University Course

Physics 5041
Mathematical Methods for Physics

University of Minnesota, Twin Cities
Spring 2019

My Class Notes

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Spring 2019

Contents

Chapter 1

Introduction

1.1 syllabus

Phys 5041.001

Mathematical Methods for Physics

CLASS HOME SYLLABUS POSTINGS MY SCORES

Topics Covered *modified 23-Jan-2019 at 11:22AM by Joseph Kapusta*

Topics to be covered include but are not limited to: ordinary differential equations; infinite series and sums; complex analysis; Fourier analysis; Laplace transforms; vectors, matrices, and tensors; special functions; Green's functions; partial differential equations; group theory.

Textbook *modified 23-Jan-2019 at 11:22AM by Joseph Kapusta*

There are many good textbooks on mathematical methods of physics. I have chosen the following one because it is inexpensive and it has nearly 1000 solved problems.

M. R. Spiegel, *Advanced Mathematics for Engineers and Scientists* (Schaum's Outlines).

Most of my lectures will not follow this text very closely but are instead based on a variety of sources. Here are a few I recommend:

R. V. Churchhill, *Complex Variables and Applications* (McGraw-Hill).

J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Benjamin/Cummings).

G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists* (Academic Press).

S. Hassani, *Foundations of Mathematical Physics* (Allyn and Bacon).

K. F. Riley, M. P. Hobson and S. J. Bence, *Mathematical Methods for Physics and Engineering* (Cambridge).

H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics* (Cambridge).

G. Goertzel and N. Tralli, *Some Mathematical Methods of Physics* (Dover).

M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover).

Grading *modified 2-Jan-2019 at 5:59PM by Joseph Kapusta*

The course grade will be determined on the basis of homework 30%, class participation 10%, two mid-terms worth 15% each, and final exam 30%. Homework will be due one week after it is assigned. A deduction of 10% will be assessed for every business day that the homework is late. The rational is to keep all students up to date in the course and to be fair to the grader. Students are expected to attend every lecture.

Homework: There will be approximately twelve homework assignments. Students are allowed to discuss the homework problems with each other. The rules are:

1. Each student must write up his or her own solutions.

2. List other students you discussed the problems with.

3. If you used any resources other than the required text, such as books, articles, web sites, past homework solutions, and so on, you must list them on your homework.

Class Participation: Every Friday, beginning with week two, I will assign an in-class problem at the beginning of the second period. Students will work in groups of three to solve the problem. After 30 minutes one of the groups will be asked to go to the board to show their solution. Notes from each group must be signed and collected, but will not be graded. That counts as class participation.

Mid-terms: Mid-term exams will be given on Wednesday February 27 and Wednesday April 10. They will be held in Tate B65.

Final Exam: Thursday May 9, time and room to be announced.

Grades will be assigned as follows (these are guaranteed, the cutoffs may turn out to be lower):

A: 90 to 100%
B: 80 to 90%
C: 70 to 80%
D: 50 to 70%
F: 0 to 50%

Office Hours *modified 2-Jan-2019 at 1:37PM by Joseph Kapusta*

My office is 375-16 Tate Hall. Due to the broad spectrum of students taking the course I doubt that there is a convenient time for regularly scheduled office hours. Instead students may either make an appointment or stop by and if I am not otherwise engaged I will be happy to help.

1.2 Links

1. class web page

Chapter 2

HWs

Local contents

2.1 HW 1

Local contents

2.1.1 HW 1 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 1 due Friday February 1. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (2 pts) Solve

$$x^2y' + y^2 = xyy'$$

2. (2 pts) Solve

$$y' = \frac{a^2}{(x+y)^2}$$

3. (2 pts) Solve

$$y'' + y'^2 + 1 = 0$$

4. (2 pts) Solve

$$xy' + y + x^4y^4e^x = 0$$

5. (3 pts) Find both a general solution and a singular solution of

$$x^2y'^2 - 2(xy-4)y' + y^2 = 0$$

Hint: Differentiate it once.

6. (4 pts) Find the general real solution to the following equation where $A(x)$ is a known function.

$$A(x)y'' + A'(x)y' + \frac{y}{A(x)} = 0$$

7. (5 pts) Find the general real solution to the equation

$$xy'' + \frac{3}{x}y = 1 + x^3$$

8. (5 pts) For what values of k does the equation

$$y'' - \left(\frac{1}{4} + \frac{k}{x}\right)y = 0$$

defined for $0 < x < \infty$ have a solution vanishing at $x = 0$ and at $x = \infty$?

2.1.2 Problem 1

Problem Solve $x^2y' + y^2 = xyy'$

Solution

Rewriting the ODE as

$$y'\left(x^2 - xy\right) + y^2 = 0 \tag{1}$$

Dividing by $x^2 \neq 0$ gives

$$\frac{dy}{dx}\left(1 - \frac{y}{x}\right) + \frac{y^2}{x^2} = 0$$

We see this is homogeneous of order 1. This can be confirmed by writing the above as

$$\begin{aligned} dy\left(x^2 - xy\right) + y^2dx &= 0 \\ dyx^2 - xydy + y^2dx &= 0 \end{aligned}$$

We want to find if a weight m can be found, so that the substitution $y = vx^m$ makes the above ODE separable. To find m, we assign weight m to both y and dy, and a weight of 1 to both x and dx, and then try to find if there is an m which makes each term sums to the same total weight. (in other words, we want each term units to be the same).

The term $\left(dy\right)\left(x^2\right)$ has total weight of $m+2$ (it is the exponents that we add). And the term $(x)\left(y\right)\left(dy\right)$ has total weight $1+2m$ and the last term $\left(y^2\right)(dx)$ has weight $2m+1$. Therefore we have this result for the weight of each term (there are 3 terms above).

$$\{m+2, 1+2m, 1+2m\}$$

We see that if $m = 1$ then each term will have the same total weight of 3 giving $\{3,3,3\}$. So this is homogenous ODE of order $m = 1$. Now that we know the weight, we use the substitution

$$\boxed{y = vx}$$

Hence $y' = v'x + v$. Substituting these back into (1) gives a new ODE in v which is separable. If it is not separable, it means we made a mistake somewhere.

$$\begin{aligned} (v'x + v)\left(x^2 - x^2v\right) + v^2x^2 &= 0 \\ v'x^3 - v'vx^3 + vx^2 - x^2v^2 + v^2x^2 &= 0 \\ v'x^3 - v'vx^3 + vx^2 &= 0 \end{aligned}$$

Dividing by x^3 for $x \neq 0$ gives

$$\begin{aligned}
v' - v'v + \frac{v}{x} &= 0 \\
v'(1-v) &= -\frac{v}{x} \\
\frac{dv}{dx}\frac{(1-v)}{v} &= -\frac{1}{x} \\
dv\frac{(1-v)}{v} &= -\frac{1}{x}dx \\
dv\frac{(v-1)}{v} &= \frac{1}{x}dx
\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}
\int v - \frac{1}{v}dv &= \int \frac{1}{x}dx \\
v - \ln v &= \ln x + C
\end{aligned}$$

Taking exponential of both sides gives

$$\begin{aligned}
e^{v - \ln v} &= Cx \\
\frac{e^v}{v} &= Cx
\end{aligned}$$

But $v = \frac{y}{x}$. Therefore the above becomes

$$\begin{aligned}
\frac{e^{\frac{y}{x}}}{\frac{y}{x}} &= Cx \\
\frac{e^{\frac{y}{x}}}{y} &= C
\end{aligned}$$

Hence the solution is

$$y = C_1 e^{\frac{y}{x}} \qquad x \neq 0 \tag{2}$$

Where C_1 is the constant of integration. y can not be solved for directly in the above. But we can solve for x in terms of y if needed as follows

$$\begin{aligned}
\ln y &= \ln C_1 + \frac{y}{x} \\
\ln y - C_2 &= \frac{y}{x} \\
x &= \frac{y}{\ln y - C_2} \qquad (3)
\end{aligned}$$

2.1.3 Problem 2

Problem Solve $y' = \frac{a^2}{(x+y)^2}$

Solution

Let $v(x) = x + y(x)$. Hence $v' = 1 + y'$ or $y' = v' - 1$. Substituting this back into the ODE gives

$$
\begin{aligned}
v' - 1 &= \frac{a^2}{v^2} \\
\frac{dv}{dx} &= \frac{a^2}{v^2} + 1
\end{aligned}
$$

This is separable.

$$
\begin{aligned}
\frac{dv}{\frac{a^2}{v^2} + 1} &= dx \\
\frac{v^2}{a^2 + v^2} dv &= dx
\end{aligned}
$$

By long division $\frac{v^2}{a^2+v^2} = 1 - \frac{a^2}{a^2+v^2}$. The above becomes

$$
\left(1 - \frac{a^2}{a^2 + v^2}\right) dv = dx
$$

Integrating both sides gives

$$
\begin{aligned}
\int \left(1 - \frac{a^2}{a^2 + v^2}\right) dv &= \int dx \\
\int dv - a^2 \int \frac{1}{a^2 + v^2} dv &= \int dx
\end{aligned}
$$

But $\int \frac{1}{a^2+v^2} dv = \frac{1}{a^2} \int \frac{1}{1+\left(\frac{v}{a}\right)^2} dv = \frac{1}{a^2} \left(a \arctan\left(\frac{v}{a}\right)\right) = \frac{1}{a} \arctan\left(\frac{v}{a}\right)$, hence the above becomes

$$
\begin{aligned}
v - a^2 \left(\frac{1}{a} \arctan\left(\frac{v}{a}\right)\right) &= x + C \\
v - a \arctan\left(\frac{v}{a}\right) &= x + C \\
a \arctan\left(\frac{v}{a}\right) &= v - x - C \\
\arctan\left(\frac{v}{a}\right) &= \frac{v - x}{a} + C_1
\end{aligned}
$$

Where $C_1 = \frac{-C}{a}$, a new constant. Taking the $\tan$ of both sides gives

$$
\frac{v}{a} = \tan\left(\frac{v - x}{a} + C_1\right)
$$

But $v = x + y$, and the above becomes

$$\frac{x+y}{a} = \tan\left(\frac{(x+y)-x}{a} + C_1\right)$$
$$\frac{x+y}{a} = \tan\left(\frac{y}{a} + C_1\right)$$

Therefore the final solution is

$$y = a\tan\left(\frac{y}{a} + C_1\right) - x \qquad a \neq 0$$

Where C_1 is arbitrary constant.

2.1.4 Problem 3

Problem Solve $y'' + (y')^2 + 1 = 0$

Solution

Since y is missing from the ODE, we can convert this to a first order using $y' = p(x)$. Therefore $y'' = \frac{dp}{dx}$ and the ODE becomes

$$\frac{dp}{dx} + p^2 + 1 = 0$$
$$\frac{dp}{dx} = -\left(1+p^2\right)$$
$$\frac{dp}{1+p^2} = -dx$$

Integrating both sides gives

$$\int \frac{dp}{1+p^2} = -\int dx$$
$$\arctan(p) = -x + C_1$$
$$p = \tan(-x + C_1)$$

But $p = y'$. Hence we need now to solve $\frac{dy}{dx} = \tan(-x + C_1)$. Integrating both sides gives

$$\begin{aligned} y &= \int \tan(-x+C_1)\,dx \\ &= \int \frac{\sin(-x+C_1)}{\cos(-x+C_1)}dx \\ &= \int \frac{-\sin(x-C_1)}{\cos(x-C_1)}dx \\ &= \int \frac{\frac{d}{dx}(\cos(x-C_1))}{\cos(x-C_1)}dx \end{aligned}$$

But $\int \frac{V'}{V} dx = \ln(V)$, hence the above becomes

$$y = \ln(\cos(x - C_1)) + C_2$$

Replacing $-C_1$ by new constant C_3, the final solution becomes

$$y = \ln(\cos(x + C_3)) + C_2$$

Where C_2, C_3 are constants of integration.

2.1.5 Problem 4

Problem Solve $xy' + y + x^4 y^4 e^x = 0$

Solution

Dividing by $x \neq 0$ and rewriting gives

$$y' + \frac{1}{x} y = \left(-x^3 e^x\right) y^4 \tag{1}$$

A Bernoulli ODE has the form $y' + a(x) y = b(x) y^n$ where $n \neq 1$. Comparing the above to Bernoulli ODE form, show it is Bernoulli ODE where $a(x) = \frac{1}{x}, b(x) = -x^3 e^x$. Dividing (1) by y^4 gives

$$\frac{1}{y^4} y' + \frac{1}{x} y^{-3} = -x^3 e^x$$

Letting $v = y^{-3}$ or $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$. Hence $\frac{dy}{dx} = -\frac{dv}{dx}\frac{y^4}{3}$. Substituting this in the above gives

$$\begin{aligned}
\frac{1}{y^4}\left(-\frac{dv}{dx}\frac{y^4}{3}\right) + \frac{1}{x} v &= -x^3 e^x \\
-\frac{1}{3}\frac{dv}{dx} + \frac{1}{x} v &= -x^3 e^x \\
\frac{dv}{dx} - \frac{3}{x} v &= 3x^3 e^x
\end{aligned}$$

This is now linear in v. The integrating factor $\mu = e^{\int -\frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}$. Multiplying both sides of the above by this integrating factor making the left side complete differential

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^3} v\right) &= \frac{1}{x^3} 3x^3 e^x \\
\frac{d}{dx}\left(\frac{1}{x^3} v\right) &= 3e^x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{1}{x^3} v &= 3e^x + C \\
v &= 3x^3 e^x + Cx^3 \\
&= x^3 (3e^x + C)
\end{aligned}$$

But $v = y^{-3}$, hence the above becomes

$$\frac{1}{y^3} = x^3\left(3e^x + C\right)$$
$$y^3 = \frac{1}{x^3\left(3e^x + C\right)}$$

This shows that there are 3 solutions since the above is a cubic equation. But we can leave the solution in implicit form

$$\begin{aligned} y &= \sqrt[\frac{1}{3}]{\frac{1}{x^3\left(3e^x + C\right)}} \\ &= \frac{1}{x}\sqrt[\frac{1}{3}]{\frac{1}{3e^x + C}} \end{aligned}$$

2.1.6 Problem 5

Problem Find both a general solution and a singular solution of

$$x^2\left(y'\right)^2 - 2\left(xy - 4\right)y' + y^2 = 0$$

Solution

Rewriting the ODE as

$$y^2 - 2xyy' + 8y' + x^2\left(y'\right)^2 = 0$$

Let $y' = p$ and the above becomes

$$y^2 + y\left(-2xp\right) + \left(8p + x^2p^2\right) = 0$$

This is quadratic in y. Solving for $y = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$

$$\begin{aligned} y &= xp \pm \frac{1}{2}\sqrt{4x^2p^2 - 4\left(8p + x^2p^2\right)} \\ &= xp \pm \sqrt{x^2p^2 - 8p - x^2p^2} \\ &= xp \pm 2\sqrt{-2p} \end{aligned}$$

case one

$$\begin{aligned} y &= xp + 2\sqrt{-2p} \\ &= xp + f\left(p\right) \end{aligned} \tag{1}$$

This can be written as

$$y = G\left(x, p\right)$$

Where $G\left(x, p\right) = xp + f\left(p\right)$. This form of ODE is called the Clairaut ODE. Taking derivative w.r.t. x gives

$$y' = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{dp}{dx}$$

But $y' = p$ and the above becomes

$$p = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{dp}{dx}$$

But $\frac{\partial G}{\partial x} = p$, hence the above reduces to

$$0 = \frac{\partial G}{\partial p}\frac{dp}{dx} \tag{2}$$

Then either $\frac{\partial G}{\partial p} = 0$ or $\frac{dp}{dx} = 0$.

When $\frac{dp}{dx} = 0$ or $y'' = 0$ therefore the solution is

$$y = C_1 x + C_2 \tag{3}$$

But we are solving a first order ODE. So we expect it to have one constant of integration only. By comparing (3) with equation (1) which is $y = xp + f\left(p\right)$ shows that

$$C_2 = f\left(C_1\right) = 2\sqrt{-2C_1}$$

Then the solution will now contain one constant of integration C_1. Hence the first solution is

$$y = C_1 x + 2\sqrt{-2C_1}$$

The second possibility comes from $\frac{\partial G}{\partial p} = 0$. This gives

$$\begin{aligned}
x + f'\left(p\right) &= 0\\
x + 2\frac{d}{dp}\left(-2p\right)^{\frac{1}{2}} &= 0\\
x + 2\frac{1}{2}\left(-2p\right)^{\frac{-1}{2}}\left(-2\right) &= 0\\
x - \frac{2}{\sqrt{-2p}} &= 0\\
x\sqrt{-2p} &= 2\\
-2px^2 &= 4\\
p &= -\frac{2}{x^2}
\end{aligned}$$

Now that we found p, we substitute it back into (1) given by $y = xp + 2\sqrt{-2p}$. Hence the second solution is found directly as follows

$$\begin{aligned}
y &= xp + 2\sqrt{-2p} \\
&= -\frac{2}{x} + 2\sqrt{-2\left(-\frac{2}{x^2}\right)} \\
&= -\frac{2}{x} + 2\sqrt{\frac{4}{x^2}} \\
&= -\frac{2}{x} + \frac{4}{x} \\
&= \frac{2}{x}
\end{aligned}$$

Summary of case one From above we obtained the following two solutions

$$\begin{aligned}
y_1 &= C_1 x + 2\sqrt{-2C_1} \\
y_2 &= \frac{2}{x}
\end{aligned}$$

Where $y_2(x)$ is the singular solution since it can't be obtained from the first solution with the constants of integrations by changing them to any value.

We now do the same steps for the case of $y = xp - 2\sqrt{-2p}$. This follows the same steps as above as the only difference is the sign and hence the steps will not be repeated. It gives the solution

$$y_3 = C_1 x - 2\sqrt{-2C_1}$$

With the same singular solution. Therefore there are three solutions to this ODE and these are summarized below

$$\begin{aligned}
y_1 &= C_1 x + 2\sqrt{-2C_1} \\
y_2 &= \frac{2}{x} \\
y_3 &= C_1 x - 2\sqrt{-2C_1}
\end{aligned}$$

With $y_2(x)$ being the singular solution. Singular solutions do not have constant of integration in them and can not be obtained from the general solution by any substitution for constants of integration. The general solution contain constant of integrations in them.

2.1.7 Problem 6

Problem

Find the general real solution to the following equation where $A(x)$ is a known function

$$A(x)\,y'' + A'(x)\,y' + \frac{y}{A(x)} = 0$$

Solution

Let us first assume $A(x)$ is constant not zero. The above reduces to

$$y'' + \frac{y}{A^2} = 0$$

This is harmonic oscillator It has the form of $y'' + \omega^2 y = 0$ with $\omega = \frac{1}{A}$ being the natural frequency. The solution to this is easily found to be

$$\begin{aligned} y(x) &= C_1 \cos(\omega x) + C_2 \sin(\omega x) \\ &= C_1 \cos\left(\frac{x}{A}\right) + C_2 \sin\left(\frac{x}{A}\right) \end{aligned} \tag{1}$$

Since A is not constant, then we can try a similar solution[1] but use $f(x)$ for the arguments of the trigonometric functions

$$y(x) = C_1 \cos\left(f(x)\right) + C_2 \sin\left(f(x)\right) \tag{2}$$

where $f(x)$ is function of x to be determined. Hence. From now on, we will write f instead of $f(x)$ to simplify notation.

$$\begin{aligned} y' &= -C_1 f' \sin\left(f\right) + C_2 f' \cos\left(f\right) \\ y'' &= -C_1 f'' \sin\left(f\right) - C_1 \left(f'\right)^2 \cos\left(f\right) + C_2 f'' \cos\left(f\right) - C_2 \left(f'\right)^2 \sin\left(f\right) \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} A^2 \left(-C_1 f'' \sin\left(f\right) - C_1 \left(f'\right)^2 \cos\left(f\right) + C_2 f'' \cos\left(f\right) - C_2 \left(f'\right)^2 \sin\left(f\right)\right) + \\ AA' \left(-C_1 f' \sin\left(f\right) + C_2 f' \cos\left(f\right)\right) + C_1 \cos\left(f\right) + C_2 \sin\left(f\right) = 0 \end{aligned}$$

Collecting terms gives

$$\cos\left(f\right)\left(-C_1 A^2 \left(f'\right)^2 + C_2 A^2 f'' + C_2 AA' f' + C_1\right) + \sin\left(f\right)\left(-C_1 A^2 f'' - C_2 A^2 \left(f'\right)^2 - C_1 AA' f' + C_2\right) = 0$$

Since this is zero for all $\sin$ and $\cos$ then

$$\begin{aligned} -C_1 A^2 \left(f'\right)^2 + C_2 A^2 f'' + C_2 AA' f' + C_1 &= 0 \\ -C_1 A^2 f'' - C_2 A^2 \left(f'\right)^2 - C_1 AA' f' + C_2 &= 0 \end{aligned}$$

Multiplying the first equation by C_2 and the second by C_1 gives

$$\begin{aligned} -C_2 C_1 A^2 \left(f'\right)^2 + C_2^2 A^2 f'' + C_2^2 AA' f' + C_1 C_2 &= 0 \\ -C_1^2 A^2 f'' - C_1 C_2 A^2 \left(f'\right)^2 - C_1^2 AA' f' + C_1 C_2 &= 0 \end{aligned}$$

[1]Thanks to hint from the Professor.

Subtracting the second equation from the first gives

$$\left(-C_2C_1A^2\left(f'\right)^2+C_2^2A^2f''+C_2^2AA'f'+C_1C_2\right)-\left(-C_1^2A^2f''-C_1C_2A^2\left(f'\right)^2-C_1^2AA'f'+C_1C_2\right)=0$$
$$-C_2C_1A^2\left(f'\right)^2+C_2^2A^2f''+C_2^2AA'f'+C_1C_2+C_1^2A^2f''+C_1C_2A^2\left(f'\right)^2+C_1^2AA'f'-C_1C_2=0$$
$$C_2^2A^2f''+C_2^2AA'f'+C_1^2A^2f''+C_1^2AA'f'=0$$
$$f''\left(C_2^2A^2+C_1^2A^2\right)+f'\left(C_2^2AA'+C_1^2AA'\right)=0$$

Let us call $C_2^2A^2+C_1^2A^2=\mu$ and $C_2^2AA'+C_1^2AA'=\beta$ for the moment. The above becomes

$$\mu f''+\beta f'=0$$

Since f is missing, then we can solve the above by assuming $f'=v$. The above becomes $v'+\frac{\beta}{\mu}v=0$. This is linear in v. The integrating factor is $I=e^{\int\frac{\beta}{\mu}dx}$. Hence the ode becomes

$$\frac{d}{dx}\left(e^{\int\frac{\beta}{\mu}dx}v\right)=0$$
$$v=C_3e^{-\int\frac{\beta}{\mu}dx}$$

Since the proposed solution in (2) contains integration of constants already, we can choose $C_3=1$ without affecting the final solution. Hence

$$f'(x)=e^{-\int\frac{\beta}{\mu}dx}$$

Therefore

$$\begin{aligned} f(x)&=\int e^{-\int\frac{\beta}{\mu}dx}dx+C_4\\ &=\int\left(C_3e^{-\int\frac{C_2^2AA'+C_1^2AA'}{C_2^2A^2+C_1^2A^2}dx}dx\right)dx+C_4 \end{aligned}\tag{3}$$

Again, since the proposed solution in (2) contains integration of constants already, we can choose $C_4=0$. The above becomes

$$\begin{aligned} f(x)&=\int e^{-\int\frac{\beta}{\mu}dx}dx\\ &=\int e^{-\int\frac{C_2^2AA'+C_1^2AA'}{C_2^2A^2+C_1^2A^2}dx}dx \end{aligned}$$

The expression $\frac{C_2^2AA'+C_1^2AA'}{C_2^2A^2+C_1^2A^2}$ can be simplified as follows

$$\frac{C_2^2AA'+C_1^2AA'}{C_2^2A^2+C_1^2A^2}=\frac{AA'\left(C_2^2+C_1^2\right)}{A^2\left(C_2^2+C_1^2\right)}=\frac{AA'}{A^2}$$

Hence (3) becomes

$$
\begin{aligned}
f(x) &= \int e^{-\int \frac{AA'}{A^2}dx} dx \\
&= \int e^{-\int \frac{A'}{A}dx} dx \\
&= \int e^{-\ln A} dx \\
&= \int \frac{1}{A(x)} dx
\end{aligned}
$$

Therefore the solution from (2) is

$$
\begin{aligned}
y(x) &= C_1 \cos\left(f(x)\right) + C_2 \sin\left(f(x)\right) \\
&= C_1 \cos\left(\int \frac{1}{A(x)} dx\right) + C_2 \sin\left(\int \frac{1}{A(x)} dx\right) \qquad (4)
\end{aligned}
$$

Let us now try to verify this solution by substituting it back into the ODE. From (4), where we now write A instead of $A(x)$ everywhere to simplify the notation

$$
\begin{aligned}
y'(x) &= -C_1 \sin\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' + C_2 \cos\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' \\
&= -C_1 \sin\left(\int \frac{1}{A}dx\right)\frac{1}{A} + C_2 \cos\left(\int \frac{1}{A}dx\right)\frac{1}{A}
\end{aligned}
$$

And $y''(x)$ becomes

$$
\begin{aligned}
y''(x) = -C_1 &\left(\cos\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' \frac{1}{A} + \sin\left(\int \frac{1}{A}dx\right)\left(\frac{-A'}{A^2}\right)\right) \\
&+ C_2\left(-\sin\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' \frac{1}{A} + \cos\left(\int \frac{1}{A}dx\right)\left(\frac{-A'}{A^2}\right)\right)
\end{aligned}
$$

or

$$
\begin{aligned}
y''(x) &= -C_1\left(\cos\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} - \sin\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2}\right) + C_2\left(-\sin\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} - \cos\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2}\right) \\
&= -C_1 \cos\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} + C_1 \sin\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2} - C_2 \sin\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} - C_2 \cos\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2}
\end{aligned}
$$

Substituting the above expression for y, y', y'' into the original ODE $A^2 y'' + AA'y' + y = 0$ gives

$$
\begin{aligned}
&A^2\left(-C_1 \cos\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} + C_1 \sin\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2} - C_2 \sin\left(\int \frac{1}{A}dx\right)\frac{1}{A^2} - C_2 \cos\left(\int \frac{1}{A}dx\right)\frac{A'}{A^2}\right) \\
&+AA'\left(-C_1 \sin\left(\int \frac{1}{A}dx\right)\frac{1}{A} + C_2 \cos\left(\int \frac{1}{A}dx\right)\frac{1}{A}\right) + C_1 \cos\left(\int \frac{1}{A(x)}dx\right) + C_2 \sin\left(\int \frac{1}{A(x)}dx\right) = 0
\end{aligned}
$$

Simplifying gives

$$-C_1\cos\left(\int\frac{1}{A}dx\right)+C_1\sin\left(\int\frac{1}{A}dx\right)A'-C_2\sin\left(\int\frac{1}{A}dx\right)-C_2\cos\left(\int\frac{1}{A}dx\right)A'$$
$$-C_1A'\sin\left(\int\frac{1}{A}dx\right)+C_2A'\cos\left(\int\frac{1}{A}dx\right)+C_1\cos\left(\int\frac{1}{A}dx\right)+C_2\sin\left(\int\frac{1}{A(x)}dx\right)=0$$

Canceling C_1terms gives

$$-C_2\sin\left(\int\frac{1}{A}dx\right)-C_2\cos\left(\int\frac{1}{A}dx\right)A'+C_2A'\cos\left(\int\frac{1}{A}dx\right)+C_2\sin\left(\int\frac{1}{A}dx\right)=0$$

Which simplifies to

$$-C_2\cos\left(\int\frac{1}{A}dx\right)A'+C_2A'\cos\left(\int\frac{1}{A}dx\right)=0$$

Or

$$0=0$$

Solution (4) has been verified.

2.1.8 Problem 7

Problem Find the general real solution to the equation

$$xy''+\frac{3}{x}y=1+x^3$$

Solution

We start by writing the ODE as

$$x^2y''+3y=x+x^4 \tag{1}$$

The solution is given by

$$y=y_h+y_p$$

where y_h is solution to homogeneous ODE $x^2y_h''+3y_h=0$ and y_p is a particular solution to $x^2y_p''+3y_p=x+x^4$. We start by solving the homogeneous

$$x^2y''+3y=0$$

This is Euler type ODE. Using the standard substitution $y=Ax^r$, then $y'=Arx^{r-1}, y''=Ar(r-1)x^{r-2}$ and the above becomes

$$x^2Ar(r-1)x^{r-2}+3Ax^r=0$$
$$Ar(r-1)x^r+3Ax^r=0$$

Since $x^r\neq 0$ and $A\neq 0$ then the above simplifies to

$$r(r-1)+3=0$$
$$r^2-r+3=0$$

Hence

$$\begin{aligned} r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} \\ &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1-12} \\ &= \frac{1}{2} \pm \frac{1}{2}i\sqrt{11} \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= C_1 x^{\frac{1}{2}+\frac{i}{2}\sqrt{11}} + C_2 x^{\frac{1}{2}-\frac{i}{2}\sqrt{11}} \\ &= C_1 x^{\frac{1}{2}} x^{\frac{i}{2}\sqrt{11}} + C_2 x^{\frac{1}{2}} x^{\frac{-i}{2}\sqrt{11}} \\ &= C_1 \sqrt{x} e^{\ln x^{\frac{i}{2}\sqrt{11}}} + C_2 \sqrt{x} e^{\ln x^{\frac{-i}{2}\sqrt{11}}} \\ &= C_1 \sqrt{x} e^{\frac{i}{2}\sqrt{11}\ln x} + C_2 \sqrt{x} e^{\frac{-i}{2}\sqrt{11}\ln x} \end{aligned}$$

Using Euler formula the above can now be written in terms of $\sin$ and $\cos$

$$\begin{aligned} y &= \sqrt{x}\left(C_1 e^{\frac{i}{2}\sqrt{11}\ln x} + C_2 e^{\frac{-i}{2}\sqrt{11}\ln x}\right) \\ y_h &= \sqrt{x}\left(C_3 \cos\left(\frac{1}{2}\sqrt{11}\ln x\right) + C_4 \sin\left(\frac{1}{2}\sqrt{11}\ln x\right)\right) \end{aligned} \tag{2}$$

Now we find the particular solution using the method of undetermined coefficients. Since the RHS is polynomial $x + x^4$ then we guess

$$y_p = A + Bx + Cx^2 + Dx^3 + Ex^4$$

Then $y' = B + 2Cx + 3Dx^2 + 4Ex^3$ and $y'' = 2C + 6Dx + 12Ex^2$. Substituting these back in (1)

$$\begin{aligned} x^2\left(2C + 6Dx + 12Ex^2\right) + 3\left(A + Bx + Cx^2 + Dx^3 + Ex^4\right) &= x + x^4 \\ 2Cx^2 + 6Dx^3 + 12Ex^4 + 3A + 3Bx + 3Cx^2 + 3Dx^3 + 3Ex^4 &= x + x^4 \\ 3A + x(3B) + x^2(2C + 3C) + x^3(6D + 3D) + (3E + 12E)x^4 &= x + x^4 \\ 3A + x(3B) + x^2(5C) + x^3(9D) + 15Ex^4 &= x + x^4 \end{aligned}$$

By comparing coefficients the following equations are generated

$$\begin{aligned} A &= 0 \\ 3B &= 1 \\ 5C &= 0 \\ 9D &= 0 \\ 15E &= 1 \end{aligned}$$

Hence $A = 0, B = \frac{1}{3}, C = 0, D = 0, E = \frac{1}{15}$. Therefore

$$y_p = \frac{1}{3}x + \frac{1}{15}x^4$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \sqrt{x}\left(C_3 \cos\left(\frac{1}{2}\sqrt{11}\ln x\right) + C_4 \sin\left(\frac{1}{2}\sqrt{11}\ln x\right)\right) + \frac{1}{3}x + \frac{1}{15}x^4 \end{aligned}$$

2.1.9 Problem 8

Problem For what values of k does the equation

$$y'' - \left(\frac{1}{4} + \frac{k}{x}\right) y = 0 \tag{1}$$

defined for $0 < x < \infty$ have a solution vanishing at $x = 0$ and at $x = \infty$?

Solution

Let us look what happens at $x \to \infty$, then the term $\frac{1}{4} \gg \frac{k}{x}$ and the ODE simplifies to

$$y'' - \frac{1}{4}y = 0$$

Which has the solutions $y = \left\{e^{\frac{1}{2}x}, e^{\frac{-1}{2}x}\right\}$. We reject the first one since it does not vanish at $x \to \infty$, and use $y = e^{\frac{-1}{2}x}$. Now we assume the solution to (1) is of the form

$$y = P(x)\, e^{\frac{-1}{2}x} \tag{2}$$

And we now try to find $P(x)$. Substituting this solution back into (1), given that

$$\begin{aligned} y' &= P'e^{\frac{-x}{2}} - \frac{1}{2}Pe^{\frac{-x}{2}} \\ y'' &= P''e^{\frac{-x}{2}} - \frac{1}{2}P'e^{\frac{-x}{2}} - \frac{1}{2}P'e^{\frac{-x}{2}} + \frac{1}{4}Pe^{\frac{-x}{2}} \\ &= P''e^{\frac{-x}{2}} - P'e^{\frac{-x}{2}} + \frac{1}{4}Pe^{\frac{-x}{2}} \end{aligned}$$

Substituting the above into (1) and canceling common term $e^{\frac{-x}{2}}$ gives

$$\begin{aligned} \left(P'' - P' + \frac{1}{4}P\right) - \left(\frac{1}{4} + \frac{k}{x}\right)P &= 0 \\ P'' - P' - \frac{k}{x}P &= 0 \\ xP'' - xP' - kP &= 0 \end{aligned} \tag{3}$$

To solve this for $P(x)$, we use Frobenius series . Assuming

$$
\begin{aligned}
P(x) &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
P'(x) &= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\
P''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}
\end{aligned}
$$

Hence (3) becomes

$$
\begin{aligned}
x\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - x\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - k\sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} - k\sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r-1) c_{n-1} x^{n+r-1} - k\sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} &= 0
\end{aligned}
$$

For $n=0$, and assuming $c_0 \neq 0$ then

$$
\begin{aligned}
(n+r)(n+r-1) c_n &= 0 \\
(r)(r-1) c_0 &= 0 \\
r(r-1) &= 0
\end{aligned}
$$

Hence $r=1$ or $r=0$.

Case $r=1$

$$
\begin{aligned}
P &= \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{n=1}^{\infty} c_{n-1} x^{n}
\end{aligned}
$$

Hence

$$
\begin{aligned}
P' &= \sum_{n=1}^{\infty} n c_{n-1} x^{n-1} \\
P'' &= \sum_{n=1}^{\infty} (n)(n-1) c_{n-1} x^{n-2}
\end{aligned}
$$

And now (3) becomes

$$
\begin{aligned}
x\sum_{n=1}^{\infty}(n)(n-1)c_{n-1}x^{n-2}-x\sum_{n=1}^{\infty}nc_{n-1}x^{n-1}-k\sum_{n=1}^{\infty}c_{n-1}x^{n}&=0\\
\sum_{n=1}^{\infty}(n)(n-1)c_{n-1}x^{n-1}-\sum_{n=1}^{\infty}nc_{n-1}x^{n}-k\sum_{n=1}^{\infty}c_{n-1}x^{n}&=0\\
\sum_{n=2}^{\infty}(n)(n-1)c_{n-1}x^{n-1}-\sum_{n=1}^{\infty}nc_{n-1}x^{n}-k\sum_{n=1}^{\infty}c_{n-1}x^{n}&=0\\
\sum_{n=1}^{\infty}(n+1)(n)c_{n}x^{n}-\sum_{n=1}^{\infty}nc_{n-1}x^{n}-k\sum_{n=1}^{\infty}c_{n-1}x^{n}&=0
\end{aligned}
$$

Hence for $n\geq 1$ we obtain

$$
\begin{aligned}
(n+1)(n)c_n-nc_{n-1}-kc_{n-1}&=0\\
c_n&=\frac{(n+k)c_{n-1}}{n(n+1)}
\end{aligned}
$$

For $n=1$

$$c_1=\frac{(k+1)c_0}{2}$$

For $n=2$

$$c_2=\frac{(k+2)c_1}{2(3)}=\frac{(k+2)}{2(3)}\frac{(k+1)}{2}c_0=\frac{(k+1)(k+2)}{(2)(2)(3)}c_0$$

For $n=3$

$$c_3=\frac{(k+3)c_2}{3(4)}=\frac{(k+3)}{(3)(4)}\frac{(k+1)(k+2)}{(2)(2)(3)}c_0=\frac{(k+1)(k+2)(k+3)}{(2)(2)(3)(3)(4)}c_0$$

For $n=4$

$$c_4=\frac{(k+4)c_{n-1}}{(4)(5)}=\frac{(k+4)c_3}{(4)(5)}=\frac{(k+1)(k+2)(k+3)(k+4)}{(2)(2)(3)(3)(4)(4)(5)}c_0$$

And so on. Hence

$$
\begin{aligned}
P(x)&=\sum_{n=1}^{\infty}c_{n-1}x^n\\
&=c_0x+c_1x^2+c_2x^3+c_3x^4+\cdots\\
&=c_0\left(x+\frac{(k+1)}{2}x^2+\frac{(k+1)(k+2)}{(2)(2)(3)}x^3+\frac{(k+1)(k+2)(k+3)}{(2)(2)(3)(3)(4)}x^4+\frac{(k+1)(k+2)(k+3)(k+4)}{(2)(2)(3)(3)(4)(4)(5)}x^5+\cdots\right)\\
&=c_0\left(x+(k+1)\frac{x^2}{2!}+\frac{(k+1)(k+2)}{2!}\frac{x^3}{3!}+\frac{(k+1)(k+2)(k+3)}{3!}\frac{x^4}{4!}+\frac{(k+1)(k+2)(k+3)(k+4)}{4!}\frac{x^5}{5!}+\cdots\right)
\end{aligned}
\tag{4}
$$

But $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$. Or $e^x-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$. So there is an exponential term inside (4). Hence to make (4) vanish at $x\to\infty$, then k needs to be a negative integer. Taking $k=-1$ makes all terms with k in them vanish, leaving

$$P(x)=c_0x$$

So now the solution from (2) becomes

$$y(x) = c_0 x e^{-\frac{x}{2}}$$

Which goes to zero as $x \to \infty$ since an exponential decays to zero faster that x going to infinity.

We now need to check if negative k integer value (specifically $k = -1$ which we picked from above) will also make the solution vanish as $x \to 0$. When $x \to 0$ the ODE becomes

$$y'' - \frac{k}{x} y = 0 \tag{5}$$

Since $\frac{k}{x} \gg \frac{1}{4}$ close to $x = 0$. Since k is negative integer -1 then the above becomes

$$y'' + \frac{k}{x} y = 0$$

To see this will go to zero as $x \to 0$, Intuitively since $\frac{k}{x}$ is now positive and very large, then this is like a harmonic oscillator with very large stiffness. (Spring mass system). When the stiffness becomes very large, the solution goes to zero (the natural frequency goes to infinity, since $\omega = \sqrt{\frac{k}{x}}$ which means the period goes to zero since $\omega = 2\pi T$) which implies no motion. So this shows that negative integer value of k found from first part makes the solution vanish at both $x \to \infty$ and at $x \to 0$. Actually for $x \to 0$ we just needed k to be negative in order to change the sign. But for $x \to \infty$ we found we needed k to be a negative integer which we choose -1. So this will work for $x = 0$ and $x = \infty$.

2.1.9.1 Appendix

I first tried to solve the give ODE directly using series method. I left this here as an appendix, not to be graded but as a reference.

x is singular point. But it is a regular singular point since $\lim_{x \to 0} x^2 \frac{k}{x} = x$ and hence the limit exist. Therefore assuming solution is Frobenius series

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Therefore $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$, then (1) becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - \left(\frac{1}{4} + \frac{k}{x}\right) \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - \frac{k}{x} \sum_{n=0}^{\infty} c_n x^{n+r} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - k \sum_{n=0}^{\infty} c_n x^{n+r-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

But $k \sum_{n=0}^{\infty} c_n x^{n+r-1} = k \sum_{n=1}^{\infty} c_{n-1} x^{n+r-2}$ and $\sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=2}^{\infty} c_{n-2} x^{n+r-2}$ and the above becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - k\sum_{n=1}^{\infty}c_{n-1}x^{n+r-2} - \frac{1}{4}\sum_{n=2}^{\infty}c_{n-2}x^{n+r-2} = 0 \tag{2}$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from $n = 0$ in (2) with the assumption that $c_0 \neq 0$. This leads to

$$\begin{aligned}(n+r)(n+r-1)c_n &= 0\\ r(r-1)c_0 &= 0\end{aligned}$$

c_0 is always taken as non-zero. This leads to

$$r(r-1) = 0$$

With solutions $r_1 = 1$ or $r_2 = 0$. (We take r_1 as the larger root first, since Frobenius series solution can only guarantee solution for the larger root, when the roots differ by an integer as this is the case).

Since $r_1 - r_2$ is an integer, then this tells us we can obtain a first solution $y_1(x)$ associated with $r_1 = 1$ from the Frobenius series

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \tag{3}$$

But to find the second solution $y_2(x)$ associated with $r_2 = 0$ we can try either reduction of order method or use

$$y_2(x) = Ay_1(x)\ln(x) + \sum_{n=0}^{\infty} d_n x^n \tag{4}$$

Where A is some constant, which can be zero, and d_n are the coefficients for the second series. We have to do the above when the roots of the indicial equation differ by integer. Otherwise, the second solution would have been found using Frobenius series $y_2(x)\sum_{n=0}^{\infty} c_n x^{n+r_2}$ like with the first solution.

OK, Now we will first find $y_1(x)$ from (3)

case $r_1 = 1$

Using (3)

$$\begin{aligned}y' &= \sum_{n=0}^{\infty}(n+1)c_n x^n\\ y'' &= \sum_{n=0}^{\infty} n(n+1)c_n x^{n-1}\\ &= \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1}\end{aligned}$$

Substituting the above into (1) gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \left(\frac{1}{4} + \frac{k}{x}\right) \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+1} - \frac{k}{x} \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+1} - k \sum_{n=0}^{\infty} c_n x^{n} &= 0 \\
\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+1} x^{n} - \frac{1}{4} \sum_{n=1}^{\infty} c_{n-1} x^{n} - k \sum_{n=0}^{\infty} c_n x^{n} &= 0
\end{aligned}
$$

For $n = 0$

$$
\begin{aligned}
(1)(2) c_1 - k c_0 &= 0 \\
c_1 &= \frac{k}{2} c_0
\end{aligned}
$$

For $n > 0$ we obtain the recursion equation

$$
\begin{aligned}
(n+1)(n+2) c_{n+1} - \frac{1}{4} c_{n-1} - k c_n &= 0 \\
c_{n+1} &= \frac{\frac{1}{4} c_{n-1} + k c_n}{(n+1)(n+2)}
\end{aligned}
$$

For $n = 1$

$$
c_2 = \frac{\frac{1}{4} c_0 + k c_1}{(2)(3)} = \frac{\frac{1}{4} c_0 + k \left(\frac{k}{2} c_0\right)}{6} = \frac{\frac{1}{4} c_0 + \frac{k^2}{2} c_0}{6} = c_0 \frac{\frac{1}{4} + \frac{k^2}{2}}{6} = c_0 \frac{1 + 2k^2}{24}
$$

For $n = 2$

$$
\begin{aligned}
c_3 &= \frac{\frac{1}{4} c_1 + k c_2}{(3)(4)} \\
&= \frac{\frac{1}{4} \frac{k}{2} c_0 + k c_0 \frac{1+2k^2}{24}}{12} \\
&= \frac{\frac{k}{8} c_0 + k c_0 \frac{1+2k^2}{24}}{12} \\
&= c_0 \frac{\frac{3k+k+2k^3}{24}}{(12)} \\
&= c_0 \frac{4k + 2k^3}{288}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_1(x) &= c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + \cdots \\
&= c_0 x + \frac{k}{2} c_0 x^2 + c_0 \frac{1+2k^2}{24} x^3 + c_0 \frac{4k+2k^3}{288} x^4 + \cdots \\
&= c_0 x \left(1 + \frac{k}{2} x + \frac{1+2k^2}{24} x^2 + \frac{4k+2k^3}{288} x^3 + \cdots\right) \\
&= c_0 x \left(1 + \frac{k}{2} x + \left(\frac{1}{24} + \frac{1}{12} k^2\right) x^2 + \frac{k}{288}\left(4+2k^2\right) x^3 + \cdots\right)
\end{aligned}
$$

I could not find closed form function for the above.

Now that we found $y_1(x)$, then $y_2(x)$ is, from (4), repeated here

$$
y_2(x) = A y_1(x) \ln(x) + \sum_{n=0}^{\infty} d_n x^n \tag{4}
$$

Since we want the solution to vanish at $x = 0$ then we set $A = 0$ and $y_2(x)$ simplifies to

$$
y_2(x) = \sum_{n=0}^{\infty} d_n x^n \tag{4}
$$

Where $d_0 \neq 0$. Hence $y'(x) = \sum_{n=0}^{\infty} n d_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) d_n x^{n-2}$. Rewriting the ODE as $xy'' - \left(\frac{x}{4} + k\right) y = 0$ and now substituting the derivatives into this gives

$$
\begin{aligned}
x \sum_{n=0}^{\infty} n(n-1) d_n x^{n-2} - \left(\frac{x}{4} + k\right) \sum_{n=0}^{\infty} d_n x^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1) d_n x^{n-1} - \frac{x}{4} \sum_{n=0}^{\infty} d_n x^n - k \sum_{n=0}^{\infty} d_n x^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1) d_n x^{n-1} - \frac{1}{4} \sum_{n=0}^{\infty} d_n x^{n+1} - k \sum_{n=0}^{\infty} d_n x^n &= 0 \\
\sum_{n=1}^{\infty} (n+1)(n) d_{n+1} x^n - \frac{1}{4} \sum_{n=1}^{\infty} d_{n-1} x^n - k \sum_{n=0}^{\infty} d_n x^n &= 0
\end{aligned}
$$

For $n = 0$ we obtain $kd_0 = 0$ which implies $d_0 = 0$ since $k \neq 0$.

For $n > 0$

$$
\begin{aligned}
(n+1)(n) d_{n+1} - \frac{1}{4} d_{n-1} - k d_n &= 0 \\
d_{n+1} &= \frac{\frac{1}{4} d_{n-1} + k d_n}{(n)(n+1)}
\end{aligned}
$$

For $n = 1$

$$
d_2 = \frac{\frac{1}{4} d_0 + k d_1}{2} = \frac{k}{2} d_1
$$

For $n=2$

$$d_3 = \frac{\frac{1}{4}d_1 + kd_2}{(2)(3)} = \frac{\frac{1}{4}d_1 + k\left(\frac{k}{2}d_1\right)}{6} = \frac{d_1 + 2k^2 d_1}{32} = d_1 \frac{1+2k^2}{32}$$

For $n=3$

$$d_4 = \frac{\frac{1}{4}d_2 + kd_3}{(3)(4)} = \frac{d_2 + 4kd_3}{48} = \frac{\frac{k}{2}d_1 + 4k\left(d_1\frac{1+2k^2}{32}\right)}{48} = d_1 \frac{\frac{1}{8}k\left(2k^2+5\right)}{48} = d_1 \frac{\left(2k^3+5k\right)}{384}$$

And so on. Hence the second solution is

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} d_n x^n \\
&= d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + \cdots \\
&= d_1 x + \frac{k}{2} d_1 x^2 + d_1 \frac{1+2k^2}{32} x^3 + d_1 \frac{\left(2k^3+5k\right)}{384} x^4 + \cdots \\
&= d_1 x \left(1 + \frac{k}{2} x + d_1 \frac{1+2k^2}{32} x^2 + d_1 \frac{\left(2k^3+5k\right)}{384} x^3 + \right)
\end{aligned}$$

I am not sure if the above solution for $y_2(x)$ is correct. I need to check this again later.

2.1.10 Key solution for HW 1

① $x^2 y' + y^2 = x y y'$ $\quad \dim(y) = \dim(x)$

Try $y = x v \rightarrow \frac{dv}{dx} = \frac{v}{v-1}\frac{1}{x}$ separable

$x = \frac{c}{v} e^{v} \rightarrow \boxed{y = c e^{y/x}}$

② $y' = \frac{a^2}{(x+y)^2}$ Try $u = x+y$

Leads to $dx = \frac{du}{1 + \frac{a^2}{u^2}}$ separable

$$\Rightarrow x - c = \int du \left[1 - \frac{a^2}{u^2+a^2} \right] = u - a \tan^{-1}\left(\frac{u}{a}\right)$$
$$= x+y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\boxed{y = a \tan\left(\frac{y+c}{a}\right) - x}$$

③ $y'' + y'^2 + 1 = 0$ Lacks 2 variables.

$p = y' \quad p' = -(p^2+1)$ separable

$p = -\tan(x+c) = \frac{dy}{dx}$

$$\boxed{y = a + \ln\left[\cos(x+c)\right]}$$

(4) $xy' + y + x^4 y^4 e^x = 0$

Let $v = xy$ $\quad \frac{dv}{dx} = -v^4 e^x$ separable

$$v^3 = \frac{1}{3(e^x + c)} \qquad \boxed{y = \frac{1}{x}\frac{1}{\left[3(e^x + c)\right]^{1/3}}}$$

(5) $x^2 y'^2 - 2(xy-4)y' + y^2 = 0$

Differentiate: $2xy'^2 + 2x^2 y' y'' - 2(y + xy')y'$

$- 2(xy-4)y'' + 2yy' = 0$

Factorize: $y''\left[x^2 y' - xy + 4\right] = 0$

(i) $y'' = 0 \Rightarrow y = ax + b$ Substitute into original equation which is solved only if $a = -\frac{1}{8} b^2$

$$\boxed{y = -\frac{1}{8} b^2 x + b}$$

(ii) $y' - \frac{1}{x} y = \frac{-4}{x^2}$ Solve by method of integrating factor.

$y = \frac{2}{x} + cx$

Substitute into original equation which is solved only if $c = 0$

$$\boxed{y = \frac{2}{x}}$$

(6) $A^2(x)\, y'' + A(x)A'(x)\, y' + y = 0$

Notice that dim (A) = dim (x)

Look for solution of the form

$$y = c_1 \cos f(x) + c_2 \sin f(x)$$ 2 linearly independent solutions

$$y' = -c_1 f' \sin f + c_2 f' \cos f$$

$$y'' = \cos f\left[-c_1 f'^2 + c_2 f''\right] + \sin f\left[-c_1 f'' - c_2 f'^2\right]$$

Original equation becomes

$$\cos f\left\{ c_1\left[1 - A^2 f'^2\right] + c_2 A\left[A f'' + A' f'\right]\right\}$$
$$+ \sin f\left\{ c_2\left[1 - A^2 f'^2\right] + c_1 A\left[-A f'' - A' f'\right]\right\} = 0$$

satisfied if $f' = \frac{1}{A}$ or $\boxed{f(x) = \int_{x_0}^{x} \frac{dx'}{A(x')}}$

(sign doesn't matter)

$$\boxed{y = c_1 \cos f(x) + c_2 \sin f(x)}$$

(7) $x y'' + \frac{3}{x} y = 1 + x^3$

$\boxed{y = y_H + y_p}$ $\quad x^2 y_H'' + 3 y_H = 0$

$x = 0$ is a regular singular point. Try $y_H = x^s$

Then $s(s-1) + 3 = 0 \Rightarrow s = \frac{1}{2} \pm \frac{\sqrt{11}}{2} i$

Now $x^{\pm \frac{\sqrt{11}}{2} i} = \exp\left[\ln x^{\pm \frac{\sqrt{11}}{2} i}\right] = \exp\left[\pm \frac{\sqrt{11}}{2} i \ln x\right]$

or $\cos\left(\frac{\sqrt{11}}{2} \ln x\right)$ & $\sin\left(\frac{\sqrt{11}}{2} \ln x\right)$

$$\boxed{y_H = \sqrt{x}\left[a \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + b \sin\left(\frac{\sqrt{11}}{2} \ln x\right)\right]}$$

Try $y_p = \alpha x^4 + \beta x^3 + \gamma x^2 + \rho x$

Substitution $\Rightarrow$ $\alpha = \frac{1}{15}$ $\quad \rho = \frac{1}{3}$ $\quad \beta = \gamma = 0$

$$\boxed{y_p = \frac{1}{15} x^4 + \frac{1}{3} x}$$

(8) $y'' - \left(\frac{1}{4} + \frac{k}{x}\right) y = 0$

As $x \to \infty$ neglect $\frac{k}{x}$ compared to $\frac{1}{4}$.

Then $y \to e^{\pm \frac{1}{2}x}$. We want $y \to e^{-\frac{1}{2}x}$.

Now write $y = f(x)\, e^{-\frac{1}{2}x}$

f: polynomial in x?

$$f'' - f' - \frac{k}{x} f = 0$$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \qquad f' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad f'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} a_n$$

Substitution: $\displaystyle\sum_{n=1}^{\infty} \left\{ n(n+1) a_{n+1} - n a_n - k a_n \right\} x^{n-1} = 0$

$$\Rightarrow \quad a_{n+1} = \frac{n+k}{n(n+1)} a_n \qquad \frac{a_{n+1}}{a_n} \underset{n \to \infty}{\longrightarrow} \frac{1}{n} \quad \text{so this}$$

would lead to a factor e^{x}, and the full solution would go as $y \underset{x \to \infty}{\longrightarrow} e^{-\frac{1}{2}x} e^{x} = e^{\frac{1}{2}x}$.

Series terminates if $n + k = 0$ for some $n = 1, 2, 3, \ldots$

$$\boxed{k = -1, -2, -3, \ldots}$$

2.2 HW 2

Local contents

2.2.1 HW 2 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 2 due Monday February 11. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (4 pts) Sum the series

$$1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - \cdots$$

2. (4 pts) Sum the series

$$\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \cdots$$

3. (5 pts) Sum the following series assuming that $0 < \theta < \pi$ for definiteness.

$$f(\theta) = \sin(\theta) + \frac{1}{3}\sin(2\theta) + \frac{1}{5}\sin(3\theta) + \frac{1}{7}\sin(4\theta) + \cdots$$

4. (5 pts) Evaluate the series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = x - \frac{4x^3}{3!} + \frac{9x^5}{5!} - \cdots$$

in closed form by comparing with

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

5. The Euler numbers are defined by

$$\sec(x) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$$

(a) (1 pt) What is E_0?

(b) (4 pts) Find a recursion relation for the E_{2n} when $n \geq 1$. Determine E_2, E_4, E_6, E_8 explicitly.

(c) (2 pts) The partial fraction expansion of the secant is

$$\sec(k\pi) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)^2 - 4k^2}$$

Expand the right hand side in a power series in k and use it to evaluate the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}}$$

in terms of one or more Euler numbers.

2.2.2 Problem 1

Find the sum of $1+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}-\cdots$

Solution

We would like to combine each two consecutive negative terms and combine each two consecutive positive terms in the series in order to obtain an alternating series which is easier to work with. But to do that, we first need to check that the series is absolutely convergent. The $|a_n|$ term is $\frac{1}{4^n}$, therefore

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{4^{n+1}}}{\frac{1}{4^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{4^n}{4^{n+1}}\right| \\
&= \left|\frac{1}{4}\right|
\end{aligned}$$

Since $|L|<1$ then the series is absolutely convergent so we are allowed now to group (or rearrange) terms as follows

$$\begin{aligned}
S &= \left(1+\frac{1}{4}\right)-\left(\frac{1}{16}+\frac{1}{64}\right)+\left(\frac{1}{256}+\frac{1}{1024}\right)-\left(\frac{1}{4096}+\frac{1}{16384}\right)+\cdots \\
&= \frac{5}{4}-\frac{5}{64}+\frac{5}{1024}-\frac{5}{16\,384}+\cdots \\
&= \frac{5}{4}\left(1-\frac{1}{16}+\frac{1}{256}-\frac{1}{4096}+\cdots\right) \\
&= \frac{5}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n}} \\
&= \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n \qquad (1)
\end{aligned}$$

But $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n$ has the form $\sum_{n=0}^{\infty}(-1)^n r^n$ where $r=\frac{1}{16}$ and since $|r|<1$ then by the binomial series

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n r^n &= 1-r+r^2-r^3+\cdots \\
&= \frac{1}{1+r}
\end{aligned}$$

Therefore the sum in (1) becomes, when using $r = \frac{1}{16}$ the following

$$
\begin{aligned}
S &= \frac{5}{4}\left(\frac{1}{1+\frac{1}{16}}\right) \\
&= \frac{5}{4}\left(\frac{16}{17}\right)
\end{aligned}
$$

Hence

$$\boxed{S = \frac{20}{17}}$$

Or

$$S \approx 1.176$$

2.2.3 Problem 2

Find the sum of $\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \cdots$

<u>Solution</u>

$$
\begin{aligned}
S &= \sum_{n=0} \frac{n+1}{n!} \\
&= \sum_{n=0} \frac{n}{n!} + \sum_{n=0} \frac{1}{n!} \\
&= \sum_{n=0} \frac{n}{(n)(n-1)!} + e \\
&= \sum_{n=0} \frac{1}{(n-1)!} + e \\
&= \sum_{n=-1} \frac{1}{n!} + e \\
&= \frac{1}{(-1)!} + \sum_{n=0} \frac{1}{n!} + e \\
&= \frac{1}{(-1)!} + e + e \\
&= \frac{1}{(-1)!} + 2e
\end{aligned}
$$

Now to handle $\frac{1}{(-1)!}$, we use Gamma function definition for factorials $\Gamma(n) = (n-1)!$ for positive integers, and the generalized $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ for non positive integers. By definition $\Gamma(-k)$ where k is negative integer is ∞. (Gamma function is defined only for negative values other than the negative integers).

Hence $\frac{1}{(-1)!} = \frac{1}{\infty} = 0$. So the above result now simplifies to

$$S = 2e$$

2.2.4 Problem 3

Sum the following series assuming that $0 < \theta < \pi$ for definiteness.

$$f(\theta) = \sin(\theta) + \frac{1}{3}\sin(2\theta) + \frac{1}{5}\sin(3\theta) + \frac{1}{7}\sin(4\theta) + \cdots$$

Solution

Since $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ then the above is the same as writing

$$f(\theta) = \operatorname{Im}(e^{i\theta} + \frac{1}{3}e^{2i\theta} + \frac{1}{5}e^{3i\theta} + \frac{1}{7}e^{4i\theta} + \cdots) \tag{1}$$

Let $e^{i\frac{\theta}{2}} = x$, then the above becomes

$$\begin{aligned} f(\theta) &= \operatorname{Im}(x^2 + \frac{1}{3}x^4 + \frac{1}{5}x^6 + \frac{1}{7}x^8 + \cdots) \\ &= \operatorname{Im}\left(x\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right)\right) \end{aligned}$$

Let $g(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots$, hence the above becomes

$$\begin{aligned} f(\theta) &= \operatorname{Im}\left(xg(x)\right) \\ &= \operatorname{Im}\left(x\int g'(x)\,dx\right) \end{aligned} \tag{2}$$

But $g'(x) = 1 + \frac{3x^2}{3} + \frac{5}{5}x^4 + \cdots = 1 + x^2 + x^4 + x^6 + \cdots$. Now for $|x| < 1$ and using Binomial series this has the sum

$$g'(x) = \frac{1}{1 - x^2}$$

Substituting the above into (2) gives

$$f(\theta) = \operatorname{Im}\left(x\int\frac{1}{1 - x^2}dx\right) \tag{3}$$

But

$$\int\frac{1}{1 - x^2}dx = \int\frac{1}{(1 - x)(1 + x)}dx$$

Let $\frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$. Hence $A(1 + x) + B(1 - x) = 1$ or $A + Ax + B - Bx = 1$ or $x(A - B) + (A + B) = 1$. Therefore $A = 1 - B$ and $A = B$. Hence $2B = 1$ or $B = \frac{1}{2}$ and also

$A = \frac{1}{2}$. It follows that the above integral becomes

$$\begin{aligned}
\int \frac{1}{1-x^2} dx &= \int \frac{A}{1-x} + \frac{B}{1+x} dx \\
&= \frac{1}{2} \int \frac{1}{1-x} + \frac{1}{1+x} dx \\
&= \frac{1}{2} \left(\ln(1+x) - \ln(1-x) \right) \\
&= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)
\end{aligned}$$

Substituting the above into (3) gives

$$f(\theta) = \operatorname{Im} \left(\frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) \right)$$

Now, replacing x back by $e^{i\frac{\theta}{2}}$ gives

$$f(\theta) = \frac{1}{2} \operatorname{Im} \left(e^{i\frac{\theta}{2}} \ln \left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}} \right) \right)$$

Multiplying the numerator and denominator inside the $\ln$ by $e^{\frac{-i\theta}{4}}$ gives

$$\begin{aligned}
f(\theta) &= \frac{1}{2} \operatorname{Im} \left(e^{i\frac{\theta}{2}} \ln \left(\frac{e^{\frac{-i\theta}{4}} + e^{i\frac{\theta}{4}}}{e^{\frac{-i\theta}{4}} - e^{i\frac{\theta}{4}}} \right) \right) \\
&= \frac{1}{2} \operatorname{Im} \left(e^{i\frac{\theta}{2}} \ln \left(\frac{e^{i\frac{\theta}{4}} + e^{\frac{-i\theta}{4}}}{-\left(e^{i\frac{\theta}{4}} - e^{\frac{-i\theta}{4}} \right)} \right) \right) \qquad (4)
\end{aligned}$$

But $\cos\left(\frac{\theta}{4}\right) = \frac{e^{i\frac{\theta}{4}} + e^{\frac{-i\theta}{4}}}{2}$ and $\sin\left(\frac{\theta}{4}\right) = \frac{e^{\frac{-i\theta}{4}} - e^{i\frac{\theta}{4}}}{2i}$, therefore

$$\begin{aligned}
e^{i\frac{\theta}{4}} + e^{\frac{-i\theta}{4}} &= 2\cos\left(\frac{\theta}{4}\right) \\
e^{i\frac{\theta}{4}} - e^{\frac{-i\theta}{4}} &= 2i\sin\left(\frac{\theta}{4}\right)
\end{aligned}$$

Using these in (4) gives

$$\begin{aligned}
f(\theta) &= \frac{1}{2} \operatorname{Im} \left(e^{i\frac{\theta}{2}} \ln \left(\frac{2\cos\left(\frac{\theta}{4}\right)}{-2i\sin\left(\frac{\theta}{4}\right)} \right) \right) \\
&= \frac{1}{2} \operatorname{Im} \left(e^{i\frac{\theta}{2}} \ln \left(i \frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)} \right) \right) \qquad (5)
\end{aligned}$$

Using $\ln z = \ln|z| + i\arg(z)$, where the principal argument is used. Here $z = i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$. This gives $|z| = \frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$ and

$$\arg(z) = \arg\left(i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right)$$

Since since $\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)} > 0$ for all θ in the range $0 < \theta < \pi$ then $i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$ is complex in the positive i direction. Hence

$$\arg(z) = \frac{\pi}{2}$$

Therefore we can write that

$$\ln\left(i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) = \ln\left(\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) + i\frac{\pi}{2}$$

But we can simplify the above more using

$$\begin{aligned}
\ln\left(\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) &= \ln\left(\frac{1}{\tan\left(\frac{\theta}{4}\right)}\right)\\
&= \ln 1 - \ln\tan\left(\frac{\theta}{4}\right)\\
&= -\ln\tan\left(\frac{\theta}{4}\right)
\end{aligned}$$

Substituting all the above back into (5) gives

$$\begin{aligned}
f(\theta) &= \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\left[-\ln\tan\frac{\theta}{4} + i\frac{\pi}{2}\right]\right)\\
&= \frac{1}{2}\operatorname{Im}\left(\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(-\ln\tan\frac{\theta}{4} + i\frac{\pi}{2}\right)\right)\\
&= \frac{1}{2}\operatorname{Im}\left(-\cos\frac{\theta}{2}\ln\tan\frac{\theta}{4} - i\sin\frac{\theta}{2}\ln\tan\frac{\theta}{4} + i\frac{\pi}{2}\cos\frac{\theta}{2} - \frac{\pi}{2}\sin\frac{\theta}{2}\right)\\
&= \frac{1}{2}\operatorname{Im}\left(i\left[-\sin\frac{\theta}{2}\ln\tan\frac{\theta}{4} + \frac{\pi}{2}\cos\frac{\theta}{2}\right] + \left[-\cos\frac{\theta}{2}\ln\tan\frac{\theta}{4} + \frac{\pi}{2}\sin\frac{\theta}{2}\right]\right)
\end{aligned}$$

Now we can take the imaginary part, giving the final answer as

$$f(\theta)=\frac{1}{2}\left(\frac{\pi}{2}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\ln\tan\left(\frac{\theta}{4}\right)\right)$$

2.2.5 Problem 4

Evaluate the series $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2x^{2n-1}}{(2n-1)!}=x-\frac{4x^3}{3!}+\frac{9x^5}{5!}+\cdots$ by comparing it with $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

Solution

Since

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2x^{2n-1}}{(2n-1)!}$$

And since

$$\sin(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

Then we start by taking derivative of $\sin(x)$ twice, which gives

$$\frac{d}{dx}\sin(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)x^{2n-2}}{(2n-1)!} \tag{1}$$

And differentiating one more time

$$\begin{aligned}\frac{d^2}{dx^2}\sin(x)&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)(2n-2)x^{2n-3}}{(2n-1)!}\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(4n^2-6n+2\right)x^{2n-3}}{(2n-1)!}\\&=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2x^{2n-3}}{(2n-1)!}-6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}nx^{2n-3}}{(2n-1)!}+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-3}}{(2n-1)!}\end{aligned}$$

Multiplying both sides by x^2 gives

$$x^2\frac{d^2}{dx^2}\sin(x)=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2x^{2n-1}}{(2n-1)!}-6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}nx^{2n-1}}{(2n-1)!}+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

$$=4f(x)-6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}nx^{2n-1}}{(2n-1)!}+2\sin(x) \tag{2}$$

Let

$$g(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}nx^{2n-1}}{(2n-1)!}$$

Then (2) becomes

$$\begin{aligned} x^2 \frac{d^2}{dx^2} \sin(x) &= 4f(x) - 6g(x) + 2\sin(x) \\ -x^2 \cos x &= 4f(x) - 6g(x) + 2\sin(x) \end{aligned} \tag{3}$$

So we just need to find $g(x)$. For this we can use (1). Writing (1) as

$$\begin{aligned} \frac{d}{dx}\sin(x) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-2}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{(2n-1)!} \\ x\frac{d}{dx}\sin(x) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-1}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} \\ x\frac{d}{dx}\sin(x) &= 2g(x) - \sin(x) \end{aligned}$$

Hence

$$g(x) = \frac{x\frac{d}{dx}\sin(x) + \sin(x)}{2}$$

Using the above in (3) gives

$$\begin{aligned} x^2 \frac{d^2}{dx^2}\sin(x) &= 4f(x) - 6\left(\frac{x\frac{d}{dx}\sin(x) + \sin(x)}{2}\right) + 2\sin(x) \\ -x^2 \sin(x) &= 4f(x) - 3(x\cos x + \sin x) + 2\sin x \end{aligned}$$

Solving for $f(x)$

$$\begin{aligned} f(x) &= \frac{-x^2 \sin + 3x\cos x + \sin x}{4} \\ &= \frac{\left(1 - x^2\right)\sin x + 3x\cos x}{4} \end{aligned}$$

Or

$$x - \frac{4x^3}{3!} + \frac{9x^5}{5!} + \cdots = \frac{1}{4}\left(1 - x^2\right)\sin x + \frac{3}{4} x\cos x$$

2.2.6 Problem 5

The Euler numbers are defined by

$$\sec(x) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$$

(a) What is E_0?

(b) Find recursion expansion for E_{2n} when $n \geq 1$. Determine E_2, E_4, E_6, E_8 explicitly.

(c) The partial fraction expansion of secant is

$$\sec(k\pi) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)}{(2m+1)^2-4k^2}$$

Expand the right side in a power series in k and use it to evaluate the sum

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}}$$

In terms of one or more Euler numbers.

Solution

2.2.6.1 Part (a)

Using the formula given, we see that

$$\sec(x) = E_0 - \frac{E_2}{2!}x^2 + \frac{E_4}{4!}x^4 - \frac{E_6}{6!}x^6 + \cdots$$

When $x = 0$ the above gives

$$\sec(0) = E_0$$

Hence

$$E_0 = 1$$

2.2.6.2 Part (b)

Since $\cos(x)\sec(x) = 1$ then

$$1 = \cos(x)\left(\sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}\right)$$

Using power series expansion for $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$, then the above becomes

$$1 = \left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}\right)$$

To see the pattern, so that we can combine the product above, let us multiply few terms, and collect on powers of x

$$\begin{aligned}
1 &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(E_0 - \frac{E_2}{2!}x^2 + \frac{E_4}{4!}x^4 - \frac{E_6}{6!}x^6\cdots\right)\\
&= x^0(E_0) + x^2\left(-\frac{E_2}{2!} - \frac{E_0}{2!}\right) + x^4\left(\frac{E_4}{4!} + \frac{E_2}{2!2!} + \frac{E_0}{4!}\right) + x^6\left(-\frac{E_6}{6!} - \frac{E_4}{2!4!} - \frac{E_2}{4!2!} - \frac{E_0}{6!}\right) + \cdots\\
&= x^0(E_0) - x^2\left(\frac{E_0}{2!} + \frac{E_2}{2!}\right) + x^4\left(\frac{E_0}{4!} + \frac{E_2}{2!2!} + \frac{E_4}{4!}\right) - x^6\left(\frac{E_0}{6!} + \frac{E_2}{4!2!} + \frac{E_4}{2!4!} + \frac{E_6}{6!}\right) + \cdots
\end{aligned}$$

Therefore the above can be written as

$$1=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)(-1)^{n}\,x^{2n}$$

When $n=0$ then the RHS $\sum_{k=0}^{n}\frac{1}{(2n-2k)!(2k)!}E_{2k}=E_0=1$. Hence we can rewrite the above by starting sum from $n=1$ as follows

$$1=1+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)(-1)^{n}\,x^{2n}$$

$$0=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)(-1)^{n}\,x^{2n}$$

Equating terms of powers of x on both sides: since left side has no x, then this implies the coefficient of x in the RHS must be zero. This implies

$$\sum_{k=0}^{n}\frac{(-1)^{n}}{(2n-2k)!\,(2k)!}E_{2k}=0$$

$$\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}=0$$

Since we want E_{2n}, then we make the sum stop at $n-1$ to isolate that term. Hence the above becomes

$$\begin{aligned}\left(\sum_{k=0}^{n-1}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)+\frac{1}{(2n-2n)!\,(2n)!}E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)+\frac{1}{(2n)!}E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\frac{(2n)!}{(2n-2k)!\,(2k)!}E_{2k}\right)+E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k}\right)+E_{2n}&=0\end{aligned}$$

Therefore the recursion formula is finally found as

$$E_{2n}=-\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k}\tag{4}$$

Using (4), we now calculate E_2,E_4,E_6,E_8.

<u>For $n = 1$</u> then (4) becomes

$$\begin{aligned} E_2 &= -\sum_{k=0}^{0} \frac{(2)!}{(2-2k)!\,(2k)!} E_{2k} \\ &= \frac{(2)!}{(2)!} E_0 \\ &= -E_0 \\ &= -1 \end{aligned}$$

<u>For $n = 2$</u> then (5) becomes

$$\begin{aligned} E_4 &= -\sum_{k=0}^{1} \frac{(4)!}{(4-2k)!\,(2k)!} E_{2k} \\ &= -\left(\frac{(4)!}{(4)!} E_0 + \frac{(4)!}{(4-2)!\,(2)!} E_2 \right) \\ &= -\left(E_0 + \frac{(4)\,(3)\,(2)}{(2)\,(2)} E_2 \right) \\ &= -\left(1 + (2)\,(3)\,(-1)\right) \\ &= -(1-6) \\ &= 5 \end{aligned}$$

<u>For $n = 3$</u> then (5) becomes

$$\begin{aligned} E_6 &= -\sum_{k=0}^{2} \frac{(6)!}{(6-2k)!\,(2k)!} E_{2k} \\ &= -\left(\frac{(6)!}{(6)!} E_0 + \frac{(6)!}{(6-2)!\,(2)!} E_2 + \frac{(6)!}{(2)!\,(4)!} E_4 \right) \\ &= -\left(E_0 + \frac{(6)\,(5)}{2} E_2 + \frac{(6)\,(5)}{2} E_4 \right) \\ &= -\left(E_0 + 15E_2 + 15E_4\right) \\ &= -\left(1 + 15\,(-1) + 15\,(5)\right) \\ &= -61 \end{aligned}$$

For $n = 4$ then (5) becomes

$$\begin{aligned}
E_6 &= -\sum_{k=0}^{3} \frac{(8)!}{(8-2k)!\,(2k)!} E_{2k} \\
&= -\left(\frac{(8)!}{(8)!}E_0 + \frac{(8)!}{(8-2)!\,(2)!}E_2 + \frac{(8)!}{(4)!\,(4)!}E_4 + \frac{(8)!}{(8-6)!\,(6)!}E_6\right) \\
&= -\left(E_0 + \frac{(8)!}{(6)!\,(2)!}E_2 + \frac{(8)\,(7)\,(6)\,(5)}{(4)!}E_4 + \frac{(8)!}{(2)!\,(6)!}E_6\right) \\
&= -\left(E_0 + \frac{(8)\,(7)}{(2)!}E_2 + \frac{(8)\,(7)\,(6)\,(5)}{(4)\,(3)\,(2)}E_4 + \frac{(8)\,(7)}{(2)}E_6\right) \\
&= -\left(E_0 + 28E_2 + 70E_4 + 28E_6\right) \\
&= -\left(1 + 28\,(-1) + 70\,(5) + 28\,(-61)\right) \\
&= 1385
\end{aligned}$$

Summary

n	E_{2n}
0	$E_0 = 1$
1	$E_2 = -1$
2	$E_4 = 5$
3	$E_6 = -61$
4	$E_8 = 1385$

2.2.6.3 Part c

$$\begin{aligned}
\sec(k\pi) &= \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m\,(2m+1)}{(2m+1)^2 - (2k)^2} \\
&= \frac{4}{\pi}\sum_{m=0}^{\infty} (-1)^m \frac{(2m+1)}{(2m+1)^2 - (2k)^2} \\
&= \frac{4}{\pi}\sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1) - \frac{(2k)^2}{(2m+1)}} \\
&= \frac{4}{\pi}\sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)\left(1 - \frac{(2k)^2}{(2m+1)^2}\right)} \\
&= \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \frac{1}{\left(1 - \frac{(2k)^2}{(2m+1)^2}\right)} \\
&= \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \frac{1}{\left(1 - \left(\frac{2k}{2m+1}\right)^2\right)}
\end{aligned}$$

Assuming $\left|\frac{2k}{2m+1}\right| < 1$ then $\frac{1}{\left(1-\left(\frac{2k}{2m+1}\right)^2\right)} = \sum_{n=0}^{\infty}\left(\frac{2k}{2m+1}\right)^{2n}$. From Binomial series. Then the above can be written as

$$\sec(k\pi) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\left(\sum_{n=0}^{\infty}\left(\frac{2k}{2m+1}\right)^{2n}\right)$$

Interchanging the order of summation in order to combine m terms

$$\begin{aligned}\sec(k\pi) &= \frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\left(\frac{2}{2m+1}\right)^{2n}\right)\\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}}\right) \qquad (1)\end{aligned}$$

But since $\sec(x) = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}$, then when $x = k\pi$, this becomes

$$\begin{aligned}\sec(k\pi) &= \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}(k\pi)^{2n}\\ &= \sum_{n=0}^{\infty}k^{2n}(-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n} \qquad (2)\end{aligned}$$

Comparing (1) and (2), we see this correspondence

$$\frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}}\right) = \sum_{n=0}^{\infty}k^{2n}(-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n}$$

Hence

$$\begin{aligned}\frac{4}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}} &= (-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n}\\ \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} &= (-1)^n\left(\frac{\pi}{4}\right)\left(\frac{1}{2^{2n}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n}\\ &= (-1)^n\left(\frac{1}{2^2}\right)\left(\frac{1}{2^{2n}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n+1}\end{aligned}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} = (-1)^n\left(\frac{1}{2^{2(n+1)}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n+1}$$

Where E_{2n} are the Euler numbers found above.

2.2.7 Key solution for HW 2

① $S = 1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - \cdots$

Series converges absolutely since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$

$$S = \left(1 - \frac{1}{16} + \frac{1}{256} - \cdots\right) + \underbrace{\left(\frac{1}{4} - \frac{1}{64} + \frac{1}{1024} - \cdots\right)}_{\frac{1}{4}\left(1 - \frac{1}{16} + \frac{1}{256} - \cdots\right)}$$

$$= \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n = \frac{5}{4}\,\frac{1}{1+\frac{1}{16}} = \boxed{\frac{20}{17} = S}$$

② $S = \frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \cdots$ Looks related to e^x.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \frac{d}{dx}\left(xe^x\right) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty}\frac{(n+1)x^n}{n!} =$$

$$= \frac{1}{0!} + \frac{2}{1!}x + \frac{3}{2!}x^2 + \cdots \qquad S \text{ has } x = 1$$

$$\frac{d}{dx}(xe^x) = (1+x)e^x \qquad \boxed{S = 2e}$$

(3) $f(\theta) = \sin(\theta) + \frac{1}{3}\sin(2\theta) + \frac{1}{5}\sin(3\theta) + \frac{1}{7}\sin(4\theta) + \cdots$

$= \operatorname{Im}\left\{ e^{i\theta} + \frac{1}{3}e^{i2\theta} + \frac{1}{5}e^{i3\theta} + \frac{1}{7}e^{i4\theta} + \cdots \right\}$

$= \operatorname{Im}\left\{ e^{\frac{i\theta}{2}}\left[e^{\frac{i\theta}{2}} + \frac{1}{3}e^{\frac{i3\theta}{2}} + \frac{1}{5}e^{\frac{i5\theta}{2}} + \frac{1}{7}e^{\frac{i7\theta}{2}} + \cdots \right]\right\}$

$= \operatorname{Im} g$ where $g = z\left(z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \cdots\right) = zh$

and $z = e^{\frac{i\theta}{2}}$

$h' = 1 + z^2 + z^4 + z^6 + \cdots = \frac{1}{1-z^2}$

Now $h = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \underbrace{a + ib}_{\text{real constants}}$

$\ln\left(\frac{1+z}{1-z}\right) = \ln\left(\frac{1+e^{\frac{i\theta}{2}}}{1-e^{\frac{i\theta}{2}}}\right) = \ln\left(\frac{e^{\frac{i\theta}{4}} + e^{-i\frac{\theta}{4}}}{e^{-i\frac{\theta}{4}} - e^{i\frac{\theta}{4}}}\right) =$

$= \ln\left(\frac{\cos\frac{\theta}{4}}{-i\sin\frac{\theta}{4}}\right) = \ln\left(\frac{e^{\frac{i\pi}{2}}}{\tan\frac{\theta}{4}}\right) = i\frac{\pi}{2} - \ln\left(\tan\frac{\theta}{4}\right)$

$g = \frac{1}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left[i\frac{\pi}{2} - \ln\left(\tan\frac{\theta}{4}\right) + 2a + 2ib\right]$

$f = \operatorname{Im} g = \cos\frac{\theta}{2}\left[b + \frac{\pi}{4}\right] - \frac{1}{2}\sin\frac{\theta}{2}\ln\left(\tan\frac{\theta}{4}\right) + a\sin\frac{\theta}{2}$

Now $f(0) = f(\pi) = 0 \Rightarrow a = 0,\ b = -\frac{\pi}{4}$

$$\boxed{f(\theta) = -\frac{1}{2}\sin\frac{\theta}{2}\ln\left(\tan\frac{\theta}{4}\right)}$$

(4) $$f = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!}$$ Looks like 2 derivatives to get n^2 factor.

$$g = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n-1)!} \qquad x\frac{dg}{dx} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n}}{(2n-1)!}$$

$$\frac{d}{dx}\left(x\frac{dg}{dx}\right) = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = 4f$$

$$g = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = x\sin x$$

$$\frac{dg}{dx} = x\cos x + \sin x \qquad \frac{d}{dx}\left(x\frac{dg}{dx}\right) = \frac{d}{dx}\left(x^2\cos x + x\sin x\right) =$$

$$= 2x\cos x - x^2\sin x + \sin x + x\cos x$$

$$= 3x\cos x + (1-x^2)\sin x$$

$$\boxed{f(x) = \frac{3}{4}x\cos x + \frac{(1-x^2)}{4}\sin x}$$

(5) $$\sec(x) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$$

(a) $x = 0$ $\quad \sec(0) = \boxed{1 = E_0}$

(b) $$1 = \cos(x) \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n} = \left[\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}\right]\left[\sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!}\right]$$

The $m = n = 0$ term gives 1. The rest must vanish order by order. For $n \geq 1$

$$\frac{(-1)^n E_{2n}}{(2n)!} - \frac{1}{2!} \frac{(-1)^{n-1} E_{2n-2}}{(2n-2)!} + \frac{1}{4!} \frac{(-1)^n E_{2n-4}}{(2n-4)!} - \cdots \frac{(-1)^n E_0}{(2n)!} = 0$$

$$\boxed{E_{2n} + \frac{(2n)!}{2!(2n-2)!} E_{2n-2} + \frac{(2n)!}{4!(2n-4)!} E_{2n-4} + \cdots + E_0 = 0}$$

Note that the coefficients are binomial coefficients.

$$\boxed{E_2 = -1 \quad E_4 = 5 \quad E_6 = -61 \quad E_8 = 1385}$$

(c) For small k we can expand

$$\frac{1}{(2m+1)^2 - 4k^2} = \frac{1}{(2m+1)^2}\left[1 - \left(\frac{2k}{2m+1}\right)^2\right]^{-1} =$$

$$= \frac{1}{(2m+1)^2} \sum_{n=0}^{\infty} \frac{(4k^2)^n}{(2m+1)^{2n}}$$

$$\sec(k\pi) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\sum_{n=0}^{\infty}\frac{(2k)^{2n}}{(2m+1)^{2n}}$$

$$= \frac{4}{\pi}\sum_{n=0}^{\infty}(2k)^{2n}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}}$$

Compare to $\sec(k\pi) = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}(k\pi)^{2n}$

order by order in k.

$$\frac{4}{\pi}\cdot 2^{2n}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} = \frac{(-1)^n\pi^{2n}E_{2n}}{(2n)!}$$

$$\Rightarrow \boxed{\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} = \frac{(-1)^n}{2}\left(\frac{\pi}{2}\right)^{2n+1}\frac{E_{2n}}{(2n)!}}$$

2.3 HW 3

Local contents

2.3.1 HW 3 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 3 due Monday February 18. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (4 pts) Consider the function $f(z) = z^{1/n}$ where n is a positive integer. The branch point is at $z = 0$ and the branch cut is chosen to be along the positive x axis. How many sheets are there? What is the range of θ corresponding to each sheet?

2. (5 pts) Derive the formula

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$$

3. (5 pts) Using the formula for $\tan^{-1} z$ from the previous problem, find the real functions $u(x,y)$ and $v(x,y)$ in the expression $\tan^{-1} z = u(x,y) + iv(x,y)$.

4. (6 pts) In the domain $r > 0$, $0 < \theta < 2\pi$, show that the function $u = \ln r$ is harmonic and find its harmonic conjugate. Do this in both Cartesian and polar coordinates.

5. (5 pts) Find the value of $\int_C f(z)\, dz$ where $f(z) = e^z$ for two different contours. C_1 is a straight line from the origin to the point (2,1). C_2 is a straight line from the origin to the point (2,0) followed by another straight line from (2,0) to (2,1).

2.3.2 Problem 1

Consider the function $f(z) = z^{\frac{1}{n}}$ where n is a positive integer. The branch point is at $z = 0$ and the branch cut is chosen to be along the positive x axis. How many sheets are there? What is the range of θ corresponding to each sheet?

Solution

Following the example in the class handout, where it showed how to find the number of sheets for $z^{\frac{1}{2}}$, the same method is used here, which is to keep adding a multiple of 2π angles until the same result for the original principal value of the function $g(z)$ evaluated at θ is obtained. This gives the number of sheets.

Let

$$\begin{aligned} g(z) &= z^{\frac{1}{n}} \\ g(r,\theta) &= \left(re^{i\theta}\right)^{\frac{1}{n}} \\ g(r,\theta) &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \end{aligned} \tag{1}$$

In the above, θ is called principal argument. And now the idea is to find how many times 2π needs to be added to θ in order to get back the same value of original of $g(r,\theta)$ at the starting θ that one picks. Adding one time 2π to θ, equation (1) becomes

$$\begin{aligned} g(r,\theta+2\pi) &= r^{\frac{1}{n}} e^{i\frac{(\theta+2\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{2\pi}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{2\pi}{n}} \end{aligned}$$

And we add another 2π, or now a total of 4π

$$\begin{aligned} g(r,\theta+4\pi) &= r^{\frac{1}{n}} e^{i\frac{(\theta+4\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{4\pi}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{4\pi}{n}} \end{aligned}$$

And so on. We keep adding 2π, or a total of $k(2\pi)$ such that the last term above, which in term of k is $e^{\frac{k(2\pi)i}{n}}$ simplifies to 1 which implies getting back original function value at $g(r,\theta)$. Hence for k times we have

$$\begin{aligned} g(r,\theta+k(2\pi)) &= r^{\frac{1}{n}} e^{i\frac{(\theta+k(2\pi))}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{k(2\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{k(2\pi)}{n}} \end{aligned}$$

We see from the above, is that only $\underline{\text{when } k=n}$, then $r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{k(2\pi)}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{2\pi i}$. But $e^{2\pi i} = 1$, therefore it reduces to

$$\begin{aligned} g(r,\theta+n(2\pi)) &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \\ &= g(r,\theta) \end{aligned}$$

Which is the original value of the function. Therefore there are $\underline{n \text{ sheets}}$.

The formula that can also be used to obtain all values for this multivalued function is

$$g(r,\theta) = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)} \qquad k = 0,1,\cdots n-1$$

Now to answer the angle θ range question. From the above, we see the range of the angle for each sheet is as follows

$$\begin{aligned} &R_1 : 0 < \theta < 2\pi \\ &R_2 : 2\pi < \theta < 4\pi \\ &R_3 : 4\pi < \theta < 6\pi \\ &\vdots \\ &R_n : (n-1)\,2\pi < \theta < n\,(2\pi) \end{aligned}$$

Sheet R_1 is called the principal sheet associated with $k = 0$.

2.3.3 Problem 2

Derive the formula

$$\arctan z = \frac{i}{2} \ln \left(\frac{i+z}{i-z} \right)$$

Solution

Let $w = \arctan(z)$ hence

$$\begin{aligned} z &= \tan(w) \\ z &= \frac{\sin w}{\cos w} \end{aligned}$$

But $\sin w = \frac{e^{iw}-e^{-iw}}{2i}$ and $\cos w = \frac{e^{iw}+e^{-iw}}{2}$, hence the above simplifies to

$$\begin{aligned} z &= \frac{\frac{e^{iw}-e^{-iw}}{2i}}{\frac{e^{iw}+e^{-iw}}{2}} \\ &= \frac{1}{i} \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} \end{aligned}$$

Or

$$iz = \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

Multiplying the numerator and denominator of the right side by e^{iw} gives

$$iz = \frac{e^{2iw}-1}{e^{2iw}+1}$$

Let $e^{iw} = x$ then the above is the same as

$$\begin{aligned}
iz &= \frac{x^2-1}{x^2+1} \\
iz\left(x^2+1\right) &= x^2-1 \\
x^2 iz + iz &= x^2-1 \\
x^2 iz + iz - x^2 + 1 &= 0 \\
x^2\left(iz-1\right) + \left(1+iz\right) &= 0 \\
x^2 &= \frac{-\left(1+iz\right)}{\left(iz-1\right)} \\
&= \frac{\left(1+iz\right)}{\left(1-iz\right)}
\end{aligned}$$

Simplifying gives

$$\begin{aligned}
x^2 &= \frac{i\left(-i+z\right)}{i\left(-i-z\right)} \\
&= \frac{\left(z-i\right)}{\left(-i-z\right)}
\end{aligned}$$

Hence

$$x = \pm\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

But $x = e^{iw}$, and the above becomes

$$e^{iw} = \pm\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

We need now to decide which sign to take. Since $z = \tan(w)$, then when $w = 0$, $z = 0$ because $\tan(0) = 0$. Putting $w = 0, z = 0$ in the above gives

$$\begin{aligned}
1 &= \pm\left(\frac{i}{i}\right)^{\frac{1}{2}} \\
&= \pm\left(1\right)^{\frac{1}{2}} \\
&= \pm 1
\end{aligned}$$

Hence we need to choose the + sign so both sides is positive. Hence

$$e^{iw} = \left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

Now, taking the natural log of both sides gives

$$
\begin{aligned}
iw &= \ln\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}} \\
iw &= \frac{1}{2}\ln\left(\frac{z-i}{-i-z}\right) \\
w &= \frac{1}{2i}\ln\left(\frac{z-i}{-i-z}\right) \\
&= \frac{-i}{2}\ln\left(\frac{z-i}{-i-z}\right) \\
&= \frac{i}{2}\ln\left(\left(\frac{z-i}{-i-z}\right)^{-1}\right) \\
&= \frac{i}{2}\ln\left(\frac{-i-z}{z-i}\right) \\
&= \frac{i}{2}\ln\left(\frac{-(z+i)}{-(i-z)}\right) \\
&= \frac{i}{2}\ln\left(\frac{z+i}{i-z}\right)
\end{aligned}
$$

But $w = \arctan(z)$, hence the final result is

$$
\arctan(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)
$$

2.3.4 Problem 3

Using the formula for $\arctan z$ from the previous problem, find the real functions $u(x,y)$ and $v(x,y)$ in the expression $\arctan z = u(x,y) + iv(x,y)$

Solution

Let

$$
\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) = u + iv
$$

where $u \equiv u(x,y), v \equiv v(x,y)$ are the real and imaginary parts of $\arctan(z)$. Therefore

$$
\begin{aligned}
\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) &= \frac{i}{2}\left(\ln\left|\frac{i+z}{i-z}\right| + i\left(\arg\left(\frac{i+z}{i-z}\right) + 2n\pi\right)\right) \qquad n = 0, \pm 1, \pm 2, \cdots \\
&= \frac{i}{2}\ln\left|\frac{i+z}{i-z}\right| - \frac{1}{2}\left(\arg\left(\frac{i+z}{i-z}\right) + 2n\pi\right) \qquad (1)
\end{aligned}
$$

Where $\arg\left(\frac{i+z}{i-z}\right)$ is the principal argument. But since $z = x + iy$ then we see that

$$\begin{aligned}
\left|\frac{i+z}{i-z}\right| &= \left|\frac{i+(x+iy)}{i-(x+iy)}\right| \\
&= \left|\frac{i+x+iy}{i-x-iy}\right| \\
&= \left|\frac{x+i(1+y)}{-x+i(1-y)}\right| \\
&= \frac{\sqrt{x^2+(1+y)^2}}{\sqrt{x^2+(1-y)^2}} \\
&= \sqrt{\frac{x^2+(1+y)^2}{x^2+(1-y)^2}} \qquad (2)
\end{aligned}$$

And the principal argument is

$$\begin{aligned}
\arg\left(\frac{i+z}{i-z}\right) &= \arg(i+z) - \arg(i-z) \\
&= \arg(i(1-iz)) - \arg(i(1+iz)) \\
&= \arg i + \arg(1-iz) - \arg i + \arg(1+iz) \\
&= \arg(1-iz) + \arg(1+iz)
\end{aligned}$$

Letting $z = x + iy$ in the above results in

$$\begin{aligned}
\arg\left(\frac{i+z}{i-z}\right) &= \arg\left(1-i\left(x+iy\right)\right) - \arg\left(1+i\left(x+iy\right)\right) \\
&= \arg\left(1-ix+y\right) - \arg\left(1+ix-y\right) \\
&= \arg\left(\left(1+y\right)-ix\right) - \arg\left(\left(1-y\right)+ix\right) \\
&= \arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) &= \frac{i}{2}\left(\ln\sqrt{\frac{x^2+(1+y)^2}{x^2+(1-y)^2}} + i\left(\arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) + 2n\pi\right)\right) \qquad n = 0, \pm 1, \pm 2, \cdots \\
&= \frac{i}{4}\ln\left(\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right) - \frac{1}{2}\left(\arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) + 2n\pi\right)
\end{aligned}$$

Setting the above equal to $u+iv$ shows that <u>the real part and the imaginary parts are</u>

$$u=-\frac{1}{2}\left(\arctan\left(\frac{-x}{1+y}\right)-\arctan\left(\frac{x}{1-y}\right)+2n\pi\right)\qquad n=0,\pm 1,\pm 2,\cdots$$
$$v=\frac{1}{4}\ln\left(\frac{x^2+\left(y+1\right)^2}{x^2+\left(1-y\right)^2}\right)$$

Therefore

$$\begin{aligned}\arctan\left(z\right)&=\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)\\&=u+iv\end{aligned}$$

Where u,v are given above. We see that $\arctan\left(z\right)$ is multivalued as it depends on the value of n.

For illustration of $u\left(x,y\right)$ and $v\left(x,y\right)$, the following is a plot of the above found solution showing the real part $u\left(x,y\right)$ for $n=0$ (principal sheet)

```
Plot3D[-1/2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x]),
 {x, -Pi/2, Pi/2}, {y, -Pi/2, Pi/2},
 AxesLabel → {"x", "y", "u(x,y)"},
 BaseStyle → 14, Ticks → {{-Pi/2, 0, Pi/2}, {-Pi/2, 0, Pi/2},
   Automatic}]
```

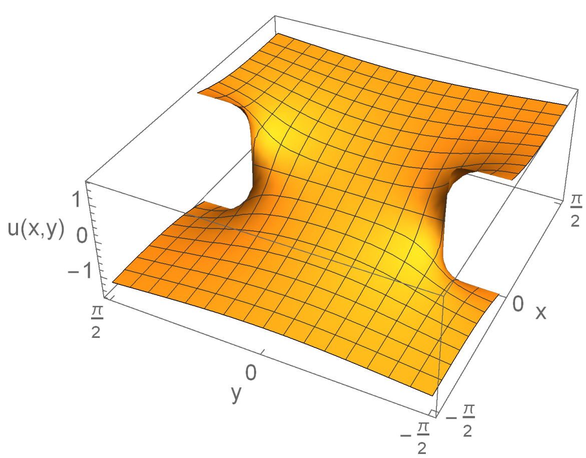


Figure 2.1: Real part $u(x,y)$ using principal sheet

And the following shows $u\left(x,y\right)$ with both $n=0$ and $n=1$ on the same plot showing two sheets

```
In[*]:= Plot3D[{-1 / 2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x]),
   -1 / 2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x] + 2 Pi)},
  {x, -Pi / 2, Pi / 2}, {y, -Pi / 2, Pi / 2},
  AxesLabel → {"x", "y", "u(x,y)"},
  BaseStyle → 14, Ticks → {{-Pi / 2, 0, Pi / 2}, {-Pi / 2, 0, Pi / 2},
    Automatic}]
```

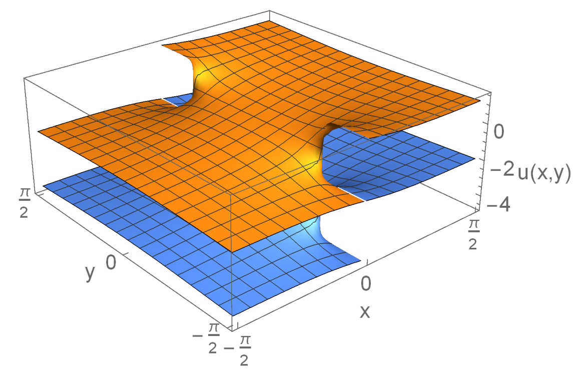


Figure 2.2: Real part $u(x,y)$ showing $n=0,n=1$ on same plot

And the following plot shows the imaginary part $v\left(x,y\right)$

```
In[*]:= Plot3D[1 / 4 Log[(x^2 + (y + 1)^2) / (x^2 + (1 - y)^2)], {x, -Pi / 2, Pi / 2}, {y, -1.5, 1.5},
  AxesLabel → {"x", "y", "u(x,y)"}, BaseStyle → 14,
  Ticks → {{-Pi / 2, 0, Pi / 2}, {-Pi / 2, 0, Pi / 2}, Automatic}]
```

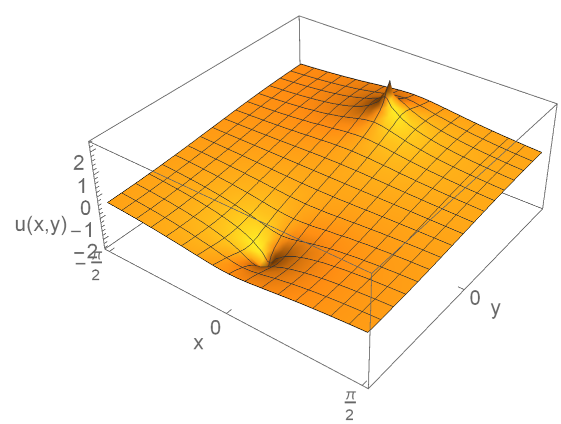


Figure 2.3: Imaginary part $v(x,y)$

2.3.5 Problem 4

In the domain $r > 0, 0 < \theta < 2\pi$. show that the function $u = \ln r$ is harmonic and find its conjugate. Do this in both Cartesian and polar coordinates.

2.3.5.1 Part (a) Using Cartesian

A function $u\left(x,y\right)$ is harmonic if it satisfies the Laplace PDE $u_{xx} + u_{yy} = 0$. Since

$$r = \sqrt{x^2 + y^2}$$

Then

$$\begin{aligned} u &= \ln r \\ &= \ln \sqrt{x^2 + y^2} \\ &= \frac{1}{2} \ln\left(x^2 + y^2\right) \end{aligned}$$

We now need to calculate u_{xx} and u_{yy}.

$$\begin{aligned} u_x &= \frac{1}{2}\frac{\partial}{\partial x} \ln\left(x^2 + y^2\right) \\ &= \frac{1}{2}\frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

And

$$u_{xx} = \frac{\partial}{\partial x}\frac{x}{x^2 + y^2}$$

Applying the integration rule $\frac{\partial}{\partial x}\frac{f(x)}{g(x)} = \frac{f'g - fg}{g^2}$ to the above, where $f = x$ and $g = x^2 + y^2$ results in

$$\begin{aligned} u_{xx} &= \frac{x^2 + y^2 - x\left(2x\right)}{\left(x^2 + y^2\right)^2} \\ &= \frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2} \\ &= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \end{aligned} \tag{1}$$

Similarly

$$\begin{aligned}
u_y &= \frac{1}{2}\frac{\partial}{\partial y}\ln\left(x^2+y^2\right)\\
&= \frac{1}{2}\frac{2y}{x^2+y^2}\\
&= \frac{y}{x^2+y^2}
\end{aligned}$$

Applying the integration rule $\frac{\partial}{\partial y}\frac{f(y)}{g(y)} = \frac{f'g-fg}{g^2}$ to the above, where $f = y$ and $g = x^2+y^2$ results in

$$\begin{aligned}
u_{yy} &= \frac{x^2+y^2-y\left(2y\right)}{\left(x^2+y^2\right)^2}\\
&= \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \qquad (2)
\end{aligned}$$

Now that we found u_{xx} and u_{yy}, we need to verify that $u_{xx}+u_{yy}=0$. Adding (1,2) gives

$$\begin{aligned}
u_{xx}+u_{yy} &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2} + \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}\\
&= 0
\end{aligned}$$

Hence $u = \ln r$ is <u>harmonic</u>.

To find its conjugate. Let the conjugate be $v\left(x,y\right)$. Let u be the real part of analytic function

$$f = u + iv$$

Applying Cauchy Riemann equations to f results in

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (4)$$

From (3) and using the earlier result found for u_x gives

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

Integrating the above w.r.t. y gives

$$\begin{aligned} v &= \int \frac{x}{x^2+y^2} dy + \Phi(x) \\ &= x \int \frac{1}{x^2+y^2} dy + \Phi(x) \\ &= \frac{1}{x} \int \frac{1}{1+\left(\frac{y}{x}\right)^2} dy + \Phi(x) \end{aligned}$$

The above is integrated using substitution. Let $u = \frac{y}{x}$, then $\frac{du}{dy} = \frac{1}{x}$ and the integral becomes

$$\begin{aligned} v &= \frac{1}{x}\left(\int \frac{1}{1+u^2}(x du)\right) + \Phi(x) \\ &= \int \frac{1}{1+u^2} du + \Phi(x) \end{aligned}$$

But $\int \frac{1}{1+u^2} du = \arctan(u) = \arctan\left(\frac{y}{x}\right)$, therefore the above becomes

$$v = \arctan\left(\frac{y}{x}\right) + \Phi(x) \tag{5}$$

Taking derivative of (5) w.r.t. x gives an ODE to solve for $\Phi(x)$

$$\frac{\partial v}{\partial x} = \frac{d}{dx}\left(\arctan\left(\frac{y}{x}\right)\right) + \Phi'(x) \tag{5A}$$

To find $\frac{d}{dx}\arctan\left(\frac{y}{x}\right)$, let

$$w = \arctan\left(\frac{y}{x}\right)$$

Now the goal is to find $\frac{dw}{dx}$. The above is the same as

$$\tan(w) = \frac{y}{x} \tag{6}$$

Taking derivative of both sides of the above w.r.t. x gives

$$\frac{d}{dx}\tan(w) = -\frac{y}{x^2}$$

But $\frac{d}{dx}\tan(w) = \sec^2(w)\frac{dw}{dx}$, and the above can be written as

$$\begin{aligned} \sec^2(w)\frac{dw}{dx} &= -\frac{y}{x^2} \\ \frac{dw}{dx} &= -\frac{y}{x^2}\frac{1}{\sec^2(w)} \end{aligned} \tag{7}$$

But $\sec^2(w) = \frac{1}{\cos^2 w}$ and $\cos^2 w + \sin^2 w = 1$. Therefore dividing by $\cos^2 w$ gives $1 + \frac{\sin^2 w}{\cos^2 w} = \sec^2(w)$ or $1 + \tan^2 w = \sec^2(w)$. But from (6) we know that $\tan(w) = \frac{y}{x}$, therefore $1 + \left(\frac{y}{x}\right)^2 =$

$\sec^2(w)$. Replacing this expression for $\sec^2(w)$ in (7) gives

$$\begin{aligned}\frac{dw}{dx} &= -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}\\ &= -\frac{y}{x^2}\frac{x^2}{x^2+y^2}\\ &= \frac{-y}{x^2+y^2}\end{aligned}$$

Now that we found $\frac{dw}{dx}$ which is $\frac{d}{dx}\arctan\left(\frac{y}{x}\right)$, then 5A becomes

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} + \Phi'(x)$$

But from Cauchy Riemann equation (4) above, we know that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, therefore the above is the same as

$$\frac{\partial u}{\partial y} = -\left(\frac{-y}{x^2+y^2} + \Phi'(x)\right)$$

We know what $\frac{\partial u}{\partial y}$ is. We found this earlier which is $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$. Hence the above equation becomes

$$\begin{aligned}\frac{y}{x^2+y^2} &= \frac{y}{x^2+y^2} - \Phi'(x)\\ \Phi'(x) &= 0\end{aligned}$$

Therefore Φ is constant, say C_1. Equation (5) becomes

$$\boxed{v(x,y) = \arctan\left(\frac{y}{x}\right) + C_1} \tag{8}$$

Which is the conjugate of $u = \frac{1}{2}\ln\left(x^2+y^2\right)$. To verify the result in (8), we now check that $v(x,y)$ is indeed harmonic by checking that it satisfies the Laplace PDE.

$$\begin{aligned}v_x &= \frac{-y}{x^2+y^2}\\ v_{xx} &= \frac{y(2x)}{\left(x^2+y^2\right)^2}\end{aligned}$$

And

$$\begin{aligned}v_y &= \frac{x}{x^2+y^2}\\ v_{yy} &= \frac{-x(2y)}{\left(x^2+y^2\right)^2}\end{aligned}$$

Using the above we see that

$$\begin{aligned} v_{xx} + v_{yy} &= \frac{y\,(2x)}{\left(x^2+y^2\right)^2} - \frac{x\left(2y\right)}{\left(x^2+y^2\right)^2} \\ &= 0 \end{aligned}$$

This shows that $v\left(x,y\right)$ obtained above is harmonic. It is the conjugate of $u\left(x,y\right)$.

$v\left(x,y\right)$ is not a unique conjugate of $u\left(x,y\right)$, since the constant C_1 is arbitrary.

2.3.5.2 Part (b) Using Polar coordinates

Here $z = re^{i\theta}$ and we are told that $u\,(r,\theta) = \ln r$. To show this is harmonic in polar coordinates, we need to show it satisfies Laplacian in polar coordinates, which is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

But $u_r = \frac{d}{dr}\ln r = \frac{1}{r}$ and $u_{rr} = -\frac{1}{r^2}$ and $u_{\theta\theta} = 0$. Substituting these into the above gives

$$\begin{aligned} -\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore $u = \ln r$ is harmonic since it satisfies the Laplacian in polar coordinates. To find its conjugate, we use C-R in polar coordinates, and these are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \tag{2}$$

From (1), and since we know that $\frac{\partial u}{\partial r} = \frac{1}{r}$, then this gives

$$\begin{aligned} \frac{1}{r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial \theta} &= 1 \end{aligned}$$

Or by integration w.r.t. θ

$$v = \theta + \Phi\,(r)$$

Where $\Phi\,(r)$ is the constant of integration (a function). Taking derivative of the above w.r.t. r gives

$$\frac{\partial v}{\partial r} = \Phi'\,(r)$$

But from (2) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0$. (Because u does not depend on θ). Hence the above results in $\Phi'(r) = 0$ or $\Phi = C_1$ a constant. Therefore the conjugate harmonic function is

$$\boxed{v(r,\theta) = \theta + C_1}$$

Now we verify this satisfies Laplacian in Polar. From

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$$

We see since $v_r = 0$ and $v_{rr} = 0$ and $v_\theta = 1$ and $v_{\theta\theta} = 0$, therefore we obtain $0 = 0$ also. Hence $v = \theta + C_1$ satisfies the Laplacian.

2.3.6 Problem 5

Find the value of $\int_C f(z)\,dz$ where $f(z) = e^z$ for two different contours. C_1 is straight line from the origin to the point $(2,1)$. C_2 is a straight line from the origin to the point $(2,0)$ followed by another straight line from $(2,0)$ to $(2,1)$

Solution

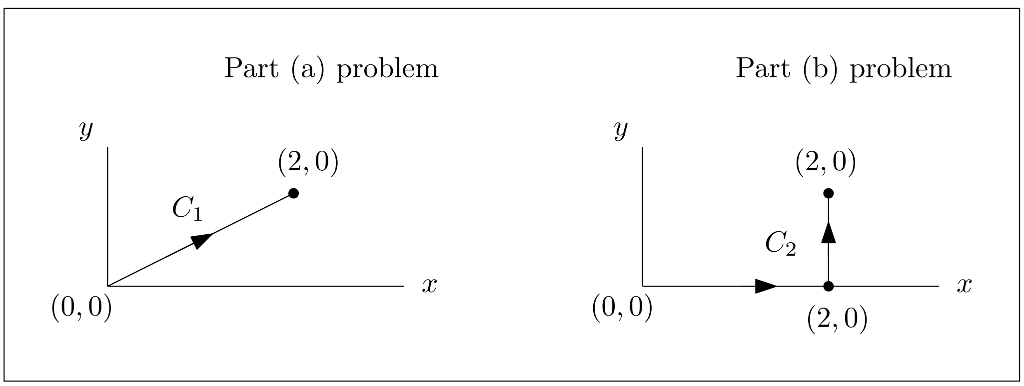


Figure 2.4: Showing contours for part(a) and pat (b)

2.3.6.1 Part a

Using contour C_1. The line starts from $(x_0, y_0) = (0,0)$ and ends at $(x_1, y_1) = (2,1)$. Hence the parametrization for this line is given by

$$\begin{aligned} x(t) &= (1-t)\,x_0 + t x_1 \\ &= 2t \end{aligned}$$

And

$$\begin{aligned} y(t) &= (1-t)\,y_0 + t y_1 \\ &= t \end{aligned}$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t this becomes

$$\begin{aligned} f(t) &= e^{2t+it} \\ &= e^{t(2+i)} \end{aligned}$$

Hence

$$\begin{aligned} \int_{C_1} f(z)\,dz &= \int_{t=0}^{t=1} f(t)\,z'(t)\,dt \\ &= \int_0^1 e^{t(2+i)} z'(t)\,dt \end{aligned}$$

But $z(t) = x(t) + iy(t) = 2t + it$, hence $z'(t) = 2 + i$ and the above becomes

$$\begin{aligned} \int_{C_1} f(z)\,dz &= \int_0^1 e^{t(2+i)}(2+i)\,dt \\ &= (2+i)\int_0^1 e^{t(2+i)} dt \\ &= (2+i)\left(\frac{e^{t(2+i)}}{(2+i)}\right)_0^1 \\ &= \left(e^{t(2+i)}\right)_0^1 \end{aligned}$$

Hence the final result is

$$\int_{C_1} f(z)\,dz = e^{2+i} - 1$$

2.3.6.2 Part b

Using C_2. The first line starts from $(x_0, y_0) = (0,0)$ and ends at $(x_1, y_1) = (2,0)$. Hence the parametrization for this line is given by

$$\begin{aligned} x(t) &= (1-t)\,x_0 + tx_1 \\ &= 2t \end{aligned}$$

And

$$\begin{aligned} y(t) &= (1-t)\,y_0 + ty_1 \\ &= 0 \end{aligned}$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t the function $f(z)$ becomes

$$f(t) = e^{2t}$$

Hence, for the line from $(0,0)$ to $(2,0)$ we have

$$\begin{aligned} \int_{C_{2_1}} f(z)\,dz &= \int_{t=0}^{t=1} f(t)\,z'(t)\,dt \\ &= \int_0^1 e^{2t} z'(t)\,dt \end{aligned}$$

But $z = x + iy = 2t$ since $y(t) = 0$. hence $z'(t) = 2$ and the above becomes

$$\begin{aligned}\int_{C_{2_1}} f(z)\,dz &= 2\int_0^1 e^{2t} dt \\ &= 2\left(\frac{e^{2t}}{2}\right)_0^1 \\ &= e^2 - 1 \qquad (1)\end{aligned}$$

The <u>second line</u> starts from $(x_0, y_0) = (2, 0)$ and ends at $(x_1, y_1) = (2, 1)$. Hence the parametrization for this line is given by

$$\begin{aligned}x(t) &= (1-t)\,x_0 + tx_1 \\ &= (1-t)\,2 + 2t \\ &= 2\end{aligned}$$

And

$$\begin{aligned}y(t) &= (1-t)\,y_0 + ty_1 \\ &= t\end{aligned}$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t this becomes

$$f(t) = e^{2+it}$$

Hence, for the line from $(2, 0)$ to $(2, 1)$ we have

$$\begin{aligned}\int_{C_{2_2}} f(z)\,dz &= \int_{t=0}^{t=1} f(t)\,z'(t)\,dt \\ &= \int_0^1 e^{2+it} z'(t)\,dt\end{aligned}$$

But $z = x + iy = 2 + it$. hence $z'(t) = i$ and the above becomes

$$\begin{aligned}\int_{C_{2_2}} f(z)\,dz &= \int_0^1 ie^{2+it} dt \\ &= i\left(\frac{e^{2+it}}{i}\right)_0^1 \\ &= \left(e^{2+it}\right)_0^1 \\ &= e^{2+i} - e^2 \qquad (2)\end{aligned}$$

Therefore the total is the sum of (1) and (2)

$$\int_{C_2} f(z)\,dz = e^2 - 1 + e^{2+i} - e^2$$

Hence <u>the final result</u> is

$$\int_{C_2} f(z)\,dz = e^{2+i} - 1 \qquad (3)$$

To verify this, since e^z is analytic then $\int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz$ should come out to be zero (By Cauchy theorem). This is because $\oint f(z)\,dz = 0$ around the closed contour, going clockwise. Let us see if this is true:

$$\begin{aligned}\int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz &= \left[e^{2+i} - 1\right] - \left[e^{2+i} - 1\right]\\ &= 0\\ &= \oint f(z)\,dz\end{aligned}$$

Verified. A small note: $\oint_C f(z)\,dz = 0$ does not necessarily mean that $f(z)$ is analytic on and inside C as some non analytic function can give zero, depending on C. But if $f(z)$ happened to be analytic, then $\oint_C f(z)\,dz$ is always zero. But here we now that e^{az} is analytic.

2.3.7 Key solution for HW 3

① $f(z) = z^{1/n} = r^{1/n} e^{i\theta/n}$ n sheets

R_1: $0 < \theta < 2\pi$

R_2: $2\pi < \theta < 4\pi$ $f(r, \theta + 2\pi n) = f(r, \theta)$

$\vdots$ $\vdots$

R_n $2\pi(n-1) < \theta < 2\pi n$

② $w = \tan^{-1} z$ $z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} =$

$= \frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1} \Rightarrow iz(e^{i2w} + 1) = e^{i2w} - 1$

$e^{i2w}(1 - iz) = 1 + iz$

$e^{i2w} = \frac{1 + iz}{1 - iz} = \frac{i - z}{i + z}$

$2iw = \ln\left(\frac{i - z}{i + z}\right)$

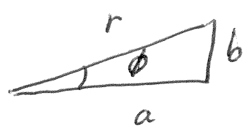
$w = \frac{1}{2i} \ln\left(\frac{i - z}{i + z}\right)$

$$\boxed{w = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)}$$

(3) $$w = \tan^{-1} z = \frac{i}{2}\left[\ln\left(x + i(y+1)\right) - \ln\left(-x + i(1-y)\right)\right]$$

$$\ln(a + ib) = \ln r + i(\phi + 2\pi n)$$

any integer



$$r = \sqrt{a^2 + b^2} \qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\ln\left(x + i(y+1)\right) = \frac{1}{2}\ln\left[x^2 + (1+y)^2\right] + i\left[\tan^{-1}\left(\frac{1+y}{x}\right) + 2\pi k\right]$$

$$\ln\left(-x + i(1-y)\right) = \ln\left(x + i(y-1)\right) + \underbrace{\ln(-1)}_{i\pi(2m+1)}$$

$$= \frac{1}{2}\ln\left[x^2 + (y-1)^2\right] + i\left[\tan^{-1}\left(\frac{y-1}{x}\right) + \pi(2m+1)\right]$$

any integer

$$\boxed{\tan^{-1} z = -\frac{1}{2}\tan^{-1}\left(\frac{y+1}{x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1-y}{x}\right) + \pi\left(n + \frac{1}{2}\right) + \frac{i}{4}\ln\left[\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}\right]}$$

The principle value of $\tan^{-1}$ is usually taken to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(4) $u = \ln r = \frac{1}{2} \ln r^2 = \frac{1}{2} \ln(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{r^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

$$\frac{\partial u}{\partial y} = \frac{y}{r^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4} = 0 \quad \checkmark$$

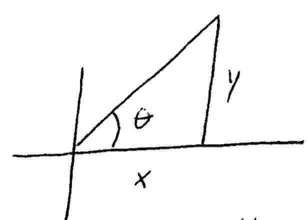
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right) + \phi(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2}$$

$$\shortparallel$$

$$\frac{-y}{x^2+y^2} + \phi'(x) \Rightarrow \phi = c \quad \text{constant}$$

$$\boxed{v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + c}$$



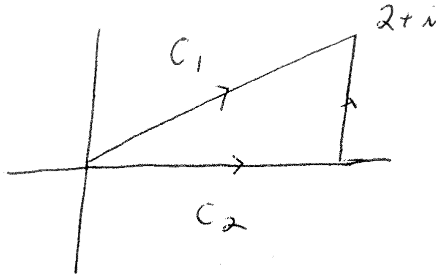
$$\tan\theta = \frac{y}{x}$$

$$\boxed{v = \theta + c}$$

Polar coordinates $\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ $\checkmark$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r} \quad \checkmark$$

5



Along C_1 $\quad x = 2t$, $y = t$ $\quad 0 \le t \le 1$

$$\int_{C_1} e^z dz = \int_{C_1} e^{x+iy}(dx + i dy) = \int_0^1 e^{(2+i)t}(2+i)dt$$

$$= e^{(2+i)t}\Big|_0^1 = e^2 e^i - 1 = e^2(\cos(1) + i\sin(1)) - 1$$

Along C_2 $\quad \int_{C_2} e^z dz = \int_0^2 e^x dx + i\int_0^1 e^2 e^{iy} dy$

$$= (e^2 - 1) + i e^2 \frac{e^i - 1}{i} = e^2 e^i - 1$$ Same answer.

2.4 HW 4

Local contents

2.4.1 HW 4 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 4 due Monday February 25. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (6 pts) Let C denote the square contour with corners at $\pm 2 \pm 2i$ and which is taken in a counterclockwise direction. Use the residue theorem to evaluate the integral $\int_C f(z)\,dz$ for the following functions.

$$\text{(a)} \quad \frac{e^{-z}}{z - i\pi/2}$$

$$\text{(b)} \quad \frac{\cos z}{z(z^2+8)}$$

$$\text{(c)} \quad \frac{z}{2z+1}$$

2. (4 pts) Assume that $f(z)$ is analytic on and interior to a closed contour C and that the point z_0 lies inside C. Show that

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$$

3. (5 pts) Give the Laurent series expansion, both in powers of z and in powers of $z-1$, for the following function.

$$\frac{1}{z^2(1-z)}$$

4. (5 pts) Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^4}$$

5. (5 pts) Evaluate the following integral when $a > |b|$

$$\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{a + b\cos\theta}$$

2.4.2 Problem 1

Let C denote the square contour with corners at $\pm 2, \pm 2i$ and which is taken in counter clockwise direction. Use the residue theorem to evaluate the integral $\int_C f(x)\, dz$ for the following functions

(a) $\frac{e^{-z}}{z-i\frac{\pi}{2}}$, (b) $\frac{\cos z}{z\left(z^2+8\right)}$ (c) $\frac{z}{z+1}$

<u>Solution</u>

2.4.2.1 Part (a)

The function $f(z) = \frac{e^{-z}}{z-i\frac{\pi}{2}}$ has a simple pole at $z = i\frac{\pi}{2} \approx 1.57i$, hence it is inside the contour.

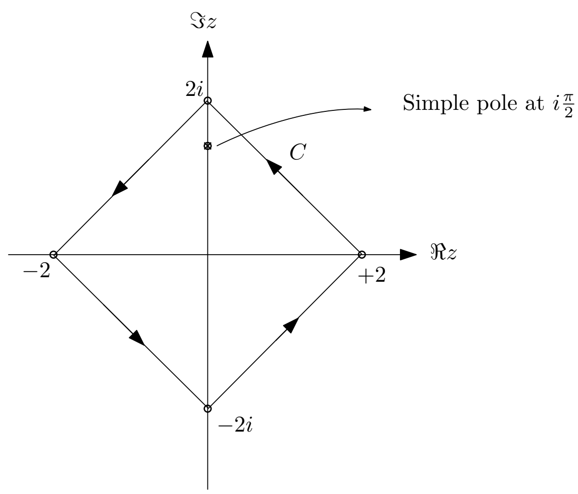


Figure 2.5: Location of pole relative to contour

Hence by residue theorem

$$\oint_C f(z)\, dz = 2\pi i \ \text{Residue}\left(f(z)\right)_{z=z_0}$$

So we just need to find the residue of $f(z)$ at $z = z_0 = i\frac{\pi}{2}$. Since this is a simple plot, then the residue is given by

$$\begin{aligned}
\text{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0}(z-z_0)\,f(z)\\
&= \lim_{z\to i\frac{\pi}{2}}\left(z-i\frac{\pi}{2}\right)\frac{e^{-z}}{z-i\frac{\pi}{2}}\\
&= \lim_{z\to i\frac{\pi}{2}} e^{-z}\\
&= e^{-i\frac{\pi}{2}}\\
&= \cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\\
&= -i
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{e^{-z}}{z-i\frac{\pi}{2}}dz &= 2\pi i\,(-i)\\
&= 2\pi
\end{aligned}$$

2.4.2.2 Part (b)

The function $f(z)=\frac{\cos z}{z\left(z^2+8\right)}$ has one simple pole at $z=0$ which is inside the contour, and a poles at $z=\pm i\sqrt{8}=\pm 2i\sqrt{2}\approx\pm 2.83i$ but these are outside the contour.

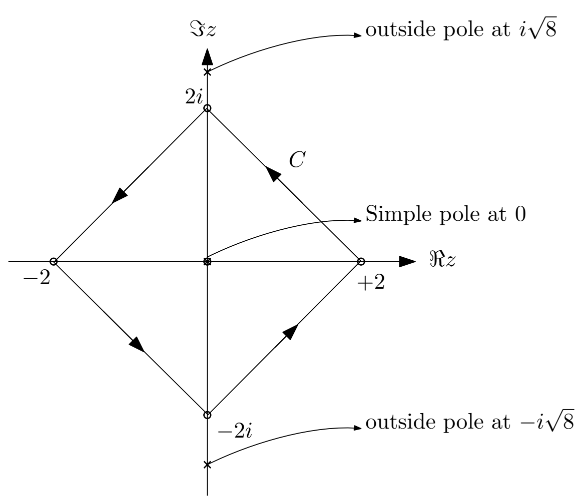


Figure 2.6: Location of pole relative to contour

Therefore by residue theorem, only the pole inside the contour which is at $z=0$ will contribute to the integral. So we just need to find residue at $z = 0$

$$\begin{aligned}
\text{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0}(z-z_0)\,f(z)\\
&= \lim_{z\to 0}(z)\frac{\cos z}{z\left(z^2+8\right)}\\
&= \lim_{z\to 0}\frac{\cos z}{\left(z^2+8\right)}\\
&= \frac{1}{8}
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{\cos z}{z\left(z^2+8\right)}dz &= 2\pi i\left(\frac{1}{8}\right)\\
&= i\frac{\pi}{4}
\end{aligned}$$

2.4.2.3 Part (c)

The function $f(z) = \frac{z}{2(z+1)}$ has one simple pole at $z = -1$ which is inside the contour.

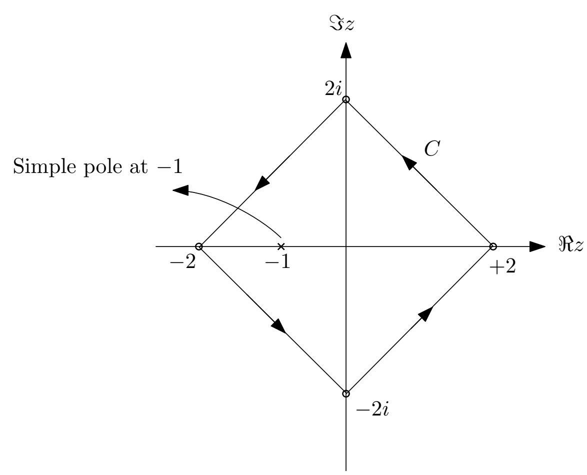


Figure 2.7: Location of pole relative to contour

So we just need to find residue at $z=-1$

$$\begin{aligned}
\operatorname{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0}(z-z_0)f(z)\\
&= \lim_{z\to -1}(z+1)\frac{z}{2(z+1)}\\
&= \lim_{z\to -1}\frac{z}{2}\\
&= \frac{-1}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{z}{2(z+1)}dz &= 2\pi i\left(\frac{-1}{2}\right)\\
&= -i\pi
\end{aligned}$$

2.4.3 Problem 2

Assume that $f(z)$ is analytic on and interior to a closed contour C and that the point z_0 lies inside C. Show that

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}dz$$

Solution

We see that $g(z)=\frac{f'(z)}{z-z_0}$ has a simple pole at $z=z_0$. Therefore

$$\oint_C g(z)\,dz = 2\pi i\,(b_1) \tag{1}$$

Where b_1 is the Residue of $g(z)$ at z_0. By definition the residue of a simple pole is found as follows

$$\begin{aligned}
b_1 &= \lim_{z\to z_0}(z-z_0)g(z)\\
&= \lim_{z\to z_0}(z-z_0)\frac{f'(z)}{z-z_0}\\
&= \lim_{z\to z_0}f'(z)\\
&= f'(z_0)
\end{aligned}$$

Hence (1) becomes

$$\oint_C g(z)\,dz = (2\pi i)f'(z_0)$$

$$\oint_C \frac{f'(z)}{z-z_0}dz = (2\pi i)f'(z_0) \tag{2}$$

But per lecture notes, page 46 on complex analysis, it shows that

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

Substituting the above back into RHS of (2) results in

$$\oint_C \frac{f'(z)}{z-z_0}dz = 2\pi i\left(\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz\right)$$

Therefore

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}dz$$

QED.

2.4.4 Problem 3

Give the Laurent series expansion both in powers of z and in powers of $(z-1)$ for the function $\frac{1}{z^2(1-z)}$

Solution

There is a pole of order 2 at $z=0$ and a pole of order one at $z=1$. Therefore, there is a Laurent series expansion about $z=0$ which is valid in inside a disk or radius 1 centered at $z=0$. Around $z=1$ there is another Laurent series expansion of the function, which is valid inside a disk centered at $z=0$ of radius 1.

Laurent series expansion around $z=0$

$$\begin{aligned}
\frac{1}{z^2(1-z)} &= \frac{1}{z^2}\frac{1}{(1-z)}\\
&= \frac{1}{z^2}\left(1+z+z^2+z^3+\cdots\right) \qquad |z|<1\\
&= \frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\cdots
\end{aligned}$$

We see from the above that the residue at $z=0$ is 1 which is the coefficient of $\frac{1}{z}$ term.

Laurent series expansion around $z=1$

Let $u=z-1$, hence $z=u+1$ and the function $\frac{1}{z^2(1-z)}$ in terms of u becomes

$$\frac{1}{(1+u)^2(-u)} = \frac{-1}{u}\frac{1}{(1+u)^2} \tag{1}$$

But $\frac{1}{(1+u)^2} = (1+u)^{-2}$. Applying Binomial expansion $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+$

$\cdots$ which is valid for $|x| < 1$ then we see that for $n = -2$ we obtain

$$(1+u)^{-2} = 1 + (-2)\,u + \frac{(-2)(-2-1)}{2!}u^2 + \frac{(-2)(-2-1)(-2-2)}{3!}u^3 + \cdots$$

The above is valid for $|u| < 1$ or $|z-1| < 1$ or $0 < z < 2$. Simplifying the above gives

$$\frac{1}{(1-u)^2} = 1 - 2u + 3u^2 - 4u^3 + \cdots$$

Substituting the above back into (1) gives

$$\begin{aligned}\frac{-1}{u}\frac{1}{(1+u)^2} &= \frac{-1}{u}\left(1 - 2u + 3u^2 - 4u^3 + \cdots\right)\\ &= \frac{-1}{u} + 2 - 3u + 4u^2 - \cdots\end{aligned}$$

But since $u = z-1$ then the above becomes

$$\frac{1}{z^2(1-z)} = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - 5(z-1)^3 + \cdots$$

We see from the above that the residue of $f(z)$ is -1 at $z = 1$.

In summary

1. Laurent series around $z = 0$ is $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$ which is valid inside disk centered at $z = 0$ of radius 1
2. Laurent series around $z = 1$ is $\frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - 5(z-1)^3 + \cdots$ which is valid inside disk centered at $z = 1$ of radius 1

Note that there is another Laurent series expansions that can be found, which is for the region $1 < |z| < \infty$, which is outside a disk of radius 1 centered at $z = 0$. But the problem is asking for the above two expansions only.

2.4.5 Problem 4

Evaluate the integral $\int_0^\infty \frac{dx}{1+x^4}dx$

solution

Since the integrand is even, then

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$$

Now we consider the following contour

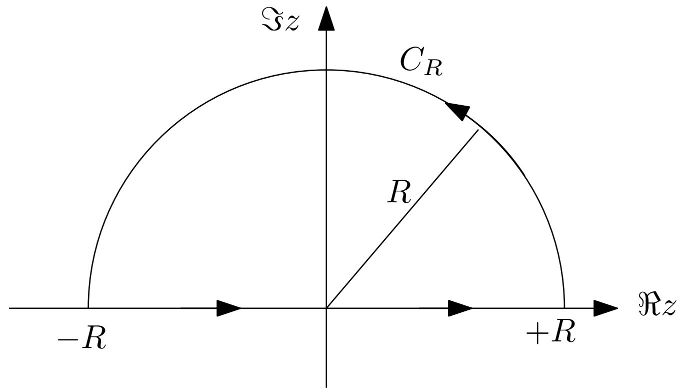


Figure 2.8: contour used for problem 4

Therefore

$$\oint_C f(z)\,dz = \left(\lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty}\int_{0}^{\tilde{R}} f(x)\,dx\right) + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned}\oint_C f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,d\\ &= 2\pi i\sum \text{Residue}\end{aligned}$$

Where $\sum \text{Residue}$ is sum of residues of $\frac{1}{z^4+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx &= 2\pi i\sum \text{Residue} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz\\ \int_{-\infty}^{\infty}\frac{1}{x^4+1}dx &= 2\pi i\sum \text{Residue} - \lim_{R\to\infty}\int_{C_R}\frac{1}{z^4+1}dz \qquad (1)\end{aligned}$$

Now we will show that $\lim_{R\to\infty}\int_{C_R}\frac{1}{z^4+1}dz = 0$. Since

$$\begin{aligned}\left|\int_{C_R}\frac{1}{z^4+1}dz\right| &\leq ML\\ &= \left|f(z)\right|_{\max}(\pi R) \qquad (2)\end{aligned}$$

But

$$f(z) = \frac{1}{\left(z^2-i\right)\left(z^2+i\right)}$$

Hence, and since $z = R\, e^{i\theta}$ then

$$\left|f(z)\right|_{\max} \leq \frac{1}{\left|z^2 - i\right|_{\min}\left|z^2 + i\right|_{\min}}$$

Using the inverse triangle inequality then $\left|z^2 - i\right| \geq |z|^2 + 1$ and $\left|z^2 + i\right| \geq |z|^2 - 1$, and because $|z| = R$ then the above becomes

$$\begin{aligned}\left|f(z)\right|_{\max} &\leq \frac{1}{\left(R^2+1\right)\left(R^2-1\right)}\\ &= \frac{1}{R^4-1}\end{aligned}$$

Therefore (2) becomes

$$\left|\int_{C_R} \frac{1}{z^4+1} dz\right| \leq \frac{\pi R}{R^4-1}$$

Then it is clear that as $R \to \infty$ the above goes to zero since $\lim_{R\to\infty} \frac{\pi R}{R^4-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}}{1-\frac{1}{R^4}} = \frac{0}{1} = 0$. Equation (1) now simplifies to

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \sum \text{Residue} \tag{2A}$$

We just now need to find the residues of $\frac{1}{z^4+1}$ located in upper half plane. The zeros of the denominator $z^4 + 1 = 0$ are at $z = -1^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}}$, then the first zero is at $e^{i\frac{\pi}{4}}$, and the second zero at $e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = e^{i\left(\frac{3}{4}\pi\right)}$ and the third zero at $e^{i\left(\frac{3}{4}\pi+\frac{\pi}{2}\right)} = e^{i\left(\frac{5}{4}\pi\right)}$ and the fourth zero at $e^{i\left(\frac{5}{4}\pi+\frac{\pi}{2}\right)} = e^{i\frac{7}{4}\pi}$. Hence poles are at

$$\begin{aligned} z_1 &= e^{i\frac{\pi}{4}}\\ z_2 &= e^{i\frac{3}{4}\pi}\\ z_3 &= e^{i\frac{5}{4}\pi}\\ z_4 &= e^{i\frac{7}{4}\pi}\end{aligned}$$

Out of these only the first two are in upper half plane. Hence since these are simple poles, we can use the following to find the residues

$$\begin{aligned}\text{Residue}(z_1) &= \lim_{z\to z_1} (z - z_1) f(z)\\ &= \lim_{z\to z_1} (z - z_1) \frac{1}{z^4-1}\end{aligned}$$

Applying L'Hopitals rule, the above becomes

$$\begin{aligned}
\operatorname{Residue}(z_1) &= \lim_{z\to z_1} \frac{\frac{d}{dz}(z-z_1)}{\frac{d}{dz}\left(z^4-1\right)}\\
&= \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{4z^3}\\
&= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}\\
&= \frac{1}{4e^{i\frac{3\pi}{4}}}
\end{aligned}$$

Similarly for the other residue

$$\begin{aligned}
\operatorname{Residue}(z_2) &= \lim_{z\to z_2} (z-z_2)\, f(z)\\
&= \lim_{z\to z_2} (z-z_2)\, \frac{1}{z^4-1}
\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}
\operatorname{Residue}(z_2) &= \lim_{z\to e^{i\frac{3}{4}\pi}} \frac{1}{4z^3}\\
&= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3}\\
&= \frac{1}{4e^{i\frac{9\pi}{4}}}\\
&= \frac{1}{4e^{i\frac{\pi}{4}}}
\end{aligned}$$

Now that we found all the residues, then (2A) becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \left(\frac{1}{4e^{i\frac{3\pi}{4}}} + \frac{1}{4e^{i\frac{\pi}{4}}}\right)\\
&= 2\pi i \left(\frac{\sqrt{2}}{4i}\right)\\
&= \frac{1}{2}\sqrt{2}\pi
\end{aligned}$$

But $\int_0^\infty \frac{1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^4+1}dx$, therefore

$$\begin{aligned}\int_0^\infty \frac{1}{x^4+1}dx &= \frac{1}{2}\left(\frac{1}{2}\sqrt{2}\pi\right)\\ &= \frac{1}{4}\sqrt{2}\pi\\ &= \frac{2}{4\sqrt{2}}\pi\\ &= \frac{1}{2\sqrt{2}}\pi\end{aligned}$$

2.4.6 Problem 5

Evaluate the following integral $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta$ when $a > |b|$

solution

This is converted to complex integration by using $z = re^{i\theta} = e^{i\theta}$ since $r = 1$. Therefore $dz = ie^{i\theta}d\theta$ or

$$dz = izd\theta$$

In addition,

$$\begin{aligned}\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\\ &= \frac{z+z^{-1}}{2}\end{aligned}$$

And

$$\begin{aligned}\sin^2\theta &= \frac{1}{2}-\frac{1}{2}\cos 2\theta\\ &= \frac{1}{2}-\frac{1}{2}\left(\frac{e^{i2\theta}+e^{-i2\theta}}{2}\right)\\ &= \frac{1}{2}-\frac{1}{2}\left(\frac{z^2+z^{-2}}{2}\right)\\ &= \frac{1}{2}-\frac{1}{4}\left(z^2+z^{-2}\right)\end{aligned}$$

Using all of the above back in the original integral gives

$$\begin{aligned}I &= \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta\\ &= \oint_C \frac{\frac{1}{2}-\frac{1}{4}\left(z^2+z^{-2}\right)}{a+b\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz}\end{aligned}$$

Where the contour C is around the unit circle in counter clockwise direction. Therefore

$$
\begin{aligned}
I &= \frac{1}{i}\oint_C \frac{\frac{1}{2}-\frac{1}{4}\left(z^2+\frac{1}{z^2}\right)}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{z}\\
&= \frac{1}{i}\oint_C \frac{\frac{1}{2}-\frac{1}{4}\left(\frac{z^4+1}{z^2}\right)}{a+\frac{b}{2}\left(\frac{z^2+1}{z}\right)}\frac{dz}{z}\\
&= \frac{1}{i}\oint_C \frac{\frac{\frac{z^2}{2}-\frac{1}{4}\left(z^4+1\right)}{z^2}}{\frac{az+\frac{b}{2}\left(z^2+1\right)}{z}}\frac{dz}{z}\\
&= \frac{1}{i}\oint_C \frac{\frac{z^2}{2}-\frac{1}{4}\left(z^4+1\right)}{az+\frac{b}{2}\left(z^2+1\right)}\frac{dz}{z^2}\\
&= \frac{1}{i}\oint_C \frac{\frac{2z^2}{4}-\frac{1}{4}\left(z^4+1\right)}{\frac{2az}{2}+\frac{b}{2}\left(z^2+1\right)}\frac{dz}{z^2}\\
&= \frac{1}{i}\oint_C \frac{2z^2-z^4-1}{4az+2bz^2+2b}\frac{dz}{z^2}\\
&= \frac{1}{i}\oint_C \frac{1}{z^2}\frac{2z^2-z^4-1}{2bz^2+4az+2b}dz\\
&= \frac{1}{i}\oint_C \frac{1}{z^2}\frac{\frac{1}{b}z^2-\frac{1}{2b}z^4-\frac{1}{2b}}{z^2+\frac{2a}{b}z+1}dz\\
&= \frac{1}{2bi}\oint_C \frac{1}{z^2}\frac{2z^2-z^4-1}{\left(z^2+\frac{2a}{b}z+1\right)}dz
\end{aligned}
$$

Now we can use the residue theorem. There is a pole at $z=0$ of order 2 and two poles which are the roots of $z^2+\frac{2a}{b}z+1=0$. Hence

$$
I = 2\pi i \sum \text{Residue}
$$

First we find the roots of $z^2+\frac{2a}{b}z+1=0$ to see the location of the poles and if there are inside the unit circle or not. These are

$$
\begin{aligned}
-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} &= -\frac{\frac{2a}{b}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{2a}{b}\right)^2 - 4} \\
&= -\frac{a}{b} \pm \frac{1}{2}\sqrt{4\frac{a^2}{b^2} - 4} \\
&= -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}
$$

Since $a > |b|$ then $\frac{a^2}{b^2} > 1$ and the value under the square root is real. Hence both roots are real. Roots are

$$
\begin{aligned}
z_1 &= -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} \\
z_2 &= -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}
$$

Now we need to decide the location of these poles. Let $\frac{a}{b} = x$. Where $x > 1$ since $a > |b|$. Then the roots can be written as

$$
\begin{aligned}
z_1 &= -x + \sqrt{x^2 - 1} \\
z_2 &= -x - \sqrt{x^2 - 1}
\end{aligned}
$$

Now $\sqrt{x^2-1}$ is always smaller than x but $\left(\sqrt{x^2-1} - x\right)$ can not be larger than 1 in magnitude. Hence z_1 will always be inside the unit disk. On the other hand, $\left(\sqrt{x^2-1} + x\right)$ will always be larger than 1 in magnitude (the sign is not important, we just wanted to know which pole is smaller or larger than 1 only. Therefore we conclude that z_1 is inside the unit disk and z_2 is outside.

Therefore, we need to find residue at $z = 0$ and $z = z_1$ and not at $z = z_2$. The function $f(z)$ is from above is

$$
\begin{aligned}
f(z) &= \frac{1}{z^2}\frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} \\
&= \frac{1}{z^2}\frac{2z^2 - z^4 - 1}{(z - z_1)(z - z_2)}
\end{aligned}
$$

Residue of $f(z)$ at $z = 0$

Since this pole is of order $n = 2$, then

$$
\begin{aligned}
\text{Residue} &= \lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{(z-z_0)^n f(z)}{(n-1)!}\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left((z-z_0)^2\frac{1}{z^2}\frac{2z^2-z^4-1}{\left(z^2+\frac{2a}{b}z+1\right)}\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left(z^2\frac{1}{z^2}\frac{2z^2-z^4-1}{\left(z^2+\frac{2a}{b}z+1\right)}\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left(\frac{2z^2-z^4-1}{z^2+\frac{2a}{b}z+1}\right)\\
&= \lim_{z\to 0}\frac{\left(4z-4z^3\right)\left(z^2+\frac{2a}{b}z+1\right)-\left(2z^2-z^4-1\right)\left(2z+\frac{2a}{b}\right)}{\left(z^2+\frac{2a}{b}z+1\right)^2}\\
&= -(-1)\left(\frac{2a}{b}\right)\\
&= \frac{2a}{b}
\end{aligned}
$$

<u>Residue at $z_1 = -\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}$</u>

Since this pole is of order 1, then the reside is

$$
\begin{aligned}
\text{Residue} &= \lim_{z\to z_1}\left((z-z_1)f(z)\right)\\
&= \lim_{z\to z_1}\left((z-z_1)\frac{1}{z^2}\frac{2z^2-z^4-1}{(z-z_1)(z-z_2)}\right)\\
&= \lim_{z\to z_1}\left(\frac{1}{z^2}\frac{2z^2-z^4-1}{z-z_2}\right)\\
&= \frac{1}{z_1^2}\frac{2z_1^2-z_1^4-1}{z_1-z_2}\\
&= \frac{-1}{z_1^2}\frac{z_1^4-2z_1^2+1}{z_1-z_2}\\
&= \frac{-1}{z_1^2}\frac{\left(z_1^2-1\right)^2}{z_1-z_2}
\end{aligned}
$$

Let $\frac{a}{b} = x$, hence $\sqrt{\frac{a^2}{b^2}-1} = \sqrt{x^2-1}$. Therefore we can write $z_1 = -x+\sqrt{x^2-1}$ and $z_2 =$

$-x-\sqrt{x^2-1}$ and now the above becomes

$$
\begin{aligned}
\text{Residue} &= \frac{-1}{\left(-x+\sqrt{x^2-1}\right)^2}\frac{\left(\left(-x+\sqrt{x^2-1}\right)^2-1\right)^2}{\left(-x+\sqrt{x^2-1}\right)-\left(-x-\sqrt{x^2-1}\right)}\\
&= \frac{-1}{2}\frac{\left(\left(-x+\sqrt{x^2-1}\right)^2-1\right)^2}{\left(-x+\sqrt{x^2-1}\right)^2\sqrt{x^2-1}}
\end{aligned}
$$

But $\left(-x+\sqrt{x^2-1}\right)^2 = x^2+\left(x^2-1\right)-2x\sqrt{x^2-1} = 2x^2-2x\sqrt{x^2-1}-1$ and the above becomes

$$
\begin{aligned}
\text{Residue} &= \frac{-1}{2}\frac{\left(2x^2-2x\sqrt{x^2-1}-1-1\right)^2}{\left(2x^2-2x\sqrt{x^2-1}-1\right)\sqrt{x^2-1}}\\
&= \frac{-1}{2}\frac{\left(2x^2-2x\sqrt{x^2-1}-2\right)^2}{\left(2x^2-2x\sqrt{x^2-1}-1\right)\sqrt{x^2-1}}\\
&= \frac{-1}{2}\frac{4\left(x^2-x\sqrt{x^2-1}-1\right)^2}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}\\
&= -2\frac{\left(\left(x^2-1\right)-x\sqrt{x^2-1}\right)^2}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}
\end{aligned}
$$

Expanding gives

$$\begin{aligned}
\text{Residue} &= -2\frac{\left(x^2-1\right)^2+\left(x\sqrt{x^2-1}\right)^2-2\left(x^2-1\right)x\sqrt{x^2-1}}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}\\
&= -2\frac{\left(x^2-1\right)^2+x^2\left(x^2-1\right)-2\left(x^2-1\right)x\sqrt{x^2-1}}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}\\
&= -2\frac{\left(x^2-1\right)\left(x^2-1+x^2-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}\\
&= -2\frac{\left(x^2-1\right)\left(2x^2-1-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}
\end{aligned}$$

Dividing numerator and denominator by $\left(x^2-1\right)$

$$\begin{aligned}
\text{Residue} &= -2\frac{\sqrt{x^2-1}\left(2x^2-1-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)}\\
&= -2\sqrt{x^2-1}
\end{aligned}$$

Since $x=\frac{a}{b}$ then the above becomes

$$\text{Residue} = -2\sqrt{\frac{a^2}{b^2}-1}$$

We found all residues. The sum is

$$\sum \text{Residue} = \frac{2a}{b}-2\sqrt{\frac{a^2}{b^2}-1}$$

From the above we see now that

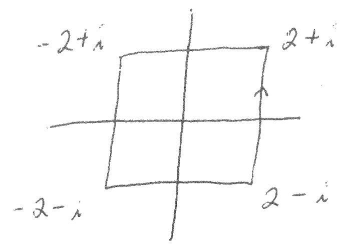
$$
\begin{aligned}
I &= \frac{1}{2bi}\oint_C \frac{1}{z^2}\frac{2z^2-z^4-1}{\left(z^2+\frac{2a}{b}z+1\right)}dz \\
&= \frac{1}{2bi}\left(2\pi i\sum \text{Residue}\right) \\
&= \frac{1}{2bi}\left(2\pi i\left(\frac{2a}{b}-2\sqrt{\frac{a^2}{b^2}-1}\right)\right) \\
&= \frac{\pi}{b}\left(\frac{2a}{b}-2\sqrt{\frac{a^2}{b^2}-1}\right) \\
&= \frac{\pi}{b}\left(\frac{2a}{b}-\frac{2}{b}\sqrt{a^2-b^2}\right) \\
&= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)
\end{aligned}
$$

Hence the final result is

$$
\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)
$$

2.4.7 Key solution for HW 4

(1)



C

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0}$$

(a) $f = e^{-z}$

$$\oint_C \frac{f(z)\,dz}{z - \frac{i\pi}{2}} = 2\pi i\, e^{-\frac{i\pi}{2}} = \boxed{2\pi}$$

z_0 is within C

(b)

$$\oint_C \frac{\cos z\,dz}{z\,(z + i\,2\sqrt{2})(z - i\,2\sqrt{2})} = \frac{2\pi i \cos(0)}{8} = \boxed{\frac{\pi}{4} i}$$

inside C

outside C

(c)

$$\oint_C \frac{\frac{1}{2} z\,dz}{z + \frac{1}{2}} = 2\pi i \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{2} i}$$

inside C

(2) Cauchy integral formula for first derivative is

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^2}$$

Then apply the Cauchy integral formula to $f'(z)$.

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f'(z)\,dz}{z - z_0}$$

Equate them:

$$\boxed{\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}}$$

(3) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$

Expansion about $z = 0$:

$$\frac{1}{z^2}\frac{1}{1-z} = \boxed{\sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots}$$

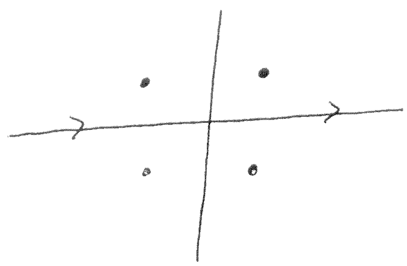
Expansion about $z = 1$: $\frac{1}{z^2} = \frac{1}{[1+(z-1)]^2} =$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} = 1 - 2(z-1) + 3(z-1)^2 - \cdots$$

$$\frac{1}{z^2}\frac{1}{1-z} = -\frac{1}{z-1}\sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} =$$

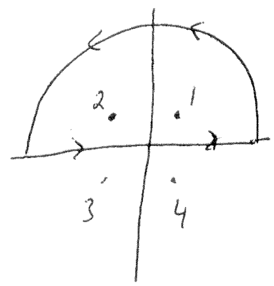
$$= \boxed{\sum_{n=1}^{\infty} (-1)^n n (z-1)^{n-2} = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 + \cdots}$$

(4) $I = \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{1+z^4}$ Poles $z = e^{\pm i\frac{\pi}{4}},\ e^{\pm i \frac{3\pi}{4}}$



Close the contour in either the upper half or lower half planes with a semi-circle of radius $R \to \infty$.

Choose upper half, use residue theorem.



$$e^{\pm i\frac{\pi}{4}} = \frac{1 \pm i}{\sqrt{2}} \qquad e^{\pm i\frac{3\pi}{4}} = \frac{-1 \pm i}{\sqrt{2}}$$

$$1 + z^4 = (z^2+i)(z^2-i) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$$

Relevant residues of $\frac{1}{1+z^4}$ are

$$K_1 = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{1}{8}\frac{1}{i-1}\cdot 2\sqrt{2}$$

$$K_2 = \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{1}{8}\frac{1}{i+1}\cdot 2\sqrt{2}$$

$$I = \frac{1}{2}\cdot 2\pi i\,(K_1+K_2) = \frac{\pi i}{8}\underbrace{\left(\frac{1}{i+1} + \frac{1}{i-1}\right)}_{\frac{2i}{i^2-1} = -i}\cdot 2\sqrt{2}$$

$$\boxed{I = \frac{\pi}{2\sqrt{2}}}$$

5 $$I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}$$

$$z = e^{i\theta} \quad dz = iz\,d\theta \quad \sin\theta = \frac{z - \frac{1}{z}}{2i} \quad \cos\theta = \frac{z + \frac{1}{z}}{2}$$

$$I = \int_C \frac{\left(z - \frac{1}{z}\right)^2}{(2i)^2} \frac{dz}{iz} \frac{1}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{i}{2b} \int_C \frac{\left(z^2 - 2 + \frac{1}{z^2}\right)}{\left(z^2 + 2\frac{a}{b}z + 1\right)} dz$$

C ↑ unit circle

There are two singularities inside the unit circle.

They are $z = 0$ and one of the roots $z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$.

For definiteness assume $b \geq 0$. Then $+$ sign.

Residues are $-\frac{ia}{b^2}$ and $\frac{i}{4b}\sqrt{\frac{a^2}{b^2} - 1}$.

Then $I = \frac{2\pi}{b}\left[\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right]$ or

$$\boxed{I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{a + \sqrt{a^2 - b^2}}}$$

2.5 HW 5

Local contents

2.5.1 HW 5 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 5 due Monday March 4. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (6 pts) Evaluate the following integral for $t > 0$ and for $t < 0$ when $\omega_0 > 0$ and $\epsilon \to 0^+$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t}\, d\omega}{(\omega - i\epsilon)^2 - \omega_0^2}\,.$$

2. (9 pts) Evaluate the following integrals

$$\int_0^\infty \frac{\ln x\, dx}{1+x^2} \quad \text{and} \quad \int_0^\infty \frac{(\ln x)^2\, dx}{1+x^2}\,.$$

In order to find the second one you need to consider the integral

$$\int_0^\infty \frac{(\ln x)^3\, dx}{1+x^2}\,.$$

2.5.2 Problem 1

Evaluate the following integral for $t > 0$ and for $t < 0$ when $\omega_0 > 0$ and $\epsilon \to 0^+$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega$$

<u>Solution</u>

<u>Case $t > 0$</u>

We select the upper half for contour C since when $t > 0$ the integral on upper half will vanish as will be shown below.

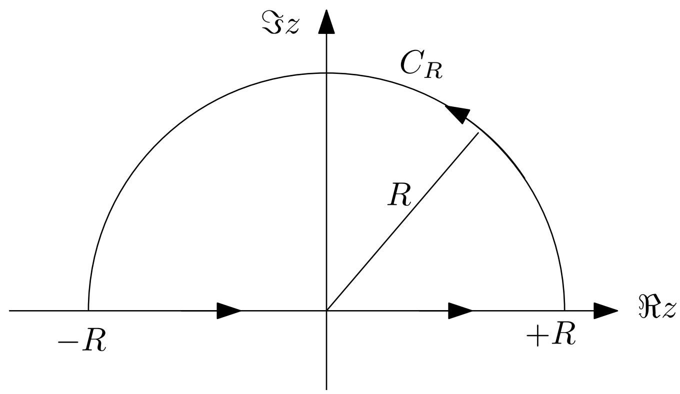


Figure 2.9: Contour used for $t > 0$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega &= \oint_C \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\
&= \lim_{R\to\infty} (P.V.) \int_{-R}^{R} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz + \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\
&= 2\pi i \sum \text{Residue}
\end{aligned}$$

Therefore, if we can show that $\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz = 0$, then the above implies that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 2\pi i \sum \text{Residue} \tag{1}$$

Now we need to find the residues inside the contour shown. There is a pole when $(\omega - i\epsilon)^2 = \omega_0^2$ or $\omega - i\epsilon = \pm\omega_0$ or $\omega = i\epsilon \pm \omega_0$. Hence there are two simple poles, they are

$$\begin{aligned}
z_1 &= i\epsilon + \omega_0 \\
z_2 &= i\epsilon - \omega_0
\end{aligned}$$

They are both in upper half, inside the contour (since $\omega_0 > 0$ and ϵ is positive).

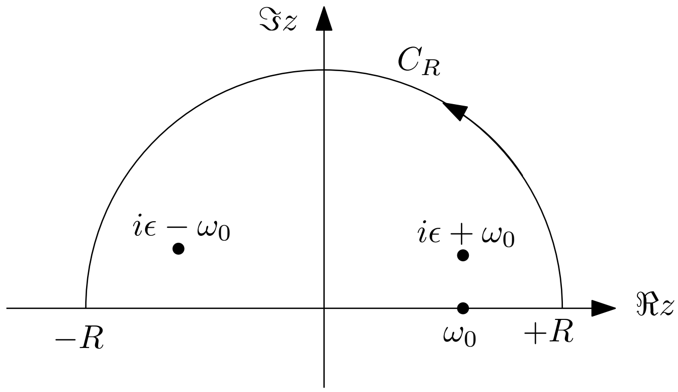


Figure 2.10: Locations of poles

Now we find the residues

$$
\begin{aligned}
\text{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)\frac{e^{izt}}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_1}\frac{e^{izt}}{(z-z_2)}\\
&= \frac{e^{it(i\epsilon+\omega_0)}}{(i\epsilon+\omega_0)-(i\epsilon-\omega_0)}\\
&= \frac{e^{-t\epsilon}e^{it\omega_0}}{2\omega_0} \qquad (2)
\end{aligned}
$$

And

$$
\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2)\frac{e^{izt}}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_2}\frac{e^{izt}}{(z-z_1)}\\
&= \frac{e^{it(i\epsilon-\omega_0)}}{(i\epsilon-\omega_0)-(i\epsilon+\omega_0)}\\
&= \frac{e^{-t\epsilon}e^{-it\omega_0}}{-2\omega_0} \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega &= 2\pi i \left(\frac{e^{-t\epsilon} e^{it\omega_0}}{2\omega_0} + \frac{e^{-t\epsilon} e^{-it\omega_0}}{-2\omega_0} \right) \\
&= \frac{2\pi i}{2\omega_0} e^{-t\epsilon} \left(e^{it\omega_0} - e^{-it\omega_0} \right) \\
&= \frac{2\pi}{\omega_0} e^{-t\epsilon} \left(\frac{e^{it\omega_0} - e^{-it\omega_0}}{-2i} \right) \\
&= -\frac{2\pi}{\omega_0} e^{-t\epsilon} \left(\frac{e^{it\omega_0} - e^{-it\omega_0}}{2i} \right) \\
&= -\frac{2\pi}{\omega_0} e^{-t\epsilon} \sin(t\omega_0)
\end{aligned}
$$

Now, to finish the solution, we must show that $\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} dz = 0$. But

$$
\begin{aligned}
\int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz &\leq \left| \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \right|_{\max} \\
&\leq \int_{CR} \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} dz \\
&= \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \int_{CR} dz \\
&= \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \int_0^{\pi} R d\theta \\
&= R\pi \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \qquad (4)
\end{aligned}
$$

But

$$\begin{aligned}
\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} &\leq \frac{\left|e^{izt}\right|_{\max}}{\left|(z-z_1)(z-z_2)\right|_{\min}} \\
&\leq \frac{\left|e^{izt}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{it(x+iy)}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{itx-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{itx}\right|_{\max}\left|e^{-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&\leq \frac{\left|e^{-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}}
\end{aligned}$$

Now, since $y>0$ (we are in the upper half) and also since $t>0$, then $\left|e^{-ty}\right|_{\max}=1$, which occurs when $y=0$. Hence the above becomes

$$\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} \leq \frac{1}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}}$$

By inverse triangle inequality $\left|(z-z_1)\right|_{\min}\geq |z|^2+|z_1|^2=R^2+\left|\epsilon^2+\omega_0^2\right|^2$ and $\left|(z-z_2)\right|_{\min}\geq |z|^2+|z_2|^2=R^2+\left|\epsilon^2+\omega_0^2\right|^2$. The above becomes

$$\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} \leq \frac{1}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}$$

Substituting the above in (4) gives

$$\begin{aligned}
\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz &\leq \lim_{R\to\infty} R\pi\left(\frac{1}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}\right) \\
&= \pi\lim_{R\to\infty}\frac{R}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}
\end{aligned}$$

But $2\left|\epsilon^2+\omega_0^2\right|^2$ is a finite value, say β so the above is

$$\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz \leq \pi\lim_{R\to\infty}\frac{R}{2R^2+\beta}$$

And it is clear now that the above limit goes to zero. In other words, $\lim_{R\to\infty}\frac{R}{2R^2+\beta}=$

$\lim_{R\to\infty} \frac{\frac{1}{R}}{2+\frac{\beta}{R^2}} = \frac{0}{2} = 0.$

Hence The final solution is

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = -\frac{2\pi}{\omega_0} e^{-t\epsilon} \sin(t\omega_0)$$

Case $t < 0$

Here, we must use the lower half for the contour in order for the half circle contour integral to vanish.

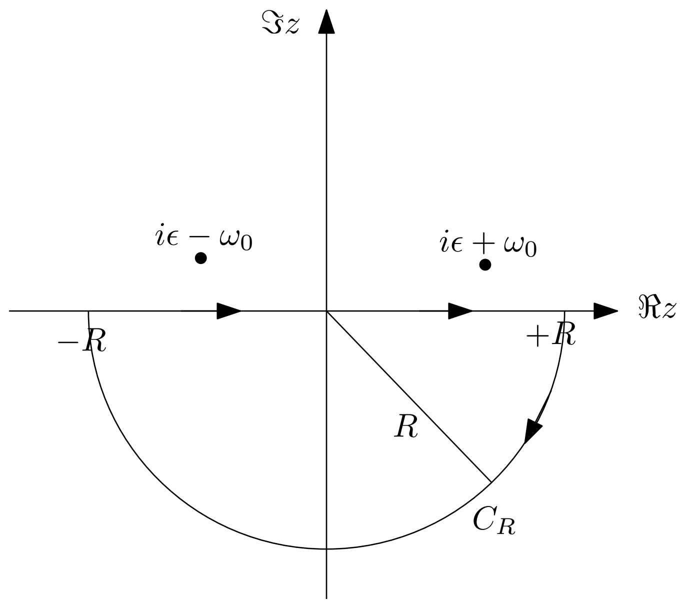


Figure 2.11: Contour for $t < 0$

In this case the sum of residues is zero (since both poles are in the upper half), then we see right away that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 0 \qquad t < 0$$

But we must show that $\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz = 0$ here as well for the above result to be valid. Similar to what was done earlier:

$$\begin{aligned}
\int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz &\leq \left| \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz \right|_{\max} \\
&\leq \int_{CR} \left| \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} \right|_{\max} dz \qquad (2.1) \\
&= R\pi \left| \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} \right|_{\max} \qquad (4)
\end{aligned}$$

But

$$\begin{aligned}
\left| \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} \right|_{\max} &\leq \frac{\left|e^{izt}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&\leq \frac{\left|e^{it(x+iy)}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&= \frac{\left|e^{itx-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&= \frac{\left|e^{itx}\right|_{\max} \left|e^{-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&\leq \frac{\left|e^{-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}}
\end{aligned}$$

Since $y < 0$ (we are now in lower half) and also since $t < 0$, then $\left|e^{-ty}\right|_{\max} = 1$, which occurs when $y = 0$. Hence

$$\left| \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} \right|_{\max} \leq \frac{1}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}}$$

But by inverse triangle inequality $|(z-z_1)|_{\min} \geq |z|^2 + |z_1|^2 = R^2 + \left|\epsilon^2 + \omega_0^2\right|^2$ and $|(z-z_2)|_{\min} \geq |z|^2 + |z_2|^2 = R^2 + \left|\epsilon^2 + \omega_0^2\right|^2$. Hence the above becomes

$$\left| \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} \right|_{\max} \leq \frac{1}{2R^2 + 2\left|\epsilon^2+\omega_0^2\right|^2}$$

The rest follows what was done in first part. Therefore

$$\begin{aligned}
\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz &\leq \lim_{R\to\infty} R\pi \left(\frac{1}{2R^2 + 2\left|\epsilon^2+\omega_0^2\right|^2} \right) \\
&= \pi \lim_{R\to\infty} \frac{R}{2R^2 + 2\left|\epsilon^2+\omega_0^2\right|^2}
\end{aligned}$$

But $2\left|\epsilon^2+\omega_0^2\right|^2$ is finite number, say β so the above is

$$\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz\leq\pi\lim_{R\to\infty}\frac{R}{2R^2+\beta}$$

And it is clear now that the above limit goes to zero.

The final solution is

$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-i\epsilon)^2-\omega_0^2}d\omega=0\qquad t<0$$

2.5.3 Problem 2

Evaluate the following integrals $\int_0^\infty\frac{\ln x}{1+x^2}dx$ and $\int_0^\infty\frac{\ln^2 x}{1+x^2}dx$. In order to find the second one you need to consider the integral $\int_0^\infty\frac{\ln^3 x}{1+x^2}dx$

Solution

2.5.3.1 Part (a)

There are two ways to find $\int_0^\infty\frac{\ln x}{1+x^2}dx$. One uses a substitution method and requires no complex contour integration and the second method uses $\int_0^\infty\frac{\ln^2 x}{1+x^2}dx$ with complex integration to find $\int_0^\infty\frac{\ln x}{1+x^2}dx$.

Method one

Let $x=\frac{1}{y}$. Hence $dx=-\frac{1}{y^2}dy$. When $x=0\to y=\infty$ and when $x=\infty\to y=0$. Hence the integral $\int_0^\infty \frac{\ln x}{1+x^2} dx$ becomes

$$
\begin{aligned}
\int_0^\infty \frac{\ln x}{1+x^2} dx &= \int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{1+\frac{1}{y^2}} \left(-\frac{1}{y^2} dy\right) \\
&= -\int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{\frac{y^2+1}{y^2}} \left(\frac{1}{y^2} dy\right) \\
&= -\int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{y^2+1} dy \\
&= \int_\infty^0 \frac{\ln\left(y\right)}{y^2+1} dy \\
&= -\int_0^\infty \frac{\ln\left(y\right)}{y^2+1} dy
\end{aligned}
$$

Since on the RHS y is arbitrary integration variable, we can rename it back to x. Hence the above becomes

$$
\begin{aligned}
\int_0^\infty \frac{\ln x}{1+x^2} dx &= -\int_0^\infty \frac{\ln(x)}{x^2+1} dy \\
2\int_0^\infty \frac{\ln x}{1+x^2} dx &= 0
\end{aligned}
$$

Therefore

$$
\boxed{\int_0^\infty \frac{\ln x}{1+x^2} dx = 0}
$$

Method two

In this method will use complex integration on $\int_0^\infty \frac{\ln^2 z}{1+z^2} dz$ to show that $\int_0^\infty \frac{\ln z}{1+z^2} dz = 0$. The following contour will be used.

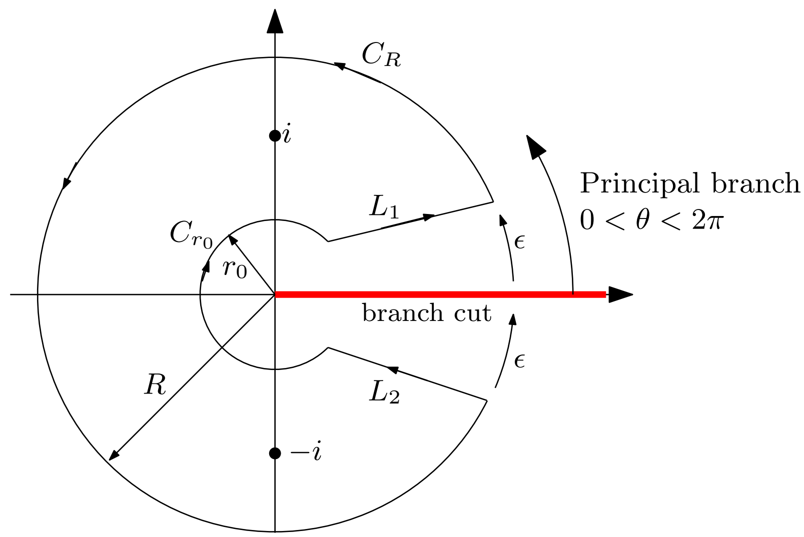


Figure 2.12: Contour for problem 2, showing location of poles at $\pm i$

$$
\begin{aligned}
\oint \frac{\ln^2 z}{1+z^2} dz &= \oint f(z)\, dz \\
&= \int_{L_2} f(z)\, dz + \int_{C_{r_0}} f(z)\, dz + \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz \\
&= 2\pi i \sum \text{Residue}
\end{aligned}
$$

Hence

$$
\int_{L_2} f(z)\, dz + \int_{C_{r_0}} f(z)\, dz + \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz = 2\pi i \sum \text{Residue} \tag{1}
$$

There are two poles in $\frac{\ln^2 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$
\begin{aligned}
\operatorname{Residue}(i) &= \lim_{z\to i}(z-i)\frac{\ln^2 z}{(z-i)(z+i)} \\
&= \lim_{z\to i}\frac{\ln^2 z}{(z+i)} \\
&= \frac{\ln^2 i}{2i} \\
&= \frac{\left(\ln(1) + i\frac{\pi}{2}\right)^2}{2i} \\
&= \frac{\left(i\frac{\pi}{2}\right)^2}{2i} \\
&= \frac{-\frac{\pi^2}{4}}{2i} \\
&= \frac{-\pi^2}{8i}
\end{aligned} \tag{2}
$$

And

$$
\begin{aligned}
\operatorname{Residue}(-i) &= \lim_{z\to -i}(z+i)\frac{\ln^2 z}{(z-i)(z+i)} \\
&= \lim_{z\to -i}\frac{\ln^2 z}{(z-i)} \\
&= \frac{\ln^2(-i)}{-2i}
\end{aligned}
$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$
\begin{aligned}
\operatorname{Residue}(-i) &= \frac{\left(\ln(1) + i\frac{3}{2}\pi\right)^2}{-2i} \\
&= \frac{-\frac{9}{4}\pi^2}{-2i} \\
&= \frac{9\pi^2}{8i}
\end{aligned} \tag{3}
$$

Adding (2+3) and substituting in (1) gives

$$
\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{-\pi^2}{8i} + \frac{9\pi^2}{8i}\right)
$$

$$
\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi^3
$$

We will show at the end that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$. Given this, the above simplifies to only two integrals to evaluate

$$\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz = 2\pi^3 \tag{3A}$$

We will now work on finding $\int_{L_1} f(z)\,dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$\begin{aligned}
\int_{L_1} \frac{\ln^2 z}{1+z^2}dz &= \int_0^\infty \frac{\ln^2\left(re^{i\epsilon}\right)}{1+\left(re^{i\epsilon}\right)^2}dre^{i\epsilon}\\
&= e^{i\epsilon}\int_0^\infty \frac{(\ln r + i\epsilon)^2}{1+r^2e^{2i\epsilon}}dr\\
&= e^{i\epsilon}\int_0^\infty \frac{\ln^2 r + i^2\epsilon^2 + 2i\epsilon\ln r}{1+r^2e^{2i\epsilon}}dr
\end{aligned}$$

Now taking the limit as $\epsilon\to 0$ the above becomes

$$\int_{L_1} \frac{\ln^2 z}{1+z^2}dz = \int_0^\infty \frac{\ln^2 r}{1+r^2}dr \tag{4}$$

We will now work on finding $\int_{L_2} f(z)\,dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^2 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^2\left(re^{i(2\pi-\epsilon)}\right)}{1+\left(re^{i(2\pi-\epsilon)}\right)^2}dre^{i(2\pi-\epsilon)}\\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{(\ln(r) + i(2\pi-\epsilon))^2}{1+r^2e^{2i(2\pi-\epsilon)}}dr\\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{\ln^2(r) - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr\\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{\ln^2(r) - \left(4\pi^2+\epsilon^2-4\pi\epsilon\right) + 2i(2\pi-\epsilon)\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr
\end{aligned}$$

Taking the limit as $\epsilon\to 0$ the above becomes

$$\int_{L_2} \frac{\ln^2 z}{1+z^2}dz = e^{i2\pi}\int_\infty^0 \frac{\ln^2(r) - 4\pi^2 + 4\pi i\ln r}{1+r^2e^{i4\pi}}dr$$

But $e^{i2\pi}=1$ and $e^{i4\pi}=1$ then the above becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^2 z}{1+z^2}dz &= \int_{\infty}^{0}\frac{\ln^2(r)-4\pi^2+4\pi i\ln r}{1+r^2}dr\\ &= \int_{\infty}^{0}\frac{\ln^2 r}{1+r^2}dr-\int_{\infty}^{0}\frac{4\pi^2}{1+r^2}dr+4\pi i\int_{\infty}^{0}\frac{\ln r}{1+r^2}dr\\ &= -\int_{0}^{\infty}\frac{\ln^2 r}{1+r^2}dr+\int_{0}^{\infty}\frac{4\pi^2}{1+r^2}dr-4\pi i\int_{0}^{\infty}\frac{\ln r}{1+r^2}dr \qquad (5)\end{aligned}$$

Using (4,5) in (3A) gives

$$\begin{aligned}-\int_{0}^{\infty}\frac{\ln^2 r}{1+r^2}dr+\int_{0}^{\infty}\frac{4\pi^2}{1+r^2}dr-4\pi i\int_{0}^{\infty}\frac{\ln r}{1+r^2}dr+\int_{0}^{\infty}\frac{\ln^2 r}{1+r^2}dr &= 2\pi^3\\ 4\pi^2\int_{0}^{\infty}\frac{1}{1+r^2}dr-4\pi i\int_{0}^{\infty}\frac{\ln r}{1+r^2}dr &= 2\pi^3\end{aligned}$$

But $\int_0^\infty \frac{1}{1+r^2}dr=\arctan(r)_0^\infty=\arctan(\infty)-\arctan(0)=\frac{\pi}{2}$, hence the above becomes

$$\begin{aligned}4\pi^2\left(\frac{\pi}{2}\right)-4\pi i\int_{0}^{\infty}\frac{\ln r}{1+r^2}dr &= 2\pi^3\\ -4\pi i\int_{0}^{\infty}\frac{\ln r}{1+r^2}dr &= 0\end{aligned}$$

Which implies

$$\boxed{\int_0^\infty \frac{\ln r}{1+r^2}dr=0}$$

Which is the same result obtained using method one above.

2.5.3.1.1 Appendix Here we will show that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz=0$ and $\lim_{R\to\infty}\int_{C_R} f(z)\,dz=0$.

For $\underline{\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz}$, let $z=r_0e^{i\theta}$. Hence $dz=r_0ie^{i\theta}d\theta$ and the integral becomes

$$\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta=\lim_{r_0\to 0} i\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}r_0d\theta$$

As $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta &= \lim_{r_0\to 0} i\int_{2\pi}^{0}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}r_0d\theta\\
&\leq \lim_{r_0\to 0}\left| i\int_{2\pi}^{0}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}r_0d\theta\right|_{\max}\\
&\leq \lim_{r_0\to 0}\int_{2\pi}^{0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max} r_0d\theta\\
&\leq \lim_{r_0\to 0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max}\int_{2\pi}^{0} r_0d\theta\\
&= 2\pi r_0\lim_{r_0\to 0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max}\\
&\leq 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln^2\left(r_0e^{i\theta}\right)\right|_{\max}}{\left|1+r_0^2e^{i\theta}\right|_{\min}}\\
&= 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln r_0+i\theta\right|_{\max}^2}{\left|1+r_0^2e^{i\theta}\right|_{\min}}\\
&= 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln^2 r_0+(i\theta)^2+2i\theta\ln r_0\right|_{\max}}{1-r_0^2}\\
&= 2\pi\lim_{r_0\to 0}\frac{r_0\ln^2 r_0-4\pi^2r_0+4\pi r_0\ln r_0}{1-r_0^2}\\
&= 2\pi\lim_{r_0\to 0}\left(\frac{r_0\ln^2 r_0}{1-r_0^2}-4\pi^2\frac{r_0}{1-r_0^2}+4\pi\frac{r_0\ln r_0}{1-r_0^2}\right)
\end{aligned}$$

But $\lim_{r_0\to 0}\frac{r_0\ln^2 r_0}{1-r_0^2}=0$ and $\lim_{r_0\to 0}\frac{r_0}{1-r_0^2}=0$ and $\lim_{r_0\to 0}\frac{r_0\ln r_0}{1-r_0^2}=0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta &= \lim_{r_0\to 0}\int_{C_{r_0}}\frac{\ln^2 z}{1+z^2}dz\\
&= 0
\end{aligned}$$

Now we will do the same $\underline{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}$, let $z=Re^{i\theta}$. Hence $dz=Rie^{i\theta}d\theta$ and the integral becomes

$$\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty} i\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta$$

As $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty} i\int_0^{2\pi}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta \\
&\leq \lim_{R\to\infty}\left| i\int_0^{2\pi}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\right|_{\max} \\
&\leq \lim_{R\to\infty}\int_0^{2\pi}\left|\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}R\right|_{\max} d\theta \\
&\leq \lim_{R\to\infty}\left|\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}\right|\int_0^{2\pi} Rd\theta \\
&= 2\pi \lim_{R\to\infty} R\frac{\left|\ln^2\left(\mathrm{Re}^{i\theta}\right)\right|_{\max}}{\left|1+R^2e^{2i\theta}\right|_{\min}} \\
&= 2\pi \lim_{R\to\infty} R\frac{\left|\ln R+i\theta\right|^2_{\max}}{1-R^2} \\
&= 2\pi \lim_{R\to\infty} R\frac{\left|\ln^2 R-\theta^2+2i\theta\ln R\right|_{\max}}{1-R^2} \\
&\leq 2\pi \lim_{R\to\infty}\frac{R\ln^2 R-4\pi^2R+4\pi R\ln R}{1-R^2} \\
&= 2\pi \lim_{R\to\infty}\left(\frac{R\ln^2 R}{1-R^2}-4\pi^2\frac{R}{1-R^2}+4\pi\frac{R\ln R}{1-R^2}\right)
\end{aligned}$$

But $\lim_{R\to\infty}\frac{R\ln^2 R}{1-R^2}=0$ and $\lim_{R\to\infty}\frac{R}{1-R^2}=0$ and $\lim_{R\to\infty}\frac{R\ln R}{1-R^2}=0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty}\int_{C_R}\frac{\ln^2 z}{1+z^2}dz=0 \\
&= 0
\end{aligned}$$

2.5.3.2 Part (b)

We will now find $\int_0^\infty \frac{\ln^3 z}{1+z^2}dz$ in order to determine $\int_0^\infty \frac{\ln^2 z}{1+z^2}dz$. We will use the same contour integration as part (a) above.

$$\int_{L_2} f(z)\,dz+\int_{C_{r_0}} f(z)\,dz+\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz = 2\pi i\sum \text{Residue} \tag{1}$$

There are two poles in $\frac{\ln^3 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$\begin{aligned}
\operatorname{Residue}(i) &= \lim_{z\to i}(z-i)\frac{\ln^3 z}{(z-i)(z+i)} \\
&= \lim_{z\to i}\frac{\ln^3 z}{(z+i)} \\
&= \frac{\ln^3 i}{2i} \\
&= \frac{\left(\ln(1) + i\frac{\pi}{2}\right)^3}{2i} \\
&= \frac{\left(i\frac{\pi}{2}\right)^3}{2i} \\
&= \frac{-i\frac{\pi^3}{8}}{2i} \\
&= \frac{-\pi^3}{16} \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\operatorname{Residue}(-i) &= \lim_{z\to -i}(z+i)\frac{\ln^3 z}{(z-i)(z+i)} \\
&= \lim_{z\to -i}\frac{\ln^3 z}{(z-i)} \\
&= \frac{\ln^3(-i)}{-2i}
\end{aligned}$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$\begin{aligned}
\operatorname{Residue}(-i) &= \frac{\left(\ln(1) + i\frac{3}{2}\pi\right)^3}{-2i} \\
&= \frac{\left(i\frac{3}{2}\pi\right)^3}{-2i} \\
&= \frac{-i\frac{27}{8}\pi^3}{-2i} \\
&= \frac{27\pi^3}{16} \qquad (3)
\end{aligned}$$

Adding (2+3) and substituting in (1) gives

$$\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{-\pi^3}{16} + \frac{27\pi^3}{16}\right)$$
$$\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = \frac{13}{4}\pi^4 i$$

We will show below that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$, which simplifies the above to

$$\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz = \frac{13}{4}\pi^4 i \tag{3A}$$

We will now work on finding $\int_{L_1} f(z)\,dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$\begin{aligned}
\int_{L_1} \frac{\ln^3 z}{1+z^2}dz &= \int_0^\infty \frac{\ln^3\left(re^{i\epsilon}\right)}{1+\left(re^{i\epsilon}\right)^2}dre^{i\epsilon}\\
&= e^{i\epsilon}\int_0^\infty \frac{(\ln r + i\epsilon)^3}{1+r^2e^{2i\epsilon}}dr\\
&= e^{i\epsilon}\int_0^\infty \frac{\left(\ln^2 r + i^2\epsilon^2 + 2i\epsilon\ln r\right)(\ln r + i\epsilon)}{1+r^2e^{2i\epsilon}}dr\\
&= e^{i\epsilon}\int_0^\infty \frac{\left(\ln^3 r + i^2\epsilon^2\ln r + 2i\epsilon\ln^2 r\right) + \left(i\epsilon\ln^2 r + i^3\epsilon^3 + 2i^2\epsilon^2\ln r\right)}{1+r^2e^{2i\epsilon}}dr
\end{aligned}$$

Now taking the limit as $\epsilon \to 0$ the above becomes

$$\int_{L_1} \frac{\ln^3 z}{1+z^2}dz = \int_0^\infty \frac{\ln^3 r}{1+r^2}dr \tag{4}$$

We will now work on finding $\int_{L_2} f(z)\,dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^3\left(re^{i(2\pi-\epsilon)}\right)}{1+\left(re^{i(2\pi-\epsilon)}\right)^2}dre^{i(2\pi-\epsilon)}\\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{(\ln(r) + i(2\pi-\epsilon))^3}{1+r^2e^{2i(2\pi-\epsilon)}}dr
\end{aligned}$$

But $\lim_{\epsilon\to 0} e^{i(2\pi-\epsilon)} = e^{2\pi i} = 1$ and the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\left(\ln^2 r - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln r\right)(\ln(r) + i(2\pi-\epsilon))}{1+r^2e^{2i(2\pi-\epsilon)}}dr\\
&= \int_\infty^0 \frac{\ln^3 r - \ln r(2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln^2 r + i(2\pi-\epsilon)\ln^2 r - i(2\pi-\epsilon)^3 + 2i^2(2\pi-\epsilon)^2\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_\infty^0 \frac{\ln^3 r - 4\pi^2 \ln r + 4\pi i \ln^2 r + 2\pi i \ln^2 r - i\left(2\pi-\epsilon\right)^2 \left(2\pi-\epsilon\right) + 2i^2\left(4\pi^2+\epsilon^2-4\pi\epsilon\right)\ln r}{1+r^2 e^{4\pi i}} dr \\
&= \int_\infty^0 \frac{\ln^3 r - 4\pi^2 \ln r + 4\pi i \ln^2 r + 2\pi i \ln^2 r - i\left(4\pi^2+\epsilon^2-4\pi\epsilon\right)\left(2\pi-\epsilon\right) - 8\pi^2 \ln r}{1+r^2} dr \\
&= \int_\infty^0 \frac{\ln^3 r - 4\pi^2 \ln r + 6\pi i \ln^2 r - i\left(8\pi^3 + 2\pi\epsilon^2 - 8\pi^2\epsilon\right) - \left(4\pi^2\epsilon + \epsilon^3 - 4\pi\epsilon^2\right) - 8\pi^2 \ln r}{1+r^2} dr
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_\infty^0 \frac{\ln^3(r) - 4\pi^2 \ln r + 6\pi i \ln^2 r - 8i\pi^3 - 8\pi^2 \ln r}{1+r^2} dr \\
&= \int_\infty^0 \frac{\ln^3(r) - 12\pi^2 \ln r + 6\pi i \ln^2 r - 8i\pi^3}{1+r^2} dr
\end{aligned}$$

Hence the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_\infty^0 \frac{\ln^3(r)}{1+r^2} dr - 12\pi^2 \int_\infty^0 \frac{\ln r}{1+r^2} dr + 6\pi i \int_\infty^0 \frac{\ln^2 r}{1+r^2} dr - 8i\pi^3 \int_\infty^0 \frac{1}{1+r^2} dr \\
&= -\int_0^\infty \frac{\ln^3 r}{1+r^2} dr + 12\pi^2 \int_0^\infty \frac{\ln r}{1+r^2} dr - 6\pi i \int_0^\infty \frac{\ln^2 r}{1+r^2} dr + 8i\pi^3 \int_0^\infty \frac{1}{1+r^2} dr
\end{aligned}$$

But $\int_0^\infty \frac{\ln r}{1+r^2} dr = 0$ from part (a) and $\int_0^\infty \frac{1}{1+r^2} dr = \frac{\pi}{2}$, hence the above becomes

$$\int_{L_2} \frac{\ln^3 z}{1+z^2} dz = -\int_0^\infty \frac{\ln^3 r}{1+r^2} dr - 6\pi i \int_0^\infty \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4 \tag{5}$$

Using (4,5) in (3A) gives

$$\begin{aligned}
\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz &= \frac{13}{4}\pi^4 i \\
\left(-\int_0^\infty \frac{\ln^3 r}{1+r^2} dr - 6\pi i \int_0^\infty \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4\right) + \left(\int_0^\infty \frac{\ln^3 r}{1+r^2} dr\right) &= \frac{13}{4}\pi^4 i \\
-6\pi i \int_0^\infty \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4 &= \frac{13}{4}\pi^4 i \\
\int_0^\infty \frac{\ln^2 r}{1+r^2} dr &= \frac{\frac{13}{4}\pi^4 i - 4i\pi^4}{-6\pi i} \\
\int_0^\infty \frac{\ln^2 r}{1+r^2} dr &= \frac{13\pi^4 i - 16i\pi^4}{-24\pi i} \\
&= \frac{-3\pi^4 i}{-24\pi i} \\
&= \frac{\pi^3}{8}
\end{aligned}$$

Which implies

$$\boxed{\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = \frac{\pi^3}{8}}$$

2.5.3.2.1 Appendix Here we will show that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$.

For $\underline{\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz}$, let $z = r_0 e^{i\theta}$. Hence $dz = r_0 i e^{i\theta} d\theta$ and the integral becomes

$$\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0\to 0} i\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta$$

As $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0\to 0} i\int_{2\pi}^{0} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta \\
&\leq \lim_{r_0\to 0}\left| i\int_{2\pi}^{0} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta\right|_{\max} \\
&\leq \lim_{r_0\to 0}\int_{2\pi}^{0} \left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max} r_0 d\theta \\
&\leq \lim_{r_0\to 0}\left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max} \int_{2\pi}^{0} r_0 d\theta \\
&= 2\pi r_0 \lim_{r_0\to 0}\left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max} \\
&\leq 2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln^3\left(r_0 e^{i\theta}\right)\right|_{\max}}{\left|1+r_0^2 e^{i\theta}\right|_{\min}} \\
&\leq 2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln r_0 + i\theta\right|_{\max}^2 \left|\ln r_0 + i\theta\right|_{\max}}{\left|1+r_0^2 e^{i\theta}\right|_{\min}}
\end{aligned}$$

But from part (a) we showed that $2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln r_0+i\theta\right|_{\max}^2}{\left|1+r_0^2 e^{i\theta}\right|_{\min}} = 0$, hence it follows that the RHS above goes to zero. Therefore

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0\to 0}\int_{C_{r_0}} \frac{\ln^3 z}{1+z^2} dz \\
&= 0
\end{aligned}$$

Now we will do the same $\lim_{R\to\infty}\int_{C_R} f(z)\,dz$, let $z = Re^{i\theta}$. Hence $dz = Rie^{i\theta}d\theta$ and the integral becomes

$$\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty} i\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta$$

As $\epsilon\to 0$ the above becomes

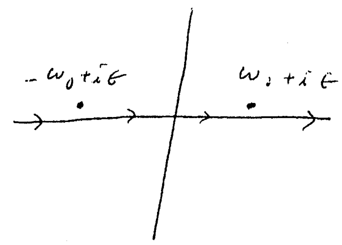
$$\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty} i\int_{0}^{2\pi}\frac{\ln^3\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\\
&\leq \lim_{R\to\infty}\left|i\int_{0}^{2\pi}\frac{\ln^3\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\right|_{\max}\\
&\leq \lim_{R\to\infty}\int_{0}^{2\pi}\left|\frac{\ln^3\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}R\right|_{\max}d\theta\\
&\leq \lim_{R\to\infty}\left|\frac{\ln^3\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}\right|\int_{0}^{2\pi}Rd\theta\\
&= 2\pi\lim_{R\to\infty} R\frac{\left|\ln^3\left(\mathrm{Re}^{i\theta}\right)\right|_{\max}}{\left|1+R^2e^{2i\theta}\right|_{\min}}\\
&\leq 2\pi\lim_{R\to\infty} R\frac{\left|\ln R+i\theta\right|^2_{\max}\left|\ln\left(\mathrm{Re}^{i\theta}\right)\right|_{\max}}{1-R^2}
\end{aligned}$$

But from part (a) we showed that $2\pi\lim_{R\to\infty}R\frac{|\ln R+i\theta|^2_{\max}}{1-R^2} = 0$, hence it follows that the RHS above goes to zero. Therefore

$$\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty}\int_{C_R}\frac{\ln^3 z}{1+z^2}dz = 0\\
&= 0
\end{aligned}$$

2.5.4 Key solution for HW 5

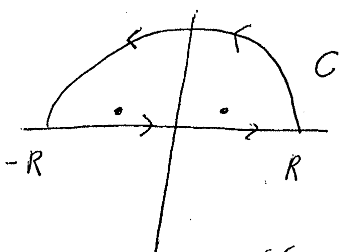
(1) $$I = \int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{(\omega - i\epsilon)^2 - \omega_0^2}$$



Poles $\omega = \pm\omega_0 + i\epsilon$

When $t > 0$ we can add a semi-circle in the upper half-plane.

That contribution is zero as $R \to \infty$.



$$I = \int_C \frac{e^{i\omega t}\, dt}{(\omega - \omega_0 - i\epsilon)(\omega + \omega_0 - i\epsilon)} = 2\pi i \left[\frac{e^{i(\omega_0 + i\epsilon)t}}{2\omega_0} + \frac{e^{i(-\omega_0 + i\epsilon)t}}{-2\omega_0} \right]$$

$$= 2\pi i \cdot \frac{2i \sin(\omega_0 t)}{2\omega_0} = -\frac{2\pi}{\omega_0} \sin(\omega_0 t)$$

When $t < 0$ we can add a semi-circle in the lower half-plane. No poles are enclosed so then $I = 0$.

$$\boxed{I = -\frac{2\pi}{\omega_0} \sin(\omega_0 t)\, \theta(t)}$$

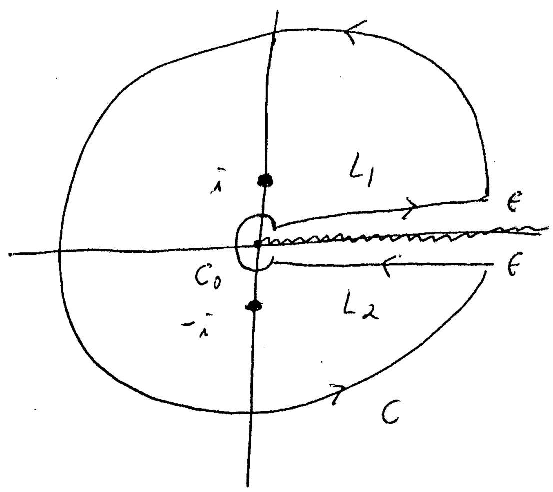
(2) The integral $\int_0^\infty \frac{\ln x \, dx}{1+x^2}$ is elementary.

$$\int_0^\infty \frac{\ln x \, dx}{1+x^2} = \int_\infty^0 \frac{\ln y^{-1}}{1+\frac{1}{y^2}}\left(\frac{-dy}{y^2}\right) = -\int_0^\infty \frac{\ln y \, dy}{1+y^2}$$

$$x = \frac{1}{y} \quad dx = \frac{-dy}{y^2}$$

Hence $\boxed{\int_0^\infty \frac{\ln x \, dx}{1+x^2} = 0}$

Next $\int_0^\infty \frac{(\ln x)^2 dx}{1+x^2} = \int_0^\infty \frac{(\ln z)^2 dz}{(z+i)(z-i)} = \int_0^\infty f(z)\,dz$



Just as in lecture.

$$\int_C f\,dz + \int_{C_0} f\,dz + \int_{L_1} f\,dz + \int_{L_2} f\,dz = 2\pi i\left[\frac{\left(\ln e^{\frac{i\pi}{2}}\right)^2}{2i} + \frac{\left(\ln e^{\frac{i3\pi}{2}}\right)^2}{-2i}\right]$$

$$= \pi\left[\left(i\frac{\pi}{2}\right)^2 - \left(i\frac{3\pi}{2}\right)^2\right] = 2\pi^3$$

$$\int_C f\,dz = 0 \quad \text{when } R \to \infty$$

$$\int_{C_0} f\,dz = 0 \quad \text{when } r_0 \to 0$$

$$\int_{L_1} f\,dz + \int_{L_2} f\,dz = \int_0^\infty \frac{\left(\ln re^{i\epsilon}\right)^2}{\left(re^{i\epsilon}\right)^2+1}\,dr + \int_\infty^0 \frac{\left(\ln re^{i(2\pi-\epsilon)}\right)^2}{\left(re^{i(2\pi-\epsilon)}\right)^2+1}\,dr$$

$$= \int_0^\infty \frac{(\ln r + i\epsilon)^2\,dr}{r^2+1} - \int_0^\infty \frac{(\ln r + i(2\pi-\epsilon))^2\,dr}{r^2+1}$$

$$= -2\pi i \int_0^\infty \frac{\ln r\,dr}{r^2+1} + 4\pi^2 \underbrace{\int_0^\infty \frac{dr}{r^2+1}}_{\frac{\pi}{2}}$$

This gives once again $\displaystyle\int_0^\infty \frac{\ln r\,dr}{r^2+1} = 0$

which is nothing new. Note the cancelation of the $(\ln r)^2$ terms.

Repeat the calculation for $\int_0^\infty \frac{(\ln z)^3 dz}{z^2+1} = \int_0^\infty g\,dz$

$$\int_C g\,dz + \int_{C_0} g\,dz + \int_{L_1} g\,dz + \int_{L_2} g\,dz = 2\pi i\left[\frac{\left(i\frac{\pi}{2}\right)^3}{2i} + \frac{\left(i\frac{3\pi}{2}\right)^3}{-2i}\right] =$$

$$= \frac{13}{4}\pi^4 i$$

$$\int_{L_1} g\,dz + \int_{L_2} g\,dz = \int_0^\infty \frac{dx}{x^2+1}\left[(\ln x)^3 - (\ln x + 2\pi i)^3\right] =$$

$$= -2\pi i\int_0^\infty \frac{3(\ln x)^2 dx}{x^2+1} - (2\pi i)^3 \underbrace{\int_0^\infty \frac{dx}{x^2+1}}_{\frac{\pi}{2}}$$

Putting it all together gives

$$\boxed{\int_0^\infty \frac{(\ln x)^2 dx}{x^2+1} = \frac{\pi^3}{8}}$$

2.6 HW 6

Local contents

2.6.1 HW 6 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 6 due Wednesday March 13. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (5 pts) The rate of nuclear reactions in a star is given by the formula

$$R = N \int_0^\infty dE\, E\, \mathrm{e}^{-\beta E}\, \mathrm{e}^{-\alpha E^{-1/2}}$$

where E is energy, $\beta = 1/k_B T$, α is a constant, and N is a normalization. Evaluate this integral using the saddle point approximation when $(\beta\alpha^2)^{1/3} \gg 1$. This is the low temperature limit appropriate for conditions in the star.

2. (5 pts) Assume that $g(x_0) = 0$ for $a < x_0 < b$ and that $g^{-1}(x)$ exists in that range of x. Show that

$$\int_a^b f(x)\delta(g(x))dx = \frac{f(x_0)}{|g'(x_0)|}$$

3. (5 pts) Find the Fourier series that represents the periodic function

$$f(x) = 1 + \frac{2x}{L} \quad \text{when} \quad -\frac{L}{2} \leq x \leq 0$$

$$f(x) = 1 - \frac{2x}{L} \quad \text{when} \quad 0 \leq x \leq \frac{L}{2}$$

4. (10 pts) Consider the Fourier series for the function $f(\theta) = 1$ when $0 < \theta < \pi$ and $f(\theta) = -1$ when $\pi < \theta < 2\pi$. Just to the right of $\theta = 0$ the first n terms in the series exhibit a local maximum of $1+\delta_n$. For large n, $\delta_n \approx 0.2$. Using computer software, make plots of the series for 4 representative values of n of your choosing for $0 < \theta < \pi/2$ for illustration. What is the limit of the overshoot δ_n as $n \to \infty$ to 4 significant figures? Include printouts of the programs you wrote to make the plots and to find the limit. This is called the Gibbs phenomenon.

2.6.2 Problem 1

1. (5 pts) The rate of nuclear reactions in a star is given by the formula

$$R = N\int_0^\infty dE\, E\, \mathrm{e}^{-\beta E}\, \mathrm{e}^{-\alpha E^{-1/2}}$$

where E is energy, $\beta = 1/k_B T$, α is a constant, and N is a normalization. Evaluate this integral using the saddle point approximation when $(\beta\alpha^2)^{1/3} \gg 1$. This is the low temperature limit appropriate for conditions in the star.

Figure 2.13: Problem statement

<u>Solution</u>

The first step in saddle point method is to write the integral as $\int_0^\infty e^{f(E)}dE$. Hence

$$\begin{aligned} R &= N\int_0^\infty e^{\left(-\beta E-\alpha E^{\frac{-1}{2}}+\ln E\right)}dE \\ &= N\int_0^\infty e^{f(E)}dE \qquad \text{(A)} \end{aligned}$$

Where

$$f(E) = -\beta E - \alpha E^{\frac{-1}{2}} + \ln E \qquad (1)$$

The next step is to determine where $f(E)$ is maximum. Therefore we need to solve $f'(E) = 0$ in order to determine E_0, where $f(E_0)$ is maximum.

$$\begin{aligned} f'(E) &= -\beta + \frac{1}{2}\alpha E^{\frac{-3}{2}} + \frac{1}{E} \\ &= 0 \end{aligned}$$

We need to make this dimensionless. Multiplying both sides of the above by α^2 gives

$$-\alpha^2\beta + \frac{1}{2}\alpha^3 E^{\frac{-3}{2}} + \frac{\alpha^2}{E} = 0$$

Let $E = x\alpha^2$, then the above becomes

$$\begin{aligned} -\alpha^2\beta + \frac{1}{2}\alpha^3\left(x\alpha^2\right)^{\frac{-3}{2}} + \frac{\alpha^2}{\left(x\alpha^2\right)} &= 0 \\ -\alpha^2\beta + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}} + \frac{1}{x} &= 0 \qquad (2) \end{aligned}$$

<u>Case 1</u> Ignoring the term $\frac{1}{x^{\frac{3}{2}}}$ in (2) results in

$$
\begin{aligned}
-\alpha^2\beta + \frac{1}{x} &= 0 \\
\frac{1}{x} &= \alpha^2\beta \\
x &= \frac{1}{\alpha^2\beta}
\end{aligned}
$$

Using this value for x we check if this is larger than or smaller than the term we ignored which is $\frac{1}{x^{\frac{3}{2}}}$.

$$
\left[\frac{1}{x^{\frac{3}{2}}}\right]_{x=\frac{1}{\alpha^2\beta}} = \frac{1}{\left(\frac{1}{\alpha^2\beta}\right)^{\frac{3}{2}}} = \frac{1}{\left(\frac{1}{\alpha\beta^2}\right)^{3}} = \left(\beta^2\alpha\right)^3
$$

Since $\left(\alpha^2\beta\right)^{\frac{1}{3}} \gg 1$, then $\alpha^2\beta \gg 1$ and hence $x = \frac{1}{\alpha^2\beta}$ is much smaller than $\left(\beta^2\alpha\right)^3$. So our choice of ignoring $\frac{1}{x^{\frac{3}{2}}}$ was wrong. Hence we need to ignore the term $\frac{1}{x}$ from (2)

<u>Case 2</u> Ignoring the term $\frac{1}{x}$ results in

$$
\begin{aligned}
-\alpha^2\beta + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}} &= 0 \\
\frac{-2x^{\frac{3}{2}}\alpha^2\beta + 1}{2x^{\frac{3}{2}}} &= 0 \\
-2x^{\frac{3}{2}}\alpha^2\beta + 1 &= 0 \\
x^{\frac{3}{2}} &= \frac{-1}{-2\alpha^2\beta}
\end{aligned}
$$

Solving gives

$$
x = \left(\frac{1}{2\alpha^2\beta}\right)^{\frac{2}{3}}
$$

But $E = x\alpha^2$, and from the above we the energy E_0 which makes $f(E)$ maximum as

$$\begin{aligned} E_0 &= \alpha^2 \left(\frac{1}{2\alpha^2\beta}\right)^{\frac{2}{3}} \\ &= \frac{\alpha^{2-\frac{4}{3}}}{2^{\frac{2}{3}}\beta^{\frac{2}{3}}} \\ &= \frac{\alpha^{\frac{2}{3}}}{2^{\frac{2}{3}}\beta^{\frac{2}{3}}} \end{aligned}$$

Hence

$$\boxed{E_0 = \left(\frac{\alpha}{2\beta}\right)^{\frac{2}{3}}}$$

Now that we found which value of E makes $f(E)$ maximum, we can expand $f(E)$ in Taylor series around E_0

$$f(E) = f(E_0) + f'(E_0)(E - E_0) + \frac{f''(E_0)}{2!}(E - E_0)^2 + H.O.T$$

But $f'(E_0) = 0$ then the above becomes, after ignoring H.O.T.

$$f(E) = f(E_0) + \frac{f''(E_0)}{2!}(E - E_0)^2 \tag{3}$$

Since $f'(E) = -\beta + \frac{1}{2}\alpha E^{\frac{-3}{2}} + \frac{1}{E}$ then

$$f''(E_0) = -\frac{3}{4}\alpha E_0^{\frac{-5}{2}} - E_0^{-2}$$

Since $E_0^{\frac{-5}{2}} \gg E_0^{-2}$ the above becomes

$$\begin{aligned} f''(E_0) &= -\frac{3}{4}\alpha E_0^{\frac{-5}{2}} \\ &\simeq -\frac{3}{2}\frac{\beta^2}{E_0} \end{aligned} \tag{4}$$

Equation (A) now becomes

$$\begin{aligned} R &= N\int_0^\infty e^{f(E)} dE \\ &= N\int_0^\infty e^{f(E_0)+\frac{f''(E_0)}{2!}(E-E_0)^2} dE \\ &= Ne^{f(E_0)}\int_0^\infty e^{\frac{f''(E_0)}{2!}(E-E_0)^2} dE \end{aligned}$$

We would like to write the above as $\int_0^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. Therefore, assuming $u = E - E_0$, hence $\frac{du}{dE} = 1$. When $E = 0$ then $u = -E_0$ and when $E = \infty$ then $u = \infty$. Hence the above becomes

$$
\begin{aligned}
R &= Ne^{f(E_0)} \int_{-E_0}^{\infty} e^{\frac{f''(E_0)}{2!}u^2} du \\
&= Ne^{f(E_0)} \int_{-E_0}^{\infty} e^{-\frac{3}{4}\frac{\beta^2}{E_0}u^2} du
\end{aligned}
$$

Since E_0 is positive, then contribution from lower limit $u = -E_0$ to the value of the integral is Negligible. We can then let lower limit go to $-\infty$ without affecting the overall result of the integral. The above becomes

$$
R = Ne^{f(E_0)} \int_{-\infty}^{\infty} e^{-\frac{3}{4}\frac{\beta^2}{E_0}u^2} du
$$

This is now in the form of Gaussian $\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. Hence we can write the above, using $a = \frac{3}{4}\frac{\beta^2}{E_0}$

$$
\begin{aligned}
R &= Ne^{f(E_0)} \sqrt{\frac{\pi}{\frac{3}{4}\frac{\beta^2}{E_0}}} \\
&= Ne^{f(E_0)} \sqrt{\frac{4\pi E_0}{3\beta^2}}
\end{aligned}
$$

But $f(E_0)$ from (1) is $f(E_0) = -\beta E_0 - \alpha E_0^{\frac{-1}{2}} + \ln E_0$, hence the above becomes

$$
\begin{aligned}
R &= NE_0 e^{-\beta E_0 - \alpha E_0^{\frac{-1}{2}}} \sqrt{\frac{4\pi E_0}{3\beta^2}} \\
&= NE_0 e^{-\beta E_0 - \alpha E_0^{\frac{-1}{2}}} \sqrt{\frac{4\pi}{3\alpha E_0^{\frac{-5}{2}}}}
\end{aligned}
$$

But $E_0 = \left(\frac{\alpha}{2\beta}\right)^{2/3}$, therefore the above becomes, after some more simplifications

$$
R = N\left(\frac{\alpha}{2\beta}\right)^{2/3} \exp\left(-\beta\left(\frac{\alpha}{2\beta}\right)^{2/3} - \alpha\left(\frac{\alpha}{2\beta}\right)^{-2/6}\right) \sqrt{\frac{4\pi}{3\alpha\left(\frac{\alpha}{2\beta}\right)^{-10/6}}}
$$

Simplifies to

$$
R = \sqrt{\frac{\pi}{3}} N \left(k_\beta T\right)^{\frac{3}{2}} \alpha e^{-\left(\frac{\alpha^2}{4}k_\beta T\right)^{\frac{1}{3}}}
$$

This was a hard problem. See key solution.

2.6.3 Problem 2

2. (5 pts) Assume that $g(x_0) = 0$ for $a < x_0 < b$ and that $g^{-1}(x)$ exists in that range of x. Show that

$$\int_a^b f(x)\delta(g(x))dx = \frac{f(x_0)}{|g'(x_0)|}$$

Figure 2.14: Problem statement

Solution

Let $u = g(x)$, hence

$$\frac{du}{dx} = g'(x) \tag{1}$$

But

$$\begin{aligned} x &= g^{-1}\left(g(x)\right) \\ &= g^{-1}(u) \end{aligned}$$

Replacing x in (1) by the above results (so everything is in terms of u) gives

$$\frac{du}{dx} = g'\left(g^{-1}(u)\right)$$

Now we take care of the limits of integration. When $x = a$ then $u = g(a)$ and when $x = b$ then $u = g(b)$. Now the integral I becomes in terms of u the following

$$\begin{aligned} I &= \int_{g(a)}^{g(b)} f\left(g^{-1}(u)\right)\delta(u)\frac{du}{g'\left(g^{-1}(u)\right)} \\ &= \int_{g(a)}^{g(b)} \delta(u)\left[\frac{f\left(g^{-1}(u)\right)}{g'\left(g^{-1}(u)\right)}\right]du \end{aligned} \tag{2}$$

Since we do not know the sign of $g'(x_0)$, as it can be positive or negative, so we take its absolute value in the above, so that the limits of integration do not switch. Hence (2) becomes

$$I = \int_{g(a)}^{g(b)} \delta(u)\left[\frac{f\left(g^{-1}(u)\right)}{\left|g'\left(g^{-1}(u)\right)\right|}\right]du \tag{3}$$

We are given that there is one point x_0 between $g(a)$, and $g(b)$ where $g(x_0) = 0$ which is the same as saying $u = 0$ at that point. Hence by applying the standard property of Dirac delta function, which says that $\int_a^b \delta(0)\,\phi(z)\,dz = \phi(0)$ to equation (3) gives

$$I = \frac{f\left(g^{-1}(0)\right)}{\left|g'\left(g^{-1}(0)\right)\right|}$$

But $g^{-1}(0) = x_0$, therefore the above becomes

$$\int_a^b f(x)\,\delta\left(g(x)\right)dx = \frac{f(x_0)}{\left|g'(x_0)\right|}$$

Which is the result required to show.

2.6.4 Problem 3

3. (5 pts) Find the Fourier series that represents the periodic function

$$f(x) = 1 + \frac{2x}{L} \quad \text{when} \quad -\frac{L}{2} \le x \le 0$$

$$f(x) = 1 - \frac{2x}{L} \quad \text{when} \quad 0 \le x \le \frac{L}{2}$$

Figure 2.15: Problem statement

<u>Solution</u>

A plot of the function to approximate is (using $L = 1$) for illustration

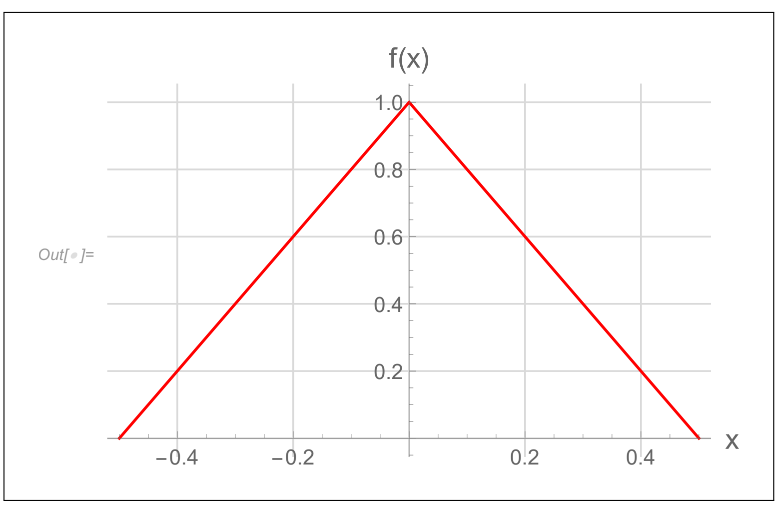


Figure 2.16: The function $f(x)$ to find its Fourier series

The function period is $T = L$. Hence the Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L} n x\right) + b_n \cos\left(\frac{2\pi}{L} n x\right)$$

Since $f(x)$ is an even function, then $b_n = 0$ and the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L} n x\right)$$

Where

$$a_0 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, dx$$

We can calculate this integral, but it is easier to find a_0 knowing that $\frac{a_0}{2}$ represent the average of the area under the function $f(x)$.

We see right away that the area is $2\left(\frac{1}{2}\frac{L}{2}\right) = \frac{L}{2}$. Hence, solving $\frac{a_0}{2} L = \frac{L}{2}$ for a_0 gives $a_0 = 1$. Now we find a_n

$$a_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos\left(\frac{2\pi}{L} n x\right) dx$$

Since $f(x)$ is even and $\cos\left(\frac{2\pi}{L} n x\right)$ is even, then the above simplifies to

$$\begin{aligned}
a_n &= \frac{4}{L} \int_0^{\frac{L}{2}} f(x) \cos\left(\frac{2\pi}{L} n x\right) dx \\
&= \frac{4}{L} \int_0^{\frac{L}{2}} \left(1 - \frac{2x}{L}\right) \cos\left(\frac{2\pi}{L} n x\right) dx \\
&= \frac{4}{L} \left(\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi}{L} n x\right) dx - \frac{2}{L} \int_0^{\frac{L}{2}} x \cos\left(\frac{2\pi}{L} n x\right) dx \right) \qquad (1)
\end{aligned}$$

But

$$\begin{aligned}
\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi}{L} n x\right) dx &= \frac{1}{\frac{2n\pi}{L}} \left[\sin\left(\frac{2\pi}{L} n x\right) \right]_0^{\frac{L}{2}} \\
&= \frac{L}{2n\pi} \left(\sin\left(\frac{2\pi}{L} n \frac{L}{2}\right) \right) \\
&= \frac{L}{2n\pi} \sin(\pi n) \\
&= 0
\end{aligned}$$

And $\int_0^{\frac{L}{2}} x \cos\left(\frac{2\pi}{L} n x\right) dx$ is integrated by parts. Let $u = x, dv = \cos\left(\frac{2\pi}{L} n x\right)$, hence $du = 1$ and

$v = \frac{1}{\frac{2n\pi}{L}} \sin\left(\frac{2\pi}{L}nx\right)$. Therefore

$$
\begin{aligned}
\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx &= uv - \int vdu \\
&= \frac{1}{\frac{2n\pi}{L}}\left[x\sin\left(\frac{2\pi}{L}nx\right)\right]_0^{\frac{L}{2}} - \frac{1}{\frac{2n\pi}{L}}\int \sin\left(\frac{2\pi}{L}nx\right)dx \\
&= -\frac{L}{2n\pi}\int \sin\left(\frac{2\pi}{L}nx\right)dx \\
&= \frac{L}{2n\pi}\left[\frac{\cos\left(\frac{2\pi}{L}nx\right)}{\frac{2\pi}{L}n}\right]_0^{\frac{L}{2}} \\
&= \left(\frac{L}{2n\pi}\right)^2\left(\cos\left(\frac{2\pi}{L}n\frac{L}{2}\right) - 1\right) \\
&= \left(\frac{L}{2n\pi}\right)^2(\cos(n\pi) - 1) \\
&= \left(\frac{L}{2n\pi}\right)^2\left((-1)^n - 1\right)
\end{aligned}
$$

Substituting these results in (1) gives

$$
\begin{aligned}
a_n &= -\frac{4}{L}\left(\frac{2}{L}\left(\frac{L}{2n\pi}\right)^2\left((-1)^n - 1\right)\right) \\
&= -\frac{2}{n^2\pi^2}\left((-1)^n - 1\right)
\end{aligned}
$$

When n is even we see that $a_n = 0$ and when n is odd, then $a_n = \frac{4}{n^2\pi^2}$. Therefore

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L}nx\right) \\
&= \frac{1}{2} + \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{4}{n^2\pi^2}\right)\cos\left(\frac{2\pi}{L}nx\right) \\
&= \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{2\pi}{L}(2n-1)x\right)
\end{aligned}
$$

2.6.5 Problem 4

4. (10 pts) Consider the Fourier series for the function $f(\theta) = 1$ when $0 < \theta < \pi$ and $f(\theta) = -1$ when $\pi < \theta < 2\pi$. Just to the right of $\theta = 0$ the first n terms in the series exhibit a local maximum of $1 + \delta_n$. For large n, $\delta_n \approx 0.2$. Using computer software, make plots of the series for 4 representative values of n of your choosing for $0 < \theta < \pi/2$ for illustration. What is the limit of the overshoot δ_n as $n \to \infty$ to 4 significant figures? Include printouts of the programs you wrote to make the plots and to find the limit. This is called the Gibbs phenomenon.

Figure 2.17: Problem statement

<u>Solution</u>

A plot of the above function is

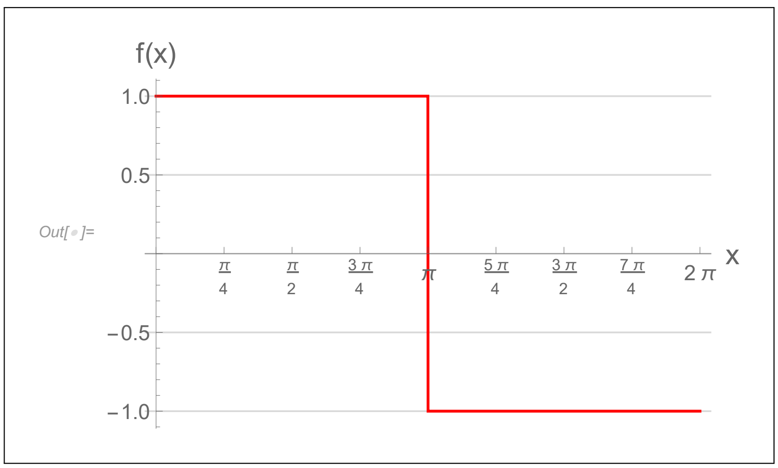


Figure 2.18: The function $f(x)$ over one period

We first need to find the Fourier series of the function $f(x)$. Since the function is odd, then we only need to determine b_n

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

Since $f(x)$ is odd, and $\sin$ is odd, then the product is even, and the above simplifies to

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx\\
&= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx\\
&= \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}\\
&= \frac{-2}{n\pi}(\cos nx)_0^{\pi}\\
&= \frac{-2}{n\pi}(\cos n\pi - 1)\\
&= \frac{-2}{n\pi}\left((-1)^n - 1\right)\\
&= \frac{2}{n\pi}\left(1 - (-1)^n\right)
\end{aligned}$$

When n is even, then $b_n = 0$ and when n is odd then $b_n = \frac{4}{n\pi}$, therefore

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n}\sin(nx)$$

Which can be written as

$$f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\sin((2n-1)x) \tag{1}$$

Next, 4 plots were made to see the approximation for $n = 1, 5, 10, 20$.

Out[]=

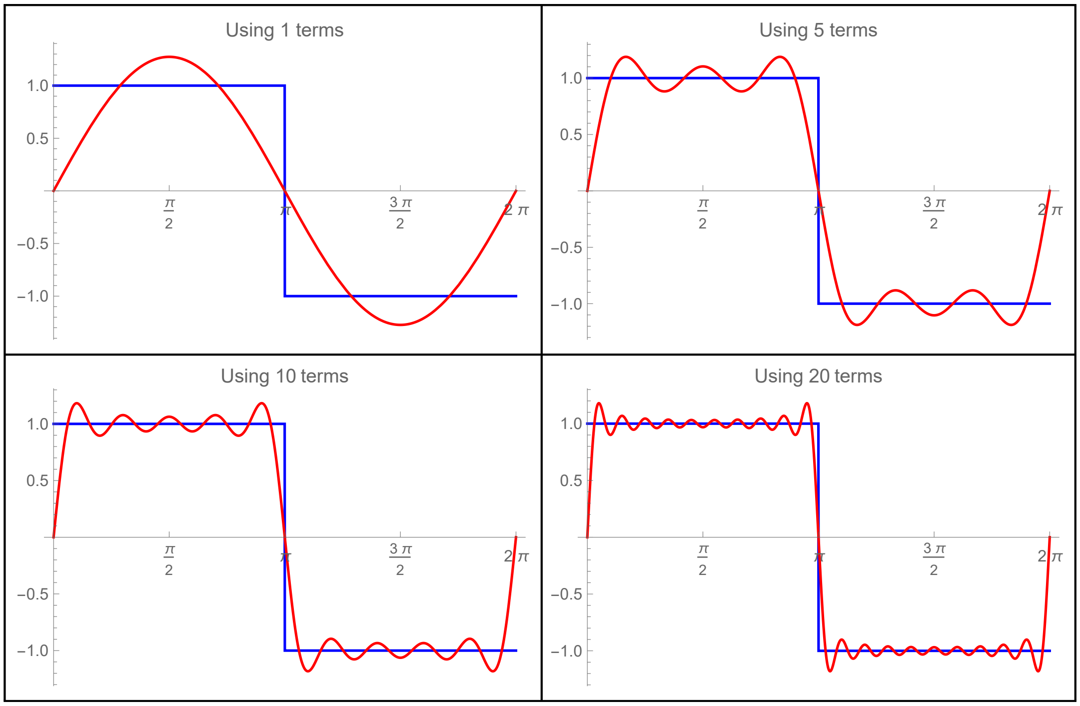


Figure 2.19: Fourier series approximation for different n values

The source code used is

```
In[]:= ClearAll[f, x, n];
f[x_ /; 0 ≤ x ≤ 2 Pi] := Piecewise[{{1, 0 ≤ x < Pi}, {-1, Pi ≤ x ≤ 2 Pi}}];
fApprox[x_, nTerms_] := 4/Pi Sum[1/(2 n - 1) Sin[(2 n - 1) x], {n, 1, nTerms}];
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, 0, 2 Pi},
    PlotStyle → {Blue, Red}, PlotLabel → Row[{"Using ", n, " terms"}],
    ImageSize → 320, Ticks → {Range[0, 2 Pi, Pi / 2], Automatic}
   ],
   {n, {1, 5, 10, 20}}], 2], Frame → All, Alignment → Center, Spacings → {1,  1}]
```

Figure 2.20: Source code used to generate the above plot

The partial sum of (1) is

$$f_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{(2n-1)}\sin\left((2n-1)\,x\right) \tag{2}$$

To determine the overshoot, we need to first find x_0 where the local maximum near $x = 0$ is. This is an illustration, showing the Fourier series approximation to the right of $x = 0$. This plot uses $n = 100$.

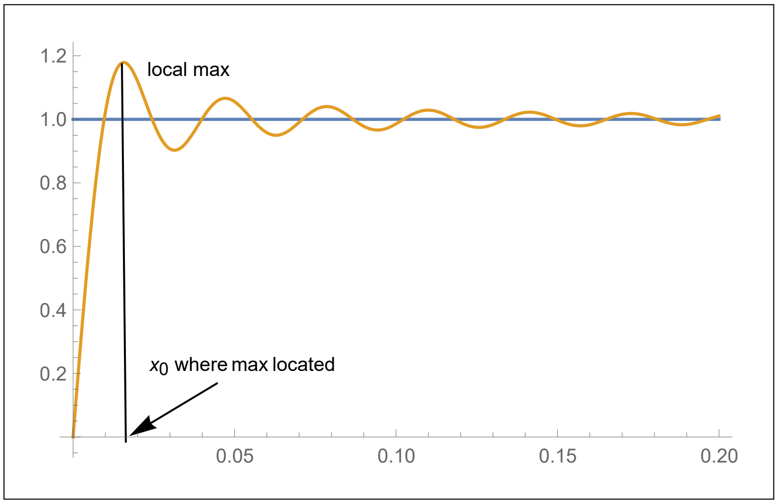


Figure 2.21: Finding x_0 where maximum overshoot is located

Hence we need to determine $f'(x)$ and then solve for $f'(x)=0$ in order to find x_0

$$
\begin{aligned}
f_N'(x) &= \frac{4}{\pi}\sum_{n=1}^{N}\cos\left((2n-1)x\right)\\
&= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin x}
\end{aligned}
$$

Derivation that shows the above is included in the appendix of this problem. Therefore solving $\frac{\sin(2Nx)}{\sin x}=0$ implies $\sin(2Nx)=0$ or $2Nx=\pi$ (since we want to be on the right side of $x=0$, we do not pick 0, but the next zero, this means π is first value). This implies that local maximum to the right of $x=0$ is located at

$$
\boxed{x_0 = \frac{\pi}{2N}}
$$

Therefore we need to determine $f_N(x_0)$ to calculate the overshoot due to the Gibbs effect to the right of $x=0$. From (2) and using x_0 now instead of x gives

$$
\begin{aligned}
f_N\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{(2n-1)}\sin\left((2n-1)\frac{\pi}{2N}\right)\\
&= \frac{4}{\pi}\left(\frac{\sin\left(\frac{\pi}{2N}\right)}{1}+\frac{\sin\left(3\frac{\pi}{2N}\right)}{3}+\frac{\sin\left(5\frac{\pi}{2N}\right)}{5}+\cdots+\frac{\sin\left((2N-1)\frac{\pi}{2N}\right)}{2N-1}\right)
\end{aligned}
$$

But $\frac{\sin(\pi z)}{\pi z} = \operatorname{sinc}(z)$, therefore we rewrite the above as

$$
\begin{aligned}
f_N\left(\frac{\pi}{2N}\right) &= 4\left(\frac{\sin\left(\frac{\pi}{2N}\right)}{\pi} + \frac{\sin\left(3\frac{\pi}{2N}\right)}{3\pi} + \frac{\sin\left(5\frac{\pi}{2N}\right)}{5\pi} + \cdots + \frac{\sin\left((2N-1)\frac{\pi}{2N}\right)}{(2N-1)\pi}\right) \\
&= 4\left(\frac{1}{2N}\frac{\sin\left(\pi\frac{1}{2N}\right)}{\pi\frac{1}{2N}} + \frac{1}{2N}\frac{\sin\left(\pi\frac{3}{2N}\right)}{3\pi\frac{1}{2N}} + \frac{1}{2N}\frac{\sin\left(\pi\frac{5}{2N}\right)}{5\pi\frac{1}{2N}} + \cdots + \frac{1}{2N}\frac{\sin\left(\pi\frac{(2N-1)}{2N}\right)}{(2N-1)\pi\frac{1}{2N}}\right) \\
&= 4\left(\frac{1}{2N}\operatorname{sinc}\left(\frac{1}{2N}\right) + \frac{1}{2N}\operatorname{sinc}\left(\frac{3}{2N}\right) + \frac{1}{2N}\operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \frac{1}{2N}\operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_N\left(\frac{\pi}{2N}\right) &= 2\left(\frac{1}{N}\operatorname{sinc}\left(\frac{1}{2N}\right) + \frac{1}{N}\operatorname{sinc}\left(\frac{3}{2N}\right) + \frac{1}{N}\operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \frac{1}{N}\operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right) \\
&= 2\left\{\left[\operatorname{sinc}\left(\frac{1}{2N}\right) + \operatorname{sinc}\left(\frac{3}{2N}\right) + \operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right]\frac{1}{N}\right\}
\end{aligned}
$$

Therefore, if we consider a length of 1 and $\frac{1}{N}$ is partition length, then the sum inside {} above is a Riemann sum and the above becomes In the limit, as $N \to \infty$

$$
\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) = 2\int_0^1 \operatorname{sinc}(x)\,dx
$$

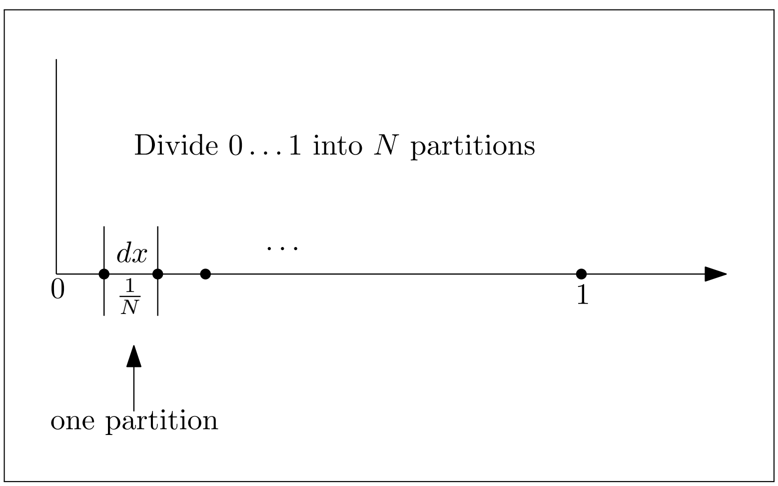


Figure 2.22: Converting Riemman sum to an integral

Therefore

$$
\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) = 2\int_0^1 \frac{\sin(\pi x)}{\pi x}dx
$$

The $\int_0^1 \frac{\sin(\pi x)}{\pi x}dx$ is known as Si. I could not solve it analytically. It has numerical value of

0.5894898772. Therefore

$$\begin{aligned}\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) &= 2\,(0.5894898772)\\ &= 1.17897974\end{aligned}$$

Since $f(x)=1$ between 0 and π, then we see that the overshoot is the difference, which is

$$\begin{aligned}\lim_{N\to\infty}\delta_N &= 1.17897974-1\\ &= 0.1789\end{aligned}$$

For 4 decimal places. The above result gives good agreement with the plot showing that the overshoot is a little less than 0.2 when viewed on the computer screen. The only use for computation used by the computer for this part of the problem was the evaluation of $\int_0^1 \frac{\sin(\pi x)}{\pi x}dx$. The code is

```
In[ ]:= Integrate[Sin[Pi x] / (Pi x), {x, 0, 1}]
Out[ ]= SinIntegral[π]
        --------------
              π
In[ ]:= N[%, 16]
Out[ ]= 0.5894898722360836
```

Figure 2.23: Finding the limit

2.6.5.1 Appendix

Here we show the following result used in the above solution.

$$\frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x)=\frac{2}{\pi}\frac{\sin(2Nx)}{\sin x}$$

Since $\cos z=\operatorname{Re}\left(e^{iz}\right)$, then $\cos((2n-1)x)=\operatorname{Re}\left(e^{i(2n-1)x}\right)$. Hence the above is the same as

$$\frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x)=\frac{4}{\pi}\operatorname{Re}\sum_{n=1}^{N}e^{i(2n-1)x} \tag{1}$$

But

$$\begin{aligned}\sum_{n=1}^{N}e^{i(2n-1)x} &= \sum_{n=1}^{N}e^{2ixn-ix}\\ &= e^{-ix}\sum_{n=1}^{N}e^{2ixn}\\ &= e^{-ix}\sum_{n=1}^{N}\left(e^{2ix}\right)^n\end{aligned}$$

Using partial sum property $\sum_{n=1}^{N} r^n = r\frac{1-r^N}{1-r}$, then we can write the above using $r = e^{2ix}$ as

$$\begin{aligned}
\sum_{n=1}^{N} e^{i(2n-1)x} &= e^{-ix}\left(e^{2ix}\frac{1-e^{2iNx}}{1-e^{2ix}}\right) \\
&= e^{ix}\frac{1-e^{2iNx}}{1-e^{2ix}} \\
&= \frac{1-e^{2iNx}}{e^{-ix}-e^{ix}} \\
&= \frac{e^{2iNx}-1}{e^{ix}-e^{-ix}} \\
&= \frac{e^{2iNx}-1}{2i\sin(x)} \\
&= \frac{\cos(2Nx)+i\sin(2Nx)-1}{2i\sin(x)}
\end{aligned}$$

Multiplying numerator and denominator by i gives

$$\begin{aligned}
\sum_{n=1}^{N} e^{i(2n-1)x} &= \frac{i\cos(2Nx)-\sin(2Nx)-i}{-2\sin(x)} \\
&= i\frac{(\cos(2Nx)-1)}{-2\sin x} + \frac{\sin(2Nx)}{2\sin(x)}
\end{aligned}$$

The real part of the above is $\frac{\sin(2Nx)}{2\sin(x)}$, hence (1) becomes

$$\begin{aligned}
\frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x) &= \frac{4}{\pi}\operatorname{Re}\sum_{n=1}^{N} e^{i(2n-1)x} \\
&= \frac{4}{\pi}\left(\frac{\sin(2Nx)}{2\sin(x)}\right) \\
&= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin(x)}
\end{aligned}$$

Which is the result was needed to show.

2.6.6 Key solution for HW 6

① $R = N\int_0^\infty dE\, E\, e^{-\beta E} e^{-\alpha E^{-1/2}} = N\int_0^\infty dE\, e^{f(E)}$

$f(E) = -\beta E - \alpha E^{-1/2} + \ln E$

Find the saddle point: $f' = -\beta + \frac{\alpha}{2} E^{-3/2} + \frac{1}{E} = 0$

$1 = \beta E - \frac{1}{2}\frac{\alpha}{\sqrt{E}}$ Solution referred to as E_0.

Define $x = \beta E$ which is dimensionless.

$1 = x - \frac{1}{2}\frac{\alpha\sqrt{\beta}}{\sqrt{x}}$ When $\alpha^2\beta >> 1$ the 2 terms on the right side must balance each other.

$x_0^{3/2} \approx \frac{\alpha\sqrt{\beta}}{2}$ $x_0^3 \approx \frac{\alpha^2\beta}{4}$ $x_0 \approx \left(\frac{\alpha^2\beta}{4}\right)^{1/3} >> 1$

Thus justifies ignoring the 1 on the left side.

$f(x_0) = -x_0 - \frac{\alpha\sqrt{\beta}}{x_0} + \ln(x_0/\beta)$

$\uparrow$ small compared to other 2 terms

$f''(E) = -\frac{3}{4}\alpha E^{-5/2} - \frac{1}{E^2}$

$$f''(x_0) = -\frac{3}{4}\frac{\alpha\beta^{5/2}}{x_0^{5/2}} - \frac{\beta^2}{x_0^2} = -\frac{\beta^2}{x_0^2}\left(\frac{3}{4}\frac{\alpha\sqrt{\beta}}{\sqrt{x_0}} + 1\right) =$$

$$= -\frac{\beta^2}{x_0^2}\left(\frac{3}{4}\underset{\substack{\uparrow \\ \text{This term is much larger}}}{\frac{2x_0^{3/2}}{x_0^{1/2}}} + 1\right) \simeq -\frac{3}{2}\frac{\beta^2}{x_0}$$

Thus $R \simeq N e^{f(x_0)} \underbrace{\int_0^\infty dE\, e^{\frac{1}{2}f''(x_0)(E-E_0)^2}}_{\sqrt{\frac{2\pi}{-f''(x_0)}} = \sqrt{\frac{4\pi}{3}}\frac{\sqrt{x_0}}{\beta}}$

$$= \sqrt{\frac{4\pi}{3\beta}}\, N \left(\frac{x_0}{\beta}\right)^{3/2} e^{-x_0} = \sqrt{\frac{\pi}{3}}\, 2N \frac{x_0^{3/2}}{\beta^2} e^{-x_0}$$

Now $x_0^{3/2} = \frac{\alpha\sqrt{\beta}}{2}$ so we get

$$\boxed{R \simeq \sqrt{\frac{\pi}{3}}\, N (k_B T)^{3/2} \alpha\, e^{-(\alpha^2/4k_BT)^{1/3}}}$$

Note that $R \to 0$ rapidly as $T \to 0$.

(2) $$I = \int_a^b f(x)\,\delta(g(x))\,dx$$

$g(x_0) = 0$ and $a < x_0 < b$ Let $y = g(x)$.

$dy = g'(x)\,dx$ Then $I = \int_{y_{min}}^{y_{max}} f(g^{-1}(y))\,\delta(y)\,\frac{dy}{|g'(g^{-1}(y))|}$

where $g^{-1}(y) = x$. We use the absolute value of g' in the Jacobian and make the lower limit of the y integration smaller than the upper limit. This takes into account both the possibility that $g'(x) > 0$ and $g'(x) < 0$ in that interval. Then

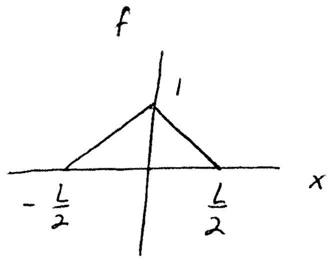
$$I = \frac{f(g^{-1}(0))}{|g'(g^{-1}(0))|} = \frac{f(x_0)}{|g'(x_0)|}$$

$$\boxed{\int_a^b f(x)\,\delta(g(x))\,dx = \frac{f(x_0)}{|g'(x_0)|}}$$

This generalizes straight forwardly if $g(x) = 0$ at multiple points in the interval.

3)

$$f(x) = \begin{cases} 1 + \frac{2x}{L} & -\frac{L}{2} \le x \le 0 \\ 1 - \frac{2x}{L} & 0 \le x \le \frac{L}{2} \end{cases} \quad \text{periodic}$$



$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi n x}{L} + B_n \sin\frac{2\pi n x}{L}\right)$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{2\pi n x}{L}\,dx = 0 \quad \text{because } f(x) \text{ is even}$$

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\,dx = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)dx = \frac{4}{L}\left(\frac{L}{2} - \frac{L}{4}\right) = 1$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{2\pi n x}{L}\,dx = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)\cos\frac{2\pi n x}{L}\,dx$$

Use $\frac{d}{dk}\int \sin kx\,dx = \int x\cos kx\,dx \qquad k_n \equiv \frac{2\pi n}{L}$

$$\int_0^{L/2} x\cos k_n x\,dx = \frac{d}{dk_n}\int_0^{L/2}\sin k_n x\,dx = -\frac{d}{dk_n}\left.\frac{\cos k_n x}{k_n}\right|_0^{L/2} =$$

$$= -\frac{d}{dk_n}\left[\frac{\cos\left(\frac{k_n L}{2}\right) - 1}{k_n}\right] = \frac{\left[\cos\left(\frac{k_n L}{2}\right) - 1\right]}{k_n^2} + \frac{L}{2k_n}\sin\left(\frac{k_n L}{2}\right) =$$

$$= \frac{L^2}{(2\pi n)^2}\left[\cos(\pi n) - 1\right] + \frac{L^2}{4\pi n}\sin(\pi n)$$

$$\int_0^{L/2} \cos k_n x \, dx = \left.\frac{\sin k_n x}{k_n}\right|_0^{L/2} = \frac{L}{2\pi n}\sin(\pi n)$$

$$A_n = \frac{4}{L}\left\{\frac{L}{2\pi n}\sin(\pi n) - \frac{L}{2\pi^2 n^2}\left[\cos(\pi n) - 1\right] - \frac{L}{2\pi n}\sin(\pi n)\right\}$$

$$= \frac{2}{\pi^2 n^2}\left[1 - \cos(\pi n)\right] = \frac{2}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

$$= \begin{cases} \frac{4}{\pi^2 n^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Let $n = 2m+1, \quad m = 0, 1, 2, \ldots$

$$\boxed{f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos\left[\frac{2\pi(2m+1)x}{L}\right]}$$

2.7 HW 7

Local contents

2.7.1 HW 7 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 7 due Monday April 1. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (5 pts) Evaluate the integral

$$\int_0^\pi dx \int_1^2 dy\, \delta(\sin x)\delta(x^2 - y^2)$$

2. (5 pts) Consider the linear response formula

$$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t')dt'$$

When the input is $F(t) = \mathrm{e}^{-\lambda t}\theta(t)$ the output is $x(t) = (1-\mathrm{e}^{-\alpha t})\mathrm{e}^{-\lambda t}$. What is $\tilde{G}(\omega)$? What is the output if $F(t) = F_0\delta(t)$?

3. (5 pts) By using the integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

find the Laplace transform of J_0.

4. (5 pts) A reasonably accurate description of the atomic contribution to the dielectric function is

$$\epsilon(\omega) = 1 + \omega_P^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

There are f_j electrons per molecule with binding frequency ω_j and damping constant γ_j. The oscillator strengths f_j obey the sum rule $\sum_j f_j = Z$ which is the total number of electrons per molecule. Using the imaginary part of ϵ in the dispersion relation, show that the real part is correctly reproduced.

2.7.2 Problem 1

1. (5 pts) Evaluate the integral

$$\int_0^{\pi} dx \int_1^2 dy\, \delta(\sin x)\delta(x^2 - y^2)$$

Figure 2.24: Problem statement

<u>Solution</u>

$$I = \int_0^{\pi} \left(\int_1^2 \delta(\sin(x))\, \delta\left(x^2 - y^2\right) dy \right) dx$$

Since $\delta(\sin(x))$ does not depend on y we can move it from the inner integral to the outside integral

$$I = \int_0^{\pi} \delta(\sin(x)) \left(\int_1^2 \delta\left(x^2 - y^2\right) dy \right) dx \tag{1}$$

Now we need to evaluate.

$$I_2 = \int_1^2 \delta\left(x^2 - y^2\right) dy$$

This is in the form of $\int_1^2 f\left(y\right) \delta\left(g\left(y\right)\right) dy$ where now $f\left(y\right) = 1$ and $g\left(y\right) = x^2 - y^2$. Therefore the roots of $g\left(y\right)$ are $\pm x$. We see that x has to be in the range of $1 \cdots 2$, since that is where y is defined over. Hence the root $-x$ is outside this range and can not be used. So there is only one root which is $+x$. Now, using the result obtained from last HW which says

$$\int_1^2 f\left(y\right) \delta\left(g\left(y\right)\right) du = \frac{f\left(y_0\right)}{\left|g'\left(y_0\right)\right|}$$

Therefore integral I_2 becomes

$$I_2 = \lim_{y \to y_0} \frac{f\left(y\right)}{\left|g'\left(y\right)\right|}$$

Where $y_0 = x$ is the root and where $g'\left(y\right) = -2y$ and where $f\left(y\right) = 1$. Hence the above becomes

$$I_2 = \frac{1}{2|x|} \left(\theta(x-1) - \theta(x-2)\right)$$

Where we added $(\theta(x-1)-\theta(x-2))$ to insure that x is $1<x<2$. Using this result in (1) gives (we do not need to write $|x|$ any more since $x>0$)

$$\begin{aligned} I &= \int_0^{\pi} \frac{1}{2x}(\theta(x-1)-\theta(x-2))\,\delta(\sin(x))\,dx \\ &= \int_1^2 \frac{1}{2x}\delta(\sin(x))\,dx \end{aligned}$$

Let $f(x)=\frac{1}{2x}, g(x)=\sin(x)$, then the above in the form

$$I=\int_1^2 f(x)\,\delta\left(g(x)\right)dx=\sum_{x_0}\frac{f(x_0)}{\left|g'(x_0)\right|}$$

Where x_0 are the zero of $g(x)=\sin(x)$ inside the range $x=1\cdots 2$. But there are no zeros of $\sin(x)$ in this range. Therefore this leads to

$$I=0$$

In other words

$$\int_0^{\pi}\left(\int_1^2 \delta(\sin(x))\,\delta\left(x^2-y^2\right)dy\right)dx=0$$

2.7.3 Problem 2

2. (5 pts) Consider the linear response formula

$$x(t)=\int_{-\infty}^{\infty} G(t-t')F(t')dt'$$

When the input is $F(t)=\mathrm{e}^{-\lambda t}\theta(t)$ the output is $x(t)=(1-\mathrm{e}^{-\alpha t})\mathrm{e}^{-\lambda t}$. What is $\tilde{G}(\omega)$? What is the output if $F(t)=F_0\delta(t)$?

Figure 2.25: Problem statement

Solution

2.7.3.1 Part (a)

Since

$$\tilde{G}(\omega)=\frac{\text{Fourier transform of output}}{\text{Fourier transform of input}} \tag{1}$$

Assuming causal system, then the output $x(t)$ is $x(t)=\left(1-e^{-\alpha t}\right)e^{-\lambda t}\theta(t)$. In other words, we added unit step $\theta(t)$ to indicate it also starts at $t=0$, since the input starts at $t=0$.

Therefore the above definition becomes

$$
\begin{aligned}
G(\omega) &== \frac{\int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt}{\int_{-\infty}^{\infty} F(t)\, e^{-i\omega t} dt} \\
&= \frac{\int_{-\infty}^{\infty} \left(1 - e^{-\alpha t}\right) e^{-\lambda t} \theta(t)\, e^{-i\omega t} dt}{\int_{-\infty}^{\infty} e^{-\lambda t} \theta(t)\, e^{-i\omega t} dt} \\
&= \frac{\int_{0}^{\infty} \left(1 - e^{-\alpha t}\right) e^{-\lambda t} e^{-i\omega t} dt}{\int_{0}^{\infty} e^{-\lambda t} e^{-i\omega t} dt} \qquad (2)
\end{aligned}
$$

But

$$
\begin{aligned}
\int_0^{\infty} \left(1 - e^{-\alpha t}\right) e^{-\lambda t} e^{-i\omega t} dt &= \int_0^{\infty} e^{-\lambda t} e^{-i\omega t} dt - \int_0^{\infty} e^{-\alpha t} e^{-\lambda t} e^{-i\omega t} dt \\
&= \int_0^{\infty} e^{-t(\lambda + i\omega)} dt - \int_0^{\infty} e^{-t(\alpha + \lambda + i\omega)} dt \\
&= \left[\frac{e^{-t(\lambda + i\omega)}}{-(\lambda + i\omega)}\right]_0^{\infty} + \left[\frac{e^{-t(\alpha + \lambda + i\omega)}}{\alpha + \lambda + i\omega}\right]_0^{\infty} \\
&= \frac{-1}{(\lambda + i\omega)} \left[e^{-t(\lambda + i\omega)}\right]_0^{\infty} + \frac{1}{\alpha + \lambda + i\omega} \left[e^{-t(\alpha + \lambda + i\omega)}\right]_0^{\infty} \\
&= \frac{-1}{(\lambda + i\omega)} \left[e^{-t\lambda} e^{-it\omega}\right]_0^{\infty} + \frac{1}{\alpha + \lambda + i\omega} \left[e^{-t(\alpha + \lambda)} e^{-it\omega}\right]_0^{\infty}
\end{aligned}
$$

With the assumptions[2] that $\lambda > 0, \alpha > 0$, then the above simplifies to

$$
\begin{aligned}
\int_0^{\infty} \left(1 - e^{-\alpha t}\right) e^{-\lambda t} e^{-i\omega t} dt &= \frac{-1}{(\lambda + i\omega)} [0 - 1] + \frac{1}{\alpha + \lambda + i\omega} [0 - 1] \\
&= \frac{1}{(\lambda + i\omega)} - \frac{1}{(\alpha + \lambda + i\omega)} \\
&= \frac{(\alpha + \lambda + i\omega) - (\lambda + i\omega)}{(\lambda + i\omega)(\alpha + \lambda + i\omega)} \\
&= \frac{\alpha}{(\lambda + i\omega)(\alpha + \lambda + i\omega)} \qquad (3)
\end{aligned}
$$

And

$$
\begin{aligned}
\int_0^{\infty} e^{-\lambda t} e^{-i\omega t} dt &= \int_0^{\infty} e^{-t(\lambda + i\omega)} dt \\
&= \left[\frac{e^{-t(\lambda + i\omega)}}{-(\lambda + i\omega)}\right]_0^{\infty} \\
&= \frac{-1}{(\lambda + i\omega)} \left[e^{-t(\lambda + i\omega)}\right]_0^{\infty}
\end{aligned}
$$

[2]So that input does not blow up with time, and its follows that output also decays with time, hence $\alpha > 0$

Since we assumed that $\lambda > 0$, then the above simplifies to

$$\begin{aligned}\int_0^\infty e^{-\lambda t}e^{-i\omega t}dt &= \frac{-1}{(\lambda + i\omega)}\left[0-1\right]\\ &= \frac{1}{(\lambda+i\omega)} \qquad (4)\end{aligned}$$

Substituting (3,4) into (2) gives the transfer function

$$\tilde{G}(\omega) = \frac{\frac{\alpha}{(\lambda+i\omega)(\alpha+\lambda+i\omega)}}{\frac{1}{(\lambda+i\omega)}}$$

Therefore

$$\boxed{\tilde{G}(\omega) = \frac{\alpha}{\alpha+\lambda+i\omega}}$$

2.7.3.2 Part (b)

If the input is $F(t) = F_0\delta(t)$ then the output is

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} G(t-t')F_0\delta(t')\,dt\\ &= F_0 G(t) \qquad (4A)\end{aligned}$$

Hence we just need to find $G(t)$ which is the inverse Fourier transform of $\tilde{G}(\omega)$ we found above.

$$\begin{aligned}G(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{\alpha+\lambda+i\omega}e^{i\omega t}d\omega\\ &= \frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\alpha+\lambda)+i\omega}d\omega\end{aligned}$$

To integrate the the above, we will use complex contour integration. Let $\omega = z$, hence the above becomes

$$G(t) = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz$$

Therefore $f(z) = \frac{e^{izt}}{(\alpha+\lambda)+iz}$. The pole is at $iz = -(\alpha+\lambda)$ or $z = i(\alpha+\lambda)$. Since $\alpha+\lambda > 0$, then the pole is in upper half plane. Lets find out where we will put the half circle, if it will go on the upper half or lower half. Since numerator is $e^{izt} = e^{i(x+iy)t} = e^{iz}e^{-yt}$ and therefore, since $t > 0$, then we want to choose the upper half circle, since there y is positive, which will cause the numerator to go to zero as $R \to \infty$. This implies there is one pole inside the upper half plane, we all what we need to do is find the residue at $z_0 = i(\alpha+\lambda)$.

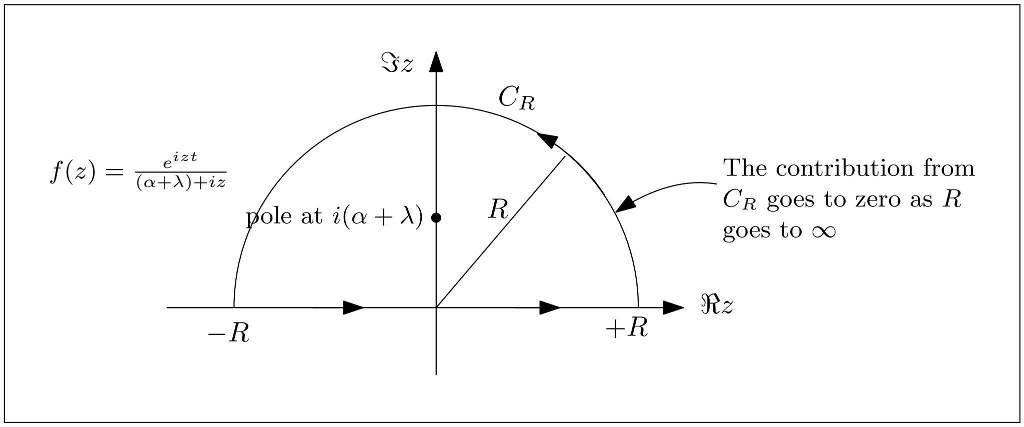


Figure 2.26: Contour integration used for finding inverse Fourier transform

Hence

$$\left(\frac{\alpha}{2\pi}\right)\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz = \left(\frac{\alpha}{2\pi}\right)2\pi i\sum \text{Residue} \tag{5}$$

But, since $z_0 = i(\alpha+\lambda)$, then

$$\begin{aligned}\text{Residue}(z_0) &= \lim_{z\to z_0}(z-z_0)f(z)\\ &= \lim_{z\to i(\alpha+\lambda)}(z-i(\alpha+\lambda))\frac{e^{izt}}{(\alpha+\lambda)+iz}\\ &= \left(\lim_{z\to i(\alpha+\lambda)} e^{izt}\right)\left(\lim_{z\to i(\alpha+\lambda)}\frac{z-i(\alpha+\lambda)}{(\alpha+\lambda)+iz}\right)\end{aligned}$$

Applying L'Hopitals gives

$$\begin{aligned}\text{Residue}(z_0) &= \left(\lim_{z\to i(\alpha+\lambda)} e^{izt}\right)\left(\lim_{z\to i(\alpha+\lambda)}\frac{1}{i}\right)\\ &= -i\lim_{z\to i(\alpha+\lambda)} e^{izt}\\ &= -ie^{-(\alpha+\lambda)t}\end{aligned}$$

Now that we found the residue, then from (5)

$$\begin{aligned}\left(\frac{\alpha}{2\pi}\right)\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz &= \left(\frac{\alpha}{2\pi}\right)2\pi i\left(-ie^{-(\alpha+\lambda)t}\right)\\ &= \alpha e^{-(\alpha+\lambda)t}\end{aligned}$$

We have found $G(t)$

$$G(t) = \alpha e^{-(\alpha+\lambda)t} \qquad t>0$$

From (4A), the response is

$$\begin{aligned}x(t) &= F_0 G(t)\\ &= \alpha F_0 e^{-(\alpha+\lambda)t}\theta(t)\end{aligned}$$

2.7.4 Problem 3

3. (5 pts) By using the integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

find the Laplace transform of J_0.

Figure 2.27: Problem statement

<u>Solution</u>

Using

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

Hence Laplace transform is

$$\begin{aligned}
\hat{J}_0(s) &= \int_0^{\infty} J_0(x)\, e^{-sx} dx \\
&= \frac{1}{2\pi}\int_0^{\infty}\left(\int_0^{2\pi}\cos(x\cos\theta)\, d\theta\right) e^{-sx} dx
\end{aligned}$$

Changing order of integration

$$\hat{J}_0(s) = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\infty}\cos(x\cos\theta)\, e^{-sx} dx\right) d\theta \tag{1}$$

Let $I = \int_0^{\infty}\cos(x\cos\theta)\, e^{-sx} dx$. This is solved by applying integration by parts twice

Let $u = \cos(x\cos\theta), dv = e^{-sx}$, hence $du = -\cos\theta\sin(x\cos\theta), v = -\frac{e^{-sx}}{s}$. Therefore

$$\begin{aligned}
I &= [uv]_0^{\infty} - \int_0^{\infty} v du \\
&= -\frac{1}{s}\left[\cos(x\cos\theta)\, e^{-sx}\right]_0^{\infty} - \frac{1}{s}\cos\theta\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx \\
&= -\frac{1}{s}[0-1] - \frac{\cos\theta}{s}\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx \\
&= \frac{1}{s} - \frac{\cos\theta}{s}\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx
\end{aligned}$$

Integration by parts again, let $\sin(x\cos\theta) = u, du = \cos\theta\cos(x\cos\theta), dv = e^{-sx}, v = -\frac{e^{-sx}}{s}$, and the above becomes

$$
\begin{aligned}
I &= \frac{1}{s} - \frac{\cos\theta}{s}\left([uv]_0^\infty - \int_0^\infty v du\right)\\
&= \frac{1}{s} - \frac{\cos\theta}{s}\left(-\frac{1}{s}\left[\sin\left(x\cos\theta\right)e^{-sx}\right]_0^\infty + \frac{\cos\theta}{s}\int_0^\infty \cos\left(x\cos\theta\right)e^{-sx} du\right)\\
&= \frac{1}{s} - \frac{\cos\theta}{s}\left(-\frac{1}{s}\left[0\right] + \frac{\cos\theta}{s}I\right)\\
&= \frac{1}{s} - \frac{\cos\theta}{s}\left(\frac{\cos\theta}{s}I\right)\\
&= \frac{1}{s} - \frac{\cos^2\theta}{s^2}I
\end{aligned}
$$

Solving for I gives

$$
\begin{aligned}
I + \frac{\cos^2\theta}{s^2}I &= \frac{1}{s}\\
I\left(1 + \frac{\cos^2\theta}{s^2}\right) &= \frac{1}{s}\\
I\left(\frac{s^2 + \cos^2\theta}{s^2}\right) &= \frac{1}{s}\\
I\left(\frac{s^2 + \cos^2\theta}{s}\right) &= 1\\
I &= \frac{s}{s^2 + \cos^2\theta}
\end{aligned}
$$

Therefore

$$
\int_0^\infty \cos\left(x\cos\theta\right)e^{-sx} dx = \frac{s}{s^2 + \cos^2\left(\theta\right)} \tag{2}
$$

Substituting (2) in (1) gives

$$
\hat{J}_0\left(s\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s}{s^2 + \cos^2\left(\theta\right)} d\theta
$$

Since the above is an even function, we can rewrite as

$$
\hat{J}_0\left(s\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{s}{s^2 + \cos^2\left(\theta\right)} d\theta
$$

The above can be solved using contour integration or using standard method of integration using substitution which I think is simpler here.

Multiplying numerator and denominator of $\hat{J}_0\left(s\right)$ above by $\sec^2\left(\theta\right)$ gives

$$
\hat{J}_0\left(s\right) = \frac{2s}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\theta\right)}{s^2\sec^2\left(\theta\right) + 1} d\theta
$$

Let $u = \tan\left(\theta\right)$. When $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{2}, u = \infty$. Since $du = d\theta\sec^2\left(\theta\right)$. Hence the above integral becomes, since $\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + u^2$

$$\hat{J}_0(s) = \frac{2s}{\pi}\int_0^\infty \frac{1}{s^2\sec^2(\theta)+1}du$$

But $\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + u^2$ therefore the above becomes

$$\begin{aligned}
\hat{J}_0(s) &= \frac{2s}{\pi}\int_0^\infty \frac{1}{s^2\left(1+u^2\right)+1}du \\
&= \frac{2s}{\pi}\int_0^\infty \frac{1}{\left(1+s^2\right)+s^2u^2}du \\
&= \frac{2s}{\pi}\int_0^\infty \frac{1}{s^2}\frac{1}{\left(\frac{1+s^2}{s^2}\right)+u^2}du \\
&= \frac{2s}{\pi}\int_0^\infty \frac{1}{\left(\frac{1+s^2}{s^2}\right)+u^2}du
\end{aligned}$$

Let $\left(\frac{1+s^2}{s^2}\right) = A$, so the integral in the form $\int \frac{1}{A+u^2}du = \frac{1}{\sqrt{A}}\arctan\left(\frac{u}{\sqrt{A}}\right)$, hence the above becomes

$$\begin{aligned}
\hat{J}_0(s) &= \frac{2s}{\pi}\left[\frac{1}{\sqrt{\frac{1+s^2}{s^2}}}\arctan\left(\frac{u}{\sqrt{\frac{1+s^2}{s^2}}}\right)\right]_0^\infty \\
&= \frac{2}{\sqrt{1+s^2}}\frac{1}{\pi}\left[\arctan\left(\frac{s}{\sqrt{1+s^2}}u\right)\right]_0^\infty \\
&= \frac{2}{\sqrt{1+s^2}}\frac{1}{\pi}\left[\frac{\pi}{2}-0\right] \\
&= \frac{1}{\sqrt{1+s^2}}
\end{aligned}$$

2.7.4.1 Appendix

This part contains attempt made using contour integration. For reference and not for grading.

Solve

$$\hat{J}_0(s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s}{s^2+\cos^2(\theta)}d\theta$$

Let $z = e^{i\theta}$, then $dz = izd\theta$, and $\cos(\theta) = \frac{z+z^{-1}}{2}$, hence the above integral becomes

$$\begin{aligned}
\hat{J}_0(s) &= \frac{1}{2\pi}\oint \frac{s}{s^2 + \left(\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz} \\
&= \frac{1}{2\pi}\oint \frac{4s}{4s^2 + \left(z + \frac{1}{z}\right)^2}\frac{dz}{iz} \\
&= -\frac{i4s}{2\pi}\oint \frac{1}{4s^2 + \left(z + \frac{1}{z}\right)^2}\frac{dz}{z} \\
&= -\frac{i4s}{2\pi}\oint \frac{1}{4s^2 + \left(\frac{z^2+1}{z}\right)^2}\frac{dz}{z} \\
&= -\frac{i4s}{2\pi}\oint \frac{z}{4s^2z^2 + \left(z^2+1\right)^2}dz
\end{aligned}$$

Did not complete.

Alternative solution

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)\,d\theta$$

Let $\cos\theta = u$, hence $\frac{du}{d\theta} = -\sin\theta$. But $\cos^2\theta + \sin^2\theta = 1$, therefore $\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$. Hence $\sin\theta = \sqrt{1-u^2}$. When $\theta = 0, u = 1$ and when $\theta = \pi, u = -1$, therefore the above integral now can be written as

$$\begin{aligned}
J_0(x) &= \frac{1}{\pi}\int_1^{-1}\cos(xu)\left(\frac{-du}{\sqrt{1-u^2}}\right) \\
&= \frac{1}{\pi}\int_{-1}^{1}\cos(xu)\frac{du}{\sqrt{1-u^2}}
\end{aligned}$$

Since the integrand is even, then the above becomes

$$J_0(x) = \frac{2}{\pi}\int_0^1\cos(xu)\frac{du}{\sqrt{1-u^2}}$$

And the above is what will be used as starting point. I could not solve this using complex contour integration, which is probably would have been easier if I knew how to do it, but instead solved it using substitution as follows.

Changing the argument from x to α gives

$$J_0(\alpha) = \frac{2}{\pi}\int_0^1\cos(\alpha u)\frac{du}{\sqrt{1-u^2}}$$

u is arbitrary inside the integral so we can rename it back to x and the above becomes

$$J_0(\alpha) = \frac{2}{\pi}\int_0^1 \cos(\alpha x)\frac{dx}{\sqrt{1-x^2}}$$

Which is the same as (by renaming the argument again, since it better to use t with Laplace by convention, just for notation sake)

$$J_0(\alpha t) = \frac{2}{\pi}\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}$$

Now, the Laplace transform of $J_0(\alpha t)$ is

$$\begin{aligned}
\hat{J}_0(s) &= \int_0^\infty J_0(\alpha t) e^{-st} dt \\
&= \int_0^\infty \left(\frac{2}{\pi}\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}\right) e^{-st} dt \\
&= \frac{2}{\pi}\int_0^\infty \left(\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}\right) e^{-st} dt
\end{aligned}$$

Changing order of integration gives

$$\hat{J}_0(s) = \frac{2}{\pi}\int_0^1 \left(\int_0^\infty \cos(\alpha t x) e^{-st} dt\right)\frac{1}{\sqrt{1-x^2}} dx$$

But $\int_0^\infty \cos(\alpha t x) e^{-st} dt$ is the Laplace transform of $\cos(\alpha t x)$ which is from tables $\frac{s}{s^2+a^2x^2}$. Hence the above simplifies to

$$\begin{aligned}
\hat{J}_0(s) &= \frac{2}{\pi}\int_0^1 \frac{s}{s^2+\alpha^2x^2}\frac{1}{\sqrt{1-x^2}} dx \\
&= \frac{2s}{\pi}\int_0^1 \frac{1}{\left(s^2+\alpha^2x^2\right)\sqrt{1-x^2}} dx \\
&= \frac{2s}{\pi}\frac{\alpha\frac{\pi}{s}}{2\alpha\sqrt{\alpha^2+s^2}} \\
&= \frac{1}{\sqrt{\alpha^2+s^2}}
\end{aligned}$$

But we did the Laplace transform of $J_0(\alpha t)$, which is the same as $J_0(\alpha x)$ and to get Laplace transform of $J_0(x)$, we just need to set $\alpha = 1$ in the above result, which gives

$$\hat{J}_0(s) = \frac{1}{\sqrt{1+s^2}}$$

2.7.5 Problem 4

4. (5 pts) A reasonably accurate description of the atomic contribution to the dielectric function is

$$\epsilon(\omega) = 1 + \omega_P^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

There are f_j electrons per molecule with binding frequency ω_j and damping constant γ_j. The oscillator strengths f_j obey the sum rule $\sum_j f_j = Z$ which is the total number of electrons per molecule. Using the imaginary part of ϵ in the dispersion relation, show that the real part is correctly reproduced.

Figure 2.28: Problem statement

Solution

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

It is enough to work with one term in the sum above and verify what is being asked on that term. Then it will be valid for the sum. Hence we will use the following as the starting relation

$$\begin{aligned}\epsilon(\omega) &= 1 + \frac{\omega_p^2 f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega} \qquad j = 1,2,3,\cdots \\ &= 1 - \frac{\omega_p^2 f_j}{\left(\omega^2 - \omega_j^2\right) - 2i\gamma_j\omega} \qquad (1)\end{aligned}$$

It is assumed that γ is much smaller than ω. In the above ω is the variable quantity and $\omega_j, \omega_p, \gamma_j$ are given parameters with known values for the problem

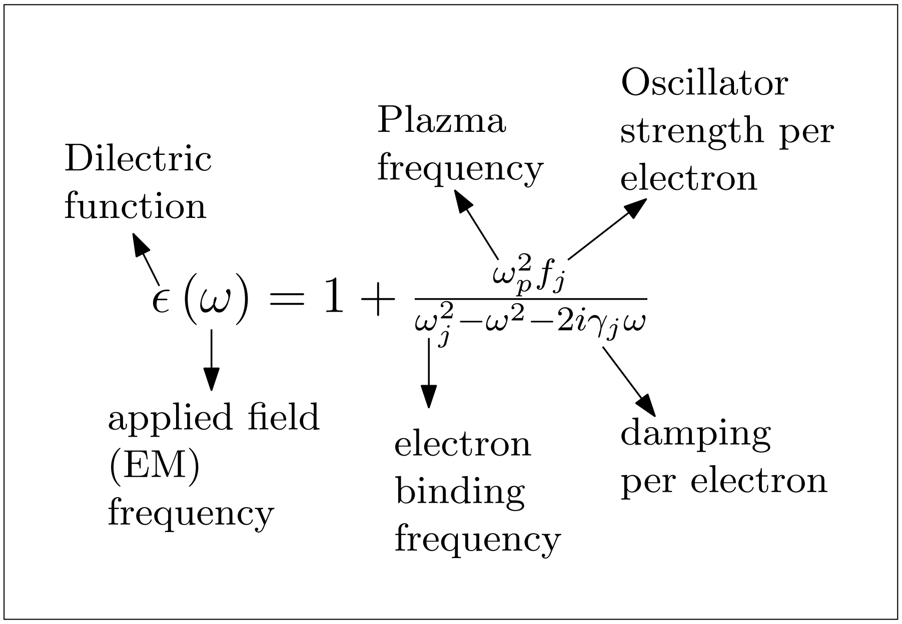


Figure 2.29: Physical meaning of terms involved

The real and imaginary parts are found by multiplying numerator and denominator by complex conjugate of denominator

$$
\begin{aligned}
\epsilon(\omega) &= 1 - \frac{\omega_p^2 f_j}{\left(\left(\omega^2-\omega_j^2\right) - 2i\gamma_j\omega\right)} \frac{\left(\omega^2-\omega_j^2\right) + 2i\gamma_j\omega}{\left(\left(\omega^2-\omega_j^2\right) + 2i\gamma_j\omega\right)} \\
&= 1 - \frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right) + 2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2 - \left(2i\gamma_j\omega\right)^2} \\
&= 1 - \frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right) + 2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma_j^2\omega^2} \\
&= 1 - \frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma_j^2\omega^2} - \frac{2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma_j^2\omega^2} \\
&= \left(1 - \frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma_j^2\omega^2}\right) - i\left(\frac{2\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma^2\omega^2}\right)
\end{aligned}
$$

Hence we see that

$$
\operatorname{Re}\left(\epsilon(\omega)\right) = 1 - \frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma_j^2\omega^2} \qquad j = 1,2,3,\cdots \tag{1}
$$

$$
\operatorname{Im}\left(\epsilon(\omega)\right) = -\frac{2\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2 + 4\gamma^2\omega^2} \qquad j = 1,2,3,\cdots \tag{2}
$$

Now, the dispersion relations for the above are, as derived in class notes

$$\operatorname{Re}(\epsilon(\omega)) = 1 + \frac{1}{\pi}(P.V.)\int_{-\infty}^{\infty}\frac{\operatorname{Im}(\epsilon(\omega'))}{\omega'-\omega}d\omega' \tag{3}$$

$$\operatorname{Im}(\epsilon(\omega)) = -\frac{1}{\pi}(P.V.)\int_{-\infty}^{\infty}\frac{\operatorname{Re}(\epsilon(\omega'))}{\omega'-\omega}d\omega' \tag{4}$$

The question is asking to use (2) in (3) in order to obtain and verify (1).

Substituting (2) into (3) gives

$$\begin{aligned}\operatorname{Re}(\epsilon(\omega)) &= 1 - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega'-\omega}\overbrace{\left(\frac{2\omega_p^2\gamma_j\omega' f_j}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}\right)}^{\operatorname{Im}(\epsilon(\omega'))}d\omega' \\ &= 1 - \frac{2\gamma\omega_p^2 f_j}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega'-\omega)}\left(\frac{\omega'}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}\right)d\omega' \end{aligned}\tag{5}$$

To find the poles in (5), it is easier to start from the original function

$$\frac{\omega_p^2 f_j}{\omega^2 - 2i\gamma_j\omega - \omega_j^2}$$

The roots of the denominator are $r_{1,2} = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} = \frac{2i\gamma_j}{2} \pm \frac{1}{2}\sqrt{\left(-2i\gamma_j\right)^2+4\omega_j^2} = i\gamma_j \pm \frac{1}{2}\sqrt{-4\gamma_j^2+4\omega_j^2} = i\gamma_j \pm \sqrt{\omega_j^2-\gamma_j^2}$. Hence after multiplying by the complex conjugate as we did above, we obtain the new term which is $\omega^2 + 2i\gamma_j\omega - \omega_j^2$. This one has roots $r_{3,4} = \frac{-2i\gamma_j}{2} \pm \frac{1}{2}\sqrt{\left(2i\gamma_j\right)^2+4\omega_j^2} = -i\gamma_j \pm \sqrt{\omega_j^2-\gamma_j^2}$. Therefore, we see that the poles for the term $\frac{\omega'}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}$ are

$$\begin{aligned} r_1 &= i\gamma_j + \sqrt{\omega_j^2-\gamma_j^2} \\ r_2 &= i\gamma_j - \sqrt{\omega_j^2-\gamma_j^2} \\ r_3 &= -i\gamma_j + \sqrt{\omega_j^2-\gamma_j^2} \\ r_4 &= -i\gamma_j - \sqrt{\omega_j^2-\gamma_j^2} \end{aligned}$$

We now need to handle the term $\frac{1}{(\omega'-\omega)}$ in (5) in order to find all the poles. To do this, we use

$$\begin{aligned}\frac{1}{\omega'-\omega-i\Delta} &= \frac{1}{\omega'-\omega} + i\pi\delta(\omega'-\omega) \\ \frac{1}{\omega'-\omega+i\Delta} &= \frac{1}{\omega'-\omega} - i\pi\delta(\omega'-\omega)\end{aligned}$$

Where Δ is very small quantity. Adding the above two equations gives

$$\frac{1}{\omega' - \omega - i\Delta} + \frac{1}{\omega' - \omega + i\Delta} = \frac{2}{\omega' - \omega}$$
$$\frac{1}{\omega' - \omega} = \frac{1}{2}\left(\frac{1}{\omega' - (\omega + i\Delta)} + \frac{1}{\omega' - (\omega - i\Delta)}\right)$$

Where in the above final steps we let $\Delta^n \to 0$ for $n > 1$ since Δ is very small. The above is what we will use in (6). Hence (5) becomes

$$\begin{aligned}
\operatorname{Re}(\epsilon(\omega)) &= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\omega' - (\omega + i\Delta)} + \frac{1}{\omega' - (\omega - i\Delta)}\right)\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\frac{\omega' - (\omega - i\Delta) + \omega' - (\omega + i\Delta)}{(\omega' - (\omega + i\Delta))(\omega' - (\omega - i\Delta))}\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\frac{(2\omega' - 2\omega)}{(\omega' - (\omega + i\Delta))(\omega' - (\omega - i\Delta))}\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{\pi}\int_{-\infty}^{\infty}\frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)(\omega' - r_6)(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}d\omega'
\end{aligned} \tag{5A}$$

There are 6 poles in total

$$\begin{aligned}
r_1 &= i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_2 &= i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} \\
r_3 &= -i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_4 &= -i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} \\
r_5 &= \omega + i\Delta \\
r_6 &= \omega - i\Delta
\end{aligned}$$

Three of the above poles are in lower half plane, and three are in the upper half plane. Here is a diagram which shows the location of the poles. Recalling that Δ is small quantity.

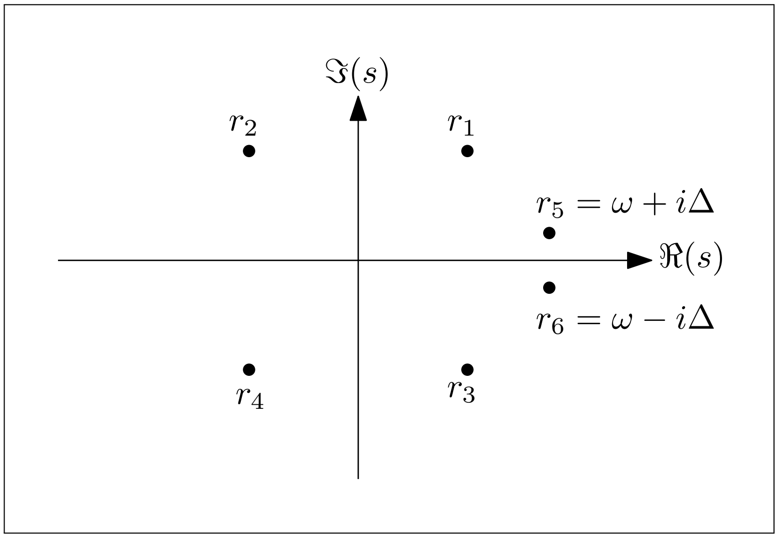


Figure 2.30: Location of the 6 poles

We will use the following contour

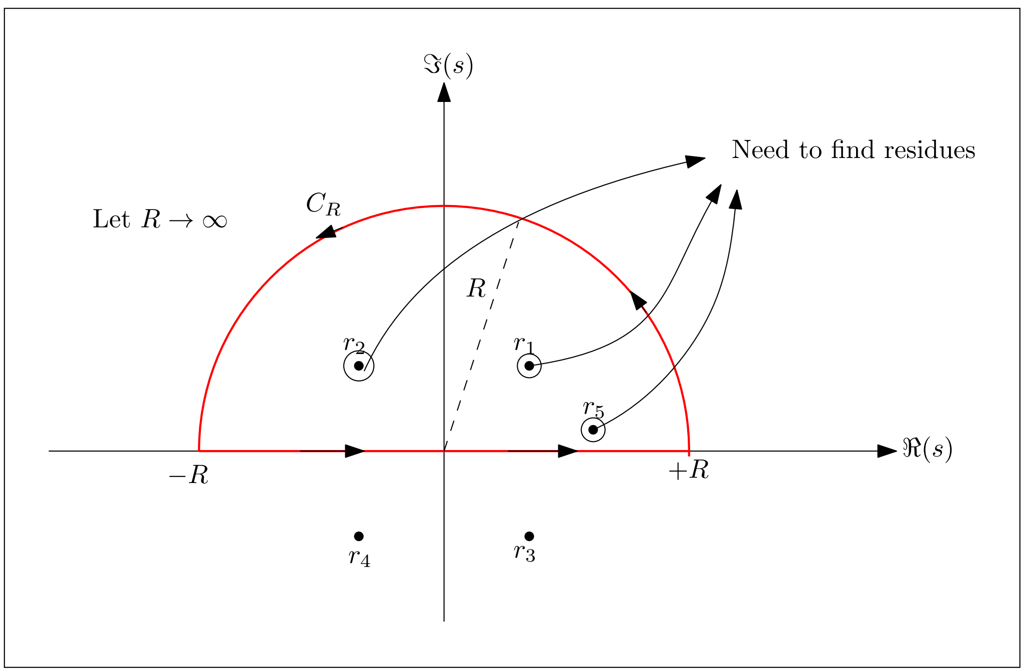


Figure 2.31: Countour used for the integral

The integrand which is a function of ω' is analytic except for the 3 poles in the upper half. Let the integrand be $g(\omega')$, then using residue theorem gives

$$\begin{aligned}\lim_{R\to\infty}\oint g(z)\,dz &= \lim_{R\to\infty}(P.V.)\int_{-R}^{R} g(\omega')\,d\omega' + \lim_{R\to\infty}\int_{C_R} g(z)\,dz\\ &= 2\pi i\sum \text{Residue}\end{aligned}$$

Hence

$$\lim_{R\to\infty}(P.V.)\int_{-R}^{R} g(\omega')\,d\omega' = 2\pi i\sum \text{Residue} - \lim_{R\to\infty}\int_{C_R} g(z)\,dz$$

Since the denominator in (5A) has higher powers of ω' than in the numerator (6^{th} order vs. 2^{nd} order), then this shows that $\lim_{R\to\infty}\int_{C_R} g(z)\,dz \to 0$, and the above reduces to

$$(P.V.)\int_{-\infty}^{\infty} g(\omega')\,d\omega' = 2\pi i\sum \text{Residue} \tag{8}$$

Therefore we just need to find the three residues at r_1, r_2, r_5 in order to find the integral above.

$$\begin{aligned}
\text{Residue}(r_1) &= \lim_{\omega'\to r_1}(\omega'-r_1)\frac{(\omega')^2-\omega\omega'}{(\omega'-(\omega+i\Delta))(\omega'-(\omega-i\Delta))(\omega'-r_1)(\omega'-r_2)(\omega'-r_3)(\omega'-r_4)}\\
&= \lim_{\omega'\to r_1}\frac{(\omega')^2-\omega\omega'}{(\omega'-(\omega+i\Delta))(\omega'-(\omega-i\Delta))(\omega'-r_2)(\omega'-r_3)(\omega'-r_4)}\\
&= \frac{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-r_2\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-r_3\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-r_4\right)}\\
&= \frac{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-\left(-i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-\left(-i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\right)}\\
&= \frac{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}+i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}+i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}
\end{aligned}$$

Hence

$$\text{Residue}(r_1) = \frac{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(2\sqrt{\omega_j^2-\gamma_j^2}\right)\left(2i\gamma_j\right)\left(2i\gamma_j+2\sqrt{\omega_j^2-\gamma_j^2}\right)} \tag{9}$$

And

$$\begin{aligned}
\text{Residue}(r_2) &= \lim_{\omega'\to r_2}(\omega'-r_2)\frac{(\omega')^2-\omega\omega'}{(\omega'-(\omega+i\Delta))(\omega'-(\omega-i\Delta))(\omega'-r_1)(\omega'-r_2)(\omega'-r_3)(\omega'-r_4)}\\
&= \lim_{\omega'\to r_2}\frac{(\omega')^2-\omega\omega'}{(\omega'-(\omega+i\Delta))(\omega'-(\omega-i\Delta))(\omega'-r_1)(\omega'-r_3)(\omega'-r_4)}\\
&= \frac{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-r_1\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-r_3\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-r_4\right)}\\
&= \frac{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-\left(i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-\left(-i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-\left(-i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\right)}\\
&= \frac{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)^2-\omega\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)}{\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega+i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-(\omega-i\Delta)\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}-i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}+i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}\right)\left(i\gamma_j-\sqrt{\omega_j^2-\gamma_j^2}+i\gamma_j+\sqrt{\omega_j^2-\gamma_j^2}\right)}
\end{aligned}$$

Hence

$$\operatorname{Residue}(r_2) = \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(-2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j - 2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)} \tag{10}$$

And finally

$$\begin{aligned}
\operatorname{Residue}(r_5) &= \lim_{\omega' \to r_5} (\omega' - r_5) \frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)(\omega' - r_6)(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)} \\
&= \lim_{\omega' \to r_5} \frac{(\omega')^2 - \omega\omega'}{(\omega' - r_6)(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)} \\
&= \frac{(\omega + i\Delta)^2 - \omega(\omega + i\Delta)}{\left(\omega + i\Delta - \left(\omega_j - i\Delta\right)\right)(\omega + i\Delta - r_1)(\omega + i\Delta - r_2)(\omega + i\Delta - r_3)(\omega + i\Delta - r_4)} \\
&= \frac{\omega^2 - \Delta^2 + 2i\omega\Delta - \omega^2 - i\Delta\omega}{(2i\Delta)\left(\omega + i\Delta - \left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(-i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(-i\gamma_j - \sqrt{\omega \cdots}\right)\right)} \\
&= \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta - i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}
\end{aligned} \tag{11}$$

We found all residues for I. Hence

$$\int_{-\infty}^{\infty} \frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)(\omega' - r_6)(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)} d\omega' = 2\pi i \sum \operatorname{Residue}$$

Where $\sum \operatorname{Residue}$ is given by adding (9,10,11) giving

$$\begin{aligned}
\sum \operatorname{Residue} = &\frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)\left(2i\gamma_j + 2\sqrt{\omega_j^2 - \gamma_j^2}\right)} \\
&+ \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(-2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j - 2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)} \\
&+ \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta - i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}
\end{aligned}$$

Therefore (5A) becomes

$$\operatorname{Re}(\epsilon(\omega)) = 1 - \frac{\gamma\omega_p^2 f_j}{\pi}(2\pi i) \sum \operatorname{Residue}$$

To make some progress, I had to simplify the $\sum$Residue by assuming γ is very small compared to ω_j and hence terms such as $\sqrt{\omega_j^2-\gamma_j^2}\to\omega_j$. Using this gives

$$\begin{aligned}\sum \text{Residue} =& \frac{\left(i\gamma_j+\omega_j\right)^2-\omega\left(i\gamma_j+\omega_j\right)}{\left(i\gamma_j+\omega_j-(\omega+i\Delta)\right)\left(i\gamma_j+\omega_j-(\omega-i\Delta)\right)\left(2\omega_j\right)\left(2i\gamma_j\right)\left(2i\gamma_j+2\omega_j\right)}\\
&+\frac{\left(i\gamma_j-\omega_j\right)^2-\omega\left(i\gamma_j-\omega_j\right)}{\left(i\gamma_j-\omega_j-(\omega+i\Delta)\right)\left(i\gamma_j-\omega_j-(\omega-i\Delta)\right)\left(-2\omega_j\right)\left(2i\gamma_j-2\omega_j\right)\left(2i\gamma_j\right)}\\
&+\frac{i\omega\Delta-\Delta^2}{(2i\Delta)\left(\omega+i\Delta-i\gamma_j-\omega_j\right)\left(\omega+i\Delta-i\gamma_j+\omega_j\right)\left(\omega+i\Delta+i\gamma_j-\omega_j\right)\left(\omega+i\Delta+i\gamma_j+\omega_j\right)}\end{aligned}$$

Or

$$\begin{aligned}\sum \text{Residue} =& \frac{-\gamma_j^2+\omega_j^2+2i\gamma_j\omega_j-\omega i\gamma_j-\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j-2\omega\omega_j-\gamma_j^2+2i\gamma_j\omega_j+\omega_j^2\right)\left(8i\gamma_j\omega_j^2-8\gamma_j^2\omega_j\right)}\\
&+\frac{-\gamma_j^2+\omega_j^2-2i\gamma_j\omega_j-i\omega\gamma_j+\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j+2\omega\omega_j-\gamma_j^2-2i\gamma_j\omega_j+\omega_j^2\right)\left(8\gamma_j^2\omega_j+8i\gamma_j\omega_j^2\right)}\\
&+\frac{i\omega\Delta-\Delta^2}{(2i\Delta)\left(\omega+i\Delta-i\gamma_j-\omega_j\right)\left(\omega+i\Delta-i\gamma_j+\omega_j\right)\left(\omega+i\Delta+i\gamma_j-\omega_j\right)\left(\omega+i\Delta+i\gamma_j+\omega_j\right)}\end{aligned}$$

Expanding the denominator in 3rd term above, lots of terms cancel since they contain higher powers of Δ. Removing all terms that contain Δ^2 or higher gives

$$\begin{aligned}\sum \text{Residue} =& \frac{-\gamma_j^2+\omega_j^2+2i\gamma_j\omega_j-\omega i\gamma_j-\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j-2\omega\omega_j-\gamma_j^2+2i\gamma_j\omega_j+\omega_j^2\right)\left(8i\gamma_j\omega_j^2-8\gamma_j^2\omega_j\right)}\\
&+\frac{-\gamma_j^2+\omega_j^2-2i\gamma_j\omega_j-i\omega\gamma_j+\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j+2\omega\omega_j-\gamma_j^2-2i\gamma_j\omega_j+\omega_j^2\right)\left(8\gamma_j^2\omega_j+8i\gamma_j\omega_j^2\right)}\\
&+\frac{i\omega\Delta}{2i\Delta\omega^4+4i\Delta\omega^2\gamma_j^2-4i\Delta\omega^2\omega_j^2+2i\Delta\gamma_j^4+4i\Delta\gamma_j^2\omega_j^2+2i\Delta\omega_j^4}\end{aligned}$$

Removing terms that contain only γ_j^2 since γ_j is small gives

$$\begin{aligned}\sum \text{Residue} =& \frac{\omega_j^2+2i\gamma_j\omega_j-\omega i\gamma_j-\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j-2\omega\omega_j+2i\gamma_j\omega_j+\omega_j^2\right)\left(8i\gamma_j\omega_j^2-8\gamma_j^2\omega_j\right)}\\
&+\frac{\omega_j^2-2i\gamma_j\omega_j-i\omega\gamma_j+\omega\omega_j}{\left(\omega^2-2i\omega\gamma_j+2\omega\omega_j-2i\gamma_j\omega_j+\omega_j^2\right)\left(8\gamma_j^2\omega_j+8i\gamma_j\omega_j^2\right)}\\
&+\frac{\omega\Delta}{2\Delta\omega^4+4\Delta\omega^2\gamma_j^2-4\Delta\omega^2\omega_j^2+2\Delta\gamma_j^4+4\Delta\gamma_j^2\omega_j^2+2\Delta\omega_j^4}\end{aligned}$$

Canceling all terms with $\Delta\gamma_j^2, \Delta\gamma_j^4$ in them, since both are small gives

$$\begin{aligned}\sum \text{Residue} &= \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)} \\ &+ \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)} \\ &+ \frac{\omega\Delta}{2\Delta\omega^4 + 4\Delta\omega^2\gamma_j^2 - 4\Delta\omega^2\omega_j^2 + 2\Delta\omega_j^4}\end{aligned}$$

Canceling Δ in last term gives

$$\begin{aligned}\sum \text{Residue} &= \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)} \\ &+ \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)} \\ &+ \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}\end{aligned}$$

Expanding

$$\begin{aligned}\sum \text{Residue} &= \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{-8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 16i\omega\gamma_j^3\omega_j + 32\omega\gamma_j^2\omega_j^2 - 16i\omega\gamma_j\omega_j^3 - 16i\gamma_j^3\omega_j^2 - 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} \\ &+ \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 - 16i\omega\gamma_j^3\omega_j + 32\omega\gamma_j^2\omega_j^2 + 16i\omega\gamma_j\omega_j^3 - 16i\gamma_j^3\omega_j^2 + 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} \\ &+ \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}\end{aligned}$$

Removing terms with γ_j^3 and higher, since γ is small gives

$$\begin{aligned}\sum \text{Residue} &= \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{-8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 32\omega\gamma_j^2\omega_j^2 - 16i\omega\gamma_j\omega_j^3 - 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} \\ &+ \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 32\omega\gamma_j^2\omega_j^2 + 16i\omega\gamma_j\omega_j^3 + 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} \\ &+ \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}\end{aligned}$$

Or

$$\sum \text{Residue} = -\overbrace{\frac{\left(2i\omega^2\gamma_j^2 + i\omega^2\omega_j^2 + 2\omega\gamma_j\omega_j^2 - 6i\gamma_j^2\omega_j^2 - i\omega_j^4\right)}{4\gamma_j\left(\omega^2 - \omega_j^2\right)\left(-\omega^2\gamma_j^2 - \omega^2\omega_j^2 + 4i\omega\gamma_j\omega_j^2 + 9\gamma_j^2\omega_j^2 + \omega_j^4\right)}}^{\text{This term needs to be simplified. Error somewhere}} + \frac{\omega}{2\left(\omega^2 - \omega_j^2\right)^2 + 4\gamma_j^2\omega_j^2}$$

Hence the result becomes

$$\operatorname{Re}\left(\epsilon\left(\omega\right)\right)=1-\frac{\gamma\omega_p^2 f_j}{\pi}\left(2\pi i\right)\sum\text{Residue}$$

The above should come out to be as shown in (1) which is

$$\operatorname{Re}\left(\epsilon\left(\omega\right)\right)=1-\omega_p^2 f_j\frac{\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}$$

I was not able to fully simplify the first term in $\sum\text{Residue}$ above, I seem to have made an error somewhere and not able to find it now, but the second terms looks OK. All complex i terms should cancel out since the result must be real.

2.7.6 Key solution for HW 7

(1) $$I = \int_0^{\pi} dx \int_1^2 dy\, \delta(\sin x)\,\delta(x^2 - y^2)$$

First $$\int_1^2 dy\, \delta(x^2-y^2) = \frac{1}{\left|\frac{\partial}{\partial y}(x^2-y^2)\right|_{y=|x|}} = \frac{1}{2|x|}$$

This is true if $1<|x|<2$, otherwise the zero of the δ-function is outside the range of the y integration and so we would get 0.

$$I = \int_1^2 \frac{dx}{2x}\, \delta(\sin x) \underbrace{\theta(x-1)\,\theta(2-x)}_{\substack{1 \text{ if } 1<x<2 \\ 0 \text{ otherwise}}}$$

Now $\sin x \neq 0$ for $1 < x < 2$

Thus $\boxed{I = 0}$

(2) $$x(t) = \int_{-\infty}^{\infty} G(t-t')\,F(t')\,dt'$$

$$\tilde{x}(\omega) = \tilde{G}(\omega)\,\tilde{F}(\omega)$$

Input $F(t) = e^{-\lambda t}\theta(t)$ so that

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} e^{-\lambda t}\theta(t)\,dt = \int_{0}^{\infty} e^{-(\lambda+i\omega)t}\,dt = \frac{1}{\lambda+i\omega}$$

Output $x(t) = e^{-\lambda t} - e^{-(\alpha+\lambda)t}$ so that

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\left[e^{-\lambda t} - e^{-(\alpha+\lambda)t}\right]\theta(t)\,dt =$$

no output before input

$$= \frac{1}{\lambda+i\omega} - \frac{1}{\alpha+\lambda+i\omega} = \frac{\alpha}{(\lambda+i\omega)(\alpha+\lambda+i\omega)} = \tilde{G}(\omega)\,\underbrace{\tilde{F}(\omega)}_{\frac{1}{\lambda+i\omega}}$$

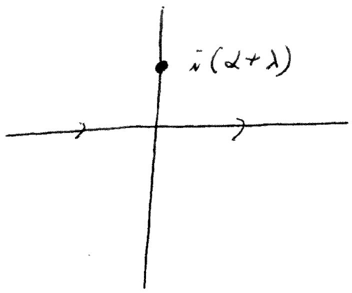
$$\boxed{\tilde{G}(\omega) = \frac{\alpha}{\alpha+\lambda+i\omega}}$$

When $F(t) = F_0\,\delta(t)$ we have

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F_0\,\delta(t)\,dt = F_0$$

$$\Rightarrow\quad x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\,\tilde{G}(\omega)\,\tilde{F}(\omega) = \frac{\alpha F_0}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega\, e^{i\omega t}}{\lambda+\alpha+i\omega}$$

This has a pole at $\omega = i(\alpha+\lambda)$



If $t<0$ we can add a semi-circle in the lower half plane. No pole is enclosed so $x(t<0) = 0$.

If $t>0$ we can add a semi-circle in the upper half plane. Residue theorem gives

$$\int_C \frac{d\omega\, e^{i\omega t}}{i\left[\omega - i(\alpha+\lambda)\right]} = \frac{2\pi i}{i} e^{i\left[i(\alpha+\lambda)\right]t} = 2\pi e^{-(\alpha+\lambda)t}$$

Then $\boxed{x(t) = \alpha F_0 e^{-(\alpha+\lambda)t} \theta(t)}$

(3) $$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\,d\theta$$

Laplace transform $$\hat{J}_0(s) = \int_0^\infty J_0(x) e^{-xs} dx =$$

$$= \int_0^\infty dx\, e^{-xs}\, \mathrm{Re} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ix\cos\theta} = \mathrm{Re}\int_0^{2\pi}\frac{d\theta}{2\pi}\underbrace{\int_0^\infty dx\, e^{-(s-i\cos\theta)x}}_{\frac{1}{s-i\cos\theta}}$$

$$= \mathrm{Re}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{s-i\cos\theta}$$

Use $z = e^{i\theta}$ $\quad dz = iz\,d\theta \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\hat{J}_0(s) = \frac{1}{2\pi}\mathrm{Re}\int_C \frac{dz}{iz}\frac{1}{s-\frac{i}{2}\left(z+\frac{1}{z}\right)} = \frac{1}{2\pi}\mathrm{Re}\int_C \frac{2\,dz}{z^2+2isz+1}$$

C ↑ unit circle

$$= \frac{1}{2\pi}\mathrm{Re}\int_C \frac{2\,dz}{(z-z_+)(z-z_-)} \qquad z_\pm = -is \pm i\sqrt{s^2+1}$$

One can check that z_+ lies inside the unit circle while z_- lies outside. Use residue theorem.

$$\hat{J}_0(s) = \frac{1}{\pi}\mathrm{Re}\frac{2\pi i}{z_+ - z_-} = \frac{1}{\pi}\mathrm{Re}\frac{2\pi i}{2i\sqrt{s^2+1}}$$

$$\boxed{\hat{J}_0(s) = \frac{1}{\sqrt{s^2+1}}}$$

4 $$\epsilon(\omega) - 1 = \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

$$\text{Im}\,\epsilon(\omega) = 2\omega_p^2 \sum_j \frac{f_j \gamma_j \omega}{(\omega^2-\omega_j^2)^2 + 4\gamma_j^2\omega^2}$$

$$\text{Re}\,\epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega^2-\omega_j^2)^2 + 4\gamma_j^2\omega^2}$$

The dispersion relation is

$$\text{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\text{Im}\,\epsilon(\omega')\,d\omega'}{\omega' - \omega}$$

The poles are located in both upper & lower half planes.

When $\gamma_j < \omega_j$ they are located at

$\pm \nu_j \pm i\gamma_j$ where $\nu_j = \sqrt{\omega_j^2 - \gamma_j^2}$.

Because $\text{Im}\,\epsilon(-\omega) = -\text{Im}\,\epsilon(\omega)$ we can rewrite the formula as

$$\text{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi} P\int_0^{\infty} d\omega'\,\text{Im}\,\epsilon(\omega')\left[\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega}\right]$$

which shows that $\text{Re}\,\epsilon(\omega)$ is even in ω.

If we had $\text{Im}\,\epsilon(\omega)$ for $\omega \geq 0$ in numerical form we would just evaluate the integral numerically. But we have an analytical expression for $\text{Im}\,\epsilon(\omega)$. One way to proceed would be to use the method of partial fractions. The denominator is a fifth order polynomial (we know all the roots) so that method would be extremely tedious. Instead we proceed as follows. In class we derived

$$\frac{1}{\omega'-\omega-i\varepsilon} = P\frac{1}{\omega'-\omega} + i\pi\,\delta(\omega'-\omega)$$

assuming that ω' and ω are real. We can take the complex conjugate to get

$$\frac{1}{\omega'-\omega+i\varepsilon} = P\frac{1}{\omega'-\omega} - i\pi\,\delta(\omega'-\omega).$$

This formula can also be derived similar to what we did in class. Add them together

$$P\frac{1}{\omega'-\omega} = \frac{1}{2}\left[\frac{1}{\omega'-\omega+i\varepsilon} + \frac{1}{\omega'-\omega-i\varepsilon}\right]$$

The RHS can now be analytically continued in ω'.

Then we get

$$Re\ \epsilon(\omega) - 1 = \frac{\omega_p^2}{\pi} \sum_j f_j \gamma_j \int_{-\infty}^{\infty} \frac{d\omega' \omega'}{(\omega'^2 - \omega_j^2)^2 + 4\gamma_j^2 \omega'^2}$$

$$\cdot \left[\frac{1}{\omega' - \omega + i\epsilon} + \frac{1}{\omega' - \omega - i\epsilon} \right]$$

close contour in upper half plane for convenience

close contour in lower half plane for convenience

Then application of the residue theorem gives

$$\int_C \frac{d\omega' \ \omega'}{(\omega' - \nu_j - i\gamma_j)(\omega' + \nu_j - i\gamma_j)(\omega' - \nu_j + i\gamma_j)(\omega' + \nu_j + i\gamma_j)(\omega' - \omega + i\epsilon)}$$

$$= \frac{-\pi}{2\gamma_j} \frac{1}{\omega^2 - \omega_j^2 - i2\gamma_j\omega}$$ while the term with

$$\frac{1}{\omega' - \omega - i\epsilon}$$ gives $$\frac{-\pi}{2\gamma_j} \frac{1}{\omega^2 - \omega_j^2 + i2\gamma_j\omega}.$$

Their sum is $$\frac{-\pi}{\gamma_j} \frac{\omega^2 - \omega_j^2}{(\omega^2 - \omega_j^2)^2 + 4\gamma_j^2\omega^2}$$

Putting this back into the dispersion relation gives

$$\operatorname{Re} \epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega^2 - \omega_j^2)^2 + 4\gamma_j^2 \omega^2}$$

which agrees with the original expression for $\epsilon(\omega)$.

2.8 HW 8

Local contents

2.8.1 HW 8 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 8 due Monday April 8. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (10 pts) Prove the following relations.

$$\begin{aligned}(AB)^T &= B^T A^T\\ (AB)^\dagger &= B^\dagger A^\dagger\\ \mathrm{Tr}(AB) &= \mathrm{Tr}(BA)\\ \det A^T &= \det A\\ \det(AB) &= \det(A)\cdot\det(B)\end{aligned}$$

For the last one you may assume that A and B are diagonal.

2. (7 pts) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}\end{pmatrix}$$

3. (5 pts) Let U be a unitary matrix and let x_1 and x_2 be two eigenvectors of U with eigenvalues λ_1 and λ_2, respectively. Show that $|\lambda_1| = |\lambda_2| = 1$. Also show that if $\lambda_1 \neq \lambda_2$ then $x_1^\dagger x_2 = 0$.

4. (3 pts) Calculate the determinant of the sparse matrix (sparse means that most of the entries are zero)

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0\\ i & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & i\\ 0 & 0 & 0 & i & 1\end{pmatrix}$$

2.8.2 Problem 1

1. (10 pts) Prove the following relations.

$$
\begin{aligned}
(AB)^T &= B^T A^T \\
(AB)^\dagger &= B^\dagger A^\dagger \\
\mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \\
\det A^T &= \det A \\
\det(AB) &= \det(A) \cdot \det(B)
\end{aligned}
$$

For the last one you may assume that A and B are diagonal.

Figure 2.32: Problem statement

2.8.2.1 part 1 $(AB)^T = B^T A^T$

Let A be an n,m matrix and B be an m,p matrix. Hence $AB = C$ is an n,p matrix. By definition of matrix product which is rows of A multiply columns of B then the ij element of C is

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Then $(AB)^T = C^T$. Hence from above, elements of C^T are given by

$$c_{ji} = \sum_{k=1}^{m} a_{jk} b_{ki} \tag{1}$$

Now let $B^T A^T = Q$. Where now B^T is order $p \times m$ and A^T is order $m \times n$, hence Q is $p \times n$.

$$
\begin{aligned}
q_{ij} &= \sum_{k=1}^{m} (b_{ik})^T \left(a_{kj}\right)^T \\
&= \sum_{k=1}^{m} b_{ki} a_{jk}
\end{aligned}
$$

But $\sum b_{ki} a_{jk}$ means to multiply column i of B by row j in A, which is the same as multiplying row j of A by column i of B. Hence we can change the order of multiplication above as

$$q_{ij} = \sum_{k=1}^{m} a_{jk} b_{ki} \tag{2}$$

Comparing (1) and (2) shows they are the same. Hence

$$C^T = Q$$

Or

$$(AB)^T = B^T A^T$$

2.8.2.2 Part 2 $(AB)^\dagger = B^\dagger A^\dagger$

By definition $A^\dagger = \left(A^T\right)^*$. Which means we take the transpose of A and then apply complex conjugate to its entries. Hence the solution follows the above, but we just have to apply complex conjugate at the end of each operation

Let A be an $n \times m$ matrix and B be $m \times p$ matrix. Hence $AB = C$ which is $n \times p$ matrix. By definition of matrix product which is row of A multiplies columns of B then the ij element of C is

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Then $(AB)^\dagger_{ij} = \left(C^T_{ij}\right)^* = c^*_{ji}$. Hence from above

$$c^*_{ji} = \sum_{k=1}^{m} \left(a_{jk} b_{ki}\right)^*$$

But complex conjugate of product is same as product of complex conjugates, hence the above is same as

$$c^*_{ji} = \sum_{k=1}^{m} a^*_{jk} b^*_{ki} \tag{1}$$

Now let $B^\dagger A^\dagger = Q$. Then

$$\begin{aligned} q_{ij} &= \sum_{k=1}^{m} \left(b^T_{ik}\right)^* \left(a^T_{kj}\right)^* \\ &= \sum_{k=1}^{m} b^*_{ki} a^*_{jk} \end{aligned}$$

But $\sum_{k=1}^{m} b^*_{ki} a^*_{jk}$ means to multiply complex conjugate of column i of B by complex conjugate of row j in A, which is the same as multiplying complex conjugate complex of row j of A by complex conjugate of column i of B. Hence the above can be written as

$$q_{ij} = \sum_{k=1}^{m} a^*_{jk} b^*_{ki} \tag{2}$$

Comparing (1) and (2) shows they are the same. Hence

$$\left(C^T\right)^* = Q$$

Or

$$(AB)^\dagger = B^\dagger A^\dagger$$

2.8.2.3 Part 3 $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

The trace Tr of a matrix is the sum of elements on the diagonal matrix (and this applies only to square matrices). Let A be $n \times m$ And B be an $m \times n$ matrix. Hence AB is $n \times n$ matrix and BA is $m \times m$ matrix.

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} b_{ji} \right) \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} b_{ji} a_{ij} \right) \\
&= \sum_{j=1}^{m} (BA)_{jj} \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

2.8.2.4 Part 4 $\det\left(A^T\right) = \det A$

Proof by induction. Let base be $n = 1$. Hence $A_{1\times 1}$. It is clear that $\det(A) = \det\left(A^T\right)$ in this case. We could also have selected base case to be $n = 2$. Any base case will work in proof by induction.

We now assume it is true for the $n-1$ case. i.e. $\det\left(A_{(n-1)\times(n-1)}\right) = \det\left(A^T_{(n-1)\times(n-1)}\right)$ is assumed to be true. This is called the induction hypothesis step.

We need now to show it is true for the case of n, i.e. we need to show that $\det(A_{n\times n}) = \det\left(A^T_{n\times n}\right)$. Let

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Therefore

$$A^T_{n\times n} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Now we take $\det(A)$ and expand using cofactors along the first row which gives

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + \cdots + (-1)^{n+1} a_{1n} \det(A_{1n}) \tag{1}$$

Where A_{ij} in the above means the matrix of dimensions $(n-1, n-1)$ taken from $A_{n\times n}$ by removing the i^{th} row and the j^{th} column. Now we do the same for A^T above, but instead of expanding using first row, we expend using first column of A^T since we can pick any row or any column to expand around in order find the determinant. This gives

$$\det\left(A^T\right) = a_{11}\det\left(A^T\right)_{11} - a_{12}\det\left(A^T\right)_{21} + \cdots + (-1)^{n+1} a_{1n}\det\left(A^T\right)_{n1} \tag{2}$$

For (1) to be the same as (2) we need to show that $\det(A_{11}) = \det\left(A^T\right)_{11}$ and $\det(A_{12}) = \det\left(A^T\right)_{21}$ and all the way to $\det(A_{1n}) = \det\left(A^T\right)_{n1}$. But this is true by assumption. Since we assumed that $\det\left(A_{(n-1)\times(n-1)}\right) = \det\left(A^T_{(n-1)\times(n-1)}\right)$. In other words, by the induction hypothesis $\det\left(A_{ij}\right) = \det\left(A^T\right)_{ji}$ since both are $(n-1)\times(n-1)$ order. Hence (1) is the same as (2). This completes the proof.

2.8.2.5 Part 5 $\det(AB) = \det(A)\det(B)$

Since the matrices are diagonal they must be square. And since product AB is defined, then they must both be same dimension, say $n \times n$.

Since A, B are diagonal, then

$$\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_i^n a_{ii}$$
$$\det(B) = b_{11}b_{22}\cdots b_{nn} = \prod_i^n b_{jj}$$

Now since A, B are diagonals, then the product is diagonal. Using definition of a row from A multiplies a column in B, we get

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}\begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 & 0 & 0 \\ 0 & a_{22}b_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn}b_{nn} \end{pmatrix}$$

Then we see that

$$\begin{aligned} \det(AB) &= (a_{11}b_{11})(a_{22}b_{22})\cdots(a_{nn}b_{nn}) \\ &= (a_{11}a_{22}\cdots a_{nn})(b_{11}b_{22}\cdots b_{nn}) \\ &= \prod_i^n a_{ii}\prod_i^n b_{jj} \\ &= \det(A)\det(B) \end{aligned}$$

2.8.3 Problem 2

2. (7 pts) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix}$$

Figure 2.33: Problem statement

We first need to find the eigenvalues λ by solving

$$\det\left(A - \lambda I\right) = 0$$

The above gives a polynomial of order 3.

$$\left|\begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} \frac{5}{2} - \lambda & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} - \lambda & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{5}{2} - \lambda\right)\begin{vmatrix} \frac{7}{3} - \lambda & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{1}{18}} & \frac{13}{6} - \lambda \end{vmatrix} - \sqrt{\frac{3}{2}}\begin{vmatrix} \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \frac{13}{6} - \lambda \end{vmatrix} + \sqrt{\frac{3}{4}}\begin{vmatrix} \sqrt{\frac{3}{2}} & \frac{7}{3} - \lambda \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} \end{vmatrix} = 0$$

Hence

$$\begin{aligned}\left(\frac{5}{2} - \lambda\right)\left(\left(\frac{7}{3} - \lambda\right)\left(\frac{13}{6} - \lambda\right) - \sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}\right) \\ - \sqrt{\frac{3}{2}}\left(\sqrt{\frac{3}{2}}\left(\frac{13}{6} - \lambda\right) - \sqrt{\frac{1}{18}}\sqrt{\frac{3}{4}}\right) \\ + \sqrt{\frac{3}{4}}\left(\sqrt{\frac{3}{2}}\sqrt{\frac{1}{18}} - \left(\frac{7}{3} - \lambda\right)\sqrt{\frac{3}{4}}\right) = 0\end{aligned}$$

Or

$$
\begin{aligned}
\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)-\sqrt{\frac{3}{2}}\left(\sqrt{6}-\frac{1}{2}\sqrt{2}\sqrt{3}\lambda\right)+\sqrt{\frac{3}{4}}\left(\sqrt{3}\left(\frac{1}{2}\lambda-1\right)\right) &= 0\\
\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)+\left(\frac{3}{2}\lambda-3\right)+\left(\frac{3}{4}\lambda-\frac{3}{2}\right) &= 0\\
\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)+\frac{9}{4}\lambda-\frac{9}{2} &= 0\\
-\lambda^3+7\lambda^2-14\lambda+8 &= 0\\
\lambda^3-7\lambda^2+14\lambda-8 &= 0
\end{aligned}
$$

By inspection we see that $\lambda = 2$ is a root. Then by long division $\frac{\lambda^3-7\lambda^2+14\lambda-8}{\lambda-2} = \lambda^2-5\lambda+4$. Therefore the above polynomial can be written as

$$
\begin{aligned}
\left(\lambda^2-5\lambda+4\right)(\lambda-2) &= 0\\
(\lambda-1)(\lambda-4)(\lambda-2) &= 0
\end{aligned}
$$

Hence the eigenvalues are

$$
\begin{aligned}
\lambda_1 &= 1\\
\lambda_2 &= 2\\
\lambda_3 &= 4
\end{aligned}
$$

For each eigenvalue there is one corresponding eigenvector (unless it is degenerate). The eigenvectors are found by solving the following

$$
\begin{aligned}
Av_i &= \lambda_i v_i\\
(A-\lambda_i I)\, v_i &= 0\\
\begin{pmatrix} \frac{5}{2}-\lambda_i & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3}-\lambda_i & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-\lambda_i \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}_i &= \begin{pmatrix}0\\0\\0\end{pmatrix}
\end{aligned}
$$

<u>For $\lambda_1 = 1$</u>

$$
\begin{aligned}
\begin{pmatrix} \frac{5}{2}-1 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3}-1 & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-1 \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix} \frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{4}{3} & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{7}{6} \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}
\end{aligned}
$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} \frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{4}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{7}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$\begin{aligned} \frac{3}{2} + \sqrt{\frac{3}{2}} v_2 + \sqrt{\frac{3}{4}} v_3 &= 0 \\ \sqrt{\frac{3}{2}} + \frac{4}{3} v_2 + \sqrt{\frac{1}{18}} v_3 &= 0 \end{aligned}$$

From the first equation above

$$v_2 = \frac{-\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \tag{4}$$

Substituting in the second equation gives

$$\begin{aligned} \sqrt{\frac{3}{2}} + \frac{4}{3}\left(\frac{-\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \right) + \sqrt{\frac{1}{18}} v_3 &= 0 \\ -\frac{1}{2}\sqrt{2} v_3 - \frac{1}{6}\sqrt{2}\sqrt{3} &= 0 \\ v_3 &= -\frac{\frac{1}{6}\sqrt{2}\sqrt{3}}{\frac{1}{2}\sqrt{2}} \\ &= -\frac{2\sqrt{3}}{6} \\ &= -\frac{\sqrt{3}}{3} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

Hence from (4)

$$\begin{aligned} v_2 &= \frac{-\frac{3}{2} - \sqrt{\frac{3}{4}}\left(-\frac{1}{\sqrt{3}}\right)}{\sqrt{\frac{3}{2}}} \\ &= -\frac{\sqrt{2}}{\sqrt{3}} \end{aligned}$$

Therefore the eigenvector associated with $\lambda_1 = 1$ is $\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$ or by scaling it all by $-\sqrt{3}$ it becomes

$$\vec{v}_1 = \begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

We now do the same for the second eigenvalue.

For $\lambda_2 = 2$

$$\begin{pmatrix} \frac{5}{2} - 2 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} - 2 & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} - 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$\frac{1}{2} + \sqrt{\frac{3}{2}} v_2 + \sqrt{\frac{3}{4}} v_3 = 0$$
$$\sqrt{\frac{3}{2}} + \frac{1}{3} v_2 + \sqrt{\frac{1}{18}} v_3 = 0$$

From the first equation above

$$v_2 = \frac{-\frac{1}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \tag{4A}$$

Substituting in the second equation gives

$$\sqrt{\frac{3}{2}}+\frac{1}{3}\left(\frac{-\frac{1}{2}-\sqrt{\frac{3}{4}}v_3}{\sqrt{\frac{3}{2}}}\right)+\sqrt{\frac{1}{18}}v_3=0$$

$$\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{18}}v_3-\frac{1}{18}\sqrt{2}\sqrt{3}+\sqrt{\frac{1}{18}}v_3=0$$

$$0=\sqrt{\frac{3}{2}}+\frac{1}{18}\sqrt{2}\sqrt{3}$$

This is not possible. So out choice of setting $v_1=1$ does not work. Let us try to set $v_2=1$ and repeat the process

$$\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}v_1\\ 1\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Again, we only need the first two equations. This results in

$$\frac{1}{2}v_1+\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{4}}v_3=0$$

$$\sqrt{\frac{3}{2}}v_1+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3=0$$

From the first equation above

$$v_1=\frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}v_3}{\frac{1}{2}} \tag{4A}$$

Substituting in the second equation gives

$$
\begin{aligned}
\sqrt{\frac{3}{4}}\left(\frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}v_3}{\frac{1}{2}}\right)+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3 &= 0\\
-\frac{3}{2}v_3-\frac{3}{2}\sqrt{2}+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3 &= 0\\
\frac{1}{6}\sqrt{2}v_3-\frac{3}{2}v_3-\frac{3}{2}\sqrt{2}+\frac{1}{3} &= 0\\
v_3\left(\frac{1}{6}\sqrt{2}-\frac{3}{2}\right) &= \frac{3}{2}\sqrt{2}-\frac{1}{3}\\
v_3 &= \frac{\frac{3}{2}\sqrt{2}-\frac{1}{3}}{\frac{1}{6}\sqrt{2}-\frac{3}{2}}\\
&= -\sqrt{2}
\end{aligned}
$$

Hence from (4A) $v_1=\frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}(-\sqrt{2})}{\frac{1}{2}}=\frac{-\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}}}{\frac{1}{2}}=0$. Therefore the eigenvector associated with $\lambda_2=2$ is $\begin{pmatrix}0\\1\\-\sqrt{2}\end{pmatrix}$ or by scaling it all by $-\frac{1}{\sqrt{2}}$ it becomes

$$
\vec{v}_2=\begin{pmatrix}0\\-\frac{1}{\sqrt{2}}\\1\end{pmatrix}
$$

We now do the same for the final eigenvalue

<u>For $\lambda_3=4$</u>

$$
\begin{pmatrix}\frac{5}{2}-4 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3}-4 & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-4\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}
$$

$$
\begin{pmatrix}-\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & -\frac{5}{3} & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & -\frac{11}{6}\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}
$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} -\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & -\frac{5}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & -\frac{11}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$\begin{aligned} -\frac{3}{2} + \sqrt{\frac{3}{2}} v_2 + \sqrt{\frac{3}{4}} v_3 &= 0 \\ \sqrt{\frac{3}{2}} - \frac{5}{3} v_2 + \sqrt{\frac{1}{18}} v_3 &= 0 \end{aligned}$$

From the first equation above

$$v_2 = \frac{\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \tag{4B}$$

Substituting in the second equation gives

$$\begin{aligned} \sqrt{\frac{3}{2}} - \frac{5}{3}\left(\frac{\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}}\right) + \sqrt{\frac{1}{18}} v_3 &= 0 \\ \frac{5}{6}\sqrt{2} v_3 - \frac{1}{3}\sqrt{2}\sqrt{3} + \sqrt{\frac{1}{18}} v_3 &= 0 \\ \sqrt{2} v_3 - \frac{1}{3}\sqrt{2}\sqrt{3} &= 0 \\ v_3 &= \frac{\frac{1}{3}\sqrt{2}\sqrt{3}}{\sqrt{2}} \\ &= \frac{1}{3}\sqrt{3} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

Hence from (4B) $v_2 = \frac{\frac{3}{2} - \sqrt{\frac{3}{4}}\left(\frac{1}{\sqrt{3}}\right)}{\sqrt{\frac{3}{2}}} = \frac{1}{3}\sqrt{2}\sqrt{3} = \frac{\sqrt{2}}{\sqrt{3}}$. Therefore the eigenvector associated with

$\lambda_3 = 4$ is $\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ or by scaling it all by $\sqrt{3}$ it becomes

$$\vec{v}_3 = \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Therefore the final solution is

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 4 \end{aligned}$$

And

$$\vec{v}_1 = \begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

2.8.4 Problem 3

3. (5 pts) Let U be a unitary matrix and let x_1 and x_2 be two eigenvectors of U with eigenvalues λ_1 and λ_2, respectively. Show that $|\lambda_1| = |\lambda_2| = 1$. Also show that if $\lambda_1 \neq \lambda_2$ then $x_1^\dagger x_2 = 0$.

Figure 2.34: Problem statement

A unitary matrix U means $U^{-1} = U^\dagger$. Let λ, x be the eigenvalue and the associated eigenvector. We also assume that the eigenvalue is not zero. Hence

$$Ux = \lambda x \tag{1}$$

Applying $\dagger$ operation (i.e. Transpose followed by complex conjugate) on the above gives

$$\begin{aligned} (Ux)^\dagger &= (\lambda x)^\dagger \\ x^\dagger U^\dagger &= x^\dagger \lambda^* \end{aligned} \tag{2}$$

Multiplying (2) by (1) gives

$$x^\dagger U^\dagger U x = x^\dagger \lambda^* \lambda x$$

But U is unitary, hence $U^{\dagger} = U^{-1}$ and the above becomes after replacing $\lambda^{*}\lambda$ by $|\lambda|^2$

$$
\begin{aligned}
x^{\dagger}U^{-1}Ux &= |\lambda|^2\left(x^{\dagger}x\right)\\
x^{\dagger}x &= |\lambda|^2\left(x^{\dagger}x\right)
\end{aligned}
$$

Hence $|\lambda|^2 = 1$ or $|\lambda| = 1$ since this is a length, and so can not be negative. But since λ is an arbitrary eigenvalue, then any complex eigenvalue has absolute value of 1. Therefore

$$
|\lambda_1| = |\lambda_2| = 1
$$

Now we consider the specific case when $\lambda_1 \neq \lambda_2$ but we still require that $|\lambda_1| = 1$ and $|\lambda_2| = 1$ which was shown in first part above. We also assume for generality that the eigenvalues are not zero.

Given that

$$
\begin{aligned}
Ux_1 &= \lambda_1 x_1 \qquad (1)\\
Ux_2 &= \lambda_2 x_2 \qquad (2)
\end{aligned}
$$

From (1) we obtain

$$
\begin{aligned}
(Ux_1)^{\dagger} &= (\lambda_1 x_1)^{\dagger}\\
x_1^{\dagger}U^{\dagger} &= x_1^{\dagger}\lambda_1^{*} \qquad (3)
\end{aligned}
$$

Multiplying (3) by (2) gives

$$
\begin{aligned}
x_1^{\dagger}U^{\dagger}Ux_2 &= x_1^{\dagger}\lambda_1^{*}\lambda_2 x_2\\
x_1^{\dagger}U^{-1}Ux_2 &= \left(\lambda_1^{*}\lambda_2\right)\left(x_1^{\dagger}x_2\right)\\
x_1^{\dagger}x_2 &= \left(\lambda_1^{*}\lambda_2\right)\left(x_1^{\dagger}x_2\right)
\end{aligned}
$$

Since $|\lambda_1| = |\lambda_2| = 1$ but $\lambda_1 \neq \lambda_2$, therefore $\left(\lambda_1^{*}\lambda_2\right) \neq 1$. From the above this implies that $x_1^{\dagger}x_2 = 0$.

2.8.5 Problem 4

4. (3 pts) Calculate the determinant of the sparse matrix (sparse means that most of the entries are zero)

$$
\begin{pmatrix}
0 & -i & 0 & 0 & 0\\
i & 0 & 0 & 0 & 0\\
0 & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 1 & i\\
0 & 0 & 0 & i & 1
\end{pmatrix}
$$

Figure 2.35: Problem statement

$$A=\begin{pmatrix}0 & -i & 0 & 0 & 0\\ i & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & i\\ 0 & 0 & 0 & i & 1\end{pmatrix}$$

We want to expand using a row or column which has most zeros in it since this leads to lots of cancellations and more efficient. Expanding using first row, then

$$\begin{aligned}\det(A) &= 0+i\det\begin{pmatrix}i & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 1 & i\\ 0 & 0 & i & 1\end{pmatrix}+0+0+0\\ &= i\left(i\det\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & i\\ 0 & i & 1\end{pmatrix}\right)\\ &= i\left(i\left(3\det\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\right)\right)\\ &= i\left(i\left(3\left(1-i^2\right)\right)\right)\\ &= 3i^2\left(1-i^2\right)\\ &= -3(1+1)\\ &= -6\end{aligned}$$

To verify this, we will now do expansion along the second row. To get the sign of a_{21} we use $(-1)^{2+1} = -1^3 = -1$. Hence

$$\begin{aligned}
\det(A) &= -i \det \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} \\
&= -i \left(-i \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} \right) \\
&= -i \left(-i \left(3 \det \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \right) \\
&= -i \left(-i \left(3 \left(1 - i^2 \right) \right) \right) \\
&= 3i^2 \left(1 - i^2 \right) \\
&= -3 (1 + 1) \\
&= -6
\end{aligned}$$

Which is the same as the expansion using the first row. Verified OK.

2.8.6 Key solution for HW 8

① (a) $(AB)^T_{ij} = \sum_\ell A_{j\ell} B_{\ell i} = \sum_\ell (B^T)_{i\ell} (A^T)_{\ell j} = (B^T A^T)_{ij}$

ij component of $(AB)^T$ $\Rightarrow$ $\boxed{(AB)^T = B^T A^T}$

(b) $(AB)^\dagger_{ij} = \sum_\ell A^*_{j\ell} B^*_{\ell i} = \sum_\ell (B^\dagger)_{i\ell} (A^\dagger)_{\ell j} = (B^\dagger A^\dagger)_{ij}$

$\Rightarrow \boxed{(AB)^\dagger = B^\dagger A^\dagger}$

(c) $\mathrm{Tr}\,(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ji} A_{ij}$

$= \sum_j (BA)_{jj} = \mathrm{Tr}\,(BA)$ $\boxed{\mathrm{Tr}(AB) = \mathrm{Tr}(BA)}$

(d) $\det A^T = \det A$ is clearly true when A is 1×1.

Proceed by induction. Assume it is true for $n \times n$.

Then for $(n+1) \times (n+1)$ $\det A^T = \sum_{j=1}^{n+1} A^T_{ij} (-1)^{i+j} \det A^T(i,j)$

$= \sum_{j=1}^{n+1} A_{ji} (-1)^{i+j} \det A(j,i) = \det A$

(e) $A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$ $B = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}$ $AB = \begin{pmatrix} a_1 b_1 & & & 0 \\ & a_2 b_2 & & \\ & & \ddots & \\ 0 & & & a_n b_n \end{pmatrix}$

$\det AB = \prod_{i=1}^n a_i b_i = \left(\prod_{i=1}^n a_i\right)\left(\prod_{i=1}^n b_i\right) = \det A \cdot \det B$

(2) $$T = \begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix}$$ Note: symmetric matrix

$$\det(T-\lambda I) = \left(\frac{5}{2}-\lambda\right)\left(\frac{7}{3}-\lambda\right)\left(\frac{13}{6}-\lambda\right) + 2\cdot\sqrt{\frac{3}{2}\cdot\frac{1}{18}\cdot\frac{3}{4}}$$

$$- \frac{3}{4}\left(\frac{7}{3}-\lambda\right) - \frac{1}{18}\left(\frac{5}{2}-\lambda\right) - \frac{3}{2}\left(\frac{13}{6}-\lambda\right) = -\lambda^3+7\lambda^2-14\lambda+8$$

$$= -(\lambda-1)(\lambda-2)(\lambda-4)$$

Eigenvalues: $\boxed{\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 4}$

Find the eigenvectors by solving the algebraic equations $(T-\lambda_i I)x_i = 0$. The results are

$$x_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right)$$

$$x_2 = \left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$$

$$x_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$$

These are orthogonal and chosen to be normalized to one.

(3) $Ux = \lambda x \qquad U^{\dagger}U = UU^{\dagger} = I$

Take Hermitian conjugate $\Rightarrow x^{\dagger}U^{\dagger} = \lambda^{*}x^{\dagger}$

Multiply $(x^{\dagger}U^{\dagger})(Ux) = (\lambda^{*}x^{\dagger})(\lambda x)$

$$x^{\dagger}\underbrace{(U^{\dagger}U)}_{I}x = \lambda^{*}(x^{\dagger}x)\lambda \Rightarrow \lambda^{*}\lambda = 1$$

$\Rightarrow \boxed{|\lambda| = 1}$

$$x_1^{\dagger}x_2 = x_1^{\dagger}(U^{\dagger}U)x_2 = (x_1^{\dagger}U^{\dagger})(Ux_2) = (\lambda_1^{*}x_1^{\dagger})(\lambda_2 x_2)$$

$= \lambda_1^{*}\lambda_2 \, x_1^{\dagger}x_2 \Rightarrow$ Either $\lambda_1^{*}\lambda_2 = 1$ or $x_1^{\dagger}x_2 = 0$

Since $|\lambda| = 1$ write $\lambda_1 = e^{i\theta_1}$ and $\lambda_2 = e^{i\theta_2}$.

Then $\lambda_1^{*}\lambda_2 = e^{i(\theta_2 - \theta_1)}$. If $\lambda_1^{*}\lambda_2 = 1$ then

$\theta_2 - \theta_1 = 2\pi n \Rightarrow \lambda_1 = \lambda_2$.

integer

$\boxed{\text{Thus if } \lambda_1 \neq \lambda_2 \text{ then } x_1^{\dagger}x_2 = 0.}$

(4) Expand around the first row.

$$\det\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & 1 \end{pmatrix} = \underbrace{(-1)^{1+2}(-i)}_{i}\det\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix}$$

$$= \underbrace{i(-1)^{1+1}i}_{-1}\det\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} = -(-1)^{1+1}3\det\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = -6$$

As a check expand around the second row.

$$\underbrace{(-1)^{2+1}i}_{-i}\det\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} = -i(-i)\det\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix}$$

$$= -[3+3] = -6$$

$$\boxed{\text{determinant} = -6}$$

2.9 HW 9

Local contents

2.9.1 HW 9 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 9 due Monday April 15. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (5 pts) Calculate the metric in elliptic coordinates

$$\begin{aligned} x &= \frac{a}{2}\cosh\mu\,\cos\theta \\ y &= \frac{a}{2}\sinh\mu\,\sin\theta \end{aligned}$$

where a is a constant.

2. (5 pts) Show that in a general coordinate system $\epsilon_{i_1\cdots i_N} = g\,\epsilon^{i_1\cdots i_N}$ where the covariant form is obtained by lowering the indices on the contravariant form.

3. (5 pts) Compute all components of the affine connection in polar coordinates.

4. (5 pts) Calculate the gradient, curl, divergence, and Laplacian in spherical coordinates using tensor analysis.

2.9.2 Problem 1

Problem Calculate the metric in elliptical coordinates

$$
\begin{aligned}
x &= \frac{a}{2}\cosh\mu\cos\theta \\
y &= \frac{a}{2}\sinh\mu\sin\theta
\end{aligned}
$$

Solution

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y$ and the coordinates in the other system (Elliptic) are $x^1 = \mu, x^2 = \theta$. The relation between these must be known and invertible also, meaning $\zeta \equiv \zeta(x)$ and $x \equiv x(\zeta)$. This relation is given to use above as

$$
\begin{aligned}
\zeta^1 &= \frac{a}{2}\cosh\mu\cos\theta \\
\zeta^2 &= \frac{a}{2}\sinh\mu\sin\theta
\end{aligned}
$$

The first step is to determine the metric tensor g_{ij} for the Polar coordinates. This is given by

$$
g_{kl} = \delta_{ij}\frac{\partial\zeta^i}{\partial x^k}\frac{\partial\zeta^j}{\partial x^l}
$$

The above using Einstein summation notation.

$$
\begin{aligned}
g_{11} &= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^1} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^1} \\
&= \frac{\partial\zeta^1}{\partial\mu}\frac{\partial\zeta^1}{\partial\mu} + \frac{\partial\zeta^2}{\partial\mu}\frac{\partial\zeta^2}{\partial\mu} \\
&= \left(\frac{\partial\zeta^1}{\partial\mu}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\mu}\right)^2 \\
&= \left(\frac{a}{2}\sinh\mu\cos\theta\right)^2 + \left(\frac{a}{2}\cosh\mu\sin\theta\right)^2 \\
&= \frac{a^2}{4}\left(\sinh^2\mu\cos^2\theta + \cosh^2\mu\sin^2\theta\right) \\
&= \frac{a^2}{4}\left(\left(\cosh^2\mu - 1\right)\cos^2\theta + \cosh^2\mu\left(1-\cos^2\theta\right)\right) \\
&= \frac{a^2}{4}\left(\cosh^2\mu\cos^2\theta - \cos^2\theta + \cosh^2\mu - \cosh^2\mu\cos^2\theta\right) \\
&= \frac{a^2}{4}\left(\cosh^2\mu - \cos^2\theta\right)
\end{aligned}
$$

And

$$\begin{aligned}
g_{12} &= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^2} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^2} \\
&= \frac{\partial\zeta^1}{\partial\mu}\frac{\partial\zeta^1}{\partial\theta} + \frac{\partial\zeta^2}{\partial\mu}\frac{\partial\zeta^2}{\partial\theta} \\
&= \left(\frac{a}{2}\sinh\mu\cos\theta\right)\left(-\frac{a}{2}\cosh\mu\sin\theta\right) + \left(\frac{a}{2}\cosh\mu\sin\theta\right)\left(\frac{a}{2}\sinh\mu\cos\theta\right) \\
&= 0
\end{aligned}$$

The above is as expected since the coordinate system is orthogonal. And

$$\begin{aligned}
g_{21} &= \frac{\partial\zeta^1}{\partial x^2}\frac{\partial\zeta^1}{\partial x^1} + \frac{\partial\zeta^2}{\partial x^2}\frac{\partial\zeta^2}{\partial x^1} \\
&= \frac{\partial\zeta^1}{\partial\theta}\frac{\partial\zeta^1}{\partial\mu} + \frac{\partial\zeta^2}{\partial\theta}\frac{\partial\zeta^2}{\partial\mu} \\
&= \left(-\frac{a}{2}\cosh\mu\sin\theta\right)\left(\frac{a}{2}\sinh\mu\cos\theta\right) + \left(\frac{a}{2}\sinh\mu\cos\theta\right)\left(\frac{a}{2}\cosh\mu\sin\theta\right) \\
&= 0
\end{aligned}$$

The above is as expected since the coordinate system is orthogonal. It is also because g_{ij} is symmetric and we already found that $g_{12} = 0$. And finally

$$\begin{aligned}
g_{22} &= \frac{\partial\zeta^1}{\partial x^2}\frac{\partial\zeta^1}{\partial x^2} + \frac{\partial\zeta^2}{\partial x^2}\frac{\partial\zeta^2}{\partial x^2} \\
&= \frac{\partial\zeta^1}{\partial\theta}\frac{\partial\zeta^1}{\partial\theta} + \frac{\partial\zeta^2}{\partial\theta}\frac{\partial\zeta^2}{\partial\theta} \\
&= \left(\frac{\partial\zeta^1}{\partial\theta}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\theta}\right)^2 \\
&= \left(-\frac{a}{2}\cosh\mu\sin\theta\right)^2 + \left(\frac{a}{2}\sinh\mu\cos\theta\right)^2 \\
&= \frac{a^2}{4}\left(\cosh^2\mu\sin^2\theta + \sinh^2\mu\cos^2\theta\right) \\
&= \frac{a^2}{4}\left(\cosh^2\mu\left(1-\cos^2\theta\right) + \left(\cosh^2\mu - 1\right)\cos^2\theta\right) \\
&= \frac{a^2}{4}\left(\cosh^2\mu - \cosh^2\mu\cos^2\theta + \cosh^2\mu\cos^2\theta - \cos^2\theta\right) \\
&= \frac{a^2}{4}\left(\cosh^2\mu - \cos^2\theta\right)
\end{aligned}$$

From the above we see that

$$
\begin{aligned}
g_{ij} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \\
&= \frac{a^2}{4} \begin{pmatrix} \cosh^2\mu - \cos^2\theta & 0 \\ 0 & \cosh^2\mu - \cos^2\theta \end{pmatrix}
\end{aligned}
$$

That there are different ways to write the above, and they are all the same. For example, we can write

$$
\begin{aligned}
g_{ij} &= \frac{a^2}{4} \begin{pmatrix} \left(1+\sinh^2\mu\right) - \left(1-\sin^2\theta\right) & 0 \\ 0 & \left(1+\sin^2\mu\right) - \left(1-\sin^2\theta\right) \end{pmatrix} \\
&= \frac{a^2}{4} \begin{pmatrix} \sinh^2\mu + \sin^2\theta & 0 \\ 0 & \sinh^2\mu + \sin^2\theta \end{pmatrix}
\end{aligned}
$$

Or we could use the double angle relations $\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$ and $\cosh^2\mu = \frac{1}{2}\left(1+\cosh(2\theta)\right)$ to obtain

$$
\begin{aligned}
g_{ij} &= \frac{a^2}{4} \begin{pmatrix} \frac{1}{2}\left(1+\cosh(2\theta)\right) - \frac{1}{2}\left(1+\cos(2\theta)\right) & 0 \\ 0 & \frac{1}{2}\left(1+\cosh(2\theta)\right) - \frac{1}{2}\left(1+\cos(2\theta)\right) \end{pmatrix} \\
&= \frac{a^2}{8} \begin{pmatrix} \cosh(2\theta) - \cos(2\theta) & 0 \\ 0 & \cosh(2\theta) - \cos(2\theta) \end{pmatrix}
\end{aligned}
$$

2.9.3 Problem 2

Problem Show that in a general coordinates system $\epsilon_{i_1\cdots i_N} = g\epsilon^{i_1\cdots i_N}$ where the covariant form is obtained by lowering the indices on the contravariant form.

Solution

In tensor analysis, contravariant components of a tensor uses upper indices and covariant components uses lower indices. Given a tensor in contravariant form ϵ^i then the covariant form ϵ_i is obtained using

$$
\epsilon_i = g_{ij}\epsilon^j
$$

Where on the right side the sum is taken over j since it is the repeated index. This operation is called index contracting.

Therefore extending the above to all indices in $\epsilon_{i_1\cdots i_N}$ results in

$$
\epsilon_{i_1 i_2\cdots i_N} = g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_N j_N} \epsilon^{j_1 j_2\cdots j_N} \tag{1}
$$

But we know that, from page 123 in the Matrices notes, that the determinant of the metric can be written using Levi-Civita tensor as

$$
g = \sum_{i_1 i_2\cdots i_N} g_{1 i_1} g_{2 i_2} \cdots g_{N i_N} \epsilon^{i_1 i_2\cdots i_N} \tag{2}
$$

Comparing (1) and (2) shows that

$$\begin{aligned}\epsilon_{123\cdots N} &= g_{1i_1}g_{2i_2}\cdots g_{Ni_N}\epsilon^{i_1 i_2\cdots i_N}\\ &= k\epsilon^{i_1 i_2\cdots i_N}\end{aligned}$$

Where k is constant, which in the case of $\epsilon_{123\cdots N}$, this constant is g. Now need to show that the constant is g for all cases of indices in $\epsilon_{i_1 i_2\cdots i_N}$ and not for the case $\epsilon_{123\cdots N}$.

Looking at the case of $N=2$, and let us see what happens if we change the order of the indices.

$$\epsilon_{i_1 i_2} = g_{i_1 j_1}g_{i_2 j_2}\epsilon^{j_1 j_2}$$

And

$$\epsilon_{i_2 i_1} = g_{i_2 j_2}g_{i_1 j_1}\epsilon^{j_2 j_1}$$

But $g_{i_1 j_1}g_{i_2 j_2}$ is the same as $g_{i_2 j_2}g_{i_1 j_1}$. So the ordering of indices does not change the constant k. And since we found that this constant is g from above, therefore we conclude that

$$\epsilon_{i_1 i_2\cdots i_N} = g\epsilon^{j_1 j_2\cdots j_N} \tag{3}$$

2.9.4 Problem 3

Problem Compute all components of the affine connection in polar coordinates.

Solution

In polar coordinates $x^1=r, x^2=\theta$, the relation to the Cartesian coordinates is

$$\begin{aligned}x &= r\cos\theta\\ y &= r\sin\theta\end{aligned}$$

Using

$$\Gamma^i_{jk} = \frac{1}{2}g^{li}\left(\frac{\partial g_{kl}}{\partial x^i}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right) \tag{1}$$

We know that in polar coordinates the metric tensor is $g_{11}=g_{rr}=1$, and $g_{12}=g_{r\theta}=0$, and $g_{21}=g_{\theta r}=0$, and $g_{22}=g_{\theta\theta}=0$ or in matrix form

$$g_{ij} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}$$

Hence g^{ij} is its inverse

$$g^{ij} = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{r^2}\end{pmatrix}$$

Using (1), let $i=r, j=r, k=r$ then

$$\Gamma^r_{rr} = \frac{1}{2}g^{lr}\left(\frac{\partial g_{rl}}{\partial r}+\frac{\partial g_{rl}}{\partial r}-\frac{\partial g_{rr}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$
\begin{aligned}
\Gamma_{rr}^{r} &= \frac{1}{2}g^{rr}\left(\frac{\partial g_{rr}}{\partial r}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial r}\right)+\frac{1}{2}g^{\theta r}\left(\frac{\partial g_{rr}}{\partial r}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial \theta}\right)\\
&= \frac{1}{2}(1)(0+0-0)+\frac{1}{2}(0)\left(\frac{\partial g_{rr}}{\partial r}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial \theta}\right)\\
&= 0 \qquad (2)
\end{aligned}
$$

Using (1), let $i=r,j=\theta,k=r$ then

$$
\Gamma_{\theta r}^{r}=\frac{1}{2}g^{lr}\left(\frac{\partial g_{rl}}{\partial r}+\frac{\partial g_{\theta l}}{\partial r}-\frac{\partial g_{\theta r}}{\partial x^{l}}\right)
$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$
\begin{aligned}
\Gamma_{\theta r}^{r} &= \frac{1}{2}g^{rr}\left(\frac{\partial g_{rr}}{\partial r}+\frac{\partial g_{\theta r}}{\partial r}-\frac{\partial g_{\theta r}}{\partial r}\right)+\frac{1}{2}g^{\theta r}\left(\frac{\partial g_{r\theta}}{\partial r}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial \theta}\right)\\
&= \frac{1}{2}(1)(0+0-0)+\frac{1}{2}(0)\left(\frac{\partial g_{r\theta}}{\partial r}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial \theta}\right)\\
&= 0 \qquad (3)
\end{aligned}
$$

Using (1), now let $i=r,j=\theta,k=\theta$ then

$$
\Gamma_{\theta\theta}^{r}=\frac{1}{2}g^{lr}\left(\frac{\partial g_{\theta l}}{\partial \theta}+\frac{\partial g_{\theta l}}{\partial \theta}-\frac{\partial g_{\theta\theta}}{\partial x^{l}}\right)
$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$
\begin{aligned}
\Gamma_{\theta\theta}^{r} &= \frac{1}{2}g^{rr}\left(\frac{\partial g_{\theta r}}{\partial \theta}+\frac{\partial g_{\theta r}}{\partial \theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}g^{\theta r}\left(\frac{\partial g_{\theta\theta}}{\partial \theta}+\frac{\partial g_{\theta\theta}}{\partial \theta}-\frac{\partial g_{\theta\theta}}{\partial \theta}\right)\\
&= \frac{1}{2}(1)\left((0)+(0)-\frac{\partial r^{2}}{\partial r}\right)+\frac{1}{2}(0)\left(\frac{\partial g_{r\theta}}{\partial r}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial \theta}\right)\\
&= \frac{1}{2}(-2r)\\
&= -r \qquad (4)
\end{aligned}
$$

Using (1), now let $i=r,j=r,k=\theta$. Hence we need to find $\Gamma_{r\theta}^{r}$. But due to symmetry in lower indices, then $\Gamma_{r\theta}^{r}=\Gamma_{\theta r}^{r}$ which we found in (3) to be zero. Hence

$$
\Gamma_{r\theta}^{r}=0 \qquad (5)
$$

Using (1), now let $i=\theta,j=r,k=r$ then

$$
\Gamma_{rr}^{\theta}=\frac{1}{2}g^{l\theta}\left(\frac{\partial g_{rl}}{\partial \theta}+\frac{\partial g_{rl}}{\partial r}-\frac{\partial g_{rr}}{\partial x^{l}}\right)
$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^{\theta}_{rr} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{r\theta}}{\partial\theta}+\frac{\partial g_{r\theta}}{\partial r}-\frac{\partial g_{rr}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial r}\right)+\frac{1}{2}\left(\frac{1}{r^2}\right)(0+0-0)\\ &= 0 \end{aligned} \tag{6}$$

Using (1), now let $i=\theta, j=\theta, k=r$ then

$$\Gamma^{\theta}_{\theta r} = \frac{1}{2}g^{l\theta}\left(\frac{\partial g_{rl}}{\partial\theta}+\frac{\partial g_{\theta l}}{\partial r}-\frac{\partial g_{\theta r}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^{\theta}_{\theta r} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial r}-\frac{\partial g_{\theta r}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{r\theta}}{\partial\theta}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial r}-\frac{\partial g_{\theta r}}{\partial r}\right)+\frac{1}{2}\frac{1}{r^2}\left(0+\frac{\partial r^2}{\partial r}-0\right)\\ &= \frac{1}{2}\frac{1}{r^2}(2r)\\ &= \frac{1}{r}\end{aligned} \tag{7}$$

Using (1), now let $i=\theta, j=r, k=\theta$ which finds $\Gamma^{\theta}_{r\theta}$ but due to symmetry this is the same as $\Gamma^{\theta}_{\theta r}$ which is found above. Hence

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r} \tag{8}$$

Using (1), now let $i=\theta, j=\theta, k=\theta$ then

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}g^{l\theta}\left(\frac{\partial g_{\theta l}}{\partial\theta}+\frac{\partial g_{\theta l}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^{\theta}_{\theta\theta} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{\theta r}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{\partial g_{\theta\theta}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{\theta r}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}\frac{1}{r^2}(0+0-0)\\ &= 0\end{aligned} \tag{9}$$

This completes the computation. In summary

$$
\begin{aligned}
\Gamma^{r}_{rr} &= 0\\
\Gamma^{r}_{\theta r} &= 0\\
\Gamma^{r}_{\theta\theta} &= -r\\
\Gamma^{r}_{r\theta} &= 0\\
\Gamma^{\theta}_{rr} &= 0\\
\Gamma^{\theta}_{\theta r} &= \frac{1}{r}\\
\Gamma^{\theta}_{r\theta} &= \frac{1}{r}\\
\Gamma^{\theta}_{\theta\theta} &= 0
\end{aligned}
$$

2.9.5 Problem 4

Problem Calculate the gradient curl and divergence and Laplacian in spherical coordinates using tensor analysis.

Solution

The following coordinates system convention is used

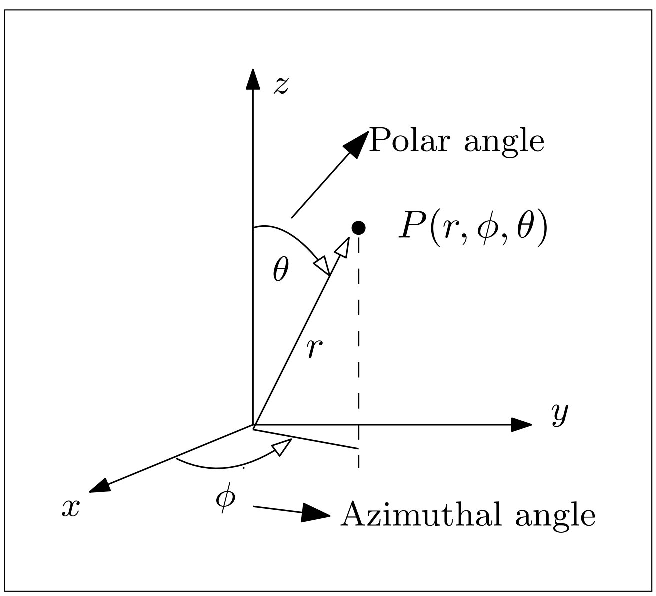


Figure 2.36: Spherical Coordinates system

2.9.5.1 Finding metric tensor g_{ij}

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Spherical system are $x^1 = \phi, x^2 = r, x^3 = \theta$. The relation between these is known as

(Note that the following depends on convention used for which is θ and which is ϕ. Physics convention as shown in the diagram above is used here).

$$
\begin{aligned}
\zeta^1 &= r\sin\theta\cos\phi\\
\zeta^2 &= r\sin\theta\sin\phi\\
\zeta^3 &= r\cos\theta
\end{aligned}
$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$
g_{kl} = \delta_{ij}\frac{\partial\zeta^i}{\partial x^k}\frac{\partial\zeta^j}{\partial x^l}
$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$
\begin{aligned}
g_{11} &= g_{\phi\phi}\\
&= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^1} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^1} + \frac{\partial\zeta^3}{\partial x^1}\frac{\partial\zeta^3}{\partial x^1}\\
&= \frac{\partial\zeta^1}{\partial\phi}\frac{\partial\zeta^1}{\partial\phi} + \frac{\partial\zeta^2}{\partial\phi}\frac{\partial\zeta^2}{\partial\phi} + \frac{\partial\zeta^3}{\partial\phi}\frac{\partial\zeta^3}{\partial\phi}\\
&= \left(\frac{\partial\zeta^1}{\partial\phi}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\phi}\right)^2 + \left(\frac{\partial\zeta^3}{\partial\phi}\right)^2\\
&= \left(-r\sin\theta\sin\phi\right)^2 + \left(r\sin\theta\cos\phi\right)^2 + (0)^2\\
&= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\\
&= r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)\\
&= r^2\sin^2\theta
\end{aligned}
$$

And

$$
\begin{aligned}
g_{22} &= g_{rr}\\
&= \frac{\partial\zeta^1}{\partial x^2}\frac{\partial\zeta^1}{\partial x^2} + \frac{\partial\zeta^2}{\partial x^2}\frac{\partial\zeta^2}{\partial x^2} + \frac{\partial\zeta^3}{\partial x^2}\frac{\partial\zeta^3}{\partial x^2}\\
&= \frac{\partial\zeta^1}{\partial r}\frac{\partial\zeta^1}{\partial r} + \frac{\partial\zeta^2}{\partial r}\frac{\partial\zeta^2}{\partial r} + \frac{\partial\zeta^3}{\partial r}\frac{\partial\zeta^3}{\partial r}\\
&= \left(\frac{\partial\zeta^1}{\partial r}\right)^2 + \left(\frac{\partial\zeta^2}{\partial r}\right)^2 + \left(\frac{\partial\zeta^3}{\partial r}\right)^2\\
&= \left(\sin\theta\cos\phi\right)^2 + \left(\sin\theta\sin\phi\right)^2 + (\cos\theta)^2\\
&= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\\
&= \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\\
&= \sin^2\theta + \cos^2\theta\\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
g_{33} &= g_{\theta\theta} \\
&= \frac{\partial\zeta^1}{\partial x^3}\frac{\partial\zeta^1}{\partial x^3} + \frac{\partial\zeta^2}{\partial x^3}\frac{\partial\zeta^2}{\partial x^3} + \frac{\partial\zeta^3}{\partial x^3}\frac{\partial\zeta^3}{\partial x^3} \\
&= \frac{\partial\zeta^1}{\partial\theta}\frac{\partial\zeta^1}{\partial\theta} + \frac{\partial\zeta^2}{\partial\theta}\frac{\partial\zeta^2}{\partial\theta} + \frac{\partial\zeta^3}{\partial\theta}\frac{\partial\zeta^3}{\partial\theta} \\
&= \left(\frac{\partial\zeta^1}{\partial\theta}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\theta}\right)^2 + \left(\frac{\partial\zeta^3}{\partial\theta}\right)^2 \\
&= \left(r\cos\theta\cos\phi\right)^2 + \left(r\cos\theta\sin\phi\right)^2 + (-r\sin\theta)^2 \\
&= r^2\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^2\theta \\
&= r^2\cos^2\theta + r^2\sin^2\theta \\
&= r^2
\end{aligned}
$$

Hence ds^2 in Spherical coordinates is

$$
\begin{aligned}
ds^2 &= g_{kl}dx^k dx^l \\
&= g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + g_{33}\left(dx^3\right)^2 \\
&= g_{11}\left(d\phi\right)^2 + g_{22}\left(dr\right)^2 + g_{33}\left(d\theta\right)^2 \\
&= r^2\sin^2\theta\left(d\phi\right)^2 + (dr)^2 + r^2\left(d\theta\right)^2
\end{aligned}
$$

From the above we see that, using the order ϕ, r, θ for the rows and columns

$$
\begin{aligned}
g_{ij} &= \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \\
&= \begin{pmatrix} r^2\sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}
\end{aligned}
$$

Therefore the determinant is $g = r^4\sin^2\theta$ and h_i are given by the square root of the diagonal elements of g_{ij}

$$
\begin{aligned}
h_1 &= r\sin\theta \\
h_2 &= 1 \\
h_3 &= r
\end{aligned}
\tag{A}
$$

2.9.5.2 Finding Gradient

$$
\nabla = \left(\frac{1}{h_1}\frac{\partial}{\partial x^1}, \frac{1}{h_2}\frac{\partial}{\partial x^2}, \frac{1}{h_3}\frac{\partial}{\partial x^3}\right)
$$

Where h_i are given in (A) and $x^1 = \phi, x^2 = r, x^3 = \theta$. Therefore

$$\nabla = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta} \right)$$

Hence given a function scalar $f\left(\phi, r, \theta\right)$ then

$$\nabla f = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi + \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta$$

2.9.5.3 Finding Curl

Using h_i in (A) and $x^1 = \phi, x^2 = r, x^3 = \theta$ then

$$\begin{aligned}
\left(\vec{\nabla}\times\vec{V}\right)_1 &= \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x^2}\left(h_3 V_3\right) - \frac{\partial}{\partial x^3}\left(h_2 V_2\right)\right) \\
\left(\vec{\nabla}\times\vec{v}\right)_\phi &= \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)
\end{aligned}$$

And

$$\begin{aligned}
\left(\vec{\nabla}\times\vec{V}\right)_2 &= \frac{1}{h_3 h_1}\left(\frac{\partial}{\partial x^3}\left(h_1 V_1\right) - \frac{\partial}{\partial x^1}\left(h_3 V_3\right)\right) \\
\left(\vec{\nabla}\times\vec{V}\right)_r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(r\sin\theta V_\phi\right) - \frac{\partial}{\partial\phi}\left(rV_\theta\right)\right) \\
&= \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta V_\phi\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right)
\end{aligned}$$

And

$$\begin{aligned}
\left(\vec{\nabla}\times\vec{V}\right)_3 &= \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x^1}\left(h_2 V_2\right) - \frac{\partial}{\partial x^2}\left(h_1 V_1\right)\right) \\
\left(\vec{\nabla}\times\vec{V}\right)_\theta &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\left(V_r\right) - \frac{\partial}{\partial r}\left(r\sin\theta V_\phi\right)\right) \\
&= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right)
\end{aligned}$$

Therefore given a vector $\vec{V}$, its curl is

$$\vec{\nabla}\times\vec{V} = \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)\hat{e}_\phi + \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta V_\phi\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right)\hat{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right)\hat{e}_\theta$$

2.9.5.4 Finding Divergence

$$\nabla\cdot V = \nabla_i V^i = \frac{\partial}{\partial x^i}V^i + \Gamma^i_{ij}V^j \tag{1}$$

Where $\Gamma^i_{ij} = \frac{1}{2}g^{li}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) = \frac{1}{2}g^{li}\left(\frac{\partial g_{il}}{\partial x^j}\right)$ which simplifies to as shown in class notes page 143 to hence above becomes

$$\Gamma^i_{ij} = \frac{1}{\sqrt{g}}\frac{\partial}{x^j}\left(\sqrt{g}\right)$$

Hence (1) becomes

$$\begin{aligned}\nabla\cdot V &= \frac{\partial}{\partial x^i}V^i + \frac{1}{\sqrt{g}}\frac{\partial}{x^j}\left(\sqrt{g}\right)V^j \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{x^i}\left(\sqrt{g}V^i\right)\end{aligned}$$

Using the covariant form the above becomes

$$\nabla\cdot V = \frac{1}{\sqrt{g}}\frac{\partial}{x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{ii}}}V_i\right)$$

Where in class notes h_i is used in place of $\sqrt{g_{ii}}$, but it is it the same.

The sum is over i. From above, the spherical coordinates are $x^1 = \phi, x^2 = r, x^3 = \theta$. And $g = r^4\sin^2\theta$. Hence the above becomes after expanding

$$\begin{aligned}\nabla . V &= \frac{1}{\sqrt{r^4\sin^2\theta}}\left(\frac{\partial}{\partial\phi}\left(\frac{\sqrt{r^4\sin^2\theta}}{\sqrt{g_{\phi\phi}}}V_\phi\right) + \frac{\partial}{\partial r}\left(\frac{\sqrt{r^4\sin^2\theta}}{\sqrt{g_{rr}}}V_r\right) + \frac{\partial}{\partial\theta}\left(\frac{\sqrt{r^4\sin^2\theta}}{\sqrt{g_{\theta\theta}}}V_\theta\right)\right) \\ &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\phi}\left(\frac{r^2\sin\theta}{r\sin\theta}V_\phi\right) + \frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{1}V_r\right) + \frac{\partial}{\partial\theta}\left(\frac{r^2\sin\theta}{r}V_\theta\right)\right) \\ &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\phi}\left(rV_\phi\right) + \frac{\partial}{\partial r}\left(r^2\sin\theta V_r\right) + \frac{\partial}{\partial\theta}\left(\sin\theta V_\theta\right)\right) \\ &= \frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}V_\phi\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta V_\theta\right)\end{aligned}$$

2.9.5.5 Finding Laplacian

The Laplacian is given by

$$\nabla^2 = \frac{1}{\sqrt{\det\left(g\right)}}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{\det\left(g\right)}}{g_{ii}}\frac{\partial}{\partial x^i}\right)$$

Hence

$$
\begin{aligned}
\nabla^2 &= \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{11}}\frac{\partial}{\partial x^1}\right) + \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_2}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{22}}\frac{\partial}{\partial x^2}\right) + \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x_3}\left(\frac{\sqrt{r^4\sin^2\theta}}{g_{33}}\frac{\partial}{\partial x^3}\right)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{r^2\sin\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{r^2\sin\theta}{r^2}\frac{\partial}{\partial\theta}\right)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\\
&= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right) + \frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial^2}{\partial\theta^2}\right)\\
&= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\partial^2 u}{\partial\theta^2}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\\
&= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta + u_{\theta\theta}\right) + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}
\end{aligned}
$$

2.9.6 Key solution for HW 9

(1) $x = \frac{a}{2}\cosh\mu\cos\theta \qquad y = \frac{a}{2}\sinh\mu\sin\theta$

Using notation in lecture $\xi^1 = x,\ \xi^2 = y,\ x^1 = \mu,\ x^2 = \theta$

$$g_{kl} = \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^l}\delta_{ij}$$

$$g_{\mu\mu} = \left(\frac{\partial x}{\partial\mu}\right)^2 + \left(\frac{\partial y}{\partial\mu}\right)^2 = \frac{a^2}{4}\left[\sinh^2\mu\cos^2\theta + \underbrace{\cosh^2\mu}_{1+\sinh^2\mu}\sin^2\theta\right]$$

$$= \frac{a^2}{4}\left[\sinh^2\mu + \sin^2\theta\right] = \frac{a^2}{4}\left[\cosh^2\mu - \cos^2\theta\right]$$

$$g_{\theta\theta} = \left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 = \frac{a^2}{4}\left[\cosh^2\mu\sin^2\theta + \sinh^2\mu\cos^2\theta\right] = g_{\theta\theta}$$

$$g_{\mu\theta} = g_{\theta\mu} = \frac{\partial x}{\partial\mu}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\mu}\frac{\partial y}{\partial\theta} = \frac{a^2}{4}\Big[(\sinh\mu\cos\theta)(-\cosh\mu\sin\theta)$$

$$+ (\cosh\mu\sin\theta)(\sinh\mu\cos\theta)\Big] = 0$$

$$\boxed{\begin{gathered} g_{\mu\mu} = g_{\theta\theta} = \frac{a^2}{4}\left[\sinh^2\mu + \sin^2\theta\right] = \frac{a^2}{4}\left[\cosh^2\mu - \cos^2\theta\right] \\ g_{\mu\theta} = g_{\theta\mu} = 0 \end{gathered}}$$

(2) $$\Gamma^i_{jk} = \frac{1}{2} g^{li}\left[\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right]$$

From lecture we know that $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{r\theta} = 0$.

Here $x^1 = r$ and $x^2 = \theta$. Since g_{ij} is diagonal

we have $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2}$, $g^{r\theta} = 0$.

Hence $$\Gamma^i_{jk} = \frac{1}{2} g^{ii}\left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right]$$ no sum on i

$$\Gamma^r_{rr} = \frac{1}{2}\left[0 + 0 - 0\right] = 0 \qquad \boxed{\Gamma^r_{rr} = 0}$$

$$\Gamma^r_{\theta r} = \frac{1}{2}\left[0 + 0 - 0\right] = 0 \qquad \boxed{\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}\left[0 + 0 - 2r\right] = -r \qquad \boxed{\Gamma^r_{\theta\theta} = -r}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2}\frac{1}{r^2}\left[0 + 0 - 0\right] = 0 \qquad \boxed{\Gamma^\theta_{\theta\theta} = 0}$$

$$\Gamma^\theta_{rr} = \frac{1}{2}\frac{1}{r^2}\left[0 + 0 - 0\right] = 0 \qquad \boxed{\Gamma^\theta_{rr} = 0}$$

$$\Gamma^\theta_{\theta r} = \frac{1}{2}\frac{1}{r^2}\left[0 + 2r - 0\right] = \frac{1}{r} \qquad \boxed{\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}}$$

(3) $\epsilon_{i_1 \cdots i_n} = g_{i_1 i_1'} g_{i_2 i_2'} \cdots g_{i_n i_n'} \epsilon^{i_1' i_2' \cdots i_n'}$

which convert contravariant to covariant.

By definition $\epsilon^{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if even permutation} \\ -1 & \text{if odd permutation} \\ 0 & \text{if any pair of indices are equal} \end{cases}$

Then the same is true of $\epsilon_{i_1 i_2 \cdots i_n}$, because $g_{ij} = g_{ji}$.

Thus $\epsilon_{i_1 \cdots i_n} = \text{constant} \cdot \epsilon^{i_1 \cdots i_n}$

To find the constant choose $i_1 \cdots i_n = 1 2 \cdots n$.

$\epsilon_{12 \cdots n} = \text{constant} \underbrace{\epsilon^{12 \cdots n}}_{1} = \text{constant}$

$\parallel$

$g_{1 i_1} g_{2 i_2} \cdots g_{n i_n} \epsilon^{i_1 i_2 \cdots i_n} = \det g_{ij} = g$

$\Rightarrow \boxed{\epsilon_{i_1 i_2 \cdots i_n} = g \, \epsilon^{i_1 i_2 \cdots i_n}}$

(4) Spherical coordinates are orthogonal so g_{ij} is diagonal.

$x = r\cos\phi\sin\theta \quad y = r\sin\phi\sin\theta \quad z = r\cos\theta$

$$h_r^2 = g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = 1$$

$$h_\theta^2 = g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2$$

$$h_\phi^2 = g_{\phi\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2\sin^2\theta$$

Ordinary vectors with components ordered by $\hat{r}, \hat{\theta}, \hat{\phi}$.

gradient components are $\frac{1}{h_i}\frac{\partial}{\partial x^i}$

$$\boxed{\vec{\nabla}\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)}$$

divergence $\vec{\nabla}\cdot\vec{V} = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial x^i}\left(\frac{h_1h_2h_3}{h_i}\bar{V}_i\right)$

$$\boxed{\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\bar{V}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\bar{V}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\bar{V}_\phi}{\partial\phi}}$$

curl $\left(\vec{\nabla}\times\vec{V}\right)_k = \frac{h_k}{h_1h_2h_3}\sum_{ij}\epsilon^{ijk}\frac{\partial}{\partial x^j}\left(h_i\bar{V}_i\right)$

$$\boxed{\vec{\nabla}\times\vec{V} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\bar{V}_\phi\right) - \frac{\partial \bar{V}_\theta}{\partial\phi}\right], \frac{1}{r\sin\theta}\left[\frac{\partial \bar{V}_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}\left(r\bar{V}_\phi\right)\right], \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,\bar{V}_\theta\right) - \frac{\partial \bar{V}_r}{\partial\theta}\right]\right)}$$

Laplacian $\nabla^2\Phi = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial x^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\Phi}{\partial x^i}\right)$

$$\boxed{\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}}$$

2.10 HW 10

Local contents

2.10.1 HW 10 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 10 due Monday April 22. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (8 pts) Show that

$$\int_0^\infty J_m(x)\, J_n(x) \frac{dx}{x} = \frac{2}{\pi} \frac{\sin[(m-n)\frac{\pi}{2}]}{m^2 - n^2}$$

where $m+n > 0$. Suggestion: Multiply the differential equations satisfied by J_m and J_n by x and subtract. Then use the asymptotic expression of $J_n(x)$ for large values of x.

2. (8 pts) What linear second order differential equation does the function $x^m J_n(ax^k)$ solve? Are there any required relationships among m, n, k? Use this to solve $y'' + x^2 y = 0$.

3. (3 pts) Prove that $|J_n(x)| \leq 1$ for all integer n.

4. (6 pts) Starting with the integral formula for the hypergeometric function express the following in terms of elementary functions

$$_2F_1(1, 1; 2; x) \quad \text{and} \quad {_2F_1}(a, 1; 1; x) \tag{1}$$

2.10.2 Problem 1

Problem Show that

$$\int_0^\infty \frac{1}{x} J_m(x) J_n(x)\,dx = \frac{2}{\pi}\frac{\sin\left((m-n)\frac{\pi}{2}\right)}{m^2-n^2}$$

Solution

$$\begin{aligned} x^2 J_n''(x) + xJ_n'(x) + \left(x^2-n^2\right) J_n(x) &= 0\\ x^2 J_m''(x) + xJ_m'(x) + \left(x^2-m^2\right) J_m(x) &= 0 \end{aligned}$$

Dividing both equations by x^2 gives

$$\begin{aligned} J_n''(x) + \frac{1}{x}J_n'(x) + \left(1-\frac{n^2}{x^2}\right) J_n(x) &= 0\\ J_m''(x) + \frac{1}{x}J_m'(x) + \left(1-\frac{m^2}{x^2}\right) J_m(x) &= 0 \end{aligned}$$

Multiplying the first ODE by $xJ_m(x)$ and the second by $xJ_n(x)$ gives (multiplying by just x did not lead to a result that I could use).

$$\begin{aligned} xJ_mJ_n'' + J_mJ_n' + x\left(1-\frac{n^2}{x^2}\right) J_mJ_n &= 0\\ xJ_nJ_m'' + J_nJ_m' + x\left(1-\frac{m^2}{x^2}\right) J_nJ_m &= 0 \end{aligned}$$

Subtracting gives

$$\begin{aligned} \left(xJ_mJ_n'' + J_mJ_n' + x\left(1-\frac{n^2}{x^2}\right) J_mJ_n\right) - \left(xJ_nJ_m'' + J_nJ_m' + x\left(1-\frac{m^2}{x^2}\right) J_nJ_m\right) &= 0\\ x\left(J_mJ_n'' - J_nJ_m''\right) + J_mJ_n' - J_nJ_m' - xJ_mJ_n\left(\left(1-\frac{n^2}{x^2}\right)-\left(1-\frac{m^2}{x^2}\right)\right) &= 0 \end{aligned}$$

Or

$$x\left(J_mJ_n'' - J_nJ_m''\right) + J_mJ_n' - J_nJ_m' = xJ_mJ_n\left(\left(1-\frac{m^2}{x^2}\right)-\left(1-\frac{n^2}{x^2}\right)\right) \tag{1}$$

But the LHS above is complete differential[3]

$$x\left(J_mJ_n'' - J_nJ_m''\right) + J_mJ_n' - J_nJ_m' = \left(x\left(J_mJ_n' - J_nJ_m'\right)\right)' \tag{2}$$

[3]

$$\begin{aligned} \left(x\left(J_mJ_n' - J_nJ_m'\right)\right)' &= \left(J_mJ_n' - J_nJ_m'\right) + x\left(J_mJ_n' - J_nJ_m'\right)'\\ &= J_mJ_n' - J_nJ_m' + x\left(J_m'J_n' + J_mJ_n'' - J_n'J_m' - J_nJ_m''\right)\\ &= J_mJ_n' - J_nJ_m' + x\left(J_mJ_n'' - J_nJ_m''\right) \end{aligned}$$

Hence using (2) in (1), then (1) simplifies to

$$
\begin{aligned}
\left(x\left(J_m J_n' - J_n J_m'\right)\right)' &= x J_m J_n \left(\left(1 - \frac{m^2}{x^2}\right) - \left(1 - \frac{n^2}{x^2}\right)\right)\\
&= x J_m J_n \left(\frac{n^2}{x^2} - \frac{m^2}{x^2}\right)\\
&= \frac{J_m J_n}{x}\left(n^2 - m^2\right)
\end{aligned}
$$

Integrating both sides above gives

$$
\left[x\left(J_m J_n' - J_n J_m'\right)\right]_0^\infty = \left(n^2 - m^2\right)\int_0^\infty \frac{J_m J_n}{x} dx
$$

Therefore

$$
\int_0^\infty \frac{J_m(x) J_n(x)}{x} dx = \frac{1}{\left(m^2 - n^2\right)}\left[x\left(J_n(x) J_m'(x) - J_m(x) J_n'(x)\right)\right]_0^\infty \tag{3}
$$

At $x = 0$ the expression $x\left(J_n(x) J_m'(x) - J_m(x) J_n'(x)\right) = 0$. And at $x = \infty$ we can use the asymptotic approximation given by

$$
\begin{aligned}
J_n(x) &= \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\\
J_n'(x) &= -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}
$$

And similarly for $J_m(x)$

$$
\begin{aligned}
J_m(x) &= \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\\
J_m'(x) &= -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}
$$

Therefore

$$
J_n(x) J_m'(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left(-\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right)
$$

Let $x-\frac{n\pi}{2}-\frac{\pi}{4}=\alpha$, and let $x-\frac{m\pi}{2}-\frac{\pi}{4}=\beta$, then the above becomes

$$\begin{aligned}
J_n(x)J_m'(x) &= \sqrt{\frac{2}{\pi x}}\cos(\alpha)\left(-\sqrt{\frac{2}{\pi x}}\sin(\beta)-\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\beta)\right)\\
&= -\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\sqrt{\frac{2}{\pi x}}\sqrt{\frac{1}{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\alpha)\cos(\beta)\\
&= -\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{\frac{1}{2}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\alpha)\cos(\beta)\\
&= -\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{2}\cos(\alpha)\cos(\beta) \qquad (4)
\end{aligned}$$

Similarly

$$J_m(x)J_n'(x) = -\frac{2}{\pi x}\cos(\beta)\sin(\alpha)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{2}\cos(\beta)\cos(\alpha) \qquad (5)$$

Substituting (4,5) into (3) gives (only the term as $x\to\infty$ remains)

$$\begin{aligned}
\int_0^\infty \frac{J_m(x)J_n(x)}{x}dx &= \frac{1}{(m^2-n^2)}\left[x\left(J_n(x)J_m'(x)-J_m(x)J_n'(x)\right)\right]_0^\infty\\
&= \frac{x}{(m^2-n^2)}\left(\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\alpha)\cos(\beta)\right)-\left(-\frac{2}{\pi x}\cos(\beta)\sin(\alpha)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\beta)\cos(\alpha)\right)\right)\\
&= \frac{x}{(m^2-n^2)}\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\alpha)\cos(\beta)+\frac{2}{\pi x}\cos(\beta)\sin(\alpha)+\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\beta)\cos(\alpha)\right)\\
&= \frac{x}{(m^2-n^2)}\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)+\frac{2}{\pi x}\cos(\beta)\sin(\alpha)\right)\\
&= \frac{2}{\pi}\frac{1}{(m^2-n^2)}\left(\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\right) \qquad (6)
\end{aligned}$$

But

$$\begin{aligned}
\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) &= \sin(\alpha-\beta)\\
&= \sin\left(\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)-\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\right)\\
&= \sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}-x+\frac{m\pi}{2}+\frac{\pi}{4}\right)\\
&= \sin\left(\frac{m\pi}{2}-\frac{n\pi}{2}\right)\\
&= \sin\left((m-n)\frac{\pi}{2}\right)
\end{aligned}$$

Using the above in (6) gives

$$\int_0^\infty \frac{J_m(x)\,J_n(x)}{x}dx = \frac{2}{\pi}\frac{\sin\left((m-n)\frac{\pi}{2}\right)}{\left(m^2-n^2\right)}$$

Which is the result required to show. QED.

2.10.3 Problem 2

Problem What linear second order ODE does the function $x^m J_n\left(ax^k\right)$ solves? Are there any required relationships among m, n, k? Use this to solve $y'' + x^2 y = 0$

Solution

2.10.3.1 Part (a)

We know that the Bessel ODE

$$t^2 z''(t) + t z'(t) + \left(t^2 - \left(\frac{\alpha}{\beta}\right)^2\right) z(t) = 0 \tag{1}$$

I am using the order as $\frac{\alpha}{\beta}$ instead of n to make it more general. At the end, $\frac{\alpha}{\beta}$ can always be replaced back by n.

The ODE above has solution

$$z(t) = J_{\frac{\alpha}{\beta}}(t)$$

Hence using the transformation

$$t = ax^k \tag{2}$$

The solution $y(x) \equiv z\left(ax^k\right)$ will becomes

$$y(x) = J_{\frac{\alpha}{\beta}}\left(ax^k\right)$$

Therefore the question now is, how does ODE (1) transforms under (2)? From (2)

$$x = \left(\frac{t}{a}\right)^{\frac{1}{k}}$$

Hence

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{k}\left(\frac{t}{a}\right)^{\frac{1}{k}-1} \\ &= \frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\end{aligned} \tag{3}$$

Now

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{dt}\\ &= \frac{dz}{dx}\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1} \end{aligned} \tag{5}$$

And

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(\frac{dz}{dt}\right)\\ &= \frac{d}{dt}\left(\frac{dz}{dx}\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)\\ &= \frac{d^2z}{dx^2}\frac{dx}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) + \frac{dz}{dx}\frac{d}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)\\ &= \frac{d^2z}{dx^2}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + \frac{dz}{dx}\left(\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right) \end{aligned} \tag{6}$$

Using (5,6) then ODE (1) becomes

$$t^2\left(z''\left(ax^k\right)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + z'\left(ax^k\right)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right)+t\left(z'\left(ax^k\right)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)+\left(t^2-\left(\frac{\alpha}{\beta}\right)^2\right)z\left(ax^k\right)=0$$

Writing $y(x) \equiv z\left(ax^k\right)$ so we do not have to keep writing $z\left(ax^k\right)$, the above becomes

$$t^2\left(y''(x)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right)+t\left(y'(x)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)+\left(t^2-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

But $t = ax^k$ and the above becomes

$$a^2x^{2k}\left(y''(x)\left(\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right)^2 + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-2}\right)+ax^k\left(y'(x)\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right)+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

Which is simplified more as follows

$$
\begin{aligned}
a^2x^{2k}\left(y''(x)\left(\frac{1}{ak}\left(\frac{x}{x^k}\right)\right)^2 + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right) + ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right) + \left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x) &= 0\\
a^2x^{2k}\left(y''(x)\frac{1}{a^2k^2}\left(\frac{x^2}{x^{2k}}\right) + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right) + ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right) + \left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x) &= 0\\
x^{2k}\left(y''(x)\frac{1}{k^2}\left(\frac{x^2}{x^{2k}}\right) + y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right) + y'(x)\frac{x}{k} + \left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x) &= 0\\
\frac{x^2}{k^2}y''(x) + y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)x + y'(x)\frac{x}{k} + \left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x) &= 0\\
x^2y''(x) + y'(x)k\left(\frac{1}{k}-1\right)x + y'(x)kx + \left(k^2a^2x^{2k}-k^2\left(\frac{\alpha}{\beta}\right)^2\right)y(x) &= 0\\
x^2y''(x) + xy'(x) + \left(k^2a^2x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)y(x) &= 0
\end{aligned}
\tag{7}
$$

We know that the above ODE has one solution as $y(x) = J_{\frac{\alpha}{\beta}}\left(ax^k\right)$ because this is how the above was constructed. Now assuming that

$$
\begin{aligned}
w(x) &= x^m y(x)\\
&= x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)
\end{aligned}
$$

Then $w(x)$ is the solution we want. This means we need to express (7) in terms of $w(x)$ instead of $y(x)$ in order to find the ODE whose solution is $x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$.

Since $y(x) = w(x)x^{-m}$ then

$$
\begin{aligned}
y'(x) &= \frac{d}{dx}\left(x^{-m}w\right)\\
&= -mx^{-m-1}w + x^{-m}w'
\end{aligned}
$$

And

$$
\begin{aligned}
y''(x) &= \frac{d}{dx}\left(-mx^{-m-1}w + x^{-m}w'\right)\\
&= -m(-m-1)x^{-m-2}w - mx^{-m-1}w' - mx^{-m-1}w' + x^{-m}w''\\
&= m(m+1)x^{-m-2}w - 2w'mx^{-m-1} + x^{-m}w''
\end{aligned}
$$

Substituting the above results back into (7) gives

$$
x^2\left(m(m+1)x^{-m-2}w - 2w'mx^{-m-1} + x^{-m}w''\right) + x\left(-mx^{-m-1}w + x^{-m}w'\right) + \left(k^2a^2x^{2k} - \frac{k^2\alpha^2}{\beta^2}\right)wx^{-m} = 0
$$

Dividing by x^{-m}

$$x^2\left(m(m+1)x^{-2}w-2w'mx^{-1}+w''\right)+x\left(-mx^{-1}w+w'\right)+\left(k^2a^2x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)w=0$$

$$m(m+1)w-2xw'm+x^2w''-mw+xw'+\left(k^2a^2x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)w=0$$

$$x^2w''+w'(-2xm+x)+\left(k^2a^2x^{2k}+m(m+1)-m-\frac{k^2\alpha^2}{\beta^2}\right)w=0$$

$$x^2w''+(1-2m)xw'+\left(k^2a^2x^{2k}+m^2-\frac{k^2\alpha^2}{\beta^2}\right)w=0 \tag{8}$$

Hence the above ODE (8) will have the solution $x^mJ_{\frac{\alpha}{\beta}}\left(ax^k\right)$. We can now let $n=\frac{\alpha}{\beta}$ and the above ODE becomes

$$x^2w''+(1-2m)xw'+\left(k^2a^2x^{2k}+m^2-k^2n^2\right)w=0 \tag{9}$$

Has the required solution $x^mJ_n\left(ax^k\right)$.

To answer the final part about the relation between n,m,k. One restriction is that $m=\frac{1}{2}$. One relation between the order n and k is that $m^2-k^2n^2$ being a rational number. This means

$$m^2-k^2n^2=\frac{N}{M}$$

Where N,M are integers.

2.10.3.2 Part (b)

$$y''(x)+x^2y(x)=0 \tag{1}$$

Comparing this ODE to one found in part (a), written below again, now using $y(x)$ to make it easier to compare

$$x^2y''(x)+(1-2m)xy'(x)+\left(k^2a^2x^{2k}+m^2-k^2n^2\right)y(x)=0$$

$$y''(x)+\frac{(1-2m)}{x}y'(x)+\frac{1}{x^2}\left(k^2a^2x^{2k}+m^2-k^2n^2\right)y(x)=0 \tag{2}$$

To make (2) same as (1), we want $(1-2m)=0$ or $m=\frac{1}{2}$. Also need $2k=4$ or $k=2$. Using these the above reduces to

$$y''(x)+\left(4a^2x^2+\frac{\frac{1}{4}-4n^2}{x^2}\right)y(x)=0$$

Therefore, we need also that $n^2=\frac{1}{16}$ in order to cancel extra term above. Hence $n=\frac{1}{4}$. Now the above becomes

$$y''(x)+4a^2x^2y(x)=0$$

Finally, if we let $a^2 = \frac{1}{4}$ or $a = \frac{1}{2}$, then the above becomes

$$y''(x) + x^2 y(x) = 0$$

Therefore, we found that

$$\begin{aligned} n &= \frac{1}{4} \\ a &= \frac{1}{2} \\ k &= 2 \\ m &= \frac{1}{2} \end{aligned}$$

Hence the following solves the ODE

$$\begin{aligned} y(x) &= x^m J_n\left(ax^k\right) \\ &= \sqrt{x} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) \end{aligned}$$

2.10.3.3 Appendix

To verify the above result, it is solved again directly. We first need to convert this ODE to Bessel ODE. Let

$$y = x^{\frac{1}{2}} z(x)$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}}z + x^{\frac{1}{2}}z' \\ \frac{d^2y}{dx^2} &= -\frac{1}{4}x^{-\frac{3}{2}}z + \frac{1}{2}x^{-\frac{1}{2}}z' + \frac{1}{2}x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z'' \\ &= -\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z'' \end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned} \left(-\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z''\right) + x^2 x^{\frac{1}{2}} z &= 0 \\ x^{\frac{1}{2}}z'' + x^{-\frac{1}{2}}z' + \left(x^{\frac{5}{2}} - \frac{1}{4}x^{-\frac{3}{2}}\right)z &= 0 \end{aligned}$$

Multiplying both sides by $x^{\frac{3}{2}}$ gives

$$x^2 z'' + xz' + \left(x^4 - \frac{1}{4}\right)z = 0 \tag{2}$$

Where the derivatives above is with respect to x. Now let $t = \frac{x^2}{2}$. Then

$$\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = x\frac{dz}{dt}$$

And

$$\frac{d^2z}{dx^2} = \frac{d^2z}{dt^2}\left(\frac{dt}{dx}\right)(x) + \frac{dz}{dt}$$
$$= \frac{d^2z}{dt^2}x^2 + \frac{dz}{dt}$$

Substituting the above into (2) gives

$$x^2\left(x^2z'' + z'\right) + x\left(xz'\right) + \left(x^4 - \frac{1}{4}\right)z = 0$$

Where the derivatives above is with respect to t now. This simplifies to

$$x^4z'' + 2x^2z' + \left(x^4 - \frac{1}{4}\right)z = 0$$

But $t = \frac{x^2}{2}$, hence the above becomes

$$4t^2z'' + 4tz' + \left(4t^2 - \frac{1}{4}\right)z = 0$$
$$t^2z'' + tz' + \left(t^2 - \frac{1}{16}\right)z = 0$$

This now in the form of Bessel ODE

$$t^2z'' + tz' + \left(t^2 - n^2\right)z = 0$$

Where $n = \frac{1}{4}$. Hence one solution is

$$z(t) = J_n(t)$$
$$= J_{\frac{1}{4}}(t)$$

But $y(x) = \sqrt{x}z(x)$ and $t = \frac{x^2}{2}$, therefore the above becomes

$$y(x) = \sqrt{x}J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) \tag{3}$$

Which is the same as found in part (b)

2.10.4 Problem 3

Problem Prove that $|J_n(x)| \leq 1$ for all integers n

Solution

From the integral representation of $J_n(x)$ for integer n

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta$$

Then

$$\begin{aligned}
|J_n(x)| &\le \frac{1}{\pi}\left|\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta\right|_{\max} \\
&\le \frac{1}{\pi}\int_0^\pi |\cos(n\theta - x\sin\theta)|_{\max}\,d\theta \\
&= \frac{1}{\pi}|M|_{\max}\int_0^\pi d\theta \\
&= \frac{1}{\pi}|M|_{\max}\,\pi \\
&= |M|_{\max}
\end{aligned}$$

Where $|M|_{\max} = |\cos(n\theta - x\sin\theta)|_{\max}$ over $\theta = 0\cdots\pi$. But this is 1 for the cosine function. Hence

$$|J_n(x)| \le 1$$

2.10.5 Problem 4

Problem Starting with the integral formula for hypergeometric function, express the following in terms of elementary functions ${}_2F_1(1,1,2;x)$ and ${}_2F_1(a,1,1;x)$

Solution

$$\begin{aligned}
{}_2F_1(a,b,c;x) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt \qquad (1)\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!} \qquad (2)
\end{aligned}$$

2.10.5.1 Part (a)

Here $a=1, b=1, c=2$. Therefore, using (1) representation gives

$$\begin{aligned}
{}_2F_1(1,1,2;x) &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(2-1)}\int_0^1 t^{1-1}(1-t)^{2-1-1}(1-tx)^{-1}\,dt \\
&= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 \frac{dt}{1-tx}
\end{aligned}$$

But $\Gamma(2) = 1, \Gamma(1) = 0$, therefore the above becomes

$$
\begin{aligned}
{}_2F_1(1,1,2;x) &= \int_0^1 \frac{dt}{1-tx} \\
&= \left[\frac{-\ln(1-tx)}{x}\right]_0^1 \\
&= -\left(\frac{\ln(1-x)}{x} - \frac{-\ln(1-0)}{x}\right) \\
&= -\frac{\ln(1-x)}{x}
\end{aligned}
$$

2.10.5.2 Part (b)

Here $a = a, b = 1, c = 1$. Therefore (2) representation gives

$$
\begin{aligned}
{}_2F_1(a,1,1;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!} \\
&= \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1+n)}{\Gamma(1+n)} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{x^n}{n!}
\end{aligned}
$$

Looking at few values

n	${}_2F_1(a,1,1;x)$
0	$\frac{\Gamma(a)}{\Gamma(a)} = 1$
1	$\frac{\Gamma(a+1)}{\Gamma(a)} x$
2	$\frac{\Gamma(a+2)}{\Gamma(a)} \frac{x^2}{2!}$
3	$\frac{\Gamma(a+3)}{\Gamma(a)} \frac{x^3}{3!}$
$\vdots$	$\vdots$

Using the recursive relation $\Gamma(a+1) = a\Gamma(a)$, which works for integer and non integer a, then we see that

$$\Gamma(a+1) = a\Gamma(a)$$

And

$$
\begin{aligned}
\Gamma(a+2) &= \Gamma((a+1)+1) \\
&= (a+1)\Gamma(a+1) \\
&= (a+1)\, a\Gamma(a)
\end{aligned}
$$

And

$$\begin{aligned}\Gamma(a+3) &= \Gamma((a+2)+1)\\ &= (a+2)\Gamma((a+2))\\ &= (a+2)(a+1)a\Gamma(a)\end{aligned}$$

And so on. Hence the above now becomes

n	${}_2F_1(a,1,1;x)$
0	1
1	$\frac{a\Gamma(a)}{\Gamma(a)}x = ax$
2	$\frac{(a+1)a\Gamma(a)}{\Gamma(a)}\frac{x^2}{2!} = a(a+1)\frac{x^2}{2!}$
3	$\frac{(a+2)(a+1)a\Gamma(a)}{\Gamma(a)}\frac{x^3}{3!} = a(a+1)(a+2)\frac{x^3}{3!}$
$\vdots$	$\vdots$

We see from the above the pattern of the sequence is as follows

$$ {}_2F_1(a,1,1;x) = 1 + ax + a(a+1)\frac{x^2}{2!} + a(a+1)(a+2)\frac{x^3}{3!} + \cdots \tag{1}$$

Comparing the above to the Binomial expansion given by

$$(1+z)^n = 1 + nz + n(n-1)\frac{z^2}{2!} + n(n-1)(n-2)\frac{z^3}{3!} + \cdots \tag{2}$$

By replacing $z \to -x$ and $n \to -a$, the above becomes

$$\begin{aligned}(1-x)^{-a} &= 1 + (-a)(-x) + (-a)((-a)-1)\frac{(-x)^2}{2!} + (-a)((-a)-1)((-a)-2)\frac{(-x)^3}{3!} + \cdots\\ &= 1 + ax + (a)(a+1)\frac{x^2}{2!} + (a)(a+1)(a+2)\frac{x^3}{3!} + \cdots\end{aligned}$$

Comparing the above to (1) shows it is the same series. Hence

$$ {}_2F_1(a,1,1;x) = (1-x)^{-a}$$

2.10.6 Key solution for HW 10

(1) Start with $J_m'' + \frac{1}{x} J_m' + \left(1 - \frac{m^2}{x^2}\right) J_m = 0$

and $J_n'' + \frac{1}{x} J_n' + \left(1 - \frac{n^2}{x^2}\right) J_n = 0$

Multiply the first by $x^l J_n$ and the second by $x^l J_m$ and subtract to get

$$x^l \left(J_m J_n'' - J_n J_m''\right) + x^{l-1}\left(J_m J_n' - J_n J_m'\right) + x^{l-2}\left(m^2 - n^2\right) J_m J_n = 0$$

The integral we are looking for suggests that we choose $l = 1$. Then

$$(m^2 - n^2) \frac{J_m J_n}{x} = x\left(J_n J_m'' - J_m J_n''\right) + \left(J_n J_m' - J_m J_n'\right)$$

$$= \frac{d}{dx}\left[x\left(J_n J_m' - J_m J_n'\right)\right]$$ which is a total derivative!

Then $(m^2 - n^2) \int_0^\infty J_m(x) J_n(x) \frac{dx}{x} =$

$$= \int_0^\infty \frac{d}{dx}\left[x\left(J_n J_m' - J_m J_n'\right)\right] dx =$$

$$= x\left(J_n J_m' - J_m J_n'\right)\Big|_0^\infty$$

Because $J_\ell(x) \sim x^\ell$ as $x \to 0$ the lower limit of integration will contribute zero as long as $m+n>0$. For the upper limit we use the asymptotic expressions

$$J_\ell(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \ell\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$J_\ell'(x) \sim -\sqrt{\frac{2}{\pi x}} \sin\left(x - \ell\frac{\pi}{2} - \frac{\pi}{4}\right) \quad \text{to get}$$

$$x\left(J_n J_m' - J_m J_n'\right)\Big|_0^\infty = \frac{2}{\pi}\left[\cos\theta_m \sin\theta_n - \sin\theta_m \cos\theta_n\right]$$

$$= \frac{2}{\pi}\sin(\theta_n - \theta_m) \quad \text{where} \quad \theta_\ell = x - \ell\frac{\pi}{2} - \frac{\pi}{4}.$$

$$= \frac{2}{\pi}\sin\left[(m-n)\frac{\pi}{2}\right]$$

Finally

$$\boxed{\int_0^\infty J_m(x)\, J_n(x)\, \frac{dx}{x} = \frac{2}{\pi}\,\frac{\sin\left[(m-n)\frac{\pi}{2}\right]}{m^2-n^2}}$$

This is valid for $m+n>0$.

(2) $J_n(z)$ satisfies $z^2 J_n''(z) + z J_n'(z) + (z^2-n^2)J_n(z) = 0$

Let $z = ax^k$. Then $\frac{d}{dx}J_n(z) = J_n'(z)\cdot\frac{dz}{dx} = akx^{k-1}J_n'(z)$

$$\frac{d^2}{dx^2}J_n(z) = \frac{d}{dx}\left[akx^{k-1}J_n'(z)\right] = ak(k-1)x^{k-2}J_n'(z) + a^2k^2x^{2(k-1)}J_n''(z)$$

$$\Rightarrow \quad J_n'(z) = \frac{1}{ak}x^{-k+1}\frac{d}{dx}J_n(z)$$

and
$$J_n''(z) = \frac{1}{a^2k^2}x^{-2(k-1)}\frac{d^2}{dx^2}J_n(z) - \frac{k-1}{ak}\frac{1}{x^k}\underbrace{J_n'(z)}_{\frac{1}{ak}x^{-k+1}\frac{d}{dx}J_n(z)}$$

$$= \frac{1}{a^2k^2}x^{-2k}\left[x^2\frac{d^2}{dx^2}J_n(z) - (k-1)x\frac{d}{dx}J_n(z)\right]$$

Now let $\boxed{\psi = x^m J_n(z)}$, $\quad \frac{d\psi}{dx} = mx^{m-1}J_n(z) + x^m\frac{d}{dx}J_n(z)$

$\Rightarrow \boxed{J_n(z) = x^{-m}\psi}$ and

$$\frac{d}{dx}J_n(z) = x^{-m}\frac{d\psi}{dx} - \frac{m}{x}J_n(z) = x^{-m}\frac{d\psi}{dx} - mx^{-(m+1)}\psi$$

$$\overset{\shortparallel}{akx^{k-1}J_n'(z)} \quad \Rightarrow \quad \boxed{J_n'(z) = \frac{1}{ak}x^{-(m+k-1)}\frac{d\psi}{dx} - \frac{m}{ak}x^{-(m+k)}\psi}$$

$$\frac{d^2}{dx^2} J_n(z) = \frac{d}{dx}\left[x^{-m}\frac{d\varphi}{dx} - m\,x^{-(m+1)}\varphi\right]$$

$$\shortparallel$$

$$a^2 k^2 x^{2(k-1)} J_n''(z) + a k(k-1)\, x^{k-2} J_n'(z)$$

$$= x^{-m}\frac{d^2\varphi}{dx^2} - 2m\,x^{-(m+1)}\frac{d\varphi}{dx} + m(m+1)\,x^{-(m+2)}\varphi$$

Solve for $J_n''(z)$.

substitute for this too

$$J_n''(z) = \frac{1}{a^2k^2} x^{-2(k-1)}\Big\{ -ak(k-1)\,x^{k-2} J_n'(z)$$

$$+ x^{-m}\frac{d^2\varphi}{dx^2} - 2m\,x^{-(m+1)}\frac{d\varphi}{dx} + m(m+1)\,x^{-(m+2)}\varphi\Big\}$$

$$= \frac{1}{a^2k^2} x^{-2(k-1)}\Big\{ -ak(k-1)\,x^{k-2}\Big[\frac{1}{ak} x^{-(m+k-1)}\frac{d\varphi}{dx}$$

$$- \frac{m}{ak} x^{-(m+k)}\varphi\Big] + x^{-m}\frac{d^2\varphi}{dx^2} - 2m\,x^{-(m+1)}\frac{d\varphi}{dx}$$

$$+ m(m+1)\,x^{-(m+2)}\varphi\Big\}$$

$$\boxed{J_n''(z) = \frac{1}{a^2k^2} x^{-(2k+m-2)}\frac{d^2\varphi}{dx^2} - \frac{(2m+k-1)}{a^2k^2} x^{-(2k+m-1)}\frac{d\varphi}{dx} + \frac{m(m+k)}{a^2k^2} x^{-(2k+m)}\varphi}$$

Now substitute the above into

$$z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) = 0$$

$$a^2 x^{2k} \left\{ \frac{1}{a^2 k^2} x^{-(2k+m-2)} \psi''(x) - \frac{(2m+k-1)}{a^2 k^2} x^{-(2k+m-1)} \psi'(x) \right.$$

$$\left. + \frac{m(m+k)}{a^2 k^2} x^{-(2k+m)} \psi \right\} + a x^k \left\{ \frac{1}{ak} x^{-(m+k-1)} \psi'(x) \right.$$

$$\left. - \frac{m}{ak} x^{-(m+k)} \psi(x) \right\} + (a^2 x^{2k} - n^2) x^{-m} \psi(x) = 0$$

Multiply by $k^2 x^m$ and group terms to get

$$\boxed{x^2 \psi''(x) + (1-2m) x \psi'(x) + \left[m^2 + k^2 (a^2 x^{2k} - n^2) \right] \psi(x) = 0}$$

When $m \to 0$ and $a \to 1$ and $k \to 1$ we recover the standard form of Bessel's equation.

We can solve $\psi'' + x^2 \psi = 0$ by choosing

$m = \frac{1}{2}$, $k = 2$, $a = \frac{1}{2}$, $n = \frac{1}{4}$

$$\Rightarrow \boxed{\psi = \sqrt{x}\, J_{\frac{1}{4}}\left(\frac{1}{2} x^2\right)}$$

(3) Use the representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$$

Since $|\cos\phi| \leq 1$ we have

$$|J_n(x)| \leq \frac{1}{\pi}\int_0^{\pi} d\theta = 1$$

$$\boxed{|J_n(x)| \leq 1}$$

(4) $${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt$$

$${}_2F_1(1,1;2;x) = \underbrace{\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}}_{1}\int_0^1 \frac{dt}{1-tx} = -\frac{1}{x}\ln(1-tx)\Big|_0^1$$

$$= -\frac{\ln(1-x)}{x}$$

$$\boxed{{}_2F_1(1,1;2;x) = -\frac{1}{x}\ln(1-x)}$$

$${}_2F_1(a,1;1;x) = \frac{\Gamma(1)}{\Gamma(1)\Gamma(0)}\int_0^1 (1-t)^{-1}(1-tx)^{-a}\,dt$$

But $\Gamma(0)$ is infinite so the integral representation is not well defined. Instead use

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}$$

to get $${}_2F_1(a,1;1;x) = \frac{\Gamma(1)}{\Gamma(a)\Gamma(1)}\sum_{n=0}^{\infty}\Gamma(a+n)\frac{x^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{x^n}{n!} = 1 + ax + \frac{a(a+1)}{2!}x^2 + \frac{a(a+1)(a+2)}{3!}x^3 + \cdots$$

$$\boxed{{}_2F_1(a,1;1;x) = (1-x)^{-a}}$$

2.11 HW 11

Local contents

2.11.1 HW 11 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 11 due Monday April 29. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (6 pts) Find the normal modes of a rectangular drum head with sides of length L_x and L_y.

2. (6 pts) Find the normal modes for acoustic waves in a hollow sphere of radius R. The wave equation is

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

with boundary conditions $\partial\psi/\partial r = 0$ at $r = 0$ and at $r = R$. What is the lowest frequency?

3. (6 pts) A sphere of radius R is at temperature $T = 0$. At time $t = 0$ it is immersed in a heat bath of temperature T_0. What is the temperature distribution $T(r, t)$ as a function of time?

4. (7 pts) Consider the Helmholz equation

$$\nabla^2 u(r,\theta) + k^2 u(r,\theta) = 0$$

inside the circle $r = R$ with the boundary condition $u(R,\theta) = f(\theta)$. The solution can be written in the form

$$u(r,\theta) = \int_0^{2\pi} f(\theta')G(r,\theta;\theta')d\theta'$$

Find the Green function G.

2.11.2 Problem 1

Find the normal modes of a rectangular drum with sides of length L_x and L_y

solution

The geometry of the problem is

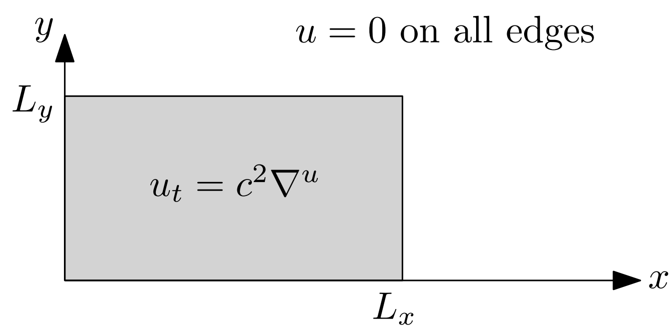


Figure 2.37: Problem to solve

Using Cartesian coordinates. Wave displacement is $u \equiv u\left(x, y, t\right)$ (out of page).

$$
\begin{gathered}
\frac{\partial^2 u\left(x, y, t\right)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
0 < x < L_x \\
0 < y < L_y
\end{gathered}
$$

Boundary conditions on x

$$
\begin{aligned}
u\left(0, y, t\right) &= 0 \\
u\left(L_x, y, t\right) &= 0
\end{aligned}
$$

And boundary conditions on y

$$
\begin{aligned}
u\left(x, 0, t\right) &= 0 \\
u\left(x, L_y, t\right) &= 0
\end{aligned}
$$

Solution

Let $u = X(x)\,Y\left(y\right)T(t)$. Substituting into the PDE gives

$$
\begin{gathered}
\frac{1}{c^2}T''XY = X''YT + Y''XT \\
\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}
\end{gathered}
$$

Hence, using λ as first separation constant we obtain

$$
\begin{aligned}
\frac{1}{c^2}\frac{T''}{T} &= -\lambda \\
\frac{X''}{X} + \frac{Y''}{Y} &= -\lambda
\end{aligned}
$$

The time ODE becomes

$$T'' + c^2\lambda T = 0$$

And the space ODE becomes

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Separating the space ODE again

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

Where μ is the new separation variable. This gives two new separate ODE's

$$\frac{X''}{X} = -\mu$$

$$-\lambda - \frac{Y''}{Y} = -\mu$$

Or

$$X'' + \mu X = 0$$

$$Y'' + Y\left(\lambda - \mu\right) = 0$$

Solving for X ODE first, and knowing that $\mu > 0$ from nature of boundary conditions, we obtain

$$X(x) = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$

Applying B.C. at $x = 0$

$$0 = A$$

Hence $X(x) = B\sin\left(\sqrt{\mu}x\right)$. Applying B.C. at $x = L_x$

$$0 = B\sin\left(\sqrt{\mu}L_x\right)$$

Hence

$$\sqrt{\mu}L_x = n\pi$$

$$\mu_n = \left(\frac{n\pi}{L_x}\right)^2 \qquad n = 1,2,3,\cdots \tag{1}$$

Therefore the $X_n(x)$ solution is

$$X_n(x) = B_n\sin\left(\frac{n\pi}{L_x}x\right) \qquad n = 1,2,3,\cdots \tag{2}$$

Solving the $Y\left(y\right)$ ODE using the same eigenvalues found above

$$Y'' + Y\left(\lambda - \left(\frac{n\pi}{L_x}\right)^2\right) = 0$$

The solution is

$$Y(y) = C\cos\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}y\right) + D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}y\right)$$

Applying first B.C. $Y(0) = 0$ gives

$$0 = C$$

Hence

$$Y(y) = D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}y\right)$$

Applying second B.C. $Y(L_y) = 0$

$$0 = D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}L_y\right)$$

Hence

$$\begin{aligned}
\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}L_y &= m\pi \qquad m = 1,2,3,\cdots \\
\lambda_{nm} - \left(\frac{n\pi}{L_x}\right)^2 &= \left(\frac{m\pi}{L_y}\right)^2 \\
\lambda_{nm} &= \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_x}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots
\end{aligned}$$

Hence the Y_{nm} solution is

$$Y_{nm} = D_{nm}\sin\left(\frac{m\pi}{L_y}y\right) \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots$$

We notice that $X_n(x)$ solution depends on n only, while $Y_{nm}(y)$ solution depends on n and m. Now that we found λ we can we solve the time $T(t)$ ode

$$\begin{aligned}
T''_{nm} + c^2\lambda_{nm}T_{nm} &= 0 \\
T_{nm}(t) &= E_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + F_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)
\end{aligned}$$

Combining all solution , and merging all constants into two, we find

$$\begin{aligned}
u_{nm}\left(x,y,t\right) &= X_n\left(x\right)Y_{nm}\left(y\right)T_{nm}\left(t\right)\\
&= \left(B_nX_n\right)\left(D_{nm}\sin\left(\frac{m\pi}{L_y}y\right)\right)\left(E_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)\\
&= B_nX_n\sin\left(\frac{m\pi}{L_y}y\right)\left(E'_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F'_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)\\
&= X_n\sin\left(\frac{m\pi}{L_y}y\right)\left(E''_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F''_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)
\end{aligned}$$

Where E''_{nm}, F''_{nm} are the new constants after merging them with the other constants. Renaming $E''_{nm} = A_{nm}, F''_{nm} = B_{nm}$ the above solution can be written as

$$\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}X_n\left(x\right)Y_{mn}\left(y\right)T_{mn}\left(t\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\cos\left(c\sqrt{\lambda_{nm}}t\right)\\
&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}\sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\sin\left(c\sqrt{\lambda_{nm}}t\right)
\end{aligned}\tag{3}$$

To solve this completely, we apply initial conditions to find A_{nm}, B_{nm}. But the problem is just asking for the normal modes. These are given by $X_n\left(x\right)Y_{mn}\left(y\right)$. Therefore for $n=1$, we have the modes $\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right), \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right), \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right), \cdots$ and for $n=2$ we have $\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right), \sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right), \sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right), \cdots$ and so on.

n	$m=1$	2	3	4
1	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
2	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
3	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

To draw these modes, let us assume that $L_x = 1, L_y = 1$. This gives

n	$m=1$	2	3	4
1	$\sin(\pi x)\sin(\pi y)$	$\sin(\pi x)\sin(2\pi y)$	$\sin(\pi x)\sin(3\pi y)$	$\cdots$
2	$\sin(2\pi x)\sin(\pi y)$	$\sin(2\pi x)\sin(2\pi y)$	$\sin(2\pi x)\sin(3\pi y)$	$\cdots$
3	$\sin(3\pi x)\sin(\pi y)$	$\sin(3\pi x)\sin(2\pi y)$	$\sin(3\pi x)\sin(3\pi y)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The following is a plot of the above modes for illustrations with the code used to generate these plots.

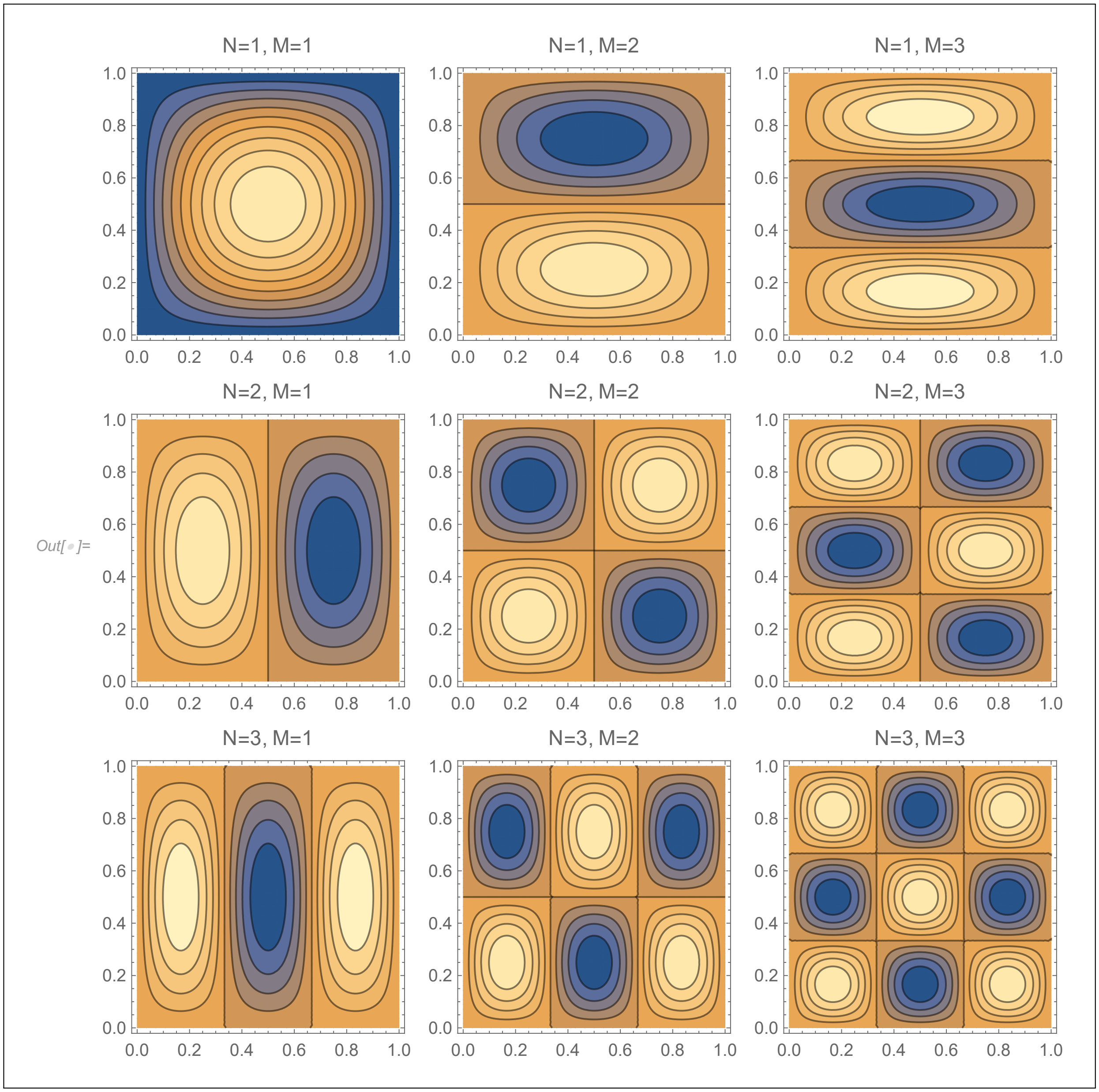


Figure 2.38: Modes using $L_x = 1, L_y = 1$

```
makePlot[n_, m_] :=
    ContourPlot[Sin[n Pi x] * Sin[m Pi y], {x, 0, 1}, {y, 0, 1},
      PlotLegends → None,
      Frame → True, FrameLabel → {{None, None}, {None, Style[Row[{"N=", n, ", M=", m}], 12]}}];
Grid@Table[makePlot[n, m], {n, 1, 3}, {m, 1, 3}]
```

Figure 2.39: Code used to draw above plot

The following is 3D view of the above modes.

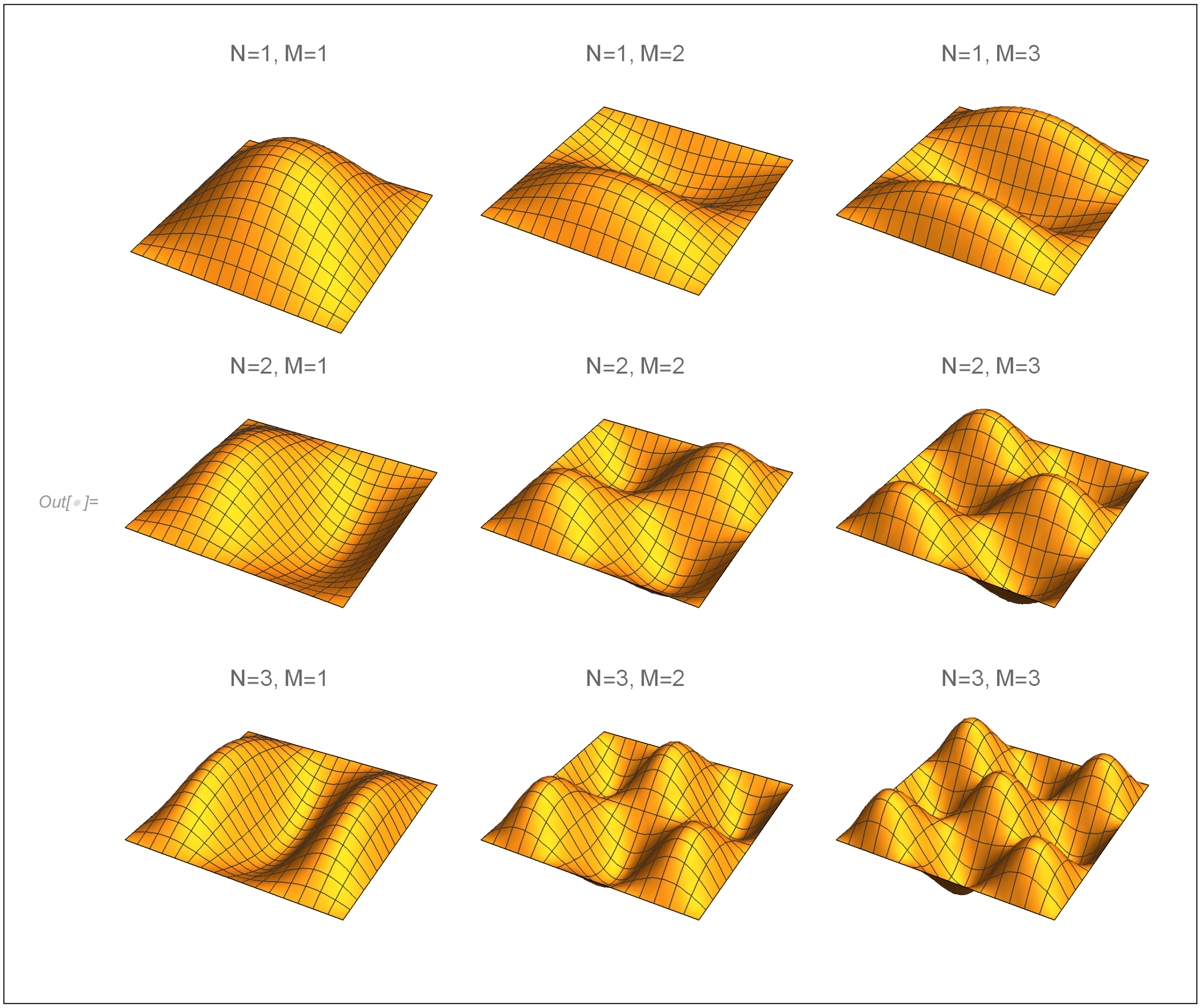


Figure 2.40: 3D view of the modes using $L_x = 1, L_y = 1$

```
In[ ]:= makePlot[n_, m_] :=
          Plot3D[Sin[n Pi x] * Sin[m Pi y], {x, 0, 1}, {y, 0, 1},
            PlotLabel → Style[Row[{"N=", n, ", M=", m}], 12],
            Boxed -> False, Axes → False
          ];
        Grid@Table[makePlot[n, m], {n, 1, 3}, {m, 1, 3}]
```

Figure 2.41: Code used to draw above plot

2.11.3 Problem 2

Find the normal modes of an acoustic waves in a hollow sphere of radius R. The wave equation is

$$\nabla^2\psi\left(r,\theta,\phi,t\right)=\frac{1}{c^2}\psi_{tt}$$

With boundary conditions $\psi_r=0$ at $r=0$ and at $r=r_0$. (I used r_0 in place of R because wanted to use $R(r)$ for separation of variables).

What is the lowest frequency?

solution

Let

$$\psi\left(r,\theta,\phi,t\right)=u\left(r,\theta,\phi\right)e^{-i\omega t}$$

Substituting this back in the original PDE gives

$$\nabla^2 u\left(r,\theta,\phi\right)+\frac{\omega^2}{c^2}u\left(r,\theta,\phi\right)=0$$

Let $k=\frac{\omega}{c}$ (wave number) and the above becomes

$$\nabla^2 u+k^2u=0 \tag{1}$$

The above is called the Helmholtz PDE. In spherical coordinates it becomes

$$\overbrace{u_{rr}+\frac{2}{r}u_r}^{\text{Radial part}}+\overbrace{\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta+u_{\theta\theta}\right)+\frac{1}{r^2\sin^2\theta}u_{\phi\phi}}^{\text{Angular part}}+k^2u=0$$

Let $u\left(r,\theta,\phi\right)=R(r)\Theta(\theta)\Phi\left(\phi\right)$ and the above becomes

$$R''T\Theta\Phi+\frac{2}{r}R'T\Theta\Phi+\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\Theta'RT\Phi+\Theta''RT\Phi\right)+\frac{1}{r^2\sin^2\theta}\Phi''R\Theta T+k^2R\Theta T=0$$

Dividing by $R\Theta\Phi\neq 0$ gives

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}+k^2=0$$

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=-\frac{\Phi''}{\Phi}$$

The left side depends only on r,θ and the right side depends only on ϕ. Let the second separation constant be m^2 and the above becomes

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=-\frac{\Phi''}{\Phi}=m^2 \tag{2}$$

Which gives the first angular ODE as

$$\Phi''+m^2\Phi=0 \tag{2A}$$

We now go back to (2) to obtain the rest of the solutions. We now have

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=m^2$$

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)+\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)=\frac{m^2}{\sin^2\theta}$$

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)=-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}$$

The left side depends on r and the right side depends on θ only. Let the separation constant be $l(l+1)$ where l is integer which results in

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)=-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}=l(l+1) \tag{3}$$

Therefore the next angular ODE is

$$-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}=l(l+1)$$

$$-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}-l(l+1)=0$$

$$\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)-\frac{m^2}{\sin^2\theta}+l(l+1)=0$$

$$\Theta''+\frac{\cos\theta}{\sin\theta}\Theta'+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0 \tag{4}$$

Let $z=\cos\theta$, then $\frac{d\Theta}{d\theta}=\frac{d\Theta}{dz}\frac{dz}{d\theta}=-\frac{d\Theta}{dz}\sin\theta$ and

$$\begin{aligned}\frac{d^2\Theta}{d\theta^2}&=\frac{d}{d\theta}\left(-\frac{d\Theta}{dz}\sin\theta\right)\\&=-\frac{d^2\Theta}{dz^2}\frac{dz}{d\theta}\sin\theta-\frac{d\Theta}{dz}\cos\theta\\&=\frac{d^2\Theta}{dz^2}\sin^2\theta-\frac{d\Theta}{dz}\cos\theta\end{aligned}$$

But $\sin^2\theta=1-\cos^2\theta=1-z^2$ and the above becomes

$$\frac{d^2\Theta}{d\theta^2}=\frac{d^2\Theta}{dz^2}\left(1-z^2\right)-\frac{d\Theta}{dz}z$$

Using these in (4) gives

$$\frac{d^2\Theta}{dz^2}\left(1-z^2\right)-\frac{d\Theta}{dz}z+\frac{z}{\sin\theta}\left(-\frac{d\Theta}{dz}\sin\theta\right)+\left(l(l+1)-\frac{m^2}{1-z^2}\right)\Theta(z)=0$$

$$\left(1-z^2\right)\Theta''-2z\Theta'+\left(l(l+1)-\frac{m^2}{1-z^2}\right)\Theta(z)=0 \tag{3A}$$

And finally, we obtain the final ODE, which is the radial ODE from (3)

$$
\begin{aligned}
k^2r^2 + r^2\left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R}\right) &= l\left(l+1\right)\\
k^2r^2R + r^2\left(R'' + \frac{2}{r}R'\right) - l\left(l+1\right)R &= 0\\
r^2R'' + 2rR' + \left(k^2r^2 - l\left(l+1\right)\right)R &= 0\\
R'' + \frac{2}{r}R' + \left(k^2 - \frac{l\left(l+1\right)}{r^2}\right)R &= 0 \qquad \text{(4A)}
\end{aligned}
$$

In summary we have obtained the following 4 ODE's to solve (1A,2A,3A,4A)

$$
\begin{aligned}
\Phi'' + m^2\Phi &= 0 \qquad \text{(2A)}\\
\left(1-z^2\right)\Theta'' - 2z\Theta' + \left(l\left(l+1\right) - \frac{m^2}{1-z^2}\right)\Theta\left(z\right) &= 0 \qquad \text{(3A)}\\
R'' + \frac{2}{r}R' + \left(k^2 - \frac{l\left(l+1\right)}{r^2}\right)R &= 0 \qquad \text{(4A)}
\end{aligned}
$$

Solution to (2A) requires m to be integer due to periodicity requirements of solution. The solution is $\Phi\left(\phi\right) = e^{\pm im\phi}$. Equation (3A) is the associated Legendre ODE. Since we are taking l as integer then the solution is known to be $\Theta\left(z\right) = P_l^m\left(z\right) + Q_l^m\left(z\right)$ where $P_l^m\left(z\right)$ is called the associated Legendre polynomial and Q_l^m is the Legendre function of the second kind. Finally (4A) can be converted to Bessel ODE as shown in class notes using the transformation $R\left(r\right) = \frac{u(r)}{\sqrt{r}}$ which results in

$$
u'' + \frac{1}{r}u' + \left(k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right)u = 0
$$

Which has solution $J_{l+\frac{1}{2}}\left(kr\right)$. The second solution $J_{-\left(l+\frac{1}{2}\right)}\left(kr\right)$ is rejected since it is not finite at zero and hence makes the solution blow up at center of sphere. Therefore solution to (4A) is

$$
\begin{aligned}
R\left(r\right) &= C\sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}\left(kr\right)\\
&= Cj_l\left(kr\right)
\end{aligned}
$$

Where C is arbitrary constant. Putting all the above together, then the final solution is

$$
\psi\left(r,\theta,\phi,t\right) = \left\{ e^{-i\omega t} \right. \left\{ \begin{array}{c} e^{im\phi} \\ e^{-im\phi} \end{array} \right. \left\{ \begin{array}{c} P_l^m\left(\cos\theta\right) \\ Q_l^m\left(\cos\theta\right) \end{array} \right. \left\{ j_l\left(kr\right) \right.
$$

Where $j_l\left(kr\right)$ are the spherical Bessel functions. Now we need to satisfy the boundary conditions. Since only $j_l\left(kr\right)$ depends on r, then $\psi_r = 0$ at $r = 0$ and at $r = r_0$ are equivalent to looking at $R'\left(r\right) = 0$ at $r = 0$ and $r = r_0$. Therefore we need to find the smallest l,k which satisfy both conditions. This will give the lowest frequency.

I found from DLMF that the series expansion of $j_l(kr)$ is

$$j_l(kr) = \frac{(kr)^l}{(2l+1)!!}\left(1 - \frac{(kr)^2}{2(2l+3)} + \frac{(kr)^4}{8(2l+5)(2l+3)} + \cdots\right) \tag{5}$$

Hence for $r \to 0$, we can approximate the above as the following by ignoring all higher order terms

$$\lim_{r\to 0} j_l(kr) = \frac{(kr)^l}{(2l+1)!!}$$

Which means for small r, the derivative is

$$\frac{d}{dr} j_l(kr) = \frac{l(kr)^{l-1}}{(2l+1)!!}$$

At $r = 0$ then setting $\left[\frac{d}{dr} j_l(kr)\right]_{r\to 0} = 0$ is satisfied for all l. Now taking derivative of (5) gives

$$\frac{d}{dr} j_l(kr) = \frac{l(kr)^{l-1}}{(2l+1)!!}\left(1 - \frac{(kr)^2}{2(2l+3)} + \frac{(kr)^4}{8(2l+5)(2l+3)} + \cdots\right) + \frac{(kr)^l}{(2l+1)!!}\left(1 - \frac{2(kr)}{2(2l+3)} + \frac{4(kr)^3}{8(2l+5)(2l+3)} + \cdots\right)$$

At $r = r_0$ the above becomes

$$\left[\frac{d}{dr} j_l(kr)\right]_{r\to r_0} = \frac{l(kr_0)^{l-1}}{(2l+1)!!}\left(1 - \frac{(kr_0)^2}{2(2l+3)} + \frac{(kr_0)^4}{8(2l+5)(2l+3)} + \cdots\right) + \frac{(kr_0)^l}{(2l+1)!!}\left(1 - \frac{2(kr_0)}{2(2l+3)} + \frac{4(kr_0)^3}{8(2l+5)(2l+3)} + \cdots\right)$$

Now we ask, for which values of l is the above zero? If we let $l \to \infty$ then we obtain

$$\begin{aligned}\left[\frac{d}{dr} j_l(kr)\right]_{\substack{r\to r_0\\ l\to\infty}} &= \lim_{l\to\infty} \frac{l(kr_0)^{l-1}}{(2l+1)!!} + \frac{(kr_0)^l}{(2l+1)!!}\\ &= 0\end{aligned}$$

Therefore, to satisfy both $\left[\frac{d}{dr} j_l(kr)\right]_{r\to 0} = 0$ and $\left[\frac{d}{dr} j_l(kr)\right]_{r\to r_0} = 0$ we need $l \to \infty$. In other words, a very large integer. The larger l is, the lower the radial frequency. In addition, increasing k while keeping l fixed will increase the frequency. And decreasing k while keeping l fixed decreases the frequency. And for fixed k, increasing l decreases the frequency.

2.11.4 Problem 3

A sphere of radius R is at temperature $u = 0$. At time $t = 0$ it is immersed in a heat bath of temperature u_0. What is the temperature distribution $u(r,t)$ as function of time?

solution

Note: I Used $u(r,t)$ instead of $T(r,t)$ as the dependent variable to allow using $T(t)$ for separation of variables without confusing it with the original $T(r,t)$.

The PDE specification is, solve for $u(r,t)$

$$u_t = k\nabla^2 u \qquad t>0, 0<r<R$$

With initial conditions

$$u(r,0)=0$$

And boundary conditions

$$u(R,t)=u_0$$
$$|u(0,t)|<\infty$$

Where the second B.C. above means the temperature u is bounded at origin (center of sphere). In spherical coordinates, the PDE becomes (There are no dependency on θ,ϕ due to symmetry), and only radial dependency.

$$\frac{1}{k}u_t = \frac{1}{r}(ru)_{rr} \tag{1}$$

To simplify the solution, let

$$U(r,t) = ru(r,t)$$

And we obtain a new PDE

$$\frac{1}{k}U_t = U_{rr} \tag{2}$$

And the boundary conditions $u(R,t)=u_0$ becomes $U(R,t)=Ru_0$ and the initial conditions becomes $U(r,0)=0$. So we will solve (2) and not (1). But since the boundary conditions are not homogenous, we can not use separation of variables. We introduce a reference function $w(r)$ which need to satisfy the nonhomogeneous boundary conditions only. Let $w(r)=Br$. When $r=R$ then $Ru_0=BR$ or $B=u_0$ When $r=0$ then $w=0$ which is bounded. Hence

$$w(r)=u_0 r$$

Therefore, the solution now can be written as

$$U(r,t) = v(r,t) + u_0 r \tag{3}$$

Where $v(r,t)$ now satisfies the PDE but with homogenous B.C. Substituting (3) into (2) gives

$$v_t = k\frac{\partial^2}{\partial r^2}(v(r,t)+u_0 r)$$
$$v_t = kv_{rr}(r,t) \tag{4}$$

We need to solve the above but with homogenous boundary conditions

$$v(R,t)=0$$
$$|v(0,t)|<\infty$$

This is standard PDE, who can be solved by separation of variables. let $v=F(r)T(t)$, hence

(4) becomes

$$T'F = kF''T$$
$$k\frac{T'}{T} = \frac{F''}{F} = -\lambda^2$$

Which gives

$$F'' + \lambda^2 F = 0$$

Due to boundary conditions only $\lambda > 0$ is eigenvalues. Hence solution is

$$F(r) = A\cos(\lambda r) + B\sin(\lambda r)$$

At $r = 0$, since bounded, say 0, then we can take $A = 0$, leaving the solution

$$F(r) = B\sin(\lambda r)$$

At $r = R$

$$0 = B\sin(\lambda R)$$

For nontrivial solution

$$\lambda R = n\pi \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = \frac{n\pi}{R}$$

Hence eigenfunctions are

$$F_n(r) = \sin\left(\frac{n\pi}{R}r\right) \qquad n = 1, 2, 3, \cdots$$

The time ODE is therefore $T' + \lambda^2 kT = 0$ with solution $T_n(t) = A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt}$. Hence the solution to (4) is

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)$$

Therefore from (3)

$$U(r,t) = \left(\sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)\right) + u_0 r$$

But $U(r,t) = ru(r,t)$, hence

$$u(r,t) = \left(\frac{1}{r}\sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)\right) + u_0 \tag{5}$$

Now we find A_n from initial conditions. At $t = 0$

$$0 = u_0 + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{R}r\right)$$
$$-ru_0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{R}r\right)$$

Therefore A_n are the Fourier series coefficients of $-ru_0$

$$\begin{aligned}
\frac{R}{2}A_n &= -\int_0^R ru_0 \sin\left(\frac{n\pi}{R}r\right) dr \\
A_n &= -\frac{2u_0}{R}\int_0^R r \sin\left(\frac{n\pi}{R}r\right) dr \\
&= -\frac{2u_0}{R}(-1)^{n+1}\frac{R^2}{n\pi} \\
&= (-1)^n \frac{2R}{n\pi}u_0
\end{aligned}$$

Hence the solution (5) becomes

$$\begin{aligned}
u(r,t) &= u_0 + u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right) \\
&= u_0\left(1 + \frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right)\right) \qquad (7)
\end{aligned}$$

Verification of solution

Verification that (7) satisfies the PDE $u_t = k\nabla^2 u$. Taking time derivative of (7) gives

$$u_t = -u_0\frac{2R}{r\pi}k\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}\left(\frac{n\pi}{R}\right)^2 e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right) \qquad (8)$$

And taking space derivatives of (7) gives

$$\begin{aligned}
u_x &= u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\frac{n\pi}{R}\cos\left(\frac{n\pi}{R}r\right) \\
u_{xx} &= -u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\left(\frac{n\pi}{R}\right)^2\sin\left(\frac{n\pi}{R}r\right)
\end{aligned}$$

Hence ku_{xx} becomes

$$ku_{xx} = -u_0\frac{2R}{r\pi}k\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\left(\frac{n\pi}{R}\right)^2\sin\left(\frac{n\pi}{R}r\right) \qquad (9)$$

Comparing (8) and (9) shows they are the same expressions.

Verification that (7) satisfies the boundary condition.

When $r = R$, therefore (7) gives, when replacing r by R

$$\begin{aligned}
u(R,t) &= u_0\left(1 + \frac{2R}{R\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}R\right)\right) \\
&= u_0\left(1 + \frac{2R}{R\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin(n\pi)\right) \\
&= u_0(1+0) \\
&= u_0
\end{aligned}$$

But n is integer. Hence $\sin(n\pi) = 0$ for all n. And the above becomes

$$\begin{aligned} u(R,t) &= u_0(1+0) \\ &= u_0 \end{aligned}$$

Verified.

Verification that (7) satisfies the initial conditions $u(r,0) = 0$ for $r < R$.

At $t = 0$ (7) becomes

$$\begin{aligned} u(r,0) &= u_0\left(1 + \frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\sin\left(\frac{n\pi}{R}r\right)\right) \\ &= u_0 + \frac{2R}{r\pi}u_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{R}r\right) \\ &= u_0 + \frac{2R}{r\pi}u_0\left(-\sin\left(\frac{\pi}{R}r\right) + \frac{1}{2}\sin\left(\frac{2\pi}{R}r\right) - \frac{1}{3}\sin\left(\frac{3\pi}{R}r\right) + \frac{1}{4}\sin\left(\frac{4\pi}{R}r\right) - \cdots\right) \end{aligned}$$

I could not simplify the above by hand, but using the computer, I verified numerically it is zero for $0 < r < R$ for a given R and given u_0.

```
In[∘]:= ClearAll[R, r]
        R = 1; (*radius*)
        u0 = 10; (*B.C. value*)
        s = Sum[(-1)^n 1/n Sin[n Pi/R r], {n, 1, Infinity}] (*obtain sum*)
        Table[Chop[u0 + (2 R)/(r Pi) u0*s], {r, 0.05, R, .05}]

Out[∘]= -1/2 ⅈ (-Log[1 + ⅇ^(ⅈ π r)] + Log[ⅇ^(-ⅈ π r) (1 + ⅇ^(ⅈ π r))])

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Figure 2.42: Obtaining the sum using the computer

2.11.5 Problem 4

Consider the Helmholtz equation

$$\nabla^2 u(r,\theta) + k^2 u(r,\theta) = 0 \tag{1}$$

inside the circle $r = r_0$ with the boundary condition $u(r_0,\theta) = f(\theta)$. The solution can be written in the form $u(r,\theta) = \int_0^{2\pi} f(\theta') G(r,\theta;\theta')\,d\theta'$. Find the Green function G.

solution

I will solve (1) directly and then compare the solution obtain to $u(r,\theta)=\int_0^{2\pi} f(\theta')\,G(r,\theta;\theta')\,d\theta'$ in order to read off the Green function expression. (1) in polar coordinates becomes

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+k^2u=0$$

Writing $u(r,\theta)=R(r)\Theta(\theta)$, the above PDE becomes

$$\begin{aligned}
R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}\Theta''R+k^2R\Theta&=0\\
\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+k^2&=0\\
r^2\frac{R''}{R}+r\frac{R'}{R}+r^2k^2&=-\frac{\Theta''}{\Theta}=m
\end{aligned}$$

Where m is the separation constant. The eigenvalue problem is taken as

$$\Theta''+m\Theta=0$$

Due to periodicity of the solution on the disk, then $\Theta(-\pi)=\Theta(\pi)$ and $\Theta'(-\pi)=\Theta'(\pi)$. These boundary conditions restrict m to only positive integer values. Hence let $m=n^2$ and the solution to the above becomes

$$\Theta_\alpha(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)$$

Now the radial ODE is

$$\begin{aligned}
r^2\frac{R''}{R}+r\frac{R'}{R}+r^2k^2&=\alpha^2\\
r^2R''+rR'+\left(r^2k^2-n^2\right)R&=0\\
R''+\frac{1}{r}R'+\left(k^2-\frac{n^2}{r^2}\right)R&=0
\end{aligned}$$

This is Bessel ODE whose solutions are (since n are integers) is

$$R_\alpha(r)=C_nJ_n(kr)+E_nY_n(kr)$$

But $Y_n(kr)$ blows up at $r=0$, hence it is rejected leaving solution $R_n(r)=C_nJ_n(kr)$. Hence the final solution is

$$u(r,\theta)=\sum_{m=1}^{\infty}\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)J_n(kr) \tag{2}$$

Where the constant C_n is merged with the other two constants. Now, at $r=r_0$ we are told that $u(r_0,\theta)=f(\theta)$. Hence the above becomes

$$f(\theta)=\sum_{m=1}^{\infty}\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)J_n(kr_0)$$

By orthogonality of $\cos(n\theta),\sin(n\theta)$ we find the Fourier cosine and Fourier sine coefficients

A_n, B_n as

$$A_n J_n(kr_0)\frac{1}{\pi} = \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$
$$B_n J_n(kr_0)\frac{1}{\pi} = \int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

Substituting the above back into the solution found in (2) results in

$$\begin{aligned}
u(r,\theta) &= \sum_{m=1}^{\infty}\left[\left(\frac{\pi}{J_n(kr_0)}\int_0^{2\pi} f(\theta')\cos(n\theta')\,d\theta'\right)\cos(n\theta) + \left(\frac{\pi}{J_n(kr_0)}\int_0^{2\pi} f(\theta')\sin(n\theta')\,d\theta'\right)\sin(n\theta)\right]J_n(kr)\\
&= \sum_{m=1}^{\infty}\frac{\pi}{J_n(kr_0)}\left(\int_0^{2\pi} f(\theta')\cos(n\theta')\cos(n\theta)\,d\theta' + \int_0^{2\pi} f(\theta')\sin(n\theta')\sin(n\theta)\,d\theta'\right)J_n(kr)
\end{aligned} \tag{3}$$

Using trig relations

$$\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$$
$$\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$$

Then (3) becomes

$$u(r,\theta) = \sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left(\int_0^{2\pi} f(\theta')(\cos(n(\theta'+\theta))+\cos(n(\theta'-\theta)))\,d\theta' + \int_0^{2\pi} f(\theta')(\cos(n(\theta'-\theta))-\cos(n(\theta'+\theta)))\,d\theta'\right)J_n(kr)$$

Which is simplified to, after combining both integrals to one

$$\begin{aligned}
u(r,\theta) &= \sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left(\int_0^{2\pi} f(\theta')(\cos(n(\theta'+\theta))+\cos(n(\theta'-\theta))+\cos(n(\theta'-\theta))-\cos n(\theta'+\theta))\,d\theta'\right)J_n\\
&= \sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left[\int_0^{2\pi} f(\theta')\,2\cos(\theta'-\theta)\,d\theta'\right]J_n(kr)\\
&= \sum_{m=1}^{\infty}\left[\int_0^{2\pi} f(\theta')\frac{\pi}{J_n(kr_0)}\cos(\theta'-\theta)\,d\theta'\right]J_n(kr)
\end{aligned}$$

Exchanging integration with summation gives

$$u(r,\theta) = \int_0^{2\pi} f(\theta')\left(\sum_{m=1}^{\infty}\frac{\pi}{J_n(kr_0)}\cos(\theta'-\theta)J_n(kr)\right)d\theta'$$

Comparing the above to

$$u(r,\theta) = \int_0^{2\pi} f(\theta')\,G(r,\theta;\theta')\,d\theta'$$

Shows that Green function is

$$G(r,\theta;\theta') = \sum_{m=1}^{\infty}\frac{\pi}{J_n(kr_0)}\cos(\theta'-\theta)J_n(kr)$$

Where r_0 is radius of disk. It is symmetric in θ as expected.

2.11.6 Key solution for HW 11

① $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$ $\quad 0 \le x \le L_x$

$0 \le y \le L_y$

$\psi(x,y,t) = f(x)\,g(y)\,T(t)$ separation of variables

$$\frac{1}{fgT}\left[f''gT + fg''T - \frac{1}{c^2}fgT''\right] = 0$$

$$\frac{f''}{f} + \frac{g''}{g} = \frac{1}{c^2}\frac{T''}{T} = \text{constant} = -k^2$$

$$T'' = -k^2c^2T \Rightarrow T(t) = A_1\cos\omega t + B_1\sin\omega t \qquad \omega = kc$$

$$\frac{f''}{f} = -\frac{g''}{g} + \text{constant} = -p^2$$

$$\Rightarrow f(x) = A_2\cos px + B_2\sin px$$

and $g(y) = A_3\cos qy + B_3\sin qy$

Now k, p, and q are related by $p^2+q^2 = k^2$

Boundary conditions ; $f(0) = f(L_x) = 0 \Rightarrow A_2 = 0$

$g(0) = g(L_y) = 0 \Rightarrow A_3 = 0$

$$f(L_x) = B_2\sin(pL_x) = 0 \Rightarrow pL_x = m\pi \qquad m = 1,2,3,\ldots$$

$$g(L_y) = B_3\sin(qL_y) = 0 \Rightarrow qL_y = n\pi \qquad n = 1,2,3,\ldots$$

Thus $k^2 = \left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right)\pi^2$

$$\boxed{\omega_{mn} = \sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}}\,\pi c}$$

$$\boxed{\Psi_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right) \cdot \left[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right]}$$

(2) $$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

We know that the solution which is zero or finite at the origin is

$$\psi(r,\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr)\, Y_{lm}(\theta,\phi)\left[A_{lm}\cos(\omega t) + B_{lm}\sin(\omega t)\right]$$ where $\omega = kc$.

We only need to apply the boundary conditions.

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{4}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+l+\frac{3}{2})}\left(\frac{x}{2}\right)^{l+2n}$$

$$\frac{d}{dx}j_l(x) = \frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(l+2n)}{n!\,\Gamma(n+l+\frac{3}{2})}\left(\frac{x}{2}\right)^{l+2n-1} \quad \text{for } l \ge 1$$

$$\frac{d}{dx}j_0 = \frac{d}{dx}\frac{\sin x}{x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Thus $\left.\frac{d}{dx}j_0(x)\right|_{x=0} = 0$

$$\left.\frac{d}{dx}j_1(x)\right|_{x=0} = \frac{1}{3} \qquad \left.\frac{d}{dx}j_l(x)\right|_{x=0} = 0 \quad \text{for } l \ge 2$$

Hence $l = 1$ is excluded.

For the rest we require that

$\frac{d}{dx} j_l(x)\Big|_{x=kR} = 0$. There are an infinite number of x's which satisfy this condition. Label them $x_{l,n}$ where n is the n'th one.

Finally
$$\boxed{\Psi(r,\theta,\phi,t) = \sum_{\substack{l=0 \\ l\neq 1}}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} j_l(k_{l,n} r)\, Y_{lm}(\theta,\phi) \cdot \left[A_{lmn} \cos(k_{l,n} c t) + B_{lmn} \sin(k_{l,n} c t) \right]}$$

Lowest frequency corresponds to $l = 0$, $n = 1$.

$\frac{d}{dx} j_0(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$ so $\tan x_{0,1} = x_{0,1}$

This is the same as encountered in lecture.

$x_{0,1} \approx 4.49$ so $\boxed{\omega_{0,1} = 4.49 \frac{c}{R}}$

(3) In class we showed that

$$T(\vec{x},t) = \int d^3x' \, T(\vec{x}',0)\, G(\vec{x},t;\vec{x}') \quad \text{with}$$

$$G(\vec{x},t;\vec{x}') = \frac{e^{-(\vec{x}-\vec{x}')^2/4\kappa t}}{(4\pi\kappa t)^{3/2}}$$

Now $T(\vec{x}',0) = \begin{cases} 0 & \text{if } |\vec{x}'| < R \\ T_0 & \text{if } |\vec{x}'| > R \end{cases} = T_0\,\theta(r'-R)$

$$T(\vec{x},t) = T_0 \int d^3x' \, \theta(r'-R) \frac{e^{-(\vec{x}-\vec{x}')^2/4\kappa t}}{(4\pi\kappa t)^{3/2}} =$$

$$= \frac{2\pi T_0}{(4\pi\kappa t)^{3/2}} \int_R^\infty dr' \, r'^2 \underbrace{\int_{-1}^{1} d(\cos\theta)\, e^{-(r^2+r'^2)/4\kappa t}\, e^{\frac{rr'}{2\kappa t}\cos\theta}}_{\frac{2\kappa t}{rr'}\left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right]}$$

$$= \frac{T_0}{r\sqrt{4\pi\kappa t}} \int_R^\infty dr' \, r' \left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right]$$

Now $\displaystyle\int_R^\infty dr' r' e^{-(r'-r)^2/4\kappa t} = \sqrt{4\kappa t}\int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du\left(\sqrt{4\kappa t}\,u + r\right)e^{-u^2} =$

$$u = \frac{r'-r}{\sqrt{4\kappa t}} \qquad r' = \sqrt{4\kappa t}\,u + r$$

$$= 4\kappa t\int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du\, u e^{-u^2} + \sqrt{4\kappa t}\, r\int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du\, e^{-u^2} =$$

$$= 2\kappa t\, e^{-(R-r)^2/4\kappa t} + \sqrt{4\kappa t}\, r\,\frac{\sqrt{\pi}}{2}\left[1 - \operatorname{erf}\left(\frac{R-r}{\sqrt{4\kappa t}}\right)\right]$$

This is for $r \le R$. The other integral is the same after replacing r with $-r$.

$$\boxed{\frac{T(r,t)}{T_0} = \frac{1}{r}\sqrt{\frac{\kappa t}{\pi}}\left[e^{-(R-r)^2/4\kappa t} - e^{-(R+r)^2/4\kappa t}\right] + 1 - \frac{1}{2}\operatorname{erf}\left(\frac{R-r}{\sqrt{4\kappa t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{R+r}{\sqrt{4\kappa t}}\right)}$$

(4) $$\nabla^2 u + k^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0$$

This is the same as the drum head problem worked out in lecture except that now the boundary condition is $u(R,\theta) = f(\theta)$ instead of $u(R,\theta) = 0$.

Look for a solution of the form

$$u(r,\theta) = \int_0^{2\pi} f(\theta')\, G(r,\theta;\theta')\, d\theta'$$

(i) $G(r,\theta;\theta')$ must satisfy the original differential equation

(ii) $G(R,\theta;\theta') = \delta(\theta-\theta')$ so that $u(R,\theta) = f(\theta)$

The general solution - without applying boundary conditions and which is finite at $r=0$ - is

$$G(r,\theta;\theta') = \sum_{n=0}^{\infty} J_n(kr)\left[A_n(\theta')\, e^{in\theta} + B_n(\theta')\, e^{-in\theta}\right]$$

$$G(R,\theta;\theta') = \sum_{n=0}^{\infty} J_n(kR)\left[A_n(\theta')\, e^{in\theta} + B_n(\theta')\, e^{-in\theta}\right]$$

$$= \delta(\theta-\theta')$$

Let's choose the representation

$$\delta(\theta-\theta') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta')} = \lim_{M\to\infty} \frac{1}{2\pi}\sum_{m=-M}^{M} e^{im(\theta-\theta')}$$

But to use this we need to sum over both positive and negative n so we write

$$G(r,\theta;\theta') = \sum_{n=-\infty}^{\infty} C_n(\theta')\, J_n(kr)\, e^{in\theta}$$

where $J_{-n}(x) = (-1)^n J_n(x)$.

$$G(R,\theta;\theta') = \sum_{n=-\infty}^{\infty} J_n(kR)\, C_n(\theta')\, e^{in\theta}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')}$$

$$\Rightarrow \quad C_n(\theta') = \frac{e^{-in\theta'}}{2\pi\, J_n(kR)}$$

$$\boxed{G(r,\theta;\theta') = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{J_n(kr)}{J_n(kR)}\, e^{in(\theta-\theta')}}$$

This can also be written as

$$G(r,\theta;\theta') = \frac{1}{2\pi}\frac{J_0(kr)}{J_0(kR)} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{J_n(kr)}{J_n(kR)} \cos\left[n(\theta-\theta')\right]$$

2.12 HW 12

Local contents

2.12.1 HW 12 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Homework 12 due Monday May 6. Show all work. Use of Mathematica, MatLab, or similar software is not allowed.

1. (5 pts) Consider the following two elements of S_5

$$\begin{aligned} g_1 &= [54123] \\ g_2 &= [21534] \end{aligned}$$

Find a third element g of this group such that $g^{-1}g_1g = g_2$.

2. (5 pts) Do the following matrices form a group?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $z = e^{i2\pi/3}$. If not, add the minimum number of 2x2 matrices to form a group. Then make a list of all possible subgroups.

3. (5 pts) The Lorentz transformation with velocity v along the x axis is described by

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = M(v) \begin{pmatrix} x \\ t \end{pmatrix} \quad \text{where} \quad M(v) = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$$

Show that the product of two such Lorentz transformations is again a Lorentz transformation, i.e. $M(v_2)M(v_1) = M(v_{12})$ and find v_{12}. Using this result, show that these transformations form a group.

4. (10 pts) Using $[X_i, X_j] = c_{ij}^k X_k$ where c_{ij}^k are the structure constants and a summation over k is implied
(a) Show that $c_{ji}^k = -c_{ij}^k$.
(b) Prove the Jacobi identity

$$[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] = 0$$

(c) Show that the Jacobi identity implies

$$c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0$$

Conditions (a) and (c) are the only conditions on the structure constants. Any set of real numbers c_{ij}^k obeying these two conditions defines a Lie algebra.

2.12.2 Problem 1

Problem Consider the following two elements of S_5

$$g_1 = [54123]$$
$$g_2 = [21534]$$

Find a third element g of this group such that $g^{-1}g_1g = g_2$

Solution

When $g^{-1}xg = y$, we say that y is conjugate to x using g.

$$\begin{aligned} gg^{-1}g_1g &= gg_2 \\ g_1g &= gg_2 \end{aligned} \tag{1}$$

But the class of conjugate pairs is symmetric. This means that

$$\begin{aligned} g^{-1}g_2g &= g_1 \\ gg^{-1}g_2g &= gg_1 \\ g_2g &= gg_1 \end{aligned} \tag{2}$$

We have two equations (1,2). Let us now apply g_1, g_2 on them. Let $g = [abcde]$ and the goal is to determine the unknowns a, b, c, d, e. Equation (1) becomes

$$\begin{aligned} [54123]\,[abcde] &= [abcde]\,[21534] \\ [edabc] &= [abcde]\,[21534] \end{aligned} \tag{1A}$$

Similarly for (2)

$$\begin{aligned} [21534]\,[abcde] &= [abcde]\,[54123] \\ [baecd] &= [abcde]\,[54123] \end{aligned} \tag{2A}$$

OK, this is some progress. But how are we going to find a, b, c, d, e?. Let use try $\underline{a = 1}$ and see what we get. If $a = 1$ then (1A) implies $e = 2$ and (2A) implies $b = 5$. Now, if $b = 5$ then (1A) gives $d = 4$ and (2A) gives $a = 3$. Which is conflict with our assumption that $a = 1$ we started with.

Let us next assume that $\underline{a = 2}$ and see if we get a conflict or not. If $a = 2$ then (1A) gives $e = 1$ and (2A) gives $b = 4$. Now, if $b = 4$ then (1A) gives $d = 3$ and (2A) gives $a = 2$. Good no conflict so far. Now taking $d = 3$ then (1A) gives $b = 5$, which is a conflict of what we found so far. So our starting guess of $a = 2$ is not correct.

Let us next assume that $\underline{a = 3}$ and see if we get a conflict or not. If $a = 3$ then (1A) gives $e = 5$ and (2A) gives $b = 1$. Now using $b = 1$ then (1A) gives $d = 2$ and (2A) gives $a = 5$, which is conflict with our assumption that $a = 3$.

Let us next assume that $\underline{a = 4}$ and see if we get a conflict or not. If $a = 4$ then (1A) gives $e = 3$ and (2A) gives $b = 2$. Now using $b = 2$ then (1A) gives $d = 1$ and (2A) gives $a = 4$. Good. No conflict so far. So far we found $a, b, e, d = 4, 2, 3, 1$. It must mean this case that $c = 5$ since it it only entry left. Let us check if this works or not.

From above we have a candidate element to check which is

$$g = [42513]$$

Trying it on (1,2). From (1)

$$\begin{aligned} g_1 g &= g g_2 \\ [54123]\,[42513] &= [42513]\,[21534] \\ [31425] &= [31425] \end{aligned}$$

OK. Let us check (2)

$$\begin{aligned} g_2 g &= g g_1 \\ [21534]\,[42513] &= [42513]\,[54123] \\ [24351] &= [24351] \end{aligned}$$

Verified. Hence one element is $\underline{g = [42513]}$.

This means that

$$[42513]^{-1}\,[54123]\,[42513] = [21534]$$

2.12.3 Problem 2

Do the following matrices for a group?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $z = e^{i\frac{2\pi}{3}}$. If not, add the minimum number of 2×2 matrices to form a group. Then make a list of all possible subgroups.

Solution

The group G with elements g_i must have the following properties (using matrix multiplication as the binary operation $\circ$)

1. $g_i \circ g_j$ is also an element in the group G
2. Binary operation is associative: $\left(g_i \circ g_j\right) \circ g_k = g_i \circ \left(g_j \circ g_k\right)$
3. There is element I called the identity element such that $I \circ g_i = g_i \circ I = g_i$ for all $g_i \in G$
4. Each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$

Checking the first property. Let $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then since g_1 is the identity element, all products with it will also be in G. Looking at products with g_2

$$
\begin{aligned}
g_2 g_3 &= \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}
\end{aligned}
$$

But $\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$ is not in G. Hence it is not a group since not closed under the matrix multiplication.

Adding this as new element and calling it g_4

$$
g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}
$$

But now we see that

$$
g_2 g_4 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}
$$

Is not in G. Calling the g_5.

$$
g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}
$$

Check again if closed

$$
\begin{aligned}
g_2 g_5 &= \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i2\pi} \\ e^{i2\pi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3
\end{aligned}
$$

Which is in G. Now checking all products with g_3 to see if they are in G.

$$
g_3 g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5
$$

Which is in G. And

$$
g_3 g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}
$$

But this is not in G. Adding the above as new element g_6

$$
g_6 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}
$$

Checking again from the start that the group we have now is closed, which now contains $g_1, g_2, g_3, g_4, g_5, g_6$.

<u>Checking all products with g_2</u>

$$g_2g_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z^4 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{2\pi}{3}(4)} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{\pi}{3}} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_2g_3 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_2g_4 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_2g_5 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_2g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} e^{i2\pi} & 0 \\ 0 & e^{i2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

<u>Checking all products with g_3</u>

$$g_3g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_3g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_3g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_3g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_3g_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

Checking all products with g_4

$$g_4g_2 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_4g_3 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_4g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_4g_5 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z^4 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{2\pi}{3}(4)} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{\pi}{3}} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_4g_6 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

Checking all products with g_5

$$g_5g_2 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^4 \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_5g_3 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_5g_4 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^4 & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_5g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_5g_6 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

Checking all products with g_6

$$g_6g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_6g_3 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_6g_4 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_6g_5 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^4 \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_6g_6 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^4 & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

Therefore the group

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

Is closed under matrix multiplication. To check the associative property, which says that $g_i \circ \left(g_j \circ g_k\right) = \left(g_i \circ g_j\right) \circ g_k$ for all i, j, k in G. But from the property of matrix multiplication, we know this property is already satisfied since the matrices are all of same order which is 2×2. Checking that There is element I called the identity element such that $I \circ g_i = g_i \circ I = g_i$, then we see that $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is clearly I in this case. Checking the last property: Each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$. In this case g_i^{-1} is the inverse.

For g_1 then g_1^{-1} is itself.

Checking g_2

$$g_2^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_2^{-1} \circ g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_3

$$g_3^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_3^{-1} \circ g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking for g_4

$$g_4^{-1} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_4^{-1} \circ g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_5

$$g_5^{-1} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_5^{-1} \circ g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_6

$$g_6^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_6^{-1} \circ g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

OK. All elements checked. Hence G is indeed a group.

$$G = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{g_1}, \overbrace{\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}}^{g_2}, \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{g_3}, \overbrace{\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}}^{g_4}, \overbrace{\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}}^{g_5}, \overbrace{\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}}^{g_6}$$

Setting up the Group table. In this table $g_1 = I$ the identity element.

$\circ$	I	g_2	g_3	g_4	g_5	g_6
I	I	g_2	g_3	g_4	g_5	g_6
g_2	g_2	g_6	g_4	g_5	g_3	I
g_3	g_3	g_5	I	g_6	g_2	g_4
g_4	g_4	g_3	g_2	I	g_6	g_5
g_5	g_5	g_4	g_6	g_2	I	g_3
g_6	g_6	I	g_5	g_3	g_4	g_2

Now we need to find all subgroups. By Lagrange theorem, we know for finite group such as G above, all subgroups are of order that divides the order of G. This means the order of the subgroups (if they exist) must be 2 or 3. (not counting order 1 which is just I and order 6 which is the group G itself).

Let us consider possible subgroups of order 2 first. Since subgroup must include the identity element $g_1 = I$, then all possible subgroups of order 2 are the following

$$[I, g_2], [I, g_3], [I, g_4], [I, g_5], [I, g_6]$$

Clearly each one of these is closed under $\circ$. Since $I \circ g_i = g_i \circ I = g_i \in G_{sub}$. But when checking for the property that each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$, then this fails unless each element is the same as its inverse. From earlier we found that

$$g_3^{-1} = g_3$$
$$g_4^{-1} = g_4$$
$$g_5^{-1} = g_5$$

Only. This implies that out of the above 6 candidate subgroups of order 2 only the following are subgroups

$$[I, g_3], [I, g_4], [I, g_5]$$

We found 3 subgroups so far. Now we need to consider all possible subgroups of order 3. Candidates are

$$[I, g_2, g_3], [I, g_2, g_4], [I, g_2, g_5], [I, g_2, g_6], [I, g_3, g_4], [I, g_3, g_5], [I, g_3, g_6], [I, g_4, g_5], [I, g_4, g_6], [I, g_5, g_6]$$

There are 10 candidates subgroups of order 3 above that we need to check. Easiest check is if the subgroup is closed. We know they satisfy the associative property.

<u>Checking $[I, g_2, g_3]$</u>

$$g_2 \circ g_3 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_4$$

Not closed.

<u>Checking $[I, g_2, g_4]$</u>

$$g_2 \circ g_4 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_5$$

Not closed.

<u>Checking $[I, g_2, g_5]$</u>

$$g_2 \circ g_5 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_3$$

Not closed.

Checking $[I, g_2, g_6]$

$$g_2 \circ g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = I$$

$$g_6 \circ g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = I$$

Closed. Associativity is met since these are matrices of same order. Let check inverse property: Each subgroup element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$. In this case g_i^{-1} is the inverse matrix.

For g_2

$$g_2^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

And $g_2^{-1} \circ g_2 = I$. OK. And

$$g_6^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

And $g_6^{-1} \circ g_6 = I$. OK. Therefore $[I, g_2, g_6]$ is indeed a subgroup.

Checking $[I, g_3, g_4]$

$$g_3 \circ g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

But g_6 is not in this subgroup. Hence not closed.

Checking $[I, g_3, g_5]$

$$g_3 \circ g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

But g_2 is not in this subgroup. Hence not closed.

Checking $[I, g_3, g_6]$

$$g_3 \circ g_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

But g_4 is not in this subgroup. Hence not closed.

Checking $[I, g_4, g_5]$

$$g_4 \circ g_5 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

But g_5 is not in this subgroup. Hence not closed.

Checking $[I, g_4, g_6]$

$$g_4 \circ g_6 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

But g_5 is not in this subgroup. Hence not closed.

Checking $[I, g_5, g_6]$

$$g_5 \circ g_6 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

But g_3 is not in this subgroup. Hence not closed.

All subgroups of order 3 are checked. Therefore the following are the subgroups found. There are 4 in total

$$[I, g_3], [I, g_4], [I, g_5], [I, g_2, g_6]$$

Or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

2.12.4 Problem 3

The Lorentz transformation with velocity v along the x axis is described by

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = M(v) \begin{pmatrix} x \\ t \end{pmatrix}$$

Where $M(v) = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$. Show that the product of two such transformations is again a Lorentz transformation. i.e. $M(v_2) M(v_1) = M(v_{12})$ and find v_{12}. Using this result, show that these transformations form a group.

solution

The following diagram is used to help in understanding what we are trying to show.

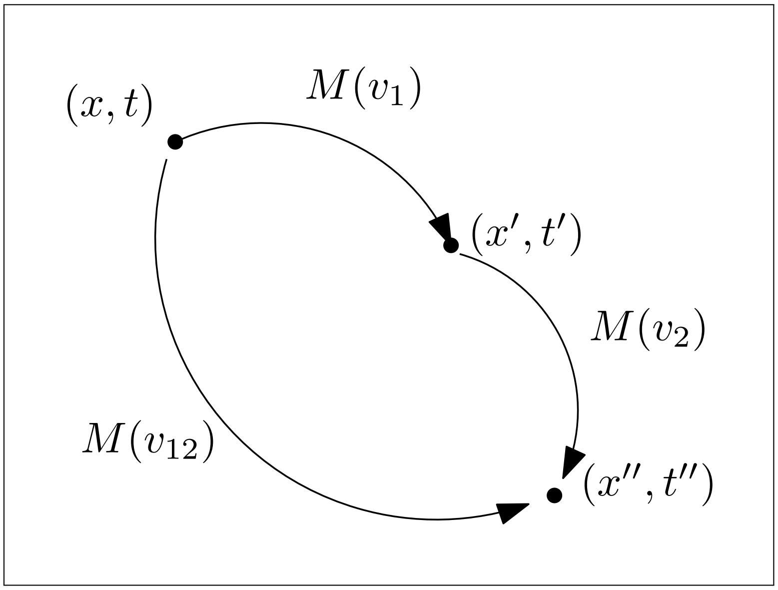


Figure 2.43: Lorentz transformations involved

Given

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = M(v_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

And

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_2) \begin{pmatrix} x' \\ t' \end{pmatrix}$$

We need to show that, with the help of the diagram above, that

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_2) \begin{pmatrix} x' \\ t' \end{pmatrix} = M(v_2) M(v_1) \begin{pmatrix} x \\ t \end{pmatrix} = M(v_{12}) \begin{pmatrix} x \\ t \end{pmatrix}$$

So we need to find $M(v_{12})$ and see if it is a Lorentz transformation also. In other words, to see if $M(v_{12})$ has the form of $\frac{1}{\sqrt{1-v_{12}^2}} \begin{pmatrix} 1 & v_{12} \\ v_{12} & 1 \end{pmatrix}$ and need to find what v_{12} is. Starting by finding $M(v_2)$. Given that

$$\begin{aligned} \begin{pmatrix} x' \\ t' \end{pmatrix} &= M(v_1) \begin{pmatrix} x \\ t \end{pmatrix} \\ &= \frac{1}{\sqrt{1-v_1^2}} \begin{pmatrix} 1 & v_1 \\ v_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \\ &= \frac{1}{\sqrt{1-v_1^2}} \begin{pmatrix} x + v_1 t \\ v_1 x + t \end{pmatrix} \end{aligned}$$

The above gives

$$x' = \frac{1}{\sqrt{1-v_1^2}}\left(x+v_1 t\right)$$
$$t' = \frac{1}{\sqrt{1-v_1^2}}\left(v_1 x+t\right)$$

Applying the transformation again on the above result gives

$$\begin{aligned}
\begin{pmatrix} x'' \\ t'' \end{pmatrix} &= M\left(v_2\right)\begin{pmatrix} x' \\ t' \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\begin{pmatrix} 1 & v_2 \\ v_2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-v_1^2}}\left(x+v_1 t\right) \\ \frac{1}{\sqrt{1-v_1^2}}\left(v_1 x+t\right) \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\begin{pmatrix} \frac{1}{\sqrt{1-v_1^2}}\left(x+v_1 t\right)+\frac{v_2}{\sqrt{1-v_1^2}}\left(v_1 x+t\right) \\ \frac{v_2}{\sqrt{1-v_1^2}}\left(x+v_1 t\right)+\frac{1}{\sqrt{1-v_1^2}}\left(v_1 x+t\right) \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\frac{1}{\sqrt{1-v_1^2}}\begin{pmatrix} \left(x+v_1 t\right)+v_2\left(v_1 x+t\right) \\ v_2\left(x+v_1 t\right)+\left(v_1 x+t\right) \end{pmatrix} \\
&= \frac{1}{\sqrt{\left(1-v_2^2\right)\left(1-v_1^2\right)}}\begin{pmatrix} x+v_1 t+v_2 v_1 x+v_2 t \\ v_2 x+v_2 v_1 t+v_1 x+t \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_1^2-v_2^2+v_2^2 v_1^2}}\begin{pmatrix} x\left(1+v_2 v_1\right)+t\left(v_1+v_2\right) \\ x\left(v_2+v_1\right)+t\left(1+v_2 v_1\right) \end{pmatrix} \\
&= \frac{\left(1+v_2 v_1\right)}{\sqrt{1-v_1^2-v_2^2+v_2^2 v_1^2}}\begin{pmatrix} x+t\frac{\left(v_1+v_2\right)}{\left(1+v_2 v_1\right)} \\ x\frac{\left(v_2+v_1\right)}{\left(1+v_2 v_1\right)}+t \end{pmatrix} \\
&= \frac{1}{\sqrt{\frac{1-v_1^2-v_2^2+v_2^2 v_1^2}{\left(1+v_2 v_1\right)^2}}}\begin{pmatrix} 1 & \frac{\left(v_1+v_2\right)}{\left(1+v_2 v_1\right)} \\ \frac{\left(1+v_2 v_1\right)}{\left(1+v_2 v_1\right)} & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}
\end{aligned}$$

But

$$\frac{1-v_1^2-v_2^2+v_2^2 v_1^2}{\left(1+v_2 v_1\right)^2} = \frac{\left(1+v_1 v_2\right)^2-\left(v_1+v_2\right)^2}{\left(1+v_2 v_1\right)^2}$$

Therefore

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{(v_1+v_2)^2}{(1+v_2v_1)^2}}} \begin{pmatrix} 1 & \frac{(v_1+v_2)}{(1+v_2v_1)} \\ \frac{(1+v_2v_1)}{(1+v_2v_1)} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{1}$$

Now it is in the form of Lorentz transformation. $\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_{12}) \begin{pmatrix} x \\ t \end{pmatrix}$. Comparing this (1) shows that

$$M(v_{12}) = \frac{1}{\sqrt{1-\frac{(v_1+v_2)^2}{(1+v_2v_1)^2}}} \begin{pmatrix} 1 & \frac{(v_1+v_2)}{(1+v_2v_1)} \\ \frac{(1+v_2v_1)}{(1+v_2v_1)} & 1 \end{pmatrix}$$

But $M(v) = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$. By comparing to the above shows that

$$v_{12} = \frac{v_1 + v_2}{1 + v_2 v_1}$$

Therefore what we did above is apply Lorentz transformation again $M(v_2)$ on result we obtained from $M(v_1)$ and we obtained a result which also a valid Lorentz transformation. This means the group is closed under this transformation. We need to show associativity. Which means

$$\begin{aligned} M(v_3)[M(v_2)M(v_1)] &= [M(v_3)M(v_2)]M(v_1) \\ M(v_3)M(v_{12}) &= M(v_{23})M(v_1) \end{aligned} \tag{3}$$

But we found from the above that $M(v_2)M(v_1) = M(v_{12})$ results in $v_{12} = \frac{v_1+v_2}{1+v_2v_1}$. Therefore we can conclude that left side of (2) which is $M(v_3)M(v_{12})$ will also result in

$$v_{321} = \frac{v_{12} + v_3}{1 + v_3 v_{12}}$$

But $v_{12} = \frac{v_1+v_2}{1+v_2v_1}$, therefore the above simplifies to

$$\begin{aligned} v_{321} &= \frac{\frac{v_1+v_2}{1+v_2v_1} + v_3}{1 + v_3 \frac{v_1+v_2}{1+v_2v_1}} \\ &= \frac{v_1 + v_2 + v_3(1 + v_2v_1)}{1 + v_2v_1 + v_3v_1 + v_2} \\ &= \frac{v_1 + v_2 + v_3 + v_3v_2v_1}{1 + v_2v_1 + v_3v_1 + v_3v_2} \end{aligned} \tag{3A}$$

And the right side of (3) which is $M(v_{23})M(v_1)$ also gives

$$v_{123} = \frac{v_1 + v_{23}}{1 + v_1 v_{23}}$$

But again, $v_{23} = \frac{v_2+v_3}{1+v_3v_2}$ and the above simplifies to

$$\begin{aligned}
v_{123} &= \frac{v_1 + \frac{v_2+v_3}{1+v_3v_2}}{1 + v_1\frac{v_2+v_3}{1+v_3v_2}} \\
&= \frac{v_1(1+v_3v_2)+v_2+v_3}{1+v_3v_2+v_1v_2+v_1v_3} \\
&= \frac{v_1+v_3v_2v_1+v_2+v_3}{1+v_3v_2+v_1v_2+v_1v_3} \qquad \text{(3B)}
\end{aligned}$$

By comparing (3A) and (3B) we see they are the same. Hence associativity is satisfied. Next we need to check the inverse property. What this means that for each $M(v_i)$ there exist $M^{-1}(v_i)$ such that $M(v_i)M^{-1}(v_i) = I$. where the identity in this case is $M(0) = I$ since

$$\begin{aligned}
M(0) &= \frac{1}{\sqrt{1-0}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Since $M(v) = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$ then $M(-v) = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$ and

$$\begin{aligned}
M(v)M(-v) &= \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\
&= \frac{1}{1-v^2}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\
&= \frac{1}{1-v^2}\begin{pmatrix} 1-v^2 & 0 \\ 0 & 1-v^2 \end{pmatrix} \\
&= \frac{1-v^2}{1-v^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= M(0)
\end{aligned}$$

Which is the identity. Hence we showed that for each $M(v_i)$ there exists an inverse $M(-v_i)$. All properties of group have been satisfied. Hence the given Lorentz transformation forms a group.

2.12.5 Problem 4

Using $[X_i, X_j] = c_{ij}^k X_k$ where c_{ij}^k are the structure constants and a summation over k is implied.

1. Show that $c_{ji}^k = -c_{ij}^k$

2. Prove the Jacobi identity $\left[\left[X_i, X_j\right], X_k\right] + \left[\left[X_j, X_k\right], X_i\right] + \left[\left[X_k, X_i\right], X_j\right] = 0$
3. Show that the Jacobi identity implies $c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0$

Conditions (1,3) are the only conditions on the structure constants. Any set of real numbers c_{ij}^k obeying these two conditions defines a Lie algebra.

solution

2.12.5.1 Part (1)

The commutator of 2 generators (X_i, X_j) is linear combination of the generators. Hence

$$\left[X_i, X_j\right] = X_i X_j - X_j X_i = c_{ij}^k X_k \tag{1}$$

Therefore, we also have

$$\left[X_j, X_i\right] = X_j X_i - X_i X_j = c_{ji}^k X_k \tag{2}$$

Adding (1) and (2) gives

$$\begin{aligned}
\left(X_i X_j - X_j X_i\right) + \left(X_j X_i - X_i X_j\right) &= c_{ij}^k X_k + c_{ji}^k X_k \\
0 &= X_k \left(c_{ij}^k + c_{ji}^k\right) \\
0 &= c_{ij}^k + c_{ji}^k \\
c_{ij}^k &= -c_{ji}^k
\end{aligned}$$

2.12.5.2 Part (2)

Applying the commutator relation

$$\left[X_i, X_j\right] = X_i X_j - X_j X_i$$

Let LHS of the Jacobi identity be Δ. Applying the above to each term in Δ gives

$$\Delta = \left[\left(X_i X_j - X_j X_i\right), X_k\right] + \left[\left(X_j X_k - X_k X_j\right), X_i\right] + \left[\left(X_k X_i - X_i X_k\right), X_j\right] \tag{1}$$

We want to show that $\Delta = 0$. Now, applying commutator relation again each term of the above gives for the first term

$$\begin{aligned}
\left[\left(X_i X_j - X_j X_i\right), X_k\right] &= \left(X_i X_j - X_j X_i\right) X_k - X_k \left(X_i X_j - X_j X_i\right) \\
&= X_i X_j X_k - X_j X_i X_k - X_k X_i X_j + X_k X_j X_i
\end{aligned} \tag{2}$$

And for the second term in (1)

$$\begin{aligned}
\left[\left(X_j X_k - X_k X_j\right), X_i\right] &= \left(X_j X_k - X_k X_j\right) X_i - X_i \left(X_j X_k - X_k X_j\right) \\
&= X_j X_k X_i - X_k X_j X_i - X_i X_j X_k + X_i X_k X_j
\end{aligned} \tag{3}$$

And for the third term in (1)

$$\begin{aligned}
\left[\left(X_k X_i - X_i X_k\right), X_j\right] &= \left(X_k X_i - X_i X_k\right) X_j - X_j \left(X_k X_i - X_i X_k\right) \\
&= X_k X_i X_j - X_i X_k X_j - X_j X_k X_i + X_j X_i X_k
\end{aligned} \tag{4}$$

Substituting (2,3,4) back into (1) gives

$$\begin{aligned}\Delta &= \left(X_iX_jX_k - X_jX_iX_k - X_kX_iX_j + X_kX_jX_i\right)\\ &+ \left(X_jX_kX_i - X_kX_jX_i - X_iX_jX_k + X_iX_kX_j\right)\\ &+ \left(X_kX_iX_j - X_iX_kX_j - X_jX_kX_i + X_jX_iX_k\right)\end{aligned}$$

We see that all terms cancel each other. Hence $\Delta = 0$ which is what we wanted to show.

2.12.5.3 Part (3)

The Jacobi identity is

$$\left[\left[X_i, X_j\right], X_k\right] + \left[\left[X_j, X_k\right], X_i\right] + \left[\left[X_k, X_i\right], X_j\right] = 0$$

Applying $\left[X_i, X_j\right] = c_{ij}^{l} X_l$ on each term in the LHS above gives, where the summation index l is used in each term, which is OK to do since the terms are separated from each others

$$\begin{aligned}0 &= \left[c_{ij}^{l} X_l, X_k\right] + \left[c_{jk}^{l} X_l, X_i\right] + \left[c_{ki}^{l} X_l, X_j\right]\\ &= c_{ij}^{l}\left[X_l, X_k\right] + c_{jk}^{l}\left[X_l, X_i\right] + c_{ki}^{l}\left[X_l, X_j\right]\end{aligned}$$

Now, applying $\left[X_i, X_j\right] = c_{ij}^{m} X_m$ again on each term above and now using m as the summation index gives

$$\begin{aligned}0 &= c_{ij}^{l} c_{lk}^{m} X_m + c_{jk}^{l} c_{li}^{m} X_m + c_{ki}^{l} c_{lj}^{m}\\ &= \left(c_{ij}^{l} c_{lk}^{m} + c_{jk}^{l} c_{li}^{m} + c_{ki}^{l} c_{lj}^{m}\right) X_m\\ &= c_{ij}^{l} c_{lk}^{m} + c_{jk}^{l} c_{li}^{m} + c_{ki}^{l} c_{lj}^{m}\end{aligned}$$

Which is what the problem asked to show.

2.12.6 Key solution for HW 12

① $g_1 = [5\,4\,1\,2\,3]$ $g_2 = [2\,1\,5\,3\,4]$

What g will give $g^{-1} g_1 g = g_2$ or $g_1 g = g g_2$?

Answer: $\boxed{g = [3\,1\,4\,2\,5]}$

check:

$g_1\, g = [5\,4\,1\,2\,3][3\,1\,4\,2\,5] = [5\,2\,3\,1\,4]$

$g\, g_2 = [3\,1\,4\,2\,5][2\,1\,5\,3\,4] = [5\,2\,3\,1\,4]$

(2) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$

Using the fact that $z^3 = 1$ we have

$A_2 A_2 = \boxed{\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} \equiv A_3}$ $A_1 A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$A_1 A_2 = \boxed{\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \equiv A_4}$ $A_2 A_1 = \boxed{\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \equiv A_5}$

Let's make a table.

$A_1 A_1 = I$	$A_2 A_1 = A_5$	$A_3 A_1 = A_4$	$A_4 A_1 = A_3$	$A_5 A_1 = A_2$
$A_1 A_2 = A_4$	$A_2 A_2 = A_3$	$A_3 A_2 = I$	$A_4 A_2 = A_5$	$A_5 A_2 = A_1$
$A_1 A_3 = A_5$	$A_2 A_3 = I$	$A_3 A_3 = A_2$	$A_4 A_3 = A_1$	$A_5 A_3 = A_4$
$A_1 A_4 = A_1$	$A_2 A_4 = A_1$	$A_3 A_4 = A_5$	$A_4 A_4 = I$	$A_5 A_4 = A_3$
$A_1 A_5 = A_3$	$A_2 A_5 = A_4$	$A_3 A_5 = A_1$	$A_4 A_5 = A_2$	$A_5 A_5 = I$

The inverses are: $A_2^{-1} = A_3$, $A_1^{-1} = A_1$, $A_4^{-1} = A_4$, $A_5^{-1} = A_5$

Hence this is a group.

The only <u>proper</u> subgroups are

$\{I, A_1\}$ $\{I, A_4\}$ $\{I, A_5\}$ $\{I, A_2, A_3\}$

③ $$M(v_2)M(v_1) = \frac{1}{\sqrt{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1 & v_2 \\ v_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & v_1 \\ v_1 & 1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1+v_1v_2 & v_1+v_2 \\ v_1+v_2 & 1+v_1v_2 \end{pmatrix}$$

$$= \sqrt{\frac{(1+v_1v_2)^2}{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1 & \frac{v_1+v_2}{1+v_1v_2} \\ \frac{v_1+v_2}{1+v_1v_2} & 1 \end{pmatrix}$$ Define $\boxed{v_{12} = \frac{v_1+v_2}{1+v_1v_2}}$

$$1 - v_{12}^2 = 1 - \frac{(v_1+v_2)^2}{(1+v_1v_2)^2} = \frac{(1+2v_1v_2+v_1^2v_2^2) - (v_1^2+2v_1v_2+v_2^2)}{(1+v_1v_2)^2}$$

$$= \frac{1+v_1^2v_2^2 - v_1^2 v_2^2}{(1+v_1v_2)^2} = \frac{(1-v_1^2)(1-v_2^2)}{(1+v_1v_2)^2}$$

or $$\frac{1}{1-v_{12}^2} = \frac{(1+v_1v_2)^2}{(1-v_1^2)(1-v_2^2)}$$

Thus $\boxed{M(v_2)M(v_1) = M(v_{12})}$ with v_{12} as above.

This is closure, Associativity follows from matrix multiplication. $I = M(0)$ is the identity.

The inverse for each element is $M^{-1}(v) = M(-v)$.

(4) $[X_i, X_j] = c_{ij}^k X_k$

$$\| $$

$X_i X_j - X_j X_i$

(a) $[X_i, X_j] = -[X_j, X_i] = -c_{ji}^k$

$\Rightarrow \boxed{c_{ji}^k = -c_{ij}^k}$

(b) $[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] =$

$= [X_i X_j - X_j X_i, X_k] + [X_j X_k - X_k X_j, X_i] + [X_k X_i - X_i X_k, X_j]$

$= X_i X_j X_k - X_j X_i X_k - X_k X_i X_j + X_k X_j X_i$

$+ X_j X_k X_i - X_k X_j X_i - X_i X_j X_k + X_i X_k X_j$

$+ X_k X_i X_j - X_i X_k X_j - X_j X_k X_i + X_j X_i X_k$

These all cancel pairwise to give 0.

(c) $[\underbrace{[X_i, X_j]}_{c_{ij}^l X_l}, X_k] + [\underbrace{[X_j, X_k]}_{c_{jk}^l X_l}, X_i] + [\underbrace{[X_k, X_i]}_{c_{ki}^l X_l}, X_j] = 0$

$$0 = C_{ij}^{\ l}\underbrace{[X_l, X_k]}_{C_{lk}^{m} X_m} + C_{jk}^{l}\underbrace{[X_l, X_i]}_{C_{li}^{m} X_m} + C_{ki}^{l}\underbrace{[X_l, X_j]}_{C_{lj}^{m} X_m}$$

$$= \left(C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} \right) X_m$$

Since none of the X_m are zero we must have

$$\boxed{C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} = 0}$$

Chapter 3

Exams

Local contents

3.1 Exam 1

Local contents

3.1.1 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Quiz 1: Wednesday 27 February 2018 from 1:25 to 2:15 pm.

1. (5 pts) Solve the differential equation

$$y' = \frac{x-y}{x+y}$$

2. (5 pts) Find the harmonic conjugate to the function

$$u = \frac{y}{x^2+y^2}$$

3. (5 pts) Evaluate

$$\int_C \frac{z+1}{z^2-2z}\, dz$$

where the contour C is the unit circle.

4. (5 pts) Find the poles and their residues of the function

$$\frac{\mathrm{e}^z}{z^2+\pi^2}$$

Figure 3.1: Questions, exam 1

3.2 Exam 2

Local contents

3.2.1 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Quiz 2: Wednesday 10 April 2019 from 1:25 to 2:15 pm. Only pencil or pen and paper are allowed.

1. (5 pts) Evaluate the integral

$$f(x) = \int_0^{2\pi} d\theta\, \delta(\sin^2\theta - x)$$

2. (5 pts) Evaluate the integral

$$g(x) = \int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x+a)^2 + b^2}$$

where a and b are real constants.

3. (5 pts) A function $f(x)$ is defined on the interval $-L \le x \le L$ such that it equals $1/(2\epsilon)$ when $|x| < \epsilon$ and is zero otherwise, with $\epsilon < L$. (In the limit $\epsilon \to 0$ this represents a δ-function, but do not take this limit.) What is its Fourier ~~transform~~ series?

4. (5 pts) Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

What is the eigenvalue corresponding to the eigenvector $(1, 0, -1)$?

Figure 3.2: Questions, exam 2

3.3 Final exam

Local contents

3.3.1 questions

UNIVERSITY OF MINNESOTA
School of Physics and Astronomy

Physics 5041 – Mathematical Methods for Physics

Final Exam: Thursday 9 May 2019 from 12:30 to 3:30 pm. Only pencil or pen and paper are allowed.

1. (5 pts) Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$$

2. (8 pts) The second order differential equation $x^2 u'' + f(x)u' + g(x)u = 0$ with f and g real functions is satisfied by $u = x^m \exp(ix^n)$. What are $f(x)$ and $g(x)$?

3. (5 pts) The metric for the surface of a globe of the earth can be read off from the distance formula $ds^2 = a^2 d\lambda^2 + a^2 \cos^2\lambda \, d\phi^2$ where λ is the latitude and ϕ is the longitude. The metric of a flat map of the world with Cartesian coordinates x and y would be $ds^2 = dx^2 + dy^2$. However, this does not properly represent the geometry of the globe. Therefore we make a cylindrical projection defined by $x = a\phi$, $y = a\sin\lambda$. Find the metric for the x and y coordinates. Where is the distortion of the globe the greatest and where is it the least?

4. (5 pts) Evaluate the following integral by contour integration when $k^2 < 1$

$$\int_0^{2\pi} \frac{d\theta}{1 + k\cos\theta}$$

5. (5 pts) Starting with the series representation

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n}$$

prove the following identity

$$x^m \frac{d}{dx}\left[x^{-m} J_m(x)\right] = -J_{m+1}(x)$$

6. (8 pts) Let c_{ij}^k be the structure constants for a group G. Define a set of matrices by $(M_i)_{jk} = -c_{ij}^k = c_{ji}^k$, meaning the matrix M_i with rows labeled by j and columns represented by k. Show that these matrices satisfy the same commutation relations as the generators of the group. This is called the adjoint representation of the Lie algebra. It may be useful to recall a relationship that you derived in homework 12, namely $c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0$.

7. (4 pts) Consider a group consisting of the matrics

$$A = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} \qquad B = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$$

where $z = e^{i2\pi/3}$, along with the identity I. What are the characters of the three matrices? How many classes are there?