < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega| < \pi \end{cases}$ and $\arg H(\Omega) = \frac{-3\Omega}{2}$. Use your answer to determine the values of n for which $x[n] = 0$.

solution

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{4}} e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\frac{-3\Omega}{2}} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\Omega\left(\frac{-3}{2}+n\right)} d\Omega \\
&= \frac{1}{2\pi}\frac{1}{j\left(\frac{-3}{2}+n\right)}\left[e^{j\Omega\left(\frac{-3}{2}+n\right)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2\pi}\frac{1}{j\left(\frac{-3}{2}+n\right)}\left[e^{j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)} - e^{-j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)}\right] \\
&= \frac{1}{\pi}\frac{1}{\left(\frac{-3}{2}+n\right)}\left[\frac{e^{j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)} - e^{-j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)}}{2j}\right] \\
&= \frac{1}{\pi}\frac{1}{\left(\frac{-3}{2}+n\right)}\sin\left(\frac{\pi}{4}\left(\frac{-3}{2}+n\right)\right) \\
&= \frac{1}{\pi}\frac{\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right)}{n-\frac{3}{2}}
\end{aligned}
$$

Now the above is zero when $\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right) = 0$ or $\frac{\pi}{4}\left(n-\frac{3}{2}\right) = m\pi$ for integer m. Hence $n-\frac{3}{2} = 4m$. Or $n = 4m+\frac{3}{2}$. Since m is integer, and since n must be an integer as well, then there is <u>no finite n</u> where $\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right) = 0$. The other option is to look at denominator of $\frac{\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right)}{n-\frac{3}{2}}$ and ask where is that ∞. This happens when $n \to \pm\infty$ and only then $x[n] = 0$.

4.5.3 Problem 5.9, Chapter 5

The following four facts are given about a real signal $x[n]$ with Fourier transform $X(\Omega)$

1. $x[n] = 0$ for $n > 0$
2. $x[0] > 0$

3. $\operatorname{Im}(X(\Omega)) = \sin\Omega - \sin(2\Omega)$
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\Omega)|^2\,d\Omega = 3$

Determine $x[n]$

solution

From tables we know that the odd part of $x[n]$ has Fourier transform which is $j\operatorname{Im}(X(\Omega))$. Hence using (3) above, this means that odd part of $x[n]$ has Fourier transform of $j(\sin\Omega - \sin(2\Omega))$ or $j\left(\frac{e^{j\Omega}-e^{-j\Omega}}{2j} - \frac{e^{j2\Omega}-e^{-j2\Omega}}{2j}\right)$ or $\frac{1}{2}\left(e^{j\Omega} - e^{-j\Omega} - e^{j2\Omega} + e^{-j2\Omega}\right)$. From tables, we know find the inverse Fourier transform of this. Hence odd part of $x[n]$ is $\frac{1}{2}(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$. So now we know what the odd part of $x[n]$ is.

But since $x[n] = 0$ for $n > 0$ then the odd part of $x[n]$ reduces to $\frac{1}{2}(\delta[n+1] - \delta[n+2])$.

But we also know that any function can be expressed as the sum of its odd part and its even part. But since $x[n] = 0$ for $n > 0$ then this means $x[n] = 2\left(\frac{1}{2}(\delta[n+1] - \delta[n+2])\right)$ for $n < 0$. Hence

$$x[n] = \delta[n+1] - \delta[n+2] \qquad n < 0$$

Finally, using (4) above,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\Omega)|^2\,d\Omega = 3 = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \sum_{n=-\infty}^{0}|x[n]|^2$$

Hence

$$\begin{aligned}3 &= |\delta[-1]|^2 + |\delta[-2]|^2 + |x[0]|^2\\ &= 1 + 1 + x[n]^2\\ x[n]^2 &= 3 - 2\\ &= 1\end{aligned}$$

Therefore $x[n] = 1$ or $x[n] = -1$. But from (2) $x[0] > 0$. Hence $x[0]$. Therefore

$$x[n] = \delta[n+1] - \delta[n+2] + \delta[n] \qquad n \leq 0$$

4.5.4 Problem 5.13, Chapter 5

An LTI system with impulse response $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel

interconnection has the frequency response

$$H(\Omega) = \frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}}$$

Determine $h_2[n]$.

solution

Since the connection is parallel, then $h[n] = h_1[n] + h_2[n]$. Or $H(\Omega) = H_1(\Omega) + H_2(\Omega)$. Hence

$$H_2(\Omega) = H(\Omega) - H_1(\Omega) \tag{1}$$

But

$$\begin{aligned} H_1(\Omega) &= \sum_{n=-\infty}^{\infty} h_1[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\Omega}\right)^n \\ &= \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = \frac{1}{1-\frac{1}{3}e^{-j\Omega}} \\ &= \frac{3}{3-e^{-j\Omega}} \end{aligned}$$

Therefore from (1)

$$H_2(\Omega) = \frac{-12+5e^{-j\Omega}}{12-7e^{-j\Omega}+e^{-j2\Omega}} - \frac{3}{3-e^{-j\Omega}}$$

Let $e^{-j\Omega} = x$ to simplify notation. The above becomes

$$\begin{aligned} H_2(\Omega) &= \frac{-12+5x}{12-7x+x^2} - \frac{3}{3-x} \\ &= \frac{-12+5x}{(x-3)(x-4)} + \frac{3}{(x-3)} \\ &= \frac{-12+5x+3(x-4)}{(x-3)(x-4)} \\ &= \frac{-12+5x+3x-12}{(x-3)(x-4)} \\ &= \frac{8x-24}{(x-3)(x-4)} \\ &= \frac{8(x-3)}{(x-3)(x-4)} \\ &= \frac{8}{(x-4)} \\ &= -2\left(\frac{1}{1-\frac{1}{4}x}\right) \end{aligned}$$

Hence

$$H_2(\Omega) = -2\left(\frac{1}{1-\frac{1}{4}e^{-j\Omega}}\right)$$

from tables, $a^n u[n] \iff \frac{1}{1-ae^{-j\Omega}}$ for $|a| < 1$. Comparing this to the above gives

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

4.5.5 Problem 5.19, Chapter 5

Consider a causal and stable LTI system S whose input $x[n]$ and output $y[n]$ are related through the second-order difference equation

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

(a) Determine the frequency response $H(\Omega)$ for the system S. (b) Determine the impulse response $h[n]$ for the system S.

solution

4.5.5.1 part a

Taking DFT of the difference equation gives

$$\begin{aligned} Y(\Omega) - \frac{1}{6}e^{-j\Omega}Y(\Omega) - \frac{1}{6}e^{-j2\Omega}Y(\Omega) &= X(\Omega) \\ Y(\Omega)\left(1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}\right) &= X(\Omega) \\ \frac{Y(\Omega)}{X(\Omega)} &= \frac{1}{1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}} \end{aligned}$$

Let $e^{-j\Omega} = x$ to simplify the notation, then

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{6}x - \frac{1}{6}x^2} = \frac{6}{6 - x - x^2} = \frac{-6}{x^2 + x - 6} = \frac{-6}{(x-2)(x+3)}$$

Hence

$$\begin{aligned} H(\Omega) &= \frac{Y(\Omega)}{X(\Omega)} \\ &= \frac{-6}{\left(e^{-j\Omega} - 2\right)\left(e^{-j\Omega} + 3\right)} \end{aligned}$$

4.5.5.2 part b

Applying partial fractions

$$H(\Omega) = \frac{-6}{\left(e^{-j\Omega} - 2\right)\left(e^{-j\Omega} + 3\right)} = \frac{A}{(x-2)} + \frac{B}{(x+3)}$$

Hence $A=-\frac{6}{5}, B=\frac{6}{5}$. Therefore

$$\begin{aligned}
H(\Omega) &= -\frac{6}{5}\frac{1}{e^{-j\Omega}-2}+\frac{6}{5}\frac{1}{e^{-j\Omega}+3} \\
&= -\frac{3}{5}\frac{1}{\frac{1}{2}e^{-j\Omega}-1}+\frac{2}{5}\frac{1}{\frac{1}{3}e^{-j\Omega}+1} \\
&= \frac{3}{5}\frac{1}{1-\frac{1}{2}e^{-j\Omega}}+\frac{2}{5}\frac{1}{1+\frac{1}{3}e^{-j\Omega}}
\end{aligned}$$

Taking the inverse DFT using tables gives

$$\begin{aligned}
h[n] &= \frac{3}{5}\left(\frac{1}{2}\right)^n u[n]+\frac{2}{5}\left(-\frac{1}{3}\right)^n u[n] \\
&= \left(\frac{3}{5}\left(\frac{1}{2}\right)^n+\frac{2}{5}\left(-\frac{1}{3}\right)^n\right)u[n]
\end{aligned}$$

4.5.6 Problem 5.30, Chapter 5

In Chapter 4, we indicated that the continuous-time LTI system with impulse response

$$h(t)=\frac{W}{\pi}\sin c\left(\frac{Wt}{\pi}\right)=\frac{\sin(Wt)}{\pi t}$$

plays a very important role in LTI system analysis. The same is true of the discrete time LTI system with impulse response

$$h(n)=\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right)=\frac{\sin(Wn)}{\pi n}$$

(a) Determine and sketch the frequency response for the system with impulse response $h[n]$.
(b) Consider the signal $x[n]=\sin\left(\frac{\pi n}{8}\right)-2\cos\left(\frac{\pi n}{4}\right)$. Suppose that this signal is the input to LTI systems with the following impulse responses. Determine the output in each case (i) $h[n]=\frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$. (ii) $h[n]=\frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}+\frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$

solution

4.5.6.1 Part a

Given $h(n) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin(Wn)}{\pi n}$. We will show that $H(\Omega)$ is the rectangle function by reverse. Assuming that $H(\Omega) = \begin{matrix} 1 & |\Omega| < 2W \\ 0 & \text{otherwise} \end{matrix}$ therefore

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-W}^{W} X(\Omega)\, e^{j\Omega n} d\omega \\
&= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n} d\omega \\
&= \frac{1}{2\pi}\frac{e^{jWn} - e^{-jWn}}{jn} \\
&= \frac{1}{\pi n}\sin(Wn)
\end{aligned}
$$

Which is the $h[n]$ given. Therefore, the above shows that $\frac{\sin(Wn)}{\pi n}$ has DFT of $H(\Omega)$ as the rectangle function. Here is sketch

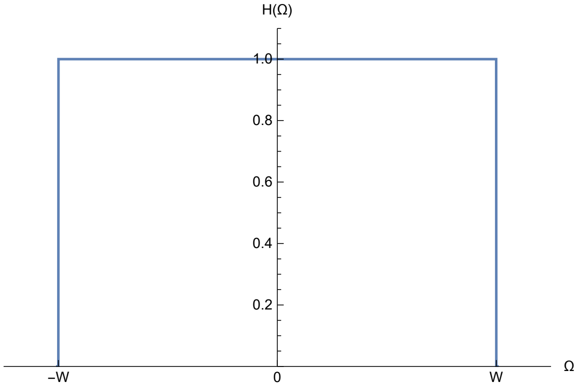


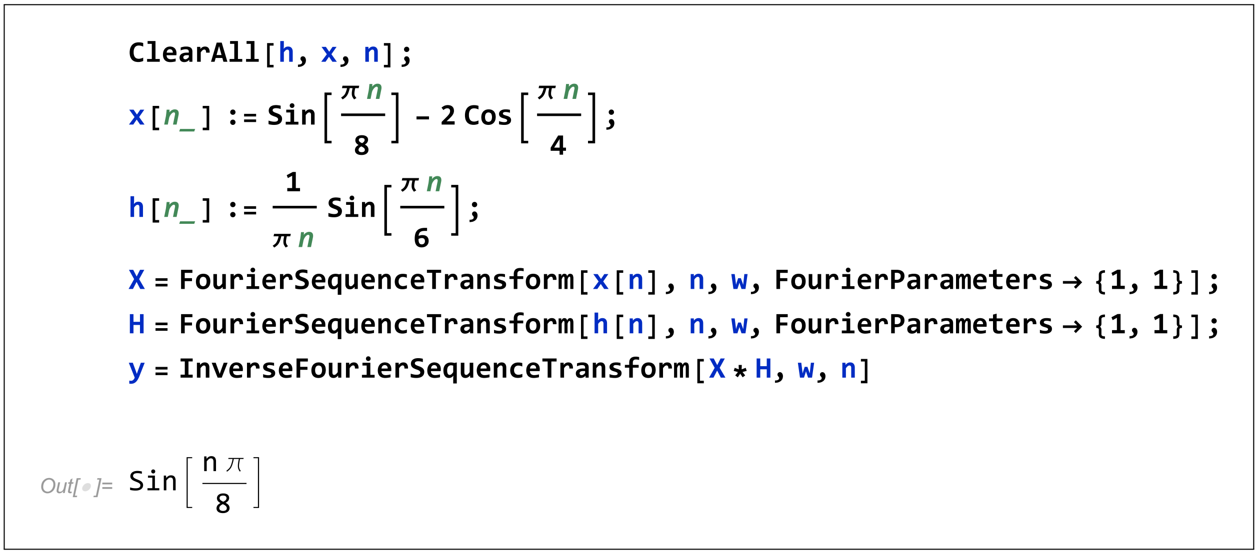
Figure 4.61: Plot of $H(\Omega)$

4.5.6.2 Part b

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$$

(i) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$. Hence $y[n] = x[n] \circledast h[n]$. Or $Y(\Omega) = X(\Omega)H(\Omega)$, and then we find $y[n]$ by taking the inverse discrete Fourier transform. Here is the result and the code used. The result is

$$y[n] = \sin\left(\frac{n\pi}{8}\right)$$



```
ClearAll[h, x, n];
x[n_] := Sin[π n/8] - 2 Cos[π n/4];
h[n_] := 1/(π n) Sin[π n/6];
X = FourierSequenceTransform[x[n], n, w, FourierParameters → {1, 1}];
H = FourierSequenceTransform[h[n], n, w, FourierParameters → {1, 1}];
y = InverseFourierSequenceTransform[X * H, w, n]

Out[•]= Sin[n π/8]
```

Figure 4.62: Code used to generate $y[n]$

Here is plot of $y[n]$ for $n = -8 \cdots 8$

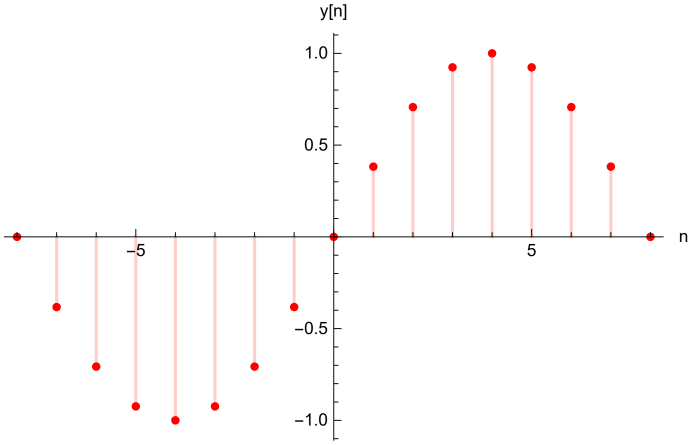
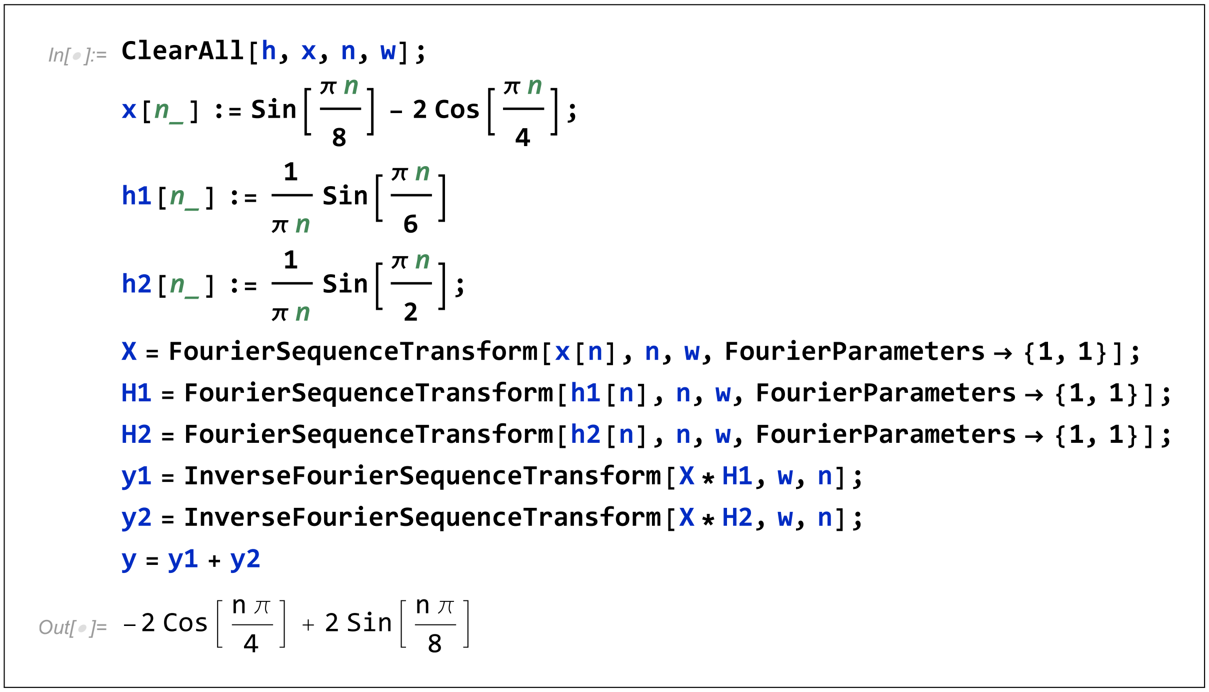


Figure 4.63: Plot of above $y[n]$

(ii) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n} + \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$. Here is the result and the code used. The result is

$$y[n] = 2\sin\left(\frac{n\pi}{8}\right) - 2\cos\left(\frac{n\pi}{4}\right)$$



```
In[∘]:= ClearAll[h, x, n, w];
        x[n_] := Sin[π n/8] - 2 Cos[π n/4];
        h1[n_] := 1/(π n) Sin[π n/6]
        h2[n_] := 1/(π n) Sin[π n/2];
        X = FourierSequenceTransform[x[n], n, w, FourierParameters → {1, 1}];
        H1 = FourierSequenceTransform[h1[n], n, w, FourierParameters → {1, 1}];
        H2 = FourierSequenceTransform[h2[n], n, w, FourierParameters → {1, 1}];
        y1 = InverseFourierSequenceTransform[X * H1, w, n];
        y2 = InverseFourierSequenceTransform[X * H2, w, n];
        y = y1 + y2

Out[∘]= -2 Cos[n π/4] + 2 Sin[n π/8]
```

Figure 4.64: Code used to generate $y[n]$

Here is plot of $y\,[n]$ for $n = -8 \cdots 8$

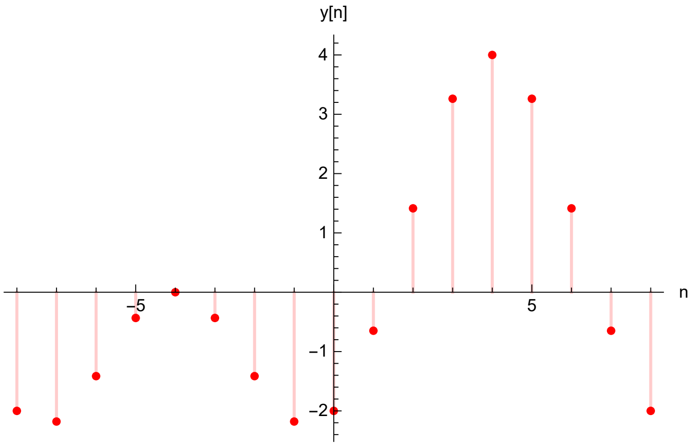
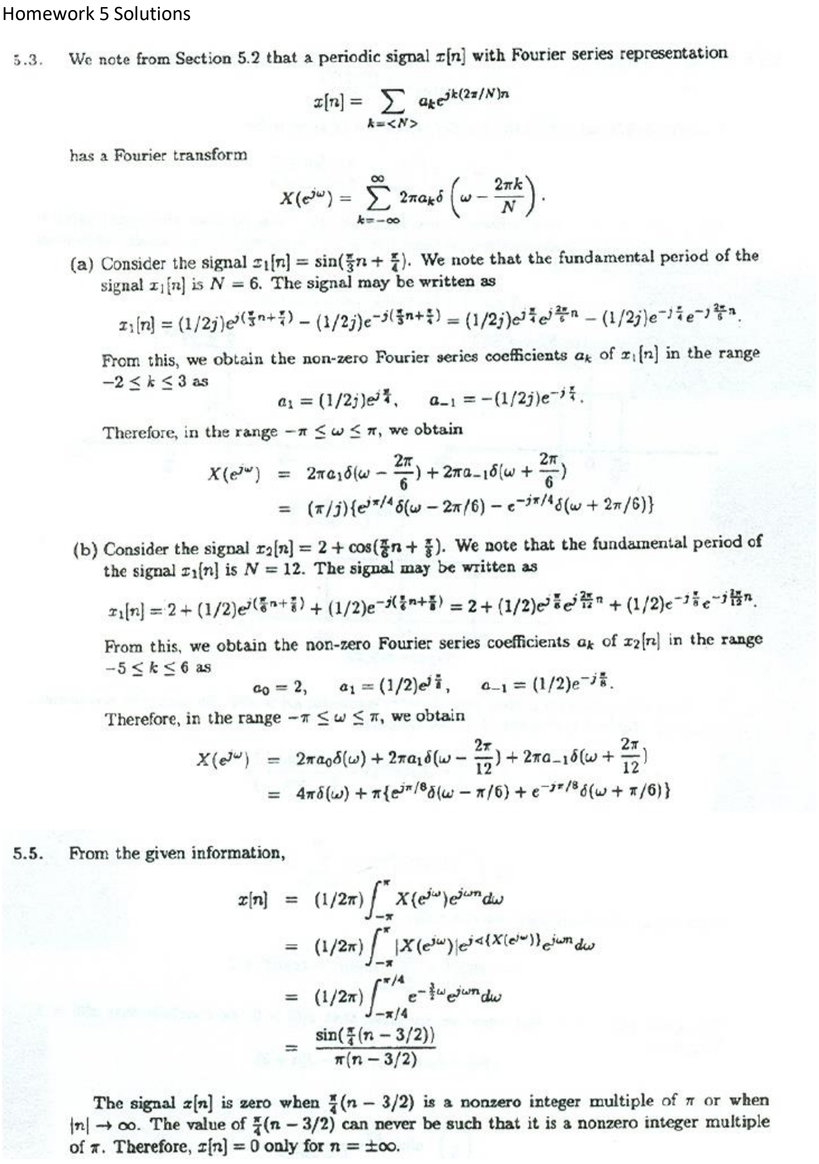


Figure 4.65: Plot of above $y[n]$

4.5.7 key solution



Homework 5 Solutions

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ in the range $-2 \leq k \leq 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \quad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \leq \omega \leq \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1 \delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6}) \\ &= (\pi/j)\{e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\} \end{aligned}$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 12$. The signal may be written as

$$x_1[n] = 2 + (1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{2\pi}{12}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{2\pi}{12}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \leq k \leq 6$ as

$$a_0 = 2, \quad a_1 = (1/2)e^{j\frac{\pi}{8}}, \quad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

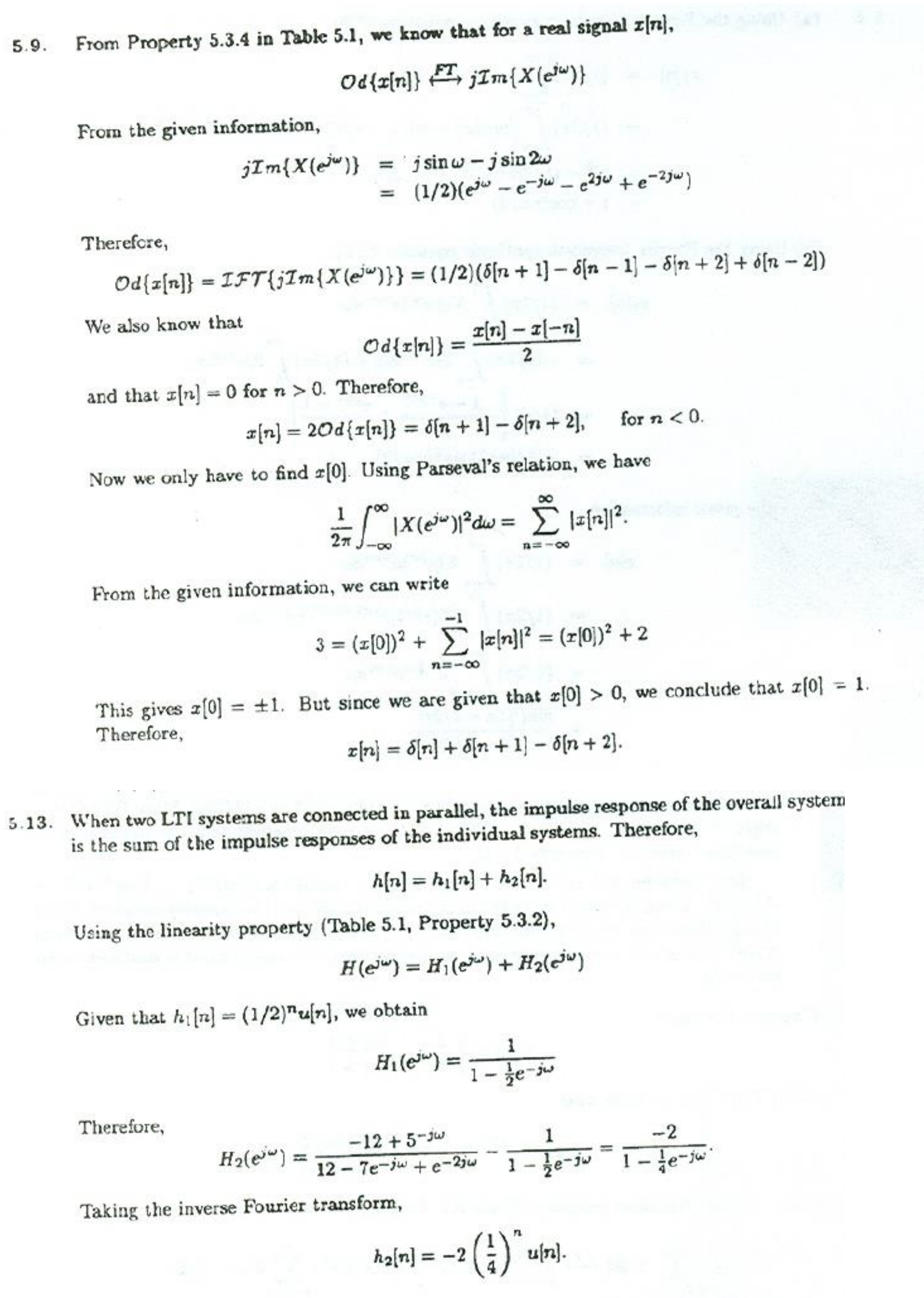
Therefore, in the range $-\pi \leq \omega \leq \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12}) \\ &= 4\pi\delta(\omega) + \pi\{e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\} \end{aligned}$$

5.5. From the given information,

$$\begin{aligned} x[n] &= (1/2\pi)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\sphericalangle\{X(e^{j\omega})\}}e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi/4}^{\pi/4} e^{-\frac{3}{2}\omega}e^{j\omega n}d\omega \\ &= \frac{\sin(\frac{\pi}{4}(n-3/2))}{\pi(n-3/2)} \end{aligned}$$

The signal $x[n]$ is zero when $\frac{\pi}{4}(n - 3/2)$ is a nonzero integer multiple of π or when $|n| \to \infty$. The value of $\frac{\pi}{4}(n - 3/2)$ can never be such that it is a nonzero integer multiple of π. Therefore, $x[n] = 0$ only for $n = \pm\infty$.



5.9. From Property 5.3.4 in Table 5.1, we know that for a real signal $x[n]$,

$$\mathcal{O}d\{x[n]\} \xleftrightarrow{FT} j\mathcal{I}m\{X(e^{j\omega})\}$$

From the given information,

$$\begin{aligned} j\mathcal{I}m\{X(e^{j\omega})\} &= j\sin\omega - j\sin 2\omega \\ &= (1/2)(e^{j\omega} - e^{-j\omega} - e^{2j\omega} + e^{-2j\omega}) \end{aligned}$$

Therefore,

$$\mathcal{O}d\{x[n]\} = \mathcal{IFT}\{j\mathcal{I}m\{X(e^{j\omega})\}\} = (1/2)(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$$

We also know that

$$\mathcal{O}d\{x[n]\} = \frac{x[n] - x[-n]}{2}$$

and that $x[n] = 0$ for $n > 0$. Therefore,

$$x[n] = 2\mathcal{O}d\{x[n]\} = \delta[n+1] - \delta[n+2], \qquad \text{for } n < 0.$$

Now we only have to find $x[0]$. Using Parseval's relation, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

From the given information, we can write

$$3 = (x[0])^2 + \sum_{n=-\infty}^{-1} |x[n]|^2 = (x[0])^2 + 2$$

This gives $x[0] = \pm 1$. But since we are given that $x[0] > 0$, we conclude that $x[0] = 1$. Therefore,

$$x[n] = \delta[n] + \delta[n+1] - \delta[n+2].$$

5.13. When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore,

$$h[n] = h_1[n] + h_2[n].$$

Using the linearity property (Table 5.1, Property 5.3.2),

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Given that $h_1[n] = (1/2)^n u[n]$, we obtain

$$H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Therefore,

$$H_2(e^{j\omega}) = \frac{-12 + 5^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}} - \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{-2}{1 - \frac{1}{4}e^{-j\omega}}.$$

Taking the inverse Fourier transform,

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n].$$

5.19. (a) Taking the Fourier transform of both sides of the difference equation, we have

$$Y(e^{j\omega})\left[1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}\right]=X(e^{j\omega}).$$

Therefore,

$$H(ej\omega)=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}}=\frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}.$$

(b) Using Partial fraction expansion,

$$H(ej\omega)=\frac{3/5}{1-\frac{1}{2}e^{-j\omega}}+\frac{2/5}{1+\frac{1}{3}e^{-j\omega}}.$$

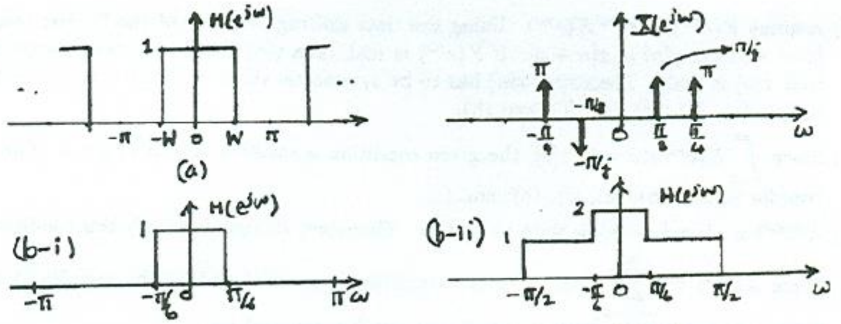
Using Table 5.2, and taking the inverse Fourier trasform, we obtain

$$h[n]=\frac{3}{5}\left(\frac{1}{2}\right)^n u[n]+\frac{2}{5}\left(-\frac{1}{3}\right)^n u[n].$$

(a) The frequency response of the system is as shown in Figure S5.30.

(b) The Fourier transform $X(e^{j\omega})$ of $x[n]$ is as shown in Figure S5.30.

- (i) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n] = \sin(\pi n/8)$.
- (ii) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n] = 2\sin(\pi n/8) - 2\cos(\pi n/4)$.



4.6 HW 6

Local contents

4.6.1 Problem 6.2

Consider a discrete-time LTI system with frequency response $H(\Omega) = |H(\Omega)| e^{j \arg H(\Omega)}$ and real impulse response $h[n]$. Suppose that we apply the input $x[n] = \sin\left(\Omega_0 n + \phi_0\right)$ to this system. The resulting output can be shown to be of the form $y[n] = \left|H(\Omega_0)\right| x[n - n_0]$ provided that $\arg H(\Omega_0)$ and Ω_0 are related in a particular way. Determine this relationship.

solution

From standard LTI theory, the output is given by

$$y[n] = \left|H(\Omega_0)\right| \sin\left(\Omega_0 n + \phi_0 + \arg\left(H(\Omega_0)\right)\right) \tag{1}$$

Comparing the above to

$$\left|H(\Omega_0)\right| x[n - n_0] = \left|H(\Omega_0)\right| \sin\left(\Omega_0 (n - n_0) + \phi_0\right) \tag{2}$$

Shows that

$$\begin{aligned}\sin\left(\Omega_0 (n - n_0) + \phi_0\right) &= \sin\left(\Omega_0 n + \phi_0 + \arg\left(H(\Omega_0)\right)\right)\\ \sin\left(\Omega_0 n - \Omega_0 n_0 + \phi_0\right) &= \sin\left(\Omega_0 n + \phi_0 + \arg\left(H(\Omega_0)\right)\right)\end{aligned}$$

Hence we need

$$-\Omega_0 n_0 = \arg\left(H(\Omega_0)\right)$$

Since the input is periodic of period $2\pi k$ for k integer, then the above can also be written as

$$-\Omega_0 n_0 + 2\pi k = \arg\left(H(\Omega_0)\right)$$

This is the relation needed.

4.6.2 Problem 6.5

Consider a continuous-time ideal bandpass filter whose frequency response is

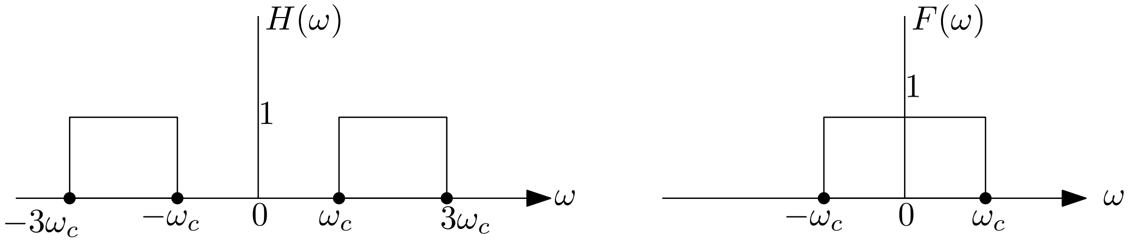
$$H(\omega) = \begin{matrix} 1 & \omega_c \leq |\omega| \leq 3\omega_c \\ 0 & \text{elsewhere} \end{matrix}$$

(a) If $h(t)$ is the impulse response of this filter, determine a function $g(t)$ such that $h(t) = \left(\frac{\sin \omega_c t}{\pi t}\right) g(t)$. (b) As ω_c is increased, does the impulse response of the filter get more concentrated or less concentrated about the origin?

solution

4.6.2.1 Part a

Let $f(t) = \frac{\sin \omega_c t}{\pi t}$ which is a sinc function. The following is a sketch of $H(\omega)$ and of the CTFT of $f(t)$, i.e. $F(\omega)$ which we know will be rectangle since Fourier transform of rectangle is sinc.

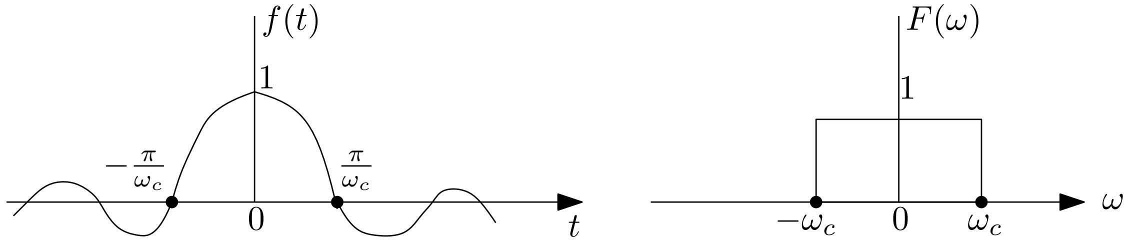


Fourier Transform of the given filter $h(t)$

Fourier Transform of $f(t) = \frac{1}{\pi t}\sin(\omega_c t)$

Figure 4.66: Sketch of CTFT of filter and sinc function

The relation between $\frac{\sin \omega_c t}{\pi t}$ and its CTFT is given in this sketch



$f(t) = \frac{1}{\pi t}\sin(\omega_c t)$

Fourier Transform of $f(t) = \frac{1}{\pi t}\sin(\omega_c t)$

Figure 4.67: Sketch of sinc function and its CTFT

Now, we know that, since $h(t) = f(t)g(t)$, then by modulation theory, this results in $2\pi H(\omega) = F(\omega) \circledast G(\omega)$. In this, we are given $H(\omega)$ and $F(\omega)$ but we do not know $G(\omega)$. But looking at $H(\omega)$ and $F(\omega)$, we see that if $G(\omega)$ happened to be two Dirac impulses, one at $-2\omega_c$ and one at $+2\omega_c$, then convolving $F(\omega)$ with it, will give $2\pi H(\omega)$. So we need $G(\omega)$ to be the following

$$G(\omega) = 2\pi\left(\delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c)\right)$$

And now $F(\omega) \circledast G(\omega)$ will result in $H(\omega)$. The factor 2π was added to cancel the 2π from the definition of modulation theory. But $\cos(2\omega_c t)$ has the CTFT of $\pi(\delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c))$. This shows that $G(\omega)$ is the Fourier transform of $2\cos(2\omega_c t)$. Therefore

$$g(t) = 2\cos(2\omega_c t)$$

4.6.2.2 Part b

Since $f(t) = \frac{\sin\omega_c t}{\pi t}$ then we see as ω_c increases, the sinc function becomes more concentrated at origin, since the first loop cut off is given by $\frac{\pi}{\omega_c}$. So this gets closer to origin. And since $h(t) = f(t)(2\cos(2\omega_c t))$ then $h(t)$ becomes <u>more concentrated</u> around the origin as well.

4.6.3 Problem 6.7

A continuous-time lowpass filter has been designed with a passband frequency of 1000 Hz, a stopband frequency of 1200 Hz, passband ripple of 0.1, and stopband ripple of 0.05. Let the impulse response of this lowpass filter be denoted by $h(t)$. We wish to convert the filter into a bandpass filter with impulse response

$$g(t) = 2h(t)\cos(4000\pi t)$$

Assuming that $|H(\omega)|$ is negligible for for $|\omega| > 4000\pi$, answer the following questions. (a) If the passband ripple for the bandpass filter is constrained to be 0.1, what are

the two passband frequencies associated with the bandpass filter? (b) If the stopband ripple for the bandpass filter is constrained to be 0.05, what are the two stopband frequencies associated with the bandpass filter?

<u>solution</u>

4.6.3.1 Part a

Let $f(t) = 2\cos(4000\pi t)$. By modulation theory multiplication in time becomes convolution in frequency (with 2π factor)

$$g(t) = h(t)f(t) \to \frac{1}{2\pi}H(\omega) \circledast F(\omega) \tag{1}$$

Where $H(\omega)$ is the CTFT of $h(t)$ and $F(\omega)$ is the CTFT of $2\cos(4000\pi t)$ which is given by (since it is periodic)

$$F(\omega) = 2\sum_{n=-\infty}^{\infty} 2\pi a_k \delta(\omega - n\omega_0)$$

Where a_k are the Fourier series coefficients of $\cos(4000\pi t)$. In the above $\omega_0 = 4000\pi$. But we know that $a_{-1} = \frac{1}{2}$ and $a_1 = \frac{1}{2}$ from Euler relation. Hence the above becomes

$$
\begin{aligned}
F(\omega) &= 2\left(2\pi\frac{1}{2}\delta(\omega + 4000\pi) + 2\pi\frac{1}{2}\delta(\omega - 4000\pi)\right)\\
&= 2\pi\left(\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi)\right)
\end{aligned}
$$

Now that we know $F(\omega)$ we go back to (1) and find the CTFT of $g(t)$ which is

$$
\begin{aligned}
G(\omega) &= \frac{1}{2\pi}H(\omega) \circledast \left(2\pi\left(\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi)\right)\right)\\
&= H(\omega) \circledast \left(\left(\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi)\right)\right)
\end{aligned}
$$

The above shows that bandpass filter is the lowpass filter spectrum but shifted to the right and to the left by 4000π. This is because convolution with impulse causes shifting. The following diagram shows the result

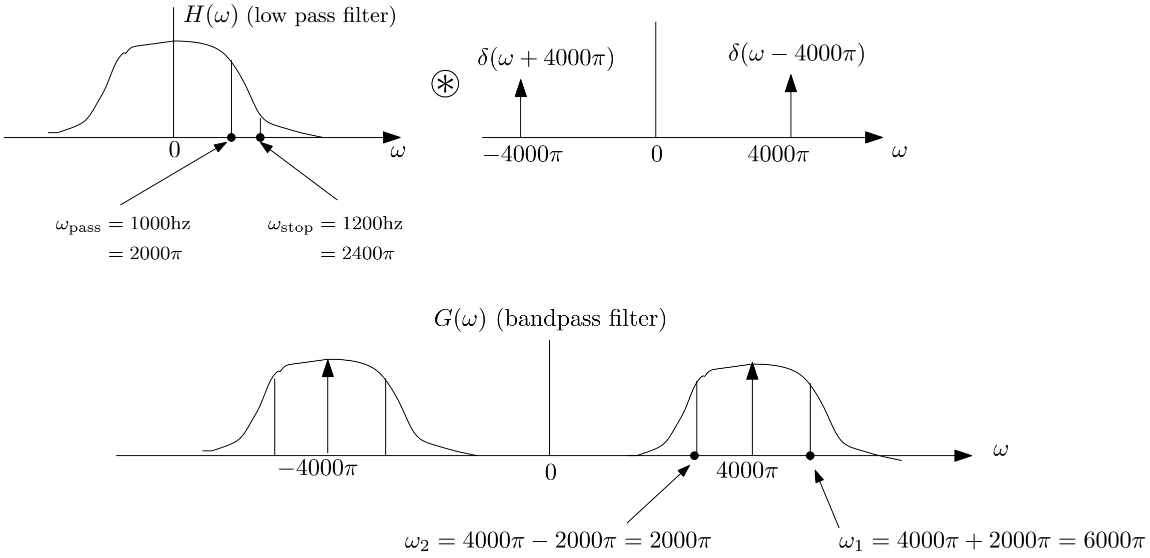


Figure 4.68: Sketch of bandpass filter

Therefore the two bandpass stop frequencies are

$$
\begin{aligned}
\omega_1 &= 6000\pi = 3000 \text{ hz}\\
\omega_2 &= 2000\pi = 1000 \text{ hz}
\end{aligned}
$$

The same on the negative side.

4.6.3.2 Part b

Per instructor, we do not need to account for ripple effect in this problem. Therefore this is the same as part (a).

4.6.4 Problem 6.17

For each of the following second-order difference equations for causal and stable LTI systems, determine whether or not the step response of the system is oscillatory: (a) $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]$ (b) $y[n]-y[n-1]+\frac{1}{4}y[n-2]=x[n]$

solution

4.6.4.1 Part a

Taking the DFT of the difference equation $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]$ gives

$$Y(\Omega)+e^{-j\Omega}Y(\Omega)+\frac{1}{4}e^{-2j\Omega}Y(\Omega)=X(\Omega)$$

A step response means that the input is a step function. Hence $x[n]=u[n]$. Therefore $X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=0}^{\infty}e^{-j\Omega n}=\frac{1}{1-e^{-j\Omega}}$. The above becomes

$$\begin{aligned}Y(\Omega)\left(1+e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}\right)&=\frac{1}{1-e^{-j\Omega}}\\ Y(\Omega)&=\frac{1}{1-e^{-j\Omega}}\frac{1}{1+e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}}\end{aligned}$$

Let $e^{-j\Omega}=x$ for now to make it easier to factor the RHS. The above becomes

$$\begin{aligned}Y(\Omega)&=\frac{1}{1-x}\frac{1}{1+x+\frac{1}{4}x^2}\\ &=\frac{1}{1-x}\frac{4}{4+4x+x^2}\\ &=\frac{4}{(1-x)(x+2)^2}\end{aligned}$$

Using partial fractions on the RHS gives

$$\begin{aligned}\frac{4}{(1-x)(x+2)^2}&=\frac{A}{1-x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\\ &=\frac{A(x+2)^2+B(x+2)(1-x)+C(1-x)}{(1-x)(x+2)^2}\end{aligned}$$

Hence

$$\begin{aligned}4&=A(x+2)^2+B(x+2)(1-x)+C(1-x)\\ &=A\left(x^2+4+4x\right)+B\left(x-x^2+2-2x\right)+C-Cx\\ &=Ax^2+4A+4xA-Bx-Bx^2+2B+C-Cx\\ &=(4A+2B+C)+x(4A-B-C)+x^2(A-B)\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 4A+2B+C &= 4\\ 4A-B-C &= 0\\ A-B &= 0 \end{aligned}$$

Solving gives $A=\frac{4}{9},B=\frac{4}{9},C=\frac{4}{3}$. Hence

$$\begin{aligned} Y(\Omega) &= \frac{4}{(1-x)(x+2)^2}\\ &= \frac{A}{1-x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\\ &= \frac{4}{9}\frac{1}{1-x}+\frac{4}{9}\frac{1}{x+2}+\frac{4}{3}\frac{1}{(x+2)^2} \end{aligned}$$

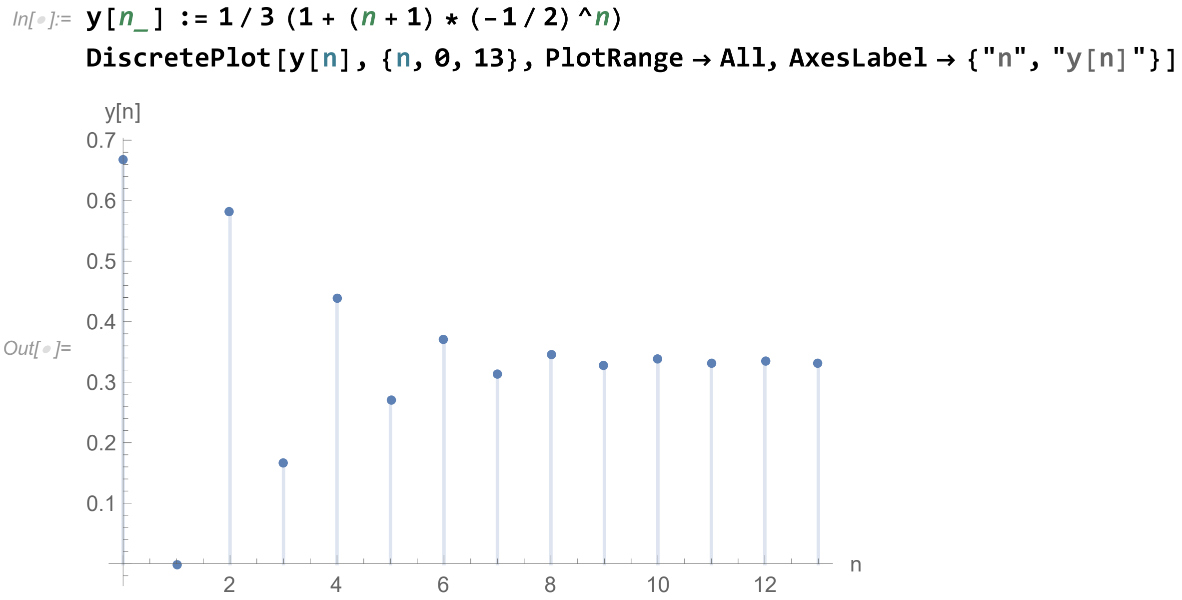
But $x=e^{-j\Omega}$. The above becomes

$$\begin{aligned} Y(\Omega) &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}}+\frac{4}{9}\frac{1}{2+e^{-j\Omega}}+\frac{4}{3}\frac{1}{\left(2+e^{-j\Omega}\right)^2}\\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}}+\frac{4}{9}\frac{1}{2\left(1+\frac{1}{2}e^{-j\Omega}\right)}+\frac{4}{3}\frac{1}{\left(2\left(1+\frac{1}{2}e^{-j\Omega}\right)\right)^2}\\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}}+\frac{4}{18}\frac{1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{4}{3}\frac{1}{4\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}\\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}}+\frac{4}{18}\frac{1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{1}{3}\frac{1}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2} \end{aligned}$$

From tables, using $au[n] \iff \frac{1}{1-ae^{-j\Omega}}$ and $(n+1)a^n u[n] \iff \frac{1}{\left(1-ae^{-j\Omega}\right)^2}$. Applying these to the above gives

$$\begin{aligned} y(n) &= \frac{4}{9}u[n]-\frac{4}{18}\frac{1}{2}u[n]+\frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n u[n]\\ &= \left(\frac{4}{9}-\frac{4}{36}+\frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n\right)u[n]\\ &= \left(\frac{1}{3}+\frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n\right)u[n]\\ &= \frac{1}{3}\left(1+(n+1)\left(-\frac{1}{2}\right)^n\right)u[n] \end{aligned}$$

The following is a plot of $y[n]$



```
In[•]:= y[n_] := 1/3 (1 + (n + 1) * (-1/2)^n)
        DiscretePlot[y[n], {n, 0, 13}, PlotRange → All, AxesLabel → {"n", "y[n]"}]
Out[•]=
```

Figure 4.69: Plot of reponse $y[n]$ to step input

The above shows the response is $\underline{\text{oscillatory}}$.

4.6.4.2 Part b

This is similar to part (a) except for sign difference. Taking the DFT of the difference equation $y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$ gives

$$Y(\Omega) - e^{-j\Omega}Y(\Omega) + \frac{1}{4}e^{-2j\Omega}Y(\Omega) = X(\Omega)$$

A step response means that the input is a step function. Hence $x[n] = u[n]$. Therefore $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-j\Omega n} = \frac{1}{1-e^{-j\Omega}}$. The above becomes

$$\begin{aligned} Y(\Omega)\left(1 - e^{-j\Omega} + \frac{1}{4}e^{-2j\Omega}\right) &= \frac{1}{1-e^{-j\Omega}} \\ Y(\Omega) &= \frac{1}{1-e^{-j\Omega}}\frac{1}{1-e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}} \end{aligned}$$

Let $e^{-j\Omega} = x$ for now to make it easier to factor the RHS. The above becomes

$$\begin{aligned} Y(\Omega) &= \frac{1}{1-x}\frac{1}{1-x+\frac{1}{4}x^2} \\ &= \frac{1}{1-x}\frac{4}{4-4x+x^2} \\ &= \frac{4}{(1-x)(x-2)^2} \end{aligned}$$

Using partial fractions on the RHS gives

$$\begin{aligned}\frac{4}{(1-x)(x-2)^2} &= \frac{A}{1-x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ &= \frac{A(x-2)^2+B(x-2)(1-x)+C(1-x)}{(1-x)(x-2)^2}\end{aligned}$$

Hence

$$\begin{aligned}4 &= A(x-2)^2+B(x-2)(1-x)+C(1-x)\\ &= A\left(x^2+4-4x\right)+B\left(x-x^2-2+2x\right)+C-Cx\\ &= Ax^2+4A-4xA+3Bx-Bx^2-2B+C-Cx\\ &= (4A-2B+C)+x(-4A+3B-C)+x^2(A-B)\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}4A-2B+C&=4\\ -4A+3B-C&=0\\ A-B&=0\end{aligned}$$

Solving gives $A=4,B=4,C=-4$. Hence

$$\begin{aligned}Y(\Omega) &= \frac{4}{(1-x)(x-2)^2}\\ &= \frac{A}{1-x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ &= 4\frac{1}{1-x}+4\frac{1}{x-2}-4\frac{1}{(x-2)^2}\end{aligned}$$

But $x=e^{-j\Omega}$. The above becomes

$$\begin{aligned}Y(\Omega) &= 4\frac{1}{1-e^{-j\Omega}}+4\frac{1}{e^{-j\Omega}-2}-4\frac{1}{\left(e^{-j\Omega}-2\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}}+4\frac{1}{-2\left(1-\frac{1}{2}e^{-j\Omega}\right)}-4\frac{1}{\left(-2\left(1-\frac{1}{2}e^{-j\Omega}\right)\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}}-2\frac{1}{1-\frac{1}{2}e^{-j\Omega}}-4\frac{1}{4\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}}-2\frac{1}{1-\frac{1}{2}e^{-j\Omega}}-\frac{1}{\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}\end{aligned}$$

From tables, using $au[n] \Longleftrightarrow \frac{1}{1-ae^{-j\Omega}}$ and $(n+1)a^n u[n] \Longleftrightarrow \frac{1}{\left(1-ae^{-j\Omega}\right)^2}$. Applying these to the above gives

$$\begin{aligned} y(n) &= 4u[n] - 2\frac{1}{2}u[n] - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n u[n] \\ &= \left(4 - 1 - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n\right)u[n] \\ &= \left(3 - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n\right)u[n] \end{aligned}$$

The following is a plot of $y[n]$

```
In[∘]:= y[n_] := (3 - 1 / 3 (n + 1) * (1 / 2) ^n)
        DiscretePlot[y[n], {n, 0, 40}, PlotRange → All, AxesLabel → {"n", "y[n]"}]
```

Out[∘]=

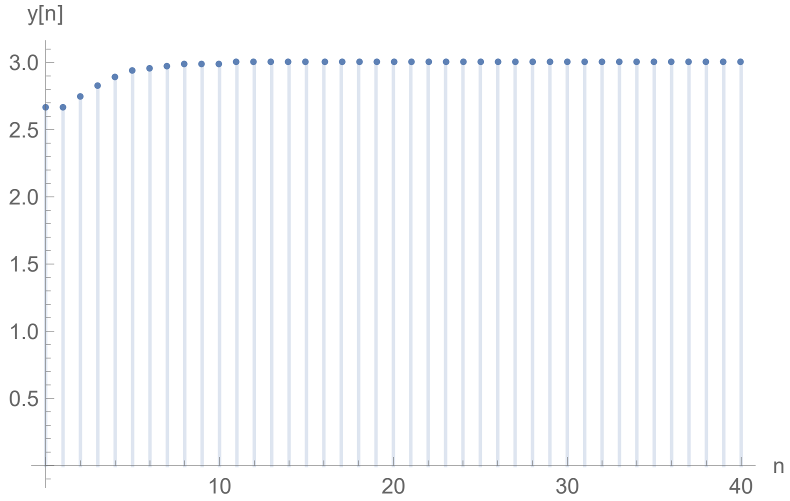


Figure 4.70: Plot of reponse $y[n]$ to step input

The above shows the response is <u>not oscillatory</u>. The reason is that sign difference in $y[n-1]$ term in the difference equation. $y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$.

4.6.5 Problem 6.22

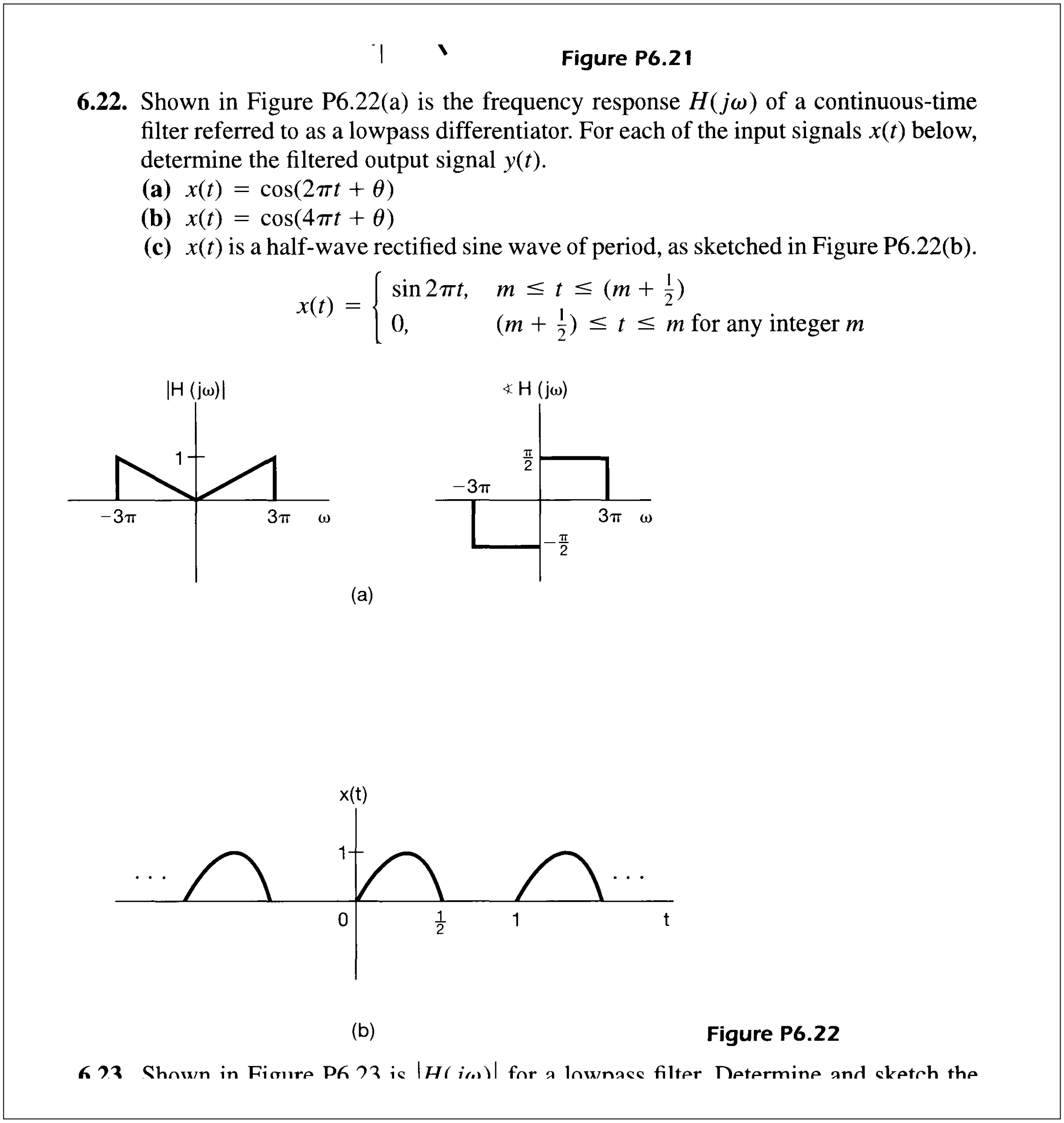


Figure P6.21

6.22. Shown in Figure P6.22(a) is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals $x(t)$ below, determine the filtered output signal $y(t)$.

(a) $x(t) = \cos(2\pi t + \theta)$

(b) $x(t) = \cos(4\pi t + \theta)$

(c) $x(t)$ is a half-wave rectified sine wave of period, as sketched in Figure P6.22(b).

$$x(t) = \begin{cases} \sin 2\pi t, & m \le t \le (m + \frac{1}{2}) \\ 0, & (m + \frac{1}{2}) \le t \le m \text{ for any integer } m \end{cases}$$

(b) **Figure P6.22**

6.23. Shown in Figure P6.23 is $|H(j\omega)|$ for a lowpass filter. Determine and sketch the

Figure 4.71: Problem description

solution

4.6.5.1 Part a

The relation between the input and output is given by

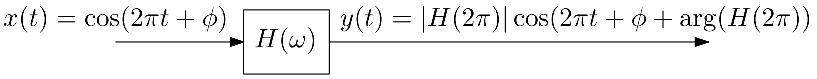


Figure 4.72: Output of LTI when input is sinuosidal

Since $|H(\omega)| = \frac{1}{3\pi}\omega$ then $|H(2\pi)| = \frac{2}{3}$ and from the phase diagram $\arg(H(2\pi)) = \frac{\pi}{2}$. Therefore

$$\begin{aligned} y(t) &= |H(2\pi)| \cos(2\pi t + \theta + \arg(H((2\pi)))) \\ &= \frac{2}{3}\cos\left(2\pi t + \theta + \frac{\pi}{2}\right) \end{aligned}$$

But $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$, hence the above can be simplified to

$$y(t) = -\frac{2}{3}\sin(2\pi t + \theta)$$

4.6.5.2 Part b

Since $|H(\omega)| = 0$ for $\omega = 4\pi$, then $y(t) = 0$

4.6.5.3 Part c

$$X(\omega) = 2\pi \sum a_k \delta(\omega - k\omega_0)$$

Looking at $x(t)$ shows that its period is $T_0 = 1$. Hence $\omega_0 = 2\pi$. The above becomes

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k2\pi)$$

Hence

$$Y(\omega) = X(\omega) H(\omega)$$

But $|H(\omega)| = 0$ outside $|\omega| = 3\pi$. Then only $k = 0, k = -1, k = +1$ will go through the filter. Hence

$$\begin{aligned} Y(\omega) &= \left(2\pi \sum_{k=-1}^{1} a_k \delta(\omega - k2\pi)\right) H(\omega) \\ &= (2\pi(a_0\delta(\omega) + a_{-1}\delta(\omega + 2\pi) + a_1\delta(\omega - 2\pi))) H(\omega) \qquad (1) \end{aligned}$$

To find a_0, a_{-1}, a_1. From

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} dt \\ &= \int_0^1 x(t)\, e^{-jk2\pi t} dt \end{aligned}$$

Looking at $x(t)$ shows that its period is $T_0 = 1$. Hence $\omega_0 = 2\pi$. From above

$$\begin{aligned} a_0 &= \int_0^1 x(t)\, dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, dt \\ &= -\left[\frac{\cos(2\pi t)}{2\pi}\right]_0^{\frac{1}{2}} \\ &= -\frac{1}{2\pi}(\cos\pi - 1) \\ &= \frac{1}{\pi} \end{aligned}$$

And

$$\begin{aligned} a_{-1} &= \int_0^1 x(t)\, e^{j2\pi t} dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, e^{j2\pi t} dt \\ &= \frac{j}{4} \end{aligned}$$

And

$$\begin{aligned} a_1 &= \int_0^1 x(t)\, e^{-j2\pi t} dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, e^{-j2\pi t} dt \\ &= \frac{-j}{4} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} Y(\omega) &= \left(2\pi\left(\frac{1}{\pi}\delta(\omega) + \frac{j}{4}\delta(\omega + 2\pi) - \frac{j}{4}\delta(\omega - 2\pi)\right)\right) H(\omega) \\ &= \left(2\pi\left(\frac{1}{\pi}\delta(\omega) + \frac{j}{4}\delta(\omega + 2\pi) - \frac{j}{4}\delta(\omega - 2\pi)\right)\right) |H(\omega)|\, e^{j\arg H(\omega)} \end{aligned}$$

At $\omega = 0, Y(\omega) = 0$ since $|H(\omega)| = 0$ at $\omega = 0$. And at $\omega = 2\pi$,

$$\begin{aligned}
Y(\omega) &= \left(2\pi\left(-\frac{j}{4}\right)\right)|H(2\pi)|\, e^{j\arg H(2\pi)} \\
&= \left(2\pi\left(-\frac{j}{4}\right)\right)\frac{2}{3}e^{j\frac{\pi}{2}} \\
&= -j\pi\frac{1}{3}e^{j\frac{\pi}{2}} \\
&= -j\pi\frac{1}{3}j \\
&= \frac{1}{3}\pi
\end{aligned}$$

And at $\omega = -2\pi$,

$$\begin{aligned}
Y(\omega) &= \left(2\pi\left(\frac{j}{4}\right)\right)|H(-2\pi)|\, e^{j\arg H(2\pi)} \\
&= \left(2\pi\left(\frac{j}{4}\right)\right)\frac{2}{3}e^{-j\frac{\pi}{2}} \\
&= j\pi\frac{1}{3}e^{-j\frac{\pi}{2}} \\
&= j\pi\frac{1}{3}(-j) \\
&= \frac{1}{3}\pi
\end{aligned}$$

Hence the spectrum of $Y(\omega)$ is

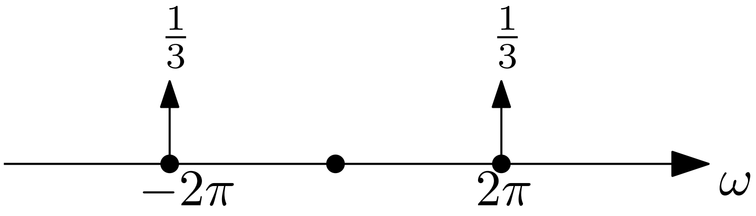


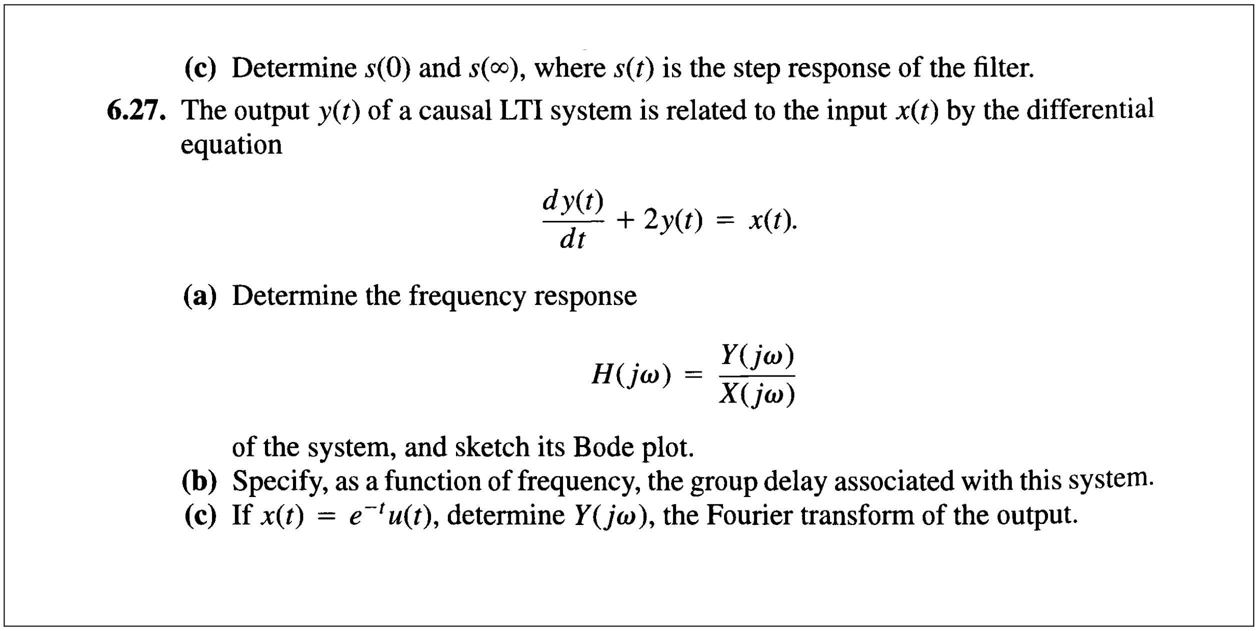
Figure 4.73: $Y(\omega)$

But the above is the Fourier transform of

$$y(t) = \frac{1}{3}\cos(2\pi t)$$

Which is therefore the output of the filter.

4.6.6 Problem 6.27 (a,b,c,d)



(c) Determine $s(0)$ and $s(\infty)$, where $s(t)$ is the step response of the filter.

6.27. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Determine the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

of the system, and sketch its Bode plot.

(b) Specify, as a function of frequency, the group delay associated with this system.

(c) If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output.

Figure 4.74: Problem description

solution

4.6.6.1 Part a

$$y'(t) + 2y(t) = x(t)$$

Taking the Fourier transform gives

$$\begin{aligned} j\omega Y(\omega) + 2Y(\omega) &= X(\omega) \\ Y(\omega)\left(2 + j\omega\right) &= X(\omega) \end{aligned}$$

Hence

$$\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\ &= \frac{1}{2 + j\omega} \end{aligned}$$

Using Matlab, the following is the Bode plot

```
clear all;
s   = tf('s');
sys = 1 / (2+s);
bode(sys);
grid
```

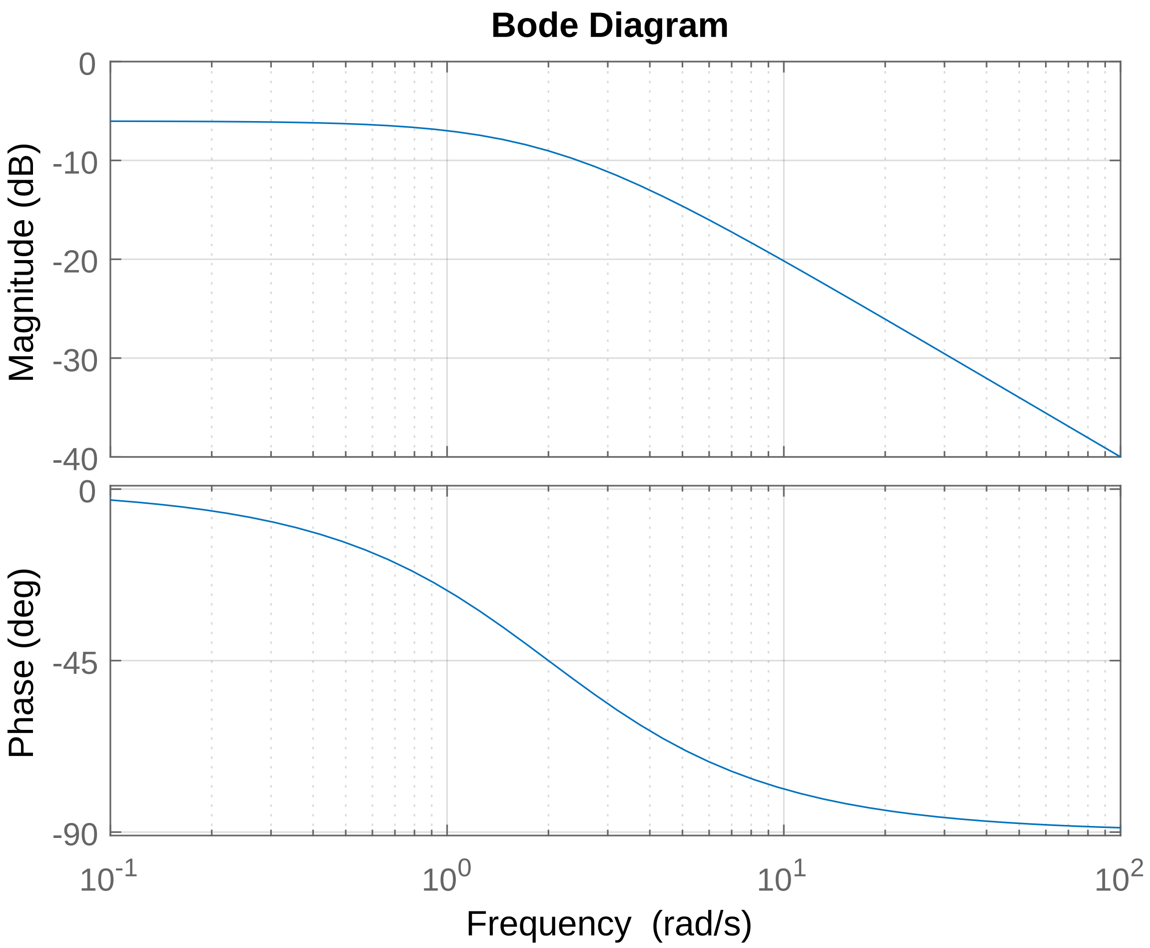


Figure 4.75: Bode plot

4.6.6.2 Part b

In class, it was mentioned that group delay is given by derivative of the Phase of the Fourier transform. Since $H(\omega)=\frac{1}{2+j\omega}$, then

$$\arg(H(\omega)) = -\arctan\left(\frac{\omega}{2}\right)$$

Hence, using the rule $\frac{d}{dx}\arctan(ax)=\frac{a}{1+a^2x^2}$, then using $a=\frac{1}{2}$ the derivative of the above becomes

$$\begin{aligned}\frac{d}{d\omega}\arg(H(\omega)) &= -\frac{\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2\omega^2}\\ &= -\frac{2}{4+\omega^2}\end{aligned}$$

4.6.6.3 Part c

Since $x(t)=e^{-t}u(t)$ then now

$$Y(\omega)=X(\omega)H(\omega) \tag{1}$$

But

$$\begin{aligned}X(\omega)&=\int_0^\infty x(t)e^{-j\omega t}dt\\&=\int_0^\infty e^{-t}e^{-j\omega t}dt\\&=\int_0^\infty e^{-t\left(1+j\omega\right)}dt\\&=\left[\frac{e^{-t\left(1+j\omega\right)}}{-\left(1+j\omega\right)}\right]_0^\infty\\&=\frac{1}{-\left(1+j\omega\right)}\left[e^{-t\left(1+j\omega\right)}\right]_0^\infty\\&=\frac{1}{-\left(1+j\omega\right)}(0-1)\\&=\frac{1}{1+j\omega}\end{aligned}$$

Assuming $\operatorname{Re}(\omega)>1$. Hence (1) becomes

$$Y(\omega)=\left(\frac{1}{1+j\omega}\right)\left(\frac{1}{2+j\omega}\right)$$

4.6.6.4 Part d

To find $y(t)$

$$\left(\frac{1}{1+j\omega}\right)\left(\frac{1}{2+j\omega}\right)=\frac{A}{1+j\omega}+\frac{B}{2+j\omega}$$

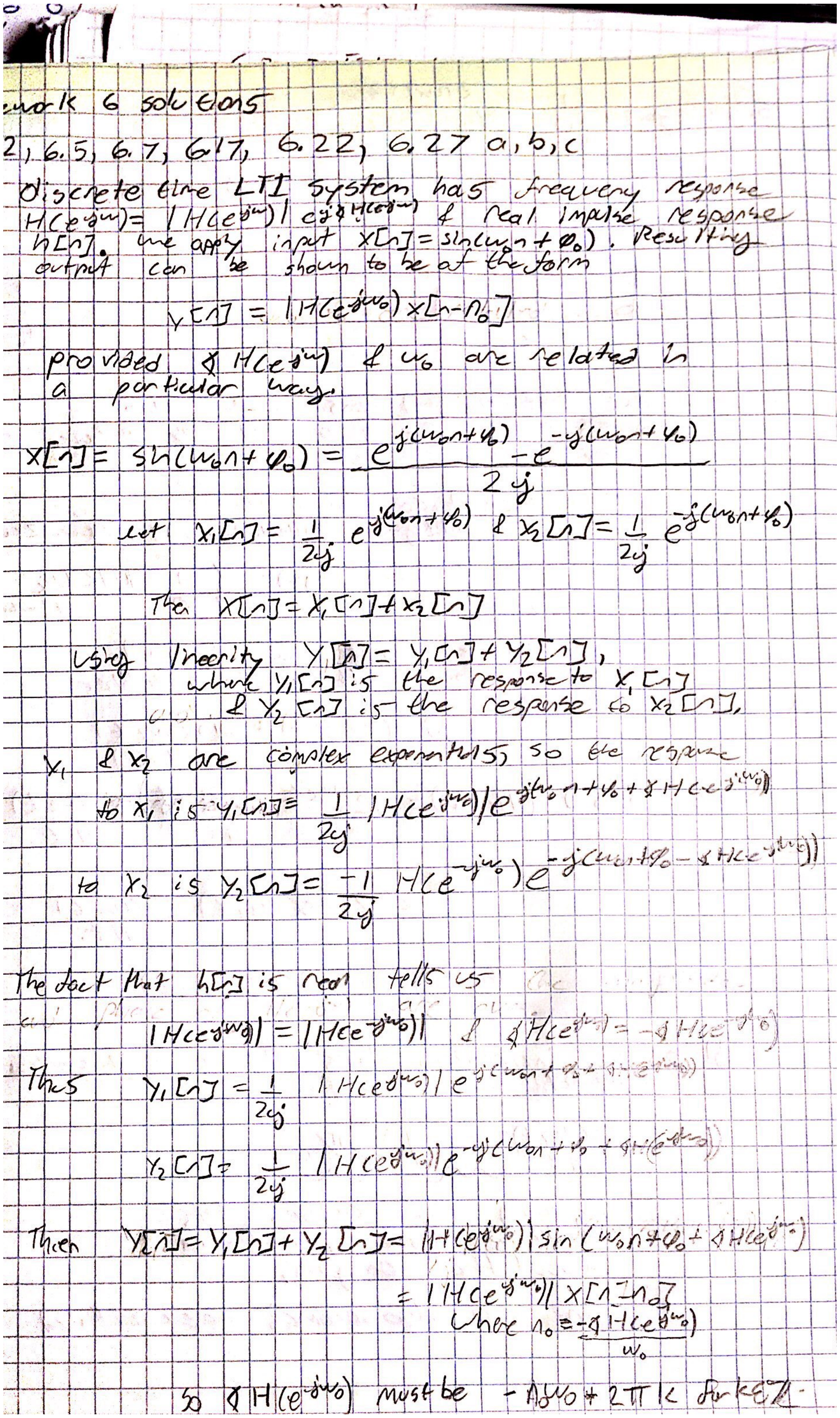
Hence $A=\left(\frac{1}{2+j\omega}\right)_{\omega=j}=1$ and $B=\left(\frac{1}{1+j\omega}\right)_{\omega=2j}=-1$. Therefore

$$Y(\omega)=\frac{1}{1+j\omega}-\frac{1}{2+j\omega}$$

From tables

$$y(t)=\left(e^{-t}-e^{-2t}\right)u(t)$$

4.6.7 key solution



work 6 solutions

2, 6.5, 6.7, 6.17, 6.22, 6.27 a, b, c

Discrete time LTI system has frequency response $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\sphericalangle H(e^{j\omega})}$ & real impulse response $h[n]$. We apply input $x[n] = \sin(\omega_0 n + \phi_0)$. Resulting output can be shown to be of the form

$$y[n] = |H(e^{j\omega_0}) x[n-n_0]$$

provided $\sphericalangle H(e^{j\omega})$ & ω_0 are related in a particular way.

$$x[n] = \sin(\omega_0 n + \phi_0) = \frac{e^{j(\omega_0 n + \phi_0)} - e^{-j(\omega_0 n + \phi_0)}}{2j}$$

let $x_1[n] = \frac{1}{2j} e^{j(\omega_0 n + \phi_0)}$ & $x_2[n] = \frac{1}{2j} e^{-j(\omega_0 n + \phi_0)}$

Then $x[n] = x_1[n] + x_2[n]$

Using linearity $y[n] = y_1[n] + y_2[n]$,
where $y_1[n]$ is the response to $x_1[n]$
& $y_2[n]$ is the response to $x_2[n]$,

x_1 & x_2 are complex exponentials, so the response

to x_1 is $y_1[n] = \frac{1}{2j} |H(e^{j\omega_0})| e^{j(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0}))}$

to x_2 is $y_2[n] = \frac{-1}{2j} H(e^{-j\omega_0}) e^{-j(\omega_0 n + \phi_0 - \sphericalangle H(e^{-j\omega_0}))}$

The fact that $h[n]$ is real tells us

$$|H(e^{j\omega_0})| = |H(e^{-j\omega_0})| \quad \& \quad \sphericalangle H(e^{j\omega_0}) = -\sphericalangle H(e^{-j\omega_0})$$

Thus $y_1[n] = \frac{1}{2j} |H(e^{j\omega_0})| e^{j(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0}))}$

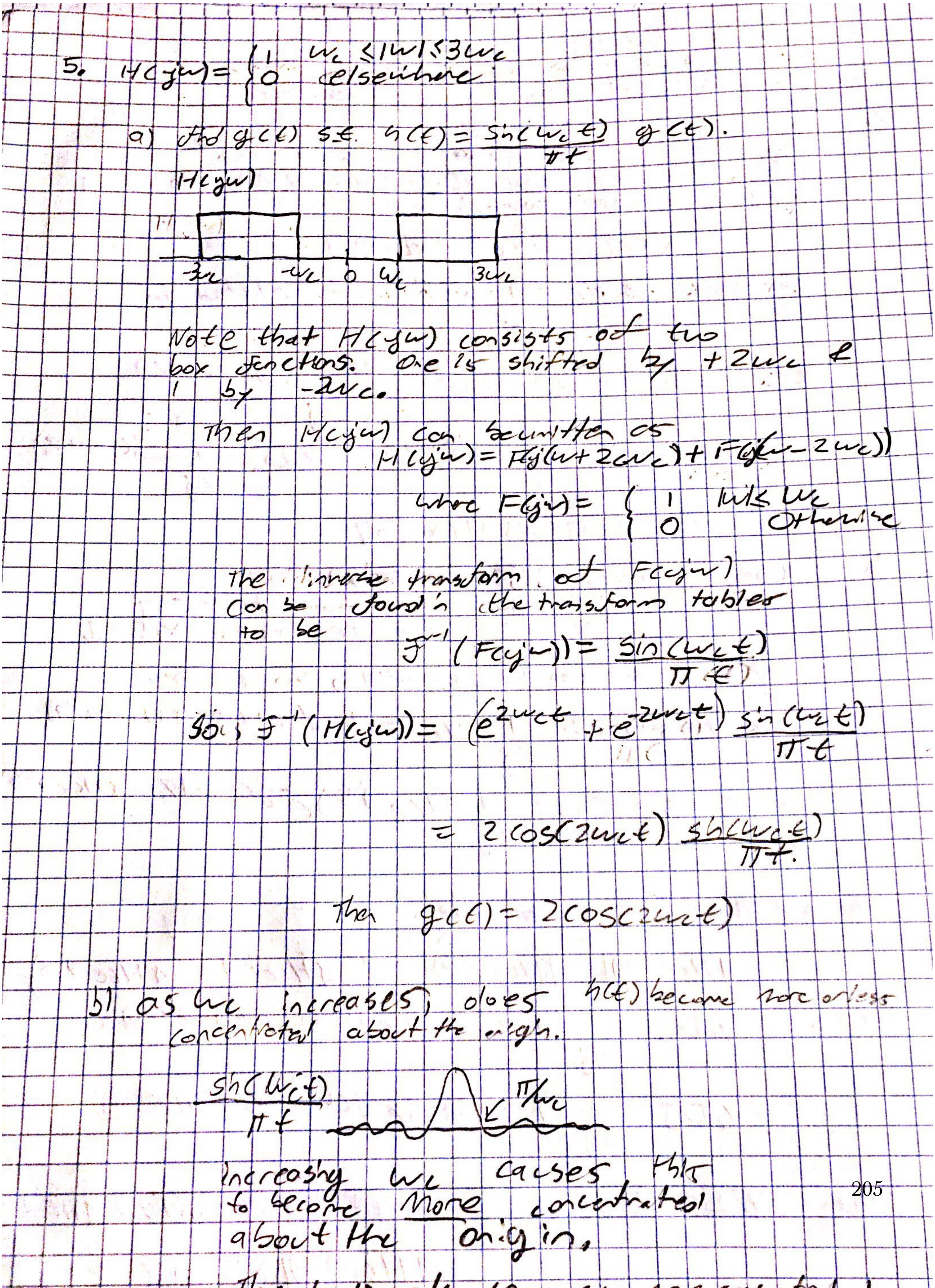
$y_2[n] = \frac{1}{2j} |H(e^{j\omega_0})| e^{-j(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0}))}$

Then $y[n] = y_1[n] + y_2[n] = |H(e^{j\omega_0})| \sin(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0}))$

$= |H(e^{j\omega_0})| x[n - n_0]$

where $n_0 = \frac{-\sphericalangle H(e^{j\omega_0})}{\omega_0}$

So $\sphericalangle H(e^{j\omega_0})$ must be $-n_0\omega_0 + 2\pi k$ for $k \in \mathbb{Z}$



5. $H(j\omega) = \begin{cases} 1 & \omega_c \leq |\omega| \leq 3\omega_c \\ 0 & \text{elsewhere} \end{cases}$

a) Find $g(t)$ s.t. $h(t) = \frac{\sin(\omega_c t)}{\pi t}\, g(t)$.

Note that $H(j\omega)$ consists of two box functions. One is shifted by $+2\omega_c$ & 1 by $-2\omega_c$.

Then $H(j\omega)$ can be written as

$$H(j\omega) = F(j(\omega + 2\omega_c)) + F(j(\omega - 2\omega_c))$$

where $F(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{Otherwise} \end{cases}$

The inverse transform of $F(j\omega)$ can be found in the transform tables to be

$$\mathcal{F}^{-1}(F(j\omega)) = \frac{\sin(\omega_c t)}{\pi (t)}$$

So, $\mathcal{F}^{-1}(H(j\omega)) = \left(e^{2\omega_c t} + e^{-2\omega_c t}\right) \frac{\sin(\omega_c t)}{\pi t}$

$$= 2\cos(2\omega_c t)\, \frac{\sin(\omega_c t)}{\pi t}.$$

Then $g(t) = 2\cos(2\omega_c t)$

b) as ω_c increases, does $h(t)$ become more or less concentrated about the origin.

Increasing ω_c causes this to become More concentrated about the origin.

.17 LPF has passband at 1000Hz & stopband at 1200Hz w/ passband ripple of 0.1 & stopband ripple of 0.05.

The impulse response of this filter is denoted $h(t)$.

we want a filter with impulse response

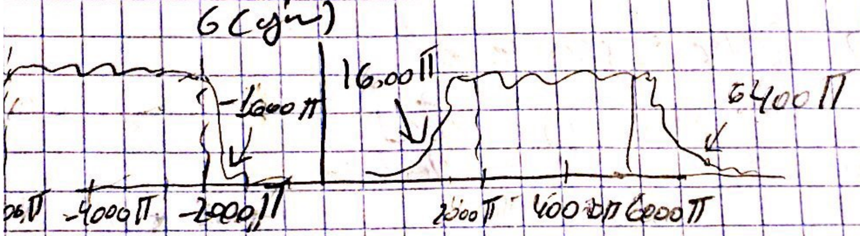
$g(t)=2h(t)\cos(4000\pi t)$

Note that the $\mathcal{F}\{\cos(4000\pi t)\} = \pi[\delta(\omega-4000\pi)+\delta(\omega-4000\pi)]$

Then $G(j\omega) = 2H(j\omega) * [\pi\delta(\omega-4000\pi)+\pi\delta(\omega+4000\pi)]$

$= 2\pi[H(j(\omega-4000\pi))+H(j(\omega+4000\pi))]$

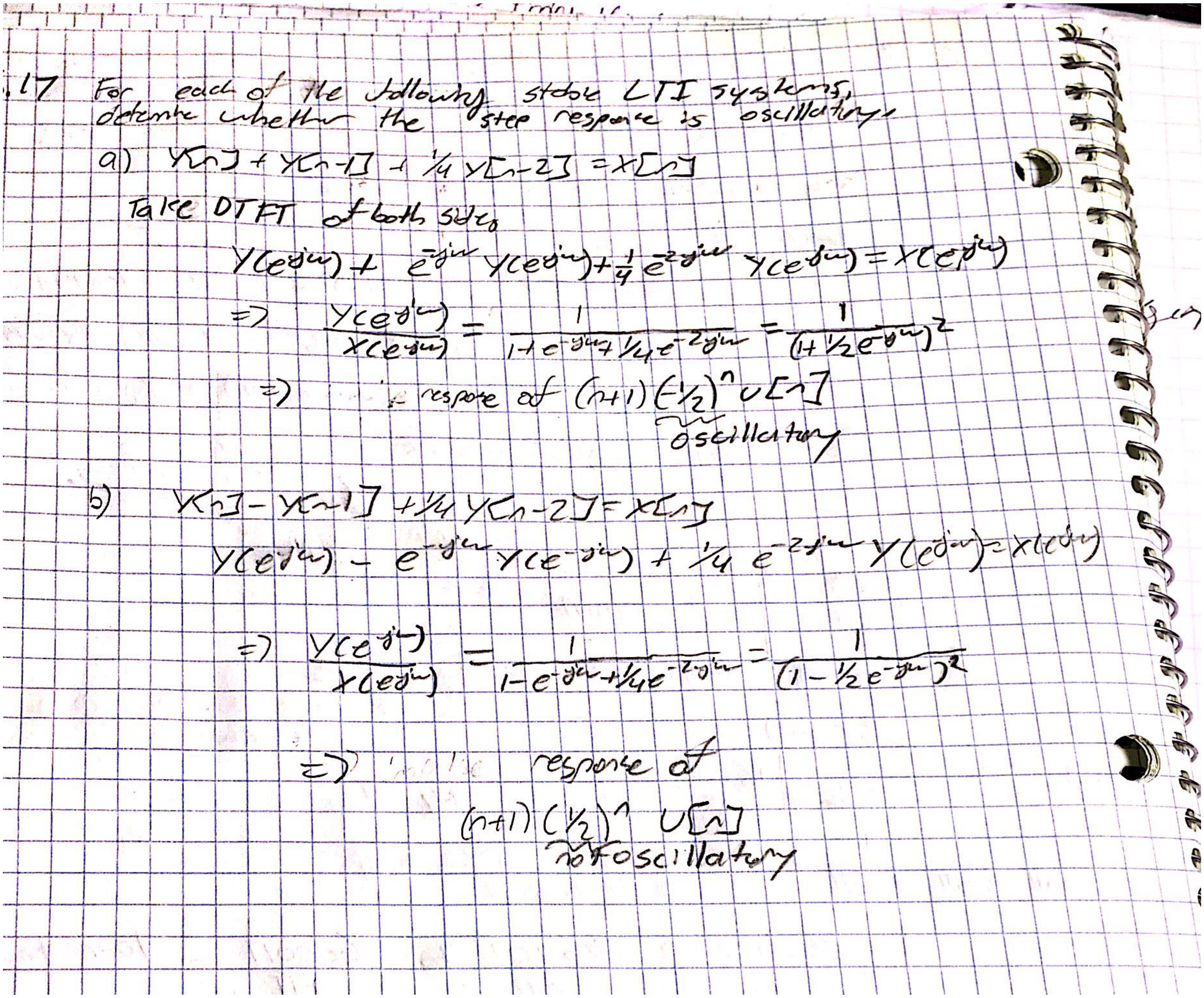
So $G(j\omega)$ just consists of 2 $H(j\omega)$'s, scaled & shifted in opposite directions.



⇒ passband is $\frac{2000\pi}{2\pi}$ to $\frac{6000\pi}{2\pi}$ = 1000 to 3000 H

Stopband is $\frac{1600\pi}{2\pi} = 800$ Hz

or $\frac{6400}{2\pi} = 3200$ Hz.



.17 For each of the following stable LTI systems, determine whether the step response is oscillatory.

a) $y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n]$

Take DTFT of both sides

$$Y(e^{j\omega}) + e^{-j\omega} Y(e^{j\omega}) + \frac{1}{4} e^{-2j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+e^{-j\omega}+\frac{1}{4}e^{-2j\omega}} = \frac{1}{(1+\frac{1}{2}e^{-j\omega})^2}$$

$$\Rightarrow \text{response of } (n+1)\underbrace{(-\tfrac{1}{2})^n}_{\text{oscillatory}} u[n]$$

b) $y[n] - y[n-1] + \frac{1}{4} y[n-2] = x[n]$

$$Y(e^{j\omega}) - e^{-j\omega} Y(e^{-j\omega}) + \frac{1}{4} e^{-2j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

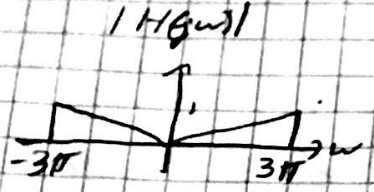
$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-e^{-j\omega}+\frac{1}{4}e^{-2j\omega}} = \frac{1}{(1-\frac{1}{2}e^{-j\omega})^2}$$

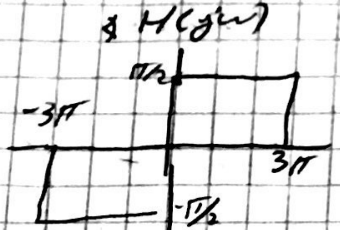
$\Rightarrow$ response of

$$(n+1)(\tfrac{1}{2})^n u[n]$$

not oscillatory

22.





determine filtered output

a) $x(t) = \cos(2\pi t + \theta)$

$$y(t) = |H(j\omega)|\Big|_{\omega=2\pi} \cos(2\pi t + \theta + \measuredangle H(j\omega)\Big|_{\omega=2\pi})$$

$$\Rightarrow y(t) = \frac{2}{3} \cos(2\pi t + \theta + \pi/2)$$

b) $x(t) = \cos(4\pi t + \theta)$

$$y(t) = |H(j\omega)|\Big|_{\omega=4\pi} \cos(2\pi t + \theta + \measuredangle H(j\omega)\Big|_{\omega=4\pi})$$

$$= 0.$$

c) $$x(t) = \begin{cases} \sin(2\pi t) & n < t \leq n + 1/2 \\ 0 & n+1/2 \leq t \leq n \quad \text{for } n \in \mathbb{Z} \end{cases}$$

$$y(t) = \begin{cases} |H(j\omega)|_{\omega=2\pi} \sin(2\pi t + \measuredangle H(j\omega))\Big|_{\omega=2\pi} \\ 0 \end{cases}$$

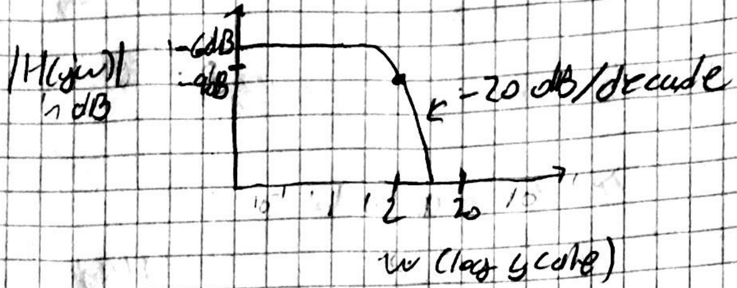
$$= \begin{cases} 2/3 \sin(2\pi t + \pi/2) & n \leq t \leq n+1/2 \\ 0 & n+1/2 \leq t \leq n \quad n \in \mathbb{Z}. \end{cases}$$

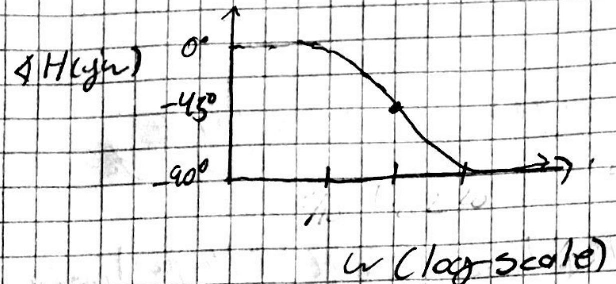
27. $\frac{dy(t)}{dt} + 2y(t) = x(t)$

Take the fourier transform at both sides to get

$$j\omega Y(j\omega) + 2Y(j\omega) = X(j\omega)$$

a) $\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega+2}$





b) group delay $= -\frac{d}{d\omega}(\measuredangle H(j\omega)) = -\frac{d}{d\omega}\left(-\tan^{-1}\left(\frac{\omega}{2}\right)\right)$

$$= -\frac{1}{2} \cdot \frac{-1}{1+(\omega/2)^2}$$

$$= \frac{1}{2+\omega^2/2} = \frac{2}{4+\omega^2}$$

c) $x(t) = e^{-t}u(t)$, $X(j\omega) = \mathcal{F}(x(t)) = \frac{1}{1+j\omega}$

$\Rightarrow Y(j\omega) = \frac{1}{(j\omega+2)(j\omega+1)}$

4.7 HW 7

Local contents

4.7.1 Problem 7.2

A continuous-time signal $x(t)$ is obtained at the output of an ideal lowpass filter with cutoff frequency $\omega_c = 1000\pi$ rad/sec. If impulse-train sampling is performed on $x(t)$, which of the following sampling periods would guarantee that $x(t)$ can be recovered from its sampled version using an appropriate lowpass filter? (a) $T = 0.5 \times 10^{-3}$ sec. (b) $T = 2 \times 10^{-3}$ sec (c) $T = 10^{-4}$ sec

solution

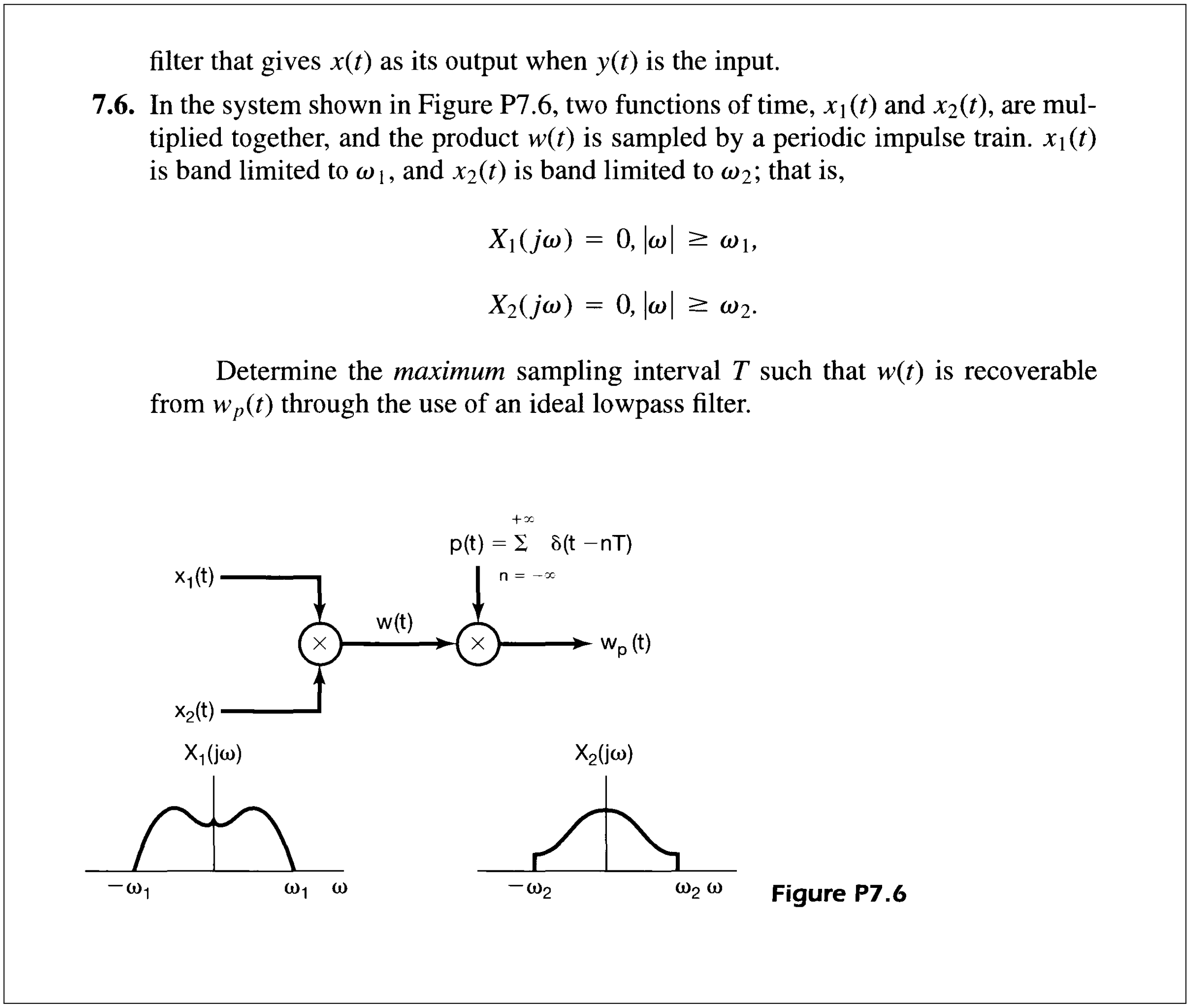
Note: In all these problems, I will use Ω for the digital frequency and ω for the continuous frequency.

We want the Nyquist frequency to be larger than twice ω_c. Hence Nyquist frequency should be larger than 2000π rad/sec or larger than 1000 Hz.

Translating the given periods to hertz using $f = \frac{1}{T}$ relation, shows that (a) is $\frac{1}{0.5\times 10^{-3}} = 2000$ Hz, (b) is $\frac{1}{2\times 10^{-3}} = 500$ Hz, (c) is $\frac{1}{10^{-4}} = 10000$ Hz.

Therefore (a) and (c) would guarantee that $x(t)$ can be recovered.

4.7.2 Problem 7.6



filter that gives $x(t)$ as its output when $y(t)$ is the input.

7.6. In the system shown in Figure P7.6, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1,$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2.$$

Determine the *maximum* sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

Figure P7.6

Figure 4.76: Problem description

solution

The multiplication of $x_1(t) \times x_2(t)$ becomes convolution in frequency domain $X_1(j\omega) \circledast X_2(j\omega)$. But we know when doing convolution the width of the result is the sum of each function width. This means the frequency spectrum of $w(t)$ will have width of $\omega_1 + \omega_2$.

Now by Nyquist theory, we know that the sampling frequency should be at least twice the largest frequency in the signal being sampled. This means

$$\omega_{sampling} > 2(\omega_1 + \omega_2)$$

Since $\omega_1 + \omega_2$ is now the largest frequency present in $w(t)$. But $\omega_{sampling} = \frac{2\pi}{T_{sampling}}$. Hence the above becomes

$$\frac{2\pi}{T_{sampling}} > 2(\omega_1 + \omega_2)$$
$$\frac{1}{T_{sampling}} > \frac{\omega_1 + \omega_2}{\pi}$$

Or

$$T_{sampling} < \frac{\pi}{\omega_1 + \omega_2}$$

This means the maximum possible sampling period is

$$T_{\max} = \frac{\pi}{\omega_1 + \omega_2}$$

In seconds.

4.7.3 Problem 7.11

Let $X_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(\omega) = 0$ for $|\omega| \geq 2000\pi$. A discrete-time signal

$$x_d[n] = x_n\left(n\left(0.5 \times 10^{-3}\right)\right)$$

Is obtained. For each of the following constraints on the Fourier transform $X_d(\Omega)$ of $x_d[n]$ determine the corresponding constraint on $X_c(\omega)$.

a $X_d(\Omega)$ is real

b The maximum value of $X_d(\Omega)$ over all Ω is 1.

c $X_d(\Omega) = 0$ for $\frac{3\pi}{4} \leq |\Omega| \leq \pi$

d $X_d(\Omega) = X_d(\Omega - \pi)$

solution

The main relation to translate between continuous time frequency ω (radians per second) and digital frequency Ω (radians per sample) which is used in all of these parts is the following

$$\Omega = \omega T$$

Where T is the sampling period (in seconds per sample). i.e. number of seconds to obtain one sample.

4.7.3.1 Part a

Since $X_d(\Omega)$ is the same as $X_c(\omega)$ (except for scaling factor) which contains replicated copies of $X_c(\omega)$ spaced at sampling frequencies intervals, then if $X_d(\Omega)$ is real, then this means $X_c(\omega)$ must also be real, since $X_d(\Omega)$ is just copies of $X_c(\omega)$.

4.7.3.2 Part b

If maximum value of $X_d(\Omega)$ is A then maximum value of $X_c(\omega)$ is given by AT where T is sampling period. Hence since $A = 1$ in this problem, then the maximum value of $X_c(\omega)$ will be 0.5×10^{-3}.

4.7.3.3 Part c

$X_d(\Omega) = 0$ for $\frac{3\pi}{4} \leq |\Omega| \leq \pi$ is translated to $X_c(\omega) = 0$ for $\frac{3\pi}{4} \leq |\omega T| \leq \pi$ since $\Omega = \omega T$. Therefore

$$\frac{3\pi}{4}\frac{1}{T} \leq |\omega| \leq \frac{\pi}{T}$$

But $T = 0.5 \times 10^{-3}$, hence the above becomes

$$\begin{aligned} \frac{3\pi}{4}(2000) &\leq |\omega| \leq \pi(2000) \\ 1500\pi &\leq |\omega| \leq 2000\pi \end{aligned}$$

Hence $X_c(\omega) = 0$ for $1500\pi \leq |\omega| \leq 2000\pi$. Actually, since $X_c(\omega) = 0$ for $|\omega| \geq 2000\pi$ from the problem statement, this can be simplified to

$$X_c(\omega) = 0 \qquad |\omega| \geq 500\pi$$

4.7.3.4 Part d

$X_d(\Omega) = X_d(\Omega - \pi)$ is translated to $X_c(\omega) = X_c\left(\omega - \frac{\pi}{T}\right) = X_c\left(\omega - \frac{\pi}{0.5\times10^{-3}}\right) = X_c(\omega - 2000\pi)$

Therefore

$$X_c(\omega) = X_c(\omega - 2000\pi)$$

4.7.4 Problem 7.15

Impulse-train sampling of $x[n]$ is used to obtain

$$g[n] = \sum_{k=-\infty}^{\infty} x[n]\,\delta[n - kN]$$

If $X(\Omega) = 0$ for $\frac{3\pi}{7} \leq |\Omega| \leq \pi$, determine the largest value for the sampling interval N which ensures that no aliasing takes place while sampling $x[n]$.

solution

This is similar to problem 7.6 above, but using digital frequency. By Nyquist theory, the sampling frequency must be larger than twice the largest frequency in the signal. We are given that $\frac{3\pi}{7} \leq |\Omega| \leq \pi$. Hence the largest frequency is $\frac{3\pi}{7}$. Hence,

$$\Omega_{sampling} > 2\left(\frac{3\pi}{7}\right) = \frac{6}{7}\pi$$

Therefore

$$\frac{2\pi}{N_{sampling}} > \frac{6}{7}\pi$$

Where $N_{sampling}$ is the discrete sampling period, which is number of samples. Therefore from above

$$\frac{1}{N_{sampling}} > \frac{3}{7}$$

$$N_{sampling} < \frac{7}{3}$$

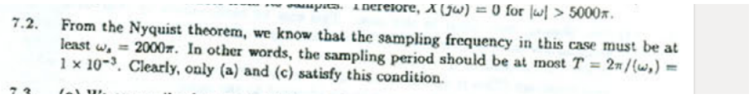
But $N_{sampling}$ must be an integer (since it is number of samples, hence

$$N_{sampling} < 2$$

Therefore the maximum is

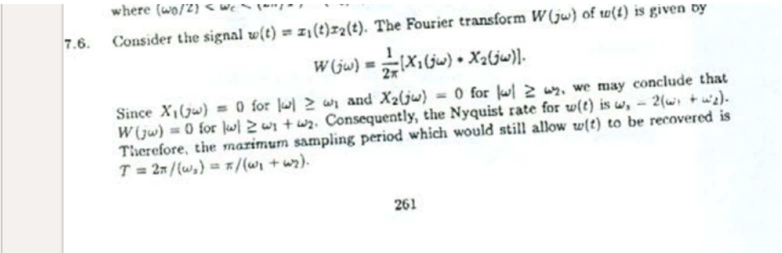
$$N = 2$$

4.7.5 key solution



Therefore, $X(j\omega) = 0$ for $|\omega| > 5000\pi$.

7.2. From the Nyquist theorem, we know that the sampling frequency in this case must be at least $\omega_s = 2000\pi$. In other words, the sampling period should be at most $T = 2\pi/(\omega_s) = 1 \times 10^{-3}$. Clearly, only (a) and (c) satisfy this condition.



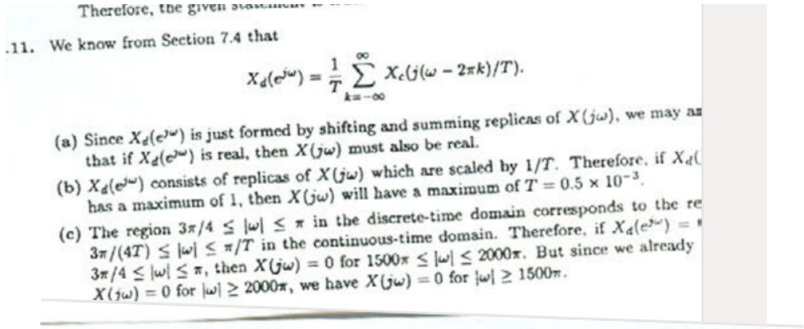
where $(\omega_0/2) < \omega_c$

7.6. Consider the signal $w(t) = x_1(t)x_2(t)$. The Fourier transform $W(j\omega)$ of $w(t)$ is given by

$$W(j\omega) = \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)].$$

Since $X_1(j\omega) = 0$ for $|\omega| \geq \omega_1$ and $X_2(j\omega) = 0$ for $|\omega| \geq \omega_2$, we may conclude that $W(j\omega) = 0$ for $|\omega| \geq \omega_1 + \omega_2$. Consequently, the Nyquist rate for $w(t)$ is $\omega_s = 2(\omega_1 + \omega_2)$. Therefore, the *maximum* sampling period which would still allow $w(t)$ to be recovered is $T = 2\pi/(\omega_s) = \pi/(\omega_1 + \omega_2)$.

261



Therefore, the given

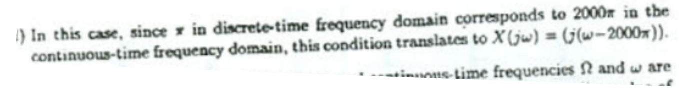
.11. We know from Section 7.4 that

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - 2\pi k)/T).$$

(a) Since $X_d(e^{j\omega})$ is just formed by shifting and summing replicas of $X(j\omega)$, we may as that if $X_d(e^{j\omega})$ is real, then $X(j\omega)$ must also be real.

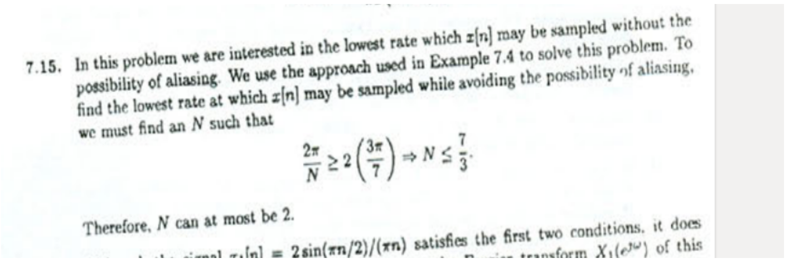
(b) $X_d(e^{j\omega})$ consists of replicas of $X(j\omega)$ which are scaled by $1/T$. Therefore, if $X_d($ has a maximum of 1, then $X(j\omega)$ will have a maximum of $T = 0.5 \times 10^{-3}$.

(c) The region $3\pi/4 \leq |\omega| \leq \pi$ in the discrete-time domain corresponds to the re $3\pi/(4T) \leq |\omega| \leq \pi/T$ in the continuous-time domain. Therefore, if $X_d(e^{j\omega}) =$ $3\pi/4 \leq |\omega| \leq \pi$, then $X(j\omega) = 0$ for $1500\pi \leq |\omega| \leq 2000\pi$. But since we already $X(j\omega) = 0$ for $|\omega| \geq 2000\pi$, we have $X(j\omega) = 0$ for $|\omega| \geq 1500\pi$.



) In this case, since π in discrete-time frequency domain corresponds to 2000π in the continuous-time frequency domain, this condition translates to $X(j\omega) = (j(\omega - 2000\pi))$.

continuous-time frequencies Ω and ω are

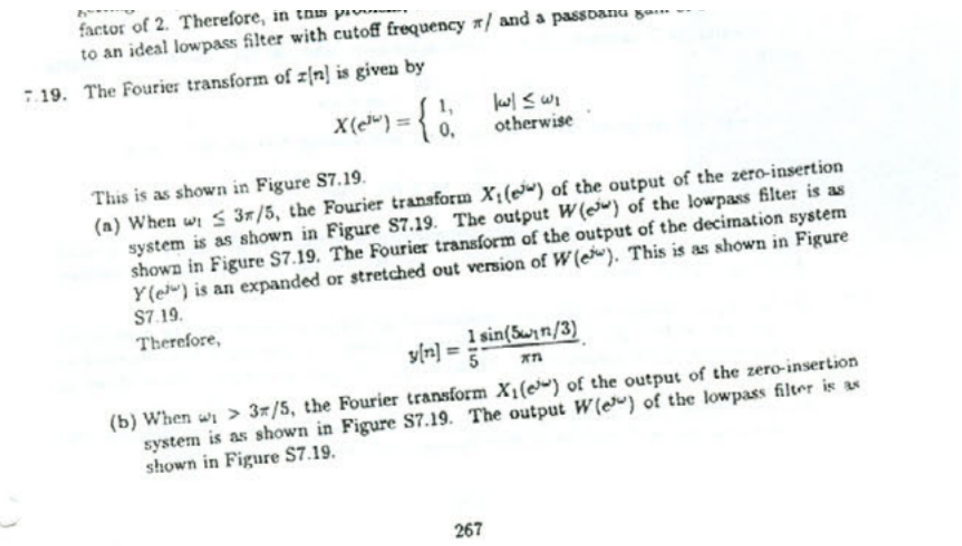


7.15. In this problem we are interested in the lowest rate which $x[n]$ may be sampled without the possibility of aliasing. We use the approach used in Example 7.4 to solve this problem. To find the lowest rate at which $x[n]$ may be sampled while avoiding the possibility of aliasing, we must find an N such that

$$\frac{2\pi}{N} \geq 2\left(\frac{3\pi}{7}\right) \Rightarrow N \leq \frac{7}{3}.$$

Therefore, N can at most be 2.

$x_1[n] = 2\sin(\pi n/2)/(\pi n)$ satisfies the first two conditions, it does transform $X_1(e^{j\omega})$ of this



factor of 2. Therefore, in this problem … to an ideal lowpass filter with cutoff frequency $\pi/$ and a passband gain …

7.19. The Fourier transform of $x[n]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_1 \\ 0, & \text{otherwise} \end{cases}.$$

This is as shown in Figure S7.19.

(a) When $\omega_1 \le 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19. The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an expanded or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19.

Therefore,

$$y[n] = \frac{1}{5}\frac{\sin(5\omega_1 n/3)}{\pi n}.$$

(b) When $\omega_1 > 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19.

267

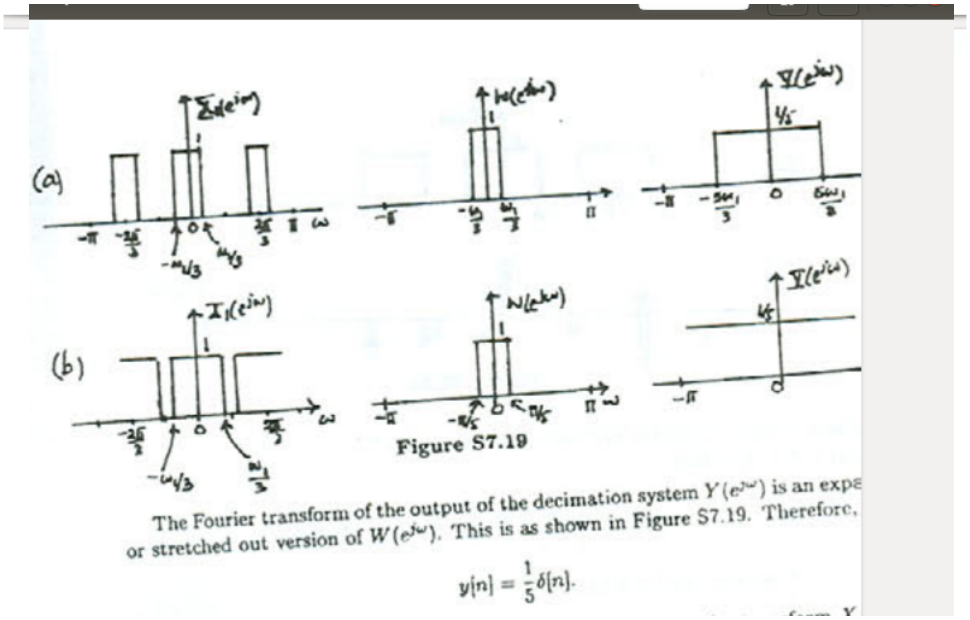
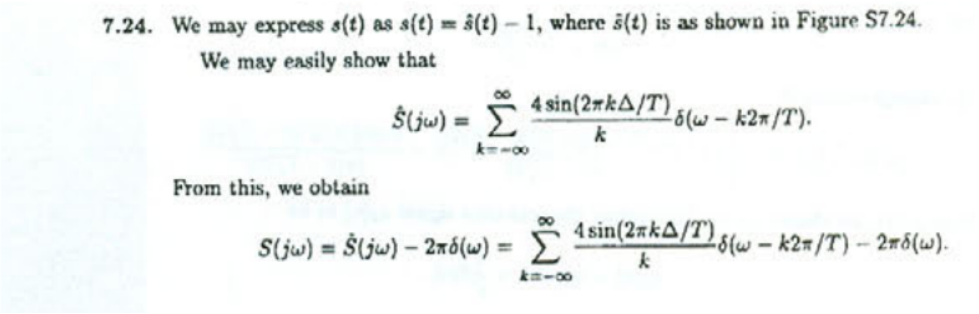


Figure S7.19

The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an expa or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19. Therefore,

$$y[n] = \frac{1}{5}\delta[n].$$



7.24. We may express $s(t)$ as $s(t) = \hat{s}(t) - 1$, where $\hat{s}(t)$ is as shown in Figure S7.24.

We may easily show that

$$\hat{S}(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T).$$

From this, we obtain

$$S(j\omega) = \hat{S}(j\omega) - 2\pi\delta(\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

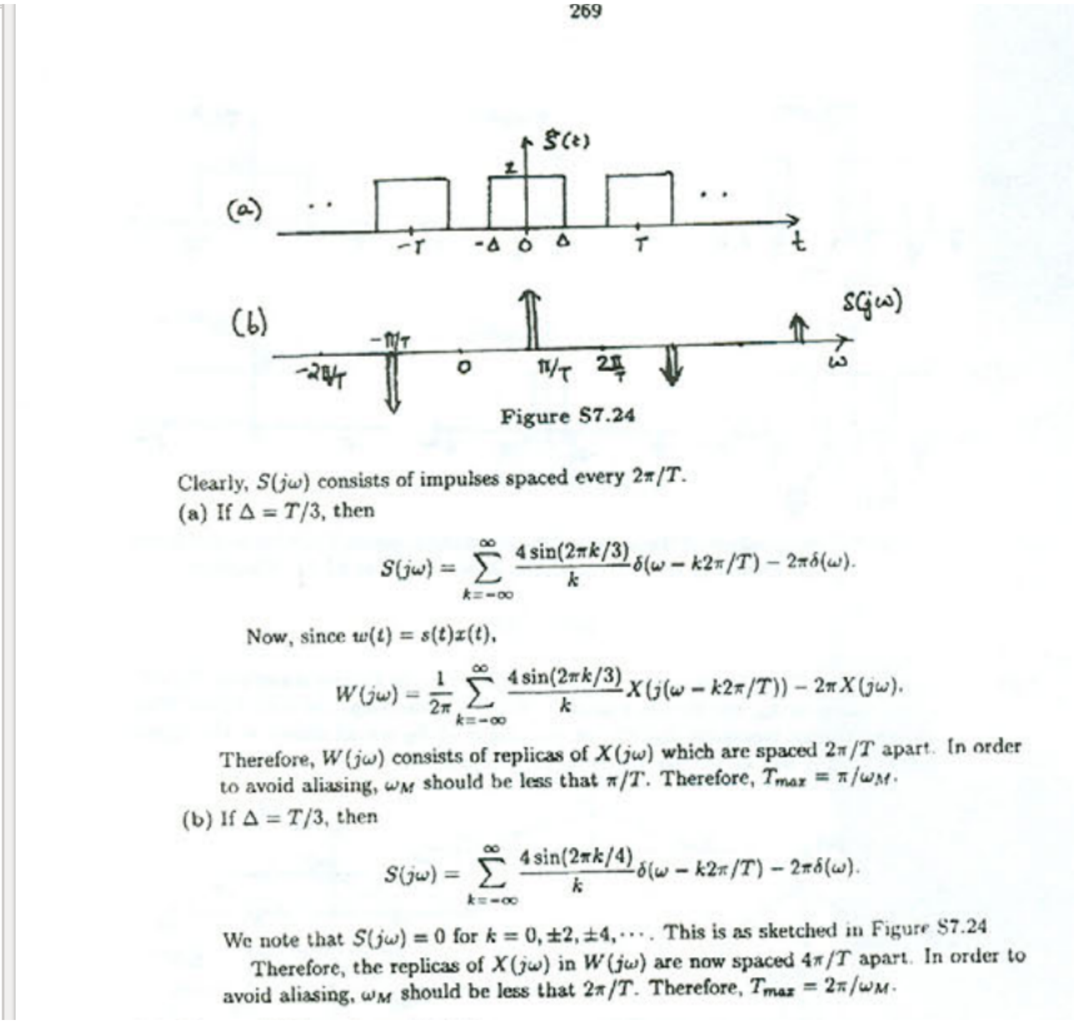


Figure S7.24

Clearly, $S(j\omega)$ consists of impulses spaced every $2\pi/T$.

(a) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

Now, since $w(t) = s(t)x(t)$,

$$W(j\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}X(j(\omega - k2\pi/T)) - 2\pi X(j\omega).$$

Therefore, $W(j\omega)$ consists of replicas of $X(j\omega)$ which are spaced $2\pi/T$ apart. In order to avoid aliasing, ω_M should be less that π/T. Therefore, $T_{max} = \pi/\omega_M$.

(b) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/4)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

We note that $S(j\omega) = 0$ for $k = 0, \pm 2, \pm 4, \cdots$. This is as sketched in Figure S7.24

Therefore, the replicas of $X(j\omega)$ in $W(j\omega)$ are now spaced $4\pi/T$ apart. In order to avoid aliasing, ω_M should be less that $2\pi/T$. Therefore, $T_{max} = 2\pi/\omega_M$.

4.8 HW 8

Local contents

4.8.1 Problem 9.3

Consider the signal $x(t) = e^{-5t}u(t) + e^{-\beta t}u(t)$ and denote its Laplace transform by $X(s)$ What are the constraints placed on the

real and imaginary parts of β if the region of convergence of $X(s)$ is $\operatorname{Re}(s) > -3$?

solution

The Laplace transform is

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \\
&= \int_{0}^{\infty} \left(e^{-5t} + e^{-\beta t}\right) e^{-st} dt \\
&= \int_{0}^{\infty} e^{-5t} e^{-st} dt + \int_{0}^{\infty} e^{-\beta t} e^{-st} dt \\
&= \int_{0}^{\infty} e^{-t(5+s)} dt + \int_{0}^{\infty} e^{-t\left(\beta+s\right)} dt \\
&= \frac{1}{-(s+5)} \left[e^{-t(5+s)}\right]_0^{\infty} - \frac{1}{\beta+s} \left[e^{-t\left(\beta+s\right)}\right]_0^{\infty}
\end{aligned}$$

For the first term $\frac{1}{-(s+5)}\left[e^{-t(5+s)}\right]_0^{\infty} = \frac{1}{-(s+5)}\left[e^{-\infty(5+s)} - 1\right]$. For this term to converge we need $5 + \operatorname{Re}(s) > 0$ or

$$\operatorname{Re}(s) > -5$$

For the second term, let $\beta = a + ib$ and let $s = \sigma + j\omega$, hence the second term becomes

$$\begin{aligned}
\frac{1}{\beta+s}\left[e^{-t\left(\beta+s\right)}\right]_0^{\infty} &= \frac{1}{\beta+s}\left[e^{-t\left(\left(a+ib\right)+\left(\sigma+j\omega\right)\right)}\right]_0^{\infty} \\
&= \frac{1}{\beta+s}\left[e^{-t\left(a+\sigma+j(b+\omega)\right)}\right]_0^{\infty} \\
&= \frac{1}{\beta+s}\left[e^{-t(a+\sigma)} e^{-jt(b+\omega)}\right]_0^{\infty} \\
&= \frac{1}{\beta+s}\left[e^{-\infty(a+\sigma)} e^{-j\infty(b+\omega)} - 1\right]
\end{aligned}$$

The complex exponential terms always converges since its norm is bounded by 1. For the real exponential term, we need $a+\sigma>0$ or $a+\operatorname{Re}(s)>0$ or $\operatorname{Re}(s)>-a$. Since we are told that $\operatorname{Re}(s)>-3$, then

$$a=3$$

Is the requirement on real part of β. There is $\underline{\text{no restriction on the imaginary part}}$ of β.

4.8.2 Problem 9.9

Given that $e^{-at}u(t)\Longleftrightarrow\frac{1}{s+a}$ for $\operatorname{Re}(s)>\operatorname{Re}(-a)$, determine the inverse Laplace transform of

$$X(s)=\frac{2(s+2)}{s^2+7s+12}\qquad \operatorname{Re}(s)>-3$$

solution

Writing $X(s)$ as

$$\begin{aligned}X(s)&=\frac{2(s+2)}{(s+4)(s+3)}\\&=\frac{A}{(s+4)}+\frac{B}{(s+3)}\end{aligned}$$

Hence $A=\left.\frac{2(s+2)}{(s+3)}\right|_{s=-4}=\frac{2(-4+2)}{(-4+3)}=4$ and $B=\left.\frac{2(s+2)}{(s+4)}\right|_{s=-3}=\frac{2(-3+2)}{(-3+4)}=-2$, therefore the above becomes

$$X(s)=\frac{4}{(s+4)}-\frac{2}{(s+3)}$$

Using $e^{-at}u(t)\Longleftrightarrow\frac{1}{s+a}$ gives the inverse Laplace transform as

$$\begin{aligned}x(t)&=4e^{-4t}u(t)-2e^{-3t}u(t)\\&=\left(4e^{-4t}-2e^{-3t}\right)u(t)\end{aligned}$$

With $\operatorname{Re}(s)>-4$ and also $\operatorname{Re}(s)>-3$. Therefore the ROC for both is $\operatorname{Re}(s)>-3$.

4.8.3 Problem 9.15

Consider the two right-sides signals $x(t),y(t)$ related through the differential equations

$$\begin{aligned}\frac{dx(t)}{dt}&=-2y(t)+\delta(t)\\\frac{dy(t)}{dt}&=2x(t)\end{aligned}$$

Determine $Y(s),X(s)$ along with their ROC.

solution

The Laplace transform of $\delta(t)$ is 1. Taking the Laplace transform of both the ODE's above, and assuming zero initial conditions gives

$$sX(s) = -2Y(s) + 1 \tag{1}$$
$$sY(s) = 2X(s) \tag{2}$$

Using the second equation in the first gives

$$\begin{aligned} sX(s) &= -2\left(\frac{2X(s)}{s}\right) + 1 \\ &= \frac{-4X(s) + s}{s} \\ s^2X(s) &= -4X(s) + s \\ \left(s^2 + 4\right)X(s) &= s \\ X(s) &= \frac{s}{\left(s^2 + 4\right)} \end{aligned}$$

Using the above in (2) gives $Y(s)$

$$\begin{aligned} sY(s) &= 2\frac{s}{\left(s^2 + 4\right)} \\ Y(s) &= \frac{2}{\left(s^2 + 4\right)} \end{aligned}$$

Considering $X(s)$ to find its ROC, let us write it as

$$X(s) = \frac{s}{\left(s^2 + 4\right)} = \frac{s}{\left(s + 2j\right)\left(s - 2j\right)} = \frac{A}{\left(s + 2j\right)} + \frac{B}{\left(s - 2j\right)}$$

We see that the ROC for first term is $\operatorname{Re}(s) > -\operatorname{Re}\left(2j\right)$ which means $\operatorname{Re}(s) > 0$ since real part is zero. Same for the second term. Hence we see that for $X(s)$ the ROC is $\operatorname{Re}(s) > 0$. Similarly for $Y(s)$. Therefore the overall ROC is

$$\operatorname{Re}(s) > 0$$

4.8.4 Problem 9.32

A causal LTI system with impulse response $h(t)$ has the following properties: (1) When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{1}{6}e^{2t}$ for all t. (2) The impulse response $h(t)$ satisfies the differential equation

$$\frac{dh(t)}{dt} + 2h(t) = e^{-4t}u(t) + bu(t)$$

Where b is unknown constant. Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in the answer

solution

First $H(s)$ is found from the differential equation. Taking Laplace transform gives (assuming zero initial conditions)

$$\begin{aligned} sH(s) + 2H(s) &= \frac{1}{s+4} + \frac{b}{s} \\ H(s)(2+s) &= \frac{1}{s+4} + \frac{b}{s} \\ H(s) &= \frac{1}{(s+4)(s+2)} + \frac{b}{s(s+2)} \\ &= \frac{s+b(s+4)}{s(s+4)(s+2)} \qquad (1) \end{aligned}$$

Now we are told when the input is e^{2t} then the output is $\frac{1}{6}e^{2t}$. In Laplace domain this means $Y(s) = X(s)H(s)$. Therefore

$$\begin{aligned} Y(s) &= \frac{1}{6}\frac{1}{s-2} \qquad \operatorname{Re}(s) > 2 \\ X(s) &= \frac{1}{s-2} \qquad \operatorname{Re}(s) > 2 \end{aligned}$$

Hence

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ H(s) &= \frac{\frac{1}{6}\frac{1}{s-2}}{\frac{1}{s-2}} \\ &= \frac{1}{6} \qquad (2) \end{aligned}$$

Comparing (1,2) then

$$\frac{1}{6} = \frac{s+b(s+4)}{s(s+4)(s+2)}$$

Solving for b gives

$$\begin{aligned} \frac{s(s+4)(s+2)}{6} &= s + b(s+4) \\ \frac{s(s+4)(s+2)}{6(s+4)} - \frac{s}{(s+4)} &= b \\ b &= \frac{s(s+2)}{6} - \frac{s}{(s+4)} \\ &= \frac{s(s+2)(s+4) - 6s}{6(s+4)} \\ &= \frac{s\left(s^2+6s+2\right)}{6(s+4)} \end{aligned}$$

This is true for $\operatorname{Re}(s) > 2$. Hence for $s = 2$ the above reduces to

$$\begin{aligned} b &= \frac{2(4+12+2)}{6(2+4)} \\ &= 1 \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned} H(s) &= \frac{s+(s+4)}{s(s+4)(s+2)} \\ &= \frac{2s+4}{s(s+4)(s+2)} \\ &= \frac{2(s+2)}{s(s+4)(s+2)} \\ &= \frac{2}{s(s+4)} \end{aligned}$$

4.8.5 Problem 9.40

Consider the system S characterized by the differential equation

$$y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = x(t)$$

(a) Determine the zero-state response of this system for the input $x(t) = e^{-4t}u(t)$ (b) Determine the zero-input response of the system for $t > 0^-$ given the initial conditions $y(0^-) = 1$, $\left.\frac{dy}{dt}\right|_{t=0^-} = -1$, $\left.\frac{d^2y}{dt^2}\right|_{t=0^-} = 1$. (c) Determine the output of S when the input is $x(t) = e^{-4t}u(t)$ and the initial conditions are the same as those specified in part (b).

Solution

4.8.5.1 Part a

Applying Laplace transform on the ODE and using zero initial conditions gives

$$\begin{aligned} s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) &= \frac{1}{s+4} \\ Y(s)\left(s^3+6s^2+11s+6\right) &= \frac{1}{s+4} \\ Y(s) &= \frac{1}{(s+4)\left(s^3+6s^2+11s+6\right)} \\ &= \frac{1}{(s+4)(s+1)(s+2)(s+3)} \qquad (1) \end{aligned}$$

Using partial fractions

$$\frac{1}{(s+4)(s+1)(s+2)(s+3)} = \frac{A}{(s+4)} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+3)}$$

Hence

$$\begin{aligned}
A &= \left.\frac{1}{(s+1)(s+2)(s+3)}\right|_{s=-4} = \frac{1}{(-4+1)(-4+2)(-4+3)} = -\frac{1}{6}\\
B &= \left.\frac{1}{(s+4)(s+2)(s+3)}\right|_{s=-1} = \frac{1}{(-1+4)(-1+2)(-1+3)} = \frac{1}{6}\\
C &= \left.\frac{1}{(s+4)(s+1)(s+3)}\right|_{s=-2} = \frac{1}{(-2+4)(-2+1)(-2+3)} = \frac{-1}{2}\\
D &= \left.\frac{1}{(s+4)(s+1)(s+2)}\right|_{s=-3} = \frac{1}{(-3+4)(-3+1)(-3+2)} = \frac{1}{2}
\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
Y(s) &= \frac{A}{(s+4)} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+3)}\\
&= -\frac{1}{6}\frac{1}{(s+4)} + \frac{1}{6}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+2)} + \frac{1}{2}\frac{1}{(s+3)} \qquad \operatorname{Re}(s) > -1
\end{aligned}$$

From tables, the inverse Laplace transform is (one sided) is

$$y(t) = -\frac{1}{6}e^{-4t}u(t) + \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

4.8.5.2 Part b

Applying Laplace transform on the ODE $y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = 0$ and using the non-zero initial conditions given above gives

$$\begin{aligned}
\left(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right) + 6\left(s^2Y(s) - sy(0) - y'(0)\right) + 11\left(sY(s) - y(0)\right) + 6Y(s) &= 0\\
\left(s^3Y(s) - s^2 + s - 1\right) + 6\left(s^2Y(s) - s + 1\right) + 11\left(sY(s) - 1\right) + 6Y(s) &= 0\\
s^3Y(s) - s^2 + s - 1 + 6s^2Y(s) - 6s + 6 + 11sY(s) - 11 + 6Y(s) &= 0\\
Y(s)\left(s^3 + 6s^2 + 11s + 6\right) - s^2 + s - 1 - 6s + 6 - 11 &= 0
\end{aligned} \tag{1}$$

Hence

$$\begin{aligned}
Y(s)\left(s^3 + 6s^2 + 11s + 6\right) &= s^2 - s + 1 + 6s - 6 + 11\\
Y(s) &= \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6}\\
&= \frac{(s+3)(s+2)}{(s+1)(s+2)(s+3)}\\
&= \frac{1}{s+1} \qquad \operatorname{Re}(s) > -1
\end{aligned}$$

Hence the inverse Laplace transform (one sided) gives

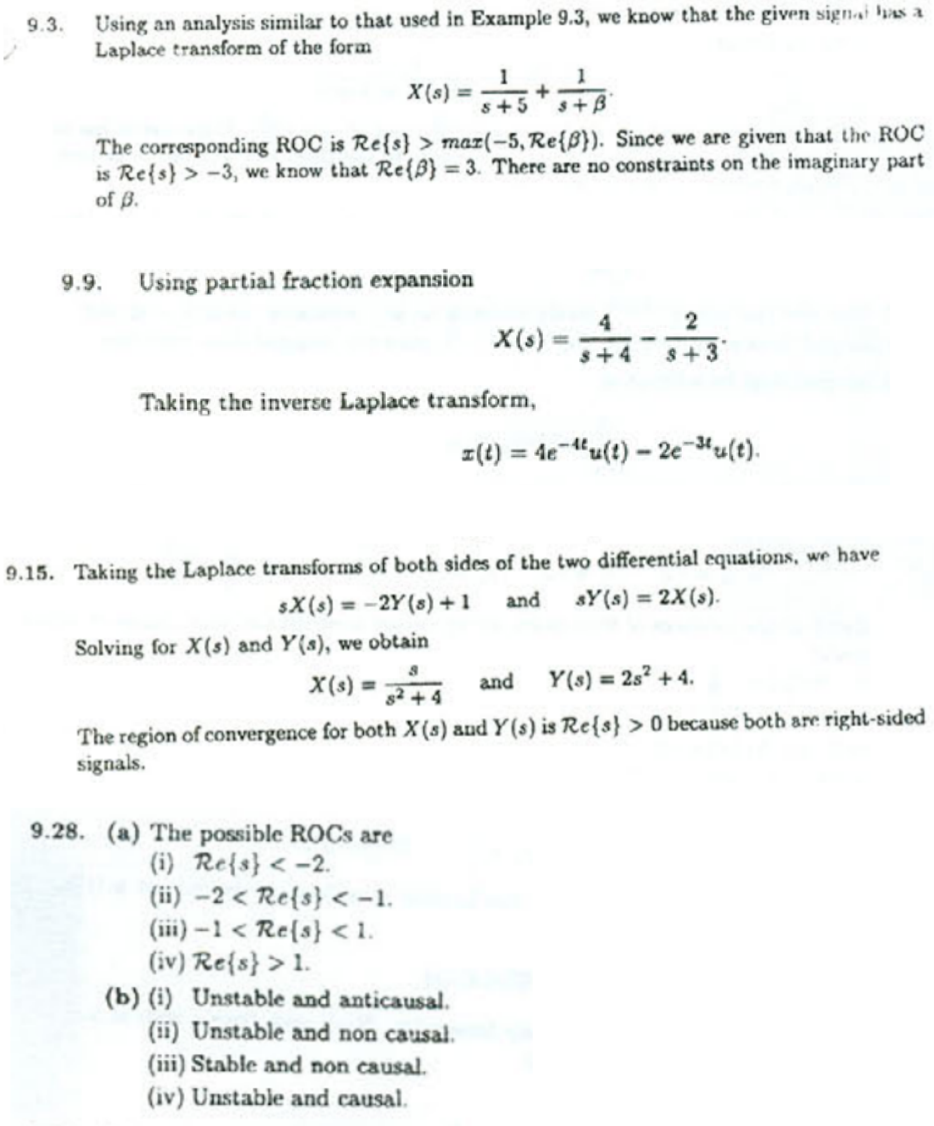
$$y(t) = e^{-t}u(t)$$

4.8.5.3 Part c

This is the sum of the response of part(a) and part(b) since the system is linear ODE. Hence

$$\begin{aligned}
y(t) &= -\frac{1}{6}e^{-4t}u(t) + \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) + e^{-t}u(t) \\
&= \left(-\frac{1}{6}e^{-4t} + \frac{1}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t} + e^{-t}\right)u(t) \\
&= \left(-\frac{1}{6}e^{-4t} + \frac{7}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t}\right)u(t)
\end{aligned}$$

4.8.6 key solution



9.3. Using an analysis similar to that used in Example 9.3, we know that the given signal has a Laplace transform of the form

$$X(s) = \frac{1}{s+5} + \frac{1}{s+\beta}.$$

The corresponding ROC is $\mathcal{R}e\{s\} > max(-5, \mathcal{R}e\{\beta\})$. Since we are given that the ROC is $\mathcal{R}e\{s\} > -3$, we know that $\mathcal{R}e\{\beta\} = 3$. There are no constraints on the imaginary part of β.

9.9. Using partial fraction expansion

$$X(s) = \frac{4}{s+4} - \frac{2}{s+3}.$$

Taking the inverse Laplace transform,

$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t).$$

9.15. Taking the Laplace transforms of both sides of the two differential equations, we have

$$sX(s) = -2Y(s) + 1 \quad \text{and} \quad sY(s) = 2X(s).$$

Solving for $X(s)$ and $Y(s)$, we obtain

$$X(s) = \frac{s}{s^2+4} \quad \text{and} \quad Y(s) = 2s^2 + 4.$$

The region of convergence for both $X(s)$ and $Y(s)$ is $\mathcal{R}e\{s\} > 0$ because both are right-sided signals.

9.28. **(a)** The possible ROCs are
- (i) $\mathcal{R}e\{s\} < -2$.
- (ii) $-2 < \mathcal{R}e\{s\} < -1$.
- (iii) $-1 < \mathcal{R}e\{s\} < 1$.
- (iv) $\mathcal{R}e\{s\} > 1$.

(b)
- (i) Unstable and anticausal.
- (ii) Unstable and non causal.
- (iii) Stable and non causal.
- (iv) Unstable and causal.

9.40. Taking the unilateral Laplace transform of both sides of the given differential equation, we get

$$\begin{aligned} s^3\mathcal{Y}(s) \; - \; & s^2y(0^-) - sy'(0^-) - y''(0^-) + 6s^2\mathcal{Y}(s) - 6sy(0^-) \\ & -6y(0^-) + 11s\mathcal{Y}(s) - 11y(0^-) + 6\mathcal{Y}(s) = \mathcal{X}(s). \end{aligned} \qquad (S9.40-1)$$

(a) For the zero state response, assume that all the initial conditions are zero. Furthermore, from the given $x(t)$ we may determine

$$\mathcal{X}(s) = \frac{1}{s+4}, \qquad \mathcal{R}e\{s\} > -4.$$

From eq. (S9.40-1), we get

$$\mathcal{Y}(s)[s^3 + 6s^2 + 11s + 6] = \frac{1}{s+4}.$$

Therefore,

$$\mathcal{Y}(s) = \frac{1}{(s+4)(s^3 + 6s^2 + 11s + 6)}.$$

Taking the inverse unilateral Laplace transform of the partial fraction expansion of the above equation, we get

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

(b) For the zero-input response, we assume that $\mathcal{X}(s) = 0$. Assuming that the initial conditions are as given, we obtain from (S9.40-1)

$$\mathcal{Y}(s) = \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform of the above equation, we get

$$y(t) = e^{-t}u(t).$$

(c) The total response is the sum of the zero-state and zero-input responses.

$$y(t) = \frac{7}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

4.9 HW 9

Local contents

4.9.1 Problem 10.2

Consider the signal

$$x[n] = \left(\frac{1}{5}\right)^n u[n-3]$$

Use eq. (10.3)

$$X[z] = \sum_{n=-\infty}^{n=\infty} x[n] z^{-n} \tag{10.3}$$

to evaluate the Z-transform of this signal, and specify the corresponding region of convergence.

solution

$$X[z] = \sum_{n=-\infty}^{n=\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n}$$

But $u[n-3]$ is zero for $n < 3$ and 1 otherwise. Hence the above becomes

$$X[z] = \sum_{n=3}^{n=\infty} \left(\frac{1}{5}\right)^n z^{-n}$$

Let $m = n - 3$. When $n = 3, m = 0$ therefore the above can be written as

$$\begin{aligned} X[z] &= \sum_{m=0}^{m=\infty} \left(\frac{1}{5}\right)^{m+3} z^{-(m+3)} \\ &= \left(\frac{z^{-1}}{5}\right)^3 \sum_{m=0}^{m=\infty} \left(\frac{1}{5}\right)^m z^{-m} \\ &= \frac{z^{-3}}{125} \sum_{m=0}^{m=\infty} \left(\frac{1}{5}\right)^m z^{-m} \end{aligned}$$

Renaming back to n

$$X[z] = \frac{z^{-3}}{125} \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \tag{1}$$

Now, looking at $\sum_{n=0}^{n=\infty}\left(\frac{1}{5z}\right)^n$ then assuming $|5z|>1$ and using the formula $\sum_{n=0}^{n=\infty}a^n=\frac{1}{1-a}$, where $a=\frac{1}{5z}$ in this case gives

$$\sum_{n=0}^{n=\infty}\left(\frac{1}{5z}\right)^n=\frac{1}{1-\frac{1}{5}z^{-1}}$$

Hence (1) becomes

$$X[z]=\frac{z^{-3}}{125}\left(\frac{1}{1-\frac{1}{5}z^{-1}}\right)$$

The above shows a pole at $\frac{1}{5}z^{-1}=1$ or $z=\frac{1}{5}$ and a pole at $z=0$. Since this is right handed signal, then the ROC is outside the outer most pole. Therefore ROC is

$$|z|>\frac{1}{5}$$

Which means the region is outside a circle of radius $\frac{1}{5}$. Since this ROC includes the unit circle, meaning a DTFT exist, it shows that this is a stable signal.

4.9.2 Problem 10.9

Using partial-fraction expansion and the fact that

$$a^n u[n]\overset{Z}{\longleftrightarrow}\frac{1}{1-az^{-1}}\qquad |z|>|a|$$

Find the inverse Z-transform of

$$X(z)=\frac{1-\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)\left(1+2z^{-1}\right)}\qquad |z|>2$$

solution

Let

$$\frac{1-\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)\left(1+2z^{-1}\right)}=\frac{A}{\left(1-z^{-1}\right)}+\frac{B}{\left(1+2z^{-1}\right)}$$

Hence $A=\left(\frac{1-\frac{1}{3}z^{-1}}{1+2z^{-1}}\right)_{z^{-1}=1}=\frac{1-\frac{1}{3}}{1+2}=\frac{2}{9}$ and $B=\left(\frac{1-\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)}\right)_{z^{-1}=-\frac{1}{2}}=\frac{1-\frac{1}{3}\left(-\frac{1}{2}\right)}{\left(1-\left(-\frac{1}{2}\right)\right)}=\frac{7}{9}$ Therefore the above becomes

$$X(z)=\frac{2}{9}\frac{1}{\left(1-z^{-1}\right)}+\frac{7}{9}\frac{1}{\left(1+2z^{-1}\right)}$$

The pole of first term at $z^{-1}=1$ or $z=1$ and the pole for second term is $2z^{-1}=-1$ or $z=-2$. Since the ROC is outside the out most pole, then this is right handed signal. Hence

$$\begin{aligned}x[n]&=\frac{2}{9}u[n]+\frac{7}{9}(-2)^n u[n]\\&=\left(\frac{2}{9}+\frac{7}{9}(-2)^n\right)u[n]\end{aligned}$$

Which is valid when $X(z)$ defined for $|z|>2$ since this is the common region for $|z|>1$ and $|z|>2$ at the same time. We notice the ROC does not include the unit circle and hence it is not stable signal. This is confirmed by looking at the term $(-2)^n$ which grows with n with no limit.

4.9.3 Problem 10.26

Consider a left-sided sequence $x[n]$ with z-transform

$$X(z)=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

a Write $X(z)$ as a ratio of polynomials in z instead of z^{-1}

b Using a partial-fraction expression, express $X(z)$ as a sum of terms, where each term represents a pole from your answer in part (a).

c Determine $x[n]$

solution

4.9.3.1 Part a

$$\begin{aligned}X(z)&=\frac{z}{z\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}\\&=\frac{z}{\left(z-\frac{1}{2}\right)\left(1-z^{-1}\right)}\\&=\frac{z^2}{z\left(z-\frac{1}{2}\right)\left(1-z^{-1}\right)}\\&=\frac{z^2}{\left(z-\frac{1}{2}\right)(z-1)}\\&=\frac{z^2}{z^2-\frac{3}{2}z+\frac{1}{2}}\end{aligned}$$

One pols at $z=\frac{1}{2}$ and one pole at $z=1$.

4.9.3.2 Part b

$$X(z)=\frac{z^2}{\left(z-\frac{1}{2}\right)(z-1)}$$

To do partial fractions, the degree in numerator must be smaller than in the denominator, which is not the case here. Hence we start by factoring out a z which gives

$$\begin{aligned} X(z) &= z^2\left(\frac{1}{\left(z-\frac{1}{2}\right)(z-1)}\right) \\ &= z^2\left(\frac{A}{z-\frac{1}{2}}+\frac{B}{z-1}\right) \end{aligned}$$

Hence

$$\frac{1}{\left(z-\frac{1}{2}\right)(z-1)} = \frac{A}{z-\frac{1}{2}}+\frac{B}{z-1}$$

Therefore $A = \left(\frac{1}{(z-1)}\right)_{z=\frac{1}{2}} = \frac{1}{\left(\frac{1}{2}-1\right)} = -2$ and $B = \left(\frac{1}{z-\frac{1}{2}}\right)_{z=1} = \frac{1}{1-\frac{1}{2}} = 2$. Hence the above becomes

$$\begin{aligned} X(z) &= z^2\left(-\frac{2}{z-\frac{1}{2}}+\frac{2}{z-1}\right) \\ &= 2z^2\left(-\frac{1}{z-\frac{1}{2}}+\frac{1}{z-1}\right) \end{aligned}$$

Pole at $z=\frac{1}{2}$ and one at $z=1$.

4.9.3.3 Part c

Writing the above as

$$X(z) = 2z^2 X_1(z)$$

Where $x_1[n] \iff X_1(z)$ where ROC for $X_1(z)$ is inside the inner most pole (since left sided). Hence ROC for $X_1(z)$ is $|z| < \frac{1}{2}$. What is left is to find $x_1[n]$ which is the inverse Z transform of $\frac{-1}{z-\frac{1}{2}}+\frac{1}{z-1}$. We want to use $a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$ so rewriting this as

$$\begin{aligned} X_1(z) &= \frac{-1}{z-\frac{1}{2}}+\frac{1}{z-1} \\ &= \frac{-z^{-1}}{1-\frac{1}{2}z^{-1}}+\frac{z^{-1}}{1-z^{-1}} \end{aligned}$$

Hence

$$\begin{aligned} X(z) &= 2z^2 X_1(z) \\ &= 2z^2\left(\frac{-z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-z^{-1}}\right) \\ &= 2z\left(\frac{-1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-z^{-1}}\right) \qquad (1) \end{aligned}$$

Then (since left handed) then $\frac{-1}{1-\frac{1}{2}z^{-1}} \longleftrightarrow \left(\frac{1}{2}\right)^n u[-n-1]$. Similarly for $\frac{1}{1-z^{-1}} \longleftrightarrow -u[-n-1]$. Hence

$$x[n] = \left(\frac{1}{2}\right)^n u[-n-1] - u[-n-1]$$

Substituting the above in (1) gives

$$x[n] = 2\left(\left(\frac{1}{2}\right)^n u[-n-2] - u[-n-2]\right)$$

Where $u[-n-1]$ is changed to $u[-n-2]$ because of the extra z in (1) outside, which causes extra shift and same for $u[-n-1]$ changed to $u[-n-2]$. Therefore the final answer is

$$x[n] = 2\left(\frac{1}{2}\right)^n u[-n-2] - 2u[-n-2]$$

4.9.4 Problem 10.34

A causal LTI system is described by the difference equation

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

a Find the system function $H(z) = \frac{Y(z)}{X(z)}$ for this system. Plot the poles and zeros of $H(z)$ and indicate the region of convergence.

b Find the unit sample response of the system.

c You should have found the system to be unstable. Find a stable (non causal) unit sample response that satisfies the difference equation.

solution

4.9.4.1 Part a

Taking the Z transform of the difference equation gives

$$\begin{aligned} Y(z) &= z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \\ Y(z)\left(1 - z^{-1} - z^{-2}\right) &= z^{-1}X(z) \\ \frac{Y(z)}{X(z)} &= \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \\ &= \frac{z}{z^2 - z - 1} \\ &= \frac{z}{\left(z - \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)\right)\left(z - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)\right)} \end{aligned}$$

Hence a pole at $z = \frac{1}{2}\sqrt{5} + \frac{1}{2} = 1.618$ and a pole at $z = \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right) = -0.618$ and zero at $z = 0$

Since this is a causal $H(z)$ then ROC is always to the right of the right most pole. Hence ROC is

$$|z| > \frac{1}{2}\sqrt{5} + \frac{1}{2} = 1.618$$

Here is a plot of the poles and zeros. The ROC is all the region to the right of 1.618 pole.

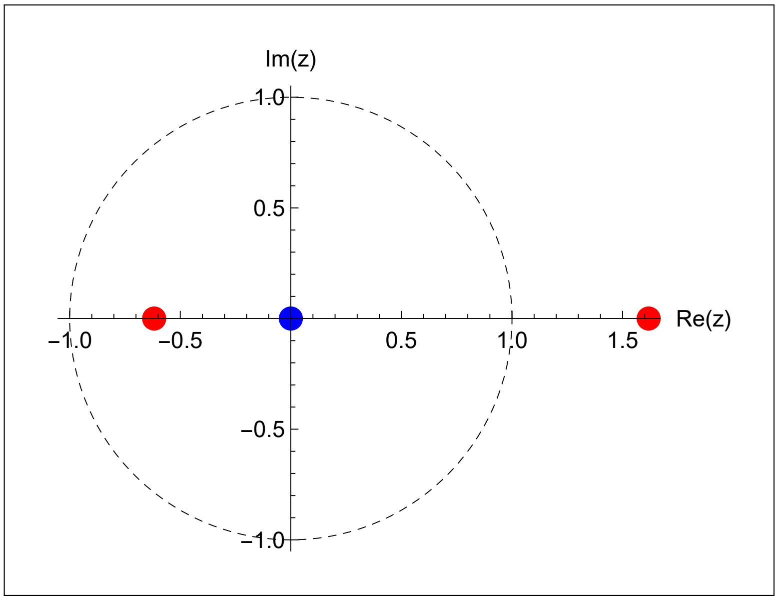
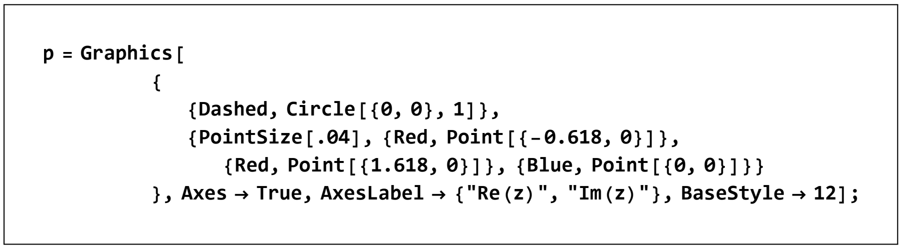


Figure 4.77: $H(z)$ Pole Zero plot. Red points are poles. Blue is zeros



```
p = Graphics[
        {
           {Dashed, Circle[{0, 0}, 1]},
           {PointSize[.04], {Red, Point[{-0.618, 0}]},
              {Red, Point[{1.618, 0}]}, {Blue, Point[{0, 0}]}}
        }, Axes → True, AxesLabel → {"Re(z)", "Im(z)"}, BaseStyle → 12];
```

Figure 4.78: Code used for the above

4.9.4.2 Part b

If the input $x[n] = \delta[n]$ then the difference equation is now

$$y[n] = y[n-1] + y[n-2] + \delta[n-1]$$

Hence taking the Z transform gives

$$\begin{aligned} Y(z) &= z^{-1}Y(z) + z^{-2}Y(z) + z^{-1} \\ Y(z)\left(1 - z^{-2} - z^{-1}\right) &= z^{-1} \\ Y(z) &= \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \\ &= \frac{-z^{-1}}{z^{-2} + z^{-1} - 1} \\ &= \frac{-z^{-1}}{\left(z^{-1} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)\right)\left(z^{-1} - \left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)\right)} \end{aligned} \tag{1}$$

Applying partial fractions gives

$$\frac{-z^{-1}}{\left(z^{-1} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)\right)\left(z^{-1} - \left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)\right)} = \frac{A}{z^{-1} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)} + \frac{B}{z^{-1} - \left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)}$$

Hence

$$A = \left(\frac{-z^{-1}}{\left(z^{-1} - \left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)\right)}\right)_{z^{-1}=\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)} = \frac{-\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)}{\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) - \left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)} = \frac{1}{10}\sqrt{5} - \frac{1}{2}$$

And

$$B = \left(\frac{-z^{-1}}{z^{-1} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)}\right)_{z=\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)} = \frac{-\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)}{\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right) - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)} = -\frac{1}{10}\sqrt{5} - \frac{1}{2}$$

Therefore (1) becomes

$$
\begin{aligned}
Y(z) &= \left(\frac{1}{10}\sqrt{5}-\frac{1}{2}\right)\frac{1}{z^{-1}-\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)}-\left(\frac{1}{10}\sqrt{5}+\frac{1}{2}\right)\frac{1}{z^{-1}-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)}\\
&= \frac{\left(\frac{1}{10}\sqrt{5}-\frac{1}{2}\right)}{-\frac{1}{2}+\frac{1}{2}\sqrt{5}}\frac{1}{\frac{1}{-\frac{1}{2}+\frac{1}{2}\sqrt{5}}z^{-1}-1}-\frac{\left(\frac{1}{10}\sqrt{5}+\frac{1}{2}\right)}{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)}\frac{1}{\frac{1}{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)}z^{-1}-1}\\
&= \frac{1}{5}\sqrt{5}\frac{1}{1-\left(\frac{2}{-1+\sqrt{5}}\right)z^{-1}}-\frac{1}{5}\sqrt{5}\frac{1}{1-\frac{2}{\left(-1-\sqrt{5}\right)}z^{-1}}\\
&= \frac{1}{5}\sqrt{5}\frac{1}{1-\left(\frac{1}{2}\sqrt{5}+\frac{1}{2}\right)z^{-1}}-\frac{1}{5}\sqrt{5}\frac{1}{1-\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)z^{-1}}
\end{aligned}
$$

Now we can use the table $\frac{1}{1-az^{-1}} \rightarrow a^n u[n]$ for $|z| > a$. Taking the inverse Z transform of the above gives

$$
\begin{aligned}
y[n] &= -\left(\frac{1}{5}\sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n u[n]+\left(\frac{1}{5}\sqrt{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n u[n]\\
&= \left(-(0.447\,21)(1.618)^n+(0.447\,21)(-0.618)^n\right)u[n]
\end{aligned}
$$

This is <u>unstable</u> response $y[n]$ due to the term $(1.618)^n$ which grows with no limit as $n \rightarrow \infty$.

4.9.4.3 Part c

Using the ROC where $0.618 < |z| < 1.618$ instead of $|z| > 1.618$, then

$$
\begin{aligned}
y[n] &= \left(\frac{1}{5}\sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n u[-n-1]+\left(\frac{1}{5}\sqrt{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n u[n]\\
&= \left((0.447\,21)(1.618)^n u[-n-1]+(0.447\,21)(-0.618)^n\right)u[n]
\end{aligned}
$$

which is now stable since the index on 1.618^n run is <u>negative</u> instead of positive.

4.9.5 Problem 10.36

Consider the linear, discrete-time, shift-invariant system with input $x[n]$ and output $y[n]$ for which

$$
y[n-1]-\frac{10}{3}y[n]+y[n+1]=x[n]
$$

is stable. Determine the unit sample response.

<u>solution</u>

Taking the Z transform of the difference equation gives

$$
\begin{aligned}
z^{-1}Y(z)-\frac{10}{3}Y(z)+zY(z) &= X(z)\\
Y(z)\left(z^{-1}-\frac{10}{3}+z\right) &= X(z)
\end{aligned}
$$

Hence the unit sample is when $x[n]=\delta[n]$. Hence $X(z)=1$. Therefore the impulse response is

$$
\begin{aligned}
H(z) &= \frac{1}{z^{-1}-\frac{10}{3}+z}\\
&= \frac{z^{-1}}{z^{-2}-\frac{10}{3}z^{-1}+1}\\
&= \frac{z^{-1}}{\left(z^{-1}-3\right)\left(z^{-1}-\frac{1}{3}\right)}
\end{aligned}
$$

Applying partial fractions

$$
H(z)=\frac{A}{\left(z^{-1}-3\right)}+\frac{B}{\left(z^{-1}-\frac{1}{3}\right)}
$$

Hence $A=\left(\frac{z^{-1}}{\left(z^{-1}-\frac{1}{3}\right)}\right)_{z^{-1}=3}=\frac{3}{\left(3-\frac{1}{3}\right)}=\frac{9}{8}$ and $B=\left(\frac{z^{-1}}{\left(z^{-1}-3\right)}\right)_{z^{-1}=\frac{1}{3}}=\frac{\frac{1}{3}}{\left(\frac{1}{3}-3\right)}=-\frac{1}{8}$. Therefore

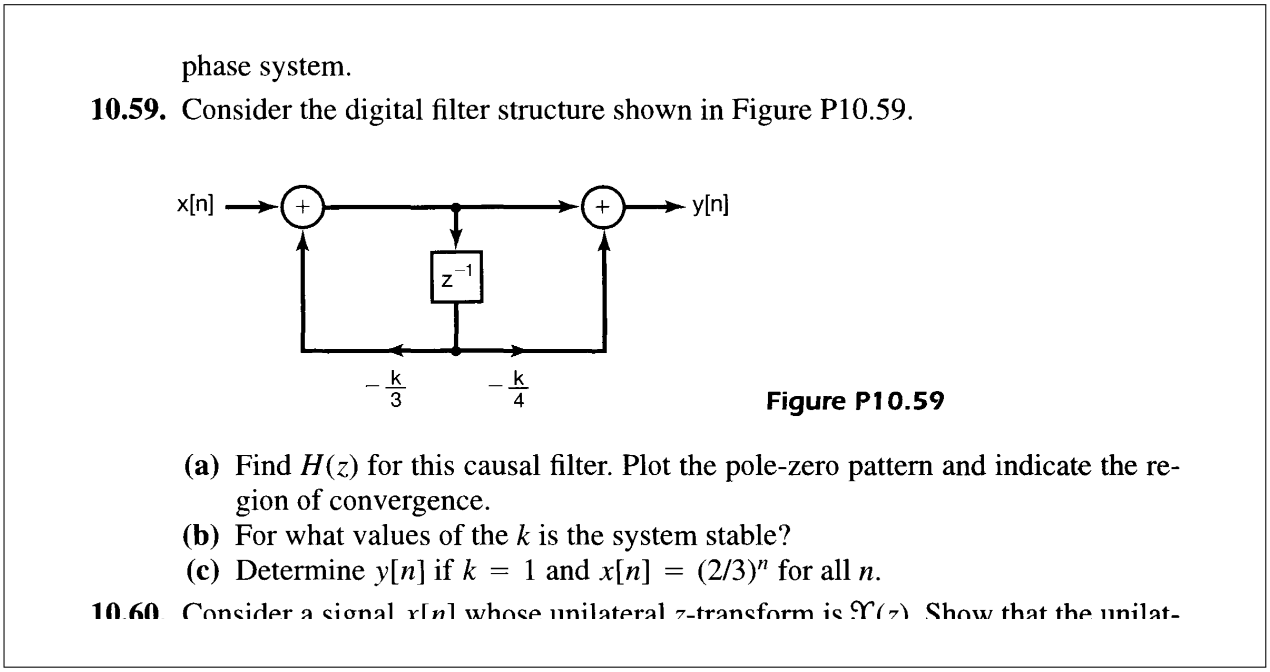
$$
\begin{aligned}
H(z) &= \frac{9}{8}\frac{1}{\left(z^{-1}-3\right)}-\frac{1}{8}\frac{1}{\left(z^{-1}-\frac{1}{3}\right)}\\
&= \frac{3}{8}\frac{1}{\left(\frac{1}{3}z^{-1}-1\right)}-\frac{3}{8}\frac{1}{\left(3z^{-1}-1\right)}\\
&= \frac{3}{8}\frac{1}{1-3z^{-1}}-\frac{3}{8}\frac{1}{1-\frac{1}{3}z^{-1}} \qquad (1)
\end{aligned}
$$

We see a $\underline{\text{pole at}}$ $z=3$ and a $\underline{\text{pole at}}$ $z=\frac{1}{3}$.

For $\frac{1}{1-3z^{-1}}$, this is stable only for a left sided signal, this is because a which is 3 here is larger than 1. Hence its inverse Z transform is of this is $x_1[n]=-\frac{3}{8}3^n u[-n-1]$ and for the second term $\frac{1}{1-\frac{1}{3}z^{-1}}$ is stable for right sided signal, since $\frac{1}{3}<1$. Hence its inverse Z transform is $-\frac{3}{8}\left(\frac{1}{3}\right)^n u[n]$. Therefore

$$
h[n]=-\frac{3}{8}(3)^n u[-n-1]-\frac{3}{8}\left(\frac{1}{3}\right)^n u[n]
$$

4.9.6 Problem 10.59



phase system.

10.59. Consider the digital filter structure shown in Figure P10.59.

Figure P10.59

(a) Find $H(z)$ for this causal filter. Plot the pole-zero pattern and indicate the region of convergence.

(b) For what values of the k is the system stable?

(c) Determine $y[n]$ if $k = 1$ and $x[n] = (2/3)^n$ for all n.

10.60 Consider a signal $x[n]$ whose unilateral z-transform is $\mathcal{X}(z)$. Show that the unilat-

Figure 4.79: Problem description

solution

4.9.6.1 Part (a)

Let the value at the branch just to the right of $x[n]$ summation sign be called $A[z]$.

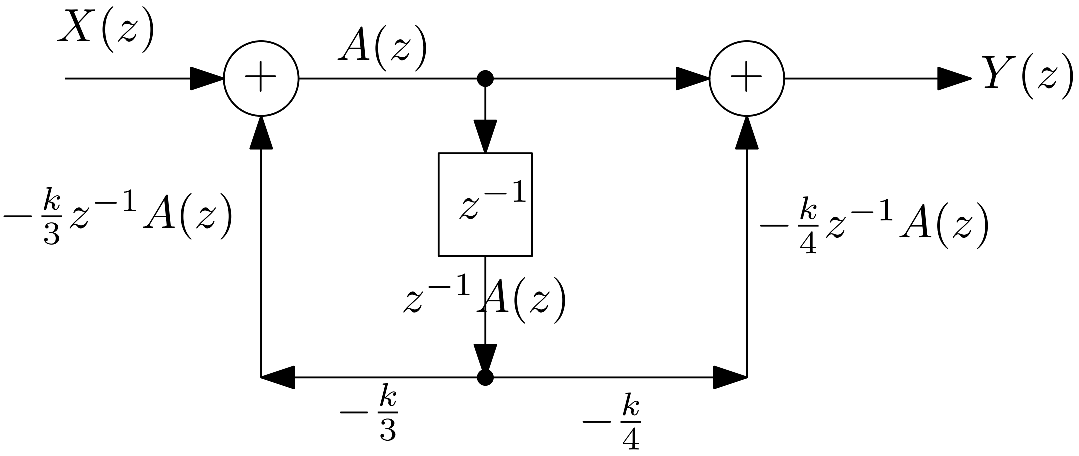


Figure 4.80: Filter diagram

Then we see that

$$Y(z) = A(z) - \frac{k}{4} z^{-1} A(z)$$

We just need to find $A(z)$. We see that $A(z) = X(z) - \frac{k}{3} z^{-1} A(z)$. Hence $A(z)\left(1 + \frac{k}{3} z^{-1}\right) = X(z)$ or $A(z) = \frac{X(z)}{1+\frac{k}{3}z^{-1}}$. Therefore the above becomes

$$\begin{aligned} Y(z) &= \frac{X(z)}{1+\frac{k}{3}z^{-1}} - \frac{k}{4} z^{-1} \frac{X(z)}{1+\frac{k}{3}z^{-1}} \\ &= X(z)\left(\frac{1}{1+\frac{k}{3}z^{-1}} - \frac{k}{4} \frac{z^{-1}}{1+\frac{k}{3}z^{-1}}\right) \end{aligned}$$

Hence

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1}{1+\frac{k}{3}z^{-1}} - \frac{k}{4} \frac{z^{-1}}{1+\frac{k}{3}z^{-1}} \\ &= \frac{1-\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}} \end{aligned}$$

The pole is when $\frac{k}{3} z^{-1} = -1$ or $z = -\frac{k}{3}$. Zero is when $1 - kz^{-1} = 0$ or $kz^{-1} = 1$ or $z = k$. Since this causal system, then the ROC is to the right of the most right pole. Hence $|z| > \frac{|k|}{3}$ is the ROC.

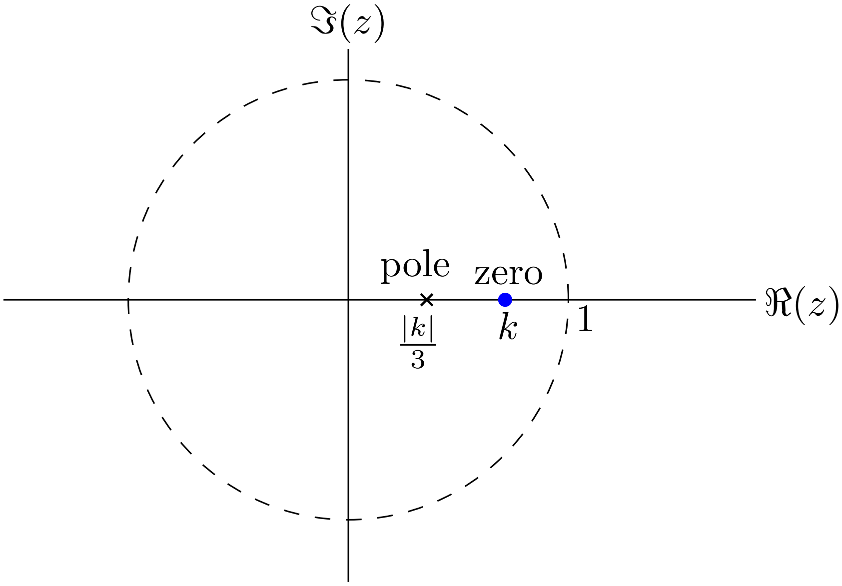


Figure 4.81: Pole zero polt. ROC is $|z| >]frac|k|3$

4.9.6.2 Part (b)

System is stable if it has a Discrete time Fourier transform. This implies the ROC must include the unit circle. Hence $\frac{|k|}{3} < 1$ or $|k| < 3$.

4.9.6.3 Part (c)

From part (a), the unit sample response is $H(z) = \frac{1-\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}$. When $k = 1$ this becomes $H(z) = \frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{3}z^{-1}}$

Since $x[n] = \left(\frac{2}{3}\right)^n$ for all n and this is casual system, then this means $x[n] = \left(\frac{2}{3}\right)^n u[n]$. Therefore

$$X(z) = \frac{1}{1-\frac{2}{3}z^{-1}}$$

Hence from part (a)

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{3}z^{-1}}\frac{1}{1-\frac{2}{3}z^{-1}} \\ &= \frac{1-\frac{1}{4}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)} \\ &= \frac{A}{1+\frac{1}{3}z^{-1}} + \frac{B}{1-\frac{2}{3}z^{-1}} \end{aligned}$$

Therefore $A = \left(\frac{1-\frac{1}{4}z^{-1}}{\left(1-\frac{2}{3}z^{-1}\right)}\right)_{z^{-1}=-3} = \frac{1-\frac{1}{4}(-3)}{\left(1-\frac{2}{3}(-3)\right)} = \frac{7}{12}$ and $B = \left(\frac{1-\frac{1}{4}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)}\right)_{z^{-1}=\frac{3}{2}} = \left(\frac{1-\frac{1}{4}\left(\frac{3}{2}\right)}{\left(1+\frac{1}{3}\left(\frac{3}{2}\right)\right)}\right) = \frac{5}{12}$. Hence

$$Y(z) = \frac{7}{12}\frac{1}{1+\frac{1}{3}z^{-1}} + \frac{5}{12}\frac{1}{1-\frac{2}{3}z^{-1}}$$

Therefore

$$y[n] = \frac{7}{12}\left(-\frac{1}{3}\right)^n u[n] + \frac{5}{12}\left(\frac{2}{3}\right)^n u[n]$$

The following is a plot of the solution

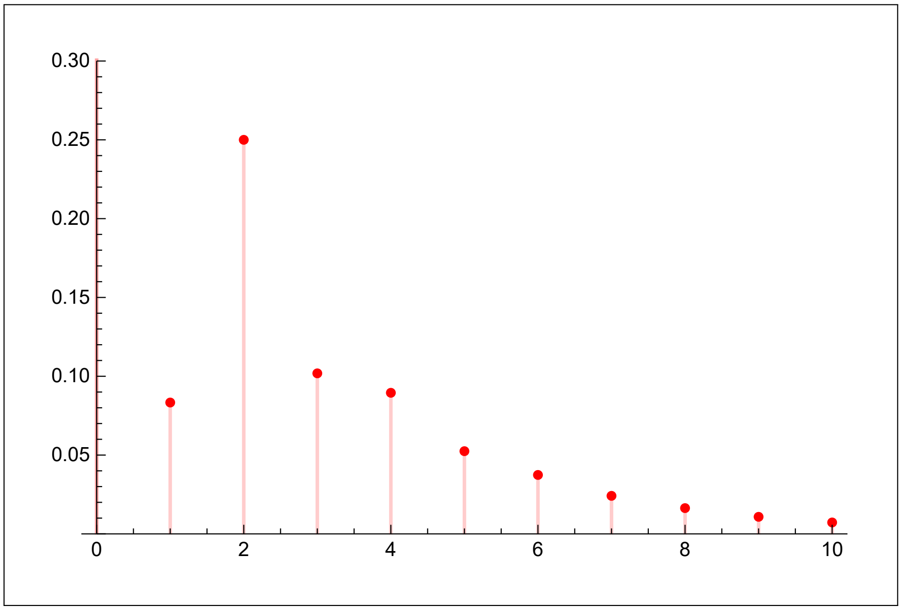
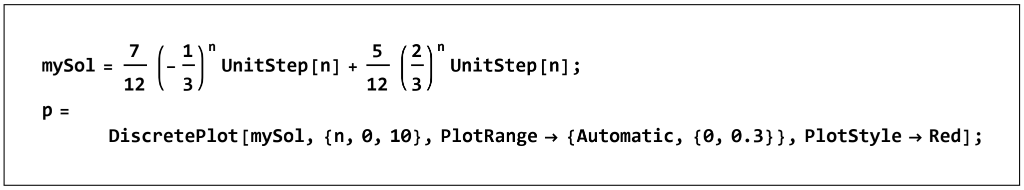


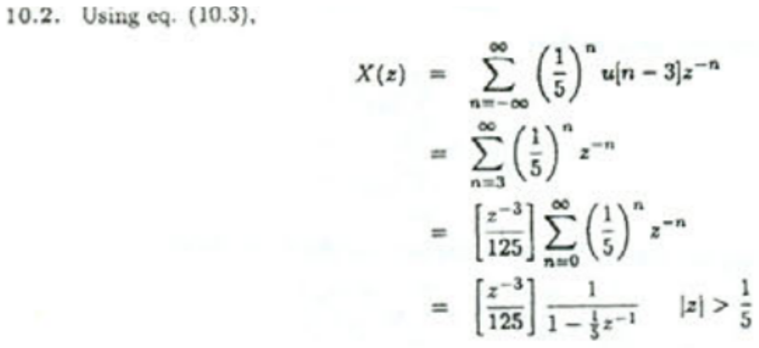
Figure 4.82: Plot of $y[n]$



```
mySol = 7/12 (-1/3)^n UnitStep[n] + 5/12 (2/3)^n UnitStep[n];
p =
      DiscretePlot[mySol, {n, 0, 10}, PlotRange → {Automatic, {0, 0.3}}, PlotStyle → Red];
```

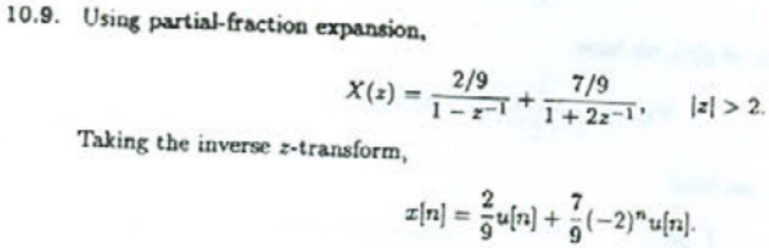
Figure 4.83: Code used

4.9.7 key solution



10.2. Using eq. (10.3),

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n} \\
&= \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \\
&= \left[\frac{z^{-3}}{125}\right] \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \\
&= \left[\frac{z^{-3}}{125}\right] \frac{1}{1-\frac{1}{5}z^{-1}} \qquad |z| > \frac{1}{5}
\end{aligned}
$$

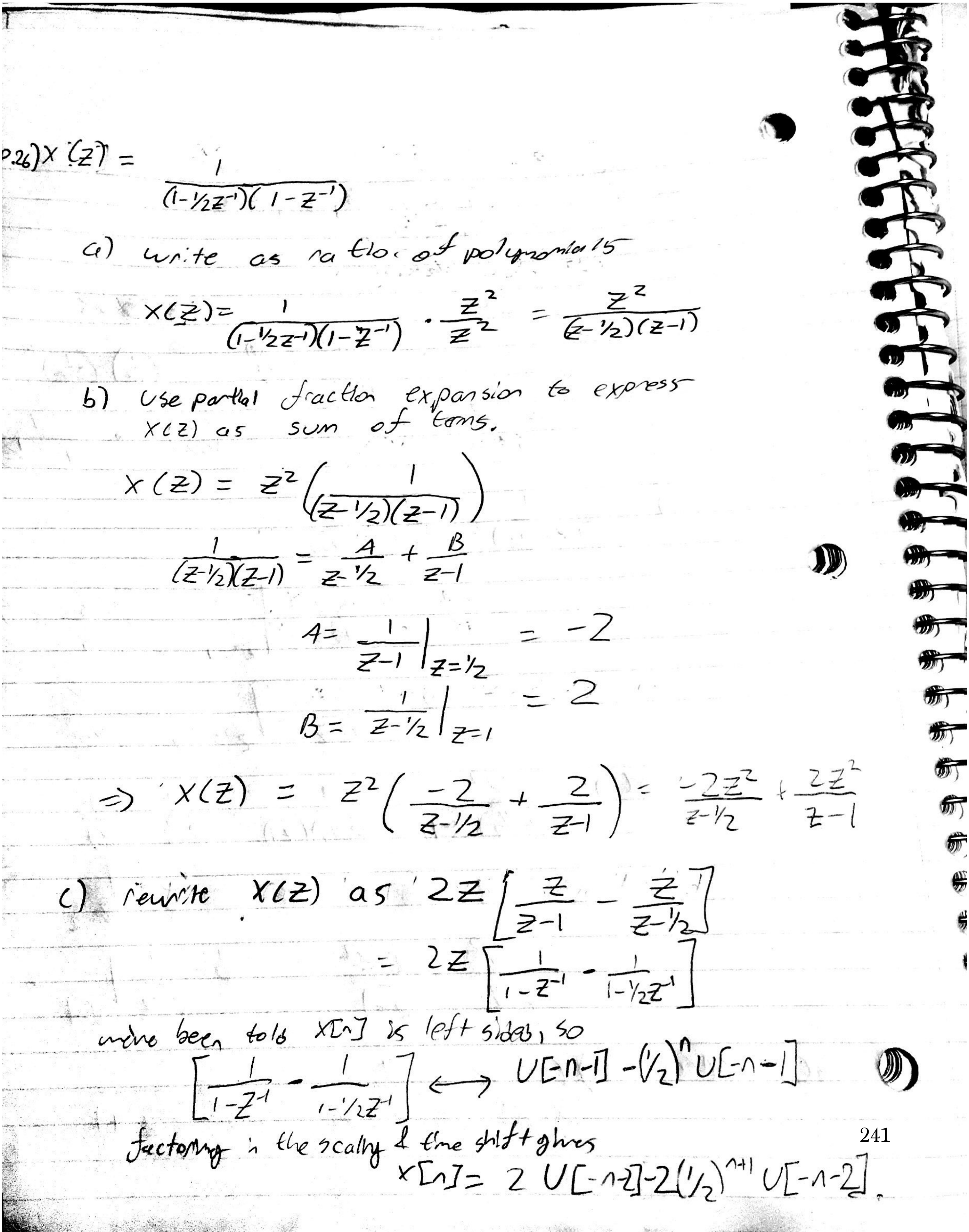


10.9. Using partial-fraction expansion,

$$X(z) = \frac{2/9}{1-z^{-1}} + \frac{7/9}{1+2z^{-1}}, \qquad |z| > 2.$$

Taking the inverse z-transform,

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n].$$



.26) $X(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$

a) write as ratio of polynomials

$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \cdot \frac{z^2}{z^2} = \frac{z^2}{(z-\frac{1}{2})(z-1)}$$

b) Use partial fraction expansion to express X(z) as sum of terms.

$$X(z) = z^2\left(\frac{1}{(z-\frac{1}{2})(z-1)}\right)$$

$$\frac{1}{(z-\frac{1}{2})(z-1)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-1}$$

$$A = \frac{1}{z-1}\bigg|_{z=\frac{1}{2}} = -2$$

$$B = \frac{1}{z-\frac{1}{2}}\bigg|_{z=1} = 2$$

$$\Rightarrow X(z) = z^2\left(\frac{-2}{z-\frac{1}{2}} + \frac{2}{z-1}\right) = \frac{-2z^2}{z-\frac{1}{2}} + \frac{2z^2}{z-1}$$

c) rewrite $X(z)$ as $2z\left[\dfrac{z}{z-1} - \dfrac{z}{z-\frac{1}{2}}\right]$

$$= 2z\left[\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right]$$

we've been told $x[n]$ is left sided, so

$$\left[\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right] \longleftrightarrow U[-n-1] - (\tfrac{1}{2})^n U[-n-1]$$

factoring in the scaling & time shift gives

$$x[n] = 2\,U[-n-2] - 2(\tfrac{1}{2})^{n+1} U[-n-2].$$

10.34. (a) Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}.$$

The poles of $H(z)$ are at $z = (1/2) \pm (\sqrt{5}/2)$. $H(z)$ has a zero at $z = 0$. The pole-zero plot for $H(z)$ is as shown in Figure S10.34. Since $h[n]$ is causal, the ROC for $H(z)$ has to be $|z| > (1/2) + (\sqrt{5}/2)$.

(b) The partial fraction expansion of $H(z)$ is

$$H(z) = -\frac{1/\sqrt{5}}{1 - (\frac{1+\sqrt{5}}{2})z^{-1}} + \frac{1/\sqrt{5}}{1 - (\frac{1-\sqrt{5}}{2})z^{-1}}.$$

Therefore,

$$h[n] = -\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

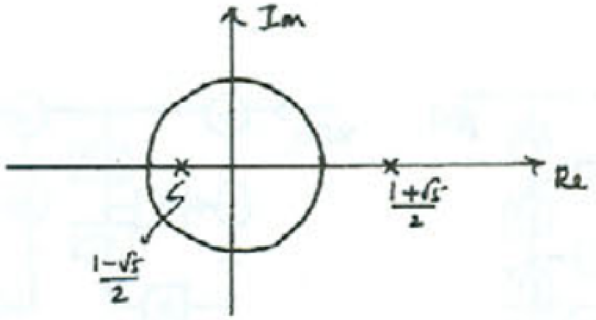


Figure S10.34

(c) Now assuming that the ROC is $(\sqrt{5}/2) - (1/2) < |z| < (1/2) + (\sqrt{5}/2)$, we get

$$h[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[-n-1] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

10.36. Taking the z-transform of both sides of the given difference equation and simplifying, we get

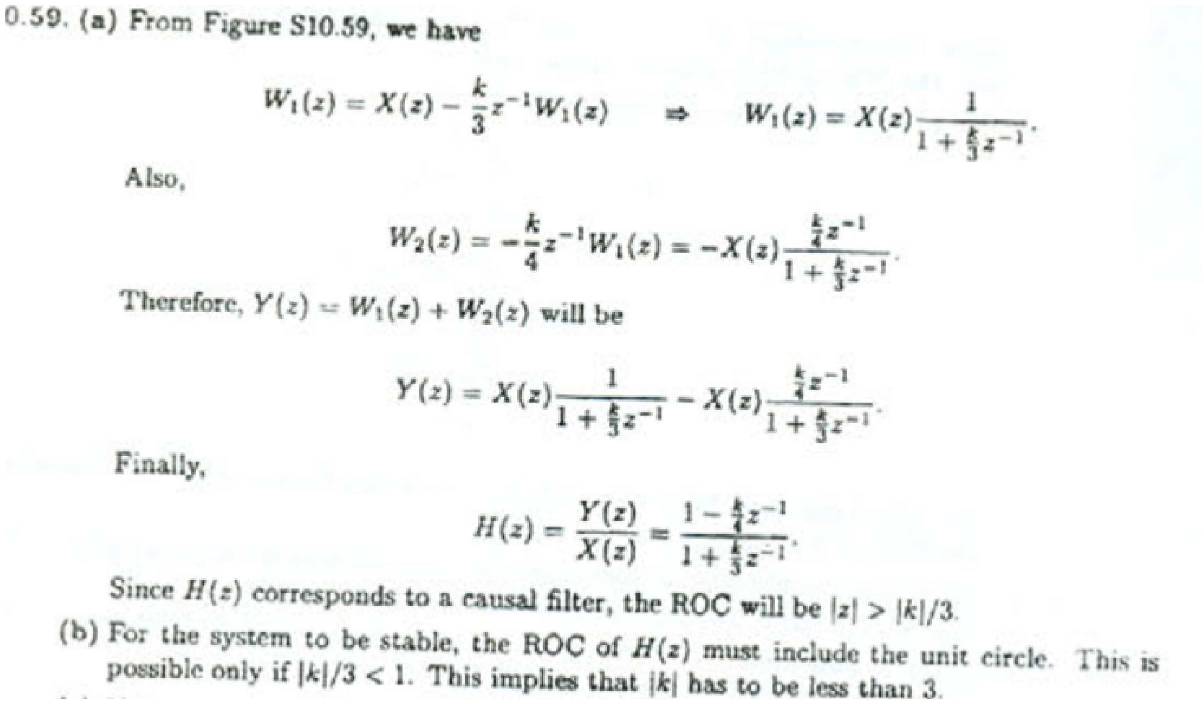
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}}.$$

The partial fraction expansion of $H(z)$ is

$$H(z) = -\frac{3/8}{1 - \frac{1}{3}z^{-1}} + \frac{3/8}{1 - 3z^{-1}}.$$

Since $H(z)$ corresponds to a stable system, the ROC has to be $(1/3) < |z| < 3$. Therefore,

$$h[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}(3)^n u[-n-1].$$



0.59. (a) From Figure S10.59, we have

$$W_1(z) = X(z) - \frac{k}{3}z^{-1}W_1(z) \quad \Rightarrow \quad W_1(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}}.$$

Also,

$$W_2(z) = -\frac{k}{4}z^{-1}W_1(z) = -X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Therefore, $Y(z) = W_1(z) + W_2(z)$ will be

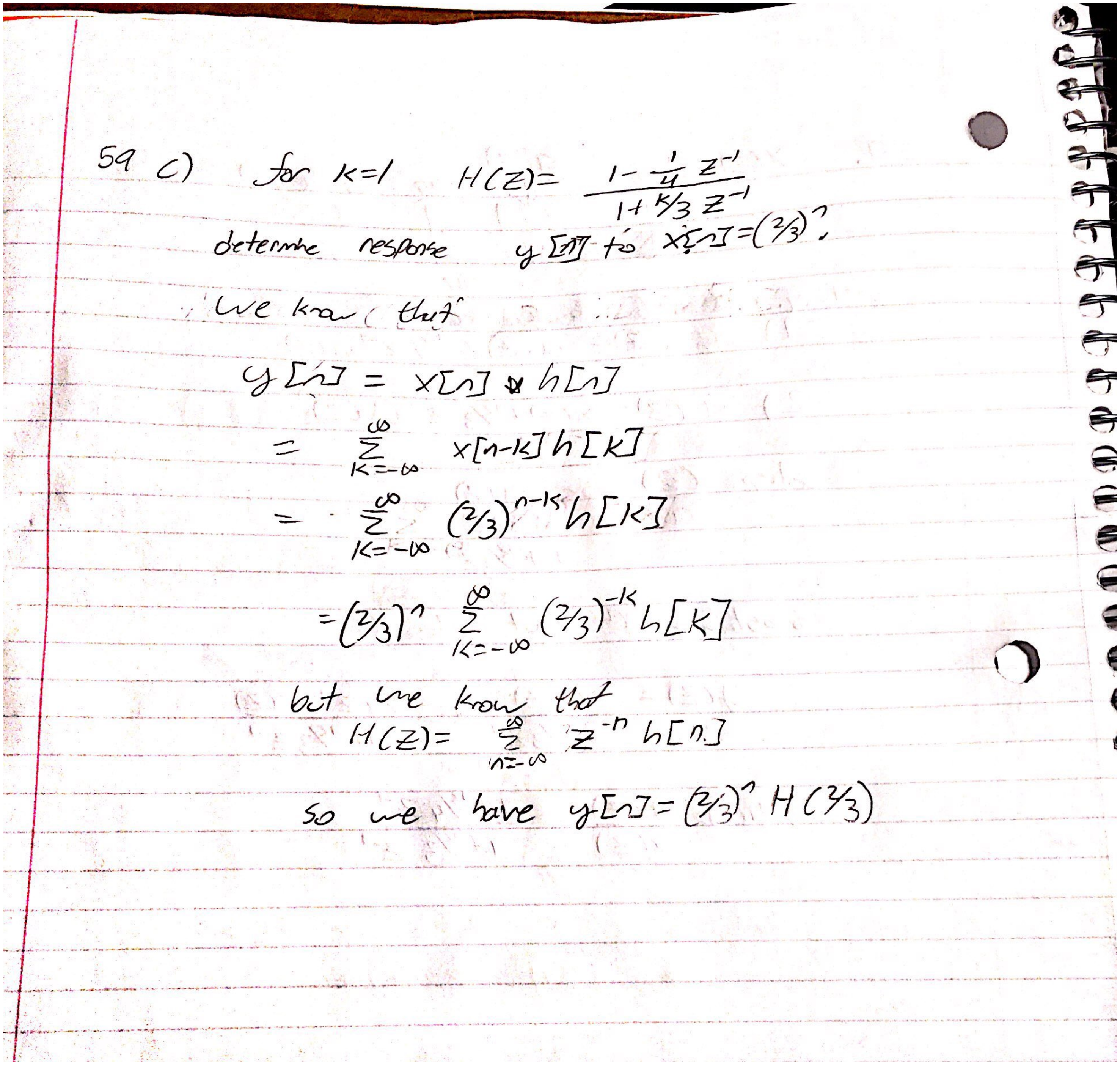
$$Y(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}} - X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Finally,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Since $H(z)$ corresponds to a causal filter, the ROC will be $|z| > |k|/3$.

(b) For the system to be stable, the ROC of $H(z)$ must include the unit circle. This is possible only if $|k|/3 < 1$. This implies that $|k|$ has to be less than 3.



59 c) for $K=1$ $H(Z)= \frac{1-\frac{1}{4}Z^{-1}}{1+K/3\,Z^{-1}}$

determine response $y[n]$ to $x[n]=(2/3)^n$.

We know that

$$y[n] = x[n] * h[n]$$

$$= \sum_{K=-\infty}^{\infty} x[n-K]\,h[K]$$

$$= \sum_{K=-\infty}^{\infty} (2/3)^{n-K}\,h[K]$$

$$=(2/3)^n \sum_{K=-\infty}^{\infty} (2/3)^{-K}\,h[K]$$

but we know that

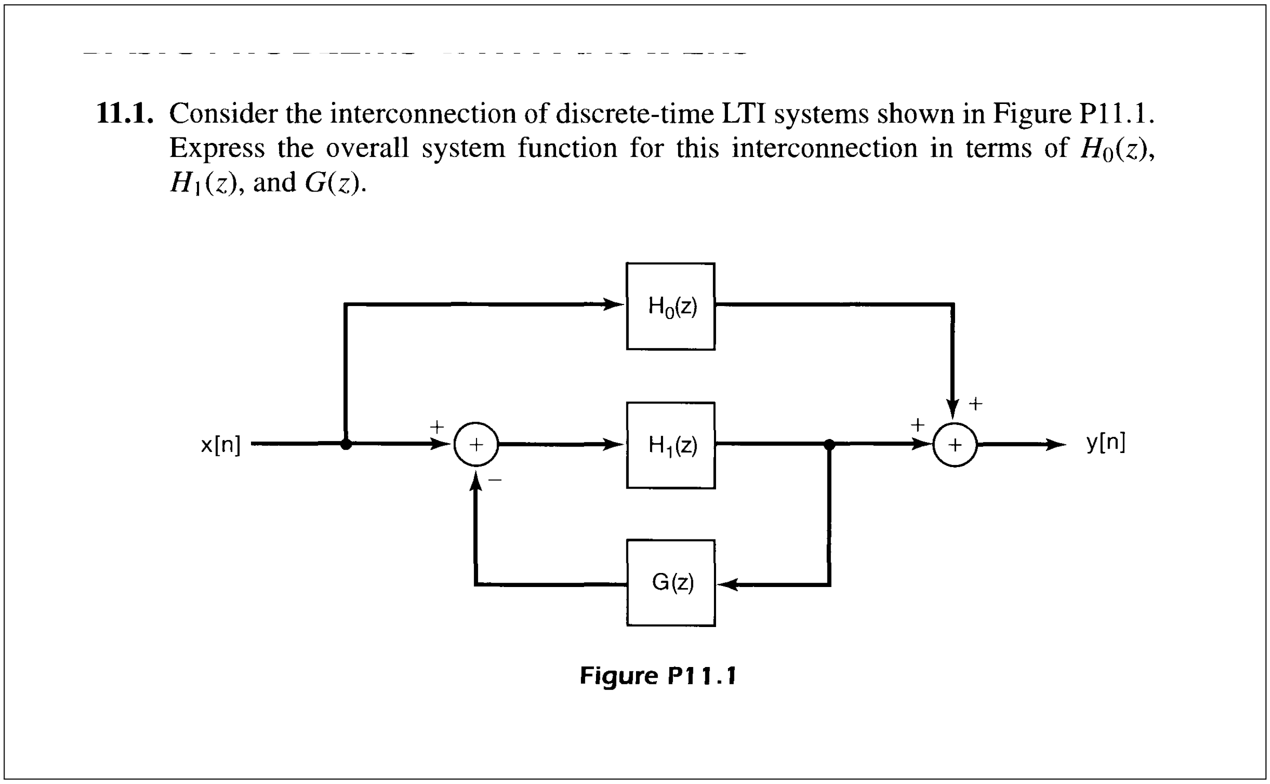
$$H(Z)= \sum_{n=-\infty}^{\infty} Z^{-n}\,h[n]$$

so we have $y[n]=(2/3)^n\,H(2/3)$

4.10 HW 10

Local contents

4.10.1 Problem 11.1



11.1. Consider the interconnection of discrete-time LTI systems shown in Figure P11.1. Express the overall system function for this interconnection in terms of $H_0(z)$, $H_1(z)$, and $G(z)$.

Figure P11.1

Figure 4.84: Problem description

<u>solution</u>

Adding the following notations on the diagram to make it easy to do the computation

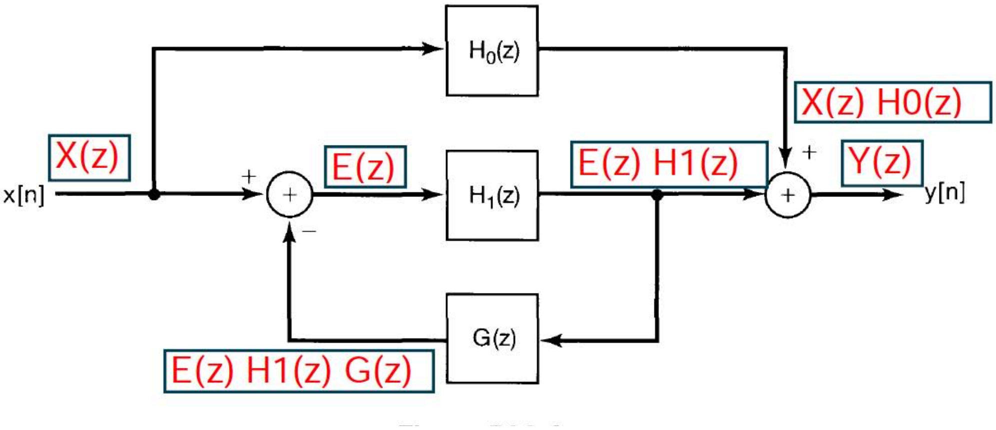


Figure 4.85: Annotations added

Therefore we see that

$$Y(z) = X(z) H_0(z) + E(z) H_1(z) \tag{1}$$

So we just need to determine $E(z)$. But $E(z) = X(z) - E(z) H_1(z) G(z)$. Hence $E(z)(1 + H_1(z) G(z)) = X(z)$ or

$$E(z) = \frac{X(z)}{1 + H_1(z) G(z)}$$

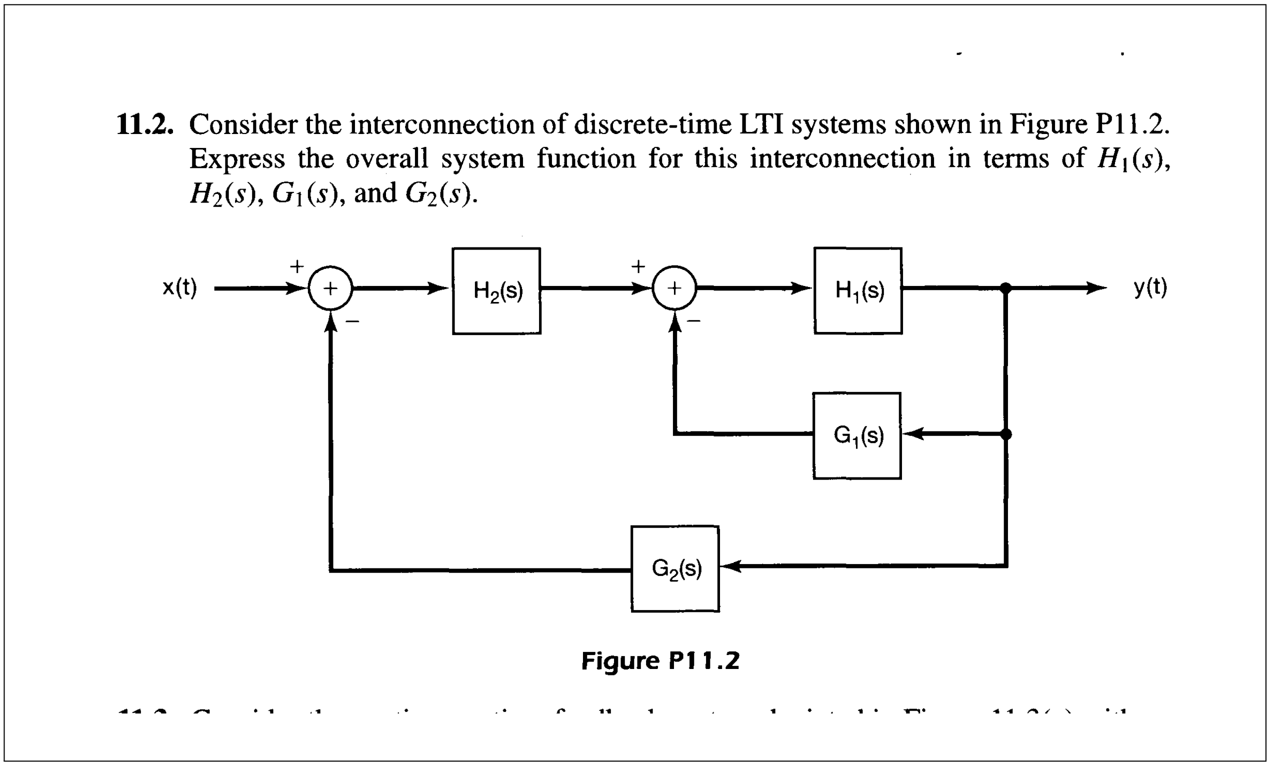
Substituting this into (1) gives

$$Y(z) = X(z) H_0(z) + \frac{X(z)}{1 + H_1(z) G(z)} H_1(z)$$
$$Y(z) = X(z) \left(H_0(z) + \frac{H_1(z)}{1 + H_1(z) G(z)} \right)$$

Hence

$$\frac{Y(z)}{X(z)} = H_0(z) + \frac{H_1(z)}{1 + H_1(z) G(z)}$$

4.10.2 Problem 11.2



11.2. Consider the interconnection of discrete-time LTI systems shown in Figure P11.2. Express the overall system function for this interconnection in terms of $H_1(s)$, $H_2(s)$, $G_1(s)$, and $G_2(s)$.

Figure P11.2

Figure 4.86: Problem description

<u>solution</u>

Adding the following notations on the diagram to make it easy to do the computation

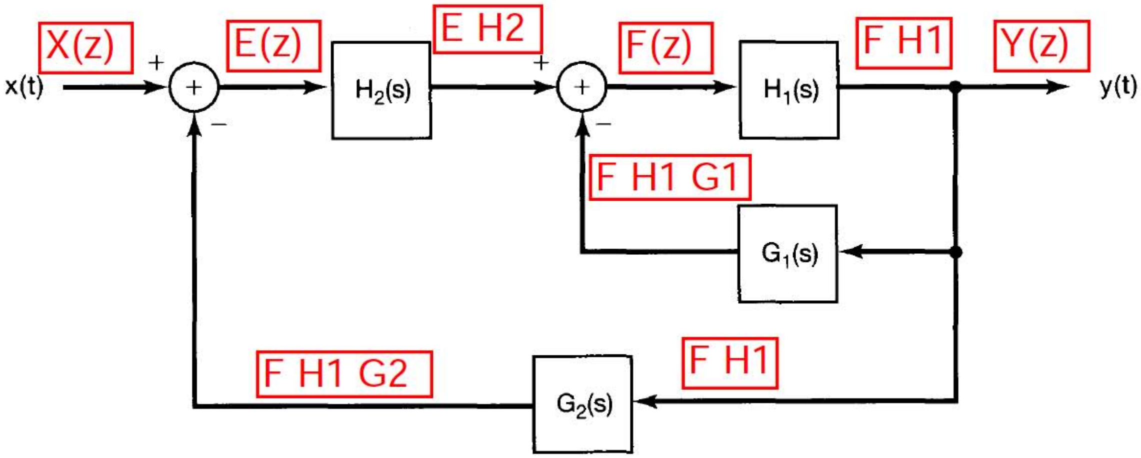


Figure 4.87: Annotations added

Therefore we see that

$$E = X - FH_1G_2 \tag{1}$$
$$F = EH_2 - FH_1G_1 \tag{2}$$

We have 2 equations with 2 unknowns E, F. Substituting first equation into the second gives

$$\begin{aligned}
F &= (X - FH_1G_2)H_2 - FH_1G_1\\
F &= XH_2 - FH_1G_2H_2 - FH_1G_1\\
F(1 + H_1G_2H_2 + H_1G_1) &= XH_2\\
F &= \frac{XH_2}{1 + H_1G_2H_2 + H_1G_1} \qquad (3)
\end{aligned}$$

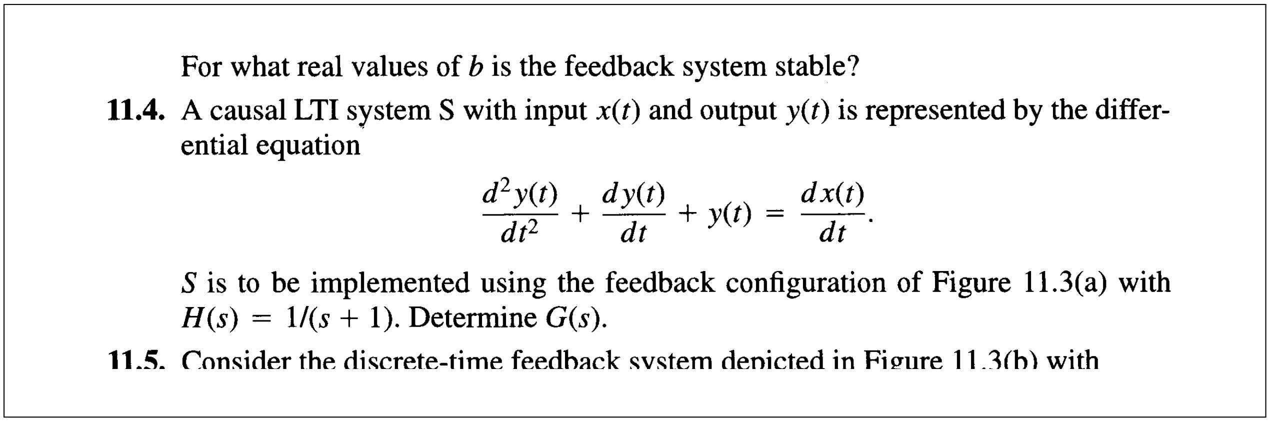
But

$$Y(z) = F(z)H_1(z)$$

Hence using (3) into the above gives

$$\begin{aligned}
Y(z) &= \frac{XH_2}{1 + H_1G_2H_2 + H_1G_1}H_1\\
\frac{Y(z)}{X(z)} &= \frac{H_2H_1}{1 + H_1G_2H_2 + H_1G_1}
\end{aligned}$$

4.10.3 Problem 11.4



For what real values of b is the feedback system stable?

11.4. A causal LTI system S with input $x(t)$ and output $y(t)$ is represented by the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}.$$

S is to be implemented using the feedback configuration of Figure 11.3(a) with $H(s) = 1/(s+1)$. Determine $G(s)$.

11.5. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

Figure 4.88: Problem description

solution

Figure 11.3 a is the following

ure 11.3(a) and that of a discrete-time LTI feedback system ir

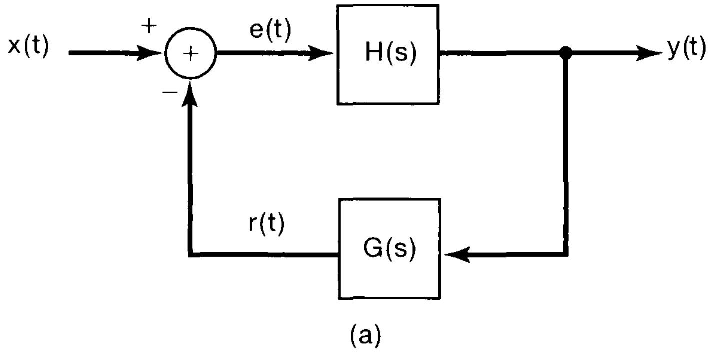


Figure 4.89: figure from book 11.3(a)

Taking the Laplace transform of the ODE gives (assuming zero initial conditions)

$$\begin{aligned} s^2Y(s)+sY(s)+Y(s) &= sX(x) \\ \frac{Y(s)}{X(s)} &= \frac{s}{s^2+s+1} \end{aligned} \tag{1}$$

From the diagram, we see that

$$Y(s)=E(s)H(s) \tag{2}$$

But $E(s)=X(s)-R(s)$ and $R(s)=E(s)H(s)G(s)$. Hence

$$\begin{aligned} E(s) &= X(s)-(E(s)H(s)G(s)) \\ E(s)(1+H(s)G(s)) &= X(s) \\ E(s) &= \frac{X(s)}{1+H(s)G(s)} \end{aligned}$$

Substituting the above in (2) gives

$$\begin{aligned} Y(s) &= \frac{X(s)}{1+H(s)G(s)}H(s) \\ \frac{Y(s)}{X(s)} &= \frac{H(s)}{1+H(s)G(s)} \end{aligned} \tag{3}$$

Comparing (3) and (1) shows that

$$\frac{H(s)}{1+H(s)G(s)}=\frac{s}{s^2+s+1}$$

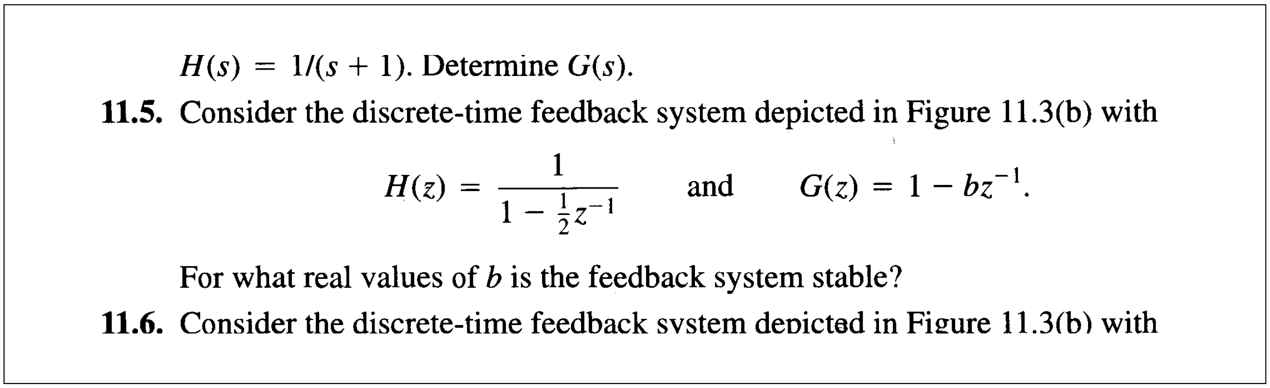
But we are given that $H(s) = \frac{1}{s+1}$. Hence the above becomes

$$\frac{\frac{1}{s+1}}{1+\frac{1}{s+1}G(s)} = \frac{s}{s^2+s+1}$$

Now we solve for $G(s)$

$$\begin{aligned}
\frac{\frac{1}{s+1}}{\frac{s+1+G(s)}{s+1}} &= \frac{s}{s^2+s+1}\\
\frac{1}{s+1+G(s)} &= \frac{s}{s^2+s+1}\\
s^2+s+sG(s) &= s^2+s+1\\
sG(s) &= s^2+s+1-s^2-s\\
G(s) &= \frac{1}{s}
\end{aligned}$$

4.10.4 Problem 11.5



$H(s) = 1/(s+1)$. Determine $G(s)$.

11.5. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{and} \qquad G(z) = 1 - bz^{-1}.$$

For what real values of b is the feedback system stable?

11.6. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

Figure 4.90: Problem description

solution

Figure 11.3 b is the following

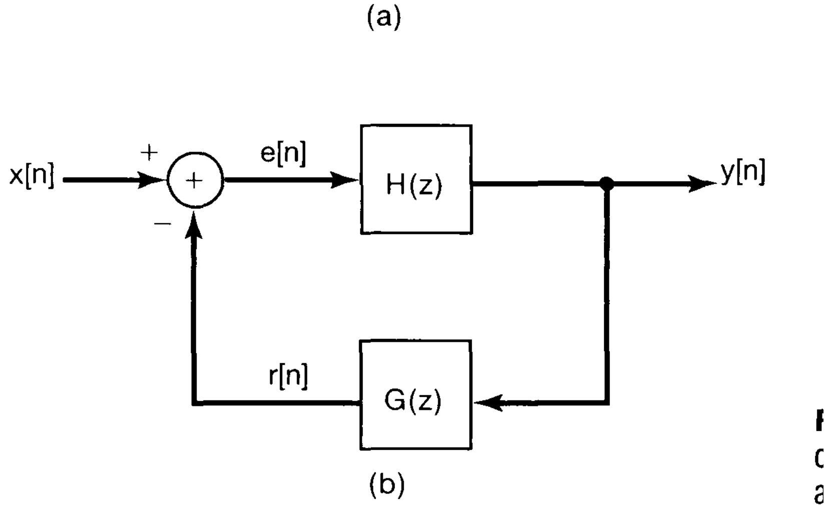


Figure 4.91: figure from book 11.3(b)

From the diagram $Y(z) = E(z)H(z)$ but $E(z) = X(z) - R(z)$ and $R(z) = E(z)H(z)G(z)$, hence

$$\begin{aligned} E(z) &= X(z) - E(z)H(z)G(z) \\ E(z)(1 + H(z)G(z)) &= X(z) \\ E(z) &= \frac{X(z)}{(1 + H(z)G(z))} \end{aligned}$$

Therefore

$$\begin{aligned} Y(z) &= E(z)H(z) \\ &= \frac{X(z)}{(1 + H(z)G(z))}H(z) \\ \frac{Y(z)}{X(z)} &= \frac{H(z)}{1 + H(z)G(z)} \end{aligned}$$

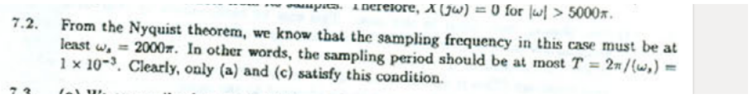
But $H(z)=\frac{1}{1-\frac{1}{2}z^{-1}}$ and $G(z)=1-bz^{-1}$. Hence the above becomes

$$\begin{aligned}\frac{Y(z)}{X(z)} &= \frac{\frac{1}{1-\frac{1}{2}z^{-1}}}{1+\frac{1}{1-\frac{1}{2}z^{-1}}\left(1-bz^{-1}\right)}\\ &= \frac{1}{1-\frac{1}{2}z^{-1}+1-bz^{-1}}\\ &= \frac{1}{2-\frac{1}{2}z^{-1}-bz^{-1}}\\ &= \frac{1}{2-\left(\frac{1}{2}+b\right)z^{-1}}\\ &= \frac{1}{2}\frac{1}{1-\left(\frac{1}{4}+\frac{b}{2}\right)z^{-1}}\end{aligned}$$

The pole is $\left(\frac{1}{4}+\frac{b}{2}\right)z^{-1}=1$ or $z=\frac{1}{4}+\frac{b}{2}$. For causal system the pole should be inside the unit circle for stable system (so that it has a DFT). Therefore

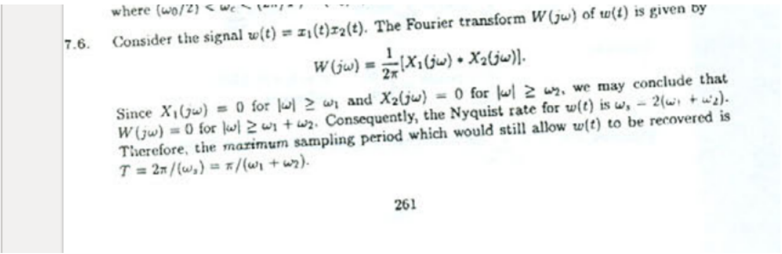
$$\begin{aligned}\left|\frac{1}{4}+\frac{b}{2}\right| &< 1\\ -1<\frac{1}{4}+\frac{b}{2} &< 1\\ -1-\frac{1}{4}<\frac{b}{2} &< 1-\frac{1}{4}\\ -\frac{5}{4}<\frac{b}{2} &< \frac{3}{4}\\ -\frac{10}{4}<b &< \frac{6}{4}\\ -\frac{5}{2}<b &< \frac{3}{2}\end{aligned}$$

4.10.5 key solution



Therefore, $X(j\omega) = 0$ for $|\omega| > 5000\pi$.

7.2. From the Nyquist theorem, we know that the sampling frequency in this case must be at least $\omega_s = 2000\pi$. In other words, the sampling period should be at most $T = 2\pi/(\omega_s) = 1 \times 10^{-3}$. Clearly, only (a) and (c) satisfy this condition.



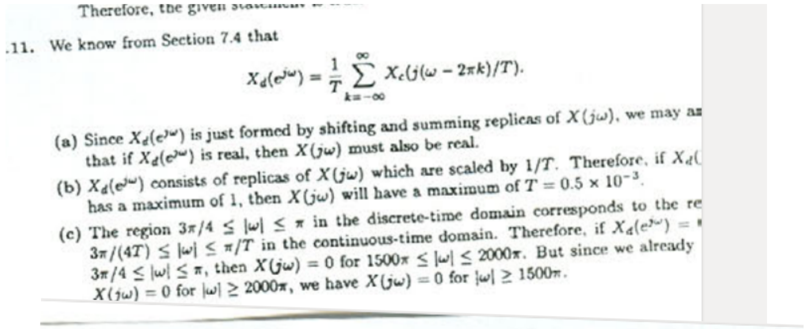
where $(\omega_0/2) < \omega_c$

7.6. Consider the signal $w(t) = x_1(t)x_2(t)$. The Fourier transform $W(j\omega)$ of $w(t)$ is given by

$$W(j\omega) = \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)].$$

Since $X_1(j\omega) = 0$ for $|\omega| \geq \omega_1$ and $X_2(j\omega) = 0$ for $|\omega| \geq \omega_2$, we may conclude that $W(j\omega) = 0$ for $|\omega| \geq \omega_1 + \omega_2$. Consequently, the Nyquist rate for $w(t)$ is $\omega_s = 2(\omega_1 + \omega_2)$. Therefore, the *maximum* sampling period which would still allow $w(t)$ to be recovered is $T = 2\pi/(\omega_s) = \pi/(\omega_1 + \omega_2)$.

261



Therefore, the given

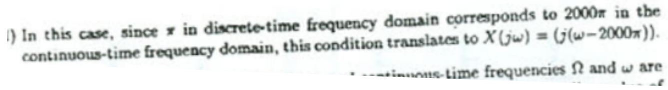
.11. We know from Section 7.4 that

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - 2\pi k)/T).$$

(a) Since $X_d(e^{j\omega})$ is just formed by shifting and summing replicas of $X(j\omega)$, we may as that if $X_d(e^{j\omega})$ is real, then $X(j\omega)$ must also be real.

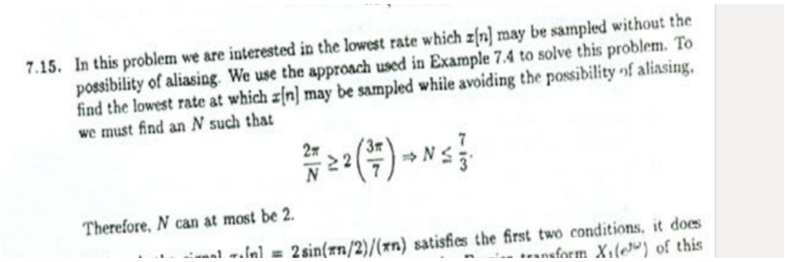
(b) $X_d(e^{j\omega})$ consists of replicas of $X(j\omega)$ which are scaled by $1/T$. Therefore, if $X_d($ has a maximum of 1, then $X(j\omega)$ will have a maximum of $T = 0.5 \times 10^{-3}$.

(c) The region $3\pi/4 \leq |\omega| \leq \pi$ in the discrete-time domain corresponds to the re $3\pi/(4T) \leq |\omega| \leq \pi/T$ in the continuous-time domain. Therefore, if $X_d(e^{j\omega}) =$ $3\pi/4 \leq |\omega| \leq \pi$, then $X(j\omega) = 0$ for $1500\pi \leq |\omega| \leq 2000\pi$. But since we already $X(j\omega) = 0$ for $|\omega| \geq 2000\pi$, we have $X(j\omega) = 0$ for $|\omega| \geq 1500\pi$.



) In this case, since π in discrete-time frequency domain corresponds to 2000π in the continuous-time frequency domain, this condition translates to $X(j\omega) = (j(\omega - 2000\pi))$.

continuous-time frequencies Ω and ω are

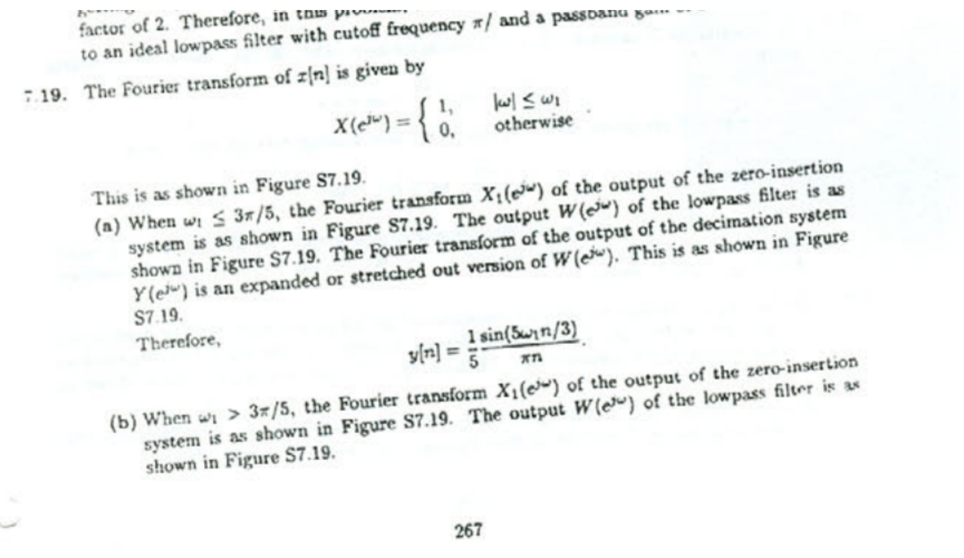


7.15. In this problem we are interested in the lowest rate which $x[n]$ may be sampled without the possibility of aliasing. We use the approach used in Example 7.4 to solve this problem. To find the lowest rate at which $x[n]$ may be sampled while avoiding the possibility of aliasing, we must find an N such that

$$\frac{2\pi}{N} \geq 2\left(\frac{3\pi}{7}\right) \Rightarrow N \leq \frac{7}{3}.$$

Therefore, N can at most be 2.

$x_1[n] = 2\sin(\pi n/2)/(\pi n)$ satisfies the first two conditions, it does transform $X_1(e^{j\omega})$ of this



factor of 2. Therefore, in this
to an ideal lowpass filter with cutoff frequency $\pi/$ and a passband

7.19. The Fourier transform of $x[n]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_1 \\ 0, & \text{otherwise} \end{cases}.$$

This is as shown in Figure S7.19.

(a) When $\omega_1 \le 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19. The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an expanded or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19.

Therefore,

$$y[n] = \frac{1}{5}\frac{\sin(5\omega_1 n/3)}{\pi n}.$$

(b) When $\omega_1 > 3\pi/5$, the Fourier transform $X_1(e^{j\omega})$ of the output of the zero-insertion system is as shown in Figure S7.19. The output $W(e^{j\omega})$ of the lowpass filter is as shown in Figure S7.19.

267

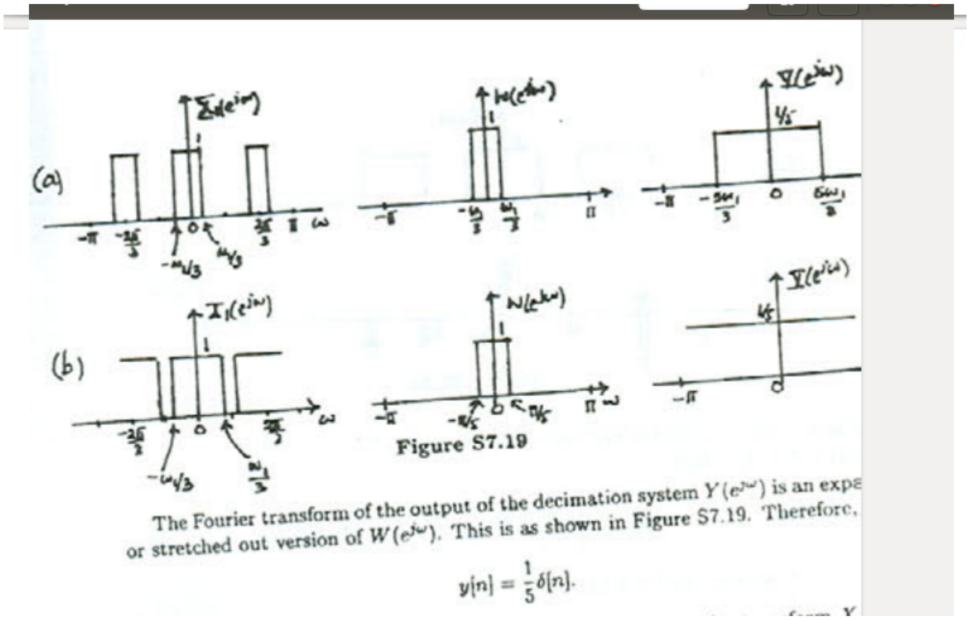


Figure S7.19

The Fourier transform of the output of the decimation system $Y(e^{j\omega})$ is an expa
or stretched out version of $W(e^{j\omega})$. This is as shown in Figure S7.19. Therefore,

$$y[n] = \frac{1}{5}\delta[n].$$

7.24. We may express $s(t)$ as $s(t) = \hat{s}(t) - 1$, where $\hat{s}(t)$ is as shown in Figure S7.24.

We may easily show that

$$\hat{S}(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T).$$

From this, we obtain

$$S(j\omega) = \hat{S}(j\omega) - 2\pi\delta(\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k\Delta/T)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

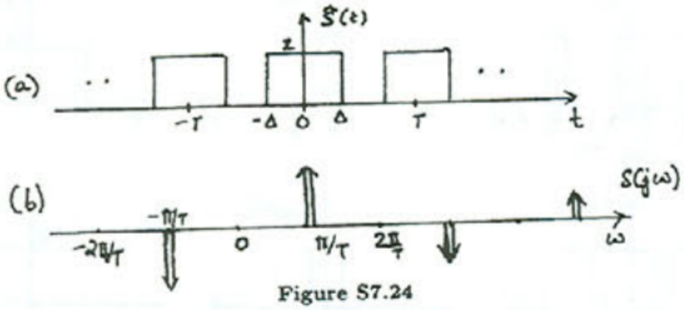


Figure S7.24

Clearly, $S(j\omega)$ consists of impulses spaced every $2\pi/T$.

(a) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

Now, since $w(t) = s(t)x(t)$,

$$W(j\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/3)}{k}X(j(\omega - k2\pi/T)) - 2\pi X(j\omega).$$

Therefore, $W(j\omega)$ consists of replicas of $X(j\omega)$ which are spaced $2\pi/T$ apart. In order to avoid aliasing, ω_M should be less that π/T. Therefore, $T_{max} = \pi/\omega_M$.

(b) If $\Delta = T/3$, then

$$S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin(2\pi k/4)}{k}\delta(\omega - k2\pi/T) - 2\pi\delta(\omega).$$

We note that $S(j\omega) = 0$ for $k = 0, \pm 2, \pm 4, \cdots$. This is as sketched in Figure S7.24

Therefore, the replicas of $X(j\omega)$ in $W(j\omega)$ are now spaced $4\pi/T$ apart. In order to avoid aliasing, ω_M should be less that $2\pi/T$. Therefore, $T_{max} = 2\pi/\omega_M$.

Chapter 5

study notes, cheat sheet

Local contents

5.1 Transform tables

5.1.1 Fourier table

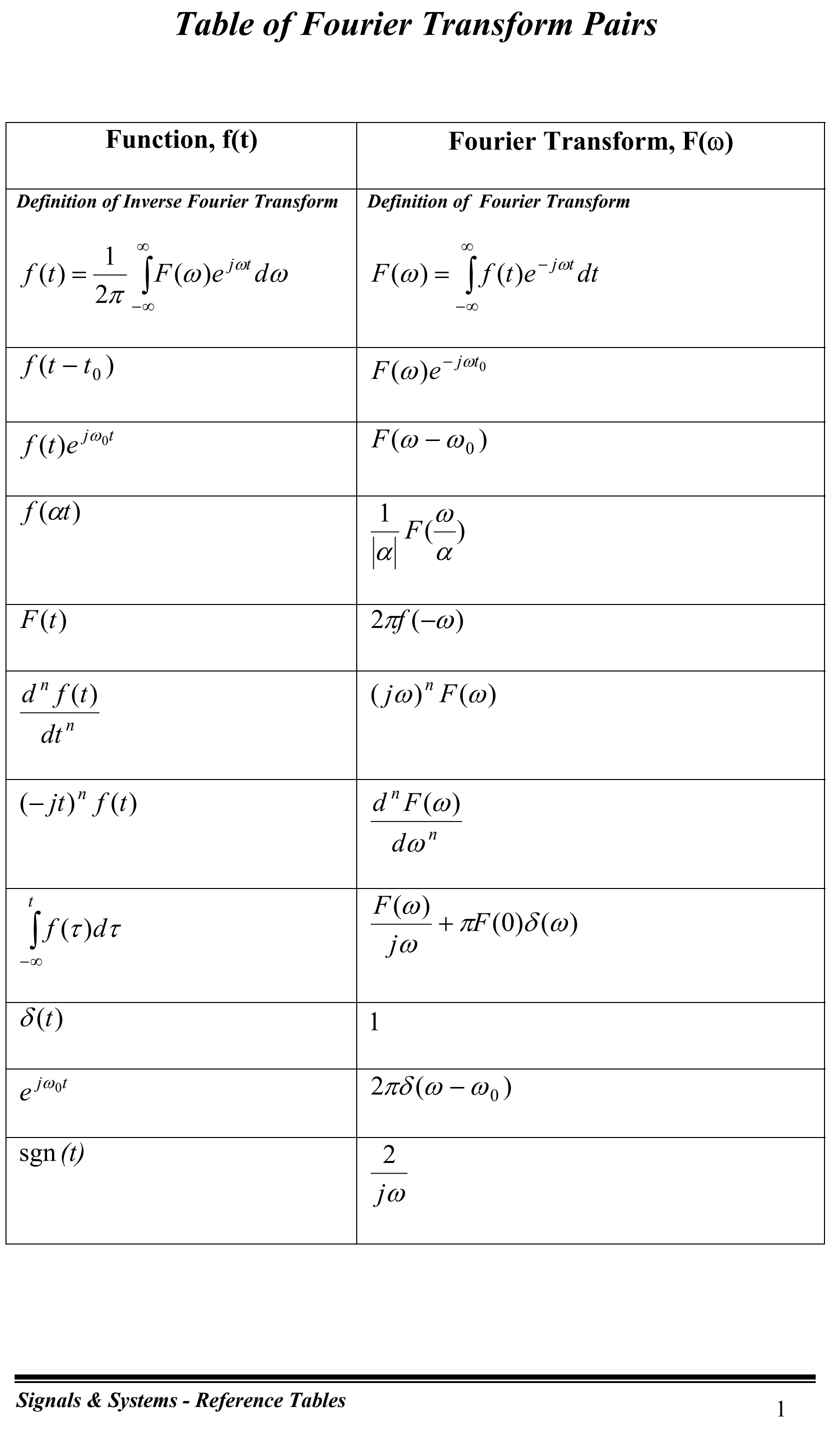


Table of Fourier Transform Pairs

Function, f(t)	**Fourier Transform, F(ω)**
Definition of Inverse Fourier Transform $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}\,d\omega$	***Definition of Fourier Transform*** $F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}\,dt$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F(\frac{\omega}{\alpha})$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{F(\omega)}{j\omega}+\pi F(0)\delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
sgn *(t)*	$\frac{2}{j\omega}$

Fourier Transform Table

UBC M267 Resources for 2005

$F(t)$	$\widehat{F}(\omega)$	Notes	(0)
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	Definition.	(1)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega t}\,d\omega$	$\widehat{f}(\omega)$	Inversion formula.	(2)
$\widehat{f}(-t)$	$2\pi f(\omega)$	Duality property.	(3)
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	a constant, $\Re e(a) > 0$	(4)
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	a constant, $\Re e(a) > 0$	(5)
$\beta(t) = \begin{cases} 1, & \text{if } \|t\| < 1, \\ 0, & \text{if } \|t\| > 1 \end{cases}$	$2\,\text{sinc}(\omega) = 2\frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi}\,\text{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)
$f'(t)$	$i\omega\widehat{f}(\omega)$	Derivative in time.	(8)
$f''(t)$	$(i\omega)^2\widehat{f}(\omega)$	Higher derivatives similar.	(9)
$tf(t)$	$i\frac{d}{d\omega}\widehat{f}(\omega)$	Derivative in frequency.	(10)
$t^2f(t)$	$i^2\frac{d^2}{d\omega^2}\widehat{f}(\omega)$	Higher derivatives similar.	(11)
$e^{i\omega_0 t}f(t)$	$\widehat{f}(\omega-\omega_0)$	Modulation property.	(12)
$f\left(\frac{t-t_0}{k}\right)$	$ke^{-i\omega t_0}\widehat{f}(k\omega)$	Time shift and squeeze.	(13)
$(f*g)(t)$	$\widehat{f}(\omega)\widehat{g}(\omega)$	Convolution in time.	(14)
$u(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t > 0 \end{cases}$	$\frac{1}{i\omega} + \pi\delta(\omega)$	Heaviside step function.	(15)
$\delta(t-t_0)f(t)$	$e^{-i\omega t_0}f(t_0)$	Assumes f continuous at t_0.	(16)
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	Useful for $\sin(\omega_0 t)$, $\cos(\omega_0 t)$.	(17)

Convolution:
$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\,du = \int_{-\infty}^{\infty} f(u)g(t-u)\,du.$$

Parseval:
$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right|^2 d\omega.$$

$j\frac{1}{\pi t}$	$\mathrm{sgn}(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty}F_n\delta(\omega-n\omega_0)$
$rect(\frac{t}{\tau})$	$\tau Sa(\frac{\omega\tau}{2})$
$\frac{B}{2\pi}Sa(\frac{Bt}{2})$	$rect(\frac{\omega}{B})$
$tri(t)$	$Sa^2(\frac{\omega}{2})$
$A\cos(\frac{\pi t}{2\tau})rect(\frac{t}{2\tau})$	$\frac{A\pi}{\tau}\frac{\cos(\omega\tau)}{(\pi/2\tau)^2-\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)\cos(\omega_0 t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
$u(t)\sin(\omega_0 t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$
$u(t)e^{-\alpha t}\cos(\omega_0 t)$	$\frac{(\alpha+j\omega)}{\omega_0^2+(\alpha+j\omega)^2}$

$u(t)e^{-\alpha t}\sin(\omega_0 t)$	$\dfrac{\omega_0}{\omega_0^2+(\alpha+j\omega)^2}$
$e^{-\alpha\|t\|}$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$
$u(t)e^{-\alpha t}$	$\dfrac{1}{\alpha+j\omega}$
$u(t)te^{-\alpha t}$	$\dfrac{1}{(\alpha+j\omega)^2}$

- ***Trigonometric Fourier Series***

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(\omega_0 nt)+b_n\sin(\omega_0 nt)\right)$$

where

$$a_0=\frac{1}{T}\int_0^T f(t)dt \quad , \quad a_n=\frac{2}{T}\int_0^T f(t)\cos(\omega_0 nt)dt \quad \text{, and}$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(\omega_0 nt)dt$$

- ***Complex Exponential Fourier Series***

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{j\omega nt} \quad \text{, where} \quad F_n=\frac{1}{T}\int_0^T f(t)e^{-j\omega_0 nt}dt$$

Some Useful Mathematical Relationships

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$2\cos^2(x) = 1 + \cos(2x)$$

$$2\sin^2(x) = 1 - \cos(2x)$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$2\cos(x)\cos(y) = \cos(x - y) + \cos(x + y)$$

$$2\sin(x)\sin(y) = \cos(x - y) - \cos(x + y)$$

$$2\sin(x)\cos(y) = \sin(x - y) + \sin(x + y)$$

Useful Integrals

$\int \cos(x)dx$	$\sin(x)$
$\int \sin(x)dx$	$-\cos(x)$
$\int x\cos(x)dx$	$\cos(x) + x\sin(x)$
$\int x\sin(x)dx$	$\sin(x) - x\cos(x)$
$\int x^2\cos(x)dx$	$2x\cos(x) + (x^2 - 2)\sin(x)$
$\int x^2\sin(x)dx$	$2x\sin(x) - (x^2 - 2)\cos(x)$
$\int e^{\alpha x}dx$	$\frac{e^{\alpha x}}{a}$
$\int xe^{\alpha x}dx$	$e^{\alpha x}\left[\frac{x}{a} - \frac{1}{a^2}\right]$
$\int x^2e^{\alpha x}dx$	$e^{\alpha x}\left[\frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3}\right]$
$\int \frac{dx}{\alpha + \beta x}$	$\frac{1}{\beta}\ln\lvert\alpha + \beta x\rvert$
$\int \frac{dx}{\alpha^2 + \beta^2 x^2}$	$\frac{1}{\alpha\beta}\tan^{-1}(\frac{\beta x}{\alpha})$

Engineering Tables/Fourier Transform Table 2

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< Engineering Tables
Jump to: navigation, search

Signal	**Fourier transform unitary, angular frequency**	**Fourier transform unitary, ordinary frequency**	**Remarks**
$g(t) \equiv$ $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega$	$G(\omega) \equiv$ $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt$	$G(f) \equiv$ $\int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt$	
10 $\mathrm{rect}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot \mathrm{sinc}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\|a\|}\cdot \mathrm{sinc}\left(\frac{f}{a}\right)$	The rectangular pulse and the normalized sinc function
11 $\mathrm{sinc}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot \mathrm{rect}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\|a\|}\cdot \mathrm{rect}\left(\frac{f}{a}\right)$	Dual of rule 10. The rectangular function is an idealized low-pass filter, and the sinc function is the non-causal impulse response of such a filter.
12 $\mathrm{sinc}^2(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot \mathrm{tri}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\|a\|}\cdot \mathrm{tri}\left(\frac{f}{a}\right)$	*tri* is the triangular function
13 $\mathrm{tri}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot \mathrm{sinc}^2\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\|a\|}\cdot \mathrm{sinc}^2\left(\frac{f}{a}\right)$	Dual of rule 12.
14 $e^{-\alpha t^2}$	$\frac{1}{\sqrt{2\alpha}}\cdot e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\alpha}}\cdot e^{-\frac{(\pi f)^2}{\alpha}}$	Shows that the Gaussian function exp(- at^2) is its own Fourier transform. For this to be integrable we must have Re(a) > 0.

$e^{iat^2} = e^{-\alpha t^2}\Big\|_{\alpha=-ia}$	$\frac{1}{\sqrt{2a}} \cdot e^{-i\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)}$	$\sqrt{\frac{\pi}{a}} \cdot e^{-i\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)}$	common in optics
$\cos(at^2)$	$\frac{1}{\sqrt{2a}} \cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$	$\sqrt{\frac{\pi}{a}} \cos\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)$	
$\sin(at^2)$	$\frac{-1}{\sqrt{2a}} \sin\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$	$-\sqrt{\frac{\pi}{a}} \sin\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)$	
$e^{-a\|t\|}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2}$	$\frac{2a}{a^2+4\pi^2 f^2}$	*a>0*
$\frac{1}{\sqrt{\|t\|}}$	$\frac{1}{\sqrt{\|\omega\|}}$	$\frac{1}{\sqrt{\|f\|}}$	the transform is the function itself
$J_0(t)$	$\sqrt{\frac{2}{\pi}} \cdot \frac{\mathrm{rect}\left(\frac{\omega}{2}\right)}{\sqrt{1-\omega^2}}$	$\frac{2 \cdot \mathrm{rect}(\pi f)}{\sqrt{1-4\pi^2 f^2}}$	$J_0(t)$ is the Bessel function of first kind of order 0, *rect* is the rectangular function
$J_n(t)$	$\sqrt{\frac{2}{\pi}} \frac{(-i)^n T_n(\omega)\mathrm{rect}\left(\frac{\omega}{2}\right)}{\sqrt{1-\omega^2}}$	$\frac{2(-i)^n T_n(2\pi f)\mathrm{rect}(\pi f)}{\sqrt{1-4\pi^2 f^2}}$	it's the generalization of the previous transform; $T_n(t)$ is the Chebyshev polynomial of the first kind.
$\frac{J_n(t)}{t}$	$\sqrt{\frac{2}{\pi}} \frac{i}{n}(-i)^n \cdot U_{n-1}(\omega)$ $\cdot \sqrt{1-\omega^2}\mathrm{rect}\left(\frac{\omega}{2}\right)$	$\frac{2i}{n}(-i)^n \cdot U_{n-1}(2\pi f)$ $\cdot \sqrt{1-4\pi^2 f^2}\mathrm{rect}(\pi f)$	$U_n(t)$ is the Chebyshev polynomial of the second kind

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Category: Engineering Tables

Views

5.1.2 Properties tables, Feb 27, 2020

Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Property	**Periodic Signal**	**Fourier Series Coefficients**
	$\left.\begin{array}{r} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Table 2: **Properties of the Discrete-Time Fourier Series**

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

Property	**Periodic signal**	**Fourier series coefficients**
	$\left.\begin{array}{r} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{r} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shift	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shift	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \left(\begin{array}{l} \text{viewed as} \\ \text{periodic with} \\ \text{period } mN \end{array}\right)$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)}) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \left(\begin{array}{l} \text{finite-valued and} \\ \text{periodic only if } a_0 = 0 \end{array}\right)$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Table 3: **Properties of the Continuous-Time Fourier Transform**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Property	**Aperiodic Signal**	**Fourier transform**
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time-shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t)=\mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$$

Table 4: **Basic Continuous-Time Fourier Transform Pairs**

Signal	**Fourier transform**	**Fourier series coefficients (if periodic)**
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0, \quad \text{otherwise}$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0, \quad \text{otherwise}$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0, \quad \text{otherwise}$
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ $\left(\text{this is the Fourier series representation for any choice of } T > 0\right)$
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc} \left(\frac{k\omega_0 T_1}{\pi} \right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta \left(\omega - \frac{2\pi k}{T} \right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert \omega \rvert < W \\ 0, & \lvert \omega \rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t),$ $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Table 5: **Properties of the Discrete-Time Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Property	**Aperiodic Signal**	**Fourier transform**
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ Periodic with period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time-Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansions	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$$

Table 6: **Basic Discrete-Time Fourier Transform Pairs**

Signal	**Fourier transform**	**Fourier series coefficients (if periodic)**
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1+1}{N}, \quad k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Table 7: **Properties of the Laplace Transform**

Property	**Signal**	**Transform**	**ROC**
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	Shifted version of R [i.e., s is in the ROC if $(s-s_0)$ is in R]
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Table 8: **Laplace Transforms of Elementary Functions**

Signal	**Transform**	**ROC**
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3. $-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4. $\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5. $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6. $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\Re e\{\alpha\}$
7. $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\Re e\{\alpha\}$
8. $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\Re e\{\alpha\}$
9. $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\Re e\{\alpha\}$
10. $\delta(t-T)$	e^{-sT}	All s
11. $[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12. $[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13. $[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
14. $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
15. $u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16. $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Table 9: **Properties of the z-Transform**

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R except for the possible addition or deletion of the origin
Scaling in the z-Domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R =$ the set of points $\{\|a\|z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$ where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
Differentiation in the z-Domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, then
$x[0] = \lim_{z \to \infty} X(z)$

Table 10: **Some Common z-Transform Pairs**

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$

5.2 cheat sheet

■ Fourier series. Periodic signals, Continuous time

Let $\omega_0 = \frac{2\pi}{T_0}$ be the fundamental frequency (rad/sec), and T_0 the fundamental period, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

■ Fourier series. Periodic signals, Discrete time

Let $\Omega_0 = \frac{2\pi}{N}$ be the fundamental frequency (rad/sample), and N the fundamental period, then

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$$

■ Fourier transform. Non periodic signal, Continuous time.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

It is also possible to obtain a Fourier transform for periodic signal. For $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$ its Fourier transform becomes ($\omega_0 = \frac{2\pi}{T_0}$)

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

■ Fourier transform. Non periodic signal, Discrete time.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

It is also possible to obtain a Fourier transform for periodic discrete signal, where $\Omega_0 = \frac{2\pi}{N}$

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

■ When input to LTI system is $x(t) = e^{j\omega t}$ and system has impulse response $h(t)$ then output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)} d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau \\ &= e^{j\omega t} H(\omega) \end{aligned}$$

Where $H(\omega)$ is the Fourier transform of $h(t)$. In the above $e^{j\omega t}$ is called eigenfucntions of the system and $H(\omega)$ the eigenvalues.

■ If input $x(t) = a\cos(5\omega_0 t + \theta)$ and $H(\omega)$ is the Fourier transform of the system, then

$$y(t) = a\left|H(5\omega_0)\right|\cos(5\omega_0 t + \theta + \arg H(5\omega_0))$$

Same for discrete time.

■ Modulation. $y(t) = x(t)\, h(t)$ in CTFT becomes $Y(\omega) = \frac{1}{2\pi} X(\omega) \circledast H(\omega)$ where $X(\omega) \circledast H(\omega) = \int_{-\infty}^{\infty} X(z)\, H(\omega - z)\, dz$. Notice the extra $\frac{1}{2\pi}$ factor.

■ To find discrete period given a signal, write $x[n] = x[n+N]$ and then solve for N. See HW's.

■ $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ and $\sum_{n=N}^{\infty} a^n = \frac{a^N}{1-a}$ and $\sum_{n=0}^{N} a^n = \frac{a^{1+N}-1}{a-1}$, and $\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1}-a^{N_2+1}}{1-a}$

■ Fourier transform relations. $y(t) \Longleftrightarrow Y(\omega)$ then $y(at) \Longleftrightarrow \frac{1}{a} Y\left(\frac{\omega}{a}\right)$

■ Euler relations. $\cos x = \frac{e^{jx}+e^{-jx}}{2}, \sin x = \frac{e^{jx}-e^{-jx}}{2j}$

■ Circuit. Voltage cross resistor R is $V(t) = Ri(t)$. Voltage cross inductor L is $V(t) = L\frac{di}{dt}$ and current across capacitor C is $i(t) = C\frac{dV}{dt}$

■ Partial fractions.

$\frac{f(x)}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{f(x)}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{f(x)}{(x-a)\left(x^2+bx+c\right)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
$\frac{f(x)}{(x-a)(x+d)^2}$	$\frac{A}{x-a} + \frac{B}{x+d} + \frac{C}{(x+d)^2}$
$\frac{f(x)}{(x+d)^2}$	$\frac{A}{x+d} + \frac{B}{(x+d)^2}$
$\frac{f(x)}{(x-a)\left(x^2-b^2\right)}$	$\frac{A}{x+d} + \frac{Bx+C}{x^2-b^2}$
$\frac{f(x)}{\left(x^2-a\right)\left(x^2-b\right)}$	$\frac{Ax+B}{x^2-a} + \frac{Cx+D}{x^2-b}$
$\frac{f(x)}{\left(x^2-a\right)^2}$	$\frac{Ax+B}{x^2-a} + \frac{Cx+D}{\left(x^2-a\right)^2}$

■ Parsevel's. For non-periodic cont. time: $\int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$. For periodic cont. time : $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|a_k|^2$. For discrete: $\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\Omega)|^2\,d\Omega = \sum_{n=-\infty}^{\infty}|x[n]|^2$.

■ Properties Fourier series. If $a_k = a_{-k}^*$ then $x(t)$ is real. If a_k is even, then $x(t)$ is even. For $x(t)$ real and odd, then a_k are pure imaginary and odd. i.e. $a_k = -a_{-k}$, and $a_0 = 0$.

■ More Fourier transform relations. Continuos time

$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$x(t)\,e^{-j\omega_0 t}$	$X(\omega+\omega_0)$
$x(t)\,e^{j\omega_0 t}$	$X(\omega-\omega_0)$
$\frac{\sin(a\omega)}{\omega}$	Box from $t = -a \cdots a$

Discrete time

$u[n]$	$\frac{1}{1-e^{-j\Omega}}$
$u[n-1]$	$e^{-j\Omega}U(\Omega) = e^{-j\Omega}\frac{1}{1-e^{-j\Omega}}$
$a^n u[n]$	$\frac{1}{1-ae^{-j\Omega}}$
$e^{j\Omega_0 n}x[n]$	$X(\Omega-\Omega_0)$

From above we see that unit delay in discrete time means multiplying by $e^{-j\Omega}$.

■ Difference equations. $y[n-1] \iff e^{-j\Omega}Y(\Omega)$. For example, given $y[n]-ay[n-1]=x[n]$ then applying DFT gives $Y(\Omega)-ae^{-j\Omega}Y(\Omega)=X(\Omega)$ or $H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{1}{1-ae^{-j\Omega}}$. From tables, the inverse DFT of this is $a^n u[n]$. Need to know partial fractions sometimes. For example given $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$ then

$$
\begin{aligned}
Y(\Omega)-\frac{3}{4}e^{-j\Omega}Y(\Omega)+\frac{1}{8}e^{-j2\Omega}Y(\Omega) &= 2X(\Omega)\\
H(\Omega) &= \frac{Y(\Omega)}{X(\Omega)}\\
&= \frac{2}{\left(1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-j2\Omega}\right)}\\
&= \frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}
\end{aligned}
$$

And using partial fractions gives $H(\Omega)=\frac{4}{1-\frac{1}{2}e^{-j\Omega}}-\frac{2}{1-\frac{1}{4}e^{-j\Omega}}$. Hence using above table gives

$$h[n]=\left(4\left(\frac{1}{2}\right)^n-2\left(\frac{1}{4}\right)^n\right)u[n]$$

■ $|X(\omega)|^2$ may be interpreted as the energy density spectrum of $x(t)$. This means $\frac{1}{2\pi}|X(\omega)|^2\,d\omega$ is amount of energy in $d\omega$ range of frequencies. i.e. between ω and $\omega+d\omega$. $|X(\omega)|$ is called the gain of the system and $\arg(H(\omega))$ is called the phase shift of the system. When $\arg(H(\omega))$ is linear function in ω then the effect in time domain is time shift. (delay).

■ z transforms $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$. If $x[n] \rightarrow X(z)$ then $x[n-1] \rightarrow z^{-1}X(z)$.

■ $\frac{\sin(ax)}{ax} = \operatorname{sinc}\left(\frac{ax}{\pi}\right)$ and $\frac{\sin(x)}{x} = \operatorname{sinc}\left(\frac{x}{\pi}\right)$. In class we use $\frac{\sin(\omega_c t)}{\pi t}$. This has FT as rectangle from $-\omega_c$ to ω_c and amplitude 1.

■ in digital, sampling rate is in hz, but units is samples per second and not cycles per second as with analog.

■

$$\Omega = \frac{\omega}{F_s}$$

where F_s is sampling rate in samples per second, and Ω is unnormalized digital frequency (radians per sample) and ω is analog frequency (radians per second). This can also be written as

$$\Omega = \omega T_s$$

where here T_s is seconds per sample (i.e. number of seconds to obtain one sample). Per sample is used to make the units come out OK.

■ Trig identities

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)\\
\cos A \cos B &= \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)\\
\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)
\end{aligned}$$

■ Group delay is given by $-\frac{d}{d\omega}(\arg(H(\omega)))$. For example, if $H(\omega) = \frac{1}{2+j\omega}$ then $\arg(H(\omega)) = -\arctan\left(\frac{\omega}{2}\right)$ which leads to group delay being $\frac{2}{4+\omega^2}$.

■ FT of $\cos(\omega_c t)$ has delta at $\pm\omega_c$ each of amplitude π. And FT of $\sin(\omega_c t)$ has delta at ω_c of amplitude $\frac{\pi}{j}$ and has delta at $-\omega_c$ of amplitude $\frac{-\pi}{j}$ and $\frac{\sin(\omega_c t)}{\pi t}$ has FT as rectangle of amplitude 1 and width from $-\omega_c$ to $+\omega_c$.

$$\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}\\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right) u[n] e^{-j\Omega n}\\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right) e^{-j\Omega n}
\end{aligned}$$

But $\cos\left(\frac{\pi n}{2}\right) = \frac{1}{2}\left(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right)$ and the above becomes

$$
\begin{aligned}
X(\Omega) &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{1}{2}\left(e^{j\frac{\pi n}{2}}+e^{-j\frac{\pi n}{2}}\right)e^{-j\Omega n}\\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}e^{j\frac{\pi n}{2}}e^{-j\Omega n}+\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}e^{-j\frac{\pi n}{2}}e^{-j\Omega n}\right)\\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)}\right)^{n}+\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\left(-\frac{\pi}{2}-\Omega\right)}\right)^{n}\right)
\end{aligned}
$$

Since $\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)} < 1$ then we can use $\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$ for both terms and the above becomes

$$
\begin{aligned}
X(\Omega) &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)}}+\frac{1}{1-\frac{1}{2}e^{j\left(-\frac{\pi}{2}-\Omega\right)}}\right)\\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{2}}e^{-j\Omega}}+\frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}e^{-j\Omega}}\right)
\end{aligned}
$$

But $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$ and the above becomes

$$
\begin{aligned}
X(\Omega) &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}je^{-j\Omega}}+\frac{1}{1+\frac{1}{2}je^{-j\Omega}}\right)\\
&= \frac{1}{2}\left(\frac{1+\frac{1}{2}je^{-j\Omega}+1-\frac{1}{2}je^{-j\Omega}}{\left(1-\frac{1}{2}je^{-j\Omega}\right)\left(1+\frac{1}{2}je^{-j\Omega}\right)}\right)\\
&= \frac{1}{2}\left(\frac{2}{1+\frac{1}{2}je^{-j\Omega}-\frac{1}{2}je^{-j\Omega}-\frac{1}{4}j^{2}e^{-2j\Omega}}\right)\\
&= \frac{1}{1+\frac{1}{4}e^{-2j\Omega}}
\end{aligned}
$$

■ Z transforms

$u[n]$	Z
$a^{n}u[n]$	$\frac{1}{1-az^{-1}}$
$a^{n-1}u[n-1]$	$z^{-1}\frac{1}{1-az^{-1}}$
$a^{n-2}u[n-2]$	$z^{-2}\frac{1}{1-az^{-1}}$

If the ROC outside the out most pole, then right-handed signal. (Causal). If the ROC isinside the inner most pole, then left-handed signal (non causal).

5.3 Study notes

5.3.1 When input is complex exponential

When input is $x[n] = e^{j\Omega_0 n}$ and system is given by $H(\Omega)$ then the output is $y[n] = e^{j\Omega_0 n} H(\Omega_0)$ which is the same as $y[n] = e^{j\Omega_0 n} \left|H(\Omega_0)\right| e^{j \arg H(\Omega_0)}$.

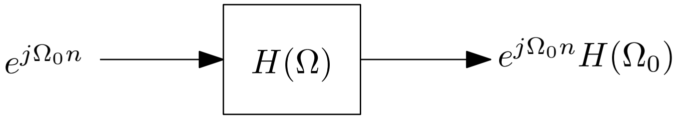

Figure 5.1: Output when input is complex exponential

Hence when the input is linear combination of complex exponentials

$$\begin{aligned} A\cos(\Omega_0 n + \theta) &= \frac{A}{2}\left(e^{j(\Omega_0 n + \theta)} + e^{-j(\Omega_0 n + \theta)}\right) \\ &= \left(\frac{A}{2} e^{j\theta}\right) e^{j\Omega_0 n} + \left(\frac{A}{2} e^{-j\theta}\right) e^{-j\Omega_0 n} \end{aligned}$$

Then, and since the system is linear, then the output will be scaled and linear sum of each output corresponding to each term above. In other words, when the input is $\left(\frac{A}{2} e^{j\theta}\right) e^{j\Omega_0 n}$ then the output is

$$\begin{aligned} y_1[n] &= \left(\frac{A}{2} e^{j\theta}\right) e^{j\Omega_0 n} \left|H(\Omega_0)\right| e^{j \arg H(\Omega_0)} \\ &= \left|H(\Omega_0)\right| \frac{A}{2} e^{j(\Omega_0 n + \theta + \arg H(\Omega_0))} \qquad (1) \end{aligned}$$

And when the input is $\left(\frac{A}{2} e^{-j\theta}\right) e^{-j\Omega_0 n}$ then the output is

$$\begin{aligned} y_2[n] &= \left(\frac{A}{2} e^{-j\theta}\right) e^{-j\Omega_0 n} \left|H(-\Omega_0)\right| e^{j \arg H(-\Omega_0)} \\ &= \left|H(-\Omega_0)\right| \frac{A}{2} e^{-j(\Omega_0 n + \theta - \arg H(-\Omega_0))} \end{aligned}$$

But for real input, which is the case here, $\left|H(\Omega_0)\right|$ is symmetrical. Hence $\left|H(\Omega_0)\right| = \left|H(-\Omega_0)\right|$ and $\arg H(-\Omega_0) = -\arg H(\Omega_0)$ (see table 4.6 for these properties). Hence

$$y_2[n] = \left|H(\Omega_0)\right| \frac{A}{2} e^{-j(\Omega_0 n + \theta + \arg H(\Omega_0))} \qquad (2)$$

Therefore, by linearity, $y[n] = y_1[n] + y_2[n]$ or by adding (1) and (2)

$$\begin{aligned}
y[n] &= \left|H(\Omega_0)\right| \frac{A}{2} e^{j(\Omega_0 n + \theta + \arg H(\Omega_0))} + \left|H(\Omega_0)\right| \frac{A}{2} e^{-j(\Omega_0 n + \theta + \arg H(\Omega_0))} \\
&= \left|H(\Omega_0)\right| A \left(\frac{e^{j(\Omega_0 n + \theta + \arg H(\Omega_0))} + e^{-j(\Omega_0 n + \theta + \arg H(\Omega_0))}}{2} \right) \\
&= \left|H(\Omega_0)\right| A \cos\left(\Omega_0 n + \theta + \arg H(\Omega_0)\right)
\end{aligned}$$